Wave properties using displaced phase amplitude

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Abstract. Displaced phase-amplitude variable in polar form. This variable is used to investigate changes in amplitude in complex fields with phases that depend only on the position in the propagation. Soliton on Finite Background (SFB) which is an exact solution of Nonlinear Schrodinger (NLS) equation has been widely used in investigating wave propagation dynamics so that it is the basic for the proposed displaced phase-amplitude. Using displaced phase-amplitude, the results obtained can be described in Argand Diagrams. Wave equation used as a model is the Benjamin Bona Mahony (BBM) equation where the envelope of this wave evolves following the NLS equation. This wave is unidirectional long wave on the surface and has low amplitude characteristics. Step by step to obtain a SFB solution that contains displaced phase-amplitude described and displayed in an argand diagram. In additional, the envelope graph is given.

1. Introduction
Modulation instability is one of mechanism that needs to be considered in an effort to generate large and unbreakable waves called extreme waves in the hydrodynamic laboratory [1]. This mechanism can causes an increase in amplitude of wave in their propagations [1-7]. Investigation of water wave instability in the propagation can be carried out through Soliton on Finite Background (SFB). The SFB found by Akhmediev and Ankiewiez (1987) is an exact solution of the Non Linear Schrodinger (NLS) equation [8,9]. Water wave which is modelled by Korteweq de Vries (KdV) and Benjamin Bona Mahony (BBM) equations can be investigated using SFB. Therefore, both of these have modulation instability in their propagation and envelope that evolves following the NLS equation [1,9-15].

More specifically, to investigate changes in amplitude in complex fields where phases that only depend on position can be carried out by transforming SFB in complex numbers into polar forms called displaced phase-amplitude variables [16]. This variable has been used to investigate the waves of the KdV equation [1] and in this article we will investigate the properties of the water surface of the BBM equation using displaced phase-amplitude.
The next section describe the relationship between the BBM and NLS equations and the derivation in obtaining the SFB solution using the displaced phase-amplitude variable. The conclusions of this research are the closing section.

2. Relationship between BBM and NLS

2.1. BBM equation
The BBM equation models unidirectional wave propagation with a large wave number and a small amplitude. This equation shows soliton stability as an improvement of the KdV equation [17]. The equation for BBM in dimensionless form is as follows.

\[ \eta_t + \eta_x + \gamma \eta \eta_x - \eta_{xxx} = 0, \]  

(1)

\( \eta(x,t) \) is wave elevation, \( x \) spatial variable, and \( t \) time variable. Here, ansatz of the equation is \( \eta(x,t) = a(x,t)e^{i\theta} + c.c \) where \( \theta = (kx - \omega t) \), \( a(x,t) \), \( k \) and \( \omega \) are wave amplitude, wave number, and frequency, respectively, and c.c. is a complex conjugate.

2.2. NLS equation
The NLS equation has been applied in various fields of mathematics-physics, such as nonlinear optics, many particle quantum systems, plasma physics, superconductivity and quantum mechanics, and surface water waves. The application of the NLS equation to the physical system makes it possible to study the system with high accuracy. If we want to predict the evolution of the envelope in space, the NLS equation used is the spatial NLS equation [1,15].

\[ A_\xi + i\beta A_{\tau\tau} + i\gamma |A|^2 A = 0 \]  

(2)

where \( \beta \) and \( \gamma \) is constants that depend on frequency or wave number. Independent variable \( \xi \) describes spatial variables (space) and \( \tau \) represents time variables.

2.3. NLS as an equation for BBM envelopes
The derivation in the envelope of BBM was carried out by multiple scale method, before the transformation was carried out first \( \theta = (kx - \omega t) \) as fast to slow variable into slow variable; \( \xi = \varepsilon^2 x \), \( \tau = \varepsilon (t - x/p) \), and \( a(x,t) = \varepsilon A(\xi,\tau) \) where \( p = (1 + 2k\omega)/(1 + k^2) \) so that the NLS in equation (2) is obtained. NLS-BBM equation has constants \( \beta = -(2\omega p + \omega^2/k)/p^3 \) and \( \gamma = \omega(2/(p-1) - k/(8k^2\omega + 2\omega - 2k))/p \) that represents BBM as carrier wave [14].

The NLS equation has one of the exact solutions called SFB found by Akhmediev et al [8]. Spatial SFB solutions are expressed as follows.

\[ A(\xi,\tau) = A_0(\xi)A_{SFB}(\xi,\tau) \]  

(3)

SFB is a nonlinear interaction of a monochromatic amplitude \( r_0 \) signal that is \( A_0(\xi) = r_0e^{-i\gamma r_0^2\xi} \) which is disturbed by a modulating wave with a small wave number interval and causing instability.

3. Displaced phase-amplitude variables
Transformation of SFB variables in the form of complex amplitude to displaced phase-amplitude variables in polar forms intended to investigate changes in amplitude in complex fields with phases that depend only on positions that can be presented through polar coordinates [1,17].

\[ \left. A(\xi,\tau) = A_0(\xi)F(\xi,\tau) \right| \]  

(4)

with

\[ F(\xi,\tau) = G(\xi,\tau)e^{i\phi(\xi,\tau)} - 1. \]  

(5)
Solution in equation (4) is a form of SFB solution with phase \( \phi (\xi, \tau) \), amplitude \( G (\xi, \tau) \) and assumed \((\xi, \tau) = (0,0)\) as the maximum position and time. Displaced phase-amplitude variables are derived based on the variational formula and can be written as [1].

\[
G(\xi, \tau) = \frac{p(\xi)}{q(\xi) - \cos(\nu \tau)} \quad (6)
\]

\[
P(\xi) = \sqrt{\frac{\beta^2 + \sinh^2 \sigma(\xi)}{1 - \frac{\nu^2}{2}}} \quad (7)
\]

\[
Q(\xi) = \frac{\cosh \sigma(\xi)}{\sqrt{1 - \frac{\nu^2}{2}}} \quad (8)
\]

\[
\phi(\xi) = \tan^{-1} \left( \frac{-\frac{\nu}{\beta^2}}{\tan \sigma(\xi)} \right) \quad (9)
\]

by substitution equations (7) and (8) into equation (6) the following form of displaced amplitude \( G (\xi, \tau) \) will be obtained.

\[
G(\xi, \tau) = \frac{\nu \sqrt{\beta^2 + \sinh^2 \sigma(\xi)}}{\cosh \sigma(\xi) - \sqrt{1 - \frac{\nu^2}{2}} \cos(\nu \tau)} \quad (10)
\]

Then displaced phase \( \phi(\xi) \) as in equation (9) is substitution into equation (5), then the displaced phase-amplitude variable will be obtained in exponential form. Therefore, the form \( e^{i\phi} \) where \( \phi(\xi) \) is a displaced phase will be transformed into a polar form. It would be assumed that \( x = \left( -\frac{\nu}{\beta^2} \right) \tan \sigma(\xi) \) to simplify, then equation (9) would be \( \phi(\xi) = \tan^{-1} x \). So that,

\[
e^{i\phi} = e^{i \tan^{-1} x} = \cos(\tan^{-1} x) + i \sin(\tan^{-1} x)
\]

\[
= \frac{1}{\sqrt{1 + x^2}} + i \frac{x}{\sqrt{1 + x^2}}
\]

substitution \( x \) and the following phase displaced will be obtained.

\[
e^{i\phi} = \frac{\beta^2 \cosh \sigma(\xi) - i \nu \sinh \sigma(\xi)}{\sqrt{\beta^2 \cosh^2 \sigma(\xi) + \nu^2 \sinh^2 \sigma(\xi)}} \quad (11)
\]

from equations (10) and (11) the SFB solution is obtained as follows

\[
Ge^{i\phi} = \frac{\nu^2 \cosh \sigma(\xi) - \nu \sinh \sigma(\xi)}{\cosh(\sigma(\xi) - \sqrt{1 - \frac{\nu^2}{2}} \cos(\nu \tau))} \cdot \frac{\sqrt{\beta^2 \cosh^2 \sigma(\xi) + \nu^2 \sinh^2 \sigma(\xi)}}{\sqrt{\beta^2 \cosh^2 \sigma(\xi) + \nu^2 \sinh^2 \sigma(\xi)}} \quad (12)
\]

The forms of SFB solutions from equations (4), (5), and (12) are written as follows

\[
A(\xi, \tau) = A_0(\xi) \left[ \frac{\beta^2 \cosh \sigma(\xi) - \nu \sinh \sigma(\xi)}{\cosh(\sigma(\xi) - \sqrt{1 - \frac{\nu^2}{2}} \cos(\nu \tau))} \cdot \frac{\sqrt{\beta^2 \cosh^2 \sigma(\xi) + \nu^2 \sinh^2 \sigma(\xi)}}{\sqrt{\beta^2 \cosh^2 \sigma(\xi) + \nu^2 \sinh^2 \sigma(\xi)}} \right] - 1 \quad (13)
\]

where \( \xi \) represent the position and \( \tau \) represent the time with a period of \( 2\pi n / \nu \) for \( n = 0, \pm 1, \pm 2, \ldots \) The value of \( \nu = r_0 \sqrt{\gamma / \beta^2} \), \( 0 < \nu < \sqrt{2} \) is the modulation frequency, \( \sigma = \gamma r_0 \beta \), \( \hat{\sigma} = \nu \sqrt{2 - \nu^2} \) denote the Benjamin Feir Instability (BFI) rate and \( A_0(\xi) \) is the initial wave [18-20].
The form of $Ge^{i\phi}$ in equation (12) is different from Karjanto et al. [1, 2]. For $\xi = 0$ then the value of 
\[
\left( \bar{\theta} \sqrt{\bar{\theta}^2 + \sinh^2(\sigma \xi)} \right) / \left( \sqrt{\bar{\theta}^4 \cosh^2(\sigma \xi) + \bar{\sigma}^2 \sinh^2(\sigma \xi)} \right)
\] is 1, so in general the form $Ge^{i\phi}$ in equation (12) will be the same as Karjanto et al [1, 21] as follows

\[
Ge^{i\phi} = \frac{\theta^2 \cosh(\sigma \xi) - i \sigma \sinh(\sigma \xi)}{\cosh(\sigma \xi) - \frac{\bar{\theta}^2}{2} \cos(\nu \tau)} \tag{14}
\]

This applies to each modulation frequency value $\bar{\theta}$ at the interval of instability is selected. The SFB solution in equation (14) in polar form can be described in Argand diagrams.

**Figure 1.** Representing complex numbers in polar form in Argand diagram.

Argand diagrams are used to represent complex numbers in a geometrical plane where real numbers are horizontal axis and imaginary numbers as a vertical axis. The blue lines in Figure 1 are referred to as modulus complex numbers are phasors that have magnitudes and directions. The length of the phasor is obtained by calculating the modulus of a complex number and the direction of the phasor represents the direction of the wave. Arguments of complex numbers as phase angle is an angle to the real axis. The greater the phase angle, the waveform is getting steeper and the smaller the phase angle, the waveform is getting sloping. Phasors and phase angles represent the wave phase.

**Figure 2.** Argand diagram of SFB for $(\bar{r}_0, \beta, y, \bar{\theta}) = (1, 1, 1, \sqrt{1/2})$.

**Figure 3.** Graph of envelope $|A_{SFB}|$ for $(\bar{r}_0, \beta, y, \bar{\theta}) = (1, 1, 1, \sqrt{1/2})$. 
Figure 2 presents the argand diagram of SFB and Figure 3 presents the envelope $|A_{SFB}|$ for normalize conditions with $f_0 = 1$, the dispersive coefficient $\beta = 1$, the nonlinear coefficient $\gamma = 1$, and selected $\Delta = \sqrt{1/2}$ at the interval of modulation frequency instability, which is $0 < \Delta < \sqrt{2}$ for one period $0 < \tau < 2\pi/\nu$. In argand diagrams for $\xi > 0$, the line is below the real axis and is above the real axis for $\xi < 0$ and $\xi = 0$, the line on the real axis and it is also found that this line is the longest line. While for $\xi \to \pm \infty$, the length of the line decreases until it appears only as a point. Observed from the shape of the envelope, the increase in amplitude is symmetrical with respect to position and amplitude peaked when $\xi = 0$ and $\tau = 0$ (blue). It can be seen on the argand diagram, the position has the longest line at $(\xi, \tau) = (0,0)$. This SFB solution can be used to analyze the properties of the envelope of the BBM wave group in its propagation and to obtain an overview of the evolution and phenomena of phase singularity as described in Halfiani et al [14,15].

4. Conclusion and remarks

The SFB solution from NLS that is suitable for investigating the envelope of the BBM wave group has been derived by applying displaced phase-amplitude variables. Results in a polar form can be presented in argand diagrams, modulus complex numbers are represented as phasors and complex number arguments are represented by phase angles. According to the initial assumption that the occurrence (extreme position) occurs when $(\xi, \tau) = (0,0)$ with a symmetrical increase and decrease. This is known from the line when $\xi = 0$ which is the longest line and also can be clearly seen in the envelope graph is given.

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