Semi-classical dispersive estimates

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Abstract. We prove dispersive estimates for the wave group $e^{it\sqrt{P(h)}}$ and the Schrödinger group $e^{itP(h)}$, where $P(h)$ is a self-adjoint, elliptic second-order differential operator depending on a parameter $0 < h \leq 1$, which is supposed to be a short-range perturbation of $-h^2\Delta$, $\Delta$ being the Euclidean Laplacian. In particular, applications are made to non-trapping metric perturbations and to perturbations by a magnetic potential.

1 Introduction and statement of results

Denote by $P_0(h)$ the self-adjoint realization of $-h^2\Delta$ on $L^2(\mathbb{R}^n)$, $n \geq 2$, and let $\phi \in C_0^\infty((0, +\infty))$ be independent of $h$. It is well known (see the appendix) that the free wave and Schrödinger groups satisfy the following dispersive estimates

$$
\left\| \langle x \rangle^{-\sigma} e^{it\sqrt{P_0(h)}} \phi(P_0(h)) \langle x \rangle^{-\sigma} \right\|_{L^1 \rightarrow L^\infty} \leq Ch^{-n-\sigma} t^{-\frac{n-1}{2}-\sigma}, \quad (1.1)
$$

$$
\left\| \langle x \rangle^{-\sigma} e^{itP_0(h)} \phi(P_0(h)) \langle x \rangle^{-\sigma} \right\|_{L^1 \rightarrow L^\infty} \leq Ch^{-n-\sigma} t^{-\frac{n}{2}-\sigma}, \quad (1.2)
$$

for all $t > 0$, $0 < h \leq 1$, $\sigma \geq 0$, with a constant $C > 0$ independent of $t$ and $h$. The purpose of this paper is to prove analogues of (1.1) and (1.2) for more general second-order operators of the form

$$
P(h) = \sum_{i,j=1}^n D_x a_{ij}(x,h) D_x + \sum_{j=1}^n \left( b_j(x,h) D_x + D_x b_j(x,h) \right) + V(x,h),
$$

with real-valued coefficients $a_{ij}, b_j \in C^1(\mathbb{R}^n)$ and $V \in L^\infty(\mathbb{R}^n)$, where $D_x := -ih\partial_x$, $0 < h \leq 1$ is a semi-classical parameter (not necessarily small). More precisely, the coefficients are of the form

$$
a_{ij}(x,h) = a_{ij}^0(x) + ha_{ij}^1(x,h), \quad b_j(x,h) = b_j^0(x) + hb_j^1(x,h), \quad V(x,h) = V^0(x) + hV^1(x,h),
$$

where $a_{ij}^0, b_j^0, V^0 \in C^1(\mathbb{R}^n)$ are independent of $h$, and $a_{ij}^1, b_j^1, V^1 \in L^\infty(\mathbb{R}^n)$ uniformly in $h$. So, the principal symbol of $P(h)$ is given by

$$
p(x, \xi) = \sum_{i,j=1}^n a_{ij}^0(x) \xi_i \xi_j + 2 \sum_{j=1}^n b_j^0(x) \xi_j + V^0(x).
$$

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We suppose that this operator admits a self-adjoint realization on the Hilbert space $L^2(\mathbb{R}^n)$ (which will be again denoted by $P(h)$) satisfying the ellipticity condition
\[
\sum_{0 \leq |\alpha| \leq 2} \left\| D_x^\alpha (P(h) \pm i)^{-1} \right\|_{L^2 \to L^2} \leq C,
\] (1.3)
with a constant $C > 0$ independent of $h$. We also suppose that $P(h)$ is a short-range perturbation of $P_0(h)$, namely
\[
\sum_{i,j=1}^n \left| \partial_x^\alpha \left( a_{ij}(x, h) - a_{ij}^b \right) \right| + \sum_{j=1}^n \left| \partial_x^\alpha b_j(x, h) \right| + \left| \partial_x^\alpha V^0(x) \right| + \left| V^1(x, h) \right| \leq C|x|^{-\delta},
\] (1.4)
for $0 \leq |\alpha| \leq 1$, with constants $C > 0$, $\delta > 1$ independent of $h$, where $a_{ij}^b = 1$ if $i = j$, $a_{ij}^b = 0$ if $i \neq j$. When $n \geq 4$ we suppose that there exists a sufficiently small constant $\gamma > 0$, independent of $h$, such that
\[
\sup_{x \in \mathbb{R}^n} \sum_{i,j=1}^n \left| a_{ij}(x, h) - a_{ij}^b \right| \leq \gamma.
\] (1.5)

Given a $z \in \mathbb{C}$, $\text{Im} \, z \neq 0$, set
\[
R_s(z, h) := \langle x \rangle^{-s} (P(h) - z)^{-1} \langle x \rangle^{-s}.
\]

Finally, we suppose that there exist an energy level $E > 0$ and a constant $0 < \varepsilon_0 < E$, both independent of $h$, so that for every $z \in [E - \varepsilon_0, E + \varepsilon_0]$, $s > 1/2$, the limits
\[
R_s^\pm(z, h) := \lim_{\varepsilon \to 0^+} R_s(z \pm i\varepsilon, h) : L^2 \to L^2
\]
exist as continuous functions in $z$ and satisfy the bound
\[
\left\| R_s^\pm(z, h) \right\|_{L^2 \to L^2} \leq C \mu(h), \quad \forall z \in [E - \varepsilon_0, E + \varepsilon_0],
\] (1.6)
with a constant $C > 0$ and a function $\mu(h) \geq h^{-1} \geq 1$. If the coefficients are smooth and if $E$ is a non-trapping energy level, i.e. all bicharacteristics belonging to $\{(x, \xi) \in T^* \mathbb{R}^n : p(x, \xi) = E\}$ escape to infinity, it is well known that (1.6) holds with $\mu(h) = h^{-1}$ provided $\varepsilon_0$ is taken small enough independent of $h$. More generally, it is proved in [16] that (1.6) holds with $\mu(h) = h^{-1} \log (h^{-1})$ if all periodic bicharacteristics belonging to $\{(x, \xi) \in T^* \mathbb{R}^n : p(x, \xi) = E\}$ are of hyperbolic type satisfying a topological condition. On the other hand, without any geometrical condition we have that $\mu(h) = e^{\beta/h}$, $\beta > 0$ a constant, still for smooth coefficients (e.g. see [2], [3]). Hence, in this case the function $\mu$ satisfies
\[
h^{-1} \leq \mu(h) \leq e^{\beta/h}, \quad \beta > 0.
\] (1.7)

It is largely expected that (1.7) holds true under the assumptions above.

Let $\varphi \in C_0^\infty ((E - \varepsilon_0, E + \varepsilon_0))$ be independent of $h$. In the present paper we are interested in bounding from above uniformly in $h$ the following quantities
\[
A_1(h, \sigma) = h^{n+\sigma} \sup_{f \in \langle x \rangle^{-s} L^2, \| \langle x \rangle^s f \|_{L^2} = 1} \sup_{t > 0} t^{(n-1)/2 + \sigma} \left\| \langle x \rangle^{-\sigma} e^{it\sqrt{P(h)}} \varphi(P(h)) \langle x \rangle^{-\sigma} f \right\|_{L^\infty},
\]

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where $s > n/2$,
\[
A_2(h, \sigma) = h^{n+\sigma} \sup_{f \in L^1, \|f\|_{L^1} = 1} \sup_{t > 0} t^{(n-1)/2+\sigma} \left\| \langle x \rangle^{-\sigma} e^{it\sqrt{P(h)}} \varphi(P(h)) \langle x \rangle^{-\sigma} f \right\|_{L^\infty},
\]
\[
B_1(h, \sigma) = h^{n+\sigma} \sup_{f \in \Re(x)^{-n} L^2, \|f\|_{L^2} = 1} \sup_{t > 0} t^{n/2+\sigma} \left\| \langle x \rangle^{-\sigma} e^{itP(h)} \varphi(P(h)) \langle x \rangle^{-\sigma} f \right\|_{L^\infty},
\]
where $s > (n+1)/2$,
\[
B_2(h, \sigma) = h^{n+\sigma} \sup_{f \in L^1, \|f\|_{L^1} = 1} \sup_{t > 0} t^{n/2+\sigma} \left\| \langle x \rangle^{-\sigma} e^{itP(h)} \varphi(P(h)) \langle x \rangle^{-\sigma} f \right\|_{L^\infty}.
\]
In view of the estimates (1.1) and (1.2), we have that in the case of the free operator $P_0(h)$ all these quantities are bounded by a constant independent of $h$. In the present paper we will show that in the general case of the operator $P(h)$ these quantities can be bounded from above in terms of the function $\mu(h)$, provided $\delta$ is big enough. To state our main result more precisely, we define the number $\nu \in \{0, 1, 2\}$ as follows: $\nu = 2$ if $a_{ij}^0(x) - a_{ij}^2(x) \equiv b_{ij}^0(x) \equiv V^0(x) \equiv 0$, and the functions $a_{ij}^1(x, h), b_{ij}^1(x, h), V^1(x, h)$ are $O(h)$ as $h \to 0$; $\nu = 1$ if $a_{ij}^0(x) - a_{ij}^2 \equiv b_{ij}^0(x) \equiv V^0(x) \equiv 0$; $\nu = 0$ otherwise. In other words, the quantity $2 - \nu$ can be viewed as the order of the perturbation $P(h) - P_0(h)$. We have the following

**Theorem 1.1** Suppose the conditions (1.3)-(1.6) satisfied with $\delta > \frac{n+2}{2} + \sigma$ in the case of the wave group and $\delta > \frac{n+3}{2} + \sigma$ in the case of the Schrödinger group with some $\sigma \geq 0$. Then the following bounds hold true:

\[
A_1(h, \sigma) \leq C_\varepsilon h^{\nu + \sigma + \frac{n-1}{2}} \mu(h)^{\frac{n+1}{2} + \sigma + \varepsilon} + C,
\]
\[
A_2(h, \sigma) \leq C_\varepsilon h^{2\nu + \sigma + \frac{3}{2}} \mu(h)^{\frac{n+1}{2} + \sigma + \varepsilon} + C h^{\nu - \frac{n+1}{2}} + C,
\]
\[
B_1(h, \sigma) \leq C_\varepsilon h^{\nu + \sigma + \frac{n-1}{2}} \mu(h)^{\frac{n+2}{2} + \sigma + \varepsilon} + C,
\]
\[
B_2(h, \sigma) \leq C_\varepsilon h^{2\nu + \sigma + \frac{3}{2}} \mu(h)^{\frac{n+2}{2} + \sigma + \varepsilon} + C h^{\nu - \frac{n+1}{2}} + C,
\]
for every $0 < \varepsilon \ll 1$.

We will apply these estimates to operators of the form $P(h) = h^2 G$, where $G$ is the self-adjoint realization of a second-order operator of the form
\[
G = -\sum_{i,j=1}^n \partial_x a_{ij}(x) \partial_x j + \sum_{j=1}^n \left( b_j(x) \partial_x j \partial_x j + \partial_x j b_j(x) \right) + V(x),
\]
with real-valued coefficients $a_{ij}, b_j \in C^1(\Re^n), V \in L^\infty(\Re^n)$ independent of $h$, satisfying
\[
\sum_{i,j=1}^n \left| \partial_x^{\alpha} \left( a_{ij}(x) - a_{ij}^2 \right) \right| + \sum_{j=1}^n \left| \partial_x^{\alpha} b_j(x) \right| + |V(x)| \leq C(x)^{-\delta},
\]
for $0 \leq |\alpha| \leq 1$, with constants $C > 0$, $\delta > 1$. In other words, $G$ is supposed to be a short-range perturbation of the self-adjoint realization, $G_0$, of the free Laplacian $-\Delta$. When $n \geq 4$ we suppose that there exists a sufficiently small constant $\gamma > 0$ such that
\[
\sup_{x \in \Re^n} \sum_{i,j=1}^n \left| a_{ij}(x) - a_{ij}^2 \right| \leq \gamma.
\]
We also suppose that $G$ is elliptic, that is,

$$\partial_x^\alpha (G \pm i)^{-1} \in \mathcal{L}(L^2), \quad (1.14)$$

for all $|\alpha| \leq 2$, where $\mathcal{L}(L^2)$ denotes the set of the bounded operators on $L^2$. We finally suppose that there exist constants $C, \lambda_0 > 0$ and $k \geq 0$ such that

$$\left\| \langle x \rangle^{-s} (G - \lambda^2 \pm i0)^{-1} \langle x \rangle^{-s} \right\|_{L^2 \to L^2} \leq C\lambda^{-1+k}, \quad \forall \lambda \geq \lambda_0, \ s > 1/2. \quad (1.15)$$

This implies that the operator $P(h) = h^2 G$ satisfies (1.6) with $\mu(h) = h^{-1-k}$. Set

$$p_n(\sigma) = \max \left\{ 0, \frac{n+1}{2} - \nu, \frac{n+4}{2} - 2\nu + \frac{k(n+1)}{2} + k\sigma \right\},$$

$$q_n(\sigma) = \max \left\{ 0, \frac{n+5}{2} - 2\nu + \frac{k(n+2)}{2} + k\sigma \right\},$$

where $\sigma \geq 0$ and $2 - \nu$ is the order of the differential operator $G - G_0$. Let $\chi \in C^\infty(\mathbb{R})$, supp $\chi \subset (\lambda_0, +\infty)$, $\chi(\lambda) = 1$ for $\lambda \geq \lambda_0 + 1$. As a consequence of the above theorem we get the following $\langle x \rangle^\sigma L^1 \to \langle x \rangle^{-\sigma} L^\infty$ dispersive estimates for the perturbed wave (resp. Schrödinger) group with a loss of $p_n(\sigma)$ (resp. $q_n(\sigma) + \varepsilon$) derivatives.

**Theorem 1.2** Suppose the conditions (1.12)-(1.15) satisfied with $\delta > \frac{n+2}{2} + \sigma$ in the case of the wave group and $\delta > \frac{n+3}{2} + \sigma$ in the case of the Schrödinger group with some $\sigma \geq 0$. Then, the following dispersive estimates hold true:

$$\left\| \langle x \rangle^{-\sigma} e^{it\sqrt{G}} (\sqrt{G})^{-\frac{n+1}{2} - p_n(\sigma) - \varepsilon} \chi (\sqrt{G}) \langle x \rangle^{-\sigma} \right\|_{L^1 \to L^\infty} \leq C\varepsilon |t|^{-\frac{n+1}{2} - \sigma}, \quad \forall t \neq 0, \quad (1.16)$$

$$\left\| \langle x \rangle^{-\sigma} e^{it\sqrt{G}} \sqrt{G}^{-\frac{n+1}{2} - q_n(\sigma) - \varepsilon} \chi (\sqrt{G}) \langle x \rangle^{-\sigma} \right\|_{L^1 \to L^\infty} \leq C\varepsilon |t|^{-\frac{n+3}{2} - \sigma}, \quad \forall t \neq 0, \quad (1.17)$$

for every $0 < \varepsilon \ll 1$. Moreover, if $k < 1$ and

$$\delta > \frac{n+3}{2} + \frac{q_n(0)}{1-k},$$

then for all $\sigma$ satisfying

$$\frac{q_n(0)}{1-k} < \sigma < \delta - \frac{n+3}{2},$$

we have the estimate

$$\left\| \langle x \rangle^{-\sigma} e^{it\sqrt{G}} \chi (\sqrt{G}) \langle x \rangle^{-\sigma} \right\|_{L^1 \to L^\infty} \leq C|t|^{-\frac{n+3}{2} - \sigma}, \quad \forall t \neq 0. \quad (1.18)$$

In the particular case of non-trivial non-trapping metric perturbations we have (1.15) with $k = 0$ as well as $\nu = 0$, so $p_n(\sigma) = \frac{n+4}{2}$, $q_n(\sigma) = \frac{n+5}{2}$. Thus, in this case we obtain $\langle x \rangle^\sigma L^1 \to \langle x \rangle^{-\sigma} L^\infty$ dispersive estimates for the perturbed wave (resp. Schrödinger) group with a loss of $\frac{n+4}{2}$ (resp. $\frac{n+5}{2} + \varepsilon$) derivatives. The same conclusion remains true for more general metric perturbations with infinitely many periodic geodesics of hyperbolic type. Indeed, for such perturbations the bound (1.15) with $k = \varepsilon$, $\forall 0 < \varepsilon \ll 1$, has been proved in [16] under...
some natural topological conditions. We get a better result for perturbations by a magnetic potential, namely for operators of the form

\[ G = (i\nabla + b(x))^2 + V(x), \]

where \( b(x) = (b_1(x), ..., b_n(x)) \in C^1(\mathbb{R}^n; \mathbb{R}^n) \) is a vector-valued function and \( V \in L^\infty(\mathbb{R}^n; \mathbb{R}) \). When \( n \geq 3 \) it is proved in [10] (see Proposition 4.3) that in this case (1.15) holds with \( k = 0 \). Since \( \nu = 1 \), we have in this case \( p_n(\sigma) = \frac{3}{2}, q_n(\sigma) = \frac{n+1}{2} \). When \( b(x) \equiv 0 \), we have \( \nu = 2 \) and hence in this case \( p_n(\sigma) = q_n(\sigma) = \frac{n-3}{2} \). This latter case, however, has already been studied in [4], [6], [17], [18] under a little bit weaker assumption on the potential \( V \).

To our best knowledge, it is the first time dispersive estimates are proved for perturbations different from a potential. Our estimates are not optimal (i.e. we are obliged to loose derivatives), but one could hardly do better without assuming a stronger regularity of the coefficients. Indeed, it was shown in [12] in the context of the Schrödinger equation with a potential that if \( n \geq 4 \), it is not possible to have optimal \( L^1 \rightarrow L^\infty \) dispersive estimates for potentials \( V \in C^k_0(\mathbb{R}^n) \), \( \forall k < \frac{n-2}{2} \). In contrast, when \( n \leq 3 \) no regularity of the potential is needed in order to have optimal \( L^1 \rightarrow L^\infty \) dispersive estimates for both the wave and the Schrödinger groups (e.g. see [14] when \( n = 2 \) and [8], [11] when \( n = 3 \)). When \( n \geq 4 \) it is expected that optimal dispersive estimates hold true for potentials \( V \in C^{\frac{n-2}{2}}_0(\mathbb{R}^n) \). Indeed, such results have been recently proved in [9] when \( n = 5, 7 \), (see also [7]) in the case of the Schrödinger equation and in [5] when \( 4 \leq n \leq 7 \) in the case of the wave equation. For potentials with stronger regularity optimal dispersive estimates were proved in [1] in the case of the wave equation with Schwartz class potentials and in [13] (see also [15]) in the case of the Schrödinger equation with potentials satisfying \( \tilde{V} \in L^1 \). To our best knowledge, no optimal dispersive estimates have been proved so far in the more general context of the operator \( G \) above when the function

\[ \sum_{i,j=1}^n |a_{ij}(x) - a_{ij}^0| + \sum_{j=1}^n |b_j(x)| \]

is not identically zero, even if we suppose that \( a_{ij} - a_{ij}^0, b_j, V \in C^k_0(\mathbb{R}^n) \). In general, proving optimal dispersive estimates turns out to be a very tough problem.

To prove the main result we extend to more general perturbations the method developed in [17], [18], which consists of deriving the dispersive estimates from decay estimates on weighted \( L^2 \) spaces. This analysis is based on a careful study of the regularity of the resolvent on weighted \( L^2 \) spaces (see Proposition 3.2 below). Note that the assumption (1.5) is only used in the proof of the estimate (2.5) which plays a crucial role in our approach. It might be possible, however, that (2.5) could hold without (1.5). It becomes clear from the proof that the reason why we need (1.5) is due to the fact that when \( n \geq 4 \) the singularity at zero of the Hankel functions is too strong, which in turn implies a very strong singularity on the diagonal of the kernel of the free resolvent. Consequently, \( (P_0(h) - z)^{-1} : L^2 \rightarrow L^\infty, \Im z \neq 0 \), is no longer bounded when \( n \geq 4 \). This difficulty is overcome by Lemma 2.2 below. Note finally that we expect that the above estimates hold true for \( \delta > \frac{n+1}{2} \), but this is much harder to prove especially in the case of the Schrödinger group.

2 Study of the operator \( \varphi(P(h)) \)

In this section we will prove the following
Proposition 2.1 Assume (1.3),(1.4) and (1.5) fulfilled. Then, for all $0 \leq s, s_1, s_2, s_1 + s_2 \leq \delta$, we have the bounds
\begin{align}
\| \langle x \rangle^{-s} \varphi(P_0(h)) \langle x \rangle^s \|_{L^2 \to L^2} & \leq C, \quad (2.1) \\
\| \langle x \rangle^{-s} \varphi(P(h)) \langle x \rangle^s \|_{L^2 \to L^2} & \leq C, \quad (2.2) \\
\| \langle x \rangle^{s_1} (\varphi(P(h)) - \varphi(P_0(h))) \langle x \rangle^{s_2} \|_{L^2 \to L^2} & \leq Ch^\nu, \quad (2.3) \\
\| \langle x \rangle^{s_1} \varphi(P_0(h)) (P(h) - P_0(h)) \varphi(P(h)) \langle x \rangle^{s_2} \|_{L^2 \to L^2} & \leq Ch^\nu, \quad (2.4) \\
\| (\varphi(P(h)) - \varphi(P_0(h))) \langle x \rangle^s \|_{L^2 \to L^\infty} & \leq Ch^{\nu-n/2}, \quad (2.5)
\end{align}
with a constant $C > 0$ independent of $h$.

Proof. The estimate (2.1) is well known, while (2.2) follows from (2.1) and (2.3). It is also easy to see that (2.4) follows from (2.3). To prove (2.3) and (2.5) we will use the Helffer-Sjöstrand formula
\begin{equation}
\varphi(P(h)) = \frac{1}{\pi} \int_C \frac{\partial \tilde{\varphi}}{\partial \bar{z}} (z) (P(h) - z)^{-1} L(dz), \quad (2.6)
\end{equation}
where $L(dz)$ denotes the Lebesgue measure on $C$, $\tilde{\varphi} \in C_0^\infty(C)$ is an almost analytic continuation of $\varphi$ supported in a small complex neighbourhood of $\text{supp} \varphi$ and satisfying
\begin{equation}
\left| \frac{\partial \tilde{\varphi}}{\partial \bar{z}} (z) \right| \leq C_N |\text{Im} z|^N, \quad \forall N \geq 1. \quad (2.7)
\end{equation}
It is well known that the free resolvent satisfies the estimate (e.g. see the proof of Lemma 2.3 of [17])
\begin{equation}
\| \langle x \rangle^{-s} D_x^\alpha (P_0(h) - z)^{-1} \langle x \rangle^s \|_{L^2 \to L^2} \leq C |\text{Im} z|^{-q}, \quad |\alpha| \leq 2, \quad (2.8)
\end{equation}
for $z \in \text{supp} \tilde{\varphi}$, $\text{Im} z \neq 0$, with a constant $C > 0$ independent of $z$ and $h$. Let us see that a similar estimate holds true for the perturbed resolvent. Recall first that by assumption
\begin{equation}
P(h) - P_0(h) = h^\nu \sum_{|\alpha| \leq 2} r_\alpha(x,h) D_x^\alpha, \quad (2.9)
\end{equation}
with coefficients satisfying
\begin{equation}
|r_{\alpha}(x,h)| \leq C |x|^{-\delta}, \quad (2.10)
\end{equation}
with a constant $C > 0$ independent of $x$ and $h$. Note also that (1.3) implies
\begin{equation}
\| D_x^\alpha (P(h) - z)^{-1} \|_{L^2 \to L^2} \leq C |\text{Im} z|^{-1}, \quad |\alpha| \leq 2, \quad (2.11)
\end{equation}
for $z \in \text{supp} \tilde{\varphi}$, $\text{Im} z \neq 0$. Using (2.8)-(2.11) together with the resolvent identity, we obtain
\begin{align}
\| \langle x \rangle^{-s} D_x^\alpha (P(h) - z)^{-1} \langle x \rangle^s \|_{L^2 \to L^2} & \leq \| \langle x \rangle^{-s} D_x^\alpha (P_0(h) - z)^{-1} \langle x \rangle^s \|_{L^2 \to L^2} \\
& + C \sum_{|\beta| \leq 2} \| D_x^\beta (P(h) - z)^{-1} \|_{L^2 \to L^2} \| \langle x \rangle^{-\delta} D_x^\beta (P_0(h) - z)^{-1} \langle x \rangle^s \|_{L^2 \to L^2} \\
& \leq C |\text{Im} z|^{-q-1}. \quad (2.12)
\end{align}
On the other hand, using (2.6), (2.9), (2.10) and the resolvent identity, we get
\[
\| \langle x \rangle^{s_1} (\varphi(P(h)) - \varphi(P_0(h))) \langle x \rangle^{s_2} \|_{L^2 \to L^2} \leq C h^\nu \sum_{|\alpha| \leq 2} \int C \left| \frac{\partial \varphi}{\partial \bar{z}}(z) \right| \left\| \langle x \rangle^{s_1} (P(h) - z)^{-1} \langle x \rangle^{-s_1} \right\|_{L^2 \to L^2} \times \left\| \langle x \rangle^{s_1-\delta} D_x^\alpha (P(h) - z)^{-1} \langle x \rangle^{s_2} \right\|_{L^2 \to L^2} L(dz). \tag{2.13}
\]
Clearly, (2.3) follows from (2.7), (2.8), (2.12) and (2.13).

To prove (2.5) we will first consider the case \( n = 2, 3 \). Then it is well known that the free resolvent satisfies the estimate
\[
\left\| (P_0(h) - z)^{-1} \right\|_{L^2 \to L^\infty} \leq C h^{-n/2} |\text{Im} z|^{-q}, \tag{2.14}
\]
for \( z \in \text{supp} \varphi \), \( \text{Im} z \neq 0 \), with constants \( C, q > 0 \) independent of \( z \) and \( h \). On the other hand, using (2.6), (2.9), (2.10) and the resolvent identity, we get
\[
\left\| (\varphi(P(h)) - \varphi(P_0(h))) \langle x \rangle^s \right\|_{L^2 \to L^\infty} \leq C h^\nu \sum_{|\alpha| \leq 2} \int C \left| \frac{\partial \varphi}{\partial \bar{z}}(z) \right| \left\| (P_0(h) - z)^{-1} \right\|_{L^2 \to L^\infty} \left\| \langle x \rangle^{-\delta} D_x^\alpha (P(h) - z)^{-1} \langle x \rangle^s \right\|_{L^2 \to L^2} L(dz). \tag{2.15}
\]
In this case (2.5) follows from (2.7), (2.12), (2.14) and (2.15). Let now \( n \geq 4 \). Then (2.14) is no longer true because the singularity of the kernel of the free resolvent on the diagonal gets too strong. In this case we will derive (2.5) from the following

**Lemma 2.2** Given any \( 0 < \varepsilon \ll 1 \), the free resolvent can be decomposed as
\[
(P_0(h) - z)^{-1} = \sum_{j=1}^{3} \mathcal{B}_{\varepsilon}^{(j)}(z, h),
\]
where \( \mathcal{B}_{\varepsilon}^{(j)}(z, h), j = 1, 3, \) are analytic on \( \text{supp} \varphi \). Moreover, for \( s \geq 0, z \in \text{supp} \varphi \), we have the estimates
\[
\left\| D_x^\alpha \mathcal{B}_{\varepsilon}^{(1)}(z, h) \right\|_{L^1 \to L^1} \leq C \varepsilon^{1-|\alpha|}, \quad |\alpha| \leq 2, \tag{2.16}
\]
\[
\left\| \langle x \rangle^s \mathcal{B}_{\varepsilon}^{(2)}(z, h) \langle x \rangle^{-s} \right\|_{L^1 \to L^2} \leq C \varepsilon h^{-\frac{s}{2}} |\text{Im} z|^{-q}, \tag{2.17}
\]
\[
\left\| \mathcal{B}_{\varepsilon}^{(3)}(z, h) \right\|_{L^1 \to L^2} \leq C \varepsilon h^{-\frac{s}{2}}, \tag{2.18}
\]
with constants \( C, q > 0 \) independent of \( z, h \) and \( \varepsilon \), and a constant \( C_{\varepsilon} > 0 \) independent of \( z \) and \( h \).

Note that by assumption we have \( r_\alpha = O(\gamma) \) for \( |\alpha| = 2 \). It follows from (2.16) together with (2.9) and (2.10) that
\[
\left\| (P(h) - P_0(h)) \mathcal{B}_{\varepsilon}^{(1)}(z, h) \right\|_{L^1 \to L^1} \leq C (\varepsilon^{\frac{3}{2}} + \gamma), \tag{2.19}
\]
with a constant $C > 0$ independent of $\varepsilon$, $h$ and $\varepsilon$. Clearly, (2.19) implies that the operator $1 + (P(h) - P_0(h)) B_\varepsilon^{(1)}(z, h)$ is invertible on $L^1$, provided $\varepsilon, \gamma > 0$ are taken small enough, independent of $h$, with an inverse analytic on supp $\tilde{\varphi}$. Therefore, we can write

$$(P(h) - z)^{-1} - (P_0(h) - z)^{-1} = \sum_{j=1}^{4} \mathcal{F}_j(z, h),$$  

where

$$\mathcal{F}_j(z, h) = -B_\varepsilon^{(j)}(z, h) (P(h) - P_0(h)) B_\varepsilon^{(1)}(z, h) \left( 1 + (P(h) - P_0(h)) B_\varepsilon^{(1)}(z, h) \right)^{-1},$$

$j = 1, 2, 3$, and

$$\mathcal{F}_4(z, h) = -(P(h) - z)^{-1} (P(h) - P_0(h)) \left( B_\varepsilon^{(2)}(z, h) + B_\varepsilon^{(3)}(z, h) \right) \left( 1 + (P(h) - P_0(h)) B_\varepsilon^{(1)}(z, h) \right)^{-1}.$$  

Clearly, $\mathcal{F}_j(z, h), j = 1, 3$, are analytic on supp $\tilde{\varphi}$, so in view of (2.21) we can write

$$\varphi(P(h)) - \varphi(P_0(h)) = \sum_{j=2,4} \frac{1}{\pi} \int_C \frac{\partial \tilde{\varphi}}{\partial z}(z) \mathcal{F}_j(z, h) L(dz).$$  

By (2.17) and (2.20),

$$\|\langle x \rangle^s \mathcal{F}_2(z, h)\|_{L^1 \to L^2} \leq C h^\nu \|\langle x \rangle^s B_\varepsilon^{(2)}(z, h) \langle x \rangle^{-\delta}\|_{L^1 \to L^2} \leq C_\varepsilon h^{\nu - \frac{s}{2}} |\text{Im} z|^{-q_1}. \quad (2.23)$$

By (2.9), (2.10), (2.12), (2.17) and (2.18),

$$\|\langle x \rangle^s \mathcal{F}_4(z, h)\|_{L^1 \to L^2} \leq C \|\langle x \rangle^s (P(h) - z)^{-1} (P(h) - P_0(h))\|_{L^2 \to L^2} \left( \|B_\varepsilon^{(2)}(z, h)\|_{L^1 \to L^2} + \|B_\varepsilon^{(3)}(z, h)\|_{L^1 \to L^2} \right) \leq C_\varepsilon h^{\nu - \frac{s}{2}} |\text{Im} z|^{-q_2}. \quad (2.24)$$

By (2.7), (2.22), (2.23) and (2.24), we conclude

$$\|\langle x \rangle^s (\varphi(P(h)) - \varphi(P_0(h)))\|_{L^1 \to L^2} \leq C_\varepsilon h^{\nu - \frac{s}{2}},$$

which is equivalent to (2.5).

Proof of Lemma 2.2. Let $\phi \in C_0^\infty([1, 2])$ be such that $\int \phi(\theta) d\theta = 1$. Given any $0 < \varepsilon \ll 1$, write $[0, +\infty) = \bigcup_{j=1}^{3} I_j(\varepsilon)$, where $I_1(\varepsilon) = [0, \varepsilon]$, $I_2(\varepsilon) = [\varepsilon, \varepsilon^{-1}]$, $I_3(\varepsilon) = [\varepsilon^{-1}, +\infty)$. Set

$$\chi_\varepsilon^{(j)}(\sigma) = \sigma \int_{I_j(\varepsilon)} \phi(\sigma \theta) d\theta, \quad B_\varepsilon^{(j)}(z, h) = (P_0(h) - z)^{-1} \chi_\varepsilon^{(j)}(P_0(h)) = \int_{I_j(\varepsilon)} \psi(\theta P_0(h), \theta z) d\theta,$$
where
\[ \psi(\lambda, w) = \lambda(\lambda - w)^{-1}\phi(\lambda). \]
Since \( \text{supp} \tilde{\varphi} \) is a compact disjoint from zero, taking \( \varepsilon > 0 \) small enough, we can arrange that \( \theta z \) does not belong to the support of \( \phi \) as long as \( \theta \in I_1(\varepsilon) \cup I_2(\varepsilon) \) and \( z \in \text{supp} \tilde{\varphi} \). Therefore, the operator-valued functions \( B^{(j)}_{\varepsilon}(\cdot, h) \), \( j = 1, 3 \), are analytic on \( \text{supp} \tilde{\varphi} \). We also have the bounds
\[
\left| \partial^k_\lambda \left( \lambda^{[\alpha]/2}\psi(\lambda, \theta z) \right) \right| \leq C_k, \quad \theta \in I_1(\varepsilon), \tag{2.25}
\]
\[
\left| \partial^k_\lambda \psi(\lambda, \theta z) \right| \leq C_k |\text{Im} z|^{-k-1}, \quad \theta \in I_2(\varepsilon), \tag{2.26}
\]
\[
\left| \partial^k_\lambda \psi(\lambda, \theta z) \right| \leq C_k \theta^{-1}, \quad \theta \in I_3(\varepsilon), \tag{2.27}
\]
for \( z \in \text{supp} \tilde{\varphi} \) and all integers \( k \geq 0 \). Recall now that given any function \( f \in C_0^\infty(\mathbb{R}) \) and any \( h > 0 \), the operator \( f(P_0(h)) \) satisfies the estimates (e.g. see Lemma A.1 of \[15\])
\[
\|f(P_0(h))\|_{L^1\to L^1} \leq \tilde{C} \sum_{k=0}^{N} \sup_{\lambda} \left| \partial^k_\lambda f(\lambda) \right|, \tag{2.28}
\]
\[
\|\langle x \rangle^{s} f(P_0(h)) \langle x \rangle^{-s}\|_{L^1\to L^2} \leq \tilde{C} h^{-n/2} \|f\|_{H^{|s|}} \sum_{k=0}^{N_s} \sup_{\lambda} \left| \partial^k_\lambda f(\lambda) \right|, \quad \forall s \in \mathbb{R}, \tag{2.29}
\]
where \( N \) and \( N_s \) are integers independent of \( f \) and \( h \), while \( \tilde{C} > 0 \) is a constant depending only on the support of \( f \). If \( |\alpha| \leq 1 \), by (2.25) and (2.28), we get
\[
\left\| D_x^\alpha B^{(1)}_{\varepsilon}(z, h) \right\|_{L^1\to L^1} \leq C \left\| P_0(h)^{[\alpha]/2} B^{(1)}_{\varepsilon}(z, h) \right\|_{L^1\to L^1} \leq C \int_{-\varepsilon}^{\varepsilon} \left\| P_0(h)^{[\alpha]/2} \psi(\theta P_0(h), \theta z) \right\|_{L^1\to L^1} d\theta \leq C \int_{-\varepsilon}^{\varepsilon} \theta^{-|\alpha|/2} d\theta \leq C \varepsilon^{1-|\alpha|/2}.
\]
Using (2.29) together with (2.26) and (2.27), we also get
\[
\left\| \langle x \rangle^{s} B^{(2)}_{\varepsilon}(z, h) \langle x \rangle^{-s} \right\|_{L^1\to L^2} \leq \int_{-\varepsilon}^{\varepsilon} \left\| \langle x \rangle^{s} \psi(\theta P_0(h), \theta z) \langle x \rangle^{-s} \right\|_{L^1\to L^2} d\theta \leq C \varepsilon^{-n/2} |\text{Im} z|^{-N_s-1} \int_{-\varepsilon}^{\varepsilon} \theta^{-n/4} (1 + \theta)^s d\theta \leq C \varepsilon^{-n/2} |\text{Im} z|^{-N_s-1},
\]
\[
\left\| B^{(3)}_{\varepsilon}(z, h) \right\|_{L^1\to L^2} \leq \int_{-\varepsilon}^{\varepsilon} \|\psi(\theta P_0(h), \theta z)\|_{L^1\to L^2} d\theta \leq C \varepsilon^{-n/2} \int_{-\varepsilon}^{\varepsilon} \theta^{-1-n/4} d\theta \leq C \varepsilon^{-n/2}.
\]
It remains to prove (2.16) for \( |\alpha| = 2 \). Clearly, it suffices to show that the operator \( \chi_{\varepsilon}^{(1)}(P_0(h)) \) is bounded on \( L^1 \) uniformly in \( \varepsilon \) and \( h \). Since \( \chi_{\varepsilon}^{(1)}(\sigma) = \chi_{1}^{(1)}(\varepsilon \sigma) \), we need to show that the operator \( \chi_{1}^{(1)}(-\varepsilon h^2 \Delta) \) is bounded on \( L^1 \) uniformly in \( \varepsilon \) and \( h \). To see this observe that the kernel of \( \chi_{1}^{(1)}(-\varepsilon h^2 \Delta) \) is of the form \( (\varepsilon^2 h^2)^{-n} K(|x-y|/\varepsilon h) \), where \( K(|x-y|) \) is the kernel of \( \chi_{1}^{(1)}(-\Delta) \). Hence \( \chi_{1}^{(1)}(-\varepsilon h^2 \Delta) \) is bounded on \( L^1 \) if and only if so is \( \chi_{1}^{(1)}(-\Delta) \) and the norms coincide. On the other hand, we have \( \chi_{1}^{(1)} \in C_0^\infty(\mathbb{R}) \), \( \chi_{1}^{(1)}(\sigma) = 0 \) for \( \sigma \leq 1 \), \( \chi_{1}^{(1)}(\sigma) = 1 \) for \( \sigma \geq 2 \), which implies that \( \chi_{1}^{(1)}(-\Delta) \) is bounded on \( L^1 \). \( \square \)
3 Uniform estimates on weighted $L^2$ spaces

We will prove the following

**Theorem 3.1** Assume (1.3), (1.4) and (1.6) fulfilled. Let $0 \leq s < \delta - 1$, $0 < \epsilon \ll 1$. Then, we have the estimates

$$
\int_{-\infty}^{\infty} (t)^{2s} \| \langle x \rangle^{-1/2-s-\epsilon} e^{it \sqrt{\epsilon} P(h)} \varphi(P(h)) \langle x \rangle^{-1/2-s-\epsilon} f \|^2_{L^2} dt \leq C \epsilon \mu(h)^{2+2s+2\epsilon} \| f \|^2_{L^2},
$$

(3.1)

$$
\left\| \langle x \rangle^{-1/2-s-\epsilon} e^{it \sqrt{\epsilon} P(h)} \varphi(P(h)) \langle x \rangle^{-1/2-s-\epsilon} \right\|_{L^2 \rightarrow L^2} \leq C \epsilon \mu(h)^{1+s+\epsilon} (t)^{-s}, \quad \forall t,
$$

(3.2)

$$
\int_{-\infty}^{\infty} (t)^{2s} \left\| \langle x \rangle^{-1/2-s-\epsilon} e^{it \sqrt{\epsilon} P(h)} \varphi(P(h)) \langle x \rangle^{-1/2-s-\epsilon} f \right\|^2_{L^2} dt \leq C \epsilon \mu(h)^{2+2s+2\epsilon} \| f \|^2_{L^2},
$$

(3.3)

$$
\left\| \langle x \rangle^{-1/2-s-\epsilon} e^{it \sqrt{\epsilon} P(h)} \varphi(P(h)) \langle x \rangle^{-1/2-s-\epsilon} \right\|_{L^2 \rightarrow L^2} \leq C \epsilon \mu(h)^{1+s+\epsilon} (t)^{-s}, \quad \forall t,
$$

(3.4)

for every $0 < \epsilon \ll 1$.

**Proof.** Let us first see that (3.2) follows from (3.1). Given any $f \in L^2$, set

$$
u(x, t) = \langle x \rangle^{-1/2-s-\epsilon} e^{it \sqrt{\epsilon} P(h)} \varphi(P(h)) \langle x \rangle^{-1/2-s-\epsilon} f.
$$

It follows from (3.1) that there exists a sequence $t_k \to \infty$ such that

$$
\lim_{t_k \to \infty} \| \nu(\cdot, t_k) \|_{L^2} = 0.
$$

(3.5)

Let $\varphi_1 \in C_0^\infty((E - \epsilon_0, E + \epsilon_0))$, $\varphi_1 = 1$ on supp $\varphi$. Using (2.2) we have

$$
\left| \frac{d}{dt} \| \nu(\cdot, t) \|^2_{L^2} \right| = 2 \text{Re} \left\langle \partial_t \nu(\cdot, t), \nu(\cdot, t) \right\rangle_{L^2}
$$

$$
= 2 \text{Im} \left( \langle x \rangle^{-1/2-s-t} \sqrt{\epsilon} P(h) \varphi_1 (P(h)) \langle x \rangle^{1/2+s+\epsilon} u(x, t), u(x, t) \right)_{L^2} \leq C \| \nu(\cdot, t) \|^2_{L^2}
$$

with a constant $C > 0$ independent of $h$ and $t$. Hence given any $t > 0$, we get

$$
\| \nu(\cdot, t) \|^2_{L^2} \leq \| \nu(\cdot, t_k) \|^2_{L^2} + C \int_t^{t_k} \| \nu(\cdot, \tau) \|^2_{L^2} d\tau,
$$

which together with (3.5) imply

$$
\| \nu(\cdot, t) \|^2_{L^2} \leq C \int_t^{\infty} \| \nu(\cdot, \tau) \|^2_{L^2} d\tau.
$$

(3.6)

By (3.6)

$$
i^{2s} \| \nu(\cdot, t) \|^2_{L^2} \leq C \int_0^{\infty} \tau^{2s} \| \nu(\cdot, \tau) \|^2_{L^2} d\tau \leq C \epsilon \mu(h)^{2s+2+2\epsilon} \| f \|^2_{L^2},
$$

which is the desired bound. The fact that (3.3) implies (3.4) can be proved in precisely the same way. We will next derive (3.1) and (3.3) from the following
Proposition 3.2 Assume (1.3), (1.4) and (1.6) fulfilled. Let $0 \leq s < \delta - 1$, $0 < \varepsilon \ll 1$. Then,

$$D^a_x R^\pm_{1/2+s+\varepsilon}(\cdot, h) \in C^s \left((E - \varepsilon_0, E + \varepsilon_0); L(L^2)\right)$$

and

$$\left\| D^a_x R^\pm_{1/2+s+\varepsilon}(\cdot, h) \right\|_{C^s} \leq C\mu(h)^{1+s}, \tag{3.7}$$

where $0 \leq |a| \leq 2$.

Observe first that it suffices to bound the integral in the left-hand side of (3.1) over the interval $[1, \infty)$ only, since over $(-\infty, 1]$ it can be treated similarly while over $[-1, 1]$ it is trivial. Thus, it suffices to prove the bound

$$\int_{2^k}^{2^{k+1}} \left\| u(\cdot, t) \right\|^2_{L^2} dt \leq C2^{-2k(s+\varepsilon)} \mu(h)^{2+2s+2\varepsilon} \| f \|^2_{L^2}, \tag{3.8}$$

for every integer $k \geq 0$ and every $0 \leq \varepsilon \ll 1$. Let $\phi \in C^\infty(\mathbb{R})$, $0 \leq \phi \leq 1$, $\phi(t) = 0$ for $t \leq 1/3$, $\phi(t) = 1$ for $t \geq 1/2$. We have

$$\left(\partial_t^2 + P(h)\right) (x)^{1/2+s+\varepsilon} \phi(t) u(x,t) = 2i\phi'(t) \sqrt{P(h)} (x)^{1/2+s+\varepsilon} u(x,t) + \phi''(t) (x)^{1/2+s+\varepsilon} u(x,t)$$

$$=: \varphi_1(P(h)) (x)^{-1/2-s-\varepsilon} v(x,t).$$

In view of (2.2) we have

$$\left\| v(\cdot, t) \right\|_{L^2} \leq C \left\| f \right\|_{L^2}, \quad \forall t,$$

with a constan $C > 0$ independent of $t$ and $h$. By Duhamel’s formula we get

$$\phi(t) u(x, t) = \int_0^t (x)^{-1/2-s-\varepsilon} \frac{\sin \left((t - \tau) \sqrt{P(h)} \right)}{\sqrt{P(h)}} \varphi_1(P(h)) (x)^{-1/2-s-\varepsilon} v(x, \tau) d\tau. \tag{3.10}$$

Taking the Fourier transform with respect to $t$, we deduce from (3.10)

$$\hat{\phi} u(x, \lambda) = T(\lambda, h) \hat{v}(x, \lambda), \quad \lambda \in \mathbb{R}, \tag{3.11}$$

where

$$T(\lambda, h) = (x)^{-1/2-s-\varepsilon} \left( P(h) - \lambda^2 + i0 \right)^{-1} \varphi_1(P(h)) (x)^{-1/2-s-\varepsilon}.$$
Let now \( \rho \in C_0^\infty([1/3, 1/2]) \), \( \rho \geq 0 \), such that \( \int \rho(\sigma)d\sigma = 1 \). Given a parameter \( 0 < \theta \leq 1 \), set
\[
T_\theta(\lambda, h) = \theta^{-1} \int T(\lambda + \sigma, h) \rho(\sigma/\theta)d\sigma.
\]
It follows from (3.12) that the operator-valued function \( T_\theta(\lambda, h) \) satisfies the bounds
\[
\left\| \partial^j_\lambda T_\theta(\lambda, h) \right\|_{L^2\to L^2} \leq C \mu(h)^{1+j}, \quad 0 \leq j \leq [s],
\]
(3.13)
\[
\left\| \partial^j_\lambda (T - T_\theta)(\lambda, h) \right\|_{L^2\to L^2} \leq \theta^{-1} \int \left\| \partial^j_\lambda T(\lambda + \sigma, h) - \partial^j_\lambda T(\lambda, h) \right\|_{L^2\to L^2} \rho(\sigma/\theta)d\sigma
\leq C \mu(h)^{1+s-\theta[s]} \quad 0 \leq j \leq [s],
\]
(3.14)
\[
\left\| \partial^j_\lambda T_\theta(\lambda, h) \right\|_{L^2\to L^2} \leq \theta^{-2} \int \left\| \partial^j_\lambda T(\lambda + \sigma, h) - \partial^j_\lambda T(\lambda, h) \right\|_{L^2\to L^2} \rho'(\sigma/\theta)\ d\sigma
\leq C \mu(h)^{1+s-\theta[s]-1}, \quad j = [s] + 1.
\]
(3.15)
Define the function \( u_\theta(t, x) \) by the relation
\[
\hat{u}_\theta(x, \lambda) = T_\theta(\lambda, h)\hat{v}(x, \lambda).
\]
Using (3.9), (3.14) together with Plancherel’s identity, we obtain
\[
\int_{-\infty}^\infty |t|^{2[s]} \|\phi u(\cdot, t) - u_\theta(\cdot, t)\|_{L^2}^2\ dt = \int_{-\infty}^\infty \left\| \partial^s_\lambda \left( \hat{\phi} u(\cdot, \lambda) - \hat{u}_\theta(\cdot, \lambda) \right) \right\|_{L^2}^2\ d\lambda
\leq C \sum_{j=0}^s \int_{-\infty}^\infty \left\| \partial^j_\lambda (T - T_\theta)(\lambda, h)\hat{\phi}_\lambda^{s-j}\hat{v}(\cdot, \lambda) \right\|_{L^2}^2\ d\lambda
\leq C \mu(h)^{2+2s} \theta^{2s-2[s]} \sum_{j=0}^s \int_{1/3}^{1/2} \left\| \hat{v}(\cdot, t) \right\|_{L^2}^2\ dt \leq C \mu(h)^{2+2s} \theta^{2s-2[s]} \|f\|_{L^2}^2.
\]
Hence, given a parameter \( M \geq 1 \), we get
\[
\int_{M}^{2M} \|u(\cdot, t) - u_\theta(\cdot, t)\|_{L^2}^2\ dt \leq C \mu(h)^{2+2s} \theta^{2s-2[s]} M^{-2[s]} \|f\|_{L^2}^2.
\]
(3.16)
Similarly, using (3.9), (3.13), (3.15) together with Plancherel’s identity, we obtain
\[
\int_{-\infty}^\infty |t|^{2[s]+2} \|u_\theta(\cdot, t)\|_{L^2}^2\ dt = \int_{-\infty}^\infty \left\| \partial^{[s]+1}_\lambda \hat{u}_\theta(\cdot, \lambda) \right\|_{L^2}^2\ d\lambda
\leq C \sum_{j=0}^{[s]+1} \int_{-\infty}^\infty \left\| \partial^j_\lambda T_\theta(\lambda, h)\hat{\phi}_\lambda^{[s]+1-j}\hat{v}(\cdot, \lambda) \right\|_{L^2}^2\ d\lambda
\leq C \mu(h)^{2+2s} \theta^{2s-2[s]-2} \sum_{j=0}^{[s]+1} \int_{-\infty}^\infty \left\| \hat{\phi}_\lambda^{[s]+1-j}\hat{v}(\cdot, \lambda) \right\|_{L^2}^2\ d\lambda.
\]
which we rewrite as follows
\[ \int_{1/3}^{2M} \|u_\theta(\cdot, t)\|^2_{L^2_x} dt \leq C \mu(h)^{2+2s} \theta^{2s} \|f\|_{L^2}^2, \]
which implies
\[ \int_{M}^{2M} \|u_\theta(\cdot, t)\|^2_{L^2_x} dt \leq C \mu(h)^{2+2s} \theta^{2s} \|M^{-2s}\|f\|_{L^2}^2. \]  
Taking \( \theta = M^{-1} \) we deduce from (3.16) and (3.17)
\[ \int_{M}^{2M} \|u_\theta(\cdot, t)\|^2_{L^2_x} dt \leq C \mu(h)^{2+2s} M^{-2s} \|f\|_{L^2}^2. \]  
Observe finally that the estimates (3.13)-(3.15) hold true with \( s \) replaced by \( s + \varepsilon \), \( \forall 0 \leq \varepsilon \ll 1 \), and hence so does (3.18), which in turn proves (3.8).

The estimate (3.3) can be proved in the same way. The only difference is that the function
\[ w(x, t) = \langle x \rangle^{-1/2-s-\varepsilon} e^{itP(h)} \varphi(P(h)) \langle x \rangle^{-1/2-s-\varepsilon} f \]
satisfies the identity
\[ \hat{w}(x, \lambda) = \hat{T}(\lambda, h) \hat{v}(x, \lambda), \quad \lambda \in \mathbb{R}, \]
where
\[ \hat{T}(\lambda, h) = \langle x \rangle^{-1/2-s-\varepsilon} (P(h) - \lambda + i0)^{-1} \varphi_1(P(h)) \langle x \rangle^{-1/2-s-\varepsilon} \]
belongs again to \( C^s(\mathbb{R}; \mathcal{L}(L^2)) \) and satisfies (3.12), while the function \( v(x, t) \) is compactly supported in \( t \) and satisfies (3.9).

\[ \text{Proof of Proposition 3.2.} \]
We will use the commutator identity
\[ \Delta + \frac{1}{2} [x \cdot \nabla, \Delta] = 0, \]
which we rewrite as follows
\[ P(h) + \frac{1}{2} [x \cdot \nabla, P(h)] = P(h) - P_0(h) + \frac{1}{2} [x \cdot \nabla, P(h) - P_0(h)] =: Q(h). \]  
Given any \( z \in \mathbb{C} \), \( \text{Im} z \neq 0 \), we deduce from (3.19)
\[ P(h) - z + \frac{1}{2} [x \cdot \nabla, P(h) - z] = -z + Q(h), \]
which yields the identity
\[ (P(h) - z)^{-1} - \frac{1}{2} [x \cdot \nabla, (P(h) - z)^{-1}] \]
\[ = -z (P(h) - z)^{-2} + (P(h) - z)^{-1} Q(h) (P(h) - z)^{-1}. \]  
We will first consider the case \( s = m \), where \( 0 \leq m < \delta - 1 \) is an integer. We will proceed by induction. When \( m = 0 \) the assertion is true by assumption. Suppose it is true for all integers \( 0 \leq k \leq m - 1 \). We differentiate \( m - 1 \) times the identity (3.20) with respect to \( z \) to get
\[ z (P(h) - z)^{-m-1} = \tilde{c}_m (P(h) - z)^{-m} + \frac{1}{2} [x \cdot \nabla, (P(h) - z)^{-m}] \]
A simple computation shows that

\[
+ \sum_{k=1}^{m} c_k (P(h) - z)^{-k} Q(h) (P(h) - z)^{k-m-1},
\]

which in turn leads to the identity

\[
z \frac{d^m}{dz^m} R_{1/2+m+\epsilon}(z, h) = \tilde{c}_m(x)^{-1} R_{1/2+m+\epsilon}(z, h) (x)^{-1}
\]

\[
+ \frac{1}{2} (x)^{-1/2-m-\epsilon} x \cdot \nabla (x)^{-1/2-m+\epsilon} x \nabla (x)^{-1/2-m-\epsilon}
\]

\[
- \frac{1}{2} (x)^{-1} R_{1/2+m+\epsilon}(z, h) (x)^{-1/2-m+\epsilon} x \cdot \nabla (x)^{-1/2-m-\epsilon}
\]

\[
+ \sum_{k=1}^{m} c_k (x)^{k-m-1} \frac{d^{k-1}}{dz^{k-1}} R_{1/2+k+\epsilon}(z, h) \tilde{Q}_k(h) \frac{d^{m-k}}{dz^{m-k}} R_{1/2+m-k+\epsilon}(z, h) (x)^{-k},
\]

where

\[
\tilde{Q}_k(h) = (x)^{-1/2+k+\epsilon} Q(h) (x)^{1/2+m-k+\epsilon}.
\]

A simple computation shows that

\[
Q(h) = \frac{1}{2} \sum_{j=1}^{n} (b_j(x, h) D_{x_j} + D_{x_j} b_j(x, h)) + V(x, h) + \frac{1}{2} \left[ x \cdot h \nabla, V^1(x, h) \right]
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{n} D_{x_i} x \cdot \nabla a_{ij}(x, h) D_{x_j} + \frac{1}{2} \sum_{j=1}^{n} (x \cdot \nabla b_j(x, h) D_{x_j} + D_{x_j} x \cdot \nabla b_j(x, h)) + \frac{1}{2} x \cdot \nabla V^0(x).
\]

Hence, in view of (1.4), we have that the operators \( \tilde{Q}_k(h) \) are of the form

\[
\tilde{Q}_k(h) = \sum_{\alpha, \beta \in \Omega} D^\alpha_x \bar{q}^{(k)}_{\alpha, \beta}(x, h) D^\beta_x
\]

where \( \Omega \) is the set of all multi-indices such that \( 0 \leq |\alpha| \leq 2, 0 \leq |\beta| \leq 2, |\alpha| + |\beta| \leq 3 \), and the coefficients satisfy

\[
|q^{(k)}_{\alpha, \beta}(x, h)| \leq C,
\]

with a constant \( C > 0 \) independent of \( x \) and \( h \). By (3.22) and (3.23) we obtain

\[
|z| \left\| \frac{d^m}{dz^m} R_{1/2+m+\epsilon}(z, h) \right\|_{L^2 \rightarrow L^2} \leq C h^{-1} \sum_{0 \leq |\alpha| \leq 1} \left\| D^\alpha_x \frac{d^{m-1}}{dz^{m-1}} R_{1/2+m+\epsilon}(z, h) \right\|_{L^2 \rightarrow L^2}
\]

\[
+ C h^{-1} \sum_{0 \leq |\alpha| \leq 1} \left\| D^\alpha_x \frac{d^{m-1}}{dz^{m-1}} R_{1/2+m+\epsilon}(\bar{z}, h) \right\|_{L^2 \rightarrow L^2}
\]

\[
+ C \sum_{k=1}^{m} \sum_{\alpha, \beta \in \Omega} \left\| D^\alpha_x \frac{d^{k-1}}{dz^{k-1}} R_{1/2+k+\epsilon}(\bar{z}, h) \right\|_{L^2 \rightarrow L^2} \left\| D^\beta_x \frac{d^{m-k}}{dz^{m-k}} R_{1/2+m-k+\epsilon}(z, h) \right\|_{L^2 \rightarrow L^2},
\]

with a constant \( C > 0 \) independent of \( z \) and \( h \). Applying (3.24) with \( z \) replaced by \( z \pm i\epsilon \), \( z \in [E - \epsilon_0, E + \epsilon_0] \), \( 0 < \epsilon \ll 1 \), and taking the limit as \( \epsilon \to 0 \), we get

\[
\left\| \frac{d^m}{dz^m} R_{1/2+m+\epsilon}(z, h) \right\|_{L^2 \rightarrow L^2} \leq C h^{-1} \sum_{0 \leq |\alpha| \leq 1} \left\| D^\alpha_x \frac{d^{m-1}}{dz^{m-1}} R_{1/2+m+\epsilon}(z, h) \right\|_{L^2 \rightarrow L^2}
\]
Indeed, (3.25) implies
\[
\left\| \frac{d^{m+1}}{dz^{m+1}} R_{1/2}^{+}(z \pm i \varepsilon, h) \right\|_{L^{2} \rightarrow L^{2}} \leq C \mu(h)^{m+\nu+1} \varepsilon^{1+\nu}, \quad \forall z \in [E - \varepsilon_0, E + \varepsilon_0].
\] (3.25)

Now, given any \( z_1, z_2 \in [E - \varepsilon_0, E + \varepsilon_0], \) \( 0 < |z_1 - z_2| \leq 1, \) by (3.25) and (3.26), we get
\[
\left\| \frac{d^{m}}{dz^{m}} R_{1/2}^{+}(z_1, h) - \frac{d^{m}}{dz^{m}} R_{1/2}^{+}(z_2, h) \right\|_{L^{2} \rightarrow L^{2}} \leq \sum_{j=1}^{2} \left\| \frac{d^{m}}{dz^{m}} R_{1/2}^{+}(z_j \pm i \varepsilon, h) - \frac{d^{m}}{dz^{m}} R_{1/2}^{+}(z_j, h) \right\|_{L^{2} \rightarrow L^{2}}
\]
\[
+ \left\| \frac{d^{m}}{dz^{m}} R_{1/2}^{+}(z_1 \pm i \varepsilon, h) - \frac{d^{m}}{dz^{m}} R_{1/2}^{+}(z_2 \pm i \varepsilon, h) \right\|_{L^{2} \rightarrow L^{2}} \leq C \mu(h)^{m+\nu+1} (\varepsilon^{\nu} + |z_1 - z_2|^{1+\nu}) \leq C \mu(h)^{m+\nu+1} |z_1 - z_2|^\nu
\] (3.27)

if we take \( \varepsilon = |z_1 - z_2|. \) So, in this case (3.7) with \( \alpha = 0 \) follows from (3.27). For any multi-index \( |\alpha| \leq 2, \) it follows from (1.3).

Using (3.21) and proceeding by induction as above, it is easy to see that (3.25) follows from the following

**Lemma 3.3** Let \( z \in [E - \varepsilon_0, E + \varepsilon_0], \) \( 0 < \varepsilon, \varepsilon \ll 1, \) \( 0 \leq \nu \leq 1. \) Then
\[
\left\| \langle x \rangle^{-\nu/2 - \varepsilon} (P(h) - z \pm i \varepsilon)^{-1} \langle x \rangle^{-1/2 - \varepsilon} \right\|_{L^{2} \rightarrow L^{2}} \leq C \mu(h)^{\frac{\nu}{2}} \varepsilon^{\frac{\nu}{2}}.
\] (3.28)

**Proof.** When \( \nu = 1 \) (3.28) follows from (1.6). To prove (3.28) for \( \nu = 0 \) we will use the identity
\[
(P(h) - z \mp i \varepsilon)^{-1} (P(h) - z \pm i \varepsilon)^{-1} = \frac{\pm 2}{\varepsilon} \left( (P(h) - z \pm i \varepsilon)^{-1} - (P(h) - z \mp i \varepsilon)^{-1} \right).
\]
Hence, the operator
\[
A = (P(h) - z \pm i \varepsilon)^{-1} \langle x \rangle^{-1/2 - \varepsilon}
\]
satisfies
\[ \|A^*A\|_{L^2 \to L^2} \leq 2\varepsilon^{-1} \sum_{\pm} \| (x)^{-1/2-\epsilon} (P(h) - z \pm i\varepsilon)^{-1} (x)^{-1/2-\epsilon} \|_{L^2 \to L^2} \leq C\mu(h)\varepsilon^{-1}, \]
where we have also used (1.6). Let now \( 0 < \nu < 1 \). Given a set \( M \subset \mathbb{R}^n \), denote by \( \eta(M) \) the characteristic function of \( M \). Let \( M > 1 \) be a parameter to be fixed later on. We have
\[
\left\| \langle x \rangle^{-\nu/2-\epsilon} (P(h) - z \pm i\varepsilon)^{-1} \right\|_{L^2 \to L^2} \\
\leq \left\| \langle x \rangle^{-\nu/2-\epsilon} \eta(\langle x \rangle \geq M) (P(h) - z \pm i\varepsilon)^{-1} \right\|_{L^2 \to L^2} \\
+ \left\| \langle x \rangle^{-\nu/2-\epsilon} \eta(\langle x \rangle \leq M) (P(h) - z \pm i\varepsilon)^{-1} \right\|_{L^2 \to L^2} \\
\leq M^{-\nu/2} \left\| (P(h) - z \pm i\varepsilon)^{-1} \right\|_{L^2 \to L^2} \\
+ M^{(1-\nu)/2} \left\| (P(h) - z \pm i\varepsilon)^{-1} \right\|_{L^2 \to L^2} \\
\leq CM^{-\nu/2} \mu(h)^{1/2} \varepsilon^{-1/2} + CM^{(1-\nu)/2} \mu(h) \leq C(\mu(h)\varepsilon^{-1}),
\]
if we choose \( M = \mu(h)\varepsilon^{-1} \).

\[ \square \]

\section{4 Dispersive estimates}

We will first prove the following

\textbf{Proposition 4.1} Under the assumptions of Theorem 1.1, for all \( t \neq 0, 0 < \epsilon, \varepsilon \ll 1, 0 \leq s \leq \frac{n-1}{2} \), we have the estimates
\[
\left\| \langle x \rangle^{-\sigma} \left( e^{it\sqrt{P(h)}} \varphi(P(h)) - e^{it\sqrt{P_0(h)}} \varphi(P_0(h)) \right) \right\|_{L^2 \to L^\infty} \\
\leq C\varepsilon h^{\nu - \frac{n+1}{2}} \mu(h)^{1/2} |t|^{-s-\sigma}.
\]
(4.1)
\[
\left\| \langle x \rangle^{-\sigma} \left( e^{itP(h)} \varphi(h) - e^{itP_0(h)} \varphi(P_0(h)) \right) \right\|_{L^2 \to L^\infty} \\
\leq C\varepsilon h^{\nu - \frac{n+1}{2}} \mu(h)^{3/2} |t|^{-s-\sigma-1/2}.
\]
(4.2)

\textbf{Proof.} Recall first that the free groups satisfy the estimates (see the appendix)
\[
\left\| \langle x \rangle^{-\sigma} e^{it\sqrt{P_0(h)}} \varphi(P_0(h)) \right\|_{L^2 \to L^\infty} \leq Ch^{-s-\sigma - \frac{n+1}{2}} |t|^{-s-\sigma},
\]
(4.3)
\[
\int_{-\infty}^{\infty} |t|^{2s+2\sigma} \left\| \langle x \rangle^{-1/2-s-\sigma} e^{it\sqrt{P_0(h)}} \varphi(P_0(h)) \right\|_{L^2}^2 dt \leq Ch^{n-1-2s-2\sigma} \|f\|_{L^1}^2,
\]
(4.4)
\[
\left\| \langle x \rangle^{-1/2-s-\sigma} \right\|_{L^2 \to L^\infty} \leq Ch^{-s-\sigma - \frac{n+1}{2}} |t|^{-s-\sigma-1/2},
\]
(4.5)
\[
\int_{-\infty}^{\infty} |t|^{2s+2\sigma} \left\| \langle x \rangle^{-1/2-s-\sigma} e^{itP_0(h)} \varphi(P_0(h)) \right\|_{L^2}^2 dt \leq Ch^{n-1-2s-2\sigma} \|f\|_{L^1}^2.
\]
(4.6)
Without loss of generality we may suppose that $t > 0$. To prove (4.1) observe first that Duhamel’s formula for the wave equation implies the identity (e.g. see Section 3 of [17])

$$e^{it\sqrt{\mathcal{P}(h)}}\varphi(P(h)) - e^{it\sqrt{\mathcal{P}_0(h)}}\varphi(P_0(h)) = \Phi_1(t, h) + \Phi_2(t, h),$$  

(4.7)

where

$$\Phi_1(t, h) = (\varphi_1(P(h)) - \varphi_1(P_0(h))) e^{it\sqrt{\mathcal{P}(h)}}\varphi(P(h))$$

$$+ \varphi_1(P_0(h)) \cos \left(t\sqrt{\mathcal{P}_0(h)}\right) (\varphi(P(h)) - \varphi(P_0(h)))$$

$$+ i\varphi_1^\sharp(P_0(h)) \sin \left(t\sqrt{\mathcal{P}_0(h)}\right) (\varphi^\sharp(P(h)) - \varphi^\sharp(P_0(h))),$$

$$\Phi_2(t, h) = - \int_0^t \varphi_1^\sharp(P_0(h)) \sin \left((t - \tau)\sqrt{\mathcal{P}_0(h)}\right) \mathcal{P}(h) e^{i\tau\sqrt{\mathcal{P}(h)}}\varphi(P(h)) d\tau,$$

where

$$\mathcal{P}(h) = \varphi_2(P_0(h)) (P(h) - P_0(h)) \varphi_2(P(h)),$$

$\varphi_1, \varphi_2 \in C_0^\infty([E - \varepsilon_0, E + \varepsilon_0])$, $\varphi_1 = 1$ on supp $\varphi$, $\varphi_2 = 1$ on supp $\varphi_1$, $\varphi_1^\sharp(z) = z^{1/2}\varphi(z)$, $\varphi_1^\sharp(z) = z^{-1/2}\varphi_1(z)$. By (2.3), (2.5), (3.2) and (4.3), we get

$$\left\| \langle x \rangle^{-\sigma} \Phi_1(t, h) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} \right\|_{L^2 \rightarrow L^\infty}$$

$$\leq \left\| (\varphi_1(P(h)) - \varphi_1(P_0(h))) \langle x \rangle^{n/2 + \sigma + \epsilon} \right\|_{L^2 \rightarrow L^\infty}$$

$$\times \left\| \langle x \rangle^{-n/2 - \sigma - \epsilon} e^{it\sqrt{\mathcal{P}(h)}}\varphi(P(h)) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} \right\|_{L^2 \rightarrow L^2}$$

$$+ \left\| \langle x \rangle^{-\sigma} \cos \left(t\sqrt{\mathcal{P}_0(h)}\right) \varphi_1(P_0(h)) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} \right\|_{L^2 \rightarrow L^\infty}$$

$$\times \left\| \langle x \rangle^{1/2 + s + \sigma + \epsilon} (\varphi(P(h)) - \varphi(P_0(h))) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} \right\|_{L^2 \rightarrow L^2}$$

$$+ \left\| \langle x \rangle^{-\sigma} \sin \left(t\sqrt{\mathcal{P}_0(h)}\right) \varphi_1^\sharp(P_0(h)) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} \right\|_{L^2 \rightarrow L^\infty}$$

$$\times \left\| \langle x \rangle^{1/2 + s + \sigma + \epsilon} (\varphi^\sharp(P(h)) - \varphi^\sharp(P_0(h))) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} \right\|_{L^2 \rightarrow L^2}$$

$$\leq C\varepsilon h^{\nu - n/2} \mu(h)^{1 + s + \sigma + \epsilon} L^{s - \sigma} + Ch^{\nu - s - \sigma - (n+1)/2} L^{s - \sigma}.$$  

(4.8)

Furthermore, given any $f \in L^2$, $g \in L^1$, using (2.4), (3.1) and (4.4), we get

$$t^{s+\sigma} \left| \langle \Phi_2(t, h) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} f, \langle x \rangle^{-\sigma} g \rangle \right| \leq C \int_0^{t/2} (t - \tau)^{s+\sigma}$$

$$\times \left| \mathcal{P}(h) e^{i\tau\sqrt{\mathcal{P}(h)}}\varphi(P(h)) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} f, \sin \left((t - \tau)\sqrt{\mathcal{P}_0(h)}\right) \varphi_1^\sharp(P_0(h)) \langle x \rangle^{-\sigma} g \right| d\tau$$

$$+ C \int_{t/2}^t \tau^{s+\sigma} \left| \mathcal{P}(h) e^{i\tau\sqrt{\mathcal{P}(h)}}\varphi(P(h)) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} f, \sin \left((t - \tau)\sqrt{\mathcal{P}_0(h)}\right) \varphi_1^\sharp(P_0(h)) \langle x \rangle^{-\sigma} g \right| d\tau$$

$$\leq C h^{\nu} \int_0^{t/2} (t - \tau)^{s+\sigma} \left\| \langle x \rangle^{-1/2 - \epsilon} e^{i\tau\sqrt{\mathcal{P}(h)}}\varphi(P(h)) \langle x \rangle^{-1/2 - s - \sigma - \epsilon} f \right\|_{L^2}.$$  

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\[
\times \left\| (x)^{-1/2-s-\sigma-\epsilon} \sin \left( (t-\tau) \sqrt{P_0(h)} \right) \phi_1^2(P_0(h)) (x)^{-\sigma} g \right\|_{L^2} d\tau \\
+ C h^\nu \int_{t/2}^t \tau^{s+\sigma} \left\| (x)^{-1/2-s-\sigma-\epsilon} e^{i\tau \sqrt{P(h)} \varphi(P(h)) (x)^{-1/2-s-\sigma-\epsilon} f} \right\|_{L^2} \\
\times \left\| (x)^{-1/2-\epsilon} \sin \left( (t-\tau) \sqrt{P_0(h)} \right) \phi_1^2(P_0(h)) (x)^{-\sigma} g \right\|_{L^2} d\tau \\
\leq C h^\nu \left( \int_0^\infty \tau^{2s+2\sigma} \left\| (x)^{-1/2-s-\sigma-\epsilon} \sin \left( \tau \sqrt{P_0(h)} \right) \phi_1^2(P_0(h)) (x)^{-\sigma} g \right\|^2_{L^2} d\tau \right)^{1/2} \\
+ C h^\nu \left( \int_0^\infty \tau^{2s+2\sigma} \left\| (x)^{-1/2-s-\sigma-\epsilon} e^{i\tau \sqrt{P(h)} \varphi(P(h)) (x)^{-1/2-s-\sigma-\epsilon} f} \right\|^2_{L^2} d\tau \right)^{1/2} \\
\times \left( \int_0^\infty \left\| (x)^{-1/2-\epsilon} \sin \left( \tau \sqrt{P_0(h)} \right) \phi_1^2(P_0(h)) (x)^{-\sigma} g \right\|^2_{L^2} d\tau \right)^{1/2} \\
\leq C c h^\nu - s - \sigma - \frac{\alpha+1}{2} \mu(h) \|f\|_{L^2} \|g\|_{L^1} + C c h^\nu - \frac{\alpha+1}{2} \mu(h) \|f\|_{L^2} \|g\|_{L^1}.
\]

Clearly, (4.1) follows from (4.7), (4.8) and (4.9). To prove (4.2) we rewrite Duhamel’s formula for the Schrödinger equation as follows

\[
e^{itP(h)} \varphi(P(h)) - e^{itP_0(h)} \varphi(P_0(h)) = \Psi_1(t, h) + \Psi_2(t, h),
\]

where

\[
\Psi_1(t, h) = (\varphi_1(P(h)) - \varphi_1(P_0(h))) e^{itP_0(h)} \varphi(P(h)) + \varphi_1(P_0(h)) e^{itP_0(h)} (\varphi(P(h)) - \varphi(P_0(h))),
\]

\[
\Psi_2(t, h) = i \int_0^t e^{i(t-\tau)P_0(h)} \varphi_1(P_0(h)) P(h) e^{i\tau P(h)} \varphi(P(h)) d\tau.
\]

Using (4.10) together with (4.5) and (4.6), it is easy to see that (4.2) can be proved in the same way as (4.1) above. \(\square\)

Clearly, (1.8) (resp. (1.10)) follows from (4.1) and (4.3) (resp. (4.2) and (4.5)) applied with \(s = \frac{n-1}{2}\). To prove (1.9) we will use once again the identity (4.7). By (2.5), (4.1) and (4.3), we get

\[
\left\| (x)^{-\sigma} \Phi_1(t, h) (x)^{-\sigma} \right\|_{L^1 \to L^\infty} \\
\leq \left\| (\varphi_1(P(h)) - \varphi_1(P_0(h))) (x)^{n/2+\sigma+\epsilon} \right\|_{L^2 \to L^\infty} \\
\times \left\| (x)^{-n/2-\sigma-\epsilon} e^{it\sqrt{P(h)} \varphi(P(h)) (x)^{-\sigma}} \right\|_{L^1 \to L^2} \\
+ \left\| (x)^{-\sigma} \cos \left( t \sqrt{P_0(h)} \right) \varphi_1(P_0(h)) (x)^{-n/2-\sigma-\epsilon} \right\|_{L^2 \to L^\infty} \\
\times \left\| (x)^{n/2+\sigma+\epsilon} (\varphi(P(h)) - \varphi(P_0(h))) \right\|_{L^1 \to L^2}
\]

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\[
+ \left\| \langle x \rangle^{-\sigma} \sin \left( t \sqrt{P_0(h)} \right) \varphi_1^\varepsilon(P_0(h)) \langle x \rangle^{-n/2-\sigma-\varepsilon} \right\|_{L^2 \rightarrow L^\infty} \\
\times \left\| \langle x \rangle^{n/2+\sigma+\varepsilon} \left( \varphi^\varepsilon(P(h)) - \varphi^\varepsilon(P_0(h)) \right) \right\|_{L^1 \rightarrow L^2} \\
\leq C \varepsilon t^{2\nu-n-1/2} \mu(h)^{(n+1)/2+\varepsilon} t^{-(n-1)/2-\sigma} + C t^{\nu-3n/2} t^{-(n-1)/2-\sigma} .
\] (4.11)

Furthermore, given any \( f, g \in L^1 \), using (2.4), (3.1) and (4.4), we get
\[
\left| t^{(n-1)/2+\sigma} \left| \langle \Phi_2(t, h) \rangle \langle x \rangle^{-\sigma} f, \langle x \rangle^{-\sigma} g \right| \right| \leq C \int_0^{t/2} (t - \tau)^{(n-1)/2+\sigma}
\times \left| \langle P(h) e^{i\tau \sqrt{P(h)} \varphi(P(h))} \rangle \langle x \rangle^{-\sigma} f, \sin \left( t - \tau \right) \langle P(h) \rangle \varphi_1^\varepsilon(P_0(h)) \langle x \rangle^{-\sigma} g \right| d\tau
+ C \int_0^{t/2} (t - \tau)^{(n-1)/2+\sigma}
\times \left| \langle P(h) e^{i\tau \sqrt{P(h)} \varphi(P(h))} \rangle \langle x \rangle^{-\sigma} f, \sin \left( t - \tau \right) \langle P(h) \rangle \varphi_1^\varepsilon(P_0(h)) \langle x \rangle^{-\sigma} g \right| d\tau
\leq C t^{\nu} \int_0^{t/2} (t - \tau)^{(n-1)/2+\sigma}
\times \left| \langle x \rangle^{-n/2-\sigma-\varepsilon} \sin \left( t - \tau \right) \langle P(h) \rangle \varphi_1^\varepsilon(P_0(h)) \langle x \rangle^{-\sigma} g \right| d\tau
+ C t^{\nu} \int_0^{t/2} (t - \tau)^{(n-1)/2+\sigma}
\times \left| \langle x \rangle^{-n/2-\sigma-\varepsilon} \sin \left( t - \tau \right) \langle P(h) \rangle \varphi_1^\varepsilon(P_0(h)) \langle x \rangle^{-\sigma} g \right| d\tau
\leq C t^{\nu} \left( \int_0^{\infty} \left| \langle \tau \rangle^{-1/2-\varepsilon} e^{i\tau \sqrt{P(h)} \varphi(P(h))} \right|_{L^2}^2 d\tau \right)^{1/2}
\times \left( \int_0^{\infty} \tau^{n+1} \left| \langle \tau \rangle^{-n/2-\sigma-\varepsilon} \sin \left( \tau \sqrt{P(h)} \right) \varphi_1^\varepsilon(P_0(h)) \right|_{L^2}^2 d\tau \right)^{1/2}
+ C t^{\nu} \left( \int_0^{\infty} \tau^{n+1} \left| \langle \tau \rangle^{-n/2-\sigma-\varepsilon} \sin \left( \tau \sqrt{P(h)} \right) \varphi_1^\varepsilon(P_0(h)) \right|_{L^2}^2 d\tau \right)^{1/2}
\times \left( \int_0^{\infty} \left| \langle \tau \rangle^{-1/2-\varepsilon} \sin \left( \tau \sqrt{P(h)} \right) \varphi_1^\varepsilon(P_0(h)) \right|_{L^2}^2 d\tau \right)^{1/2}
\]
Clearly, (1.9) follows from (4.7), (4.11) and (4.12) and we have

\begin{align}
+C\varepsilon h^{2\nu-(n+1)/2} \frac{\mu(h) (n+1/2) + \varepsilon \mu(h)}{2} + C\varepsilon h^{2\nu - \frac{3n+1}{2}} \mu(h) + C\varepsilon h^{2\nu - \frac{n+1}{2}} \mu(h)^{\frac{n+1}{2} + \varepsilon} \|f\|_{L^1} \|g\|_{L^1}.
\end{align}

(4.12)

Clearly, (1.9) follows from (4.7), (4.11) and (4.12). The bound (1.11) can be proved in a similar way using (4.2), (4.5) and (4.10).

**Proof of Theorem 1.2** Let \( \varphi \in C_0^\infty((0, +\infty)) \). It follows from (1.9) and (1.11) that we have the estimates

\begin{align}
\|\langle x \rangle^{-\sigma} e^{it\sqrt{G}} \varphi(h^2 G) \langle x \rangle^{-\sigma} \|_{L^1 \rightarrow L^\infty} &\leq C\varepsilon h^{-(n+1)/2 - p_n(\sigma) - \varepsilon/2} |t|^{-(n-1)/2 - \sigma}, \\
\|\langle x \rangle^{-\sigma} e^{it\sqrt{G}} \varphi(h^2 G) \langle x \rangle^{-\sigma} \|_{L^1 \rightarrow L^\infty} &\leq C\varepsilon h^{\sigma - q_n(\sigma) - \varepsilon/2} |t|^{-n/2 - \sigma}.
\end{align}

(4.13, 4.14)

We now write

\begin{align}
\left( \sqrt{G} \right)^{-(n+1)/2 - p_n(\sigma) - \varepsilon} \chi \left( \sqrt{G} \right) = \int_0^1 \psi \left( h \sqrt{G} \right) h^{(n+1)/2 + p_n(\sigma) - \varepsilon - 1} dh,
\end{align}

(4.15)

where \( \psi(\lambda) = \lambda^{-(n+1)/2 - p_n(\sigma) - \varepsilon} \chi'(\lambda) \in C_0^\infty((0, +\infty)) \). By (4.13) and (4.15) we get

\begin{align}
\|\langle x \rangle^{-\sigma} e^{it\sqrt{G}} \left( \sqrt{G} \right)^{-(n+1)/2 - p_n(\sigma) - \varepsilon} \chi \left( \sqrt{G} \right) \langle x \rangle^{-\sigma} \|_{L^1 \rightarrow L^\infty}
\leq \int_0^1 \|\langle x \rangle^{-\sigma} e^{it\sqrt{G}} \psi(h \sqrt{G}) \langle x \rangle^{-\sigma} \|_{L^1 \rightarrow L^\infty} h^{(n+1)/2 + p_n(\sigma) + \varepsilon - 1} dh
\leq C\varepsilon |t|^{-(n-1)/2 - \sigma} \int_0^1 h^{-1+\varepsilon/2} dh \leq C\varepsilon |t|^{-(n-1)/2 - \sigma}.
\end{align}

Similarly, using (4.14) we get

\begin{align}
\|\langle x \rangle^{-\sigma} e^{it\sqrt{G}} \left( \sqrt{G} \right)^{\sigma - q_n(\sigma) - \varepsilon} \chi \left( \sqrt{G} \right) \langle x \rangle^{-\sigma} \|_{L^1 \rightarrow L^\infty}
\leq \int_0^1 \|\langle x \rangle^{-\sigma} e^{it\sqrt{G}} \psi_1(h \sqrt{G}) \langle x \rangle^{-\sigma} \|_{L^1 \rightarrow L^\infty} h^{-\sigma + q_n(\sigma) + \varepsilon - 1} dh
\leq C\varepsilon |t|^{-n/2 - \sigma} \int_0^1 h^{-1+\varepsilon/2} dh \leq C\varepsilon |t|^{-n/2 - \sigma}.
\end{align}

To prove (1.18) observe that, if \( k < 1 \) and \( \delta \) and \( \sigma \) satisfy the conditions of Theorem 1.2, we have \( \sigma > q_n(\sigma) \). Hence

\begin{align}
\|\langle x \rangle^{-\sigma} e^{it\sqrt{G}} \chi \left( \sqrt{G} \right) \langle x \rangle^{-\sigma} \|_{L^1 \rightarrow L^\infty}
\leq \int_0^1 \|\langle x \rangle^{-\sigma} e^{it\sqrt{G}} \psi_2(h \sqrt{G}) \langle x \rangle^{-\sigma} \|_{L^1 \rightarrow L^\infty} h^{-1} dh
\leq C\varepsilon |t|^{-n/2 - \sigma} \int_0^1 h^{\sigma - q_n(\sigma) - 1 - \varepsilon/2} dh \leq C\varepsilon |t|^{-n/2 - \sigma}.
\end{align}

\( \square \)
Appendix

In this appendix we will sketch the proof of the estimates (4.3)-(4.6). To this end, we will use the fact that the kernels of the operators $e^{it\sqrt{f_0(h)}}\varphi(P_0(h))$ and $e^{itP_0(h)}\varphi(P_0(h))$ are of the form $K_h(|x-y|,t)$ and $\widetilde{K}_h(|x-y|,t)$, respectively, where

\begin{align}
K_h(w, t) &= \frac{w^{2-n}}{(2\pi)^{n/2}} \int_0^\infty e^{ith\lambda} \varphi(h^2\lambda^2) J_{n-2}(w\lambda) \lambda d\lambda = h^{-n}K_1(w/h, t), \\
\widetilde{K}_h(w, t) &= \frac{w^{2-n}}{(2\pi)^{n/2}} \int_0^\infty e^{ith^2\lambda^2} \varphi(h^2\lambda^2) J_{n-2}(w\lambda) \lambda d\lambda = h^{-n}\widetilde{K}_1(w/h, t),
\end{align}

where $J_{n-2}(z) = z^{(n-2)/2}J_{n-2}(z)$, $J_{n-2}(z)$ being the Bessel function of order $\frac{n-2}{2}$. In view of the inequality

$$\langle x \rangle^{-\sigma} \langle y \rangle^{-\sigma} \geq \langle x - y \rangle^{-\sigma}, \quad \forall \sigma \geq 0,$$

it is easy to see that the estimates (4.3)-(4.6) follow from the following

**Lemma A.1.** For all $w > 0, t \neq 0, 0 < h \leq 1, s \geq 0$, we have

\begin{align}
|K_h(w, t)| &\leq C|t|^{-s}h^{-s-(n+1)/2}g_s(w), \\
\int_{-\infty}^\infty |t|^{2s} |K_h(w, t)|^2 dt &\leq Ch^{-2s-n-1}g_s(w)^2, \\
|\widetilde{K}_h(w, t)| &\leq C|t|^{-s-1/2}h^{-s-(n+1)/2}g_s(w), \\
\int_{-\infty}^\infty |t|^{2s} |\widetilde{K}_h(w, t)|^2 dt &\leq Ch^{-2s-n-1}g_s(w)^2,
\end{align}

where $g_s(w) = w^{s-(n-1)/2}$ if $s \leq (n-1)/2$, $g_s(w) = \langle w \rangle^{s-(n-1)/2}$ if $s \geq (n-1)/2$.

**Proof.** In view of the identities (A.1) and (A.2), it is clear that it suffices to prove (A.3)-(A.6) for $h = 1$. Let first $w \leq 1$. Recall that near $z = 0$ the function $J_{n-2}(z)$ is equal to $z^{n-2}$ times an analytic function. Using this and integrating by parts, it is easy to see that in this case the functions $K_1$ and $\widetilde{K}_1$ satisfy the bounds

\begin{align}
|K_1(w, t)| &\leq C|t|^{-s}, \\
|\widetilde{K}_1(w, t)| &\leq C|t|^{-s-1/2},
\end{align}

for every $s \geq 0$. Clearly, when $w \leq 1$ the estimates (A.3)-(A.6) follow from (A.7) and (A.8). Let now $w \geq 1$. In this case we will use the fact that for $z \gg 1$ the function $J_{n-2}(z)$ is of the form $e^{iz}b^+(z) + e^{-iz}b^-(z)$, where $b^\pm(z)$ are symbols of order $\frac{n-3}{2}$. Given any integers $k, \ell \geq 0$, set

$$b_k^\pm(z) = e^{iz} \frac{d^k}{dz^k} \left( e^{\pm iz}b^\pm(z) \right), \quad b_{k,\ell}^\pm(z) = \frac{d^{\ell}}{dz^{\ell}} b_k^\pm(z).$$

Clearly, $b_k^\pm(z)$ are also symbols of order $\frac{n-3}{2}$. Hence

\begin{align}
\left| b_{k,\ell}^\pm(z) \right| &\leq C_{k,\ell}z^{\frac{n-3}{2}-\ell}, \quad \forall z \geq 1.
\end{align}
Let $m, N \geq 0$ be integers. Integrating $m$ times by parts, we can write

$$K_1(w, t) = \frac{w^{2-n}}{(2\pi)^{n/2}} (it)^{-m} \sum_{k=0}^{m} \int_0^\infty e^{i(t\pm \omega)} \psi_k^+ b_k^\pm (\omega \lambda) \varphi_{k,m}(\lambda) d\lambda,$$

with some functions $\varphi_{k,m} \in C_0^\infty((0, +\infty))$ independent of $w$ and $t$. We now integrate $N$ times by parts to obtain

$$K_1(w, t) = \frac{w^{2-n}}{(2\pi)^{n/2}} (it)^{-m} \sum_{k=0}^{m} \sum_{\ell=0}^{N} (t \pm w)^{-N} \int_0^\infty e^{i(t\pm \omega)} \psi_k^+ b_k^\pm (\omega \lambda) \varphi_{k,m,N}(\lambda) d\lambda.$$

Hence, in view of (A.9), we get the bound

$$|K_1(w, t)| \leq C_{m,N} w^{m-n-\frac{3}{2}} |t|^{-m} \left( |t-w|^{-N} + |t+w|^{-N} \right). \quad (A.10)$$

By interpolation, (A.10) holds for all real $m \geq 0$. It is easy to see now that the estimates (A.3) and (A.4) (with $h = 1$) follow from (A.10).

Integrating by parts $m$ times with respect to the variable $\lambda^2$ we can write the function $\tilde{K}_1$ as follows

$$\tilde{K}_1(w, t) = \frac{w^{2-n}}{(2\pi)^{n/2}} (it)^{-m} \sum_{k=0}^{m} \int_0^\infty e^{i\lambda^2} \tilde{\varphi}_{k,m}(\lambda) \frac{d^k}{d\lambda^k} \mathcal{J}_{w-\frac{1}{2}}(\lambda w) d\lambda$$

$$= \frac{w^{2-n}}{(2\pi)^{n/2}} (it)^{-m} \sum_{k=0}^{m} \int_0^\infty e^{i\lambda^2} f_{k,m}(w, \lambda) d\lambda,$$

where

$$f_{k,m}(w, \lambda) = \varphi_{k,m}^+(\lambda) \frac{d^k}{d\lambda^k} \mathcal{J}_{w-\frac{1}{2}}(\lambda w).$$

We now apply the inequality

$$\left| \int_0^\infty e^{i\lambda^2} f(\lambda) d\lambda \right| \leq C |t|^{-1/2} \left\| f \right\|_{L^1}, \quad \forall f \in C_0^\infty(\mathbb{R}),$$

to get

$$\left| \tilde{K}_1(w, t) \right| \leq C_{m} w^{2-n} |t|^{m-1/2} \sum_{k=0}^{m} \left\| \tilde{f}_{k,m}(\cdot, w) \right\|_{L^1}. \quad (A.11)$$

On the other hand, as above one can see that the function $\tilde{f}_{k,m}$ satisfies the bound

$$\left| \tilde{f}_{k,m}(\tau, w) \right| \leq C_N w^{k+\frac{n-3}{2}} \left( |\tau-w|^{-N} + |\tau+w|^{-N} \right) \quad (A.12)$$

for every integer $N \geq 0$. By (A.12)

$$\left\| \tilde{f}_{k,m}(\cdot, w) \right\|_{L^1} \leq C w^{k+\frac{n-3}{2}}. \quad (A.13)$$

By (A.11) and (A.13)

$$\left| \tilde{K}_1(w, t) \right| \leq C_m w^{m-\frac{n-1}{2}} |t|^{-m-1/2}$$

(A.14)
for every integer $m \geq 0$, and hence by interpolation for all real $m \geq 0$, which in turn proves (A.5). It is easy also to see that (A.6) (with $h = 1$) follows from (A.14). Indeed, applying (A.14) with $m = s - \epsilon$ and $m = s + \epsilon$, we have

$$\int_{-\infty}^{\infty} |t|^{2s-2\epsilon-n+1} \left| \widetilde{K}_h(w,t) \right|^2 dt \leq Cw^{2s-2\epsilon-n+1} \int_{|t| \leq w} |t|^{-1+2\epsilon} dt + Cw^{2s+2\epsilon-n+1} \int_{|t| \geq w} |t|^{-1-2\epsilon} dt \leq Cw^{2s-n+1}.$$

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