CONNECTIVE CONSTANT FOR A WEIGHTED SELF-VOIDING WALK ON $\mathbb{Z}^2$

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Abstract. We consider a self-avoiding walk on $\mathbb{Z}^2$. The weight of each square depends on the way the walk passes through it and the weight of the whole walk is calculated as a product of these weights. We consider a family of critical weights parametrized by $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$. For $\theta = \frac{\pi}{3}$, this can be mapped to the self-avoiding walk on the honeycomb lattice. The connective constant in this case was rigorously proved to be equal to $\sqrt{2 + \sqrt{2}}$ by Duminil-Copin and Smirnov in [6]. We generalize their proof.

1. Introduction

Self-avoiding walks (i.e. visiting each vertex at most once) were proposed by P. Flory and W. J. C. Orr [8, 15]. These walks turned out to be a very interesting object leading to rich mathematical theories, see [12, 2], and raising important challenges (it is difficult to understand because of its non-markovity). There are not so many rigorous statements in this field, one of the main conjectures being convergence to SLE$(8/3)$. Some progress in this direction was achieved by G. Lawler, O. Schramm and W. Werner, who proved in [10] that if the scaling-limit of self-avoiding walk exists and is conformally invariant, then it is SLE$(8/3)$. In 1984, B. Nienhuis nonrigorously derived in [14] that the connective constant for the hexagonal lattice equals to $\sqrt{2 + \sqrt{2}}$. This has been proved recently by H. Duminil-Copin and S. Smirnov in [6]. Since the self-avoiding walk on the square lattice does not seem to be integrable, it is not reasonable to expect any explicit formula for the connective constant in this case. Nevertheless, one can study natural variations of the model, for instance by introducing additional weights.

We fix $\theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ and consider the self-avoiding walk on $\Lambda$ — the skewed $\mathbb{Z}^2$ lattice with edges having length 1 and all plaquets having angles $\theta$ and $\pi - \theta$.
To be precise this will be a curve starting and ending at the middle of edges, intersecting edges at right angles and having in each plaquet either one straight line connecting two opposite edges or two arcs surrounding opposite vertices or one arc or just no arcs (see fig. 1). Each rhombus has a weight according to the configuration of arcs inside it:

- empty plaquet has weight 1
- plaquet with a short arc has weight $u_1$
- plaquet with a long arc has weight $u_2$
- plaquet with a straight line has weight $v$
- plaquet with a two short arcs has weight $w_1$
- plaquet with a two long arcs has weight $w_2$

![Figure 1. Different ways of passing a rhombus with their weights.](image)

The weight of the whole walk is calculated as the product of weights of the plaquets. The partition function is equal to the sum of the weights of all self-avoiding walks on $\Lambda$ starting at 0:

$$\omega(\gamma) = \prod_{r\text{-rhombus}} \omega(r)$$

$$Z(u_1, u_2, v, w_1, w_2) = \sum_\gamma \omega(\gamma).$$

Let us consider

$$\tilde{c}_n = \frac{1}{u_1^n} \sum_{|\gamma|=n} \omega(\gamma).$$

By definition, we find

$$Z(u_1, u_2, v, w_1, w_2) = \sum_{n=0}^{\infty} \tilde{c}_n u_1^n.$$

We are now in a position to state our main result:
Theorem 1. There exists a family of weights \((u_1^c, u_2^c, v^c, w_1^c, w_2^c)_\theta\) parametrized by \(\theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]\) such that for these weights \(\lim_{n \to \infty} n^{\sqrt{\tilde{c}_n}}\) exists and is equal to \(\frac{1}{u_1^c}\).

Furthermore, these weights can be calculated explicitly:

\begin{align}
\tag{1}
\begin{align*}
\frac{1}{2}
\end{align}
\end{align}

\begin{align}
\tag{2}
\begin{align*}
\frac{1}{2}
\end{align}
\end{align}

\begin{align}
\tag{3}
\begin{align*}
\frac{1}{2}
\end{align}
\end{align}

\begin{align}
\tag{4}
\begin{align*}
\frac{1}{2}
\end{align}
\end{align}

\begin{align}
\tag{5}
\begin{align*}
\frac{1}{2}
\end{align}
\end{align}

These weights were discovered by B. Nienhuis [13] in 1990 as solutions of the Yang-Baxter equation. They were rediscovered by J. Cardy and Y. Ikhlef [?] in 2009 as the weights for which the parafermionic observable satisfies some particular equations. For the connection between these two approaches, see [?, 9]. We should just mention here that in [13] and [?] more general case is considered — \(O(n)\) model with \(n \in [-2, 2]\) (self-avoiding walk is a particular case of this model for \(n = 0\)). Unfortunately, the weights written there contain some minor misprints, so for completeness we include correct version of the weights in Appendix.

The parafermionic observable will be the core of our proof. In order to simplify its definition, we consider self-avoiding walks on a skewed \(\mathbb{Z}^2\), though we could also work directly with self-avoiding walks embedded to a normal square grid.

We will prove Theorem 1 following the scheme proposed by H. Duminil-Copin and S. Smirnov in [6]. It would be natural to try to generalize another statement [3] concerning the self-avoiding on the hexagonal lattice and find the critical value of fugacity for all weights described above. Though we conjecture that the same should hold, i.e. that the critical fugacity is equal to \(1/(1 + (u_1^c)^2)\), we cannot prove this at the moment.

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Proof

To analyse the behaviour of the self-avoiding walk, we will use the following observable introduced in [6]. Let \( \Omega \) be a simply connected domain, \( a \in \partial \Omega \) and \( z \in \Omega \):

\[
F_a(z) = \sum_{\gamma : a \to z} \omega(\gamma) e^{-i\sigma W(\gamma)},
\]

where the sum runs over self-avoiding walks starting at \( a \), ending at \( z \) and staying in \( \Omega \). Above, \( W(\gamma) \) denotes the winding of \( \gamma \). The value \( \sigma \) will be fixed later. Observables for other models were introduced in [7, 15, 16], see also [7] for a survey.

We will find the weights for which our observable satisfies version of Cauchy-Riemann equation for any rhombus

\[
F_a(z_{SE}) - F_a(z_{NW}) = e^{i\theta}(F_a(z_{SW}) - F_a(z_{NE})),
\]

where \( SENW \) is a rhombus with angles at \( N \) and \( S \) equal to \( \theta \) and \( z_{SE}, z_{NE}, z_{NW}, z_{SW} \) are midpoints of its edges (see fig. 2). It will be more convenient to rewrite the equation with all of the terms on the left side:

\[
F_a(z_{SE}) + e^{i\theta}F_a(z_{NE}) - F_a(z_{NW}) - e^{i\theta}F_a(z_{SW}) = 0.
\]

Lemma 2. If \( \sigma = \frac{\ell}{8} \), where \( \ell \) is some odd number, and the weights are given by \( u_1^c, u_2^c, v^c, w_1^c \) and \( w_2^c \):

\[
\begin{align*}
  u_1^c &= \frac{1}{t_c} \sin(-2\sigma\pi) \sin(\sigma\pi + (1 - \sigma)\theta) \\
  u_2^c &= \frac{1}{t_c} \sin(-2\sigma\pi) \sin((1 - \sigma)\theta) \\
  v^c &= \frac{1}{t_c} \sin(\sigma\pi + (1 - \sigma)\theta) \sin((1 - \sigma)\theta) \\
  w_1^c &= \frac{1}{t_c} \sin(\sigma\pi + (1 - \sigma)\theta) \sin((1 - \sigma)\theta - 2\sigma\pi) \\
  w_2^c &= \frac{1}{t_c} \sin(3\sigma\pi + (1 - \sigma)\theta) \sin((1 - \sigma)\theta)
\end{align*}
\]

where \( t_c = \sin(2\sigma\pi + (1 - \sigma)\theta) \sin(\sigma\pi - (1 - \sigma)\theta) \), then \( F_a(u_1^c, u_2^c, v^c, w_1^c, w_2^c) \) satisfies (6).

Proof. Let us consider all paths contributing to this equation for some fixed rhombus \( SENW \). We plan to divide walks into small groups and to prove that for each of these groups, the sum of the contributions is 0.
Without loss of generality, let us focus on walks visiting $SENW$ first by $z_{SE}$ (and possibly some other edges of $SENW$ afterwards). They can be divided into the following groups according to the behaviour outside $SENW$ (see fig. 2):

- outside $SENW$ the walk $\gamma$ is just the path from $a$ to $z_{SE}$
- outside $SENW$ the walk $\gamma$ is the union of the path from $a$ to $z_{SE}$ and the path from $z_{NW}$ to $z_{NE}$
- outside $SENW$ the walk $\gamma$ is the union of the path from $a$ to $z_{SE}$ and the path from $z_{SW}$ to $z_{NE}$
- outside $SENW$ the walk $\gamma$ is the union of the path from $a$ to $z_{SE}$ and the path from $z_{SW}$ to $z_{NE}$

Note that if the total contribution of paths in each group is zero then $F$ satisfies equation (6). At the same time, in each of these groups, paths differ one from another only inside the rhombus $SENW$. Hence, if the
Following equations hold, we obtain \([6]\):

\[
\begin{align*}
  &1 + \lambda \bar{\mu} e^{i\theta} u_2 - v - \lambda e^{i\theta} u_1 = 0 \\
  &\lambda \bar{\mu}^2 e^{i\theta} v - \mu u_2 - \lambda e^{i\theta} w_2 = 0 \\
  &-\lambda \mu e^{i\theta} v - \bar{\mu} u_1 + \lambda \bar{\mu} e^{i\theta} w_1 = 0, \\
  &-\lambda \mu e^{i\theta} u_2 - \mu^2 w_2 + \lambda \bar{\mu}^2 e^{i\theta} u_1 - \bar{\mu}^2 w_1 = 0
\end{align*}
\]

where \(\lambda = e^{-i\sigma \theta}\), \(\mu = e^{-i\sigma \pi}\). Solving this linear system, we obtain that \(\sigma = \frac{\ell}{8}\) where \(\ell\) is some odd number. For each \(\sigma\) and \(\theta\) parameters, weights satisfying equations \([12]-[15]\) are given by \([7]-[11]\). \(\square\)

For reasons which will become clear later, we are interested in \(\sigma = \frac{5}{8}\). In this case, weights given by \([7]-[11]\) can be rewritten as the weights given by \([1]-[5]\). For \(\frac{\pi}{3} < \theta \leq \frac{\pi}{2}\), all of them are positive and

\[
\begin{align*}
  &u_1^2 > w_1 \\
  &u_2^2 > w_2
\end{align*}
\]

For \(\theta < \frac{\pi}{3}\) we have \(u_1^2 < w_1\) and \(w_2 < 0\).

For \(\theta = \frac{\pi}{3}\) we have \(u_1 = \frac{1}{\sqrt{2+\sqrt{2}}}, u_2 = v = w_1 = u_1^2\) and \(w_2 = 0\). This case is in direct correspondence with the self-avoiding walk on the honeycomb lattice. We can divide a rhombus with angle \(\frac{\pi}{3}\) into two equilateral triangles. Then, all possible states of a rhombus can be viewed as states of these two triangles (see fig. [3]). If we do not permit our curve to have two long arcs in the same rhombus, there will be no more than one arc of our curve in each triangle. A rhombus is divided into triangles both of which have arcs if and only if there were either two short arcs in a rhombus or one long arc or one straight line.

\[\text{Figure 3. The bijection between self-avoiding walks on rhombi with angle 60° and self-avoiding walks on honeycomb lattice. Weights are indicated just below the corresponding mapping.}\]
Therefore, we just obtained the self-avoiding on the honeycomb lattice.

Let us take $x_c = u_1$, $u_1(x) = x$, $u_2(x) = x \cdot \frac{w_e}{x_c}$, $v(x) = x \cdot \frac{v_e}{x_c}$, $w_1(x) = x^2 \cdot \frac{w_1}{(x_c)^2}$ and $w_2(x) = x^2 \cdot \frac{w_2}{(x_c)^2}$. One can observe that for any $x$ and any self-avoiding walk $\gamma$ of length $n$ holds $\omega(\gamma) = x^n \omega_c(\gamma)$, where $\omega_c(\gamma)$ is the weight of $\gamma$ for $x = x_c$. In order to prove Theorem 1 we need just to show that the radius of convergence of $Z(x) = Z(x, u_2(x), v(x), w_1(x), w_2(x))$ is $x_c$. To prove that $Z(x) < \infty$ for $x < x_c$, we will use a standard bridge decomposition. The arguments proving $Z(x_c) = \infty$ are more subtle and use local relations. It is clear that $F_a$ with parameters $x, u_2(x), v(x), w_1(x), w_2(x)$ satisfies equation (6) at $x = x_c$. Usually these equations are referred to as half of discrete Cauchy-Riemann equations. The reason is that equations (6) say that the function has rotor equal to zero, which is a part of Cauchy-Riemann equations. And the part which is missing is that the divergence of the function is equal to zero, i.e. some equations around vertices.

![Figure 4. Parallelogram R and self-avoiding walks to its sides.](image-url)

Now, let us consider a parallelogram $R$ with angles $\theta$ and $\pi - \theta$, divided into congruent rhombuses (see fig. 4). By $\alpha, \beta, \delta, \varepsilon$ we denote the left, the right, the top and the bottom sides of $R$ (angle between alpha and $\delta$ is equal to $\theta$). Let $2L + 1$ be the number of rhombuses touching $\alpha$ and $T$ be the number of rhombuses touching $\delta$. The origin $a$ will be in
the middle of its left side. We will use the following notations:

\[ A_{T,L}(x) = \sum_{\gamma:a \to z \in \alpha} \omega_x(\gamma) \]
\[ B_{T,L}(x) = \sum_{\gamma:a \to z \in \beta} \omega_x(\gamma) \]
\[ D_{T,L}(x) = \sum_{\gamma:a \to z \in \delta} \omega_x(\gamma) \]
\[ E_{T,L}(x) = \sum_{\gamma:a \to z \in \varepsilon} \omega_x(\gamma), \]

where \( \omega_x(\gamma) \) is the weight of \( \gamma \) if the weights of the plaquettes are \( x, u_2(x), v(x), w_1(x) \) and \( w_2(x) \).

**Lemma 3.** For \( c_\alpha = \cos \frac{3\pi}{8}, c_\delta = \cos \left( -\frac{3\pi}{8} \right) \) and \( c_\varepsilon = \cos \left( \frac{3\pi}{8}(\pi - \theta) \right) \)

(18) \quad c_\alpha A_{T,L}(x_c) + B_{T,L}(x_c) + c_\delta D_{T,L}(x_c) + c_\varepsilon E_{T,L}(x_c) = 1.

**Proof.** To obtain this equation we need sum 6 over all rhombi in the parallelogram \( R \). \( \square \)

Note that all three coefficients \( c_\alpha, c_\delta \) and \( c_\varepsilon \) are positive (here we use that \( \sigma = \frac{\pi}{8} \)).

For \( x \leq x_c \) consider \( A_T(x), B_T(x), D_T(x) \) and \( E_T(x) \):

\[ A_T(x) = \lim_{L \to \infty} A_{T,L}(x) \]
\[ B_T(x) = \lim_{L \to \infty} B_{T,L}(x) \]
\[ E_T(x) = \lim_{L \to \infty} c_\delta D_{T,L}(x) + c_\varepsilon E_{T,L}(x). \]

The two first limits exist because \( A_{T,L}(x) \) and \( B_{T,L}(x) \) are increasing in \( L \) and bounded by 1 (see [18]), and the third one because [18] relates it to the limits of \( A_{T,L}(x) \) and \( B_{T,L}(x) \). We thus obtain

\[ c_\alpha A_T(x_c) + B_T(x_c) + E_T(x_c) = 1. \]

**Lemma 4.** \( Z(x_c) = \infty. \)

**Proof.** Note that if \( E_T(x_c) > 0 \), then \( Z(x_c) > \sum_L \frac{E_{T,L}(x_c)}{c_\varepsilon} = \infty. \) From now on, we may assume that \( E_T(x_c) = 0. \) In this case,

(19) \quad c_\alpha A_T(x_c) + B_T(x_c) = 1.

Note that \( B_T(x_c) - B_{T+1}(x_c) = c_\alpha (A_{T+1}(x_c) - A_T(x_c)) \). But \( A_{T+1}(x_c) - A_T(x_c) \) is the sum of weights of all the self-avoiding paths in the strip with width \( T + 1 \) beginning at \( a \) and ending on the left side of the strip, which also touch the right side. Each of these paths \( \gamma \) can be
Proof. Let us consider we split \( \gamma \) right after the first time \( \gamma \) touches the right side.

At this point, one must be aware that the weight of \( \gamma \) is not the product of weights of \( \gamma_1 \) and \( \gamma_2 \) since squares having two arcs of \( \gamma \) may have one arc of \( \gamma_1 \) and one arc of \( \gamma_2 \). These squares contribute \( w_c \) to \( \omega(\gamma) \) and \( x^2 \) to \( \omega(\gamma_1) \omega(\gamma_2) \). Nevertheless, the inequalities (16) and (17) imply that \( \omega(\gamma) \leq \omega(\gamma_1) \omega(\gamma_2) \).

Note that \( \gamma_1 \) has winding equal to either \( \theta \) or \( \pi - \theta \). In the first case, we obtain \( \gamma_1' \) by concatenating a short arc to \( \gamma_1 \) and \( \gamma_2' \) — by concatenating \( \gamma_2 \) to a long arc. In the second case, we concatenate a long arc to \( \gamma_1 \) and \( \gamma_2 \) to a short arc. In both cases, \( \omega(\gamma_1') \omega(\gamma_2') = \omega(\gamma_1) \omega(\gamma_2) x_2 u_2^c \leq \omega(\gamma) x_2 u_2^c \). Paths \( \gamma_1' \) and \( \gamma_2' \) are self-avoiding bridges starting at some particular points — \( \gamma_1' \) starts at \( a \) and \( \gamma_2' \) starts at the end point of \( \gamma_1' \). This leads to the inequality

\[
A_{T+1}(x_c) - A_T(x_c) \leq (B_{T+1}(x_c))^2/(x_c u_2^c).
\]

Using (19), we obtain a lower bound on the growth of \( B_T(x_c) \):

\[
B_T(x_c) - B_{T+1}(x_c) \leq \frac{c_0}{x_c u_2^c} \cdot (B_{T+1}(x_c))^2
\]

\[
\frac{c_0}{x_c u_2^c} \cdot (B_{T+1}(x_c))^2 + B_{T+1}(x_c) \geq B_T(x_c)
\]

\[
B_T(x_c) \geq \frac{1}{T} \min(B_1(x_c), \frac{x_c u_2^c}{c_0}).
\]

Hence, \( Z(x_c) \geq \sum_T B_T(x_c) = \infty \) since the harmonic series diverges.

We are now ready to prove Theorem 1.

Proof. Let us consider \( x < x_c \). Any self-avoiding walk can be decomposed into bridges, no three of which have the same length (see 11). At the same time, the weight of the walk is not greater than the product of weights of these bridges. Hence

\[
Z(x) = Z(x, u_2(x), v(x), w_1(x), w_2(x)) \leq 2 \prod_{T>0} (1 + B_T(x))^2.
\]

It is clear that \( B_T(x) \leq (\frac{x}{x_c})^T \cdot B(x_c) \leq (\frac{x}{x_c})^T \). Thus \( Z(x) \leq \prod_{T>0} (1 + (\frac{x}{x_c})^T) < \infty \). Hence, we know that \( Z(x) < \infty \) for \( x < x_c \). From Lemma 1 we obtain \( Z(x_c) = \infty \).
It is clear that \( Z(x) = Z(x, u_2(x), v(x), w_1(x), w_2(x)) = \sum_{n \geq 0} \tilde{c}_n x^n. \) Because of \( \tilde{c}_n \geq (1 + \frac{n}{x})^n \) (one can walk always up-right), and the sub-multiplicativity \( \tilde{c}_{n+m} \leq \tilde{c}_n \tilde{c}_m \) (any path of length \( n + m \) can be divided into a path of length \( n \) and a path of length \( m \): the weight of original path is not greater than the product of weights of these shorter paths since \( \theta \geq \frac{\pi}{3} \)), there exists \( \tilde{\mu} \in (0, \infty) \) such that \( \tilde{\mu} = \lim_{n \to \infty} \tilde{c}_n^{1/n}. \)

From \( Z(x_c) = \infty \) and \( Z(x) < \infty \) for \( x < x_c \), it follows that \( \tilde{\mu} = \frac{1}{x_c}. \)

**Appendix: Critical weights for \( O(n) \) model.**

We consider a loop representation of \( O(n) \) model on any finite simply connected rhombic tiling. The configuration in each rhombus is one of those mentioned in fig. 1 and we consider only the configurations which can be decomposed into several loops. In this case the weight of the configuration is:

\[
\omega(\text{conf}) = \prod_{\text{rhombus } r} \omega(r) \cdot n^{\#\text{loops}},
\]

where \( \omega(r) \) is the weight of rhombus \( r \), i.e. either 1 or one of \( u_1, u_2, v, w_1, w_2 \). We can also add boundary conditions — allow paths going from one particular edge on the boundary to another.

Now let us take any \( s \) and \( n = -2 \cos \frac{4\pi}{3} s \). We consider the following family of weights parametrized by \( s \) and angle \( \theta \) of the rhombus:

\[
\begin{align*}
u_1 &= \frac{1}{t} \cdot \sin (\pi - \theta)s \cdot \sin \frac{2\pi}{3} s \\
u_2 &= \frac{1}{t} \cdot \sin \theta s \cdot \sin \frac{2\pi}{3} s \\
v &= \frac{1}{t} \cdot \sin \theta s \cdot \sin (\pi - \theta)s \\
w_1 &= \frac{1}{t} \cdot \sin (\frac{2\pi}{3} - \theta)s \cdot \sin (\pi - \theta)s \\
w_2 &= \frac{1}{t} \cdot \sin (\theta - \frac{\pi}{3})s \cdot \sin \theta s,
\end{align*}
\]

where

\[
t = \frac{\sin^3 \frac{2\pi}{3} s}{\sin \frac{\pi}{3} s} + \sin (\theta - \frac{\pi}{3})s \cdot \sin (\frac{2\pi}{3} - \theta)s.
\]

**Proposition 5.** These weights satisfy Yang-Baxter equation and parafermionic observable \( F \) with spin \( \sigma \) equal to \( s+1 \) on any rhombus \( SENW \) satisfies the following equation:

\[
F_a(z_{SE}) + e^{i\theta} F_a(z_{NE}) - F_a(z_{NW}) - e^{i\theta} F_a(z_{SW}) = 0.
\]
The proof can be done by local transformations in the same way as the proof of Lemma 6. One should just keep in mind that there are more different local configuration in this case, because we can have loops.

**Remark 6.** The weights are symmetric in $\theta$ — if one takes $\pi - \theta$ instead of $\theta$ then $v$ is the same, $x_1$ and $x_2$ are exchanged and $w_1$ and $w_2$ are exchanged.

One can see that for $\theta = \frac{\pi}{3}$ and any $s$ the weights can be factorized, i.e. $w_1 = v = u_2 = u_1^2$, $w_2 = 0$ and $u_1 = x_c = \frac{1}{\sqrt{2+\sqrt{2-n}}}$. So this is just the $O(n)$ model on the honeycomb lattice with the weight for each edge equal to $x_c = \frac{1}{\sqrt{2+\sqrt{2-n}}}$. Nienhuis nonrigorously derived this to be the critical value for $O(n)$ model on the honeycomb lattice.

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