The output entropy of quantum channels and operations

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1 Introduction

The output von Neumann entropy of a quantum channel is an important characteristic of this channel used in study of its information properties. This is a concave lower semicontinuous function on the set of input states of the channel, taking values in $[0, +\infty]$. In applications, in particular, in analysis of the classical capacity of a quantum channel it is necessary to have conditions for continuity of the output entropy of this channel on subsets of input state space. The natural questions arising in this direction are the following:

1) What are the conditions under which the output entropy of a quantum channel is continuous on the whole space of input states?

2) What are the conditions under which the output entropy of a quantum channel is continuous on any set of input states with continuous entropy?

The first part of this paper is devoted to study of these and some other questions in the general context of positive linear maps between Banach spaces of trace-class operators with the special attention to the classes of quantum channels and operations. In Section 3 it is shown that finiteness of the output entropy of a positive linear map on the whole space of input states is equivalent to its continuity and the sufficient conditions of this property for a quantum operation and its complementary operation expressed in terms of Kraus operators are obtained. In Section 4 the characterization of a positive linear map, for which the property in the second above-stated question holds, is obtained and its applications to the class of quantum operations are considered. The special relation between continuity properties of the output entropies of a pair of complementary quantum operations and its corollaries are presented in Section 5.

In the second part of this paper the properties of the output entropy considered as a function of a pair (map, input state) are investigated. Such analysis is a necessary tool for exploring continuity of information characteristics of a quantum channel as functions of a channel, it also can be used in study of quantum channels by means of their approximation by quantum operations [10, 14]. In Section 6 the general continuity condition and the continuity condition based on the complementary relation are obtained and their corollaries are considered.
In Section 7 the possibilities to prove continuity of the output entropy of a quantum operation and of a converging sequence of quantum operations on a given set of states, based on the results of the previous sections, are discussed.

Some applications of the above continuity conditions in analysis of the Holevo capacity of a quantum channel and of the Entanglement of Formation of a state of a composite infinite dimensional quantum system are considered in Section 8.

2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ – the Banach space of all bounded operators in $\mathcal{H}$ with the operator norm $\| \cdot \|$, $\mathfrak{T}(\mathcal{H})$ – the Banach space of all trace-class operators in $\mathcal{H}$ with the trace norm $\| \cdot \|_1$, containing the cone $\mathfrak{T}_+(\mathcal{H})$ of all positive trace-class operators. The closed convex subsets

$$\mathfrak{T}_1(\mathcal{H}) = \{ A \in \mathfrak{T}_+(\mathcal{H}) \mid \text{Tr} A \leq 1 \} \quad \text{and} \quad \mathfrak{S}(\mathcal{H}) = \{ A \in \mathfrak{T}_+(\mathcal{H}) \mid \text{Tr} A = 1 \}$$

are complete separable metric spaces with the metric defined by the trace norm. Operators in $\mathfrak{S}(\mathcal{H})$ are called density operators or states since each density operator uniquely defines a normal state on $\mathfrak{B}(\mathcal{H})$.

We will use the Dirac notations $|\varphi\rangle$, $|\varphi\rangle \langle \psi|$, ... for vectors $\varphi, \psi, ...$ in a Hilbert space with arbitrary norms (including the zero vector).

In what follows $\mathcal{A}$ is a subset of the cone of positive trace class operators.

We denote by $\cl(\mathcal{A})$, $\co(\mathcal{A})$, $\overline{\co}(\mathcal{A})$ and $\extr(\mathcal{A})$ the closure, the convex hull, the convex closure and the set of all extreme points of a set $\mathcal{A}$ correspondingly [13, 20].

The set of all Borel probability measures on a closed set $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ endowed with the topology of weak convergence is denoted $\mathcal{P}(\mathcal{A})$. This set can be considered as a complete separable metric space [3, 19]. The barycenter $b(\mu)$ of the measure $\mu$ in $\mathcal{P}(\mathcal{A})$ is the state in $\overline{\co}(\mathcal{A})$ defined by the Bochner integral

$$b(\mu) = \int_A \rho \mu(d\rho).$$

For arbitrary subset $\mathcal{B} \subseteq \overline{\co}(\mathcal{A})$ let $\mathcal{P}_\mathcal{B}(\mathcal{A})$ be the subset of $\mathcal{P}(\mathcal{A})$ consisting of all measures with barycenter in $\mathcal{B}$. 

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Let $\mathcal{P}^a(A)$ be the subset of $\mathcal{P}(A)$ consisting of atomic measures and let $\mathcal{P}^f(A)$ be the subset of $\mathcal{P}^a(A)$ consisting of measures with a finite number of atoms. Each measure in $\mathcal{P}^a(A)$ corresponds to a collection of states $\{\rho_i\} \subset A$ with probability distribution $\{\pi_i\}$ conventionally called ensemble and denoted $\{\pi_i, \rho_i\}$, its barycenter coincides with the average state $\sum_i \pi_i \rho_i$ of this ensemble.

The identity operator in a Hilbert space $H$ and the identity transformation of the Banach space $\mathfrak{S}(H)$ are denoted respectively $I_H$ and $\text{Id}_H$.

Let $H$ and $H'$ be separable Hilbert spaces which we call correspondingly input and output space. Let $\Phi : T$ of this ensemble.

The convex subset of $\mathfrak{S}(H)$ consisting of quantum operations (correspondingly maps called quantum channels, is denoted $\mathfrak{F}_{\leq 1}(H, H')$. The convex subset of $\mathfrak{F}_{\leq 1}(H, H')$, consisting of trace preserving maps called quantum channels, is denoted $\mathfrak{F}_{= 1}(H, H')$.

We assume that the set $\mathfrak{F}_{\leq 1}(H, H')$ is endowed with the topology generated on this set by the strong operator topology on the set of all linear operators between Banach spaces $\mathfrak{S}(H)$ and $\mathfrak{S}(H')$. We call it the strong convergence topology. It is this topology that makes the sets $\mathfrak{F}_{\leq 1}(H, H')$ and $\mathfrak{F}_{= 1}(H, H')$ to be isomorphic to the particular subsets of the cone $\mathfrak{S}_{+}(H \otimes H')$ (the generalized Choi-Jamiolkowski isomorphism) [10]. Convergence of a sequence $\{\Phi_n\} \subset \mathfrak{F}_{\leq 1}(H, H')$ to a map $\Phi_0 \in \mathfrak{F}_{\leq 1}(H, H')$ in the strong convergence topology means that

$$\lim_{n \to +\infty} \Phi_n(\rho) = \Phi_0(\rho), \quad \forall \rho \in \mathfrak{S}(H).$$

An arbitrary quantum operation (correspondingly channel) $\Phi \in \mathfrak{F}_{\leq 1}(H, H')$ has the following Kraus representation

$$\Phi(\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^*, \quad (1)$$

where $\{V_i\}_{i=1}^{+\infty}$ is a set of bounded linear operators from $H$ into $H'$ such that $\sum_{i=1}^{+\infty} V_i^* V_i \leq I_H$ (correspondingly $\sum_{i=1}^{+\infty} V_i V_i^* = I_H$).

For natural $k$ we denote by $\mathfrak{F}_{\leq 1}^k(H, H')$ (correspondingly by $\mathfrak{F}_{= 1}^k(H, H')$) the subset of $\mathfrak{F}_{\leq 1}(H, H')$ consisting of quantum operations (correspondingly
of quantum channels) having the Kraus representation with $\leq k$ nonzero summands.

If $\Phi$ is a quantum operation (correspondingly channel) in $\mathcal{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ then by the Stinespring dilation theorem there exist a Hilbert space $\mathcal{H}''$ and a contraction (correspondingly isometry) $V : \mathcal{H} \to \mathcal{H}' \otimes \mathcal{H}''$ such that

$$\Phi(A) = \text{Tr}_{\mathcal{H}''} V A V^*, \quad \forall A \in \mathcal{F}(\mathcal{H}).$$

The quantum operation (correspondingly channel) $\mathcal{F}(\mathcal{H}) \ni A \mapsto \tilde{\Phi}(A) = \text{Tr}_{\mathcal{H}''} V A V^* \in \mathcal{F}(\mathcal{H}'')$ (3)

is called complementary to the operation (correspondingly channel) $\Phi$ \footnote{The operation $\tilde{\Phi}$ is also called conjugate or canonically dual to the operation $\Phi$. [16, 27].}

We will use the following extension of the von Neumann entropy $H(\rho) = -\text{Tr}\rho \log \rho$ of a state $\rho \in \mathcal{S}(\mathcal{H})$ to the cone $\mathcal{T}(\mathcal{H})^+$ (cf. [15])

$$H(A) = \text{Tr}_\eta(A) - \eta(\text{Tr} A), \quad \forall A \in \mathcal{T}_+(\mathcal{H}), \quad \text{where } \eta(x) = -x \log x.$$

In what follows the function $A \mapsto H(A)$ on the cone $\mathcal{T}_+(\mathcal{H})$ is called the quantum entropy while the function $\{x_i\} \mapsto H(\{x_i\}) = \sum_i \eta(x_i) - \eta(\sum_i x_i)$ on the positive cone of the space $\ell_1$, coinciding with the Shannon entropy on the set $\mathcal{P}_+\infty$ of probability distributions, is called the classical entropy.

Nonnegativity, concavity and lower semicontinuity of the von Neumann entropy on the set $\mathcal{S}(\mathcal{H})$ imply the same properties of the quantum entropy on the set $\mathcal{T}_+(\mathcal{H})$. By definition we have

$$H(\lambda A) = \lambda H(A), \quad A \in \mathcal{T}_+(\mathcal{H}), \ \lambda \geq 0.$$ (4)

This relation and proposition 6.2 in [18] imply

$$H(A) + H(B - A) \leq H(B) \leq H(A) + H(B - A) + \text{Tr}h_2\left(\frac{\text{Tr} A}{\text{Tr} B}\right), \quad (5)$$

where $A, B \in \mathcal{T}_+(\mathcal{H}), \ A \leq B$, and $h_2(x) = \eta(x) + \eta(1 - x)$.

By using theorem 11.10 in [17] and a simple approximation it is easy to obtain the following inequality

$$\sum_{i=1}^n \lambda_i H(A_i) \leq H\left(\sum_{i=1}^n \lambda_i A_i\right) \leq \sum_{i=1}^n \lambda_i H(A_i) + H(\{\lambda_i\}_{i=1}^n), \quad (6)$$
valid for any set \( \{ A_i \}_{i=1}^n \) of operators in \( \mathfrak{T}_1(H) \) and any probability distribution \( \{ \lambda_i \}_{i=1}^n \), where \( n \leq +\infty \). This inequality implies the following one
\[
\sum_{i=1}^n H(A_i) \leq H \left( \sum_{i=1}^n A_i \right) \leq \sum_{i=1}^n H(A_i) + H \left( \{ \text{Tr}A_i \}_{i=1}^n \right),
\]
(7)
valid for any set \( \{ A_i \}_{i=1}^n \) of operators in \( \mathfrak{T}_1(H) \) with finite \( \sum_{i=1}^n \text{Tr}A_i \).

Following [9] an arbitrary positive unbounded operator in a separable Hilbert space with discrete spectrum of finite multiplicity is called \( \mathfrak{H} \)-operator. Let \( g(H) = \inf \{ \lambda > 0 \mid \text{Tr}e^{-\lambda H} < +\infty \} \) assuming that \( g(H) = +\infty \) if \( \text{Tr}e^{-\lambda H} = +\infty \) for all \( \lambda > 0 \). For given \( \mathfrak{H} \)-operator \( H \) in a Hilbert space \( \mathcal{H} \) and positive \( h \) consider the convex set
\[
\mathcal{K}_{H,h} = \{ A \in \mathfrak{T}_1(H) \mid \text{Tr}AH \leq h \}.
\]

We will use the following generalizations\(^2\) of proposition 1a in [22] and proposition 6.6. in [18].

**Proposition 1.** Let \( H \) be a \( \mathfrak{H} \)-operator in a Hilbert space \( \mathcal{H} \) and \( h > 0 \).

A) The quantum entropy is bounded on the set \( \mathcal{K}_{H,h} \) if and only if \( g(H) < +\infty \);

B) The quantum entropy is continuous on the set \( \mathcal{K}_{H,h} \) if and only if \( g(H) = 0 \).

We will also use the following result easily derived from corollaries 3 and 4 in [25].

**Lemma 1.** Let \( \{ A_n \} \) and \( \{ B_n \} \) be sequences of operators in \( \mathfrak{T}_+(\mathcal{H}) \) converging respectively to operators \( A_0 \) and \( B_0 \). Then
\[
\lim_{n\to+\infty} H(A_n + B_n) = H(A_0 + B_0)
\]
if and only if \( \lim_{n\to+\infty} H(A_n) = H(A_0) \) and \( \lim_{n\to+\infty} H(B_n) = H(B_0) \).

The quantum entropy of an arbitrary operator \( A \in \mathfrak{T}_+(\mathcal{H}) \) and the classical entropy of the sequence of its diagonal values in any orthonormal basis \( \{ |i\rangle \}_{i=1}^{+\infty} \) of the space \( \mathcal{H} \) are related as follows
\[
H(A) \leq H \left( \{ |i \langle A|i \rangle \}_{i=1}^{+\infty} \right)
\]
(8)

\(^2\)These generalizations can be easily obtained by using the construction from the proof of Lemma 2 below.
(this inequality can be proved by using nonnegativity of the relative entropy).

By using relations (4) and (8) it is easy to derive from proposition 5E in [22] the following continuity condition for the quantum entropy.

**Proposition 2.** Let \{\{i\}\}^{\infty}_{i=1} be an orthonormal basis of a Hilbert space \(\mathcal{H}\). Continuity of the quantum entropy on a set \(A \subset \mathcal{T}_+(\mathcal{H})\) follows from continuity of the classical entropy on the set \(\{(\langle i|A|i\rangle)\}^{\infty}_{i=1} | A \in \mathcal{A}\} \subset (\ell_1)_+\).

We will use the triangle inequality

\[
H(C) \geq |H(\text{Tr}_K C) - H(\text{Tr}_H C)|
\]

valid for any operator \(C\) in \(\mathcal{T}_+(\mathcal{H} \otimes \mathcal{K})\) [17].

For arbitrary map \(\Phi \in \mathcal{L}_+^{\leq 1}(\mathcal{H}, \mathcal{H}')\) and operator \(A \in \mathcal{T}_+(\mathcal{H})\) the following estimation holds

\[
H(\Phi(A)) \leq \left[ \sup_{\rho \in \text{extr} \mathcal{S}(\mathcal{H})} H(\Phi(\rho)) \right] \text{Tr} A + H(A),
\]

which is proved by using the spectral decomposition of \(A\) and inequality (6).

The relative entropy for two operators \(A\) and \(B\) in \(\mathcal{T}_+(\mathcal{H})\) is defined as follows (cf. [15])

\[
H(A \parallel B) = \sum^{\infty}_{i=1} \langle i| (A \log A - A \log B + B - A) |i\rangle,
\]

where \(\{|i\rangle\}^{\infty}_{i=1}\) is the orthonormal basis of eigenvectors of the operator \(A\) and it is assumed that \(H(A \parallel B) = +\infty\) if \(\text{supp} A\) is not contained in \(\text{supp} B\).

For natural \(k\) we denote by \(\mathcal{T}_+^k(\mathcal{H})\) (correspondingly by \(\mathcal{S}_k(\mathcal{H})\)) the set of positive trace class operators (correspondingly states) having rank \(\leq k\).

For natural \(k\) and lower bounded Borel function \(f\) on the set \(\mathcal{S}_k(\mathcal{H})\) consider the functions

\[
\mathcal{S}(\mathcal{H}) \ni \rho \mapsto \hat{f}_k^\rho(\rho) = \sup_{\{\pi_i, \rho_i\} \in \mathcal{P}_k^\rho(\mathcal{S}_k(\mathcal{H}))} \sum \pi_i f(\rho_i)
\]

and

\[
\mathcal{S}(\mathcal{H}) \ni \rho \mapsto \hat{f}_k^\mu(\rho) = \sup_{\mu \in \mathcal{P}_k(\mathcal{S}_k(\mathcal{H}))} \int_{\mathcal{S}_k(\mathcal{H})} f(\sigma) \mu(d\sigma)
\]
(the first supremum is over all decompositions of the state \( \rho \) into countable convex combination of states of rank \( \leq k \), the second one is over all probability measures with the barycenter \( \rho \) supported by states of rank \( \leq k \)). Properties of these functions are studied in [25], where it is shown that lower semicontinuity of the function \( f \) on the set \( \mathcal{G}_k(\mathcal{H}) \) implies lower semicontinuity and coincidence of the functions \( \hat{f}_k^\rho \) and \( \hat{f}_k^\mu \) on the set \( \mathcal{G}(\mathcal{H}) \).

Following [25] we say that a subset \( \mathcal{A} \) of the cone \( \mathcal{T}_+(\mathcal{H}) \) of positive trace class operators has the uniform approximation property (the UA-property) if

\[
\lim_{k \to +\infty} \sup_{A \in \mathcal{A}} \Delta_k(A) = 0,
\]

where

\[
\Delta_k(A) = \inf_{\{\pi_i, A_i\} \in \mathcal{P}_a(\mathcal{T}_+(\mathcal{H}))} \sum_i \pi_i H(A_i \| A)
\]

for each natural \( k \) (the infimum is over all decompositions of the operator \( A \) into countable convex combination of operators of rank \( \leq k \)). The UA-property of a set \( \mathcal{A} \) is a sufficient condition for continuity of the quantum entropy on this set, if the set \( \mathcal{A} \) is compact then this condition is also necessary.

We will use the following simple observation

\[
\lim_{y \to 0} \sup_{x \in [0,1]} (x + y) h_2 \left( \frac{y}{x + y} \right) = 0.
\]

(13)

**Note:** In what follows continuity of a function \( f \) on a set \( \mathcal{A} \subset \mathcal{T}_+(\mathcal{H}) \) implies its finiteness on this set (in contrast to lower (upper) semicontinuity).

### 3 On continuity of the output entropy of a positive linear map

#### 3.1 The general case

Let \( \Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}') \) be a positive linear map. The output entropy \( H_\Phi = H \circ \Phi \) of this map is a concave nonnegative lower semicontinuous function on the set \( \mathcal{G}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H}) \). The following theorem shows that this function can not be finite and discontinuous simultaneously.
Theorem 1. Let $\Phi$ be a map in $\Sigma_{\leq 1}(\mathcal{H}, \mathcal{H}')$. The following properties are equivalent:

(i) the function $\rho \mapsto H_\Phi(\rho)$ is finite on $\mathcal{S}(\mathcal{H})$;
(ii) the function $\rho \mapsto H_\Phi(\rho)$ is continuous and bounded on $\mathcal{S}(\mathcal{H})$;
(iii) there exists an orthonormal basis $\{|i\rangle\}_{i=1}^{+\infty}$ of the space $\mathcal{H}'$ such that the function $\rho \mapsto H \{\langle i | \Phi(\rho) | i \rangle\}_{i=1}^{+\infty}$ is continuous and bounded on $\mathcal{S}(\mathcal{H})$;
(iv) there exist an orthonormal basis $\{|i\rangle\}_{i=1}^{+\infty}$ of the space $\mathcal{H}'$ and a sequence $\{h_i\}_{i=1}^{+\infty}$ of nonnegative numbers such that
\[
\left\| \sum_{i=1}^{+\infty} h_i \Phi^* (|i\rangle\langle i|) \right\| < +\infty \text{ and } \sum_{i=1}^{+\infty} e^{-h_i} < +\infty,
\]
where $\Phi^* : \mathcal{B}(\mathcal{H}') \to \mathcal{B}(\mathcal{H})$ is the dual map to the map $\Phi$.

The set $\mathcal{S}(\mathcal{H})$ in (i) can be replaced by arbitrary convex closed bounded subset $A \subset \Sigma_+(\mathcal{H})$ such that $\sup_{A \in A} \lim_{n \to +\infty} \text{Tr} A B_n < +\infty \Rightarrow \sup_n \|B_n\| < +\infty$ for any increasing sequence $\{B_n\}$ of positive operators in $\Phi^*(\mathcal{B}(\mathcal{H}'))$.

Proof. (i) $\Rightarrow$ (ii) Validity of the discrete Jensen inequality for the concave finite nonnegative function $\rho \mapsto H_\Phi(\rho)$ implies its boundedness. Indeed, if for each natural $n$ there exists a state $\rho_n$ such that $H_\Phi(\rho_n) \geq 2^n$ then
\[
H_\Phi \left( \sum_{n=1}^{+\infty} 2^{-n} \rho_n \right) \geq \sum_{n=1}^{+\infty} 2^{-n} H_\Phi(\rho_n) = +\infty.
\]

Lemma 2 below implies existence of a $\mathcal{H}$-operator $H = -\log T$ such that $g(H) < +\infty$ and $\text{Tr} H \Phi(\rho) \leq h$ for all $\rho \in \mathcal{S}(\mathcal{H})$ and some $h > 0$. Let $H = \sum_{i=1}^{+\infty} h_i |i\rangle\langle i|$. Since the function
\[
\rho \mapsto \text{Tr} H \Phi(\rho) = \sum_{i=1}^{+\infty} h_i \langle i | \Phi(\rho) | i \rangle = \text{Tr} \left[ \sum_{i=1}^{+\infty} h_i \Phi^* (|i\rangle\langle i|) \right] \rho
\]
is bounded on $\mathcal{S}(\mathcal{H})$, the linear operator in the square brackets is bounded on $\mathcal{H}$. Thus the above function is continuous on $\mathcal{S}(\mathcal{H})$. For arbitrary compact subset $\mathcal{C}$ of $\mathcal{S}(\mathcal{H})$ Dini’s lemma implies uniform convergence of the

\[\text{By Proposition 2 this property is formally stronger than the previous one.}\]
series \( \sum_{i=1}^{+\infty} h_i \text{Tr } \Phi^*(|i\rangle\langle i|) \rho \) on the set \( C \) and hence existence of a nondecreasing sequence \( \{y_i^C\}_{i=1}^{+\infty} \) of positive numbers converging to \(+\infty\) such that \( \sup_{\rho \in C} \sum_{i=1}^{+\infty} y_i^C h_i \text{Tr } \Phi^*(|i\rangle\langle i|) \rho < +\infty \). Let \( H^C = \sum_{i=1}^{+\infty} y_i^C h_i |i\rangle\langle i| \) be a \( \mathcal{H} \)-operator with \( g(H^C) = 0 \). Thus we have

\[
\sup_{\rho \in C} \text{Tr} H^C \Phi(\rho) = \sup_{\rho \in C} \sum_{i=1}^{+\infty} y_i^C h_i \text{Tr } \Phi^*(|i\rangle\langle i|) \rho < +\infty. \tag{14}
\]

By proposition \( \Box \) the function \( \rho \mapsto H(\Phi(\rho)) \) is continuous on the set \( C \) and hence on the set \( \mathcal{S}(\mathcal{H}) \) (since \( C \) is an arbitrary compact subset of \( \mathcal{S}(\mathcal{H}) \)).

(i) \( \Rightarrow \) (iv): In the proof of (i) \( \Rightarrow \) (ii) existence of the basis \( \{|i\rangle\}_{i=1}^{+\infty} \) and of the sequence \( \{h'_i = \lambda h_i\}_{i=1}^{+\infty} \), \( \lambda > 0 \), with the desired properties is shown.

(iv) \( \Rightarrow \) (iii) follows from the proof of (i) \( \Rightarrow \) (ii) since (14) and Proposition \( \Box \) implies continuity of the function \( \rho \mapsto H(\{\langle i|\Phi(\rho)|i\rangle\}_{i=1}^{+\infty}) \) on the set \( C \).

(iii) \( \Rightarrow \) (i) follows from relation (8).

The last assertion of the theorem is a corollary of the proof of (i) \( \Rightarrow \) (ii).

□

Lemma 2. For an arbitrary convex set \( A \subset \mathcal{T}_1(\mathcal{H}) \), on which the quantum entropy is bounded, there exists an operator \( T \in \mathcal{T}_1(\mathcal{H}) \) such that

\[
\sup_{A \in A} \text{Tr} A (\log T) < +\infty \quad \text{and} \quad UT = TU
\]

for any unitary \( U \) in \( \mathcal{B}(\mathcal{H}) \) such that \( UAU^* \in A \) for all \( A \in A \).

Proof. Let \( \mathcal{K} \) be the one dimensional space generated by the unit vector \( |0\rangle \). Consider the convex set

\[
\mathcal{A}^e = \{ \rho_A = A \oplus (1 - \text{Tr} A) |0\rangle\langle 0| \mid A \in A \}
\]

of states in \( \mathcal{S}(\mathcal{H} \oplus \mathcal{K}) \). For arbitrary \( A \in A \) we have

\[
H(\rho_A) = -\text{Tr} A \log A + \eta(1 - \text{Tr} A) = H(A) + \eta(\text{Tr} A) + \eta(1 - \text{Tr} A) \leq H(A) + 1.
\]

Thus the von Neumann entropy is bounded on the convex set \( \mathcal{A}^e \). Hence the \( \chi \)-capacity \( \tilde{C}(\mathcal{A}^e) \) of this set is finite \( \Box \). Theorem 1 in \( \Box \) implies existence of the unique state \( \Omega(\mathcal{A}^e) \) in \( \text{cl}(\mathcal{A}^e) \) (called the optimal average state of the set \( \mathcal{A}^e \)) such that

\[
H(\rho||\Omega(\mathcal{A}^e)) \leq \tilde{C}(\mathcal{A}^e)
\]
for all $\rho \in A^e$. The state $\Omega(A^e)$ has the form $T \oplus \lambda |0\rangle \langle 0|$, where $T \in \mathfrak{S}_1(\mathcal{H})$ and $\lambda > 0$.

For arbitrary unitary $U$ in $\mathfrak{B}(\mathcal{H})$ such that $UAU^* = A$ corollary 8 in [22] implies $(U \oplus I_K)\Omega(A^e) = \Omega(A^e)(U \oplus I_K)$ and hence $UT = TU$. □

**Remark 1.** Theorem 1 do not assert that finiteness of the quantum entropy on the set $\Phi(\mathfrak{S}(\mathcal{H}))$ implies its continuity on this set since continuity of the function $\rho \mapsto H(\Phi(\rho))$ on the noncompact set $\mathfrak{S}(\mathcal{H})$ does not imply continuity of the function $A \mapsto H(A)$ on the set $\Phi(\mathfrak{S}(\mathcal{H}))$. To show this consider the following example.

Let $A$ be a convex closed subset of $\mathfrak{S}(\mathcal{H}')$ on which the von Neumann entropy is bounded but not continuous (see the examples in [22]). Let $\{\sigma_n\}_{n=1}^{+\infty}$ be a sequence of states in $A$ converging to a state $\sigma_0$ such that $\lim_{n \to +\infty} H(\sigma_n) \neq H(\sigma_0)$. Consider the map $\Phi: \rho \mapsto \sum_{n \geq 0} \langle n | \rho | n \rangle \sigma_n$, where $\{|n\rangle\}_{n \geq 0}$ is a particular orthonormal basis in $\mathcal{H}$. By Theorem 1 the function $\rho \mapsto H_\Phi(\rho)$ is continuous on the set $\mathfrak{S}(\mathcal{H})$ but the function $A \mapsto H(A)$ is not continuous on the set $\Phi(\mathfrak{S}(\mathcal{H}))$ containing the sequence $\{\sigma_n\}_{n=1}^{+\infty}$ and the state $\sigma_0$.

Continuity of the function $\rho \mapsto H_\Phi(\rho)$ on the set $\mathfrak{S}(\mathcal{H})$ means continuity of the function $A \mapsto H(A)$ on each set of the form $\Phi(C)$, where $C$ is a compact subset of $\mathfrak{S}(\mathcal{H})$.

**Remark 2.** The main assertion of Theorem 1 (the implication (i) $\Rightarrow$ (ii)) is based on the specific property of the von Neumann entropy, it can not be proved by using only such general properties of entropy-type functions as concavity, lower semicontinuity and nonnegativity. The simplest example showing this is given by the function $\rho \mapsto R_0(\Phi(\rho)) \doteq \|\Phi(\rho)\|_1 \log \text{rank}(\Phi(\rho))$ – the output 0-order Renyi entropy of the map $\Phi$.

Theorem 1 can be used to obtain a condition of continuity of the output entropy for the following class of quantum channels.

**Example 1.** Let $G$ be a compact group, $\{V_g\}_{g \in G}$ be an unitary representation of $G$ on $\mathcal{H}'$, $M$ be a positive operator-valued measure (POVM) on $G$ taking values in $\mathfrak{B}(\mathcal{H})$. For given arbitrary state $\sigma$ in $\mathfrak{S}(\mathcal{H}')$ consider the quantum channel

$$
\Phi_\sigma(\rho) = \int_G V_g \sigma V_g^* \text{Tr} \rho M(dg).
$$

The channel of this type was used in [11] as an example of entanglement-
breaking channel which has no Kraus representation with operators of rank 1.\footnote{An arbitrary finite dimensional entanglement-breaking channel has Kraus representation with operators of rank 1 \cite{12}.}

By Theorem 1 the channel $\Phi_\sigma$ has continuous output entropy if the state $\omega(G, V, \sigma) = \int_G V g \sigma V_\sigma^* \mu_H(dg)$, where $\mu_H$ is the Haar measure on $G$, has finite entropy. This condition is necessary if the set of probability measures $\{\text{Tr}_\rho M(\cdot)\}_{\rho \in S(H)}$ is weakly dense in the set of all probability measures on $G$.

Theorem 1 and inequality (6) imply the following observation (which can be directly proved by using Lemma 1).

**Corollary 1.** Let $\Phi$ and $\Psi$ be maps in $\mathcal{L}_{\leq 1}(H, H')$ and $\lambda \in (0, 1)$. The map $\lambda \Phi + (1 - \lambda) \Psi$ has continuous output entropy if and only if the maps $\Phi$ and $\Psi$ have continuous output entropy.

Thus the set of all positive maps with continuous output entropy is convex and forms a face of the convex set $\mathcal{L}_{\leq 1}(H, H')$. It is easy to show that this face is dense in $\mathcal{P}_{\leq 1}(H, H')$ (in the strong convergence topology).

We will use the following corollary of Theorem 1 and inequality (7).

**Corollary 2.** Let $\{\Phi_i\}_{i \in I}$ be a finite or countable family of maps in $\mathcal{L}_{\leq 1}(H', H')$ such that $\sup_{\rho \in S(H)} \sum_{i \in I} \text{Tr}\Phi_i(\rho) < +\infty$. The output entropy of the map $\sum_{i \in I} \Phi_i$ is continuous if

$$\sum_{i \in I} H(\Phi_i(\rho)) < +\infty \quad \text{and} \quad H\left(\{\text{Tr}\Phi_i(\rho)\}_{i \in I}\right) < +\infty \quad \forall \rho \in S(H).$$

This condition is necessary if either $\text{supp} \Phi_i(\rho) \perp \text{supp} \Phi_j(\rho)$ for all $i \neq j$ and all $\rho \in S(H)$ or the set $I$ is finite.

Theorem 1 provides a simple proof of the following result.\footnote{It can be also proved by using Proposition 15 in the Appendix and corollary 4 in \cite{25}.}

**Corollary 3.** Let $\Phi : \mathcal{S}(H) \to \mathcal{S}(H')$ and $\Psi : \mathcal{S}(K) \to \mathcal{S}(K')$ be two positive linear bounded maps having continuous output entropy. If the map $\Phi \otimes \Psi : \mathcal{S}(H \otimes K) \to \mathcal{S}(H' \otimes K')$ is positive then it has continuous output entropy.

**Proof.** We may assume that the maps $\Phi$ and $\Psi$ are trace non-increasing. By Theorem 1 it is sufficient to prove that $H(\Phi \otimes \Psi(\omega)) < +\infty$ for any $\omega \in S(H \otimes K)$. This follows from subadditivity of the quantum entropy since $\text{Tr}_K \Phi \otimes \Psi(\omega) \leq \Phi(\text{Tr}_K \omega)$ and $\text{Tr}_K \Phi \otimes \Psi(\omega) \leq \Psi(\text{Tr}_H \omega)$. $\square$
In Section 4 we will use the following corollary of Theorem 1.

**Corollary 4.** Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H')$ and $\Psi : \mathcal{L}(K) \to \mathcal{L}(K')$ be positive linear bounded maps such that the map $\Phi \otimes \Psi : \mathcal{L}(H \otimes K) \to \mathcal{L}(H' \otimes K')$ is positive ($H, H', K, K'$ are separable Hilbert spaces). If the map $\Psi$ is trace preserving and has finite (and hence continuous) output entropy then the following properties are equivalent:

(i) $H(\Phi \otimes \Psi(|\varphi\rangle\langle\varphi|)) < +\infty$ for any unit vector $\varphi \in H \otimes K$;

(ii) the map $\Phi$ has continuous output entropy;

(iii) the map $\Phi \otimes \Psi$ has continuous output entropy.

If the map $\Phi$ is trace preserving then the condition of finiteness of the output entropy of the map $\Psi$ can be replaced by the condition

$$\min \{H_{\Phi}(\text{Tr}_K|\varphi\rangle\langle\varphi|), H_{\Psi}(\text{Tr}_H|\varphi\rangle\langle\varphi|)\} < +\infty \quad \forall \varphi \in H \otimes K.$$ 

**Proof.** We may assume that the map $\Phi$ is trace non-increasing.

(i) $\Rightarrow$ (ii) Let $\rho$ be an arbitrary state in $\mathcal{S}(H)$ and $|\varphi\rangle$ be a vector in $H \otimes K$ such that $\rho = \text{Tr}_K|\varphi\rangle\langle\varphi|$. Since the map $\Psi$ is trace preserving we have $\text{Tr}_K\Phi \otimes \Psi(|\varphi\rangle\langle\varphi|) = \Phi(\rho)$. By noting that $\text{Tr}_H\Phi \otimes \Psi(|\varphi\rangle\langle\varphi|) \leq \Psi(\sigma)$, where $\sigma = \text{Tr}_H|\varphi\rangle\langle\varphi|$, and by using finiteness of $H(\Psi(\sigma))$ with (5) and (9) we conclude that $H(\Phi(\rho)) < +\infty$. By Theorem 1 the map $\Phi$ has continuous output entropy.

(ii) $\Rightarrow$ (iii) follows from Corollary 3

(iii) $\Rightarrow$ (i) is obvious.

The last assertion of the corollary is proved by the similar argumentation.

$\square$

Property (iv) in Theorem 1 can be considered as a criterion of continuity of the output entropy of a map $\Phi$ in terms of its dual map $\Phi^*$. It will be used in the proof of Proposition 3 in the next subsection.

### 3.2 The case of quantum operation

The simplest example of quantum operation (completely positive trace non-increasing linear map) from $\mathcal{L}(H)$ to $\mathcal{L}(H')$ is the map $(\cdot) \mapsto V(\cdot)V^*$, where $V$ is a linear contraction from $H$ to $H'$. Theorem 1 implies the following result.
Proposition 3. Let $V$ be a linear operator from $\mathcal{H}$ to $\mathcal{H}'$. The function $\mathcal{S}(\mathcal{H}) \ni \rho \mapsto H(V\rho V^*)$ is continuous if and only if the operator $V$ is compact and has such sequence $\{\nu_i\}$ of singular values (eigenvalues of $\sqrt{V^*V}$) that $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} < +\infty$ for some $\lambda > 0$. If this condition holds then

$$\sup_{\rho \in \mathcal{S}(\mathcal{H})} H(V\rho V^*) = \lambda^*(V)$$

where $\lambda^*(V)$ is either the unique solution of the equation $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} = 1$ if it exists or equal to $g(\{\nu_i^{-2}\}) = \inf\{\lambda > 0 \mid \sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} < +\infty\}$ otherwise.\footnote{The equation $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} = 1$ has no solution if and only if $\sum_{i=1}^{+\infty} e^{-g(\{\nu_i^{-2}\})/\nu_i^2} < 1$.}

In what follows we will use the parameter $\lambda^*(V)$ defined in Proposition 3 for arbitrary operator $V \in \mathcal{B}(\mathcal{H})$ assuming that $\lambda^*(V) = +\infty$ if the operator $V$ either is not compact or has such sequence of singular values $\{\nu_i\}$ that $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} = +\infty$ for all $\lambda > 0$.

Proof. We can assume that $\mathcal{H} = \mathcal{H}'$, $V = |V|$, $\|V\| \leq 1$ and $\text{Ker} V = \{0\}$.

Let $V = \sum_{i=1}^{+\infty} \nu_i|i\rangle\langle i|$. If $\sum_{i=1}^{+\infty} e^{-\lambda/\nu_i^2} < +\infty$ for $\lambda > 0$ then property (iv) in Theorem 4 holds with the basis $\{|i\rangle\}_{i=1}^{+\infty}$ and the sequence $\{h_i = \lambda
u_i^{-2}\}_{i=1}^{+\infty}$ (since in this case $\Phi^*(\cdot) = V(\cdot)V$ and hence $\Phi^*(|i\rangle\langle i|) = \nu_i^2|i\rangle\langle i|$).

The assertion concerning the supremum of the function $\rho \mapsto H(V\rho V)$ is easily derived from Lemma 7 in the Appendix by using relation (8).

Suppose the function $\rho \mapsto H(V\rho V)$ is continuous on the set $\mathcal{S}(\mathcal{H})$. Then the entropy is bounded on the convex set $\{V\rho V \mid \rho \in \mathcal{S}(\mathcal{H})\}$ and hence this set is relatively compact by corollary 5 in [22] (used with the construction from the proof of Lemma 2). Thus the operator $V$ is compact (since otherwise there exists a sequence of unit vectors $\{|\varphi_n\rangle\}$ such that the sequence $\{V|\varphi_n\rangle\}$ is not relatively compact). Lemma 2 implies existence of an operator $T \in \mathcal{S}_1(\mathcal{H})$ such that $\sup_{\rho \in \mathcal{S}(\mathcal{H})} \text{Tr} V\rho V(-\log T) < +\infty$ and $UT = TU$ for arbitrary unitary $U$ commuting with the operator $V$. It follows from the last property of the operator $T$ that this operator is diagonalizable in the basis $\{|i\rangle\}$, i.e. $T = \sum_{i=1}^{+\infty} \tau_i|i\rangle\langle i|$, where $\{\tau_i\}_{i=1}^{+\infty}$ is a sequence of nonnegative numbers such that $\sum_{i=1}^{+\infty} \tau_i \leq 1$. Thus we have

$$\sup_{\rho \in \mathcal{S}(\mathcal{H})} \text{Tr} V\rho V(-\log T) = \sup_{\rho \in \mathcal{S}(\mathcal{H})} \sum_{i=1}^{+\infty} \langle i|\rho|i\rangle \nu_i^2(-\log \tau_i) = \lambda < +\infty$$

and hence $\nu_i^2(-\log \tau_i) \leq \lambda$ for all $i$. This implies $\lambda^*(V) < +\infty$. $\square$
An arbitrary quantum operation $\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}')$ has Kraus representation (1) determined by a set $\{V_i\}_{i=1}^{+\infty}$ of linear operators from $\mathcal{H}$ into $\mathcal{H}'$ such that $\sum_{i=1}^{+\infty} V_i^* V_i \leq I_H$. The following proposition contains the sufficient conditions for continuity of the output entropy of a quantum operation $\Phi$ expressed in terms of the set $\{V_i\}_{i=1}^{+\infty}$ of its Kraus operators.

**Proposition 4.** Let $\Phi$ be a quantum operation in $\mathcal{S}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ and $\{V_i\}_{i=1}^I$ be the corresponding set of Kraus operators. Let $d_i = \text{rank} V_i \leq +\infty$.

A) If the set $I$ is finite then the operation $\Phi$ has continuous output entropy if and only if

$$\lambda^* (V_i) < +\infty \quad \forall i \in I,$$

which in this case is equivalent to

$$\lambda^* \left( \sqrt{\sum_{i \in I} V_i^* V_i} \right) < +\infty \quad \text{(16)}$$

In general case (15) is a necessary condition of continuity of the output entropy of the operation $\Phi$ (in contrast to (16)).

B) If $I = \mathbb{N}$ then the operation $\Phi$ has continuous output entropy if one of the following conditions is valid:

a) $d_i < +\infty$ for all $i$ and there exists a sequence $\{h_i\}_{i=1}^{+\infty}$ of nonnegative numbers such that

$$\| \sum_{i=1}^{+\infty} h_i V_i^* V_i \| < +\infty \quad \text{and} \quad \sum_{i=1}^{+\infty} d_i e^{-h_i} < +\infty;$$

b) $H \left( \{\text{Tr} V_i \rho V_i^*\}_{i=1}^{+\infty} \right) < +\infty$ for all $\rho \in \mathcal{S}(\mathcal{H})$ and condition (16) holds.

Condition (16) in b) can be replaced by the condition of finiteness of one of the series $\sum_{i=1}^{+\infty} \lambda^* (V_i)$ and $\sum_{i=1}^{+\infty} \log d_i \|V_i\|^2$.

If the sequence $\{V_i\}_{i=1}^{+\infty}$ consists of scalar multiples of mutually orthogonal projectors then a) is a necessary condition of continuity of the output entropy of the operation $\Phi$.

---

7In contrast to the set $\{V_i\}_{i=1}^I$ the operator $\sum_{i \in I} V_i^* V_i = \Phi^*(I_{H'})$ is uniquely determined by the operation $\Phi$.

8The sense of this condition and its "variations" are considered in Proposition 5 below.
If RanVi ⊥ RanVj for all i ≠ j then b) is a necessary condition of continuity of the output entropy of the operation Φ.

Proof. A) This directly follows from Corollary 2, Proposition 3 and Corollary 11 in Section 5 below (since ∑1≤i≤n Vi*Vi = Φ*(IₜHₜ)).

B) Suppose condition a) holds. Let K = ⨁1≤i≤n RanVi and Ui be a partial isometry from H into K such that UₗVi is the projector onto RanVi ⊂ H' and UₗUᵢ* is the projector onto RanVᵢ ⊂ K. Consider the quantum operation Φ in أشكال (H, K) defined by the sequence of Kraus operators {Vi = UᵢVᵢ}₁≤i≤n.

We have RanVi ⊥ RanVj for all i ≠ j. Let Pi be the dᵢ-rank projector onto the subspace RanVi. Consider the isometry from K into K such that the condition ∑₁≤i≤n dᵢe⁻ʰᵢ < +∞ means that g(H) < +∞. The condition ∥∑₁≤i≤n hᵢVi*Vi∥ < h < +∞ implies

\[ \text{Tr} H \hat{\Phi}(\rho) = \sum_{i=1}^{+∞} h_i \text{Tr} P_i \hat{V}_i \rho \hat{V}_i^* = \text{Tr} \sum_{i=1}^{+∞} h_i V_i^* V_i \rho \leq h, \quad \forall \rho \in \mathcal{S}(H). \]

By Proposition 1A the quantum entropy is bounded on the set \( \hat{\Phi}(\mathcal{S}(H)) \). Since

\[ H(\hat{\Phi}(\rho)) = \sum_{i=1}^{+∞} H(V_i \rho V_i^*) + H(\{\text{Tr} V_i \rho V_i^*\})_{i=1}^{+∞} \]

inequality (17) implies boundedness of the function \( \rho \mapsto H(\Phi(\rho)) \). By Theorem 1A this function is continuous.

If condition b) holds then Proposition 3 below and Corollary 11 in Section 5 imply continuity of the output entropy of the operation Φ. Possibility to replace condition (16) in b) by one of the conditions ∑₁≤i≤n λᵢ(Vᵢ) < +∞ and ∑₁≤i≤n log dᵢ∥Vᵢ∥² < +∞ follows from Corollary 2, since each of these conditions implies finiteness of the series ∑₁≤i≤n H(VᵢρVᵢ*) for any ρ in \( \mathcal{S}(H) \).

To prove the assertion concerning necessity of condition a) assume that Φ(·) = ∑₁≤i≤n cᵢPᵢ(·)Pᵢ, where \{Pᵢ\} is a sequence of mutually orthogonal projectors. Lemma 2 implies existence of a trace class operator of the form T = ∑₁≤i≤n λᵢPᵢ such that

\[ \sup_{\rho \in \mathcal{S}(H)} \text{Tr} \Phi(\rho) (-\log T) = \sup_{\rho \in \mathcal{S}(H)} \text{Tr} \sum_{i=1}^{+∞} c_i (-\log \lambda_i) P_i \rho < +∞. \]

Since TrT = ∑₁≤i≤n dᵢλᵢ, condition a) holds with the sequence \{hᵢ = −\log λᵢ\}.
To prove the assertion concerning necessity of condition b) it is sufficient to note that the condition $\text{Ran} V_i \perp \text{Ran} V_j$ for all $i \neq j$ implies

$$H(V_\rho V^*) \leq H(\Phi(\rho)) = \sum_{i=1}^{+\infty} H(V_i \rho V_i^*) + H \left( \{ \text{Tr} V_i \rho V_i^* \}_{i=1}^{+\infty} \right) \quad \forall \rho \in \mathcal{G}(\mathcal{H}),$$

where $V$ is the Stinespring contraction of the operation $\Phi$ defined via the set $\{ V_i \}_{i=1}^{+\infty}$ (see the proof of Proposition 5 below), and to apply Theorem 1 and Proposition 3 (by using $V^* V = \sum_{i=1}^{+\infty} V_i^* V_i$).

Example 2. Let $\{ V_i \}_{i=1}^{+\infty}$ be a sequence of finite rank operators in $\mathfrak{B}(\mathcal{H})$ such that $\sum_{i=1}^{+\infty} V_i^* V_i \leq I_{\mathcal{H}}$, $\text{Ran} V_i^* \perp \text{Ran} V_j^*$ for all sufficiently large $i \neq j$ and $\| V_i \|^2 \leq C \log^{-\alpha}(i)$ for all $i$, where $\alpha \geq 0$ and $C > 0$. Since $V_i^* V_i \leq C \log^{-\alpha}(i) P_i$, where $P_i$ is the projector on the subspace $\text{Ran} V_i^*$, condition a) in Proposition 4B holds for the operation $\Phi_\alpha(\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^*$ for all $\alpha \geq 1$ provided the rate of increase of the sequence $\{ \text{rank} V_i \}_i$ does not exceed the polynomial rate: $\text{rank} V_i \leq i^n$ for some natural $n$ and all sufficiently large $i$ (this can be shown by using the sequence $\{ h_i = (n+2) \log(i) \}$). Hence the output entropy of the operation $\Phi_\alpha$ is continuous in this case.

The last assertion of Proposition 4 shows that the output entropy of the operation $\Phi_\alpha$ is not continuous if $\alpha < 1$ and $V_i = \sqrt{C \log^{-\alpha}(i) P_i}$ even for bounded sequence $\{ \text{rank} V_i \}_i$. □

The following proposition contains the sufficient conditions for continuity of the output entropy of the complementary operation expressed in terms of the set $\{ V_i \}_{i=1}^{+\infty}$ of Kraus operators of the initial operation.

**Proposition 5.** Let $\Phi$ be a quantum operation in $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ and $\{ V_i \}_{i=1}^{+\infty}$ be the corresponding set of Kraus operators. The complementary operation $\tilde{\Phi}$ has continuous output entropy if one of the following conditions (related by c) $\Rightarrow$ b) $\Rightarrow$ a)) is valid:

a) $H \left( \{ \text{Tr} V_i \rho V_i^* \}_{i=1}^{+\infty} \right) < +\infty$ for all $\rho \in \mathcal{G}(\mathcal{H})$;

b) there exists a sequence $\{ h_i \}_{i=1}^{+\infty}$ of nonnegative numbers such that

$$\| \sum_{i=1}^{+\infty} h_i V_i^* V_i \| < +\infty \quad \text{and} \quad \sum_{i=1}^{+\infty} e^{-h_i} < +\infty;$$

c) $H \left( \{ \| V_i \|^2 \}_{i=1}^{+\infty} \right) < +\infty.$
If Ran\(V_i \perp \text{Ran}V_j\) for all \(i \neq j\) then \(a) \iff b)\) is a necessary condition of continuity of the output entropy of the operation \(\tilde{\Phi}\).

**Proof.** Show first that \(a)\) implies continuity of the function \(\rho \mapsto H(\tilde{\Phi}(\rho))\).

Let \(\mathcal{H}''\) be a separable Hilbert space and \(\{|i\}\) be an orthonormal basis in \(\mathcal{H}''\). Then the operator \(V : \mathcal{H} \ni |\varphi\rangle \mapsto \sum_{i=1}^{+\infty} |V_i\varphi\rangle \otimes |i\rangle \in \mathcal{H}' \otimes \mathcal{H}''\) is the Stinespring contraction for the operation \(\Phi\), i.e.

\[
\Phi(A) = \text{Tr}_{\mathcal{H}''} V^* A V, \quad A \in \mathcal{B}(\mathcal{H}).
\]

So we have

\[
\tilde{\Phi}(A) = \text{Tr}_{\mathcal{H}''} V^* A V^* = \sum_{i,j=1}^{+\infty} \text{Tr} [V_i A V_j^*] |i\rangle \langle j|, \quad A \in \mathcal{B}(\mathcal{H}).
\]

By relation (8) condition \(a)\) implies

\[
H(\tilde{\Phi}(\rho)) \leq H\left(\left\{|i| \tilde{\Phi}(\rho)|i\rangle\right\}_{i=1}^{+\infty}\right) = H\left(\left\{\text{Tr} V_i \rho V_i^*\right\}_{i=1}^{+\infty}\right) < +\infty, \quad \rho \in \mathcal{S}(\mathcal{H}).
\]

By Theorem \(\text{[1]}\) the output entropy of the operation \(\tilde{\Phi}\) is continuous.

Since finiteness of the function \(\mathcal{S}(\mathcal{H}) \ni \rho \mapsto H\left(\left\{\text{Tr} V_i \rho V_i^*\right\}_{i=1}^{+\infty}\right)\) implies its boundedness, equivalence of conditions \(a)\) and \(b)\) can be proved by noting that the last condition can be rewritten as follows

\[
\sup_{\rho \in \mathcal{S}(\mathcal{H})} \sum_{i=1}^{+\infty} h_i \text{Tr} V_i \rho V_i^* < +\infty
\]

and by using the classical versions of Proposition \([\text{1}A]\) and Lemma \([\text{2}2]\).

The implication \(c) \Rightarrow b)\) is obvious.

If \(\text{Ran}V_i \perp \text{Ran}V_j\) for all \(i \neq j\) then

\[
\tilde{\Phi}(\rho) = \sum_{i=1}^{+\infty} \text{Tr} [V_i \rho V_i^*] |i\rangle \langle i|, \quad \rho \in \mathcal{S}(\mathcal{H}),
\]

and hence the function in \(a)\) coincides with the output entropy of the operation \(\tilde{\Phi}\). \(\square\)

**Example 3.** Let \(\Phi_{\alpha}\) be the quantum operation described in Example \([\text{2}2]\) with no rank restrictions on its Kraus operators. By Proposition \([\text{5}2]\) the output entropy of the operation \(\tilde{\Phi}_{\alpha}\) is continuous if \(\alpha \geq 1\) but it is not continuous if \(\alpha < 1\) and \(V_i = \sqrt{C \log^{-\alpha}(i)} P_i\). \(\square\)
4 The PCE-property a positive linear map

Continuity of the output entropy of a positive linear map on the whole set of input states is a very strong requirement. In this section we consider the substantially weaker property of a positive linear map consisting in continuity of the output entropy on each subset of input states on which the von Neumann entropy is continuous.

4.1 The general case

Let $\Phi : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ be a positive linear map. Since the output entropy $H_\Phi = H \circ \Phi$ of this map is a concave nonnegative lower semicontinuous function on the cone $\mathcal{T}_+(\mathcal{H})$, we can apply the results of section 4 in [25] to this function as follows.

For each natural $k$ consider the concave function

$$H_k^\Phi(A) = \sup_{\{\pi_i, A_i\} \in \mathcal{P}^k(\mathcal{T}_+(\mathcal{H}))} \sum_i \pi_i H_\Phi(A_i)$$

on the cone $\mathcal{T}_+(\mathcal{H})$ (the supremum is over all decompositions of the operator $A$ into countable convex combination of operators of rank $\leq k$). By using (1) it is easy to see that the restriction of the above function $H_k^\Phi$ to the set $\mathcal{S}(\mathcal{H})$ coincides with the function $(\hat{H}_\Phi)^{\sigma}_k$ defined by (11) with $f = H_\Phi$ and that

$$H_k^\Phi(\lambda A) = \lambda H_k^\Phi(A), \quad A \in \mathcal{T}_+(\mathcal{H}), \quad \lambda \geq 0.$$ 

Hence by using propositions 1 and 3 in [25] it is easy to show that the function $H_k^\Phi$ is lower semicontinuous on the cone $\mathcal{T}_+(\mathcal{H})$ and that the monotonic sequence $\{H_k^\Phi\}$ pointwise converges to the function $H_\Phi$. Following [25] we will call the function $H_k^\Phi$ the $k$-order approximator of the output entropy of the map $\Phi$.

By using spectral decompositions one can prove uniform convergence of the sequence $\{H_k^\Phi\}$ to the function $H_\Phi$ on compact subsets of $\mathcal{T}_+(\mathcal{H})$ on which the quantum entropy is continuous.

**Lemma 3.** If the quantum entropy is continuous on a compact subset $A$ of $\mathcal{T}_+(\mathcal{H})$ then

$$\lim_{k \to +\infty} \sup_{A \in A, \Phi \in \mathcal{L}_{\leq 1}^{+}(\mathcal{H}, \mathcal{H}')} (H_\Phi(A) - H_k^\Phi(A)) = 0.$$  

(18)
Proof. We may assume that \( A \subset \mathcal{F}_1(\mathcal{H}) \). Let \( \lambda_k(A) \) be the sum of the eigenvalues \( \lambda_{i-1}k+1, \ldots, \lambda_{ik} \) of the operator \( A \) (arranged in non-increasing order) and \( P_k \) be the spectral projector of this operator corresponding to the above set of eigenvalues. Since the ensemble \{\( \pi_k^i, (\pi_k^i)^{-1}P_k^iA \)\}, where \( \pi_k^i = \|A\|^{-1}\lambda_k^i(A) \), lies in \( \mathcal{P}_a(\Sigma_k^+(\mathcal{H})) \), by using inequalities (5) and (7) we obtain

\[
H_\Phi(A) - H_k^\Phi(A) \leq H(\Phi(A)) - \sum_i \pi_k^i H(\Phi((\pi_k^i)^{-1}P_k^iA))
\]

for arbitrary map \( \Phi \) in \( \mathcal{L}_{\leq 1}(\mathcal{H}, \mathcal{H}') \). Hence the assertion of the lemma follows from lemma 9 in [25], which implies

\[
\lim_{k \to +\infty} \sup_{A \in A} \Delta_k(A) = \lim_{k \to +\infty} \sup_{A \in A} H(\{\lambda_k^i(A)\}) = 0. \quad \square
\]

Note that concavity of the function \( \eta(x) = -x \log x \) implies the inequality

\[
H_\Phi(A) - H_k^\Phi(A) \leq \inf_{\{\pi_i, A_i\} \in \mathcal{P}_a(\Sigma_k^+(\mathcal{H}))} \sum_i \pi_i H(\Phi(A_i)\|\Phi(A)), \quad A \in \mathcal{F}_+^+(\mathcal{H}),
\]

showing that (18) holds for arbitrary subset \( A \) of \( \Sigma_+(\mathcal{H}) \) having the UA-property (not necessarily compact) provided that the set \( \mathcal{L}_{\leq 1}(\mathcal{H}, \mathcal{H}') \) is replaced by the set \( \mathcal{F}_{\leq 1}(\mathcal{H}, \mathcal{H}') \) of all quantum operations (or by any other subset of positive linear maps for which monotonicity of the relative entropy holds).

Remark 3. Since the function \( A \mapsto H_k^\Phi(A) \) is lower semicontinuous on the set \( \Sigma_+(\mathcal{H}) \) for each \( k \), the generalized Dini’s lemma\footnote{The condition of continuity of functions of the increasing sequence in the standard Dini’s lemma can be replaced by the condition of their lower semicontinuity (provided that the condition of continuity of the limit function is valid).} shows that continuity of the function \( A \mapsto H_\Phi(A) \) on a compact subset \( A \) of \( \Sigma_+(\mathcal{H}) \) implies

\[
\lim_{k \to +\infty} \sup_{A \in A} (H_\Phi(A) - H_k^\Phi(A)) = 0.
\]

The converse implication obviously holds if the function \( A \mapsto H_k^\Phi(A) \) is continuous on the set \( A \) for each \( k \). \( \square \)

The above observations imply the following answer on the second question stated in the Introduction.
**Theorem 2.** Let $\Phi$ be a map in $\mathcal{L}^+_{\leq 1}(\mathcal{H},\mathcal{H}')$. The following properties are equivalent:

(i) the function $A \mapsto H_\Phi(A)$ is continuous on $\mathfrak{T}^1_+(\mathcal{H})$;\(^{10}\)

(ii) the function $A \mapsto H^k_\Phi(A)$ is continuous on $\mathfrak{T}_+(\mathcal{H})$ for each $k$;

(iii) the function $A \mapsto H_\Phi(A)$ is continuous on an arbitrary subset of $\mathfrak{T}_+(\mathcal{H})$ on which the quantum entropy is continuous.

Property (i) is equivalent to continuity and boundedness of the function $\rho \mapsto H_\Phi(\rho)$ on the set $\extr \mathfrak{S}(\mathcal{H})$ and hence it follows from the UA-property of the set $\Phi(\extr \mathfrak{S}(\mathcal{H}))$.

**Proof.** (i) $\Rightarrow$ (ii) Show first that (i) implies continuity of the function $A \mapsto H_\Phi(A)$ on the set $\mathfrak{T}^k_+(\mathcal{H})$ for each $k$. Suppose there exists a sequence $\{A_n\} \subset \mathfrak{T}^1_+(\mathcal{H})$ converging to an operator $A_0$ such that

$$\lim_{n \to +\infty} H_\Phi(A_n) > H_\Phi(A_0).$$

For each $n \in \mathbb{N}$ we have $A_n = \sum_{i=1}^k A^n_i$, where $\{A^n_i\}_{i=1}^k$ is a subset of $\mathfrak{T}^1_+(\mathcal{H})$. Since the set $\{A_n\}_{n \geq 0}$ is compact the compactness criterion for subsets of $\mathfrak{T}_+(\mathcal{H})$ (see the Appendix in [25]) implies relative compactness of the sequence $\{A^n_i\}_n$ for each $i = 1, k$. Hence we may consider that there exists $\lim_{n \to +\infty} A^n_i = A_0^i \in \mathfrak{T}^1_+(\mathcal{H})$ for each $i = 1, k$. It is clear that $\sum_{i=1}^k A_0^i = A_0$. It follows from (i) that $\lim_{n \to +\infty} H_\Phi(A^n_i) = H_\Phi(A_0^i)$. Hence Lemma 1 implies a contradiction to (19).

Continuity of the function $H_\Phi$ on the set $\mathfrak{T}^k_+(\mathcal{H})$ implies its boundedness on the set $\mathfrak{G}_k(\mathcal{H})$. By corollary 1 in [25] the function $H^k_\Phi$ is continuous and bounded on the set $\mathfrak{G}(\mathcal{H})$ and hence it is continuous on the set $\mathfrak{T}_+(\mathcal{H})$.

(ii) $\Rightarrow$ (iii) directly follows from Lemma 3. (iii) $\Rightarrow$ (i) is obvious.

The last assertion of the theorem follows from theorem 2A in [25] (since the UA-property of a bounded set obviously implies boundedness of the quantum entropy on this set). □

**Remark 4.** The main assertion of Theorem 2 (the implication (i) $\Rightarrow$ (iii)) is based on the special properties of the von Neumann entropy, it can not be

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10 $\mathfrak{T}^1_+(\mathcal{H})$ is the set of 1-rank positive operators in $\mathcal{H}$. 

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proved by using only such general properties of entropy-type functions as concavity, lower semicontinuity and nonnegativity. The simplest example showing this is given by the function $A \mapsto R_0(\Phi(A)) = \|\Phi(A)\|_1 \log \text{rank}(\Phi(A))$ – the output 0-order Renyi entropy of the map $\Phi$. The essential roles in the proof of Theorem 2 are played by the second inequality in (5) and the part "if" of the assertion of Lemma 1.

If property (iii) in Theorem 2 holds for a positive linear map $\Phi$ then we may say roughly speaking that this map "preserves continuity of the entropy". This motivates the following definition.

**Definition 1.** A positive linear map (correspondingly quantum operation or quantum channel) $\Phi$, for which property (iii) in Theorem 2 holds, is called PCE-map (correspondingly PCE-operation or PCE-channel).

The simplest examples of PCE-maps are completely positive linear maps with finite Kraus representations, for which property (i) in Theorem 2 obviously holds.

By the last assertion of Theorem 2 to prove the PCE-property of a map $\Phi$ it is sufficient to show that

$$\Phi(\text{extr}\mathcal{G}(\mathcal{H})) \subseteq \Lambda(\mathcal{A}),$$

where $\Lambda$ is a finite composition of set-operations preserving the UA-property (see proposition 4 in [25]) and $\mathcal{A}$ is a compact set on which the entropy is continuous. This implies the following PCE-condition.

**Corollary 5.** A map $\Phi$ in $\mathfrak{L}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ is a PCE-map if there exist separable Hilbert space $\mathcal{K}$, family $\{A_{\psi}\}_{\psi \in \mathcal{H}, \|\psi\|=1}$ of operators belonging to a particular compact subset $\mathcal{A}$ of $\mathfrak{T}_+(\mathcal{K})$, on which the quantum entropy is continuous, and family $\{V_{\psi}\}_{\psi \in \mathcal{H}, \|\psi\|=1}$ of linear contractions from $\mathcal{K}$ to $\mathcal{H}'$ such that $\Phi(|\psi\rangle\langle\psi|) = V_{\psi} A_{\psi} V_{\psi}^*$ for each unit vector $\psi$ in $\mathcal{H}$.

If $\Phi$ is a quantum operation having the Kraus representation with $k$ nonzero summands then the condition of Corollary 5 trivially holds (with $k$-dimensional Hilbert space $\mathcal{K}$). Nontrivial application of Corollary 5 is the proof of the PCE-property for the following family of quantum channels.

\footnote{Indeed, if $\Phi(A) = \frac{1}{2}(A + UAU^*)$, where $U$ is an unitary having no eigenvectors, then $R_0(\Phi(A)) = \|\Phi(A)\|_1 \log 2$ for all $A \in \mathfrak{T}_{\leq 1}(\mathcal{H})$ but it is easy to see that the function $A \mapsto R_0(\Phi(A))$ is not continuous on the set $\mathfrak{T}_{\leq 1}(\mathcal{H})$ on which $R_0(A) = \|A\|_1 \log 2$.}
Example 4. Let $\mathcal{H}_a$ be the Hilbert space $\mathcal{L}_2([-a, +a])$, where $a < +\infty$, and $\{U_t\}_{t \in \mathbb{R}}$ be the group of unitary operators in $\mathcal{H}_a$ defined as follows

$$(U_t\varphi)(x) = e^{-ixt}\varphi(x), \quad \forall \varphi \in \mathcal{H}_a.$$ 

For given probability density function $p(t)$ consider the quantum channel

$$\Phi_a^p : \mathfrak{T}(\mathcal{H}_a) \ni A \mapsto \int_{-\infty}^{+\infty} U_tAU_t^* p(t)dt \in \mathfrak{T}(\mathcal{H}_a).$$

In [23] it is shown that the condition of Corollary 5 holds for the channel $\Phi_a^p$ (with some set of unitary operators $\{V_{\psi}\}$) provided that the differential entropy of the distribution $p(t)$ is finite and that the function $p(t)$ is bounded and monotonic on $(-\infty, -b]$ and on $[+b, +\infty)$ for sufficiently large $b$. □

If property (iii) in Theorem 2 holds for two positive maps then it obviously holds for their composition, hence this theorem implies the following result.

**Corollary 6.** If property (i) in Theorem 2 holds for positive linear bounded maps $\Phi : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}')$ and $\Psi : \mathfrak{T}(\mathcal{H}') \to \mathfrak{T}(\mathcal{H}'')$ then it holds for the map $\Psi \circ \Phi : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}'')$.

In quantum information theory the notion of the Convex Closure of the Output Entropy (CCoOE) of a quantum channel (considered as a function on the set of input states of this channel) is widely used [1, 23]. By generalizing the proof of proposition 2 in [23] it is easy to show that property (i) of Theorem 2 of a positive linear map $\Phi$ is equivalent to continuity and boundedness of the CCoOE of this map on the set $\mathfrak{S}(\mathcal{H})$. Hence Corollary 6 shows that continuity and boundedness of the CCoOE of positive linear bounded maps $\Phi : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}')$ and $\Psi : \mathfrak{T}(\mathcal{H}') \to \mathfrak{T}(\mathcal{H}'')$ imply continuity and boundedness of the CCoOE of the map $\Psi \circ \Phi : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}'')$.

Lemma 1 implies the PCE-analog of Corollary 1.

**Corollary 7.** Let $\Phi$ and $\Psi$ be maps in $\mathcal{L}_{\leq 1}^+(\mathcal{H}, \mathcal{H}')$ and $\lambda \in (0, 1)$. $\lambda\Phi + (1 - \lambda)\Psi$ is a PCE-map if and only if $\Phi$ and $\Psi$ are PCE-maps.

Thus the set of all PCE-maps is convex and forms a face of the convex set $\mathcal{L}_{\leq 1}^+(\mathcal{H}, \mathcal{H}')$. This face obviously contains the face of all maps in $\mathcal{L}_{\leq 1}^+(\mathcal{H}, \mathcal{H}')$ with continuous output entropy.

### 4.2 The case of quantum operation

Theorem 2 implies the following observation.
Proposition 6. Let \( \Phi \) be a quantum operation in \( \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}') \) and \( \tilde{\Phi} \) be its complementary operation. The following properties are equivalent:

(i) \( \Phi \) is a PCE-operation;

(ii) \( \tilde{\Phi} \) is a PCE-operation.

In terms of the set \( \{V_i\}_i^{+\infty} \) of Kraus operators of the operation \( \Phi \) a sufficient condition for (i) – (ii) can be expressed in one of the following forms:

a) the function \( \varphi \mapsto H \left( \{\|V_i|\varphi\|^2\}_i^{+\infty} \right) \) is continuous on the space \( \mathcal{H} \);

b) one of the conditions in Proposition 5 holds for the sequence \( \{V_i\}_i^{+\infty} \).

If \( \Phi \) is a PCE-operation and \( \text{Ran} V_i \perp \text{Ran} V_j \) for all \( i \neq j \) then a) holds.

Proof. By Theorem 2 equivalence of (i) and (ii) follows from coincidence of the output entropies of the operations \( \Phi \) and \( \tilde{\Phi} \) on the set of pure states.

Sufficiency of condition a) follows from the corresponding assertion of Corollary 3 in Section 6 below (with \( V_n = \{V_i\}_i^{+\infty} \) for all \( n \)).

Sufficiency of condition b) follows from the first assertion of this proposition (since continuity of the output entropy implies the PCE-property).

Necessity of condition a) in the case \( \text{Ran} V_i \perp \text{Ran} V_j \) for all \( i \neq j \) is obvious since in this case

\[
H \left( \sum_{i=1}^{+\infty} V_i|\varphi\rangle \langle \varphi|V_i^* \right) = H \left( \{\|V_i|\varphi\|^2\}_i^{+\infty} \right). \quad \Box
\]

By Corollary 3 the tensor product of two quantum operations with continuous output entropy is a quantum operation with continuous output entropy. The PCE-property is not preserved in general with respect to tensor products. The simplest example showing this is the quantum channel \( \text{Id}_\mathcal{H} \otimes \text{Id}_\mathcal{H} \), which is not a PCE-channel if \( \dim \mathcal{H} = +\infty \). By Corollary 3 and Proposition 6 \( \Phi \otimes \Psi \) is a PCE-operation if either the operations \( \Phi \) and \( \Psi \) or the operations \( \tilde{\Phi} \) and \( \tilde{\Psi} \) have continuous output entropy. The following proposition contains several observations concerning the PCE-property of the map \( \Phi \otimes \Psi \).

Proposition 7. Let \( \Phi : \mathfrak{I}(\mathcal{H}) \to \mathfrak{I}(\mathcal{H}') \) and \( \Psi : \mathfrak{I}(\mathcal{K}) \to \mathfrak{I}(\mathcal{K}') \) be quantum operations.
A) If $\Psi$ is a finite-dimensional operation ($\max\{\dim \mathcal{K}, \dim \mathcal{K}'\} < +\infty$) then $\Phi \otimes \Psi$ is a PCE-operation if and only if $\Phi$ is a PCE-operation.\(^{12}\)

B) If $\dim \mathcal{H} = \dim \mathcal{K} = +\infty$ and $\Psi$ is a quantum channel such that its complementary channel $\tilde{\Psi}$ has finite output entropy then the following properties are equivalent:

(i) $H(\Phi \otimes \Psi(|\varphi\rangle\langle\varphi|)) < +\infty$ for any unit vector $\varphi \in \mathcal{H} \otimes \mathcal{K}$;

(ii) $H(A) < +\infty \Rightarrow H(\Phi \otimes \Psi(A)) < +\infty$ for any operator $A \in \mathcal{T}_+(\mathcal{H} \otimes \mathcal{K})$;

(iii) $\Phi \otimes \Psi$ is a PCE-operation;

If $\Phi$ is a quantum channel then the condition of finiteness of the output entropy of the channel $\tilde{\Psi}$ can be replaced by the condition

$$\min \left\{ H_{\tilde{\Phi}}(\text{Tr}_\mathcal{K}|\varphi\rangle\langle\varphi|), H_{\tilde{\Psi}}(\text{Tr}_\mathcal{H}|\varphi\rangle\langle\varphi|) \right\} < +\infty \quad \forall \varphi \in \mathcal{H} \otimes \mathcal{K}.$$  

Remark 5. By Theorem 2 $\Phi \otimes \Psi$ is a PCE-operation if and only if the function $\varphi \mapsto H(\Phi \otimes \Psi(|\varphi\rangle\langle\varphi|))$ is continuous and bounded on the unit sphere of $\mathcal{H} \otimes \mathcal{K}$. Proposition 7 shows that continuity and boundedness of this function follows from its finiteness (provided the condition of this proposition holds). Proposition 7 also shows that the operation $\Phi \otimes \Psi$ preserves continuity of the entropy (the PCE-property) if it preserves finiteness of the entropy (property (ii)).

Proof. A) The PCE-property of the operation $\Phi \otimes \Psi$ obviously implies the same property of the operation $\Phi$. To prove the converse implication it is sufficient to show that $\text{Id}_{\mathcal{H}'} \otimes \Psi$ and $\Phi \otimes \text{Id}_\mathcal{K}$ are PCE-operations.

The operation $\text{Id}_{\mathcal{H}'} \otimes \Psi$ has the PCE-property since it has a finite Kraus representation. By Theorem 2 the PCE-property of the operation $\Phi \otimes \text{Id}_\mathcal{K}$ follows from continuity and boundedness of the function

$$\text{extr} \mathcal{G}(\mathcal{H} \otimes \mathcal{K}) \ni \omega \mapsto H_{\Phi \otimes \text{Id}_\mathcal{K}}(\omega) = H_{\tilde{\Phi} \otimes \tilde{\text{Id}}_\mathcal{K}}(\omega) = H \left( \tilde{\Phi} \left( \text{Tr}_\mathcal{K} \omega \right) \right), \quad (20)$$

Since the map $\omega \mapsto \text{Tr}_\mathcal{K} \omega$ is a continuous surjection from $\text{extr} \mathcal{G}(\mathcal{H} \otimes \mathcal{K})$ onto $\mathcal{G}_k(\mathcal{H})$, where $k = \dim \mathcal{K}$, the function (20) is continuous and bounded if the function $H_{\tilde{\Phi}}$ is continuous and bounded on the set $\mathcal{G}_k(\mathcal{H})$. By Theorem 2 and Proposition 6 this holds if (and only if) $\Phi$ is a PCE-operation.

\(^{12}\)The condition $\dim \mathcal{K} < +\infty$ is essential. This can be shown by the example $\Phi = \text{Id}_\mathcal{H}$ and $\Psi = \text{Id}_\mathcal{K}$.
B) By applying Corollary 4 to the operation \( \tilde{\Phi} \) and to the channel \( \tilde{\Psi} \) we obtain that (i) implies continuity of the output entropy of the operation \( \tilde{\Phi} \otimes \tilde{\Psi} \) (since \( H(\tilde{\Phi} \otimes \tilde{\Psi}(|\varphi\rangle\langle\varphi|)) = H(\tilde{\Phi} \otimes \tilde{\Psi}(|\varphi\rangle\langle\varphi|)) \)). Hence (iii) follows from Proposition 6. It is clear that (iii) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (i). □

**Remark 6.** Proposition 7 with \( \Psi = \text{Id}_K \) gives the conditions for the PCE-property of the operation \( \Phi \otimes \text{Id}_K \). The PCE-property for the tensor product of two quantum operations \( \Phi \in \mathcal{F}_{\leq 1}(H, H') \) and \( \Psi \in \mathcal{F}_{\leq 1}(K, K') \) can be proved by showing that either \( \Phi \otimes \text{Id}_{K'} \) and \( \text{Id}_H \otimes \Psi \) or \( \Phi \otimes \text{Id}_K \) and \( \text{Id}_H \otimes \Psi \) are PCE-operations since \( \Phi \otimes \Psi = \text{Id}_{H'} \otimes \Psi \circ \Phi \otimes \text{Id}_K = \Phi \otimes \text{Id}_{K'} \circ \text{Id}_H \otimes \Psi \).

Note that the PCE-property of the operation \( \Phi \otimes \Psi \) does not imply the PCE-property of the above components. To show this it is sufficient to consider the example \( \Phi = \tilde{\text{Id}}_H \) and \( \Psi = \tilde{\text{Id}}_K \) with \( \dim H = \dim K = +\infty \).

5 **The output entropies of a pair of complementary quantum operations**

The output entropies of two complementary quantum operations coincide on the set of pure input states but they are substantially different functions on the whole space of input states. Nevertheless the following relation between continuity properties of these functions can be established.

**Proposition 8.** Let \( \Phi : \mathcal{T}(H) \to \mathcal{T}(H') \) be a quantum operation and \( \tilde{\Phi} \) be its complementary operation. Let \( A \) be a subset of \( \mathcal{T}_+(H) \) such that \( \min \{ H_\Phi(A), H_{\tilde{\Phi}}(A) \} < +\infty \) for all \( A \in A \). Then continuity of the quantum entropy on the set \( A \) implies continuity of the function \( A \mapsto (H_\Phi(A) - H_{\tilde{\Phi}}(A)) \) on the set \( A \).

The assertion of Proposition 8 follows from the more general assertion of Proposition 10 in Section 6 below.

**Remark 7.** If \( \Phi \) is a quantum channel then \( H_\Phi(\rho) - H_{\tilde{\Phi}}(\rho) \) is the coherent information \( I_c(\rho, \Phi) \) of this channel at a state \( \rho \) [7, 17].

**Corollary 8.** Let \( \Phi : \mathcal{T}(H) \to \mathcal{T}(H') \) be a quantum channel and \( \tilde{\Phi} \) be its complementary channel. If any two functions from the triple \( \{ H, H_\Phi, H_{\tilde{\Phi}} \} \) are continuous on a particular set \( A \subset \mathcal{T}_+(H) \) then the third one is also continuous on this set.
The above assertion holds for a quantum operation $\Phi : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}')$ if
\[
\lambda^* \left( \sqrt{I_{\mathcal{H}} - \Phi^*(I_{\mathcal{H}'})} \right) < +\infty \tag{21}
\]

**Proof.** By representations (2) and (3) the first assertion of the corollary follows from Proposition 8 and Proposition 15 in the Appendix.

The second assertion of the corollary is derived from the first one by means of Lemma 4 below since by representations (2) and (3) we have $\Phi = \Theta \circ \Lambda$ and $\tilde{\Phi} = \tilde{\Theta} \circ \Lambda$, where $\Theta(\cdot) = \text{Tr}_{\mathcal{H}''}(\cdot)$ is a quantum channel from $\mathcal{S}(\mathcal{H}' \otimes \mathcal{H}''')$ into $\mathcal{S}(\mathcal{H}')$ and $\Lambda(\cdot) = V(\cdot)V^*$ is a quantum operation from $\mathcal{S}(\mathcal{H})$ into $\mathcal{S}(\mathcal{H}' \otimes \mathcal{H}'')$.

**Lemma 4.** Let $V$ be a linear contraction from $\mathcal{H}$ into $\mathcal{H}'$ such that
\[
\lambda^* \left( \sqrt{I_{\mathcal{H}} - V^*V} \right) < +\infty.
\]
Then continuity of the function $A \mapsto H(VAV^*)$ on a particular set $A \subset \mathcal{S}_+(\mathcal{H})$ implies continuity of the quantum entropy on this set.

**Proof.** Consider the quantum channel
\[
\mathcal{S}(\mathcal{H}) \ni A \mapsto \Psi(A) = VAV^* \oplus \sqrt{I_{\mathcal{H}} - V^*V} A \sqrt{I_{\mathcal{H}} - V^*V} \in \mathcal{S}(\mathcal{H}' \oplus \mathcal{H}).
\]
By Proposition 8 the function $A \mapsto H(\sqrt{I_{\mathcal{H}} - V^*V} A \sqrt{I_{\mathcal{H}} - V^*V})$ is continuous on the set $\mathcal{S}_+(\mathcal{H})$. Hence the function $A \mapsto H_{\Phi}(A)$ is continuous on the set $A$. Since the complementary channel $\tilde{\Psi}$ has two dimensional output space the assertion of the lemma follows from the first part of Corollary 8.

**Corollary 9.** Let $\Phi : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}')$ be a quantum channel (or quantum operation satisfying condition (21)) such that the complementary channel (operation) $\tilde{\Phi}$ has finite output entropy. Then the function $A \mapsto H_{\Phi}(A)$ is continuous on a set $A \subset \mathcal{S}_+(\mathcal{H})$ if and only if the quantum entropy is continuous on this set.

The simplest class of quantum channels for which the condition of Corollary 9 holds consists of channels with finite Kraus representation, for which complementary channels have finite dimensional output.

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13The parameter $\lambda^*(\cdot)$ is defined in Proposition 3 in Section 3.2.
14In fact, "is equivalent to", since the converse implication holds for an arbitrary contraction $V$ by Theorem 2.
15The sufficient conditions for this property expressed in terms of the Kraus operators of the operation $\Phi$ is presented in Proposition 5 in Section 3.2.
Corollary 10. \[\Phi(\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^*\] be a quantum channel (or quantum operation satisfying condition (21)) in $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H'})$ such that $\text{Ran} V_i \perp \text{Ran} V_j$ for all sufficiently large $i \neq j$. Then continuity of the function $A \mapsto H_\Phi(A)$ on a set $A \subset \mathfrak{F}_+^\infty(\mathcal{H})$ implies continuity of the quantum entropy on the set $A$.

Proof. Note first that the complementary operation $\tilde{\Phi}$ can be represented as follows
\[
\tilde{\Phi}(A) = \sum_{i,j=1}^{+\infty} \text{Tr} [V_iAV_j^*] |i\rangle \langle j|, \quad A \in \mathfrak{F}(\mathcal{H}),
\]
where $\{|i\rangle\}$ is an orthonormal basis in the output space $\mathcal{H''}$ of this operation.

Suppose $\text{Ran} V_i \perp \text{Ran} V_j$ for all $i, j \geq n, i \neq j$. Let $\Phi_1(\cdot) = \sum_{i=1}^{n-1} V_i(\cdot)V_i^*$ and $\Phi_2(\cdot) = \sum_{i=n}^{+\infty} V_i(\cdot)V_i^*$ be quantum operations. By Lemma 1 continuity of the function $A \ni A \mapsto H(\Phi(A)) = H(\Phi_1(A) + \Phi_2(A))$ implies continuity of the function $A \ni A \mapsto H(\Phi_2(A))$. By the condition
\[
H(\Phi_2(A)) = \sum_{i=n}^{+\infty} H(V_iAV_i^*) + H(\{\text{Tr} V_iAV_i^*\}_{i=n}^{+\infty}) = \sum_{i=n}^{+\infty} H(V_iAV_i^*) + H(\tilde{\Phi}_2(A)),
\]
where $\tilde{\Phi}_2(A) = \sum_{i=n}^{+\infty} \text{Tr} [V_iAV_i^*] |i\rangle \langle i|$. Since the both terms in the right side of the above expression are lower semicontinuous functions of $A$, continuity of the function $A \ni A \mapsto H(\Phi_2(A))$ implies continuity of the function $A \ni A \mapsto H(\tilde{\Phi}_2(A))$.

Consider the quantum channel $\Pi(\cdot) = P(\cdot)P + (I_{\mathcal{H''}} - P)(\cdot)(I_{\mathcal{H''}} - P)$ in $\mathfrak{F}_{\leq 1}(\mathcal{H''}, \mathcal{H''})$, where $P = \sum_{i=1}^{n-1} |i\rangle \langle i|$. Since
\[
\Pi(\tilde{\Phi}(A)) = \sum_{i,j=1}^{n-1} \text{Tr} [V_iAV_j^*] |i\rangle \langle j| + \tilde{\Phi}_2(A),
\]
continuity of the function $A \ni A \mapsto H(\Phi_2(A))$ implies continuity of the function $A \ni A \mapsto H(\Pi(\tilde{\Phi}(A)))$ by Lemma 1 which is equivalent to continuity of the function $A \ni A \mapsto H(\tilde{\Phi}(A))$ by Corollary 9. Hence the function $A \ni A \mapsto H(A)$ is continuous by Corollary 8. \[\square\]

Remark 8. The assertions of Corollaries 8, 9 and 10 do not hold for a quantum operation $\Phi$ not satisfying condition (21). This follows from the next corollary.

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\[\text{This corollary can be considered as a generalization of Proposition 2 since its application to the channel } \Phi(A) = \sum_{i=1}^{+\infty} \langle i|A|i\rangle |i\rangle \langle i| \text{ implies the assertion of that proposition.}\]
Corollary 11. Let $\Phi$ be a quantum operation in $\mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ and
$$ T_{\Phi} = \{ A \in T_1(\mathcal{H}) \mid \min\{H_{\Phi}(A), H_{\tilde{\Phi}}(A)\} < +\infty \}. $$

If $\lambda^* \left( \sqrt{\Phi^*(I_{\mathcal{H}'})} \right) < +\infty$ then the function $A \mapsto (H_{\Phi}(A) - H_{\tilde{\Phi}}(A))$ is continuous on the set $T_{\Phi}$ and its absolute value does not exceed $\lambda^* \left( \sqrt{\Phi^*(I_{\mathcal{H}'})} \right)$.

If the functions $\rho \mapsto H_{\Phi}(\rho)$ and $\rho \mapsto H_{\tilde{\Phi}}(\rho)$ are continuous on the set $\mathfrak{S}(\mathcal{H})$ then the operator $\Phi^*(I_{\mathcal{H}'})$ satisfies the above condition.

Proof. Since $\Phi^*(I_{\mathcal{H}'}) = V^*V$, where $V$ is the Stinespring contraction for the operation $\Phi$, the first assertion of the corollary follows from Propositions 3 and 8 while the second one – from Proposition 8 and Proposition 15 in the Appendix. □

6 The output entropy as a function of a pair (map, input state)

In analysis of continuity of information characteristics of a quantum channel as functions of a channel it is necessary to consider the output entropy as a function of a pair (channel, input state) and to explore continuity of this function with respect to the Cartesian product (coordinate-wise) topology on the set of such pairs [10, 14]. The same problem appears in study of quantum channels by means of their approximation by quantum operations [10].

6.1 The general continuity condition

The central result of this subsection is the following proposition.

Proposition 9. Let $\{\Phi_n\}$ be a sequence of maps in $\mathfrak{S}_{\leq 1}^{+}(\mathcal{H}, \mathcal{H}')$ converging to a map $\Phi_0$. The following properties are equivalent:

(i) $\lim_{n \to +\infty} H_{\Phi_n}(A_n) = H_{\Phi_0}(A_0) < +\infty$ for any sequence $\{A_n\} \subset T_1^+(\mathcal{H})$ converging to an operator $A_0$.[17]

(ii) $\lim_{n \to +\infty} H(A_n) = H(A_0) < +\infty \Rightarrow \lim_{n \to +\infty} H_{\Phi_n}(A_n) = H_{\Phi_0}(A_0) < +\infty$ for any sequence $\{A_n\} \subset T_+(\mathcal{H})$ converging to an operator $A_0$.

[17] $T_1^+(\mathcal{H})$ is the set of 1-rank positive operators in $\mathcal{H}$. 

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Proof. It suffice to show that (i) $\Rightarrow$ (ii). For given natural $k$ the obvious modification of the first part of the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 2 implies validity of (ii) for any sequence $\{A_n\} \subset \mathfrak{T}_+(\mathcal{H})$.

Suppose there exist $\varepsilon > 0$ and a sequence $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H})$ converging to a state $\rho_0$ such that

$$\lim_{n \to +\infty} H(\rho_n) = H(\rho_0) < +\infty \quad \text{and} \quad \lim_{n \to +\infty} H_{\Phi_n}(\rho_n) > H_{\Phi_0}(\rho_0) + 3\varepsilon. \quad (22)$$

It is easy to see that (i) implies $\lim\sup_{n \to +\infty} \sup_{\rho \in \mathfrak{extr}(\mathcal{H})} H_{\Phi_n}(\rho) < +\infty$. Hence we may consider that

$$\sup_{n > 0} \sup_{\rho \in \mathfrak{extr}(\mathcal{H})} H_{\Phi_n}(\rho) < +\infty. \quad (23)$$

By Lemma 6 in the Appendix used with (10), (13) and (23) we may assume existence of a sequence $\{A_n^{k}\}_n \subset \mathfrak{T}_+(\mathcal{H})$, converging to an operator $A_0^k$ ($k \in \mathbb{N}$), such that $B_n^k = \rho_n - A_n^k \geq 0$,

$$H(\Phi_n(B_n^k)) < \varepsilon \quad \text{and} \quad \gamma_n = \text{Tr}\Phi_n(\rho_n) h_2 \left( \frac{\text{Tr}\Phi_n(B_n^k)}{\text{Tr}\Phi_n(\rho_n)} \right) < \varepsilon \quad \forall n \geq 0.$$

By the remark at the begin of the proof $|H(\Phi_n(A_n^k)) - H(\Phi_0(A_n^k))| < \varepsilon$ for all sufficiently large $n$. Since $\Phi_n(\rho_n) = \Phi_n(A_n^k) + \Phi_n(B_n^k)$ for each $n \geq 0$, by using inequality (5) we obtain

$$H(\Phi_n(\rho_n)) - H(\Phi_0(\rho_0)) \leq H(\Phi_n(A_n^k)) - H(\Phi_0(A_0^k)) + H(\Phi_n(B_n^k)) + \gamma_n < 3\varepsilon$$

for all sufficiently large $n$, contradicting to (22).

By this contradiction and lower semicontinuity of the quantum entropy property (ii) holds for an arbitrary sequence $\{A_n\} \subset \mathfrak{S}(\mathcal{H})$. Its validity for an arbitrary sequence $\{A_n\} \subset \mathfrak{T}_+(\mathcal{H})$ can be easily shown by using (10) and (23). $\square$

**Corollary 12.** For an arbitrary subset $\mathcal{A}$ of $\mathfrak{T}_+(\mathcal{H})$, on which the quantum entropy is continuous, the function $(\Phi, A) \mapsto H_{\Phi}(A)$ is continuous on the set $\mathfrak{T}^k_{\geq 1}(\mathcal{H}, \mathcal{H}') \times \mathcal{A}$ for each natural $k$.\footnote{$\mathfrak{T}^k_{\leq 1}(\mathcal{H}, \mathcal{H}')$ is the set of all quantum operations from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ having the Kraus representation consisting of $\leq k$ summands.}
Let $\mathfrak{W}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ be the set of all sequences $\{V_i\}_{i=1}^{+\infty}$ of linear bounded operators from $\mathcal{H}$ into $\mathcal{H}'$ such that $\sum_{i=1}^{+\infty} V_i^* V_i \leq I_{\mathcal{H}}$ endowed with the Cartesian product of the strong operator topology (the topology of coordinatewise strong operator convergence). In what follows a sequence $\{V_i\}_{i=1}^{+\infty}$ in $\mathfrak{W}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ will be called $\mathfrak{W}$-vector and will be denoted $\nabla$, the corresponding operator $\sum_{i=1}^{+\infty} V_i^* V_i$ will be denoted $|\nabla|$.

Let $\{|i\rangle\}_{i=1}^{+\infty}$ be an orthonormal basis in a separable Hilbert space $\mathcal{H}''$. Consider the maps

$$\mathfrak{W}_{\leq 1}(\mathcal{H}, \mathcal{H}') \ni \nabla \mapsto \Phi[\nabla](\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^* \in \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}')$$  \hspace{1cm} (24)

and

$$\mathfrak{W}_{\leq 1}(\mathcal{H}, \mathcal{H}') \ni \nabla \mapsto \tilde{\Phi}[\nabla](\cdot) = \sum_{i,j=1}^{+\infty} \text{Tr}(V_i(\cdot)V_j^*) |i\rangle\langle j| \in \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}'')$$  \hspace{1cm} (25)

The following lemma shows, in particular, continuity of these maps on the subset $\mathfrak{W}_{=1}(\mathcal{H}, \mathcal{H}') = \{\nabla | |\nabla| = I_{\mathcal{H}}\}$ of $\mathfrak{W}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ corresponding to the set of quantum channels.

**Lemma 5.** Let $\{\nabla_n\}$ be a sequence of $\mathfrak{W}$-vectors in $\mathfrak{W}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ converging to a $\mathfrak{W}$-vector $\nabla_0$. The following properties are equivalent:

(i) the sequence $\{|\nabla_n|\}$ weakly converges to the operator $|\nabla_0|$;

(ii) the sequence $\{\Phi[\nabla_n]\}$ strongly converges to the map $\Phi[\nabla_0]$;

(iii) the sequence $\{\tilde{\Phi}[\nabla_n]\}$ strongly converges to the map $\tilde{\Phi}[\nabla_0]$.

**Proof.** (i) $\Rightarrow$ (ii) It suffice to show that the sequence $\{\Phi[\nabla_n](|\varphi\rangle\langle \varphi|)\}$ tends to the operator $\Phi[\nabla_0](|\varphi\rangle\langle \varphi|)$ for arbitrary unit vector $\varphi \in \mathcal{H}$. This can be done by noting that $\lim_{n \rightarrow +\infty} \langle \varphi||V_n||\varphi\rangle = \langle \varphi||\nabla_0||\varphi\rangle$ implies

$$\lim_{m \rightarrow +\infty} \sup_n \text{Tr} \sum_{i \geq m} V_n^i |\varphi\rangle\langle \varphi|V_n^i| = 0 \quad \forall \varphi \in \mathcal{H}.$$  \hspace{1cm} (i) $\Rightarrow$ (iii) It suffice to show that the sequence $\{\tilde{\Phi}[\nabla_n](|\varphi\rangle\langle \varphi|)\}$ tends to the operator $\tilde{\Phi}[\nabla_0](|\varphi\rangle\langle \varphi|)$ for arbitrary unit vector $\varphi \in \mathcal{H}$. This can be
done by noting that (i) means \( \lim_{n \to +\infty} \text{Tr} \tilde{\Phi}[V_n](|\varphi\rangle\langle\varphi|) = \text{Tr} \tilde{\Phi}[V_0](|\varphi\rangle\langle\varphi|) \) and that weak convergence of a sequence \( \{A_n\} \subset \mathcal{T}_+(\mathcal{H}^\prime) \) to an operator \( A_0 \in \mathcal{T}_+(\mathcal{H}^\prime) \) such that \( \lim_{n \to +\infty} \text{Tr} A_n = \text{Tr} A_0 \) implies its convergence in the trace norm \[\|\cdot\|\].

(ii) \( \Rightarrow \) (i) and (iii) \( \Rightarrow \) (i) are obvious since

\[
\langle \varphi | V | \varphi \rangle = \text{Tr} \Phi[|\varphi\rangle\langle\varphi|] = \text{Tr} \tilde{\Phi}[|\varphi\rangle\langle\varphi|] \quad \forall \varphi \in \mathcal{H}. \quad \square
\]

Proposition \[\] implies the following observation.

**Corollary 13.** Let \( \{V_n\} \) be a sequence of \( \mathcal{U} \)-vectors in \( \mathcal{U}_{\leq 1}(\mathcal{H}, \mathcal{H}^\prime) \) converging to a \( \mathcal{U} \)-vector \( V_0 \) such that property (i) in Lemma \[\] holds and

\[
\lim_{n \to +\infty} H \left( \left\{ \|V_i^n|\varphi_n\rangle\|^2 \right\}_{i=1}^{+\infty} \right) = H \left( \left\{ \|V_i^0|\varphi_0\rangle\|^2 \right\}_{i=1}^{+\infty} \right) < +\infty \quad (26)
\]

for any sequence \( \{\varphi_n\} \) of vectors in \( \mathcal{H} \) converging to a vector \( \varphi_0 \). Then

\[
\lim_{n \to +\infty} H \left( \Phi[V_n](A_n) \right) = H \left( \Phi[V_0](A_0) \right) < +\infty
\]

and

\[
\lim_{n \to +\infty} H \left( \tilde{\Phi}[V_n](A_n) \right) = H \left( \tilde{\Phi}[V_0](A_0) \right) < +\infty
\]

for any sequence \( \{A_n\} \subset \mathcal{T}_+(\mathcal{H}) \) converging to an operator \( A_0 \) such that \( \lim_{n \to +\infty} H(A_n) = H(A_0) < +\infty \).

The above requirements on the sequence \( \{V_n\} \) can be replaced by one of the following conditions:

a) there exists a sequence \( \{h_i\}_{i=1}^{+\infty} \) of nonnegative numbers such that

\[
\sup_{n \geq 0} \left\| \sum_{i=1}^{+\infty} h_i V_i^n V_i^n \right\| < +\infty \quad \text{and} \quad \sum_{i=1}^{+\infty} e^{-\lambda h_i} < +\infty \quad \text{for all } \lambda > 0;
\]

b) property (i) in Lemma \[\] holds for the sequence \( \{V_n\} \) and

\[
\lim_{m \to +\infty} \sup_{n \geq 0} H \left( \left\{ \|V_i^n\|^2 \right\}_{i>m} \right) = 0.
\]
Proof. By using Proposition 2 it is easy to show that (26) implies
\[ \lim_{n \to +\infty} H \left( \widetilde{\Phi} \left( V_n \right)(|\varphi_n\rangle\langle \varphi_n|) \right) = H \left( \widetilde{\Phi} \left( V_0 \right)(|\varphi_0\rangle\langle \varphi_0|) \right) < +\infty. \]
for any sequence \( \{\varphi_n\} \) of vectors in \( \mathcal{H} \) converging to a vector \( \varphi_0 \). Hence the main assertion of the corollary follows from Proposition 9 and Lemma 5 (since the output entropies of complementary quantum operations coincide on the set of 1-rank operators).

Condition a) means that
\[ \sup_{n \geq 0} \sup_{\varphi \in \mathcal{H}, \|\varphi\| \leq 1} \sum_{i=1}^{+\infty} h_i \| V_i^n \| \varphi \|^2 < +\infty, \]
which implies \( \lim_{m \to +\infty} \sup_{n \geq 0} \sum_{i=m}^{+\infty} \| V_i^n \| \varphi \|^2 = 0 \) for each \( \varphi \in \mathcal{H} \), showing that property (i) in Lemma 5 holds for the sequence \( \{\nabla_n\} \). By Proposition 1 it also implies (26) for any sequence \( \{\varphi_n\} \) of vectors in \( \mathcal{H} \) converging to a vector \( \varphi_0 \).

Condition b) implies
\[ \lim_{m \to +\infty} \sup_{n \geq 0} \left[ H \left( \{ \| V_i^n \| \varphi_n \|^2 \}_{i=1}^{+\infty} \right) - H \left( \{ \| V_i^n \| \varphi_n \|^2 \}_{i=1}^{m} \right) \right] = 0 \]
for any sequence \( \{\varphi_n\} \) of vectors in \( \mathcal{H} \) converging to a vector \( \varphi_0 \). Indeed, since weak convergence of the sequence \( \{\nabla_n\} \) to the operator \( \nabla_0 \) implies \( \lim_{n \to +\infty} \langle \varphi_n | \nabla_n | \varphi_n \rangle = \langle \varphi_0 | \nabla_0 | \varphi_0 \rangle \) and hence
\[ \lim_{m \to +\infty} \sup_{n \geq 0} \sum_{i>m} \| V_i^n \| \varphi_n \|^2 = 0, \]
the above assertion can be proved by using (5) and (13). □

Example 5. Let \( \{\nabla_n = \{ V_i^n \}\} \) be a sequence of \( \mathfrak{N} \)-vectors in \( \mathfrak{N}_{\leq 1}(\mathcal{H}, \mathcal{H}') \) converging to a \( \mathfrak{N} \)-vector \( \nabla_0 = \{ V_i^0 \} \) such that \( \text{Ran}(V_i^n) \perp (\text{Ran}V_j^n)^* \) and \( \| V_i^n \|^2 \leq x_i \log^{-1}(i) \) for each \( n \) and all \( i, j > m, i \neq j \), where \( m \in \mathbb{N} \) and \( \{ x_i \}_{i>m} \) is a given sequence of positive numbers converging to zero. Then condition a) in Corollary 13 holds for the sequence \( \{\nabla_n\} \) with the sequence \( \{ h_i \}_{i=1}^{+\infty} \), where \( h_i = 0 \) if \( i \leq m \) and \( h_i = x_i^{-1} \log(i) \) if \( i > m \).
6.2 The continuity condition based on the complementary relation

By using the relation between complementary quantum operations the following result can be established.

**Proposition 10.** Let \( \{\Phi_n\} \) be a sequence of operations in \( \mathcal{F}_{\leq 1}(\mathcal{H}, \mathcal{H}') \) converging to an operation \( \Phi_0 \) and \( \{\tilde{\Phi}_n\} \) be a sequence of operations in \( \mathcal{F}_{\leq 1}(\mathcal{H}, \mathcal{H}'') \) converging to an operation \( \tilde{\Phi}_0 \) such that \((\Phi_n, \tilde{\Phi}_n)\) is a complementary pair for each \( n = 0, 1, 2, \ldots \). Let \( \{A_n\} \) be a sequence of operators in \( \mathcal{T}_+(\mathcal{H}) \) converging to an operator \( A_0 \) such that

\[
\lim_{n \to +\infty} H(A_n) = H(A_0) < +\infty \quad \text{and} \quad \min\{H_{\Phi_n}(A_n), H_{\tilde{\Phi}_n}(A_n)\} < +\infty, \quad n \geq 0.
\]

Then

\[
\lim_{n \to +\infty} (H_{\Phi_n}(A_n) - H_{\tilde{\Phi}_n}(A_n)) = H_{\Phi_0}(A_0) - H_{\tilde{\Phi}_0}(A_0) < +\infty.
\]

**Proof.** Finiteness of the values \( H_{\Phi_n}(A_n) \) and \( H_{\tilde{\Phi}_n}(A_n) \) for each \( n \geq 0 \) follows from the definition of a complementary operation and inequality (9).

Let \( \{A_n = \rho_n\} \) be a sequence of states in \( \mathcal{S}(\mathcal{H}) \) converging to a state \( A_0 = \rho_0 \) such that \( \lim_{n \to +\infty} H(\rho_n) = H(\rho_0) < +\infty \).

Let \( a_n = H_{\Phi_n}(\rho_n) - H_{\tilde{\Phi}_n}(\rho_n) \) for each \( n \geq 0 \). By symmetry to prove that \( \lim_{n \to +\infty} a_n = a_0 \) it is sufficient to show that

\[
\liminf_{n \to +\infty} a_n \geq a_0.
\]

Let \( K \) be a separable Hilbert space and \( \{\varphi_n\} \) be a sequence of unit vectors in \( \mathcal{H} \otimes K \) converging to a vector \( |\varphi_0\rangle \) such that \( \text{Tr}_K |\varphi_n\rangle \langle \varphi_n| = \rho_n \) for each \( n \geq 0 \). Finiteness of the values \( H_{\Phi_n}(\rho_n) \) and \( H_{\tilde{\Phi}_n}(\rho_n) \) implies

\[
b_n = H(\Phi_n \otimes \text{Id}_K(|\varphi_n\rangle \langle \varphi_n|))\|\Phi_n(\rho_n) \otimes \rho_n\| = a_n + c_n,
\]

where \( c_n = \text{Tr} \Phi_n \otimes \text{Id}_K(|\varphi_n\rangle \langle \varphi_n|) \cdot I_{\mathcal{H}'} \otimes (- \log \rho_n) \).

Since the sequence \( \{\Phi_n \otimes \text{Id}_K\} \) strongly converges to the operation \( \Phi_0 \otimes \text{Id}_K \) we have \( \liminf_{n \to +\infty} b_n \geq b_0 \) by lower semicontinuity of the relative entropy. Hence to prove (27) we have to show that

\[
\limsup_{n \to +\infty} c_n \leq c_0.
\]
For each $n$ consider the quantum channel $\Psi_n = \Phi_n + \Delta_n$ in $\mathcal{F}_1(\mathcal{H}, \mathcal{H}')$, where $\Delta_n(\cdot) = \sigma \text{Tr}((I_{\mathcal{H}} - \Phi_n(\mathcal{I}_{\mathcal{H}}))(\cdot))$ is a quantum operation defined by means of some state $\sigma$ in $\mathcal{G}(\mathcal{H}')$. It is clear that the sequence $\{\Delta_n\}$ strongly converges to the operation $\Delta_0$. Since $\lim_{n \to +\infty} H(\rho_n) = H(\rho_0) < +\infty$ and $H(\rho_n) = \text{Tr} \Psi_n \otimes \text{Id}_K(|\varphi_n\rangle\langle\varphi_n|) \cdot I_{\mathcal{H}'} \otimes (-\log \rho_n) = c_n + d_n$, $n = 0, 1, 2, \ldots$, where $d_n = \text{Tr} \Delta_n \otimes \text{Id}_K(|\varphi_n\rangle\langle\varphi_n|) \cdot I_{\mathcal{H}'} \otimes (-\log \rho_n)$, to prove (28) it suffice to show that
\[
\lim_{n \to +\infty} \inf d_n \geq d_0.
\]
We have $d_n = \text{Tr} B_n (-\log \rho_n)$, where $B_n = \text{Tr}_{\mathcal{H}'} \Delta_n \otimes \text{Id}_K(|\varphi_n\rangle\langle\varphi_n|)$. Since $B_n \leq B_n + \text{Tr}_{\mathcal{H}'} \Phi_n \otimes \text{Id}_K(|\varphi_n\rangle\langle\varphi_n|) = \rho_n$, we have $H(B_n) < +\infty$ and hence $d_n = H(B_n) + H(B_n|\rho_n) + \eta(\text{Tr} B_n) + \text{Tr} B_n - 1$. Lower semicontinuity of the quantum entropy and of the relative entropy implies (29).

Thus the assertion of the proposition is proved in the case $\{A_n\} \subset \mathcal{G}(\mathcal{H})$. The general assertion is easily derived from the above observation by noting that for arbitrary sequence $\{A_n\}$ converging to zero inequality (9) and the inequality $H(V^n A_n (V^n)^*) \leq H(A_n)$, where $V^n$ is the Stinespring contraction for the operations $\Phi_n$ and $\Phi_0$, show that
\[
\lim_{n \to +\infty} H(A_n) = 0 \implies \lim_{n \to +\infty} \left( H_{\Phi_n}(A_n) - H_{\Phi_0}(A_n) \right) = 0. \quad \square
\]

**Corollary 14.** Let $\{\nabla_n\}$ be a sequence of $\mathcal{V}$-vectors in $\mathcal{V}_{\leq 1}(\mathcal{H}, \mathcal{H}')$ converging to a $\mathcal{V}$-vector $\nabla_0$ such that property (i) in Lemma 7 holds, and $\{A_n\}$ be a sequence of operators in $\mathcal{S}_+(\mathcal{H})$ converging to an operator $A_0$ such that $\lim_{n \to +\infty} H(A_n) = H(A_0) < +\infty$. The following properties are equivalent:

(i) $\lim_{n \to +\infty} H \left( \Phi[\nabla_n](A_n) \right) = H \left( \Phi[\nabla_0](A_0) \right) < +\infty$;

(ii) $\lim_{n \to +\infty} H \left( \tilde{\Phi}[\nabla_n](A_n) \right) = H \left( \tilde{\Phi}[\nabla_0](A_0) \right) < +\infty$.

These properties hold if
\[
\lim_{n \to +\infty} H \left( \left\{ \text{Tr} V^n_i A_n (V^n_i)^* \right\}_{i=1}^{+\infty} \right) = H \left( \left\{ \text{Tr} V^0_i A_0 (V^0_i)^* \right\}_{i=1}^{+\infty} \right) < +\infty.
\]

**Proof.** By Lemma 7 the main assertion directly follows from Proposition 10. The second assertion is proved by using Proposition 2 $\square$
7 On continuity of the output entropy of quantum operations on a given set of states

7.1 The case of a single operation

In analysis of quantum channels and operations the question of continuity of their output entropy on a given set of input states naturally arises (see Section 8). By summarizing the results of the previous sections we consider below the possibilities to prove continuity of the output entropy of a quantum operation \( \Phi(\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^* \) on a given set of states \( \mathcal{A} \) in the two cases (distinguished by accessible information about properties of this set).

A) If the set \( \mathcal{A} \) is arbitrary then the function \( \rho \mapsto H_\Phi(\rho) \) is continuous on this set provided one of the following conditions is valid:

1) the quantum entropy is continuous on the set \( \Phi(\mathcal{A}) \);\(^{19}\)

2) the operation \( \Phi \) has continuous output entropy (sufficient conditions are presented in Section 3);

3) \( \lambda^* \left( \sqrt{\Phi^*(I_{\mathcal{H}})} \right) < +\infty \) and the function \( \rho \mapsto H_{\tilde{\Phi}}(\rho) \) is continuous on the set \( \mathcal{A} \) (the last property can be verified by using Propositions \( \Xi \) and \( \Pi \)).

B) If the von Neumann entropy is continuous on the set \( \mathcal{A} \) then the function \( \rho \mapsto H_\Phi(\rho) \) is continuous on this set provided one of the following conditions is valid:

1) \( \Phi \) is a PCE-operation (sufficient conditions are presented in Section 4);

2) the function \( \rho \mapsto H_{\tilde{\Phi}}(\rho) \) is continuous on the set \( \mathcal{A} \) (Proposition \( \Pi \) below).

Conditions A-3 and B-2 follow respectively from Corollary \( \Pi \) and Proposition \( \Xi \). To verify continuity of the function \( \rho \mapsto H_{\tilde{\Phi}}(\rho) \) in these conditions one can use either Proposition \( \Xi \) or the following proposition.

**Proposition 11.** Let \( \Phi(\cdot) = \sum_{i=1}^{+\infty} V_i(\cdot)V_i^* \) be a quantum operation and \( \mathcal{A} \) be a subset of \( \mathcal{G}(\mathcal{H}) \). The function \( \rho \mapsto H_{\tilde{\Phi}}(\rho) \) is continuous on the set \( \mathcal{A} \) if one of the following conditions (related by b) \( \Rightarrow \) a)) is valid:

a) the function \( \rho \mapsto H \left( \{\text{Tr}V_i\rho V_i^*\}_{i=1}^{+\infty} \right) \) is continuous on the set \( \mathcal{A} \);

---

\(^{19}\)This condition is obviously sufficient but it is not necessary (see Remark \( \Pi \)).
b) there exists a sequence \( \{h_i\}_{i=1}^{+\infty} \) of nonnegative numbers such that

\[
\sup_{\rho \in \mathcal{A}} \sum_{i=1}^{+\infty} h_i V_i^* V_i \rho < +\infty \quad \text{and} \quad \sum_{i=1}^{+\infty} e^{-\lambda h_i} < +\infty \quad \text{for all } \lambda > 0.
\]

**Proof.** Sufficiency of condition a) follows from the last assertion of Corollary 14. The implication b) \( \Rightarrow \) a) is proved by using Proposition 1B. \( \square \)

**Example 6.** Let \( \mathcal{A} = \mathcal{K}_{H,h}^s = \{ \rho \in \mathcal{S}(\mathcal{H}) | \text{Tr} H \rho \leq h \} \) be the set of quantum states with "bounded mean energy" defined by a \( H \)-operator \( H \) with \( g(H) = \inf \{ \lambda > 0 | \text{Tr} e^{-\lambda H} < +\infty \} = 0 \) and positive \( h \) (exceeding the minimal eigenvalue of \( H \)). The von Neumann entropy is continuous on the set \( \mathcal{K}_{H,h}^s \) by Proposition 1B. The above condition B-2 and condition b) in Proposition 11 show that the sufficient condition of continuity of the function \( \rho \mapsto H_\mathcal{A}(\rho) \) on the set \( \mathcal{K}_{H,h}^s \) consists in existence of a sequence \( \{h_i\}_{i=1}^{+\infty} \) of nonnegative numbers such that

\[
\sum_{i=1}^{+\infty} h_i V_i^* V_i \leq H \quad \text{and} \quad \sum_{i=1}^{+\infty} e^{-\lambda h_i} < +\infty \quad \text{for all } \lambda > 0.
\]

It is possible to show\(^{20}\) that this condition is also necessary if the operator \( H \) is strictly positive and \( \text{Ran} V_i \perp \text{Ran} V_j \) for all sufficiently large \( i \neq j \).

### 7.2 The case of a converging sequence of quantum operations

In analysis of continuity of information characteristics of a quantum channel with respect to perturbations of this channel we have to study continuity of the output entropy as a function of a pair (channel, input state). Practically,

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\(^{20}\)This can be done by using the following observation: for an arbitrary closed convex set \( \mathfrak{P}_0 \) of probability distributions, on which the Shannon entropy is continuous, there exists a sequence \( \{h_i\}_{i=1}^{+\infty} \) of nonnegative numbers such that

\[
\sup_{\{\pi_i\} \in \mathfrak{P}_0} \sum_{i=1}^{+\infty} h_i \pi_i < +\infty \quad \text{and} \quad \sum_{i=1}^{+\infty} e^{-\lambda h_i} < +\infty \quad \text{for all } \lambda > 0.
\]

This observation can be proved by using Lemma 2 and the arguments from the proof of the implication (i) \( \Rightarrow \) (ii) in Theorem 1 based on Dini’s lemma.
the following problem naturally arises in this analysis: for a given sequence \( \{\Phi_n\} \) of quantum operations in \( F_{\leq 1}(\mathcal{H}, \mathcal{H}') \) converging to an operation \( \Phi_0 \) and for a given closed subset \( \mathcal{A} \subset \mathcal{G}(\mathcal{H}) \) to show that

\[
\lim_{n \to +\infty} H_{\Phi_n}(\rho_n) = H_{\Phi_0}(\rho_0) \quad \forall \{\rho_n\} \subset \mathcal{A} \quad \text{such that} \quad \lim_{n \to +\infty} \rho_n = \rho_0. \quad (30)
\]

Summarizing the results of Section 6 we obtain the following observation.

If the von Neumann entropy is continuous on the set \( \mathcal{A} \) then property (30) holds provided one of the following conditions is valid:

1) \( \lim_{n \to +\infty} H_{\Phi_n}(|\varphi_n\rangle \langle \varphi_n|) = H_{\Phi_0}(|\varphi_0\rangle \langle \varphi_0|) \) for any sequence \( \{\varphi_n\} \) of vectors in \( \mathcal{H} \) converging to a vector \( \varphi_0 \);

2) property (30) holds for the sequence \( \{\tilde{\Phi}_n\} \) of quantum operations strongly converging to the operation \( \tilde{\Phi}_0 \) such that \( (\Phi_n, \tilde{\Phi}_n) \) is a complementary pair for each \( n \geq 0 \).

If \( \Phi_n(\cdot) = \sum_{i=1}^{+\infty} V_i^n(\cdot)(V_i^n)^* \) for each \( n \geq 0 \), where the sequence \( \{V_i^n\}_n \) strongly converges to the operator \( V_i^0 \) for each \( i \), then the above conditions 1) and 2) can be replaced respectively by the following ones:

1)’ \( \lim_{n \to +\infty} H \left( \left\{ \|V_i^n|\varphi_n\rangle\| \right\}^{+\infty}_{i=1} \right) = H \left( \left\{ \|V_i^0|\varphi_0\rangle\| \right\}^{+\infty}_{i=1} \right) \)

for any sequence \( \{\varphi_n\} \) of vectors in \( \mathcal{H} \) converging to a vector \( \varphi_0 \);

2)’ \( \lim_{n \to +\infty} H \left( \left\{ \text{Tr}_i V_i^n \rho_n (V_i^n)^* \right\}^{+\infty}_{i=1} \right) = H \left( \left\{ \text{Tr}_i V_i^0 \rho_0 (V_i^0)^* \right\}^{+\infty}_{i=1} \right) \).

The sufficient conditions for 1’) can be found in the second part of Corollary 13. The sufficient condition for 2’) looks like condition b) in Proposition 11 with \( V_i^n \) instead of \( V_i \) and ” \( \sup_{n \geq 0} \) ” added to ” \( \sup_{\rho \in \mathcal{A}} \) ”.

**Example 7.** The above condition 2’) makes it possible to replace the strong* operator topology by the strong operator topology in the assertion in example 3 in [25], concerning quantum measurements with a countable number of outcomes and stating that *continuity of the Shannon entropy of the outcomes probability distribution with respect to a priori state and to a measurement implies continuity of the von Neumann entropy of the mean posteriori state with respect to the same variables provided a priori state varies within a set on which the von Neumann entropy is continuous.*
8 Some applications

8.1 The Holevo capacity of quantum channels

Let $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}')$ be a quantum channel and $\mathcal{A}$ be a subset of $\mathcal{S}(\mathcal{H})$. The Holevo capacity of the $\mathcal{A}$-constrained channel $\Phi$ is defined as follows (cf. [8, 9])

$$\tilde{C}(\Phi, \mathcal{A}) = \sup_{\{\pi_i, \rho_i\} \in \mathcal{P}_A(\mathcal{S}(\mathcal{H}))} \sum_i \pi_i H(\Phi(\rho_i) \parallel \Phi(b(\{\pi_i, \rho_i\})))$$ (31)

(the supremum is over all finite ensembles of states with the average in $\mathcal{A}$).

8.1.1 On existence of continuous optimal ensembles

The well known fact concerning the Holevo capacity of a finite dimensional quantum channel $\Phi$ constrained by a closed subset $\mathcal{A}$ consists in existence of an optimal ensemble at which the supremum in (31) is achieved [21]. Since

$$\tilde{C}(\Phi, \mathcal{A}) = \sup_{\mu \in \mathcal{P}_A(\mathcal{S}(\mathcal{H}))} \int_{\mathcal{S}(\mathcal{H})} H(\Phi(\rho) \parallel \Phi(b(\mu))) \mu(d\rho)$$ (32)

(the supremum is over all probability measures with the barycenter in $\mathcal{A}$) the notion of an optimal ensemble is naturally generalized to the infinite dimensional case leading to the notion of an optimal measure (generalized or continuous optimal ensemble) at which the supremum in (32) is achieved [9]. In contrast to the finite dimensional case we can not claim existence of an optimal measure for an arbitrary quantum channel constrained by closed or even compact subset of states (see the example in [9]). By the theorem in [9], containing a sufficient condition for existence of an optimal measure, we have the following result.

**Proposition 12.** Let $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}')$ be a quantum channel and let $\mathcal{A}$ be a compact subset of $\mathcal{S}(\mathcal{H})$. If one of the conditions of continuity of the function $\mathcal{A} \ni \rho \mapsto H_{\Phi}(\rho)$ presented in Section 7.1 holds then there exists an optimal measure for the $\mathcal{A}$-constrained channel $\Phi$ supported by pure states.

This proposition implies existence of an optimal measure for an arbitrary quantum channel with finite dimensional environment (with finite Kraus representation) constrained by a compact subset of states on which the entropy is continuous.
8.1.2 On continuity of the Holevo capacity as a function of a channel

In the finite dimensional case the Holevo capacity $\bar{C}(\Phi, A)$ is a continuous function of $\Phi$ on the set of all quantum channels $\mathfrak{F}_1(\mathcal{H}, \mathcal{H}')$, but it is not continuous in infinite dimensions even with respect to the norm of complete boundedness [14]. By proposition 7 in [10], containing a sufficient condition for continuity of the function $\Phi \mapsto \bar{C}(\Phi, A)$ on subsets of $\mathfrak{F}_1(\mathcal{H}, \mathcal{H}')$ with respect to the topology of strong convergence, we have the following result.

**Proposition 13.** Let $\{\Phi_n\}$ be a sequence of quantum channels strongly converging to a quantum channel $\Phi_0$ and let $A$ be a compact subset of $\mathcal{S}(\mathcal{H})$ on which the von Neumann entropy is continuous. If one of the conditions of validity of (30) presented in Section 7.2 holds then

$$\lim_{n \to +\infty} \bar{C}(\Phi_n, A) = \bar{C}(\Phi_0, A).$$

**Corollary 15.** For arbitrary compact subset $A$ of $\mathcal{S}(\mathcal{H})$, on which the von Neumann entropy is continuous, the function $\Phi \mapsto \bar{C}(\Phi, A)$ is continuous on the set $\mathfrak{F}_k(\mathcal{H}, \mathcal{H}')$ for each $k$.\footnote{\text{\textsuperscript{21}}}$\mathfrak{F}_k(\mathcal{H}, \mathcal{H}')$ is the set of all quantum channels from $\mathfrak{F}(\mathcal{H})$ to $\mathfrak{F}(\mathcal{H}')$ having the Kraus representation consisting of $\leq k$ summands.

Note that the set of channels having finite Kraus representation is dense in the set $\mathfrak{F}_1(\mathcal{H}, \mathcal{H}')$ of all channels in the topology of strong convergence.

8.2 On continuity of the Entanglement of Formation

The notion of the Entanglement of Formation as a quantitative characteristic of entanglement of a state in a composite quantum system is introduced in [2] in the finite dimensional case. The possible infinite dimensional generalizations of this notion are based respectively on the $\sigma$-convex roof and the $\mu$-convex roof constructions [6, 24]. Comparison of these constructions, in particular, the sufficient conditions for their coincidence on subsets of states of composite system are presented in [24], where the arguments showing preferability of the $\mu$-convex roof construction are also considered.\footnote{\text{\textsuperscript{22}}}$\text{\textsuperscript{22}}$The question of coincidence of the $\mu$-convex roof and the $\sigma$-convex roof constructions of the EoF on the whole state space is open (as far as I know). In [24] it is shown that this question can not be solved by using only such general properties of the entropy as concavity.
what follows we will use the generalization of the EoF based on this construction, t.i.

$$E_F(\omega) = (H \circ \Theta)^\mu(\omega) = \inf \int_{\text{extr}\mathcal{S}(H \otimes K)} H(\Theta(\omega)) \mu(d\omega), \quad \omega \in \mathcal{S}(H \otimes K),$$

where $\Theta(\cdot) = \text{Tr}_K(\cdot)$ and the infimum is over all probability measures $\mu$ on the set $\text{extr}\mathcal{S}(H \otimes K)$ with the barycenter $\omega$.

By proposition 8 in [24] to show continuity of the function $\omega \mapsto E_F(\omega)$ on a particular subset of $\mathcal{S}(H \otimes K)$ it is sufficient to show continuity of one of the functions $\omega \mapsto H(\text{Tr}_K(\omega))$ and $\omega \mapsto H(\text{Tr}_H(\omega))$ on this subset. Thus by using the results of the previous sections one can obtain continuity conditions for the function $\omega \mapsto E_F(\omega)$.

**Proposition 14.** Let $\{\omega_n\}$ be a sequence of states in $\mathcal{S}(H \otimes K)$, converging to a state $\omega_0$, such that $\lim_{n \to +\infty} H(\rho_n) = H(\rho_0) < +\infty$, where $\rho_n = \text{Tr}_K \omega_n$ for $n = 0, 1, 2, \ldots$. Let $\{\Phi_n\}$ and $\{\Psi_n\}$ be sequences of operations in $\mathcal{F}_{\leq 1}(H)$ and in $\mathcal{F}_{\leq 1}(K)$ strongly converging to operations $\Phi_0$ and $\Psi_0$ correspondingly. If one of the conditions of validity of (30) for the sequences $\{\rho_n\}$ and $\{\Phi_n\}$ presented in Section 7.2 holds and $\text{Tr} \Phi_0 \otimes \Psi_0(\omega_0) > 0$ then

$$\lim_{n \to +\infty} E_F\left( \frac{\Phi_n \otimes \Psi_n(\omega_n)}{\text{Tr} \Phi_n \otimes \Psi_n(\omega_n)} \right) = E_F\left( \frac{\Phi_0 \otimes \Psi_0(\omega_0)}{\text{Tr} \Phi_0 \otimes \Psi_0(\omega_0)} \right).$$

**Proof.** By the remark before the proposition it is sufficient to show that

$$\lim_{n \to +\infty} H(\text{Tr}_K \Phi_n \otimes \Psi_n(\omega_n)) = H(\text{Tr}_K \Phi_0 \otimes \Psi_0(\omega_0)) < +\infty.$$  

Since by the condition we have $\lim_{n \to +\infty} H(\Phi_n(\rho_n)) = H(\Phi_0(\rho_0)) < +\infty$ and $\text{Tr}_K \Phi_n \otimes \Psi_n(\omega_n) \leq \Phi_n(\rho_n)$ for $n = 0, 1, 2, \ldots$, the above relation follows from corollary 4 in [25].

The assertion of this proposition is valid in the following cases:

- $\Phi_n = \Phi_0$ is a PCE-operation (see Section 4);
• \( \{ \Phi_n \} \subseteq \mathcal{F}_{\leq 1}^k (\mathcal{H}) \) (the set of quantum operations having the Kraus representation consisting of \( \leq k \) summands).

Proposition 14 can be used in analysis of continuity of the EoF under local operations.

9 Appendix

9.1 Continuity of the entropy on subsets of \( \mathcal{T}_+ (\mathcal{H} \otimes \mathcal{K}) \)

Proposition 10 in [22] can be generalized to subsets of \( \mathcal{T}_+ (\mathcal{H} \otimes \mathcal{K}) \) as follows.

**Proposition 15.** Let \( \mathcal{C} \) be a subset of \( \mathcal{T}_+ (\mathcal{H} \otimes \mathcal{K}) \). Continuity of the quantum entropy on the sets \( \text{Tr}_K \mathcal{C} \subseteq \mathcal{T}_+ (\mathcal{H}) \) and \( \text{Tr}_H \mathcal{C} \subseteq \mathcal{T}_+ (\mathcal{K}) \) implies continuity of the quantum entropy on the set \( \mathcal{C} \).

**Proof.** Let \( \{ C_n \} \subseteq \mathcal{C} \) be a sequence converging to an operator \( C_0 \in \mathcal{C} \). If \( C_0 \neq 0 \) then by the condition we have

\[
\lim_{n \to +\infty} H \left( \frac{C_n^H}{\text{Tr} C_n} \right) = H \left( \frac{C_0^H}{\text{Tr} C_0} \right) \quad \text{and} \quad \lim_{n \to +\infty} H \left( \frac{C_n^K}{\text{Tr} C_n} \right) = H \left( \frac{C_0^K}{\text{Tr} C_0} \right)
\]

where \( C_n^H = \text{Tr}_K C_n \) and \( C_n^K = \text{Tr}_H C_n \) for all \( n \). Proposition 10 in [22] implies

\[
\lim_{n \to +\infty} H \left( \frac{C_n}{\text{Tr} C_n} \right) = H \left( \frac{C_0}{\text{Tr} C_0} \right) \quad \text{and hence} \quad \lim_{n \to +\infty} H (C_n) = H (C_0).
\]

If \( C_0 = 0 \) then convergence to zero of the sequence \( \{ H (C_n) \} \) follows from convergence to zero of the sequences \( \{ H \left( C_n^H \right) \} \) and \( \{ H \left( C_n^K \right) \} \) by means of the inequality \( H (C_n) \leq H \left( C_n^H \right) + H \left( C_n^K \right) \), \( n \in \mathbb{N} \). \( \square \)

9.2 Auxiliary results

**Lemma 6.** Let \( \{ \rho_n \} \) be a sequence of states converging to a state \( \rho_0 \) such that \( \lim_{n \to +\infty} H (\rho_n) = H (\rho_0) \). For given natural \( k \) let \( P_n^k \) be a \( k \)-rank spectral projector of the state \( \rho_n \) corresponding to its \( k \) maximal eigenvalues and let \( A_n^k = P_n^k \rho_n \) for all \( n \).\(^{23}\) For arbitrary \( \varepsilon > 0 \) there exist a natural \( k_\varepsilon \) and

\(^{23}\)The projector \( P_n^k \) is uniquely defined if the state \( \rho_n \) has no multiple eigenvalues. In any case all variants of \( P_n^k \) lead to isomorphic operators \( A_n^k \) and we assume that one of them is chosen.
a subsequence \( \{A_{n_t}^{k_{\varepsilon}}\} \) converging to the operator \( A_0^{k_{\varepsilon}} = \bar{P}_0^{k_{\varepsilon}} \rho_0 \), where \( \bar{P}_0^{k_{\varepsilon}} \) is a particular \( k_{\varepsilon} \)-rank spectral projector of the state \( \rho_0 \) corresponding to its \( k_{\varepsilon} \) maximal eigenvalues, such that

\[
\text{Tr} B_{n_t}^{k_{\varepsilon}} < \varepsilon \quad \text{and} \quad H(B_{n_t}^{k_{\varepsilon}}) < \varepsilon,
\]

where \( B_{n_t}^{k_{\varepsilon}} = \rho_{n_t} - A_{n_t}^{k_{\varepsilon}} \) is a positive operator, for all sufficiently large \( t \).

**Proof.** Despite possible multiple meaning of the operator \( P_0^k \rho_0 \) the values \( \text{Tr} P_0^k \rho_0 \) and \( H(P_0^k \rho_0) \) are uniquely defined by the state \( \rho_0 \). By lemma 4 in [25] the sequence \( \{H(P_0^k \rho_0)\} \) is nondecreasing and tends to \( H(\rho_0) \) as \( k \to +\infty \). Let \( k_{\varepsilon} \) be such that \( \text{Tr} P_0^{k_{\varepsilon}} \rho_0 > 1 - \varepsilon/2 \) and \( |H(\rho_0) - H(P_0^{k_{\varepsilon}} \rho_0)| < \varepsilon/3 \). Since \( A_{n_t}^{k_{\varepsilon}} \leq \rho_{n_t} \) for all \( n \), the compactness criterion for subsets of \( \mathfrak{S}_+(\mathcal{H}) \) (see the Appendix in [25]) shows relative compactness of the sequence \( \{A_{n_t}^{k_{\varepsilon}}\} \) and hence existence of a subsequence \( \{A_{n_t}^{k_{\varepsilon}}\} \) converging to an operator \( A_0^{k_{\varepsilon}} \). By using coincidence of \( \text{Tr} A_{n_t}^{k_{\varepsilon}} \) with \( \sup P \text{Tr} P \rho_{n_t} \), where \( P \) runs over the set of all \( k_{\varepsilon} \)-rank projectors, it is easy to show that \( A_0^{k_{\varepsilon}} = P_0^k \rho_0 \), where \( P_0^k \) is a particular \( k_{\varepsilon} \)-rank spectral projector of the state \( \rho_0 \) corresponding to its \( k_{\varepsilon} \) maximal eigenvalues.

Since \( \lim_{t \to +\infty} \text{Tr} A_{n_t}^{k_{\varepsilon}} = \text{Tr} A_0^{k_{\varepsilon}} \) and \( \lim_{t \to +\infty} H(A_{n_t}^{k_{\varepsilon}}) = H(A_0^{k_{\varepsilon}}) \) we have

\[
\text{Tr} B_{n_t}^{k_{\varepsilon}} = 1 - \text{Tr} A_{n_t}^{k_{\varepsilon}} \leq |1 - \text{Tr} A_0^{k_{\varepsilon}}| + |\text{Tr} A_{n_t}^{k_{\varepsilon}} - \text{Tr} A_0^{k_{\varepsilon}}| < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]

and

\[
H(B_{n_t}^{k_{\varepsilon}}) \leq H(\rho_{n_t}) - H(A_{n_t}^{k_{\varepsilon}}) \leq |H(\rho_{n_t}) - H(\rho_0)|
\]

\[
+ |H(\rho_0) - H(A_{n_t}^{k_{\varepsilon}})| + |H(A_{n_t}^{k_{\varepsilon}}) - H(A_0^{k_{\varepsilon}})| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\]

for all sufficiently large \( t \), where the second inequality is obtained by using inequality [13]. \( \square \)

**Lemma 7.** Let \( \{\pi_i\}_{i=1}^{+\infty} \) be a sequence of positive numbers. Then

\[
\sup_{\{x_i\} \in \mathfrak{S}_+^{+\infty}} H\left(\{\pi_i x_i\}_{i=1}^{+\infty}\right) = \lambda^* \text{,}
\]

where \( \lambda^* \) is either the unique finite solution of the equation \( \sum_{i=1}^{+\infty} e^{-\lambda/\pi_i} = 1 \) if it exists or equal to \( g(\{\pi_i^{-1}\}) = \inf \{\lambda > 0 \mid \sum_{i=1}^{+\infty} e^{-\lambda/\pi_i} < +\infty\} \) otherwise\footnote{It is assumed that \( \inf \emptyset = +\infty \). The equation \( \sum_{i=1}^{+\infty} e^{-\lambda/\pi_i} = 1 \) has no solution if and only if \( g(\{\pi_i^{-1}\}) = +\infty \) or \( \sum_{i=1}^{+\infty} e^{-g(\{\pi_i^{-1}\})/\pi_i} < 1 \).}

\[44\]
Proof. By using the Lagrange method it is easy to show that the function \( \mathcal{P}_n \ni \{x_i\}_{i=1}^n \mapsto H(\{\pi_i x_i\}_{i=1}^n) \) attains its maximum at the vector \( \{x_i^* = c \pi_i^{-1} e^{-\lambda_n^*/\pi_i}\} \), where \( \lambda_n^* \) is the solution of the equation \( \sum_{i=1}^n e^{-\lambda/\pi_i} = 1 \) and \( c = \left[ \sum_{i=1}^n \pi_i^{-1} e^{-\lambda_n^*/\pi_i} \right]^{-1} \). Hence

\[
\max_{\{x_i\} \in \mathcal{P}_n} H(\{\pi_i x_i\}_{i=1}^n) = \lambda_n^*.
\] (33)

The assertion of the lemma can be derived from (33) by noting that the sequence \( \{\lambda_n^*\} \) tends to \( \lambda^* \) as \( n \to +\infty \) and by using lower semicontinuity of the classical entropy. □

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