On the quantifier-free dynamic complexity of Reachability

Thomas Zeume and Thomas Schwentick
TU Dortmund University
Germany

Abstract. The dynamic complexity of the reachability query is studied in the dynamic complexity framework of Patnaik and Immerman, restricted to quantifier-free update formulas. It is shown that, with this restriction, the reachability query cannot be dynamically maintained, neither with binary auxiliary relations nor with unary auxiliary functions, and that ternary auxiliary relations are more powerful with respect to graph queries than binary auxiliary relations. Further results are obtained including more inexpressibility results for reachability in a different setting, inexpressibility results for some other queries and normal forms for quantifier-free update programs.

1 Introduction

In modern data management scenarios, data is subject to frequent changes. In order to avoid costly re-computations from scratch after each small update, one can try to (re-)use auxiliary data structures that has been already computed before to keep the information about the data up-to-date. However, the auxiliary data structures need to be updated dynamically whenever the data changes.

The descriptive dynamic complexity framework (short: dynamic complexity) introduced by Patnaik and Immerman [1] models this setting. It was mainly inspired by relational databases. For a relational database subject to change, auxiliary relations are maintained with the intention to help answering a query \( \mathcal{Q} \). When an update to the database, an insertion or deletion of a tuple, occurs, every auxiliary relation is updated through a first-order query that can refer to the database as well as to the auxiliary relations. A particular auxiliary relation shall always represent the answer to \( \mathcal{Q} \). The class of all queries maintainable in this way, and thus also in the core of SQL, is called DYNFO.

Beyond query or view maintenance in databases we consider it an important goal to understand the dynamic complexity of fundamental algorithmic problems. Reachability in directed graphs is the most intensely investigated problem in dynamic complexity (and also much studied in dynamic algorithms and other dynamic contexts) and the main query studied in this paper. It is one of the simplest inherently recursive queries and thus serves as a kind of drosophila in the study of the dynamic maintainability of recursive queries by non-recursive means. It can be maintained with first-order update formulas supplemented by
counting quantifiers on general graphs [2] and with plain first-order update formulas on both acyclic graphs and undirected graphs [1]. However, it is not known whether Reachability on general graphs is maintainable with first-order updates. This is one of the major open questions in dynamic complexity.

All attempts to show that Reachability cannot be maintained in DynFO have failed. In fact, there are no general inexpressibility results for DynFO at all. This seems to be due to a lack of understanding of the underlying mechanisms of DynFO. To improve the understanding of dynamic complexity, mainly two kinds of restrictions of DynFO have been studied: (1) limiting the information content of the auxiliary data by restricting the arity of auxiliary relations and functions and (2) reducing the amount of quantification in update formulas.

A study of bounded arity auxiliary relations was started in [3] and it was shown that unary auxiliary relations are not sufficient to maintain the reachability query with first-order updates. Further inexpressibility results for unary auxiliary relations were shown and an arity hierarchy for auxiliary relations was established. However, to separate level $k$ from higher levels, database relations of arity larger than $k$ were used. Thus the hierarchy has not yet been established for queries on graphs. In [4] it was shown that unary auxiliary relations are not sufficient to maintain Reachability for update formulas of any local logic. The proofs strongly use the “static” weakness of local logics and do not fully exploit the dynamic setting, as they only require update sequences of constant length.

The second line of research was initiated by Hesse [5]. He invented and studied the class DynProp of queries maintainable with quantifier-free update formulas. He proved that Reachability on deterministic graphs (i.e. graphs of unary functions) can be maintained with quantifier-free first-order update formulas.

There is still no proof that Reachability on general graphs cannot be maintained in DynProp. However, some inexpressibility results for DynProp have been shown in [3]: the alternating reachability query (on graphs with $\land$- and $\lor$-nodes) is not maintainable in DynProp. Furthermore, on strings, DynProp exactly contains the regular languages (as Boolean queries on strings).

**Contributions.** The high-level goal of this paper is to achieve a better understanding of the dynamic maintainability of Reachability and dynamic complexity in general. Our main result is that the reachability query cannot be dynamically maintained by quantifier-free updates with binary auxiliary relations. This result is weaker than that of [3] in terms of the logic (quantifier-free vs. general first-order) but it is stronger with respect to the information content of the auxiliary data (binary vs. unary). We establish a strict hierarchy between DynProp for unary, binary and ternary auxiliary relations (this is still open for DynFO).

We further show that Reachability is not maintainable with unary auxiliary functions (plus unary auxiliary relations). Although unary functions provide less information content than binary relations, they offer a very weak form of

---

1 Of course, a query maintainable in DynFO can be evaluated in polynomial time and thus queries that cannot be evaluated in polynomial time cannot be maintained in DynFO either.
quantification in the sense that more elements of the domain can be taken into account by update formulas.

All these results hold in the setting of Patnaik and Immerman where update sequences start from an empty database as well as in the setting that starts from an arbitrary database, where the auxiliary data is initialized by an arbitrary function. We show that if, in the latter setting, the initialization mapping is permutation-invariant, quantifier-free updates cannot maintain Reachability even with auxiliary functions and relations of arbitrary arity. Intuitively a permutation-invariant initialization mapping maps isomorphic databases to isomorphic auxiliary data. A particular case of permutation-invariant initialization mappings, studied in [7], is when the initialization is specified by logical formulas. In this case, lower bounds for first-order update formulas have been obtained for several problems [7].

We transfer many of our inexpressibility results to the $k$-CLIQUE query for fixed $k \geq 3$ and the $k$-COL query for fixed $k \geq 2$.

Finally, we show two normal form results: every query in DynProp is already maintainable with negation-free quantifier-free formulas only as well as with conjunctive quantifier-free formulas only. Thus, one approach to inexpressibility proofs could be to use these syntactically restricted update formulas.

**Related Work.** We already mentioned the related work that is most closely related to our results. As said before, the reachability query has been studied in various dynamic frameworks, one of which is the Cell Probe model. In the Cell Probe model, one aims for lower bounds for the number of memory accesses of a RAM machine for static and dynamic problems. For dynamic reachability, lower bounds of order $\log n$ have been proved [8].

**Outline.** In Section 2 we fix our notation and in Section 3 we define our dynamic setting more precisely. The lower bound results for Reachability are presented in Section 4 (for auxiliary relations) and in Section 5 (for auxiliary functions). In Section 6 we transfer the lower bounds to other queries. Finally, we establish the two normal forms for DynProp in Section 7.

**Acknowledgement.** We thank Ahmet Kara and Martin Schuster for careful proofreading. We acknowledge the financial support by the German DFG under grant SCHW 678/6-1.

## 2 Preliminaries

In this section, we repeat some basic notions and fix some of our notation.

A *domain* is a finite set. For $k$-tuples, $\vec{a} = (a_1, \ldots, a_k)$ and $\vec{b} = (b_1, \ldots, b_k)$ over some domain $D$, the $2k$-tuple obtained by concatenating $\vec{a}$ and $\vec{b}$ is denoted by $(\vec{a}, \vec{b})$. The tuple $\vec{a}$ is $\prec$-ordered with respect to an order $\prec$ of $D$, if $a_1 \prec \ldots \prec a_k$. If $\pi$ is a function on $D$, we denote $(\pi(a_1), \ldots, \pi(a_k))$ by $\pi(\vec{a})$. We slightly abuse set theoretic notations and write $c \in \vec{a}$ if $c = a_i$ for some $c \in D$ and some $i$, and $\vec{a} \cup \vec{b}$ for the set \{a_1, \ldots, a_k, b_1, \ldots, b_k\}.

[^2]: Throughout the paper all functions are total.
A schema (or signature) \( \tau \) consists of a set \( \tau_{rel} \) of relation symbols, a set \( \tau_{fun} \) of function symbols and a set \( \tau_{const} \) of constant symbols together with an arity function \( \text{Ar} : \tau_{rel} \cup \tau_{fun} \rightarrow \mathbb{N} \). A schema is relational if \( \tau_{fun} = \emptyset \). A database \( \mathcal{D} \) of schema \( \tau \) with domain \( D \) is a mapping that assigns to every relation symbol \( R \in \tau_{rel} \) a relation of arity \( \text{Ar}(R) \) over \( D \), to every \( k \)-ary function symbol \( f \in \tau_{fun} \) a \( k \)-ary function, and to every constant symbol \( c \in \tau_{const} \) a single element (called constant) from \( D \). The size of a database is the size of its domain.

Unless otherwise stated (as, e.g., in Section 5), we always consider relational schemas.

A \( \tau \)-structure \( S \) is a pair \((D, \mathcal{D})\) where \( \mathcal{D} \) is a database with schema \( \tau \) and domain \( D \). Sometimes we omit the schema when it is clear from the context. If \( S \) is a structure over domain \( D \) and \( D' \) is a subset of \( D \) that contains all constants of \( S \), then the substructure of \( S \) induced by \( D' \) is denoted by \( S \upharpoonright D' \).

Let \( S \) and \( T \) be two structures of schema \( \tau \) and over domains \( D \) and \( T \), respectively. A mapping \( \pi : S \rightarrow T \) preserves a relation symbol \( R \in \tau \) of arity \( m \), when \( \bar{a} \in R^S \) if and only if \( \pi(\bar{a}) \in R^T \), for all \( m \)-tuples \( \bar{a} \). It preserves a constant symbol \( c \in \tau \), if \( c^T = \pi(c^S) \). The mapping is \( \tau \)-preserving, if it preserves all relation symbols and all constant symbols from \( \tau \). Two \( \tau \)-structures \( S \) and \( T \) are isomorphic via \( \pi \), denoted by \( S \simeq \pi T \), if \( \pi \) is a bijection from \( S \) to \( T \) which is \( \tau \)-preserving. We define \( \text{id}[\bar{a}, \bar{b}] : S \rightarrow S \) to be the bijection that maps, for every \( i \), \( a_i \) to \( b_i \) and \( b_i \) to \( a_i \), and maps all other elements to themselves.

The \( k \)-ary atomic type \( (\langle S, \bar{a} \rangle) \) of a tuple \( \bar{a} = (a_1, \ldots, a_k) \) over \( D \) with respect to a \( \tau \)-structure \( S \) is the set of all atomic formulas \( \varphi(\bar{x}) \) with \( \bar{x} = (x_1, \ldots, x_k) \) for which \( \varphi(\bar{a}) \) holds in \( S \), where \( \varphi(\bar{a}) \) is short for the substitution of \( \bar{x} \) by \( \bar{a} \) in \( \varphi \). We note that the atomic formulas can use constant symbols. The \( \sigma \)-type \( (\langle S, \bar{a} \rangle)_{\sigma} \) is the set of atomic formulas of \( (\langle S, \bar{a} \rangle) \) with relation symbols from \( \sigma \). If \( \prec \) is a linear order on \( D \) we call a subset \( D' \subseteq D \) \( \prec \)-homogeneous (or homogeneous, if \( \prec \) is clear from the context) if, for every \( \ell \), the type of all \( \prec \)-ordered \( \ell \)-tuples over \( D' \) is the same, that is if \( (\langle S, \bar{a} \rangle) = (\langle S, \bar{b} \rangle) \) for all ordered \( \ell \)-tuples \( \bar{a} \) and \( \bar{b} \). It is easy to observe, that a set \( D' \) is already \( \prec \)-homogeneous if the condition holds for every \( \ell \) up to the maximal arity of \( \tau \).

An \( s \)-\( t \)-graph is a graph \( G = (V, E) \) with two distinguished nodes \( s \) and \( t \). A \( k \)-layered \( s \)-\( t \)-graph \( G \) is a directed graph \( (V, E) \) in which \( V \) \( \setminus \{s, t\} \) is partitioned into \( k \) layers \( A_1, \ldots, A_k \) such that every edge is from \( s \) to \( A_1 \), from \( A_k \) to \( t \) or from \( A_i \) to \( A_{i+1} \), for some \( i \in \{1, \ldots, k-1\} \). The reachability query \( \text{REACH} \) on graphs is defined as usual, that is \( (a, b) \) is in \( \text{REACH}(G) \) if \( b \) can be reached from \( a \) in \( G \). The \( s \)-\( t \)-reachability query \( s \)-\( t \)-\( \text{REACH} \) is a Boolean query that is true for an \( s \)-\( t \)-graph \( G \), if and only if \( (s, t) \in \text{REACH}(G) \).

Formally, an \( s \)-\( t \)-graph is a structure over a schema with one binary relation symbol (interpreted by the set of edges \( E \)) and two constant symbols (interpreted by the two distinguished nodes \( s \) and \( t \)).

---

3 As we only consider atomic types in this paper, we will often simply say type instead of atomic type.
3 Dynamic Queries and Programs

The following presentation follows [9] and [10].

Informally a dynamic instance of a static query $Q$ is a pair $(D, \alpha)$, where $D$ is a database and $\alpha$ is a sequence of updates, i.e. a sequence of tuple insertions and deletions into $D$. The dynamic query $\text{Dyn}(Q)$ yields as result the relation that is obtained by first applying the updates from $\alpha$ to $D$ and evaluating query $Q$ on the resulting database. We formalize this as follows.

Definition 1. (Abstract and concrete updates) The set $\Delta$ of abstract updates of a schema $\tau$ contains the terms $\text{INS}_R$ and $\text{DEL}_R$, for every relation symbol $R \in \tau$. For a database $D$ over schema $\tau$ with domain $D$, a concrete update is a term of the form $\text{INS}_R(a_1, \ldots, a_k)$ or $\text{DEL}_R(a_1, \ldots, a_k)$ where $R \in \tau$ is a relation symbol of arity $k$ and $a_1, \ldots, a_k$ are elements from $D$.

Applying an update $\text{INS}_R(a_1, \ldots, a_k)$ to a database $D$ replaces relation $R^D$ by $R^D \cup \{(a_1, \ldots, a_k)\}$. Analogously, applying an update $\text{DEL}_R(a_1, \ldots, a_k)$ replaces $R^D$ by $R^D \setminus \{(a_1, \ldots, a_k)\}$. All other relations remain unchanged. The database resulting from applying an update $\delta$ to a database $D$ is denoted by $\delta(D)$. The result $\alpha(D)$ of applying a sequence of updates $\alpha = \delta_1 \ldots \delta_m$ to a database $D$ is defined by $\alpha(D) = \delta_m(\ldots(\delta_1(D))\ldots)$.

Definition 2. (Dynamic Query) A dynamic instance is a pair $(D, \alpha)$ consisting of an input database $D$ and an update sequence $\alpha$. For a static query $Q$ with schema $\tau$, the dynamic query $\text{Dyn}(Q)$ is the mapping that yields $Q(\alpha(D))$, for every dynamic instance $(D, \alpha)$.

Our main interest in this work is the dynamic version $\text{Dyn}(s.t.-\text{REACH})$ of the $s.t.$-reachability query.

Dynamic programs, to be defined next, consist of an initialization mechanism and an update program. The former yields, for every database $D$ an initial state of $P$ with initial auxiliary data (and possibly with further built-in data). The latter defines the new state, for each update in $\alpha$. Built-in data never changes. In general, built-in data can be “simulated” by auxiliary data yet this does not (seem to) hold for all of the restricted kinds of auxiliary data studied in this paper.

A dynamic schema is a triple $(\tau_{\text{in}}, \tau_{\text{aux}}, \tau_{\text{bi}})$ of schemas of the input database, the built-in database and the auxiliary database, respectively. We always let $\tau \overset{\text{def}}{=} \tau_{\text{in}} \cup \tau_{\text{aux}} \cup \tau_{\text{bi}}$. Throughout the paper, $\tau_{\text{in}}$ has to be relational. In our basic setting we also require that the two other schemas, $\tau_{\text{aux}}$ and $\tau_{\text{bi}}$ are relational (this will be relaxed in Section 5).

Definition 3. (Update program) An update program $P$ over dynamic schema $(\tau_{\text{in}}, \tau_{\text{bi}}, \tau_{\text{aux}})$ is a set of first-order formulas (called update formulas in the following) that contains, for every $R \in \tau_{\text{aux}}$ and every abstract update $\delta$ of some $S \in \tau_{\text{in}}$, an update formula $\phi_R^S(x_1, \ldots, x_l; y_1, \ldots, y_m)$ over the schema $\tau$ where $l$ is the arity of $S$ and $m$ is the arity of $R$.

\text{In this work we do not allow updates of constants, for simplicity.}
A program state $S$ over dynamic schema $(\tau_{\text{in}}, \tau_{\text{aux}}, \tau_{\text{bi}})$ is a structure $(D, I, A, B)$ where $D$ is the domain, $I$ is a database over the input schema (the current database), $A$ is a database over the auxiliary schema (the auxiliary database) and $B$ is a database over the built-in schema (the built-in database).

The semantics of update programs is as follows. For an update $\delta(\vec{a})$ and program state $S = (D, I, A, B)$ we denote by $P_\delta(S)$ the state $(D, \delta(I), A', B)$, where $A'$ consists of relations $R' \equiv \{ \vec{b} \mid S \models \phi^R(\vec{a}; \vec{b}) \}$. The effect $P_\alpha(S)$ of an update sequence $\alpha = \delta_1 \ldots \delta_m$ to a state $S$ is the state $P_{\delta_m}(\ldots (P_{\delta_1}(S)) \ldots)$.

**Definition 4.** (Dynamic program) A dynamic program is a triple $(P, \text{Init}, Q)$, where

- $P$ is an update program over some dynamic schema $(\tau_{\text{in}}, \tau_{\text{bi}}, \tau_{\text{aux}})$,
- the tuple $\text{Init} = (\text{Init}_{\text{aux}}, \text{Init}_{\text{bi}})$ consists of a function $\text{Init}_{\text{aux}}$ that maps $\tau_{\text{in}}$-databases to $\tau_{\text{aux}}$-databases and a function $\text{Init}_{\text{bi}}$ that maps domains to $\tau_{\text{bi}}$-databases, and
- $Q \in \tau_{\text{aux}}$ is a designated query symbol.

A dynamic program $P = (P, \text{Init}, Q)$ maintains a dynamic query $\text{Dyn}(Q)$ if, for every dynamic instance $(D, \alpha)$, the relation $Q(\alpha(D))$ coincides with the query relation $Q^S$ in the state $S = P_\alpha(S_{\text{Init}}(D))$, where $S_{\text{Init}}(D)$ is the initial state, i.e., $S_{\text{Init}}(D) \equiv (D, \text{Init}_{\text{aux}}(D), \text{Init}_{\text{bi}}(D))$.

Several dynamic settings and restrictions of dynamic programs have been studied in the literature [11] [11] [7] [10]. Possible parameters are, for instance:

- the logic in which update formulas are expressed;
- whether in dynamic instances $(D, \alpha)$, the initial database $D$ is always empty;
- whether the initialization mapping $\text{Init}_{\text{aux}}$ is permutation-invariant (short: invariant), that is, whether $\pi(\text{Init}_{\text{aux}}(D)) = \text{Init}_{\text{aux}}(\pi(D))$ holds, for every database $D$ and permutation $\pi$ of the domain; and
- whether there are any built-in relations at all.

In [12], Dyn-FO is defined as the class of (Boolean) queries that can be maintained for empty initial databases with first-order update formulas, first-order definable initialization mapping and without built-in data. Furthermore, a larger class with polynomial-time computable initialization mapping was considered. Also [11] considers empty initial databases without built-in data. In [7], general instances (with non-empty initial databases) are allowed, but the initialization mapping has to be defined by logical formulas and is thus always invariant; and there is no built-in data. In [10] update formulas are restricted to be quantifier-free, the initial database is empty and a built-in order is available.

**Definition 5.** (DynFO, DynProp) DynFO is the class of all dynamic queries maintainable by dynamic programs with first-order update formulas. DynProp is the subclass of DynFO, where update formulas do not use quantifiers. A dynamic program is $k$-ary if the arity of its auxiliary relation symbol is at most $k$. By $k$-ary DynProp (resp. DynFO) we refer to dynamic queries that can be maintained with $k$-ary dynamic programs.

---

5 We note that this restriction does not apply to the built-in relations.
Thus in our basic setting the initialization mappings can be arbitrary. We will explicitly state when we relax this most general setting. Now we sketch important relaxations. Figure 1 illustrates the relationships between the various settings.

First we note that for arbitrary initialization mappings, the same queries can be maintained regardless whether one starts from an empty or from a non-empty initial database.\(^6\) Restricting the setting for non-empty initial databases to invariant auxiliary data initialization and to no built-in data leads to the initialization used in \(^7\) (called *invariant initialization* in the following). For empty initial databases, allowing empty initial auxiliary data only and no built-in data leads to the initialization model of \(^7\) (called *empty initialization* in the following).

It is easy to see that applying an invariant initialization mapping to an empty database is pretty much useless, as, all tuples with the same constants at the same positions are treated in the same way. Therefore, queries maintainable in \textsc{DynFO} or \textsc{DynProp} with empty initial database and invariant initialization can also be maintained with empty initialization.\(^7\) This statement also holds in the presence of arbitrary built-in relations.

From now on we restrict our attention to quantifier-free update programs. Next, we give an example for such a program.

**Example 1.** We provide a \textsc{DynProp}-program \(P\) for the dynamic variant of the Boolean query \textsc{NonEmptySet}, where, for a unary relation \(U\) subject to insertions and deletions of elements, one asks whether \(U\) is empty. It illustrates a technique to maintain lists with quantifier-free dynamic programs, introduced in \[^{10}\] Proposition 4.5, which is used in some of our upper bounds.

The program \(P\) is over auxiliary schema \(\tau_{\text{aux}} = \{Q, \text{FIRST}, \text{LAST}, \text{LIST}\}\), where \(Q\) is the query bit (i.e. a 0-ary relation symbol), \(\text{FIRST}\) and \(\text{LAST}\) are unary relation symbols, and \(\text{LIST}\) is a binary relation symbol. The idea is to store in a program state \(S\) a list of all elements currently in \(U\). The list structure is stored in the binary relation \(\text{LIST}^S\) such that \(\text{LIST}^S(a, b)\) holds for all elements

\[^{6}\] The initialization for a non-empty database can be obtained as the auxiliary relations obtained after inserting all tuples of the database into the empty one.

\[^{7}\] We do not formally prove this here.
a and b that are adjacent in the list. The first and last element of the list are stored in First⁵ and Last⁵, respectively. We note that the order in which the elements of U are stored in the list depends on the order in which they are inserted into the set.

For a given instance of NONEMPTYSET the initialization mapping initializes the auxiliary relations accordingly.

**Insertion of a into U.** A newly inserted element is attached to the end of the list Therefore the First-relation does not change except when the first element is inserted into an empty set U. Furthermore, the inserted element is the new last element of the list and has a connection to the former last element. Finally, after inserting an element into U, the query result is 'true':

\[ φ_{\text{First ins}}(a; x) \overset{\text{def}}{=} (¬Q \land a = x) \lor (Q \land \text{First}(x)) \]
\[ φ_{\text{Last ins}}(a; x) \overset{\text{def}}{=} a = x \]
\[ φ_{\text{List ins}}(a; x, y) \overset{\text{def}}{=} \text{List}(x, y) \lor (\text{Last}(x) \land a = y) \]
\[ φ_{\text{Q ins}}(a) \overset{\text{def}}{=} \top. \]

**Deletion of a from U.** How a deleted element a is removed from the list, depends on whether a is the first element of the list, the last element of the list or some other element of the list. The query bit remains 'true', if a was not the first and last element of the list.

\[ φ_{\text{First del}}(a; x) \overset{\text{def}}{=} (\text{First}(x) \land a \neq x) \lor (\text{First}(a) \land \text{List}(a, x)) \]
\[ φ_{\text{Last del}}(a; x) \overset{\text{def}}{=} (\text{Last}(x) \land a \neq x) \lor (\text{Last}(a) \land \text{List}(x, a)) \]
\[ φ_{\text{List del}}(a; x, y) \overset{\text{def}}{=} x \neq a \land y \neq a \land (\text{List}(x, y) \lor (\text{List}(x, a) \land \text{List}(a, y))) \]
\[ φ_{\text{Q del}}(a) \overset{\text{def}}{=} ¬(\text{First}(a) \land \text{Last}(a)) \]

4 Lower Bounds for Dynamic Reachability

In this section we prove lower bounds for the maintainability of the dynamic s-t-reachability query Dyn(s-t-Reach).

First we introduce a tool for proving lower bounds for quantifier-free formulas. Afterwards we prove that

- Dyn(s-t-Reach) is not in binary DynProp; and
- when only invariant initialization mappings are allowed, then Dyn(s-t-Reach) is not in DynProp.

For simplicity we assume that only elements that are not already in U are inserted, the formulas given can be extended easily to the general case. Similar assumptions are made whenever necessary.
The first result is used to obtain an arity hierarchy up to arity three for quantifier-free updates and binary queries.

The proofs use the following tool which is a slight variation of Lemma 1 from [10]. The intuition is as follows. When updating an auxiliary tuple \( \mathbf{c} \) after an insertion or deletion of a tuple \( \mathbf{d} \), a quantifier-free update formula has access to \( \mathbf{c}, \mathbf{d} \), and the constants only. Thus, if a sequence of updates changes only tuples from a substructure \( S' \) of \( S \), the auxiliary data of \( S' \) is not affected by information outside \( S' \). In particular, two isomorphic substructures \( S' \) and \( T' \) should remain isomorphic, when corresponding updates are applied to them.

We formalize the notion of corresponding updates as follows. Let \( \pi \) be an isomorphism from a structure \( S \) to a structure \( T \). Two updates \( \delta(\mathbf{a}) \) on \( S \) and \( \delta(\mathbf{b}) \) on \( T \) are said to be \( \pi \)-respecting if \( \mathbf{b} = \pi(\mathbf{a}) \). Two sequences \( \alpha = \delta_1 \ldots \delta_m \) and \( \beta = \delta'_1 \ldots \delta'_m \) of updates respect \( \pi \) if, for every \( i \leq m \), \( \delta_i \) and \( \delta'_i \) are \( \pi \)-respecting.

**Lemma 2 (Substructure Lemma).** Let \( \mathcal{P} \) be a \( \text{DynProp} \) program and \( S \) and \( T \) states of \( \mathcal{P} \) with domains \( S \) and \( T \), respectively. Further, let \( S' \subseteq S \) and \( T' \subseteq T \) such that \( S \upharpoonright S' \) and \( T \upharpoonright T' \) are isomorphic via \( \pi \). Then \( P_\alpha(S) \upharpoonright S' \) and \( P_\beta(T) \upharpoonright T' \) are isomorphic via \( \pi \) for all \( \pi \)-respecting update sequences \( \alpha \), \( \beta \) on \( S' \) and \( T' \).

*Proof.* The lemma can be shown by induction on the length of the update sequences. To this end, it is sufficient to prove the claim for a pair of \( \pi \)-respecting updates \( \delta(\mathbf{a}) \) and \( \delta(\mathbf{b}) \) on \( S' \) and \( T' \). We abbreviate \( S \upharpoonright S' \) and \( T \upharpoonright T' \) by \( S' \) and \( T' \), respectively.

Since \( \pi \) is an isomorphism from \( S' \) to \( T' \), we know that \( R_{\pi}(\mathbf{d}) \) holds if and only if \( R_{\pi}(\mathbf{d}) \) holds, for every \( m \)-tuple \( \mathbf{d} \) over \( S' \) and every relation symbol \( R \in \tau \). Therefore, \( \varphi(\mathbf{x}) \) evaluates to true in \( S' \) under \( \mathbf{d} \) if and only if it does so in \( T' \) under \( \pi(\mathbf{d}) \), for every quantifier-free formula \( \varphi(\mathbf{x}) \) over schema \( \tau \). Thus all update formulas from \( \mathcal{P} \) yield the same result for corresponding tuples \( \mathbf{d} \) and \( \pi(\mathbf{d}) \) from \( S' \) and \( T' \), respectively. Hence \( P_\delta(\mathbf{a}) \upharpoonright S' \) is isomorphic to \( P_\delta(\mathbf{a}) \upharpoonright S' \), yielding the claim.

The following corollary is implied by Lemma 2 since the 0-ary auxiliary relations of two isomorphic structures coincide.

**Corollary 3.** Let \( \mathcal{P} \) be a \( \text{DynProp} \)-program with designated Boolean query symbol \( Q \), and let \( S \) and \( T \) be states of \( \mathcal{P} \) with domains \( S \) and \( T \). Further let \( S' \subseteq S \) and \( T' \subseteq T \) such that \( S \upharpoonright S' \) and \( T \upharpoonright T' \) are isomorphic via \( \pi \). Then \( Q \) has the same value in \( P_\alpha(S) \upharpoonright S' \) and \( P_\beta(T) \upharpoonright T' \) for all \( \pi \)-respecting sequences \( \alpha \), \( \beta \) of updates on \( S' \) and \( T' \).

The Substructure Lemma can be applied along the following lines to prove that \( \text{Dyn}(s-t\text{-Reach}) \) cannot be maintained in some settings with quantifier-free updates. Towards a contradiction, assume that there is a quantifier-free program \( \mathcal{P} = \langle P, \text{Init}, Q \rangle \) that maintains \( \text{Dyn}(s-t\text{-Reach}) \). Then, find

- two states \( S \) and \( T \) occurring as states of \( \mathcal{P} \) with current graphs \( G_S \) and \( G_T \);

\[ \text{I.e. } S = P_\delta(S_{\text{init}}(G)) \text{ for some } s-t\text{-graph } G, \text{ and likewise for } T. \]
Theorem 4 (Ramsey’s Theorem for Structures). For every schema \( \tau \) and all natural numbers \( k \) and \( n \) there exists a number \( R_{\tau,k}(n) \) such that, for every \( \tau \)-structure \( S \) with domain \( A \) of size \( R_{\tau,k}(n) \), every \( \vec{d} \in A^k \) and every order \( \prec \) on \( A \), there is a subset \( B \) of \( A \) of size \( n \) with \( B \cap \vec{d} = \emptyset \), such that, for every \( l \), the type of \( (\vec{a}, \vec{d}) \) in \( S \) is the same, for all \( \prec \)-ordered \( l \)-tuples \( \vec{a} \) over \( B \).

The proof of Theorem 4 uses the well-known Ramsey’s Theorem for hypergraphs (see, e.g., [13, p. 7]) and is based on the proof of Observation 1’ in [10, p. 11].

A \( k \)-hypergraph \( G \) is a pair \((V, E)\) where \( V \) is a set and \( E \) is a set of \( k \)-element subsets of \( V \). If \( E \) contains all \( k \)-element subsets of \( V \), then \( G \) is called complete. A \( k \)-hypergraph \( G' = (V', E') \) is a sub-\( k \)-hypergraph of a \( k \)-hypergraph \( G = (V, E) \), if \( V' \subseteq V \) and \( E' \) contains all edges \( e \in E \) with \( e \subseteq V' \). A \( C \)-coloring \( \text{col} \) of \( G \), where \( C \) is a finite set of colors, is a mapping that assigns to every edge in \( E \) a color from \( C \), that is, \( \text{col} : E \rightarrow C \). A \( C \)-colored \( k \)-hypergraph is a pair \((G, \text{col})\) where \( G \) is a \( k \)-hypergraph and \( \text{col} \) is a \( C \)-coloring of \( G \). If the name of the \( C \)-coloring is not important we also say \( G \) is \( C \)-colored.

Theorem 5. (Ramsey’s Theorem for Hypergraphs) For every set \( C \) of colors and natural numbers \( n \) and \( k \) there exists a number \( R_{C,n}(k) \) such that, if the edges of a complete \( k \)-hypergraph of size \( R_{C,n}(k) \) are \( C \)-colored, then the hypergraph contains a complete sub-\( k \)-hypergraph with \( n \) nodes whose edges are all colored with the same color.

Proof (of Theorem 4). Given a schema \( \tau \) and natural numbers \( k, n \). Let \( R_{\tau,k}(n) \) be chosen sufficiently large with respect to \( k, n, \) and \( \tau \) such that the following argument works. Further let \( S \) be a \( \tau \)-structure with domain \( A \) of size greater than \( R_{\tau,k}(n) \) and \( \prec \) an arbitrary order on \( A \). Denote by \( m \) the maximal arity in \( \tau \) and by \( \vec{c} \) the constants of \( S \) in some order. Further denote by \( C \) the set of all constants and all elements occurring in \( \vec{d} \).

Observe that proving the claim for \( l \leq m \) is sufficient.

We first prove the claim for \(|C| = 0\), by constructing inductively sets \( B_l \) that satisfy the condition for \( l \) with \( l \leq m \). Let \( B_0 = A \). The set \( B_l, l \leq m, \) is obtained from \( B_{l-1} \) as follows. From \( B_{l-1} \) a coloring \( \text{col} \) of the complete \( l \)-hypergraph \( G \) with node set \( B_{l-1} \) is constructed. The coloring \( \text{col} \) uses \( l \)-ary \( \tau \)-types as colors. An edge \( e = \{e_1, \ldots, e_l\} \) with \( e_1 \prec \ldots \prec e_l \) is colored by the type \( \langle S, e_1, \ldots, e_l \rangle \).

Because \( B_{l-1} \) is large, it has, by Ramsey’s theorem, a subset \( B_l \) such that all
edges \( e \subseteq B_l \) of size \( l \) are colored with the same color by \( \text{col} \). But then, by the definition of \( \text{col} \), all \( \prec \)-ordered \( l \)-tuples over \( B_l \) have the same type in \( S \). By this construction we obtain a set \( B_m \) such that for every \( l \leq m \) the type of all \( \prec \)-ordered \( l \)-tuples over \( B_m \) is the same. Setting \( B := B_m \) proves the claim for \( |C| = 0 \).

The idea for the case \( |C| \neq 0 \) is to construct from \( S \) a new structure \( S' \) of an extended schema over domain \( A' = A \setminus C \) such that \( S' \) encodes all information about \( C \) contained in \( S \) and then use the case \( |C| = 0 \) for \( S' \).

The structure \( S' \) is of schema \( \tau \cup \tau' \), where \( \tau' \) contains for every \( l \leq m \) and every \((l + |C|)\)-ary \( \tau \)-type \( t \), an \( l \)-ary relation symbol \( R_t \). An \( l \)-tuple \( \bar{a} \) is in \( R_t^{S'} \) if and only if \( t \) is the \( \tau \)-type of \((\bar{a}, \bar{C})\). Application of the case \( |C| = 0 \) to \( S' \) yields a huge homogeneous subset \( B' \) with respect to \( \prec \) and schema \( \tau \cup \tau' \). Then, for every \( l \leq m \), the type of \((\bar{a}, \bar{C})\) in \( S \) is the same, for all \( \prec \)-ordered \( l \)-tuples \( \bar{a} \) over \( B' \). This proves the claim.

A word \( u \) is a subsequence of a word \( v \), in symbols \( u \subseteq v \), if \( u = u_1 \ldots u_k \) and \( v = v_0u_1v_1 \ldots v_{k-1}u_kv_k \) for some words \( u_1, \ldots, u_k \) and \( v_0, \ldots, v_k \).

**Theorem 6 (Higman’s Lemma).** For every infinite sequence \( (w_i)_{i \in \mathbb{N}} \) of words over an alphabet \( \Sigma \) there are \( l \) and \( k \) such that \( l < k \) and \( w_l \sqsubseteq w_k \).

We will actually make use of the following stronger result. See e.g. [14, Proposition 2.5, page 3] for a proof.

**Theorem 7.** For every alphabet of size \( c \) and function \( g : \mathbb{N} \to \mathbb{N} \) there is a natural number \( H(c) \) such that in every sequence \( (w_i)_{1 \leq i \leq H(c)} \) of \( H(c) \) many words with \( |w_i| \leq g(i) \) there are \( l \) and \( k \) with \( l < k \) and \( w_l \sqsubseteq w_k \).

In the following we will refer to both results as Higman’s Lemma.

### 4.1 A Binary Lower Bound

As already mentioned in the introduction, the proof that \( \text{DYN}(s,t\text{-REACH}) \) is not in unary \( \text{DYNFO} \) in [3] uses constant-length update sequences, and is mainly an application of a locality-based static lower bound for monadic second order logic. This technique does not seem to generalize to binary \( \text{DYNFO} \). We prove the first unmaintainability result for \( \text{DYN}(s,t\text{-REACH}) \) with respect to binary auxiliary relations. We recall that binary \( \text{DYNPROP} \) can have built-in relations of arbitrary arity.

**Theorem 8.** \( \text{DYN}(s,t\text{-REACH}) \) is not in binary \( \text{DYNPROP} \).

The proof of Theorem [3] will actually show that binary \( \text{DYNPROP} \) cannot even maintain \( \text{DYN}(s,t\text{-REACH}) \) on 2-layered \( s,t \)-graphs. These restricted graphs will then help us to separate binary \( \text{DYNPROP} \) from ternary \( \text{DYNPROP} \). This separation shows that the lower bound technique for binary \( \text{DYNPROP} \) does not immediately transfer to ternary \( \text{DYNPROP} \). At the moment we do not know whether it is possible to adapt the technique to full \( \text{DYNPROP} \).

Before proving Theorem [3] we show the following corresponding result for unary \( \text{DYNPROP} \) whose proof uses the same techniques in a simpler setting.
Proposition 9. The dynamic s-t-reachability query is not in unary DynProp, not even for 1-layered s-t-graphs.

Proof. Towards a contradiction, assume that \( \mathcal{P} = (P, \text{Init}, Q) \) is a dynamic program over schema \( \tau = (\tau_{\text{in}}, \tau_{\text{aux}}, \tau_{\text{bi}}) \) with unary \( \tau_{\text{aux}} \) that maintains the s-t-reachability query for 1-layered s-t-graphs. Let \( n' \) be sufficiently large\(^{10} \) with respect to \( \tau \) and \( n \) be sufficiently large with respect to \( n' \). Further let \( m \) be the highest arity of a relation symbol from \( \tau_{\text{bi}} \).

Let \( G = (V, E) \) be a 1-layered s-t-graph such that \( V = \{s, t\} \cup A \) with \( n = |A| \) and \( E = \emptyset \). Further let \( S = (V, E, A, B) \) be the state obtained by applying \( \text{Init} \) to \( G \).

Here and in the following, we do not explicitly represent the constants \( s \) and \( t \) in \( S \), as they never change during the application of an update sequence (but, of course, tuples containing constants might change in the graph and in the auxiliary relations).

First, we identify a subset of \( A \) on which the built-in relations are homogeneous. By Ramsey’s Theorem for structures (choosing \( \vec{c} = (s, t) \)) and because \( n = |A| \) is sufficiently large with respect to \( n' \) there is a set \( A' \subseteq A \) of size \( n' \) and an order \( \prec \) on \( A' \) such that all \( \prec \)-ordered \( m \)-tuples \( a_1 \) and \( a_2 \) over \( A' \) are of equal \( \tau_{\text{bi}} \)-type.

Let \( S' \equiv (V, E', A', B) \) be the state of \( \mathcal{P} \) that is reached from \( S \) after application of the following updates to \( G \) (in some arbitrary order):

(a) For every node \( a \in A' \), insert edges \((s, a)\) and \((a, t)\).

Let \( a_1 \prec \ldots \prec a_{n'} \) be an enumeration of the elements of \( A' \). For every \( i \in \{1, \ldots, n'\} \), we define \( a_i \) to be the update sequence that deletes the edges \((s, a_{n'-1}), \ldots, (s, a_{i+1})\), in this order. Let \( S'_i \) be the state reached by applying \( a_i \) to \( S' \). Thus, in state \( S'_i \) only nodes \( a_1, \ldots, a_i \) have edges to node \( s \). For every \( i \), we construct a word \( w_i \) of length \( i \), that has a letter for every node \( a_1, \ldots, a_i \) and captures all relevant information about those nodes in \( S'_i \).

The words \( w_i \) are over the set of all unary types of \( \tau_{\text{aux}} \). More precisely, the \( j \)-th letter \( \sigma_j \) of \( w_i \) is the unary \( \tau_{\text{aux}} \)-type of \( a_j \) in \( S'_i \). We recall that the unary type of \( a_j \) captures all information about the tuple \((s, a_j, t)\).

Since \( n' = |A'| \) was chosen sufficiently large with respect to \( \tau \), it follows by Higman’s Lemma, that there are \( k \) and \( l \) such that \( k < l \) and \( a_k \subseteq w_1 \), that is, \( w_k = \sigma_1 \sigma_2 \ldots \sigma_k = \sigma_1^{i_1} \sigma_2^{i_2} \ldots \sigma_k^{i_k} \) for suitable numbers \( i_1 < \ldots < i_k \).

We argue that the structures \( S'_k \mid \{s, t, a_1, \ldots, a_k\} \) and \( S'_l \mid \{s, t, a_1, \ldots, a_k\} \) are isomorphic via the mapping \( \pi \) with \( \pi(a_j) = a_{i_j} \) for all \( j \), \( \pi(s) = s \) and \( \pi(t) = t \). By definition of \( A' \) and because built-in relations do not change, the mapping \( \pi \) preserves \( \tau_{\text{bi}} \). The schema \( \tau_{\text{aux}} \) is preserved since \( a_j \) and \( a_{i_j} \) are of equal unary type, by the definition of \( w_k \) and \( w_1 \). Thus \( \pi \) is indeed an isomorphism.

Therefore, by Corollary \(^{3} \) the program \( \mathcal{P} \) computes the same query result for the following \( \pi \)-respecting update sequences \( \beta_1 \) and \( \beta_2 \):

(\( \beta_1 \)) Delete edges \((s, a_1), \ldots, (s, a_k)\) from \( S'_k \).

\(^{10} \) Explicit numbers are given at the end of the proof.
Fig. 2. The structures $S'_l$ and $S'_k$ from the proof of Proposition 9. Edges still present are solid; deleted edges are dotted. The isomorphic substructures are highlighted in blue.

$(\beta_2)$ Delete edges $(s, a_{i_1}), \ldots, (s, a_{i_k})$ from $S'_l$.

We refer to Figure 2 for an illustration.

However, applying the update sequence $\beta_1$ yields a graph where $t$ is not reachable from $s$, whereas by $\beta_2$ a graph is obtained where $t$ is reachable from $s$, the desired contradiction.

We now make the numbers $n$ and $n'$ that were chosen in the beginning of the proof more precise. In order to apply Higman’s Lemma, the set $A'$ needs to be of size at least $n' \overset{\text{def}}{=} H(|n''|)$ where $n''$ is the number of unary types of $\tau$. Therefore, the set $A$ has to be of size $n \overset{\text{def}}{=} R_\tau(n')$.

Now we are ready to prove Theorem 8 i.e. that $\text{Dyn}(s-t-\text{Reach})$ is not in binary $\text{DynProp}$. In the proof, we will again first choose a homogeneous subset with respect to the built-in relations. The notation introduced next and the following lemma prepare that step.

We refine the notion of homogeneous sets. Let $S$ be a structure of some schema $\tau$ and $A$, $B$ disjoint subsets of the domain of $S$. We say that $B$ is $A$-homogeneous up to arity $m$, if for every $l \leq m$, all tuples $(a, \vec{b})$, where $a \in A$ and $\vec{b}$ is an $\prec$-ordered $l$-tuple over $B$, have the same type. We may drop the order $\prec$ from the notation if it is clear from the context, and we may drop $A$ if $A = \emptyset$. 


We observe that if the maximal arity of $\tau$ is $m$ and $B$ is $A$-homogeneous up to arity $m$, then $B$ is $A$-homogeneous up to arity $m'$ for every $m'$. In this case we simply say $B$ is $A$-homogeneous.

**Lemma 10.** For every schema $\tau$ and natural number $n$, there is a natural number $R_{\tau}^{\hom}(n)$ such that for any two disjoint subsets $A$, $B$ of the domain of a $\tau$-structure $S$ with $|A|, |B| \geq R_{\tau}^{\hom}(n)$, there are subsets $A' \subseteq A$ and $B' \subseteq B$ such that $|A'|, |B'| = n$ and $B'$ is $A'$-homogeneous in $S$.

**Proof.** Let $\tau$ be a schema with maximal arity $m$. Choose $k'$ to be a large number with respect to $\tau$ and $n$; and let $k$ be a large number with respect to $k'$. In particular $k$ is large with respect to the number of constant symbols in $\tau$. Further let $A$, $B$ be disjoint subsets of the domain of a $\tau$-structure $S$ with $|A|, |B| > k$. Since $k$ is large with respect to the number of constants in $S$, we assume, without loss of generality, that neither $A$ nor $B$ contains a constant.

Fix a $k'$-tuple $\vec{a} = (a_1, \ldots, a_{k'})$ of $A$. Further let $\prec$ be an arbitrary order on $B$. Because $|B|$ is large with respect to $k'$, $n$ and $\tau$, and by Ramsey’s theorem on structures (choose $\vec{c} = \vec{a}$), there is a subset $B'$ of $B$ of size $n$ such that for every $l \leq m$ the type of $(\vec{a}, \vec{b})$ in $S$ is the same, for all $\prec$-ordered $l$-tuples $\vec{b}$ over $B'$.

Since $k'$ is large with respect to $\tau$ and because there is only bounded number of $(m+1)$-ary $\tau$-types, there is an increasing sequence $i_1, \ldots, i_n$ such that for all $l \leq m$ the $\tau$-types of tuples $(a_{i_j}, \vec{b})$ are equal, for all $\prec$-ordered $l$-tuples $\vec{b}$ over $B'$ and $j \in \{1, \ldots, n\}$. We choose $A' := \{a_{i_1}, \ldots, a_{i_n}\}$. Then $B'$ is $A'$-homogeneous up to arity $m$ and therefore $A'$-homogeneous.

It remains to give explicit numbers. For the sequence $i_1, \ldots, i_n$ to exist in $1, \ldots, k'$, the number $k'$ has to be at least $nM + 1$ where $M$ is the number of $(m+1)$-ary $\tau$-types. Thus $k$ has to be at least $R_{\tau,k'}(k') + c$ where $c$ is the number of constants in $\tau$. Define $R_{\tau}^{\hom}(n) \overset{\text{def}}{=} k$.

**Proof (of Theorem 8).** Let us assume, towards a contradiction, that the dynamic program $(P, \text{INIT}, Q)$ over schema $\tau = (\tau_{in}, \tau_{aux}, \tau_{bi})$ with binary $\tau_{aux}$ maintains the dynamic $s$-$t$-reachability query for 2-layered $s$-$t$-graphs. We choose numbers $n, n_1, n_2$ and $n_3$ such that $n_3$ is sufficiently large with respect to $\tau$, $n_2$ is sufficiently large with respect to $n_3$, $n_2$ is sufficiently large with respect to $n_1$ and $n$ is sufficiently large with respect to $n_1$.

Let $G = (V, E)$ be a 2-layered $s$-$t$-graph with layers $A$, $B$, where $A$ and $B$ are both of size $n$ and $E = \{(b,t) \ | \ b \in B\}$. Further, let $S = (V, E, A, B)$ be the state obtained by applying $\text{INIT}$ to $G$.

We will first choose homogeneous subsets. By Lemma 10 and because $n$ is sufficiently large, there are subsets $A_1$ and $B_1$ such that $|A_1| = |B_1| = n_1$ and $B_1$ is $A_1$-homogeneous in $S$, for some order $\prec$. Next, let $A_2$ and $B_2$ be arbitrarily chosen subsets of $A_1$ and $B_1$, respectively, of size $|B_2| = n_2$ and $|A_2| = 2^{|B_2|}$, respectively. We note that $B_2$ is still $A_2$-homogeneous. In particular, $B_2$ is still

---

11 Again, explicit numbers can be found at the end of the proof.
A₂-homogeneous with respect to schema τ₃. We associate with every subset $X \subseteq B_2$ a unique vertex $a_X$ from $A_2$ in an arbitrary fashion.

Now, we define the update sequence $\alpha$ as follows.

(a) For every subset $X$ of $B_2$ and every $b \in X$ insert an edge $(a_X, b)$, in some arbitrarily chosen order.

Let $S' \overset{\text{def}}{=} (V', E', A', B')$ be the state of $\mathcal{P}$ after applying $\alpha$ to $S$, i.e. $S' = P_\alpha(S)$. We observe that the built-in data has not changed, but the auxiliary data might have changed. In particular, $B_2$ is not necessarily $A_2$-homogeneous with respect to schema $\tau_{aux}$ in state $S'$.

Our plan is to exhibit two sets $X, X'$ such that $X \subsetneq X' \subseteq B_2$ such that the restriction of $S'$ to $\{s, t, a_X\} \cup X'$ contains an isomorphic copy of $S'$ restricted to $\{s, t, a_X\} \cup X$. Then the Substructure Lemma will easily give us a contradiction.

By Ramsey’s Theorem and because $|B_2|$ is sufficiently large with respect to $n_2$, there is a subset $B_3 \subseteq B_2$ of size $n_3$ such that $B_3$ is $S'$-homogeneous in $S'$. Let $b_1 < \ldots < b_{n_3}$ be an enumeration of the elements of $B_3$ and let $X_i \overset{\text{def}}{=} \{b_1, \ldots, b_i\}$, for every $i \in \{1, \ldots, n_3\}$.

Let $S'_i$ denote the restriction of $S'$ to $X_i \cup \{s, t, a_X\}$. For every $i$, we construct a word $w_i$ of length $i$, that has a letter for every node in $X_i$ and captures all relevant information about those nodes in $S'_i$. More precisely, $w_i \overset{\text{def}}{=} \sigma_i^1 \cdots \sigma_i^l$, where for every $i$ and $j$, $\sigma_j^l$ is the binary type of $(a_{X_i}, b_j)$.

Since $B_3$ is sufficiently large with respect to $\tau_{aux}$, it follows, by Higman’s Lemma, that there are $k$ and $l$ such that $k < l$ and $w_k \subseteq w_l$, that is $w_k = \sigma_1^1 \sigma_2^2 \cdots \sigma_k^k = \sigma_1^{i_1} \sigma_2^{i_2} \cdots \sigma_k^{i_k}$ for suitable numbers $i_1 < \ldots < i_k$. Let $\vec{b} \overset{\text{def}}{=} (b_1, \ldots, b_k)$ and $\vec{\sigma}_{\vec{b}} \overset{\text{def}}{=} (b_1, \ldots, b_k)$. Further, let $T_k \overset{\text{def}}{=} S'_i \upharpoonright T_k$ where $T_k = \{s, t, a_{X_i}\} \cup \vec{b}$, and $\vec{T}_i \overset{\text{def}}{=} S'_i \upharpoonright T_i$ where $T_i \overset{\text{def}}{=} \{s, t, a_{X_i}\} \cup \vec{\sigma}_{\vec{b}}$. We refer to Figure 3 for an illustration of the substructure $T_k$ and $\vec{T}_i$ of $S'$.

We show that $\vec{T}_i \cong_{\vec{\tau}_i} T_i$, where $\pi$ is the isomorphism that maps $s$ and $t$ to themselves, $a_{X_i}$ to $a_{X_i}$, and $b_j$ to $b_j$, for every $j \in \{1, \ldots, k\}$. We argue that $\pi$ fulfills the requirements of an isomorphism, for every relation symbol $R$ from $\tau_{in} \cup \tau_{bi} \cup \tau_{aux}$:

- For the input relation $E$ this is obvious. In $S'$ there are no edges from $s$ to nodes in $A_2$ and all nodes from $B_2$ have an edge to $t$. Further $X_i$ is connected to all nodes in $b$ and $X_k$ is connected to all nodes in $\vec{\sigma}_{\vec{b}}$.
- For $R \in \tau_{bi}$, the requirement follows because $B_2$ is $A_2$-homogeneous for schema $\tau_{bi}$.
- For $R \in \tau_{aux}$ of arity 2 and two 2-tuples $\vec{c}$ and $\pi(\vec{c})$ we distinguish two cases. First, if $\vec{c}$ and $\pi(\vec{c})$ contain elements from $B_3$ only, then $R^{T_k}(\vec{c})$ if and only if $R^{\vec{T}_i}(\pi(\vec{c}))$ because $B_3$ is homogeneous in $S'$. Second, if $\vec{c}$ contains $s$, $t$ or $a_{X_i}$, then $R^{T_k}(\vec{c})$ if and only if $R^{\vec{T}_i}(\pi(\vec{c}))$ because of the construction of $w_k$ and $w_l$.

Thus, by the Substructure Lemma, application of the following two update sequences to $S'$ results in the same query result:

(β₁) Deleting edges $(a_{X_k}, b_1), \ldots, (a_{X_k}, b_k)$ and adding an edge $(s, a_{X_k})$. 


Fig. 3. The structure $\mathcal{S}'$ from the proof of Theorem 8. The isomorphic substructures $\mathcal{T}_k$ and $\mathcal{T}_l$ are highlighted in blue.

$(\beta_2)$ Deleting edges $(a_{X_1}, b_{i_1}), \ldots, (a_{X_k}, b_{i_k})$ and adding an edge $(s, a_{X_l})$.

However, applying $\beta_1$ yields a graph in which $t$ is not reachable from $s$, whereas by applying $\beta_2$ a graph is obtained in which $t$ is reachable from $s$. This is the desired contradiction.

It remains to make the sizes of the sets precise. To apply Higman’s Lemma, $|B_3|$ has to be of size at least $n_3 \overset{\text{def}}{=} H(m)$ where $m$ is the number of binary types over $\tau_{\text{aux}}$. Hence, for applying Ramsey’s theorem, $|B_2|$ has to be of size $n_2 \overset{\text{def}}{=} R_r(n_3)$. Thus it is sufficient if $|B_1|$ and $|A_1|$ contain $n_1 \overset{\text{def}}{=} 2^{n_2}$ elements. Therefore, by Lemma 10 the sets $A$ and $B$ can be chosen of size $n \overset{\text{def}}{=} R_{\text{hom}}(n_1)$.

$\square$
4.2 Separating Low Arities

An arity hierarchy for $\text{DynFO}$ was established in [3]. The dynamic queries $Q_{k+1}$ used to separate $k$-ary and $(k + 1)$-ary $\text{DynFO}$ can already be maintained in $(k + 1)$-ary $\text{DynProp}$, thus the hierarchy transfers to $\text{DynProp}$ immediately. However, $Q_{k+1}$ is a $k$-ary query and has an input schema of arity $6k+1$ (improved to $3k+1$ in [15]). Here we establish a strict arity hierarchy between unary, binary and ternary $\text{DynProp}$ for Boolean queries and binary input schemas.

We use the following problems $s$-$t$-$\text{TwoPath}$ and $s$-$\text{TwoPath}$

**Query:** $s$-$t$-$\text{TwoPath}$

**Input:** An $s$-$t$-graph $G = (V, E)$.

**Question:** Is there a path of length two from $s$ to $t$?

**Query:** $s$-$\text{TwoPath}$

**Input:** A graph $G = (V, E)$ with one distinguished node $s \in V$.

**Question:** Is there a path of length two starting from $s$?

**Proposition 11.** The dynamic query $\text{Dyn}(s$-$t$-$\text{TwoPath})$ is in binary $\text{DynProp}$, but not in unary $\text{DynProp}$.

**Proof sketch.** That $\text{Dyn}(s$-$t$-$\text{TwoPath})$ is not in unary $\text{DynProp}$ follows immediately from Proposition 9 as such a program would also maintain the dynamic $s$-$t$-reachability query for 1-layered graphs.

In order to prove that $\text{Dyn}(s$-$t$-$\text{TwoPath})$ is in binary $\text{DynProp}$, we sketch a $\text{DynProp}$-program $(P, \text{Init}, Q)$ whose auxiliary schema contains unary relation symbols $\text{In}$, $\text{Out}$, $\text{First}$, and $\text{Last}$ and a binary relation symbol $\text{List}$. The idea is to store, in a program state $S$, a list of all nodes $a$ such that $(s, a, t)$ is a path in $E^S$. The relation $\text{In}^S$ contains all nodes with an incoming edge from $s$, and $\text{Out}^S$ contains all nodes with an outgoing edge to $t$. The relations $\text{First}^S$, $\text{Last}^S$, $\text{List}^S$ maintain the actual list, similarly to Example 1. The current query bit is maintained in $Q^S$.

For a given instance of $s$-$t$-$\text{TwoPath}$ the initialization mapping initializes the auxiliary relations accordingly.

**Insertion of $(a, b)$ into $E$.** We note that edges $(a, b)$ where $a \neq s$ and $b \neq t$ can be ignored, as they cannot contribute to any path of length 2 from $s$ to $t$. Furthermore, paths of length 2 involving only nodes $s$ and $t$ can be easily handled by $\text{DynProp}$ formulas, and therefore will be ignored as well.

If $a = s$ and $b \neq t$, then $b$ is inserted into $\text{In}$, otherwise if $a \neq s$ and $b = t$ then $a$ is inserted into $\text{Out}$.

Afterwards $a$ or $b$ is inserted into $\text{List}_1$, if it is now contained in both $\text{In}$ and $\text{Out}$. In that case the query bit is set true.
Formally:

\[ \phi_{\text{ins}}^\text{IN}(a, b; x) = \text{IN}(x) \lor (x = b \land a = s \land b \neq s \land b \neq t) \]

\[ \phi_{\text{ins}}^\text{OUT}(a, b; x) = \text{OUT}(x) \lor (x = a \land a \neq s \land a \neq t \land b = t) \]

\[ \phi_{\text{ins}}^\text{FIRST}(a, b; x) = \text{FIRST}(x) \lor (\neg Q \land \varphi_n(x)) \]

\[ \phi_{\text{ins}}^\text{LAST}(a, b; x) = (\text{LAST}(x) \land \neg \varphi_n(a) \land \neg \varphi_n(b)) \lor \varphi_n(x) \]

\[ \phi_{\text{ins}}^\text{LIST}(a, b; x, y) = (\text{LIST}(x, y) \land \neg \varphi_n(a) \land \neg \varphi_n(b)) \lor (\text{LAST}(x) \land \varphi_n(y)) \]

\[ \phi_{\text{ins}}^Q(a, b) = Q \lor \varphi_n(a) \lor \varphi_n(b) \]

Here, \( \varphi_n(x) \) is an abbreviation for

\[ \phi_{\text{ins}}^\text{IN}(a, b; x) \land \phi_{\text{ins}}^\text{OUT}(a, b; x) \land (\neg \text{IN}(x) \lor \neg \text{OUT}(x)) \]

expressing that \( x \) is becoming newly inserted into \( \text{LIST} \).

**Deletion of \((a, b)\) from \(E\).** First, if \( a = s \), then \( b \) is removed from \( \text{IN} \). Further if \( b = t \) then \( a \) is removed from \( \text{OUT} \).

Afterwards \( a \) or \( b \) is removed from \( \text{LIST} \), if it has been removed from \( \text{IN} \) or \( \text{OUT} \). If \( \text{LIST} \) is empty now, then the query bit is set to false. The precise formulas are along the lines of the formulas of Example 11.

**Proposition 12.** The dynamic query \( \text{Dyn}(s\text{-TwoPath}) \) is in ternary \( \text{DynProp} \), but not in binary \( \text{DynProp} \).

**Proof Sketch.** For proving that \( \text{Dyn}(s\text{-TwoPath}) \) is not in binary \( \text{DynProp} \), assume to the contrary that there is a binary \( \text{DynProp} \)-program \( P = (P, \text{INIT}, Q) \) for \( \text{Dyn}(s\text{-TwoPath}) \). With the help of \( P \) one can, for the graphs from the proof of Proposition 9, maintain whether there is a path from \( s \) to some node of \( B \). However, this yields a correct answer for \( s\text{-I-Reach} \) for those graphs, since in the proof all nodes of \( B \) have an edge to \( t \).

In order to prove that \( \text{Dyn}(s\text{-TwoPath}) \) is in ternary \( \text{DynProp} \), we sketch a \( \text{DynProp} \)-program \( (P, \text{INIT}, Q) \) whose auxiliary schema contains unary relation symbols \( \text{IN}, \text{OUT}, \text{FIRST}_1, \text{LAST}_1 \) and \( \text{EMPTY}_1 \), binary relation symbols \( \text{LIST}_1 \), \( \text{FIRST}_2 \), \( \text{LAST}_2 \) and \( \text{EMPTY}_2 \), and a ternary relation symbol \( \text{LIST}_2 \). The idea is that in a state \( S \), the binary relation \( \text{LIST}_1^S \) contains a list of all nodes \( a \) on a path \((s, a, b)\) in \( E^S \), for some node \( b \). The relation \( \text{IN}^S \) contains all nodes with an incoming edge from \( s \), and \( \text{OUT}^S \) contains all nodes with an outgoing edge. In order to update \( \text{OUT}^S \), the projection \( \text{LIST}_2^S(a, \cdot, \cdot) \) of the ternary relation \( \text{LIST}_2^S \) stores a list of nodes \( b \) with \((a, b) \in E^S \), for every node \( a \). The lists \( \text{LIST}_1^S \) and \( \text{LIST}_2^S(a, \cdot, \cdot) \) are maintained by using the technique from Example 11 and by using the auxiliary relations stored in \( \text{FIRST}_1^S, \text{LAST}_1^S, \text{EMPTY}_1^S, \text{FIRST}_2^S, \text{LAST}_2^S \) and \( \text{EMPTY}_2^S \). The current query bit is maintained in \( Q^S \).

For a given instance of \( s\text{-TwoPath} \) the initialization mapping initializes the auxiliary relations accordingly.

**Insertion of \((a, b)\) into \(E\).** First, if \( a = s \) then \( b \) is inserted into \( \text{IN} \). Otherwise, \( a \) is inserted into \( \text{OUT} \) and \( b \) is inserted into \( \text{LIST}_2(a, \cdot, \cdot) \).
Afterwards $a$ or $b$ is inserted into List, if it is now contained in both In and Out. If one of them is inserted, then the query bit is set true.

**Deletion of $(a, b)$ from $E$.** First, if $a = s$ then $b$ is removed from In. Otherwise, $b$ is removed from List$_2(a, \cdot)$ and if List$_2(a, \cdot)$ is empty afterwards, then $a$ is removed from Out.

Afterwards $a$ or $b$ is removed from List$_1$, if it has been removed from In or Out. The query bit is set to false, if the list List$_1$ is empty now. □

### 4.3 Invariant Initialization

We now turn to the setting with invariant initialization. Recall that an initialization mapping Init with $\text{Init} = (\text{Init}_{\text{aux}}, \text{Init}_{\text{bi}})$ is invariant if $\pi(\text{Init}_{\text{aux}}(D)) = \text{Init}_{\text{aux}}(\pi(D))$ holds, for every database $D$ and permutation $\pi$ of the domain of $D$.

First-order logic, second-order logic and other logics considered in computer science can only define queries, i.e. mappings that are invariant under permutations. Therefore the following result applies, in particular, for all initialization mappings defined in those logics.

**Theorem 13.** $\text{Dyn}(s\text{-t-REACH})$ cannot be maintained in $\text{DynProp}$ with invariant initialization mapping and empty built-in schema. This holds even for 1-layered $s$-$t$-graphs.

**Proof.** Towards a contradiction, assume that the dynamic program $(P, \text{Init}, Q)$ with schema $\tau = \tau_{\text{in}} \cup \tau_{\text{aux}}$ and invariant initialization mapping Init maintains the $s$-$t$-reachability query for 1-layered $s$-$t$-graphs. Let $n$ be the number of types of tuples of arity up to $m$ for $\tau_{\text{aux}} \cup \{E\}$ where $m$ is the highest arity of relation symbols in $\tau_{\text{aux}} \cup \{E\}$.

We consider the 1-layered $s$-$t$-graphs $G_i = (V_i, E_i)$, for every $i \in \{1, \ldots, n + 1\}$, with $V_i = \{s, t\} \cup A_i$ where $A_i = \{a_0, \ldots, a_i\}$ and $E = \{\{s\} \times A_i \cup A_i \times \{t\}\}$. Further, we let $S_i = (V_i, E_i, A_i)$ be the state obtained by applying Init to $G_i$.

Our goal is to find $S_k$ and $S_l$ with $k < l$ such that $S_k$ is isomorphic to $S_l \mid V_k$ (see Figure 3 for an illustration). Then, by Corollary 3, the program $P$ computes the same query result for the following update sequences:

1. $(\beta_1)$ Delete edges $(s, a_0), \ldots, (s, a_k)$ from $S_k$.
2. $(\beta_2)$ Delete edges $(s, a_0), \ldots, (s, a_k)$ from $S_l$.

However, applying the update sequence $\beta_1$ yields a graph where $t$ is reachable from $s$, whereas by $\beta_2$ a graph is obtained where $t$ is not reachable from $s$, a contradiction.

Thus it remains to find such states $S_k$ and $S_l$. A tuple is *diverse*, if all components are pairwise different. For arbitrary $m' \leq m$, diverse tuples $\vec{a}, \vec{b} \in A^{m'}$ and $i \leq n$, we observe that $G_i \simeq_{\text{id}[\vec{a}, \vec{b}]} G_i$ where $\text{id}[\vec{a}, \vec{b}]$ is the bijection that maps $a_i$ to $b_i$, $b_i$ to $a_i$ and every other element from $S$ to itself. Therefore $S_i \simeq_{\text{id}[\vec{a}, \vec{b}]} S_l$ by the invariance of Init. Thus $(S_i, \vec{a}) = (S_l, \vec{a})$, and therefore all diverse $m'$ tuples are of the same type in $S_i$. 

Fig. 4. The structures $S_k$ and $S_l$ from the proof of Theorem 13. The isomorphic substructures are highlighted in blue.

Since $n$ is the number of types up to arity $m$, there are two states $S_k$ and $S_l$ such that, for every $m' \leq m$, all diverse $m'$-tuples are of the same type in $S_k$ and $S_l$. But then $S_k \simeq S_l|V_k$.

The proof of the previous result does not extend to DynFO, since reachability in graphs of depth three is expressible even in (static) predicate logic. The proof fails, because the substructure lemma does not hold for DynFO-programs. At first glance, layered graphs with many layers look like a good candidate for proving that DynFO cannot maintain $s$-$t$-Reach in this setting. However, in [7] it is shown that DynFO with FO+TC-definable initialization mappings can express $s$-$t$-Reach for arbitrary acyclic graphs.

5 Lower Bounds with Auxiliary Functions

In this section we consider the extension of the quantifier-free update formalism by auxiliary functions. Recall that DynProp-update formulas have access only to the inserted or deleted tuple $\vec{a}$ and the currently updated tuple of an auxiliary relation $\vec{b}$. When auxiliary functions are allowed in update formulas, tuples obtained by applying an auxiliary function to $\vec{a}$ and $\vec{b}$ can be accessed, too. Furthermore, auxiliary functions can be updated. The class of dynamic queries that can be maintained with quantifier-free update formulas and auxiliary functions is denoted DynQF.

After the formalization of this setting and amplifying the Substructure Lemma to DynQF, we prove that

– Dyn($s$-$t$-Reach) is not in unary DynQF; and
– when only invariant initialization mappings are allowed, then Dyn($s$-$t$-Reach) is not in DynQF.

Allowing auxiliary functions can be seen as adding weak quantification to quantifier-free formulas, since applying a function yields a new element. When full first-order updates are available, auxiliary functions can be simulated in a straightforward way by auxiliary relations. However, without quantifiers this is
not possible. Auxiliary functions are quite powerful. While only regular languages can be maintained in DYNPROP, all Dyck languages, among other non-regular languages, can be maintained in DYNQF \[6\]. Further undirected reachability can be maintained in DYNQF with built-in relations \[5\].

We extend our schemata to allow also function symbols. From now on, a schema (or signature) \(\tau\) consists of a set \(\tau_{rel}\) of relation symbols, a set \(\tau_{fun}\) of function symbols and a set \(\tau_{const}\) of constant symbols together with an arity function \(\text{Ar} : \tau_{rel} \cup \tau_{fun} \to \mathbb{N}\). A schema is relational if \(\tau_{fun} = \emptyset\). A database \(D\) of schema \(\tau\) with domain \(D\) is a mapping that assigns to every relation symbol \(R \in \tau_{rel}\) a relation of arity \(\text{Ar}(R)\) over \(D\), to every \(k\)-ary function symbol \(f \in \tau_{fun}\) a \(k\)-ary function, and to every constant symbol \(c \in \tau_{const}\) a single element (called constant) from \(D\).

In the following, we extend our definition of update programs for the case of auxiliary schemata with functions. It is straightforward to extend the definition of update formulas for auxiliary relations: they simply can make use of function terms. However, following the spirit of DYNPROP, we allow a more powerful update mechanism for auxiliary functions that allows case distinctions in addition to composition of function terms.

The following definitions are adapted from \[6\].

**Definition 14.** (Update term) Update terms are inductively defined by the following.

1. Every variable and every constant is an update term.
2. If \(f\) is a \(k\)-ary function symbol and \(t_1, \ldots, t_k\) are update terms, then \(f(t_1, \ldots, t_k)\) is an update term.
3. If \(\phi\) is a quantifier-free update formula (possibly using update terms) and \(t_1\) and \(t_2\) are update terms, then \(\text{ITE}(\phi, t_1, t_2)\) is an update term.

The semantics of update terms associates with every update term \(t\) and interpretation \(I = (S, \beta)\), where \(S\) is a state and \(\beta\) a variable assignment, a value \(\llbracket t \rrbracket_I\) from \(S\). The semantics of (1) and (2) is straightforward. If \(S \models \phi\) is true, then \(\llbracket \text{ITE}(\phi, t_1, t_2) \rrbracket_I = \llbracket t_1 \rrbracket_I\) else \(\llbracket t_2 \rrbracket_I\).

Now we can extend the notion of update programs for auxiliary schemata with function symbols as follows. An update program still has an update formula \(\phi^R_\delta\), for every relation symbol \(R \in \tau_{aux}\) and every abstract update \(\delta\). Furthermore, it has, for every abstract update \(\delta\) and every function symbol \(f \in \tau_{aux}\), an update term \(t^f_\delta(\vec{a}; \vec{y})\). For a concrete update \(\delta(\vec{a})\) it redefines \(f\) for each tuple \(\vec{b}\) by evaluating \(t^f_\delta(\vec{a}; \vec{b})\) in \(S\).

We emphasize that, for simplicity, we allow update terms only to update auxiliary functions, not in update formulas for relations.

**Definition 15.** (DYNQF) DYNQF is the class of queries maintainable by quantifier-free update programs with (possibly) auxiliary functions. The class \(k\)-ary DYNQF is defined via update programs that use auxiliary functions and relations of arity at most \(k\).
Lists can be represented by unary functions in a straightforward way. Therefore, it is not surprising that the upper bound of Proposition 11 already holds for unary DYNProp with unary built-in functions.

**Proposition 16.** DYN(s-t-Reach) on 1-layered s-t-graphs can be maintained in unary DYNProp with unary built-in functions.

**Proof sketch.** We construct a DYNProp-program $P$ over relational auxiliary schema $\{Q,S,T,C\}$ and functional built-in schema $\{\text{Pred}, \text{Succ}\}$, where $Q$ is the query bit (i.e. a 0-ary relation symbol), $S$, $T$ and $C$ are unary relation symbols and $\text{Pred}$ and $\text{Succ}$ are unary function symbols.

For a state $S$ assume without loss of generality that its domain is of the form $D = \{0, \ldots, n-1\}$ with $s = 0$ and $t = n-1$. The built-in function $\text{Succ}^S$ is the standard successor function on $D$ (with $\text{Succ}^S(n-1) = n-1$) and $\text{Pred}^S$ is its corresponding predecessor function (with $\text{Pred}^S(0) = 0$).

The idea is to store the number of vertices connected to both $s$ and $t$ in the unary relation $C^S$. If $i$ nodes are connected to $s$ and $t$ in $S$ then $C^S$ shall be true exactly for node $i \in D$. If an edge-insertion connects an element to $s$ and $t$ then $C^S$ is shifted to $i+1$ by using $\text{Pred}^S$ and $\text{Succ}^S$. Analogously $C^S$ is shifted to $i-1$ for edge-removals that disconnect an element from $s$ or $t$. The relations $S^S$ and $T^S$ store the elements currently connected to $s$ and $t$, respectively.

For a given instance of the $s$-t-reachability query on 1-layered s-t-graphs the initialization mapping initializes the auxiliary relations accordingly.

**Insertion of $(a,b)$ into $E$.** If $a = s$ then node $b$ is inserted into $S$; if $b = t$ then node $a$ is inserted into $T$. Further, if $b$ is now in both $S$ and $T$ then the counter is incremented by 1:

\[
\begin{align*}
\phi_{\text{Ins}}^{S}(a,b;x) & \overset{\text{def}}{=} (a = s \land x = b) \lor S(x) \\
\phi_{\text{Ins}}^{T}(a,b;x) & \overset{\text{def}}{=} (b = t \land x = a) \lor T(x) \\
\phi_{\text{Ins}}^{C}(a,b;x) & \overset{\text{def}}{=} (a = s \land T(b) \land C(\text{Pred}(x))) \\
& \lor (b = t \land S(a) \land C(\text{Pred}(x))) \\
& \lor (\neg S(a) \land \neg T(b) \land C(x)) \\
\phi_{\text{Ins}}^{Q}(a,b) & \overset{\text{def}}{=} \neg \phi_{\text{Ins}}^{C}(a,b;s)
\end{align*}
\]

Deletions can be maintained in a similar way.\hfill\Box

In the following we work towards lower bounds for DYNQF.

The *nesting depth* $d(t)$ of an update term $t$ is the nesting depth of its functions, that is, it is defined inductively as follows. If $t$ is a variable, then $d(t) = 0$; if $t$ is of the form $f(t_1, \ldots, t_k)$ then $d(t) = \max\{d(t_1), \ldots, d(t_k)\} + 1$; and if $t$ is of the form $\text{ite}(\varphi, t_1, t_2)$ then $d(t) = \max\{d(t_1), d(t_2)\}$. The nesting depth of an update formula $\varphi$ is the maximal nesting depth of all update terms occurring in $\varphi$.

The *nesting depth* of $P$ is the maximal nesting depth of an update term occurring in $P$. 
Lemma 17. If an \(l\)-ary query \(Q\) can be maintained by a \(\text{DynQF}\)-program, then \(Q\) can be maintained by a \(k\)-ary \(\text{DynQF}\)-program with only one \(l\)-ary auxiliary relation (used for storing the query result) on databases with at least two elements.

Proof sketch. A \(k\)-ary relation \(R\), for \(k \geq 1\) can be easily encoded by a \(k\)-ary function \(f_R\) via \(R(a_1, \ldots, a_k)\) if and only if \(f(a_1, \ldots, a_k) = a_1\). A Boolean relation \(R\) can be encoded via by a unary function that is the identity if \(R\) holds and fulfills \(f(a) \neq a\), for every \(a\), otherwise. \(\Box\)

We extend the Substructure Lemma to non-relational structures. If an update changes a tuple from a substructure \(S'\) of a structure \(S\), then the update of the auxiliary data of \(S'\) can depend on elements obtained from applying functions to elements in \(S'\). Those elements are called neighbourhood and defined next.

For a schema \(\tau\), let \(\text{TERMS}_k^\tau\) be the set of terms of depth at most \(k\) with function symbols from \(\tau\). Let \(\vec{t}_\tau = (t_1, \ldots, t_l)\) be the lexicographic enumeration of \(\text{TERMS}_\tau^k\), based on some fixed order of the function symbols.

Definition 18. (Neighbourhoods) The \(k\)-neighbourhood vector \(N_S^k(a)\) of an element \(a\) in structure \(S\) over schema \(\tau\) is the tuple \(\vec{t}_\tau(a) = (a, t_1(a), \ldots, t_l(a))\). For a tuple \(\vec{a} = (a_1, \ldots, a_m)\) the \(k\)-neighbourhood vector \(N_S^k(\vec{a})\) of \(\vec{a}\) is the tuple \((N_S^k(a_1), \ldots, N_S^k(a_m))\). The \(k\)-neighbourhood \(N_S^k(a)\) of an element \(a\) is the set of elements of \(S\) that occur in \(N_S^k(a)\). The \(k\)-neighbourhood \(N_S^k(S')\) of a subset \(S' \subseteq S\) is the union of all \(k\)-neighbourhoods of all elements in \(S'\). A subset \(S'\) of \(S\) is closed if \(N_S^1(S') = S'\).

A bijection \(\pi\) between (the domains \(S\) and \(T\) of) two structures \(S\) and \(T\) over \(\tau\) is an isomorphism, if it is a \(\tau_{\text{rel}}\)-isomorphism and \(\pi(f^S(\vec{a})) = f^T(\pi(\vec{a}))\) for all \(k\)-ary function symbols \(f \in \tau_{\text{aux}}\) and \(k\)-tuples \(\vec{a}\) over \(S\).

Two subsets \(S' \subseteq S, T' \subseteq T\) are \(k\)-similar, if there is a bijection \(\pi : N_S^k(S') \rightarrow N_T^k(T')\) such that

- the restriction of \(\pi\) to \(S'\) is a bijection of \(S'\) and \(T'\),
- \(\pi\) satisfies \(\pi(t^S(\vec{a})) = t^T(\pi(\vec{a}))\) for all \(t \in \text{TERMS}_T^k\) and \(\vec{a} \in S'\),
- \(\pi\) preserves \(\tau_{\text{rel}}\) on \(N_S^k(S')\), and

We write \(S' \approx_k^{\pi,\tau} T'\) to indicate that \(S'\) and \(T'\) are \(k\)-similar via \(\pi\) in \(S\) and \(T\). We drop \(S\) and \(T\) from this notation if they are clear from context and we drop \(\pi\) to indicate that there is such \(\pi\). We also write \((a_1, \ldots, a_p) \approx_k^{\tau,\pi} (b_1, \ldots, b_p)\) to indicate that \(\{a_1, \ldots, a_p\} \approx_k^{\tau,\pi} \{b_1, \ldots, b_p\}\) via an isomorphism that maps \(a_i\) to \(b_i\), for every \(i \in \{1, \ldots, p\}\). Note that if \(S' \approx_0 T'\), then they \(S' \cap S\) and \(T' \cap T\) are \(\tau_{\text{rel}}\)-isomorphic by the second property.

The following lemma is a slightly generalized variation of Lemma 4 from [10] and a generalization of the Substructure lemma.

Lemma 19. (Substructure Lemma for DynQF) Let \(\mathcal{P}\) be a \(\text{DynQF}\) program, let \(k\) be the nesting depth of \(\mathcal{P}\) and let \(l\) be some number. Furthermore let \(S\) and \(T\) be states of \(\mathcal{P}\) and let \(S\) and \(T\) be subsets of the domains of \(S\) and \(T\), respectively.
There is a number $m \in \mathbb{N}$ such that if $S \approx_{m}^{\pi,S,T} T$, then $S \approx_{0}^{\pi,P_{\alpha}(S),P_{\beta}(T)} T$, for all $\pi$-respecting update sequences $\alpha$, $\beta$ on $S'$ and $T'$ of length $l$.

Proof. The proof is an extension of the proof of Lemma 2. The lemma follows by an induction over the length $l$ of the update sequence. The induction step follows easily with Claim (C) below, for sufficiently large $m$ depending only on $k$ and $l$.

Let $\delta(\bar{a})$ and $\delta(\bar{b})$ be two $\pi$-respecting updates on $S'$ and $T'$, respectively, i.e. $\bar{b} = \pi(\bar{a})$. Let $S' \equiv P_{\delta(\bar{a})}(S)$ and $T' \equiv P_{\delta(\bar{b})}(S)$. We prove the following claims for arbitrary $r \in \mathbb{N}$:

(A) If $S \approx_{r+k}^{\pi,S,T} T$, then for all $\bar{c}$ over $N_{S}(S)$:

(i) $R^{S'}(\bar{c})$ if and only if $R^{T'}(\pi(\bar{c}))$ for all relation symbols $R \in \tau_{\text{aux}}$.

(ii) $f^{S'}(\bar{c}) \in N_{S}^{r+k}(S)$ and $\pi(f^{S'}(\bar{c})) = f^{T'}(\pi(\bar{c}))$ for all function symbols $f \in \tau_{\text{aux}}$.

(B) If $S \approx_{r-k}^{\pi,S,T} T$, then $t^{S'}(\bar{c}) \in N_{S}^{r-k}(S)$ and $\pi(t^{S'}(\bar{c})) = t^{T'}(\pi(\bar{c}))$ for all terms $t \in \text{Terms}_{\tau_{\text{aux}}}^{r}$ and $\bar{c}$ over $S$.

(C) If $S \approx_{r-k+k}^{\pi,S,T} T$, then $S \approx_{r}^{\pi,S,T} T$.

We prove Claim (A) first. We recall that $R^{S'}(\bar{c})$ holds if and only if $S \models \phi_{R}(\bar{a};\bar{c})$, and that $f^{S'}(\bar{c})$ is $[t]^{f}_{(\bar{x},\bar{y})}(\bar{S},\gamma)$ where $\gamma$ maps $(\bar{x},\bar{y})$ to $(\bar{a},\bar{c})$. Since $\bar{a}$ and $\bar{c}$ are tuples over $N_{S}(S)$ it is sufficient to prove, for every tuple $\bar{d}$ over $N_{S}(S)$, that $\varphi(\bar{d})$ holds in $S$ if and only if $\varphi(\pi(\bar{d}))$ holds in $T$, and that $\pi([t]_{(\bar{x},\bar{y})}(\bar{S},\gamma)) = [t]_{(\bar{t},\pi(\bar{a}))}$, for every quantifier-free formula $\varphi$ and every update term $t$ with nesting depth at most $k$.

The proof is by induction on the nesting depth of $\varphi$. We start with the base case. The case, where the update term is a single variable, is trivial because $\pi$ witnesses the $(r + k)$-similarity of $S$ and $T$ in $S$ and $T$. Further, $R^{S}(\bar{d})$ holds if and only if $R^{T}(\pi(\bar{d}))$ holds, for every relation symbol $R$ and $\bar{d} \subseteq \bar{d}$, since $\pi$ preserves $\tau_{\text{rel}}$ on $N_{S}(S)$. Therefore, the claim holds for every update term and update formula of nesting depth 0.

For the induction step, we consider update terms and update formulas with nesting depth $k' \in \{1, \ldots, k\}$. If an update term $t$ with $d(t) = k'$ is of the form $f(\bar{s})$ with $\bar{s} = (s_{1}, \ldots, s_{n})$, then, by induction hypothesis, for every $i$ it holds $\pi([s_{i}]_{(\bar{x},\bar{y})}(\bar{S},\gamma)) = [s_{i}]_{(\bar{x},\bar{y})}(\bar{S},\gamma)$ and $s_{i}(\bar{c}_{i}) \in N_{S}^{r+k'-1}(S)$ where $\bar{c}_{i}$ is a vector consisting of elements from $\bar{d}$. Thus, $\pi([f(\bar{s})]_{(\bar{x},\bar{y})}(\bar{S},\gamma)) = [f(\bar{s})]_{(\bar{x},\bar{y})}(\bar{S},\gamma)$ because $S$ and $T$ are $(r + k)$-similar and $k' \leq k$. The other cases are analogous. This concludes the proof of Claim (A).

Claim (B) can be proved by an induction over the nesting depth of $t$. The induction step uses Claim (A ii).

For Claim (C) we have to prove that $\pi$ is witnessing the $r$-similarity of $S$ and $T$ in $S'$ and $T'$. The first property of similarity is trivial and the second follows

12 Here, we use $\bar{d}$ to denote the variable assignment mapping the free variables of $t$ to the components of $\bar{d}$.
Lemma 20. Let \( \mathcal{P} \) be a DynQF program and \( S \) and \( T \) be states of \( \mathcal{P} \) with domains \( S \) and \( T \). Further let \( S' \subseteq S \) and \( T' \subseteq T \) be closed. If \( S \upharpoonright S' \) and \( T \upharpoonright T' \) are isomorphic via \( \pi \) then \( P_\alpha(S) \upharpoonright S' \) and \( P_\beta(T) \upharpoonright T' \) are isomorphic via \( \pi \) for all \( \pi \)-respecting update sequences \( \alpha, \beta \) on \( S' \) and \( T' \).

Proof. Observe that when \( S' \) and \( T' \) are closed and \( S \upharpoonright S' \) and \( T \upharpoonright T' \) are isomorphic via \( \pi \) then \( S' \) and \( T' \) are \( k \)-similar via \( \pi \) for arbitrary \( k \). Thus the claim follows from Lemma 19.

We now prove that unary DynQF cannot maintain \( s-t \)-reachability. Intuitively unary functions cannot store the transitive closure relation of a directed path in such a way, that the information can be extracted by a quantifier-free formula.

Theorem 21. Dyn(s-t-Reach) is not in unary DynQF.

Proof. Towards a contradiction, we assume that \( \mathcal{P} = (P, \text{Init}, Q) \) is a unary DynQF-program that maintains \( s-t \)-reachability over schema \( \tau = \tau_m \cup \tau_{aux} \) with unary \( \tau_{aux} \). By Lemma 17 we can assume that \( \tau_{aux} \) contains only unary function symbols and one 0-ary relation symbol \( Q \) for storing the query result. The graphs used in this proof do not have self-loops and every node has at most one outgoing edge. Therefore we can assume, in order to simplify the presentation, that \( \tau_{aux} \) contains a unary function symbol \( e \), such that in every state \( S \) the function \( e^S \) encodes the edge relation \( E \) as follows. If the single outgoing edge from \( u \) is \((u, v)\) then \( e(u) = v \) and if \( u \) has no outgoing edge then \( e(u) = u \).

Let \( k \) be the nesting depth of \( \mathcal{P} \) and let \( n \) be chosen sufficiently large with respect to \( \tau \) and \( k \). Let \( G = (V, E) \) be a graph where \( V = \{s, t\} \cup A \) with \( A = \{a_1, \ldots, a_n\} \) and \( E = \{(a_i, a_{i+1}) \mid i \in \{1, \ldots, n - 1\}\} \), i.e. \( G \upharpoonright A \) is a path of length \( n - 1 \). Further let \( S = (V, E, A) \) be the state obtained by applying \( \text{Init} \) to \( G \).

Our goal is to find \( i \) and \( j \) with \( i < j \) such that for the two nodes \( a \overset{\text{def}}{=} a_i \) and \( b \overset{\text{def}}{=} a_j \) it holds \((a, b, s, t) \approx_m (b, a, s, t)\), where \( m \) is the number from Lemma 19 for update sequences of length 2 and nesting depth \( k \).

Then, by the Substructure lemma for auxiliary functions, the program \( \mathcal{P} \) computes the same query result for the following two update sequences:

(\( \beta_1 \)) Insert edges \((s, a)\) and \((b, t)\).
(\( \beta_2 \)) Insert edges \((s, b)\) and \((a, t)\).

However, applying the update sequence \( \beta_1 \) yields a graph in which \( t \) is reachable from \( s \), whereas by \( \beta_2 \) yields a graph in which \( t \) is not reachable from \( s \) (see Figure 5 for an illustration). This is the desired contradiction.
Thus it remains to exhibit $i$ and $j$. We recall that $\mathcal{N}_S^n(\vec{a})$ is the neighbourhood vector of a $p$-tuple $\vec{a}$ with respect to function terms over $\tau_e \overset{df}{=} \tau_{\text{aux}} \cup \{e\}$ of nesting depth at most $q$. The number of equality types of such neighbourhood vectors is finite and bounded by a number that only depends on $p$, $q$ and $\tau_e$.

By applying Ramsey’s Theorem on the graph over $\{1, \ldots, n\}$, where each pair $(i, j)$ with $i < j$ is colored by the equality type of $\mathcal{N}_S^{m+1}(a_i, a_j, s, t)$, we obtain numbers $i_1 < i_2 < i_3$ such that the equality types of $\mathcal{N}_S^{m+1}(a_{i_1}, a_{i_2}, s, t)$, $\mathcal{N}_S^{m+1}(a_{i_1}, a_{i_3}, s, t)$, and $\mathcal{N}_S^{m+1}(a_{i_2}, a_{i_3}, s, t)$ are equal. In particular, as all function symbols are unary, the equality types of $\mathcal{N}_S^{m+1}(a_{i_1}, s, t)$, and $\mathcal{N}_S^{m+1}(a_{i_2}, s, t)$ and finally those of $\mathcal{N}_S^{m+1}(a_{i_1}, a_{i_2}, s, t)$ and $\mathcal{N}_S^{m+1}(a_{i_2}, a_{i_1}, s, t)$ are equal.

For the latter conclusion, we observe that, if for two terms $t_1$ and $t_2$ of depth $m+1$ it holds $t_1(a_{i_1}) = t_2(a_{i_2})$ then also $t_1(a_{i_3}) = t_2(a_{i_3})$ and $t_1(a_{i_2}) = t_2(a_{i_2})$. Hence, $t_1(a_{i_2}) = t_2(a_{i_2})$ and therefore $t_1(a_{i_2}) = t_2(a_{i_2}) = t_1(a_{i_1}) = t_2(a_{i_1})$. The latter equality follows as the equality types of $\mathcal{N}_S^{m+1}(a_{i_1}, s, t)$, and $\mathcal{N}_S^{m+1}(a_{i_2}, s, t)$ are equal.

To prove $\langle S, \mathcal{N}_S^n(a, b, s, t) \rangle = \langle S, \mathcal{N}_S^n(b, a, s, t) \rangle$ it only remains to show that $E(u, v)$ if and only if $E(u', v')$, for two components $u$ and $v$ from $\mathcal{N}_S^n(a, b, s, t)$ and their corresponding components $u'$ and $v'$ from $\mathcal{N}_S^n(b, a, s, t)$. However, $E(u, v)$ if and only if $e(u) = v$, and analogously $E(u', v')$ if and only if $e(u') = v'$. Thus this claim follows already from the fact that $\mathcal{N}_S^{m+1}(a_{i_1}, a_{i_2}, s, t)$ and $\mathcal{N}_S^{m+1}(a_{i_2}, a_{i_1}, s, t)$ have the same equality type.
We now extend the lower bound of Theorem 13 to quantifier-free programs with auxiliary functions.

Invariant initialization is still weak in the presence of auxiliary functions in the sense that functions initialized by invariant initialization can only point to ‘distinguished’ nodes, as formalized by the following lemma.

**Lemma 22.** Let $\mathcal{P} = (P, \text{Init}, Q)$ be a DynQF-program with invariant initialization mapping $\text{Init}$ and auxiliary schema $\tau_{aux}$. Further let $I$ be an input structure for $\mathcal{P}$ whose domain contains $b$ and $b'$ with $b \neq b'$. If $\text{id}[b, b']$ is an isomorphism of $I$, then $f^{\text{Init}(I)}(\vec{a}) \neq b$ for all $k$-ary function symbols $f \in \tau_{aux}$ and all $k$-tuples $\vec{a}$.

**Proof.** The claim follows immediately from the invariance of the initialization mapping.

**Theorem 23.** Dyn(s-t-Reach) cannot be maintained in DynQF with invariant initialization mapping and empty built-in schema. This holds even for 1-layered s-t-graphs.

**Proof.** We follow the argumentation of the proof of Theorem 13.

Towards a contradiction, assume that $\mathcal{P}$ is a DynQF-program $\mathcal{P}$ with auxiliary schema $\tau_{aux}$ and invariant initialization mapping $\text{Init}$ which maintains the s-t-reachability query for 1-layered s-t-graphs. Let $m$ be the maximum arity of relation or function symbols in $\tau_{aux} \cup \{E\}$. Further let $n$ be the number of isomorphism types of structures with at most $m + 2$ elements.

We consider the complete 1-layered s-t-graphs $G_i = (V_i, E_i), 2 \leq i \leq n + 2$, with $V_i = \{s, t\} \cup A_i$ and $A_i = \{a_1, \ldots, a_i\}$. Further let $S_i = (V_i, E_i, A_i)$ be the state obtained by applying $\text{Init}$ to $G_i$.

We observe that $\text{id}[b, b']$ is an automorphism of $G_i$ for all pairs $(b, b')$ of nodes in $A_i$ with $b \neq b'$. Thus, by Lemma 22 $s$ and $t$ are the only values that the auxiliary functions in $S_i$ can assume, and therefore $S_i \upharpoonright A \cup \{s, t\}$ is closed for any subset $A$ of $A_i$. Hence, by Lemma 20 it is sufficient to find $S_k$ and $S_l$ with $k < l$ such that $S_k$ is isomorphic to $S_l \upharpoonright V_k$. Then, we can apply the same sequences of updates as in Theorem 13 to reach a contradiction.

Recall that a tuple is diverse, if all components differ pairwise. Since $G_i \simeq_{\text{id}[\vec{a}, \vec{b}]} G_i$, for two diverse $m'$-tuples $\vec{a}$ and $\vec{b}$ over $A_i$ with $m' \leq m$, also $S_i \simeq_{\text{id}[\vec{a}, \vec{b}]} S_i$ by the invariance of $\text{Init}$. In particular $(s, t, \vec{a})$ and $(s, t, \vec{b})$ are of the same isomorphism type.

Since $n$ is the number of isomorphism types of structures of at most $m + 2$ elements, there are two states $S_k$ and $S_l$ such that, all diverse $m$-tuples over $A_k$ and $A_l$ extended by $s$ and $t$ are of the same isomorphism type in $S_k$ and $S_l$, respectively. But then $S_k \simeq S_l \upharpoonright V_k$. 

6 Lower Bounds for Other Dynamic Queries

In this section we use the lower bounds obtained for the dynamic $s$-$t$-reachability query for shallow graphs to establish lower bounds for the dynamic variants of the following Boolean queries

**Query:** $k$-Clique  
**Input:** A graph $G$  
**Question:** Does $G$ contain a $k$-clique?

**Query:** $k$-Col  
**Input:** A graph $G$  
**Question:** Is $G$ $k$-colorable?

where $k$ is a fixed natural number.

Lower bounds for the dynamic variants of the $k$-Clique and $k$-Col problems (where $k$ is fixed) can be established via reductions to the dynamic $s$-$t$-reachability query for shallow graphs.

**Proposition 24.** The dynamic query $\text{Dyn}(k\text{-Clique})$, for $k \geq 3$, and the dynamic query $\text{Dyn}(k\text{-Col})$, for $k \geq 2$, are not in binary $\text{DynProp}$.

**Proof.** We prove that $\text{Dyn}(3\text{-Clique})$ cannot be maintained in binary $\text{DynProp}$. Afterwards we sketch the proof for $\text{Dyn}(k\text{-Clique})$, for arbitrary $k \geq 3$. The graphs used in the proof have a $k$-Clique if and only if they are not $(k-1)$-colorable. Therefore it follows that $\text{Dyn}(k\text{-Col})$ cannot be maintained in binary $\text{DynProp}$.

More precisely, we show that from a binary $\text{DynProp}$ dynamic program $P'$ for $\text{Dyn}(3\text{-Clique})$ one can construct a dynamic program $P$ that maintains the $s$-$t$-reachability query for 2-layered $s$-$t$-graphs. As the latter does not exist thanks to Proposition 24, we can conclude that the former does not exist either.

Let us thus assume that $P' = (P', \text{Init}', Q')$ is a dynamic program for $\text{Dyn}(3\text{-Clique})$ with binary auxiliary schema $\tau'_{\text{aux}}$ and built-in schema $\tau'_{\text{bi}}$.

The reduction is very simple. For a 2-layered graph $G = (\{s, t\} \cup A \cup B, E)$, let $G'$ be the graph obtained from $G$ by identifying $s$ and $t$. Clearly, $G$ has a path from $s$ to $t$ if and only if $G'$ has a 3-clique.

The dynamic program $P$ uses the same auxiliary schema as $P'$, the same initialization mapping and the same built-in schema relations. However, edges $(u, t)$ in $E$ are interpreted as if they were edges $(u, s)$ in $E'$. More precisely, the update formulas of $P$ are obtained from those in $P'$ by replacing every atomic formula $E'(x, y)$ with $(y = s \land E(x, t)) \lor (y \neq s \land E(x, y))$. Obviously, $P$ is a dynamic program for $s$-$t$-reachability for 2-layered $s$-$t$-graphs if $P'$ is a dynamic program for $\text{Dyn}(3\text{-Clique})$, as desired.

For arbitrary $k$, the construction is similar. The idea is that $P$ simulates on a graph $G$ the behaviour of $P'$ on $G \otimes K_{k-3}$, that is, a graph that results from $G$ by adding a $(k-3)$-clique and completely connecting it with every node of $G$. 
Interestingly, the update formulas of $\mathcal{P}$ are exactly as in the previous reduction to $\text{DYN}(3\text{-CLIQUE})$, as the “virtual” additional $k - 3$ nodes are never involved in changes of the graph. However, $\text{Init}$ is not the same as $\text{Init}'(G)$ but rather the projection of $\text{Init}'(G \otimes K_{k-3})$ to the nodes of $G$.

**Proposition 25.** The dynamic query $\text{DYN}(k\text{-CLIQUE})$, for $k \geq 3$, and the dynamic query $\text{DYN}(k\text{-COL})$, for $k \geq 2$, cannot be maintained in $\text{DynQF}$ with invariant initialization mapping.

*Proof.* The proof approach is the same as for the previous proposition. We prove that $\text{DYN}(3\text{-CLIQUE})$ cannot be maintained in $\text{DynQF}$ with invariant initialization. Afterwards we sketch the proof for $\text{DYN}(k\text{-CLIQUE})$, for arbitrary $k \geq 3$. The graphs used in the proof have a $k$-Clique if and only if they are not $(k - 1)$-colorable. Therefore it follows that $\text{DYN}(k\text{-COL})$ cannot be maintained in $\text{DynQF}$ with invariant initialization mapping.

More precisely, we show that from $\text{DynQF}$ dynamic program $\mathcal{P}'$ with invariant initialization that maintains $\text{DYN}(3\text{-CLIQUE})$ one can construct a dynamic program $\mathcal{P}'$ that maintains the $s$-$t$-reachability query for 1-layered $s$-$t$-graphs. As the latter does not exist thanks to Theorem 23, we can conclude that the former does not exist either.

Let us thus assume that $\mathcal{P}' = (P', \text{Init}', Q')$ is a dynamic program for $\text{DYN}(3\text{-CLIQUE})$ with invariant initialization mapping $\text{Init}'$ and auxiliary schema $\tau_{\text{aux}}'$.

The reduction is very simple. For a 1-layered graph $G = (\{s, t\} \cup A, E)$, let $G'$ be the graph obtained from $G$ by adding an edge $(s, t)$. Clearly, $G$ has a path from $s$ to $t$ if and only if $G'$ has a 3-clique.

The dynamic program $\mathcal{P}$ uses the same auxiliary schema as $\mathcal{P}'$ and the same initialization mapping. The update formulas of $\mathcal{P}$ are obtained from those in $\mathcal{P}'$ by replacing every atomic formula $E'(x, y)$ with $(E(x, y) \lor (x = s \land y = t))$. Obviously, $\mathcal{P}$ is a dynamic program for $s$-$t$-reachability for 2-layered $s$-$t$-graphs if $\mathcal{P}'$ is a dynamic program for $\text{DYN}(3\text{-CLIQUE})$, as desired.

For arbitrary $k$, the construction is similar. The idea is that $\mathcal{P}$ simulates on a graph $G$ the behaviour of $\mathcal{P}'$ on $G \otimes (K_{k-3}, K_{k-3})$, that is, a graph that results from $G$ by adding two $(k - 3)$-cliques and completely connecting them with every node of $G$. Interestingly, the update formulas of $\mathcal{P}$ are exactly as in the previous reduction to $\text{DYN}(3\text{-CLIQUE})$, as the “virtual” additional $2(k - 3)$ nodes are never involved in changes of the graph. However, $\text{Init}$ is not the same as $\text{Init}'(G)$ but rather the projection of $\text{Init}'(G \otimes (K_{k-3}, K_{k-3}))$ to the nodes of $G$. By Lemma 22, auxiliary functions in $\text{Init}(G)$ do not take values from $(K_{k-3}, K_{k-3})$. Thus $\mathcal{P}$ is a dynamic program for $s$-$t$-reachability for 2-layered $s$-$t$-graphs if $\mathcal{P}'$ is a dynamic program for $\text{DYN}(k\text{-CLIQUE})$.

### 7 Normal forms for Dynamic Programs

In this section we introduce normal forms for dynamic programs. The study of normal forms has a long tradition in logics, and usually offers useful insights for
proofs based on the structure of formulas as well as insights for the construction of algorithms. However, the main reason for us to study normal forms of dynamic programs is slightly different. Assume that every dynamic program with update formulas from class $C$ of formulas can be converted to a dynamic program that uses only update formulas from a syntactically 'easier' class $C'$ of formulas, i.e. there is a $C'$ normal form for $C$-formulas. Then properties that cannot be maintained by $C'$-formulas cannot be maintained by $C$-formulas, either. In other words, proving dynamic inexpressibility for $C'$ is already as hard as for $C$.

In the whole section we allow arbitrary initialization and no auxiliary functions.

A formula is negation-free if it does not use negation at all. A formula is conjunctive if it is a conjunction of (positive or negated) literals. A dynamic program is negation-free (conjunctive, respectively) if all its update formulas are negation-free (conjunctive, respectively).

Two dynamic programs $P$ and $P'$ are equivalent, if they maintain the same query.

The first of the following theorems is a straightforward generalization of Theorem 6.6 from [5] which states this observation for a subclass of DynProp.

**Theorem 26.** (a) Every DynFO-program has an equivalent negation-free DynFO-program.

(b) Every DynProp-program has an equivalent negation-free DynProp-program.

**Proof.** We prove (a) first. The construction we use does not introduce quantifiers and can therefore be used for (b) as well.

Let $P = (P, \text{Init}, Q)$ be a DynFO-program over schema $\tau$. We assume, without loss of generality, that $P$ is in negation normal form. Further we assume, for ease of presentation, that the input relations have update formulas as well\(^\text{13}\).

We construct a negation-free DynFO-program equivalent to $P$ that uses the schema $\tau \cup \tilde{\tau}$ where $\tilde{\tau}$ contains for every relation symbol $R \in \tau$ a fresh relation symbol $\tilde{R}$ of equal arity. Recall that $\tau$ includes the input schema, the built-in schema and the auxiliary schema. The idea is to maintain in $\tilde{R}S$ the negation of $RS$, for all states $S$.

In a first step we construct a DynFO-program $P' = (P', \text{Init}', Q)$ in negation normal form over $\tau \cup \tilde{\tau}$ that maintains $\tilde{R}S$ and $\tilde{R}S$ (but still uses negations). The update formulas for relation symbols $R \in \tau$ are as in $P$. For every $\tilde{R} \in \tilde{\tau}$ and every abstract update $\delta$, the update formula $\phi_S^\delta(\tilde{x}; \tilde{y})$ is the negation normal form of $\neg \phi_S^\delta(\tilde{x}; \tilde{y})$. The initialization mapping $\text{Init}'$ initializes $\tilde{R}$ with the complement of $\text{INIT}(R)$.

From $P'$ we construct a negation-free DynFO-program $P'' = (P'', \text{Init}', Q)$. An update formula $\phi_S^\delta(\tilde{x}; \tilde{y})$ for $P''$ is obtained from the update formula $\phi_S^\delta(\tilde{x}; \tilde{y})$ for $P'$ by replacing all negative literals $\neg S$ by $\tilde{S}$. The initialization mapping of $P''$ is the same as for $P'$.

\(^{13}\) E.g. if the input database is a graph, then $\phi_E^\text{ins}(a, b; x, y) = E(x, y) \lor (a = x \land b = y)$ etc.
The equivalence of \( \mathcal{P} \) and \( \mathcal{P}'' \) can be proved by an induction over the length of update sequences.

**Theorem 27.** Every \textsc{DynProp}-program has an equivalent conjunctive \textsc{DynProp}-program.

*Proof.* Let \( \mathcal{P} = (P, \text{Init}, Q) \) be a \textsc{DynProp}-program over schema \( \tau \). We assume, without loss of generality, that \( \tau \) contains, for every relation symbol \( R \), a relation symbol \( \hat{R} \) and that \( \mathcal{P} \) ensures that \( \hat{R}^S \) is the complement of \( R^S \) for every state \( S \). This can be achieved by using the same technique as in Theorem 26. Further we assume that all update formulas of \( \mathcal{P} \) are in conjunctive normal form.

The conjunctive \textsc{DynProp}-program we are going to construct is over schema \( \tau \cup \tau' \) where \( \tau' \) contains a fresh relation symbol \( \delta_{-C} \) for every clause \( C \) occurring in some update formula of \( \mathcal{P} \). The goal of the construction is to ensure that \( R_{-C}^S(\vec{z}) \) holds if and only if \( \neg C(\vec{z}) \) is true in state \( S \). Then an update formula \( \phi = C_1(\vec{x}_1) \land \ldots \land C_k(\vec{x}_k) \) with clauses \( C_1(\vec{x}_1), \ldots, C_k(\vec{x}_k) \) can be replaced by the conjunctive formula \( \neg R_{-C_1}(\vec{x}_1) \land \ldots \land \neg R_{-C_k}(\vec{x}_k) \).

In a first step we construct a \textsc{DynProp}-program \( \mathcal{P}' = (P', \text{Init}', Q) \) in conjunctive normal form that maintains the relations \( R_{-C}^S \). To this end, let \( C \) be a clause with \( k \) variables and let \( \vec{z} \) be the \( k \)-tuple that contains those variables in the order in which they occur in \( C \). Assume that \( C = L_1(\vec{z}_1) \lor \ldots \lor L_l(\vec{z}_l) \) where \( \vec{z}_i \subseteq \vec{z} \) and each \( L_i \) is an atom or a negated atom. Thus \( \neg C \equiv \neg L_1(\vec{z}_1) \land \ldots \land \neg L_l(\vec{z}_l) \).

The relation symbol \( \delta_{-C} \) is of arity \( k \). For an abstract update \( \delta \) the update formula for \( R_{-C} \) is

\[
\phi^{R_{-C}}_{\delta}(\vec{x}; \vec{z}) = \phi^{X_1}_{\delta}(\vec{x}; \vec{z}_1) \land \ldots \land \phi^{X_l}_{\delta}(\vec{x}; \vec{z}_l)
\]

where \( X_i \) is the relation symbol \( R \) if \( L_i = \neg R \) and \( X_i = \hat{R} \) if \( L_i = R \). Observe that \( \phi^{R_{-C}}_{\delta}(\vec{x}; \vec{z}) \) is in conjunctive normal form, because each \( \phi^{X_i}_{\delta}(\vec{x}; \vec{z}_i) \) is in conjunctive normal form; further \( \phi^{R_{-C}}_{\delta}(\vec{x}; \vec{z}) \) does not use new clauses. The initialization mapping \( \text{Init}' \) extends the initialization mapping \( \text{Init} \) to the schema \( \tau' \) in a natural way. For a clause \( C \) and input database \( I \), a tuple \( \vec{a} \) is in \( \text{Init}'(R_{-C}) \) if and only if \( C \) evaluates to true in \( \text{Init}(I) \) for \( \vec{a} \).

The second step is to construct from \( \mathcal{P}' \) the desired conjunctive \textsc{DynProp}-program \( \mathcal{P}'' \): every clause \( C \) in every update formula of \( \mathcal{P}' \) is replaced by \( \neg R_{-C} \). This construction yields a conjunctive program \( \mathcal{P}'' \). The initialization mapping of \( \mathcal{P}'' \) is the same as for \( \mathcal{P}' \).

We sketch the proof that \( \mathcal{P}'' \) is equivalent to \( \mathcal{P} \). The dynamic program \( \mathcal{P}' \) updates relations from \( \tau \) exactly as program \( \mathcal{P} \). By an induction over the length of update sequences, one can prove that \( R_{-C}^S(\vec{a}) \) holds if and only if \( \neg C(\vec{a}) \) is true in state \( S \) for all tuples \( \vec{a} \). Thus corresponding update formulas of \( \mathcal{P} \) and \( \mathcal{P}'' \) always yield the same result.

8 Future Work

The question whether Reachability is maintainable with first-order updates remains one of the major open questions in dynamic complexity. Proving that
Reachability cannot be maintained with quantifier-free updates with arbitrary auxiliary data seems to be a worthwhile intermediate goal, but it appears non-trivial as well.

We contributed to the intermediate goal by giving a first lower bound for binary auxiliary relations. Whether the strictness of the arity hierarchy for DynProp extends beyond arity three is another open question.

For (full) first-order updates a major challenge is the development of lower bound tools. Current techniques are in some sense not fully dynamic: either results from static descriptive complexity are applied to constant-length update sequences; or non-constant but very regular update sequences are used. In the latter case, the updates do not depend on previous changes to the auxiliary data (as, e.g., in [7] and in this paper). Finding techniques that adapt to changes could be a good starting point.

The normal forms obtained for DynProp give hope that some fragments of DynFO collapse. Therefore, we plan to study normal forms for DynFO extensively. One interesting question being which fragments of DynFO can be captured by a conjunctive query normal form.

References

[1] Patnaik, S., Immerman, N.: Dyn-FO: A parallel, dynamic complexity class. In: PODS, ACM Press (1994) 210–221
[2] Hesse, W.: The dynamic complexity of transitive closure is in DynTC$^0$. In: ICDT. (2001) 234–247
[3] Dong, G., Su, J.: Arity bounds in first-order incremental evaluation and definition of polynomial time database queries. J. Comput. Syst. Sci. 57(3) (1998) 289–308
[4] Dong, G., Libkin, L., Wong, L.: Incremental recomputation in local languages. Inf. Comput. 181(2) (2003) 88–98
[5] Hesse, W.: Dynamic Computational Complexity. PhD thesis, University of Massachusetts Amherst (2003)
[6] Gelade, W., Marquardt, M., Schwentick, T.: The dynamic complexity of formal languages. In: STACS. (2009) 481–492
[7] Grädel, E., Siebertz, S.: Dynamic definability. In: ICDT. (2012) 236–248
[8] Patrascu, M., Demaine, E.D.: Lower bounds for dynamic connectivity. In Babai, L., ed.: STOC, ACM (2004) 546–553
[9] Weber, V., Schwentick, T.: Dynamic complexity theory revisited. Theory Comput. Syst. 40(4) (2007) 355–377
[10] Gelade, W., Marquardt, M., Schwentick, T.: The dynamic complexity of formal languages. ACM Trans. Comput. Log. 13(3) (2012) 19
[11] Etessami, K.: Dynamic tree isomorphism via first-order updates. In: PODS, ACM Press (1998) 235–243
[12] Patnaik, S., Immerman, N.: Dyn-FO: A parallel, dynamic complexity class. J. Comput. Syst. Sci. 55(2) (1997) 199–209
[13] Graham, R., Rothschild, B., Spencer, J.: Ramsey Theory. Wiley Series in Discrete Mathematics and Optimization. Wiley (1990)
[14] Schmitz, S., Schnoebelen, P.: Multiply-recursive upper bounds with Higman’s lemma. In: ICALP vol. 2. (2011) 441–452
[15] Dong, G., Zhang, L.: Separating auxiliary arity hierarchy of first-order incremental evaluation systems using (3k+1)-ary input relations. Int. J. Found. Comput. Sci. 11(4) (2000) 573–578