Pointwise upper estimates for transition probability of continuous time random walks on graphs

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Abstract
Let \( X \) be a continuous time random walk on a weighted graph. Given the on-diagonal upper bounds of transition probabilities at two vertices \( x_1 \) and \( x_2 \), we use an adapted metric initiated by Davies, and obtain Gaussian upper estimates for the off-diagonal transition probability \( \mathbb{P}_{x_1}(X_t = x_2) \).

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1 Introduction
Let \( \Gamma = (V, E) \) be a connected, locally finite graph without double edges. The graph \( \Gamma \) can be either finite or infinite. Let \( \mu \) be an edge weight function on \( E \), such that \( \mu_{xy} = \mu_{yx} > 0 \) for each \( (x, y) \in E \), while \( \mu_{xy} = 0 \) for each \( (x, y) \notin E \). Let \( \nu = (\nu_x) \) be a positive vertex weights on \( V \). Denote by \( X = \{X_t: t \geq 0\} \) a continuous time random walk on \( \Gamma \) with generator

\[
\mathcal{L} f(x) = \frac{1}{\nu_x} \sum_{y \in V} (f(y) - f(x))\mu_{xy}.
\]

Write \( \mathbb{P}_x \) for the probability measure of \( X \) starting from \( x \).

If \( \nu_x = \sum \mu_{xy} \) for all \( x \), then the process \( X \) is called the constant speed random walk or CSRW on \( V \). It is a process that waits an exponential time mean 1 at each vertex and then jumps along one of its neighbor. If \( \nu_x \equiv 1 \), then the expected waiting time of each jump may vary greatly. Moreover, such a process may explode in finite time.

In this paper, we fix vertices \( x_1, x_2 \in V \) and functions \( f_1, f_2 \) on \( \mathbb{R}_+ \) such that for any \( i = 1, 2 \) and \( t \geq 0 \),

\[
\mathbb{P}_{x_i}(X_t = x_i) \leq \frac{1}{f_i(t)}. \tag{1.1}
\]

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Our interest is, under what circumstance \( P_{x_1}(X_t = x_2) \) will have Gaussian upper bounds. Let \( d_\nu(\cdot, \cdot) \) be a metric of \( \Gamma \) such that

\[
\begin{cases}
\frac{1}{\nu_x} \sum_y d_\nu(x, y)^2 \mu_{xy} \leq 1 & \text{for all } x \in V, \\
d_\nu(x, y) \leq 1 & \text{whenever } x, y \in V \text{ and } x \sim y.
\end{cases}
\] (1.2)

Metrics satisfying (1.2) are called adapted metrics. Such metrics were initiated by Davies [5] and [6], and are closely related to the intrinsic metric associated with a given Dirichlet form. Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \). We say that \( f \) is \((A, \gamma)\)-regular on \([a, b]\), if the function \( f \) is increasing on \( \mathbb{R}_+ \) and satisfies that

\[
\frac{f(\gamma s)}{f(s)} \leq A \frac{f(\gamma t)}{f(t)} \text{ for all } a \leq s < t < \gamma^{-1} b.
\] (1.3)

In particular, if \( a = 0 \) and \( b = \infty \) then we say that \( f \) is \((A, \gamma)\)-regular, which was introduced by Grigor’yan [8]. Here and hereafter, \( A \geq 1 \) and \( \gamma > 1 \).

**Theorem 1.1** Let \( \delta \geq 1 \). If each \( f_i \) is \((A, \gamma)\)-regular and satisfies

\[
f_i(t) \leq Ae^{\delta t} \text{ for all } t \in \mathbb{R}_+,
\] (1.4)

then there exist universal positive constants \( C_1 \) and \( \theta \), such that for any \( t \geq d_\nu(x_1, x_2) \) we have

\[
P_{x_1}(X_t = x_2) \leq C_1 A^\beta \left( \frac{\nu_{x_2}/\nu_{x_1}}{\sqrt{f_1(\alpha t)f_2(\alpha t)}} \right)^{1/2} \exp \left( -\theta \frac{d_\nu(x_1, x_2)^2}{t} \right),
\]

where \( \alpha = \min\{(2\gamma)^{-1}, (64\delta)^{-1}\} \) and \( \beta = \lceil \log \gamma \log 2 \rceil \).

The problem of getting a Gaussian upper bound from two point estimates was introduced in the manifold case by Grigor’yan [8]. In the subsequent researches, Coulhon, Grigor’yan & Zucca [4] studied the problem for discrete time random walks on graphs, while Folz [7] studied in the continuous time random walks. The current paper considers the same problem, however, it improves the result of [7] by no longer requiring a lower bound on \( \nu_x \). The improvement comes from imposing conditions on the transition probabilities \( P_x(X_t = x) \) instead of the heat kernels \( p_t(x, x) \). Note that the transition probabilities are invariant under the transformation from \((\mu, \nu)\) to \((c\mu, c\nu)\), where \((c\mu)_{xy} = c\mu_{xy}\) and \((c\nu)_x = c\nu_x\).

**Remark 1.1** The condition (1.4) is quite natural. Note that \( P_x(X_t = x) \geq \exp \left( -\frac{\mu_x t}{\nu_x} \right) \), where \( \mu_x = \sum_y \mu_{xy} \). It implies that (1.4) holds if \( A = 1 \) and \( \delta = \max\{\nu_{x_1}, \nu_{x_2}\} \). In particular, for CSRW one can take \( \delta = 1 \).

**Remark 1.2** One can also trace the values of \( C_1 \) and \( \theta \). Indeed, we select \( \theta = 10^{-7} \) in our proof.
Theorem 1.2 Let $\delta \geq 1$. If each $f_i$ is $(A, \gamma)$--regular on $[T_1, T_2)$ and satisfies
\[ f_i(t) \leq A e^{\delta t} \text{ for all } t \in [T_1, T_2), \]
then there exist universal positive constants $C_1$ and $\theta$, such that for any $t \in \tilde{T}_1$, $T_2$ we have
\[ \mathbb{P}_{x_1}(X_t = x_2) \leq C_1 A^{\beta \nu(x_2) / \nu(x_1)} \exp \left( -\theta \frac{d_\nu(x_1, x_2)^2}{t} \right). \]
Here $\alpha = \min\{(2\gamma)^{-1}, (64\delta)^{-1}\}$, $\beta = \lceil \frac{\log \gamma}{\log 2} \rceil$ and $\tilde{T}_1 = (8\alpha^{-2}T_1^2) \lor d_\nu(x_1, x_2)$.

Remark 1.3 Theorems 1.1 and 1.2 are potentially very useful for random walks in random environments where one may lack global regularity. See [1], [2] and [3] for their application.

In Section 2, we show the Integral Maximum Principle for a positive subsolution function on $\mathbb{R}_+ \times \mathbb{V}$. From this, we get the initiate estimates of the transition probabilities, the case $t \leq d_\nu(x, y)$ included. In Section 3, we add some regular condition, so that, we may update those results in the previous section. In Section 4, we give the proof of Theorem 1.1. In the final section, We discuss functions which are regular only on an interval and have rate of growth in different ways; in the meantime, Theorem 1.2 will be got.

2 Integral maximum principle

For any functions $f, g$ on $\mathbb{V}$, define
\[ \langle f, g \rangle = \sum_{x \in \mathbb{V}} f(x)g(x)\nu_x. \]
Then $\langle \cdot, \cdot \rangle$ induces an inner product space. Denote by $\| \cdot \|$ the induced norm. Let $\mathbb{I}$ be an interval of $\mathbb{R}_+$. We say that $u : \mathbb{I} \times \mathbb{V} \mapsto \mathbb{R}_+$ is a positive subsolution on $\mathbb{I} \times \mathbb{V}$ if
\[ \frac{\partial}{\partial t} u \leq \mathcal{L} u \text{ on } \mathbb{I} \times \mathbb{V}. \]
Furthermore, we define a set of functions:
\[ \mathcal{H}(\mathbb{I}) = \{ u : u \text{ is a positive subsolution on } \mathbb{I} \times \mathbb{V} \text{ and } |\{ z \in \mathbb{V} : u(t, z) \neq 0, t \in \mathbb{I} \}| < \infty \}. \]
Let $o \in B \subseteq \mathbb{V}$ with $|B| < \infty$. Set
\[ u_B(t, z) = \nu_z^{1/2} \mathbb{P}_o(X_t = z, \inf\{ s \geq 0 : X_s \not\in B \} > t). \] (2.1)
Then $u_B = 0$ on $\mathbb{R}_+ \times (\mathbb{V} \setminus B)$. Since $\Gamma$ is a locally finite graph, $u_B$ is a positive subsolution on $\mathbb{R}_+ \times \mathbb{V}$ and so $u_B \in \mathcal{H}(\mathbb{R}_+)$. Now we show the Integral Maximum Principle.
Theorem 2.1 Let \( h \) be a positive function on \( I \times V \) and \( u \in H(I) \). If for each \( t \in I \) one has

\[
\frac{1}{t_y} \sum_x \left| \frac{h(t,x) - h(t,y)}{4h(t,x)h(t,y)} \right|^2 \mu_{xy} \leq -\frac{\partial}{\partial t} \log h(t,y) \quad \text{for all} \quad y \in V, \tag{2.2}
\]

then \( J(t) = \langle u^2(t,\cdot), h(t,\cdot) \rangle \) is decreasing on \( I \).

Proof. For brevity, we frequently omit the notation \( t \). Write \( \nabla_{xy} g = g(t,y) - g(t,x) \) for any function \( g \) on \( I \times V \) and get

\[
\langle 2u \mathcal{L} u, h \rangle = 2 \langle uh, \mathcal{L} u \rangle
\]

\[
= - \sum_{x,y} \nabla_{xy}(uh) \cdot \nabla_{xy} u \cdot \mu_{xy} \quad \text{since} \quad |\{ z \in V : u(t,z) \neq 0, t \in I \}| < \infty
\]

\[
= - \sum_{x,y} (h(x) \nabla_{xy} u + u(y) \nabla_{xy} h) \cdot \nabla_{xy} u \cdot \mu_{xy}
\]

\[
= - \sum_{x,y} ((\nabla_{xy} u)^2 h(x) + u(y) \nabla_{xy} u \cdot \nabla_{xy} h) \mu_{xy}
\]

\[
= \sum_{x,y} \left[ - \left( \sqrt{h(x)} \nabla_{xy} u + \frac{u(y) \nabla_{xy} h}{2 \sqrt{h(x)}} \right)^2 + \frac{(u(y) \nabla_{xy} h)^2}{4h(x)} \right] \mu_{xy} \quad \text{since} \quad h \text{ is positive}
\]

\[
\leq \sum_{x,y} u(y)^2 \frac{|\nabla_{xy} h|^2}{4h(x)} \mu_{xy}
\]

\[
= \sum_y u(y)^2 \left( \sum_x \frac{|\nabla_{xy} h|^2}{4h(x)} \mu_{xy} \right).
\]

By (2.2), \( \sum_x \frac{|\nabla_{xy} h|^2}{4h(x)} \mu_{xy} \leq -\nu_y \frac{\partial}{\partial t} h(y) \) and hence

\[
\langle 2u \mathcal{L} u, h \rangle \leq - \sum_y u(y)^2 \nu_y \frac{\partial}{\partial t} h(y) = -\langle u^2, \frac{\partial}{\partial t} h \rangle.
\]

On the other hand, by the condition that \( u \) is a positive subsolution on \( I \times V \), we have

\[
\frac{d}{dt} J = \frac{\partial}{\partial t} \langle u^2, h \rangle = \langle 2u \frac{\partial}{\partial t} u, h \rangle + \langle u^2, \frac{\partial^2}{\partial t^2} h \rangle \leq \langle 2u \mathcal{L} u, h \rangle + \langle u^2, \frac{\partial}{\partial t} h \rangle \leq 0.
\]

Therefore, \( J \) is decreasing. \( \square \)

Owing to the metric \( d_\nu \) satisfying (1.2), Theorem 2.1 leads immediately to Corollary 2.2 as follows. Define a set of functions:

\[
\mathcal{F}(I) = \{ h : h \text{ is a positive function on } I \times V \text{ and for each } t \in I, x, y \in V \text{ with } x \sim y, \}
\]

\[
\frac{|h(t,x) - h(t,y)|^2}{4h(t,x)h(t,y)} \leq -d_\nu(x,y)^2 \frac{\partial}{\partial t} \log h(t,y) \}\}.
\]
Corollary 2.2 Let \( u \in \mathcal{H}(\mathbb{I}) \) and \( h \in \mathcal{F}(\mathbb{I}) \). Then \( J(t) = \langle u^2(t, \cdot), h(t, \cdot) \rangle \) is decreasing on \( \mathbb{I} \).

Next, several useful examples of functions in \( \mathcal{F}(\mathbb{I}) \) will be given. Let \( \rho(\cdot) \) be any nonnegative function on \( \mathbb{V} \) such that

\[
|\rho(x) - \rho(y)| \leq d_\nu(x, y) \quad \text{for any } x, y \in \mathbb{V} \text{ with } x \sim y.
\]

(In practice, one often choose \( \rho(\cdot) = d_\nu(o, \cdot) \land R \) for some \( o \in \mathbb{V} \) and \( R \geq 0 \).

Lemma 2.3 Let \( \tau > 0 \). For each \( t \geq 0 \) and \( z \in \mathbb{V} \), set

\[
h(t, z) = \exp \left\{ \left( \rho(z) - 4^{-1}e^{(t + \tau)} \right) \log \left( 1 \lor \frac{\rho(z)}{4^{-1}e^{(t + \tau)}} \right) - \frac{t}{\tau} \right\}.
\]

Then \( h(t, z) \in \mathcal{F}(\mathbb{R}_+) \).

Proof. We first show that for any \( x \in [0, \infty) \) and \( \varepsilon \in [0, 1] \), there have

\[
e^{\varepsilon x} + e^{-\varepsilon x} - 2 \leq \varepsilon^2(e^x + e^{-x} - 2); \quad \text{and}
\]

\[
1 - e^{-\varepsilon x} \geq \varepsilon(1 - e^{-x}).
\]

By the Mean Value Theorem,

\[
\frac{e^{x_2} + e^{-x_2} - 2}{\varepsilon^2(e^x + e^{-x} - 2)} = \frac{e^{x_1} - e^{-x_1}}{\varepsilon(e^x_1 - e^{-x_1})} = \frac{e^{x_2} + e^{-x_2}}{e^{x_2} + e^{-x_2}} \leq 1,
\]

where \( x > x_1 > x_2 > 0 \) and \((2.4)\) follows. In the same way, we can obtain \((2.5)\).

Fix \( y \sim z \) and \( \varepsilon = d_\nu(y, z) \). Then \( |\rho(y) - \rho(z)| \leq \varepsilon \leq 1 \) by \((2.3)\). Write \( t^+ = t + \tau \) and

\[
b = \left| (\rho(y) - 4^{-1}e^{t^+}) \log \left( 1 \lor \frac{\rho(y)}{4^{-1}e^{t^+}} \right) - (\rho(z) - 4^{-1}e^{t^+}) \log \left( 1 \lor \frac{\rho(z)}{4^{-1}e^{t^+}} \right) \right|.
\]

Then

\[
\frac{|h(t, z) - h(t, y)|^2}{4h(t, z)h(t, y)} = \frac{e^b + e^{-b} - 2}{4}.
\]

We shall consider three cases.

Case I: \( \rho(z), \rho(y) \leq 4^{-1}e^{t^+} \). Then \( b = 0 \) and

\[
\frac{|h(t, z) - h(t, y)|^2}{4h(t, z)h(t, y)} = \frac{e^b + e^{-b} - 2}{4} = 0.
\]

Case II: \( \rho(z), \rho(y) \geq 4^{-1}e^{t^+} \). By the Mean Value Theorem,

\[
b = |\rho(y) - \rho(z)| \left( \log \left( \frac{\xi}{4^{-1}e^{t^+}} \right) + \frac{\xi - 4^{-1}e^{t^+}}{\xi} \right),
\]
where $\xi$ is some value between $\rho(y)$ and $\rho(z)$. Such, $4^{-1}et^+ \leq \xi \leq \rho(y) + \varepsilon$. Furthermore,

$$b \leq \varepsilon \log \left( \frac{4\xi}{t^+} e^{-4^{-1}et^+ / \xi} \right)$$

$$\leq \varepsilon \log \left( \frac{4\xi}{t^+} \left( 1 - (1 - e^{-1})4^{-1}et^+ / \xi \right) \right) \quad \text{since (2.5)}$$

$$= \varepsilon \log \left( \frac{4\xi}{t^+} - e + 1 \right) \leq \varepsilon \log \left( \frac{4(\rho(y) + \varepsilon)}{t^+} - e + 1 \right).$$

As a result,

$$e^b + e^{-b} - 2 \leq \exp \left( \varepsilon \log \left( \frac{4(\rho(y) + \varepsilon)}{t^+} - e + 1 \right) \right) + \exp \left( -\varepsilon \log \left( \frac{4(\rho(y) + \varepsilon)}{t^+} - e + 1 \right) \right) - 2.$$

Using (2.4) we get

$$e^b + e^{-b} - 2 \leq \varepsilon^2 \left\{ \left( \frac{\rho(y) + \varepsilon}{t^+} - e + 1 \right) + \left( \frac{\rho(y) + \varepsilon}{t^+} - e + 1 \right)^{-1} - 2 \right\}$$

$$\leq \varepsilon^2 \left\{ 4 \left( \frac{\rho(y) + \varepsilon}{t^+} - e \right) \right\},$$

and hence

$$\frac{|h(t, z) - h(t, y)|}{4h(t, z)h(t, y)} \leq \varepsilon^2 \left( \frac{\rho(y) + \varepsilon}{t^+} - \frac{e}{4} \right) \leq \varepsilon^2 \left( \frac{\rho(y)}{t^+} + \frac{1}{\tau} - \frac{e}{4} \right). \quad (2.6)$$

Case III: $((\rho(z) - 4^{-1}et^+) (\rho(y) - 4^{-1}et^+) < 0$. Then $0 \leq \rho(z) \lor \rho(y) - 4^{-1}et^+ \leq \varepsilon$. So,

$$b \leq \varepsilon \log \left( \frac{\rho(y) + \varepsilon}{4^{-1}et^+} \right) \leq \varepsilon \log \left( \frac{4\rho(y) + \varepsilon}{t^+} - e + 1 \right).$$

Similarly, we have (2.6) for this case.

On the other hand, note that $h(\cdot, y)$ is differentiable on $\mathbb{R}^+$ and satisfies

$$-\frac{\partial}{\partial t} \log h(t, y) = -\frac{\partial}{\partial t} \left( (\rho(y) - 4^{-1}et^+) \log \left( 1 \lor \frac{\rho(y)}{4^{-1}et^+} \right) - \frac{t}{\tau} \right)$$

$$= \frac{1}{\tau} + 4^{-1}e \log \left( 1 \lor \frac{\rho(y)}{4^{-1}et^+} \right) + \frac{(\rho(y) - 4^{-1}et^+) \lor 0}{t^+}$$

$$\geq \frac{1}{\tau} + \left( \frac{\rho(y)}{t^+} - \frac{e}{4} \right) \lor 0.$$

Therefore, in any case we have

$$\frac{|h(t, z) - h(t, y)|}{4h(t, z)h(t, y)} \leq \varepsilon^2 \left( \frac{1}{\tau} + \left( \frac{\rho(y)}{t^+} - \frac{e}{4} \right) \lor 0 \right) \leq -\varepsilon^2 \frac{\partial}{\partial t} \log h(t, y).$$
which implies \( h \in \mathcal{F}(\mathbb{R}_+) \).

The following two examples can be obtained in a similar way as Lemma 2.3 and we leave it to the reader. See the examples in [4] Proposition 2.5 and Theorem 4.1] for a reference.

**Example 2.4** Fix \( a \in [0, \frac{1}{t}] \). Let \( h_1(t, x) = e^{ap(x) - \frac{a^2}{2} t} \). Then \( h_1 \in \mathcal{F}(\mathbb{R}_+) \).

**Example 2.5** Fix \( D \geq 5 \), \( R \geq 1 \), \( \Delta \geq \frac{24R}{D} \) and \( s > 0 \). For each \( t \in [0, s] \) and \( x \in \mathbb{V} \), set \( h_2(t, x) = \exp \left( -\frac{\rho(x)^2}{D(s-t+\Delta)} \right) \). If \( 1 \leq \rho(x) \leq R \) for each \( x \in \mathbb{V} \), then \( h_2 \in \mathcal{F}([0, s]) \).

Now, fix \( o \in \mathbb{V} \) and for each \( R \geq 0 \) set
\[
\mathcal{G}_R(\mathbb{I}) = \{ g : g \text{ is a function on } \mathbb{I} \times \mathbb{R}_+, \ g(t, r) \text{ is increasing in } r, \ g(\cdot, d_o(o, \cdot) \wedge R) \in \mathcal{F}(\mathbb{I}) \}.
\]

For brevity, we write \( B_R = \{ z \in \mathbb{V} : d_o(o, z) < R \} \). The lemma below shows the way we use Theorem 2.1.

**Lemma 2.6** Let \( T \geq \tau \geq 0 \) and \( R \geq r \geq 0 \). Let \( u \in \mathcal{H}([\tau, T]) \) and \( g \in \mathcal{G}_R([\tau, T]) \). Then
\[
\langle u(T, \cdot)^2, 1 - 1_{B_R} \rangle \leq \frac{g(\tau, r)}{g(T, R)} \| u(\tau, \cdot) \|^2 + \frac{g(\tau, R)}{g(T, R)} \langle u(\tau, \cdot)^2, 1 - 1_{B_r} \rangle.
\]

**Proof.** Let \( \rho(z) = \min \{ d_o(o, z), R \} \) for each \( z \in \mathbb{V} \). Then \( \rho = R \) on \( \mathbb{V} \setminus B_R \) and hence
\[
\langle u(T, \cdot)^2, 1 - 1_{B_R} \rangle \leq \langle u(T, \cdot)^2, g(T, \rho(\cdot)) \rangle g(T, R)^{-1}.
\]

By Theorem 2.1 and the hypothesis \( u \in \mathcal{H}([\tau, T]) \) and \( g(\cdot, \rho(\cdot)) \in \mathcal{F}([\tau, T]) \), we have
\[
\langle u(T, \cdot)^2, g(T, \rho) \rangle \leq \langle u(\tau, \cdot)^2, g(\tau, \rho) \rangle.
\]

Using the condition that \( g(t, \cdot) \) is an increasing function, we get
\[
\langle u(\tau, \cdot)^2, g(\tau, \rho) \rangle \leq \langle u(\tau, \cdot)^2, 1_{B_r} \rangle g(\tau, r) + \langle u(\tau, \cdot)^2, 1 - 1_{B_r} \rangle g(\tau, R) \leq g(\tau, r) \| u(\tau, \cdot) \|^2 + g(\tau, R) \langle u(\tau, \cdot)^2, 1 - 1_{B_r} \rangle,
\]
proving the lemma.

Furthermore, we set
\[
\mathcal{H}_o = \{ u \in \mathcal{H}(\mathbb{R}_+) : u(0, z) = \nu_o^{-1/2} 1_{\{ o \}}(z) \text{ for each } z \in \mathbb{V} \}.
\]

**Proposition 2.7** Let \( u \in \mathcal{H}_o \). For any \( t, R > 0 \), we have
\[
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq \begin{cases} 
\exp \left( -\frac{R^2}{2t} \right) & \text{ if } t \geq R, \\
\exp \left( -R \log \left( \frac{1.01R}{t} \right) + 120 \right) & \text{ if } t \leq R.
\end{cases}
\]
Proof. Consider \( t \geq R \) first. Take \( a = \frac{R}{ct} \) then \( a \in (0, \frac{1}{4}) \). For each \( s \geq 0 \) and \( r \geq 0 \), set

\[
g_1(s, r) = e^{ar} - \frac{s^2}{2}
\]

By Example 2.4, \( g_1 \in \mathcal{G}_R(\mathbb{R}_+) \). Use Lemma 2.6 and get for any \( R \geq r > 0 \),

\[
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq \frac{g_1(0, r)}{g_1(t, R)} \| u(0, \cdot) \|^2 + \frac{g_1(0, R)}{g_1(t, R)} \langle u(0, \cdot)^2, 1 - 1_{B_r} \rangle.
\]

From \( u(0, z) = \nu_0^{-1/2}1_{\{0\}}(z) \), it follows immediately that

\[
\langle u(0, \cdot)^2, 1 - 1_{B_r} \rangle = 0 \quad \text{and} \quad \| u(0, \cdot) \|^2 = 1.
\]

So,

\[
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq \lim_{r \to 0^+} \frac{g_1(0, r)}{g_1(t, R)} = \frac{g_1(0, 0)}{g_1(t, R)}.
\]

Obviously, \( g_1(0, 0) = 1 \) and hence

\[
\langle u^2(t, \cdot), 1 - 1_{B_R} \rangle \leq e^{-aR + \frac{s^2}{2}t}.
\]

Substituting the value of \( a \) into the above, we get the first inequality of the proposition.

Next, suppose \( t \leq R \). Choose \( \tau = (4c/e - 1)t \), where \( b = (4c/e - 1)^{-1} \approx 117.6 \) and \( c = e^{-e^{-1}}/1.01 \). For each \( s \geq 0 \) and \( r \geq 0 \), set

\[
g_2(s, r) = \exp \left\{ (r - 4^{-1}e)(s + \tau) \log \left( 1 + 4^{-1}e \frac{r}{s + \tau} \right) - \frac{s}{\tau} \right\}.
\]

Obviously, \( g_2(0, 0) = 1 \). By Lemma 2.3 we have \( g_2 \in \mathcal{G}_R(\mathbb{R}_+) \). Since \( x \log(R/x) \leq e^{-1}R \) for any \( x > 0 \), we get

\[
\log (g_2(t, R)) = (R - ct) \log \left( \frac{R}{ct} \right) - b
\]

\[
=R \log \left( \frac{1.01R}{t} \right) + R \log \left( \frac{1}{1.01c} \right) - ct \log \left( \frac{R}{ct} \right) - b
\]

\[
\geq R \log \left( \frac{1.01R}{t} \right) + R \log \left( \frac{1}{1.01c} \right) - e^{-1}R - 120
\]

\[
=R \log \left( \frac{1.01R}{t} \right) - 120. \tag{2.7}
\]

From (2.7) and \( g_2 \in \mathcal{G}_R(\mathbb{R}_+) \), we prove the second inequality as the first result. \( \square \)

Corollary 2.8 For any \( z \in \mathbb{V} \),

\[
\mathbb{P}_o(\mathbb{X}_t = z) \leq \begin{cases} 
(\nu_z/\nu_o)^{1/2} \exp \left\{ -\frac{s^2}{16t} \right\} & \text{if } t \geq r > 0; \\
(\nu_z/\nu_o)^{1/2} \exp \left\{ -\frac{s}{2} \log \left( \frac{0.01s}{t} \right) + 60 \right\} & \text{if } r \geq t > 0,
\end{cases}
\]

where \( r = d_o(o, z) \).
Proof. Recall the definition $u_B$ in (2.1). Denote by $d(\cdot, \cdot)$ the graph distance of $\Gamma$. Set $S_n = \{z : d(o, z) < n\}$. Then $S_n$ is a finite set since $\Gamma$ is a locally finite graph and hence $u_{S_n} \in H_o$. Clearly, $u_{S_n}$ converges pointwise to $u$ as $n$ tends to infinity, where

$$u(t, z) = \frac{\nu_o^{1/2}}{\nu_z} \mathbb{P}_o(X_t = z).$$

Note that $\langle u_{S_n}(t, \cdot)^2, 1 - 1_{B_r} \rangle \geq u_{S_n}(t, z)^2 \nu_z$, provided $z \notin B_r$. So,

$$u(t, z)^2 \nu_z = \lim_{n \to \infty} u_{S_n}(t, z)^2 \nu_z \leq \sup_n \langle u_{S_n}(t, \cdot)^2, 1 - 1_{B_r} \rangle.$$

Combining with Propositions 2.7, we get the desired result.

The intuition of Theorem 1.1 can be seen from Corollary 2.8, if $f_1 = f_2 \equiv 1$ are selected as the trivial upper bounds. Compared with [7, Theorems 2.1 and 2.2], which also give upper bounds of the transition probabilities, Corollary 2.8 works more efficiently when $t \in [0.9r, 1.1r]$ and $r = d_o(o, z)$ is large.

3 Regular functions and integral estimates

Fix $u \in H_o$ and $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ which satisfy that

$$\|u(t, \cdot)\|^2 \leq \frac{1}{f(2t)} \text{ for all } t \in \mathbb{R}_+. \quad (3.1)$$

In this section, our interest is to extend Proposition 2.7 into a result which can be used to prove Theorem 1.1.

Recall that $A \geq 1$ and $\gamma > 1$. Fix $\beta = \lceil \frac{\log 2}{\log \gamma} \rceil$.

Lemma 3.1 Let $f$ be an $(A, \gamma)$-regular function. Then for any $k \in \mathbb{N}$ and $t > 0$,

$$f(2^{-k}t) \geq \left( A^\beta \frac{f(t)}{f(\gamma^{-\beta}t)} \right)^{-k} f(t).$$

Proof. By the regularity, for any $t \geq s > 0$ we have

$$\frac{f(\gamma^j s)}{f(s)} = \prod_{j=0}^{\beta-1} \frac{f(\gamma^{j+1} s)}{f(\gamma^j s)} \leq \prod_{j=0}^{\beta-1} \left( A \frac{f(\gamma^{j+1} t)}{f(\gamma^j t)} \right) = A^\beta \frac{f(\gamma^j t)}{f(t)}.$$  \hspace{3cm} (3.2)

In other words, an $(A, \gamma)$-regular function is also $(A^\beta, \gamma^\beta)$-regular. Furthermore, by the monotonicity we get

$$\frac{f(t)}{f(2^{-k}t)} \leq \frac{f(t)}{f(\gamma^{-\beta}kt)} \leq \prod_{j=-k}^{-1} \frac{f(\gamma^{j+1} t)}{f(\gamma^j t)} \leq \prod_{j=-k}^{-1} \left( A^\beta \frac{f(t)}{f(\gamma^{-\beta}t)} \right)^k.$$
**Lemma 3.2** Let \( \delta \geq 1 \). If \( f(t) \leq Ae^{\delta t} \) for all \( t \in \mathbb{R}_+ \), then there exists a universal constant \( c \) such that for any \( t > 0 \) and \( R \in [t, 64t] \),

\[
\langle u(t, \cdot), 1 - 1_{B_R} \rangle \leq \frac{cA}{f(R)} e^{-10^{-4}R}.
\]

**Proof.** Fix \( t > 0 \), \( R \in [t, 64t] \), \( x = t/R \) and \( a = (64\delta)^{-1} \). Then \( a \leq x \leq 1 \). Write \( a_1 = 4^{-1}e \) \((a + 0.45)\) and \( b = 4^{-1}e(x + 0.45) \). Such,

\[ a_1 \geq 4^{-1}e \cdot 0.45 \geq 0.3 \quad \text{and} \quad a_1 \leq b \leq 4^{-1}e(1 + 0.45) \leq 0.99. \]

For each \( s \geq 0 \) and \( r \geq 0 \), we define

\[
g(s, r) = \exp \left\{ \left( r - 4^{-1}e (s + 0.45R) \right) \log \left( 1 \vee \frac{r}{4^{-1}e (s + 0.45R)} \right) - \frac{s}{0.45R} \right\}.
\]

By Lemma 2.3 we have \( g \in \mathcal{G}_R(\mathbb{R}_+) \). By Lemma 2.0, we get

\[
\langle u(t, \cdot), 1 - 1_{B_R} \rangle \leq \frac{g(aR, a_1R)}{g(xR, R)} \| u(aR, \cdot) \|^2 + \frac{g(aR, R)}{g(xR, R)} \langle u(aR, \cdot)^2, 1 - 1_{B_{a_1R}} \rangle. \tag{3.3}
\]

By direct calculation, we have \( g(aR, a_1R) \leq 1 \),

\[
\log (g(aR, R)) \leq R(1 - a_1) \log \left( 1 \vee \frac{1}{a_1} \right) \leq R(1 - 0.3) \log \left( \frac{1}{0.3} \right) \leq 0.8428R;
\]

and

\[
\log (g(xR, R)) \leq R(1 - b) \log \left( \frac{1}{b} \right) - \frac{x}{0.45} \geq R(1 - 0.99) \log \left( \frac{1}{0.99} \right) - 3 \geq 0.0001R - 3.
\]

Thus, (3.3) becomes

\[
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq e^{-0.0001R+3} \| u(aR, \cdot) \|^2 + e^{0.843R+3} \langle u(aR, \cdot)^2, 1 - 1_{B_{a_1R}} \rangle.
\]

By (3.1) and the hypothesis \( \log (f(s)) \leq Ae^{\delta s} \), we obtain,

\[
\langle u(t, \cdot)^2, 1 - 1_{B_{a_1R}} \rangle \leq \frac{1}{f(2aR)} e^{-0.0001R+3} + \frac{Ae^{2a\delta R}}{f(2aR)} e^{0.843R+3} \langle u(aR, \cdot)^2, 1 - 1_{B_{a_1R}} \rangle.
\]

By Proposition 2.7

\[
\langle u(aR, \cdot)^2, 1 - 1_{B_{a_1R}} \rangle \leq \exp \left( -a_1 R \log \left( \frac{a_1}{a} \right) + 120 \right).
\]

Therefore,

\[
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq \frac{Ae^{123}}{f(2aR)} \left( e^{-0.0001R} + \exp \left( 2a\delta R + 0.843R - a_1 R \log \left( \frac{a_1}{a} \right) \right) \right).
\]
Substitute \( a = (64\delta)^{-1} \) and get,

\[
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq \frac{Ae^{123}}{f\left(\frac{R}{328}\right)} \left( e^{-0.0001R} + e^{-RC} \right),
\]

where \( C = a_1 \log (64a_1\delta) - 0.8743 \). Since \( a_1 \geq 0.3 \) and \( \delta \geq 1 \), we have \( C \geq 0.01 \). So,

\[
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq \frac{2Ae^{123}}{f\left(\frac{R}{328}\right)} e^{-0.0001R}.
\]

Proposition 3.3 Let \( \delta \geq 1 \). Suppose that \( f \) is \((A, \gamma)\)-regular and satisfies \( f(t) \leq Ae^{at} \) for all \( t \in \mathbb{R}_+ \). Then there exist universal positive constants \( C_0 \) and \( \theta_1 \) such that

\[
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq C_0 A^{\theta_1} \exp \left( -\theta_1 \frac{R^2}{t} \right) \text{ for all } t \geq R \geq 10^3,
\]

where \( \alpha = \min\{ (2\gamma)^{-1}, (64\delta)^{-1} \} \).

Proof. Fix \( L = \log \left( \frac{A^{\theta_1} f(2t)}{f(2t/\gamma^\beta)} \right) \), \( D = 100 \), \( \Delta = \frac{R}{4} \) and \( \theta_1 = 10^{-6} \). If \( \theta_1 \frac{R^2}{t} - L - \frac{1}{D\Delta} < \theta_1 \), then we complete the proof since

\[
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq \| u(t, \cdot) \|^2 \leq \frac{1}{f(2t)} \leq e^{\theta_1} \exp \left( L + \frac{1}{D\Delta} - \theta_1 \frac{R^2}{t} \right)
\]

\[
= \frac{e^{\theta_1} A^{\theta_1} \exp \left( \frac{1}{D\Delta} \right)}{f(2t/\gamma^\beta)} \exp \left( -\theta_1 \frac{R^2}{t} \right)
\]

\[
\leq \frac{e^{\theta_1} A^{\theta_1} \exp (1/100)}{f(t/\gamma)} \exp \left( -\theta_1 \frac{R^2}{t} \right),
\]

where the last inequality uses the monotonicity of \( f \). Therefore, we may assume that

\[
t \geq R \geq 10^3 \quad \text{and} \quad \theta_1 \frac{R^2}{t} - L - \frac{1}{D\Delta} \geq \theta_1.
\]

This implies that \( R \leq t \leq R^2 \) and \( L \leq \theta_1 \frac{R^2}{t} \).

Let \( \rho(x) = (R - d(x, o)) \vee 1 \) for any \( x \in \mathbb{V} \). Then \( \rho \) satisfies (2.3) and \( 1 \leq \rho(x) \leq R \). For each \( s \in [0, t] \) and \( r \geq 0 \), set

\[
g(s, r) = \exp \left( -\left( \frac{(R - r) \vee 1}{D(t - s + \Delta)} \right)^2 \right).
\]
Then $g \in \mathcal{G}_R([0, t])$ by Example 2.5 and the argument above about $\rho$. From Lemma 2.6, we get that for any $r \in [0, R]$ and $s \in [0, t]$,

$$
\langle u(t, \cdot)^2, 1 - 1_{B_R} \rangle \leq \frac{g(s, r)}{g(t, R)} \|u(s, \cdot)\|^2 + \frac{g(s, R)}{g(t, R)} \langle u(s, \cdot)^2, 1 - 1_{B_r} \rangle
\leq \exp(\frac{\rho}{D\Delta}) \exp\left(-\frac{(R - r)^2}{D(t - s + \Delta)}\right) + \exp(\frac{1}{D\Delta}) \langle u(s, \cdot)^2, 1 - 1_{B_r} \rangle.
$$

(3.5)

We shall iterate using (3.5). Let us build a sequence $\{(t_j, R_j) : 0 \leq j \leq j_0\}$. Take $t_j = t/2^{j-1}$, $R_j = R/2 + R/(j + 1)$ for each $0 \leq j \leq j_0$; and

$$
\begin{align*}
&j_0 = \min\{j : R_j \geq t_j\}.
&\text{Then } j_0 \geq 1 \text{ and for all } 0 \leq j < j_0 \text{ we have } t_j > R_j > R/2 > 1. \text{ Hence }
&t_j - t_{j+1} = t_j/2 \geq R/4 = \Delta.
&\text{From } t_{j_0 - 1} > R/2, \text{ we get }
&j_0 < \frac{\log(8t/R)}{\log 2}.
&\text{Using the identity } (R_j - R_{j+1})^2 = \frac{R^2}{(j+1)^2 j^2}, \text{ we obtain }
&(R_j - R_{j+1})^2 \geq \frac{(R_j - R_{j+1})^2}{Dt_j} = \frac{2^{j-1} R^2}{D(j+1)^2(j+2)^2}. 
\end{align*}
$$

Note that

$$
\min\left\{\frac{2^{j-1}}{100(j+1)^3(j+2)^2} : j \geq 1\right\} = \frac{2^{6-1}}{100(6+1)^3(6+2)^2} \approx 1.5 \times 10^{-5}.
$$

Combined with $\theta_1 = 10^{-6}$, it follows immediately that

$$
\frac{(R_j - R_{j+1})^2}{D(t_j - t_{j+1} + \Delta)} \geq (j + 1)\theta_1 \frac{R^2}{t}.
$$

Iterating (3.5), we obtain

$$
\langle u(t, \cdot)^2, 1 - 1_{B_{R_j}} \rangle = \langle u(t_1, \cdot)^2, 1 - 1_{B_{R_1}} \rangle
\leq \sum_{j=1}^{j_0-1} \exp\left(\frac{j}{f(2t_{j+1})}\right) \exp\left(-\frac{(R_j - R_{j+1})^2}{D(t_j - t_{j+1} + \Delta)}\right) + \exp\left(\frac{j_0 - 1}{D\Delta}\right) \langle u(t_{j_0}, \cdot)^2, 1 - 1_{B_{R_{j_0}}} \rangle
:= \Lambda_1 + \Lambda_2.
$$

By Lemma 3.1, we have

$$
f(2t_{j+1}) \geq f(2t)e^{-jL}.
$$

(3.6)
Together with (3.4), we conclude
\[ \Lambda_1 \leq \frac{1}{f(2t)} \exp \left( -\theta_1 \frac{R^2}{t} \right) \sum_{j=1}^{j_0-1} \exp \left( -j \left( \theta_1 \frac{R^2}{t} - L - \frac{1}{D\Delta} \right) \right) \]
\[ \leq \frac{1}{f(2t)} \exp \left( -\theta_1 \frac{R^2}{t} \right) \sum_{j=1}^{j_0-1} \exp(-j\theta_1) \]
\[ \leq \frac{e^{-\theta_1} (1 - e^{-\theta_1})^{-1}}{f(2t)} \exp \left( -\theta_1 \frac{R^2}{t} \right). \]  
(3.7)

On the other hand, since \(2t_{j_0} = t_{j_0-1} > R_{j_0-1} > R_{j_0} \geq t_{j_0}\), we use Lemma 3.2 and get
\[ \langle u(t_{j_0}, \cdot)^2, 1 - 1_{B_{R_{j_0}}} \rangle \leq \frac{cA}{f(\frac{R_{j_0}}{32\delta})} e^{-10^{-4}R_{j_0}}, \]  
(3.8)

where \(c\) is a universal constant. By Lemma 3.1 and (3.2), we also have
\[ f \left( \frac{R_{j_0}}{32\delta} \right) \geq f \left( \frac{t_{j_0}}{25 \delta} \right) \geq \left( \frac{A^3 f \left( \frac{t_{j_0}}{25 \delta} \right)}{f \left( \frac{t_{j_0}}{25 \delta} \right)^{\gamma^2}} \right)^{-j_0+1} f \left( \frac{R_{j_0}}{32\delta} \right) \]
\[ \geq \left( \frac{A^2 f \left( \frac{2t}{25 \delta} / \gamma^2 \right)}{f \left( \frac{2t}{25 \delta} / \gamma^2 \right)} \right)^{-j_0+1} f \left( \frac{t_{j_0}}{25 \delta} \right) \]
\[ \geq f \left( \frac{t}{25 \delta} \right) e^{-2j_0 L}. \]  
(3.9)

So,
\[ \Lambda_2 = \exp \left( \frac{j_0 - 1}{D\Delta} \langle u(t_{j_0}, \cdot)^2, 1 - 1_{B_{R_{j_0}}} \rangle \right) \leq \exp \left( \frac{j_0 - 1}{D\Delta} \right) \frac{cA}{f(\frac{R_{j_0}}{32\delta})} e^{-10^{-4}R_{j_0}} \]
\[ \leq \frac{cA}{f(\frac{t}{32\delta})} \exp \left( \frac{j_0}{D\Delta} + 2j_0 L - 10^{-4}R/2 \right). \]

Note that
\[ 10^3 \leq R \leq t \leq R^2, \quad j_0 < \frac{\log(8t/R)}{\log 2}, \quad D\Delta = 25R, \quad \text{and} \quad L \leq \theta_1 \frac{R^2}{t}. \]

From these inequalities, we calculate
\[ \frac{j_0}{D\Delta R} < \frac{\log(8t/R)}{25R^2 \log 2} \leq \frac{\log(8R)}{25R^2 \log 2} = \frac{\log(8 \cdot 10^3)}{25 \cdot 10^6 \log 2} < 5.2 \times 10^{-7}; \]
\[ \frac{2j_0 L}{R} < 2\theta_1 \frac{\log(8t/R) R}{log 2} \leq 2\theta_1 \frac{8}{e \log 2} < 8.5 \times 10^{-6}. \]

So, \( \frac{j_0}{D\Delta} + 2j_0 L - 10^{-4}R/2 < -\theta_1 R \) and hence
\[ \Lambda_2 \leq \frac{cA}{f(\frac{t}{32\delta})} e^{-\theta_1 R} \leq \frac{cA}{f(\frac{t}{32\delta})} e^{-\theta_1 R^2/t}. \]  
(3.10)
Finally, we choose
\[ C_0 = e^{\theta_1 + 0.01} + e^{-\theta_1}(1 - e^{-\theta_1})^{-1} + c. \]
and complete the proof. \(\square\)

Now, we give a result which prepares for the proof of Theorem 1.1 in the next Section. Set \(\theta_1\) and \(\alpha\) as in Proposition 3.3 and write \(\theta_2 = \theta_1/5\).

**Lemma 3.4** Under the condition of Proposition 3.3, there exists a universal constant \(C_1\) such that for each \(t > 0\),
\[
\langle u(t, \cdot)^2, \exp \left( \frac{\theta_2 (d_\nu(o, \cdot) \wedge (2t))^2}{t} \right) \rangle \leq \frac{C_1 A^3}{f(2\alpha t)}. \tag{3.11}
\]

**Proof.** Write \(\rho(z) = d_\nu(o, z) \wedge (2t)\) for short. If \(t \leq 10^6\), then the result is trivial since
\[
\langle u(t, \cdot)^2, e^{\theta_2 \rho^2/t} \rangle \leq e^{4\theta_2 t}\|u(t, \cdot)\|^2 \leq \frac{e^{4\cdot 10^6\theta_2}}{f(2t)}.
\]
So, we may assume that \(t \geq 10^6\) in the following.

Fix \(R = t^{1/2}\) and \(n = \left\lceil \frac{\log(t/R)}{\log 2} \right\rceil\). Then \(2^n R \geq t\), and \(t \geq 2^j R \geq 10^3\) for each \(1 \leq j \leq n\). Write
\[
\Upsilon_0 = \langle u(t, \cdot)^2, e^{\theta_2 \rho^2/t} \rangle, \quad \Upsilon_\infty = \langle u(t, \cdot)^2, e^{\theta_2 \rho^2/t} (1 - 1_{B_R}) \rangle
\]
and for each \(1 \leq j \leq n\), set
\[
\Upsilon_j = \langle u(t, \cdot)^2, e^{\theta_2 \rho^2/t} (1_{B_{2^j R}} - 1_{B_{2^{j-1} R}}) \rangle.
\]
Then
\[
\langle u(t, \cdot)^2, \exp \left( \frac{\theta_2 \rho^2}{t} \right) \rangle \leq \Upsilon_0 + \sum_{j=1}^n \Upsilon_j + \Upsilon_\infty.
\]
We estimate each \(\Upsilon_j\) separately.

The first term admits the estimate
\[
\Upsilon_0 \leq \langle u(t, \cdot)^2, e^{\theta_2} 1_{B_R} \rangle \leq e^{\theta_2} \|u(t, \cdot)\|^2 \leq \frac{e^{\theta_2}}{f(2t)}.
\]
Next, for each \(1 \leq j \leq n\), we have
\[
\Upsilon_j \leq \langle u(t, \cdot)^2, e^{2\theta_2 (2^j)^2} (1_{B_{2^j R}} - 1_{B_{2^{j-1} R}}) \rangle \leq e^{4\theta_2} \langle u(t, \cdot)^2, 1 - 1_{B_{2^{j-1} R}} \rangle. \tag{3.12}
\]
From Proposition 3.3, we obtain
\[
\langle u(t, \cdot)^2, 1 - 1_{B_{2^j-1 R}} \rangle \leq \frac{C_0 A^3}{f(2\alpha t)} \exp \left( -\theta_1 \cdot 4^{j-1} \right).
\]
By definition $\theta_2 = \theta_1/5$; therefore we get

$$\Upsilon_j \leq \frac{C_0 A^\beta}{f(2\alpha t)} \exp (-\theta_2 \cdot 4^{j-1}).$$

For the remaining term,

$$\Upsilon_\infty \leq e^{4\theta_2 t} \langle u(t, \cdot)^2, (1 - 1_{B_t}) \rangle.$$ Use Proposition 3.3 again and get

$$\Upsilon_\infty \leq e^{4\theta_2 t} \leq C_1 A^\beta + C_0 \sum_{j=1}^{\infty} \exp (-\theta_2 \cdot 4^{j-1}) + C_0.$$}

\[\square\]

4 Proof of Theorem 1.1

Proof of Theorem 1.1. Fix $t \geq d^\nu(x_1, x_2)$ and $s = t/2$. For each $z \in V$ and $i \in \{1, 2\}$, set

$$\rho_i(s, z) = d^\nu(x_i, z) \wedge (2s) \quad \text{and} \quad h_i(s, z) = \exp \left( \frac{1}{2} \cdot \frac{\theta_2 \rho_i(s, z)^2}{s} \right).$$

Then $2\rho_1(s, z)^2 + 2\rho_2(s, z)^2 \geq d^\nu(x_1, x_2)^2$ and so

$$h_1(s, z)h_2(s, z) \geq \exp \left( \frac{\theta_2}{2} \cdot \frac{d^\nu(x_1, x_2)^2}{t} \right).$$

(4.1) Let $d(\cdot, \cdot)$ be the graph distance of $\Gamma$. As in Corollary 2.8, we define

$$u_{ij}(s, z) = \frac{\nu^i_{x_i}}{\nu_z} P_{x_i}(X_s = z, \ inf\{l \in \mathbb{R}_+ : d(x_i, X_l) \geq j\} > s)$$

and $u_i(s, z) = \frac{\nu^i_{x_i}}{\nu_z} P_{x_i}(X_s = z)$. Then $\{u_{ij}(s, z) : j = 1, 2, \ldots \}$ is an increasing sequence and satisfies

$$\lim_{j \to \infty} u_{ij}(s, z) = u_i(s, z).$$
By (1.1), for any $l \geq 0$ we have
\[
\|u_{ij}(l, \cdot)\|^2 \leq \|u_i(\cdot)\|^2 = \mathbb{P}_{x_i}(X_{2l} = x_i) \leq \frac{1}{f_i(2l)}.
\]
Since $u_{ij} \in \mathcal{H}_{x_i}$, we use Lemma 3.4 and get
\[
\|u_{ij}(s, \cdot)h_i(s, \cdot)\|^2 = \langle u_{ij}(s, \cdot)^2, \exp\left(\frac{\rho_i(s, \cdot)^2}{s}\right)\rangle \leq \frac{C_1 A^\beta}{f_i(2s)} = \frac{C_1 A^\beta}{f_i(\alpha t)}.
\]
By the Monotone Convergence Theorem,
\[
\|u_i(s, \cdot)h_i(s, \cdot)\|^2 = \lim_{j \to \infty} \|u_{ij}(s, \cdot)h_i(s, \cdot)\|^2 \leq \frac{C_1 A^\beta}{f_i(\alpha t)}.
\]
By (4.1) and the Cauchy-Schwarz inequality, we obtain
\[
\mathbb{P}_{x_i}(X_t = x_2) = \sum_{z \in \mathcal{V}} \mathbb{P}_{x_1}(X_s = z)\mathbb{P}_{x_2}(X_s = z) \frac{\nu_{x_2}}{\nu_z}
\]
\[
= (\nu_{x_2}/\nu_{x_1})^{1/2} \langle u_1(s, \cdot), u_2(s, \cdot) \rangle
\]
\[
\leq (\nu_{x_2}/\nu_{x_1})^{1/2} \langle u_1(s, \cdot)h_1(s, \cdot), u_2(s, \cdot)h_2(s, \cdot) \rangle \exp\left(-\frac{\theta_2}{2} \frac{d_{\nu}(x_1, x_2)^2}{t}\right)
\]
\[
\leq (\nu_{x_2}/\nu_{x_1})^{1/2} \|u_1(s, \cdot)h_1(s, \cdot)\| \|u_2(s, \cdot)h_2(s, \cdot)\| \exp\left(-\frac{\theta_2}{2} \frac{d_{\nu}(x_1, x_2)^2}{t}\right)
\]
\[
\leq \frac{C_1 A^\beta (\nu_{x_2}/\nu_{x_1})^{1/2}}{\sqrt{f_i(\alpha t)f_2(\alpha t)}} \exp\left(-\frac{\theta_2}{2} \frac{d_{\nu}(x_1, x_2)^2}{t}\right).
\]
Set $\theta = \theta_2/2$ and we complete the proof.

5 Regularity on an interval

If each $f_i$ is regular only on an interval and has rate of growth in a different way, the main question is to distinguish when the transition probabilities begin having the Gaussian upper bounds. In this section, we discuss three kinds of rate of growth: exponential, sub-exponential and polynomial.

Proof of Theorem 1.2. We just outline the proof here. All we need are some results similar to Lemma 3.4 and Proposition 3.3 and then complete the proof as Theorem 1.1. However we should check the inequalities (3.6), (3.8) and (3.9), which use the regular condition and the rate of growth of $f_i$. So that, we should ensure that $t < T_2/2$ and $2\alpha t_j_0 \geq T_1$ there. Note that we have $2t_j_0 \geq R/2$ in Proposition 3.3 and use $R = t^{1/2}$ in Lemma 3.4 Therefore, (3.11) holds for each $t \in [(2\alpha^{-1}T_1)^2, T_2/2)$. 

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**Theorem 5.1**  Let $\delta \geq 0$ and $\varepsilon \in [0,1)$. If each $f_i$ is $(A,\gamma)$-regular on $[T_1,T_2)$ and satisfies
\[
f_i(t) \leq Ae^{\delta t \varepsilon} \quad \text{for all } t \in [T_1,T_2),
\] (5.1)
then there exist a constant $C_1(A,\gamma,\delta,\varepsilon) > 0$ and a universal constant $\theta > 0$ such that for each $t \in [\tilde{T}_1,T_2)$,
\[
\mathbb{P}_{x_1}(X_t = x_2) \leq \frac{C_1(\nu_{x_2}/\nu_{x_1})^{1/2}}{f_1(\frac{t}{2\gamma})f_2(\frac{t}{2\gamma})} \exp\left(-\theta \frac{d_\nu(x_1,x_2)^2}{t}\right).
\] (5.2)

Here $\tilde{T}_1 = (2^9 \delta T_1^{1+\varepsilon}) \vee d_\nu(x_1,x_2)$.

As above, we first prove some results which are similar to Proposition 3.3 and Lemma 3.4.

**Proposition 5.2**  Let $\delta > 0$ and $\varepsilon \in (0,1)$. Let $u, f$ be defined as in Section 3. Suppose further that $f$ is $(A,\gamma)$-regular on $[T_1,T_2)$ and satisfies
\[
f(t) \leq Ae^{\delta t \varepsilon} \quad \text{for all } t \in [T_1,T_2).
\] (5.3)
Then there exist a constant $C_0(A,\gamma,\delta,\varepsilon) > 0$ and a universal constant $\theta_1 > 0$ such that for any $t < T_2/2$, $R \geq \max\{4, 2\kappa T_1^{1+\varepsilon} / \nu, 2(\kappa T_1)^{1+\varepsilon} / \beta\}$ with $\kappa t \geq R^{2/(1+\varepsilon)}$, we have
\[
\langle u(t, \cdot)^2, 1 - 1_B \rangle \leq \frac{C_0}{f(\frac{t}{\gamma})} \exp\left(-\theta_1 \frac{R^2}{t}\right).
\]
Here, $\kappa = (64\delta)^{1/(1+\varepsilon)}$.

**Proof.** We only show the part of the proof which is different from that of Proposition 3.3.

Let $t < T_2/2$ and $R \geq \max\{2\kappa T_1^{1+\varepsilon} / \nu, 2(\kappa T_1)^{1+\varepsilon} / \beta\}$ with $\kappa t \geq R^{2/(1+\varepsilon)}$. Take $L, D, \Delta, R_j, t_j, \theta_1$ as in Proposition 3.3 and we can still assume that $\theta_1 R_j^2 - L - \frac{1}{\nu \kappa} \geq \theta_1$. (Hence $t \leq R^2$ and $L \leq \theta_1 R^2 \kappa t$.) However, we set
\[
j_0 = \min\{j : R_j^{2/(1+\varepsilon)} \geq \kappa t_j\}.
\]
By $R \geq \max\{4, 2\kappa^{(1+\varepsilon)/(1-\varepsilon)}\}$, for each $j < j_0$ we have
\[
t_j > \kappa^{-1} R_j^{2/(1+\varepsilon)} > \kappa^{-1} (R/2)^{2/(1+\varepsilon)} \geq R/2 \geq 2.
\]
Hence
\[
t_j - t_{j+1} = t_j / 2 \geq R/4 = \Delta.
\]
Use $R \geq 2(\kappa T_1)^{(1+\varepsilon)/2}$ and get
\[
t_{j_0} = t_{j_0-1} / 2 \geq \kappa^{-1} (R/2)^{2/(1+\varepsilon)} / 2 \geq T_1/2.
\]
Since \( f \) is \((A, \gamma)\)-regular on \([T_1, T_2]\) and \(T_1 \leq 2t_{j+1} \leq 2t < T_2\) for each \(j < j_0\), one can get
\[
f(2t_{j+1}) \geq f(2t)e^{-jL}
\]
in the same way as Lemma 3.1. Hence (3.7) holds under this circumstance, too. That is,
\[
\Lambda_1 := \sum_{j=1}^{j_0-1} \exp\left(\frac{j}{D\Delta}\right) \exp \left(-\frac{(R_j - R_{j+1})^2}{D(t_j - t_{j+1} + \Delta)}\right) \leq \frac{e^{-\theta_1}(1 - e^{-\theta_1})^{-1}}{f(2t)} \exp \left(-\theta_1 \frac{R^2}{t}\right).
\]

Next, by Proposition 2.7 we obtain
\[
\langle u(t_0, \cdot)^2, 1 - 1_{B_{R_{j_0}}} \rangle \leq e^{-\frac{\kappa^2}{8j_{j_0}}} \leq \exp \left(-\frac{\kappa}{8} R_{j_0}^{2\varepsilon/(1+\varepsilon)}\right) \leq \exp \left(-\frac{\kappa}{16} R_{j_0}^{2\varepsilon/(1+\varepsilon)}\right).
\]
From \(R_{j_0}^{2/(1+\varepsilon)} \geq \kappa t_{j_0}\) and \(R_{j_0-1}^{2/(1+\varepsilon)} < \kappa t_{j_0-1}\), we get the following inequalities respectively:
\[
t_{j_0} \leq R^{2/(1+\varepsilon)} / \kappa \quad \text{and} \quad j_0 < \frac{1}{\log 2} \log \left(\frac{16\kappa t}{R^{2/(1+\varepsilon)}}\right).
\]
Hence
\[
f(2t) \leq f(2t_{j_0}) e^{j_0 L} \leq f(2R^{2/(1+\varepsilon)} / \kappa) \exp \left\{ \frac{1}{\log 2} \log \left(\frac{16\kappa t}{R^{2/(1+\varepsilon)}}\right) \cdot L \right\}
\]
By (5.3) and the assumption \(L \leq \theta_1 \frac{R^2}{t}\),
\[
f(2t) \leq A \exp \left(\frac{2\varepsilon \delta}{\kappa \varepsilon} R^{2\varepsilon/(1+\varepsilon)}\right) \cdot \exp \left\{ \frac{1}{\log 2} \log \left(\frac{16\kappa t}{R^{2/(1+\varepsilon)}}\right) \cdot \theta_1 \frac{R^2}{t} \right\}
\]
\[
\leq A \exp \left(\frac{2\varepsilon \delta}{\kappa \varepsilon} R^{2\varepsilon/(1+\varepsilon)}\right) \exp \left\{ \frac{\theta_1}{e \log 2} \frac{1}{25 R} 16\kappa R^{2\varepsilon/(1+\varepsilon)} \right\},
\]
Since \(t \leq R^2\) and \(R \geq 4\), there exists a constant \(c_1\) which depends on \(\kappa\) and \(\varepsilon\) such that
\[
\exp\left(\frac{j_0 - 1}{D\Delta}\right) \leq \exp \left(\frac{1}{\log 2} \log \left(\frac{16\kappa t}{R^{2/(1+\varepsilon)}}\right) \cdot \frac{1}{25 R}\right) \leq \exp \left(\frac{1}{\log 2} \log \left(\frac{16\kappa R^2}{R^{2/(1+\varepsilon)}}\right) \cdot \frac{1}{25 R}\right) \leq c_1.
\]
Now, substituting \(\kappa = (64\delta)^{1/(1+\varepsilon)}\) and using the condition \(\kappa t \geq R^{2/(1+\varepsilon)}\), we have
\[
\Lambda_2 := \exp\left(\frac{j_0 - 1}{D\Delta}\right) \langle u(t_0, \cdot)^2, 1 - 1_{B_{R_{j_0}}} \rangle
\]
\[
\leq c_1 \exp \left(-\frac{\kappa}{16} R^{2\varepsilon/(1+\varepsilon)}\right)
\]
\[
\leq c_1 \exp \left(-\frac{\kappa}{16} R^{2\varepsilon/(1+\varepsilon)}\right) \cdot \frac{1}{f(2t)} \cdot A \exp \left(\frac{2\varepsilon \delta}{\kappa \varepsilon} R^{2\varepsilon/(1+\varepsilon)}\right) \exp \left\{ \frac{\theta_1}{e \log 2} \frac{1}{25 R} 16\kappa R^{2\varepsilon/(1+\varepsilon)} \right\}
\]
\[
= \frac{c_1 A}{f(2t)} \exp \left(\left(-\frac{\kappa}{16} + \frac{2\varepsilon \kappa}{64} + \frac{\theta_1}{e \log 2} 16\kappa\right) R^{2\varepsilon/(1+\varepsilon)}\right)
\]
\[
\begin{align*}
&\leq \frac{c_1 A}{f(2t)} \exp \left( -\frac{\kappa}{64} R^{2/(1+\varepsilon)} \right) \\
&\leq \frac{c_1 A}{f(2t)} \exp \left( -\frac{1}{64} \frac{R^2}{t} \right).
\end{align*}
\]

This completes the proof. \(\square\)

**Lemma 5.3** Under the condition of Proposition 5.2, there exist \(C_0(A, \gamma, \delta, \varepsilon) > 0\) and a universal constant \(\theta_2 > 0\) such that for any \(t \in [2^8\delta T_1^{1+\varepsilon}, T_2/2]\), we have

\[
\left\langle u(t, \cdot)^2, \exp \left( \theta_2 \frac{d_n(a, \cdot) \wedge (2t)'}{t} \right) \right\rangle \leq \frac{C_0}{f(t/\gamma)}.
\]

**Proof.** We only show the difference from Lemma 3.4. Let \(t \in [2^8\delta T_1^{1+\varepsilon}, T_2/2]\). Fix \(\kappa = (64\delta)^{1/(1+\varepsilon)}\) and \(t_0 = \max\{16, 4\kappa^{(2+2\varepsilon)/(1-\varepsilon)}, \kappa^{-(1+\varepsilon)/\varepsilon}\}\). If \(t \leq t_0\), then as before the result is trivial. So, we may assume further \(t \geq t_0\). Fix \(R = \frac{t_1}{2}\). Then \(R \geq \max\{4, 2\kappa^{1+\varepsilon}, 2(\kappa T_1^{(1+\varepsilon)/2})/2\}\) and \(\kappa t \geq R^{2/(1+\varepsilon)}\).

Define \(\theta_2, \Upsilon_j\) and \(n\) as in Lemma 3.4. However, we set \(m = \max\{j : \kappa t \geq (2^j R)^{2/(1+\varepsilon)}\}\). So \(\kappa t < (2^{m+1} R)^{2/(1+\varepsilon)} = (2^{m+1} t^{1/2})^{2/(1+\varepsilon)}\). It deduces

\[
4^{m+1} > \kappa^{1+\varepsilon} t^{\varepsilon} = 64\delta t^{\varepsilon}.
\]

Using Proposition 5.2 as before we can still get for each \(1 \leq j \leq m\),

\[
\Upsilon_j \leq \frac{C_0}{f(t/\gamma)} \exp \left( -\theta_2 \cdot 4^{j-1} \right).
\]

If \(m + 1 \leq j \leq n\) then use Proposition 2.7 and get

\[
\Upsilon_j \leq e^{\theta_2} \langle u(t, \cdot)^2, 1 - B_{2j-1} R \rangle \leq e^{\theta_2} \cdot \exp \left( -\frac{(2^{j-1})^2}{8} \right) \leq \exp \left( -\frac{4^{j-1}}{12} \right).
\]

By (5.3) and (5.4), we have

\[
\Upsilon_j \leq \frac{A e^{4\varepsilon}}{f(t)} \exp \left( -\frac{4^{j-1}}{12} \right) \leq \frac{A}{f(t)} \exp \left( 4^{m-2} - \frac{4^{j-1}}{12} \right) \leq \frac{A}{f(t)} \exp \left( 4^{j-3} - \frac{4^{j-1}}{12} \right) = \frac{A}{f(t)} \exp \left( -\frac{4^{j-1}}{48} \right).
\]

For the rest terms \(\Upsilon_0\) and \(\Upsilon_\infty\), we can estimate in the same way as Lemma 3.4 and so we finish the proof. \(\square\)

**Proof of Theorem 5.1.** If \(\varepsilon \delta = 0\), then the problem is reduced to Corollary 2.8 since each \(f_i\) has a constant upper bound on \([T_1, T_2]\). Otherwise, if \(\delta > 0\) and \(\varepsilon \in (0, 1)\) then we can get the proof as Theorem 1.1 by using Lemma 5.3 and the Cauchy-Schwarz inequality. \(\square\)
Theorem 5.4 Let $\varepsilon \geq 0$. If each $f_i$ is $(A, \gamma)$–regular on $[T_1, T_2)$ and satisfies

$$f_i(t) \leq At^\varepsilon$$

for all $t \in [T_1, T_2)$, then there exist a constant $C_1(A, \gamma, \varepsilon) > 0$ and a universal constant $\theta > 0$ such that for each $t \in [\tilde{T}_1, T_2)$,

$$\mathbb{P}_{x_1}(X_t = x_2) \leq \frac{C_1(\nu_{x_2}/\nu_{x_1})^{1/2}}{\sqrt{f_1(\frac{t}{2\gamma})f_2(\frac{t}{2\gamma})}} \exp \left(-\theta \frac{d_\nu(x_1, x_2)^2}{t}\right).$$

Here, $\tilde{T}_1 = \left(2^{10}\varepsilon T_1 \log(T_1 \vee 1)\right) \vee d_\nu(x_1, x_2)$.

Proof. We obtain a similar result as Proposition 5.2 just by setting

$$j_0 = \min\{j : R_j^2/\log R_j \geq \kappa t_j\},$$

and then prove the theorem as above. \qed

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