Parallel Texture Structures with Cofactor Zeros in Leptonic Sector

Weijian Wang

\textsuperscript{1}Department of Physics, North China Electric Power University, Baoding 071003, P. R. China

Abstract

In this paper we investigate the parallel texture structures with cofactor zeros in the charged lepton and neutrino sectors. These textures can not be obtained from arbitrary leptonic matrices by making weak basis transformations, which therefore have physical meaning. The 15 parallel textures are grouped as 4 classes where each class have the same physical implications. It is founded that one of them is not phenomenological viable and another is equivalent to the texture zero structures extensively explored in previous literature. Thus we focus on the other two class of parallel texture structures and study the their phenomenological implications of the correlations between physical variables such as the mixing angle, the Dirac CP-violated phase and Majorana effective neutrino mass. The constraint on these physical variables are obtained for each class which is essential for the model selection and can be measured by future experiments.

\textsuperscript{*}Electronic address: wjnwang96@gmail.com
I. INTRODUCTION

The discovery of neutrino oscillations have provided us with convincing evidences for massive neutrinos and leptonic flavor mixing with high degree of accuracy. The recent measurement of large reactor mixing angle $\theta_{13}$ has not only open the door for us to explore the leptonic Dirac-CP violation and the mass hierarchy in the future experiments, but also highlight the flavor puzzle of neutrino mass and mixing pattern which appears to be rather different from the distinct mass hierarchy and the small mixing angles shown by quarks. Although a full theory is still missing, several ideas have been proposed by introducing various flavor symmetries to explain the observed leptonic mixing pattern. By reducing the number of free parameters in Yukawa sector, these flavor symmetries often lead to specific structures of leptonic mass matrices including texture zeros, hybrid textures, zero trace, zero determinant, vanishing minors, two traceless submatrices, equal elements or cofactors, hybrid $M_{\nu}^{-1}$ textures. Among these models, the textures with zero elements or zero minors are particularly interesting, which is not only because the textures can be naturally realized by introducing discrete flavor symmetry within seesaw mechanisms, but also they are stable against the one-loop quantum corrections as the running of RGEs from seesaw scale $\Lambda$ to the electroweak scale $\mu \simeq M_Z$. The phenomenological analysis of neutrino mass matrices with texture zeros or cofactor zeros in flavor basis have been widely investigated in earlier literature.

However, there is no priori requirement that the analysis must be done in flavor basis. The more general situation should be considered in the basis where both charged lepton and neutrino mass textures are nondiagonal. In this spirit, the parallel Ansätze has been proposed where the charged lepton mass matrix $M_l$ and the Majorana neutrino mass matrix $M_\nu$ having the same structure (We denote them “parallel structure”). A popular parallel structures appears as the Fritzsch-like matrix with zero elements and firstly applied to understand the quark mixing pattern. Subsequently the idea is generalized to the leptonic sector. A systematic and completed search on the parallel structure with texture zeros in lepton mass matrix...
are reported in Ref. [18]. It is shown that some sets of texture zeros have no physical meaning by themselves, since they can be obtained by making suitable weak basis (WB) transformation from arbitrary mass matrix and leaving the gauge currents invariant. The minimal no trivial case is the four texture zero model. Recently, a similar investigation is done in the context of parallel hybrid textures with one zero and two equal elements [19].

In this work, we study the parallel structures with two cofactor zeros in both charged lepton and neutrino mass texture. There exist \( C_6^2 = 15 \) logically possible patterns. Furthermore, we assume the mass matrices to be Hermitian and all neutrinos are massive, which indicates \( \text{det} M_\nu \neq 0 \) and existence of \( M_\nu^{-1} \). Thus the mass textures \( M_\nu \) with cofactor zeros are equivalent to the \( M_\nu^{-1} \) with texture zeros. As the texture zero case [18], the 15 textures structures can be grouped into 4 classes and the textures belonging to the specific class have the same physical implications. Among the 4 classes, we will find that one of them is not viable phenomenologically and another class is equal to the matrix with texture zeros. In this sense, we pay our attention to the other two classes which having been studied before.

The paper is organized as follows. In Sec. II, we discuss the classification of textures and relate them to the experimental results. In Sec. III, we diagonalize the mass matrices, confront the numerical results with the experimental data and discuss their predictions. A summary is given in Sec. IV.

II. FORMALISM

A. Weak basis equivalent classes

We assume the neutrinos to be Majorana fermions. The most general WB transformations leaving gauge currents invariant is given by

\[
M_l \to M'_l = W^\dagger M_l W_R \quad M_\nu \to M'_\nu = W^T M_\nu W
\]

(1)
where $M_l, M_\nu$ denote the charged lepton mass matrices and neutrino mass matrices respectively and $W, W_R$ are $3 \times 3$ unitary matrices. Therefore the parallel texture with cofactor zeros located at different positions can be related by permutation matrix $P$ as the WB transformation

$$M'_l = P^T M_l P \quad M'_\nu = P^T M_\nu P$$

which change the position of cofactor zero element but preserve the parallel structure for both charged lepton and neutrino mass textures. It is noted that $P$ belongs to the group of 6 permutations and are isomorphic to $S_3$. Then the four cofactor zeros texture can be classified into 4 classes as following:

Class I:

$$\begin{pmatrix}
\Delta \times \Delta \\
\times \times \times \\
\Delta \times \times \\
\times \times \times \\
\times \Delta \Delta \\
\times \Delta \times
\end{pmatrix}, \quad \begin{pmatrix}
\Delta \Delta \times \\
\times \times \times \\
\times \times \times \\
\times \times \Delta \\
\times \times \times \\
\times \Delta \Delta
\end{pmatrix}, \quad \begin{pmatrix}
\times \Delta \times \\
\times \times \times \\
\times \times \times \\
\times \times \Delta \\
\times \times \times \\
\times \Delta \Delta
\end{pmatrix}$$

Class II:

$$\begin{pmatrix}
\Delta \times \times \\
\times \times \Delta \\
\times \Delta \times \\
\times \Delta \times
\end{pmatrix}, \quad \begin{pmatrix}
\times \times \Delta \\
\times \Delta \times \\
\Delta \times \times \\
\Delta \times \times
\end{pmatrix}, \quad \begin{pmatrix}
\times \Delta \times \\
\times \times \times \\
\times \times \times \\
\times \times \Delta
\end{pmatrix}$$

Class III:

$$\begin{pmatrix}
\Delta \times \times \\
\times \Delta \times \\
\times \times \times
\end{pmatrix}, \quad \begin{pmatrix}
\Delta \times \times \\
\times \times \times \\
\times \Delta \times
\end{pmatrix}, \quad \begin{pmatrix}
\times \times \times \\
\times \times \times \\
\times \times \Delta
\end{pmatrix}$$
Class IV:
\[
\begin{pmatrix}
\times \triangle \triangle \\
\triangle \times \times \\
\triangle \times \times 
\end{pmatrix}
\begin{pmatrix}
\times \triangle \times \\
\triangle \times \triangle \\
\times \times \triangle 
\end{pmatrix}
\begin{pmatrix}
\times \times \triangle \\
\times \times \triangle \\
\triangle \triangle \times 
\end{pmatrix}
\]
(6)

where ”△” at (i, j) position denotes its the zero cofactor \( C_{ij} = 0 \) while ”×” stands for arbitrary element. Since the \( M_{l,\nu} \) with zero cofactor is equivalent to \( M_{l,\nu}^{-1} \) with zero elements, the classification given above is the same as the texture zero ones shown in Ref. [18] except for replacing ”△” with ”0”. Like the matrices with texture zeros, the class IV leads to the decoupling of a generation of lepton from mixing and thus not experimentally viable. On the other hand, one can easily check that the textures of class I correspond to the texture zero cases, which has already studied in previous literature [16–18]. As an example, for the first matrix of class I, we have
\[
M_{l,\nu} = \begin{pmatrix}
\triangle \times \triangle \\
\times \times \times \\
\triangle \times \times 
\end{pmatrix}
\Rightarrow M_{l,\nu}^{-1} = \begin{pmatrix}
0 \times 0 \\
\times \times \times \\
0 \times \times 
\end{pmatrix}
\Rightarrow M_{l,\nu} = \begin{pmatrix}
\times \times \times \\
\times 0 0 \\
\times 0 \times 
\end{pmatrix}
\]
(7)

Therefore only class II and class III have notivial physical implication. In this work we concentrate on these two classes.

B. Some useful notations

As mentions above, we only need investigate two mass matrices as the representations of class II and class III. In this work, we choose
\[
M_{l,\nu}^{II} = \begin{pmatrix}
\triangle \times \times \\
\times \times \triangle \\
\times \triangle \times 
\end{pmatrix}
M_{\nu}^{III} = \begin{pmatrix}
\triangle \times \times \\
\times \triangle \times \\
\times \times \times 
\end{pmatrix}
\]
(8)

The charged leptonic mass texture \( M_l \) is the complex Hermitian matrix and the Majorana neutrino mass texture \( M_\nu \) is a complex symmetric matrix. Thus \( M_l \) and \( M_\nu \) can be diagonalized by unitary matrix \( V_l \) and \( V_\nu \)
\[
M_l = V_l M_l^D V_l^\dagger \\
M_\nu = V_\nu M_\nu^D V_\nu^T
\]
(9)
where $M^D_l = \text{Diag}(m_e, m_\mu, m_\tau)$, $M^D_\nu = \text{Diag}(m_1, m_2, m_3)$. The Pontecorvo-Maki-Nakagawa-Sakata matrix [20] $U_{PMNS}$ is given by

$$U_{PMNS} = V^\dagger \nu$$

and can be parameterized as

$$U_{PMNS} = U_P \nu = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - c_{23}s_{12}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i(\beta+\delta)} \end{pmatrix}$$

(11)

where the abbreviation $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$ is used. The $\alpha$ and $\beta$ in $P_\nu$ denote two Majorana CP-violating phases and $\delta$ in $U$ denotes the Dirac CP-violating phase. In order to most easily diagonalize the leptonic mass matrices and simplify our calculation, it is better to start from $M^{-1}_l$ rather than $M_l$. From (9), we get

$$M^{-1}_l = V_l(M^D_l)^{-1}V_l^\dagger$$

(12)

So the $V_l$ can not only diagonalize $M_l$ but also $M^{-1}_l$. Furthermore, we treat the Hermitian matrix $M^{-1}_l$ factorisable. i.e

$$M^{-1}_l = K_l(M^{-1}_l)^r K_l^\dagger$$

(13)

where $K_l$ is the unitary phase matrix and can be parameterised as $K_l = \text{diag}(1, e^{i\phi_1}, e^{i\phi_2})$. The $(M^{-1}_l)^r$ becomes a real symmetric matrix which can be diagonalized by real orthogonal matrix $O_l$. Then we have

$$V_l = K_l O_l$$

(14)

and

$$U_{PMNS} = O_l^T K_l^\dagger V_\nu$$

(15)

From (9), (10) and (15), the neutrino mass matrix $M_\nu$ is given by

$$M_\nu = K_l V P_\nu M^D_\nu P_\nu V^T K_l^\dagger$$

(16)

where $V \equiv O_l U$. From (16) The restriction of two cofactor zeros on $M_\nu$

$$M_{\nu(q)\nu(r\ell)} - M_{\nu(q)\nu(r\ell)} = 0 \quad M_{\nu(p)\nu(r\ell)} - M_{\nu(p)\nu(r\ell)} = 0$$

(17)
induces two equations

\[ m_1 m_2 K_3 e^{2 i \alpha} + m_2 m_3 K_1 e^{2 i (\alpha + \beta + \delta)} + m_3 m_1 K_2 e^{2 i (\beta + \delta)} = 0 \]  

(18)

\[ m_1 m_2 L_3 e^{2 i \alpha} + m_2 m_3 L_1 e^{2 i (\alpha + \beta + \delta)} + m_3 m_1 L_2 e^{2 i (\beta + \delta)} = 0 \]  

(19)

where

\[ K_i = (V_{pj} V_{qj} V_{rk} V_{sk} - V_{ij} V_{qj} V_{vk} V_{wk}) + (j \leftrightarrow k) \]  

(20)

\[ L_i = (V_{pj} V_{qj} V_{r'k} V_{s'k} - V_{ij} V_{q'j} V_{s'k} V_{w'k}) + (j \leftrightarrow k) \]  

(21)

with \((i, j, k)\) a cyclic permutation of \((1, 2, 3)\). After solving Eq. (18) and (19), we arrive at

\[ \frac{m_1}{m_2} e^{-2 i \alpha} = \frac{K_3 L_1 - K_1 L_3}{K_2 L_3 - K_3 L_2} \]  

(22)

\[ \frac{m_1}{m_3} e^{-2 i \beta} = \frac{K_2 L_1 - K_1 L_2}{K_3 L_2 - K_2 L_3} e^{2 i \delta} \]  

(23)

With the help of Eq. (22) and (23), we obtain the magnitudes of mass ratios

\[ \rho = \left| \frac{m_1}{m_3} e^{-2 i \beta} \right| \]  

(24)

\[ \sigma = \left| \frac{m_1}{m_2} e^{-2 i \alpha} \right| \]  

(25)

as well as the two Majorana CP-violating phases

\[ \alpha = -\frac{1}{2} \text{arg} \left( \frac{K_3 L_1 - K_1 L_3}{K_2 L_3 - K_3 L_2} \right) \]  

(26)

\[ \beta = -\frac{1}{2} \text{arg} \left( \frac{K_2 L_1 - K_1 L_2}{K_3 L_2 - K_2 L_3} e^{2 i \delta} \right) \]  

(27)

The results of Eq. (21), (25), (26) and (27) imply that the two mass ratio \((\rho \text{ and } \sigma)\) and two Majorana CP-violating phases \((\alpha \text{ and } \beta)\) are fully determined in terms of the real orthogonal matrix \(O_l\) and \(U(\theta_{12}, \theta_{23}, \theta_{13} \text{ and } \delta)\). The neutrino mass ratios \(\rho\) and \(\sigma\) are related to the ratios of two neutrino mass-squared ratios obtained from the solar and atmosphere oscillation experiments as

\[ R_\nu \equiv \frac{\delta m^2}{\Delta m^2} = \frac{2 \rho^2 (1 - \sigma^2)}{|2 \sigma^2 - \rho^2 - \rho^2 \sigma^2|} \]  

(28)
and to the three neutrino mass as

\[ m_2 = \sqrt{\frac{\delta m^2}{1 - \sigma^2}} \quad m_1 = \sigma m_2 \quad m_3 = \frac{m_1}{\rho} \]  

(29)

where \( \delta m^2 \equiv m_2^2 - m_1^2 \) and \( \Delta m^2 \equiv |m_3^2 - \frac{1}{2}(m_1^2 + m_2^2)| \). In the numerical analysis, we will use the latest global-fit neutrino oscillation experimental data, at 3\( \sigma \) confidential level, which is listed in Ref. [22]

\[ \sin^2 \theta_{12}/10^{-1} = 3.08^{+0.51}_{-0.49} \quad \sin^2 \theta_{23}/10^{-1} = 4.25^{+2.16}_{-0.68} \quad \sin^2 \theta_{13}/10^{-2} = 2.34^{+0.63}_{-0.57} \]  

\[ \delta m^2/10^{-5} = 7.54^{+0.64}_{-0.55} \text{eV}^2 \quad \Delta m^2/10^{-3} = 2.44^{+0.22}_{-0.21} \text{eV}^2 \]  

(30)

for normal hierarchy (NH) and

\[ \sin^2 \theta_{12}/10^{-1} = 3.08^{+0.51}_{-0.49} \quad \sin^2 \theta_{23}/10^{-1} = 4.25^{+2.22}_{-0.74} \quad \sin^2 \theta_{13}/10^{-2} = 2.34^{+0.61}_{-0.61} \]  

\[ \delta m^2/10^{-5} = 7.54^{+0.64}_{-0.55} \text{eV}^2 \quad \Delta m^2/10^{-3} = 2.40^{+0.21}_{-0.23} \text{eV}^2 \]  

(31)

for inverted hierarchy (IH). There is no constraint on the Dirac-CP violated phase \( \delta \) at 3\( \sigma \) level, however, the recent global fit tends to give \( \delta \approx 1.40\pi \). In neutrino oscillation experiments, CP violation effect is usually reflected by the Jarlskog rephasing invariant quantity [21] defined as

\[ J_{CP} = s_{12}s_{23}s_{13}c_{12}c_{23}c_{13}^2 \sin \delta \]  

(32)

The Majorana nature of neutrino can be determined if any signal of neutrinoless double decay is observed, implying the violation of leptonic number violation. The decay ratio is related to the effective of neutrino \( m_{ee} \), which is written as

\[ m_{ee} = |m_1 c_{12}^2 c_{13}^2 + m_2 s_{12}^2 c_{15}^2 e^{2i\alpha} + m_3 s_{13}^2 e^{2i\beta}| \]  

(33)

Although a 3\( \sigma \) result of \( m_{ee} = (0.11 - 0.56) \text{ eV} \) is reported by the Heidelberg-Moscow Collaboration [23], this result is criticized in Ref [24] and shall be checked by the forthcoming experiment. It is believed that that the next generation \( 0\nu\beta\beta \) experiments, with the sensitivity of \( m_{ee} \) being up to 0.01 eV [25], will open the window to not only the absolute neutrino mass scale but also the Majorana-type CP violation. Besides the \( 0\nu\beta\beta \) experiments, a more severe constraint was set from the recent cosmology observation. Recently, an upper bound on the sum of neutrino mass \( \sum m_i < 0.23 \text{ eV} \) is reported by Plank Collaboration [26] combined with the WMAP, high-resolution CMB and BAO experiments.
III. PHENOMENOLOGICAL IMPLICATIONS OF PARALLEL COFACTOR ZERO TEXTURES

A. Class II

In this section, we study the phenomenological implications of parallel cofactor zero matrices of class II. Let’s start with the factorisable inverse charged leptonic matrix \((M_l^{-1})^r\) parametrised as

\[
(M_l^{-1})_{II}^r = \begin{pmatrix}
0 & a & c \\
-1 & b & 0 \\
0 & c & d \\
\end{pmatrix}
\]  

(34)

and can be diagonalized by an orthogonal matrix \(O_l\)

\[
O_l^T (M_l^{-1})_{II}^r O_l = \text{diag}(m_e^{-1}, -m_\mu^{-1}, m_\tau^{-1})
\]

(35)

where the coefficients \(a, c, d\) are real and positive; The \(m_e, m_\mu\) and \(m_\tau\) denote the masse eigenvalues of charged leptons for three generations. The minus sign in (35) has been introduced to facilitate the analytical calculation and have no physical meaning since the charged lepton is Dirac fermions. Using the invariant \(\text{Tr}(M_l^{-1})^r\), \(\text{Det}(M_l^{-1})^r\) and \(\text{Tr}(M_l^{-1})^r^2\) the nozero elements of \((M_l^{-1})^r\) can be expressed in terms of three mass eigenvalues \(m_e, m_\mu, m_\tau\) and \(d\)

\[
a = \sqrt{-\frac{(m_e^{-1} - m_\mu^{-1} - d)(m_e^{-1} + m_\tau^{-1} - d)(-m_\mu^{-1} + m_\tau^{-1} - d)}{m_e^{-1} - m_\tau^{-1} + m_\tau^{-1} - 2d}}
\]

(36)

\[
b = m_e^{-1} - m_\mu^{-1} + m_\tau^{-1} - d
\]

(37)

\[
c = \sqrt{\frac{(d - m_e^{-1})(d + m_\mu^{-1})(d - m_\tau^{-1})}{m_e^{-1} - m_\tau^{-1} + m_\tau^{-1} - 2d}}
\]

(38)

where the parameter \(d\) is allowed in the range of \(0 < d < m_\tau^{-1}\) and \(m_e^{-1} - m_\tau^{-1} < d < m_e^{-1}\). Then the \(O_l\) can be easily constructed as

\[
O_l = \begin{pmatrix}
\frac{(b - m_e^{-1})(d - m_\tau^{-1})}{N_1} & \frac{(b + m_\mu^{-1})(d + m_\mu^{-1})}{N_2} & \frac{(b - m_\tau^{-1})(d - m_\tau^{-1})}{N_3} \\
-\frac{a(d - m_\mu^{-1})}{N_1} & -\frac{a(d + m_\mu^{-1})}{N_2} & -\frac{a(d - m_\tau^{-1})}{N_3} \\
-\frac{c(b - m_e^{-1})}{N_3} & -\frac{c(b + m_\mu^{-1})}{N_3} & -\frac{c(b - m_\tau^{-1})}{N_3}
\end{pmatrix}
\]

(39)
where the $a, b$ and $c$ in (39) is given in (36), (37) and (38); The $N_1, N_2$ and $N_3$ are the normalized coefficient given by

\[
N_1^2 = (b - m_e^{-1})^2(d - m_e^{-1})^2 + a^2(d - m_e^{-1})^2 + c^2(b - m_\tau^{-1})^2
\]  
(40)

\[
N_2^2 = (b + m_\mu^{-1})^2(d + m_\mu^{-1})^2 + a^2(d + m_\mu^{-1})^2 + c^2(b + m_\mu^{-1})^2
\]  
(41)

\[
N_3^2 = (b - m_\tau^{-1})^2(d - m_\tau^{-1})^2 + a^2(d - m_\tau^{-1})^2 + c^2(b - m_\tau^{-1})^2
\]  
(42)

Substitute the $O_i$ we obtained in (39) to (24), (25), (26), (27) and (28), one can see that the ratio of mass and the ratio of mass squared difference can be expressed via eight parameters. i.e three mixing angle($\theta_{12}, \theta_{23}, \theta_{13}$), one Dirac-CP violated phase $\delta$, three charged lepton mass ($m_e, m_\mu, m_\tau$) and a parameter $d$. Here we choose three charged leptonic mass at the electroweak scale($\mu \simeq M_Z$) i.e

\[
m_e = 0.486570154MeV \quad m_\mu = 102.7181377MeV \quad m_\tau = 1746.17MeV
\]  
(43)

In the numerical analysis, We randomly vary the three mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$) in their $3\sigma$ range and parameter $d$ in its proper range. Up to now, no bound was set on Dirac CP-violating phase $\delta$ at $3\sigma$ level, so we vary it randomly in the range of $[0, 2\pi]$. Using Eq. (28), the mass-squared difference ratio $R_\nu$ is determined. Then the input parameters is empirically acceptable when the $R_\nu$ falls inside the the $3\sigma$ range of experimental data, otherwise they are excluded. Finally, we get the value of neutrino mass and Majorana CP-violating $\alpha$ and $\beta$ though Eq. (24), (25). Since we have already obtained the absolute neutrino mass $m_{1,2,3}$, the further constraint from cosmology should be considered. In this work, we set the upper bound on the sum of neutrino mass $\Sigma m_i$ less than 0.23 eV.

We present the numerical results of class II in Fig.1 for the NH and in Fig.2 for the IH of neutrino masses. One can see from the figures that different mass spectra exhibit different correlations between physical variables. For the NH case, the Dirac CP-violated phase $\delta$ is highly restricted in the range of $60^\circ \sim 70^\circ$ and leads to the Jarlskog rephasing invariant $|J_{\text{CP}}| > 0.02$ which is promising to be detected in the future long baseline neutrino oscillation experiments. On the other hand, there exists a bound of $\theta_{23} > 48^\circ$. Although accepted at $3\sigma$ level, this result is phenomenologically
ruled out at 2σ level since recent experiments tend to give $\theta_{23} < \pi/4$. We obtain the bound on the lightest neutrino mass $M_1$, $0.025eV < m_1 < 0.075eV$ and the effective Majorana neutrino mass $m_{ee} 0.04eV < m_{ee} < 0.10eV$ which reaches the accuracy of future neutrinoless double beta decay ($0\nu\beta\beta$) experiments. For the IH case, the three mixing angle $\theta_{12}, \theta_{23}$, and $\theta_{13}$ are fully covered its 3σ range while the constrained Dirac CP-violated phase $\delta$ lies in the range of $70^\circ \sim 290^\circ$, leading to the $|J_{CP}| \sim (0) - (0.04)$. Interestingly, $m_{ee}$ and the lightest neutrino mass $m_3$ exhibit a strong dependence on $\delta$. Such correlations is essential for the model selection and could be tested by experiments. There also exists a bound of $0.005eV < m_{ee} < 0.095eV$ which could be in principle tested by future $0\nu\beta\beta$ experiments.
B. Class III

Let’s consider another class of textures which is phenomenologically interesting. In the factorisable case, the real matrix \((M^{-1}_I)^r\) is parametrised as

\[
(M^{-1}_I)^r_{III} = \begin{pmatrix}
0 & a & b \\
0 & 0 & c \\
b & c & d
\end{pmatrix}
\] (44)

where \(a, b, c\) and \(d\) are real number. Without loss of generality, the parameter \(b, c\) are set to be positive. The matrix can be diagonalized by the orthogonal matrix \(O_l\)

\[
O_l^T (M^{-1}_I)^r_{III} O_l = \text{diag}(m_e^{-1}, -m_\mu^{-1}, m_\tau^{-1})
\] (45)

Different from class II, we choose \(a\) as the free parameter since the trace of \((M^{-1}_I)^r\) has already fixed \(d\) to be

\[
d = m_e^{-1} - m_\mu^{-1} + m_\tau^{-1}
\] (46)
Figure 3: The correlation plots for class III(NH).

Using the invariant $\text{Det}(M_l^{-1})^r$ and $\text{Tr}(M_l^{-1})^r^2$, the parameter $b, c$ can be expressed by three charged leptonic mass eigenvalues($m_e, m_\mu, m_\tau$) and $a$

$$(b \pm c)^2 = -(m_e^{-1}m_\mu^{-1}m_\tau^{-1} - m_e^{-1}m_\tau^{-1} - m_\mu^{-1}m_\tau^{-1}) - a^2 \pm a(m_e^{-1} - m_\mu^{-1} + m_\tau^{-1}) - m_e^{-1}m_\mu^{-1}m_\tau^{-1}$$

With the help of Eq. (46) and Eq. (47), one can construct the diagonalization matrix $O_l$ to be

$$\left( M_l^{-1} \right)_{\text{III}} = \begin{pmatrix} O(11) & O(12) & O(13) \\ N_1 & N_2 & N_3 \\ O(21) & O(22) & O(23) \\ N_1 & N_2 & N_3 \\ O(31) & O(32) & O(33) \\ N_1 & N_2 & N_3 \end{pmatrix}$$

(48)
where

\[ O(11) = am_e^{-1}(bm_e^{-1} + ca^{-1}) + bm_e^{-1}(am_e^{-1} - m_ea^{-1}) \]
\[ O(12) = am_e^{-1}(-bm_\mu^{-1} + ca^{-1}) - bm_\mu^{-1}(-am_\mu^{-1} + m_\mu a^{-1}) \]
\[ O(13) = am_\tau^{-1}(bm_\tau^{-1} + ca^{-1}) + bm_\tau^{-1}(am_\tau^{-1} - m_\tau a^{-1}) \]
\[ O(21) = bm_e^{-1} + ca^{-1} \]
\[ O(22) = -bm_\mu^{-1} + ca^{-1} \]
\[ O(23) = bm_\tau^{-1} + ca^{-1} \]
\[ O(31) = am_e^{-1} - m_ea^{-1} \]
\[ O(32) = -am_\mu^{-1} + m_\mu a^{-1} \]
\[ O(33) = am_\tau^{-1} - m_\tau a^{-1} \]
and the normalized coefficients is given by

\[ N_1^2 = O(11)^2 + O(21)^2 + O(31)^2 \]
\[ N_2^2 = O(12)^2 + O(22)^2 + O(32)^2 \]  \hspace{1cm} (50)
\[ N_3^2 = O(13)^2 + O(23)^2 + O(33)^2 \]

From the condition that \( b, c \) are real and positive, we have the free parameter \( a \) allowed in the range of

\[-\left( \frac{m_e^{-1}m_\mu^{-1}m_\tau}{m_e^{-1} - m_\mu^{-1} + m_\tau^{-1}} \right)^{\frac{1}{2}} < a < 0 \] \hspace{1cm} (51)

or

\[ \left( \frac{m_e^{-1}m_\mu^{-1}m_\tau}{m_e^{-1} - m_\mu^{-1} + m_\tau^{-1}} \right)^{\frac{1}{2}} < a < \left( m_e^{-1}m_\mu^{-1} + m_\mu m_\tau - m_e^{-1}m_\tau^{-1} \right)^{\frac{1}{2}} \] \hspace{1cm} (52)

We present the allowed region for class III in Fig. 3 and Fig. 4. For NH case, no bound is set on the Dirac CP-violated phase \( \delta \). However, the numerical results shows a strong preference for the \( \delta \) in the range of \( 0^\circ \sim 70^\circ (290^\circ \sim 360^\circ) \). In this region, the \( 0.04eV < m_{ee} < 0.20eV \) is obtained unless \( \delta \) is fine-tuned around \( 0^\circ (180^\circ) \) where there exists a possibility for \( m_{ee} \approx 0eV \). On the other hand, when \( \delta \) are located at \( 70^\circ \sim 290^\circ \), we have a highly suppressed \( m_{ee} \approx 0eV \), implying the underlying cancellation of three neutrino masses in \( m_{ee} \). There also exists the lower bound on the lightest neutrino mass \( m_1 > 0.05eV \). For IH case, we get the Dirac CP-violated phase \( \delta \) constrained to the range of \( 110^\circ \sim 250^\circ \). No constrained parameter space are obtained for three mixing angle, leading to the \( |J_{CP}| \sim (0) - (0.035) \). Similar to the class II , there exists interesting correlations between \( \delta \), the lightest neutrino mass \( m_3 \) and the effective Majorana neutrino mass \( m_{ee} \). In particular, we obtain the bound \( 0.01eV < m_{ee} < 0.05eV \) for \( 110^\circ < \delta < 160^\circ (200^\circ < \delta < 250^\circ) \), the values which is in the scope of the accuracy of \( 0\nu\beta\beta \) experiments near the future.

**IV. CONCLUSION AND DISCUSSION**

In this work, we have studied the parallel structures with cofactor zeros in lepton mass matrices. These matrices with two cofactor zeros are equivalent to the ones with two texture zeros and thus can not obtained from arbitrary Hermitian texture
by making WB transformations. Using the permutation transformation, the 15 possible textures are grouped into 4 classes where the matrices in each class lead to the same physical implications. Among the 4 classes, one of them is not compatible with experimental results and another is equivalent to the texture zero structures extensively explored in previous literature. We concentrate our attention on the other two classes (class II and class III). Using the new results from the neutrino oscillation and cosmology experiments, a systematic and phenomenological analysis are proposed for each class and mass hierarchy. We have demonstrated that some predictions for the atmosphere mixing angle $\theta_{23}$, the Dirac CP-violated phase $\delta$ and the Majorana effective neutrino mass $m_{ee}$ are rather interesting and deserve to be explored in the future experiments. It is noted that our analysis is totally phenomenological. The flavor symmetry realization for the parallel texture structures deserves further study. A recent progress is presented in Ref. [17] that the $Z_6 \times Z_9$ flavor symmetry is introduced within the framework of inverse seesaw mechanism to produce the parallel structure with texture zeros. It is expected that a cooperation between phenomenological study and the flavor symmetry point will help us real the structure of leptonic texture.

Acknowledgments

The author would like to thank Z. Z. Xing for the useful discussion during this work.

[1] Q.R. Ahmad et al. (SNO Collaboration), Phys. Rev. Lett. 89, 011301 (2002); K. Eguchi et al. (KamLAND Collaboration), Phys. Rev. Lett. 90, 021802 (2003); M.H. Ahn et al. (K2K Collaboration), Phys. Rev. Lett. 90, 041801 (2003).

[2] F.P. An et al. (DAYA-BAY Collaboration), Phys. Rev. Lett. 108, 171803 (2012).

[3] J.K. Ahn et al. (RENO Collaboration), Phys. Rev. D108, 191802 (2012).

[4] P.H. Frampton, S. L. Glashow, and D. Marfatia, Phys. Lett. B536, 79 (2002); Z.-z. Xing, Phys. Lett. B530, 159 (2002); M. Randhawa, G. Ahuja, and M. Gupta, Phys.
Let. B643, 175(2006); A. Merle, and W. Rodejohann, Phys. Rev. D73, 073012(2006); S. Dev, S. Kumar, S. Verma, and S. Gupta, Phys. Rev. D76, 013002(2007); S. Dev, S. Kumar, S. Verma, and S. Gupta, Nucl. Phys. B784, 103(2007); G. Ahuja, S. Kumar, M. Randhawa, M. Gupta, and S. Dev, Phys. Rev. D76, 013006(2007); S. Dev, S. Kumar, Mod. Phys. Lett. A22, 1401(2007); S. Kumar, Phys. Rev. D84, 077301(2011); P.O. Ludl, S. Morisi, and E. Peinado, Nucl. Phys. B857, 411(2012); W. Grimus. and P.O. Ludl, [arXiv:1208.4515]; D. Meloni, and G. Blankenburg, Nucl. Phys. B867, 749(2013); H. Fritzsch, Z.-z. Xing, and S. Zhou, J. High Energy Phys. 09 (2011)083.

[5] S. Kaneko, H. Sawanaka, and M. Tanimoto, J. High Energy Phys. 08 (2005)073; S. Dev, S. Verma, and S. Gupta, Phys. Lett. B687, 53(2010); S. Goswami, S. Khan, and A. Watanable, Phys. Lett. B687, 53(2010), W. Grimus, and P. O. Ludl, arXiv: 1208.4515.

[6] J.-Y. Liu and S. Zhou, Phys. Rev. D87, 093010(2013).

[7] X.-G. He and A. Zee, Phys. Rev. D68, 037302(2003).

[8] G.C. Branco, R. Gonzalez Felipe, F.R. Joaquim, and T. Yanagida, Phys. Lett. B562, 265(2003); B.C. Chauhan, J. Pulido, and M. Picariello, Phys. Rev. D73, 053003(2006).

[9] L. Lavoura, Phys. Lett. B609, 317(2005); E.I. Lashin and N. Chamoun, Phys. Rev. D78, 073002(2008); E.I. Lashin and N. Chamoun, Phys. Rev. D80, 093004(2009); S. Dev, S. Gupta, and R.R. Gautam, Mod. Phys. Lett. A26, 501(2011); S. Dev, S. Gupta, R.R. Gautam, and L. Singh, Phys. Lett. B706, 168(2011); T. Araki, J. Heeck, and J. Kubo, J. High Energy Phys. 07 (2012)083; S. Verma, Nucl. Phys. B854, 340(2012); S. Dev, R.R. Gautam, and L. Singh, arXiv: 1309.4219;

[10] S. Dev, S. Verma, S. Gupta, and R.R. Gautam, Phys. Rev. D81, 053010(2010); J. Liao,D. Marfatia, K. Whisnant, arXiv: 1311.2639.

[11] H.A. Alhendi, E.I. Lashin, and A.A. Mudlej, Phys. Rev. D77, 013009(2008).

[12] S. Dev, R.R. Gautam, and L. Singh, Phys. Rev. D87, 073011(2013).

[13] S. Dev, R.R. Gautam, and L. Singh, Phys. Rev. D88, 033008(2013); W. Wang, Eur. Phys. J. C73, 2551(2013).

[14] H. Fritzsch, M. Gell-Mann, and P. Minkowski, Phys. Lett. B59, 256(1975); P.
Minkowski, Phys. Lett. B67, 421(1977); T. Yanagida, in Proceedings of Workshop on Unified Theory and the Baryon Number of the Universe, edited by O. Sawada and A. Sugamoto(KEK, Tsukuba, 1979), p. 95; M. Gell-Mann, P. Ramond, and Slansky, in Supergravity, edited by P. van. Nieuwenhuizen and D.Z. Freeman (North-Holland, Amsterdam,1979), p. 315; R.N. Mohapatra and G. Senjanovic, Phys, Rev. Lett. 44, 912(1980); J. Schechter and J. W. F. Valle, Phys. Rev. D22, 2227(1980); J. Schechter and J. W. F. Valle, Phys. Rev. D25, 774(1982).

[15] H. Fritzsch, Phys. Lett. B73, 317(2978); Nucl. Phys. B155, 189(2979).

[16] Z. Z. Xing. Phys. Lett. B550, 178(2002); Z. Z. Xing, and S. Zhou Phys. Lett. B593, 156(2004); S. Zhou, and Z. Z. Xing, Eur. Phys. J. C38, 495(2005); G. Ahuja, M. Gupta, M. Randhawa, and R. Verma, Phys. Rev. D79, 093006(2009);

[17] Y. L. Zhou, Phys. Rev. D86, 093001(2012);

[18] G. C. Branco, D. Emmannuel-Costa, R. González Felipe, and H. Serôdio, Phys. Lett. B670, 340(2009);

[19] S. Dev, S. Gupta, and R.R. Gautam, Phys. Rev. D82, 073015(2010);

[20] B. Pontecorvo, Zh. Eksp. Teor. Fiz. 33, 549(1957); Z. Maki, M. Nakagawa, and N. Sakata, Prog. Theor. Phys. 28, 870(1962).

[21] C. Jarlskog, Phys, Rev. Lett. 55, 1039(1985).

[22] F. Copazzi, G.L. Fogli, E. Lisi, A. Marrone, D. Montanino, and A. Palazzo, arXiv: 1312.2878

[23] H.V. Klapdor-Kleingrothaus, A. Dietz, H.L. Harney, and I.V. Krivosheina, Mod. Phys. Lett. A16, 2409(2001).

[24] C.E. Aalseth et al. Mod. Phys. Lett. A17, 1475(2002); F. Feruglio, A. Strumia, and F. Vissani, Nucl. Phys. B637, 345(2002).

[25] S.M. Bilenky and C. Giunti, Mod. Phys, Lett. A16, 1230015(2012).

[26] P.A.R. Ade et al. (Planck Collaboration), arXiv: 1303.5076.

[27] Z. Z. Xing, H. Zhang, and S. Zhou, Phys. Rev. D86, 013013(2012).