Representations of product systems over semigroups and dilations of commuting CP maps

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Abstract

We prove that every pair of commuting CP maps on a von Neumann algebra $M$ can be dilated to a commuting pair of endomorphisms (on a larger von Neumann algebra). To achieve this, we first prove that every completely contractive representation of a product system of $C^*$-correspondences over the semigroup $\mathbb{N}^2$ can be dilated to an isometric (or Toeplitz) representation.

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1 Introduction

A $C^*$-correspondence $E$ over a $C^*$-algebra $A$ is a (right) Hilbert $C^*$-module over $A$ that carries also a left action of $A$ (by adjointable operators). It is also called a Hilbert bimodule in the literature. A c.c. representation of $E$ on a Hilbert space $H$ is a pair $(\sigma, T)$ where $\sigma$ is a representation of $A$ on

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$H$ and $T : E \to B(H)$ is a completely contractive linear map that is also a bimodule map (that is, $T(a \cdot \xi \cdot b) = \sigma(a)T(\xi)\sigma(b)$ for $a, b \in A$ and $\xi \in E$). The representation is said to be isometric (or Toeplitz) if $T(\xi)^*T(\eta) = \sigma(\langle \xi, \eta \rangle)$ for every $\xi, \eta \in E$.

In [28], Pimsner associated with such a correspondence two $C^*$-algebras ($\mathcal{O}(E)$ and $\mathcal{T}(E)$) with certain universal properties. In [20] we studied a universal operator algebra (called the tensor algebra) $T_{+}(E)$ associated with such a correspondence.

A product system $X$ of $C^*$-correspondences over a semigroup $P$ is, roughly speaking, a family $\{X_s : s \in P\}$ of $C^*$-correspondences (over the same $C^*$-algebra $A$), with $X_e = A$, such that $X_s \otimes X_t$ is isomorphic to $X_{st}$ for all $s, t \in P \setminus \{e\}$ (See Section 3 for the precise definition). A c.c. (respectively, isometric) representation of $X$ is a family $\{T_s\}$ such that, for all $s \in P$, $(T_e, T_s)$ is a c.c. (respectively, isometric) representation of $X_s$ for all $s \in P$ and such that, whenever $x \in X_s$ and $y \in X_t$, $T_{st}(\theta_{s,t}(x \otimes y)) = T_s(x)T_t(y)$ (where $\theta_{s,t} : X_s \otimes X_t \to X_{st}$ is the isomorphism).

If $A = \mathbb{C}$, a $C^*$-correspondence over $A$ is simply a Hilbert space. In [1], Arveson introduced product systems of Hilbert spaces over the semigroup $\mathbb{R}_+$ (in order to study semigroups of of endomorphisms of $B(H)$). When the semigroup is not discrete, one usually assumes certain continuity or measurability conditions on the product system. Product systems of $C^*$-correspondences over $\mathbb{R}_+$ or subsemigroups of $\mathbb{R}_+$ were studied by various authors (e.g. [5], [8], [22], [18], [32], [12] and others). Of course, a single correspondence can also be thought of as a product system over the semigroup $\mathbb{N}$.

In [10], Fowler studied product systems over more general (discrete) semigroups $P$. He proved the existence of a $C^*$-algebra $\mathcal{T}(X)$ that is universal with respect to Toeplitz representations. In fact, in most of the work done on operator algebras associated with product systems of correspondences (on semigroups other than $\mathbb{N}$), the operator algebras that were studied are $C^*$-algebras. Two exceptions that we are aware of are [14] and [7]. The fast growing body of literature dealing with $C^*$-algebras associated with $k$-graphs (see [31] and the references there) can also be viewed as the study of certain product systems of correspondences over the semigroup $\mathbb{N}^k$. Among other works on product systems over semigroups, see [9], [11], [19] and [16].

In Section 3 we associate, with every product system $X$ of $C^*$-correspondences over a discrete semigroup $P$ (with a unit and left-cancellation), an operator algebra $\mathcal{T}_{+}(X)$ (called the universal tensor algebra) which is uni-
universal with respect to completely contractive representations. (See Proposition 3.2).

For the rest of the paper (Sections 4 and 5) we concentrate on the case where the semigroup $P$ is $\mathbb{N}^2$.

One of the main results of the paper (Theorem 5.13) is a dilation result for a pair of commuting (contractive, normal) completely positive maps on a von Neumann algebra $M$ (to a pair of commuting normal $\ast$-endomorphisms on a larger von Neumann algebra $R$). A special case ($M = B(H)$) was proved by Bhat in [4] but the methods used here are very different and the emphasis here is on the relationship between representations of product systems and semigroups of CP maps (as explained below).

Over the years there have been numerous studies wherein the authors dilate CP maps or semigroups of CP maps. One can find in the literature several approaches to dilation theory (of semigroups of CP maps) with different properties. For a recent account and a list of references see [2, Chapter 8]. We shall concentrate here on the dilations of the kind that proved effective in the study of CP-semigroups and $E_0$-semigroups initiated by Powers and Arveson.

Suppose $M$ is a von Neumann algebra acting on a Hilbert space $H$ and $\Theta$ is a contractive, normal, completely positive map on $M$. A quadruple $(K, R, \alpha, W)$, consisting of a Hilbert space $K$, a von Neumann algebra $R$, a $\ast$-endomorphism $\alpha$ and an isometric embedding $W$ of $H$ into $K$ will be called an endomorphic dilation of $(M, \Theta)$ if $\alpha(WW^*) = \alpha(I)WW^*$ (i.e., $WW^*$ is coinvariant under $\alpha$), $W^*RW = M$ (i.e. $M$ embeds as a corner of $R$) and, for all $a \in M$,

$$\Theta(a) = W^*\alpha(WaW^*)W.$$  

Similarly one defines an endomorphic dilation of a semigroup $\{\Theta_t : t \in P\}$ of CP maps on $M$. If the semigroup is not discrete, one usually requires that certain continuity properties of the CP-semigroup would hold also for the endomorphism semigroup dilating it.

In [3], Bhat proved that every (unital) CP-semigroup $\{\Theta_t : t \geq 0\}$ on the von Neumann algebra $B(H)$ can be dilated to a (unital) semigroup of $\ast$-endomorphisms on $B(K)$ for some larger Hilbert space $K$. For general von Neumann algebras $M$ this was proved by Bhat and Skeide in [5]. A different proof was provided in [22]. Both proofs used product systems of correspondences but in a different way. In fact, the correspondences in [5]
are over \( M \) while the correspondences in [22] are over \( M' \). They are related by “duality”. (Since we shall not need it here, we will not elaborate on this concept of duality but refer the reader to [24] or [32]).

Since the methods of this paper will use some results and ideas from [22], we shall now describe the approach taken there (for a single CP map). Before we proceed, we note that, although it was assumed in [22] that the CP maps are unital, the results we use here hold also for non unital maps.

Given a CP map \( \Theta \) on a von Neumann algebra \( M \subseteq B(H) \), we write \( M \otimes \Theta H \) for the Hilbert space obtained by the Hausdorff completion of the algebraic tensor product \( M \otimes H \) with respect to

\[
\langle a \otimes h, b \otimes k \rangle = \langle h, a^*bk \rangle_H, \quad a, b \in M, \; h, k \in H.
\]

A “typical” element of \( M \otimes \Theta H \) will be written \( a \otimes \Theta h \) and there is a natural action of \( M \) on this space where \( a \in M \) sends \( b \otimes \Theta h \) to \( ab \otimes \Theta h \) (and we write \( a \otimes I_H \) for this operator). Now set

\[
E_\Theta = \{ X : H \to M \otimes \Theta H : Xa = (a \otimes I_H)X, \; a \in M \}.
\]

As was shown in [22, Proposition 2.5], this space is, in fact, a \( W^* \)-correspondence over the von Neumann algebra \( M' \) (see Definition 2.1) and there is a natural completely contractive representation associated to it. The representation is \( (\sigma, T_\Theta) \) where \( \sigma = id \), the identity representation of \( M' \), and \( T(X) = W_\Theta X \in B(H) \) where \( W_\Theta : H \to M \otimes \Theta H \) is defined by \( W_\Theta h = I \otimes \Theta h \). One can check that \( T_\Theta \) is an injective map (and so is \( \sigma \)).

To summarize, to every (contractive, normal) CP map on \( M \) we associated a pair \( (E_\Theta, (\sigma, T_\Theta)) \) consisting of a \( W^* \)-correspondence and a completely contractive representation (and both \( \sigma \) and \( T_\Theta \) are injective).

This construction can be “reversed”. Given a \( W^* \)-correspondence \( E \) over \( M' \) and a completely contractive representation \( (\sigma, T) \) of \( E \) on \( H \) (such that the maps \( \sigma \) and \( T \) are injective), we can define a (contractive, normal) CP map on \( M \) by setting \( \Theta_T(a) = \tilde{T}(I_E \otimes a)\tilde{T}^*, \; a \in M \). (Here we use the Hilbert space \( E \otimes_\sigma H \) defined by the Hausdorff completion of the algebraic tensor product with respect to

\[
\langle \xi \otimes h, \eta \otimes k \rangle = \langle h, \sigma((\xi, \eta))k \rangle
\]

and we let \( \tilde{T} \) be the map \( \tilde{T} : E \otimes_\sigma H \to H \) defined by \( \tilde{T}(\xi \otimes h) = T(\xi)h \) and \( I_E \otimes a \) be the map sending \( \xi \otimes_\sigma h \) to \( \xi \otimes_\sigma ah \).

The two constructions are the inverse of each other up to isomorphisms of pairs \( (E, (\sigma, T)) \) (that is, an isomorphism of the correspondences that carries
one representation to the other one). One direction of this statement is \[22\] Corollary 2.23. The other direction was proved in \[25\].

Moreover, this bijection (between CP maps and pairs \((E, (\sigma, T))\)) carries \(^*-\)endomorphisms to representations that are isometric (and vice versa). (See \[22\] Proposition 2.21).

The dilation of a single CP map can then be proved combining the bijection described above with the dilation result for c.c. representations (to isometric representations) in \[20\] Theorem 3.3. For the details, see \[22\] Theorem 2.24.

In this paper we study to what extent we can apply these ideas to product systems over \(\mathbb{N}^2\) (in place of \(\mathbb{N}\)) and a pair of commuting CP maps. The first result we need is the dilation theorem for completely contractive representations of product system over \(\mathbb{N}^2\). This is achieved in Theorem 4.4. Applied to the case where \(M = \mathbb{C}\) and each “fiber” of the product system is \(\mathbb{C}\), this theorem yields Ando’s Theorem (for dilations of a pair of commuting contractions to a pair of commuting isometries). Since it is known that, in general, one cannot dilate simultaneously a commuting triple of contractions to a commuting triple of isometries (see \[27\] Chapter 5), one cannot hope to have a general isometric dilation result for representations of product systems over \(\mathbb{N}^k\) for \(k > 2\).

A consequence of Theorem 4.4 (Corollary 4.5) is that two row contractions that, in some general sense, commute with each other, can be simultaneously dilated to two isometric row contractions preserving the commutation relation. (Giving up the commutation relation, this result can be found in \[30\]. For a single row contraction, the dilation result was proved by Popescu in \[29\].)

Trying to extend the bijection described above (between CP maps and pairs \((E, (\sigma, T))\)) from the case \(P = \mathbb{N}\) to the case \(P = \mathbb{N}^2\), one runs into a problem. It turns out that one has to require that the two commuting CP maps \(\Theta\) and \(\Phi\) satisfy a stronger condition (see Definition 5.1). A pair of CP maps satisfying this condition is said to commute strongly. The condition is needed so that we can find a product system \(X_{\Theta,\Phi}\) and a representation of it that will play the role played by \((E_{\Theta}, (\sigma, T_{\Theta}))\) in the case of a single CP map \(\Theta\) (see Proposition 5.6). Assuming that this stronger condition holds, we establish the required bijection (see Proposition 5.7 and the discussion preceding it). This bijection, together with Theorem 4.4, implies that every pair of CP maps that commute strongly can be simultaneously dilated to a commuting pair of \(^*-\)endomorphisms.
However, the dilation result holds even if the CP maps commute but not strongly. In order to prove it, we first have to show that every pair of commuting CP maps can be “realized” using some representation of a product system over $\mathbb{N}^2$. This is proved in Proposition 5.11. What we lose here (if the maps do not commute strongly) is the uniqueness of the product system and the representation. Proposition 5.11 is then applied to dilate a general pair of commuting CP maps (Theorem 5.13).

As is shown in Proposition 5.15, knowing that the maps commute strongly has the additional advantage that, for each of the CP maps, the correspondences associated with the map and with its dilation are isomorphic. This was proved useful, for single CP maps, in studying the index and the curvature of a CP map in [23].

2 Preliminaries: Correspondences and representations

We begin by recalling the notions of a $C^*$-correspondence and a $W^*$-correspondence. For the general theory of Hilbert $C^*$-modules which we use, we will follow [15]. In particular, a Hilbert $C^*$-module $E$ over a $C^*$-algebra $A$ will be a right Hilbert $C^*$-module. We write $\mathcal{L}(E)$ for the algebra of continuous, adjointable $A$-module maps on $E$. It is known to be a $C^*$-algebra.

Definition 2.1

1. A $C^*$-correspondence over a $C^*$-algebra $A$ is a Hilbert $C^*$-module $E$ over $A$ endowed with the structure of a left $A$-module via a $^*$-homomorphism $\varphi_E : A \to \mathcal{L}(E)$.

2. A Hilbert $W^*$-module over a von Neumann algebra $M$ is a Hilbert $C^*$-module over $M$ that is self dual (i.e., every continuous $M$-module map from $E$ to $M$ is implemented by an element of $E$).

3. A $W^*$-correspondence over a von Neumann algebra $M$ is a Hilbert $W^*$-module $E$ that is a $C^*$-correspondence over $M$ and the map $\varphi_E$ is a normal $^*$-homomorphism. (When $E$ is a Hilbert $W^*$-module, $\mathcal{L}(E)$ is known to be a von Neumann algebra [26]).

When dealing with a specific $C^*$-correspondence $E$ it will be convenient to write $\varphi$ (instead of $\varphi_E$) or even to suppress it and write $a\xi$ or $a \cdot \xi$ for $\varphi(a)\xi$. 
If $E$ and $F$ are $C^*$-correspondences over $A$, then the balanced tensor product $E \otimes_A F$ is a $C^*$-correspondence over $A$. It is defined as the Hausdorff completion of the algebraic balanced tensor product with the internal inner product given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \varphi_F(\langle \xi_1, \xi_2 \rangle_E) \eta_2 \rangle_F$$

for all $\xi_1, \xi_2 \in E$ and $\eta_1, \eta_2 \in F$. The left and right actions of $a \in M$ are defined by

$$\varphi_{E \otimes F}(a)(\xi \otimes \eta)b = \varphi_E(a)\xi \otimes \eta b$$

for all $a, b \in M$, $\xi \in E$ and $\eta \in F$.

If $E$ and $F$ are $W^*$-correspondences over the von Neumann algebra $M$, the tensor product $E \otimes_M F$ is understood to be the self-dual extension ([26]) of that Hausdorff completion. The left and right actions are as in (2) and, since $\varphi_{E \otimes F}$ is now a normal $^*$-homomorphism, $E \otimes_M F$ is a $W^*$-correspondence.

**Definition 2.2** An isomorphism of $C^*$-correspondences (or $W^*$-correspondences) $E$ and $F$ is a surjective, bimodule map that preserves the inner products. We write $E \cong F$ if such an isomorphism exists.

If $E$ is a $C^*$-correspondence over $A$ and $\sigma$ is a representation of $A$ on a Hilbert space $H$ (which is assumed to be normal if $E$ is a $W^*$-correspondence) then $E \otimes_\sigma H$ is the Hilbert space obtained as the Hausdorff completion of the algebraic tensor product with respect to $\langle \xi \otimes h, \eta \otimes k \rangle = \langle h, \sigma(\langle \xi, \eta \rangle_E)k \rangle_H$. Given an operator $X \in \mathcal{L}(E)$ and an operator $S \in \sigma(M)'$, the map $\xi \otimes h \mapsto X\xi \otimes Sh$ defines a bounded operator $X \otimes S$ on $E \otimes_\sigma H$. When $S = I_E$ and $X = \varphi_E(a)$ (for $a \in A$) we get a representation of $A$ on this Hilbert space. (If $E$ is a $W^*$-correspondence and $\sigma$ is a normal representation, so is $a \mapsto \varphi(a) \otimes I_H$.) We frequently write $a \otimes I_H$ for $\varphi(a) \otimes I_H$.

**Definition 2.3** Let $E$ be a $C^*$-correspondence over a $C^*$-algebra $A$. Then a completely contractive covariant representation of $E$ (or, simply, a c.c. representation of $E$) on a Hilbert space $H$ is a pair $(T, \sigma)$, where

1. $\sigma$ is a $^*$-representation of $A$ in $B(H)$.
2. $T$ is a linear, completely contractive map from $E$ to $B(H)$.
3. $T$ is a bimodule map in the sense that $T(a\xi b) = \sigma(a)T(\xi)\sigma(b)$, $\xi \in E$, and $a, b \in A$. 

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If $A$ is a von Neumann algebra and $E$ is a $W^*$-correspondence, we require also that

(4) $\sigma$ is a normal representation.

It should be noted that there is a natural way to view $E$ as an operator space (by viewing it as a subspace of its linking algebra) and this defines the operator space structure of $E$ to which the Definition 2.3 refers when it is asserted that $T$ is completely contractive.

As we noted in the introduction and developed in [20, Lemmas 3.4–3.6] and in [24], if a completely contractive covariant representation, $(T, \sigma)$, of $E$ in $B(H)$ is given, then it determines a contraction $\tilde{T} : E \otimes_{\sigma} H \to H$ defined by the formula $\tilde{T}(\eta \otimes h) := T(\eta)h$, $\eta \otimes h \in E \otimes_{\sigma} H$. The operator $\tilde{T}$ satisfies

$$\tilde{T}(\varphi(\cdot) \otimes I) = \sigma(\cdot)\tilde{T}. \tag{3}$$

In fact we have the following lemma from [24, Lemma 2.16].

**Lemma 2.4** The map $(T, \sigma) \to \tilde{T}$ is a bijection between all completely contractive covariant representations $(T, \sigma)$ of $E$ on the Hilbert space $H$ and contractive operators $\tilde{T} : E \otimes_{\sigma} H \to H$ that satisfy equation (3). Given $\sigma$ and a contraction $\tilde{T}$ satisfying the covariance condition (3), we get a completely contractive covariant representation $(T, \sigma)$ of $E$ on $H$ by setting $T(\xi)h := \tilde{T}(\xi \otimes h)$.

**Remark 2.5** In addition to $\tilde{T}$ we also require the “generalized higher powers” of $\tilde{T}$. These are maps $\tilde{T}_n : E^\otimes n \otimes H \to H$ defined by the equation $\tilde{T}_n(\xi_1 \otimes \ldots \otimes \xi_n \otimes h) = T(\xi_1) \cdots T(\xi_n)h$, $\xi_1 \otimes \ldots \otimes \xi_n \otimes h \in E^\otimes n \otimes H$. One checks easily that $\tilde{T}_n = \tilde{T} \circ (I_E \otimes \tilde{T}) \circ \cdots \circ (I_{E^\otimes n-1} \otimes \tilde{T})$, $n > 1$.

## 3 Representations of product systems and the universal algebra

In the following we follow the notation of Fowler [10]. Suppose $P$ is a left-cancellative, countable, semigroup with an identity $e$ and $p : X \to P$ is a family of $C^*$-correspondences over $A$. Write $X_s$ for the correspondence $p^{-1}(s)$ for $s \in P$ and $\varphi_s : A \to \mathcal{L}(X_s)$ for the left action of $A$ on $X_s$. We say that $X$ is a product system over $P$ if $X$ is a semigroup, $p$ is a semigroup
homomorphism and, for each $s, t \in P \setminus \{e\}$, the map $(x, y) \in X_s \times X_t \to X_{st}$ extends to an isomorphism $\theta_{s, t}$ of correspondences from $X_s \otimes X_t$ onto $X_{st}$. We also require that $X_e = A$ and that the multiplications $X_e \times X_s \to X_s$ and $X_s \times X_e \to X_s$ are given by the left and right actions of $A$ on $X_s$. The associativity of the multiplication means that, for every $s, t, r \in P$,

$$\theta_{st, r}(\theta_{s, t} \otimes I_{X_r}) = \theta_{s, tr}(I_{X_s} \otimes \theta_{t, r}).$$

(4)

**Definition 3.1** Suppose $H$ is a Hilbert space and $T : X \to B(H)$. Write $T_s$ for the restriction of $T$ to $X_s$ and for $s = e$ write $\sigma$ for $T_e$. We call $T$ (or $(\sigma, T)$) a completely contractive representation of $X$ (and we write “a c.c. representation”) if

1. For each $s$, $(\sigma, T_s)$ is a c.c. representation of $X_s$ (as in Definition 2.3).
2. $T(xy) = T(x)T(y)$ for all $x, y \in X$.

Such a representation is said to be an isometric (or a Toeplitz) representation if we also have

3. $T(x)^*T(y) = \sigma(\langle x, y \rangle)$ whenever $x, y \in X_s$ for some $s \in P$.

An important representation is the Fock representation. It is defined as in [10]. We write

$$\mathcal{F}(X) = \sum_{s \in P} \oplus X_s.$$  

As mentioned in [10], this is a $C^*$-correspondence over $A$ with left action given by

$$\varphi_\infty(a)(\oplus x_s) = (\oplus \varphi_s(a)x_s).$$

We can define a representation $L$ of $X$ on $\mathcal{F}(X)$ by setting

$$L(x)(\oplus x_s) = \oplus (x \otimes x_s), \quad \oplus x_s \in \mathcal{F}(X).$$

It is clear that $L$ is completely contractive. In fact, it is completely isometric (i.e., $L_s$ is completely isometric for every $s \in P$). This can be seen even by considering the restriction of $L(x)$ to $A \subseteq \mathcal{F}(X)$.

Note that, strictly speaking this is not what we defined as a representation above (since $\mathcal{F}(X)$ is not a Hilbert space) but we can “fix” it by representing $\mathcal{L}(\mathcal{F}(X))$ on a Hilbert space.
As was shown in [10, Proposition 2.8], the representation \( L \) gives rise to a \( C^* \)-representation of a certain \( C^* \)-algebra containing, for every \( s \in P \), a copy of \( X_s \) and the representation, restricted to this copy is equal to \( L \). This \( C^* \)-algebra, \( \mathcal{T}(X) \), (called the Toeplitz algebra of \( X \)) has a universal property with respect to isometric (or Toeplitz) representations of \( X \).

The next proposition shows that there is (a unique) operator algebra \( \mathcal{T}_+(X) \) which is universal with respect to c.c. representations of \( X \). The proof is standard and is omitted.

**Proposition 3.2** Let \( X \) be a product system over \( P \) of \( C^* \)-correspondences over \( A \). Then there is a (closed) operator algebra \( \mathcal{T}_+(X) \), called the tensor algebra of \( X \), and a c.c. representation \((i_A, i_X)\) of \( X \) into \( \mathcal{T}_+(X) \) such that

(a) \( \mathcal{T}_+(X) \) is generated by the image of \((i_A, i_X)\).

(b) For every c.c. representation \((\sigma, T)\) of \( X \) on \( H \), there is a completely contractive representation \( T \times \sigma \) of \( \mathcal{T}_+(X) \) into \( B(H) \) such that \((T \times \sigma) \circ i_A = \sigma \) and \((T \times \sigma) \circ (i_X)_s = T_s \) (for \( s \in P \)).

We shall refer to the maps \((i_A, i_X)\) as the universal maps.

The triple \( (\mathcal{T}_+(X), i_A, i_X) \) is unique up to a canonical completely isometric isomorphism and, for every \( s \in P \), \((i_X)_s \) is a complete isometry.

**Remark 3.3** For \( P = \mathbb{N} \), it follows from [24] that \( \mathcal{T}_+(X) \) is the tensor algebra defined there. Hence, in this case, it can be realized as a subalgebra of \( \mathcal{L}(\mathcal{F}(X)) \).

Note that for \( A = \mathbb{C} \) and a product system \( X \) with one-dimensional fibers (i.e., \( X_s = \mathbb{C} \)) and multiplication induced from the multiplication of \( P \), the algebra \( \mathcal{T}_+(X) \) is the algebra \( OA(P) \) of Blecher and Paulsen [6].

## 4 Product systems over \( \mathbb{N}^2 \)

Now we consider the case \( P = \mathbb{N}^2 \) (where \( \mathbb{N} = \{0, 1, \ldots\} \)) and prove a dilation result which can be viewed as the analogue of Ando’s dilation theorem (for two commuting contractions).

We start with setting some notation. For \((m, n) \in \mathbb{N}^2 \) and a product system of correspondences \( X \) on \( \mathbb{N}^2 \), it will be convenient to write \( X(m, n) \)
(instead of $X_{(m,n)}$) for the fiber at $(m, n)$. If we set $E = X(1, 0)$ and $F = (0, 1)$, then $X(m, n) \simeq E^m \otimes F^n$. For convenience, we shall write $E^m$ for $E^m \otimes F^0$ (and similarly for $F$) and write $X(m, n) = E^m \otimes F^n$. (In other words, we shall take this isomorphism to be the identity.) In the notation of the previous section, this implies that $\theta_{(m,0)(0,n)} = \text{id}$ and, more generally, $\theta_{(k,0)(m,n)}$ and $\theta_{(k,l)(0,n)}$ are identity maps (for $k, l, m, n \in \mathbb{N}$). Now, $X(m, n)$ is also isomorphic to $F^n \otimes E^m$. This isomorphism will be written $t_{m,n}$, so that $t_{m,n} : E^m \otimes F^n \rightarrow F^n \otimes E^m$.

In fact, $t_{m,n} = \theta_{(0,n)(m,0)}^{-1}$ and we write $t$ for $t_{1,1}$. Then, the associativity requirement enables one to write each $t_{m,n}$ in terms of $t$. Straightforward computation shows that we have

$$t_{1,n} = (I_{E^{n-1}} \otimes t)(I_{E^{n-2}} \otimes t \otimes I_F) \cdots (t \otimes I_{F^{n-1}})$$

and

$$t_{m,n} = (t_{1,n} \otimes I_{E^{m-1}})(I_E \otimes t_{n,1} \otimes I_{F^{m-2}}) \cdots (I_{E^{m-1}} \otimes t_{1,n}).$$

Also, given an isomorphism $t : E \otimes F \rightarrow F \otimes E$, we can define $t_{m,n}$ (using (4) and (6)) and use it to define $\theta_{(m,n)(k,l)}$ (for all $k, l, m, n \in \mathbb{N}$) such that (4) holds. Thus, defining a product system over $\mathbb{N}^2$ amounts to defining a triple $(E, F, t)$ where $E$ and $F$ are $C^*$-correspondences over the same $C^*$-algebra and $t : E \otimes F \rightarrow F \otimes E$ is an isomorphism of correspondences.

Every completely contractive representation of $X$ on $H$ is determined by its restrictions to $A$, to $E$ and to $F$. Thus we write such a representation as a triple $(\sigma, T, S)$ where $T$ and $S$ are the restrictions to $E$ and $F$ respectively. The image of $x = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_m \otimes \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n \in E^m \otimes F^n$ under the representation would then be $T(\xi_1) \cdots T(\xi_m)S(\eta_1) \cdots S(\eta_n)$.

Using Lemma 2.4 and Remark 2.5, we can write the last expression as $\tilde{T}_m(I_{E^m} \otimes \tilde{S}_n)(x)$. We have

$$\tilde{T}_m(I_{E^m} \otimes \tilde{S}_n) : X(m, n) \otimes_\sigma H = E^m \otimes F^n \otimes H \rightarrow H$$

and, using condition (2) of Definition 3.1 we get the following “commutation” relation

$$\tilde{T}_m(I_{E^m} \otimes \tilde{S}_n) = \tilde{S}_n(I_{F^n} \otimes \tilde{T}_m) \circ (t_{m,n} \otimes I_H).$$
For $m = n = 1$ we have

$$
\tilde{T}(I_E \otimes \tilde{S}) = \tilde{S}(I_F \otimes \tilde{T}) \circ (t \otimes I_H).
$$

(9)

In fact, a tedious computation, using (6), (which we omit) shows that (9) implies (8) for all $n, m \in \mathbb{N}$. Reversing the arguments, one also verifies the following lemma.

**Lemma 4.1** If $(\sigma, T)$ and $(\sigma, S)$ are completely contractive representations of $E$ and $F$ respectively that satisfy (9), then (7) defines a (completely contractive) representation of $X$.

**Remark 4.2** So far we dealt with a product system of $C^*$-correspondences over a $C^*$-algebra $A$. In Section 5 we shall be interested in a product system of $W^*$-correspondences over a von Neumann algebra $M$. For such a product system, a c.c. representation $T$ is assumed to have the property that $\sigma (= T_e)$ is a normal representation of $M$. (Note that then, using [24, Remark 2.6], each $T_s$ will, automatically, be continuous with respect to the $\sigma$-topology on $X_s$ and the $\sigma$-weak topology on $B(H)$).

Now we discuss isometric dilations of completely covariant representations. In the following we fix the product system $X$ and we use the notation set above.

**Definition 4.3** Let $(\sigma, T, S)$ be a completely contractive covariant representation of $X$ on a Hilbert space $H$. An isometric dilation of $(\sigma, T, S)$ is an isometric representation $(\rho, V, U)$ of $X$ on a Hilbert space $K$ containing $H$, such that

1. $H$ reduces $\rho$ and $\rho(a)\vert H = P_H\rho(a)\vert H = \sigma(a)$, for all $a \in M$,
2. $K \ominus H$ is invariant under each $V(\xi)$ and each $U(\eta)$ (for $\xi \in E$, $\eta \in F$); that is, $P_HV(\xi)\vert K \ominus H = P_HU(\eta)\vert K \ominus H = 0$, and
3. for all $\xi \in E$ and $\eta \in F$, $P_HV(\xi)|H = T(\xi)$ and $P_HU(\eta)|H = S(\eta)$.

We shall say that such a dilation is minimal in case the smallest subspace of $K$ containing $H$ and invariant under every $V(\xi)$, $\xi \in E$, and every $U(\eta)$, $\eta \in F$, is all of $K$. 

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Note that, if \((\rho, V, U)\) is an isometric dilation of \((\sigma, T, S)\) as above, then, for every \(\xi_1, \xi_2, \ldots, \xi_n \in E\) and \(\eta_1, \eta_2, \ldots, \eta_m \in F\),

\[
P_H V(\xi_1) \cdots V(\xi_n) U(\eta_1) \cdots U(\eta_m) |H = T(\xi_1) \cdots T(\xi_n) S(\eta_1) \cdots S(\eta_m).
\]

Also, a similar statement holds for all “mixed” products; e.g.

\[
P_H V(\xi_1) U(\eta_1) V(\xi_2) |H = T(\xi_1) S(\eta_1) T(\xi_2).
\]

**Theorem 4.4** Let \((\sigma, T, S)\) be a c.c. representation of \(X\) on \(H\) as above. Then there is a Hilbert space \(K\) containing \(H\) and a minimal isometric representation \((\rho, V, U)\) of \(X\) on \(K\) that dilates \((\sigma, T, S)\).

If \(\sigma\) is non degenerate and \(E\) and \(F\) are essential (where \(E\) is essential if the subspace spanned by \(\varphi(A)E\) is dense in \(E\)), then \(\rho\) is nondegenerate.

If \(X\) is a product system of \(W^\ast\)-correspondences and \(\sigma\) is assumed to be normal then \(\rho\) is also a normal representation.

**Proof.** We write \(H_0\) for the Hilbert space \(H\) together with the representation \(\sigma\) on it (we refer to it as a Hilbert module over \(A\)) and define a sequence \(\{H_k\}\) of Hilbert modules over \(A\) inductively by \(H_{k+1} = H_k \oplus H_{k}^{(\infty)}\) where \(H_{k}^{(\infty)}\) is the direct sum of infinitely many copies of \(H_k\) (as Hilbert modules over \(A\)). We write \(\sigma_k\) for the representation of \(A\) on \(H_k\) and we think of \(H_k\) as contained in \(H_{k+1}\) where the inclusion map sends \(h\) to \(h \oplus 0^\infty\). Also, given a correspondence \(Y\) over \(A\), we get an inclusion \(Y \otimes \sigma_k H_k \subseteq Y \otimes \sigma_{k+1} H_{k+1}\).

The space \(K\) that we need is

\[
K = \sum_{(m,n)} \oplus (X(m,n) \otimes_{\sigma_{\max\{m,n\}}} H_{\max\{m,n\}}).
\]

There is a natural representation of \(A\) on \(K\). We shall write \(\rho\) for it (and we shall also write \(\rho\) for its restriction to various \(\rho(A)\)-invariant subspaces of \(K\)). Note that \(K = \sum_{l=0}^{\infty} K(l)\) where we write

\[
K(l) = \sum_{\max\{m,n\} = l} E^m \otimes F^n \otimes_{\sigma_l} H_l.
\]

The dilation will constructed in several steps.

We first define \(V_2 : E \otimes K \to K\) and \(U_2 : F \otimes K \to K\) by their restrictions to \(E \otimes E^m \otimes F^n \otimes H_{\max\{n,m\}}\) and \(F \otimes E^m \otimes F^n \otimes H_{\max\{n,m\}}\) as follows.
For a fixed $n \geq 0$, we write $q_0$ for the projection of $H_n$ onto $H = H_0$ (which is contained in $H_n$) and, for $h_0 \in H$ and $e \in E$, we set $D_T(e)h_0 = \Delta_T(e \otimes h_0) \in E \otimes H \subseteq E \otimes H_n$ where $\Delta_T = (I_E \otimes H - \overline{T^*T})^{1/2} \in B(E \otimes H)$.

We then let $V_0 : E \otimes H_n \to H_n \oplus (E \otimes H_n)$ be defined by

$$V_0(e \otimes h) = (T(e)q_0h) \oplus (D_T(e)q_0h \oplus (e \otimes (I - q_0)h)). \quad (11)$$

Now, for $n = m = 0$, we define $V_2|E \otimes H$ to be $V_0$ (with $n = 0$) and, for $m = 0$ and $n > 0$, we set

$$V_2|E \otimes F^n \otimes_{\sigma_n} H_n = (I_{F^n \otimes H_n} \oplus (t_{1,n}^{-1} \otimes I_{H_n})) \circ (I_{F^n} \otimes V_0) \circ (t_{1,n} \otimes I_{H_n}).$$

Thus, for $n \geq 0$, $V_2$ maps $E \otimes F^n \otimes_{\sigma_n} H_n$ into $(F^n \otimes_{\sigma_n} H_n) \oplus (E \otimes F^n \otimes_{\sigma_n} H_n)$.

Since $\|T(e)h_0 \oplus D_T(e)h_0\| = \|\overline{T(e \otimes h_0)} \oplus \Delta_T(e \otimes h_0)\| = \|e \otimes h_0\|$, for $h_0 \in H$, the map $V_0$ is an isometry. It is also straightforward to check that $V_0$ is an $A$-module map (where $a \in A$ acts on $H_n$ by $\sigma_n(a)$ and on $E \otimes H_n$ by $\varphi_E(a) \otimes I_{H_n}$). Thus $I_{F^n} \otimes V_0$ is an isometry from $F^n \otimes E \otimes H_n$ into $(F^n \otimes H_n) \oplus (F^n \otimes E \otimes H_n)$. It follows that $V_2|E \otimes F^n \otimes_{\sigma_n} H_n$ is a composition of three isometries. Thus it is an isometry into $(F^n \otimes H_n) \oplus (E \otimes F^n \otimes H_n)$.

For $m > 0$ we let $V_2|E \otimes F^n \otimes H_{\max\{n,m\}}$ be the inclusion map into $E^{m+1} \otimes F^n \otimes H_{\max\{n,m+1\}}$ (where $E \otimes E^m$ is identified with $E^{m+1}$ and $H_{\max\{n,m\}}$ is identified as a subspace of $H_{\max\{n,m+1\}}$).

For different $n, m$ the ranges of $V_2|E \otimes F^n \otimes H_{\max\{n,m\}}$ are orthogonal to each other and, thus, it follows that $V_2$ defines an isometry from $E \otimes K$ into $K$.

The definition of $U_2$ is similar. For $n = 0$ we let

$$U_2|F \otimes E^m \otimes_{\sigma_m} H_m = (I_{E^m \otimes H_m} \oplus (t_{1,m} \otimes I_{H_m})) \circ (I_{E^m} \otimes U_0) \circ (t_{1,m}^{-1} \otimes I_{H_m})$$

where $U_0 : F \otimes H_m \to H_m \oplus (F \otimes H_m)$ is defined by

$$U_0(f \otimes h) = (S(f)q_0h) \oplus (D_S(f)q_0h \oplus (f \otimes (I - q_0)h)). \quad (12)$$

For $n > 0$ we let $U_2|F \otimes E^m \otimes F^n \otimes H_{\max\{n,m\}}$ be the map $t_{m,1}^{-1} \otimes I_{F^n} \otimes I_{H_{\max\{n,m\}}}$ composed with the inclusion map of $E^m \otimes F \otimes F^n \otimes H_{\max\{n,m\}}$ into $E^m \otimes F^{n+1} \otimes H_{\max\{n+1,m\}}$. Clearly, $U_2$ is an isometry from $F \otimes K$ into $K$.

It is easy to check that we have, for $a \in A$,

$$V_2(\varphi_E(a) \otimes I_K) = \rho(a)V_2 \quad (13)$$

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and
\[ U_2(\varphi_F(a) \otimes I_K) = \rho(a)U_2. \] (14)

In general, the isometries \( V_2, U_2 \) do not necessarily satisfy a commutation relation as in Equation 13. In fact, the maps \( V_2(I_E \otimes U_2) \) and \( U_2(I_F \otimes V_2) (t \otimes I_H) \) (where here \( t = t_{11} \)), defined on \( E \otimes F \otimes H \), may differ. However, both maps map \( E \otimes F \otimes H \) into \( H \oplus (E \otimes H) \oplus (F \otimes H) \oplus (E \otimes F \otimes H) \subseteq K(0) \oplus K(1) \).

For every \( l \geq 0 \), we write \( P_l \) for the projection of \( K \) onto \( K(l) \) (so that \( P_0 \) is the projection onto \( H = H_0 \)). A simple computation shows that
\[ P_lV_2(I_E \otimes U_2) = \tilde{T}(I_E \otimes \tilde{S}) = \tilde{S}(I_F \otimes \tilde{T})(t \otimes I_H) = P_0U_2(I_F \otimes V_2)(t \otimes I_H). \]

Recall that \( K(1) \) is \( (E \otimes H_1) \oplus (F \otimes H_1) \oplus (E \otimes F \otimes H_1) \) and \( P_1 \) is the projection onto it. Write \( G_1 \) for the closure of the range of \( P_1V_2(I_E \otimes U_2) \) and \( G_2 \) for the closure of the range of \( P_1U_2(I_F \otimes V_2)(t \otimes I_H) \).

Since both maps \( V_2(I_E \otimes U_2) \) and \( U_2(I_F \otimes V_2)(t \otimes I_H) \) are isometries, we have, for \( \xi \in E \otimes F \otimes H \),
\[
\|P_1V_2(I_E \otimes U_2)\xi\|^2 = \|V_2(I_E \otimes U_2)\xi\|^2 - \|P_0V_2(I_E \otimes U_2)\xi\|^2 = \|U_2(I_F \otimes V_2)(t \otimes I_H)\xi\|^2 - \|P_0U_2(I_F \otimes V_2)(t \otimes I_H)\xi\|^2.
\]

Thus, the map sending \( P_1V_2(I_E \otimes U_2)\xi \) to \( P_1U_2(I_F \otimes V_2)(t \otimes I_H)\xi \) defines a unitary map from \( G_1 \) onto \( G_2 \). We write \( W(1)' \) for this map. For every \( a \in A, \rho(a) \) maps \( G_i \) \((i = 1, 2)\) into itself; so that we can view \( G_i \) as a left \( A \)-module. Moreover, the map \( W(1)' \) is an isomorphism of \( A \)-modules (i.e., the representations of \( A \) associated with \( G_1 \) and \( G_2 \) are equivalent). Now write \( \tau \) for the representation of \( A \) on \( (E \otimes H) \oplus (F \otimes H) \oplus (E \otimes F \otimes H) \) (i.e., \( \tau \) is the restriction of \( \rho \) to this space) and \( \tau_\infty \) for the representation of \( A \) on \( (E \otimes H^{(\infty)}) \oplus (F \otimes H^{(\infty)}) \oplus (E \otimes F \otimes H^{(\infty)}) \) (where \( H^{(\infty)} = H_1 \otimes H \)). Clearly, \( \tau_\infty \) is the sum of infinitely many copies of \( \tau \). Also write \( \pi_i \) \((i = 1, 2)\) for the representation of \( A \) on \( ((E \otimes H) \oplus (F \otimes H) \oplus (E \otimes F \otimes H)) \otimes G_i \). Then \( \pi_i \leq \tau \) and, thus, \( \pi_1 \oplus \tau_\infty = \pi_2 \oplus \tau_\infty \). Write \( W(1)'' : ((E \otimes H_1) \oplus (F \otimes H_1) \oplus (E \otimes F \otimes H_1)) \otimes G_1 \rightarrow ((E \otimes H_1) \oplus (F \otimes H_1) \oplus (E \otimes F \otimes H_1)) \otimes G_2 \) for the unitary implementing this equivalence and, forming \( W(1) := W(1)' \oplus W(1)'' \), we get a unitary operator on \( K(1) \) that commutes with the restriction of \( \rho \) to \( K(1) \). Also write \( W(0) \) for the identity map on \( H \). Then we have, for \( \xi \in E \otimes F \otimes H \),
\[
W(1)V_2(I_E \otimes U_2)(I_E \otimes I_F \otimes W(0)^{-1})\xi = U_2(I_F \otimes V_2)(t \otimes I_H)\xi.
\]
Next, we shall define, inductively, unitary operators \( W(k) \), on \( K(k) \), such that, writing \( K[k] \) for the direct sum \( \sum_{i=0}^{k} \oplus K(l) \) and \( W[k] \) for \( \sum_{i=0}^{k} \oplus W(l) \) for every \( k \geq 0 \), we get

\[
W[k + 1]V_2(I_E \otimes U_2)(I_E \otimes I_F \otimes W[k]^{-1})\xi = U_2(I_F \otimes V_2)(t \otimes I_{K[k]})\xi
\]

for every \( \xi \in E \otimes F \otimes K[k] \). Once this is done, we write \( W = \sum \oplus W(k), V = WV_2 \) and \( \bar{U} = U_2(I_F \otimes W^{-1}) \) to get

\[
\bar{V}(I_E \otimes \bar{U}) = \bar{U}(I_F \otimes \bar{V})(t \otimes I_K).
\]

So we now assume that \( W(l) \) has been defined for all \( 0 \leq l \leq k \). Write \( G_1 \) for the subspace \( V_2(I_E \otimes U_2)(I_E \otimes I_F \otimes W[k]^{-1})(E \otimes F \otimes K(k)) \). Since it is an isometric image of \( E \otimes F \otimes K(k) \), it is closed. Using the definition of \( V_2 \) and \( U_2 \) one can easily check that

\[
G_1 \subseteq \sum_{\max\{m,n\}=k+1} E^m \otimes F^n \otimes H_k \subseteq K(k + 1).
\]

Similarly, write \( G_2 \) for the (closed) subspace \( U_2(I_F \otimes V_2)(t \otimes I_{K(k)})(E \otimes F \otimes K(k)) \) and note that

\[
G_2 \subseteq \sum_{\max\{m,n\}=k+1} E^m \otimes F^n \otimes H_k \subseteq K(k + 1).
\]

The map \( W(k + 1)' \) sending \( V_2(I_E \otimes U_2)(I_E \otimes I_F \otimes W[k]^{-1})\xi, \) in \( G_1 \) (for \( \xi \in E \otimes F \otimes K(k) \)) to \( U_2(I_F \otimes V_2)(t \otimes I_{K(k)})\xi, \) in \( G_2, \) is a unitary operator from \( G_1 \) onto \( G_2 \) intertwining \( \rho \). Now write \( \pi_i \) for the restriction of \( \rho \) to \( (\sum_{\max\{m,n\}=k+1} E^m \otimes F^n \otimes H_k) \oplus G_i, \) \( \tau \) for the restriction of \( \rho \) to \( \sum_{\max\{m,n\}=k+1} E^m \otimes F^n \otimes H_k \) and \( \tau_\infty \) for the restriction of \( \rho \) to \( \sum_{\max\{m,n\}=k+1} E^m \otimes F^n \otimes H_k^\infty. \) Now, argue as above (the case \( k = 0 \)) to find the unitary \( W(k + 1), \) on \( K(k + 1), \) satisfying

\[
W(k + 1)V_2(I_E \otimes U_2)(I_E \otimes I_F \otimes W(k)^{-1})\xi = U_2(I_F \otimes V_2)(t \otimes I_H)\xi = U_2(I_F \otimes V_2)(t \otimes I_{K(k)})\xi
\]

for each \( \xi \in E \otimes F \otimes K(k) \). This, together with the induction hypothesis, implies (15) and, after setting \( \bar{V} = WV_2 \) and \( \bar{U} = U_2(I_F \otimes W)^{-1}, \) we get (16).
Both $\tilde{V}$ and $\tilde{U}$ are isometries and it follows from (13) and (14) and the fact that $W$ commutes with $\rho(A)$, that, for $a \in A$,

$$\tilde{V}(\varphi_E(a) \otimes I_K) = \rho(a)\tilde{V}$$

and

$$\tilde{U}(\varphi_F(a) \otimes I_K) = \rho(a)\tilde{U}.$$ 

Setting $V(\xi)_k = \tilde{V}(\xi \otimes k)$ and $U(\eta)_k = \tilde{U}(\eta \otimes k)$ (for $\xi \in E$, $\eta \in F$ and $k \in K$), the triple $(\rho, V, U)$ defines an isometric representation of $X$ on $K$. To see that it is a dilation of $(\sigma, T, S)$ note that parts (1) and (2) of Definition 4.3 are easy to verify. To check part (3), fix $\xi \in E$ and $\eta \in H \subseteq K$ and compute $P_H V(\xi)h = P_H \tilde{V}(\xi \otimes h) = P_H W V(\xi \otimes h) = P_H W V_2(\xi \otimes h) = P_H((T(\xi)q_0h) \oplus (D_T(\xi)q_0h \oplus (\xi \otimes (I - q_0)h))) = T(\xi)h$. The computation for $U$ is similar.

The statement about the non degeneracy of $\rho$ is clear from its definition. It is also clear that, if $E$ and $F$ are $W^*$-correspondences over a von Neumann algebra $M$ and $\sigma$ is normal, so is $\rho$ (as both $\varphi_E$ and $\varphi_F$ are assumed to be normal homomorphisms).

Finally, the dilation that we get in this way may not be minimal but, restricting $(\rho, U, V)$ to the closed subspace of $K$ spanned by $H$ and by the vectors of the form $R_1 R_2 \cdots R_n h$, where $R_i \in U(F) \cup V(E)$ and $h \in H$, we get a minimal isometric dilation. □

We now apply the theorem to obtain a dilation result for two “commuting” row contractions. We note that, if one gives up the commutativity condition in the next corollary, the dilation result was obtained by Popescu in [30].

**Corollary 4.5** Suppose $(T_1, \ldots T_n)$ and $(S_1, \ldots S_m)$ are an $n$-tuple and an $m$-tuple of operators on a Hilbert space $H$ satisfying

(i) $\sum T_i T_i^* \leq I$ and $\sum S_j S_j^* \leq I$.

(ii) There is a unitary matrix $u = (u_{(i,j),(k,l)})$ (whose rows and columns are indexed by $\{1, 2, \ldots n\} \times \{1, \ldots m\}$) such that, for all $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$T_i S_j = \sum_{k,l} u_{(i,j),(k,l)} S_l T_k.$$
Then there is a larger Hilbert space $K$, containing $H$, an $n$-tuple of isometries $(V_1, \ldots, V_n)$ in $B(K)$ and an $m$-tuple of isometries $(U_1, \ldots, U_m)$ in $B(K)$ such that

(a) $\sum V_i V_i^* \leq I$ and $\sum U_j U_j^* \leq I$ (that is, in each tuple the isometries have pairwise orthogonal ranges).

(b) For every $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$V_i U_j = \sum_{k,l} u_{i,j}(k,l) U_l V_k.$$ 

(c) Each $V_i$ and $U_j$ leave $K \ominus H$ invariant.

(d) For every $1 \leq i \leq n$ and $1 \leq j \leq m$, $P_H V_i | H = T_i$ and $P_H U_j | H = S_j$ (and, together with (c), this implies that each product involving $V_i$’s and $U_j$’s dilates the corresponding product with $T_i$’s and $S_j$’s).

**Proof.** Let $M = \mathbb{C}$, $E = \mathbb{C}^n$, $F = \mathbb{C}^m$ (with orthonormal bases $\{e_i\}$ and $\{f_j\}$ respectively) and $t : \mathbb{C}^n \otimes \mathbb{C}^m \to \mathbb{C}^m \otimes \mathbb{C}^n$ be defined by $t(e_i \otimes f_j) = \sum_{k,l} u_{i,j}(k,l) f_l \otimes e_k$. An $n$-tuple $(T_1, \ldots, T_n)$ satisfying $\sum T_i T_i^* \leq I$ defines a completely contractive linear map $T : E \to B(H)$ by $T(e_i) = T_i$. Similarly we define $S : F \to B(H)$ and (ii) implies that they satisfy the commutation relation \[(a)\]. Letting $\sigma$ be the obvious representation of $\mathbb{C}$ on $H$, we get a representation $(\sigma, T, S)$ of the product system $X$ defined by $E$, $F$ and $t$. Applying Theorem 4.4 we get a Hilbert space $K$ and maps $V : E \to B(K)$ and $U : F \to B(K)$ defining isometric representations (that dilate $T$ and $S$ respectively). We now let $V_i$ be $V(e_i)$ and $U_j$ be $U(f_j)$. The fact that these are isometric representations imply that the operators $V_i$ and $U_j$ are all isometries. The rest of (a)-(d) follows immediately. $\Box$

A special case of the following corollary (for $\alpha$ and $\beta$ that are automorphisms) can be found in [17].

**Corollary 4.6** Let $\alpha$ and $\beta$ be commuting $^*$-endomorphisms of a $C^*$-algebra $A$ that extend to the multiplier algebra $M(A)$ (as commuting endomorphisms $\overline{\alpha}$ and $\overline{\beta}$). Suppose $\sigma$ is a non degenerate representation of $A$ on $H$ and $T_0$, $S_0$ are contractions in $B(H)$ satisfying

(i) $\sigma(a) T_0 = T_0 \sigma(\alpha(a))$ and $\sigma(a) S_0 = S_0 \sigma(\beta(a))$ for all $a \in A$, and
(ii) $T_0 S_0 = S_0 T_0$.

Then there is a Hilbert space $K$, containing $H$, a non degenerate representation $\rho$ of $A$ on $K$ and partial isometries $V_0$ and $U_0$ in $B(K)$ such that

1. $\rho(a) V_0 = V_0 \rho(\alpha(a))$ and $\rho(a) U_0 = U_0 \rho(\beta(a))$ for all $a \in A$,
2. $V_0 U_0 = U_0 V_0$,
3. $U_0^* U_0 = \overline{\rho}(\overline{\beta}(I))$ and $V_0^* V_0 = \overline{\rho}(\overline{\alpha}(I))$,
4. $H$ reduces $\rho$ and $\rho(a) | H = \sigma(a)$,
5. $H$ is invariant for $U_0^*$ and for $V_0^*$, and
6. $P_H V_0 | H = T_0$ and $P_H U_0 | H = S_0$.

**Proof.** In the notation of Theorem 4.4 let $E = \overline{\sigma}(I) A = \alpha(A)\overline{A}$ and $F = \overline{\beta}(I) A = \beta(A)\overline{A}$. The correspondence structure of $E$ is defined by $\langle \xi, \eta \rangle = \xi^* \eta$ and $\varphi_E(a) \xi b = \alpha(a) \xi b$, for $a, b \in A$ and $\xi, \eta \in E$ (and similarly for $F$). Then one can easily check that $E \otimes F$ is isomorphic to the correspondence $\overline{\sigma}(I) A$ (via $\xi \otimes \eta \mapsto \beta(\xi) \eta$) and $F \otimes E$ is isomorphic to $\overline{\alpha}(I) A$. Combining these isomorphisms, we get an isomorphism $t : E \otimes F \rightarrow F \otimes E$ which can be written $t(\alpha(a_1)a_2 \otimes \beta(I)b) = \beta(I)\beta(a_1) \otimes \overline{\alpha}(I)\beta(a_2)b$ for $a_1, a_2, b \in A$.

A triple $(\sigma, T_0, S_0)$ satisfying (i) and (ii) defines a representation $(\sigma, T, S)$ by setting $T(\overline{\sigma}(I)a) = T_0 \sigma(a)$ and $S(\overline{\beta}(I)a) = S_0 \sigma(a)$. Let $(\rho, V, U)$ be a minimal isometric dilation. Then, for $a, b \in A$ and $k \in K$, $\langle V(\overline{\sigma}(I)a)k, V(\overline{\beta}(I)b)g \rangle = \langle \rho(b^* \overline{\sigma}(I)a)k, g \rangle = \langle \overline{\rho}(\overline{\sigma}(I))\rho(a)k, \rho(b^*)g \rangle$. Thus, there is a partial isometry $V_0$ with $V_0^* V_0 = \overline{\rho}(\overline{\alpha}(I))$ satisfying $V_0 \rho(a)k = V(\overline{\sigma}(I)a)$. Similarly one defines $U_0$ and properties (1) and (3) follow. Properties (4)-(6) follow from the dilation properties and (2) follows from Equation 9 (for $V$ and $U$). We omit the details.

□

**Remark 4.7** For c.c. representations of a single $C^*$-correspondence it was shown in [20], Theorem 4.4] that commutant lifting holds for the minimal isometric dilation. When $A = \mathbb{C}$, this was proved in [20, Theorem 3.2] generalizing the commutant lifting theorem of Sz.-Nagy and Foias. It is known in the classical case that the commutant lifting theorem of Sz.-Nagy and Foias can be derived from Ando’s dilation theorem. It is not hard to see that Theorem 4.4 can be used to give a different proof of the commutant lifting theorem of [20]. Since we shall not use it in this paper, we omit the details.
The following corollary shows that, when \( P = \mathbb{N}^2 \), the universal tensor algebra \( T_+(X) \) of Proposition 3.2 is contained in the universal Toeplitz \( C^* \)-algebra \( T(X) \) of [10].

**Corollary 4.8** Let \( X \) be a product system of \( C^* \)-correspondences (over a \( C^* \)-algebra \( A \)) with \( P = \mathbb{N}^2 \). Let \( T_+(X) \), \( i_A \) and \( i_X \) be the universal tensor algebra and the universal maps as in Proposition 3.2. Let \( T(X) \), \( k_A \) and \( k_X \) be the universal Toeplitz algebra and the universal maps as in [10, Proposition 2.8]. Then there is a completely isometric homomorphism

\[
\Psi : T_+(X) \rightarrow T(X)
\]

such that \( \Psi \circ i_A = k_A \) and \( \Psi \circ i_X = k_X \).

**Proof.** Write \( B \) for the norm-closed subalgebra of \( T(X) \) generated by \( k_A(A) \) and \( k_X(X) \). We will show that \((B, k_A, k_X)\) has the universal property (b) of Proposition 3.2. Since it also satisfies (a), the uniqueness of the universal algebra will complete the proof.

So suppose that \((\sigma, T)\) is a c.c. representation of \( X \) on \( H \). It can be dilated to an isometric (i.e., Toeplitz) representation \((\rho, V)\) on \( K \). Then \((\rho, V)\) defines a \( C^* \)-representation \( V \times \rho \) of \( T(X) \) with \((V \times \rho) \circ k_A = \rho \) and \((V \times \rho) \circ k_X = V \). Set \( \pi(b) = P_H(V \times \rho)(b)|H \) for \( b \in B \). Since all \( V(x) \) (for \( x \in X \)) and \( \rho(a) \) (for \( a \in A \)) leave \( K \ominus H \) invariant, \( P_H \) is a semiinvariant projection for \((V \times \rho)(B)\) and, thus, \( \pi \) is a completely contractive representation of \( B \) on \( H \). We also have, for \( a \in A \) and \( x \in X \), \( \pi(k_A(a)) = P_H \rho(a)|H = \sigma(a) \) and \( \pi(k_X(x)) = P_H V(x)|H = T(x) \). Thus \( \pi \) is \( T \times \sigma \), completing the proof.

\( \square \)

## 5 Commuting CP maps

In this section we study commuting pairs of contractive, normal, completely positive maps on von Neumann algebras. The term “CP map” will always refer here to a contractive, normal, completely positive map on a von Neumann algebra.

Let \( \Theta \) and \( \Phi \) be two normal CP maps on a given von Neumann algebra \( M \). We assume that \( M \subseteq B(H) \) and consider two Hilbert spaces defined as
follows. On the algebraic tensor product $M \otimes M \otimes H$ we define a sesquilinear form

$$\langle a_1 \otimes b_1 \otimes h_1, a_2 \otimes b_2 \otimes h_2 \rangle = \langle h_1, \Theta(b_1^* \Phi(a_1^* a_2) b_2) h_2 \rangle.$$ 

We write $H_{\Phi, \Theta}$ (or $M \otimes \Phi \otimes \Theta H$) for the Hilbert space obtained by the Hausdorff completion of the algebraic tensor product with respect to this semi inner product. A “typical” element in $H_{\Phi, \Theta}$ will be written $a \otimes b \otimes \Theta h$. The Hilbert space $M \otimes \Phi H$ has a natural (normal) representation of $M$ on it. It is defined simply by $\Theta : M \to M \otimes \Phi H$. We write $H = M \otimes \Phi H$. We now introduce a condition on the pair $(\Theta, \Phi)$ that is stronger than the commutation relation $\Theta \Phi = \Phi \Theta$. Its significance will be made clear later.

**Definition 5.1** Given $\Theta$ and $\Phi$ as above, we say that they commute strongly if there is a unitary $u : H_{\Phi, \Theta} \to H_{\Theta, \Phi}$ such that

(i) $u(a \otimes \Phi I \otimes \Theta h) = a \otimes \Phi I \otimes \Theta h$ for $a \in M$ and $h \in H$.

(ii) $u(ca \otimes \Phi b \otimes \Theta h) = (c \otimes I_M \otimes I_H) u(a \otimes \Phi b \otimes \Theta h)$ for $a, b, c \in M$ and $h \in H$ (that is, $u$ intertwines the actions of $M$).

(iii) $u(a \otimes \Phi b \otimes \Theta dh) = (I_M \otimes I_M \otimes d) u(a \otimes \Phi b \otimes \Theta h)$ for $a, b \in M$, $d \in M'$ and $h \in H$ (that is, $u$ intertwines the actions of $M'$).

**Remark 5.2** Note that, for $a \in M$ and $h \in H$, we have $\|a \otimes \Phi I \otimes \Theta h\|^2 = \langle h, \Theta(\Phi(a^* a)) h \rangle$ while $\|a \otimes \Phi I \otimes \Theta h\|^2 = \langle h, \Phi(\Theta(a^* a)) h \rangle$. Thus, the existence of a unitary $u$ satisfying (i) of Definition 5.1 is equivalent to the assumption that $\Theta$ and $\Phi$ commute. It follows that, if $\Theta$ and $\Phi$ commute strongly, then they commute. The converse is false, as we shall see in Example 5.3.

Given a von Neumann algebra $M \subseteq B(H)$ and a normal CP map $\Theta : M \to M$, we write

$$E_\Theta = \{ X : H \to M \otimes \Theta H : X a = (a \otimes I) X, \ a \in M \}.$$
(Recall that $M \otimes \Theta H$ was defined in the introduction). In [22] we wrote $\mathcal{L}_M(H, M \otimes \Theta H)$ for it and showed that it has a structure of a $W^*$-correspondence over $M'$ ([22 Proposition 2.5]). In fact, the right action of $d \in M'$ is given by $Xd = X \circ d$, the left action of $d$ is $\varphi_{E_{\Theta}}(d)X = (I_M \otimes d) \circ X$ (where $I_M \otimes d$ sends $a \otimes \Theta h$ to $a \otimes \Theta dh$ and the inner product is $\langle X_1, X_2 \rangle = X_1^*X_2$, for $X_1, X_2 \in E_{\Theta}$. We also defined (see [22 Equation (2.7)]) the identity representation of this correspondence to be the pair $(\sigma, T_{\Theta})$ where $\sigma$ is the identity representation of $M'$ on $H$ and $T_{\Theta}(X) = W_{\Theta}^*X$ (for $X \in E_{\Theta}$) where $W_{\Theta} : H \to M \otimes \Theta H$ is defined by $W_{\Theta}h = I \otimes h$ (and, consequently, $W_{\Theta}^*(a \otimes \Theta h) = \Theta(a)h$). Note that $T_{\Theta} : E_{\Theta} \to B(H)$ is an injective map (because, for all $h, g \in H$ and $a \in M$, $\langle W_{\Theta}Xa^*h, g \rangle = \langle Xa^*h, I \otimes \Theta g \rangle = \langle (I \otimes a^*)Xh, I \otimes \Theta g \rangle = \langle Xh, a \otimes \Theta g \rangle$).

We also write (for normal CP maps $\Theta$ and $\Phi$)

$$E_{\Phi, \Theta} = \{Z : H \to H_{\Phi, \Theta} : Za = (a \otimes \Phi I \otimes \Theta I)Z, a \in M\}.$$ 

Then $E_{\Phi, \Theta}$ is a $W^*$-correspondence over $M'$ where the right action is by composition, the left action is by $\varphi(d)Z = (I \otimes I \otimes d) \circ Z$ and the inner product is $\langle Z_1, Z_2 \rangle = Z_1^*Z_2$. Recall ([22 Proposition 2.12]) that the map $X \otimes Y \mapsto (I \otimes X)Y$ is an isomorphism from the correspondence $E_{\Theta} \otimes_{M'} E_{\Phi}$ onto the correspondence $E_{\Phi, \Theta}$. We write $\Gamma_{\Phi, \Theta}$ for this isomorphism. Proposition 2.12 of [22] also shows that there is an isometry $V$ from $E_{\Theta\Phi}$ into $E_{\Phi, \Theta}$ such that $m := V^*\Gamma_{\Phi, \Theta}$ is a coisometry mapping $E_{\Theta} \otimes E_{\Phi}$ onto $E_{\Phi \Theta}$. Similarly, one has a coisometry $n : E_{\Phi} \otimes E_{\Theta} \to E_{\Phi \Theta}$.

**Remark 5.3** It is easily seen from [22 Proposition 2.12] that (for commuting maps $\Phi$ and $\Theta$) $\Theta$ and $\Phi$ strongly commute if and only if the partial isometry $m$ can be extended to an isometry (of correspondences) from $E_{\Theta} \otimes E_{\Phi}$ onto $E_{\Phi} \otimes E_{\Theta}$. In the case where $M = B(H)$, these correspondences are Hilbert spaces (isomorphic to Arveson’s metric operator spaces, [2]) and the maps commute strongly if and only if $\dim(Ker(m)) = \dim(Ker(n))$.

Using the remark above, the following lemma follows from [22 Proposition 2.14].

**Lemma 5.4** (1) If $\Theta$ and $\Phi$ are (normal) endomorphisms that commute then they commute strongly.

(2) If $\Theta$ is a normal CP map and $\alpha$ is a normal automorphism of $M$ that commutes with it then $\Theta$ and $\alpha$ commute strongly.
(3) If $\Theta$ is a normal CP map, $\alpha$ is a normal automorphism of $M$ that commutes with $\Theta$ and $\Phi := \Theta \circ \alpha$ commutes with $\Theta$, then $\Theta$ and $\Phi$ commute strongly.

**Example 5.5** There are pairs of commuting normal CP maps that do not commute strongly.

Let $H$ be a Hilbert space, $P$ be a non trivial projection in $B(H)$ and $S \in B(H)$ a coisometric map with $S^*S = P$ and such that $S$ has some unit vector $k \in H$ with $S^*k = k$. Let $\Theta : B(H) \to B(H)$ be the normal CP map $\Theta(a) = \langle ak, k \rangle I_H$ and $\Phi : B(H) \to B(H)$ be defined by $\Phi(a) = SaS^*$. Then, for $a \in B(H)$, $\Phi(\Theta(a)) = \Phi(\langle ak, k \rangle I) = \langle ak, k \rangle SS^* = \langle aS^*k, S^*k \rangle I = \Theta(\Phi(a))$ so that the maps commute and, in fact, $\Phi \circ \Theta = \Theta \circ \Phi = \Theta$. A straightforward calculation shows that, for every $a, b \in B(H)$ and $h \in H$, $a \otimes_\Theta b \otimes_\Theta h = aS^*bS \otimes_\Phi I \otimes_\Theta h$ in $H_{\Theta, \Phi}$. Thus, $H_{\Theta, \Phi}$ is equal to the closed subspace spanned by vectors of the form $c \otimes_\Theta I \otimes_\Phi g$. On the other hand, if we choose $b \in B(H)$ and $h \in H$ such that $(I - P)bP h \neq 0$ and set $x = I \otimes_\Theta (I - P)bP \otimes_\Phi h \in H_{\Theta, \Phi}$, then $x \neq 0$ and is orthogonal to the closed subspace of $H_{\Theta, \Phi}$ spanned by the vectors of the form $c \otimes_\Theta I \otimes_\Phi g$. This shows that the maps do not commute strongly.

The importance of knowing whether two commuting normal CP maps commute strongly follows from the next proposition. First, recall that a (single) normal CP map on a von Neumann algebra $M$ always “comes” from an (injective) representation of some $W^*$-correspondence $E$. More precisely, given such CP map $\Theta$ on $M \subseteq B(H)$, there is a $W^*$-correspondence $E$ over $M'$ and a completely contractive covariant representation $(\sigma, T)$ of $E$ on $H$ (where $T$ is injective and $\sigma = id$) such that

$$\Theta(a) = \tilde{T}(I_E \otimes a)\tilde{T}^*, \quad a \in M.$$  

(For the proof, see [22, Corollary 2.23].) As the following proposition shows, a similar statement holds for a commuting pair of CP maps if and only if they commute strongly.

**Proposition 5.6** Suppose $\Theta$ and $\Phi$ are commuting normal CP maps on $M \subseteq B(H)$. Then the following are equivalent.

(1) $\Theta$ and $\Phi$ commute strongly.
(2) There is an isomorphism \( t = t_{\Theta, \Phi} : E_\Theta \otimes_{M'} E_\Phi \to E_\Phi \otimes_{M'} E_\Theta \) (defining a product system \( X_{\Theta, \Phi} \) over \( \mathbb{N}^2 \)) such that the identity representations \( \Theta \) and \( \Phi \) commute strongly. It follows that there is an isomorphism \( u \) of \( \mathcal{U} \) such that \( \Phi \) intertwines the left actions of \( \Theta \) and \( \Phi \).

\[
\tilde{T}_{\Theta}(I_{E_\Theta} \otimes \tilde{T}_{\Phi}) = \tilde{T}_{\Phi}(I_{E_\Phi} \otimes \tilde{T}_{\Theta}) \circ (t_{\Theta, \Phi} \otimes I_H) \tag{17}
\]

(definition of a representation of the resulting product system such that, for every \( n, m, \tilde{T}_{\Theta m}(I_{E_n} \otimes \tilde{T}_{\Phi n})(I_{E_n} \otimes I_{E_n} \otimes a)(I_{E_n} \otimes \tilde{T}_{\Phi n})^* \tilde{T}_{\Theta m} = \Theta^n(\Phi^m(a)) \), \( a \in M \).

(3) There is a product system \( X(m, n) \) (\( (m, n) \in \mathbb{N}^2 \)) of \( W^* \)-correspondences over a von Neumann algebra \( \mathcal{N} \) (with \( E = X(1, 0) \) and \( F = X(0, 1) \)) and a representation \( (\sigma, T, S) \) of \( X \) on \( H \) such that \( \sigma \) is injective, \( M = \sigma(\mathcal{N})' \), \( T \) and \( S \) are injective maps (of \( E \) or \( F \) into \( B(H) \)) and, for \( a \in M \), \( \tilde{T}(I_E \otimes a)\tilde{T}^* = \Theta(a) \) and \( \tilde{S}(I_F \otimes a)\tilde{S}^* = \Phi(a) \).

**Proof.** We start by proving that (1) implies (2). Thus, we assume that \( \Theta \) and \( \Phi \) commute strongly. It follows that there is an isomorphism \( u : H_{\Phi, \Theta} \to H_{\Theta, \Phi} \) that maps \( I \otimes_{\Theta} I \otimes_{\Theta} h \) to \( I \otimes_{\Phi} I \otimes_{\Phi} h \) and satisfies the conditions of Definition 5.1. Write \( \Psi \) for the map taking \( Z \in E_{\Phi, \Theta} \) to \( u \circ Z \in E_{\Theta, \Phi} \). The fact that \( u \) intertwines the representations of \( M \) shows that \( \Psi(Z) \) is indeed in \( E_{\Theta, \Phi} \). It is clearly an isomorphism of \( W^* \)-modules. To see that it also intertwines the left actions of \( M' \) on \( E_{\Phi, \Theta} \) and on \( E_{\Theta, \Phi} \), we compute, for \( d \in M' \), \( \varphi(d)(u \circ Z) = (I_M \otimes I_M \otimes d)u \circ Z = u(I_M \otimes I_M \otimes d)Z = u \circ (\varphi(d)Z) \).

Recall that \( \Gamma_{\Phi, \Theta} \) is the isomorphism of \( E_{\Theta} \otimes_{M'} E_{\Phi} \) onto \( E_{\Phi} \otimes_{M'} E_{\Theta} \) mapping \( X \otimes Y \) to \( (I \otimes X)Y \) ([22] Proposition 2.12]) and write \( t = t_{\Theta, \Phi} : E_\Theta \otimes_{M'} E_\Phi \to E_\Phi \otimes_{M'} E_\Theta \) for the isomorphism defined by

\[
t_{\Theta, \Phi} = \Gamma^{-1}_{\Phi, \Theta} \circ \Psi \circ \Gamma_{\Phi, \Theta}. \tag{18}
\]

We shall now turn to prove (9).

First, let \( U_\Theta \) be the map from \( M \otimes_{\Phi} H \) to \( M \otimes_{\Theta} M \otimes_{\Phi} H \) defined by \( U_\Theta(b \otimes_{\Phi} h) = I \otimes_{\Theta} b \otimes_{\Phi} h \). Then \( U_\Theta \) is a well defined contractive map and its adjoint is \( U_\Theta^*(a \otimes_{\Theta} b \otimes_{\Phi} h) = \Theta(a)b \otimes_{\Phi} h \). Also, we have

\[
W_\Phi^* U_\Theta^* u = W_\Theta^* U_\Phi^*. \tag{19}
\]

To see this, we compute, for \( h \in H \),

\[
u^* U_\Theta W_\Phi h = u^*(I \otimes_{\Theta} I \otimes_{\Phi} h) = I \otimes_{\Phi} I \otimes_{\Theta} h = U_\Phi W_\Theta h
\]
and (13) follows.

Note also that, for $X \in E_\Theta$, $W_\Phi X^*(a \otimes_\Theta h) = I \otimes_\Phi (X^*(a \otimes_\Theta h)) = (I_M \otimes X^*)(I \otimes_\Phi a \otimes_\Theta h) = (I_M \otimes X^*)U_\Phi (a \otimes_\Theta h)$. Thus, $U_\Phi^*(I_M \otimes X) = XW_\Phi^*$ and, consequently, for $X \in E_\Theta$ and $Y \in E_\Phi$,

$$U_\Phi^* \Gamma_{\Phi,\Theta} (X \otimes Y) = U_\Phi^* (I_M \otimes X)Y = XW_\Phi^* Y. \tag{20}$$

It follows that, for $h \in H$,

$$W_\Phi^* U_\Phi^* (\Gamma_{\Phi,\Theta}(X \otimes Y)) h = T_\Theta(X)T_\Phi(Y) h = \tilde{T}_\Theta(I \otimes \tilde{T}_\Phi)(X \otimes Y \otimes h). \tag{21}$$

Thus, $\tilde{T}_\Theta(I \otimes \tilde{T}_\Phi)(X \otimes Y \otimes h) = W_\Phi^* U_\Phi^* (\Gamma_{\Phi,\Theta}(X \otimes Y)) h = W_\Phi^* U_\Phi^* (\Gamma_{\Phi,\Theta}(t(X \otimes Y)) h = \tilde{T}_\Phi(I \otimes \tilde{T}_\Theta)(t(X \otimes Y) \otimes h) = \tilde{T}_\Phi(I \otimes \tilde{T}_\Theta)(t \otimes I_H)(X \otimes Y \otimes h)$. This proves (17).

Finally, the equation $\tilde{T}_\Theta_m(I_{E_m} \otimes \tilde{T}_\Phi_m)(I_{E_m} \otimes I_{F_m} \otimes a)(I_{E_m} \otimes \tilde{T}_\Phi_m)^*\tilde{T}_\Phi_m = \Theta^m(\Phi^m(a))$ for $a \in M$ and arbitrary $m, n$ follows easily from the cases $n = 1, m = 0$ and $m = 1, n = 0$. These, in turn, follow from [22, Corollary 2.23].

This completes the proof that (1) implies (2). Since (2) obviously implies (3) (using [22, Corollary 2.23]), we now assume that (3) holds and turn to prove (1). As $\sigma$ is assumed to be injective and $M = \sigma(N)'$, we can replace $N$ by $\sigma(N)$ and assume $\sigma = id$ (and $N = M'$).

We start by defining the map $\Lambda_{\Theta,\Phi} : M \otimes_\Theta M \otimes_\Phi H \rightarrow F \otimes_\Phi E \otimes_\Theta H$ by

$$\Lambda_{\Theta,\Phi}(a \otimes_\Theta b \otimes_\Phi h) = (I_F \otimes (I_E \otimes a)\tilde{T}^*b)\tilde{S}^*h$$

and the map $\Lambda_{\Phi,\Theta} : M \otimes_\Phi M \otimes_\Theta H \rightarrow E \otimes_\Theta F \otimes_\Phi N$ by

$$\Lambda_{\Phi,\Theta}(a \otimes_\Phi b \otimes_\Theta h) = (I_E \otimes (I_F \otimes a)\tilde{S}^*b)\tilde{T}^*h.$$}

We shall show that these maps are (well defined, surjective) unitary maps and the map $u := \Lambda_{\Theta,\Phi}^{-1} \circ (t \otimes I_H) \circ \Lambda_{\Phi,\Theta}$ (22)

where $t : E \otimes M', F \rightarrow F \otimes M', E$ is an isomorphism satisfying $\tilde{T}(I_E \otimes \tilde{S}) = \tilde{S}(I_F \otimes \tilde{T})(t \otimes I_H)$, satisfies the conditions of Definition 5.11.

We first compute, for every $a, b, c, d \in M$ and $h, k \in H$, $(a \otimes_\Theta b \otimes_\Phi h, c \otimes_\Phi d \otimes_\Theta k) = \langle h, \Phi(b'^*\Theta(a^*c)d)k \rangle = \langle h, \tilde{S}(I_F \otimes b'^*\tilde{T}(I_E \otimes a^*c)\tilde{T}^*b)\tilde{S}^*k \rangle = \langle (I_F \otimes (I_E \otimes a)\tilde{T}^*b)\tilde{S}^*h, (I_F \otimes (I_E \otimes c)\tilde{T}^*d)\tilde{S}^*k \rangle$.

This shows that $\Lambda_{\Theta,\Phi}$ is a well defined isometric map. Note also that, for $c \in M$,

$$\Lambda_{\Theta,\Phi} \circ (c \otimes I_M \otimes I_H) = (I_F \otimes I_E \otimes c)\Lambda_{\Theta,\Phi}. \tag{23}$$
To show that the map is surjective, note first that the subspace of $F \otimes_N H$ spanned by vectors of the form $(I_F \otimes b)\tilde{S}^*h$, for $h \in H$ and $b \in M$, is invariant under $I_F \otimes M$ and, thus, the projection onto this subspace lies in $(I_F \otimes M)' = \mathcal{L}(F) \otimes I_H$. Write $q \otimes I_H$ for it. If $q \neq I$, there is some $\xi = (I - q)\xi \neq 0$ in $F$. But then, for all $h, k \in H$ and $b \in M$, $0 = \langle \xi \otimes k, (I_F \otimes b)\tilde{S}^*h \rangle = \langle \tilde{S}(\xi \otimes bk), h \rangle = \langle S(\xi)bk, h \rangle$ contradicting the assumed injectivity of $S$. Thus the closed subspace spanned by vectors of the form $(I_F \otimes b)\tilde{S}^*h$ is all of $F \otimes H$. Applying a similar argument to $T$ completes the proof that $\Lambda_{\Theta, \Phi}$ is surjective.

Thus $\Lambda_{\Theta, \Phi}$ and $\Lambda_{\Phi, \Theta}$ are unitary maps. Since $(t \otimes I_H)$ is unitary, so is $u$. Using (23), we find that $u$ intertwines the representations of $M$. For $h \in H$, $u(I \otimes_\Theta I \otimes_\Theta h) = \Lambda_{\Theta, \Phi}^{-1} \circ (t \otimes I_H) \circ \Lambda_{\Phi, \Theta}(I \otimes_\Phi I \otimes_\Theta h) = \Lambda_{\Phi, \Theta}^{-1} \circ (t \otimes I_H)(I_E \otimes \tilde{S}^*)\tilde{T}^*h = \Lambda_{\Theta, \Phi}^{-1}(I_F \otimes \tilde{T}^*)\tilde{S}^*h = I \otimes_\Theta I \otimes_\Theta h$, proving that $u$ satisfies conditions (i) and (ii) of Definition 5.1. To see that it satisfies condition (iii), note that $\Lambda_{\Phi, \Theta}(I \otimes I \otimes d) = (\varphi_E(d) \otimes I_F \otimes I_H)\Lambda_{\Phi, \Theta}$ (for $d \in M'$) and $\varphi_E(d) \otimes I_F = (\varphi_E(d) \otimes I_E)t$.

Now fix a von Neumann algebra $M \subseteq B(H)$.

Suppose $\Theta$ and $\Phi$ are normal CP maps on $M$ that commute strongly. Then, by Proposition 5.6 we get a product system $X_{\Theta, \Phi}$ over $M'$, defined by $(E_{\Theta, \Phi}, E_{\Theta, \Phi}, t_{\Theta, \Phi})$, and a representation $(id, T_{\Theta, \Phi})$ of $X_{\Theta, \Phi}$ on $H$ (and $T_{\Theta}$ and $T_{\Phi}$ are injective maps). It will be convenient to refer to this construction by $\tilde{X}$; that is, $\tilde{X}(\Theta, \Phi) = (X_{\Theta, \Phi}, T_{\Theta, \Phi})$.

Conversely, suppose we start with a product system $X$ (of $W^*$-correspondences over $M'$), defined by $(E, F, t)$ and suppose $(id, T, S)$ is a c.c. representation of $X$ on $H$ (and $T$ and $S$ are injective maps). Then we get normal CP maps $\Theta$ and $\Phi$ on $M$ by setting $\Theta(a) = \tilde{T}(I_E \otimes a)\tilde{T}^*$ and $\Phi(a) = \tilde{S}(I_F \otimes a)\tilde{S}^*$ for $a \in M$. It follows from Proposition 5.6 that $\Theta$ and $\Phi$ commute strongly. We shall refer to this construction as $\tilde{\Theta}$; that is, $\tilde{\Theta}(X, T, S) = (\Theta, \Phi)$.

Now, it follows from Proposition 5.6 ((1) implies (2)) that $\tilde{\Theta} \circ \tilde{X} = id$.

The following proposition shows that $\tilde{X} \circ \tilde{\Theta}$ is an isomorphism. So that, up to isomorphisms of product systems (more precisely, of product systems with representations), these two constructions are the inverses of each other.
Proposition 5.7 Let $M \subseteq B(H)$ be a von Neumann algebra. Suppose $X$ is a product system (of $W^*$-correspondences over $M'$) defined by $(E, F, t)$ and suppose $(id, T, S)$ is a c.c. representation of $X$ on $H$ (and $T$ and $S$ are injective maps). Let $\Theta$ and $\Phi$ be the normal CP maps defined on $M$ by $\Theta(a) = \tilde{T}(I_E \otimes a)\tilde{T}^*$ and $\Phi(a) = \tilde{S}(I_F \otimes a)\tilde{S}^*$ for $a \in M$. Let $X_{\Theta, \Phi}$ be the product system constructed in the proof of 
(1) implies (2) of Proposition 5.6 (so that it is defined by $(E_\Theta, E_\Phi, t_{\Theta, \Phi})$ and $t_{\Theta, \Phi}$ is as in [18]) and let $(id, T_\Theta, T_\Phi)$ be the identity representation of $X_{\Theta, \Phi}$.

Then there are (surjective) isomorphisms $w_E : E_\Theta \to E$ and $w_F : E_\Phi \to F$ such that

1. $t \circ (w_E \otimes w_F) = (w_F \otimes w_E) \circ t_{\Theta, \Phi}$, and

2. $T \circ w_E = T_\Theta$ and $S \circ w_F = T_\Phi$.

Proof. Let $v_E : M \otimes_\Theta H \to E \otimes H$ be defined by $v_E(b \otimes h) = (I \otimes_\Theta b)\tilde{T}^*h$ and $v_F : M \otimes_\Phi H \to F \otimes H$ is defined similarly (using $S$). The argument we gave in the proof of Proposition 5.6 to show that the map $\Lambda_{\Theta, \Phi}$ is a unitary map shows also that $v_E$ and $v_F$ are well defined unitary maps. (Note that this uses the injectivity of $T$ and $S$). It was shown in [25, Theorem 2.14], using the self duality of $E$, that, for every $R \in E_\Theta$ one can find a (unique) $w_E(R) \in E$ such that, for $\xi \in E$ and $h \in H$, $\langle w_E(R), \xi \rangle h = R^*v_E^*(\xi \otimes h)$. It follows that, for every $h \in H$ and $R \in E_\Theta$,

$$w_E(R) \otimes h = v_E Rh.$$  \hspace{1cm} (24)

It is also shown there that $w_E$ is a unitary, surjective, map from $E_\Theta$ onto $E$ and that part (2) holds.

Now we turn to prove part (1). We first claim that, for every $R \in E_\Theta$, $Y \in E_\Phi$ and $g \in H$, we have

$$\Lambda_{\Phi, \Theta}(\Gamma_{\Phi, \Theta}(R \otimes Y)g) = w_E(R) \otimes w_F(Y) \otimes g.$$  \hspace{1cm} (25)

Recalling the definition of $\Lambda_{\Phi, \Theta}$, we compute, for $a, b \in M$ and $h \in H$, $\Lambda_{\Phi, \Theta}(a \otimes_\Theta b \otimes_\Theta h) = (I_E \otimes I_F \otimes a)(I_E \otimes \tilde{S}^*)(I_E \otimes b)\tilde{T}^*h = (I_E \otimes I_F \otimes a)(I_E \otimes \tilde{S}^*)(w_E(b \otimes_\Theta h)).$ This holds for every $b \otimes_\Theta h \in M \otimes_\Theta H$. In particular, it holds with $Rf$ ($R \in E_\Theta, f \in H$) in place of $b \otimes_\Theta h$. Thus $\Lambda_{\Phi, \Theta}(I_M \otimes R)(a \otimes_\Theta f) = \Lambda_{\Phi, \Theta}(a \otimes_\Phi Rf) = (I_E \otimes I_F \otimes a)(I_E \otimes \tilde{S}^*)(w_E(R) \otimes f) = w_E(R) \otimes (I_F \otimes a)\tilde{S}^*f = w_E(R) \otimes v_E(a \otimes_\Phi f)$. Now write $Yg$ (for $Y \in E_\Phi$ and $g \in H$) in place of $a \otimes_\Phi f$ to get $\Lambda_{\Phi, \Theta}(I_M \otimes R)Yg =
\(w_E(R) \otimes v_E(Yg) = w_F(R) \otimes w_F(Y) \otimes g\). Since \(\Gamma_{\Phi,\Theta}(R \otimes Y) = (I_M \otimes R)Y\), this proves (25).

For \(Z \in E_{\Phi,\Theta}, \Gamma_{\Phi,\Theta}^{-1}(Z)\) lies in \(E_{\Theta} \otimes E_{\Phi}\) and we can apply (25) to it (in place of \(R \otimes Y\)) and get \(\Lambda_{\Phi,\Theta}(Zg) = (w_E \otimes w_F)(\Gamma_{\Phi,\Theta}^{-1}(Z)) \otimes g\). Interchanging \(\Phi\) and \(\Theta\), we have \(\Lambda_{\Theta,\Phi}(R \otimes Y) = (I_M \otimes R)Y\), this proves (25).

For \(Z \in E_{\Phi,\Theta}, \Gamma_{\Phi,\Theta}^{-1}(Z)\) lies in \(E_{\Theta} \otimes E_{\Phi}\) and we can apply (25) to it (in place of \(R \otimes Y\)) and get \(\Lambda_{\Phi,\Theta}(Zg) = (w_E \otimes w_F)(\Gamma_{\Phi,\Theta}^{-1}(Z)) \otimes g\). Interchanging \(\Phi\) and \(\Theta\), we get \(\Lambda_{\Theta,\Phi}(Gg) = (w_F \otimes w_E)(\Gamma_{\Phi,\Theta}^{-1}(G)) \otimes g\). But, using (25), the left hand side of (26) is equal to \((t \otimes I_H)(w_E(R) \otimes w_F(Y) \otimes g)\).

This completes the proof of (1) \(\square\)

**Proposition 5.8** Let \(\Theta\) and \(\Phi\) be commuting normal CP maps on \(B(H)\). then they commute strongly if and only if there are \(n \leq \infty\) and \(m \leq \infty\) and operators \(T_i, S_j\) in \(B(H)\) \((1 \leq i \leq n, 1 \leq j \leq m)\) such that

\[
\Theta(a) = \sum_{i=1}^{n} T_i a T_i^*, \ a \in B(H),
\]

\[
\Phi(a) = \sum_{i=1}^{m} S_i a S_i^*, \ a \in B(H)
\]

(where, if the sum is infinite, it is assumed to converge in the weak operator topology) and \(\{T_i\}\) and \(\{S_j\}\) satisfy the following conditions.

(i) \(\sum T_i T_i^* \leq I\) and \(\sum S_j S_j^* \leq I\).

(ii) \((l^2\text{-independence})\) \(\sum \alpha_i T_i \neq 0\) whenever \(\alpha = \{\alpha_i\} \in l^2\) is nonzero (and similarly for \(\{S_j\}\)).

(iii) There is a unitary matrix \(u = (u_{k,l}^{(i,j)})_{(i,j)(k,l)}\) (whose rows and columns are indexed by the set of pairs \((i, j)\) with \(i \leq n, j \leq m\)) such that, for all \(i, j\),

\[
T_i S_j = \sum_{k,l} u_{k,l}^{(i,j)} S_i T_k.
\]
Proof. This is, in fact, a restatement of the equivalence of (1) and (3) in Proposition \[5.6\] for the case when \( M = B(H) \).

\hspace{1cm} \square

**Lemma 5.9** Suppose \( E \) and \( F \) are \( W^* \)-correspondences over a von Neumann algebra \( N \) and \( t : E \to F \) is a partial isometry in \( L(E, F) \) that intertwines the left actions of \( N \). (We shall refer to such a map as a bimodule partial isometry). Then there are projections \( z_1 \) and \( z_2 \) (in the center of \( \mathcal{L}(E) \cap \varphi_E(N)' \) and the center of \( \mathcal{L}(F) \cap \varphi_F(N)' \) respectively) and two bimodule partial isometries \( t_1, t_2 \) in \( \mathcal{L}(E, F) \) such that

(i) \( t_1^* t_1 = z_1 \) and \( t_1 t_1^* \leq z_2 \) (so that we can view it as a bimodule isometry from \( z_1 E \) into \( z_2 F \)).

(ii) \( t_2^* t_2 \leq I_E - z_1 \) and \( t_2 t_2^* = I_F - z_2 \) (so that we view it as a bimodule coisometry from \( (I_E - z_1)E \) onto \( (I_F - z_2)F \)).

(iii) \( t_1 \) extends \( t_0 z_1 \) and \( t_2 \) extends \( t_0 (I_E - z_1) \).

(iv) \( (t_1 + t_2)z_1 = z_2 (t_1 + t_2) \).

**Proof.** View \( t_0 \) as a partial isometry from \( E \oplus F \) into \( E \oplus F \) (by letting it be \( 0 \) on \( F \)). Then it is a partial isometry in the von Neumann algebra \( R := L(E \oplus F) \cap \varphi_{E \oplus F}(N)' \) (since it is a bimodule map). Apply the Comparison Theorem (\[13\] Theorem 6.2.7) to the projections \( f_1 := I_E - t_0^* t_0 \) and \( f_2 := I_F - t_0^* t_0 \) to find a central projection \( z \) in \( R \) and partial isometries \( v_1 \) and \( v_2 \) in \( R \) with \( v_1^* v_1 = f_1 z, v_1 v_1^* \leq f_2 z, v_2^* v_2 \leq f_1 (I - z) \) and \( v_2 v_2^* = f_2 (I - z) \). Finally, set \( z_1 = z I_E, z_2 = z I_F, t_1 = t_0 z_1 + v_1 \) and \( t_2 = t_0 (I_E - z_1) + v_2 \).

\hspace{1cm} \square

**Lemma 5.10** Let \( E_0 \) and \( F_0 \) be two \( W^* \)-correspondences over a von Neumann algebra \( N \) and \( t_0 : E_0 \otimes F_0 \to F_0 \otimes E_0 \) be a partial isometry (of \( W^* \)-correspondences; that is, it is an adjointable bimodule map). Then there is a partial isometry (of \( W^* \)-correspondences) \( t : E_0 \otimes F_0 \to F_0 \otimes E_0 \) that extends \( t_0 \) and there are \( W^* \)-correspondences \( E \) and \( F \) over \( N \) containing \( E_0 \) and \( F_0 \) respectively (as subcorrespondences), an isomorphism (of correspondences) \( s : E \otimes F \to F \otimes E \) and projections \( e_1 \) and \( e_2 \) such that (writing \( q_E \) and \( q_F \) for the projections of \( E \) and \( F \) onto \( E_0 \) and \( F_0 \) respectively) we have

(i) \( e_1 \) lies in the center of \( \mathcal{L}(E \otimes F) \cap \varphi_{E \otimes F}(N)' \) and \( e_2 \) lies in the center of \( \mathcal{L}(F \otimes E) \cap \varphi_{F \otimes E}(N)' \).

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\( \text{(ii)} \) \( se_1(q_E \otimes q_F) = te_1(q_E \otimes q_F) = (q_F \otimes q_E)te_1(q_E \otimes q_F) \) and this map is an isometry from \( e_1(E_0 \otimes F_0) \) into \( e_2(F_0 \otimes E_0) \).

\( \text{(iii)} \) \( (I-e_2)(q_F \otimes q_E)s = t(I-e_1)(q_E \otimes q_F) = (q_F \otimes q_E)t(I-e_1)(q_E \otimes q_F) \) and this map is a coisometry from \( (I-e_1)(E_0 \otimes F_0) \) onto \( (I-e_2)(F_0 \otimes E_0) \).

\( \text{(iv)} \) \( te_1 = e_2t \).

**Proof.** Applying Lemma 5.9 to \( t_0 \), we get projections \( z_1 \) (in the center of \( \mathcal{L}(E_0 \otimes F_0) \cap \varphi_{E_0 \otimes F_0}(N)' \)) and \( z_2 \) (in the center of \( \mathcal{L}(F_0 \otimes E_0) \cap \varphi_{F_0 \otimes E_0}(N)' \)) and partial isometries \( t_1 \) and \( t_2 \) (that are bimodule maps) satisfying the conditions of that lemma. Write \( t = t_1 + t_2 \). Then \( t \) is a partial isometry, \( t(z_1) \) is an isometry from \( z_1(E_0 \otimes F_0) \) into \( z_2(F_0 \otimes E_0) \) and \( t(I_{E_0 \otimes F_0} - z_1) \) is a coisometry from \( (I_{E_0 \otimes F_0} - z_1)(E_0 \otimes F_0) \) onto \( (I_{F_0 \otimes E_0} - z_2)(F_0 \otimes E_0) \).

Let \( E_1 \) be a \( W^* \)-correspondence over \( N \) that is isomorphic to \( E_0 \) and let \( F_1 \) be isomorphic to \( F_0 \). These isomorphisms induce (surjective) isomorphisms \( \tau : E_0 \otimes F_0 \to E_1 \otimes F_1 \), \( \theta : F_1 \otimes E_1 \to F_0 \otimes E_0 \) and \( \gamma : (E_0 \otimes E_1) \oplus (F_1 \otimes F_0) \to (E_1 \otimes E_0) \oplus (F_0 \otimes F_1) \). Write \( E = E_0 \oplus F_1 \) and \( F = E_1 \oplus F_0 \) and let \( q_E \) and \( q_F \) be the projections of \( E \) onto \( E_0 \) and \( F \) onto \( F_0 \) respectively. Also write \( q_1 \) for the projection \( q_E \otimes q_F \) (from \( E \otimes F \) onto \( E_0 \otimes F_0 \)) and write \( q_2 \) for \( q_F \otimes q_E \). Clearly \( q_1 \in \mathcal{L}(E \otimes F) \cap \varphi_{E \otimes F}(N)' \) and \( q_2 \in \mathcal{L}(F \otimes E) \cap \varphi_{E \otimes F}(N)' \).

The isomorphism \( s : E \otimes F \to F \otimes E \) will be written matricially with respect to the decompositions \( E \otimes F = (E_0 \otimes F_0) \oplus (F_1 \otimes E_1) \oplus (E_0 \otimes E_1) \oplus (F_1 \otimes F_0) \) and \( F \otimes E = (F_0 \otimes E_0) \oplus (E_1 \otimes F_1) \oplus (E_1 \otimes E_0) \oplus (F_0 \otimes F_1) \) as

\[
\begin{pmatrix}
  t & \theta - tt^*\theta & 0 \\
  \tau - \tau t^*t & \tau t^*\theta & 0 \\
  0 & 0 & \gamma
\end{pmatrix}
\]

Clearly, \( s \) is an isomorphism of correspondences.

Note that \( \mathcal{L}(E_0 \otimes F_0) \cap \varphi_{E_0 \otimes F_0}(N)' = q_1(\mathcal{L}(E \otimes F) \cap \varphi_{E \otimes F}(N)'q_1. Thus, there is a projection \( e_1 \) in the center of \( \mathcal{L}(E \otimes F) \cap \varphi_{E \otimes F}(N)' \) such that \( z_1 = q_1e_1q_1 \) (see [13] Proposition 5.5.6 and Corollary 5.5.7)]. Similarly we get \( e_2 \) in the center of \( \mathcal{L}(F \otimes E) \cap \varphi_{F \otimes E}(N)' \) satisfying \( q_2e_2q_2 = z_2 \).

Since \( t^*z_1 = t_1^*t_1z_1 = z_1 \), we see that \( se_1q_1 = sz_1 = t_1e_1q_1 = q_2te_1q_1 \) is an isometry from \( z_1(E_0 \otimes F_0) \) into \( z_2(F_0 \otimes E_0) \). This proves (ii) and a similar argument works for (iii). Part (iv) here follows from part (iv) of Lemma 5.9.

\( \square \)
Proposition 5.11 Let $\Theta$ and $\Phi$ be two commuting normal CP maps on $M \subseteq B(H)$. Then there is a product system $X(m, n)$ ($(m, n) \in \mathbb{N}^2$) of $W^*$-correspondences over the von Neumann algebra $M'$ (with $E = X(1, 0)$ and $F = X(0, 1)$) and a representation $(\text{id}, T, S)$ of $X$ on $H$ such that, for $a \in M$, $\hat{T}(I_E \otimes a)\hat{T}^* = \Theta(a)$ and $\hat{S}(I_F \otimes a)\hat{S}^* = \Phi(a)$.

Proof. We shall follow the idea of the proof of (1) implies (2) in Proposition 5.6 making changes when necessary. Since $\Theta$ and $\Phi$ commute, it follows from Remark 5.2 that there is a partial isometry $u_0$ in $B(H_{\Phi, \Theta}, H_{\Theta, \Phi})$ that is defined by the formula $u_0(a \otimes \Phi I \otimes \Theta h) = a \otimes \Theta I \otimes \Phi h$ (for $a \in M$ and $h \in H$) and vanishes on the orthogonal complement of the space spanned by the vectors $a \otimes \Phi I \otimes \Theta h$, $a \in M$, $h \in H$. It is easy to check that $u_0$ intertwines both the actions of $M$ on $H_{\Phi, \Theta}$ and on $H_{\Theta, \Phi}$ and the actions of $M'$ on these spaces. We now write $\Psi_0$ for the map taking $Z \in E_{\Phi, \Theta}$ to $u_0 \circ Z \in E_{\Theta, \Phi}$. Then $\Psi_0$ is a partial isometry and a bimodule map. As in (19), we have $W_{\Phi}U_{\Theta}^* = W_{\Theta}U_{\Phi}^*u_0^*$ and $W_{\Phi}U_{\Theta}^*u_0 = W_{\Theta}U_{\Phi}^*$. Thus we have

$$W_{\Phi}U_{\Theta}^*Z = W_{\Theta}U_{\Phi}^*\Psi_0(Z), \ Z \in E_{\Phi, \Theta}$$

(27)

and

$$W_{\Phi}U_{\Theta}^*\Psi_0(R) = W_{\Theta}U_{\Phi}^*R, \ R \in E_{\Theta, \Phi}.$$  

(28)

We now set

$$t_0 = \Gamma_{\Theta, \Phi}^{-1} \circ \Psi_0 \circ \Gamma_{\Phi, \Theta}$$

where $\Gamma_{\Phi, \Theta}$ is the isomorphism of $E_{\Theta} \otimes_{M'} E_{\Phi}$ onto $E_{\Phi, \Theta}$ mapping $X \otimes Y$ to $(I \otimes X)Y$. The map $t_0$ is a partial isometry (of correspondences) from $E_{\Theta} \otimes_{M'} E_{\Phi}$ to $E_{\Theta} \otimes_{M'} E_{\Phi}$. Applying Lemma 5.10 to $t_0$, we get $W^*$-correspondences $E$ and $F$ over $M'$, projections $e_1$ and $e_2$, a partial isometry $t$ extending $t_0$ and an isomorphism $s : E \otimes_{M'} F \rightarrow F \otimes_{M'} E$ such that $E_{\Theta} \subseteq E$, $E_{\Phi} \subseteq F$ and we have (writing $q_{\Theta}$ and $q_{\Phi}$ for the projections onto $E_{\Theta}$ and $E_{\Phi}$ respectively),

(i) $e_1$ lies in the center of $\mathcal{L}(E \otimes F) \cap \varphi_{E \otimes F}(M')'$ and $e_2$ lies in the center of $\mathcal{L}(F \otimes E) \cap \varphi_{F \otimes E}(M')'$.

(ii) $s e_1(q_{\Theta} \otimes q_{\Phi}) = te_1(q_{\Theta} \otimes q_{\Phi}) = (q_{\Phi} \otimes q_{\Theta})te_1(q_{\Theta} \otimes q_{\Phi})$ and this map is an isometry from $e_1(E_{\Theta} \otimes E_{\Phi})$ into $e_2(E_{\Phi} \otimes E_{\Theta})$.

(iii) $(I - e_2)(q_{\Phi} \otimes q_{\Theta})s = t(I - e_1)(q_{\Theta} \otimes q_{\Phi}) = (q_{\Phi} \otimes q_{\Theta})t(I - e_1)(q_{\Theta} \otimes q_{\Phi})$ and this map is a coisometry from $(I - e_1)(E_{\Theta} \otimes E_{\Phi})$ onto $(I - e_2)(E_{\Phi} \otimes E_{\Theta})$.

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(iv) \( te_1 = e_2 t \).

Define \( \Psi = \Gamma_{\Theta,F} \circ t \circ \Gamma^{-1}_{\phi,F} \). Then \( \Psi \) extends \( \Psi_0 \) (that is, \( \Psi_0 = \Psi_0 \Psi_0 \)). and we have, using (27) and (28),

\[
W^*_{\Theta} U^*_\Psi Z = W^*_{\Theta} U^*_\Psi \Psi_0(Z) = W^*_{\Theta} U^*_\Psi \Psi_0 \Psi^*(Z) = W^*_{\Theta} U^*_\Psi \Psi^*(Z)
\]

for \( Z \in E_{\Theta,F} \).

Now let \( T = T_\Theta q_\Theta, S = T_\Phi q_\Phi \) and \( \sigma = \text{id} \). Then \( (\sigma, T) \) and \( (\sigma, S) \) are c.c. representations of \( E \) and \( F \) respectively. (To see this, note that \( q_\Theta \) and \( q_\Phi \) are bimodule maps since they project onto submodules.)

Note that (21) (in the proof of Proposition 5.6) still holds whenever \( X = q_\Theta(X) \) and \( Y = q_\Phi(Y) \). So we can compute, for \( X \in E, Y \in F \) and \( h \in H \),

\[
S(I \otimes T)(e_2 \otimes I_H)(Y \otimes X \otimes h) = \tilde{T}_\phi(I \otimes \tilde{T}_\phi)(e_2 \otimes I_H)(q_\Phi(Y) \otimes q_\Theta(X) \otimes h) = W^*_{\Theta} U^*_\Psi (\Gamma_{\Theta,F} e_2(q_\Phi(Y) \otimes q_\Theta(X))) h = W^*_{\Theta} U^*_\Psi (\Gamma_{\Theta,F} e_2(q_\Phi(Y) \otimes q_\Theta(X))) h = W^*_{\Theta} U^*_\Psi (\Gamma_{\Theta,F}(t^* e_2(q_\Phi(Y) \otimes q_\Theta(X))) h = \tilde{T}_{\Theta}(I \otimes \tilde{T}_{\Theta})(t^* e_2(q_\Phi(Y) \otimes q_\Theta(X))) h = \tilde{T}_{\Theta}(I \otimes \tilde{T}_{\Theta})(t^* \otimes I_H)(e_2 \otimes I_H)(q_\Phi(Y) \otimes q_\Theta(X) \otimes h) = \tilde{T}_{\Theta}(I \otimes \tilde{T}_{\Theta})(t^* \otimes I_H)(e_2 \otimes I_H)(q_\Phi(Y) \otimes q_\Theta(X) \otimes h).
\]

Using (ii) above, we find that \( te_1(q_\Theta \otimes q_\Phi) = e_2t(q_\Theta \otimes q_\Phi) \) and, therefore, \( e_1(q_\Theta \otimes q_\Phi)t^* = (q_\Theta \otimes q_\Phi)t^* e_2 \).

Thus \( S(I \otimes T)(e_2 \otimes I_H)(Y \otimes X \otimes h) = \tilde{T}_{\Theta}(I \otimes \tilde{T}_{\Theta})(e_1 \otimes I_H)(q_\Theta \otimes q_\Phi \otimes I_H)(t^* \otimes I_H)(q_\Theta(Y) \otimes q_\Theta(X) \otimes h) = \tilde{T}_{\Theta}(I \otimes \tilde{T}_{\Theta})(e_1 \otimes I_H)(q_\Theta \otimes q_\Phi \otimes I_H)(s^* \otimes I)(Y \otimes X \otimes h) = \tilde{T}(I \otimes \tilde{T})(e_1 \otimes I_H)((s^* \otimes I)(Y \otimes X \otimes h)). \) (Here we used (ii) and the fact that \( \tilde{T}(I \otimes \tilde{T}) = \tilde{T}(I \otimes \tilde{T})(q_\Theta \otimes q_\Phi \otimes I_H) \).

Since this holds for every \( Y \times X \in F \times X \), it holds for \( s(X \otimes Y) \).

Thus \( \tilde{T}(I \otimes \tilde{T})(e_1 \otimes I_H)(X \otimes Y \otimes h) = \tilde{s}(I \otimes \tilde{T})(e_2 \otimes I_H)(s \otimes I_H)(X \otimes Y \otimes h) \). (29)

This dealt with the “isometric” part. Now we turn to the “coisometric” one and we compute, for \( X, Y \) and \( h \) as above,

\[
\tilde{T}(I \otimes \tilde{T})(((I-e_1) \otimes I_H)(X \otimes Y \otimes h) = \tilde{T}_\phi(I \otimes \tilde{T}_\phi)((I-e_1) \otimes I_H)(q_\Theta(X) \otimes q_\Phi(Y) \otimes h) = W^*_{\Theta} U^*_\Psi (\Gamma_{\Theta,F}(e_1 \otimes I_H)(q_\Theta(X) \otimes q_\Phi(Y))) = W^*_{\Theta} U^*_\Psi (\Gamma_{\Theta,F}(t(I - e_1)(q_\Theta(X) \otimes q_\Phi(Y)))) = \tilde{T}_\phi(I \otimes \tilde{T}_\phi)(t(I - e_1)(q_\Theta(X) \otimes q_\Phi(Y))) = \tilde{T}_\phi(I \otimes \tilde{T}_\phi)(t \otimes I_H)(q_\Theta(X) \otimes q_\Phi(Y) \otimes h) = \tilde{T}_\phi(I \otimes \tilde{T}_\phi)((I-e_1) \otimes I_H)(q_\Theta(X) \otimes q_\Phi(Y) \otimes h) = \tilde{s}(I \otimes \tilde{T})(((I-e_2) \otimes I)(s \otimes I)(X \otimes Y \otimes h). \)

Thus, \( \tilde{T}(I \otimes \tilde{T})(((I-e_1) \otimes I_H)(X \otimes Y \otimes h) = \tilde{s}(I \otimes \tilde{T})(((I-e_2) \otimes I_H)(s \otimes I_H)(X \otimes Y \otimes h). \) (30)

Adding up Equations (29) and (30), we get

\[
\tilde{T}(I \otimes \tilde{T}) = \tilde{s}(I \otimes \tilde{T})(s \otimes I).
\] (31)
This shows that \((\sigma, T, S)\) is indeed a representation of the system defined by \((E, F, s)\).

Finally, for \(a \in M\),
\[
\tilde{T}(I_E \otimes a)\tilde{T}^* = \tilde{T}_\Theta(q_\Theta \otimes I_H)(I_E \otimes a)(q_\Theta \otimes I_H)\tilde{T}_\Theta^* = \tilde{T}_\Theta(I_{E_\Theta} \otimes a)\tilde{T}_\Theta^* = \Theta(a)
\]
and a similar computation applies to \(\Phi\).

\[\square\]

**Definition 5.12** Let \(M \subseteq B(H)\) be a von Neumann algebra and let \(\Phi\) and \(\Theta\) be two normal CP maps on \(M\). An endomorphic dilation of the pair \((\Phi, \Theta)\) is a pair \((\alpha, \beta)\) of normal, commuting, \(*\)-endomorphisms of a von Neumann algebra \(R \subseteq B(K)\) and an isometry \(W : H \to K\) such that

\[(i) \ M = W^*RW, \]
\[(ii) \ \alpha(WW^*)WW^* = \alpha(I)WW^* \text{ and } \beta(WW^*)WW^* = \beta(I)WW^*, \]
\[(iii) \ \Phi(a) = W^*\alpha(WaW^*)W \text{ and } \Theta(a) = W^*\beta(WaW^*)W \text{ for all } a \in M.\]

**Theorem 5.13** Let \(M \subseteq B(H)\) be a von Neumann algebra and \(\Theta\) and \(\Phi\) be two commuting normal CP maps on \(M\). Then the pair \((\Theta, \Phi)\) has an endomorphic dilation.

**Proof.** Let \(X\) and \((id, T, S)\) be as in Proposition 5.11. Using Theorem 4.4 we find a Hilbert space \(K\), an isometric map \(W : H \to K\) and an isometric representation \((\rho, V, U)\) of \(X\) on \(K\) that dilates \((id, T, S)\). Write \(R = \rho(M')' \subseteq B(K)\) and let
\[
\alpha(b) = \tilde{V}(I_E \otimes b)\tilde{V}^*, \ b \in R
\]
and
\[
\beta(b) = \tilde{U}(I_F \otimes b)\tilde{U}^*, \ b \in R.
\]

Then, as was shown in Proposition 5.6 \(\alpha\) and \(\beta\) are two commuting, normal, CP maps on \(R\). It follows from [22, Proposition 2.21] that \(\alpha\) and \(\beta\) are \(*\)-endomorphisms of \(R\).

Since \((\rho, V, U)\) is an isometric dilation of \((id, T, S)\), we can write \(W : H \to K\) for the isometric embedding of \(H\) into \(K\) (so that \(WW^*\) is the projection of \(K\) onto \(H\)) and get
\[
W^*\tilde{V}(I_E \otimes W) = \tilde{T}
\]
and
\[
\tilde{V}^*WW^* = (I_E \otimes WW^*)\tilde{V}^*WW^*.
\]
Thus, for $a \in M$,
\[ \Theta(a) = \tilde{T}(I_E \otimes a)\tilde{T}^* = W^*\tilde{V}(I_E \otimes W)(I_E \otimes a)(I_E \otimes W^*)\tilde{V}^*W = \]
\[= W^*\alpha(WaW^*)W \quad (34) \]
and a similar equality holds for $\Phi$. Also, $\alpha(I)WW^* = \tilde{V}\tilde{V}^*WW^* = \tilde{V}(I_E \otimes WW^*)\tilde{V}^*WW^* = \alpha(WW^*)WW^*$. (And similarly for $\beta$). □

**Remark 5.14** Consider the commuting CP maps of Example 5.5. Let $S$ and $k$ be as assumed there. Note that $\Theta(a) = \langle ak, k \rangle I = \sum_{i=1}^{\infty}(e_i \otimes k^*)a(e_i \otimes k^*)^*$ and $\Phi(a) = SaS^*$ (where $\{e_i\}$ is a fixed orthonormal basis of $H$ and we write $x \otimes y^*$ for the rank-one operator $(x \otimes y^*)h = \langle h, y \rangle x$). Write $T_i = e_i \otimes k^*$. We can dilate $\Theta$ to an endomorphism (of some $B(K)$) by using Popescu’s isometric dilation of the row contraction $\{T_i\}$ and we can dilate $\Phi$ using an isometric dilation of the contraction $S$. But these two endomorphisms will not commute. The proof of Theorem 5.13 shows that in order to get a commuting pair, one needs to add zeroes to both families ($\{T_i\}$ and $\{S\}$) before dilating them.

In the case where $\Theta$ and $\Phi$ commute strongly we can say more.

**Proposition 5.15** If $\Theta$ and $\Phi$ are two normal CP maps on $M$ that commute strongly then $(\Theta, \Phi)$ has an endomorphic dilation $(\alpha, \beta)$ such that

1. $E_{\Theta}$ is isomorphic (as a correspondence) to $E_{\alpha}$ and $E_{\Phi}$ is isomorphic to $E_{\beta}$.

2. If $M$ is a semifinite factor and (using the terminology of [23, Definition 4.9]), the index of $\Theta$ is finite, then so is the index of $\alpha$ and the two indices are equal. (Similar statement holds for $\Phi$ and $\beta$).

**Proof.** Since $\Theta$ and $\Phi$ commute strongly, we can use, in the proof of Theorem 5.13 the product system $X_{\Theta, \Phi}$ and the representation $(id, T_\theta, T_\phi)$ (instead of $X$ and $(id, T, S)$ that were obtained from Proposition 5.11). Writing $(\rho, V, U)$ for an isomorphic dilation of this representation, we get $\alpha$ and $\beta$ as in the proof of the theorem.

In the notation of the discussion preceding Proposition 5.14 we have $(\alpha, \beta) = \tilde{\Theta}(X_{\Theta, \Phi}, V, U)$. It follows from Proposition 5.6 (2) (since, as we
have shown in Lemma 5.4 (1), \( \alpha \) and \( \beta \) commute strongly) that \((\alpha, \beta) = \tilde{\Theta}(X_{\alpha,\beta}, T_\alpha, T_\beta)\). It now follows from Proposition 5.7 that \(X_{\Theta,\Phi}\) is isomorphic to \(X_{\alpha,\beta}\). In particular, (1) follows. To be more precise, \(E_\Theta\) and \(E_\Phi\) are correspondences over \(M'\) while \(E_\alpha\) and \(E_\beta\) are over \(\rho(M')\). Thus the bimodule isomorphism and the inner-product preservation are satisfied “up to \(\rho\)”. (Note that \(\rho\) is injective). Part (2) follows immediately from (1) and the fact that the index of a normal CP map \(\Theta\) depends only on \(E_\Theta\).

\(\Box\)

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