Dispersive evaluation of the second-class amplitude $\tau \to \eta\pi\nu_\tau$
in the standard model

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Abstract

We reevaluate the two form factors relevant for the $\eta\pi$ second-class $\tau$ decay mode, making systematic use of analyticity, unitarity, combined with updated inputs to the NLO chiral constraints. We focus in particular on the shape of the $\rho$ resonance peak which is a background-free signature of a second-class current. Its dispersive construction requires the $\eta\pi \to \pi\pi$ scattering amplitude which we derive from a family of Khuri-Treiman equations solutions constrained with accurate recent results on the $\eta \to 3\pi$ Dalitz plot.

Keywords:

1. Introduction

Weinberg remarked that exact isospin conservation by the strong interactions would imply selection rules for semi-leptonic weak decays [1]. Isospin conservation, of course, is only approximate, being broken both in QCD because $m_u \neq m_d$ and in QED because $q_u \neq q_d$. Yet, no quantitative experimental evidence for these so-called second-class currents have been reported to date. Processes of this type in $\tau$ decays are the $\eta\pi$ or $\eta'\pi$ modes: because of parity conservation these decays must proceed through the $I = 1$ vector current which is even under a $G$-parity rotation, while $\eta\pi$, $\eta'\pi$ are odd eigenstates of $G$-parity. The sensitivity of these modes to physics beyond the standard model (SM) has been discussed in refs [2,3]. Within the standard model, measuring these $\tau$ decay amplitudes would provide non trivial informations on matrix elements of the scalar operator $\partial_\mu \bar{u} \gamma^\mu d$ and a related determination of the $q\bar{q}$ content of the scalar $a_0(980)$, $a_0(1450)$ resonances. The Babar collaboration has published upper bounds for the $\eta\pi$, $\eta'\pi$ modes [4,5].

We reconsider here the theoretical expectation for the $\tau \to \eta\pi\nu_\tau$ mode in the SM. This problem was first addressed in ref. [6]. We attempt to refine the theoretical prediction by exploiting the analyticity properties of the two form factors involved and combining them with chiral symmetry constraints. This proves particularly fruitful for the vector form factor, for which unitarity provides a simple relation with the isospin violating $\eta \to 3\pi$ decay amplitude. We will show how considerable recent progress in measuring this amplitude impacts the determination of the $\eta\pi$ vector form factor. It is usually expected that the integrated branching fraction should be dominated by the scalar rather than the vector form factor. However, the main experimental obstacle to the observation of second-class amplitudes at B-factories is the pollution from first-class background contributions (e.g. from $\tau^\pm \to \eta\pi^0\nu$, $\eta\pi^\pm K^0\nu$ where the extra neutral particle escapes detection). While a peak in the $\eta\pi$ invariant mass at the $a_0(980)$ mass may be present in background modes, a peak at the $\rho(770)$ mass unmistakingly signals a second-class contribution.

2. Dispersion relations and chiral symmetry

The $\eta\pi$ form factors satisfy simple analyticity properties, in exactly the same way as the more familiar $\pi\pi$ or
The two form factors were computed at NLO in the chiral expansion, the two form factors are defined

\[ F^{\pi}(s) = F^{\pi}(0) + s F^{\pi}(0) \pm \frac{s^2}{\pi} \int_{4m^2_\pi}^{\infty} ds' \frac{\text{disc}[f^{\pi}(s')]}{(s')^2(s' - s)} \]  

(1)

Thanks to the cutoff function, the integrand is dominated by the energy region below 1 GeV where we can evaluate the discontinuity, using unitarity, with only a few channels contributing (essentially, only a single channel). The price to pay is that we have to provide the values of the form factor and its derivative at \( s = 0 \). For this purpose, we can rely on three flavour chiral symmetry, since the \( \pi \) and the \( \eta \) are both pseudo-Nambu-Goldstone bosons in this framework.

The vector and scalar \( \pi \eta \) form factors are defined starting from the matrix element of the vector current

\[ \langle \pi^\pm | i \gamma^\mu d(0) | 0 \rangle = -\sqrt{2} \left[ f^{\pi}(s)(p_\eta - p_\pi) + f^{\pi}(s)(p_\eta + p_\pi) \right] \]

and

\[ f_0^{\pi}(s) = f_0^{\pi}(s) \pm \frac{s}{\Delta_{\pi\eta}} f_0^{\pi}(s), \quad \Delta_{\pi\eta} = m_\eta^2 - m_\pi^2 \]  

(2)

At LO in the chiral expansion, the two form factors are constant and equal,

\[ f_0^{\pi}(s) = f_0^{\pi}(s)|_{LO} = \epsilon = \frac{\sqrt{3}(m_d - m_u)}{4(m_s - m_u)} = 0.99 \times 10^{-2}. \]

There is no electromagnetic contribution at this order and the numerical estimate uses the chiral expansion of the mass differences \( m_{\pi^0}^2 - m_{\eta^0}^2, m_{\eta^0}^2 - m_{\eta^+}^2 \), also at LO.

The two form factors were computed at NLO in the chiral expansion by Neufeld and Rupertberger [7], including also the EM contributions at order \( e^2 \). A remarkably simple expression emerges from their results, relating the value of the \( \eta \pi \) form factors at \( s = 0 \) to the ratio of the \( K^* \pi^0 \) and \( K^0\pi^+ \) form factors

\[ f^{\pi}_{\pi^0}(0) = \frac{1}{\sqrt{3}} \left[ \frac{f^{K^*\pi^0}(0)}{f^{K^0\pi^+}(0)} - 1 - \frac{3e^2}{4(4\pi)^2} \log \frac{m_K^2}{m_{\eta^0}^2} \right] \]  

(3)

Exploiting the recent results on \( K^* \) and \( K^0 \) decays from \( K \) factories (see e.g. [8]) yields the most precise evaluation of the \( \eta \pi \) form factors at \( s = 0 \),

\[ f_0^{\pi^+}(0) = f_0^{\pi^0}(0) = (1.49 \pm 0.23) \times 10^{-2} \]  

(4)

which is significantly enhanced from its LO estimate.

3. Vector form factor and \( \eta \to 3\pi \)

The discontinuity of \( f^{\eta\pi^+} \) can be associated with a sum over intermediate states of the matrix element of the vector current

\[ \text{Im}(\eta\pi^+ | i \gamma^\lambda d | 0) = \frac{1}{2} \sum_n T_{\eta\pi^+ \eta \eta}(n | i \gamma^\lambda d | 0). \]  

(5)

The derivation is valid in the unphysical situation where the \( \eta \) meson is stable and we will assume than an analytic continuation as a function of \( m_\eta \) is possible. Below 1 GeV, the sum in (5) is essentially saturated by the contribution of the lightest state \( n = \pi^0 \pi^\pm \) (as it is strongly enhanced by its coupling to the \( \rho \) resonance). This leads to the following estimate of the discontinuity (for \( 4m_\pi^2 \leq s \leq 1 \ GeV^2 \))

\[ \text{disc}[f^{\eta\pi^+}(s)] = -\theta(s - 4m_\pi^2) \times \frac{s - 4m_\pi^2}{32\pi \sqrt{\lambda_{\eta\pi}(s)}} F_\lambda^{\pi\pi}(s) \int_{s_1}^{1} dz z T_{\pi\pi \eta \eta}(s, t(z)) \]

(6)

with \( \lambda_{\eta\pi}(s) = (s - m_\pi^2)(s - m_\pi^2), m_\pi = m_\eta \pm m_\pi \). In this equation, \( F_\lambda^{\pi\pi} \) is the pion vector form factor, which is precisely known experimentally. In addition, one needs to evaluate the \( \pi\pi \to \eta\pi \) amplitude projected on the \( P \)-wave, partly in an unphysical region (\( s < m_\pi^2 \)). It can be determined using its analyticity properties together with experimental constraints on \( \eta \to 3\pi \) decay.

Combining analyticity with elastic unitarity for \( \pi\pi \) (re)scattering leads to a system of Khuri-Treiman (KT) equations [9]. The solutions of these equations in their full generality were first discussed in refs. [10][11]. A subtle point, in particular, concerns the treatment of the singularities of the partial wave projected \( \pi\pi \to \eta\pi \) amplitude (thus, \( T_{\pi\pi \eta \eta} \to 1/(s - m_\pi^2)^{1/2} \) when \( s \to m_\pi^2 \)) in the integrals. Eq. (6) shows that these singularities affect also the computation of the vector form factor and must be treated by the same method. In practice, this leads to a distortion of the shape of the \( \rho \) resonance as compared to a naive vector meson dominance (VMD) approach.

The authors of ref. [11] argue that a four-parameter family of solutions are relevant for the \( \eta \to 3\pi \) decay, which we can also use for our problem. Assuming a sufficiently fast convergence of the three-flavour chiral expansion, they propose to determine all these four parameters by matching the dispersive and NLO chiral amplitude in a region around the Adler zero. Unfortunately, the amplitude obtained in this manner turns out not to be in agreement with the experimental results on the Dalitz plot parameters (see table [1] below). One must thus determine the KT solution parameters partly from
matching to the NLO amplitude and partly from fitting the experimental Dalitz plot data [12]. We perform here a fit analogous to ref. [12] but constraining the four KT parameters to be exactly real. From the NLO amplitude, we use the position of the Adler zero but not the value of the amplitude slope at this point. Thus, we do not attempt to determine the value of the quark mass ratio from the $\eta$ decay rate as in [12] (see also [13, 14]) but take this ratio from the PDG (which leads to a value of the slope at the Adler zero differing from the NLO prediction by approximately 20%).

$$\left| \frac{f_{\eta\pi\pi} + (s)}{\sqrt{s}} \right|$$

Table 1: Comparison of the $\eta \to 3\pi$ Dalitz plot parameters obtained from KT solutions with experiment. The Dalitz parameters $a, b, d, f$ refer to the charged decay mode and are taken from [15], while $\alpha$ refers to the neutral mode and the quoted value is taken from the PDG.

| param | experimental | NLO Match. | Fit |
|-------|-------------|------------|-----|
| a     | $-1.090 \pm 0.005^{+0.008}_{-0.019}$ | $-1.300$ | $-1.065$ |
| b     | $0.124 \pm 0.006 \pm 0.010$ | $0.463$ | $0.159$ |
| d     | $0.057 \pm 0.006^{+0.007}_{-0.016}$ | $0.069$ | $0.066$ |
| f     | $0.14 \pm 0.01 \pm 0.02$ | $0.001$ | $0.107$ |
| $\alpha$ | $-0.0315 \pm 0.0015$ | $0.015$ | $-0.0355$ |

The results of using a KT solution in the discontinuity relation (6) and then computing the form factor from the DR with weight functions $1/s$ and $1/s^2$ and using NLO chiral constraints like (3) is illustrated in fig. 1. The stability with respect to the weight functions is satisfactory below 1 GeV. The results are also compared with the naive VMD where $f_{\eta\pi\pi}^d$ would be simply proportional to the pion form factor normalized to the value $\sqrt{s}$ at the origin. The dispersive calculation is seen to yield a significantly reduced resonance peak as compared to a naive VMD modelling. The influence of the KT parameters is also rather significant and provide an idea of the uncertainties of this calculation.

4. Scalar form factor model

The scalar form factor (see [2]) coincides with the $\eta\pi$ matrix element of the derivative operator $i\partial^\mu\partial^\nu d$ and a discontinuity relation analogous to eq. (5) can be written. Since $i\partial^\mu\partial^\nu d$ is itself isospin suppressed as can be seen from the Ward identity $i\partial^\mu\partial^\nu d = (m_d - m_u)\bar{u}d - eA_\mu\bar{u}\gamma^\mu d$, the sum over states [4] can be resticted to

$$\text{Im} f_0^{\eta\pi}(s) = \theta(s - m^2_{\eta\pi}) \times$$

$$\frac{\sqrt{s}}{16\pi s} f_0^{\eta\pi}(s) \times \frac{1}{2} \int_{-1}^{1} dz T_{\eta\pi\to\eta\pi}(s, t(z))$$

For such a unitarity relation Watson’s theorem implies that the phase of the form factor coincides with the $\eta\pi$ scattering phase shift in the region of elastic scattering. It is then natural to employ a phase dispersive representation for the form factor, e.g.

$$f_0^{\eta\pi}(s) = f_0^{\eta\pi}(0) \left( \frac{f_0^{\eta\pi}(\Delta_{\eta\pi})}{f_0^{\eta\pi}(0)} \right)^{\frac{s}{\pi}} \times$$

$$\exp \left( \frac{\phi^{\eta\pi}(s')}{\pi} \int_{m_{\eta\pi}}^{\infty} ds' \frac{\phi^{\eta\pi}(s')}{s' - \Delta_{\eta\pi}(s' - s)} \right)$$

(which uses the weight function $1/s(s - \Delta_{\eta\pi})$ following [16]). The values of the form factor at $s = 0$ and the Dashen-Weinstein point $s = \Delta_{\eta\pi}$ must be provided from NLO ChPT. A difficulty at this point is that the $\eta\pi$ scattering phase shift is not measurable by the same methods as used for $\pi\pi$ or $KK$. The experimental information concerns the properties of the resonances which couple to $\eta\pi$ and the phase shift is constrained near the threshold by chiral symmetry. We will use a simple model proposed in ref. [17] which interpolates between these pieces of information. This model makes the plausible prediction that the global features of $\eta\pi$ scattering are fairly similar to those of $\pi\pi$ and $KK$ scattering. The phase shift is a steadily raising function and inelasticity sets in rather sharply at a two-particle threshold ($KK$) close to a resonance ($a_0(980)$). We can make use of this analogy to make a guess for the behaviour of the
form factor phase in the inelastic energy region. In the case of $\pi\pi$ or $\pi K$ the analogous phase can be determined by solving a system of coupled Muskhelishvili-Omnès equations using a known set of $T$-matrix elements. A common feature is that the phase drops sharply by approximately $\pi$ close to the inelastic threshold, which causes a dip in the modulus of the form factor. The exact point $s_{dip}$ where this happens in the case of $\pi\pi$ cannot, of course, be known without actually solving the analogous equations, but it seems plausible that this should be somewhere in between the two resonances $a_0(980)$ and $a_0(1450)$. If one of these resonances can be interpreted as a tetraquark state (i.e. having a suppressed coupling to the $u\bar{d}$ operator) then, from the analyticity point of view, this corresponds to $s_{dip}$ lying close to the corresponding resonance. Fig. 2 illustrates different values for $s_{dip}$ and fig. 3 shows the complete spectral function for $\tau \rightarrow \eta\pi\nu$ assuming $s_{dip}$ to be midway between $a_0(980)$ and $a_0(1450)$.

Finally, integrating over the spectral functions, we obtain the branching fractions shown in the last line of table 4. The results are preliminary and the errors quoted reflect only the uncertainty associated with the variation of $s_{dip}$. Our results tend to be in the lower range of previous evaluations.

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