A generalised trapezoid type inequality for convex functions

S.S. Dragomir

Abstract. A generalised trapezoid inequality for convex functions and applications for quadrature rules are given. A refinement and a counterpart result for the Hermite-Hadamard inequalities are obtained and some inequalities for pdf’s and \((HH)\)–divergence measure are also mentioned.

1. Introduction

The following integral inequality for the generalised trapezoid formula was obtained in \([2]\) (see also \([1,\text{p. 68}]\)):

**Theorem 1.** Let \(f : [a, b] \to \mathbb{R}\) be a function of bounded variation. We have the inequality

\[
\left| \int_a^b f(t) \, dt - [(x - a) f(a) + (b - x) f(b)] \right| 
\leq \left[ \frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right] \sum_a^b (f),
\]

holding for all \(x \in [a, b]\), where \(\sum_a^b (f)\) denotes the total variation of \(f\) on the interval \([a, b]\).

The constant \(\frac{1}{2}\) is the best possible one.

This result may be improved if one assumes the monotonicity of \(f\) as follows (see \([1, \text{p. 76}]\))

**Theorem 2.** Let \(f : [a, b] \to \mathbb{R}\) be a monotonic nondecreasing function on \([a, b]\). Then we have the inequality:

\[
\left| \int_a^b f(t) \, dt - [(x - a) f(a) + (b - x) f(b)] \right| 
\leq (b - x) f(b) - (x - a) f(a) + \int_a^b \text{sgn} (x - t) f(t) \, dt
\]

\[
\leq (x - a) [f(x) - f(a)] + (b - x) [f(b) - f(x)]
\]

\[
\leq \left[ \frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right] [f(b) - f(a)]
\]
for all $x \in [a, b]$. The above inequalities are sharp.

If the mapping is Lipschitzian, then the following result holds as well $\mathbb{R}$ (see also [11 p. 82]).

**Theorem 3.** Let $f : [a, b] \rightarrow \mathbb{R}$ be an $L$-Lipschitzian function on $[a, b]$, i.e., $f$ satisfies the condition:

$$(L) \quad |f(s) - f(t)| \leq L|s - t| \quad \text{for any } s, t \in [a, b] \quad (L > 0 \text{ is given}).$$

Then we have the inequality:

$$(1.3) \quad \left| \int_a^b f(t) \, dt - [(x-a)f(a) + (b-x)f(b)] \right| \leq \left[ \frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] L$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is best in (1.3).

If we would assume absolute continuity for the function $f$, then the following estimates in terms of the Lebesgue norms of the derivative $f'$ hold [1, p. 93].

**Theorem 4.** Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then for any $x \in [a, b]$, we have

$$(1.4) \quad \left| \int_a^b f(t) \, dt - [(x-a)f(a) + (b-x)f(b)] \right| \leq \left\{ \begin{array}{ll} \frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \|f'\|_\infty & \text{if } f' \in L_\infty [a, b]; \\ \frac{1}{(q+1)^p} \left[ (x-a)^{q+1} + (b-x)^{q+1} \right]^{\frac{1}{q}} \|f'\|_p & \text{if } f' \in L_p [a, b], \\ \left[ \frac{1}{2} (b-a) + \left|x - \frac{a+b}{2}\right| \right] \|f'\|_1, & \end{array} \right.$$

where $\|\cdot\|_p (p \in [1, \infty])$ are the Lebesgue norms, i.e.,

$$\|f'\|_\infty = \text{ess sup}_{s \in [a, b]} |f'(s)|$$

and

$$\|f'\|_p := \left( \int_a^b |f'(s)| \, ds \right)^{\frac{1}{p}}, \quad p \geq 1.$$
2. The Results

The following theorem providing a lower bound for the difference

\[(x - a) f (a) + (b - x) f (b) - \int_a^b f (t) dt\]

holds.

**Theorem 5.** Let \(f : [a, b] \to \mathbb{R}\) be a convex function on \([a, b]\). Then for any \(x \in (a, b)\) we have the inequality

\[
\frac{1}{2} \left[ (b - x)^2 f'_+ (x) - (x - a)^2 f'_- (x) \right] 
\leq (x - a) f (a) + (b - x) f (b) - \int_a^b f (t) dt.
\]  

(2.1)

The constant \(\frac{1}{2}\) in the left hand side of (2.1) is sharp in the sense that it cannot be replaced by a larger constant.

**Proof.** It is easy to see that for any locally absolutely continuous function \(f : (a, b) \to \mathbb{R}\), we have the identity

\[
(x - a) f (a) + (b - x) f (b) - \int_a^b f (t) dt = \int_a^b (t - x) f' (t) dt
\]

for any \(x \in (a, b)\), where \(f'\) is the derivative of \(f\) which exists a.e. on \([a, b]\).

Since \(f\) is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any \(x \in (a, b)\), we have the inequalities:

\[
f'(t) \leq f'_- (x) \text{ for a.e. } t \in [a, x]
\]

and

\[
f'(t) \geq f'_+ (x) \text{ for a.e. } t \in [x, b].
\]

If we multiply \(2.3\) by \(x - t \geq 0, t \in [a, x]\) and integrate on \([a, x]\), we get

\[
\int_a^x (x - t) f' (t) dt \leq \frac{1}{2} (x - a)^2 f'_- (x)
\]

(2.5)

and if we multiply \(2.4\) by \(t - x \geq 0, t \in [x, b]\) and integrate on \([x, b]\), we also have

\[
\int_x^b (t - x) f' (t) dt \geq \frac{1}{2} (b - x)^2 f'_+ (x).
\]

(2.6)

Finally, if we subtract (2.5) from (2.6) and use the representation (2.2), we deduce the desired inequality

\[
\int_a^b \int_a^x (x - t) f' (t) dt \leq \frac{1}{2} (x - a)^2 f'_- (x)
\]

(2.7)

Now, assume that \(2.1\) holds with a constant \(C > 0\) instead of \(\frac{1}{2}\), i.e.,

\[
C \left[ (b - x)^2 f'_+ (x) - (x - a)^2 f'_- (x) \right] 
\leq (x - a) f (a) + (b - x) f (b) - \int_a^b f (t) dt.
\]
Consider the convex function \( f_0(t) := k \left| t - \frac{a+b}{2} \right|, \ k > 0, t \in [a, b] \). Then

\[
\begin{align*}
\frac{d^+}{d} \left( \frac{a+b}{2} \right) &= k, & \frac{d^-}{d} \left( \frac{a+b}{2} \right) &= -k, \\
\left. f_0 \right|_{a} &= \left. \frac{k(b-a)}{2} = f_0 \right|_{b}, & \int_{a}^{b} f_0(t) \, dt &= \frac{1}{4} k (b-a)^2.
\end{align*}
\]

If in (2.7) we choose \( f_0 \) as above and \( x = \frac{a+b}{2} \) then we get

\[
\left[ \frac{1}{4} (b-a)^2 k + \frac{1}{4} (b-a)^2 k \right] \leq \frac{1}{4} k (b-a)^2
\]

giving \( C \leq \frac{1}{2} \), and the sharpness of the constant is proved. \( \blacksquare \)

Now, recall that the following inequality which is well known in the literature as the *Hermite-Hadamard inequality* for convex functions holds

\[
(H-H) \quad f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \leq \frac{f(a) + f(b)}{2}.
\]

The following corollary gives a sharp lower bound for the difference

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt.
\]

**Corollary 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex function on \([a, b]\). Then

\[
0 \leq \frac{1}{8} \left[ f' \left( \frac{a+b}{2} \right) - f' \left( \frac{a+b}{2} \right) \right] (b-a) \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt.
\]

The constant \( \frac{1}{8} \) is sharp.

The proof is obvious by the above theorem. The sharpness of the constant is obtained for \( f_0(t) = k \left| t - \frac{a+b}{2} \right|, \ t \in [a, b], \ k > 0 \).

When \( x \) is a point of differentiability, we may state the following corollary as well.

**Corollary 2.** Let \( f \) be as in Theorem 6. If \( x \in (a, b) \) is a point of differentiability for \( f \), then

\[
(b-a) \left( \frac{a+b}{2} - x \right) f' \left( x \right) \leq (x-a) f \left( a \right) + (b-x) f \left( b \right) - \int_{a}^{b} f(t) \, dt.
\]

**Remark 1.** If \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is convex on \( I \) and if we choose \( x \in \hat{I} \) (\( \hat{I} \) is the interior of \( I \)), \( b = x + \frac{h}{2}, \ a = x - \frac{h}{2}, \ h > 0 \) is such that \( a, b \in I \), then from (2.7) we may write

\[
0 \leq \frac{1}{8} h^2 \left[ f' \left( x \right) - f' \left( x \right) \right] \leq \frac{f(a) + f(b)}{2} \cdot h - \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t) \, dt
\]

and the constant \( \frac{1}{8} \) is sharp in (2.10).
The following result providing an upper bound for the difference
\[(x - a) f(a) + (b - x) f(b) - \int_a^b f(t) \, dt\]
also holds.

**Theorem 6.** Let \(f: [a, b] \to \mathbb{R}\) be a convex function on \([a, b]\). Then for any \(x \in [a, b]\), we have the inequality:
\[(x - a) f(a) + (b - x) f(b) - \int_a^b f(t) \, dt \leq \frac{1}{2} \left[(b - x)^2 f'_-(b) - (x - a)^2 f'_+(a)\right].\]

The constant \(\frac{1}{2}\) is sharp in the sense that it cannot be replaced by a smaller constant.

**Proof.** If either \(f'_+(a) = -\infty\) or \(f'_-(b) = +\infty\), then the inequality (2.11) evidently holds true.

Assume that \(f'_+(a)\) and \(f'_-(b)\) are finite.

Since \(f\) is convex on \([a, b]\), we have
\[(2.12) \quad f'(t) \geq f'_+(a) \quad \text{for \ a.e. \ } t \in [a, x]\]
and
\[(2.13) \quad f'(t) \leq f'_-(b) \quad \text{for \ a.e. \ } t \in [x, b].\]
If we multiply (2.12) by \((x - t) \geq 0, t \in [a, x]\) and integrate on \([a, x]\), then we deduce
\[(2.14) \quad \int_a^x (x - t) f'(t) \, dt \geq \frac{1}{2} (x - a)^2 f'_+(a)\]
and if we multiply (2.13) by \(t - x \geq 0, t \in [x, b]\) and integrate on \([x, b]\), then we also have
\[(2.15) \quad \int_x^b (t - x) f'(t) \, dt \leq \frac{1}{2} (b - x)^2 f'_-(b).\]
Finally, if we subtract (2.14) from (2.15) and use the representation (2.22), we deduce the desired inequality (2.11).

Now, assume that (2.11) holds with a constant \(D > 0\) instead of \(\frac{1}{2}\), i.e.,
\[(2.16) \quad (x - a) f(a) + (b - x) f(b) - \int_a^b f(t) \, dt \leq D \left[(b - x)^2 f'_-(b) - (x - a)^2 f'_+(a)\right].\]
If we consider the convex function \(f_0: [a, b] \to \mathbb{R}, f_0(t) = k \left|t - \frac{a+b}{2}\right|\), then we have \(f'_-(b) = k, f'_+(a) = -k\) and by (2.10) we deduce for \(x = \frac{a+b}{2}\) that
\[\frac{1}{4} k (b-a)^2 \leq D \left[\frac{1}{4} k (b-a)^2 + \frac{1}{4} k (b-a)^2\right]\]
giving \(D \geq \frac{1}{2}\), and the sharpness of the constant is proved.

The following corollary related to the Hermite-Hadamard inequality is interesting as well.
Corollary 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be convex on $[a, b]$. Then

\begin{equation}
0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{1}{8} [f'_-(b) - f'_+(a)] (b - a)
\end{equation}

and the constant $\frac{1}{8}$ is sharp.

Remark 2. Denote $B := f'_-(b)$, $A := f'_+(a)$ and assume that $B \neq A$, i.e., $f$ is not constant on $(a, b)$. Then

\begin{equation}
(b - x)^2 B - (x - a)^2 A = (B - A) \left[ x - \left( \frac{bB - aA}{B - A} \right) \right]^2 - \frac{AB}{(B - A)^2} (b - a)^2
\end{equation}

and by (2.14) we get

\begin{equation}
(x - a) f(a) + (b - x) f(b) - \int_a^b f(t) \, dt \leq (B - A) \left[ x - \left( \frac{bB - aA}{B - A} \right) \right]^2 - \frac{AB}{(B - A)^2} (b - a)^2
\end{equation}

for any $x \in [a, b]$.

If $A \geq 0$, then $x_0 = \frac{bB - aA}{B - A} \in [a, b]$, and by (2.18) for $x = \frac{bB - aA}{B - A}$ we get that

\begin{equation}
0 \leq \frac{1}{2} \frac{AB}{B - A} (b - a) \leq \frac{Bf(a) - Af(b)}{B - A} - \frac{1}{b - a} \int_a^b f(t) \, dt
\end{equation}

which is an interesting inequality in itself as well.

3. The Composite Case

Consider the division $I_n : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ and denote $h_i := x_{i+1} - x_i$ ($i = 0, n - 1$). If $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, n - 1$) are intermediate points, then we will denote by

\begin{equation}
G_n (f; I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})]
\end{equation}

the generalised trapezoid rule associated to $f$, $I_n$ and $\xi$.

The following theorem providing upper and lower bounds for the remainder in approximating the integral $\int_a^b f(t) \, dt$ of a convex function $f$ in terms of the generalised trapezoid rule holds.

Theorem 7. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function and $I_n$ and $\xi$ be as above. Then we have:

\begin{equation}
\int_a^b f(t) \, dt = G_n (f; I_n, \xi) - S_n (f; I_n, \xi),
\end{equation}
where $G_n (f; I_n, \xi)$ is the generalised Trapezoid Rule defined by (3.2) and the remainder $S_n (f; I_n, \xi)$ satisfies the estimate:

$$
\frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_+ (\xi_i) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_- (\xi_i)
$$

(3.3) \quad \leq \quad S_n (f; I_n, \xi)

\leq \quad \frac{1}{2} \left[ (b - \xi_{n-1})^2 f'_- (b) + \sum_{i=1}^{n-1} (x_i - \xi_{i-1})^2 f'_- (x_i) - (\xi_0 - a)^2 f'_+ (a) \right].

Proof: If we write the inequalities (2.1) and (2.11) on the interval $[x_i, x_{i+1}]$ for the intermediate points $\xi_i \in [x_i, x_{i+1}]$, then we have

$$
\frac{1}{2} \left[ (x_{i+1} - \xi_i)^2 f'_+ (x_i) - (\xi_i - x_i)^2 f'_- (\xi_i) \right]
$$

\leq \quad (\xi_i - x_i) f (x_i) + (x_{i+1} - \xi_i) f (x_{i+1}) - \int_{x_i}^{x_{i+1}} f (t) \, dt

\leq \quad \frac{1}{2} \left[ (x_{i+1} - \xi_i)^2 f'_- (x_{i+1}) - (\xi_i - x_i)^2 f'_+ (x_i) \right].

Summing the above inequalities over $i$ from 0 to $n - 1$, we deduce

$$
\frac{1}{2} \sum_{i=0}^{n-1} \left[ (x_{i+1} - \xi_i)^2 f'_+ (\xi_i) - (\xi_i - x_i)^2 f'_- (\xi_i) \right]
$$

(3.4) \quad \leq \quad G_n (f; I_n, \xi) - \int_a^b f (t) \, dt

\leq \quad \frac{1}{2} \left[ \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_- (x_{i+1}) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+ (x_i) \right].

However,

$$
\sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_- (x_{i+1}) = \quad (b - \xi_{n-1})^2 f'_- (b) + \sum_{i=0}^{n-2} (x_{i+1} - \xi_i)^2 f'_- (x_{i+1})
$$

$$
= \quad (b - \xi_{n-1})^2 f'_- (b) + \sum_{i=0}^{n-1} (x_i - \xi_{i-1})^2 f'_- (x_i)
$$

and

$$
\sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+ (x_i) = \sum_{i=1}^{n-1} (\xi_i - x_i)^2 f'_+ (x_i) + (\xi_0 - a)^2 f'_+ (a)
$$

and then, by (3.3), we deduce the desired estimate (3.3). \quad \square

The following corollary may be useful in practical applications.
Corollary 4. Let \( f : [a, b] \to \mathbb{R} \) be a differentiable convex function on \([a, b]\). Then we have the representation (3.2) and \( S_n(f; I_n, \xi) \) satisfies the estimate:

\[
\sum_{i=0}^{n-1} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right) h_i f''(\xi_i) \leq S_n(f; I_n, \xi) \leq \frac{1}{2} \left[ (b - \xi_{n-1})^2 f_+(b) - (\xi_0 - a)^2 f_+(a) \right.
\]
\[
+ \sum_{i=1}^{n-1} \left[ \left( x_i - \frac{\xi_i + \xi_{i-1}}{2} \right) (\xi_i - \xi_{i-1}) f'(x_i) \right].
\]

We may also consider the trapezoid quadrature rule:

\[
T_n(f; I_n) := \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i.
\]

Using the above results, we may state the following corollary.

Corollary 5. Assume that \( f : [a, b] \to \mathbb{R} \) is a convex function on \([a, b]\) and \( I_n \) is a division as above. Then we have the representation (3.7)

\[
\int_a^b f(t) \, dt = T_n(f; I_n) - Q_n(f; I_n)
\]

where \( T_n(f; I_n) \) is the mid-point quadrature formula given in (3.6) and the remainder \( Q_n(f; I_n) \) satisfies the estimates

\[
0 \leq \frac{1}{8} \sum_{i=0}^{n-1} \left[ f'_+ \left( \frac{x_i + x_{i+1}}{2} \right) - f'_- \left( \frac{x_i + x_{i+1}}{2} \right) \right] h_i^2 \leq Q_n(f; I_n) \leq \frac{1}{8} \sum_{i=0}^{n-1} \left[ f'_+ (x_{i+1}) - f'_- (x_i) \right] h_i^2.
\]

The constant \( \frac{1}{8} \) is sharp in both inequalities.

4. Applications for P.D.F.s

Let \( X \) be a random variable with the probability density function \( f : [a, b] \subset \mathbb{R} \to [0, \infty) \) and with cumulative distribution function \( F(x) = \text{Pr}(X \leq x) \).

The following theorem holds.

Theorem 8. If \( f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+ \) is monotonically increasing on \([a, b]\), then we have the inequality:

\[
\frac{1}{2} \left[ (b - x)^2 f_+(x) - (x - a)^2 f_-(x) \right] + x \leq E(X)
\]
\[
\leq \frac{1}{2} \left[ (b - x)^2 f_+(b) - (x - a)^2 f_-(a) \right] + x
\]

for any \( x \in (a, b) \), where \( f_{\pm}(\alpha) \) represent respectively the right and left limits of \( f \) in \( \alpha \) and \( E(X) \) is the expectation of \( X \).

The constant \( \frac{1}{2} \) is sharp in both inequalities.

The second inequality also holds for \( x = a \) or \( x = b \).
Proof. Follows by Theorem 5 and 6 applied for the convex cdf function $F(x) = \int_a^x f(t) \, dt$, $x \in [a, b]$ and taking into account that
\[
\int_a^b F(x) \, dx = b - E(X).
\]

Finally, we may state the following corollary in estimating the expectation of $X$.

**Corollary 6.** With the above assumptions, we have
\[
\frac{1}{8} \left[ f_+ \left( \frac{a + b}{2} \right) - f_- \left( \frac{a + b}{2} \right) \right] \left( b - a \right)^2 + \frac{a + b}{2} \leq E(X) \leq \frac{1}{8} \left[ f_+ \left( b \right) - f_- \left( a \right) \right] \left( b - a \right)^2 + \frac{a + b}{2}.
\]

5. Applications for $HH$–Divergence

Assume that a set $\chi$ and the $\sigma$–finite measure $\mu$ are given. Consider the set of all probability densities on $\mu$ to be
\[
\Omega := \left\{ p \mid p : \Omega \to \mathbb{R}, \ p(x) \geq 0, \ \int_\chi p(x) \, d\mu(x) = 1 \right\}.
\]

Csiszár’s $f$–divergence is defined as follows
\[
D_f(p, q) := \int_\chi p(x) f\left( \frac{q(x)}{p(x)} \right) \, d\mu(x), \ p, q \in \Omega,
\]
where $f$ is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived.

In [5], Shioya and Da-te introduced the generalised Lin-Wong $f$–divergence $D_f \left( p, \frac{1}{2}p + \frac{1}{2}q \right)$ and the Hermite-Hadamard ($HH$) divergence
\[
D^{HH}_f(p, q) := \int_\chi \frac{p^2(x)}{q(x) - p(x)} \left( \int_1^{\frac{q(x)}{p(x)}} f(t) \, dt \right) \, d\mu(x), \ p, q \in \Omega,
\]
and, by the use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality
\[
D_f \left( p, \frac{1}{2}p + \frac{1}{2}q \right) \leq D^{HH}_f(p, q) \leq \frac{1}{2}D_f(p, q),
\]
provided that $f$ is convex and normalised, i.e., $f(1) = 0$.

The following result in estimating the difference
\[
\frac{1}{2}D_f(p, q) - D^{HH}_f(p, q)
\]
holds.
Theorem 9. Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be a normalised convex function and \( p, q \in \Omega \). Then we have the inequality:

\[
0 \leq \frac{1}{8} \left[ D_{f^+} \left| \frac{a+b}{2} \right| (p, q) - D_{f^-} \left| \frac{a+b}{2} \right| (p, q) \right] \\
\leq \frac{1}{2} D_f (p, q) - D_{f^H} (p, q) \\
\leq \frac{1}{8} D_{f^+} (1) (p, q).
\]

Proof. Using the double inequality

\[
0 \leq \frac{1}{8} \left[ f_+ \left( \frac{a+b}{2} \right) - f_- \left( \frac{a+b}{2} \right) \right] |b - a| \\
\leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \\
\leq \frac{1}{8} \left[ f_-(b) - f_+ (a) \right] (b - a)
\]

for the choices \( a = 1, b = \frac{q(x)}{p(x)}, x \in \chi \), multiplying with \( p(x) \geq 0 \) and integrating over \( x \) on \( \chi \) we get

\[
0 \leq \frac{1}{8} \int_{\chi} \left[ f_+ \left( \frac{p(x) + q(x)}{2p(x)} \right) - f_- \left( \frac{p(x) + q(x)}{2p(x)} \right) \right] |q(x) - p(x)| \, d\mu(x) \\
\leq \frac{1}{2} D_f (p, q) - D_{f^H} (p, q) \\
\leq \frac{1}{8} \int_{\chi} \left[ f_+ \left( \frac{q(x)}{p(x)} \right) - f_+ (1) \right] (q(x) - p(x)) \, d\mu(x),
\]

which is clearly equivalent to (5.5).

Corollary 7. With the above assumptions and if \( f \) is differentiable on \((0, \infty)\), then

\[
0 \leq \frac{1}{2} D_f (p, q) - D_{f^H} (p, q) \leq \frac{1}{8} D_{f^+} (1) (p, q).
\]

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, 8001, Victoria, Australia.
E-mail address: sever@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html