Holographic dark energy in Gauss-Bonnet gravity with Granda-Oliveros cut-off

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February 15, 2022

Abstract

In this paper, we have studied a model of holographic dark energy (HDE) with a homogeneous and anisotropic Bianchi type-I Universe in the framework of Gauss-Bonnet (GB) gravity or $f(G)$ gravity. To find an exact solution of the field equations, we assume that the deceleration parameter varies with cosmic time ($t$). We plot some physical quantities and give interpretations of the results obtained. Finally, we compared our model with the $Λ$CDM model by analyzing the Jerk parameter, and we examine the equivalence between the HDE of the present work and the generalized HDE.

Keywords: Bianchi type-I Universe, $f(G)$ gravity, Holographic dark energy, Granda-Oliveros cut-off.

1 Introduction

According to recent astronomical observations [1–11], our Universe has entered a phase of accelerated expansion. This discovery leads to the presence of a component of unknown nature, called Dark Energy (DE), which has negative pressure and represents 68% of the total density of the Universe, it behaves like a repulsive gravity. Its nature remains unknown today. The simple proposition for DE is the cosmological constant ($Λ$), which Einstein introduced into the field equations in General Relativity (GR) in another context. This cosmological constant has an equation of state (EoS) parameter $ω = −1$ and is considered
to be very consistent with the observation data. Faced with the difficulties linked to its theoretically predicted order of magnitude with respect to that of the observed vacuum energy \[12\], other dynamic models of DE have been proposed, such as quintessence \[13,14\], phantom \[15\], k-essence \[16\], tachyons \[17\], chaplygin gas \[18\] and so forth. There is another class of dynamic DE models, in which we do not need to introduce any other form of energy, this class is called modified gravity theories, that is, an accelerated expansion can be caused by a modification in action. In other words, GR is no longer valid on cosmological scales. Such as \( f(T) \) gravity, \( f(R,G) \) gravity, \( f(R,T) \) gravity, \( f(R,T,R_{\mu\nu}T^{\mu\nu}) \) gravity and \( f(T,T) \) gravity, where \( T \) is the trace of the energy-momentum tensor, \( G \) is the Gauss-Bonnet (GB) invariant and \( R_{\mu\nu} \) is the Ricci tensor \[19–22\]. The modified gravity of GB or the \( f(G) \) gravity, is considered among the modified versions of GR in which we replace \( R \) by \( R + f(G) \) (where \( f(G) \) is an arbitrary function of the GB invariant \( G \)) in the Einstein-Hilbert action \[23,24\]. Several authors have studied the applications of this theory. The generalized second law of thermodynamics in cosmology in the framework of the modified GB theory of gravity is investigated by Sadjadi \[25\]. Myrzakulov et al. \[26\] studied cosmological solutions, especially the well-known \( \Lambda \)CDM model. It is shown that the DE contribution and even the inflationary epoch can be explained in the frame of this kind of theory with no need for any other kind of component. The cosmological application of holographic DE (HDE) in the framework of \( f(G) \) modified gravity was discussed by Jawad et. al. \[27\]. Although these works strongly inspire us to propose a new cosmological model in the framework of \( f(G) \) gravity and to discuss the most important problems of the Standard Model of cosmology like DE, there are other more important motivations, as this theory can describe the current cosmic acceleration by passing the solar system tests for some specific choices of \( f(G) \) gravity models \[23\].

The holographic principle (HP) is another alternative to solve the problem of DE, as it is known in the literature because it has great potential to solve many long-standing problems in various physical fields. This principle was first proposed by Hooft \[28\] in the context of black hole physics, then in a cosmological context another version of HP was proposed by Fischler and Susskind \[29\]. In the context of the DE problem, the HP tells us that all physical quantities in the Universe, including the density of DE \( (\rho_\Lambda) \), can be described by a few quantities on the boundary of the Universe. It is clear that in terms of two physical quantities, namely the reduced Planck mass \( (M_p) \) and the cosmological length scale \( (L) \), as follows \( \rho_\Lambda \approx c^2M_p^2L^{-2} \) \[30\]. Subsequently, a relation was proposed which combines the HDE density \( (\rho_\Lambda) \) and the Hubble parameter \( (H) \) as \( \rho_\Lambda \propto H^2 \), it does not contribute to the current accelerated expansion of the Universe \[31\]. For purely dimensional reasons, Granda and Oliveros \[32\] proposed a new infrared (IR) cut-off for the holographic DE density of the form \( \rho_\Lambda \approx \alpha H^2 + \beta H \) where \( \alpha \) and \( \beta \) are constants. They show that this new model of DE represents the accelerated expansion of the Universe and is consistent with current observational data. Sarkar in numerous works, studied the holographic model of DE in various contexts \[33–35\]. Samanta studied the homogeneous and anisotropic Bianchi type-V Universe filled with matter and HDE components, and established a correspondence between the HDE and quintessence DE \[36\]. Recently, locally rotationally symmetric (LRS) Bianchi type-I models with HDE within the framework of \( f(G) \) theory of gravitation are studied by Shaikh et al. \[37\].

Recently, the anisotropic Universe has attracted the attention of many researchers, because anisotropy played an important role in the early time of cosmic evolution. In addition,
the possibility of an anisotropy phase at the beginning of the Universe followed by an isotropy phase was supported by the observations i.e. the cosmic microwave background (CMB) anomalies from the results obtained by Planck [49]. Several researchers have studied homogeneous and anisotropic Bianchi models, such as the spatially homogeneous and anisotropic Bianchi type-I model, which is a direct generalization of the FLRW Universe with a scale factor in each spatial direction [38][39]. In this paper, motivated by the work [35], we study the holographic model of DE under $f(G)$ gravity in the Bianchi type-I Universe, to find solutions of the field equations and some physical quantities, we will assume that the deceleration parameter (DP) varies with time. The paper is organized as follows: Sect. 1 is an introduction. In Sect. 2, we write the action of $f(G)$ gravity and the field equation. We have derived the Bianchi type-I metric and defined some physical and geometrical parameters to solve the field equations in Sect. 3. In Sect. 4, we solve the field equations by considering the time-varying deceleration parameter. Sect. 5 we discuss the jerk’s parameter. Also, we examine the equivalence between the HDE of the present work and the generalized HDE in Sect. 6. The last section is devoted to a conclusion.

2 Basic equations in Gauss-Bonnet gravity

The modified Einstein–Hilbert action of the $f(G)$ gravity is given by [23]

$$S = \frac{1}{2k^2} \int d^4x \sqrt{-g} \left[ R + f(G) \right] + S_M (g^\mu{}^\nu, \psi),$$

(1)

where $k^2 = 8\pi G$, $R$ is the Ricci scalar, and $f(G)$ is a general differentiable function of $GB$ invariant, $S_M$ is a matter action that depends on a space-time metric $g^\mu{}^\nu$ and matter fields $\psi$. The $GB$ invariant quantity is

$$G = R^2 - 4R^\mu{}_\rho R^\nu{}^\rho + R^\mu{}_{\rho\sigma\beta} R^{\nu\rho\sigma\beta},$$

(2)

where $R^\mu{}_{\rho\sigma\beta}$ is the Ricci tensor and $R^\mu{}_{\rho\sigma\beta}$ is the Riemannian tensor.

The variation of the action (1) with respect to $g^\mu{}^\nu$ leads to the following equation

$$G^\mu{}^\nu + 8 \left[ R^\mu{}_{\rho\sigma\rho} + R^\mu{}_{\rho\sigma\nu} - R^\rho{}_{\rho\sigma} g^\nu{}^\rho - R^\rho{}_{\rho\nu} g^\mu{}^\sigma + R^\rho{}_{\mu\sigma \rho} + \frac{1}{2} R (g^\mu{}^\rho g^\sigma{}^\rho - g^\mu{}^\sigma g^\rho{}^\rho) \right] \nabla^\rho \nabla^\sigma f_G + (Gf_G - f) g^\mu{}^\nu = k^2 T^\mu{}^\nu,$$

(3)

where $\nabla^\mu$ denotes covariant differentiation, $G^\mu{}^\nu = R^\mu{}^\nu - \frac{1}{2} g^\mu{}^\nu$ is the Einstein tensor, $T^\mu{}^\nu$ is the energy-momentum tensor of a matter fluid and the subscript $G$ in $f_G$ represents the derivative of $f$ with respect to $G$.

3 Metric and field equations

In this article, we will focus on a spatially homogeneous and anisotropic Bianchi type-I Universe with different scale factors in each spatial direction
\[ ds^2 = dt^2 - A(t)^2 \, dx^2 - B(t)^2 \left( dy^2 + dz^2 \right), \quad (4) \]

where \( A(t) \) and \( B(t) \) are the directional metric potentials.

The Ricci scalar and GB invariant for Bianchi type-I Universe are as follows

\[ R = -2 \left[ \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right] \quad (5) \]

\[ G = 8 \left[ \frac{\ddot{A}}{A} + 2 \frac{\ddot{A}B}{AB} \right] \quad (6) \]

The energy-momentum tensor for matter and the HDE are defined as

\[ \tilde{T}_{\mu\nu} = \rho_m u_{\mu} u_{\nu}, \quad (7) \]

and

\[ T_{\mu\nu} = (\rho_{\Lambda} + p_{\Lambda}) u_{\mu} u_{\nu} + g_{\mu\nu} p_{\Lambda}, \quad (8) \]

where \( \rho_m, \rho_{\Lambda} \) are the energy densities of matter and the HDE respectively and \( p_{\Lambda} \) is the pressure of the HDE.

The field equations (3), with (7) and (8) for the metric (4) leads to the following system of field equations

\[ -2 \frac{\ddot{B}}{B} - \frac{\dot{B}}{B}^2 + 16 \frac{\dot{B}B}{B^2} \ddot{f}_G + 8 \frac{\dot{B}}{B}^2 \dddot{f}_G - G \dddot{f}_G + f = k^2 p_{\Lambda}, \quad (9) \]

\[- \frac{\ddot{A}}{A} - \frac{\dot{B}}{B} \frac{\dot{A}}{AB} + 8 \left( \frac{\dddot{A}B}{AB} + \frac{\ddot{A}B}{AB} \right) \dddot{f}_G + 8 \frac{\ddot{A}B}{AB} \dddot{f}_G - G \dddot{f}_G + f = k^2 p_{\Lambda}, \quad (10) \]

\[ 2 \frac{\ddot{A}B}{AB} + \frac{\dot{B}}{B}^2 - 24 \frac{\ddot{A}B}{AB^2} \dddot{f}_G + G \dddot{f}_G - f = k^2 (\rho_m + \rho_{\Lambda}), \quad (11) \]

with an over dot (\( \cdot \)) denote derivative with respect to the cosmic time \( t \).

The average scale factor \( a \) of the Bianchi type-I Universe is defined as

\[ a = \left( AB^2 \right)^{\frac{1}{3}}. \quad (12) \]

The spatial volume \( V \) of the Universe is given by

\[ V = a^3 = AB^2. \quad (13) \]

The average Hubble’s parameter \( H \) is defined as

\[ H = \frac{\dot{a}}{a} = \frac{1}{3} (H_1 + 2H_2), \quad (14) \]
where $H_1 = \frac{\dot{A}}{A}$ and $H_2 = H_3 = \frac{\dot{B}}{B}$ are directional Hubble parameter along $x$, $y$ and $z$ axes respectively.

For the Bianchi type-I Universe [41], the scalar expansion ($\theta$), deceleration parameter ($q$) and the shear scalar ($\sigma^2$) have the form

$$\theta = 3H = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B},$$

(15)

$$q = -\frac{a\ddot{a}}{a^2} = \frac{d}{dt} \left( \frac{1}{H} \right) - 1,$$

(16)

$$\sigma^2 = \frac{1}{2} \left[ \sum_{i=1}^{3} H_i^2 - \frac{1}{3} \theta^2 \right].$$

(17)

The average anisotropy parameter ($A_m$) is defined by

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2 = 6 \left( \frac{\sigma}{\theta} \right)^2,$$

(18)

where $\Delta H_i = H_i - H$ and $H_i (i = 1, 2, 3)$ represent the directional Hubble parameters.

By combining the HP and dimensional analysis, we find the HDE density as follows [30]

$$\rho_{HDE} = \frac{3C^2}{k^2 L_{IR}^2},$$

(19)

where $C$ is a numerical constant that acts as a free parameter, $L_{IR}$ is the infrared (IR) cut-off and $k^2 = 8\pi G$. In this work, we use a new holographic Ricci dark energy model proposed by Granda and Oliveros [32] for the HDE density

$$\rho_\Lambda = 3 \left( \alpha H^2 + \beta \dot{H} \right),$$

(20)

where $H$ is the average Hubble’s parameter and $\alpha$, $\beta$ are constants which must satisfy the constraints imposed by the current observational data.

Combining (9)–(11) the continuity equation can be obtained as

$$\dot{\rho}_m + \dot{\rho}_\Lambda + 3H (\rho_m + \rho_\Lambda + p_\Lambda) = 0.$$

(21)

The continuity equation of the matter is

$$\dot{\rho}_m + 3H \rho_m = 0.$$

(22)

The continuity equation of the HDE is

$$\dot{\rho}_\Lambda + 3H (\rho_\Lambda + p_\Lambda) = 0.$$

(23)

Using Eqs. (20)–(23) and the barotropic equation of state $p_\Lambda = \omega_\Lambda \rho_\Lambda$, the equation of state HDE parameter is obtained as

$$\omega_\Lambda = -1 - \frac{2\alpha H \dot{H} + \beta \ddot{H}}{3H \left( \alpha H^2 + \beta \dot{H} \right)}.$$

(24)
4 Solutions of Field Equations

The above field equations are nonlinear and complicated differential equations, in order to solve these equations we assume that the deceleration parameter varies with the time, which is of the form

\[ q = -1 + \gamma e^{-\gamma t}, \]  

(25)

where \( \gamma > 0 \) is a constant.

Using the relation \((16)\) and solving Eq. \((25)\), we find

\[ a = \left( e^{\gamma t} - 1 \right)^{\frac{1}{\gamma}}. \]  

(26)

Using the definition of the Hubble parameter we get

\[ H = e^{\gamma t} \left( e^{\gamma t} - 1 \right)^{-1}. \]  

(27)

Using the relation \( a(t) = \frac{1}{(1+z)} \), with \( z \) being the redshift, gives us the following relation

\[ t(z) = \frac{1}{\gamma} \log \left[ 1 + \frac{1}{(1+z)^{\gamma}} \right]. \]  

(28)

Also, the deceleration parameter \( (q) \) can be written in terms of redshift \( (z) \) as follows

\[ q(z) = -1 + \frac{\gamma (1 + z)^{\gamma}}{1 + (1 + z)^{\gamma}}. \]  

(29)

Figure 1: Deceleration parameter versus redshift with \( \gamma \geq 1.2 \).
The deceleration parameter is the quantity that describes the evolution of the expansion of the Universe. This parameter is positive \((q > 0)\) when the Universe is decelerated over time, and is negative \((q < 0)\) otherwise, that is, when the expansion of the Universe is accelerating. The current observational data \([41,42]\) indicates that the Universe is accelerating and that the value of the deceleration parameter is in the range \(-1 \leq q < 0\). Fig. 1 shows the behavior of the deceleration parameter in terms of redshift, it can be seen in Fig. 1 that the deceleration parameter contains two phases in the Universe, the initial deceleration phase and the current acceleration phase. In this work, to produce both phases, we need \(\gamma \geq 1.2\). Also, the transition from the early deceleration phase to the current accelerated phase is done with a certain redshift, called a transition redshift \(z_{tr}\). From the figure, the value of the transition redshift for \(\gamma = 1.5\) is \(z_{tr} = 0.62\). This transition redshift value is therefore consistent with the results of the observation \([43–45]\).

Similarly, we get the Hubble parameter \((H)\) in terms of the redshift \((z)\) as

\[
H(z) = 1 + (1 + z)^\gamma. \tag{30}
\]

The evolution of the Hubble parameter in terms of redshift with different \(\gamma\) (i.e. \(\gamma \geq 1.2\)) are shown in Fig. 2. It appears from this figure that the Hubble parameter is a positive function in terms of redshift. At present \((z = 0)\) and early \((z > 0)\), the Hubble parameter is strictly positive and increases with increasing \(z\) value. Also, the present HDE model does not admit any turning point in \(H(z)\). Hence, our model has the same behavior in Ref. \([46]\).

From the two Eqs. \([13]\) and \([26]\) above, the values of the metric potentials \(A\) and \(B\) are obtained as

\[
A = \left(e^{\gamma t} - 1\right)^{\frac{2}{\gamma}}, \tag{31}
\]
\[ B = (e^{\gamma t} - 1)^{\frac{1}{\gamma}}. \]  

(32)

Here, we have used a condition so that the shear scalar is proportional to the expansion scalar \( (\theta \propto \sigma^2) \), and from it we can write a relationship that sums the metric potentials as follows: \( A = B^m \), where \( m \neq 1 \) is an arbitrary constant. The model suit an isotropic model for \( m = 1 \), or else it suits anisotropic. In this paper we have taken \( m = 4 \), as a result, we get the metric potentials in Eqs. (30) and (31). The main reasons behind the assumptions that led to this condition are discussed in detail here [54]. Observations of the velocity redshift relation for extragalactic sources indicate that the Hubble expansion of the Universe can reach isotropic when \( \frac{\sigma^2}{\theta} \) is constant. The condition has been used in many studies, see [33–36].

Using Eqs. (31) and (32), the Bianchi type-I Universe takes the form

\[ ds^2 = dt^2 - (e^{\gamma t} - 1)^{\frac{4}{\gamma}} dx^2 - \left( e^{\gamma t} - 1 \right)^{\frac{1}{\gamma}} (dy^2 + dz^2). \]  

(33)

We suppose that the model of the function \( f(G) \) follows the following power-law models proposed by [47]

\[ f(G) = \eta G^{n+1}, \]  

(34)

where \( \eta \) and \( n \) are arbitrary constants. The motivations behind this choice are numerous as mentioned by Shaikh et al. [37]. For example, the chances of the emergence of the Big-Rip singularity disappear, and also the prediction of the existence of the transient phantom era consistent with astrophysical observations are considered among the attracting factors in the power-law model of \( f(G) \) gravity. Some authors have worked on the power-law of \( f(G) \), such as [48,49].

The scalar expansion \( (\theta) \), shear scalar \( (\sigma^2) \) and the average anisotropy parameter \( (A_m) \) are therefore obtained as

\[ \theta = 3e^{\gamma t} \left( e^{\gamma t} - 1 \right)^{-1}, \]  

(35)

\[ \sigma^2 = \frac{3}{4}e^{2\gamma t} \left( e^{\gamma t} - 1 \right)^{-2}, \]  

(36)

\[ A_m = \frac{1}{2}. \]  

(37)

From Eqs. (35) and (36) it appears that the scalar expansion and the shear scalar diverge at \( t \to 0 \) and they become finite when \( t \to \infty \). However, from Eq. (37) it is observed that the anisotropic parameter remains constant throughout cosmic evolution, which indicates that our model is anisotropic from the initial era of the Universe to the final era.

Using Eq. (27) in (20), we get the HDE density as

\[ \rho_A = 3e^{\gamma t} \left( \alpha e^{\gamma t} - \beta \gamma \right) \left( e^{\gamma t} - 1 \right)^{-2}. \]  

(38)

Again, using Eq. (27) in (22), we get the matter energy density as

\[ \rho_m = c_1 \left( e^{\gamma t} - 1 \right)^{-\frac{3}{2}}, \]  

(39)
with $c_1$ is a constant of integration.

The coincidence parameter ($r$) can be defined as the ratio between the HDE density ($\rho_\Lambda$) and the matter energy density ($\rho_m$), therefore for Eqs. (38) and (39) the coincidence parameter becomes

$$r = \frac{\rho_\Lambda}{\rho_m} = \frac{3}{c_1} e^{\gamma t} (\alpha e^{\gamma t} - \beta \gamma)(e^{\gamma t} - 1)^{\frac{3}{2}(3-2\gamma)}.$$  

(40)

From Fig. 3, it is shown that the HDE density ($\rho_\Lambda$) is a decreasing function of cosmic time and positive throughout the evolution of the Universe, its value being larger at the initial epoch and then disappear later, which leads to the Universe ruled by a vacuum. The evolution of the matter energy density ($\rho_m$) is illustrated in Fig. 4, where it starts at a positive value, but also disappears later, which represents the expansion of the Universe. The energy density of the Universe has not been the same since the Big Bang and the beginning of the expansion of the Universe. When the energy density is large, the radiation is more diffuse. This period is called the radiation-dominated era and when the density is low, it is the energy of the vacuum that dominates the Universe. Also, the coincidence parameter ($r$) as a function of cosmic time, it turns out that the latter is an increasing function of cosmic time. Therefore, at the beginning time, the Universe is dominated by HDE and later the Universe is dominated by matter. This result is consistent with the current Universe.

![Figure 3: An evolution of HDE density versus cosmic time with $\alpha = 0.01$, and $\beta = 0.001$.](image)

Now, using Eq. (27) in (24), we get the DE equation of state parameter as

$$\omega_\Lambda = -1 - \frac{\gamma [e^{\gamma t}(\beta \gamma - 2\alpha) + \beta \gamma]}{e^{\gamma t}(3\alpha e^{\gamma t} - 3\beta \gamma)}.$$  

(41)
Fig. 5 clearly shows that the equation of state parameter evolves with negative values in an appropriate range \((-1 \leq \omega_\Lambda \leq 0)\), which is in good agreement with astronomical observations. Our studied model is therefore realistic. From Fig. 5, we notice that at the beginning of cosmic time the equation of state parameter starts close to zero (that is to say the Universe dominated by matter) and then at the end of cosmic time it takes a close negative value of \(-1\) (i.e. when the Universe dominated by the HDE). If \(\omega_\Lambda = -1\), it represents the \(\Lambda CDM\) model, \(-1 < \omega_\Lambda < -1/3\), represents the quintessence model and \(\omega_\Lambda < -1\) indicates the phantom behavior of the model. Also, we can observe that in the early Universe \(-1 < \omega_\Lambda < 0\), it indicates the quintessential model and in the current Universe, \(\omega_\Lambda\) tends to \(-1\), i.e. the model \(\Lambda CDM\). We conclude from Fig. 6 that the current value of the equation of state parameter of our model is in rough agreement with recent observational data from Planck + WMAP [50].

The HDE pressure is obtained as

\[
p_\Lambda = -\left[ e^{\gamma t} \left( 3\alpha e^{\gamma t} + \beta \gamma^2 - 2\alpha \gamma - 3\beta \gamma \right) + \beta \gamma^2 \right] (e^{\gamma t} - 1)^{-2}. \tag{42}
\]

It is useful to use yet another notation, the abundances, also called the density parameters, it represents the proportion of each element in the Universe. The total energy density parameter \((\Omega = \Omega_m + \Omega_\Lambda)\) takes three values \(\Omega > 1, \Omega = 1, \Omega < 1\) correspond respectively to the open, flat and closed Universe. The matter density parameter \((\Omega_m)\) and HDE density parameter \((\Omega_\Lambda)\) are respectively given by

\[
\Omega_m = \frac{\rho_m}{3H^2} = \frac{1}{3} c_1 e^{-2\gamma t} (e^{\gamma t} - 1)^{\frac{1}{\gamma}(2\gamma - 3)}, \tag{43}
\]

and
\[ \Omega_\Lambda = \frac{\rho_\Lambda}{3H^2} = \alpha - \beta \gamma e^{-\gamma t}. \]  
(44)

Figure 5: An evolution of equation of state parameter versus cosmic time with \( \alpha = 0.01 \) and \( \beta = 0.001 \).

We obtain the total density parameter yields as

\[ \Omega = \frac{1}{3} c_1 e^{-2\gamma t} \left( e^{\gamma t} - 1 \right)^{\frac{1}{2}(2\gamma-3)} + \alpha - \beta \gamma e^{-\gamma t}. \]  
(45)

Fig. 7 represents the evolution of the energy density parameter as a function of time, and it appears that its value is large in the initial era of the Universe, but we begin to approach \( \Omega \sim 1 \) in the last era of Universe, which causes our model to predict a flat Universe at a later time, as recent astronomical observations indicate.

We should note that under the assumption (26), the GB invariant \( G \) and Ricci scalar \( R \) behave as

\[ G = 12 \left( (1 - \gamma) e^{4\gamma t} (e^{\gamma t} - 1)^{-4} + \gamma e^{3\gamma t} (e^{\gamma t} - 1)^{-3} \right), \]  
(46)

\[ R = \left( 6\gamma - \frac{27}{2} \right) e^{2\gamma t} (e^{\gamma t} - 1)^{-2} - 6\gamma e^{\gamma t} (e^{\gamma t} - 1)^{-1}. \]  
(47)

Using Eqs. (34) and (46), the function \( f(G) \) is obtained as

\[ f(G) = \eta \left[ 12 (1 - \gamma) e^{4\gamma t} (e^{\gamma t} - 1)^{-4} + 12\gamma e^{3\gamma t} (e^{\gamma t} - 1)^{-3} \right]^{n+1}. \]  
(48)
Figure 6: An evolution of equation of state parameter versus redshift with $\alpha = 0.01$, and $\beta = 0.001$.

Fig. 8 represents the function $f(G)$ in terms of time for a range of values $n < 0$. It shows that the function $f(G)$ is positive over cosmic time and contains a transitory behavior. At the beginning of time, the function $f(G)$ starts with large values, then approaches zero, then increases, and finally takes a constant value.

5 The jerk parameter

Among the simple tools to find deviations from the $\Lambda CDM$ concord model, there is the Jerk parameter which is an important quantity to describe the dynamic evolution of the Universe. We know that the models close to the $\Lambda CDM$ model can be described by the jerk parameter, for example for the flat $\Lambda CDM$ model the value of this parameter is constant $j = 1$. A deceleration to acceleration transition occurs for models with a positive value of $j_0$ and negative $q_0$. The jerk parameter is a third dimensionless derivative of the scale factor with respect to cosmic time, in cosmology is defined as $[51,53]$:

$$j(t) = \frac{1}{H^3 a} \frac{\dddot{a}}{a} = q + 2q^2 - \frac{q}{H}. \quad (49)$$

For our model, the jerk parameter is given as follows

$$j(t) = 1 - 3\gamma e^{-\gamma t} + 2\gamma^2 e^{-2\gamma t} + \gamma^2 e^{-2\gamma t} (e^{\gamma t} - 1). \quad (50)$$

Fig. 9 represents the evolution of the jerk parameter in terms of cosmic time and shows that this parameter is positive throughout the evolution of the Universe and for a large cosmic
time, the jerk parameter tends towards 1, it approaches the \( \Lambda CDM \) model.

\section{Generalized Holographic Dark Energy}

In this section, we effort to establish that our dark energy model has a direct equivalence to the generalized holographic dark energy model. As shown in Eq. \( \text{[19]} \), the holographic dark energy density is inversely proportional to the squared infrared cut-off \( L_{IR} \). The IR cut-off is supposed to be the particle horizon \( L_P \) or the future event horizon \( L_F \), which are determined respectively as \( \text{[55]} \)

\begin{align}
L_P &\equiv a \int_0^t \frac{dt}{a}, \quad L_F \equiv a \int_t^\infty \frac{dt}{a}. \tag{51}
\end{align}

Adopted the time derivative of the above equation, we get the expressions for the Hubble parameter in terms of \( L_P, L_F \), and their time derivatives as follows

\begin{align}
H \left( L_P, \dot{L}_P \right) &\equiv \frac{\dot{L}_P}{L_P} - \frac{1}{L_P}, \quad H \left( L_F, \dot{L}_F \right) \equiv \frac{\dot{L}_F}{L_F} + \frac{1}{L_F}. \tag{52}
\end{align}

The general form of the cut-off is suggested in this work \( \text{[56]} \)

\begin{align}
L_{IR} &= L_{IR} \left( L_P, \dot{L}_P, \ddot{L}_P, ..., L_F, \dot{L}_F, \ddot{L}_F, ..., a \right). \tag{53}
\end{align}

Also, the IR cut-off depends on other parameters such as the Hubble parameter, the Ricci scalar, and their derivatives. Though, can be transformed to either \( L_P \) and their derivatives
or $L_F$ and their derivatives. The above-mentioned cut-off might be selected to be equivalent to a general covariant gravity model

$$S = \int d^4 \sqrt{-g} F \left( R, R_{\mu\nu} R^{\mu\nu}, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \Box R, \Box^{-1} R, \nabla_\mu R \nabla^\mu R, ... \right).$$  \hspace{1cm} (54)

In the following, using the above expressions and with the help of the generalized cut-off, we will show that the HDE of the present work has direct equivalence to the generalized HDE model.

From Eq. (24) we get

$$\omega_\Lambda = -1 + \frac{(1 + z)}{3} \frac{d}{dz} \left[ \ln \left( \alpha H^2 + \beta \dot{H} \right) \right],$$  \hspace{1cm} (55)

where, we used $dt = \frac{dz}{1 + z}$, and $H$ is given by Eq. (30). The comparison of Eq. (19) with Eq. (20) and using Eq. (52) immediately conduct to the equivalence holographic cut-off (in terms of $L_P$ and its derivatives or in terms of $L_F$ and its derivatives) corresponds to the HDE as

$$\frac{3C^2}{k^2 L_R^2} = 3 \left\{ \alpha \left( \frac{\ddot{L}_P}{L_P} - \frac{1}{L_P} \right)^2 + \beta \left( \frac{\ddot{L}_P}{L_P} - \frac{\dot{L}_P^2}{L_P^2} + \frac{\dot{L}_P^2}{L_P^2} \right) \right\},$$  \hspace{1cm} (56)

$$= 3 \left\{ \alpha \left( \frac{\ddot{L}_F}{L_F} + \frac{1}{L_F} \right)^2 + \beta \left( \frac{\ddot{L}_F}{L_F} - \frac{\dot{L}_F^2}{L_F^2} - \frac{\dot{L}_F^2}{L_F^2} \right) \right\}. $$
The equation of state parameter can be derived from the conservation equation corresponds to the HDE density $\rho_{HDE}$

$$\omega_{HDE}^{(RD)} = -1 - \frac{\dot{\rho}_{HDE}}{3H\rho_{HDE}} = -1 + \frac{2}{3HL_R} \frac{dL_R}{dt},$$

(57)

where $L_R$ is given by Eq. (56). Hence, we conclude that $\omega_{HDE}^{(RD)}$ is equivalent to $\omega_\Lambda$ as derived in Eq. (55).

### 7 Conclusion

In this paper, we have studied a model of HDE with a homogeneous and anisotropic Universe of Bianchi type-I in the framework of $f(G)$ gravity. In order to find exact solutions of the field equations, we assume that the deceleration parameter (DP) varies with cosmic time. We discussed some physical and geometric quantities of the model and got the following results:

- The deceleration parameter contains two phases in the Universe, the initial deceleration phase and the current acceleration phase. The value of the transition redshift for our model is $z_{tr} = 0.62$, this value corresponds to the observational data.

- The scalar expansion and the shear scalar diverge at $t \to 0$ and they become finite when $t \to \infty$. The anisotropic parameter remains constant throughout cosmic evolution, which indicates that our model is puvely anisotropic from the initial era of the Universe to the final era.
• From the evolution of the equation of state (EoS) parameter, we can observe that in the primitive Universe $-1 < \omega_\Lambda < 0$, it indicates the quintessential model and in the current Universe, $\omega_\Lambda$ tends to $-1$, that is, the $\Lambda CDM$ model. Thus, the current value of the equation of state parameter for our model is in good agreement with recent observations.

• The value of the density parameter is large in the first era of the Universe, but it started to approach $\Omega \sim 1$ in the last era of the Universe, which leads our model to predict a flat Universe at a later time, as recent astronomical observations indicate.

• The cosmic jerk parameter is positive throughout the evolution of the Universe and tends towards 1 at late times.

In the literature, it is known that in the Standard Model, the cosmological constant is the mechanism responsible for the current cosmic acceleration, that is, the dominated phase of DE, which has negative pressure. In $f(G)$ gravity, responsible for this, are the additional terms of $f(G)$ next to the scalar curvature $R$.

Acknowledgments We are very much grateful to the editor and anonymous referee for illuminating suggestions that have significantly improved our article.

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