Chance-Constrained AC Optimal Power Flow: Reformulations and Efficient Algorithms

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Abstract—Higher levels of renewable electricity generation increase uncertainty in power system operation. To ensure secure system operation, new tools that account for this uncertainty are required. In this paper, we formulate a chance-constrained AC optimal power flow problem, which guarantees that generation, power flows and voltages remain within their bounds with a predefined probability. We then propose an accurate, yet tractable analytical reformulation of the chance constraints. The reformulation maintains the full, non-linear AC power flow equations for the forecasted operating point, and models the impact of uncertainty through a linearization around this point. We discuss different solution algorithms, including one-shot optimization with and without recourse, and an iterative algorithm which enables scalable implementations. We further discuss how more general chance constraint reformulations can be incorporated within the iterative solution algorithm.

In a case study based on four different IEEE systems, we compare the performance of the solution algorithms, and demonstrate scalability of the iterative scheme. We further show that the analytical reformulation accurately and efficiently enforces chance constraints in both in- and out-of-sample tests, and that the analytical approach outperforms two alternative, sample based chance constraint reformulations.

I. INTRODUCTION

Over the last decade, energy production from wind and solar power has reached significant levels in many countries across the world. For power system operators, the integration of renewable energy poses a variety of challenges, from long-term generation adequacy to a reduction of system rotational inertia. In this paper, we address the question of how to assess and mitigate the impact of forecast uncertainty from renewable generation in day-to-day operational planning.

A fundamental problem in the operational planning is the Optimal Power Flow (OPF) problem, an optimization problem commonly involved in market clearing and security assessment processes. Most OPF problems aim at minimizing operational cost while ensuring secure operation by enforcing constraints such as transmission capacity, voltage and generation limits. Traditionally, the OPF has been formulated as a deterministic problem. However, with more renewable generation, it becomes increasingly important to account for forecast uncertainty and treat the OPF as a stochastic problem.

Consequently, a wide variety of approaches and methods to account for uncertainty within the OPF have recently been proposed in literature. These include, among others, robust and worst-case methods [1]–[4], two- and multi-stage stochastic programming based on samples [5]–[8] or stochastic approximation techniques [9], and chance-constrained formulations [10]–[13]. In this paper, we work with chance constraints, which ensure that the system constraints will be satisfied with a desired probability. Discussions with transmission system operators [19] revealed that choosing a probabilistic security level is perceived as relatively intuitive and transparent. Chance constraints also align well with several industry practices, such as the probabilistic reserve dimensioning in ENTSO-E [20] or the reliability margins in the European market coupling [21].

While chance constraints offer an intuitive way of limiting risk from forecast uncertainty, the resulting optimization problems are generally hard to solve. Most literature has so far considered the linear DC power flow approximation, however, many applications, such as distribution system optimization or transmission system security assessment, require the more accurate AC power flow equations. Formulations that have attempted to solve the full Chance-Constrained AC OPF (AC CC-OPF) include [13]–[18]. In [13], the problem is solved using an iterative gradient calculation and numerical integration, while [14] employs an iterative approach based on the cumulant method and Cornish-Fischer expansion. AC CC-OPF based on linearized equations has been used to find optimal redispatch schedules [15] and solve voltage-constrained OPF in distribution grids [16]. The probably the most comprehensive AC CC-OPF formulation to date is provided in [17], which is based on a convex relaxation of the AC power flow equations and the sample-based approach to reformulate the chance constraint. However, the high computational requirements limits the approach to small systems. Indeed, all of the above AC CC-OPF methods have only been applied to small test cases, signalling a need for scalable approaches [22].

In this paper, we formulate the AC CC-OPF based on partial linearization of the AC power flow equations. We include the full AC power flow equations for the forecasted operating point. Assuming that the forecast errors are small, we then model the impact of uncertainty using a linearization around this point. With this partial linearization, we apply the analytical chance constraint reformulation proposed in [11], [12], which is applicable to both Gaussian and partially unknown uncertainty distributions [23]. Note that this approach is more accurate than the full linearization in [13]–[16], and does not require a relaxation as in [17]. Similar partial linearization methods were applied to stochastic load flow, see [24] and discussion therein, and were also adopted in the risk-based OPF in [25]. The linearization method was first proposed for chance-constraints in [26], and preliminary results for the application to the AC CC-OPF were presented in [18].

In addition to the analytical chance constraint reformulation, we also discuss different algorithms to solve the problem. We show that a one-shot optimization allows us to determine the

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optimal system response to uncertainty, which has been shown to reduce cost of integrating uncertain resources [27]. We also demonstrate that the iterative algorithm from [18] both allows for scalable implementations, and that it enables more general reformulations of the chance constraints. In particular, we propose two alternative reformulations, based on a Monte-Carlo simulation and the sample-based scenario approach. We use these implementations to benchmark the performance of the proposed analytical reformulation.

In summary, the contributions of this paper are the following: 1) We use a more comprehensive model than [18], including uncertainty of generator active and reactive power outputs. 2) We propose and compare different ways of solving the AC CC-OPF. We implement a one-shot optimization with and without optimal uncertainty response, and compare it with the iterative algorithm from [18]. We demonstrate the scalability of the iterative algorithm by solving the large Polish test case with 941 uncertain loads in 32s on a desktop computer. 3) We extend the iterative algorithm from [18] to more general chance-constraint reformulations based on a Monte-Carlo simulation and the scenario approach. In the case study, we show that the analytical reformulation based on a Gaussian uncertainty is accurate both for in- and out-of-sample tests, and outperforms the sample-based reformulations.

The remainder of the paper is structured as follows. We describe the uncertainty modelling in Section II and the AC OPF formulation with chance constraints and the analytical reformulation in Section III. Section IV describes the one-shot and iterative solution approach, while Section V discusses alternative chance constraint reformulations. In Section VI, we present the case study to assess the performance of the proposed method, before Section VII summarizes and concludes.

II. POWER SYSTEM MODELING UNDER UNCERTAINTY

We consider a power system where $\mathcal{N}$, $\mathcal{L}$ denote the set of nodes and lines, and $|\mathcal{N}| = m$ and $|\mathcal{L}| = l$. The set of nodes with uncertain demand or production of energy is given by $\mathcal{U} \subseteq \mathcal{N}$, while the set of conventional generators is denoted by $\mathcal{G}$. To simplify notation, we assume that there is one conventional generator with active and reactive power outputs $p_{G,i}$, $q_{G,i}$, one composite uncertainty source $p_{L,i}$, $q_{L,i}$, and one demand $p_{D,i}$, $q_{D,i}$ per node, such that $|\mathcal{G}| = |\mathcal{U}| = |\mathcal{N}| = m$. To model generation and control at different types of buses, we add subscripts $PQ$, $PV$, and $\theta V$ to distinguish between PQ, PV and $\theta V$ (reference) buses.

1) Uncertainty Modelling: The deviations in demand or production at any given node can be due to, e.g., load fluctuations, forecast errors for wind or PV or intra-day electricity trading. We model the deviations in active power as the sum of the forecasted value $p_U$ and a zero mean fluctuation $\omega$.

$$\tilde{p}_U = p_U + \omega .$$

We assume a constant power factor $\cos \phi$ for the uncertain injections, such that the reactive power injections are given by

$$\tilde{q}_U = q_U + \gamma \omega , \quad \text{where } \gamma = \sqrt{(1 - \cos^2 \phi) / \cos^2 \phi} .$$

The variable $\gamma$ will be referred to as the power ratio in the following. The reactive power injections could also be modelled in other ways, e.g. assuming that the reactive power injections remain constant, that the reactive power can be dispatched (at least partially) independent of the active power production or that some uncertainty sources (e.g., large wind farms) participate in controlling the voltage at their point of connection. These types of control can be included in the formulation without any conceptual changes.

2) Generation and Voltage Control: The generation dispatch $p_U$, $q_U$ and voltage magnitudes $v$ are scheduled by the system operator. The controllable generators further adjust their reactive and active power outputs to ensure power balance and maintain the desired voltage profile during fluctuations. We assume that active power is balanced by the Automatic Generation Control (AGC) [28], and that the total power mismatch $\Omega = \sum_{i \in \mathcal{U}} \omega_i$ is divided among generators according to participation factors $\alpha$. The change in the active power losses, which is a-priori unknown and denoted by $\delta p$, is balanced out by the generator at the reference bus.

$$\tilde{p}_{G,i}(\omega) = p_{G,i} - \alpha_i \Omega, \quad \forall i \in \mathcal{G}, p_Q, q_V,$$

$$\tilde{q}_{G,i}(\omega) = q_{G,i} - \alpha_i \Omega + \delta q_i, \quad \forall i \in \mathcal{G}, q_V.$$

Conversely, the voltage magnitude is fixed at the reference and PV buses, but varies at PQ buses:

$$\tilde{v}_j(\omega) = v_j + \delta v_j, \quad \forall j \in \mathcal{N}_{PQ},$$

$$\tilde{v}_j(\omega) = v_j, \quad \forall j \in \mathcal{N}_{PV}, \mathcal{N}_{\theta V} .$$

3) Power flows: Assuming a thermally constrained power system, we model the power flows in terms of currents, with $i_{ij}$ being the current magnitude on line $ij$. Changes in the power injections induce a change $\delta i_{ij}$ in the current magnitudes, i.e.,

$$i_{ij}(\omega) = i_{ij} + \delta i_{ij}, \quad \forall i \in \mathcal{L} .$$

III. CHANCE-CONSTRAINED AC OPTIMAL POWER FLOW

This section first presents the AC CC-OPF and discusses the interpretation of the chance constraints. We then explain how a closed-form, approximate reformulation can be obtained, based on the full AC power flow for the forecasted operating point and a linearization around that point for the deviations.

A. Original Chance-Constrained Problem

We state the full AC CC-OPF as

$$\min_{p_{G},q_{G},v,\theta,\omega,\gamma,\alpha} \sum_{i \in \mathcal{G}} \left( c_2, p_{G,i}^2 + c_1, p_{G,i} + c_0,i \right) \tag{3a}$$

subject to

$$f \left( \tilde{\theta}(\omega), \tilde{\theta}(\omega), \tilde{p}(\omega), \tilde{q}(\omega) \right) = 0, \quad \forall \omega \in \mathcal{D} \tag{3b}$$

$$P \left( \tilde{p}_{G,i}(\omega) \leq p_{G,i}^{max} \right) \geq 1 - \epsilon_p, \quad \forall i \in \mathcal{G} \tag{3c}$$

$$P \left( \tilde{p}_{G,i}(\omega) \geq p_{G,i}^{min} \right) \geq 1 - \epsilon_p, \quad \forall i \in \mathcal{G} \tag{3d}$$
\[
\begin{align*}
\mathbb{P}(\hat{q}_{G,i}(\omega) & \le q_{G,i}^{\text{max}}) \ge 1 - \epsilon_Q, \quad \forall i \in G \quad (3e) \\
\mathbb{P}(q_{G,i}(\omega) & \ge q_{G,i}^{\text{min}}) \ge 1 - \epsilon_Q, \quad \forall i \in G \quad (3f) \\
\mathbb{P}(\hat{v}_j(\omega) \le v_j^\text{max}) & \ge 1 - \epsilon_V, \quad \forall j \in N \quad (3g) \\
\mathbb{P}(\hat{v}_j(\omega) \ge v_j^\text{min}) & \ge 1 - \epsilon_V, \quad \forall j \in N \quad (3h) \\
\mathbb{P}(i_{ij}(\omega) \le i_{ij}^\text{max}) & \ge 1 - \epsilon_I, \quad \forall i,j \in E \quad (3i) \\
\theta_{\text{bus}} &= 0 \quad (3j)
\end{align*}
\]

Eq. (3a) minimizes the cost of active power generation, with \( c_2, c_1 \) and \( c_0 \) being the quadratic, linear and constant cost coefficients. Eq. (3b) are the AC power balance constraints for all possible \( \omega \) within the uncertainty set \( \mathcal{D} \). These are functions of the nodal voltage angles \( \theta(\omega) \) and magnitudes \( v(\omega) \), as well as the nodal injections of active \( \hat{p}(\omega) \) and reactive power \( \hat{q}(\omega) \). The nodal power injections are the sum of the power injections from generator, loads and uncertainty sources,
\[
\begin{align*}
\hat{p}(\omega) &= \hat{p}_G(\omega) + p_D + u + \omega, \quad (4) \\
\hat{q}(\omega) &= \hat{q}_G(\omega) + q_D + q_W + \gamma \omega. \quad (5)
\end{align*}
\]

The remaining constraints are generation constraints for active and reactive power \((3c)-(3j)\), constraints on the voltage magnitudes at each bus \(3g, 3h\) and transmission constraints in the form of limits on the current magnitudes \( i(\omega) \) \(3i\). All these constraints are formulated as chance constraints with acceptable violation probabilities of \( \epsilon_P, \epsilon_Q, \epsilon_V \) and \( \epsilon_I \), respectively. For the generation constraints, that are in reality hard constraints, a constraint violation would indicate a situation in which regular control actions would leave the system unbalanced and manual intervention would be needed. For the current and voltage constraints, a constraint violation would indicate the occurrence of under- or over-voltages and transmission line overloads, which can either be tolerated if the magnitude and duration are not too large, or removed through additional control actions. Finally, the voltage angle at the reference bus is set to zero by \(3j\).

### B. Tractable Problem Reformulation

The problem \((3)\) is not tractable in its current form. First, \((3b)\) is semi-infinite, as the set of fluctuations \(\mathcal{D}\) is uncountable. Further, the chance constraints \(3e-3j\) must be reformulated into deterministic constraints. We now suggest a reformulation of \((3a)-(3j)\), based on the following main ideas:

- For the forecasted operating point, given by \(\omega = 0\) and the scheduled nodal power injections \(p, q\), we solve the full AC power flow equations. This ensures an accurate, AC feasible solution for the forecasted system state.
- The impact of the uncertainty \(\omega\) is modelled using a first-order Taylor expansion around the forecasted operating point, which can be assumed to be reasonably accurate when the fluctuations \(\omega\) are small compared to the overall scheduled power injections \(|\omega|\ll\sum_{i\in N} P_i\).

Following the above ideas, we replace \((3a)-(3j)\) by a single set of deterministic equations for the forecasted operating point,
\[
f(\theta, v, p, q) = 0, \quad (6)
\]
where \(\theta, v\) are the voltage angles and magnitudes corresponding to the scheduled injections \(p, q\). In the following, we will denote this operating point by \(x = (\theta, v, p, q)\). We then define sensitivity factors with respect to the fluctuations \(\omega\), i.e.,
\[
\Gamma_P = \frac{\partial p}{\partial \omega}(x), \quad \Gamma_V = \frac{\partial v}{\partial \omega}(x), \quad \Gamma_Q = \frac{\partial q}{\partial \omega}(x), \quad \Gamma_I = \frac{\partial i}{\partial \omega}(x).
\]

Here, \(\Gamma_P, \Gamma_Q\) denote the sensitivity factors for the active and reactive power injections, respectively, while \(\Gamma_V, \Gamma_I\) are the sensitivity factors for the voltage and current magnitudes. The derivation of the sensitivity factors \(\Gamma_P, \Gamma_Q, \Gamma_V, \Gamma_I\) can be found in \([29]\). Note that they depend non-linearly on the forecasted operating point \(x\), and linearly on the participation factors \(\alpha\) and the power ratio \(\gamma\).

The sensitivity factors allow us to approximate the chance constraints as linear functions of the random variables \(\omega\), e.g., for the current constraint \((3i)\) we obtain
\[
\mathbb{P}(i_{ij} + \Gamma_{l(i,j)}\omega \le i_{ij}^{\text{max}}) \ge 1 - \epsilon_I, \quad \forall i,j \in E \quad (7)
\]

The linear dependence on \(\omega\) enables the use of an analytical chance constraint reformulation \([11, 18, 26]\), even though the sensitivity factors \(\Gamma_P, \Gamma_Q, \Gamma_V, \Gamma_I\) and the current magnitudes \(i_{ij}\) are non-linear functions of the decision variables. Assuming that the fluctuations \(\omega\) follows a multivariate normal distribution \([11]\) with zero mean and covariance matrix \(\Sigma_W\), we obtain the following expression for \((7)\),
\[
i_{ij} + \Phi^{-1}(1-\epsilon_I) \|\Gamma_{l(i,j)}\Sigma_W^{1/2}\|_2 \le i_{ij}^{\text{max}}. \quad (8)
\]

Here, \(\Phi^{-1}(1-\epsilon_I)\) represents the inverse cumulative distribution function of the standard normal distribution, evaluated at \(1 - \epsilon_I\). We observe that the consideration of uncertainty introduce an uncertainty margin, i.e., a tightening of the forecasted current constraint which is necessary to secure the system against uncertainty. Denoting this uncertainty margin by \(\lambda_I\), we rewrite \((8)\) as
\[
i_{ij} \le i_{ij}^{\text{max}} - \lambda_I, \quad \text{with} \quad \lambda_I = \Phi^{-1}(1-\epsilon_I) \|\Gamma_{l(i,j)}\Sigma_W^{1/2}\|_2 \quad (9)
\]

### C. Reformulated Chance-Constrained Problem

Applying the analytical reformulation to all chance constraints and using the definition of the uncertainty margins from above, we express the AC CC-OPF \((3)\) as
\[
\min_{x, p_G, q_G, \alpha, \gamma} \sum_{i \in G} (c_2, p_{G,i}^2 + c_1, p_{G,i} + c_0,i) \quad (11a)
\]
\[
\text{s.t.} \quad f(\theta, v, p, q) = 0 \quad (11b)
\]
\[
p_{G,i}^{\text{min}} \le p_G \le p_{G,i}^{\text{max}} - \lambda_P \quad (11c)
\]
\[
q_{G,i}^{\text{min}} + \lambda_Q \le q_G \le q_{G,i}^{\text{max}} - \lambda_Q \quad (11d)
\]
\[
v_{G,i}^{\text{min}} + \lambda_V \le v \le v_{G,i}^{\text{max}} - \lambda_V \quad (11e)
\]
\[
i_i \le i_{ij}^{\text{max}} - \lambda_I \quad (11f)
\]
\[
\theta_{\text{bus}} = 0 \quad (11g)
\]

The uncertainty margins for currents \(\lambda_I\) are defined by \((10)\), while the voltage uncertainty margins \(\lambda_V\) are given by
\[
\lambda_{V,j} = f_P^{-1}(1 - \epsilon_V) \|\Gamma_{V(j,\cdot)}\Sigma_W^{1/2}\|_2 \quad \forall j \in N_{PV}, \quad (12)
\]
\[
\lambda_{V,j} = 0 \quad \forall j \in N_{PV}, \quad (13)
\]

More general distributions can be accounted for without significant changes to the approach \([28]\).
and the $\lambda_P$, $\lambda_Q$ for active and reactive power are given by

$$\lambda_{P,i} = \alpha_i f_P(1 - \epsilon_P) \sigma_{P,i}, \quad \forall i \in G_{PQ}, \; g_{PV}, \quad (14a)$$

$$\lambda_{P,i} = f_P(1 - \epsilon_P) \|-(\alpha_i 1_{1,i} + \Gamma_{P(i,i)})^T W^1/2 \|_2, \quad \forall i \in G_{PV}, \quad (14b)$$

$$\lambda_{Q,i} = f_P(1 - \epsilon_P) \|\Gamma_{Q(i,i)}^T W^1/2 \|_2, \quad \forall i \in G_{PV}, \; g_{sv}, \quad (14c)$$

$$\lambda_{Q,i} = 0, \quad \forall i \in G_{PQ}, \quad (14d)$$

### IV. Solution Algorithms

We now discuss two solution algorithms for the AC CC-OPF described above.

#### A. One-Shot Optimization

The reformulated AC CC-OPF given by (11), (14) is a continuous, non-convex optimization problem, which can be solved directly by a suitable local solver. The solver optimizes the scheduled generation dispatch $p_G q_G$, while inherently accounting for the dependency of the sensitivity factors $\Gamma_P$, $\Gamma_Q$, $\Gamma_P$ and $\Gamma_V$ on the forecasted operating point $x$ and the participation factors $\alpha$, $\gamma$. By including $\alpha$ and $\gamma$ as optimization variables, it is thus possible to optimize not only the scheduled dispatch, but also the procurement of reserves and voltage control during deviations.

The drawback of attempting a one-shot solution is the problem complexity, which might lead to long solution times. The deterministic AC OPF is already a non-convex problem, and adding additional terms with complex dependencies on the decision variables only increases the difficulty of obtaining a solution. This can be a bottleneck for adoption in more realistic settings, where scalability and robustness of the OPF are important criteria.

#### B. Iterative Solution Algorithm

To address the problem of increased computational complexity, we use an iterative solution algorithm from [18] which allows us to obtain a solution to the AC CC-OPF (11), (14) using any existing AC OPF tool. The iterative solution algorithm is based on the observation that the uncertainty margins only occur in the uncertainty margins. The scheme thus alternates between solving the AC CC-OPF (11) and evaluating $\lambda$ based on (14). It is deemed to have converged when the uncertainty margins $\lambda$ no change between iterations.

Specifically, the algorithm consists of the following steps:

1) **Initialization:** Set uncertainty margins $\lambda^U_P = \lambda^L_Q = \lambda^L_P = \lambda^U_Q = 0$, and iteration count $\kappa = 0$.

2) **Solve AC OPF:** Solve the AC OPF defined by (11), and increase iteration count $\kappa = \kappa + 1$.

3) **Evaluate uncertainty margins:** Compute the uncertainty margins of the current iteration $\lambda^U_P$, $\lambda^L_Q$, $\lambda^U_Q$ and $\lambda^L_P$ based on (14). Then, evaluate the maximum difference to the last iteration for the power injections, voltages and currents,

$$\eta^U_P = \max \{|\lambda^U_P - \lambda^{\kappa-1}_P|\}, \quad \eta^L_P = \max \{|\lambda^L_P - \lambda^{\kappa-1}_P|\},$$

$$\eta^U_Q = \max \{|\lambda^U_Q - \lambda^{\kappa-1}_Q|\}, \quad \eta^L_Q = \max \{|\lambda^L_Q - \lambda^{\kappa-1}_Q|\},$$

4) **Check convergence:** Compare maximum difference with the stopping criterion $\eta$:

$$\eta^U_P \leq \eta^L_P, \quad \eta^L_Q \leq \eta^U_Q, \quad \eta^U_Q \leq \eta^L_V, \quad \eta^L_V \leq \eta^U_I . \quad (15)$$

If (15) holds, the algorithm has converged. If not, move back to step 2).

While the above algorithm is straightforward to implement and offers scalability, it has some drawbacks.

First, the uncertainty margins can only be computed as part of the outer loop, which takes the forecasted operating point and values for $\alpha$ and $\gamma$ as inputs. Since the uncertainty margins are not part of the inner loop optimization, the iterative AC CC-OPF does not define the operating point, $\alpha$ or $\gamma$ in a way that minimizes the uncertainty margins for congested lines or for buses with tight voltage constraints.

Second, the solution is not guaranteed to converge. In the simulations conducted for this paper, the algorithm converged within a few iterations when $\hat{\eta}_P$, $\hat{\eta}_Q$, $\hat{\eta}_V$ and $\hat{\eta}_I$ are chosen not smaller than 0.001 MVA for active and reactive power, and $10^{-5}$ p.u. for voltage and current magnitudes.

### V. Alternative Chance Constraint Formulations

When solving the problem using the iterative approach, the uncertainty margins $\lambda^U_P$, $\lambda^L_Q$, $\lambda^U_Q$ and $\lambda^L_I$ are calculated only in the outer iteration. This opens for the possibility of using other methods than the analytical reformulation to calculate the uncertainty margins in each iteration. Note that the convergence criterion remains the same, i.e., the iterative solution stops when the uncertainty margins no longer change.

1) **Uncertainty Margins from Monte Carlo Simulation:** The analytical uncertainty margins (14) represent an approximate quantile of the current, voltage and generation distributions. By running a Monte-Carlo simulation where we sample the uncertainty vector $\omega$ and calculate the resulting power flows for a large number of samples, we can compute an empirical distribution function and the corresponding quantiles to define Monte Carlo uncertainty margins. Since the AC power flow equations are non-linear and the samples might not follow a symmetrical distribution, we calculate the upper and lower uncertainty margins separately.

Using a voltage constraint as an example, a Monte Carlo simulation or numerical integration is used to determine the distribution function of the voltage around the forecast solution $v_i(x)$. We then determine the upper $(1 - \epsilon)$ and lower $\epsilon$ quantiles of the distribution, denoted by $v_i^{1-\epsilon}$, $v_i^\epsilon$, and calculate the constraint tightenings by

$$\lambda_{x,i}^U = v_i^{1-\epsilon} - v_i(x) \quad \text{and} \quad \lambda_{x,i}^L = v_i(x) - v_i^\epsilon. \quad (16)$$

2) **Uncertainty Margins for Joint Chance Constraints:** While the uncertainty margins (14) represent the quantiles of the respective constraints, they can more generally be interpreted as the margin which is necessary to secure the system against uncertainty. A similar constraint tightening can be observed in other stochastic and robust formulations of the OPF problem, e.g. in chance-constrained formulations based on joint chance constraints [10], [17], which ensure that all
constraints will hold jointly with a pre-described probability \( \epsilon_J \).

To reformulate the joint chance constraints, these papers rely on the scenario approach [30] or a robust version of it [31]. The scenario approach guarantees that the joint chance constraint will hold if all constraints are satisfied for a defined number \( N_S \) of randomly drawn samples. The required number of samples depends on the number of decision variables \( N_X \) and the acceptable joint violation probability \( \epsilon_J \) [32].

\[
N_S \geq \frac{2}{\epsilon_J} \left( \ln \frac{1}{\beta} + N_X \right).
\] (17)

The value \( 1 - \beta \) represents the confidence level for satisfying the chance constraint, and can be chosen to be very small.

Representing uncertainty sets based on samples, only applies when the underlying problem is convex, since convexity guarantees that the linear combination of feasible points remain within the convex, feasible set. In [17], this problem is solved by representing the AC power flow constraints through a convex relaxation. However, while the scenario approach is applicable for the relaxed problem, it does not provide guarantees for the actual AC problem.

Here, we take a different approach. In each iteration, we first solve (11) applying the full, non-convex AC power flow equations. We use a set of samples \( \mathcal{S} \), \(|\mathcal{S}| = N_S \) as prescribed by the scenario approach (17), and run an AC power flow simulation for each of the scenarios \( \hat{\omega} \in \mathcal{S} \). While the overall problem is not convex, we assume that the AC power flow solutions for each of the samples remain in vicinity of the local optimum, where the problem is known to be locally convex and the scenario approach holds. To ensure that the constraints hold for all samples, we define the uncertainty margin as the difference between the forecasted voltage magnitude and the highest/lowest observed magnitude, i.e.,

\[
\lambda^U_{V,i} = \max_{\omega \in \mathcal{S}} \{ v_i(\hat{\omega}) \} - v_i(x), \quad (18a)
\]

\[
\lambda^L_{V,i} = v_i(x) - \min_{\omega \in \mathcal{S}} \{ v_i(\hat{\omega}) \}. \quad (18b)
\]

3) Comparison of Uncertainty Margins: Fig. 1 compares the margins calculated with the analytical approach [14], the Monte Carlo simulation [16] and the scenario approach [18]. The histogram represents the empirical distribution of the voltage magnitudes, as calculated based on the samples used in the Monte Carlo simulation and the scenario approach. The analytical margins are symmetric, leading to a larger upper margin and a smaller lower margin compared with the empirical quantiles used for the Monte Carlo margins. The scenario approach has much larger uncertainty margins than either of the two other approaches, representing worst-case among the drawn scenarios. The uncertainty margins for the scenario approach are however not directly comparable to the analytical and Monte Carlo approach, since the scenario approach enforces a joint violation probability. In this example, the scenario approach guarantees a joint violation probability of \( \epsilon_J \leq 0.1 \), while the analytical and Monte Carlo approach limit the violation probability of each constraint to \( \epsilon \leq 0.01 \).

VI. CASE STUDY

In this case study, we demonstrate the performance of the AC CC-OPF in terms of operating cost, computational time and chance constraint satisfaction.

1) Investigated approaches: We compare the following versions of the AC CC-OPF:

A Standard AC OPF without consideration of uncertainty.

B One-Shot AC CC-OPF with both pre-determined and optimized uncertainty response \( \alpha, \gamma \).

B1 Fixed \( \alpha, \gamma \) where \( \alpha, \gamma \) are pre-defined.

B2 Optimized \( \alpha, \gamma \) where \( \alpha \) and \( \gamma \) are optimized with the generation dispatch.

C Iterative AC CC-OPF with \( \alpha, \gamma \) pre-defined, and different uncertainty margin definitions:

C1 Analytical with uncertainty margins defined by the closed-form expressions [14].

C2 Monte Carlo with empirical uncertainty margins obtained from a Monte Carlo simulation [16].

C3 Scenario Approach with empirical uncertainty margins obtained from the limiting scenarios [17].

2) Test systems: For all test cases, we assume that uncertainty is observed as fluctuations in the net load, with standard deviations given as a percentage of forecasted load. The uncertainty levels are chosen to obtain congested, but feasible test cases. We do not consider unit commitment, and hence set the lower generation limits to zero. We run simulations for four different test systems of different size:

IEEE RTS96 One Area Test Case provided with Matpower 5.1 [34], with the maximum generation limits increased by a factor of 1.5. All 17 loads are uncertain, with standard deviations equal to 10\% and zero correlation between loads.

IEEE 118 Bus Test System from the NICTA Energy System Test Archive [35]. The generation limits are increased 1.5, and the system is split into three zones as in [27]. We assume that all 99 loads are uncertain with standard deviation of 5\%, a correlation coefficient of \( \rho = 0.3 \) within each zone, and zero correlation between loads in different zones.

IEEE 300 Bus Test System from the NICTA Energy System Test Archive [35]. Loads with consumption between 0 and 100

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\textsuperscript{2}The relaxation might not be tight, in which case the solution to the relaxed power flow equations are not physical solutions to the AC power flow problem.

\textsuperscript{3}We might apply solution methods based on convex relaxation to find the solution for the AC OPF [31], but would check AC power flow feasibility of the resulting solution. Tightness of the relaxation can be assured, e.g., by applying higher-order moment relaxations proposed in [33].
MW are assumed to be uncertain, with standard deviations of 5% and zero correlation. This corresponds to 131 uncertain loads, and 21% of the total system demand.

3) Polish 2383 Bus, Winter Peak Test Case: from Matpower 5.1 [34]. The upper generation limits for active power are increased by a factor of 2, and the reactive power capability of generators at PV buses is increased by +/- 10 MVAR for all generators. All loads with consumption between 10 and 50 MW are assumed to be uncertain, with standard deviations equal to 10% and zero correlation. This corresponds to 941 uncertain loads, and 67% of the total system demand.

As a base case, we enforce all chance constraints with violation probabilities \( \epsilon_P = \epsilon_Q = \epsilon_V = \epsilon_I = 0.01 \). For the iterative solution algorithms we set \( \tilde{\eta}_P = \tilde{\eta}_Q = 0.001 \text{MVA}, \tilde{\eta}_V = 10^{-5} \text{p.u.} \) and \( \tilde{\eta}_I = 0.001 \text{kA} \). When \( \alpha, \gamma \) are predefined, we assume that \( \gamma \) is the forecasted power factor of the uncertainty sources and that the participation factors \( \alpha \) are given by \( \alpha_i = p_{G,i}^{\max} / \sum_{g \in G} p_{G,g}^{\max} \) as in [11].

A. Comparison of Solution Algorithms

We first compare the solutions obtained with the deterministic AC OPF (A), the iterative approach (B1), the one-shot optimization with fixed participation factors \( \alpha, \gamma \) (C1) and the one-shot optimization with \( \alpha, \gamma \) as decision variables (C2) in terms of cost and solution time. The results are listed in Table I. The computational times are based on our own AC OPF implementation, solved using KNITRO.

Due to the introduction of the uncertainty margin, the AC CC-OPF formulations have higher operational cost than the deterministic solution. Co-optimizing the participation factors \( \alpha, \gamma \) with the one-shot optimization (B2), leads to a lower increase in cost. Solving the problem using the one-shot solution approach can lead to better solutions, particularly when \( \alpha, \gamma \) are co-optimized. However, co-optimizing the uncertainty margins does however also significantly increases computational complexity and solution time. Already for the modestly sized 118 bus system, the one-shot optimization (B1) has a longer solution time than the iterative approach. The optimization (B2) does not converge, even after running for 13h on a desktop computer.

B. Scalability of the Iterative Approach

We now demonstrate how the iterative approach addresses the issue of scalability by utilizing existing OPF tools. We implement the iterative AC CC-OPF as an outer iteration on the standard Matpower 5.1 “runopf” function [34] with the default MIPS solver, and solve the problem for all four test systems. The resulting times and the number of iterations are shown in Table II. We also show the evolution of the generation cost between first, second and last iterations.

The problems converge within 4-5 iterations, and the solutions are obtained within half a minute on a standard desktop computer, even for the Polish test case with 941 uncertain loads and 2383 buses. Further, the main change in cost happens between the first and the second iteration, with only minor adjustments until final convergence. This implies that even if the problem does not converge, the solution at intermediate iterations might be valuable in itself.

C. Evaluation of Chance Constraint Reformulation

The approximate AC CC-OPF has two main sources of inaccuracy. First, the impact of uncertainty is approximated by a linearization. Second, the analytical reformulation assumes a normal distribution, which might be an inaccurate description of the true uncertainty distribution. To assess the accuracy of the analytical reformulation, we solve the problem and compare the pre-described acceptable violation probability with observed violation probabilities from a Monte Carlo simulation. The comparisons are based on the RTS96 system, and the AC CC-OPF is solved using the iterative approach with Matpower and MIPS.

1) Accuracy of the Linearization: In-Sample Test: To assess the accuracy of the linearization, we assume perfect knowledge of the distribution and run an in-sample test based on 10’000 samples from the multivariate normal distribution. Based on these samples, we assess the empirical violation probabilities through a Monte Carlo simulation. We perform the assessment for different standard deviations \( \sigma_W = \{0.075, 0.1, 0.125\} \) corresponding to fluctuations of different size, and different acceptable violation probabilities \( \epsilon_V = \epsilon_I = \{0.01, 0.05, 0.1\} \) (where a smaller \( \epsilon \) implies the estimation of quantiles that are further into the tail of the distribution and thus further away from the operating point). The acceptable violation probabilities of the generator constraints on active and reactive power are kept constant at \( \epsilon_P = \epsilon_Q = 0.01 \), as these constraints are not heavily influenced by the power flow equations.

Table III shows the results of the in-sample testing, with the maximum observed empirical violation probability \( \epsilon_{emp} \) for any constraint, as well as the joint violation probability (calculated based on the number of samples that have at least

| RTS96 | 118 Bus | 300 Bus | Polish |
|-------|---------|---------|--------|
| Buses | 24 | 118 | 300 | 2383 |
| Uncertain loads | 17 | 99 | 131 | 941 |
| Solution time | 0.54s | 1.15s | 3.37s | 31.89s |
| Iterations | 5 | 4 | 5 | 4 |
| Cost (1st) | 36 771 | 3504 1 | 16 779 | 787 987 |
| Cost (2nd) | 40 249 | 3575 5 | 17 173 | 802 529 |
| Cost (final) | 40 127 | 3575 3 | 17 143 | 802 238 |

| RTS96 | 118 Bus | 300 Bus | Polish |
|-------|---------|---------|--------|
| Cost (final) | 40 127 | 3575 3 | 17 143 | 802 238 |

Table II For the four different test cases (listed with system size and number of uncertain loads): solution times, number of iterations and generation costs in the first, second and final iterations.
one violated constraint). The upper part of the table shows the results for different standard deviations, and the lower part shows the results for varying values of $\epsilon$. Since the linearization is a better approximation close to the forecasted operating point, we observe that maximum violation probability is closer to the acceptable value for small standard deviations $\sigma_W$ and large acceptable violation probabilities $\epsilon$. For large standard deviations and small $\epsilon$, the maximum violation probability is higher than the acceptable value ($\epsilon_{\text{emp}} > \epsilon$) implying a violation of the chance constraint, whereas the method actually lead to conservative solutions ($\epsilon_{\text{emp}} < \epsilon$) for larger values of $\epsilon$. However, the empirical violation probability $\epsilon_{\text{emp}}$ is always within $\pm 0.01$ of the pre-described acceptable $\epsilon$.

We further observe that the AC CC-OPF limits the joint violation probability $\epsilon_j$ (i.e., the probability of observing at least one violation) to a relatively small percentage, even though it only aims at enforcing the separate chance constraints.

2) Accuracy of Normal Distribution: Out-Of-Sample Test:
To check whether the assumption of a normal distribution leads to accurate results, we perform an out-of-sample test with power injection samples based on the historical data from Austrian Power Grid. We assign one set of historical samples, in total 8492 data points, to each uncertain load. The samples are then rescaled to match the assumed standard deviation $\sigma_W$ and assumed correlation coefficient $\rho = 0$. We use 5000 samples for the evaluation of constraint violation probabilities.

The results of the out-of-sample testing based on APG historical data is shown in Table [V] including both the maximum violation probability for any individual constraint and the joint violation probability as above. Since the out-of-sample test includes inaccuracies both due to linearization errors and non-normally distributed samples, the maximum observed violation probability is higher in the out-of-sample test than in the in-sample test. The difference is however not particularly large, and the empirical violation probability $\epsilon_{\text{emp}}$ is still within $\pm 0.01$ of the pre-described acceptable $\epsilon$.

Since the inaccuracy due to the distribution assumption is on the same order of magnitude as the inaccuracy due to the linearization error, the normal distribution appears to be a reasonable model for current and voltage magnitudes, even though the power injections are not normally distributed.

Similarly, the joint violation probability observed with the APG data is slightly higher than in the in-sample test, but remains in the same range.

D. Comparison with Sample-Based Reformulations
Finally, we compare the analytical reformulation with the sample-based methods to define the uncertainty margins. For the analytical approach (C1), we assume a normal distribution and enforce $\epsilon = 0.01$. For the Monte Carlo simulation (C2), we enforce $\epsilon = 0.01$ and use 1000 samples from the APG data to calculate the empirical uncertainty margins. For the scenario approach (C3), we enforce $\epsilon_j$, which corresponds to a prescribed number of $N_S = 2465$ samples, which were taken from the APG data. To evaluate the actual violation probability, we use 5000 samples from the APG historical data.

In Table [V] the generation cost, solution time and number of iterations are listed. We also show the maximum empirical violation probability $\epsilon_{\text{emp}}$, and joint violation probability $\epsilon_J$.

We observe that the analytical (C1) and Monte Carlo (C2) approaches lead to relatively similar solutions. The Monte Carlo solution has slightly lower cost, but higher violation probabilities. The Monte Carlo thus leads to a more significant violation of the acceptable violation probability than the analytical reformulation, despite being based on the full AC power flow equations and no explicit assumption about the distribution. This highlights the sensitivity of the Monte Carlo approach to the availability sufficiently many, high quality samples. A larger number of samples would lead to a better solution, but would also increase computational time. The Monte Carlo already requires a lot more computational resources than the analytical reformulation, with a solution time of 2min 7s compared with 0.5s.

The scenario approach (C3) has a higher cost than the other two solutions, but also significantly lowers violation probabilities. While the pre-described $\epsilon_j = 0.1$, the actual joint violation probability was only $\epsilon_{J,\text{emp}} = 0.007$. This
shows an important drawbacks of the scenario approach: While it guarantees chance constraint feasibility, the solution might be far from cost optimal.

VII. CONCLUSIONS

In this paper, we describe a model for the AC CC-OPF problem, as well as a method to reformulate the chance constraints into tractable, closed-form expressions. The reformulation method uses the full, non-linear AC power flow equations for the forecasted operating point, and a linearization around this point to model the impact of uncertainty. This partial linearization leads to a linear dependence on the random fluctuations and enables an analytical chance constraint reformulation based on a normal distribution assumption. We show that this analytical reformulation approach is reasonably accurate for the AC CC-OPF, even when the uncertain power injections are not normally distributed.

Different methods to solve the problem were proposed and demonstrated in the case study. Solving the problem in one-shot allows us to co-optimize the reaction to the fluctuations, which reduces cost, but also significantly increases computational complexity. To address the issue of scalability, we use an iterative solution method, where the chance constraints are enforced through an outer iteration on the deterministic AC OPF. By utilizing the standard Matpower package, we were able to solve the AC CC-OPF for the Polish test case with 2383 buses and 941 uncertain loads in half a minute.

We further used the iterative approach to enable two alternative, sample-based reformulations of the chance constraints, based on Monte Carlo simulations and the scenario approach. In the case study, we showed that the analytical reformulation had lower solution times and outperformed the Monte Carlo reformulation in terms of enforcing the chance constraint, and the scenario approach in terms of cost.

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