LONG TIME BEHAVIOR OF THE FRACTIONAL
KORTEweg-de Vries equation WITH
CubIC NONLINEARITY

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Abstract. We prove global existence and modified scattering for the solutions
of the Cauchy problem to the fractional Korteweg-de Vries equation with cubic
nonlinearity for small, smooth and localized initial data.

1. Introduction. We consider the fractional Korteweg-de Vries equation with cubic nonlinearity:

\[ \partial_t u - |D|^{\alpha} \partial_x u = -u^2 \partial_x u, \quad -1 < \alpha < 0, \]

where \( u \) maps \( \mathbb{R}_t \times \mathbb{R}_x \) to \( \mathbb{R} \) and \( |D|^{\alpha} \) is the usual Fourier multiplier operator with the symbol \( |\xi|^{\alpha} \). The initial data is given by

\[ u(0, x) = u_0(x). \]

Although it does not seem to appear in a specific physical context, this equation, which will be from now on referred to as the modified fractional KdV equation (modified fKdV), is a good toy model to understand the influence of a weak dispersion on the dynamics of a scalar conservation law such as the modified Burgers equation.

We restrict to negative values of \( \alpha \) since when \( \alpha > 0 \) the dispersive effects dominate, excluding the existence of shocks for instance and with the possibility of global existence of arbitrary large solutions which is conjectured in the quadratic case when \( \alpha > 1/2 \) (see [17, 18]) and actually proven when \( \alpha > 6/7 \), see [19]. Also the question of global existence of small solutions is much easier when \( \alpha > 0 \) due to the stronger dispersive effects.

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When $\alpha = -\frac{1}{2}$, (1) is reminiscent for large frequencies, of a modified Whitham equation

$$\partial_t u + \mathcal{L} \partial_x u = -u^2 \partial_x u,$$

where the Fourier multiplier operator $\mathcal{L}$ has symbol $\left(\frac{\tanh \xi}{\xi}\right)^{1/2}$ which is the dispersion of gravity waves in finite depth.

Actually the dispersion in (1) when $\alpha = -1/2$ is that of purely gravity waves in infinite depth.

We refer for instance to [16] for various issues and results on the usual (quadratic) Whitham equation, in particular on its KdV, long wave limit. It is straightforward to check that Theorem 1 in [16] extends to the modified Whitham equation, proving its long wave limit to the modified KdV equation. More precisely we consider the rescaled modified Whitham equation

$$u_t + \mathcal{L}_\epsilon u_x + \epsilon u^2 u_x = 0,$$

where the non local operator $\mathcal{L}_\epsilon$ is related to the dispersion relation of the (linearized) water waves system and is defined by

$$\mathcal{L}_\epsilon = \ell(\sqrt{\epsilon}D) := \left(\frac{\tanh \sqrt{\epsilon}|D|}{\sqrt{\epsilon}|D|}\right)^{1/2}$$

and $D = -i \nabla = -i \frac{\partial}{\partial x}$, that we want to compare to the modified KdV equation

$$v_t + v_x + \epsilon v^2 v_x + \epsilon v_{xxx} = 0.$$

(5)

It is obvious that for any initial data $\phi \in H^s(\mathbb{R}), s > \frac{3}{2}$, (4) admits a unique solution $u$ in $C([0, T_\epsilon); H^s(\mathbb{R}))$ where $T_\epsilon = O(1/\epsilon)$. Denoting $v$ the solution of (5) with the same initial data $\phi$, one obtains, proceeding as in the proof of Theorem 1 in [16] which considered the Whitham equation (86):

**Theorem 1.1.** Let $\phi \in H^\infty(\mathbb{R})$. Then, for all $j \in \mathbb{N}$, $j \geq 0$, there exists $M_j = M_j(\|\phi\|_{H^{j+s}}) > 0$ such that

$$\|(u - v)(t)\|_{H^j} \leq M_j \epsilon^2 t,$$

for all $0 \leq t \lesssim \epsilon^{-1}$.

It is well known that the modified Burgers equation undergoes shock formation, even for arbitrary small smooth initial data, provided the square of the initial has a negative slope at some point. In particular no global solutions exist for arbitrary small smooth initial data in Sobolev spaces, such as a small gaussian.

The question we address here is whether this property is still true when adding a weakly dispersive term as in (1). In fact we prove that adding this dispersive term allows the existence and (modified) scattering of small solutions.

Throughout the paper, we will always use the notation $f(t) = e^{-t|D|^\alpha \partial_x} u(t)$ to denote the profile of $u$. By time reversibility we need only to consider the existence for positive time.

Our main result can be stated precisely as follows:

**Theorem 1.2.** Let $\alpha \in (-1, 0)$ and define the $Z$-norm

$$\|g\|_Z = \|(1 + |\xi|)^{10} \hat{g}(\xi)\|_{L^\infty_\xi}.$$
Assume that \( N_0 = 100, \ p_0 \in (0, \frac{1}{100}] \cap (0, -\frac{\alpha}{100}] \) are fixed, and \( u_0 \in H^{N_0}(\mathbb{R}) \) satisfies
\[
\|u_0\|_{H^{N_0}} + \|u_0\|_{H^{1,1}} + \|u_0\|_Z = \varepsilon_0 \leq \varepsilon,
\]
for some constant \( \varepsilon \) sufficiently small (depending only on \( \alpha \) and \( p_0 \)). Then the Cauchy problem (1)-(2) admits a unique global solution \( u \in C(\mathbb{R} : H^{N_0}(\mathbb{R})) \) satisfying the following uniform bounds for \( t \geq 1 \)
\[
t^{-p_0}\|u\|_{H^{N_0}} + t^{-p_0}\|f\|_{H^{1,1}} + \|f\|_Z \lesssim \varepsilon_0.
\]
Moreover, there exists \( w_\infty \in L^\infty(\mathbb{R}) \) such that for \( t \geq 1 \)
\[
t^{p_0}\left\| \exp \left( \frac{-i\xi|\xi|^{1-\alpha}}{\alpha(\alpha + 1)} \int_1^t \|\hat{f}(\xi, s)\|^2 \frac{ds}{s} \right) (1 + |\xi|)^{10} \hat{f}(\xi) - w_\infty(\xi) \right\|_{L_\xi^\infty} \lesssim \varepsilon_0.
\]

The local well-posedness on the time interval \([0, 1]\) for (1)-(2) is standard provided \( \|u_0\|_{H^2} \) is small enough, in particular under the smallness assumption (7). Then the existence and uniqueness of global solutions may be constructed by a bootstrap argument which allows to extend the local solutions. More precisely, assume that the following X-norm is a priori small:
\[
\|u\|_X = \sup_{t \geq 1} \left( t^{-p_0}\|u\|_{H^{N_0}} + t^{-p_0}\|f\|_{H^{1,1}} + \|f\|_Z \right) \leq \varepsilon_1
\]
with \( \varepsilon_1 = \varepsilon_0^{1/3} \), we then aim to show that the above a priori assumption may be improved to
\[
\|u\|_X \leq C(\varepsilon_0 + \varepsilon_1^3),
\]
for some absolute constant \( C > 1 \).

The literature dealing with the problem of global existence and scattering of small solutions to nonlinear dispersive equations is vast and we refer to \([6, 7, 8, 13, 15]\) and to the references therein.

The present work is close to \([4, 14, 15]\) in methodology. In comparison with \([15]\), the presence of a derivative on the nonlinearity in the equation (1) plays a crucial role in our proof in the sense that on the one hand it eliminates part of resonances in low frequencies and on the other hand it avoids using \( \partial_x \hat{f}_1 \) in \( L^\infty \)-norm in some frequency regimes, which allows us to extend the estimates to all \( \alpha \in (-1, 0) \).

To end this section, we list the notations frequently used throughout the paper. We denote by \( \mathcal{F}(g) \) or \( \hat{g} \) the Fourier transform of a Schwartz function \( g \) whose formula is given by
\[
\mathcal{F}(g)(\xi) = \hat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x)e^{-ix\xi} \, dx
\]
with inverse
\[
\mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi)e^{ix\xi} \, d\xi,
\]
and by \( m(\partial_x) \) the Fourier multiplier with symbol \( m \) via the relation
\[
\mathcal{F}(m(\partial_x)g)(\xi) = m(i\xi)\hat{g}(\xi).
\]

Let \( \varphi \) be a smooth function satisfying
\[
\varphi(\xi) = \begin{cases} 1, & |\xi| \leq 1 \\ 0, & |\xi| > 2. \end{cases}
\]
Set
\[ \psi(\xi) = \varphi(\xi) - \varphi(2\xi), \quad \psi_j(\xi) = \varphi(2^{-j}\xi), \quad \varphi_j(\xi) = \varphi(2^{-j}\xi), \]
we then may define the Littlewood-Paley projections \( P_j, P_{\leq j}, P_{> j} \) via
\[ \widehat{P_j} g(\xi) = \psi_j(\xi) \widehat{g}(\xi), \quad \widehat{P_{\leq j}} g(\xi) = \varphi_j(\xi) \widehat{g}(\xi), \quad P_{> j} = 1 - P_{\leq j}, \]
and also \( P_{= j}, P_{\leq j}, P_{< j} \) by
\[ P_{= j} = \sum_{2^j \leq 2^k} P_k, \quad P_{\leq j} = \sum_{2^k \leq 2^{j+c}} P_k, \quad P_{< j} = \sum_{2^k < 2^j} P_k, \]
and the obvious notation for \( P_{[a,b]} \). We will also denote \( g_j = P_j g, g_{\leq j} = P_{\leq j} g \), and so on, for convenience.

The notation \( C \) always denotes a nonnegative universal constant which may be different from line to line but is independent of the parameters involved. Otherwise, we will specify it by the notation \( C(a,h,\ldots) \). We write \( g \lesssim h \) when \( g \leq Ch \) (\( g \gtrsim h \) when \( g \geq h \)) and \( g \sim h \) when \( g \lesssim h \lesssim g \). We also write \( \sqrt{1 + x^2} = (x) \), \( \|g\|_{H^{1,1}} = \|(x)g\|_{H^{1,1}} \), and \( P_{[k-2,k+2]} := P_k^2 \) for simplicity.

2. Decay estimates. This section is devoted to presenting some decay estimates of the solution of the equation (1). We first recall a dispersive linear estimate from [4, Lemma 2.3]:

**Lemma 2.1.** For any \( t \geq 1 \), the following linear dispersive estimates hold:
\[ \left\| e^{t[D]^{\alpha}} P_k g \right\|_{L^\infty} \lesssim t^{-\frac{1}{2}} 2^{\frac{1-\alpha k}{2}} \|g\|_{L^\infty} + t^{-\frac{3}{4}} 2^{-\frac{1+3\alpha}{4} k} (\|\widehat{g}\|_{L^2} + 2^k \|\partial_x \widehat{g}\|_{L^2}), \]
and
\[ \left\| e^{t[D]^{\alpha}} P_k g \right\|_{L^\infty} \lesssim t^{-\frac{3}{4}} 2^{\frac{1-\alpha k}{2}} \|g\|_{L^1}, \]
for \( \alpha \in (-1,1) \setminus \{0\} \).

We next show the following decay estimates for the solution:

**Lemma 2.2.** Let \( \alpha \in (-1,0) \) and \( u \) be the solution to (1). Assume that
\[ t^{-p_0} \|u\|_{H^{\alpha_0}} + (1 + t)^{-p_0} \|f\|_{H^{1,1}} + \|f\|_Z \leq 1, \]
for any \( t \geq 1 \), then we have
\[ \|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty} \lesssim t^{-1/2}, \]
for any \( t \geq 1 \).

**Proof.** Case \( \alpha \in (-1, -\frac{1}{2}) \). Observe \( \frac{1-\alpha}{2} \in [\frac{2}{3}, 1) \) and \( -\frac{1+3\alpha}{4} \in [0, \frac{1}{2}) \). To prove that \( \|u\|_{L^\infty} \) satisfy (14), we first show
\[ \sup_{k \in \mathbb{Z}} (2^{2k} \|P_k u\|_{L^\infty}) \lesssim t^{-\frac{1}{2}}. \]

In the frequency regime \( 2^k \geq t^{(1-4p_0)/10} \), we use (13) and (88) to deduce that
\[ 2^{2k} \|P_k u\|_{L^\infty} \lesssim t^{-\frac{1}{2}} 2^{2k} 2^{\frac{1-\alpha k}{2}} \|P^f_k f\|_{L^1} \]
\[ \lesssim t^{-\frac{1}{2}} 2^{2k} 2^{\frac{1-\alpha k}{2}} 2^{-\frac{1}{2}} \|P^f_k f\|_{L^2}^{1/2} (\|P^f_k f\|_{L^2} + 2^k \|\partial_x P^f_k f\|_{L^2})^{1/2} \]
\[ \lesssim t^{-\frac{1}{2}} 2^{2k} 2^{\frac{1-\alpha k}{2}} 2^{-\frac{1}{2}} 2^{-\frac{1}{2}} \|P^f_k f\|_{H^{\alpha_0}}^{1/2} (\|P^f_k f\|_{L^2} + 2^k \|\partial_x P^f_k f\|_{L^2})^{1/2} \]
\[ \lesssim t^{-\frac{1}{2}} 2^{1-4p_0 + \frac{1-\alpha}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2}} P_0, \]
which gives a stronger bound than what we need. We then consider the frequency regime \(2^k \leq t^{(1-4\rho_0)/10}\). It follows from (12) that
\[
\|P_k u\|_{L^\infty} \lesssim t^{-\frac{1}{2}} 2^{\frac{1}{2} - \frac{n}{2} k} \|P_k f\|_{L^\infty} + t^{-\frac{3}{2}} 2^{-\frac{1+3\alpha}{4} k} \|P_k^3 f\|_{L^2} + 2^k \|\partial P_k^2 f\|_{L^2}.
\] (17)
Now the desired bound (15) is a consequence of (17) and of the following facts
\[
2^k \left(\frac{1+\alpha}{2} + 2k\right) \lesssim 1, \quad 2^k \left(-\frac{1+3\alpha}{4} + 2k\right) \lesssim t^{\frac{1}{2} - \rho_0},
\]
and
\[
(1 + 2^{10k}) \left|\left| P_k^3 f \right|\right|_{L^\infty} \lesssim 1, \quad \left|\left| P_k^2 f \right|\right|_{L^2} + 2^k \left|\left| \partial P_k^2 f \right|\right|_{L^2} \lesssim t^{\rho_0}.
\]
We are ready to show \(\|u\|_{L^\infty} \) satisfy (14). It first follows from (15) that
\[
\sum_{0 \leq k \in \mathbb{Z}} \|P_k u\|_{L^\infty} = \sum_{0 \leq k \in \mathbb{Z}} 2^{-2k} (2^k \|P_k u\|_{L^\infty}) \lesssim t^{-\frac{1}{2}} \sum_{0 \leq k \in \mathbb{Z}} 2^{-2k} \lesssim t^{-\frac{1}{2}}.
\]
Thus it remains to estimate \(\sum_{0 > k \in \mathbb{Z}} \|P_k u\|_{L^\infty}\). When \(2^k \leq t^{-1}\), the desired bound follows from the following inequality after summation in \(k\)
\[
\|P_k u\|_{L^\infty} \lesssim 2^{k/2} \|u\|_{L^2},
\]
where we have used Bernstein inequality. For \(1 \geq 2^k \geq t^{-1}\), it follows from (17) that
\[
\sum_{1 \geq 2^k \geq t^{-1}} \|P_k u\|_{L^\infty} \lesssim t^{-\frac{1}{2}} \sum_{1 \geq 2^k \geq t^{-1}} \frac{2^{\frac{1+\alpha}{2} k}}{1 + 2^{10k}} (1 + 2^{10k}) \left|\left| P_k^3 f \right|\right|_{L^\infty} + t^{-\frac{1}{2}} \rho_0 \sum_{1 \geq 2^k \geq t^{-1}} 2^{-\frac{1+3\alpha}{4} k}
\]
\[
\lesssim t^{-\frac{1}{2}} + t^{-\frac{1}{4}} \rho_0 (1 + \log t) \lesssim t^{-\frac{1}{2}}.
\] (18)
We finally prove the bound for \(\|\partial_x u\|_{L^\infty}\). One has
\[
\|\partial_x u\|_{L^\infty} \leq \sum_{k \in \mathbb{Z}} \|P_k \partial_x u\|_{L^\infty} \lesssim \sum_{k \in \mathbb{Z}} 2^k \|P_k u\|_{L^\infty}
\]
\[
= \sum_{0 \geq k \in \mathbb{Z}} 2^k \|P_k u\|_{L^\infty} + \sum_{0 \leq k \in \mathbb{Z}} 2^{-k} (2^k \|P_k u\|_{L^\infty})
\]
\[
\lesssim \|u\|_{L^\infty} \sum_{0 \geq k \in \mathbb{Z}} 2^k + t^{-\frac{1}{2}} \sum_{0 \leq k \in \mathbb{Z}} 2^{-k} \lesssim t^{-\frac{1}{2}}.
\]
**Case \(\alpha \in (-\frac{1}{3}, 0)\).** Observe that \(\frac{1+\alpha}{2} \in (\frac{1}{2}, \frac{3}{2})\) and \(-\frac{1+3\alpha}{4} \in (-\frac{1}{4}, 0)\). We now split the frequencies into \(2^k \geq t^{(1-4\rho_0)/8}\) and \(2^k \leq t^{(1-4\rho_0)/8}\). In a similar fashion as before, one still has
\[
\sup_{k \in \mathbb{Z}} (2^{2k} \|P_k u\|_{L^\infty}) \lesssim t^{-\frac{1}{2}},
\]
and furthermore shows
\[
\sum_{0 \leq k \in \mathbb{Z}} \|P_k u\|_{L^\infty} \lesssim t^{-\frac{1}{2}}.
\]
The desired bound for $\sum_{2^k \leq t^{-1}} ||P_k u||_{L^\infty}$ may be obtained as before. We now consider the frequency regime $1 \geq 2^k \geq t^{-1}$. It follows from (17) that
\[
\sum_{1 \leq 2^k \leq t^{-1}} ||P_k u||_{L^\infty} \lesssim t^{-\frac{1}{2}} \sum_{1 \leq 2^k \leq t^{-1}} 2^{\frac{k}{2}} \|1 + 2^{10k} P_k^* f\|_{L^\infty} + t^{-\frac{3}{2} + \epsilon} \sum_{1 \leq 2^k \leq t^{-1}} 2^{-\frac{k}{2} - \frac{3}{2}} t^{\frac{k}{2} - \frac{3}{2}} \leq t^{-\frac{1}{2}} + t^{-\frac{3}{2} + \epsilon} t^{\frac{1}{2} + \epsilon} \leq t^{-\frac{1}{2}},
\]
where we have used the assumption $p_0 \leq -\frac{\alpha}{10}$ in the last inequality.

The desired bound for $||\partial_x u||_{L^\infty}$ can be estimated as before. \qed

3. Estimates on $||u||_{H^{\infty}}$ and $||f||_{H^{1,1}}$. In this section we will prove uniform bounds for the energy part in (10). More precisely:

**Theorem 3.1.** Let $u$ be a solution of (1)-(2) satisfying the a priori bounds (10). Then the following estimates hold true:
\[
||u(t, \cdot)||_{H^{\infty}} \leq C\varepsilon_0 (t)^{C\varepsilon^2}, \quad (19)
\]
and
\[
||f(t, \cdot)||_{H^{1,1}} \leq C(\varepsilon_0 + \varepsilon^2) (t)^{C\varepsilon^2}. \quad (20)
\]

**Proof.** Let
\[
\mathcal{L} = \partial_t - |D|^\alpha \partial_x, \quad J = x + (\alpha + 1)t |D|^\alpha, \quad \Lambda = \partial_x^{-1}((\alpha + 1)t \partial_t + x \partial_x + 1).
\]
Since $[\mathcal{L}, \partial_x] = 0$, the solution $u$ of (1)-(2) satisfies
\[
\mathcal{L} \partial_x^k u = -\partial_x^k (u^2 \partial_x u), \quad k = 1, 2, \ldots, N_0.
\]
This gives the energy identity
\[
\frac{1}{2} \frac{d}{dt} \int |\partial_x^k u|^2 \, dx = - \int (\partial_x^k (u^2 \partial_x u) - u^2 \partial_x^{k+1} u) \partial_x^k u \, dx - \int u^2 \partial_x^{k+1} u \partial_x^k u \, dx.
\]
Notice that
\[
||\partial_x^k (u^2 \partial_x u) - u^2 \partial_x^{k+1} u||_{L^2} \lesssim ||\partial_x (u^2)||_{L^\infty} ||\partial_x^k u||_{L^2} + ||\partial_x u||_{L^\infty} ||\partial_x^k (u^2)||_{L^2} \lesssim ||u||_{L^\infty} ||\partial_x u||_{L^\infty} ||\partial_x^k u||_{L^2},
\]
and
\[
- \int u^2 \partial_x^{k+1} u \partial_x^k u \, dx = \frac{1}{2} \int \partial_x (u^2) |\partial_x^k u|^2 \, dx \lesssim ||u||_{L^\infty} ||\partial_x u||_{L^\infty} ||\partial_x^k u||_{L^2}^2.
\]
Hence
\[
\frac{d}{dt} ||\partial_x^k u||_{L^2}^2 \lesssim ||u||_{L^\infty} ||\partial_x u||_{L^\infty} ||\partial_x^k u||_{L^2}^2 \lesssim \varepsilon_0^2 (t)^{-1} ||\partial_x^k u||_{L^2}^2,
\]
where we have used (14), which entails (19).

A small calculation shows
\[
\mathcal{L} \Lambda u = u^2 \partial_x \Lambda u + (\alpha - 2)u^3/3.
\]
It follows that
\[
\frac{1}{2} \frac{d}{dt} ||\Lambda u||_{L^2}^2 = \int u^2 \partial_x \Lambda u \Lambda u \, dx + \frac{\alpha - 2}{3} \int u^3 \Lambda u \, dx \lesssim ||u||_{L^\infty} ||\partial_x u||_{L^\infty} ||\Lambda u||_{L^2}^2 + ||u||_{L^\infty} ||\Lambda u||_{L^2} ||\Lambda u||_{L^2},
\]
which combined with (14) lead to
\[ \|Au\|_{L^2} \lesssim \varepsilon_0(t) C\varepsilon_t^2. \] (21)
Notice that
\[ J u = \Lambda u + (\alpha + 1)tu^3/3, \] (22)
then we employ (14), (21) and (22) to deduce
\[ \|xf\|_{L^2} = \|Ju\|_{L^2} \lesssim \varepsilon_0(t) C\varepsilon_t^2 + t\|u\|_{L^\infty}^2 \lesssim (\varepsilon_0 + \varepsilon_1^2)(t) C\varepsilon_t^2. \] (23)
Using \([\mathcal{L}, \partial_x] = 0\) and \([\Lambda, \partial_x] = -\text{id}\), one calculates that
\[ \mathcal{L}\partial_x u = u^2\partial_x\Lambda\partial_x u + 2u\partial_x u\Lambda\partial_x u + (\alpha + 2)u^2\partial_x u. \]
This results into the estimate
\[ \frac{d}{dt}\|\Lambda\partial_x u\|_{L^2} \lesssim \|u\|_{L^\infty}\|\Lambda\partial_x u\|_{L^2} \lesssim (\varepsilon_0 + \varepsilon_1^2)(t) C\varepsilon_t^2. \] (24)
We use \([\mathcal{J}, \partial_x] = -\text{id}\) to calculate
\[ \mathcal{J}\partial_x u = -\Lambda u - (\alpha + 1)tu^3/3 + (\Lambda\partial_x + 1)u + (\alpha + 1)tu^2\partial_x u. \] (25)
Therefore the estimates (14), (19), (21) and (24), together with (25) yield
\[ \|xf\partial_x f\|_{L^2} = \|J\partial_x u\|_{L^2} \lesssim (\varepsilon_0 + \varepsilon_1^2)(t) C\varepsilon_t^2. \] (26)
The estimate (20) is a consequence of (23) and (26). 

4. Estimates on \(\|f\|_{L^2}\). Let
\[ \Phi(\xi, \eta, \sigma) = |\xi|^{\alpha}\xi - |\xi - \eta - \sigma|^{\alpha}(\xi - \eta - \sigma) - |\eta|^{\alpha}\eta - |\sigma|^{\alpha}\sigma. \]
Taking the Fourier transform of (1) gives
\[ \partial_t \hat{f}(\xi, t) = -i(6\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^2} e^{-i\Phi(\xi, \eta, \sigma)} \hat{f}(\xi - \eta - \sigma, t) \hat{f}(\eta, t) d\eta d\sigma \]
\[ = -i(6\pi)^{-\frac{1}{2}} I(\xi, t). \] (27)
Set
\[ H(\xi, t) := \frac{-\xi|\xi|^{1-\alpha}}{\alpha(\alpha + 1)} \int_1^t |\hat{f}(\xi, s)|^2 ds, \]
and
\[ g(\xi, t) := e^{iH(\xi, t)} \hat{f}(\xi, t). \]
Then \(g\) satisfies the following evolutionary equation
\[ \partial_t g(\xi, t) = -i(6\pi)^{-\frac{1}{2}} e^{iH(\xi, t)} (I(\xi, t) - 3\hat{\sigma}t^{-\frac{1}{2}}|\xi|^{1-\alpha}|\hat{f}(\xi, t)|^2 \hat{f}(\xi, t)), \] (28)
where \(\hat{\sigma} := -2\pi/\alpha(\alpha + 1)\).

This section is aimed to show the following theorem:

**Theorem 4.1.** It holds that
\[ t_1^\alpha \| (1 + |\xi|)^{10}(g(\xi, t_2) - g(\xi, t_1)) \|_{L_\xi^\infty} \lesssim c_0, \] (29)
for any \(t_1 \leq t_2 \in [1, T]\).
The $Z$-norm part of (11) is an immediate consequence of the estimate (29) which also entails (9).

4.1. **Proof of Theorem 4.1.** We decompose in frequencies

$$I(\xi, t) = \sum_{k_1,k_2,k_3 \in \mathbb{Z}} I_{k_1,k_2,k_3}(\xi, t),$$

in which

$$I_{k_1,k_2,k_3}(\xi, t) := \xi \int_{\mathbb{R}^2} e^{-iH(\xi, \eta, \sigma)} \hat{f}_{k_1}(\xi - \eta, \sigma) \hat{f}_{k_2}(\eta) \hat{f}_{k_3}(\sigma) \, d\eta d\sigma.$$  \hspace{1cm} (31)

For (29), it suffices to show that if $t_1 \leq t_2 \in [2^m - 1, 2^{m+1}] \cap [1, T]$, for some $m \in \{1, 2, \ldots \}$, then

$$\|(1 + |\xi|^3)(g(\xi, t_2) - g(\xi, t_1))\|_{L^p_{\xi}} \lesssim \epsilon_0 2^{-p \alpha m}. \hspace{1cm} (32)$$

Let $k \in \mathbb{Z}$ and $|\xi| \in [2^k, 2^{k+1}]$ and $s \in [2^m - 1, 2^{m+1}] \cap [1, T]$. Using the interpolation (88), similar to (16), it is easy to see that

$$\|(1 + |\xi|^3) f_k(\xi)\| \lesssim \epsilon_0 2^{-p_0 m}, \hspace{1cm} \text{for } k \in [p_0 m, \infty) \cap \mathbb{Z},$$

which entails (32) in this frequency regime.

By the assumptions (10), for $s \in [2^m - 1, 2^{m+1}] \cap [1, T]$ and any $l \in \mathbb{Z}$, we have

$$\|\hat{f}(s)|\|_{L^2} \lesssim \epsilon_1 2^{p_0 m} 2^{-N_0 l},$$

$$\|\partial^l \hat{f}(s)|\|_{L^2} \lesssim \epsilon_1 2^{p_0 m} 2^{-l},$$

$$\|\hat{f}(s)|\|_{L^\infty} \lesssim \epsilon_1 2^{-10 l},$$

$$\|e^{s D^l} \partial_x \hat{f}(s)|\|_{L^\infty} \lesssim \epsilon_1 2^{-m/2}, \hspace{1cm} \text{(34)}$$

where $l_+ = \max(l, 0)$.

It remains to consider (32) for $|\xi| \in [2^k, 2^{k+1}]$ with $k \in (-\infty, p_0 m]$. By the equation (28) and the decomposition (30)-(31), it suffices to show that

$$\sum_{k_1,k_2,k_3 \in \mathbb{Z}} \int_{t_1}^{t_2} e^{iH(\xi, s)} \left(I_{k_1,k_2,k_3}(\xi, s) - \tilde{c} s^{-1} |\xi|^{1-\alpha} \hat{f}_{k_1}(\xi, s) \hat{f}_{k_2}(-\xi, s) \hat{f}_{k_3}(-\xi, s)ight. \quad \text{d}s$$

$$\left. - \tilde{c} s^{-1} |\xi|^{1-\alpha} \hat{f}_{k_1}(\xi, s) \hat{f}_{k_2}(-\xi, s) \hat{f}_{k_3}(\xi, s)ight) \lesssim \epsilon_1 2^{-p_0 m} 2^{-10 k_+}, \hspace{1cm} \text{(35)}$$

for $t_1 \leq t_2 \in [2^m - 1, 2^{m+1}] \cap [1, T]$.

Using (34), it is straightforward to show

$$|I_{k_1,k_2,k_3}(\xi, s)| \lesssim \epsilon_1^3 2^{3p_0 m} 2^{k_0 \min(k_1,k_2,k_3)/2} 2^{-N_0 (k_1^+ + k_2^+ + k_3^+)}, \hspace{1cm} \text{(36)}$$

and

$$|I_{k_1,k_2,k_3}(\xi, s)| \lesssim \epsilon_1^3 2^{2k_0 \min(k_1,k_2,k_3)} 2^{k_0 \med(k_1,k_2,k_3)} 2^{-10 \max(k_1^+, k_2^+, k_3^+)}. \hspace{1cm} \text{(37)}$$
and
\[ s^{-1} |\xi|^{2-a} \left( |\hat{f}_{k_1}(\xi, s)\hat{f}_{k_2}(\xi, s)\hat{f}_{k_3}(-\xi, s)| + |\hat{f}_{k_1}(-\xi, s)\hat{f}_{k_2}(\xi, s)\hat{f}_{k_3}(\xi, s)| \right) \]
\[ \lesssim \epsilon_{1}^{2-m} 2^{-(2-a)k_2-30k_1} 1_{[0,t_0]}(\max(|k_1-k|, |k_2-k|, |k_3-k|)), \]
where \(1_A(x)\) is a characteristic function which equals 1 for \(x \in A\), and equals 0 otherwise.

With the bounds (36)-(38) at hand, one easily verifies (35) if one of the following conditions holds
\[ \min(k_1, k_2, k_3) \leq -4m, \]
\[ \max(k_1, k_2, k_3) \geq \frac{p_0m}{10}, \]
\[ \min(k_1, k_2, k_3) + \med(k_1, k_2, k_3) \leq -(1+10p_0)m. \]

Therefore, to complete the proof (35), we are reduced to show

**Theorem 4.2.** Assume that \(k, k_1, k_2, k_3 \in \mathbb{Z}, \ m \in \mathbb{Z} \cap [100, \infty), \ |\xi| \in [2^k, 2^{k+1}], \) and \(t_1 \leq t_2 \in [2^m-1, 2^{m+1}] \cap [1, T].\) If
\[ k \in (-\infty, p_0m], \]
\[ k_1, k_2, k_3 \in [-4m, p_0m/10], \]
\[ \min(k_1, k_2, k_3) + \med(k_1, k_2, k_3) \geq -(1+10p_0)m, \]
then
\[ \left| \int_{t_1}^{t_2} e^{iH(\xi, s)} \left( I_{k_1, k_2, k_3}(\xi, s) - \hat{c} s^{-1} |\xi|^{1-a} \hat{f}_{k_1}(\xi, s)\hat{f}_{k_2}(\xi, s)\hat{f}_{k_3}(-\xi, s) \right) ds \right| \lesssim \epsilon_{1}^{2-m} 2^{-10k_1}. \]

**4.2. Proof of Theorem 4.2.** We divide the proof into several propositions.

**Proposition 1.** Let \(k, k_1, k_2, k_3 \in \mathbb{Z} \) and \(k_1, k_2, k_3 \in [-20, k+20],\) then the bound (40) holds.

It is easy to check (40) if \(k \leq -m/2.\) Hence we will assume \(k \geq -m/2\) in the following. Without loss of generality, we may assume that \(\xi > 0\) and \(\xi \in [2^k, 2^{k+1}].\) We split the integral \(I_{k_1, k_2, k_3}\) as follows:
\[ I_{k_1, k_2, k_3}(\xi, s) = \sum_{i_1, i_2, i_3 \in \{+, -\}} I_{i_1, i_2, i_3}^{k_1, k_2, k_3}(\xi, s), \]
with
\[ I_{i_1, i_2, i_3}^{k_1, k_2, k_3}(\xi, s) = \xi \int_{\mathbb{R}^2} e^{-i\Phi(\xi, \eta, \sigma)} \hat{f}_{i_1}^{k_1}(\xi - \eta - \sigma, s) \hat{f}_{i_2}^{k_2}(\eta, s) \hat{f}_{i_3}^{k_3}(\sigma, s) d\eta d\sigma, \]

where \(\hat{f}_{i}^{k}(\mu) := \hat{f}(\mu)1_{i}(\mu), 1_{+} := 1_{[0, \infty)}, 1_{-} := 1_{(-\infty, 0)}\). We first observe that
\[ I_{-1, -1, -1}^{k_1, k_2, k_3}(\xi, s) = 0, \]
so that to finish the proof of (40) under the assumption of Proposition 1, we are then left to show the following two lemmas:
Lemma 4.3. It holds that
\[
|I_{k_1,k_2,k_3}^{+,+,+}(\xi,s) - \tilde{c} s^{-1} \xi|^{1-\alpha} \hat{f}_{k_1}(\xi,s) \hat{f}_{k_2}(\xi,s) \hat{f}_{k_3}(-\xi,s)|
\]
\[
+ |I_{k_1,k_2,k_3}^{+,-,+}(\xi,s) - \tilde{c} s^{-1} \xi|^{1-\alpha} \hat{f}_{k_1}(\xi,s) \hat{f}_{k_2}(-\xi,s) \hat{f}_{k_3}(\xi,s)|
\]
\[
+ |I_{k_1,k_2,k_3}^{-,-,+}(\xi,s) - \tilde{c} s^{-1} \xi|^{1-\alpha} \hat{f}_{k_1}(-\xi,s) \hat{f}_{k_2}(\xi,s) \hat{f}_{k_3}(\xi,s)|
\]  
(41)
\[
\lesssim \varepsilon_1^3 2^{-m} 2^{-3p_0m} 2^{-10k_+}.
\]

Lemma 4.4. We have
\[
\left| \int_{t_1}^{t_2} e^{iH(\xi,s)} I_{k_1,k_2,k_3}^{t_1,t_2,t_3}(\xi,s) \, ds \right| \lesssim \varepsilon_1^3 2^{-2p_0m} 2^{-10k_+},
\]  
(42)
for \((t_1,t_2,t_3) \in \{ (+,+,+), (+,-,-), (-,+,+), (-,-,-) \} \).

We start by showing (41).

Proof of Lemma 4.3. We only prove that the third term in LHS of (41) is bounded by \(\varepsilon_1^3 2^{-3p_0m} 2^{-10k_+} \), and the other two terms may be handled similarly.

Let \(\bar{l} \) be the smallest integer with the property that
\[
2^{\bar{l}} \geq 2^{(1-\alpha)k/2} - 2^{49(m/100)}.
\]
Since \(k \geq -m/2\), one has \(\bar{l} \leq k - 20\). We may decompose
\[
I_{k_1,k_2,k_3}^{-,+,+}(\xi,s) = \sum_{l_1,l_2=\bar{l}}^{k+20} J_{l_1,l_2}(\xi,s),
\]
with
\[
J_{l_1,l_2}(\xi,s) = \xi \int_{\mathbb{R}^2} e^{-i\varphi_\xi(\xi,\eta,\sigma)} \hat{f}_{k_1}(\xi-\eta-\sigma,s) \hat{f}_{k_2}(\eta,s) \hat{f}_{k_3}(\sigma,s)
\]
\[
\times \varphi^{(l_1)}_\xi(\xi-\eta) \varphi^{(l_2)}_\eta(\xi-\sigma) \, d\eta d\sigma,
\]
for any \(l_1, l_2 \geq \bar{l}\), and where
\[
\varphi^{(l)}_\xi(x) := \varphi(x/2^l), \quad \text{if} \quad k = l,
\]
and
\[
\varphi^{(l)}_\xi(x) := \varphi(x/2^k) - \varphi(x/2^{k-1}), \quad \text{if} \quad k \geq l + 1.
\]

Step 1. \(l_2 \geq \max(l_1, \bar{l} + 1)\) or \(l_1 \geq \max(l_2, \bar{l} + 1)\). We only consider the case \(l_2 \geq \max(l_1, \bar{l} + 1)\), a similar argument may apply to the other case.

We will show
\[
|J_{l_1,l_2}(\xi,s)| \lesssim \varepsilon_1^3 2^{-m} 2^{-3p_0m} 2^{-10k_+}.
\]
On the support of the integral, one has \(|\xi-\eta-\sigma| \approx |\eta| \approx 2^k\) and \(|\xi-\sigma| \approx 2^{l_2}\), and then one finds
\[
|\partial_\xi \Phi(\xi,\eta,\sigma)| = (\alpha + 1) |\xi-\eta-\sigma|^\alpha - |\eta|^\alpha \gtrsim 2^k 2^{(\alpha-1)k}.
\]  
(43)
Using the identity
\[
e^{-i\varphi_\xi(\xi,\eta,\sigma)} = \frac{\partial_\xi e^{-i\varphi_\xi(\xi,\eta,\sigma)}}{-i\partial_\eta \Phi(\xi,\eta,\sigma)},
\]
we use integration by parts in \( \eta \) to obtain

\[
|J_{1,1,2}(\xi,s)| \leq |J_{1,1,2,1}(\xi,s)| + |F_{1,1,2,1}(\xi,s)| + |G_{1,1,2,1}(\xi,s)|,
\]

where

\[
J_{1,1,2,1}(\xi,s) = \xi \int_{\mathbb{R}^2} \partial_\eta m_1(\eta,\sigma)e^{-i\Phi(\xi,\eta,\sigma)}\frac{\hat{f}_{\xi_1}}{\hat{G}_\xi}(\xi - \eta - \sigma,s)\hat{f}_{\xi_2}(\eta,\sigma)\hat{f}_{\xi_3}(\sigma,s)\,d\eta d\sigma,
\]

\[
F_{1,1,2,1}(\xi,s) = \xi \int_{\mathbb{R}^2} m_1(\eta,\sigma)e^{-i\Phi(\xi,\eta,\sigma)}\partial_\eta \hat{f}_{\xi_1}(\xi - \eta - \sigma,s)\hat{f}_{\xi_2}(\eta,\sigma)\hat{f}_{\xi_3}(\sigma,s)\,d\eta d\sigma,
\]

\[
G_{1,1,2,1}(\xi,s) = \xi \int_{\mathbb{R}^2} m_1(\eta,\sigma)e^{-i\Phi(\xi,\eta,\sigma)}\hat{f}_{\xi_1}(\xi - \eta - \sigma,s)\partial_\eta \hat{f}_{\xi_2}(\eta,\sigma)\hat{f}_{\xi_3}(\sigma,s)\,d\eta d\sigma,
\]

with

\[
m_1(\eta,\sigma) := \frac{\varphi^{(i)}_{\xi_1}(\xi - \eta)\varphi^{(i)}_{\xi_2}(\xi - \sigma)}{s\partial_\eta \Phi(\xi,\eta,\sigma)}\varphi'_{\xi_1}(\xi - \eta - \sigma)\varphi'_{\xi_2}(\eta)\varphi'_{\xi_3}(\sigma).
\] (44)

Following (43) and (44), a straightforward calculation shows that

\[
|\partial_\eta^{a+b} m_1(\eta,\sigma)| \lesssim (2^{-m}2^{-l_2}2^{(1-\alpha)k})(2^{-a}2^{-b}2^{l_2})
\times 1_{[0,2^{l_2}]}(|\xi - \eta|)1_{[2^{l_2-4},2^{l_2+4}]}(|\xi - \sigma|),
\]

for \( a, b \in [0, 20] \cap \mathbb{Z} \). Hence

\[
\|\mathcal{F}^{-1}(m_1)\|_{L^1} \lesssim 2^{-m}2^{-l_2}2^{(1-\alpha)k}.
\]

We first estimate the term \( F_{1,1,1,1} \). Fix \( \xi \) and \( s \), let

\[
\hat{f}(\theta) := e^{i\xi|\xi - \theta|}\partial_\eta \hat{f}_{\xi_1}(\xi - \theta,s),
\]

\[
\hat{g}(\eta) := e^{i\xi|\eta|}\hat{f}_{\xi_2}(\eta,s),
\]

\[
\hat{h}(\sigma) := e^{i\sigma|\sigma|}\hat{f}_{\xi_3}(\sigma,s),
\]

which in light of (34) gives

\[
\|f\|_{L^2} \lesssim \epsilon_1 2^{-k}2^{p_0 m},
\]

\[
\|g\|_{L^\infty} \lesssim \epsilon_1 2^{-m/2},
\]

\[
\|h\|_{L^2} \lesssim \epsilon_1 2^{p_0 m}2^{-N_0 k_+}.
\]

It then follows from Lemma 6.2 that

\[
|F_{1,1,1,1}(\xi,s)| \lesssim \epsilon_1 2^k\|\mathcal{F}^{-1}(m_1)\|_{L^1}\|f\|_{L^2}\|g\|_{L^\infty}\|h\|_{L^2}
\lesssim \epsilon_1 2^{-3m/2+2p_0 m}2^{-l_2}2^{(1-\alpha)k}2^{-N_0 k_+}
\lesssim \epsilon_1 3^2 2^{-3m/2+2p_0 m+49/100}\gamma_0(1-\alpha)k/22^{-N_0 k_+}
\lesssim \epsilon_1 3^2 2^{-m/2}2^{200 m}/2^{-10k_+},
\]

which is stronger than what we need. A similar argument yields

\[
|G_{1,1,1,1}(\xi,s)| \lesssim \epsilon_1 3^2 2^{-m/2}2^{200 m}/2^{-10k_+}.
\]

To estimate the term \( J_{1,1,1,1} \), we integrate by parts in \( \eta \) again to deduce

\[
|J_{1,1,1,1}(\xi,s)| \leq |J_{1,1,2,1}(\xi,s)| + |F_{1,1,2,1}(\xi,s)| + |G_{1,1,2,1}(\xi,s)|,
\]
in which
\[ J_{1,l_2}(\xi, s) = \xi \int_{\mathbb{R}^2} \partial_\eta m_2(\eta, \sigma) e^{-i\Phi(\xi, \eta, \sigma)} \widehat{f_{k_1}}(\xi - \eta - \sigma, s) \widehat{f_{k_2}}(\eta, s) \widehat{f_{k_3}}(\sigma, s) \, d\eta d\sigma, \]
\[ F_{1,l_2}(\xi, s) = \xi \int_{\mathbb{R}^2} m_2(\eta, \sigma) e^{-i\Phi(\xi, \eta, \sigma)} \partial_\eta \widehat{f_{k_1}}(\xi - \eta - \sigma, s) \widehat{f_{k_2}}(\eta, s) \widehat{f_{k_3}}(\sigma, s) \, d\eta d\sigma, \]
\[ G_{1,l_2}(\xi, s) = \xi \int_{\mathbb{R}^2} m_2(\eta, \sigma) e^{-i\Phi(\xi, \eta, \sigma)} \partial_\eta \widehat{f_{k_1}}(\xi - \eta - \sigma, s) \widehat{f_{k_2}}(\eta, s) \widehat{f_{k_3}}(\sigma, s) \, d\eta d\sigma, \]
with
\[ m_2(\eta, \sigma) := \frac{\partial_\eta m_1(\eta, \sigma)}{s \partial_\eta \Phi(\xi, \eta, \sigma)}. \] (45)

It follows from (43) and (45) that \( m_2 \) satisfies the following stronger estimate
\[
|\partial_\eta \partial_\sigma m_2(\eta, \sigma)| \lesssim \left(2^{-m_2-1}2^{-1}\alpha k(2^{-m_2-1}2^{-1}k(2^{-m_2-1}2^{-1}k)^2 a l_1 l_2\right) \times 1_{[0,2^{l+1}]}(\xi - \eta) 1_{[2^{l-4},2^{l+4}]}(|\xi - \sigma|),
\]
for \( a, b \in [0, 10] \cap \mathbb{Z} \). In a similar fashion as \( F_{1,l_1}(\xi, s) \) and \( G_{1,l_1}(\xi, s) \), we employ Lemma 6.2 to obtain
\[
|F_{1,l_1,1}(\xi, s)| + |G_{1,l_1,1}(\xi, s)| \lesssim \xi^{-3} 2^{-m_2-2p_0} 2^{-10k\xi},
\]
We finally estimate the left term \( J_{1,l_1,2}(\xi, s) \) as follows:
\[
|J_{1,l_1,2}(\xi, s)| \lesssim 2^{l_1} 2^{l_2} \| \partial_\eta m_2 \|_{L^\infty} \| \widehat{f_{k_1}} \|_{L^\infty} \| \widehat{f_{k_2}} \|_{L^\infty} \| \widehat{f_{k_3}} \|_{L^\infty} \lesssim \xi^{-3} 2^{-m_2-1} 2^{-1} 2^{-\alpha k} 2^{-m_2} 2^{-2-30k\xi} \lesssim \xi^{-3} 2^{-m_2-2p_0} 2^{-10k\xi}.
\]

**Step 2.** \( l_1 = l_2 = \tilde{l} \). In this case, it suffices to prove that
\[
|J_{1,l}(\xi, s) - \tilde{c} s^{-1} \xi^{-2} \alpha f_{k_1}(\xi - \eta, s) f_{k_2}(\eta, s) f_{k_3}(\sigma, s)| \lesssim \xi^{-3} 2^{-m_2-2p_0} 2^{-10k\xi}. \] (46)
Define
\[
\tilde{J}_{1,l}(\xi, s) = \xi \int_{\mathbb{R}^2} e^{-i\alpha(\alpha+1)(\xi - \eta)\xi^{-\alpha-1} \xi^{-1-\alpha} f_{k_1}(\xi - \eta - \sigma, s) f_{k_2}(\eta, s) f_{k_3}(\sigma, s) \times \varphi(2^{-l}(\xi - \eta)) \varphi(2^{-l}(\xi - \sigma)) \, d\eta d\sigma,
\]
and observe
\[
\Phi(\xi, \eta, \sigma) = \alpha(\alpha+1)(\xi - \eta)\xi^{-\alpha-1} \xi^{-1-\alpha} f_{k_1}(\xi - \eta - \sigma, s) f_{k_2}(\eta, s) f_{k_3}(\sigma, s) \lesssim 2^{a k} 2^{5/2} \xi^{-30k\xi} \lesssim 2^{a k} 2^{-m_2/5} 2^{-10k\xi}.
\]
Notice that
\[
\| \tilde{f}_{k_1}(\xi + r, s) - \tilde{f}_{k_1}(\xi, s) \| \lesssim 2^{l/2} 2^{-k} 2^{p_0 m}, \quad \text{for } |r| \leq 2^{\tilde{l}},
\]
we thus obtain
\[
|\tilde{f}_{k_1}(\xi - \eta - \sigma, s) f_{k_2}(\eta, s) f_{k_3}(\sigma, s) - \tilde{f}_{k_1}(\xi, s) f_{k_2}(\eta, s) f_{k_3}(\xi, s)| \lesssim \xi^{-3} 2^{l/2} 2^{p_0 m} 2^{-k} 2^{-20k\xi}, \quad \text{for } |\xi - \eta| + |\xi - \sigma| \leq 2^{l+4}.
\]
It then follows that
\[
\left| \tilde{f}_{k_l}(\xi, s) - \xi \int_{\mathbb{R}^2} e^{-i\alpha(\alpha+1)(\xi-\eta)(\xi-\sigma)/\xi^1 - \alpha} \tilde{f}_{k_1}(-\xi, s) \tilde{f}_{k_2}(\xi, s) \tilde{f}_{k_3}(\xi, s) \times \varphi(2^{-l}(\xi - \eta)) \varphi(2^{-l}(\xi - \sigma)) \, dyd\sigma \right| \leq e^{3 \gamma_{l/2}^2 \gamma_{m}^2 2^{2l_2} 2^{-10k_+}} \lesssim e^{3 \gamma_{l/2} - 6m/52 - 10k_+}.
\]

One calculates
\[
\int e^{-ixy} e^{-x^2/N^2} e^{-y^2/N^2} \, dx dy = \frac{2\pi N}{\sqrt{4N^2 - 1}} = 2\pi + \mathcal{O}(N^{-1}),
\]
in which we have used the formula
\[
\int e^{-ax^2 - bx} \, dx = e^{b^2/(4a)} \sqrt{\pi/a}, \quad \text{for } a, b \in \mathbb{C}, \Re a > 0.
\]

It follows that
\[
\int e^{-ixy} \varphi(x/N) \varphi(y/N) \, dx dy = 2\pi + \mathcal{O}(N^{-1/2}), \quad \text{for } N \geq 1.
\]

We then may estimate
\[
\left| \int e^{-i\alpha^2(\alpha+1)(\xi-\eta)(\xi-\sigma)/\xi^1 - \alpha} \varphi(2^{-l}(\xi - \eta)) \varphi(2^{-l}(\xi - \sigma)) \, dyd\sigma + \frac{2\pi\xi^{1-\alpha}}{\alpha(\alpha+1)} \right| \leq 2^{(1-\alpha)k_2^2} 2^{-m} \left(2^{m/100}\right)^{-1/2} \lesssim 2^{-m/2 - m/300}.
\]

Hence
\[
\left| \int_{\mathbb{R}^2} e^{-i\alpha^2(\alpha+1)(\xi-\eta)(\xi-\sigma)/\xi^1 - \alpha} \tilde{f}_{k_1}(-\xi, s) \tilde{f}_{k_2}(\xi, s) \tilde{f}_{k_3}(\xi, s) \varphi(2^{-l}(\xi - \eta)) \times \varphi(2^{-l}(\xi - \sigma)) \, dyd\sigma + \frac{2\pi\xi^{2-\alpha}}{\alpha(\alpha+1)} \tilde{f}_{k_1}(-\xi, s) \tilde{f}_{k_2}(\xi, s) \tilde{f}_{k_3}(\xi, s) \varphi(2^{-l}(\xi - \eta)) \right| \leq e^{3 \gamma_{l/2} - 2 - m/300} 2^{-2 m/300} 2^{-10k_+} \lesssim e^{3 \gamma_{l/2} - 2 - m/300} 2^{-2 m/300} 2^{-10k_+}.
\]

We finally conclude (46) from (47)-(49). \hfill \square

**Proof of Lemma 4.4.** Since \( \alpha \in (-1, 0) \), it is straightforward to check
\[
a^\alpha + b^\alpha + c^\alpha + (a + b + c)^\alpha - a^\alpha + b^\alpha \geq c^\alpha + 1,
\]
if \( a \geq b \geq c \in (0, \infty) \). We only present the proof of the case \((\ell_1, \ell_2, \ell_3) = (+, -, -)\), since the other cases can be handled in a similar fashion. Use (50) and recall \( k_1, k_2, k_3 \in [k - 20, k + 20] \), we then have
\[
|\Phi(\xi, \eta, \sigma)| \gtrsim 2^{(\alpha+1)k}.
\]

Observing
\[
e^{-is\Phi(\xi, \eta, \sigma)} = \frac{\partial_s e^{-is\Phi(\xi, \eta, \sigma)}}{-i\Phi(\xi, \eta, \sigma)},
\]
we integrate by parts in $s$ to deduce

\[
\left| \xi \int_{t_1}^{t_2} e^{iH(\xi,s)} e^{-i\Phi(\xi,\eta,\sigma)} \hat{f}_{k_1}^1 (\xi - \eta, s) \hat{f}_{k_2}^2 (\eta, s) \hat{f}_{k_3}^3 (\sigma, s) \, d\eta d\sigma ds \right|
\]

\[
\lesssim \int_{t_1}^{t_2} \left| \xi \int_{\mathbb{R}^2} \frac{e^{-iH(\xi,s)}}{\Phi(\xi,\eta,\sigma)} \frac{d}{ds} \left[ e^{-iH(\xi,s)} \hat{f}_{k_1}^1 (\xi - \eta, s) \hat{f}_{k_2}^2 (\eta, s) \hat{f}_{k_3}^3 (\sigma, s) \right] \, d\eta d\sigma \right| ds
\]

\[
+ \sum_{j=1}^{2} \left| \xi \int_{\mathbb{R}^2} \frac{e^{iH(\xi,\tau_j)} e^{-i\Phi(\xi,\eta,\sigma)}}{\Phi(\xi,\eta,\sigma)} \hat{f}_{k_1}^1 (\xi - \eta, \tau_j) \hat{f}_{k_2}^2 (\eta, \tau_j) \hat{f}_{k_3}^3 (\sigma, \tau_j) \, d\eta d\sigma \right|
\]

\[=: A^0(\xi) + \sum_{j=1}^{2} A_j(\xi, \tau_j). \]  

(52)

We first handle the two terms $A_1(\xi, t_1)$ and $A_2(\xi, t_2)$. Let

\[m_3(\eta, \sigma) := \frac{1}{\Phi(\xi, \eta, \sigma)} \varphi'_{k_1} (\xi - \eta - \sigma) \varphi'_{k_2} (\eta) \varphi'_{k_3} (\sigma). \]  

(53)

It thus follows from (51) and (53) that

\[\| F^{-1}(m_3) \|_{L^1} \lesssim 2^{-(\alpha+1)k}. \]

We therefore, in light of Lemma 6.2, may estimate

\[A_j(\xi, \tau_j) \lesssim \epsilon_1^2 2^{-\alpha k/2} (2^{\alpha_0 - 1/2})^{m_2 - 2N_0 k_+} \lesssim \epsilon_1^2 2^{-m/4} 2^{-10k_+}, \]

for $j = 1, 2$, where we have used $\alpha \in (-1, 0)$ in the last inequality.

We now come to estimate the term $A^0(\xi)$. For this, we expand $d/ds$ to deduce

\[A^0(\xi) \lesssim \sup_{s \in [t_1, t_2]} \left( A_0^0(\xi, s) + A_1^0(\xi, s) + A_2^0(\xi, s) + A_3^0(\xi, s) \right), \]  

(54)

where

\[
A_0^0(\xi, s) = \xi \int_{\mathbb{R}^2} m_4(\eta, \sigma) e^{-i\Phi(\xi, \eta, \sigma)} \hat{f}_{k_1}^1 (\xi - \eta - \sigma, s) \hat{f}_{k_2}^2 (\eta, s) \hat{f}_{k_3}^3 (\sigma, s) \, d\eta d\sigma,
\]

\[
A_1^0(\xi, s) = \xi \int_{\mathbb{R}^2} m_3(\eta, \sigma) e^{-i\Phi(\xi, \eta, \sigma)} \partial_s \hat{f}_{k_1}^1 (\xi - \eta - \sigma, s) \hat{f}_{k_2}^2 (\eta, s) \hat{f}_{k_3}^3 (\sigma, s) \, d\eta d\sigma,
\]

\[
A_2^0(\xi, s) = \xi \int_{\mathbb{R}^2} m_3(\eta, \sigma) e^{-i\Phi(\xi, \eta, \sigma)} \hat{f}_{k_1}^1 (\xi - \eta - \sigma, s) \partial_s \hat{f}_{k_2}^2 (\eta, s) \hat{f}_{k_3}^3 (\sigma, s) \, d\eta d\sigma,
\]

\[
A_3^0(\xi, s) = \xi \int_{\mathbb{R}^2} m_3(\eta, \sigma) e^{-i\Phi(\xi, \eta, \sigma)} \hat{f}_{k_1}^1 (\xi - \eta - \sigma, s) \hat{f}_{k_2}^2 (\eta, s) \partial_s \hat{f}_{k_3}^3 (\sigma, s) \, d\eta d\sigma,
\]

(55)

with

\[m_4(\eta, \sigma) := \partial_s H(\xi, \eta) m_3(\eta, \sigma). \]  

(56)

To control $A_0^0(\xi, s)$, one observes

\[|\partial_s H(\xi, \eta, s)| \lesssim \epsilon_1^2 2^{(2-\alpha)k} 2^{-m} 2^{-20k_+}, \]  

and thus obtains

\[\| F^{-1}(m_4) \|_{L^1} \lesssim \epsilon_1^2 2^{(1-2\alpha)k} 2^{-m} 2^{-20k_+}. \]

We apply Lemma 6.2 again to obtain

\[\sup_{s \in [t_1, t_2]} A_0^0(\xi, s) \lesssim \epsilon_1^2 2^{-m} 2^{-10k_+}. \]
We next consider the term $A_1^0(\xi, s)$. For this, in view of (27), (30) and (34), we first easily see that
\[ \|\partial_s f_k(\xi)\|_{L^2} \lesssim \epsilon_3^{\frac{1}{2} - m} 2^{3p_0 m} 2^{20k_1}. \] (58)
Let
\[
\tilde{f}(\theta) := e^{i s (\xi - \theta)} f_{k_1}(\xi - \theta, s),
\]
\[
\tilde{g}(\eta) := e^{i s |\eta|^\alpha} f_{k_2}(\eta, s),
\]
\[
\tilde{h}(\sigma) := e^{i s |\sigma|^\alpha} f_{k_3}(\sigma, s),
\]
we then use (34) and (58) to get
\[ \|f\|_{L^2} \lesssim \epsilon_1 2^{3p_0 m} 2^{-20k_1} 2^{-m}, \]
\[ \|g\|_{L^2} \lesssim \epsilon_1 2^{-N_0 k_1} 2^{p_0 m}, \]
\[ \|h\|_{L^2} \lesssim \epsilon_1 2^{-m/2}. \]
Recall $\alpha \in (-1, 0)$, it then follows from Lemma 6.2 that
\[ \sup_{s \in [\xi, \eta]} A_1^0(\xi, s) \lesssim \epsilon_3^{\frac{1}{2} - m} 2^{3p_0 m} 2^{-20k_1} 2^{-20k_2} \lesssim \epsilon_3^{\frac{1}{2} - m} 2^{-2} 2^{-10k_1}. \]
Similarly, one has
\[ \sup_{s \in [\xi, \eta]} (A_2^0(\xi, s) + A_3^0(\xi, s)) \lesssim \epsilon_3^{\frac{1}{2} - m} 2^{-m/2} 2^{-10k_1}. \]

\[ \square \]

**Proposition 2.** Let
\[ \min(k_1, k_2, k_3) \geq -\frac{(1 - 20\rho_0)m}{2(\alpha + 1)}, \]
\[ \max(|k_1 - k|, |k_2 - k|, |k_3 - k|) \geq 21, \] (59)
then the bound (40) holds.

**Proof.** It suffices to show
\[ |I_{k_1, k_2, k_3}(\xi, s)| \lesssim \epsilon_1^{\frac{1}{2} - m} 2^{3p_0 m} 2^{-10k_1}. \]

**Step 1.** max\(|k_1 - k_2|, |k_1 - k_3|, |k_2 - k_3|\) \geq 5. Without loss of generality, we may assume that \(|k_1 - k_2| \geq 5\). On the support \(|\xi - \eta - \sigma| \in [2^{k_1 - 2}, 2^{k_1 + 2}]\) and \(|\eta| \in [2^{k_2 - 2}, 2^{k_2 + 2}]\), one has
\[ |\partial_\eta \Phi(\xi, \eta, \sigma)| = (\alpha + 1)|\xi - \eta - \sigma^\alpha - |\eta^\alpha| \gtrsim 2^\alpha \min(k_1, k_2), \] (60)
and then integrate by parts in \(\eta\) to control
\[ |I_{k_1, k_2, k_3}(\xi, s)| \leq |J_1(\xi, s)| + |F_1(\xi, s)| + |G_1(\xi, s)|, \] (61)
where
\[ J_1(\xi, s) = \xi \int_{R^2} \partial_\eta m_5(\eta, \sigma) e^{-i s \Phi(\xi, \eta, \sigma)} \tilde{f}_{k_1}(\xi - \eta - \sigma, s, \tilde{\eta}_{k_2}(\eta, s) \tilde{f}_{k_3}(\sigma, s) d\eta d\sigma, \]
\[ F_1(\xi, s) = \xi \int_{R^2} m_5(\eta, \sigma) e^{-i s \Phi(\xi, \eta, \sigma)} \partial_\eta \tilde{f}_{k_1}(\xi - \eta - \sigma, s, \tilde{\eta}_{k_2}(\eta, s) \tilde{f}_{k_3}(\sigma, s) d\eta d\sigma, \] (62)
\[ G_1(\xi, s) = \xi \int_{R^2} m_5(\eta, \sigma) e^{-i s \Phi(\xi, \eta, \sigma)} \tilde{f}_{k_1}(\xi - \eta - \sigma, s, \partial_\eta \tilde{\eta}_{k_2}(\eta, s) \tilde{f}_{k_3}(\sigma, s) d\eta d\sigma, \]
with
\[ m_5(\eta, \sigma) := \frac{1}{s \partial_s \Phi(\xi, \eta, \sigma)} \varphi'_{k_1}(\xi - \eta - \sigma) \varphi'_{k_2}(\eta) \varphi'_{k_3}(\sigma). \]  
(63)

Using (60) and (63), one finds that \( m_5 \) satisfies the following estimates
\[ \| \mathcal{F}^{-1}(m_5) \|_{L^1} \lesssim 2^{-m_2 - \alpha \min(k_1, k_2)}, \]  
(64)
and
\[ \| \mathcal{F}^{-1}(\partial_\eta m_5) \|_{L^1} \lesssim 2^{-m_2 - (\alpha + 1) \min(k_1, k_2)}. \]  
(65)

Applying Lemma 6.2, we use (34) and (65) to see
\[ |J_1(\xi, s)| \lesssim \epsilon_{1}^{2} k_2^{2 - (\alpha + 1) \min(k_1, k_2)} g_2(2p_0 - 3/2) m_2 - N_0 \max(k_1, k_2, k_3), \]
and instead use (34) and (64) to find
\[ |F_1(\xi, s)| \lesssim \epsilon_{1}^{2} k_2^{2 - k_2 - \alpha \min(k_1, k_2)} g_2(2p_0 - 3/2) m_2 - N_0 \max(k_1, k_2, k_3), \]
and
\[ |G_1(\xi, s)| \lesssim \epsilon_{1}^{2} k_2^{2 - k_2 - \alpha \min(k_1, k_2)} g_2(2p_0 - 3/2) m_2 - N_0 \max(k_1, k_2, k_3). \]

We finally conclude that
\[ |J_1(\xi, s)| + |F_1(\xi, s)| + |G_1(\xi, s)| \lesssim \epsilon_{1}^{2} k_2^{2 - (\alpha + 1) \min(k_1, k_2)} g_2(2p_0 - 3/2) m_2 - 10k + (2^{10} \max(k_1, k_2, k_3) + 1) \lesssim \epsilon_{1}^{2} k_2^{2 - m_2 - 5p_0 m_2 - 10k}, \]
where we have used (39) and (59) in the last inequality.

**Step 2.** \( \max(|k_1 - k_2|, |k_1 - k_3|, |k_2 - k_3|) \leq 4. \) In this case \( \partial_\eta \Phi \neq 0 \) or \( \partial_\sigma \Phi \neq 0. \) Without loss of generality, we assume \( \partial_\eta \Phi \neq 0. \) On the support \( |\xi - \eta - \sigma| \in [2^{k_2 - 2}, 2^{k_2 + 2}], |\eta| \in [2^{k_1 - 2}, 2^{k_1 + 2}] \) and \( |\sigma| \approx 2^{k_3} \), recalling \( 2^{k_1} \approx 2^{k_2} \approx 2^{k_3} \), we have
\[ |\partial_\eta \Phi(\xi, \eta, \sigma)| = (\alpha + 1)|\xi - \eta - \sigma|^\alpha - |\eta|^\alpha \gtrsim 2^{\alpha k_2}. \]  
(66)

One may use integration by parts in \( \eta \) as (61)-(62) to control \( I_{k_1, k_2, k_3}. \) Due to (66), we instead have
\[ \| \mathcal{F}^{-1}(m_5) \|_{L^1} \lesssim 2^{-m_2 - \alpha k_2}, \]  
(67)
and
\[ \| \mathcal{F}^{-1}(\partial_\eta m_5) \|_{L^1} \lesssim 2^{-m_2 - (\alpha + 1) k_2}. \]  
(68)

Recall \( 2^{k_1} \approx 2^{k_2} \approx 2^{k_3} \) and repeat the arguments of **Step 1** using instead (67)-(68), we finally conclude that
\[ |J_2(\xi, s)| + |F_2(\xi, s)| + |G_2(\xi, s)| \lesssim \epsilon_{1}^{2} k_2^{2 - (\alpha + 1) k_2} g_2(2p_0 - 3/2) m_2 - N_0 k_2 + \lesssim \epsilon_{1}^{2} k_2^{2 - m_2 - 5p_0 m_2 - 10k}, \]
in which we have used the assumption
\[ -(\alpha + 1) k_2 \leq \left( \frac{1}{2} - 10p_0 \right) m. \]
Proposition 3. Let
\[ \min(k_1, k_2, k_3) \leq -\frac{(1 - 20p_0)m}{2(\alpha + 1)}, \]
\[ \max(|k_1 - k|, |k_2 - k|, |k_3 - k|) \geq 21, \] (69)
then the bound (40) holds.

Proof. It suffices to show
\[ \left| \int_{t_1}^{t_2} e^{iH(\xi, s)} f_{k_1, k_2, k_3}^{(2)}(\xi, s) \, ds \right| \lesssim \varepsilon_{k_1}^3 2^{-2p_0m} 2^{-10k_+.} \]

Case 1. \((t_1, t_2, t_3) = (+, +, +).\) In this case, one may estimate
\[ -\Phi(\xi, \eta, \sigma) = (\xi - \eta - \sigma)^\alpha + \eta^\alpha + \sigma^\alpha - \xi^\alpha \]
\[ \gtrsim 2^{(\alpha+1) \text{med}(k_1, k_2, k_3)}. \]

It follows from (39) and (69) that
\[ \text{med}(k_1, k_2, k_3) \geq -(1 + 10p_0)m + \frac{(1 - 20p_0)m}{2(\alpha + 1)}. \]

Therefore
\[ -\Phi(\xi, \eta, \sigma) \gtrsim 2^{-(1 + 10p_0)(\alpha + 1)m + (\frac{1}{2} - 10p_0)m} \gtrsim 2^{-(\frac{1}{2} - 20p_0)m} \] (70)

where we have used the assumption \(p_0 \leq -\frac{\alpha}{10m}.\)

Due to (70), we may integrate by parts in \(s\) as (52)-(55) and need to control the terms \(A_j(\xi, t_j), A_j^0(\xi, s), j = 1, 2; l = 0, 1, 2, 3.\) Using (70) and (53), one calculates that
\[ \|F^{-1}(m_j)\|_{L^1} \lesssim 2^{(\frac{1}{2} - 20p_0)m}, \]
\[ \|F^{-1}(m_4)\|_{L^1} \lesssim 2^{2(\frac{1}{2} - 20p_0)m} 2^{(2 - \alpha)k_2 - m_2 - 20k_+}. \]
\[ \|F^{-1}(m_4)\|_{L^1} \lesssim 2^{2(\frac{1}{2} - 20p_0)m} 2^{(2 - \alpha)k_2 - m_2 - 20k_+}. \]
\[ \|F^{-1}(m_4)\|_{L^1} \lesssim 2^{2(\frac{1}{2} - 20p_0)m} 2^{(2 - \alpha)k_2 - m_2 - 20k_+}. \]
\[ \|F^{-1}(m_4)\|_{L^1} \lesssim 2^{2(\frac{1}{2} - 20p_0)m} 2^{(2 - \alpha)k_2 - m_2 - 20k_+}. \]

The symbol-type estimate (71) together with Lemma 6.2 yields
\[ A_j(\xi, t_j) \lesssim c_1 2^{k_2 - 20p_0} 2^{2(\frac{1}{2} - 20p_0)m} 2^{(2 - \alpha)k_2 - m_2 - 20k_+}, \]
\[ A_j^0(\xi, s) \lesssim c_1 2^{k_2 - 20p_0} 2^{(4 - 3\alpha)k_2 - m_2 - 20k_+}, \]
and
\[ \sup_{s \in [t_1, t_2]} A_j^0(\xi, s) \lesssim c_1 2^{\frac{3}{2} - 5p_0 + 2 - 10k_+}, \]
and the same bound for \(A_j^0(\xi, s)\) and \(A_j^0(\xi, s).\) Applying Lemma 6.2 with (72), we may estimate
\[ \sup_{s \in [t_1, t_2]} A_j^0(\xi, s) \lesssim c_1 2^{\frac{3}{2} - 5p_0 + 2 - 10k_+}, \]
\[ \sup_{s \in [t_1, t_2]} A_j^0(\xi, s) \lesssim c_1 2^{\frac{3}{2} - 5p_0 + 2 - 10k_+}, \]

Case 2. \((t_1, t_2, t_3) \in \{ (+, -, -), (-, +, -), (-, -, +) \}.\) Since they are similar, we only analyze the case \((t_1, t_2, t_3) = (+, -, -).\) In this case, we have
\[ \Phi(\xi, \eta, \sigma) = \xi^\alpha + (-\eta)^\alpha + (-\sigma)^\alpha - (\xi - \eta - \sigma)^\alpha \]
\[ \gtrsim 2^{(\alpha+1) \text{med}(k_2, k_3)}. \]
There are three sub-cases to consider:

(i) If \( \text{med}(k, k_2, k_3) = k \), then
\[
\Phi(\xi, \eta, \sigma) \gtrsim 2^{(\alpha+1)k}.
\]

(ii) If \( \text{med}(k, k_2, k_3) = k_2 \), then \( k \leq k_2 \leq k_3 \) or \( k_3 \leq k_2 \leq k \).
When \( k \leq k_2 \leq k_3 \), one has
\[
\Phi(\xi, \eta, \sigma) \gtrsim 2^{(\alpha+1)k_2} \geq 2^{(\alpha+1)k}.
\]

For \( k_3 \leq k_2 \leq k \), it holds
\[
\Phi(\xi, \eta, \sigma) \gtrsim 2^{(\alpha+1)k_2} \geq 2^{(\alpha+1)\text{med}(k_1, k_2, k_3)}.
\]

(iii) If \( \text{med}(k, k_2, k_3) = k_3 \), then, similarly to (ii), one may show \( \Phi \) enjoys the same bounds as (ii).

We conclude from (i)-(iii) that
\[
\Phi(\xi, \eta, \sigma) \gtrsim 2^{(\alpha+1)k}, \quad (73)
\]
or
\[
\Phi(\xi, \eta, \sigma) \gtrsim 2^{(\alpha+1)\text{med}(k_1, k_2, k_3)} \gtrsim 2^{-\left(\frac{1}{2} - 20p_0\right)m}. \quad (74)
\]

For the case of (73), the Phase \( \Phi \) enjoys the same bound as (51), thus the terms \( A_j(\xi, t_j), A_j(\xi, s), j = 1, 2, l = 0, 1, 2, 3 \) can be estimated as it in Lemma 4.4. The latter case (74) can be handled identically to Case 1.

**Case 3.** \( (t_1, t_2, t_3) \in \{(+, +, -), (+, +, +), (-, +, +)\} \). We only consider the case \( (t_1, t_2, t_3) = (+, +, -) \), and the other cases may be handled in a similar fashion. In this case, one has
\[
\Phi(\xi, \eta, \sigma) = \xi^{\alpha+1} - (\xi - \eta - \sigma)^{\alpha+1} - \eta^{\alpha+1} + (-\sigma)^{\alpha+1}.
\]

We shall divide it into two sub-cases.

(i) \( k_3 = \min(k_1, k_2, k_3) \). Recalling the assumption \( k_1, k_2, k_3 \in [-4m, p_0m/10] \), we have
\[
k_3 \in \left[-(1 + 20p_0)m, \frac{- (1 - 20p_0)m}{2(\alpha + 1)}\right], \quad k_1, k_2 \in \left[-\frac{(1 - 40p_0)m}{2(\alpha + 1)}, \frac{p_0m}{10}\right].
\]
Notice that
\[
a^{\alpha+1} + b^{\alpha+1} - (a + b)^{\alpha+1} \gtrsim b^{\alpha+1},
\]
if \( a \geq b \in (0, \infty) \). We then may estimate
\[
-\Phi(\xi, \eta, \sigma) = \left(-\xi^{\alpha+1} + (\xi - \eta - \sigma)^{\alpha+1} + (\eta + \sigma)^{\alpha+1}\right) + \left(\eta^{\alpha+1} - (\eta + \sigma)^{\alpha+1}\right) - (-\sigma)^{\alpha+1} \gtrsim 2^{(\alpha+1)\min(k_1, k_2)} \gtrsim 2^{-\left(\frac{1}{2} - 20p_0\right)m},
\]
provided that \( |\xi - \eta - \sigma| \in [2^{k_1-2}, 2^{k_1+2}], |\eta| \in [2^{k_2-2}, 2^{k_2+2}] \) and \( |\sigma| \in [2^{k_3-2}, 2^{k_3+2}] \).

Proceed as (52)-(55) by integration by parts in \( s \). It follows from (75) and (53) that
\[
\|\mathcal{F}^{-1}(m_3)\|_{L^1} \lesssim 2^{\left(\frac{1}{2} - 20p_0\right)m}, \quad (76)
\]
and
\[
\|\mathcal{F}^{-1}(m_4)\|_{L^1} \lesssim \epsilon_1^2 2^{2(\alpha-1)k} 2^{\left(\frac{1}{2} - 20p_0\right)m} 2^{-m_2 2^{-20k_+}}. \quad (77)
\]
With the symbol-type bounds (76) and (77) at hand, repeating the argument of Case 1, one may estimate
\[
A_j(\xi, t_j) \lesssim \epsilon_1^3 2^{-5p_0m} 2^{-10k_+}, \quad \text{for } j = 1, 2,
\]

**Case 1.**
and
\[
\sup_{s \in [t_1, t_2]} \left( A_0^0(\xi, s) + A_1^0(\xi, s) + A_2^0(\xi, s) + A_3^0(\xi, s) \right) \lesssim \epsilon_1^3 2^{-m} 2^{-5p_0m} 2^{-10k_+}.
\]

(ii) \( k_3 \neq \min(k_1, k_2, k_3) \). By symmetry we may assume \( k_1 = \min(k_1, k_2, k_3) \). In this case, we have
\[
k_1 \in \left[ -\left(1 + 20p_0 \right)m, -\frac{1 - 20p_0}{2(\alpha + 1)} \right], \quad k_2, k_3 \in \left[ -\frac{1 - 40p_0}{2(\alpha + 1)}, \frac{p_0m}{10} \right].
\]
Define
\[
\chi_{k,m}(\eta, \sigma) = \begin{cases} 1, & \text{if } |k - k_1| \geq 11, \\ 1 - \varphi(2^{(1+20p_0)m}(\eta + \sigma)), & \text{if } |k - k_1| \leq 10. \end{cases}
\]
We first observe that
\[
\left| \xi \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \left( 1 - \chi_{k,m}(\eta, \sigma) \right) e^{iH(\xi, s)} e^{-i\Phi(\xi, \eta, \sigma)} \widehat{f}_{k_1}(\xi - \eta - \sigma, s) \times \widehat{f}_{k_2}(\eta, \sigma) \widehat{f}_{k_3}(\sigma, s) \ d\eta d\sigma ds \right| \leq 2^m 2 \left| \int_{\mathbb{R}^2} \varphi(2^{(1+20p_0)m}(\eta + \sigma)) e^{-i\Phi(\xi, \eta, \sigma)} \widehat{f}_{k_1}(\xi - \eta - \sigma, s) \times \widehat{f}_{k_2}(\eta, \sigma) \widehat{f}_{k_3}(\sigma, s) \ d\eta d\sigma \right|
\]
\[
\lesssim \epsilon_1^3 2^m 2^{k_2 - (1/2 + 10p_0)m} 2^{-m/2} 2^{p_0m} 2^{-10k_+}
\]
\[
\lesssim \epsilon_1^3 2^{-5p_0m} 2^{-10k_+}.
\]
To complete the proof it remains to show
\[
\left| \xi \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \chi_{k,m}(\eta, \sigma) e^{iH(\xi, s)} e^{-i\Phi(\xi, \eta, \sigma)} \widehat{f}_{k_1}(\xi - \eta - \sigma, s) \times \widehat{f}_{k_2}(\eta, \sigma) \widehat{f}_{k_3}(\sigma, s) \ d\eta d\sigma ds \right| \tag{78}
\]
\[
\lesssim \epsilon_1^3 2^{-5p_0m} 2^{-10k_+}.
\]

The crucial ingredient in showing (78) is that the phase \( \Phi \) satisfies the following weakly elliptic bound
\[
|\Phi(\xi, \eta, \sigma)| \geq \lambda := \begin{cases} 2^{(2+1)k_1 - 100}, & \text{if } k \leq k_1 - 11, \\ 2^{2(\alpha + 1) \min(k, k_2, k_3) - 100}, & \text{if } k \geq k_1 + 11, \\ 2^{2 - (1+20p_0)m} 2^{\alpha k_1 - 100}, & \text{if } |k - k_1| \leq 10, \end{cases} \tag{79}
\]
provided that \(|\xi - \eta - \sigma| \in [2^{k_1 - 2}, 2^{k_1 + 2}], |\eta| \in [2^{k_2 - 2}, 2^{k_2 + 2}], |\sigma| \in [2^{k_3 - 2}, 2^{k_3 + 2}]\)
and \( \chi_{k,m}(\eta, \sigma) \neq 0 \). We now show (79) as follows:
If \( k \leq k_1 - 11 \), we then estimate
\[
-\Phi(\xi, \eta, \sigma) \geq -\xi^{\alpha + 1} + (\xi - \eta - \sigma)^{\alpha + 1} - |\eta|^{\alpha + 1} - (-\sigma)^{\alpha + 1} \geq -2^{(2+1)k_1 - 11 - 2^{k_1 + 2k_2 + 10}}
\]
\[
\geq 2^{(2+1)k_1 - 2}.
\]
If $k \geq k_1 + 11$, we then deduce
\[
\Phi(\xi, \eta, \sigma) \geq (\xi^{\alpha+1} + (-\sigma)^{\alpha+1} - (\xi - \eta - \sigma)^{\alpha+1} - |\eta|^{\alpha+1} - (\xi - \eta - \sigma)|^{\alpha+1})
- 2(\alpha+1)\min(k,k_3) - 2 - g(\alpha+1)k_1 + 1 - 2k_1 + \alpha k_2 + 10
\geq 2(\alpha+1)\min(k,k_3) - 2.
\]
If $|k - k_1| \leq 10$, first notice $2^{-(1+20p_0)m} \leq |\eta + \sigma| \leq 2^{k_1+11}$, then one has
\[
|\Phi(\xi, \eta, \sigma)| = (\alpha + 1)(\mu\xi + (1 - \mu)(\xi - \eta - \sigma))^\alpha - (\theta\eta + (1 - \theta)(-\sigma))^\alpha ||\eta + \sigma|
\geq 2^{\alpha k_1} 2^{-(1+20p_0)m},
\]
for some $\mu, \theta \in (0,1)$.

To show (78) in view of (79), we integrate by parts in $s$ to get
\[
\left| \xi \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \chi_{k,m}(\eta, \sigma)e^{iH(\xi,s)}e^{-i\Phi(\xi,\eta,\sigma)} \hat{f}_{k_3}^+(\xi - \eta - \sigma, s)
\times \hat{f}_{k_2}^+(\eta, \sigma) \hat{f}_{k_3}(\sigma, s) d\eta d\sigma ds \right|
\leq \int_{t_1}^{t_2} \left| \xi \int_{\mathbb{R}^2} \chi_{k,m}(\eta, \sigma) e^{-i\Phi(\xi,\eta,\sigma)} \frac{d}{ds} \left[ e^{iH(\xi,s)} \hat{f}_{k_3}^+(\xi - \eta - \sigma, s)
\times \hat{f}_{k_2}^+(\eta, \sigma) \hat{f}_{k_3}(\sigma, s) \right] d\eta d\sigma \right| ds
+ \sum_{j=1}^{2} \left| \xi \int_{\mathbb{R}^2} \chi_{k,m}(\eta, \sigma) e^{iH(\xi,t_j)} e^{-i\eta \Phi(\xi,\eta,\sigma)} \hat{f}_{k_3}^+(\xi - \eta - \sigma, t_j)
\times \hat{f}_{k_2}^+(\eta, t_j) \hat{f}_{k_3}(\sigma, t_j) d\eta d\sigma \right|
=: B_0^0(\xi) + \sum_{j=1}^{2} B_j(\xi, t_j),
\]
and the term $B_0^0(\xi)$ may be bounded by
\[
B_0^0(\xi) \lesssim 2^{\alpha m} \sup_{s \in [t_1, t_2]} \left( B_0^0(\xi, s) + B_1^0(\xi, s) + B_2^0(\xi, s) + B_3^0(\xi, s) \right),
\]
in which each term is defined by
\[
B_0^0(\xi, s) = \left| \xi \int_{\mathbb{R}^2} m_7(\eta, \sigma)e^{-i\Phi(\xi,\eta,\sigma)} \hat{f}_{k_3}^+(\xi - \eta - \sigma, s) \hat{f}_{k_2}^+(\eta, \sigma) \hat{f}_{k_3}(\sigma, s) d\eta d\sigma \right|
B_1^0(\xi, s) = \left| \xi \int_{\mathbb{R}^2} m_6(\eta, \sigma)e^{-i\Phi(\xi,\eta,\sigma)} \partial \hat{f}_{k_3}^+(\xi - \eta - \sigma, s) \hat{f}_{k_2}^+(\eta, \sigma) \hat{f}_{k_3}(\sigma, s) d\eta d\sigma \right|
B_2^0(\xi, s) = \left| \xi \int_{\mathbb{R}^2} m_6(\eta, \sigma)e^{-i\Phi(\xi,\eta,\sigma)} \hat{f}_{k_3}^+(\xi - \eta - \sigma, s) \partial \hat{f}_{k_2}^+(\eta, \sigma) \hat{f}_{k_3}(\sigma, s) d\eta d\sigma \right|
B_3^0(\xi, s) = \left| \xi \int_{\mathbb{R}^2} m_6(\eta, \sigma)e^{-i\Phi(\xi,\eta,\sigma)} \hat{f}_{k_3}^+(\xi - \eta - \sigma, s) \hat{f}_{k_2}^+(\eta, \sigma) \partial \hat{f}_{k_3}(\sigma, s) d\eta d\sigma \right|
\]
with
\[
m_6(\eta, \sigma) := \chi_{k,m}(\eta, \sigma) \frac{\phi_{k_1}'(\xi - \eta - \sigma)}{\Phi(\xi, \eta, \sigma)} \phi_{k_2}'(\eta) \phi_{k_3}'(\sigma),
\]
If we apply Lemma 6.2 instead of using (83) to obtain
\[ ||F^{-1}(m_6)||_{L^1} \lesssim \lambda^{-1}, \tag{82} \]
and from (79) and (81) that
\[ ||F^{-1}(m_7)||_{L^1} \lesssim \epsilon_1^2 \lambda^{-2(2-\alpha)k} 2^{-m} 2^{-20k_+}. \tag{83} \]
Applying Lemma 6.2 with (82) yields
\[ B_j(\xi, t_j) \lesssim \epsilon_1^2 \lambda^{-1} 2^k 2^{(1+20\alpha/100)k_+} 2^{-10k_+}, \quad \text{for } j = 1, 2. \]
So we need to estimate \( \lambda^{-1} 2^k \). If \( k \leq k_1 - 11 \), then
\[ \lambda^{-1} 2^k \lesssim 2^{-(\alpha+1)k} 2^k \lesssim 2^{-\alpha-10k}. \]
If \( k \geq k_1 + 11 \), then when \( k \leq \min(k_2, k_3) \)
\[ \lambda^{-1} 2^k \lesssim 2^{-(\alpha+1)k} 2^k \lesssim 2^{-\alpha-10k}, \]
and when \( k \geq \min(k_2, k_3) \)
\[ \lambda^{-1} 2^k \lesssim 2^{-(\alpha+1)k} 2^k \lesssim 2^{(1+20\alpha/100)k_+}. \]
If \( |k - k_1| \leq 10 \), then
\[ \lambda^{-1} 2^k \lesssim 2^{(1+20\alpha/100)k_+} 2^{-\alpha-10k}. \]
where we have used the assumption \( \alpha \leq -\frac{\alpha}{100} \). We finally conclude
\[ B_j(\xi, t_j) \lesssim \epsilon_1^2 2^{-5p_0m} 2^{-10k_+}, \quad \text{for } j = 1, 2. \]
Similarly, we obtain
\[ \sup_{s \in [t_1, t_2]} (B^0_j(\xi, s) + B^0_j(\xi, s) + B^0_j(\xi, s)) \lesssim \epsilon_1^2 \lambda^{-1} 2^k 2^{(4p_0-3/2)2^{-m} 2^{-10k_+}} \lesssim \epsilon_1^2 2^{-m} 2^{-5p_0m} 2^{-10k_+}. \tag{84} \]
We apply Lemma 6.2 instead of using (83) to obtain
\[ \sup_{s \in [t_1, t_2]} B^0_j(\xi, s) \lesssim \epsilon_1^2 \lambda^{-1} 2^k 2^{(3-\alpha)k} 2^{-m} 2^{-10k_+} \lesssim \epsilon_1^2 2^{-m} 2^{-5p_0m} 2^{-10k_+}. \]

5. Concluding remarks. We conclude this paper by some related and open questions. Some previous results on long time behavior or finite time blow-up for solutions to fractional KdV type equations concerned equations with quadratic nonlinearities, namely the fractional KdV equation (fKdV)
\[ \partial_t u - |D|^\alpha \partial_x u = -u \partial_x u, \quad -1 < \alpha < 0, \tag{85} \]
or the Whitham equation
\[ \partial_t u + L \partial_x u = -u \partial_x u. \tag{86} \]
It was proven in [5] that the lifespan of solutions to (85) with \(-1 < \alpha < 1\), \( \alpha \neq 0 \) and with initial data of size \( O(\epsilon) \) in \( H^N(\mathbb{R}), N \geq 3 \) is \( O(1/\epsilon^2) \).
This result is specially striking when \(-1 < \alpha < 0\) since for \( \alpha > 0 \) one expects global existence of small solutions. Note that for the inviscid Burgers equation this lifespan is \( O(1/\epsilon) \).
Proving a global existence result of small solutions of the fKdV equation in the range \(-1 < \alpha < 0\) or for the Whitham equation (and for the modified Whitham equation) is a challenging open question.

On the other hand, a finite time blow-up by shock formation was observed via numerical approach and conjectured in \([16, 17]\). This has been now proven in \([11, 12]\) both for the fKdV equation when \(-\frac{1}{3} < \alpha < 0\) and for the Whitham equation but is still an open question when \(-\frac{1}{3} < \alpha < 0\).

Note that \(\alpha = -1\) corresponds to the Burgers-Hilbert equation, see \([1]\)

\[ u_t + uu_x + \mathcal{H}u_x = 0 \] (87)

where \(\mathcal{H}\) is the Hilbert transform (convolution with p.v. \(\frac{1}{x}\)), which is closer to a conservation law and for which the global existence (and scattering) of small smooth solutions is not expected. Actually the global existence of weak \(L^2\) solutions was proven in \([2]\), the blow-up of a \(C^{1+\delta}\) norm with \(\delta \in (0, 1)\) was shown in \([3]\), and the formation of shock solutions was strongly suggested in \([1]\) but only has been rigorously established recently in \([20, 21]\). Note however the spectacular result in \([9, 10]\) where it was proven that the lifespan of solutions with smooth initial data of size \(O(\epsilon)\) has the enhanced lifespan \(O(1/\epsilon^2)\) instead of the \(O(1/\epsilon)\) lifespan of the corresponding solutions of the Burgers equation.

6. Appendix. For the reader’s convenience, we list two technical lemmas proven in \([14]\) and \([15]\) respectively. The first one is the following interpolation inequality:

**Lemma 6.1** (\([14]\)). It holds that

\[
\|P_k g\|_{L^\infty} \lesssim \|P_k g\|_{L^1}^{2} \lesssim 2^{-k} \|\hat{P_k g}\|_{L^2} (\|\hat{P_k g}\|_{L^2} + 2^{k} \|\partial \hat{P_k g}\|_{L^2}).
\] (88)

The other one is the bound on pseudo-product operators satisfying certain strong integrability conditions:

**Lemma 6.2** (\([15]\)). Assume that \(m \in L^1(\mathbb{R} \times \mathbb{R})\) satisfies

\[
\left\| \int_{\mathbb{R}^2} m(\eta, \sigma)e^{i\eta \eta' e^{i\sigma \sigma'}} d\eta d\sigma \right\|_{L^1_{\eta, \sigma}} \lesssim A,
\]

for some \(A \in (0, \infty)\). Then for any \((p, q, r) \in \{(2, 2, \infty), (2, \infty, 2), (\infty, 2, 2)\}\),

\[
\left| \int_{\mathbb{R}^2} m(\eta, \sigma)\hat{f}(\eta)\hat{g}(\sigma)\hat{h}(-\eta - \sigma) d\eta d\sigma \right| \lesssim A\|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.
\]

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**REFERENCES**

[1] J. Biello and J. K. Hunter, Nonlinear Hamiltonian waves with constant frequency and surface waves on vorticity discontinuities, *Comm. Pure Appl. Math.*, 63 (2010), 303–336.

[2] A. Bressan and K. T. Nguyen, Global existence of weak solutions for the Burgers-Hilbert equation, *SIAM J. Math. Anal.*, 46 (2014), 2884–2904.

[3] A. Castro, D. Córdoba and F. Gancedo, Singularity formations for a surface wave model, *Nonlinearity*, 23 (2010), 2835–2847.

[4] D. Córdoba, J. Gómez-Serrano and A. D. Ionescu, Global solutions for the generalized SQG patch equation, *Arch. Ration. Mech. Anal.*, 233 (2019), 1211–1251.

[5] M. Ehrnstöm and Y. Wang, Enhanced existence time of solutions to the fractional Korteweg-de Vries equation, *SIAM J. Math. Anal.*, 51 (2019), 3298–3323.
[6] P. Germain, N. Masmoudi and J. Shatah, Global solutions for 3D quadratic Schrödinger equations, Int. Math. Res. Not. IMRN., 2009 (2009), 414–432.
[7] S. Gustafson, K. Nakanishi and T.-P. Tsai, Global dispersive solutions for the Gross-Pitaevskii equation in two and three dimensions, Ann. Henri Poincaré, 8 (2007), 1303–1331.
[8] N. Hayashi and P. I. Naumkin, Large time behavior of solutions for the modified Korteweg-de Vries equation, Internat. Math. Res. Notices, 1999 (1999), 395–418.
[9] J. K. Hunter and M. Ifrim, Enhanced life span of smooth solutions of a Burgers-Hilbert equation, SIAM J. Math. Anal., 44 (2012), 2039–2052.
[10] J. K. Hunter, M. Ifrim, D. Tataru and T. K. Wong, Long time solutions for a Burgers-Hilbert equation via a modified energy method, Proc. Amer. Math. Soc., 143 (2015), 3407–3412.
[11] V. M. Hur and L. Tao, Wave breaking for the Whitham equation with fractional dispersion, Nonlinearity, 27 (2014), 2937–2949.
[12] V. M. Hur, Wave breaking in the Whitham equation, Adv. Math., 317 (2017), 410–437.
[13] A. D. Ionescu and F. Pusateri, Global solutions for the gravity water waves system in 2d, Invent. Math., 199 (2015), 653–804.
[14] A. D. Ionescu and F. Pusateri, Global analysis of a model for capillary water waves in two dimensions, Comm. Pure Appl. Math., 69 (2016), 2015–2071.
[15] A. D. Ionescu and F. Pusateri, Nonlinear fractional Schrödinger equations in one dimension, J. Funct. Anal., 266 (2014), 139–176.
[16] C. Klein, F. Linares, D. Pilod and J.-C. Saut, On Whitham and related equations, Stud. Appl. Math., 140 (2018), 133–177
[17] C. Klein and J.-C. Saut, A numerical approach to blow-up issues for dispersive perturbations of Burgers’ equation, Phys. D, 295/296 (2015), 46–65.
[18] F. Linares, D. Pilod and J.-C. Saut, Dispersive perturbations of Burgers and hyperbolic equations I: Local theory, SIAM J. Math. Anal., 46 (2014), 1505–1537.
[19] L. Molinet, D. Pilod and S. Vento, On well-posedness for some dispersive perturbations of Burgers’ equation, Ann. Inst. H. Poincaré Anal. Non Linéaire, 35 (2018), 1719–1756.
[20] J.-C. Saut and Y. Wang, The wave breaking for Whitham-type equations revisited, preprint, arXiv:2006.03803.
[21] R. Yang, Shock formation for the Burgers-Hilbert equation, preprint, arXiv:2006.05668.

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