Simplicial gravity with coordinates

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Received 15 October 2020, revised 20 March 2021
Accepted for publication 9 April 2021
Published 7 May 2021

Abstract
We present a formulation of Regge calculus where arbitrary coordinates are associated to each vertex of the simplicial complex and the fundamental degrees of freedom are given by the metric $g_{\mu\nu}(\alpha)$ on each simplex $\alpha$. The lengths of the edges, which are the usual degrees of freedom of Regge calculus, are thus determined and are left invariant under arbitrary transformations of the discrete set of coordinates, provided the metric transforms accordingly. Invariance under coordinate transformations entails tensor calculus and our formulation then follows closely the usual formalism of the continuum theory. This includes a definition of partial derivative which stems from a generalization to simplicial lattices of the symmetric finite difference operator on a cubic lattice. The definitions of parallel transport, Christoffel symbol, covariant derivatives and Riemann curvature tensor follow in a rather natural way establishing a kind of dictionary between continuum and simplicial lattice quantities. In this correspondence Einstein action becomes Regge action with the deficit angle $\theta$ replaced by $\sin \theta$. The correspondence with the continuum theory can be extended to actions with higher powers of the curvature tensor, to the vielbein formalism and to the coupling of gravity with matter fields (scalars, fermionic fields including spin $3/2$ fields and gauge fields) which are then determined unambiguously and discussed in the paper. An action on the simplicial lattice for $N = 1$ supergravity in four dimensions is derived in this context. Another relevant result is that Yang–Mills actions on a simplicial lattice consist, even in absence of gravity, of two plaquette terms, unlike the one plaquette Wilson action on the hypercubic lattice. An attempt is also made to formulate a discrete differential calculus to include differential forms of higher order and the gauging of free differential algebras in this scheme. However this leads to form products that do not satisfy associativity and distributive law with respect to the $d$ operator. A proper formulation of theories that contain higher order differential forms in the context of Regge calculus is then still lacking.

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Keywords: gravity, Regge calculus, simplicial

(Some figures may appear in colour only in the online journal)

1. Introduction

The title of Regge’s seminal paper of 1961 [1] ‘general relativity without coordinates’ emphasizes a crucial aspect of his approach to discrete gravity, namely that it does away with the notion of coordinates and formulates general relativity purely in terms of geometrical quantities: lengths, volumes, angles, etc.

This was in itself an extraordinary achievement. In the continuum theory absolute differential calculus, or tensor calculus, plays a fundamental role in the mathematical formulation of general relativity. Invariance under general coordinate transformations follows directly from the principle of equivalence in its most general form, namely that all reference frames are equivalent in the description of the physical world and that the only real observables are the underlying geometric properties of space-time, which are the building blocks of Regge’s formulation.

The basic ideas of Regge calculus are well known: the smooth $d$-dimensional space-time manifold of the continuum formulation is replaced by a triangulated manifold made of piecewise flat $d$-dimensional simplices glued together by identifying in pairs their $d - 1$ dimensional faces. The geometrical properties of this manifold are determined by the lengths of all its one dimensional edges: in fact each $d$-dimensional simplex is completely fixed by the lengths of its $\binom{d+1}{2}$ edges.

The curvature is associated to the $d - 2$ dimensional subspaces, the hinges, and is given for each hinge $h$ by the deficit angle $\theta_h$ defined as $2\pi$ minus the sum of the dihedral angles between the faces of the simplices which the hinge $h$ belongs to. In a flat space $\theta_h$ is zero for all $h$, as clearly shown by the two dimensional case, where the hinges are points (dimension zero) and $\theta_h$ the complement to $2\pi$ of the sum of the angles meeting at that point.

The discrete version of Einstein action is then given by:

$$S_R = \sum_h |V_h| \theta_h,$$

where $|V_h|$ is the volume of the hinge $h$.

Following Regge’s original paper a great number of different formulations and approaches to Regge calculus appeared. We shall not even try to go over the huge literature on the subject, which can be found in the review paper of reference [2], recently updated in reference [3].

Some of the new proposals maintained the same purely geometrical approach of the original Regge paper, like the so called area Regge calculus [4] where in four dimensions the areas of the triangles are chosen as fundamental degrees of freedom in place of the edges’ lengths.

Coupling gravity with matter fields, and in particular with fermions, requires however the introduction of vielbeins, and hence of some kind of local coordinates, on the simplicial complex.

This was done in reference [5, 6], where a Euclidean reference frame is introduced in each simplex, and the degrees of freedom are defined on the links of the dual lattice as the Poincaré transformations needed to rotate the reference frame defined on a simplex $\alpha$ into the one of a contiguous simplex $\beta$. The action is the one of a gauge theory on the dual lattice with a local Poincaré invariance, but it is eventually equivalent to Regge action of equation (1.1) although with the deficit angle $\theta_h$ replaced by $\sin(\theta_h)$. 

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A first order formalism is possible in this framework, and the presence of a local Lorentz group makes the coupling of fermions to gravity possible.

In this paper, while keeping the original Regge’s triangulation of space-time, we reintroduce space time coordinates trying to keep the formalism as close as possible to the continuum formulation. This is done by associating arbitrary space time coordinates to each vertex of the simplicial complex and a constant metric tensor $g_{\mu\nu}(\alpha)$ to each simplex $\alpha$.\(^1\) The length of all the edges, which are the degrees of freedom of Regge calculus, are then entirely fixed and are preserved by arbitrary transformations of the coordinates provided the metric tensor in each simplex is transformed accordingly (section 2).

With this choice of degrees of freedom gravity can be formulated on a simplicial complex following step by step the classical textbook formalism of continuum general relativity. This includes a discretized version of tensor calculus which can be formalized to assure invariance under coordinate transformations (section 3). Another fundamental step is the definition of partial derivative on the simplicial lattice, which generalizes in a non trivial way the symmetric finite difference operator on the hypercubic lattice (section 4). Parallel transport can then be defined to make derivatives covariant (section 5) and eventually the analogue of the Riemann curvature tensor is obtained (section 6).

As a result a kind of dictionary is established that allows to translate any gravitational action in the continuum into a corresponding action on the simplicial complex. Within this correspondence the Einstein action is naturally translated into Regge’s action but with the deficit angle $\theta$ replaced by $\sin \theta$ as in reference [5] (section 6).

Gravitational actions with higher derivatives terms and Brans–Dicke type of actions can also be included in this scheme, and a definite prescription for their formulation on a simplicial lattice is given in section 7.

Brans–Dicke action involves the coupling of the gravitational field to scalar fields. The coupling of gravity to matter fields with higher spin, such as gauge fields and fermions is the subject of the last sections of the paper. Gauge fields are defined on the links of the dual lattice and the field strength on the dual lattice plaquettes (i.e. the hinges of the simplicial lattice) whose number of sides is not fixed. Yang–Mills action is obtained by coupling with the metric tensor two plaquettes that have a site (that is a simplex of the original lattice) in common and is therefore rather different, even in absence of gravity, from the one plaquette term of the standard formulation on an hypercubic lattice (section 8).

The vielbain formalism is introduced in section 9 by following the same approach used in reference [5], that is by introducing in each simplex $\alpha$ a Euclidean reference frame defined up to an arbitrary Lorentz rotation. The vielbeins in $\alpha$ are identified with the components of the local coordinate transformation from the general frame originally defined in $\alpha$ by the coordinate choice to the local Euclidean frame. As in the continuum theory the vielbein transform under both the general coordinate transformation and the local Lorentz transformations, which constitute a local symmetry group of the theory and can be treated according to the scheme already outlined in section 8. As in reference [5] the Lorentz connections are defined on the links of the dual lattice and are the gauge fields associated to the local Lorentz rotations.

The introduction of the vielbains and of the local Lorentz group makes the coupling of fermionic fields to gravity on a simplicial lattice possible exactly as in the continuum case. This is discussed in section 10. Having established a discrete version of tensor calculus this coupling can be easily extended to fermionic fields that transform as vectors under general

\(^1\) A similar parameterization has been used by Khatsymovsky in reference [7].
coordinate transformations, like for instance the gravitino. It is then possible to write a discrete action that corresponds to $D = 4$ and $N = 1$ supergravity in the continuum.

In order to have a complete correspondence between continuum and simplicial lattice theories one should include one more set of fields, namely the $p$-form potentials (with $p > 1$) that arise from the gauging of free differential algebras. These fields play an important role for instance in higher dimensional supergravity theories. In section 11 we discuss this point and find that a straightforward extension to these fields of the correspondence established in the previous sections leads to field strengths ($p + 1$ forms in the continuum) that are not gauge invariant.

This is probably related to the fact that, in spite of the invariance under coordinate transformations (which however involves only a discrete set of points), our formulation is equivalent to Regge calculus and does not have invariance under diffeomorphisms. A consistent formulation of differential forms on a simplicial lattice\(^2\) would probably be the answer to the problem of including $p$-form potential. Although this was beyond the original purpose of the paper an attempt was made in this direction leading to a definition of discrete differential forms that, although elegant, has a non-associative product. More seriously, the product does not obey the usual distribution law with respect to the $d$ operator. This is discussed for completeness in the appendix A.

2. Simplicial gravity with coordinates

Let $\alpha$ be a $d$-dimensional simplex and $i, j, \ldots$ labels for its $d + 1$ vertices. In Regge calculus the simplex is completely identified by giving the $\frac{d(d + 1)}{2}$ lengths $l_{ij}$ of the edges joining the vertices $i$ and $j$. The lengths $l_{ij}$ have to satisfy triangular inequalities, but are otherwise arbitrary. They constitute the fundamental degrees of freedom of Regge’s discrete gravity.

There are alternative ways to identify the simplex $\alpha$. One is to associate a coordinate $x^{\mu}_i$ ($\mu = 1, 2, \ldots, d$) to each vertex $i$ and a constant metric $g_{\mu\nu}(\alpha)$, in general not Euclidean, to each simplex $\alpha$. The lengths $l_{ij}$ of the edges are then determined and given by:

$$l^2_{ij} = g_{\mu\nu}(\alpha) \left( x^{\mu}_i - x^{\mu}_j \right) \left( x^{\nu}_i - x^{\nu}_j \right).$$

(2.1)

Conversely, if the lengths $l_{ij}$ are given and the coordinates $x^{\mu}_i$ of the vertices are chosen in an arbitrary way, then equation (2.1) provide a set of $d(d + 1)/2$ equations in the $d(d + 1)/2$ unknown components of the metric $g_{\mu\nu}(\alpha)$. These equations have a unique solution provided the determinant of the $d(d + 1)/2 \times d(d + 1)/2$ matrix $\Delta_{ij,\mu\nu}$ of their coefficients is not vanishing:

$$\det \Delta_{ij,\mu\nu} \equiv \det \left\{ \left( x^{\mu}_i - x^{\mu}_{d + 1} \right) \left( x^{\nu}_i - x^{\nu}_j \right) \right\} \neq 0.$$  

(2.2)

The determinant of $\Delta_{ij,\mu\nu}$ can be calculated and it is given by:

$$\det \Delta_{ij,\mu\nu} = \left[ \det \left( x^{\mu}_i - x^{\mu}_{d + 1} \right) \right]^{d + 1}, \quad i = 1, \ldots, d$$

(2.3)

so that the condition (2.2) is satisfied iff the determinant of the differences $x^{\mu}_i - x^{\mu}_{d + 1}$ is different from zero:

$$\det \left( x^{\mu}_i - x^{\mu}_{d + 1} \right) \neq 0 \quad i = 1, \ldots, d.$$

(2.4)

The last condition insures that the simplex $\alpha$ is not degenerate in $d$ dimensions. So a simplicial manifold can be characterized by assigning, instead of the edges’ lengths as in Regge calculus,

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\(^2\) Actually the precise correspondence would be with the lattice dual to the original simplicial lattice.
the coordinates of all the vertices and the metric of each simplex. This is essentially the same as in the usual formulation of Einstein’s gravity in the continuum. As in the continuum the choice of the coordinates is arbitrary, provided for each simplex the determinant condition (2.4) is satisfied, and we expect the theory to be invariant under general coordinate transformations, that is to depend only on geometrical invariants such as \( l_{ij} \).

Notice however that the components of the metric tensors belonging to different simplices are not all independent. Consider in fact two simplices \( \alpha \) and \( \beta \) which have in common \( p + 1 \) vertices, namely a \( p \)-dimensional sub-simplex. They have \( \frac{d(d+1)}{2} \) edges in common whose lengths \( l_{ij} \) cannot depend upon the fact of being considered as part of \( \alpha \) or as part of \( \beta \). Then from (2.1) we have the \( \frac{d(d+1)}{2} \) relations:

\[
[g_{\mu\nu}(\alpha) - g_{\mu\nu}(\beta)] (x'^{\mu}_i - x''^{\mu}_i) (x'^{\nu}_j - x''^{\nu}_j) = 0 \quad i, j \in \alpha \cap \beta. \tag{2.5}
\]

Equation (2.5) should be regarded as constraints to be implemented (which may not be easy) whenever the metric is varied or is integrated upon in the functional integral. As a result the number of degrees of freedom per simplex associated to the metric is much smaller than the expected \( \frac{d(d+1)}{2} \).

In fact, consider a simplex \( \alpha \) and suppose that its metric \( g_{\mu\nu}(\alpha) \) is given. The metric \( g_{\mu\nu}(\beta) \) of any adjacent simplex \( \beta \) that has a \( d - 1 \)-dimensional face in common with \( \alpha \) has to satisfy equation (2.5). Since \( \alpha \) and \( \beta \) have \( d \) vertices in common the number of constraints is \( \frac{d(d-1)}{2} \) and only \( d \) components of \( g_{\mu\nu}(\beta) \) (the ones with one index in an orthogonal direction with respect to \( \alpha \cap \beta \)) can be chosen independently from \( g_{\mu\nu}(\alpha) \). This argument can be generalised to any simplicial sub-manifold \( \mathcal{R} \), which we assume for simplicity to have the topology of a sphere, made of an arbitrarily large number on simplices. Let \( N \) be the number of degrees of freedom associated to the metric in \( \mathcal{R} \). Consider now an additional simplex \( \gamma \) attached to \( \mathcal{R} \) by one of its \( d - 1 \) dimensional faces. By the previous argument only \( d \) components of \( g_{\mu\nu}(\gamma) \) can be chosen arbitrarily.

In conclusion, by adding a simplex to \( \mathcal{R} \) its number of degrees of freedom increases of \( d \), except for the cases mentioned in the footnote. So we may conclude that the number of degrees of freedom per simplex associated to the metric is not \( \frac{d(d+1)}{2} \), as one would naively expect from a correspondence with the continuum case, but it is of order \( d \).

The previous argument shows that the degrees of freedom associated to the metric of a given simplex are ‘spread’ over a region whose size is proportional to the dimension \( d \). In fact the condition for two simplices to have completely independent metrics is from equation (2.5) to have no more than one vertex in common. If we define the ‘distance’ between two simplices \( \alpha \) and \( \beta \) the length (measured in number of links) of the shortest path on the dual lattice\(^4\) connecting the vertex \( \alpha \) to the vertex \( \beta \), then the minimum distance for \( \alpha \) and \( \beta \) to have completely independent metric is \( d \). This simply follows from the fact that the number of vertices in common with the starting point \( \alpha \) can only decrease of 1 for each one-link step.

\(^3\) It is possible that \( \gamma \) has more than one face in common with \( \mathcal{R} \) and in this case its metric is completely fixed. This happens whenever \( \gamma \) is the only simplex of a given hinge that does not belong to \( \mathcal{R} \).

\(^4\) Here and in the following we define the dual lattice as the lattice obtained by a Voronoi tassellation of the simplicial complex. The vertices of the dual lattice are then the circumcenters \( c(\alpha) \) of the simplices \( \alpha \) and its links are the lines joining the circumcenters of neighbouring simplices. Although it is not strictly necessary we shall assume that the circumcenters are always inside the corresponding simplex, namely that the simplicial complex is a Delaunay triangulation.
3. General coordinates transformations and tensor calculus

Let us consider now a general coordinate transformation on the simplicial manifold:

$$x_i^\mu \Rightarrow x_i'^\mu \quad i = 1, \ldots, N$$

(3.1)

where $N$ is now the total number of vertices in the manifold. The metric $g_{\mu\nu}(\alpha)$ of each simplex $\alpha$ should transform under (3.1) into a new metric $g'_{\mu\nu}(\alpha)$ in such a way to leave all the edges lengths $l_{ij}$ invariant.

Let now $x_i^\mu$ be the coordinates on the vertices of a specific simplex $\alpha$. Then the general coordinate transformation (3.1), restricted to the vertices in $\alpha$ can always be written as:

$$x_i'^\mu = \Lambda^\mu_\nu(\alpha)x_i^\nu + \Lambda^\mu(\alpha) \quad i \in \alpha$$

(3.2)

which also implies:

$$x_i'^\mu - x_j'^\mu = \Lambda^\mu_\nu(\alpha) (x_i^\nu - x_j^\nu) \quad i, j \in \alpha.$$  

(3.3)

The matrix $\Lambda^\mu_\nu(\alpha)$ is the discrete analogue of $\frac{\partial x'^\mu}{\partial x^\nu}$ and equation (3.3) can be used to define the transformation properties of a contravariant vector under general coordinate transformation on the simplicial manifold:

$$A'^\mu(\alpha) = \Lambda^\mu_\nu(\alpha)A^\nu(\alpha).$$

(3.4)

We shall assume that $\det \Lambda^\mu_\nu(\alpha) \neq 0$ for all $\alpha$. In fact it is clear from (3.3) and (2.4) that this is the necessary and sufficient condition for (2.4) to be preserved under (3.1). Notice that if $i$ and $j$ in (3.3) belong to both simplices $\alpha$ and $\beta$ then from (3.3) we have:

$$\left[\Lambda^\mu_\nu(\alpha) - \Lambda^\mu_\nu(\beta)\right] (x_i^\nu - x_j^\nu) = 0 \quad i, j \in \alpha \cap \beta.$$  

(3.5)

Equation (3.5) follows automatically from the restriction of (3.1) to the simplices $\alpha$ and $\beta$. However an alternative way of defining a general coordinate transformation is to assign, in place of (3.1), the matrices $\Lambda^\mu_\nu(\alpha)$ and $\Lambda^\nu(\alpha)$ for each simplex $\alpha$. In that case equation (3.5) should be regarded as constraints to be implemented on $\Lambda^\mu_\nu(\alpha)$.

Tensor calculus can now be formulated on the lattice: tensors with covariant and contravariant indices can be defined as quantities that transform with $\Lambda$ for each contravariant index and with $\Lambda^{-1}$ for each covariant index:

$$A'^{\mu_1 \cdots \mu_\nu}_{\nu_1 \cdots \nu_\mu}(\alpha) = \Lambda^{\mu_1}_{\rho_1}(\alpha) \cdots \Lambda^{\mu_\nu}_{\rho_\nu}(\alpha)(\Lambda^{-1})^{\nu_1}_{\rho_1}(\alpha) \cdots (\Lambda^{-1})^{\nu_\mu}_{\rho_\mu}(\alpha) A^{\rho_1 \cdots \rho_\nu}_{\sigma_1 \cdots \sigma_\mu}(\alpha).$$

(3.6)

The metric $g_{\mu\nu}(\alpha)$ transforms as a covariant tensor of rank 2. In fact from (3.3) the requirement that the lengths $l_{ij}$ given in (2.1) are invariant under general coordinate transformations gives:

$$g'^{\mu\nu}(\alpha) = (\Lambda^{-1})^{\rho\nu}_{\mu}(\alpha) (\Lambda^{-1})^{\rho\mu}_{\nu}(\alpha) g_{\mu\nu}(\alpha).$$

(3.7)

Quantities that are invariant under general coordinate transformations, and hence have an intrinsic geometrical meaning can now be constructed. The simplest is the volume $V(\alpha)$ of the simplex, which is the discrete analogue of the invariant volume $\sqrt{g} \, d^d x$ of the continuum theory, and is given by:

$$V(\alpha) = \frac{1}{d!} g_{\mu_1 \nu_1 \cdots \mu_d \nu_d} \left( x_{\mu_1}^{\nu_1} - x_{\nu_1}^{\mu_1} \right) \cdots \left( x_{\mu_d}^{\nu_d} - x_{\nu_d}^{\mu_d} \right) \sqrt{\det g_{\mu\nu}(\alpha)}.$$  

(3.8)
Notice that the value of $V(\alpha)$ changes sign, due to the antisymmetry of the $\epsilon$ symbol, if an odd permutation of the vertices is performed. We shall assume in the future that the order of the $x_i$’s in (3.8) is such that $V(\alpha)$ is positive. The invariance of $V(\alpha)$ under general coordinate transformations follows from (3.5) and (3.7).

Given a simplex $\alpha$ there are $d + 1$ neighbouring simplices that have a $d - 1$ dimensional face in common with $\alpha$. We shall denote such simplices as $\alpha_i$, where the index $i$ denotes the vertex of $\alpha$ which is not in $\alpha_i$: $x_i \notin \alpha_i$.

Let us denote by $\alpha \cap \alpha_i$ the $d - 1$ dimensional face that $\alpha$ and $\alpha_i$ have in common.

We can define then the following covariant vector:

$$V^{(\alpha \cap \alpha_i)}(\alpha) = d \frac{\partial V(\alpha)}{\partial x^\mu_i}$$

$$\quad = \sqrt{\text{det} g_{\mu\nu}(\alpha)} (d - 1)! \epsilon_{\mu \ldots \nu i} \left(x^{\mu_1}_i - x^{\mu_1}_{i+1}\right) \ldots \left(x^{\mu_s}_i - x^{\mu_s}_{i+1}\right)$$

where the index $\mu$ in the $\epsilon$ symbol is in the $i$th position and the $\langle i \rangle$ bracket means that the term $\left(x^{\mu_i}_i - x^{\mu_i}_{i+1}\right)$ in the product is missing.

The vector $V^{(\alpha \cap \alpha_i)}(\alpha)$ is orthogonal to the face $\alpha \cap \alpha_i$:

$$\left(x^\mu_i - x^\mu_s\right) V^{(\alpha \cap \alpha_i)}(\alpha) = 0 \quad r, s \neq i$$

and it can be shown to be pointing towards the outside of $\alpha$. Equation (3.9) also implies:

$$\left(x^\mu_i - x^\mu_s\right) V^{(\alpha \cap \alpha_i)}(\alpha) = dV(\alpha) \quad r \neq i.$$  \quad (3.11)

The modulus of $V^{(\alpha \cap \alpha_i)}(\alpha)$ is equal to the $d - 1$ dimensional volume $V(\alpha \cap \alpha_i)$ of $\alpha \cap \alpha_i$, so that we can write:

$$V^{(\alpha \cap \alpha_i)}(\alpha) = V(\alpha \cap \alpha_i) n^{(\alpha \cap \alpha_i)}(\alpha).$$

where $n^{(\alpha \cap \alpha_i)}(\alpha)$ is a vector orthogonal to $\alpha \cap \alpha_i$, pointing to the outside of $\alpha$ and with modulus 1:

$$n^{(\alpha \cap \alpha_i)}(\alpha) g^{\mu\nu}(\alpha) n^{(\alpha \cap \alpha_i)}(\alpha) = 1.$$  \quad (3.13)

A unit vector $n^{(\alpha \cap \alpha_i)}(\alpha)$ orthogonal to $\alpha \cap \alpha_i$ can be obtained starting from $\alpha_i$ instead of $\alpha$. It can be shown then from (3.8), (3.9) and (3.12) that $n^{(\alpha \cap \alpha_i)}(\alpha_i)$ is proportional to $n^{(\alpha \cap \alpha_i)}(\alpha)$. More precisely we have:

$$\frac{n^{(\alpha \cap \alpha_i)}(\alpha)}{\sqrt{\text{det} g_{\mu\nu}(\alpha)}} = - \frac{n^{(\alpha \cap \alpha_i)}(\alpha_i)}{\sqrt{\text{det} g_{\mu\nu}(\alpha_i)}}.$$  \quad (3.14)

where the minus sign is due to the orientation convention. Equation (3.14) can be read directly from (3.9). In fact the coordinates appearing on the rhs of (3.9) are only the ones of the vertices that are common to $\alpha$ and $\alpha_i$ ($x^\mu_i$ does not appear), so the only difference between $V^{(\alpha \cap \alpha_i)}(\alpha)$ and $V^{(\alpha \cap \alpha_i)}(\alpha_i)$ is the argument of the determinant under the square root.

5 Alternatively the absolute value can be taken at the rhs. Notice also that the choice of the label $d + 1$ for the reference vertex is irrelevant modulo a sign factor coming from the antisymmetric tensor.

6 This follows from the observation that the modulus does not depend on the choice of the coordinates and it is easily seen by choosing the coordinates of vertices of $\alpha$ in such a way that $g_{\mu\nu}(\alpha) = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Euclidean metric.
The simplex $\alpha$ and any two neighbouring simplices $\alpha_i$ and $\alpha_j$ have a $d - 2$ dimensional simplex (hinge) $h_{ij}$ in common. The set of the $d - 1$ vertices of $h_{ij}$ is the set of the vertices of $\alpha_i$ where the vertices labelled $i$ and $j$ have been removed. Since there are $\frac{(d+1)!}{2!}$ ways of removing two vertices from $\alpha$ there are $\frac{(d+1)!}{2}$ distinct hinges belonging to $\alpha$.

In analogy with what we have done for the faces, we can associate to the hinge $h_{ij}$ the covariant tensor of equation (3.15) can then be written as:

$$V^{(h_{ij})}_{\nu \mu} (\alpha) = d(d - 1) \frac{\partial^2 V(\alpha)}{\partial x^\mu r \partial x^\nu s}$$

$$= \frac{\sqrt{\det g_{\mu \nu}(\alpha)}}{(d - 2)!} (x^\mu_{i1} - x^\mu_{i2}) \ldots (x^\mu_{ij} - x^\mu_{jd+1}) \ldots (x^\nu_{j1} - x^\nu_{j2}) \ldots (x^\nu_{jd} - x^\nu_{jd+1})$$

(3.15)

where the indices $\mu_1$ and $\mu_2$ in the $\epsilon$ antisymmetric tensor are respectively in the $i$th and the $j$th position and the symbol $\langle i j \rangle$ means that the terms $(x^\mu_i - x^\mu_{jd+1})$ and $(x^\nu_j - x^\nu_{jd+1})$ are omitted in the product at the rhs of (3.15).

Notice that $V^{(h_{ij})}_{\nu \mu} (\alpha)$ is not only antisymmetric in the tensor indices $\mu_1$ and $\mu_2$ but also under exchange of $i$ and $j$:

$$V^{(h_{ij})}_{\nu \mu} (\alpha) = -V^{(h_{ij})}_{\mu \nu} (\alpha).$$

(3.16)

$V^{(h_{ij})}_{\nu \mu} (\alpha)$ and $V^{(h_{ij})}_{\mu \nu} (\alpha)$ correspond to the two different orientations of the hinge, which are better viewed by going to the dual lattice where the $d - 2$ dimensional hinge corresponds to a two-dimensional plaquette.

It can be easily seen from (3.15) that $V^{(h_{ij})}_{\nu \mu} (\alpha)$ is orthogonal to the hinge $h_{ij}$:

$$\langle x^\mu_i - x^\nu_s \rangle V^{(h_{ij})}_{\nu \mu} (\alpha) = 0 \quad \forall \ r, s \neq i, j.$$

(3.17)

Also, in analogy to equation (3.11), we have:

$$\langle x^\mu_i - x^\nu_s \rangle \langle x^\nu_j - x^\nu_r \rangle V^{(h_{ij})}_{\mu \nu} (\alpha) = d(d - 1)V(\alpha) \quad \forall \ r \neq i, j.$$

(3.18)

Notice also that $V^{(h_{ij})}_{\mu \nu} (\alpha)$ is entirely contained in the two-dimensional subspace spanned by $n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha)$ and $n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha)$. This base can be made orthonormal by defining:

$$n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha) = n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha); \quad n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha) = \frac{1}{\sqrt{1 - c^2}} n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha) - \frac{c}{\sqrt{1 - c^2}} n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha)$$

(3.19)

where $c = n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha) n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha)$. The vectors $n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha)$ and $n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha)$ satisfy now orthonormality relation with respect to the metric $g_{\mu \nu}(\alpha)$:

$$n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha) g^{\mu \nu}(\alpha) n^{(\alpha \nu \alpha \gamma)}_\mu (\alpha) = \delta^{ab} \quad a, b \in \{i, j\}.$$

(3.20)

The covariant tensor of equation (3.15) can then be written as:

$$V^{(h_{ij})}_{\mu \nu} (\alpha) = n^{(h_{ij})}_{\mu \nu} (\alpha) V(h_{ij})$$

(3.21)

where

$$n^{(h_{ij})}_{\mu \nu} (\alpha) = (n^{(h_{ij})}_{\mu \nu} (\alpha) - n^{(h_{ij})}_{\nu \mu} (\alpha) n^{(h_{ij})}_{\mu \nu} (\alpha)) \bigg|_{\mu \nu}.$$
and \( V(h_{ij}) \) is the absolute value\(^7\) of the \( d - 2 \) dimensional volume of the hinge.

4. Derivatives on a simplicial lattice

Derivatives are replaced on a lattice by finite differences. This is rather straightforward on regular hypercubic lattices which can be regarded as a discretization of a Euclidean coordinate system where all coordinates are integer multiples of the lattice spacing. Regge calculus on the other hand is defined on a simplicial complex which is in general not regular, and the \( d + 1 \) faces of each simplex point into different directions which are not related to any coordinate system.

Defining on a simplicial lattice the analogue of the partial derivative \( \partial_\mu \) with the further requirement that it transforms as a covariant vector under the coordinate transformations defined in the previous sections is not a trivial problem and is the object of the present section.

To start with, we shall consider only derivatives of scalar quantities; derivatives of vectors and tensors need to be made covariant and require the notion of parallel transport. They will be discussed in the following sections.

Let \( \varphi(x) \) be a scalar field of the continuum theory and \( \varphi(\alpha) \) the corresponding field on a simplicial lattice. The partial derivative \( \partial_\mu \varphi(\alpha) \) transforms as a covariant vector, so we want to construct on the simplicial lattice a new field \( \hat{\partial}_\mu \varphi(\alpha) \) that transforms as a covariant vector under the coordinate transformations defined in (3.2), depends on the value of \( \varphi(\alpha) \) in the simplex \( \alpha \) and in its neighbouring simplices and reduces to \( \partial_\mu \varphi(x) \) in the continuum limit.

Let us denote by \( \alpha_i \) with \( i = 1, 2, \ldots, d + 1 \) the \( d + 1 \) simplices that have one face in common with \( \alpha \). We shall use the following conventions: if \( P_1, P_2, \ldots, P_{d+1} \) are the vertices of \( \alpha \) with coordinates \( x^{\mu}_{P_1}, \ldots, x^{\mu}_{P_{d+1}} \), then the simplex \( \alpha_i \) denotes the simplex that has in common with \( \alpha \) the \( d - 1 \)-dimensional face that does not contain the vertex \( P_i \).

We then define the derivative of a scalar field \( \varphi(\alpha) \) on a simplicial lattice as follows\(^8\):

\[
\hat{\partial}_\mu \varphi(\alpha) = \frac{1}{2} \sum_{i=1}^{d+1} \left[ \varphi(\alpha_i) - \varphi(\alpha) \right] \frac{V^{(\alpha \cap \alpha_i)}(\alpha)}{V(\alpha)}, \tag{4.1}
\]

where \( V^{(\alpha \cap \alpha_i)}(\alpha) \), defined in (3.9), is a covariant vector whose modulus is the \( d - 1 \)-dimensional volume \( V(\alpha \cap \alpha_i) \) of the face \( \alpha \cap \alpha_i \) and whose direction is orthogonal to \( \alpha \cap \alpha_i \) (see equation (3.12)).

We now associate to the simplices \( \alpha \) and \( \alpha_i \) a length \( l(\alpha | \alpha_i) \) defined as\(^9\)

\[
l(\alpha | \alpha_i) = \frac{V(\alpha)}{V(\alpha \cap \alpha_i)} = \frac{V^{(\alpha \cap \alpha_i)}(\alpha)}{d} (x^{\mu}_i - x^{\mu}_j) \quad j \neq i, \tag{4.2}
\]

then the derivative takes the natural form:

\[
\hat{\partial}_\mu \varphi(\alpha) = \frac{1}{2} \sum_{i=1}^{d+1} \left[ \varphi(\alpha_i) - \varphi(\alpha) \right] \frac{l(\alpha | \alpha_i)}{l(\alpha)} \delta^{(\alpha \cap \alpha_i)}(\alpha), \tag{4.3}
\]

where the length \( l(\alpha | \alpha_i) \) plays locally the role of a lattice spacing.

---

\(^7\) This implies that \( V(h_{ij}) \) is independent of the orientation: \( V(h_{ij}) = V(h_{ji}) > 0 \).

\(^8\) We shall denote with \( \partial_\mu \) the partial derivative on the simplicial lattice to distinguish it from the one in the continuum \( \partial_\mu \).

\(^9\) Notice that \( l(\alpha | \alpha_i) \) is not symmetric: in general \( l(\alpha | \alpha_i) \neq l(\alpha_i | \alpha) \).
Equations (4.1) and (4.3) can be further simplified by noticing that the area vectors $V_{\mu}(\alpha_i)$ of a simplex $\alpha$ are not linearly independent and satisfy the well known relation:

$$
\sum_{i=1}^{d+1} V_{\mu}(\alpha_i) = 0.
$$

(4.4)

By using (4.4) we have then:

$$
\hat{\partial}_\mu \varphi(\alpha) = \frac{1}{2} \sum_{i=1}^{d+1} (\hat{x}^\nu(\alpha_i) - \hat{x}^\nu(\alpha)) \partial_\nu \varphi_c(\hat{x}(\alpha)) \frac{V_{\mu}(\alpha_i)}{V(\alpha)} = \frac{1}{2} \sum_{i=1}^{d+1} \frac{\varphi(\alpha_i)}{r(\alpha|\alpha_i)} d_{\mu}(\alpha_i) (\alpha).
$$

(4.5)

Consider now a field $\varphi_c(x)$ of the continuum theory and define the field $\varphi(\alpha)$ on the simplicial complex as:

$$
\varphi(\alpha) = \varphi_c(\hat{x}(\alpha)),
$$

(4.6)

where $\hat{x}^\nu(\alpha)$ are the coordinates of a point inside $\alpha$ that may be considered the ‘centre’ of $\alpha$. By inserting (4.6) into (4.5) and expanding around $\hat{x}^\nu(\alpha)$ we have:

$$
\hat{\partial}_\mu \varphi(\alpha) = \frac{1}{2} \sum_{i=1}^{d+1} (\hat{x}^\nu(\alpha_i) - \hat{x}^\nu(\alpha)) \partial_\nu \varphi_c(\hat{x}(\alpha)) \frac{V_{\mu}(\alpha_i)}{V(\alpha)} + O \left( (\hat{x}^\nu(\alpha_i) - \hat{x}^\nu(\alpha))^2 \right).
$$

(4.7)

If the simplices $\alpha$ and $\alpha_i$ are generic, the rhs of (4.7) cannot be calculated due to the ambiguity implicit in the choice of $\hat{x}^\nu(\alpha_i)$ and in general, even neglecting terms of second order in $\hat{x}^\nu(\alpha_i) - \hat{x}^\nu(\alpha)$, the partial derivatives $\hat{\partial}_\mu \varphi(\alpha)$ and $\partial_\nu \varphi_c(\hat{x}(\alpha))$ will not coincide. However if all the simplices involved are regular, then the sum at the rhs of (4.7) can be calculated and gives:

$$
\sum_{i=1}^{d+1} (\hat{x}^\nu(\alpha_i) - \hat{x}^\nu(\alpha)) V_{\mu}(\alpha_i) = 2V(\alpha)\delta^\nu_\mu,
$$

(4.8)

which implies:

$$
\hat{\partial}_\mu \varphi(\alpha) = \partial_\mu \varphi_c(\hat{x}(\alpha)) + O \left( (\hat{x}(\alpha_i) - \hat{x}(\alpha))^2 \right).
$$

(4.9)

It should also be noticed that in the case of regular hypercubic lattice, where $\alpha$ and $\alpha_i$ are $d$-dimensional hypercubes, $r(\alpha|\alpha_i)$ coincides with the lattice spacing and the derivative (4.5) reduces to the usual symmetric finite difference operation on the lattice. So, in a sense, the derivative $\hat{\partial}_\mu \varphi(\alpha)$ is a generalization to a simplicial lattice of the symmetric finite difference on a cubic lattice.

---

10 A possible choice is the circumcenter of $\alpha$, but this choice is not unique unless $\alpha$ is a regular simplex, which is the case considered below.

11 The explicit calculation is rather lengthy and is better done by choosing the same Euclidean coordinates and metric in all the simplices involved.
Given the form (4.5) for the derivative on a simplicial lattice it is immediate to write the action for a scalar field coupled with the metric tensor. In the continuum the action is:

\[ S_\varphi = \int d^d x \sqrt{g(x)} \left[ g^{\mu\nu}(x) \partial_\mu \varphi(x) \partial_\nu \varphi(x) + V(\varphi(x)) \right]. \]  

(4.10)

where \( V \) is an arbitrary potential. The corresponding simplicial action is simply obtained by replacing \( \int d^d x \sqrt{g(x)} \) with \( \sum_\alpha V(\alpha) \), the continuum variable \( x \) with the label \( \alpha \) of the simplex and the partial derivative \( \partial_\mu \) with \( \hat{\partial}_\mu \):

\[ S_\varphi = \sum_\alpha V(\alpha) \left[ g^{\mu\nu}(\alpha) \hat{\partial}_\mu \varphi(\alpha) \hat{\partial}_\nu \varphi(\alpha) + V(\varphi(\alpha)) \right]. \]  

(4.11)

The derivative \( \hat{\partial}_\mu \varphi(\alpha) \) in (4.11) can be now replaced by the rhs of (4.5), and the kinetic term becomes:

\[ S_\varphi^{\text{kin}} = \frac{1}{4} \sum_\alpha \sum_{i,j=1}^{d+1} \frac{g^{\mu\nu}(\alpha) V^{(\mu\nu\alpha\beta)}(\alpha) V^{(\mu\nu\alpha\beta)}(\alpha)}{V(\alpha)} \varphi(\alpha) \varphi(\alpha). \]  

(4.12)

The kinetic term (4.12) has a coupling between a simplex \( \alpha_i \) and a simplex \( \alpha_j \) that for \( i \neq j \) have in common only a \( d - 2 \) dimensional hinge (not a \( d - 1 \) dimensional face). This is different from the actions for scalar fields on a simplicial lattice previously used in the literature, where either the scalar fields were defined on the sites of the simplicial lattice [9, 10] or they were defined as in the present case on the simplices (the sites of the dual lattice) but with couplings only between simplices with a face in common [8].

5. Parallel transport and Christoffel symbol

In order to proceed in analogy with the Einstein theory of gravity we have now to introduce the notion of parallel transport. Consider a contravariant vector \( A^\mu(\alpha) \). According to (4.1) the derivative of \( A^\mu(\alpha) \) involves the differences \( A^\mu(\alpha) \rightarrow A^\mu(\alpha_i) \), which however are not vectors since the two terms of the difference transform with different matrices, respectively \( \Lambda(\alpha) \) and \( \Lambda(\alpha_i) \), under general coordinate transformations.

In order to define covariant differences (and then a covariant derivative) that transforms like vectors we need to introduce, as in the continuum case, the notion of parallel transport of a contravariant vector \( A^\mu(\beta) \) from a simplex \( \beta \) onto a neighbouring simplex \( \alpha \). We define the transported vector \( A^\mu_{\alpha(\beta)}(\beta) \) as:

\[ A^\mu(\beta) \rightarrow A^\mu_{\alpha(\beta)}(\beta) = K^\mu_{\alpha(\beta)} A^\mu(\beta) \]  

(5.1)

where \( K^\mu_{\alpha(\beta)}(\alpha | \beta) \) is entirely determined by the following properties:

- If \( g_{\mu\nu}(\alpha) = g_{\mu\nu}(\beta) \) then \( K^\mu_{\alpha(\beta)}(\alpha | \beta) = \delta^\mu_{\nu} \).
- \( A^\mu_{\alpha(\beta)}(\beta) \) transforms as a contravariant vector in \( \alpha \), namely it transforms with \( \Lambda^\nu_{\beta}(\alpha) \) under general coordinate transformations:

\[ A^\mu_{\alpha(\beta)}(\beta) = \Lambda^\nu_{\beta}(\alpha) A^\nu_{\alpha(\beta)}(\beta). \]  

(5.2)

The matrix \( K^\mu_{\alpha(\beta)}(\alpha | \beta) \) does not transform as a tensor but rather as link variable on the dual lattice. In fact from (5.2) and (5.1) one easily finds:

\[ K^\mu_{\alpha(\beta)}(\alpha | \beta) = \Lambda^\nu_{\beta}(\alpha) K^\nu_{\beta(\alpha)}(\alpha | \beta) \Lambda^{-1}_{\nu}(\beta). \]  

(5.3)
Notice that if we start in (5.3) from a coordinate system where \( g_{\mu\nu}(\alpha) = g_{\mu\nu}(\beta) \), then \( K^\mu_\nu(\alpha|\beta) = \delta^\mu_\nu \) and in a generic coordinate system \( K^\mu_\nu(\alpha|\beta) \) can always be written in the form:

\[
K^\mu_\nu(\alpha|\beta) = \Lambda^\mu_\rho|_\nu(\alpha)\Lambda^{-1}_\rho(\beta).
\]

(5.4)

It follows from (5.4) that \( K^\mu_\nu(\beta|\alpha) \) is the inverse of \( K^\mu_\nu(\alpha|\beta) \):

\[
K^{\mu}_{\nu}(\alpha|\beta)K^{\nu}_{\rho}(\beta|\alpha) = \delta^\mu_\rho.
\]

(5.5)

Scalar quantities are obviously invariant under parallel transform. This determines, together with (5.1), the parallel transport of a covariant vector:

\[
A_\rho(\beta) \Rightarrow A_{(\alpha)\rho}(\beta) = A_\rho(\beta)K^\rho_{\mu}(\beta|\alpha).
\]

(5.6)

Equations (5.1) and (5.6) can easily be generalised to tensors of arbitrary rank: in particular it follows from the definition of parallel transport that the parallel transport of \( g_{\mu\nu}(\beta) \) to a neighbour simplex \( \alpha \) coincides with \( g_{\mu\nu}(\alpha) \), so that the following identity holds:

\[
g_{\mu\nu}(\alpha) = g_{\rho\sigma}(\beta)K^\rho_{\mu}(\beta|\alpha)K^\sigma_{\nu}(\beta|\alpha).
\]

(5.7)

The last equation defines \( K^\rho_{\mu}(\alpha|\beta) \) implicitly as a function of the metric tensor, but unlike the continuum case it is quadratic in \( K^\rho_{\mu}(\alpha|\beta) \). Therefore \( K^\rho_{\mu}(\alpha|\beta) \) cannot be expressed, as in the continuum, by linear combinations of the derivatives of the metric tensor unless the equation is linearized by neglecting higher order terms in the lattice constant \( l(\alpha|\beta) \) (see discussion below).

In a hypothetical first order formulation, analogue of the Palatini formalism of the continuum theory, \( K^\rho_{\mu}(\alpha|\beta) \) and \( g_{\mu\nu}(\alpha) \) would be treated as independent dynamical variables and equation (5.7) should arise from the eq.s of motion. We are not going to discuss this formulation in the present paper.

The variation of \( A^\nu(\beta) \) as a result of the parallel transport from \( \beta \) to \( \alpha \) is then given by:

\[
\delta_{\beta\rightarrow\alpha}A^\nu(\beta) = A^\nu_{(\alpha)}(\beta) - A^\nu(\beta) = [K^\rho_{\mu}(\alpha|\beta) - \delta^\rho_\nu]A^\nu(\beta).
\]

(5.8)

Notice that in (5.8) only the component of \( A^\nu(\beta) \) orthogonal to the face \( \alpha \cap \beta \) contributes to the variation. In fact from (5.4) and (3.5) we have:

\[
[K^\rho_{\mu}(\alpha|\beta) - \delta^\rho_\nu](x_j^\gamma - x_j^\beta) = 0 \quad i, j \in \alpha \cap \beta.
\]

(5.9)

Consider now a contravariant vector \( A^\mu(\alpha) \). The covariant difference between two neighbouring simplices \( \alpha \) and \( \beta \) is defined as:

\[
\hat{DA}^\mu(\alpha|\beta) = A^\mu_{(\alpha)}(\beta) - A^\mu(\beta) = A^\mu(\beta) - A^\mu(\alpha) + \delta_{\beta\rightarrow\alpha}\Lambda^\mu_\nu A^\nu(\beta)
\]

(10.10)

and it transforms as a contravariant vector in \( \alpha \):

\[
\hat{DA}^\mu(\alpha|\beta)' = \Lambda^\mu_\rho(\alpha)\hat{DA}^\rho(\alpha|\beta).
\]

(10.11)

The covariant difference is not antisymmetric under exchange of \( \alpha \) and \( \beta \), but rather it satisfies the relation:

\[
\hat{DA}^\mu(\beta|\alpha) = -K^\rho_{\mu}(\beta|\alpha)\hat{DA}^\rho(\alpha|\beta).
\]

(10.12)
The covariant derivative of a contravariant vector is obtained by replacing in equation (4.3) the differences with the corresponding covariant differences:

$$
\hat{D}_\mu A^\nu(\alpha) = \frac{1}{2} \sum_{i=1}^{d+1} K^\nu_\mu(\alpha|\alpha_i) A^\alpha(\alpha_i) - A^\nu(\alpha) \eta^\alpha_{(\alpha_i)}(\alpha)
$$

(5.13)

where $\eta(\alpha|\alpha_i)$ is given by (4.2). As in the case of ordinary derivatives equation (5.13) can be further simplified according to equation (4.4):

$$
\hat{D}_\mu A^\nu(\alpha) = \frac{1}{2} \sum_{i=1}^{d+1} K^\nu_\mu(\alpha|\alpha_i) A^\alpha(\alpha_i)
$$

(5.14)

The covariant derivative (5.14) can be written as the sum an ordinary derivative plus a term involving a discrete analogue $\Gamma^\nu_{\rho\mu}(\alpha|\alpha_i)$ of the Christoffel symbol:

$$
\hat{D}_\mu A^\nu(\alpha) = \hat{\partial}_\mu A^\nu(\alpha) + \frac{1}{2} \sum_{i=1}^{d+1} \Gamma^\nu_{\rho\mu}(\alpha|\alpha_i) A^\rho(\alpha_i),
$$

(5.15)

where

$$
\Gamma^\nu_{\rho\mu}(\alpha|\alpha_i) = \frac{K^\nu_{\rho\mu}(\alpha|\alpha_i) - \delta^\nu_\rho \eta^\mu_{(\alpha_i)}(\alpha)}{\eta(\alpha|\alpha_i)}.
$$

(5.16)

Notice that according to equation (5.9) $\Gamma^\nu_{\rho\mu}(\alpha|\alpha_i)$ is different from zero only when the index $\rho$ is orthogonal to the face $\alpha \cap \alpha_i$, hence the Christoffel symbol can be written in the form:

$$
\Gamma^\nu_{\rho\mu}(\alpha|\alpha_i) = \tilde{\Gamma}^\nu_{\rho\mu}(\alpha|\alpha_i) n^\mu_{(\alpha_i)}(\alpha),
$$

(5.17)

where

$$
\tilde{\Gamma}^\nu_{\rho\mu}(\alpha|\alpha_i) = \frac{K^\nu_{\rho\mu}(\alpha|\alpha_i) - \delta^\nu_\rho \eta^\mu_{(\alpha_i)}(\alpha)}{\eta(\alpha|\alpha_i)}.
$$

(5.18)

The symmetry of $\Gamma^\nu_{\rho\mu}(\alpha|\alpha_i)$ in the indices $\nu$ and $\rho$ is obvious from equation (5.17).

In the continuum theory the Christoffel symbol can be expressed in terms of the derivatives of the metric tensor. We can try to do the same thing here by writing equation (5.7) in terms of the Christoffel symbol (5.16). We find:

$$
g_{\alpha \beta}(\alpha) = g_{\mu \nu}(\beta)
\frac{\eta(\alpha|\beta)}{\eta(\alpha|\beta)} = \eta^{\alpha(\nu)\beta(\mu)}(\alpha) \left[ \Gamma^\rho_{\mu\nu}(\alpha|\beta) g_{\rho \nu}(\beta) + \Gamma^\rho_{\nu\mu}(\alpha|\beta) g_{\rho \mu}(\beta) \right]
+ \eta(\alpha|\beta) \eta^{\alpha(\nu)\beta(\mu)}(\alpha) \Gamma^\rho_{\nu\mu}(\alpha|\beta) \Gamma^\sigma_{\nu\rho}(\alpha|\beta) g_{\sigma \mu}(\beta).
$$

(5.19)

The lhs of (5.19) is essentially the derivative of the metric tensor along a direction orthogonal to the face $\alpha \cap \beta$. The rhs consists of a linear term, that resembles the one of the continuum theory, and of a quadratic term which however is of order $l(\alpha|\beta)$ and hence vanishes in the continuum limit.

On the lattice the Christoffel symbol depends only on the $d$ components of $\Gamma^\nu(\alpha|\beta)$ as...
defined in (5.18), and it is then convenient to express (5.19) in terms of $\Gamma^\mu(\alpha|\beta)$:

$$\frac{g_{\mu\nu}(\alpha) - g_{\mu\nu}(\beta)}{l(\alpha|\beta)} = n^{(\alpha|\beta)}_\mu(\alpha)g_{\mu\nu}(\beta)\Gamma^\nu(\alpha|\beta) + n^{(\alpha|\beta)}_\nu(\alpha)g_{\mu\nu}(\beta)\Gamma^\mu(\alpha|\beta)$$

$$+ l(\alpha|\beta)n^{(\alpha|\beta)}_\mu(\alpha)n^{(\alpha|\beta)}_\nu(\alpha)g_{\mu\nu}(\beta)\Gamma^\nu(\alpha|\beta)\Gamma^\mu(\alpha|\beta).$$

(5.20)

If we contract equation (5.20) with $n^{(\alpha|\beta)_\mu(\alpha)}$ and with $(x'_i - x'_j)$ $(i, j \in \alpha \cap \beta)$, then equation (5.20) becomes linear:

$$n^{(\alpha|\beta)_\mu(\alpha)}(x'_i - x'_j) \frac{g_{\mu\nu}(\alpha) - g_{\mu\nu}(\beta)}{l(\alpha|\beta)} = (x'_i - x'_j) g_{\mu\nu}(\beta)\Gamma^\nu(\alpha|\beta).$$

(5.21)

This equation shows that all components of $g_{\mu\nu}(\beta)\Gamma^\nu(\alpha|\beta)$ with the index $\nu$ belonging to the $d - 1$ dimensional subspace $\alpha \cap \beta$ can be expressed also on the lattice as derivatives of the metric tensor. Instead the perpendicular component $n^{(\alpha|\beta)_\mu(\alpha)}g_{\mu\nu}(\beta)\Gamma^\nu(\alpha|\beta)$ is solution of the quadratic equation

$$n^{(\alpha|\beta)_\mu(\alpha)}n^{(\alpha|\beta)_\nu(\alpha)}g_{\mu\nu}(\alpha) - g_{\mu\nu}(\beta) = 2n^{(\alpha|\beta)_\mu(\alpha)}g_{\mu\nu}(\beta)\Gamma^\nu(\alpha|\beta)$$

$$+ l(\alpha|\beta)g_{\mu\nu}(\beta)\Gamma^\nu(\alpha|\beta)\Gamma^\nu(\alpha|\beta)$$

(5.22)

which becomes linear only in the limit $l(\alpha|\beta) \to 0$.

We conclude this section with the proof that the divergence theorem for an arbitrary contravariant vector $A^\mu(\alpha)$, defined on a $d$-dimensional simplicial complex with boundary, is exactly satisfied.

In the continuum the divergence theorem states that given a contravariant vector $A^\mu(x)$ on a $d$-dimensional manifold $\mathcal{M}$ the following identities hold:

$$\int_{\mathcal{M}} \, dx \sqrt{g(x)} D_\mu A^\mu(x) = \int_{\partial \mathcal{M}} \, dx \sqrt{h(x)} n_\mu(x) A^\mu(x),$$

(5.23)

where $h(x)$ is the determinant of the metric on $\partial \mathcal{M}$ induced by pulling back the metric from $\mathcal{M}$ and $n_\mu(x)$ is the unit vector orthogonal to $\partial \mathcal{M}$ in $x$.

We consider now a triangulation of $\mathcal{M}$, namely a simplicial complex $\hat{\mathcal{M}}$ whose boundary $\partial \hat{\mathcal{M}}$ is made of $d - 1$ dimensional simplices. Each simplex in $\partial \mathcal{M}$ is a face of some simplex in $\hat{\mathcal{M}}$. We also associate to the boundary $\partial \mathcal{M}$ a layer of $d$-dimensional simplices, which we shall denote $\mathcal{M}_B$, defined as the ensemble of simplices of $\hat{\mathcal{M}}$ which have at least one face belonging to the boundary $\partial \mathcal{M}$.

We can now write the simplicial analogue of the lhs of (5.23) as:

$$\sum_{\alpha \in \mathcal{M}} V(\alpha) \hat{D}_\mu A^\mu(\alpha) = \frac{1}{2} \sum_{\alpha, \beta \in \mathcal{M}} \left[ K^\mu_\rho(\alpha|\beta) A^\rho(\beta) - A^\mu(\alpha) \right] V^{(\alpha|\beta)}_\rho(\alpha),$$

(5.24)

where the sum at the rhs is understood to extend over all pairs of simplices $\alpha$ and $\beta$ that have a $d - 1$ dimensional face in common. We can now use on the rhs of (5.24) the identity

$$V^{(\alpha|\beta)}_\rho(\alpha) K^\mu_\rho(\alpha|\beta) = - V^{(\alpha|\beta)}_\rho(\beta)$$

(5.25)

which holds for any pair of neighbouring simplices $\alpha$ and $\beta$. Equation (5.25) can be easily proved by just going from generic coordinates to a choice of coordinates where $g_{\mu\nu}(\alpha) = g_{\mu\nu}(\beta)$.
By applying (5.25) in (5.24) one finds that the two terms at the rhs of (5.24) become identical, modulo an irrelevant exchange of the labels $\alpha$ and $\beta$. We have:

$$\sum_{\alpha \in \mathcal{M}} V(\alpha) \dot{D}_\mu A^\mu(\alpha) = - \sum_{\alpha \in \mathcal{M}} A^\mu(\alpha) \sum_{\beta \in \mathcal{M}} V^{(\alpha \beta)}(\alpha),$$

(5.26)

where the sum over $\beta$ only extends to the simplices in $\mathcal{M}$ that have a face in common with $\alpha$. If none of the faces of $\alpha$ belongs to the boundary $\partial \mathcal{M}$, namely if $\alpha \notin \mathcal{M}_B$, then the sum over $\beta$ at the rhs of (5.26) vanishes identically (see equation (4.4)).

On the other hand if $\alpha \in \mathcal{M}_B$, namely if one (or more) of the faces of $\alpha$ belong to $\partial \mathcal{M}$, then the sum over $\beta$ at the rhs of (5.26) does not vanish and is given by the sum\(^{12}\) (with the sign changed) of the terms missing in the sum over $\beta$, which obviously correspond to $d - 1$ dimensional simplices belonging to the boundary $\partial \mathcal{M}$.

In order to write the final form of equation (5.26) it is convenient to introduce some new notation. Let us denote the $d - 1$ dimensional simplices belonging to $\partial \mathcal{M}$ with $\bar{\alpha}$, $\bar{\beta}$, etc and the corresponding $d$ dimensional simplices by the same greek letters without bar: $\bar{\alpha} \subset \alpha$, $\bar{\beta} \subset \beta$, etc. We can also define the contravariant vector $A^\mu$ on the boundary by simply putting:

$$A^\mu(\bar{\alpha}) = A^\mu(\alpha) \quad \bar{\alpha} \in \alpha.$$  

(5.27)

Then we can write equation (5.26) as:

$$\sum_{\alpha \in \mathcal{M}} V(\alpha) \dot{D}_\mu A^\mu(\alpha) = \sum_{\bar{\alpha} \in \partial \mathcal{M}} A^\mu(\bar{\alpha}) V^{(\bar{\alpha} \alpha)}(\alpha) = \sum_{\bar{\alpha} \in \partial \mathcal{M}} V(\bar{\alpha}) \eta^{(\bar{\alpha} \alpha)}(\alpha) A^\mu(\bar{\alpha})$$  

(5.28)

where $V(\bar{\alpha})$ is the $d - 1$ dimensional volume of $\bar{\alpha}$. Equation (5.28) is the divergence theorem on a simplicial lattice. It is remarkable that it is an exact result on the lattice and that it has a precise correspondence with the continuum case given in equation (5.23).

### 6. Riemann curvature tensor, Bianchi identities and Einstein action

We shall now introduce the Riemann curvature tensor, which in a piecewise flat simplicial manifold is localized on the $d - 2$ dimensional hinges.

Following the notation introduced at the end of section 3, given a simplex $\alpha$ and two neighbouring simplices $\alpha_i$ and $\alpha_j$ we denote by $h_{ij}$ the hinge that they have in common.

We now define a closed path $\gamma_{h_{ij}}$ that starting from $\alpha$ goes all round the hinge $h_{ij}$, more precisely:

$$\gamma_{h_{ij}} \equiv \alpha \to \alpha_1 \to \beta_1 \to \ldots \to \beta_k \to \alpha_j \to \alpha$$  

(6.1)

where $\beta_r$ with $r = 1, 2, \ldots, h$ are the other simplices that have $h_{ij}$ as a hinge. Notice that $\gamma_{h_{ij}}$ corresponds to a plaquette in the dual lattice and that $\gamma_{h_{ij}}$ denotes the same path taken in opposite direction.

Given a contravariant vector $A^\mu(\alpha)$ we can define, according to the definitions of the previous section, the parallel transport of $A^\mu(\alpha)$ around the hinge $h_{ij}$ along $\gamma_{h_{ij}}$ starting and arriving in $\alpha$. The variation of $A^\mu(\alpha)$ under parallel transport along $\gamma_{h_{ij}}$ is then given by:

$$\delta_{\gamma_{h_{ij}}} A^\mu(\alpha) = R^\mu_{\mu'}(\gamma_{h_{ij}}) A^{\mu'}(\alpha)$$  

(6.2)

\(^{12}\) In general the sum is made of a single term, but it is possible for a simplex to have more than one boundary face.
with
\[ R^\mu_\rho(\gamma_{h_1}) = K^\mu_{\nu_1}(\alpha|\alpha_i)K^{\nu_1}_{\nu_2}(\alpha|\beta_1) \ldots K^{\nu_{h-1}}_{\nu_h}(\beta_h|\alpha_j)K^{\nu_h}_{\nu_{h+1}}(\alpha_j|\alpha) - \delta^\mu_\rho. \] (6.3)

From the transformation property of \( K^\mu_\rho(\alpha|\beta) \) given in (5.3) it follows immediately that \( R^\mu_\rho(\gamma_{h_1}) \) transforms as a mixed tensor with one covariant and one contravariant index. The index \( \mu \) in \( R^\mu_\rho(\gamma_{h_1}) \) can be lowered to define a covariant tensor of rank two:
\[ R_{\mu\rho}(\gamma_{h_1}) = g_{\mu\nu}(\alpha)R^\nu_\rho(\gamma_{h_1}). \] (6.4)

The curvature tensor \( R_{\mu\rho}(\gamma_{h_1}) \) satisfies the symmetry relation:
\[ R_{\mu\rho}(\gamma_{h_1}) = R_{\rho\mu}(\gamma_{h_1}), \] (6.5)
where \( \gamma_{h_1} \) is the same path as \( \gamma_{h_1} \) but taken in the opposite direction. This property follows from an analogue property of \( K^\mu_\rho(\alpha|\beta) \), namely:
\[ K_{\mu\nu}(\alpha|\beta) = K_{\nu\mu}(\beta|\alpha), \] (6.6)
where the metric tensor has been used to lower indices in \( K \) and equations (5.7) and (5.5) have been applied. Repeated use of (6.6) leads to equation (6.5).

In equation (6.2) only the components of \( A^\mu(\alpha) \) orthogonal to the hinge are modified under parallel transport along \( \gamma_{h_1} \), that is the only non vanishing elements of the curvature matrix \( R^\mu_\rho(h_{1}) \) are the ones where the index \( \rho \) is orthogonal to the hinge \( h_{1} \). This is a direct consequence of equation (5.9) and thanks to the symmetry (6.5) this property applies to both covariant indices in \( R_{\mu\rho}(\gamma_{h_1}) \):
\[ R_{\mu\rho}(\gamma_{h_1})(x_{\nu}^i - x_{\nu}^j) = (x_{\nu}^i - x_{\nu}^j)R_{\mu\rho}(\gamma_{h_1}) = 0 \quad r, s \in h_{1} \rightarrow r, s \neq i, j. \] (6.7)

The curvature \( R_{\mu\rho}(\gamma_{h_1}) \) is then entirely contained in the two dimensional subspace spanned by the orthonormal base vectors \( n_{\mu}^{(1)}(\alpha) \) and \( n_{\mu}^{(2)}(\alpha) \) introduced in (3.19). Since the orthonormality relations are preserved under parallel transport, the effect of a parallel transport along \( \gamma_{h_1} \) can only be a rotation by an angle \( \theta_{ij} \) of the orthonormal base vectors in this two dimensional subspace. The angle \( \theta_{ij} \) can be identified as the deficit angle of the Regge calculus associated to the hinge \( h_{1} \).

An explicit expression for \( R_{\mu\rho}(\gamma_{h_1}) \) in terms of the deficit angle and of the vectors \( n_{\mu}^{(a)}(\alpha) \) \((a \in \{i, j\})\) can then be written, and reads:
\[ R_{\mu\rho}(\gamma_{h_1}) = (\cos \theta_{ij} - 1) \sum_{a \in \{i, j\}} n_{\rho}^{(a)}(\alpha) n_{\mu}^{(a)}(\alpha) + \sin \theta_{ij} n_{\mu\rho}^{(ij)}(\alpha), \] (6.8)
where \( n_{\mu\rho}^{(ij)}(\alpha) \) is given in (3.22). Notice that the first term at the rhs of (6.8), which is of order \( \theta_{ij}^2 \) for small deficit angles, is independent of the orientation of the hinge, whereas the second term (order \( \theta_{ij} \)) changes sign if the orientation of the hinge is reversed. However equation (6.8) is consistent with (6.5) because \( n_{\mu\rho}^{(ij)}(\alpha) \) is antisymmetric in both pairs of indices \( ij \) and \( \mu\rho \).

The term in \( \sin \theta_{ij} \) is the relevant one in Regge calculus and for that reason we shall use the antisymmetric combination
\[ R_{\mu\rho}^{\perp}(\gamma_{h_1}) = \frac{1}{2} [R_{\mu\rho}(\gamma_{h_1}) - R_{\rho\mu}(\gamma_{h_1})] = \sin \theta_{ij} n_{\mu\rho}^{(ij)}(\alpha). \] (6.9)
As shown below the use of \( R_{\mu\nu}^{\rho\sigma}(\gamma_{hij}) \) in place of \( R_{\mu\nu}(\gamma_{hij}) \), besides eliminating the higher order term in \( \cos \theta_{ij} \), leads to a Riemann tensor which is independent of orientation of the hinge, an important feature for an unambiguous definition of the lattice action.

The first set of indices of the Riemann tensor can be identified with the two covariant indices in \( R_{\mu\nu}^{\rho\sigma}(\gamma_{hij}) \) and describe the rotation of a vector under parallel transport around the loop \( \gamma_{hij} \). The second set of indices describe the spatial orientation of the loop. They are contracted with the area element (in the continuum: \( dx^\mu \wedge dx^\nu \)) and on the simplicial lattice they should be orthogonal to the hinge \( h_{ij} \) and hence proportional to \( V_{\mu\nu}(\alpha) \) as defined in (3.15).

In conclusion, the curvature tensor with four covariant indices should have the form:

\[
R_{\mu\nu,\rho\sigma}(\gamma_{hij}) = R_{\mu\nu}(\gamma_{hij}) V_{\rho\sigma}(\alpha) = \sin \theta_{ij} V(h_{ij}) n^{\mu\nu}_{\rho\sigma}(\alpha) n^{\rho\sigma}_{\mu\nu}(\alpha).
\]  

For dimensional reasons the quantity \( v(h_{ij}) \) is a \( d \)-volume and can be identified with the support volume of the hinge \( h_{ij} \). A precise and rigorous definition of the support volume can be found in [16], it will suffice here to know that a point \( P \) of the simplicial complex belongs to the support of a hinge \( h_{ij} \) if its minimal distance from a point of \( h_{ij} \) is less than that from any other hinge of the complex. It is also useful, as we shall see later on in this section, to define the volume \( v(h_{ij}|\alpha) \), namely the volume of the part of the support of \( h_{ij} \) that belongs to a given simplex \( \alpha \). The following relations then obviously hold:

\[
\sum_{\alpha|\alpha\ni h_{ij}} v(h_{ij}|\alpha) = v(h_{ij}),
\]

\[
\sum_{h_{ij}|h_{ij}\in \alpha} v(h_{ij}|\alpha) = V(\alpha).
\]

It is easy to check that the usual algebraic symmetries of the Riemann tensor are identically satisfied, namely the invariance under exchange of the two pairs of indices and the first Bianchi identity:

\[
R_{\mu\nu,\rho\sigma}(\gamma_{hij}) + R_{\mu\nu,\sigma\nu}(\gamma_{hij}) + R_{\mu\nu,\nu\rho}(\gamma_{hij}) = 0.
\]

We shall briefly discuss now, in the context of our approach, the second Bianchi identity. This is a differential identity, whose formulation on a simplicial complex was already outlined in the original Regge paper [1] and discussed in detail in [11].

First we prove that there is one Bianchi identity associated to each \( d-3 \) dimensional sub-simplex, which we name \( \sigma_{d-3} \), of the simplicial complex. Let us consider the dual lattice, namely the Voronoi tassellation generated by the vertices of the simplicial lattice. The dual of \( \sigma_{d-3} \) is a three-dimensional polytope (polyhedron) \( \star\sigma_{d-3} \) whose faces are the plaquettes which are dual of the hinges that contain \( \sigma_{d-3} \) as a sub-simplex.

Let \( f \), \( v \) and \( s \) be respectively the number of faces, vertices and edges of the boundary of \( \star\sigma_{d-3} \), which we shall assume has the topology of a sphere. Then the Euler relation holds, that we write as:

\[
f = (s - v) + 2.
\]
Figure 1. A three dimensional polytope $\ast\sigma_{d-3}$ with seven plaquettes showing the basic steps to derive the second Bianchi identity (see text).

(links), and by choosing the coordinate system in one of the simplices at the ends of the link the transport matrix can be made equal to the identity. This can be described as one link collapsing to a point with the two vertices at the ends becoming a single vertex. This new vertex does not correspond to a $d$-simplex anymore, but to the union of two simplices with the same metric. This procedure can be repeated $v-1$ times, until there is only one vertex left, a coordinate transformation on this last vertex being just an overall transformation. If we denote by $s'$ the number of links left, namely the number of links where the parallel transport is non trivial, we have:

$$f = s' + 1.$$  \hspace{1cm} (6.15)

The curvatures (and the corresponding deficit angles) associated to each of the $f$ plaquettes (hinges) are obtained from products of transport matrices of the $s'$ links and since $f - s' = 1$ they cannot be independent and must be related by one (and only one) identity (second Bianchi identity).

In order to find a more explicit form for the second Bianchi identity let us follow the track of Regge’s original paper. The procedure is illustrated in figure 1 where an example of $\ast\sigma_{d-3}$ with seven plaquettes is represented. Let us denote by $y_i (i = 1, \ldots, f)$ the centres of the $f$ plaquettes
in $\ast\sigma_{d-3}$. By joining $y_i$ to $y_{i+1}$ modulo $f$ let us now construct a closed path $\gamma$ that divides the boundary of $\ast\sigma_{d-3}$ in two regions $\mathcal{M}$ and $\mathcal{M}'$. Let us choose in $\mathcal{M}$ (resp. $\mathcal{M}'$) a vertex of $\ast\sigma_{d-3}$ that corresponds to a given simplex $\alpha$ (resp. $\beta$) and denote by $\gamma_i$ a path that goes from $\alpha$ to $\beta$ along a sequence of links and crosses $\gamma$ in the section between $y_i$ and $y_{i+1}$. Let us now define $a_i = \gamma_i^{-1} \gamma_i$ and notice that $a_i$ is a closed path that starts and ends in $\alpha$ and encircles the $i$th plaquette. Consider now the rotation matrix that describes the parallel transport along $a_i$:

$$S^\rho_\nu(a_i) = K^\rho_\nu(\alpha|\alpha_1^{(i)}) K^\rho_\nu(\alpha_2^{(i)}|\alpha_1^{(i)}) \cdots K^\rho_\nu(\alpha_1^{(i)}|\alpha) = \delta^\rho_\nu + R^\rho_\nu(h_i|\alpha)$$  \hspace{1cm} (6.16)

where $\alpha_k^{(i)}$ ($k = 1, \ldots, l$) are the $l$th simplices along the path $a_i$ and $R^\rho_\nu(h_i|\alpha)$ is the curvature matrix associated to the $i$th plaquette. The latter is obtained by going around the $i$th plaquette following the path $a_i$ starting and ending in $\alpha$; this is the same as going around the plaquette starting and ending in a simplex $\alpha'$ on the plaquette and then performing the parallel transport of the resulting curvature matrix from $\alpha'$ to $\alpha$ always following the path of $a_i$.

The second Bianchi identity on the simplicial lattice is then given by the identity;

$$S^\rho_\nu(a_1) S^\nu_\mu(a_2) \cdots S^\rho_\nu(a_f) = (\delta^\rho_\nu + R^\rho_\nu(h_1|\alpha)) \cdots (\delta^\rho_\nu + R^\rho_\nu(h_f|\alpha)) = \delta^\rho_\nu.$$  \hspace{1cm} (6.17)

Notice that (6.17) is not linear in the curvatures, and it becomes linear only in the limit of small curvatures, namely in the limit of small deficit angles, where it takes the form:

$$\sum_{i=1}^{f} R^\rho_\nu(h_i|\alpha) \approx 0.$$  \hspace{1cm} (6.18)

Let us go back now to the curvature tensor $R_{\mu\nu,\rho\sigma}(\gamma_{h_j})$ given in equation (6.10). By contracting pairs of indices in $R_{\mu\nu,\rho\sigma}(\gamma_{h_j})$ with the inverse metric $g^{\mu\rho}(\alpha)$ one obtains the Ricci tensor. This is given by:

$$R_{\nu\sigma}(\gamma_{h_j}) = \frac{\sin \theta_{h_j} V(h_j)}{v(h_j)} \left[ n_{ij}^{\mu}(\alpha)n_{ij}^{\nu}(\alpha) + n_{ij}^{\nu}(\alpha)n_{ij}^{\mu}(\alpha) \right] \frac{n_{ij}^{\mu}(\alpha)n_{ij}^{\nu}(\alpha)}{2v(h_j)}.$$  \hspace{1cm} (6.19)

By further contraction of the Ricci tensor with $g^{\nu\sigma}(\alpha)$ we obtain the curvature scalar:

$$R(\gamma_{h_j}) = \frac{\sin \theta_{h_j} V(h_j)}{v(h_j)}.$$  \hspace{1cm} (6.20)

The above expressions for the Riemann curvature tensor (6.10), for the Ricci tensor (6.19) and for the curvature scalar (6.20) give the contribution to the curvature coming from a particular hinge $h_j$.

In order to have a direct correspondence with the continuum case we can define a Riemann tensor associated to each simplex $\alpha$ by taking a weighted sum over all the $\frac{d(d+1)}{2}$ hinges $h_{ij}$ that belong to $\alpha$:

$$R_{\mu\nu,\rho\sigma}(\alpha) = \sum_{h_{ij} \in \alpha} \frac{v(h_{ij}|\alpha)}{V(\alpha)} R_{\mu\nu,\rho\sigma}(\gamma_{h_{ij}})$$  \hspace{1cm} (6.21)

13 Each plaquette is dual to the hinge $h_i \equiv [x_i^1, \ldots, x_i^{d-2}, y_i^d]$ where $y_i^d$ is the additional vertex of $h$ that does not belong to $\sigma_{d-3}$ which is given in this notation by $\sigma_{d-3} \equiv [x_i^1, \ldots, x_i^{d-2}]$. The centre of the $i$th plaquette corresponds to the position of $y_i^d$. 

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where \( v(h_{ij} | \alpha) \) has been defined in the discussion following equation (6.10) and the ratio \( \frac{v(h_{ij} | \alpha)}{V(\alpha)} \) corresponds to the fraction of the volume \( V(\alpha) \) that belongs to the support of \( h_{ij} \).

Similarly for the curvature scalar we can define:

\[
\mathcal{R}(\alpha) = \sum_{h_{ij} \in \alpha} \frac{v(h_{ij} | \alpha)}{V(\alpha)} \mathcal{R}(\gamma_{h_{ij}}) = \frac{1}{V(\alpha)} \sum_{h_{ij} \in \alpha} \frac{v(h_{ij} | \alpha)V(h_{ij})}{v(h_{ij})} \sin \theta_{h_{ij}}.
\]

(6.22)

\( \mathcal{R}(\alpha) \) is the true lattice analogue of the curvature scalar \( R(x) \) of Einstein continuum theory. Given the correspondence between the simplex volume \( V(\alpha) \) (see equation (3.8)) and the integration volume of the continuum \( \sqrt{g} \, d^d x \) the lattice equivalent of the Einstein–Hilbert action is then:

\[
S_l = k \sum_{\alpha} \mathcal{R}(\alpha) V(\alpha) = k \sum_h \sin \theta_h V(h)
\]

(6.23)

where \( k \) is the Newton constant and the sum at the rhs is over all hinges of the simplicial complex. Notice also that equation (6.11) has been used after exchanging summations in obtaining the last expression in (6.23).

The action at the rhs of (6.23) coincides with Regge’s action only in the limit of small deficit angles as it contains \( \sin \theta_h \) in place of \( \theta_h \). This is not a new feature (see reference [5, 6]) and it seems a natural one in the present approach where \( \theta_h \) is a compact parameter.

However replacing \( \theta_h \) with \( \sin \theta_h \) is not without consequences, particularly in 2 and 3 space–time dimensions.

Let us consider the two dimensional case first. In two dimensions Regge’s action is simply

\[
S_R^{(d=2)} = \sum_h \theta_h
\]

(6.24)

and like in the continuum theory it is a topological invariant being proportional to the Euler characteristic of the two dimensional simplicial complex. This is not true anymore if \( \theta_h \) is replaced in the action by \( \sin \theta_h \), and the topological nature of the action is only recovered in the limit of small deficit angle, namely in the continuum limit.

Consider now the variation of Regge’s action in an arbitrary dimension (1.1) with respect to the lengths \( l_{ij} \) of the edges which are the dynamical variables in standard Regge calculus. It was shown in Regge’s original paper that the variation, in analogy to what happens in the continuum, is given simply by:

\[
\delta S_R = \sum_h \theta_h \delta |V|h
\]

(6.25)

This is because of the identity

\[
\sum_h \delta \theta_h |V|h = 0
\]

(6.26)

which was also proved in [1].

In three dimensions the volumes \( |V|h \) of the one dimensional hinges are just the lengths \( l_h \) of the edges, and the variational principle \( \delta S_R^{(d=3)} = 0 \) leads to the equation \( \theta_h = 0 \) in agreement with the continuum result and with the fact that pure gravity has no dynamics in three dimensions.
The last result does not hold anymore if the action has the \( \sin \theta_h \) term, since its variation with respect to \( \theta_h \) does not vanish identically thanks to equation (6.26) but only up to terms quadratic in the deficit angles.

The construction of the boundary term for Regge’s action is another problem where the identity (6.26) plays a fundamental role. This problem was solved by Hartle and Sorkin [12] and the vanishing of the variation of the action with respect to the deficit angles \( \theta_h \) is crucial to the derivation.

Finding the analogue of the Gibbons–Hawking–York boundary term for the action (6.23), is a non-trivial (if solvable at all) problem, and we are not going to address it in this paper.

7. Higher derivatives and Brans–Dicke actions

Gravitational theories that contain higher derivative terms, namely terms that are quadratic or of higher order in the curvature scalar or in the Riemann tensor, have received a lot of attention in recent years (see for instance [13] and references therein).

In the original Regge calculus the curvature is associated to the \( d - 2 \) dimensional hinges of the simplicial complex. However we have shown in the previous section that the Riemann tensor can be associated to each simplex \( \alpha \) by taking a suitable average over all the hinges belonging to \( \alpha \) (see equation (6.21)).

This gives a straightforward prescription for writing on the simplicial lattice any arbitrary gravitational action in any dimension. It is sufficient to replace the \( d \)-dimensional invariant integration volume with the sum of the volume \( V(\alpha) \) over all simplices and replace the Riemann tensor (and its contractions) with its discrete counterpart (6.21) on the simplicial lattice:

\[
\int d^d x \sqrt{g(x)} R \Rightarrow \sum_{\alpha} V(\alpha), \quad (7.1)
\]

\[
R_{\mu\nu\rho\sigma}(x) \Rightarrow R_{\mu\nu\rho\sigma}(\alpha). \quad (7.2)
\]

This correspondence has already been shown in (6.23) to reproduce Regge’s original action\textsuperscript{14}, and the rhs of equation (7.1) obviously gives the simplicial version of the cosmological term.

In the case of gravitational actions with quadratic or higher order terms in the curvature the correspondence between the action in the continuum and the one on a simplicial lattice expressed in terms of the deficit angles \( \theta_h \) is not unique and the prescription given in (7.2) provides a well defined and consistent way of constructing the lattice action.

For instance for quadratic terms in the curvature equation (7.2) gives:

\[
\int d^d x \sqrt{g(x)} R^2 \Rightarrow \sum_{\alpha} V(\alpha) R^2(\alpha), \quad (7.3)
\]

\[
\int d^d x \sqrt{g(x)} R^\mu{}^{\nu}(x) R_{\mu\nu}(x) \Rightarrow \sum_{\alpha} V(\alpha) R^\mu{}^{\nu}(\alpha) R_{\mu\nu}(\alpha), \quad (7.4)
\]

\[
\int d^d x \sqrt{g(x)} R^\mu{}^{\nu\rho\sigma}\mu\nu\rho\sigma \Rightarrow \sum_{\alpha} V(\alpha) R^\mu{}^{\nu\rho\sigma}(\alpha) R_{\mu\nu\rho\sigma}(\alpha). \quad (7.5)
\]

\textsuperscript{14}It is always understood that the deficit angle \( \theta_h \) is replaced here by \( \sin \theta_h \).
An explicit expression of the rhs in (7.3)–(7.5) in terms of the deficit angles can be easily obtained by replacing in them the Riemann curvature and its contractions as given in equations (6.21) and (6.22).

We are not interested here in the detailed expressions, except for remarking that they contain mixed terms in the deficit angles of the form $\sin \theta_h \sin \theta_{h'}$ where $h$ and $h'$ are neighbouring hinges, namely hinges that have a simplex in common.

This is different from the simplest way of expressing $R^2$ actions in terms of the deficit angles $\theta_h$, which would be to associate a factor $\sin^2 \theta_h$ (or $\theta_h^2$ in the small angle limit) to each hinge and sum, with suitable weights, over all hinges [14]. Without mixed terms however all quadratic actions look the same when expressed in terms of the deficit angles. This difficulty was recognized already in [15] where mixed terms were introduced very much along the same lines as the ones presented here, namely by weighting the hinges in the curvature proportionally to their support in the simplex.

$f(R)$ theories of gravity, namely theories where the curvature scalar $R$ in Einstein action is replaced by an arbitrary function $f(R)$, can be written on the simplicial lattice by the usual replacement:

$$\int d^d x \sqrt{g(x)} f(R(x)) \Rightarrow \sum_\alpha V(\alpha) f(R(\alpha)).$$  

(7.6)

In the continuum it is convenient to write the $f(R)$ gravity action as a linear action in $R$ by introducing an auxiliary scalar field $\Phi(x)$ (see the detailed discussion at page 12 in [13]). The same applies, with the usual replacement rules, in the simplicial lattice case. The action at the rhs of (7.6) is equivalent to

$$S_f = \sum_\alpha V(\alpha) [\Phi(\alpha) R(\alpha) - W(\Phi(\alpha))]$$  

(7.7)

where the potential $W(\Phi)$ is related to the original function $f$ that appears in (7.6) by the equations:

$$R(\alpha) = W'(\Phi(\alpha)),$$

(7.8)

$$f(R(\alpha)) = \Phi(\alpha) R(\alpha) - W(\Phi(\alpha)).$$

(7.9)

The scalar field of equation (7.7) may be endowed with a kinetic term, following the results of section 4, and equation (7.7) with $W(\Phi(\alpha)) = 0$ becomes the simplicial lattice version of Brans–Dicke theory:

$$S_{BD} = \sum_\alpha V(\alpha) \left[ \Phi(\alpha) R(\alpha) - \frac{\omega}{\Phi(\alpha)} \gamma^\mu(\alpha) \partial_\mu \Phi(\alpha) \partial_\nu \Phi(\alpha) \right].$$

(7.10)

8. Coupling of gauge theories to gravity on a simplicial lattice

Let us consider first a scalar field $\Phi^a(x)$ that transforms under a certain irreducible representation of a local symmetry group $G$. In the continuum a finite gauge transformation reads:

$$\Phi^a(x) \Rightarrow (e^{i \eta^a(x) T_A})_b^a \Phi^b(x)$$

(8.1)

where $\eta^a(x)$ are the local gauge parameters and $T_A$ the generators of the gauge group $G$. 


On a simplicial lattice the space–time label \( x \) is replaced by a label \( \alpha \) that runs over the simplices of the lattice. A gauge transformation can then be written by replacing \( x \) with \( \alpha \) everywhere in (8.1):

\[
\Phi^{\mu}(\alpha) \rightarrow (e^{i\theta^{\mu}(\alpha)\hat{T}^{x}_{\mu}})_{b}^{a}\Phi^{b}(\alpha).
\]

(8.2)

Notice that the simplices \( \alpha \) are the sites of the dual lattice, hence the gauge transformation (8.2) is local in the dual lattice as it is usual in lattice gauge theories.

The lattice derivative of \( \Phi^{\mu}(\alpha) \), as defined in (4.1) or (4.5), does not transform according to (8.2) and has to be replaced by a covariant derivative \( \hat{D}_{\mu} \) defined as:

\[
\hat{D}_{\mu}\Phi^{\mu}(\alpha) = \frac{1}{2}\sum_{j=1}^{d+1} U(\alpha|\alpha_{j})^{\alpha}_{b}^{\beta}\Phi^{\beta}(\alpha)\frac{\hat{V}^{(\alpha\alpha_{j})}(\alpha)}{V(\alpha)}
\]

where \( U(\alpha|\alpha_{j}) \) is an element of the gauge group \( G \), defined as usual on the link \( (\alpha, \alpha_{j}) \) of the dual lattice, and transforms under a gauge transformation (8.2) as:

\[
U(\alpha|\alpha_{j})^{\alpha}_{b}^{\beta} \rightarrow (e^{i\theta^{(\alpha\alpha_{j})\hat{T}^{x}_{\mu}}})_{b}^{\beta}U(\alpha|\alpha_{j})^{\alpha}_{f}^{\beta}(e^{-i\theta^{(\alpha\alpha_{j})\hat{T}^{x}_{\mu}}})_{f}^{\beta}.
\]

(8.4)

The transformation properties of the covariant derivative (8.3) follow directly from (8.4):

\[
\hat{D}_{\mu}\Phi^{\mu}(\alpha) \rightarrow (e^{i\theta^{(\alpha\alpha_{j})\hat{T}^{x}_{\mu}}})_{b}^{\beta}\hat{D}_{\mu}\Phi^{b}(\alpha).
\]

(8.5)

The covariant derivative (8.3) can be written as the sum of an ordinary derivative and of a term containing the lattice equivalent of the gauge field \( A_{\mu}(x) \):

\[
\hat{D}_{\mu}\Phi^{\mu}(\alpha) = \hat{\partial}_{\mu}\Phi^{\mu}(\alpha) + \sum_{j=1}^{d+1} A_{\mu}^{a\beta}(\alpha|\alpha_{j})\Phi^{\beta}(\alpha_{j})
\]

(8.6)

where according to (8.3) the gauge field \( A_{\mu}^{a\beta}(\alpha|\alpha_{j}) \) is given by:

\[
A_{\mu}^{a\beta}(\alpha|\alpha_{j}) = \left[U(\alpha|\alpha_{j})^{\alpha}_{b}^{\beta} - \delta^{\alpha}_{b}\right]\frac{\hat{V}^{(\alpha\alpha_{j})}(\alpha)}{V(\alpha)}.
\]

(8.7)

Notice that the gauge field (8.7) has a link nature and is not a function of the simplex \( \alpha \), as one would expect from a naive correspondence with the continuum field \( A_{\mu}(x) \), but of the \( d-1 \) dimensional face \( \alpha \cap \alpha_{j} \) which in the dual lattice is the link joining the simplex \( \alpha \) to \( \alpha_{j} \).

Let \( \Phi_{a}(\alpha) \) be the conjugate scalar field of \( \Phi^{\mu}(\alpha) \). The gauge transformation of \( \Phi_{a}(\alpha) \) and its covariant derivative are obviously given by:

\[
\Phi_{a}(\alpha) \rightarrow \Phi_{b}(\alpha)(e^{-i\theta^{\mu}(\alpha)\hat{T}^{x}_{\mu}})_{b}^{a},
\]

(8.8)

\[
\hat{D}_{\mu}\Phi_{a}(\alpha) = \frac{1}{2}\sum_{j=1}^{d+1} \Phi_{b}(\alpha_{j})U(\alpha_{j}|\alpha)_{b}^{a}\frac{\hat{V}^{(\alpha\alpha_{j})}(\alpha)}{V(\alpha)}.
\]

(8.9)

We can now write the action for the kinetic term of \( \Phi_{a}(\alpha) \) which is both invariant under gauge transformations and general coordinate transformations:

\[
S_{\text{kin}} = \sum_{\alpha} V(\alpha)g^{\mu\nu}(\alpha)\hat{D}_{\mu}\Phi_{a}(\alpha)\hat{D}_{\nu}\Phi_{a}(\alpha).
\]

(8.10)
A more explicit expression for $S_{\text{kin}}$ can be obtained by inserting in (8.10) the explicit form of the covariant derivatives:

$$S_{\text{kin}} = \frac{1}{4} \sum_{\alpha} \sum_{i \neq j}^{d+1} V(\alpha)g^{ij}(\alpha)\Phi_{\mu}(\alpha) \frac{U(\alpha|\alpha)^{\mu}U(\alpha|\alpha)^{\nu}}{\ell(\alpha|\alpha)|\ell(\alpha|\alpha)} g_{\mu\nu}(\alpha)$$  (8.11)

where $\ell(\alpha|\alpha)$ is given in equation (4.2) and we denote by $g^{ij}(\alpha)$ the metric in $\alpha$ projected in the directions orthogonal to the faces $i$ and $j$:

$$g^{ij}(\alpha) = g^{\mu\nu}(\alpha) n_\mu^{(\alpha\nu)}(\alpha) n_\nu^{(\alpha\nu)}(\alpha).$$  (8.12)

It should be remarked at this point that the kinetic term (8.11), unlike the standard kinetic term of a scalar field on an hypercubic lattice, involves scalar fields separated by two links, and hence it is quadratic in the gauge variable $U$. This occurs also in an hypercubic lattice if a symmetric lattice difference is used as a lattice derivative.

In the present formulation the two links coupling is required by the choice of the derivative (4.5) and it seems a necessary ingredient to couple scalar fields to the metric.

We consider now the Yang–Mills action coupled to a curved metric $g_{\mu\nu}(x)$:

$$S_{\text{YM}} = \int d^d x \sqrt{g(x)} \Tr[F_{\mu\nu}(x)F^{\mu\nu}(x)] g^{\mu\sigma}(x)g^{\nu\rho}(x).$$  (8.13)

In order to put this action on a simplicial lattice we first proceed to construct the lattice analogue of the gauge curvature $F_{\mu\nu}(x)$ following a procedure similar to the one used for the Riemann curvature in section 6.

Let $\alpha$ be a simplex and $\alpha_i$ the $d + 1$ simplices that have with $\alpha$ a face in common. We define, as in section 6, the hinge $h_i$ as the hinge intersection of $\alpha_i$, $\alpha$, and $\alpha_j$ and the path $\gamma_{h_i}$ as the closed path around $h_i$ starting and ending in $\alpha$. A precise definition is given in (6.1).

We consider now the product of the link variables $U(\alpha|\beta)^\mu_\nu$ along the path $\gamma_{h_i}$ and define:

$$U(\gamma_{h_i}|\alpha)^\mu_\nu = U(\alpha|\alpha)^\mu_\nu U(\alpha|\beta_1)^\mu_\nu \ldots U(\beta_{k+1}|\alpha)^\mu_\nu U(\alpha|\alpha)^\mu_\nu.$$  (8.14)

The path $\gamma_{h_i}$ begins and ends in $\alpha$ so that $U(\gamma_{h_i}|\alpha)^\mu_\nu$ transforms as follows:

$$U(\gamma_{h_i}|\alpha)^\mu_\nu \Rightarrow e^{i\omega^{\mu\rho}(\alpha)T_\rho} U(\gamma_{h_i}|\alpha)^\mu_\nu e^{-i\omega^{\mu\rho}(\alpha)T_\rho}$$  (8.15)

so that its trace is invariant under gauge transformations. Notice also that the orientation of the path is relevant and

$$U(\gamma_{h_i}|\alpha) = U^{-1}(\gamma_{h_i}|\alpha).$$  (8.16)

Following the same procedure already used for the Riemann curvature tensor we proceed to write the gauge curvature tensor $F_{\mu\nu}(x)$ on the simplicial lattice. First we define the field strength associated to a single hinge $h_i$ as:

$$F^\mu_{\mu\nu}(h_i) = \frac{V(h_i)}{v(h_i)} U^{\gamma_{h_i}}(\gamma_{h_i}|\alpha)^\mu_\nu n_{\mu\nu}^c(\alpha),$$  (8.17)

where

$$U^{\gamma_{h_i}}(\gamma_{h_i}|\alpha)^\mu_\nu = U(\gamma_{h_i}|\alpha)^\mu_\nu - U(\gamma_{h_i}|\alpha)^\mu_\nu$$  (8.18)
and then we define \( F^{a}_{\mu\nu}(\alpha) \) by summing, with a suitable weight, over all the hinges that belong to the simplex \( \alpha \):

\[
F^{a}_{\mu\nu}(\alpha) = \sum_{h_{ij} \in \alpha} \frac{v(h_{ij}|\alpha)}{V(\alpha)} F^{a}_{\mu\nu}(h_{ij}).
\] (8.19)

The weights \( \frac{v(h_{ij}|\alpha)}{V(\alpha)} \) are the same used in defining the Riemann curvature in equation (6.21) and correspond to the ratio of the support volume in \( \alpha \) of \( h_{ij} \) and the total volume of \( \alpha \).

Notice finally that \( U(\gamma h_{ij}|\alpha) a_{b} \) changes sign when the orientation of the hinge is reversed but that is compensated by the antisymmetry of \( n^{ij}_{\rho\sigma}(\alpha) \) under exchange of \( i \) and \( j \), so in the end each term in the sum at the rhs of (8.19) does not depend on the orientation of the hinge.

Given the field strength (8.19), the Yang–Mills action (8.13) can be formulated on the simplicial lattice by doing the replacements already used to write higher order gravity actions in section 7:

\[
S_{\text{YM}} \Rightarrow S_{\text{(latt)}} = \sum_{\alpha} V(\alpha) \text{Tr} \left[ F_{\mu\nu}(\alpha) F_{\rho\sigma}(\alpha) \right] g^{\mu\rho}(\alpha) g^{\nu\sigma}(\alpha). \] (8.20)

It is important to remark that each field strength in (8.20) contains a plaquette variable, so that the action is a sum of terms involving two plaquettes associated in general to different hinges, and so with different orientations. This is very different from the usual one plaquette action of lattice gauge theories on flat hypercubic lattices. The coupling of two plaquettes seems to be an essential ingredient if the Yang–Mills action has to be embedded in a curved metric and coupled with gravity.

We conclude this section with some remarks about the topological invariant \( \theta \) term in four dimensions, namely:

\[
S_{\theta} = \frac{1}{16\pi^2} \int d^4x \text{Tr} \left[ F_{\mu\nu}(x) F_{\rho\sigma}(x) \right] \epsilon^{\mu\nu\rho\sigma}. \] (8.21)

By following the same correspondence already used for Yang–Mills action we can at least formally write a simplicial lattice analogue of (8.21) as:

\[
S_{\theta} \Rightarrow S_{\theta}^{\text{(latt)}} = \frac{1}{16\pi^2} \sum_{\alpha} \frac{V(\alpha)}{\sqrt{\det g(\alpha)}} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left[ F_{\mu\nu}(\alpha) F_{\rho\sigma}(\alpha) \right]. \] (8.22)

The correspondence is purely formal in the sense that we cannot expect the topological nature of the continuum term to be preserved on the lattice, nor there is a guarantee, without further investigation, that it will be recovered in the continuum limit.

However the action has some interesting features which are worth describing. Consider first the following identity, which can be easily verified by using the explicit expression of \( V^{\mu\rho}_{\nu\sigma}(\alpha) \) given in (3.15)

\[
\epsilon^{\mu\nu\rho\sigma} V^{\mu\rho}_{\nu\sigma}(\alpha) V^{\nu\sigma}_{\mu\rho}(\alpha) = 12 \sqrt{\det g(\alpha)} V(\alpha) \epsilon^{ijkl},
\] (8.23)

where \( \epsilon^{ijkl} \) with the indices \( i, j, k, l \) running from 1 to 5 is completely antisymmetric and further defined by the relation:

\[
\sum_{i=1}^{5} \epsilon^{ijkl} = 0 \] (8.24)
and by:

\[ \bar{\epsilon}^{1234} = \pm 1 \]

where the sign is determined by the sign of \( V(\alpha) \) in (3.8) where \( d \) has been set to four.

By replacing (8.23) into the action at the rhs of (8.22) we obtain:

\[ S_1^{\text{latt}} = \sum_\alpha \sum_{i,j,k,l} \bar{\epsilon}^{ijkl} \text{Tr} \left[ U^{-1}(\gamma_{h_{ij}\alpha})U^{-1}(\gamma_{h_{kl}\alpha}) \right] \frac{v(h_{ij}\alpha)v(h_{kl}\alpha)}{v(h_{ij})v(h_{kl})}, \quad (8.26) \]

The action (8.26) does not contain the metric \( g_{\mu\nu}(\alpha) \) explicitly, but a residual dependence on the metric is present in the weight function \( v(h_{ij}\alpha) \) and \( v(h_{kl}) \). So if we require complete metric independence, as in the original topological action of the continuum theory, the weight factor at the rhs of (8.26) should be modified. A possible improvement, in this respect, of the action (8.26) is to replace the ratio \( \frac{v(h_{ij}\alpha)}{v(h_{ij})} \) with \( \frac{1}{n_{h_{ij}}} \) where \( n_{h_{ij}} \) is the number of simplices \( \alpha \) that insist on the hinge \( h_{ij} \). With this choice the action becomes metric independent and reads:

\[ S_1^{\text{latt}} = \sum_\alpha \sum_{i,j,k,l} \bar{\epsilon}^{ijkl} \text{Tr} \left[ U^{-1}(\gamma_{h_{ij}\alpha})U^{-1}(\gamma_{h_{kl}\alpha}) \right] \frac{1}{n_{h_{ij}}n_{h_{kl}}}, \quad (8.27) \]

### 9. Vierbein and local Lorentz invariance

Given a simplex \( \alpha \) it is always possible to perform a change of coordinates of the form (3.2) that transforms the metric \( g_{\mu\nu}(\alpha) \) into the flat diagonal metric \( \eta_{ab} \) (the flat indices will from now on be denoted by the letters \( a, b, \ldots \) to distinguish them from the ‘curved’ indices \( \mu, \nu, \ldots \)). If we name \( \xi^\mu_\alpha \) the new coordinates of the vertices of \( \alpha \) the transformation (3.2) now reads:

\[ \xi^\mu_\alpha = \Lambda^\mu_\alpha(\alpha) \chi^\mu + \Lambda^\mu_\alpha(\alpha) \eta_{ab} \]

and according to equation (3.7) the metric is given in terms of \( \Lambda^\mu_\alpha(\alpha) \) by:

\[ g_{\mu\nu}(\alpha) = \Lambda^\mu_\mu(\alpha) \Lambda^\nu_\nu(\alpha) \eta_{ab}. \quad (9.2) \]

It is clear from (9.2) that \( \Lambda^\mu_\mu(\alpha) \) can be interpreted as a vierbein (or \( d \)-bein), and that given the metric \( g_{\mu\nu}(\alpha) \) the vierbein \( \Lambda^\mu_\mu(\alpha) \) is determined only up to a Lorentz transformation (or rotation in Euclidean space-time) acting on the flat index \( \alpha \). Similarly the coordinates \( \xi^\mu_\alpha \) are determined up to a Poincaré transformation whose translational part is given by \( \Lambda^\alpha(\alpha) \) in (9.1).

We assume that the transformation (9.1) is done separately and independently in each simplex, so that the coordinates \( \xi^\mu_\alpha \) of a vertex \( i \) regarded as part of a simplex \( \alpha \) are in general different from the coordinates of the same vertex regarded as part of a neighbouring simplex \( \beta \). In the following we shall denote these coordinates \( \xi^\mu_\alpha \) to avoid ambiguities.

At the end of this procedure through a transformation of the form (9.1) each simplex of the simplicial manifold is endowed with a Euclidean reference frame determined up to a Poincaré

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15 The choice of the weights \( v(h_{ij}|\alpha) \) and \( v(h_{ij}) \) has a certain degree of arbitrariness. An different choice, alternative to the one given in section 6, is for instance given by \( v(h|\alpha) = \frac{1}{\sqrt{h}} \) which satisfy (6.11) and (6.12) with \( v(h) = \frac{\sqrt{h}}{\sqrt{\sum_{h_{ij}}h_{ij}} \sqrt{\sum_{h_{kl}}h_{kl}}} \). With this choice the rhs of (8.26) would depend only on the volumes of the simplices and hence only on the determinant of the metric.

16 This means that the transformation (9.1) is not the restriction to the simplex \( \alpha \) of a general coordinate transformation (3.1).
transformation. Geometrical entities are not affected by the choice of the local frame and the resulting theory will exhibit a local Poincaré invariance.\footnote{This point of view was first developed in reference [5, 6] and many of the results of the present section can be already found there.}

Consider now two neighbouringsimplices $\alpha$ and $\beta$ with a face in common. The transition from the Euclidean reference frame in $\alpha$ to the one in $\beta$ is described by a Poincaré transformation, so that the coordinates of a generic point in the reference frame of $\alpha$ and $\beta$ are related by:

$$\xi^\mu(\beta) = \Omega^\mu_\nu(\beta|\alpha)\xi^\nu(\alpha) + \Omega^\mu(\beta|\alpha).$$  \hspace{1cm} (9.3)

Let $i$ and $j$ be two vertices that belong to both $\alpha$ and $\beta$. Then from (9.3) we have:

$$\left(\xi^\mu_i(\beta) - \xi^\mu_j(\beta)\right) - \Omega^\mu_\nu(\beta|\alpha)\left(\xi^\nu_i(\alpha) - \xi^\nu_j(\alpha)\right) = 0. \hspace{1cm} (9.4)$$

By replacing in (9.4) the Euclidean reference frame coordinates $\xi^\mu_i$ with their value given in equation (9.1) we obtain the following identity for the vielbeins in $\alpha$ and $\beta$:

$$\left(\Lambda^\mu_\nu(\beta) - \Omega^\mu_\nu(\beta|\alpha)\Lambda^\nu_i(\alpha)\right)(x^\mu_i - x^\mu_j) = 0 \hspace{1cm} i, j \in \alpha \cap \beta. \hspace{1cm} (9.5)$$

Equation (9.5) provides the constraints to which the vielbeins belonging to neighbouringsimplices have to satisfy and in fact its square reproduces the analogue constraints (2.5) satisfied by $g_{\alpha\beta}$.

The rotation matrix $\Omega^\mu_\nu(\beta|\alpha)$ (which is a Lorentz rotation in Minkowski metric) is closely related to the matrix $K^\mu_\nu(\beta|\alpha)$ that defines the parallel transform. In fact if in equation (5.3) we replace the curved indices $\mu$ and $\nu$ with flat ones we have:

$$\Omega^\mu_\nu(\beta|\alpha) = \Lambda^\mu_\nu(\beta|\alpha)\Lambda^{\nu}_i(\alpha)\Lambda^{-1}_i(\alpha), \hspace{1cm} (9.6)$$

where $\Lambda^{-1}_i(\alpha)$ is the inverse of the vierbein. Equation (9.6) can be written in the form

$$\Lambda^\mu_\nu(\beta) = \Omega^\mu_\nu(\beta|\alpha)\Lambda^{\nu}_i(\alpha)K^\nu_\mu(\alpha|\beta) \hspace{1cm} (9.7)$$

which is the analogue in the vielbein formalism of equation (5.7). The Lorentz connection $\Omega^\mu_\nu(\beta|\alpha)$ transforms under local Lorentz transformations (or rotations in a Euclidean space) as the gauge field in (8.4), namely:

$$\Omega(\beta|\alpha)^\mu_\nu = (e^{J^\mu_\nu(\alpha)T_A})^\mu_\nu \Omega(\beta|\alpha)^\mu_\nu (e^{-J^\mu_\nu(\alpha)T_A})^\mu_\nu \hspace{1cm} (9.8)$$

where $T_A$ are the generators of the group which are supposed here to be in the adjoint representation.

Given a field $\Phi^a(\alpha)$ that transforms under a non trivial representation $R$ of the local Lorentz group its covariant derivative is of the form (8.3) but with the gauge field $U(\beta|\alpha)$ replaced by the Lorentz rotation $\Omega(\beta|\alpha)$ in the representation $R$:

$$\hat{D}_\mu \Phi^a(\alpha) = \frac{1}{2} \sum_{i=1}^{d+1} \Omega^a_\mu(\alpha)\Phi^b(\alpha)\frac{V^a_b(\alpha)}{V(\alpha)} \hspace{1cm} (9.9)$$
Notice however that if the field is also a tensor under general coordinate transformations then a parallel transport has to be done at the same time. For instance the covariant derivative of the vierbein $\Lambda^a_\mu(\alpha)$ is given by:

$$
\hat{D}_\mu \Lambda^a_\nu(\alpha) = \frac{1}{2} \sum_{i=1}^{d+1} \Omega(\alpha|\alpha_i) y^a_b(\alpha_i) K^a_\mu(\alpha_i) \frac{V^{(\alpha|\alpha_i)}(\alpha)}{V(\alpha)}. \tag{9.10}
$$

If we apply (9.6) and then (4.4) in (9.10) we find that the rhs is identically zero, namely that the covariant derivative of the $d$-beinvanishes as expected:

$$
\hat{D}_\mu \Lambda^a_\nu(\alpha) = 0. \tag{9.11}
$$

In analogy to what was done in (8.7), we can define the Lorentz connection as the gauge field associated to $\Omega(\alpha|\alpha_i)$:

$$
\omega_{\mu}(\alpha|\alpha_i) y^a_b = [\Omega(\alpha|\alpha_i) y^a_b - \delta^a_b] \frac{V^{(\alpha|\alpha_i)}(\alpha)}{V(\alpha)}. \tag{9.12}
$$

It is then easy to write the antisymmetric part of (9.11) in terms of the Lorentz connection, using the fact that the contribution coming from the Christoffel symbol is symmetric and disappears. We have then, in total analogy with the continuum case:

$$
\hat{\partial}_\mu \Lambda^a_\nu(\alpha) - \frac{1}{2} \sum_{i=1}^{d+1} \omega_{\mu}(\alpha|\alpha_i) y^a_b(\alpha_i) = 0 \tag{9.13}
$$

where the square brackets denote the antisymmetrization in the indices $\mu$ and $\nu$.

The curvature $\mathcal{R}^a_{\beta \gamma}(\alpha)$ associated to the Lorentz connection $\Omega_{\beta}(\gamma|\alpha)$ is defined following exactly the same prescriptions (8.14) and (8.19) used for gauge theories. Given a hinge $h_{ij}$ and the path $\gamma_{h_{ij}}$ defined in (6.1) that goes around $h_{ij}$ starting and ending in the simplex $\alpha$, we can define:

$$
\Omega(\gamma_{h_{ij}}) y^a_b = \Omega(\alpha|\alpha_i) y^a_{\beta_1}(\alpha_i) \Omega(\alpha_i|\beta_2) \ldots \Omega(\beta_{k-1}|\alpha_j) \Omega(\alpha_j|\alpha) \tag{9.14}
$$

and, in agreement with (8.19):

$$
\mathcal{R}^a_{\beta \gamma}(\alpha) = \sum_{h_{ij}, \alpha} \frac{V(h_{ij})}{V(h_{ij})} (\Omega(\gamma_{h_{ij}}) y^a_b - \Omega(\gamma_{h_{ij}}) y^a_b) \eta^{\beta \gamma}(\alpha). \tag{9.15}
$$

The curvature $\mathcal{R}^a_{\beta \gamma}(\alpha)$ is directly related to the Riemann curvature tensor. In fact from (9.6) it is easy to show that

$$
\mathcal{R}^a_{\beta \gamma}(\alpha) = \Lambda^a_\rho(\alpha) \Lambda^\rho_\beta(\alpha) \mathcal{R}_{\beta \gamma}(\alpha), \tag{9.16}
$$

where curved (resp. flat) indices have been raised with $g^{\mu \nu}(\alpha)$ (resp $\eta^{\mu \nu}$). The curvature scalar is obviously given by:

$$
\mathcal{R}(\alpha) = \Lambda^{-1}_a(\mu) \Lambda^{-1}_b(\nu) \mathcal{R}^a_{\mu \nu}(\alpha). \tag{9.17}
$$

As already shown in reference [5] the action (6.23) can be written in terms of the vielbeins.
in a form that exhibits a local Lorentz invariance:

\[ S_l = \frac{k}{2(d-2)!} \sum_{\alpha} \mathcal{R}^{\mu_1 \mu_2}_{\mu_3 \mu_4} (\alpha) \Lambda_{\mu_3}^a (\alpha) \cdots \Lambda_{\mu_d}^a (\alpha) \]

\[ (x_{i_1} - x_{d+1})^{\mu_1} \cdots (x_{i_d} - x_{d+1})^{\mu_d} \epsilon_{\alpha \alpha_2 \cdots \alpha_d} \epsilon^{i_1 \cdots i_d}. \]  

(9.18)

In showing the equivalence of (6.23) and (9.18) one uses a trivial consequence of (9.2), namely:

\[ \det \Lambda^\alpha_\mu (\alpha) = \sqrt{\det g_{\mu \nu} (\alpha)}. \]  

(9.19)

Notice the analogy of (9.18) with the continuum action written in terms of the vielbein and differential form, with the differences \((x_i - x_{d+1})^{\mu_i}\) playing the role of the differentials \(dx^\mu\) of the continuum.

However, as discussed in detail in section 11 and in the appendix A, equation (9.18) is not part of a consistent formulation of exterior calculus on a simplicial lattice. In particular some operations of the continuum theory, like partial integration, have no precise correspondence here.

10. Coupling of gravity with fermions

The local Lorentz (rotational) symmetry was introduced in the previous section alongside with the vielbein formalism for the metric by endowing each simplex with an independent Euclidean reference frame. This allows us to introduce fields that transform as spinors under the local Lorentz transformations, which is indeed a necessary step if one wants to couple fermionic fields to the metric.

Let \(\psi^\alpha(\alpha)\) be a fermionic field which transforms as a spinor\(^{18}\) under local \(d\)-dimensional rotations but is invariant under general coordinate transformations.

Its covariant derivative is then a particular case of equation (9.9), namely:

\[ \hat{D}_\mu \psi^\alpha (\alpha) = \frac{1}{2} \sum_{i=1}^{d+1} \Omega^i_\alpha (\alpha) \eta^\alpha \psi^\beta (\alpha) \frac{V^{(\gamma_\alpha \gamma_\beta)} (\alpha)}{V (\alpha)}. \]  

(10.1)

where the label \(S\) denotes that the rotation \(\Omega (\alpha | \alpha_i)\) is now in a spinorial representation.

The action of a free fermion coupled to the metric is given in the continuum by:

\[ S_f = \int d^d x \ \psi^\dagger (x) \gamma^a \Lambda^{-1}_a (x) D_\mu \psi (x) \]  

(10.2)

where \(\gamma^a\) are \(d\)-dimensional \(\gamma\) matrices and spinorial indices are understood.

Following the correspondence used already in previous sections for other matter fields we can write (10.2) on the lattice as:

\[ S_f^{(\text{lat})} = \sum_{\alpha} V (\alpha) \ \bar{\psi}^\dagger (\alpha) \gamma^a \Lambda^{-1}_a (\alpha) \hat{D}_\mu \psi (\alpha), \]  

(10.3)

where the covariant derivative at the rhs is given by (10.1).

The covariant derivative (10.1), and correspondingly the action (10.3), can be easily generalized to the case where the fermion transforms also under a representation \(R\) of some internal

\(^{18}\) Spinorial indices will be denoted with dotted italic letters.
gauge group \( G \). If \( U(\alpha, \beta) \) is the gauge field associated this gauge symmetry, then covariant derivative reads:

\[
D_{\mu} \psi_\mu^{\alpha}(\alpha) = \frac{1}{2} \sum_{j=1}^{d+1} \Omega_{\beta}^{\alpha}(\alpha) \bar{U}_{\beta}^{\alpha}(\alpha) Y_{j} \psi_{\mu}^{\beta}(\alpha) \frac{V^{(\alpha)}_{\mu}(\alpha)}{V(\alpha)},
\]

where group elements corresponding to the direct product of \( G \) and of the local Lorentz group appear. The action is a direct generalization of (10.3) with the covariant derivative (10.1) replaced by (10.4) and the index structure accordingly rearranged.

Another interesting case of fermionic field coupled to gravity is that of a spin 3/2 field. This will in fact provide the fermionic (gravitino) term of the \( N = 1 \) four dimensional supergravity. Let \( \psi_\mu^{\alpha}(\alpha) \) be the spin 3/2 field on the simplicial lattice. It is a covariant vector under general coordinates transformations and transforms as a spinor under local Lorentz transformations. Hence its covariant derivatives is (see also equation (9.10)):

\[
D_{\mu} \psi_\mu^{\alpha}(\alpha) = \frac{1}{2} \sum_{j=1}^{d+1} \Omega_{\beta}^{\alpha}(\alpha) \bar{U}_{\beta}^{\alpha}(\alpha) K_{\mu}^{\beta}(\alpha) \frac{V^{(\alpha)}_{\mu}(\alpha)}{V(\alpha)}.
\]

Let us restrict ourselves now to the four dimensional case. The gravitino term of the \( N = 1 \) supergravity action is in the continuum:

\[
S_{\text{gravit}} = \int d^4 x \det \Lambda_{\mu}^{\nu}(x) \tilde{\psi}_{\mu}(x) \gamma^{\mu(\nu;\rho;q)}(x) \Lambda_{\nu}^{\rho}(x) \Lambda_{\rho}^{q}(x) D_{\mu} \psi_{\nu}(x)
\]

\[
= \int \tilde{\psi}_{\mu}(x) \epsilon_{\mu\nu\rho\sigma} \gamma^{\nu(\rho;\sigma)}(x) D_{\mu} \psi_{\nu}(x) dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} \wedge dx^{\mu},
\]

where \( \gamma^{\nu(\rho;\sigma)} \) is the antisymmetrized product of three gamma matrices and \( \Lambda_{\mu}^{\nu} \) is the inverse of the vierbain \( \Lambda_{\mu}^{\nu} \). Notice that due to the antisymmetrization of the covariant indices in the covariant derivative the Christoffel symbol does not contribute and the covariant derivative is simply given by:

\[
D_{\mu} \psi_{\nu}(x) = \partial_{\mu} \psi_{\nu}(x) + \omega_{\mu}^{\rho}(x) \Sigma_{ab} \psi_{\rho}(x).
\]

where \( \omega_{\mu}^{\rho}(x) \) is the Lorentz connection and \( \Sigma_{ab} \) the generators of the Lorentz group in spinorial representation.

The lattice version of the action (10.6) in the two forms given above can be easily derived by the usual replacements:

\[
S_{\text{gravit}}^{\text{latt}} = \sum_{\alpha} V(\alpha) \tilde{\psi}_{\mu}(\alpha) \gamma^{\mu(\nu;\rho;q)}(\alpha) \Lambda_{\nu}^{\rho}(\alpha) \Lambda_{\rho}^{q}(\alpha) D_{\mu} \psi_{\nu}(\alpha)
\]

\[
= \sum_{\alpha} \tilde{\psi}_{\mu}(\alpha) \epsilon_{\mu\nu\rho\sigma} \gamma^{\nu(\rho;\sigma)}(\alpha) D_{\mu} \psi_{\nu}(\alpha) u^{(\nu(\rho;\sigma))}_{\mu}(\alpha)
\]

where

\[
u^{(\rho;\sigma))}_{\nu}(\alpha) = \frac{1}{4} \epsilon^{(\rho;\sigma);\nu}(\alpha)
\]

As in the continuum case the Christoffel symbol in the covariant derivative does not contribute.
due to the antisymmetrization of the indices, so the expression (10.5) can be replaced in (10.8) by:

\[
\hat{D}_\mu \psi^a_{\nu(\alpha)} = \frac{1}{2} \sum_{j=1}^{d+1} \Omega(\alpha_1) \gamma^\mu_a \gamma^\nu b \frac{V^{(\alpha-\alpha_1)}(\alpha)}{V(\alpha)}. \tag{10.10}
\]

By adding the action for pure gravity given in (9.18) to the gravitino action as given above in (10.8) one can write a lattice action that corresponds in the continuum to the \(N = 1\) supergravity in four dimensions:

\[
S_{\text{sugra}} = \frac{k}{4} \sum_\alpha \epsilon_{\mu_1 \mu_2 \mu_3} \left[ R_{\mu_1 \mu_2}^{\alpha_1 \alpha_2}(\alpha) \Lambda^{\alpha_2}_{\mu_3}(\alpha) \Lambda^{\alpha_3}_{\mu_4}(\alpha) + \gamma^{\alpha_1 \mu_2 \alpha_3} \Lambda^{\alpha_2}_{\mu_3}(\alpha) \hat{D}_{\mu_3} \psi^a_{\mu_4(\alpha)} \right] V^{(\mu_1 \mu_2 \mu_3 \mu_4)}(\alpha) \tag{10.11}
\]

where \(V^{(\mu_1 \mu_2 \mu_3 \mu_4)}\) is given in (10.9). It is important to remark that although the action (10.11) is formally analogue in the present formalism to the continuum \(N = 1\) four dimensional supergravity, exact supersymmetry is certainly broken\(^{19}\) on the lattice and there is no guarantee at this stage that it would be recovered in the continuum limit. This should be the object of an independent investigation.

11. Coupling of gravity to differential \(p\)-forms

In the previous sections we described the coupling of different types of matter fields (scalar fields, gauge fields, fermions) with gravity within the simplicial lattice framework of the Regge calculus. This has provided us with a dictionary to translate any continuum action containing those fields into a simplicial lattice action.

For the correspondence to be complete however it would still be necessary to find a lattice description of fields that are differential \(p\)-forms (with \(p > 1\)). These fields play an important role in many relevant theories, for instance a three-form field \(A_{[\mu \nu \rho]}(x)\) is one of the fundamental fields of supergravity in 11 dimensions.

In general a differential \(p\)-form field \(A_{[\mu_1 \mu_2 ... \mu_p]}(x)\) is associated to an abelian gauge invariance of the form:

\[
A_{[\mu_1 \mu_2 ... \mu_p]}(x) \Rightarrow A_{[\mu_1 \mu_2 ... \mu_p]}(x) + \partial_{[\mu_1} \Lambda_{\mu_2 ... \mu_p]}(x) \tag{11.1}
\]

where the gauge parameter \(\Lambda_{[\mu_1 ... \mu_{p-1}]}(x)\) is a \(p - 1\) form.

The gauge invariant field strength is then a \(p + 1\) form and is given by:

\[
F_{[\mu_1 \mu_2 ... \mu_{p+1}]}(x) = \partial_{[\mu_1} A_{\mu_2 ... \mu_p \mu_{p+1}]}(x) \tag{11.2}
\]

where the square brackets denote antisymmetrization of the indices.

The most direct way to write a \(p\) form on a simplicial lattice following the approach described in the previous sections would be to replace the continuum field \(A_{[\mu_1 \mu_2 ... \mu_p]}(x)\) with a completely antisymmetric tensor \(A_{[\mu_1 \mu_2 ... \mu_p]}(\alpha)\) of rank \(p\) associated to each simplex \(\alpha\), and to define its field strength as its covariant derivative, which in this case would coincide with

\(^{19}\)For instance it is crucial in the continuum that the commutator of two covariant derivatives is proportional to the curvature, which is not true here, since a whole loop around a hinge is needed to reproduce the curvature.
the ordinary derivative due to the antisymmetrization of the indices. In short, equation (11.2) would be replaced by:

\[ F_{[\mu_1 \mu_2 \ldots \mu_{p+1}]}(\alpha) = \hat{\partial}_{[\mu_1} A_{\mu_2 \mu_3 \ldots \mu_{p+1}]}(\alpha). \]  

(11.3)

However with this definition a gauge transformation of the type (11.1), with \( \alpha \) replaced by \( \alpha \) and the partial derivative by the lattice derivative (4.5), would not be a symmetry of the field strength. In fact one can easily see from the definition of the partial derivative on a simplicial lattice given in (4.5) that derivatives in different directions do not commute, namely:

\[ \hat{\partial}_\mu \left( \hat{\partial}_\nu \phi(\alpha) \right) \neq \hat{\partial}_\nu \left( \hat{\partial}_\mu \phi(\alpha) \right). \]  

(11.4)

This is a consequence of the simplicial lattice structure: derivatives are associated to one link moves on the dual lattice, which is made of Voronoi cells, and on such lattice the result of two moves depends on their order, unlike what happens on a hypercubic lattice.

In the previous sections the one forms describing gauge fields have been associated to the \((d - 1)\)-dimensional faces of the simplices, that is to the links of the dual Voronoi tesselation. Similarly the two form describing curvatures or field strengths were associated to the \(d - 2\) dimensional hinges, namely to the two dimensional plaquettes of the dual lattice.

It is clear then that the natural way to describe a \(p\)-form field on a simplicial lattice would be to associate it to a \(p\)-dimensional cell of the dual Voronoi tesselation. This is completely identified by the \(d - p + 1\) vertices of its dual \(d - p\) dimensional simplex\(^{20}\).

The problem is then to formulate on the simplicial lattice a discrete exterior calculus, endowed with a wedge product of forms and of a nilpotent differential \(\hat{d}\) operator that satisfy, as much as possible, the usual algebraic properties of the exterior calculus.

This problem has been investigated (see for instance \cite{16} and references therein) but mostly in the more direct way of associating a \(p\)-form to a \(p\)-dimensional simplex of the simplicial complex.

It was shown in reference \cite{16} that a wedge product of a \(p\) and a \(q\) form can be defined as a quantity associated to \(p + q\) dimensional simplices. This product is commutative (in a graded sense) as in the continuum, but it is not associative, although the non-associative terms can be shown to vanish in the continuum limit. Finally a \(d\) operator can be defined, that satisfies the nilpotency relation \(\hat{d}^2 = 0\) and the graded distributive property with respect to the wedge product.

However the case we are interested in is different: a \(p\)-form has now to be associated to a \(p\) dimensional Voronoi cell, which is dual to a \(d - p\) dimensional simplex within the simplicial complex. A \(p\)-form is then a field defined on the \(d - p\) dimensional simplices, and the wedge product of a \(p\)-form and a \(q\)-form should be associated to \(d - p - q\) dimensional simplices, which are dual to \(p + q\) dimensional Voronoi cells.

Vertices in a Voronoi cell, which correspond to \(d\)-dimensional simplices on the original lattice, can be several links apart, unlike what happens on a simplex where all pairs of vertices are connected by a link. This can make defining a wedge product and a differential \(d\) operator that satisfy the algebraic rules of exterior calculus even more difficult than in the case where the forms are defined directly on simplices.

This is indeed the case. We succeeded in defining a wedge product for forms (see the appendix A for details) defined on Voronoi cells and also a nilpotent differential operator \(\hat{d}\) that maps a \(p\) form into a \(p + 1\) form.

\(^{20}\) For a more precise definition of this simplex-cell duality see for instance reference \cite{16}.
This ensures that by operating with \( \hat{d} \) on a given \( p \) form gauge field one obtains a \( p + 1 \) form—the field strength—which is an invariant under gauge transformations whose parameters are \( p - 1 \) forms thus overcoming the problem discussed at the beginning of this section.

However the wedge product defined in this way has some rather severe shortcomings. For a start, as in the case mentioned above of the wedge product of forms defined directly on the simplices, it is not associative. More worryingly the differential operator \( \hat{d} \) does not satisfy the Leibnitz rule when applied to the wedge product of forms.

This implies that, although gauge invariance is preserved, partial integration is not allowed\(^{21}\) and different forms of an action, which are equivalent in the continuum up to surface terms may become different on the lattice.

This is particularly important in actions like the Chern-Simons action in three dimensions or the \( \int F A \) term (with \( A \) the above mentioned three form and \( F \) its field strength) in eleven dimensional supergravity. In the continuum these actions can be written as surface terms of gauge invariant actions in respectively three and twelve dimensions, but this property breaks down if the Leibnitz rule is violated.

In spite of its shortcomings the above mentioned wedge product is interesting and may be the base for future investigations in the subject, particularly concerning the recovery of the fundamental algebraic properties of the exterior calculus in the continuum limit. For this reason the details of its definition and of its main properties are given in the appendix A.

### 12. Some final remarks

This paper started as an attempt to answer a perhaps naive question: ‘is it possible to have a formulation of simplicial gravity where the fundamental degrees of freedom are, as in the continuum theory, the components of the metric tensor?’.

Since the metric tensor must depend on the choice of coordinates, we had to attach coordinates to the vertices of the simplices and require invariance under coordinates transformations. This discrete version of invariance under coordinate transformations does not imply invariance under diffeomorphisms, as the vertices form a discrete set, their adjacency matrix is kept fixed and the model is ultimately equivalent to Regge calculus. However the invariance under coordinate transformations provides the basis for a discrete tensor calculus which, in turn, makes the correspondence with the continuum theory much more strict and suitable for extension to the coupling of gravity to different types of matter fields.

One crucial ingredient of this correspondence is the definition of partial derivative on the lattice defined in section 4. This can be regarded as a generalization to simplicial lattices of the symmetric finite difference operation on an hypercubic lattice and is strongly motivated by the requirement that it transforms as a covariant vector under general coordinate transformation.

The original aim turns then into a more ambitious one, namely finding a precise correspondence, a kind of dictionary, between actions in the continuum and actions on a simplicial lattice, thus allowing to write the coupling of any matter field to gravity within the framework of Regge calculus. Following this correspondence we were for instance able to write an action on the simplicial lattice that corresponds in the continuum to supergravity in four dimensions.

The problem of coupling scalars, fermions and gauge fields to discrete gravity has obviously been discussed in the literature before (see references in the different sections) but mostly on

\(^{21}\) The Leibnitz rule, and hence partial integration, might be recovered in the continuum limit, but further investigation is needed in that respect.
a case by case basis, without a unifying scheme as the one we developed here. However, as discussed in the last section and in the appendix A, the correspondence with the continuum is not complete: actions that contain $p$-form potentials gauging free differential algebras do not seem to fit in this scheme and coupling them consistently to gravity within the framework of Regge calculus is still an open problem.

Much work still needs to be done. We have not checked for instance the continuum limit, even at the classical level, of the actions of the different kinds of matter coupled to gravity. This is particularly relevant for gauge theories. In fact the simplicial lattice action for pure Yang–Mills theory is quite different, even in absence of gravity, from the traditional Wilson action as it consists of two plaquette terms rather than of the usual one plaquette term.

Although the correspondence with the continuum theory is quite compelling there are some fundamental differences that would also need further investigation. The fundamental degrees of freedom in our approach are the component of the metric tensor on each simplex, but these are not independent degrees of freedom as they are constrained to coincide on their common $d-1$ dimensional faces. As a result the $d(d+1)/2$ degrees of freedom of the components of $g_{\mu\nu}(x)$ at the point of coordinates $x^\mu$ are spread on the simplicial lattice over a number of neighbouring simplices which is of order $d$. This would obviously be relevant in any attempt to find a correct measure of integration in a functional integral for quantum gravity. We have not addressed this problem here.

A lattice length $l(\alpha|\beta)$ has been defined in (4.2) and some symmetries of the continuum theory are broken by higher order terms in $l(\alpha|\beta)$ and are recovered in the limit where $l(\alpha|\beta)$ tends to zero. As already remarked this is the case of some symmetries of the Riemann tensor which are violated on the lattice by higher order terms in the deficit angle $\theta_h$. The presence of higher order terms makes it also apparently impossible to invert equation (5.19) and express the Christoffel symbol in terms of derivatives of the metric tensor.

All through this paper we considered the metric to be Euclidean. However the formalism can be easily generalized to the case where the metric associated to each simplex $\alpha$ is a Lorentzian metric.

There would be some obvious changes, notably the fact that the rotations associated to the parallel transport around a space-like hinge become Lorentz rotations, and the corresponding deficit angles are imaginary. So for those hinges the term $\sin \theta_h$ in the action is replaced by $\sinh \theta_h$.

The casual dynamical triangulation approach (for a review see [17]) developed by J Ambjørn, Loll and J Jurkiewicz fits in the present scheme.

Consider in the original $d$-dimensional simplicial complex a $d-1$ dimensional spacelike surface $\Sigma_0$ made up entirely by faces of the $d$-dimensional simplices. The unit vector orthogonal to each of these faces is then a timelike vector. Consider the layer of $d$-simplices that are on the same side of $\Sigma_0$ and have at least one vertex on $\Sigma_0$. This layer defines a new $d-1$ dimensional surface $\Sigma_1$, and the procedure can then be iterated to define after $t$ iteration a surface $\Sigma_t$. The discrete index $t$ in $\Sigma_t$ can be thought of as a time evolution parameter. Causality will require that all surfaces $\Sigma_t$ are spacelike, and some conditions may have to be imposed on the metrics of the $d$-simplices. Initial conditions, say at $t = 0$, are given by specifying the metric on those $d$-simplices of the aforementioned layer whose faces make up $\Sigma_0$. They include the metric on $\Sigma_0$, but also the components of the metric along the ‘time’ direction, they are a complete set of degrees of freedom at $t = 0$ and are completely independent from the ones corresponding to different values of $t$. They are however still constrained by equation (2.5). The above construction is essentially the same as the one proposed by Ambjørn et al and it may prove a useful tool also in the present approach to separate and count the degrees of freedom and to formulate a Hamiltonian type of formalism.
Acknowledgments

I wish to thank M Billo, M Caselle and N Kawamoto for discussions and critical reading of the manuscript.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. An attempt of constructing a discrete exterior calculus within the Regge calculus framework

In describing the interaction of matter fields with gravity within the framework of Regge calculus scalar fields (zero forms) have been associated to the $d$-dimensional simplices and the gauge fields (one forms) to their $(d - 1)$ dimensional faces, namely they have been respectively associated to the sites and the links of the dual Voronoi lattice. The Voronoi tassellation generated by the vertices of the simplicial lattice consists of $d$-dimensional cells which are dual to the vertices. The $p$-dimensional faces of the Voronoi cells are dual to the $d - p$ dimensional simplices of the original lattice, and each of them is completely identified by the $d - p + 1$ vertices of the dual simplex. The natural generalization of the $p = 0$ and $p = 1$ cases to arbitrary $p$ is to associate a $p$-form field of the continuum theory to the $p$-dimensional cells of the dual Voronoi tassellation, namely, by duality, to the $d - p$ dimensional simplices of the original simplicial lattice. More precisely if we denote by $\sigma_{d-p}$ the $d - p$ dimensional simplices and by $\star \sigma_{d-p}$ the $p$ dimensional cells dual to them, we can define a discrete $p$-form as a map from $\star \sigma_{d-p}$ onto the real numbers.

The simplex $\sigma_{d-p}$ is identified by its $d - p + 1$ vertices $P_0, P_1, \ldots, P_{d-p}$:

$$\sigma_{d-p} \equiv [P_0, P_1, \ldots P_{d-p}].$$

(A.1)

Similarly we shall identify $\star \sigma_{d-p}$ as:

$$\star \sigma_{d-p} \equiv \star[P_0, P_1, \ldots P_{d-p}].$$

(A.2)

A $p$-form field $A_{[\mu_1\mu_2\ldots\mu_p]}(x)$ of the continuum theory will have the following correspondence on the simplicial lattice:

$$A_{[\mu_1\mu_2\ldots\mu_p]}(x) \Rightarrow A(\star[P_0, P_1, \ldots P_{d-p}]).$$

(A.3)

Notice that the simplex $\sigma_{d-p}$ and the cell $\star \sigma_{d-p}$ are oriented, so the map defined by equation (A.3) is antisymmetric under permutations of the vertices, for instance:

$$A(\star[P_0, P_1, P_2 \ldots P_{d-p}]) = -A(\star[P_1, P_0, P_2 \ldots P_{d-p}]).$$

(A.4)

In order to proceed with the construction of the discrete theory we need to set up and define at least the basic ingredients of the discrete exterior calculus\(^{22}\).

\(^{22}\) The discrete exterior calculus that we try to construct here is different from the one extensively discussed for instance in reference [16], since we associate $p$-forms to $d - p$ dimensional simplices (or $p$ dimensional Voronoi cells) rather than to $p$ dimensional simplices as in reference [16].
Let us first introduce the notion of discrete exterior derivative. The exterior derivative of a $p$-form is a $p + 1$ form, hence it is defined on the $p + 1$ dimensional cells of the dual Voronoi tassellation or equivalently by duality on the $d - p - 1$ dimensional simplices of the original lattice.

Given the $p$-form $A$ at the rhs of (A.3) its exterior derivative $\hat{d}A$ is then a function of the ordered $d - p$ vertices of a $d - p - 1$ dimensional simplex, and it can be defined as:

$$\hat{d}A(*[P_1, P_2, \ldots, P_{d-p}]) = \sum_Q A(*[P_1, P_2, \ldots, P_{d-p}, Q])$$  \hspace{1em} (A.5)

where the sum is extended to all vertices $Q$ such that $[P_1, P_2, \ldots, P_{d-p}, Q]$ is a simplex that has $[P_1, P_2, \ldots, P_{d-p}]$ as a proper face. In terms of the dual lattice the sum at the rhs of (A.5) is over the $p - 1$ dimensional cells $*[P_1, P_2, \ldots, P_{d-p}, Q]$ that form the boundary of the $p$ dimensional cell $*[P_1, P_2, \ldots, P_{d-p}]$.

Equation (A.5) can be generalized to the $p$-chains $\omega_p$ defined as finite formal sums of the $p$-cells with coefficients $l_i$ in $\mathbb{Z}$:

$$\omega_p = \sum_i l_i \ast \sigma^i_{d-p} = \sum_i l_i \ast [P_0^i, P_1^i, P_2^i, \ldots, P_{d-p}^i].$$  \hspace{1em} (A.6)

The boundary operator $\partial$ acts on $\omega_p$ as:

$$\partial \omega_p = \sum_i l_i \sum_{Q^i} \ast [P_1^i, P_2^i, \ldots, P_{d-p}^i, Q^i]$$  \hspace{1em} (A.7)

Assuming that the map defining the $p$-form $A$ is a linear one, we can generalize (A.5) to the form:

$$\hat{d}A(\omega_p) = A(\partial \omega_p)$$  \hspace{1em} (A.8)

which is the discrete equivalent of

$$\int_M \partial A_p = \int_{\partial M_p} A_p.$$  \hspace{1em} (A.9)

From the definition (A.7) and the antisymmetry (A.4) it follows immediately that the square of the boundary operator is zero, and consequently that also $\hat{d}^2 = 0$.

The next step is define a wedge product of two discrete forms trying to preserve as much as possible the algebraic properties of the product of forms in the continuum. The product of a $p$-form and a $q$-form is a $(p + q)$-form, so in our discrete formalism it should be of the form:

$$A(* \sigma_{d-p}) \wedge B(* \sigma_{d-q}) \Rightarrow (A \wedge B)(* \sigma_{d-p-q}).$$  \hspace{1em} (A.10)

The best definition of discrete wedge product we could find has the form$^{23}$:

$$(A \wedge B)(*[P_0, P_1, \ldots, P_{d-p-q}]) = \sum_{R_1, \ldots, R_p, S_1, \ldots, S_q} A(*[P_0, \ldots, P_{d-p-q}, S_1, \ldots, S_q])$$

$$B(*[P_0, \ldots, P_{d-p-q}, R_1, \ldots, R_p]) E([P_0, \ldots, P_{d-p-q}, S_1, \ldots, S_q, R_1, \ldots, R_p])$$  \hspace{1em} (A.11)

$^{23}$This wedge product is also considered in [16] as the ‘discrete dual–dual wedge product’, but its properties are not studied there.
where $E([P_1, P_2, \ldots, P_{d+1}]) = \pm 1$ if the $d + 1$ vertices $P_1, P_2, \ldots, P_{d+1}$ form a $d$-dimensional simplex, otherwise it is zero.

The symbol $E([P_1, P_2, \ldots, P_{d+1}])$ is completely antisymmetric in its arguments and the $\pm$ sign may be chosen to coincide with the sign of the volume in equation (3.8). While an overall sign in the definition of $E([P_1, P_2, \ldots, P_{d+1}])$ is essentially a matter of principle, the relative sign between two neighbouring simplices is crucial and is given by:

$$E([P_1, P_2, \ldots, P_p, P_{p+1}, \ldots, P_{d+1}]) = -E([P_1, P_2, \ldots, P', P_{p+1}, \ldots, P_{d+1}])$$  \hspace{1cm} (A.12)

where $P_p$ and $P'_p$ are the vertices which are not shared by the two simplices, which have a $d - 1$ dimensional face in common. Repeated use of (A.12) determines in principle the signs of the symbol for all simplices of the simplicial complex (assuming it is simply connected).

It follows immediately from (A.11) that even and odd forms (anti)commute according to the usual rule:

$$\Lambda(\star\sigma_{d-p}) \wedge \Lambda(\star\sigma_{d-q}) = (-1)^{pq} \Lambda(\star\sigma_{d-q}) \wedge \Lambda(\star\sigma_{d-p}).$$  \hspace{1cm} (A.13)

However some important properties of the wedge product in the continuum are not preserved by (A.11). First of all the product defined in (A.11) is not associative. This was to be expected: the wedge product introduced in [16], where $p$-forms are directly associated to $\sigma_p$ rather than to $\star\sigma_{d-p}$ as in our case, was shown not to be associative, although it was proved in the same paper that associativity is recovered in the continuum limit.

In order to show the non associativity of (A.11) it is enough to write explicitly the product of the forms $\Lambda(\star\sigma_{d-p})$, $\Lambda(\star\sigma_{d-q})$ and $\Lambda(\star\sigma_{d-r})$:

$$(k(A \wedge B) \wedge C)(\star[T_0, T_1 \ldots T_{d-p-q-r}])$$

$$= \sum_{P_., Q_., R_., S_} A(\star[T_., R_., Q_., S_]) B(\star[T_., R_., P_., S_]) C(\star[T_., S_])$$

$$E([T_., R_., Q_., P_., S_]) \cdot E([T_., R_., S_])$$  \hspace{1cm} (A.14)

where $T_.$ stands for the set of points $T_0, T_1, \ldots, T_{d-p-q-r}$. Similarly $R_., Q_., P_.$ and $S_.$ stand for sets of respectively $r, q, p$ and $p + q$ points.

The non commutativity is apparent from the asymmetry of (A.14) in the three forms $A$, $B$ and $C$. The symmetry would be restored if the set of points $\{S_\}$ coincided with the union of $\{P_\}$ and $\{Q_\}$, so the associative terms correspond to a subset of the terms appearing at the rhs of (A.14). We do not have any argument at the moment to argue that associativity would be restored in the continuum limit, further investigation is needed in that respect.

The other property which is not satisfied by the wedge product (A.14) is the distributive law (Leibnitz rule) with respect to the exterior derivative defined in (A.5).

As in the case of the non associativity this can be checked directly. Let us consider the wedge product of a $p$ and a $q$ form defined in (A.11) and take its exterior derivative. We have:

$$\tilde{d}(A \wedge B)((T_\cdot)) \sum_{P_., Q_., R_., S_} A(T_., R_., Q_., S_) B(T_., R_., P_., S_) E(T_., R_., Q_., P_., S_).$$  \hspace{1cm} (A.15)

where $T_.$ has now only $d - p - q$ entries, as $\tilde{d}(A \wedge B)$ is a $p + q + 1$ form, while $P_.$ and $Q_.$ are defined as above. The sum over the single vertex $R$ is the result of the exterior derivative operation.
We shall compare the result of (A.15) with what one would expect if the Leibnitz rule were valid, namely:

\[
\left( \hat{d}A \wedge B + (-1)^p A \wedge \hat{d}B \right) \left( [T\ldots] \right) = \sum_{P\ldots,R\ldots,S\ldots} A([T\ldots,R\ldots]) \cdot B([T\ldots,S\ldots,P\ldots]) \cdot \left\{ E([T\ldots,R\ldots,Q\ldots,P\ldots]) + E([T\ldots,S\ldots,Q\ldots,P\ldots]) \right\} . \tag{A.16}
\]

If we compare the rhs of equation (A.16) with the rhs of (A.15) we see that in the former there is an extra sum over the vertex \( S \) that was not present in (A.15). The two expressions have the same structure only in a subset of terms, namely if in (A.16) we set \( R = S \). In fact, while at the rhs of (A.15) the forms \( A \) and \( B \) take value on simplices that are both contained in the same \( d \) dimensional simplex (i.e. the argument of the \( E \) function) this is not generally true in equation (A.16) unless \( S \) and \( R \) are set to be equal.

The lack of associativity is not a problem in three dimensional Chern Simons theory and in 11 dimensional supergravity. For instance if \( A \) is the three form field of supergravity in 11 dimension it is immediate to see that even without assuming associativity the two forms \((A \wedge dA) \wedge dA \) and \( A \wedge (dA \wedge dA) \) only differ for a total differential, provided the distributive law with respect to \( d \) is satisfied.

The violation, by a large number of terms, of the Leibnitz rule is a much more serious problem because it prevents from using partial integration and from writing the Chern Simons action and the FFA term in 11 dimensional supergravity as boundary terms of topological actions in one higher dimension.

In the discrete exterior calculus described in [16] \( p \) forms are associated to \( p \) dimensional simplices rather than to the \( p \) dimensional Voronoï cells of the dual lattice. The wedge product defined there is not associative but satisfies the distribution law (Leibnitz rule) with respect to the exterior derivative.

However defining \( p \) forms on the \( p \) dimensional simplices does not seem to fit in the Regge calculus scheme outlined in this paper. At the root of the difficulty, which seems of difficult solution, is the asymmetry between simplicial lattice and dual lattice, which does not allow a consistent definition of a discrete dual Hodge operator.

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