Rigorous QCD Evaluation of Spectrum and Other Properties of Heavy $q\bar{q}$ Systems.

II. Bottomium with $n = 2$, $l = 0, 1$.

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Abstract

We calculate the Lamb, fine and hyperfine shifts in $b\bar{b}$ with $n = 2$, $l = 0, 1$. Radiative corrections as well as leading nonperturbative corrections (known to be due to the gluon condensate) are taken into account. The calculation is parameter-free, as we take $\Lambda$, $\langle \alpha_s G^2 \rangle$ from independent sources. Agreement with experiment is found at the expected level $\sim 30\%$. Particularly interesting is a prediction for the hyperfine splitting, $M_{\text{average}}(2^3 P) - M(2^1 P_1) = 1.7 \pm 0.9$ MeV, opposite in sign to the $c\bar{c}$ one ($\approx -0.9$ MeV), and where the nonzero value of $\langle \alpha_s G^2 \rangle$ plays a leading role.

* This work is partially supported by the U.S Department of Energy and CICYT, Spain.
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1 Introduction

In a previous paper \cite{TY} (hereafter to be referred as TY) we presented an evaluation of the potential for heavy $q\bar{q}$ systems \cite{1, 2}. The evaluation included relativistic effects, one–loop radiative corrections and (for the spin–independent part) the dominating two–loop ones. With this we evaluated a number of quantities, taking into account also leading nonperturbative corrections, which are known \cite{3} to be due to the contributions of the gluon condensate. It was shown that a very good account could be given of the lowest lying $b\bar{b}$ bound states (some features of $c\bar{c}$ were also discussed). Notably, both the energy and wave function (this last through $e^+e^-$ decay) of the states with $n = 1$ were given; the splittings between these states and those with $n = 2$, $l = 0, 1$ were reproduced in what is essentially a zero parameter calculation using only the known values of the basic QCD parameters,

\begin{equation}
\Lambda(n_f = 4, 2 \text{ loops}) = 200^{+80}_{-60} \text{ MeV} \\
\langle \alpha_s G^2 \rangle = 0.042 \pm 0.020 \text{ GeV}^4 \\
m_b = 4906^{+69}_{-51} (-4, +11)_{+40} \text{ MeV.}
\end{equation}

Actually we preferred in TY to deduce $m_b$ from the mass of the $\Upsilon(1S)$ state. The errors given for this quantity in \cite{TY} correspond to that in $\Lambda$ (the first), to that in the gluon condensate (the second); the third is an estimated systematic error.

The value of $m_b$ given in \cite{TY} is for the pole mass, which is the appropriate quantity to be used in a Schrödinger equation. It corresponds to a running

\footnote{We will freely use the notation of TY}
mass value of
\[ \overline{m}_b(m_b^2) = 4397^{+7}_{-2}^{+3}_{-4}^{+16}_{-32} \text{MeV}, \tag{2} \]
which compares favorably with the SVZ estimate \([4]\) of 4250 ± 100 MeV.

For some of the states with \(n = 2, l = 1, 0\) no result could be given; only the perturbative contributions were presented and they failed to reproduce the experimental values. This was because the nonperturbative corrections, more involved than for the \(n = 1\) case, had not been calculated at the time.

In the present paper we finish the calculation of the leading nonperturbative (NP, henceforth) contributions to the \(n = 2\) states. We are thus able to present a complete, rigorous and parameter–free QCD evaluation of the full \(n = 1\) and \(n = 2, l = 1, 0\) bottomium system. For some of the quantities the NP corrections (which are always large) are under control; for some others the calculation loses reliability. By and large, nevertheless, a coherent picture and good agreement with experiment are obtained.

NP corrections grow very fast with \(n\) so for \(n \geq 3\) they get so large (for \(b\bar{b}\)) that a QCD calculation based on leading effects becomes meaningless as was indeed to be expected. However, we present some results for \(n = 3, 4, 5\) with a view to future applications to the \(t\bar{t}\) system for which NP corrections remain small up to \(n \sim 5\).

This paper is organized as follows: the perturbative \(q\bar{q}\) hamiltonian is reproduced in Sec. 2 for ease of reference. The NP corrections to the interaction are evaluated in Sec. 3. Sec. 4 contains the ensuing shifts in energies and wave functions, which are then applied in Sec. 5 to the complete evaluation of \(n = 1, 2, l = 0, 1, j = 0, 1, 2\) and spin \(s = 0, 1\) bound states of \(b\bar{b}\). The
article is finished in Sec. 6 with numerical results and Conclusions.

## 2 The perturbative QCD Potential.

We present here the Hamiltonian for the $q \bar{q}$ system for ease of reference. We write it separating the spin–independent, LS, tensor and hyperfine pieces as follows:

\[ H_{\text{s.i.}} = H^{(0)} - \frac{C_F \beta_0 \alpha_s(\mu^2)}{2\pi} \frac{\ln r \mu}{r}, \]  

\[ H^{(0)} = -\frac{1}{m} \Delta - \frac{C_F \bar{\alpha}_s(\mu^2)}{r}, \]  

\[ \bar{\alpha}_s(\mu^2) = \left[ 1 + \frac{\alpha_1 + \gamma_E \beta_0 / 2}{\pi} \alpha_s(\mu^2) \right] \alpha_s(\mu^2); \]  

\[ V_{\text{LS}}(\vec{r}) = \frac{3C_F \alpha_s(\mu^2)}{2m^2 r^3} \vec{L} \cdot \vec{S} \times \left\{ 1 + \left[ \beta_0 \frac{1}{2} (\ln r \mu - 1) + 2(1 - \ln mr) + \frac{125 - 10 n_f}{36} \right] \alpha_s(\mu^2) \right\} \]  

\[ V_{\text{T}}(\vec{r}) = \frac{C_F \alpha_s(\mu^2)}{4m^2 r^3} S_{12}(\vec{r}) \times \left\{ 1 + \left[ D + \frac{\beta_0}{2} \ln r \mu - 3 \ln m r \right] \alpha_s(\mu^2) \right\} \]  

\[ V_{\text{hf}}(\vec{r}) = \frac{4\pi C_F \alpha_s(\mu^2)}{3m^2} S^2 \left\{ \delta(\vec{r}) + \left[ \frac{\beta_0}{2} \left( \frac{1}{4\pi} \text{reg} \frac{1}{r^3} + (\ln \mu) \delta(\vec{r}) \right) \right] \right. \]  

Here,

\[ C_A = 3, \quad T_F = 1/2, \quad \beta_0 = 11 - \frac{2n_f}{3}, \quad \beta_1 = 102 - \frac{38n_f}{3} \]

\[ a_1 = \frac{31 C_A - 20 T_F n_f}{36}, \]

\[ B = \frac{3}{2} (1 - \ln 2) T_F - \frac{5}{9} T_F n_f + \frac{11 C_A - 9 C_F}{18}, \]
\[ D = \frac{4}{3} \left( 3 - \frac{\beta_0}{2} \right) + \frac{65}{12} - \frac{5 n_f}{18}. \]

\( \vec{S} = \vec{S}_1 + \vec{S}_2 \) is the total spin, \( \vec{L} \) the orbital angular momentum and

\[ S_{12}(\vec{r}) = 2 \sum_{ij} S_i S_j \left( \frac{3}{r^2} r_i r_j - \delta_{ij} \right). \]

\( n_f \) is the number of active flavours. The running coupling constant we take to two loops,

\[ \alpha_s(\mu^2) = \frac{4\pi}{\beta_0 \ln \mu^2/\Lambda^2} \left\{ 1 - \frac{\beta_1 \ln \ln \mu^2/\Lambda^2}{\beta_0^2 \ln \mu^2/\Lambda^2} \right\}. \]

We have lumped the constant piece of the one–loop correction into \( \tilde{\alpha}_s \) (Eq. (5)) because the ensuing potential is still Coulombic and therefore \( H^{(0)} \) may still be solved exactly. The relativistic, full one loop and leading two loop corrections to the spin–independent piece are known; see TY for details. We will not need them now. The total Hamiltonian is of course

\[ H_p = H_{s.i.} + V_{LS} + V_T + V_{hf}, \hspace{1cm} (9) \]

where the index \( p \) emphasizes that only perturbative contributions are taken into account.

A result that we take over from TY is the form of the (spin–independent) wave functions \( \Psi_{nl}^{(0)} \) pertaining to the Hamiltonian \( H_{s.i.} \). They are easiest obtained with a variational method; one finds that they are given by a formula like that for the wave functions of the Coulombic Hamiltonian \( H^{(0)} \) with the replacement of the ”Bohr radius”,

\[ a = \frac{2}{m C_F \tilde{\alpha}_s}, \]
by
\[ b(n,l) = a \left\{ 1 + \frac{\ln(n\mu/mC\bar{\alpha}_s) + \psi(n + l + 1) - 1}{2\pi} \beta_0 \alpha_s \right\}^{-1} \]. \quad (10)

\[ \Psi_{nl}^{(0)}(\vec{r}) = \Psi_{nl}^{(0)}(\vec{r}; a \to b) \].

A few explicit expressions may be found in Appendix II.

In particular the wave function at the origin becomes
\[ \Psi_{nl}^{(0)}(0) \to \bar{\Psi}_{nl}^{(0)}(0) = \{1 + \delta_{nl}(n,l)\} \Psi_{nl}^{(0)}(0), \]
\[ \delta_{nl}(n,l) = \frac{3\beta_0}{4\pi} \left[ \ln\left(\frac{n\mu}{mC\bar{\alpha}_s}\right) + \psi(n + l + 1) - 1 \right] \alpha_s. \] \quad (11)

As stated, \( \Psi_{nl}^{(0)} \) is the solution of the equation
\[ H^{(0)} \Psi_{nl}^{(0)} = E_{nl}^{(0)} \Psi_{nl}^{(0)}, \quad E_{nl}^{(0)} = -\frac{(C_F\bar{\alpha}_s)^2}{4n^2} m. \] \quad (12)

When taking into account the full \( H_{s,i} \), the energies are shifted to \( \bar{E}_{nl}^{(0)} \),
\[ \bar{E}_{nl}^{(0)} = E_{nl}^{(0)} - \frac{C_F\beta_0}{4\pi n^2} \alpha_s^2 \left[ \ln\left(\frac{n\mu}{mC\bar{\alpha}_s}\right) + \psi(n + l + 1) \right] m. \] \quad (13)

A last word about the notation: the superindex \( (0) \) in say, \( \Psi^{(0)} \), \( E^{(0)} \)
means ”of zero order with respect to nonperturbative (NP) effects”.

3 The Nonperturbative Interactions.

It can be shown (TY and \cite{3, 4, 5}) that the leading NP interactions, at short
distances, are those associated with the gluon condensate; and, of these, the
dominant ones are those where two gluons are attached to the quarks. These
interactions are equivalent, in the nonrelativistic limit (including first order
relativistic corrections) to those obtained assuming the quarks to move inside
a medium of constant, random chromoelectric, $\vec{E}$ and chromomagnetic, $\vec{B}$ fields. Because the fields are constant they may be considered to be classical; and because they are random we may take them of zero average value

$$\langle \vec{E} \rangle = \langle \vec{B} \rangle = 0.$$ 

The average is taken in the physical vacuum. Quadratic averages are non–vanishing and may be related to the gluon condensate. With $i, j$ spatial indices and $a, b$ color ones one has (for $N_c = 3$ colors)

$$\langle g^2 B_i^a B_j^b \rangle = -\langle g^2 E_i^a E_j^b \rangle = \frac{\pi \delta_{ij} \delta_{ab}}{3(N_c^2 - 1)} \langle \alpha_s G^2 \rangle . \quad (14)$$

The relativistic interaction of a quark (labeled with index 1) with classical vector fields may be described by the Dirac Hamiltonian

$$H_{D1} = i\vec{\alpha} \cdot \vec{\nabla}_1 - g\gamma \cdot A(\vec{r}_1) + \beta_1 m , \quad (15)$$

$A^\mu = \sum_a t^a A_a^\mu$ being gluon fields (in matrix notation). A convenient gauge is that in which

$$A_1^0 = -\vec{r}_1 \cdot \vec{E} , \quad A_1 = -\frac{1}{2} \vec{r}_1 \times \vec{B} .$$

To solve our problem one can apply a Foldy–Wouthuysen transformation \[6\] to obtain the Hamiltonian (correct including first order relativistic effects)

$$H_{FW1} = m + \frac{1}{2m}(\vec{p}_1 - g\vec{A}_1)^2 - \frac{1}{8m^3} \vec{p}_1^4$$

$$- \frac{g}{m} \vec{S}_1 \cdot \vec{B} - \frac{g}{2m^2} \vec{S}_1 \cdot (\vec{E} \times \vec{p}_1) , \quad (16)$$

$\vec{S}_1$ the spin operator and $\vec{p}_1 = -i\vec{\nabla}_1$. Adding to this the Hamiltonian of the antiquark ($g \rightarrow -g, \vec{r}_1 \rightarrow \vec{r}_2$) and their interactions given in the previous
section we find the full hamiltonian, which now includes leading NP effects,

\[ H = H^{(0)} - \frac{C_F \beta_0 \alpha_s^2}{2\pi} \ln \frac{r\mu}{r} + V_{LS} + V_T + V_{mf} \]

\[ -g \vec{r} \cdot \vec{E} + \frac{g}{2m^2} (\vec{S} \times \vec{p}) \cdot \vec{E} - \frac{g}{m} (\vec{S}_1 - \vec{S}_2) \cdot \vec{B}. \]

\( H^{(0)}, V_{LS}, V_T, V_{mf} \) are given by Eqs. (4) to (8). Some of the peculiarities of Eq. (17), in particular the absence of an \( \vec{L} \cdot \vec{S} \) interaction as well as the presence of a term involving the differences of the spins, had been noted in the similar case of the Zeeman effect in positronium \([7]\). In Eq. (17) we have omitted a term obtained when expanding the square \((\vec{p}_1 - gA_1)^2\) in Eq. (16), viz., the piece \(A_1^2\). It would have produced a term \(\pi \langle \alpha_s G^2 \rangle r^2 / (48 N_c m)\), to be added to Eq. (17). The reason for its omission is that it gives subleading corrections to all processes (as compared to the contributions of the other terms).

Before embarking upon detailed calculations, let us elaborate on this matter of leading and subleading corrections. Because

\[ \langle r \rangle \sim a = \frac{2}{mC_F \tilde{\alpha}_s}, \]

\[ \langle p \rangle \sim m v \sim mC_F \tilde{\alpha}_s, \]

it follows that the NP terms in Eq. (17) are

\[ -g \vec{r} \cdot \vec{E} \sim \frac{1}{\tilde{\alpha}_s}, \frac{g}{2m^2} (\vec{S} \times \vec{p}) \cdot \vec{E} \sim \tilde{\alpha}_s, \]

\[ -\frac{g}{m} (\vec{S}_1 - \vec{S}_2) \cdot \vec{B} \sim (\tilde{\alpha}_s)^0. \]

This simplifies enormously the calculation at the leading order as seldom more than one, and at most two terms need to be considered. A further
simplification is that, with the only exception of the hyperfine splitting for
\( n = 2, \, l = 1 \), only the tree level piece of \( H_p \) has to be taken into account
when evaluating leading NP effects.

4 Energy and Wave Function Shifts.

4.1 Spin–independent Shifts.

Although most of the spin–independent shifts of energies and wave functions
were discussed in TY and [3], we give here a detailed calculation for ease of
reference, to correct an error common to TY and [3], to present the results
for the \( n = 2 \) wave functions and to explain in this simple case the way the
calculation works.

The effects of the nonzero condensate are evaluated with the help of
perturbation theory. The perturbation consists of the terms (cf. Eq. (17)),

\[-g_\sigma \cdot \vec{E}, \quad \frac{g}{2m^2} (\vec{S} \times \vec{p}) \cdot \vec{E}, \quad -\frac{g}{m} (\vec{S}_1 - \vec{S}_2) \cdot \vec{B}.
\]

Because, for spin independent
effects, the first term gives a nonzero result we may neglect the others which
would contribute corrections of higher order in \( \alpha_s \), cf. Eq. (18). Second
order perturbation theory is required as only quadratic terms in \( \vec{E} \) will give a
nonvanishing contribution, as discussed in the previous section, Eq. (14) and
above. The method of evaluation, for this particular case, has been developed
by Leutwyler, Ref. [3], and is related to Kotani’s treatment of the second
order Stark effect [8], up to color and angular momentum complications that
we now discuss.
We denote the solutions of the unperturbed Hamiltonian by

\[
H^{(0)} (0) |\Psi^{(0)}_{nlM} \rangle = E^{(0)}_n |\Psi^{(0)}_{nlM} \rangle ,
\]

\[
E^{(0)}_n = - \frac{1}{m a^2 n^2} = - \frac{C_F^2 \tilde{\alpha}_s}{4 n^2} m ,
\]

\[
\Psi^{(0)}_{nlM} = Y^l_M (\vec{r} / r) R^{(0)}_{nl} (r) .
\] (19)

(We have omitted the trivial rest mass energy term). The \( R^{(0)}_{nl} (r) \) are identical to the standard Coulombic wave functions for the hydrogen atom with the replacement of the "Bohr radius" by \( a = \frac{2}{m C_F \tilde{\alpha}_s} \). Second order perturbation theory yields immediately the energy and wave function shifts:

\[
E = E^{(0)} + E^{NP} ; \Psi = \Psi^{(0)} + \Psi^{NP}
\]

with

\[
E^{NP}_{nl} = - \sum_{ij,ab} \left( \langle \Psi^{(0)}_{nlM} | g r_i E^i_a t^a \frac{1}{H^{(0)} - E^{(0)}_n} g r_j E^j_b t^b | \Psi^{(0)}_{nlM} \rangle \right)
\] (20)

and

\[
|\Psi^{NP}_{nlM} \rangle = \sum_{ij,ab} P_{nl} \frac{1}{H^{(0)} - E^{(0)}_n} P_{nl} g r_i E^i_a t^a \frac{1}{H^{(0)} - E^{(0)}_n} g r_j E^j_b t^b |\Psi^{(0)}_{nlM} \rangle .
\] (21)
Here
\[ P_{nl} = 1 - \left| \Psi_{nlM}^{(0)} \right\rangle \left\langle \Psi_{nlM}^{(0)} \right| \]
is the projector orthogonal to the \( nl \) state. It does not appear in Eq. (24) because
\[ \left\langle \Psi_{nlM}^{(0)} \right| \vec{r} \cdot \vec{E} \left| \Psi_{nlM}^{(0)} \right\rangle = 0. \]

The expressions (20), (21) are first simplified by replacing
\[ g E^i_a \ldots g E^j_b \rightarrow -\delta_{ij} \delta_{ab} \frac{\pi}{24} \langle \alpha_s G^2 \rangle, \quad (22) \]
recalling Eq. (14).

Next we take care of the color algebra. The one–gluon exchange potential is given, when acting on arbitrary color states by
\[ -\frac{\alpha_s}{r} \sum_a t^a_{ii'} t^b_{kk'} \cdot (23) \]
If the initial (and final) states are color singlets we may average
\[ \frac{1}{\sqrt{N_c}} \sum_{ik} \delta_{ik} \frac{1}{\sqrt{N_c}} \sum_{k'k} \delta_{i'k'} \cdot \]
and then we get the potential, and Hamiltonian,
\[ -\frac{C_F \tilde{\alpha}_s}{r}, \quad H^{(0)} = -\frac{1}{m} \Delta - \frac{C_F \tilde{\alpha}_s}{r}; \]
we have incorporated, as we always do everywhere, the Coulombic piece of the one–loop corrections into \( \tilde{\alpha}_s \).

In Eqs. (20) and (21), however, the states \( | \Psi_{nlM}^{(0)} \rangle \) are certainly color singlets: hence the matrices \( t^b \) (for example) when acting on them will produce a color octet state. For a color octet the potential and Hamiltonian are
\[ \frac{\tilde{\alpha}_s}{2N_c r}, \quad H^{(0)} = -\frac{1}{m} \Delta + \frac{\tilde{\alpha}_s}{2N_c r}. \quad (24) \]
One then finds
\[
\sum_{ab} \delta_{ab} t^a \frac{1}{H^{(0)} - E_n^{(0)}} t^b |\text{singlet}\rangle = C_F \frac{1}{H^{(0)} - E_n^{(0)}} |\text{singlet}\rangle. \tag{25}
\]
Putting this together with Eq. (22) into Eqs. (20) and (21) gives the formulas

\[
E_{nP}^{nl} = \frac{\pi \langle \alpha_s G^2 \rangle}{6N_c} \sum_i \left| \langle \psi_{nlM}^{(0)} | r_i \frac{1}{H'(0) - E_n^{(0)}} r_i | \psi_{nlM}^{(0)} \rangle \right|,
\]

(26)

\[
|\psi_{nP}^{nlM}\rangle = -\frac{\pi \langle \alpha_s G^2 \rangle}{6N_c} P_{nl} \frac{1}{H(0) - E_n^{(0)}} P_{nl} \sum_i r_i \frac{1}{H'(0) - E_n^{(0)}} r_i |\psi_{nlM}^{(0)}\rangle,
\]

(27)

which takes care of color complications, so we turn to deal with angular momentum. Obviously the perturbation is rotationally invariant so the third component of angular momentum, \( M \), is not affected by it; but the total angular momentum algebra is not entirely trivial. We write

\[
\sum_i r_i \frac{1}{H'(0) - E_n^{(0)}} r_i = \sum_\lambda r_\lambda^* \frac{1}{H'(0) - E_n^{(0)}} r_\lambda,
\]

where \( \lambda = 0, \pm 1 \), and the \( r_\lambda \)'s are spherical components,

\[
r_{\pm 1} = \frac{1}{\sqrt{2}} (r_1 \pm ir_2), \quad r_0 = r_3.
\]

Using the formulas

\[
\frac{1}{r} r_\lambda = \sqrt{\frac{4\pi}{3}} Y_\lambda^1(r/r) ; \quad \frac{1}{r} r_\lambda^* = (-1)^\lambda \sqrt{\frac{4\pi}{3}} Y_{-\lambda}^1(r/r); \quad (28)
\]

and the addition theorem for spherical harmonics we get

\[
r_\lambda Y_M^l = r \sum_{l'=|l-1|,|l+1|} C_M(l, l', \lambda) Y_{M+l}^{l'},
\]

\[
C_M(l, l', \lambda) = \sqrt{\frac{2l + 1}{2l' + 1}} (l, M; 1, \lambda|l') (l, 0; 1, 0|l')
\]

with \((\ldots|\ldots)\) the standard Clebsch–Gordan coefficients.
When acting on a function with well-defined angular momentum \( l \) we have
\[
\frac{1}{H^{(0)} - E_n^{(0)}} |l\rangle = \frac{1}{H^{(0)}_l - E_n^{(0)}} |l\rangle,
\]
\[
\frac{1}{H^{(0)} - E_n^{(0)}} |l\rangle = \frac{1}{H^{(0)}_l - E_n^{(0)}} |l\rangle,
\]
where
\[
H_l = -\frac{1}{m} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{mr^2} + \kappa \tilde{\alpha}_s r,
\]
with \( \kappa = -C_F \) for \( H^{(0)}_l \) and \( \kappa = 1/(2N_c) \) for \( H^{(0)}_l \). Using this and the explicit values of the Clebsch–Gordan coefficients we find that Eqs. \((26), (27)\) become
\[
E_{nl}^{NP} = \frac{\pi \langle \alpha_s G^2 \rangle}{6N_c} \frac{1}{2l+1} \times \left\langle R_{nl}^{(0)} \right| r \left\{ \frac{l}{H^{(0)}_{l-1} - E_n^{(0)}} + \frac{l+1}{H^{(0)}_{l+1} - E_n^{(0)}} \right\} r \left| R_{nl}^{(0)} \right\rangle,
\]
\[
|R_{nl}^{NP}\rangle = -\frac{\pi \langle \alpha_s G^2 \rangle}{6N_c} \frac{1}{2l+1} P_{nl} \frac{1}{H^{(0)}_l - E_n^{(0)}} P_{nl} \times \left\{ \frac{l}{H^{(0)}_{l-1} - E_n^{(0)}} + \frac{l+1}{H^{(0)}_{l+1} - E_n^{(0)}} \right\} r \left| R_{nl}^{(0)} \right\rangle.
\]
We have succeeded in separating the color and angular variables to obtain equations involving only the radial variable and radial wave functions. To finish the calculations all that is needed is to find the inverses
\[
\frac{1}{H^{(0)}_l - E_n^{(0)}} \cdots \frac{1}{H^{(0)}_l - E_n^{(0)}} \cdots \frac{1}{H^{(0)}_l - E_n^{(0)}} \cdots \frac{1}{H^{(0)}_l - E_n^{(0)}}.
\]
This is described in Appendix I. The ensuing expressions for the \( E_{nl}^{NP} \) and \( R_{nl}^{NP} \) are collected in Appendix II for a few values of \( n, l \) and will be employed later.
The expression we get for $E_{nl}^{NP}$ agrees with that found by Leutwyler [3] and also $R_{10}^{NP}$, the only wave function calculated in Ref. [3], agrees with our evaluation.

We have not succeeded in obtaining a closed general formula for $R_{nl}^{NP}$ (for $E_{nl}^{NP}$ one is given in Ref. [3]) but a few general properties may be inferred from Eqs. (31), (32). Because

$$\langle r \rangle_{nl} = \frac{a}{2} (3n^2 - l(l + 1)) \sim \frac{n^3}{\bar{\alpha}_s}$$

and each energy denominator yields a factor $\frac{1}{\bar{\alpha}_s n^2}$ (see Appendix I) we expect

$$E_{nl}^{NP} \sim \frac{n^6}{\bar{\alpha}_s^4}, \quad R_{nl}^{NP} \sim \frac{n^8}{\bar{\alpha}_s^6}.$$  

It thus follows that the importance of nonperturbative effects grows very rapidly with $n$. Moreover we expect them to be smaller for energies than for wave functions and, generally, to be larger when $l = 0$ than for $l \neq 0$ (for the same value of $n$). These properties may be verified explicitly in the expressions collected in Appendix II.

The energies and wave functions correct to leading order in NP effects and including one–loop corrections are then

$$E_{nl} = \bar{E}_{nl}^{(0)} + E_{nl}^{NP},$$
$$R_{nl}(r) = \bar{R}_{nl}^{(0)}(r) + R_{nl}^{NP}(r),$$
$$\Psi_{nlM} = Y_M^l(\vec{r}/r) R_{nl}(r),$$

the $\bar{R}^{(0)}$, $\bar{E}^{(0)}$ being as given in Eqs. (10), (11).
4.2 Hyperfine splittings

The hyperfine splittings are caused by the interactions that depend only on spin; they are $V_{hf}$ in Eq. (8), and the piece

$$-\frac{g}{m} (\vec{S}_1 - \vec{S}_2) \cdot \vec{B}$$

in Eq. (17). Besides the splitting caused directly by the last term, there is a nonperturbative contribution indirectly generated by $-g\vec{r} \cdot \vec{E}$. This contribution that we will call ”internal”, comes about because, when evaluating the expectation values

$$\langle \Psi | V_{hf} | \Psi \rangle$$

we should use the wave function including the NP corrections discussed in the previous Subsection:

$$\langle \Psi_{nl} | V_{hf} | \Psi_{nl} \rangle = \langle \bar{\Psi}_{nl}^{(0)} + \Psi_{nl}^{NP} | V_{hf} | \bar{\Psi}_{nl}^{(0)} + \Psi_{nl}^{NP} \rangle$$

$$\simeq \langle \bar{\Psi}_{nl}^{(0)} | V_{hf} | \bar{\Psi}_{nl}^{(0)} \rangle + 2 \langle \Psi_{nl}^{(0)} | V_{hf} | \Psi_{nl}^{NP} \rangle .$$

The ”internal” NP splitting is the last term in Eq. (35):

$$\Delta_{in}^{\text{hf}} E_{nl} = 2 \langle \Psi_{nl,s}^{(0)} | V_{hf} | \Psi_{nl,s}^{NP} \rangle .$$

To evaluate this to leading order we use the expression

$$V_{hf} \simeq \frac{4\pi C_F \alpha_s}{3m^2} \delta(\vec{r}) \vec{S}^2$$

and thus we get

$$\Delta_{in}^{\text{hf}} E_{n0s} = 2s(s + 1) \frac{4\pi C_F \alpha_s}{3m^2} \frac{1}{4\pi} R_{n0}^{(0)}(0) R_{n0}^{NP}(0) .$$

15
For $l \neq 0$ the leading piece of $V_{hf}$ gives zero, because $R_{nl}^{(0)}(0)$ vanishes. We have to take into account the radiative correction to $V_{hf}$ and then

$$
\Delta_{hl}^{in} E_{nls} = 2s(s+1) \frac{4\pi C_F \alpha_s}{3m^2} \left( \frac{\beta_0}{2} - \frac{21}{4} \right) \frac{1}{4\pi} \alpha_s
\times \int_0^\infty dr \, r^2 R_{nl}^{(0)}(r) \frac{1}{r^3} R_{nl}^{NP}(r) , \quad l \neq 0 .
$$

(38)

It will turn out that, for $l \neq 0$, this internal shift will be subleading. This fact is very interesting because this is one of the few cases where a rigorous QCD analysis yields results qualitatively different from the calculations based on phenomenological potentials. This we will discuss in detail elsewhere.

The contribution to hyperfine splitting of the interaction $-\frac{g}{m} (\vec{S}_1 - \vec{S}_2) \cdot \vec{B}$ we will call "external" §. It may be calculated as we calculated $E_{nl}^{NP}$ in the previous Subsection. We find,

$$
\Delta_{hl}^{ex} E_{nls} = [s(s+1) - 3] \frac{\pi (\alpha_s C^2)}{6N_c m^2} \langle R_{nl}^{(0)} \left| \frac{1}{H_i^{(0)} - E_n^{(0)}} \right| R_{nl}^{(0)} \rangle .
$$

(39)

The inverse is obtained with the formulas of Appendix I.

To the NP contributions we have to add tree level (relativistic) and radiative ones, that we collectively label perturbative: from TY,

$$
\Delta_{hl}^{p} E_{n0s} = \frac{s(s+1)}{2} \frac{C_F^4 \alpha_s (\mu^2) \tilde{\alpha}_s^2 (\mu^2)}{3n^3} \left[ 1 + \delta_{\text{ew}}(n,0) \right] \frac{1}{m} \left\{ 1 + \left[ \frac{\beta_0}{2} \left( \ln - \frac{n \mu}{m C_F \alpha_s} - \sum_{k=1}^{n} \frac{1}{k} - 1 + \gamma_E - \frac{n-1}{2n} \right) \right.ight.

\left. - \frac{21}{4} \left( \ln \frac{n}{C_F \alpha_s} - \sum_{k=1}^{n} \frac{1}{k} - \frac{n-1}{2n} \right) + B \right\} \frac{\alpha_s}{\pi} ,
$$

(40)

$$
\Delta_{hl}^{p} E_{nls} = \frac{s(s+1)}{2} \frac{C_F^4 \alpha_s^2 \tilde{\alpha}_s^3}{6\pi n^3 l(l+1)(2l+1)} \left( \frac{\beta_0}{2} - \frac{21}{4} \right) \frac{1}{m} , \quad l \neq 0 .
$$

§In the case of hyperfine splittings the internal contribution is chromoelectric and the external one chromomagnetic, but this is not true in other splittings.
The constants are as in Eq. (8). The full splitting is thus
\[
\Delta_{nf} E_{nls} = \Delta_{np} E_{nls} + \Delta_{in} E_{nls} + \Delta_{ex} E_{nls},
\]
with the various pieces given in Eqs. (36) to (40).

### 4.3 Fine splittings

Also here we have ”internal” and ”external” contributions. The internal ones are, as before, induced by the NP modification of the wave function. The calculation is somewhat complicated because now two operators, the LS and Tensor ones (Eqs. (3) and (7) ) contribute. We find

\[
\Delta_{nf} E_{nlj} = 2 \delta_{NP}(n,l) \left\{ \langle V_{LS}^{(0)} \rangle_{nlj} + \langle V_{T}^{(0)} \rangle_{nlj} \right\},
\]

where
\[
\delta_{NP}(n,l) = \frac{\langle R_{nl}^{(0)} | r^{-3} | R_{nl}^{NP} \rangle}{\langle R_{nl}^{(0)} | r^{-3} | R_{nl}^{(0)} \rangle};
\]

\[R_{nl}^{NP}\] is given in Eq. (32) and \[V_{LS}^{(0)}, V_{T}^{(0)}\] are the leading (tree level) pieces of \[V_{LS}, V_{T}\]. Using the explicit expressions for these we have,

\[
\langle V_{LS}^{(0)} \rangle_{nlj} = \left[ j(j+1) - l(l+1) - 2 \right] \frac{3 C_F^4 \alpha_s \bar{\alpha}_s^3}{16 n^3 l(l+1)(2l+1)} m;
\]

\[
\langle V_{T}^{(0)} \rangle_{nlj} = \frac{1}{2} S_{12} \rangle_{jl} \frac{C_F^4 \alpha_s \bar{\alpha}_s^3}{8 n^2 l(l+1)(2l+1)} m;
\]

with
\[
\langle \frac{1}{2} S_{12} \rangle_{jl} = \begin{cases} -\frac{j+1}{2l+1}, & j = l \pm 1 \\ +1, & j = l \\ -\frac{l}{2l+3}, & j = l \pm 1. \end{cases}
\]

\(\text{We consider that the states correspond to total spin } s = 1. \text{ For } s = 0, \Delta_{nf} E_{nlj} = 0.\)
The leading "external" fine structure shift, $\Delta_{E}^{ex} E_{nlj}$, is caused by the crossed combination of the perturbations

$$-g \vec{r} \cdot \vec{E}, \quad \frac{g}{2m^2} (\vec{S} \times \vec{p}) \cdot \vec{E}.$$

In this case the external shift is also chromoelectric; the chromomagnetic perturbation $-\frac{g}{m} (\vec{S}_1 - \vec{S}_2) \cdot \vec{B}$ does not contribute to the fine structure. The color algebra is now like the one for the spin–independent shift, Subsection 4.1.

Thus,

$$\Delta_{E}^{ex} E_{nlj} = -2 \frac{\pi \langle \alpha_s G^2 \rangle}{6N_c} \frac{1}{2m^2} \sum_i \left\langle \Psi_{nlj}^{(0)} \left| \left( \vec{S} \times \vec{p} \right)_i \frac{1}{H_1^{(0)} - E_1^{(0)}} r_i \right| \Psi_{nlj}^{(0)} \right\rangle. \quad (47)$$

The angular momentum algebra, on the other hand, is somewhat complicated. It is developed in detail in Appendix III for $n = 2, l = 1$. One gets

$$\begin{align*}
\Delta_{E}^{ex} E_{21j} &= -\frac{\pi \langle \alpha_s G^2 \rangle}{6N_c m^2} \left\{ \frac{j(j+1)}{2} - 4 \right\} \left( \frac{1}{r H_2^{(0)} - E_2^{(0)}} \right) \left( \frac{1}{r H_2^{(0)} - E_2^{(0)}} \right) \left\langle R_{21}^{(0)} \left| R_{21}^{(0)} \right\rangle \\
&\quad + \nu(j) \left( \frac{\partial}{\partial r} \left( \frac{1}{H_2^{(0)} - E_2^{(0)}} \right) \right) \left( \frac{1}{r H_2^{(0)} - E_2^{(0)}} \right) \right\},
\end{align*}$$

$$\frac{1}{2} \nu(0) = \nu(1) = -\nu(2) = \frac{1}{3}.$$

The calculation is finished using the inverses of Appendix I. The result is

$$\Delta_{E}^{ex} E_{21j} = \frac{1780 [j(j+1) - 4] - 2784 \nu(j)}{9945} \frac{\pi \langle \alpha_s G^2 \rangle}{m^3(C_F \bar{\alpha}_s)^2} \equiv K(j) \frac{\pi \langle \alpha_s G^2 \rangle}{m^3(C_F \bar{\alpha}_s)^2}; \quad (48)$$
with
\[ K(0) = -\frac{8976}{9945}, \quad K(1) = -K(2) = \frac{1}{2} K(0). \]

The perturbative fine splitting is (for \( s = 1 \); the splitting should be considered to vanish for \( s = 0 \))
\[ \Delta_p E_{nlj} = \frac{3C_F^4\alpha_s(\mu^2)\bar{\alpha}_s^3(\mu^2)}{16n^3l(l+1)(2l+1)} \left[ j(j+1) - l(l+1) - 2 \right] \left[ 1 + \delta_{\text{wt}}(n,0) \right]^2 \]
\[ \times \left\{ 1 + \left[ \left( \frac{\beta_0}{2} - 2 \right) \ln n - 1 - \psi(n + l + 1) + \psi(2l + 3) + \psi(2l) - \frac{n - l - 1/2}{n} \right] + \frac{125 - 10n_f}{36} + \frac{\beta_0}{2} \ln \frac{\mu}{m C_F \bar{\alpha}_s} + 2 \ln C_F \bar{\alpha}_s \right\} \frac{\alpha_s}{\pi} \]
\[ + \frac{C_F^4\alpha_s(\mu^2)\bar{\alpha}_s^3(\mu^2)}{8n^3l(l+1)(2l+1)} m \left( \frac{1}{2} S_{12} \right)_{ij} \left[ 1 + \delta_{\text{wt}}(n,0) \right]^2 \]
\[ \times \left\{ 1 + \left[ D + \left( \frac{\beta_0}{2} - 3 \right) \ln n - \psi(n + l + 1) + \psi(2l + 3) + \psi(2l) - \frac{n - l - 1/2}{n} \right] + \frac{\beta_0}{2} \ln \frac{\mu}{m C_F \bar{\alpha}_s} + 3 \ln C_F \bar{\alpha}_s \right\} \frac{\alpha_s}{\pi}. \]

The constants as in Eqs. (6), (7) and (11).

The full, relativistic plus radiative plus NP fine splitting is then
\[ \Delta_f E_{nlj} = \Delta_p E_{nlj} + \Delta^\text{in} E_{nlj} + \Delta^\text{ex} E_{nlj}, \]
the various terms given in Eqs. (42), (48) and (50).

### 4.4 Decays into \( e^+e^- \)

For a state with \( l = 0 \) the decay rate into \( e^+e^- \) is given by
\[ \Gamma(\Upsilon(nS) \to e^+e^-) = \frac{2}{n^3} \left[ \frac{Q_b \alpha}{M(\Upsilon(nS))} \right]^2 \left[ m C_F \bar{\alpha}_s(\mu^2) \right]^3 \times \left( 1 + \delta_r \right) \left[ 1 + \delta_{\text{wt}}(n,0) + \rho_{\text{NP}}(n) \right]^2. \]
Here $\delta_r$ is a "hard" radiative correction \cite{9},

$$\delta_r = -\frac{4C_F\alpha_s}{\pi}, \quad (53)$$

$\delta_{wf}(n, 0)$ is given in Eq. (11) and $\rho_{NP}(n)$ is the ratio of NP to unperturbed wave functions at the origin:

$$\rho_{NP}(n) = \frac{R_{NP}(n)}{R_{n0}(0)}. \quad (54)$$

It is to be calculated with the expressions of Appendix II.

5 Properties of Bottomium in States with $n = 1, 2$.

We will use spectroscopic notation: states will be labeled $n^{2s+1}l_j$, $l = 0, 1, 2 \ldots$ or $S, P, D, \ldots$. The somewhat whimsical notation of the Particle Data Tables (PDT) \cite{10} will also be indicated. For $n = 1, 2, 3$ mixing does not occur.

5.1 States with $n = 1$.

From TY we have

$$M(1^{3}S_1) = M(\Upsilon) = 2m \left\{ 1 - \frac{C_F^2 \tilde{\alpha}_s^2(\mu^2)}{8} - \frac{C_F^2 \tilde{\beta}_0 \alpha_s^2(\mu^2) \tilde{\alpha}_s(\mu^2)}{8\pi} \right\} \times \left( \ln \frac{\mu}{mC_F\tilde{\alpha}_s} + 1 - \gamma_E \right) + \frac{\epsilon_{10}\pi\langle\alpha_sG^2\rangle}{(mC_F\tilde{\alpha}_s)^4} m, \quad (55)$$

$$\epsilon_{10} = \frac{1872}{1275} \approx 1.468.$$
The order $\alpha_s^4$ is partially known; it adds to the right-hand-side of Eq. (55) a term

$$2m \left[ -\frac{3 C_F^4}{16} \left( 1 + \left( a_1 + \frac{\gamma E \beta_0}{2} \right) \frac{\alpha_s}{\pi} \right) \delta \tilde{\alpha}_s^3 + \frac{C_F^3 a_2}{8} \alpha_s^2 \tilde{\alpha}_s^2 
- \frac{5 C_F^4}{128} \delta \tilde{\alpha}_s^4 - \frac{3 C_F^4 \beta_0^2}{16 \pi^2} \left( \ln \frac{\mu}{m C_F \tilde{\alpha}_s} - 1 - \gamma_E \right) \alpha_s^2 \tilde{\alpha}_s + \frac{C_F^4}{6} \alpha_s \tilde{\alpha}_s^3 \right].$$

(56)

We will use both Eq. (55) alone and Eqs. (55) plus (56).

The hyperfine splitting is obtained from Eq. (41), $\Delta_{hf} (\text{Eq. (37)})$ evaluated with the expressions for the $R$’s of Appendix II, and the inverse in Eq. (39) with those of Appendix I. The result is

$$M(1^3 S_1) - M(1^1 S_0) = M(\Upsilon) - M(\eta_b)$$

$$= \frac{C_F^4 \alpha_s (\mu^2) \tilde{\alpha}_s^3 (\mu^2)}{3} m$$

$$\times \left\{ 1 + \left[ \frac{\beta_0}{2} \left( \ln \frac{\mu}{m C_F \tilde{\alpha}_s} + \gamma_E \right) - \frac{21}{4} \left( \ln \frac{1}{C_F \tilde{\alpha}_s} - 1 \right) + B \right] \frac{\alpha_s}{\pi} \right\}$$

$$+ \frac{C_F^4 \alpha_s (\mu^2) \tilde{\alpha}_s^3 (\mu^2)}{3} m \left[ \frac{270459}{108800} + \frac{1161}{8704} \right] \pi \frac{(\alpha_s G^2)}{m^4 \tilde{\alpha}_s^6}. \quad (57)$$

In the NP contribution the first term is the internal, the second the external which is, as is generally the case, substantially smaller than the first. The term in square brackets is, after multiplying by $\pi$, $7.81 + 0.42 = 8.23$, slightly smaller than the value given by Leutwyler [3] which was also used in TY and equal to 10.2. The difference in the value of the hyperfine splitting, however, is fairly small. The corrected value, following from Eq. (57), will be given below.

\[\text{It includes leading relativistic corrections } \mathcal{O}(\alpha_s^4), \text{ one–loop radiative ones } \mathcal{O}(\alpha_s^4 / \pi \ln \mu^2) \text{ and } \mathcal{O}(\alpha_s^4 / \pi), \text{ and leading logarithm two–loop corrections } \mathcal{O}(\alpha_s^4 / \pi^2 \ln^2 \mu^2). \text{ The error of Eq. (57) should be at the 10 to 20 } \% \text{ level.}\]
For the $e^+e^-$ decay Eq. (52) gives us

$$\Gamma(1^3S_1 \rightarrow e^+e^-) = \Gamma(\Upsilon \rightarrow e^+e^-)$$

$$= 2 \left[ \frac{Q_b \alpha}{M(\Upsilon)} \right]^2 \left[ m C_F \alpha_s (\mu^2) \right]^3 \left( 1 - \frac{4 C_F \alpha_s}{\pi} \right)$$

$$\times \left[ 1 + 3 \beta_0 \left( \ln \frac{2 \mu}{m C_F \alpha_s} + \frac{1}{2} - \gamma_E \right) \frac{\alpha_s}{4 \pi} + \frac{270459}{217600} \pi \langle \alpha_s G^2 \rangle \right] \left( 1 - 4 C_F \alpha_s / \pi \right)^2$$

and we have inserted the explicit values for $\delta_r, \delta_{wf}, \rho_{NP}$.

5.2 States with $n = 2$. Spin–independent shifts. Decay into $e^+e^-$. We will denote by $\overline{M}(2^3P)$ the average of the masses of the states** $2^3P_j, j = 0, 1, 2$:

$$\overline{M}(2^3P) = \frac{1}{9} \left\{ 5 M(2^3P_2) + 3 M(2^3P_1) + M(2^3P_0) \right\} = 9900 \pm 1 \text{MeV}.$$  

(59)

From the analysis of TY and Ref. [3] we have,

$$M(2^3S_1) - M(1^3S_1) = M(\Upsilon(2S)) - M(\Upsilon(1S))$$

$$= 2 m \left\{ \frac{3 C_F^2 \alpha_s^2 (\mu^2)}{32} \right.$$

$$+ \frac{C_F^2 \beta_0 \alpha_s \tilde{\alpha}_s}{32} \left[ 3 \ln \frac{\mu}{C_F m \tilde{\alpha}_s} + \frac{5}{2} - 3 \gamma_E - \ln 2 \right] \frac{\alpha_s}{\pi} \right\}$$

$$+ m \frac{(2e_20 - \epsilon_{10}) \pi \langle \alpha_s G^2 \rangle}{C_F^2 m^4 \tilde{\alpha}_s^4}, e_{20} = \frac{2102}{1326} \approx 1.585 ;(60)$$

$$\overline{M}(2^3P) - M(1^3S_1) = 2 m \left\{ \frac{3 C_F^2 \alpha_s^2 (\mu^2)}{32} \right.$$

$$+ \frac{C_F^2 \beta_0 \alpha_s \tilde{\alpha}_s}{32} \left[ 3 \ln \frac{\mu}{C_F m \tilde{\alpha}_s} + \frac{13}{6} - 3 \gamma_E - \ln 2 \right] \frac{\alpha_s}{\pi} \right\}$$

** Denoted by $\chi_{bj}(1P)$ by the PDT people, Ref. [10].
It is interesting to consider on its own the "Lamb shift", difference between Eqs. (60) and (61), as here only the states with \(n = 2\) are involved:

\[
M(2\,^3S_1) - \overline{M}(2\,^3P) = 2m \frac{C_F^2 \beta_0 \alpha_s^2 \bar{\alpha}_s}{96\pi} + m \frac{2^6 (\epsilon_{20} - \epsilon_{21}) \pi \langle \alpha_s G^2 \rangle}{C_F^4 m^4 \bar{\alpha}_s^4}. \tag{62}
\]

As for the decay \(\Upsilon(2\,^3S) \rightarrow e^+ e^-\), Eq. (52) gives

\[
\Gamma(2\,^3S_1 \rightarrow e^+ e^-) = \frac{1}{4} \left[ \frac{Q_b \alpha_s}{M(\Upsilon(2\,^3S))} \right]^2 [mC_F \bar{\alpha}_s(\mu^2)]^3 \times \left( 1 - \frac{4C_F \alpha_s}{\pi} \right) \left[ 1 + 3 \beta_0 \left( \ln \frac{2\mu}{mC_F \bar{\alpha}_s} + \frac{1}{2} - \gamma_E \right) \frac{\alpha_s}{4\pi} + \frac{302859}{884} \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 \bar{\alpha}_s^6} \right] \tag{63}
\]

5.3 States with \(n = 2\). Fine splittings.

From Eq. (51) and after some work we get the fine structure splittings\[1\]

\[
M(2\,^3P_j) - \overline{M}(2\,^3P) = m C_F^4 \beta_0 \alpha_s(\mu^2) \bar{\alpha}_s^3(\mu^2) \times \left[ 1 + 3 \beta_0 \left( \ln \frac{2\mu}{mC_F \bar{\alpha}_s} + \frac{5}{6} - \gamma_E \right) \frac{\alpha_s}{4\pi} \right] \left( 1 + \frac{111699}{221} \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 \bar{\alpha}_s^6} \right) \times \left\{ \frac{j(j+1)-4}{256} \left[ 1 + \left( \frac{\beta_0}{2} - 3 \right) \left( \ln \frac{2\mu}{mC_F \bar{\alpha}_s} - \gamma_E \right) + 2 \ln \frac{\mu}{m} + \frac{125 - 10 n_f}{36} \right] \frac{\alpha_s}{\pi} \right\} + \frac{K(j) \pi \langle \alpha_s G^2 \rangle}{384} \times \left\{ 1 + \left( \frac{\beta_0}{2} - 3 \right) \left( \ln \frac{2\mu}{mC_F \bar{\alpha}_s} + 1 - \gamma_E \right) + 3 \ln \frac{\mu}{m} + D \right\} \frac{\alpha_s}{\pi} \right\} + m K(j) \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 (C_F \bar{\alpha}_s)^6}. \tag{64}
\]

The first term containing \(\langle \alpha_s G^2 \rangle\) is the "internal" NP shift (corresponding to Eq. (42)); the last term is the "external" piece, Eq. (48). The experimental

\[1\] Because \(\delta_{\text{NP}}, \delta_{\text{e}\text{f}}\) are large we have included them in a factor \([1 + \delta_{\text{e}\text{f}}] (1 + 2\delta_{\text{NP}})\) in Eq. (64). This form or the equivalent one of a factor \([1 + \delta_{\text{e}\text{f}} + \delta_{\text{NP}}]^2\) are the ones that give more stable numerical results.
shifts are

\[ M(2^3P_2) - M(2^3P_1) = 21 \pm 1 \text{ MeV} \]
\[ M(2^3P_1) - M(2^3P_0) = 32 \pm 2 \text{ MeV}. \]

5.4 Hyperfine splittings for states with \( n = 2, l = 1. \)

The hyperfine splitting \( M(2^3P) - M(2^1P_1) \) has not been measured experimentally for bottomium. For charmonium,

\[ \overline{M}_{c\bar{c}}(2^3P) - M_{c\bar{c}}(2^1P_1) = -0.9 \pm 0.2 \text{ MeV}. \]  \hfill (65)

The theoretical calculation has been displayed in Subsection 4.2. After substituting the explicit expressions for the various pieces we get

\[ \overline{M}(2^3P) - M(2^1P_1) = m \left( \frac{\beta_0}{2} - \frac{21}{4} \right) \frac{C_1^4\alpha_s^2\tilde{\alpha}_s^3}{288\pi} \]
\[ + m \frac{61 \pi \langle \alpha_s G^2 \rangle}{117 m^4\tilde{\alpha}_s^2}. \]  \hfill (66)

This effect is remarkable. The coefficient \( \frac{\beta_0}{2} - \frac{21}{4} \) is negative; hence the perturbative and all internal NP contributions (which are, however, subleading) will be negative. On the other hand, the external NP correction is positive.

For the (relatively) light quarks \( c\bar{c} \), the perturbative piece dominates; but for \( bb \), because it decreases like \( \alpha_s^5 \), and the NP one grows like \( \alpha_s^{-2} \), the situation is reversed and we will get

\[ \overline{M}_{bb}(2^3P) - M_{bb}(2^1P_1) > 0. \]

This is of importance for calculations based on phenomenological potentials (see e.g. Refs \[2, 11\]), a matter that will be discussed in a separate publication.
6 Numerical Results.

The numerical results which correspond to the formulas given in the previous sections are presented in Table I. Before discussing them a few words have to be said about the calculational procedure. The quantities pertaining exclusively to $\bar{b}b$ in states with $n = 1$ have been taken from TY with the only modification of the hyperfine $\Upsilon - \eta_b$ mass difference where we have incorporated the (minute) modification following our corrected evaluation of the NP contribution. The criterion adopted in TY to choose the renormalization point $\mu$, was to require that radiative and NP contributions be equal in absolute value. Most results were in fact little dependent on the actual value of $\mu$ chosen. The reason is that, for $n = 1$ the quark mass (as a function of $M(\Upsilon)$ taken as input) begins at order $\alpha_s^0$ and the first corrections are $O(\alpha_s^2)$. For the decay $\Upsilon \to e^+e^-$, the leading contribution is order $\alpha_s^3$; finally the "Balmer" mass differences $M(\Upsilon nS) - M(\Upsilon 1S)$ start at order $\alpha_s^2$. By contrast the Lamb shift $M(2^3S_1) - M(3^3P)$ starts at $O(\alpha_s^3)$, the fine splittings among $2^3P_j$ states begin at order $\alpha_s^4$ (as does the $n = 1$ hyperfine splitting) and, finally, the hyperfine splitting $M(2^3P) - M(1^3P_1)$ is an effect of $O(\alpha_s^5)$. This means that for all these quantities the choice of $\mu$ is essential as small variations in $\mu$ get amplified. Because of this we have chosen to fit the value of $\mu$. We have considered three possibilities: fit the two fine splittings, and then the Lamb shift and Balmer splitting $M(2^3S_1) - M(1^3S_1)$ come out as predictions; include the Lamb effect in the fit; or fit all four processes. We present results in all three cases; we consider the last possibility to give the optimum calculation. A remarkable fact that lends credence to our results
is that the values of $\mu$ obtained with the three methods, as well as with the
criterion of TY (for the Lamb shift and Balmer splitting that was considered
also there) are extremely close one to the other.

The values of $\Lambda, \langle \alpha_s G^2 \rangle$ were not fitted. We chose, as already mentioned,

$$\Lambda(n_f = 3, 2 \text{ loops}) = 250 \pm 80 \text{ MeV},$$

$$\langle \alpha_s G^2 \rangle = 0.042 \pm 0.020 \text{ GeV}^4. \quad (67)$$

Because we take $M(\Upsilon)$ as input, we deduce $m_b$ (and $\bar{m}_b(\bar{m}_b^2)$). For the pole
mass, Eq. (67) implies, according to the analysis in TY,

$$m_b = 4906 \pm 69 - 51(A) \pm 4 - 4 \pm 0.9 \text{ MeV}, \quad (68)$$

the first variation in Eq. (68) tied to the variation of $\Lambda$ in Eq. (67), the second
tied to that of the gluon condensate also in Eq. (67).

The agreement between theory and experimental data is remarkable, as
is remarkable the stability of the predictions of the (as yet unmeasured)
hyperfine splittings. The deviations are of the expected order of the higher
corrections, $O(\alpha_s) \sim 30\%$. As drawbacks, however, let us mention the fact
that some of the NP corrections, notably the ratio $\delta_{NP}$, do actually exceed
unity$^\dagger$. This makes the results of the fine splittings less impressive than
what they look at first sight. Nevertheless, the choice of $\mu$ as well as the way
to write our equations certainly allow a control of the results.

The process $\Upsilon(2^3 S_1) \to e^+e^-$ merits a special discussion. If we take the
central value $\mu = 976 \text{ MeV}$ (Table I, column(c)) and consider the leading

$^\dagger$A list of some radiative and NP contributions is given in Table II.
expression of the width, i.e., we neglect radiative and NP corrections, we get
\[
\Gamma^{(0)} = \frac{1}{4} \left[ \frac{Q_b \alpha}{M(2S)} \right]^2 C_F m^3 \tilde{\alpha}_s^3 = 0.77 \text{keV} .
\] (69)

This is the value reported in Table I, and it compares favorably with experiment. Unfortunately the corrections involve the factors
\[
(1 + \delta_r), \ (1 + \delta_{\text{wf}}(2,0))^2, \ (1 + \rho_{\text{NP}}(2))^2
\]
(see Eq. (63) for the expressions for the \(\delta, \rho\)) and one has
\[
\delta_r = -0.61 , \ \delta_{\text{wf}} = -0.53 , \ \rho_{\text{NP}} = 3.6 .
\]

The prediction then becomes meaningless since the corrections are much larger than the nominally leading term, Eq. (63); although here, as it happens in the \(c\bar{c}\) case (see TY) this leading term yields a reasonable evaluation, considered as an order of magnitude estimate.

Taken all together, our results here as well as those of TY, constitute a coherent description of the lowest lying states of heavy quark systems, using only rigorously derived QCD properties and without need to have recourse to phenomenological potentials or adjustable parameters.
Appendix I.
We evaluate the inverses

\[
\frac{1}{H_\kappa^l - E_n^{(0)}} \rho^\nu e^{-\rho/2} \equiv p_{\nu}(\rho)e^{-\rho/2}.
\]

Here

\[
\rho \equiv \frac{2r}{na}, \quad E_n^{(0)} = -\frac{1}{ma^2n^2} = -m\frac{C_F^2\tilde{\alpha}_s^2}{4n^2}, \quad a = \frac{2}{mC_F\tilde{\alpha}_s},
\]

and

\[
H_\kappa^l = -\frac{1}{mr^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{mr^2} + \frac{\kappa\tilde{\alpha}_s}{r}.
\]

For \( \nu \) integer \( p_{\nu} \) turns out to be a polynomial:

\[
p_{\nu}(\rho) = \sum_{j=0}^{\nu+1} c_j \rho^j,
\]

and

\[
c_{\nu+1} = \frac{C_F}{\kappa n + (\nu + 2)C_F} \frac{mn^2a^2}{4},
\]

\[
c_{j-1} = \frac{C_F}{\kappa n + jC_F} \left[ j(j+1) - l(l+1) \right] c_j, \quad j = l, l+1, \ldots, \nu+1;
\]

\[
c_j = 0, \quad j < l.
\]

When \( H_\kappa^l = H_\kappa^{l(0)} \) those equations give a unique well-defined \( p_{\nu} \). For \( H_\kappa^l = H_l^{(0)} \) one should replace \( n \) by \( n + \epsilon \). Then \( p_{\nu} \) contains a singular coefficient, proportional to \( 1/\epsilon \). However, when evaluating

\[
\frac{1}{H_l^{(0)} - E_n^{(0)}} P_{nl} \rho^\nu e^{-\rho/2}
\]

with \( P_{nl} \) the projector orthogonal to the solution of

\[
(H_l^{(0)} - E_n^{(0)}) R_{nl}^{(0)} = 0,
\]

the singular term drops out and the limit \( \epsilon \to 0 \) may be taken.
Appendix II.

Here we list some nonperturbative energy shifts and wave functions (spin-independent). We write

\[ E_{nl}^{NP} = \epsilon_{nl} n^6 \frac{\pi \langle \alpha_s G^2 \rangle}{(mC_F\bar{\alpha}_s)^4} m. \]

Then,

\[ \begin{align*}
\epsilon_{10} &= 624 \frac{425}{9.929} \\
\epsilon_{20} &= 1051 \frac{1.091}{663} \\
\epsilon_{21} &= 9.945 \frac{9.929}{11.562} \\
\epsilon_{30} &= 769.456 \frac{463.239}{101.509} \\
\epsilon_{31} &= 11.562 \frac{272}{8.492} \\
\epsilon_{40} &= 101.509 \frac{60.060}{443.288} \\
\epsilon_{50} &= 443.288 \frac{260.175}{260.175} \\
\end{align*} \]

For the wave functions, and with \( \rho \equiv \frac{2r}{n a} \),

\[ R_{10}^{NP} = \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 (C_F\bar{\alpha}_s)^6} \frac{2}{a^{3/2}} e^{-\rho/2} \]
\[ \times \left\{ \frac{2.968}{425} - \frac{104}{425} \rho^2 - \frac{52}{1275} \rho^3 - \frac{1}{225} \rho^4 \right\} \]

\[ R_{20}^{NP} = \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 (C_F\bar{\alpha}_s)^6} \frac{1}{\sqrt{2} a^{3/2}} e^{-\rho/2} \]
\[ \times \left\{ \frac{3.828736}{1989} - \frac{1914.368}{1989} \rho - \frac{134528}{1989} \rho^2 + \frac{67264}{5967} \rho^3 + \frac{376}{663} \rho^4 + \frac{16}{153} \rho^5 \right\} \]

\[ R_{21}^{NP} = \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 (C_F\bar{\alpha}_s)^6} \frac{1}{\sqrt{4} a^{3/2}} \rho e^{-\rho/2} \]
\[ \times \left\{ \frac{3.299840}{1989} - \frac{149.888}{5967} \rho^2 + \frac{5248}{1989} \rho^3 - \frac{32}{153} \rho^4 \right\} \]

\[ R_{30}^{NP} = \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 (C_F\bar{\alpha}_s)^6} \frac{1}{\sqrt{3} a^{3/2}} e^{-\rho/2} \]
\[ \times \left\{ \frac{189.965808}{5719} - \frac{189.965808}{5719} \rho + \frac{24735.864}{5719} \rho^2 + \frac{3462.552}{5719} \rho^3 - \frac{1302}{43} \rho^4 \right\} \]
\[ R_{31}^{NP} = \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 (C_F \bar{\alpha_s})^6} \frac{1}{\sqrt{6} a^{3/2}} e^{-\rho/2} \]
\[ \times \left\{ \frac{1}{62909} \left[ \frac{1325287104}{62909} \rho - \frac{331321776}{62909} \rho^2 - \frac{124833216}{314545} \rho^3 + \frac{49872}{1505} \rho^4 \right] \\
+ \frac{3672}{1505} \rho^5 + \frac{9}{43} \rho^6 \right\} \]
\[ R_{40}^{NP} = \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 (C_F \bar{\alpha_s})^6} \frac{1}{a^{3/2}} e^{-\rho/2} \]
\[ \times \left\{ \frac{5609365504}{45045} - \frac{2804682752}{15015} \rho + \frac{57706496}{1001} \rho^2 - \frac{20160512}{15015} \rho^3 \\
- \frac{93551104}{135135} \rho^4 + \frac{59392}{3861} \rho^5 + \frac{256}{429} \rho^6 + \frac{32}{351} \rho^7 \right\} \]
\[ R_{50}^{NP} = \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 (C_F \bar{\alpha_s})^6} \frac{1}{\sqrt{5} a^{3/2}} e^{-\rho/2} \]
\[ \times \left\{ + \frac{37087558150000}{31221081} - \frac{74175116300000}{31221081} \rho + \frac{35702282000000}{31221081} \rho^2 \\
- \frac{13695312550000}{93663243} \rho^3 - \frac{561983427500}{93663243} \rho^4 + \frac{138527387500}{93663243} \rho^5 \\
- \frac{4827500}{261873} \rho^6 - \frac{1250}{9699} \rho^7 - \frac{625}{6588} \rho^8 \right\}. \]

For ease of reference we also give the first \( R^{(0)} \)'s
\[ R_{10}^{(0)}(r) = \frac{2}{a^{3/2}} e^{-r/a} \]
\[ R_{20}^{(0)}(r) = \frac{1}{\sqrt{2} a^{3/2}} \left( 1 - \frac{r}{a} \right) e^{-r/2a} \]
\[ R_{21}^{(0)}(r) = \frac{1}{2 \sqrt{6} a^{3/2}} \frac{r}{a} e^{-r/2a} \]

For the \( \bar{R}_{nl}^{(0)} \)'s, replace \( a \) by \( b(n, l) \) given in Eq. (10).
Appendix III.

We evaluate the matrix element (21 stands for $nl$)

$$\mathcal{M} = \sum_i \left\langle \Psi_{21j}^{(0)} \left| (\vec{S} \times \vec{P})_i \, \frac{1}{H^{(0)} - E_2^{(0)}} r_i \right| \Psi_{21j}^{(0)} \right\rangle.$$ 

It is convenient to use a Cartesian basis for the spin–angular momentum piece of $\Psi_{21j}^{(0)}$, so that

$$\Psi_{21j}(\vec{r}) = \sum_{ik} \xi_{ik}^{(a)}(j) \hat{r}_i \chi_k R_{21}^{(0)}(r).$$ (III.1)

Here $\hat{r} = \vec{r}/r$, the $\chi_k$ are column spin 1 wave functions and the coefficients $\xi_{ik}^{(a)}(j)$ are

$$\begin{align*}
\xi_{ik}^{(0)}(0) &= \frac{1}{\sqrt{4\pi}} \delta_{ik}, \\
\xi_{ik}^{(a)}(1) &= \frac{3}{\sqrt{8\pi}} \epsilon_{aik}, \\
\xi_{ik}^{(ab)}(2) &= \frac{3}{\sqrt{4\pi}} \left\{ \delta_{ia} \delta_{kb} - \frac{1}{3} \delta_{ik} \delta_{ab} \right\}.
\end{align*}$$

The last expression valid for $a \neq b$. The indices $0, a, ab$, collectively denoted by $\alpha$ in (III.1) give the (Cartesian) third component of total angular momentum. The spin–angular momentum wave functions

$$\xi^{(a)}(j) = \sum_{ik} \xi_{ik}^{(a)}(j) \hat{r}_i \chi_k$$

form an orthonormal set:

$$\int d\Omega \, \xi^{(a)}(j) \xi^{(\beta)}(j') = \delta_{jj'} \delta_{\alpha\beta}.$$ 

We have

$$\mathcal{M} = \sum_a \left\langle R_{21}^{(0)} \xi^{(a)}(j) \left| (\vec{S} \times \vec{P})_a \, \frac{1}{H^{(0)} - E_2^{(0)}} r_a \right| R_{21}^{(0)} \xi^{(a)}(j) \right\rangle.$$
If we write identically
\[ \hat{r}_a \hat{r}_i = (\hat{r}_a \hat{r}_i - \frac{1}{3} \delta_{ai}) + \frac{1}{3} \delta_{ai} , \]
then the first term corresponds to angular momentum 2, and the second to angular momentum zero. Therefore, when acting on the first we may replace \( H'_{(0)} \) by \( H'_{2(0)} \), and when acting on the second \( H'_{(0)} \) by \( H'_{0(0)} \). Hence,

\[
\mathcal{M} = \sum_{ik'k'} \left\langle R_{21}^{(0)} \left| \int d\Omega \, \xi_{i'k'}(j) \hat{r}_{i'} \chi_{k'}^\dagger \epsilon_{abc} S_b P_c \right. \right. \\
\times \left. \left. \frac{1}{H'_{(0)} - E_{2}^{(0)}} \right. r_{a} \hat{r}_{i} \xi_{ik} \chi_{k} \left| R_{21}^{(0)} \right. \right. \\
+ \frac{1}{3} \sum_{ik'k'} \left\langle R_{21}^{(0)} \left| \int d\Omega \, \xi_{i'k'}(j) \hat{r}_{i'} \chi_{k'}^\dagger \epsilon_{abc} S_b P_c \delta_{ai} \frac{1}{H'_{0(0)} - E_{2}^{(0)}} \xi_{ik} \chi_{k} \right| R_{21}^{(0)} \right. \right. \\
+ \frac{1}{3} \sum_{ik'k'} \left( \delta_{is} \delta_{ck} - \delta_{ik} \delta_{cs} \right) \left\langle R_{21}^{(0)} \left| \int d\Omega \, \xi_{i'k'}(j) \hat{r}_{i'} \chi_{k'}^\dagger \xi_{ik} \hat{r}_{c} \chi_{s} \right. \right. \\
\times \left. \left. \frac{\partial}{\partial r} \left( \frac{1}{H'_{0(0)} - E_{2}^{(0)}} - \frac{1}{H'_{2(0)} - E_{2}^{(0)}} \right) r \left| R_{21}^{(0)} \right. \right. \right. \\
and, after straightforward substitutions and arrangements,

\[
\mathcal{M} = \sum_{ik'k'} \left\langle R_{21}^{(0)} \left| \int d\Omega \, \xi_{i'k'}(j) \hat{r}_{i'} \chi_{k'}^\dagger \hat{S} \cdot \hat{L} \right. r_{i} \frac{1}{r} \frac{1}{H'_{2(0)} - E_{2}^{(0)}} \xi_{ik} \chi_{k} \left| R_{21}^{(0)} \right. \right. \\
+ \frac{1}{3} \sum_{ik'k'} \left( \delta_{is} \delta_{ck} - \delta_{ik} \delta_{cs} \right) \left\langle R_{21}^{(0)} \left| \int d\Omega \, \xi_{i'k'}(j) \hat{r}_{i'} \chi_{k'}^\dagger \xi_{ik} \hat{r}_{c} \chi_{s} \right. \right. \\
\times \left. \left. \frac{\partial}{\partial r} \left( \frac{1}{H'_{0(0)} - E_{2}^{(0)}} - \frac{1}{H'_{2(0)} - E_{2}^{(0)}} \right) r \left| R_{21}^{(0)} \right. \right. \right. \\
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The only noteworthy aspects of the derivation are first, that, because $H_t^{(0)}$ only acts on the radial variable, and the $\hat{r}_i$ only depend on the angular ones,

$$\frac{1}{H_t^{(0)} - E_2^{(0)}} \hat{r}_i = \hat{r}_i \frac{1}{H_t^{(0)} - E_2^{(0)}},$$

and, second, that for any $f(r)$,

$$P_k f(r) = -i \hat{r}_k \frac{\partial f(r)}{\partial r}.$$ 

The calculation is readily finished. Because

$$\sum_{ik} \xi_{ik}(j) \hat{r}_i \chi_k$$

corresponds to total angular momentum $\vec{j}$,

$$\vec{S} \cdot \vec{L} \sum_{ik} \xi_{ik}(j) \hat{r}_i \chi_k = \frac{j(j + 1) - l(l + 1) - s(s + 1)}{2} \sum_{ik} \xi_{ik}(j) \hat{r}_i \chi_k,$$

with $l = s = 1$. Defining also

$$\nu(j) = \frac{4\pi}{9} \sum_{ij} (\xi_{ik}(j)\xi_{ki}(j) - \xi_{ii}(j)\xi_{kk}(j)),$$

$$\frac{1}{2} \nu(0) = \nu(1) = -\nu(2) = \frac{1}{3},$$

we finally get

$$\mathcal{M} = \frac{4 - j(j + 1)}{2} \left\langle R_{21}^{(0)} \left| \frac{1}{r} \frac{1}{H_t^{(0)} - E_2^{(0)}} r \right| R_{21}^{(0)} \right\rangle$$

$$+ \nu(j) \left\langle R_{21}^{(0)} \left| \frac{\partial}{\partial r} \left( \frac{1}{H_t^{(0)} - E_2^{(0)}} - \frac{1}{H_2^{(0)} - E_2^{(0)}} \right) r \right| R_{21}^{(0)} \right\rangle.$$
References

[1] S. Titard and F. J. Ynduráin, preprint UM-TH 93-25 (1993), in press in Phys. Rev. D.

[2] A. Billoire, Phys. Lett. 92 B, 343 (1980)
   S. N. Gupta and S. F. Radford, Phys. Rev. D24, 2309 (1981); ibid., D25, 3430 (1982);
   S. N. Gupta, S. F. Radford and W. W. Repko, Phys. Rev. D26, 3305 (1982)
   W. Buchmuller, Y. J. Ng and S.–H. H. Tye, Phys. Rev. D24, 3003 (1981)
   J. Pantalone, S.–H. H. Tye and Y. J. Ng, Phys. Rev. D33, 777 (1986)

[3] H. Leutwyler, Phys. Lett. 98 B, 447 (1981)

[4] M. A. Shifman, A. I. Vainshtein and V. I. Zacharov, Nucl. Phys B147, 385 and 448 (1979)
   J. Gasser and H. Leutwyler, Phys. Reports C87, 77 (1982)

[5] F. J. Yndurain, The Theory of Quark and Gluon Interactions, Springer, 1993

[6] J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics, Mc Graw Hill, 1964

[7] A. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics, Wiley, 1963
[8] M. Kotani, *Quantum Mechanics*, vol. I, p. 27, (Tokyo, 1951); see also
L. I. Schiff, *Quantum Mechanics*, p.264, (McGraw-Hill, 1968)

[9] R. Barbieri, R. Gatto, R. Kogerler and Z. Kunszt, Phys. Lett. **57B**, 455
(1975);
R. Barbieri, G. Curci, E. d’Emilio and E. Remiddi, Nucl. Phys **B154**, 535 (1979)

[10] Particle Data Tables, Phys. Rev **D45**, Part 2, 1992

[11] F. Halzen et al., Phys. Lett. **B283**, 379 (1992)
Theoretical predictions, and experimental values for $b\bar{b}$ states with $n = 2, 1$ and $l = 1, 0, s = 1, 0, j = 0, 1, 2$.

(a) The parameter $\mu$ obtained by fitting $2^3P_j$.

(b) Fit including also $2^3S_1 - \overline{2^3P}$.

(c) Fit with the former and $2^3S_1 - 1^3S_1$.

\[ \chi^2/\text{degrees of freedom} = \left( \frac{0.33 \, ^{+0.15}_{-0.15} \, ^{+0.29}_{-0.23}}{3} \right) \]

(d) Result from TY.

(e) Result with analysis from TY with corrected NP contribution (N. B.: old result, 35 MeV).

(f) Values obtained from $e^+e^- \rightarrow$ hadrons via QCD sum rules, see Refs. 4.

| Quantity | (a) | (b) | (c) | Experiment |
|----------|-----|-----|-----|------------|
| $\mu$ (MeV) | $990 \, ^{+213}_{-198} \, ^{-43}_{+90}$ | $968 \, ^{+231}_{-224} \, ^{-53}_{+104}$ | $976 \, ^{+238}_{-228} \, ^{-54}_{+107}$ |
| $\alpha_s(\mu^2)$ | $0.36 \pm 0.03 \, ^{+0.01}_{-0.02}$ | $0.37 \pm 0.02 \, ^{+0.01}_{-0.03}$ | $0.36 \, ^{+0.03}_{-0.01} \, ^{+0.02}_{-0.02}$ |
| $\tilde{\alpha}_s(\mu^2)$ | $0.54 \, ^{+0.07}_{-0.06} \, ^{+0.02}_{-0.04}$ | $0.55 \, ^{+0.06}_{-0.03} \, ^{+0.03}_{-0.05}$ | $0.55 \, ^{+0.05}_{-0.03} \, ^{+0.03}_{-0.05}$ |
| $2^3P_2 - 2^3P_1$ | $22.2 \, ^{-0.4}_{+0.2} \, ^{+0}_{-0}$ | $18.9 \, ^{+2.3}_{-6.1} \, ^{+2.6}_{-2.5}$ | $20.0 \, ^{+1.2}_{-6.7} \, ^{-2.5}_{+2.6}$ | $21 \pm 1 \text{ MeV}$ |
| $2^3P_1 - 2^3P_0$ | $30.0 \pm 0.6 \, ^{+0}_{-0.2}$ | $25.6 \, ^{+4.2}_{-9.0} \, ^{+3.6}_{-3.1}$ | $27.2 \, ^{+5.2}_{-9.8} \, ^{-3.6}_{+3.2}$ | $32 \pm 2 \text{ MeV}$ |
| $2^3S_1 - \overline{2^3P}$ | $193 \, ^{-49}_{+92} \, ^{+92}_{-50}$ | $183 \, ^{+31}_{-42}$ | $186 \, ^{-39}_{+40} \, ^{+32}_{-42}$ | $123 \pm 1 \text{ MeV}$ |
| $2^3S_1 - 1^3S_1$ | $487 \, ^{-148}_{+222} \, ^{+68}_{-69}$ | $436 \, ^{-105}_{+74} \, ^{+14}_{-28}$ | $455 \, ^{-97}_{+68} \, ^{+17}_{-32}$ | $563 \pm 0.4 \text{ MeV}$ |
| $2^3P - 1^3P_1$ | $1.7 \, ^{-0.6}_{+0.7} \, ^{+0}_{-0.7}$ | $1.6 \, ^{-0.5}_{+0.4} \, ^{+0}_{-0.6}$ | $1.7 \, ^{-0.6}_{+0.4} \, ^{+0}_{-0.6} \text{ MeV}$ |
| $2^3S_1 \rightarrow e^+e^-$ | | | | $\sim 0.77$ | $0.56 \pm 0.10 \text{ MeV}$ |
| $\overline{m}_b(m_b^2)$ | | $4397 \, ^{+7}_{-2} \, ^{-3}_{+4}$ MeV (d) | | | $4250 \pm 100$ (f) |
| $1^3S_1 - 1^1S_0$ | | | | | |
| $1^3S_1 \rightarrow e^+e^-$ | | | | | |
| $1^3S_1 \rightarrow 2\gamma$ | | | | | |

Table 1: Compilation of results.
Table 2: Sample set of contributions.

| Quantity           | tree (a) | tree + rad. (b) | NP ext. (c) | δwf | δNP | Total |
|--------------------|----------|-----------------|-------------|-----|-----|-------|
| $2^3 P_2 - 2^3 P_1$| 11.5     | 3.4             | 1.9         | −0.27 | 2.2 | 20.0  |
| $2^3 P_1 - 2^3 P_0$| 14.4     | 4.9             | 0.95        | −0.27 | 2.2 | 27.0  |

with $\mu = 976$ MeV; $\Lambda(n_f = 3, \text{2 loops}) = 250$ MeV; $\langle \alpha_s G^2 \rangle = 0.042$ GeV$^4$.

(a) With tree level potential (including relativistic corrections).

(b) One loop radiative corrections.

(c) External NP corrections.

All dimensional numbers in MeV.