Global attractor and asymptotic dynamics in the Kuramoto model for coupled noisy phase oscillators

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Abstract

We study the dynamics of the large \(N\) limit of the Kuramoto model of coupled phase oscillators, subject to white noise. We introduce the notion of shadow inertial manifold and we prove their existence for this model, supporting the fact that the long-term dynamics of this model is finite dimensional. Following this, we prove that the global attractor of this model takes one of two forms. When coupling strength is below a critical value, the global attractor is a single equilibrium point corresponding to an incoherent state. Otherwise, when coupling strength is beyond this critical value, the global attractor is a two-dimensional disc composed of radial trajectories connecting a saddle-point equilibrium (the incoherent state) to an invariant closed curve of locally stable equilibria (partially synchronized state). Our analysis hinges, on the one hand, upon sharp existence and uniqueness results and their consequence for the existence of a global attractor, and, on the other hand, on the study of the dynamics in the vicinity of the incoherent and coherent (or synchronized) equilibria. We prove in particular nonlinear stability of each synchronized equilibrium, and normal hyperbolicity of the set of such equilibria. We explore mathematically and numerically several properties of the global attractor, in particular we discuss the limit of this attractor as noise intensity decreases to zero.

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1. Introduction

The Kuramoto model is a classical model of interacting phase oscillators, coupled through a mean-field term, widely used to investigate synchronization. It has been applied in various
fields such as physics, chemistry and biology (see review [1] and reference therein). In different contexts the same model is also referred to as the Sakaguchi model [25], the Smoluchowski model [7, 8, 27], or the mean-field classical spin XY model [24]. One remarkable feature of this model is that, for large populations, it undergoes a transition from an incoherent state to a (partially) synchronous one as the coupling strength between oscillators is increased. Many numerical and theoretical studies have analysed this transition under various hypotheses [25]. In this work, we are interested in the long-term dynamics of infinitely many interacting identical noisy phase oscillators. For such an infinite population, the following mean-field model can be derived from the microscopic description (see [4] and references therein)

\[
\begin{align*}
\frac{\partial q(t, \theta)}{\partial t} &= \frac{1}{2} \frac{\partial^2 q(t, \theta)}{\partial \theta^2} - K \frac{\partial q(t, \theta)}{\partial \theta} (J \ast q)(t, \theta) \\
q(0, \theta) &= q_0(\theta) \\
q(t, -\pi) &= q(t, \pi) \\
\frac{\partial q(t, -\pi)}{\partial \theta} &= \frac{\partial q(t, \pi)}{\partial \theta} 
\end{align*}
\]  

(1.1)

where

\[
J \ast q(t, \theta) = \int_{-\pi}^{\pi} \sin(\varphi - \theta) q(t, \varphi) d\varphi,
\]

(1.2)

\(K\) is a real constant representing the coupling strength \(q \geq 0\) such that \(\int_{-\pi}^{\pi} q(t, \theta) d\theta = 1\). The variable \(\theta \in [-\pi, \pi]\) accounts for the phase of the oscillators, and the unknown \(q(t, \theta)\) for the density of oscillators at phase \(\theta\) at time \(t\). We have assumed that oscillators are homogeneous, i.e. they all have the same intrinsic frequency \(\omega = 0\), and without loss of generality we have assumed that the intensity of the noise perturbing each oscillator is one.

For this system one encounters two distinct regimes depending on the ratio between the noise intensity and the coupling strength. When the coupling strength \(K\) is smaller than a critical value \(K_c\) the noise dominates, a uniform state is the only equilibrium of (1.1), and the population always tends to this incoherent state. When \(K > K_c\) instead the coupling dominates, a family of non-trivial coherent (or synchronized) equilibria exists, and the population tends to synchronize. When \(K > K_c\), for a different coupling potential \(\tilde{J} \ast q(t, \theta) = \int_{-\pi}^{\pi} \sin^2(\varphi - \theta) q(\theta) d\theta\), existence and uniqueness of a one-dimensional circle \(\mathcal{C} = \{f^* (+\varphi), \varphi \in [0, 2\pi]\}\), of non-trivial equilibria has been established in [18]. In a more general setting, estimates on the number of equilibria, and asymptotics in the large coupling limit \(K \rightarrow +\infty\), have been established in [6]. For equation (1.1) with (1.2), when \(K > K_c\), existence and uniqueness of a circle of non-trivial equilibria \(\mathcal{C}\) has been established in [8] (see also [4] and references therein). A spectral gap estimate and linear stability of trivial equilibria \(\tilde{q}(\cdot + \varphi)\) have also been shown in [4].

Several major advances have recently been made in the understanding of the global dynamics of Kuramoto models, in the case of deterministic oscillators and in the case of noisy oscillators. For an infinite number of identical oscillators subject to no noise, equivalent to equation (1.1) with no diffusion, Ott and Antonsen have shown thanks to a well chosen ansatz that a two-dimensional invariant manifold exists, and they have provided an explicit expression for the orbits on this manifold [21]. These orbits are the paths followed by the oscillator population during synchronization, as they describe the trajectories that connect the incoherent state to synchronized ones. Generalizing these ideas to finite-dimensional systems, Mirollo and Strogatz [20] have shown that regardless of the (finite) number of oscillators, the dynamics of the homogeneous Kuramoto model can be reduced to three dimensions, through an action of a three-dimensional Möbius group on the torus \(\mathbb{T}^N\). While these constitute important breakthroughs in our understanding of the dynamics of coupled phase oscillators, they do not deal with the situation where noise perturbs the dynamics of the units.
In the case of an infinite number of noisy identical oscillators (1.1), Vukadinovic has established in [27] the existence of finite-dimensional invariant exponential attractors (inertial manifolds) in the invariant subspace of symmetric solutions $q(t, \theta) = q(t, -\theta)$ of (1.1). Inertial manifolds existence theorems do not apply directly to equations of this form, but Vukadinovic has developed methods to show their existence for a Smoluchowski equation on the circle [27] and on the sphere [28], and for a Burgers equation [29] for example. Although the dynamics of (1.1) is infinite dimensional, inertial manifolds show that it has some typical properties of finite-dimensional systems.

In this work, we are interested in infinite populations of noisy coupled oscillators (1.1) and the finite-dimensional behaviour of their dynamics. This work is in line with previous findings and it extends them to give a complete rigorous description of the long-term dynamics of the Kuramoto model (1.1). This paper is organized so as to move progressively from general results concerning the solutions of equation (1.1) to more refined descriptions of the dynamics on its attractor and ending with a number of numerical investigations opening the way for some conjectures.

In section 2, we start with sharp existence and regularity results for solutions of (1.1). Existence and uniqueness of solutions for (1.1) in $L^2$ or Sobolev $H^s$ spaces is a classical result. Combining a result of [4] and a method of [7], we show that for any initial condition $q_0$ in a measure class the (unique) solution $q(t)$ of (1.1) is in a analytical functions space for all times $t > 0$. One consequence of the regularizing properties of equation (1.1) is that many convergence phenomena of solutions towards equilibria or invariant manifolds happen in an analytic function space and not just in the classical $L^2$ space.

In section 3, we prove the existence of shadow inertial manifolds for the Kuramoto model (1.1). Like inertial manifolds, which have been introduced to overcome some flaws of global attractors [12, 19], shadow inertial manifolds attract all solutions exponentially fast, the dynamics on a shadow inertial manifold is given by an ODE system, and each solution of the system has a phase on an asymptotically complete shadow inertial manifold. In that respect inertial manifolds and shadow inertial manifolds give finite dimensional and accurate reduction of the long-term behaviour of a dynamical system. Our proof uses original ideas of [27], while avoiding some of its technical difficulties.

These previous results establish essentially that the long-term dynamics of equation (1.1) can be captured by finite-dimensional ODEs. From this point on, we focus on more specific properties of its asymptotic dynamics. In section 4, we analyse the stability (or lack of stability) of equilibria. For readers’ convenience, we start by briefly recalling existing results for equilibria of equation (1.1) of [8, 4] mentioned above. Following this we perform local stability analyses of equilibria. We write linearization theorems at the incoherent equilibrium $q(\theta) = \frac{1}{2\pi}$ in a rigorous mathematical setting, and we show that they confirm what has been largely expected in the literature. This thorough analysis also gives global stability information and estimates of escape times when the incoherent equilibrium is unstable, and it will be a basic ingredient in the study on the global attractor in section 5. At the synchronized equilibria, an ad hoc Hilbert structure and associated spectral gap estimates have recently been found [4]. Thanks to these we show that, when the coupling strength is large enough, $K > K_c$, the family of synchronized equilibria is asymptotically stable. Due to rotational invariance of (1.1), this family forms a circle like closed invariant curve. We give an estimate of the phase-shifting effect of a small perturbation of a synchronized equilibrium on this invariant curve. Finally, another important consequence of these local stability results is the uniform normal hyperbolicity of the family of synchronized equilibria that is of interest on its own. More on this can be found in [14].
In section 5, we establish several properties of the global attractor of (1.1). The existence of a global attractor is a classical consequence of regularizing properties of equation (1.1), such as proved in [4, 5, 7]. This is briefly recalled in section 5.1. In section 5.2, relying on (in)stability analysis of equilibria in section 4, we show that, just as without the diffusion term, equation (1.1) possesses a two-dimensional invariant manifold. Interestingly, this invariant manifold coincides with the global attractor of the model, whereas in the case of equation (1.1) without diffusion, the global attractor, if it exists, cannot be reduced to the two dimensional invariant manifold of [21]: there are many unstable equilibria in the no-diffusion case that the global attractor would have to contain. Global attractors are classical tools to obtain a finite-dimensional reduction of the asymptotic behaviour of a large system [15], this means that Kuramoto model (1.1) can asymptotically be reduced to a simple two-dimensional dynamical system on a disc. We characterize the trajectories on this manifold as orbits connecting the incoherent state to a synchronized one and further provide an algorithm for their numerical computation. Several properties of the global attractor of (1.1), the dynamics on it and in its neighbourhoods are also discussed numerically in section 5.3. In the strong-coupling limit (or equivalently in the small-diffusion limit ) \( K \to +\infty \), we show that the global attractor of (1.1) converges formally to the two-dimensional invariant manifold found by Ott and Antonsen [21] in the no-diffusion case of equation (1.1) and supported by numerical evidence, we conjecture that this convergence holds in analytical function spaces.

In this way, we derive a full description at different levels of the long-term dynamics of the mean-field coupled Kuramoto model composed of infinitely many identical phase oscillators in the presence of noise.

2. Existence, uniqueness and regularity results

In the following we consider Sobolev spaces \( H^s(\mathbb{S}) = \{ q \in L^2(\mathbb{S}) : \sum_{n=1}^{+\infty} (1+k^2)(a_n^2 + b_n^2) < \infty \} \) for \( s \geq 0 \), and the Gevrey spaces \( G_a = \{ q \in L^2(\mathbb{S}) : \| q \|_{G_a} = \sum_{k=0}^{+\infty} a^k (a_n^2 + b_n^2) < +\infty \} \). When \( a > 1 \), the Gevrey space \( G_a \) is a subspace on the space of real analytic functions, for any \( k \) the natural injections \( G_a \to H^k \) are continuous and compact, and if \( 1 < a_1 < a_2 \) then the injection \( G_{a_1} \to G_{a_2} \) is continuous and compact. Note that in a different context Gevrey functions are usually between \( C^\infty \) regularity and analytic regularity, but for partial differential equations the Gevrey spaces \( G_a \) above are classical too [11].

If \( q(t, \theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=0}^{+\infty} x_n(t) \cos(n\theta) + y_n(t) \sin(n\theta) \) is a solution of (1.1) in \( C^1([0, +\infty[, L^2(\mathbb{S})) \), taking \( z_n = x_n + iy_n \) one finds that for all \( n \geq 1 \) and \( t \geq 0 \),

\[
z_n' = -\frac{n^2}{2} z_n + \frac{K}{2} [z_{n-1} - \bar{z}_{n+1}],
\]

where \( z_0(t) = 1 \) for all times. This ODE system is the key point to show that solutions are in a Gevrey space for any positive time, details of computations are similar to those of theorem 3.1 in [7] for the symmetric case \( y_n = 0 \) (\( \forall n \geq 0 \) and \( t \geq 0 \)). In [4], it was shown that for \( q_0 \) in a measure class, the solution \( q(t) \) of (1.1) is \( C^\infty([0, +\infty[ \times \mathbb{S}^1) \). Combining these, we can show that Gevrey regularization occurs in the general case \( (y_n \neq 0) \) too, and regardless of the coupling strength \( K \) the solution of the Kuramoto equation is in \( G_a \) for any \( a > 1 \) (for all \( t > 0 \)). This proof also inspires the following lemma 2.2, establishing that equation (1.1) defines a continuous semiflow \( L^2 \to G_a \), which will be used several times in the next sections.

**Theorem 2.1.** Let \( q_0 \) be a probability measure on \( \mathbb{S} \). There is a unique weak solution of the Kuramoto equation \( q_t \in C^0([0, +\infty[, M(\mathbb{S})) \), absolutely continuous with respect to the Lebesgue measure for any \( t > 0 \), and denoting its density \( q(t, \cdot) \) we have \( q(t, \cdot) > 0 \) for all \( t > 0 \) and \( \theta \in [0, 2\pi] \), we have \( q \in C^\infty([0, +\infty[ \times [0, 2\pi]) \) and \( q(t, \cdot) \in G_a \) for any constant
Asymptotic dynamics for the Kuramoto model

Furthermore, for any \( a > 1 \) there is a bounded absorbing set \( B_a \) in Gevrey space \( G_a \) for equation (1.1).

From now on we will denote by \( S_t \) the semiflow associated with equation (1.1). A bounded subset \( B_a \) is an absorbing set for a semiflow \( S_t \), defined in a metric space \( X \), if for any bounded set \( B \subset X \) the trajectories initiated in \( B \) enter \( B_a \) in finite time and remain in that set thereafter, i.e. \( \exists \delta \exists \delta \in B \forall t \geq t_0, x(t) \in B_a \). In theorem 2.1, \( B_a \) is a compact absorbing set for \( S_t \) in \( X = H^s \) for any \( s \in \mathbb{N} \) and in \( X = G_a' \) for any \( 1 < a' < a \).

The following lemma will be used several times in the following sections to show convergence results in Gevrey spaces. Its proof can be found in the appendix.

**Lemma 2.2 (Gevrey-convergence lemma).** For any positive time \( \epsilon > 0 \) and any constant \( a \geq 1 \), the semiflow \( S_{\epsilon} : L^2 \rightarrow G_a \) associated with equation (1.1) is Lipschitz continuous on bounded sets. In particular, if \( q \) and \( \tilde{q} \) are two solutions of equation (1.1) bounded in \( L^2 \), then for any \( \epsilon > 0 \), there is a Lipschitz constant \( L \) (depending only on \( \epsilon, \|q_0\|_{L^2} \text{ and } \|\tilde{q}_0\|_{L^2} \)) such that for any time \( t \geq 0 \),

\[
\|q(t+\epsilon) - \tilde{q}(t+\epsilon)\|_{G_a} \leq L \|q(t) - \tilde{q}(t)\|_{L^2}.
\]

(2.2)

### 3. Asymptotically complete shadow inertial manifolds

Inertial manifolds are finite dimensional, Lipschitz and positively invariant manifolds that exponentially attract solutions of evolution equations. There is an extensive literature on the existence of inertial manifolds and related invariant sets such as inertial sets for partial differential equations ([9, 10, 12, 22, 23]). However, these general results do not apply directly to nonlinear Fokker–Planck equations similar to the Kuramoto model. Indeed, existence theorems for inertial manifolds require a spectral gap condition on the spectrum of the linearized equation that FPEs such as the Kuramoto model do not satisfy. This hurdle notwithstanding, Vukadinovic made significant progress by establishing the existence of inertial manifolds for a number of nonlinear FPEs including the Kuramoto model [27–29]. For the latter, his result is stated in the case of even solutions. Our aim in this section is to show that a structure reminiscent of inertial manifolds exists for the Kuramoto model whether solutions are even or not. As some of the key steps of our proof are similar to Vukadinovic’s work, we briefly review his strategy. This short review also clarifies why we modify his approach in order to analyse the situation where initial data are not necessarily even functions.

For a linear Fokker–Planck equation of the form

\[
\frac{\partial}{\partial t} q = \frac{\partial^2}{\partial \theta^2} q + \frac{\partial}{\partial \theta}(q \frac{\partial}{\partial \theta}(V)),
\]

(3.1)
a standard method to obtain a regular reaction term is to set \( u = e^V q \), which gives

\[
\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial \theta^2} u + F[V]u.
\]

(3.2)

The Kuramoto equation (1.1) can be written in the form of (3.1), but is not linear because the ‘potential’ \( V(\theta) = -K \int_{-\pi}^{\pi} \cos(\theta - \phi)q(\phi) d\phi \) depends on \( q \). Nevertheless, in [27] Vukadinovic has shown that in the invariant subspace of even solutions of (1.1), the map \( q \mapsto e^V q \) is a bi-Lipschitz bijection onto its image, and so that equation (3.2) can be written in closed form in \( u \). The reaction term \( u \mapsto F[V(u)]u \) is then well defined \( H \rightarrow H \) for the appropriate Hilbert space \( H \), and standard theorems for the existence of inertial manifolds hold for (3.2). Using the inverse transform \( u = e^V q \mapsto q \), Vukadinovic was able to establish the first inertial manifold existence results for equations of the form (3.1). Proving that the transform \( q \mapsto u = e^V q \) is one to one, establishing its regularity and the regularity of its inverse are the most difficult and specific parts of [27].
This method cannot be readily extended to the non-symmetric case, because it is not clear whether the map \( q \mapsto V(q) \) is one to one for non-even \( q \in L^2 \). This motivates us to modify the approach to circumvent this question by embedding the system in a larger one where it is possible to prove the existence of inertial manifolds. This is sketched below with proofs left to the appendices. Prior to that we introduce the following definition of asymptotically complete shadow inertial manifold (AcSIM) and comment on it.

**Definition.** Suppose that there exists a compact absorbing set \( B_\delta \) in \( X \) for the semiflow \( S_t \). Let \( M \) be a (finite Hausdorff dimensional) subset of \( X \). The set \( M \) an asymptotically complete shadow inertial manifold (AcSIM) for \( S_t \) on \( X \) if

- there exists a smooth function \( \Phi : \mathbb{R}^n \to X \) such that \( M = \Phi(\mathbb{R}^n) \) is the graph of \( \Phi \),
- \( M \) attracts exponentially all trajectories of \( S_t \): there is a \( \delta > 0 \)
  \[
  d(S_t x_0, M) = O(e^{-\delta t}) \quad \forall x_0 \in X,
  \]
- there is a flow \( \tilde{S}_t \) on \( \mathbb{R}^n \) (associated with an ODE system) such that and for any \( x_0 \in X \), there is a unique phase \( \phi_0 \in \mathbb{R}^n \) such that \( d(S_t x_0, \Phi(\tilde{S}_t \phi_0)) = O(e^{-\delta t}) \).

An AcSIM gives a finite-dimensional reduction of the dynamics of \( S_t \), to which the trajectories converge exponentially fast, and from a practical point of view this reduction to \( M \) gives as much information as an inertial manifold would do. The flow \( \tilde{S}_t \) is a shadow inertial form. An AcSIM is an (asymptotically complete) inertial set as soon as it is positively invariant by \( S_t \). An AcSIM is a \( C^1 \) manifold as soon as the function \( \Phi \) is \( C^1 \) and injective, and if it is both a manifold and an inertial set, an AcSIM is a classical AcIM. The uniqueness of the phase \( \phi_0 \) implies that the trajectories \( \tilde{S}_t \) on \( M \) capture the slow dynamics of \( S_t \) on \( X \) (dynamics at speed less than \( O(e^{-\delta t}) \)). When they exist, inertial manifolds, inertial sets or approximate inertial manifolds contain the global attractor of a dynamical system. The same type of result holds for asymptotically complete shadow inertial manifolds.

We introduce now several notations and we sketch the main steps of proof of the existence of AcSIMs for the Kuramoto model. Naturally, several of these steps are similar to [27], notably when it comes to checking that the transformed system admits an inertial manifold. We consider the injection \( I : q \mapsto (q, x_1, y_1) \) where \( x_1 = \int_0^\delta q(\theta) \cos(\theta) \, d\theta \) and \( y_1 = \int_0^\delta q(\theta) \sin(\theta) \, d\theta \), and we denote the projection \( P : (q, a, b) \mapsto q \). We define the transform

\[
\mathcal{F} : \begin{pmatrix} q \\ a \\ b \end{pmatrix} \mapsto \mathcal{U} \begin{pmatrix} e^V q \\ a \\ b \end{pmatrix}
\]

where \( V = -\frac{K}{\pi} (a \cos(\theta) + b \sin(\theta)) \). We will also denote \( u = e^V q \). We will show that if \( q \) is a solution of equation (1.1), then \( \mathcal{U} = \mathcal{F} \circ I(q) \) is a solution of a reaction–diffusion problem

\[
\frac{d\mathcal{U}}{dt} + A\mathcal{U} = N(\mathcal{U}),
\]

where \( N \) is well defined \( H^s \times \mathbb{R}^2 \to H^s \times \mathbb{R}^2 \) for any \( s \geq 0 \). We will then check that equation (3.4) is well posed, that it has a bounded absorbing set, and that the eigenvalues of \( A \) satisfy the spectral gap condition ensuring the existence of inertial manifolds \( M^* \) for equation (3.4). We will denote by \( S'_t \) the semiflow generated by equation (3.4) in the following. Then, we will show that the transform \( \mathcal{F} \) is regular and bijective, and we will deduce the existence of an AcSIM for equation (1.1) from the existence of \( M^* \). More precisely, we will show the existence of a finite dimensional \( C^1 \) graph \( M \) which attracts exponentially any solution of
equation (1.1). For any initial condition \( q_0 \in H^s \), the corresponding solution \( S_t q_0 \) of (1.1) has (for any large enough \( t_0 \)) a unique phase \( v_0 \in M^* \), characterized by

\[
\| S_{t_0} q_0 - P \mathcal{F}^{-1} \tilde{S}_t v_0 \|_{H^s} = O(e^{-\eta t}),
\]

where \( \tilde{S}_t \) is the restriction of \( S_t^* \) to (the finite-dimensional manifold) \( M^* \). This graph \( M \) is an asymptotically complete shadow inertial manifold.

This strategy makes use of a new kind of attractor, i.e. AcSIM. It does not rely on the injectivity of the potential \( q \mapsto V \), which makes it simpler. It improves the function spaces in which exponential convergence is proved from \( L^2 \) to \( H^s \) for any \( s \geq 0 \), and it allows the definition of inertial form \( \tilde{S}_t = S_t^*|_{M^*} \) (the ODE system ruling the dynamics on \( M^* \)), which allows numerical finite-dimensional approximation of the dynamics of (1.1) in principle. We will also see that the transform \( q \mapsto (e^V q, x_1, y_1) \) is \( C^1 : H^s \to H^s \times \mathbb{R}^2 \) so that results of Rosa and Temam [22] may be applied to equation (3.4), and show that the inertial manifold \( M^* \) (and the dynamics on it) are \( C^1 \) and uniformly normally hyperbolic. The proof of this theorem is detailed in the appendix.

**Theorem 3.1.** For any \( s \in \mathbb{N} \), the set \( M = P \mathcal{F}^{-1}(M^*) \) is a \( C^1 \) finite dimensional graph in \( H^s \), there is a \( \eta > 0 \) and there is a flow \( \tilde{S}_t = S_t^*|_{M^*} \) such that for any \( q_0 \in H^s \) and \( t_0 \) large enough, there is a unique \( v_0 \in M^* \) such that

\[
\| S_{t_0} q_0 - P \mathcal{F}^{-1} \tilde{S}_t v_0 \|_{H^s} = O(e^{-\eta t}).
\]

We have in particular \( \text{dist}_{H^s}(S_{t_0} q_0, M) \leq \| S_{t_0} q_0 - P \mathcal{F}^{-1} \tilde{S}_t v_0 \|_{H^s} = O(e^{-\eta t}) \). The graph \( M \) is exponentially attractive for the flow \( S_t \) generated by equation (1.1), \( \tilde{S}_t v_0 \) is the phase associated with the solution \( S_t q_0 \) of (1.1), and \( P \mathcal{F}^{-1} \tilde{S}_t v_0 \) is its shadow on \( M \).

Furthermore, the set of equilibria of equation (3.4) is the image of the set of equilibria of (1.1) by the transform \( \mathcal{F} \circ I \), and the global attractor of equation (3.4) is the image of the global attractor of (1.1) by the transform \( \mathcal{F} \circ I \).

4. Equilibria of the system: stability analysis

4.1. On the stationary solutions to (1.1)

Up to a rotation all stationary solutions of (1.1) which are probability densities—we call them *equilibria*—can be written as

\[
\hat{q}(\theta) = ce^{2Kr \cos(\theta)},
\]

(4.1)

with \( r \) a solution of the equation

\[
r = \frac{I_1(2Kr)}{I_0(2Kr)}, \quad \text{with } I_i(s) = \frac{1}{2\pi} \int_0^{2\pi} (\cos(\theta))^i \exp(s \cos(\theta)) \, d\theta \quad \text{for } s \in \mathbb{R},
\]

(4.2)

and \( c \) such that \( \int_0^{2\pi} \hat{q}(\theta) \, d\theta = 1 \). We refer to [8, 27] for this, and to [4] for this and a review of the mathematical physics literature on this issue, and a number of side facts. See [18] for similar results in the case of equation (1.1) with \( J * q(t, \theta) = \int_{-\pi}^{\pi} \sin^2(\theta - \varphi) q(\varphi) \, d\varphi \), and see [6] for asymptotic estimates when \( K \to +\infty \) in a three-dimensional case similar to (1.1).

Equations (4.1) and (4.2) directly imply that the fixed point equation (4.2) can have at most three solutions: \( r = 0 \), which is always present, and gives \( \hat{q}(\cdot) = \frac{1}{2\pi} \): this corresponds to the incoherent state of the system; and two non-trivial solutions \( \pm r \) (we denote by \( r \) the positive solution) that exist if and only if when \( K > 1 \). If we set \( \hat{q}_\varphi(\theta) := \hat{q}(\theta + \varphi) \), where \( \hat{q} \) is given in (4.1) with \( r \) the non-trivial positive solution of (4.2), then \( \hat{q}_\varphi \) is an equilibrium for every choice of \( \varphi \), so that we have a circle \( \mathcal{C} = \{ \hat{q}_\varphi : \varphi \in \mathbb{S} \} \) of equilibria where each \( q_\varphi \) describes the (partially) synchronized state around the phase \( \varphi \).
Equation (1.1) can be written in a gradient-flow form \( \partial_t q = \partial_\theta(q \partial_\theta \delta F / \delta q(\theta)) \), where the functional \( F \) is not increasing along solutions of equation (1.1). The existence of a Lyapunov functional \( F \) gives information about the asymptotic behaviour of the system, and the stability of equilibria, see for example [2, 13].

In the following sections we will go beyond these results, by exploiting properties of the system linearized at the equilibria and by using invariant manifold theorems. Additionally, the following stability and unstability results will also be useful for the study of the global attractor (section 5). We do this separately for the incoherent stationary solution and for the synchronized solutions.

4.2. The incoherent equilibrium \( \frac{1}{2\pi} \)

When \( K \leq 1 \), the constant \( \frac{1}{2\pi} \) is the only stationary solution, and the existence of a Lyapunov functional implies that it is globally attractive.

**Proposition 4.1.** For any value of \( K \) and any initial condition \( q_0 \in \mathcal{M}(\mathbb{S}) \), the o-limit set \( \omega(q_0) = \cap_{T \geq 0} \mathcal{S}_T q_0, t \geq T \) is included in the set of equilibria of equation (1.1). In particular if \( K \leq 1 \), for any \( q_0 \in \mathcal{M}(\mathbb{S}) \) we have \( \| q(t) - \frac{1}{2\pi} \|_{\mathcal{G}_a} \rightarrow 0 \) as \( t \rightarrow +\infty \).

**Proof.** The convergence to \( \frac{1}{2\pi} \) is a consequence of both the existence of a Lyapunov functional, and the fact that \( \frac{1}{2\pi} \) is the unique equilibrium of (1.1). Lemma 2.2 implies that this convergence happens in Gevrey spaces \( \mathcal{G}_a \). \( \square \)

We now use invariant manifolds to describe precisely the dynamics around \( \frac{1}{2\pi} \) and to have estimates of the speed of convergence to \( \frac{1}{2\pi} \).

**Proposition 4.2.** When \( K < 1 \), there are constants \( \epsilon > 0 \) and \( M < \infty \) such that if \( \| q_0 - \frac{1}{2\pi} \|_{H^1} \leq \epsilon \) then the solution of equation (1.1) satisfies
\[
\| q(t) - \frac{1}{2\pi} \|_{\mathcal{G}_a} \leq M \| q_0 - \frac{1}{2\pi} \|_{H^1} e^{-\frac{1}{2(1-K)} t} \quad \text{for every } t \geq 0.
\]

Moreover, we have
\[
q(t) = \frac{1}{2\pi} + P(q_0) e^{-\frac{1}{2(1-K)} t} + \epsilon(t),
\]
with \( \| \epsilon(t) \|_{H^1} \leq C \| q_0 - \frac{1}{2\pi} \|_{H^1} e^{-\gamma t} \) for any \( \frac{1}{2(1-K)} < \gamma < 1 - K \) and \( C = C(\gamma) > 0 \), and \( P \) is continuous from a neighbourhood of \( \frac{1}{2\pi} \) in \( H^1 \) into \( E_{\frac{1-K}{2}} \) with the property \( P(q) = \Pi(q - \frac{1}{2\pi}) + \mathcal{O}(\| q - \frac{1}{2\pi} \|_{H^1}) \), where \( \Pi \) is the orthogonal projection onto \( E_{\frac{1-K}{2}} \) with kernel \( E_{\frac{1}{2}} \).

**Proof.** The linearized operator \( L_{\frac{1}{2\pi}} v = \frac{1}{2\pi} \partial_\theta^2 v - \frac{K}{2} J^* v \) is diagonal in the classical Fourier basis of \( L^2(\mathbb{S}) \), and self-adjoint with compact resolvent. Classical proof of local stable manifold existence (see [17] chapter 5 section 1 for example) can be adapted here, where there are two identical largest eigenvalues instead of one only. \( \square \)

When \( K = 1 \) a bifurcation occurs at \( \frac{1}{2\pi} \). While \( \frac{1}{2\pi} \) still attracts every initial condition (cf proposition 4.1), solutions do not approach \( \frac{1}{2\pi} \) exponentially fast. Nevertheless there is an exponentially attractive (in Gevrey norm) two-dimensional central manifold, along which a phase is defined: for any solution \( q(t) \) of equation (1.1), there is a solution on the central manifold \( \overline{q}(t) \)—the phase—such that \( q(t) \) converges to \( \overline{q}(t) \) exponentially fast.
Proposition 4.3. Suppose \( K = 1 \). Let \( k \geq 2 \), \( k \in \mathbb{N} \). There is a map \( \Psi \in C^2(\mathcal{O}, E_+) \rightarrow E_+ \) equipped with the \( H^k \) norm—with \( \Psi(0) = 0 \) and \( D\Psi(0) = 0 \), and a neighbourhood \( \mathcal{O} \) of the origin in \( E_{-\infty} \) such that the manifold
\[
\mathcal{M}_0 = \{ u_0 + \Psi(u_0) : u_0 \in \mathcal{O} \}
\] (4.5)
has the following properties:

(i) The central manifold \( \mathcal{M}_0 \) is locally invariant: if \( q(t) \) is a solution of equation (1.1) on \( [0, T] \) with \( q(0) \in \mathcal{O} \cap \mathcal{M}_0 \) and \( q(t) \in \mathcal{O} \) for all \( t \in [0, T] \), then \( q(t) \in \mathcal{M}_0 \) for all \( t \in [0, T] \).

(ii) \( \mathcal{M}_0 \) contains all solutions which stay close enough to \( \frac{1}{2\pi} \) for all time \( t \in \mathbb{R} \): if \( q(t) \) is a solution of equation (1.1) for \( t \in \mathbb{R} \) such that \( q(t) \in \mathcal{O} \) for all \( t \in \mathbb{R} \), then \( q(0) \in \mathcal{M}_0 \) and \( q(t) \in \mathcal{M}_0 \) for all \( t \in \mathbb{R} \).

(iii) \( \mathcal{M}_0 \) is invariant under reflections and rotations: if \( q \in \mathcal{M}_0 \) then \( \theta \mapsto q(-\theta) \) and \( \theta \mapsto q(\theta + \phi) \) (for any \( \phi \in [0, 2\pi] \)) are in \( \mathcal{M}_0 \).

(iv) \( \mathcal{M}_0 \) is locally attractive: for any \( q_0 \in \mathcal{M}(\mathcal{O}) \), there is \( \mathcal{O}_0 \in \mathcal{M}_0 \) and constants \( C, \delta > 0 \) such that
\[
\left\| S_t q_0 - S_t \mathcal{O}_0 \right\|_{\mathcal{C}_0} \leq C e^{-\delta t}.
\] (4.6)

Proof. The results (i) to (iv) are consequences of theorems 2.9, 3.13 and 3.22 in [16] and the Gevrey-convergence lemma (lemma 2.2). These theorems apply in the setting \( L^2 \) is linear continuous and the nonlinear part \( R \in C^2(H^2, H^1) \) where \( H^k \) are the classical Sobolev spaces.

When \( K > 1 \), the incoherent equilibrium at \( \frac{1}{2\pi} \) is no longer stable. Classical local stable and unstable invariant manifold theorems apply here, and due to the specific form of equation (1.1), we also have more precise results about invariant manifolds, particularly about how they stabilize.

Proposition 4.4. Suppose \( K > 1 \). The stable manifold at \( \frac{1}{2\pi} \) is
\[
W^s = W^s \left( \frac{1}{2\pi} \right) = \frac{1}{2\pi} + E_+ \left\{ q \in L^2 : q \geq 0, \int q(\theta) \, d\theta = 1, \int q(\theta) e^{i\theta} \, d\theta = 0 \right\}.
\] (4.7)

(i) For any \( q_0 \in W^s \), we have \( \| q(t) - \frac{1}{2\pi} \|_{\mathcal{C}_0} \leq C \| q_0 - \frac{1}{2\pi} \|_{L^2} e^{-\frac{K^2}{2\pi^2} t} \) for all \( t \geq 0 \).

(ii) Set \( \mathcal{O} = B_{L^2} \left( \frac{1}{2\pi}, \delta \right) \cap \{ \int q(\theta) \, d\theta = 1 \} \). If \( \delta < 1 - \frac{1}{K} \) and if a solution of (1.1) satisfies \( q(t) \in \mathcal{O} \) for all times \( t \geq 0 \), then \( q(0) \in W^s \).

(iii) More precisely, if for a \( \delta \in [0, 1 - 1/K] \) we have \( q(t) \in \mathcal{O} \) for every \( t \in [0, T] \), then, with \( q_0 = q(0) \), we have
\[
\left| \int q_0(\theta) e^{i\theta} \, d\theta \right| = \text{dist} (q_0, W^s) \leq \delta e^{-\frac{1}{2}[K(1-\delta)-1]T}.
\] (4.8)

or, conversely, if \( \left| \int q_0(\theta) e^{i\theta} \, d\theta \right| \geq \epsilon \) for a value of \( \epsilon \in (0, \delta) \), then there is a time \( t \leq T_{0, \epsilon} = \frac{2|\text{ln}(\epsilon)|}{K(1-\delta)-1} \), such that \( q(t) \notin B_{L^2} \left( \frac{1}{2\pi}, \delta \right) \).

Moreover, for any \( \alpha \in [0, \min(2, (K - 1)/2)] \), there are constants \( C \) and a neighbourhood \( \mathcal{O} \) of \( \frac{1}{2\pi} \) in \( H^1 \) such that there is a two-dimensional local unstable manifold \( W^u_{loc} \) Lipschitz continuous in \( H^1 \) with the properties

(iv) \( \frac{1}{2\pi} \in W^u \) and \( W^u \) has a tangent space at \( \frac{1}{2\pi} \) which is \( E_{-\infty} \).
(v) For any $u_0 \in W^u$, equation (1.1) has a solution for $t \in [\theta_0, 0]$ with $q(0) = \frac{1}{2\pi} + u_0$ such that
\[
\left\| q(t) - \frac{1}{2\pi} \right\|_{H^1} \leq C \left\| q(0) - \frac{1}{2\pi} \right\|_{H^1} e^{Kt},
\]
for all $t \leq 0$.

(vi) And for any $q_0 \in \tilde{O}$ with $\int q_0(\theta) \, d\theta = 1$, if there is a solution $q(t)$ of equation (1.1) which is defined on $[\theta_0, 0]$ and satisfies $q(t) \xrightarrow{t \to -\infty} \frac{1}{2\pi}$, then $q_0 \in W^u$.

The fact that $W^u(\frac{1}{2\pi})$ is two dimensional will in particular be used in section 5.2.

**Proof.** The elementary, but key point is to realize that the dynamics on $W^u(\frac{1}{2\pi})$ is linear: it is just $\partial_t u = \frac{1}{2\partial_t^2} q$. So (i) is a direct consequence of standard results on the linear heat equation or of lemma 2.2.

To prove that any solution staying in a neighbourhood of $\frac{1}{2\pi}$ is in the set $W^u$, that is (ii), we consider the Fourier ODE system (2.1) equivalent to equation (1.1). If we set $z_n = \rho_n e^{i\theta_n} \in \mathbb{C}$, we have
\[
\rho_n' = -n^2 \rho_n + \frac{Kn}{2} \rho_1 \left[ \rho_{n-1} \cos(\theta_1 + \theta_{n-1} - \theta_n) - \rho_{n+1} \cos(-\theta_1 + \theta_{n+1} - \theta_n) \right],
\]
\[
\rho_n e^{i\theta_n} = \frac{Kn}{2} \rho_1 \left[ \rho_{n-1} \sin(\theta_1 + \theta_{n-1} - \theta_n) - \rho_{n+1} \sin(-\theta_1 + \theta_{n+1} - \theta_n) \right].
\]

In particular, $\rho_1' = -\frac{1}{2} \rho_1 + \frac{K}{2} \rho_1 [1 - \rho_2 \cos(\theta_2 - 2\theta_1)]$. Now, observe that $q \in O$ implies $\rho_n \leq 1$ for all $n \geq 1$ and $\rho_2(q) \leq \delta$. So, if $q(t) \in O$ for all $t \geq T_0$, we have $\rho_1' \geq \frac{1}{2} (K(1 - \delta) - 1) \rho_1$ on $[T_0, +\infty[$. Since $\delta < 1 - \frac{1}{2}$ and $K > 1$ we have $\rho_1(t) \geq \rho_1(T_0) \exp(\epsilon t)$, with $\epsilon > 0$, so if $\rho_1(T_0) > 0$ we have that $\rho_1(t)$ grows arbitrarily large, which is absurd, so we must have $\rho_1(T_0) = 0$, that is $q(T_0) \in W^u$.

For what concerns (iii) we use once again that $q \in O$ implies $\rho_2 \leq \delta$. Thanks to the Hilbert structure, we also see that $\text{dist}(q, W^u) = \rho_1 = (x_1^2 + y_1^2)^{1/2}$. Now, suppose that $q(t) \in B(1/2\pi, \delta)$ for all $t \in [0, T]$; like before we obtain $\rho_1' \geq \frac{1}{2} (K(1 - \delta) - 1)$, and $\rho_1(t) \geq \rho_1(0) e^{-K(1-\delta)T}$. Therefore $\rho_1(0) \leq \rho_1(T_0) e^{-K(1-\delta)T}$. For the second statement in (iii) it suffices to choose $\rho_1(0) \geq \epsilon$ and to suppose that $q(t) \in B(1/2\pi, \delta)$ for all $t \in [0, T]$; this leads to a contradiction for $T$ sufficiently large.

Statements (iv) to (vi) are classical results on the existence of an unstable manifold in a neighbourhood of $\frac{1}{2\pi}$ (see for example [23]). See section 5 for further global results on the unstable manifold $W^u(\frac{1}{2\pi})$. \hfill \square

### 4.3. The non-trivial equilibria

We assume $K > 1$. Several results of [4] will be used in this section, which we briefly sum up here for the reader’s convenience. Recall (cf section 4.1) the notation(s) $\hat{q}(\theta) = \hat{q}_0(\theta) = \frac{1}{2\pi} e^{iK\theta}(\cos(\theta))$ for one of the non-trivial equilibria of (1.1) when $K > 1$. We consider the linearized operator $L_\theta$ at $\hat{q}$ $L_\theta v = \frac{1}{2} \partial_\theta^2 v - K \partial_\theta [(J \ast \hat{q}) v + (J \ast v) \hat{q}]$, and the scalar product $\langle \langle u, v \rangle \rangle = \int_{\mathbb{S}} \frac{u(\theta) v(\theta)}{\theta} \, d\theta$ defined for $u$ and $v$ such that there are a $U, V \in L^2(\mathbb{S}, \frac{1}{\theta} \, d\theta)$ with $U = u$ and $V = v$, and $\int_{\mathbb{S}} \frac{u(\theta) v(\theta)}{\theta} \, d\theta = \int_{\mathbb{S}} \frac{U(\theta) V(\theta)}{\theta} \, d\theta = 0$. The corresponding Hilbert space is denoted $H_\theta^{-1}$.

It is not difficult to see that $H_{\hat{q}^{-1}}$ is equivalent to $H^{-1}$, that is the space with weight one (see details on this issue in [14, section 2]).
Theorem 4.5 (cf [4]). The operator $L_\hat{q}$ is essentially self-adjoint in $H^{-1}_0(\hat{q})$. Its spectrum is pure point and lies in $]-\infty, \lambda_1[ \cup [0]$, where $\lambda_1 < 0$ and 0 is a simple eigenvalue of $L_\hat{q}$ with eigenvector $\delta_0\hat{q}$. Furthermore, for $u, v \in D(L_\hat{q})$, we have

$$\langle \langle L_\hat{q}u, v \rangle \rangle = \langle \langle u, L_\hat{q}v \rangle \rangle = \frac{1}{2} \int_\mathbb{S} \frac{uv}{q} \, d\theta + \int_\mathbb{S} v \cdot (J * u) \, d\theta,$$

and for $K$ fixed we have $D(L_\hat{q}^{1/2}) \approx L^2_0(\mathbb{S}) \approx L^2(\mathbb{S})$, meaning that the scalar products $\langle \langle L_\hat{q}u, v \rangle \rangle, \langle u, v \rangle = \int_\mathbb{S} \frac{uv}{q} \, d\theta$ and $(u, v) = \int_\mathbb{S} uv \, d\theta$ are equivalent on $R(L_\hat{q}) = \{ v \in L^2(\mathbb{S}) : \int_\mathbb{S} v \, d\theta = 0 \}$. Of course $\lambda_1 = \lambda_1(K)$ depends on $K$. In the following we will content ourselves with the fact that $\lambda_1(K) < 0$ for every $K > 1$, see [4] for a sharper explicit bound on $\lambda_1(K)$.

The next theorem shows that the only long-term effect of a small $L^2$-perturbation of any synchronized equilibria $\hat{q}_\psi \in \mathcal{C}$ is a small rotational shift.

Theorem 4.6. There is $\delta > 0$ such that if $q_0 \in L^2$ with $q_0 \geq 0$ and $\int_\mathbb{S} q_0(\theta) \, d\theta = 1$ and if there exists $\psi \in \mathfrak{S}$ such that $\|q_0 - \hat{q}_\psi\|_{L^2} \leq \delta$ then, for $q(t)$ the solution of equation (1.1) with $q(0) = q_0$, we have $\|q(t) - \hat{q}_\psi\|_{L^2} \leq \delta$ for all $t \geq 0$. Moreover, there exists a $\varphi_\infty \in \mathfrak{S}$ such that for any $0 < \beta < |\lambda_1|

$$\|q(t) - \hat{q}_\psi + \varphi_\infty\|_{L^2} = \mathcal{O}(e^{-\beta t}),$$

and we have

$$|\varphi_\infty - \varphi_0| = o \left( \|q_0 - \hat{q}_\psi\|_{L^2} \right) \quad \text{as} \quad \|q_0 - \hat{q}_\psi\|_{L^2} \to 0.$$

From the quoted theorem of [4] of from 4.6, we can deduce that $\mathcal{C}$ is an invariant and stable normally hyperbolic curve for (1.1). (see definition in [3] or [23] for instance).

Corollary 4.7. The smooth and invariant curve $\mathcal{C}$ is stable normally hyperbolic. More precisely, at each point $q_\psi \in \mathcal{C}$, we consider the splitting

$$L^2 = T_{q_\psi} \mathcal{C} \oplus N^\psi_{q_\psi},$$

with $T_{q_\psi} \mathcal{C} = \mathbb{R} \delta_0\hat{q}_\psi$ is the kernel of the linearized operator $L_{\hat{q}_\psi}$ at $\hat{q}_\psi$ and $N^\psi_{q_\psi}$ is its orthogonal with respect to the $H^{-1}_0(\hat{q}_\psi)$ scalar product ($N^\psi_{q_\psi}$ is also the range of the operator $L_{\hat{q}_\psi}$). This splitting depends continuously on $\varphi$ and there is a $T > 0$ such that

$$\|DS^{\psi^T}(\hat{q}_\psi).v\| \leq \|v\| \quad \text{for any} \quad v \in T_{q_\psi} \mathcal{C} \text{ and } n \in \mathbb{Z},$$

and

$$\|DS^{\psi^T}(\hat{q}_\psi).v\| \leq \frac{2}{10^n} \|v\| \quad \text{for any} \quad v \in N^\psi_{q_\psi}, \text{ and } n \geq 1.$$

The proof of the theorem is outlined here, details and the proof of the corollary are postponed to the appendix. We mention that using that the invariant curve $\mathcal{C}$ is in fact not only invariant but composed of equilibria only, its normal hyperbolicity can also be derived from the spectral gap result of [4] directly, see [14].

Proof. Thanks to rotation symmetry of equation (1.1), without loss of generality we can assume that $\psi = 0$, i.e. $\hat{q}_\psi = \hat{q}$ and $\|q_0 - \hat{q}\|_{L^2} < \delta$.

Lemma 4.8. Consider $u(t, \theta) = q(t, \theta) - \hat{q}(\theta)$ and $x_\psi(\theta) = \hat{q}_\psi(\theta) - \hat{q}(\theta) = \hat{q}(\theta + \varphi) - \hat{q}(\theta)$.

There is a neighbourhood $\mathcal{V}$ of 0 in $L^2$ and to smooth projections

$$\mathcal{V} \to [-\epsilon, \epsilon] \subset \mathbb{R} \quad \text{and} \quad \mathcal{V} \to R(L_\hat{q}) \subset L^2$$

$$u \mapsto \varphi(u) \quad \text{and} \quad u \mapsto y_u,$$
such that \( u = x_\nu(u) + y_u \) for all \( u \in \mathcal{V} \), and \( u + \hat{q} \) is a solution of equation (1.1) if and only if \( \varphi = \varphi(u) \) and \( y = y_u \) are solutions of
\[
\frac{d\varphi}{dt} = \Phi(\varphi, y) \quad \text{and} \quad \frac{dy}{dt} + A_2 y = g(\varphi, y),
\]
where
\[
\Phi(\varphi, y) = \frac{1}{\langle v, \xi \rangle} \left( \langle v, f(x_\nu + y) - f(x_\nu) \rangle \right),
\]
and
\[
g(\varphi, y) = f(x_\nu + y) - f(x_\nu) - \partial_x f(x_\nu) \Phi(\varphi, y).
\]
are smooth, with \( \Phi(0, 0) = 0 \) and \( D\Phi(0, 0) = 0 \), and \( A_2 \) is the restriction of \( L_q \) on \( R(L_q) \).

**Proof.** Details of proof of this lemma can be found in the appendix.

From this we can deduce a local stability result for the \( y \) variable. Let us introduce the natural Hilbert basis of eigenvectors \( e_k \in D(\hat{L}_q) \) so that \( L_q = \sum_{k \geq 0} \lambda_k \langle \cdot, e_k \rangle e_k \), with \( \lambda_0 = 0 \) and \( 0 > \lambda_1 \geq \lambda_n \geq \lambda_{n+1} \to -\infty \) for \( n \geq 1 \). We have \( A_2 = \sum_{k \geq 1} \lambda_k \langle \cdot, e_k \rangle e_k \) and \( e^t A_2 = \sum_{k \geq 1} e^{t \lambda_k} \langle \cdot, e_k \rangle e_k \), so that
\[
\|A_2^t e^{tA_2} v\|_{H^{-1}} \leq e^{(1-\epsilon)\lambda t} \max_k (|\lambda_k|^2 e^{-2\epsilon |\lambda_k| t}) \|v\|_{H^{-1}},
\]
for any \( \epsilon \in [0, 1] \) and from this we directly infer
\[
\|e^{tA_2} v\|_{D(A_2^*)} \leq \frac{C_{\alpha, \epsilon}}{t^\alpha} e^{(1-\epsilon)\lambda t} \|v\|_{H^{-1}}.
\]
for any \( t > 0 \) and \( \alpha > 0 \) (we have used \( \|v\|_{D(A_2^*)} \) for \( \|A_2^t v\|_{H^{-1}} \)).

**Lemma 4.9.** For every \( \beta \in [0, 1] \) there exist \( \rho_0 > 0 \) and \( M > 0 \) such that, if \( \|\varphi_0\| + \|y_0\|_{L^2} = \frac{1}{2} \rho \leq \frac{1}{2} \rho_0 \), then for all \( t \geq 0 \)
\[
|\varphi(t)| + \|y(t)\|_{L^2} \leq \rho.
\]
Moreover, for all \( t \geq 0 \) we have
\[
\|y(t)\|_{L^2} \leq 2 \|y_0\|_{L^2} e^{-\beta t}.
\]

**Proof.** Details of the proof of this lemma can be found in the appendix.

Since \( \Phi \) is smooth, we have \( \frac{d\varphi}{dt}(t) \leq C \|y_0\| e^{-\beta t} \), and \( \varphi(t) \) converges: \( |\varphi(t) - \varphi_\infty| = O(e^{-\beta t}) \). So we have
\[
\|y(t)\|_{L^2} + |\varphi(t) - \varphi_\infty| = O \left( e^{-\beta t} \right),
\]
which, combined with the Gevrey-convergence lemma, gives (4.11).

For (4.12) we argue instead that \( \frac{d\varphi}{dt} \leq 2\gamma(\rho)\|y_0\|_{L^2} e^{-\beta t} \) and that \( |\varphi_\infty| \) is bounded by \( 2\gamma(\rho)\|y_0\|_{L^2} \frac{1}{\beta} = O(\|y_0\|_{L^2}) \), for some \( \gamma(\rho) \) satisfying \( \gamma(\rho) \to 0 \) (see the proof of lemma 4.9 in the appendix), and \( o(|\varphi_0| + \|y_0\|) = o(||q_0 - \hat{q}||) \) since \( u \mapsto (\varphi, y) \) is Lipschitz and its inverse is Lipschitz too. This completes the proof of theorem 4.6.

\[\Box\]

5. The global attractor

This section is entirely devoted to the study of the global attractor of equation (1.1). It starts with general results and progressively moves to more refined descriptions of this set. The end of the section proposes conjectures supported by numerical investigations.
5.1. Existence

The existence of a global attractor is a classical consequence of the existence of a compact absorbing set, and regularizing properties of equation (1.1) imply its existence (see [4, 5, 7] for example). We state it here:

**Theorem 5.1.** There is a set \( A \) bounded in Gevrey space \( G_\alpha \), such that for any space \( E \) such that the injection \( G_\alpha \to E \) is compact, for any bounded \( B \) set in \( E \), we have

\[
\text{dist}(S_t B, A) \xrightarrow{t \to +\infty} 0
\]

where \( \text{dist} \) is measured in the natural \( E \) norm. This includes the cases \( E = L^2, E = H^s \) for any \( s > 0 \), \( E = C^k \) for any \( k \geq 1 \), and even \( E = G_\alpha \) if \( a' < a \).

When \( K \leq 1 \), the global attractor is \( \{ \frac{1}{\sqrt{7}} \} \). In the remainder of section 5, we discuss the global attractor in the case \( K > 1 \).

5.2. Description of the global attractor

Equation (1.1) is equivariant under \( O(2) \), and the subspace of even functions is invariant by the dynamics. However for \( q \in L^2(\mathbb{S}) \) there is in general no \( \alpha \in [0, 2\pi[ \) such that \( q(\cdot + \alpha) \) is even. More precisely, for \( q = \frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \sum x_k \cos(k\theta) + y_k \sin(k\theta) \), such an \( \alpha \) exists if and only if the quantities \( \frac{x_k}{\sqrt{x_k^2+y_k^2}} \) and \( \frac{y_k}{\sqrt{x_k^2+y_k^2}} \) are independent of \( k \). So the whole dynamics of (1.1) in \( L^2 \) are not captured by its restriction on the subspace of even solutions. Nevertheless, we show below that the dynamics on the global attractor \( A \) are characterized by its restriction on the subspace of even functions. In fact the global attractor is radial: it is composed of one even heteroclinic solution \( q_h \), with \( q_h(-\infty) = \frac{1}{\sqrt{7}} \) and \( q_h(+\infty) = \hat{q} \), and its rotations \( q_h(t, \cdot + \varphi) \).

**Proposition 5.2.** The two-dimensional unstable manifold \( W^u(\frac{1}{\sqrt{7}}) \) is the global attractor of the system.

The two-dimensional unstable manifold of the constant equilibrium \( \frac{1}{\sqrt{7}} \) consists in a family of heteroclinic orbits, each one connecting \( \frac{1}{\sqrt{7}} \) to a non-trivial equilibrium \( \hat{q}(\cdot + \varphi)(\varphi \in [0, 2\pi[) \). All these orbits are obtained by rotation from one heteroclinic connection in the even functions subspace. In particular, for any solution \( q \) in \( W^u(\frac{1}{\sqrt{7}}) \) of equation (1.1), there is an angle \( \varphi \) independent of the time \( t \), such that \( q(t, \cdot + \varphi) \) is even for any \( t \in ]-\infty, +\infty[ \).

**Proof.** In dissipative systems unstable manifolds of equilibria are included in the global attractor [15]. The converse inclusion is a rather general property, it relies essentially on the existence of a Lyapunov functional and on the fact that \( \frac{1}{\sqrt{7}} \) is the only equilibrium with unstable directions. It implies that the global attractor \( A \) is exactly the unstable manifold \( W^u(\frac{1}{\sqrt{7}}) \) here.

We show now that in the even subspace, the one-dimensional unstable manifold of the constant equilibrium \( \frac{1}{\sqrt{7}} \) consists of two heteroclinic orbits, connecting \( \frac{1}{\sqrt{7}} \) to \( \hat{q} \) and \( \hat{q}(\cdot + \pi) \). Up to the symmetry \( \theta \mapsto -\theta \) and time translations, with no loss of generality we can consider only one solution \( q(t) \) escaping from \( \frac{1}{\sqrt{7}} \) with \( x_1(t) > 0 \) when \( t \to -\infty \). The dissipativity and regularity properties imply that the trajectory \( \{q(t), \ t \in \mathbb{R}\} \) is compact and that \( q(t) \) converges to one of the three equilibria of the system \( \hat{q}, \ \frac{1}{\sqrt{7}} \) or \( \hat{q}(\cdot + \pi) \). One has the estimate \( |x_k(t)| \leq 1 \) for all Fourier coefficients along solutions of our equation, and we can easily see that \( x_1(t) > 0 \) for all time \( t \in \mathbb{R} \). From the Fourier ODEs system we see that \( \lim_{t \to +\infty} x_1(t) = 0 \) is absurd if \( x_1 \) is not 0 for all times. Hence the solution \( q(t) \) is not an homoclinic connection of \( \frac{1}{\sqrt{7}} \), and it is a heteroclinic solution. The \( O(2) \) invariance of equation (1.1) eventually implies the second part of proposition 5.2. \( \square \)
5.3. Global attractor and numerical explorations

As in the previous section, we denote \( q_h \) the heteroclinic solution of (1.1) such that \( q_h(t) \xrightarrow{t \to +\infty} \frac{1}{2} \) and \( q_h(t) \xrightarrow{t \to -\infty} \hat{q} \), where \( \hat{q} \) is the unique equilibrium of (1.1) with \( x_1 > 0 \) in the even functions subspace. Since the structure of the global attractor is radial, without loss of generality, we can restrict ourselves to working in the subspace of even functions.

This section is organized as a sketch of a proof of two conjectures, which are partially proved and partially supported by numerical evidence. The first one is that the global attractor is not only a two-dimensional manifold homeomorphic to a disc, but it is an analytical graph over the disc \( \{ x_1^2 + y_1^2 \leq 1 \} \) in \( L^2 \), and a recursive procedure for the Taylor series coefficient is given. The second conjecture is that this graph, which depends on the coupling constant \( K \), converges in analytic function space when \( K \to +\infty \) to Ott–Antonsen two-dimensional invariant manifold.

The global attractor is an analytic graph.  Consider the global attractor as a geometrical curve \( \mathcal{H} = [q_h(t), t \in ]-\infty, +\infty[ \) .

**Proposition 5.3.** There is a neighbourhood \( \mathcal{V} \) of \( \frac{1}{2\pi} \) in \( L^2 \) such that \( \mathcal{H} \cap \mathcal{V} \) is a graph on \( \text{Span} \{ \cos \} \). That is to say there is \( \epsilon > 0 \) such that

\[
\mathcal{H} \cap \mathcal{V} \rightarrow ]0, \epsilon[ \qquad q \mapsto x_1 = \int_0^{\pi} q(\theta) \cos(\theta) \, d\theta
\]

is a smooth bijection.

Moreover, if \( t \mapsto x_1(t) = \int_0^{\pi} q_h(t, \theta) \cos(\theta) \, d\theta \) is increasing on \([-\infty, +\infty[ \) (where \( q_h \) denotes the heteroclinic solution of (1.1)), then \( \mathcal{H} \) is globally a \( C^\infty \) graph on \( \text{Span} \{ \cos \} \).

**Proof.** By unstable manifold theorem at \( \frac{1}{2\pi} \), we know that \( \mathcal{H} \) is a \( C^\infty \)-graph over \( \mathbb{R} \cos \) in a neighbourhood of \( \frac{1}{2\pi} \) and there is a \( T_0 > -\infty \) such that \( t \mapsto x_1(t) \) is increasing on \([-\infty, T_0[ \) .

The first part of the proposition holds in particular for \( \mathcal{V} = \{ q \in L^2, \ 0 < \int_0^{\pi} q(\theta) \cos(\theta) \, d\theta < x_1(T_0) \} \). If we have \( T_0 = +\infty \), then \( t \mapsto x_1(t) \) is a bijection from \( \mathbb{R} \) to its image, and \( \mathcal{H} \) is globally a graph on \( \text{Span} \{ \cos \} \). \( \square \)

In the following we denote the graph function by \( \Phi_{\mathcal{H}} : \begin{cases} ]0, \epsilon[ \to \mathcal{H} \\ x_1 \mapsto q \end{cases} \) inverse of the bijection (5.2).

**Numerical result.** For all \( t \in \mathbb{R}, x_1^0(t) > 0 \). (see figure 1)

The property \( x_1^0(t) > 0 \) is numerically true for any \( n \geq 1 \) and any time \( t \in \mathbb{R} \) along the solution \( q_h \). All these curves have a unique inflection point, they are convex for large negative times \( t \) and concave when \( t \to +\infty \). Their limit, the Fourier coefficients of \( \hat{q} \), are given by the Bessel modified functions: \( \lim_{t \to +\infty} x_n(t) = \frac{L(2K\pi)}{L(2K\pi)} \) so that all of them tend to \( \lim_{x \to +\infty} \frac{L(x)}{L(x)} = 1 \) when \( K \to +\infty \), but for any finite \( K \) we have \( \lim_{t \to +\infty} x_n(t) \leq C \frac{1}{\pi} \) for some constants \( C > 0 \), \( a > 1 \) and all \( n \) large enough.

Recall that \( \int_{-\pi}^{\pi} \hat{q}(\theta) \cos(\theta) \, d\theta = r \) (see section (4.1)), so that \( \lim_{t \to +\infty} \int_{-\pi}^{\pi} q_h(\theta) \cos(t, \theta) \, d\theta = \lim_{t \to +\infty} x_1(t) = r \). If proved, the monotonicity of \( x_1 \) would imply the following.

**Conjecture.** The graph function \( \Phi_{\mathcal{H}} \) is \( C^\infty \{(0, r)\} \), and the global attractor \( W^{\Sigma} \) is a \( C^\infty \) graph on the disc \( \{(x_1, y_1), \ x_1^2 + y_1^2 < r \} \).
Figure 1. The heteroclinic solution $q_h$ is a graph on $x_1$. Plots of $x_1(t)$ along the heteroclinic solution $q_h$ for $K = 2.0$ (bottom) and $K = 4.0$ (top). These curves were obtained using the Fourier ODE system equivalent to (1.1), truncated with $N = 200$ equations, Euler scheme and $dt = 0.00001$. Numerics were checked using $N = 100$ and $dt = 0.0001$ and classical Runge–Kutta fourth-order scheme, the maximum difference between the two is of order $10^{-5}$. The first Fourier coefficient $x_1$ is increasing along the heteroclinic solution. The same holds for all the $x_n(t), n \geq 1$ (figures not shown).

With the notations of proposition 5.3, there are functions $\varphi_n \in C^\infty$ such that for all $x_1 \in [0, \epsilon]$, we have $x_n(t) = \int_{-\pi}^{\pi} q_h(t, \theta) \cos(n \theta) \, d\theta = \varphi_n(x_1(t))$. So for $t \in ]-\infty, T_0[$ equation (1.1) is equivalent to

$$x_1' = -\frac{1}{2}x_1 + \frac{K}{2}x_1(1 - \varphi_2(x_1)) \quad \text{and} \quad x_n = \varphi_n(x_1), \quad \forall n \geq 2. \tag{5.3}$$

**Proposition 5.4.** The functions $\varphi_n$ are $C^\infty$ in a neighbourhood of 0, their Taylor series are of the form

$$\varphi_n(x) = x^n \sum_{p \geq 0} \alpha_{n,2p} x^{2p},$$

where the coefficients $\alpha_{n,2p}$ can be computed recursively.

**Proof.** We use the notation $\varphi_1(x) = x$, so that $\alpha_{1,0} = 1$ and $\alpha_{1,p} = 0$ for all $p \geq 1$. We use the Fourier ODE system $x_n' = -\frac{K}{2}x_n + \frac{K}{2}x_1(x_{n-1} - x_{n+1}) \quad (\forall n \geq 1)$, equivalent to equation (1.1), and the Taylor expansion of the $\varphi_n$ to obtain relations between the $\alpha_{n,p}$. First one finds that $p < n$ implies $\varphi_n^{(p)} = 0$, so that $\varphi_n(x) = x^n \sum a_{n,p} x^{2p}$ and then all odd coefficients $a_{n,2p+1}$ are zero, so that the series has the form $\varphi_n(x) = x^n \sum_{p \geq 0} a_{n,2p} x^{2p}$.

Then one finds $a_{2,0} = \frac{K}{1+1}$ and $a_{n,0} = \frac{K}{K + (n-1)} a_{n-1,0} = \prod_{j=2}^{n-1} \frac{K}{K+j}$ and the recurrence relations

$$((1 + p)K + (1 - p))\alpha_{2,2p} = \frac{K}{2} \left( \sum_{i+j=p-1, i \geq 0, j \geq 0} (2 + 2j)\alpha_{2,2j}\alpha_{2,2i} \right) - K\alpha_{3,2(p-1)} \tag{5.4}$$

for $n = 2$ and all $p \geq 1$ and

$$((n + 2p)K + (n^2 - n - 2p))\alpha_{n,2p}$$

$$= K \left( \sum_{i+j=p-1} (n + 2j)\alpha_{n,2j}\alpha_{2,2i} \right) + Kn(\alpha_{n-1,2p} - \alpha_{n+1,2(p-1)}) \tag{5.5}$$
Numerical result.

For any sum of the Taylor series \( \sum l x_n \) are for all \( n \). The no-resonance conditions \((n + 2p) + \frac{n^2 - n - 2p}{K} \neq 0\) are satisfied as soon as \( K > 1 \).

We introduce the order \( \ll \) on \( \mathbb{N}^2 \): we have \((m_1, q_1) \ll (m_2, q_2)\) iff \((q_1 < q_2 \text{ or } q_1 = q_2 \text{ and } m_1 < m_2)\). This is the lexicographical order for the inverted couples \((q, m)\). One can check that for that order, the recursion relation defining \( \alpha_n, q\) uses only \( \alpha_{m, q}\) with \((m, q) \ll (n, 2p)\). There are no infinite strictly decreasing sequences in \( \mathbb{N}^2 \) for the order \( \ll \), and so one can solve recursively the relations defining \( \alpha_n, q\).

Consider \( q_n \) an heteroclinic solution of (1.1) and \( x_n(t) \) its Fourier coefficients. Since \( t \mapsto x_1(t) \) is increasing and bijective \( \mathbb{R} \to ]0, r[ \), we can define \( X_n(x) \) for all \( x \in ]0, r[ \) by \( X_n(x_1(t)) = x_n(x_1^{-1}(x_1(t))) \) for all \( t \in \mathbb{R} \).

**Numerical result.** For any \( K \) and any \( n \), the Taylor series defining \( \phi_n \) has a positive radius, which is \( r \) (see figure 2). For any \( K \) and any \( n \), the sum of the Taylor series defining \( \phi_n \) equals \( X_n \) on \( ]0, r[ \) (see figure 3).

For a series \( \sum a_p x^p \), the convergence radius \( R \) is given by \( R^{-1} = \limsup_{p \to +\infty} |a_p|^\frac{1}{l} \). In figure 2, we see that \( r_{n, p} = \log((\alpha_{n, 2p} R^2)) \to l > 0 \), where \( l > 0 \) is a real number depending on \( K \) but not on \( n \). That is to say all series \( \sum \alpha_{n, 2p} x^2p \) have a positive radius \( R = e^{-l/2} \), which depends on \( K \) but does not depend on \( n \). We also observe that for all \( K \), the limit is \( l \approx -2 \log(r) \), suggesting that the radius \( R = r \). Since the Taylor series of \( \phi_n \) are \( x^n \sum \alpha_{n, 2p} x^{2p} \) the same results hold for them. We have also checked that d’Alembert’s ratio test is numerically consistent with this: for all \( n \geq 2 \) and \( K > 1 \) we have \( \frac{\alpha_{n+2p, q}}{\alpha_{n, q}} \to \frac{l}{\pi} \), and \( l \approx \frac{1}{\pi} \) suggesting that \( R = r \) too (figures not shown). Figure 3 shows that the truncated sum of the Taylor series \( \phi_n \approx \sum a_{n, 2p} x^{2p} \) equals \( X_n \) on \( ]0, r[ \). Note that the verification \( \sum a_{n, 2p} x^{2p} = X_n(x) \) is necessary since some functions, like \( x \mapsto e^{-x^2} \), have converging Taylor series without being analytical.
The sum $\sum p \alpha_n x_n^2 + 2 p x_n$ coincides with the function $x_n(x_1)$. Plots of functions $y_2 = \varphi_2(x)$ and $y(x) = x_2(x_1, x_2)$. The sum of the series was computed with $N = 200$ terms, and the drawn curve $\{(x, y(x)), x \in [0, r]\}$ was computed with the Fourier ODE system with $N = 200$ equations, Euler scheme and $dt = 0.001$. The numerical difference between the two is less than $10^{-5}$ for $x_1 \in [0, r]$. The equilibrium $\tilde{q}$ is at $x = r$, characterized by $\varphi_2(r) = 1 - \frac{1}{2} = \frac{1}{2}$ here. The sum of the Taylor series coincides with the function $x_n(x_1)$ along the heteroclinic solution $q_h$ for all $n \geq 2$. In particular, this heteroclinic solution seen to have a graph on $x_1$ is an analytic function on $[0, r]$. The previous numerical observations suggest the following. The graph function satisfies

$$\Phi_q(x) = 2\pi \sum_{n \geq 1} \varphi_n(x) \cos(n \cdot) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n \geq 1} \left( x^n \sum_{p=1}^{+\infty} \alpha_{n,2p} x^{2p} \right) \cos(n \cdot),$$

and it is analytical on $[0, r]$ in the sense that all $\varphi_n$ are.

**Conjecture.** The global attractor $\mathcal{H}$ is an analytical graph on the disc $\{(x_1, y_1), x_1^2 + y_1^2 < r\}$.

These Taylor series can also be used to estimate the eigenvalues at the incoherent equilibrium $\frac{1}{r}$ and at the synchronized equilibrium $\tilde{q}$. At $\frac{1}{r}$ one can analytically compute the eigenvalues, and we have checked that the numerical estimates given by the Taylor series are accurate. At $\tilde{q}$, the eigenvalues can also be computed by linearizing the truncated Fourier ODE system (2.1), and we have checked that the two methods agree there too.

The recurrence equations (5.4) and (5.5) can actually be used to compute recursively the Taylor coefficients $\alpha_{n,2p}$, and so one can compute directly the heteroclinic solution $q_n(x_1) = \frac{1}{2\pi} + \frac{1}{\pi} \sum \varphi_n(x_1) \cos(n \theta)$ and the whole global attractor with arbitrary precision without discretizing the partial differential equation (1.1) or the equivalent infinite Fourier system (2.1).

**Convergence to Ott–Antonsen ansatz for large $K$.**

**Proposition 5.5.** Considering the Taylor series $\varphi_n(x) = x^n \sum_{p=0}^{+\infty} \alpha_{n,2p} x^{2p}$, we have $\alpha_{n,0} \rightarrow 1$ and $\alpha_{n,2p} \rightarrow 0$ for all $n \geq 2$ and all $p \geq 1$.

**Proof.** The formulae $\alpha_{n,0} = \prod_{j=0}^{n-1} \frac{K}{K+j}$ found above show that $\alpha_{n,0} \rightarrow 1$ when $K \rightarrow +\infty$. 

![Figure 3. The sum $\sum p \alpha_n x_n^2 + 2 p x_n$ coincides with the function $x_n(x_1)$. Plots of functions $y = \varphi_2(x)$ and $y(x) = x_2(x_1, x_2)$. The sum of the series was computed with $N = 200$ terms, and the drawn curve $\{(x, y(x)), x \in [0, r]\}$ was computed with the Fourier ODE system with $N = 200$ equations, Euler scheme and $dt = 0.001$. The numerical difference between the two is less than $10^{-5}$ for $x_1 \in [0, r]$. The equilibrium $\tilde{q}$ is at $x = r$, characterized by $\varphi_2(r) = 1 - \frac{1}{2} = \frac{1}{2}$ here. The sum of the Taylor series coincides with the function $x_n(x_1)$ along the heteroclinic solution $q_h$ for all $n \geq 2$. In particular, this heteroclinic solution seen to have a graph on $x_1$ is an analytic function on $[0, r]$.](image-url)
When \( p = 1 \) we have
\[
\left( 1 + \frac{2}{n} + \frac{n^2 - n - 2}{Kn} \right) \alpha_n,2 = \alpha_n,0 \alpha_2,0 + (\alpha_{n-1,2} - \alpha_{n+1,0})
\]  
(5.6)
and knowing \( \alpha_{n0} \to 1 \), the recursion hypothesis \( \alpha_{n,2} \to 0 \) for all \( n \leq N \) implies \( \alpha_{N+1,2} \to 0 \). One can directly check that \( \alpha_{2,2} \to 0 \), and so deduce \( \alpha_{n,2} \to 0 \) for all \( n \geq 2 \).

When \( p \geq 2 \) (and \( n \geq 2 \)), using what we know about the case \( p = 0 \) and \( p = 1 \), and the relation
\[
\left( 1 + \frac{2}{n} + \frac{n^2 - n - 2}{Kn} \right) \alpha_n,2p = \left( \sum_{i+j=p-1} \left( 1 + \frac{j}{n} \right) \alpha_{i,j} \alpha_{2,i+j} \right) + (\alpha_{n-1,2p} - \alpha_{n+1,2p-1}),
\]
(5.7)
we see that the recursion hypothesis
\[
\forall (m, 2q) \ll (n, 2p) \quad \text{with} \quad q \geq 1, \quad \alpha_{m,2q} \to 0
\]
implies \( \alpha_{n,2p} \to 0 \) and so this is true for all \( p \geq 2 \) and \( n \geq 2 \) by the induction principle. \( \square \)

In the no-diffusion case of equation (1.1), the graph \( H_{OA} = \{ q_{OA}(x), x \in [0, 1] \} \subset L^2 \), where \( q_{OA}(x) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} x^n \cos(n\theta) \) (\( x \in [0, 1] \)), is an invariant manifold for the dynamics (see [21]). In that case, the two-dimensional invariant manifold of Ott and Antonsen is obtained from \( q_{OA} \) by rotations
\[
\mathcal{M}_{OA} = \left\{ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} x^n \cos(n(\theta + \alpha)) \middle| x \in [0, 1], \alpha \in [0, 2\pi] \right\} \subset L^2.
\]

Proposition 5.5 implies that \( \psi_n(x) \to x^n \) and that \( \Phi_{H} \to q_{OA} \) formally.

**Proposition 5.6.** Assume that for all \( K \) large enough, we have \( x_1(t) > 0 \) for all \( t \in ]-\infty, +\infty[ \). Consider \( P_N : L^2 \to \text{Span}\{\cos(k\theta), k \leq N\} \) the natural \( L^2 \)-orthogonal projection. For all \( z \in [0, 1] \) and \( N < +\infty \), we have
\[
P_N \Phi_H(z) \to P_N q_{OA}(z)
\]
with convergence in \( G_{a} \) for any \( a > 0 \).

**Proof.** Recall that (see section 4.1) \( r = \int_{-\pi}^{\pi} \hat{q}(\theta) \cos(\theta) \, d\theta = \lim_{K \to +\infty} x_1(t) \to 1 \). For any \( K \) large enough, we have \( x_1 : \mathbb{R} \to [0, 1] \) is a bijection. For any \( a \in [0, 1] \) and \( K \) large enough, we have \( 0 < a < r \), and there is a unique \( t_{a,K} \in ]-\infty, +\infty[ \) such that \( \int_{-\pi}^{\pi} q_{t_{a,K}, \theta}(\cos(\theta) \, d\theta = x_1(t_{a,K}) = a \). The previous proposition implies that
\[
P_N \Phi_H(z) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{N} \left( \sum_{0 \leq n, 2p \leq N} \alpha_{n,2p} \zeta^{2np} \right) \cos(n-\frac{1}{2\pi}) \xrightarrow[K \to +\infty]{} \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{N} \zeta^n \cos(n\cdot),
\]
uniformly on \( [0, 2\pi] \). The degree of \( P_N \Phi_H(z) \) (trigonometric polynomial) is independent of \( K \), and so convergence occurs in any \( G_{a} \), \( a > 0 \). \( \square \)

We recall now a classical property of analytic functions.

**Proposition 5.7.** Let \( f_K (K \in \mathbb{N}) \) be a sequence of analytic functions with radius \( R_K > a \) for some \( a > 0 \) independent of \( K \). Suppose that \( g \) is analytic with radius \( R_g > a \) and \( f_K \to g \) uniformly on \( [0, a] \). Then we have \( f_K \to g \) in analytic function space, and in particular for any \( l \geq 0 \), \( \frac{d^l}{dx^l} f_K \to \frac{d^l}{dx^l} g \) uniformly on \( [0, a] \).
Asymptotic dynamics for the Kuramoto model

Figure 4. Uniform convergence $\varphi_n(x) \xrightarrow{K \to +\infty} x^n$. Plots of functions $\varphi_n^2(x)$ for $x \in [0, r]$, for different values of $K$: $K = 1.5$ (a), $K = 2.0$ (b), $K = 4.0$ (c) and $K = 16.0$ (d). All values were computed with $N = 100$ terms in the entire series defining $\varphi_n$. The topmost curve in all plots is $y = x^2$. The function $\varphi_n^2$ converges to the limit $y = x^2$ when $K \to +\infty$, uniformly on any closed set of $[0, 1]$. The difference $x^n - \varphi_n(x)$ is increasing for $x \in [0, r]$, so that $\sup_{0 \leq x \leq r} |x^n - \varphi_n(x)| = r^n - \varphi_n(r) = (\int q(\theta) \cos(\theta) \, d\theta)^n - \int q(\theta) \cos(n\theta) \, d\theta$. One has $r = 1 - \frac{1}{2K} + O(\frac{1}{K^2})$ and $\varphi_n(r) = \frac{\sqrt{2\pi}}{\sqrt{x^*(r)}} = 1 - \frac{x^2}{2K} + O(\frac{1}{K^2})$. Thus one expects a convergence towards the Ott and Antonsen two-dimensional invariant manifold at speed $\frac{1}{K}$.

As explained above, figures 2 and 3 suggest that $\varphi_n$ are analytical for any $n$, and their Taylor series $\sum_n \alpha_n x^n$ have the same radius $R = r \xrightarrow{K \to +\infty} 1$.

Numerical result. For any $n$ we have $\varphi_n(x) \xrightarrow{K \to +\infty} x^n$ uniformly on any compact subset of $[0, 1]$ (see figure 4).

Figure 4 shows convergence $\varphi_n \to x^n$ uniformly on any compact of $[0, 1]$ when $K \to +\infty$. The difference $\varphi_n - x_n$ is increasing on $[0, r]$ (for any $K$ and $n$), and at $x = r$ we have $\lim_{K \to +\infty} \varphi_n(r) - r^n = 0$, which shows that the convergence is uniform. Furthermore we expect theoretically $|\varphi_n(r) - r^n| = \frac{c}{K} + o(\frac{1}{K})$ when $K \to 0$, and this is confirmed numerically on figure 4 too, suggesting that the convergence happens with speed $\frac{1}{K}$.

Assuming that these numerical observations are true, we have

$$
\Phi_N(z) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^N \left( \sum_{0 \leq n + 2p \leq +\infty} \alpha_{n, 2p} z^{2p} \right) \cos(nz).
$$
with \( \sum_{0 \leq n \leq 2^p \leq +\infty} \alpha_n 2^p \zeta^{2^p} \rightarrow 1 \) uniformly on any compact subset of \([0, 1]\) (for any \( n \)), and
\[ | \sum_{0 \leq n \leq 2^p \leq +\infty} \alpha_n 2^p \zeta^{2^p} | \leq 1 \text{ for all } n, K \text{ and } z \in [0, r]. \]
This would imply \(| \varphi_n(z) | \leq \zeta^n \) (for all \( n \geq 1 \) and \( z \in [0, r] \)), and \( \Phi_n(z) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^N \varphi_n(z) \cos(n\cdot) \) is in Gevrey space \( G_\alpha \) uniformly on \( K \), for all \( z \leq \alpha \) and \( a < \frac{1}{\alpha} \).

**Conjecture.** For any \( \alpha \in (0, 1] \) and any \( z \in (0, \alpha] \), we have
\[ \Phi_n(z) \rightarrow q(t_A(z)) \quad \text{in } G_\alpha \quad \forall \alpha < \frac{1}{\alpha}. \]

The two-dimensional global attractor \( W^u(\frac{1}{2\pi}) \) of (1.1) converges to the two-dimensional invariant manifold \( M_{0A} \) of the Kuramoto equation with no diffusion, in analytic functions space, with speed \( \frac{1}{K} \) when \( K \rightarrow +\infty \).

The dynamics on \( \mathcal{H} \) are given by \( x_1' = -\frac{1}{K} x_1 + K x_1 (1 - \varphi_2(x_1)) \). Considering \( \tilde{x}_1(t) = x_1(\frac{t}{K}) \), in the limit \( K \rightarrow +\infty \), we find \( \tilde{x}_1' = x_1(1 - \tilde{x}_1^2) \), which is the equation ruling the dynamics on \( M_{0A} \) in the no diffusion case.

**Conjecture.** Appropriately time-rescaled by a factor \( \frac{1}{K} \), the solutions of equation (1.1) on \( W^u(\frac{1}{2\pi}) \) converge to solutions of the no-diffusion case on \( M_{0A} \) in Gevrey space on all intervals \([-\infty, T] \), with speed \( \frac{1}{K} \).

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**Appendix**

**Proof of lemma 2.2**

We will use the classical fact that for any positive time \( \eta > 0 \), the semiflow \( S_\eta \) of equation (1.1) is well defined \( L^2 \rightarrow L^2 \) and Lipschitz continuous on bounded sets, and the Lipschitz constant can be chosen uniformly for \( \eta \) in bounded intervals \([0, \epsilon] \) (see [17] for example).

Consider \( q_0 \) and \( q_0 \) in \( L^2 \), and \( q(t) \) and \( \tilde{q}(t) \) the associated solutions of equation (1.1). The equation for the difference \( w = q - \tilde{q} \) is linear:
\[ \dot{w} = \frac{1}{2} \partial_\theta (Fw) - K \partial_\theta (Fw) - K \partial_\theta ((J \ast w) \tilde{q}), \]
where \( F(t) = J \ast q(t) = x_1(t) \cos(\theta) - y_1(t) \sin(\theta) \).

Noting \( w = \frac{1}{2} \sum (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)) \), \( \gamma_\eta = \alpha_n + i \beta_n \), \( z_\eta = x_1 + iy_1 \) and \( \tilde{z}_\eta = \tilde{x}_1 + iy_1 \), the previous equation implies the Fourier system:
\[ \gamma_\eta' = -\frac{n^2}{2} \gamma_\eta + \frac{Kn}{2} (z_\eta \gamma_{\eta+1} - \tilde{z}_\eta \gamma_{\eta+1}) + \frac{Kn}{2} (\gamma_\eta \zeta_{\eta+1} - \gamma_{\eta+1} \zeta_{\eta+1}) \quad (A.2) \]
with \( \gamma_\eta = 0 \) for all \( \eta > 0 \) and taking \( A_n = z_\eta \gamma_{\eta+1} + \tilde{z}_\eta \gamma_{\eta+1} - 1 \) and \( B_n = \gamma_\eta \zeta_{\eta+1} - \gamma_{\eta+1} \zeta_{\eta+1} \) we have (similarly to the previous proof)
\[ \frac{d}{dt} |\gamma_\eta|^2 = -n^2 |\gamma_\eta| + \frac{Kn}{2} (A_n - A_{n+1}) + \frac{Kn}{2} (B_n - B_{n+1}). \quad (A.3) \]

Note that for any time \( \eta \geq 0 \) and for any \( n \geq 1 \),
\[ |\gamma_\eta(\eta)| \leq \|w(\eta)\|_{L^2} \leq \|q(\eta) - \tilde{q}(\eta)\|_{L^2} \leq \text{Lip}(S_\eta) \|q_0 - \tilde{q}_0\|_{L^2}. \quad (A.4) \]
so for \( f(n) = a^{n \min(t,1)} \), using \(|z_1| \leq 2\), we have
\[
\frac{d}{dt} \left( \sum_{n=N}^{M} \frac{f(n)}{n} \right) + \left( 1 - \frac{2 \log(a)}{N} - \frac{C}{\sqrt{N(N+1)}} \right) \sum_{n=N}^{M} \frac{nf(n)}{n} \leq C f(N) |\gamma_n| |\gamma_{n-1}| + C |\gamma_1| \sum_{n=N+1}^{M} f(n) |\gamma_n| |\zeta_{n-1}| \quad (A.5)
\]
\[
\leq C(t) \|q_0 - \tilde{q}_0\|_{L^2}^2 + \frac{C}{N(N+1)} \sum_{n=N}^{M} nf(n) |\gamma_n| |\zeta_{n-1}| \quad (A.6)
\]
\[
\leq C(t) \|q_0 - \tilde{q}_0\|_{L^2}^2 + \frac{C}{N(N+1)} \sum_{n=N}^{M} nf(n) |\gamma_n| |\zeta_{n-1}| \quad (A.7)
\]
\[
\leq C(t) \|q_0 - \tilde{q}_0\|_{L^2}^2 + \frac{C}{N(N+1)} \sum_{n=N}^{M} nf(n) |\gamma_n| |\zeta_{n-1}| + C(t) \|q_0 - \tilde{q}_0\|_{L^2}^2 \quad (A.8)
\]

From theorem 2.1, we have that \( |\hat{q}(t)|_{G_{\text{trans}}(t)} \) is bounded on \([0, \epsilon]\), for any \( a \geq 1\), and so \( \|\hat{q}(t)|_{G_{\text{trans}}(t)} \) is bounded on \([0, \epsilon]\) too (for any \( a \geq 1\)). The constants \( C \) above are all independent of \( M \), so there is a \( N \) large enough such that for all \( t \in [0, \epsilon] \),
\[
2 \frac{d}{dt} \left( \sum_{n \geq N} \frac{f(n)}{n} \right)^{1/2} + C \left( \sum_{n \geq N} nf(n) \right)^{1/2} \leq C(t) \|q_t(0) - q(0)\|_{L^2} \quad (A.9)
\]

This proves that for any \( q_0 \) in \( L^2 \), for any \( \epsilon > 0 \) and \( \delta > 0 \), for any \( a \geq 1 \), there is a \( M < +\infty \) such that for any \( \tilde{q}_0 \in B_{L^2}(q_0, \epsilon) \) we have \( \|q(\delta) - \hat{q}(\delta)|_{G_{\text{trans}}(t)} \leq M \|q_0 - \tilde{q}_0\|_{L^2} \).

**Proof of theorem 3.1**

**Definition of the transform \( \mathcal{F} \) and the new equation (3.4).** In the following we slightly abuse notations by using the same letter \( V \) to denote both
\[
V = V(q)(\theta) = -K \left( x_1(q) \cos(\theta) + y_1(q) \sin(\theta) \right),
\]
\[
\text{or} \quad V = V(a, b)(\theta) = -K \left( a \cos(\theta) + b \sin(\theta) \right)
\]
depending on context. For \( q(t) \) a solution of (1.1), the ordinary differential equations satisfied by \( x_1 = x_1(q(t)) \) and \( y_1 = y_1(q(t)) \) are
\[
x_1' = -\frac{1}{2} x_1 + \frac{k}{2} \left[ x_1(1-x_2) - y_1 y_2 \right]
\]
\[
y_1' = -\frac{1}{2} y_1 + \frac{k}{2} \left[ -x_1 y_2 + y_1(1+x_2) \right]
\]
and this gives in particular
\[
\partial_\gamma V = -\frac{V}{2} - \frac{K}{2} \left[ F_1(u, x_1, y_1) \cos(\theta) + G_1(u, x_1, y_1) \sin(\theta) \right]. \quad (A.11)
\]

Along a solution of the Kuramoto equation (1.1), where \( u = e^{\sigma t} q \) and
\[
F_1(u, a, b) = [a(1 - X_2(u, a, b)) - b Y_2(u, a, b)]
\]
\[
G_1(u, a, b) = [-a Y_2(u, a, b) + b(1 + X_2(u, a, b))],
\]
with
\[
X_2(u, a, b) = \int_{-\pi}^{\pi} u(\theta) e^{\xi (a \cos(\theta) + b \sin(\theta))} \cos(2\theta) \, d\theta,
\]
\[
Y_2(u, a, b) = \int_{-\pi}^{\pi} u(\theta) e^{\xi (a \cos(\theta) + b \sin(\theta))} \sin(2\theta) \, d\theta,
\]
we have that for any solution \( q \) of equation (1.1), \( \mathcal{U} \) satisfies (3.4) where
\[
A \mathcal{U} = \left( \begin{array}{c} -\frac{1}{2} \Delta & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{array} \right) \mathcal{U} = \left( \begin{array}{c} -\frac{1}{2} \partial_\gamma^2 u \\ \frac{1}{2} x_1 \\ \frac{1}{2} y_1 \end{array} \right), \quad (A.14)
\]
We will show now that inertial manifolds exist for equation (3.4). There are several inertial manifolds existence theorem (see [26, 29, 22] for instance), such as the following.

**Theorem 1.8.** Consider an abstract evolution equation on a Hilbert space $H$
\[ \frac{du}{dt} + Au = f(u) \]  
where $A$ is a positive self-adjoint operator with compact resolvent and $f : H \to H$ is Lipschitz continuous on bounded sets. The spectrum of $A$ consists in a sequence of eigenvalues $\lambda_n$ such that $\lambda_n \leq \lambda_{n+1} \to +\infty$. Suppose that equation (A.16) has a bounded absorbing set in $H$. Then there exists a constant $C$, such that if
\[ \lambda_{n+1} - \lambda_n > C \]  
for some $n \geq 0$, then there exists an asymptotically complete inertial manifold $M^*$ for equation (A.16), which is the graph of a Lipschitz function $\Phi$ on $E_n = \bigcap_{i \geq n} \ker(A - \lambda_i I)$. Furthermore, if $f$ is $C^1(H, H)$, then the graph function $\Phi$ and the inertial manifold $M^*$ are $C^1$ too.

During the next two steps of this proof, we show that theorem 1.8 applies to equation (3.4). The Cauchy problem for equation (3.4) is well posed. The operator $A$ is diagonal in the natural Hilbert basis of $E := H^s \times \mathbb{R} \times \mathbb{R}$, and the spectral gap condition of theorem 1.8 holds for the operator $A$. The maps $X_2, Y_2, F_1$, and $G_1 : L^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are $C^\infty$. Since the first line of the nonlinear term $N(\mathcal{U})$ is the product of a Fourier polynomial (whose coefficients are $C^\infty(H^s \times \mathbb{R} \times \mathbb{R})$) and $u$, we have that $N$ is $C^\infty : H^s \times \mathbb{R}^2 \to H^s \times \mathbb{R}^2$. In particular, the nonlinear term $N$ is Lipschitz on bounded subsets of any $H^s \times \mathbb{R} \times \mathbb{R}$, and classical semilinear parabolic equation results in Hilbert spaces apply (see for example [5]).

**Proposition 1.9.** Let $s \in \mathbb{N}$. For any $\mathcal{U}_0 \in H^{s+2} \times \mathbb{R} \times \mathbb{R}$, there is a unique solution $\mathcal{U}(t)$ of (3.4) with $\mathcal{U} \in C^0([0, T_{\max}); H^{s+2} \times \mathbb{R}] \cap C^1([0, T_{\max}); H^s \times \mathbb{R} \times \mathbb{R})$, and we have either $T_{\max} = +\infty$ or $\lim_{t \to T_{\max}^-} \|\mathcal{U}(t)\|_{H^s \times \mathbb{R} \times \mathbb{R}} = +\infty$.

Furthermore, if $|\mathcal{U}_0| \geq 0$, then we have $\mathcal{U}(t) \geq 0$ for all $t \in [0, T_{\max}]$.

In the following we denote by $S_t^\ast$ the semiflow generated by equation (3.4). Thanks to the regularizing properties of equation (1.1), without loss of generality we may assume that $q_0 \in H^2$ and $\mathcal{U}_0 \in H^{s+2} \times \mathbb{R} \times \mathbb{R}$ (for any fixed $s \geq 0$).

**Existence of a bounded absorbing set for equation (3.4).** We have already established that all hypotheses of theorem 1.8 hold for equation (3.4), but the existence of a bounded absorbing set.

For any $\mathcal{U}$ solution of (3.4), $(q, a, b) = \mathcal{F}^{-1}(\mathcal{U})$ satisfies
\[ \partial_\tau q = \frac{1}{2} \partial_\tau^2 q + K \partial_\tau \left[ (-a \sin(\cdot) + b \cos(\cdot))q \right], \]
\[ a' = -\frac{1}{4} a + \frac{1}{2} K F_1(e^{-K(a \cos(\cdot) + b \sin(\cdot))}) q, \quad a, b \]
\[ b' = -\frac{1}{4} b + \frac{1}{2} K G_1(e^{-K(a \cos(\cdot) + b \sin(\cdot))}) q, \quad a, b. \]  
As for Kuramoto equation (1.1), we prove that equations (A.18) and (A.19) have a bounded absorbing set in a Gevrey space, by mean of an equivalent Fourier ODE system.
Denoting \( w_1(t) = a(t) + ib(t) \), and \( z_n(t) = x_n(t) + iy_n(t) = \int_{-\pi}^{\pi} e^{in\theta} q(t, \theta) \, d\theta \) we have
\[
 z_n' = -\frac{n^2}{2} z_n + \frac{Kn}{2} [w_1 z_{n-1} - \overline{w_1} z_{n+1}] .
\] (A.20)

With the method of theorem 2.1 and lemma 2.2, we obtain
\[
 \frac{d}{dt} |q|_{f,N}^2 + \sum_{n} \int_{\pi}^{\pi} f'(n,t) \, d\theta \leq 2K f(N,t) + \frac{2K}{\sqrt{N(N-1)}} (a^2 - 1) |w_1(t)||q|_{f,N}^2 .
\] (A.21)

If \( w_1 \) is bounded on some time interval \([0, T]\), then for \( N \) large enough we have \( \frac{d}{dt} |q|_{f,N}^2 + \|q\|_{f,N}^2 \leq C \), and for any solution \( U \in C^1([0, T], L^2 \times \mathbb{R} \times \mathbb{R}) \), we have \( u(t) \in H_f \) for all \( t \in [0, T] \). We also see from the previous argument that if we have a uniform bound \( |w_1| \leq C_1 \) for all \( t \in [0, +\infty[ \), then \( N \) can be chosen large enough such that \( \frac{d}{dt} |q|_{f,N}^2 + \|q\|_{f,N}^2 \leq C \) and the bounded ball \( B_{H_f}(0, 2C) \) in Gevrey space is an absorbing set for our system.

We now prove a uniform bound for \( w_1 \) on \([0, +\infty[ \). Equation (A.18) preserves the integral of \( q \), \( \int_{\pi}^{\pi} q(t, \theta) \, d\theta \) along solutions. Since \( q_0 \geq 0 \) implies \( q(t) \geq 0 \) for solutions of equations (A.18) and (A.19), we have \( |x_1| = \int_{\pi}^{\pi} q_0(\theta) \, d\theta = C \) and \( |y_1| = C \) as long as the solution exists (where \( x_1 = \int_{\pi}^{\pi} q(t, \theta) \cos(\theta) \, d\theta \) and \( y_1 = \int_{\pi}^{\pi} q(t, \theta) \sin(\theta) \, d\theta \)). Consider \((q_0, a_0, b_0) \in L^2 \times \mathbb{R} \times \mathbb{R}\) and the associated solution \((q(t), a(t), b(t))\) of equations (A.18) and (A.19). We have
\[
 x_1' = -\frac{i}{2} x_1 + \frac{K}{2} K_1 (e^{-K(a \cos(\theta) + b \sin(\theta))} q, a, b),
 y_1' = -\frac{i}{2} y_1 + \frac{K}{2} K_1 (e^{-K(a \cos(\theta) + b \sin(\theta))} q, a, b),
\] (A.22)

so that \( \frac{d}{dt}(x_1 - a) = -(x_1 - a) \) and \( \frac{d}{dt}(y_1 - b) = -(y_1 - b) \). (A.23)

The quantities \( |x_1 - a| \) and \( |y_1 - b| \) are exponentially decreasing. Since \( |x_1(t)| \leq 1 \) and \( |y_1(t)| \leq 1 \) for all time \( t \geq 0 \), there is a time \( t_0 \) such that \( |w_1| \leq |a| + |b| \leq 4 \) for all \( t \geq t_0 \).

As seen in the previous paragraph, this implies that
\[
 \frac{d}{dt} |q|_{f,N}^2 + \|q\|_{f,N}^2 \leq C_1 \quad \forall t \in [0, +\infty[ ,
\] (A.24)

and the bounded ball \( B_{H_f}(0, 2C) \) in Gevrey space is an absorbing set for our equations (A.18) and (A.19) and so for equation (3.4).

Existence of an AcIM for equation (3.4) and an AcSIM for equation (1.1). By theorem 1.8, there exists an inertial manifold \( M^* \) for equation (3.4). Furthermore, since the nonlinear term \( N(U) \) is \( C^1 (k \geq 1) \), the inertial manifold \( M^* \) is \( C^1 \), and for any \( U_0 \in H^* \times \mathbb{R}^2 \) and \( t_0 \) large enough there is a unique phase \( \varphi(t) \in M^* \) such that
\[
 \|S_{t_0}^* U_0 - S_{\varphi}^* \varphi \| = O(e^{-\eta t})
\] (A.25)

and the leaves
\[
 L_{\varphi} = \{U_0, \|S_{t_0}^* U_0 - S_{\varphi}^* \varphi \| = O(e^{-\eta t})\}
\] (A.26)
form a \( C^1 \) foliation of space \( H^* \times \mathbb{R}^2 \).

To deduce properties of the flow of equation (1.1) from results for solutions of equation (3.4), it is essential that the transform \( F \circ I : q \mapsto U \) and the inverse \( F^{-1} \) are smooth. With the definition \( u(\theta) = q(\theta)e^{+\frac{K}{2} K_1 (x_1 \cos(\theta) + y_1 \sin(\theta))} \), one sees that
\[
 \|u\|_{L^2} \leq C_K \|q\|_{L^2} e^{-\frac{1}{2} K_1 (x_1 \cos(\theta) + y_1 \sin(\theta))} \leq C_K C(K, x_1, y_1) \|q\|_{L^2}
\] (A.27)
and also for any $s \in \mathbb{N}$ \(\|u\|_{H^s} \leq C_{K,s}C(K, s, x_1, y_1)\|q\|_{H^s}\), so that \(q \mapsto u\) is well defined \(H^s \to H^s\). In addition to the mappings \(q \mapsto (x_1, y_1) \mapsto V(x_1, y_1, \cdot) \mapsto e^{\frac{1}{2}q} V\) are smooth and one finally finds that \(F : q \mapsto \mathcal{U} = C^\infty(H^s; H^s \times \mathbb{R} \times \mathbb{R})\) for any \(s \in \mathbb{N}\). For the same reasons, the inverse mapping \(\mathcal{F}^{-1}\mathcal{U} = (u, a, b) \mapsto (q, a, b) = (ue^{\frac{1}{2}K(u, \cos \cdot, \sin \cdot))}\) is also well defined and \(C^\infty\) smooth \(H^s \times \mathbb{R} \times \mathbb{R} \to H^s \times \mathbb{R} \times \mathbb{R}\) for any \(s \geq 0\).

For any solution \(q\) of equation (1.1), \(\mathcal{U} = \mathcal{F} \circ I(q)\) is a solution of (3.4), and \([(q, a, b) \in H^s \times \mathbb{R}^2, x_1(q) = a, y_1(q) = b]\) is an invariant manifold for equation (3.4). With any \(q_0 \in H^s\), we associate \(\mathcal{U}_0 = \mathcal{F}(q_0)\), which has a phase on the inertial manifold \(M^s\) of equations (A.18) and (A.19), i.e. there is a \(\phi(u)\) with any \(q \in \mathcal{U}\) we have \(\mathcal{L}_{x}\) the origin.

Let us denote by \(A\) the origin. In a neighbourhood of the origins, the equality \(\partial_{\theta} q(?) = \partial_{\theta} q\) is normal, and we have the orthogonal sum \(\hat{q} = \hat{q}\). For any solution \(\mathcal{U}\), we can now write the dynamics for the new variables \(\phi(u)\) and variable change \(u \mapsto (x_{\phi(a)}, y = z_a - z_{\phi(a)})\) will be well defined.

Since \(x_{\phi(a)} = \hat{q}(\cdot + \phi) - \hat{q}(\cdot)\), and \(\frac{dx_{\phi(a)}}{d\phi} \big|_{\phi = 0} = \partial_{\theta} \hat{q} \neq 0\), the set \(\{x_{\phi(a)} : \phi \in [-\epsilon, \epsilon]\}\) is a graph above \(\mathbb{R}\). In particular, for \(u\) close enough to zero, there is a unique \(\phi(u)\) such that \(\mathcal{U} = \mathcal{U}_{\phi(a)}\). Moreover, \(\phi \mapsto x_{\phi}\) is \(C^k([-\epsilon, \epsilon], H^{-k}_{1/q})\) \((k \geq 0)\) and the decomposition \(u \mapsto (x_{\phi(a)}, y = z_a - z_{\phi(a)})\) is \(C^k\) in a neighbourhood of \(u = 0\) in \(H^{-k}_{1/q}\). The mapping \(u \mapsto (\phi, y)\) is well defined and \(C^k : H^{-k}_{1/q} \to [-\epsilon, \epsilon] \times H^{-k}_{1/q}\) on a neighbourhood of the origin.

We can now write the dynamics for the new variables \(\phi\) and \(y\). The \(x_{\phi}\) are equilibria, i.e. we have \(L_{\phi}x_{\phi} = f(x_{\phi})\), for \(\phi \in [0, 2\pi]\). If \(u\) is a solution of \(\frac{dz}{dt} + L_{\phi}u = f(u)\), we have

\[
\left(\partial_{\phi} x_{\phi}\right) \frac{d\phi}{dt} + \frac{dy}{dt} + L_{\phi}y = f(x_{\phi} + y) - f(x_{\phi}).
\]

Let us denote by \(A_2\) the restriction of \(L_{\phi}\) on \(R(L_{\phi})\), i.e. \(A_2 = L_{\phi}|_{R(L_{\phi})}\), and take \(v = \partial_{\phi} q/\|\partial_{\phi} q\|^2\) (so that \(L_{\phi}v = 0\) and \(\langle (v, \partial_{\phi} x_{\phi}) \rangle = \langle (v, \partial_{\phi} \hat{q}) \rangle = 1\)). Then we get the system

\[
\frac{d\phi}{dt} = \Phi(\phi, y) \quad \text{and} \quad \frac{dy}{dt} + A_2 y = g(\phi, y).
\]

where

\[
\Phi(\phi, y) = \frac{1}{\langle (v, \partial_{\phi} x_{\phi}) \rangle} \langle (v, f(x_{\phi} + y) - f(x_{\phi})) \rangle
\]
and
\[ g(\psi, y) = f(x_\psi + y) - f(x_\psi) - \partial_x x_\psi \Phi(x_\psi). \] (A.31)

We note that the nonlinearity is well defined \( L^2(S) \to H^{-1}(S) \), and since \( f \) is bilinear, we have \( f \in C^k(L^2, H^{-1}) \) (as before, \( k \) is arbitrary). Since \( f \in C^k(L^2, H^{-1}) \), \( x_\psi \in C^k([0, 2\pi]; L^2) \) and \( \partial_x x_\psi |_{\psi=0} = \partial_\psi \hat{q} \neq 0 \), it is easy to check that there is an \( \epsilon > 0 \) and an \( \eta > 0 \) such that \( \Phi: [-\epsilon, \epsilon] \times B_L(0, \eta) \to \mathbb{R} \) and \( g: [-\epsilon, \epsilon] \times B_L(0, \eta) \to H^{-1} \) are \( C^k \). The Frechet derivative of \( f \) at the origin is null, i.e., \( Df(0) = 0 \), and the expressions for \( \Phi \) and \( g \) give \( D\Phi(0, 0) = 0 \) and \( Dg(0, 0) = 0 \).

**Proof of lemma 4.9**

Inequality (4.20) also holds if we put \( D(A_2^\alpha) \) and \( D(A_\gamma^\beta) \) instead of \( D(A_2^\alpha) \) and \( H_{1/2} \). For \( \alpha = 0 \), one directly has \( \|D^{A_\gamma^\beta}v\|_{D(A_\gamma^\beta)} \leq e^{-\beta t}\|v\|_{D(A_\gamma^\beta)} \) for any \( t \geq 0 \). We now choose \( \alpha = 1/2 \) and a value for \( \epsilon \).

Since the Frechet derivative of \( \Phi \) and \( g \) are smooth, for any \( \delta > 0 \), there is an \( \eta > 0 \) such that \( |\psi| + \|y\| \leq \eta \) implies
\[ \|D\Phi(\psi, y)\| \leq \delta \quad \text{and} \quad \|Dg(\psi, y)\| \leq \delta, \] (A.32)
where the norms are the appropriate operator norms. We also see that for any \( \psi \in [-\epsilon, \epsilon] \), we have \( \Phi(\psi, 0) = 0 \) and \( g(\psi, 0) = 0 \). Thus for any \( \rho > 0 \), there is a \( \gamma(\rho) \) such that on the set \( \{(\psi, y), |\psi| + \|y\| \leq \rho\} \), we have
\[ |\Phi(\psi, y)| + \|g(\psi, y)\|_{H^{-1}} \leq \gamma(\rho)\|y\|_{L^2} \quad \text{and} \quad \gamma(\rho) \to 0. \] (A.33)

By the hypothesis on the initial condition and by the continuity of \( t \mapsto |\psi| + \|y\|_{L^2} \), we have that there exists \( T > 0 \) such that \( |\psi| + \|y\|_{L^2} \leq \rho \). We check first that the estimates of the lemma are true on \([0, T]\).

For \( y_0 \in R(L_\beta) \), we have
\[ y(t) = e^{-tA_\gamma}y_0 + \int_0^t e^{-t(s-A_\gamma)}g(\psi(s), y(s)) \, ds, \] (A.34)
which implies on \([0, T]\)
\[ \|y(t)\|_{L^2} \leq \|y_0\|_{L^2}e^{-\|\lambda_1\|^2} + \int_0^T \|e^{-t(s-A_\gamma)}\|_{L^1(H^{-1}, L^2)}\|g(\psi(s), y(s))\|_{H^{-1}} \, ds \]
\[ \leq \|y_0\|_{L^2}e^{-\|\lambda_1\|^2} + C_{1/2, \epsilon} \int_0^T \frac{1}{\sqrt{T-s}} \|e^{-\|\lambda_1\|^2|s-t|}g(\psi(s), y(s))\|_{L^2} \, ds, \] (A.35)
where \( \epsilon \) is arbitrarily chosen in \([0, 1/2]\).

Then we consider \( u(t) := \sup_{0 \leq s \leq t} \|y(s)\|_{L^2}e^{(1-2\epsilon)|\lambda_1|} \). We have for \( t \in [0, T] \)
\[ \|y(t)\|_{L^2}e^{(1-2\epsilon)|\lambda_1|} \leq \|y_0\|_{L^2} + \|\gamma(\rho)\|_{C_{1/2, \epsilon}} \int_0^T \frac{1}{\sqrt{T-s}} e^{-\|\lambda_1\|s} u(s) \, ds \]
\[ \leq \|y_0\|_{L^2} + \|\gamma(\rho)\|_{M_\epsilon} u(T), \] (A.36)
where \( M_\epsilon := C_{1/2, \epsilon} \int_0^\infty \frac{1}{\sqrt{s}} e^{-\|\lambda_1\|s} \, ds \) and this means (choose \( \rho \) so that \( \gamma(\rho) \leq 1/(2M_\epsilon) \))
\[ u(T) \leq 2\|y_0\|_{L^2}, \] (A.37)
which directly implies (4.22) with \( \beta := (1-2\epsilon)|\lambda_1| \), still for \( t \in [0, T] \).

We now want to relax the condition \( t \in [0, T] \). For this we first observe that since \( \|\frac{d\Phi}{dt}\| = \|\Phi(\psi, y)\| \leq \gamma(\rho)\|y\|_{L^2} \), we have
\[ \left| \frac{d\psi}{dt} \right| \leq 2\gamma(\rho)\|y_0\|_{L^2}e^{-\beta t}, \] (A.38)
and therefore \(|\varphi(t)| \leq |\varphi_0| + \gamma(\rho) \frac{\rho}{2\beta}\). If we now choose \(\rho_0\) such that we have \(\gamma(\rho) \leq \beta\), then on \([0, T]\) we have
\[
|\varphi(t)| \leq |\varphi_0| + \frac{1}{2}\rho \leq \frac{3}{2}\rho
\]
\[\text{(A.39)}\]
and
\[
\|y(t)\|_{L^2} + |\varphi(t)| \leq 2\|y_0\|_{L^2} + \frac{3}{2}\rho \leq \rho
\]
\[\text{(A.40)}\]
This implies that there is a \(T_1 > T\) such that \(\|y(t)\|_{L^2} + |\varphi(t)| \leq \rho\) on \([0, T_1]\) and therefore that \(T\) can be chosen arbitrarily large. In fact, if we set \(T_* := \sup[t : |\varphi(t)| + \|y(t)\|_{L^2} \leq \rho]\) and if \(T_* < \infty\) we can take \(T = T_*\) in the first part of the proof and we arrive at a contradiction.

**Proof of corollary 4.7**

Because of rotation symmetry, it is enough to study the system around \(\dot{\varphi} = q_{\varphi=0}\). We consider the splitting \(E = L^2 = \ker(L_{\varphi}) \oplus R(L_{\varphi}) = T_qC \oplus \mathcal{N}_{q}^\perp\), setting \(\mathcal{N}_{q}^\perp = R(L_{\varphi})\). This splitting extends continuously to all \(\dot{\varphi}(\cdot + \varphi) \in C\) by rotations.

Any point of \(C\) is an equilibrium, so that for any time \(t\) and integer \(n\), we have \(S_{nt}(\hat{\varphi}) = Id|C\), and so \(DS_{nt}(\hat{\varphi}) = v = v\) for any \(v \in T_qC\). This implies that \(\|DS^{nt}(\hat{\varphi}).v\| \leq \|v\|\) for any \(T > 0\) and \(n \in \mathbb{Z}\).

We consider now \((\varphi_0, y_0)\) and \((\varphi(t), y(t))\) the associated solution. The estimates from the lemma 4.9 imply \(|\varphi(t) - \varphi_0| \leq 2\gamma(\rho) \frac{\rho}{2\beta}\|y_0\|\), and \(\|y(t)\| \leq 2\|y_0\|e^{-\beta t}\), where \(\rho = 8(|\varphi_0| + \|y_0\|)\).

For \(v \in R(L_{\varphi}) = \mathcal{N}_{q}^\perp\), we have \(v = (0, y_v) = (0, v)\) in the new coordinates, and if we consider \(q(0) = (\varphi_0, y_0) = \hat{\varphi} + \epsilon v\), we find
\[
|\varphi(t)| + \|y(t)\| \leq \left[\frac{2\gamma(\rho)}{\beta} + 2e^{-\beta t}\right]\|y_0\|.
\]
\[\text{(A.41)}\]

With \(\|y_0\| = \frac{1}{2}\rho = \epsilon\|v\|\), we have \(\gamma(\rho) \xrightarrow{\epsilon \to 0} 0\), and so there is a \(\epsilon_1\) such that for any \(\epsilon < \epsilon_1\) we have \(\frac{2\gamma(\rho)}{\beta} \leq \frac{1}{10}\). And choosing \(T\) such that \(e^{-\beta T} \leq \frac{1}{20}\) gives
\[
\|S^{T}(\hat{\varphi} + \epsilon v) - S^{T}(\hat{\varphi})\| \leq \left[\frac{1}{10} + 2\frac{1}{20}\right]\|v\| \leq \frac{\epsilon}{10}\|v\|.
\]
\[\text{(A.42)}\]
and \(\|DS^{T}(\hat{\varphi}).v\| \leq \frac{\epsilon}{10}\|v\|\).

In the same way, for any \(n \geq 1\), there is a \(\epsilon_n > 0\) such that for all \(\epsilon < \epsilon_n < \epsilon_1\), we have \(\frac{2\gamma(\rho)}{\beta} \leq \frac{1}{10n}\), and then
\[
\|S^{nt}(\hat{\varphi} + \epsilon v) - S^{nt}(\hat{\varphi})\| \leq \left[\frac{1}{10n} + 2\frac{1}{20n}\right]\|v\| \leq \frac{2\epsilon}{10n}\|v\|.
\]
\[\text{(A.43)}\]
for any \(v \in R(L_{\varphi}) = \mathcal{N}_{q}^\perp\), and this implies \(\|DS^{nt}(\hat{\varphi}).v\| \leq \frac{2\epsilon}{10n}\|v\|\). This completes the proof of normal hyperbolicity of \(C\).

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