QUANTUM EXTENDED CRYSTAL SUPER PDE’S

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ABSTRACT. We generalize our geometric theory on extended crystal PDE’s and their stability, to the category $Q_S$ of quantum supermanifolds. By using algebraic topologic techniques, obstructions to the existence of global quantum smooth solutions for such equations are obtained. Applications are given to encode quantum dynamics of nuclear nuclides, identified with graviton-quark-gluon plasmas, and study their stability. We prove that such quantum dynamical systems are encoded by suitable quantum extended crystal Yang-Mills super PDE’s. In this way stable nuclear-charged plasmas and nuclides are characterized as suitable stable quantum solutions of such quantum Yang-Mills super PDE’s. An existence theorem of local and global solutions with mass-gap, is given for quantum super Yang-Mills PDE’s, $(\tilde{YM})$, by identifying a suitable constraint, $(\text{Higgs}) \subset (\tilde{YM})$, Higgs quantum super PDE, bounded by a quantum super partial differential relation $(\text{Goldstone}) \subset (\tilde{YM})$, quantum Goldstone-boundary. A global solution $V \subset (\tilde{YM})$, crossing the quantum Goldstone-boundary acquires (or loses) mass. Stability properties of such solutions are characterized.

1. Introduction

The mathematical heritage of the last century is essentially that Physics is Geometry and nothing else. This is, in fact, the main issue by the Einstein’s General Relativity theory [17], (previously supported also by the Maxwell’s theory of electromagnetism [53, 54]). Unfortunately this message was not well understood at the quantum level! The principal motivation was, we believe, that Mathematics was not ready to extend such a philosophy in the formulation of a geometric quantum theory. In fact, at the beginning of the last century the Ricci-Curbastro’s tensor calculus [102, 103, 108] was just enough developed to allow to Einstein his formulation of general gravitation. Instead, in order to formulate a “quantum general relativity”, it was necessary yet to build a new geometric theory of quantum PDE’s!

1Some results of this paper were partially announced in [95].
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At the beginning of last century, however, was just well understood that the concept of "mass" is synonymous of concentrated energy. In fact, the General Relativity Theory proved that big masses are able to deform space-time geometry. Nowadays we can state that also at quantum level, high concentration of energy modifies geometry and produces noncommutative geometry. Thus the mass-energy, is nothing else that a property of the geometry involved to describe "particles". This becomes conjectured also after the first middle of the last century, thanks to the famous formula $E = mc^2$, and nuclear energy production experiments. To this purpose, let us recall the well known J. A. Wheeler's slogans: "mass without mass, charge without charge, field without field" and his pioneering works on the "geometrodynamics". (See, e.g., Refs.[130, 131, 132, 58].)

Our PDE's Algebraic Topology, developed in the category of (non)commutative manifolds, aims to follow this philosophy, building a new mathematics just able to allow a full geometrization of Physics.

In some previous works we have characterized PDE’s as extended crystals, in the sense that their integral bordism groups can be considered as extensions of suitable crystallographic subgroups. For such structures a geometric formulation of stability theory has been developed also. More recently such theory has been extended also to quantum (super) PDE’s, i.e., PDE’s built in the category $\mathcal{Q}_S$ of quantum supermanifolds, as previously introduced by us. (See Refs.[86, 88, 89, 90, 91, 95].)

Aim of the present paper is to specialize our study to quantum supergravity Yang-Mills PDE’s (quantum SG-Yang-Mills PDE’s). This type of equations have been previously introduced by us in some recent works and appears very useful to encode quantum dynamics unifying, just at quantum level, gravity with the other fundamental forces of Nature, i.e., electromagnetic, weak and strong forces.[71, 72, 75, 76, 77, 78, 82, 83, 84, 85, 92, 93, 94, 95]. These equations extend, at quantum level, some superclassical ones, well known in literature about supergravity. (See, e.g., Refs.[14, 25, 123, 127, 128, 133].) In fact supergravity, as has been usually considered, is a classical field theory, that, in some sense comes from a generalization of Charles Ehresmann and Élie Cartan’s differential geometry [110].

Then classical supergravity requires to be quantized. But in this way one discards nonlinear phenomena. In fact this quantization is obtained by means of so-called quantum propagators, that are just associated to linearizations of classical PDE’s. Our formulation, instead, works directly on noncommutative manifolds (quantum supermanifolds), and the quantization is not more necessary. In fact, whether it is performed in this noncommutative framework, it can bee seen as a linear approximation of a more general nonlinear integration. In some previous papers this important aspect has been carefully proved. (See Refs.[85, 94].)

This paper, after Introduction, splits in two more sections. Section 2. Here we characterize quantum super PDE’s like extended crystals, in the sense that their integral bordism groups can be considered as extensions of crystallographic subgroups. This approach generalizes our previous one for commutative PDE’s, and allows us to identify an algebraic topologic obstruction to the existence of global smooth solutions for PDE’s in the category $\mathcal{Q}_S$. Furthermore, for such solutions we study their stability properties from a geometric point of view. Section 3. Here we consider "quantum gravity" in the category $\mathcal{Q}_S$, and encoded by suitable quantum
Yang-Mills equations (quantum SG-Yang-Mills PDE’s), say $\hat{(YM)}$. In this way we are able to characterize quantum (super)gravity like a secondary object, associated to some geometric fundamental objects (fields), solutions of $\hat{(YM)}$. Then mass properties of such solutions are directly pointed-out, without the necessity to directly assume symmetry breaking Higgs-mechanisms. However, we recognize a constraint in $\hat{(YM)}$, that gives a pure quantum geometrodynamic mechanism able to justify mass acquisition (or loss) to a quantum solution of $\hat{(YM)}$. Furthermore, nuclear particles and nuclides can be seen as suitable $p$-chain solutions of $\hat{(YM)}$, and their energy-thermodynamic contents and stability properties characterized.

The main results of this paper are the following. Theorem 2.3 characterizes the crystal structure of quantum super PDE’s. Corollary 2.18 identifies the algebraic topologic obstruction to the existence of global smooth solutions of PDE’s in the category $\mathcal{Q}_S$. Theorem 2.34 and Theorem 2.40 characterize the stability of quantum super PDE’s and their solutions. Theorem 2.46 gives a criterion to average stability. Theorem 3.4 and Theorem 3.8 encode dynamic for quantum (super)gravity and characterize the quantum Cartan geometry induced by solutions of quantum SG-Yang-Mills PDE’s, shortly denoted by $\hat{(YM)}$. Theorem 3.11 gives a criterion, founded on the quantum Higgs-symmetry breaking mechanism, to recognize solutions of $\hat{(YM)}$ whose quantum Levi-Civita connections induce zero covariant derivative on the corresponding quantum metric. Theorem 3.12 and Theorem 3.14 characterize the quantum crystal structure of $\hat{(YM)}$. Theorem 3.19 gives a local mass-formula for solutions of $\hat{(YM)}$. Theorem 3.20 identifies important quantum observed fields by means of a quantum relativistic frame. Theorem 3.21 characterizes the stability properties of $\hat{(YM)}$ and its solutions. Theorem 3.23 identifies thermodynamic functions and thermodynamic equations associated to observed solutions of $\hat{(YM)}$. Theorem 3.28, and Corollary 3.38 prove existence of a formally integrable and completely integrable quantum super PDE, $(\hat{Higgs}) \subset \hat{(YM)}$, where for any point, initial condition, there exist solutions with mass-gap.\(^3\) We call such a constraint *Higgs quantum super PDE*. A global solution, $V \subset \hat{(YM)}$, crossing the boundary, (denoted $\hat{\text{Goldstone}}$), of $(\hat{Higgs})$ in $\hat{(YM)}$, *quantum Goldstone-boundary*, acquires (resp. loses) mass going inside, (resp. outside), $(\hat{Higgs})$. So that the quantum Goldstone-boundary can be considered as the quantum integral situs for mass-creation, or, vice versa, mass-destruction, according that one considers the solution going inside $(\hat{Higgs})$, or outgoing from $(\hat{Higgs})$. The stability of such solutions are studied and identified the corresponding stabilized quantum extended crystal super PDE, where all the smooth solutions have mass-gap and are stable at finite-times.

## 2. EXTENDED CRYSTAL SUPER PDE’s STABILITY IN $\mathcal{Q}_S$

In this section we resume some recent characterizations of quantum PDE’s as quantum extended crystal PDE’s, made by us in the framework of our geometric theory

\(^3\)Here and in the following, talking about quantum super PDE’s, we simply say formally integrable, (resp. completely integrable), instead of formally quantum superintegrable, (resp. completely quantum superintegrable). (Compare with previous works on the same subjects.)
of quantum PDE’s. Furthermore, we shall consider the stability of quantum super PDE’s in the framework of the geometric theory of quantum super PDE’s. We will follow the line just drawn in some our previous papers on this subject for commutative PDE’s, where we have interpreted stability of PDE’s on the ground of their integral bordism groups and related the quantum bordism of PDE’s to Ulam stability too.

**Definition 2.1.** We say that a quantum super PDE $\tilde{E}_k \subset \tilde{J}^k_{m|n}(W)$ is a quantum extended 0-crystal super PDE, if its weak integral bordism group $\Omega_{m-1|n-1,w}^{\tilde{E}_k}$ is zero.

**Theorem 2.2.** (Criterion to recognize quantum extended 0-crystal super PDE’s). Let $\tilde{E}_k \subset \tilde{J}^k_{m|n}(W)$ be a formally integrable and completely integrable quantum super PDE such that $W$ is contractible. If $m-1 \neq 0$ and $n-1 \neq 0$, then $\tilde{E}_k$ is a quantum extended 0-crystal super PDE.

**Proof.** In fact, one has the following isomorphisms, (see [84]):

\[
\Omega_{m-1|n-1,w}^{\tilde{E}_k} \cong H_{m-1|n-1}(W; A)
\]

\[
\cong (A_0 \otimes_{\mathbb{K}} H_{m-1}(W; \mathbb{K})) \oplus (A_1 \otimes_{\mathbb{K}} H_{n-1}(W; \mathbb{K})).
\]

Thus, when $W$ is contractible, and $m-1 \neq 0$, $n-1 \neq 0$, one has $H_{m-1}(W; \mathbb{K}) = H_{n-1}(W; \mathbb{K}) = 0$, hence we get $\Omega_{m-1|n-1,w}^{\tilde{E}_k} = 0$. 

**Theorem 2.3.** (Crystal structure of quantum super PDE’s). Let $\tilde{E}_k \subset \tilde{J}^k_{m|n}(W)$ be a formally integrable and completely integrable quantum super PDE. Then its integral bordism group $\Omega_{m-1|n-1}^{\tilde{E}_k}$ is an extension of some crystallographic subgroup $G \triangleleft G(d)$. We call $d$ the crystal dimension of $\tilde{E}_k$ and $G(d)$ its crystal structure or crystal group.

**Proof.** We resume here the proof given in [95]. The first step is to note that there is a relation between lower dimensions integral bordisms in a commutative PDE.

**Lemma 2.4.** (Relations between lower dimensions integral bordisms in commutative PDE’s). Let $E_k \subset J^k_{m|n}(W)$ be a PDE on the fiber bundle $\pi : W \to M$, $\dim W = m + n$, $\dim M = n$. Let $S C_p(E_k)$ be the set of all compact $p$-dimensional admissible integral smooth manifolds of $E_k$. The disjoint union gives an addition on $S C_p(E_k)$ with $\emptyset$ as the zero element. Let us consider the homomorphisms $\partial_p : S C_p(E_k) \to S C_{p-1}(E_k)$ that associates to any element $a \in S C_p(E_k)$ its boundary $\partial a = \partial_p(a)$. So we obtain the chain complex (2) of abelian groups (integral smooth bordisms chain complex):

\[
S C_n(E_k) \xrightarrow{\partial_n} S C_{n-1}(E_k) \xrightarrow{\partial_{n-1}} S C_{n-2}(E_k) \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_2} S C_0(E_k).
\]

4For general informations on Algebraic (Co)homology see, e.g., the following Refs.[1, 6, 7, 8, 10, 24, 32, 41, 42, 46, 48, 49, 50, 51, 52, 55, 56, 59, 100, 105, 106, 113, 114, 116, 117, 119, 120, 121, 125, 126]. For general informations on crystallography, as used in this paper, see, e.g., Refs.[19, 28, 62, 101, 107, 109, 115]. For the geometric theory of PDE’s, see, e.g., Refs.[9, 20, 21, 26, 36, 37, 38, 39, 40, 43, 45, 61, 64, 65, 66, 67, 99, 111, 112, 118]. For the Algebraic Topology of PDE’s, quantum PDE’s and quantum super PDE’s see Refs.[68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95]. See also the following Refs.[2, 3, 4, 5, 96, 97, 98], where interesting applications of the PDE’s Algebraic Topology are given.
Then the $p$-bordism groups $\Omega^E_k$, $0 < p < n$, can be represented by means of the homology of the chain complex (2).

**Proof.** Let us denote by $\{S^kC\bullet(E_k), \partial\bullet\}$ the chain complex in (2). Then, we can build the exact commutative diagram (3).

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & SB\bullet(E_k) & \longrightarrow & SZ\bullet(E_k) & \longrightarrow & SH\bullet(E_k) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
SC\bullet(E_k) & \longrightarrow & SC\bullet(E_k) & \longrightarrow & 0 & & & & \\
\downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & S\text{Bor}\bullet(E_k) & \longrightarrow & S\text{Cyc}\bullet(E_k) & \longrightarrow & 0 & & \\
\end{array}
\]

where

\[
\begin{cases}
SB\bullet(E_k) = \ker(\partial\bullet); \\
SZ\bullet(E_k) = \text{im}(\partial\bullet); \\
SH\bullet(E_k) = \frac{SZ\bullet(E_k)}{SB\bullet(E_k)};
\end{cases}
\]

\[
b \in [a] \in S\text{Bor}\bullet(E_k) \Rightarrow a - b = \partial c; \\
b \in [a] \in S\text{Cyc}\bullet(E_k) \Rightarrow \partial(a - b) = 0; \\
b \in [a] \in \Omega^E_k \Rightarrow \partial a = \partial b = 0
\]

Then from (3) it follows directly that $\Omega^E_k \cong SH_p(E_k)$, $0 < p < n$. \hfill \square

**Lemma 2.5.** (Relations between integral bordisms groups in commutative PDEs). One has the following canonical isomorphism:

\[
\mathbb{Z} \bigotimes_{\Omega^E_k} \mathbb{Z}^S\text{Bor}\bullet(E_k) \cong \mathbb{Z}^S\text{Cyc}\bullet(E_k).
\]

**Proof.** Follows directly from the extension of groups given at the bottom of the commutative exact diagram (3) and some properties between extension of groups. (See, e.g., [71].) \hfill \square

**Lemma 2.6.** (Relations between integral bordisms groups in commutative PDEs-2). If $H^2(S\text{Cyc}\bullet(E_k), \Omega^E_k) = 0$ one has the following canonical isomorphism:

\[
S\text{Bor}\bullet(E_k) \cong \Omega^E_k \times S\text{Cyc}\bullet(E_k).
\]

**Proof.** Follows directly from the extension of groups given at the bottom of the commutative exact diagram (3) and some properties between extension of groups. (See, e.g., [71].) \hfill \square

**Lemma 2.7.** (Integral ringoid of PDE). A ringoid is a structure $(A, +, \cdot)$, where $A$ is a set and $+$ is a binary operation such that $(A, +)$ is an abelian additive group with zero $0 \in A$; $\cdot$ is a partially binary operation, i.e., it is defined only for some couples $(a, b) \in A \times A$, such that it is associative, and distributive with respect to...
+, i.e., if \(a \cdot b\) and \(a \cdot c\) are defined, then it is defined also \(a \cdot (b + c) = a \cdot b + a \cdot c\). A graded ringoid is a set \(A = \bigoplus A_n\), where each \(A_n\) is an abelian additive group and there is a partial binary operation \(\cdot\), associative, and distributive with respect to +, such that if \(a \in A_n, b \in A_m\), then \(a \cdot b \in A_{m+n}\), whenever it is defined.

Let \(E_k \subset J^k_n(W)\) be a PDE, with \(\pi: W \to M\) a fiber bundle, \(\dim W = m + n\), \(\dim M = n\). Then the integral bordism groups \(\Omega^E_k\), \(0 \leq p \leq n - 1\), identify a graded ringoid \(\Omega^E_k\), that we call integral ringoid of \(E_k\), that is an extension of a graded ringoid contained in the nonoriented bordism ring \(\Omega_k\). One has the commutative diagram (6).

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{} & \text{Hom}_{\text{ringoid}}(\overline{K}^E_k; \mathbb{R}) & \xrightarrow{} & \text{Hom}_{\text{ringoid}}(\Omega^E_k; \mathbb{R}) & \xrightarrow{} & \text{Hom}_{\text{ringoid}}((n-1)\Omega^E_k; \mathbb{R}) & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & H_\ast(E_k) & & & & & & \\
\end{array}
\]

where \(H_\ast(E_k) \equiv \bigoplus_{0 \leq p \leq n-1} H_p(E_k)\), that allows us to represent differential conservation laws of order \(k\) by means of ringoid homomorphisms \(\Omega^E_k \to \mathbb{R}\).

Proof. Set

\[
\Omega^E_k \equiv \bigoplus_{0 \leq p \leq n-1} \Omega^E_p.
\]

Each \(\Omega^E_p\) are additive abelian groups, with addition induced by disjoint union, \(\sqcup\). Furthermore, there is a natural product induced by the cartesian product, i.e., \([X_1] \cdot [X_2] = [X_1 \times X_2] \in \Omega^E_{p_1+p_2}\), for \([X_i] \in \Omega^E_{p_i}, 0 \leq p_i \leq n - 1, i = 1, 2, 0 \leq p_1 + p_2 \leq n - 1\). This product it is not always defined for all couples \((X_1, X_2)\) of closed admissible integral manifolds \(X_i, i = 1, 2\), but only for ones such that \(X_1 \times X_2\) is a closed integral admissible manifold. Therefore \(\Omega^E_k\) is a graded ringoid. Set \((n-1)\Omega = \bigoplus_{0 \leq p \leq n-1} \Omega_p\). It has in a natural way a graded ringoid structure, with respect the same operations with respect to which \(\Omega_k\) is a graded ring. Furthermore, for any \(0 \leq p \leq n - 1\), one has the exact sequence (8), (see proof of Theorem 3.16 in [85]).

\[
0 \xrightarrow{} \overline{K}^E_k \xrightarrow{} \Omega^E_k \xrightarrow{} \Omega_p \xrightarrow{} 0
\]

As a by-product one has also the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{} & \overline{K}^E_k & \xrightarrow{} & \Omega^E_k & \xrightarrow{} & (n-1)\Omega_k & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & \Omega_k & & & & & & \\
\end{array}
\]

where \(\overline{K}^E_k \equiv \bigoplus_{0 \leq p \leq n-1} \overline{K}^E_p\).

A full \(p\)-conservation law is any function \(f: \Omega^E_k \to \mathbb{R}\), \(0 \leq p \leq n - 1\). These, identify elements of \(H_\ast(E_k) \equiv \bigoplus_{0 \leq p \leq n-1} H_p(E_k)\) in a natural way. In \(H_\ast(E_k)\) are
contained also ones identified by means of differential conservation laws of order \(k\), belonging to the following quotient space, \(\text{(space of characteristic integral q-forms on } E_k)\): \(\mathcal{J}(E_k)\) \(\equiv\) \(\bigoplus_{0 \leq q \leq n-1} \mathcal{J}(E_k)^q\), with
\[
\mathcal{J}(E_k)^q \equiv \frac{\Omega^q(E_k) \cap d^{-1}(C\Omega^{q+1}(E_k))}{d\Omega^{q-1}(E_k) \oplus \{C\Omega^{q}(E_k) \cap d^{-1}(C\Omega^{q+1}(E_k))\}}.
\]

Here, \(\Omega^q(E_k)\) is the space of smooth q-differential forms on \(E_k\) and \(C\Omega^q(E_k)\) is the space of Cartan q-forms on \(E_k\), that are zero on the Cartan distribution \(E_k\) of \(E_k\).

\(\beta \in C\Omega^q(E_k)\) iff \(\beta(\zeta_1, \ldots, \zeta_q) = 0\), for all \(\zeta_i \in C^\infty(E_k)\). Any \([\alpha] \in \mathcal{J}(E_k)^p\) identifies a ringoid homomorphism \(f[\alpha] : \Omega^E_k \to \mathbb{R}\). More precisely one has \(f[\alpha](\{[X_1]+[X_2]\}) = \alpha(X_1) + \alpha(X_2)\), for \([\alpha] \in \mathcal{J}(E_k)^p\), \([X_1], [X_2] \in \Omega^E_k\), and \(f[\alpha](\{[X_1] \cdot [X_2]\}) = \alpha(X_1) \cdot \alpha(X_2)\), for \([\alpha] = [\alpha_1]+[\alpha_2] \in \mathcal{J}(E_k)^p \oplus \mathcal{J}(E_k)^p\), \([X_1] \in \Omega_{p_1}, [X_2] \in \Omega_{p_2}\). \(\square\)

The next step is to extend above results to PDE’s in the category \(\mathcal{Q}_S\) of quantum supermanifolds.

**Lemma 2.8.** (Relations between lower order integral bigraded-bordisms in quantum super PDE’s). Let \(\hat{E}_k \subset \hat{j}_{m,n}(W)\) be a quantum super PDE on the fiber bundle \(\pi : W \to M\), \(\dim_B W = (m,n,r)\), \(\dim_A M = m,n, B = A \times E, E\) a quantum superalgebra that is also a \(Z\)-module, with \(Z = Z(A)\) the centre of \(A\). Let \(S^C_{p|q}(\hat{E}_k)\) be the set of all compact \(p|q\)-dimensional, (with respect to \(A\), admissible integral smooth manifolds of \(\hat{E}_k\), \(0 \leq p \leq m, 0 \leq q \leq n\). The disjoint union gives an addition on \(S^C_{p|q}(\hat{E}_k)\) with \(\emptyset\) as the zero element. Let us consider the homomorphisms \(\partial_{p|q} : S^C_{p|q}(\hat{E}_k) \to S^C_{p-1|q-1}(\hat{E}_k)\) that associates to any element \(a \in S^C_{p|q}(\hat{E}_k)\) its boundary \(\partial a = \partial_{p|q}(a)\). So we obtain the chain complex (11) of abelian groups (integral smooth bigraded-bordisms chain complex) of \(\hat{E}_k \subset \hat{j}_{m,n}(W)\).

\[
S^C_{m|n}(\hat{E}_k) \xrightarrow{\partial_{m|n}} S^C_{m-1|n-1}(\hat{E}_k) \xrightarrow{\partial_{m-1|n-1}} S^C_{m-2|n-2}(\hat{E}_k) \xrightarrow{\partial_{m-2|n-2}} \cdots \xrightarrow{\partial_{m-r|n-r}} S^C_{m-r|n-r}(\hat{E}_k)
\]

where \(r = \min\{m,n\}\). Then the \(p|q\)-integral bordism groups \(\Omega^E_{p|q}\), \((m-r) < p < m, (n-r) < q < n\), can be represented by means of the homology of the chain complex (11).

One has the following canonical isomorphism:
\[
\mathbb{Z} \bigotimes \mathbb{Z}^S \text{Bor}_{\bullet}^\bullet(\hat{E}_k) \cong \mathbb{Z}^S \text{Cyc}_{\bullet}^\bullet(\hat{E}_k).
\]

Furthermore, if \(H^2(S^C_{\text{Cyc}}^\bullet(\hat{E}_k); \Omega^E_{\bullet|\bullet}) = 0\) one has the following canonical isomorphism:
\[
S^\text{Bor}_{\bullet}^\bullet(\hat{E}_k) \cong \mathbb{Z}^S \times S^C_{\text{Cyc}}^\bullet(\hat{E}_k).
\]

\(^5\)The space of conservation laws of \(E_k\), \(\text{Cons}(E_k)\), can be identified with the spectral term \(E_1^{[0:n-1]}\) of the spectral sequence associated to the filtration induced in the graded algebra \(\Omega^*(E_\infty) \equiv \bigoplus_{q \geq 0} \Omega^q(E_\infty)\), by the subspaces \(C\Omega^*(E_\infty) \subset \Omega^*(E_\infty)\). (For abuse of language we shall call “conservation laws of k-order”, characteristic integral \((n-1)\)-forms too. Note that \(C\Omega^*(E_k) = 0\). See also Refs.[71, 73, 75, 82].)
Then the admissible integral quantum smooth manifolds of any element $0 \leq \pi$:  

Proof. The proof can be conducted similarly to the ones for Lemma 2.4, Lemma 2.5 and Lemma 2.6.  

Similarly we can prove the following lemma concerning the total analogue of the complex (11) too.

**Lemma 2.9.** (Relations between lower order integral total-bordisms in quantum super PDE's). Let $\hat{E}_k \subset \hat{J}^k_{m|n}(W)$ be a quantum super PDE on the fiber bundle $\pi : W \rightarrow M$, $\dim_B W = (m|n,r|s)$, $\dim_A M = m|n$, $B = A \times E$, $E$ a quantum superalgebra that is also a $Z$-module, with $Z = Z(A)$ the centre of $A$. Let $^S C_p(\hat{E}_k)$, $0 \leq p \leq m + n$, be the set of all compact $uv$-dimensional, (with respect to $A$), admissible integral quantum smooth manifolds of $\hat{E}_k$, such that $u + v = p$. The disjoint union gives an addition on $^S C_p(\hat{E}_k)$ with $\emptyset$ as the zero element. Thus we can write

$$^S C_p(\hat{E}_k) = \bigoplus_{u,v:u+v=p} ^S C_{u|v}(\hat{E}_k) = ^{Tot,}^S C_p(\hat{E}_k).$$

Let us consider the homomorphisms $\partial_p : ^S C_p(\hat{E}_k) \rightarrow ^S C_{p-1}(\hat{E}_k)$ that associates to any element $a \in ^S C_p(\hat{E}_k)$ its boundary $\partial a = \partial_p(a)$, i.e., one has:

$$\begin{align*}
\partial_p a &= \partial_p (a_p|0,1_{a-1}|1,1_2|2,\ldots,a_{0|p}) \\
&= (\partial_p a|0,\partial_{p-1}|1,1_{a-1}|1,\partial_{p-2}|2a_{p-2}|2,\ldots,\partial_{0|p}a_{0|p}) \\
e &\in \bigoplus_{u,v:u+v=p-1} ^S C_{u|v}(\hat{E}_k) = ^S C_{p-1}(\hat{E}_k).
\end{align*}$$

One has $\partial_{p-1} \circ \partial_p = 0$. So we get the chain complex (16) of abelian groups (integral smooth bigraded-bordisms chain complex) of $\hat{E}_k \subset \hat{J}^k_{m|n}(W)$.

$$^S C_n(\hat{E}_k) \xrightarrow{\partial_n} ^S C_{n-1}(\hat{E}_k) \xrightarrow{\partial_{n-1}} ^S C_{n-2}(\hat{E}_k) \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} ^S C_0(\hat{E}_k).$$

Then the $p$-integral total bordism groups $^S \Omega^{\hat{E}_k}_p$, $0 < p < m + n$, can be represented by means of the homology of the chain complex (16).

One has the following canonical isomorphism:

$$\mathbb{Z} \bigotimes \mathbb{Z}^S \text{Bor}_\bullet(\hat{E}_k) \equiv \mathbb{Z}^S \text{Cyc}_\bullet(\hat{E}_k).$$

Furthermore, if $H^2(^S \text{Cyc}_\bullet(\hat{E}_k), \Omega^{\hat{E}_k}_\bullet) = 0$ one has the following canonical isomorphism:

$$^S \text{Bor}_\bullet(\hat{E}_k) \cong \Omega^{\hat{E}_k}_\bullet \times ^S \text{Cyc}_\bullet(\hat{E}_k).$$

Proof. The proof is similar to the one of Lemma 2.8.  

**Lemma 2.10.** (Integral ringoid of PDE’s in $Q_S$ and quantum conservation laws). Let $\hat{E}_k \subset \hat{J}^k_{m|n}(W)$ be a PDE in the category $Q_S$ as defined in Lemma 2.9. Then, $\Omega^{\hat{E}_k}_\bullet = \bigoplus_{0 \leq p \leq m+n} \Omega^{\hat{E}_k}_p$, has a natural structure of graded ringoid, with respect to the (partial) binary operations similar to the commutative case. We call $\Omega^{\hat{E}_k}_\bullet$ the integral ringoid of $\hat{E}_k$. Furthermore, quantum conservation laws of order $k$, $\hat{f} \in \text{Map}(\Omega^{\hat{E}_k}_p, B_k) \equiv H_p(\hat{E}_k)$, can be projected on their classic limits $\hat{f} \mapsto \hat{f}_C \equiv$
\[ c \circ \hat{f} \in \text{Map}(\Omega_{\hat{E}_k}^{p|q}, \mathbb{K}) \equiv \text{H}_{p|q}(\hat{E}_k)_{\text{C}}. \] By passing to the corresponding total spaces, we get the exact commutative diagram (19).

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{H}_{p|q}(\hat{E}_k) \\
\downarrow & & \downarrow \\
\text{H}_{p|q}(\hat{E}_k)_{\text{C}} & \longrightarrow & 0 \\
\text{H}_*(\hat{E}_k) & \longrightarrow & \text{H}_*(\hat{E}_k)_{\text{C}} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\]

Moreover, graded ringoid homomorphisms \( \hat{h} \in \text{Hom}_{\text{ringoid}}(\Omega_{\hat{E}_k}, \mathbb{K}) \), can be identified by means of classic limit quantum conservation laws of \( \hat{E}_k \). One has the exact commutative diagram (20).

\[
\begin{array}{ccc}
0 & \longrightarrow & R\text{H}_*(\hat{E}_k) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{H}_*(\hat{E}_k)_{\text{C}}
\end{array}
\]

that defines a subalgebra \( R\text{H}_*(\hat{E}_k) \) of \( \text{H}_*(\hat{E}_k) \), whose elements we call rigid quantum conservation laws, and whose classic limit can be identified with ringoid homomorphisms \( \Omega_{\hat{E}_k} \rightarrow \mathbb{K} \). In particular, quantum conservation laws arising by full quantum differential form classes

\[
(21) \quad \{ [\alpha] \in \bigoplus_{p,q \geq 0} \hat{\mathcal{J}}(\hat{E}_k)^p|q : \hat{\mathcal{J}}(\hat{E}_k)^p|q = \frac{\Omega^p|q(\hat{E}_k) \cap d^{-1}(C\Omega^{p+1|q+1}(\hat{E}_k))}{d\Omega^{p-1|q-1}(\hat{E}_k) \oplus (C \Omega^{p+1|q+1}(\hat{E}_k))} \}
\]

belong to \( R\text{H}_*(\hat{E}_k) \).

**Proof.** The proof follows directly from above lemmas. (For details on spaces \( \hat{\mathcal{J}}(\hat{E}_k)^p|q \) see Refs. [77, 82, 84].) \( \square \)

Let us, now, denote \( \Omega_{\hat{E}_k}^{p|q} \) (or \( \Omega_{\hat{E}_k}^{c|p+q} \)), the classic limit of integral \((p,q)\)-bordism group of \( \hat{E}_k \), i.e., the \((p+q)\)-bordism group of classic limits of integral supermanifolds \( N \subset \hat{E}_k \), such that \( \dim_A N = p|q \).\(^6\) Furthermore, let us denote by \( \Omega_{\hat{E}_k}^{c|p+q} \) the classic limit of total integral \((p+q)\)-bordism group of \( \hat{E}_k \), i.e., the \((p+q)\)-bordism group of classic limits of integral supermanifolds \( N \subset \hat{E}_k \), such that \( \dim_A N = u|v \), with \( u + v = p + q \). One has the exact commutative diagram (22).

\(^6\)Let us recall that a \((p|q)\)-dimensional integral quantum supermanifold \( V \) of \( \hat{E}_k \), \( 0 \leq p \leq m \), \( 0 \leq q \leq n \), with boundary \( \partial V \) (or eventually with \( \partial V = \emptyset \)), we mean an element \( V \in C^{p|q}_{c\partial}(\hat{E}_{k+h}, A) \), \( h \geq 0 \), such that \( TV \subset \hat{E}_{k+h} \). So, if \( V = \sum_{i} a_i u_i + \sum_{j} b_j v_j \), \( a_i \in A_0 \), \( b_j \in A_1 \), one has \( \partial V = \sum_{i} (-1)^i a_i \partial_i u_i + \sum_{j} (-1)^j b_j \partial_j v_j \). (For more details see [84].)
Taking into account Theorem 3.6 in [84] we get a relation between $\Omega^k_{m-1|n-1}$ and the bordism group $\Omega_{m+n-2}$. In fact, we can see that there is a relation between integral bordism groups in quantum super PDEs and Reinhart integral bordism groups of commutative manifolds. More precisely, let $N_0, N_1 \subset \hat{E}_k \subset \hat{J}^k_{m|n}(W)$ be closed admissible integral quantum supermanifolds of a quantum super PDE $\hat{E}_k$, of dimension $(m-1|n-1)$ over $A$, such that $N_0 \cup N_1 = \partial V$, for some admissible integral quantum supermanifold $V \subset \hat{E}_k$, of dimension $(m|n)$ over $A$. Then $(N_0)_C \cup (N_1)_C = \partial V_C$: iff $(N_0)_C$ and $(N_1)_C$ have the same Stiefel-Whitney and Euler characteristic numbers. In fact, by denoting $\Omega^+_p$ the Reinhart $p$-bordism groups and $\Omega_p$ the $p$-bordism group for closed smooth finite dimensional manifolds respectively, one has the exact commutative diagram (23).

(23)

This has as a consequence that if $N_0 \cup N_1 = \partial V$, then $(N_0)_C \cup (N_1)_C = \partial V_C$: iff $(N_0)_C$ and $(N_1)_C$ have the same Stiefel-Whitney and Euler characteristic numbers. From above exact commutative diagram one has that $\Omega^k_{m-1|n-1}$ is an extension of a subgroup of $\Omega_{m+n-2}$.

Let us consider, now, the following lemmas.

**Lemma 2.11.** [87] Bordism groups, $\Omega_p$, relative to smooth manifolds can be considered as extensions of some crystallographic subgroup $G \triangleleft G(d)$.

**Lemma 2.12.** If the group $G$ is an extension of $H$, any subgroup $\hat{G} \triangleleft G$ is an extension of a subgroup $\hat{H} \triangleleft H$.

**Proof.** In fact $\hat{G}$ is an extension of $p(\hat{G}) \triangleleft H$, with respect to the following short exact sequence: $0 \rightarrow K \rightarrow G \xrightarrow{p} H \rightarrow 0$.

Therefore by using above two lemmas, we get also that $\Omega^k_{m-1|n-1}$ is an extension of some crystallographic subgroup $G \triangleleft G(d)$.

7Note that for $p + q = 3$ one has $K^+_3 = 0$, hence one has $\Omega^+_3 = \Omega_3$. 
Theorem 2.14. [95] Let $B_k$ be the model quantum superalgebra of $J^k_{m|n}(W)$, $k \geq 0$. (See [83, 84].) We denote also by $B_\infty = \lim_k B_k$. Let $\hat{E}_k \subset J^k_{m|n}(W)$ be a formally integrable and completely integrable quantum super PDE. Then, in the algebra $H_{m-1|n-1}(\hat{E}_k) \equiv \text{Map}(\Omega^k_{m-1|n-1}; B_k)$, Hopf quantum superalgebra of $\hat{E}_k$, there is a quantum sub-superalgebra, (crystal Hopf quantum superalgebra) of $\hat{E}_k$. On such an algebra we can represent the quantum superalgebra $B^{G(d)}$ associated to the quantum crystal supergroup $G(d)$ of $\hat{E}_k$. (This justifies the name.) We call quantum crystal conservation superlaws of $\hat{E}_k$ the elements of its quantum Hopf crystal superalgebra. Then, the obstruction to find global smooth solutions of $\hat{E}_k$, for integral boundaries with orientable classic limit, can be identified with the quotient $H_{m-1|n-1}(\hat{E}_{\infty})/B^{\Omega_{m+n-2}}_{\infty}$.

Definition 2.15. We define crystal obstruction of $\hat{E}_k$ the above quotient of algebras, and put: $\text{cry}(\hat{E}_k) = H_{m-1|n-1}(\hat{E}_{\infty})/B^{\Omega_{m+n-2}}_{\infty}$. We call quantum 0-crystal super PDE a quantum super PDE $\hat{E}_k \subset J^k_{m|n}(W)$ such that $\text{cry}(\hat{E}_k) = 0$.

Remark 2.16. A quantum extended 0-crystal super PDE $\hat{E}_k \subset J^k_{m|n}(W)$ does not necessitate to be a quantum 0-crystal super PDE. In fact $\hat{E}_k$ is an extended 0-crystal quantum super PDE if $\Omega^k_{m-1|n-1,w} = 0$. This does not necessarily implies that $\Omega^k_{m-1|n-1} = 0$. In fact, the different types of integral bordism groups of PDE’s in the category $\Omega_S$, are related by the following proposition.

Proposition 2.17. (Relations between integral bordism groups). [95][84] The different types of integral bordism groups for a quantum super PDE, are related by the exact commutative diagram reported in (24).

---

8We also adopt the notation $B_k(A)$ and $B_\infty(A)$, whether it is necessary to specify the starting original quantum super algebra $A$.

9Recall that with the term quantum Hopf superalgebra we mean an extension $A \longrightarrow C = A \otimes_\mathbb{Z} H \longrightarrow D \longrightarrow D/C \longrightarrow 0$, where $H$ is an Hopf $\mathbb{Z}$-algebra and $A$ is a quantum superalgebra. (For more details on generalized Hopf algebras, associated to PDE’s, see Refs.[72, 73, 84].)
One has the canonical isomorphisms:

\[
\begin{cases}
K_{m-1|n-1,w/(s,w)}^\hat{E}_k \cong K_{m-1|n-1,s}^\hat{E}_k \\
\Omega_{m-1|n-1}^\hat{E}_k/K_{m-1|n-1,s}^\hat{E}_k \cong \Omega_{m-1|n-1,s}^\hat{E}_k \\
\Omega_{m-1|n-1,s}^\hat{E}_k/K_{m-1|n-1,s,w}^\hat{E}_k \cong \Omega_{m-1|n-1,w}^\hat{E}_k \\
\Omega_{m-1|n-1,w}^\hat{E}_k/K_{m-1|n-1,w}^\hat{E}_k \cong \Omega_{m-1|n-1,\hat{E}_k}
\end{cases}
\]

Corollary 2.18. Let \( \hat{E}_k \subset J_{\hat{m}|\hat{n}}^k(W) \) be a quantum 0-crystal super PDE. Let \( N_0, N_1 \subset \hat{E}_k \) be two closed initial and final Cauchy data of \( \hat{E}_k \) such that \( X \equiv N_0 \cup N_1 \in [0] \in \Omega_{m-1|n-1}, \) and such that \( X_C \) is orientable. Then there exists a smooth solution \( V \subset \hat{E}_k \) such that \( \partial V = X. \)

Let us, now, revisit some definitions and results about stability of mappings and their relations with singularities of mappings, adapting them to the category \( \Omega_S. \)

Definition 2.19. Let \( X, \) (resp. \( Y \)), be a quantum supermanifold of dimension \( m|n, \) (resp. \( r|s \)), with respect to a quantum superalgebra \( A = A_0 \oplus A_1, \) (resp. \( B = B_0 \oplus B_1 \). We shall assume that the centre \( Z = Z(A) \) of \( A, \) acts on \( B \) that becomes a \( Z \)-module.\(^{10} \) Let \( f \in Q^\infty_w(X,Y). \) Then \( f \) is stable if there is a neighborhood \( W_f \subset Q^\infty_w(X,Y) \) of \( f, \) in the natural Whitney-type topology of \( Q^\infty_w(X,Y), \) such that every \( W_f \) is contained in the orbit of \( f, \) via the action of the group \( \hat{Diff}(X) \times \hat{Diff}(Y). \)\(^{11} \)

This is equivalent to say that for any \( f' \in W_f \) there exist quantum diffeomorphisms \( g : X \to X \) and \( h : Y \to Y \) such that \( h \circ f = f' \circ g. \) Furthermore, \( f \) is called infinitesimally stable if there exist a map \( \zeta : X \to TY, \) such that \( \pi_Y \circ \zeta = f, \) where \( \pi_Y : TY \to Y \) is the canonical map, and integrable vector fields \( \nu : Y \to TY, \)

\(^{10} \)In the following, whether it is not differently specified, \( X \) and \( Y \) are such quantum supermanifolds.

\(^{11} \)Here \( \hat{Diff}(X) \) denotes the group of quantum diffeomorphisms of a quantum super manifold \( X. \)
\( \xi : X \to TX \), such that \( \zeta = T(f) \circ \xi + \nu \circ f \). Thus the diagram (26) is commutative.

(26)

\[
\begin{array}{ccc}
TX & \xrightarrow{T(f)} & TY \\
\downarrow{\xi} & & \downarrow{\pi_Y} \\
X & \xrightarrow{f} & Y
\end{array}
\]

\[\xrightarrow{\pi_X} \]

\[\xrightarrow{\nu} \]

Thus the diagram (26) is commutative.

Theorem 2.20. Let \( X \) be a compact quantum supermanifold and \( f : X \to Y \) be quantum smooth. Then \( f \) is stable iff \( f \) is infinitesimally stable. Furthermore, if \( f \) is a proper mapping, then does not necessitate assume that \( X \) is compact.\(^{12}\)

Proof. Note that the infinitesimal stability, requires existence of flows \( g_t : X \to X \), \( \partial g_t = \xi \), \( h_t : Y \to Y \), \( \partial h_t = \nu \), such that for the infinitesimal variation \( \zeta \) of \( f_t = h_t \circ f \circ g_t \) one has \( \zeta = T(f) \circ \xi + \nu \circ f \). In fact, one has the following lemma.

Lemma 2.21. Let \((W, \pi_W; B)\) be a bundle of geometric objects in the category \( Q^{\infty}_W \) and in the intrinsic sense \[63, 64\]. Let \( \phi : \mathbb{R} \times V \to V \) be a one-parameter group of \( Q^{\infty}_W \) transformations of \( V \), \( \xi = \partial \phi \) its infinitesimal generator and \( s : V \to W \) a field of geometric objects, i.e. a section of \( \pi_W \). Then, \( \phi \) induces a deformation \( \tilde{s} \) of \( s \) defined by means of the commutative diagram (27).

(27)

\[
\begin{array}{ccc}
\mathbb{R} \times \mathbb{R} \times V & \xrightarrow{\tilde{s}} & W \\
\downarrow{(id_\mathbb{R}, \phi)} & & \downarrow{\phi_\lambda} \\
\mathbb{R} \times V & \xrightarrow{(id_\mathbb{R}, s)} & \mathbb{R} \times W
\end{array}
\]

where \( \phi_\lambda \equiv B(\phi^{-1}_\lambda) \), \( \forall \lambda \in \mathbb{R} \). One has \( \tilde{s}(0,0) = s \). Then, for the infinitesimal variation of \( \tilde{s} \) (Lie derivative of \( s \) with respect to the integrable field \( \xi \)), \( \partial(\tilde{s} \circ d) : V \to s^*vTW \), one has:

(28)

\[\partial(\tilde{s} \circ d) = \partial(s \circ \phi) + \partial(\phi) \circ s = T(s) \circ \xi + \nu \circ s.\]

Proof. This lemma can be proved by copying the intrinsic proof for the commutative case given in \[63\].

In our case we can consider the following situation, with respect to Lemma 2.21, \( W \equiv X \times Y \), \( V \equiv X \), \( B(g_\lambda) = h_\lambda \) and \( s = (id_X, f) \).

Furthermore, in the case that \( X \) is compact, the proof follows the same lines of the proof given by Mather for commutative manifolds \[49\].

Theorem 2.22. Stable maps \( f : X \to Y \) do not necessitate to be dense in \( Q^{\infty}_W(X, Y) \).

Proof. This is just a corollary of the corresponding theorem for commutative manifolds given by Thom-Levine \[41, 42\].

\(^{12}\)Recall that a map \( f : X \to Y \) between topological spaces is a proper map if for every compact subset \( K \subset Y \), \( f^{-1}(K) \) is a compact subset of \( X \).

\(^{13}\)See also Refs.\[71\] for related subjects.
Example 2.23. (Submersions and stability). Let $X$ be a compact quantum supermanifold. Let $f : X \to Y$ be a quantum differentiable mapping of maximum possible super-rank. If $m \geq r > 1$, $n \geq s > 1$, $f$ is a quantum submersion and it is (infinitesimally) stable.

Example 2.24. (Immersions and stability). Let $X$ be a compact quantum supermanifold. Let $f : X \to Y$ be a quantum differentiable mapping of maximum possible super-rank. If $m \leq r$, $n \leq s$, $f$ is an immersion and if it is $1 : 1$ then it is also stable. (Not all immersions are stable.)

Definition 2.25. (Singular solutions of quantum super PDE’s). Let $\pi : W \to M$ be a fiber bundle, where $M$ is a quantum supermanifold of dimension $(m|n)$ on the quantum superalgebra $A$ and $W$ is a quantum supermanifold of dimension $(m,n,r|s)$ on the quantum superalgebra $B = A \times E$, where $E$ is also a $Z$-module, with $Z = \mathbb{Z}(A)$ the centre of $A$.

Let $E_k \subset J^D(W)$ be a quantum super PDE. By using the natural embedding $J^D(W) \subset \check{J}^k_{m|n}(W)$, we can consider quantum super PDEs $\check{E}_k \subset J^D(W)$ like quantum super PDEs $\hat{E}_k \subset J^k_{m|n}(W)$, hence we can consider solutions of $E_k$ as $(m|n)$-dimensional, (over $A$), quantum supermanifolds $V \subset \check{E}_k$ such that $V$ can be represented in the neighborhood of any of its points $q \in V$, except for a nowhere dense subset $\Sigma (V) \subset V$, of dimension $\leq (m-1)|n-1|$, as $N^{(b)}$, where $N^{(b)}$ is the $k$-quantum prolongation of a $(m|n)$-dimensional (over $A$) quantum supermanifold $N \subset W$. In the case that $\Sigma (V) = \emptyset$, we say that $V$ is a regular solution of $\check{E}_k \subset \check{J}^k_{m|n}(W)$.

Solutions $V$ of $\check{E}_k \subset \check{J}^k_{m|n}(W)$, even if regular ones, are not, in general diffeomorphic to their projections $\pi_k(V) \subset M$, hence are not representable by means of sections of $\pi : W \to M$. $\Sigma (V) \subset V$ is the singular points set of $V$. Then $V \setminus \Sigma (V) = \bigcup_r V_r$ is the disjoint union of connected components $V_r$.

For every of such components $\pi_{k,0} : V_r \to W$ is an immersion and can be represented by means of $k$-prolongation of some quantum supermanifold of dimension $m|n$ over $A$, contained in $W$. Whether we consider $\check{E}_k$ as contained in $J^D(W)$ then regular solutions are locally obtained as image of $k$-derivative of sections of $\pi : W \to M$. So we can (locally) represent such solutions by means of mapping $f : M \to E_k$, such that $f = D^k s$, for some section $s : M \to W$.

We shall also consider solutions of $\check{E}_k \subset \check{J}^k_{m|n}(W)$, any subset $V \subset \check{E}_k$, that can be obtained as projections of ones of the previous type, but contained in some $s$-prolongation $\check{E}_{k+s} \subset \check{J}^k_{m|n}(W)$, $s > 0$.

We define weak solutions, solutions $V \subset \check{E}_k$, such that the set $\Sigma (V)$ of singular points of $V$, contains also discontinuity points, $q,q' \in V$, with $\pi_{k,0}(q) = \pi_{k,0}(q') = a \in W$, or $\pi_k(q) = \pi_k(q') = p \in M$. We denote such a set by $\Sigma (V)_S \subset \Sigma (V)$, and, in such cases we shall talk more precisely of singular boundary of $V$, like $(\partial V)_S = \partial V \setminus \Sigma (V)_S$. However for abuse of notation we shall denote $(\partial V)_S$, (resp. $\Sigma (V)_S$), simply by $(\partial V)$, (resp. $\Sigma (V)$), also if no confusion can arise.

Definition 2.26. (Stable solutions of quantum super PDE’s). Let us consider a quantum super PDE $\hat{E}_k \subset J^D(W)$, and let us denote $\text{Std}(\hat{E}_k)$ the set of regular solutions of $\hat{E}_k$. This has a natural structure of locally convex manifold. Let $f : X \to E_k$ be a regular solution, where $X \subset M$ is a smooth $(m|n)$-dimensional compact manifold with boundary $\partial X$. Then $f$ is stable if there is a neighborhood
$W_f$ of $f$ in $\text{Sol}(\hat{E}_k)$, such that each $f' \in W_f$ is equivalent to $f$, i.e., $f$ is transformed in $f'$ by some integrable vertical symmetries of $\hat{E}_k$.

\begin{equation}
(D^k \alpha)^*: vT \hat{E}_k \to \hat{E}_k[\alpha] = (D^k \alpha)(vT \hat{E}_k) \sim \hat{E}_k[\alpha]
\end{equation}

**Theorem 2.27.** Let $\hat{E}_k \subset J\hat{D}^k(W)$ be a $k$-order quantum super PDE on the fiber bundle $\pi : W \to M$ in the category of quantum smooth supermanifolds. Let $s : M \to W$ be a section, solution of $\hat{E}_k$, and let $\nu : M \to s^*vTW \equiv \hat{E}[s]$ be an integrable solution of the linearized equation $\hat{E}_k[s] \subset J\hat{D}^k(\hat{E}[s])$. Then to $\nu$ it is associated a flow $\{\phi_\lambda\}_{\lambda \in J}$, where $J \subset \mathbb{R}$ is a neighborhood of $0 \in \mathbb{R}$, that transforms $V$ into a new solution $\hat{V} \subset \hat{E}_k$.

**Proof.** Let $(x^\alpha, y^I)$ be fibered coordinates on $W$. Let $\nu = \partial y_j(\nu^I) : M \to s^*vTW$ a vertical vector field on $W$ along the section $s : M \to W$. Then $\nu$ is a solution of $\hat{E}_k[s]$ iff the diagram (29) is commutative. Then $D^k\nu(p)$ identifies, for any $p \in M$, a vertical vector on $\hat{E}_k$ in the point $q = D^k\nu(p) \in V = D^k\nu(M) \subset \hat{E}_k$. On the other hand infinitesimal vertical symmetries on $\hat{E}_k$ are locally written in the form

\begin{equation}
\begin{cases}
\zeta = \sum_{0 \leq |\alpha| \leq k} \partial y^\alpha_j (Y^I_\alpha), & 0 = \zeta \cdot F = < dF, \left( \sum_{0 \leq r \leq k} \partial y^\alpha_1 \cdots \partial y^\alpha_r (Y^I_{\alpha_1 \cdots \alpha_r}) \right) > \\
Y^I_\alpha = Z^{(0)}_\alpha (Y^I), & Z^{(0)}_\alpha = \partial x_\alpha + \partial y^\alpha, \\
Y^I_{\alpha_1 \cdots \alpha_r} = Z^{(r)}_I (Y^I_{\alpha_1 \cdots \alpha_r}), & Z^{(r)}_I = Z^{(r-1)}_I + \partial y^\alpha_1 \cdots \partial y^\alpha_{\alpha_1 \cdots \alpha_r}
\end{cases}
\end{equation}

where $Y^I_\alpha \in Q^\infty_U (U \subset J\hat{D}^k(W); \hat{A}(E))$, $\hat{A}(E) \equiv Hom_Z (\hat{A} \otimes Z \cdots \otimes Z \hat{A}(E))$, $0 \leq |\alpha| \leq k$, $\partial y^\alpha_j (q) \in Hom_Z (\hat{A}(E); T_q \hat{D}^k(W))$, $y^I_{\alpha_1 \cdots \alpha_r} \in Q^\infty_U (U; \hat{A}(E))$. Then we can see that solutions of $\hat{E}_k[s]$ are vertical vector fields $\nu : M \to s^*vTW \equiv \hat{E}[s]$, such that their prolongations $D^k\nu = \zeta \circ D^k s$, for some vertical symmetry $\zeta$ of $\hat{E}_k$. Therefore, the flows of above integrable vertical vector fields, transform regular solutions $V$ of $\hat{E}_k$ into new solutions of $\hat{E}_k$. Solutions of the linearized equation $\hat{E}_k[s]$ give initial conditions for the determination of such vertical flows.

The following lemmas are also important to understand how the structure of solutions of $\hat{E}_k[s]$ are related to the vertical symmetries of $\hat{E}_k$. (For complementary informations on the contact structure of $J^k_{m,n}(W)$, see [83].)
Lemma 2.28. (Symmetries of horizontal \(k\)-order contact ideals). Let \(\tilde{J}_{m,n}^k(W)\) be a quantum \((k+1)\)-connection on \(W\), i.e., a \(Q_{m,n}^\infty\)-section of \(\pi_{k+1,k}\). (The restriction of \(\tilde{J}\) to \(\tilde{J}_{m,n}^k(W)\) is also called quantum \((k+1)\)-connection.)

Let \(\mathcal{H}_k(\cdot)\) be the quantum horizontal \(k\)-order contact ideal of \(\tilde{\Omega}^0(\tilde{J}_{m,n}^k(W))\) given by \(\mathcal{H}_k(\cdot) \equiv \tilde{\mathcal{C}}_{k+1}(W)\), where \(\tilde{\mathcal{C}}_{k+1}(W)\) is the contact ideal of \(\tilde{J}_{m,n}^{k+1}(W)\). Locally one can write \(\mathcal{H}_k(\cdot) = \langle \omega^j, \ldots, \omega^j_{\alpha_1\ldots\alpha_k-1}, \tilde{H}_j^1 \rangle_{\alpha_1\ldots\alpha_k}\), where

\[
\begin{align*}
\tilde{H}_j^{1(\alpha_1\ldots\alpha_k)} &\equiv \left[ \omega_{\alpha_1\ldots\alpha_k} \right] = \left[ (dy^j_{\alpha_1\ldots\alpha_k} - y^j_{\alpha_1\ldots\alpha_k\beta} dx^\beta) 
\right. \\
&= dy^j_{\alpha_1\ldots\alpha_k} - y^j_{\alpha_1\ldots\alpha_k\beta} dx^\beta \in \tilde{\Omega}^1(\tilde{J}_{m,n}^k(W)),
\end{align*}
\]

with \(\left[ y^j_{\alpha_1\ldots\alpha_k} \right] \equiv y^j_{\alpha_1,\ldots,\alpha_k} v^\alpha \in \tilde{\Omega}^0(\tilde{J}_{m,n}^k(W))\). \(\tilde{\mathcal{C}}_k(W)\) is a subideal of \(\mathcal{H}\). Then the quantum horizontal \(k\)-order Cartan distribution \(\mathcal{H}_k(\cdot) \subset TJ_{m,n}^k(W)\) (identified by a \((k+1)\)-connection \(\tilde{J}\)) is the Cauchy characteristic distribution associated to \(\mathcal{H}_k(\cdot)\).

\(\mathcal{D}(\mathcal{H}_k(\cdot))\) admits the following local (canonical) basis:

\[
\begin{align*}
\zeta_\alpha &= \partial x_\alpha + \partial y_j y^j_{\alpha_1\ldots\alpha_k} \\
&\quad + \cdots + \partial y_j y^j_{\alpha_1\ldots\alpha_k-1} y^j_{\alpha_1\ldots\alpha_k-1} + \partial y_j y^j_{\alpha_1\ldots\alpha_k} y^j_{\alpha_1\ldots\alpha_k},
\end{align*}
\]

For any quantum \((k+1)\)-connection \(\tilde{J}\) on \(W\), one has the following direct sum decompositions:

\[
\begin{align*}
E_{m,n}^k(W)_q \cong H_k(\cdot)_q \oplus H_{0,m,n}^k(S^k(T_nN); \nu_\alpha) \\
\tilde{\Omega}^1(\tilde{J}_{m,n}^k(W)) \cong \tilde{\Omega}^1(\tilde{J}_{m,n}^k(W))_v \oplus \mathcal{H}_k(\cdot)_v
\end{align*}
\]

with \(\nu \equiv \pi_{k,0}(q) \in W\), \(\gamma(q) = [N^k_{\alpha_1}]\), and \(H_k(\cdot)_q \equiv T_q \mathcal{H}_k(\cdot)_q \equiv \mathcal{H}_k(\cdot) \cap \tilde{\Omega}^1(\tilde{J}_{m,n}^k(W))_v\). The connection \(\tilde{J}\) is flat, i.e., with zero curvature, iff the differential ideal \(\mathcal{H}_k(\cdot)\) is closed, or equivalently, iff \(\mathcal{H}_k(\cdot)\) is involutive. If \(\tilde{J}\) is a flat quantum \((k+1)\)-connection on \(W\), then one has the following:

\[
\begin{align*}
\tilde{\mathcal{C}}_k(W) \subset \mathcal{H}_k(\cdot) &\quad \text{as a closed subideal} \\
\mathcal{H}_k(\cdot) &\equiv C_{\text{har}}(\mathcal{H}_k(\cdot)); \quad \text{char}(\mathcal{H}_k(\cdot)) \subset s(\mathcal{H}_k(\cdot)).
\end{align*}
\]

\(\tilde{J}_{m,n}^k(W)\) is foliated by regular solutions \(Z\) such that \(\mathcal{H}_k(\cdot)|_Z = \{0\}\). The leaves of the foliation are given in implicit form by the following equations: \(f^I(x^\alpha, y^j, \ldots, y^j_{\alpha_1\ldots\alpha_k}) = \kappa^I \in B_k, 1 \leq I \leq p + q, \dim J_{m,n}^k(W) - (p+q) = m|n\), where \(f^I\) represent a complete independent system of primitive integrals of the linear system of PDEs \((\zeta_\alpha, f) = 0, 1 \leq \alpha \leq m + n\), \(\zeta_\alpha\) is a basis (e.g., the canonical basis) of the horizontal
distribution $H_k(\cdot)$.\footnote{\textnormal{(local) section $s$ of $\pi$ identifies a flat (local) $(k+1)$-connection $\lambda_{\a_1\ldots\a_{k+1}} = (\partial x_{\a_1} \ldots \partial x_{\a_{k+1}}, s)$.)} Any $\zeta \in \mathfrak{g}(H_k(\cdot))$ has the following local representation:

\begin{equation}
\zeta = \zeta_{\alpha}(X^\alpha) + \partial y_j(Y^j) + \partial y_j^{\alpha_1\ldots\alpha_k}(\zeta_{\alpha}, \zeta_{\alpha_2}, Y^j) + \partial y_j^{\alpha_1\ldots\alpha_k}(\zeta_{\alpha_1} \ldots \zeta_{\alpha_k}, Y^j),
\end{equation}

for any choice of $x^\alpha \in Q^\infty(W \subset j^k_{m|n}(W), A)$, $1 \leq \alpha \leq m + n$, and $Y^j \in Q^\infty(W \subset j^k_{m|n}(W), E)$, $1 \leq j \leq r + s$, such that

\begin{equation}
(\zeta_{\alpha_1} \ldots \zeta_{\alpha_k}, Y^j) = (\partial y_j^{\gamma_1\ldots\gamma_k}, \zeta_{\gamma_1} \ldots \zeta_{\gamma_k}, Y^j).
\end{equation}

The space $\mathfrak{g}(\hat{\mathcal{H}}_k(\cdot))$ admits the following direct sum decomposition:

\begin{equation}
\mathfrak{g}(\hat{\mathcal{H}}_k(\cdot)) \cong \mathfrak{d}(H_k(\cdot)) \bigoplus \mathfrak{v}_k(\cdot),
\end{equation}

where $\mathfrak{v}_k(\cdot)$ is the collection of all vectors of the form

\begin{equation}
\xi = \zeta - \zeta_{\alpha}(X^\alpha) = \partial y_j(Y^j) + \partial y_j^{\alpha_1\beta}(\zeta_{\alpha}, Y^j) + \partial y_j^{\alpha_1\ldots\alpha_k}(\zeta_{\alpha_1} \ldots \zeta_{\alpha_k}, Y^j),
\end{equation}

for any choice of $Y^j \in Q^\infty(W \subset j^k_{m|n}(W), E)$, $1 \leq j \leq r + s$, such that conditions (34) are satisfied. $\mathfrak{g}(\hat{\mathcal{H}}_k(\cdot))$ is a Lie algebra that admits the subalgebra $\mathfrak{d}(H_k(\cdot))$ as an ideal.

The general local expression for the symmetries of the $(m|n)$-dimensional involutive Cartan distribution $\mathbb{E}^\infty(W) \subset Tj^\infty_{m|n}(W)$, can be also obtained by equations (33) with all $k > 0$, and forgetting conditions (34).\footnote{In fact the Cartan distribution on $j^\infty_{m|n}(W)$ can be considered an horizontal distribution induced by the canonical connection identified by the local canonical basis $\zeta_{\alpha} = \partial x_{\alpha} + \Sigma_{|\beta| \geq 0} y_{\alpha,\beta} \partial y_j^{\beta}$ just generating $\mathbb{E}^\infty_{m|n}(W)$.} So we get the following expression for $\zeta \in \mathfrak{g}(\mathbb{E}^\infty_{m|n}(W))$:

\begin{equation}
\zeta = \partial_{\alpha}(X^\alpha) + \sum_{r \geq 0} \partial y_j^{\alpha_1\ldots\alpha_r}(Y_{\alpha_1}^{\ldots}\alpha_r),
\end{equation}

\begin{equation}
\partial_{\alpha} = \partial x_{\alpha} + \sum_{r \geq 0} \partial y_j^{\alpha_1\ldots\alpha_r}(y_{\alpha_1}^{\ldots}\alpha_r),
\end{equation}

\begin{equation}
Y_{\alpha_1}^{\ldots}\alpha_r = (\partial_{\alpha_1} \ldots \partial_{\alpha_r} Y^j),
\end{equation}

\begin{equation}
Y^j \in Q^\infty(W \subset j^k_{m|n}(W), E), 1 \leq j \leq r + s.
\end{equation}

Then the canonical splitting $Tq^\infty_{m|n}(W) \cong (\mathbb{E}^\infty_{m|n}(W)) \bigoplus Tq^\infty_{j^k_{m|n}(W)}(W)$, $q \in j^k_{m|n}(W)$, gives the following splitting in $\mathfrak{g}(\mathbb{E}^\infty_{m|n}(W)) = \mathfrak{d}(\mathbb{E}^\infty_{m|n}(W)) \bigoplus \mathfrak{v}_k(\cdot)$, $\zeta = \zeta_{\alpha} + \zeta_{\nu}$, with $\zeta_{\alpha} = \partial_{\alpha}(X^\alpha)$ and $\zeta_{\nu} = \sum_{r \geq 0} \partial y_j^{\alpha_1\ldots\alpha_r}(Y_{\alpha_1}^{\ldots}\alpha_r)$, where $Y_{\alpha_1}^{\ldots}\alpha_r$ are given in (36).

\textbf{Definition 2.29.} Let $\hat{E}_k \subset j^k_{m|n}(W)$, where $\pi : W \rightarrow M$ is a fiber bundle, in the category of quantum smooth supermanifolds. We say that $\hat{E}_k$ is functionally stable if for any compact regular solution $V \subset \hat{E}_k$, such that $\partial V = N_0 \cup P \cup \hat{N}_1$ one has quantum solutions $\hat{V} \subset j^{k+s}(W)$, $s \geq 0$, such that $\pi_{k+s,0}(\hat{N}_0 \cup \hat{N}_1) = \pi_{k,0}(N_0 \cup N_1) \equiv X \subset W$, where $\partial \hat{V} = \hat{N}_0 \cup \hat{P} \cup \hat{N}_1$.\footnote{\textnormal{(local) section $s$ of $\pi$ identifies a flat (local) $(k+1)$-connection $\lambda_{\alpha_1\ldots\alpha_{k+1}} = (\partial x_{\alpha_1} \ldots \partial x_{\alpha_{k+1}}, s)$.)}
We call the set \( \Omega[V] \) of such solutions \( \tilde{V} \) the full quantum situs of \( V \). We call also each element \( \tilde{V} \in \Omega[V] \) a quantum fluctuation of \( V \).

**Definition 2.30.** We call infinitesimal bordism of a regular solution \( V \subset \hat{E}_k \subset J^D \hat{K}(W) \) an element \( \tilde{V} \in \Omega[V] \), defined in the proof of Theorem 2.27. We denote by \( \Omega_0[V] \subset \Omega[V] \) the set of infinitesimal bordisms of \( V \). We call \( \Omega_0[V] \) the infinitesimal situs of \( V \).

Let \( \hat{E}_k \subset \hat{j}^{k}_{m|n}(W) \), where \( \pi : W \to M \) is a fiber bundle, in the category of quantum smooth supermanifolds. We say that a regular solution \( V \subset \hat{E}_k \), \( \partial V = N_0 \cup P \cup N_1 \), is functionally stable if the infinitesimal situs \( \Omega_0[V] \subset \Omega[V] \) of \( V \) does not contain singular infinitesimal bordisms.

**Theorem 2.31.** Let \( \hat{E}_k \subset \hat{j}^{k}_{m|n}(W) \), where \( \pi : W \to M \) is a fiber bundle, in the category of quantum smooth supermanifolds. If \( \hat{E}_k \) is formally integrable and completely integrable, then it is functionally stable as well as Ulam-extended superstable. A regular solution \( V \subset \hat{E}_k \) is stable iff it is functionally stable.

**Proof.** In fact, if \( \hat{E}_k \) is formally integrable and completely integrable, we can consider, for any compact regular solution \( V \subset \hat{E}_k \), its \( s \)-th prolongation \( V^{(s)} \subset \hat{j}^{k+s}_{m|n}(W) \). Since one has the following short exact sequence

\[
\begin{array}{c}
\Omega^{(\hat{E}_k)_{+,s}}_m \longrightarrow \Omega^{(\hat{E}_k)_{+,s}}_m (\hat{E}_k)_{+,s} \longrightarrow 0
\end{array}
\]

where \( \Omega^{(\hat{E}_k)_{+,s}}_m \), (resp. \( \Omega^{(\hat{E}_k)_{+,s}}_m (\hat{E}_k)_{+,s} \)), is the integral bordism group, (resp. quantum bordism group),\(^1\) we get that there exists a solution \( \tilde{V} \subset \hat{j}^{k+s}_{m|n}(W) \) such that

\[
\begin{align*}
&\partial \tilde{V} = \tilde{N}_0 \cup \tilde{P} \cup \tilde{N}_1; \\
&\partial V^{(s)} = N^{(s)}_0 \cup P^{(s)} \cup N^{(s)}_1
\end{align*}
\]

Then, as a by-product we get also: \( \pi_{k+s,0}(\tilde{N}_0 \cup \tilde{N}_1) = \pi_{k,0}(N_0 \cup N_1) \subset W \). Therefore, \( \hat{E}_k \) is functionally stable. Furthermore, \( \hat{E}_k \) is also Ulam-extended superstable, since the integral bordism group \( \Omega^{\hat{E}_k}_{m-1|n-1} \) for smooth solutions and the integral bordism group \( \Omega^{\hat{E}_k}_{m-1|n-1,s} \) for singular solutions, are related by the following short exact sequence:

\[
\begin{array}{c}
0 \longrightarrow \hat{E}_k \longrightarrow \Omega^{\hat{E}_k}_{m-1|n-1,s} \longrightarrow \Omega^{\hat{E}_k}_{m-1|n-1} \longrightarrow 0
\end{array}
\]

This implies that in the neighborhood of each smooth solution there are singular solutions.

Finally a regular solution \( V \subset \hat{E}_k \) is stable iff the set of solutions of the corresponding linearized equation \( \hat{E}_k[V] \) does not contains singular solutions. But this is just the requirement that \( \Omega_0[V] \) does not contains singular solutions. Therefore, \( V \) is stable if it is functionally stable and vice versa. More precisely if \( f = D^{k+s} : X \to \hat{E}_k \)

\(^{18}\) Let us emphasize that to \( \Omega[V] \) belong also (non necessarily regular) solutions \( V' \subset E_k \) such that \( N_0 \cup N_1 = N_0' \cup N_1 \), where \( \partial V' = N_0' \cup P' \cup N_1' \).

\(^{19}\) Here the considered bordism groups are for admissible non-necessarily closed Cauchy hypersurfaces.
is a stable solution of $\hat{E}_k$, then there exists an open set $W_s \subset \text{Sol}(\hat{E}_k)$ such that for any $s' \in W_s$, $s'$ is equivalent to $s$.\(^{20}\) Let us consider the tangent space $T_s\text{Sol}(\hat{E}_k)$.

One has the following isomorphism
\[
T_s\text{Sol}(\hat{E}_k) \cong \left\{ \xi \in (Q^\infty_w)_0((D^k s)^* v T \hat{E}_k) \mid \exists \xi \in T_s Q^\infty_w(W), \xi = |k \circ D^k \xi| \right\} \cong \Omega_0[V]
\]
where $|k$ is the canonical isomorphism $(D^k s)^* v T \hat{E}_k \cong (D^k s)^* v T J \hat{E}_k$, and $V = D^k s(X) \subset \hat{E}_k$. Since $W_s$ is open in $\text{Sol}(\hat{E}_k)$, one has also the following isomorphism $T_s\text{Sol}(\hat{E}_k)$, thus also to $s'$ there correspond vector fields $\xi \in T_s W_s$ that must be regular ones, i.e., without singular points. Therefore $\Omega_0[V]$ cannot contain singular solutions, hence $V$ is functionally stable. Vice versa, if $V$ is functionally stable, then we can find an open neighborhood $W_s \subset \text{Sol}(\hat{E}_k)$ built by perturbing $V$ with all the flows induced by the regular vector fields belonging to $\Omega_0[V]$. This set is an open set of $W_s \subset \text{Sol}(\hat{E}_k)$ since its tangent space at any of its point $s'$ is isomorphic to $T_s\text{Sol}(\hat{E}_k)$, since this last is isomorphic to $\Omega_0[V]$. Furthermore, any two of such points of such an open set are equivalent since they can be related both to $s$ by local diffeomorphisms. Therefore, $V$ that is functionally stable, is also stable. \(\square\)

**Remark 2.32.** Let us emphasize that the definition of functionally stable quantum super PDE interprets in pure geometric way the definition of Ulam superstable functional equation just adapted to PDE’s.\(^{21}\)

**Definition 2.33.** We say that $\hat{E}_k \subset J\hat{D}^k(W)$ is a stable extended crystal quantum super PDE if it is an extended crystal quantum super PDE that is functionally stable and all its regular quantum smooth solutions are (functionally) stable.

We say that $\hat{E}_k \subset J\hat{D}^k(W)$ is a stabilizable extended crystal quantum super PDE if it is an extended crystal quantum super PDE and to $\hat{E}_k$ can be canonically associated a stable extended crystal quantum super PDE $^{(S)}\hat{E}_k \subset J\hat{D}^{k+r}(W)$. We call $^{(S)}\hat{E}_k$ just the stable extended crystal quantum super PDE of $\hat{E}_k$.

We have the following criteria for functional stability of solutions of quantum super PDE’s and to identify stable extended crystal quantum super PDE’s.

**Theorem 2.34.** (Functional stability criteria). Let $\hat{E}_k \subset J\hat{D}^k(W)$ be a $k$-order formally integrable and completely integrable quantum super PDE on the fiber bundle $\pi : W \rightarrow M$.

1) If the symbol $\hat{g}_k = 0$, then all the quantum smooth regular solutions $V \subset \hat{E}_k \subset J\hat{D}^k(W)$ are functionally stable, with respect to any non-weak perturbation. So $\hat{E}_k$ is a stable extended crystal.

2) If $\hat{E}_k$ is of finite type, i.e., $\hat{g}_{k+r} = 0$, for $r > 0$, then all the quantum smooth regular solutions $V \subset \hat{E}_{k+r} \subset J\hat{D}^{k+r}(W)$ are functionally stable, with respect to any non-weak perturbation. So $\hat{E}_k$ is a stabilizable extended crystal with stable extended crystal $^{(S)}\hat{E}_k = \hat{E}_{k+r}$.

3) If $V \subset (\hat{E}_k)_{(+)} \subset J\hat{D}^\infty(W)$ is a smooth regular solution, then $V$ is functionally stable, with respect to any non-weak perturbation. So any formally integrable end

---

\(^{20}\)Recall that $Q^\infty_w(W)$ has a natural structure of quantum smooth supermanifold modeled on locally convex topological vector fields. $\text{Sol}(\hat{E}_k)$ is a closed submanifold of $\hat{\text{Sol}}(\hat{E}_k) \subset Q^\infty_w(W)$. (For details see ref.[71].)

\(^{21}\)For informations on the Ulam stability see Refs.[33, 98, 122].
completely integrable quantum super PDE $\hat{E}_k \subset J^k(W)$, is a stabilizable quantum extended crystal PDE, with stable quantum extended crystal PDE $(S)\hat{E}_k = (\hat{E}_k)_{+\infty}$.

Proof. We shall use the following lemmas.

**Lemma 2.35.** Let $\hat{E}_k \subset J^k(W)$ be a formally integrable and completely integrable quantum super PDE the fiber bundle $\pi : W \to M$. Then for any quantum smooth regular solution $s : M \to W$, one has the following canonical isomorphism: $(\hat{E}_k[s])_{+h} \cong ((\hat{E}_k)_{+h})[s], \forall h \geq 1, \infty$.

**Proof.** In fact one has the following commutative diagram.

\[
\begin{align*}
(41) \quad \begin{cases}
(\hat{E}_k[s])_{+h} = J^h((D^k s)^* vT \hat{E}_k) \bigcap J^h s^* vTW \\
\cong (D^h s)^* vT J^h (\hat{E}_k) \bigcap (D^h s)^* vT J^h (W) \\
\cong (D^h s)^* vT ((\hat{E}_k)_{+h}) = ((\hat{E}_k)_{+h})[s].
\end{cases}
\]

\[\square\]

**Lemma 2.36.** Let $\hat{E}_k \subset J^k(W)$ be a formally integrable and completely integrable quantum super PDE the fiber bundle $\pi : W \to M$. Let $\hat{g}_k = 0$. Then also the prolonged equations $(\hat{E}_k)_{+r}, \forall r \geq 1, \infty$, have their symbols zero: $(\hat{g}_k)_{+r} = 0, \forall r \geq 1, \infty$.

**Proof.** In fact, from the definition of symbol and prolonged symbols, it follows that the prolonged symbols coincide with the symbols of the corresponding prolonged equations. \[\square\]

1) This follows from Lemma 2.35 and from the fact that if $\hat{g}_k = 0$ is also $\hat{g}_k[s] = 0$. This excludes that $\hat{E}_k[s]$ could have singular solutions. Furthermore, Lemma 2.36 excludes also that there are singular (nonweak) solutions in the prolonged equations $\hat{E}_k[s]_{+r}, \forall r \geq 1, \infty$.

2) If $\hat{E}_k$ is of finite type, with $\hat{g}_k[r] = 0$, then it is also $\hat{g}_k+r[s] = 0$. Then $\hat{E}_k+r[s]$ cannot have singular (nonweak) solutions.

3) $\hat{E}_\infty$ has zero symbol, hence also $\hat{E}_\infty[s]$ has zero symbol and cannot have singular (nonweak) solutions.

(No the proof follows the same lines drawn for commutative PDE’s.) \[\square\]

**Theorem 2.37.** (Functional stable solutions and $(k+1)$-connections). Let $\hat{E}_k \subset J^k(W)$ be a formally integrable and completely integrable quantum super PDE. Let $\hat{J}$ be a quantum flat $(k+1)$-connection, such that $\hat{J}|_{\hat{E}_k}$ is a $Q^w$-section of the affine fiber bundle $\pi_{k+1,k} : (\hat{E}_k)_{+1} \to \hat{E}_k$. Then, the sub-equation $\hat{E}_k \subset \hat{E}_k$ identified, by means of the ideal $\hat{J}(\hat{E}_k)$, is formally integrable and completely integrable sub-equation with zero symbol $\hat{g}_k$. Then $\hat{E}_k \subset \hat{E}_k$ is functionally stable and Ulam-extended superstable. Furthermore any regular quantum smooth solution $V \subset \hat{E}_k$ is also functionally stable in $\hat{E}_k$, with respect to any non weak perturbation.

**Proof.** In fact, one has the commutative diagram (42) of exact lines.
Furthermore, since $\hat{g}_k = 0$, $\hat{E}_k$ is of finite type, hence its smooth regular solutions are functionally stable. □

Taking into account the meaning that connections assume in any physical theory, we can give the following definition.

**Definition 2.38.** Let $\hat{E}_k \subset \hat{J}^k_{\mathbb{Z}_m,n}(W)$ be a formally integrable and completely integrable quantum super PDE. Let $\hat{E}_k$ be a flat quantum $(k+1)$-connection, such that $\hat{E}_k$ is a $Q^\infty$-section of the affine fiber bundle $\pi_{k+1} : (\hat{E}_{k+1})_+ \to \hat{E}_k$. We call the couple $(\hat{E}_k,\hat{E}_k)$ a polarized quantum super PDE. We call also polarized quantum super PDE, a couple $(\hat{E}_k,\hat{E}_k)$, where $\hat{E}_k \subset \hat{E}_k$, is defined in Theorem 2.37. We call $\hat{E}_k$ a polarization of $\hat{E}_k$.

**Corollary 2.39.** Any quantum smooth regular solutions of a polarization of a polarized couple $(\hat{E}_k,\hat{E}_k)$, is functionally stable, with respect to any non-weak perturbation.

**Theorem 2.40.** (Finite stable quantum extended crystal super PDE’s). Let $\hat{E}_k \subset \hat{J}^k_{\mathbb{Z}_m,n}(W)$ be a formally integrable and completely integrable quantum super PDE, such that the centre $Z(A)$ of the quantum superalgebra $A$, model for $M$, is Noetherian. Then, under suitable finite ellipticity conditions, there exists a stable extended crystal super PDE $(\hat{E}_k)_{k \in \mathbb{Z}}$ canonically associated to $\hat{E}_k$, i.e., $\hat{E}_k$ is a stabilizable extended crystal.

**Proof.** In fact, we can use the following lemma.

**Lemma 2.41.** (Finite stability criterion). Let $\hat{E}_k \subset \hat{J}^k_{\mathbb{Z}_m,n}(W)$ be a formally integrable and completely integrable quantum super PDE, such that the centre $Z(A)$ of the quantum superalgebra $A$, model for $M$, is Noetherian. Then there exists an integer $s_0$ such that, under suitable finite ellipticity conditions, any regular quantum smooth solution $V \subset (\hat{E}_k)_{k \in \mathbb{Z}}$ is functionally stable.

**Proof.** On the assumption that $Z(A)$ is Noetherian, the proof can be conducted by following the same lines of the commutative case. (See [88].)

Let us, now, use the hypothesis that $\hat{E}_k$ is formally integrable and completely integrable. Then all its regular quantum smooth solutions are all that of $(\hat{E}_k)_{k \in \mathbb{Z}}$. In fact, these are all the solutions of $(\hat{E}_k)_{\infty} \subset J\hat{D}^\infty(W)$. However, even if a smooth regular solution $V \subset \hat{E}_k$, and their $s_0$-prolongations, $V^{(s_0)} \subset (\hat{E}_k)_{k \in \mathbb{Z}}$, are equivalent as solutions, they cannot be considered equivalent from the stability point of view!!! In fact, $\hat{E}_k$ can admit singular solutions, instead for $(\hat{E}_k)_{k \in \mathbb{Z}}$ these
are forbidden. Therefore, for $\hat{E}_k(s)$ singular perturbations are possible, i.e. are possible infinitesimal vertical symmetries of $\hat{E}_k$, in a neighborhood of the solution $s$, having singular points. Instead for $(\hat{E}_k)_s^0[s]$ all solutions are without singular points, hence $s$ considered as solution of $(\hat{E}_k)_s^0$ necessitates to be functionally stable.

By conclusions, $\hat{E}_k$, under the finite ellipticity conditions is a stabilizable extended crystal quantum super PDE, and its stable extended crystal quantum super PDE is $(S)\hat{E}_k = (\hat{E}_k)_s^0$, for a suitable finite number $s_0$.

**Remark 2.42.** With respect to a quantum frame [77, 83, 84], we can consider the perturbation behaviours of global solutions for $t \to \infty$, where $t$ is the proper time of the quantum frame. Then, we can talk about asymptotic stability by reproducing similar situations for commutative PDE’s. (See Refs. [86, 91].) In particular we can consider the concept of ”averaged stability” also for solutions of quantum (super) PDE’s. With this respect, let us recall the following definition and properties of quantum (pseudo)Riemannian supermanifold given in [77, 85].

**Definition 2.43.** [77, 85] A quantum (pseudo)Riemannian supermanifold $(M, \hat{\Lambda})$ is a quantum supermanifold $M$ of dimension $(m | n)$ over a quantum superalgebra $A$, endowed with a $Q^\infty_\omega$ section $\hat{g} : M \to \text{Hom}_Z(TM \otimes_Z TM; A)$ such that the induced homomorphisms $T_pM \to (T_pM)^\ast$, $\forall p \in M$, are injective.

**Proposition 2.44.** [77, 85] In quantum coordinates $\hat{g}(p)$ is represented by a matrix $\hat{g}_{\alpha\beta}(p) \in \hat{A}_{00}(A) \times \hat{A}_{10}(A) \times \hat{A}_{01}(A) \times \hat{A}_{11}(A)$. The corresponding dual quantum metric gives $\hat{g}^{\alpha\beta}(p) \in \hat{A}^{00}(A) \times \hat{A}^{10}(A) \times \hat{A}^{01}(A) \times \hat{A}^{11}(A)$, with $\hat{A}^{ij}(A) = \text{Hom}_Z(A; A_i \otimes_Z A_j)$, $i, j \in \mathbb{Z}_2$, such that $\hat{g}_{\alpha\beta}(p)\hat{g}^{\alpha\beta}(p) = \delta^\alpha_\gamma \in \hat{A}$, $\hat{g}^{\alpha\beta}(p)\hat{g}_{\gamma\beta}(p) = \delta^\alpha_\gamma \in \text{Hom}_Z(A \otimes_Z A; A \otimes_Z A)$.

In fact we have the following definition.

**Definition 2.45.** Let $\hat{E}_k \subset J^k(W)$ be a formally integrable and completely integrable quantum super PDE on the fiber bundle $\pi : W \to M$, and let $V = D^k_s(M) \subset E_k$ be a regular smooth solution of $\hat{E}_k$. Let $\xi : M \to E_k[s]$ be the general solution of $E_k[s]$. Let us assume that there is an Euclidean structure on the fiber of $E_k[s] \to M$. Let $(\psi : \mathbb{R} \times N \to N; i : N \to M)$ be a quantum frame [77, 83, 84]. Then, we say that $V$ is average asymptotic stable, with respect to the quantum frame, if the function of time $p[i](t)$ defined by the formula:

\begin{equation}
(43) \quad p[i](t) = \frac{1}{2\text{vol}(B_t)} \int_{B_t} i^*\xi^2 \eta
\end{equation}

has the following behaviour: $< p[i](t) > = < p[i](0) > e^{-ct}$ for some real number $c > 0$. Here $B_t \equiv N_t \cap \text{supp}(i^*\xi^2)$, where $N = \bigcup_{t \in T} N_t$, is the fiber structure of $N$, over the proper-time of the quantum frame, and the cuspidated bracket $<$, $>$ denotes expectation value, (or evaluation with respect to any quantum state of the corresponding quantum (super)algebra). We call $\tau_0 = 1/c_0$ the characteristic stability time of the solution $V$. If $\tau_0 = \infty$ it means that $V$ is average unstable.\footnote{In the following, if there are not reasons of confusion, we shall call also stable solution a smooth regular solution of a quantum super PDE $\hat{E}_k \subset J^k(W)$ that is average asymptotic stable. In this paper, as specified in Lemma 2.8, we shall in general assume that the fiber bundle}
We have the following criterion of average asymptotic stability.

**Theorem 2.46.** (Criterion of average asymptotic stability). A regular global smooth solution $s$ of $\hat{E}_k$ is average stable, with respect to the quantum frame $(\psi : \mathbb{R} \times N \to N; t : N \to M)$, if the following conditions are satisfied:

\begin{equation}
<\mathbf{p}[i](t) > \leq -c <\mathbf{p}[i](t) >, \quad c \in \mathbb{R}^+, \forall t.
\end{equation}

where

\begin{equation}
\mathbf{p}[i](t) = \frac{1}{2 \text{vol}(B_t)} \int_{B_t} i^* \xi^2 \eta
\end{equation}

and

\begin{equation}
\dot{\mathbf{p}}[i](t) = \frac{1}{2 \text{vol}(B_t)} \int_{B_t} \left( \frac{\delta i^* \xi^2}{\delta t} \right) \eta = \frac{1}{\text{vol}(B_t)} \int_{B_t} \left( \frac{\delta i^* \xi}{\delta t} \xi^2 \right) \eta.
\end{equation}

Here $i^* \xi$ represents the integrable general solution of the linearized equation $\hat{E}_k[s[i]$ of $\hat{E}_k$ at the solution $s$, and with respect to the quantum frame. Let us denote by $c_0$ the infimum of the positive constants $c$ such that inequality (44) is satisfied. Then we call $\tau_0 = 1/c_0$ the characteristic stability time of the solution $V$. If $\tau_0 = \infty$ means that $V$ is unstable.

Furthermore, Let $s$ be a smooth regular solution of a formally integrable and completely integrable quantum super PDE $\hat{E}_k \subset J \hat{D}^k(W)$, where $\pi : W \to M$. There exists a differential operator $\mathcal{P}[s[i](\xi)]$, on $\hat{\pi} : E[s[i] \equiv i^* v_TW \to N$, canonically associated to the solution $s$, and with respect to the quantum frame, such that $s$ is average stable in $\hat{E}_k$, or in some suitable prolongation $(\hat{E}_k)+h, k + h = 2s \geq k$, if the following conditions are verified:

(i) $\mathcal{P}[s[i](\xi)]$ is self-adjoint (or symmetric) on the constraint

\begin{equation}
(\hat{E}_k)(_{+r})[s[i] \subset J \hat{D}^{k+r}(\hat{E}[s[i]),
\end{equation}

for some $r \geq 0$.

(ii) The smallest eigenvalue $\lambda_1 = \lambda_1(t)$ of $\mathcal{P}[s[i](\xi)]$ is positive for any $t \in T$ and lower bounded: $\lambda_1 \geq \lambda_1 > 0$.

Furthermore, average stability can be also translated into a variational problem constrained by $(\hat{E}_k)(_{+h})[s],$ for some $h \geq 0$, such that $k + h = 2s$.

Proof. We shall use Theorem 2.27 and the following lemma.

**Lemma 2.47.** (Grönwall’s lemma)[27] Suppose $f(t)$ is a real function whose derivative is bounded according to the following inequality: $\frac{df(t)}{dt} \leq g(t)f + h(t),$ for some real functions $g(t)$ and $h(t)$. Then, $f(t)$ is bounded pointwise in time according to $f(t) \leq f(0)e^{G(t)} + \int_{[0,t]} e^{G(t-s)}h(s)ds$, where $G(t) = \int_{[0,t]} g(r)dr$.

Then a sufficient condition for the solution $V$ stability, with respect to the quantum frame, is that inequality (44) should be satisfied. In fact it is enough to use Lemma 2.47 with $g(t) = -c$ and $h(t) = 0$, to have $<\mathbf{p}[i](t) > = <\mathbf{p}[i](0) > e^{-ct}$.

$\pi : W \to M$, in the category $\mathcal{D}_E$, has $\dim_{A} M = m|n$ and $\dim_{B} W = (m|n, r|s)$, with respect to a quantum superalgebra $B = A \times E$, where $E$ is a quantum superalgebra that is also a $Z$-module, with $Z = Z(A)$ the centre of $A$. Therefore, $i^* \xi^2$ is a $E$-valued function on $N$ and the expectation value $< i^* \xi^2 >$ is a numerical function on $N$. Similar remarks hold for $<\mathbf{p}[i](t) >$ and $<\dot{\mathbf{p}}[i](t) >$. 

$\tau_0$ has just the physical dimension of a time.
Furthermore, condition (44) is satisfied iff

\begin{equation}
I[\xi|i] \equiv -2 \int_{B_T} \left< \frac{\delta i^* \xi}{\delta t} + c i^* \xi, i^* \xi > \eta \right> \geq 0,
\end{equation}

for some constant \( c > 0 \) and for any integrable solution \( i^* \xi \) of \( \bar{E}_k|s|i \). (The large cuspidated brackets \( <,> \) in (48) denotes expectation value) So the problem is converted to study the spectrum of the differential operator, \( \mathcal{P}[s|i](\xi) \equiv \frac{\delta^2 i^* \xi}{\delta t^2} \), on \( \bar{E}[s|i] \to N \), constrained by \( (\bar{E}_k)(s)|s|i \), for some \( r \geq 0 \), since \( \mathcal{P}[s|i](\xi) \) is of order \( \geq k \). If this is self-adjoint, (or symmetric), it follows that it has real spectrum and the stability of the solution is related to the sign of the smallest eigenvalue.\(^{24}\)

If such an eigenvalue \( \lambda_1(t) \) is positive, \( \forall t \in T \), and \( \lambda_1 = \inf_{t \in T} \lambda_1(t) > 0 \), then the ratio \( < - \frac{\delta i^* \xi}{\delta t} | s > / < i^* \xi | s > \) is higher than a positive constant, hence the solution \( s \) is average stable. In fact, we get

\begin{equation}
\left\{ \begin{array}{l}
- \frac{\delta i^* \xi}{\delta t} - \lambda_1 i^* \xi = \int_{B_T} (\mathcal{P}[s|i](\xi) - \lambda_1 i^* \xi^2) \eta \\
= \int_{B_T} (\bar{\lambda}_1(t) - \lambda_1) i^* \xi^2 \eta = (\bar{\lambda}_1(t) - \lambda_1) \int_{B_T} i^* \xi^2 \eta
\end{array} \right.
\end{equation}

for any \( t \in T \). Thus we also have, (for any \( \int_{B_T} i^* \xi^2 \eta \neq 0 \),

\begin{equation}
\frac{< - \frac{\delta i^* \xi}{\delta t} | s > - \lambda_1 = (\bar{\lambda}_1(t) - \lambda_1) \geq 0 \Rightarrow \frac{< - \frac{\delta i^* \xi}{\delta t} | s >}{< i^* \xi | s >} \geq \lambda_1, \ \forall t \in T.
\end{equation}

So condition (44) is satisfied, hence the solution \( s \) is average stable. In order to complete the proof of Theorem 2.46, let us emphasize that in general \( \mathcal{P}[s|i](\xi) = \delta i^* \xi \) is not identified with the quantum Euler-Lagrange operator for some quantum Lagrangian. In fact, in general, the differential order of such an operator does not necessitate to be even. By the way, since \( \bar{E}_k \) is assumed formally integrable and completely integrable, we can identify any smooth solution \( V \subset \bar{E}_k \), with its \( h \)-prolongation \( V^{(h)} \subset J \bar{E}^{k+h}(W) \), such that \( k + h = 2s \). Thus the problem of average stability can be translated in a variational problem, constrained by solutions of \( (\bar{E}_k)^{(h)}|s|i \).

\begin{equation}
\left\{ \begin{array}{l}
- \frac{\delta i^* \xi}{\delta t} = 2 \lambda(t) i^* \xi, \quad F^i[s|i] = 0, \quad 0 \leq |\alpha| \leq h, \quad k + h = 2s
\end{array} \right\}_t=\text{const}
\end{equation}

on the fiber bundle \( \bar{\pi} : \bar{E}[s|i] \equiv i^*(s^*vTW) \to N \). Here \( F^i[s|i] = 0 \) are the equations encoding \( \bar{E}_k|s|i \). This can be made not only locally but also globally. In fact one has the following lemmas. (See for the terminology [80, 94] and references quoted there.)

**Lemma 2.48.** [94] Let \( \pi : W \to M \), a fiber bundle in the category \( \Omega_S \), \( \dim A M = m|n, \dim B W = (m|n, r|s) \). Let \( L : J^k_{m|n}(W) \to \hat{A} \) be a \( k \)-order quantum Lagrangian function and \( \theta = L \eta \in \bar{\Omega}^{m+n}_{m|n}(J^k_{m|n}(W)) \), locally given by \( \theta = Ldx^1 \Delta \cdots \Delta dx^{m+n} = \)

\(^{24}\) Really it should be enough to require that \( \mathcal{P}[s|i] \) is a symmetric operator in the Hilbert space \( \mathcal{H}_t \), canonically associated to \( \bar{E}[s]|s|_i \). In fact the point spectrum \( \mathcal{S}(A)_p \) of a symmetric linear operator \( A \) on \( \mathcal{H}_t \) is real \( \mathcal{S}(A)_p \subset \mathbb{R} \). (This is true also for its continuous spectrum: \( \mathcal{S}(A)_c \subset \mathbb{R} \).) In our case it is enough that \( \mathcal{P}[s|i] \) should symmetric on the space of \( \bar{E}_k|s|i \) solutions. However, it is well known in functional analysis that every symmetric operator has a self-adjoint extension, on a possibly larger space [16].
\[ l\mu_\ast \circ dx^1 \triangle \cdots \triangle dx^{m+n}, \] where \((x^a, y^i)\) are fibered quantum coordinates on \(W\), and \(\hat{\mu}_\ast : T_0^{m+n}(A) \rightarrow A\) is the \(Z\)-homomorphism induced by the product on \(A\). Then, extremals for \(\theta\), constrained by \(\hat{E}_k\), are solutions \(f : X \rightarrow \hat{E}_k\), with \(X\) a quantum supermanifold of dimension \(m|n\) with respect to \(A\), such that the following condition is satisfied:

\[ \left\langle \sum_{1 \leq j \leq r+s} \nu^j \sum_{0 \leq |i| \leq k} (-1)^{|i|} \partial_i \left( \frac{\partial L}{\partial y^j_i} \right), \eta, X \right\rangle = 0, \]

for any \(\nu = \nu^j \partial y^j\), solution of the linearized equation of \(\hat{E}_k\) at the solution \(s\). In particular, if \(\hat{E}_k = \hat{j}_k^{m|n}(W)\), then extremals are solutions of the following equation (quantum Euler-Lagrange super equation):

\[ \hat{E}[\theta] : \hat{j}_k^{2k}(W) : \left\{ \sum_{0 \leq |i| \leq k} (-1)^{|i|} \partial_i \left( \frac{\partial L}{\partial y^j_i} \right) = 0 \right\} \]

This completes the proof.

3. SUPERGRAVITY YANG-MILLS PDE's IN \(\Omega_S\)

In a previous paper we have encoded quantum supergravity as suitable quantum super Yang-Mills PDE’s. Nowadays, there are experimental evidences that nuclides can be considered as quark-gluon plasmas. For example, in order to justify spin-nuclides, it is not enough to consider them as simply made by quarks. In fact, collective effects appear necessary to justify nuclides properties. (See, e.g., Refs. [11, 12, 104].) With this respect, we shall encode nuclear nuclides with suitable quantum supergravity Yang-Mills PDE’s as formally introduced in [95]. Then stable nuclides are stable solutions with mass-gap. An existence theorem for solutions with mass-gap is given. A quantum super partial differential relation, \((\text{Goldstone})\), \((\text{quantum Goldstone-boundary})\), contained in a quantum super Yang-Mills equation, having the property to create, or destroy, mass is recognized and characterized. \((\text{Goldstone})\) bounds an open constraint \((\text{Higgs}) \subset (\hat{Y}M)\), where live all the solutions with mass-gap. A stable quantum super PDE, where all the smooth solutions have mass-gap and are stable in finite times, is obtained.

Let us introduce some fundamental geometric objects to encode quantum supergravity. (See also our previous works on this subjects that formulate quantum supergravity in the framework of our geometric theory of quantum super PDE’s [75, 80, 83, 90, 92].) The first geometric object to consider is an affine \(m\)-dimensional Minkowski space-time \((N, N, \alpha; g)\), where \(N\) is a \(m\)-dimensional \(\mathbb{R}\)-vector space, endowed with an hyperbolic metric \(g \in S^2(N)\), with signature \((+\ldots-\ldots-)\). \(\alpha : N \times N \rightarrow N\) is the translation mapping. Then, we consider also a quantum Riemannian \((\text{super})\)manifold, \((M, \hat{g})\), of dimension \(m\), \((m|n)\), with respect to a quantum algebra \(A\), where \(\hat{g} : M \rightarrow \text{Hom}_Z(T^2_M; A)\) is a quantum metric. We shall assume that \(M\) is locally quantum \((\text{super})\) Minkowskian, i.e., there is a \(Z\)-isomorphism, \((\text{quantum vierbein})\):

\[ \hat{\theta}(p) : T_pM \cong A \otimes_R N, \forall p \in M, \]
where $T_pM$ is the tangent space at $p \in M$ to $M$. Equivalently a quantum vierbein is a section $\hat{\theta} : M \to Hom_{R}(TM; E) \cong \hat{E} \otimes \hat{\mathcal{A}}(TM)^{+}$, where $E$ is the trivial fiber bundle $\hat{\mathcal{A}} : E \equiv M \times A \otimes_R \mathbb{N}$. Let us denote by $\hat{g}$ induces a $A$-valued scalar product, $\hat{g}$, on $A \otimes_R \mathbb{N}$, given by $\hat{g}(a \otimes u, b \otimes v) = ab \hat{g}(u, v) \in A$. By using the canonical splitting $\text{Hom}_{R}(\hat{T}_0^2(A \otimes_R \mathbb{N}; A) \equiv \text{Hom}_{R}(\hat{S}_0^2(A \otimes_R \mathbb{N}; A) \oplus \text{Hom}_{R}(\hat{\Lambda}_0^2(A \otimes_R \mathbb{N}; A)$, we get also the split representation $\hat{g} = \hat{g}_{(s)} + \hat{g}_{(a)}$. More precisely one has

$$\hat{g}_{a}(a \otimes u, b \otimes v) = [a, b]_+ \hat{g}(u, v), \quad \hat{g}_{a}(a \otimes u, b \otimes v) = [a, b]_- \hat{g}(u, v).$$

Furthermore, if $(e_{\alpha})$ is a basis in $\mathbb{N}$, and $(e^{\beta})$ is its dual, characterized by the conditions $e_{\alpha}e^{\beta} = \delta^{\beta}_{\alpha}$, let us denote respectively by $(\hat{1} \otimes e_{\alpha})$ and $((1 \otimes e^{\beta})^{\perp})$ the induced dual bases on the spaces $A \otimes_R \mathbb{N}$ and $(A \otimes_R \mathbb{N})^{+}$ respectively. Then one has the following representations

$$\hat{\mathcal{A}}_{a \beta}(1 \otimes e^{\alpha})^{\perp} \otimes (1 \otimes e^{\beta})^{\perp}, \quad \hat{\mathcal{A}}_{a \beta} \in \hat{\mathcal{A}}$$

$$\hat{\mathcal{A}}_{\alpha \beta}(1 \otimes e^{\alpha})^{\perp} \bullet (1 \otimes e^{\beta})^{\perp}, \quad \hat{\mathcal{A}}_{\alpha \beta} \in \hat{\mathcal{A}}$$

By means of the isomorphism $\hat{\theta}^{\otimes}$, we can induce on $M$ a quantum metric, i.e., the quantum Minkowskian metric of $M$, $\hat{g} = \hat{g} \circ \hat{\theta}^{\otimes}$. Conversely any quantum metric $\hat{g}$ on $M$, induces on the space $A \otimes_R \mathbb{N}$, scalar products, for any $p \in M$: $\hat{g}(p) = \hat{g}(p) \circ (\hat{\theta}^{\otimes}(p))^{-1}$. As a by-product, we get that any quantum metric $\hat{g}$ on $M$, induces a quantum metric on the fiber bundle $\hat{\mathcal{A}} : E \to M$, that we call the deformed quantum metrics of $\hat{\mathcal{A}} : E \to M$. Therefore, when we talk about locally Minkowskian quantum manifold $M$, we mean that on $M$ is defined a Minkowskian quantum metric. Since $\text{Hom}_{R}(TM; A \otimes_R \mathbb{N}) \equiv A \otimes_R \mathbb{N} \otimes_{\hat{\mathcal{A}}} (TM)^{+}$, we can locally represent a quantum vierbein in the following form:

$$\hat{\theta} = \hat{1} \otimes e_{\beta} \otimes \hat{\theta}_{\alpha}^{\beta} \, dx^{\alpha},$$

where $\hat{1} \otimes e_{\beta} \in \text{Hom}_{R}(A; A \otimes_R \mathbb{N})$, is the full quantum extension of a basis $(e_{\alpha})_{0 \leq \alpha \leq m-1}$ of $\mathbb{N}$, i.e., $\hat{1} \otimes e_{\beta}(a) = a \otimes e_{\beta}$. Furthermore, $\hat{\theta}_{a}^{\beta}(p) \in \hat{\mathcal{A}}$. Then, if $\zeta : M \to \hat{T}M \equiv \text{Hom}_{R}(A; TM)$ is a full quantum vector field on $M$, locally represented by $\zeta = \partial x_{\alpha}\zeta^{\alpha}$, we get that its local representation by means of quantum vierbein, is given by the following formula:

$$\hat{\theta}(\zeta) = \hat{1} \otimes e_{\beta} \hat{\theta}_{a}^{\beta} \zeta^{\alpha},$$

where the product is given by composition:

$$\begin{array}{ccc}
A & \xrightarrow{\zeta^{\alpha}} & A \\
\downarrow & \hat{\theta}_{a}^{\beta} \circ & \downarrow \hat{1} \otimes e_{\beta} \\
\Sigma_{\alpha, \beta} = & \hat{\theta} & (\zeta)
\end{array}$$

(For abuse of notation we can also denote $\hat{\theta}(\zeta)$ by $\zeta$ yet.) Whether $\hat{g} = \hat{g}_{a \beta} dx^{\alpha} \otimes dx^{\beta}$, is the quantum Minkowskian metric of $M$, then its local representation by means
of the quantum vierbein is the following:

\[
\begin{align*}
\tilde{g} &= g_{\alpha\omega} dx^\alpha \otimes dx^\omega \\
\tilde{g}_{\alpha\omega} &= \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma g_{\beta\gamma}, \quad \hat{g}_{\alpha\omega}(p) \in \mathbb{R}, \quad \forall p \in M.
\end{align*}
\]

where \(\hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma(p)\), can be identified with \(\hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma(p) \in \text{Hom}_Z(\mathcal{T}_0^2(A); \mathcal{T}_0^2(A))\). In fact, one has the following extension \(\hat{\theta}^\otimes\) of \(\hat{\theta}\):

\[
\begin{align*}
\hat{\theta}^\otimes &\in \text{Hom}_Z(TM \otimes TM; (A \otimes \mathbb{R} N) \otimes Z (A \otimes \mathbb{R} N)) \\
&\cong (A \otimes \mathbb{R} N) \otimes Z (A \otimes \mathbb{R} N) \otimes \mathcal{A}(TM \otimes TM)^	op.
\end{align*}
\]

Locally one can write

\[
\hat{\theta}^\otimes = (1 \otimes \epsilon_\gamma) \otimes (1 \otimes \epsilon_\omega) \otimes \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma \ dx^\alpha \otimes dx^\beta.
\]

In fact, we have

\[
\tilde{g}(\zeta, \xi) = \tilde{g}(1 \otimes \epsilon_\beta \hat{\theta}_\alpha^\beta \zeta^\alpha, 1 \otimes \epsilon_\gamma \hat{\theta}_\omega^\gamma \xi^\omega) = \hat{\theta}_\alpha^\beta \zeta^\alpha \hat{\theta}_\omega^\gamma \xi^\omega \ g(e_\beta, e_\gamma) = \hat{\theta}_\alpha^\beta \zeta^\alpha \hat{\theta}_\omega^\gamma \xi^\omega \ g_{\beta\gamma}.
\]

In the particular case that \((e_\beta)\) is an orthonormal basis, then we get the following quantum Minkowskian representation for \(\tilde{g}\)

\[
\begin{align*}
\tilde{g}_{\alpha\omega} &= \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma \eta_{\beta\gamma}, \quad \langle \eta_{\beta\gamma} \rangle = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & -1
\end{pmatrix}.
\end{align*}
\]

The splitting in symmetric and skew-symmetric part of \(\tilde{g}\), i.e.,

\[
\tilde{g} = \tilde{g}(s) + \tilde{g}(a) = \tilde{g}_{\alpha\beta} \ dx^\alpha \otimes dx^\beta + \tilde{g}_{\alpha\beta} \ dx^\alpha \triangledown dx^\beta
\]

can be written in term of quantum vierbein in the following way:

\[
\begin{align*}
\hat{\theta}^\otimes &= \hat{\theta}^\otimes + \hat{\theta}^\wedge \\
\hat{\theta}^\otimes &= (1 \otimes \epsilon_\gamma) \otimes (1 \otimes \epsilon_\omega) \otimes \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma \ dx^\alpha \otimes dx^\beta \\
\hat{\theta}^\wedge &= (1 \otimes \epsilon_\gamma) \otimes (1 \otimes \epsilon_\omega) \otimes \hat{\theta}_\alpha^\beta \ dx^\alpha \triangledown dx^\beta \\
\tilde{g}(s)(\zeta, \xi) &= [\hat{\theta}_{\beta\gamma}^\zeta \otimes \hat{\theta}_\omega^\gamma \xi^\omega] + \tilde{g}_{\beta\gamma}, \ \Rightarrow \ \tilde{g}(s)_{\alpha\omega} = \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma \tilde{g}_{\beta\gamma} \\
\tilde{g}(a)(\zeta, \xi) &= \hat{\theta}_\alpha^\beta \hat{\theta}_\omega^\gamma \tilde{g}_{\beta\gamma} - \tilde{g}_{\beta\gamma}, \ \Rightarrow \ \tilde{g}(a)_{\alpha\omega} = \hat{\theta}_\alpha^\beta \triangledown \hat{\theta}_\omega^\gamma \tilde{g}_{\beta\gamma}.
\end{align*}
\]

Conversely, the local expression of the quantum deformed metrics on \(\tilde{\pi} : E \to M\), induced by a quantum metrics \(\tilde{\theta}\) on \(M\), is given by the following formulas:

\[
\begin{align*}
(\hat{\theta}^\otimes)^{-1} &= (\hat{\theta}^\otimes)^{-1} + (\hat{\theta}^\wedge)^{-1} \\
(\hat{\theta}^\otimes)^{-1} &= \partial_x^\alpha \otimes \partial_x^\beta \otimes \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma (1 \otimes \epsilon^\gamma)^+ \otimes (1 \otimes \epsilon^\omega)^+ \\
(\hat{\theta}^\wedge)^{-1} &= \partial_x^\alpha \triangledown \partial_x^\beta \otimes \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma (1 \otimes \epsilon^\gamma)^+ \triangle (1 \otimes \epsilon^\omega)^+ \\
\tilde{g}(s)(\zeta^\alpha \otimes \epsilon_\alpha, \xi^\beta \otimes \epsilon_\beta) &= \tilde{g}_{\alpha\omega} \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma \xi^\beta, \ \Rightarrow \ \tilde{g}(s)^{\alpha\beta} = \tilde{g}_{\alpha\omega} \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma \\
\tilde{g}(a)(\zeta^\alpha \otimes \epsilon_\alpha, \xi^\beta \otimes \epsilon_\beta) &= \tilde{g}(s)^{\alpha\omega} \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma + \tilde{g}_{\alpha\omega} \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma, \ \Rightarrow \ \tilde{g}(a)^{\alpha\beta} = \tilde{g}(s)^{\alpha\omega} \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma.
\end{align*}
\]

In the particular case that \(\tilde{g}\) is Minkowskian, then we can use for \(\tilde{g}_{\gamma\omega}\), \(\tilde{g}(s)^{\gamma\omega}\) and \(\tilde{g}(a)^{\gamma\omega}\) the corresponding expressions in (60), and by using the property that
\[ \hat{\theta}^a \hat{\theta}^b = \delta^b_a, \] we get \( \hat{g}_{\gamma \omega} = \hat{\theta}^\gamma_{\omega}, \hat{g}_{(\sigma)\gamma \omega} = \hat{\theta}^\sigma_{\gamma \omega} \) and \( \hat{g}_{(\sigma)\gamma \omega} = \hat{\theta}^\sigma_{\gamma \omega} \). The contra-variant full quantum metric \( \hat{g} \) of \( \hat{g} : M \to \text{Hom}(\hat{T}_0^2 M; A) \) is a section \( \hat{g} : M \to \text{Hom}_{\hat{Z}}(\hat{A}; \hat{T}_0^2 M) \) such that the following conditions are satisfied:

\[ \begin{aligned}
\hat{g} = \partial_x \otimes \partial_x \hat{g}^{\alpha \beta}, & \quad \hat{g} = \hat{g}_{\omega \gamma}(dx^\gamma \otimes dx^\omega), & \quad \hat{g}^{\alpha \beta}(p) \in \text{Hom}_{\hat{Z}}(A; A \otimes \hat{Z} A), & \quad \hat{g}_{\alpha \beta}(p) \in \hat{A} = \hat{g}_{\alpha \beta}(p) \in \hat{Z} A, & \quad \hat{g}^{\alpha \beta}(p) \hat{g}_{\gamma \omega}(p) = \delta^\alpha_\beta \in \hat{Z} A. & \quad \hat{g}^{\alpha \beta}(p) \hat{g}_{\gamma \omega}(p) = \delta^\alpha_\beta \in \hat{Z} A. \\
\end{aligned} \]

The products in (68) are meant by composition:

\[ \begin{array}{c}
\begin{array}{c}
A \xrightarrow{g^{\alpha \beta}(p)} A \otimes Z A \quad A \xrightarrow{\hat{g}_{\alpha \beta}(p)} A \quad A \xrightarrow{\hat{g}^{\alpha \beta}(p)} A \otimes Z A.
\end{array}
\end{array} \]

In the commutative diagram (70) it is shown the pairing working between the fiber bundles \( (\hat{T}_0^2 M)^+ \) and \( \hat{T}_0^2 M \) over \( M \).

\[ \begin{array}{c}
\begin{array}{c}
\hat{T}_0^2 M \times_M (\hat{T}_0^2 M)^+ \xrightarrow{<,>} \hat{A} \xrightarrow{\text{pr}_2} \hat{T}_0^2 M \otimes (\hat{T}_0^2 M)^+ \xrightarrow{\text{tr}} M \times \hat{A}
\end{array}
\end{array} \]

In particular, one has:

\[ \begin{array}{c}
\begin{array}{c}
1 \quad \frac{1}{s} < \hat{g}, \hat{g} > = \frac{1}{s} g^{\alpha \beta} g_{\alpha \beta} = \frac{1}{s} \delta^\alpha_\beta \delta^\beta_\alpha = 1, & \quad s = \begin{cases} m, & \text{dim}_A M = m \\
 m + n, & \text{dim}_A M = m|n. \end{cases}
\end{array}
\end{array} \]

It is direct to verify that \( \hat{g}^{\alpha \beta} = \hat{\theta}^\alpha_\omega \otimes \hat{\theta}^\beta_\omega \hat{g}^{\omega \gamma} \) is the contra-variant expression of the full quantum metric \( \hat{g}_{\alpha \beta} = \hat{\theta}^\alpha_\gamma \otimes \hat{\theta}^\beta_\gamma \hat{g}^{\gamma \omega} \), when \( \hat{g}^{\omega \gamma} \) is the contra-variant one of \( \hat{g}_{\gamma \omega} \). In other words if \( \hat{g}_{\alpha \beta} \hat{g}^{\omega \gamma} = \delta^\alpha_\beta, \) then \( g_{\alpha \beta} g^{\alpha \beta} = \delta^\beta_\gamma \). This means that the full quantum metric \( \hat{g} \), induced on \( A \otimes \hat{N} \) by \( \hat{g} \), is not degenerate, i.e. one has the following short exact sequence:

\[ \begin{array}{c}
\begin{array}{c}
0 \quad A \otimes \hat{N} \xrightarrow{\hat{g}} (A \otimes \hat{N})^+.
\end{array}
\end{array} \]

In fact, one can see that \( \text{ker}(\hat{g}) = \{ 0 \} \). Really, \( \hat{g}(a \otimes v)(b \otimes u) = abg(v, u) = 0 \), for all \( b \in A \) and \( u \in \hat{N} \) if \( a = 0 \) or \( v = 0 \). In fact we can take \( b = 1 \) and \( u = 0 \) any vector of \( \hat{N} \). So, since \( \hat{g} \) is not degenerate, it follows that cannot be \( g(v, u) = 0 \), for a non zero \( v \), and \( \forall u \in \hat{N} \). The nondegeneration of \( \hat{g} \) induces also the following isomorphism \( \hat{A} \otimes \hat{N} \cong (A \otimes \hat{N})^+ \).

**Definition 3.1.** (Quantum SG-Yang-Mills PDE’s). A quantum supergravity Yang-Mills PDE, (quantum SG-Yang-Mills PDE), is a quantum super Yang-Mills PDE where the quantum super Lie algebra \( \mathfrak{g} \) in the configuration bundle \( \pi : W \equiv \text{Hom}_{\hat{Z}}(TM; \hat{g}) \rightarrow M \) is a quantum superextension of the Poincaré Lie algebra and admits the following splitting of vector spaces:

\[ \begin{aligned}
\mathfrak{g} = \mathfrak{g}_@ + \mathfrak{g}_@ + \mathfrak{g}_*,
\end{aligned} \]

where \( \mathfrak{g}_@ = A \otimes \hat{N}, \) (resp. \( \mathfrak{g}_@ \) is the quantum superextension of the Lorentz part of the Poincaré algebra). Here \( A \) is a quantum (super)algebra on which is modeled the
quantum (super)manifold $M$, and $N$ is the 4-dimensional Minkowsky vector space. Furthermore, one assumes that there exists a non-degenerate metric $\mathfrak{g}$ on $\mathfrak{g}$. Taking into account the canonical splitting:

$$ (74) \quad \text{Hom}_Z(TM; \mathfrak{g}) \cong \text{Hom}_Z(TM; \mathfrak{g}_\oplus) \times \text{Hom}_Z(TM; \mathfrak{g}_\ominus) \times \text{Hom}_Z(TM; \mathfrak{g}_\Phi) $$

we get that the fundamental field $\hat{\mu} : M \to W$, in a quantum supergravity Yang-Mills PDE, admits the following canonical splitting:

$$ (75) \quad \hat{\mu} = \hat{\mu}_\oplus + \hat{\mu}_\ominus + \hat{\mu}_\Phi. $$

**Definition 3.2.** We say that $\hat{\mu}$ is non-degenerate if $\hat{\mu}_\oplus$ identifies, for any $p \in M$, an isomorphism $\hat{\mu}_\oplus(p) : T_pM \cong A \otimes \mathbb{R} N$, hence $\hat{\mu}_\oplus$ can be identified with a quantum vierbein on $M$: $\hat{\mu}_\oplus \equiv \hat{\theta}$. Then we define $\hat{\mu}_\ominus$, (resp. $\hat{\mu}_\ominus$, resp. $\hat{\mu}_\Phi$), the vierbein-component, (resp. Lorentz-component, resp. deviatory-component), of $\hat{\mu}$.

**Definition 3.3.** Similarly to the quantum connection $\hat{\mu}$, i.e., the fundamental quantum field, we get the following splitting of the quantum curvature:

$$ (76) \quad \hat{R} = \oplus \hat{R} + \ominus \hat{R} + \Phi \hat{R}. $$

We call also quantum torsion the component $\oplus \hat{R}$ of the quantum curvature. With this respect the $\oplus \mu$-component of the quantum field equation is also called quantum torsion equation.

**Theorem 3.4.** (SG-Yang-Mills PDE's) The dynamic equation $(\hat{YM})$ for a second order SG-Yang-Mills PDE assumes in quantum coordinates the expression reported in Tab.1. In Tab.2 there is also its unified expression.

---

**Tab.1 - Quantum Dynamic Equation $(\hat{YM}) \subset J \bar{D}^2(W)$ and Quantum Bianchi identity.**

| Quantum Fields Equation | $(\partial \oplus \hat{\mu}_A^L, L) - \partial B (\partial \oplus \hat{\mu}_B^L, L) = 0$ (Quantum curvature-Lorentz equation) |
|-------------------------|------------------------------------------------------------------------------------------------------------------|
|                         | $(\partial \ominus \hat{\mu}_A^L, L) - \partial B (\partial \ominus \hat{\mu}_B^L, L) = 0$ (Quantum curvature-vierbein equation) |
|                         | $(\partial \Phi \hat{\mu}_A^L, L) - \partial B (\partial \Phi \hat{\mu}_B^L, L) = 0$ (Quantum curvature-deviatory equation) |
| Quantum Bianchi Identity | $(\partial x_H \cdot R^K_{AB}) + \frac{1}{2} C^K_{IJ}[\hat{R}^I_{HL}, R^L_{AB}] = 0$ |
|                         | $(\partial x_H \cdot R^K_{AB}) + \frac{1}{2} C^K_{IJ}[\hat{\mu}^I_H, R^L_{AB}] = 0$ |
|                         | $(\partial x_H \cdot R^K_{AB}) + \frac{1}{2} C^K_{IJ}[\hat{\mu}^I_H, R^L_{AB}] = 0$ |
| Quantum Fields          | $\oplus R^K_{BA} = (\partial x_B \cdot \hat{\mu}_A^K) + \oplus C^K_{IJ}[\hat{\mu}_B^I, \hat{\mu}_A^K]$ (Quantum vierbein-curvature) |
|                         | $\ominus R^K_{BA} = (\partial x_B \cdot \hat{\mu}_A^K) + \ominus C^K_{IJ}[\hat{\mu}_B^I, \hat{\mu}_A^K]$ (Quantum Lorentz-curvature) |
|                         | $\Phi R^K_{BA} = (\partial x_B \cdot \Phi \hat{\mu}_A^K) + \Phi C^K_{IJ}[\hat{\mu}_B^I, \Phi \hat{\mu}_A^K]$ (Quantum deviatory-curvature) |

---

25We shall use also the following notation $\hat{\mu} = \oplus \hat{\mu} + \ominus \hat{\mu} + \Phi \hat{\mu}$, that can be useful when one must add some indexes, e.g. $\oplus \hat{\mu}_A^K$.

26For example for the case of quantum gravity-Yang-Mills PDE, corresponding to systems considered in Example 3.9, one obtains a quantum gravity with quantum torsion.
Proof. Let $(Z_K \in Hom_Z(A; \mathfrak{g}))$ be the basis for the quantum extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$. Let us denote $(Z_K) = (\otimes Z_R \otimes \otimes Z_S \otimes \otimes Z_T)$ the split induced by the one in (74). Similarly we get an induced notation on the quantum structure constants:

\[(77) \quad [Z_I, Z_J] = \otimes \hat{C}^K_{IJ} \otimes Z_K + \otimes \hat{C}^K_{IJ} \otimes Z_K + \otimes \hat{C}^K_{IJ} \otimes Z_K.\]

Then this property is represented, in local quantum coordinates, by the fact that in the following formula

\[(78) \quad \hat{\mu} = Z_K \otimes \hat{\mu}^K dA = \otimes Z_R \otimes \otimes \hat{\mu}^R dA + \otimes Z_S \otimes \otimes \hat{\mu}^S dA + \otimes \hat{\mu}^T dA\]

one has $(\otimes \hat{\mu}^K(p) \in GL(\hat{A}; 4))$.

The curvature, corresponding to $\hat{\mu}$, can be locally written in the form: $\hat{R} = Z_K \otimes \hat{R}^K_{AB} dx^A \wedge dx^B$, with $\hat{R}^K_{AB} = (\partial_x B \hat{\mu}^K) + \hat{C}^K_{IJ} [\hat{\mu}^I_B, \hat{\mu}^J_A]$. The quantum curvature also admits the splitting induced by the quantum lie algebra, as well as the corresponding Bianchi identities. Furthermore, we shall assume a first order quantum Lagrangian $L : JD(W) \rightarrow \hat{A}$, $L \circ Ds = \frac{1}{2} \hat{R}^K_{AB} \hat{R}^A_{BC}$, $\forall s \in Q_{\infty}^\infty(W)$. The local expression of $(\hat{Y} \hat{M})$ is given in Tab.2. Note that the quantum super Yang-Mills equation is now $\hat{\mu}^K_{\mu}, L = \hat{\mu}^{AB}_{\mu} L = 0$. Furthermore, it results $\hat{\mu}^K_{\mu}, L \equiv [\hat{C}^K_{AB} \hat{H}^C_{AB}], +$ and $(\partial \hat{\mu}^{AB}_{\mu}, L) = \hat{R}^K_{AB}$. (For more details on quantum gauge theories see also Refs. [71, 77, 85, 92, 94, 95].)

\[\square\]

Remark 3.5. So in a quantum SG-Yang-Mills PDE, the quantum Riemannian metric $\hat{\mathfrak{g}}$ is not a fundamental field, but a secondary field, obtained by means of the quantum vierbein $\hat{\theta} = \hat{\mu}_{\mathfrak{g}}$, that, instead is a fundamental dynamic field. Of course since there is a relation one-to-one between quantum vierbein and quantum metric, on a locally Minkowskian quantum (super)manifold, one can choose also quantum metric as a fundamental field, instead of the quantum vierbein. However, in a quantum SG-Yang-Mills PDE it is more natural to adopt quantum vierbein as independent field, since it is just enclosed in the fundamental field $\hat{\mu}$. The dynamic equation are resumed in Tab.2.

| Tab.2 | Local expression of $(\hat{Y} \hat{M}) \subset JD^2(W)$ and Bianchi identity $(B) \subset JD^2(W)$. |
| --- | --- |
| (Field equations) $F^A_{K1} = -(\partial_x B \hat{R}^{AB}_K + \langle \hat{C}^{AC}_K, \hat{R}^A_W \hat{R}^W_{BC} \rangle)_+ = 0$ | $(\hat{Y} \hat{M})$ |
| (Fields) $F^K_{A12} \equiv ^{\otimes} \hat{R}^K_{A12} - [\partial_x \hat{R}^{AB}_K, \hat{R}^{AC}_K]_+ = 0$ | |
| (Bianchi identities) $B^K_{A12} \equiv (\partial_x \hat{R}^{AB}_K, \hat{R}^{AC}_K)_+ = 0$ | $(B)$ |

Definition 3.6. We call quantum graviton a quantum metric $\hat{\mathfrak{g}}$ obtained by a solution $\hat{\mu}$ of $(\hat{Y} \hat{M})$, via the corresponding quantum vierbein.

Definition 3.7. In relation to the splitting (76), and with respect to the possible triviality of such quantum curvatures, we can classify solutions of $\hat{E}_2$, as reported in Tab.3.

---

Footnotes:

27 The rising and lowering of indexes is obtained by means of the full quantum metrics $\hat{\mathfrak{g}}$ on $M$ and $\mathfrak{g}$ on $\hat{\mathfrak{g}}$ respectively.

28 Another suitable name for $\hat{\theta}$ could be quantum dynamical fundamental solder form. In fact, it solders the quantum Minkowsky vector space $A \otimes \mathfrak{g} \otimes N$ at all the points $p \in M$. But the previous name is more handable.
quantum fundamental field $\hat{\mu}$ comes from a quantum principal connection on such a principal bundle, as it results from the commutative diagram (79).

**Theorem 3.8.** (Quantum Cartan geometry). Any non-degenerate solution $\mu$ of a quantum SG-Yang-Mills PDE, identifies on the base quantum supermanifold $M$ a quantum Cartan supergeometry, i.e., a Cartan geometry in the category $\Omega_S$.

**Proof.** A quantum Cartan supergeometry on a the quantum supermanifold $M$, is the natural extension, in the category $\Omega_S$, of Cartan geometry in the category of smooth finite dimensional manifolds [110]. More precisely it is a principal fiber bundle $\pi : G \to M$, with structure group $H$, where $G$ is a group in the category $\Omega_S$, such that the following conditions are satisfied: (i) $M$ admits as quantum model the quantum Klein geometry $(G,H)$, i.e., $T_xM \cong \mathfrak{g}/\mathfrak{h}$, for any $x \in M$, and $(G,H)$ is a quantum Klein model in $\Omega_S$, i.e., $G/H$ is a homogeneous space in $\Omega_S$, with $G$ containing the subgroup $H$; (ii) there exists a section $\omega : M \to Hom_\mathbb{Z}(TG;\mathfrak{g})$, of class $Q^\omega_\mathbb{Z}$, such that $\omega(p)$ is an isomorphism $\omega(p) : T_pG \to \mathfrak{g}$, for all $p \in G$; (iii) $\omega(p)|_{vT_xG} : vT_pP \cong \mathfrak{h}$; (iv) $(R_h)^*\omega = Ad(h^{-1})\omega$, $\forall h \in H$, where $R_h$ denotes right multiplication translation for $h$. Then a quantum SG-Yang-Mills PDE identifies the following quantum Cartan supergeometry $\pi : G \to M$, where $G$ is any quantum superextension Lie group of the Poincaré group, such that its quantum super Lie algebra is just $\mathfrak{g}$, and containing a subgroup $H$, with quantum super Lie algebra $\mathfrak{h} = \oplus \mathfrak{g} \oplus \bigoplus \mathfrak{g}$, then $M$ admits as model the quantum Klein geometry $(G,H)$, since one has the isomorphism $T_xM \cong \oplus \mathfrak{g} \cong \mathfrak{g}/(\oplus \mathfrak{g} \oplus \bigoplus \mathfrak{g})$, $\forall x \in M$. Furthermore, any quantum fundamental field $\mu$ comes from a quantum principal connection on such a principal bundle, as it results from the commutative diagram (79).

$$
\begin{array}{ccc}
G & \xrightarrow{\omega} & Hom_\mathbb{Z}(TG;\mathfrak{g}) \\
\pi \downarrow & & \downarrow \pi_* \\
M & \xrightarrow{\hat{\mu}} & Hom_\mathbb{Z}(TM;\mathfrak{g})
\end{array}
$$

**Example 3.9.** (Quantum gravity-Yang-Mills PDE’s in $D = 4$). This is the most simple situation where $\mathfrak{g} = \oplus \mathfrak{g} \oplus \bigoplus \mathfrak{g}$. In such a case the quantum Klein geometry is $(P(\mathcal{N}), SO(A \otimes_\mathbb{R} \mathbf{N}))$, where $A \otimes_\mathbb{R} \mathbf{N}$ is the 4-dimensional quantum Minkowsky vector space, extension of the Minkowsky vector space $\mathbf{N}$, with respect to the quantum algebra $A$, endowed with the quantum metric $\mathfrak{g} \equiv 1 \otimes \mathfrak{g}$, natural extension of the quantum algebra $A$.

$\mu$Let us emphasize that a section $\omega$, considered in the above point (ii), is just a quantum pseudoconnection in the sense introduced in Refs. [77, 84]. (See also Refs. [70, 99] for superclassical analogous ones.) Then, one can see that $\omega$ is just a principal connection, (Ehresmann connection [18]), on the $\mathfrak{g}$-principal fiber bundle $P \equiv G \times \mathfrak{g} \to G$. The proof can be copied from an intrinsic previous one given in [99] for the superclassical case.) Recall that in a Cartan geometry $\pi : G \to M$, the torsion is obtained by composition $T = T \circ R : \Omega^2G \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, where $R$ is the curvature associated to the connection $\omega$ and $\pi$ is the canonical projection.
Minkowsky metric $g$ on $N$. $SO(A \otimes N)$, is the symmetry group of $(A \otimes N, \hat{g})$. One can see that this is just isomorphic to the classic Lorentz group $SO(A \otimes N) \cong SO(1, 3)$. In fact, any element $\hat{\Lambda} \in SO(A \otimes N)$ is necessarily of the type $\hat{\Lambda} = 1_A \otimes \Lambda$, with $\Lambda \in SO(N)$. In fact, by the condition $\hat{g}(a \otimes u, b \otimes v) = \hat{g}(\Lambda(a \otimes u), \Lambda(b \otimes v))$, we get

$$
\begin{align}
\hat{g}(a \otimes u, b \otimes v) &= \hat{g}(a^\alpha \otimes e_\alpha, b^\beta \otimes e_\beta) = a^\alpha b^\beta \hat{g}(e_\alpha, e_\beta) = a^\alpha b^\beta g_{\alpha \beta} \\
\Rightarrow \hat{g}(\Lambda(a \otimes u), \Lambda(b \otimes v)) &= \hat{g}(\hat{\Lambda}(a^\alpha \otimes e_\alpha), \hat{\Lambda}(b^\beta \otimes e_\beta)) \\
&= \hat{g}(\hat{\Lambda}_\alpha(a^\alpha) 1 \otimes e_\gamma, \hat{\Lambda}_\beta(b^\beta) 1 \otimes e_\delta) \\
&= \hat{\Lambda}_\alpha(a^\alpha) \hat{\Lambda}_\beta(b^\beta) g_{\gamma \delta} \\
&= \hat{\Lambda}_\alpha(a^\alpha) \hat{\Lambda}_\beta(b^\beta) g_{\gamma \delta}.
\end{align}
$$

with $a^\alpha, b^\beta \in A$, $\hat{\Lambda}_\alpha \in \hat{\Lambda}$, $g_{\alpha \beta} \in \mathbb{R}$. Therefore we get

$$
a^\alpha b^\beta g_{\alpha \beta} = \hat{\Lambda}_\alpha(a^\alpha) \hat{\Lambda}_\beta(b^\beta) g_{\gamma \delta}.
$$

For the arbitrariness of $a^\alpha$ and $b^\beta$, we get that must be also

$$
g_{\alpha \beta} = \hat{\Lambda}_\alpha \hat{\Lambda}_\beta g_{\gamma \delta}.
$$

Since $g_{\alpha \beta} \in \mathbb{R}$, must necessarily be $\hat{\Lambda}_\gamma \in \mathbb{R}$. This means that it is $\hat{\Lambda} = 1_A \otimes \Lambda$, with $\Lambda \in SO(N)$. Therefore one has the following isomorphisms:

$$
SO(A \otimes N) \cong SO(N) \cong SO(1, 3) \Rightarrow (\hat{\Lambda}_\gamma) = (\Lambda_\gamma).
$$

$P(\hat{N})$ is the symmetry group of the 4-dimensional affine quantum Minkowsky space-time $(\hat{N}, A \otimes \hat{g})$. One has the following isomorphisms:

$$
P(\hat{N}) \cong A \otimes N \times SO(A \otimes N) \cong A \otimes N \times SO(N) \cong A \otimes R^{1, 3} \times SO(1, 3).
$$

One has the following short exact sequence:

$$
0 \rightarrow A \otimes N \rightarrow A \otimes N \times SO(A \otimes N) \rightarrow SO(A \otimes N) \rightarrow 0
$$

The semidirect product means that the product in $P(\hat{N})$ is given by the following:

$$
\begin{align}
(a \otimes u, \hat{\Lambda}).(b \otimes v, \hat{\Lambda}') &= (a \otimes u + \hat{\Lambda}(b \otimes v), \hat{\Lambda}\hat{\Lambda}') \\
&= (a \otimes u + b \otimes \Lambda(v), (1 \otimes \Lambda)(1 \otimes \Lambda')) \\
&= (a \otimes u + b \otimes \Lambda(v), 1 \otimes \Lambda\Lambda').
\end{align}
$$

The quantum Cartan geometry is given by the $SO(A \otimes N)$-principal group, in the category $\Omega$, $\pi : P(\hat{N}) \rightarrow M$, where $M$ is a 4-dimensional quantum manifold, with respect to the quantum algebra $A$, whose tangent spaces $T_pM \cong A \otimes \mathbb{N} \cong P(\hat{N})/SO(A \otimes N)$. Therefore $M$ is locally quantum Minkowskian. The quantum Lie algebra $\mathfrak{g}$ of $P(\hat{N})$ has the following splitting:

$$
\mathfrak{g} \cong A \otimes \mathbb{N} \oplus so(1, 3) \equiv \mathfrak{g} \oplus \mathfrak{g}.
$$

Let us denote by $\hat{P}_u = 1 \otimes P_u$ the generators of $\mathfrak{g}$, where $P_u$ are the corresponding translation-generators in the Poincaré algebra. The can see that one has

$$
[\hat{P}_u, \hat{P}_v] = 1 \otimes [P_u, P_v] = 1 \otimes C^\alpha_{\mu \nu} P_\alpha = C^\alpha_{\mu \nu} 1 \otimes P_\alpha = C^\alpha_{\mu \nu} \hat{P}_\alpha
$$

with $C^\alpha_{\mu \nu} \in \mathbb{R}$. In fact, one can put on $A \otimes \mathbb{N}$ quantum coordinates $\hat{x}^A : A \otimes \mathbb{N} \rightarrow A$, adapted to the structure $A \otimes \mathbb{N}$, i.e., $\hat{x}^A(a \otimes u) = ax^A(u) = au^A \in A$, where
$x^A : N \to \mathbb{R}$ are coordinates on the 4-dimensional affine Minkowsky space-time $N$. Then, for any function $f : A \otimes \mathbb{R} N \to A$ of class $Q^2_N$, one has:

\[
\begin{align*}
P_A.f &= (\partial x_A.f) = (\partial x_A.\hat{x}^B) \\
&= (\partial x_B.f)(1 \otimes \delta^B_A) = (\partial \hat{x}_A.f) \\
&= (P_A.f)
\end{align*}
\]

since $(\partial x_A.\hat{x}^B) = (\partial x_A.(1 \otimes x^B)) = (\partial x_A.1) \otimes x^B + 1 \otimes (\partial x_A.x^B) = 1 \otimes \delta^B_A$.

By resuming all the quantum structure constants, for this quantum gravity Yang-Mills PDE’s, are real numbers. So all the generators of the quantum Poincaré algebra just coincide with the ones of the Poincaré algebra. The situation is summarized in Tab.4.

| Tab.4 - Quantum Poincaré algebra in $D=4$. |
|------------------------------------------|
| $[J_{\alpha \beta}, J_{\gamma \delta}] = \eta_{\alpha \gamma} J_{\beta \delta} - \eta_{\alpha \delta} J_{\beta \gamma}$ |
| $[P_\alpha, P_\beta] = 0$, $[J_{\alpha \beta}, P_\gamma] = \eta_{\alpha \gamma} P_\beta - \eta_{\alpha \delta} P_\gamma$ |

Boost: $K_1 = J_{03}$; Rotations: $J_i = e_{ij} K^j$, $i,j \in \{1,2,3\}$

Example 3.10. (Quantum $N$-superextensions of the Poincaré algebra in $D = 4$).

In $D = 4$, the usual $N$-supersymmetric extension $g$ of the Poincaré algebra $p = so(1,3) \oplus t$, is a $\mathbb{Z}_2$-graded vector space $g = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with a graded Lie bracket, such that $\mathfrak{g}_0 = p \oplus b$, where $b$ is a reductive Lie algebra, such that its self-adjoint part is the tangent space to a real compact Lie group.\(^{30}\) Furthermore $\mathfrak{g}_1 = (\frac{1}{2},0) \oplus s \oplus (0,\frac{1}{2}) \oplus \tilde{s}^*$, where $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ are specific representations of the Poincaré algebra. Both components are conjugate to each other under the $*$ conjugation. $s$ is a $N$-dimensional complex representation of $b$ and $s^*$ its dual representation.\(^{31}\) Note also that the Lie bracket for the odd part is usually denoted by $\{,\}$ in theoretical physics.

Then with such a notation one has

\[
\{Q^i_\alpha, Q^j_\beta\} = \delta^i_\beta \eta^{\mu \nu} C_{\alpha \beta} P_\mu + U^{ij}(C)_{\alpha \beta} \epsilon^{\nu \mu} + V^{ij} C_{\gamma \delta} \alpha \beta
\]

where $U^{ij} = -U^{ji}$, $V^{ij} = -V^{ji}$ are the $(N-1)N$ central charges, $C$ is the (antisymmetric) charge conjugation matrix, $(Q^i_\alpha)_{i=1,...,N}$, are the $N$ Majorana spinor supersymmetry charge generators. The dynamical components $\tilde{\mu}^\nu$, $i = 1,\ldots, N$, of the quantum fundamental field, corresponding to the generators $Q^i$, are called quantum gravitinos. So in a quantum $N$-SG-Yang-Mills $D=4$, one distinguishes $N$ quantum gravitino types, and $(N-1)N$ central charges.\(^{32}\)

The more simple case are ones with $N = 1$ and $N = 2$. More precisely, for $N = 1$, with $b = u(1)$ and $s$ the 1D representation of $u(1)$. In such a case one has an electric charge, (i.e., $u(1)$-charge), but there are not central charges. In Tab.5 are reported the brackets in the case $N = 1$ and $N = 2$.\(^{33}\) Then a quantum superextension of $p$ is $A \otimes \mathbb{R} \mathfrak{g}$, where $A$ is a quantum superalgebra. This can be taken $A \subseteq L(\mathcal{H})$, where

---

\(^{30}\)A reductive Lie algebra is the sum of a semisimple and an abelian Lie algebra. Since a semisimple Lie algebra is the direct sum of simple algebras, i.e., non-abelian Lie algebras, $l_i$, where the only ideals are $\{0\}$ and $\{l_i\}$, it follows that $b$ can be represented in the form $b = a \oplus \sum l_i$.

\(^{31}\)If $\rho : g \to L(V)$ is a representation of Lie algebra, its dual $\hat{\rho} : g \to L(V)$, working on the dual space $V$, is defined by $\hat{\rho}(g) = -\rho(g), \forall \in g$.

\(^{32}\)Since the central charges in (90) have physical dimension of mass, they cannot be carried by massless solutions.

\(^{33}\)For the case $N = 2$, and with respect to equation (90), one should also have $[Q_{3i}, Q_{3j}] = (C_{\gamma \delta})_{\beta \gamma} \epsilon^{\mu \nu} P_\mu + C_{\beta \gamma} \epsilon^{\mu \nu} + \epsilon_{ij} (C_{\gamma \delta})_{\delta \nu} \epsilon^{\mu \nu}$. But the term $\epsilon_{ij} (C_{\gamma \delta})_{\beta \nu} \epsilon^{\mu \nu}$ can always be rotated into $C_{\beta \nu} \epsilon_{ij} \tilde{B}$ by a chiral transformations, and therefore does not represent a further charge.
$\mathcal{H}$ is a super-Hilbert space. (See also Refs. [85, 94].) With respect to the splitting (75) we get:

(91)

$$(N = 1) : \begin{cases} \hat{\mu} = P_\alpha \hat{\theta}^\alpha \, dx^\gamma \\ \hat{\mu} = J_{\alpha \beta} \hat{\omega}^{\alpha \beta} \, dx^\gamma \\ \star \hat{\mu} = Q_\alpha \phi_\gamma \, dx^\gamma . \end{cases}$$

\hspace{1cm} (N = 2) : \begin{cases} \hat{\mu} = P_\alpha \hat{\theta}^\alpha \, dx^\gamma \\ \hat{\mu} = J_{\alpha \beta} \hat{\omega}^{\alpha \beta} \, dx^\gamma \\ \star \hat{\mu} = [Z A_\gamma + Q_{\alpha i} \phi^{\alpha i}] \, dx^\gamma . \end{cases}

---

| Tab.5 - N=1,2 Super Poincaré algebra in $D=4$. |
|---|
| **$N=1$** |
| $[J_{\alpha \beta}, J_{\gamma \delta}] = \eta_{\alpha \gamma} J_{\beta \delta} - \eta_{\alpha \delta} J_{\beta \gamma}$ |
| $[P_\alpha, P_\beta] = 0$, $[J_{\alpha \beta}, P_\gamma] = \eta_{\alpha \gamma} P_\beta - \eta_{\beta \gamma} P_\alpha$, $[P_\alpha, Q_\gamma] = 0$ |
| $[J_{\alpha \beta}, Q_\gamma] = (\sigma_{\alpha \beta})^c_\gamma Q_\mu$, $[Q_\alpha, Q_\beta] = \eta_{\alpha \beta}$ |
| **$N=2$** |
| $[J_{\alpha \beta}, J_{\gamma \delta}] = \eta_{\alpha \gamma} J_{\beta \delta} + \eta_{\alpha \delta} J_{\beta \gamma} - \eta_{\beta \gamma} J_{\alpha \delta} - \eta_{\beta \delta} J_{\alpha \gamma}$ |
| $[P_\alpha, P_\beta] = 0$, $[J_{\alpha \beta}, P_\gamma] = \eta_{\alpha \gamma} P_\beta - \eta_{\beta \gamma} P_\alpha$, $[P_\alpha, Q_\gamma] = 0$ |
| $[J_{\alpha \beta}, Q_\gamma] = (\sigma_{\alpha \beta})^c_\gamma Q_\mu$, $[Q_\alpha, Q_\beta] = (\eta_{\alpha \beta}) a_\alpha \phi_\beta P_\gamma + C_{\alpha \beta \gamma} Z$ |

---

| Tab.6 - Supersymmetric semi-simple tensor extension Poincaré algebra in $D=4$. |
|---|
| $[J_{\alpha \beta}, J_{\gamma \delta}] = \eta_{\alpha \gamma} J_{\beta \delta} + \eta_{\alpha \delta} J_{\beta \gamma} - \eta_{\beta \gamma} J_{\alpha \delta} - \eta_{\beta \delta} J_{\alpha \gamma}$ |
| $[P_\alpha, P_\beta] = 0$, $[J_{\alpha \beta}, P_\gamma] = \eta_{\alpha \gamma} P_\beta - \eta_{\beta \gamma} P_\alpha$, $[P_\alpha, Q_\gamma] = 0$ |
| $[J_{\alpha \beta}, Q_\gamma] = (\sigma_{\alpha \beta})^c_\gamma Q_\mu$, $[Q_\alpha, Q_\beta] = a(\gamma_{\alpha \beta} Q_\gamma)$ |

In Tab.6 are reported supersymmetric semi-simple tensor extensions of Poincaré algebra in $D = 4$ too. There $a$, $b$ and $c$ are constants. This algebra admits the following splitting: $so(3, 1) \oplus osp(1, 4)$, where $so(3, 1)$ is the 4-dimensional Lorentz algebra and $osp(1, 4)$ is the orthosymplectic algebra. Then, by considering the quantum superextension $A \otimes_R [so(3, 1) \oplus osp(1, 4)]$, where $A$ is a quantum superalgebra, and with respect to the splitting (75) we get:

(92)

\hspace{1cm} \begin{cases} \hat{\mu} = P_\alpha \hat{\theta}^\alpha \, dx^\gamma \\ \hat{\mu} = J_{\alpha \beta} \hat{\omega}^{\alpha \beta} \, dx^\gamma \\ \star \hat{\mu} = [Z A_\gamma + Q_{\alpha i} \phi^{\alpha i}] \, dx^\gamma . \end{cases}

**Theorem 3.11.** (Quantum Levi-Civita connection and quantum Higgs fields in $(\hat{YM})$). Quantum Levi-Civita connections $\hat{\omega}$, belonging to a solution $\hat{\mu}$ of $(\hat{YM})$, identify covariant derivative on the quantum metric $\hat{g}$, corresponding to $\hat{\mu}$, such that $\hat{\omega} \nabla \hat{g} = 0$, when $\hat{\omega}$ comes from a quantum Higgs-symmetry breaking mechanism, where $\hat{g}$ can be identified with a quantum Higgs field.

**Proof.** Quantum Higgs fields and symmetry breaking are two aspects of an unique mathematical phenomenon: reduction of a $G$-principal fiber bundle to a closed subgroup $H \subset G$, considered in the category $\mathcal{Q}_S$. When this happens one has the commutative diagram (93) of fiber bundles.
\( \pi_H : P \to P/H \) is a principal bundle with structure group \( H \) and \( \pi/_{H} : P/H \to M \) is a fiber bundle associated to \( \pi : P \to M \), with the natural action of \( G \) on the fiber type \( G/H \). \( \pi_h : \hat{h}P \to M \) is a \( H \)-principal bundle, reduction of \( P \). Any of such reduction is identified with a global section \( \hat{h} : M \to \hat{P}/H \) of \( \pi/_{H} \), such that \( \hat{h}P = \pi^{-1}_{H}(\hat{h}(M)) \cong \hat{h}P \). Any principal connection \( \hat{h}\omega \) on \( \hat{h}P \) identifies a principal connection on \( P \), hence a covariant derivative \( \hat{h}\nabla \) on sections of \( \pi/_{H} \), such that \( \hat{h}\omega \nabla \hat{h} = 0 \). Conversely a principal connection \( \hat{\omega} \) on \( P \) is projected onto \( \hat{h}P \) iff \( \hat{h}\omega \nabla \hat{h} = 0 \). Furthermore, if the quantum Lie (super)algebra \( g \) of \( G \), splits into \( g = h \oplus a \), where \( h \) is associated to \( H \) and \( a \) is a subspace of \( g \) on which \( G \) acts for adjointness, then \( \hat{h} \omega_h : \hat{h}P \to \text{Hom}_Z(T\hat{h}P; h) \) is a principal connection on \( \hat{h}P \), as defined by the commutative diagram (94).

\[\text{(94)} \quad \text{Hom}_Z(TP; g) \to \text{Hom}_Z(TP; h) \to \text{Hom}_Z(T\hat{h}P; h) \]

In the case that \( P = \mathcal{E}(M) \) is the principal bundle of linear frames on \( M \), with structure group \( GL(4,A) \), that is reducible to \( SO(1,3) \), then global sections of \( \mathcal{E}(M)/SO(1,3) \to M \) are related to quantum metrics on \( M \), as it is shown by the commutative and exact diagram (95).

\[\text{(95)} \quad 0 \to \mathcal{E}(M)/SO(1,3) \xrightarrow{\lambda} \text{Hom}_Z(T^2_{\tilde{g}} M; A) \]

such that

\[\text{(96)} \quad i \circ \hat{h} = \tilde{g} = g_{\alpha\beta} dx^\alpha \otimes dx^\beta = \theta^a_\alpha \theta^b_\beta \eta_{ab} dx^\alpha \otimes dx^\beta.\]

So, in this case, a quantum Higgs field is identified with a locally Minkowskian quantum metric. Since any principal connection \( \hat{h}\omega \) on \( \hat{h}\mathcal{E}(M) \subset \mathcal{E}(M) \) is extendable to a principal connection \( \hat{\omega} \) on \( \mathcal{E}(M) \), and identifies a linear quantum connection on \( TM \), and on other associated vector bundles, such that \( \hat{\omega} \nabla \tilde{g} = 0 \), when \( \tilde{g} \) is just the quantum metric-Higgs field defined in (96). This proves that quantum Levi-Civita connections, corresponding to solutions of \( (YM) \), such that condition...
\[ \hat{\omega} \nabla \hat{g} = 0, \] are particular cases, related to the quantum Higgs-symmetry breaking mechanisms.

**Theorem 3.12.** (Quantum crystal structure of \((YM)\)). If \(H_3(M; \mathbb{K}) = 0\) the dynamic equation \((YM)\) is a quantum extended crystal super PDE. Moreover, under the full admissibility hypothesis, it becomes a quantum 0-crystal PDE.

**Proof.** In Refs.[75, 83] it is proved that \((YM) \subset J\tilde{D}^2(W)\) is formally integrable and also completely integrable.\(^{34}\) That proof works well also in this situation, since it is of local nature, and remains valid also for quantum supermanifolds that are only locally quantum super-Minkowskian ones. Then, by using Theorem 2.2 we get \([97]\)

\[ K_{3|3}(YM) = \Omega_{3|3}(YM) \equiv \Omega_{3|3}(YM) = A_0 \otimes_{\mathbb{K}} H_3(W; \mathbb{K}) \oplus A_1 \otimes_{\mathbb{K}} H_3(W; \mathbb{K}). \]

Thus, under the condition that \(H_3(M; \mathbb{K}) = 0\), one has \(\Omega_{3|3}(YM) = \Omega_{3|3}(YM) = 0\), hence \((YM)\) becomes a quantum extended crystal super PDE. This is surely the case when \(M\) is globally quantum super Minkowskian. (See Refs.[76, 80, 83, 90, 92].) In such a case one has \(\Omega_{3|3}(YM) = K_{3|3}(YM)\), where

\[ (97) \quad K_{3|3}(YM) = \left\{ [N](YM) \in \Omega_{3|3}(YM) \mid N = \partial V, \text{for some (singular)} \left(4|4\right)-\text{dimensional quantum supermanifold } V \subset W \right\}. \]

So \((YM)\) is not a quantum 0-crystal super PDE. However, if we consider admissible only integral boundary manifolds, with orientable classic limit, and with zero characteristic quantum supernumbers, \((\text{full admissibility hypothesis})\), one has: \(\Omega_{3|3}(YM) = 0\), and \((YM)\) becomes a quantum 0-crystal super PDE. Hence we get existence of global \(Q_w^\infty\) solutions for any boundary condition of class \(Q_w^\infty\).

With respect to the commutative exact diagram in (23) we get the exact commutative diagram (98).

\[ (98) \quad 0 \longrightarrow K_{3|3}(YM) \longrightarrow \Omega_{3|3}(YM) \longrightarrow \Omega_6(YM) \longrightarrow 0 \]

\[ 0 \longrightarrow K_6^+ \longrightarrow \Omega_6^+ \longrightarrow \Omega_6 \longrightarrow 0 \]

Taking into account the result by Thom on the unoriented cobordism groups [119], we can calculate \(\Omega_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\). Then, we can represent \(\Omega_6\) as a subgroup of a 3-dimensional crystallographic group type \([G(3)]\). In fact, we can consider the

\(^{34}\)We shall assume that the \(A\) is a quantum (super)algebra, over \(K = \mathbb{R}\), or \(K = \mathbb{C}\), with Noetherian centre \(Z \equiv Z(A)\). In general \(A\) is a subalgebra of \(L(H)\), where \(H\) is a (super)Hilbert space. Then \(A\) is Noetherian since \(L(H)\) is so. In fact, \(L(H)\) is Morita equivalent to \(K\). (Two rings \(R\) and \(S\) are Morita equivalent, \(R \sim S\), if there is a \(R\)-module \(W_R\), \(\text{progenerator}\), such that \(S \equiv \text{End}(W_R)\).) If \(R\) is Noetherian, then \(\mathbb{M}^2(R)\) is Noetherian too. As a by-product, it follows that also the centre \(Z \subset A\) is a Noetherian ring. Note also that the derived quantum algebra \(\hat{A} \equiv \text{Hom}_Z(A; A)\) is a Noetherian ring. In fact, in this case \(\hat{A} \sim Z\), with progenerator the \(Z\)-module \(A\).
amalgamated subgroup \(D_2 \times \mathbb{Z}_2 \ast D_2 D_4\), and monomorphism \(\Omega_6 \to D_2 \times \mathbb{Z}_2 \ast D_2 D_4\), given by \((a, b, c) \mapsto (a, b, b, c)\). Alternatively we can consider also \(\Omega_6 \to D_1 \ast D_2 D_4\). (See Appendix C in [87] for amalgamated subgroups of \([G(3)]\).) In any case the crystallographic dimension of \((YM)\) is 3 and the crystallographic space group type are \(D_{2d}\) or \(D_{4h}\) belonging to the tetragonal syngony. (See Tab.6 in [87] and, for further informations, [28].)

**Example 3.13.** If \(M\) is the quantum superextension, with respect to the quantum superalgebra \(A\), of the 4-dimensional affine Minkowsky space-time, then \((YM)\) is a quantum extended crystal PDE.

**Theorem 3.14.** (Quantum crystal structure of \((YM)[i]\)). The observed dynamic equation \((YM)[i]\), by means of a quantum relativistic frame, is a quantum extended crystal super PDE. Moreover, under the full admissibility hypothesis, it becomes a quantum 0-crystal super PDE.

**Proof.** The evaluation of \((YM)\) on a macroscopic shell \(i(MC) \subset M\) is given by the equations reported in Tab.7.

| Tab.7 - Local expression of \((YM)[i] \subset J\hat{D}^2(i^*W)\) and Bianchi identity \((B)[i] \subset J\hat{D}^2(i^*W)\). |
|---|
| (Field equations) \((\partial_\alpha R^{\alpha\beta\gamma\delta}) + [\dot{C}^\alpha_{\beta\gamma\delta}]_+ = 0\) | \((YM)[i]\) |
| (Fields) \(R^i_{a1a2} = (\partial_\alpha R^i_{a\alpha}) + \frac{1}{2} \dot{C}_{ij}^\alpha R^i_{a\alpha a2} + R^i_{a1a2}\) | \((YM)[i]\) |
| (Bianchi identities) \((\partial_\alpha \dot{R}^i_{a1a2}) + \frac{1}{2} \dot{R}^i_{a1a2} R^i_{a2a1} + \frac{1}{2} \dot{C}_{ij}^\alpha R^i_{a\alpha a1} = 0\) | \((B)[i]\) |

This equation is also formally integrable and completely integrable. Furthermore, the 3-dimensional integral bordism group of \((YM)[i]\) and its infinity prolongation \((YM)[i]_+\) are trivial, under the full admissibility hypothesis: \(\Omega^3_3(YM)[i] \cong \Omega^3_3(YM)[i]_+ \cong 0\). So equation \((YM)[i] \subset J\hat{D}^2(i^*W)\) becomes a quantum 0-crystal super PDE and it admits global (smooth) solutions for any fixed time-like 3-dimensional (smooth) boundary conditions.

**Proposition 3.15.** The quantum vierbein curvature \(\hat{\theta} \hat{R}\) identifies, by means of the quantum vierbein \(\hat{\theta}\) a quantum field \(\hat{S}: M \to Hom_Z(\mathbb{A}_0^2 M; TM)\), that we call quantum torsion, associated to \(\hat{\mu}\). In quantum coordinates one can write

\[
\hat{S} = \partial x_C \otimes \hat{S}^C_{AB} dx^A \Delta dx^B, \quad \hat{S}^C_{AB} = \hat{\theta}^C_{\hat{R}K} \hat{R}^K_{AB}.
\]

Furthermore, with respect to a quantum relativistic frame \(i: N \to M\), the quantum torsion \(\hat{S}\) identifies a \(A\)-valued \((1,2)\)-tensor field on \(N\), \(\hat{S} = i^*\hat{S}: N \to A \otimes_R \mathbb{A}_0^2 N \otimes_R TN\), that we call quantum torsion of the observed solution.

**Proof.** In fact \(\hat{S} = \hat{\theta}^{-1} \circ \hat{\theta}\), i.e., the diagram (100) is commutative.

\[
\begin{array}{ccc}
M & \xrightarrow{\theta} & \text{Hom}_Z(\mathbb{A}_0^2 M; TM) \\
\hat{\theta}^{-1} & \downarrow & \\
\hat{S} & \xrightarrow{\hat{\theta}} & \text{Hom}_Z(\mathbb{A}_0^2 M; \mathfrak{g})
\end{array}
\]

where \(\hat{\theta}^{-1}(p) \equiv Hom_Z(1_{\mathbb{A}_0^2(T_p M)}; \theta^{-1}(p)), \forall p \in M\).
Definition 3.16. Furthermore, we say that an observed solution has a quantum spin, if the observed solution has an observed torsion

\[ \bar{S} \equiv i^* \bar{S} = \partial x^\gamma \otimes \sum_{0 \leq \alpha < \beta \leq 3} \bar{S}^\gamma_{\alpha \beta} dx^\alpha \wedge dx^\beta : N \to A \otimes_R N \otimes_R A^0_2(N) \cong A \otimes_R A^0_2(N) \otimes_R TN \]

with \( \bar{S}^\gamma_{\alpha \beta}(p) = -\bar{S}^\gamma_{\beta \alpha}(p) \in A, p \in N, \) that satisfies the following conditions, (quantum-spin-conditions):

\[
\begin{cases}
\bar{S} = s \otimes \psi \\
\bar{S} = \sum_{0 \leq \alpha < \beta \leq 3} \bar{s}_{\alpha \beta} dx^\alpha \wedge dx^\beta : N \to A \otimes_R A^0_2(N), \\
\bar{s}_{\alpha \beta}(p) = -s_{\beta \alpha}(p) \in A, p \in N, \\
\bar{\psi}|_{\bar{S}} = 0 \\
\bar{S}^\lambda_{\alpha \beta} = \bar{s}_{\alpha \beta} \bar{\psi}^\lambda \\
\bar{S}^\alpha_{\alpha \beta} = 0
\end{cases}
\]

(102)

where \( \bar{\psi} \) is the velocity field on \( N \) of the time-like foliation representing the quantum relativistic frame on \( N \). When conditions (102) are satisfied, we say that the solution considered admits a quantum spin-structure, with respect to the quantum relativistic frame. We call \( s \) the quantum 2-form spin of the observed solution. Let \( \{\xi^\alpha\}_{0 \leq \alpha \leq 3} \) be coordinates on \( N \), adapted to the quantum relativistic frame. Then one has the following local representations:

\[
\{ \bar{s} = \bar{s}_{ij} d\xi^i \wedge d\xi^j \}.
\]

We define quantum spin-vector-field of the observed solution

\[
\bar{\xi} = \langle \epsilon, \bar{S} \rangle = [\epsilon_{\mu \nu \lambda \rho} \bar{s}_{\mu \nu}^{\lambda \rho}] d\xi^\rho \equiv \bar{s}_{\rho} d\xi^\rho = \bar{s}_{k} d\xi^k \Rightarrow \bar{s} = \partial \xi_s \bar{s}_{k} g^{ki} = \partial \xi_s \bar{s}^i
\]

where \( \epsilon_{\mu \nu \lambda \rho} = \sqrt{|g|_{\mu \nu \lambda \rho}} \) is the completely antisymmetric tensor density on \( N \). One has \( \bar{s}_{\rho}(p) \in A, p \in N \). The classification of the observed solution on the ground of the spectrum of \( |\bar{s}|^2 = \bar{s}^i \bar{s}_i \), and its (quantum helicity), i.e., component \( \bar{s}_z \), is reported in Tab. 8.35

![Tab.8 - Local quantum spectral-spin-classification of (YM) solutions.](image)

| Definition | Name |
|------------|------|
| \( S(p) \) \in \{ \{ n \in \{0, 1, 2, \ldots \} \} \} | bosonic-polarized |
| \( S(p) \) \in \{ \{ 0 \} \} | fermionic-polarized |
| \( S(p) \) \in \{ \{ \} \} | unpolarized |
| \( S(p) \) \in \{ \{ n \in \{0, 1, 2, \ldots \} \} \} | mixt-polarized |

Remark 3.17. May be useful to emphasize that since here the quantum spin content of an observed solution is a geometric object of local nature, does not necessitate that its property, in relation to the classification in Tab. 7, should be constant in any

\[ |s(p)|^2 \equiv \bar{s}^i (p) \bar{s}_i (p), p \in N. \]

\( s \) = spin quantum number; \( m_s \) = spin orientation quantum number.

35In particle physics with the term *helicity* one usually means the component of the spin along the momentum. In this way one should obtain only the contribution by the “intrinsic spin”, decoupled by the angular momentum. However, in our geometric formulation one talks of spin of an observed solution, since the macroscopic model may be inadequate, and also undesirable.
part of the solution. In fact, e.g., for solutions representing particles reactions, can happen that the spin-content changes during the reaction process. For example the spins of composite particles, such as protons and atomic nucleides, are not just the sum of their constituent particles. The usual justification is ascribed to the contribution of the total angular momenta, in some mechanical simulations. On the other hand this game does not work well. For example for the proton there are experimental evidences that such mechanical models are inadequate. (See e.g. [104].) Some approaches similar to graviton-quark-gluon plasma appear more appropriate. But it is unknown if gluons have spin... So, instead to insist with naive and unjustified mechanical models, it is more appropriate describe quantum particles like p-chains solutions of suitable quantum Yang-Mills PDE’s.\textsuperscript{36}

In some previous works we have proved the following important result.

**Theorem 3.18.** (Obstruction-mass-gap-existence)\textsuperscript{78, 85, 94} A quantum full-flat solution cannot have mass-gap. In order that an observed solution admits mass-gap it is enough that the following conditions should be satisfied:

$$\left\{ \begin{array}{c}
\text{tr} \left( \tilde{R}_\beta^K(p) \tilde{R}_H(p) - \tilde{\mu}_\alpha^K(p) \tilde{R}_H(p) \right) = 0; \\
\frac{1}{\text{tr} (R_{\beta\alpha}^K(p) R_{\gamma\delta}^\alpha(p) - \mu_{\beta\gamma}^K(p) R_{\delta\gamma}^\alpha(p))} \in A.
\end{array} \right\}_{p \in N}$$

**Theorem 3.19.** (Local mass-formula). If an observed solution of $\hat{Y}M$ is with mass-gap, one has the following local mass-formula:

$$m(p) = \otimes m(p) + \otimes m(p) + \star m(p), \ p \in N.$$ 

We call $\otimes m(p)$, (resp. $\otimes m(p)$, resp. $\star m(p)$), the local torsion mass, (resp. local Lorentz-mass, resp. local deviatory-mass), of the observed solution.\textsuperscript{37}

**Proof.** In fact, the splitting on the quantum Lie superalgebra $\mathfrak{g}$, induces the following splitting on the quantum Hamiltonian observed by the quantum relativistic frame:

$$H = \otimes H + \otimes H + \star H \left\{ \begin{array}{l}
\otimes H = \otimes \tilde{R}_\beta^K \tilde{R}_H^\beta \otimes - \otimes \tilde{\mu}_\alpha^K \tilde{R}_H^\alpha \otimes \\
\otimes H = \otimes \tilde{R}_\beta^\alpha \tilde{R}_H^\beta \otimes - \otimes \tilde{\mu}_\alpha^\beta \tilde{R}_H^\alpha \otimes \\
\star H = \star \tilde{R}_\beta^\alpha \star \tilde{R}_H^\beta \otimes - \star \tilde{\mu}_\alpha^\beta \star \tilde{R}_H^\alpha \otimes .
\end{array} \right.$$ 

Therefore, if the observed solution is with mass gap, we get that spectrum of the hamiltonian has a splitting induced by the corresponding hamiltonian splitting (107).

**Theorem 3.20.** (Observed objects and splitting formulas). To any observed solution of $\hat{Y}M$ one can associate the following geometric objects:

$$\begin{cases}
\text{(quantum-electric-charge-field):} \\
\tilde{E} = \hat{\psi} \tilde{R} = Z_K \otimes \tilde{E}_\alpha^K dx^\alpha = Z_K \otimes \tilde{R}_\alpha^K \psi^\alpha d\omega : N \rightarrow \mathfrak{g} \otimes_R T^* N \\
\text{(quantum-magnetic-charge-field):} \\
\tilde{B} = \hat{\psi} (\star \tilde{R}) = Z_K \otimes \tilde{H}_{\mu}^K dx^\mu = Z_K \otimes \epsilon_{\mu\nu\lambda\kappa} \tilde{R}_\alpha^K \psi^\alpha d\omega : N \rightarrow \mathfrak{g} \otimes_R T^* N.
\end{cases}$$

\textsuperscript{36}With a language nearer to physicists, p-chains can be called $p$-dimensional extendons.

\textsuperscript{37}Note that the formula (106) interprets the local mass $m(p)$ as a local-charge, emphasizing the role played by the different charge-components of the systems.
In adapted coordinates \( \{ \xi^a \}_{0 \leq a \leq 3} \), to the relativistic quantum frame, \( \hat{E} \) and \( \hat{B} \) have the following representations:

\[
\begin{align*}
\hat{E} &= Z_K \otimes \hat{E}_i^K d\xi^i = Z_K \otimes \hat{R}_i^K d\xi^i, \quad \hat{E}_i^K(p) \in A, \quad \forall p \in N \\
\hat{B} &= Z_K \otimes \hat{B}_i^K d\xi^i = Z_K \otimes \varepsilon_{\mu\nu}^a \hat{R}_r^K d\xi^m, \quad \hat{B}_i^K(p) \in A, \quad \forall p \in N.
\end{align*}
\]

(109)

This means that in the quantum relativistic frame system \( \hat{E} \) and \( \hat{B} \) are \( A \)-valued space-like objects, whose components belong to the quantum superalgebra \( A \). Therefore, their quantum content is given by the spectra of \( \hat{E}_i^K(p) \) and \( \hat{B}_i^K(p) \), for \( p \in N \). One has the following splittings:

\[
\begin{align*}
\hat{E} &= \hat{E}_0 + \hat{E} + \hat{E}_e \\
\hat{B} &= \hat{B}_0 + \hat{B} + \hat{B}_e.
\end{align*}
\]

(110)

We call \( \hat{E}_0, \) (resp. \( \hat{E}_e \), resp. \( \hat{E}_e \)), the torsion-quantum electric-charge-field, (resp. Lorentz-quantum electric-charge-field, resp. deviatory-quantum electric-charge-field), of the observed solution. Similarly, we call \( \hat{B}_0, \) (resp. \( \hat{B}_e \), resp. \( \hat{B}_e \)), the torsion-quantum magnetic-charge-field, (resp. Lorentz-quantum magnetic-charge-field, resp. deviatory-quantum magnetic-charge-field), of the observed solution.

If the solution has a spin-structure, then \( \hat{E}_0 = 0 \) and \( \hat{B}_0 \) is related to the quantum spin-vector field \( \tilde{s} \) by the following formula:

\[
\hat{E}_e = \hat{E}_0 + \hat{E} + \hat{E}_e \quad \Rightarrow \quad (\text{in frame-adapted coordinates}): \quad \hat{E}_e = \tilde{s}.
\]

Therefore, in an observed solution with spin-structure, it is recognized that the opposite torsion-quantum magnetic-charge-field identifies a distribution of quantum spin-vector fields, \( \tilde{s}_k = g^{i\kappa} \tilde{s}_i \), with \( \tilde{s}_i \) given in (111). Thus, \( \hat{B}_0 \) determines the spectral-spin-classification of the solution, according to Tab. 7.

\[
\begin{align*}
g_{\alpha\beta} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & g_{11} & g_{12} & g_{13} \\
0 & g_{21} & g_{22} & g_{23} \\
0 & g_{31} & g_{32} & g_{33} \end{pmatrix}, \\
(g_{\alpha\beta}) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & g_{11} & g_{12} & g_{13} \\
0 & g_{21} & g_{22} & g_{23} \\
0 & g_{31} & g_{32} & g_{33} \end{pmatrix}, \\
g^{ij} g_{jk} &= \delta_k. \tag{112}
\end{align*}
\]

Furthermore taking into account that \( \epsilon_{\mu\nu}^a = g^{a\alpha} g^{a\beta} \epsilon_{\alpha\beta\mu\nu} \), with \( \epsilon_{a\beta\mu\nu} = \sqrt{|g|} \delta_{a}^{(23)} \), the formulas (109) follow directly. When the observed solution admits spin-structure, then, since \( \tilde{S}_{\alpha\beta} = \hat{g}_K^0 \tilde{R}_K^0 \), we get also \( \tilde{S}_{\alpha\beta} = \hat{g}_K^0 \hat{E}_K^0 \). Thus we have

\[
\begin{align*}
0 &= \hat{g}_K^0 \hat{g}_K^0 \hat{E}_K^0 + \hat{g}_K^0 \hat{E}_K^0 \\
\text{Therefore must necessarily be} \quad \hat{g}_K^0 \hat{E}_K^0 = 0.
\end{align*}
\]

Finally we can write \( \hat{g}_K^0 \hat{B}_K^0 = \epsilon_{\mu\nu}^a \hat{g}_K^0 \tilde{R}_K^0 \hat{\psi}_{\mu} \). If the observed solution admits a spin-structure, then \( \hat{g}_K^0 \tilde{R}_K^0 \hat{\psi}_{\mu} \). Therefore, we can write

\[
\begin{align*}
\hat{g}_K^0 \hat{B}_K^0 &= \epsilon_{\mu\nu}^a \hat{g}_K^0 \tilde{R}_K^0 \hat{\psi}_{\mu} \hat{\psi}_{\nu} = g^{\alpha\lambda} g^{\beta\lambda} \epsilon_{\alpha\beta\mu\nu} \hat{\psi}_{\mu} \hat{\psi}_{\nu} \\
&= -\epsilon_{\mu\alpha\beta} \hat{\psi}_{\mu} \hat{\psi}_{\nu} \hat{\psi}_{\lambda} = -\hat{s}_\mu \hat{\psi}_{\lambda}.
\end{align*}
\]

\[
\square
\]
Theorem 3.21. (Stability properties of \( \hat{YM} \)). \( \hat{YM} \) is a functional stable quantum super PDE. In general a global solution \( V \subset (YM) \) is unstable and the corresponding observed solution, by means of a quantum relativistic frame, is unstable at finite times. However, \( \hat{YM} \) admits a stable quantum extended crystal PDE. There all the observed smooth solutions are stable at finite times. Furthermore, to study the asymptotic stability of a global solution \( V \subset (YM) \), with respect to a quantum relativistic frame, we can apply Theorem 2.46.

Proof. \( \hat{YM} \) is a functional stable quantum super PDE since it is completely integrable and formally integrable. (See Theorem 2.34). For the same reason it admits \((\hat{YM})_\infty \subset J\hat{D}_\infty(W)\) like stable quantum extended crystal PDE. Furthermore, since its symbol \( \hat{g} \) is not trivial, any global solution \( V \subset (YM) \) can be unstable, and the corresponding observed solution, can appear unstable in finite times. However, global smooth solution, result stable in finite times in \((\hat{YM})_\infty\). Finally the asymptotic stability study of global solutions of \((YM)\), with respect to a quantum relativistic frame, can be performed by means of Theorem 2.46, since, for any section \( s : M \rightarrow W \), on the fibers of \( \hat{E}[s] \rightarrow M \) there exists a non-degenerate scalar product. In fact, \( \hat{E}[s] \cong W \), as \( W \) is a vector bundle over \( M \). Furthermore, for any section \( s \), we can identify on \( M \) a non degenerate metric \( \hat{g} \) that beside the rigid metric \( g \) on \( g \), identifies a non-degenerate metric on each fiber \( \hat{E}[s]_p \cong W_p = Hom_Z(T_pM; g) \), \( \forall p \in M \). In fact we get \( \xi(p) : \xi(p)' = g_{K,H} \xi^A(p) \xi^B(p) \otimes \xi^H_B(p) \in A. \)

Example 3.22. (Stable nuclear-charged plasmas and nuclides). Of particular relevance are solutions of \((YM)\) that encode nuclear-charged plasmas, or nuclides, dynamics. These are described by solutions that, when observed by means of a quantum relativistic frame have at any \( t \in T \), i.e., frame-proper time, compact sectional support \( B_t \subset N \). The global mass at the time \( t \), i.e. the evaluation \( m_t = \int_{B_t} m(t, \xi^A) \sqrt{\det(g_{ij})} d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \) of such mass on the space-like section \( B_t \), gives the global mass-contents of the nuclear-plasmas or nuclides, in their ground-eigen-states, at the proper time \( t \). Whether such solutions are asymptotically stable, then they interpret the meaning of stable nuclear-plasmas or nuclides.

The following theorem shows how thermodynamic functions can be associated to solutions of \((YM)\).

Definition 3.23. (Thermodynamic states). A thermodynamic state is characterized by a set of independent parameters \( (s, c_\alpha) \) \( \leq \alpha \leq n \), that identify the internal energy: \( e = e(s, c_\alpha, a^i) \), where \( a^i \) identifies the \( i \)-th material particle, \( s \) is the specific entropy, \( (\text{with physical dimension energy/temperature mass}) \), and \( c_\alpha \) are electric or mechanical parameters. The system is said to be thermodynamically homogeneous if the following equations are satisfied: \( (\partial a_\alpha, e) = 0. \) Important thermodynamic functions and thermodynamic equations are given in Tab.9. \( ^{38} \) It should be more precise to denote \( \hat{g} \) with the symbol \( \hat{g}[s] \), since it is identified by means of the section \( s \).

\(^{39} \) One has \( t = \theta(s, c^\alpha, a^i), t^\alpha = t^\alpha(s, c^\alpha, a^i) \). Of course one can choose \( \theta \) and \( c^\alpha \) as independent thermodynamic functions: \( s = s(\theta, c^\alpha, a^i), t^\alpha = t^\alpha(\theta, c^\alpha, a^i) \) and \( e = e(\theta, c^\alpha, a^i) \).

\(^{40} \) With respect to the quantum relativistic frame it is identified on \( N \) a time-like vector field \( v \), such that for any scalar \( \mathbb{R} \)-valued thermodynamic function \( f \) one has \( \frac{df}{dt} = (\partial_t f) + v^k (\partial x_k, f) \).
| Name                          | Definition                                                                 |
|-------------------------------|---------------------------------------------------------------------------|
| temperature                   | $\theta \equiv (\partial s \cdot \nu)$                                    |
| thermodynamic stresses        | $t^n \equiv (\partial \nu^n \cdot \nu)$                                   |
| thermodynamic pressure        | $p \equiv t^1 = -\left( \partial (\frac{1}{\gamma} \nu) \cdot \nu \right)$  |
| chemical potential            | $t^u \equiv p^u \equiv (\partial \nu^u \cdot \nu) (**)                  |
| specific heats                | $C \equiv \frac{4\pi}{2} \lambda \left(\mathbf{v} - \nu \nu \cdot \mathbf{v} \right)$ |
| latent heats                  | $\hat{e}^n \equiv \frac{2}{\gamma} \lambda \left(\mathbf{v} - \nu \nu \cdot \mathbf{v} \right)$ |
| specific heat at $e^u = \text{cost.}$ | $C_t = C_c + \left( \partial \nu^u \cdot \nu \cdot \mathbf{v} \right)$, fixed $a.$ |
| free energy density           | $f = e - \theta s$                                                        |
| hentalpy density              | $h = e - t^n c_n$                                                         |
| free henthalpy density        | $g = h - \theta s$                                                        |
| Gibbs-equations of first type | $dc = \theta ds + t^n dc_m$, fixed $a.$                                   |
| Gibbs-equations of second type| $\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} + t^n \frac{\partial}{\partial \nu}$ |
| Thermodynamic evolution       | $C = C + \left( \partial \nu \cdot \nu \right)$                         |
| along thermodynamic curve     | $\tilde{e}_n = \frac{1}{\gamma} \lambda \left( \mathbf{v} - \nu \nu \cdot \mathbf{v} \right)$ |
|                               | $\tilde{m}_{m} = \frac{1}{\gamma} \lambda \left( \mathbf{v} - \nu \nu \cdot \mathbf{v} \right)$ |

(*) One denotes $-t^1$ if $c_1 = \frac{1}{\gamma}$.

(**) $c_n = c_m$ is the concentration of the $n$-component in a system with different components.

An observed thermodynamic curve, $\lambda = \lambda(t)$ is identified at fixed $a^u$.

**Theorem 3.24.** (Thermodynamics of $(\hat{Y}M)$). There exists a canonical way to characterize thermodynamic states of observed solutions of $(\hat{Y}M)$. Thermodynamic functions and equations of observed solutions of $(\hat{Y}M)$ can be encoded as scalar-valued differential operator on the extended fiber bundle $W[i] \times N \cong W[i] \times \mathbb{R} \equiv W[i]$, over $N$, whose sections $(\hat{\mu}, \hat{\beta})$ over $N$, represent an observed quantum fundamental field, $\hat{\mu}$, and a function $\beta : N \rightarrow \mathbb{R}$, thermal function. If $\beta = \frac{1}{\kappa_B \theta}$, where $\kappa_B$ is the Boltzmann constant and $\theta$ is the temperature, then the observed solution encodes a system in equilibrium with a heat bath.

**Proof.** Let $H$ be the Hamiltonian corresponding to an observed solution of $(\hat{Y}M)$.\(^{41}\) Let us denote by $E \in Sp(H)$. If $N(E) = \text{tr} \delta(E - \bar{H})$ denotes the degeneracy of $E$, let us define local partition function of the observed solution the Laplace transform of the degeneracy $N(E)$, with respect the spectrum $Sp(H)$ of $\bar{H}$. We get

$$Z(\beta) = \int_{Sp(H)} e^{-\beta E} N(E) dE$$

$$= \int_{Sp(H)} e^{-\beta E} \text{tr} \delta(E - \bar{H}) dE$$

$$= \text{tr} e^{-\beta \bar{H}}.$$  

So we get the following formula

$$Z(\beta) = \text{tr} e^{-\beta \bar{H}}$$  

\(^{41}\)Let us recall that $\bar{H}$ is a $\hat{A}$-valued function on the 4-dimensional space-time $N$, considered in the quantum relativistic frame.
where $\beta$ is the Laplace transform variable and it does not necessitate to be interpreted as the "inverse temperature", i.e., $\beta = \frac{1}{\kappa_B \theta}$, where $\kappa_B$ is the Boltzmann’s constant.\textsuperscript{42} Note that all above objects are local functions on the space-time $N$. The same holds for $\beta$. We can interpret $Z(\beta)$ as a normalization factor for the local probability density

\begin{equation}
P(E) = \frac{1}{Z} N(E) e^{-\beta E}
\end{equation}

that the system, encoded by the observed solution, should assume the local energy $E$, with degeneration $N(E)$. In fact we have:

\begin{equation}
1 = \int_{\mathcal{S}_p(\mathcal{H})} P(E) dE = \frac{1}{Z} \int_{\mathcal{S}_p(\mathcal{H})} N(E) e^{-\beta E} dE = \frac{Z}{Z}.
\end{equation}

As a by-product we get that the local average energy $\langle E \rangle$ can be written, by means of the partition function, in the following way:

\begin{equation}
e = -(\partial \beta \ln Z).
\end{equation}

In fact, one has

\begin{equation}
\begin{aligned}
e &= \int_{\mathcal{S}_p(\mathcal{H})} EP(E) dE = \frac{1}{Z} \int_{\mathcal{S}_p(\mathcal{H})} EN(E) e^{-\beta E} dE \\
e &= \frac{1}{2} \int_{\mathcal{S}_p(\mathcal{H})} \text{tr} (E - H) e^{-\beta E} dE \\
e &= \frac{Z}{2} \text{tr} (\mathcal{H} e^{-\beta \mathcal{H}}) = -\frac{1}{2} (\partial \beta. Z) = -(\partial \beta. \ln Z).
\end{aligned}
\end{equation}

When we can interpret $\beta = \frac{1}{\kappa_B \theta}$, then one can write

\begin{equation}
e = \kappa_B \theta^2 (\partial \ln Z).
\end{equation}

Then we get also that the local energy fluctuation is expressed by means of the variance of $e$:

\begin{equation}< (\Delta E)^2 > = < (E - e)^2 > = (\partial \beta \partial \beta. \ln Z).
\end{equation}

Furthermore, we get for the local heat capacity $C_v$ the following formula:

\begin{equation}
C_v = (\partial \beta. e) = \frac{1}{\kappa_B \theta^2} < (\Delta E)^2 >.
\end{equation}

We can define the local entropy by means of the following formula:

\begin{equation}
s = -\kappa_B \int_{\mathcal{S}_p(\mathcal{H})} P(E) \ln P(E) dE.
\end{equation}

In fact one can prove that one has the usual relation by means of the energy. (See Tab.9.) Really we get:

\begin{equation}
\begin{aligned}
s &= -\kappa_B \int_{\mathcal{S}_p(\mathcal{H})} P(E) \ln P(E) dE \\
&= \kappa_B (\ln Z + \beta e) = (\partial \theta. (\kappa_B \theta \ln Z)).
\end{aligned}
\end{equation}

Then from the relation $s = \kappa_B (\ln Z + \beta e)$ we get $e = \theta s - \kappa_B \ln Z$, hence also $(\partial s. e) = \theta$. This justifies the definition of entropy given in (122). Furthermore, from (123) we get also $\frac{1}{\theta} \ln Z = e - \theta s = f$, where $f$ is the Helmholtz free energy. It

\textsuperscript{42}If $\beta = \frac{1}{\kappa_B \theta}$ then the system encoded by the observed solution of $(\mathcal{Y} \mathcal{M})$, is in equilibrium with a heat bath (canonical system).
follows the following expression of the local Helmoltz free energy, by means of the local partition function $Z$:

$$f = e - \theta s = -\kappa_B \theta \ln Z.$$  

(124)

Conversely, from (124) it follows that the partition function can be expressed by means of the local Helmoltz free energy

$$Z = e^{-f}.$$  

(125)

So we see that the local thermodynamic functions, can be expressed as scalar-valued differential operators on the fiber bundle $W[i] \times N \rightarrow N$. The situation is resumed in Tab.10.

| Name                  | Definition                | Remark         | Order |
|-----------------------|---------------------------|----------------|-------|
| partition function    | $Z = \text{tr} (e^{-\beta_0})$ | $Z = Z(\beta, \rho)$ | 1     |
| interior energy       | $e = -(\beta \delta, \ln Z)$ | $e = \kappa_B \theta^2 (\delta \theta, \ln Z)$ | 1     |
| fluctuation interior energy | $\langle (\Delta E)^2 \rangle = \langle (E - e)^2 \rangle$ | $\langle (\Delta E)^2 \rangle = (\beta \delta \theta, \ln Z)$ | 2     |
| entropy               | $s = \kappa_B (\ln Z + \beta e)$ | $s = \beta \theta (\kappa_B \ln Z)$ | 1     |
| free energy           | $f = e - \theta s$        | $f = -\kappa_B \theta \ln Z$ | 1     |

Table 10 - Local thermodynamics functions of $(\hat{Y}, \hat{M})$ solutions.

It is assumed a Lagrangian of first derivation order.

The following lemmas relate spectral measures identified by $\hat{H}(p)$, $p \in N$, and the local partition function introduced in (114). It is useful to recall some definitions and results about quantum states. More precisely a quantum state of a quantum algebra $A$, is a function $S : A \rightarrow C$, that satisfies the following properties: (i) $S$ is $C$-linear; (ii) $S$ is self-adjoint; (iii) normalized by the constraint $\sup_{\|a\| \leq 1} S(a^\ast a) = 1$. A quantum pure state is one which is not a linear combination with positive coefficients of two other states, otherwise it is called mixed. A mixed state is described by its associated density operator $\rho = \sum p_s |\psi_s \rangle \langle \psi_s|$, where $p_s$ is the fraction of the set in each pure state $|\psi_s \rangle$. A criterion to see whether a density operator is describing a pure or mixed state is that $\text{tr} (\rho^2) = 1$ for pure state, and $\text{tr} (\rho^2) < 1$ for mixed state.

**Lemma 3.25.** (Gelfand-Naimark-Segal construction). If $A$ is a $C^\ast$-algebra, then every state on $A$ is of the following type $a \mapsto \langle \xi, \pi(a)(\xi) \rangle$, where $\pi : A \rightarrow L(\mathcal{H})$ is a representation of $A$ in an Hilbert space $\mathcal{H}$, and $\xi \in \mathcal{H}$ is a cyclic vector for $\pi$, i.e., $\pi(A) (\xi)$ is norm dense in $\mathcal{H}$, hence $\pi$ is a cyclic representation.

A set of quantum states of $A$ is called complete if the only element of $A$ which vanishes in every state of the set is zero. The variance of a self-adjoint element $a \in A$, in a quantum state $S$, is defined by $S(a^2) - S(a)^2$. $a$ is said to have the exact value $S(a)$, in the state $S$, in the case its variance vanishes in this state. A quantum value of $\hat{H}(p)$ is a point $\lambda(p) \in Sp(\hat{H}(p))$. The probability to find a quantum value of $\hat{H}(p)$ in the Borel set $U \subset \mathbb{R}$, if the system is in the quantum state $S$, is given by the following formula

$$p(\hat{H}(p); S, U) = \text{tr} (E_{\hat{H}(p)}(U) S) = \text{tr} (SE_{\hat{H}(p)}(U)),$$

where $E_{\hat{H}(p)}$ is the spectral measure of $E_{\hat{H}(p)}$. If $S = \psi \otimes \psi$, we have

$$p(\hat{H}(p); S, U) = \int_U d(E_{\hat{H}(p)}) \psi(\lambda),$$

(127)
where \((E_{\tilde{H}(p)})_{\psi} : \mathcal{B}(\mathbb{R}) \to \mathbb{R}\) is the measure on the \(\sigma\)-algebra of Borel subsets of \(\mathbb{R}\), given by \((E_{\tilde{H}(p)}(U))_{\psi} = \langle \tilde{H}(p)(U), \psi \rangle\). Then the mean value, or expectation value, of \(\tilde{H}(p)\) in the state \(S\) is given by the following formula:

\[
\begin{align*}
< \tilde{H}(p) >_S &= \int_{\text{spec}(\tilde{H}(p))} \lambda dE_{\tilde{H}(p)}(\lambda)S \\
&= \text{tr}(\tilde{H}(p)S) = \text{tr}(S\tilde{H}(p)).
\end{align*}
\]

If \(S = \psi \otimes \psi\) we have

\[
< \tilde{H}(p) >_S = \int_{\text{spec}(\tilde{H}(p))} \lambda d(E_{\tilde{H}(p)}(\lambda))_\psi.
\]

The corresponding variance, in the state \(S\), is given by the following:

\[
\begin{align*}
\sigma^2(\tilde{H}(p))_S &= \int_{\text{spec}(\tilde{H}(p))} (\lambda - < \tilde{H}(p) >)^2 d(E_{\tilde{H}(p)})(\lambda) \\
&= \text{tr}(\tilde{H}(p)^2 S) - < \tilde{H}(p) >^2.
\end{align*}
\]

If \(S = \psi \otimes \psi\) we have

\[
\begin{align*}
\sigma^2(\tilde{H}(p))_S &= \int_{\text{spec}(\tilde{H}(p))} (\lambda - < \tilde{H}(p) >)^2 d(E_{\tilde{H}(p)}(\lambda))_\psi \\
&= \|\tilde{H}(p)(\psi)\|^2 - < \tilde{H}(p)(\psi)|\psi >^2.
\end{align*}
\]

**Lemma 3.26.** (Gibbs canonical quantum state). The local partition function given in (113) is normalization factor of a quantum state called Gibbs canonical quantum state.

**Proof.** Let \(E_{\tilde{H}(p)} : (\mathbb{R}, \mathcal{B}) \to \tilde{A}\) be the spectral measure on the \(\sigma\)-algebra \(\mathcal{B} = \mathcal{B}(\mathbb{R})\) of Borel subsets of \(\mathbb{R}\), uniquely identified by \(\tilde{H}(p), p \in N\). Then, given a state \(S\), the distribution of \(\tilde{H}(p)\) under \(S\) is the probability measure on \(\mathcal{B}\), given by

\[
D_{\tilde{H}(p)}(U) = \text{tr}(E_{\tilde{H}(p)}(U)S)
\]

and the expected value, in the state \(S\), of \(\tilde{H}(p)\), is

\[
< \tilde{H}(p) >_S = \int_{\mathbb{R}} \lambda dD_{\tilde{H}(p)}(\lambda).
\]

One has \(< \tilde{H}(p) >_S = \text{tr}(\tilde{H}(p)S) = \text{tr}(S\tilde{H}(p))\). If \(S\) is a pure state corresponding to the vector \(\psi\), then \(< \tilde{H}(p) >_S = < \psi | \tilde{H}(p) | \psi >\). Furthermore, let \(Sp(\tilde{H}(p)) = Sp(\tilde{H}(p))_p\), i.e., let us assume that \(\tilde{H}(p)\) has a pure point-spectrum with eigenvalues \(E_n\), that go to \(+\infty\) as sufficiently fast, then \(e^{-\beta(\tilde{H}(p))}\) will be a non-negative trace-class operator for every positive \(\beta(p) \in \mathbb{R}\). One defines *Gibbs canonical quantum state*

\[
S = \frac{e^{-\beta\tilde{H}}}{\text{tr}(e^{-\beta\tilde{H}})} = \frac{e^{-\beta\tilde{H}}}{\sum_n e^{-\beta E_n}}.
\]

Then, \(\text{tr}(S\tilde{H}) = e\), as obtained in (114). Furthermore, the definition of entropy given in (123) just coincides with the von Neumann entropy of the state \(S\): \(s_{\text{von-Neumann}} = \text{tr}(S \ln S)\) [124].

\[\footnote{The state \(S\) can be diagonalized and one can write \(s_{\text{von-Neumann}} = -\sum \lambda_i \ln \lambda_i\), with the convention \(0 \ln 0 = 0\). This is a real number belonging to \([0, +\infty]\). \(s_{\text{von-Neumann}}\) measures the amount of randomness in the state \(S\). (Larger entropy corresponds to more dispersed eigenvalues.)}
From above results it follows that the way to define local thermodynamic functions, by means of local partition function and then characterizes associated thermodynamic functions, coincides with the expectation value of energy in a Gibbs canonical state, when the spectrum of the observed Hamiltonian is only a point spectrum. □

Remark 3.27. The approach given here to formulate the local thermodynamics of quantum systems, differs from one actually adopted in the literature. In fact, this last, usually necessitates to quantize classical systems and then characterizes associated thermodynamic functions by means of the energy-momentum tensor anomaly \(< \tilde{T}_\mu^\mu > \equiv \epsilon \). (See, e.g., Refs. [11, 12].) Really the trace of the classic energy-momentum tensor is zero, but in the process of quantization this conservation is not more assured and the trace of the quantized classic energy-momentum tensor can take a non-zero value: \( \epsilon \neq 0 \). This discrepancy between classic and quantum formulation is to ascribe to the fact that the process of quantization is performed by means of a linearization of the dynamic in the neighborhood of the classic solution, hence it discards non-linear effects. However, in our non-commutative framework, there is not discrepancy in the covariant description of quantum systems, with respect to eventual classic or superclassic analogues. Therefore, we can directly implement thermodynamic function on the observed quantum solutions, characterizing their spectral properties.

The concept of quantum states can be also related to a proof for existence of solutions with mass-gap. In fact, we have the following theorem, that completes Theorem 3.18, and recognizes an interior constraint in \((Y.M)\), where live solutions with mass gap.

Theorem 3.28. (Existence of \((Y.M)\) solutions with mass-gap.) Equation \((Y.M)\) admits local and global solutions with mass-gap. These are contained into a sub-equation, \((Higgs-quantum super PDE)\), \((Higgs) \subset (Y.M)\), that is formally integrable and completely integrable, and also a stable quantum super PDE. If \(H_3(M; K) = 0\), \((Higgs)\) is also a quantum extended crystal super PDE. In general solutions contained in \((Higgs)\) are not stable in finite times. However there exists an associated stabilized quantum super PDE, (resp. quantum extended crystal super PDE), where all global smooth solutions are stable in finite times. Furthermore, there exists a quantum super partial differential relation, (quantum Goldstone-boundary), \((Goldstone) \subset (Y.M)\), bounding \((Higgs)\), such that any global solution of \((Y.M)\), loses/acquires mass, by crossing \((Goldstone)\).

Proof. The Hamiltonian of \((Y.M)\) can be identified with a \(Q^\omega\)-function \(\dot{H} : JD(W) \rightarrow \dot{A}\). Therefore, the set of solutions, with mass-gap, of the trivial equation \(JD(W) \subseteq JD(W)\) can be identified with the subset \(\dot{H}_{(Higgs)} = H^{-1}(G(\dot{A})) \subset JD(W)\), where \(G(\dot{A})\) is the abelian group of units of \(\dot{A}\). In general one has \(G(\dot{A}) \subseteq \dot{A}\), where the quantum state \(S\) is called maximally mixed quantum state if it maximizes \(s_{von-Neumann}(S) = 0\) if \(S\) is a pure state, i.e., \(S = |\psi><\psi|\).

The symmetries properties of the Higgs-quantum super PDE \((Higgs)\) are different from the ones of the quantum super Yang-Mills equation \((Y.M)\), we can consider this theorem like a quantum dynamic generalization of the usual Higgs-breaking-symmetry mechanism. This justifies the name given, and the symbol, used to denote the constraint in \((Y.M)\) where live the solutions with mass-gap.

\[\text{\textsuperscript{44}}\text{Since the symmetries properties of the Higgs-quantum super PDE (Higgs) are different from the ones of the quantum super Yang-Mills equation (Y.M), we can consider this theorem like a quantum dynamic generalization of the usual Higgs-breaking-symmetry mechanism. This justifies the name given, and the symbol, used to denote the constraint in (Y.M) where live the solutions with mass-gap.}\]
equality happens iff $\hat{A}$ is a division algebra. It is well known that the only associative division algebras are the real $\mathbb{R}$, complex $\mathbb{C}$, and quaternion $\mathbb{H}$ numbers. Let us exclude these cases.\textsuperscript{45} Therefore solutions of $(\hat{YM})$ with mass-gap are all the solutions of the following augmented partial differential relations:

$$
\left\{(\hat{\text{Higgs}}) \equiv (\hat{YM}) \bigcap \hat{H}_{(\text{Higgs})} \subset JD^2(W) \right\}
$$

In order to study the topological-differential structure of $\hat{H}_{(\text{Higgs})}$, and hence that of $(\hat{\text{Higgs}})$, let us consider the quantum states. In fact, between quantum states, useful to characterize energetic contents of $(\hat{YM})$ solutions, there are ones coming from characters of quantum algebras. More precisely we give the following definition.

**Definition 3.29.** We define character-quantum state a character of the quantum algebra $\hat{A}$, i.e., a unitary multiplicative linear function $\chi : \hat{A} \to \mathbb{C}$. We denote by $\text{Ch}(\hat{A})$ the set of such quantum states.\textsuperscript{46}

One has the following lemma that gives an alternative way to identify solutions of $(\hat{YM})$ with mass-gap.

**Lemma 3.30.** (Character-quantum states and mass-gap). A solution of $(\hat{YM})$ has mass-gap in $p \in M$, iff for any character-quantum-state $\chi \in \text{Ch}(\hat{A})$, one has $\chi(H(p)) \neq 0$.

**Proof.** It is a direct consequence of the following lemma.

**Lemma 3.31.** Let $B$ be a $\mathbb{K}$-algebra. Then $b \in G(B)$ iff $\chi(b) \neq 0$, $\forall \chi \in \text{Ch}(B)$, where $\text{Ch}(B)$ is the set of characters of $B$. (One has also $\chi(b) \in \text{Sp}(b)$, $\forall b \in B$.)

**Proof.** It is standard. (See e.g., Ref.\textsuperscript{[8]}.)

In fact we have $H(p) \in \hat{A}$, for any $p \in N$. \hfill $\square$

**Lemma 3.32.** (Topology of $G(\hat{A})$). $G(\hat{A})$ is an open set in $\hat{A}$.

**Proof.** Let us emphasize that the group of units of a topological ring may not be a topological group using the subspace topology, as inversion on the unit group need not be continuous with the subspace topology. However, we have the following lemma.

**Lemma 3.33.** The operation of taking an inverse is continuous on quantum algebras.

**Proof.** In fact, a quantum (super)algebra is a Fréchet algebra, and for such algebras it is well known that the operation of taking an inverse is continuous [29]. \hfill $\square$

Let us use, now, the following lemma.

**Lemma 3.34.** The operation of taking an inverse is continuous for a general $F$-algebra $A$ iff the group $G(A)$ of its invertible elements is a $G_\delta$-set.\textsuperscript{47}

\textsuperscript{45}If $A$ is an associative division algebra, identified with $\mathbb{K} \equiv \mathbb{R}$ or $\mathbb{K} \equiv \mathbb{C}$, the constraint $\hat{H}_{(\text{Higgs})}$ is trivially an open set $\hat{H}_{(\text{Higgs})} \subset JD(W)$. In fact, when $A = \mathbb{K}$, one has $Z(A) = \mathbb{K}$ and $\mathbb{K} = \mathbb{K} = A$, hence $G(\hat{A})$ is the open set $G(\hat{A}) = A \setminus \{0\}$. Therefore, $\hat{H}_{(\text{Higgs})} \equiv H^{-1}(A \setminus \{0\})$, is necessarily an open quantum submanifold of $JD(W)$.

\textsuperscript{46}We call $\text{Ch}(\hat{A})$ also spectrum of $\hat{A}$ and we write $\text{Sp}(\hat{A}) = \text{Ch}(\hat{A})$.

\textsuperscript{47}In a topological space, a $G_\delta$-set is one countable intersection of open sets.
Proof. See, e.g., [29] and references quoted there.

Then taking into account that a quantum (super)algebra is a Fréchet algebra, hence must necessarily $G(A)$ be a $G_δ$-set, therefore an open set in $A$. Note that this result could directly follow from Lemma 3.31 if we could state that all characters of $\hat{A}$ are continuous functions. But this is not assured for a topological algebra. (Michael-Mazur’s problem [57].) By the way, we can state that there exists a countable subset $Ch(\hat{A})_c^0 \subseteq Ch(\hat{A})$ of continuous characters such that $G(\hat{A}) = \bigcap_{\chi \in Ch(\hat{A})_c} G(\hat{A})_\chi$, where $G(\hat{A})_\chi \equiv \chi^{-1}(\mathbb{C} \setminus \{0\})$ is an open set in $\hat{A}$. This open set cannot be empty, since it surely contains the identity map $1_A \in \hat{A}$. □

Lemma 3.35. (Topology of $\hat{H}(\text{Higgs})$). $\hat{H}(\text{Higgs})$ is an open quantum submanifold of $J\hat{D}(W)$.

Proof. From above lemma it follows that $\hat{H}(\text{Higgs})$ is an open set in $J\hat{D}(W)$ since $\hat{H}$ is a continuous map as it is of class $Q_{\text{w}}$. □

Let us, now, remark that the restriction $\pi_{2,1}$ to $(\hat{Y}M)$ of the canonical projection $\pi_{2,1} : J\hat{D}^2(W) \rightarrow J\hat{D}(W)$, gives a surjective mapping $\pi_{2,1} : (\hat{Y}M) \rightarrow J\hat{D}(W)$ too. Therefore, we can identify the following constraint $(\text{Higgs}) \equiv \pi_{2,1}(\hat{H}(\text{Higgs})) \subset (\hat{Y}M)$. One has the following commutative diagram, with vertical exact lines.

\[
\begin{array}{cccc}
\hat{H}(\text{Higgs}) & \rightarrow & (\hat{Y}M) & \rightarrow J\hat{D}^2(W) \\
\downarrow & & \downarrow & \\
0 & \rightarrow & J\hat{D}(W) & \rightarrow J\hat{D}(W)
\end{array}
\]

Taking into account that $\pi_{2,1}$ is a continuous mapping and that $\hat{H}(\text{Higgs})$ is an open subset of $J\hat{D}(W)$, it follows that also $(\text{Higgs})$ is an open quantum submanifold of $(\hat{Y}M)$ over the open quantum submanifold $\hat{H}(\text{Higgs})$ of $J\hat{D}(W)$. Since $(\hat{Y}M)$ is formally integrable and completely integrable, it follows that also $(\text{Higgs})$ is so. By conclusion, the situs of solutions of $(\hat{Y}M)$ with mass-gap is the formally integrable and completely integrable quantum super PDE $(\text{Higgs}) \subset (\hat{Y}M)$. For any point $q \in (\text{Higgs})$ there exists a solution of $(\hat{Y}M)$, having mass-gap. The characterization of global solutions is made by means of integral bordism groups of $(\text{Higgs})$. Since this last equation is an open quantum submanifold of $(\hat{Y}M)$ its integral bordism groups coincide with the ones of $(\hat{Y}M)$. (See Theorem 3.12.) Therefore, if $H_3(M; \mathbb{K}) = 0$, equation $(\text{Higgs})$ is a quantum extended crystal super PDE. Moreover, under the full admissibility hypothesis, it becomes a quantum 0-crystal super PDE. From Theorem 2.34 it follows that $(\text{Higgs})$ is a stable quantum extended crystal super PDE. Since its symbol is not trivial, we get also that in general such solution with mass-gap are not stable in finite times. However, we get also that for the infinity
Figure 1. Quantum matter-solution and elementary bordism-decompositions. a, b, c and d are quantum-matter solutions. r, s and t are quantum-matter-free solutions. All the solution $V \equiv a \cup r \cup b \cup s \cup t \cup c \cup d$ is a quantum-matter-solution.

prolongation of such an equation, one has $(\widehat{\text{Higgs}})_{+\infty} \subset (\widehat{\text{YM}})_{+\infty}$. Hence we can state that $(\widehat{\text{Higgs}})$ is a stabilizable quantum extended crystal super PDE, with stable quantum extended crystal super PDE just $(\widehat{\text{Higgs}})_{+\infty}$. There all global smooth solutions have mass-gap, and are stable in finite times. In order to fix ideas, let us, before to complete the proof of Theorem 3.28, consider the following example containing also some useful definitions.

Example 3.36. We call a global solution $V \subset (\widehat{\text{YM}})$ a quantum matter-solution when $V \cap (\widehat{\text{Higgs}}) \neq \emptyset$, otherwise we say that $V$ is a quantum matter-free-solution. Of the first type are solutions describing, e.g., the following particles or nuclear reactions: $\gamma + p \rightarrow \pi^{\pm} + p$, $\pi^{\pm} \rightarrow \mu^{\pm} + \nu_{\mu}$, $Li + H \rightarrow 2He e + 22.4MeV$. Instead examples of quantum matter-free-solutions are ones encoding electro-magnetic fields or gluons fields. A generic global solution can be made by pieces that are completely inside $(\widehat{\text{Higgs}})$, quantum pure-matter-solution, and other ones that are outside $(\widehat{\text{Higgs}})$, quantum matter-free-solution. The case of photoproduction of pions is an example of solutions of this type. Another one is a monopole-vortex chain in $SU(2)$ gauge theory. Then Theorem 2.19 in [93] gives a decomposition of such solutions as union of a finite number of (integral) bordisms between elementary bordisms. Some of these are quantum pure-matter-bordisms (resp. quantum matter-free-bordisms), since completely inside, (resp. outside) $(\widehat{\text{Higgs}})$. (See Fig.1.) Let $V \subset (\widehat{\text{YM}})$ be such a quantum matter-solution, such that $\partial V = N_{0} \cup P \cup N_{1}$, with $N_{0} \subset (\widehat{\text{Higgs}})$ and $N_{1} \subset (\widehat{\text{Higgs}})$. Set $\partial V \equiv V \cap (\widehat{\text{Higgs}})$ and $\bullet V \equiv V \setminus \partial V \subset (\widehat{\text{Higgs}})$. Then we can write $V = \partial V \cup \bullet V$. Since $(\widehat{\text{Higgs}})$ is open in $(\widehat{\text{YM}})$, it follows that $\partial V \cap ((\widehat{\text{Higgs}}) \setminus (\widehat{\text{Higgs}})) = \emptyset$ and $\bullet V \cap ((\widehat{\text{Higgs}}) \setminus (\widehat{\text{Higgs}})) \equiv V_{0} \subset (\widehat{\text{Higgs}})$. Furthermore one has $V_{0} \in [N_{1}]$ and $N_{0} \in \text{Bor}_{m-1|n-1}((\widehat{\text{YM}}); A)$. 
Let us conclude the proof of the theorem by considering the boundary \((\text{Goldstone}) \equiv \partial(\text{Higgs})\) of \((\text{Higgs})\), i.e., \((\text{Goldstone}) \equiv (\text{Higgs}) \setminus (\text{Higgs})\). Since \((\text{Goldstone}) \subset (\text{YM})\), for any point \(q \in (\text{Goldstone})\) there exists a solution \(r \subset (\text{YM})\) of \((\text{YM})\).

\(r\) can be a quantum matter-free-solution. However, since \((\text{Goldstone})\) is dense in \((\text{Higgs})\), \(q\) is limit point of a sequences of points \(q_i \in (\text{Higgs})\), \(i \in J\). To each point \(q_i\) there corresponds a quantum matter-solution \(s_i \subset (\text{Higgs})\). This means that for any quantum matter-free-condition \(q \in (\text{Goldstone})\), we can always find quantum matter-solutions \(s_i \subset (\text{Higgs})\) converging to \(q\). Then by a surgery technique we can prolong such quantum matter-solutions to a solution \(V\), across \((\text{Goldstone})\), soldering with the quantum matter-free solution \(r\). Since \((\text{YM})\) and \((\text{Higgs})\) are formally integrable and completely integrable, we can repeat this surgery technique to the infinity prolongations of these equations, i.e., by considering the sequence: \((\text{Higgs})_{+\infty} \subset (\text{Goldstone})_{+\infty} \subset (\text{YM})_{+\infty}\), by obtaining also quantum smooth solutions that passing across \((\text{Goldstone})_{+\infty}\) acquire/lose mass. Such solutions, are therefore stable in finite times.

**Definition 3.37.** Let \(V \subset (\text{YM})\) be a global solution crossing the quantum Goldstone-boundary \((\text{Goldstone})\). Let us call \(V \partial \equiv V \cap (\text{Goldstone})\) the Goldstone-piece of \(V\).

**Corollary 3.38.** (Goldstone-piece characterization of \((\text{YM})\) solutions). In any connected global solution \(V \subset (\text{YM})\) bording Cauchy data \(N_0 \subset (\text{YM}) \setminus (\text{Higgs})\) and \(N_1 \subset (\text{Higgs})\), there exists a Goldstone piece.\(^{48}\)

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\(^{48}\) Theorem 3.28 and Corollary 3.37, thanks to our geometric theory of quantum (super)PDE's, give a full quantum dynamical justification to mass production/destruction mechanism in quantum solutions of \((\text{YM})\). These results can be considered generalizations, in the geometric theory of quantum PDE’s, of the well-known Goldstone’s theorem, and “Higgs symmetry-breaking mechanism” conjectured, in the second middle of the last century, in order to justify mass in a gauge theory. (See Refs.[22, 23, 30, 31].)
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