NILPOTENT ORBITS AND FINITE $W$-ALGEBRAS

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Abstract. In recent years, the finite $W$-algebras associated to a semisimple Lie algebra $\mathfrak{g}$ and a nilpotent element of $\mathfrak{g}$ have been studied intensively from different viewpoints. In this lecture series, we shall present some basic constructions, connections, and applications of finite $W$-algebras. The topics include:

- various equivalent definitions of $W$-algebras
- independence of $W$-algebras on choices of Lagrangian subspaces
- $W$-algebras as quantization of Slodowy slices
- classification of good $\mathbb{Z}$-gradings in type $A$
- independence of $W$-algebras on choices of good gradings
- category equivalence between $W$-algebra modules and Whittaker $\mathfrak{g}$-modules
- higher level Schur duality between $W$-algebras and Hecke algebras
- $W$-(super)algebras in prime characteristic

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1. Introduction

1.1. The finite $W$-algebras are certain associative algebras associated to a complex semisimple Lie algebra $\mathfrak{g}$ and a nilpotent element $e \in \mathfrak{g}$. In the full generality, they were introduced by Premet [Pr1, Pr2] into mathematics in a different terminology. Some important special cases go back to Lynch’s thesis [Ly], which is in turn a generalization of Kostant’s construction in the case when $e$ is regular nilpotent. For $e = 0$, a finite $W$-algebra is simply the universal enveloping algebra $U(\mathfrak{g})$.  

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On the other extreme, the finite $W$-algebra for regular nilpotent $e$ is isomorphic to the center of $U(\mathfrak{g})$ [Ko]. In the literature of mathematical physics, the finite $W$-algebras appeared in the work of de Boer and Tjin [BT] from the viewpoint of BRST quantum hamiltonian reduction. The history is further complicated as there is an enormous amount of work on affine $W$-algebras in the 1990’s preceding the recent interest of finite $W$-algebras (and we apologize for inadequate references in this regard). See the book of Bouwknegt and Schoutens [BS] and the vast references therein; also see some more precise comments below.

In the last few years, the finite $W$-algebras and their representations have been extensively studied in mathematics with various new connections developed. The goal of the lecture series is to provide an introduction to some basic aspects of the fast-growing area of finite $W$-algebras. We sometimes sketch the main ideas of proofs when it is not practical to reproduce complete proofs. These lecture notes are purely expository in nature and contain no original results.

1.2. Premet [Pr1] started out with a characteristic $p$ version of the finite $W$-algebras, and used it to settle a long-standing conjecture of Kac and Weisfeiler [WK] (cf. the review of Jantzen [Ja1] on modular representations of Lie algebras). Subsequently, Premet [Pr2] made a transition to characteristic zero and showed that, with respect to the so-called Kazhdan filtration on the finite $W$-algebras (cf. [Ko, Mo]), the associated graded algebra is the algebra of functions on the Slodowy slice through $e$ [Slo]. In Premet’s definition, one starts with a $\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ which is induced by a standard $\mathfrak{sl}_2$-triple $\{e, h, f\}$ (nowadays such a $\mathbb{Z}$-grading is referred to as a Dynkin $\mathbb{Z}$-grading). The definition of finite $W$-algebras $W_{e, \Gamma}$ depends on the choice of a Dynkin $\mathbb{Z}$-grading $\Gamma$ and also the choice of a Lagrangian subspace $l$ of $\mathfrak{g}^{-1}$ (which is known to be a symplectic vector space).

Gan and Ginzburg [GG] developed a purely characteristic zero treatment in a generalized form of Premet’s theorem above which allows us to regard finite $W$-algebras as quantizations of the Slodowy slices. Moreover, Gan and Ginzburg modified Premet’s definition by starting with an arbitrary isotropic (instead of Lagrangian) subspace $l$ of $\mathfrak{g}^{-1}$ and showed that different choices of isotropic (in particular, Lagrangian) subspaces $l$ gave rise to isomorphic finite $W$-algebras $W_{e, \Gamma}$. The approach of Gan and Ginzburg also provides a new and more transparent proof of Skryabin’s equivalence of categories [Sk1] between a module category of $W_{e, \Gamma}$ and a category of Whittaker modules corresponding to $e$. The Whittaker $\mathfrak{g}$-module was earlier studied in Kostant’s paper [Ko] when $e$ is regular nilpotent.

Various key ingredients for finite $W$-algebras actually appeared earlier in the 1990’s in the setting of (classical and quantum) affine $W$-algebras. In particular, Fehér et. al. [FRTW1] Section III] formulated the classical affine $W$-algebras in the generality of good $\mathbb{Z}$-gradings of $\mathfrak{g}$ (as a natural generalization of the Dynkin $\mathbb{Z}$-gradings) for an arbitrary nilpotent element $e \in \mathfrak{g}$, though the terminology of good $\mathbb{Z}$-gradings was formalized later by Kac, Roan and Wakimoto [KRW]. Fehér et. al. [FRTW2] Section 3.4] also recognized the key role played by a Lagrangian subspace of $\mathfrak{g}^{-1}$ in classical affine $W$-algebras. The quantum affine $W$-algebras associated to a regular nilpotent $e$ from the BRST approach (also called quantized
Drinfeld-Sokolov reduction in the affine setup) were studied from the viewpoint of vertex algebras by Frenkel, Kac, and Wakimoto [FKW]. The finite and affine $W$-algebras are related via a fundamental construction of Zhu algebra in the theory of vertex algebras [Zhu] (cf. Arakawa [Ar], De Sole and Kac [DK]). The good $Z$-gradings were subsequently classified by Elashvili and Kac [EK]; also see Baur and Wallach [BW] for closely related work and further clarification.

Brundan and Goodwin [BG] showed that different good $Z$-gradings $\Gamma$ on $\mathfrak{g}$ for $e$ led to isomorphic finite $W$-algebras $W_e$ (where we drop the dependence on $\Gamma$). Actually, they worked and established the results in a somewhat more general and flexible setting of good $\mathbb{R}$-gradings, which allowed them to introduce some suitable geometric object called good grading polytopes.

It is clear from the definition that two nilpotent elements in $\mathfrak{g}$ which are conjugate under the adjoint group $G$ lead to isomorphic finite $W$-algebras. Hence, the isomorphism classes of finite $W$-algebras ultimately depend only on the nilpotent orbits in $\mathfrak{g}$.

The representation theory of finite $W$-algebras has been developed most adequately in type $A$ in a series of papers by Brundan and Kleshchev [BK1, BK2, BK3]. In particular, there exists a remarkable higher level duality between finite $W$-algebras and cyclotomic Hecke algebras, which recovers the classical Schur duality at level one [BK3].

The constructions of finite $W$-algebras afford a natural superalgebra generalization. In characteristic $p$, they are developed in Lei Zhao’s Virginia dissertation (see [WZ1, WZ2, Zh]), in connection with the formulation and proof of a super analogue of the Kac-Weisfeiler conjecture when $\mathfrak{g}$ is a basic classical Lie superalgebra or a queer Lie superalgebra. Many aspects of the representation theory of finite $W$-algebras over $\mathbb{C}$ afford a superalgebra generalization, and they are currently being investigated.

1.3. For lack of expertise and space, we have left out many interesting topics. We refer to Section 10 for somewhat more detailed comments on these topics and discussions of some open questions. We also refer to a more recent survey by Losev [Lo5] which covers various complementary topics in detail.

Here is a quick outline on various topics which are (regretfully) not to be covered in details in these lecture notes. Losev [Lo1] has developed a powerful deformation quantization approach toward finite $W$-algebras. Connections between shifted Yangians and finite $W$-algebras of type $A$ [BK1] (also cf. RS in some special cases) allow Brundan and Kleshchev [BK2] to classify the finite-dimensional irreducible $W_e$-modules and obtain Kazhdan-Lusztig type character formula in type $A$ (see [VD] for formulation of Kazhdan-Lusztig conjecture for finite $W$-algebras). Deep connections between finite-dimensional modules of finite $W$-algebras and primitive ideals have been developed [Pr3, Pr4, Pr5, Gl, Lo1, Lo2, Lo4, GRU], and there has also been interesting work on category $\mathcal{O}$ of finite $W$-algebras, see [Pr3, BGK, Lo3, Go, We].

1.4. The plan of the lecture notes is as follows.
In Section 2, a bijection between $\mathfrak{sl}_2$-triples modulo $G$-conjugation and nonzero nilpotent orbits in $\mathfrak{g}$ is formulated. We introduce the Dynkin and good $\mathbb{Z}$-gradings, and then explain some of their basic properties.

In Section 3, the finite $W$-algebras are defined in several different ways and these definitions are shown to be equivalent.

In Section 4, we formulate the finite $W$-algebras based on an isotropic subspace $l$ of $\mathfrak{g}_{-1}$. Then we introduce the Slodowy slices and the Kazhdan filtration on finite $W$-algebras, with respect to which the associated graded algebra is isomorphic to the algebra of functions on Slodowy slices. The proof of the independence of the choices of $l$ for the isoclasses of finite $W$-algebras is reduced to some Lie algebra cohomology calculation, which is completed in Section 5. The same type of argument is used to derive Skryabin’s equivalence in Section 5.

In Section 6, the classification of good $\mathbb{Z}$-gradings in type $A$ is presented, in terms of some combinatorial objects called pyramids.

In Section 7, different good $\mathbb{Z}$-gradings are shown to lead to isomorphic finite $W$-algebras.

In Section 8, we formulate and outline the higher level Schur duality between finite $W$-algebras of type $A$ and cyclotomic Hecke algebras.

In Section 9, we formulate $W$-superalgebras in characteristic $p$, and its role in proof of super Kac-Weisfeiler conjecture. This includes the Lie algebra counterpart as its most important special cases.

In Section 10, we briefly outline some major topics on finite $W$-algebras which are not covered in these lecture notes, and then list some open problems.

2. Nilpotent orbits, Dynkin and good $\mathbb{Z}$-gradings

2.1. Basic setups for Lie algebras. For a Lie algebra $\mathfrak{a}$, we will denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$. Let $\mathfrak{a}_e$ denote the centralizer of $e \in \mathfrak{a}$. Sometimes it is convenient to regard $\mathfrak{a}_e = \ker(\text{ad } e)$, the kernel of the adjoint operator $\text{ad } e : \mathfrak{a} \to \mathfrak{a}$.

Let $\mathfrak{g}$ be a finite dimensional semisimple or reductive Lie algebra over $\mathbb{C}$ equipped with a non-degenerate invariant symmetric bilinear form $(\cdot | \cdot)$. Throughout the lectures, we almost always take $\mathfrak{g}$ either to be a simple Lie algebra or the general linear Lie algebra $\mathfrak{gl}_N$. For $\mathfrak{g} = \mathfrak{gl}_N$, we may take the bilinear form by letting $(a|b) = \text{tr}(ab)$ where $ab$ denotes the matrix multiplication of $a, b \in \mathfrak{gl}_N$. An element $e \in \mathfrak{g}$ is nilpotent means that $\text{ad } e$ is a nilpotent endomorphism on $\mathfrak{g}$. For example, a nilpotent element in $\mathfrak{gl}_N$ is exactly an $N \times N$ matrix with all eigenvalues 0.

A $\mathbb{Z}$-grading of Lie algebra $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ which is always assumed to satisfy the graded commutation relation: $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, \forall i, j$.

2.2. Nilpotent orbits. Denote by $G$ the adjoint group associated to the Lie algebra $\mathfrak{g}$, and by $\mathcal{O}_e$ the $G$-adjoint orbit of $e$ in $\mathfrak{g}$. We refer to Collingwood and McGovern [CMc] for more on nilpotent orbits.

Denote by $\mathcal{N}$ the null cone which consists of all nilpotent elements in $\mathfrak{g}$. There exists a unique dense open orbit $\mathcal{O}_{\text{reg}}$ in $\mathcal{N}$, called the regular nilpotent orbit. A
nilpotent element $e \in \mathfrak{g}$ is regular nilpotent if and only if $\mathfrak{g}_e$ attains the minimal dimension (which equals the rank of $\mathfrak{g}$). The regular nilpotent orbit $O_{\text{reg}}$ consists of all the regular nilpotent elements in $\mathfrak{g}$. There is a unique dense open orbit in $\mathcal{N}\setminus O_{\text{reg}}$, called the subregular orbit and denoted by $O_{\text{sub}}$. There exists a unique minimal orbit $O_{\text{min}}$ in $\mathcal{N}$ of smallest positive dimension. The dimension of $O_{\text{min}}$ is known [Wa] to be $2h^\vee - 2$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Incidentally, the dual Coxeter number plays a key role in representation theory of affine Lie algebras.

The set of nilpotent orbits in $\mathfrak{g}$ is naturally a poset $\mathcal{P}$ with partial order $\leq$ as follows: $O' \leq O$ if and only if $O' \subseteq O$ (the closure of $O$). Then $O_{\text{reg}}$ is the maximum and the zero orbit is the minimum in the poset $\mathcal{P}$. Moreover, $O_{\text{sub}}$ is the maximum in the poset $\mathcal{P}\setminus \{O_{\text{reg}}\}$, while $O_{\text{min}}$ is the minimum in the poset $\mathcal{P}\setminus \{0\}$.

**Example 1.** Let $\mathfrak{g} = \mathfrak{gl}_n$. Every nilpotent matrix in $\mathfrak{gl}_n$ is conjugate to a matrix in Jordan canonical form. It follows that the nilpotent orbits in $\mathfrak{g}$ are parameterized by the partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of $n$, where the orbit $O_\lambda$ contains the matrix in standard Jordan form $\text{diag}(J_{\lambda_1}, J_{\lambda_2}, \ldots)$, and we have denoted by $J_k$ the $k \times k$ Jordan block.

Associated to the four distinguished partitions $(n)$, $(n-1, 1)$, $(2, 1^{n-2})$, $(1^n)$ are the orbits $O_{\text{reg}}$, $O_{\text{sub}}$, $O_{\text{min}}$ and the zero orbit $\{0\}$, respectively.

### 2.3. Jacobson-Morozov Theorem.

Associated to a nonzero nilpotent element $e \in \mathfrak{g}$, there always exists $\{e, h, f\} \subseteq \mathfrak{g}$ which satisfy the $\mathfrak{sl}_2$ commutation relation:

\[
[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.
\]

We will refer to this as a standard $\mathfrak{sl}_2$-triple or simply an $\mathfrak{sl}_2$-triple.

This statement is known as the Jacobson-Morozov Theorem, and it can be proved by induction on the dimension of $\mathfrak{g}$. See Carter [Ca, 5.3] or [CMc, 3.3].

**Example 2.** Let $\mathfrak{g} = \mathfrak{gl}_n$, $e = J_n$, $h = \text{diag}(n-1, n-3, \ldots, 3-n, 1-n)$, and let

\[
f = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\]

with exactly one nonzero (sub)diagonal with entries $a_i = i(n - i)$ for $1 \leq i \leq n - 1$. Then $\{e, h, f\}$ form a standard $\mathfrak{sl}_2$-triple.

### 2.4. The Dynkin gradings.

By the representation theory of $\mathfrak{sl}_2 = \mathbb{C}\{e, h, f\}$, the eigenspace decomposition of the adjoint action $\text{ad} h : \mathfrak{g} \to \mathfrak{g}$ provides a $\mathbb{Z}$-grading

\[
\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j,
\]

that is, $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \text{ad} h(x) = jx\}$. A $\mathbb{Z}$-grading of $\mathfrak{g}$ arising from $\mathfrak{sl}_2$ this way will be referred to as a Dynkin $\mathbb{Z}$-grading. A Dynkin grading affords the following.
favorable properties:

\[ e \in g_2, \quad \text{ad } e : g_j \to g_{j+2} \text{ is injective for } j \leq -1, \]

\[ \text{ad } e : g_j \to g_{j+2} \text{ is surjective for } j \geq -1, \]

\[ g_e \subseteq \bigoplus_{j \geq 0} g_j, \]

\[ (g_i | g_j) = 0 \text{ unless } i + j = 0, \]

\[ \dim g_e = \dim g_0 + \dim g_1. \]

Note that (1) follows from the \( \mathfrak{sl}_2 \)-triple definition. Claims (2) and (3) are immediate consequences of the highest weight theory of \( \mathfrak{sl}_2 \) when applied to the the adjoint module \( g \). Claim (4) follows from (2) and (3). Formulas (5) and (6) are also easy, and they will be proved in a generalized setting in Proposition 5 below using only (1), (2), and (3).

For \( k \in \mathbb{Z} \), we shall denote \( g_{>k} = \bigoplus_{j > k} g_j \). Similarly, we define \( g_{\leq k}, g_{= k}, g_{\geq k} \).

2.5. The good \( \mathbb{Z} \)-gradings. In this subsection, we recall the notion of good \( \mathbb{Z} \)-gradings (see [FRTW1, (3.3)] and [KRW]) and its basic properties [EK].

**Definition 3.** A \( \mathbb{Z} \)-grading \( \Gamma : g = \bigoplus_{j \in \mathbb{Z}} g_j \) for \( g \) semisimple is called a good \( \mathbb{Z} \)-grading for \( e \) if it satisfies the conditions (1)-(3) above.

For \( g \) reductive, a good \( \mathbb{Z} \)-grading \( g = \bigoplus_{j \in \mathbb{Z}} g_j \) for \( e \) is required to satisfy (1)-(3) and an additional condition that the center \( z(g) \) of \( g \) is contained in \( g_0 \).

In particular, \( \text{ad } e : g_{-1} \to g_1 \) is always a bijection.

The following simple lemma will be repeatedly used.

**Lemma 4.** For any \( \mathbb{Z} \)-grading \( \Gamma : g = \bigoplus_{j \in \mathbb{Z}} g_j \), there exists a semisimple element \( h_\Gamma \in [g, g] \) such that \( \Gamma \) coincides with the eigenspace decomposition of \( \text{ad } h_\Gamma \), i.e.

\[ g_j = \{ x \in g \mid [h_\Gamma, x] = jx \}. \]

**Proof.** The degree operator \( \partial : g \to g \) which sends \( x \mapsto jx \) for \( x \in g_j \) is a derivation on the semisimple Lie algebra \( [g, g] \), hence an inner derivation on \( [g, g] \), given by \( \text{ad } h_\Gamma \) for some semisimple element \( h_\Gamma \in [g, g] \). Then \( \partial = \text{ad } h_\Gamma \) as derivations on \( g = [g, g] \oplus z(g) \) since the equality holds on \( z(g) \) too. \( \square \)

**Proposition 5.** Properties (4)-(6) remain to be valid for every good \( \mathbb{Z} \)-grading \( \Gamma : g = \bigoplus_{j \in \mathbb{Z}} g_j \).

**Proof.** Clearly \( g_e \) is a \( \mathbb{Z} \)-graded Lie subalgebra of \( g \), and (4) follows from (2).

For \( x \in g_i, y \in g_j \), we have \( -i(x|y) = ([x, h_\Gamma]|y) = (x|[h_\Gamma, y]) = j(x|y) \) by Lemma 4. This implies (5).

Finally, (6) follows by an exact sequence of vector spaces:

\[ 0 \to g_e \to g_{-1} + g_0 + g_{>0} \xrightarrow{\text{ad } e} g_{>0} \to 0 \]

which is well-defined by (2) and (3). \( \square \)

**Example 6.** Associated to \( e = 0 \), we have a good \( \mathbb{Z} \)-grading with the whole \( g \) concentrated on degree zero, i.e. \( g_0 = g \).
Example 7. Let $g = gl_3$ and $e = E_{13}$.

(1) Let $h = \text{diag}(1,0,-1)$ and $f = E_{31}$. Then $\{e, h, f\}$ forms an $sl_2$-triple, and it gives rise to a Dynkin $\mathbb{Z}$-grading of $gl_3$ for $e \in g_2$ whose degrees on the matrix units $E_{ij}$ are listed in the matrix below:

\[
\begin{pmatrix}
0 & 1 & 2 \\
-1 & 0 & 1 \\
-2 & -1 & 0
\end{pmatrix}
\]

A basis of the (5-dimensional) centralizer $g_e$ consists of 2 degree-zero elements $E_{11} + E_{33}, E_{22}$, 2 degree-one elements $E_{12}, E_{23}$, and a degree-two element $E_{13}$.

(2) The eigenspace decomposition of the diagonal element diag $(1,1,-1)$ gives rise to a good (but non-Dynkin) $\mathbb{Z}$-grading for $e = E_{13}$ whose degrees on the matrix units $E_{ij}$ are listed in the matrix below:

\[
\begin{pmatrix}
0 & 0 & 2 \\
0 & 0 & 2 \\
-2 & -2 & 0
\end{pmatrix}
\]

Note now $g_{-1} = 0$.

In short, Dynkin is good, but being good is not good enough for being Dynkin.

Lemma 8. Properties (2) and (3) are equivalent for any $\mathbb{Z}$-grading $g = \oplus_{j \in \mathbb{Z}} g_j$.

Proof. Note the proof of (3) in Proposition 5 only uses the $\mathbb{Z}$-grading of $g$, but does not use (2) and (3). Then we have a non-degenerate pairing between $g_i$ and $g_{-i}$ induced from the non-degenerate form $(\cdot | \cdot)$ on $g$. The lemma follows by this non-degeneracy and the equation $\langle x_i | [e, y_j] \rangle = \langle x_i, e \rangle y_j$ for $i + j = -2$ and $i \leq -1 \leq j$. □

Let $\Gamma : g = \oplus_{j \in \mathbb{Z}} g_j$ be a good $\mathbb{Z}$-grading of $g$. Note that $h_\Gamma$ defined in Lemma 4 actually lies in $g_0$. Then $g_0 = g_{h_\Gamma}$ is a Levi (reductive) subalgebra of $g$. Choose $h$ to be a Cartan subalgebra of $g_0$, and hence also a Cartan subalgebra of $g$. Let $g = h \oplus \oplus_{\alpha \in \Phi} g\alpha$ be a root space decomposition. Fix a system of positive roots $\Delta^+_0$ of $(g_0, h)$.

Lemma 9. Let $\Gamma : g = \oplus_{j \in \mathbb{Z}} g_j$ be a good $\mathbb{Z}$-grading of $g$. Then,

1. each root subspace $g\alpha$ lies in $g_j$ for some $j \in \mathbb{Z}$;
2. $\Delta^+ := \Delta^+_0 \cup \{\alpha \mid g\alpha \subseteq g_{>0}\}$ is a system of positive roots of $(g, h)$.

Proof. (1) Take $h_\Gamma \in g$ as in Lemma 4. Then $[h_\Gamma, x] = \alpha(h_\Gamma)x$ for $x \in g\alpha$ implies that $g\alpha \subseteq g_j$ for $j = \alpha(h_\Gamma) \in \mathbb{Z}$.

(2) Choose a regular element $r \in h$. Then $c\epsilon + h_\Gamma$ for small $\epsilon > 0$ is also regular, and so $\Delta^+ = \{\alpha \in \Phi \mid \alpha(c\epsilon + h_\Gamma) > 0\}$ is a system of positive roots of $g$. □

Let $\Pi$ be the set of simple roots for $\Delta^+$, and denote $\mathbb{N} = \{0, 1, 2, \ldots\}$. Then,

$$\Pi = \bigcup_{j \in \mathbb{N}} \Pi_j, \quad \text{where } \Pi_j = \{\alpha \in \Pi \mid g\alpha \subseteq g_j\}.$$
Lemma 10. For a good \( \mathbb{Z} \)-grading \( \Gamma \), we have \( \Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2 \).

Proof. Assume to the contrary that there exists \( \beta \in \Pi_s \) for \( s > 2 \). Since \( e \in \mathfrak{g}_2 \) and \( \mathfrak{g}_2 \) is contained in the subalgebra generated by \( \mathfrak{g} \) with \( \alpha \in \Pi_0 \cup \Pi_1 \cup \Pi_2 \), we have \( [e, \mathfrak{g}^{-\beta}] = 0 \), or in other words, \( \mathfrak{g}^{-\beta} \subseteq \mathfrak{g}_e \). This contradicts with the fact (1) that \( \mathfrak{g}_e \) has \( \mathbb{N} \)-grading as \( \mathfrak{g}^{-\beta} \subseteq \mathfrak{g}_{-s} \).

2.6. A bijective map \( \Omega \). Clearly, the group \( G \) acts on the collection of \( \mathfrak{sl}_2 \)-triples in \( \mathfrak{g} \) by conjugation. Define a map

\[
\Omega : \{ \text{\( \mathfrak{sl}_2 \)-triples} \} / G \rightarrow \{ \text{nonzero nilpotent orbits} \},
\]

\( \Omega(\{e, h, f\}) = O_e \).

Theorem 11. The map \( \Omega \) is a bijection.

Sketch of a proof. According to Jacobson-Morozov Theorem in 2.3, the map \( \Omega \) is surjective. A theorem of Kostant asserts that the map \( \Omega \) is also injective, as we sketch below (see [CMc, 3.4] for details).

To that end, take two \( \mathfrak{sl}_2 \)-triples which are mapped by \( \Omega \) to the same nilpotent orbit \( O_e \). Applying a \( G \)-conjugation, we may assume without loss of generality that the two triples are \( \{e, h, f\} \) and \( \{e, h', f'\} \), that is, they share the same \( e \).

Let \( u_e := \mathfrak{g}_e \cap [\mathfrak{g}, e] \). Note that \( h' - h \in u_e \) and \( u_e \) is an \( \text{ad} h \)-invariant, nilpotent ideal of \( \mathfrak{g}_e \). Actually \( u_e = \oplus_{j > 0} \mathfrak{g}_e \cap \mathfrak{g}_j \). Denote by \( U_e \) the subgroup of \( G \) with Lie algebra \( u_e \).

By using the fact that \( u_e \) is a \( \mathbb{N} \)-graded nilpotent ideal, one shows easily that \( \text{Ad} U_e(h) = h + u_e \). Hence, there exists \( x \in U_e \) such that \( \text{Ad} x(h) = h' \).

Note that we automatically have \( \text{Ad} x(e) = e \) for \( x \in U_e \). Now the injectivity of \( \Omega \) follows from the rigidity of \( \mathfrak{sl}_2 \) below.

Lemma 12. Let \( \{e, h, f\} \) and \( \{e, h, f'\} \) be standard \( \mathfrak{sl}_2 \)-triples of \( \mathfrak{g} \). Then \( f = f' \).

Exercise 13. Prove Lemma 12.

Remark 14. If one works a bit harder (cf. [CMc, 3.4]), one recovers a theorem of Molcev which asserts the injectivity of the map

\[
\Theta : \{ \text{\( \mathfrak{sl}_2 \)-triples} \} / G \rightarrow \{ \text{nonzero semisimple orbits} \},
\]

\( \Theta(\{e, h, f\}) = O_h \).

Given an \( \mathfrak{sl}_2 \)-triple \( \{e, h, f\} \), by Lemma 10 (which applies to the special case of Dynkin grading \( \text{ad} h \)) we have a system of simple roots of \( \mathfrak{g} \) whose weights on \( h \) only take values from \( \{0, 1, 2\} \). Therefore the number of semisimple orbits with such weight restriction is \( \leq 3^{\text{rank}(\mathfrak{g})} \). It now follows by the properties of \( \Omega \) and \( \Theta \) above that the number of nilpotent orbits in \( \mathfrak{g} \) is finite.

An alternative way of obtaining the finiteness of nilpotent orbits of \( \mathfrak{g} \) of classical type is to use the parametrization by partitions (cf. [Ja2, Chapter 1]). But such an alternative ceases to work well for Lie algebras \( \mathfrak{g} \) of exceptional type.
3. Definitions of $W$-algebras

3.1. The endomorphism algebra definition. Assume we are given a reductive Lie algebra $\mathfrak{g}$ with a non-degenerate invariant bilinear form $(\cdot | \cdot)$, an nilpotent element $e \in \mathfrak{g}$, and a good $\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ for $e$.

There exists a unique $\chi \in \mathfrak{g}^*$ such that $\chi(x) = (x|e)$. Define a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_{-1}$ as follows:

$$\langle \cdot, \cdot \rangle : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{C}, \quad \langle x, y \rangle := ([x,y]|e) = \chi([x,y]).$$

The following lemma was noticed earlier [FRTW2, 3.4] in the setting for classical affine $W$-algebras.

**Lemma 15.** The bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_{-1}$ is skew-symmetric and non-degenerate.

**Proof.** The skew-symmetry follows by definition. The non-degeneracy follows by the bijection $\text{ad} : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ and the identity $\langle x, y \rangle = \langle x[y, e] \rangle$.

It follows that $\mathfrak{g}_{-1}$ is even-dimensional. Pick a Lagrangian (= maximal isotropic) subspace $\mathfrak{l}$ of $\mathfrak{g}_{-1}$ with respect to the form $\langle \cdot, \cdot \rangle$. Then $\dim \mathfrak{l} = \frac{1}{2} \dim \mathfrak{g}_{-1}$. Introduce the following important nilpotent subalgebra of $\mathfrak{g}$:

$$\mathfrak{m} := \mathfrak{l} \bigoplus_{j \leq -2} \mathfrak{g}_j.$$

**Exercise 16.** Prove that

$$\dim \mathfrak{m} = \frac{1}{2} \dim \mathfrak{o}_e.$$

The restriction of $\chi$ to $\mathfrak{m}$, denoted by $\chi : \mathfrak{m} \to \mathbb{C}$, defines a one-dimensional representation $\mathbb{C}_\chi$ of $\mathfrak{m}$, thanks to the Lagrangian condition on $\mathfrak{l}$. Let $I^\mathfrak{m}_\chi$ denote the kernel of the corresponding associative algebra homomorphism $U(\mathfrak{m}) \to \mathbb{C}$, i.e. the two-sided ideal of $U(\mathfrak{m})$ generated by $a - \chi(a)$ for $a \in \mathfrak{m}$. Let $I_\chi$ denote the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a)$ for $a \in \mathfrak{m}$. Define the induced $\mathfrak{g}$-module (called generalized Gelfand-Graev module)

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi.$$

The *finite $W$-algebra* (or simply $W$-algebra in this paper) $W_\chi$ is defined to be the endomorphism algebra

$$W_\chi := \text{End}_{U(\mathfrak{g})}(Q_\chi)^{\text{op}}.$$

This is the original definition of Premet [Pr2].

Actually, the above notions depend in addition on $\Gamma, \mathfrak{l}$, and it would be more precise to write $Q_{\chi, \Gamma, \mathfrak{l}}, I_{\chi, \Gamma, \mathfrak{l}}, W_{\chi, \Gamma, \mathfrak{l}}$, etc. But we choose to use the simplified notations, partly because we will eventually show that the isoclasses of finite $W$-algebras do not depend on $\Gamma$ and $\mathfrak{l}$. At different occasions later on, we may use some other indices among $\chi, \Gamma, \mathfrak{l}$ to put an emphasis on the dependence of those indices.

**Example 17.** Let $e = 0$. Then $\chi = 0$, $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{m} = 0$, $Q_\chi = U(\mathfrak{g})$, and $W_\chi = U(\mathfrak{g})$. 

3.2. The Whittaker model definition. Since \( Q_\chi = U(g)/I_\chi \) is a cyclic module, any endomorphism of the \( g \)-module \( Q_\chi \) is determined by the image of \( \bar{I} \), where \( \bar{y} \) denotes the coset \( y + I_\chi \) of \( y \in U(g) \). The image of \( \bar{I} \) must be annihilated by \( I_\chi \). Hence we obtain the following identification of \( W_\chi \) (as the space of Whittaker vectors in \( U(g)/I_\chi \) in the terminology introduced in a later Section 5.2):

\[
W_\chi = \{ \bar{y} \in U(g)/I_\chi \mid (a - \chi(a))y \in I_\chi, \forall a \in m \}.
\]  
\( (8) \)

By definition of \( I_\chi \), \( W_\chi \) can be further identified with the subspace of \( \text{ad}m \)-invariants in \( Q_\chi \):

\[
W_\chi \cong (Q_\chi \text{ad}m) := \{ \bar{y} \in U(g)/I_\chi \mid \forall a \in m \},
\]  
\( (9) \)

Transferred via the above identification, the algebra structure on \( W_\chi \) is given by

\[
\bar{y}_1 \bar{y}_2 = \bar{y}_1 y_2
\]

for \( y_i \in U(g) \) such that \([a, y_i] \in I_\chi \) for all \( a \in m \) and \( i = 1, 2 \).

**Exercise 18**. Without using the identification with the definition (7), check directly that

1. the ideal \( I_\chi \) is \( \text{ad}m \)-invariant, hence \((Q_\chi \text{ad}m)\) in (9) as a vector space is well-defined;
2. for \( y_i \) satisfying \([a, y_i] \in I_\chi \) for all \( a \in m \) and \( i = 1, 2 \), we have \([a, y_1 y_2] \in I_\chi \) for all \( a \in m \). Hence, \((Q_\chi \text{ad}m)\) in (9) as an algebra is well-defined.

Hence, we may regard \((9)\), or \((8)\), as a second definition of \( W \)-algebras.

3.3. A simplified definition for even good gradings. In this subsection, we will assume \( g \) is equipped with an *even* good \( \mathbb{Z} \)-grading. A good \( \mathbb{Z} \)-grading \( g = \bigoplus_{j \in \mathbb{Z}} g_j \) is called *even*, if \( g_j = 0 \) unless \( j \) is an even integer.

All the complications of choice of a Lagrangian/isotropic subspace \( I \) disappear for even \( \mathbb{Z} \)-gradings, since \( g_{-1} = 0 \). Now we have \( m = \bigoplus_{j \leq -2} g_j \), and a parabolic submodule

\[
p := \bigoplus_{j \geq 0} g_j.
\]  
\( (10) \)

It follows by the PBW theorem that

\[
U(g) = U(p) \bigoplus I_\chi.
\]

The projection \( \text{pr}_\chi : U(g) \to U(p) \) along this direct sum decomposition induces an isomorphism

\[
\text{pr}_\chi : U(g)/I_\chi \simeq U(p).
\]

By the above observation, the algebra \( W_\chi \) can be regarded as a subalgebra of \( U(p) \). Consider a \( \chi \)-twisted adjoint action of \( m \) on \( U(p) \) by \( a.y := \text{pr}_\chi([a, y]) \) for \( a \in m \) and \( y \in U(p) \). We identify \( W_\chi \) as

\[
W_\chi = U(p)^{\text{ad}m} := \{ y \in U(p) \mid [a, y] \in I_\chi, \forall a \in m \}.
\]  
\( (11) \)

Since \( \text{pr}_\chi : U(g)/I_\chi \simeq U(p) \) is an isomorphism of \( m \)-modules, the equivalence of (11) with the second definition (for even \( \mathbb{Z} \)-gradings) is clear.
Hence, we may take (11) as a third definition of the $W$-algebras. This is the original definition used by Kostant and Lynch [Ko, Ly] (who only considered even Dynkin $\mathbb{Z}$-gradings). As observed by Brundan-Kleshchev [BK1], it works equally well in the generality of even good gradings by dropping “Dynkin”. Brundan-Goodwin-Kleshchev [BGK, Section 2] also has a (necessarily) more complicated version of this definition without assuming the grading to be even.

**Example 19.** According to a theorem of Kostant (see Theorem 31 below), for a regular nilpotent element $e$ in $\mathfrak{g}$, $\mathcal{W}_\chi$ is isomorphic to $Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$.

Let $e = E_{12} \in \mathfrak{gl}_2$, which is regular nilpotent. Then $\mathfrak{m} = \mathbb{C}f$ with $f = E_{21}$, and $\mathfrak{p} = \mathbb{C}e + \mathbb{C}E_{11} + \mathbb{C}E_{22}$. A direct computation shows that $E_{11} + E_{22}$ and $e + \frac{1}{3}h^2 - \frac{1}{3}h$ lie in $U(\mathfrak{p})_{ad}$. (If we combine with Kostant’s theorem, then $\mathcal{W}_\chi = U(\mathfrak{p})_{ad}$ is the polynomial algebra generated by these two elements.)

Let $\mathfrak{g} = \mathfrak{gl}_n$ and $e = J_n$ as in Example 2. Let

$$\Omega(u) := \begin{pmatrix}
E_{11} + u + 1 & E_{12} & \cdots & E_{1n} \\
1 & E_{22} + u + 2 & E_{23} & \vdots \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & E_{n-1,n} \\
0 & \cdots & 0 & 1 & E_{nn} + u + n
\end{pmatrix}$$

The row determinant of a matrix $A = (a_{ij})_{n \times n}$ with non-commutative entries is defined to be

$$\text{rdet } A = \sum_{\pi \in S_n} \text{sgn}(\pi)a_{1,\pi 1} \cdots a_{n,\pi n}. \quad (12)$$

Write $\text{rdet } \Omega(u) = u^n + \sum_{i=1}^n w_i u^{n-i}$ for $w_i \in U(\mathfrak{p})$. As shown in [BK1, Section 12], $w_i, 1 \leq i \leq n$ are commuting elements which lie in $U(\mathfrak{p})_{ad}$, and they freely generate $\mathcal{W}_\chi = U(\mathfrak{p})_{ad}$.

3.4. **The BRST definition.** There is yet another definition of $W$-algebras using the BRST complex.

For the sake of simplicity, we will work under the assumption of an even good $\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ for $e \in \mathfrak{gl}_2$. In this case, we have $\mathfrak{m} = \bigoplus_{j \leq -2} \mathfrak{g}_j$.

Let us take another copy of $\mathfrak{m}$, which will be denoted by $\hat{\mathfrak{m}}$.

Endow the vector space $\mathfrak{m}^* \oplus \hat{\mathfrak{m}}$ with the symmetric bilinear form induced by the pairing between $\mathfrak{m}^*$ and $\hat{\mathfrak{m}}$. The corresponding Clifford algebra on $\mathfrak{m}^* \oplus \hat{\mathfrak{m}}$ can be naturally identified with $\land(\mathfrak{m}^*) \otimes \land(\hat{\mathfrak{m}})$. Consider the tensor algebra of this Clifford algebra with $U(\mathfrak{g})$

$$\mathcal{B}^* := \land(\mathfrak{m}^*) \otimes U(\mathfrak{g}) \otimes \land(\hat{\mathfrak{m}}).$$

It admits a BRST (cohomological) $\mathbb{Z}$-grading with the assignment of degrees to generators in $\mathfrak{m}^*, \mathfrak{g}, \hat{\mathfrak{m}}$ to be 1, 0, -1, respectively. This $\mathbb{Z}$-grading is compatible with the superalgebra structure on $\mathcal{B}^*$ by declaring generators in $\mathfrak{g}$ to be even and generators in $\mathfrak{m}^* \oplus \hat{\mathfrak{m}}$ odd.
Take a basis \( \{b_i\} \) of \( m \) and let \( \{f^i\} \) be its dual basis for \( m^* \). Let \( d = [\phi, -] \) be the super-derivation of \( B^* \) (of BRST degree 1), which is defined to be the supercommutator with the following odd element
\[
\phi = f^i(b_i - \chi(b_i)) - \frac{1}{2} f^i f^j[b_i, b_j]^\wedge
\]
where \( a^\wedge \) denotes the corresponding element in \( \hat{m} \) for \( a \in m \). Here and below we have adopted the convention of summation over repeated indices. It is easy to check that \( \phi \) is independent of the choices of the dual bases.

In more concrete terms, one finds that, for \( x \in g, f \in m^* \) and \( a^\wedge \in \hat{m} \) corresponding to \( a \in m \subseteq g \),
\[
d(x) = f^i[b_i, x], \quad d(f) = \frac{1}{2} f^i \text{ad}^* b_i(f),
\]
\[
d(a^\wedge) = a - \chi(a) + f^i[b_i, a]^\wedge,
\]
(13)
where we have denoted by \( \text{ad}^* \) the coadjoint action.

**Exercise 20.** Verify that \( d^2 = 0 \).

So we have defined the so-called BRST complex \((B^*, d)\) and we can define as usual its cohomology \( H^*(B^*) \). The BRST definition of the \( W \)-algebra (cf. de Boer and Tjin [BT]) is
\[
W_\chi = H^0(B^*, d).
\]
This is justified by the following isomorphism theorem, which confirms a conjecture of Premet [Pr2, 1.10].

**Theorem 21.** [DHK] The BRST cohomology \( H^*(B^*) \) is concentrated in cohomological degree 0 and there is an algebra isomorphism between \( H^0(B^*, d) \) and \((Q_\chi)^{adm}\).

**Sketch of a proof.** Note that in the definition of the BRST complex \((B^*, d)\), \( B^* \) does not depend on \( \chi \), but \( d \) does. We introduce a complex \((B^*_\chi, d')\) which is isomorphic to the BRST complex \((B^*, d)\) but with a shift of the dependence of \( \chi \) from the differential to the complex.

The complex \( B^*_\chi := \wedge(m^*) \otimes (U(g) \otimes C_\chi) \otimes \wedge(\hat{m}) \) is obtained from twisting \( B^* \) by \( \chi \), and \( U(g) \otimes C_\chi \) is naturally an \( m \)-bimodule; the differential \( d' \) equals the sum of two (anti)commuting differentials \( d_m \) and \( d^m \), where \( d_m \) (and respectively, \( d^m \)) is a \( m \)-homology (respectively, \( m \)-cohomology) differential. Then \((B^*_\chi, d')\) can be regarded as a double complex, and the first term of a spectral sequence computation gives us \( H^0(B^*_\chi, d_m) \cong \wedge^*(m^*) \otimes Q_\chi \) (the \( m \)-cohomology complex with coefficient in \( Q_\chi \)); moreover, \( H^{p,q}(B^*_\chi, d_m) = 0 \) whenever \( q \neq 0 \). The spectral sequence stabilizes at the second term, which shows \( H^i(B^*_\chi, d) \cong \text{gr} H^i(B^*_\chi, d) = H^i(m, Q_\chi) \), for all \( i \).

We shall see from Theorem 33 later on that \( H^0(m, Q_\chi) = Q_\chi^{adm} \) and that \( H^i(m, Q_\chi) = 0 \) for \( i > 0 \).

A quasi-isomorphism \( \wedge^*(m^*) \otimes Q_\chi \rightarrow B^*_\chi \) can then be constructed explicitly, and it induces the algebra isomorphism on 0th cohomology. Putting all things together, we have obtained the theorem. \( \square \)
We shall describe such an isomorphism in a more concrete fashion. Note by the PBW theorem that $\mathcal{B}^0 = U(\mathfrak{g}) \oplus m^* \mathcal{B}^0 \hat{m}$, with $m^* \mathcal{B}^0 \hat{m}$ being a two-sided ideal. So we can consider the composition $q$ of the two natural maps $\mathcal{B}^0 \rightarrow U(\mathfrak{g}) \rightarrow Q_\chi$, which fits into the following commutative diagram:

$$
\begin{array}{c}
0 & \longrightarrow & I_\chi \oplus m^* \mathcal{B}^0 \hat{m} & \longrightarrow & \mathcal{B}^0 & \longrightarrow & q & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \text{pr} & & \downarrow \\
0 & \longrightarrow & I_\chi & \longrightarrow & U(\mathfrak{g}) & \longrightarrow & Q_\chi & \longrightarrow & 0
\end{array}
$$

By (13), we check that $d$ maps $\mathcal{B}^{-1}$ into $I_\chi \oplus m^* \mathcal{B}^0 \hat{m}$. So $q$ induces a well-defined linear map $q: H^0(\mathcal{B}^{*},d) \rightarrow Q_\chi$. Then, $q$ is the algebra isomorphism between $H^0(\mathcal{B}^{*},d)$ and $Q^m_\chi$ claimed in Theorem 21.

Remark 22. The BRST formulation of $W$-algebras works for an arbitrary good $\mathbb{Z}$-gradings without the assumption of “even”. One needs to add an additional “neutral fermion” tensor factor to $\mathcal{B}^{*}$ which corresponds to the presence of nonzero $\mathfrak{g}^{-1}$ (cf. [KRW, DHK]). Again here, one has the flexibility of choosing a Lagrangian/isotropic subspace $l$ of $\mathfrak{g}^{-1}$ in defining the modified BRST complex.

Remark 23. There is yet another equivalent definition of finite $W$-algebras, due to Losev [Lo1], via deformation quantization.

4. Quantization of the Slodowy slices

In this section, we will explain the independence of $l$ in the definition of finite $W$-algebras, denoted now by $\mathcal{W}_l$, associated to an isotropic subspace $l \subseteq \mathfrak{g}^{-1}$, following Gan and Ginzburg [GG]. To that end, we explain the geometric picture behind the $W$-algebras in terms of Slodowy slices, due to Premet [Pr2] (and generalized in [GG]). Everything works in a general good $\mathbb{Z}$-grading setting, as remarked in the Introduction of [BK1].

4.1. The isotropic subspace definition of $W$-algebras. Fix an isotropic subspace $l$ of $\mathfrak{g}^{-1}$ with respect to $\langle \cdot, \cdot \rangle$, which means that $\langle l, l \rangle = 0$, and let

$$
l' = \{ x \in \mathfrak{g}^{-1} \mid \langle x, l \rangle = 0 \}.$$

Clearly we have $l \subseteq l'$. In this section, we will introduce and study the generalized Gelfand-Graev modules and $W$-algebras (using notations $Q_l$ and $\mathcal{W}_l$ to emphasize the dependence on $l$) in such a setting.

Define the following nilpotent subalgebras of $\mathfrak{g}$ (where $m \subseteq m'$):

$$
m = \bigoplus_{i \leq -2} \mathfrak{g}_i \quad \text{and} \quad m' = l' \bigoplus_{i \leq -2} \mathfrak{g}_i.$$

Note that $\chi$ restricts to a character on $m$ (actually this is equivalent to the requirement that $l$ is isotropic in $\mathfrak{g}^{-1}$). Denote by $\mathbb{C}_\chi$ the corresponding 1-dimensional $U(m)$-module, and let

$$
Q_t = U(\mathfrak{g}) \otimes_{U(m)} \mathbb{C}_\chi = U(\mathfrak{g})/I_t
$$
be the induced $U(\mathfrak{g})$-module, where $I_l$ denotes the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a), \forall a \in \mathfrak{m}$. The same proof for Exercise 18 shows that $I_l$ is ad $\mathfrak{m}'$-invariant. Thus, there is an induced ad $\mathfrak{m}'$-action on $Q_l$. Let

$$W_l := (U(\mathfrak{g})/I_l)^{\text{ad} \mathfrak{m}'} \equiv \{ \bar{y} \in U(\mathfrak{g})/I_l \mid [a, y] \in I_l, \forall a \in \mathfrak{m}' \}. $$

In the same spirit of Exercise 18 letting $\bar{y}_1\bar{y}_2 = \bar{y}_1\bar{y}_2$, for $\bar{y}_1, \bar{y}_2 \in W_l$, provides a well-defined algebra structure on $W_l$.

Clearly, the above definition of $Q_l$ and $W_l$ reduces to the earlier one when $l$ is Lagrangian.

**Theorem 24.** \[\text{CG}\] The algebras $W_l$ are all isomorphic for different choices of isotropic (in particular, Lagrangian) subspaces $l \subseteq \mathfrak{g}_{-1}$.

The proof of this theorem requires some preparations.

4.2. $\Gamma$-graded $\mathfrak{sl}_2$-triples.

**Lemma 25.** Let $\Gamma : \mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a good $\mathbb{Z}$-grading for $0 \neq e \in \mathfrak{g}_2$. Then there exists $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-2}$ such that $\{e, h, f\}$ form an $\mathfrak{sl}_2$-triple.

This will be referred to as a $\Gamma$-graded $\mathfrak{sl}_2$-triple associated to $e$. This $\Gamma$-graded $\mathfrak{sl}_2$-triple will be used often.

**Proof.** By Jacobson–Morozov Theorem in 23, there exists an $\mathfrak{sl}_2$-triple $\{e, h', f'\}$ in $\mathfrak{g}$. Denote by $h' = \sum_{j \in \mathbb{Z}} h_j$, $f' = \sum_{j \in \mathbb{Z}} f'_j$ the decompositions with respect to the $\mathbb{Z}$-grading $\Gamma$, and take $h = h_0 \in \mathfrak{g}_0$. Then $[h, e] = 2e$ and $h = [e, f'_{-2}] \in [e, \mathfrak{g}]$. Taking $\bar{f}$ to be the degree $-2$ component of $f_{-2}$ with respect to ad $h$, we obtain a new $\mathfrak{sl}_2$-triple $\{e, h, \bar{f}\}$. Finally, taking $f$ to be the degree $-2$ component of $\bar{f}$ with respect to $\Gamma$, we obtain the desired $\mathfrak{sl}_2$-triple $\{e, h, f\}$. (Indeed, $\bar{f} = f$ by Lemma 12 in other word, $\bar{f}$ is already $\Gamma$-homogeneous.)

Given a subspace $V$ of $\mathfrak{g}$, we let

$$V^\perp = \{ x \in \mathfrak{g} \mid (x|v) = 0, \forall v \in V \}, \quad V^{* \perp} = \{ \xi \in \mathfrak{g}^* \mid \xi(v) = 0, \forall v \in V \}. $$

**Lemma 26.** Let $\{e, h, f\}$ be a $\Gamma$-graded $\mathfrak{sl}_2$-triple. Then, we have $\mathfrak{m}^\perp = [\mathfrak{m}', e] \oplus \mathfrak{g}_f$.

**Proof.** We have properties (1)-(6) associated to $e$ and their $f$-counterparts available. Now the lemma follows from the four facts below:

(i) $\mathfrak{m}^\perp \supseteq \mathfrak{g}_f$. This follows from $\mathfrak{m}^\perp \supseteq \mathfrak{g}_{\leq 0}$ by (5) and the $f$-counterpart to (1) which says that $\mathfrak{g}_f \subseteq \mathfrak{g}_{\leq 0}$.

(ii) $\mathfrak{m}^\perp \supseteq [\mathfrak{m}', e]$. This can be seen by a direct computation: $\langle m|[m', e] \rangle = \langle m, m' \rangle |e \rangle = 0$ for $m \in \mathfrak{m}$ and $m' \in \mathfrak{m}'$.

(iii) $[\mathfrak{m}', e] \cap \mathfrak{g}_f = 0$. This follows by the $\mathfrak{sl}_2$ representation theory.

(iv) $\dim \mathfrak{m}^\perp = \dim \mathfrak{m}' + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{-1} = \dim [\mathfrak{m}', e] + \dim \mathfrak{g}_f$. This follows by the bijection $\mathfrak{m}' \to [\mathfrak{m}', e]$, $x \mapsto [x, e]$, by (2), and the $f$-counterpart to (6).
4.3. **Some $\mathbb{C}^*$-actions.** The embedding of the $\Gamma$-graded $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ exponentiates to a rational homomorphism $\tilde{\gamma} : \text{SL}_2(\mathbb{C}) \to G$. We put

$$\gamma : \mathbb{C}^* \to G, \quad t \mapsto \tilde{\gamma}(\text{diag}(t, t^{-1})).$$

Note that $(\text{Ad} \gamma(t))(e) = t^2 e$. The desired action of $\mathbb{C}^*$ on $\mathfrak{g}$, to be denoted by $\rho$, is defined by

$$\rho(t)(x) = e^2 \cdot (\text{Ad} \gamma(t^{-1}))(x), \quad \forall t \in \mathbb{C}^*, x \in \mathfrak{g}.$$

Note that $\rho(t)(e + x) = e + \rho(t)(x)$. Thus, since $\rho(t)$ stabilizes $\mathfrak{g}_f$ and $\mathfrak{m}^\perp$, it also stabilizes $e + \mathfrak{g}_f$ and $e + \mathfrak{m}^\perp$, respectively. Note that $\lim_{t \to 0} \rho(t)(x) = e, \forall x \in e + \mathfrak{m}^\perp$, i.e. the $\mathbb{C}^*$-action on $e + \mathfrak{m}^\perp$ is contracting.

Let $M'$ denote the closed subgroup of $G$ whose Lie algebra is $\mathfrak{m}'$. Define a $\mathbb{C}^*$-action on $M' \times (e + \mathfrak{g}_f)$ by

$$t \cdot (g, x) = (\gamma(t^{-1})g \gamma(t), \rho(t)(x)).$$

Note that for any $(g, x) \in M' \times (e + \mathfrak{g}_f)$, we have $\lim_{t \to 0} t \cdot (g, x) = (1, e)$.

4.4. Denote by $\kappa : \mathfrak{g} \to \mathfrak{g}^*$ the isomorphism induced by the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$. Following the terminology of Gan and Ginzburg, we will call $S := \chi + \ker \text{ad}^* f \equiv \kappa(e + \mathfrak{g}_f)$ the Slodowy slice (through $\chi$), or $e + \mathfrak{g}_f$ the Slodowy slice (through $e$). Note that $e + \mathfrak{g}_f$ is a transversal slice to the adjoint orbit through $e$.

**Lemma 27.** The adjoint action map

$$\alpha : M' \times (e + \mathfrak{g}_f) \longrightarrow e + \mathfrak{m}^\perp$$

is an isomorphism of affine varieties.

**Sketch of a proof.** The action map $\alpha$ is $\mathbb{C}^*$-equivariant.

By Lemma 26, the differential map of $\alpha$ is an isomorphism between the tangent spaces at the $\mathbb{C}^*$-fixed points $(1, e)$ and $e$.

Now, Lemma 27 follows from the following general nonsense: an equivariant morphism $\alpha : X_1 \to X_2$ of smooth affine $\mathbb{C}^*$-varieties with contracting $\mathbb{C}^*$-actions which induces an isomorphism between the tangent spaces of the $\mathbb{C}^*$-fixed points must be an isomorphism. □

4.5. The remainder of this section is to make sense the following commutative diagram:

$$\begin{array}{cccccccc}
\text{gr}U(\mathfrak{g}) & \cong & \mathbb{C}[\mathfrak{g}^*] & \longrightarrow & \mathbb{C}[\mathfrak{g}] \\
\downarrow & & \downarrow & & \downarrow \\
\text{gr}Q_l & \cong & \mathbb{C}[\chi + \mathfrak{m}^*] & \longrightarrow & \mathbb{C}[e + \mathfrak{m}^\perp] \\
\uparrow & & \downarrow & & \downarrow \\
\text{gr}W_l & \sim \longrightarrow & \mathbb{C}[S] & \longrightarrow & \mathbb{C}[e + \mathfrak{g}_f]
\end{array} \quad (14)$$
Remark 28. When reading (14), it is instructive to keep in mind the isomorphism $\mathbb{C}[e + \mathfrak{g}] \cong \mathbb{C}[e + \mathfrak{m}^\perp \mathfrak{ad} \mathfrak{m}']$ by Lemma 27 and the identity $\mathcal{W}_l = Q_l^{\mathfrak{ad} \mathfrak{m}'}$ by definition. Another way to rephrase this isomorphism is that $\mathcal{S}$ is obtained by the symplectic reduction for the coadjoint action of $\mathfrak{m}'$ on $\mathfrak{g}^*$ at the point $\chi$.

Some parts of the diagram (14) are easy to explain. The vertical arrows in the third column are restriction maps (e.g., $\mathcal{g}_f \subseteq \mathfrak{m}^\perp$ by Lemma 26). The second column (which is more conceptual from the coadjoint orbit philosophy) is transferred from the (simpler) third one by the isomorphism $\kappa : \mathfrak{g} \to \mathfrak{g}^*$.

It remains to explain the first column (where $\mathcal{S}$ stands for the associated graded space/algebra for a Kazhdan-filtered space/algebra $A$ and its identification with the second one. See Section 4.6 below.

Granting the diagram (14), we can complete the proof of Theorem 24.

Proof of Theorem 24. Given two isotropic subspaces $l_1, l_2$ of $\mathfrak{g}_{-1}$ such that $l_1 \subseteq l_2$, we have a natural $U(\mathfrak{g})$-homomorphism $Q_{l_1} \to Q_{l_2}$, which gives rise to an algebra homomorphism $\mathcal{W}_{l_1} \to \mathcal{W}_{l_2}$. The associated graded map $\text{gr}\mathcal{W}_{l_1} \to \text{gr}\mathcal{W}_{l_2}$ is an algebra isomorphism, by Theorem 30. Hence the canonical map $\mathcal{W}_{l_1} \to \mathcal{W}_{l_2}$ is indeed an isomorphism.

Taking $l_1 = 0$ implies that the isoclass of the algebra $\mathcal{W}_{l_2}$ is independent of the choice of an isotropic subspace $l$. This completes the proof of the theorem. \hfill \Box

4.6. Kazhdan grading and filtration. Let $\{U_j(\mathfrak{g})\}_{j \geq 0}$ be the standard PBW filtration on $U(\mathfrak{g})$. Recall that $h_\Gamma$ is introduced in Lemma 4. The action of $\mathfrak{ad} h_\Gamma$ induces a grading on each $U_j(\mathfrak{g})$ by

$$U_j(\mathfrak{g})_i = \{x \in U_j(\mathfrak{g}) \mid \mathfrak{ad} h_\Gamma(x) = ix\}.$$ 

The Kazhdan filtration on $U(\mathfrak{g})$ is defined by letting

$$F_n U(\mathfrak{g}) = \sum_{i+2k \leq n} U_k(\mathfrak{g})_i,$$

and it enjoys the following favorable properties:

(a) The canonical map $\text{gr} U(\mathfrak{g}) \to S[\mathfrak{g}] = \mathbb{C}[\mathfrak{g}^*]$ is an isomorphism of graded commutative algebras. Indeed, for $x \in \mathfrak{g}_i, y \in \mathfrak{g}_j$, we have $x \in F_{i+2} U(\mathfrak{g}), y \in F_{j+2} U(\mathfrak{g})$, and $[x, y] \in F_{i+j+2} U(\mathfrak{g})$.

The Kazhdan grading on $\mathfrak{g}$ and so on $S(\mathfrak{g})$ (which is compatible with the one on $\text{gr} U(\mathfrak{g})$) can be described directly.

(b) There is a Kazhdan filtration $\{F_n Q_l\}$ on $Q_l = U(\mathfrak{g})/I_l$ induced from $U(\mathfrak{g})$.

- $F_n Q_l = 0$ unless $n \geq 0$. Indeed, the generators $\{a - \chi(a) \mid \forall a \in \mathfrak{m}\}$ of the ideal $I_l$ contains all the negative-degree generators of $U(\mathfrak{g})$.

- $\text{gr} Q_l = \text{gr} U(\mathfrak{g})/\text{gr} I_l$ is a commutative $\mathbb{N}$-graded algebra.

- The ideal $\text{gr} I_l$ in $\text{gr} U(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ can be identified with the ideal of polynomial functions on $\mathfrak{g}^*$ which vanish on $\chi + \mathfrak{m}^\perp$.

- The canonical map $\text{gr} Q_l \to \mathbb{C}[\chi + \mathfrak{m}^\perp]$ is an algebra isomorphism.

(c) There is an induced Kazhdan filtration on the subspace $\mathcal{W}_l$ of $Q_l$ such that $F_n \mathcal{W}_l = 0$ unless $n \geq 0$. 

Remark 29. One could define the graded algebra structures on the second and third columns directly by some canonical $C^*$-action on $\mathfrak{g}$ which preserves $e + \mathfrak{m}^\perp$ and $e + \mathfrak{g}_f$ (similar to Section 4.3), and then show that the horizontal maps between the last two columns in (14) are isomorphisms of graded algebras.

We have chosen not to discuss the canonical Poisson algebra structures for various graded algebras above (see [GG] for details).

4.7. To complete the commutative diagram (14), we define the map $\nu : \text{gr} \mathcal{W}_l \to \mathbb{C}[S]$ to be the composite of the three natural maps

$$\text{gr} \mathcal{W}_l \to \text{gr} \mathcal{Q}_l \to \mathbb{C}[\chi + \mathfrak{m}^\perp, \mathfrak{m}] \to \mathbb{C}[S].$$

The last piece for the completion of the diagram (14) is the following result (due to Premet [Pr2] for Lagrangian $l$ and Gan-Ginzburg [GG] for isotropic $l$), whose proof will be postponed to the next section.

Theorem 30. The map $\nu : \text{gr} \mathcal{W}_l \to \mathbb{C}[S]$ is an isomorphism of [graded] algebras.

4.8. We have the following fundamental theorem of Kostant [Ko]. Here we follow the approach of Premet [Pr2].

Theorem 31. Let $e$ be a regular nilpotent element in $\mathfrak{g}$. Then $\mathcal{W}_l \cong \mathcal{Z}(\mathfrak{g})$, the center of $U(\mathfrak{g})$.

Sketch of a proof. The algebra of invariants $S(\mathfrak{g}^*)^G$ is known to be a polynomial algebra in $r = \text{rank}(\mathfrak{g})$ variables. Let $f_1, \ldots, f_r$ denote the algebraically independent homogeneous generators of $S(\mathfrak{g}^*)^G$, with $\deg f_i = m_i + 1$, where $m_i$ denotes the exponents of $\mathfrak{g}$. Since the associated graded algebra of $\mathcal{Z}(\mathfrak{g})$ with respect to the PBW filtration $\{U_j(\mathfrak{g})\}_{j \geq 0}$ is isomorphic to $S(\mathfrak{g}^*)^G$ (where we have identified $\mathfrak{g} \cong \mathfrak{g}^*$), we can choose lifts $\tilde{f}_i \in U_{m_i+1}(\mathfrak{g}) \cap \mathcal{Z}(\mathfrak{g})$ of each $f_i$.

Restricting the adjoint quotient map $\mathfrak{g} \to \mathfrak{g}/G = \mathbb{C}^r$, $x \mapsto (f_1(x), \ldots, f_r(x))$ to the Slodowy slice gives rise to a $\mathbb{C}^r$-equivariant isomorphism of affine varieties from $e + \mathfrak{g}_f$ to $\mathbb{C}^r$ [Ko, Slo], where the $\mathbb{C}^r$-action $\tilde{\rho}$ on $e + \mathfrak{g}_f$ is such that $\tilde{\rho}(t^2) = \rho(t)$ for $t \in \mathbb{C}^r$. Hence, $\text{gr} \mathcal{W}_l$ is generated by (the restrictions of) $f_i$ for $1 \leq i \leq r$.

One can show that the restriction of $U(\mathfrak{g}) \to \text{End}_\mathbb{C}(\mathcal{Q}_l)$ to the center $\mathcal{Z}(\mathfrak{g})$ is always injective, hence we obtain a monomorphism $\mathcal{Z}(\mathfrak{g}) \to \mathcal{W}_l = \text{End}_\mathbb{C}(\mathcal{Q}_l)$ by (7), whose associated graded map is an isomorphism. It follows from a filtered algebra argument that this has to be an isomorphism. \hfill $\square$

Remark 32. The $W$-algebras associated to subregular nilpotent elements are particularly interesting, as it can be viewed as a noncommutative deformation of the simple singularities. See Premet [Pr2] and Gordon-Rumynin [GR] for details.

5. AN EQUIVALENCE OF CATEGORIES

In this section, we formulate an equivalence of categories due to Skryabin. The proof of this category equivalence here, due to [GG], uses similar ideas of the proof of Theorem [30], which we will first complete.
5.1. **Proof of Theorem 30.** Recall that $\mathfrak{m}'$ is graded with respect to the grading $\Gamma$. We view $U(\mathfrak{g})$ and $Q_l$ as $\mathfrak{m}'$-modules via the adjoint $\mathfrak{m}'$-action. Then, $U(\mathfrak{g})$ and $Q_l$ are Kazhdan-filtered $\mathfrak{m}'$-modules and the canonical map $p : U(\mathfrak{g}) \rightarrow Q_l$ is $\mathfrak{m}'$-module homomorphism. Thus, $grU(\mathfrak{g})$ and $grQ_l$ are Kazhdan-graded $\mathfrak{m}'$-modules, and $gr p : grU(\mathfrak{g}) \rightarrow grQ_l$ is an $\mathfrak{m}'$-module homomorphism.

Note that $\mathcal{W}_l = H^0(\mathfrak{m}', Q_l)$, the 0th Lie algebra cohomology of $\mathfrak{m}'$ with coefficient in $Q_l$. We reformulate and prove Theorem 30 as follows.

**Theorem 33.** [GG] The map $\nu : gr\mathcal{W}_l \rightarrow \mathbb{C}[S]$ is equal to the composite $\nu_2\nu_1$:

$$grH^0(\mathfrak{m}', Q_l) \xrightarrow{\nu_1} H^0(\mathfrak{m}', grQ_l) \xrightarrow{\nu_2} \mathbb{C}[S]$$

where $\nu_1$ and $\nu_2$ are isomorphisms. Moreover, $H^i(\mathfrak{m}', Q_l) = H^i(\mathfrak{m}', grQ_l) = 0$ for $i > 0$.

*Proof.* By (b) of [14.6] and Lemma 27, we obtain isomorphisms of vector spaces

$$grQ_l \cong \mathbb{C}[\chi + m^{*-1}] \cong \mathbb{C}[\mathfrak{m}'] \otimes \mathbb{C}[S].$$

These isomorphisms are actually on the level of $\mathfrak{m}'$-modules, where the $\mathfrak{m}'$-module structure on the third space comes from the $\mathfrak{m}'$-adjoint action on its first tensor factor $\mathbb{C}[\mathfrak{m}']$. Now

$$H^i(\mathfrak{m}', \mathbb{C}[\mathfrak{m}']) = \delta_{i,0} \mathbb{C}$$

since the standard cochain complex for Lie algebra cohomology with coefficients in $\mathbb{C}[\mathfrak{m}']$ is just the algebraic de Rham complex for $\mathfrak{m}'$ which admits trivial cohomology for an affine space such as $\mathfrak{m}'$. Putting all things together, we have proved that $\nu_2$ is an isomorphism and that $H^i(\mathfrak{m}', grQ_l) = 0$ for $i > 0$.

Recall from (b) in [14.6] that the Kazhdan-filtration on $Q_l$ has no negative-degree component. Note in addition that $\mathfrak{m}'$ is a negatively graded subalgebra of $\mathfrak{g}$ with respect to the grading $\Gamma$, and so its dual $\mathfrak{m}'^*$ is positively graded (with respect to $\Gamma$). We write this graded decomposition as $\mathfrak{m}'^* = \bigoplus_{i \geq 1} \mathfrak{m}'_{i}^*$.

Consider the standard cochain complex for computing the $\mathfrak{m}'$-cohomology of $Q_l$:

$$0 \rightarrow Q_l \rightarrow \mathfrak{m}'^* \otimes Q_l \rightarrow \ldots \rightarrow \wedge^k \mathfrak{m}'^* \otimes Q_l \rightarrow \ldots$$

(15)

A filtration on $\wedge^k \mathfrak{m}'^* \otimes Q_l$ is defined by letting $F_p(\wedge^k \mathfrak{m}'^* \otimes Q_l)$ be the subspace of $\wedge^k \mathfrak{m}'^* \otimes Q_l$ spanned by $(x_1 \wedge \ldots \wedge x_k) \otimes v$, for all $x_i \in \mathfrak{m}'_{i}^*$, $x_k \in \mathfrak{m}'_{i_k}^*$ and $v \in F_q Q_l$ such that $i_1 + \ldots + i_n + j \leq p$. This defines a filtered complex structure on (15).

Taking the associated graded complex of (15) gives the standard cochain complex for computing the $\mathfrak{m}'$-cohomology of $grQ_l$.

Now consider the spectral sequence with

$$E_0^{p,q} = F_p(\wedge^{p+q} \mathfrak{m}'^* \otimes Q_l)/F_{p-1}(\wedge^{p+q} \mathfrak{m}'^* \otimes Q_l).$$

Then $E_1^{p,q} = H^{p+q}(\mathfrak{m}', gr_{p}Q_l)$, and the spectral sequence converges to $E_\infty^{p,q} = F_p H^{p+q}(\mathfrak{m}', Q_l)/F_{p-1} H^{p+q}(\mathfrak{m}', Q_l)$. The remaining parts of Theorem 33 follow from this and the parts about $grQ_l$ established above. \qed
5.2. The Whittaker functor. As we have established the independence of the $W$-algebras from the choice of isotropic subspaces $I$, we will switch the notations for the generalized Gelfand-Graev module and the $W$-algebra back to $Q_X$, $W_X$, to emphasize the crucial dependence on $\chi$.

In the remainder of this section, we will set up the connections between $W$-algebras and the category of Whittaker modules. To that end, we shall fix an nilpotent element $e$ (and hence $\chi$) and a Lagrangian subspace $I$ of $g_{-1}$ once for all.

**Definition 34.** A $g$-module $E$ is called a Whittaker module if $a - \chi(a)$, $\forall a \in m$, acts on $E$ locally nilpotently. A Whittaker vector in a Whittaker $g$-module $E$ is a vector $x \in E$ which satisfies $(a - \chi(a))x = 0$, $\forall a \in m$.

Let $g$-$\text{mod}^\chi$ be the category of finitely generated Whittaker $g$-modules.

**Lemma 35.**

1. Given a Whittaker $g$-module $E$ with an action map $g$, $\text{Wh}(E)$ is naturally a $W_X$-module by letting $\bar{y}.v = \phi(y)v$ for $v \in \text{Wh}(E)$ and $\bar{y} \in W_X$.

2. For $V \in W_X$-$\text{mod}$, $Q_X \otimes_{W_X} V$ is a Whittaker $g$-module by letting $y.(q \otimes v) = (y.q) \otimes v$, for $y \in U(g), q \in Q_X = U(g)/I_X$, $v \in V$.

**Proof.**

1. Let $v \in \text{Wh}(E)$. The formula $\bar{y}.v = \phi(y)v$ is well-defined since $\phi(y)v = 0$ for all $y \in I_X$ and $v \in \text{Wh}(E)$. Being $\text{ad } m$-invariants, $\bar{y} \in W_X$ satisfies $[\phi(y), a] \in I_X$ and so $\bar{y}.v \in \text{Wh}(E)$. It follows from the $g$-module homomorphism $\phi$ that the formula defines an action of $W_X$.

2. When we use the first definition $W_X = \text{End}_{U(g)}(Q_X)^{op}$, it is trivial to see that $Q_X$ is a $(g, W_X)$-bimodule and then $Q_X \otimes_{W_X} V$ is a $g$-module.

To check that $Q_X \otimes_{W_X} V$ lies in $g$-$\text{mod}^\chi$, it suffices to check that $a - \chi(a)$, $\forall a \in m$ acts locally nilpotently on $Q_X$ by the definition of the $U(g)$-action on $Q_X \otimes_{W_X} V$. Since $m$ is negatively graded with respect to the grading $\Gamma$, $a = \text{ad } (a - \chi(a))$ for any $a \in m$ acts locally nilpotently on $g$ and so locally nilpotently on $U(g)$ (and also on $Q_X = U(g)/I_X$) by induction on the PBW filtration length for $U(g)$. \hfill \square

Let $W_X$-$\text{mod}$ be the category of finitely generated $W_X$-modules. We define the Whittaker functor

$$\text{Wh} : g$-$\text{mod}^\chi \rightarrow W_X$-$\text{mod}, \quad E \mapsto \text{Wh}(E).$$

We define another functor

$$Q_X \otimes_{W_X} - : W_X$-$\text{mod} \rightarrow g$-$\text{mod}^\chi, \quad V \mapsto Q_X \otimes_{W_X} V.$$

5.3. The Skryabin equivalence. The following theorem is due to Skryabin [Sk1], and here we follow the new proof of [GG].

**Theorem 36.** The functor $Q_X \otimes_{W_X} - : W_X$-$\text{mod} \rightarrow g$-$\text{mod}^\chi$ is an equivalence of categories, with $\text{Wh} : g$-$\text{mod}^\chi \rightarrow W_X$-$\text{mod}$ as its quasi-inverse.
Proof. (1) We first prove that $Wh(Q_{\chi} \otimes_{W_x} V) = V$.

Assume $V$ is generated as a $W_x$-module by a finite-dimensional subspace $V_0$. This gives rise to a $W_x$-filtered module structure on $V$ by letting $F_nV = (F_n W_x) V_0$. Note that $H^0(m, Q_{\chi} \otimes_{W_x} V) = Wh(Q_{\chi} \otimes_{W_x} V)$, where we regard $Q_{\chi} \otimes_{W_x} V$ as an $m$-module with $\chi$-twisted action.

We shall establish the following Claim in Lie algebra cohomology; this readily implies that $H^0(m, Q_{\chi} \otimes_{W_x} V) \cong V$, which is equivalent to $Wh(Q_{\chi} \otimes_{W_x} V) = V$.

Claim 1. $grH^0(m, (Q_{\chi} \otimes_{W_x} V)) \cong H^0(m, gr(Q_{\chi} \otimes_{W_x} V)) \cong grV$; moreover, $H^i(m, (Q_{\chi} \otimes_{W_x} V)) = H^i(m, gr(Q_{\chi} \otimes_{W_x} V)) = 0$ for $i > 0$.

The two isomorphisms $\mu_1, \mu_2$ in the Claim are parallel to the two isomorphisms $\nu_1, \nu_2$ in Theorem 33 and they will be proved in the same strategy as in the proof of Theorem 33.

By Lemma 27 and the diagram (14), we have

$$gr(Q_{\chi} \otimes_{W_x} V) \cong grQ_{\chi} \otimes_{grW_x} grV \cong (C[M] \otimes grW_{\chi}) \otimes_{grW_x} grV \cong C[M'] \otimes grV.$$  

The isomorphism $\mu_2$ follows quickly from this.

The isomorphism $\mu_1$ follows exactly as the argument for the isomorphism $\nu_1$ in the proof of Theorem 33 once we note that $Q_{\chi} \otimes_{W_x} V$ is filtered with no negative-degree components and $m^*$ is $\mathbb{N}$-graded, and then apply a spectral sequence argument.

(2) We shall show that, for any $E \in g{\text{-}}\text{mod}^\chi$, the canonical map

$$\gamma : Q_{\chi} \otimes_{W_x} Wh(E) \rightarrow E, \quad \bar{y} \otimes_v \mapsto y.v$$

is an isomorphism, where $\bar{y} \in Q_{\chi} = U(g)/I_{\chi}$ is associated to $y \in U(g)$.

We first note that $\gamma$ is well-defined since $Q_{\chi} = U(g)/I_{\chi}$ and $I_{\chi}$ is generated by $a - \chi(a), \forall a \in m$. Also observe that $Wh(E) = 0$ implies that $E = 0$.

Take an exact sequence of $g$-modules:

$$0 \rightarrow E' \rightarrow Q_{\chi} \otimes_{W_x} Wh(E) \rightarrow E \rightarrow E'' \rightarrow 0. \quad (16)$$

Let us show that $\gamma$ is injective (i.e. $E' = 0$). Indeed,

$$Wh(E') = E' \cap Wh(Q_{\chi} \otimes_{W_x} Wh(E)) = E' \cap Wh(E) = 0$$

where the second equality follows from Part (1) above and the third equality follows by definition of $\gamma$. Hence $E' = 0$.

Now we prove that $\gamma$ is surjective (i.e. $E'' = 0$). (16) reduces now to a short exact sequence

$$0 \rightarrow Q_{\chi} \otimes_{W_x} Wh(E) \rightarrow E \rightarrow E'' \rightarrow 0$$

which gives rise to a long exact sequence

$$0 \rightarrow H^0(m, Q_{\chi} \otimes_{W_x} Wh(E)) \rightarrow H^0(m, E) \rightarrow H^0(m, E'') \rightarrow H^1(m, Q_{\chi} \otimes_{W_x} Wh(E))$$

where $H^1(m, Q_{\chi} \otimes_{W_x} Wh(E)) = 0$ by Claim 1. Then, this exact sequence can be rewritten as

$$0 \rightarrow Wh(Q_{\chi} \otimes_{W_x} Wh(E)) \rightarrow Wh(E) \rightarrow Wh(E'') \rightarrow 0$$
By (1), \( \text{Wh}(Q_x \otimes_{W_x} \text{Wh}(E)) = \text{Wh}(E) \) and \( \gamma^* \) is an isomorphism. So, \( \text{Wh}(E'') = 0 \), whence \( E'' = 0 \). \( \square \)

6. Good \( \mathbb{Z} \)-gradings in type \( A \)

In this section, we will describe the classification of good \( \mathbb{Z} \)-gradings on \( g = \mathfrak{gl}_N \) or \( \mathfrak{sl}_N \), due to Elashvili and Kac [EK].

6.1. Pyramids of shape \( \lambda \). Given a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of \( N \), we construct a combinatorial object, called pyramids (of shape \( \lambda \)). We will illustrate the process by building pyramids of shape \( \lambda = (3, 2, 2) \).

We start with a (first) row of \( \lambda_1 = 3 \) boxes of size 2 units by 2 units, with column numbers \( 1 - \lambda_1, 3 - \lambda_1, \ldots \lambda_1 - 1 \) (which is \(-2, 0, 2\) in this example). We mark by • to indicate the column number 0.

Then, we add a (second) row of \( \lambda_2 = 2 \) boxes on top of the row 1. The rule is: keep the stair shape with permissible shifts by integer units. In this example, we have three possibilities, and this gives rise to 3 pyramids of shape \( (3, 2) \).

Then we repeat the process with the same rule by adding now a (third) row of \( \lambda_3 = 2 \) boxes on top of row 2. In this example, we have only one permissible way of doing so, and so obtain three pyramids of shape \( (3, 2, 2) \) below (where the column numbers are also indicated).

**Exercise 37.** There are 7 pyramids of shape \( (4, 1) \) as there are 7 permissible way of putting one box on top of

6.2. Given a pyramid \( P \) of shape \( \lambda \), let us fix a labeling by numbers \( \{1, 2, \ldots, N\} \) of the \( N \) boxes in \( P \). A convenient choice is to label downward from left to right in an increasing order.
Let us take the second pyramid of shape \((3, 2, 2)\) as an example with \(N = 7\). Our labeled pyramid reads

\[
\begin{array}{ccc}
1 & 4 & \\
2 & 5 & \\
3 & 6 & 7 \\
\end{array}
\]

We fix a standard basis \(v_i (1 \leq i \leq N)\) of \(\mathbb{C}^N\) associated to the above labelling. Let

\[
e = e^P = E_{14} + E_{25} + E_{36} + E_{67}
\]

be the nilpotent element in \(g\) which sends a vector \(v_i\) to \(v_{R(i)}\) where \(R(i)\) denotes the label to the right of \(i\) in the labelled pyramid \(P\) (by convention \(v_{R(i)} = 0\) whenever \(R(i)\) is not defined). Note that \(e\) has \(J_\lambda\) as its Jordan form.

A \(\mathbb{Z}\)-grading \(\Gamma^P\) of \(g\) is determined by letting the degree of a root vector for a simple root \(\alpha_i = \varepsilon_i - \varepsilon_{i+1}\) to be \(\text{col}_{i+1} - \text{col}_i\), where \(\text{col}_i\) denotes the column number of the box labelled by \(i\) in \(P\). In other words, the grading operator is

\[
h^P = h_\Gamma^P = -\text{diag} (\text{col}_1, \text{col}_2, \ldots, \text{col}_N)
\]

(note that \(h^P\) is unique up to a shift of a scalar multiple of the identity matrix \(I_N\), and there is a unique shift which results to a traceless matrix).

The grading \(\Gamma\) is even if and only if a pyramid is even in the sense that every box in \(P\) lies immediately on top of at most one box. For example, for \(\lambda = (3, 2, 2)\) above, the first and second pyramids are even while the third one is not.

**Example 38.** Let \(g = \mathfrak{gl}_3\) and \(e = E_{13}\). Then the good grading for \(e\) in Example 2 corresponds to the first pyramid and the Dynkin grading for \(e\) corresponds to the third pyramid below:

\[
\begin{array}{ccc}
1 & 3 & \\
2 & 3 & \\
\end{array}
\]

**Exercise 39.** Describe explicitly the good \(\mathbb{Z}\)-grading corresponding to the second pyramid.

**Theorem 40.** [EK] Let \(g = \mathfrak{gl}_N\) or \(\mathfrak{sl}_N\), and let \(\lambda\) be a partition of \(N\). There exists a one-to-one correspondence between the set of pyramids of shape \(\lambda\) and the set of good \(\mathbb{Z}\)-gradings for a nilpotent matrix of Jordan shape \(\lambda\) up to \(\text{GL}_N\)-conjugation,

\[
\{\text{Pyramids of shape } \lambda\} \longrightarrow \{\text{good } \mathbb{Z}\text{-gradings for a nilpotent matrix in } \Theta_\lambda\}/\text{GL}_N
\]

by sending \(P\) to \(\Gamma^P\).

**Proof.** It is easy to see the map is well-defined: different choices of labelling of \(P\) leads to another nilpotent matrix in the same nilpotent orbit.

To see that \(\Gamma^P\) is a good \(\mathbb{Z}\)-grading for \(e = e^P\), we will actually construct a \(\mathbb{N}\)-graded homogeneous basis for the centralizer \(g_e\) explicitly. Let \(\ell = \lambda'_1\). We will choose to label the first box of row \(1, \ldots, \ell\) of \(P\) upward by \(1, \ldots, \ell\). Note that
\{ e^{k_i}v_i \mid 1 \leq i \leq \ell, 0 \leq k_i \leq \lambda_i - 1 \} is a linear basis of \( \mathbb{C}^N \), and indeed each such \( e^{k_i}v_i\) is equal to \( v_j \) for some \( 1 \leq j \leq N \) which appears in the same row as \( i \) in \( P \).

An element \( z \) in \( g_e \) is determined by the values \( z(v_i) \) for \( 1 \leq i \leq \ell \), since \( z(e^{k_i}v_i) = e^{k_i}z(v_i) \). Consider for now \( z_{j,i;k_j} \in g_e(1 \leq i,j \leq \ell) \) such that \( z_{j,i;k_j}(v_i) = e^{k_j}v_j \) and \( z_{j,i;k_j}(v_{i'}) = 0 \) for \( 1 \leq i' \leq \ell, i' \neq i \). Then since \( e^{\lambda_j + k_j}v_j = z_{j,i;k_j}(e^{\lambda_j}v_i) = 0 \) and recall \( 0 \leq k_j < \lambda_j \), we see that the \( k_j \)'s have to satisfy the inequality:

\[ \lambda_j > k_j \geq \max(0, \lambda_j - \lambda_i), \quad \forall i,j \]

and this condition is sufficient for \( z_{j,i;k_j} \in g_e \) to be well-defined.

Since \( \min(\lambda_i, \lambda_j) = \lambda_j - \max(0, \lambda_j - \lambda_i) \), there are in total \( \sum_{1 \leq i,j \leq \ell} \min(\lambda_i, \lambda_j) \) of such \( z_{j,i;k_j} \in g_e \). These elements \( z_{j,i;k_j} \)'s in \( g_e \) are manifestly homogeneous, \( \mathbb{N} \)-graded and linearly independent. Hence, they must form a basis for \( g_e \), thanks to the well-known identity \( \dim g_e = \sum_{1 \leq i,j \leq \ell} \min(\lambda_i, \lambda_j) \) (which is also equal to \( \sum_{i \geq 1} \lambda_i' \) in terms of the conjugate partition \( \lambda' \)).

Now we shall construct an inverse map. Let \( e \) be a nilpotent element in the nilpotent orbit \( O_\lambda \) and let \( \Gamma = \text{ad} h \) be a good \( \mathbb{Z} \)-grading for \( e \). Up to \( GL_N \)-conjugation, we can assume that \( e = J_\lambda \), the canonical Jordan canonical form, and that \( h \) is a diagonal matrix. We can suitably relabel the standard basis vectors \( v_i \)'s of \( \mathbb{C}^N \) (or by a conjugation by some permutation matrix in \( GL_N \)), so that the standard basis of \( \mathbb{C}^N \) consists of \( \{ e^{k_i}v_i \mid 1 \leq i \leq \ell, 0 \leq k_i \leq \lambda_i - 1 \} \). In particular, \( E_{ji} \) for all \( i,j \) are homogeneous with respect to the grading \( \Gamma \). Now we visualize a labeled “generalized” pyramid \( P \) of shape \( \lambda \) as follows (here generalized means that the staircase conditions on both sides of \( P \) are not verified at the moment): the row one of \( P \) is fixed with column numbers \( 1 - \lambda_1, 3 - \lambda_1, \ldots, \lambda_1 - 1 \) with the leftmost box labelled by 1, and the remaining rows are fixed by letting the leftmost box of row \( j \) to have a column number \( \deg E_{j1} + 1 - \lambda_1 \).

Take the basis \( z_{j,i;k_j} \) for \( g_e \) constructed earlier. The good grading \( \Gamma \) implies that \( z_{j,i;0} = E_{ji} + \ldots \) have non-negative integral degrees, and so do \( E_{ji} \), for \( 1 \leq j < i \leq \ell \). This ensures that the left-hand side of the generalized pyramid \( P \) is of staircase shape. Similarly, the consideration of \( z_{j,i;\lambda_j - \lambda_i} \) for \( 1 \leq j < i \leq \ell \) shows the right-hand side of \( P \) is of staircase shape.

Hence, \( P \) is indeed a pyramid. It follows by the construction of \( P \) that \( e = e^P \).

**Corollary 41.** There exists a bijection between even pyramids of size \( N \) and even good \( \mathbb{Z} \)-gradings of \( g \) up to \( GL_N \)-conjugation (by sending \( P \) to \( \Gamma^P \)).

**Proof.** Follows from the way a grading is defined using a pyramid. \( \square \)

**Corollary 42.** Given a nilpotent element \( e \in g \), there exists an even good \( \mathbb{Z} \)-grading of \( g \) for \( e \).

**Proof.** Assume the Jordan form of \( e \) corresponds to a partition \( \lambda \). The Young diagram \( \lambda \) in French fashion is an even pyramid of shape \( \lambda \). \( \square \)

A pyramid \( P \) is call symmetric if \( P \) is self-dual with respect to the reflection along the vertical line passing through the mid-point of row one of \( P \). For example, the third pyramid of shape \( (3,2,2) \) in Section 6.1 is symmetric.
Corollary 43. There exists a bijection from the set of symmetric pyramids of size $N$ to the set of Dynkin gradings of $\mathfrak{g}$ up to $GL_N$-conjugation (by sending $P$ to $\Gamma^P$).

Proof. Let $P$ be a symmetric pyramid of size $N$. Then the good $\mathbb{Z}$-grading $\Gamma^P$ of $\mathfrak{g}$ is Dynkin, since we can take $\mathfrak{sl}_2$ to be the diagonal subalgebra of the direct sum of the $\mathfrak{sl}_2$'s associated to each row of $P$ (which corresponds to a Jordan block) as constructed in Example 2.

Any two Dynkin gradings for a given nilpotent element $e$ are conjugated by $GL_N$, as we have seen in the course of establishing the bijection $\Omega$ in 2.6. □

Remark 44. The classification of good $\mathbb{Z}$-gradings for other types was also carried out in [EK]. See Baur and Wallach [BW] for closely related classification of nice parabolic subalgebras and further clarification.

7. $W$-algebras and independence of good gradings

In this section, we shall prove that the isoclasses of finite $W$-algebras are independent of the choices of the good $\mathbb{R}$-gradings $\Gamma$, following Brundan and Goodwin [BG].

7.1. In this subsection, we fix notations which will be used throughout Section 7.

Let $e \in \mathfrak{g}$ be a nilpotent element. Fix an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ in $\mathfrak{g}$, and denote this copy of $\mathfrak{sl}_2$ by $\mathfrak{s}$.

Recall that the ad $h$-eigenspace decomposition of $\mathfrak{g}$ defines the Dynkin $\mathbb{Z}$-grading

$$\Gamma^\emptyset : \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j,$$

where $\mathfrak{g}_0 = \mathfrak{g}_h$.

Let $c = \mathfrak{g}_h$ and let $C$ be the corresponding closed connected subgroup of the adjoint group $G$. Also let $r = \bigoplus_{j > 0} \mathfrak{g}_j$ (associated to $\Gamma^\emptyset$) and let $R$ be the corresponding closed connected subgroup of $G_e$, with Lie algebra $\mathfrak{r}_e$, and that $R_e$ is the unipotent radical of $G_e$, with Lie algebra $\mathfrak{r}_e$.

Fix a maximal torus $T$ of $G$ contained in $C$ and containing a maximal torus of $C_e$. An important role is played by the centralizer $\mathfrak{h}_e$ of $e$ in the Lie algebra $\mathfrak{h}$ of $T$. It is a Cartan subalgebra of the reductive part $\mathfrak{c}_e$ of the centralizer $\mathfrak{g}_e$.

7.2. An $\mathbb{R}$-grading

$$\Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}_j$$

of $\mathfrak{g}$ is called a good grading (or good $\mathbb{R}$-grading) for $e$ if the conditions (1)-(3) in 2.4 holds (where the indices are taken in $\mathbb{R}$). Lemma 8 (i.e. (2) $\iff$ (3)) remains to be valid for $\mathbb{R}$-gradings.

We can associate to any $\mathbb{R}$-grading a semisimple element $h_\Gamma$, as in Lemma 4.

Definition 45. We say that a grading $\Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}_j$ is compatible with $\mathfrak{h}$ if $\mathfrak{h} \subseteq \mathfrak{g}_0$, or equivalently, if $h_\Gamma \in \mathfrak{h}$.
Remark 46. We have $h = h_{r^0} \in \mathfrak{h}$ since $\mathfrak{h} \subseteq \mathfrak{c}$ implies $[h, \mathfrak{h}] = 0$ and $\mathfrak{h}$ is maximal abelian. Hence, the Dynkin grading $\Gamma^0$ is compatible with $\mathfrak{h}$.

Lemma 47. Every $\mathbb{R}$-grading $\Gamma$ is $G$-conjugate to a $\mathbb{R}$-grading that is compatible with $\mathfrak{h}$.

Proof. Every semisimple element of $\mathfrak{g}$, in particular $h_{r^0}$, is $G$-conjugate to an element of $\mathfrak{h}$. The lemma follows. □

Let $E_e$ be the $\mathbb{R}$-form for $\mathfrak{h}_e$ consisting of all $p \in \mathfrak{h}_e$ such that the eigenvalues of $\text{ad} \ p$ on $\mathfrak{g}$ are real.

Lemma 48. (1) Any good $\mathbb{R}$-grading for $e$ is $G_e$-conjugate to a good $\mathbb{R}$-grading $\Gamma$ for $e$ compatible with $\mathfrak{h}$. (2) For such a $\Gamma$, we have that $h \in \mathfrak{g}_0$, $f \in \mathfrak{g}_{-2}$, and $h_{r^0} = h + p$ for some point $p \in E_e$.

Proof. (1) Recall that any two $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ sharing the same $e$ are conjugate by $G_e$, as seen from the proof of Theorem [11]. By Lemma 25, we may assume that the $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e, h, f\}$ we start with is $\Gamma^0$-graded, that is, $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-2}$.

Take the semisimple element $h_{r^0} \in \mathfrak{g}_0 = \mathfrak{h}_h$ as in Lemma 45. Let $p' = h_{r^0} - h$, which is a semisimple element lying in $\mathfrak{c}_e$ (which is the centralizer of $h$ and $e$), since $[p', e] = [h_{r^0}, e] - [h, e] = 2e - 2e = 0$. Then $p'$ is $C_e$-conjugate to an element in $\mathfrak{h}_e$, a Cartan subalgebra of $\mathfrak{c}_e$. Write $\text{Ad}g(p') = p \in \mathfrak{h}_e$ for $g \in C_e \subseteq G_e$. Recall $h \in \mathfrak{h}$ from Remark 46. Hence, $\text{Ad}g(h_{r^0}) = \text{Ad}g(h + p') = h + p \in \mathfrak{h}$, and the good grading $\Gamma := \text{Ad}g(\Gamma^0)$ for $e$ is compatible with $\mathfrak{h}$.

(2) For such a $\Gamma$ as in (1), we have $h \in \mathfrak{g}_0$ since $\mathfrak{h} \subseteq \mathfrak{g}_0$ and $h \in \mathfrak{h}$. Then $p = h_{r^0} - h$ centralizes $e$ and $h$, hence also $f$, so $[h_{r^0}, f] = [h, f] = -2f$ and $f \in \mathfrak{g}_{-2}$. Finally, $p$ belongs to $\mathfrak{h}_e$, hence to $E_e$, since $\Gamma$ is an $\mathbb{R}$-grading. □

7.3. For $p \in E_e$, we let $\Gamma^p$ denote the $\mathbb{R}$-grading of $\mathfrak{g}$ defined by $\text{ad} (h + p)$ (which is consistent with the notation for the Dynkin grading $\Gamma^0$). Such a grading $\Gamma^p$ is automatically compatible with $\mathfrak{h}$, but it may not be a good grading.

Note that $\mathfrak{h}_e$ commutes with $h$ and $e$, and hence with $\mathfrak{s}$. For $\alpha \in \Phi^*_e$ and $i \geq 0$, let $L(\alpha, i)$ denote the irreducible $\mathfrak{h}_e \oplus \mathfrak{s}$-module of dimension $(i + 1)$ on which $\mathfrak{h}_e$ acts by weight $\alpha$. Decompose $\mathfrak{g}$ as an $\mathfrak{h}_e \oplus \mathfrak{s}$-module

$$\mathfrak{g} \cong \bigoplus_{\alpha \in \Phi^*_e \cup \{0\}} \bigoplus_{i \geq 0} m(\alpha, i)L(\alpha, i)$$

where $\Phi^*_e$ denotes the set of nonzero weights of $\mathfrak{h}_e$ (or $E_e$) of $\mathfrak{g}$. One can show that $\Phi^*_e$ is a so-called restricted root system on $E_e$ (cf. [13]), but for our purpose we only need to record two simple facts: $\alpha \in \Phi^*_e$ implies $\pm \alpha \in \Phi^*_e$ and $\Phi^*_e$ spans $E^*_e$. For $\alpha \in \Phi^*_e$, let

$$d(\alpha) = 1 + \min \{i \geq 0 \mid m(\alpha, i) \neq 0\}$$

be the minimal dimension of a simple $\mathfrak{s}$-submodule of the $\alpha$-weight space of $\mathfrak{g}$.

Proposition 49. Let $p \in E_e$. The grading $\Gamma^p$ is a good grading for $e$ if and only if $|\alpha(p)| < d(\alpha)$ for all $\alpha \in \Phi^*_e$. 

Proof. Recall that $\Gamma^p : g = \bigoplus_{j \in \mathbb{Z}} g_j$ is a good grading for $e$ if and only if $g_e \subseteq \bigoplus_{j > -1} g_j$. Note that $h + p$ acts on the highest weight vector of $L(\alpha, i)$ as the scalar $\alpha(p) + i$ for $\alpha \in \Phi_e$. So, $\Gamma^p$ is a good grading for $e$ if and only if $g_e \subseteq \bigoplus_{j > -1} g_j$. Note that $h + p$ acts on the highest weight vector of $L(\alpha, i)$ as the scalar $\alpha(p) + i$ for $\alpha \in \Phi_e$. So, $\Gamma^p$ is a good grading for $e$ if and only if $\alpha(p) + i > -1$ for all $\alpha \in \Phi_e$; the latter (by considering $\pm \alpha \in \Phi_e$ simultaneously) is equivalent to $|\alpha(p)| < 1 + i$ for all $\alpha \in \Phi_e$. □

The good grading polytope for $e$ is defined to be

$$\mathcal{P}_e = \{ p \in E_e \mid \Gamma^p \text{ is a good } \mathbb{R}\text{-grading for } e \}.$$ 

Since $\Phi_e$ spans $E^*_e$, $\mathcal{P}_e$ is an open convex polytope in $E_e$ by Proposition 49: indeed, $|\alpha(tp_1 + (1-t)p_2)| \leq t|\alpha(p_1)| + (1-t)|\alpha(p_2)| < d(\alpha)$, for $p_1, p_2 \in \mathcal{P}_e$ and $0 \leq t \leq 1$.

7.4. Two good $\mathbb{R}$-gradings $\Gamma : g = \bigoplus_{i \in \mathbb{R}} g_i$ and $\Gamma' : g = \bigoplus_{i \in \mathbb{R}} g'_i$ are adjacent if

$$g = \bigoplus_{i^- \leq j \leq i^+} g_i \cap g'_j,$$

where $i^-$ denotes the largest integer strictly smaller than $i$ and $i^+$ denotes the smallest integer strictly greater than $i$.

Lemma 50. The following conditions are equivalent for $i, j \in \mathbb{R}$:

1. $i^- \leq j \leq i^+$;
2. $i$ and $j$ lie in the same unit interval $[a, a+1]$ for some $a \in \mathbb{Z}$;
3. $j^- \leq i \leq j^+$.

Exercise 51. Prove Lemma 50.

The next lemma allows us to reduce the study of adjacent good $\mathbb{R}$-gradings to the good grading polytope $\mathcal{P}_e$.

Lemma 52. Every pair of adjacent good $\mathbb{R}$-gradings for $e$ is $G_e$-conjugate to a pair of adjacent good $\mathbb{R}$-gradings for $e$ compatible with $\mathfrak{h}$.

Sketch of a proof. For adjacent gradings $\Gamma$ and $\Gamma'$, $\text{ad} h \Gamma$ and $\text{ad} h \Gamma'$ are by definition simultaneously diagonalizable, and hence $[h \Gamma, h \Gamma'] = 0$.

We can refine Lemma 25 with essentially the same proof to find an $\mathfrak{sl}_2$-triple $s = \{e, h, f\}$ which is doubly $\Gamma$-graded and $\Gamma'$-graded, that is, $h \in g_0 \cap g'_0$ and $f \in g_{-2} \cap g'_{-2}$.

The lemma can now be viewed as a “double” analogue of Lemma 48 (1), and it can be proved by refining the arguments therein with the help of the doubly graded $\mathfrak{sl}_2$-triple above. □

Exercise 53. Fill in the details of the proof of Lemma 52.

Consider the hyperplanes

$$H_{\alpha,k} = \{ p \in E_e \mid \alpha(p) = k \}, \quad \alpha \in \Phi_e, k \in \mathbb{Z}.$$ 

The connected components of $E_e \setminus \bigcup_{\alpha,k} H_{\alpha,k}$ will be referred to as (open) alcoves.

Lemma 54. Let $p, p' \in \mathcal{P}_e$. Then, $\Gamma^p$ and $\Gamma'^p$ are adjacent if and only if $p$ and $p'$ belong to the closure of the same alcove.
Lemma 56. Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}_j$ and $\mathfrak{g}' = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}'_j$ be adjacent good gradings for $e$. Then there exist Lagrangian subspaces $\mathfrak{l}$ of $\mathfrak{g}_{-1}$ and $\mathfrak{l}'$ of $\mathfrak{g}'_{-1}$ such that
\[
\mathfrak{l} \bigoplus \left( \bigoplus_{i < -1} \mathfrak{g}_i \right) = \mathfrak{l}' \bigoplus \left( \bigoplus_{j < -1} \mathfrak{g}'_j \right).
\]

Proof. We have that
\[
\mathfrak{g}_{-1} = (\mathfrak{g}_{-1} \cap \mathfrak{g}_{<-1}) \bigoplus (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{<-1}) \bigoplus (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1}),
\]
\[
\mathfrak{g}'_{-1} = (\mathfrak{g}_{-1} \cap \mathfrak{g}_{>-1}) \bigoplus (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{>-1}) \bigoplus (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1}).
\]
Note the last summand $\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1}$ is orthogonal to the first two summands in either identity above with respect to $\langle \cdot, \cdot \rangle = (e[\cdot, \cdot])$. Hence $\langle \cdot, \cdot \rangle$ regarded as a (skew-symmetric) bilinear form on $\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1}$ remains to be non-degenerate. Take a Lagrangian subspace $\mathfrak{t}$ of $\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1}$, and set $\mathfrak{l} = \mathfrak{t} \oplus (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{<-1})$ and $\mathfrak{l}' = \mathfrak{t} \oplus (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1})$. The lemma follows by using the assumption of adjacency to check the inclusion of subspaces $\mathfrak{l} \oplus (\oplus_{i < -1} \mathfrak{g}_i) \subseteq \mathfrak{l}' \oplus (\oplus_{j < -1} \mathfrak{g}'_j)$ and the opposite inclusion by symmetry. □

Theorem 57. [BG] The finite $W$-algebras associated to any two good gradings $\Gamma$ and $\Gamma'$ for $e$ are isomorphic.

Proof. In this proof, we shall write a finite $W$-algebra as $\mathcal{W}_{l,\Gamma}$, indicating its dependence on a Lagrangian $l$ and a good $\mathbb{R}$-grading $\Gamma$.

Fix a chain $\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_n = \Gamma'$ of good gradings for $e$ such that $\Gamma_i$ is adjacent to $\Gamma_{i+1}$ for each $i$, as in Proposition 55.

By Lemma 56, we can suitably choose Lagrangians $l_i$ and $l'_i$ with respect to the gradings $\Gamma_i$ and $\Gamma_{i+1}$, respectively, which result in an equality of the corresponding Lie subalgebras $\mathfrak{m}_i = \mathfrak{m}_{i+1}$. So by the Whittaker model definition of $W$-algebras, we have $\mathcal{W}_{l_i,\Gamma_i} = \mathcal{W}_{l'_i,\Gamma_{i+1}}$. Accordingly to Gan-Gunzburg’s Theorem 24 (or rather its generalization in the setting of $\mathbb{R}$-gradings), the algebras $\mathcal{W}_{l,\Gamma}$ are isomorphic for a fixed grading $\Gamma$ and different choices of isotropic subspaces $l \subseteq \mathfrak{g}_{-1}$. Hence,
the theorem follows by composing a sequence of algebra isomorphisms:
\[ W_{1,1} = W_{l_1} \cong W_{1,2} = W_{l_2} \cong \cdots \cong W_{l_{n-1},1} = W_{l_n} \]
\[ \square \]

8. Higher level Schur duality

In this section, we let \( g = \mathfrak{gl}_N \). We present a duality between finite \( W \)-algebras and degenerate cyclotomic Hecke algebras, due to Brundan and Kleshchev [BK3].

8.1. Schur duality. Let \( V = \mathbb{C}^N \). Then the tensor space \( V \otimes d \) is naturally a \((U(g), S_d)\)-bimodule, which will be expressed as
\[ U(g) \curvearrowright V \otimes d \curvearrowleft S_d. \]

The celebrated Schur duality states that the images of \( U(g) \) and \( \mathbb{C}S_d \) form mutual centralizers in \( \text{End}(V \otimes d) \). Since the \( S_d \)-module \( V \otimes d \) is semisimple, one further obtains a multiplicity-free decomposition of \( V \otimes d \) as a \((U(g), S_d)\)-bimodule.

8.2. A duality of graded algebras. Let \( e \in g \) be nilpotent, and let \( \chi \in g^* \) be associated to \( e \) as before by \((\cdot | \cdot)\). Assume the Jordan canonical form of \( e \) is given by a partition \( \lambda \) of \( N \). We choose the (even) pyramid to be the Young diagram \( \lambda \) in the French fashion, labelled by \{1, \ldots, N\} down the columns from left to right.

Denote by \( \ell = \lambda_1 \). Then \( e^\ell = 0 \).

Denote by \( \overline{H}_d := \mathbb{C}[x_1, \ldots, x_d] \rtimes \mathbb{C}S_d \) the semi-direct product algebra formed by the natural action of \( S_d \) on the polynomial algebra \( \mathbb{C}[x_1, \ldots, x_d] \). We denote by \( \mathcal{C}_\ell[x_1, \ldots, x_d] \) the truncated polynomial algebra \( \mathbb{C}[x_1, \ldots, x_d]/\langle x_1^\ell, \ldots, x_n^\ell \rangle \), and then the algebra \( \overline{H}_d^\ell := \mathcal{C}_\ell[x_1, \ldots, x_d] \rtimes \mathbb{C}S_d \) is a quotient of \( \overline{H}_d \). Both \( \overline{H}_d \) and \( \overline{H}_d^\ell \) are \( \mathbb{N} \)-graded algebras with \( \text{deg} \sigma = 0, \forall \sigma \in S_n \), and \( \text{deg} x_i = 2, \forall i \).

We define an \( \overline{H}_d \)-module structure on \( V \otimes d \) by letting
\[ x_i = 1^\otimes (i-1) \otimes e \otimes 1^\otimes (d-i), \quad 1 \leq i \leq d. \]

The action of \( \overline{H}_d \) factors through \( \overline{H}_d^\ell \). Letting \( g_e \) act as the subalgebra of \( g \), we clearly have a \((U(g_e), \overline{H}_d^\ell)\)-bimodule structure on \( V \otimes d \):
\[ U(g_e) \overline{\Phi}_d \overline{V} \otimes d \overline{\Psi}_d \overline{\overline{H}}_d^\ell. \]

(17)

The Vust duality (cf. [BK3, Theorem 2.4]) states that the images of \( U(g_e) \) and \( \overline{H}_d^\ell \) form mutual centralizers in \( \text{End}(V \otimes d) \), i.e.
\[ \overline{\Phi}_d(U(g_e)) = \text{End}_{\overline{H}_d^\ell}(V \otimes d), \quad \text{End}_{U(g_e)}(V \otimes d)^{\text{op}} = \overline{\Psi}_d(\overline{H}_d^\ell). \]

In general, these images are not semisimple algebras.
8.3. Higher level Schur duality. Recall from Section 3.3 the definition of $W_\chi$ as $m$-invariants in $U(p)$, where $p$ is given in [10]. Let $\eta : U(p) \to U(p)$ be the algebra automorphism defined by

$$\eta(e_{i,j}) = e_{i,j} + \delta_{i,j}(\lambda_1 - \lambda'_{col(j)} - \lambda'_{col(j)+1} - \cdots - \lambda'_d), \quad e_{i,j} \in p,$$

where $\lambda'$ denotes the conjugate partition of $\lambda$. For simplicity of some later formulas, we will adopt the definition of $W_\chi$ by a twist of automorphism $\eta$ as follows:

$$W_\chi = \{ y \in U(p) \mid [a, \eta(y)] \in I_\chi, \forall a \in m \}.$$

Clearly, this new definition is isomorphic to the one in Section 3.3.

The higher level Schur duality is a filtered deformation of the Vust duality (17) of the following form:

$$W_\chi \overset{\Phi_d}{\underset{\Psi_d}{\rightleftharpoons}} V^\otimes d \overset{\Phi_d}{\underset{\Psi_d}{\rightleftharpoons}} H_d^{[\lambda]}$$

where $U(g_e)$ is replaced by the $W$-algebra $W_\chi$ and $H_d^\ell$ by a cyclotomic Hecke algebra $H_d^{[\lambda]}$. Recall that the degenerate affine Hecke algebra $H_d$ (introduced by Drinfeld [Dr]) is generated by its polynomial subalgebra $\mathbb{C}[x_1, \ldots, x_d]$ and a subalgebra $\mathbb{C}S_d$, and that it satisfies the additional relation for $1 \leq i, j \leq d$:

$$s_i x_j = x_j s_i \quad (j \neq i, i + 1), \quad s_i x_{i+1} = x_i s_i + 1.$$

The cyclotomic Hecke algebra $H_d^{[\lambda]}$ is defined to be the quotient of $H_d$ by the two-sided ideal generated by $\prod_{i=1}^d (x_1 - \lambda'_i + \lambda_1)$.

The remainder of this section is to explain the two (commuting) actions on $V^\otimes d$ of $W_\chi$ and $H_d$; the action of $H_d$ factors through $H_d^{[\lambda]}$.

8.4. A theorem of Arakawa and Suzuki. Given a $g$-module $M$, we consider the endomorphism of $g$-module $x : M \otimes V \to M \otimes V$, where $x$ acts by $\Omega = \sum_{1 \leq i, j \leq N} E_{ij} \otimes E_{ji}$. We also let $s = 1 \otimes \Omega : M \otimes V \otimes V \to M \otimes V \otimes V$, which can be easily seen to coincide with the permutation of the two copies of $V$. Abstractly, we may regard $x$ as an endomorphism of the functor $- \otimes V$ and $s$ as an endomorphism of the functor $(- \otimes V)^2$. This provides an example of $\mathfrak{sl}_2$-catégorification of Chuang and Rouquier [CR].

Now let $x_i = 1^{d-i} x_1^{i-1} (1 \leq i \leq d)$ and $s_j = 1^{d-j-1} s_1^{j-1} (1 \leq j \leq d - 1)$ be the endomorphisms of the $d$th power $(- \otimes V)^d$. Equivalently, by the natural isomorphism $(- \otimes V)^d \cong - \otimes V^\otimes d$, we have the corresponding endomorphisms

$$\hat{x}_i = \sum_{k=0}^{i-1} \Omega(k,i) = \Omega(0,i) \sum_{1 \leq k < i} s_{ki}, \quad \hat{s}_j = \Omega(j,j+1)$$

on $M \otimes V^\otimes d$ for a $g$-module $M$, where $s_{ki}$ denotes the permutation of the $k$th and $i$th copies of $V$. Here we have labeled the tensor factors from 0 to $d$, and the notation $\Omega^{(a,b)}$ means $\prod_{i,j} \Omega_{ij} \otimes \cdots \otimes \Omega_{ji}$ with the two nontrivial factors at $a$th and $b$th places.

According to a theorem of Arakawa and Suzuki [AS], these $\hat{x}_i$’s and $\hat{s}_j$’s satisfy the relations of the degenerate affine Hecke algebra $H_d$. Hence we have obtained a $(g, H_d)$-bimodule structure on $M \otimes V^\otimes d$ for any $g$-module $M$. 
Exercise 58. Prove that $\hat{x}_i$ for $1 \leq i \leq d$ and $\hat{s}_j$ for $1 \leq j \leq d - 1$ satisfy the defining relations of the degenerate affine Hecke algebra $H_d$.

8.5. **Higher level Schur duality.** Recall the Skryabin equivalence of categories from Section 5.2. Wh : $\mathfrak{g}^\text{-mod}^X \to \mathcal{W}_\chi^\text{-mod}$ and $Q_\chi \otimes_{\mathcal{W}_\chi} : \mathcal{W}_\chi^\text{-mod} \to \mathfrak{g}^\text{-mod}^X$.

Via such an equivalence, we can transfer functors on the category $\mathfrak{g}^\text{-mod}^X$ to functors on $\mathcal{W}_\chi^\text{-mod}$.

Given a finite-dimensional $\mathfrak{g}$-module $X$, the functor $- \otimes X : \mathfrak{g}^\text{-mod}^X \to \mathfrak{g}^\text{-mod}^X$ gives rise to, via the Skryabin equivalence, an exact functor

$$- \otimes X : \mathcal{W}_\chi^\text{-mod} \to \mathcal{W}_\chi^\text{-mod}, \quad M \mapsto \text{Wh}((Q_\chi \otimes_{\mathcal{W}_\chi} M) \otimes X).$$

These functors satisfy the naturality $(M \otimes X) \otimes Y \sim M \otimes (X \otimes Y)$ for another finite dimensional $\mathfrak{g}$-module $Y$. For our purpose, we will mainly consider the cases when $X$ and $Y$ are tensor powers of $V$.

By using the functors $- \otimes V^\otimes d$ on $\mathcal{W}_\chi^\text{-mod}$ in place of the functors $- \otimes V^\otimes d$ on $\mathfrak{g}^\text{-mod}^X$, we obtain via the Skryabin equivalence, for any $\mathcal{W}_\chi$-module $M$, a $(\mathcal{W}_\chi,H_d)$-bimodule structure on $M \otimes V^\otimes d$. We will be mostly interested in the case when $M = \mathbb{C}$, the restriction to $\mathcal{W}_\chi$ of the trivial $U(\mathfrak{p})$-module, but the arguments below are equally valid for a $U(\mathfrak{p})$-module $M$.

Suppose that $M$ is actually a $\mathfrak{p}$-module, and view $M$ and $M \otimes X$ as $\mathcal{W}_\chi$-modules by restricting from $U(\mathfrak{p})$ to $\mathcal{W}_\chi$. Consider the map

$$\mu_{M,X} : (Q_\chi \otimes_{\mathcal{W}_\chi} M) \otimes X \to M \otimes X,$$

which sends $(\eta(u)1_\chi \otimes m) \otimes x \mapsto um \otimes x$ for $u \in U(\mathfrak{p})$, $m \in M$ and $x \in X$. The restriction of this map to the subspace $\text{Wh}((Q_\chi \otimes_{\mathcal{W}_\chi} M) \otimes X)$ obviously defines a homomorphism of $\mathcal{W}_\chi$-modules

$$\mu_{M,X} : M \otimes X \to M \otimes X.$$

It turns out that this homomorphism is actually an isomorphism \[\text{BK3}\] (also cf. \[\text{Go}\]). Roughly speaking, the isomorphism can be obtained by using the following $\mathfrak{m}$-module isomorphism of Skryabin:

$$Q_\chi \otimes_{\mathcal{W}_\chi} M \cong M \otimes_{\mathbb{C}} U(\mathfrak{m})^*$$

where $U(\mathfrak{m})^* = \lim_{k} U(\mathfrak{m})/I^m_{\chi,k}$, and $I^m_{\chi,k}$ is the left ideal of $U(\mathfrak{m})$ generated by $(a - \chi(a))^k$, $\forall a \in \mathfrak{m}$.

In case when $M = \mathbb{C}$, we have obtained an isomorphism of $(\mathcal{W}_\chi,H_d)$-bimodules

$$\mathbb{C} \otimes V^\otimes d \cong V^\otimes d.$$ \hspace{1cm} (18)

Recall the standard basis $v_i$, $1 \leq i \leq N$ for $V = \mathbb{C}^N$. Then $V^\otimes d$ has a basis $v_i = v_i \otimes \ldots \otimes v_i$, where $i = (i_1, \ldots, i_d)$ with $1 \leq i_1, \ldots, i_d \leq N$. Denote by $L(i)$ the number in the box immediately to the left of the $i$th box in the pyramid $\lambda$ if it exists.
Lemma 59. The element $x_1 \in H_d$ sends each $v_i \in V^\otimes d$ to
$$v_{\Delta} + (\lambda_{\text{col}(i_1)} - \lambda_1)v_i - \sum_{1 < k \leq d \atop \text{col}(i_k) < \text{col}(i_1)} v_{i(1,k)},$$
where $v_{\Delta} = v_{L(i_1)} \otimes v_{i_2} \otimes \ldots \otimes v_{i_d}$ if $L(i_1)$ is defined; otherwise, set $v_{\Delta} = 0$.

Sketch of a proof. By examining carefully step by step how the isomorphism (18) is obtained, the action of $x_1$ is traced back to the corresponding action in Section 8.4.

□

Theorem 60. [BK3] The action of $H_d$ on $V^\otimes d$ factors through $H_d^{[\lambda]}$ and this gives rise to commuting actions:
$$W \times e_{\Delta} V^\otimes d \xrightarrow{\Phi_d} H_d^{[\lambda]}.$$

The maps $\Phi_d$ and $\Psi_d$ satisfy the double centralizer property, i.e.
$$\Phi_d(W \times e_{\Delta} V^\otimes d) = \text{End}_{H_d^{[\lambda]}}(V^\otimes d),$$
$$\text{End}_{W \times e_{\Delta} V^\otimes d} = \Psi_d(H_d^{[\lambda]}).$$

Sketch of a proof. The first statement follows from a direct computation of the minimal polynomial of $x_1$ by using Lemma 59.

The commutativity of the two actions follows from the theorem of Arakawa and Suzuki via the Skryabin equivalence.

The double centralizer property follows from the associated graded version (i.e. the Vust duality in Section 8.2) and a standard filtered algebra argument.

□

Remark 61. The higher level Schur duality has applications to the BGG category $\mathcal{O}$ for $g$; see [BK3] for more details.

9. $W$-(super)algebras in positive characteristic

In this section, we will present the $W$-algebras and $W$-superalgebras in positive characteristic, following [Pr1, WZ1, WZ2].

9.1. Restricted Lie superalgebras. Let $F$ be an algebraically closed field of characteristic $p > 2$. A Lie superalgebra $g = g_0 \oplus g_1$ over $F$ is a restricted Lie superalgebra if there is a $p$th power map $g_0 \to g_0, x \mapsto x^p$, such that the even subalgebra $g_0$ is a restricted Lie algebra and the odd part $g_1$ is a restricted module by the adjoint action of $g_0$. The notion of restricted Lie algebras was introduced by Jacobson, cf. Jantzen [Ja1] for an excellent review of modular representations of Lie algebras.

Example 62. The $p$-power map on the general linear Lie algebra $gl_n$ is given by $x \mapsto x^p$, the $p$th product for the underlying associative algebra.

The general linear Lie superalgebra $gl_{m|n}$, which consists of $(m+n) \times (m+n)$ matrices and whose even subalgebra is isomorphic to $gl_m \oplus gl_n$, is a restricted Lie superalgebra.
All the Lie (super)algebras considered here will be restricted. If one is only interested in Lie algebras, one can always set $g_1 = 0$ below.

We list all the basic classical Lie superalgebras over $F$ with restrictions on $p$ as follows (the general linear Lie superalgebra, though not simple, is also included).

| Lie superalgebra | Characteristic of $K$ |
|------------------|-----------------------|
| $gl_{m|n}$       | $p > 2$               |
| $sl_{m|n}$       | $p > 2, p \nmid (m - n)$ |
| $B(m, n), C(n), D(m, n)$ | $p > 2$ |
| $D(2, 1; \alpha)$ | $p > 3$ |
| $F(4)$           | $p > 15$              |
| $G(3)$           | $p > 15$              |

The queer Lie superalgebra $q_n$ is a Lie subalgebra of the general linear Lie superalgebra $gl_{n|n}$, which consists of matrices of the form:

$$
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix},
$$

where $A$ and $B$ are arbitrary $n \times n$ matrices. Note that the even subalgebra $(q_n)_{\overline{0}}$ is isomorphic to $gl_n$, and the odd part $(q_n)_{\overline{1}}$ is another isomorphic copy of $gl_n$ under the adjoint action of $(q_n)_{\overline{0}}$.

9.2. Reduced enveloping superalgebras. Let $g$ be a restricted Lie superalgebra. For each $x \in g_{\overline{0}}$, the element $x^p - x^{[p]} \in U(g)$ is central by definition of the $p$th power map. We refer to $Z_p(g) = K\langle x^p - x^{[p]} \mid x \in g_{\overline{0}} \rangle$ as the $p$-center of $U(g)$. Let $x_1, \ldots, x_s$ (resp. $y_1, \ldots, y_t$) be a basis of $g_{\overline{0}}$ (resp. $g_{\overline{1}}$). As a consequence of the PBW theorem for $U(g)$, $Z_p(g)$ is a polynomial algebra isomorphic to $K[x_i^p - x_i^{[p]} \mid i = 1, \ldots, s]$, and the enveloping superalgebra $U(g)$ is free over $Z_p(g)$ with basis

$$
\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t} \mid 0 \leq a_i < p; b_j = 0, 1 \text{ for all } i, j\}.
$$

**Exercise 63.** Prove that the $p$-center $Z_p(g)$ is a polynomial algebra isomorphic to $K[x_i^p - x_i^{[p]} \mid i = 1, \ldots, s]$, and that $U(g)$ is free over $Z_p(g)$ with basis

$$
\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t} \mid 0 \leq a_i < p; b_j = 0, 1 \text{ for all } i, j\}.
$$

Let $V$ be a simple $U(g)$-module. By Schur’s lemma, the central element $x^p - x^{[p]}$ for $x \in g_{\overline{0}}$ acts by a scalar $\zeta(x)$, which can be written as $\chi_V(x)^p$ for some $\chi_V \in g^*_0$. We call $\chi_V$ the $p$-character of the module $V$. We often regard $\chi \in g^*$ by letting $\chi(g_1) = 0$.

Fix $\chi \in g^*_0$. Let $J_\chi$ be the ideal of $U(g)$ generated by the even central elements $x^p - x^{[p]} - \chi(x)^p$. The quotient algebra $U_\chi(g) := U(g)/J_\chi$ is called the reduced enveloping superalgebra with $p$-character $\chi$. It follows from Exercise 63 that the superalgebra $U_\chi(g)$ has a linear basis

$$
\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t} \mid 0 \leq a_i < p; b_j = 0, 1 \text{ for all } i, j\}.$$
In particular, \( \dim U_\chi(g) = p^{\dim g_0} q^{\dim g_1} \).

A superalgebra analogue of Schur’s Lemma states that the endomorphism ring of an irreducible module of an associative superalgebra is either one-dimensional or two-dimensional (in the latter case it is isomorphic to a Clifford algebra), cf. e.g. Kleshchev [Kle, Chap. 12]. An irreducible module is of type \( M \) if its endomorphism ring is one-dimensional and it is of type \( Q \) otherwise.

9.3. **The good \( Z \)-gradings.** Let \( g \) be one of the basic classical Lie superalgebras. The Lie superalgebra \( g \) admits a nondegenerate invariant even bilinear form \( \langle \cdot \mid \cdot \rangle \), whose restriction on \( g_0 \) gives an isomorphism \( g_0 \to g_0^* \). Let \( \chi \in g_0^* \) be a nilpotent element, that is, it is the image of some nilpotent element \( e \in g_0 \) under the above isomorphism. Then \( g_\chi := \{ x \in g \mid \chi([x, y]) = 0, \forall y \in g \} \) is equal to the usual centralizer \( g_e \). We shall denote \( \dim g = \dim g_0 + \dim g_1 \).

**Proposition 64.** Let \( g \) be a basic classical Lie superalgebra or a queer Lie superalgebra. Then there exists a good \( Z \)-grading \( g = \bigoplus_{j \in \mathbb{Z}} g_j \) as defined in 2.2. This \( Z \)-grading is compatible with the \( \mathbb{Z}_2 \)-grading, that is, \( g_j = g_{j,0} \oplus g_{j,1} \) where \( g_{j,a} = g_j \cap g_a \) for \( a \in \mathbb{Z}_2, j \in \mathbb{Z} \). In particular, we have a refinement of (18):

\[
\dim g_e = \dim g_0 + \dim g_1.
\]

**Sketch of a proof.** For Lie superalgebras of type \( A, B, C, D \) and type \( Q \), we use the classification of nilpotent orbits for classical Lie algebras which leads to an elementary way of defining \( Z \)-gradings, cf. Jantzen [Ja2, Chapter 1]. For the exceptional Lie superalgebras, the assumptions on \( p \) in the Table of 2.1 ensure the existence of an \( sl_2 \)-triple \( \{e, h, f\} \) and the semisimplicity of the adjoint action on \( g \) by the \( sl_2 \)-triple (see Carter [Ca]). The action of \( Fh \) on \( g \) lifts to a torus action which provides the desired grading on \( g \). \( \square \)

9.4. **The subalgebra \( m \).** Let \( g \) be one of the basic classical Lie superalgebras or \( q_n \) with a good \( Z \)-grading \( g = \bigoplus_{j \in \mathbb{Z}} g_j \). Define a bilinear form \( \langle \cdot, \cdot \rangle \) on \( g_{-1} \) by

\[
\langle x, y \rangle := \chi([x, y]), \quad x, y \in g_{-1}.
\]

The following is a super analogue of Lemma 15.

**Exercise 65.** Show that (20) defines an even, non-degenerate, skew-supersymmetric form \( \langle \cdot, \cdot \rangle \) on \( g_{-1} \), that is, a non-degenerate bilinear form such that \( \langle g_{-1,0}, g_{-1,1} \rangle = 0 \), and that the restriction of the form to \( g_{-1,0} \) (resp. \( g_{-1,1} \)) is skew-symmetric (resp. symmetric).

It follows that \( \dim g_{-1,0} \) is even. Take \( I_0 \subseteq g_{-1,0} \) to be a maximal isotropic subspace with respect to \( \langle \cdot, \cdot \rangle \). It satisfies \( \dim I_0 = \frac{1}{2} \dim g_{-1,0} \).

Denote \( r = \dim g_{-1,1} \). There is a basis \( v_1, \ldots, v_r \) of \( g_{-1,1} \) under which the symmetric form \( \langle \cdot, \cdot \rangle \) has matrix form \( E_{1r} + E_{2,r-1} + \ldots + E_{r1} \). If \( r \) is even, take \( I_1 \subseteq g_{-1,1} \).
to be the subspace spanned by \( v_1, \ldots, v_r \). If \( r \) is odd, take \( I_1 \subseteq g_{-1,1} \) to be the subspace spanned by \( v_1, \ldots, v_{r-1} \). Set \( I = I_0 \oplus I_1 \) and introduce the subalgebras

\[
\begin{align*}
\mathfrak{m} &= \bigoplus_{j \geq 2} g_{-j}, \\
\mathfrak{m}' &= \begin{cases} 
\mathfrak{m} \oplus F v_{r+1}, & \text{for } r \text{ odd} \\
\mathfrak{m}, & \text{for } r \text{ even}
\end{cases}
\end{align*}
\]

The subalgebra \( \mathfrak{m} \) is \( p \)-nilpotent, and the linear function \( \chi \) vanishes on the \( p \)-closure of \([\mathfrak{m}, \mathfrak{m}]\). It follows that \( U_\chi(\mathfrak{m}) \) has the trivial module, denoted by \( F \chi \), as its only irreducible module and \( U_\chi(\mathfrak{m})/N_m \cong F \chi \) where \( N_m \) is the Jacobson radical of \( U_\chi(\mathfrak{m}) \).

**Proposition 66.** Let \( g \) be a basic classical Lie superalgebra or \( q_n \). Then every \( U_\chi(g) \)-module is \( U_\chi(\mathfrak{m}) \)-free.

Proposition 66 in the case when \( g \) is the Lie algebra of a reductive algebraic group was first established \([Pr1]\) using the machinery of support varieties (cf. \([FP, Ja1]\)) in an essential way. The proof of Proposition 66 in \([WZ1]\) follows an idea of Skryabin \([Sk2]\), completely bypassing the machinery of support varieties (which has yet to be developed for modular Lie superalgebras).

### 9.5. The super Kac-Weisfeiler conjecture

The following theorem in the present form is due to \([WZ1, WZ2]\), which generalize the original Kac-Weisfeiler conjecture \([WK]\) (theorem of Premet \([Pr1]\)) when \( g \) is a Lie algebra of a reductive groups. Moreover, based on a deformation approach of Premet and Skryabin, a new proof \([Zh]\) has optimally improved the assumption on \( p \) (that is, one may assume \( p > 3 \) for the last two exceptional superalgebras in the Table of \([JL]\)).

**Theorem 67.** Let \( g \) be a basic classical Lie superalgebra or \( q_n \), and let \( \chi \in g^*_0 \) be nilpotent. Let \( d_i = \dim g_i - \dim (g_\chi)_i, \) \( i \in \mathbb{Z}_2 \). Then the dimension of every \( U_\chi(g) \)-module \( M \) is divisible by \( p^{\frac{d_0}{2}} \lfloor \frac{d_1}{2} \rfloor \), where \( \lfloor \frac{d_1}{2} \rfloor \) denotes the least integer which is \( \geq \frac{d_1}{2} \).

**Proof.** By \((19)\) and since \( \dim g_j = \dim g_{-j} \), we have

\[
\dim g - \dim g_\chi = 2 \sum_{j \geq 2} \dim g_{-j} + \dim g_{-1}.
\]

In particular, \( r := \dim g_{-1,1} \) and \( d_1 \) have the same parity. It follows now from the definition of \( \mathfrak{m} \) that either \( (1) \frac{d_0}{2} | \frac{d_1}{2} = \dim \mathfrak{m} \) when \( d_1 \) is even, or \( (2) \frac{d_0}{2} | \frac{d_1-1}{2} = \dim \mathfrak{m} \) when \( d_1 \) is odd.

In case (1), the theorem follows immediately from Proposition 66. Note that \( d_1 = d_0 \) and hence the case (2) never occurs for \( g = q_n \).

In case (2), the induced \( U_\chi(m') \)-module \( V = U_\chi(m') \otimes_{U_\chi(m)} F \) is two-dimensional, irreducible, and admits an odd automorphism of order 2 induced from \( v_{r+1} \). By Frobenius reciprocity, it is the only irreducible \( U_\chi(m') \)-module. Thus, \( U_\chi(m')/N_{m'} \)
is isomorphic to the simple associative superalgebra \( q_1^{\text{ss}} \) (which coincides with \( q_1 \) as a vector superspace), and the unique simple \( U_\chi(m') \)-module \( V \) is of type \( Q \).

Hence, for each \( U_\chi(g) \)-module \( M \), the subspace \( M^m \) of \( m \)-invariants in \( M \), which coincides with \( M^{m'} \), is a module over the superalgebra \( U_\chi(m')/N_m \cong q_1^{\text{ss}} \). Since the (unique) simple module of \( q_1^{\text{ss}} \) is two-dimensional, \( \dim M^m \) is divisible by 2. Now the isomorphism \( M \cong U_\chi(m')^* \otimes M^m \) implies the desired divisibility. \( \square \)

9.6. **The modular \( W \)-superalgebras.** Denote by \( Q_\chi \) the induced \( U_\chi(g) \)-module \( U_\chi(g) \otimes_{U_\chi(m)} F_\chi \).

**Definition 68.** The (modular) \( W \)-superalgebra associated to the pair \( (g, \chi) \) is the following associative \( F \)-superalgebra

\[
W_\chi = \text{End}_{U_\chi(g)}(Q_\chi)^{\text{op}}.
\]

Theorem 67 can be somewhat strengthened in the following form \( \text{[WZ1] [WZ2]} \), which is a superalgebra generalization of a theorem of Premet \( \text{[Pr2, Theorem 2.3]} \).

**Theorem 69.** Set \( \delta = \dim U_\chi(m) \). Then \( Q_m \) is a projective \( U_\chi(g) \)-module and

\[
U_\chi(g) \cong M_\delta(W_\chi).
\]

**Proof.** Let \( V_1, \ldots, V_s \) (resp. \( W_1, \ldots, W_t \)) be all inequivalent simple \( U_\chi(g) \)-modules of type \( M \) (resp. of type \( Q \)). Let \( P_i \) (resp. \( Q_j \)) denote the projective cover of \( V_i \) (resp. \( W_j \)). By Proposition 66 \( V_i \) and \( W_j \) are free over \( U_\chi(m) \). It follows by Frobenius reciprocity that

\[
\dim \text{Hom}_g(Q_\chi, V_i) = \dim \text{Hom}_m(F_\chi, V_i) =: a_i.
\]

By Frobenius reciprocity,

\[
\dim \text{Hom}_g(Q_\chi, W_j) = \dim \text{Hom}_m(F_\chi, W_j) = \dim \text{Hom}_m(U_\chi(m), W_j)
\]

which has to be an even number, say \( 2b_j \), since as a type \( Q \) module \( W_j \) admits an odd involution commuting with \( m \). It follows that the ranks of the free \( U_\chi(m) \)-modules \( V_i \) and \( W_j \) are \( a_i \) and \( 2b_j \) respectively. Put

\[
P = (\bigoplus_{i=1}^s P_i^{a_i}) \bigoplus (\bigoplus_{j=1}^t Q_j^{b_j}).
\]

Then \( P \) is projective and has the same head as \( Q_m \). So there is a surjective homomorphism \( \psi : P \to Q_\chi \).

Since \( \dim V_i = \delta a_i \) and \( \dim W_j = 2\delta b_j \), by Wedderburn theorem for superalgebras (cf. Kleshchev \( \text{[Kle, Theorem 12.2.9]} \)) the left regular \( U_\chi(g) \)-module is isomorphic to \( P^\delta \). The equality of dimensions

\[
\dim P = \dim U_\chi(g)/\delta = \dim Q_m
\]

implies that \( \psi \) is an isomorphism. Finally,

\[
U_\chi(g) \cong \text{End}_{U_\chi(g)}(U_\chi(g))^{\text{op}} \cong \text{End}_{U_\chi(g)}(P^\delta)^{\text{op}}
\]

\[
\cong (M_\delta(\text{End}_{U_\chi(g)}(P)))^{\text{op}} \cong M_\delta(W_\chi).
\]

This completes the proof of the theorem. \( \square \)
Remark 70. In light of Gan-Ginzburg’s definition of $W$-algebras over $\mathbb{C}$, it makes better sense to define a (modular) $W$-superalgebra over $F$ as

$$W'_\chi := \frac{U_\chi(g)}{J_\chi \langle [a, y] \in J_\chi, \forall a \in m' \rangle},$$

where $J_\chi$ denotes the left ideal of $U_\chi(g)$ generated by $N_m$, with multiplication $y_1 y_2 = \overline{y}_1 \overline{y}_2$, for $y_1, y_2 \in W'_\chi$. In the case when $d_1$ (or equivalently, $\dim g_{-1,1}$) is odd, $W'_\chi$ is in general a subalgebra of $W_\chi \cong \frac{U_\chi(g)}{J_\chi \langle a \rangle}$.

Similar subtle differences appear in various constructions of $W$-superalgebras over $\mathbb{C}$.

10. Further work and open problems

10.1. Representations of $W_\chi$ of type $A$. The representation theory of finite $W$-algebras has been most adequately developed for $g = gl_N$. Brundan and Kleshchev [BK1] showed that the finite $W$-algebras were quotient algebras of what they introduced and called shifted Yangians, associated to a given pyramid $\pi$. In the special case when $\pi$ is of rectangular shape, such a connection between finite $W$-algebras and the truncated Yangians (which are quotients of the Yangians introduced by Drinfeld) was found by Ragoucy and Sorba [RS], and the truncated Yangians were studied by Cherednik [Ch]. Also see Brown [Br] for a generalization of [RS] to other classical Lie algebras.

Such a connection with shifted Yangians was subsequently explored systematically in [BK2], where the finite-dimensional $W_\chi$-modules are classified. Moreover, a highest weight representation theory of $W_\chi$ is developed, and a Kazhdan-Lusztig solution to the finite-dimensional irreducible character problem of $W_\chi$ is obtained. A formulation of Kazhdan-Lusztig conjectures for general finite $W$-algebras was given by de Vos and van Driel [VD].

Remark 71. The affine Yangian associated to the affine algebra of $g$ have been studied by Guay [Gu]. We remark that the affine Yangian for $gl_1$ should be identified with the $W_{1+\infty}$ algebra. The main result of [FKRW] can be reformulated as follows: The vertex algebra for the truncated affine Yangian of $gl_1$ at level $N$ is isomorphic to the affine $W$-algebra associated to $gl_N$ and a regular nilpotent element at level $N$.

10.2. Finite-dimensional $W_\chi$-modules and primitive ideals. The importance of finite-dimensional $W_\chi$-modules was realized by Premet [Pr3] in connection to primitive ideals for $U(g)$, where he showed that the corresponding $g$-modules under the Skryabin equivalence have the closure $\overline{O}_e$ of a nilpotent orbit as their associated varieties and $\frac{1}{2} \dim \overline{O}_e$ as their Gelfand-Kirillov dimensions. Further different approaches toward connections between finite-dimensional $W_\chi$-modules and primitive ideals are developed in [Gi, Lo1, Lo2, Pr4, Pr5] (see [Pr3, Conjecture 3.2] for motivation).

Among all finite-dimensional $W_\chi$-modules, the one-dimensional modules stand out (cf. [Pr3, Conjecture 3.1]), and they are studied intensively in [Lo1, Lo4, Pr5, GRU].
The upshot of all these developments is that finite-dimensional modules are shown to exist for every $W_\chi$, and one-dimensional modules are shown to exist (modulo some open cases in $E_8$). The existence of one-dimensional $W$-modules implies that the dimensional bound provided by Kac-Weisfeiler conjecture is optimal for characteristic $p \gg 0$ [Pr5].

10.3. Category $O$. The category $O$ for finite $W$-algebras was well studied in [BK2] in type $A$. One needs to overcome substantial technical difficulties to formulate the category $O$ for finite $W$-algebras in other types, see [BGK, Lo3, Go, Pr3, We] for research in this direction.

10.4. Open problems. As in any reasonable representation theory, one of the basic problems in $W$-algebras is the following.

Problem 72. Find a (Kazhdan-Lusztig type) solution to the irreducible character problem for category $O$ of $W_\chi$-modules.

As we have seen that finite $W$-superalgebras are natural subjects of study in connections to the natural yet nontrivial superalgebra generalization of the Kac-Weisfeiler conjecture [WZ1, WZ2]. We remark that the queer Lie superalgebra $q_n$ is often referred to as the truly superanalogue of $gl_n$.

Problem 73. Develop systematically the structures and representation theory of finite $W$-superalgebras over $\mathbb{C}$, for both basic classical Lie superalgebras and the queer Lie superalgebras.

Various aspects of finite $W$-superalgebras over $\mathbb{C}$ are currently being developed in the Virginia dissertation of Yun-Ning Peng.

Problem 74. Formulate the notion of quantum finite $W$-algebras associated to quantum groups $U_q(g)$ and then develop its representation theory (for both $q$ being generic and $q$ being a root of unity).

In the case of regular nilpotent elements, there has been a very interesting work of Sevostyanov [Sev].

It is shown [PPY] that the algebra of invariants of the centralizer $g_e$ in the symmetric algebra of $g_e$ is often a polynomial algebra in rank($g$) generators, even though $g_e$ is in general non-reductive; also see [BrB] for a constructive approach in type $A$. One should compare with the fact that the center of $W_\chi$ is always isomorphic to $Z(g)$, which is independent of $\chi$ (see [BK3] for type $A$ and [F3] Footnote 2 in general).

We pose the problem of finding a skew analogue of the above results.

Problem 75. When is the algebra of $g_e$-invariants in the exterior algebra of $g_e$ an exterior algebra in rank($g$) generators?

Note that the answer is affirmative when $e = 0$ or $e$ is regular nilpotent by classical results. A counterpart of this problem for $W_\chi$ would naturally lead to the following.
Problem 76. Develop a cohomology theory of finite $W$-algebras.

The connection between shifted Yangians and finite $W$-algebras of type $A$ is provided by explicit formulas [BK1]. Being explicit, these formulas greatly facilitate the detailed study in type $A$, but on the other hand, they seem to be too rigid to allow direct generalizations to other types or to the affine setting. This leads to the following.

Problem 77. Redevelop the connections between Yangians and finite $W$-algebras in type $A$ from the BRST approach. Formulate a generalization of Remark 71, which should be viewed as an affine analogue of the above connections in type $A$.

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