Non-uniform approximations for sums of discrete m-dependent random variables

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Abstract

Non-uniform estimates are obtained for Poisson, compound Poisson, translated Poisson, negative binomial and binomial approximations to sums of of m-dependent integer-valued random variables. Estimates for Wasserstein metric also follow easily from our results. The results are then exemplified by the approximation of Poisson binomial distribution, 2-runs and m-dependent \((k_1, k_2)\)-events.

Key words: Poisson distribution, compound Poisson distribution, translated Poisson distribution, negative binomial distribution, binomial distribution, m-dependent variables, Wasserstein norm, non-uniform estimates.

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1 Introduction

Nonuniform estimates for normal approximation are well known, see the classical results in Chapter 5 of [12] and the references [9], [10] and [19] for some recent developments. On the other hand, nonuniform estimates for discrete approximations are only a few. For example, the Poisson approximation to Poisson binomial distribution has been considered in [18] and translated Poisson approximation for independent lattice summands via the Stein method has been discussed in [2]. Some general estimates for independent summands under assumption of matching of pseudomoments were obtained in [6]. For possibly dependent Bernoulli variables, nonuniform estimates for Poisson approximation problems were discussed in [20]. However, the estimates obtained had a better accuracy than estimates in total variation only for $x$ larger than exponent of the sum’s mean. In [7], 2-runs statistic was approximated by compound Poisson distribution. In this paper, we obtain nonuniform estimates for Poisson, compound Poisson, translated Poisson, negative binomial and binomial approximations, under a quite general set of assumptions.

We recall that the sequence of random variables $\{X_k\}_{k \geq 1}$ is called $m$-dependent if, for $1 < s < t < \infty$, $t - s > m$, the sigma algebras generated by $X_1, \ldots, X_s$ and $X_t, X_{t+1}, \ldots$ are independent. Without loss of generality, we can reduce the sum of $m$-dependent variables to the sum of 1-dependent ones, by grouping consecutive $m$ summands. Therefore, we consider henceforth, without loss of generality, the sum $S_n = X_1 + X_2 + \cdots + X_n$ of non-identically distributed 1-dependent random variables concentrated on nonnegative integers.

We denote the distribution function and the characteristic function of $S_n$ by $F_n(x)$ and $\hat{F}_n(t)$, respectively. Similarly, for a signed measure $M$ concentrated on the set $N$ of nonnegative integers, we denote by $M(x) = \sum_{k=0}^x M\{k\}$ and $\hat{M}(t) = \sum_{k=0}^\infty e^{itk}M\{k\}$, the analogues of distribution function and Fourier-Stieltjes transform, respectively. Though our aim is to obtain the non-uniform estimates, we obtain also estimates for Wasserstein norm defined as

$$\|M\|_W = \sum_{j=0}^\infty |M(j)|.$$  

Note that Wasserstein norm is stronger than total variation norm defined by $\|M\| = \sum_{j=0}^\infty |M\{j\}|$. 
Next we introduce the approximations considered in this paper. Let

\[ \lambda = E S_n, \quad \Gamma_2 = \frac{1}{2} (\text{Var} S_n - E S_n). \]

For brevity, let \( z(t) = e^{it} - 1 \). Also, let \( \Pi \) and \( \Pi_1 \) respectively denote the Poisson distribution with parameter \( \lambda \) and its second order difference multiplied by \( \Gamma_2 \). More precisely,

\[ \widehat{\Pi}(t) = \exp\{\lambda z\}, \quad \widehat{\Pi}_1(t) = \widehat{\Pi}(t) \Gamma_2 z^2. \]

It is clear that \( \Pi + \Pi_1 \) is second-order (and, consequently, two-parametric) Poisson approximation. As an alternative to the Poisson based two-parametric approximation, we choose compound Poisson measure \( G \) with the following Fourier-Stieltjes transform

\[ \widehat{G}(t) = \exp\{\lambda z + \Gamma_2 z^2\}. \]

The approximation \( G \) was used in many papers, see \[1\], \[3\], \[16\] and the references therein. If \( \Gamma_2 < 0 \), then \( G \) becomes signed measure, which is not always convenient and natural for approximation to nonnegative \( S_n \). Therefore, we define next three distributional approximations. Translated Poisson \((TP)\) approximation has the following characteristic function:

\[ \widehat{TP}(t) = \exp\{[-2\Gamma_2] it + (\lambda + 2\Gamma_2 + \tilde{\delta})z\} = \exp\{\lambda z + (2\Gamma_2 + \tilde{\delta})(z - it)\}. \]

Here \([-2\Gamma_2]\) and \(\tilde{\delta}\) are respectively the integer part and the fractional part of \(-2\Gamma_2\), so that \(-2\Gamma_2 = [-2\Gamma_2] + \tilde{\delta}, \quad 0 \leq \tilde{\delta} < 1\). The TP approximation was investigated in numerous papers, see, for example, \[1\], \[2\], \[13\] and \[14\]. If \( E S_n < \text{Var} S_n \), then one can apply the negative binomial \((NB)\) approximation, which is defined in the following way:

\[ \text{NB}\{j\} = \frac{\Gamma(r + j)}{j!\Gamma(r)} q^r (1 - q)^j, \quad (j \in \mathbb{Z}_+), \quad \frac{r(1 - \frac{\theta}{q})}{q} = \lambda, \quad \frac{1 - \frac{\theta}{q}}{q} = 2\Gamma_2. \]

Note that

\[ \widehat{\text{NB}}(t) = \left(\frac{\frac{\theta}{1 - \frac{\theta}{q}}e^{it}}{\frac{\theta}{1 - \frac{\theta}{q}}e^{it}}\right)^r = \left(1 - \frac{1 - \frac{\theta}{q}}{\frac{\theta}{q}}\right)^{-r}. \]
If \( \text{Var} S_n < E S_n \), the more natural approximation is the binomial one defined as follows:

\[
\hat{\text{Bi}}(t) = (1 + \bar{p} z)^N, \quad N = \lfloor \tilde{N} \rfloor, \quad \tilde{N} = \frac{\chi^2}{2|\Gamma_2|}, \quad \bar{p} = \frac{\lambda}{N}.
\]

Note that symbols \( \overline{q} \) and \( \overline{p} \) are not related and, in general, \( \overline{q} + \overline{p} \neq 1 \).

Finally, we introduce some technical notations, related to the method of proof. Let \( \{Y_k\}_{k \geq 1} \) be a sequence of arbitrary real or complex-valued random variables. We assume that \( \hat{E}(Y_1) = E Y_1 \) and, for \( k \geq 2 \), define \( \hat{E}(Y_1, Y_2, \cdots, Y_k) \) by

\[
\hat{E}(Y_1, Y_2, \cdots, Y_k) = E Y_1 Y_2 \cdots Y_k - \sum_{j=1}^{k-1} \hat{E}(Y_1, \cdots, Y_j)EY_{j+1} \cdots Y_k.
\]

Let

\[
\hat{E}^+(X_1) = EX_1, \quad \hat{E}^+(X_1, X_2) = EX_1 X_2 + EX_1 EX_2,
\]

\[
\hat{E}^+(X_1, \ldots, X_k) = EX_1 \cdots X_k + \sum_{j=1}^{k-1} \hat{E}^+(X_1, \ldots, X_j)EX_{j+1}X_{j+2} \cdots X_k,
\]

\[
\hat{E}^+_2(X_{k-1}, X_k) = \hat{E}^+(X_{k-1}(X_{k-1} - 1), X_k) + \hat{E}^+(X_{k-1}, X_k(X_k - 1)),
\]

\[
\hat{E}^+_2(X_{k-2}, X_{k-1}, X_k) = \hat{E}^+(X_{k-2}(X_{k-2} - 1), X_{k-1}, X_k) + \hat{E}^+(X_{k-2}, X_{k-1}(X_{k-1} - 1), X_k).
\]

We define the \( j \)-th factorial moment of \( X_k \) by \( \nu_j(k) = \mathbb{E}X_k(X_k - 1) \cdots (X_k - j + 1), \) \( k = 1, 2, \ldots, n, \) \( j = 1, 2, \ldots \). For the sake of convenience, we assume that \( X_k \equiv 0 \) and \( \nu_j(k) = 0 \) if \( k \leq 0 \) and \( \sum_k^n = 0 \) if \( k > n \). Next we define remainder terms \( R_0 \) and \( R_1 \), which appear in the main results, as

\[
R_0 = \sum_{k=1}^{n} \left\{ \nu_2(k) + \nu_1^2(k) + \mathbb{E}X_{k-1}X_k \right\},
\]

\[
R_1 = \sum_{k=1}^{n} \left\{ \nu_3^2(k) + \nu_1(k)\nu_2(k) + \nu_3(k) + [\nu_1(k - 2) + \nu_1(k - 1) + \nu_1(k)]\mathbb{E}X_{k-1}X_k 
\]

\[
+ \hat{E}^+_2(X_{k-1}, X_k) + \hat{E}^+(X_{k-2}, X_{k-1}, X_k) \right\}.
\]

We use symbol \( C \) to denote (in general different) positive absolute constants.
2 The Main Results

All the results are obtained under the following conditions:

\begin{eqnarray}
\nu_1(k) & \leq & 1/100, \quad \nu_2(k) \leq \nu_1(k), \quad |X_k| \leq C_0, \quad (k = 1, 2, \ldots, n), \\
\lambda & \geq & 1, \quad \sum_{k=1}^n \nu_2(k) \leq \frac{\lambda}{20}, \quad \sum_{k=2}^n |\text{Cov}(X_{k-1}, X_k)| \leq \frac{\lambda}{20}.
\end{eqnarray}

Assumptions (1) and (2) are rather restrictive. However, they (a) allow to include independent random variables as partial case of general results and (b) are satisfied for many cases of \(k\)-runs and \((k_1, k_2)\) events. The method of proof does not allow to get small constants. Therefore, we have concentrated our efforts on the order of the accuracy of approximation. Next, we state the main results of this paper.

**Theorem 2.1** Let conditions (3) and (2) be satisfied. Then, for any \(x \in \mathbb{N}\),

\begin{eqnarray}
\left(1 + \frac{(x - \lambda)^2}{\lambda}\right)|F_n(x) - \Pi(x)| & \leq & C_1 \frac{R_0}{\lambda}, \\
\left(1 + \frac{(x - \lambda)^2}{\lambda}\right)|F_n(x) - \Pi(x) - \Pi_1(x)| & \leq & C_2 \left(\frac{R_0^2}{\lambda^2} + \frac{R_1}{\lambda^2 \lambda}\right), \\
\left(1 + \frac{(x - \lambda)^2}{\lambda}\right)|F_n(x) - G(x)| & \leq & C_3 \frac{R_1}{\lambda \lambda \lambda}, \\
\left(1 + \frac{(x - \lambda)^2}{\lambda}\right)|F_n(x) - \text{TP}(x)| & \leq & C_4 \left(\frac{R_1}{\lambda \lambda \lambda} + \frac{\Gamma_2}{\lambda} + \frac{\tilde{\delta}}{\lambda}\right).
\end{eqnarray}

If in addition \(\Gamma_2 > 0\), then

\begin{equation}
\left(1 + \frac{(x - \lambda)^2}{\lambda}\right)|F_n(x) - \text{NB}(x)| \leq C_5 \left(\frac{R_1}{\lambda \lambda \lambda} + \frac{\Gamma_2^2}{\lambda^2 \lambda^2 \lambda}\right).
\end{equation}

If instead \(\Gamma_2 < 0\), then

\begin{equation}
\left(1 + \frac{(x - \lambda)^2}{\lambda}\right)|F_n(x) - \text{Bi}(x)| \leq C_6 \left(\frac{R_1}{\lambda \lambda \lambda} + \frac{\Gamma_2^2}{\lambda^2 \lambda^2 \lambda}\right).
\end{equation}

**Remark 2.1** Nonuniform normal estimates usually match estimates in Kolmogorov metric. Sim-
ilarly, the bounds in (3)-(8) match estimates in total variation:

\[ \|F_n - \Pi\| \leq C_7 \frac{R_0}{\lambda}, \quad \|F_n - \Pi - \Pi_1\| \leq C_8 \left( \frac{R_0^2}{\lambda^2} + \frac{R_1}{\lambda \sqrt{\lambda}} \right), \quad \|F_n - G\| \leq C_9 \frac{R_1}{\lambda \sqrt{\lambda}}, \]

and etc., see [8].

Estimates for Wasserstein metric easily follow by summing up nonuniform estimates.

**Theorem 2.2** Let conditions (1) and (2) be satisfied. Then,

\[ \|F_n - \Pi\|_W \leq C_{10} \frac{R_0}{\sqrt{\lambda}}, \]

(9)

\[ \|F_n - \Pi - \Pi_1\|_W \leq C_{11} \left( \frac{R_0^2}{\lambda \sqrt{\lambda}} + \frac{R_1}{\lambda} \right), \]

(10)

\[ \|F_n - G\|_W \leq C_{12} \frac{R_1}{\lambda}, \]

(11)

\[ \|F_n - TP\|_W \leq C_{13} \left( \frac{R_1 + |\Gamma_2|}{\lambda} + \frac{\delta}{\sqrt{\lambda}} \right). \]

(12)

When in addition \( \Gamma_2 > 0 \), we have

\[ \|F_n - NB\|_W \leq C_{14} \left( \frac{R_1}{\lambda} + \frac{\Gamma_2^2}{\lambda^2} \right), \]

(13)

and when \( \Gamma_2 < 0 \), we have

\[ \|F_n - Bi\|_W \leq C_{15} \left( \frac{R_1}{\lambda} + \frac{\Gamma_2^2}{\lambda^2} \right). \]

(14)

Observe that the local nonuniform estimates have better order of accuracy.

**Theorem 2.3** Let conditions (1) and (2) hold. Then, for any \( x \in \mathbb{N} \),

\[ \left( 1 + \frac{(x - \lambda)^2}{\lambda} \right) |F_n\{x\} - \Pi\{x\}| \leq C_{16} \frac{R_0}{\lambda \sqrt{\lambda}}, \]

(15)

\[ \left( 1 + \frac{(x - \lambda)^2}{\lambda} \right) |F_n\{x\} - \Pi\{x\} - \Pi_1\{x\}| \leq C_{17} \left( \frac{R_0^2}{\lambda^2 \sqrt{\lambda}} + \frac{R_1}{\lambda^2} \right), \]

(16)

\[ \left( 1 + \frac{(x - \lambda)^2}{\lambda} \right) |F_n\{x\} - G\{x\}| \leq C_{18} \frac{R_1}{\lambda^2}, \]

(17)

\[ \left( 1 + \frac{(x - \lambda)^2}{\lambda} \right) |F_n\{x\} - TP\{x\}| \leq C_{19} \left( \frac{R_1 + |\Gamma_2|}{\lambda^2} + \frac{\delta}{\lambda \sqrt{\lambda}} \right). \]

(18)
If in addition $\Gamma_2 > 0$, then
\[
\left(1 + \frac{(x - \lambda)^2}{\lambda}\right)|F_n\{x\} - \text{NB}\{x\}| \leq C_{20}\left(\frac{R_1}{\lambda^2} + \frac{\Gamma_2^2}{\lambda^3}\right). \tag{19}
\]

If instead $\Gamma_2 < 0$, then
\[
\left(1 + \frac{(x - \lambda)^2}{\lambda}\right)|F_n\{x\} - \text{Bi}\{x\}| \leq C_{21}\left(\frac{R_1}{\lambda^2} + \frac{\Gamma_2^2}{\lambda^3}\right). \tag{20}
\]

Remark 2.2 (i) Estimates in (15)-(20) match estimates in local metric, see [8].

(ii) Consider the case of independent Bernoulli variables with $p \leq 1/20$ and $\lambda \geq 1$. Then, for all integers $x$, Poisson approximation is of the order
\[
C\sum_{j=1}^n p_j^2 \frac{1}{(1 + (x - \lambda)^2/\lambda)\lambda\sqrt{\lambda}},
\]
which is usually much better than
\[
\min(x^{-1}, \lambda^{-1}) \sum_{j=1}^n p_j^2
\]
from [17].

3 Some Applications

(i): Asymptotically sharp constant for Poisson approximation to Poisson binomial distribution. Formally, independent random variables make a subset of 1-dependent variables. Therefore, one can rightly expect that results of the previous section apply to independent summands as well. We exemplify this fact by considering one of the best known cases in Poisson approximation theory. Let $W = \xi_1 + \xi_2 + \cdots + \xi_n$, where $\xi_i$ are independent Bernoulli variables with $P(\xi_i = 1) = 1 - P(\xi_i = 0) = p_i$. Let $\lambda = \sum_1^n p_i$, $\lambda_2 = \sum_1^n p_i^2$. As shown in [4] (see equation (1.8)),
\[
\|\mathcal{L}(W) - \Pi\|_W \leq \frac{1.1437\lambda_2}{\sqrt{\lambda}}. \tag{21}
\]
Though absolute constant in (21) is small, we shall show that asymptotically sharp constant is much smaller. Let $\max_i p_i \to 0$ and $\lambda \to \infty$, as $n \to \infty$. Then

$$\lim_{n \to \infty} \frac{\sqrt{\lambda}}{\lambda^2} \|\mathcal{L}(W) - \Pi\|_W = \frac{1}{\sqrt{2\pi}} \leq 0.399.$$  \hfill (22)

Indeed, we have

$$\left|\|\mathcal{L}(W) - \Pi\|_W - \frac{\lambda_2}{\sqrt{2\pi\lambda}}\right| \leq \|\mathcal{L}(W) - \Pi - \Pi_1\|_W + \left|\left\|\Pi_1\right\|_W - \frac{\lambda_2}{\sqrt{2\pi\lambda}}\right|.$$  

If $\max_i p_i \leq 1/20$ and $\lambda \geq 1$, then it follows from (10) that

$$\|\mathcal{L}(W) - \Pi - \Pi_1\|_W \leq \frac{C\lambda_2}{\sqrt{\lambda}} \left(\max_j \eta_j + \frac{1}{\sqrt{\lambda}}\right).$$

For the estimation of the second difference, we require some notations for measures. Let $Z$ be a measure, corresponding to Fourier-Stieltjes transform $z(t) = (e^{it} - 1)$. Let product and powers of measures be understood in the convolution sense. Then, by the properties of norms and Proposition 4 from [15] (see also Lemma 6.2 in [8]), we get

$$\left|\left\|\Pi_1\right\|_W - \frac{\lambda_2}{\sqrt{2\pi\lambda}}\right| = \left|\frac{\lambda_2}{2} \left\|\Pi Z^2\right\|_W - \frac{\lambda_2}{\sqrt{2\pi\lambda}}\right| = \left|\frac{\lambda_2}{2} \left\|\Pi Z^2\right\|_W - \frac{\sqrt{2/\pi}}{\sqrt{\lambda}}\right| = \frac{\lambda_2}{2} \left\|\Pi Z\right\| - \frac{\sqrt{2/\pi}}{\sqrt{\lambda}} \leq \frac{C\lambda_2}{2\lambda} = \frac{\lambda_2}{\sqrt{\lambda}2\sqrt{\lambda}}.$$  

Thus, for $\max_i p_i \leq 1/20$ and $\lambda \geq 1$, we obtain asymptotically sharp norm estimate

$$\left|\|\mathcal{L}(W) - \Pi\|_W - \frac{\lambda_2}{\sqrt{2\pi\lambda}}\right| \leq \frac{C\lambda_2}{\sqrt{\lambda}} \left(\max_j \eta_j + \frac{1}{\sqrt{\lambda}}\right),$$

which is even more general than (22).

(ii): **Negative binomial approximation to 2-runs.** The k-runs (and especially 2-runs) statistic is one of the best investigated cases of sums of dependent discrete random variables, see [22] and the references therein. Let $S_n = X_1 + X_2 + \cdots + X_n$, where $X_i = \eta_i\eta_{i+1}$ and $\eta_j \sim Be(p)$, $(j = 1, 2, \ldots, n + 1)$ are independent Bernoulli variables. Then $S_n$ is called 2-runs statistic. It is
known that then
\[ \lambda = np^2, \quad \Gamma_2 = \frac{np^3(2 - 3p) - 2p^3(1 - p)}{2}. \]

Let \( p \leq 1/20 \) and \( np^2 \geq 1 \). Then, from (7) it follows for any \( x \in \mathbb{N} \),
\[ \left( 1 + \frac{(x - \lambda)^2}{\lambda} \right) |F_n(x) - NB(x)| \leq C \frac{p}{\sqrt{n}}. \]
This estimate has the same order as the estimate in total variation, see and [5] and [8].

(iii): Binomial approximation to \((k_1, k_2)\)-events. Let \( \eta_i \) be independent Bernoulli \( Be(p) \) \((0 < p < 1)\) variables and let \( Y_j = (1 - \eta_{j-m+1}) \cdots (1 - \eta_{j-k_2}) \eta_{j-k_2+1} \cdots \eta_{j-1} \eta_j, j = m, m+1, \ldots, n, k_1 + k_2 = m \). Further, we assume that \( k_1 > 0 \) and \( k_2 > 0 \). Then \( N(n; k_1, k_2) = Y_m + Y_{m+1} + \cdots + Y_n \) denote the number of \((k_1, k_2)\)-events and we denote its distribution by \( H \). The Poisson approximation to \( H \) has been considered in [21]. Let \( a(p) = (1 - p)^{k_1} p^{k_2} \).

Note that \( Y_1, Y_2, \ldots \) are \( m \)-dependent. However, one can group summands in the following natural way:
\[
N(n; k_1, k_2) = (Y_m + Y_{m+1} + \cdots + Y_{2m-1}) + (Y_{2m} + Y_{2m+1} + \cdots + Y_{3m-1}) + \cdots
\]
\[
= X_1 + X_2 + \ldots.
\]
Each \( X_j \), with probable exception of the last one, contains \( m \) summands. It is not difficult to check that \( X_1, X_2, \ldots \) are 1-dependent Bernoulli variables. Then all parameters can be written explicitly.
Set \( N = \lfloor \tilde{N} \rfloor \) be the integer part of \( \tilde{N} \) defined by
\[
\tilde{N} = \frac{(n - m + 1)^2}{(n - m + 1)(2m - 1) - m(m - 1)}, \quad \bar{p} = \frac{(n - m + 1)a(p)}{N}.
\]
It is known (see [8]) that
\[
\lambda = (n - m + 1)a(p), \quad \Gamma_2 = -\frac{a^2(p)}{2}[(n - m + 1)(2m - 1) - m(m - 1)], \quad R_1 \leq C(n - m + 1)m^2 a^3(p).
\]
Let now \( \lambda \geq 1 \) and \( ma(p) \leq 0.01 \). Then it follows from \( \Box \) that, for any \( x \in \mathbb{N} \),

\[
\left(1 + \frac{(x-\lambda)^2}{\lambda}\right) |H(x) - Bi(x)| \leq C \frac{a^{3/2}(p)m^2}{\sqrt{n - m + 1}}.
\]

4 Auxiliary results

Let \( \theta \) to denote a real or complex quantity satisfying \( |\theta| \leq 1 \). Moreover, let \( Z_j = \exp\{itX_j\} - 1 \), \( \Psi_{j,k} = \hat{E}(Z_jZ_{j+1},\ldots,Z_k) \). As before, we assume that \( \nu_j(k) = 0 \) and \( X_k = 0 \) for \( k \leq 0 \) and \( z(t) = e^{it} - 1 \). Also, we omit the argument \( t \), wherever possible and, for example, write \( z \) instead of \( z(t) \). Hereafter, the primes denote the derivatives with respect to \( t \).

Lemma 4.1 Let \( X \) be concentrated on nonnegative integers and \( \nu_3 < \infty \). Then, for all \( t \in \mathbb{R} \),

\[
E\exp\{itX\} = 1 + \nu_1z + \nu_2\frac{z^2}{2} + \theta\nu_3|z|^3 + \frac{6}{6},
\]

\[
E(\exp\{itX\})' = \nu_1z' + \nu_2\frac{(z^2)'}{2} + \theta\nu_3|z|^2 + \frac{2}{2},
\]

\[
E(\exp\{itX\})'' = \nu_1z'' + \nu_2\frac{(z^2)''}{2} + \theta\nu_3|z|. \]

Proof. First equality is well known expansion of characteristic function in factorial moments. The other two equalities also easily follow from expansions in powers of \( z \). For example,

\[
(e^{itX})'' = i^2X^2e^{itX} = i^2X(X - 1)(e^{it})^2e^{it(X - 2)} + i^2e^{it}Xe^{it(X - 1)}
\]

\[
= i^2(e^{it})^2X(X - 1)[1 + \theta(X - 2)|z|] + i^2e^{it}X[1 + (X - 1)z + \theta(X - 1)(X - 2)|z|^2/2]
\]

\[
= Xz'' + \frac{X(X - 1)}{2}(z^2)' + \theta 2X(X - 1)(X - 2)|z|. \quad \Box (23)
\]

Lemma 4.2 ([14]) Let \( Y_1, Y_2, \ldots, Y_k \) be 1-dependent complex-valued random variables with \( E|Y_m|^2 < \infty \), \( 1 \leq m \leq k \). Then

\[
|\hat{E}(Y_1, Y_2, \ldots, Y_k)| \leq 2^{k-1} \prod_{m=1}^{k} (E|Y_m|^2)^{1/2}.
\]
Lemma 4.3 Let conditions (I) be satisfied and \( j < k - 1 \). Then, for all \( t \),

\[
|\Psi_{j,k}| \leq 4^{k-j}|z| \prod_{l=j}^{k} \sqrt{\nu_1(l)},
\]  

(24)

\[
|\Psi'_{j,k}| \leq 4^{k-j}|z|(k-j+1) \prod_{l=j}^{k} \sqrt{\nu_1(l)},
\]  

(25)

\[
|\Psi''_{j,k}| \leq \sqrt{2}C_04^{k-j}|z|(k-j)(k-j) \prod_{l=j}^{k} \sqrt{\nu_1(l)}.
\]  

(26)

**Proof.** First two estimates follow from more general estimates in (47) and Lemma 7.5 in [8]. Note also the following inequalities:

\[
|z| \leq 2, \quad |Z_k| \leq 2, \quad |Z_k| \leq X_k|z|, \quad \text{EX}_i^2 = \nu_2(i) + \nu_1(i) \leq 2\nu_1(i).
\]  

(27)

Therefore, by Lemma 4.2 and for \( m \leq k \),

\[
|\hat{E}(Z_j,\ldots,Z'_m,\ldots,Z'_i,\ldots Z_k)| \leq 2^{k-j} \sqrt{E|Z'_m|^2E|Z'_i|^2} \prod_{l=j,l\neq m,i}^{k} \sqrt{2}|z|\nu_1(l)
\]  

\[
\leq 2^{k-j} \sqrt{2\nu_1(m)2\nu_1(i)}2^{(k-j-1)/2}|z|(k-j-1)^{1/2} \prod_{l=j,l\neq m,i}^{k} \sqrt{\nu_1(l)} \leq 4^{k-j}2^{-1}|z| \prod_{l=j}^{k} \sqrt{\nu_1(l)}.
\]

Similarly,

\[
|\hat{E}(Z_j,\ldots,Z''_m,\ldots,Z''_i,\ldots Z_k)| \leq 2^{k-j} \sqrt{E|Z''_i|^2} \prod_{l=j,l\neq i}^{k} \sqrt{2}|z|\nu_1(l)
\]  

\[
\leq 2^{k-j} \sqrt{\text{EX}_i^42^{(k-j)/2}2^{(k-j)/2}} \prod_{l=j,l\neq i}^{k} \sqrt{\nu_1(l)}
\]  

\[
\leq 4^{k-j}2^{-1}|z|C_0\sqrt{\text{EX}_i^2} \prod_{l=j,l\neq i}^{k} \sqrt{\nu_1(l)} \leq 4^{k-j}2^{-1/2}C_0 \prod_{l=j}^{k} \sqrt{\nu_1(l)}.
\]
Thus,

\[ |\Psi''_{j,k}| \leq \sum_{i=j}^{k} |\hat{E}(Z_j, \ldots, Z'_i, \ldots, Z_k)| + \sum_{i=j}^{k} \sum_{m=j, m \neq i} \sum_{m=j}^{k} |\hat{E}(Z_j, \ldots, Z'_m, \ldots, Z'_i, \ldots, Z_k)| \]

\[ \leq (k - j + 1)4^{k-j}C_02^{-1/2}|z| \prod_{l=j}^{k} \sqrt{\nu_1(l)} + (k - j + 1)(k - j)4^{k-j}2^{-1}|z| \prod_{l=j}^{k} \sqrt{\nu_1(l)} \]

\[ \leq \sqrt{2}C_04^{k-j}(k - j + 1)(k - j)|z| \prod_{l=j}^{k} \sqrt{\nu_1(l)}. \quad \square \]

In the following Lemmas 4.4–4.5, we present some facts about characteristic function \( \hat{F}_n(t) \) from [8]. Here again we assume (1), though many relations hold also under weaker assumptions, see [8]. We begin from Heinrich’s representation of \( \hat{F}_n(t) \) as product of functions.

**Lemma 4.4** Let (1) hold. Then \( \hat{F}_n(t) = \varphi_1(t)\varphi_2(t) \ldots \varphi_n(t) \), where \( \varphi_1(t) = Ee^{itX_1} \) and, for \( k = 2, \ldots, n \),

\[ \varphi_k = 1 + EZ_k + \sum_{j=1}^{k-1} \frac{\Psi_{j,k}}{\varphi_j \varphi_{j+1} \cdots \varphi_{k-1}}. \quad (28) \]

Let

\[ g_j(t) = \exp \left\{ \nu_1(j)z + \left( \frac{\nu_2(j) - \nu_1^2(j)}{2} + \hat{E}(X_{j-1}, X_j) \right)z^2 \right\}, \]

\[ \lambda_k = 1.6\nu_1(k) - 0.3\nu_1(k - 1) - 2\nu_2(k) - 0.1EX_{k-2}X_{k-1} - 15.58EX_{k-1}X_k, \]

\[ \gamma_2(k) = \frac{\nu_2(k)}{2} + \hat{E}(X_{k-1}, X_k), \]

\[ \gamma_2(k) = \frac{\nu_2(k)}{2} + \hat{E}(X_{k-1}, X_k), \]

\[ \gamma_2(k) = \nu_3(k) + \sum_{l=0}^{5} \nu_1^{(l)}(k - l) + [\nu_1(k - 1) + \nu_1(k - 2)]EX_{k-1}X_k + \hat{E}^+(X_{k-1}, X_k) \]

\[ + \hat{E}^+(X_{k-2}, X_{k-1}, X_k), \]

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Lemma 4.5 Let the conditions in (1) hold. Then

\[ \frac{1}{|\varphi_k|} \leq \frac{10}{9}, \]  
(29)

\[ |\varphi_k| \leq \exp\{-\lambda_k \sin^2(t/2)\}, \quad |g_k| \leq \exp\{-\lambda_k \sin^2(t/2)\} \]  
(30)

\[ \frac{1}{\varphi_{k-1}} = 1 + C\theta |z| \{\nu_1 (k - 2) + \nu_1 (k - 1)\}; \]  
(31)

\[ \varphi_k' = 33\theta [\nu_1 (k) + \nu_1 (k - 1)], \]  
(32)

\[ \sum_{k=1}^{n} |\varphi_k - g_k| \leq CR_1 |z|^3, \quad \sum_{k=1}^{n} |\varphi_k' - g_k'| \leq CR_1 |z|^2. \]  
(33)

Similar estimates hold for the second derivative, as seen in the next lemma.

Lemma 4.6 Let (1) hold. Then, for \( k = 1, 2, \ldots, n \),

\[ \varphi_k'' = \theta C_{22} [\nu_1 (k) + \nu_1 (k - 1)], \]  
(34)

\[ \varphi_k'' = \nu_1 (k) z'' + \gamma_2 (k)(z^2)'' + \theta C |z| r_1 (k). \]  
(35)

Proof. From Lemma 4.4 it follows that

\[ \varphi_k'' = (EZ_k)'' + \sum_{j=1}^{k-1} \frac{\Psi_{j,k}}{\varphi_j \cdots \varphi_{k-1}} - 2 \sum_{j=1}^{k-1} \frac{\Psi_{j,k}'}{\varphi_j \cdots \varphi_{k-1}} \sum_{i=j}^{k-1} \frac{\varphi_i'}{\varphi_i} \]

\[ + \sum_{j=1}^{k-1} \frac{\Psi_{j,k}}{\varphi_j \cdots \varphi_{k-1}} \left( \sum_{i=j}^{k-1} \frac{\varphi_i'}{\varphi_i} \right)^2 + \sum_{j=1}^{k-1} \frac{\Psi_{j,k}}{\varphi_j \cdots \varphi_{k-1}} \sum_{i=j}^{k-1} \left( \frac{\varphi_i'}{\varphi_i} \right)^2 \]

\[ - \sum_{j=1}^{k-1} \frac{\Psi_{j,k}}{\varphi_j \cdots \varphi_{k-1}} \sum_{i=j}^{k-1} \frac{\varphi_i'}{\varphi_i}. \]  
(36)

We prove (34) by mathematical induction. Note that by Lemma 4.1 \((EZ_k)'' = C\theta \nu_1 (k)\). Moreover, for \( j \leq k - 2 \),

\[ \prod_{l=j}^{k} \sqrt{\nu_1 (l)} = \sqrt{\nu_1 (k) \nu_1 (k - 1)} \prod_{l=j}^{k-2} \sqrt{\nu_1 (l)} \leq \frac{\nu_1 (k) + \nu_1 (k - 1)}{2} 10^{-(k-j-1)}. \]  
(37)

Applying (37) to (24), for all \( j \leq k - 2 \), we prove

\[ |\Psi_{j,k}| \leq 10 \left( \frac{4}{10} \right)^{k-j} [\nu_1 (k) + \nu_1 (k - 1)]. \]  
(38)
Taking into account (27) and (1), it is easy to check that

\[
|\hat{E}(Z_{k-1}, Z_k)| \leq E|Z_{k-1}Z_k| + E|Z_{k-1}|E|Z_k| = E|Z_{k-1}Z_k|/2 + E|Z_{k-1}|E|Z_k|/2 + E|Z_{k-1}|E|Z_k|/2 \\
+ E|Z_{k-1}|E|Z_k|/2 \leq E|Z_{k-1}| + E|Z_k| + 0.01E|Z_{k-1}| + 0.01E|Z_k| \\
\leq 2.02[\nu_1(k-1) + \nu_1(k)].
\]

Therefore, we see that (38) holds also for \( j = k - 1 \). From inductional assumption, (29), (32) and (1), it follows

\[
|\varphi_i''| / |\varphi_i| \leq C_{22} [\nu_1(i-1) + \nu_1(i)]^{10/9} \leq 2C_{22} / 90.
\]

Using (29) and the previous estimate, we obtain

\[
\left| \sum_{j=1}^{k-1} \frac{\Psi_{j,k}}{\varphi_{j,k-1}} \sum_{i=j}^{k-1} |\varphi_i''| / |\varphi_i| \right| \leq \sum_{j=1}^{k-1} 10 \left( \frac{4}{9} \right)^{k-j} [\nu_1(k) + \nu_1(k-1)](k-j) \frac{2C_{22}}{90} \leq \frac{8C_{22}}{25} [\nu_1(k) + \nu_1(k-1)].
\]

Estimating all other sums (without using induction arguments) in a similar manner, we finally arrive at the estimate

\[
|\varphi_k''| \leq C_{23} [\nu_1(k-1) + \nu_1(k)] + \frac{8C_{22}}{25} [\nu_1(k) + \nu_1(k-1)].
\]

It remains to choose \( C_{22} = 25C_{23}/17 \) to complete the proof of (34).

Since the proof of (35) is quite similar, we give only a general outline of it. First, we assume that \( k \geq 6 \). Then in (36) split all sums into \( \sum_{j=1}^{k-5} + \sum_{j=k-4}^{k-1} \). Next, note that

\[
\prod_{l=j}^{k} \sqrt{\nu_1(l)} \leq \prod_{l=k-5}^{k} \sqrt{\nu_1(l)} \prod_{l=j}^{k-6} \left( \frac{1}{10} \right) \leq \sum_{l=k-5}^{k} \nu_1^3(l) 10^{-(k-j-5)} \leq r_1(k) 10^{-(k-j-5)}.
\]

Therefore, applying (24)–(26) and using (29), (32) and (34), we easily prove that all sums \( \sum_{j=1}^{k-5} \) are by absolute value less than \( C|z|r_1(k) \). The cases \( j = k - 4, k - 3, k - 2 \) all contain at least three
Combining the last estimate with (40) and (39), we complete the proof of (35). The case $k = 6$ is easily verifiable estimates $|\hat{E}(Z_{k-1}, Z_k)| \leq C|z| r_1(k)$. Consequently, we obtain

$$2\hat{E}(Z'_{k-1}, Z'_k) = 2z'\hat{E}(X_{k-1}, Z'_k) + \theta \hat{E}^+_2(X_{k-1}, X_k)|z| = 2(z')^2\hat{E}(X_{k-1}, X_k) + \theta C \hat{E}^+_2(X_{k-1}, X_k)|z|.$$ 

Similarly, $Z = X_k z + \theta X_k (X_k - 1)|z|^2/2$ and

$$\hat{E}(Z''_{k-1}, Z_k) + \hat{E}(Z_{k-1}, Z''_k) = z(\hat{E}(Z''_{k-1}, X_k) + \hat{E}(X_{k-1}, Z''_k)) + \theta C|z|\hat{E}^+_2(X_{k-1}, X_k).$$

Applying (23), we prove $\hat{E}(Z''_{k-1}, X_k) = z''\hat{E}(X_{k-1}, X_k) + \theta C \hat{E}^+_2(X_{k-1}, X_k)$. Consequently,

$$(\hat{E}(Z_{k-1}, Z_k))'' = (z')^2\hat{E}(X_{k-1}, X_k) + \theta C|z|\hat{E}^+_2(X_{k-1}, X_k).$$

Combining the last estimate with (40) and (39), we complete the proof of (35). The case $k < 6$ is proved exactly by the same arguments. □
Let $\tilde{\varphi}_k = \varphi_k \exp\{-it\nu_1(k)\}$, $\tilde{g}_k = g_k \exp\{-it\nu_1(k)\}$, $\psi = \exp\{-0.1\lambda \sin^2(t/2)\}$.

**Lemma 4.7** Let (11) hold. Then

\[
\sum_{l=1}^{n} |\varphi'_l| \leq C\lambda|z|, \quad \sum_{l=1}^{n} |\tilde{g}'_l| \leq C\lambda|z|, \quad \sum_{l=1}^{n} |\tilde{\varphi}''_l| \leq C\lambda,
\]

\[
\sum_{l=1}^{n} |\tilde{g}''_l| \leq C\lambda, \quad \left| \prod_{l=1}^{n} \tilde{\varphi}_l - \prod_{l=1}^{n} \tilde{g}_l \right| \leq CR_1|z|^3\psi,
\]

\[
\left| \left( \prod_{l=1}^{n} \tilde{\varphi}_l - \prod_{l=1}^{n} \tilde{g}_l \right)' \right| \leq C\lambda \psi^2, \quad \left| \left( \prod_{l=1}^{n} \tilde{\varphi}_l - \prod_{l=1}^{n} \tilde{g}_l \right)'' \right| \leq CR_1|z|\psi.
\]

**Proof.** The first four estimates follow from Lemmas 4.5 and 4.6 and trivial estimate $EX_{k-1}X_k \leq C_0\nu_1(k)$. Also, using (11) and (30), we get

\[
\prod_{l=1, l \neq k}^{n} \exp\{-\lambda_l \sin^2(t/2)\} \leq C \prod_{l=1}^{n} \exp\{-\lambda_l \sin^2(t/2)\} \leq C\psi^2.
\]

Therefore, by (30) and (33),

\[
\left| \prod_{l=1}^{n} \tilde{\varphi}_l - \prod_{l=1}^{n} \tilde{g}_l \right| = \left| \prod_{l=1}^{n} \varphi_l - \prod_{l=1}^{n} g_l \right| \leq \sum_{j=1}^{n} |\varphi_j - g_j| \prod_{l=1}^{j-1} |g_l| \prod_{l=j+1}^{n} |\varphi_l| \leq C\psi^2 \sum_{j=1}^{n} |\varphi_j - g_j| \leq CR_1|z|^3\psi^2.
\]

From (11) and trivial estimate $ze^{-x} \leq 1$, for $x > 0$, we get

\[
|\Gamma_2| \leq 0.08\lambda, \quad \lambda|z|^2\psi \leq C.
\]

Therefore,

\[
\left| \left( \prod_{l=1}^{n} \tilde{\varphi}_l - \prod_{l=1}^{n} \tilde{g}_l \right)' \right| \leq \sum_{l=1}^{n} |\varphi'_l - \tilde{g}'_l| \prod_{k \neq l} \tilde{\varphi}_k + \sum_{l=1}^{n} |\tilde{g}'_l| \prod_{k \neq l} \tilde{\varphi}_k - \prod_{l \neq k} \tilde{g}_k \leq C\psi^2[R_1|z|^2 + \lambda|z|R_1|z|^3] \leq C\psi R_1|z|^2.
\]

The proof of last estimate is very similar and therefore omitted. $\square$
5 Proof of Theorems

Proof of Theorem 2.1 Hereafter, $x \in \mathbb{N}$, the set of nonnegative integers. The beginning of the proof is almost identical to the proof of Tsaregradsky’s inequality. Let $M$ be concentrated on integers. Then summing up the formula of inversion

$$
M\{k\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{M}(t)e^{-itk}dt
$$

(41)

, we get

$$
\sum_{k=m}^{x} M\{k\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{M}(t)\frac{e^{-it(m-1)} - e^{-itx}}{z}dt.
$$

If $|\hat{M}(t)/z|$ is bounded, then as $m \to -\infty$ and by Riemann-Lebesgue theorem, we get

$$
M(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{M}(t)e^{-itx}dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{M}(t)e^{-it/2}e^{-itx}dt.
$$

(42)

The Tsaregradsky’s inequality

$$
|M(x)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\hat{M}(t)\right|\frac{1}{|z|}dt
$$

(43)

now follows easily. Let next $M = F_n - G$. Then expressing $\hat{M}(t)$ in powers of $z$, we get

$$
\hat{M}(t) = \sum_{k=2}^{\infty} a_k z^k,
$$

for some coefficients $a_k$ which depend on factorial moments of $S_n$. Therefore, $\hat{M}(\pi)/z(\pi) = \hat{M}(-\pi)/z(-\pi)$. Consequently, integrating (42) by parts, we obtain, for $x \neq \lambda$,

$$
M(x) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{M}(t)e^{-it(\lambda+1/2)}\frac{e^{-it(x-\lambda)}}{2i\sin(t/2)}dt = \frac{1}{2\pi(x-\lambda)^2} \int_{-\pi}^{\pi} u''(t)e^{-it(x-\lambda)}dt,
$$

where

$$
u(t) = e^{-(\lambda+1/2)it} \frac{\hat{M}(t)}{2i\sin(t/2)} = \frac{\Pi_{j=1}^{\lambda} \tilde{\varphi}_j - \Pi_{j=1}^{\lambda} \tilde{g}_j}{z}.
$$

Thus, for all $x \in \mathbb{N},$

$$
(x-\lambda)^2 M(x) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|u''(t)\right|dt.
$$

(44)

Using Lemma 4.7, equations (43), (44) and the trivial estimate

$$
\int_{-\pi}^{\pi} |z|^k \psi(t)dt \leq \frac{C(k)}{\lambda^{(k+1)/2}}.
$$

(45)
the proof of (5) follows.

All other approximations are compared to compound Poisson measure $G$ and then the triangle inequality is applied. We begin from the negative binomial distribution. Due to the assumptions,

$$\Gamma_2 \leq \frac{3}{40} \lambda, \quad \frac{1 - \bar{q}}{\bar{q}} = \frac{2\Gamma_2}{\lambda} \leq 0.15,$$

see [8]. Therefore, $\hat{NB}(t) \exp\{-\lambda it\} = \exp\{A\}$, where

$$A = \lambda z - it + \Gamma_2 z^2 + \sum_{j=3}^{\infty} \frac{r}{j} \left(\frac{1 - \bar{q}}{\bar{q}}\right)^j z^j = \lambda(z - it) + \Gamma_2 z^2 + \theta CT_2^2 \lambda^{-1} |z|^3.$$ 

Moreover,

$$|A'| \leq C\lambda |z|, \quad |A''| \leq C\lambda, \quad |e^A| \leq \psi^2.$$ 

Let $B = \lambda(z - it) + \Gamma_2 z^2$ so that $\hat{G}(t) \exp\{-\lambda it\} = \exp\{B\}$ and $u_1(t) = (e^A - e^B)/z$. Then

$$|u_1| \leq \frac{|e^A - e^B|}{|z|} \leq \psi^2 \frac{|A - B|}{|z|} \leq C\psi^2 \frac{\Gamma_2^2 |z|^2}{\lambda}, \quad \int_{-\pi}^{\pi} |u_1| |dt| \leq C \frac{\Gamma_2^2}{\lambda^2 \sqrt{\lambda}} \quad (46)$$

Also,

$$|(e^A - e^B)'|^2 \leq |A''| |e^A - e^B| + |(A')^2| |e^A - e^B| + |A'' - B''||e^B| + |(A')^2 - (B')^2||e^B|$$

$$\leq C\psi^2 \left\{ \lambda^2 |z|^3 + \lambda^2 |z|^2 \Gamma_2^2 \frac{\Gamma_2^2}{\lambda} |z|^3 + \Gamma_2^2 \frac{\Gamma_2^2}{\lambda} |z| + \lambda |z| \Gamma_2^2 \frac{\Gamma_2^2}{\lambda} |z|^2 \right\} \leq C\psi |z| \frac{\Gamma_2^2}{\lambda}.$$ 

Similarly,

$$|(e^A - e^B)| \leq |A'||e^A - e^B| + |e^B||A' - B'| \leq C\psi |z|^2 \frac{\Gamma_2^2}{\lambda}$$

and we obtain finally

$$|u_1''| \leq C\psi \frac{\Gamma_2^2}{\lambda}, \quad \int_{-\pi}^{\pi} |u_1''| |dt| \leq C \frac{\Gamma_2^2}{\lambda^2 \sqrt{\lambda}} \quad (47)$$

Estimates in (46) and (47) allow us to write

$$\left(1 + \frac{(x - \lambda)^2}{\lambda}\right) |G(x) - NB(x)| \leq C \frac{\Gamma_2^2}{\lambda^2 \sqrt{\lambda}},$$

which combined with (5) proves (7).
For the proof of translated Poisson approximation, let \( B \) be defined as in above,

\[
T = \lambda(z - it) + (2\Gamma_2 + \tilde{\delta})(z - it), \quad D = \lambda(z - it) + (\Gamma_2 + \tilde{\delta}/2)z^2,
\]

and

\[
u_2 = (e^D - e^T)/z, \quad u_3 = (e^B - e^D)/z.
\]

Note that, for \(|t| \leq \pi\), we have \(|t|/\pi \leq |\sin(t/2)| \leq |t|/2\). Therefore, arguing similarly as in above, we obtain

\[
\int_{-\pi}^{\pi} |u_2|dt \leq \frac{C(|\Gamma_2| + \tilde{\delta})}{\lambda \sqrt{\lambda}}, \quad \int_{-\pi}^{\pi} |u_2''|dt \leq \frac{C(|\Gamma_2| + \tilde{\delta})}{\sqrt{\lambda}}.
\] (48)

Observe next that

\[
u_3 = \frac{e^B}{z}(e^{\delta z^2/z} - 1) = \frac{e^B}{z} \int_0^1 (\delta z^2/2)e^{\tau \delta z^2/2}d\tau = \int_0^1 \frac{\delta z}{2} e^{B + \tau \delta z^2/2}d\tau.
\]

Consequently,

\[
\int_{-\pi}^{\pi} |u_3|dt \leq C \int_{-\pi}^{\pi} \psi^2 \tilde{\delta} |z|dt \leq \frac{C\tilde{\delta}}{\lambda}.
\] (49)

Similarly,

\[
u_3'' = \frac{\delta}{2} \int_0^1 e^{B + \tau \delta t}[z'' + 2z'(B' + \tau \tilde{\delta} z') + z(B'' + \tau \tilde{\delta} (z')') + z(B' + \tau \tilde{\delta} z'')2]d\tau
\]

and using \( \tilde{\delta} \leq 1 \leq \lambda \), we get

\[
|u_3''| \leq C\psi^2 \tilde{\delta}(1 + \lambda|z| + \tilde{\delta}|z| + |z|(|\lambda|z| + \tilde{\delta}|z|)^2) \leq C\tilde{\delta}\psi \sqrt{\lambda}.
\]

Consequently,

\[
\int_{-\pi}^{\pi} |u_3''|dt \leq C\tilde{\delta}.
\]

Combining the last estimate, the inequalities in (48), (49) and the estimate for \( \hat{G} = e^B \), the result in (6) is proved.
For binomial approximation, note first that
\[ e^{-\lambda t} \hat{B}_i = e^E, \quad E = \lambda(z - it) + \Gamma_2 z^2 + z^2 \theta \frac{50 \Gamma_2^2}{21 \lambda^2} \epsilon + \theta \frac{5N \Psi^3 |z|^3}{9}, \]
\[ \Psi \leq \frac{50 |\Gamma_2|}{21 \lambda} < \frac{1}{5}, \quad |\Gamma_2| \leq 0.08 \lambda, \quad |N \Psi^3| \leq C \frac{\Gamma_2^2}{\lambda}, \]
see [8]. Let
\[ L = \lambda(z - it) + \Gamma_2 z^2 + z^2 \theta \frac{50 \Gamma_2^2}{21 \lambda^2} \epsilon, \quad u_4 = (e^L - e^E)/z, \quad u_5 = (e^B - e^L)/z. \]
Next,
\[ u_5 = \int_0^1 e^B \tau \exp \left\{ \tau \frac{\theta}{21 \lambda^2} e \right\} \frac{50 \Gamma_2^2}{21 \lambda^2} \epsilon d\tau. \]
Now the proof is practically identical to that of (6) and is, therefore, omitted.
The proofs of (3) and (4) are also very similar and use the facts
\[ \frac{e^B - e^{-\lambda t} \hat{\Pi} + \hat{\Pi}_1}{z} = \int_0^1 (1 - \tau) \Gamma_2^2 z^3 \exp \{ \lambda(z - it) + \tau \Gamma_2 z^2 \} d\tau, \]
\[ \frac{e^B - e^{-\lambda t} \hat{\Pi}}{z} = \int_0^1 \Gamma_2 z \exp \{ \lambda(z - it) + \tau \Gamma_2 z^2 \} d\tau. \]

**Proof of Theorem 2.3.** Let \( M \) be a measure concentrated on integers and \( \hat{M}(t) = \sum_{k=1}^{\infty} M\{k\} e^{i tk} \).
Then from formula (41) of inversion, we get
\[ |M\{x\}| \frac{1}{2\pi} \leq \int_{-\pi}^{\pi} |\hat{M}(t)| dt. \]
Moreover, integrating (41) by parts, we obtain
\[ (x - \lambda)^2 |M\{x\}| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |(\hat{M}(t) \exp\{-\lambda t\})''| dt. \]
The rest of the proof is a simplified version of the proof of Theorem 2.1 and hence omitted. □

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