A DIRECT METHOD BASED ON THE CLENSHAW-CURTIS FORMULA FOR FRACTIONAL OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we present a new method based on the Clenshaw-Curtis formula to solve a class of fractional optimal control problems. First, we convert the fractional optimal control problem to an equivalent problem in the fractional calculus of variations. Then, by utilizing the Clenshaw-Curtis formula and the Chebyshev-Gauss-Lobatto points, we transform the problem to a discrete form. By this approach, we get a nonlinear programming problem by solving of which we can approximate the optimal solution of the main problem. We analyze the convergence of the obtained approximate solution and solve some numerical examples to show the efficiency of the method.

1. Introduction. Fractional calculus has taken place in many fields of study such as engineering and scientific applications. Since the appearance of the fractional calculus, a great advancement has been occurred and today many considerable works have been done on this matter. There are rich books in fractional calculus such as [11], [19], [15], [16].

Optimal control (OC) problems have been applied in various branches of science such as economics, mechanics, aerospace, etc. Researchers have presented many numerical methods to solve OC problems. All of these approaches occur in the two direct and indirect methods. Direct methods are based on “discritize and then optimize”, and indirect methods are based on the Pontryagin minimum principle and Hamiltonian equations (For more details see [14] and references therein).

Fractional optimal control (FOC) problems are the generalization of the OC problems with fractional dynamical systems. FOC problems have devoted a lot of attention, because the solution of these problems for many natural systems are more accurate than the classical OC ones. Since obtaining the analytical solution of the FOC problems is a complex work, the researchers try to solve these problems numerically. In recent years, noticeable researches have been performed. For instance, Sweilam et al in [20] have considered two different direct and indirect methods to solve FOC problems. The indirect method is based on the necessary optimality

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conditions which gives a system of nonlinear equations. In direct method, the FOC problem is reduced to a nonlinear programming (NLP) problem. Bhrawy et al [5] reported a new operational technique to solve FOC problems. They have used the Chebyshev polynomial and the Legendre-Gauss quadrature formula to have the operational matrix of the Caputo fractional derivatives. Then, in order to reduce the FOC problem to an algebraic equations system, they have employed the Lagrange multiplier scheme. In [3] Almeida and Torres have used a discrete method to solve FOC problems. They have converted the FOC problem to a new one that contains integer derivatives only. Lotfi et al [10] solved the FOC problem directly. Their method is based on the Legendre orthogonal polynomials. Keshavarz et al [9] also used a direct method based on the Bernouli polynomials. By the properties of the Bernouli polynomials, they reduced the given optimization problem to an algebraic equations system. Rabiei et al [18] have used Boubaker polynomials to solve the FOC problems directly and have converted the problem in to a discrete form. Among other works on FOC problems we can mention [22], [4], [12] and [13]. Also, FOC have been appeared in aerospace [1], robotic [8] and medical [7] models.

In this paper, we implement a new direct method to approximate the optimal solution of the FOC problems. First we convert the problem into an equivalent calculus of variations problem which has less constraints in comparison to the main problem. Then we utilize the Chebyshev-Gauss-Lobatto (CGL) points as interpolating nodes and use the Clenshaw-Curtis (CC) formula to approximate the optimal solution.

The organization of this paper is arranged as follows. In Section 2, we introduce the CC formula. In section 3, we introduce the FOC problems and some preliminaries which are needed. Section 4 is devoted to the method of the paper. In Section 5, we analyze the convergence of the method. In Section 6, we show the efficiency and applicability of the method by solving some numerical examples. Finally, the conclusion of the paper is presented in section 7.

2. Clenshaw-Curtis formula. Assume that \( f : [-1, 1] \rightarrow \mathbb{R} \) is a given continuous function. The following integral approximation is called CC formula (see [6] and [21])

\[
\int_{-1}^{1} f(t) dt \simeq \sum_{j=0}^{N} w_{j} f(t_{j}),
\]

where \( t_{j}, j = 0, 1, \ldots, N \) are the roots of \((1-t^{2}) \frac{d}{dt} P_{N}(t)\) and \( P_{N}(t) = \cos(N \cos^{-1}(t)) \) is the Chebyshev polynomial of order \( N \) which satisfies the following recursive formula

\[
\begin{align*}
P_{i}(t) &= 2tP_{i-1}(t) - P_{i-2}(t), & i = 2, 3, \ldots, \\
P_{0}(t) &= 1, & P_{1}(t) = t.
\end{align*}
\]

Also, \( w_{j}, j = 0, 1, \ldots, N \) are the quadrature weights of the numerical approximation (1). For \( N \) even, the weights are

\[
\begin{align*}
w_{0} &= w_{N} = \frac{1}{(N-1)}, \\
w_{s} &= w_{N-s} = \frac{4}{N} \sum_{j=0}^{\frac{N}{2}} \frac{1}{1+4j^{2}} \cos\left(\frac{2\pi js}{N}\right), & s = 1, 2, \ldots, \frac{N}{2},
\end{align*}
\]
and for $N$ odd, the weights are given by
\[
\begin{cases}
  w_0 = w_N = \frac{1}{N}, \\
  w_s = w_{N-s} = \frac{2}{N} \sum_{j=0}^{\frac{N-1}{2}} \cos\left(\frac{2\pi j s}{N}\right), & s = 1, 2, \ldots, \left(\frac{N-1}{2}\right).
\end{cases}
\]
The double prime in the weights formula denotes the first and the last elements have to be halved. The formula (1) is exact for any polynomials $f(\cdot)$ of order $N$. One notable advantage of using Clenshaw-Curtis formula (1) compared with other formulas is the high degree of accuracy.

Lemma 2.1. [6, 21] Let $t_k = \cos\left(\frac{Nk}{N} \pi\right)$ for $k = 0, 1, \ldots, N$ and $w_k, k = 0, 1, \ldots, N$ be the weights of CC formula. For any continuous function $f(\cdot)$ on $[-1, 1]$, we have
\[
\int_{-1}^{1} f(t) dt = \lim_{N \to \infty} \sum_{k=0}^{N} w_k f(t_k).
\]

3. Fractional optimal control problem. The main purpose of this paper is to solve the following class of the FOC problem
\[
\begin{align*}
\text{Minimize} & \quad J(X, U) = \int_{0}^{T} F(\tau, X(\tau), U(\tau)) d\tau, \\ 
\text{subject to} & \quad M \dot{X}(\tau) + N_0^C D_\tau^\alpha X(\tau) = G(\tau, X(\tau)) + H(\tau, X(\tau)) U(\tau), \\ & \quad E(X(0), X(T)) = 0,
\end{align*}
\]
where $\alpha \in (0, 1)$, $M, N, X_0$ and $X_N$ are in $\mathbb{R}^n$, $(M, N) \neq (0, 0)$, $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $G, H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $E : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ are differentiable functions. We assume that $H(\tau, X(\tau)) \neq 0$ for all $(\tau, X(\tau))$ where $\tau \in [0, T]$. Also, $X(\cdot)$ and $U(\cdot)$ are the state and control variables, respectively. Here, $C^\alpha D_\tau^\alpha X(\cdot)$ is the left Caputo fractional derivative which is defined below.

Definition 3.1. Assume that function $f(\cdot)$ is defined on interval $[a, b]$. The left and right Caputo fractional derivatives of $f(\cdot)$ of order $\alpha > 0$ are denoted by $C^\alpha_a D_\tau^\alpha$ and $C^\alpha_b D_\tau^\alpha$, respectively and defined by
\[
C^\alpha_a D_\tau^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \right),
\]
and
\[
C^\alpha_b D_\tau^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left( \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s) ds \right),
\]
respectively, where $n = [\alpha] + 1$.

When $0 < \alpha < 1$ we have
\[
C^\alpha_a D_\tau^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \int_a^t (t-s)^{-\alpha} f'(s) ds \right), \quad t > a,
\]
and
\[
C^\alpha_b D_\tau^\alpha f(t) = \frac{-1}{\Gamma(1-\alpha)} \left( \int_t^b (s-t)^{-\alpha} f'(s) ds \right), \quad t < b.
\]
If $\alpha = 0$, $C^\alpha_a D_\tau^0 f(t) = C^\alpha_b D_\tau^0 f(t) = f(t), \quad a < t < b,$

and if $\alpha = 1$, we have
\[
C^\alpha_a D_\tau^1 f(t) = C^\alpha_b D_\tau^1 f(t) = f'(t), \quad a < t < b.
\]
4. Implementing the method. In this section, we illustrate and implement a method based on the CC formula to solve the FOC problem (2)-(4) which contains the left Caputo fractional derivative and the usual one. In this method, we utilize the CGL points \( t_k = \cos \left( \frac{k\pi}{N} \right) \) for \( k = 0, 1, \ldots, N \). Since the CGL points are on the interval \([-1, 1]\), we first convert the main problem (2)-(4) to the interval \([-1, 1]\). To this end, we use the transformation \( \tau = \frac{T}{2}(t + 1) \) where \( t \in [-1, 1] \). Here, we need the following theorem.

**Theorem 4.1.** Assume that \( X : [0, T] \to \mathbb{R} \) is a given differentiable function and define \( x(t) = X\left(\frac{T}{2}(t + 1)\right), \quad t \in (-1, 1) \). Then

\[
^C_0 D^\alpha_\tau X(\tau) = \left(\frac{2}{T}\right)^\alpha C_1 D^\alpha_t x(t), \quad t \in (-1, 1), \quad \tau \in (0, T). \tag{5}
\]

**Proof.** Let \( t \in (-1, 1) \) is given and define \( \tau = \frac{T}{2}(t + 1) \). For all \( \xi \in (-1, t) \), define \( s = \frac{T}{1+t}(\xi + 1) \). We get

\[
X'(s) = \frac{dX}{ds} = \frac{d\xi}{ds} \frac{dX}{d\xi} = \frac{1+t}{\tau} \frac{dx}{d\xi} = \frac{1+t}{\tau} x'(\xi) \tag{6}
\]

and

\[
ds = \frac{\tau}{1+t} d\xi. \tag{7}
\]

Hence

\[
^C_0 D^\alpha_\tau X(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\tau} X'(s) ds = \frac{1}{\Gamma(1-\alpha)} \int_{-1}^{t} \frac{1+t}{\tau} x'(\xi) \left(\frac{T}{1+t}(\xi + 1)\right) d\xi \]
\[
= \frac{1}{\Gamma(1-\alpha)} \int_{-1}^{t} x'(\xi) d\xi = \left(\frac{2}{T}\right)^\alpha \frac{1}{\Gamma(1-\alpha)} \int_{-1}^{t} x'(\xi) d\xi = \left(\frac{2}{T}\right)^\alpha C_1 D^\alpha_t x(t). \]

By employing (5) and the transformation \( \tau = \frac{T}{2}(t + 1) \), the FOC problem (2)-(4) is converted into the following FOC problem

Minimize

\[
J(x, u) = \frac{T}{2} \int_{-1}^{1} f(t, x(t), u(t)) dt, \tag{8}
\]

subject to

\[
\frac{2}{T} M\dot{x}(t) + \left(\frac{2}{T}\right)^\alpha N^C_1 D^\alpha_t x(t) = g(t, x(t)) + h(t, x(t)) u(t), \quad e(x(-1), x(1)) = 0, \tag{9}
\]

where

\[
x(t) = X(\tau), \quad u(t) = U(\tau),
\]
\[
g(t, x(t)) = G(\tau, X(\tau)), \quad h(t, x(t)) = H(\tau, X(\tau)), \quad e(x(-1), x(1)) = E(X(0), X(T)).
\]

Now, by (9) we get

\[
u(t) = \frac{\frac{2}{T} M\dot{x}(t) + \left(\frac{2}{T}\right)^\alpha N^C_1 D^\alpha_t x(t) - g(t, x(t))}{h(t, x(t))}, \quad t \in (-1, 1), \tag{11}
\]

where \( h(t, x(t)) \neq 0 \) for all \( t \in (-1, 1) \). We note that

\[
\dot{X}(\tau) = \frac{dX}{d\tau} = \frac{dt}{d\tau} \frac{dX}{dt} = \frac{2}{T} \frac{dx}{dt} = \frac{2}{T} \dot{x}(t). \tag{12}
\]
Now, by substituting (11) into (8), we get the following fractional problem in the calculus of variations

\begin{align*}
\text{Minimize} \quad & I(x(\cdot)) = \int_{-1}^{1} \tilde{f}(t, x(t), \dot{x}(t), \tilde{x}(t)) dt, \\
\text{subject to} \quad & e(x(-1), x(1)) = 0,
\end{align*}

where
\begin{equation}
\tilde{f}(t, x(t), \dot{x}(t), \tilde{x}(t)) = \frac{T}{2} f(t, x(t), \frac{\partial}{\partial t} M \dot{x}(t) + (\frac{\partial}{\partial x})^{\alpha} N C_{-1} D_{t}^{\alpha} x(t) - g(t, x(t))) h(t, x(t)) \right),
\end{equation}

Now, we apply the CC formula to approximate the integral in problem (13) and (14). We first use the following fractional Lagrange interpolating polynomial
\begin{equation}
h_{j}^{\alpha}(t) = \prod_{i=0, i \neq j}^{N} \frac{(t + 1)^{\alpha} - (t_{i} + 1)^{\alpha}}{(t_{j} + 1)^{\alpha} - (t_{i} + 1)^{\alpha}}, \quad j = 0, 1, \ldots, N,
\end{equation}

where \( \alpha \in (0, 1) \) and \( t_{i}, i = 0, 1, \ldots, N \) are the CGL points. It has the worth to mention that the fractional Lagrange interpolating polynomials has the delta Kronecker property, i.e., \( h_{j}^{\alpha}(t_{k}) = 0 \) if \( j \neq k \) and otherwise \( h_{j}^{\alpha}(t_{k}) = 1 \). Now, we approximate the state variable as follows
\begin{equation}
x(t) \simeq x_{N\alpha}(t) = \sum_{j=0}^{N} \bar{x}_{j} h_{j}^{\alpha}(t), \quad -1 \leq t \leq 1.
\end{equation}

By implementing the interpolation property, we get
\begin{equation}
x(t_{k}) \simeq x_{N\alpha}(t_{k}) = \bar{x}_{k}, \quad k = 0, 1, \ldots, N.
\end{equation}

Also, we have
\begin{equation}
\dot{x}(t) \simeq \dot{x}_{N\alpha}(t) = \sum_{j=0}^{N} \bar{x}_{j} h_{j}^{\alpha'}(t),
\end{equation}

where
\begin{equation}
h_{j}^{\alpha'}(t) = \sum_{r=0, r \neq j}^{N} \frac{\alpha(t + 1)^{\alpha-1}}{(t_{j} + 1)^{\alpha} - (t_{r} + 1)^{\alpha}} \prod_{i=0, i \neq r, i \neq j}^{N} \frac{(t + 1)^{\alpha} - (t_{i} + 1)^{\alpha}}{(t_{j} + 1)^{\alpha} - (t_{i} + 1)^{\alpha}}.
\end{equation}

Moreover
\begin{equation}
C_{-1} D_{t}^{\alpha} x(t) \simeq C_{-1} D_{t}^{\alpha} x_{N\alpha}(t) = \sum_{j=0}^{N} \bar{x}_{j} C_{-1} D_{t}^{\alpha} h_{j}^{\alpha}(t),
\end{equation}

where
\begin{equation}
C_{-1} D_{t}^{\alpha} h_{j}^{\alpha}(t) = \frac{1}{\Gamma(1 - \alpha)} \int_{-1}^{t} \frac{h_{j}^{\alpha}(\xi) d\xi}{(t - \xi)^{\alpha}}.
\end{equation}

We define
\begin{equation}
D_{kj}^{\alpha} = h_{j}^{\alpha'}(t_{k}), \quad \tilde{D}_{kj}^{\alpha} = C_{-1} D_{j}^{\alpha} h_{j}(t_{k}), \quad k, j = 0, 1, \ldots, N.
\end{equation}
Thus
\[ x'(t_k) \approx x'_{N\alpha}(t_k) = \sum_{j=0}^{N} \bar{x}_j D_{kj}^\alpha, \quad k = 0, 1, \ldots, N, \]
\[ C_1 D_t^\alpha x(t_k) \approx C_1 D_t^\alpha x_{N\alpha}(t_k) = \sum_{j=0}^{N} \bar{x}_j \hat{D}_{kj}^\alpha, \quad k = 0, 1, \ldots, N. \]

Hence, by implementing the CC formula and CGL points we get the following NLP problem

\[ \begin{align*}
\text{Minimize} & \quad I_N(\bar{x}) = \sum_{k=0}^{N} w_k \tilde{f}(t_k, \bar{x}_k, \sum_{j=0}^{N} \bar{x}_j D_{kj}^\alpha, \sum_{j=0}^{N} \bar{x}_j \hat{D}_{kj}^\alpha) \\
\text{subject to} & \quad e(\bar{x}_0, \bar{x}_N) = 0,
\end{align*} \tag{24} \]

where the variables are \( \bar{x} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N) \). By solving the NLP problem (24) and (25), we approximate the optimal state \( x(\cdot) \) as (17) and then by (11) we obtain the approximate optimal control for the FOC problem (2)-(4).

5. **Convergence analysis of the method.** In this section, we show that the sequence of optimal solutions of NLP problem (24)-(25) converges to an optimal solution of the problem (13)-(14) (or equivalent problem (8)-(10)).

We assume that function \( \tilde{f} \), defined by (15), has Lipschitz property with respect to its second and third components, i.e., there is a constant \( M > 0 \) such that
\[ || \tilde{f}(t, x, y, z) - \tilde{f}(t, \bar{x}, \bar{y}, \bar{z}) || \leq M (|| x - \bar{x} || + || y - \bar{y} ||). \tag{26} \]

Let \( \tilde{x}^* = (\tilde{x}^*_0, \tilde{x}^*_1, \ldots, \tilde{x}^*_N) \) be an optimal solution of the NLP problem (24)-(25). Define
\[ x_{N\alpha}(t) = \sum_{j=0}^{N} \bar{x}_j h^\alpha_k(t), \quad -1 \leq t \leq 1, \tag{27} \]

where \( h^\alpha_k(t), k = 0, 1, \ldots, N \) are the fractional Lagrange polynomials defined by (16). Note that we have a sequence of discrete optimal solutions \( \{ \tilde{x}^* = (\tilde{x}^*_0, \tilde{x}^*_1, \ldots, \tilde{x}^*_N) \}_{N=K} \) and their interpolating polynomials \( \{ x_{N\alpha}(\cdot) \}_{N=K} \).

**Theorem 5.1.** Consider problems (13)-(14) and (24)-(25) with condition \( e(x(-1), x(1)) = x(-1) = 0 \). Assume that \( x_{N\alpha}(\cdot) \) for any \( N \geq K \) satisfies relation (27). Also, suppose that \( \{ \tilde{x}_{N\alpha}(\cdot) \}_{N=K} \) has a subsequence \( \{ \tilde{x}_{N_{i}\alpha}(\cdot) \}_{i=1}^\infty \) such that \( \lim_{i \to \infty} N_i = \infty \) and \( \lim_{i \to \infty} || \tilde{x}_{N_{i}\alpha}(\cdot) - \phi(\cdot) ||_{\infty} = 0 \) where \( \phi(\cdot) \) is a continuous function on \([-1,1]\). Then
\[ x^*(t) = \int_{-1}^{t} \phi(s)ds, \quad -1 \leq t \leq 1, \tag{28} \]
is an optimal solution for problem (13)-(14).

**Proof.** By relation (28) and \( \lim_{i \to \infty} \tilde{x}_{N_{i}\alpha}(\cdot) = \phi(\cdot), \) we get
\[ \lim_{i \to \infty} x_{N_{i}\alpha}(\cdot) = x^*(\cdot). \tag{29} \]

Moreover,
\[ \lim_{i \to \infty} \tilde{x}_{N_{i}\alpha}(t_k) = \lim_{i \to \infty} \sum_{j=0}^{N_i} \bar{x}_j D_{kj} = \phi(t_k), \quad k = 0, 1, \ldots, N. \tag{30} \]
Thus, by (32) and (33) we achieve (31), i.e.,

\[
\lim_{i \to \infty} \sum_{k=0}^{N_i} w_k \tilde{f}(t_k, \tilde{x}^*_k) = \sum_{j=0}^{N_i} \tilde{x}^*_j \dot{D}^o_{k_j} + \sum_{j=0}^{N_i} \tilde{x}^*_j \ddot{D}^o_{k_j} = \int_{-1}^{1} \tilde{f}(t, \tilde{x}^*(t), \tilde{x}^*(t), \dot{D}^o_{t} \tilde{x}^*(t)) dt.
\]

By Lemma 2.1, we get

\[
\int_{-1}^{1} \tilde{f}(t, \tilde{x}^*(t), \tilde{x}^*(t), \dot{D}^o_{t} \tilde{x}^*(t)) dt = \lim_{i \to \infty} \sum_{k=0}^{N_i} w_k \tilde{f}(t_k, \tilde{x}^*(t_k), \tilde{x}^*(t_k), \dot{D}^o_{t} \tilde{x}^*(t_k)).
\]

Also, by \( \sum_{k=0}^{N_i} w_k = 2 \) and relations (15), (29) and (30), we gain

\[
0 \leq \lim_{i \to \infty} \sum_{k=0}^{N_i} w_k \left( \tilde{f}(t_k, \tilde{x}^*(t_k), \tilde{x}^*(t_k), \dot{D}^o_{t} \tilde{x}^*(t_k)) - \tilde{f}(t_k, \tilde{x}^*_k, \sum_{j=0}^{N_i} \tilde{x}^*_j \dot{D}^o_{k_j} + \sum_{j=0}^{N_i} \tilde{x}^*_j \ddot{D}^o_{k_j}) \right) \leq M \lim_{i \to \infty} \sum_{k=0}^{N_i} w_k \left( \| \tilde{x}^*(t_k) - \tilde{x}^*_k \| + \| \tilde{x}^*(t_k) - \sum_{j=0}^{N_i} \tilde{x}^*_j \dot{D}^o_{k_j} \| \right) = 0.
\]

Thus, by (32) and (33) we achieve (31), i.e.,

\[
\lim_{i \to \infty} I_{N_i}(\tilde{x}^*) = I(x^*(\cdot)).
\]

Now, we show that \( x^*(t) \) defined by (28) is an optimal solution for (13)-(14). Suppose that \( x^{**}(t) \) is an optimal solution of problem (13)-(14). We define \( \hat{x} = x^{**}(t_k), k = 0, 1, \ldots, N \) and \( \hat{x} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N) \). So

\[
\lim_{N \to \infty} I_N(\hat{x}) = I(x^{**}(\cdot)).
\]

Now, from optimality of \( \tilde{x}^* \) and \( x^{**}(\cdot) \), we get

\[
I(x^{**}(\cdot)) \leq I(x^*(\cdot)) \leq \lim_{i \to \infty} I_{N_i}(\tilde{x}^*) \leq \lim_{i \to \infty} I_{N_i}(\tilde{x}) = \lim_{N \to \infty} I_N(\hat{x}) = I(x^{**}(\cdot)).
\]

Hence, \( I(x^*(\cdot)) = I(x^{**}(\cdot)) \). Therefore, \( x^*(\cdot) \) is an optimal solution of the problem (13)-(14).

Notice that Theorem 5.1 shows that, under relatively mild conditions, if the sequence of optimal solutions of the discrete-time problem (24)-(25) is convergent, it must converges to an optimal solution of the continuous-time problem (13)-(14) (or equivalent problem (8)-(10)).

6. **Numerical examples.** In this section, we apply the method to solve three examples and show the performance of the method. In the following examples, we use the FMINCON function in MATLAB software to solve the corresponding NLP problem (24) and (25). Also, we calculate the absolute error of the approximate optimal state and control variables as

\[
E_X(\tau) = |X^*(\tau) - X(\tau)|, \quad E_U(\tau) = |U^*(\tau) - U(\tau)|, \quad \tau \in [0, T],
\]

\[34\]
where \((X^*(\cdot), U^*(\cdot))\) and \((\bar{X}(\cdot), \bar{U}(\cdot))\) are the exact and approximate optimal solutions of the FOC problem, respectively.

**Example 6.1.** Consider the following FOC problem from [2]

\[
\begin{align*}
\text{Minimize} & \quad J(X, U) = \frac{1}{2} \int_0^1 \left[ (X(\tau) - \tau^\alpha)^2 + (U(\tau) - \tau^\alpha - \Gamma(\alpha + 1))^2 \right] d\tau, \\
\text{subject to} & \quad \partial D^\alpha_\tau X(\tau) = -X(\tau) + U(\tau), \quad X(0) = 0,
\end{align*}
\]

(35)

where the exact solution of this FOC problem is \((X^*(\tau), U^*(\tau)) = (\tau^\alpha, \tau^\alpha + \Gamma(\alpha + 1))\) and the corresponding optimal value of objective function is \(J^* = 0\). The corresponding FOC problem (8)-(10) is as follows

\[
\begin{align*}
\text{Minimize} & \quad J(x, u) = \frac{1}{4} \int_{-1}^1 \left[ (x(t) - \left(\frac{1}{2}(t + 1)\right)^\alpha)^2 + \left( u(t) - \left(\frac{1}{2}(t + 1)\right)^\alpha - \Gamma(\alpha + 1) \right)^2 \right] dt, \\
\text{subject to} & \quad 2^\alpha \partial C_{-1}^\alpha x(t) = -x(t) + u(t), \quad x(-1) = 0.
\end{align*}
\]

(37)

(38)

(39)

From (39) we have

\[ u(t) = 2^\alpha \partial C_{-1}^\alpha x(t) + x(t). \]

(40)

By substituting (40) into (37) we get the following calculus of variations problem

\[
\begin{align*}
\text{Minimize} & \quad I(x) = \frac{1}{4} \int_{-1}^1 \left[ (x(t) - \left(\frac{1}{2}(t + 1)\right)^\alpha)^2 + \left( 2^\alpha \partial C_{-1}^\alpha x(t) + x(t) - \left(\frac{1}{2}(t + 1)\right)^\alpha - \Gamma(\alpha + 1) \right)^2 \right] dt, \\
\text{subject to} & \quad x(-1) = 0.
\end{align*}
\]

Thus by proposed method, we achieve the following NLP problem

\[
\begin{align*}
\text{Minimize} & \quad I_N(\bar{x}) = \frac{1}{4} \sum_{k=0}^N w_k \left[ (\bar{x}_k - \left(\frac{1}{2}(t_k + 1)\right)^\alpha)^2 + \left( 2^\alpha \sum_{j=0}^{k} \bar{x}_j \partial C_{-1}^\alpha + \bar{x}_k - \left(\frac{1}{2}(t_k + 1)\right)^\alpha - \Gamma(\alpha + 1) \right)^2 \right], \\
\text{subject to} & \quad \bar{x}_0 = 0,
\end{align*}
\]

where \(t_k\) for \(k = 0, 1, \ldots, N\) are the CGL points. By solving the above NLP problem for \(N = 4\) and \(\alpha = 0.5, 0.6\) and \(0.7\), we get the approximate optimal state variable which has been illustrated in Figure 1. Also, by (40) we gain the approximate optimal control which is shown in Figure 2. Figures 3 and 4 show the absolute errors of state and control variables, respectively. In Tables 1 and 2, we can see the maximum of absolute errors of the obtained approximate optimal state and control variables and the optimal value of objective functional, respectively.
**Figure 1.** The exact and approximate optimal state for $N = 4$ in Example 6.1

**Figure 2.** The exact and approximate optimal control for $N = 4$ in Example 6.1

**Table 1.** The maximum absolute error for $N = 4$ in Example 6.1.

| $\alpha$ | Max $|E_x(t)|$ | Max $|E_u(t)|$ |
|----------|----------------|----------------|
| 0.5      | $6.077585 \times 10^{-5}$ | $2.031847 \times 10^{-5}$ |
| 0.6      | $2.236463 \times 10^{-5}$ | $6.593832 \times 10^{-5}$ |
| 0.7      | $1.662597 \times 10^{-5}$ | $4.616561 \times 10^{-5}$ |

**Table 2.** The optimal value of the objective functional for $N = 4$ in Example 6.1.

| $\alpha$ | $J^*$ |
|----------|-------|
| 0.5      | $7.986571 \times 10^{-19}$ |
| 0.6      | $1.200881 \times 10^{-8}$  |
| 0.7      | $6.080480 \times 10^{-1}$  |
Example 6.2. Consider the following FOC problem \cite{20}

\[
\text{Minimize } J(X,U) = \int_0^1 (U(\tau) - X(\tau))^2 d\tau
\]

subject to

\[
\dot{X}(\tau) + \frac{C}{0} D^\alpha X(\tau) = U(\tau) - X(\tau) + \frac{6\tau^{\alpha+2}}{\Gamma(\alpha+3)} + \tau^3,
\]

\[
X(0) = 0, \quad X(1) = \frac{6}{\Gamma(\alpha+4)}.
\]

The exact optimal solution of this FOC problem is \((X^*(\tau), U^*(\tau)) = \left(\frac{6\tau^{\alpha+3}}{\Gamma(\alpha+4)}, \frac{6\tau^{\alpha+3}}{\Gamma(\alpha+4)}\right)\) and the corresponding optimal value of objective functional is \(J^* = 0\). The
corresponding FOC problem (8)-(10) is as follow

Minimize \( J(x, u) = \frac{1}{2} \int_{-1}^{1} (u(t) - x(t))^2 dt \), \( (44) \)

subject to \( 2\dot{x}(t) + 2^\alpha C_{-1}D_t^\alpha x(t) = u(t) - x(t) + \frac{6(\frac{1}{2}(t + 1))^{\alpha+2}}{\Gamma(\alpha + 3)} - \frac{1}{2}(t + 1)^3, \)

\( x(-1) = 0, \quad x(1) = \frac{6}{\Gamma(\alpha + 3)}. \)

(45)

From (45) we have

\( u(t) = 2\dot{x}(t) + 2^\alpha C_{-1}D_t^\alpha x(t) + x(t) - \frac{6(\frac{1}{2}(t + 1))^{\alpha+2}}{\Gamma(\alpha + 3)} - \frac{1}{2}(t + 1)^3. \)

(47)

By substituting (47) into (44) we get

Minimize \( I(x) = \frac{1}{2} \int_{-1}^{1} \left(2\dot{x}(t) + 2^\alpha C_{-1}D_t^\alpha x(t) - \frac{6(\frac{1}{2}(t + 1))^{\alpha+2}}{\Gamma(\alpha + 3)} - \frac{1}{2}(t + 1)^3\right)^2 dt, \)

subject to \( x(-1) = 0, \quad x(1) = \frac{6}{\Gamma(\alpha + 3)}. \)

(49)

By implementing the method we obtain

Minimize \( I_N(\bar{x}) = \frac{1}{2} \sum_{k=0}^{N} w_k \left(2 \sum_{j=0}^{N} \bar{x}_j D_{k_j}^\alpha + 2^\alpha \sum_{j=0}^{N} \bar{x}_j \hat{D}_{k_j}^\alpha - \frac{6(\frac{1}{2}(t_k + 1))^{\alpha+2}}{\Gamma(\alpha + 3)} - \frac{1}{2}(t_k + 1)^3\right)^2 \)

subject to \( \bar{x}_0 = 0, \quad \bar{x}_N = \frac{6}{\Gamma(\alpha + 3)}. \)

(50)

We assume \( N = 6 \) and \( \alpha = 0.5, \ 0.6 \) and \( 0.7 \), and solve the above NLP problem. The gained approximate optimal state and control variables have been shown in Figures 5 and 6, respectively. Further, the absolute errors of approximate state and control variables are illustrated in Figures 7 and 8, respectively. Also, Tables 3 and 4 show the maximum of absolute error and the optimal value of the objective functional, respectively.

**Table 3.** The maximum absolute error for \( N = 6 \) in Example 6.2.

| \( \alpha = 0.5 \) | \( \alpha = 0.6 \) | \( \alpha = 0.7 \) |
|------------------|------------------|------------------|
| Max\(|E_x(t)|\) | \(1.128957 \times 10^{-3}\) | \(2.967422 \times 10^{-4}\) | \(1.393032 \times 10^{-3}\) |
| Max\(|E_u(t)|\) | \(1.508986 \times 10^{-2}\) | \(5.269326 \times 10^{-3}\) | \(2.011737 \times 10^{-3}\) |

**Example 6.3.** Consider the following FOC problem \([17]\)

Minimize \( J(X, U) = \int_0^1 (\tau U(\tau) - (\alpha + 2)X(\tau))^2 d\tau \)

subject to \( \dot{X}(\tau) + C_0 D_\tau^\alpha X(\tau) = U(\tau) - \tau^2, \)

(51)

(52)
The exact and approximate optimal state for $N = 6$ in Example 6.2

The exact and approximate optimal control for $N = 6$ in Example 6.2

|   | $\alpha = 0.5$ | $\alpha = 0.6$ | $\alpha = 0.7$ |
|---|----------------|----------------|----------------|
| $J^*$ | $4.699837 \times 10^{-6}$ | $5.516687 \times 10^{-7}$ | $8.165173 \times 10^{-8}$ |

The exact optimal solution of this FOC problem is $(X^*(\tau), U^*(\tau)) = \left( \frac{2^\gamma \alpha^{\alpha+2}}{\Gamma(\alpha+3)}, \frac{2^\gamma \alpha^{\alpha+1}}{\Gamma(\alpha+2)} \right)$, and the corresponding optimal value of objective function is $J^* = 0$. The
corresponding FOC problem (8)-(10) is as follows

Minimize  \[ J(x, u) = \frac{1}{2} \int_{-1}^{1} (tu(t) - (\alpha + 2)x(t))^2 dt, \]  \hspace{1cm} (54)

subject to  \[ 2\dot{x}(t) + 2^\alpha C_{-1}D_t^\alpha x(t) = u(t) + \left(\frac{1}{2}(t + 1)\right)^2 \]  \hspace{1cm} (55)

\[ x(-1) = 0, \quad x(1) = \frac{2}{\Gamma(\alpha + 3)}. \]  \hspace{1cm} (56)

From (55) we have

\[ u(t) = 2\dot{x}(t) + 2^\alpha C_{-1}D_t^\alpha x(t) - \left(\frac{1}{2}(t + 1)\right)^2. \]  \hspace{1cm} (57)
By substituting (57) into (54) we get

\[ \text{Minimize} \quad I(x) = \frac{1}{2} \int_{-1}^{1} \left( \frac{1}{2} (t + 1) (2 \dot{x}(t) + 2^{\alpha} C_{-1} D_{t}^{\alpha} x(t) \right)^{2} dt, \]

subject to \( x(-1) = 0, \quad x(1) = \frac{2}{\Gamma(\alpha + 3)}. \)

Therefore, we have the following NLP problem

\[ \text{Minimize} \quad I_{N}(\bar{x}) = \frac{1}{2} \sum_{k=0}^{N} w_{k} \left[ \frac{1}{2} (t_{k} + 1) \left( 2 \sum_{j=0}^{N} \bar{x}_{j} D_{t}^{\alpha} \bar{x}_{j} + 2^{\alpha} \sum_{j=0}^{N} \bar{x}_{j} \hat{D}_{t}^{\alpha} \bar{x}_{j} \right) \right] \]

subject to \( \bar{x}_{0} = 0, \quad \bar{x}_{N} = \frac{2}{\Gamma(\alpha + 3)}. \)

Figure 9 shows the obtained approximate optimal state variable and Figure 10 displays the gained approximate optimal control variable. The absolute error of approximate optimal state and control variables are demonstrated in Figures 11 and 12, respectively. Also, Tables 5 and 6 show the maximum of absolute error and the optimal value of the objective functional, respectively.

\[ \begin{array}{ccc}
\lambda & \alpha = 0.5 & \alpha = 0.6 & \alpha = 0.7 \\
\text{Max}_{t} \left| E_{x}(t) \right| & 2.475186 \times 10^{-3} & 3.593088 \times 10^{-4} & 1.005558 \times 10^{-5} \\
\text{Max}_{t} \left| E_{u}(t) \right| & 5.873404 \times 10^{-2} & 5.118654 \times 10^{-3} & 5.399343 \times 10^{-4}
\end{array} \]
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Figure 10. The exact and approximate optimal control for $N = 6$ in Example 6.3

Figure 11. The absolute error of the approximate optimal state in Example 6.3

Table 6. The optimal value of the objective functional for $N = 6$ in Example 6.3.

|   | $\alpha = 0.5$ | $\alpha = 0.6$ | $\alpha = 0.7$ |
|---|----------------|----------------|----------------|
| $J^*| 1.080409 \times 10^{-7}$ | $2.074337 \times 10^{-9}$ | $4.013053 \times 10^{-9}$ |

7. Conclusion. In this study, we considered a class of FOC problems and presented a new direct method to solve them based on the CC formula. In this method, we first transmitted the FOC problem to an equivalent calculus of variations problem. Then, we approximated the integral in objective function by using the CC formula and converted the obtained calculus of variation problem into an NLP one. We analyzed the convergence of the approximate optimal solutions and showed the
efficiency of the method by solving the attained NLP problem. Also, numerical results demonstrated the validity of the proposed method to solve the FOC problems.

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