ON PRIMES REPRESENTED BY QUADRATIC POLYNOMIALS

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Abstract. This is a survey article on the Hardy-Littlewood conjecture about primes in quadratic progressions. We recount the history and quote some results approximating this hitherto unresolved conjecture.

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1. The Conjecture

It is attributed to Dirichlet that any linear polynomial with integer coefficients represents infinitely many primes provided the coefficients are co-prime. The next natural step seems to be establishing a similar statement for quadratic polynomials. G. H. Hardy and J. E. Littlewood [20] gave the following conjecture in 1922 based on their circle method.

Conjecture. Suppose $a$, $b$ and $c$ are integers with $a > 0$, $\gcd(a, b, c) = 1$, $a + b$ and $c$ are not both even, and $D = b^2 - 4ac$ is not a square. Let $P_f(x)$ be the number of primes $p \leq x$ of the form $p = f(n) = an^2 + bn + c$ with $n \in \mathbb{Z}$. Then

$$P_f(x) \sim \gcd(2, a + b) \frac{\zeta(D)}{\sqrt{a}} \frac{\sqrt{x}}{\log x} \prod_{p \mid a, p > 2} \frac{p}{p - 1},$$

where

$$\zeta(D) = \prod_{p \mid a, p > 2} \left(1 - \frac{\left(\frac{D}{p}\right)}{p - 1}\right).$$

Here and after, $\left(\frac{D}{p}\right)$ denotes the Legendre symbol, i.e. its value is 1 if $D$ is a quadratic residue modulo $p$, $-1$ if $D$ is a quadratic non-residue modulo $p$ and 0 if $p$ divides $D$.

The conjecture has thus far resisted attack to the extent that its simplest case for the polynomial $n^2 + 1$ is not even resolved. Indeed, no polynomial of degree two or higher is known to represent infinitely many primes.

In a related problem, L. Euler and A.-M. Legendre were the first to observe that $n^2 + n + 41$ is prime for all $0 \leq n \leq 39$. G. Rabinowitsch [36] showed that $n^2 + n + A$ is prime for $0 \leq n \leq A - 2$ if and only if $4A - 1$ is square-free and the ring of integers of the number field $\mathbb{Q}(\sqrt{1 - 4A})$ has class number one. This question was further studied by A. Granville and R. A. Mollin in [19] and the works, particularly those of Mollin, referred to therein. It is most note-worthy that an upper bound for $P_f(x)$ of the order of magnitude predicted by [19] was proved in [19] unconditionally uniform in $f$, and uniform in $x$ under the Riemann hypothesis for the Dirichlet $L$-function $L(s, (D/\cdot))$. Furthermore, it was shown unconditionally in [19] that for large $R$ and $N$ with $R^2 < N < \sqrt{R},$

$$\# \{n \leq N : n^2 + n + A \in \mathbb{P}\} \asymp L \left(1, \left(\frac{1 - 4A}{\cdot}\right)^{-1} \frac{N}{\log N}\right).$$

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holds for a positive proportion of integers \( A \) in the range \( R < A < 2R \). They also proved in [19] that an asymptotic formula for \( P_f(x) \), with \( f \) belonging to certain families of quadratic polynomials, holds for \( x \) in some range under the assumption of the existence of a Siegel zero for the relevant Dirichlet \( L \)-function. The methods used come from a paper of J. B. Friedlander and A. Granville [16] in the study of irregularities in the distributions of primes represented by polynomials. The ideas in [16] originated from the work of H. Maier [31] on irregularities of the distribution of primes in short intervals.

It is noteworthy that certain cases of the asymptotics in (1.1) would follow from a part of another unsolved conjecture due to S. Lang and H. Trotter [29] regarding elliptic curves. To explain the contents of this conjecture, we need some further notation. Let \( E \) be be an elliptic curve over \( \mathbb{Q} \). If \( E \) has good reduction at a prime \( p \) (that is, the reduced curve \( E_p \) modulo \( p \) is non-singular), then a well-known theorem of H. Hasse states that the number of points on \( E_p \) differs from \( p + 1 \) by an integer \( \lambda_E(p) \) (the trace of the Frobenius morphism of \( E/\mathbb{F}_p \)) satisfying the bound \( |\lambda_E(p)| \leq 2\sqrt{p} \). The Lang-Trotter conjecture predicts an asymptotic formula for the number of primes \( p \leq x \) such that \( \lambda_E(p) = r \) for some unspecified positive constant \( r \). If \( E \) has “complex multiplication” and \( r \neq 0 \), then the primes \( p \) satisfying \( \lambda_E(p) = r \) lie in quadratic progression. Therefore the Lang-Trotter conjecture is related to the Hardy-Littlewood conjecture stated above. For example, consider the elliptic curve \( E : y^2 = x^3 - x \) whose endomorphism ring is isomorphic to \( \mathbb{Z}[i] \). It turns out that \( p = n^2 + 1 \) for some integer \( n \) if and only if \( \lambda_E(p) = \pm 2 \). See for example [29] for the details.

Conjectures similar to (1.1) also exist for polynomials of higher degree. Hypothesis H of A. Schinzel and W. Sierpiński [37] gives that if \( f \) is an irreducible polynomial with integer coefficients that is not congruent to zero modulo any prime, then \( f \) is prime for infinitely many integers \( n \). P. T. Bateman and R. A. Horn [10] gave a more explicit version, with an asymptotic formula, of the last-mentioned conjecture.

The following notations and conventions are used throughout paper.

\( \mathbb{Z}, \mathbb{N}, \mathbb{P} \) and \( \mathbb{Q} \) denote the sets of integers, natural numbers, primes and rational numbers, respectively.

\( f = O(g) \) means \( |f| \leq cg \) for some unspecified positive constant \( c \).

\( f \ll g \) means \( f = O(g) \).

\( f \asymp g \) means \( c_1g \leq f \leq c_2g \) for some unspecified positive constants \( c_1 \) and \( c_2 \).

\( f(x) \sim g(x) \) means \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \).

\( \{x\} \) denotes the fractional part of a real number \( x \).

2. THE CONJECTURE ON AVERAGE

The von Mangoldt function \( \Lambda(n) \), the usual weight with which primes are counted, is defined as follows.

\[
\Lambda(n) = \begin{cases} 
\log p, & \text{if } n = p^l \text{ for some } p \in \mathbb{P} \text{ and } l \in \mathbb{N}, \\
0, & \text{otherwise}.
\end{cases}
\]

For the quadratic polynomials of the form \( n^2 + k \) for some fixed \( k \in \mathbb{N} \) together with the weight of the van Mangoldt function, the conjecture (1.1) takes the following simpler form.

\[
(2.1) \quad \sum_{n \leq x} \Lambda(n^2 + k) \sim \Theta(-4k)x.
\]

The asymptotic formula in (2.1) was studied on average by the authors in [2] and it was established that (2.1) holds true for almost all natural numbers \( k \leq K \) if \( x^{1+\varepsilon} \leq K \leq x^{2}/2 \). In particular, we have the following.

**Theorem 1.** Suppose that \( z \geq 3 \). Given \( B > 0 \), we have, for \( z^{1/2+\varepsilon} \leq K \leq z/2 \),

\[
(2.2) \quad \sum_{1 \leq k \leq K} \left| \sum_{z < n^2 + k \leq 2z} \Lambda(n^2 + k) - \Theta(-4k) \right|^2 \ll \frac{Kz}{(\log z)^B}.
\]

From Theorem 1 the following corollary can be deduced immediately.
Corollary. Given $A,B > 0$ and $\mathcal{S}(k)$ as defined above, we have, for $z^{1/2+\varepsilon} \leq K \leq z/2$, that

\[
\sum_{z < n^2 + k \leq 2z} \Lambda(n^2 + k) = \mathcal{S}(-4k) \sum_{z < n^2 + k \leq 2z} 1 + O\left(\frac{\sqrt{z}}{(\log z)^B}\right)
\]

holds for all natural numbers $k$ not exceeding $K$ with at most $O(K(\log z)^{-A})$ exceptions.

It can be easily shown, as done in section 1 of [3], that $\mathcal{S}(-4k)$ converges and

\[
\mathcal{S}(-4k) \gg \frac{1}{\log k} \gg \frac{1}{\log K} \gg \frac{1}{\log z}.
\]

The above inequality shows that the main terms in (2.2) and (2.3) are indeed dominating for the $k$’s under consideration if $B > 1$ and that we truly have an “almost all” result.

Actually, the following sharpened version of Theorem 1 for short segments of quadratic progressions on average was proved in [2].

Theorem 2. Suppose that $z \geq 3$, $z^{2/3+\varepsilon} \leq \Delta \leq z^{1-\varepsilon}$ and $z^{1/2+\varepsilon} \leq K \leq z/2$. Then, given $B > 0$, we have

\[
\int_{z}^{2z} \left| \sum_{1 \leq k \leq K} \sum_{t < n^2 + k \leq t + \Delta} \Lambda(n^2 + k) - \mathcal{S}(-4k) \sum_{t < n^2 + k \leq t + \Delta} 1 \right|^2 dt \ll \frac{\Delta^2 K}{(\log z)^B}.
\]

Moreover, we noted in [2] that under the generalized Riemann hypothesis (GRH) for Dirichlet $L$-functions, the $\Delta$-range in Theorem 2 can be extended to $z^{1/2+\varepsilon} \leq \Delta \leq z^{1-\varepsilon}$. It is noteworthy that for $\Delta = z^{1/2+\varepsilon}$ the segments of quadratic progressions under consideration are extremely short; that is, they contain only $O(z^\varepsilon)$ elements. Theorem 2 can be interpreted as saying that the asymptotic formula

\[
\sum_{t < n^2 + k \leq t + \Delta} \Lambda(n^2 + k) \sim \mathcal{S}(-4k) \sum_{t < n^2 + k \leq t + \Delta} 1
\]

holds for almost all $k$ and $t$ in the indicated ranges.

These results improve some earlier results of the authors [5] where we used the circle method together with some lemmas in harmonic analysis due to P. X. Gallagher [18] and H. Mikawa [33] and the sieve for real characters of D. R. Heath-Brown [23]. In [5] $k$ is restricted to be square-free and $K$ can only be in the much smaller range of $z(\log z)^{-A} \leq K \leq z/2$. Unlike in [5], our approach in the proof of Theorem 2 in [2] is a variant of the dispersion method of J. V. Linnik [30], similar to that used by H. Mikawa in the study of the twin primes problem in [34].

3. Approximating $n^2 + 1$

One may find several results on approximations to the problem of detecting primes of the form $n^2 + 1$ in the literature. Note that $n^2 + 1$ is a prime if and only if $n + i$ is a Gaussian prime. Hence the problem is equivalent to counting Gaussian primes on the line $\Re z = 1$. Therefore, the problem can be approximated by counting Gaussian primes in narrow strips or sectors which can be studied using Hecke $L$-functions. In this direction, C. Ankeny [1] and P. Kubilius [27] showed independently that under the Riemann hypothesis for Hecke $L$-functions for $\mathbb{Q}[i]$ there exist infinitely many Gaussian primes of the form $\pi = m + ni$ with $n < c \log |\pi|$, where $c$ is some positive constant. From this, one infers the infinitude of primes of the form $p = m^2 + n^2$ with $n < c \log p$. Using sieve methods for $\mathbb{Z}[i]$, G. Harman and P. Lewis [21] showed unconditionally that there exist infinitely many primes of the above form with $n \leq p^{0.119}$.

Moreover, it is easy to see that $n^2 + 1$ represents an infinitude of primes if and only if there are infinitely many primes $p$ such that the fractional part of $\sqrt{p}$ is very small, namely $< 1/\sqrt{p}$. A. Balog, G. Harman and the first-named author [6,11,22] dealt with the following related question. Given $0 \leq \lambda \leq 1$ and a real number $\theta$, for what positive numbers $\tau$ can one prove that there exist infinitely many primes $p$ for which the inequality

\[
\{p^\lambda - \theta\} < p^{-\tau}
\]

is satisfied? Roughly speaking, three different methods were used to study this problem depending on whether \( \lambda \) lies in the lower, middle or upper part of \([0, 1]\). These methods are zero density estimates for the Riemann zeta-function for the lower, approximate functional equation for the Riemann zeta-function for the middle, and estimation of exponential sums over primes for the upper. This problem in turn is related to estimating the number of primes of the form \([n^c]\), where \( c > 1 \) is fixed and \( n \) runs over the positive integers. Primes of this form are referred to as Pyateckii-Šapiro primes [7, 35].

It was established by C. Hooley [24] that if \( D \) is not a perfect square then the greatest prime factor of \( n^2 - D \) exceeds \( n^\theta \) infinitely often if \( \theta < \theta_0 = 1.1001 \ldots \). J.-M. Deshouillers and H. Iwaniec [12] improved this to the effect that \( n^2 + 1 \) has infinitely often a prime factor greater than \( n^{6-\varepsilon} \), where \( \theta_0 = 1.202 \ldots \) satisfies

\[
2 - \theta_0 - 2 \log(2 - \theta_0) = \frac{4}{5}.
\]

The improvement comes from utilizing mean-value estimates of Kloosterman sums of J.-M. Deshouillers and H. Iwaniec [11]. The result in [12] can also be generalized to \( n^2 - D \) by Hooley's arguments.

Moreover, H. Iwaniec [20] also showed that there are infinitely many integers \( n \) such that \( n^2 + 1 \) is the product of at most two primes. The result improves a previous one of P. Kuhn [28] that \( n^2 + 1 \) is the product of at most three primes for infinitely many integers \( n \) and can be extended to any irreducible polynomial \( an^2 + bn + c \) with \( a > 0 \) and \( c \) odd.

The results mentioned in the last two paragraphs were based on sieve methods. It is also note-worthy that J. B. Friedlander and H. Iwaniec [17], using results on half-dimensional sieve of H. Iwaniec [25], obtained lower bounds for the number of integers with no small prime divisors representing the set of integers of the form \( n^2 + 1 \) but is still very sparse. The number of such integers not exceeding \( x \) is \( O(x^{3/4}) \). It is generally very difficult to detect primes in sparse sets.

In [3], we approximate the problem of representation of primes by \( m^2 + 1 \) in the following way. For a natural number \( n \) let \( s(n) \) be the square-free kernel of \( n \); i.e. \( s(n) = n/m^2 \), where \( m^2 \) is the largest square dividing \( n \). We note that \( s(n) = 1 \) if and only if \( n \) is a perfect square. We consider primes of the form \( n + 1 \), where \( s(n) \) is small. More precisely, we have the following.

**Theorem 3.** Let \( \varepsilon > 0 \). Then there exist infinitely many primes \( p \) such that \( s(p - 1) \leq p^{5/9+\varepsilon} \).

The set of natural numbers \( n \) with \( s(n) \leq n^{5/9+\varepsilon} \) is also very sparse. More precisely, the number of \( n \leq x \) with \( s(n) \leq n^{5/9+\varepsilon} \) is \( O(x^{7/9+\varepsilon/2}) \) as the following calculation shows.

\[
|\{n \leq x : s(n) \leq n^{5/9+\varepsilon}\}| \leq \left| \{(a,m) \in \mathbb{N}^2 : a \leq x^{5/9+\varepsilon}, am^2 \leq x\} \right| = \sum_{a \leq x^{5/9+\varepsilon}} \sum_{m \leq \sqrt{x/a}} 1 = O(x^{7/9+\varepsilon/2}).
\]

**Theorem 3** can be reformulated as follows.

**Theorem 3.** Let \( \varepsilon > 0 \). Then there exist infinitely many primes of the form \( p = am^2 + 1 \) such that \( a \leq p^{5/9+\varepsilon} \).

**Theorem 3** can be deduced from a Bombieri-Vinogradov type theorem for square moduli, which is as follows.

**Theorem 4.** For any \( \varepsilon > 0 \) and fixed \( A > 0 \), we have

\[
(3.1) \quad \sum_{q \leq x^{2/9-\varepsilon}} \max_{\gcd(a,q)=1} q \left| \psi(x; q^2, a) - \frac{x}{\phi(q^2)} \right| \ll \frac{x}{(\log x)^4},
\]
where
\[ \psi(x; q, a) = \sum_{n \leq x, n \equiv a \mod q} \Lambda(n) \]
and \( \varphi(q) \) is the number of units in \( \mathbb{Z}/q\mathbb{Z} \).

Theorem 4 improves some results of H. Mikawa and T. P. Peneva [32] and P. D. T. A. Elliott [14]. The key ingredient in the proof of Theorem 4 is the large sieve for square moduli which was studied both independently and jointly by the authors [4, 8, 38].

The classical Bombieri-Vinogradov theorem gives
\[ \sum_{q \leq \sqrt{x}/(\log x)^{4+\varepsilon}} \max_{\gcd(a, q) = 1} \left| \psi(x; q, a) - \frac{x}{\varphi(q)} \right| \ll \frac{x}{(\log x)^A}. \]

Hence the analogous statement for square moduli should have \( q \leq x^{1/4}(\log x)^{-A} \) in the sum over \( q \) in [3, 4]. Therefore Theorem 4 is not the complete analogue of the classical theorem. This is due to the fact that in [4] we established a result weaker than the expected analogue of the classical large sieve in the large sieve for square moduli. The latter imperfection is caused by the fact that only a result weaker than the expected was established concerning the spacing of Farey fractions with square denominators. See [4][8][38] for the details.

Furthermore, if any of the above-mentioned expectations can be established (spacing of special Farey fractions, large sieve for square moduli or [3, 4] with the extended range for \( q \) with \( q \leq x^{1/4-\varepsilon} \)), it would follow that there exist infinitely many primes \( p \) such that \( s(p - 1) \leq p^{1/2+\varepsilon} \). We can get the same result under the assumption of the generalized Riemann hypothesis for Dirichlet \( L \)-functions. We note that the set of \( n \) such that \( s(n) \leq n^{1/2+\varepsilon} \) is “almost” as sparse as the set of numbers \( m^2 + n^4 \) considered by Friedlander and Iwaniec [15]. Indeed, the number of \( n \leq x \) such that \( s(n) \leq n^{1/2+\varepsilon} \) is \( O(x^{3/4+\varepsilon/2}) \).

It is conceivable that an Elliott-Halberstam [13] type hypothesis holds for primes in arithmetic progressions to square moduli, i.e., that [3, 4] holds with the exponent \( 1/2 - \varepsilon \) in place of \( 2/9 - \varepsilon \). This would imply that there exist infinitely many primes \( p \) such that \( s(p - 1) \leq p^\varepsilon \). A result of this kind comes very close to the conjecture that there exist infinitely many primes of the form \( n^2 + 1 \) since the number of \( n \leq x \) such that \( s(n) \leq n^\varepsilon \) is \( O(x^{1/2+\varepsilon/2}) \).

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