Generalized principal eigenvalues on $\mathbb{R}^d$ of second order
elliptic operators with rough nonlocal kernels

ARI ARAPOSTATHIS†, ANUP BISWAS‡, AND PRASUN ROYCHOWDHRY†

Abstract. We study the generalized eigenvalue problem on the whole space for a class of integro-
differential elliptic operators. The nonlocal operator is over a finite measure, but this has no
particular structure. Some of our results even hold for singular kernels. The first part of the
paper presents results concerning the existence of a principal eigenfunction. Then we present
various necessary and/or sufficient conditions for the maximum principle to hold, and use these to
characterize the simplicity of the principal eigenvalue.

1. Introduction

The analysis of eigenvalues and eigenfunctions is a central topic in the study of operator theory,
partial differential equations and probability. The importance of eigenvalue theory is evident from
its wide range of applications including maximum principles, bifurcation theory, stability analysis
of nonlinear pde, large deviation principle, risk-sensitive control etc. For important early work on
the generalized principal eigenvalue of elliptic operators we refer the reader to the works of Protter-
Weinberger [35], Donsker-Varadhan [24], Nussbaum [31], Nussbaum-Pinchover [32]. In their seminal
work Berestycki–Nirenberg–Varadhan [10] study the properties of generalized Dirichlet principal
eigenvalue of uniformly elliptic operators in bounded domains and show that the validity of max-
imum principle in bounded domains is equivalent to the positivity of the principal eigenvalue.
This work has been extended for different kinds operators both in bounded and unbounded do-
"omains. See for instance, Armstrong [8], Quaas-Sirakov [37], Ishii-Yoshimura [27], Juutinen [28] for
bounded domains and Berestycki-Hamel-Rossi [11], Berestycki-Rossi [12], Biswas-Roychowdhury
[15], Nyguen-Vo [30] for unbounded domains. In this article we are interested in the eigenvalue
theory of nonlocal operators taking the form

$$
I f(x) = \text{Tr}(a(x) \nabla^2 f) + b(x) \cdot \nabla f(x) + c(x) f(x) + I[f, x],
$$

where

$$
I[f, x] = \int_{\mathbb{R}^d} \left( f(x + z) - f(x) \right) \nu(x, dz),
$$

and $\nu(x, \cdot)$ is a finite, non-negative Borel measure on $\mathbb{R}^d$. It is easily seen that $I$ belongs to a large
family of integro-differential operators. Very recently, generalized principal eigenvalues of integro-
differential operators have been studied in bounded domains, see for instance, Arapostathis-Biswas
[3], Quaas-Salort-Xia [36], Biswas [16], Biswas-Lőrinczi [17], Pinsky [33, 34], Dipierro-Proi"et normi
Lippi-Valdinoci [23] and references therein. To the best of our knowledge, there are only few works in non-
local setting dealing with the eigenvalue problems in unbounded domains. Berestycki-Roquejoffre-
Rossi [13] discuss eigenvalue problems for the fractional Laplacian in $\mathbb{R}^d$ for a periodic patch model.

† Department of ECE, The University of Texas at Austin, EER 7.824, Austin, TX 78712
‡ Department of Mathematics, Indian Institute of Science Education and Research, Dr. Homi
Bhabha Road, Pune 411008, India

E-mail addresses: ari@utexas.edu, anup@iiserpune.ac.in, prasun.roychowdhury@students.iiserpune.ac.in

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inequality.
There are quite a few works on generalized eigenvalue problems in unbounded domains with dispersal nonlocal kernels, see for example, Berestycki-Coville-Vo [11], Coville-Hamel [19], Coville [20], Rawal-Shen [38], Shen-Xie [40] and references therein.

The very first question that one would ask while studying the eigenvalue problem in $\mathbb{R}^d$ is the existence of a principal eigenfunction. In the case of nondegenerate local elliptic pde (i.e., when $\nu = 0$) this is obtained by passing to the limit in the Dirichlet eigenvalue problems over an increasing sequence of balls covering $\mathbb{R}^d$, with the help of Harnack inequality (cf. [14]). In the case of integro-differential equations, one also needs to control the tail behaviour of the limiting eigenfunction to justify the passage to the limit in the nonlocal integration (cf. [3, 11, 19]). These two key factors (i.e., Harnack inequality and tail behaviour of the eigenfunction) make the nonlocal eigenvalue problem in unbounded domains difficult (see also the discussions in [11, p 2711] in the context of nonlocal dispersal kernels). For the particular operator $\mathcal{I}$ above it is known that the Harnack inequality does not hold in general [6, Example 1.1]. Therefore the existence of principal eigenfunctions becomes non-obvious. In Theorems 2.1 and 2.2 we identify a large family of operators for which a Harnack type inequality holds, and therefore, existence of a principal eigenfunction can be proved Theorem 2.3. We also show that if $\nu(x, \mathbb{R}^d) \to 0$ at infinity then under some additional mild hypotheses we can assert the existence of a principal eigenfunction. In Section 3 we discuss the maximum principle for the operator $\mathcal{I}$ in $\mathbb{R}^d$ and its relation with generalized eigenvalues. These results are in the spirit of [14]. Let us also mention Theorem 3.8, where we prove an equivalence relation between minimal growth at infinity (Definition 3.2) and the strict monotonicity property of principal eigenvalue on the right (Definition 3.3). In Appendix A we gather some known results concerning Dirichlet eigenvalue problems in bounded domains.

2. Existence of principal eigenfunction

We deal with a second order, linear nonlocal operator given by

$$\mathcal{I} f(x) = \text{Tr}(a(x) \nabla^2 f) + b(x) \cdot \nabla f(x) + c(x) f(x) + I[f, x],$$

(2.1)

where

$$I[f, x] = \int_{\mathbb{R}^d} (f(x + z) - f(x)) \nu(x, dz).$$

Let us also denote by

$$\mathcal{A} f(x) = \text{Tr}(a(x) \nabla^2 f) + b(x) \cdot \nabla f(x) + c(x) f(x) - \nu(x) f(x),$$

(2.2)

where $\nu(x) = \nu(x, \mathbb{R}^d)$. It should be observed that the value $\nu(x, \{0\})$ has no effect on the equation, and thus without any loss of generality we assume that $\nu(x, \{0\}) = 0$. Throughout this article we impose the following assumptions, unless we state otherwise.

**Assumption 2.1.** The coefficients satisfy the following conditions.

(i) $b : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \to \mathbb{R}$ are locally bounded.

(ii) $a : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is continuous and locally uniformly elliptic.

(iii) $\nu$ is a nonnegative, Borel measure satisfying the following:

- The map $x \mapsto \nu(x, \mathbb{R}^d) := \nu(x)$ locally bounded.

- $\nu$ has locally compact support in the sense that for every compact set $K \subset \mathbb{R}^d$ there exists a compact set $K_1 \subset \mathbb{R}^d$ so that support($\nu(x, \cdot)$) $\subset K_1$ for all $x \in K$.

We are interested in the notion of generalized principal eigenvalue of $\mathcal{I}$ in $\mathbb{R}^d$. Generalizing Berestycki-Rossi [14], which is actually in the spirit of [10, 32], we define the principal eigenvalue as follows

$$\lambda_1(\mathcal{I}) = \sup \{ \lambda \in \mathbb{R} : \exists \text{ positive } \phi \in W^{2, d}_0(\mathbb{R}^d) \text{ satisfying } \mathcal{I}\phi + \lambda \phi \leq 0 \text{ in } \mathbb{R}^d \}. \quad (2.3)$$
This definition can also been as a generalization of the Dirichlet principal eigenvalue in bounded domains. Let \( D \) be a smooth bounded domain. Then the Dirichlet principal eigenvalue of \( \mathcal{I} \) in \( D \) is defined as

\[
\lambda(\mathcal{I}, D) := \sup \left\{ \lambda \in \mathbb{R} : \exists \phi \in C_+(\mathbb{R}^d) \cap W^{2,d}_{\text{loc}}(D) \text{ satisfying } \mathcal{I}\phi + \lambda\phi \leq 0 \text{ in } D \\
\text{and } \phi > 0 \text{ in } D \right\},
\]

where \( C_+(\mathbb{R}^d) \) denotes the subspace of \( C(\mathbb{R}^d) \) consisting of nonnegative functions. The existence of principal eigenfunction in \( D \) follows from Theorem A.1 in Appendix A. Also, note that \( \lambda(\mathcal{I}, D) \) is decreasing with respect to increasing domains i.e. for \( D_1 \subset D_2 \) we have \( \lambda(\mathcal{I}, D_1) \geq \lambda(\mathcal{I}, D_2) \). It is also evident from this definition that \( \lambda(\mathcal{I}, D) \geq \lambda_1(\mathcal{I}) \) for all \( D \subset \mathbb{R}^d \).

Now the following questions emerge in a natural manner.

**Q1.** Is \( \lim_{n \to \infty} \lambda(\mathcal{I}, B_n) \) equal to \( \lambda_1(\mathcal{I}) \)?

**Q2.** Does there exist a principal eigenfunction attaining the value \( \lambda_1(\mathcal{I}) \)?

The above questions are quite related to each other. Recall that for nondegenerate second order elliptic operators (i.e., \( \nu = 0 \)) the answers to the above questions are affirmative. In fact, the equality in the question Q1 implies existence of a principal eigenfunction. The main machinery in obtaining these results is the Harnack inequality (cf. [14]). It is known from [6, Example 1.1] that the Harnack inequality does not hold true for \( \mathcal{I} \), in general. Also, a recent work of Mou [29] confirms only a weak-Harnack inequality for \( \mathcal{I} \). Very recently, Harnack inequality has been studied for various integro-differential operators. For instance, Foudun [25] obtains Harnack inequality for the bounded harmonic functions of second order integro-differential operators, Bass-Levin [9], Caffarelli-Silvestre [18] consider Harnack estimate for the fractional Laplacian type operators, Di Castro et al. [22] establish similar estimate for the fractional p-Laplacian whereas Coville [21], Coville-Hamel [19] derive Harnack estimate for dispersal type nonlocal kernels. One of the main contributions of this article is to produce a large family of kernel \( \nu \) for which a suitable Harnack type inequality holds, and thus resolve Q1 and Q2 for this family of kernels.

In (H1) and (H2) which follow, we describe two classes of kernels for which we can obtain a Harnack type estimate.

**H1** The measure \( \nu \) takes the form \( \nu(x, dy) = g(x, y)dy \) for some measurable function \( g : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty) \). There exists a nondecreasing function \( \gamma : (0, \infty) \to (0, \infty) \) and positive functions \( M_1, M_2 \) defined on \( (0, \infty) \), such that for all \( R > 0 \) the following hold:

\[
\begin{align*}
g(x, y) &= 0 \quad \forall (x, y) \in B_R \times \bar{B}_\gamma(R), \quad (2.5a) \\
g(x, y) &\leq M_1(R) \quad \forall (x, y) \in B_R \times \bar{B}_\gamma(R), \quad (2.5b) \\
\int_{B_\gamma(R)} g(y, x-y) \, dy &\geq M_2(R) \quad \forall x \in B_R. \quad (2.5c)
\end{align*}
\]

**H2** For \( d = 1 \), \( \nu \) is locally compactly supported, that is, for some function \( \gamma \) as in (H1) we have 

\[
\text{support}(\nu(x, \cdot)) \subset B_\gamma(R) \quad \text{for all } x \in B_R.
\]

Here, and in the rest of the paper, we use the notation \( B_r \) to denote the ball of radius \( r \) centered at \( 0 \).

**Remark 2.1.** Note that (2.5a) is equivalent to the statement that \( \nu \) has locally compact support (see Assumption 2.1 (iii)), and (2.5b) implies that \( \nu \) is locally bounded. Hypothesis (2.5c) is a local positivity assumption. It is satisfied, for example, if \( g(x, y) \geq c\mathbb{1}_{B_R}(x-y) \) for some positive constants \( c \) and \( \epsilon \) depending on \( R \). An example of a measure \( \nu \) that does not satisfy (2.5c) is given by

\[
g(x, y) = \mathbb{1}_{B_{2|x|}\setminus B_{|x|}}(x+y).
\]
This is because, if we evaluate (2.5c) at \( x = 0 \), we obtain
\[
\int_{B_{\gamma}(R)} g(y, x - y) \, dy = \int_{B_{\gamma}(R)} 1_{B_{2\gamma}|B|}(0) \, dy = 0.
\]
On the other hand, if we let
\[
g(x, y) = 1_{B_{\gamma}|x|}(x + y), \quad \text{for some } \kappa > 0,
\]
then setting \( \gamma(R) = (1 + \kappa)R \), it can be easily checked that (H1) holds.

**Example 2.1.** If \( \nu \) is translation invariant and has a density, that is, \( \nu(x, dy) = g(y)dy \), then (2.5c) is always satisfied unless \( \nu \equiv 0 \). Therefore, translation invariant measures \( \nu \) which have a bounded density with compact support satisfy (H1). Then, in view of Theorem 2.3 which appears later in this section, questions Q1 and Q2 have an affirmative answer for this class of operators. The reader should note that an even larger class of measures satisfying (2.5c) are those that can be minorized by a translation invariant measure with density. For instance, a typical such example of measure \( \nu \) satisfying (H1) would be \( \nu(x, dy) = 1_{B_{\xi|x|}(y)}(0)k(x, y)g(y)dy \) where the function \( g : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty) \) and \( \xi : \mathbb{R}^d \to (0, \infty) \) are locally bounded, \( k \) is bounded from above and below by positive constants on every compact sets, and for some \( \delta > 0 \) we have
\[
\int_{B_{\delta}} g(y)dy > 0.
\]

Next we prove a Harnack type estimate.

**Theorem 2.1.** Let \( a \) be locally Lipschitz, and (H1) or (H2) hold. Set \( \gamma(R) := \gamma(2R + \gamma(2R)) \vee 2R \). Then, for every \( R > 0 \) there exists a positive constant \( C(R) \) such that every nonnegative solution \( u \) of \( Au = 0 \) in \( B_{2\gamma(R)} \) satisfies
\[
\sup_{B_R} u \leq C(R)u(0). \tag{2.6}
\]

**Proof.** Let
\[
\mathcal{A}(x) := \int_{\mathbb{R}^d} u(x + z)\nu(x, dz),
\]
and recall the definition of \( \mathcal{A} \) in (2.2). Thus
\[
\mathcal{A}u(x) = -\mathcal{J}(x) \quad \text{in } B_{2\gamma(R)} \tag{2.7}
\]
by the hypothesis of the theorem. Applying [26, Theorem 9.20 and 9.22] to (2.7), we obtain
\[
\sup_{B_R} u \leq C_0 \left( \inf_{B_R} u + \|\mathcal{J}\|_{L^1(B_{2R})} \right) \leq C_0 \left( u(0) + \|\mathcal{J}\|_{L^1(B_{2R})} \right), \tag{2.8}
\]
for some positive constant \( C_0 \) depending only on \( R \). We continue by estimating \( \|\mathcal{J}\|_{L^1(B_{2R})} \). Let \( \{X_t\}_{t \geq 0} \) be the diffusion process with generator
\[
\mathcal{A}_c f = \text{Tr}(a(x)\nabla^2 f) + b(x) \cdot \nabla f(x),
\]
and \( p^{B_{2\gamma(R)}}(t, x, y) \) denote the killed transition kernel in \( B_{2\gamma(R)} \), that is,
\[
\mathbb{P}_x(X_t \in A : t < \hat{\tau}(R)) = \int_A p^{B_{2\gamma(R)}}(t, x, y) \, dy \quad \text{for all } Borel } A \subset B_{2\gamma(R)},
\]
where \( \hat{\tau}(R) = \tau_{B_{2\gamma(R)}} \) denotes the first exit time of \( X \) from \( B_{2\gamma(R)} \). Let
\[
\kappa_1(R) := \sup_{x \in B_{2\gamma(R)}} |c(x) - \nu(x)|.
\]
Applying Itô’s formula to (2.7) we obtain
\[ u(0) = \mathbb{E}_0\left[u\left(X_{2\wedge \hat{t}(R)}\right)\right] + \mathbb{E}_0\left[\int_0^{2\wedge \hat{t}(R)} e^{\frac{1}{2} \left(c\left(X_s\right) - \nu\left(X_s\right)\right) ds} \mathcal{J}(X_t) dt\right] \]
\[ \geq e^{-2\kappa_1(R)} \mathbb{E}_0\left[\int_0^{2\wedge \hat{t}(R)} \mathbb{1}_{\{t < \hat{t}(R)\}} \mathcal{J}(X_t) dt\right]. \]  

(2.9)

Since \( p^B_{2\gamma(R)}(t, x, y) \) solves a parabolic equation, it is known from the parabolic Harnack’s inequality (see [39, 41]) that there exists a positive constant \( \kappa_2(R) \) such that
\[ \inf_{t \in [1, 2]} \inf_{y \in B_{\delta(R)}} p^B_{2\gamma(R)}(t, 0, y) \geq \kappa_2(R). \]

Therefore, we have
\[ \mathbb{E}_0\left[\int_0^{2\wedge \hat{t}(R)} \mathbb{1}_{\{t < \hat{t}(R)\}} \mathcal{J}(X_t) dt\right] \geq \int_1^2 dt \int_{B_{\delta(R)}} \mathcal{J}(z) p^B_{2\gamma(R)}(t, 0, z) dz \]
\[ \geq \kappa_2(R) \int_{B_{\delta(R)}} \mathcal{J}(z) dz. \]  

(2.10)

Combining (2.9) and (2.10), we obtain
\[ \int_{B_{\delta(R)}} \mathcal{J}(z) dz \leq \left(\kappa_2(R) e^{-2\kappa_1(R)}\right)^{-1} u(0). \]  

(2.11)

First assume \((H1)\). Then, using (2.11), we have
\[ \left(\kappa_2(R) e^{-2\kappa_1(R)}\right)^{-1} u(0) \geq \int_{B_{\delta(R)}} \mathcal{J}(z) dz \]
\[ = \int_{B_{\delta(R)}} \left(\int_{\mathbb{R}^d} u(y) g(z, y - z) dy\right) dz \]
\[ \geq \int_{B_{\delta(R)}} \left(\int_{B_{2R+\gamma(2R)}} u(y) g(z, y - z) dy\right) dz \]
\[ \geq M_2(2R + \gamma(2R)) \int_{B_{2R+\gamma(2R)}} u(y) dy, \]  

(2.12)

where in the third inequality we use Fubini’s theorem and (2.5c) for the last inequality. Thus, by (2.5a), (2.5b), and (2.12), we obtain for \( x \in B_{2R} \) that
\[ \mathcal{J}(x) \leq M_1(R) \int_{x + B_M(2R)} u(y) dy \]
\[ \leq M_1(R) \int_{B_{2R+\gamma(2R)}} u(y) dy \]
\[ \leq \frac{M_1(R)}{M_2(2R + \gamma(2R)) \kappa_2(R)} e^{2\kappa_1(R)} u(0) \quad \forall x \in B_{2R}. \]  

(2.13)

It is clear then that (2.6) follows from (2.8) and (2.13).

Under \((H2)\), since \( d = 1 \), (2.6) follows from (2.11). This completes the proof. \(\square\)

Remark 2.2. Under \((H1)\) or \((H2)\) a slightly more general estimate holds. Suppose that a nonnegative \( u \) satisfies \( \mathcal{L} u = -f \) in \( B_{2\gamma(R)} \), for some nonnegative \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \). Then there exists a positive constant \( C_R \) such that
\[ \sup_{B_{2R}} u \leq C_R \left(\inf_{B_{2R}} u + \|f\|_{L^1(B_{2R})}\right). \]
We continue with the following definition.

**Definition 2.1.** We say that $\nu$ points inwards in a bounded domain $D$ if there exists a domain $D' \supseteq D$ such that $\text{support}(\nu(x, \cdot)) \subset D - x$ for all $x \in D'$.

**Theorem 2.2.** Let $a$ be locally Lipschitz, and $D$ a domain on which $\nu$ points inwards. Then, for any bounded domain $\bar{D} \supseteq D$, there exists a constant $C_H$ such that every nonnegative solution $u$ of $\mathcal{L}u = 0$ in $\bar{D}$ satisfies

$$u(x) \leq C_H u(y) \quad \forall x, y \in D.$$  

**Proof.** Let $D'$ be as in Definition 2.1. As in (2.9), we have

$$\sup_{\bar{D}} u \leq C_0 \left( \inf_{\bar{D}} u + \|\mathcal{J}\|_{L^d(D' \cap \bar{D})} \right)$$

for some positive constant $C_0$. Thus, either $\sup_{\bar{D}} u \leq 2C_0 \inf_{\bar{D}} u$, in which case (2.14) holds with $C_H = 2C_0$, or

$$\sup_{\bar{D}} u \leq 2C_0 \|\mathcal{J}\|_{L^d(D' \cap \bar{D})}.$$  

Since $\nu$ points inwards in $D$, (2.15) implies that

$$\sup_{D' \cap \bar{D}} \mathcal{J} \leq C'_0 \|\mathcal{J}\|_{L^d(D' \cap \bar{D})}$$

for some positive constant $C'_0$. From (2.16), using the Minkowski inequality, we obtain

$$\sup_{D' \cap \bar{D}} \mathcal{J} \leq C'_0 \|\mathcal{J}\|_{L^d(D' \cap \bar{D})} \leq C'_0 \left( \sup_{D' \cap \bar{D}} \mathcal{J} \right)^{\frac{d}{d-1}} \|\mathcal{J}\|_{L^1(D' \cap \bar{D})}^\frac{1}{d},$$

which implies that

$$\sup_{D' \cap \bar{D}} \mathcal{J} \leq C'_0 \|\mathcal{J}\|_{L^1(D' \cap \bar{D})}.$$  

Equations (2.7) and (2.17) imply a Harnack estimate of the form (2.14) by [7, Corollary 2.2]. \hfill \square

Consider the following hypothesis.

(H3) There exists an increasing sequence of bounded domains $\{D_n\}_{n \in \mathbb{N}}$ which covers $\mathbb{R}^d$ and $\nu$ points inwards on $D_n$ for all $n \in \mathbb{N}$.

Recall the definitions in (2.3) and (2.4). We have the following theorem concerning the existence of a principal eigenfunction for $\lambda_1(I)$ on $\mathbb{R}^d$.

**Theorem 2.3.** Grant any of the hypotheses (H1)–(H3), Then, the following hold:

(a) We have $\lim_{R \to \infty} \lambda(I, B_R) = \lambda_1(I)$ under (H1) or (H2), and

$$\lim_{n \to \infty} \lambda(I, D_n) = \lim_{n \to \infty} \lambda(I, B_n) = \lambda_1(I)$$

under (H3).

(b) Suppose that $\lambda_1(I) > -\infty$. Then, for any $\lambda \leq \lambda_1(I)$, there exists a positive $\varphi \in W^{2,p}_{\text{loc}}(\mathbb{R}^d)$, $p > d$, satisfying

$$\mathcal{L}\varphi + \lambda \varphi = 0 \quad \text{in } \mathbb{R}^d.$$  

**Proof.** We start with part (a). Since we have a Harnack inequality under the above hypotheses, the proof follows a standard argument (cf. [14]). We show that $\lim_{n \to \infty} \lambda(I, B_n) = \lambda_1(I)$, and there exists a positive eigenfunction corresponding to the eigenvalue $\lambda_1(I)$. Without loss of generality, we assume that $D_n = B_n$. Define $\hat{\lambda} := \lim_{n \to \infty} \lambda(I, B_n)$. It follows from this definition that $\hat{\lambda} \geq \lambda_1(I)$.

If $\hat{\lambda} = -\infty$, then $\lambda_1(I) = -\infty$ and there is nothing to prove. So we suppose that $\hat{\lambda} \in (-\infty, \infty)$. Then, we have...
Let \( \psi_n \) be the Dirichlet principal eigenfunction in \( B_n \) corresponding to the eigenvalue \( \lambda(\mathcal{I}, B_n) \) (see Theorem A.1) i.e.
\[
\mathcal{I}\psi_n = -\lambda(\mathcal{I}, B_n) \psi_n \quad \text{in} \quad B_n,
\]
\[
\psi_n = 0 \quad \text{in} \quad B_n^c,
\]
\[
\psi_n > 0 \quad \text{in} \quad B_n.
\] (2.18)

Applying Theorem 2.1 or Theorem 2.2 we see that for any compact set \( K \), there exists an integer \( m(K) \) such that
\[
\sup_{n \geq m(K)} \sup_K \psi_n \leq C,
\] (2.19)
for some constant \( C \) dependent on \( K \). Writing (2.18) as
\[
A\psi_n = -\mathcal{J}_n(x) = -\int_{\mathbb{R}^d} \psi_n(x + y) \nu(x, dy).
\]

Then, by Assumption 2.1 (iii) and (2.19), we obtain
\[
\sup_{n \geq m_1(K)} \sup_K \mathcal{J}_n \leq C_1
\]
for some constant \( C_1 \) and some integer \( m_1(K) \geq m(K) \). Thus, from standard elliptic estimates, we obtain a constant \( C_2 \) for every \( p > d \) satisfying
\[
\sup_{n \geq m_2(K)} \|\psi_n\|_{W^{2,p}(K)} \leq C_2
\]
for some integer \( m_2(K) \). Therefore, using a standard diagonalization argument we can extract a subsequence \( \psi_{n_k} \) converging to a nonnegative \( \psi \in W^{2,p}_{\text{loc}}(\mathbb{R}^d) \), \( p > d \), and \( \psi(x_0) = 1 \) as \( n_k \to \infty \). By Assumption 2.1 we also obtain
\[
\mathcal{I}\psi + \hat{\lambda}\psi = 0 \quad \text{in} \quad \mathbb{R}^d.
\]

Since \( (A + \hat{\lambda})\psi \leq 0 \), by the strong maximum principle we also have \( \psi > 0 \) in \( \mathbb{R}^d \). Therefore, \( \psi \) is an admissible test function in (2.3) implying that \( \hat{\lambda} \leq \lambda_1(\mathcal{I}) \). Hence \( \hat{\lambda} = \lambda_1(\mathcal{I}) \) and this completes the proof of step 1.

We continue with part (b) and show that for every \( \lambda < \lambda_1(\mathcal{I}) \) there exists a positive eigenfunction. Fix \( \lambda < \lambda_1(\mathcal{I}) \). Let \( f_n \) be a smooth, non-positive, nonzero function with support(\( f_n \)) \( \in B_n \setminus B_{n-1} \).

By Theorem A.3, since \( \lambda(\mathcal{I}, B_n) - \lambda > 0 \), there exists a unique positive \( \varphi_n \) satisfying
\[
\mathcal{I}\varphi_n + \lambda\varphi_n = f_n \quad \text{in} \quad B_n,
\]
\[
\varphi_n = 0 \quad \text{in} \quad B_n^c.
\]

Normalize \( \varphi_n \) to satisfy \( \varphi_n(x_0) = 1 \) and then we can use the Harnack inequality as in Step 1 to extract a converging subsequence of \( \varphi_n \) to some positive function \( \varphi \in W^{2,p}_{\text{loc}}(\mathbb{R}^d) \), \( p > d \). Thus, we obtain \( \mathcal{I}\varphi + \lambda\varphi = 0 \) in \( \mathbb{R}^d \). This completes the proof. \( \square \)

Our next result deals with another class of kernels not covered by Theorem 2.3. We denote by \( \lambda(A, D) \) the Dirichlet principal eigenvalue of \( A \) in a smooth domain \( D \), possibly unbounded, i.e.,
\[
\lambda(A, D) := \sup \left\{ \lambda \in \mathbb{R} : \exists \phi \in C_+(\mathbb{R}^d) \cap W^{2,d}_{\text{loc}}(D) \text{ satisfying } A\phi + \lambda\phi \leq 0 \text{ in } D \text{ and } \phi > 0 \text{ in } D \right\}.
\]

For \( D = \mathbb{R}^d \) this principal eigenvalue will be denoted as \( \lambda_1(A) \). It is evident from the definition that \( \lambda(A, D) \geq \lambda(\mathcal{I}, D) \).

**Theorem 2.4.** We assume the following:

1. The matrix \( a \) is bounded, uniformly elliptic and uniformly continuous in \( \mathbb{R}^d \).
2. For some \( \alpha \geq 0 \) we have \( |b(x)| \leq C(1 + |x|^\alpha) \) and \( |c(x)| \leq C(1 + |x|^{2\alpha}) \) for all \( x \in \mathbb{R}^d \).
(3) There exists some open ball $B_0 \subset \mathbb{R}^d$ such that $\text{supp}(\nu(x, \cdot)) \subset B_0$ for all $x \in \mathbb{R}^d$, and

$$\lim_{|x| \to \infty} \nu(x, \mathbb{R}^d) e^{|x|^\alpha} = 0.$$  \hspace{1cm} (2.20)

Then

$$\lim_{n \to \infty} \lambda(I, B_n) = \lambda_1(I).$$  \hspace{1cm} (2.21)

In addition, if we also have

$$\lim_{n \to \infty} \lambda(A, B_n^c) > \lambda_1(I) > -\infty,$$  \hspace{1cm} (2.22)

then there exists a principal eigenfunction for $\lambda_1(I)$.

Proof. We first show that $\lambda_1(A) > -\infty$ and $\sup_{x \in \mathbb{R}^d} \nu(x, \mathbb{R}^d) e^{|x|^\alpha} < \infty$ imply that $\lambda_1(I) > -\infty$. This is because if $\lambda_1(A) > -\infty$, then there exists a positive $\varphi \in W^{2,p}_{\text{loc}}(\mathbb{R}^d)$, $p > d$, satisfying

$$A \varphi + \lambda_1(A) \varphi = 0 \quad \text{in} \quad \mathbb{R}^d.$$  \hspace{1cm} (2.23)

By the assumption on the coefficients of the operator $A$, and using [2, Lemma 4.1], we obtain from (2.23) that

$$\frac{\nabla \varphi(x)}{\varphi(x)} \leq C_1 (1 + |x|^\alpha) \quad \forall x \in \mathbb{R}^d,$$  \hspace{1cm} (2.24)

for some constant $C_1$. Let $f(x) := \log \varphi(x)$. Then, it follows from (2.24) that $|\nabla f(x)| \leq C_1 (1 + |x|^\alpha)$. Hence we have

$$\sup_{y \in B_0} |f(x + y) - f(x)| \leq C_2 (1 + |x|^\alpha) \quad \forall x \in \mathbb{R}^d.$$  

This implies that

$$\sup_{x \in \mathbb{R}^d} \sup_{y \in B_0} \frac{\varphi(x + y)}{\varphi(x)} < C_3 e^{|x|^\alpha}$$

for some constant $C_3$. Thus we obtain

$$\frac{1}{\varphi} I \varphi = \frac{1}{\varphi} A \varphi + \int_{B_0} \frac{\varphi(x + y)}{\varphi(x)} \nu(x, dy)$$

$$\leq -\lambda_1(A) + C_3 \sup_{x \in \mathbb{R}^d} \nu(x, B_0) e^{|x|^\alpha},$$  \hspace{1cm} (2.25)

which shows that $\lambda_1(I) > -\infty$.

We proceed to prove (2.21). In view of the preceding paragraph, if $\lambda_1(I) = -\infty$, then we must have

$$-\infty = \lambda_1(A) = \lim_{n \to \infty} \lambda(A, B_n) \geq \lim_{n \to \infty} \lambda(I, B_n) \geq \lambda_1(I)$$

and (2.21) holds. Therefore, without loss of generality, we assume that $\lambda_1(I) > -\infty$. For $\delta \in \mathbb{R}$ we define

$$\tilde{\lambda}(\delta) = \lim_{n \to \infty} \lambda(I + \delta 1_{B_1}, B_n).$$

From this definition it follows that $\tilde{\lambda}(\delta) \geq \lambda_1(I + \delta 1_{B_1})$ for all $\delta \in \mathbb{R}$. We prove (2.21) using the argument of contradiction. Suppose that $\lambda(0) > \lambda_1(I)$. Since $\delta \mapsto \lambda(I + \delta 1_{B_1}, B_n)$ is concave and decreasing [3, Theorem 2.3], it follows that $\delta \mapsto \tilde{\lambda}(\delta)$ is concave and thus continuous. Moreover,

$$\tilde{\lambda}(\delta) \leq \lambda(I + \delta 1_{B_1}, B_1) = \lambda(I, B_1) - \delta \to -\infty$$

as $\delta \to \infty$. This means that we can select $\delta > 0$ such that

$$\tilde{\lambda}(0) > \tilde{\lambda}(\delta) > \lambda_1(I).$$  \hspace{1cm} (2.26)

Since

$$\lambda_1(A) = \lim_{n \to \infty} \lambda(A, B_n) \geq \lim_{n \to \infty} \lambda(I, B_n) = \tilde{\lambda}(0),$$

it also holds that $\lambda_1(A) > \tilde{\lambda}(\delta)$. Let $\varepsilon := \frac{1}{2} (\lambda_1(A) - \tilde{\lambda}(\delta)) > 0.$
We write (2.23) as

$$A\varphi + (\bar{\lambda}(\delta) + 2\varepsilon) \varphi = 0 \quad \text{in } \mathbb{R}^d.$$  

(2.27)

Thus, using (2.20), (2.25), and (2.27), we obtain

$$\frac{1}{\varphi} \mathcal{I} \varphi = \frac{1}{\varphi} A \varphi + \int_{B_0} \varphi(x + y) \nu(x, dy)$$

$$\leq -\tilde{\lambda}(\delta) - 2\varepsilon + C_3 \nu(x, B_0) e^{|x|^{\alpha}}$$

$$< -\tilde{\lambda}(\delta) - \varepsilon$$  

(2.28)

for all $x$ outside a ball $B_{k_0}$, for some $k_0 \in \mathbb{N}$. Now consider the Dirichlet eigenvalue problem

$$\mathcal{I} \psi_n + \delta \mathbb{1}_{B_1} \psi_n + \lambda(\mathcal{I} + \delta \mathbb{1}_{B_1}, B_n) \psi_n = 0 \quad \text{in } B_n, \quad \psi_n = 0 \quad \text{on } B_n^c.$$  

(2.29)

Let

$$\kappa_n := \min_{B_n} \varphi_n, \quad \varphi_n := \kappa_n \psi_n.$$  

Fix $n_0$ large so that $\lambda(\mathcal{I} + \delta \mathbb{1}_{B_1}, B_n) < \tilde{\lambda}(\delta) + \varepsilon$ for all $n \geq n_0$. By (2.28) and (2.29), we have

$$\mathcal{I} (\varphi - \varphi_n) + \lambda(\mathcal{I} + \delta \mathbb{1}_{B_1}, B_n) (\varphi - \varphi_n) < 0 \quad \forall x \in B_n \setminus B_{k_0}.$$  

(2.30)

Thus, since $\varphi \geq \varphi_n$ on $B_n$, it follows from (2.30) and the strong maximum principle that $\varphi - \kappa_n \psi_n$ cannot vanish in $B_n \setminus B_{k_0}$, which implies that the minimum of $\varphi_n$ is attained in $B_{k_0}$ for all $n > k_0 \vee n_0$. We write (2.29) as

$$A \varphi_n + \delta \mathbb{1}_{B_1} \varphi_n + \lambda(\mathcal{I} + \delta \mathbb{1}_{B_1}, B_n) \varphi_n = -J_n \quad \text{in } B_n,$$  

(2.31)

with

$$J_n(x) := \int_{\mathbb{R}^d} \varphi_n(x + z) \nu(x, dz) \leq \int_{\mathbb{R}^d} \varphi(x + z) \nu(x, dz) =: \hat{J}(x).$$

By [26, Theorem 9.20 and 9.22], we obtain from (2.31) the bound

$$\sup_{B_R} \varphi_n \leq C_R \left( \inf_{B_{2R}} \varphi_n + \|J_n\|_{L^d(B_{2R})} \right) \leq C_R \left( \sup_{B_{k_0}} \varphi + \|\hat{J}\|_{L^d(B_{2R})} \right),$$

which is valid for all $R$ and $n$ such that $n > 2R > 2k_0$. Thus, by a standard elliptic estimate, the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $W^{2,p}(B_R)$, $p > d$, for any $R > 0$, and this permits us to extract a subsequence converging weakly in $W^{2,p}_{loc}(\mathbb{R}^d)$ to some nonnegative function $\psi \in W^{2,p}_{loc}(\mathbb{R}^d)$. Passing to the limit as $n \to \infty$ in (2.31), we obtain

$$A \psi + \delta \mathbb{1}_{B_1} \psi + \lambda(\mathcal{I}) \psi = -\int_{\mathbb{R}^d} \psi(x + y) \nu(x, dy).$$  

(2.32)

Since, as we showed earlier, $\varphi_n$ touches $\varphi$ at one point from below in $B_{k_0}$, we must have $\psi > 0$ in $\mathbb{R}^d$ by the strong maximum principle. (2.32) then implies that $\mathcal{I} \psi + \lambda(\delta) \psi \leq 0$ in $\mathbb{R}^d$, which contradicts the inequality $\lambda_1(\mathcal{I}) < \tilde{\lambda}(\delta)$ in (2.26). This completes the proof of (2.21).

Next we establish the existence of a principal eigenfunction under (2.22). Choose $n_o$ large enough so that $\lambda(A, B_{n_o}^c) > \lambda_1(\mathcal{I})$, and let $\varphi_{n_o} \in W^{2,p}_{loc}(B_{n_o}^c)$ be such that

$$A \varphi_{n_o} + \lambda(A, B_{n_o}^c) \varphi_{n_o} = 0 \quad \text{in } B_{n_o}^c, \quad \varphi_{n_o} > 0 \quad \text{in } B_{n_o}^c.$$  

(2.33)

Existence of the eigenfunction $\varphi_{n_o}$ follows from [14]. Let $\zeta: \mathbb{R}^d \to [0, 1]$ be a smooth cut-off function such that $\zeta = 1$ in $B_{n_o+1}$ and $\zeta = 0$ in $B_{n_o+2}^c$. Define $\Phi := \zeta + (1 - \zeta) \varphi_{n_o} \in W^{2,p}_{loc}(\mathbb{R}^d)$. By the definition of $\zeta$ and (2.33), we have $A \Phi + \lambda(A, B_{n_o}^c) \Phi = 0$ in $B_{n_o+2}^c$. This means that (2.24) holds for all $x$ outside some compact set, and we can follow the argument in the first part of the proof (see (2.28)), combined with the fact that support($\nu(x, \cdot)$) $\subset B_0$, to find a ball $B_{k_0}$ satisfying

$$\mathcal{I} \Phi + (\lambda(A, B_{k_0}^c) - \varepsilon) \Phi < 0 \quad \text{in } B_{k_0}^c.$$
where $\varepsilon > 0$ is chosen small enough to satisfy $\lambda(A, B_{n_0}) - \varepsilon > \lambda_1(I)$. Now consider the principal Dirichlet eigenfunction in $B_n$ satisfying

$$\mathcal{I}\psi_n = -\lambda(I, B_n)\psi_n \text{ in } B_n, \quad \psi_n = 0 \text{ in } B_n^c, \quad \psi_n > 0 \text{ in } B_n. \quad (2.34)$$

Let $\rho_n = \min_{B_n} \frac{\phi}{\psi_n}$ and define $\hat{\psi}_n = \rho_n \psi_n$. Employing the argument in the first part of the proof, we can show that $\hat{\psi}_n$ touches $\Phi$ at one point from below in $B_{\rho_n}$, and that the sequence $\{\hat{\psi}_n\}_{n \in \mathbb{N}}$ is bounded in $W^{2,p}(B_R)$, $p > d$, for any $R > 0$. We extract a subsequence converging in $W^{2,p}_{loc}(\mathbb{R}^d)$, $p > d$, to some positive $\hat{\psi}$, and pass to the limit in (2.34), after multiplying this equation with $\rho_n$, to obtain $\mathcal{I}\hat{\psi} + \lambda_1(I)\hat{\psi} = 0$ in $\mathbb{R}^d$. This completes the proof. \hfill \Box

Remark 2.3. Note that (2.22) is used to construct a suitable super-solution that allows us to pass the limit. This can be replaced by the following condition: there exists a positive $V \in W^{2,d}_{loc}(\mathbb{R}^d)$ satisfying

$$IV + (\lambda_1(I) + \varepsilon)V \leq 0 \text{ in } K^c,$$

for some compact set $K$ and $\varepsilon > 0$. For instance, if $\lim_{|x| \to \infty} c(x) = -\infty$, then we can take $V = 1$.

Remark 2.4. If there exists a principal eigenfunction for $\lambda_1(I)$, and $A$ is strictly right-monotone at $\lambda_1(A)$ (see [5] and Definition 3.3) and $\nu$ is nontrivial, then $\lambda_1(I) < \lambda_1(A)$, and therefore (2.22) holds.

3. Maximum principles and simplicity of eigenvalues

Throughout this section we assume that $\lambda_1(I) > -\infty$. Following [14], we define the eigenvalues $\lambda'_1(I)$ and $\lambda''_1(I)$ by

$$\lambda'_1(I) := \inf \left\{ \lambda \in \mathbb{R} : \exists \text{ positive } \phi \in W^{2,d}_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \text{ satisfying } \mathcal{I}\phi + \lambda\phi \geq 0 \text{ in } \mathbb{R}^d \right\},$$

and

$$\lambda''_1(I) := \sup \left\{ \lambda \in \mathbb{R} : \exists \phi \in W^{2,d}_{loc}(\mathbb{R}^d) \text{ satisfying } \mathcal{I}\phi + \lambda\phi \leq 0 \text{ in } \mathbb{R}^d \text{ and } \inf_{\mathbb{R}^d} \phi > 0 \right\}.$$

First, we develop a maximum principle.

Definition 3.1. We say that the operator $\mathcal{I}$ satisfies the maximum principle (on $\mathbb{R}^d$) if for every function $u \in W^{2,d}_{loc}(\mathbb{R}^d)$ satisfying $\mathcal{I}u \geq 0$ in $\mathbb{R}^d$ and $\sup_{\mathbb{R}^d} u < \infty$, we have $u \leq 0$ in $\mathbb{R}^d$.

It is well-known that the eigenvalues $\lambda'_1(I)$ and $\lambda''_1(I)$ are closely connected to the maximum principle for the operator $\mathcal{I}$ [14,15]. We refer to the following two sets of conditions on the coefficients of $\mathcal{I}$ in Theorem 3.1 below.

$$\sup_{\mathbb{R}^d} c(x) < \infty, \quad \sup_{\mathbb{R}^d} |a_{ij}(x)| < \infty \quad \text{and} \quad \sup_{\mathbb{R}^d} \frac{b(x) \cdot x}{|x|} < \infty, \quad (3.1)$$

and

$$\sup_{\mathbb{R}^d} c(x) < \infty, \quad \limsup_{|x| \to \infty} \frac{|a_{ij}(x)|}{|x|^2} < \infty \quad \text{and} \quad \limsup_{|x| \to \infty} \frac{b(x) \cdot x}{|x|^2} < \infty. \quad (3.2)$$

Our next result provides a necessary and a sufficient condition for maximum principle.

Theorem 3.1. The following hold.

(i) If $\mathcal{I}$ satisfies the maximum principle then $\lambda'_1(I) \geq 0$.

(ii) Suppose that for some ball $B_a$ we have $\text{supp}(\nu(x, \cdot)) \subset B_a$ for all $x$, $\nu(x)$ is bounded, and either (3.1) or (3.2) hold. Then $\mathcal{I}$ satisfies the maximum principle if $\lambda''_1(I) > 0$.

(iii) Suppose $a$, $b$ and $(c - \nu)^{-}$ are bounded. Then $\mathcal{I}$ satisfies the maximum principle if $\lambda''_1(I) > 0$. 

Proof. (i) Let $\lambda_1'(I) < 0$. Then there exists a positive $\phi \in W^2_0(R^d) \cap L^\infty(R^d)$ satisfying $I\phi \geq 0$ in $R^d$. This clearly contradicts the maximum principle. Thus if $I$ satisfies the maximum principle, it must be the case that $\lambda_1'(I) \geq 0$.

(ii) Suppose, on the contrary, that $\lambda_n''(I) > 0$, but $I$ does not satisfy the maximum principle. Then there exists a $u \in W^2_0(R^d)$ which is positive somewhere in $R^d$ and $Iu \geq 0$ in $R^d$ with $\sup_{R^d} u < \infty$. Also, since $\lambda_n'(I) > 0$, there exist $\lambda > 0$ and $\psi \in W^2_0(R^d)$ satisfying $I\psi + \lambda \psi \leq 0$ in $R^d$ and $\inf_{R^d} \psi > 0$. By scaling appropriately we may also assume that $\psi \geq u$ in $R^d$. Now choose a smooth positive function $\chi: R^d \to R$ such that if $x \in B^c_1$, then

$$
\chi(x) = \begin{cases} 
\exp[\sigma(x)] & \text{if } I \text{ satisfies (3.1)}, \\
|x|^\sigma & \text{if } I \text{ satisfies (3.2)}. 
\end{cases}
$$

After an easy computation we can write for $x \in B^c_1$ that

$$
I\chi(x) \leq \left( \sigma(\sigma + \frac{1}{|x|})|a(x)| + \sigma \frac{b(x) \cdot x}{|x|} + c(x) - \nu(x) + \int_{B_0} \frac{\exp[|x+z|^\sigma]}{|x|^\sigma} \nu(x, dz) \right) \chi(x),
$$

if $I$ satisfies (3.1), and

$$
I\chi(x) \leq \left( \sigma^2 + d \sigma - 2\sigma \right) |a(x)| + \sigma \frac{b(x) \cdot x}{|x|^2} + c(x) - \nu(x) + \int_{B_0} \frac{|x+z|^{\sigma}}{|x|^\sigma} \nu(x, dz) \right) \chi(x),
$$

if $I$ satisfies (3.2). Now observe that for $x \in B^c_1$ there exist a positive constant $C$ such that

$$
\frac{|x+z|^\sigma}{|x|^\sigma} \leq C(1 + |z|^\sigma), \quad \text{and} \quad \frac{\exp[|x+z|^\sigma]}{|x|^\sigma} \leq \exp[|z|^\sigma].
$$

Next, using (3.3)–(3.5), and the fact that support$(\nu(x, \cdot)) \subset B_0$, we deduce that for some a positive constant $C_1$ we have

$$
I\chi(x) \leq C_1 \chi(x) \quad \forall \ x \in B^c_1.
$$

Now set $\psi_n = \psi + \frac{1}{n} \chi$ and define $\kappa_n = \sup_{R^d} \frac{u}{\psi_n}$. Note that $\kappa_n > 0$ for all $n \in N$ since $u$ is positive at some point. Using the fact that $\psi \geq u$ and the definition of $\kappa_n$ we can write $\kappa_n \leq 1$, and $\kappa_n \leq \kappa_{n+1}$ for all $n \geq 1$. Moreover, since sup$_{R^d} u < \infty$, we have

$$
\limsup_{|x| \to \infty} \frac{u(x)}{\psi_n(x)} \leq 0.
$$

Hence the supremum sup$_{R^d} \frac{u}{\psi_n}$ is attained, and there exists $x_n \in R^d$ such that $\kappa_n = \frac{u(x_n)}{\psi_n(x_n)}$. Next, we estimate the term $\frac{\chi(x_n)}{n}$. Note that

$$
\frac{1}{\kappa_{2n}} \leq \frac{\psi_{2n}(x_n)}{u(x_n)} = \frac{1}{\kappa_n} - \frac{\chi(x_n)}{2nu(x_n)},
$$

which implies that

$$
\frac{\chi(x_n)}{n} \leq 2 \left( \frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}} \right) u(x_n) \leq 2 \left( \frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}} \right) \psi(x_n).
$$

Hence, by continuity, for each $n \in N$ there exists a positive $\eta_n$ such that

$$
\frac{\chi(x)}{n} \leq \left( \frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}} \right) \psi(x) \quad \text{in } B_{\eta_n}(x_n).
$$

(3.7)

On the other hand, using the linearity of $I$ together with (3.6) and (3.7), we obtain

$$
I\psi_n = I\psi + \frac{1}{n} I\chi \leq \left[ -\lambda + C_1 \left( \frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}} \right) \right] \psi(x) \quad \forall \ x \in B_{\eta_n}(x_n).
$$
Since \( \{\kappa_n\}_{n \in \mathbb{N}} \) is a convergent sequence, we can choose \( n_0 \) large enough so that
\[
\mathcal{I}
\psi_{n_0} < 0 \quad \text{in} \quad B_{\eta_{n_0}}(x_{n_0}).
\]  
(3.8)

Note that for the nonnegative function \( w = \kappa_{n_0}\psi_{n_0} - u \), the following holds
\[
\mathcal{I}w = \kappa_{n_0}\mathcal{I}\psi_{n_0} - \mathcal{I}u < 0 \quad \text{in} \quad B_{\eta_{n_0}}(x_{n_0}),
\]
which also implies that,
\[
\mathcal{A}w \leq \mathcal{I}w < 0 \quad \text{in} \quad B_{\eta_{n_0}}(x_{n_0}).
\]
But \( w(x_{n_0}) = 0 \), and the strong maximum principle infers that \( w \equiv 0 \) in \( B_{\eta_{n_0}}(x_{n_0}) \). Hence \( \int_{\mathbb{R}^d} w(x + z)\nu(x, dz) = 0 \) in \( B_{\eta_{n_0}}(x_{n_0}) \). Using these facts and (3.8), we get
\[
\mathcal{I}u = \kappa_{n_0}\mathcal{I}\psi_{n_0} < 0 \quad \text{in} \quad B_{\eta_{n_0}}(x_{n_0}),
\]
which clearly contradicts the fact \( \mathcal{I}u \geq 0 \) in \( \mathbb{R}^d \). Therefore we must have \( u \leq 0 \).

(iii) As before, we consider \( u \in \mathcal{W}^{2,1}_\text{loc}(\mathbb{R}^d) \) satisfying
\[
\mathcal{I}u \geq 0 \quad \text{in} \quad \mathbb{R}^d, \quad \sup_{\mathbb{R}^d} u < \infty,
\]
and since \( \lambda'_I(\mathcal{I}) > 0 \), we choose \( \lambda > 0 \) and \( \psi \in \mathcal{W}^{2,1}_\text{loc}(\mathbb{R}^d) \) satisfying \( \mathcal{I}\psi + \lambda\psi \leq 0 \) in \( \mathbb{R}^d \) and \( \inf_{\mathbb{R}^d} \psi > 0 \). Scaling suitably, we may assume that \( \inf_{\mathbb{R}^d} \psi \geq 1 \) which in turn, implies that
\[
\mathcal{I}\psi \leq -\lambda \quad \text{in} \quad \mathbb{R}^d. \tag{3.9}
\]
If \( \sup_{\mathbb{R}^d} u > 0 \), then \( \kappa := (\sup_{\mathbb{R}^d} u)^{-1} \in (0, \infty) \). Using (3.9), we then get
\[
-\lambda \geq \mathcal{I}\Phi \geq \tilde{\mathcal{A}}\Phi \quad \text{in} \quad \mathbb{R}^d,
\]
where \( \Phi := \psi - \kappa u \), and
\[
\tilde{\mathcal{A}} := \sum_{i,j} a_{ij} \partial_{x_i} x_j + \sum_i b_i(x) \partial_{x_i} - (c(x) - \nu(x))^{-}.
\]
By the strong maximum principle, we have \( \Phi > 0 \) in \( \mathbb{R}^d \). Then, applying [12, Lemma 2.1(i)], it follows that \( \inf_{\mathbb{R}^d} \Phi > 0 \) which contradicts the fact that \( \inf_{\mathbb{R}^d} \Phi = 0 \). Thus \( \kappa^{-1} = 0 \) and \( \sup_{\mathbb{R}^d} u \leq 0 \). This completes the proof. \( \square \)

As an application of Theorem 3.1, we establish a relation between \( \lambda'_I \) and \( \lambda'_I' \) which extends the result in [14, Theorem 1.7].

**Theorem 3.2.** Suppose that \( \lim_{n \to \infty} \lambda(\mathcal{I}, B_n) = \lambda_1(\mathcal{I}) \). Then we have \( \lambda_1(\mathcal{I}) \geq \lambda'_I(\mathcal{I}) \). Also, under the hypotheses of Theorem 3.1 (ii) or (iii), we have \( \lambda'_I(\mathcal{I}) \geq \lambda'_I(\mathcal{I}) \).

**Proof.** The proof follows from the same argument as in [14]. We show that for any \( \lambda > \lambda_1 \) we have \( \lambda \geq \lambda'_I \). Replacing \( c \) by \( c - \lambda \) we may assume that \( \lambda = 0 \). Since \( \lim_{n \to \infty} \lambda(\mathcal{I}, B_n) = \lambda_1 \), we can find a ball \( \mathcal{B} \) large enough such that the corresponding Dirichlet eigenvalue \( \lambda(\mathcal{I}, \mathcal{B}) \) is negative. Let \( \varphi_\mathcal{B} \) be the corresponding principal eigenfunction. We normalize the eigenfunction so that it satisfies
\[
\|\varphi_\mathcal{B}\|_{L^\infty(\mathcal{B})} = \min \left\{ 1, -\frac{\lambda(\mathcal{I}, \mathcal{B})}{\|c\|_{L^\infty(\mathcal{B})}} \right\}.
\]
Then for the equation \( \mathcal{I}f = c^+ f^2 \) we see that \( \tilde{u} = 1 \) is a bounded super-solution, and \( \varphi_\mathcal{B} \) is a sub-solution in \( \mathbb{R}^d \) and also lies below 1. Now we can apply the monotone iteration method (since the comparison principle holds for above operator) to construct a positive solution which is bounded above by 1. This shows that \( \lambda'_I \leq 0 \). Hence the proof.

Next, we prove the second part of the theorem. Suppose, on the contrary, that there exists \( \lambda > 0 \) satisfying \( \lambda'_I(\mathcal{I}) > \lambda > \lambda'_I(\mathcal{I}) \). So from the definition of \( \lambda'_I(\mathcal{I}) \) there exists a positive \( \phi \in \mathcal{W}^{2,1}_\text{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) that satisfies \( \mathcal{I}\phi + \lambda\phi \geq 0 \) in \( \mathbb{R}^d \). But note that \( \lambda'_I(\mathcal{I} + \lambda) = \lambda'_I(\mathcal{I}) - \lambda > 0 \),
which means that $\mathcal{I} + \lambda$ satisfies the maximum principle in Theorem 3.1, and this contradicts the existence of such a function $\phi$. This completes the proof. \hfill \Box

The following observation is used in the sequel.

**Lemma 3.1.** For any domain $D$, it holds that $-\sup_{x \in D} c(x) \leq \lambda''(\mathcal{I}, D)$.

**Proof.** Suppose that this is not true. Then, there exists $\lambda$ such that $-\sup_D c > \lambda > \lambda''(\mathcal{I}, D)$. Now consider $\psi \equiv 1$. We deduce that $\mathcal{I}(1) + \lambda \leq \sup_D c + \lambda < 0$. Hence, the definition of $\lambda''(\mathcal{I}, D)$ implies that $\lambda''(\mathcal{I}, D) \geq \lambda$, and this contradicts the original hypothesis. \hfill \Box

We also need the following existence result in the exterior domain.

**Lemma 3.2.** Let $\mathcal{K}$ be a compact domain in $\mathbb{R}^d$ with smooth boundary. Let $\bar{v}, \underline{v} \in W^{1,2}_{\text{loc}}(\mathcal{K}^c) \cap C(\mathbb{R}^d)$, with $\underline{v} \leq \bar{v}$ in $\mathbb{R}^d$, satisfy $\mathcal{I}\bar{v} \leq f(x, \bar{v})$ in $\mathcal{K}^c$ for some function $f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$, and $\underline{v} \in C^2(\mathbb{R}^d)$ is a subsolution of $\mathcal{I}\underline{v} = f$ in $\mathcal{K}^c$. Assume that $f(x, \cdot)$ is locally Lipschitz, locally uniformly in $x$, and $f$ is locally bounded. Then there exists a solution $u \in W^{2,p}_{\text{loc}}(\mathcal{K}^c) \cap C(\mathbb{R}^d)$, $p > d$, of $\mathcal{I}u = f$ in $\mathcal{K}^c$ satisfying $\underline{v} \leq u \leq \bar{v}$.

**Proof.** Let $\{\Omega_n : n \in \mathbb{N}\}$ be an increasing sequence of bounded domains with smooth boundary which cover $\mathcal{K}^c$. Consider a solution $u_n \in W^{2,p}_{\text{loc}}(\Omega_n) \cap C(\mathbb{R}^d)$ satisfying

$$
\mathcal{I}u_n = f(x, u_n) \quad \text{in } \Omega_n,
$$

$$
u_n = \underline{v} \quad \text{in } \Omega_n^c. \tag{3.10}
$$

Existence of $u_n$ follows from a monotone iteration argument. For instance, replacing $f$ by $f - \mathcal{I}\underline{v}$ we may assume that $\underline{v} = 0$ and $\bar{v} \geq 0$. Choose $\theta$ large enough so that $\lambda_{\Omega_n}(\mathcal{I} - \theta) > 0$, $\theta \geq \|c\|_{L^\infty(\Omega_n)}$, and

$$
|f(x, s_1) - f(x, s_2)| \leq \theta|s_1 - s_2| \quad \forall \ x \in \Omega_n,
$$

with $s_1, s_2 \in [\inf_{\Omega_n} \underline{v}, \sup_{\Omega_n} \underline{v}]$. Now, we let $\xi_1 = 0 = \underline{v}$, and applying [3, Theorem 6.3], we define the sequence $\{\xi_i\}_{i \in \mathbb{N}} \subset W^{2,p}_{\text{loc}}(\Omega_n) \cap C(\mathbb{R}^d)$ via the recursion

$$
\mathcal{I}\xi_{i+1} - \theta\xi_i + f(x, \xi_i) = f(x, \xi_i) - \theta\xi_i \quad \text{in } \Omega_n, \quad \xi_{i+1} = 0 \quad \text{in } \Omega_n^c.
$$

Applying Corollary A.1 we obtain that $0 \leq \xi_1 \leq \xi_2 \leq \cdots \leq \bar{v}$ in $\Omega_n$. Then using [3, Lemma 6.1], and a standard elliptic regularity estimate, we pass to the limit $\xi_i \rightarrow u_n$ as $i \rightarrow \infty$, to get a solution of (3.10). We also have $\underline{v} \leq u_n \leq \bar{v}$.

Defining

$$
\mathcal{A}u_n = \int_{\mathbb{R}^d} u_n(x + y)\nu(x, dy),
$$

we rewrite (3.10) as

$$
\mathcal{A}u_n = -\mathcal{J}_n(x) + f(x, u_n) \quad \text{in } \Omega_n.
$$

Since $\underline{v} \leq u_n \leq \bar{v}$, it follows that $\mathcal{J}_n(x)$ and $f(x, u_n)$ are locally uniformly bounded, and thus $\{u_n\}$ is bounded in $W^{2,p}_{\text{loc}}(\mathcal{K}^c)$ for $p > d$. The behaviour of $u_n$ near the boundary $\partial \mathcal{K}$ can also be uniformly controlled (cf. [3, Lemma 6.1]). Thus we can extract a subsequence of $u_n$ converging to a function $u \in W^{2,p}_{\text{loc}}(\mathcal{K}^c) \cap C(\mathbb{R}^d)$. Finally passing the limit in (3.10) we have desired result. \hfill \Box

The theorem which follows is used later to establish equality of eigenvalues. This result should be compared with [14, Theorem 7.6].

**Theorem 3.3.** It holds that

$$
\lambda''(\mathcal{I}) = \min \left\{ \lambda_1(\mathcal{I}), \lim_{n \rightarrow \infty} \lambda''(\mathcal{I}, B^n_r) \right\},
$$

where $B^n_r$ denotes the $r$-neighborhood of $\mathcal{I}$.
where

\[
\lambda''(I, D) := \sup \left\{ \lambda \in \mathbb{R} : \exists \phi \in \mathcal{W}^{2, d}_{\text{loc}}(\mathbb{R}^d) \cap C(\mathbb{R}^d) \right. \\
\left. \text{satisfying } I \phi + \lambda \phi \leq 0 \text{ in } D \text{ and } \inf_{\mathbb{R}^d} \phi > 0 \right\}. 
\]  

(3.11)

Proof. By the monotonicity property of \( r \mapsto \lambda''(I, B^c_r) \), we have

\[
\lambda''(I) \leq \lim_{r \to \infty} \lambda''(I, B^c_r). 
\]

Using this fact and the definition of the eigenvalues we can immediately write

\[
\lambda''(I) \leq \min \left\{ \lambda_1(I), \lim_{r \to \infty} \lambda''(I, B^c_r) \right\}. 
\]

To establish the reverse inequality, we show that for any \( \lambda < \min \{ \lambda_1(I), \lim_{r \to \infty} \lambda''(I, B^c_r) \} \) we also have \( \lambda''(I) \geq \lambda \). By hypothesis, we can find a positive number \( R \) such that \( \lambda < \lambda''(I, B^c_R) \). Thus there exists \( \eta \in \mathcal{W}^{2, d}_{\text{loc}}(\overline{B^c_R}) \cap C(\mathbb{R}^d) \) satisfying \( I \eta + \lambda \eta \leq 0 \) in \( \overline{B^c_R} \) and \( \inf_{\mathbb{R}^d} \eta > 0 \). After multiplying by a suitable constant we may assume that \( \inf_{\mathbb{R}^d} \eta \geq 2 \). Choose a Lipschitz continuous function \( f : \mathbb{R} \to (-\infty, 0] \) satisfying \( f(1) = -1 \), \(-1 \leq f(t) \leq 0 \) for \( t \in [1, 2] \) and \( f(t) = 0 \) for \( t \geq 2 \).

Then \( \bar{u} = \eta \) is a supersolution to

\[
I u + \lambda u = |c(x) + \lambda| f(u(x)) \quad \text{in } \overline{B^c_R},
\]

and \( u = 1 \) is a subsolution. By Lemma 3.2, there exists \( \phi \in \mathcal{W}^{2, p}_{\text{loc}}(\overline{B^c_R}) \cap C(\mathbb{R}^d), p > d, \) satisfying \( \inf_{\mathbb{R}^d} \phi \geq 1 \) and

\[
I \phi + \lambda \phi \leq 0 \quad \text{in } \overline{B^c_R}.
\]

By Morrey’s inequality we can see that \( \phi \in C^1(\overline{B^c_{R+1}}) \). Since \( \lambda < \lambda_1(I) \), there exist \( \varepsilon_1 > 0 \) and a positive function \( \psi \in \mathcal{W}^{2, d}_{\text{loc}}(\mathbb{R}^d) \) satisfying

\[
I \psi + (\lambda + \varepsilon_1) \psi \leq 0 \quad \text{in } \mathbb{R}^d.
\]

We choose a nonnegative function \( \chi \in C^2(\mathbb{R}^d) \), taking values in \([0, 1]\), such that \( \chi = 0 \) in \( B^c_{R+1} \) and \( \chi = 1 \) in \( B^c_{R+2} \). Let \( u := \psi + \varepsilon \chi \phi \), with \( \varepsilon \) a positive constant to be chosen later. Recall from Assumption 2.1 (iii) that there exists a compact set \( K_1 \) such that \( \text{support}(\nu(x, \cdot)) \subset K_1 \) for all \( x \in B^c_{R+2} \).

We continue with some estimates of \( u \) on a partition of \( \mathbb{R}^d \). For \( x \in B^c_{R+1} \) we have

\[
I u + \lambda u \leq -\varepsilon_1 \psi + \varepsilon \int_{K_1} \chi(x + z) \phi(x + z) \nu(x, dz) \leq -\varepsilon_1 \psi + \varepsilon \kappa_1, 
\]

(3.12)

where \( \kappa_1 \) depends upon the bounds of \( \phi \) in the compact set \( B^c_{R+1} + K_1 \) and the measure of \( K_1 \). If we consider the annular region \( B^c_{R+2} \setminus B^c_{R+1} \), then for all \( x \in B^c_{R+2} \setminus B^c_{R+1} \), it holds that

\[
I u + \lambda u \leq -\varepsilon_1 \psi + \varepsilon \left[ I(\chi \phi) + \lambda \chi \phi \right] \\
\leq -\varepsilon_1 \psi + \varepsilon \chi(I + \lambda) \phi + \varepsilon \left[ 2a_{ij} \partial_i \chi \partial_j \phi + (a_{ij} \partial_i \chi + b_i \partial_i \chi) \phi \right] \\
+ \varepsilon \int_{\mathbb{R}^d} \left( \chi(x + z) - \chi(x) \right) \phi(x + z) \nu(x, dz) \\
\leq -\varepsilon_1 \psi + \varepsilon \kappa_2 + \varepsilon \int_{K_1} \left( \chi(x + z) - \chi(x) \right) \phi(x + z) \nu(x, dz) \\
\leq -\varepsilon_1 \psi + \kappa_3 \varepsilon,
\]

(3.13)
where \( \kappa_2 \) and \( \kappa_3 \) are some constants depending on the bounds of the coefficients in the set \( B_{R+2} \setminus B_{R+1} \). Now choosing \( \varepsilon \) small enough, we can make the right-hand side of (3.12) and (3.13) non-positive. Finally, when \( x \in B^c_{R+2} \), we have

\[
\mathcal{I}u + \lambda u \leq -\varepsilon_1 \psi + \varepsilon (\mathcal{I} \phi + \lambda \phi) + \varepsilon \int_{\mathbb{R}^d} (\chi(x + z) - 1) \phi(x + z) \nu(x, dz) \leq 0.
\]

Combining all the above cases, we deduce that \( \mathcal{I}u + \lambda u \leq 0 \) in \( \mathbb{R}^d \). Hence, from the definition of \( \lambda''(\mathcal{I}) \), it is evident that \( \lambda''(\mathcal{I}) \geq \lambda \), which completes the proof.

**Remark 3.1.** Suppose \( \sup_D c < \infty \). Then the admissible test functions in the definition (3.11) can be restricted to the class satisfying \( \inf_D \phi > 0 \). That is,

\[
\lambda''(\mathcal{I}, D) = \sup \left\{ \lambda \in \mathbb{R} : \exists \phi \in W^{2,\infty}_{\text{loc}}(D) \cap C(\mathbb{R}^d), \text{ satisfying } \mathcal{I} \phi + \lambda \phi \leq 0 \text{ in } D \text{ and } \inf_D \phi > 0 \right\}.
\]

To see this consider any \( \phi \in W^{2,\infty}_{\text{loc}}(D) \cap C(\mathbb{R}^d) \) with \( \inf_D \phi > 0 \) and

\[
\mathcal{I} \phi + \lambda \phi \leq 0 \text{ in } D.
\]

For some \( \varepsilon > 0 \), define \( \psi_\varepsilon = \phi + \varepsilon \) and note that for \( x \in D \),

\[
\mathcal{I} \psi_\varepsilon + \lambda \psi_\varepsilon \leq (c(x) + \lambda) \varepsilon \leq \varepsilon \left( \lambda + \sup_D c(x) \right) \left( \inf_D \phi \right)^{-1} \psi_\varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we get (3.11).

Our next result is an extension of [14, Proposition 1.11]. It shows that if the potential \( c \) is negative at infinity, then the principal eigenvalue \( \lambda_1(\mathcal{I}) \) characterizes the validity of the maximum principle.

**Theorem 3.4.** Suppose that \( \zeta := \limsup_{|x| \to \infty} c(x) < 0 \). Then the following hold:

(i) Under the hypotheses in Theorem 3.1 (ii) or (iii), the maximum principle holds if \( \lambda_1(\mathcal{I}) > 0 \).

(ii) If \( \lim_{n \to \infty} \lambda(\mathcal{I}, B_n) = \lambda_1(\mathcal{I}) \) and the maximum principle holds, then \( \lambda_1(\mathcal{I}) > 0 \).

**Proof.** (i) Suppose that \( \lambda_1(\mathcal{I}) > 0 \). Then applying Lemma 3.1 and Theorem 3.3 we deduce that \( \lambda''(\mathcal{I}) > 0 \), which implies by Theorem 3.1 that \( \mathcal{I} \) satisfies the maximum principle.

(ii) If \( \mathcal{I} \) satisfies the maximum principle, then we have \( \lambda(\mathcal{I}) \geq 0 \) by Theorem 3.1 (i), and therefore, using Theorem 3.2 we deduce that \( \lambda_1(\mathcal{I}) \geq 0 \). Suppose that \( \lambda_1(\mathcal{I}) = 0 \). Taking \( V = 1 \) we see that

\[
\mathcal{I}V = c(x) < \zeta/2 \quad \text{outside a compact set}.
\]

Hence by Remark 2.3 and the proof of Theorem 2.4 we can construct a principal eigenfunction \( \varphi \) satisfying \( \mathcal{I} \varphi = 0 \) and \( \varphi \leq V \) in \( \mathbb{R}^d \). This clearly, contradicts the maximum principle. Hence we must have \( \lambda_1(\mathcal{I}) > 0 \). \( \square \)

Our next result should be compared with [14, Theorem 1.9].

**Theorem 3.5.** \( \lambda_1(\mathcal{I}) = \lambda''(\mathcal{I}) \) holds in each of the following cases:

(i) \( \lambda_1(\mathcal{I} - \gamma) = \lambda''(\mathcal{I} - \gamma) \) for some nonnegative function \( \gamma \in L^\infty(\mathbb{R}^d) \) satisfying \( \lim_{|x| \to \infty} \gamma(x) = 0 \);

(ii) \( \lambda_1(\mathcal{I}) \leq -\limsup_{|x| \to \infty} c(x) \);

(iii) \( \|a\|_{L^\infty(\mathbb{R}^d)} \leq \lambda \), \( \lim_{|x| \to \infty} b(x) = 0 \), and

\[
\forall r > 0, \forall \beta < \limsup_{|x| \to \infty} c(x), \exists B_r(x_0) \text{ satisfying } \inf_{B_r(x_0)} (c(x) - \nu(x)) > \beta;
\]

(iv) There exists \( V \in C^2(\mathbb{R}^d) \) which is bounded from below away from 0, and satisfies \( \mathcal{I}V + \lambda_1(\mathcal{I})V \leq 0 \) outside some compact set.
Proof. Hypotheses (i) and (ii) are exactly the same as in [14, Theorem 1.9(ii)-(iii)] which only uses Theorem 3.3.

Next we show that (iii) ⇒ (ii). Note that it is enough to demonstrate that if \( \sigma < \limsup_{|x| \to \infty} c(x) \) then \( \lambda_1(I) \leq -\sigma \). Consider the function

\[
\psi(x) = \exp \left( -\frac{1}{1 - |x|^2} \right) \text{ on } B_{1/\varepsilon}, \quad \text{and } \psi(x) = 0 \text{ on } B_{1/\varepsilon}^c,
\]

for an appropriate positive constant \( \varepsilon \) to be chosen later. An easy calculation shows that

\[
D_{x_i} \psi = \frac{-2\varepsilon^2 x_i}{(1 - |x|^2)^2} \psi
\]

and

\[
D_{x_i x_j} \psi = \left( \frac{4\varepsilon^4}{(1 - |x|^2)^2} x_i x_j - \frac{2\varepsilon^2}{(1 - |x|^2)^2} \delta_{ij} - \frac{8\varepsilon^4}{(1 - |x|^2)^3} x_i x_j \right) \psi.
\]

For \( x_0 \in \mathbb{R}^d \), define \( \phi(x) = \psi(x - x_0) \). Suppose that we show that we can choose \( \varepsilon \) and \( x_0 \) in such a way that

\[
I \phi - \sigma \phi > 0 \quad \text{in } B_{1/\varepsilon}(x_0).
\]

(3.14)

Since the principal eigenvalues \( \lambda(I, D) \) and \( \lambda(I, D) \), the later defined by (A.3), are equal for a bounded domain \( D \) (see Theorem A.2), we see that

\[
-\sigma \geq \lambda(I, B_{1/\varepsilon}(x_0)) = \lambda(I, B_{1/\varepsilon}(x_0)) \geq \lambda_1(I)
\]

by (3.14). Thus it remains to establish (3.14). From the calculations of \( D_{x_i} \psi \) and \( D_{x_i x_j} \psi \) we see that

\[
I \phi(x) - \sigma \phi(x) \geq \left( \frac{4\varepsilon^2 |\varepsilon(x-x_0)|^2}{(1 - |\varepsilon(x-x_0)|^2)^4} - \frac{2d\varepsilon^2}{(1 - |\varepsilon(x-x_0)|^2)^2} - \frac{8\varepsilon^4 |\varepsilon(x-x_0)|^2}{(1 - |\varepsilon(x-x_0)|^2)^3}

- \frac{2\varepsilon^2 |x-x_0|^2}{(1 - |x-x_0|^2)^2} + c(x) - \nu(x) - \sigma \right) \phi(x).
\]

(3.15)

Given \( \varepsilon > 0 \) we first choose \( R \) large enough such that \( |b(x)| \leq \varepsilon \) for \( |x| \geq R \), and then choose \( x_0 \in \mathbb{R}^d \) satisfying \( |x_0| \geq R + 2\varepsilon^{-1} \). Furthermore, due to our hypothesis, we can choose \( x_0 \) is such a fashion that

\[
\inf_{B_{1/\varepsilon}(x_0)} (c(x) - \nu(x)) > \sigma.
\]

(3.16)

Next, we estimate (3.15) in two steps.

**Step 1.** Suppose that \( 1 - \delta < |\varepsilon(x-x_0)|^2 < 1 \) where \( \delta \) is a small positive number such that

\[
4\Lambda(1-\delta) - 2d\Lambda\delta^2 - 8\Lambda(1-\delta)^2 - 2\delta^2 > 0.
\]

It then follows from (3.15) that

\[
I \phi - \sigma \phi \geq \frac{\varepsilon^2 (4\Lambda(1-\delta) - 2d\Lambda\delta^2 - 8\Lambda(1-\delta)^2 - 2\delta^2)}{(1 - |\varepsilon(x-x_0)|^2)^4} \phi + (c(x) - \nu(x) - \sigma) \phi.
\]

This proves (3.14) in the annulus \( 1 - \delta < |\varepsilon(x-x_0)|^2 < 1 \).

**Step 2.** We are left with the region \( 0 \leq |\varepsilon(x-x_0)|^2 \leq 1 - \delta \), where \( \delta \) is as chosen in Step 1. Here we deduce that

\[
I \phi - \sigma \phi \geq \left( (c(x) - \nu(x) - \sigma) - \frac{2d\Lambda\varepsilon^2}{\delta^2} - \frac{8\Lambda(1-\delta)^2}{\delta^3} - \frac{2\varepsilon^2}{\delta^2} \right) \phi.
\]

Using (3.16), we can choose \( \varepsilon \) small enough to make the right-hand side of the preceding equation positive. Combining the above steps we obtain (3.14), which completes the proof.
For (iv) we again use Theorem 3.3. Choose a large \( r \) such that \( B_r^c \subset \mathcal{K}^c \). Then \( \lambda_1(\mathcal{I}) \leq \lambda''(\mathcal{I}, \bar{B}_r^c) \) by the definition. Now letting \( r \to \infty \) and using Theorem Theorem 3.3, we deduce that \( \lambda''(\mathcal{I}) = \lambda_1(\mathcal{I}) \).

In the remaining part of this article we discuss the simplicity of the principal eigenvalue. As well known from [14, Proposition 8.1], \( \lambda_1(\mathcal{I}) \) need not be simple, in general. The following definition is the extension of Agmon’s minimal growth at infinity [1] criterion in the nonlocal situation.

**Definition 3.2.** A positive function \( u \in W^{2,d}_{loc}(\mathbb{R}^d) \) satisfying \( \mathcal{I}u = 0 \) in \( \mathbb{R}^d \) is said to be a solution of minimal growth at infinity if for any \( \rho > 0 \) and any positive function \( v \in W^{2,d}_{loc}(B_{\rho}^c) \cap C(\mathbb{R}^d) \) satisfying \( \mathcal{I}v \leq 0 \) in \( B_{\rho}^c \), there exist constants \( R \geq \rho \) and \( k > 0 \) such that \( ku \leq v \) in \( B_R^c \).

**Theorem 3.6.** Let \( u \in W^{2,d}_{loc}(\mathbb{R}^d) \) be a positive solution of minimal growth at infinity of the equation \( \mathcal{I}u = 0 \) in \( \mathbb{R}^d \). Then, for any positive function \( v \in W^{2,d}_{loc}(\mathbb{R}^d) \) satisfying \( \mathcal{I}v \leq 0 \) in \( \mathbb{R}^d \), there exists \( \kappa > 0 \) such that \( v \equiv \kappa u \) in \( \mathbb{R}^d \).

**Proof.** Define the quantity \( \kappa = \inf_{\mathbb{R}^d} \frac{v}{u} \). Clearly \( v - \kappa u \geq 0 \). First, we claim that \( v - \kappa u > 0 \) can not be positive in \( \mathbb{R}^d \). Suppose, on the contrary, that \( v - \kappa u > 0 \) in \( \mathbb{R}^d \). Since \( \mathcal{I}(v - \kappa u) \leq 0 \) in \( \mathbb{R}^d \), by the definition of solution of minimal growth at infinity of \( u \), there exist positive constants \( R \) and \( \kappa_1 \) such that \( \kappa_1 u \leq v - \kappa u \) in \( B_R^c \). Note that \( \mathcal{I}u = 0 \) in \( \mathbb{R}^d \) immediately gives \( \lambda(\mathcal{I}, B_{R+1}^c) \geq 0 \), and the use of [3, Corollary 2.1] implies that \( \lambda(\mathcal{I}, B_{R}) > 0 \). Applying Corollary A.1 we deduce that \( \kappa_1 u \leq v - \kappa u \) in \( \mathbb{R}^d \) and this contradicts the definition of \( \kappa \). Thus \( v - \kappa u \) must vanish somewhere in \( \mathbb{R}^d \), proving the claim. Since 

\[
\mathcal{A}(v - \kappa u) \leq 0 \quad \text{in} \quad \mathbb{R}^d,
\]

by the strong maximum principle we get \( v = \kappa u \). This completes the proof. \( \square \)

From the above result it is evident that only principal eigenfunctions can have minimal growth at infinity. Our next result is an extension of [14, Proposition 8.4] which establishes a sufficient condition for minimal growth at infinity of the eigenfunctions.

**Theorem 3.7.** Suppose that

\[
\lim_{r \to \infty} \lambda(\mathcal{I}, \bar{B}_r^c) > \lambda_1(\mathcal{I}), \quad \lim_{n \to \infty} \lambda(\mathcal{I}, B_n) = \lambda_1(\mathcal{I}),
\]

and \( \text{support}(\nu(x, \cdot)) \subset B \) for some ball \( B \), for all \( x \in \mathbb{R}^d \). Then there exists an eigenfunction \( \psi \) for \( \lambda_1(\mathcal{I}) \) which has minimal growth at infinity. In particular, \( \lambda_1(\mathcal{I}) \) is simple in the class of positive functions.

**Proof.** Without loss of generality we can assume \( \lambda_1(\mathcal{I}) = 0 \). In view of Theorem 3.6, it is enough to show that there exists a principal eigenfunction with minimal growth at infinity. For every \( n \in \mathbb{N} \), let \( \psi_n \in W^{2,d}_{loc}(B_n) \cap C(\mathbb{R}^d) \) be an eigenfunction associated with \( \lambda(\mathcal{I}, B_n) \) i.e.

\[
\mathcal{I}\psi_n = -\lambda(\mathcal{I}, B_n)\psi_n \quad \text{in} \quad B_n, \quad \psi_n > 0 \quad \text{in} \quad B_n, \quad \text{and} \quad \psi_n = 0 \quad \text{in} \quad B_n^c.
\]

By hypothesis, there exist a large positive number \( R \) and \( n_0 \in \mathbb{N} \) with \( n_0 > R \), such that for all \( n \geq n_0 \), we have \( \lambda(\mathcal{I}, \bar{B}_R^c) > \lambda > \lambda(\mathcal{I}, B_n) > 0 \), for some \( \lambda > 0 \).

From the definition of \( \lambda(\mathcal{I}, \bar{B}_R^c) \) there exists a nonnegative function \( \phi \in W^{2,d}_{loc}(\bar{B}_R^c) \cap C(\mathbb{R}^d) \) satisfying

\[
\mathcal{I}\phi \leq -\lambda\phi \quad \text{in} \quad \bar{B}_R^c, \quad \phi > 0 \quad \text{in} \quad \bar{B}_R^c.
\]

Let \( \chi : \mathbb{R}^d \to [0,1] \) be a \( C^2 \) cut-off function satisfying \( \chi = 1 \) in \( B_R \) and \( \chi = 0 \) in \( B_{R+1}^c \). We define \( \varphi = \chi + \phi \). Note that \( \varphi > 0 \) in \( \mathbb{R}^d \) and

\[
\mathcal{I}\varphi \leq -\lambda\varphi \quad \text{in} \quad \bar{B}_{R_1}^c, \quad \varphi > 0 \quad \text{in} \quad \mathbb{R}^d,
\]

for some large \( R_1 \) satisfying \( \mathcal{B} \cap (\mathcal{B} - x) = \emptyset \) for all \( x \in \bar{B}_{R_1}^c \).
Fix any \( n > \max\{R_1, n_0\} \). Let \( \kappa_n > 0 \) such that \( \kappa_n \psi_n \leq \varphi \) in \( B_n \) and \( \kappa_n \psi_n \) touches \( \varphi \) at some point in \( B_n \). We claim that \( \kappa_n \psi_n \) has to touch \( \varphi \) inside \( B_{R_1} \). Note that in \( \bar{B}_{R_1}^c \cap B_n \) we have,

\[
A(\varphi - \kappa_n \psi_n) \leq \mathcal{I}(\varphi - \kappa_n \psi_n) \leq -\lambda \varphi + \lambda(\mathcal{I}, B_n) \kappa_n \psi_n \leq (\lambda + \lambda(\mathcal{I}, B_n)) \varphi < 0.
\]

If \( \kappa_n \psi_n \) touches \( \varphi \) outside \( B_{R_1} \), then applying the strong maximum principle we have \( \kappa_n \psi_n \equiv \varphi \) in \( \bar{B}_{R_1}^c \). But since \( \psi_n = 0 \) on \( \partial B_n \), we must have \( \varphi = 0 \) on \( \partial B_n \), which is not possible. Normalizing, we work with the eigenfunction \( \varphi_n := \kappa_n \psi_n \) instead of \( \psi_n \). Exploiting a similar method as in Theorem 2.4 and Remark 2.3 we can show that \( \varphi_n \) converges along some subsequence, to a positive eigenfunction \( \psi \in W^{2,p}_0(\mathbb{R}^d) \), \( p > d \), associated to the eigenvalue \( \lambda_1(\mathcal{I}) = 0 \).

Now let \( v \) be any positive supersolution as in Definition 3.2 satisfying \( \mathcal{I}v \leq 0 \) in \( B_{\rho'} \). Fix some \( R' > \max\{R_1, \rho\} \). We scale \( v \) by multiplying it with a positive constant \( \kappa \) in such a way that \( \sup_{B_{R'}} \varphi_n \leq v \) in \( B_{R'} \). Then to complete the proof it is enough to show that \( \psi \leq v \) in \( \mathbb{R}^d \). For \( \varepsilon > 0 \) we define \( \zeta_\varepsilon = v + \varepsilon \varphi \) and we note that \( \mathcal{I}\zeta_\varepsilon \leq -\lambda \varepsilon \varphi \) in \( B_{R'}^c \). Hence in \( \bar{B}_{R'}^c \cap B_n \) we have

\[
\mathcal{I}(\zeta_\varepsilon - \varphi_n) \leq -\varepsilon \lambda \varphi + \lambda(\mathcal{I}, B_n)\varphi_n \leq (\varepsilon \lambda + \lambda(\mathcal{I}, B_n)) \varphi < 0
\]

for all large enough \( n \), since \( \lim_{n \to \infty} \lambda(\mathcal{I}, B_n) = \lambda_1(\mathcal{I}) = 0 \). Since \( \lambda(\mathcal{I}, B_{R'}^c \cap B_n) > \lambda_1(\mathcal{I}) = 0 \) by Corollary A.1, we have \( \psi_n \leq \zeta_\varepsilon \) in \( \mathbb{R}^d \). Now we first let \( n \to \infty \), and then \( \varepsilon \to 0 \), to conclude that \( \psi \leq v \) in \( \mathbb{R}^d \).

\[\square\]

Remark 3.2. Theorem 3.7 should be compared with the second assertion in Theorem 2.4. At first sight, Theorem 3.7 seems to be stronger since hypotheses (1), (2) and (2.20) of Theorem 2.4 are not enforced, but keep in mind that \( \lambda_1(\mathcal{I}) > -\infty \) is a blanket assumption in Section 3.

Remark 3.3. Suppose that for some ball \( B_0 \) we have \( \sup\{\nu(x, \cdot)\} \subset B_0 \) for all \( x \) and, \( \nu(x) \) is bounded. Also, assume that the coefficients satisfy

\[
\|a\|_{L^\infty(\mathbb{R}^d)} < \infty, \quad \lim_{|x| \to \infty} \frac{b(x) \cdot x}{|x|} = \pm \infty, \quad \text{and} \quad \sup_{\mathbb{R}^d} c(x) < \infty.
\]

This implies the condition \( \lim_{r \to \infty} \lambda(\mathcal{I}, \bar{B}_r^c) > \lambda_1(\mathcal{I}) \) in Theorem 3.7. To see this, define the function \( \phi_r(x) := \exp(\mp |x|) \) in \( \bar{B}_r^c, r > 0 \), where \( \mp \) matches with the sign \( \pm \) in the hypothesis. Now by direct calculation we obtain for \( x \in \bar{B}_r^c \),

\[
(\mathcal{I} + \lambda_1(\mathcal{I}) + 1)\phi_r(x) = \left( \frac{a_{ij} x_i x_j}{|x|^2} \mp \left( \frac{\text{Tr}(a_{ij})}{|x|} - \frac{a_{ij} x_i x_j}{|x|} \mp \frac{b(x) \cdot x}{|x|} \right) \right)
+ c(x) - \nu(x) + \lambda_1(\mathcal{I}) + 1 + \int_{\partial B_0} \left( \frac{e^{\mp |x| z}}{e^{\mp |z|}} \nu(x, dz) \right) \phi_r(x).
\]

Now using the given hypotheses and choosing large \( r \), we deduce that \( (\mathcal{I} + \lambda_1(\mathcal{I}) + 1)\phi_r < 0 \) for \( x \in \bar{B}_r^c \). This implies that \( \lambda(\mathcal{I}, \bar{B}^c) \geq \lambda_1(\mathcal{I}) + 1 \) for all large enough \( r \) and completes the assertion.

The result that follows establishes the equivalence between the minimal growth at infinity and a certain monotonicity property of the principal eigenvalue. We need the following notion of monotonicity from [5, Section 2.2]. To express explicitly the dependency of the eigenvalue of the potential \( c \) we write the principal eigenvalue \( \lambda_1(\mathcal{I}) \) as \( \lambda_1(c) \).

**Definition 3.3.** We say \( c \mapsto \lambda_1(c) \) is strictly monotone on the right at \( c \) if for any non-zero, nonnegative bounded function \( h \) we have \( \lambda_1(c + h) < \lambda_1(c) \).

When \( \nu = 0 \), it is shown in [4, Theorem 2.1] that Agmon’s minimal growth at infinity is equivalent to the monotonicity property on the right. The argument in [4] is based on a probabilistic method which uses the stochastic representation of the principal eigenfunction. Our next result extends this equivalence for nonlocal operators, and also supplies a simpler proof.
Theorem 3.8. The following hold.

(i) Suppose that the principal eigenfunction \( \psi \) satisfying \( \mathcal{I} \psi + \lambda_1(\psi) = 0 \) in \( \mathbb{R}^d \) has minimal growth at infinity. In addition, assume that for every bounded \( h \geq 0 \) there exists a positive supersolution of \( \mathcal{I} + h + \lambda_1(c+h) \) in \( \mathbb{R}^d \). Then \( \lambda_1(c) = \lambda_1 \) is strictly monotone on the right at \( c \).

(ii) Assume that \( \lambda_1(\psi) \) is strictly monotone on the right at \( c \) and there exists a positive function \( \phi \in W^{2,d}_{\text{loc}}(\mathbb{R}^d) \cap C(\mathbb{R}^d) \) satisfying

\[
\mathcal{I} \phi + \lambda_1(\psi) \phi \leq 0 \quad \text{in} \quad \mathbb{R}^d,
\]

where \( B \subset \mathbb{R}^d \) is some ball. Then there exists a principal eigenfunction for \( \lambda_1(\mathcal{I}) \), which has minimal growth at infinity. In particular, \( \lambda_1(\mathcal{I}) \) is a simple eigenvalue by Theorem 3.6.

Proof. (i) Let \( h \geq 0 \) be a bounded function. From the definition of the principal eigenvalue it is evident that \( \lambda_1(\psi + h) \leq \lambda_1(c) \). Arguing by contradiction, suppose that \( \lambda_1(c+h) = \lambda_1 \). Consider a positive \( \varphi \in W^{2,d}_{\text{loc}}(\mathbb{R}^d) \) satisfying

\[
\mathcal{I} \varphi + h \varphi + \lambda_1(c+h) \varphi \leq 0 \quad \text{in} \quad \mathbb{R}^d.
\]

Since \( h \geq 0 \) it follows from above that

\[
\mathcal{I} \varphi + \lambda_1(\psi) \varphi \leq 0 \quad \text{in} \quad \mathbb{R}^d.
\]

Using Theorem 3.6 we then obtain \( \varphi = \kappa \psi \) for some \( \kappa > 0 \). This of course, implies that \( h \varphi = 0 \) in \( \mathbb{R}^d \) which contradicts the fact \( h \neq 0 \). Hence we must have \( \lambda_1(c+h) < \lambda_1(c) \).

(ii) We construct a principal eigenfunction with minimal growth at infinity. With no loss of generality we may assume that \( B = B_1 \), the unit ball centered at 0. Let \( f(x) = \mathbb{1}_{B_1}(x) \). Let \( \varphi_n \in W^{2,p}_{\text{loc}}(B_1) \cap C(\mathbb{R}^d) \) be the unique solution of

\[
\mathcal{I} \varphi_n + \lambda_1(\psi) \varphi_n = -f \quad \text{in} \quad B_1, \quad \text{and} \quad \varphi_n = 0 \text{ on } B^c_1.
\]

Existence of \( \varphi_n \) follows from Theorem A.3, and we have \( \varphi_n > 0 \) in \( B_1 \).

We claim that \( \beta_n := \max_{B_1} \varphi_n \to \infty \) as \( n \to \infty \). To prove the claim, suppose that, on the contrary, \( \beta_{n_k} \) is bounded along some subsequence \( \{n_k\} \). Define

\[
\kappa_n = \max \{ t : \phi - t \varphi_n > 0 \text{ in } B_1 \} \land 1.
\]

It is evident that

\[
\kappa_n \geq \left[ \frac{\beta_{n_k}^{-1} \min_{B_1} \phi}{\varphi_n} \right] \land 1.
\]

Hence \( \kappa_{n_k} \geq \kappa > 0 \) for all \( n_k \). Letting \( \psi_n = \kappa_n \varphi_n \), we obtain

\[
\mathcal{I} \psi_n + \lambda_1(\psi) \psi_n = -\kappa_n f \quad \text{in} \quad B_1, \quad \text{and} \quad \psi_n = 0 \text{ on } B^c_1.
\]

Applying the comparison principle in Corollary A.1 in \( B_n \setminus B_1 \) we get \( \psi_n \leq \phi \) in \( \mathbb{R}^d \). Next, applying an argument similar to Theorem 2.4 (see the arguments after (2.31)) we can extract a subsequence of \( \{\psi_{n_k}\} \) and \( \{\kappa_{n_k}\} \) converging to \( \psi \in W^{2,p}_{\text{loc}}(\mathbb{R}^d) \), \( p > d \), and \( \kappa' > 0 \), respectively. Furthermore, \( \psi \geq 0 \) and

\[
\mathcal{I} \psi + \lambda_1(\psi) \psi = -\kappa f \quad \text{in} \quad \mathbb{R}^d. \tag{3.17}
\]

Using the strong maximum principle we either have \( \psi > 0 \) in \( \mathbb{R}^d \) or \( \psi \equiv 0 \). But \( \psi \equiv 0 \) is not possible since \( \kappa' f \neq 0 \). Hence, \( \psi > 0 \) in \( \mathbb{R}^d \). We write (3.17) as

\[
\mathcal{I} \psi + (\kappa' \mathbb{1}_{B_1} \psi^{-1} + \lambda_1(\psi)) \psi = 0 \quad \text{in} \quad \mathbb{R}^d,
\]

which also implies \( \lambda_1(\psi + \kappa' \mathbb{1}_{B_1} \psi^{-1}) \geq \lambda_1 \). But this contradicts the assumption of strict monotonicity on the right. This establishes the claim that \( \beta_n := \max_{B_1} \varphi_n \to \infty \). It also implies that \( \kappa_n \to 0 \) as \( n \to \infty \), and thus the constant \( \kappa' \) in (3.17) equals 0. This shows that the subsequence
ψ_{n_k} converges to a positive ψ ∈ W^{2,p}_{loc}(\mathbb{R}^d) satisfying \mathcal{I}\psi + \lambda_1(c)\psi = 0 \text{ in } \mathbb{R}^d. Next we show that ψ has minimal growth property at infinity. Let \nu ∈ W^{2,1}_{loc}(\mathring{B}_\rho^c) ∩ C(\mathbb{R}^d) be a positive function and
\[
\mathcal{I}v + \lambda_1(c)v \leq 0 \text{ in } \bar{B}_\rho^c.
\]
Without any loss of generality we may assume that \rho ≥ 1. Let \kappa > 0 be such that \kappa v - \psi_{n_k} ≥ 0 in \mathring{B}_\rho for all n_k ≥ \rho. Now
\[
\mathcal{I}(\kappa v - \psi_{n_k}) + \lambda_1(c)(\kappa v - \psi_{n_k}) ≤ 0 \text{ in } B_{n_k} \cap \bar{B}_\rho^c.
\]
Using Corollary A.1 we then have ψ_{n_k} ≤ κv in \mathbb{R}^d. Letting n_k → ∞ we then have ψ ≤ κv in \mathbb{R}^d. This completes the proof.

**Appendix A. The Dirichlet Principal eigenvalue and its properties**

Recall from (2.1) and (2.2) the operators
\[
\mathcal{I}f(x) = \text{Tr}(a(x)\nabla^2 f) + b(x) \cdot \nabla f(x) + c(x)f(x) + I[f, x],
\]
where
\[
I[f, x] = \int_{\mathbb{R}^d} (f(x + z) - f(x))\nu(x, dz),
\]
and
\[
\mathcal{A}f(x) = \text{Tr}(a(x)\nabla^2 f) + b(x) \cdot \nabla f(x) + c(x)f(x) - \nu(x)f(x),
\]
with \nu(x) = ν(x, \mathbb{R}^d). Let D be a bounded smooth domain. The following assumption on the coefficients is enforced throughout this section, without further mention.

**Assumption A.1.** The following hold.

(A1) \nu(x, \cdot) is a nonnegative Borel measure and the map x → ν(x, \mathbb{R}^d) is locally bounded.

(A2) The map x → a(x) is continuous in D, and there exists a positive constant κ such that \kappa I ≤ a(x) ≤ κ^{-1}I for all x ∈ D, where I denotes the identity matrix.

(A3) b: D → \mathbb{R}^d and c: D → \mathbb{R} are bounded.

By \mathcal{C}_{b,+}(\mathbb{R}^d) we denote the set of all bounded, nonnegative continuous functions on \mathbb{R}^d. Recall from (2.4) that the Dirichlet principal eigenvalue of \mathcal{I} in D is defined as follows:
\[
\lambda(\mathcal{I}, D) := \sup \left\{ \lambda ∈ \mathbb{R} : \Psi(\lambda) ≠ \emptyset \right\},
\]
where
\[
\Psi(\lambda) := \left\{ ψ ∈ \mathcal{C}_{b,+}(\mathbb{R}^d) \cap W^{2,1}_{loc}(D) : ψ > 0 \text{ in } D, \text{ and } \mathcal{I}\psi(x) + \lambda ψ ≤ 0 \text{ in } D \right\}.
\]

The following result is proved in [3, Theorem 2.1].

**Theorem A.1.** There exists a unique ψ_D ∈ W^{2,p}_{loc}(D) ∩ \mathcal{C}_{b,+}(\mathbb{R}^d), p > d, satisfying
\[
\begin{align*}
\mathcal{I}\psi_D &= -\lambda(\mathcal{I}, D) ψ_D \quad \text{in } D, \\
ψ_D &= 0 \quad \text{in } D^c, \\
ψ_D(0) &= 1, \quad ψ_D > 0 \quad \text{in } D.
\end{align*}
\tag{A.1}
\]

We also have the following characterization of the principal eigenvalue.
Theorem A.2. It holds that
\[ \lambda(I, D) = \sup \left\{ \lambda \in \mathbb{R} : \exists \psi \in C_b(\mathbb{R}^d) \cap W^{2, d}_{\text{loc}}(D) \text{ satisfying } \inf_{\mathbb{R}^d} \psi > 0 \right\} \]

\[ = \inf \left\{ \lambda \in \mathbb{R} : \exists \psi \in C_b(\mathbb{R}^d) \cap W^{2, d}_{\text{loc}}(D) \text{ satisfying } \sup_D \psi > 0, \psi \leq 0 \text{ in } D^c, \right\} \]

\[ + \mathcal{I}\psi(x) + \lambda \psi \leq 0 \text{ in } D \} \right\} \quad \text{(A.2)} \]

\[ = \inf \left\{ \lambda \in \mathbb{R} : \exists \psi \in C_b(\mathbb{R}^d) \cap W^{2, d}_{\text{loc}}(D) \text{ satisfying } \sup_D \psi > 0, \psi \leq 0 \text{ in } D^c, \right\} \]

\[ + \mathcal{I}\psi(x) + \lambda \psi \geq 0 \text{ in } D \} \right\} \quad \text{(A.3)} \]

Proof. Let \( \lambda_1 \) denote the rhs of (A.2). It is then obvious from the definition that \( \lambda(I, D) \geq \lambda_1 \). Let \( D_n \) be a sequence of strictly decreasing domains of \( C^2 \) type that converges to \( D \). Then it follows from [3, Theorem 2.2] that \( \lambda(I, D_n) \to \lambda(I, D) \) as \( n \to \infty \). For a given \( \varepsilon > 0 \) we fix \( n \) large enough to satisfy \( \lambda(I, D) \leq \lambda(I, D_n) + \varepsilon \). Let \( \psi_n \) be the Dirichlet principal eigenfunction corresponding to \( \lambda(I, D_n) \) and \( \chi \) is a smooth cut-off function satisfying \( \chi = 0 \) in \( D \) and \( \chi = 1 \) in \( D^c_n \). Let \( \xi_\delta(x) = \psi_n(x) + \delta \chi(x) \). It then follows that \( \inf_{\mathbb{R}^d} \xi_\delta > 0 \) for every \( \delta > 0 \). In addition, for \( \delta \) sufficiently small, we have

\[ \mathcal{I}\xi_\delta = \mathcal{I}\psi_n + \delta \int_{\mathbb{R}^d} \chi(x + z) \nu(x, d(z)) \]

\[ = -\lambda(I, D_n) \xi_\delta + \delta \left[ \lambda_n \chi(x) + \int_{\mathbb{R}^d} \chi(x + z) \nu(x, d(z)) \right] \]

\[ \leq (-\lambda(I, D) + \varepsilon) \xi_\delta + \delta \xi_\delta(x) \sup_{x \in D} \frac{1}{\xi_\delta(x)} \left[ \lambda_n \chi(x) + \int_{\mathbb{R}^d} \chi(x + z) \nu(x, d(z)) \right] \]

\[ \leq (-\lambda(I, D) + \varepsilon) \xi_\delta(x) + \varepsilon \xi_\delta(x) \quad \forall x \in D. \]

Hence \( \lambda_1 \geq \lambda(I, D) - 2\varepsilon \). Since \( \varepsilon \) is arbitrary, we have \( \lambda(I, D) = \lambda_1 \), proving (A.2).

Next, let \( \lambda_2 \) denote the rhs of (A.3). Note that the principal eigenfunction in (A.1) is a valid member of the admissible functions in (A.3). Thus we have \( \lambda(I, D) \geq \lambda_2 \). To establish the equality we show that for any \( \mu < \lambda(I, D) \) there exists no \( \psi \in C_b(\mathbb{R}^d) \cap W^{2, d}_{\text{loc}}(D) \) with \( \sup_D \psi > 0 \) and \( \psi \leq 0 \) in \( D^c \) satisfying

\[ \mathcal{I}\psi + \mu \psi \geq 0 \quad \text{in } D. \quad \text{(A.4)} \]

Suppose, on the contrary, that such \( \psi \) exists. Using the characterization in (A.2) we can find \( \varphi \) with \( \inf_{\mathbb{R}^d} \varphi > 0 \) and

\[ \mathcal{I}\varphi + \lambda \varphi \leq 0 \quad \text{in } D, \]

for some \( \lambda \in (\mu, \lambda(I, D)) \). Define

\[ \kappa = \inf \{ t > 0 : t \varphi - \psi > 0 \text{ in } D \}, \quad \varphi_\kappa = \kappa \varphi - \psi. \]

Since \( \sup_D \psi > 0 \), we have \( \kappa > 0 \), and \( \varphi_\kappa \) vanishes at some point in \( D \). Also, from (A.4), we have

\[ \mathcal{I}\varphi_\kappa + \mu \varphi_\kappa \leq \kappa (\mu - \lambda) \varphi \leq 0 \quad \text{in } D. \]

This of course, implies that

\[ \text{Tr}(a(x) \nabla^2 \varphi_\kappa) + b(x) \cdot \nabla \varphi_\kappa(x) - (c(x) - \nu(x) + \mu) \varphi_\kappa(x) \leq 0 \quad \text{in } D. \]

Since \( \varphi_\kappa \) attains its minimum 0 inside \( D \), it follows from the strong maximum principle that \( \varphi_\kappa \equiv 0 \) in \( D \). But this contradicts that fact that \( \varphi_\kappa > 0 \) on \( \partial D \). Hence there is no such \( \psi \) satisfying (A.4). Thus we have \( \lambda_2 \geq \lambda(I, D) \), giving us (A.3).

Note that (A.3) gives a refined maximum principle.
Corollary A.1. Suppose that $\lambda(I, D) > 0$. Then any $\psi \in C_b(\mathbb{R}^d) \cap W^{2,d}_{\text{loc}}(D)$ satisfying
\[ I\psi \geq 0 \quad \text{in} \quad D, \quad \text{and} \quad \psi \leq 0 \quad \text{in} \quad D^c, \]
is nonpositive in $\mathbb{R}^d$.

Corollary A.1 gives us the next existence result.

Theorem A.3. Suppose that $\lambda(I, D) > 0$ and $f \in C(\bar{D})$. Then there exists a unique $u \in C_b(\mathbb{R}^d) \cap W^{2,d}_{\text{loc}}(D)$ satisfying
\[ Iu = f \quad \text{in} \quad D, \]
\[ u = 0 \quad \text{in} \quad D^c. \]
Furthermore, if $f \leq 0$, then $u > 0$ in $D$.

Proof. Uniqueness follows from Corollary A.1. Existence follows from a standard monotone iteration method (cf. [37, Proposition 4.5]). The last conclusion follows from the strong maximum principle.

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