Incommensurate nematic fluctuations in two dimensional metals

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To assess the strength of nematic fluctuations with a finite wave vector in a two-dimensional metal, we compute the static d-wave polarization function for tight-binding electrons on a square lattice. At Van Hove filling and zero temperature the function diverges logarithmically at q = 0. Away from Van Hove filling the ground state polarization function exhibits finite peaks at finite wave vectors. A nematic instability driven by a sufficiently strong attraction in the d-wave charge channel thus leads naturally to a spatially modulated nematic state, with a modulation vector that increases in length with the distance from Van Hove filling. Above Van Hove filling, for a Fermi surface crossing the magnetic Brillouin zone boundary, the modulation vector connects antiferromagnetic hot spots with collinear Fermi velocities.

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I. INTRODUCTION

Metallic two-dimensional electron systems frequently exhibit multiple enhanced fluctuations in the charge, magnetic and pairing channels, which lead to a variety of competing instabilities. In the last decade, nematic order and nematic fluctuations in two-dimensional metals have attracted increasing interest. In a homogeneous nematic state an orientational symmetry of the system is spontaneously broken, without breaking the translation invariance. A nematic state can be formed by partial melting of stripe order in a doped antiferromagnetic Mott insulator. Alternatively, a nematic state can be obtained from a Pomeranchuk instability generated by forward scattering interactions in a normal metal. On a square lattice, the most natural candidate for a Pomeranchuk instability has a dₓ²−y² symmetry, which breaks the equivalence between x- and y-directions.

A direct experimental signature of nematic order is a pronounced in-plane anisotropy in transport or spectroscopic measurements, in the absence of additional Bragg peaks indicating a broken translation invariance. Evidence for nematic order with a dₓ²−y² symmetry has been observed in several strongly interacting electron compounds. A nematic phase with a sharp phase boundary has been established in a series of experiments on ultrapure Sr₃Ru₂O₇ crystals in a strong magnetic field. Nematic order has also been observed in the high temperature superconductor YBa₂Cu₃O₇ in transport experiments and neutron scattering. Due to the slight orthorhombicity of the CuO₂ planes one cannot expect a sharp nematic phase transition in YBa₂Cu₃O₇. However, the strong temperature dependence of the observed in-plane anisotropy indicates that the system develops an intrinsic electronic nematicity, which enhances the in-plane anisotropy imposed by the structure.

Nematic fluctuations close to a continuous nematic quantum phase transition induce non-Fermi liquid behavior with a strongly momentum dependent decay rate of electronic excitations. Mean-field theories for nematic transitions driven by forward scattering interactions in two dimensions indicate that the transition is typically first order at low temperatures such that a quantum critical point remains elusive. However, order parameter fluctuations can change the order of the transition, in favor of a continuous transition even at zero temperature.

So far, only homogeneous nematic states have been considered. However, Metlitski and Sachdev recently found a tendency toward formation of a modulated nematic state in a two-dimensional metal with strong antiferromagnetic spin-density wave fluctuations. In that state the nematic order oscillates spatially across the crystal, with a small and generally incommensurate wave vector that points along the Brillouin zone diagonal and increases in length with the distance of the Fermi surface from the Van Hove points. The modulated nematic order occurs as a secondary instability generated by the antiferromagnetic spin fluctuations, and is related by an emergent pseudospin symmetry to the familiar d-wave pairing instability.

In this paper, we investigate the possibility of a modulated nematic state from a different starting point. We consider a model of tight-binding electrons on a square lattice with an interaction that has an attractive d-wave component for forward scattering in the charge channel. Within a random phase approximation (RPA), we analyze at which wave vector the modulated nematic fluctuations are maximal. For electron densities above Van Hove filling we find the same diagonal wave vector as Metlitski and Sachdev in their spin-density wave model. Below Van Hove filling a modulation parallel to the x- or y-axis can be favored.

In realistic models nematic fluctuations compete with other potential instabilities such as magnetism and superconductivity. Here we do not address this competition. Our model interaction is restricted to a specific d-wave charge channel. The competition of modulated nematic fluctuations with antiferromagnetism, superconductivity and charge density wave order in the two-dimensional (extended) Hubbard model has been the subject of a recent renormalization group analysis.
The paper is structured as follows. In Sec. II we define our model and introduce the d-wave charge susceptibility. In Sec. III we analyze the momentum dependence of the static d-wave charge susceptibility. We identify the peaks which determine the wave vector of the leading instability in the ground state, and compute the singular momentum dependence around the peaks. We also determine the shift of the peaks at low finite temperature. We finally conclude in Sec. IV.

II. MODEL AND SUSCEPTIBILITY

We consider a one-band model of electrons on a square lattice with a dispersion $\epsilon_k$ and an interaction of the form \[ H_I = \frac{1}{2L} \sum_{\mathbf{q}} g(\mathbf{q}) n_d(\mathbf{q}) n_d(-\mathbf{q}), \] where $L$ is the number of lattice sites, and
\[ n_d(\mathbf{q}) = \sum_{\mathbf{k} \sigma = \uparrow, \downarrow} d^\dagger_{\mathbf{k} \sigma} c_{\mathbf{k}-\mathbf{q}/2, \sigma} c_{\mathbf{k}+\mathbf{q}/2, \sigma} \] is a d-wave density fluctuation operator; $d^\dagger_{\mathbf{k}}$ is a form factor with $d_{x^2-y^2}$ symmetry such as $d^\dagger_{\mathbf{k}} = \cos k_x - \cos k_y$. The coupling function $g(\mathbf{q})$ is negative and may depend on the momentum transfer $\mathbf{q}$. An interaction of the form $H_I$ appears in the d-wave charge channel of the two-particle vertex in microscopic models such as the Hubbard or t-J model. Note that the fermionic operators in Eq. (2) are taken at two momenta $\mathbf{k} \pm \mathbf{q}/2$ situated symmetrically around the momentum $\mathbf{k}$ appearing in the form factor $d^\dagger_{\mathbf{k}}$. Hence, for $\mathbf{q} = \mathbf{Q} = (\pi, \pi)$, the operator $n_d(\mathbf{q})$ differs significantly from the so-called d-density wave order parameter $\sum_{\mathbf{k},\sigma} d^\dagger_{\mathbf{k}} c_{\mathbf{k}+\mathbf{Q} \sigma} c_{\mathbf{k}+\mathbf{Q} \sigma}$.

For the kinetic energy we use a tight-binding dispersion of the form
\[ \epsilon_k = -2t(\cos k_x + \cos k_y) - 4t' \cos k_x \cos k_y - 2t'' \cos 2k_x + \cos 2k_y, \] where $t$, $t'$, and $t''$ are the hopping amplitudes between nearest, next-to-nearest, and third-nearest neighbors on the square lattice, respectively. We assume $t > 0$, $t' < 0$ and $t'' \geq 0$ as is adequate for cuprate superconductors. The dispersion has saddle points at $(\pi, 0)$ and $(0, \pi)$ for $t'' \leq t''' = (t + 2t')/4$. For $t'' > t'''$ the saddle points are shifted to $\pm (\pi - \arccos \frac{t''}{t'''}$, 0) and $\pm (0, \pi - \arccos \frac{t''}{t'''})$. A special situation (Van Hove filling) arises when the Fermi surface touches the saddle points, so that the density of states diverges logarithmically at the Fermi level. The chemical potential corresponding to Van Hove filling is given by
\[ \mu_{\text{vh}} = \begin{cases} 4t' - 4t'' & \text{for } t'' \leq t''' \\ \left(\frac{1}{4}t'^2 + tt' + t'^2 - 2tt''\right)/t'' & \text{for } t'' > t'''. \end{cases} \] (4)

The amount of d-wave charge fluctuations can be quantified by the dynamical d-wave charge (density) susceptibility
\[ N_d(\mathbf{q}, \omega) = -i \int_0^\infty dt e^{i\omega t} \langle [n_d(\mathbf{q}, t), n_d(-\mathbf{q}, 0)] \rangle, \] where $n_d(\mathbf{q}, t)$ is the time dependent operator corresponding to $n_d(\mathbf{q})$ in the Heisenberg picture. Within RPA, the d-wave charge susceptibility in the model defined above is given by
\[ N_d(\mathbf{q}, \omega) = \frac{2\Pi_d^\text{d}(\mathbf{q}, \omega)}{1 - 2g(\mathbf{q})\Pi_d^\text{d}(\mathbf{q}, \omega)}, \] with the bare d-wave polarization function (particle-hole bubble)
\[ \Pi_d^\text{d}(\mathbf{q}, \omega) = -\int \frac{d^2p}{(2\pi)^2} \frac{f(\xi_{p+\mathbf{q}/2}) - f(\xi_{p-\mathbf{q}/2})}{\omega + i0^+ - (\xi_{p+\mathbf{q}/2} - \xi_{p-\mathbf{q}/2})} q^2_p. \] (7)

Here and in the following $f$ is the Fermi function and $\xi_k = \epsilon_k - \mu$. The factor 2 in Eq. (6) is due to the spin.

An instability toward an ordered state with an order parameter $(n_d(\mathbf{q}^*)) \neq 0$ is signalled by a divergence of the static susceptibility $N_d(\mathbf{q}, 0)$ for a certain wave vector $\mathbf{q}^*$. Within RPA the instability is reached once $2g(\mathbf{q}^*)\Pi_d^\text{d}(\mathbf{q}^*, 0) = 1$ while $2g(\mathbf{q})\Pi_d^\text{d}(\mathbf{q}, 0) < 1$ for $\mathbf{q} \neq \mathbf{q}^*$. For $\mathbf{q}^* = 0$ the ordered state is a homogeneous nematic with unbroken translation invariance. The case $\mathbf{q}^* \neq 0$ leads to a modulated nematic state with a modulation vector $\mathbf{q}^*$. In previous studies of the above model it was always assumed that the coupling function $g(\mathbf{q})$ is sufficiently strongly peaked at $\mathbf{q} = 0$ such that the leading instability is at $\mathbf{q}^* = 0$. In the present paper we consider the case where $g(\mathbf{q})$ exhibits no or only a weak dependence on $\mathbf{q}$ in some region around $\mathbf{q} = 0$. The leading instability in the d-wave charge channel then occurs at wave vectors at which $\Pi_d^\text{d}(\mathbf{q}, 0)$ has a peak. In the following we therefore study the structure of the d-wave particle-hole bubble, paying particular attention to its extrema.

III. STATIC PARTICLE-HOLE BUBBLE

In this section we analyze the momentum dependence of the d-wave particle-hole bubble $\Pi_d^\text{d}$ at zero frequency. We will also consider the usual (s-wave) particle-hole bubble $\Pi_0^\text{d}$ for comparison. The latter is given by Eq. (7) without the d-wave form factor. The structure depends significantly on the electron density which is determined by the chemical potential. There are three qualitatively different cases: below, at, and above Van Hove filling. We will first analyze $\Pi_d^\text{d}(\mathbf{q}, 0)$ in the ground state, and treat finite temperatures subsequently.
A. Global structure and peaks

In Fig. 1 we show $\Pi_0^d(q,0)$ in the ground state as a function of $q$, for hopping parameters $t = 1, t' = -1/4$, and $t'' = 0$, with three different choices for the electron density below, at, and above Van Hove filling, corresponding to $\mu = \mu_{vh} - 0.01$, $\mu = \mu_{vh}$, and $\mu = \mu_{vh} + 0.05$, respectively. In all cases there are enhanced (negative) values along the lines in the Brillouin zone given by the condition

$$\xi((q+G)/2) = 0,$$  

where $G$ is a reciprocal lattice vector. Geometrically these lines can be constructed by expanding the Fermi surface by a factor two and then backfolding pieces outside the first Brillouin zone into the first zone, as illustrated in Fig. 2. The momentum transfers $q$ satisfying the condition (8) are special in that they connect Fermi points with parallel tangents. Eq. (8) is the lattice generalization of the condition $|q| = 2k_F$ in a continuum system with a circular Fermi surface. With this in mind, one may call such momentum transfers $2k_F$-momenta on the lattice, too, although the Fermi momenta in a lattice system do not have a common modulus $k_F$. In the following we will refer to the lines defined by Eq. (8) as $2k_F$-lines.

In Fig. 3 we show the $s$-wave particle-hole bubble $\Pi_0^s(q,0)$ for comparison. There is again some structure tracing the lines given by $\xi((q+G)/2)$. However, the strongest weight now lies in a region near $(\pi,\pi)$ in the Brillouin zone, which is generated by particle-hole excitations connecting the saddle point regions near $(\pi,0)$ and $(0,\pi)$. In the $d$-wave bubble these contributions are suppressed by the $d$-wave form factor, since $d_p = 0$ for $p = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$. Note that this suppression hinges upon the symmetric choice of the momentum in $d_q$ at the center of the fermionic momenta $p \pm q/2$. To obtain the polarization bubble describing $d$-density wave fluctuations as defined by Chakravarty et al. one would have to replace $p - q/2$ by $p$ and $p + q/2$ by $p + q$ in Eq. (7). For $q = Q = (\pi,\pi)$ one then picks up large contributions from the saddle point region.

Incommensurate charge density waves with conventional $s$-wave symmetry have also been discussed for two-dimensional electron systems, especially in the context of cuprate superconductors. In that scenario, the incommensurate modulation vector is however determined by a competition between Coulomb energy and phase separation tendencies, not by peaks in the polarization function $\Pi^d(q,0)$. Peaks in $\Pi^d(q,0)$ at small incommensurate wave vectors occurring for large doping (far from half-
They are not linked to any symmetry axis of the lattice. Further away from Van Hove filling, the global maxima of $|\Pi_0(q, 0)|$ can be situated at inflection points of the $2k_F$-lines, which are inherited from inflection points of the Fermi surface. Explicit expressions for these points are rather lengthy so that we refrain from reporting them. They are not linked to any symmetry axis of the lattice.

Above Van Hove filling, the distance of the lines given by $\xi(q + G)/2 = 0$ from the origin is given by

$$q_a = 2 \arccos \left( \frac{t + 2t' + \sqrt{(t + 2t')^2 - 4t''(2t + \mu)}}{4t''} \right)$$

$$= 2 \arccos \left( \frac{2t + \mu}{2t + 4t''} \right) \text{ for } t'' = 0 . \quad (9)$$

However, the points $(q_a, 0)$ etc. are not the global extrema. Above Van Hove filling, the lines Eq. (8) intersect on the diagonals in the Brillouin zone, and the highest weight is reached at these intersection points. The leading instability is thus a modulated nematic with a diagonal modulation vector $q^* = (\pm q_d, \pm q_d)$, or a superposition of such modulations. Solving $\xi(q + G)/2 = 0$ for $q = (q_d, q_d)$ and $G = -(2\pi, 0)$ or $G = -(0, 2\pi)$, one obtains

$$q_d = 2 \arccos \left( \frac{\mu - 4t''}{4t''} \right) \text{ for } t'' = 0 . \quad (10)$$

Close to Van Hove filling, $q_d$ is small. Note that $q_d$ is defined only for $\mu_{ch} \leq \mu \leq 4t''$. For $\mu > 4t''$, no intersection points exist.

The wave vector $q_d = (q_d, q_d)$ connects antiferromagnetic hot spots on the Fermi surface (see Fig. 4). This is because these hot spots lie on the magnetic Brillouin zone boundary and have parallel Fermi surface tangents, and are therefore connected by a diagonal $2k_F$-vector. The length of $q_d$ is thus twice the distance between the hot spots and the nearest $M$-point $(\pi, 0)$ or $(0, \pi)$. An instability toward a modulated nematic state with the same modulation vector $q_d$ has been obtained in a recent study.
of secondary instabilities generated by antiferromagnetic fluctuations\textsuperscript{21,23} In that analysis, q\textsubscript{d} arises naturally as a vector connecting hot spots since these are the points where (commensurate) antiferromagnetic fluctuations couple most strongly to electronic excitations at the Fermi surface. It is remarkable that the same modulation vector emerges as a peak in the d-wave particle-hole bubble, where it is selected by the Fermi surface geometry without any relation to antiferromagnetism.

B. Expansion around q = 0

Away from Van Hove filling, there is a region around q = 0 where Π\textsubscript{0}\textsuperscript{d}(q, 0) is relatively flat and isotropic (see Fig. 1). This region is clearly limited by the 2k\textsubscript{F}-lines ξ(q\textsubscript{a} + q)/2 = 0. For |q| ≪ q\textsubscript{a} one may expand around q = 0. The leading terms are\textsuperscript{13}

\[ \Pi_{0}^{d}(q, 0) = a + c|q|^2 + O(|q|^4). \]  

(12)

The first coefficient is given by a weighted density of states at the Fermi level,

\[ a = -N_{\Delta^2}(\mu) = -\int \frac{d^2k}{(2\pi)^2} d^2k_\Delta(\mu - \epsilon_k), \]  

with a minus sign, so that a is always negative. The second coefficient can be written in the form\textsuperscript{23}

\[ c = \frac{1}{16} N_{\Delta^2}(\mu) - \frac{1}{48} N_{\Delta^2,v}(\mu). \]  

Here

\[ N_{\Delta^2}(\epsilon) = \int \frac{d^2k}{(2\pi)^2} d^2k_\Delta \epsilon_k \delta(\epsilon - \epsilon_k), \]  

with \( \Delta = \partial_{k_x} \theta + \partial_{k_y} \), and

\[ N_{\Delta^2,v}(\epsilon) = \int \frac{d^2k}{(2\pi)^2} d^2k_v \epsilon_k^2 \delta(\epsilon - \epsilon_k), \]  

with \( v_k = |\nabla \epsilon_k| \), and the primes denote derivatives. Near Van Hove filling, the coefficient c is dominated by the first term in Eq. (14), since \( N_{\Delta^2}(\mu) \) diverges for \( \mu \rightarrow \mu_{\text{vh}} \), while \( N_{\Delta^2,v}(\mu) \) remains finite. The sign of c is typically positive for \( \mu < \mu_{\text{vh}} \) and negative for \( \mu > \mu_{\text{vh}} \).

Hence, away from Van Hove filling, the interaction term \( H_{\text{I}} \) from Eq. (1) generates a homogeneous nematic instability only if \( \bar{g}(q) \) decays sufficiently rapidly at finite q. That is, below Van Hove filling, \( g(q_\text{a}, 0)\Pi_{0}^{d}(q_\text{a}, 0) \) with \( q_\text{a} = (q_\text{a}, 0) \) needs to be smaller than \( g(0)\Pi_{0}^{d}(0, 0) \). Furthermore, above Van Hove filling, \( g(q_d, 0)\Pi_{0}^{d}(q_d, 0) \) with \( q_d = (q_d, q_d) \) needs to be smaller than \( g(0)\Pi_{0}^{d}(0, 0) \), and the curvature of \( \bar{g}(q) \) around the origin has to compensate for the negative curvature of \( \Pi_{0}^{d}(q, 0) \). The RPA analysis therefore indicates that an instability toward a modulated nematic state is more natural. On the other hand, fluctuations beyond RPA may smear out the peaks resulting from the Fermi surface geometry, which could favor a homogeneous nematic state over a modulated one. In principle, such fluctuations may also wipe out a nematic instability completely\textsuperscript{21}

C. Shape of ridges and expansion around q′

The maxima of |Π\textsubscript{d}\textsuperscript{2}(q, 0)| lie on ridges following the 2k\textsubscript{F}-lines given by \( \xi(q + q)/2 = 0 \) in the Brillouin zone. The height of the ridges evolves regularly along these lines, with few exceptions. The shape of the ridge determined by the q-dependence of Π\textsubscript{d}\textsuperscript{2}(q, 0) perpendicular to the lines is generically singular. To discuss the form of this singularity, we introduce a coordinate q\textsubscript{r} describing the oriented distance from the closest 2k\textsubscript{F}-line, see Fig. 5. We denote points on the 2k\textsubscript{F}-line by q\textsubscript{2kF}, and the radius of curvature at q\textsubscript{2kF} by 2m\textsubscript{F}v\textsubscript{F}, where v\textsubscript{F} is the electron velocity at the corresponding point k\textsubscript{F} on the Fermi surface (the radius of curvature of the Fermi surface at k\textsubscript{F} is m\textsubscript{F}v\textsubscript{F}). We consider the generic case where the curvature at q\textsubscript{2kF} is finite. Exceptional cases with vanishing curvature exist at Van Hove points and at inflection points of non-convex Fermi surfaces. The coordinate q\textsubscript{r} is defined positively on the “outer” side of the 2k\textsubscript{F}-line, that is, the side containing the tangent to the line at q\textsubscript{2kF}, and negatively on the “inner” side.

![2kF-line and coordinate qr](image)

Figure 5. 2k\textsubscript{F}-line and coordinate q\textsubscript{r}.

Before describing the 2k\textsubscript{F}-singularity of Π\textsubscript{d}\textsuperscript{2}(q, 0), it is instructive to recall the behavior of the s-wave bubble Π\textsuperscript{0}(q, 0) for a quadratic dispersion \( \epsilon_k = |k|^2/2m \), which can be expressed in term of elementary functions\textsuperscript{21}

\[ \Pi_{0}(q, 0) = -\frac{m}{2\pi} + \Theta(|q| - 2k_{F}) \frac{mk_{F}}{\pi|q|} \sqrt{\left( \frac{|q|}{2k_{F}} \right)^2 - 1}. \]  

(17)

Note that Π\textsuperscript{0}(q, 0) is constant for all momenta satisfying |q| ≤ 2k\textsubscript{F}. At the 2k\textsubscript{F}-line, here given by |q| = 2k\textsubscript{F}, the s-wave bubble has a square-root singularity with infinite slope on the outer side. For small q\textsubscript{r} = |q| − 2k\textsubscript{F}, the bubble has the form

\[ \Pi_{0}(q, 0) = -\frac{m}{2\pi} + \Theta(q_{r}) \sqrt{mq_{r}/v_{F}} + O(q_{r}^{3/2}). \]  

(18)

The momentum dependence of the d-wave particle-hole bubble near a 2k\textsubscript{F}-line for two-dimensional lattice electrons has a similar form,

\[ \Pi_{d}(q, 0) = \Pi_{d}(q_{2kF}, 0) + \Theta(q_{r}) \frac{d_{2kF}}{2\pi} \sqrt{m_{F}q_{r}/v_{F} + b_{F}q_{r}}. \]  

(19)

where k\textsubscript{F} is the Fermi momentum corresponding to the point q\textsubscript{2kF} on the 2k\textsubscript{F}-line, v\textsubscript{F} and m\textsubscript{F}v\textsubscript{F} are the the Fermi velocity and the Fermi surface curvature at k\textsubscript{F},
respectively, and $b_F$ is another constant. The square-root singularity in this expression is essentially the same as in Eq. (18). The absence of a term linear in $q_r$ in Eq. (18) is a peculiarity of $\Pi^0$ for a quadratic dispersion relation in two dimensions. Usually $b_F$ is a negative number, such that $|\Pi^0(q, 0)|$ decreases with increasing $|q_r|$ in both directions.

The momentum dependence perpendicular to the ridge in Eq. (19) is closely analogous to the corresponding behavior of the s-wave bubble $\Pi^0$ in a two-dimensional lattice system[20] the only difference being the factor $d^2_F$ for the d-wave case. Altshuler et al.[20] also specified the momentum dependence for small tangential shifts, in analogy to the isotropic case. That dependence, however, is generally modified by a non-universal variation of the bubble along the ridge.

At points of zero curvature, in particular inflection points, $m_F$ diverges and the expression (19) is not applicable. For $t'/t < 0$ such points exist typically below Van Hove filling, and they can host the global extrema of $\Pi^0(q, 0)$. Above Van Hove filling the global extrema are situated at the crossing points $q^0 = (\pm q_d, \pm q_d)$ of two $2k_F$-lines on the Brillouin zone diagonal (see Sec. III.A). The momentum dependence of $\Pi^0(q, 0)$ near $q^0$ is then given by a superposition of two ridges of the form (19), which are mirror symmetric with respect to the diagonal. The momentum dependence is linear in the edge bounded by the inner side of both ridges, while it is dominated by a square-root singularity with infinite slope in the other three edges formed by the crossing ridges.

### D. Finite temperature

At finite temperatures the singularities of the particle-hole bubble are smoothed and the peaks are generally shifted with respect to their ground state position. In this section we analyze these effects at low finite $T$.

The particle-hole bubble at $T > 0$ can be written as a convolution of the bubble at $T = 0$ with the energy-derivative of the Fermi function:

$$\Pi^0(q; T, \mu) = \int_{-\infty}^{\infty} d\mu' h(\mu - \mu') \Pi^0(q; 0, \mu'),$$

(20)

where

$$h(\xi) = -f'(\xi) = \frac{1}{4T \cosh^2 \left( \frac{\xi}{2T} \right)}.$$

(21)

Note that we have suppressed the frequency variable ($\omega = 0$) in the argument of $\Pi^0$ while making the dependences on $T$ and $\mu$ explicit. Note also that $\int d\xi h(\xi) = 1$.

We now determine how the ridges along the $2k_F$-lines are shifted and smoothed at $T > 0$. To this end, we parametrize the momentum dependence by the oriented distance $q_r$ from the $2k_F$-line defined at fixed $\mu$ and $T = 0$ as in the preceding section. We denote the shift of the $2k_F$-line near $q_{2k_F}$ at $T = 0$ corresponding to a chemical potential $\mu' \neq \mu$ by $q_{0}^0(\mu')$. Eq. (19) then yields

$$\Pi^0(q; 0, \mu') = \Pi^0(q_{2k_F}; 0, \mu') + b_F q_r - q_0^0(\mu') + a_F \Theta(q_r - q_0^0(\mu')) \sqrt{q_r - q_0^0(\mu')} ,$$

(22)

where $a_F = d^2_F m_F^{1/2}/(2\pi T)^{1/2}$. Neglecting the weak and regular $\mu$-dependence of the height of the ridge one can approximate $\Pi^0(q_{2k_F}; 0, \mu')$ on the right hand side by $\Pi^0(q_{2k_F}; 0, \mu)$. Inserting Eq. (22) into Eq. (20), and using the antisymmetry of $q_0^0(\mu')$ around $\mu$, that is, $q_0^0(\mu + \delta \mu') = -q_0^0(\mu - \delta \mu')$ for small $\delta \mu'$, one obtains

$$\Pi^0(q; T, \mu) = \Pi^0(q_{2k_F}; 0, \mu) + b_F q_r + a_F \int_{-\infty}^{\infty} d\mu' h(\mu - \mu') \times \Theta(q_r - q_0^0(\mu')) \sqrt{q_r - q_0^0(\mu')} .$$

(23)

The shift of the ridge at $T > 0$ is given by the solution $q_r^0$ of the equation $\partial_{\mu} \Pi^0(q; T, \mu) = 0$, that is,

$$b_F + \frac{a_F}{2} \int_{-\infty}^{\infty} d\mu' h(\mu - \mu') \frac{\Theta(q_r - q_0^0(\mu'))}{\sqrt{q_r^0 - q_0^0(\mu')}} = 0 .$$

(24)

A qualitative contemplation of Eq. (24) reveals that $q_r^0$ is negative. For $\mu' \neq \mu$, $q_r^0(\mu')$ is a monotonic function of $\mu'$. We denote its inverse function by $\mu'(q_r^0)$ and linearize $q_r^0 - q_0^0(\mu') \approx D[\mu'(q_r^0) - \mu]$ where $D = \frac{\partial q^0_0}{\partial \mu} |_{\mu' = \mu}$. In case that $q^0_0(\mu')$ increases with $\mu'$, Eq. (24) can then be written as

$$b_F + \frac{a_F}{2 D^{1/2}} \int_{-\infty}^{\mu'(q_r^0)} d\mu' \frac{h(\mu - \mu')}{\sqrt{\mu'(q_r^0) - \mu'}} = 0 .$$

(25)

Introducing the variable $\delta \mu' = \mu' - \mu$, and substituting $\delta \mu' = \delta \mu'(q_r^0) u$, where $\delta \mu'(q_r^0) < 0$, the integral in Eq. (25) can be written as

$$\int_{-\infty}^{\mu'(q_r^0)} d\mu' \frac{h(\mu - \mu')}{\sqrt{\mu'(q_r^0) - \mu'}} = \sqrt{-\delta \mu'(q_r^0)} \int_{-\infty}^{\infty} du \frac{1}{\sqrt{u - 1}} \frac{1}{4T \cosh^2 \left( \frac{\delta \mu'(q_r^0) u}{2T} \right)} .$$

(26)

One can see that a solution of Eq. (24) requires $|\delta \mu'(q_r^0)| T \gg 1$ for small $T$. Therefore, we can approximate $\cosh^2 \frac{\delta \mu'(q_r^0) u}{4T} \approx 1 + \frac{\delta \mu'(q_r^0) u}{4T}$. The remaining integral is elementary and yields

$$\int_{-\infty}^{\mu'(q_r^0)} d\mu' \frac{h(\mu - \mu')}{\sqrt{\mu'(q_r^0) - \mu'}} \approx \sqrt{T} e^{-|\delta \mu'(q_r^0)/T|} .$$

(27)

Inserting this in Eq. (25) yields

$$|\delta \mu'(q_r^0)| = \frac{1}{2} T \ln \frac{T_0}{T} .$$

(28)

where $T_0 = \pi a_F^2/(4b_F^2 D)$. Note that indeed $|\delta \mu'(q_r^0)| \gg T$.
for small $T$, that is, for $T \ll T_0$. For the shift $q_r^d$ one thus obtains

$$ q_r^d = -\frac{D}{2} T \ln \frac{T_0}{T} . $$

Replacing $D$ by $|D|$, the last two equations are valid also in the case where $q_r^d(\mu')$ decreases with $\mu'$. In summary, the ridge is shifted toward the inner side of the $2k_F$-line by an amount of order $T|\log T|$. Hence, the peaks of $\Pi^0_d(q,0)$ at $q^* = (\pm q_a, 0)$ and $(0, \pm q_a)$ below Van Hove filling, and at $q^* = (\pm q_d, \pm q_d)$ above Van Hove filling, are also subject to a shift of order $T|\log T|$.

To quantify the smoothing of the peak at the $2k_F$-line at finite temperature, we evaluate $\partial^2 \Pi^0_d$ at $q_r^d$. Substituting $q_r - q_r^d(\mu') = D(\mu'(q_r) - \mu')$ and performing a partial integration, the second derivative of $\Pi^0_d$ with respect to $q_r$ can be written as

$$ \partial^2 q_r \Pi^0_d = \frac{a_F}{2D^{3/2}} \int_{-\infty}^{\infty} d\mu' h'(\mu' - \mu) \frac{\Theta[\mu'(q_r) - \mu']}{\sqrt{\mu'(q_r) - \mu'}} . $$

For large $|\mu'(q_r)|/T$ one can approximate $h'(\mu' - \mu) \approx \text{sgn}(D) f'_{\mu'/T}$ and perform the integral explicitly, to obtain

$$ \partial^2 q_r \Pi^0_d = \frac{a_F \sqrt{\pi}}{2D^{3/2}T^{3/2}} e^{-|\mu'(q_r)|/T} . $$

Inserting $\delta \mu'(q_r^d)$ from Eq. (28), one obtains the curvature at the shifted peak position

$$ \partial^2 q_r \Pi^0_d|_{q_r = q_r^d} = \frac{a_F \sqrt{\pi}}{2D^{3/2} \sqrt{T_0}} \frac{1}{T} = \frac{|b_F|}{D} \frac{1}{T} . $$

The last two equations are valid for any sign of $D$. Hence, the radius of curvature of the peak is proportional to $T$ at small temperatures, with a remarkably simple prefactor.

**IV. CONCLUSION**

We have analyzed the strength of nematic fluctuations with a finite wave vector in a two-dimensional metal. To this end we have computed the bare static $d$-wave polarization function $\Pi^0_d(q,0)$ as a function of the wave vector $q$ for electrons with a tight-binding dispersion on a square lattice. Peaks in $\Pi^0_d(q,0)$ indicate at which wave vectors a (modulated) nematic instability occurs in presence of a sufficiently strong attraction in the $d$-wave charge channel.

At Van Hove filling, $\Pi^0_d(q,0)$ is strongly peaked at $q = 0$, so that the leading nematic instability is homogeneous in this case. Below and close to Van Hove filling, the largest peaks are on the $q_x$- and $q_y$-axes, leading to a modulated nematic state with a small modulation vector along one of the crystal axes. Above Van Hove filling, the largest peaks of $\Pi^0_d(q,0)$ are situated at diagonal wave vectors $q^* = (\pm q_d, \pm q_d)$, so that the dominant instability leads to a spatially modulated nematic state with a diagonal modulation vector. The same modulated nematic state has been found by Metlitski and Sachdev in a recent study of second-order instabilities generated by antiferromagnetic fluctuations in a two-dimensional metal. In that context the wave vector $q^*$ is favored because it connects intersections of the Fermi surface with the antiferromagnetic Brillouin zone boundary (hot spots). Remarkably, the peak at the same $q^*$ in the $d$-wave polarization function is determined purely by the Fermi surface geometry, without any influence from antiferromagnetic fluctuations.

In all cases the peaks of $\Pi^0_d(q,0)$ lie on lines defined by the condition $\epsilon(q+G)/2 = \mu$, where $G$ is a reciprocal lattice vector, which is the lattice analogue of the condition $|q| = 2k_F$ in a continuum system. Generically, the momentum dependence of $\Pi^0_d(q,0)$ exhibits a square root singularity at these lines, which therefore characterizes also the behavior around the peaks of $\Pi^0_d(q,0)$. At low finite temperatures the peaks in the polarization function are smoothed and shifted by an amount of order $T|\log T|$.

In view of the above results it seems worthwhile to search for modulated nematic instabilities in two-dimensional Hubbard-type models. In a recent functional renormalization group study of the one-band Hubbard model, a modulated nematic instability was found to be typically favorable compared to a homogeneous one, but in any case weaker than antiferromagnetism or $d$-wave superconductivity. Adding a nearest-neighbor repulsion strengthens the nematic fluctuations. There is more room for nematic instabilities in multi-band systems. A search for modulated nematic states in such systems would therefore be particularly promising.

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