QUANTITATIVE BORELL-BRASCAMP-LIEB INEQUALITIES
FOR COMPACTLY SUPPORTED POWER CONCAVE
FUNCTIONS (AND SOME APPLICATIONS)

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Abstract. We strengthen, in two different ways, the so-called Borell-Brascamp-Lieb inequality in the class of power concave functions with compact support. As examples of applications we obtain two quantitative versions of the Brunn-Minkowski inequality and of the Urysohn inequality for torsional rigidity.

1. Introduction

Throughout the paper $u_0$ and $u_1$ will be real non-negative bounded functions belonging to $L^1(\mathbb{R}^n)$ ($n \geq 1$) with compact supports $\Omega_0$ and $\Omega_1$ respectively. To avoid triviality, we will also assume that

$$I_i = \int_{\mathbb{R}^n} u_i \, dx > 0 \quad \text{for } i = 0, 1.$$

The aim of this paper is to prove some refinements of the so-called Borell-Brascamp-Lieb inequality (BBL inequality below) for power concave functions with compact support. Then let us first recall the BBL inequality.

**Theorem 1.1 (BBL inequality).** Let $0 < \lambda < 1$, $-\frac{1}{n} \leq p \leq \infty$, $0 \leq h \in L^1(\mathbb{R}^n)$ and assume the following holds

$$(1) \quad h((1-\lambda)x + \lambda y) \geq M_p(u_0(x), u_1(y), \lambda)$$

for every $x \in \Omega_0$, $y \in \Omega_1$. Then

$$(2) \quad \int_{\mathbb{R}^n} h(x) \, dx \geq M_{\frac{p}{np+1}}(I_0, I_1, \lambda).$$

Here the number $p/(np + 1)$ has to be interpreted in the obvious way in the extremal case, i.e. it is equal to $-\infty$ when $p = -1/n$ and to $1/n$ when $p = \infty$, while, for $q \in [-\infty, +\infty]$ and $\mu \in (0, 1)$, the quantity $M_q(a, b, \mu)$ represents the $(\mu$-weighted) $q$-mean of two non-negative numbers $a$ and $b$, which is defined as follows:

$$M_q(a, b; \mu) = \begin{cases} 
(1 - \mu)a^q + \mu b^q & \text{if } 0 \neq q \in \mathbb{R} \text{ and } ab > 0 \\
\frac{a^{1-\mu}b^\mu}{\mu} & \text{if } q = 0 \\
\max\{a, b\} & \text{when } q \in \mathbb{R} \text{ and } ab = 0 \\
\min\{a, b\} & q = +\infty \\
\min\{a, b\} & q = -\infty.
\end{cases}$$

The BBL inequality was first proved in a slightly different form for $p > 0$ by Henstock and Macbeath (with $n = 1$) in [21] and by Dinghas in [12]. In its generality it is stated and proved by Brascamp and Lieb in [7] and by Borell in [1]. The case $p = 0$ was previously proved by Prékopa [28] and Leindler [26] (and rediscovered by
Brascamp and Lieb in [1] and it is usually known as the Prékopa-Leindler inequality (PL inequality in the following). Noticeably, the PL inequality can be considered a functional form of the Brunn-Minkowski inequality (see §2.3 and refer to [17] for details) and the same can be said at some extent for the BBL inequality for every $p$, as the case $p = 1$ clearly show (see §7 for details). The equality conditions of BBL inequalities are discussed in [13], while stability questions for the PL inequality are investigated in [8, 9, 11].

Our main results are a stability result for the BBL inequality, which we will state in §4 (see Theorem 4.1), and two consequent "quantitative" versions of Theorem 1.1 which apply when $u_0$ and $u_1$ are power concave functions with compact supports and $p > 0$. With the adjective "quantitative", we mean that we will strengthen (2) in terms of some distance between the functions $u_0$ and $u_1$, precisely in terms of some distance between their support sets $\Omega_0$ and $\Omega_1$.

In fact for $p \geq 1$ it is possible to obtain easily some stronger result which suggests the possibility to obtain an $L^\infty$ stability for the BBL inequality. As a final remark and suggestion for future work, we will see in detail the case $p \geq 1$ in §7.

The first quantitative result is written in terms of the Hausdorff distance between (two suitable homothetic copies of) $\Omega_0$ and $\Omega_1$.

We recall that the Hausdorff distance $H(K, L)$ between two sets $K, L \subseteq \mathbb{R}^n$ is defined as follows:

$$H(K, L) := \inf \{ r \geq 0 : K \subseteq L + rB_n, K \subseteq L + rB_n \},$$

where $B_n = \{ x \in \mathbb{R}^n : |x| < 1 \}$ is the (open) unitary ball in $\mathbb{R}^n$. Then we set

$$H_0(K, L) = H(\tau_0K, \tau_1L),$$

where $\tau_1, \tau_0$ are two homotheties (i.e. translation plus dilation) such that $|\tau_0K| = |\tau_1L| = 1$ and such that the centroids of $\tau_0K$ and $\tau_1L$ coincide.

We also recall that a function $u \geq 0$ is said $p$-concave for some $p \in [-\infty, +\infty]$ if

$$u((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p(u(x), u(y) ; \lambda)$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$ (see §2.4 for more details).

Now we are ready to state our first result.

**Theorem 1.2.** In the same assumptions and notation of Theorem 1.1 assume furthermore that $p > 0$ and

$$u_0 \text{ and } u_1 \text{ are } p\text{-concave functions}$$

(with convex compact supports $\Omega_0$ and $\Omega_1$ respectively). Then, if $H_0(\Omega_0, \Omega_1)$ is small enough, it holds

$$\int_{\Omega_\lambda} h(x) \, dx \geq \mathcal{M}_p \left( \frac{1}{p+1}, I_0, I_1, \lambda \right) + \beta H_0(\Omega_0, \Omega_1)^{(n+1)/(p+1)}$$

where $\beta$ is a constant depending only on $n, \lambda, p$ and on the diameters and the measures of $\Omega_0$ and $\Omega_1$.

Our second quantitative version of the BBL inequality is written in terms of the relative asymmetry of $\Omega_0$ and $\Omega_1$; we recall that the relative asymmetry of two sets $K$ and $L$ is defined as follows

$$A(K, L) := \inf_{x \in \mathbb{R}^n} \left\{ \frac{|K \triangle (x + \lambda F)|}{|K|} , \lambda = \left( \frac{|K|}{|L|} \right)^{\frac{1}{p+1}} \right\}.$$
For $\Omega \subseteq \mathbb{R}^n$, here and throughout $|\Omega|$ denotes the Lebesgue measure of the set $\Omega$, while $\Delta$ denotes the operation of symmetric difference, i.e. $\Omega \Delta B = (\Omega \setminus B) \cup (B \setminus \Omega)$.

**Theorem 1.3.** In the same assumptions and notation of Theorem 1.2, if $A(\Omega_0, \Omega_1)$ is small enough it holds

$$
\int_{\Omega_1} h(x) \, dx \geq M \frac{1}{p+1} (I_0, I_1, \lambda) + \delta A(\Omega_0, \Omega_1)^{\frac{2(p+1)}{p}},
$$

where $\delta$ is a constant depending only on $n, \lambda, p$ and on the measures of $\Omega_0$ and $\Omega_1$.

**Remark 1.** We can provide explicit (but not optimal) estimates for the constants $\beta$ and $\delta$ in Theorem 1.2 and Theorem 1.3. To this aim and for further use, it is convenient to introduce the following notation:

$$
d_i = d(\Omega_i) = \text{diameter of } \Omega_i, \quad \nu_i = |\Omega_i|^{1/n} \text{ for } i = 0, 1,
$$

$$
\bar{d} = \max \left\{ \frac{d_0}{\nu_0}, \frac{d_1}{\nu_1} \right\}, \quad M = \max \{\nu_0, \nu_1\}, \quad m = \min \{\nu_0, \nu_1\}.
$$

Then (6) holds with

$$
\beta = \left[ \gamma_n \left( \frac{M}{m} \frac{1}{\sqrt{1 - \lambda}} + \frac{2}{\nu} \right) \right]^{\frac{1}{p+1}} \left[ \frac{m}{2(1 + 2nM^n)} \right]^{\frac{1}{p+1}},
$$

where

$$
\gamma_n = \left( 1 + \frac{1}{3} 2^{-13} 3^{\frac{n-1}{2}} 2^{\frac{n+2}{n}} n \right) < 6.00025n.
$$

Similarly, we can observe that (8) holds with

$$
\delta = \left[ \frac{n m^n M_\frac{n}{p} (|\Omega_0|, |\Omega_1|, \lambda)}{2 M^n (1 + n 2 M^n)} \right]^{\frac{1}{p+1}} \left[ \frac{362n^7}{(2 - 2 \frac{n-1}{n})^2} \right]^{\frac{2(p+1)}{p}}.
$$

Notice that

$$
\delta \leq \left[ \frac{n m^n}{2 (1 + n 2 M^n)} \right]^{\frac{1}{p+1}} \left[ \frac{362n^7}{(2 - 2 \frac{n-1}{n})^2} \right]^{\frac{2(p+1)}{p}}.
$$

**Remark 2.** Theorem 1.2 states that (6) holds if $H_0(\Omega_0, \Omega_1)$ is small enough; this precisely means

$$
H_0(\Omega_0, \Omega_1) < (2n)^{-\frac{1}{p+1}} \beta^{-\frac{(n+1)(p+1)}{p}}.
$$

To avoid this request, we could write (6) as follows:

$$
\int_{\Omega_1} h(x) \, dx \geq M \frac{1}{p+1} \left( \int_{\Omega_0} u_0(x) \, dx, \int_{\Omega_1} u_1(x) \, dx, \lambda \right) + \min \left\{ B, \beta H_0(\Omega_0, \Omega_1)^{\frac{(n+1)(p+1)}{p}} \right\}
$$

where

$$
B = \left( \frac{1}{2n} \right)^{\frac{p+1}{p}}.
$$

A similar remark can be made for Theorem 1.3. In particular (8) holds when

$$
A(\Omega_0, \Omega_1) < (2n)^{-\frac{1}{p+1}} \delta^{-\frac{p+1}{p}},
$$

but we could remove any limitation on the size of $A(\Omega_0, \Omega_1)$ and write

$$
\int_{\Omega_1} h(x) \, dx \geq M \frac{1}{p+1} \left( \int_{\Omega_0} u_0(x) \, dx, \int_{\Omega_1} u_1(x) \, dx, \lambda \right) + \min \left\{ B, \delta A(\Omega_0, \Omega_1)^{\frac{2(p+1)}{p}} \right\}
$$
where $B$ is defined in (10).

**Remark 3.** As it is easily seen from the previous remarks, the estimates in Theorem 1.2 and Theorem 1.3 deteriorate quickly as the dimension increases. We notice that the same feature is shared by most of the known stability estimates for the Brunn-Minkowski inequality. However, R. Eldan and B. Klartag recently made in [14] a new step towards a dimension-sensitive theory for the Brunn-Minkowski inequality, giving rise to the possibility that the stability actually improves as the dimension increases. See §6 for further comments and an application of the results of [14] to the present case.

The crucial part in the proofs of Theorem 1.2 and Theorem 1.3 relies on an estimate of the measures of the supports sets of the involved functions; this estimate is contained in Theorem 4.1 (which can be in fact considered the main result of this paper). There we prove that if we are close to equality in (2), then the measure of $(1 - \lambda)\Omega_0 + \lambda \Omega_1$ is close to $M_1^n(\|\Omega_0\|, |\Omega_1|, \lambda)$. Then we can apply different quantitative versions of the classical Brunn-Minkowski inequality to get Theorem 1.2 and Theorem 1.3.

Then we can derive interesting quantitative versions of some interpolation inequalities for functionals that can be written in terms of the solutions of suitable elliptic boundary value problems (this is in fact the original reason for we tackled the stability of the BBL inequality). To be more precise, we recall that, under suitable assumptions on the operator $F$, it is possibile to compare the solutions of fully nonlinear equations

\[
F(x, u, Du, D^2u) = 0
\]

in different convex domains using the infimal convolution and a new kind of rearrangement technique, the so-called mean width rearrangement, introduced in [33].

For the sake of simplicity and clearness of exposition, we will analyze in detail as a toy model the torsion problem, that is

\[
\begin{aligned}
\Delta u &= -2 & \text{in } \Omega, \\
\quad u &= 0 & \text{on } \partial\Omega.
\end{aligned}
\]

We recall that the torsional rigidity $\tau(\Omega)$ of $\Omega$ is defined as follows (we refer to §6 for further details)

\[
\frac{1}{\tau(\Omega)} = \inf \left\{ \frac{\int_\Omega |Du|^2 \, dx}{\left( \int_\Omega |w| \, dx \right)^2} : w \in W^{1,2}_0(\Omega), \int_\Omega |w| \, dx < 0 \right\}
\]

and that in general, when a solution $u$ to problem (12) exists, we have

\[
\tau(\Omega) = \frac{\left( \int_\Omega |u| \, dx \right)^2}{\int_\Omega |Du|^2 \, dx} = \int_\Omega |u| \, dx.
\]

Borell [2] (see also [10]) proved the following Brunn-Minkowski inequality for the torsional rigidity of convex bodies (i.e. compact convex sets with non-empty interior):

\[
\tau(\Omega_\lambda) \geq M_{1/\lambda}^{\lambda} (\tau(\Omega_0), \tau(\Omega_1), \lambda),
\]

where

\[
\Omega_\lambda = (1 - \lambda)\Omega_0 + \lambda \Omega_1;
\]

equality holds in (14) if and only if $\Omega_0$ and $\Omega_1$ coincide up to a homothety.

Now we can refine this inequality as follows.
Theorem 1.4. Let $\Omega_0$ and $\Omega_1$ be convex bodies in $\mathbb{R}^n$, let $\lambda \in (0, 1)$ and set $\Omega_\lambda = (1 - \lambda)\Omega_0 + \lambda\Omega_1$. Then the following strengthened versions of (14) hold:

\begin{align*}
\tau(\Omega_\lambda) &\geq M_{\frac{1}{n+2}}(\tau(\Omega_0), \tau(\Omega_1), \lambda) + \beta H_0(\Omega_0, \Omega_1)^{3(n+1)}, \\
\tau(\Omega_\lambda) &\geq M_{\frac{1}{n+2}}(\tau(\Omega_0), \tau(\Omega_1), \lambda) + \delta A(\Omega_0, \Omega_1)^6,
\end{align*}

where $\beta$ and $\delta$ are as in Remark 1 with $p = 1/2$.

The proof of Theorem 1.4, which follows almost straightforward from Theorem 1.2 and Theorem 1.3, will be presented in §6.

Furthermore [33] shows that it is possible to use (14) to compare $u$ with the solution $v$ of the same problem in the ball $\Omega^\#$ with the same mean-width of $\Omega$ and this leads to the following Urysohn’s type inequality for the torsional rigidity

\begin{equation}
\tau(\Omega) \leq \tau(\Omega^\#) \quad \text{for every convex set } \Omega,
\end{equation}

which can be rephrased as follows: among convex sets with given mean width, the torsional rigidity is maximized by the ball.

In §6 we prove two quantitative versions of (17), one in terms of the Hausdorff distance of $\Omega$ from $\Omega^\#$ (Theorem 1.5) and another one in terms of the relative asymmetry of $\Omega$ (Theorem 1.6), as applications respectively of Theorem 1.2 and Theorem 1.3. The statements are the following.

Theorem 1.5. Let $\Omega$ be an open bounded convex set of $\mathbb{R}^n$, $n \geq 2$ with centroid in the origin. Let $d(\Omega)$ be the diameter of $\Omega$ and $\Omega^\#$ be the ball with the same mean-width of $\Omega$ with center in the origin. Then, if $H(\Omega, \Omega^\#)$ is small enough,

\begin{equation}
\tau(\Omega^\#) \geq \tau(\Omega) + \mu H(\Omega, \Omega^\#)^{3(n+1)}
\end{equation}

where $\mu$ is a constant depending on $n$, the diameter and the measure of $\Omega$. Precisely

\begin{equation}
\mu = 2^{-3}(4\gamma_n d(\Omega))^{-3(n+1)}|\Omega|^\frac{3n+5}{n}(1 + 2n|\Omega|)^{-3}
\end{equation}

where $\gamma_n < 6.00025n$ is defined in (44).

Theorem 1.6. Let $\Omega$ be an open bounded convex set of $\mathbb{R}^n$, $n \geq 2$ with centroid in the origin. Let $d(\Omega)$ be the diameter of $\Omega$ and $\Omega^\#$ be the ball with the same mean-width of $\Omega$ with center in the origin. Then, if $A(\Omega, \Omega^\#)$ is small enough,

\begin{equation}
\tau(\Omega^\#) \geq \tau(\Omega) + \nu A(\Omega, \Omega^\#)^6
\end{equation}

where $\nu$ is a constant depending on $n$ and the measure of $\Omega$. Precisely

\begin{equation}
\nu = 2^{-3}|\Omega|^3n^3(1 + 2n|\Omega|)^{-3}\left[\frac{362n^7}{(2 - 2\frac{n}{\pi})^2}\right]^{-6}.
\end{equation}

We observe that the same kind of remarks as Remark 1 and Remark 2 can be done for Theorem 1.5 and Theorem 1.6.

Finally, we remark that similar results to Theorem 1.4, Theorem 1.5 and Theorem 1.6 can be obtained for many other operators with similar properties as $\tau$ and satisfying suitable Brunn-Minkowski inequality, see §6 for some examples.

The paper is organized as follows. In §2 we introduce some notation and recall some useful known results. In §3 we give a proof of Theorem 1.1 (in the case $p \in (0, \infty)$) whose argument will be useful for the proof of Theorem 4.1. §4 is devoted to Theorem 4.1 while the proofs of Theorem 1.2 and Theorem 1.3 are
In such a case, we also say that the hyperplane \( H \).

We say that \( \mu \) then \( H \) that

\[ (22) \quad M_p(a, b; \mu) \leq M_q(a, b; \mu) \quad \text{if } p \leq q. \]

We also notice that for every \( \mu \in (0, 1) \) it holds

\[ \lim_{p \to \infty} M_p(a, b; \mu) = \max\{a, b\} \quad \text{and} \quad \lim_{p \to -\infty} M_p(a, b; \mu) = \min\{a, b\}. \]

Finally we recall the following technical lemma (for a proof, refer to [17]):

**Lemma 2.1.** Let \( 0 < \lambda < 1 \) and \( a, b, c, d \) be nonnegative numbers. If \( p + q > 0 \), then

\[ M_p(a, b, \lambda)M_q(c, d, \lambda) \geq M_s(ac, bd, \lambda) \]

where \( s = \frac{pa}{p+q} \). The same is true with \( s = 0 \) if \( p = q = 0 \).

### 2.2. Convex bodies and convex functions

Throughout the paper \( \Omega \) and \( K \), possibly with subscripts, will be bounded convex sets, most often convex bodies, that is compact convex sets with non-empty interior. We denote by \( K_0^n \) the class of convex bodies in \( \mathbb{R}^n \).

Next we recall some classical notions of convex geometry, for further details see [34]. Let \( L \subseteq \mathbb{R}^n \) be a convex set, \( p \in \mathbb{R}^n \setminus \{0\} \) and \( \alpha \in \mathbb{R} \); we set

\[ H_{p, \alpha} = \{ x \in \mathbb{R}^n : \langle x, p \rangle = \alpha \} \quad \text{and} \quad H^{-}_{p, \alpha} = \{ x \in \mathbb{R}^n : \langle x, p \rangle \leq \alpha \}. \]

We say that \( p \) is an exterior normal vector of \( L \) at \( x_0 \) if \( x_0 \in L \cap H_{p, \alpha} \) and \( L \subseteq H_{p, \alpha}^- \); in such a case, we also say that the hyperplane \( H_{p, \alpha} \) is a support hyperplane and that \( H_{p, \alpha}^- \) is a supporting halfspace (with exterior normal vector \( p \)) of \( L \).

The support function of \( L \) is defined in the following way:

\[ h(L, x) = \sup\{ \langle x, y \rangle : y \in L \}, \quad x \in \mathbb{R}^n. \]

If \( K \in K_0^n \), the latter supremum is in fact a maximum and we can write:

\[ h(K, x) = \max\{ \langle x, y \rangle : y \in K \}, \quad x \in \mathbb{R}^n. \]

For any unit vector \( \xi \in S^{n-1} \), \( h(K, \xi) \) represents the signed distance from the origin of the support plane to \( K \) with exterior normal vector \( \xi \). The support function satisfies the following properties:

(i) \( h(K, \lambda x) = \lambda h(K, x) \quad \forall \lambda \geq 0 \).

(ii) \( h(K, x + y) \leq h(K, x) + h(K, y) \).

In fact, the latter properties characterize support functions in the following sense: if \( f : \mathbb{R}^n \to \mathbb{R} \) is a function that satisfies (i) and (ii), then there is one (and only one) convex body with support function equal to \( f \).

Other useful properties of the support function are the following: let \( K, K_1, K_2 \in K_0^n \), then
In particular, let $\lambda(24) \Omega$
the Minkowski addition of convex sets is defined as follows:

$$M \text{inkowski addition} \nu V(\cdot, \cdot)$$

of convex bodies and states that

$$2.3.$$  
The Brunn-Minkowski inequality. As already mentioned in the intro-
duction, the original form of the Brunn–Minkowski inequality involve s volumes

$$K(25) V(\cdot, \cdot)$$

and several other important inequalities, e.g. the isoperimetric inequality, can be

deduced from it. It can be extended to measurable sets and it holds also, with the

right exponent, for the other quermassintegrals. We refer the interested reader to

and to the survey paper [17] for this topic.

We recall two quantitative version of (25) which will be used later.
The first proposition is due to Groemer [19].

\[ (\text{iii}) \ h(K + x_0, \cdot) = h(K, \cdot) + \langle x_0, \cdot \rangle \quad \forall x_0 \in \mathbb{R}^n; \]

\[ (\text{iv}) \ h(\lambda K, \cdot) = \lambda h(K, \cdot) \quad \forall \lambda \geq 0; \]

\[ (\text{v}) \ h(K_1 + K_2, \cdot) = h(K_1, \cdot) + h(K_2, \cdot). \]

\[ (\text{vi}) \ h(K_1, \cdot) \leq h(K_2, \cdot) \text{ if and only if } K_1 \subseteq K_2. \]

If $K \in K^n_0$ the number

$$w(K, \xi) = h(K, \xi) + h(K, -\xi), \quad \xi \in S^{n-1}$$

is the width of $K$ in the direction $\xi$, that is the distance between the two support

hyperplanes of $K$ orthogonal to $\xi$. The maximum of the width function

$$d(K) = \max\{w(K, \xi) | \xi \in S^{n-1}\}$$

is the diameter of $K$.

The mean width of $K$ is the average of the width of $K$ over all $\xi \in S^{n-1}$, that is

$$w(K) = \frac{1}{n\omega_n} \int_{S^{n-1}} w(K, \xi) d\xi = \frac{2}{n\omega_n} \int_{S^{n-1}} h(K, \xi) d\xi.$$  

Urysohn’s inequality states

$$|K| \leq \omega_n \left( \frac{w(K)}{2} \right)^n,$$

equality holding if and only if $K$ is a ball.

2.3. The Brunn-Minkowski inequality. As already mentioned in the intro-
duction, the original form of the Brunn–Minkowski inequality involves volumes of

convex bodies and states that $V(\cdot)^{1/n}$ is a concave function with respect to

Minkowski addition, where $V(\cdot)$ denotes the $n$-dimensional Lebesgue measure and

the Minkowski addition of convex sets is defined as follows:

$$A + B = \{x + y \mid x \in A, y \in B\}$$

In particular, let $\lambda \in [0, 1]$ and let $\Omega_0$ and $\Omega_1$ be convex subsets of $\mathbb{R}^n$; we define their Minkowski linear combination $\Omega_\lambda$ as

$$\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1 = \{(1-\lambda)x_0 + \lambda x_1 : x_i \in \Omega_i, i = 0, 1\}.$$  

With this notation, the classical Brunn-Minkowski inequality reads

$$V(K_\lambda)^\frac{1}{n} \geq (1-\lambda)V(K_0)^\frac{1}{n} + \lambda V(K_1)^\frac{1}{n},$$

for $K_0, K_1 \in K^n_0$ and $\lambda \in [0, 1]$ and it can be also written in the following equivalent multiplicative form

$$V(K_\lambda) \geq V(K_0)^{1-\lambda}V(K_1)^{\lambda}.$$  

As it is well known, the Brunn-Minkowski inequality and the PL inequality are
equivalent (notice that the way from the latter to the former is almost straight-
forward by taking $u_0 = \chi_{K_0}, u_1 = \chi_{K_1}$ and $h = \chi_{K_\lambda}$, where $\chi_A$ represents the characteristic function of the set $A$).

Inequality $25$ is one of the fundamental results in the theory of convex bodies and
several other important inequalities, e.g. the isoperimetric inequality, can be deduced
from it. It can be extended to measurable sets and it holds also, with the

right exponent, for the other quermassintegrals. We refer the interested reader to

[14] and to the survey paper [17] for this topic.

We recall two quantitative version of (25) which will be used later.
The first proposition is due to Groemer [19].
Proposition 2.2. Let $K_0, K_1 \in \mathcal{K}^n_0$, $n \geq 2$, $\lambda \in (0, 1)$ and let

$$K_\lambda = (1 - \lambda)K_0 + \lambda K_1.$$ 

Set $\nu_i = |K_i|^{\frac{1}{p}}$. Let $\tilde{d} = \max\{\frac{d(K_0)}{\nu_0}, \frac{d(K_1)}{\nu_1}\}$ and $M = \max\{\nu_0, \nu_1\}, m = \min\{\nu_0, \nu_1\}$. Then

$$|K_\lambda| \geq \mathcal{M}_\lambda(|K_0|, |K_1|, \lambda) + \omega H_0(K_0, K_1)^{(n + 1)}$$

where

$$\omega = \left(\gamma_n \left(\frac{M}{m} \sqrt{\lambda(1 - \lambda)} + 2\right)^\tilde{d}\right)^{-(n + 1)} m,$$

$H_0$ is defined as in (1) and

$$\gamma_n = (1 + \frac{1}{3}2^{-13})3^{\frac{n-1}{2}} 2^{\frac{2n+2}{n}} n < 6.00025n.$$ 

The second proposition is due to Figalli, Maggi, Pratelli [15, 16].

Proposition 2.3. Let $K_0, K_1 \in \mathcal{K}^n_0$, $\lambda \in (0, 1)$ and let

$$K_\lambda = (1 - \lambda)K_0 + \lambda K_1.$$ 

Set $\sigma(K_0, K_1) = \max\left\{\frac{|K_0|}{|K_1|}, |K_1|\right\}$. Then

$$|K_\lambda| \geq \mathcal{M}_\lambda(|K_0|, |K_1|, \lambda) \left(1 + \frac{1}{\sigma(K_0, K_1)} \left(\frac{A(K_0, K_1)}{\theta_n}\right)^2\right)^n,$$

where $A(K_0, K_1)$ is defined in (7) and $\theta_n$ is a constant depending on $n$ with polynomial growth. In particular

$$\theta_n = \frac{362n^7}{(2 - 2^{\frac{2n+2}{n}})^2}.$$ 

2.4. Power concave functions. Let $\Omega$ be a convex set in $\mathbb{R}^n$ and $p \in [-\infty, \infty]$. A nonnegative function $u$ defined in $\Omega$ is said $p$-concave if

$$u((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p(u(x), u(y); \lambda)$$

for all $x, y \in \Omega$ and $\lambda \in (0, 1)$. In the cases $p = 0$ and $p = -\infty$, $u$ is also said log-concave and quasi-concave in $\Omega$, respectively. In other words, a non-negative function $u$, with convex support $\Omega$, is $p$-concave if:
- it is a non-negative constant in $\Omega$, for $p = +\infty$;
- $u^p$ is concave in $\Omega$, for $p > 0$;
- log $u$ is concave in $\Omega$, for $p = 0$;
- $u^p$ is convex in $\Omega$, for $p < 0$;
- it is quasi-concave, i.e., all of its superlevel sets are convex, for $p = -\infty$.

Notice that $p = 1$ corresponds to usual concavity. Notice also that from (22) it follows that if $u$ is $p$-concave, then $u$ is $q$-concave for every $q \leq p$ (this in particular means that quasi-concavity is the weakest concavity property one can imagine).

The solutions of elliptic Dirichlet problems in convex domains are often power concave. Two famous results state for instance that the first positive eigenfunction of the Laplace operator in a convex domain is log-concave [6] and that the square root of the solution to the torsion problem in a convex domain is concave [23, 24, 25]. For recent results and updated references (in the elliptic and parabolic cases), see for instance [5, 22].
The concavity properties of a function \( u \) can be expressed in terms of its level sets. Precisely it is easily seen that a function \( u \) is concave if and only if
\[
\{ u \geq (1 - \lambda)t_0 + \lambda t_1 \} \supseteq (1 - \lambda)\{ u \geq t_0 \} + \lambda\{ u \geq t_1 \}
\]
for every \( t_0, t_1 \in \mathbb{R} \) and every \( \lambda \in (0, 1) \).

More generally, we have the following characterization of power concave functions, which easily follows from the above property.

**Proposition 2.4.** A non-negative function \( u \) is \( p \)-concave in a convex domain \( \Omega \) for some \( p \in [-\infty, +\infty) \) if and only if
\[
\{ x \in \Omega : u(x) \geq \mathcal{M}_p(t_0, t_1, \lambda) \} \supseteq (1 - \lambda)\{ x \in \Omega : u(x) \geq t_0 \} + \lambda\{ x \in \Omega : u(x) \geq t_1 \}
\]
for every \( t_0, t_1 \geq 0 \) and every \( \lambda \in (0, 1) \).

Let \( \mu \) be the distribution function of \( u \), i.e.
\[
(26) \quad \mu(t) = |\{ u \geq t \}|.
\]
Then, as a direct consequence of the Brunn-Minkowski inequality and Proposition 2.4 we have the following.

**Proposition 2.5.** If \( u \) is \( p \)-concave for some \( p \neq 0 \), then
\[
\mu(t^{1/p})^{1/n} \text{ is concave in } t.
\]
If \( u \) is log-concave (corresponding to \( p = 0 \)), then
\[
\mu(e^{t})^{1/n} \text{ is concave in } t.
\]

2.5. The \((p, \lambda)\)-convolution of non-negative functions. Let \( p \in \mathbb{R} \), \( \mu \in (0, 1) \), and \( u_0, u_1 \) non-negative functions with compact convex support \( \Omega_0 \) and \( \Omega_1 \), as usual in this paper.

The \((p, \lambda)\)-convolution of \( u_0 \) and \( u_1 \) is the function defined as follows:
\[
(27) \quad u_{p, \lambda}(x) = \sup \left\{ \mathcal{M}_p \left( u_0(x_0), u_1(x_1); \lambda \right) : x = (1 - \lambda)x_0 + \lambda x_1, \ x_i \in \overline{\Omega_i}, \ i = 0, 1 \right\}.
\]

The above definition can be extended to the case \( p = \pm \infty \), but we do not need here. Notice that (22) yields
\[
(28) \quad u_{q, \lambda} \leq u_{p, \lambda} \quad \text{if } q \leq p.
\]

It is easily seen that the support of \( u_{p, \lambda} \) is \( \Omega_{\lambda} = (1 - \lambda)\Omega_0 + \lambda\Omega_1 \), and that the continuity of \( u_0 \) and \( u_1 \) yields the continuity of \( u_{p, \lambda} \), in particular if \( u_i \in C(\overline{\Omega}_i) \) for \( i = 0, 1 \), then \( u_{p, \lambda} \in C(\overline{\Omega}_{\lambda}) \).

Let \( p \neq 0 \); then, roughly speaking, the graph of \( u_{p, \lambda}^p \) is obtained as the Minkowski convex combination (with coefficient \( \lambda \)) of the hypographs of \( u_0^p \) and \( u_1^p \); precisely we have
\[
K_{\lambda}^{(p)} = (1 - \lambda)K_0^{(p)} + \lambda K_1^{(p)},
\]
where
\[
(29) \quad K_{\lambda}^{(p)} = \{ (x, t) \in \mathbb{R}^{n+1} : x \in \Omega_{\lambda}, \ 0 \leq t \leq u_{p, \lambda}(x)^p \},
\]
\[
(30) \quad K_i^{(p)} = \{ (x, t) \in \mathbb{R}^{n+1} : x \in \Omega_i, \ 0 \leq t \leq u_i(x)^p \}, \quad i = 0, 1.
\]

In other words, the \((p, \lambda)\)-convolution of \( u_0 \) and \( u_1 \) corresponds to the \((1/p)\)-power of the supremal convolution (with coefficient \( \lambda \)) of \( u_0^p \) and \( u_1^p \). When \( p = 0 \), the above geometric considerations continue to hold with logarithm in place of power.
and exponential in place of power $1/p$. When $p = 1$, $u_{1,\lambda}$ is just the usual supremal convolution of $u_0$ and $u_1$ (see for instance [31, §3]). For more details on infimal/supremal convolutions of convex/concave functions, see [29, §5].

From the definition of $u_{p,\lambda}$ and the monotonicity of $p$-means with respect to $p$, we get

\[(31) \quad u_{p,\lambda} \leq u_{q,\lambda} \quad \text{for} \quad -\infty \leq p \leq q \leq +\infty.\]

3. A proof of Theorem 1.1

Before giving the proof of Theorem 4.1, we recall here an alternative proof of Theorem 1.1 for power concave functions. The argument will be useful for the proof of Theorem 4.1.

**Proof.** First of all, we define $u_{p,\lambda}$ as in (27) and notice that (1) implies

\[h \geq u_{p,\lambda} \quad \text{in} \quad \mathbb{R}^n.\]

Let

\[I_i = \int_{\Omega_i} u_i \ dx \quad i = 0, 1,\]

and

\[I_\lambda = \int_{\Omega_\lambda} u_{p,\lambda} \ dx.\]

As declared at the beginning, we assume

\[I_i > 0 \quad i = 0, 1,\]

and

\[L_i = \sup_{\Omega_i} u_i < \infty \quad i = 0, 1.\]

Notice that the very definition of $u_{p,\lambda}$ yields

\[L_\lambda = \sup_{\Omega_\lambda} u_{p,\lambda} = M_p(L_0, L_1, \lambda).\]

Let

\[\mu_i(s) = |\{u_i \geq s\}| \quad i = 0, 1, \quad \mu_\lambda(s) = |\{u_{p,\lambda} \geq s\}|.\]

Then

\[I_i = \int_0^{L_i} \mu_i(s) \ ds \quad i = 0, 1, \lambda.\]

The definition of $u_{p,\lambda}$ yields

\[\{u_{p,\lambda} \geq M_p(s_0, s_1; \lambda)\} \supseteq (1 - \lambda)\{u_0 \geq s_0\} + \lambda\{u_1 \geq s_1\}\]

for $s_0 \in [0, L_0]$, $s_1 \in [0, L_1]$. Then, using the Brunn-Minkowski inequality, we get

\[(32) \quad \mu_\lambda(M_p(s_0, s_1; \lambda)) \geq M_\lambda(\mu_0(s_0), \mu_1(s_1), \lambda).\]

Define the functions $s_i : [0, 1] \to [0, L_i]$ for $i = 0, 1$ such that

\[(33) \quad s_i(t) : \frac{1}{L_i} \int_0^{s_i(t)} \mu_i(s) \ ds = t \quad \text{for} \ t \in [0, 1].\]
Notice that \(s_i\) is strictly increasing, then it is differentiable almost everywhere and differentiating (33) we obtain

\[
\frac{s_i'(t)\mu_i(s_i(t))}{I_i} = 1 \text{ a.e. } t \in [0, 1], \quad i = 0, 1.
\]

Set

\[
s_\lambda(t) = M_p(s_0(t), s_1(t), \lambda) \quad t \in [0, 1]
\]

and calculate

\[
s'_\lambda(t) = \left((1 - \lambda)s'_0(t)s_0(t)^{p-1} + \lambda s'_1(t)s_1(t)^{p-1}\right)s_\lambda(t)^{1-p} \quad \text{a.e. } t \in [0, 1].
\]

Notice that the map \(s_\lambda : [0, 1] \to [0, L_\lambda]\) is strictly increasing, then invertible; let us denote by \(t_\lambda : [0, L_\lambda] \to [0, 1]\) its inverse map.

Then

\[
I_\lambda = \int_0^{L_\lambda} \mu_\lambda(s) \, ds = \int_0^1 \mu_\lambda(s_\lambda(t))s'_\lambda(t) \, dt
\]

and

\[
= \int_0^1 \mu_\lambda(s_\lambda(t))M_1(s_0'(t)s_0(t)^{p-1}, s_1'(t)s_1(t)^{p-1})s_\lambda(t)^{1-p} \, dt.
\]

Thanks to (32), we get

\[
\mu_\lambda(s_\lambda(t)) \geq M_{\frac{p}{n+1}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) \quad t \in [0, 1]
\]

and coupling (34) and (37) we arrive to

\[
I_\lambda \geq \int_0^1 M_{\frac{p}{n+1}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda)M_1(s_0'(t)s_0(t)^{p-1}, s_1'(t)s_1(t)^{p-1}, \lambda)s_\lambda(t)^{1-p} \, dt.
\]

Next we use Lemma 2.11 with \(p = \frac{1}{n}\) and \(q = 1\) to obtain

\[
M_{\frac{p}{n+1}}(\mu_0(s_0), \mu_1(s_1), \lambda)M_1(s_0'(s_0)^{p-1}, s_1'(s_1)^{p-1}, \lambda) \geq M_{\frac{n+1}{n+q}}(\mu_0(s_0)^{p-1}, s_0'(s_1)^{p-1}, \lambda)
\]

for \(s_0 \in [0, L_0], s_1 \in [0, L_1]\). Then (33) yields

\[
I_\lambda \geq \int_0^1 M_{\frac{n+1}{n+q}}(\mu_0(s_0(t))s_0(t)^{p-1}s_0'(t), \mu_1(s_1(t))s_1(t)^{p-1}s_1'(t), \lambda)s_\lambda(t)^{1-p} \, dt.
\]

Since

\[
s_\lambda^{1-p} = M_p(s_0, s_1, \lambda)^{1-p} = M_{\frac{p}{n+1}}(s_0^{1-p}, s_1^{1-p}, \lambda),
\]

using again Lemma 2.11 with \(p = \frac{1}{n+1}\) and \(q = \frac{1}{1-p}\) we get

\[
\frac{n+1}{n+q}(\mu_0(s_0)^{p-1}, s_0'(s_1)^{p-1}, \lambda)M_{\frac{n+1}{n+q}}(s_0^{1-p}, s_1^{1-p}, \lambda)
\]

\[
\geq M_{\frac{n+1}{n+q}}(\mu_0(s_0)^{p-1}, s_0'(s_1)^{p-1}, \lambda).
\]

Then coupling (41) with (39) we obtain

\[
I_\lambda \geq \int_0^1 M_{\frac{n+1}{n+q}}(\mu_0(s_0(t))s_0'(t), \mu_1(s_1(t))s_1'(t), \lambda) \, dt
\]

whence, thanks to (33), we finally arrive to

\[
I_\lambda \geq \int_0^1 M_{\frac{n+1}{n+q}}(I_0, I_1, \lambda) \, dt = M_{\frac{n+1}{n+q}}(I_0, I_1, \lambda)
\]

This concludes the proof. \(\square\)
4. The Main Result

Theorem 1.2 and Theorem 1.3 essentially stems from the following stability result for the BBL inequality, which we will prove first and can be in fact considered the main result of the paper.

**Theorem 4.1.** In the same assumptions and notation of Theorem 1.2 and Theorem 1.3. If for some \( \epsilon > 0 \) small enough

\[
\int_{\Omega_\lambda} h(x) \, dx \leq \mathcal{M}_{\frac{p}{p+n}} \left( \int_{\Omega_0} u_0(x) \, dx, \int_{\Omega_1} u_1(x) \, dx, \lambda \right) + \epsilon, \tag{42}
\]

then

\[
|\Omega_\lambda| \leq \mathcal{M}_{\frac{n}{n}} ([\Omega_0], [\Omega_1], \lambda) + \eta \sqrt[p]{\epsilon}, \tag{43}
\]

where \( \eta \) is a constant such that

\[
\eta \leq 2 \left( 1 + n M^n \right). \tag{44}
\]

**Remark 4.** We observe that we can give a slightly better estimate of the constant \( \eta \) in (43), precisely

\[
\eta \leq 2 \left( 1 + n M^n \right). \tag{45}
\]

**Remark 5.** "Small enough" (referred to \( \epsilon \) in the statement of Theorem 4.1) precisely means

\[
\epsilon \leq \left( \frac{1}{2n} \right)^{\frac{p+1}{n}}
\]

and we could make similar comments as in Remark 1 and Remark 2. This number depends on \( n \) (and tends to 0 as \( n \to \infty \)), then the result of Theorem 4.1 is dimension sensitive (see Remark 3).

**Proof.** We will use the same notation as in the proof of Theorem 1.1 given in the previous section and following the same argument we arrive again to (32). In particular for \( s_0 = s_1 = 0 \) (32) reads

\[
|\Omega_\lambda| \geq \mathcal{M}_{\frac{n}{n}} ([\Omega_0], [\Omega_1], \lambda). \tag{46}
\]

If equality holds in this inequality, there is nothing to prove. Then let us assume that

\[
|\Omega_\lambda| = \mathcal{M}_{\frac{n}{n}} ([\Omega_0], [\Omega_1], \lambda) + \tau, \tag{47}
\]

with \( \tau > 0 \). Our aim is to find and estimate on \( \tau \) depending on \( \epsilon \), that is \( \tau < f(\epsilon) \) (with \( \lim_{\epsilon \to 0} f(\epsilon) = 0 \)).

Now set

\[
F_\delta = \{ t \in [0, 1] : \mu_\lambda(s_\lambda(t)) > \mathcal{M}_{\frac{n}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) + \delta \}
\]

and

\[
\Gamma_\delta = \{ s_\lambda(t) : t \in F_\delta \}.
\]

Notice that \( F_\delta \) and \( \Gamma_\delta \) are measurable sets, thanks to Proposition 2.5 and to the monotonicity of the \( s_i \)'s.
Then we have

\[ I_\lambda = \int_0^1 \mu_\lambda(s_\lambda(t))s'_\lambda(t) \, dt \]

\[ = \int_{F_\delta} \mu_\lambda(s_\lambda(t))s'_\lambda(t) \, dt + \int_{[0,1] \setminus F_\delta} \mu_\lambda(s_\lambda(t))s'_\lambda(t) \, dt \]

\[ \geq \int_{F_\delta} \left[ M_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) + \delta \right] s'_\lambda(t) \, dt + \int_{[0,1] \setminus F_\delta} \mu_\lambda(s_\lambda(t))s'_\lambda(t) \, dt \]

\[ > \int_0^1 M_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda)s'_\lambda(t) \, dt + \delta \int_{F_\delta} s'_\lambda(t) \, dt \]

\[ = \int_0^1 M_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda)s'_\lambda(t) \, dt + \delta |\Gamma_\delta| \]

where in the first inequality we have used the definition of $F_\delta$, in the second we have used (47) and in the last equality we have used the definition of $\Gamma_\delta$ (and the change of variable $s = s_\lambda(t)$).

Continuing to argue as in the proof of Theorem 1.1 given in the previous section, we find

\[ \int_0^1 M_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda)s'_\lambda(t) \, dt \geq M_{\frac{1}{n+p+1}}(I_0, I_1, \lambda). \]

Moreover from (42) we know that

\[ M_{\frac{1}{n+p+1}}(I_0, I_1, \lambda) + \epsilon \geq I_\lambda \]

and so we can conclude

\[ M_{\frac{1}{n+p+1}}(I_0, I_1, \lambda) + \epsilon \geq I_\lambda > M_{\frac{1}{n+p+1}}(I_0, I_1, \lambda) + \delta |\Gamma_\delta| \]

which implies that

\[ |\Gamma_\delta| < \frac{\epsilon}{\delta}. \]

Take now

\[ \delta = \epsilon^\alpha / L_\lambda \]

for some $0 < \alpha < 1$. Then (48) reads

\[ |\Gamma_{\epsilon^\alpha}| < \epsilon^{1-\alpha} L_\lambda. \]

Let $u_\lambda$ be defined in (27). Then, thanks to assumption (5), $u_\lambda$ is $p$-concave, that is the following inclusion holds

\[ \{ z : u_\lambda(z) \geq M_p(s_0, s_1, \xi) \} \supseteq (1 - \xi) \{ x : u_\lambda(x) \geq s_0 \} + \xi \{ y : u_\lambda(y) \geq s_1 \}. \]

for $\xi \in [0, 1]$ and $s_0, s_1 \in \mathbb{R}$ (in fact, it is meaningful for $s_0 \in [0, L_0]$ and $s_1 \in [0, L_1]$).

Let us choose

\[ s_0 = 0, \quad s_1 = L_\lambda. \]

By (49), we can find $\tilde{t} > 0$ such that

\[ s_\lambda(\tilde{t}) \leq \epsilon^{1-\alpha} L_\lambda, \]

and

\[ \mu_\lambda(s_\lambda(\tilde{t})) \leq M_{\frac{1}{n}}(\mu_0(s_0(\tilde{t})), \mu_1(s_1(\tilde{t})), \lambda) + \epsilon^\alpha. \]
Let
\[ \xi = \left( \frac{s_1(t)}{L_\lambda} \right)^p. \]
From (52) we have
\[ \xi \leq e^{(1 - n)p}. \]
With these choices of \( s_0, s_1 \) and \( \xi \), (50) reads
\[ \{ u_\lambda \geq s_1(t) \} \supseteq (1 - \xi)\Omega_\lambda + \xi \{ u_\lambda \geq L_\lambda \}. \]
From the Brunn-Minkowski inequality we get
\[ \tau (57) \]
Then (57) reads
\[ \int h(x) \, dx \leq \mathcal{M}_n \left( \int_{\Omega_0} u_0(x) \, dx, \int_{\Omega_1} u_1(x) \, dx, \lambda \right) + \beta \mathcal{H}_0(\Omega_0, \Omega_1)^{\frac{n+1}{p-1}} \]
where \( \beta \) is defined in Remark 1. Then we apply Theorem 4.1 and we get
\[ |\Omega_\lambda| \leq \mathcal{M}_n (|\Omega_0|, |\Omega_1|, \lambda) + \eta \beta^{\frac{n+1}{p}} \mathcal{H}_0(\Omega_0, \Omega_1)^{n+1}, \]
and the proof is complete, since \( \mathcal{M}_n (|\Omega_0|, |\Omega_1|, \lambda) \leq M^n. \]

5. PROOFS OF THEOREM 1.2 AND THEOREM 1.3

Now we prove Theorem 1.2.

Proof of Theorem 1.2. We argue by contradiction. Suppose that
\[ \int h(x) \, dx \leq \mathcal{M}_n \left( \int_{\Omega_0} u_0(x) \, dx, \int_{\Omega_1} u_1(x) \, dx, \lambda \right) + \beta \mathcal{H}_0(\Omega_0, \Omega_1)^{\frac{n+1}{p-1}}, \]
where \( \beta \) is defined in Remark 1. Then we apply Theorem 4.1 and we get
\[ |\Omega_\lambda| \leq \mathcal{M}_n (|\Omega_0|, |\Omega_1|, \lambda) + \eta \beta^{\frac{n+1}{p}} \mathcal{H}_0(\Omega_0, \Omega_1)^{n+1}, \]
where $\eta$ is like in (44) or in Remark 4. Then we use Proposition 2.2 and, thanks to the definition of the constant $\beta$, we easily get a contradiction. \qed

Regarding Theorem 1.3 we notice that it can be proved precisely in the same way, using the quantitative version of the Brunn-Minkowski inequality proved by Figalli, Maggi and Pratelli, that is Proposition 2.3 in place of Proposition 2.2.

6. Some applications

In the following section we apply Theorem 4.1 to derive quantitative versions of some Urysohn inequalities for functionals that can be written in terms of the solution of a suitable elliptic boundary value problem. As a toy model, we take the torsional rigidity for which we carry out all the computations. However, as noticed in Remark 6 the same kind of quantitative results can be proved for a wide class of elliptic operators.

Let us recall the definition of the torsional rigidity $\tau(K)$ of a convex body $K$ given in (13):

$$\frac{1}{\tau(K)} = \inf \left\{ \frac{\int_K |Du|^2 \, dx}{\int_K |u| \, dx} : u \in W^{1,2}_0(\text{int}(K)), \int_K |u| \, dx < 0 \right\}.$$ 

Take $u$ the unique solution of

$$\begin{cases}
\triangle u = -2 & \text{in } \text{int}(K) \\
u = 0 & \text{on } \partial K.
\end{cases}$$

Then we have

$$\tau(K) = \int_K u \, dx.$$ 

We recall an useful geometric property satisfied by the solutions of problem (58) (see [25] and [24] for details):

**Proposition 6.1.** If $u$ is the solution to problem (58) then $u$ is $\frac{1}{2}$-concave, i.e. the function $v(x) = \sqrt{u(x)}$ is concave in $K$.

Finally we recall a comparison result for solutions of problem (58) in different domains (see [10] and [33] for details):

**Proposition 6.2.** Let $K_0, K_1$ be convex bodies, $\lambda \in [0, 1]$ and $K_\lambda = (1 - \lambda)K_0 + \lambda K_1$, Let $u_i$ be the solution of problem (58) in $K_i$, $i = 0, 1, \lambda$. Then

$$u_\lambda((1 - \lambda)x + \lambda y)^{\frac{1}{2}} \geq (1 - \lambda)u_0(x)^{\frac{1}{2}} + \lambda u_1(y)^{\frac{1}{2}} \quad \forall x \in K_0, y \in K_1.$$ 

The Brunn-Minkowski inequality for $\tau$, that is (14), essentially stems from (59) and the above propositions. Taking into account also Theorem 1.2 and Theorem 1.3 we get Theorem 1.4.

**Proof of Theorem 1.4.** Thanks to Proposition 6.1 and Proposition 6.2 it is possible to apply Theorem 1.2 and Theorem 1.3 with $p = 1/2$ and $h = u_\lambda$. Then it is easily seen that (6) and (8) precisely reads as (15) and (16) respectively, thanks to (59). \qed
6.1. An Urysohn inequality for torsional rigidity. As recalled in the Introduction, a proof of the following Theorem can be found in [33] in a more general setting. For a better understanding of our results, we give here an alternative proof.

**Theorem 6.3.** Let $\Omega$ be an open bounded convex set in $\mathbb{R}^n$ and let $\Omega^\sharp$ be the ball with the same mean-width of $\Omega$. Then it holds

\[
\tau(\Omega) \leq \tau(\Omega^\sharp)
\]

and equality holds if and only if $\Omega = \Omega^\sharp$.

**Proof.** Since $\tau$ is invariant under translations, we can translate $\Omega$ in a way that the point of Steiner $s$ of $\Omega$ coincides with the origin. We remind that the point of Steiner $s(\Omega)$ of a convex set $\Omega$ is defined as

\[
s(\Omega) = \frac{1}{\omega_n} \int_{S^{n-1}} \theta h(\Omega, \theta) \, d\mathcal{H}^{n-1}(\theta).
\]

Using Hadwiger’s Theorem (see [34]) there exists a sequence of rotations $\{\rho_k\}$ such that

\[
\Omega_k = \frac{1}{k} (\rho_1 \Omega + \cdots + \rho_k \Omega)
\]

converges, in the Hausdorff metric, to a ball.

We notice that $\Omega_k$ converges to $\Omega^\sharp$: in fact, since the mean-width is invariant under rigid motions and is additive under the Minkowski sum (see [34]), we get

\[
w(\Omega_k) = w(\Omega) = b
\]

for all $k$ and so

\[
w(\Omega^\sharp) = w(\Omega) = b.
\]

Moreover $s(\Omega_k) = 0$ for all $k$ for the same reason, and then $\Omega^\sharp$ is the ball with radius $r = \frac{b}{2}$ centered at 0.

Using (14) we get

\[
\tau(\Omega_k) \geq \tau(\Omega) \quad \text{for all } k > 0,
\]

since $\tau(\rho \Omega) = \tau(\Omega)$ for any rotation $\rho$.

Since $\Omega_k$ converges to $\Omega^\sharp$ in the Hausdorff metric when $k$ goes to infinity, there exists $m$ such that

\[
\Omega_k \subseteq B(0, r + \frac{1}{k})
\]

for all $k \geq m$. Then

\[
\tau(\Omega_k) \leq \tau(\Omega^\sharp)
\]

for all $k > 0$. Then

\[
\tau(\Omega_k) \leq \tau(\Omega_k) \leq \tau(\Omega^\sharp)
\]

for all $k > 0$ and thanks to the equality case in (14), we can conclude that $\Omega$ is a ball. \(\square\)
6.2. Proof of Theorem [1.5] and Theorem [1.6]. Let us prove Theorem [1.5]

Proof of Theorem [1.5]. Let \( \Omega_{\rho} \) be a rotation of \( \Omega \) with center in the centroid of \( \Omega \) and set
\[
\tilde{\Omega} = \frac{1}{2} \Omega + \frac{1}{2} \Omega_{\rho}.
\]
First notice that, since \( w(\tilde{\Omega}) = w(\Omega) \),
from the Urysohn inequality we get
\[
\tau(\tilde{\Omega}) \leq \tau(\Omega^\#).
\]
Then (15) gives
\[
(64) \quad \tau(\tilde{\Omega}) \geq \tau(\Omega) + \mu H(\Omega, \Omega_{\rho}) \frac{3n+1}{3}.
\]
where \( \mu \) is the constant defined in (19).
Then we just have to show that we can find a rotation \( \Omega_{\rho_0} \) of \( \Omega \) such that
\[
H(\Omega, \Omega_{\rho_0}) \geq H(\Omega, \Omega^\#)
\]
Notice that
\[
(65) \quad H(\Omega, \Omega^\#) = \max_{\theta \in S^{n-1}} \{|h_\Omega(\theta) - h_{\Omega^\#}(\theta)|\},
\]
where \( h_\Omega \) and \( h_{\Omega^\#} \) are the support functions of \( \Omega \) and \( \Omega^\# \) respectively.
Take \( \bar{\theta} \) such that the maximum in (65) is attained at \( \bar{\theta} \). We notice that, since \( \Omega^\# \) is a ball centered at the origin, we have
\[
h_{\Omega^\#}(\theta) = r \quad \forall \theta
\]
where \( r \) is the radius of \( \Omega^\# \), that is
\[
r = \frac{w(\Omega)}{2}.
\]
Then
\[
r = \frac{1}{n\omega_n} \int_{S^{n-1}} h_\Omega(\theta) d\theta
\]
and by the mean value Theorem and the continuity of \( h_\Omega \), there exists \( \theta_0 \) such that
\[
h_\Omega(\theta_0) = r.
\]
Let \( \Omega_{\rho_0} \) be a rotation of \( \Omega \) with center in the centroid of \( \Omega \) such that
\[
h_{\Omega_{\rho_0}}(\bar{\theta}) = h_\Omega(\theta_0).
\]
Then thanks to (65) we can conclude that
\[
H(\Omega, \Omega_{\rho_0}) \geq |h_\Omega(\bar{\theta}) - h_{\Omega_{\rho_0}}(\bar{\theta})| = H(\Omega, \Omega^\#),
\]
and we conclude the proof.

Theorem [1.6] can be proved precisely in the same way, using (16) in place of (15).

Remark 6. Let us denote by \( \Omega^* \) a ball with the same measure as \( \Omega \). Then we notice that (60) is weaker than the well known St Venant’s inequality (see [27])
\[
(66) \quad \tau(\Omega) \leq \tau(\Omega^*),
\]
since \( \tau \) is increasing with respect to inclusion and
\[
\Omega^* \subseteq \Omega^\#.
\]
by the classical Urysohn’s inequality between mean width and volume of convex sets. This is due to the fact that the Laplacian or other kind of operator written in divergence form works better under Schwarz symmetrization. Moreover, any quantitative version of (66) would imply immediately the same quantitative result for (17). However, to our knowledge, no quantitative version of (66) have been proved yet.

6.3. Further applications. Results like Theorem 1.4, Theorem 1.5 and Theorem 1.6 can be obtained for other elliptic operators. In particular, we could for instance derive similar results for the $p$-Laplacian, for the 2-Hessian operator in $\mathbb{R}^3$ and for the extremal Pucci’s operator $\mathcal{P}_{\Lambda_1,\Lambda_2}$; the corresponding Brunn-Minkowski inequalities, as well as the needed concavity and comparison results, similar to Proposition 6.1 and Proposition 6.2, can be explicitly found in or retrieved from [11], [32] and [5, 33], respectively.

7. The case $p \geq 1$

For $p = 1$, the function $u_{p,\lambda}$ defined by (27) simply coincides with the usual supremal convolution of $u_0$ and $u_1$, a classical operation in convex analysis, which in turn is easily seen to correspond to the Minkowski linear combination of the hypographs of $u_0$ and $u_1$ (see for instance [31, §3]). Then the corresponding BBL inequality is nothing more than a way to read the classical Brunn-Minkowski inequality for the hypographs of the involved functions and consequent stability or quantitative results for (2) immediately descends from the ones for the Brunn-Minkowski inequality.

For instance, consider two non-negative concave functions $u_0$ and $u_1$, with compact (convex) supports $\Omega_0$ and $\Omega_1$ respectively, such that

$$\int_{\Omega_0} u_0 \, dx = \int_{\Omega_1} u_1 \, dx = 1.$$  

Let $K_i = K_i^{(1)} \subseteq \mathbb{R}^{n+1}$ as defined by (30), for $i = 0, 1$; then (67) reads $|K_0| = |K_1| = 1$.

Let $\lambda$ and $h$ be as in Theorem 1.1 and assume (11) holds for some $p \geq 1$. Assume now to be close to equality in (2), precisely

$$\int_{\Omega_\lambda} h \, dx \leq 1 + \epsilon,$$

for some $\epsilon \geq 0$.

Set

$$D = \max \left\{ \sqrt{d_0^2 + \frac{(n + 1)^2}{|\Omega_0|^2}} ; \sqrt{d_1^2 + \frac{(n + 1)^2}{|\Omega_1|^2}} \right\}$$

and notice that $D$ is greater than or equal to the maximum between the diameters $d(K_0)$ and $d(K_1)$ of $K_0$ and $K_1$; indeed the concavity of $u_i$ and (67) ensure that

$$M_i = \sup_{\Omega_i} u_i \leq \frac{(n + 1)}{|\Omega_i|}$$

and it is easily seen that

$$d(K_i) \leq \sqrt{d_i^2 + M_i^2}.$$
Furthermore (1) and (28) imply $h \geq u_{p, \lambda} \geq u_{1, \lambda}$, whence
\[
\{(x, t) \in \mathbb{R}^{n+1} : x \in \Omega_{\lambda}, 0 \leq t \leq h(x)\} \supseteq K_{\lambda} = (1 - \lambda)K_0 + \lambda K_1.
\]
Then (68) gives
\[
|K_{\lambda}| \leq 1 + \epsilon
\]
and we can apply Theorem 1 of [19] to get
\[
H_0(K_0, K_1) \leq c_n \left( \frac{1}{\sqrt{\lambda(1 - \lambda)}} + 2 \right) D \epsilon^{1/(n+2)},
\]
where
\[
c_n = 3^{n/(n+1)} 2^{(n+3)/(n+2)} (n + 1)^{(n+1)/(n+2)} < 6(n + 1).
\]
This is an estimate of the Hausdorff distance between the graphs of the functions $u_0$ and $u_1$ which conveys the possibility to write an explicit $L^\infty$-stability result for the functions $u_0$ and $u_1$ when $p \geq 1$. Then we notice that, for a generic $p$, $u_{p, \lambda}$ in fact coincides with the $(1/p)$-power of the supremal convolution of $u^0_p$ and $u^1_p$ (let’s think to $p \neq 0$ for simplicity), hence we conjecture the possibility to apply again the known stability results for the Brunn-Minkowski inequality directly to the graphs of the involved functions in order to obtain some kind of strong (possibly $L^\infty$) stability for the BBL inequality for every $p$.

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