Resonant Ion Confinement Fusion Concept

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Abstract. Based on the theorized possibilities of resonant ion confinement, for a
Deuteron cloud in a Penning-Malmberg trap with a specially configured rotating wall,
the opportunity to design a new type of fusion device is prospected. It is proven that,
for some trap configurations, nuclear fusion reactions should take place and, in that
case, Lawson’s criterion for an efficient fusion reactor is met. Furthermore, the reactor
could have a compact design and, since it should not require a large facility, it can
function as a fusion cell with a pure ion thermal gas.

Keywords: Charged plasma confinement, Penning traps, Fusion reactors.

1. Introduction

For an efficient fusion reactor, the pressure \( p \) of the confined plasma times the energy
confinement time \( \tau_E \) should be greater than a given amount \( p\tau_E > L \); which was a
criterium due to Lawson (Ref. [1]). For Deuterium - Deuterium reactions this value is
of the order of one hundred \( atm \times s \). This reaction happens in the stars, where plasma
is confined by gravity. The advantage of considering Deuterium-Deuterium reactions
is that this is a stable and abundant isotope of Hydrogen on Earth, although the
cross section of these reactions is smaller than that of Deuterium-Tritium. Up to date,
the magnetic confinement of neutral plasmas has been thought as the most promising
concept to build a fusion reactor and two main concepts of devices have been developed:
tokamaks and stellarators. In tokamaks, the magnetic field confines the particles and
energy long enough for ignition to occur and it is expected that such a reactor will
generate enough energy to achieve \( Q > 10 \), which will probably be shown through the
ITER experiment in the near future (see Ref. [2], also Ref. [3]). Nevertheless, tokamaks
still have to overcome some challenges, namely the possibility of suffering disruptions,
the existence of edge localised modes (ELMs) and the intrinsic pulsed working, on
top of the possible impurity accumulation, (as from Ref. [4]). Turbulence will be also
playing a major role degrading the Tokamak plasma confinement. These problems made it necessary to explore other possible magnetic confinement fusion concepts like the stellarator one (see e.g. Ref. [5]), characterised by enabling a continuous working and by the absence of large ELMs and disruptions, since there plasmas are almost currentless. Stellarator confinement is one generation after the tokamak one, due to the fact that they have to reduce their neoclassical transport by optimizing the magnetic configuration (see e.g. Ref. [6] and references therein), as well as to solve the confinement of fast particles and to demonstrate power exhaust by a suitable divertor concept (see again Ref. [5]). Therefore, it is still necessary to overcome difficulties in both concepts to make it economically feasible for the production of electricity by fusion. Other topics to discuss in the future would be the size of the reactors, since all the scalings show an improvement of confinement with the size of the device (see e.g. Ref. [7] and references therein). Thus, to address the problems encountered in the case of magnetically confined neutral plasmas and, at the same time, to explore alternative confinement methods that minimise the resources involved, another possibility to achieve nuclear fusion is described in this work; the new method is based on the concept of resonant ion confinement (see Ref. [8]), which can be produced inside a cylindrical Penning-Malmberg ion trap with a Deuteron plasma in perfect thermal equilibrium. A remarkable feature of this fusion concept is the possibility of building small reactors, which brings with it completely different engineering challenges from those encountered in the design of magnetic confinement reactors. Indeed, as we shall see, on the grounds of these resonant ionic confinement fundamentals, a kind of compact reactor, say a fusion cell, can be conceived with, predictably, energy yields which, interestingly, are proportionally lower than those expected in ITER. These fusion cell have, therefore, complementary purposes and scopes.

In this device ions are confined by applying an electric potential difference, $V_0$, and a magnetic axial field $B$. The ion trap, additionally, has a quadrupole electric field of a frequency, $\omega$, such that it allows the Deuterium nuclei to rotate also with that same angular velocity. In this system, a fraction of ions performs trajectories that should bring them close enough to produce fusion reactions by tunnelling. The confined ion cloud takes the shape of a highly flattened spheroid. Although these characteristics are common to any Penning-Malmberg configuration, there is a very precise set of values for the trap parameters that determines the intensity of the quadrupole field for which resonant, coalescing paths, arise among all the Deuterium ion trajectories, and, in this way, nuclear fusion reactions will likely take place. Thus, the precise dependence of this quadrupole electric field as a function of the electric field potential, $V_0$, and of the magnetic field intensity, $B$, as well as the radius of the trap cylinder, $R_0$, will determine the resonance conditions required for the equipment to produce a finite number of fusion reactions. Finally, the pressure and confinement time necessary for ignition in the centre of the ionic trap should be reached only for special configurations of the confinement system parameters. What propose here, contrarily to other possible candidates for fusion devices, is to build a device that confines non-neutral plasmas in thermal equilibrium...
where, also, the ergodic theorem for the average trajectories of the confined ions is satisfied.

2. Methods

Resonant Ion Confinement.

Let us consider \( N \) Deuteron nuclei of mass \( m \) confined in a cylindrical Penning-Malmberg trap of radius \( R_0 \); we will assume perfect thermodynamic equilibrium. Let us denote the applied electrostatic potential as \( V_0 \) and the axial magnetic intensity as \( B \). A rotating wall electric quadrupolar field of intensity \( \lambda = 1/2 + \delta \) and angular frequency \( \omega \) is then added which prompts the plasma cloud to rotate collectively with the same rotating wall angular frequency \( \omega \). The latter is the guiding orbit magnetron frequency of the ion cloud. The charged plasma remains confined in the central plane of the trap and the form of the confined ion cloud is a spheroid of semi-major axis \( R_C \) and semi-minor axis \( z \).

Using the fact that ions should be in equilibrium inside the confined cloud, it is easy to see that the electric axial oscillations of ions should be related to the volume density number, \( n \), of this cloud as

\[
\omega_z = \left( \frac{n e^2}{m e_0} \right)^{1/2}.
\]

Moreover, the oscillator axial frequency also depends on the electric potential \( V_0 \) and the radius of the trap cylinder, \( R_0 \), as

\[
\omega_z = \frac{2}{R_0} \sqrt{eV_0/m}.
\]

In addition, ought to the applied axial magnetic field \( B \), the ions inside the confined spheroid have an internal fast cyclotron oscillation whose frequency is defined in terms of \( \Omega = eB/m \). Due to the diamagnetic behaviour of charge currents inside the plasma cloud, the achievable cyclotron frequency of the Deuterons is actually smaller: \( \Omega' = \Omega - \omega \), which this is called the vortex frequency. On the other hand, dynamic equilibrium of the plasma imposes that \( \omega(\Omega - \omega) \rightarrow \omega_z^2/2 \). This means that there exists a key parameter \( \vartheta \) to define confinement quality of the Penning-Malmberg trap such that

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\[
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are exponentially unstable, yet numerical calculations show that there exists a special, resonant, $\pi/\omega$ periodic orbit whenever the quadrupolar field strength $\delta$ takes the value, (see Appendix)

$$\delta \equiv \lambda - \frac{1}{2} \rightarrow \frac{1}{\sqrt{2\chi}}. \tag{1}$$

The minimum distance between the nuclei in these resonant orbits will eventually be given by the following numerical solution

$$\varrho_0 = 2R_C\kappa, \text{ with } \kappa \simeq \exp\{1/2 - 1.35\sqrt{\chi}\}, \tag{2}$$

which can reach arbitrarily small values at any temperature of the confined plasma in the limit $2\rightarrow 0$. Moreover, the kinetic energy per Deuteron in this case becomes $E \rightarrow W/2$ where $W = \frac{e^2}{4\pi\epsilon_0\varrho_0}$, which provides $\varrho_0$. On the other hand, a charged plasma in thermodynamic equilibrium at temperature $T$ should behave as a solid rotor in a Penning-Malmberg trap with a magnetron frequency oscillatory stroboscopic rotating wall quadrupolar field (see Appendix). The plasma global angular velocity $\omega$, coinciding with that of the individual ions guiding orbits, satisfies the condition $\omega = \Omega(\chi + 1)^{-1}$, we get $\chi + 1 = \Omega R_C\sqrt{k_B T/\mu}$ which, owing to the dynamic internal equilibrium of the plasma, is equivalent to saying that $\chi \simeq eV_0/(k_B T) \times (R_C/R_0)^2$. Finally, using this thermodynamic constraint and given that the axial degree of freedom is thermal, i.e., $k_B T = 1/2\mu\omega_z^2z_{\max}^2$, we conclude that the aspect ratio of the confined ion cloud minimal height spheroid is $z_{\max}/R_C \simeq 1/\sqrt{\chi}$, which is an important relation that will be used later. Moreover, the probability that two correlated ions follow the resonant orbits is (see Appendix) $\wp(\chi) = \chi - 1$. Recall that in the resonant case, all the macroscopic features of the Deuteron plasma can be written only in terms of the microscopic parameter $W$ and of the trap confinement value for $\chi$. Then, $R_C = e^2/(8\pi\epsilon_0)W^{-1}\kappa^{-1}$ and the confined volume of the ion cloud becomes $V \rightarrow \frac{4}{3}\pi(R_C/\sqrt{\chi})^3 \times \chi$, which is coincident with the volume of $\chi$ ion clump spheres of radius $R_C/\sqrt{\chi}$. The density number becomes $n = \frac{eB^2}{4\mu} \sin^2 \vartheta \rightarrow 4\omega^2\epsilon_0\mu\chi/e^2$ and the total number of confined Deuterons $N \rightarrow V \times n$.

3. Results

Fusion Conditions.

For Deuterium fusion, the occurring reactions are $^2H + ^2H \rightarrow ^3He + n$; and $^2H + ^2H \rightarrow ^3H + ^1H$ while their corresponding fusion energies are $E_{f1}[^3He + n] = 3.27$ MeV and $E_{f2}[^3H + ^1H] = 4.04$ MeV. Each reaction takes place with approximately 50% probability. Thus, together with the necessary Lawson criterium, in a self-sustaining energy balanced fusion Deuterium device the only heating terms shall be that from the kinetic energy of the confined charged fusion reactions products. The kinetic energy contribution of the neutrons, which cannot be confined in the trap, must be subtracted from the nuclear energy balance, a fraction $\eta_n = E_n/(E_{f1} + E_{f2})$. The neutron energy escapes directly to hit the reactor walls, where its energy can be extracted by a breeding blanket. On top of that, other energy losses have to be taken into account in the power balance analysis, namely the Bremsstrahlung term due to accelerated charges in the
plasma, as well as the transport term, which is assumed to be due to ion-ion Coulomb collisions. Positive power balance of the Deuteron plasma, then, requires

\[ P_f (1 - \eta_n) > P_B + P_L, \]

where \( P_f \equiv J_f V \) is the power due to fusion reactions that heats the whole plasma volume \( V \), \( P_B \) is the Bremsstrahlung term and \( P_L \) is the power loss caused by transport, assumed collisional. Again, let \( E \) be the energy of the Deuteron nuclei in the resonant orbit (this correlates with the effective temperature of the nuclei inside the density clusters) and let \( n' \) be the density number of the plasma in the resonant clusters, then the collision frequency is, (see Ref. \([10]\)) \( \nu(\chi, E) = n'E^{-3/2} \frac{3e^4}{16\pi^2\epsilon_0 m^2} \ln \Lambda(\chi, E) \). The Coulomb logarithm is \( \ln \Lambda(\chi, E) = \ln \{12\pi\epsilon_0^3 n'^{-1/2}e^{-3E^{3/2}}\} \). Now, we write \( N' = \varphi(\chi)N \) as the number of resonant nuclei in the density clusters; then \( V' = \kappa V \) is the volume of the density clump because only one of the two major axis of the spheroid is compressed due to the resonant orbits of the nuclei in it. Then every ion swirls around the centre in a vortex from \( r = R_C \) to \( r = q_0/2 \) and, therefore, the effective volume that affects all the resonant orbits can be approximated as \( V' = 4/3\pi R_C^3K \), which will be the volume available for fusion reactions in the centre of mass of the resonant complex. Then, \( n' = \varphi(\chi)/\kappa \times n \). Now, from these definitions, the reactor effective power per unit of volume becomes

\[ J = J_f - \eta \left \{ n'^2 \alpha_B E \frac{1}{2} + \frac{N'}{V} E \nu(\chi, E) \right \} \]

where the fusion power density is \( J_f = \left \{ \frac{1}{2} E_f \langle \sigma v \rangle_1 + \frac{1}{2} E_f \langle \sigma v \rangle_2 \right \} n'^2 \). Notice that in Eq. \([4]\), we consider the total volume of the plasma in the collision term, instead of that of the resonant cluster of ions; the reason is that we consider the most unfavourable case in which the collisional transport is large enough to drive the local inhomogeneities that appear in the resonant zone to the entire plasma volume. In Eq. \([4]\) \( \eta \approx 1 \) because, neglecting the \( ^3H, \, ^1p \) and \( ^3He_2 \) concentrations, i.e., those of the nuclear reaction products, there is only a single particle species in the confined plasma. For long working period of the reactor, the concentrations of these species could not be negligible and should be taken into account to develop a possible exhaust device. The trap disconnection can be used for exhausting these particles in any case. In Eq. \([4]\), \( \alpha_B = 1.4 \times 10^{-40} \left ( m_e/m_p \right )^{3/2} \left [ W/m^3 K^{1/2} \right ] \) and, for \( E(keV) < 300 \), we can make the following analytical approximation for the cross-section of the reaction: \( \langle \sigma v \rangle_i = b_i E^{b_i} \exp \left \{ c_i E^{\chi_i} \right \} \) \( b_1 = 1.198 \times 10^{-18}, b_2 = 3.5501 \times 10^{-19}, \beta_1 = -1.0759, \beta_2 = -0.9462, c_1 = -23.511, c_2 = -22.04, \chi_1 = 0.29221, \chi_2 = 0.2922 \). Using these values in Eq. \([4]\) one sees that the Bremsstrahlung and the collisional transport terms have small effects on the power balance.

**Fusion Cells.**

In order for the reactor to be energetically self-sustained, the generated energy must be greater than the sum of the emitted heat plus the magnetic back reaction pressure loss (as required from Brillouin’s theorem). Then, the actual thermodynamically available power density is \( J_C = J - \dot{W} \), where \( \dot{W} = \frac{j^2}{2\mu_0} \times \frac{\omega}{\pi} \) is the power loss per unit of
volume due to the diamagnetic currents of the confined ions calculated for the resonance frequency. Recall that the contribution to the pressure losses is written in terms of the plasma back reaction magnetic field, namely, \( b = -4(\mu/e)\omega = -2B(1 + \chi)^{-1} \). Now, the energy confinement time \( \tau_E \) may be calculated in terms of the temperature \( T \) and the density number \( n' \) for the nuclei in the resonance \( \tau_E = n'E \times J_C^{-1} \). Since the pressure is \( p = 2/3n'E \), then (see Ref. [10])

\[
p\tau_E = \frac{2}{3} n'^2 E^2 / J_C.
\]

The function \( p\tau_E \) should have a minimum value for some \( \chi \) and \( E = W/2 \). As seen in Fig. 1, Lawson’s minimum is actually reached for \( \vartheta \leq 0.105 \), and \( W = 30.7 \text{ keV} \). Indeed, this minimum is obtained for approximately the same confinement parameters, independently of the actual temperature of the confined plasma cloud. Thus \( p\tau_E \sim 90 \text{ atm} \times \text{s} \), which is expected to be experimentally achievable. Moreover, the value of the Coulomb barrier energy for the Lawson minimum computed is remarkably close to the expected experimental value (see Ref. [10]), a fact that supports the Resonant Ion Confinement reactor concept. Therefore, in order to obtain the actual power efficiency of the reactor, recall that, since the species \(^3\text{He}, \ p \) and \(^3\text{H} \) are positively charged, they will be retained in the trap but, contrarily to that, the neutrons will escape from the trap cavity and will be absorbed by the surrounding walls of the reactor chamber, where a separate equipment to absorb their energy should be installed. Their kinetic energy might be, then, transformed into heat (with an efficiency \( \eta_h \sim 1/3 \)). This heat can be used to produce electricity by means of high efficient thermoelectric materials.

![Figure 1](image.png)

**Figure 1.** Lawson triple product \( p\tau_E \) as a function of the Coulomb barrier energy achievable for D-D collisions \( W \) in keV and the parameter \( \vartheta = \arcsin(\frac{\sqrt{2}E}{W}) \). As seen here, an stable minimum does exist, \( p\tau_E \sim 90 \text{ atm} \times \text{s} \), for \( \vartheta < 0.105 \) and \( W \sim 30.7 \text{ keV} \).

Moreover, we must also take into account that every \( D - D \) collision takes place by quantum tunnelling the barrier \( W \). The Gamow-Sommerfeld probability of this transition
needed to undergo the nuclear reaction is \( \eta_f(W) \sim \exp\{-\pi \alpha c \sqrt{2\mu/W}\} \), again, the Deuteron reduced mass is \( \mu = m/2 \). It gives, for the Lawson minimum \( W \approx 30.7 \text{ keV} \), \( \eta_f \approx 1/300 \). Having this in mind, the following estimate for the electric power of the reactor can be provided \( P_f \sim \eta_n \eta\eta J_C \times V \rightarrow \eta_n \eta\eta J_C \times \{Q_t V'\} \), where we have used the fact that, for the reactants in the cell, their actual available volume, \( V \), must be calculated considering that the number of reactions in it must coincide, in average, with the sum of the "Quantum tunneling" configurations, \( Q_t \), all over the time, at some given space point; if this particular point is selected as the centre of the cell, we assume that \( V' = \kappa V \). Moreover, the quantity \( Q_t \) can be estimated as \( Q_t = \eta_f(W) \times \varphi(\chi)^2 \times N^2/2 \), in terms of the total number of available resonant ion pairs configurations times the tunnelling through the barrier probability. This taken into account, the following equation determines the key constraints for the reactor confinement configurations

\[
Q_t \times \kappa = 1, 
\]

for \( R_C \leq R_0 \), where the radius of the Penning-Malmberg trap should be \( R_0 = 2(\chi + 1)(eB)^{-1} \times (eV_0/\chi)^{1/2} \) and the maximum radius of the confined plasma cloud \( R_C = \frac{1}{2} \varrho_0 \kappa^{-1} \). The total thermoelectric power becomes

\[
P = P_f - (1 - \eta_n) aT^4[\pi/\omega]V. 
\]

Here, to obtain the realistic model of the fusion device, we have subtracted the black body radiation term \( (a \text{ is the radiation constant}) \), recall also that \( k_B T \approx eV_0/\chi \times (R_C/R_0)^2 \). In addition to that, the maximum variation of the number of Deuterons per unit of time in the reactor is \( \dot{N}_D = \frac{1}{\eta_n} P_f/(E_f) \) and, therefore, the thermonuclear reaction frequency is \( \nu_f = \dot{N}_D/N \). On the other hand, the D-D collision frequency should be of the order of the cyclotron one, \( \Omega \). Thus, for the actual resonant ion confinement device, we propose that the Deuteron fuel will enter the resonant confinement device cavity in a pulsed way. To this point, the fuel is introduced in the reactor chamber through very high frequency pulses and the achievable electric power of the ion cell will depend on the ultra high frequency and high voltage circuit relays to control the rapid pulsed fuel refilling of the reactor chamber. Then, if we denote the frequency of these circuit relays as \( \nu_{CB} \), the achievable power would be: \( 2 \pi \nu_{CB}/\Omega \times P \). A solution of the ergodic constraint in Eq. [5] does exist and is \( \chi = 477.102, B = 3.73 T, \) and \( W = 30.74 \text{ keV} \). Recall that, remarkably, the solution of the ergodic condition for the barrier energy \( W \) coincides with that provided by the stable Lawson point from the nuclear model alone. The resulting maximum achievable electric power would be \( P \sim 11 \times 2 \pi \nu_{CB}/\Omega \text{ (MW)} \) for Deuteron pulses of \( N = 2.2 \times 10^{10} \) ions, if the electric potential is within the range \( 5.9 \text{ kV} < V_0 < 10.5 \text{ kV} \). In order to estimate the actual achievable power of the cell a possible approximation should be to provide \( 2 \pi \nu_{CB} < O(\omega) \), i.e, that, in order to allow for the pulses to reach the thermal equilibrium during confinement, and to the aim to achieve the resonance, the refuelling frequency should be slightly smaller than the resonant one. In that case, a solution of Eq. [4] is represented in Fig.2 for the attainable thermoelectric power in Eq. [7] This configuration gets a maximum of approximately

\[
P \sim 20 \text{ kW}
\]

at \( V_0 \approx 8 \text{ kV} \) for a fuel rate of \( \dot{N}_D \approx 2.1 \times 10^{17} \text{ Deuteron/s} \). The confinement radius of the ion cloud would be \( R_C \approx 91 \text{ mm} \) and the trap radius is \( R_0 \approx 92.56 \text{ mm} \). The actual fusion device requires a pulsed refilling of the reactor cavity and that the
resonant configuration would be attainable only adiabatically. This kind of technological requirements will be the subject of the experimental research for real fusion facilities based on the grounds of the resonant ion confinement concept. Yet, in order to double check that the above estimates are correct, we can do a kind of Fermi analysis. As said, in the resonant cell (all over the time), the number of possible configurations between two Deuterons, leading to quantum tunneling reactions, is approximately \( Q_t \sim \eta f(W)\varphi(\chi)^2 \times N^2/2 \). Since the natural colliding frequency is \( \Omega/\pi \), the nuclear reaction rate should be \( Q_t \times \Omega/\pi \), and the number of emitted neutrons per second is half this value. Then, roughly, the actual achievable electric power must be \( P \sim \eta h 2.44 \text{ MeV} \times [\frac{1}{2} Q_t \times \Omega/\pi] \times (\omega/\Omega) \) (the last term takes into consideration that we refuel the cell at the typical resonant frequency rates). For the numbers above, i.e., \( B = 3.7 \text{ T}, N = 2 \times 10^{10} \text{ Deuterons}, \chi \sim 477, \text{ and } R_C \sim 91 \text{ mm} \) (which, for the resonant condition in Eqs. 2, is equivalent to saying that \( \varrho_0 \sim 47 \text{ fm} \) or \( W \sim 30.7 \text{ keV} \)), we get \( P \sim 26 \text{ kW} \), which is of the order of the figure that we did obtain from nuclear theory alone. This fact makes us confident in the correctness of the derived result. We can call this optimum and small size configuration of the confinement device parameters attaining the maximum power as a fusion cell.

![Figure 2. Optimal achievable power of the standard fusion device as a function of the applied confinement electric potential \( V_0 \), \( B = 3.73 \text{ T}, W = 30.7 \text{ keV} \) and \( \vartheta = 0.0915 \). For the estimation, the frequency of the Deuteron refueling pulse was taken to be exactly coincident with the resonant one.]

4. Discussion

For a compact reactor of small size, a new type of fusion technology can be developed according to the basic conditions described in this article, although some practical considerations must be taken into account. This possibility arose from imposing the parametric relations in Eqs. 1 and 2 to the standard Penning-Malmberg ion trap, which lead to the necessary resonant kind of ion confinement. On the other hand, the trap fusion conditions are the key configurations that satisfy the Eq. 6 and, very promisingly, both the resonant and the operative conditions for Deuteron confinement are found
within the range of the current state of the art of technological capabilities. In other words, due to the resonance, the Coulomb barrier can be overcome due to the movement of resonant ions. Additionally, one must expect that a pure Helium-3 resonant ion confinement device will work as another fusion possibility in exactly the same principles as the Deuterium one exposed in this article. Recall that such Helium-3 Cell will be a neutronic one and, as a safe reactor, it opens the possibility to implement a sort of thermonuclear battery in the event that it could power an autonomous device provided with some finite reservoir of Helium-3 gas. Even though the physics of both types of cells should be very alike, it does not necessarily imply that their engineering designs, which have to be associated with solving the problems of each type of reactors, should be also, in turn, similar. Therefore the Helium-3 cell must be studied separately and the results will be presented elsewhere.

5. Conclusion

In this work, the theory of resonant ion confinement has been developed for a charged gas in thermal and dynamic equilibrium. Precise relationships between the trap parameters and Deuteron-Deuteron collision probabilities have been developed for ions following coalescent trajectories, leading, this way, to fusion reactions. The analysis takes into account the statistical ergodicity criterion of the system, as well as the assumption of thermal equilibrium of the ion gas, which should be necessary, we claim, for any sustainable and stable fusion reactor. In these circumstances, the Lawson criterion is fulfilled. This device is capable of producing 20 kW of electrical power with a small reactor design that we have termed a fusion cell. Despite of the final energy production efficiency, the experimental test of this concept must be based on the eventual emitted neutron measurements.

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Appendix.

A cylindric Penning Malmberg trap is characterised by the introduction of an axial magnetic field $\mathbf{B}$ and a electrostatic potential $V_0$, in a cylindrical cavity where the ions are introduced. Additionally the full stability can be obtained with the help of a rotating electric field with angular frequency $\omega$ (see Ref. [9]) and field strength $\lambda$; in this case, the motion of the ions in the trap is decomposed into separated rotationally confined mode and an axial oscillation. Let $q$ be the charge of the ion, then, the Lagrangian is given in terms of the electrostatic quadrupole and the magnetic field frequencies of the trap, i.e., denoting the cyclotron frequency $\Omega = qB/m$

\[
L = \sum_i \frac{1}{2} m \left\{ \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 + \Omega(x_i \dot{y}_i - y_i \dot{x}_i) \right\} - U_i(r_i),
\]
\[ U_i(r) = qV_0 + \frac{1}{2}m\{\omega_z^2[z^2 - \frac{1}{2}\rho^2] + \lambda\omega_z^2[(x^2 - y^2)\cos 2\omega t - 2xy\sin 2\omega t]\}, \]

This rotating quadrupolar electric potential wall of intensity \( \lambda \) might be required for the adiabatic stability of the ions in the trap as it will be confirmed down from. Let us consider the rotating coordinate frame which rotates with angular frequency \( \omega : (x,y) \rightarrow (\xi,\zeta) \). Then, the rotating wall quadrupole perturbation becomes time independent, \((x^2 - y^2)\cos(2\omega t) - 2xy\sin(2\omega t) \rightarrow \xi^2 - \zeta^2\),

\[
L = \sum_i 2 \frac{1}{2} m \{ \dot{\xi}_i^2 + \dot{\zeta}_i^2 + \dot{z}_i^2 + (\Omega - 2\omega)(\xi_i\dot{\xi}_i - \zeta_i\dot{\zeta}_i) \} - U'(r_i), \tag{2}
\]

\[ U'(r) = qV_0 + \frac{1}{2}m\omega_z^2z^2 - \frac{m\rho^2}{4}\omega_2^2 + \frac{m\rho^2}{2}\Omega - \frac{m\rho^2}{2}\omega^2 + \frac{m}{2}\lambda\omega_2^2(\xi^2 - \zeta^2). \]

Hereafter the constant energy potential term \( qV_0 \) will be omitted. The \( z \) coordinate motion is an harmonic oscillator of frequency \( \omega_z \). The restoring force gives the effective axial component of the electric field \( eE_z = -m\omega_z^2z \). Along these lines, if the density of the plasma cloud is \( n \), \( E_z \) can be thought as that coming from the displacement of a positive charge \( q \) and from the magnetic field \( \sum_i \{ m\varphi \rho_i q + qB\rho_i^2/2 \} \), is preserved and, therefore, for instance for large \( 0 \leq \omega t \leq 2\pi \). Indeed, three forces are acting on each of the ions, namely, the centrifugal force \( +m\omega^2\rho \), the radial electric force \( +\frac{1}{2}m\omega_z^2\rho \) and that one associated with the radially electric field induced by the rotation \( \omega \) through the direction of the axial magnetic field whose value is \( -m\omega\rho \). It is this field that will provide the radial confinement.

To see how, recall that, from the symmetry of a cylindrical trap, the total angular momentum (that from the ions plus that from the magnetic field), \( \sum_i \{ m\varphi \rho_i q + qB\rho_i^2/2 \} \), is preserved and, therefore, for instance for large \( B \), \( \sum_i \rho_i^2 \) must reach some constant, which, as said, implies radial confinement; this argument is due to O’Neil (Ref. [11], Ref. [12]). With that in mind, the simplest case where the perturbed quadrupole frequency should be taken as that case where the net radial force acting on each of the charges is zero. Then, \( \omega_z^2 = 2\omega(\Omega - \omega) \). It means that a "trap angle" can be defined such that \( \omega_z^2 = \frac{1}{2}\Omega^2\sin^2\vartheta \), \( \omega = \Omega\sin^2(\vartheta/2) \). This condition corresponds to the confinement situation of a thin, rigidly rotating, spheroid of plasma at angular velocity \( \omega \) (Ref. [13] Ref. [14]). Recall also that for that confined cloud of ions of mass \( m \) and charge \( q = Ze \) a relation between the axial frequency \( \omega_z \), the cylindric radius \( R_0 \) and the applied electric potential \( V_0 \) can be found as the solution of \( V(R_0) = 0 \), where \( V(R) = V_0 - \frac{\omega_z^2}{4}\rho R^2/q \). This gives, \( \omega_z^2 = \frac{4qV_0}{mR_0^2} \). Lagrange’s equations read (denote \( \tau(t) = \Omega t \) and \( \varepsilon = \lambda\omega_z^2/\Omega^2 \))

\[
\frac{d^2\xi}{d\tau^2} - \frac{d\xi}{d\tau} + \xi\varepsilon = 0, \quad \frac{d^2\zeta}{d\tau^2} + \frac{d\zeta}{d\tau} - \zeta\varepsilon = 0. \tag{4}
\]

Elseways, since the Lagrangian is not time dependant, the energy is preserved and a first integral of motion is obtained \( E = \sum_i \{ \xi_i\partial_{\xi_i}L + \dot{\xi}_i\partial_{\dot{\xi}_i}L \} - L = \sum_i m/2 \{ \dot{\xi}_i^2 + \dot{\zeta}_i^2 + \lambda\omega_z^2(\xi^2 - \zeta^2) \} \). This means that the motion is bounded and that a time period \( t_0 \) can be
found such that $2E/(Ω'^2m) = ε \sum_i \{ξ_i(t_0)^2 - ζ_i(t_0)^2\}$. As said, the orbits of the ions are combinations of rapid bare cyclotron oscillations plus a slow guiding centre magnetron trajectory of angular velocity $ω$ which becomes stabilised by the rotating quadrupole force. Notwithstanding this, other solutions can be seen as representing binary collisions at the centre of the trap. Now, define, $η = \sqrt{1 + 4ε^2}$, $σ = \sqrt{1/2(η - 1)} \rightarrow ε$, $γ = \frac{e^{\pm γ}}{σ} \rightarrow 1$, $β = \sqrt{1/2(η + 1)} \rightarrow 1$ and $γ' = -\frac{β}{ε + γσ} \rightarrow -1$, for $ε \ll 1$. With this notation the exact solutions are

$$ξ_h = a \cosh στ, \quad ζ_h = aγ \sinh στ,$$

$$ξ_C = b \cos βτ, \quad ζ_C = bγ' \sin βτ.$$  \tag{.5}

The $ξ_C, ζ_C$ should correspond to the mentioned bare cyclotron orbits (see e.g., Hasegawa et al. Ref. 9 for the solutions of the cyclotron orbits in the rotating wall trap configuration). Thus, for instance, if the general solution is $\vec{r} = \vec{r}_h + \vec{r}_C$, the limit $ε \rightarrow 0$ is just the usual Penning trap constant radius magnetron orbit $|\vec{r}| = a$ provided with a rapid cyclotron oscillation around this guiding orbit. Incidentally, in spite of the fact that $ξ_h, ζ_h$ can be seen as spurious solutions (they are not bounded hyperbolae that do not meet the required confinement conditions), recall that, admissibly, some of the actual orbits could also be represented by linear combinations of these hyperbolic plus the bare cyclotron solutions: $ξ(i) = ξ_h(i) + ξ_C(i), \quad ζ(i) = ζ_h(i) + ζ_C(i)$. These trajectories would exist during some period of time $t_0$, say, $-t_0 ≤ t ≤ t_0$. Numerical simulation shows that, in the rotating frame, the trajectory is just an hyperbola provided with rapid cyclotron oscillations having two turning points (at which $ξ(t_0) = ζ(t_0) = 0$.) Is obvious that the physical situation corresponds to the orbits of two long range coupled ion density fluctuations that collide (and repel each other) at the centre of the trap with $ξ(i) = -ξ(2), \quad ζ(i) = -ζ(2)$. This is obviously correct because, otherwise, single ions would not preserve linear momentum individually. The only relevant degree of freedom ought be the relative distance between the two charged aggregates, $ρ = 2(ε^2 + ζ^2)^{1/2}$. For this coordinate, the collision is described from the Lagrangian Eq. 2 replacing $ξ = ρ \cos ωt, \quad ζ = ρ \cos ωt, \quad m \rightarrow μ$

$$\mathcal{L} = \frac{1}{2}μ\{\dot{ρ}^2 + \frac{1}{2}ω^2 ρ^2 - ω^2 ρ^2 λ \cos 2ωt\}.$$  \tag{.6}

Denoting $φ = ωt$ and $χ = \frac{ω^2}{2c^2} = \cot^2(\frac{φ}{2})$ the equation of motion reads

$$\frac{d^2}{dφ^2} φ - \{χ - 2λχ \cos 2φ\} φ = 0.$$  \tag{.7}

Eq. 7 is Mathieu’s equation whose time symmetric solution is obtained in terms of the Mathieu Cosine function: $φ(φ) = \frac{2Re}{c} C_c(−χ, −λχ, φ), \quad c = C_c(−χ, −λχ, 0)$. Thus, as in the case of the inverse pendulum, there are periodic stable bound solutions only within a very narrow parametric region $λ(χ)$ (see Ref. 15), they have period $π$ for the variable $φ$. For $ω_2 \gg ω$, numerically, this parametric stability constraint corresponds to a dependency between the quadrupolar electric force intensity and the parameters of the Penning trap

$$λ \rightarrow \frac{1}{2} + \frac{1}{\sqrt{2χ}}.$$  \tag{.8}

Additionally, the closest distance between the two ion bundles defines the squeezing factor, $κ$, of the coalescing orbits, i.e., numerically

$$\ln(ρ_0/2RC) \rightarrow \ln κ \rightarrow \frac{1}{2} - 1.35\sqrt{χ}.$$  \tag{.9}
here $\varrho_0 \equiv \varrho(\pi/2)$. The implication is that, if the resonant condition in Eq. 8 is satisfied, any closeness, even small, between the two positive ion clumps can be reached near the centre of the trap when $\chi \gg 1$. Eqs. 8 and 9 constitute the grounds of the Resonant Ionic Confinement Method. Owing to the existence of the quadrupole, the magnetron degree of freedom is stabilised and the plasma is macroscopically described as a rigid rotor of angular velocity $\omega$. Notwithstanding with this, microscopically, each individual ion radial velocity should be Maxwellian and the stability of the plasma requires that the solid rotor energy be thermal. Incidentally, the actual inertia momentum of every "rigidly rotating" stabilised thin disk of plasma becomes

$$I_{\text{Disk}} = \sum_{i \in \text{Disk}} m_i r_i^2 = \frac{1}{2} N_{\text{Disk}} m R_C^2,$$

providing a rotational energy giving by $E_{\text{Disk}} = \frac{1}{2} I_{\text{Disk}} \omega^2$; yet, to this rotational degree of freedom of each individual ion, a thermal energy $k_B T/2$ must be allocated. Along these lines, the condition of thermal equilibrium of the stabilised, rigidly rotating, plasma is compelled to be $N_{\text{Disk}} \times \frac{k_B T}{2} = \frac{1}{2} I_{\text{Disk}} \omega^2$. It imposes $\omega R_C = \sqrt{k_B T/\mu}$ which, since $\omega \simeq \Omega/\chi$, also implies, in the approximation $\chi \gg 1$ that $\chi \sim \frac{k_B}{m \mu} (R_C/t_0)^2$. Additionally, the density clumps in the coalescing orbits are practically confined in the centre of the trap most of the time and they move away to reach some maximum confinement radius $R_C \leq R_0$ where they bounce back again to the centre. In this case, the ions inside those large aggregates, may interact individually when, owing to the resonance, the relative distance between the clumps is reduced to a tiny minimum. It is straightforward to derive, the following relations for $\chi \gg 1$: $B \simeq \frac{2 \chi}{R_C} \sqrt{k_B T \mu}$, $n \simeq \frac{e_0}{\mu} B^2$, $z_{\text{max}} \simeq R_C/\sqrt{\chi}$, $N \simeq n \chi \times V_c$, where $V_c = V/\chi = \frac{4}{3} \pi (R_C/\sqrt{\chi})^3$ is the clump volume. Therefore, we see, the confined plasma volume can be calculated as if there were $\chi$ granular spheres of ions of radius $R_C/\sqrt{\chi}$. From these equation it is possible to derive that

$$n = \{(N/\chi)^2 \times \frac{1}{2} k_B T/E_C \} \times (N/\chi) \times \frac{4}{3} \pi R'^3$$

where $R' = R_C/\sqrt{\chi}$, whereas, $E_C = (qN/\chi)^2/(8\pi e_R R')$, corresponds to the Coulomb energy of a bubble of $N/\chi$ ions on the surface of a sphere whose radius is precisely $R'$. Yet, recall that the plasma is fully thermal, imposing that the two dimensional surface energy satisfies $2 \times \frac{1}{2} (N/\chi) k_B T = E_C$. These thermal conditions of the plasma are, indeed, fully compatible with the existence of small fluctuations in the statistics that, according to our interpretation of the two coalescing ion density clumps, in the resonant case, will orbit the cloud at periodic Mathieu trajectories. For some clump pair in the plasma, the resonant situation will hold with some probability, say $\varrho[i \in \{\text{coalescing}\}]$. To estimate this recall that, owing to the periodic radial displacement of the density bundles, analogously to the axially periodic degree of freedom, dynamical equilibrium imposes that there should be some effective radial restoring force $q \delta E_\varrho = -\mu(2\omega)^2 \delta \varrho$. Again, this corresponds to a negative effective displaced charge (a hole in the positively charged ion cloud) of $-q \delta \rho n'/\mu dS$, where $n'$ is the density of the displaced density clumps inside the plasma. Now, Gauss theorem applied to a thin disk surface of the plasma states that $n' = 2 m e_0 \omega^2 / q^2$, which, from its statistical definition $n' \equiv n \varrho$, gives

$$\varrho = \frac{\omega}{\Omega - \omega} = \chi^{-1},$$

where, given that $\omega < \Omega/2$, it is always lower than 1 as it should be. Recall that near to the centre of the trap, the minimum distance between the colliding ions can be approximated by $\varrho(t) \simeq |\varrho_0 + 2 R_C e \cos \Omega't|$, it means that the distance obtains its minimum when $\Omega' \Delta t = \pi$. 
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