On Fedosov’s approach to Deformation Quantization with Separation of Variables

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Abstract

The description of all deformation quantizations with separation of variables on a Kähler manifold from [8] is used to identify the Fedosov star-product of Wick type constructed by M. Bordemann and S. Waldmann in [3]. This star-product is shown to be the one with separation of variables which corresponds to the trivial deformation of the Kähler form in the sense of [8]. To this end a formal Fock bundle on a Kähler manifold is introduced and an associative multiplication on its sections is defined.

Introduction

For a given vector space $E$ we call formal vectors the elements of the space $E[\nu^{-1}, \nu]$ of formal Laurent series in a formal parameter $\nu$ with a finite principle part and coefficients in $E$. Thus we consider the field of formal numbers $K = \mathbb{C}[\nu^{-1}, \nu]$, formal functions, forms and differential operators.

Deformation quantization of a Poisson manifold $(M, \{\cdot, \cdot\})$, as defined by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [2], is a structure of associative algebra on the space of formal functions $\mathcal{F} = C^\infty(M)[\nu^{-1}, \nu]$. The product $\ast$ in this algebra (called a star-product) is a $K$-linear $\nu$-adically continuous product given on functions $f, g \in C^\infty(M)$ by the formula

$$f \ast g = \sum_{r=0}^{\infty} \nu^r C_r(f, g). \quad (1)$$
In (1) $C_r$ are bidifferential operators such that $C_0(f, g) = fg$, $C_1(f, g) = C_1(g, f) = i\{f, g\}$. The constant 1 is assumed to be the unit in the algebra $(\mathcal{F}, \ast)$.

Two star-products $\ast_1$ and $\ast_2$ are called equivalent if there exists an isomorphism of algebras $B : (\mathcal{F}, \ast_1) \to (\mathcal{F}, \ast_2)$ given by a formal differential operator $B = 1 + \nu B_1 + \nu^2 B_2 + \ldots$.

The problem of existence and classification up to equivalence of star-products on Poisson manifolds was first solved for symplectic manifolds (the main references are [5,6,7,12,13]; for a historical account see [14]). In the general case it was solved by Kontsevich [10].

Let $M$ be a Kähler manifold, endowed with a Kähler $(1, 1)$-form $\omega_{-1}$ and the corresponding Poisson bracket. In [8] we gave a simple geometric description of all star-products on $M$ which have the following property of separation of variables: in a local holomorphic chart the operators $C_r$ from (1) act on the first argument by antiholomorphic derivatives, and on the second argument by holomorphic ones. We have shown that these star-products are naturally parametrized by geometric objects, the formal deformations of the Kähler form $(1/\nu)\omega_{-1}$.

The interest in deformation quantization with separation of variables is explained by the fact that the Wick star-product on $\mathbb{C}^n$ and the star-products obtained from Berezin’s quantization on Kähler manifolds in [4,11,9] have the property of separation of variables. In [3] Bordemann and Waldmann constructed a star-product with separation of variables on an arbitrary Kähler manifold $(M, \omega_{-1})$, using the geometric approach developed by Fedosov in [6,7]. The goal of this letter is to identify the star-product obtained in [3], using the parametrization from [8]. We show that this star-product corresponds to the trivial deformation of the Kähler form $(1/\nu)\omega_{-1}$.

1. **Deformation quantizations with separation of variables**

For an open subset $U \subset M$ set $\mathcal{F}(U) = C^\infty(U)[\nu^{-1}, \nu]$. Since the star-product (1) is given by formal bidifferential operators, it can be localized to any open subset $U \subset M$. We denote its restriction to $\mathcal{F}(U)$ also by $\ast$.

Denote by $\mathcal{L}^\ast(U)$ and $\mathcal{R}^\ast(U)$ the sets of all operators of left and right star-multiplication in the algebra $(\mathcal{F}(U), \ast)$ respectively. All these operators are formal differential ones. The subalgebras $\mathcal{L}^\ast(U)$ and $\mathcal{R}^\ast(U)$ of the algebra of formal differential operators on $U$ are commutants of each other.

Now let $(M, \omega_{-1})$ be a Kähler manifold with the Kähler $(1, 1)$-form $\omega_{-1}$. 


Consider a star-product * on $M$ with the following property of separation of variables. For an arbitrary local coordinate chart $U \subset M$ with holomorphic coordinates $\{z^k\}$ (and antiholomorphic coordinates $\{\bar{z}^l\}$) assume that the operators from $L^*(U)$ contain only holomorphic derivatives and the operators from $R^*(U)$ contain only antiholomorphic ones. This is equivalent to the fact that the operators from $L^*(U)$ and $R^*(U)$ commute with the point-wise multiplication operators by antiholomorphic and holomorphic functions on $U$ respectively. It means that, given a holomorphic function $a$ and antiholomorphic function $b$ on $U$, the point-wise multiplication operators by $a$ and $b$ belong to $L^*(U)$ and $R^*(U)$ respectively. Therefore $L^*_a = a$ and $R^*_b = b$, so that for $f \in F(U)$ $a \ast f = af, f \ast b = bf$ holds. This property was used for the definition of quantization with separation of variables in [8].  

It was shown in [8] that the star-products with separation of variables on $(M, \omega_{-1})$ are in $1-1$ correspondence with the formal deformations of the Kähler form $(1/\nu)\omega_{-1}$, i.e., with the formal forms $\omega = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \ldots$ such that all $\omega_r$, $r \geq 0$, are closed but not necessarily nondegenerate $(1,1)$-forms on $M$.  

Given an arbitrary formal deformation $\omega$ of the Kähler form $(1/\nu)\omega_{-1}$, one can recover the corresponding star-product with separation of variables as follows. On each contractible coordinate chart $(U, \{z^k\})$ on $M$ choose a formal potential $\Phi = (1/\nu)\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \ldots$ of the form $\omega$, so that $\omega = i\partial\bar{\partial}\Phi$. Then $L^*_{\partial\Phi/\partial z^k} = \partial\Phi/\partial z^k + \partial/\partial z^k$ and $R^*_{\partial\Phi/\partial \bar{z}^l} = \partial\Phi/\partial \bar{z}^l + \partial/\partial \bar{z}^l$. Moreover, the set $L^*(U)$ consists of all formal differential operators which commute with all $R^*_{\partial\Phi/\partial z^k} = \bar{z}^l$ and $R^*_{\partial\Phi/\partial \bar{z}^l} = \partial\Phi/\partial \bar{z}^l + \partial/\partial \bar{z}^l$, and, respectively, $R^*(U)$ is the commutant of the set of all operators $L^*_a = z^k$ and $L^*_b = \partial\Phi/\partial z^k$. This completely determines the star-product.  

Remark. In [3] star-products with separation of variables on Kähler manifolds are called star-products of Wick type, since the Wick star-product is the simplest one of this kind. However, one can consider star-products with separation of variables on an arbitrary symplectic manifold endowed with a pair of transversal Lagrangean polarizations (see [1]). In the Kähler case these are the holomorphic and antiholomorphic polarizations.  

2. The formal Wick algebras bundle and the formal Fock bundle  

Consider $\mathbb{C}^n$ with holomorphic coordinates $\{\zeta^k\}$ (and antiholomorphic coordinates $\{\bar{\zeta}^l\}$) endowed with a Hermitian $(1,1)$-form $ig_{kl}d\zeta^k \wedge d\bar{\zeta}^l$ (here $g_{kl}$ are constants). Denote by $\circ$ the Wick star-product on $(\mathbb{C}^n, ig_{kl}d\zeta^k \wedge d\bar{\zeta}^l)$. This
is the star-product with separation of variables, corresponding to the trivial
deformation of the (1,1)-form \((1/\nu)ig_{kl}d\zeta^k \wedge d\bar{\zeta}^l\). The Wick star-product of
functions \(f, g \in C^\infty(C^n)\) is given by the well-known explicit formula
\[
f \circ g = \sum_{r=0}^{\infty} \frac{\nu^r}{r!} g_{1k_1} \cdots g_{rk_r} \frac{\partial^r f}{\partial \zeta^{l_1} \cdots \partial \zeta^{l_r}} \frac{\partial^r g}{\partial \zeta^{k_1} \cdots \partial \zeta^{k_r}},
\]
where \((g_{kl})\) is the matrix inverse to \((g_{kl})\). Here, as well as in the rest of
the letter we use Einstein’s summation convention.

Introduce the following gradings on the variables \(\nu, \zeta^k, \bar{\zeta}^l\) :
\(\deg_\nu(\zeta) = \deg_\nu(\bar{\zeta}) = 0; \deg_\nu'(\zeta) = 1, \deg_\nu'(\bar{\zeta}) = 0; \deg_\nu''(\zeta) = 1, \deg_\nu''(\bar{\zeta}) = 0; \deg_\nu = \deg_\nu' + \deg_\nu''; \deg_\nu' = \deg_\nu + \deg_\nu''; \deg_\nu'' = \deg_\nu + \deg_\nu''\). The Wick product \(\circ\) is a graded
product on polynomials in \(\nu, \zeta^k, \bar{\zeta}^l\) with respect to the gradings \(\deg_\nu', \deg_\nu''\) and \(\deg_\nu\). The total grading \(\deg_\nu\) is
analogous to the one on the formal Weyl algebra used by Fedosov.

The ”normal ordering” procedure establishes a 1—1 correspondence be-
tween the polynomials from \(K[\zeta^k, \bar{\zeta}^l]\) and holomorphic differential
operators on \(C^n\) with coefficients in \(K[\zeta^k]\). Set \(\hat{\zeta}^k = \zeta^k, \hat{\zeta}^l = \nu g_{lk} \partial / \partial \zeta^k\). The ”normal ordering” relates to a polynomial
\(\phi(\zeta, \bar{\zeta}) = \phi_{\alpha, \beta} \zeta^\alpha \bar{\zeta}^\beta\) the operator
\(\hat{\phi} = \phi_{\alpha, \beta} \zeta^\alpha \bar{\zeta}^\beta\). Here \(\alpha = (k_1, \ldots, k_p), \beta = (l_1, \ldots, l_q)\) are multi-indices,
\(\zeta^\alpha = \zeta^{k_1} \cdots \zeta^{k_p}, \bar{\zeta}^\beta = \bar{\zeta}^{l_1} \cdots \bar{\zeta}^{l_q}\), \(\zeta^\alpha = \hat{\zeta}^{k_1} \cdots \hat{\zeta}^{k_p}, \bar{\zeta}^\beta = \hat{\zeta}^{l_1} \cdots \hat{\zeta}^{l_q}\) and \(\phi_{\alpha, \beta} \in K\)
is symmetric with respect to \(\alpha\) and \(\beta\) separately. The polynomial \(\phi\) is called
the Wick symbol of the operator \(\hat{\phi}\). The operator product transferred to
Wick symbols provides the Wick product \(\circ\).

The Wick product \(\circ\) can be extended to the space \(W\) of formal series in
\(\nu^{-1}, \nu, \zeta^k, \bar{\zeta}^l\) with a finite principal part in \(\nu\),
\[
w = \sum_{r \geq r_0, p, q \geq 0} \nu^r \sum_{\alpha, \beta, |\alpha| = p, |\beta| = q} w_{r, \alpha, \beta} \zeta^\alpha \bar{\zeta}^\beta.
\]
Here \(r_0 \in \mathbb{Z}\), \(\alpha = (k_1, \ldots, k_p), \beta = (l_1, \ldots, l_q)\) are multi-indices, \(\zeta^\alpha = \zeta^{k_1} \cdots \zeta^{k_p}, \bar{\zeta}^\beta = \bar{\zeta}^{l_1} \cdots \bar{\zeta}^{l_q}\), and the terms of the series are ordered by
increasing degrees \(\deg_\nu = p + q + 2r\). Thus obtained algebra \((W, \circ)\) is called a
formal Wick algebra.

A formal Fock space \(V\) on \(C^n\) is the subspace of \(W\) of formal series in
\(\nu\) and \(\zeta^k\), i.e., of the formal series \(v = \sum_{r \geq r_0, \alpha} \nu^r v_{r, \alpha} \zeta^\alpha\). Denote by \(\bar{V}\) the
 subspace of \(W\) of formal series in \(\nu\) and \(\zeta^l\).
Consider the following projection operators in $W$, $\Pi' w = w|_{\zeta=0}$, $\Pi'' w = w|_{\zeta=0}$ and $\Pi w = w|_{\bar{\zeta}=0}$, $w \in W$. Then $\Pi' W = V$, $\Pi'' W = \bar{V}$ and $\Pi W = K$.

The kernels of the projections $\Pi'$ and $\Pi''$ consist of the formal series $w \in W$ with all the terms containing at least one antiholomorphic variable $\bar{\zeta}^l$ or a holomorphic variable $\zeta^k$ respectively. It is easy to check that $\text{Ker} \ \Pi'$ and $\text{Ker} \ \Pi''$ are a left and a right ideals in the Wick algebra $(W, \circ)$ respectively. It follows, in particular, that $\text{Ran} \ \Pi' = V \cong W/\text{Ker} \ \Pi'$ is a left $W$-module.

An element $w \in W$ acts on $V$ by a formal holomorphic differential operator $T_w$ on $\mathbb{C}^n$ given by the formula $T_w v = \Pi''(w \circ v)$, $v \in V$. One can show that if $w \in K[\zeta, \bar{\zeta}]$ then $T_w = \hat{w}$, i.e., $T_w$ is the differential operator with the Wick symbol $w$. We shall say for general $w \in W$ that $w$ is the Wick symbol of $T_w$ and denote $T_w = \hat{w}$. It is easy to check that the mapping $W \ni w \mapsto \hat{w}$ is an injective homomorphism of the algebra $(W, \circ)$ to the algebra of formal differential operators on $\mathbb{C}^n$.

**Lemma 1.** For $w \in W$ $\Pi' w = 0$ iff the operator $\hat{w}$ annihilates the subspace of formal constants $K \subset V$, and $\Pi'' w = 0$ iff $\text{Ran} \ \hat{w} \subset \text{Ker} \ \Pi$.

The proof of the lemma follows from elementary properties of Wick symbols.

Given a Kähler manifold $(M, \omega_\perp)$ of the complex dimension $\dim_{\mathbb{C}} M = n$, consider the unions of the formal Wick algebras and of the formal Fock spaces associated to each tangent space to $M$. Thus we obtain the bundles of formal Wick algebras $W$ and of formal Fock spaces $V$ on $M$. For an open subset $U \subset M$ denote by $W(U)$ and $V(U)$ the spaces of local sections of $W$ and $V$ on $U$ respectively. Set $W = W(M)$, $V = V(M)$.

On a coordinate chart $(U, \{z^k\})$ on $M$ introduce the following gradings on 1-forms $dz^k, d\bar{z}^l$: $\deg_a'(dz) = \deg_a''(dz) = 1$, $\deg_a'(d\bar{z}) = \deg_a''(d\bar{z}) = 0$; $\deg_a = \deg_a' + \deg_a''$. Denote $\Lambda = \oplus_r \Lambda^r$ the $\deg_a$-graded algebra of differential forms on $M$.

There exist natural inclusions of the spaces $\mathcal{F} \otimes \Lambda \subset \mathcal{V} \otimes \Lambda \subset W \otimes \Lambda$ of the (formal) scalar, $\mathcal{V}$- and $W$-valued differential forms on $M$ respectively (the tensor product is taken over $C^\infty(M)$, $\otimes = \otimes_{C^\infty(M)}$).

The fibrewise Wick product and the action of $W$ on $V$ in the first factor of the tensor product together with the wedge-product of differential forms in the second factor define the structures of $\deg_a$-graded algebra on $W \otimes \Lambda$ and of its $\deg_a$-graded module on $V \otimes \Lambda$. The product in $W \otimes \Lambda$ will be denoted also $\circ$. The projections $\Pi, \Pi'$ and $\Pi''$ define fibrewise projections in $W \otimes \Lambda$ denoted by the same symbols. The action of an element $w \in W \otimes \Lambda$
on the space $\mathcal{V} \otimes \Lambda$ is given by the operator $\hat{w}$ defined, as above, by the expression $\hat{w}v = \Pi'(w \circ v)$, where $v \in \mathcal{V} \otimes \Lambda$. We have $\Pi'(\mathcal{W} \otimes \Lambda) = \mathcal{V} \otimes \Lambda$ and $\Pi(\mathcal{W} \otimes \Lambda) = \mathcal{F} \otimes \Lambda$.

In the sequel we shall always denote by $\zeta^k, \zeta^l$ the fiber coordinates on the tangent bundle $TM$ in the frame $\{\partial/\partial z^k, \partial/\partial \bar{z}^l\}$ on a coordinate chart $(U, \{z^k\})$ on $M$.

Notice that for a local section $w(z, \bar{z}) = \sum_{r \geq t} w_{r,\alpha,\beta}(z, \bar{z}) \zeta^\alpha \bar{\zeta}^\beta \in \mathcal{W}(U)$ the coefficients $w_{r,\alpha,\beta}(z, \bar{z})$ are covariant tensor fields on $M$, symmetric with respect to $\alpha$ and $\beta$ separately.

### 3. Fedosov star-product of Wick type

Recall the construction by Bordemann and Waldmann of the Fedosov star-product of Wick type on a Kähler manifold $(M, \omega_{-1})$ from [3]. (We use, however, different conventions and notations.)

Let $\nabla$ denote the standard Kähler connection on $M$. It can be naturally extended to tensors, and thus to the bundles $\mathcal{W}$ and $\mathcal{V}$. For technical reasons it will be convenient to denote its extension to $\mathcal{W}$ also by $\nabla$, and its extension to $\mathcal{V}$ by $\hat{\nabla}$.

Express the Kähler form $\omega_{-1}$ on $M$ and the Kähler connection $\nabla$ on $\mathcal{W} \otimes \Lambda$ in local coordinates $\{z^k, \bar{z}^l, \zeta^k, \bar{\zeta}^l\} : \omega_{-1} = ig_{kl} dz^k \wedge d\bar{z}^l$, $\nabla = d - \Gamma^s_{kl} \zeta^s(\partial/\partial \zeta^s) dz^k - \Gamma^s_{ij} \bar{\zeta}^s(\partial/\partial \bar{\zeta}^s) d\bar{z}^l$, where $\Gamma^s_{ij} = \partial g_{kl} / \partial z^i$ and $\Gamma^s_{ij} = \partial g_{kl} / \partial \bar{z}^j$ are the Christoffel symbols and $(g^{kl})$ is the matrix inverse to $(g_{kl})$. Then $\hat{\nabla} = d - \Gamma^s_{kl} \zeta^s(\partial/\partial \zeta^s) dz^k$.

Introduce an element $R \in \mathcal{W} \otimes \Lambda^2$ such that it is given in local coordinates $\{z^k, \bar{z}^l, \zeta^k, \bar{\zeta}^l\}$ by the formula $R = (-g^{ls} \partial g_{kl} + \partial g_{kl} + \partial g_{kl}) \zeta^k \bar{\zeta}^l$.

The curvature of the connection $\nabla$ on the bundle $\mathcal{W}$ was calculated in [3]: $\hat{\nabla}^2 = (1/\nu) ad_{\nabla \text{Wick}}(R)$. A straightforward calculation leads to the following

**Lemma 2.** The curvature of the connection $\hat{\nabla}$ on the bundle $\mathcal{V}$ is expressed via $R$ as follows, $\hat{\nabla}^2 = (1/\nu) \hat{R}$.

Introduce Fedosov's operators $\delta$ and $\delta^{-1}$ on $\mathcal{W} \otimes \Lambda$. In local coordinates $\delta = (\partial/\partial \zeta^k) dz^k + (\partial/\partial \bar{\zeta}^l) d\bar{z}^l$ and the operator $\delta^{-1}$ is defined as follows. For an element $a \in \mathcal{W} \otimes \Lambda^0$ such that $\text{deg} a = p$ set $\delta^{-1} a = 0$ if $p + q = 0$ and $\delta^{-1} a = (p + q)^{-1}(\zeta^k i(\partial/\partial z^k) + \bar{\zeta}^l i(\partial/\partial \bar{z}^l)) a$ if $p + q > 0$.

Then $\delta = (1/\nu) ad_{\text{Wick}}(\partial)$, where $\partial = g_{kl} \zeta^l d\bar{z}^l - g_{kl} \zeta^k d\bar{z}^l$ (see [3]).

It was shown in [3] that there exists a unique element $r \in \mathcal{W} \otimes \Lambda^1$ which satisfies the equations $\delta^{-1} r = 0$ and $\delta r = R + \nabla r + (1/\nu) r \circ r$, and contains only non-negative powers of $\nu$.

In [3] a flat Fedosov's connection $D$ on $\mathcal{W}$ is defined as follows, $D = -\delta +$
\( \nabla + (1/\nu)ad_{\text{Wick}}(r) \). It is a \( \text{deg}_{a} \)-graded derivation in the algebra \((W \otimes \Lambda, \circ)\). Therefore \( W = \text{Ker} \ D \cap W \) is closed under Wick multiplication.

It was proved in [3] that the mapping \( \Pi : W_{D} \to \mathcal{F} \) is, in fact, a bijection. Transferring the product from the Fedosov algebra \((W_{D}, \circ)\) to \( \mathcal{F} \) via this bijection, one obtains a star-product \( * \) on \((M, \omega_{-1})\). Moreover, it was proved in [3] that \( * \) is a star-product with separation of variables. The proof was based on the following important statement (Lemma 4.5 in [3]): \( r \in \text{Ker} \ II^\prime \cap \text{Ker} \ II^\prime\prime \), i.e., in any local expression of \( r \) each term contains variables \( \zeta^{k} \) and \( \bar{\zeta}^{l} \) for some indices \( k, l \). We reformulate this statement using Lemma 1.

**Lemma 3.** The operator \( \hat{r} \) in \( \mathcal{V} \) annihilates the subspace \( \mathcal{F} \otimes \Lambda \subset \mathcal{V} \otimes \Lambda \). In particular, \( \hat{r}1 = 0 \). Moreover, \( \text{Ran} \ \hat{r} \subset \text{Ker} \ II \).

We are going to show that the star-product with separation of variables \( * \) constructed in [3] corresponds to the trivial deformation \( \omega = (1/\nu)\omega_{-1} \) of the Kähler form \((1/\nu)\omega_{-1} \).

**4. The Fock algebra**

Using the fact that \( \delta = (1/\nu)ad_{\text{Wick}}(\vartheta) \), one can express \( D \) as follows,

\[
D = \nabla + (1/\nu)ad_{\text{Wick}}(\gamma), \quad \text{where} \quad \gamma = -\vartheta + r.
\]

Introduce a connection \( \hat{\nabla} \) on \( \mathcal{V} \) by the formula \( \hat{\nabla} = \hat{\nabla} + (1/\nu)\hat{\gamma} \).

One can split the connections \( \nabla, D, \hat{\nabla} \), the operator \( \delta \) and the element \( r \) into the sums of their \((1,0)\)- and \((0,1)\)-components, \( \nabla = \nabla^\prime + \nabla^\prime\prime, D = D^\prime + D^\prime\prime, \hat{\nabla} = \hat{\nabla}^\prime + \hat{\nabla}^\prime\prime, \delta = \delta^\prime + \delta^\prime\prime, r = r^\prime + r^\prime\prime \).

In local coordinates denote \( \nabla_{k} = \nabla_{\partial/\partial z^{k}}, \nabla_{l} = \nabla_{\partial/\partial \bar{z}^{l}}, \) so that \( \nabla^\prime = \nabla_{k}dz^{k}, \nabla^\prime\prime = \nabla_{l}d\bar{z}^{l} \). Introduce similarly \( \hat{D}_{k}, \hat{D}_{l}, \hat{D}_{k}, \hat{D}_{l} \). Let \( r = r_{k}dz^{k} + r_{l}d\bar{z}^{l} \) be a local expression of the element \( r \). Then \( r^\prime = r_{k}dz^{k}, r^\prime\prime = r_{l}d\bar{z}^{l} \).

A simple calculation shows that \( (1/\nu)\hat{\vartheta} = \partial/\partial \zeta^{k}dz^{k} - \eta_{l}d\bar{z}^{l} \), where \( \eta_{l} = (1/\nu)g_{kl}\zeta^{k} \). Therefore,

\[
\hat{D}_{k} = \partial/\partial z^{k} - \partial/\partial \zeta^{k} + (1/\nu)\hat{r}_{k}, \quad \hat{D}_{l} = \partial/\partial \bar{z}^{l} + \eta_{l} + (1/\nu)\hat{r}_{l}.
\]  \( \text{Lemma 4.} \) Let \( f \in \mathcal{F}(U) \), where \((U, \{z^{k}\})\) is a coordinate chart on \( M \).

Then \( \hat{D}_{k}f = \partial f/\partial z^{k} \). In particular, \( \hat{D}_{k}1 = 0 \).

The lemma trivially follows from Lemma 3 and formula (2).

**Lemma 5.** For \( w \in W \otimes \Lambda \) one has \([\nabla, \hat{w}] = \hat{\nabla}w \).

Here, as well as below, the commutator is the \( \text{deg}_{a} \)-graded commutator in the graded algebra of endomorphisms of \( \mathcal{V} \otimes \Lambda \).

The lemma is an easy consequence of the fact that \( \nabla \) is a \( \text{deg}_{a} \)-graded derivation of the algebra \((W \otimes \Lambda, \circ)\). It implies the following
Proposition 1. For \( w \in \mathcal{W} \otimes \Lambda \) the formula \( [\hat{D}, \hat{w}] = \hat{D} w \) holds.

Denote \( \omega = (1/\nu)\omega_{-1} \).

Lemma 6.
(i) \([\hat{\nabla}, \hat{\vartheta}] = 0\);
(ii) \((1/\nu)[\hat{\vartheta}, \hat{r}] = \hat{\delta} r\);
(iii) \(\hat{\vartheta}^2 = i\nu^2 \omega\).

Lemma is proved by straightforward calculations. It implies the following Proposition 2.
The connection \( \hat{D} \) on \( \mathcal{V} \) has a scalar curvature, \( \hat{D}^2 = i\omega \).

The subspace \( \mathcal{W}^{D''} = \ker D'' \cap \mathcal{W} \) of the algebra \( (\mathcal{W}, \circ) \) is closed under the Wick product. We shall use the algebra \( (\mathcal{W}^{D''}, \circ) \) to define a product on the space \( \mathcal{V} \).

Introduce Fedosov’s operator \( \delta''^{-1} \) on \( \mathcal{W} \otimes \Lambda \) defining it in the local coordinates on a chart \((U, \{z^k\})\) as follows. Let \( w \in (\mathcal{W} \otimes \Lambda)(U) \) be such that \( \text{deg}''(w) = p, \text{deg}''(a) = q \). Set \( \delta''^{-1} a = 0 \) if \( p + q = 0 \) and \( \delta''^{-1} a = (p + q)^{-1} \zeta^n i(\partial/\partial z^j) a \) if \( p + q > 0 \). Then for \( w \in \mathcal{W} \otimes \Lambda \) one has \( (\delta'' \delta''^{-1} + \delta''^{-1} \delta'') w = w - w_0 \), where \( w_0 \) is the \((\text{deg}'' + \text{deg}''')\)-homogeneous component of \( w \) of the degree 0. For an element \( w \in \mathcal{W} \otimes \Lambda \) denote by \( w^{(q)} \) its \( \text{Deg}'' \)-homogeneous component of the degree \( q \).

The following proposition can be proved by Fedosov’s technique developed in [6].

Proposition 3. The mapping \( \Pi' : \mathcal{W}^{D''} \to \mathcal{V} \) is a bijection. For an element \( v \in \mathcal{V} \) such that \( \text{deg}_\nu(v) = 0 \) (i.e., which does not depend on the formal parameter \( \nu \)) the unique element \( w \in \mathcal{W}^{D''} \) such that \( v = \Pi' w \) can be calculated recursively with respect to the degree \( \text{Deg}'' \) by

\[
w^{(0)} = v;
w^{(q+1)} = \delta''^{-1}(\nabla'' w^{(q)} + (1/\nu) \sum_{p=0}^{q} a d_{\text{Wick}}(r''(p+1))w^{(q-p)}).
\]

Denote by \( \bullet \) the product in \( \mathcal{V} \) obtained by pushing forward the product in the algebra \( (\mathcal{W}^{D''}, \circ) \) by the mapping \( \Pi' \). Thus we obtain a Fock algebra \( (\mathcal{V}, \bullet) \). For \( v \in \mathcal{V} \) denote by \( L^*_v, R^*_v \) the operators of left and right multiplication by \( v \) in the algebra \( (\mathcal{V}, \bullet) \) respectively. Set \( \mathcal{L}^* = \{L^*_v | v \in \mathcal{V}\}, \mathcal{R}^* = \{R^*_v | v \in \mathcal{V}\} \).

Lemma 7. For \( w \in \mathcal{W}^{D''} \) the operator \( \hat{w} \) coincides with the left multiplication operator by the element \( v = \Pi' w \) in the Fock algebra \( (\mathcal{V}, \bullet) \), \( \hat{w} = L^*_v \).
Proof. For \( w_1, w_2 \in \mathcal{W}_{\mathcal{V}} \) set \( v_1 = \Pi'w_1, \ v_2 = \Pi'w_2 \). Then, by definition, \( v_1 \cdot v_2 = \Pi'(w_1 \circ w_2) \). Since \( \Pi' \) is a projection, \( w_2 - v_2 \in Ker \Pi' \). Taking into account that \( Ker \Pi' \) is a left ideal in the algebra \( (\mathcal{W}, \circ) \), we get \( w_1 \circ (w_2 - v_2) \in Ker \Pi' \). Therefore \( \Pi'(w_1 \circ w_2) = \Pi'(w_1 \circ v_2) = \hat{w}_1v_2 \), whence the Lemma follows. \( \square \)

Since the action of the operators \( \hat{w} \), \( w \in \mathcal{W} \), on \( \mathcal{V} \) is fibrewise, it follows from Lemma 7 that the operator of point-wise multiplication by \( f \in \mathcal{F} \) (also denoted by \( f \)) commutes with all operators from \( \mathcal{L}^* \). Therefore, \( f \in \mathcal{R}^* \), namely, \( R^*_f = f \).

Fix a coordinate chart \( (U, \{z^k\}) \) on \( M \).

**Lemma 8.** \( R^*_m = \hat{D}_l \).

**Proof.** Let \( w \in \mathcal{W}_{\mathcal{V},m}(U) \), \( v = \Pi'w \in \mathcal{V}(U) \). It follows from Lemma 7 and Proposition 1 that \([\hat{D}_l, \mathcal{L}^*_w] = [\hat{D}_l, \hat{w}] = \hat{D}_l\hat{w} = 0\), therefore \( \hat{D}_l \in \mathcal{R}^* \). Using formula (2) and Lemma 3 we get \( \hat{D}_l1 = \eta_l \), whence \( \hat{D}_l = R^*_m \). \( \square \)

Denote \( \mathcal{U} = \Pi'(\mathcal{W}_D) \subset \mathcal{V} \). Since \( \mathcal{W}_D \subset \mathcal{W}_{\mathcal{V}} \), and the projection \( \Pi' \) establishes an isomorphism of the algebras \( (\mathcal{W}_{\mathcal{V}}, \circ) \) and \( (\mathcal{V}, \bullet) \), the subspace \( \mathcal{U} \subset \mathcal{V} \) is closed under multiplication \( \bullet \) and the projection \( \Pi' \) maps the Fedosov algebra \( (\mathcal{W}_D, \circ) \) isomorphically onto the subalgebra \( (\mathcal{U}, \bullet) \) of the Fock algebra \( (\mathcal{V}, \bullet) \).

**Lemma 9.** For \( w \in \mathcal{W}_{\mathcal{V},m}(U) \) and \( v = \Pi'w \in \mathcal{V}(U) \) one has \( D_kw \in \mathcal{W}_{\mathcal{V},m}(U) \) and \([\hat{D}_k, \mathcal{L}^*_v] = \hat{D}_k\hat{w} = \mathcal{L}^*_{D_k\hat{w}} \).

**Proof.** Using Lemma 7 and Proposition 1 we obtain \([\hat{D}_k, \mathcal{L}^*_v] = [\hat{D}_k, \hat{w}] = \hat{D}_k\hat{w} \). Since Fedosov’s connection \( D \) is flat, \( D^2 = 0 \), we have \([D_k, D_l] = 0 \), whence \( \hat{D}_k\hat{D}_l\hat{w} = \hat{D}_l\hat{D}_k\hat{w} = 0 \), i.e., \( D_kw \in \mathcal{W}_{\mathcal{V},m}(U) \) and therefore \( \hat{D}_k\hat{w} \in \mathcal{L}^*(U) \). Using Lemma 4 we get \([D_k, \mathcal{L}^*_v] = \hat{D}_k\hat{v} - \mathcal{L}^*_{\hat{D}_k\hat{v}} \hat{D}_k1 = \hat{D}_k\hat{v} \) and thus \( \hat{D}_k\hat{w} = \mathcal{L}^*_{D_k\hat{w}} \), which concludes the proof. \( \square \)

Denote \( \mathcal{V}_{\mathcal{V},m}(U) = Ker \hat{D}' \cap \mathcal{V}(U) \) the space of local sections of the Fock bundle \( \mathcal{V} \) on an open subset \( U \subset M \), annihilated by \( \hat{D}' \). Set \( \mathcal{V}_{\mathcal{V},m} = \mathcal{V}_{\mathcal{V},m}(M) \).

**Proposition 4.** \( \mathcal{U} = \mathcal{V}_{\mathcal{V},m} \).

**Proof.** We have to show that on any coordinate chart \( (U, \{z^k\}) \) on \( M \), \( w \in \mathcal{W}_{\mathcal{V},m}(U) \) and \( v = \Pi'w \in \mathcal{V}(U) \) the condition \( D_kw = 0 \) holds iff \( \hat{D}_k\hat{v} = 0 \). The assertion follows immediately from the equality \( \mathcal{L}^*_{D_k\hat{v}} = \hat{D}_k\hat{w} \) proved in Lemma 9 and the fact that the mapping \( \mathcal{V} \ni w \mapsto \hat{w} \) is injective. \( \square \)

We can obtain the star-product \( \ast \) on \( M \) from the algebra \( (\mathcal{V}_{\mathcal{V},m}, \bullet) = (\mathcal{U}, \bullet) \).

Let \( v_1, v_2 \in \mathcal{V}_{\mathcal{V},m}, \ f_1 = \Pi v_1, f_2 = \Pi v_2 \in \mathcal{F} \). Then \( f_1 \ast f_2 = \Pi(v_1 \bullet v_2) \).

Let \( \Phi_{-1} \) be a local potential of the form \( \omega_{-1} = ig_{kl}dz^k \wedge d\bar{z}^l \) on a coordinate
chart \((U, \{z^k\})\) on \(M\), so that \(\partial^2 \Phi_{-1}/\partial z^k \partial \bar{z}^l = g_{kl}\). Then \(\Phi = (1/\nu)\Phi_{-1}\) is a local potential of the form \(\omega = (1/\nu)\omega_{-1}\). Set \(Q_l = \partial \Phi/\partial \bar{z}^l + \eta_l\).

**Proposition 5.** \(Q_l \in V_{D'}(U)\).

**Proof.** Using Lemma 4 we get \(\hat{D}_k \partial \Phi/\partial \bar{z}^l = \partial^2 \Phi/\partial z^k \partial \bar{z}^l = (1/\nu)g_{kl}\). It follows from Proposition 2 that \([\hat{D}_l, \hat{D}_k] = (1/\nu)g_{kl}\). Now, \(\hat{D}_k \eta_l = \hat{D}_k \hat{D}_l 1 = \hat{D}_l \hat{D}_k 1 - (1/\nu)g_{kl} = -(1/\nu)g_{kl}\) and therefore \(\hat{D}' Q_l = 0\). 

Since \(*\) is known to be a star-product with separation of variables, then \(R^*_{\bar{z}^l} = \bar{z}^l\) holds. This can be checked also directly. It follows from Lemma 4 that \(\hat{D}_k \bar{z}^l = 0\), i.e., \(\bar{z}^l \in V_{D'}(U)\). Let \(v \in V_{D'}(U)\) and \(f = \Pi v \in F(U)\). Now \(f \ast \bar{z}^l = \Pi(v \bullet \bar{z}^l) = \Pi(v \bar{z}^l) = f \bar{z}^l\), which proves the assertion.

In order to identify the star-product with separation of variables \(*\) it remains to calculate \(R^*_{\partial \Phi/\partial \bar{z}^l}\). Let \(v \in V_{D'}(U)\) and \(f = \Pi v\) as above. Calculate first \(\Pi \hat{D}_l v\). Using formula (2) we get \(\Pi \hat{D}_l v = \Pi(\partial v/\partial \bar{z}^l + \eta_l v + (1/\nu)\hat{r}_l v)\). Since \(\Pi \eta_l = 0\), we have \(\Pi(\eta_l v) = 0\). Lemma 3 implies that \(\Pi(\hat{r}_l v) = 0\). Finally we obtain that \(\Pi \hat{D}_l v = \partial f/\partial \bar{z}^l\).

Since \(\Pi Q_l = \partial \Phi/\partial \bar{z}^l\), we get \(f \ast \partial \Phi/\partial \bar{z}^l = \Pi(v \bullet Q_l) = \Pi(R^*_{Q_l} v) = \Pi((\partial \Phi/\partial \bar{z}^l + \hat{D}_l)v) = (\partial \Phi/\partial \bar{z}^l + \partial/\partial \bar{z}^l) f\). Therefore \(R^*_{\partial \Phi/\partial \bar{z}^l} = \partial \Phi/\partial \bar{z}^l + \partial/\partial \bar{z}^l\). Thus we have proved the desired

**Theorem.** The Fedosov star-product of Wick type \(*\) on a Kähler manifold \((M, \omega_{-1})\) is the star-product with separation of variables corresponding to the trivial deformation of the form \((1/\nu)\omega_{-1}\).

One might also try to generalize the construction by Bordemann and Waldmann to obtain arbitrary deformation quantizations with separation of variables.

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