1. Introduction and Statement of results

Eisenstein series play a critical role in number theory. For two hundred years they have been an essential tool in the analysis of automorphic L-functions and in studying properties of quadratic forms in one and several variables. The construction is clear and straightforward, while their properties are sometimes very surprising. The arithmetic of their Fourier coefficients, and their analytic properties are still not completely understood. There are many connections with the Riemann hypothesis and other famous unsolved problems in number theory.

Eisenstein series are named after Ferdinand Gotthold Eisenstein (1823 - 1852). Let \( k \) be an even integer larger than 2 and let \( \tau \) be in the upper complex half-space. One of the simplest Eisenstein series is defined by

\[
E_k(\tau) := \frac{1}{2}\sum_{m,n \in \mathbb{Z}, (m,n)=1} (m\tau + n)^{-k}.
\]

It has the transformation property

\[
E_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k E_k(\tau)
\]

for \((a\ b\ c\ d) \in \text{SL}_2(\mathbb{Z})\). It has a Fourier expansion with rational Fourier coefficients with bounded denominators, involving divisor functions and Bernoulli numbers, and is connected with special values of the Riemann zeta function.

To understand special values of more general types of L-functions, this simple version of Eisenstein series has been extended in many directions. Siegel and Klingen studied Eisenstein series attached to the symplectic group, in order to study, for example, quadratic forms and the structure of Siegel modular forms in several variables. This culminated in the Siegel-Weil formula [We65] and the structure theorem. Later Klingen introduced the Eisenstein series now called Klingen type [Kl90]. In another direction, Maass, Roelcke, and Selberg [Se56] studied real analytic Eisenstein series in the context of differential operators and spectral theory. Langlands [La76] succeeded in showing remarkable general analytic properties, i.e., meromorphic continuation to the whole complex plane and functional equation, for a wide range of reductive groups. This has applications in the Rankin-Selberg and the Shahidi methods to study analytic
and arithmetic properties of automorphic L-functions. The arithmetic properties of the Fourier coefficients play a fundamental role in the study of the arithmetic of the special values. Garrett’s integral representation of the triple L-function \([Ga87]\) was an unexpected example of a different sort.

Yet another direction appears in the brilliant work of the late H. Maass [Ma79], who found a new relation satisfied by the Fourier coefficients of holomorphic Eisenstein series of Siegel type of degree 2. Automorphic functions with this property he called the \textit{Spezialschar}. His beautiful work on this subject made it possible to understand and prove the main part of the Saito-Kurokawa conjecture [Za80]. Recently Skinner used results of Shimura on delicate properties of Eisenstein series to attack the Iwasawa conjecture (see also [Br07]). This brief review suggests that that new features of Eisenstein series should be fruitful.

In this paper we present a new method to study Fourier coefficients of holomorphic and non-holomorphic Eisenstein series simultaneously. This leads to a fundamental identity we state now. We mainly focus on the real analytic Eisenstein series on Siegel upper half-space \(\H_2\) of degree 2 to make our method clear and to not burden the discussion with other technical considerations.

Let \(E_k^{(2)}(Z, s)\) be the real analytic Eisenstein series of weight \(k\) and \(Z \in \H_2\) with respect to the Siegel modular group \(\text{Sp}_2(\mathbb{Z})\) and \(s \in \mathbb{C}\) with \(2 \text{Re}(s) + k > 3\). For details we refer to section 3. This function is not holomorphic as a function of \(Z\) on \(\H_2\), but does satisfy the transformation rule of a modular form. Since it is periodic with respect to the real part \(X\) of \(Z\) it has a Fourier expansion:

\[
E_k^{(2)}(X + iY, s) = \sum_N A(N, Y; s) e^{\frac{2\pi i \text{tr}(NX)}{m}}.
\]

where \(Y\) is the imaginary part of \(Z\) and \(N\) is summed over half-integral matrices \(N = \left(\begin{array}{c} n \\ r/2 \\ m \end{array}\right)\).

Then the following identity holds between the Fourier coefficients \(A(N, Y; s)\). Let \(G[H] := H^*GH\) for appropriate matrices \(G\) and \(H\). We have for all prime numbers \(p\) and for all half-integral \(N = \left(\begin{array}{c} n \\ r/2 \\ m \end{array}\right)\) the formula

\[
p^{k-1}A\left(\left(\begin{array}{c} n \\ r/2 \\ m \end{array}\right), pY; s\right) - A\left(\left(\begin{array}{c} n \\ r/2 \\ p m \end{array}\right), Y; s\right) = p^{k-1}A\left(\left(\begin{array}{c} n \\ r/2 \\ m \end{array}\right), Y \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]; s\right) - A\left(\left(\begin{array}{c} pn \\ r/2 \\ m \end{array}\right), Y \left[\begin{array}{c} p^{-1} \\ 0 \\ p^{1/2} \end{array}\right]; s\right).
\]
Here we put \( A(N, Y, s) = 0 \) if \( N \) is not half-integral.

The nature of the Fourier coefficients \( A(N, Y; s) \) is complicated, involving special values of Dirichlet L-series (Siegel series) and Bessel functions of higher order. One has to distinguish the various cases of the rank of \( N \). Nevertheless, our method works without any explicit knowledge of these formulas, and is completely explicit. Moreover it also works in the case of Hecke summation.

This paper is organized in the following way. In §2 we recall some basic aspects of Shimura’s approach to the theory of Hecke. This will be used to define a new kind of operators, which do not act on the space of modular forms, but nevertheless inherit interesting properties. In §3 we prove a decomposition of the real-analytic Eisenstein series, essentially based on consideration of two subseries \( A_k(Z, s) \) and \( B_k(Z, s) \), concerning which we prove several properties. In §4 we present the main result of this paper, namely, we show that real-analytic Eisenstein series satisfy the strong symmetry property

\[
E_k^2(Z, s) \mid \Delta T = 0
\]

for all Hecke operators \( T \), which will be explained in that paragraph in detail. We also give an example of a family of modular forms which do not have this property. Finally, we give applications, for example, the fundamental identity among the Fourier coefficients.

2. Hecke Theory á la Shimura

For \( k \in \mathbb{N} \) be even let \( M_k \) be the space of elliptic modular forms of weight \( k \) with respect to the full modular group \( \Gamma = \text{SL}_2(\mathbb{Z}) \). Let \( f \in M_k \). Hecke introduced the operators \( T_n, n \in \mathbb{N} \) given by

\[
T_n(f)(\tau) := n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f \left( \frac{n\tau + bd}{d^2} \right),
\]

which map modular forms to modular forms. These operators commute with each other. They are multiplicative and self-adjoint with respect to the Petersson scalar product on the space of cusp forms. The vector space \( M_k \) has a basis of simultaneous eigenforms. The eigenvalues \( \lambda_n(f) \) are totally real integers and are proportional to the \( n \)-th Fourier coefficients of the eigenform. Shimura [Sh71] studied systematically the underlying Hecke algebra. The realization of this Hecke algebra on the space of modular forms gives then the Hecke operators above.

We start with some basic constructions [Sh71]. Let \( (R, S) \) be a Hecke pair, meaning that \( R \) is a subgroup of the group \( S \) and for each \( s \in S \) the coset space \( R \backslash RsR \) is finite. For \( P \) be a principal ideal domain, \( R \) acts on the right on the \( P \)-module \( L_P(R, S) \)
of formal finite sums \( X = \sum_j a_j R s_j \) with \( a_j \in P, s_j \in S \). The subset \( H_p(R, S) \) of elements invariant under this action forms a ring with the multiplication

\[
(\sum_i a_i R g_i) \circ (\sum_j b_j R h_j) := \sum_{i,j} a_i b_j R g_i h_j.
\]

This ring is called the associated Hecke ring or algebra. It is convenient to identify the left coset decomposition of the double cosets \( R s R = \bigsqcup_j R s_j \) with \( \sum_j R s_j \in H_p(R, S) \) which form a basis of the \( P \)-module \( H_p(R, S) \). Hence double cosets are identified with a full system of representatives of the \( R \)-left coset decomposition of the double coset.

Now we apply this construction to our situation. For \( l \in \mathbb{N} \) put

\[
M(l) := \bigsqcup_{d | l, d \neq l} \Gamma \left( \begin{smallmatrix} d & 0 \\ 0 & l/d \end{smallmatrix} \right) \Gamma.
\]

Then we set \( M_\infty := \bigsqcup_{l=1}^\infty \left( \begin{smallmatrix} n^{-1} & 0 \\ 0 & n^{-1} \end{smallmatrix} \right) M(l) \). The following property is well-known.

**Lemma 2.1.** We have that \( (\Gamma, M_\infty) \) is a Hecke pair.

Let \( \mathcal{H} \) be the corresponding Hecke algebra of the Hecke pair \( (\Gamma, M_\infty) \) over \( \mathbb{Q} \). Then we have the Hecke pair

\[
\left( \Gamma, \cup_{l \in \mathbb{Z}} M(p^l) \right)
\]

for all prime \( p \) with corresponding Hecke algebra \( \mathcal{H}_p \). By the elementary divisor theorem

\[
\mathcal{H} = \otimes_p \mathcal{H}_p.
\]

Let \( T_l = \Gamma \backslash M(l) \). Then the Hecke algebra \( \mathcal{H}_p \) is generated by the \( T_p \), the special double cosets \( \Gamma \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \Gamma \), and \( \Gamma \left( \begin{smallmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{smallmatrix} \right) \Gamma \). Here

\[
\Gamma \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \Gamma = \Gamma \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) + \sum_{a=0}^{p-1} \Gamma \left( \begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix} \right).
\]

Let \( GL_2^+(\mathbb{R}) \) the set of \( \mathbb{R} \)-valued \( 2 \times 2 \) matrices with positive determinant. Let \( M \in GL_2^+(\mathbb{R}) \). Define \( \hat{M} := \det(M)^{-\frac{1}{2}} M \).

**Definition 2.2.** The action of the Hecke algebra \( \mathcal{H} \) on \( M_k \) is induced by double cosets. Let \( g \in GL_2^+(\mathbb{Q}) \) and \( f \in M_k \). Then

\[
f|_k[\Gamma g \Gamma] := \sum_{A \in \Gamma \backslash \Gamma g \Gamma} f|_k \hat{A}.
\]
A STRONG SYMMETRY PROPERTY OF EISENSTEIN SERIES

Here \( \mid_k \) is the Petersson slash operator. In particular, the normalized Hecke operators are defined by

\[
\mathbb{T}_n(f) := n^{\frac{k}{2} - 1} \sum_{A \in \Gamma \setminus M(n)} f_{\mid_k \tilde{A}}.
\]

Remark. The Hecke operators \( \mathbb{T}_n \) coincide with the classical Hecke operators \( T_n \) on the space \( M_k \). For \( f \) be a primitive form, the eigenvalue of \( T_n \) is the \( n \)-th Fourier coefficient of \( f \).

Shimura’s approach to Hecke theory can be generalized to introduce new operators related to classical Hecke operators, and which coincide in certain special situations.

Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \). Then

\[
A \times B := \begin{pmatrix} a & 0 \\ 0 & e \\ c & 0 \\ 0 & f \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

gives an embedding of \( \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \) into the symplectic group \( \text{Sp}_2(\mathbb{R}) \) of degree 2.

Let \( A \in \text{GL}_2(\mathbb{R}) \) with \( \det(A) = l > 1 \). We put

\[
\tilde{A}^* := \begin{pmatrix} l^{-1/2}a & l^{-1/2}b \\ l^{-1/2}c & l^{-1/2}d \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

and similarly define \( \tilde{A}^* \). Let \( F : \mathbb{H}_2 \to \mathbb{C} \) with \( F_{\mid_k g^*} = F \) for all \( g \in \Gamma \). Let \( A \in \text{GL}_2(\mathbb{Q}) \). Define the Hecke operator

\[
F_{\mid_k [\Gamma A]} := \sum_{\gamma \in \Gamma \setminus [\Gamma A]} F_{\mid_k \tilde{\gamma}^*},
\]

and similarly \( F_{\mid_k [\Gamma A]}^* \). For simplicity put \( \mid_k T^* \) and \( \mid_k T^* \) for \( T \in \mathcal{H} \).

3. Eisenstein series decomposition

In this section we state and prove a decomposition formula for \( E_k^{(2)}(Z, s) \). It is essentially constructed from two functions. The symplectic group \( \text{Sp}_n(\mathbb{R}) \) acts on the Siegel upper half-space \( \mathbb{H}_n \) of degree \( n \) via \( (A B) (Z) := (AZ + B)(CZ + D)^{-1} \). We put \( j((A B), Z) := \det(CZ + D) \). Let \( \Gamma_n := \text{Sp}_n(\mathbb{Z}) \) be the Siegel modular group and let \( \Gamma_{n,0} \) be the subgroup of all elements with \( C = 0 \).

Definition 3.1. Let \( k \) be an even integer and let \( n \in \mathbb{N} \). Define the real analytic Eisenstein series of weight \( k \) and genus \( n \) on \( \mathbb{H}_n \times \mathcal{D}_k^n \), where

\[
\mathcal{D}_k^n := \{ s \in \mathbb{C} \mid 2 \Re(s) + k > n + 1 \},
\]
by
\[ E_k^{(n)}(Z, s) := \sum_{g \in \Gamma_n, 0} j(g, Z)^{-k} \delta(g(Z))^s. \]

Here \( \delta(Z) := \text{det}(\text{Im}(Z)) \).

The infinite sum in (3.1) converges absolutely and uniformly on compacts on the set \( \mathbb{H}_n \times \mathcal{D}_k^n \). From Langlands’ theory [La76], \( E_k^{(n)}(Z, s) \) has a meromorphic continuation in \( s \) to the whole complex plane, and satisfies a functional equation. In particular, let \( k \) be an even positive integer, let \( \xi(s) := \pi^\frac{s}{2} \Gamma(\frac{s}{2}) \zeta(s) \) and \( \Gamma_n(s) := \prod_{j=1}^n \Gamma(s - \frac{j-1}{2}) \). Here \( \Gamma(s) \) is the Gamma function and \( \zeta(s) \) the Riemann zeta function. Then the function
\[ E_k^{(n)}(Z, s) := \frac{\Gamma_n(s + \frac{k}{2})}{\Gamma_n(s)} \cdot \xi(2s) \prod_{i=1}^{[n/2]} \xi(4s - 2i) E_k^{(n)} \left( Z, s - \frac{k}{2} \right) \]
is invariant under \( s \mapsto \frac{2n+1}{2} - s \) and is an entire function in \( s \) (see [Mi91]). Here \([x]\) is the largest integer smaller or equal to \( x \). When \( n = 1 \) the function
\[ E_k^{(2)}(Z, s) = \Gamma(s + \frac{k}{2}) \zeta(2s) \pi^{-s} E_k \left( \tau, s - \frac{k}{2} \right) \]
is entire and is invariant under \( s \mapsto 1-s \). Moreover, for \( n = 2 \) the function
\[ E_k^{(2)}(Z, s) = \Gamma(s + \frac{k-1}{2}) \Gamma \left( s + \frac{k-1}{2} \right) 2^{2s-3} \pi^{s-\frac{3}{2}} \zeta(2s) \zeta(4s - 2) E_k^{(2)} \left( Z, s - \frac{k}{2} \right) \]
entire and invariant under \( s \mapsto \frac{3}{2} - s \).

For a positive even integer \( k \) with \( k > n + 1 \) the function \( E_k^{(n)}(Z) := E_k^{(n)}(Z, 0) \) is the holomorphic Siegel Eisenstein series. It has a Fourier expansion with rational coefficients. Moreover the denominators are bounded. In the real analytic case the situation is somehow different. The Fourier coefficient depend on the imaginary part of \( Z \) and involve confluent hypergeometric functions. Moreover, one has to study Hecke summation if one is interested in the case \( k = n+1 \) and \( s = 0 \), for example. Let \( k \) be an even integer. Then \( \mathcal{D}_k := \{ s \in \mathbb{C} \mid 2 \text{Re}(s) + k > 3 \} \). It is well known that \( E_k^{(2)}(Z, 0) \) is finite. But we do not want to go into this topic further. We parametrize \( Z \in \mathbb{H}_2 \) by \( (\tau, z \xi) \) and define \( \varphi_k(Z) := \tau + 2z + \bar{\tau} \). For simplicity, put \( \chi_{k,s}(g, Z) := j(g, Z)^{-k}|j(g, Z)|^{-2s} \) and \( \Phi_{k,s} := \varphi_k(Z)^{-k}|\varphi_k(Z)|^{-2s} \) for \( g \in \text{Sp}_2(\mathbb{R}) \). Also let \( \Gamma_\infty = \Gamma_{1,0} \) and \( \mathbb{H} = \mathbb{H}_1 \). Let \( |k| \) be the Petersson slash operator. We drop the symbol for the weight \( k \) if it is clear from the context.
Definition 3.2. For \( k \in \mathbb{Z} \) be even we define two \( \mathbb{C} \)-valued functions \( A_k \) (resp. \( B_k \)) on \( \mathbb{H}_2 \times \mathcal{D}_k \) by
\[
(Z, s) \mapsto \delta(Z)^s \sum_{g, h \in \Gamma_{\infty} \setminus \Gamma} \chi_{k, s}(g \cdot h, Z) \quad \text{and} \\
(Z, s) \mapsto \delta(Z)^s \sum_{g \in \Gamma} \Phi_{k, s}(g \cdot (Z)) \chi_{k, s}(g, Z).
\]

These functions turn out to be subseries of the real analytic Eisenstein series of degree two, with similar convergence properties.

Theorem 3.3. Let \( k \) be an even integer. Let \( Z \in \mathbb{H}_2 \) and \( s \in \mathcal{D}_k \). Then
\[
E_k^{(2)}(Z, s) = A_k(Z, s) + \sum_{m=1}^{\infty} B_k|\Gamma \left( \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \Gamma \right)^* (Z, s) m^{-2s-k}.
\]

Proof. From Garrett [Ga84, Ga87] we know how to study coset systems of the type \( \Gamma_{2n,0} \setminus \Gamma_{2n} / \Gamma_n \times \Gamma_n \) in the context of the doubling method. Similarly, we obtain a useful \( \Gamma_{2,0} \)-left coset decomposition of \( \Gamma_2 \) given by
\[
R_0 = \Gamma_{\infty} \setminus \Gamma \times \Gamma_{\infty} \setminus \Gamma \quad \text{and} \\
R_1 = \bigsqcup_{m=1}^{\infty} g_m \left( \Gamma \times \Gamma(m) \setminus \Gamma \right).
\]
Here \( \Gamma(m) := \{ g \in \Gamma | \begin{pmatrix} 0 & 1/m \\ m & 0 \end{pmatrix} g \begin{pmatrix} 0 & 1/m \\ m & 0 \end{pmatrix} \in \Gamma \} \) and
\[
g_m := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & m & 0 & 0 \\ 0 & 0 & -m & 1 \end{pmatrix}.
\]
The subseries related to the representatives \( g_m (\Gamma \times \Gamma(m) \setminus \Gamma) \) is
\[
\delta(Z)^s \sum_{g \in \Gamma, h \in \Gamma(m) \setminus \Gamma} \chi_{k, s}(g \cdot (g \times h), Z).
\]
Let \( \mathbb{M}_m \) be the diagonal \( 4 \times 4 \) matrix with \( (1, m, 1, m^{-1}) \) on the diagonal. Then \( j(g_m, Z) = j(g, \mathbb{M}_m(Z)) \). Hence we obtain, for (3.7):
\[
\delta(Z)^s \sum_{g \in \Gamma, g \in \Gamma(m) \setminus \Gamma} \Phi_{k, s}(\mathbb{M}_m(g \times h)(Z)) \chi_{k, s}(g \times h, Z).
\]

Let \( \# \) be the automorphism of \( \text{SL}_2(\mathbb{R}) \) given by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \# = \begin{pmatrix} d & b \\ c & a \end{pmatrix} \) of \( \text{SL}_2(\mathbb{R}) \). Then we can prove in a straightforward manner the symmetric relation
\[
\Phi_{k, s}(g \cdot (Z)) \chi_{k, s}(g \cdot Z) = \Phi_{k, s}(g \# \cdot (Z)) \chi_{k, s}(g \# \cdot Z).
\]
By the elementary divisor theorem we obtain for our subseries the expression
\[(3.9) \quad \delta(Z)^s m^{(k+2s)} \sum_{\gamma \in \Gamma(m_0 m^{-1})} \Phi_{k,s}(12 \times \gamma)(Z) \chi_{k,s}(12 \times \gamma, Z).\]

Now we can apply again the symmetry relation and obtain the formula (3.4) in our theorem.

**Corollary 3.4.** Let \(k\) be an even integer. Let \(Z \in \mathbb{H}_2\) and \(s \in D_k\). Then
\[(3.10) \quad E^{(2)}_k(Z, s) = A_k(Z, s) + \sum_{m=1}^{\infty} \left( B_k \left( \Gamma \left( \begin{array}{cc} m & 0 \\ 0 & m^{-1} \end{array} \right) \Gamma \right) \right) m^{-2s-k} .\]

Let \(F\) be a complex-valued function on \(\mathbb{H}_2\). Let \(k \in \mathbb{N}_0\) be even. Then we say that \(F\) is \(\Gamma\)-modular of weight \(k\) if \(F|_{k\gamma} = F_{\gamma^*}\) for all \(\gamma \in \Gamma\).

**Corollary 3.5.** The functions \(A_k(Z, s)\) and \(B_k(Z, s)\) are \(\Gamma\)-modular.

4. **Strong symmetry of Eisenstein series**

Let \(F\) be a complex valued \(C^\infty\) function on the Siegel upper half-space of degree 2 with the transformation property of a modular form of even weight \(k\) with respect to \(\text{Sp}_2(Z)\). Let \(f(\tau, \tilde{\tau}) := F(\begin{smallmatrix} \tau & 0 \\ 0 & \tilde{\tau} \end{smallmatrix})\). Then we have the symmetry
\[(4.1) \quad f(\tau, \tilde{\tau}) = f(\tilde{\tau}, \tau),\]
since \(F\) is \(\Gamma\)-modular of weight \(k\) if \(F|_{k\gamma} = F_{\gamma^*} = F\) for all \(\gamma \in \Gamma\).

The real analytic Eisenstein series \(E^{(2)}_k(Z, s)\) of degree two has an important symmetry which had not been discovered before. Let \(T\) be an element of the Hecke algebra.
A STRONG SYMMETRY PROPERTY OF EISENSTEIN SERIES

We will show in this section that, if we apply $T$ as an operator on the Eisenstein series to the two embeddings $T^*$ and $\tilde{T}$, we get the same new function, i.e.,

\[(4.4) \quad \left( E_k(2) | T^* \right) (Z, s) = \left( E_k(2) | \tilde{T} \right) (Z, s). \]

From the viewpoint of physics this can been seen as a scattering experiment with an object $X$, in which we hit the object from outside with $T_p$ for different prime numbers and look at the reaction. For example, if we knew in advance that the object were a holomorphic Eisenstein series, then we could conclude that it is of Siegel type.

Actually we show that the subseries $A_k(Z, s)$ and

\[(4.5) \quad B_m^k(Z, s) := B_k | (\Gamma (m, 0, m^{-1}) \Gamma), (Z, s) \]

already have the strong symmetry property. Further, the function $A_k(Z, s)$ turns out to be an eigenfunction.

**Proposition 4.1.** Let $k$ be an even integer and $s \in D_k$. For $T \in \mathcal{H}$ we have

\[(4.6) \quad \left( A_k | T^* \right) (Z, s) = \left( A_k | \tilde{T} \right) (Z, s) = \lambda(T) A_k(T, s), \]

with $\lambda(T) \in \mathbb{C}$.

**Proof.** We have that

\[ A_k(Z, s) = \sum_{g, h \in \Gamma} j(g^* h^*, Z)^{-k} \delta(g^* h^*(Z))^s. \]

At this point we note that $g^* h^* = h^* g^*$ and $j(g^* h^*, Z) = j(g^*, h^*(Z)) j(h^*, Z)$. Since the series converges absolutely and uniformly on compacts in $\mathbb{H} \times D_k$ we can interchange summation to obtain

\[ A_k(Z, s) = \sum_{h \in \Gamma} j(h^*, Z)^{-k} \sum_{g \in \Gamma} j(g^*, h^*(Z))^{-k} \delta(g^*(h^*(Z))^s \]

\[ = \sum_{h \in \Gamma} E_k ((h^*(Z))^*, s) j(h, Z)^{-k}. \]

Let $Z = \begin{pmatrix} \tau & z \\ \bar{\tau} & \bar{z} \end{pmatrix}$. Here $Z^* := \tau$ and $Z^* := \bar{\tau}$. By the same procedure we obtain

\[ A_k(Z, s) = \sum_{g \in \Gamma} E_k ((g^*(Z))^*, s) j(g, Z^*)^{-k}. \]

Now let $T \in \mathcal{H}$ and $T = \sum_j a_j \Gamma t_j$. Then we have

\[ \left( A_k | T^* \right) (Z, s) = \sum_j a_j \sum_{h \in \Gamma} E_k \left( (h^* t_j^* \cdot Z)^*, s \right) j(h, \tilde{t}_j^*(Z))^s \delta \left( h^* t_j^*, \tilde{\tau} \right)^{-k}. \]
Hence,

\[
\left( A_k|\tilde{T}^* \right)(Z, s) = \sum_j a_j \sum_{h \in \Gamma_{\infty} \backslash \Gamma} E_k \left( (\tilde{t}_j \cdot h_\bullet(Z))^s, s \right) j \left( (\tilde{t}_j \cdot h_\bullet(Z))^s \right)^{-k} j(h_\bullet, Z)^{-k}
\]

(4.7)

\[
= \sum_{h \in \Gamma_{\infty} \backslash \Gamma} \left( E_k|\tilde{T}^* \right)(h_\bullet(Z)^s, s) j(h_\bullet, Z)^{-k}.
\]

It is well known that \( E_k(\tau, s) \) with \( \tau \in \mathbb{H} \) is a Hecke eigenform. This leads to \( \left( A_k|\tilde{T}^* \right)(Z, s) = \lambda(T) A_k(Z, s) \). The same argument works for \( \left( A_k|\tilde{T}^* \right)(Z, s) \) with the same eigenvalue. This proves the proposition.

**Proposition 4.2.** Let \( k \) be an even integer. Let \( m \in \mathbb{N} \) and let \( T \in \mathcal{H} \). Then we have

\[
\left( B_k^m|\tilde{T}^* \right)(Z, s) = \left( B_k^m|\tilde{T}^* \right)(Z, s)
\]

for all \((Z, s) \in \mathbb{H}_2 \times D_k\).

**Proof.** Let \( T = \sum_j a_j \Gamma g_j \) with \( a_j \in \mathbb{C} \) and \( g_j \in \text{GL}_2^+(\mathbb{Q}) \). Then we have

\[
\left( B_k^m|\tilde{T}^* \right)(Z, s) = \sum_j a_j B_k \left( (\Gamma \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}) \Gamma \right)_\bullet \bar{g}_j_\bullet(Z, s)
\]

\[
= \sum_j a_j B_k |\bar{g}_j_\bullet \left( (\Gamma \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}) \Gamma \right)_\bullet \right)(Z, s)
\]

since the Hecke algebra \( \mathcal{H} \) is commutative. Hence we can reduce our calculations to the case \( m = 1 \). Then we have for \( \left( B_k|\tilde{T}^* \right)(Z, s) \) the expression

\[
\sum_j a_j \delta(\bar{g}_j_\bullet(Z))^s \sum_{g \in \Gamma} \Phi_{k,s}((g \cdot \bar{g}_j_\bullet)(Z)) \chi_{k,s}(g_\bullet, \bar{g}_j_\bullet(Z)) j(g_\bullet, Z)^{-k}.
\]

To proceed further we use the cocycle property

\[
\chi_{k,s}(g \cdot \bar{g}_j_\bullet(Z)) = \chi_{k,s}(g_\bullet, \bar{g}_j_\bullet(Z)) \chi_{k,s}(\bar{g}_j_\bullet, Z)
\]

and the transformation property \( \delta(\bar{g}_j_\bullet(Z))^s = \delta(Z)^s |j(\bar{g}_j_\bullet, Z)|^{-2s} \). Hence \( \left( B_k|\tilde{T}^* \right)(Z, s) \) is equal to

\[
\sum_j a_j \delta(Z)^s \sum_{g \in \Gamma} \Phi_{k,s}((g \cdot \bar{g}_j_\bullet)(Z)) \chi_{k,s}((g \cdot \bar{g}_j_\bullet), Z).
\]

Now we apply the symmetry relation and note that \( T \) is invariant with respect to the automorphism \#. Then we obtain

\[
\sum_j \delta(Z)^s \sum_{g \in \Gamma} \Phi_{k,s}((\bar{g}_j_\bullet g_\bullet)(Z)) \chi_{k,s}((\bar{g}_j_\bullet g_\bullet), Z).
\]
Finally we use the $\Gamma$-invariance property of $\Phi_{k,s}$ and $\chi_{k,s}$. This leads to
\begin{equation}
\sum_j \delta(Z)^s \sum_{g \in \Gamma} \Phi_{k,s}\left(\left(g \cdot \tilde{g}_j \right)(Z)\right) \chi_{k,s}\left(\left(g \cdot \tilde{g}_j \right), Z\right).
\end{equation}
This gives the proposition. $\square$

For $T \in \mathcal{H}$ and even integer $k$ let $|_k \gg T$ be the operator $|_k \tilde{T}^\bullet - |_k \tilde{T}$. If a $\Gamma$-modular function is annihilated by this operator, we say that it satisfies the strong symmetry property. This makes sense since this property turns out to classify certain subspaces and gives a fundamental identity between Fourier coefficients. Summarizing our results, we have

**Theorem 4.3.** Let $k$ be an even integer. Let $T$ be an element of the Hecke algebra $\mathcal{H}$. Let $(Z, s) \in \mathbb{H}_2 \times D_k$. Then we have
\begin{equation}
E^{(2)}_k | \gg T(Z, s) = 0.
\end{equation}

**Corollary 4.4.** The strong symmetry (4.10) of the Eisenstein series is also preserved under meromorphic continuation.

It would be interesting to study the implication of this property for the residues in relation with the Siegel-Weil formula.

5. Applications of the strong symmetry property

In [He06] we have shown that a Siegel modular form $F$ of degree 2 with respect to the Siegel modular group $\text{Sp}_2(\mathbb{Z})$ $F$ is a Saito-Kurokawa lift if and only if $F$ has the strong symmetry property. Moreover, this can be used to study the non-vanishing of certain special values predicted by the Gross-Prasad conjecture and in the context of the Maass-Spezialschar results recently proven by Ichino. Our proof in the holomorphic case was based on the interplay between Taylor coefficients and certain differential operators. In this paper in the setting of real analytic Eisenstein series the proof does not work. That was the reason why we gave a new one and which works just because of the definition of an Eisenstein series via certain left cosets.

**Theorem 5.1.** Let $k$ be an even integer. Let $F : \mathbb{H}_2 :\rightarrow \mathbb{C}$ be a $\mathbb{C}^\infty$-function which satisfies the transformation law $F|_k \gamma = F$ for all $\gamma \in \Gamma_2$. Then we have
\begin{equation}
F|_{\gg T} = 0 \text{ for all } T \in \mathcal{H}
\end{equation}
\begin{equation}
\iff
\end{equation}
\begin{equation}
F|_{\gg T_p} = 0 \text{ for all prime numbers } p
\end{equation}
\begin{equation}
(5.3) p^{k-1} F \left( \begin{array}{c} p^2 \\ p \end{array} \right) \frac{1}{p} \sum_{\lambda \mod p} F \left( \begin{array}{c} \tau + \frac{z}{p} \\ \frac{z}{p^2} \end{array} \right) = p^{k-1} F \left( \begin{array}{c} \tau \\ p^2 \\ p \end{array} \right) + \frac{1}{p} \sum_{\mu \mod p} F \left( \begin{array}{c} \tau + \frac{z}{p} \\ \frac{z}{p^2} \end{array} \right).\end{equation}
Proof. We first show that (5.1) \(\iff\) (5.2). The direction from left to right is clear since it is a specialization. The other direction follows from the fact that the Hecke algebra \(H\) is the infinite restricted tensor product of all local Hecke algebras \(H_p\). Here \(p\) runs through the set of all primes. Hence it is sufficient to focus on the generators of \(H_p\). Here one has to be careful. This conclusion works only because everything is compatible with sums of operators and the underlying Hecke algebras are commutative. Now, since the local Hecke algebras are essentially generated by \(T_p\) we are done.

Next we show that (5.2) \(\iff\) (5.3). We have seen that \(T_p = \Gamma \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) + \sum_{\lambda \pmod{p}} \Gamma \left( \begin{smallmatrix} 1 & \lambda \\ 0 & p \end{smallmatrix} \right)\). We use this explicit description to calculate \(F|\tilde{T}_p\) and \(F|\tilde{T}_p\). Finally we make a change of variable \(z \mapsto p^{1/2} z\).

We parametrize \(Z \in \mathbb{H}_2\) with \(Z = (\tau \ z \ \tilde{\tau})\). Let \(X = (\tau y \ z_y \ \tilde{\tau} y)\) be the real part of \(Z\) and let \(Y = (\tau y \ z_y \ \tilde{\tau} y)\) be the imaginary part of \(Z\). Comparing Fourier coefficients in (5.3) we deduce the following result:

**Theorem 5.2.** Let \(k \in \mathbb{N}_0\) be even and let \(F : \mathbb{H}_2 \rightarrow \mathbb{C}\) be a \(\Gamma\)-modular function of weight \(k\). Assume that \(F\) has Fourier expansion of the form

\[
F(Z) = \sum_{N} A(N, Y) e\{NX\},
\]

summing over all half-integral symmetric \(2 \times 2\) matrices. Then \(F|_k \triangleright T = 0\) for all Hecke operators \(T \in H\) if and only if the Fourier coefficients of \(F\) satisfy for all prime numbers \(p\) the identity

\[
p^{k-1} A\left(\begin{pmatrix} n & p \\ r & 2p \end{pmatrix} \begin{pmatrix} \frac{r}{2} & \frac{p}{2} m \\ n \ p \end{pmatrix}\right) + A\left(\begin{pmatrix} p m & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}, \begin{pmatrix} \frac{r}{2} & \frac{p}{2} \tau y \\ \frac{p}{2} & \tau y \end{pmatrix}\right) = p^{k-1} A\left(\begin{pmatrix} n & \frac{r}{2} \\ r & 2p \end{pmatrix}, \begin{pmatrix} \tau y & \frac{p}{2} \tau y \\ \frac{p}{2} \tau y & \tau y \end{pmatrix}\right).
\]

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