Supermassive black holes may be limited by the Holographic Bound

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Supermassive Black Holes are the most entropic objects found in the universe. The Holographic Bound (HB) to the entropy is used to constrain their formation time with initial masses $\sim 10^6-8 M_\odot$, as inferred from observations. We find that the entropy considerations are more limiting than causality for this "direct" formation. Later we analyze the possibility of SMBHs growing from seed black holes. The growth of the initial mass is studied in the case of accretion of pure radiation and quintessence fields, and we find that there is a class of models that may allow this metamorphosis. Our analysis generalizes recent work for some models of quintessence capable of producing a substantial growth in a short time, while simultaneously obeying the causal and Holographic Bound limits.

Supermassive Black Holes in the Universe

The continued observations of galaxies has revealed a hidden population of huge massive objects in compact nuclear regions of size $\leq$ few pc. Dynamical measurements taken along the last decade point out that the compact objects have masses in the range $10^6 - 10^8 M_\odot$, and perhaps more importantly, that every galaxy seems to host a central massive object [1]. Even though some exotic alternatives have been proposed for their nature (e.g. neutrinoballs, see [2]), the simplest explanation is that the central parsecs of the galaxies are sites of residence of supermassive black holes (SMBH). Among the possible formation scenarios a hierarchical merging of smaller black holes has been suggested [3], although it is not guaranteed that the efficiency of the merging process is high enough to provide large masses. A likely alternative is that the SMBHs are primordial, i.e. preexist the galaxies, and perhaps are important for their very formation [4].

While the mass budget of the universe is not likely to be affected by the presence of the nuclear SMBHs, the total entropy will certainly be, since the entropy content of the black holes is huge. This suggests a connection between the formation of the SMBHs and the total entropy, possibly limited by the Holographic Bound, which has been proposed to limit the entropy
enclosed in a given volume and may be deeply related to the fundamental theories [5].

We discuss in this work the issue of entropic limitations to the formation of SMBHs with masses $\geq 10^6 \, M_\odot$. After a brief presentation of the Holographic Bound and related concepts in Section 2, direct formation of SMBHs is addressed in Section 3. We analyze the possibility of growing the "seed" black holes to those large values is addressed in Section 4. Conditions for fast growth due to accretion of a quintessence scalar field are addressed in Section 5. Section 6 discusses the role of causality in the process of accretion. Some general conclusions are given in Section 7.

**The Holographic Bound**

The Holographic Bound may be formulated by asserting that for a given volume $V$, the state of maximal entropy is the one containing the largest black hole that fits inside $V$, and this maximum is given by the finite area that encloses this volume. This idea generalizes a conjecture made by Bekenstein [6] in which this maximum is fixed by the non-gravitational energy within a sphere of size $R$, i.e. $S < \frac{2\pi ER}{hc}$ (now being properly called the Bekenstein limit).

Several analysis made in recent years reformulated this conjecture and proposed slightly different forms for the HB, but rather than discussing which one is correct we will base our argument on the very existence of some entropy bound, yet to be definitively identified.

To be concrete we shall assume the entropy $S$ to be bounded by the Bekenstein-Hawking value

$$S \leq \frac{A}{4} \quad (1)$$

where $A$ is the area of the enclosed system under consideration. Unless explicitly indicated, we shall use natural units throughout this paper, then the Planck length $L^2_{\text{planck}} = 1$ in eq.(1) above and so on.

Verlinde [7] observed some time ago that this bound must be modified in a cosmology with an arbitrary number of dimensions. Considering the Einstein space-time with the metric

$$ds^2 = -dt^2 + R^2 d\Omega_n^2 \quad (2)$$
where \(d\Omega_n^2\) is the line element of a unit n-dimensional sphere, the entropy of the conformal field in this space-time can be expressed in terms of its total energy \(E\) and the Casimir \(E_C\) by a generalized form of the Cardy-Verlinde formula as

\[
S = \frac{2\pi}{n} R \sqrt{E_C (2E - E_C)} \tag{3}
\]

For a \((n + 1)\) dimensional closed universe, the FRW equations are

\[
H^2 = \frac{16\pi G_n}{n(n-1)V} \frac{E}{V} - \frac{1}{R^2} \tag{4}
\]

\[
\dot{H} = -\frac{8\pi G_n}{(n-1)} \left( \frac{E}{V} + P \right) + \frac{1}{R^2} \tag{5}
\]

where \(H(t) = \frac{\dot{a}}{a}\) is the Hubble parameter (describing the expansion/contraction of the universe), the dot stands for differentiation with respect to the proper time, \(E\) is the total energy of matter filling the universe, and \(G_n\) is the Newton constant in \((n + 1)\) dimensions. \(a(t)\) describes the scale factor of the Universe and \(R(t) \propto a(t)\) its physical size.

The FRW equation can then be related to three cosmological entropy bounds; the Bekenstein-Verlinde bound \(S_{BV} = \frac{2\pi}{n} ER\), the Bekenstein-Hawking bound \(S_{BH} = (n-1)V4G_nR\) (expressing that the black hole entropy is bounded by the area of the cosmological model), and the Hubble bound \(S_H = (n-1)HV4G_n\) (a reflection of the fact that the maximal entropy is produced by black holes of the size of Hubble horizon). At a critical point defined by \(HR = 1\), all these three entropy bounds coincide with each other. Let us define \(E_{BH}\) such that \(S_{BH} = \frac{(n-1)V}{4G_nR} = \frac{2\pi}{n} E_{BH} R\). Then, the first FRW equation takes the form

\[
S_H = \frac{2\pi R}{n} \sqrt{E_{BH} (2E - E_{BH})} \tag{6}
\]

which is precisely of the same form as the Cardy-Verlinde formula. Its maximum reproduces the Hubble bound

\[
S_H \leq \frac{2\pi R}{n} E \tag{7}
\]
Therefore, in some sense it may be said that the FRW dynamics "knows" the value of the maximum entropy filling the universe. This connection between geometry and dynamics is a consequence of the Holographic Principle. However, and suggestive as these arguments are, we do not intend to further analyze them. It is just enough to keep in mind that independently of its specific form, an Holographic Bound is likely to hold for the actual 3+1 universe.

Another important concept needed for the study of SMBHs is the generalized second law of thermodynamics, formulated by J. Bekenstein [8] using a series of gedanken experiments. The generalized second law attempted to cure serious problems with the matter + radiation entropy as the latter was absorbed onto black holes (thus causing a growth of the black hole mass). Given that black holes need a few macroscopic parameters (mass, angular momentum and charge) for their description, the absorption of matter + radiation seemed to lead to a decrease of the entropy of the universe, since the matter + radiation entropy ended hidden behind the horizon. This was very problematic, since that this kind of Geroch process seems to go against the second law of thermodynamics $\Delta S > 0$. Bekenstein conjectured that the total entropy of the universe plus $N$ black holes is given by the sum of the matter + radiation entropy, plus the black hole entropy (which is proportional to the horizon area) in what is now known as the generalized second law (GSL). The GSL takes the form

$$S_{\text{total}} = S_{m+r} + \frac{1}{4} \sum_{i} A_i$$

(8)

where the first contribution is the entropy associated to usual matter and radiation, and the second term describes the black hole contribution to the total entropy. Note that the entropy of just one black hole is numerically huge, $S_{bh} \sim 10^{77}(M/M_\odot)^2$, and this fact will be very important to set astrophysical constraints. According to the GSL, as long as we deal with classical process involving black holes and matter, the total variation of entropy must be positive

$$\Delta S_{\text{total}} > 0$$

(9)

In the next Sections we will evaluate some constraints to the mass and formation time of supermassive black holes using the concepts of HB and the
SMBH and Primordial Black Holes: direct formation

While the exact origin of the SMBHs is not known, it is possible that either a primordial process contributed to form them as they are, and that they have grown explosively from a seed population.

Let us discuss direct formation first. It is well-known that big black holes have a huge entropy, and if the bounds to the cosmic entropy apply, restrictions would arise for a "direct formation" mechanism. Actually, if we impose

$$S_{smbh}(t_i) \sim 2 \times 10^{77} N (M/M_\odot)^2 < S_H(t_i)$$ \hspace{1cm} (10)$$

where $t_i$ stands for the formation time hereafter, and it is further assumed that all black holes form more or less simultaneously, we may find an upper bound to the SMBH directly formed allowed by the HB. We start by evaluating $S_H(t)$ from eq.(7) for a 3+1 FRW universe. We identify $R$ with the particle horizon $R = R_{hp}(t) \propto t$, and the total energy contained within this radius $E = \frac{4\pi}{3} \rho(t) R_{hp}^3(t)$. We further restrict the analysis to the epochs in which $a(t) \propto t^n$. Inserting all the quantities, we obtain $E_{hp}(t) \propto a^{-4}(t)t^3 \propto t$, for $n = 1/2$. We do not need consider other forms of entropy in eq.(10), since black holes actually dominate the entropy budget by a large factor.

Then, multiplying by the particle horizon, we express the HB in the radiation-dominated era as $S_{hp}(t) \sim S_{hp}(t_D)(t/t_D)^2$. For the matter-dominated era $n = 2/3$ an analogous procedure yields $E_{hp}(t) \propto \rho_m(t) R_{hp}^3(t) \propto t$. Therefore, the entropy contents become $S_{hp}(t) \sim S_{hp}(t_0)(t/t_0)^2$, with $t_0 \sim H_0^{-1}$ the present age of the universe ($\sim 10^{17} h_0^{-1} s$) and $t_D \sim 10^{13} s$ is the radiation-matter decoupling time. For the sake of generality the dimensionless Hubble constant $h_0 = (H_0/100kms^{-1}Mpc^{-1})$ has not been fixed, although recent measurements suggest $h_0 \sim 0.65$. We also know that $S_{hp}(t_0) \sim 8 \times 10^{121}$, therefore $S_{hp}(t_D) \sim \frac{8 \times 10^{121}}{(t_0/t_D)^2}$. Then, for the radiation-dominated era the condition $S_{smbh}(N,M) < S_{hp}(t)$ yields the maximum SMBH mass allowed by the entropy bound

$$M_{smbh}(t_i) < 5.6 \times 10^{-3} h_0(t_i/s)(10^{11}/N)^{1/2} M_\odot$$ \hspace{1cm} (11)$$

if their number $N$ is equal to the number of galaxies and all them have been assumed to be of the same mass. The SMBHs can form directly only
after

\[ t_{\text{min}}(M) > 5 \times 10^7 (M/10^8 M_\odot)(N/10^{11})^{1/2} s \]  (12)

because before \( t_{\text{min}} \) the entropy of these \( N \) black holes would be larger than the entropy allowed by the Holographic Bound.

The same reasoning as above can be applied to the formation in the matter-dominated era with the result

\[ M_{\text{smbh}}(t_i) < 0.6 h_0^2 (t_i/s)(10^{11}/N)^{1/2} M_\odot \]  (13)

and an earliest formation time

\[ t_{\text{min}}(M) > 1.8 \times 10^9 h_0^{-1} (M/10^8 M_\odot)(N/10^{11})^{1/2} s \]  (14)

Since the mass inside the horizon in the radiation-dominated era is just \( M_{\text{hor}}(t) \sim 7.6 \times 10^{37} (t/1s) g \), and achieves \( \sim 10^{15} M_\odot \) at its very end, we conclude that the availability of entropy is more restrictive than the demand of a causal formation. In other words, it is not sufficient to have a large horizon in which the SMBH can fit, to be allowed by the entropy of the HB seems to be even more important than that primary requirement.

A realistic and complete model would take into account an Initial Mass Function (IMF) for these black holes, with a general form given by \( \frac{dN}{dM} \). In this case we would need to replace the formula above by an integral of the form

\[ S_{\text{smbh}}(N, M) \sim 2 \times 10^{77} \int_{M_1}^{M_2} dM \left( \frac{dN}{dM} \right) (M/M_\odot)^2, \]

with \( M_1, M_2 \) the lower and upper limits to the masses determined by the specific physical conditions at formation. We shall not address this complex model here and leave it for future work. We have just considered a delta-type \( \frac{dN}{dM} \propto \delta(M - M_*) \), leading to the simple model given by \( S_{\text{smbh}}(N, M) \propto N (M/M_\odot)^2 \) in this paper.

**SMBH and Primordial Black Holes: grow of seeds to the SMBH scale**

If the direct formation of SMBHs is difficult, one may wonder if there is still a possibility of starting with black holes of masses \( M_i \ll 10^6 M_\odot \) which grow subsequently by accretion. Considering the absorption-evaporation processes of PBHs, we can identify epochs in which these objects grow or evaporate. The complete evolution of PBH mass is given semiclassically by
\[
\frac{dM}{dt} = - \frac{D}{M^2} + BM^2 c \left[ \sum_i \rho_i(t) \right]
\] (15)

where the term \( c \left[ \sum_i \rho_i(t) \right] \) describes the flux of whatever component flows through the black hole horizon, \( D \sim 10^{26} g^3 s^{-1} \) is the evaporation constant (which, strictly speaking, depends somewhat on the number of degrees of freedom of the incoming material) and \( B = \frac{27\pi G^2}{c^4} \sim 4.6 \times 10^{-55} cm^2 g^{-2} \) is the absorption constant (related to the cross-section of the black hole).

When writing down the semiclassical eq.(15) we have not ruled out any "fuel" contributing to the growth of the black hole (quintessence has been proposed by Bean and Magueijo in [9]), provided their flux is large enough to contribute to the mass balance. If the black holes formed in the radiation-dominated or matter-dominated era, the main contribution to the second term is the flux of background radiation (quintessence will be explicitly addressed later). If we consider the radiation only, the balance of the r.h.s. terms define the critical mass, (see Refs.[11-13]) for the instantaneous equilibrium between black holes and radiation, and its value is

\[
M_c(t) \sim \frac{10^{26} g}{T_{rad}(t)/T_0}
\] (16)

where the radiation temperature \( T_{rad}(t) \propto a(t)^{-1} \) falls along the cosmological expansion and \( T_0 \) is the present temperature of the CMBR. If, say, \( t_i \sim 1s \), the critical mass is then very small, and all the PBHs candidates to grow to SMBH were well above this instantaneous equilibrium mass value. Therefore, the Hawking radiation was negligible for them [12]. If the PBHs are feed, they may grow with time, and the question is whether they can gain mass until the supermassive regime \( M \geq 10^6 M_\odot \) is reached.

It is generally agreed that the gas and dust accretion are not likely to be important at early times (and do not have simple behavior with the cosmological time either). However, the radiation is always absorbed and might be important if the accretion rate is high enough [12,13]. Neglecting the gas and dust fuels, and keeping only the radiation we may obtain analytic lower limits to the growth.

A first specific question we want to address is the following: if one black hole formed initially at \( t_i \) satisfies the HB, i.e. \( S(M(t_i)) \sim 10^{77} M(t_i)^2 \leq S_{hp}(t_i) \sim 8 \times 10^{121} (t_i/t_0)^2 \), is it automatically guaranteed that this object
will satisfy the HB all the time?

To answer this question let us consider the most general flux \( F(\varrho(t)) \) absorbed by the event horizon by this black hole. Then, the mass accretion rate is given by

\[
\left( \frac{dM}{dt} \right) = \frac{27\pi}{4} r^2 g F(\varrho(t)) \tag{17}
\]

If we choose \( \varrho = \varrho_{\text{rad}} \), then \( F = c\varrho_{\text{rad}}(t) \); for quintessence accretion we use \( F(\varphi) = \frac{\dot{\varphi}^2}{2} \), see \([9]\).

Solving formally the eq.(17) above, yields

\[
M(t) = \frac{M_i}{\left[ 1 - \frac{27\pi M_i}{M_{\text{pl}}^4} \int_{t_i}^t dt' F(\varrho(t')) \right]} \tag{18}
\]

On the other hand, these black holes evolve obeying the HB if the local flux satisfies

\[
\int_{t_i}^t dt' F(\varrho(t')) < \frac{M_{\text{pl}}^4}{27\pi M_i} \left[ 1 - \frac{(M_i/M_{\odot})(t_0/t)}{2 \times 10^{22}} \right] \tag{19}
\]

For the radiation flux, \( F(t) = c\varrho_{\text{rad}}(t) \) and therefore eq.(19) requires

\[
\varrho_{\text{rad}}(t_i) < 6.3 \times 10^{28} \text{g cm}^{-3}(10^{15} \text{g}/M_i)(t_i/s) \tag{20}
\]

For \( t_i \sim 1 \text{s} \) and \( M_i < 1M_{\odot} \), typical of the radiation-dominated era, this condition is indeed satisfied.

The above general results suggest that the global constraint \( S_{\text{total}} < S_{\text{hp}} \propto t^2 \), (i.e, the HB) implies some kind of restriction to the cross-section for the local accretion onto the black hole, as if the total flux through the horizon event would have to be modified. A detailed study of the flow into the black hole is needed to address this issue.

**Quintessence models and the SMBH growth**

Recent work by R.Bean and J.Magueijo \([9]\) suggested an important growth of seed PBHs when a quintessence scalar field \( \varphi \) dominates the accretion. The key new ingredient is the role played by the kinetic term \( \frac{\dot{\varphi}^2}{2} \) in the flux onto the PBHs, which is absent in the case of absorption of pure radiation. We shall describe some growth solutions of the Bean-Magueijo model.
of quintessence around SMBHs. The evolution of the mass of these objects is given by the following formula

\[
\left( \frac{dM}{dt} \right) = \eta CM^2 \dot{\phi}^2
\]

where \( C = \frac{27\pi}{24M_{pl}} \) and \( \eta \leq 1 \) is a parameter that measures the efficiency of the accretion process. Because of the uncertainties on the value of \( \eta \) (related to the details of the depletion of material nearby the black hole, see Ref.[10] for a recent discussion), we have left it free in our calculations, so that the adoption of a different value can be easily done. The connection between the potential \( V(\phi) \) and \( \dot{\phi} \) comes because the latter regulates the expansion rate and hence the behavior of the flux \( \dot{\phi}^2 \). The complete system to solve is given by eq.(21) above together with the dynamical equations

\[
\ddot{\phi} + 3H \dot{\phi} + V' = 0 \tag{22}
\]

and

\[
H^2 = \frac{8\pi}{3M_{pl}^2} [V(\phi) + g_{\text{pbh}}] \tag{23}
\]

This set of differential equations is very difficult to solve, even in the approximation \( g_{\text{pbh}} \ll V(\phi) \), which is relevant for this work. Bean and Magueijo analyzed one particular model of quintessence in which \( V(\phi) = \lambda \exp[\lambda \phi] \), implying \( \dot{\phi}^2 \propto t^{-2} \). Within this model, in which the quintessence flux around the SMBH decreases with time, they claim that the black holes grow in time. However, the decreasing quintessence flux around these objects casts doubts on this result, since the mass gain term decreases accordingly. We pointed out elsewhere that in the radiation-dominated era the growth of black holes is actually quenched when the background flux decreases with time as \( t^{-2} \) or faster (see Ref.[12]).

The question is whether there are PBHs that gain substantial mass at asymptotic times for a given potential \( V(\phi) \) which determines the quintessence flux. Let us show a class of solutions involving quintessence accretion only which can make the PBHs grow, as an example of this general behavior. The quintessence models satisfying

\[
\dot{\phi}^2 = M_{pl}^4 (t/t_*)^n \tag{24}
\]
constitute a class of interesting growing models (we used natural units and 
t_\ast = E_\ast^{-1} \text{ is a time constant}). For this choice, the kinetic energy \( \dot{\phi}^2 = K t^n \) 
with \( K = \frac{M_i}{2} E_\ast^n \) and \( E_\ast \) measures directly the kinetic contribution of the 
field.

Inserting into eq.(21) above, and solving for \( M(t) \) we obtain, assuming \( n > 0 \)

\[
M(t) = \frac{M_i}{1 - F(t, t_i)}
\]

where

\[
F(t, t_i) = \frac{27 \pi \eta t_\ast M_i}{2(n + 1)} \left[ (t/t_\ast)^{n+1} - (t_i/t_\ast)^{n+1} \right]
\]

It is easy to show that a set of solutions parametrized by the initial masses, 
time constants \( t_\ast \) and \( n \) exist where a huge growing of the seed black holes is 
possible, provided the constraint given by eq.(19) holds all the time. SMBHs 
will arise (from initially small PBHs) at a final time \( t_f \) if

\[
0 < 1 - F(t_f, t_i) \ll 1
\]

Thus, all those PBHs with initial masses of order

\[
M_i \sim (t_f E_\ast)^{-(n+1)} \left[ \frac{2(n + 1) E_\ast}{27 \pi \eta} \right]^{-1}
\]

would end with large masses \( (M \sim 10^6 M_\odot \text{ or bigger}) \) at final times \( t_f \gg t_i \).

Numerically, the mass is

\[
M_i \sim \left( \frac{n + 1}{\eta} \right) \left[ 10^{(24 + \frac{2}{n+1})} \right]^{-(n+1)} (t_f/s)^{-(n+1)} (E_\ast/GeV)^{-n} GeV
\]

where we have absorbed a coefficient \( O(1) \) into the efficiency \( \eta \). Initial 
 masses masses may be large only if the scale \( E_\ast \) is extremely small when 
measured in GeV if \( t_f \) is inside the radiation-dominated era, according to 
eq(29).

We may invert the reasoning above and assert that if we had some initial 
black hole formed with \( M_i \) at \( t_i \), then, the constant \( E_\ast \) need to be larger than
\[ E_* \geq 10^{-(26+24/n)} \left( \frac{t_f}{s} \right)^{-(1+1/n)} \left( \frac{(n+1)}{(\omega \eta M_i/GeV)} \right)^{1/n} GeV \equiv \Theta_1 \]  

(with \( \omega \eta \equiv (27 \pi \eta/2) \)) for the black hole to grow to the SMBH regime. Note that when \( E_* \) is larger, we need smaller initial masses in order to obtain larger SMBHs at the final time \( t >> t_f \), as expected.

Eq.(28) also says that our approximations to the actual physical accretion are valid if and only if

\[ M_i < \frac{2(n+1)}{27 \pi \eta} E_* (t E_* )^{-(n+1)} \]  

and the formulae above stay valid only if the parameter \( E_* \) does not change with time. The bottomline of eq.(29) is that if HB+GSL hold for all times, then seed PBHs can not have arbitrary initial masses (independently of the details of their formation) if they had to grow by accreting a quintessential field within the proposed class. Mutatis mutandis the same conclusions hold for other fuels for accretion, where we must use eq.(17) instead, in order to obtain supermassive black holes at the final time \( t_f \). Note that this constraint gets weaker with time because \( S_{hp} \propto t^2 \), and this constraint will become at some time weaker than the geometric causal condition \( r_g < R_{hp} \). The HB is quite restrictive for large masses at black hole formation, as discussed in Section 3. Considerations on the accretion before formation must be added to this picture (see next Section).

Finally, according to eq.(19), \( E_* \) must also satisfy

\[ E_* \leq 6.6 \times 10^{-2/n} 10^{-(25+24/n)} \left( \frac{t_f}{s} \right)^{-(1+1/n)} \left( \frac{(n+1)}{(M_i/GeV)} \right)^{1/n} GeV \equiv \Theta_2 \]  

Therefore, only the PBHs contained within the range defined by \( \Theta_1(M_i) < E_* < \Theta_2(M_i) \) will satisfy the HB and become SMBHs at late times \( t \) simultaneously. This leads to the with the following constraint on a positive \( n \)

\[ n < 0.65 \log(27 \pi \eta/2) \]  

A careful examination of the \( n < 0 \) case leads us to conclude that the lower limit thus obtained is irrelevant when compared to the \( n = -1 \) case already
discussed, that is, the index is actually limited by $-1$ from below. The case of a constant flux $n = 0$ can be also worked out without complications. We conclude that a window of indexes $n$ exist for quintessence to cause the growth of seed PBHs to the SMBH regime. Such a window is independent of $M_i$. Other physical effects may be important, for example, generally speaking, the depletion of the quintessence flux around the black hole can not be ignored for large masses, an effect that has to affect the parameter $\eta$.

**Causality and Holographic requirements**

As shown in the previous Section, when eqs.(28-30) are satisfied, the energy input by the accretion of quintessence would be enough to drive a black hole with initial mass $M_i$ to values $M_f > 10^6 M_\odot$.

This energy input strongly depends on the initial mass $M_i$ and needs to be very large if the black hole was initially very small. In addition, if the accretion rate is very high, the black hole that was initially below the HB will blow that bound at some point. Then, to keep these black holes below the HB they must also obey the constraints $\dot{S}_{smbh}(t) < \dot{S}_{hp}(t)$. However, according to the eq.(10), this inequality is equivalent to

$$\dot{M}(t) < \frac{3.4 h_0^2 M_{pl}^4}{M(t)} t$$

(34)

We solve the evolution from $t_i = 0$ until $t_f > t_i$ as above and then

$$M(t_f) < M_i \left(1 + \left(\frac{t_f}{\tau}\right)^2\right)^{1/2}$$

(35)

For $t_f \gg \tau = \frac{M_i h_0^{-1}}{2.6 M_{pl}^6} \sim 2.5 \times 10^{-6} s \left(\frac{M_i}{M_\odot}\right) h_0^{-1}$, eq.(35) becomes $M(t_f) < 1.84 h_0 M_{pl}^2 t_f$.

In order to enforce a strictly causal growth $\dot{r}_g < c$, the PBHs must obey also the condition $\dot{M} < 0.5 M_{pl}^2$ at all times. Then, solving for $M(t_f)$ we obtain

$$M(t_f) < M_i \left(1 + 8 \times 10^4 (t_f/s)(M_\odot/M_i)\right)$$

(36)

In other words, any black hole with initial mass satisfying the Holographic Bound at $t_i$ will eventually be superholographic at $t = t_f$ (that is, $S_{bh}(t_f) > S_{total}(t_f)$) unless causality holds.
Since that the causality requirement is very strong, we rule out super-
holographic black holes in normal circumstances of physical accretion. If we
impose that the solution given by eqs.(25-26) must satisfy the Holographic
Bound for the maximal rate of energy gain, then combining with eq.(35)

\[
\frac{M_i}{1 - F} < M_i \left(1 + \frac{(t_f/\tau)^2}{2}\right)^{1/2}
\]

(37)

Using eq.(37), we may obtain an upper bound for the cosmological time
at which our approximations must break down. Defining the dimensionless
quantities \( \theta_1 = (t_f/\tau) \), \( \theta_2 = (t_f/\zeta t_\ast) \) and \( \zeta^{n+1} = 2(n+1)/27\pi\eta \), eq.(37)
becomes

\[
\theta_1^2 - (2\theta_1^2 + 1)\theta_2^{n+1} + (\theta_1^2 + 1)\theta_2^{2(n+1)} > 0
\]

(38)

to be solved for a given \( M_i, \eta \) and \( n \) set of values.

We must acknowledge that, in principle, seed black holes may still reach
the SMBH regime without blowing the HB even if the condition \( \dot{S}_{smbh}(t) < \dot{S}_{hp}(t) \) is not satisfied, provided its growth effectively stopped while still below
the HB value. These models, however, must be analyzed in a one-by-one basis
to check their viability.

We close this Section with the observation that both the HB requirement
and the relativistic bound \( \dot{r}_g < c \) on \( \dot{M} \) lead to essentially the same value
\( \dot{M} < 2 \times 10^{38}g\text{s}^{-1} \) within a numerical factor of the order of one.

Conclusions
We have discussed a possible form to limit the formation times of pri-
mordial black holes formed directly or grown by accretion which may be
residing at the center of most galaxies as recently identified by a variety of
observations. The huge entropy contained in these SMBH allows to limit
their formation quite efficiently, since the total content of entropy of the
universe is likely to be bounded by the HB. Even if preliminar, our analysis
of the quintessence models for the growth of seed PBHs has been found to
leave room for their formation and further growth, although not for arbitrary
fluxes. The general argument developed in Ref. [12] against fast-growing so-
lutions for PBH growth with radiation flux \( \dot{g}_{rad}(t) \propto t^{-2} \) can be directly
applied to the the particular model involving quintessence flux \( \dot{\varphi}^2 \propto t^{-2} \).
Generally speaking, the quintessence model must allow the flux to decrease
slower than $t^{-2}$ for PBHs to grow at all, and to stay constant or increase for substantial accretion to occur, as needed for achieving the SMBH condition in a short time. General conditions on $\dot{M}$ have been obtained by a combination of causal and holographic arguments and are amenable of specific applications.

Other models can be constructed to produce a population of SMBHs starting from seed PBHs. For example, accretion in a brane-world high-energy phase has been recently studied [10] and shown to allow a substantial growth in which $\dot{M} \propto M/t$. It may be possible to arrive to the end of the high-energy phase with very massive black holes, although the full consequences of this scenarios are yet to be explored.

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Figure captions

Fig. 1. The window of indexes $n$ that allow a growth of small PBHs to the SMBH regime. As explained in the text, the upper bound is set by the HB requirement on $E_*$ (eq.32); while the lower bound $n = -1$ is imposed by the requirement of having enough flux of fuel to complete the process (Ref.12). The possible modes of growth include the constant quintessence flux case $n = 0$, detailed conditions to be satisfied by these particular power-law models are given in the text.
Figure 1