Multipoles-based linear sampling method: impact of using multipole expansion

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Abstract. In this article, we present a physical interpretation of the linear sampling method (LSM). The proposed interpretation is based on the multipole expansion of the induced currents at the sampling point, thus called multipoles-based linear sampling method (MLSM). MLSM does not interpret LSM as the focusing of fields at a sampling point. It rather considers LSM as creating a current distribution that effectively radiates like a monopole at the sampling point. In this sense, MLSM brings LSM closer to the original scattering problem. We show that if infinite multipoles are considered, MLSM performs similar to LSM. This fact can be used to understand the poor performance of LSM in some particular cases, for example, circular and annular scatterers with specific permittivities. We show that it is not necessary to use infinite multipoles in MLSM. The truncation of higher order multipoles in MLSM serves as a regularization technique. As compared to the conventionally used Tikhonov regularization in LSM, the regularization in MLSM is not based on mathematical theory. It is rather a physics-based truncation technique as we truncate the multipoles which anyways do not contribute significantly to the scattered field. Such regularization is simple to understand and implement for practical engineering purposes.

1. Introduction

Linear sampling method (LSM) is a qualitative imaging method used in electromagnetic inverse scattering problems for estimating the scatterer support [1-6]. In LSM, we try to choose the amplitudes of various sources, such that the scattered field due to the chosen source amplitudes is similar to an isotropic monopole radiation pattern centered at the sampling point. If for a particular sampling point, finite source amplitudes can result into a monopole radiation pattern, the sampling point is concluded as a scattering point. Though LSM has shown good potential in practical applications, its development and understanding is based on the mathematical theory. Its relation to the forward scattering problem and its physical interpretation is very less understood.

In one physical interpretation, the basic mathematical model of LSM is interpreted as a focusing problem [7, 8], where LSM is understood as finding the source amplitudes that can focus the fields to a sampling point. However this interpretation fails in specific examples, where the so called focusing is seen when not expected or vice versa. This prompts us to take another look into the relation of LSM with the forward scattering problem and find another, more consistent physical explanation of LSM.

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We therefore propose an alternative explanation of LSM in terms of the multipole expansion of the currents induced on the scatterers at a sampling point, first introduced in [9]. Our approach is called multipoles-based linear sampling (MLSM) for ease of reference. In MLSM, we try to choose the amplitude of various sources, such that when the current induced on the scatterers is represented as effective multipoles on a sampling point, the monopole is the single most prominent multipole at the sampling point. If for a particular sampling point, finite source amplitudes can result into such induced currents, the sampling point is concluded as a scattering point.

The focus of the article is to discuss the fundamental idea behind MLSM and the impact of using multipole expansion in MLSM.

2. Introduction to the linear sampling method and the focusing interpretation

Linear sampling method (LSM) is a qualitative method used to reconstruct the support of extended scatterers. Below, we introduce the basic formulation of linear sampling method for the TM case.

Let the scatterer support be denoted by $\Sigma$. Let $\Gamma$ denote the circle on which the detectors and sources are arranged. The radius of $\Gamma$ is very large as compared to the wavelength, such that far field approximation is applicable. Accordingly, let $E^\circ(r_p, r_p)$ be the $\hat{z}$ directed scattered far-field pattern measured on $\Gamma$ in the direction of $\varphi$, when a unit amplitude plane wave impinges from the direction $\theta$. Let the far field expression of the Green’s function be denoted by $g^\circ(r_p, r_p)$.

For a generic point $\bar{r}_p$ in $\Omega$, LSM consists of solving the far-field integral equation for the unknown $h(\bar{r}_p, \bar{r}_p)$ [2, 4, 5, 7]:

$$\int_{\Gamma} E^\circ(\bar{r}_p, \bar{r}_p)h(\bar{r}_p, \bar{r}_p)d\theta = g^\circ(\bar{r}_p, \bar{r}_p).$$ (1)

It is obvious that the right hand side represents a circularly symmetric scattered field with respect to $\bar{r}_p$, which, for later convenience, is referred to as the monopole radiation pattern. The value of $h(\bar{r}_p, \bar{r}_p)$ becomes unbounded if the sampling point $\bar{r}_p$ does not belong to the scatterer support. While reconstructing, (1) is cast into a matrix equation as follows:

$$K \cdot h = \bar{g}^\circ,$$ (2)

where, $K$ is the far field multistatic matrix (an operator that maps the currents at the sources to the scattered fields at the detectors), $\bar{h}$ contains the elements $h(\bar{r}_p, \bar{r}_p)$, and $\bar{g}^\circ$ contains the elements $g^\circ(\bar{r}_p, \bar{r}_p)$. For the sake of convenience of reference, we shall refer to the vector $\bar{h}$ computed using LSM as $\bar{h}_{LSM}$, which is computed using the following equation:

$$\bar{h}_{LSM} = K^\dagger \cdot \bar{g}^\circ,$$ (3)

This above equation is solved for $\bar{h}$ typically using Tikhonov regularization. The sampling points $\bar{r}_p$ for which $|\bar{h}|$ (Euclidean norm) are significantly large are concluded to belong to the background. The remaining points are identified as the scatterer support.

The model in (2) can effectively be understood as a focusing problem [7, 8]. Considering that $h(\bar{r}_p, \bar{r}_p)$ denotes the source amplitude at an incidence angle $\theta$, the fields scattered by such current distribution is given by $K^\circ \cdot \bar{h}$ using the definition of the multistatic matrix. Then solving (2) implies finding such source distribution $h(\bar{r}_p, \bar{r}_p)$ that results into an isotropic monopole radiation pattern centered at $\bar{r}_p$. If $|\bar{h}|$ is finite and small, it implies that we can use practically finite energy sources to
focus the waves on the point \( \vec{r}_p \). On the other hand, if \( |\vec{r}| \) is very large (theoretically infinite), then it implies that there is no finite energy source distribution that can be used to focus on the point \( \vec{r}_p \).

### 3. Multipoles-based linear sampling

Another interpretation of the LSM model, closer to the scattering model, is that the solution of (2) is a source distribution that results into an induced current distribution, which in turn scatters the electric field similar to a monopole radiation from \( \vec{r}_p \). Thus, in solving (2), we are trying to generate an induced current distribution, such that it can be effectively represented as a monopole at \( \vec{r}_p \). To this end, we study the multipole expansion of the scattered field and the connection of the multipole expansion with the induced current distribution. In fact, to solve (2) is to find a superposition \( h(\vec{r}_p, \vec{r}_p) \) of \( E^\alpha(\vec{r}_p, \vec{r}_p) \) such that among all multipoles, the monopole radiation is the only dominant contributor in the resultant total radiation.

The scattered field received at a point \( \vec{r}_p’, E_x(\vec{r}_p’, \vec{r}_p) \), whose far field notation is \( E^\alpha(\vec{r}_p’, \vec{r}_p) \), is radiated by the induced currents on the scatterer, i.e.,

\[
E_x(\vec{r}_p’, \vec{r}_p) = \int \frac{\alpha_n(\vec{r}_p’, \vec{r}_p) g_n(\vec{r}_p’, \vec{r}) d\vec{r}}{} ,
\]

where, \( I^{\text{ind}}_{\vec{r}}(\vec{r}, \vec{r}_p) \) is the current induced on \( \vec{r} \). The scattered fields received at the detectors can be decomposed into various independent terms corresponding to the multipole radiation from a sampling point \( \vec{r}_p^* \). Using addition theorem \([10]\) on \( g(\vec{r}_p’, \vec{r}) \), the expression of \( E_x(\vec{r}_p’, \vec{r}_p) \) can be rewritten in terms of various multipoles corresponding to a sampling point \( \vec{r}_p^* \) as:

\[
E_x(\vec{r}_p’, \vec{r}_p) = \sum_{n=-\infty}^{\infty} \alpha_n(\vec{r}_p’, \vec{r}_p) g_n(\vec{r}_p’, \vec{r}_p) ,
\]

where,

\[
\alpha_n(\vec{r}_p’, \vec{r}_p) = \int \frac{I^{\text{ind}} (\vec{r}, \vec{r}_p) J_n \left( k \left| \vec{r} - \vec{r}_p \right| \right) e^{-i\arg(\vec{r} - \vec{r}_p)} d\vec{r}}{} ,
\]

\[
g_n(\vec{r}_p’, \vec{r}_p) = \frac{j}{4} H^{(1)}_n \left( k \left| \vec{r} - \vec{r}_p \right| \right) e^{i\arg(\vec{r} - \vec{r}_p)},
\]

and \( J_n(\bullet) \) is the Bessel function of \( n \) th order. It is evident that \( \alpha_n(\vec{r}_p’, \vec{r}_p) \) represents the \( n \) th effective multipole current at \( \vec{r}_p’ \), such that the sum of the radiated fields from all such multipoles is equal to the measured scattered field.

The above formulation suggests that in MLSM, the key physical quantity of interest is not the scattered field measured at the detectors (as in LSM), but the induced current distribution that leads to the detected scattered fields. We highlight that this is the central idea behind MLSM and enables the alternative interpretation in terms of the induced multipoles.

Using (5), the fundamental equation of LSM (1) is equivalent to:

\[
\int \alpha_n(\vec{r}_p’, \vec{r}_p) h(\vec{r}_p’, \vec{r}_p) d\theta = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}.
\]

For convenience of further use, we define:
\[ \nu_n = \int_{\Omega} \alpha_n(r_p, r_p) h(r_\theta, r_p) d\theta. \quad (9) \]

Physically, \( \nu_n \) can be understood as the multipole currents induced by the source distribution \( h(r_\theta, r_p) \). MLSM therefore implies that as long as the multipole expansion of the induced current distribution at a sampling point is such that the monopole is the only prominent contributor, the sampling point will be detected as a scatterer.

4. Reconstruction using MLSM and the physical regularization

Now, we present a reconstruction approach based on the above model. Although one needs to consider infinite number of multipoles to fully account for the scattered field in (5), usually, a sufficiently large finite number of multipoles is enough to approximate \( E_z(r_p, r_p) \), especially so in the presence of noise. Considering \( (2N + 1) \) number of multipoles, the expression for the far field (5) can be rewritten as:

\[ E_z(r_p, r_p) \approx \sum_{n=-N}^{N} \alpha_n(r_p, r_p) g_n(r_\theta, r_p). \quad (10) \]

The equation (10) suggests that the multipole radiation functions \( g_n(r_\theta, r_p) \) can be understood as a mapping from the effective multipole current at a sampling point \( r_p \) to the measured scattered electric field. Thus, by approximating (5) as (10), we have shrunk the mathematically infinite dimensional domain of this mapping to a finite \((2N + 1)\) dimensional domain. Thus, the truncation of higher order multipoles serves as the regularization scheme in MLSM.

For computing \( \overline{h}_{\text{MLSM}} \), we perform two stage inversion:

\[ A = (G^{\text{mul}})^\top \cdot K; \quad \overline{h}_{\text{MLSM}} = A^\top \cdot \overline{D}. \quad (11) \]

Here, the matrix \( A \) contains \( A(n, \theta; n = -N \text{ to } N) = \alpha_n(r_\theta, r_p) \). \( G^{\text{mul}} \) contains \( G^{\text{mul}}(\varphi, n; n = -N \text{ to } N) = g_n(r_\theta, r_p) \), and \( \overline{D} \) contains \((2N + 1)\) elements, all of which are zero, except the middle element, which corresponds to the monopole and has a value 1. In the proposed method, we have used least squares based pseudoinverse. Thus effectively, \( \overline{h}_{\text{MLSM}} \) is given as follows:

\[ \overline{h}_{\text{MLSM}} = \left( (G^{\text{mul}})^\top \cdot K \right)^\top \cdot \overline{D}. \quad (12) \]

5. Comparison of LSM and MLSM with regard to \( h(r_\theta, r_p) \)

Generally in LSM, the Tikhonov regularization parameter \( \alpha \) is chosen using the general discrepancy principle and is evaluated for each sampling point individually. Effectively, Tikhonov regularization makes the solution \( h(r_\theta, r_p) \) of (3) stable and smooth by assigning lower weights to the vectors in the signal subspace and relatively higher weights to the vectors in the noise subspace. Evidently, this regularization scheme is purely mathematical and does not have any physical background. Tikhonov regularization serves to approximate the monopole radiation in terms of the strengths given to the various spectral vectors that span the space of the scattered fields. Thus, it is a mathematical approximation which can be tuned or changed by changing the value of the Tikhonov regularization parameter \( \alpha \).

However, the nature of approximation in MLSM is entirely different. MLSM employs the approximation in terms of the truncation of higher order multipoles. This approximation is justified.
from the perspective of physics because the contribution of higher order multipoles to the scattered field is indeed small. When the value of \( N \) is very large, in the noise-free case, the solution of \( \vec{h}_{\text{LSM}} \) is close to \( \vec{h}_{\text{MLSM}} \).

Physically, it is well known that the monopole and dipoles are the most prominent sources induced on the scatterers. Not only the strengths of these multipoles is higher than the other multipoles in general, the strength of radiation from the higher order multipoles also successively diminishes. Thus, it is reasonable to use only the monopole and the dipoles. Thus, out of the infinite multipoles, our scheme uses only the monopole and dipole terms (i.e. \( N = 1 \)) for reconstruction and truncates all the higher order multipoles.

As compared to a large value of \( N \), the use of \( N = 1 \) implies the following. Now, solving (11) would mean that we seek an optimal combination of the monopole and dipole currents such that the resultant radiation fields match the scattered fields as closely as possible. The solution \( h(\vec{r}, \vec{r}') \) is such that the contribution from the determined dipole current is very small. Thus, effectively the requirement on \( h(\vec{r}, \vec{r}') \) has reduced, which would otherwise require to suppress the contribution from all higher multipoles. In terms of the subspace, considering the contribution from each multipole as an independent dimension, using \( N = 1 \) implies that we use a very small subspace from the infinite-dimensional space of induced multipoles, which serves as the data space for determining \( \vec{h}_{\text{MLSM}} \) as well as the domain of the operator \( \mathbf{G}^\text{mul} \).

6. Numerical examples

For all the examples presented in this paper, the number of sources and detectors is 13. The sources and detectors are placed uniformly along a circle of radius \( 10\lambda \), where \( \lambda \) is the wavelength of the incident wave. The region of investigation is a square region of dimension \( 2\lambda \).

We consider two examples in which the conventional LSM behaves in an anomalous manner and the focusing interpretation cannot explain the anomalous behaviour. In the first example, the scatterer is a circular scatterer of radius \( 0.3\lambda \) and relative permittivity \( 7 \), as shown in Figure 1(a). The result of LSM is plotted in Figure 1(b). It is clearly seen that LSM does not detect the origin as the scatterer support, though the origin actually belongs to the scatterer support. According to the focusing interpretation, since the origin belongs to a scatterer support, using LSM, we should be able to choose \( \vec{h}_{\text{LSM}} \) such that the fields can be focused at the origin.

(a) \hspace{2cm} (b) \hspace{2cm} (c)

Figure 1: Numerical example 1. (a) the scatterer profile: circular scatterer of radius \( 0.3\lambda \) and relative permittivity \( 7 \). (b) reconstruction using LSM, the origin is not detected as scatterer support (c) The multipole expansion of the induced current distribution, \( v_n \), at the origin.
While the focusing approach fails in explaining this, MLSM can explain this behaviour of LSM. MLSM behaves similar to LSM if sufficiently high number of multipoles are considered. Thus, we use $N = 20$ and plot the multipole expansion of the induced currents (i.e. $\nu_n$ of (9) using $\hat{F}_{LSM}$) at the origin. It is seen that the monopole is not the prominent multipole at the origin. Due to this, the LSM formulation fails to detect the origin as the scatterer support.

In the second example, we consider an annular scatterer with inner and outer radii of $0.2\lambda$ and $0.4\lambda$, and relative permittivity of 2. The scatterer is shown in Figure 2(a). The LSM result is plotted in Figure 2(b). It is seen that though the origin does not belong to the scatterer support, LSM detects origin as belonging to the scatterer support. According to the focusing interpretation, since origin does not belong to the scatterer support, finite valued $\hat{F}_{LSM}$ should not be able to focus the fields at the origin. Thus the focusing interpretation cannot explain this anomalous behaviour. Following the basic principle of MLSM, we plot the multipole expansion of the induced current at the origin in Figure 2(c). We see that the monopole is the most prominent multipole at the origin. Due to this, LSM detects the origin as belonging to the scatterer support.

Figure 2: Numerical example 2. (a) the scatterer profile: circular scatterer of radius $0.3\lambda$ and relative permittivity 7. (b) reconstruction using LSM, the origin is not detected as scatterer support (c) The multipole expansion of the induced current distribution, $\nu_n$, at the origin.

From the above examples, we can clearly see that MLSM (with large value of $N$) can indeed be used to explain the behaviour of LSM for various cases. Now, we consider the truncation of multipoles as the physical regularization. As discussed in section 4.1, using $N = 1$ in MLSM should work for most cases. Detailed examples are presented in [9]. Here we consider the two examples presented above.

The MLSM reconstruction using $N = 1$ for both the examples are presented in Figure 3. It is seen that while using $N = 1$ is sufficient for example 2 (Figure 3(b)), it is not sufficient for example 1 (Figure 3(a)). The reason is again evident in the multipole expansion. Since the octopole is the most prominent induced multipole at the origin for example 1 (see Figure 1(c)), we need to consider at least $N = 3$ for example 1. The MLSM reconstruction result using $N = 3$ is shown in Figure 4. It is seen that the origin is now correctly identified as the scatterer support. This example shows how the physical nature of the induced current can be used to determine the regularization parameter in MLSM.
Figure 3: MLSM reconstruction using $N = 1$ for (a) Example 1 and (b) Example 2. The actual scatterers are shown in the inset.

Figure 4: MLSM reconstruction using $N = 1$ for example 1.

**7. Conclusion**

Multipole based linear sampling method is discussed in the context of the popular linear sampling method. The discussion and formulation highlight that as opposed to LSM, MLSM does not consider the scattered fields at the detectors as the key physical quantity. Instead, it uses the induced current distribution as the key physical quantity that explains the linear sampling method correctly in the context of the scattering phenomenon.

To highlight the impact, we show that when very large number of multipoles is considered in MLSM, MLSM formulation is equivalent to LSM. This fact can be used to explain the reconstruction results of LSM. This becomes more important for the cases where LSM produced anomalous or incorrect reconstruction.

In addition to the above, MLSM incorporates a physical regularization scheme that involves truncation of higher order multipoles. This scheme is based on the nature of induced currents, thus strongly related to the scattering phenomenon itself. It is not strictly mathematical and can be chosen easily for most practical applications. Based on the fact that monopoles and dipoles are the most prominent induced multipoles for most scatterers, MLSM with $N = 1$ works well for most scatterers. For other scatterers, we may consider using higher values of $N$. 


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