Exact solutions of some fractional differential equations by various expansion methods

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Abstract. In this paper, we construct the exact solutions of some nonlinear space-time fractional differential equations involving modified Riemann-Liouville derivative in mathematical physics and applied mathematics; namely the fractional modified Benjamin-Bona-Mahony (mBBM) and Kawahara equations by using \((G'/G)\) and \((G'/G, 1/G)\)-expansion methods.

1. Introduction
Fractional calculus has been used to model physical and engineering sciences. Fractional differential equations (FDEs) are regarded as models of physical systems \([1, 2]\). We introduce the space-time fractional mBBM equation \([3]\)

\[D_t^\alpha u + D_x^\alpha u - vu^2 D_x^2 u + D_x^{3\alpha} u = 0,\]  
where \(v\) is a nonzero positive constant. We also consider the time fractional modified nonlinear Kawahara equation \([4]\)

\[D_t^\alpha u + u^2 u_x + pu_{xx} + qu_{xxx} = 0,\]  
where \(\alpha\) is the parameter standing for the order of the fractional time derivative, and \(0 < \alpha \leq 1\). The Jumarie’s modified Riemann-Liouville derivative of order \(\alpha\) is defined in \([5]\):

2. The \((G'/G)\)-expansion method for FDEs
We consider the following general nonlinear fractional differential equations (FDEs) of the type

\[P(u, D_t^\alpha u, D_x^\beta u, D_t^\alpha D_x^\beta u, D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta u, ... ) = 0, \quad 0 < \alpha, \beta < 1\]  
where \(u = u(x, t)\) is an unknown function. The traveling wave variable

\[u(x, t) = U(\xi), \xi = \frac{\tau x^\beta}{\Gamma(1 + \beta)} + \frac{c^\alpha}{\Gamma(1 + \alpha)},\]  
where \(\tau\) and \(c\) are nonzero arbitrary constants \([6]\). By using the chain rule

\[D_t^\alpha u = \sigma_t \frac{dU}{d\xi} D_t^\alpha \xi, D_x^\beta u = \sigma_x \frac{dU}{d\xi} D_x^\beta \xi\]  

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where \( \sigma'_t \) and \( \sigma'_x \) are called the sigma indexes see [7], without loss of generality we can take \( \sigma'_t = \sigma'_x = l_0 \), where \( l_0 \) is a constant. Eq. (3) can be reduced into an ODE:

\[
Q(U, U', U'', U''', ..., ) = 0. \tag{6}
\]

where the prime denotes the derivation with respect to \( \xi \). Suppose the solution of equation (9) can be expressed by a polynomial in \( \left( \frac{G'}{G} \right) \) as follows [8, 9]:

\[
u(\xi) = \sum_{i=0}^{m} a_i \left( \frac{G'}{G} \right)^i, \quad a_m \neq 0, \tag{7}\]

where \( a_i \) (\( i = 0, 1, ..., m \)) are constants, while \( G(\xi) \) satisfies the following second order LODE

\[
G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{8}\]

with \( \lambda \) and \( \mu \) are being constants. The positive integer \( m \) can be determined by the homogeneous balance principle in equation (6). Substituting equation (7) into equation (6) and using equation (8) collecting all terms with the same order of \( \left( \frac{G'}{G} \right) \) together. Then equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for \( a_i \) (\( i = 0, 1, ..., m \)), \( \lambda \), \( \mu \), \( \tau \) and \( c \). Solving these equations, we can get a variety of exact solutions of equation (3).

3. Algorithm of \( (G'/G, 1/G) \)-expansion method

For the second order linear ordinary differential equation (LODE)

\[
G''(\xi) + \lambda G'(\xi) = \mu, \tag{9}\]

we choose

\[
\phi = \frac{G'}{G}, \psi = \frac{1}{G} \tag{10}\]

for simplicity here and after. (9) and (10) gives

\[
\phi' = -\phi^2 + \mu \psi - \lambda, \psi' = -\phi \psi. \tag{11}\]

The details of the solution of LODE (9), concludes following three cases:

Case 1 If \( \lambda < 0 \), the general solution of the LODE (9) reads

\[
G(\xi) = A_1 \sinh(\sqrt{-\lambda} \xi) + A_2 \cosh(\sqrt{-\lambda} \xi) + \frac{\mu}{\lambda} \tag{12}\]

and we have

\[
\psi^2 = \frac{-\lambda}{\lambda^2 \sigma - \mu^2} (\phi^2 - 2\mu \psi + \lambda), \sigma = A_1^2 - A_2^2. \tag{13}\]

Case 2 If \( \lambda > 0 \), the general solution of the LODE (9) gives

\[
G(\xi) = A_1 \sin(\sqrt{\lambda} \xi) + A_2 \cos(\sqrt{\lambda} \xi) + \frac{\mu}{\lambda} \tag{14}\]

and corresponding relations are

\[
\psi^2 = \frac{\lambda}{\lambda^2 \sigma - \mu^2} (\phi^2 - 2\mu \psi + \lambda), \sigma = A_1^2 + A_2^2. \tag{15}\]

Case 3 If \( \lambda = 0 \), the general solution of the LODE (9) can be written as
\[ G(\xi) = \frac{\mu}{2} \xi^2 + A_1 \xi + A_2 \] (16)

and we have
\[ \psi^2 = \frac{1}{A_1^2 - 2\mu A_2} (\phi^2 - 2\mu \psi) \] (17)

where \( A_1 \) and \( A_2 \) are two arbitrary constants. Suppose that the solution of ODE (6) can be expressed by a polynomial in terms of \( \phi \) and \( \psi \) in the form of
\[ u(\xi) = \sum_{i=0}^{N} a_i \phi^i + \sum_{i=1}^{N} b_i \phi^{i-1} \psi \] (18)

where \( G = G(\xi) \) is the solution of the second order LODE (9), \( a_i, b_i (i = 1, \ldots, N) \), \( \lambda \) and \( \mu \) are constants and the positive integer \( N \) can be determined by balancing principle in ODE (6).

Employing (18) into Eq. (6), using (11) and (13) the left-hand side of (6) can be expressed a polynomial in terms of \( \phi \) and \( \psi \), in which the degree of \( \psi \) is not larger than 1. Coefficients of the polynomial give a system of algebraic equations in \( a_i, b_i, \lambda, \mu, A_1 \) and \( A_2 \). Solving the algebraic equations and substituting the values of \( a_i, b_i, \lambda, \mu, A_1 \) and \( A_2 \) into (18), we can obtain the travelling wave solutions expressed by the hyperbolic functions of Eq. (6).

Substituting (18) into Eq. (6), using (11) and (15) (or (11) and (17)), we obtain the travelling wave solutions of Eq. (9) expressed by trigonometric and rational functions [10, 11].

4. The space-time fractional mBBM equation

Firstly, we consider the following transformations;
\[ u(x,t) = U(\xi), \quad \xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \] (19)

where \( k \) and \( c \) are nonzero constants. Eq. (1) can be reduced into an ODE;
\[ (c+k)U - vkU_x^3 + k^3 U'' + \xi_0 = 0. \] (20)

where \( \xi_0 \) is an integration constant. Using homogeneous balance principle, we have \( m = 1 \).

Suppose that the solutions of (20) can be expressed by a polynomial in \( \left( \frac{\psi}{\xi} \right) \) as follows:
\[ U(\xi) = a_0 + a_1 \left( \frac{\psi}{\xi} \right), \quad a_1 \neq 0. \] (21)

Substituting these expressions into Eq. (21), collecting the coefficients of \( \left( \frac{\psi}{\xi} \right)^i (i = 0, \ldots, 3) \) and set it to zero we obtain a system. When this system of algebraic equations is solved, we can be written the following expansion.
\[ U(\xi) = \pm \frac{3k\lambda}{\sqrt{6c}} \pm k\sqrt{\frac{6}{c}} \left( \frac{\psi}{\xi} \right). \] (22)

We have three types of travelling wave solutions of the space-time fractional mBBM equation as follows:

When \( \lambda^2 - 4\mu > 0 \),
\[ U_{1,2}(\xi) = \pm \frac{k}{2} \sqrt{\frac{6\lambda^2 - 24\mu}{v}} \left( \frac{C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right), \] (23)
When $\lambda^2 - 4\mu < 0$, 

$$U_{3,4}(\xi) = \pm \frac{k}{2} \sqrt{\frac{24\mu - 6\lambda^2}{v}} \left( -C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi \right),$$

(24)

When $\lambda^2 - 4\mu = 0$, 

$$U_{5,6}(\xi) = \pm k \sqrt{\frac{6}{\mu}} \frac{C_2}{C_1 + C_2 \xi},$$

(25)

where $\xi = \frac{kx^\sigma}{\Gamma(1+\alpha)} + \frac{(k^3 \lambda^2 - 2k)\mu^\alpha}{2\Gamma(1+\alpha)}$. In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then $U_{1,2}(\xi)$ become

$$u_{1,2}(x,t) = \pm k \lambda \sqrt{\frac{3}{2v}} \tanh \left( \frac{\lambda}{2} \left( \frac{kx^\sigma}{\Gamma(1+\alpha)} + \frac{(k^3 \lambda^2 - 2k)\mu^\alpha}{2\Gamma(1+\alpha)} \right) \right),$$

(26)

which are the exact solutions of the space time fractional mBBM equation. According to $(G'/G, 1/G)$-expansion method, we look for the solution in the form of

$$u(\xi) = a_0 + a_1 \phi + b_1 \psi$$

(27)

where $a_0, a_1, and b_1$ are constants to be determined later. Substituting (27) into Eq. (20), using (11) and (13), (20) can be expressed as a polynomial in $\phi$ and $\psi$. Coefficients of equation yield a system of algebraic equations in $a_0, a_1, b_1, k, c, \sigma, \mu, \xi_0$ and $\lambda$. Solving the algebraic equations by Maple, we obtain hyperbolic, trigonometric and rational function solutions of Eq. (1) as follows:

$$u(\xi) = \sqrt{\frac{3}{2v}} \frac{k(A_1 \cosh(\xi \sqrt{\lambda}) \sqrt{-\lambda} + A_2 \sinh(\xi \sqrt{-\lambda}) \sqrt{-\lambda})}{A_1 \sinh(\xi \sqrt{-\lambda}) + A_2 \cosh(\xi \sqrt{-\lambda}) + \frac{\mu}{\lambda}} + \sqrt{\frac{-3\lambda^2 \sigma + 3\mu^2}{2v\lambda}} k$$

(28)

where $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{k(2+3\lambda^2)\mu^\alpha}{2\Gamma(1+\alpha)}$ and $\sigma = A_1^2 - A_2^2$

$$u(\xi) = \sqrt{\frac{3}{2v \lambda}} \frac{k(A_1 \cosh(\xi \sqrt{-\lambda}) \sqrt{-\lambda} + A_2 \sinh(\xi \sqrt{-\lambda}) \sqrt{-\lambda})}{A_1 \sinh(\xi \sqrt{-\lambda}) + A_2 \cosh(\xi \sqrt{-\lambda}) + \frac{\mu}{\lambda}} + \sqrt{\frac{-3\lambda^2 \sigma + 3\mu^2}{2v\lambda}} k$$

(29)

where $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{k(2+3\lambda^2)\mu^\alpha}{2\Gamma(1+\alpha)}$ and $\sigma = A_1^2 + A_2^2$

$$u(\xi) = \sqrt{\frac{3}{2v}} \frac{k(\mu_\xi + A_1)}{\mu_\xi^2 + A_1 \xi + A_2} + \sqrt{\frac{-3\lambda^2 \sigma + 6\mu A_2}{2v}} k \left( \frac{\mu_\xi^2 + A_1 \xi + A_2}{\mu_\xi^2 + A_1 \xi + A_2} \right)$$

(30)

where $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{k\mu^\alpha}{\Gamma(1+\alpha)}$. It is easily can be checked that our solutions are new in the literature.

5. The time fractional modified nonlinear Kawahara equation

For our goal, we present the following transformation

$$u(x,t) = U(\xi), \xi = kx - \frac{ct^\alpha}{\Gamma(1+\alpha)},$$

(31)

where $c$ and $k$ are non zero constants. Eq. (2) can be turned into an ODEs

$$-cU + \frac{U^3}{3} + pk^2 U'' + qk^3 U''' + \xi_0 = 0,$$

(32)
According to Eq. (32), collecting the coefficients of \( \left( \frac{G'}{G} \right)^i \) \((i = 0, ..., 3)\) and set it to zero we obtain a system. When this system of algebraic equations is solved, we can be written the following expansion.

\[
U(\xi) = \pm \frac{p-3qk\lambda}{\sqrt{-6q}} \pm k\sqrt{-6q} \left( \frac{G'}{G} \right).
\]

By substituting general solutions of Eq. (8) into Eq. (34) we have three types of exact solutions of the time fractional modified nonlinear Kawahara equation as follows:

When \( \lambda^2 - 4\mu > 0 \),

\[
U_{1,2}(\xi) = \pm \frac{p}{\sqrt{-6q}} \pm k\sqrt{24\mu q - 6q\lambda^2} \left( \frac{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi} \right),
\]

where \( \xi = kx - \frac{(12q^2k^3\mu - kp^2 - 3q^2k^3\lambda^2)\alpha}{6q\Gamma(1+\alpha)} \). In particular, if \( C_1 = 0, C_2 \neq 0, \lambda > 0, \mu = 0 \), then \( U_{1,2} \) becomes

\[
u_{1,2}(x, t) = \pm \frac{p}{\sqrt{-6q}} \pm \frac{k\lambda}{2}\sqrt{-6q} \tanh \left( \frac{\lambda}{2} \left( kx + \frac{(kp^2 + 3q^2k^3\lambda^2)\alpha}{6q\Gamma(1+\alpha)} \right) \right).
\]

If \( C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0 \), then \( U_{1,2} \) becomes

\[
u_{3,4}(x, t) = \pm \frac{p}{\sqrt{-6q}} \pm \frac{k\lambda}{2}\sqrt{-6q} \coth \left( \frac{\lambda}{2} \left( kx + \frac{(kp^2 + 3q^2k^3\lambda^2)\alpha}{6q\Gamma(1+\alpha)} \right) \right).
\]

When \( \lambda^2 - 4\mu < 0 \),

\[
U_{3,4}(\xi) = \pm \frac{p}{\sqrt{-6q}} \pm k\sqrt{6q\lambda^2 - 24\mu q} \left( \frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi} \right),
\]

where \( \xi = kx - \frac{(12q^2k^3\mu - kp^2 - 3q^2k^3\lambda^2)\alpha}{6q\Gamma(1+\alpha)} \). When \( \lambda^2 - 4\mu = 0 \),

\[
u_{5,6}(x, t) = \pm \frac{p}{\sqrt{-6q}} \pm \frac{k\sqrt{-6qC_2}}{C_1 + C_2 \left( kx + \frac{kp^2\alpha}{6q\Gamma(1+\alpha)} \right)},
\]

which are the exact solutions of the time fractional modified nonlinear Kawahara equation. According to \((G'/G, 1/G)\)-expansion method, we look for the solution in the form of

\[
u(\xi) = a_0 + a_1\phi + b_1\psi
\]
where $a_0, a_1$, and $b_1$ are constants to be determined later. We obtain hyperbolic, trigonometric, rational function solutions of Eq. (2) as follows:

$$u(\xi) = a_0 + \frac{\sqrt{-\frac{\lambda}{\lambda^2 - \mu^2}} b_1 (A_1 \cosh(\xi \sqrt{-\lambda}) \sqrt{-\lambda} + A_2 \sinh(\xi \sqrt{-\lambda}) \sqrt{-\lambda})}{A_1 \sinh(\xi \sqrt{-\lambda}) + A_2 \cosh(\xi \sqrt{-\lambda}) + \frac{\lambda}{\lambda}} + \frac{b_1}{A_1 \sinh(\xi \sqrt{-\lambda}) + A_2 \cosh(\xi \sqrt{-\lambda}) + \frac{\lambda}{\lambda}}$$

(41)

where $\xi = kx - \frac{k(3\alpha^2 - 3\lambda^2 + \lambda^2)}{3(\lambda^2 - \mu^2)} \frac{e^\sigma}{(1+\alpha)}$ and $\sigma = A_1^2 - A_2^2$

$$u(\xi) = a_0 + \frac{\sqrt{-\frac{\lambda}{\lambda^2 - \mu^2}} b_1 (A_1 \cos(\xi \sqrt{-\lambda}) \sqrt{-\lambda} - A_2 \sin(\xi \sqrt{-\lambda}) \sqrt{-\lambda})}{A_1 \sin(\xi \sqrt{-\lambda}) + A_2 \cos(\xi \sqrt{-\lambda}) + \frac{\lambda}{\lambda}} + \frac{b_1}{A_1 \sin(\xi \sqrt{-\lambda}) + A_2 \cos(\xi \sqrt{-\lambda}) + \frac{\lambda}{\lambda}}$$

(42)

where $\xi = kx - \frac{k(3\alpha^2 - 3\lambda^2 + \lambda^2)}{3(\lambda^2 - \mu^2)} \frac{e^\sigma}{(1+\alpha)}$ and $\sigma = A_1^2 + A_2^2$

$$u(\xi) = a_0 + \sqrt{\frac{3}{2v}} \sqrt{\frac{A_1^2 - 2\mu A_2}{\mu^2} + A_1 \xi + A_2} + \frac{b_1}{(\mu^2 + A_1 \xi + A_2)}$$

(43)

where $\xi = kx - \frac{a_2^2 k e^\sigma}{(1+\alpha)}$. It is easily can be checked that our solutions are new in the literature.

6. Conclusion

In this paper, we apply the $(G'/G)$ and $(G'/G, 1/G)$-expansion methods to solve nonlinear fractional partial differential equations. These methods for nonlinear FDEs with fractional complex transform has its own advantages: direct, succinct, basic; and so it can also be applied to other FDEs where the homogeneous balancing principle is satisfied.

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8. References

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