The maximum number of triangles in $F_k$-free graphs

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Abstract

The generalized Turán number $ex(n, K_s, H)$ is the maximum number of complete graph $K_s$ in an $H$-free graph on $n$ vertices. Let $F_k$ be the friendship graph consisting of $k$ triangles. Erdős and Sós (1976) determined the value of $ex(n, K_3, F_2)$. Alon and Shikhelman (2016) proved that $ex(n, K_3, F_k) \leq (9k-15)(k+1)n$. In this paper, by using a method developed by Chung and Frankl in hypergraph theory, we determine the exact value of $ex(n, K_3, F_k)$ and the extremal graph for any $F_k$ when $n \geq 4k^3$.

Keywords: Generalized Turán number, triangle, friendship graph

1 Introduction

One of the most basic problems in extremal Combinatorics is the study of the Turán number $ex(n, F)$, that is the largest number of edges an $n$-vertex $F$-free graph can have. A natural generalization is to count other subgraphs instead of edges. Given graphs $H$ and $G$, we let $N(H, G)$ denote the number of copies of $H$ in $G$. The generalized Turán number $ex(n, H, F)$ is the largest $N(H, G)$ among $n$-vertex $F$-free graphs $G$.

Let $T_p(n)$ denote the Turán graph; a balanced complete $p$-partite graph on $n$ vertices. Turán [23] proved that $T_p(n)$ is the unique extremal graph of $ex(n, K_{p+1})$, which is regarded as the beginning of the extremal graph theory. The famous Erdős-Stone-Simonovits Theorem [11] states if $H$ is a graph with chromatic number $\chi(H) = \chi \geq 3$, then

$$ex(n, H) = \left( \frac{\chi - 2}{\chi - 1} + o(1) \right) \binom{n}{2}.$$
That is, the Turán number \( ex(n, H) \) is determined asymptotically for any nonbipartite graph \( H \). However, it is still a challenging problem to determine the exact value of the Turán number and the extremal graphs for many nonbipartite graphs.

The friendship graph or \( k \)-fan \( F_k \) consists of \( k \) triangles all intersecting in one common vertex \( v \). Obviously, \( F_k \) is nonbipartite. Erdős, Füredi, Gould and Gunderson determined the Turán number of it. For every \( H \), in \( [3, 4, 16, 17, 18, 19, 21, 24] \). A particular line of research is to determine for a given graph \( H \) some classical results have been extended to the generalized Turán problem. One can find them (Erdős, Füredi, Gould and Gunderson\([10]\))

\[ \text{Theorem 1 (Erdős, Füredi, Gould and Gunderson\([10]\))} \]

For every \( H \), we have

\[
ex(n, F_k) = \left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}
\]

Recently, the problem of estimating generalized Turán number has received a lot of attention, some classical results have been extended to the generalized Turán problem. One can find them in \([3, 4, 10, 16, 17, 18, 19, 21, 24]\). A particular line of research is to determine for a given graph \( H \), what graphs \( F \) have the property that \( ex(n, H, F) = O(n) \). This was started by Alon and Shikhelman \([1]\), who dealt with the case \( H = K_3 \), and was continued for other graphs in \([13, 14]\).

An extended friendship graph consists of \( F_k \) for some \( k \geq 0 \) and any number of additional vertices or edges that do not create any additional cycles. Alon and Shikhelman \([1]\) showed that \( ex(n, K_3, F) = O(n) \) if and only if \( F \) is an extended friendship graph. We remark that known results easily imply that if \( F \) is not an extended friendship graph, then \( ex(n, K_3, F) = \omega(n) \) and it is also easy to see that adding further edges to \( F \) without creating any cycle does not change linearity of \( ex(n, K_3, F) \). Hence the key part of their proof is the following theorem.

\[ \text{Theorem 2 (Alon and Shikhelman\([1]\))} \]

For any \( k \) we have

\[ ex(n, K_3, F_k) < (9k - 15)(k + 1)n. \]

This upper bound for \( ex(n, K_3, F_k) \) is not tight. For instance, for \( k = 2 \), it was observed by Liu and Wang \([20]\) that a hypergraph Turán theorem of Erdős and Sós \([22]\) gives the exact result for \( ex(n, K_3, F_2) \). Let \( F_k \) denote the 3-uniform hypergraph \( (k\text{-star}) \) consisting of \( k \) hyperedges sharing exactly one vertex. Let \( ex_3(n, F_k) \) denote the largest number of hyperedges that an \( F_k \)-free \( n \)-vertex 3-uniform hypergraph can contain.

\[ \text{Theorem 3 (Erdős and Sós\([22]\))} \]

For all \( n \geq 3 \),

\[
ex_3(n, F_2) = \begin{cases} n & \text{if } n = 4m, \\ n - 1 & \text{if } n = 4m + 1, \\ n - 2 & \text{if } n = 4m + 2 \text{ or } n = 4m + 3. \end{cases}
\]

Hence, it is interesting to determine the exact value of \( ex(n, K_3, F_k) \) for any \( F_k \) \( (k \geq 3) \).

Let \( G = (V(G), E(G)) \) be a connected simple graph and \( e(G) = |E(G)| \). For any vertex \( v \in V(G) \) and subset \( S \subseteq V(G) \), let \( N_S(v) \) denote the neighbors of \( v \) in \( S \) and \( d_S(v) = |N_S(v)| \). If \( S = V(G) \), then \( N(v) = N_S(v) \) and \( d(v) = d_S(v) \). For \( X, Y \subseteq V(G) \), \([X, Y]\) denotes the set of edges with one end in \( X \) and another in \( Y \) and \([x, Y]\) if \( X = \{x\} \). Let \( \pi(G) \) denote the degree sequence of \( G \). For two graphs \( G_1 \) and \( G_2 \), \( G_1 \cup G_2 \) is the vertex disjoint union of \( G_1 \) and \( G_2 \) and \( kG \) consists of \( k \) copies of vertex disjoint union of \( G \), \( G_1 + G_2 \) is the graph obtained by taking \( G_1 \cup G_2 \) and joining all pairs \( v_1, v_2 \) with \( v_1 \in V(G_1) \) and \( v_2 \in V(G_2) \). Let \( K_n \) and \( K_n \) denote the complete graph and the empty graph on \( n \) vertices, respectively.
We first define two graphs. Let \( k \geq 4 \) be even, \( X = \{x_1, \ldots, x_{k-1}\} \) and \( Y = \{y_1, \ldots, y_{k-1}\} \). The graph \( H_k' \) is a graph obtained from a complete bipartite graph with vertex classes \( X \) and \( Y \). We subdivide the edge \( x_iy_i \) once for \( i \leq \frac{k}{2} - 1 \), and then identify the \( \frac{k}{2} - 1 \) inserted vertices into one vertex \( z \). The graph \( H_k \) is the complement of \( H_k' \) deleting the edge \( zy_{k/2} \), which is shown in Figure 1.

![Graph H_k](image)

Figure 1. The graph \( H_k \)

It is clear that \(|H_k| = |H_k'| = 2k - 1\) and \( \pi(H_k) = \pi(H_k') = (k - 1, \ldots, k - 1, k - 2) \).

The main result of this paper is the following.

**Theorem 4** Let \( k \geq 3 \) be an integer and \( n \geq 4k^3 \). If \( k \) is odd, then

\[
\text{ex}(n, K_3, F_k) = (n - 2k)k(k - 1) + 2\binom{k}{3},
\]

and \( K_{n-2k} + 2K_k \) is the unique extremal graph, and if \( k \) is even, then

\[
\text{ex}(n, K_3, F_k) = (n - 2k + 1)k \left( k - \frac{3}{2} \right) + 2\binom{k - 1}{3} + \left( \frac{k}{2} - 1 \right)^2,
\]

and \( K_{n-2k+1} + H_k \) is the unique extremal graph.

Given a graph \( G \), we let \( T(G) \) denote the 3-uniform hypergraph on the vertex set \( V(G) \) where \( \{u, v, w\} \) form a hyperedge if and only if \( uvw \) is a triangle in \( G \). The key observation is that if \( G \) is \( F_k \)-free, then \( T(G) \) is \( F_k \)-free. Therefore, \( \text{ex}(n, K_3, F_k) \leq \text{ex}_3(n, F_k) \). In the case \( k = 2 \), the upper bound obtained this way matches the lower bound provided by \( \lceil n/4 \rceil \) vertex-disjoint copies of \( K_4 \), and in the case \( n = 4m + 3 \) we also have a triangle on the remaining vertices. This gives the exact value of \( \text{ex}(n, K_3, F_2) \).

The result of Erdős and Sós [22] was extended to arbitrary \( k \) by Chung and Frankl [6], after partial results [5, 7, 8].
Theorem 5 (Chung and Frankl [6]) Let \( k \geq 3 \). If \( n \) is sufficiently large, then

\[
\text{ex}_3(n, \mathcal{F}_k) = \begin{cases} 
(n - 2k)k(k - 1) + 2\binom{k}{2} & \text{if } k \text{ is odd,} \\
(n - 2k + 1)\frac{(2k-1)(k-1)-1}{2} + (2k - 2)\binom{k-1}{2} + \left(\frac{k-2}{2}\right) - \frac{(k-2)(k-4)}{2} + \frac{k}{2} & \text{if } k \text{ is even.}
\end{cases}
\]

and \( \mathcal{F}_k := \mathcal{T}(\overline{K}_{n-2k} + 2K_k) \) is the unique extremal 3-uniform hypergraph, when \( k \) is odd.

For odd \( k \), this completes the proof of the upper bound. However, for even \( k \), the construction giving the lower bound in the above theorem is not \( \mathcal{T}(G) \) for some \( F_k \)-free graph \( G \). Still, the upper bound differs from the lower bound only by an additive constant \( c(k) \). We will heavily use the tools provided by Chung and Frankl [6] to obtain the improvement needed in Theorem 4.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we study the local structure within the neighborhood of a vertex in an \( F_k \)-free graph, and some properties of a weight function defined on the vertices of triangles which is our main method for counting the number of triangles. These results can be used to prove Theorem 4 with the best coefficient of \( n \) but a weak constant \( f(k) \). The very technical Section 4 is devoted to prove Theorem 4 precisely. In Section 5, we give some concluding remarks.

2 Preliminaries

As a preparation for proving our result, we first present some known theorems, and then we count the number of triangles in a graph with given degree sequence, which are interesting of their own right.

Let \( \nu(G) \) denote the number of edges of a maximum matching in a graph \( G \). The following is the famous result about the maximum matching,

**Theorem 6 (Berge [2])** Let \( o(G - X) \) denote the number of odd components of \( G - X \), then

\[
\nu(G) = \frac{1}{2} \min \left\{ |G| - o(G - X) + |X| : X \subseteq V(G) \right\}.
\]

**Theorem 7 (Chung and Frankl [6])** Let \( k \) be an even integer and \( H \) be a graph on \( 2k - 1 \) vertices and with \( \pi(H) = (k - 1, \ldots, k - 1, k - 2) \), then either

\[
\mathcal{N}(K_3, H) \geq \left(\frac{k}{2} - 1\right)^2 - 1,
\]

or

\[
\mathcal{N}(K_3, H) = \left(\frac{k}{2} - 2\right)\left(\frac{k}{2} - 1\right) \text{ and } H = H_k.
\]

**Theorem 8** Let \( k \) be an even integer and \( H \) be a graph on \( 2k - 1 \) vertices with \( \pi(H) = (k - 1, \ldots, k - 1, k - 2) \), then

\[
\mathcal{N}(K_3, H) \leq 2\left(\frac{k-1}{3}\right) + \left(\frac{k}{2} - 1\right)^2,
\]

equality holds if and only if \( H = H_k \).
Proof. The proof will be similar to the proof of Goodman. It is easy to see that

\[ \mathcal{N}(K_3, H_k) = 2 \binom{k-1}{3} + \binom{k/2}{2} + \binom{k/2-1}{2} = 2 \binom{k-1}{3} + \binom{k}{2} - 1. \]

Let \( \overline{H} \) be the complement of \( H \). For any triple \((x, y, z)\), if \( xyz \) is neither a triangle in \( H \) nor a triangle in \( \overline{H} \), then it is easy to check exactly two of the three, say \( x, y \) such that \(|[x, \{y, z\}]| = 1 \) and \(|[y, \{x, z\}]| = 1 \) in \( H \). Thus, we have

\[ \mathcal{N}(K_3, H) = \binom{2k-1}{3} - \mathcal{N}(K_3, \overline{H}) - \frac{1}{2} \sum_v d(v)(2k-2 - d(v)) \]

\[ = \binom{2k-1}{3} - (k-1)^3 - \frac{1}{2}k(k-2) - \mathcal{N}(K_3, \overline{H}) \]

\[ = 2 \binom{k-1}{3} + \frac{(k-2)^2}{2} \mathcal{N}(K_3, \overline{H}). \]

Obviously, it is sufficient to show \( \mathcal{N}(K_3, \overline{H}) \geq (\frac{k}{2} - 1)^2 \).

Note that \( \overline{H} \) is a graph on \( 2k - 1 \) vertices with \( \pi(\overline{H}) = (k-1, \ldots, k-1, k) \). Let \( z \) be the vertex of degree \( k \) in \( \overline{H} \).

If there is an edge \( zz' \in E(\overline{H}) \) such that \( zz' \) is contained in at least two triangles, then \( \mathcal{N}(K_3, \overline{H}) \geq \mathcal{N}(K_3, \overline{H} - zz') + 2 \). Because of \( \pi(\overline{H} - zz') = (k-1, \ldots, k-1, k-2) \), by Theorem 7 either \( \mathcal{N}(K_3, \overline{H} - zz') \geq (\frac{k}{2} - 1)^2 - 1 \) or \( \overline{H} - zz' = H_k' \). In the former case, we have

\[ \mathcal{N}(K_3, H) \leq 2 \binom{k-1}{3} + \binom{k}{2} - 1. \]

In the latter case, it is easy to check that \( \overline{H} = \overline{H}_k \), and hence \( H = H_k \).

If each edge \( zz' \) in \( E(\overline{H}) \) is contained in at most one triangle, then \( \Delta(\overline{H}[N(z)]) \leq 1 \). Let \( V_1 = N(z) \), \( V_2 = V(\overline{H}) - V_1 \), \( s = e(\overline{H}[V_1]) \) and \( t = e(\overline{H}[V_2]) \). Count the edges between \( V_1 \) and \( V_2 \) in two ways, we have

\[ k(k-1) - 2s = (k-1)^2 - 1 - 2t, \]

which implies \( s - t = \frac{k}{2} - 1 \). However, because \( s \leq \frac{k}{2} \), we must have \( s = \frac{k}{2} \) and \( t = 1 \), or \( s = \frac{k}{2} - 1 \) and \( t = 0 \). Since each edge in \( \overline{H}[V_1] \) can form a triangle with at least \( k-3 \) vertices in \( V_2 \), we get

\[ \mathcal{N}(K_3, \overline{H}) \geq \frac{k}{2}(k-3) + (k-4) > \left( \frac{k}{2} - 1 \right)^2 \]

in the former case, and

\[ \mathcal{N}(K_3, \overline{H}) \geq \left( \frac{k}{2} - 1 \right)(k-3) \geq \left( \frac{k}{2} - 1 \right)^2 \]

in the latter case with equality only if \( k = 4 \). In this case, it is not difficult to check that \( \overline{H} = \overline{H}_4 \), and so \( H = H_4 \). \( \blacksquare \)

Theorem 9 Let \( k \) be an even integer and \( H \) be a graph on \( 2k - 1 - 2s \) vertices with \( \pi(H) = (k-1, \ldots, k-1, k) \). Then

\[ \mathcal{N}(K_3, H) = 2 \binom{k-1}{3} + \binom{k}{2} - 1. \]
\[(k - 1, \ldots, k - 1, k - 2), \text{ then}

\[N(K_3, H) \leq \frac{1}{6}(2k - 1 - 2s)(k - 1)(k - 2) - (k - 1 - 2s)(2s + 1) + \frac{1}{2} - s.\]

**Proof.** Let \(A(H)\) denote the number of triples \((x, y, z)\) having exactly two edges in \(H\), say \(xy, xz \in E(H)\) and \(yz \notin E(H)\). Because \(|N(y) \cap N(z)| \geq d(y) + d(z) - (|H| - 2)\) for every nonadjacent pair \((y, z)\), and there are \(|H| - (d(y) + 1)\) nonadjacent pairs containing \(y\) for any \(y \in V(H)\), we have

\[A(H) \geq \frac{1}{2}(2k - 1 - 2s)(k - 1 - 2s) + 1(2s + 1) - (k - 2s).\]

On the other hand, since

\[(2k - 2 - 2s) \cdot \binom{k - 1}{2} + \binom{k - 2}{2} = A(H) + 3N(K_3, H),\]

we get

\[3N(K_3, H) \leq \frac{1}{2}(2k - 1 - 2s)(k - 1)(k - 2) - (k - 1 - 2s)(2s + 1) + \frac{3}{2} - 3s.\]

This completes the proof. 

3 Some properties of \(F_k\)-free graphs and a weight function

Let \(G\) be an \(F_k\)-free graph, \(uv \in E(G)\) and \(N(uv) = N(u) \cap N(v)\). Clearly, \(|N(uv)|\) is the number of triangles containing the edge \(uv\) in \(G\). We classify the edges into the following three classes:

- Heavy edges: \(H = \{uv : |N(uv)| \geq 2k - 1\}\),
- Medium edges: \(M = \{uv : k \leq |N(uv)| \leq 2k - 2\}\), and
- Light edges: \(L = \{uv : 1 \leq |N(uv)| \leq k - 1\}\).

For a fixed vertex \(u \in V(G)\), let \(G_u = G[N(u)]\) and

- \(H(u) = \{v : v \in N(u) \text{ and } uv \in H\}\),
- \(M(u) = \{v : v \in N(u) \text{ and } uv \in M\}\),
- \(L(u) = \{v : v \in N(u) \text{ and } uv \in L\}\).

This notation will be used throughout the rest of this paper.

Since \(G\) is \(F_k\)-free, then \(\nu(G_u) \leq k - 1\) for any \(u\). Thus, Theorem \ref{theorem} implies

**Observation 1** There exists some \(X \subseteq V(G_u)\) such that

\[\sum_{i=1}^{\ell} \left\lfloor \frac{|C_i|}{2} \right\rfloor + |X| \leq k - 1, \tag{3.1}\]

where \(C_1, \ldots, C_\ell\) are all the components of \(G_u - X\).
Lemma 1 Let $G$ be an $F_k$-free graph, $u \in V(G)$ and $X$ a subset of $V(G_u)$ satisfying the equation (3.1). Then we have the following:

(i) $\mathcal{H}(u) \subseteq X$. Moreover, $|\mathcal{H}(u)| \leq k - 1$ and if equality holds, then $\mathcal{M}(u) = \emptyset$.

(ii) $|\mathcal{H}(u)| + \frac{1}{2}|\mathcal{M}(u)| \leq k - \frac{1}{2}$.

Proof. Let $C_1, \ldots, C_\ell$ be the components of $G_u - X$.

(i) Let $v \in \mathcal{H}(u)$, we know that $|N(uv)| \geq 2k - 1$. If $v$ lies in some component $C_i$, then $N(uv) \subseteq V(C_i) \cup X$ and so

$$\frac{1}{2}|(C_i \cap N(uv)) \cup \{v\}| + |X \cap N(uv)| \geq k,$$

which contradicts (3.1). Hence we have $\mathcal{H}(u) \subseteq X$.

By (3.1), we have $|\mathcal{H}(u)| \leq k - 1$, and if $|\mathcal{H}(u)| = k - 1$, then $X = \mathcal{H}(u)$ and $|C_i| = 1$ for $1 \leq i \leq \ell$. Let $v$ be any vertex of $G_u - X$, then $N(uv) \subseteq X$ and hence $uv \in E$, and so $\mathcal{M}(u) = \emptyset$.

(ii) Clearly, $N(uv) \subseteq X \cup V(C_i)$ if $v \in V(C_i)$. Thus, if there are two components, say $C_1, C_2$, such that $\mathcal{M}(u) \cap V(C_i) \neq \emptyset$, then $|X| + |C_i| \geq k + 1$ for $i = 1, 2$. Hence we have $|X| + \frac{|C_1|}{2} + \frac{|C_2|}{2} \geq k$, which contradicts (3.1). Thus, we may assume $\mathcal{M}(u) \subseteq X \cup V(C_i)$. Note that $\mathcal{H}(u) \subseteq X$ as shown in (i), and

$$|\mathcal{H}(u)| + \frac{1}{2}|\mathcal{M}(u)| = |\mathcal{H}(u)| + \frac{1}{2}|\mathcal{M}(u) \cap (X - \mathcal{H}(u))| + \frac{1}{2}|\mathcal{M}(u) \cap C_i|$$

$$\leq |X| + \frac{1}{2}|C_i| \leq |X| + \frac{|C_i|}{2} + \frac{1}{2} \leq k - \frac{1}{2}.$$

The proof of the lemma is complete. $\blacksquare$

For each triangle $T = xyz$ in $G$, assign $T$ of weight 1 and define a distribution rule $w(T, \cdot)$ to distribute the weight 1 to its three vertices as below (suppose $|N(xy)| \geq |N(yz)| \geq |N(xz)|$):

$$w(T, x) = w(T, y) = w(T, z) = \frac{1}{4}, \text{ if } E(T) \cap \mathcal{H} = \emptyset \text{ or } E(T) \cap \mathcal{L} = \emptyset,$$

$$w(T, x) = w(T, z) = \frac{1}{2}, \text{ if } xy \in \mathcal{H}, \text{ } yz \in \mathcal{H} \cup \mathcal{M} \text{ and } xz \in \mathcal{L},$$

$$w(T, x) = w(T, y) = 0, \text{ if } xy \in \mathcal{H} \text{ and } yz, xz \in \mathcal{L}.$$

Now, we define a weight function $f(u)$ for each vertex $u$ of $G$ as follows.

$$f(u) = \sum_{ux \in E(G_u)} w(ux, u)$$

if $u$ lies in triangles, and $f(u) = 0$ otherwise. It is clear

$$\mathcal{N}(K_3, G) = \sum_{u \in V(G)} f(u).$$

Now, we first discuss some properties of the weight functions $w(T, \cdot)$ and $f(u)$.
Lemma 2 Let uv be an edge of an $F_k$-free graph with $k \geq 3$, then either
\[ \sum_{x \in N(uv)} w(uvx, u) = k - 1, \]
if $uv \in \mathcal{L}, |N(uv)| = k - 1$, $vx \in \mathcal{H}$ and $ux \in \mathcal{L}$ for any $x \in N(uv)$, or
\[ \sum_{x \in N(uv)} w(uvx, u) \leq k - \frac{3}{2}, \]
onlyx{otherwise.}

Proof. Let $\mathcal{H}', \mathcal{M}'(v)$ and $\mathcal{L}'(v)$ be $\mathcal{H}(v), \mathcal{M}(v)$ and $\mathcal{L}(v)$ intersecting with $N(uv)$, respectively. It is clear
\[ \sum_{x \in N(uv)} w(uvx, u) = \sum_{x \in \mathcal{H}'(v)} w(uvx, u) + \sum_{x \in \mathcal{M}'(v)} w(uvx, u) + \sum_{x \in \mathcal{L}'(v)} w(uvx, u). \]

We distinguish three cases on the number of $|N(uv)|$.

Case 1. $uv \in \mathcal{H}$

By the definition of $w(T, \cdot)$, $w(uvx, u) \leq \frac{1}{2}$ if $x \in \mathcal{H}'(v) \cup \mathcal{M}'(v)$ and $w(uvx, u) = 0$ if $x \in \mathcal{L}'(v)$. Noting that $u \in \mathcal{H}(v) - \mathcal{H}'(v)$, we have $|\mathcal{H}'(v)| + \frac{1}{2}|\mathcal{M}'(v)| \leq k - \frac{3}{2}$ by Lemma 1(iii), and hence
\[ \sum_{x \in N(uv)} w(uvx, u) \leq \frac{1}{2}|\mathcal{H}'(v)| + \frac{1}{2}|\mathcal{M}'(v)| \leq k - \frac{3}{2} - \frac{1}{2}|\mathcal{H}'(v)|, \] which implies the result holds.

Case 2. $uv \in \mathcal{M}$

In this case, $|N(uv)| \leq 2k - 2$. Since $uv \in \mathcal{M}$ implies $\mathcal{M}(v) \neq \emptyset$, by Lemma 1(i), we have $|\mathcal{H}'(v)| \leq |\mathcal{H}(v)| \leq k - 2$. By the definition of $w(T, \cdot)$, $w(uvx, u) \leq \frac{1}{2}$ if $x \in \mathcal{H}'(v)$ and $w(uvx, u) \leq \frac{1}{2} - \frac{1}{4}$ if $x \in \mathcal{M}'(v) \cup \mathcal{L}'(v)$. Thus, we have
\[ \sum_{x \in N(uv)} w(uvx, u) \leq \frac{1}{2}|\mathcal{H}'(v)| + \frac{1}{2}|\mathcal{M}'(v)| + \frac{1}{3}|\mathcal{L}'(v)| \leq k - 1 - \frac{k}{6}. \] The upper bound $k - \frac{3}{2}$ follows from the assumption $k \geq 3$.

Case 3. $uv \in \mathcal{L}$

Because $uv \in \mathcal{L}$, we have $|N(uv)| \leq k - 1$. By the definition of $w(T, \cdot)$, some triangles satisfy $w(uvx, u) = 1$ and other triangles satisfy $w(uvx, u) \leq \frac{1}{2}$. Thus we have
\[ \sum_{x \in N(uv)} w(uvx, u) \leq (k - 1) - \frac{1}{2} \left| \left\{ uvx : w(uvx, u) \leq \frac{1}{2} \right\} \right|, \] which implies $\sum_{x \in N(uv)} w(uvx, u) \leq k - \frac{3}{2}$ or $\sum_{x \in N(uv)} w(uvx, u) = k - 1$, and the latter holds if and only if $|N(uv)| = k - 1$, and all triangles $uvx$ satisfy $w(uvx, u) = 1$, that is, $vx \in \mathcal{H}$ and $ux \in \mathcal{L}$ for any $x \in N(uv)$.

\[ \]
Lemma 3 Suppose $G$ is an $F_k$-free graph and $k \geq 4$ is even. Let $u \in V(G)$, $X$ be a subset of $V(G_u)$ satisfying (3.1) and $C_1, \ldots, C_\ell$ be the components of $G_u - X$ with $|C_1| \geq \cdots \geq |C_\ell|$. Then

$$f(u) \leq k \left( k - \frac{3}{2} \right) - \frac{1}{2},$$

or $f(u) = k \left( k - \frac{3}{2} \right)$ and the following hold:

(i) If $X \neq \emptyset$, then $X$ is an independent set and $d_{G_u}(v) = k - 1$ for any $v \in X$;

(ii) $\pi(C_i) = (k - 1, \ldots, k - 1, k - 2)$, and either $G_u = C_1 \cup K_{k-1}$ with $|C_1| = k + 1$, or $G_u - X = C_1 \cup (\ell - 1)K_1$ with $|C_1| = 2k - 1 - 2|X| \geq k + 1$;

(iii) $E(G_u) \subseteq H$, $|u, G_u| \subseteq L$ and $\Delta(G_u) = k - 1$.

Remark. There is a similar lemma in Chung and Frankl’s paper [6] when they deal with function $\text{ex}_3(n, F_k)$. However, in their lemma, they overlooked the case $G_u = C_1 \cup K_{k-1}$. Using our method in Section 4, it is not difficult to complete the proof of this missed case, too.

Proof. By (3.1), $|C_i| \leq k$ for all $i \neq 1$. Let $uvw$ be any triangle. Then

$$f(u) = \sum_{i=1}^{\ell} \sum_{v \in X \subseteq E(C_i)} w(uvw, u) + \sum_{\{v, x\} \cap X \neq \emptyset} w(uvx, u).$$

If the edge $vx$ satisfies $\{v, x\} \cap X \neq \emptyset$, then by Lemma 2, we have

$$\sum_{\{v, x\} \cap X \neq \emptyset} w(uvx, u) \leq \sum_{v \in X} \sum_{x \in N(uw)} w(uvx, u) \leq |X|(k - 1). \quad (3.5)$$

If $vx \in E(C_i)$ with $|C_i| \leq k$, then noting that $k$ is even, $uvw$ is a triangle for each $vx \in E(C_i)$ and $w(uvx, u) \leq 1$, we have

$$\sum_{vx \in E(C_i)} w(uvx, u) \leq \frac{1}{2} \sum_{v \in V(C_i)} \sum_{x \in N(uw) \cap C_i} w(uvx, u)
\leq \frac{1}{2} |C_i| \left( |C_i| - 1 \right) \leq \left( \frac{|C_i|}{2} \right)^2 \leq (k - 1) \left( k - \frac{3}{2} \right) - \frac{1}{2}. \quad (3.6)$$

If $|C_1| \leq k$, that is, $|C_i| \leq k$ for $1 \leq i \leq \ell$, then by (3.1), (3.5) and (3.6), we have

$$f(u) \leq (k - 1) \left( \sum_{i=1}^{\ell} \left| \frac{|C_i|}{2} \right| + |X| \right) \leq (k - 1)^2 \left( k - \frac{3}{2} \right) - \frac{1}{2}.$$ 

The last inequality holds because of $k \geq 4$. So, we may assume that $|C_1| > k$.

If $\Delta(C_1) \geq k$, say $d_{C_1}(v) \geq k$ for some vertex $v$ in $C_1$, then $|N(uw)| \geq k$ and so $uv \notin L$. Since $H(u) \subseteq X$ by Lemma (1), we have $v \notin H(u)$ and hence $uv \in M$. By (3.3), we have $\sum_{x \in N(uw) \cap C_1} w(uvx, u) \leq (k - 1) - \frac{k}{6}$. Meanwhile, because $d_{C_1}(v) \geq k \geq 4$, there exists $v_i \in N(uw) \cap C_1$ for $1 \leq i \leq 4$. Note that $v \in M(u)$ and $v \in N(uw_i)$, by Lemma 2, we
have \( \sum_{x \in N(uv) \cap C_i} w(uv, x, u) \leq (k - 1) - \frac{1}{2} \) for \( 1 \leq i \leq 4 \), and

\[
\sum_{v \in E(C_1)} w(uvx, u) = \frac{1}{2} \sum_{v \in C_1} \sum_{x \in N(uv) \cap C_i} w(uvx, u) \leq \frac{1}{2} |C_1|(k - 1) - \left( \frac{k}{12} + 1 \right). \tag{3.7}
\]

If \( \Delta(C_1) \leq k - 1 \), then

\[
\sum_{v \in E(C_1)} w(uvx, u) = \frac{1}{2} \sum_{v \in C_1} \sum_{x \in N(uv) \cap C_i} w(uvx, u) \leq \left[ \frac{1}{2} |C_1|(k - 1) \right]. \tag{3.8}
\]

Set \( \mu(C_1) = \frac{k}{12} + 1 \) if \( \Delta(C_1) \geq k, \mu(C_1) = \frac{1}{2} \) if \( |C_1| \) is odd and \( \Delta(C_1) \leq k - 1 \) and \( \mu(C_1) = 0 \) if \( |C_1| \) is even and \( \Delta(C_1) \leq k - 1 \), then (3.5)-(3.8) imply

\[
f(u) \leq (k - 1) \left( \frac{|C_1|}{2} + \sum_{i=2}^{\ell} \left\lfloor \frac{|C_1|}{2} \right\rfloor + |X| \right) - \mu(C_1). \tag{3.9}
\]

Assume that \( f(u) > k \left( k - \frac{3}{2} \right) - \frac{1}{2} \). By (3.1) and (3.9), we have \( \mu(C_1) = \frac{1}{2} \). In this case, \( |C_1| \geq k + 1 \) is odd and \( \Delta(C_1) \leq k - 1 \). Note that if one of the equalities in (3.5), (3.6) and (3.8) does not hold, then the upper bound in (3.9) can be reduced by at least an extra \( \frac{1}{2} \). This implies the equalities in (3.5), (3.6) and (3.8) holds.

It is clear that the equalities in (3.5) hold if and only if \( X \) is an independent set and \( \sum_{x \in N(uv)} w(uvx, u) = k - 1 \) for any \( v \in X \). By Lemma 2, we get that \( d_{G_u}(v) = k - 1 \), \( uv, ux \in L \) and \( vx \in \mathcal{H} \) for any \( v \in X \).

Since \( |C_1| \geq k + 1 \) is odd, \( \Delta(C_1) \leq k - 1 \) and the equality in (3.8) holds, we can deduce that \( \pi(C_1) = (k - 1, k - 1, \ldots, k - 2) \), \( E(C_1) \subseteq \mathcal{H} \) and \( [u, C_1] \subseteq L \).

Because equality (3.6) holds, recalling \( |C_1| \geq k + 1 \) and \( k \) is even, by (3.1), we have \( |C_i| \in \{1, k - 1\} \) for \( i \geq 2 \) and each \( C_i \) is a clique with \( E(C_i) \subseteq \mathcal{H} \) and \( [u, C_i] \subseteq L \). In addition, if \( |C_i| = k - 1 \) for some \( i \geq 2 \), then by (3.1), \( |C_1| = k + 1 \) and \( X = \emptyset \), that is, \( G_u - X = C_1 \cup K_{k-1} \cup (\ell - 2)K_1 \). If \( |C_i| = 1 \) for all \( i \geq 2 \), then \( |C_1| = 2k - 1 - 2|X| \).

So, the statements (i), (ii) and (iii) hold.

\[\Box\]

**Definition 1** For any vertex \( u \in V(G) \), the loss of \( u \) is the number

\[
k \left( k - \frac{3}{2} \right) - f(u).
\]

See the following simple observations about the losses.

**Observation 2** If some vertex \( v \in X \) has \( \sum_{x \in N(uv)} w(uvx, u) \leq (k - 1) - c \), then the edge \( uv \) contributes \( c \) to the loss of \( u \).

**Proof.** It is a direct consequence of (3.5). \[\Box\]

**Observation 3** An edge \( uv \in \mathcal{H} \) contributes \( \frac{1}{2} \) to the loss of \( u \). Moreover, a triangle \( uvx \) with \( uv, ex \in \mathcal{H} \) contributes another \( \frac{1}{2} \) to the loss of \( u \).

**Proof.** Since \( uv \in \mathcal{H} \), by (3.2), \( \sum_{x \in N(uv)} w(uvx, u) \leq k - \frac{3}{2} - \frac{1}{2}|\mathcal{H}'(v)| \). Because \( v \in X \) by Lemma 1, the edge \( uv \) contributes \( \frac{1}{2} \) to the loss of \( u \) by Observation 2. Moreover, since a
triangle $uvw$ with $uv, vx \in \mathcal{H}$ satisfies $x \in \mathcal{H}'(v)$, so it contributes another $\frac{1}{2}$ to the loss of $u$ by (3.2).

**Observation 4** Let $uv \in \mathcal{M}$. If $\sum_{x \in N(uv)} w(uvx, u) \leq (k-1) - c$, then the edge $uv$ contributes at least $\min \left\{ \frac{c}{2}, \frac{k}{4} - \frac{1}{2} \right\}$ to the loss of $u$. Moreover, the edge $uv$ contributes at least $\frac{k}{12}$ to the loss of $u$.

**Proof.** If $v \in X$, then by (3.3) and Observation 2, $uv$ contributes $c$ to the loss of $u$. If $v \in V(C_i)$ and $|C_i| \geq k + 1$, then by (3.7) and (3.8), $uv$ contributes at least $\frac{c}{2}$ to the loss of $u$. If $v \in V(C_i)$ for some $i$ with $|C_i| \leq k$, then by (3.6), there is a gap between $\left\lfloor \frac{|C_i|}{2} (k-1) \right\rfloor$ and $\frac{k}{4} |C_i| (|C_i| - 1)$, and for this gap, any edge $uv'$ with $v' \in V(C_i)$ contributes $\frac{1}{2} |C_i| \left( \left\lfloor \frac{|C_i|}{2} \right\rfloor (k-1) - \frac{1}{2} |C_i| (|C_i| - 1) \right)$ to the loss of $u$. On the other hand, because

$$\sum_{x \in N(uv) \cap C_i} w(uvx, u) \leq \frac{1}{2} |C_i| - 1 = (|C_i| - 1) - \frac{1}{2} |C_i| - 1,$$

this reduces the right hand of (3.6) by an additional $\frac{1}{4} (|C_i| - 1)$. Hence the total loss of $u$ contributed by the edge $uv$ is at least

$$\frac{1}{|C_i|} \left( \left\lfloor \frac{|C_i|}{2} \right\rfloor (k-1) - \frac{1}{2} |C_i| (|C_i| - 1) \right) + \frac{1}{4} (|C_i| - 1) \geq \frac{k}{4} - \frac{1}{2}.$$

Together with (3.3), $c \geq \frac{k}{12}$, it implies that the statements of the lemma are proved.

**4 Proof of Theorem 4**

Let $G$ be an extremal graph of $ex(n, K_3, F_k)$.

If $k$ is odd, then by Theorem 5, we have

$$\mathcal{N}(K_3, G) = e(T(G)) \leq ex_3(n, F_k) = (n - 2k)k(k - 1) + 2 \binom{k}{3},$$

and the unique extremal hypergraph is $\mathcal{F}_k$ for which equality holds. Because

$$\mathcal{N}(K_3, K_{n-2k} + 2K_k) = (n - 2k)k(k - 1) + 2 \binom{k}{3},$$

and $T(K_{n-2k} + 2K_k) = \mathcal{F}_k$, we get

$$\mathcal{N}(K_3, G) = (n - 2k)k(k - 1) + 2 \binom{k}{3},$$

where equality holds if and only if $G = K_{n-2k} + 2K_k$.

The remaining part is devoted to the case when $k \geq 4$ is even. Because an edge not lying in
a triangle makes no contribution to \( N(K_3, G) \), we may assume that each edge of \( G \) is covered by some triangles.

If \( f(v) = k \left( k - \frac{3}{2} \right) \), then we call \( v \) a good vertex. Let \( U_1 = \{ v : v \text{ is good} \} \). Since \( n \geq 4k^3 \) and \( f(v) \leq k \left( k - \frac{3}{2} \right) - \frac{1}{4} \) for any \( v \notin U_1 \) by Lemma \ref{lem:triangle-free}, we have

\[
nk \left( k - \frac{3}{2} \right) - \frac{1}{2} (n - |U_1|) \geq \sum_{v \in V(G)} f(v) = N(K_3, G)
\]

\[
\geq (n - 2k + 1)k \left( k - \frac{3}{2} \right) + 2 \left( k - \frac{1}{3} \right) + \left( \frac{k}{2} - 1 \right)^2,
\]

which implies

\[
|U_1| \geq n - 2(2k - 1)k \left( k - \frac{3}{2} \right) > 2k.
\]

Moreover, if there exist \( v, v' \in U_1 \) such that \( vv' \in E(G) \), then \( vv' \in L \) by Lemma \ref{lem:triangle-free}. Let \( vv'x \) be a triangle. Applying Lemma \ref{lem:triangle-free} to \( v \), we have \( v'x \in H \) and using Lemma \ref{lem:triangle-free} to \( v' \), we have \( v'x \in L \), a contradiction. Therefore, \( U_1 \) is an independent set.

Let \( u \in U_1 \) be given, \( G_u = G[N(u)] \) as before and \( U_2 = V(G) - V(G_u) - U_1 \). We will prove Theorem \ref{thm:main} by showing \( G \) is an extremal graph only if \( U_2 = \emptyset \). Since the proof is a little complicated and long, so we sketch it first in the following two paragraphs.

In the case when \( N(u') = N(u) \) for any \( u' \in U_1 - \{ u \} \), our main idea for doing this is to partition the total weights of all vertices of \( G_u \) into two parts: One part comes from the triangles contained in \( G_u \), which is exactly \( N(K_3, G_u) \), and another part is contributed by the triangles containing one or two vertices in \( U_2 \). And then we use discharge method to transfer the latter part to the vertices in \( U_2 \) such that \( f(v) \leq k \left( k - \frac{3}{2} \right) \) is still valid for any \( v \in U_2 \) after transferring. Using this method, we show that if \( U_2 \neq \emptyset \), then the total weight of \( G \) is less than the expected number.

In the case when there is some \( u' \in U_1 \) such that \( N(u) \neq N(u') \), we transform \( G \) into a graph \( G' \) such that \( G' \) and \( G \) have the same good vertices, and all good vertices of \( G' \) have the same neighborhood as \( N(u) \), through an operation as follows: Delete all the edges between \( u' \) and \( G_u' \) and add new edges joining \( u' \) to all vertices in \( G_u \). Repeat this operation until all vertices in \( U_1 \) have the same neighborhood \( N(u) \). Let \( G' \) be the resulting graph, \( U_1' = \{ v : v \text{ is good in } G' \} \) and \( U_2' = V(G') - V(G'_u) - U_1' \). We will see that \( N(K_3, G') = N(K_3, G) \) and \( G' \) is also \( F_k \)-free and \( U_1' = U_1 \).

Firstly, since \( u' \) is good, by Lemma \ref{lem:triangle-free} we have \( f(u') = e(G_u') = k(k - \frac{3}{2}) \), which implies we destroy \( k \left( k - \frac{3}{2} \right) \) triangles first and then add \( k \left( k - \frac{3}{2} \right) \) new triangles, and so \( N(K_3, G') = N(K_3, G) \). Moreover, \( G'_u = G_u \). Secondly, since \( G_u \) has no \( kK_2 \) and \( \Delta(G_u) = k - 1 \) by Lemma \ref{lem:triangle-free} we can see that \( G' \) is also \( F_k \)-free after an easy check. Finally, because \( |U_1| > 2k \), we have \( E(G'_u) \subseteq H \), which implies \( v \notin U_1' \) for any \( v \in V(G'_u) \) by Lemma \ref{lem:triangle-free}. Furthermore, since \( \Delta(G'_u) = k - 1 \) and \( U_1' \) is an independent set, \( |U_1' G'_u| \subseteq L \). Thus, we have \( U_1 \subseteq U_1' \) by the definition of \( w(T, \cdot) \). Suppose that there is some \( v \in U_2 \) in \( G \) such that \( v \in U_1' \) in \( G' \). Let \( G'_u = G[N(v)] \) and \( X' \subseteq G'_u \) satisfy \ref{lem:triangle-free}. Since \( v \in U_1' \), \( E(G'_u) \subseteq H \) and \( v, G'_u \subseteq L \) by Lemma \ref{lem:triangle-free}. By the operation above, \( E(G'_u) \subseteq H \) in \( G \). Since \( v \in U_2 \), there is some \( v' \in G'_u \) such that \( vv' \notin L \) in \( G \), which means there is some \( u' \in U_1 \) such that \( u'v'v \) is a triangle in \( G \). Note that \( V(G'_u) \cup \{ u' \} \subseteq G(v) \). If \( v' \notin X' \), then by Lemma \ref{lem:triangle-free} it is easy to check that \( G'([u' \cup V(G'_u)]) \) contains \( kK_2 \), and so \( G \) has \( F_k \). Thus we have \( v' \in X' \). In this case, by Lemma \ref{lem:triangle-free} \( \mathcal{H}(v') = k - 1 \) in \( G \). Let \( X'' \subseteq G_{v'} \) satisfy \ref{lem:triangle-free}. By Lemma \ref{lem:triangle-free} \( \mathcal{H}(v') \subseteq X'' \).
hence $|X''| = k - 1$. Because $u'v$ is an edge in $G_v - X''$, this contradicts (3.1). Thus, we have $U'_1 = U_1$.

Since $U'_1 = U_1$ implies $U'_2 = U_2$, and $U'_2 \neq \emptyset$ in this case, $G'$ cannot be an extremal graph, and so does $G$ since $N(K_3, G') = N(K_3, G)$. Therefore, it is sufficient to show $G$ is an extremal graph only if $U_2 = \emptyset$ in the case when $N(u') = N(u)$ for any $u' \in U_1 - \{u\}$.

Let $X \subseteq G_u$ satisfy (3.1). By Lemma 3, $\Delta(G_u) = k - 1$. Moreover, $G_u$ has the following structural properties.

**Claim 1** Let $v \in V(G_i)$, where $C_i$ is some component of $G_u - X$.

1. If $d_{G_u}(v) = k - 1$ and $E(G_u) \subseteq \mathcal{H}$ by Lemma 3, we have $|\mathcal{H}(v)| \geq k - 1$. Let $X' \subseteq V(G_v)$ satisfy (3.1). By Lemma 1, $X' = \mathcal{H}(v) \subseteq V(G_u)$ which in turn implies $G_v - X'$ has no edges by (3.1), and $\mathcal{M}(v) = \emptyset$, and so $[v, U_2] \subseteq \mathcal{L}$.

2. If $d_{C}(v) = k - 2$, then $G[N_{U_2}(v)]$ is a star or a triangle, together with some isolated vertices. Moreover, if $v_1 \in N_{U_2}(v)$ and $v_1 \in \mathcal{H} \cup \mathcal{M}$, then $v_1$ is the center of the star with at least 3 vertices, or lies on the triangle.

**Proof.** (1) Since $d_{G_u}(v) = k - 1$ and $E(G_u) \subseteq \mathcal{H}$ by Lemma 3, we have $|\mathcal{H}(v)| \geq k - 1$. Let $X' \subseteq V(G_v)$ satisfy (3.1). By Lemma 1, $X' = \mathcal{H}(v) \subseteq V(G_u)$ which in turn implies $G_v - X'$ has no edges by (3.1), and $\mathcal{M}(v) = \emptyset$, and so $[v, U_2] \subseteq \mathcal{L}$.

(2) Since $d_{C}(v) = k - 2$ and $E(G_u) \subseteq \mathcal{H}$, we have $|\mathcal{H}(v)| \geq k - 2$. Let $X' \subseteq V(G_v)$ satisfy (3.1). By Lemma 1, $\mathcal{H}(v) \subseteq X'$ and so $|X' \setminus V(G_u)| \geq k - 2$. Because $|X'| \leq k - 1$, we have $|X' \setminus U_2| \leq 1$, which implies $G_v - \mathcal{H}(v)$ has no $2K_2$ by (3.1), that is, $G[N_{U_2}(v)]$ is a star or a triangle, together with some isolated vertices. Moreover, if $v_1 \in \mathcal{H} \cup \mathcal{M}$, then we have $d_{G_u}(v) = k - 2$ by Lemma 4. Since $|N(v_1)| \geq k$ and we have $|N(v_1) \cap U_2| \geq 2$, that is, $v_1$ has at least 2 neighbors in $G[N_{U_2}(v)]$. Hence, $v_1$ is the center of the star or lies on a triangle in $G[N_{U_2}(v)]$.

Let $C$ be the largest component of $G_u - X$. Then $\pi(C) = (k - 1, \ldots, k - 1, k - 2)$ by Lemma 3. Let $z \in V(C)$ be the unique vertex with $d_C(z) = k - 2$ and $N_C(z) = \{z_1, \ldots, z_{k-2}\}$. Since $\Delta(G_u) = k - 1$, we have $|V(C), X| \leq 1$ and $|V(C), X| \subseteq [z, X]$. In addition, we have the following.

**Claim 2** $|(\mathcal{H}(z) \cup \mathcal{M}(z)) \cap U_2| \leq 1$.

**Proof.** Assume $v_1, v_2 \in N_{U_2}(z)$ with $zv_1, zv_2 \notin \mathcal{L}$. By Claim 1, $v_1v_2z$ is a triangle in $N_{U_2}(z)$ for some $v$. Since $|N(zv_i)| \geq k$ for $i = 1, 2, \{z_1, \ldots, z_{k-2}\} \subseteq N(v_1) \cap N(v_2)$ which contradicts Claim 1 since $v_1, v_2 \in N_{U_2}(z)$ and $v_1v_2 \in E(G)$.

If $|(\mathcal{H}(z) \cup \mathcal{M}(z)) \cap U_2| = 1$, we assume $zv_1 \in \mathcal{H} \cup \mathcal{M}$ and $N(zv_1) \cap U_2 = \{v_2, \ldots, v_t\}$. By Claim 2, $G[\{v_1, \ldots, v_t\}]$ is a $K_3$ or a star with center $v_1$. If $N(zv_1) \cap V(C) \neq \emptyset$, let $N(zv_1) \cap V(C) = \{z_1, \ldots, z_{t'}\}$, where $t' \leq k - 2$.

Let us consider the total weight in $\sum_{v \in V(C)} f(v)$ coming from the triangles not contained in $C$. Since $[U_1, V(C)] \subseteq \mathcal{L}$, $U_1$ is an independent set, $|V(C), X| \leq 1$ and $[X, U_2] \subseteq \mathcal{L}$ by Lemma 3 and Claim 1, the weight is contributed by the triangles intersecting only with $U_2$. By Claims 1 and 2, only $z$ is contained in some triangles with two vertices in $U_2$, and so the weight coming from such triangles is $\sum_{i \leq t} w(zv_1v_i, z) + \lambda(z)$, where $\lambda(z) = w(zv_2v_3, z)$ if $v_1v_2v_3$ is a triangle and $\lambda(z) = 0$ otherwise. Furthermore, by Claims 1 and 2, for any triangle containing two vertices in $C$ and one vertex in $U_2$, only $w(zv_1z_i, z_i) \neq 0$ for $i \leq t'$ in the case $zv_1 \in \mathcal{H} \cup \mathcal{M}$.
Therefore, the weight is

\[ f_{U_2}(z) = \sum_{i \leq t} w(zv_1v_i, z) + \lambda(z) + \sum_{i=1}^{t'} w(zv_1z_i, z_i). \]

**Claim 3** If \( zv_1 \notin \mathcal{H} \), then \( f_{U_2}(z) \leq k - 1 \), and if \( zv_1 \in \mathcal{H} \), then \( f_{U_2}(z) \leq k - 1 + \frac{t'}{2} \leq \frac{3k}{2} - 2 \) and \( zv_1 \) contributes at least \( \frac{1}{2}(t' + 1) \) to the loss of \( v_1 \).

**Proof.** If \( zv_1 \in \mathcal{L} \), then \( \sum_{i \leq t'} w(zv_1v_i, z) = 0 \) and hence \( f_{U_2}(z) = \sum_{i \leq t} w(zv_1v_i, z) + \lambda(z) \leq \max\{3, k - 1\} \) by Lemma 2.

If \( zv_1 \in \mathcal{M} \), then \( |N(zv_1)| = (t - 1) + t' \leq 2k - 2 \). By the definition of \( w(T, \cdot) \) and Claim 1, we get \( f_{U_2}(z) \leq \frac{1}{2}(t - 1) + \lambda + \frac{t'}{2} \). Note that \( \lambda(z) = 0 \) if \( (t - 1) + t' = 2k - 2 \), we have \( f_{U_2}(z) \leq k - 1 \).

If \( zv_1 \in \mathcal{H} \), then \( t > k \) and so \( \lambda(z) = 0 \). By Lemma 2, \( \sum_{i \leq t} w(zv_1v_i, z) \leq k - 1 \) and so \( f_{U_2}(z) \leq k - 1 + \frac{t'}{2} \leq \frac{3k}{2} - 2 \). Moreover, since the triangle \( z_izv_1 \) satisfies \( zz_i \in \mathcal{H} \) for \( i \leq t' \), by Observation 3, the edge \( zv_1 \) contributes at least \( \frac{1}{2}(t' + 1) \) to the loss of \( v_1 \).

By Lemma 3, either \( |C| = 2k - 1 - 2|X| \geq k + 1 \) or \( |C| = k + 1 \). Moreover, since each edge of \( G \) is covered by triangles, if \( X = \emptyset \), then \( G_u \) has no isolated vertices. That is, \( G_u = C \) or \( C \cup K_{k-1} \) if \( X = \emptyset \).

We distinguish the following two cases separately according to \( |C| \).

**Case 1.** \( |C| = 2k - 1 - 2|X| \)

In this case, the structure of \( G \) are shown in Figure 2, where the thick edges are in \( \mathcal{H} \) and the thin edges are in \( \mathcal{L} \).

![Figure 2. N(u') = N(u) for any u' ∈ U_1](image)

**Case 1.1** \( X = \emptyset \).
In this case, $G_u = C$ and $|C| = 2k - 1$. If $U_2 = \emptyset$, then since $|U_1, G_u| \subseteq \mathcal{L}$ and $E(G_u) \subseteq \mathcal{H}$, by the definition of $w(T, \cdot)$, we have

$$
\sum_{v \in V(G)} f(v) = \sum_{v \in U_1} f(v) + \sum_{v \in V(C)} f(v)
= \sum_{v \in U_1} f(v) + \mathcal{N}(K_3, C) = (n - 2k + 1)k \left( \frac{k - 3}{2} \right) + \mathcal{N}(K_3, C).
$$

Because $|C| = 2k - 1$ and $\pi(C) = (k - 1, \ldots, k - 1, k - 2)$, by Theorem 8

$$
\sum_{v \in V(G)} f(v) \leq (n - 2k + 1)k \left( \frac{k - 3}{2} \right) + 2 \left( \frac{k - 1}{3} \right) + \left( \frac{k}{2} \right) + \left( \frac{k}{2} - 1 \right)
$$

and equality holds if and only if $G = \hat{K}_{n-2k+1} + H_k$, and so the result follows. Therefore, we may assume that $U_2 \neq \emptyset$.

In this case, we will try to transfer the weight $f_{U_2}(z)$ to the vertices in $U_2$ such that $f(v) \leq k(k - \frac{3}{2})$ still valid after transferring.

1. $zv_1 \in \mathcal{H}$.

By Claim 3 the edge $zv_1$ contributes at least $\frac{1}{2}(t' + 1)$ to the loss of $v_1$. Transfer the weight $\sum_{z_i \leq t'} w(z_izv_1, z_i) \leq t_1$ to $v_1$ to cover the loss of $v_1$ caused by the edge $zv_1$ and the weight $\sum_{i \leq t} w(zv_1v_i, z)$ to $v_2, \ldots, v_i(\lambda(z) = 0$ in this case). After transferring, $f_{U_2}(z) = 0$, $f(v) \leq k(k - \frac{3}{2})$ is still valid for any $v \in U_2$ and $f(v_1) \leq k(k - \frac{3}{2}) - \frac{1}{2}$. Therefore,

$$
\sum_{v \in V(G)} f(v) < (n - 2k + 1)k \left( \frac{k - 3}{2} \right) + \mathcal{N}(K_3, C).
$$

2. $zv_1 \in \mathcal{M}$.

By the definition of $w(T, \cdot)$, we have $w(z_izv_1, z_i) = \frac{1}{2}$ for $i \leq t'$, and $w(zv_1v_i, z) \leq \frac{1}{2}$ for $i \leq t$. Let $|U_2| = t = t''$.

Suppose $t'' < t' \leq k - 2$. Note that either $\lambda(z) = 1$, which implies $G[\{v_1, \ldots, v_t\}]$ is a triangle and $v_2v_3 \in \mathcal{H}$, or $\lambda(z) \leq \frac{1}{2}$. For the former case, $N(v_2v_3) \{v_1, z\} \subseteq U_2 - \{v_1, v_2, v_3\}$ and hence $|U_2 - \{v_1, v_2, v_3\}| \leq 2k - 3$ by Claim 4(71), which means $t'' > 2t'$. Thus we can transfer the weight $w(z_izv_1, z_i)$ of $z_i$ to the vertices in $U_2 - \{v_1, v_2, v_3\}$, the weight $\sum_{i \leq t} w(zv_1v_i, z) + \lambda(z)$ to the vertices $v_1, v_2, v_3$ and some others in $U_2 - \{v_1, v_2, v_3\}$. For the latter case, we transfer the weights $w(z_izv_1, z_i)$ to the vertices in $U_2 - \{v_1, \ldots, v_t\}$ and the weights $\sum_{i \leq t} w(zv_1v_i, z) + \lambda(z)$ to the vertices of $\{v_1, \ldots, v_t\}$. After the transferring, $f_{U_2}(z) = 0$, $f(v) \leq k(k - \frac{3}{2})$ is still valid for any $v \in U_2$, and (4.1) still holds.

Assume $t'' < t' \leq k - 2$. In this case, all edges of $G[\{v_1, \ldots, v_t\}]$ are in $\mathcal{L}$ for otherwise we have $t'' \geq k - 2$. Hence, $w(zv_1v_i, z) = \frac{1}{2}$ for $i \leq t$ and $\lambda(z) \leq \frac{1}{2}$. Recalling $\{z_1, \ldots, z_t'\} \subseteq N(zv_1)$, by Claim 4(1), we get $N(v_i) \{z_1, \ldots, z_t'\} = \emptyset$ for $2 \leq i \leq t$, and if $v_izv$ is a triangle such that $z' \in V(C)$ and $v \in U_2$, then $v_iz'v = vzv_1$, or $G[\{v_1, \ldots, v_t\}] = K_3$ and $v_iz'v = v_2v_3$. Thus, for
2 ≤ i ≤ t, we have
\[ f(v_i) = \sum_{z''v_i \in C} w(v_iz'z'', v_i) + \sum_{v',w',w'' \in U_2} w(v_iw'w'', v_i) + w(v_izv_i) + \eta \]
\[ \leq \frac{1}{2} \left( (2k-1-t') (k-1) - t'(k-t') \right) + \left( t'' + \frac{1}{2} \right) + \frac{1}{3} \cdot \frac{1}{3} \]
\[ = k \left( k - \frac{3}{2} \right) - \frac{1}{2} \left( t'(k+t' - 1) - (t''+1)t''-1 \right) + \frac{2}{3}, \]
where \( \eta = w(v_2zv_3, v_2) \) or \( w(v_2zv_3, v_3) \) if \( v_2zv_3 \) is a triangle, and \( \eta = 0 \) otherwise. Because the total loss of the vertices in \( U_2 \) is at least
\[ \left( \frac{1}{2} \left( t'(k+t' - 1) - (t''+1)t''-1 \right) - \frac{2}{3} \right) (t-1) + \frac{1}{2} + \frac{t''}{2}, \]
we can transfer these weights to vertices in \( U_2 \) and for \( f_{U_2}(z) = 0, f(v) \leq k \left( k - \frac{3}{2} \right) \) is still valid for any \( v \in U_2 \), and (4.1) still holds.

(3) \( zv_1 \in \mathcal{L} \).

In this case, we have \( w(z_i zv_1, z_i) = 0 \) for \( i \leq t' \) by the definition of \( w(T, \cdot) \). Since \( zv_1 \in \mathcal{L} \) and \( \{v_2, ..., v_t\} \subseteq N(zv_1) \), we have \( t \leq k - 1 \). If \( G[\{v_1, ..., v_t\}] \) has an edge \( v_iv_j \in \mathcal{H} \), then since
\[ N(v_i zv) - \{z, v_1\} \subseteq U_2 - \{v_1, ..., v_t\} \]
by Claim (1), we have \( |U_2 - \{v_1, ..., v_t\}| \geq 2k-3 \geq k-1 = t \), which means \( |U_2| > 2t \). If \( G[\{v_1, ..., v_t\}] \) has no edge in \( \mathcal{H} \), then \( w(zv_1 v_i, z) \leq \frac{1}{3} \) and \( \lambda(z) \leq \frac{1}{3} \). Thus, by Lemma 1 we can transfer the weight \( \sum_{i \leq t'} w(zv_1 v_i, z) + \lambda(z) \) to the vertices of \( U_2 \) in the former case and to the vertices of \( \{v_1, ..., v_t\} \) in the latter case. So, \( f_{U_2}(z) = 0, f(v) \leq k \left( k - \frac{3}{2} \right) \) is still valid for any \( v \in U_2 \), and (4.1) still holds.

Thus, Theorem 8 and (4.1) hold.

Case 1.2 \( X \neq \emptyset \).

Let \( X = \{x_1, ..., x_s\} \) and \( Y = V(G_u) - V(C) - X \). By Lemma 3 both \( X \) and \( Y \) are independent sets. Moreover, since \( |C| = 2k - 1 - 2|X| = 2k - 1 - 2s \geq k + 1 \), we get that
\[ s \leq \frac{k + 1}{2} - 1. \]
By Claim 3, \( f_{U_2}(z) \leq \frac{3k}{2} - 2 \). Moreover, if \( |V(C), X| = 1 \), then \( d_{G_u}(z) = k - 1 \). By Claim 1, \( zv_1 \in \mathcal{L} \) and \( N_{U_2}(z) \) is an independent set. Therefore,
\[ f_{U_2}(z) = 0 \text{ if } |V(C), X| = 1. \] (4.2)

Consider the total loss of all the vertices in \( Y \) contributed by the edges in \([X, Y]\). Let \( xy \) be any edge with \( x \in X \) and \( y \in Y \), and \( X_y \subseteq V(G_y) \) satisfy (3.1). Because \( xy \in \mathcal{H} \), by Lemma 1 \( x \in \mathcal{H}(y) \subseteq X_y \). Since \( X \) and \( Y \) are independent sets, \( N(xy) \subseteq U_1 \cup U_2 \). Thus, if \( xyv \) is a triangle, then \( xv \in \mathcal{L} \) by Claim 1, which implies \( w(xyv, y) = 0 \), and so
\[ \sum_{v \in N(xy)} w(xyv, y) = 0. \]
By Observation 2, the edge \( xy \) contributes \( k - 1 \) to the loss of \( y \). Therefore, the total loss of all the vertices in \( Y \), contributed by the edges in \([X, Y]\), is at least \(|[X, Y]| \cdot (k - 1)\).
Since \( X \) and \( Y \) are independent sets, \( N_{U_2}(x) \) is an independent set by Claim 1, \( |V(C), X| \leq 1 \) and \( [U_1, G_u] \subseteq \mathcal{L} \). So,
We try to transfer the weights \( w(xyv, x) \) of \( x \in X \) to the vertices \( y, v \), such that the new weight of \( x \) is 0, and that of each other vertex remains no more than \( k(k-\frac{3}{2}) \).

Fix an edge \( yv \) and let \( N(yv) \cap X = \{x_1, \ldots, x_{s'}\} \). Then \( x_iy \in H \) and \( x_iv \in L \) for \( 1 \leq i \leq s' \) by the arguments above. If \( yv \in L \), then \( w(x_iyv, x_i) = 0 \), and so there is nothing to transfer. If \( yv \notin L \), then \( w(x_iyv, x_i) = \frac{1}{2} \). Let \( X' \subseteq V(G_v) \) satisfy (3.1).

If \( yv \in H \), then since \( yv, x_iy \in H \), by Observation 3 the edge \( yv \) contributes at least \( \frac{1}{2}(s'+1) \) to the loss of \( v \), and so we can transfer the weight \( \sum_{i=1}^{s'} w(x_iyv, x_i) = \frac{s'}{2} \) to \( v \) to cover the loss caused by the edge \( yv \).

Suppose \( yv \in M \). Note that \( \sum_{v' \in N(x, v)} w(x_iyv', v) \leq (k-1) - \frac{1}{2} \) by (3.4) because \( w(x_iyv, v) = \frac{1}{2} \). If \( x_i \in X' \), then by Observation 2 the edge \( x_iv \) contributes \( \frac{k}{2} \) to the loss of \( v \). So we can transfer the weight \( w(x_iyv, x_i) \) to \( v \) to cover the loss contributed by the edge \( x_iv \). If \( y \in X' \), then by (3.3) and Observation 2 the edge \( yv \) contributes \( \frac{k}{6} \) to the loss of \( v \), and \( \frac{k}{12} \) to the loss of \( y \) by Observation 4 which means \( yv \) contributes at least \( \frac{k}{12} \) to the total loss of \( y \) and \( v \). Recalling \( s' \leq s \leq \frac{k}{2} - 1 \), we can transfer the weight \( \sum_{i=1}^{s'} w(x_iyv, x_i) = \frac{s'}{2} \) to \( y \) to cover the loss contributed by the edge \( yv \). If neither \( x_i \in X' \) nor \( y \in X' \), then the edge \( x_iy \) lies in some component \( C \) of \( G_v \). Rememver \( \sum_{v' \in N(x, v)} w(x_iyv', v) \leq (k-1) - \frac{1}{2} \), the edge \( x_iy \) contributes at least \( \frac{k}{12} \) to the loss of \( v \). Since \( yv \in M \), by Observation 4 it contributes at least \( \frac{k}{12} \) to the total loss of \( y \) and \( v \). Thus, the total loss of \( y \) and \( v \) is at least \( \frac{k}{3} \), and so we can transfer the weight \( \sum_{i=1}^{s'} w(x_iyv, x_i) = \frac{s'}{2} \) to \( y \) such that the weights of \( y, v \) are still no more than \( k(k-\frac{1}{2}) \).

If \( v_1 \in H \), then by Claim 3 we can transfer \( \frac{k}{3} \) from \( f_{U_2}(z) \) to \( v_1 \) to cover the loss of \( v_1 \) caused by \( zv_1 \) such that \( f_{U_2}(z) \leq k-1 \). After this possible transferring, we always have \( f_{U_2}(z) \leq k-1 \). Thus, recalling the total loss of all the vertices in \( Y \) contributed by the edges in \( [X, Y] \) is at least \( [X, Y] \cdot (k-1) \), (4.2) and \( f(x) = 0 \) for each \( x \in X \) after transferring and Theorem 9 the total weight of \( G \) is

\[
\sum_{v \in V(G)} f(v) \leq (n-2k+1+s)k \left( k - \frac{3}{2} \right) + N(K_3, C) + (k-1) - s(k-1)^2 \\
\leq (n-2k+1)k \left( k - \frac{3}{2} \right) + 2 \left( \frac{k-1}{3} \right) + \left( \frac{k}{2} - 1 \right)^2 \\
+ \frac{k^2}{4} + \left( -k^2 + \frac{7k}{2} - \frac{11}{3} \right) s + (2k-2)s^2 - \frac{4}{3}s^3 \\
= (n-2k+1)k \left( k - \frac{3}{2} \right) + 2 \left( \frac{k-1}{3} \right) + \left( \frac{k}{2} - 1 \right)^2 + \varphi(s, k).
\]

Because \( 1 \leq s \leq \frac{k}{2} - 1 \), after an easy calculation, we get \( \varphi(k, s) < 0 \) except \( \varphi(2, 6) = 1 \), \( \varphi(1, 4) = 3 \) so the result holds if \( (s, k) \neq (2, 6), (1, 4) \).

Now, consider the two exceptions. Let \( yv \) be any edge with \( y \in Y \) and \( v \in U_2 \).

Suppose that \( (s, k) = (2, 6) \). Then \( X = \{x_1, x_2\} \). Let \( U = N_{U_2}(x_1) \cup N_{U_2}(x_2) \). Assume that \( yv \in H \cup M \). If both \( x_1yv \) and \( x_2yv \) are triangles, then since \( N_{U_2}(x_1) \) and \( N_{U_2}(x_2) \) are
independent sets by Claim 1. If \( N(yv) - \{x_1, x_2\} \subseteq U_2 - U \). Clearly, \(|U_2 - U| \geq |N(yv) - \{x_1, x_2\}| \geq k - 2\). If \(|N(yv) \cap \{x_1, x_2\}| \leq 1\) for any \( yv \in \mathcal{H} \cup \mathcal{M} \), then since \( k = 6 \), \( yv \) contributes at least \( \frac{1}{2} \) to the loss of \( y \) by Observations 2 and 3, and we can transfer \( w(x_iyv, x_i) \) only to \( y \) to cover the loss of \( y \) contributed by \( yv \) if \( x_iyv \) is a triangle, such that \( f(x_i) = 0 \) for \( i = 1, 2 \) after the possible transferring. Moreover, since \(|N(yv) \cap U_2| \geq k - 1, |U_2| \geq k\), thus, note that no weights are transferred to the vertices in \( U_2 - U \) in the former case and in \( U_2 \) in the latter case. \( U_2 \) has at least \( k - 2 \) vertices whose weights are at most \( k(k - \frac{3}{2}) - \frac{1}{2} \) after transferring, which implies

\[
\sum_{v \in V(G)} f(v) \leq (n - 2k + 1)k \left( k - \frac{3}{2} \right) + 2 \left( \frac{k - 1}{3} \right) + \left( \frac{k}{2} - 1 \right)^2 + \varphi(s, k) - \frac{1}{2}(k - 2)
\]

\[
< (n - 2k + 1)k \left( k - \frac{3}{2} \right) + 2 \left( \frac{k - 1}{3} \right) + \left( \frac{k}{2} - 1 \right)^2 .
\]

If \([Y, U_2] \subseteq \mathcal{L}\), then \( f(x_i) = 0 \) for \( i = 1, 2 \) by Claim 1 and the definition of \( w(T, \cdot) \). Since \( \varphi(2, 6) = 1 \), if \(|U_2| \geq 3\), then replace \( \frac{1}{2}(k - 2) \) with \( \frac{1}{2} \cdot 3 \) in the above inequality, we get the desired result. If \(|U_2| \leq 2\), then \( f(y) \leq 1 \) for any \( y \in \mathcal{Y} \). It is easy to see the total weight of \( G \) is less than the expected number.

Suppose that \((s, k) = (1, 4)\). Then \( X = \{x_1\} \). We will transfer the weight \( f(x_1) \) and \( f_{U_2}(z) \) to other vertices in a bit different way. Note that \( f(x_1) \) comes from the triangles \( x_1yv \) with \( yv \in \mathcal{H} \cup \mathcal{M} \). Since \( x_1y \in \mathcal{H} \) for any \( y \in \mathcal{Y} \), by Lemma 1 \( x_1 \in X_y \). If \( yv \in \mathcal{H} \), then by Observation 2, the edge \( yv \) contributes \( \frac{1}{2} \) to the loss of \( y \). Assume that \( yv \in \mathcal{M} \). If \( v \in X_y \), then by (3.3) and Observation 2, \( yv \) contributes at least \( \frac{5}{6} = \frac{3}{2} > \frac{1}{2} \) to the loss of \( y \). If \( v \) lies in a component \( C' \) of \( G_y - X_y \), then \( x_1 \in X_y \) and (3.1) imply \( |C'| \leq 2k - 3 \), and so

\[
\sum_{v' \in N(yv) \cap C'} w(yv, y) \leq \frac{1}{2}(k - 2) + \frac{1}{3}(2k - 4 - (k - 2)) = (k - 1) - \left( \frac{k}{6} + \frac{2}{3} \right),
\]

and so \( yv \) contributes \( \frac{1}{2} (\frac{k}{6} + \frac{2}{3}) > \frac{1}{2} \) to the loss of \( y \) by Observation 4. Therefore, the edge \( yv \) contributes at least \( \frac{1}{2} \) to the loss of \( y \). Transfer the weight \( w(x_1yv, x_1) = \frac{1}{2} \) to \( y \) such that the new weight of \( y \) is no more than \( k (k - \frac{3}{2}) - (k - 1) \), where the loss \( k - 1 \) is contributed by the edge \( x_1y \). Transfer the weight \( f_{U_2}(z) \) to the vertices in \( U_2 \) in the same way used in Case 1.1. After the transferring, the weight of \( G \) satisfies

\[
\sum_{v \in V(G)} f(v) < (n - 2k + 1)k \left( k - \frac{3}{2} \right) + 2 \left( \frac{k - 1}{3} \right) + \left( \frac{k}{2} - 1 \right)^2 ,
\]

and so the proof of Case 1 is complete.

**Case 2.** \( G_u = C_1 \cup K_{k-1} \)

In this case, \( X = \emptyset \) and \( G_u = C \cup K_{k-1} \). Since \( \pi(C) = (k - 1, ..., k - 1, k - 2) \) by Lemma 3 and \(|C| = k + 1\), \( C \) is the complement of \( \frac{1}{2}(k - 2)K_2 \cup P_3 \), and so we have

\[
N(K_3, G_u) = \left( \frac{k - 1}{3} \right) + \left( \frac{k + 1}{3} \right) - \frac{1}{2}(k - 2)(k - 1) - 2(k - 2) - 1.
\]
Set $V(K_{k-1}) = \{p_1, \ldots, p_k\}$. We first discuss some properties of these vertices.

Claim 4  If $q_1, q_2 \in U_2$ such that $p_i q_1, p_i q_2 \in \mathcal{H} \cup \mathcal{M}$, then
(1) $G[N_{U_2}(p_i)]$ consists of a triangle $q_1 q_2 q_3$ and some isolated vertices;
(2) $|N(p_i q_1)| = |N(p_i q_2)| = k$ and $\{p_1, \ldots, p_{k-1}\} \subseteq N(q_1) \cap N(q_2)$;
(3) If $v \in U_2$ such that $p_i v \in \mathcal{H} \cup \mathcal{M}$, then $v \in \{q_1, q_2, q_3\}$. Moreover, if $v = q_3$, then $|N(p_i q_3)| = k$ for $1 \leq j \leq k-1$ and $1 \leq s \leq 3$.

Proof. (1) Since $d_{G_1}(p_i) = k - 2$ and $p_i q_1, p_i q_2 \in \mathcal{H} \cup \mathcal{M}$, by Claim 2, $G[N_{U_2}(p_i)]$ is a triangle containing $q_1, q_2$, say $q_1 q_2 q_3$, together with some isolated vertices.

(2) Since $|N(p_i q_1)| \geq k$ and $N(p_i q_1) \cap U_2 = \{q_2, q_3\}$ by (1), $\{p_1, \ldots, p_{k-1}\} \subseteq N(q_1)$. By the symmetry of $q_1$ and $q_2$, $\{p_1, \ldots, p_{k-1}\} \subseteq N(q_2)$, and so the result follows.

(3) Suppose $v \notin \{q_1, q_2, q_3\}$. By Claim 2 and (2), $G[N_{U_2}(p_i)]$ is a triangle $v q_1 q_2$ together with some isolated vertices. Because $|N(p_i v)| \geq k$ and $N(p_i v) \cap U_2 = \{q_1, q_2\}$, we have $\{p_1, \ldots, p_{k-1}\} \subseteq N(v)$. Thus, $v \in N(p_i q_1)$ and hence $|N(p_i q_1)| \geq k + 1$ which contradicts (2). Therefore, $v \notin \{q_1, q_2, q_3\}$. Moreover, if $v = q_3$, then since $|N(p_i q_3)| \geq k$ and $N(p_i q_3) \cap U_2 = \{q_1, q_2\}$, we can deduce $\{p_1, \ldots, p_{k-1}\} \subseteq N(q_3)$. After an easy check, we get that $|N(p_i q_3)| = k$ for $1 \leq j \leq k - 1$ and $1 \leq s \leq 3$.  

By Lemma 2 we have $|(\mathcal{H}(p_i) \cup \mathcal{M}(p_i)) \cap U_2| \leq 3$ for all $1 \leq i \leq k - 1$. Suppose $|(\mathcal{H}(p_i) \cup \mathcal{M}(p_i)) \cap U_2| = 3$ for some $i$. By Claim 4, $G[N_{U_2}(p_i)]$ is a triangle $q_1 q_2 q_3$ together with some isolated vertices and $p_i q_1, p_i q_2, p_i q_3 \in \mathcal{M}$. Furthermore, we have $p_i q_3 \in E(G)$ and $|N(p_i q_3)| = k$ for $1 \leq j \leq k - 1$ and $1 \leq s \leq 3$, and $q_1 q_2, q_1 q_3, q_2 q_3 \in \mathcal{H} \cup \mathcal{M}$. Because

$$\sum_{v \in N(p_i q_3)} w(p_i q_3 v, q_3) = \frac{k}{3} \leq \frac{k}{2} = k - 1 - \left(\frac{k - 1}{2}\right),$$

the edge $p_i q_3$ contributes at least $\frac{k}{2} - \frac{1}{2}$ to the loss of $q_3$ by Observation 4. Hence, all the edges $p_i q_3$, $1 \leq j \leq k - 1$ and $1 \leq s \leq 3$, contribute at least $3(k - 1)\left(\frac{k}{2} - \frac{1}{2}\right)$ to the total loss of $q_1$, $q_2$ and $q_3$. On the other hand, note that $G[N_{U_2}(p_j)]$ is the triangle $q_1 q_2 q_3$ together with some isolated vertices by Claim 4 and $[p_j, U_2 - \{q_1, q_2, q_3\}] \subseteq \mathcal{L}$ by Claim 4. So, the weight of $p_j$ contributed by the triangles not in $K_{k-1}$ is

$$\sum_{1 \leq r < s \leq 3} \sum_{p_j \neq p_i} w(p_r p_s q_3, p_i) + \sum_{1 \leq s \leq 3} w(p_j p_3 q_s, p_i) = 3 \cdot \frac{1}{3} + 3(k - 2) \cdot \frac{1}{3} = k - 1.$$

By Claim 4, $f_{U_2}(z) \leq \frac{3k}{2} - 2$. Therefore, the total weight of $G$ is at most

$$(n - 2k)k \left(\frac{k - 3}{2}\right) + N(K_3, G_n) + \left(\frac{3k}{2} - 2\right) + (k - 1)^2 - 3(k - 1) \left(\frac{k}{4} - \frac{1}{2}\right)$$

$$< (n - 2k + 1)k \left(\frac{k - 3}{2}\right) + 2 \left(\frac{k - 1}{3}\right) + \left(\frac{k}{2} - 1\right)^2,$$

a contradiction. So we assume that $|(\mathcal{H}(p_i) \cup \mathcal{M}(p_i)) \cap U_2| \leq 2$ for $1 \leq i \leq k - 1$.

Fix $p_i$ and let $N_{U_2}(p_i) = \{q_1, \ldots, q_r\}$. By Claim 4, we may assume that $G[N_{U_2}(p_i)]$ is a star.
with the center $q_1$ or a triangle $q_1q_2q_3$, and some isolated vertices. Let

$$f_{U_2}(p_i) = \sum_{v, v' \in U_2} w(p_i vv', p_i) + \sum_{p_j \neq p_i} \sum_{v \in N(p_j) \cap U_2} w(p_i p_j v, p_j).$$

It is clear that $\sum_{i=1}^{k-1} f_{U_2}(p_i)$ is the total weight of $V(K_{k-1})$ contributed by the triangles not contained in $K_{k-1}$. We will complete the proof by showing that $f_{U_2}(p_i) \leq \frac{3k}{4} - \frac{1}{2}$ and $f_{U_2}(z) < \frac{3k}{4} - \frac{1}{2}$, after some appropriate weight transferring.

If $|\{H(p_i) \cup M(p_i)\} \cap U_2| = 2$, say $p_i q_1, p_i q_2 \notin L$, then by Claim 4 $N_{U_2}(p_i)$ is a triangle $q_1q_2q_3$ together with some isolated vertices, $|N(p_i, q_1)| = |N(p_i, q_2)| = k$ and so

$$f_{U_2}(p_i) = \sum_{v, v' \in \{q_1, q_2, q_3\}} w(p_i vv', p_i) + \sum_{p_j \neq p_i} w(p_i p_j q_1, p_j) + \sum_{p_j \neq p_i} w(p_i p_j q_2, p_j).$$

Since

$$\sum_{v \in N(p_i, q_s)} w(p_i q_s v, q_s) \leq \frac{k}{2} = (k-1) - \left(\frac{k}{2} - 1\right)$$

for $s = 1, 2$,

the edge $p_i q_s$ contributes at least $\frac{k}{2} - \frac{1}{2}$ to the loss of $q_s$ for $s = 1, 2$ by Observation 4. Because $\sum_{p_j, \neq p_i} w(p_i p_j q_s, p_j) \leq \frac{k}{2}(k-2)$, transfer $\frac{k}{2} - \frac{1}{2}$ from $\sum_{p_j, \neq p_i} w(p_i p_j q_s, p_j)$ to $q_s$ for $s = 1, 2$.

After transferring, we have

$$f_{U_2}(p_i) \leq 3 + 2 \cdot \frac{1}{2} - 2 \left(\frac{k}{4} - \frac{1}{2}\right) = \frac{k + 1}{2} \leq \frac{3k}{4} - \frac{1}{2}. \tag{4.3}$$

Now, let $|\{H(p_i) \cup M(p_i)\} \cap U_2| \leq 1$. Assume $H(p_i) \cup M(p_i) \subseteq \{q_1\}$ by Claim 1.2. Because $p_i p_j \in H$ and $p_i q_s \in L, w(p_i p_j q_s, p_j) = 0$ for $2 \leq s \leq r$ and hence

$$f_{U_2}(p_i) = \sum_{j=2}^{r} w(p_i q_1 q_j, p_i) + \lambda(p_i) + \sum_{p_j \neq p_i} w(p_i p_j q_1, p_j),$$

where $\lambda(p_i) = w(p_i q_2 q_3, p_i)$ if $q_1 q_2 q_3$ is a triangle and $\lambda(p_i) = 0$ otherwise. Using the same proof as that of Claim 3 we have

Claim 5 $f_{U_2}(p_i) \leq k - 1$ if $p_i q_1 \notin H$, and $f_{U_2}(p_i) \leq k - 1 + \frac{k}{2} \leq \frac{3k}{2} - 2$ and $p_i q_1$ contributes at least $\frac{1}{2}(\ell + 1)$ to the loss of $q_1$ if $p_i q_1 \in H$, where $|N(p_i, q_1) \cap V(K_{k-1})| = \ell$.

In order to show $f_{U_2}(z) \leq \frac{3k}{4} - \frac{1}{2}$ in this case and $f_{U_2}(z) < \frac{3k}{4} - \frac{1}{2}$, we need to consider the structure of $G[U_2]$.

If $vv' \in M$ is an edge in $G[U_2]$, then by Observation 4 $vv'$ contributes $\frac{k}{6}$ to the loss of $v$ and $v'$, respectively, that is, $vv'$ contributes $\frac{k}{6}$ to the total loss of vertices in $U_2$. On the other hand, by Claims 3 and 5 we can transfer some weight from $f_{U_2}(z)$ and $f_{U_2}(p_i)$ to $v_1$ and $q_1$, respectively, such that $f_{U_2}(z) \leq k - 1$ and $f_{U_2}(p_i) \leq k - 1$, and $f(v_1) \leq k (k - \frac{3}{2})$ and $f(q_1) \leq k (k - \frac{3}{2})$ still hold. This together with (4.3) implies that after transferring some weights to the vertices in $U_2$, the total weight in $\sum_{v \in V(G_u)} f(v)$ coming from the triangles not
in $G_u$ is at most $k(k-1)$. Therefore, if $G[U_2]$ has $\frac{3k}{2} - 2$ edges in $M$, then we have

$$\sum_{v \in V(G)} f(v) \leq (n-2k)k \left( k - \frac{3}{2} \right) + N(K_3, G_u) + k(k-1) - \left( \frac{3k}{2} - 2 \right) \frac{k}{6}$$

$$< (n-2k+1)k \left( k - \frac{3}{2} \right) + 2 \left( k - \frac{1}{3} \right) + \left( \frac{k}{2} - 1 \right)^2,$$

a contradiction. Hence, $G[U_2]$ contains at most $\frac{3k}{2} - 3$ edges in $M$. Moreover, we have

Claim 6 Let $qq' \in H$ be an edge in $G[U_2]$. If $M(q) \cap M(q') \cap \{p_1, \ldots, p_{k-1}\} = \emptyset$, then $qq'$ contributes $\frac{3k}{8}$ to the total loss of $q$ and $q'$. Furthermore, $G[U_2]$ contains at most $\frac{3k}{2} - 2$ edges in $H \cup M$.

Proof By Claim 2 and the assumption, $M(q) \cap M(q') \cap \{z, p_1, \ldots, p_{k-1}\} = \emptyset$. Noting that $G[U_2]$ has at most $\frac{3k}{2} - 3$ edges in $M$, we have $|M(q)| + |M(q')| \leq k + \frac{3k}{2} - 3$. Thus, by (3.2) and Lemma 4(ii), we get

$$\sum_{v \in N(qq')} w(qq'v, q) + \sum_{v \in N(qq')} w(qq'v, q')$$

$$\leq \frac{1}{2} \left( |H(q) - \{q'\}| + |M(q)| + |H(q') - \{q\}| + |M(q')| \right) \leq 2(k-1) - \frac{3k}{8},$$

and so the conclusion follows by Observation 2.

In addition, recall $(|H(p_j) \cup M(p_j)|) \cap U_2 \leq 2$ for all $1 \leq j \leq k-1$, by Claim 4 $G[U_2]$ has at most one edge $q'_1q'_2$ such that $M(q'_1) \cap M(q'_2) \cap \{p_1, \ldots, p_{k-1}\} \neq \emptyset$. For any other $H$-edge $qq'$ in $G[U_2]$, $qq'$ contributes at least $\frac{3k}{8} > \frac{3k}{8}$ to the total loss of $q$ and $q'$, which, together with the possible edge $q'_1q'_2$, implies $G[U_2]$ contains at most $\frac{3k}{2} - 2$ edges in $H \cup M$.

Now, let us re-consider $f_{U_2}(z)$ and $f_i = f_{U_2}(p_i)$ based on Claim 6. For convenience, let $a \in \{z, p_1, \ldots, p_{k-1}\}$, $N_{U_2}(a) = \{b_1, \ldots, b_m\}$ is a star with center $b_1$ or a triangle $b_1b_2b_3$ in $G[N_{U_2}(a)]$. $(H(a) \cup M(a)) \cap U_2 \subseteq \{b_1\}$ and $N(ab_1) \cap V(G_u) = \{a_1, \ldots, a_\ell\}$. Recall the expressions of $f_{U_2}(z)$ and $f_i = f_{U_2}(p_i)$, we have

$$f_{U_2}(a) = \sum_{j=2}^{m} w(ab_1b_j, a) + \lambda(a) + \sum_{j=1}^{\ell} w(ab_1a_j, a_j).$$

If $ab_1 \in H$, then since $aa_i \in H$, we can transfer the weight $w(ab_1a_i, a_i)$ to $b_1$ to cover the loss caused by the edge $ab_1$ by Observation 3. For the weight $w(ab_1b_j, a)$, we have $w(ab_1b_j, a) \leq \frac{1}{2}$ with equality only if $b_1b_j \in H \cup M$. Thus, by Claim 6 after transferring, we have

$$f_{U_2}(a) = \sum_{j=2}^{m} w(ab_1b_j, a) + \lambda(a) \leq \max \left\{ 2, \frac{3k}{4} - 1 \right\} \leq \frac{3k}{4} - \frac{1}{2}.$$

Assume $ab_1 \in M$. Consider $w(ab_1b_j, a)$. If $b_1b_j \in H$, then $w(ab_1b_j, a) = \frac{1}{2}$, and $b_1b_j$ contributes $\frac{3k}{8}$ to the total loss of $b_1$ and $b_j$ by Claim 6. Since $a \in \{z, p_1, \ldots, p_{k-1}\}$, there are at most $k$ such triangles, and so we can transfer $\frac{3k}{8}$ of each $w(ab_1b_j, a)$ to $b_1$ and $b_j$ to cover the total loss of $b_1$ and $b_j$ caused by the edge $b_1b_j$. If $b_1b_j \in M \cup L$, then $w(ab_1b_j, a) = \frac{1}{3}$ and $ab_1$
contributes \(\frac{k}{12}\) to the loss of \(b_1\) by Observation 4. Thus, we can transfer \(\frac{1}{12}\) of each \(w(ab_1b_j, a)\) to cover the loss of \(b_1\) caused by the edge \(ab_1\). After transferring, we have \(w(ab_1b_j, a) \leq \frac{1}{4}\) and so
\[
f_{U_2}(a) \leq \frac{1}{4}|N(ab_1) \cap U_2| + \frac{1}{2}|N(ab_1) \cap V(G)| \leq \frac{3k}{4} - 1 < \frac{3k}{4} - \frac{1}{2}.
\]

If \(ab_1 \in L\), then \(w(ab_1a_j, a_j) = 0\) for \(1 \leq i \leq \ell\). If \(b_1b_j \in H\), then \(w(ab_1b_j, a) = 1\). By Claim 6 we can transfer \(\frac{1}{2}\) to cover the total loss of \(b_1\) and \(b_j\) caused by the edge \(b_1b_j\), with at most one exceptional edge in \(G[U_2]\). If \(b_1b_j \in M \cup L\), then \(w(ab_1b_j, a) = \frac{1}{3}\). After transferring, we have \(w(ab_1b_j, a) \leq \frac{5}{8}\) with at most one exception and so
\[
f_{U_2}(a) = \sum_{j=2}^{m} w(ab_1b_j, a) + \lambda(a) + \leq \frac{5}{8}(k - 2) + 1 < \frac{3k}{4} - \frac{1}{2}.
\]

By the three inequalities above, we have \(f_{U_2}(z) < \frac{3k}{4} - \frac{1}{2}\). Moreover, combining the three inequalities with (4.3), we have \(f_{U_2}(p_i) \leq \frac{3k}{4} - \frac{1}{2}\). Thus, after appropriate weight transferring, we have \(f_{U_2}(z) + \sum_{i=1}^{k-1} f_i < k(\frac{3k}{4} - \frac{1}{2})\). Hence, the total weight of \(G\) is
\[
\sum_{v \in V(G)} f(v) < (n - 2k)k \left( k - \frac{3}{2} \right) + N(K_3, G) + k \left( \frac{3k}{4} - \frac{1}{2} \right)
\]
\[
\leq (n - 2k + 1)k \left( k - \frac{3}{2} \right) + 2 \left( k - \frac{1}{3} \right) + \left( k - \frac{1}{2} \right)^2.
\]

The proof of Theorem 4 is complete.

5 Concluding Remarks

Theorem 4 is proved for \(n \geq 4k^3\), but notice that the statement does not hold for small \(n\). For example, take at most five disjoint copies of \(K_{2k}\), then the number of copies \(K_3\) is more than the extremal number in the theorem. It would be nice to determine the sharp bound for \(n\) when this generalized Turán number is correct.

It is natural to ask what happens if we count larger cliques. The third author [15] showed that \(ex(n, K_r, F_k) = O(n)\) for every \(k\) and \(r\), but the constants in the upper bound are large. We conjecture that the extremal graph for \(ex(n, K_r, F_k)\) is still \(K_{n-v(H)} + H\), where \(H\) is a graph with \(V(H) = k - 1\), \(\Delta(H) = k - 1\).

Let \(H^T\) be the graph obtained by replacing each edge of \(H\) with a triangle, e.g., the friendship graph can be considered as a \(S^T_k\). So it is also interesting to ask what if we replace each edge of any other graph \(H\) with a triangle? For example, consider the extremal function \(ex(n, K_3, P^T_k), ex(n, K_3, C^T_k)\).

We have determined the largest number of triangles in \(F\)-free graphs when \(F\) is a friendship graph, but not when \(F\) is an extended friendship graph. Alon and Shikhelman [11] showed that in that case \(c_1|V(F)|^2n \leq ex(n, K_3, F) \leq c_2|V(F)|^2n\) for absolute constants \(c_1\) and \(c_2\). Better bounds were obtained for some forests, including exact results for stars [4], paths [19] and forests consisting only of path components of order different from 3 [25].

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