On open-closed extension of boundary string field theory

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Abstract

We investigate a classical open-closed string field theory whose open string sector is given by boundary string field theory. The open-closed interaction is introduced by the overlap of a boundary state with a closed string field. With the help of the Batalin-Vilkovisky formalism, the closed string sector is determined to be the HIKKO closed string field theory. We also discuss the gauge invariance of this theory in both open and closed string sides.
1 Introduction

Off-shell formulations of string theory are indispensable for understanding its nonperturbative aspects. String field theory (SFT) is a promising off-shell formulation and recent remarkable progress in open string field theory enables us to investigate the dynamics of D-branes via tachyon condensation (for a review, see [1]). Most of these studies consider open string field theory on a D-brane. However, as D-branes are sources of closed strings, such D-brane systems should be coupled to a closed SFT to give a complete description of tachyon condensation.

There are several attempts to consider D-branes in the framework of closed SFT. One way in this direction is to construct an open-closed SFT by directly introducing both open and closed string fields [2, 3]. Another approach is to introduce a D-brane boundary state $\langle B \rvert$ as a source in a closed SFT. The action of such a system would be given by adding a source term to a closed SFT action $S_c$:

$$S_c + \langle B \rvert e^{-\int d\sigma V(\sigma)(\bar{c}_0 - \tilde{c}_0)} \lvert \Psi \rangle,$$

where $\lvert \Psi \rangle$ is a closed string field. Here $V(\sigma)$ is a marginal (on-shell) operator describing open string excitations on the D-brane and $\sigma$ parametrizes the boundary of the worldsheet. In [4], the authors considered the HIKKO closed SFT [5] with the source term in the presence of a constant electro-magnetic flux. They argued that the boundary state should transform under the closed string gauge transformation and found that the gauge invariance requires to add the Dirac-Born-Infeld (DBI) action to (1.1). The authors in [6] also considered a closed SFT with a source term representing a marginal deformation, namely (1.1). They pointed out that an (interacting) closed SFT requires a modification of the deformed boundary state; marginal deformations, which are expected to be solutions in open string theory, are no longer consistent solutions in open-closed string theory.

In most of these previous studies based on boundary states and closed SFT, the open string excitations have been taken to be on-shell. The aim of this paper is to generalize those open string fluctuations into off-shell cases. For this purpose, we adopt boundary string field theory (BSFT) [7, 8, 9, 10]. BSFT, which is a version of open string field theory, is formulated on the space of all boundary operators including off-shell ones. Although BSFT is originally defined in terms of a worldsheet sigma model with boundary interactions, it can be reformulated by using boundary states [11, 12]. This would suggest that BSFT is more suitable for introducing couplings with closed SFTs than cubic open SFT [13]. We thus seek for an open-closed SFT whose open string sector is BSFT. We expect that this open-closed SFT is a natural extension of [4, 6]. We refer to this open-closed SFT as boundary open-closed string field theory.

The organization of this paper is as follows. In the next section, we give a short review on the Batalin-Vilkovisky (BV) formalism for classical open-closed SFT and derive a gauge transformation law. In the construction of a SFT, a guiding principle is the stringy gauge invariance. The BV formalism [14] is useful for this purpose since an action satisfying the BV master equation automatically has gauge invariance. It was proposed in [15] that a classical open-closed SFT should satisfy the BV classical open-closed master equations, which
is the classical part of the full BV master equation. We examine how those master equations
determine the gauge transformation. We also provide basic ingredients for the construction of
our open-closed SFT.

In section 3, we couple BSFT to a free closed SFT with the conventional open-closed
interaction as in (1.1). We then add appropriate terms required by the BV classical open-
closed master equations. In this manner, we find that the joining-splitting type 3-closed-string
interaction is required from the BV master equation. An essential point is the utilization of
the factorization property of conformal field theory (CFT).

However, even with the joining-splitting type vertex, there also exists a region of string
length parameters where the master equation is not satisfied. For this reason, it seems difficult
to find an open-closed SFT action which completely satisfies the master equation in our
framework. Though this is somewhat disappointing, as demonstrated in section 4 we can
still show that the resulting action has gauge invariance if we restrict the string length of the
gauge parameter to negative. In the final section, we shall briefly discuss the relation between
our open-closed SFT and the D-brane soliton state in a closed SFT [16, 17]. Our conventions
and some useful formulas are summarized in appendices.

2 Batalin-Vilkovisky formalism and string field theory

In this section, we briefly review the BV formalism for classical open-closed SFT and derive
the gauge transformation law for string fields. The term ‘classical’ represents a theory without
loops, namely consisting only of diagrams with genus zero and at most one boundary. We also
review BSFT, which is our starting point in constructing our open-closed SFT.

2.1 Batalin-Vilkovisky formalism and gauge transformation

In an open-closed SFT, a closed string field $\Psi$ and an open string field $O$ are expanded in
terms of the bases $\{\Psi_I\}$ and $\{O_i\}$ respectively,

$$\Psi = \sum_I \Psi_I \psi^I, \quad O = \sum_i O_i \lambda^i, \quad (2.1)$$

where $\psi^I$ and $\lambda^i$ are target space closed and open string fields, respectively. Throughout this
paper, we denote open and closed string fields by the indices $i,j, \ldots$ and $I,J, \ldots$, respectively.
A key ingredient in the BV formalism is a fermionic two-form $\omega = \omega_c + \omega_o$,

$$\omega_c = -d\psi^I \wedge \omega^c_{IJ} d\psi^J, \quad \omega_o = -d\lambda^i \wedge \omega^o_{ij} d\lambda^j, \quad (2.2)$$

which is non-degenerate and closed. The anti-bracket is defined as the sum of those in the
open and closed string sectors

$$\{A, B\} = \{A, B\}_c + \{A, B\}_o, \quad (2.3)$$
Here $\frac{\partial}{\partial \psi}$ and $\frac{\partial}{\partial \lambda}$ are the left and right derivatives respectively, and $\omega_i^J$ and $\omega_o^j$ are the inverse matrices of $\omega_i^{cJ}$ and $\omega_o^{ij}$, respectively.

In the BV formalism, a classical closed SFT action $S_c(\psi)$ obeying the classical master equation
\[ \{S_c, S_c\}_c = 0, \]
(2.5)
is automatically guaranteed to possess a gauge invariance; the action is invariant under the closed string gauge transformation
\[ \delta_c \psi^I = \left( \omega_i^J \frac{\partial}{\partial \psi^K} + \frac{1}{2}(-1)^{JK} \frac{\partial}{\partial \psi^K} \right) \epsilon^K, \]
(2.6)

as seen from the derivative of the master equation (2.5):
\[ 0 = \frac{1}{2} \frac{\partial}{\partial \psi^K} \{S_c, S_c\}_c \epsilon^K = \frac{\partial}{\partial \psi^K} \left( \omega_i^J \frac{\partial}{\partial \psi^K} + \frac{1}{2}(-1)^{JK} \frac{\partial}{\partial \psi^K} \right) \epsilon^K. \]
(2.7)

Here, $\epsilon$ is an infinitesimal parameter and $(-1)^I$ is the Grassmann parity of $\psi^I$. Similarly, an open SFT action $S_o(\lambda)$ can be constructed to be a solution of the master equation
\[ \{S_o, S_o\}_o = 0. \]
(2.8)

The same argument as above leads to the gauge transformation law in the open string side
\[ \delta_o \lambda^i = \left( \omega_o^j \frac{\partial}{\partial \lambda^K} + \frac{1}{2}(-1)^{jk} \frac{\partial}{\partial \lambda^K} \right) \epsilon^K. \]
(2.9)

However, once open-closed interactions are taken into account, the master equations above are not a consistent classical truncation. To see this, let us consider a classical open-closed SFT action of the form
\[ S_{oc}(\psi, \lambda) = S_c(\psi) + S_D(\psi, \lambda), \]
(2.10)
where $S_c$ is a classical closed SFT action and $S_D$ corresponds to disks with and without closed string insertions. The anti-bracket of $S_{oc}$ with itself is given by
\[ \{S_{oc}, S_{oc}\} = \{S_c, S_c\}_c + 2\{S_c, S_D\}_c + \{S_D, S_D\}_o + \{S_D, S_D\}_c. \]
(2.11)

The classical master equation can be understood as a classical truncation of the quantum master equation, $\frac{1}{2}\{S, S\} + \Delta S = 0$. In the construction of classical SFT, we can ignore $\Delta = \Delta_c + \Delta_o$ because $\Delta_c$ increases the genus $g$ and $\Delta_o$ increases the number of boundaries $b$ for $b > 0$. We should note that, though the BV master equation is a strong consistency condition for SFTs, it does not necessarily guarantee a single cover of the moduli space.
The last term represents a surface with two boundaries and should be ignored in classical SFT. Therefore, as proposed in [15], the BV classical open-closed master equations are given by the following two equations:\footnote{The authors of [15] investigated the classical part of Zwiebach’s open-closed SFT [2] in terms of homotopy algebra and defined open-closed homotopy algebra (OCHA). A quantum generalization of OCHA was recently discussed in [18].}

\[
0 = \{S_c, S_c\}_c, \quad (b = 0) \tag{2.12}
\]

\[
0 = 2\{S_c, S_D\}_c + \{S_D, S_D\}_o, \quad (b = 1) \tag{2.13}
\]

where we have split (2.11) into two equations for each number of boundaries \(b\).

Similarly to the case above, we can derive gauge transformations by applying \(\frac{\partial}{\partial \psi^K} \epsilon^K + \frac{\partial}{\partial \lambda^k} \epsilon^k\) on (2.12) and (2.13). We see from (2.12) that the gauge transformation of \(S_c\) is the same as in the closed SFT. This also requires that the closed string field in \(S_D\) should transform as in (2.6). For (2.13), we have

\[
0 = \frac{\partial_r}{\partial \psi^K} \left[ \{S_c, S_D\}_c + \frac{1}{2} \{S_D, S_D\}_o \right] \epsilon^K + \frac{\partial_r}{\partial \lambda^k} \left[ \{S_c, S_D\}_c + \frac{1}{2} \{S_D, S_D\}_o \right] \epsilon^k
\]

\[
= \partial_r S_D \left\{ \omega_{ij} \frac{\partial_r}{\partial \psi^k} \frac{\partial}{\partial \lambda^l} + \frac{1}{2} (-1)^{jk} \frac{\partial}{\partial \lambda^l} \frac{\partial}{\partial \lambda^j} \right\} \epsilon^K + \omega_{ij} \left( \frac{\partial_r}{\partial \psi^k} \frac{\partial}{\partial \lambda^l} \right) \epsilon^K
\]

\[
+ \frac{\partial_r S_D}{\partial \psi^j} \left\{ \omega_{ij} \frac{\partial_{\psi^k}}{\partial \lambda^l} + \frac{1}{2} (-1)^{jk} \frac{\partial_{\psi^k}}{\partial \lambda^l} \frac{\partial}{\partial \psi^j} \right\} \epsilon^K + \omega_{ij} \left( \frac{\partial_{\psi^k}}{\partial \lambda^l} \frac{\partial}{\partial \psi^j} \right) \epsilon^K \tag{2.14}
\]

Here we have collected terms proportional to \((\partial_r S_D/\partial \lambda^i)\) in the second line, which determine the gauge transformation for open string fields. The third line is formed to obtain the consistent gauge transformation with \(S_c\). The last line is the collection of the remaining terms, which imposes a constraint on the infinitesimal parameters

\[
\left[ \omega_{ij} \frac{\partial_r}{\partial \psi^k} \frac{\partial}{\partial \lambda^l} + \frac{1}{2} (-1)^{jk} \frac{\partial_{\psi^k}}{\partial \lambda^l} \frac{\partial}{\partial \psi^j} \right] \epsilon^K + \omega_{ij} \left( \frac{\partial_r}{\partial \psi^k} \frac{\partial}{\partial \lambda^l} \right) \epsilon^K = 0. \tag{2.15}
\]

We therefore find that the total action \(S_c + S_D\) is invariant under the following gauge transformations,

\[
\delta_c \psi^j = \left[ \omega_{ij} \frac{\partial_r}{\partial \psi^k} \frac{\partial}{\partial \lambda^l} + \frac{1}{2} (-1)^{jk} \frac{\partial_{\psi^k}}{\partial \lambda^l} \frac{\partial}{\partial \psi^j} \right] \epsilon^K, \quad \delta_c \lambda^i = \omega_{ij} \left( \frac{\partial_r}{\partial \psi^k} \frac{\partial}{\partial \lambda^l} \right) \epsilon^K, \tag{2.16}
\]

\[
\delta_o \lambda^i = \left[ \omega_{ij} \frac{\partial_r}{\partial \lambda^k} + \frac{1}{2} (-1)^{jk} \frac{\partial_{\psi^k}}{\partial \lambda^l} \frac{\partial}{\partial \psi^j} \right] \epsilon^K, \quad \delta_o \psi^j = \omega_{ij} \left( \frac{\partial_r}{\partial \lambda^k} \frac{\partial}{\partial \psi^j} \right) \epsilon^K \tag{2.17}
\]

where the infinitesimal parameters must be chosen to satisfy (2.15). The second equation in (2.16) shows that open string fields get transformed by the closed string gauge transformation. We apply this formalism to construct a classical open-closed SFT whose open string part is BSFT.
2.2 Closed and open string field theories

We give some details of closed and open string fields in our construction. To begin with, it is convenient to define the following ghost zero modes

\[ b^-_0 = \frac{1}{2}(b_0 - \tilde{b}_0), \quad c^-_0 = c_0 - \tilde{c}_0, \] (2.18)

which satisfy the anti-commutation relation\(^3\)

\[ [b^-_0, c^-_0] = 1. \] (2.19)

We adopt closed string fields as a conventional one satisfying the conditions,

\[ b^-_0 |\Psi\rangle = 0, \quad L^-_0 |\Psi\rangle = 0, \] (2.20)

where \( L^-_0 \equiv L_0 - \tilde{L}_0 \). The second condition imposes the level matching condition. This condition can also be represented as \( \mathcal{P}|\Psi\rangle = |\Psi\rangle \), where the projector \( \mathcal{P} \) is defined by

\[ \mathcal{P} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(\theta L_0 - \theta \tilde{L}_0)}. \] (2.21)

On the other hand, we take open string fields as boundary operators. The ghost number of the closed string field \(|\Psi\rangle\) is 2, while that of the open string field \( O \) is 1 as usual. They are expanded in terms of target space string fields as

\[ |\Psi\rangle = \sum_I |\Psi_I\rangle \psi_I, \quad O = \sum_i O_i \lambda^i, \] (2.22)

where \(|\Psi_I\rangle\) and \( O_i \) are bases of string fields of any worldsheet ghost number. The statistics of target space fields are taken to satisfy the correct statistics of string fields, namely,

\[ (-1)^I \equiv |\psi^I| = ||\Psi_I||, \quad (-1)^i \equiv |\lambda^i| = -|O_i|. \] (2.23)

Similarly, we assign a spacetime ghost number \( g \) to target space fields so that string fields have the correct ghost number:

\[ g(\psi^I) = 2 - G(\Psi_I), \quad g(\lambda^i) = 1 - G(O_i), \] (2.24)

where \( G \) represents the worldsheet ghost number.

By using the ingredients above, we define the closed 2-form for closed strings as

\[ \omega_c = \langle d\Psi|c^-_0|d\Psi \rangle, \] (2.25)

where \(|d\Psi\rangle = |\Psi_I\rangle d\psi^I\rangle\), and \( \langle d\Psi | \) denotes its BPZ conjugate. Thus the kinetic term for closed string fields is given by the canonical one:

\[ S^\text{free}_c = \frac{1}{2} \langle \Psi|c^-_0 Q_B|\Psi \rangle. \] (2.26)

\(^3\)Throughout this paper, the bracket denotes the graded commutator \([A, B] \equiv AB - (-1)^{AB}BA\).
In the rest of this section, we focus on BSFT. We mainly follow the formulation in [11, 12], where BSFT is rewritten in terms of boundary state, since the boundary state formalism is a suitable framework to consider the coupling between open and closed strings. The fundamental object in BSFT is the operator $O$ representing a boundary operator $\mathcal{O}(\sigma)$ after $\sigma$ integration,

$$O \equiv \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} \mathcal{O}(\sigma), \quad (2.27)$$

Since $L_0^-$ generates a rotation of the disk, the boundary operator $O$ satisfies

$$[L_0^-, O] = 0. \quad (2.28)$$

With the off-shell boundary operator $O$, deformed boundary states

$$\langle B; \lambda \rangle \equiv \langle B \mid e^{2\kappa [b_0^-, O]} \rangle, \quad (2.29)$$

play an important role. Here $\langle B \mid$ represents a D-brane boundary state and $\kappa$ is the open string coupling constant. Although conformal symmetries are lost due to the off-shell deformation, $\langle B; \lambda \rangle$ is still annihilated by $b_0^-$ and $L_0^-$,

$$\langle B; \lambda \mid b_0^- = 0, \quad \langle B; \lambda \mid L_0^- = 0, \quad (2.30)$$

which takes the same form as the condition for closed string fields in (2.20). The closed two-form $\omega_o$ is defined by two-point correlation functions of the deformed worldsheet theory, which is given by

$$\omega_o = \langle B \mid \text{Sym}[e^{2\kappa [b_0^-, O]}; dO, dO] \rangle \mid_0, \quad (2.31)$$

where $dO = O_i d\lambda^i$ and $\mid \rangle$ is the $\text{SL}(2, \mathbb{C})$ vacuum. The symbol Sym[$\cdots$] is defined by

$$\text{Sym}[e^{-V}; O_1, O_2, \cdots, O_n] $$

$$= \int_0^1 dt_1 \int_{t_1}^1 dt_2 \cdots \int_{t_{n-1}}^1 dt_n e^{-t_1 V} O_1 e^{-(t_2-t_1)V} O_2 \cdots O_n e^{-(1-t_n)V} \pm (\text{perms}). \quad (2.32)$$

In the construction of BSFT, we need a fermionic vector $V^i$. As shown in [7], if the vector $V$ is nilpotent and generates a symmetry of $\omega_o$, i.e.,

$$(dV + i_V d)\omega_o = d(i_V \omega_o) = 0, \quad (2.33)$$

the BSFT action defined by $dS^\text{BSFT}_D = i_V \omega_o$ automatically satisfies the master equation (2.8).

It was shown that the vector generated by the bulk BRST operator $Q_B$,

$$\delta_V O \equiv [Q_B, O], \quad (2.34)$$

indeed satisfies the requirement above. Note that since $\delta_V O = \frac{\delta O}{\delta \lambda} V^i = O_i V^i$, the component form of (2.34) is

$$O_i V^i = [Q_B, O], \quad (2.35)$$
which we use to calculate anti-brackets in the subsequent sections. By using the vector \( V \), the BSFT action is given by

\[
dS_{D}^{\text{BSFT}} = -\langle B| \text{Sym}[e^{2ix[b_{0},O]}; dO, [Q_{B}, O]] |0 \rangle .
\]  

(2.36)

The integration of \( dS \) was performed in [11, 9], and the result is

\[
S_{D}^{\text{BSFT}} = -\frac{1}{4\kappa^{2}} \langle B| e^{2ix[b_{0},O]} c_{0}^{-} Q_{B} c_{0}^{-} |0 \rangle + \frac{i}{2\kappa} \langle B| \text{Sym}[e^{2ix[b_{0},O]}; [Q_{B}, O], c_{0}^{-}] |0 \rangle .
\]  

(2.37)

The gauge transformation can be read off from \( (2.9) \) and is given by \( [1 2] \)

\[
\delta_{\Lambda} O = [Q_{B}, \Lambda] + i\kappa \langle B| \text{Sym}[e^{2ix[b_{0},O]}; [Q_{B}, O], [b_{0}, \Lambda], O_{j}] |0 \rangle \omega_{j}^{0} O_{j} ,
\]  

(2.38)

where \( \Lambda \) is a gauge parameter of ghost number 0. As we shall see later, once we take open-closed interactions into account, \( \Lambda \) is no longer an arbitrary parameter and the constraint on \( \Lambda \) eliminates the second term of \( (2.38) \).

### 3 coupling BSFT to closed SFT

We construct an open-closed SFT action by requiring the action to satisfy the BV classical open-closed master equations. We begin with the BSFT action and the free closed SFT action as well as the conventional open-closed interaction term given by the overlap of an off-shell boundary state with a closed string field. We shall see that the joining-splitting type 3-closed-string interaction is necessary.

#### 3.1 Our ansatz and BV master equations

We start with the ansatz

\[
S_{c}^{\text{free}} + S_{D}^{\text{int}} + S_{D}^{\text{BSFT}} ,
\]  

(3.1)

and subsequently add appropriate terms required by the master equations. Here \( S_{c}^{\text{free}} \) is the kinetic term for closed string fields given in \( (2.26) \) and \( S_{D}^{\text{int}} \) describes the interaction between closed strings and boundary states\(^{4}\)

\[
S_{D}^{\text{int}} = \frac{1}{2} \langle B| e^{2ix[b_{0},O]} c_{0}^{-} |\Psi \rangle .
\]  

(3.2)

Let us check to what extent the ansatz satisfies the BV classical open-closed master equations \( (2.12) \) and \( (2.13) \). The two-form \( \omega \) has already been defined in the previous section, whose components are

\[
\omega_{I,I}^{c} = (-1)^{I+1} \langle \Psi_{I}| c_{0}^{-} |\Psi_{I} \rangle , \quad \omega_{I,J}^{0} = (-1)^{i+1} \langle B| \text{Sym}[e^{2ix[b_{0},O]}; O_{i}, O_{j}] |0 \rangle .
\]  

(3.3)

\(^{4}\)This interaction term between open and closed string fields is analogous to the ones considered in [19]. The authors introduced the interaction terms into cubic open SFT to investigate closed string cohomology.
Some useful formulas relevant for the calculation below are summarized in Appendix A. As a warm-up, let us show that the kinetic term satisfies the first equation (2.12). Using the completeness condition (A.17), we have

\[
\{S^\text{free}_c, S^\text{free}_e\}_e = \langle \Psi | e_0^- Q_B | \Psi \rangle \omega^J J (-1)^J \langle \Psi | e_0^- Q_B | \Psi \rangle \\
= -\langle \Psi | e_0^- Q_B e_0^- Q_B | \Psi \rangle \\
= -\langle \Psi | e_0^- Q_B^2 | \Psi \rangle + \frac{1}{2} \langle \Psi | e_0^- Q_B e_0^- L_0^- | \Psi \rangle - \langle \Psi | e_0^- Q_B e_0^- Q_B e_0^- Q_B | \Psi \rangle \\
= 0 ,
\]

(3.4)

where we moved the \( b_0^- \) to the right. For the second equation (2.13), we have

\[
2\{S^\text{BSFT}_D, S^\text{free}_c\}_e + 2\{S^\text{BSFT}_D, S^\text{free}_c\}_o \\
= \{S^\text{BSFT}_D, S^\text{BSFT}_o\}_o \\
+ 2\{S^\text{int}_D, S^\text{free}_c\}_e + 2\{S^\text{int}_D, S^\text{BSFT}_o\}_o \\
+ \{S^\text{int}_D, S^\text{int}_o\}_o , \quad (n = 1)
\]

(3.5)

where \( n \) denotes the number of closed string insertions in the corresponding term. The term without closed strings vanishes by construction. The terms with \( n = 1 \) are evaluated as

\[
2\{S^\text{int}_D, S^\text{free}_c\}_e + 2\{S^\text{int}_D, S^\text{BSFT}_o\}_o \\
= \langle B | e^{2i\kappa [b_0^-, O]} c_0^- | \Psi \rangle \omega^J J (-1)^J \langle \Psi | e_0^- Q_B | \Psi \rangle + 2i\kappa \langle B | \text{Sym} \left[ e^{2i\kappa [b_0^-, O]} \right] | O_i, b_0^- \rangle e_0^- | \Psi \rangle \omega^i J \omega_j e_j \psi^k V^k \\
= -\langle B | e^{2i\kappa [b_0^-, O]} c_0^- Q_B | \Psi \rangle + 2i\kappa \langle B | \text{Sym} \left[ e^{2i\kappa [b_0^-, O]} \right] | O_i, b_0^- \rangle e_0^- | \Psi \rangle V^i \\
= -\langle B | e^{2i\kappa [b_0^-, O]} c_0^- Q_B b_0^- | \Psi \rangle - 2i\kappa \langle B | \text{Sym} \left[ e^{2i\kappa [b_0^-, O]} \right] | [b_0^-, O], Q_B \rangle e_0^- | \Psi \rangle \\
= \langle B | e^{2i\kappa [b_0^-, O]} Q_B | \Psi \rangle - \langle B | e^{2i\kappa [b_0^-, O]} Q_B b_0^- | \Psi \rangle = 0 ,
\]

(3.6)

where we used (A.12) in the second line and inserted \( 1 = [b_0^-, c_0^-] \) in the second last line. It is worth noting the implication of the above equation (3.6). If one constructs open and closed string field theories with respect to different BRST operators, the BRST operator in the first term comes from the closed string side, while the second one from the open string side. Therefore (3.6) is regarded as a compatibility condition for the two BRST operators.

The remaining one is the term with \( n = 2 \),

\[
\{S^\text{int}_D, S^\text{int}_o\}_o = \kappa^2 \langle B | \text{Sym} \left[ e^{2i\kappa [b_0^-, O]} \right] | O_i \rangle | \Psi \rangle \omega^i j (-1)^{j+1} \langle B | \text{Sym} \left[ e^{2i\kappa [b_0^-, O]} \right] | O_j \rangle | \Psi \rangle ,
\]

(3.7)

which does not vanish. This implies an additional closed string interaction of the form

\[
S^\text{int}_c = \frac{\kappa^2}{3} \langle \Psi | c_0^- | \Psi \star \Psi \rangle .
\]

(3.8)

Here we do not specify the detail of the star product yet. We only assume the cyclic property

\[
\langle A | c_0^- | B \star C \rangle = (-1)^{(A+B)C} \langle C | c_0^- | A \star B \rangle = (-1)^{(B+C)A} \langle B | c_0^- | C \star A \rangle ,
\]

(3.9)
and that the product of two string fields is a string field satisfying (2.20). The contribution coming from the interaction term (3.8) to the master equation is

\[
2\{S_B^{\text{int}}, S_c^{\text{int}}\}_c = \kappa^2 \langle B | e^{2i\kappa |b_0^-, O_c|} c_0^- | \Psi_I \rangle \omega^J_F (-1)^J \langle \Psi_J | c_0^- | \Psi \rangle \\
= -\kappa^2 \langle B | e^{2i\kappa |b_0^-, O_c|} c_0^- b_0^- c_0^- | \Psi \rangle \\
= -\kappa^2 \langle B | e^{2i\kappa |b_0^-, O_c|} c_0^- | \Psi \rangle,
\]

(3.10)

and we require this to cancel (3.7):

\[
0 = (-1)^{j+1} \langle B | \text{Sym} [e^{2i\kappa |b_0^-, O_l|}; O_i] | \Psi \rangle \omega^{ij} \langle B | \text{Sym} [e^{2i\kappa |b_0^-, O_l|}; O_J] | \Psi \rangle - \langle B | e^{2i\kappa |b_0^-, O_c|} c_0^- | \Psi \rangle.
\]

(3.11)

Thus we found that three-closed-string interaction is necessary. This is consistent with [4], where the gauge transformation of open string fields is written in terms of the star product of the HIKKO closed SFT. We shall look for an appropriate star product, namely a three-string interaction vertex. We should also keep in mind that the resulting closed SFT action must satisfy the master equation (2.12).

### 3.2 Three-closed-string vertex

We have seen that the master equation requires a specific form of 3-closed-string vertex as in (3.11). In this subsection, we discuss this issue further. The first term of (3.11) is reminiscent of the factorization property in CFT. Suppose a surface with a thin neck. When it is pinched and the surface is separating into two surfaces, a correlation function factors into the product of those on two surfaces. We shall see that such a factorization indeed occurs in the second term in (3.11) if we adopt the HIKKO star product.

In the HIKKO closed SFT [5], we need to introduce the string length parameter \( \alpha \). We write it explicitly as

\[
|\Psi(\alpha)\rangle = |\Psi\rangle \otimes |\alpha\rangle,
\]

where the normalization is \( \langle \alpha_1 | \alpha_2 \rangle = 2\pi \delta(\alpha_1 - \alpha_2) \)

The HIKKO 3-closed-string vertex is defined by the overlap of three closed strings as depicted in Figure 1.

![Figure 1: Three closed string interaction in the HIKKO closed SFT for \( |\alpha_3| = |\alpha_1| + |\alpha_2| \). The parameters \( \alpha_i \) (\( i = 1, 2, 3 \)) denote the lengths of each string.](image)

In the following, we use the three-string vertex defined in CFT language, which is called the LPP vertex [20]. Since the total central charge vanishes, we implicitly use the generalized

\[5\] Note that the BPZ conjugate of \(|\alpha\rangle\) is \(|-\alpha\rangle\).
gluing and resmoothing theorem \[21, 22\]. The star product is determined through the LPP vertex \(V_{\text{LPP}}^{123}\) as follows:

\[
\langle \Psi_1(\alpha_1)|\langle \Psi_2(\alpha_2) * \Psi_3(\alpha_3) \rangle = 2\pi \delta(\alpha_1 + \alpha_2 + \alpha_3)\langle V_{\text{LPP}}^{123} | \Psi_1(\alpha_1) \rangle_1 | \Psi_2(\alpha_2) \rangle_2 | \Psi_3(\alpha_3) \rangle_3 .
\] (3.13)

In this prescription, the LPP vertex is defined in terms of conformal mappings \(h_r\) from three unit disks (with coordinates \(w_r\)) to a complex plane (with a coordinate \(z\)):

\[
\langle V_{\text{LPP}}^{123} | \Psi_1(\alpha_1) \rangle_1 | \Psi_2(\alpha_2) \rangle_2 | \Psi_3(\alpha_3) \rangle_3 = \langle h_1[\Psi_1(0)] h_2[\Psi_2(0)] h_3[\Psi_3(0)] \rangle,
\] (3.14)

where the operators \(\Psi_r(w_r = 0)\) are defined through the relation \(|\Psi_r\rangle_r = |\Psi_r(0)\rangle_r\). Here \(h_r[\Psi_r(0)]\) is the conformal transformation of \(\Psi_r(0)\) by \(h_r\). For instance, if \(\phi\) is a primary of dimension \(d\), \(f[\phi(0)] = (f'(0))^{d}\phi(f(0))\). The basic properties of the HIKKO closed SFT including the explicit forms of \(h_r\) are summarized in Appendix B.

Let us evaluate the second term in (3.11) by using the LPP vertex above:

\[
\langle V_{\text{LPP}}^{123} | \Psi(\alpha_1) \rangle_1 | \Psi(\alpha_2) \rangle_2 | B(\alpha_3); \lambda \rangle_3,
\] (3.15)

where we have assigned the length parameter to the boundary state as well,

\[
|B(\alpha); \lambda \rangle \equiv |B; \lambda \rangle \otimes |\alpha \rangle.
\] (3.16)

In the following, we focus on the parameter region with \(\alpha_1, \alpha_2 < 0\) and \(\alpha_3 = -\alpha_1 - \alpha_2 > 0\). In this setting, the strings “1” and “2” in Figure 1 represent the two closed strings, while “3” corresponds to the boundary state. Thus the two closed string worldsheets attach to each other only at a point. In the discussion below, it is important to note that the whole worldsheet can be considered as splitting into two surfaces. In order to evaluate (3.15), we follow the procedure given in [17]. We represent the boundary state as the boundary of a worldsheet with an appropriate boundary deformation.

We construct the \(\rho\)-plane by gluing the two unit disks \(|w_r| \leq 1\) \((r = 1, 2)\) with closed string insertions and the boundary \(|w_3| = 1\) together,

\[
\rho = f_r(w_r) = \begin{cases} 
\alpha_r \log w_r + T + i\beta_r & (r = 1, 2) \\
\alpha_3 \log w_3 + i\beta_3 & (r = 3)
\end{cases}, \quad (\beta_r = \pi \text{sgn}(\text{arg } w_r) \sum_{i=1}^{r-1} \alpha_i)
\] (3.17)

where \(T > 0\) is a regularization parameter eventually taken to zero. The \(\rho\)-plane and the subsequent conformal transformations are depicted in Figure 2. We take the argument of \(w_r\) to be from \(-\pi\) to \(\pi\). The \(\rho\)-plane is mapped to the upper-half plane of \(z\) by the Mandelstam mapping,

\[
\rho(z) = \alpha_1 \log \frac{z - i}{z + i} + \alpha_2 \log \frac{z - iy}{z + iy}, \quad (0 < y < 1)
\] (3.18)

where \(y\) is a parameter describing where the second closed string insertion is mapped. In this way, we obtain the LPP vertex in the current case:

\[
\langle V_{\text{LPP}}^{123}; T| \Psi(\alpha_1) \rangle_1 | \Psi(\alpha_2) \rangle_2 | B(\alpha_3); \lambda \rangle_3 = \langle \hat{h}_1[\Psi(0)] \hat{h}_2[\Psi(0)] \rangle_\lambda .
\] (3.19)
where \( \hat{h}_r(w_r) = \rho^{-1}(f_r(w_r)) \) \((r = 1, 2, 3)\) and the subscript \( \lambda \) denotes the existence of an appropriate conformal transformation of the boundary deformation. We note that (3.19) should coincide with (3.15) in the \( T \to 0 \) limit. The inverses of \( \hat{h}_r(w_r) \) are explicitly given by

\[
\begin{align*}
  w_1 &= \frac{z - iy}{z + iy} e^{-\frac{x}{\alpha_1}}, \\
  w_2 &= \frac{1 + iz}{1 - iz} \left( \frac{z - i\alpha_2}{z + i\alpha_2} \right) e^{-\frac{x}{\alpha_2}}, \\
  w_3 &= \exp \left[ \frac{2i}{\alpha_3} \arctan z + \frac{\alpha_2}{\alpha_3} \arctan \frac{z}{i} \right].
\end{align*}
\]

The interaction point \( z_I \) is a solution to \( d\rho/dz = 0 \). We easily see that the interaction point is on the imaginary axis,

\[
z_I = i \sqrt{\frac{(\alpha_1 y + \alpha_2) y}{\alpha_1 + y\alpha_2}}.
\]

Then the parameter \( y \) is determined via

\[
T = \text{Re} \rho(z_I) = \alpha_1 \log \left| \frac{z_I - i}{z_I + i} \right| + \alpha_2 \log \left| \frac{z_I - iy}{z_I + iy} \right|.
\]

Since both terms are positive, these two terms should vanish in the \( T \to 0 \) limit. This leads to \( z_I \to 0 \) and hence \( y \to 0 \). Moreover, when \( y \to 0 \), we see from (3.20) that every point on the upper-half plane has a corresponding point in the first worldsheet \( |w_1| \leq 1 \). Therefore, in this limit, the worldsheet of the second string collapses to a point, \( z = 0 \).

Along the argument in \([23]\), we find that the correlation function factors into the product of correlation functions of each closed string in this limit. As mentioned above, the surface is
splitting into two surfaces when $T$ approaches zero. The key fact here is that in the splitting limit, there exists a conformal map which is the identity for one surface and maps the other surface to a point. This is fulfilled by further mapping the upper-half plane into a unit disk as

\[ w' = g(z) = \frac{z - i}{z + i} = -\exp(2i \arctan z). \]  

(3.23)

Then the resulting composite map has the desired property,

\[ g(\hat{h}_1(w_1)) \to w_1, \quad g(\hat{h}_2(w_2)) \to -1. \quad (T \to 0) \]  

(3.24)

Therefore, in this limit, the conformal mapping leaves the first string invariant, while the second string is mapped to a point on the boundary, which can be regarded as an operator insertion on the boundary of the first worldsheet. In addition, the projector $P$ defined in (2.21), in other words, the level matching condition effectively integrates the operator insertion over the boundary of $w'$. Thus the boundary insertion must be expanded in terms of the basis of the integrated boundary operators $O_i$. If we consider the case where the boundary deformation is marginal, i.e., the deformation is invariant under the conformal transformation $g \circ \hat{h}_3$, we obtain \( \langle 3.19 \rangle \) in this limit as follows:

\[ \langle B | \text{Sym} \left[ e^{2i \kappa |b_0 \rangle \langle 0|}; O_j \right] | \Psi(\alpha_1) \rangle F_j^i(\Psi(\alpha_2)) \],

(3.25)

where $F_j^i(\Psi(\alpha_2))$ is a function of the second closed string field.

We can consider another mapping by exchanging the first and the second closed strings. In this case, the first worldsheet is mapped to a point. Then, with a function $F_j^i(\Psi(\alpha_1))$, \( \langle 3.15 \rangle \) becomes

\[ \langle B | \text{Sym} \left[ e^{2i \kappa |b_0 \rangle \langle 0|}; O_j \right] | \Psi(\alpha_2) \rangle F_j^i(\Psi(\alpha_1)) \].

(3.26)

Combining the results above, we obtain a factorized form of \( \langle 3.15 \rangle \),

\[ \langle B | \text{Sym} \left[ e^{2i \kappa |b_0 \rangle \langle 0|}; O_j \right] | \Psi(\alpha_1) \rangle f^{ij} \langle B | \text{Sym} \left[ e^{2i \kappa |b_0 \rangle \langle 0|}; O_j \right] | \Psi(\alpha_2) \rangle \].

(3.27)

This expression should be invariant under reparametrizations of the space of boundary operators, $\lambda^i \to \lambda^j(\lambda)$. Thus we require $f^{ij}$ (with an appropriate sign factor) to be a tensor such that \( \langle 3.27 \rangle \) has the correct ghost number. This uniquely determines $f^{ij}$ up to an overall constant $C$,

\[ f^{ij} = C \omega^{ij} (-1)^{i+1}, \]

(3.28)

where $(-1)^{i+1}$ arises when $O_j$ moves across $\langle B \rangle$.

We can determine the constant $C$ as follows. If we substitute \( c_0^+ | \Psi(\alpha_1) \rangle_1 \) and \( |0 \rangle_{a_2} \equiv |0 \rangle_2 \otimes |\alpha_2 \rangle_2 \) into $| \Psi(\alpha_1) \rangle_1$ and $| \Psi(\alpha_2) \rangle_2$, respectively, we can evaluate \( \langle 3.25 \rangle \) directly due to the absence of a closed string insertion on the second worldsheet:

\[ \langle V_{\text{LPP}}^{123} c_0^{-1} | \Psi(\alpha_1) \rangle_1 |0 \rangle_{a_2} B(\alpha_3); \lambda \rangle_3 = \langle B; \lambda | c_0^+ | \Psi(\alpha_1) \rangle \].

(3.29)

\[ ^6 \text{Here the boundary state is integrated over the length } \alpha, \langle B \rangle = \int \frac{d\alpha}{\alpha} \langle B(\alpha) \rangle. \text{ The symmetrization in } \langle 3.25 \rangle \text{ and subsequent equations is necessary to reproduce CFT calculations.} \]

\[ ^7 \text{To be precise, we must multiply } c_0^+ | \Psi \rangle \text{ by a spacetime field of ghost number } -1 \text{ to get the correct total ghost number. In this sense, we also multiply } |0 \rangle \text{ by a spacetime ghost field of ghost number } 2. \]
On the other hand, (3.27) becomes
\[
C(-1)^{j+1} \langle B | \text{Sym} \left[ e^{2i \kappa [b_0^-,O_i]} ; O_j \right] c_0^- | \Psi(\alpha_1) \rangle \omega_o^{ij} \langle B | \text{Sym} \left[ e^{2i \kappa [b_0^-,O_i]} ; O_j \right] | 0 \rangle \\
= C(-1)^{j+1} \langle B | \text{Sym} \left[ e^{2i \kappa [b_0^-,O_i]} ; O_j \right] c_0^- | \Psi(\alpha_1) \rangle \omega_o^{ij} \langle B | \text{Sym} \left[ e^{2i \kappa [b_0^-,O_i]} ; O_j, 1 \right] | 0 \rangle \\
= C \langle B | \text{Sym} \left[ e^{2i \kappa [b_0^-,O_i]} ; O_j \right] c_0^- | \Psi(\alpha_1) \rangle \omega_o^{ij} \omega_s^{ij} = C \langle B ; \lambda | c_0^- | \Psi(\alpha_1) \rangle ,
\]
where \( O_1 \equiv 1 \). Comparing (3.29) with (3.30), we get \( C = 1 \). We can treat the case with \( \alpha_1, \alpha_2 > 0 \) and \( \alpha_3 < 0 \) in the same way. Therefore we obtain, for \( \alpha_1 \alpha_2 > 0 \),
\[
\langle V_{\text{LPP}} \rangle \langle \Psi(\alpha_1) \rangle_1 \langle \Psi(\alpha_2) \rangle_2 \langle B(\alpha_3) ; \lambda \rangle_3 \\
= (-1)^{j+1} \langle B | \text{Sym} \left[ e^{2i \kappa [b_0^-,O_i]} ; O_j \right] | \Psi(\alpha_1) \rangle \omega_o^{ij} \langle B | \text{Sym} \left[ e^{2i \kappa [b_0^-,O_i]} ; O_j \right] | \Psi(\alpha_2) \rangle ,
\]
which leads to a conclusion that (3.11) is satisfied:
\[
\langle B ; \lambda | c_0^- | \Psi(\alpha_1) \rangle \ast \langle \Psi(\alpha_2) \rangle \\
= (-1)^{j+1} \langle B | \text{Sym} \left[ e^{2i \kappa [b_0^-,O_i]} ; O_j \right] | \Psi(\alpha_1) \rangle \omega_o^{ij} \langle B | \text{Sym} \left[ e^{2i \kappa [b_0^-,O_i]} ; O_j \right] | \Psi(\alpha_2) \rangle ,
\]
where we used (3.13) and integrated over \( \alpha_3 \).

For off-shell boundary deformations, it is in general difficult to consider the conformal transformation of the deformed boundary state because the transformation laws of general boundary operators are complicated. However, in the \( T \rightarrow 0 \) limit, the worldsheet just represents two cylinders attached each other only at a point on the boundary and hence the closed string fields \( | \Psi(\alpha_1) \rangle_1 \) and \( | \Psi(\alpha_2) \rangle_2 \) would individually feel the original off-shell boundary state \( \langle B ; \lambda \rangle \). It is therefore conceivable that (3.15) is given by the two overlaps between the original boundary state and each closed string field with an operator inserted on each boundary, which takes the same form as (3.32). To make this more concrete, we provide some consistency checks on (3.32) in Appendix C. (i) For the case of constant tachyon, (3.32) is reproduced using the result for the undeformed boundary state. (ii) The r.h.s. of (3.32) satisfies the properties of the HIKKO star product, (3.8)–(3.10). It is crucial that the off-shell boundary states appeared in the r.h.s. of (3.32) are the original one; otherwise it would break the derivation law (B.9) and the Jacobi identity (3.10). Therefore it is convincing that (3.32) holds even for off-shell boundary states, and hereafter we assume that this is the case.

Finally we comment on the case \( \alpha_1 \alpha_2 < 0 \). This corresponds to the interchange of “2” and “3” in Figure I. One can easily see that no factorization occurs since there are no pinching points on the surface. Hence even for the HIKKO vertex, the relation (3.11) does not hold in this case. One might think that non-polynomial closed SFT [25, 26] is more compatible with BSFT since there are no string length parameters. The 3-closed-string vertex in this

\[\text{There is an ambiguity to define amplitudes when we consider more than one kind of off-shell boundary. For example, in one-loop BSFT [24], the results depend on the choice of the Weyl factors in the two boundaries. In order to avoid the ambiguity consistently, one-loop BSFT is defined to reproduce standard cylinder amplitudes. In this sense, when we consider off-shell deformations, we also work on a cylinder-like coordinate such as the } \rho \text{-plane.}\]
SFT is a closed string extension of the vertex in cubic open SFT. Unlike the joining-splitting type vertex as in HIKKO, the interaction of two closed strings is not point-like. Thus non-polynomial closed SFT would not be the closed string part of our open-closed SFT.

4 Open-closed SFT action and gauge transformation

We have seen that, for $\alpha_1 \alpha_2 > 0$, the HIKKO 3-string vertex satisfies (3.11) and consequently the (second) master equation (2.13). Thus we conclude that the closed string sector of our open-closed SFT is the HIKKO closed SFT,

$$S_c = \frac{1}{2} \langle \Psi | c_0^- Q_B | \Psi \rangle + \frac{\kappa^2}{3} \langle \Psi | c_0^- | \Psi * \Psi \rangle,$$

(4.1)

where the string field $|\Psi\rangle$ should be understood as

$$\int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} |\Psi(\alpha)\rangle.$$

(4.2)

Of course, the closed SFT action (4.1) with the HIKKO 3-string interaction satisfies the “closed” master equation (2.12).

The remaining issue is the treatment for the contribution from the moduli region with $\alpha_1 \alpha_2 < 0$, which does not satisfy the master equation. One possibility might be adding more terms to the action so that it satisfies the master equation. In this paper, however, we do not pursue this possibility but just limit our attention within the region where the master equation is satisfied and investigate the implication of the results so far. As mentioned in footnote 6 the boundary state in (3.11) should be supplemented with an integral on the length parameter $\alpha$. If we restrict the integration region to $\alpha > 0$\footnote{\langle B(\alpha) | is the BPZ conjugate of $| B(\alpha) \rangle$, i.e. $\langle B(\alpha) | = \langle B | \otimes \langle -\alpha |.$},

$$\langle B \rangle \rightarrow \langle B^- \rangle \equiv \int_0^\infty \frac{d\alpha}{2\pi} \langle B(\alpha) |,$$

(4.3)

we can discard the undesired contribution in the first term of (3.11) while keeping only the contribution with $\alpha_1 < 0$ and $\alpha_2 < 0$. However, this seemingly ad hoc treatment does not solve all the problems arising from the moduli region. In the second term of (3.11), the modification (4.3) leaves the contribution with $\alpha_1 + \alpha_2 < 0$, which still contains the region with $\alpha_1 \alpha_2 < 0$. As we will see below, this results in a further constraint on the gauge parameter. Thus we consider the following open-closed SFT action:

$$S = S_c + S_D^{\mathrm{int}} + S_D^{\mathrm{BSFT}},$$

(4.4)

where $S_c$ and $S_D^{\mathrm{BSFT}}$ are already given in (4.1) and (2.37), respectively, while $S_D^{\mathrm{int}}$ is (3.2) with the replacement (4.3),

$$S_D^{\mathrm{int}} = \frac{1}{2} \langle B^- | e^{2i\pi[b_1^\alpha]} | c_0^- \rangle |\Psi\rangle.$$

(4.5)
In the rest of this section, we examine the gauge invariance of this action.

Remember that the open-closed master equation leads to the constraint (2.15) on the gauge parameter, even if the master equation is fully satisfied. Let us discuss this constraint first. Since $S_{\text{int}}$ is linear in closed string fields and the two-form $\omega_c$ is constant, the constraint reduces to

$$0 = \frac{\partial_e}{\partial \lambda^k} \left( \frac{\partial h S_{\text{int}}}{\partial \phi^J} \right) e^k = \frac{(-1)^J}{2} \frac{\partial_e}{\partial \lambda^k} \left( \langle B^- | e^{2i\kappa \hat{b}_0 \cdot \hat{O}} | c_0 \rangle \psi_J \right) e^k = -i\kappa \langle B^- | \text{Sym} \left[ e^{2i\kappa \hat{b}_0 \cdot \hat{O}}; \hat{b}_0, \Lambda_o \right] c_0 \rangle \psi_J \right) ,$$

(4.6)

where $\Lambda_o \equiv O_k e^k$ is the gauge parameter for open string fields. This constraint is easily solved by requiring

$$[b_0, \Lambda_o] = 0 ,$$

(4.7)

without imposing any constraints on $e^K$. Once this constraint is imposed, it follows that

$$\frac{\partial_e \omega_o^{ij}}{\partial \lambda^k} e^k = \frac{\partial_e \omega_o^{ij}}{\partial \lambda^k} e^k = 0 ,$$

(4.8)

which simplifies (2.17) as

$$\delta_o \lambda^i = \omega_o^{ij} \left( \frac{\partial_e \partial h S_D^{\text{BSFT}}}{\partial \lambda^k} \right) e^k = \omega_o^{ij} \frac{\partial_e (\omega_o^{ij} V^i)}{\partial \lambda^k} e^k = \partial_e (V^i) e^k .$$

(4.9)

Then the open string gauge transformation is given by

$$\delta_o \lambda^i = \omega_o^{ij} \left( \frac{\partial_e \partial h S_D^{\text{BSFT}}}{\partial \lambda^k} \right) e^k = \omega_o^{ij} \frac{\partial_e (\omega_o^{ij} V^i)}{\partial \lambda^k} e^k = \partial_e (V^i) e^k .$$

(4.10)

Despite the fact that the action (4.4) does not completely satisfy the open-closed master equation, one can easily check that the action is indeed invariant under the gauge transformation as follows. We find that the off-shell boundary state is invariant,

$$\delta_o \langle B | e^{2i\kappa \hat{b}_0 \cdot \hat{O}} \rangle = 2i\kappa \langle B | \text{Sym} \left[ e^{2i\kappa \hat{b}_0 \cdot \hat{O}}; \hat{b}_0, [Q_B, \Lambda_o] \right] c_0 \rangle = 0 ,$$

(4.11)

where we used the Jacobi identity (A.8), the constraint condition (4.7), as well as the fact that the gauge parameter $\Lambda_o$ is expanded in terms of the basis $O_i$ satisfying $[L_0, O_i] = 0$. Obviously, closed string fields do not transform under the gauge transformation by $\Lambda_o$. Thus it is straightforward to show that the action is invariant under the open string gauge transformation:

$$\delta_o S_c = 0 , \quad \delta_o S_{\text{int}} = 0 , \quad \delta_o S_D^{\text{BSFT}} = \frac{i}{2\kappa} \langle B | \text{Sym} \left[ e^{2i\kappa \hat{b}_0 \cdot \hat{O}}; [Q_B, [Q_B, \Lambda_o]] \right] c_0 \rangle = 0 .$$

(4.12)

(4.13)

Unlike Zwiebach's open-closed SFT [2], open-closed interaction terms of higher order in the closed string field would break the closed string gauge invariance since the closed string part is cubic.
Let us turn to the closed string gauge transformation. In this case, the constraint (2.15) keeps the gauge transformation intact and (2.16) gives the following gauge transformation law:

\[
\delta_{\Lambda_c} \Psi = Q_B |\Lambda_c\rangle + 2\kappa^2 |\Psi \star \Lambda_c\rangle ,
\]

\[
\delta_{\Lambda_c} O = i\kappa(-1)^{j+1} O_i \omega^i_o \langle B^\ast | \text{Sym} [e^{2i\kappa [b_0^-, O]}; O_j] |\Lambda_c\rangle .
\]

Here $|\Lambda_c\rangle \equiv -|\Psi_K\rangle e^K$ is the parameter for the closed string gauge transformation, which is of ghost number 1 and annihilated by both $b_0^-$ and $L_0^-$. In the following, we decompose the gauge parameter $|\Lambda_c\rangle$ as

\[
|\Lambda_c\rangle = |\Lambda^-_c\rangle + |\Lambda^+_c\rangle ,
\]

where

\[
|\Lambda^-_c\rangle = \int_{-\infty}^{0} \frac{d\alpha}{2\pi} |\Lambda_c(\alpha)\rangle , \quad |\Lambda^+_c\rangle = \int_{0}^{\infty} \frac{d\alpha}{2\pi} |\Lambda_c(\alpha)\rangle .
\]

Similarly, we decompose $|\Psi\rangle = |\Psi^-\rangle + |\Psi^+\rangle$.

First, let us evaluate the effect of the transformation of open string fields under the closed string gauge transformation. The gauge transformation of the BSFT action $S^\text{BSFT}_D$ is given by

\[
\delta_{\Lambda_c} S^\text{BSFT}_D = -(B| \text{Sym} [e^{2i\kappa [b_0^-, O]}; [Q_B, O], \delta_{\Lambda_c} O]|0)
\]

\[
= -i\kappa(-1)^{j+1} \langle B| \text{Sym} [e^{2i\kappa [b_0^-, O]}; [Q_B, O], O_j]|0 \rangle \omega^i_o \langle B^\ast | \text{Sym} [e^{2i\kappa [b_0^-, O]}; O_j] |\Lambda_c\rangle
\]

\[
= -i\kappa \langle B^\ast | \text{Sym} [e^{2i\kappa [b_0^-, O]}; [Q_B, O]] |\Lambda_c\rangle
\]

\[
= \frac{1}{2} \langle B^\ast; \lambda]|Q_B c_0^\ast |\Lambda^-_c\rangle .
\]

See (A.18) for the calculation details including $\omega^i_o$. On the other hand, the open-closed interaction term $S^\text{Int}_D$ is transformed by the gauge transformation of open string fields as

\[
\frac{1}{2} \langle B^\ast | e^{2i\kappa [b_0^-, O]} |0\rangle c_0^\ast |\Psi\rangle = i\kappa \langle B^\ast | \text{Sym} [e^{2i\kappa [b_0^-, O]}; \delta_{\Lambda_c} O]|\Psi^-\rangle
\]

\[
= -\kappa^2(-1)^{j+1} \langle B^\ast | \text{Sym} [e^{2i\kappa [b_0^-, O]}; O_j]|\Psi^-\rangle \omega^i_o \langle B^\ast | \text{Sym} [e^{2i\kappa [b_0^-, O]}; O_j] |\Lambda^-_c\rangle
\]

\[
= -\kappa^2(-1)^{j+1} \langle B^\ast | |\Psi^-\rangle \omega^i_o \langle B^\ast | \text{Sym} [e^{2i\kappa [b_0^-, O]}; O_j] |\Lambda^-_c\rangle
\]

\[
= -\kappa^2 \langle B^\ast; \lambda]|c_0^\ast |\Psi^- \star \Lambda^-_c\rangle .
\]

Here, the last equality comes from the following relation,

\[
(\langle B^\ast | \text{Sym} [e^{2i\kappa [b_0^-, O]}; O_j]|\Psi^-\rangle \omega^i_o \langle B^\ast | \text{Sym} [e^{2i\kappa [b_0^-, O]}; O_j] |\Lambda^-_c\rangle = \langle B^\ast; \lambda]|c_0^\ast |\Psi^- \star \Lambda^-_c\rangle ,
\]

which is obtained by applying $\delta_K e^K$ on the both sides of (3.11) with $\alpha_1 < 0$ and $\alpha_2 < 0$. Note that the transformation (4.19) implies that the gauge transformation for the boundary state is given by

\[
\delta_{\Lambda_c} \langle B^\ast; \lambda| = 2\kappa^2 \langle B^\ast; \lambda| \ast |\Lambda^-_c\rangle ,
\]

\[\text{This gauge transformation law for boundary states was first obtained in [4] for on-shell boundary deformations.}\]
which can be obtained by using the cyclic property (3.9) as well as (A.9).

Next we consider the effect of the gauge transformation of closed string fields. It is obvious that $S_c$ is invariant, whereas the contribution from $S_D^{int}$ gives

$$\frac{1}{2} \langle B^-; \lambda | c_0^\dagger \delta_{\Lambda_c} | \Psi \rangle = \frac{1}{2} \langle B^-; \lambda | c_0^\dagger Q_B | \Lambda_c \rangle + \kappa^2 \langle B^-; \lambda | c_0^\dagger | \Psi^- * \Lambda_c \rangle$$

$$= \frac{1}{2} \langle B^-; \lambda | c_0^\dagger Q_B | \Lambda^-_c \rangle + \kappa^2 \langle B^-; \lambda | c_0^\dagger | \Psi^- * \Lambda^-_c \rangle + \kappa^2 \langle B^-; \lambda | c_0^\dagger | \Psi^- * \Lambda^+_c \rangle . \quad (4.22)$$

The first two terms exactly cancel the previous contributions, (4.18) and (4.19), from the gauge transformation of open string fields. However, unfortunately, the last term is not canceled by any other term. This is because our open-closed SFT action does not completely satisfy the master equation. Thus the action (4.4) is invariant only if we restrict the closed string gauge transformation parameter $|\Lambda_c\rangle$ to $|\Lambda^+_c\rangle = 0$.

In summary, we have shown that the action (4.4) is invariant under the following gauge transformations; $\delta_{\Lambda_o} S = \delta_{\Lambda^-_c} S = 0$ with

$$\delta_{\Lambda_o} O = [Q_B, \Lambda_o] , \quad (\text{with } [b_0^- , \Lambda_o] = 0) \quad (4.23)$$

$$\delta_{\Lambda^-_c} | \Psi \rangle = Q_B | \Lambda^-_c \rangle + 2\kappa^2 | \Psi^* * \Lambda^-_c \rangle , \quad (4.24)$$

$$\delta_{\Lambda^-_c} O = i\kappa (-1)^{j+1} O_j \omega_o^{ij} \langle B^- | \text{Sym} [\epsilon^{2\kappa [b_0^-, O_j]} ; O_j ] | \Lambda^-_c \rangle . \quad (4.25)$$

It is worthwhile to note that, though (4.25) looks unfamiliar, the resultant gauge transformation for the off-shell boundary state (4.21) takes the same form as the nonlinear part of (4.24).

5 Conclusion and Discussion

We have investigated the open-closed SFT whose open string sector is BSFT. In order to determine the closed and open-closed string interactions, we used the BV classical open-closed master equations. Once we introduce the conventional open-closed interaction term $S_D^{int}$, the master equations determine the closed string sector to be the HIKKO closed SFT. We have also derived the gauge transformation law for both open and closed string fields. In solving the master equation, we have assumed that (3.32) holds even for off-shell boundary states. Though we have provided several evidences to support this, it is of course important to show the relation directly. However this is technically difficult and we leave it for future work.

In our open-closed SFT, the gauge parameters must be supplemented with the constraints $[b_0^-, \Lambda_o] = 0$ and $|\Lambda^+_c\rangle = 0$. While the former is simply due to the classical truncation of the full master equation, the latter constraint comes from the existence of the parameter region where the classical open-closed master equation is not satisfied. This is not satisfactory since the

\footnote{Note that $\langle B^-; \lambda | c_0^\dagger Q_B | \Lambda^-_c \rangle = \langle B^-; \lambda | c_0^\dagger Q_B b_0^- c_0^\dagger | \Lambda^-_c \rangle = -\langle B^-; \lambda | Q_B c_0^\dagger | \Lambda^-_c \rangle$.}
gauge invariance of the original closed SFT is half broken. One might conclude that there is no consistent way to couple BSFT to closed SFT with the conventional open-closed interaction term $S_D^{\text{int}}$. However, we can find a hint from a similar case in [16, 17], where D-branes are realized as soliton states constructed on the second quantized vacuum $|0\rangle$ in the $OSp$ invariant closed SFT. The D-brane soliton state roughly takes the form

$$\exp \left[ -S_D^{\text{int}} + \cdots \right] |0\rangle.$$

(5.1)

Here, the vacuum $|0\rangle$ is defined to be annihilated by string fields with positive length, $|\Psi(\alpha)|0\rangle = 0$ for $\alpha > 0$. Since the 3-string vertex in this closed SFT is the same as the HIKKO vertex, it might be possible to interpret our results in the context of this D-brane soliton state. Then the integration region of $\langle B(\alpha) |$ in the $S_D^{\text{int}}$ is naturally restricted to $\alpha > 0$ as in [16, 17]. Moreover, the vacuum $|0\rangle$ annihilates $|\Lambda_\alpha^-\rangle$, which would make the constraint $|\Lambda_\alpha^+\rangle = 0$ automatic and hence unnecessary [13]. In this scenario, the BSFT action would arise through the off-shell extension of the D-brane soliton state. In other words, $S_D^{\text{BSFT}}$ may appear at “···” in (5.1). If this is the case, BSFT might be considered as the effective theory describing the fluctuations of the D-brane soliton state.

Finally, it is worth mentioning the relation of the present work to [4]. The authors considered the HIKKO closed SFT with an on-shell D-brane source term and found that the gauge invariance requires the DBI action. They showed that the total action correctly reproduces the low-energy dynamics of a D-brane. Remarkably, our SFT action takes the same form of the action in the reference with the replacement of the DBI action by BSFT as well as with the off-shell extension of the source term [14]. Thus our open-closed SFT is a natural off-shell extension of their work and we expect that it describes the off-shell dynamics of string theory such as D-brane decay by tachyon condensation and closed string radiation. We hope that our formulation will provide new insights into nonperturbative aspects of string theory.

Acknowledgments

We would like to thank Hiroyuki Hata and Yuji Okawa for useful discussions at the early stage of this work.

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13 In order to show the gauge invariance of the D-brane soliton, it would be necessary to use the idempotency relation for boundary states [27, 28, 29] as in [17].

14 The open-closed interaction term here is linear in the closed string field. One might wonder how such a linear term could reproduce the non-linear coupling between the graviton and matter. Actually, a similar question arises in the original HIKKO closed SFT, because it only contains cubic interaction terms while the Einstein-Hilbert action involves all orders in the metric. A plausible scenario would be that, in these SFTs, the component fields are related to fields in the corresponding effective field theories through non-trivial field redefinition as well as integration of massive fields. In [4], the authors examined such a correspondence at the first non-trivial order.
Appendix

A Conventions and useful formulas

In this paper we mainly follow the conventions in [30] and [25]. We use the unit $\alpha' = 2$.

Virasoro generators:

$$L_m = L^m_m + L^g_m,$$  \hspace{1cm} (A.1)

where

$$L^m_m = \frac{1}{2} \sum_n \hat{\alpha}_{m-n} \alpha_{tn}, \quad L^g_m = \sum_n (2m-n) \hat{c} c_{m-n} - \delta_{m,0}.$$  \hspace{1cm} (A.2)

BRST operator:

$$Q_B = \sum_n c_n L_n + \sum_{m,n} \frac{m-n}{2} \alpha \hat{c}_m c_{\hat{c}_n} - c_0 + \text{(anti-holomorphic part)}.$$  \hspace{1cm} (A.3)

Ghost zero-modes:

$$b_0^+ \equiv b_0 + \bar{b}_0, \quad b_0^- \equiv \frac{1}{2} (b_0 - \bar{b}_0), \quad c_0^+ \equiv \frac{1}{2} (c_0 + \bar{c}_0), \quad c_0^- \equiv c_0 - \bar{c}_0.$$  \hspace{1cm} (A.4)

Normalization:

$$\langle 0 | c_{-1} \hat{c}_{-1} c_0^- c_1^+ | 0 \rangle = 1.$$  \hspace{1cm} (A.5)

Boundary state for Dp-brane (located at $x^i = 0$):

$$\langle B | = -\langle 0 | c_{-1} \hat{c}_{-1} c_0^+ \exp(2^{\phi} p(\hat{x}^i)) \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha_n^{\mu} \hat{\alpha}_m - \alpha_n^{\mu} \hat{\alpha}_m \right) - \sum_{n=1}^{\infty} \left( c_n \hat{b}_n + \hat{c}_n b_n \right) \right],$$  \hspace{1cm} (A.6)

where $\mu = 0, \ldots, p$ and $i = p+1, \ldots, 25$. This satisfies the following conditions:

$$\langle B | (\alpha_n^{\mu} + \hat{\alpha}_n^{\mu}) = \langle B | (\alpha_n^{\mu} - \hat{\alpha}_n^{\mu}) = \langle B | (c_n + \hat{c}_n) = \langle B | (c_n - \hat{c}_n) = 0.$$  \hspace{1cm} (A.7)

Jacobi identity:

$$(-1)^{AC}[A, [B, C]] + \text{cyclic perms} = 0.$$  \hspace{1cm} (A.8)

Inner products of string fields:

$$\langle A | c_0^- Q_B | B \rangle = (-1)^{(A+1)(B+1)} \langle B | c_0^+ A \rangle, \quad \langle A | c_0^- Q_B | B \rangle = (-1)^{AB} \langle B | c_0^- Q_B | A \rangle.$$  \hspace{1cm} (A.9)

Derivatives of $S_c^{\text{free}}$ and $S_D^{\text{int}}$:

$$\frac{\partial_r S_c^{\text{free}}}{\partial \psi^I} = (-1)^{r} \frac{\partial S_c^{\text{free}}}{\partial \psi^I} = \langle \Psi | c_0^- Q_B | \Psi^I \rangle = \langle \Psi^I | c_0^- Q_B | \Psi \rangle,$$  \hspace{1cm} (A.10)

$$\frac{\partial_r S_D^{\text{int}}}{\partial \lambda^i} = (-1)^{r} \frac{\partial S_D^{\text{int}}}{\partial \lambda^i} = i\kappa \langle B | \text{Sym} \left[ e^{2\kappa b_0} | O_i \rangle | \Psi \rangle.$$  \hspace{1cm} (A.11)
Component form of \(2.36\):

\[
\frac{\partial S_{\text{BFT}}}{\partial \lambda^i} = (-1)^i \frac{\partial r S_{\text{BFT}}}{\partial \lambda^i} = (-1)^{i+1} \langle B | \text{Sym} \left[ e^{2i\kappa [b_0^- , O]} ; O_i , [Q_B , O] \right] | 0 \rangle = \omega_0 \partial \lambda^i . \tag{A.12}
\]

Calculation in \(4.18\):

\[
(-1)^{j+1} \langle B | \text{Sym} \left[ e^{2i\kappa [b_0^- , O]} ; [Q_B , O] \right] | 0 \rangle \: \omega_0 \partial \lambda^i \: \langle B^- | \text{Sym} \left[ e^{2i\kappa [b_0^- , O]} ; O_j \right] | \Lambda^- \rangle = (-1)^{j+1} \langle B | \text{Sym} \left[ e^{2i\kappa [b_0^- , O]} ; O_k V^k, O_j \right] | 0 \rangle \: \omega_0 \partial \lambda^i \: \langle B^- | \text{Sym} \left[ e^{2i\kappa [b_0^- , O]} ; O_j \right] | \Lambda^- \rangle = (-1)^{j+k} V^k \omega_0 \partial \lambda^i \: \langle B^- | \text{Sym} \left[ e^{2i\kappa [b_0^- , O]} ; O_j \right] | \Lambda^- \rangle = \langle B^- | \text{Sym} \left[ e^{2i\kappa [b_0^- , O]} ; [Q_B , O] \right] | \Lambda^- \rangle . \tag{A.13}
\]

Properties of the anti-bracket:

\[
|\{S_1 , S_2\}| = |S_1| + |S_2| + 1 , \tag{A.14}
\]

\[
\{S_1 , S_2\} = -( -1)^{(S_1+1)(S_2+1)} \{S_2 , S_1\} , \tag{A.15}
\]

\[
( -1)^{(S_1+1)(S_2+1)} \{\{S_1 , S_2\} , S_3\} + \text{cyclic perms} = 0 . \tag{A.16}
\]

Completeness condition for closed string fields:

\[
|\Psi_J \rangle \omega_c^{IJ} (-1)^{J+1} \langle \Psi_J | = b_0^- \mathcal{P} , \tag{A.17}
\]

where \(\mathcal{P}\) is the projector for the level matching condition given in \(2.21\).

**Proof:** Any state \(|\Phi\rangle\) in the closed string Hilbert space can be uniquely decomposed as

\[
|\Phi\rangle = c_0 |\Psi\rangle + |\Psi_2\rangle + |\chi\rangle , \tag{A.18}
\]

where \(|\Psi\rangle\) and \(|\Psi_2\rangle\) are states annihilated by both \(b_0^-\) and \(L_0^-\), while \(|\chi\rangle\) is annihilated by \(\mathcal{P}\). Since \(\langle \Psi_J | = \langle \Psi_J | c_0 b_0^- \mathcal{P}\), by applying the l.h.s. of \(A.17\) on both sides of \(A.18\), we have

\[
(-1)^{J+1} |\Psi_J \rangle \omega_c^{IJ} \langle \Psi_J | \Phi\rangle = (-1)^{J+1} |\Psi_J \rangle \omega_c^{IJ} \langle \Psi_J | c_0 |\Psi\rangle = (-1)^{J+1} |\Psi_J \rangle \omega_c^{IJ} \langle \Psi_J | c_0 |\Psi_K\rangle \psi^K = |\Psi_I\rangle \omega_c^{IJ} \omega_{JK} |\psi^K\rangle = |\Psi_I\rangle |\psi^I\rangle = |\Psi\rangle . \tag{A.19}
\]

### B HIKKO closed string field theory

We summarize the basic properties of the HIKKO closed SFT \(5\) in CFT language. (See also \(28\).) Unlike the usual convention, we adopt a convention where string fields have ghost number 2. The explicit forms of the mappings \(a_r\) in \(3.14\) are given by

\[
h_r(w_r) = \rho^{-1}(f_r(w_r)) . \tag{B.1}
\]
Here $f_r$'s glue three unit disks (with coordinates $w_r$) together to construct the $\rho$-plane,
\[ f_r(w_r) = \alpha_r \log w_r + \tau_0 + i\beta_r, \quad (\beta_r = \pi \text{sgn}(\arg w_r) \sum_{i=1}^{r-1} \alpha_i) \] (B.2)
and $\rho^{-1}$ is the inverse of the Mandelstam mapping
\[ \rho(z) = \alpha_1 \log(z - z_1) + \alpha_2 \log(z - z_2) + \alpha_3 \log(z - z_3), \] (B.3)
where we can choose $z_1, z_2$ and $z_3$ arbitrarily. The interaction point $\tau_0$ is given by $\tau_0 = \Re \rho(z_0)$, where $z_0$ is the solution to $d\rho/dz = 0$. Using the conformal mappings above, we define the star product via (3.13) and (3.14).

The action of the HIKKO closed SFT is given by
\[ S = \frac{1}{2} \bar{\Psi} \cdot Q_B \Psi + \frac{\kappa^2}{3} \bar{\Psi} \cdot (\Psi \ast \Psi), \] (B.4)
where the dot product is defined by
\[ \Psi_1 \cdot \Psi_2 \equiv \langle \Psi_1 | c_0^* | \Psi_2 \rangle. \] (B.5)
The basic properties of the dot product and the star product are summarized as follows. Note that some sign factors are different from the conventional ones.

\[ \Psi_1 \cdot \Psi_2 = (-1)^{|i|+1}(\bar{\Psi}_2 \cdot \Psi_1), \] (B.6)

\[ (Q_B \Psi_1) \cdot \Psi_2 = (-1)^{|i|} \bar{\Psi}_2 \cdot (Q_B \Psi_1), \] (B.7)

\[ \Psi_1 \ast \Psi_2 = (-1)^{|i||2|} \bar{\Psi}_2 \ast \Psi_1, \] (B.8)

\[ Q_B(\Psi_1 \ast \Psi_2) + (Q_B \Psi_1) \ast \Psi_2 + (-1)^{|i|} \bar{\Psi}_1 \ast (Q_B \Psi_2) = 0, \] (B.9)

\[ (-1)^{|i||3|}(\bar{\Psi}_1 \ast \Psi_2) \ast \Psi_3 + (-1)^{|i||2|}(\bar{\Psi}_2 \ast \Psi_3) \ast \Psi_1 + (-1)^{|2||3|}(\bar{\Psi}_3 \ast \Psi_1) \ast \Psi_2 = 0, \] (B.10)

\[ \Psi_1 \cdot (\Psi_2 \ast \Psi_3) = (-1)^{|i||2||3|} (\bar{\Psi}_3 \cdot (\bar{\Psi}_1 \cdot \Psi_2)) = (-1)^{|2||3||1|} (\bar{\Psi}_2 \cdot (\bar{\Psi}_3 \ast \Psi_1)), \] (B.11)

where $|i|$ denotes the Grassmann parity of $\Psi_i$.

\section{C On the relation (3.32)}

We demonstrate the consistency of the relation (3.32) for off-shell boundary states. For simplicity, we put $\kappa = 1$. We first consider the off-shell boundary state in the presence of a constant tachyon $T$; $\langle B; \lambda_0 | \Psi(\alpha_1) \ast \Psi(\alpha_2) \rangle = \langle B | e^{-T} | \lambda_0 \rangle \equiv \langle B | e^{-T} \rangle$. We can factor out $e^{-T}$ from the off-shell boundary state as
\[ \langle B; \lambda_0 | c_0^* \Psi(\alpha_1) \ast \Psi(\alpha_2) \rangle = e^{-T} \langle B | c_0^* \Psi(\alpha_1) \ast \Psi(\alpha_2) \rangle, \] (C.1)
which can be evaluated by using the result for the undeformed boundary state:
\[ e^{-T}(-1)^{j+1} \langle B | O_j \Psi(\alpha_1) \rangle \omega_{ij} |_{\lambda=0} \langle B | O_j | \Psi(\alpha_2) \rangle \]
\[ = (-1)^{j+1} \langle B | \text{Sym}[e^{-T}; O_j] \Psi(\alpha_1) \rangle \omega_{ij} \langle B | \text{Sym}[e^{-T}; O_j] \Psi(\alpha_2) \rangle, \] (C.2)
where note that $\omega_{ij}^o$ is proportional to $e^T$. Therefore, though $\langle B; \lambda \rangle$ is not invariant under the conformal transformation, we found that $\langle B; 0 \rangle$ is correctly reproduced.

We next show that $\langle B; 0 \rangle$ is consistent with the properties of the HIKKO star product for any off-shell boundary state. Hereafter we use an abbreviated notation,

$$S_i \equiv {\cal S}^{\text{int}}_D \bigg|_{\Psi = \Psi_i(\alpha_i)} = \frac{1}{2} \langle B; \lambda | c_0^\dagger | \Psi_i(\alpha_i) \rangle . \quad (i = 1, 2, \ldots ) \quad (C.3)$$

where $\alpha_i$’s are taken to be negative. We now demonstrate that the properties of the HIKKO vertex $\langle B; 0 \rangle$ are reproduced, even if we use the r.h.s. of the following relation,

$$\langle B; \lambda | c_0^\dagger | \Psi_1(\alpha_1) * \Psi_2(\alpha_2) \rangle$$

$$= (-1)^{|\alpha_1|+|\alpha_2|} \langle B | \text{Sym}[e^{2i|\alpha_1|}; O_j] | \Psi_1(\alpha_1) \rangle \omega_{ij}^o \langle B | \text{Sym}[e^{2i|\alpha_1|}; O_j] | \Psi_2(\alpha_2) \rangle$$

$$= (-1)^{|\alpha_1|} \{ S_1, S_2 \}_o , \quad (C.4)$$

instead of the original HIKKO vertex. Here $(-1)^{|\alpha_1|}$ is the Grassmann parity of $|\Psi_1(\alpha_1)\rangle$, which arises when $|\Psi_1(\alpha_1)\rangle$ moves from next to $|\Psi_2(\alpha_2)\rangle$ to the position in the second line. Though closed string fields are Grassmann even, this expression is necessary for later analysis.

The commutativity $\langle B; 0 \rangle$ is obvious from $\langle A, {\cal F} \rangle$:

$$\langle B; \lambda | c_0^\dagger | \Psi_1 * \Psi_2 \rangle = \{ S_1, S_2 \}_o = \{ S_2, S_1 \}_o = \langle B; \lambda | c_0^\dagger | \Psi_2 * \Psi_1 \rangle , \quad (C.5)$$

where $|\Psi_i\rangle \equiv |\Psi_i(\alpha_i)\rangle$. The derivation law $\langle B; 0 \rangle$ is obtained as follows. Using $\langle B; 0 \rangle$ and the Jacobi identity $\langle A, {\cal F} \rangle$ as well, we have

$$\langle B; \lambda | c_0^\dagger Q_B | \Psi_1 * \Psi_2 \rangle = -\frac{\partial}{\partial \lambda} \langle B; \lambda | c_0^\dagger | \Psi_1 * \Psi_2 \rangle V^i$$

$$= -\frac{\partial}{\partial \lambda} \langle \{ S_1, S_2 \}_o \rangle \omega_{ij}^o \omega_{jk}^o V^k$$

$$= -\{ \{ S_1, S_2 \}_o, S_D^{\text{BSFT}} \}_o$$

$$= \{ \{ S_1, S_D^{\text{BSFT}} \}_o, S_2 \}_o + \{ \{ S_2, S_D^{\text{BSFT}} \}_o, S_1 \}_o . \quad (C.6)$$

The two terms are rewritten using

$$\{ S_1, S_D^{\text{BSFT}} \}_o = \frac{1}{2} \langle B; \lambda | c_0^\dagger Q_B | \Psi_i \rangle , \quad (C.7)$$

and then we find

$$\langle B; \lambda | c_0^\dagger Q_B | \Psi_1 * \Psi_2 \rangle + \langle B; \lambda | c_0^\dagger Q_B | \Psi_2 * \Psi_1 \rangle + \langle B; \lambda | c_0^\dagger | \Psi_1 * Q_B \Psi_2 \rangle = 0 . \quad (C.8)$$

In addition, as

$$\langle B; \lambda | c_0^\dagger | (\Psi_1 * \Psi_2) * \Psi_3 \rangle = -\{ \{ S_1, S_2 \}_o, S_3 \}_o , \quad (C.9)$$

the Jacobi identity $\langle A, {\cal F} \rangle$ again leads to

$$\langle B; \lambda | c_0^\dagger \left[ (\Psi_1 * \Psi_2) * \Psi_3 \right] + [(\Psi_2 * \Psi_3) * \Psi_1] + [(\Psi_3 * \Psi_1) * \Psi_2] \rangle = 0 . \quad (C.10)$$
which is consistent with (B.10). Thus we have correctly recovered the properties of the HIKKO star product (B.8)–(B.10) without referring to the original HIKKO vertex. It is important to note that, in both (C.8) and (C.10), the equalities are guaranteed by the Jacobi identity (A.16) among $S_D^{\text{int}}$ and $S_D^{\text{BSFT}}$. In other words, the form of the r.h.s. of (C.4) is crucial to reproduce the properties of the HIKKO vertex.

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