Quantum Rabi–Stark model: solutions and exotic energy spectra

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Abstract
The quantum Rabi–Stark model, where the linear dipole coupling and the nonlinear Stark-like coupling are present on an equal footing, are studied within the Bogoliubov operators approach. Transcendental functions responsible for the exact solutions are derived in a compact way, much simpler than previous ones obtained in the Bargmann representation. The zeros of transcendental functions reproduce completely the regular spectra. In terms of the explicit pole structure of these functions, two kinds of exceptional eigenvalues are obtained and distinguished in a transparent manner. Very interestingly, a first-order quantum phase transition indicated by level crossing of the ground state and the first excited state is induced by the positive nonlinear Stark-like coupling, which is however absent in any previous isotropic quantum Rabi models. When the absolute value of the nonlinear coupling strength is equal to twice the cavity frequency, this model can be reduced to an effective quantum harmonic oscillator, and solutions are then obtained analytically. The spectra collapse phenomenon is observed at a critical coupling, while below this critical coupling, infinite discrete spectra accumulate into a finite energy from below.

Keywords: Rabi–Stark model, analytic solutions, Bogoliubov operators approach

(Some figures may appear in colour only in the online journal)
1. Introduction

The quantum Rabi model (QRM), which represents the simplest interaction between a two-level atom (qubit) and a light field (cavity), continues to inspire exciting developments in many fields ranging from quantum optics, quantum information science, and condensed matter physics [1]. The Hamiltonian is given by

\[ H_R = \frac{\Delta}{2} \sigma_z + \omega a^\dagger a + g (a^\dagger + a) \sigma_x, \]  

(1)

where \( \Delta \) and \( \omega \) are frequencies of two-level system and cavity, \( \sigma_z, \sigma_x \) are usual Pauli matrices describing the two-level system, \( a (a^\dagger) \) is the annihilation (creation) bosonic operator of the cavity mode, and \( g \) is the coupling strength. In the conventional cavity quantum electrodynamics (QED) system [2], the coupling strength between the atom and the field is quite weak, \( g/\omega \sim 10^{-6} \). It can be described by the well-known Jaynes–Cummings model [3] where the rotating-wave approximation is made. Many physical phenomena can be described in this framework, such as collapse and revival of quantum state populations, vacuum Rabi splitting, and photon anti-bunching [4].

With the progress of the experimental techniques, the QRM can be implemented in enhanced parameter regimes. Some solid-state devices such as superconducting circuits [5–9], quantum wells [10], cold atoms [11] have emerged as genuine platforms for faithful representations of this model in the ultra-strong \( (g/\omega \sim 0.1) \), even deep-strong-coupling \( (g/\omega > 1) \) regime [12]. Evidence for the breakdown of the rotating-wave approximation has been provided in the qubit-oscillator system at ultra-strong coupling [5]. Many works then have been devoted to this system in the ultra-strong coupling regime [13–17]. Recently, the competition to increase the coupling strength is still on-going in different experimental systems [8, 9, 18, 19].

On the other hand, quantum simulations can engineer the interactions in a well-defined quantum system to implement the target model of interest in the infeasible parameter regime [20]. The engineered system even enables the generalization of the target model, thus more fundamental phenomena might emerge. The QRM with arbitrary parameters has been realised in quantum simulations based on Raman transitions in an optical cavity QED settings [21, 22]. In this proposed scheme [21], beside the linear dipole coupling, the following nonlinear coupling between atom and field can also emerge

\[ H_{NL} = \frac{U}{2} \sigma_x a^\dagger a, \]  

(2)

where the coupling strength \( U \) is determined by the dispersive energy shift. It is associated with the dynamical Stark shift discussed in the quantum optics [23], so this generalized model proposed by Grimsmo and Parkins is called quantum Rabi–Stark model [24]. This emergent Stark-like nonlinear interaction has no parallel in the conventional cavity QED, which adds a new member to the list of various quantum Rabi models.

Any modification to the linear QRM described by Hamiltonian (1) would possibly bring about the novel and exotic physical properties. The interaction-induced energy spectral collapse can be observed in the two-photon QRM when the normalized coupling approaches the half of cavity frequency [25–27]. The anisotropic QRM, where the coupling strength of the rotating-wave terms and counter-rotating wave terms is different, exhibits the first-order phase transitions [28]. These phenomena are obviously absent in the original linear isotropic QRM [1]. Grimsmo and Parkins conjecture that the nonlinear coupling manipulated by the dispersive energy shift would possibly induce a new superradiant phase at this single atom level if \( U < -2\omega \) [21]. Although the total Hamiltonian \( H_0 = H_R + H_{NL} \) has been studied by
the Bargmann approach [24, 29], not much attention has yet been paid to its possible novel and peculiar physical properties, to the best of our knowledge.

Analytical solutions to the linear QRM have been searched for a few decades (for a review, please refer to [30–32]). Many approximate analytical solutions have been proposed [33–44], however analytically exact solution was only found by Braak [45] using the Bargmann representations. It was shown that Braak’s solution can be constructed in terms of the mathematically well-defined Heun confluent function [46]. By Bogoliubov operators approach (BOA) [47], Braak’s solution was straightforwardly reproduced in a more transparent manner. One clear advantage is that for a discussion of the BOA, it is not required to refer to heavy mathematical terminology. It is generally accepted in the literature that the BOA is more physical [1, 48]. Moreover, BOA can be easily extended to the two-photon QRM [47], and solutions in terms of a $G$-function, which shares the common pole structure with Braak’s $G$-function for the one-photon QRM, are also found. It was demonstrated later in [26] that this two-photon $G$-function by BOA [47] allows for the desired understanding of the qualitative features of the collapse. However, the $G$-function by the direct application of the Bargmann space approach [49] has no pole structure, and thus could not give qualitative insight into the behavior of the spectral collapse [26]. To the best of our knowledge, the $G$-function with its pole structure for the two-photon QRM has only been found using the BOA and, in particular, has so far not been derived using the Bargmann space method. So in the study of the anisotropic two-photon QRM, only the BOA is employed [50]. In principle, the Bargmann space approach could still be used to recover the correct $G$-function in the two-photon QRM, which might require more mathematics.

In this work, we will study the quantum Rabi–Stark model by the BOA, and then explore some exotic physical phenomena. The paper is structured as follows: in section 2, a concise $G$-function is derived for this model by using BOA. In section 3, two kinds of exceptional solutions are obtained explicitly in terms of the pole structure of the obtained transcendental function. First-order phase transitions are then analytically detected. The energy spectral collapse is discussed by an effective one-body Hamiltonian corresponding to a quantum harmonic oscillator in section 4. The last section contains some concluding remarks and outlooks.

2. Bogoliubov operators approach and $G$-function

To facilitate the study, the Hamiltonian $H_0 = H_R + H_{NL}$ is rotated around the y-axis by an angle $\pi/2$

$$H = -\frac{1}{2} \left( \Delta + U a^\dagger a \right) \sigma_z + \omega a^\dagger a + g (a^\dagger + a) \sigma_z. \tag{3}$$

In terms of two eigenstates of $\sigma_z$, the above Hamiltonian takes the following matrix form in units of $\omega = 1$

$$H = \begin{pmatrix} a^\dagger a + g (a^\dagger + a) & -\frac{1}{2} \left( \Delta + U a^\dagger a \right) \\ -\frac{1}{2} \left( \Delta + U a^\dagger a \right) & a^\dagger a - g (a^\dagger + a) \end{pmatrix}. \tag{4}$$

Associated with this Hamiltonian is the conserved parity $\Pi = \exp \left( i \pi \hat{N} \right)$ where $\hat{N} = (1 - \sigma_z) / 2 + a^\dagger a$ is the total excitation number, such that $[\Pi, H] = 0$. $\Pi$ has two eigenvalues $\pm 1$, depending on whether $\hat{N}$ is even or odd.

We first perform the Bogoliubov transformation with displacement $w$

$$A = a + w, \tag{5}$$
where \(A\) is the new bosonic operator which obeys the commutation relation \([A, A^\dagger] = 1\), the shift \(w\) will be determined later. The transformed Hamiltonian then reads

\[
H = \begin{pmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{pmatrix},
\]

(6)

where

\[
H_{11} = A^\dagger A + (g - w) (A^\dagger A) + w^2 - 2gw,
\]

\[
H_{12} = H_{21} = -\frac{\Delta}{2} - \frac{U}{2} [A^\dagger A - w (A^\dagger A) + w^2],
\]

\[
H_{22} = A^\dagger A - (g + w) (A^\dagger A) + w^2 + 2gw.
\]

The wavefunction can be expanded in terms of the \(A\)-operators

\[
|\rangle_A = \left( \frac{\sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A}{\sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A} \right),
\]

(7)

where \(e_n\) and \(f_n\) are the expansion coefficients. \(|n\rangle_A\) is called extended coherent state [51] with the following properties

\[
|n\rangle_A = (a^\dagger + w)^n |0\rangle_A,
\]

\[
|0\rangle_A = e^{-\frac{1}{2}w^2} |0\rangle_u,
\]

(8)

where the vacuum state \(|0\rangle_A\) in Bogoliubov operators \(A\) is just well-defined as the eigenstate of one-photon annihilation operator \(a\), and known as pure coherent state.

Projecting both sides of the Schrödinger equation onto \(|m\rangle\) gives

\[
(\Gamma_m - E - 2gw) e_m + (g - w) \Lambda_m - \left( \frac{\Delta}{2} + \frac{U}{2} \Gamma_m \right) f_m + \frac{U}{2} w f_m = 0,
\]

(9)

\[
- \left( \frac{\Delta}{2} + \frac{U}{2} \Gamma_m \right) e_m + \frac{U}{2} w \Lambda_m + (\Gamma_m - E + 2gw) f_m - (g + w) \Gamma_m = 0,
\]

(10)

where

\[
\Lambda_m = (m + 1) e_{m+1} + e_{m-1},
\]

\[
\Gamma_m = (m + 1) f_{m+1} + f_{m-1},
\]

\[
\Gamma_m = m + w^2.
\]

To get one-to-one correspondence of \(e_m\) and \(f_m\), one should cancel the terms involving \(\Lambda_m\) and \(\Gamma_m\), which requires the shift \(w\) to be

\[
w = \frac{g}{\sqrt{1 - U^2/4}}.
\]

(11)

It is just equal to the value of the singularity in [24]. Then we have

\[
e_m = \Omega_m f_m,
\]

(12)

where
\[ \Omega_m = \frac{U_{\nu \pi}}{2(g + w)} \left( \Gamma_m - E + 2gw \right) - \left( \Delta + U \Gamma_m \right) - 2 \left( \Gamma_m - E - 2gw \right). \]  

(13)

Inserting equation (12) into equation (9), we obtained a three-term recurrence relation for \( f_m \)

\[
f_m = \frac{\Delta + U \Gamma_{m-1} - 2 \left( \Gamma_{m-1} - E - 2gw \right) \Omega_{m-1}}{m \left[ Uw + 2 (g - w) \Omega_m \right]} f_{m-1} 
- \frac{2 (g - w) \Omega_{m-2} + Uw}{m \left[ Uw + 2 (g - w) \Omega_m \right]} f_{m-2}
\]  

(14)

where all \( f_m \) can be obtained if set \( f_0 = 1 \).

By the opposite shift \((-w)\), we can define another Bogoliubov operator

\[ B = a - w, \]  

(15)

the wavefunction can also be expanded in the \( B \)-basis as

\[ |n\rangle_B = \left( \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f_n |n\rangle_A \right), \]  

(16)

due to the parity symmetry. \( |n\rangle_B \) is defined similar to \( |n\rangle_A \).

Assuming both wavefunctions (7) and (16) are the true eigenfunction for a nondegenerate eigenstate with eigenvalue \( E \), they should be proportional with each other, i.e. \( |\rangle_A = r |\rangle_B \).

where \( r \) is a complex constant. Projecting both sides of this identity onto the original vacuum state \( a \langle 0 | \), we have

\[
\sum_{n=0}^{\infty} \sqrt{n!} e_n a \langle 0 | n\rangle_A = r \sum_{n=0}^{\infty} \sqrt{n!} (-1)^n f_n a \langle 0 | n\rangle_B, \\
\sum_{n=0}^{\infty} \sqrt{n!} f_n a \langle 0 | n\rangle_A = r \sum_{n=0}^{\infty} \sqrt{n!} (-1)^n e_n a \langle 0 | n\rangle_B, 
\]

where

\[
\sqrt{n!} a \langle 0 | n\rangle_A = (-1)^n \sqrt{n!} a \langle 0 | n\rangle_B = e^{-w^2/2} w^n. 
\]

Eliminating the ratio constant \( r \) gives

\[
\left( \sum_{n=0}^{\infty} e_n w^n \right)^2 = \left( \sum_{n=0}^{\infty} f_n w^n \right)^2. 
\]

Immediately, we obtain the following well-defined transcendental function, so called \( G \)-function

\[
G_{\mp} (E) = \sum_{n=0}^{\infty} \left( \Omega_n f_n \pm f_n \right) w^n = 0, 
\]  

(17)

where \( \Omega_n \) and \( f_n \) can be obtained from equations (13) and (14), \( \mp \) corresponds to negative(positive) parity. The zeros of this \( G \)-function will give the regular spectrum, which should be the same as those in [24]. In principle, there are many \( G \)-functions, all have the same zeros and yield the same spectrum [52]. Note also that this \( G \)-function can be reduced to that of the original QRM [45] if set \( U = 0 \).
\[ G \]-curves for \( \Delta = 0.5, U = \pm 1, g = 0.1 \) and 0.7 are plotted in figure 1. The zeros are easily detected, and then regular energy spectra are obtained, which are exhibited in figure 2. As usual, one can check it easily with numerics, an excellent agreement can be achieved.

3. Exceptional solutions and first-order phase transitions

From equation (11), we can note that the present solution by BOA can be only applied to the Rabi–Stark model for \( |U| < 2 \). Let us now discuss novel features of the derived \( G \)-functions and the exceptional spectra.

3.1. Pole structure

We first examine the pole structure of the \( G \)-function (17). Note from equation (14) that the denominator of \( f_n \) for \( n > 0 \) vanishes, yielding the \( n \)th pole of the \( G \)-function

\[ E_{n}^{\text{pole}} = \left( 1 - \frac{U^2}{4} \right) n - \frac{U \Delta}{4} - g^2. \tag{18} \]

It is interesting to find that this pole is reduced to \( E_{n}^{\text{QRM}} = n - g^2 \), the pole of the pure QRM [45], if set \( U = 0 \).

From equation (13), one can find that \( \Omega_n \) diverges at

\[ E_{\Omega n} = \frac{E_{n}^{\text{pole}}}{\sqrt{1 - U^2/4}} + \frac{\Delta U}{4 - U^2 + 4 \sqrt{1 - U^2/4}}. \tag{19} \]

However it is not the pole of the \( G \)-function, because \( \Omega_n f_n \) appears always as a whole in the \( G \)-function (17) and is finite at \( E = E_{\Omega n} \) for \( n \neq 0 \).

Equation (18) is not suited to \( n = 0 \), because \( f_0 = 1 \). Particularly, the first term in the \( G \)-function (17), \( \Omega_0 \pm 1 \), really diverges at

\[ E_{0}^{\text{pole}} = -\frac{g^2 + \Delta U/4}{\sqrt{1 - U^2/4}} + \frac{\Delta U}{4 - U^2 + 4 \sqrt{1 - U^2/4}}, \tag{20} \]

which is just the zeroth pole of the \( G \)-function.

The poles given in equations (18) and (20) are marked with vertical lines in the \( G \)-curves of figure 1. The \( G \)-curves indeed cannot pass through these poles, therefore the whole \( G \)-curves are blocked into different smooth segments.

3.2. Exceptional solutions

3.2.1. Juddian solutions for doubly degenerate states. If the true physical system takes the energy at the zeroth pole \( E_{0}^{\text{pole}} \), the wavefunction (7) including \( e_0 = \Omega_0 f_0 \) terms should be analytic. Hence both the denominator and numerator of \( \Omega_0 \) should vanish at the same time, yielding the constrained condition for the model parameter

\[ g_c = \sqrt{\frac{(1 - U^2/4)}{U}} \Delta. \tag{21} \]

Inserting equation (21) into equation (20) gives the energy without specified parity \( E_{0}^{\text{cross}} = -\frac{\Delta}{U} \). The first energy levels for both parities thus intersect at \( g_c \). These are the doubly degenerate states, corresponding to the Juddian solution [53].
Physically, the energies for the ground-state and the first excited state cross, indicating a first-order quantum phase transition. According to equation (21), note that the qubit frequency $\Delta$ is always positive, so the finite real $g_c$ only exists for $U > 0$. No first-order phase transition exists in the present model for $U < 0$ and the linear QRM where $U = 0$. As shown in the upper panels of figure 2, two levels for the first excited state and ground state really cross once for $U > 0$. Such a crossing for the two lowest levels does not occur for $U < 0$, as shown in the lower panels.

Actually, for any $n$, if both denominator and numerator of $\Omega_n$ in equation (13) vanish, $\Omega_n$ is analytic, leading to analytic coefficients $e_n$ and $f_n$. The reason is the following. $\Omega_n = x_n/y_n$ is analytic for both $x_n = 0$ and $y_n = 0$. The denominator of $f_n$ in equation (14) is

$$Uw + 2(g - w)\Omega_n = \frac{Uwy_n + 2(g - w)x_n}{y_n},$$

we can easily find that both denominator and numerator of $f_n$ should be also zero, leading to analytic coefficients $f_n$, and therefore analytic coefficients $e_n$.

The condition that both the denominator and numerator of $\Omega_n$ in equation (13) vanish will give the coupling strength

$$g_c^{(n)} = \sqrt{\left(\frac{n + \Delta}{U}\right)\left(1 - \frac{U^2}{4}\right)}.$$  \hspace{2cm} (22)
The corresponding energy \( E_n^{\text{cross}} = -\Delta \), which is surprisingly independent of the coupling strength!

Similarly, the parity is not well defined at this energy. It is just the crossing point corresponding to doubly degenerate states. Because \( f_n \) is also analytic at this point, the pole curves (18) should also pass through these crossing points. As demonstrated in figure 2 that all these crossing points for different \( n \) (blue squares) just situate on a horizontal line \( E = -\Delta \) in the energy spectra. They are usually the last crossing points for each pair of levels with positive and negative parity and somehow hardly discerned without analytical reasonings.

Interestingly we can give a lower bound for number of states below \( E = -\Delta \) for given \( g \) by counting the level crossing points. According to equation (22), we get maximum number \( n_{\text{max}} \) for \( g^{(a)} < g \),

\[
    n_{\text{max}} = \left\lceil \frac{g^2}{(1 - \frac{U^2}{4})} - \frac{\Delta}{U} \right\rceil.
\]

where the bracket [...] denotes the Gaussian step function. There are \( n_{\text{max}} + 1 \) level crossings at the same energy \( -\Delta \) in the coupling regime \([0, g]\), as shown in figure 2. Note that those levels pass through \( n_{\text{max}} + 1 \) crossing points will lie below \( E_n^{\text{cross}} = -\Delta \) at \( g \). Then for given \( g \), we find at least \( 2(n_{\text{max}} + 1) \) states below \( -\Delta \). In the limit \( U \to \pm 2 \), \( n_{\text{max}} \to \infty \) by

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Figure 2. Energy spectra for positive (upper panels) and negative (lower panels) \( U \) where \( \Delta = 0.5 \). Red (negative) and black (positive) denote different parity. The green dashed line is \( E_n^{\text{pole}} \) and the blue dashed lines are \( E_n^{\text{pole}} \) for \( n = 1, 2, 3, 4 \). Level crossings are marked by filled symbols. First-order phase transitions are present (absent) for positive (negative) \( U \). Horizontal blue dotted lines \( E = -\Delta/U \) are guides to the eye. Exceptional solutions for nondegenerate states are not given here.

The corresponding energy \( E_n^{\text{cross}} = -\Delta \), which is surprisingly independent of the coupling strength!

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equation (23). So at $U = \pm 2$, there are possibly an infinite number of levels below or equal to $\mp \Delta/2$ for any $g$.

All other Juddian solutions for doubly-degenerate states can be figured out in terms of the $n > 0$ pole energy $E_{n}^{\text{pole}}$ (18). It is required that the numerator of $f_{n}$ vanishes. For example, for $n = 1$ pole,

$$ E_{1}^{\text{pole}} = \left(1 - \frac{U^2}{4}\right) - \frac{U \Delta}{4} - g^2. $$

$\Omega_{0}$ and $\Omega_{1}$ can be obtained through equation (13), and substitution to equation (14) yields

$$ f_{1} = \frac{2(1 + E - w^2 + 2gw) \Omega_{0} + \Delta + (w^2 - 1) U}{n[wU + 2(g - w) \Omega_{1}]} f_{0}. $$

It requires numerator to be zero, i.e.

$$ 2 \left(1 + E_{1}^{\text{pole}} - w^2 + 2gw\right) \Omega_{0} + \Delta + (w^2 - 1) U = 0, $$

this is just the constrained condition. Therefore we can obtain several $g$ for $n = 1$ pole curve in the energy spectra for fixed $\Delta$ and $U$.

The constrained condition becomes more complicated with larger $n$, but in principle can be obtained. Proceeding along this line, we can predict the values of $g$ for fixed $\Delta, U$ in the spectra for any $n > 0$. By the way, the largest $g$ for the crossing points obtained in this way should be the same as that in equation (22), as stated before. These predicted values coincide with the level crossing marked by blue filled circles and squares, as exhibited in figure 2.

So level crossings only happen in the pole curves. We want to point out that, except the level crossing points situating on the pole curves described by equations (18) and (20), there are no other true level crossings, no matter how close they are.

### 3.2.2. Exceptional solution for nondegenerate states

As in the original QRM, it is possible that the $m$th pole line can cross the energy levels away from the level crossings, leading to exceptional solutions for the non-degenerate state [29]. With fixed $m(m \geq 1)$, let $f_{n < m} = 0$ and $f_{n = m} = 1$, the nondegenerate exceptional $G$-function in $\Delta - g$ space is defined as [54]

$$ G_{m}^{\text{exc}}(g) = \sum_{n=m}^{\infty} (\Omega_{n} \pm 1) f_{n} w^{n} = 0. $$

(24)

where the energy is limited to $E = E_{m}^{\text{pole}}$ by equation (18). The zeros of exceptional $G$-function (24) will give the coupling strength.

Particularly, for the zeroth pole, see equation (20), the summation in the $G$-function is then the same as in equation (17). Note however that $\Omega_{0}$ in the first term diverges if using $E_{0}^{\text{pole}}$. In this case, we can start with $e_{0} = 1$ in recurrence relation instead, so all coefficients are well defined, including $f_{0} = 0$ due to $1/\Omega_{0} = 0$. The nondegenerate exceptional $G$-function is thus

$$ G_{0}^{\text{exc}}(g) = \sum_{n=0}^{\infty} (1 \pm 1/\Omega_{n}) e_{n} w^{n} = 0. $$

(25)

We exhibit the nondegenerate $G$-function as a function of $g$ for $\Delta = 1, U = 1.9$ in figures 3(a)–(d) for $m = 0, 1, 2, 3$. The zeros give the coupling strength where the non-degenerate exceptional solution occurs, which are marked with green circles. To show the precise location of these nondegenerate exceptional solutions, we also calculate the energy spectra with the
The same parameters, which is displayed in figure 3(e). The green circles for \( m = 0, 1, 2, 3 \) are also marked, which are just the zeros exhibited in figures 3(a)–(d).

So far, the energy spectra of the quantum Rabi–Stark model for the regular type and two kinds of exceptional ones are completely obtained.

4. Spectral accumulation and collapse at \( U = \pm 2 \)

Note that the present \( G \)-function (17) is not valid at \( U = \pm 2 \), because \( w \) in equation (13) diverges. The spectral phenomena at \( U = \pm 2 \) should be studied in another way. Here we present our analysis for \( U = 2 \) by a new approach in more detail. For the case of \( U = -2 \), the extension is achieved straightforwardly by changing \( \Delta \) into \(-\Delta\).

The bosonic components of Hamiltonian can be expressed in terms of the effective position and momentum operators of a particle of mass \( m \), defined as

\[
x = \sqrt{\frac{1}{2m\omega}} (a^\dagger + a), \quad p = i\sqrt{\frac{m\omega}{2}} (a^\dagger - a),
\]

for simplicity we can set \( m\omega = 1 \). In terms of two eigenstates of \( \sigma_z \), the Hamiltonian \( H_0 = H_K + H_{NL} \) in the matrix form then takes

\[
H_0 = \begin{pmatrix}
p^2 + x^2 - 1 + \frac{\Delta}{2} & g\sqrt{2}x \\
g\sqrt{2}x & -\frac{\Delta}{2}
\end{pmatrix}.
\]
Suppose the wavefunction is $\Psi = (\Psi_1, \Psi_2)^T$, we have two coupled Schrödinger equations

\[
\begin{align*}
(p^2 + x^2 - 1 + \frac{\Delta}{2}) \Psi_1 + g\sqrt{2}x \Psi_2 &= E \Psi_1, \\
-\frac{\Delta}{2} \Psi_2 + g\sqrt{2}x \Psi_1 &= E \Psi_2.
\end{align*}
\]

Inserting $\Psi_2 = \frac{g\sqrt{2}x}{E + \frac{\Delta}{2}} \Psi_1$ to the first equation results in the effective one-body Hamiltonian for $\Psi_1$,

\[
H_{\text{eff}} \Psi_1 = \left( E + 1 - \frac{\Delta}{2} \right) \Psi_1,
\]

where

\[
H_{\text{eff}} = 2 \left( \frac{p^2}{2} + \frac{1}{2} \omega_{\text{eff}}^2 x^2 \right),
\]

with

\[
\omega_{\text{eff}} = \sqrt{1 + \frac{2g^2}{\frac{\Delta}{2} + E}}.
\]

One can easily find the eigenvalues of this quantum harmonic oscillator

\[
E + 1 - \frac{\Delta}{2} = 2\omega_{\text{eff}} \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots \infty. \tag{29}
\]

To have the real harmonic frequency, $1 + \frac{2g^2}{\frac{\Delta}{2} + E}$ should be positive, which results in $E > -\frac{\Delta}{2}$ or $E < -\frac{\Delta}{2} - 2g^2$. For $E > -\frac{\Delta}{2}$, we have the equation for the energy

\[
\frac{\sqrt{E + \frac{\Delta}{2} (E + 1 - \frac{\Delta}{2})}}{\sqrt{\frac{\Delta}{2} + E + 2g^2}} = 2n + 1, \quad n = 0, 1, 2, \ldots \infty, \tag{30}
\]

while for $E < -\frac{\Delta}{2} - 2g^2$, we have another equation for the energy

\[
\frac{\sqrt{- \left( \frac{\Delta}{2} + E \right) (E + 1 - \frac{\Delta}{2})}}{\sqrt{- \left( \frac{\Delta}{2} + E + 2g^2 \right)}} = 2n + 1, \quad n = 0, 1, 2, \ldots \infty. \tag{31}
\]

These two equations are exactly the same as equations (39)–(41) [55] by Maciejewski et al. Our solution based on a harmonic oscillator is much simpler. It must be related to the fact that Maciejewski et al uses the Bargmann space and transform the equations first into the so-called Birkhoff form, apparently creating unnecessary complications. They obtain also Hermite polynomials for the eigenfunctions in Bargmann space, but we would say that the Hermite polynomials are the wave-functions in the ordinary Hilbert space because the system is just a harmonic oscillator with shifted frequency.

From equation (31), we can see that an infinite number of discrete energy levels is confined in the energy interval

\[
\frac{\Delta}{2} - 1 < E < -\frac{\Delta}{2} - 2g^2, \tag{32}
\]
if \( g < \sqrt{\frac{\Delta}{2}} \). For convenience, we denote \( E_c^+ = -\Delta/2 - 2g^2 \) and \( g_c^+ = \sqrt{(1 - \Delta)/2} \). The effective potential becomes flat if \( \omega_{\text{eff}} = 0 \), i.e. \( E = E_c^+ \). In this case, there are qubit states which turn the potential flat \([26, 56]\), and the spectrum collapses, like for a free particle. The infinite discrete energy levels in the low energy region for \( g < g_c^+ \) would collapse to \( E_c^+ \) for \( g = g_c^+ \).

For \( g > g_c^+ \), from equations (31) and (32), we know that no real solutions exist in this case. Then we have to resort to numerics. In figure 4, we exhibit the first several energy levels for \( U = 2, \Delta = 0.5 \) with different truncation of the Fock space by numerical exact diagonalization. \( g_c^+ = 0.5 \) in this case. All energies for \( g > g_c^+ \) become closer to \( E_c^+ \) monotonously with increasing truncated photonic number \( N_n \), although the convergence is hardly achieved by numerics. It is observed that \( E_c^+ \) is a lower bound in the regime of \( g > g_c^+ \). In the two-photon QRM \([26, 27]\), it can be easily checked that the energy in numerical diagonalization has no lower bound when coupling strength is larger than the half of cavity frequency. Although both models have a common feature of spectral collapse at a critical coupling, they display essentially different behaviour above the critical coupling.

While for \( g < g_c^+ \), one can see from the left-hand-side of each plot in figure 4 that the converging energies for low excited states in the present model are easily obtained numerically, which can be also confirmed by the solution to equation (31). However, it is extremely difficult to obtain the converging energy level by direct exact diagonalizations when energy approaches to \( E_c^+ \). Close to \( E_c^+ \), there is a quasi-continuum of states with an infinite number of discrete states.

We can analyze the average photonic number \( N \) in each eigenstates. According to the effective harmonic oscillator (28), we have the wavefunction in the \( n \)th energy level

\[
\Psi_n \propto \left( \frac{1}{g\sqrt{2\pi g_c^+}} \right) H_n \left( \omega_{\text{eff}} \right),
\]

where \( H_n \) is the Hermite polynomial of degree \( n \). Then we can calculate \( N \) straightforwardly, which is however very tedious and not shown here. When \( E \to E_c^+ \), \( N \) is approximately equal to

\[
N \approx g^2 + \frac{3g^2 (\Delta + 2g^2 - 1)}{4 (E_n - E_c^+)} + \frac{3}{8 (1 - \Delta - 2g^2)},
\]

where the use has been made of equations (26) and (29).

One can find from equation (34) that \( N \) diverges if energy level approaches to \( E_c^+ \). So it is almost impossible to use numerical exact diagonalization to calculate correctly the energy level if very close to \( E_c^+ \). If the numerical truncation of the Fock space is below \( N \), the results depend naturally on the truncation. This is a case where only an analytical treatment can give the correct answer.

The high energy levels for \( E > -\Delta/2 \) for arbitrary coupling \( g \) can be easily obtained by equation (30) analytically. Our hypothesis for the exotic energy distribution for \( E < -\Delta/2 \) at \( U = 2 \) is the following. For \( g < g_c^+ \), by equation (31), we know that infinitely many discrete levels lie below accumulation point \( E_c^+ \). Close to \( E_c^+ \), there is a quasicontinuum of states. All states are normalizable. They collapse to \( E_c^+ \) right at \( g_c^+ \). When \( g > g_c^+ \) all these energy levels could only stay in energy interval \( E_c^+ \leq E < -\frac{\Delta}{2} \), but absolutely cannot be given by equation (31). The corresponding states should be unnormalizable. In this energy interval, it is unclear whether there is a continuum of non-normalizable states, or this region is empty, which remains an open question. Obviously this issue could not be addressed by any numerics.
5. Conclusion

In this work, we have derived the $G$-function for the quantum Rabi–Stark model in a compact way by using the BOA. Zeros of the $G$-function determine the regular spectrum. Two kinds of exceptional solutions are clarified and demonstrated. For the Juddian-type solution, the true level crossing occurs at the doubly degenerate states with both parities, which exclude the previous ‘crossing’ from the same parity. The first-order phase transition is detected analytically by the pole structure of $G$-functions. The critical coupling strength of the phase transitions is obtained analytically. The exotic energy spectra at $U = \pm 2$ are analyzed within an effective quantum harmonic oscillator. Previous energy spectra by very complicated and cumbersome derivation can be very easily reproduced. Moreover, the energy spectral collapse can be attributed by the flat quadratic potential. Below the collapse critical coupling, there are infinite discrete energy levels below the collapse energy.

Both the first-order quantum phase transition and the spectral collapse can occur in the present model, and are lacking in the linear QRM. The spectral collapse also occurs in the two-photon QRM with another kind of nonlinear coupling, but the first-order quantum phase transition is absent. Spectral collapse does not occur in the anisotropic QRM where the first-order phase transitions can be induced by the anisotropy with respect to the rotating-wave and non-rotating-wave coupling strengths. It follows that the Stark-like nonlinear coupling between atom and cavity is of fundamental importance. We believe that the quantum Rabi–Stark model would exhibit various fundamental phenomena found in the various QRMs, and could even go beyond. The spectral collapse and the discrete levels below the collapse energy might be qualitatively understood in the polaron picture by the tunneling induced potential well [57]. Due to the parity symmetry, the second-order phase transition in the present model should also occur in the limit $\Delta/\omega \to \infty$, like that in the linear QRM [58–61]. We speculate that the present model would possibly experience true superradiance transition in the single-atom model at moderate frequency ration $\Delta/\omega$. Other peculiarities and novel properties in quantum Rabi–Stark model are also worthy of further explorations. The well understanding

Figure 4: The differences of the first several energy levels and $E_n^+ = -\Delta/2 - 2g^2$, i.e. $E_n + \Delta/2 + 2g^2$, as a function of $g$ by numerical exact diagonalizations with the truncation number $N_{tr} = 500$ (left), $1000$ (middle), and $2000$ (right) for $U = 2$, $\Delta = 0.5$, and accordingly $g^+_c = 0.5$. The red horizontal line corresponds to the energy value $E_n^+$. and the above analytical theory in the framework of a harmonic oscillator, therefore other rigorous study is called for.
of the closed system will lay the solid foundation for further treatment of the open quantum system [62].

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