Dielectric black hole analogues

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Alternative to the sonic black hole analogues we discuss a different scenario for modeling the Schwarzschild geometry in a laboratory – the dielectric black hole. The dielectric analogue of the horizon occurs if the velocity of the medium with a finite permittivity exceeds the speed of light in that medium. The relevance for experimental tests of the Hawking effect and possible implications are addressed.

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At present we know four fundamental interactions in physics: the strong and the weak interaction, electromagnetism and gravitation. The first three forces are described by quantum field theories whereas the fourth one is governed by the laws of general relativity – a purely classical theory. In view of the success – and the excellent agreement with experimental data – of the electroweak standard model (unifying the electromagnetic and the weak force at high energies) it is conjectured that all four interactions can be described by an underlying unified theory above the Planck scale. This fundamental description is expected to incorporate the four forces as low-energy effective theories. Despite of various investigations during the last decades a satisfactory and explicit candidate for this underlying law is still missing. At present we can just consider consistently quantum fields, e.g. electromagnetism, in the presence of classical, i.e. externally prescribed, gravitational fields. This semi-classical treatment is expected to provide some insight into the structure of the underlying theory. One of the most striking consequences of this formalism is the Hawking effect which predicts the evaporation of black holes. However, this prediction is faced with a conceptual difficulty: Its derivation is based on the assumption that the semi-classical treatment is valid at arbitrary scales. But in contrast to this presumption the decomposition into a classical gravitational sector and a quantum field sector is valid at low energies only. The investigation of high-energy effects requires some knowledge about the underlying theory including quantum gravity.

In order to elucidate this point Unruh [2] suggested a scenario which displays a close similarity to that of the Hawking effect while the underlying physical laws are completely understood – the sonic black hole analogue. These analogues are based on the remarkable observation that the propagation of sound waves in flowing fluids is under appropriate conditions equivalent to that of a scalar field in curved space-times: The dynamics of the fluid is governed by the non-linear Euler equation

\[ \dot{v} + (v \nabla)v + \nabla p/\rho = f, \]

where \( v \) denotes the local velocity of the liquid, \( \rho \) its density, \( p \) the pressure, and \( f \) the external force density, together with the equation of continuity \( \dot{\rho} + \nabla (\rho v) = 0 \). Linearizing these equations around a given flow profile \( v_0 \) via \( v = v_0 + \nabla \phi \) the scalar field \( \phi \) of the small deviations (sound waves) satisfies the Klein-Fock-Gordon equation with an appropriate (acoustic) metric \( g_{ac}^{\mu\nu} (v_0, \phi) \), see e.g. [3,4]

\[ \Box_{ac} \phi = \frac{1}{\sqrt{-g_{ac}}} \partial_\mu (\sqrt{-g_{ac}} g_{ac}^{\mu\nu} \partial_\nu \phi) = 0, \quad (1) \]

with \( g_{ac} = \det (g_{ac}^{\mu\nu}) \). The acoustic horizon occurs if the velocity of the fluid exceeds the speed of sound within the liquid. Many examinations have been devoted to this topic after the original proposal by Unruh [2], see e.g. Refs. [3,4] as well as references therein. E.g., in [5] the possibility of realizing an acoustic horizon within a Bose-Einstein condensate is addressed. More generally, Ref. [6] discusses the simulation of phenomena of curved space-times within super-fluids.

However, the sonic analogues suffer from certain conceptual difficulties and problems: In order to cast the wave equation of sound into the above form (1) it is necessary to neglect the viscosity of the liquid. This might be justified for super-fluids, but in general not for normal liquids. Since the Hawking effect is a pure quantum radiation phenomenon, it can be observed for the acoustic analogues if and only if the quantum description of sound waves is adequate. The presence of friction and the resulting decoherence destroys the quantum effects and so violates this presumption in general. In addition, the assumption of a stationary, regular, and laminar (irrotational) flow profile at the transition from subsonic to supersonic flow (which was used in the derivation) is questionable, see e.g. [7]. Finally, we would like to emphasize that the sonic analogues incorporate the scalar (spin-zero) field \( \phi \) and are not obviously generalizable to the electromagnetic (spin-one) field \( A_\mu \).

In the following we shall discuss an alternative scenario – where all these objections do not necessarily apply. Let us consider the following quantum system: Within the linear approximation the excitations of a medium are described by a bath of harmonic oscillators whose dynamics is governed by the Lagrangian density \( \mathcal{L}_\phi \). For reasons of simplicity we assume these degrees of freedom to be localized and not propagating. The linear excitations of the medium in its rest-frame are coupled to the microscopic electric field via \( \mathcal{L}_{E\phi} \). We might also consider a coupling
to the magnetic field whereby the medium would possess a non-trivial permeability in addition to its permittivity. The dynamics of the microscopic electromagnetic field itself is governed by \( \mathcal{L}_{EB} \). Hence the complete Lagrangian density \( \mathcal{L} \) of the quantum system under consideration is determined by

\[
\mathcal{L} = \mathcal{L}_{EB} + \mathcal{L}_{E\Phi} + \mathcal{L}_\Phi = \frac{1}{2} (E^2 - B^2) + \sum_\alpha E \cdot \chi_\alpha \Phi_\alpha + \frac{1}{2} \sum_\alpha \left( \Omega_\alpha^2 - \Omega^2_\alpha \Phi_\alpha^2 \right). \tag{2}
\]

\( E(t, r) \) and \( B(t, r) \) denote the microscopic electric and magnetic fields, respectively. The excitations of the medium are described by \( \Phi_\alpha(t, r) \) where \( \alpha \) labels the different vibration modes. The fundamental frequencies of the medium are indicated by \( \Omega_\alpha \). \( \chi_\alpha \) denote the vector-valued and possibly space-time dependent coupling parameters (e.g. dipole moments).

Now we turn to a macroscopic description via averaging over the microscopic degrees of freedom associated to the medium, i.e. the fields \( \Phi = \{ \Phi_\alpha \} \). Assuming that the microscopic quantum state of the medium is properly described by \( \Phi_\alpha \) and \( \Omega_\alpha\) and \( \chi_\alpha \) we may integrate out the microscopic degrees of freedom \( \Phi_\alpha \) within this formalism. As a result we obtain an effective action \( A_{\text{eff}} \) (see e.g. [8]) accounting for the remaining macroscopic degrees of freedom

\[
\exp(iA_{\text{eff}}) = \frac{1}{Z_\Phi} \int \mathcal{D}\Phi \exp(i\mathcal{A}). \tag{3}
\]

\( \mathcal{A} \) denotes the original action described in Eq. (9) and \( Z_\Phi \) is a constant normalization factor. After standard manipulations (linear substitution of \( \Phi_\alpha \) and quadratic completion, see e.g. [8]) we may accomplish the path-integration (averaging over all field configurations) with respect to \( \mathcal{D}\Phi \) explicitly, and finally we arrive at

\[
A_{\text{eff}} = A_{EB} + \frac{1}{2} \sum_\alpha \int d^4x \, E \cdot \chi_\alpha \left( \frac{\partial^2}{\partial t^2} + \Omega^2_\alpha \right)^{-1} \chi_\alpha \cdot E. \tag{4}
\]

As expected from other examples of effective field theories [6], the effective action \( A_{\text{eff}} \) is non-local owing to the occurrence of the inverse differential operator \((1 + \partial^2)^{-1}\). However, in analogy to the operator product expansion [8] we may expand this non-local quantity into a sum of local operators \((1 + \partial^2)^{-1} = \sum_n (-\partial^2)^n \). For the propagation of photons with energies much smaller than the fundamental frequencies \( \Omega_\alpha \) of the medium only the lowest term \((n = 0)\) of this asymptotic expansion yields significant contributions. Neglecting the higher order terms we obtain the local and non-dispersive low-energy effective theory of the macroscopic electromagnetic field in analogy to a heavy-mass effective theory [8]

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_{EB} + \frac{1}{2} E \cdot (\varepsilon - 1) \cdot E. \tag{5}
\]

The second term contains the permittivity tensor \( \varepsilon \) of the medium which is introduced via \( \varepsilon = 1 + \sum_\alpha \chi_\alpha \otimes \chi_\alpha / \Omega^2_\alpha \).

In contrast to the microscopic fields in Eq. (9) here \( E(t, r) \) and \( B(t, r) \) are understood as the macroscopic electric and magnetic fields, respectively. For reasons of simplicity we assume the coupling parameters \( \chi_\alpha \) of the medium to be homogeneously and isotropically distributed resulting in a constant and scalar permittivity \( \varepsilon \). Consequently the Lagrangian density assumes the simple form

\[
A_{\text{eff}} = A_{EB} + \frac{1}{2} \mathcal{L}_{\text{eff}} = \frac{1}{2} (\varepsilon E^2 - B^2). \tag{6}
\]

As expected, this is exactly the Lagrangian of the macroscopic electromagnetic field within a dielectric medium at rest possessing a permittivity \( \varepsilon \). Its dynamics is governed by the macroscopic source-free Maxwell equations \( \nabla \cdot B = 0, \ \nabla \cdot D = 0, \ \nabla \times H = D, \ \text{and} \ \nabla \times E = -B \) with \( D = \varepsilon E \) and \( H = B \) (no permeability).

Obviously the Lagrangian in Eq. (9) is not manifestly covariant. However, by virtue of Lorentz transformations it can be cast into a Poincaré invariant form via

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\varepsilon - 1}{2} F_{\mu\nu} u^\mu F^{\nu\lambda} u^\lambda. \tag{7}
\]

\( F_{\mu\nu} \) symbolizes the electromagnetic field strength tensor containing the macroscopic electric and magnetic fields \( E \) and \( B \). The signature of the Minkowski metric \( g^{\mu\nu}_M \) is chosen according to \( g^{\mu\nu}_M = \text{diag}(1,1,-1,-1) \). \( u^\mu \) denotes the four-velocity of the medium and is related to its three-velocity \( \beta \) via \( u^\mu = (1, \beta) / \sqrt{1 - \beta^2} \).

If we now assume that the dielectric properties of the medium (strictly speaking, the coupling parameters \( \chi_\alpha \)) do not change owing to a non-inertial motion (generating terms such as \( \partial_\mu \chi_\nu \)) we may generalize the above expression to arbitrarily space-time dependent four-velocities of the medium \( u^\mu \), see e.g. [10].

As we shall demonstrate now, the propagation of the electromagnetic field in a flowing dielectric medium resembles many features of a curved space-time, see also [10][12]. Introducing the effective Gordon metric

\[
g^{\mu\nu}_{\text{eff}} = g^{\mu\nu}_M + (\varepsilon - 1) u^\mu u^\nu \tag{8}
\]

we may rewrite the Lagrangian in Eq. (9) into a form associated with a curved space-time

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}. \tag{9}
\]

Inserting the expression [8] into the above equation [10] and exploiting the normalization of the four-velocity \( u_\mu \) \( u^\mu = 1 \) together with the anti-symmetry of the tensor \( F_{\mu\nu} = -F_{\nu\mu} \) we exactly recover the original formula [6].
Note that the representation of the field strength tensor via the lower indices is related to the four-vector potential $A_{\mu}$ by $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. The equation of motion assumes the compact form $\partial_{\mu}(g^{\mu\nu}_{\text{eff}} F_{\nu\rho}) = 0$. For a rigorous identification we have to consider the associated Jacobi determinant $\sqrt{-g_{\text{eff}}}$ accounting for the four-volume element. Fortunately the determinant of the metric in Eq. (8) is simply given by $|\det(g^{\mu\nu}_{\text{eff}})| = \varepsilon > 1$.

This positive and constant factor can be eliminated by a scale transformation of the distances $\dd x''$ or – alternatively – by re-scaling the four-vector potential $A_{\mu}$. In order to discuss the general properties of the Gordon metric it is convenient to calculate its inverse

$$g^{\mu\nu}_{\text{eff}} = g^{\mu}_{M\nu} - \frac{1}{\varepsilon} u_{\mu} u_{\nu}. \quad (10)$$

The condition of an ergo-sphere $g^{\mu\nu}_{\text{eff}} = 0$ (see e.g. [13]) is satisfied for $\beta^2 = 1/\varepsilon$, i.e. if the velocity of medium equals the speed of light in that medium. For a radially inward/outward flowing medium, i.e. $\beta = f(r)r$ an analysis of the geodesics (e.g. light rays) reveals that the ergosphere represents an apparent horizon, cf. [13]. An inward flowing medium corresponds to a black hole (nothing can escape) while an outward flowing medium represents a white hole (nothing can invade), see e.g. [3,4]. Assuming an eternally stationary flow the apparent horizon coincides with the event horizon, cf. [13].

For a stationary flow the effective metric in Eq. (10) describes a stationary space-time. However, this metric can be cast into a static form $ds^2_{\text{eff}} = g_{00} dt^2 - dr^2/g_{00} - r^2 d\Omega^2$ by virtue of an appropriate coordinate transformation $dt \to dt = dt + g_{01} dr/g_{00}$ as well as $dr \to dr = \sqrt{g_{11}} - g_{00} g_{11} dr$. Whereas the former expression in Eq. (14) corresponds to the Painlevé-Gullstrand-Lemaître metric (see e.g. [13]), the latter form is equivalent to the well-known Schwarzschild representation. Since the Schwarzschild metric is singular at the horizon ($g_{00} = 0$) also the transformation $t \to t$ maintains this property. Thus the coordinate $t$ should be used with special care.

Having derived the Schwarzschild representation of the dielectric black/white hole its surface gravity [13] can be calculated simply via $2\kappa = (\partial g_{00}/\partial F)|_{(g_{00}=0)}$ and yields

$$\kappa = \frac{1}{1 - \beta^2} \left( \frac{\partial \beta}{\partial r} \right)_{\text{Horizon}}. \quad (11)$$

Independently of $\varepsilon$ it coincides – up to the relativistic factor $(1 - \beta^2)^{-1}$ – with the surface gravity of the non-relativistic sonic analogues. The associated Hawking temperature [1] is (for $\beta \ll 1$) of the order of magnitude

$$T_{\text{Hawking}} = \frac{k_B c}{2\pi k_B} = \mathcal{O} \left( \frac{h c}{k_B n R} \right). \quad (12)$$

where $h$ denotes Planck’s constant, $c$ the speed of light in the vacuum, and $k_B$ Boltzmann’s constant. $n = \sqrt{\varepsilon}$ is the index of refraction of the medium and $R$ symbolizes the Schwarzschild radius of the dielectric black/white hole. Hence the Hawking temperature is proportional to the speed of light in the medium $c/n$ over the Schwarzschild radius $R$, i.e. the inverse characteristic time scale of a light ray propagating within the dielectric black/white hole.

The dielectric analogues of the Schwarzschild geometry provide a conceptual clear scenario for studying the effects of quantum fields ($\mathcal{L}_{\text{eff}}$) under the influence of external conditions ($g^{\mu\nu}_{\text{eff}}$) together with their relation to the underlying theory $\mathcal{L}$. The effective low-energy description $\mathcal{L}_{\text{eff}}$ exhibits a horizon (one-way membrane [13]) and the related thermodynamical implications (Hawking radiation [1]). Moreover, it allows for investigating the effects of a finite cut-off (trans-Planckian problem) induced by the microscopic theory $\mathcal{L}$. By means of the analogy outlined below these considerations might shed some light on the structure of quantum gravity and its semi-classical limit since for the dielectric black holes the underlying physics is understood.

Apart from the considerations above there have been already a few discussions in the literature concerning the simulation of black holes by means of dielectric properties: The solid-state analogues proposed in Ref. [12] identify the permittivity $\varepsilon^j$ and the permeability $\mu^j$ tensor directly with the (singular) Schwarzschild metric via $\varepsilon^j = \mu^j = \sqrt{-g^j g^j/\varepsilon_{00}}$. As a result the solid-state analogues require divergences of the material characteristics of the medium at rest in order to simulate a horizon ($g_{00} = 0$) – in contrast to the model described in the present article. Even though such a singular behavior might be realized for the case of a phase transition the validity of the effective theory in the presence of these divergences is questionable. In addition, the genuine difference between the black hole and the white hole horizon (see e.g. [14]) cannot be reproduced by the scenario proposed in [12].

Apart from this model a flowing dielectric medium obeying a finite permittivity was suggested in Ref. [13] in order to represent a so-called optical black hole. However, the Aharonov-Bohm scenario under consideration
in Ref. [11] – where a pure swirling of the medium is assumed – does not exhibit a horizon and therefore cannot be regarded as a model of a black/white hole. This objection has already been raised in [15], see also [18]. Consequently it is not possible to apply the concepts of surface gravity and Hawking temperature.

In contrast to the simulation of a curved space-time by a dielectric medium it is also possible to consider the inverse identification: E.g., in Ref. [10] the propagation in a gravitational background was mapped to electromagnetism within a medium in flat space-time. However, the identification in [10] requires $g_{00} > 0$ and thus does not incorporate space-times with a horizon.

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The experimental realization of the dielectric analogues might indeed become feasible in view of the recent experimental progresses in generating ultra-slow light rays, see e.g. [18–22]. These experiments exploit the phenomenon of the electromagnetically induced transparency in order to slow down the velocity of light in atomic vapor. Although the microscopic theory of this phenomenon is not properly described by the Lagrangian in Eq. (3) it generates similar macroscopic effects. If an appropriate control field (a laser) acts on the medium the group velocity of the signal field perpendicular to the control field reaches an order of magnitude of some meters per second – i.e. even below the speed of sound. In view of the non-destructive nature of the propagation, i.e. the absence of loss, dissipation, and the resulting decoherence, one might imagine the observability of quantum field theoretical effects – e.g. the Hawking effect – in an advanced experiment.

Very roughly, an experimental set-up for a dielectric black hole as depicted in the figure might be conceivable. Apart from the experimental challenge of simulating a black hole within a laboratory the scenario discussed in this article may help to understand the structure of the underlying theory including quantum gravity.

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