STABILITY OF THE MAGNETOPAUSE OF DISK-ACCRETING ROTATING STARS

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ABSTRACT

We discuss three modes of oscillation of accretion disks around rotating magnetized neutron stars which may explain the separations of the kilohertz quasi-periodic oscillations (QPOs) seen in low-mass X-ray binaries. The existence of these compressible, nonbarotropic magnetohydrodynamic (MHD) modes requires that there be a maximum in the angular velocity \(\Omega_{\phi}(r)\) of the accreting material larger than the angular velocity of the star \(\Omega_{*}\), and that the fluid be in approximately circular motion near this maximum rather than moving rapidly toward the star or out of the disk plane into funnel flows. Such a maximum in \(\Omega_{\phi}\) occurs naturally in disk accretion to a slowly rotating magnetized star due the magnetic braking of the disk rotation by the stellar field. Our MHD simulations show this type of flow and \(\Omega_{\phi}(r)\) profile. The first mode is a Rossby wave instability (RWI) mode which is radially trapped in the vicinity of the maximum of a key function \(g(r)\) at \(r_g\). The real part of \(\omega_r\) is the azimuthal mode number. We argue that the nonlinear saturation of the RWI occurs when the trapping frequency of fluid particles in the Rossby vortex equals the RWI growth rate. The second mode is a mode driven by the rotating, nonaxisymmetric component of the star’s magnetic field. It has an angular frequency equal to the star’s angular rotation rate \(\Omega_{*}\). This mode is strongly excited near the radius of the Lindblad resonance which is slightly outside of \(r_g\). The third mode arises naturally from the interaction of a flow perturbation with the rotating nonaxisymmetric component of the star’s magnetic field. It has an angular frequency \(\Omega_{*}/2\). We suggest that the first mode with \(n = 1\) is associated with the upper QPO frequency, \(\nu_{\ell}\); that the nonlinear interaction of the first and second modes gives the lower QPO frequency, \(\nu_{l} = \nu_{u} - \nu_{*}\); and that the nonlinear interaction of the first and third modes gives the lower QPO frequency \(\nu_{l} = \nu_{u} - \nu_{*}/2\), where \(\nu_{*} = \Omega_{*}/2\pi\).

Key words: accretion, accretion disks – instabilities – MHD – stars: neutron – waves – X-rays: binaries

1. INTRODUCTION

Low-mass X-ray binaries often display twin kilohertz quasi-periodic oscillations (QPOs) in their X-ray emissions (van der Klis 2006; Zhang et al. 2006). A wide variety of different models have been proposed to explain the origin and correlations of the different QPOs. These include the beat frequency model (Miller et al. 1998; Lamb & Miller 2001, 2003), the relativistic precession model (Stella & Vietri 1999), the Alfven wave model (Zhang 2004), and warped disk models (Shirakawa & Lai 2002; Kato 2004). A puzzling aspect of some of the twin QPO sources considered in this work is that the difference between the upper \(\nu_{u}\) and lower \(\nu_{l}\) QPO frequencies is roughly either the spin frequency of the star \(\nu_{*}\) (five cases where \(\nu_{u} = 191, 270, 294, 330,\) and 363 Hz) or one-half this frequency, \(\nu_{l}/2\) (five cases where \(\nu_{u} = 401, 524, 581, 600,\) and 619 Hz), for the cases where \(\nu_{u}\) is known, even though \(\nu_{u} - \nu_{l}\) varies smoothly with radial distance but are independent of \(z\), and where the magnetic field is dynamically important.

The radial profiles of these quantities are known for different conditions from MHD simulations (Romanova et al. 2002, 2008; Long et al. 2005; Kulkarni & Romanova 2008). The recent MHD studies (Romanova et al. 2008; Kulkarni & Romanova 2008) found conditions where a global Rayleigh–Taylor instability occurs and changes the nature of the accretion flow from a regular funnel flow pattern to a chaotic flow of plasma fingers. The present work is a continuation of the study of Lovelace & Romanova (2007; hereafter LR07) of the magnetohydrodynamic (MHD) stability of the compressible, nonbarotropic stability of the boundary between an accretion disk and the magnetosphere of a rotating magnetized star for conditions where \(\Omega_{\phi}, B_{\perp},\) and \(\rho \phi\) vary smoothly with radial distance but are independent of \(z\), and where the magnetic field is dynamically important. The radial profiles of these quantities are known for different conditions from MHD simulations (Romanova et al. 2002, 2008; Long et al. 2005; Kulkarni & Romanova 2008).
out a more general stability analysis of disks where $\Omega_\phi(r)$ goes through a maximum, but they did not include a self-consistent treatment of the magnetic field perturbation.

When $\Omega_\phi(r)$ has a maximum, LR07 showed that there was a Rossby wave instability (RWI) radially trapped at $r_R$ near the maximum of $\Omega_\phi(r)$. They suggested that the upper QPO frequency was $\omega_c = m\Omega_\phi(r_R)$, where $m = 1, 2, \ldots$ is the azimuthal mode number. The lower QPO frequency was suggested to be due to the interaction of the Rossby mode with the rotating nonaxisymmetric field of the star or the modulated nonaxisymmetric field. The RWI theory was developed by Lovelace et al. (1999) and Li et al. (2000) for accretion disks and earlier by Lovelace & Hohlfeld (1978) for disk galaxies. Hydrodynamic simulations of the RWI instability in disks were done by Li et al. (2001), while Sellwood & Kahn (1991) used $N$-body simulations to study the instability in galaxies. The instability has important role in the accretion–ejection instability of disks discussed by Tagger and collaborators (e.g., Tagger & Varnière 2006; Tagger & Pellat 1999).

Section 2.1 develops the general compressible, nonbarotropic MHD equations for free perturbations of an axisymmetric equilibrium flow with no $z$-dependence. Section 2.2 treats the driven perturbations due to the rotating nonaxisymmetric component of the star’s magnetic field. Section 2.3 treats the driven-modulated perturbations which result from the interaction of the nonaxisymmetric field of the star and the flow perturbation. Section 3 develops a detailed model of the axisymmetric flow/field equilibrium based on results from MHD simulations. Section 4 discusses the results obtained applying the theory of Section 2 to the model of Section 3. Section 4.1 treats the free perturbations, Section 4.2 the driven perturbations, and Section 4.3 the driven-modulated perturbations. Section 5 briefly discusses the nonlinear effect of the unstable mode. Section 6 gives the conclusions of this work.

2. THEORY

We consider the stability of the magnetopause of a rotating star with an aligned dipole magnetic field. The envisioned geometry is shown in Figure 1. We use an inertial cylindrical $(r, \phi, z)$ coordinate system. The equilibrium has $(\partial/\partial t = 0)$ and $(\partial/\partial \phi = 0)$, with the flow velocity $\mathbf{u} = u_\phi(r) \hat{\phi} = r\Omega_\phi(r) \hat{\phi}$. That is, the accretion velocity $u_\phi$ and the vertical velocity $u_z$ are assumed negligible compared with $u_\phi$. The equilibrium magnetic field is $\mathbf{B} = B(r) \hat{z}$. The equilibrium flow satisfies $\rho r (\Omega_K^2 - \Omega_\phi^2) = -d[p + B^2/(8\pi)]/dr$, where $\rho$ is the density, and $\Omega_K$ is the Keplerian angular rotation rate of a single particle.

2.1. Free Perturbations

The perturbed quantities are: the density $\tilde{\rho} = \rho + \delta \rho(r, \phi, t)$; the pressure $\tilde{p} = p + \delta p(r, \phi, t)$; the flow velocity is $\tilde{\mathbf{u}} = \mathbf{u} + \delta \mathbf{u}(r, \phi, t)$ with $\delta \mathbf{u} = (\delta u_\phi, \delta u_\phi, 0)$; and the magnetic field is $\tilde{\mathbf{B}} = \hat{\mathbf{B}} + \delta \hat{\mathbf{B}}$. We neglect the self-gravity of the disk. The equations for the perturbed flow are

$$\frac{D\tilde{\rho}}{Dt} + \tilde{\rho} \nabla \cdot \tilde{\mathbf{u}} = 0,$$

$$\frac{D\tilde{\mathbf{u}}}{Dt} = -\frac{1}{\tilde{\rho}} \nabla \left( \frac{\tilde{\rho} + B^2}{8\pi} \right) - \nabla \Phi,$$

$$\frac{DS}{Dt} = 0,$$

where $D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$, and where $S \equiv \tilde{p}/(\tilde{\rho}^\gamma)$ is the entropy of the disk matter. $\Omega_\phi(r) \equiv m\Omega_\phi(r)$ and $\Omega_\phi = u_\phi/r$. From Equation (1b) we have

$$i\Delta \omega \delta u_\phi + 2\Omega_\phi \delta u_\phi = \frac{1}{\rho} \frac{\delta p_+}{\partial r} - \frac{\delta \rho}{\rho^2} \frac{dp_+}{dr}.$$

Here, $\Omega_\phi \equiv (r^{-3}d(r^4\Omega_\phi^2)/dr)^{1/2}$ is the radial epicyclic frequency and $k_\phi \equiv m/r$ is the azimuthal wavenumber,

$$p_+ \equiv \rho + \frac{B^2}{8\pi} \text{ and } \delta p_+ \equiv \delta p + \frac{B\delta B}{4\pi}.$$

From Equations (1c) and (1d), we have

$$\delta p = c_s^2 \delta \rho - \frac{ic_s^2}{\Delta \omega L_S} \delta u_\phi \text{ and } B = \frac{\delta p}{\rho} B - \frac{i\rho \delta u_\phi}{\Delta \omega} \frac{d}{dr} \left( \frac{B}{\rho} \right).$$

Combining these expressions,

$$\delta p_+ = (c_s^2 + c_A^2)\delta \rho - \frac{ic_s^2}{\Delta \omega L_S} \left( \frac{c_s^2}{L_S} + \frac{c_A^2}{L_B} \right),$$

where

$$\frac{1}{L_S} \equiv \frac{1}{\gamma} \frac{d \ln(S)}{dr} \text{ and } \frac{1}{L_B} \equiv \frac{d \ln(B/\rho)}{dr}$$

are the length scales of the entropy $S \equiv p/\rho^\gamma$ and $B/\rho$ variations. Also,

$$c_s \equiv \left( \frac{\gamma p}{\rho} \right)^{1/2} \text{ and } c_A = \frac{|B|}{\sqrt{4\pi \rho}}.$$
are the sound and Alfvén speeds, respectively. We denote \(c_f \equiv (c_s^2 + c_A^2)^{1/2}\) as the fast magnetosonic speed.

It is useful to introduce

\[
\Psi \equiv \frac{\delta \rho_s}{\rho}. \tag{6}
\]

Equations (3) can then be written as

\[
\begin{align*}
A \delta u_r + B \delta u_\phi &= f_r, \\
C \delta u_\phi + D \delta u_r &= f_\phi,
\end{align*}
\]

where

\[
A = -i \left[ \frac{\Delta \omega + \frac{d \rho_s}{dr}}{\Delta \omega L_s} \right], \quad B = -2 \Omega_\phi, \quad C = -i \Delta \omega, \quad D = \frac{\Omega^2}{2 \Omega_\phi},
\]

Also,

\[
f_r = -\frac{\partial \Psi}{\partial r} \phi + \frac{\Psi}{L_s},
\]

\[
f_\phi = -i k_\phi \Psi,
\]

where

\[
\frac{1}{L_s} = \frac{1}{\rho c_f^2} \frac{d \rho_s}{dr} \left[ \frac{c_f^2}{L_s} + \frac{c_A^2}{L_B} \right], \tag{8a}
\]

and where

\[
\frac{1}{L_B} = \frac{d \ln(\rho)}{dr}. \tag{8b}
\]

For a strong magnetic field \(c_A \gg c_s\), we have \(L_s \rightarrow L_B\). For a weak magnetic field \(c_A \ll c_s\), we have \(L_s \rightarrow L_S\). Using the equation for the equilibrium \(\rho r(\Omega_k^2 - \Omega_\phi^2) = -\rho c_f^2/dr\), we have

\[
\frac{1}{L_s} = -\frac{r(\Omega_k^2 - \Omega_\phi^2)}{c_f^2} - \frac{1}{L_B}, \tag{8c}
\]

which is useful later.

We solve Equations (7a) and (7b) to obtain

\[
\rho \delta u_r = i \mathcal{F} \left[ \frac{\Delta \omega}{\Omega_\phi} \left( \frac{\partial \Psi}{\partial r} - \frac{\Psi}{L_s} \right) - 2 k_\phi \Psi \right], \tag{9a}
\]

\[
\rho \delta u_\phi = \mathcal{F} \left\{ \frac{\Omega^2}{2 \Omega_\phi^2} \left( \frac{\partial \Psi}{\partial r} - \frac{\Psi}{L_s} \right) - k_\phi \left( \frac{\Delta \omega}{\Omega_\phi} + \frac{d \rho_s}{dr} \right) \frac{\Psi}{L_B} \right\}. \tag{9b}
\]

Here,

\[
\mathcal{F} \equiv \frac{\rho \Omega_\phi}{D}, \tag{10a}
\]

where

\[
\mathcal{D} = \Omega^2 - (\Delta \omega)^2 - \frac{d \rho_s}{dr} \frac{\rho}{\rho L_s} \tag{10b}
\]

is the “Lindblad factor.” For real frequencies \(\omega\), \(\mathcal{D}(r)\) is equal to zero at a Lindblad resonance at \(r_L\). On the other hand at a corotation resonance \(\Re[\Delta \omega(r)] = 0\) at \(r_L\). Using Equation (8c) we find

\[
\mathcal{D} = \Omega^2 - (\Delta \omega)^2 - \frac{\rho c_f^2}{c_f^2 + \Delta \omega L_s} \frac{\rho}{\rho L_s}. \tag{10c}
\]

Equations (9a) and (9b) can now be used to obtain

\[
\nabla \cdot (\rho \Psi) = \frac{i \Delta \omega}{r} \left( \frac{r \mathcal{F}}{\Omega_\phi} \left( \frac{\partial \Psi}{\partial r} \right) + \frac{\mathcal{F} \partial \Delta \omega}{\partial r} \right)
\]

\[
- \frac{i}{r} \left[ \frac{\partial}{\partial r} \left( \frac{r \mathcal{F} \Delta \omega}{\Omega_\phi L_s} \right) \right] \Psi - \frac{i \mathcal{F} \Delta \omega}{\Omega_\phi L_s} \frac{\partial \Psi}{\partial r} - 2 i k_\phi \frac{\partial \mathcal{F}}{\partial r} \Psi
\]

\[
- 2 i k_\phi \mathcal{F} \frac{\partial \Psi}{\partial r} - i k_\phi \mathcal{F} \frac{\partial \Omega^2}{\Omega_\phi^2 L_s} + \frac{i k_\phi \mathcal{F} \Omega^2}{2 \Omega_\phi^2} \frac{\partial \Psi}{\partial r} + \frac{i k_\phi \mathcal{F} \Omega^2}{2 \Omega_\phi^2} \frac{\partial \Psi}{\partial r}
\]

\[
- i k_\phi \mathcal{F} \left[ \frac{\Delta \omega}{\Omega_\phi} + \frac{d \rho_s}{dr} \right] \frac{\rho}{\rho L_s} \Psi. \tag{11}
\]

It is useful to note that

\[
\frac{\partial \Delta \omega}{\partial r} = \frac{k_\phi (\Omega_s^2 - 4 \Omega_\phi^2)}{2 \Omega_\phi}.
\]

From Equations (4) and (9a), we have

\[
\delta \rho = \frac{\rho}{c_f^2} + \frac{\mathcal{F}}{\Delta \omega L_s} \left[ \frac{\Delta \omega}{\Omega_\phi} \left( \frac{\partial \Psi}{\partial r} - \frac{\Psi}{L_s} \right) - 2 k_\phi \Psi \right]. \tag{12}
\]

Equations (11) and (12) can now be substituted into Equation (2). When this is done we find that all of the terms involving \(\partial \Psi/\partial r\) apart from the overbraced term in Equation (11) cancel out.

Combining Equations (9a) and (9b) with Equation (2) gives

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \mathcal{F}}{\Omega_\phi} \frac{\partial \Psi}{\partial r} \right)
\]

\[
= \left[ \frac{\rho}{c_f^2} + \frac{k_\phi \mathcal{F}}{\Omega_\phi} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \mathcal{F}}{\Omega_\phi L_s} \right) + \frac{\mathcal{F}}{\Omega_\phi L_s^2} \right] \Psi
\]

\[
+ \left[ \frac{2 k_\phi \mathcal{F}}{d \ln(\rho)/dr} \right] \frac{\partial \Omega_\phi}{\partial \delta \rho} + \left[ \frac{k_\phi \mathcal{F}}{\rho L_s} \right] \frac{\partial \Psi}{\partial (\Delta \omega)^2}. \tag{13}
\]

where

\[
g = \exp(2 \int dr/L_s). \]

In the limit of no magnetic field \((c_A \rightarrow 0\) and \(L_s \rightarrow L_s)\), this equation becomes the same as the equation for the RWI (Lovelace et al. 1999; Li et al. 2000).

A quadratic form can be obtained by multiplying Equation (13) by \(\Psi^*\) (the complex conjugate of \(\Psi\)) and integrating over the disk. Assuming \(r \Psi^*(d \Psi/dr) \mathcal{F}/\Omega_\phi \rightarrow 0\) for
For corotation modes where \( |\Delta \omega^2| \ll \Omega^2 \), the third and fourth integrals on the right-hand side of Equation (14) are possibly important. Earlier work by Lovelace & Hohlfeld (1978), Lovelace et al. (1999), and Li et al. (2000) found that a corotation instability was possible if the quantity \( g \mathcal{F} \) has a maximum or minimum at the radial distance \( r_g \) where \( \delta (\Omega \omega) = 0 \). For a weak magnetic field \( c_A^2 \ll c_s^2 \) where \( L_s \approx L_g \), we find \( g \mathcal{F} = S^{1/2} \mathcal{F} \) which agrees with the result of Lovelace et al. (1999) and Li et al. (2000) where the self-gravity of the disk was neglected. With negligible self-gravity, instability is found only for conditions where \( g \mathcal{F} \) has a maximum as a function of \( r \). For a maximum, the radial group velocity of the Rossby waves is directed toward \( r_g \) (Lovelace et al. 1999). In the strong \( B \)-field limit \( (c_A^2 \gg c_s^2 \) where \( L_s \approx L_B \), \( g \mathcal{F} = (B/\rho)^2 \mathcal{F} \).

We can rewrite Equation (13) in a form more amenable to numerical analysis. That is,

\[
\Psi'' = \left( \frac{D}{\Omega} \right) \Psi' + \left( C_0 + \frac{C_1}{\Delta \omega} + \frac{C_2}{(\Delta \omega)^2} \right) \Psi,
\]

where \( D \equiv \mathcal{D}/(\rho \sigma) = (\Omega \rho/\sigma) \mathcal{F}^{-1} \), where \( \mathcal{D} \) is given by Equation (10b) and \( \mathcal{F} \) by Equation (10a). The primes denote radial derivatives. Also,

\[
C_0 = k_\phi^2 + \frac{D}{c_f^2} - \frac{D}{L_s D} + \frac{1 - L_s^2}{L_s^2},
\]

\[
C_1 = 2k_\phi \Omega \sigma [\ln(g \mathcal{F})]', \quad C_2 = k_\phi^2 \frac{B_s}{\rho L_s},
\]

where \( g \) is defined below Equation (13).

If we let \( \Psi = (\mathcal{D}/\sigma)^{1/2} \varphi \), then Equation (15) can be written in the form of a Schrödinger equation,

\[
\varphi'' = \left( \overline{C}_0 + \frac{C_1}{\Delta \omega} + \frac{C_2}{(\Delta \omega)^2} \right) \varphi \equiv \bar{U}(r) \varphi,
\]

where \( \overline{C}_0 \equiv C_0 + \frac{3}{4} \left( \frac{D}{\mathcal{D}} \right)^2 - \frac{1}{2} \left( \frac{D}{\mathcal{D}} \right)^2 \). (17)

where \( \bar{U}(r) \) is an effective potential. Both \( U \) and \( \varphi \) are in general complex for complex \( \omega \).

A quadratic form analogous to Equation (14) can be obtained by multiplying \( \varphi' \varphi' \) vanishes at small and large \( r \), we obtain

\[
- \int dr \varphi'^2 = \int dr \overline{C}_0 |\varphi|^2 + 2 \int dr k \Omega \frac{\ln(g \mathcal{F})'}{\Delta \omega} |\varphi|^2
\]

\[
+ \int dr \frac{C_2}{(\Delta \omega)^2} |\varphi|^2.
\]

(18)

The imaginary part of this equation gives

\[
0 = \cdots - \omega_i \int dr \frac{\ln(g \mathcal{F})'}{(\Delta \omega)^2} |\varphi|^2 + \cdots,
\]

where the ellipsis allow for contributions from the \( \overline{C}_0 \) and \( C_2 \) terms. If these terms are negligible as found in our numerical evaluations, it follows that a nonzero growth rate \( \omega_i \) is possible only for conditions where \( \ln(g \mathcal{F})' \) changes sign as mentioned above (Lovelace & Hohlfeld 1978). Thus, \( g \mathcal{F} \) is a key function for stability of the considered flow.

Using Equations (4), (9a), and (12) we obtain the relation of the temperature perturbation to \( \Psi \),

\[
\frac{\delta T}{T} = A_0 \Psi + A_1 \Psi',
\]

where \( A_0 = \frac{\gamma - 1}{c_f^2} + \frac{1}{\mathcal{D}} \left( \frac{\gamma - 1}{L_s^2} - \frac{\gamma}{L_s^2} \right) + \frac{2k_\phi \Omega \sigma}{\mathcal{D} \Delta \omega} \left( \frac{\gamma - 1}{L_s^2} - \frac{\gamma}{L_s^2} \right), \quad A_1 = \frac{1}{\mathcal{D}} \left( \frac{\gamma}{L_s^2} - \frac{\gamma - 1}{L_s^2} \right). \]

(20)

This equation shows that the temperature perturbation is strongly enhanced at a corotation resonance where \( |\Delta \omega| \) becomes very small and at a Lindblad resonance where \( D = 0 \).

2.2. Driven Perturbations

The influence of a small nonaxisymmetric component of the stellar magnetic field can be studied by including in Equation (1b) the small force due to this non-axisymmetry. For simplicity, we first discuss the case where there is a time-dependent vertical stellar magnetic field which penetrates the disk. In the next paragraph, we treat the case of a time-dependent transverse (\( \perp \hat{z} \)) stellar field which does not penetrate the disk. At a given radial distance, the total vertical magnetic field \( B = B^v + B^i \) consists of the vacuum component \( B^v \) due to current flow inside the star and the induced field \( B^i \) due to the current flow in the plasma outside the star. The vacuum field has the general form

\[
B^v = B^{v0}(r) + \Delta B^v(r, \phi, t),
\]

\[
\Delta B^v = B^{v0}(r) \exp(i \phi - i \Omega t)
\]

\[
+ B^{v2}(r) \exp(2i \phi - i \Omega t) + \cdots.
\]

(21)

where \( B^{v0} \) is the axisymmetric component, \( B^{v1} \ll B^{v0} \) is the quadrupole component, and \( B^{v2} \ll B^{v0} \) is the octupole component. The quadrupole term can represent a noncentered but aligned dipole field in the star. The total magnetic force or acceleration is \( \mathbf{F} = - \nabla[(\mathbf{B} + \Delta \mathbf{B}^v)^2]/(8\pi \rho) \), (because \( \mathbf{B} \cdot \nabla \mathbf{B} = 0 \)), where \( \mathbf{B} = (B^v + B^i) \) and where the average is over \( \phi \). Linearization gives \( \delta \mathbf{F} = - \nabla[(\mathbf{B} + \Delta \mathbf{B}^v)(\mathbf{B}^v + \Delta \mathbf{B}^i)]/(4\pi \rho) \). Thus, we have

\[
(\delta F_r, \delta F_\phi) = (\delta F_r, \delta F_\phi) \exp(i m \phi - i \Omega t),
\]

(22)
where \( m = 1 \) or 2 and \( \Omega \), is the angular rotation rate of the star. The calculation of Section 2.1 is modified slightly beginning with Equation (7) which are here replaced by

\[
A \delta v_r + B \delta v_\phi = f_r + \delta F_r, \\
C \delta v_\phi + D \delta v_r = f_\phi + \delta F_\phi,
\]

where \( A, \ldots, D \) are the same as defined below Equation (7).

Equation (22) also applies for the case of a rotating star with a misaligned dipole \((m = 1)\) or quadrupole \((m = 2)\) magnetic field. We discuss the dipole case where the angle between the rotation axis \( \Omega \), and the magnetic moment \( \mu \) is \( \theta \). A similar calculation applies for a misaligned quadrupole magnetic field. The time- and \( \phi \)-dependent part of the star’s field is denoted as \( \Delta B^* \). Near the disk the field components are \( \Delta B^r, \Delta B^\phi \) and they are proportional to \( \sin(\theta) \exp(i\phi - i\Omega t) \). For an infinitely conducting disk, the time-dependent component of the field vanishes inside the disk (e.g., Lai 1999). Thus, the actual field is \((\Delta B^r, \Delta B^\phi)\Theta(|z| - h)\), where \( h \) is the half-thickness of the disk \((\ll r)\) and \( \Theta \) is the unit step function. The step function gives rise to a surface current density on the two sides of the disk, \( J_F = -(c/4\pi)\Delta B^r \delta \Theta/\delta z \) and \( J_\Phi = (c/4\pi)\Delta B^\phi \delta \Theta/\delta z \). The force components arising from the surface current are simply \( F_r = J_A B/c \) and \( F_\phi = -J_A B/c \), where \( B \) is the time-independent axial field. The force on the disk’s surface is rapidly communicated over the vertical extent of the disk owing to the large value of the Alfvén velocity (see Section 3). Therefore, we can integrate the force over the disk thickness and dividing by \( h \) to obtain an average force. This gives \( \delta F_r = (\partial \Delta B^r/\partial z)h B/4\pi \) and \( \delta F_\phi = -(\partial \Delta B^\phi/\partial z)h B/4\pi \), where the zero subscript indicates evaluation at \( z = 0 \).

The forcing term \( \delta F \) gives rise to an additional contribution to \( \rho \delta v \) not included in Section 2.1. This contribution is

\[
\rho \delta v_r = \frac{1}{r \overline{D}}(-C \delta F_r + B \delta F_\phi) \equiv H_r, \\
\rho \delta v_\phi = \frac{1}{r \overline{D}}(D \delta F_r - A \delta F_\phi) \equiv H_\phi, \quad (24)
\]

where \( \overline{D} = D/(r \rho) \) and \( D \) is the Lindblad factor defined in Equation (10b). Including this contribution we find that response of the flow \( \varphi = \Psi/\overline{D}^{1/2} \) is given by

\[
\varphi'' - U(r) \varphi = i \frac{r \overline{D}^{1/2}}{\Delta \omega} \nabla \cdot \mathbf{H}, \quad (25)
\]

where \( U \) is defined in Equation (17) and \( \mathbf{H} \) is a known function determined by \( \Delta B^* \). We are interested only in the inhomogeneous solutions to this equation where \( U, \omega, \Delta \omega \), and \( \overline{D} \) are real.

### 2.3 Driven-Modulated Perturbations

Here we consider the case where the above-mentioned driving force \( \delta F \) is modulated by the flow perturbation \( \Psi \) (LR07). We discuss the vertical field case (Section 2.2), but in the next paragraph we give the result for the transverse field case. This modulation comes about naturally by including the magnetic field perturbation \( \delta B \) in the magnetic force \( \mathbf{F} \). That is, \( \mathbf{F} = -\nabla[(B^r \delta B + B^\phi \delta B^\phi)/(8\pi \rho)] \). Linearization of this force gives a contribution \( F'' = -\nabla(\delta B \Delta B^*/(4\pi \rho)) \). The equations of Section 2.1 can be used to derive an exact formula relating \( \delta B \) to \( \Psi \), but we use a simplified formula for one of the dominant terms, \( \delta B = B \Psi/(D \overline{L}_*^{1/2})^{-1} \), in order to limit the complexity of the equations. An equation for the driven-modulated perturbations is then obtained by replacing the replacement in Equation (23) of \( \delta F \rightarrow \delta F'' = \Psi \delta F/(D \overline{L}_*^{1/2}) \), where \( \delta F \) is still given by Equation (22) and is a known function determined by \( \Delta B^* \). The use of \( \Psi^* \) rather than \( \Psi \) is required due to our assumed dependence of \( \delta F \) in Equation (22). For the driven-modulated perturbations, we obtain in place of Equation (25)

\[
\varphi'' - \hat{U}(r) \varphi = i \frac{r \overline{D}^{1/2}}{\Delta \omega} \nabla \cdot \left\{ \mathbf{M} \left[ \left( \overline{D}^{1/2} \Psi \right)^* \delta \mathbf{F} \right] \right\}, \quad (26)
\]

where we have used the fact that \( \Psi = (\overline{D})^{1/2} \). The hats over different quantities indicate that they are now operators with \( \omega \rightarrow i(\partial/\partial t) \) and \( k_\phi \rightarrow -i(\partial/\partial \phi) \). The matrix \( \mathbf{M} \) allows us to write Equation (24) as \( \mathbf{H} = \mathbf{M} \cdot \delta \mathbf{F} \).

For the case of a transverse stellar field perturbation (Section 2.2), the modulated force perturbation for the dipole \((m = 1)\) is simply \( F'' = (\partial \Delta B^r/\partial z) \delta B/4\pi \) and \( \delta F'' = -(\partial \Delta B^\phi/\partial z) \delta B/4\pi \). Thus, we have \( \delta F'' = (\partial \delta B/B) \delta \mathbf{F} \), where \( \delta \mathbf{F} \) is given by Equation (22). Similar formulae apply for the quadrupole \((m = 2)\) case.

Equation (26) is a linear equation for \( \varphi \) so that in general

\[
\varphi = \varphi_0 \exp(-i\omega t) + \varphi_1 \exp(i\phi - i\omega t) + \varphi_2 \exp(2i\phi - i\omega t) + \ldots, \quad (27)
\]

where \( \omega \) is real but undetermined at this point. Clearly, the time and angle dependences on both sides of Equation (26) must match.

For the case where there is a small quadrupole field component \( \Delta B = B^{12} \exp(\varphi - i\omega t) \), Equation (26) implies

\[
\varphi'' - U_{0,0}(r) \varphi_0 = [\text{Op}(\varphi_1^*)]_{0,\omega} \quad (m = 0), \\
\varphi'' - U_{1,0}(r) \varphi_1 = [\text{Op}(\varphi_1^*')]_{1,\omega} \quad (m = 1), \quad (28)
\]

where Op stands for the linear operator on \( \varphi^* \) on the right-hand side of Equation (26), and where the new subscripts indicate the \( m \) value and the \( \omega \) value. Here, we necessarily have

\[
\omega = \frac{\Omega_s}{2}, \quad (29)
\]

as found earlier by LR07.

For the case where there is a small octupole field component \( \Delta B = B^{13} \exp(2i\phi - i\omega t) \), Equation (26) implies

\[
\varphi'' - U_{1,0}(r) \varphi_1 = [\text{Op}(\varphi_1^*)]_{1,\omega}, \quad (30)
\]

where we again have \( \omega = \Omega_s/2 \).

In contrast with the case of driven perturbations of Section 2.2, Equation (26) is linear in \( \varphi \) so that its magnitude is indeterminate. Further study is needed to determine the magnitude of \( \varphi \) in this case. One possibility is to generalize Equation (26) to include on the right-hand side of the equation the inhomogeneous driving force \( \delta \mathbf{F} \) as well as the nonlinear force \( \delta \mathbf{F}'' = -\nabla(\delta B \Delta B^*/(8\pi \rho)) \). Owing to the nonlinearity the driving force \( \delta \mathbf{F} \) at frequency \( \Omega_s \), can generate the \( 1/2 \) subharmonic oscillations of \( \varphi \) at the frequency \( \Omega_s/2 \) described by Equation (26). Analogous subharmonic generation is known in similar systems (e.g., Minorsky 1974).

For the transverse quadrupole field \((m = 2)\), the driven-modulated mode can be represented schematically as a scattering process where the conjugate wave \( \varphi^* \) scatters off the
quadrupole field $\Delta B^2 \frac{v^2}{2}$ to give $\varphi$. That is,

$$\L \varphi = \Delta B^2 \frac{v^2}{2} \varphi^2,$$

where $k_j$ is the azimuthal mode number, $\omega_j$ is the frequency, and $\L$ is a linear operator. For the transverse quadrupole field case, we have $(k_1, k_2, k_3) = (-1, 2, 1)$ and $(\omega_1, \omega_2, \omega_3) = (-\omega, \Omega_*, \omega)$. Thus, we have $\omega = \Omega_*/2$.

3. MODEL OF EQUILIBRIUM

Figure 2 shows sample measured profiles of the midplane axial magnetic field ($B$), the azimuthal frequency of the disk matter [$v_{\varphi} = v_{\varphi}/(2\pi)$], and its density ($\rho$) from MHD simulations by Romanova et al. (2008) for a globally stable case. The simulations were three-dimensional for an approximately axisymmetric case. These profiles motivate our choice of analytic functions to represent these quantities.

The fact that $v_{\varphi}(r)$ decreases as $r$ decreases close to the star is due to magnetic braking. A small twist of the star’s magnetic field transports angular momentum of the disk matter to the star. That is, for $z > 0$ there is a vertical flux of angular momentum $-r B_\varphi B_z/(4\pi) > 0$ with $|B_\varphi| \ll |B_z|$ which transports the disk angular momentum to the star along the star’s field lines (see Figure 1). This loss of angular momentum implies a mass accretion rate of the disk (outside of the region of the funnel flow) of $M_\text{d} = -r^2 B_\varphi B_z/(d l/d r) (= \text{const})$ in the absence of viscosity (see Lovelace et al. 1994), where the $h$-subscript indicates evaluation at the top surface of the disk, $l = r u_\varphi$ is the specific angular momentum, and $d l/d r$ is positive for the considered profiles. The turbulent viscosity due to the magnetorotational instability (MRI) is absent in the region of the disk where $c_A > c_s$ and/or $d S_\varphi/dr > 0$ (Balbus & Hawley 1998). Note however that the disk equilibria discussed below neglect the accretion and have $B_\varphi = 0$.

We assume a pseudo-Newtonian gravitational potential $\Phi_\varphi = -GM_*/(r - r_\text{s}),$ where $M_*$ is the star’s mass and $r_\text{s} = 2GM_*/c^2 = 4.14 \times 10^5$ cm for a 1.4 $M_\odot$ star. The angular velocity of a single particle is $\Omega_K = 2\pi v_K = (GM_*/[r(r - r_\text{s})^2])^{1/2}$ for $r > 3r_\text{s}$. Near the star, the rotation frequency of the matter is modeled following LR07 as

$$v_{\varphi}^0(r) = \frac{v_\text{s} f(r)}{1 + f(r)} + \frac{v_K(r)}{1 + f(r)},$$

which is a first approximation as explained below. Here, $v_\text{s}$ is the rotation frequency of the star and $f(r) = \exp(-(r - r_0)/\Delta)$ with $r_0$ the standoff distance of the boundary layer and $\Delta$ is its thickness. The azimuthal velocity of the matter is $v_\varphi = 2\pi rv_{\varphi}^0$. Both $r_0$ and $\Delta$ are expected to depend on the accretion rate and the star’s magnetic field.

The radial force equilibrium for the axisymmetric flow is

$$\rho(r^2)(v_\varphi^2 - v_\text{s}^2) = 0,$$

where $\rho = \rho_0 (1 + c_s/v_\varphi^0)$ is the pressure in the disk and $c_s$ is the sound speed. The density profile is modeled as

$$\rho = \rho_0 \left(1 + \epsilon \exp(-0.05r/r_\text{s})\right),$$

with $f(r)$ the same function as in Equation (31) and $\epsilon$ is a positive quantity much less than unity. The sound speed is modeled by choosing say $c_\text{s,K} = 0.1r_\text{s}\Omega_*/2$. We then go back and obtain the correction $\delta v_{\varphi}$ to the azimuthal frequency needed to account for the pressure gradient: that is, $2\rho r^2 v_\varphi^0 \delta v_{\varphi} = dp/dr$ so that the actual rotation frequency is $v_{\varphi} = v_{\varphi}^0 + \delta v_{\varphi}$. The correction is small, $|\delta v_{\varphi}/v_{\varphi}^0| \sim (c_s/v_\varphi^0)^2 \ll 1$. The radial epicyclic frequency is calculated from $\Omega_\text{e} = 2\pi v_\varphi$ with $v_\varphi^2 = r^2 d^2(r^4 v_\varphi^2)/dr$. Representative curves are shown in Figures 3 and 4. The value of the magnetic field at $r/r_\text{s} = 6$ is arbitrary, but here it is chosen so that $c_A < c_s$ which allows the MRI to grow which in turn gives rise to a turbulent viscosity in the disk (Balbus & Hawley 1998). The maximum of the disk frequency $v_\varphi$ is about $1149(4r_\text{s}/r_0)^{1.68}$ Hz for $\Delta/r_\text{s} = 0.1$, and it occurs at a distance about 0.26$r_\text{s}$ larger than $r_0$. For $\Delta/r_\text{s} = 0.2$, the maximum of $v_\varphi$ is somewhat smaller and it occurs at a distance about 0.4$r_\text{s}$ larger than $r_0$.

4. RESULTS

We solve the equations of Section 2 using the equilibrium profiles of Section 3. For this we use Maple version 12 where we define about 40 different functions of $R = r/r_\text{s}$, for example, $\rho(R), v_\varphi(R), v_\text{s}(R), \Delta\omega_0, B(R),$ and $U(R)$. Some of the functions, for example, $U$, are complex for complex $\omega$. The axisymmetric ($m = 0$) perturbation is found to be stable if the width of the boundary layer is not too narrow; that is, there is stability for $\Delta/r_\text{s} \geq 0.02$. In the following, we consider the possible instability of modes with $m = 1, 2, \ldots$. We have investigated the flow stability for a wide range of $\omega_0$ values and find that the discussed RWI is the dominant instability.
4.1. Free Perturbations

Figure 5 shows the radial dependence of the real part of the key function \( g(r)F(r) \) for the same conditions as Figure 3. This function has a maximum at \( r_R = 3.92 r_S \) indicated by the vertical arrow marked corotation resonance. The function changes sign at \( r_L = 4.07 r_S \) which is a Lindblad resonance where \( \mathcal{D}(r) \) change signs. The maximum of \( \Re[g(r)F(r)] \) appears to be a general feature for profiles similar to those in Figure 2. The dependence can be traced to the dependence of \( \mathcal{D}(r) \) (Equation (10b)). For \( r \) decreasing significantly below \( r_0 \), the term \(-(d\rho_c/dr)/(\rho L_c)\) in \( \mathcal{D} \) becomes increasingly negative. This is due to the radial dependence of the magnetic pressure and the small density. Note \( L_c \) is negative in this region. On the other hand for \( r \) increasing from \( \sim r_0 \), the positive contribution of \( \Omega^2 \) begins to dominate. The combination of these dependences gives a \( \Re[g(r)F(r)] \) profile with a maximum at a distance inside the Lindblad resonance as shown in Figure 5.

Figure 6 shows the real part of the effective potential \( U(r) \) for the same case as Figure 5. The real part of the frequency is chosen to give \( \Im[\Delta\omega(r)] = 0 \) at \( r_R/r_S = 3.92 \) of the maximum of \( \Re[g(r)F(r)] \) because this gives the maximum growth rate. For \( m = 1 \) this gives \( v_r = \omega_i/2\pi \approx 850 \) Hz. Clearly, this frequency must be larger than rotation frequency of the star \( v_c \).

The depth of the potential well shown in Figure 6 increases as the imaginary part of the frequency \( \omega_i = \Im(\omega) > 0 \) (the growth rate) decreases. We use a WKBJ treatment with \( \phi \propto k^{-1/2} \exp(\pm i\int_{r_{\infty}}^{r_{\text{in}}}) dk \) and \( k = -\Im(U)/1/2 \). The imaginary part of \( U \) is small compared with the real part. The allowed values of \( \omega_i \) are then calculated using the Bohr–Sommerfeld quantization condition,

\[
\int_{r_{\infty}}^{r_{\text{in}}} dr k = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, 1, 2, \ldots ,
\]

where \( k \equiv \sqrt{-\Im(U)} \) and \( r_{\text{in}} \) and \( r_{\text{out}} \) are the radii where \( \Im(U) = 0 \).

For the case of Figure 6, \( r_{\text{in}}/r_S = 3.83 \) and \( r_{\text{out}}/r_S = 3.97 \). The largest growth rate \( \omega_i \) corresponds to \( n = 0 \) and for the case of Figure 6 with \( m = 1 \) this gives \( v_i = \omega_i/2\pi = 89 \) Hz so that \( \omega_i/\omega_c \approx 10\% \). Modes with \( m \geq 2 \) also grow with similar \( \omega_i/\omega_c \) values but as discussed below these modes do not give time variations in the total flux from the source.

For a more gradual boundary layer with \( \Delta = 0.2 \) and \( m = 1 \), we find \( \Im[\Delta\omega(r)] = 0 \) for \( v_i = 750 \) Hz at the radius \( r_L/r_S = 3.83 \) of the maximum of \( \Re[g(r)F(r)] \). From Equation (35), we find \( v_i = 120 \) Hz. For \( \Delta \) increasing from 0.2, we find that \( v_i \) and \( r_R \) continue to decrease gradually and \( v_i \) also decreases.

Owing to the perturbation, the surface temperature of the disk is

\[
T(r, \phi, t) = T_0 + \Re[\delta T_1 \exp(i\phi - i\omega_1 t)]
+ \delta T_2(r) \exp(2i\phi - i\omega_2 t) + \cdots ,
\]

where \( \delta T_2(r) \) is the second-order term in the expansion.
where $T_0(r)$ is the unperturbed temperature, $\delta T_{1,2}(r) \ll T_0$ are the amplitudes of the $m = 1, 2$ corotation modes, and $\omega_{1,2}$ are their frequencies. We neglect the difference between the midplane and the surface temperature of the disk. The corresponding flux density from the disk surface is proportional to $S(r, \phi, t) \sim T_0^3 + \Re[4T_0^2\delta T_1 \exp(i\phi - i\omega_1 t) + 4T_0^2\delta T_2 \exp(2i\phi - i\omega_2 t) + \cdots]$. The total flux for a face-on disk, $L \sim \int r \, dr \, d\phi S$, is independent of time: the $\phi$-integration annihilates the terms dependent on $\phi$. For a more general disk orientation, with the disk angular momentum tilted say toward the line of sight by an angle $\iota$, the Doppler effect due to the disk rotation gives a boost to the frequency for say $\phi = 0$ and a decrement for $\phi = \pi$. This corresponds to multiplying $S$ by the Doppler factor $D(r, \phi) \equiv [1 + \varepsilon(r) \cos(\phi)]^3 \approx 1 + 4\varepsilon \cos(\phi)$, where $\varepsilon = [v_\phi(r)/c] \sin(\iota)$ and $\varepsilon^2 \ll 1$. For the case of Figure 6 where $r = 3.92r_S$, $v_\phi/c = 0.289$. Consequently, there is a contribution to the source flux $\delta L \sim \int r \, dr \, d\phi D(r, \phi) S(r, \phi, t) \sim 16\pi \Re[\int \, dr \, d\phi \, D(r, \phi) \frac{\delta T_1(r)}{(i\phi - i\omega_1 t)}]$. We interpret this frequency as the upper frequency component of the twin QPOs as argued earlier by LR07. Note that the higher order terms $\delta T_2, \delta T_3, \cdots$ give no contribution to the total flux due to the $\phi$-integration.

4.2. Driven Perturbations

Here, we consider the driven perturbations of the flow which result from a quadrupole or octupole component of the star’s magnetic field as discussed in Section 2.2. Inspection of Equation (25) reveals that the radial locations of the Lindblad resonances where $D$ (or $\overline{D}$) vanishes are very important for excitation of the flow (LR07). At such a resonance, the righthand side of Equation (25)—the driving term—is proportional to $D/(\overline{D})^{-1/2}$.

We first consider the quadrupole field case $B^{11}$ where $m = 1$ for $v_r = \omega_1/2\pi = 300$ Hz and $\omega_1 = 0$. We find that $\overline{D}$ goes through zero at one radius, $r_L = 4.07r_S$ for this case. Near this radius we find $\overline{D} \approx a(x - bx^2)$, where $x \equiv (r - r_L)/r_L$ and $a, b > 0$ are constants. We are interested as explained below to determine the amplitude of this driven mode near the vicinity of the corotation mode at $r_R = 3.92r_S$. For this purpose, we develop an approximate solution to Equation (25) for $x^2 \ll 1$ by retaining only the most singular term in the effective potential which is $U \approx (3/4)(\overline{D}/\overline{D})^2$. The right-hand side of the equation can be approximated as $K \overline{D}/(\overline{D})^{-3/2}$, where $K \propto B^{11}$. Thus, Equation (25) simplifies to

$$\frac{d^2\varphi}{dx^2} - \kappa^2 \varphi = \frac{K(1 - 2 bx)}{(x - bx^2)^{3/2}},$$

(37)

where $\kappa^2 = (3/4)((1 - 2 bx)/(x - bx^2))^2$. The inhomogeneous solution to this equation for $\kappa^2 \ll 1$ and $\kappa^2 \ll b^{-2}$ is

$$\varphi = \frac{K}{\sqrt{x}}.$$

(38)

Although $\varphi$ diverges at $x = 0$, note that $\Psi = \delta p_+/\rho$ is a constant. For $-x$ increasing, we find that the dependence of $\kappa^2$ changes from $\sim (3/4)x^{-2}$ to $\kappa^2 = \kappa_0^2 \approx 38$. The small value of $\kappa_0^2$ means that driven mode has a significant amplitude at the distance $r_R$ where the free mode is excited.

With both the free perturbation and the quadrupole driven perturbation present we have

$$T(r, \phi, t) = T_0 + \Re[\delta T_1'(r) \exp(i\phi - i\omega_1 t) + \delta T_2'(r) \exp(2i\phi - i\omega_2 t) + 

\delta T_3'(r) \exp(3i\phi - i\omega_3 t) + \cdots].$$

(39)

The associated flux density $S \sim T^4$ is

$$T_0^4 + 4T_0^3\Re[\delta T_1' \exp(i\phi - i\omega_1 t) + \delta T_2' \exp(i\phi - i\omega_2 t)] + 6T_0^2\Re[\delta T_1' \delta T_2' \exp[-i(\omega_1 - \Omega_\ast) t]] + \cdots,$$

(40)

where the ellipsis denotes terms of the form $O[\exp(\pm i\phi) \delta T]$ and $O[\exp(\pm i\phi) \delta T]^2$ which do not cause variations in the total flux. It follows from this equation that there are three QPO components for a generally oriented disk. Two of the components arise from the above-mentioned Doppler boost acting first on the term $\delta T_1' \exp(i\phi - i\omega_1 t)$, which gives a frequency $\omega_1$ component in the total flux, and second on the term $\delta T_2' \exp(i\phi - i\omega_2 t)$, which gives a frequency $\Omega_\ast$ (the star’s rotation frequency in the total flux. The third component arises for the final term in Equation (37) and has a frequency $\Omega_\ast - \Omega_\ast$. The component at $\Omega_\ast$ is however usually absent (van der Klis 2006) so that we do not consider this case further.

For the case of an octupole field component $B^{12}$ where $m = 2$ we find two Lindblad resonances for $\omega_2/2\pi = \Omega_\ast/2\pi = 300$ Hz and $\omega_1 = 0$ as shown in Figure 7. One resonance is at $r_{L1}/r_S = 4.11$ and the other at $r_{L0}/r_S = 4.28$. The driven motion at $r_{L1}$ is important here because this is close to the radius of the unstable free perturbation $r_R$. For this case

$$T(r, \phi, t) = T_0 + \Re[\delta T_1' \exp(i\phi - i\omega_1 t) + \delta T_2' \exp(2i\phi - i\Omega_\ast t)].$$

(41)

The associated flux density is

$$T_0^4 + 4T_0^3\Re[\delta T_1' \exp(i\phi - i\omega_1 t) + \delta T_2' \exp(2i\phi - i\Omega_\ast t)] + 6T_0^2\Re[\delta T_1' \delta T_2' \exp[-i\phi - i(\omega_1 - \Omega_\ast) t]] + \cdots.$$  

(42)

In this case, we have just two frequency components: the first is at the frequency $\omega_1$ of the unstable free perturbation as a result of the Doppler boost acting on the term $\delta T_1' \exp(i\phi - i\omega_1 t)$. The second is at the frequency $\omega_1 - \Omega_\ast$, as a result of the Doppler boost acting on the term $\delta T_1' \delta T_2' \exp[-i\phi - i(\omega_1 - \Omega_\ast) t]$. The Doppler boost acting on the remaining term in Equation (39), $\delta T_2' \exp(2i\phi - i\Omega_\ast t)$, causes no variation in the total flux.
3.5 Driven-Modulated Perturbations

Here, we consider the driven-modulated perturbations of the flow which result from the octupole component of the star’s magnetic field $B^{(3)}$ discussed in Section 2.3. Two Lindblad resonances are again found at radii similar to the case of Figure 7. The inner radius $r_L$ is important here because it is close to the radius $r_R$ of the unstable free perturbation. For the driven-modulated octupole case we have

$$ T(r, \phi, t) = T_0 + \delta T_1^d(r) \exp(i\phi - i\omega_1 t) + \delta T_2^{dm}(r) \exp[2i\phi - i(\Omega_\ast/2)t]. $$

(43)

The associated flux density is

$$ T_0^2 + 4T_0^2 \delta T_1^d(r) \exp(2i\phi - i\omega_1 t) + \delta T^{dm}_2 \exp[2i\phi - i(\Omega_\ast/2)t] + \cdots. $$

(44)

In this case, we again have just two frequency components: the first is at the frequency $\omega_1$ of the unstable free perturbation as a result of the Doppler boost acting on the term $\delta T_1^d \exp(i\phi - i\omega_1 t)$. The second is at the frequency $\Omega_\ast/2$ as a result of the Doppler boost acting on the term $\delta T^{dm}_2 \exp[-i\phi - i(\omega_1 - \Omega_\ast/2)t]$. The Doppler boost acting on the remaining term in Equation (39), $\delta T^{zm}_2 \exp[2i\phi - i(\Omega_\ast/2)t]$, causes no variation in the total flux. As mentioned in Section 2.3, the present theory does not predict the value of $\delta T^{zm}_2$.

5. NONLINEAR EFFECT OF UNSTABLE WAVE

Here, we consider the fluid particle paths in an unstable Rossby wave with azimuthal mode number $m = 1$ which have a vortex structure (Lovelace et al. 1999). Any small perturbation of the disk of finite radial extent around $r_R$ will be sheared by the differential rotation of the disk $d\Omega_\phi/dr > 0$ into a leading spiral wave. This wave will grow due to the RWI. In a reference frame uniformly rotating at the rate $\Omega_\phi(r_R)$, the fluid velocity components are $u_r \propto k^{-1} \exp(\omega_1 t) \sin(m\phi - j \cdot dr)$ corresponding to a leading spiral wave, and $u_\phi = r[\Omega_\phi(r) - \Omega_\phi(r_R)] + \delta u_\phi \approx r_R w \Delta r$, where $\Delta r = r - r_R$, $w \equiv d\Omega_\phi/dr$ at $r_R$ and $\omega_1$ is the growth rate. The approximation is valid for $|\delta u_\phi| \ll r_R w |\Delta r|$. As a simplification, we consider $u_r = u_0 \exp(\omega_1 t) \sin(m\phi - k \Delta r)$ where $k = k(r_R)$ and $u_0$ is the initial wave amplitude. The fluid particle motion is given by $d(\Delta r)/dt = u_0 \exp(\omega_1 t) \sin(m\phi - k \Delta r)$ and $d\phi/dt = w \Delta r$ (B. Newman 2009, private communication). Combining the two equations gives

$$ \frac{d^2 \phi}{dt^2} = w u_0 \exp(\omega_1 t) \sin \left( m\phi - k \frac{d\phi}{w dt} \right). $$

(45)

The term $(k/w)d\phi/dt$ is negligible compared with $m\phi$ for the small values of $\delta u_\phi$ found here.

To understand Equation (45), it is useful to linearize it about the point(s) where the sine function is zero and where the Taylor expansion gives an attractive force ($\phi = \pi$ for $m = 1$). This gives

$$ \frac{d^2 \phi}{dt^2} = -f u_0 \exp(\omega_1 t)(\phi - \pi). $$

(46a)

The relevant solution is

$$ \phi = \pi - \text{const} J_0 \left[\frac{2\omega_T}{\omega_i} \exp \left( \frac{\omega_i t}{2} \right) \right]. $$

(46b)

where $J_0$ is the usual Bessel function and $\omega_T = (mu_0 w)^{1/2}$

(46c)

is the initial trapping frequency of the fluid particle. A characteristic timescale for the fluid particle motion around the vortex center—is the trapping timescale—is taken to be the time for the Bessel function to go from unity to zero. Thus, $\tau = (2/\omega_i) \ln(1.2 \omega_0/\omega_T)$, where we have assumed $\omega_i/\omega_T >> 1$.

The exponential growth of the wave amplitude necessarily ends at the time when the fluid particle motion in the vicinity of resonant radius $r_R$ differs appreciably from the strictly azimuthal motion in the equilibrium. An estimate of this time is $\tau$. An analogous nonlinear saturation of the growth or damping of plasma waves is well known (O’Neil 1965; Krall & Trivelpiece 1973). At the saturation time, the wave amplitude is $\delta u_r = u_0 \exp(\omega_1 t)$ which gives

$$ \left| \delta u_{sats}^{\phi} \right| \approx \frac{1.44 \omega_0^2}{m w} = \frac{1.44}{m} \left( \frac{\omega_i}{\Omega_\phi} \right)^2 \Omega_\phi L_\phi, $$

(47)

where $L_\phi^{-1} \equiv \Omega_\phi^{-1}(d\Omega_\phi/dr) > 0$ evaluated at $r_R$. For the conditions of Figure 6, $L_\phi = 0.69 \phi_3$ and for $m = 1$, $\omega_i/\Omega_\phi \approx 0.1$. Thus, we find $\left| \delta u_{sats}^{\phi} \right| \approx 0.0029m_{\phi}(r_R)$. Figure 8 shows an illustrative numerical solution of Equation (45) for $m = 1$ with the radial amplitude of the motion amplified for visibility. Note that the vortex is cyclonic in contrast with the anticyclonic vortices found by Lovelace et al. (1999) and Li et al. (2000) where $d\Omega_\phi/dr < 0$.

The disk equilibrium of Section 3 does not include accretion. However, the unstable Rossby mode discussed in Section 4.1 gives rise to accretion in the vicinity of $r_R$.

$$ M_{r_0} = -2\pi r \int_{-h}^{h} dz \mathfrak{M}(\delta \rho \delta u^*), $$

(48a)

where $h$ is the half-thickness of the disk. We evaluate this by taking the most singular terms in $\Delta \omega$ in the limit where
this quantity almost vanishes at \( r = r_R \). This gives \( \delta \rho = 2k_\phi F^\Psi/(\Delta \rho L_\alpha) \) and \( \delta u_r = -2ik_\phi F^\Psi/\rho \). Thus,
\[
\dot{M}_{\text{Ro}} = 4\pi r^2 h \rho u_{\text{accr}}, \quad u_{\text{accr}} = \frac{|\delta u_r|}{\omega_1/|L_\phi|}, \tag{48b}
\]
where \( u_{\text{accr}} \) is an effective accretion speed. Here, we have taken into account that \( L_\phi < 0 \) and taken \( |\Delta \rho| = \alpha \rho^2 \). Because \( \omega_1 > 0 \) the instability acts to increase the mass accretion rate, \( \dot{M}_{\text{Ro}} > 0 \). Using Equation (47), we find
\[
u_{\text{accr}} = \frac{2.07 \alpha}{m^2} \left( \frac{\omega_1}{\Omega_\phi} \right)^3 \frac{L^2_\phi \Omega_\phi}{|L_\phi|}. \tag{48c}
\]

For \( m = 1 \) and the conditions of Figure 6, \( L_\phi = 0.668^\circ S \).
\( L_\phi = -0.45^\circ S, \omega_1/\Omega_\phi \approx 0.1 \). Equation (48c) then gives \( u_{\text{accr}} \approx 5.1 \times 10^{-4} u_\phi(r_R) \). For comparison, the accretion speed in a standard \( \alpha \)-disk is of the order of \( \alpha(h/r)^2 u_\phi \) which is \( 10^{-3} u_\phi \) for \( \alpha = 0.1 \) and \( h/r = 0.1 \).

The unstable Rossby mode also gives an inflow of angular momentum,
\[
F_{\text{Ro}} = -2\pi r \int_h^0 dz \left[ \Re(\delta \rho \delta u_\phi^* r u_\phi + \rho \delta \delta(u_\phi^* r \delta u_\phi)) \right]. \tag{49a}
\]
Using the fact that \( \rho \delta u_\phi = -k_\phi F^\Psi p_\phi^*/(\rho \Omega_\phi \Delta \rho L_\alpha) \), we find
\[
F_{\text{Ro}} = \dot{M}_{\text{Ro}} \Omega L_\phi, \tag{49b}
\]
where \( Q = (1 + u_\phi^2/2) / 2 \) and \( u_K = r \Omega_K \) is the Keplerian velocity. For the case of Figure 6, \( Q = 1.884 \). Because \( Q > 1 \), the Rossby mode transports inward more specific angular momentum \( \ell \) than exists in the equilibrium. Consequently, a nonlinear effect of the Rossby mode is to make the positive slope of \( \ell(r) \) smaller in the vicinity of \( r_R \). A sufficiently large reduction of \( d\ell/dr \) may cause the growth rate to decrease.

6. CONCLUSIONS

We have investigated three modes of accretion disks around rotating magnetized neutron stars in order to explain the separations of the kilohertz QPOs seen in low-mass X-ray binaries. This work is a continuation of the earlier work by LR07 where these modes were identified. We develop the theory of compressible, nonbarotropic MHD perturbations of an axisymmetric equilibrium flow with \( \partial \phi/\partial z = 0 \). We assume that there is a maximum in the angular velocity \( \Omega_\phi(r) \) of the accreting material larger than the angular velocity of the star \( \Omega_* \), and that the fluid is in approximately circular motion near this maximum rather than moving rapidly toward the star or out of the disk plane into funnel flows. MHD simulations by Romanova et al. (2002, 2008), Long et al. (2005), and Kulkarni & Romanova (2008) show this type of flow and \( \Omega_\phi(r) \) profile.

The first mode we find is a RWI which is radially trapped in the vicinity of the maximum of a key function \( g(r)F^\Psi(r) \) at \( r_R \). We derive a Schrödinger type equation for the perturbation, \( q'' = U(r)q \), where \( U(r) \) is the effective potential. This instability is analogous to that found earlier by Lovelace & Hohlfeld (1978), Lovelace et al. (1999), and Li et al. (2000) and in simulations by Li et al. (2001). The real part of the angular frequency of the mode is \( \omega_1 = m \Omega_\phi(r_R) \), where \( m = 1, 2, \ldots \) is the azimuthal mode number. The imaginary part of the frequency \( \omega_\ell \) (the growth rate) is determined by a Bohr–Sommerfeld quantization of the perturbation in the effective potential. We argue that for a generally oriented disk, the Doppler boost of the disk surface emission from the unstable \( m = 1 \) mode will give periodic variations in the total flux with angular frequency \( \omega_\ell \) which we suggest is the higher frequency component of the twin QPOs as proposed by LR07. We argue that the nonlinear saturation of the RWI occurs when the trapping frequency of fluid particles in the Rossby vortex equals the RWI growth rate. An analogous saturation is well known for unstable electron plasma waves.

The second mode is a mode driven by either (1) a rotating quadrupole \( \rho \exp(i\phi) \) or octupole \( \rho \exp(2i\phi) \) stellar magnetic field where the field’s vertical component penetrates the disk or (2) a misaligned dipole \( (m = 1) \) or quadrupole \( (m = 2) \) stellar field where the time-dependent transverse field does not penetrate the disk. This mode has an angular frequency equal to the star’s angular rotation rate \( \Omega_* \). This mode is strongly excited near the radius of the Lindblad resonance which is slightly outside the radius of the maximum of \( gF^\Psi \). When both the first and second modes are present the nonlinearity of the emission will in general give a product term with angular frequency \( \omega_\ell - \Omega_* \). For a quadrupole field taking into account the Doppler boost, we find that the total flux has periodic variations at three frequencies, \( \omega_\ell, \Omega_\phi, \) and \( \omega_\ell - \Omega_* \). However, the frequency \( \Omega_* \) is usually not observed (van der Klis 2006). The situation is different for an octupole field component again taking into account the Doppler boost. In this case the total flux has periodic variations at two frequencies, \( \omega_\ell \) the upper QPO and \( \omega_\ell - \Omega_* \), the lower QPO.

The third mode arises from the interaction of the flow perturbation with the rotating nonaxisymmetric components of the star’s magnetic field. This interaction arises naturally owing to the fact that the magnetic force is proportional to the gradient of the square of the magnetic field. We derive a linear differential equation for this driven-modulated perturbation. We find that the angular frequency of the perturbation is \( \Omega_\phi/2 \) for both \( m = 1 \) and \( m = 2 \) stellar field components. For the \( m = 2 \) case and taking into account the Doppler boost, we find that the total flux has periodic variations at two frequencies, \( \omega_\ell \) the upper QPO and \( \omega_\ell - \Omega_* /2 \) the lower QPO. Because the equation for the driven-modulated perturbations is linear, the amplitude of the motion is indeterminate. One possibility is that the \( \Omega_\phi/2 \) motion is excited nonlinearly by the rotating nonaxisymmetric field which has frequency \( \Omega_* \). Thus, the present theory does not determine whether the lower QPO frequency is \( \omega_\ell - \Omega_* \) or \( \omega_\ell - \Omega_* /2 \). Also, the amplitude of the second and third modes remains to be determined.
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