Semiclassical Prediction for Shot Noise in Chaotic Cavities

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We show that in clean chaotic cavities the power of shot noise takes a universal form. Our predictions go beyond previous results from random-matrix theory, in covering the experimentally relevant case of few channels. Following a semiclassical approach we evaluate the contributions of quadruplets of classical trajectories to shot noise. Our approach can be extended to a variety of transport phenomena as illustrated for the crossover between symmetry classes in the presence of a weak magnetic field.

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Ballistic chaotic cavities have universal transport properties, just as do disordered conductors. The explanation of such universality cannot rely on any disorder average but must make do with chaos in an individual clean cavity. We shall present here the semiclassical explanation of shot noise, relating the quantum properties of chaotic cavities to the interference between contributions of mutually close classical trajectories. Similar methods have recently been used for explaining universal spectral fluctuations of chaotic quantum systems \[1, 2\], and to calculate the universal mean conductance in \[3, 4\].

The transition amplitudes between ingoing channels, i.e., the angle enclosed between the initial piece and the direction of the lead, is dictated by the ingoing channel, whereas the final angle is determined by the outgoing channel. The contribution of each trajectory depends on the Heisenberg time \(T_H = \frac{\Omega}{(2\pi f)^{1/2}}\), with \(\Omega\) the volume of the energy shell and \(f\) the number of freedoms. The factor \(A_\alpha\) is determined by the stability of the trajectory, and the phase is proportional to the classical action \(S_\alpha\).

With the transition amplitudes thus semiclassically approximated, the quadratic term \((\text{tr}(tt^\dagger))\) becomes a double sum over trajectories. That double sum, which actually is the mean conductance, was evaluated in \[3\] as:

\[
\langle \text{tr}(tt^\dagger) \rangle = \frac{1}{T_H^2} \sum_{a_1, a_2} \sum_{c_1, c_2} A_{\alpha_1}^* A_{\beta_1} A_{\gamma_1}^* A_{\delta_1} e^{(S_\alpha - S_\beta + S_\gamma - S_\delta)/h}; \quad (2)
\]

here \(a_1, c_1 = 1, \ldots, N_1\) and \(a_2, c_2 = 1, \ldots, N_2\) represent ingoing and outgoing channels, connected by the trajectories \(\alpha, \beta, \gamma, \delta\), \(\alpha = (a_1 \rightarrow a_2), \beta = (c_1 \rightarrow a_2), \gamma = (c_1 \rightarrow c_2), \delta = (a_1 \rightarrow c_2)\). The sum is dominated by quadruplets where the trajectories \(\beta\) and \(\delta\) have approximately the same cumulative action as \(\alpha\) and \(\gamma\), such that the action difference \(\Delta S \equiv S_\alpha - S_\beta + S_\gamma - S_\delta\) is of the order \(h\). The contributions of other quadruplets interfere destructively.

Diagonal contribution: The simplest quadruplets have either \(\alpha = \beta, \gamma = \delta\), or \(\alpha = \delta, \beta = \gamma\); their action difference vanishes. The first case has coinciding ingoing channels \(a_1\) and \(c_1\) and contributes

\[
\langle \text{tr}(tt^\dagger) \rangle_{\alpha = \beta, \gamma = \delta} = \frac{1}{T_H} \sum_{a_1, a_2, c_2} \frac{1}{|A_{\alpha_1}|^2 |A_{\beta_1}|^2} |A_{\alpha_2}|^2 |A_{\beta_2}|^2; \quad (3)
\]

The foregoing sum can be done using ergodicity. As shown in \[3\], summing all trajectories between two fixed channels amounts to integrating over the dwell time \(T\),

\[
\sum_{\alpha(a_1 \rightarrow a_2)} |A_\alpha|^2 = \int_0^\infty dT e^{-\frac{N}{T_H} T} = \frac{T_H}{N}; \quad (4)
\]
where $e^{-\frac{2\pi}{\hbar}T}$ is the probability for the trajectory to dwell in the cavity up to the time $T$, and $\frac{N}{T_H}$ the rate of escape.

To proceed with Eq. 4, we invoke the Richter/Sieber sum rule twice and afterwards sum over all $N_1N_2^2$ possible combinations of channels with $a_1 = c_1$. Similarly, the case $\alpha = \delta$, $\beta = \gamma$ leads to $N_1^2N_2$ combinations with coinciding outgoing channels $a_2 = c_2$. Altogether, these so-called diagonal contributions sum up to $N_1^2N_2+N_1N_2^2 = N_1N_2^2$. In the unitary case, they cancel with $(\langle \mathrm{tr}(tt^\dagger) \rangle)$ such that shot noise must be entirely due to different quadruplets of trajectories.

2-encounters: The first family of such quadruplets, depicted in Fig. 1, was identified by Schanz, Puhlmann and Geisel for quantum graphs [8]. Here, the trajectories $\alpha$ and $\gamma$ approach each other in a “2-encounter”: a stretch of $\alpha$ comes so close in phase space to a stretch of $\gamma$ that the motion over the two stretches is mutually linearizable.

The remaining parts of $\alpha$ and $\gamma$ will be called “loops”. Assuming that all loops have non-vanishing length (Encounter stretches do not “stick out” into the leads), one finds two further trajectories $\beta$ and $\delta$, which practically coincide with $\alpha$ and $\gamma$ inside the loops but are differently connected in the encounters: The trajectory $\beta$ closely follows the initial loop of $\alpha$ and the final loop of $\gamma$, whereas $\delta$ follows the initial loop of $\gamma$ and the final loop of $\alpha$. Obviously, the cumulative action of $\beta$ and $\delta$ approximately coincides with the action of $\alpha$ and $\gamma$, with the action difference exclusively determined by the encounter region.

Each encounter influences the survival probability. If a particle stays in the cavity along the first encounter stretch it cannot escape during the second stretch either, since the two stretches are close to each other. The trajectories $\alpha$ and $\gamma$ are thus exposed to the danger of getting lost only on the four loops (see Fig. 1) and on one encounter stretch. Denoting the duration of the latter stretch by $t_{\text{enc}}$ we can write the overall exposure time as $T_{\text{exp}} = t_1 + t_2 + t_3 + t_4 + t_{\text{enc}}$. That exposure time is smaller than the cumulative duration $T_\alpha + T_\gamma$ of $\alpha$ and $\gamma$, by a second summand $t_{\text{enc}}$ for the second encounter stretch. The probability that both $\alpha$ and $\gamma$ stay inside the cavity reads $e^{-\frac{2\pi}{\hbar}T_{\text{exp}}}$, larger than the naive estimate $e^{-\frac{2\pi}{\hbar}(T_\alpha + T_\gamma)}$. In brief, encounters hinder escape [4].

To describe the geometry of encounters, we consider a Poincaré section $P$ in the energy shell, through an arbitrary point of $\alpha$. If $P$ cuts through an encounter as in Fig. 1 it must intersect $\gamma$ in a point close to the reference point on $\alpha$. Assuming two freedoms we decompose the separation between both points into components $s$, $u$ along the stable and unstable manifolds [2]. Both $s$ and $u$ must be small, $|u| < c$, $|s| < c$, with $c$ some classically small constant. The components $s$ and $u$ fix the action difference as $\Delta S = su$ and the encounter duration as $t_{\text{enc}} = \frac{-\lambda}{\lambda}\ln\frac{s^2}{|su|}$, where $\lambda$ is the Lyapunov constant [2].

Using ergodicity, we count the encounters within trajectory pairs. The probability density for $\gamma$ to pierce through $P$ at a specified time with phase-space separations $s$ and $u$ is uniform and given by the inverse of the volume of the energy shell $\Omega$. To capture all encounters, we integrate that density over (i) the time of piercing on $\alpha$ and (ii) the time of the reference piercing on $\alpha$. The probability for $P$ to cut an encounter is proportional to the duration $t_{\text{enc}}$, which we divide out to get the number of encounters [2]. Changing the integration variables to the loop durations $t_1$ and $t_3$ we get the weight function

$$w(s, u) = \int_0^{T_\alpha} \int_0^{T_\gamma} \Omega^{-1} t_{\text{enc}}(s, u) \frac{1}{\Omega} \frac{1}{t_{\text{enc}}(s, u)} ds du; \quad (5)$$

here the upper boundaries, depending on the dwell times $T_\alpha$, $T_\gamma$ of $\alpha$ and $\gamma$, make sure that the loop durations $t_2$ and $t_4$ remain positive. The density $w(s, u)$ is normalized such that $\int ds du w(s, u) |s| |u| = \int ds du \frac{1}{\Omega} |s| |u|$ is the number density of 2-encounters with fixed action difference $\Delta S$.

To account for all quadruplets of Fig. 1 we do the sum over $\beta$, $\delta$ in [2] by integrating with the weight $w(s, u)$,

$$\langle \mathrm{tr}(tt^\dagger) \rangle_{2-\text{enc}} = \frac{1}{T_H} \frac{1}{\Omega} \left\langle \sum_{a_1 < a_2 < c_1 < c_2} \int ds du \sum_{\alpha, \gamma} |A_\alpha|^2 |A_\gamma|^2 w(s, u) e^{isu/\hbar} \right\rangle ; \quad (6)$$

approximating $A_\beta A_\delta \approx A_\alpha A_\gamma$ [11]. Now, similarly to [4], we replace the sum over $\alpha$, $\gamma$ by an integral over the dwell times or, equivalently, over the loop durations $t_2$ and $t_4$. The integrand must be weighted with the probability $e^{-\frac{2\pi}{\hbar}T_{\text{exp}}}$ for both trajectories to remain inside the cavity. Regarding the sum over channels $a_1$, $a_2$, $c_1$, $c_2$ one might expect a factor $N_1^2N_2^2$. However, when both the ingoing channels and the outgoing channels coincide, $a_1 = c_1$, $a_2 = c_2$, each of the two dashed trajectories in Fig. 1 could be chosen as $\beta$ or $\delta$, such that the resulting contributions are doubled. Including such combinations for a second time, we get the factor $N_1N_2(N_1N_2+1)$. Altogether, we thus find

$$\langle \mathrm{tr}(tt^\dagger) \rangle_{2-\text{enc}} = \frac{N_1N_2(N_1N_2+1)}{T_H^2} \left( \int dt_1 dt_2 dt_3 dt_4 \int ds du \frac{1}{t_{\text{enc}}(s, u)} e^{-\frac{2\pi}{\hbar}(t_1+t_2+t_3+t_4+t_{\text{enc}}(s, u))} \right). \quad (7)$$
The integral factors into four independent integrals over the loop durations, \( \int_0^\infty dt e^{-\frac{tt}{N}} = \frac{T_H}{N} \), and one integral over the separations \( s, u \) inside the encounter, \( \int ds du e^{-\frac{ds du}{2T_H}} e^{isu/N} \), as shown in Fig. 4. Since all powers of \( T_H \) mutually cancel, the following diagrammatic rule arises: Each loop gives rise to a factor \( \frac{1}{N} \), and an encounter contributes a factor \( -N \). The rule yields \(-1/N^3\); upon multiplying with the number of possible combinations of channels we get

\[
-\langle \text{tr}(tt^1tt^1) \rangle_{2-\text{enc}} = \frac{N_1N_2(N_1N_2 + 1)}{N^3}, \tag{8}
\]

i.e., for \( N_1, N_2 \gg 1 \) the leading shot-noise term in \( \mathbb{H} \).

All orders: To go beyond the leading term, we have to account for all quadruplets of trajectories differing in arbitrarily many encounters, each involving arbitrarily many stretches; Fig. 2 shows a few examples. In the unitary case we must consider encounters where several stretches of either \( \alpha \) or \( \gamma \), or both, come close in phase space, whereas in presence of time-reversal invariance the stretches may also be nearly mutually time-reversed.

The contributions of all families of quadruplets obey the above diagrammatic rule. To show this, we describe each \( l \)-encounter (encounter of \( l \) stretches) by \( l-1 \) pairs of coordinates \( s_j, u_j, j = 1, \ldots, l-1 \) measuring the separations of \( l-1 \) stretches from one reference stretch. These coordinates determine both the duration of each encounter stretch and its contribution \( \sum_{j=1}^{l-1} s_j u_j \) to the action difference. The analog of the density \( w(s, u) \) in \( \mathbb{H} \) obtains a factor \( \frac{T_H}{N^3} \) from each \( l \)-encounter. The resulting product must be integrated over the durations of all loops, with integration over the final loops of \( \alpha \) and \( \gamma \) coming into play through the summation over \( \alpha \) and \( \gamma \) like in \( \mathbb{H} \). Finally, the contribution of each family factors into “loop” and “encounter” integrals similar to those in \( \mathbb{H} \). After cancellation of all powers of \( T_H \), the diagrammatic rule comes about, with a factor \( \frac{1}{N} \) from each loop and a factor \(-N \) from each encounter.

Again, we have to multiply the result with the number of possible combinations of channels. Two cases must be distinguished. First, let us consider trajectory quadruplets as in Fig. 2a–c where similarly to Fig. 1 the partner trajectories \( \beta \) and \( \delta \) connect the initial point of \( \alpha \) to the final point of \( \gamma \), and the initial point of \( \gamma \) to the final point of \( \alpha \). Such quadruplets will be called \( x \)-quadruplets. For them, the channels \( a_1, c_1, a_2, \text{ and } c_2 \) may be chosen arbitrarily and allow for \( N_1N_2(N_1N_2 + 1) \) combinations.

In contrast, Fig. 2d depicts a quadruplet where the partner trajectories connect the initial point of \( \alpha \) to the final point of \( \alpha \), and the initial point of \( \gamma \) to the final point of \( \gamma \), similarly to the diagonal contribution. We speak of a \( d \)-quadruplet then. The partner trajectories now connect the leads as \( a_1 \rightarrow a_2, c_1 \rightarrow c_2 \). Since for shot noise we need partner trajectories \( \beta(a_1 \rightarrow a_2), \delta(c_1 \rightarrow a_2) \), \( d \)-quadruplets contribute only if either the two ingoing channels, or the two outgoing channels (and thus the corresponding angles of incidence) coincide. Like for the diagonal contribution, we thus obtain \( N_1^2N_2 + N_1N_2^2 = N_1N_2 \) possible combinations.

Our diagrammatic rules determine \( \langle \text{tr}(tt^1tt^1) \rangle \) as

\[
\langle \text{tr}(tt^1tt^1) \rangle = \frac{N_1N_2(N_1N_2 + 1)}{N^2} \sum_{m=1}^{\infty} \frac{x_m}{N^m} + \frac{N_1N_2}{N} \left\{ 1 + \sum_{m=1}^{\infty} \frac{d_m}{N^m} \right\}. \tag{9}
\]

Towards explaining \( x_m \) and \( d_m \) we denote the number of encounters in a quadruplet by \( V \) and the total number of the encounter stretches by \( L \). Then \( x_m \) is the number of families of \( x \)-quadruplets with \( m = L - V \) and even \( V \), minus the number of corresponding families with odd \( V \); \( d_m \) is the analogous number of \( d \)-families. We note that the contribution of each family is proportional to \( \frac{1}{N^m} \) rather than \( \frac{1}{N} \), since there are two more loops than encounter stretches.

The leading contribution to shot noise originates from the family of Fig. 1 it gives \( x_1 = -1 \). For the next-to-leading term, we have to consider \( x \)-quadruplets with two 2-encounters or one 3-encounter, contributing to \( x_2 \) and depicted in Figs. 2a–c, and \( d \)-quadruplets which are related to a single 2-encounter and contribute to \( d_1 \) (Fig. 2d). All quadruplets in Fig. 2 involve mutually time-reversed loops and can exist in the orthogonal case only (thus \( x_2 = d_1 = 0 \) in the unitary case). Note that if we interchange the two leads, or the pairs \( (\alpha, \gamma) \) and \( (\beta, \delta) \), or the trajectories \( \alpha \) and \( \gamma \), each family of quadruplets will be either left topologically invariant or turned...
into an equivalent family making the same contribution to the shot noise. Only one representative of each such “symmetry multiplet” is shown in Fig. 2 with the number of equivalent families indicated by a multiplier. The sum of all contributions gives $x_2 = 4$, $d_1 = -2$, and

$$\langle \sigma \rangle = \frac{N_1 N_2}{N} - \frac{N_1^2 N_2^2}{N^2} + \frac{4 N_1^2 N_2^2}{N^2} - 2 \frac{N_1 N_2}{N} + O \left( \frac{1}{N^2} \right).$$

To recover the second term in (1), we get the shot noise

$$P = \frac{N_1^2 N_2^2}{N^3} + \frac{N_1 N_2 (N_1 - N_2)^2}{N^4 (1 + \xi)} + O \left( \frac{1}{N} \right), \quad (12)$$

in accordance with (11) for the limits $\xi \to 0$ and $\xi \to \infty$. If $N_1 = N_2 = N/2$ the second term in (12) vanishes, and quadruplets involving more encounter stretches yield

$$P = \frac{N}{16} + \frac{1 + 8 \xi + 4 \xi^2 + 4 \xi^3 + \xi^4}{16(1 + \xi)^4} + O \left( \frac{1}{N^2} \right); \quad (13)$$

The proof, based on a recursion derived in [2], will be given elsewhere. Summing over $m$ we get the shot noise

$$P = \left\{ \begin{array}{ll}
\frac{N_1^2 N_2^2}{N(N+1)} & \text{unitary} \\
\frac{N_1 (N_1 + 1) N_2 (N_2 + 1)}{N(N+1)(N+3)} & \text{orthogonal} \\
\end{array} \right., \quad (11)$$

valid to all orders in $\frac{1}{N}$ and thus also for a few channels.

**Weak magnetic field:** In the presence of a weak magnetic field $B$, the power of shot noise must interpolate between the orthogonal and unitary cases. A weak field increases the action of each trajectory by the line integral $\int A \cdot dq$ of the vector potential $A$. Since that increment changes sign under time reversal, a net contribution survives from all loops and encounter stretches changing direction in $B_1/\xi$ relative to $B_2/\xi$. Our above diagrammatic rule is thus modified: Each loop changing direction contributes $\frac{1}{N(1+\xi)}$ with $\xi \propto B^2$; each encounter contributes $-N(1 + \xi)^2$ with $\mu$ the number of its stretches changing direction $B_1/\xi$.

The leading contribution to shot noise, from quadruplets as in Fig. 1 remains unaffected by the magnetic field. However, all quadruplets responsible for the next-to-leading term obtain a Lorentzian factor $1/\xi^2$. (In Figs. 2a, 2b, and 2c one loop is traversed in time-reversed sense and $\mu = 0$, whereas in Fig. 2c two loops are time-reversed and one encounter has $\mu = 1$.) We thus predict

$$P = \frac{N_1^2 N_2^2}{N^3} + \frac{N_1 N_2 (N_1 - N_2)^2}{N^4 (1 + \xi)} + O \left( \frac{1}{N} \right), \quad (12)$$

in accordance with (11) for the limits $\xi \to 0$ and $\xi \to \infty$. If $N_1 = N_2 = N/2$ the second term in (12) vanishes, and quadruplets involving more encounter stretches yield

$$P = \frac{N}{16} + \frac{1 + 8 \xi + 4 \xi^2 + 4 \xi^3 + \xi^4}{16(1 + \xi)^4} + O \left( \frac{1}{N^2} \right); \quad (13)$$

the extension to $O \left( \frac{1}{N^2} \right)$ will be given elsewhere.

**Outlook:** The semiclassical approach to transport opens a large field of experimentally relevant applications. For instance, we have checked the trajectory quadruplets employed here to also yield the universal conductance variance as well as the variance of the conductance at two different energies, both within and in between the orthogonal and unitary symmetry classes.

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