Lower Bounds of the First Closed and Neumann Eigenvalues of Compact Manifolds with Positive Ricci Curvature *

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Abstract
We give new estimates on the lower bounds for the first closed or Neumann eigenvalue for a compact manifold with positive Ricci curvature in terms of the diameter and the lower bound of Ricci curvature. The results improve the previous estimates.

1 Introduction
For an n-dimensional closed Riemannian manifold whose Ricci curvature has a positive lower bound \((n - 1)K\) for some constant \(K > 0\), A. Lichnerowicz [7] gave a lower bound of the first eigenvalue \(\lambda\) of the Laplacian

\[
\lambda \geq nK.
\]

Under the same curvature assumption, Escobar [2] proved that if the compact manifold has a weakly convex boundary, the first non-zero Neumann eigenvalue of \(M\) has the above lower bound (1) as well.

The above Lichnerowicz-type lower bound (1) gives no information when the above constant \(K\) vanishes. In such case, Li-Yau [6] and Zhong-Yang [15] provided another lower bound for the first non-zero eigenvalue of a closed manifold

\[
\lambda \geq \frac{\pi^2}{d^2}.
\]

It is an interesting problem to find a unified lower bound of the first closed or Neumann eigenvalue \(\lambda\) in terms of the lower bound \((n - 1)K\) of

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the Ricci curvature and the diameter \(d\) so that the lower bound of the first non-zero eigenvalue does not vanish as \(K\) vanishes. P. Li conjectured a unified bound of the first non-zero eigenvalue should be \(\pi^2/d^2 + (n - 1)K\). There has been some work along this line, say [13] by D. Yang, and [11] by the author that improved Yang’s estimate for the first Dirichlet eigenvalue in [13]. D. Yang [13] also give an estimate on the lower bound of closed or Neumann eigenvalue

\[
\lambda \geq \frac{\pi^2}{d^2} + \frac{1}{4}(n - 1)K.
\]

In this paper, we give some new estimates on the lower bound and improve the above bound. Our main result is the following theorem.

**Theorem 1.** If \(M\) is an \(n\)-dimensional, compact Riemannian manifold that has an empty or none-empty boundary whose second fundamental form is nonnegative with respect to the outward normal (i.e., weakly convex). Suppose that Ricci curvature \(\text{Ric}(M)\) has a lower bound \((n - 1)K\) for some constant \(K > 0\), that is

\[
\text{Ric}(M) \geq (n - 1)K > 0.
\]

Then the first non-zero (closed or Neumann, which applies) eigenvalue \(\lambda\) of the Laplacian \(\Delta\) on \(M\) has the following lower bound

\[
\lambda \geq \frac{\pi^2}{d^2} + \frac{3}{8}(n - 1)K \quad \text{for } n = 2
\]

and

\[
\lambda \geq \frac{\pi^2}{d^2} + \frac{31}{100}(n - 1)K \quad \text{for } n \geq 3,
\]

where where \(d\) is the diameter of \(M\).

This estimate sharpens Yang’s bound [3]. It is a generalization of Li-Yau [6] and Zhong-Yang [15]’s result [2] for a closed manifold and it is better than Lichnerowicz’s bound [1] if the manifold is non-symmetric and has small diameter with respect to the positive lower bound of the Ricci curvature.

In order to improve the known results, we need to construct suitable test functions where detailed technical work is essential. In Section 4 we construct the test function \(\xi\). We explore the properties of the function \(\xi\), the Zhong-Yang function \(\eta\), and the ratio function \(\xi/\eta\). Those properties are essential to the construction of the suitable test functions. Because
those functions are complicated combinations of trigonometric and rational functions, the needed properties such as monotonic and convex properties are hard to prove. In the past, though we know that many nice properties might be true, only a few of them could be proven strictly in mathematics by the canonical calculus method and therefore be used in strict mathematics proof. We are able to prove those properties effectively now by studying the differential equations those functions satisfied and using the Maximum Principle. Since the constructions and proofs in that part are quite technical by nature, we put them in the last section. Readers may refer to that section when in need. We derive several preliminary estimates in the next section and prove our result in Section 3.

2 Preliminary Estimates

The first preliminary estimate is due to Lichnerowicz and Escobar. For the completeness and consistency, we use gradient estimate in [3]-[6] and [12] to derive the two estimates.

Lemma 1. Let $\lambda$ be the first non-zero (closed or Neumann, which applies) eigenvalue under the conditions in Theorem 1. Then (1) holds.

Proof. Let $u$ be a normalized eigenfunction of the first non-zero (closed or Neumann, which applies) eigenvalue $\lambda$ such that

$$\sup_M u = 1, \quad \inf_M u = -k, \quad \text{and} \quad 0 < k \leq 1,$$

and define a function $v$ by

$$(5) \quad v = \left[ u - \frac{(1 - k)}{2} \right] / \left[ \frac{(1 + k)}{2} \right].$$

Then

$$(6) \quad \max v = 1 \quad \text{and} \quad \min v = -1.$$ 

The function $v$ satisfies the following equation

$$(7) \quad \Delta v = -\lambda(v + a) \quad \text{in} \ M,$$

where

$$(8) \quad a = \frac{(1 - k)}{(1 + k)}.$$
Note that $0 \leq a < 1$. If $M$ has non-empty boundary $\partial M$, then $v$ satisfies Neumann condition on the boundary,

(9) \[ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial M, \]

where $\nu$ is the outward normal of $\partial M$.

Take an orthonormal frame $\{e_1, \ldots, e_n\}$ about $x_0 \in M$. At $x_0$

\[ \nabla_{e_j}(|\nabla v|^2)(x_0) = \sum_{i=1}^{n} 2v_i v_{ij} \]

and

\[
\begin{aligned}
\Delta(|\nabla v|^2)(x_0) &= 2 \sum_{i,j=1}^{n} v_{ij} v_{ij} + 2 \sum_{i,j=1}^{n} v_{ij} v_{jj} \\
&= 2 \sum_{i,j=1}^{n} v_{ij} v_{ij} + 2 \sum_{i,j=1}^{n} v_{ij} v_{jj} + 2 \sum_{i,j=1}^{n} R_{ij} v_i v_j \\
&= 2 \sum_{i,j=1}^{n} v_{ij} v_{ij} + 2 |\nabla v|^2 \Delta v + 2 \text{Ric}(\nabla v, \nabla v) \\
&\geq 2 \sum_{i=1}^{n} v_{ii}^2 + 2 |\nabla v|^2 \Delta v + 2(n-1)K|\nabla v|^2 \\
&\geq \frac{2}{n} (\Delta v)^2 - 2\lambda|\nabla v|^2 + 2(n-1)K|\nabla v|^2.
\end{aligned}
\]

Thus at all point $x \in M$,

(10) \[ \frac{1}{2} \Delta(|\nabla v|^2) \geq \frac{1}{n} \lambda^2 (v + a)^2 + [(n-1)K - \lambda]|\nabla v|^2. \]

On the other hand, after multiplying (7) by $v + a$ and integrating the both sides over $M$. When $M$ has non-empty boundary and $v$ satisfies Neumann condition (9), we have

\[
\begin{aligned}
\int_{M} \lambda(v + a)^2 \, dx &= - \int_{M} (v + a) \Delta v \, dx \\
&= - \int_{\partial M} (v + a) \frac{\partial}{\partial \nu} v \, ds + \int_{M} |\nabla v|^2 \, dx = \int_{M} |\nabla v|^2 \, dx.
\end{aligned}
\]
That the integral on the boundary vanishes is due to (9). Integrating (10) over $M$ and using the above equality, we get
\begin{equation}
\frac{1}{2} \int_{\partial M} \frac{\partial}{\partial \nu}(|\nabla v|^2) \, dx \geq \int_{M} (nK - \lambda) \frac{n - 1}{n} \lambda(v + a)^2 \, dx.
\end{equation}

We want to show that $\frac{\partial}{\partial \nu}(|\nabla v|^2) \leq 0$ on $\partial M$. Take any $x_0 \in \partial M$. If $\nabla v(x_0) = 0$, then it is done. Assume now that $\nabla v(x_0) \neq 0$. Choose an orthonormal frame $\{e_1, \ldots, e_n\}$ about $x_0$ such that $e_n|_{x_0}$ is the unit outward normal vector to $\partial M$ at $x_0$. Let $(h_{ij})$ be the second fundamental form of $\partial M$ with respect to the outward normal $\nu$ to $\partial M$. Now at $x_0$,
\begin{align*}
v_{in} &= e_i e_n v - (\nabla e_i e_n) v \\
&= -(\nabla e_i e_n) v \\
&= - \sum_{j=1}^{n-1} h_{ij} v_j
\end{align*}

and
\begin{align*}
\frac{\partial}{\partial \nu}(|\nabla v|^2) &= e_n |\nabla v|^2 = 2 \sum_{j=1}^{n} v_j v_{jn} \\
&= 2 \sum_{j=1}^{n-1} v_j v_{jn} = -2 \sum_{i,j=1}^{n-1} h_{ij} v_i v_j \\
&\leq 0 \quad \text{by the weak convexity of } \partial M.
\end{align*}

Putting this into (11), we get the Lichnerowicz-type bound (1) for the first non-zero Neumann eigenvalue. We get the bound (1) for the first non-zero closed eigenvalue by a similar argument as the above when $M$ has no boundary, just noticing that there are no boundary terms in such case. \qed

**Lemma 2.** Let $v$ be the same as in (5). Then $v$ satisfies the following
\begin{equation}
\frac{|\nabla v|^2}{b^2 - v^2} \leq \lambda(1 + a),
\end{equation}
where $a$ is defined in (8) and $b > 1$ is an arbitrary constant.

**Proof.** Consider the function
\begin{equation}
P(x) = |\nabla v|^2 + Av^2,
\end{equation}
where $A > 0$ is a constant. Then
\begin{align*}
P(x) &= |\nabla v|^2 + Av^2 \\
&= |\nabla v|^2 + (\nabla v) \cdot (\nabla v) + Av^2 \\
&= |\nabla v|^2 + \nabla v \cdot (\nabla v) + Av^2 \\
&= |\nabla v|^2 + \nabla v \cdot (\nabla v) + Av^2 \\
&= |\nabla v|^2 + \nabla v \cdot (\nabla v) + Av^2 \\
&= |\nabla v|^2 + \nabla v \cdot (\nabla v) + Av^2 \\
&= |\nabla v|^2 + \nabla v \cdot (\nabla v) + Av^2.
\end{align*}

We want to show that $P(x)$ is bounded from above by $\lambda(1 + a)$ on $M$. To do this, we use the weak convexity of $\partial M$. We have
\begin{align*}
P(x) &= |\nabla v|^2 + \nabla v \cdot (\nabla v) + Av^2 \\
&= |\nabla v|^2 + \nabla v \cdot (\nabla v) + Av^2 \\
&= |\nabla v|^2 + \nabla v \cdot (\nabla v) + Av^2 \\
&= |\nabla v|^2 + \nabla v \cdot (\nabla v) + Av^2.
\end{align*}

By the weak convexity of $\partial M$, we have
\begin{equation}
P(x) \leq \lambda(1 + a),
\end{equation}
where $\lambda$ is the first non-zero Neumann eigenvalue. Therefore, we have
\begin{equation}
\frac{|\nabla v|^2}{b^2 - v^2} \leq \lambda(1 + a),
\end{equation}
where $b > 1$ is an arbitrary constant. \qed
where \( v \) is the function in (5), and where \( A = \lambda (1 + a) + \epsilon \) for small \( \epsilon > 0 \). Function \( P \) must achieve its maximum at some point \( x_0 \in M \).

We claim that

\[
(15) \quad \nabla P(x_0) = 0.
\]

If \( x_0 \in M \setminus \partial M \), (15) is obviously true. Suppose that \( x_0 \in \partial M \). Choose a local orthonormal frame \( \{ e_1, e_2, \ldots, e_n \} \) of \( M \) about \( x_0 \) as in the proof of (12) such that \( e_n \) is the unit outward normal vector field near \( x_0 \in \partial M \) and \( \{ e_1, e_2, \ldots, e_{n-1} \} |_{\partial M} \) is a local frame of \( \partial M \) about \( x_0 \). Note that \( \nabla e_n e_i = 0 \) for \( i \leq n - 1 \) and \( v_n(x_0) = 0 \).

\( P(x_0) \) is the maximum implies that

\[
(16) \quad P_i(x_0) = 0 \quad \text{for } i \leq n - 1
\]

and

\[
(17) \quad P_n(x_0) \geq 0.
\]

Using (6)-(9) in the following arguments, then we have that at \( x_0 \),

\[
v_{in} = \sum_{i=1}^{n} e_i e_n v - \sum_{i=1}^{n} (\nabla e_i e_n) v
\]

\[
= - \sum_{i=1}^{n} (\nabla e_i e_n) v
\]

\[
= - \sum_{j=1}^{n-1} h_{ij} v_j
\]

and

\[
P_n = 2 \sum_{j=1}^{n} v_j v_{jn} + 2 A v v_n
\]

\[
= 2 \sum_{j=1}^{n-1} v_j v_{jn} - 2 \sum_{i,j=1}^{n-1} h_{ij} v_i v_j
\]

\[
(18) \quad \leq 0 \quad \text{by the convexity of } \partial M,
\]

where \( (h_{ij}) \) is the second fundamental form of \( \partial M \) with respect to the outward normal \( e_n \).

Now (16), (17) and (18) imply that \( P_n(x_0) = 0 \) and \( \nabla P(x_0) = 0 \).
Therefore (15) holds, no matter \( x_0 \notin \partial M \) or \( x_0 \in \partial M \). By (15) and the Maximum Principle, we have

(19) \[ \nabla P(x_0) = 0 \quad \text{and} \quad \Delta P(x_0) \leq 0. \]

There are two cases, either \( \nabla v(x_0) = 0 \) or \( \nabla v(x_0) \neq 0 \).

If \( \nabla v(x_0) = 0 \), then

\[ |\nabla v(x)|^2 + Av(x)^2 = P(x) \leq P(x_0) \leq A. \]

Let \( \epsilon \to 0 \) in the above inequality. Then (13) follows.

If \( \nabla v(x_0) \neq 0 \), then we rotate the local orthonormal frame about \( x_0 \) such that

\[ |v_1(x_0)| = |\nabla v(x_0)| \neq 0 \quad \text{and} \quad \nabla l(x_0) = 0, \quad i \geq 2. \]

From (19) we have at \( x_0 \),

(20) \[ v_{11} = -Av \quad \text{and} \quad v_{1i} = 0 \quad i \geq 2, \]

and

\[ 0 \geq \frac{1}{2} \Delta P(x_0) = \sum_{i,j=1}^{n} (v_{ji}v_{ji} + v_{j}v_{jii} + Av_{i}v_{i} + Avv_{ii}) \]

\[ = \sum_{i,j=1}^{n} v_{ji}^2 + \nabla v \nabla (\Delta v) + \text{Ric}(\nabla v, \nabla v) + A|\nabla v|^2 + Av\Delta v \]

\[ \geq v_{11}^2 + \nabla v \nabla (\Delta v) + (n-1)K|\nabla v|^2 + A|\nabla v|^2 + Av\Delta v \]

\[ = (-Av)^2 - \lambda|\nabla v|^2 + (n-1)K|\nabla v|^2 + A|\nabla v|^2 - \lambda Av(v + a) \]

\[ = (A - \lambda + (n-1)K)|\nabla v|^2 + Av^2(A - \lambda) - a\lambda Av, \]

where we have used (20) and (4). Therefore at \( x_0 \),

(21) \[ 0 \geq (A - \lambda)|\nabla v|^2 + A(A - \lambda)v^2 - a\lambda Av \]

and

\[ |\nabla v(x_0)|^2 + \lambda(1+a)v(x_0)^2 \leq \frac{a\lambda v(x_0)}{a\lambda + \epsilon}[\lambda(1+a) + \epsilon] \leq [\lambda(1+a) + \epsilon]. \]

Finally let \( \epsilon \to 0 \). So we have the estimate (13) in the second case as well. \( \square \)
We proceed to improve the above bound. Define a function $Z$ by

$$Z(t) = \max_{x \in M, t = \sin^{-1}(v(x)/b)} \frac{|\nabla v|^2}{b^2 - v^2}/\lambda.$$ 

The estimate in (13) implies that

$$Z(t) \leq 1 + a \quad \text{on } [-\sin^{-1}(1/b), \sin^{-1}(1/b)].$$

Throughout this paper, we denote $a/b$ by $c$ and set

$$\alpha = \frac{1}{2}(n - 1)K \quad \text{and} \quad \delta = \alpha/\lambda.$$ 

By (11) we have

$$\delta \leq \frac{n - 1}{2n}.$$ 

We have the following conditions on the test function.

**Theorem 2.** If the function $z : [-\sin^{-1}(1/b), \sin^{-1}(1/b)] \mapsto \mathbb{R}$ satisfies the following

1. $z(t) \geq Z(t) \quad t \in [-\sin^{-1}(1/b), \sin^{-1}(1/b)]$,
2. there exists some $x_0 \in M$ such that at point $t_0 = \sin^{-1}(v(x_0)/b)$ $z(t_0) = Z(t_0)$,
3. $z(t_0) > 0$,

then we have the following

$$0 \leq \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 + c \sin t_0 - 2\delta \cos^2 t_0$$

$$- \frac{z'(t_0)}{4z(t_0)} \cos t_0[z'(t_0) \cos t_0 - 2z(t_0) \sin t_0 + 2 \sin t_0 + 2c].$$

**Corollary 1.** If in addition to the above conditions 1-3 in Theorem 2 $z'(t_0) \geq 0$ and $1 - c \leq z(t_0) \leq 1 + a$, then we have the following

$$0 \leq \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 + c \sin t_0 - 2\delta \cos^2 t_0.$$
**Corollary 2.** If \( a = 0 \), which is defined in (5), and if in addition to the above conditions 1-3 in Theorem 2, \( z'(t_0) \sin t_0 \geq 0 \) and \( z(t_0) \leq 1 \), then we have the following
\[
0 \leq \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 - 2\delta \cos^2 t_0.
\]

**Proof of Theorem 2.** Define
\[
J(x) = \left\{ \frac{1}{b^2 - v^2} \frac{\nabla v^2}{|\nabla v|^2} - \lambda z \right\} \cos^2 t,
\]
where \( t = \sin^{-1}(v(x)/b) \). Then
\[
J(x) \leq 0 \quad \text{for} \quad x \in M \quad \text{and} \quad J(x_0) = 0.
\]
If \( \nabla v(x_0) = 0 \) then
\[
0 = J(x_0) = -\lambda z \cos^2 t.
\]
This contradicts the condition 3 in the theorem. Therefore
\[
\nabla v(x_0) \neq 0.
\]
Now if \( x_0 \in M = M \setminus \partial M \) then by the Maximum Principle, we have
\[
\Delta J(x_0) = 0 \quad \text{and} \quad \Delta J(x_0) \leq 0.
\]
If \( x_0 \in \partial M \), then the weak convexity of \( M \), the fact that \( J(x_0) \) is the maximum and an argument in the proof of Lemma 2 imply that \( J(x_0) = 0 \) and \( \Delta J(x_0) \leq 0 \). Therefore (26) holds, no matter \( x_0 \in M = M \setminus \partial M \) or \( x_0 \in \partial M \).

\( J(x) \) can be rewritten as
\[
J(x) = \frac{1}{b^2} |\nabla v|^2 - \lambda z \cos^2 t.
\]
Thus (26) is equivalent to
\[
\frac{2}{b^2} \sum_i v_i v_{ij} \bigg|_{x_0} = \lambda \cos t [z' \cos t - 2z \sin t] t_j \bigg|_{x_0}
\]
and
\[
0 \geq \frac{2}{b^2} \sum_{i,j} v^2_{ij} + \frac{2}{b^2} \sum_{i,j} v_i v_{ijj} - \lambda (z''|\nabla t|^2 + z' \Delta t) \cos^2 t
\]
\[
+ 4\lambda z' \cos t \sin t |\nabla t|^2 - \lambda z \Delta \cos^2 t \bigg|_{x_0}.
\]
Rotate the local normal frame about $x_0$ such that $v_1(x_0) \neq 0$ and $v_i(x_0) = 0$ for $i \geq 2$. Then (27) implies

$$v_{11} \big|_{x_0} = \frac{\lambda b}{2} (z' \cos t - 2z \sin t) \big|_{x_0} \quad \text{and} \quad v_{1i} \big|_{x_0} = 0 \quad \text{for} \quad i \geq 2.$$  

Now we have

$$|\nabla v|^2 \big|_{x_0} = \lambda b^2 z \cos^2 t \big|_{x_0},$$

$$|\nabla t|^2 \big|_{x_0} = \frac{|\nabla v|^2}{b^2 - v^2} = \lambda z \big|_{x_0},$$

$$\frac{\Delta v}{b} \big|_{x_0} = \Delta \sin t = \cos t \Delta t - \sin t |\nabla t|^2 \big|_{x_0},$$

$$\Delta t \big|_{x_0} = \frac{1}{\cos t} (\sin t |\nabla t|^2 + \Delta v)$$

$$= \frac{1}{\cos t} [\lambda z \sin t - \frac{\lambda}{b} (v + a)] \big|_{x_0}, \quad \text{and}$$

$$\Delta \cos^2 t \big|_{x_0} = \Delta \left( 1 - \frac{v^2}{b^2} \right) = -\frac{2}{b^2} |\nabla v|^2 - \frac{2}{b^2} v \Delta v$$

$$= -2\lambda z \cos^2 t + \frac{2}{b^2} \lambda v (v + a) \big|_{x_0}.$$  

Therefore,

$$\frac{2}{b^2} \sum_{i,j} v_{ij}^2 \big|_{x_0} \geq \frac{2}{b^2} v_{11}^2$$

$$\geq \lambda^2 (z')^2 \cos^2 t - 2\lambda^2 z z' \cos t \sin t + 2\lambda^2 z^2 \sin^2 t \big|_{x_0},$$

$$= \frac{2}{b^2} \sum_{i,j} v_{ij} v_{ij} \big|_{x_0} \geq \frac{2}{b^2} (\nabla v \nabla (\Delta v) + \text{Ric}(\nabla v, \nabla v))$$

$$\geq \frac{2}{b^2} (\nabla v \nabla (\Delta v) + (n - 1)K |\nabla v|^2)$$

$$= -2\lambda^2 z \cos^2 t + 4\alpha \lambda z \cos^2 t \big|_{x_0},$$

$$-\lambda (z'' |\nabla t|^2 + z' \Delta t) \cos^2 t \big|_{x_0}$$

$$= -\lambda^2 z z'' \cos^2 t - \lambda^2 z z' \cos t \sin t$$

$$+ \frac{1}{b} \lambda^2 z (v + a) \cos t \big|_{x_0}.$$
and
\[
4\lambda z' \cos t \sin t |\nabla t| - \lambda z \Delta t \bigg|_{x_0}
= 4\lambda^2 z z' \cos t \sin t + 2\lambda^2 z^2 \cos^2 t - \frac{2}{b} \lambda^2 z \sin t (v + a) \bigg|_{x_0}.
\]

Putting these results into (28) we get
\[
0 \geq -\lambda^2 z z'' \cos^2 t + \frac{\lambda^2}{2}(z')^2 \cos^2 t + \lambda^2 z' \cos t (z \sin t + c + \sin t)
+ 2\lambda^2 z^2 - 2\lambda^2 z - 2\lambda^2 c z \sin t + 4\alpha \lambda z \cos^2 t \bigg|_{x_0},
\]
where we used (29). Now
\[
z(t_0) > 0,
\]
by the condition 3 in the theorem. Dividing two sides of (30) by \(2\lambda^2 z \big|_{x_0}\), we have
\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + \frac{1}{2}(z'(t_0)) \cos t_0 \left( \sin t_0 + \frac{c + \sin t_0}{z(t_0)} \right) + z(t_0)
- 1 - c \sin t_0 + 2\delta \cos^2 t_0
+ \frac{1}{4z(t_0)} (z'(t_0))^2 \cos^2 t_0.
\]

Therefore,
\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + z'(t_0) \cos t_0 \sin t_0 + z(t_0) - 1 - c \sin t_0 + 2\delta \cos^2 t_0
+ \frac{z'(t_0)}{4z(t_0)} \cos t_0 [z'(t_0) \cos t_0 - 2z(t_0) \sin t_0 + 2\sin t_0 + 2c].
\]

**Proof of Corollary 1.** By Condition 2 in the theorem, \(29\), \( |\sin t_0| = |v(t_0)/b| \leq 1/b \) and \( 1 - c \leq z(t_0) \leq 1 + a \). Thus for \( t_0 \geq 0 \),
\[
-z(t_0) \sin t_0 + \sin t_0 + c \geq -\sin t_0 - a \sin t_0 + \sin t_0 + c \geq a \left( \frac{1}{b} - \sin t_0 \right) \geq 0,
\]
and for \( t_0 < 0 \),
\[
-z(t_0) \sin t_0 + \sin t_0 + c \geq -\sin t_0 + c \sin t_0 + \sin t_0 + c \geq c(1 + \sin t_0) \geq 0.
\]
In any case the last term in the (25) is non-negative.

**Proof of Corollary 2.** The last term in the (25) is nonnegative.
3 Proof of the Main Result

**Theorem 3.** If \( a > 0 \) and \( \mu \delta \leq \frac{4}{\pi^2} a \) for a constant \( \mu \in (0, 1] \), then under the conditions in Theorem 1 the first non-zero (closed or Neumann, which applies) eigenvalue \( \lambda \) has the following lower bound

\[
\lambda \geq \frac{\pi^2}{d^2} + \frac{\mu}{2} (n-1)K = \frac{\pi^2}{d^2} + \mu \alpha.
\]

**Proof.** Let \( \mu_\epsilon = \mu - \epsilon > 0 \) for small positive constant \( \epsilon \). Take \( b > 1 \) close to 1 such that \( \mu_\epsilon \delta < \frac{4}{\pi^2} c \). Let

\[
z(t) = 1 + c\eta(t) + \mu_\epsilon \delta \xi(t),
\]

where \( \xi \) and \( \eta \) are the functions defined by \((57)\) and \((65)\), respectively. Let \( \bar{I} = [-\sin^{-1}(1/b), \sin^{-1}(1/b)] \). We claim that

\[
Z(t) \leq z(t) \quad \text{for } t \in \bar{I}.
\]

By Lemma 5 and Lemma 6 we have

\[
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\]

\[
\frac{1}{2}z'' \cos^2 t - z' \cos t \sin t - z = -1 - c \sin t + 2\mu_\epsilon \cos^2 t,
\]

\[
z'(t) > 0
\]

\[
0 < 1 - \frac{a}{b} = z(-\frac{\pi}{2}) \leq z(t) \leq z(\frac{\pi}{2}) = 1 + \frac{a}{b} \leq 1 + a,
\]

where \((36)\) is due to the following.

\[
z'(t) = c\eta'(t) + \mu_\epsilon \delta \xi'(t) = \mu_\epsilon \delta \eta'(t) \left( \frac{c}{\mu_\epsilon \delta} + \frac{\xi'(t)}{\eta'(t)} \right) \\
\geq \mu_\epsilon \delta \eta'(t) \left( \frac{c}{\mu_\epsilon \delta} - \frac{\pi^2}{4} \right) > 0.
\]

Let \( P \in \mathbb{R}^1 \) and \( t_0 \in [-\sin^{-1}(1/b), \sin^{-1}(1/b)] \) such that

\[
P = \max_{t \in \bar{I}} (Z(t) - z(t)) = Z(t_0) - z(t_0).
\]

Thus

\[
Z(t) \leq z(t) + P \quad \text{for } t \in \bar{I} \quad \text{and} \quad Z(t_0) = z(t_0) + P.
\]
Suppose that $P > 0$ Then $z + P$ satisfies the inequality in Corollary 1 of Theorem 2. Then
\[
z(t_0) + P = Z(t_0)
\leq \frac{1}{2} (z + P)'(t_0) \cos t_0 - (z + P)'(t_0) \cos t_0 \sin t_0 + 1 + c \sin t_0 - 2 \delta \cos^2 t_0
\leq \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 + 1 + c \sin t_0 - 2 \mu \delta \cos^2 t_0
= z(t_0).
\]
This contradicts the assumption $P > 0$. Thus $P \leq 0$ and (34) must hold.

Now we have
\[
|\nabla t| \leq \lambda z(t) \quad \text{for } t \in \bar{I},
\]
that is
\[
(39) \quad \sqrt{\lambda} \geq \frac{|\nabla t|}{\sqrt{z(t)}}.
\]

Let $q_1$ and $q_2$ be two points in $M$ such that $v(q_1) = -1$ and $v(q_2) = 1$ and let $L$ be the minimum geodesic segment between $q_1$ and $q_2$. We integrate the both sides of (39) along $L$ and change variable and let $b \to 1$. Then
\[
(40) \quad \sqrt{\lambda} d \geq \int_L \frac{|\nabla t|}{\sqrt{z(t)}} dt = \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{z(t)}} dt \geq \left( \int_{-\pi/2}^{\pi/2} z(t) dt \right)^{-\frac{1}{2}} \geq \left( \frac{\pi^3}{\int_{-\pi/2}^{\pi/2} z(t) dt} \right)^{\frac{1}{2}}.
\]
Square the two sides. Then
\[
\lambda \geq \frac{\pi^3}{d^2 \int_{-\pi/2}^{\pi/2} z(t) dt}.
\]

Now
\[
\int_{-\pi/2}^{\pi/2} z(t) dt = \int_{-\pi/2}^{\pi/2} \left[ 1 + a \eta(t) + \mu \delta \xi(t) \right] dt = (1 - \mu \delta) \pi,
\]
where we used the facts that $\int_{-\pi/2}^{\pi/2} \eta(t) dt = 0$ since $\eta$ is an even function, and that $\int_{-\pi/2}^{\pi/2} \xi(t) dt = -\pi$ (by (60) in the Lemma 5). Therefore
\[
\lambda \geq \frac{\pi^2}{(1 - \mu \delta)d^2} \quad \text{and} \quad \lambda \geq \frac{\pi^2}{d^2} + \mu \alpha.
\]
Letting \( \epsilon \to 0 \), we get
\[
\lambda \geq \frac{\pi^2}{(1 - \mu \delta) d^2} \quad \text{and} \quad \lambda \geq \frac{\pi^2}{d^2} + \mu \alpha.
\]

**Theorem 4.** If \( a = 0 \), then under the conditions in Theorem 1 the first non-zero (closed or Neumann, which applies) eigenvalue \( \lambda \) has the following lower bound
\[
(41) \quad \lambda \geq \frac{\pi^2}{d^2} + \frac{1}{2} (n - 1)K.
\]

**Proof.** Let
\[
y(t) = 1 + \delta \xi.
\]
By Lemma 5, for \(-\pi/2 < t < \pi/2\), we have
\[
(42) \quad \frac{1}{2} y'' \cos^2 t - y' \cos t \sin t - y = -1 + 2 \delta \cos^2 t,
\]
\[
(43) \quad y'(t) \sin t \geq 0, \quad \text{and}
\]
\[
(44) \quad y(\pm \pi/2) = 1 \text{ and } 0 < y(t) < 1.
\]
We need only show that \( Z(t) \leq y(t) \) on \([-\pi/2, \pi/2]\). If it is not true, then there is \( t_0 \) and a number \( P > 0 \) such that \( P = Z(t_0) - y(t_0) = \max Z(t) - y(t) \).

Note that \( y(t) + P \geq 1 - \frac{1}{2} (\frac{n^2}{4} - 1) + P > 0 \). So \( y + P \) satisfies the inequality in the Corollary 2 in Theorem 2. Therefore
\[
y(t_0) + P = Z(t_0)
\]
\[
\leq \frac{1}{2} (y + P)'(t_0) \cos^2 t_0 - (y + P)'(t_0) \cos t_0 \sin t_0 + 1 - 2 \delta \cos^2 t_0
\]
\[
= \frac{1}{2} y''(t_0) \cos^2 t_0 - y'(t_0) \cos t_0 \sin t_0 + 1 - 2 \delta \cos^2 t_0
\]
\[
= y(t_0).
\]
This contradicts the assumption \( P > 0 \). The rest of the proof is similar to that of Theorem 3, just noticing that \( \delta \leq \frac{n - 1}{2n} < \frac{1}{2} < \frac{4}{\pi^2 - 4} \). \( \square \)

**Proof of Theorem 1 (The Main Result).** Since \( 0 \leq a < 1 \), either \( a = 0 \) or \( 0 < a < 1 \).

If \( a = 0 \), then we apply Theorem 4 to get the bound with \( \mu = 1 \),
\[
\lambda \geq \frac{\pi^2}{d^2} + \alpha = \frac{\pi^2}{d^2} + \frac{1}{2} (n - 1)K.
\]

If \( 0 < a < 1 \), then there are several cases altogether.
• (I): \( a \geq \frac{\pi^2}{4} \delta \).
• (II): \( a < \frac{\pi^2}{4} \delta \).
  - (II-a): \( a \geq 0.765 \).
  - (II-b): \( 0 < a < 0.765 \).
    * (II-b-1): \( a \geq 1.53 \delta \).
    * (II-b-2): \( a < 1.53 \delta \).

For Case (I): \( 0 < a < 1 \) and \( a \geq \frac{\pi^2}{4} \delta \), we apply Theorem 3 for \( \mu = 1 \) to get the following lower bound:

\[
\frac{\pi^2}{d^2} + \frac{1}{2} (n - 1) K.
\]

For Case (II-a): \( 0.765 \leq a < \frac{\pi^2}{4} \delta \), we apply Theorem 3 with \( \mu = \frac{4 a}{\pi^2 \delta} \) since \( \left( \frac{4 a}{\pi^2 \delta} \right) \delta \leq \frac{1}{\pi} a \) and \( 0 < \frac{4 a}{\pi^2 \delta} < 1 \). Then

\[
\lambda \geq \frac{\pi^2}{d^2} + \frac{4}{\pi^2 \delta} a \alpha = \frac{\pi^2}{d^2} + \frac{4}{\pi^2} a \lambda
\]

Thus

\[
\lambda \geq \frac{1}{1 - \frac{4 a}{\pi^2} d^2}.
\]

On the other hand we have Lichnerowicz-type lower bound (24),

\[
\lambda \geq \frac{2n}{n - 1} \alpha.
\]

The above two estimates give

\[
\lambda \geq \frac{\pi^2}{d^2} + \frac{4 a}{\pi^2 \delta} a \alpha \geq \frac{\pi^2}{d^2} + \frac{8 (0.765) n}{\pi^2 (n - 1) \alpha} > \frac{\pi^2}{d^2} + \frac{0.62 n}{n - 1} \alpha > \frac{\pi^2}{d^2} + \frac{31}{50} \alpha
\]

\[
= \frac{\pi^2}{d^2} + \frac{31}{100} (n - 1) K.
\]

The theorem is proved in this case.

For Case (II-b-1): \( 0 < a < 0.765 \), \( a < \frac{\pi^2}{4} \delta \) and \( a \geq 1.53 \delta \), we apply Theorem 3 with \( \mu = \frac{4}{\pi^2 \delta} a \) since \( \left( \frac{4}{\pi^2 \delta} a \right) \delta \leq \frac{4}{\pi^2} a \) and \( 0 < \frac{4}{\pi^2 \delta} < 1 \). Then

\[
\lambda \geq \frac{\pi^2}{d^2} + \frac{4 a}{\pi^2 \delta} \alpha \geq \frac{\pi^2}{d^2} + \frac{4}{\pi^2} \frac{153}{100} \alpha
\]

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\[ > \frac{\pi^2}{d^2} + \frac{31}{50} \alpha \]
\[ = \frac{\pi^2}{d^2} + \frac{31}{100}(n-1)K, \]
which is what we want to prove.

For the remaining Case (II-b-2): \( 0 < a < 0.765, \ a < \frac{\pi^2}{4}\delta \) and \( a < 1.53\delta \), we define a function \( z \) by

\[ z(t) = 1 + cn(t) + (\delta - \sigma c^2)\xi(t) \quad \text{on \([-\sin^{-1} \frac{1}{b}, \sin^{-1} \frac{1}{b}]\)}, \]

where

\begin{equation}
\sigma = \frac{\tau}{\left(\left[\frac{3}{2} - \frac{\pi^2}{8} - \left(\frac{\pi^2}{32} - \frac{1}{6}\right)\frac{153}{153} + \frac{200}{153} - \frac{296}{153}\right] - 1 + (12 - \pi^2)\frac{200}{153}\right) c} > 0
\end{equation}

and

\begin{equation}
\tau = \frac{2}{3\pi^2} \left(\frac{4}{3(4 - \pi)} + \frac{3(4 - \pi)}{4} - 2\right) > 0.
\end{equation}

Let \( \bar{I} = [-\sin^{-1} \frac{1}{b}, \sin^{-1} \frac{1}{b}] \).

We now show that

\begin{equation}
Z(t) \leq z(t) \quad \text{on \( \bar{I} \).}
\end{equation}

If \( \text{(47)} \) is not true, then there exists a constant \( P > 0 \) and \( t_0 \) such that

\[ Pc^2 = \frac{Z(t_0) - z(t_0)}{-\xi(t_0)} = \max_{t \in [-\sin^{-1} \frac{1}{b}, \sin^{-1} \frac{1}{b}]} \frac{Z(t) - z(t)}{-\xi(t)}.
\]

Let \( w(t) = z(t) - Pc^2\xi(t) = 1 + cn(t) + m\xi(t) \), where \( m = \delta - \sigma c^2 - Pc^2 \).

Then

\[ Z(t) \leq w(t) \quad \text{on \( \bar{I} \) and \( Z(t_0) = w(t_0) \).}
\]

By Lemma 3, \( w(t_0) > 0 \). So \( w \) satisfies (25) in Theorem 2.

\[ 0 \leq -2(\sigma + P)c^2 \cos^2 t_0 - \frac{w'(t_0)}{4w(t_0)} \cos t_0 \left(\frac{8c}{\pi} \cos t + 4mt \cos t\right).
\]

We used (58), (59), (66) and (67) to get the above inequality. Thus

\begin{equation}
\sigma + P \leq -\frac{w'(t_0)}{2c^2w(t_0)} \left(\frac{2c}{\pi} + mt\right) = -\frac{\eta'(t_0)}{\pi w(t_0)} \left(1 + \frac{m\xi'(t_0)}{\pi \eta'(t_0)}\right) \left(1 + \frac{\pi m}{2c} t_0\right).
\end{equation}
The righthand side is not positive for $t_0 \geq 0$, by Lemmas 5 and 6. Thus $t_0 < 0$, and

$$- \left(1 + \frac{m\xi'(t_0)}{\eta'(t_0)}\right) \left(1 + \frac{\pi m}{2c} t_0\right)$$

$$= \frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)} \left(\frac{\pi t_0 \eta'(t_0)}{2\xi'(t_0)} + \frac{\pi m}{2c} t_0\right) \left(-1 - \frac{\pi m}{2c} t_0\right)$$

$$\leq \frac{1}{4} \frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)} \left(\frac{\pi t_0 \eta'(t_0)}{2\xi'(t_0)} - 1\right)^2$$

$$= \frac{1}{4} \left(\frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)} + \left(\frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)}\right)^{-1} - 2\right).$$

By Lemmas 5 and 6 we have $2(3 - \pi/4) \leq \xi'(t) \leq 4/3$ and $2(4/\pi - 1) \leq \eta'(t) \leq 8/3\pi$. So

$$\frac{3(12 - \pi^2)}{8} \leq \frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)} \leq \frac{4}{3(4 - \pi)}.$$

Note that the function $f(t) = t + \frac{1}{t} - 2$ achieves its maximum on $[A, B]$ not containing 0 at an endpoint. Therefore

$$\left|\left(1 + \frac{m\xi'(t_0)}{\eta'(t_0)}\right) \left(1 + \frac{\pi m}{2c} t_0\right)\right| \leq \frac{1}{4} \left(\frac{4}{3(4 - \pi)} + \frac{3}{3(4 - \pi)} - 2\right).$$

Now (48) becomes

$$\sigma + P \leq \frac{\tau}{w(t_0)}.$$

On the other hand, by Lemma 3

$$\sigma + P < \sigma,$$ (50)

since $-P \xi(t_0) \geq 0$, we have $w(t_0) \geq z(t_0)$. This fact, (49) and (50) imply that for $P > 0$

$$\sigma + P < \sigma,$$

which is impossible.

Therefore we have the estimate (47). Now we proceed as in the proof of Theorem 3. We get the following

$$\lambda d^2 \geq \frac{\pi^3}{\pi[1 - (\delta - \sigma c^2)]}.$$
Since \( \delta - \sigma^2 > 0.625\delta \) by Lemma 3, we have
\[
\lambda \geq \frac{1}{1 - (\delta - \sigma^2)} \frac{\pi^2}{d^2} \geq \frac{1}{1 - 0.625\delta} \frac{\pi^2}{d^2}
\]
and
\[
\lambda \geq \frac{\pi^2}{d^2} + 0.625\alpha \geq \frac{\pi^2}{d^2} + \frac{31}{100}(n - 1)K.
\]

If \( n = 2 \) then we can get even better result.
If \( a = 0 \), then we apply Theorem 4 to get the lower bound \( \frac{\pi^2}{d^2} + \frac{1}{2}(n - 1)K \).
If \( a \geq \frac{\pi^2}{4}\delta \), then we apply Theorem 3 to get the lower bound \( \frac{\pi^2}{d^2} + \frac{1}{2}(n - 1)K \).
If \( a < \frac{\pi^2}{4}\delta \) and \( n = 2 \), then \( a \) satisfies
\[
a \leq \frac{(12 - \pi^2)n + \pi^2 - 4}{8n}.
\]

Otherwise that \( a < \frac{\pi^2}{4}\delta \), \( a > \frac{(12 - \pi^2)n + \pi^2 - 4}{8n} \) and \( \delta \leq \frac{n - 1}{2n} \) would yield
\[
\frac{(12 - \pi^2)n + \pi^2 - 4}{8n} < a < \frac{\pi^2}{4}\delta \leq \frac{\pi^2}{4}(n - 1) = \frac{\pi^2}{16} = \frac{\pi^2(n - 1)}{8n}.
\]

We do know the following opposite inequality holds for \( n = 2 \),
\[
\frac{(12 - \pi^2)n + \pi^2 - 4}{8n} = \frac{20 - \pi^2}{16} > \frac{\pi^2}{16} = \frac{\pi^2(n - 1)}{8n}.
\]

Therefore we may apply Theorem 5 to get the lower bound stated in the theorem, which is the least of the three lower bounds.

We now present a Lemma that is used in the proof of the Theorem 4.

**Lemma 3.** If \( a < 1.53\delta \) and \( 0 < a < 0.765 \) then
\[
z(t) = 1 + c\eta(t) + \delta\xi(t)
\]
\[
\geq \left( \frac{3}{2} - \frac{\pi^2}{8} - \frac{\pi^2}{32} \right) \frac{153}{100} \frac{200}{153} = \frac{(8\pi - \frac{\pi^2}{4})^2}{[-1 + (12 - \pi^2 \frac{100}{153})]} c > 0,
\]
for \( t \in [-\pi/2, \pi/2] \) and
\[
\delta - \sigma^2 \approx 0.625162283437 > 0.625\delta,
\]
where \( c = a/b \) and \( b > 1 \) is any constant and \( \sigma \) is the constant in 456.
Proof. By Lemmas 7, Lemma 5 and 6, the function \( z \) on \([-\pi/2, \pi/2]\) has a unique critical point \( t_1 \in (-\pi/2, 0) \) if \( 0 < a < \frac{\pi^2}{4} \delta \) and \( z \) is decreasing on \([-\pi/2, t_1]\) and increasing on \([t_1, \pi/2]\). Therefore

\[
\min_{[-\pi/2, \pi/2]} z = \min_{[-\pi/2, 0]} z = z(t_1).
\]

So we need only consider the restricted function \( z|_{[-\pi/2, 0]} \) for the minimum.

Now first consider the Taylor expansion of \( \xi \) at 0 for \( t \in [-\pi/2, 0] \). By Lemma 5, \( \xi(0) = -\frac{\pi^2}{4} + 1 \), \( \xi'(0) = 0 \) and \( \xi''(0) = 2(3 - \frac{\pi^2}{4}) \) and \( \xi'''(t) < 0 \) on \((-\pi/2, 0)\).

Thus

\[
\xi(t) = \xi(0) + \xi'(0) + \frac{\xi''(0)}{2!} t^2 + \frac{\xi'''(t_2)}{2!} t^3
\]

\[
\geq \xi(0) + \xi'(0) + \frac{\xi''(0)}{2!} t^2
\]

\[
= -\left(\frac{\pi^2}{4} - 1\right) + (3 - \frac{\pi^2}{4}) t^2,
\]

where \( t_2 \) is a constant in \((t, 0)\). Similarly, using the data \( \eta(-\pi/2) = -1 \), \( \eta'(-\pi/2) = \frac{8}{3\pi} \) and \( \eta'''(t) > 0 \) on \((-\pi/2, 0)\) (actually on \([-\pi/2, \pi/2]\)), and the Taylor expansion of \( \eta \) at \(-\pi/2\), we have for \( t \in [-\pi/2, 0] \),

\[
\eta(t) = \eta(-\frac{\pi}{2}) + \eta'(-\frac{\pi}{2})(t + \frac{\pi}{2}) + \frac{\eta''(-\frac{\pi}{2})}{2!}(t + \frac{\pi}{2})^2 + \frac{\eta'''(t_3)}{3!}(t + \frac{\pi}{2})^3
\]

\[
\geq \eta(-\frac{\pi}{2}) + \eta'(-\frac{\pi}{2})(t + \frac{\pi}{2}) + \frac{\eta''(-\frac{\pi}{2})}{2!}(t + \frac{\pi}{2})^2
\]

\[
= -1 + \frac{8}{3\pi}(t + \frac{\pi}{2}) - \frac{1}{4}(t + \frac{\pi}{2})^2
\]

\[
= -(\frac{\pi^2}{16} - \frac{1}{3}) + (\frac{8}{3\pi} - \frac{\pi}{4}) t - \frac{1}{4} t^2,
\]

where \( t_3 \) is some constant in \((-\pi/2, t)\). Therefore on \([-\pi/2, 0] \),

\[
z(t) = 1 + c\eta(t) + \delta\xi(t)
\]

\[
\geq 1 - \left(\frac{\pi^2}{16} - \frac{1}{3}\right)c - \left(\frac{\pi^2}{4} - 1\right)\delta + (\frac{8}{3\pi} - \frac{\pi}{4})ct + [-\frac{1}{4}c + (3 - \frac{\pi^2}{4})\delta]t^2.
\]

Let \( \nu = 1.53 \) and \( a_0 = 0.765 \). That \( a \leq \nu\delta \) implies \( c = a/b < \nu\delta \), where \( b > 1 \) is a constant. Using conditions \( \delta \leq \frac{a_0}{2n} < \frac{1}{2} \) and \( a \leq a_0 \), we get
\[
1 - \left( \frac{\pi^2}{16} - \frac{1}{3} \right)c - \left( \frac{\pi^2}{4} - 1 \right)\delta \\
\geq 1 - \left( \frac{\pi^2}{16} - \frac{1}{3} \right)\nu\delta - \left( \frac{\pi^2}{4} - 1 \right)\delta \\
\geq \left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right)\nu \right) \frac{1}{a_0} \nu c \\
> \left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right)\nu \right) \frac{1}{a_0} \nu c
\]

and

\[
1 + \eta(t) + \xi(t) \\
\geq \left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right)\nu \right) \frac{1}{a_0} \nu c + \left( \frac{8}{3\pi} - \frac{\pi}{4} \right) ct + \left[ -\frac{1}{4} c + (3 - \frac{\pi^2}{4}) \frac{1}{\nu} \right] |t|^2 \\
= \left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right)\nu \right) \frac{1}{a_0} \nu c + \left( \frac{8}{3\pi} - \frac{\pi}{4} \right) ct + \left[ -\frac{1}{4} c + (3 - \frac{\pi^2}{4}) \frac{1}{\nu} \right] |t|^2 c \\
\geq \left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right)\nu \right) \frac{1}{a_0} \nu c - \frac{\left( \frac{8}{3\pi} - \frac{\pi}{4} \right)^2}{4 \left[ -\frac{1}{4} c + (3 - \frac{\pi^2}{4}) \frac{1}{\nu} \right]} c \\
\geq \left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right)\nu \right) \frac{1}{a_0} \nu c - \frac{\left( \frac{8}{3\pi} - \frac{\pi}{4} \right)^2}{\left[ -1 + (12 - \pi^2) \frac{1}{\nu} \right]} c > 0.5433 > 0.
\]

Let \( \tau \) be the constant in (46). Then

\[
\sigma c^2 = \frac{\tau c}{\left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right)\nu \right) \frac{1}{a_0} \nu c - \frac{\left( \frac{8}{3\pi} - \frac{\pi}{4} \right)^2}{\left[ -1 + (12 - \pi^2) \frac{1}{\nu} \right]} c},
\]

\[
\leq \frac{\tau \nu \delta}{\left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right)\nu \right) \frac{1}{a_0} \nu c - \frac{\left( \frac{8}{3\pi} - \frac{\pi}{4} \right)^2}{\left[ -1 + (12 - \pi^2) \frac{1}{\nu} \right]} c} \approx 0.374837516563\delta
\]

and

\[
\delta - \sigma c^2 > 0.625\delta.
\]

\[\square\]
Theorem 5. If \(0 < a < \frac{\pi^2}{4}\delta\) and \(a \leq \frac{(12-\pi^2)n+\pi^2-4}{8n}\), then under the conditions in Theorem 1 the first non-zero (closed or Neumann, which applies) eigenvalue has the following lower bound

\[\lambda \geq \frac{\pi^2}{d^2} + \frac{\mu}{2}(n-1)K,\]

where

\[
(51) \quad \mu = 1 - \sqrt{\frac{\pi^2}{6(\pi^2 - 4)}} \left( \frac{4}{3(4 - \pi)} - \frac{3(4 - \pi)}{4} - 2 \right) \approx (0.765 \ldots) > 3/4.
\]

Proof. The proof is similar to that of Case (II)-b-2 in the proof of Theorem 1. Clearly, we have \(c < \frac{\pi^2}{4}\delta\), where \(c = \frac{a}{b}\) with constant \(b > 1\). Let

\[z = 1 + c\eta + (\delta - \tilde{\sigma}c^2)\xi \quad \text{on } [-\sin^{-1}\frac{1}{b}, \sin^{-1}\frac{1}{b}],\]

where \(\xi\) and \(\eta\) are functions defined in (57) and (65) respectively, \(\tau\) is the constant in (46) and

\[
(52) \quad \tilde{\sigma} = \frac{-[1 - c - (\frac{\pi^2}{4} - 1)\delta] + \sqrt{[1 - c - (\frac{\pi^2}{4} - 1)\delta]^2 + 4(\frac{\pi^2}{4} - 1)\tau c^2}}{2(\frac{\pi^2}{4} - 1)c^2}.
\]

We prove that

\[Z(t) \leq z(t) \quad \text{on } [-\sin^{-1}\frac{1}{b}, \sin^{-1}\frac{1}{b}].\]

If it is not true, then there exists a constant \(P > 0\) and \(t_0\) such that

\[Pc^2 = \frac{Z(t_0) - z(t_0)}{-\xi(t_0)} = \max_{t \in [-\sin^{-1}\frac{1}{b}, \sin^{-1}\frac{1}{b}]} \frac{Z(t) - z(t)}{-\xi(t)}.\]

Let \(\bar{I} = [-\sin^{-1}\frac{1}{b}, \sin^{-1}\frac{1}{b}]\) and \(w(t) = z(t) - Pc^2\xi(t) = 1 + c\eta(t) + m\xi(t)\), where \(m = \delta - \tilde{\sigma}c^2 - Pc^2\). Then

\[
(53) \quad Z(t) \leq w(t) \quad \text{on } \bar{I} \quad \text{and} \quad Z(t_0) = w(t_0).
\]

We want to show that \(w(t_0) > 0\). In order to do that, we now show that \(m > 0\) first.

Lemma 4. \(Z(t) \leq 1 + c\eta(t) \quad \text{on } [-\sin^{-1}\frac{1}{b}, \sin^{-1}\frac{1}{b}].\)
Proof of Lemma 4. If it is not true, then there exist \( t_0 \) and constant \( P \) such that \( P = Z(t_0) - [1 + c\eta(t_0)] = \max(Z(t) - [1 + c\eta(t)]) \). Thus \( 1 + c\eta + P \) satisfies the inequality in Corollary 1 of the Theorem 2. Therefore

\[
1 + \eta(t_0) + P = Z(t_0)
\]

\[
\leq \frac{1}{2}(1 + \eta + P)''(t_0) \cos^2 t_0 - (1 + \eta + P)'(t_0) \cos t_0 + 1 + c\sin t_0 - 2\delta \cos^2 t_0
\]

\[
= \frac{1}{2} \eta''(t_0) \cos^2 t_0 - \eta'(t_0) \cos t_0 \sin t_0 + 1 + c\sin t_0 - 2\delta \cos^2 t_0
\]

\[
= 1 + \eta(t_0) - 2\delta \cos^2 t_0
\]

\[
\leq 1 + \eta(t_0).
\]

This contradicts the assumption \( P > 0 \). The proof of the lemma is completed.

Lemma 4 implies that \( w(t_0) = 1 + c\eta(t_0) + m\xi(t_0) = Z(t_0) \leq 1 + c\eta(t_0) \). Thus \( m\xi(t_0) \leq 0 \) and \( m = \delta - \tilde{\sigma}c^2 - Pc^2 \geq 0 \).

We now show that \( w(t_0) > 0 \). By the fact \( m = 0 \), Lemmas 5 and 6, we have

\[
w(t) \geq 1 - c - \left(\frac{\pi^2}{4} - 1\right)(\delta - \tilde{\sigma}c^2 - Pc^2)
\]

\[
> 1 - c - \left(\frac{\pi^2}{4} - 1\right)\delta + \left(\frac{\pi^2}{4} - 1\right)(\tilde{\sigma}c^2 + Pc^2)
\]

\[
(54) \quad > 1 - c - \left(\frac{\pi^2}{4} - 1\right)\delta + \left(\frac{\pi^2}{4} - 1\right)\tilde{\sigma}c^2 > 1 - c - \left(\frac{\pi^2}{4} - 1\right)\delta.
\]

We claim that if \( a \leq \frac{(12 - \pi^2)n + \pi^2 - 4}{8n} \) then

\[
w(t) > 1 - c - \left(\frac{\pi^2}{4} - 1\right)\delta > 0.
\]

In fact, (54), (24), and \( a \leq \frac{(12 - \pi^2)n + \pi^2 - 4}{8n} \) imply that

\[
w(t) > 1 - c - \left(\frac{\pi^2}{4} - 1\right)\delta > 1 - a - \left(\frac{\pi^2}{4} - 1\right)\delta
\]

\[
\geq 1 - \left(\frac{12 - \pi^2}n + \frac{\pi^2}{4} - 4\right) - \left(\frac{\pi^2}{4} - 1\right)\frac{n - 1}{2n} = 0.
\]

Therefore \( w(t) > 0 \) and \( \tilde{\sigma} > 0 \). Now (53) and the fact \( w(t_0) > 0 \) imply that \( w \) satisfies (25) in Theorem 2. So

\[
0 \leq -2(\tilde{\sigma} + P)c^2 \cos^2 t_0 - \frac{w'(t_0)}{4w(t_0)} \cos t_0 \left(\frac{8c}{\pi} \cos t + 4mt \cos t\right),
\]

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where we used (58), (59), (66) and (67) to get the above inequality. Thus (55)
\[ \tilde{\sigma} + P \leq -\frac{w'(t_0)}{2c^2 w(t_0)} \left( \frac{2c}{\pi} + nt \right) = -\frac{\eta'(t_0)}{\pi w(t_0)} \left( 1 + \frac{m\xi'(t_0)}{c\eta'(t_0)} \right) \left( 1 + \frac{\pi m}{2c} t_0 \right). \]

The right-hand side is not positive as \( t_0 \geq 0 \), by Lemmas 5 and 6. Thus \( t_0 < 0 \). It is showed in the proof of Case (II)-b-2 of the proof of Theorem 1 that
\[ \left| -\left( 1 + \frac{m\xi'(t_0)}{c\eta'(t_0)} \right) \left( 1 + \frac{\pi m}{2c} t_0 \right) \right| \leq \frac{1}{4} \left( \frac{4}{3(4 - \pi)} + \frac{3}{3(4 - \pi)} - 2 \right). \]

Therefore (55) becomes
\[ (56) \quad \tilde{\sigma} + P \leq \frac{\tau}{w(t_0)}. \]

Now taking (54) into (56), we get
\[ \tilde{\sigma} + P \leq \frac{\tau}{w(t_0)} \leq \frac{\tau}{1 - c - (\frac{\pi^2}{4} - 1)\delta + (\frac{\pi^2}{4} - 1)\tilde{\sigma}c^2} = \tilde{\sigma}. \]

This contradicts \( P > 0 \). The last equality is due to the fact that \( \tilde{\sigma} \) is the positive solution of the quadratic equation
\[ -\tau + (1 - c)\tilde{\sigma} - (\frac{\pi^2}{4} - 1)\delta \tilde{\sigma} + (\frac{\pi^2}{4} - 1)\tilde{\sigma}^2 c^2 = 0. \]

Therefore
\[ Z(t) \leq z(t) = 1 + c\eta(t) + (\delta - \tilde{\sigma}c^2)\xi. \]

Note that \( \sqrt{A + B} \leq \sqrt{A} + \sqrt{B} \). By the conditions \( c < \frac{\pi^2}{4} \delta \) and \( 1 - c - (\frac{\pi^2}{4} - 1)\delta > 0 \), we have
\[ \tilde{\sigma}c^2 = \frac{-[1 - c - (\frac{\pi^2}{4} - 1)\delta] + \sqrt{[1 - c - (\frac{\pi^2}{4} - 1)\delta]^2 + 4(\frac{\pi^2}{4} - 1)\tau c^2}}{2(\frac{\pi^2}{4} - 1)} \leq \frac{c}{(\frac{\pi^2}{4} - 1)} \leq \frac{\pi^2\delta}{2} \sqrt{\frac{\tau}{\pi^2 - 4}} \approx (0.235 \ldots)\delta, \]

and
\[ \delta - \tilde{\sigma}c^2 \geq \left( 1 - \frac{\pi^2}{2} \sqrt{\frac{\tau}{\pi^2 - 4}} \right) \delta = \mu \delta, \]

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where \( \mu \) is the constant in (51). Proceeding further as in the proof of Theorem 3, we get
\[
\lambda \geq \frac{1}{1 - (\delta - \tilde{\sigma} c^2)} \frac{\pi^2}{d^2} \geq \frac{1}{1 - \mu \delta} \frac{\pi^2}{d^2}
\]
and
\[
\lambda \geq \frac{\pi^2}{d^2} + \mu \alpha.
\]

\[\square\]

4 Functions

We study the functions that are used for the construction of the test functions.

Lemma 5. Let
\[
\xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \frac{\pi^2}{4}}{\cos^2 t} \text{ on } [-\frac{\pi}{2}, \frac{\pi}{2}].
\]

Then the function \( \xi \) satisfies the following
\[
\xi''(t) = 2, \quad \xi''(0) = 2(3 - \frac{\pi^2}{4}) \quad \text{and} \quad \xi''(t) > 0 \text{ on } [-\frac{\pi}{2}, \frac{\pi}{2}],
\]
\[
\left(\frac{\xi'(t)}{t}\right)' > 0 \text{ on } (0, \pi/2) \quad \text{and} \quad 2\left(3 - \frac{\pi^2}{4}\right) \leq \frac{\xi'(t)}{t} \leq \frac{4}{3} \text{ on } [-\frac{\pi}{2}, \frac{\pi}{2}],
\]
\[
\xi''(\frac{\pi}{2}) = \frac{8\pi}{15}, \quad \xi'''(t) < 0 \text{ on } (-\frac{\pi}{2}, 0) \quad \text{and} \quad \xi'''(t) > 0 \text{ on } (0, \frac{\pi}{2}).
\]
Proof. For convenience, let \( q(t) = \xi'(t) \), i.e.,

\[
q(t) = \xi'(t) = \frac{2(2t \cos t + t^2 \sin t + \cos^2 t \sin t - \frac{\pi^2}{4} \sin t)}{\cos^3 t}.
\]

Equation 58 and the values \( \xi(\pm \frac{\pi}{2}) = 0 \), \( \xi(0) = 1 - \frac{\pi^2}{4} \) and \( \xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3} \) can be verified directly from 57 and 51. The values of \( \xi'' \) at 0 and \( \pm \frac{\pi}{2} \) can be computed via 58. By 59, \( (\xi(t) \cos^2 t)' = 4t \cos^2 t \). Therefore \( \xi(t) \cos^2 t = \int_0^t 4s \cos^2 s \, ds \), and

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \xi(t) \, dt = 2 \int_0^{\frac{\pi}{2}} \xi(t) \, dt = -8 \int_0^{\frac{\pi}{2}} \left( \frac{1}{\cos^2(t)} \int_0^{\frac{\pi}{2}} s \cos^2 s \, ds \right) \, dt
\]

\[
= -8 \int_0^{\frac{\pi}{2}} \left( \int_0^{s} \frac{1}{\cos^2(t)} \, dt \right) s \cos^2 s \, ds = -8 \int_0^{\frac{\pi}{2}} s \cos s \sin s \, ds = -\pi.
\]

It is easy to see that \( q \) and \( q' \) satisfy the following equations

\[
(62) \quad \frac{1}{2} q'' \cos t - 2q' \sin t - 2q \cos t = -4 \sin t,
\]

and

\[
(63) \quad \frac{\cos^2 t}{2(1 + \cos^2 t)} \left( q'' \right) - \frac{2 \cos t \sin t}{1 + \cos^2 t} (q')' - 2(q') = -\frac{4}{1 + \cos^2 t}.
\]

The last equation implies \( q' = \xi'' \) cannot achieve its non-positive local minimum at a point in \( (-\frac{\pi}{2}, \frac{\pi}{2}) \). On the other hand, \( \xi''(\pm \frac{\pi}{2}) = 2 \), by equation 58, \( \xi(\pm \frac{\pi}{2}) = 0 \) and \( \xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3} \). Therefore \( \xi''(t) > 0 \) on \( [-\frac{\pi}{2}, \frac{\pi}{2}] \) and \( \xi' \) is increasing. Since \( \xi'(t) = 0 \), we have \( \xi'(t) < 0 \) on \( (-\frac{\pi}{2}, 0) \) and \( \xi'(t) > 0 \) on \( (0, \frac{\pi}{2}) \). Similarly, from the equation

\[
\frac{\cos^2 t}{2(1 + \cos^2 t)} \left( q''(t) \right) - \frac{\cos t \sin t}{1 + \cos^2 t} (q'(t))' - 2(5 \cos^2 t + \cos^2 t)(q'(t)) = -\frac{8 \cos t \sin t}{(1 + \cos^2 t)^2} (q''(t))
\]

(64)

we get the results in the last line of the lemma.

Set \( h(t) = \xi''(t) t - \xi'(t) \). Then \( h(0) = 0 \) and \( h'(t) = \xi''(t) t > 0 \) in \( (0, \frac{\pi}{2}) \). Therefore \( \frac{\xi'(t)}{t} = \frac{h(t)}{t^2} > 0 \) in \( (0, \frac{\pi}{2}) \). Note that \( \frac{\xi'(0)}{t} = \frac{\xi'(\frac{\pi}{2})}{t} \), \( \xi'(t) \big|_{t=0} = \xi''(0) = 2(3 - \frac{\pi^2}{4}) \) and \( \xi'(t) \big|_{t=\pi/2} = \frac{4}{3} \). This completes the proof of the lemma. 

\( \square \)
Lemma 6. Let

\[
\eta(t) = \frac{4t + t \cos t \sin t - 2 \sin t}{\cos^2 t} \quad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}].
\]

Then the function \( \eta \) satisfies the following

\[
\frac{1}{2} \eta'' \cos^2 t - \eta' \cos t \sin t - \eta = -\sin t \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}),
\]

(66)

\[
\eta' \cos t - 2 \eta \sin t = \frac{8}{\pi} \cos t - 2 \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}),
\]

(67)

\[-1 = \eta(-\frac{\pi}{2}) \leq \eta(t) \leq \eta(\frac{\pi}{2}) = 1 \quad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}],
\]

\[
0 < 2(\frac{4}{\pi} - 1) = \eta'(0) \leq \eta'(t) \leq \eta'(\frac{\pm \pi}{2}) = \frac{8}{3\pi} \quad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}],
\]

(68)

\[
\eta''(t) > 0 \quad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}] \quad \text{and} \quad \eta''(\frac{\pm \pi}{2}) = \frac{32}{15\pi}.
\]

Proof. Let \( p(t) = \eta'(t) \), i.e.,

\[
p(t) = \eta'(t) = \frac{2(\frac{4}{\pi} \cos t + \frac{4}{\pi} t \sin t - \sin^2 t - 1)}{\cos^3 t}.
\]

Equation (66), \( \eta(\pm \frac{\pi}{2}) = \pm 1 \), \( \eta'(0) = 2(\frac{4}{\pi} - 1) \) and \( \eta'(\pm \frac{\pi}{2}) = \frac{8}{3\pi} \) can be verified directly. We get \( \eta''(\pm \frac{\pi}{2}) = \pm 1/2 \) from the above values and equation (66). By (66), \( q = \eta', q' = \eta'' \) and \( p'' = \eta''' \) satisfy the following equations in \((-\frac{\pi}{2}, \frac{\pi}{2})\)

\[
\frac{1}{2} p'' \cos t - 2 p' \sin t - 2 p \cos t = -1,
\]

(69)

\[
\frac{2 \cos^2 t}{2(1 + \cos^2 t)} p''' - 2 \frac{\cos t \sin t}{1 + \cos^2 t} p'' - 2 p' = -\frac{\sin t}{1 + \cos^2 t},
\]

and

\[
\frac{2 \cos^2 t}{2(1 + \cos^2 t)} (p'')'' - \frac{\cos t \sin t(3 + 2 \cos^2 t)}{(1 + \cos^2 t)^2} (p'')' - 2(5 \cos^2 t + \cos^4 t) (p'') = -\frac{\cos t(2 + \sin t)}{(1 + \cos^2 t)^2}.
\]

(70)

The coefficient of \( (p'')' \) in (70) is obviously negative in \((-\frac{\pi}{2}, \frac{\pi}{2})\) and the right-hand side of (70) is also negative. So \( p'' \) cannot achieve its non-positive local minimum at a point in \((-\frac{\pi}{2}, \frac{\pi}{2})\). On the other hand, \( p''(\frac{\pi}{2}) = \frac{32}{15\pi} > 0 \)
(see the proof below), $p''(t) > 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore $p'$ is increasing and $-1/2 = p'(-\frac{\pi}{2}) \leq p'(t) \leq p'(\frac{\pi}{2}) = 1/2$. Note that $p'(0) = 0$ ($p'$ is an odd function). So $p'(t) > 0$ on $(0, \frac{\pi}{2})$ and $p$ is increasing on $[0, \frac{\pi}{2}]$. Therefore $2(4/\pi - 1) = p(0) \leq p(t) = \eta'(t) \leq p(\frac{\pi}{2}) = \frac{8}{3\pi}$ on $[0, \frac{\pi}{2}]$, and on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ since $p$ is an even function. We now show that $p(\frac{\pi}{2}) = \frac{8}{3\pi}$, $p'(\frac{\pi}{2}) = 1/2$ and $p''(\frac{\pi}{2}) = \frac{32}{15\pi}$. The first is from a direct computation by using (68). By (66),

$$\frac{1}{2}p'(\frac{\pi}{2}) = \frac{1}{2}\eta'(\frac{\pi}{2}) = \lim_{t \to \frac{\pi}{2}^-} \frac{\eta'(t) \cos t + \eta(t) - \sin t}{\cos^2 t} = -\frac{1}{2}[\eta''(\frac{\pi}{2}) - 1].$$

So $p'(\frac{\pi}{2}) = 1/2$. Similarly, by (69),

$$\frac{1}{2}p''(\frac{\pi}{2}) = \lim_{t \to \frac{\pi}{2}^-} \frac{2p'(t) \sin t - 1}{\cos t} + 2p(\frac{\pi}{2}) = -2p''(\frac{\pi}{2}) + \frac{16}{3\pi}.$$

Thus $p''(\frac{\pi}{2}) = \frac{32}{15\pi}$. □

**Lemma 7.** The function $r(t) = \xi'(t)/\eta'(t)$ is an increasing function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, i.e., $r'(t) > 0$, and $|r(t)| \leq \frac{\pi}{2}$ holds on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

**Proof.** Let $p(t) = \eta'(t)$ as in (68) and $q(t) = \xi'(t)$. Then $r(t) = q(t)/p(t)$. It is easy to verify that $r(\pm \frac{\pi}{2}) = \pm \frac{\pi}{4}$. By (69) and (82),

$$(1/2)p(t)r'' \cos t + (p'(t) \cos t - 2p(t) \sin t)r' = -4 \sin t.$$

Differentiating the last equation, we get

$$[\frac{1}{2}p(t) \cos t](r'')'' + [\frac{3}{2}p'(t) \cos t - \frac{8}{2}p(t) \sin t](r')' + [p''(t) \cos t - 3p'(t) \sin t - 2p(t) \cos t - 1](r') = -4 \cos t.$$

Using (69), the above equation becomes

$$[\frac{1}{2}p(t) \cos t](r'')'' + [\frac{3}{2}p'(t) \cos t - \frac{8}{2}p(t) \sin t](r')' + [p'(t) \sin t + 2p(t) \cos t - 3](r') = -4 \cos t.$$

The coefficient of $(r')''$ in (71) is negative, for $p'(t) \sin t + 2p \cos t - 3 < \frac{1}{2} + \frac{16}{3\pi} - 3 < 0$. This fact and the negativity of the righthand side of (71) in $(-\frac{\pi}{2}, \frac{\pi}{2})$ imply that $r'$ cannot achieve its non-positive minimum on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
at a point in \((-\frac{\pi}{2}, \frac{\pi}{2})\). Now
\[
\lim_{t \to \frac{\pi}{2}^-} r'(t) = \lim_{t \to \frac{\pi}{2}^-} \frac{s(t) \cos^2 t}{(\frac{4}{\pi} \cos t + \frac{4}{\pi} t \sin t - \sin^2 t - 1)^2}
\]
\[
= \lim_{t \to \frac{\pi}{2}^-} \frac{s(t) \cos^4 t}{[(\frac{4}{\pi} \cos t + \frac{4}{\pi} t \sin t - \sin^2 t - 1)/\cos^3 t]^2}
\]
\[
= \lim_{t \to \frac{\pi}{2}^-} \frac{s(t) \cos^4 t}{[\frac{1}{2} \eta'(t)]^2}
\]
\[
= \frac{4}{3\pi} - \frac{\pi}{12}/(\frac{4}{3\pi})^2
\]
\[
> 0,
\]
where
\[
s(t) = \frac{4}{\pi} t^2 - t^2 \cos t + \frac{12}{\pi} \cos^2 t + \frac{8}{\pi} t \sin t \cos t - \cos t \sin^2 t + (\frac{\pi^2}{4} - 3) \cos t - \pi + 4t \sin t.
\]
Therefore \(r'(t) > 0\) and \(r\) is an increasing function on \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

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