A NONLOCAL CONCAVE-CONVEX PROBLEM WITH NONLOCAL MIXED BOUNDARY DATA

BOUMEDIENE ABDELLAOUI
Laboratoire d’Analyse Nonlinéaire et Mathématiques Appliquées
Université Abou Bakr Belkaïd, Tlemcen, Tlemcen 13000, Algeria

ABDELRAZEK DIEB
Département de Mathématiques
Université Ibn Khaldoun, Tiaret, Tiaret 14000, Algeria

ENRICO VALDINOCI
University of Melbourne, School of Mathematics and Statistics
Peter Hall Building, Parkville, Melbourne VIC 3010, Australia
School of Mathematics and Statistics, 35 Stirling Highway
Crawley, Perth WA 6009, Australia
Dipartimento di Matematica, Università degli studi di Milano
Via Saldini 50, 20133 Milan, Italy
and
Istituto di Matematica Applicata e Tecnologie Informatiche,
Consiglio Nazionale delle Ricerche, Via Ferrata 1, 27100 Pavia, Italy

(Communicated by Juncheng Wei)

Abstract. The aim of this paper is to study the following problem

\[(P_\lambda) \equiv \begin{cases} (-\Delta)^s u = \lambda u^q + u^p \quad &\text{in } \Omega, \\ u > 0 \quad &\text{in } \Omega, \\ B_s u = 0 \quad &\text{in } \mathbb{R}^N \setminus \Omega, \end{cases}\]

with \(0 < q < 1 < p, N > 2s, \lambda > 0, \Omega \subset \mathbb{R}^N\) is a smooth bounded domain,

\((-\Delta)^s u(x) = a_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy,

\]

\(a_{N,s}\) is a normalizing constant, and \(B_s u = u\chi_{\Sigma_1} + N_s u\chi_{\Sigma_2}\). Here, \(\Sigma_1\) and \(\Sigma_2\) are open sets in \(\mathbb{R}^N \setminus \Omega\) such that \(\Sigma_1 \cap \Sigma_2 = \emptyset\) and \(\Sigma_1 \cup \Sigma_2 = \mathbb{R}^N \setminus \Omega\).

In this setting, \(N_s u\) can be seen as a Neumann condition of nonlocal type that is compatible with the probabilistic interpretation of the fractional Laplacian, as introduced in [20], and \(B_s u\) is a mixed Dirichlet-Neumann exterior datum. The main purpose of this work is to prove existence, nonexistence and multiplicity of positive energy solutions to problem \((P_\lambda)\) for suitable ranges of \(\lambda\) and \(p\) and to understand the interaction between the concave-convex nonlinearity and the Dirichlet-Neumann data.

2000 Mathematics Subject Classification. Primary: 35R11, 35A15, 35A16; Secondary: 35J61, 60G22.

Key words and phrases. Integro differential operators, fractional Laplacian, weak solutions, mixed boundary condition, multiplicity of positive solution.

The first author is supported by research grants MTM2013-40846-P and MTM2016-80474-P, MINECO, Spain.

* Corresponding author.
1. Introduction. In [20], the authors introduced a new nonlocal Neumann condition, which is compatible with the probabilistic interpretation of the nonlocal setting related to some Lévy process in \( \mathbb{R}^N \). Motivated by this, we aim in this work to study a semilinear nonlocal elliptic problem with mixed Dirichlet-Neumann data. More precisely, we study existence and multiplicity of positive solutions to the following problem

\[
(P_{\lambda}) \equiv \begin{cases} 
(\Delta)^s u = \lambda u^q + u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
B_s u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

with \( 0 < q < 1 < p, N > 2s, \lambda > 0. \)

In our setting, \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain and \( (\Delta)^s \) is the fractional Laplacian operator, defined as

\[
(\Delta)^s u(x) = a_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy.
\]

See e.g. [23, 24, 17] and the references therein for more information about this operator. In this framework \( a_{N,s} > 0 \) is a suitable normalization constant and the exterior condition

\[
B_s u = u \chi_{\Sigma_1} + N_s u \chi_{\Sigma_2},
\]

can be seen as a nonlocal version of the classical Dirichlet-Neumann mixed boundary condition. As a matter of fact, here \( N_s \) is the non-local normal derivative introduced in [20], given by

\[
N_s u(x) = a_{N,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N \setminus \Omega.
\]

Also, \( \Sigma_1 \) and \( \Sigma_2 \) are open sets in \( \mathbb{R}^N \setminus \Omega \) such that \( \Sigma_1 \cap \Sigma_2 = \emptyset \) and \( \Sigma_1 \cup \Sigma_2 = \mathbb{R}^N \setminus \Omega. \)

As customary, in (2) we denoted by \( \chi_A \) the characteristic function of a set \( A. \)

We observe that, differently from the case of homogeneous Dirichlet conditions, the case of Neumann and mixed boundary conditions has not been much investigated in the fractional setting. This is due to the fact that the classical Neumann condition combines good geometrical properties (e.g. the normal derivative of the function vanishes, allowing symmetry and blow-up arguments) and analytic properties, while in the nonlocal case the consequences of (3) are much less intuitive and harder to deal with. This is indeed probably the first article devoted to the analysis of a nonlinear and nonlocal problem with mixed exterior data that involve the Neumann condition of [20]. We notice that recently a Hopf Lemma has been proved in [8] for such mixed exterior conditions.

Using an integration by parts formula stated in [20], one sees that problem \( (P_{\lambda}) \) can be set in a variational setting, since the requested solutions can be seen as critical points of the functional

\[
J_{\lambda}(u) = \frac{1}{2} \int_{D_{\Omega}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \frac{\lambda}{q + 1} \|u_+\|_{q+1}^{q+1} - \frac{1}{p+1} \|u_+\|_{p+1}^{p+1},
\]

where

\[
D_{\Omega} = (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Omega^c \times \Omega^c),
\]

\[
\|v\|_r^r = \int_{\Omega} |v|^r dx \quad \text{and} \quad u_+ = \max(u, 0).
\]
Such problem, in the local case of the classical Laplacian, was extensively studied in the literature, especially after the seminal work of Ambrosetti, Brezis and Cerami [3]. Similar problems with a Dirichlet-Neumann datum were studied, for the subcritical case, in [15] and, in the critical case, in [22].

In the nonlocal framework, (that is, when $s \in (0, 1)$), with Dirichlet data, the problem was dealt with in [9] for the subcritical case and in [7, 18, 19] and [11] for the critical case. See also [28, 29].

In [7] and [19], the authors use an extension method, introduced in [13], which allows them to reduce the problem to a local one. We stress that, in our case, because of the nonlocal Neumann part, we cannot use such extension and then we deal with the problem in an appropriate purely nonlocal, and somehow more general, framework. Moreover, to obtain our multiplicity result, we have to use an additional argument which was classically developed by Alama in [1].

For a series of motivations about nonlocal equations and fractional operators, see e.g. [12] and the references therein.

Our main results are the following:

**Theorem 1.1.** Let $0 < s < 1$, $0 < q < 1 < p$. Then there exists $\Lambda > 0$, such that:
1. For all $\lambda \in (0, \Lambda)$, problem $(P_\lambda)$ has a minimal solution $u_\lambda$ such that $J_\lambda(u_\lambda) < 0$. Moreover, these solutions are ordered, namely, if $\lambda_1 < \lambda_2$ then $u_{\lambda_1} < u_{\lambda_2}$.
2. If $\lambda > \Lambda$, problem $(P_\lambda)$ has no positive weak solutions.
3. If $\lambda = \Lambda$, problem $(P_\lambda)$ has at least one weak positive solution.

**Theorem 1.2.** For all $0 < s < 1$, $0 < q < 1 < p < \frac{N+2s}{N-2s}$, $\lambda \in (0, \Lambda)$, problem $(P_\lambda)$ has a second solution $v_\lambda > u_\lambda$.

The paper is organized as follows: In Section 2, we introduce the functional setting to deal with problem $(P_\lambda)$, as well as the notion of solution we will work with and some auxiliary results. Section 3 is devoted to prove the existence of minimal and extremal solutions. Finally in Section 4 we prove the existence of a second solution using Alama’s argument.

2. **Preliminaries and functional setting.** We introduce in this section a natural functional framework for our problem and we give some related properties and some useful embedding results needed when we deal with problem $(P_\lambda)$. According to the definition of the fractional Laplacian, see [17], [28], and the integration by parts formula, see [20], it is natural to introduce the following spaces. We denote by $H^s(\mathbb{R}^N)$ the classical fractional Sobolev space,

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{2} + s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}, \quad (5)$$

endowed with the norm

$$||u||^2_{H^s(\mathbb{R}^N)} = ||u||^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy. \quad (6)$$

It is clear that $H^s(\mathbb{R}^N)$ is a Hilbert space.

We recall now the classical Sobolev inequality that the proof can be found in [17]. See also [25] for an elementary proof.
Proposition 1. Let $s \in (0,1)$ with $N > 2s$. There exists a positive constant $S = S(N,s)$ such that, for any function $u \in H^s(\mathbb{R}^N)$, we have

$$S \|u\|^2_{L^{2_*}(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy,$$

where $2_*= \frac{2N}{N-2s}$.

Definition 2.1. Let $\Omega$ be a bounded domain of $\mathbb{R}^N$. For $0 < s < 1$, we define the space

$$\mathbb{H}^s(\Omega, \Sigma_1) = \{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \Sigma_1 \}.$$

It is clear that $\mathbb{H}^s(\Omega, \Sigma_1)$ is a Hilbert space endowed with the norm induced by $H^s(\mathbb{R}^N)$.

For $u \in \mathbb{H}^s(\Omega, \Sigma_1)$, we set

$$||u||_1^2 = a_{N,s} \int_{D_0} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.$$

The properties of this norm are described by the following result.

Proposition 2. The norm $||.||$ in $\mathbb{H}^s(\Omega, \Sigma_1)$ is equivalent to the one induced by $H^s(\mathbb{R}^N)$, and then $(\mathbb{H}^s(\Omega, \Sigma_1), ( , ))$ is a Hilbert space with scalar product given by

$$\langle u, v \rangle = a_{N,s} \int_{D_0} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy.$$

Proof. For $u \in \mathbb{H}^s(\Omega, \Sigma_1)$, we set

$$||u||_1^2 = ||u||^2_{L^2(\Omega)} + a_{N,s} \int_{D_0} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.$$

It is clear that $\mathbb{H}^s(\Omega, \Sigma_1)$ is a Hilbert space with the associated scalar product given by

$$\langle u, v \rangle_1 = a_{N,s} \int_{D_0} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy + \int_{\Omega} uv \, dx.$$

Notice that the completeness of $\mathbb{H}^s(\Omega, \Sigma_1)$ can be proved using exactly the same argument as in the proof of Proposition 3.1 in [20].

Now, setting

$$\lambda_1(\Omega) = \inf_{(\phi \in \mathbb{H}^s(\Omega, \Sigma_1), \phi \neq 0)} \frac{\int_{D_0} \frac{\phi(x) - \phi(y)^2}{|x-y|^{N+2s}} \, dx \, dy}{\int_{\Omega} \phi^2 \, dx},$$

then the authors in [8] proved that $\lambda_1(\Omega) > 0$. As a consequence, the previous scalar product can be reduced to the following one

$$\langle u, v \rangle = a_{N,s} \int_{D_0} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy.$$

Hence we can endowed $\mathbb{H}^s(\Omega, \Sigma_1)$ with the Gagliardo norm

$$||u||_1^2 = a_{N,s} \int_{D_0} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.$$

Now, the norm $||.||$ in $\mathbb{H}^s(\Omega, \Sigma_1)$ is bounded by the one induced by $H^s(\mathbb{R}^N)$, and so, using the Open Mapping Theorem, it holds that the norm $||.||$ in $\mathbb{H}^s(\Omega, \Sigma_1)$ is in fact equivalent to the one induced by $H^s(\mathbb{R}^N)$. Hence the result follows. □
The following result justifies our choice of $\| \cdot \|$.

**Proposition 3.** Let $s \in (0, 1)$, for all $u, v \in H^s(\Omega, \Sigma)$ we have,

$$
\int_{\Omega} v(-\Delta)^s u \, dx = \frac{a_{N,s}}{2} \int_{\partial\Omega} (u(x) - u(y))(v(x) - v(y)) \frac{dx \, dy}{|x - y|^{N+2s}} - \int_{\Sigma_2} v N_s u \, dx.
$$

The proof of this result is a direct application of the integration by parts formula, see Lemma 3.3 in [20].

In the rest of the paper, for the simplicity of typing, we shall denote the functional space introduced in Definition 2.1 by $H^s$ and we shall normalize the constant $a_{N,s}$ to be equal to 2.

Now we give a Sobolev-type result for functions in $H^s$.

**Corollary 1.** Suppose that $s \in (0, 1)$ and $N > 2s$. There exists a positive constant $C = C(N, s, \Omega, \Sigma_2)$ such that, for any function $u \in H^s$,

$$
\|u\|_{L^r(\Omega)}^2 \leq C \|u\|^2,
$$

for all $1 \leq r \leq 2^*$.\[\]

**Proof.** Since $u \in H^s \subset H^s(\mathbb{R}^N)$, then using the Sobolev inequality in (1), it holds that

$$
S\|u\|_{L^2^*(\Omega)}^2 \leq S\|u\|_{L^2^*(\mathbb{R}^N)}^2 \leq \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.
$$

Now, the result follows using Hölder inequality and Proposition 2. \[\]

Consider now the standard truncation functions given by

$$
T_k(u) = \max \{ -k, \min\{k, u\} \}
$$

and $G_k(u) = u - T_k(u)$. In this setting, the following are some useful properties of $H^s$-functions which are needed to get some regularity results for some elliptic problems in $H^s$ (see also Theorem 2.5 below).

**Proposition 4.** Let $u$ be a function in $H^s$, then

1. If $\Phi \in \text{Lip}(\mathbb{R})$ is such that $\Phi(0) = 0$, then $\Phi(u) \in H^s$. In particular for any $k > 0$, $T_k(u), G_k(u) \in H^s$.

2. For any $k \geq 0$

$$
\|G_k(u)\|^2 \leq \int_{\Omega} G_k(u)(-\Delta)^s u \, dx + \int_{\Sigma_2} G_k(u) N_s u \, dx.
$$

3. For any $k \geq 0$

$$
\|T_k(u)\|^2 \leq \int_{\Omega} T_k(u)(-\Delta)^s u \, dx + \int_{\Sigma_2} T_k(u) N_s u \, dx.
$$

**Proof.** The claim in (1) follows from the setting of the norm given in Definition 2.1. As for (2) and (3), we claim that, for any $a, b \geq 0$ and any $x \in \mathbb{R}^N$,

$$
a (G_k(u)(-\Delta)^s T_k(u))(x) + b (G_k(u) N_s T_k(u))(x) \geq 0.
$$

To check this, we can take $x \in \{G_k(u) \neq 0\}$, otherwise (8) is obvious. Then, if $x \in \{G_k(u) > 0\}$ we have that $T_k(u)(x) = k$, which is the maximum value that $T_k(u)$ attains, and therefore $(-\Delta)^s T_k(u)(x) \geq 0$ and $N_s T_k(u)(x) \geq 0$.

Conversely, if $x \in \{G_k(u) < 0\}$ we have that $T_k(u)(x) = -k$, which is the minimum value that $T_k(u)$ attains, and therefore $(-\Delta)^s T_k(u)(x) \leq 0$ and $N_s T_k(u)(x) \leq -k$.\[\]
0. By combining these observations, we obtain (8). From (8) and Proposition 3 it follows that
\[
\int_{\Omega} T_k(u)(-\Delta)^s G_k(u) \, dx + \int_{\Sigma_2} T_k(u) \mathcal{N}_s G_k(u) \, dx
= \int_{\Omega} G_k(u)(-\Delta)^s T_k(u) \, dx + \int_{\Sigma_2} G_k(u) \mathcal{N}_s T_k(u) \, dx \geq 0. \tag{9}
\]
Using the normalization condition and by Propositions 3, we reach that
\[
\|G_k(u)\|^2 = \int_{D_\Omega} \frac{(G_k(u)(x) - G_k(u)(y))^2}{|x-y|^{N+2s}} \, dx \, dy
= \int_{\Omega} G_k(u)(-\Delta)^s G_k(u) \, dx + \int_{\Sigma_2} G_k(u) \mathcal{N}_s G_k(u) \, dx \tag{10}
= \int_{\Omega} G_k(u)(-\Delta)^s (u - T_k(u)) \, dx + \int_{\Sigma_2} G_k(u) \mathcal{N}_s (u - T_k(u)) \, dx.
\]
In a similar way,
\[
\|T_k(u)\|^2 = \int_{\Omega} T_k(u)(-\Delta)^s (u - G_k(u)) \, dx + \int_{\Sigma_2} T_k(u) \mathcal{N}_s (u - G_k(u)) \, dx. \tag{11}
\]
Then, the claim in (2) follows from (10) and (9), while the claim in (3) follows from (11) and (9).

Let us now consider the following problem,
\[
\begin{cases}
(-\Delta)^s u = f & \text{in } \Omega, \\
\mathcal{B}_s u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \tag{12}
\]
where \(\Omega\) is a bounded regular domain of \(\mathbb{R}^N\), \(N > 2s\), \(\mathbb{H}^{-s}\) is the dual space of \(\mathbb{H}^s\) and \(f \in \mathbb{H}^{-s}\).

**Definition 2.2.** We say that \(u \in \mathbb{H}^s\) is an energy solution to (12) if
\[
\int_{D_\Omega} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy = (f, \varphi) \quad \forall \varphi \in \mathbb{H}^s, \tag{13}
\]
where \((, )\) represent the duality between \(\mathbb{H}^s\) and \(\mathbb{H}^{-s}\).

Notice that the existence and uniqueness of energy solutions to problem (12) follow from the Lax-Milgram Theorem. Furthermore if \(f \geq 0\) then \(u \geq 0\). Indeed for \(u \in \mathbb{H}^s\), thanks to Lemma 4, we know that \(u_- = \min\{u,0\} \in \mathbb{H}^s\). Taking \(u_-\) as a test function in (13) it follows that \(u_- = 0\).

A supersolution (respectively, subsolution) is a function that verifies (13) with equality replaced by “\(\geq\)” (respectively, “\(\leq\)” ) for every non-negative test function in \(\mathbb{H}^s\). Using a standard iterative argument we can easily prove the following result.

**Lemma 2.3.** Assume that problem (12) has a subsolution \(\underline{w}\) and a supersolution \(\overline{w}\), verifying \(\underline{w} \leq \overline{w}\). Then there exists a solution \(w\) satisfying \(\underline{w} \leq w \leq \overline{w}\).

Here we prove some regularity results when \(f\) satisfies some minimal integrability conditions. To prove the boundedness of the solution we follow the idea of Stampacchia for second order elliptic equations with bounded coefficients. The interior Hölder regularity is a consequence of continuity properties, see [20], and the regularity results in [30].
Lemma 2.4. Let $u$ be a solution to problem (12). If $f \in L^q(\Omega)$, $q > \frac{N}{2s}$, then $u \in L^\infty(\Omega)$.

Proof. We follow here a related argument presented in [24]. See also [30] and [18] for related results. Let $k > 0$ and take $\varphi = G_k(u)$ as a test function in (13). Hence, thanks to Proposition 4, we get

$$
\|G_k(u)\|^2 \leq \int_{A_k} G_k(u)f \, dx + \int_{\Sigma_2} G_k(u)N_s u \, dx,
$$

where $A_k = \{ x \in \Omega : u > k \}$. Recalling (12), we obtain

$$
\|G_k(u)\|^2 \leq \int_{A_k} G_k(u)f \, dx.
$$

Applying Corollary 1 in the left hand side and Hölder inequality in the right hand side, we obtain

$$
S^2 \|G_k(u)\|^2_{L^{2^*_s}(\Omega)} \leq \|f\|_{L^m(\Omega)} \|G_k(u)\|_{L^{2^*_s}(\Omega)} |A_k|^{1 - \frac{1}{2^*_s} - \frac{1}{m}}
$$

we have that,

$$
S^2 \|G_k(u)\|^2_{L^{2^*_s}(\Omega)} \leq \|f\|_{L^m(\Omega)} |A_k|^{1 - \frac{1}{2^*_s} - \frac{1}{m}}
$$

thus,

$$
S^2 (h - k) |A_h|^\frac{1}{2^*_s} \leq \|f\|_{L^m(\Omega)} |A_k|^{1 - \frac{1}{2^*_s} - \frac{1}{m}}
$$

and then,

$$
|A_h| \leq S^{2^*_s - 2} \|f\|_{L^m(\Omega)}^2 |A_k|^{2^*_s \left(1 - \frac{1}{2^*_s} - \frac{1}{m}\right)} (h - k)^{2^*_s}. \tag{14}
$$

Since $m > \frac{N}{2s}$ we have that

$$
2^*_s \left(1 - \frac{1}{2^*_s} - \frac{1}{m}\right) > 1.
$$

Hence we apply Lemma 14 in [24] with $\psi(\sigma) = |A_\sigma|$ and the result follows. \hfill \Box

Corollary 2. Let $u$ be an energy solution of (12) and suppose that $f \in L^\infty(\Omega)$. Then $u \in C^\gamma(\Omega)$ for some $\gamma \in (0, 1)$.

Proof. We claim that $u$ is bounded in $\mathbb{R}^N$. Then one could apply interior regularity results for the solutions to $(-\Delta)^s u = 0 \in \Omega$ and $u = g \text{ in } \Omega^c$. See e.g. [30] and [26].

To check the claim, recalling Lemma 2.4, we have to consider only the case $x \in \Sigma_2$. Then, by (3)

$$
u(x) = c(N, s)^{-1} \int_{\Omega} \frac{u(y)}{|x - y|^{N + 2s}} \, dy, \text{ where } c(N, s) = \int_{\Omega} \frac{1}{|x - y|^{N + 2s}} \, dy.
$$

Hence,

$$
|u(x)| \leq \|u\|_{L^\infty(\Omega)} \text{ for all } x \in \Sigma_2. \tag{14}
$$

Also, if $\Sigma_2$ is unbounded, using Proposition 3.13 in [20], we have

$$
\lim_{x \to \infty, x \in \Sigma_2} u(x) = \frac{1}{|\Omega|} \int_{\Omega} u(y) \, dy. \tag{15}
$$

Then the claim follows from Lemma 2.4, inequalities (14) and (15). \hfill \Box
As a variation of Lemma 2.4, we point out that if \( f = f(x,u) \) and \( f \) has the following growth

\[
|f(x,s)| \leq c(1 + |s|^p) \text{ where } p \leq \frac{N + 2s}{N - 2s},
\]

then, using a Moser iterative scheme, we can prove that:

**Theorem 2.5.** Let \( u \) be an energy solution to problem (12) with \( f \) satisfies (16), then \( u \in L^\infty(\Omega) \).

The following is a strong maximum principle for semi-linear equations, it will be used to separate minimal solution of problem \( (P_{\lambda}) \) for different values of the parameter \( \lambda \), see [16].

**Proposition 5.** Let \( N \geq 1, 0 < s < 1 \) and let \( f_1, f_2 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) be two continuous functions. Let \( \Omega \) be a domain in \( \mathbb{R}^N \) and \( v, w \in L^\infty(\mathbb{R}^N) \cap C^{2s}\gamma}(\Omega) \), for some \( \gamma > 0 \), be such that

\[
\begin{align*}
(\Delta)^s v \geq f_1(x,v) & \quad \text{in } \Omega, \\
(\Delta)^s w \leq f_2(x,w) & \quad \text{in } \Omega, \\
v \geq w & \quad \text{in } \mathbb{R}^N.
\end{align*}
\]

Suppose furthermore that

\[
f_2(x,w(x)) \leq f_1(x,w(x)) \text{ for any } x \in \Omega.
\]

If there exists a point \( x_0 \in \Omega \) at which \( v(x_0) = w(x_0) \), then \( v = w \) in the whole \( \Omega \).

**Proof.** Let \( \phi = v - w \) and set

\[ Z_\phi = \{ x \in \Omega : \phi(x) = 0 \}. \]

By assumption \( x_0 \in Z_\phi \). Moreover, thanks to the continuity of \( \phi \), we know that \( Z_\phi \) is closed. We claim now that \( Z_\phi \) is also open. Indeed, let \( \bar{x} \in Z_\phi \). Clearly \( \phi \geq 0 \) in \( \mathbb{R}^N \), \( \phi(\bar{x}) = 0 \) and

\[
(\Delta)^s \phi(\bar{x}) \geq f_1(\bar{x},v(\bar{x})) - f_2(\bar{x},w(\bar{x})) = f_1(\bar{x},w(\bar{x})) - f_2(\bar{x},w(\bar{x})) \geq 0,
\]

in view of (17). Accordingly,

\[
0 \leq (\Delta)^s \phi(\bar{x}) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{2\phi(\bar{x}) - \phi(\bar{x} + z) - \phi(\bar{x} - z)}{|z|^{N+2s}} \, dz
= \frac{1}{2} \int_{\mathbb{R}^N} \frac{-\phi(\bar{x} + z) - \phi(\bar{x} - z)}{|z|^{N+2s}} \, dz \leq 0.
\]

Hence \( \phi \) vanishes identically in \( B_\epsilon(\bar{x}) \) and then, for \( \epsilon \) small, \( B_\epsilon(\bar{x}) \subseteq Z_\phi \). That is, we have proved that \( Z_\phi \) is open, and so, by the connectedness of \( \Omega \), we get that \( Z_\phi = \Omega \).

Now we establish two important results for our purposes. The first result is a Picone-type inequality and the second is a Brezis-Kamin comparison principle for concave nonlinearities.

**Theorem 2.6.** Consider \( u, v \in H^s \), suppose that \( (\Delta)^s u \geq 0 \) is a bounded Radon measure in \( \Omega \), \( u \geq 0 \) and not identically zero, then,

\[
\int_{\Sigma_2} \frac{|v|^2}{u} N_s u \, dx + \int_{\Omega} \frac{|v|^2}{u} (\Delta)^s u \, dx \leq \int_{\Omega} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \, dx \, dy.
\]
The proof of this result is based on a punctual inequality and follows in the same way as in [24]. As a consequence, we have the next comparison principle that extends to the fractional framework the classical one obtained by Brezis and Kamin, see [10].

**Lemma 2.7.** Let \( f(x,\sigma) \) be a Carathéodory function such that \( \frac{f(x,\sigma)}{\sigma} \) is decreasing in \( \sigma \), uniformly with respect to \( x \in \Omega \). Suppose that \( u, v \in H^s \), with \( 0 < s < 1 \), are such that
\[
\begin{align*}
(-\Delta)^su & \geq f(x,u), \quad u > 0 \quad \text{in } \Omega, \\
(-\Delta)^sv & \leq f(x,v), \quad v > 0 \quad \text{in } \Omega.
\end{align*}
\]
Then \( u \geq v \) in \( \Omega \).

The proof of this result is a slight modification of the proof of Theorem 20 in [24]. Finally, we will use the following compactness lemma to get strong convergence in the space \( H^s \).

**Lemma 2.8.** Let \( \{v_n\}_n \) be a sequence of non-negative functions such that \( \{v_n\}_n \) is bounded in \( H^s \), \( v_n \rightharpoonup v \) in \( H^s \) and \( v_n \leq v \). Assume that \( (-\Delta)^sv_n \geq 0 \) then, \( v_n \to v \) strongly in \( H^s \).

**Proof.** Since \( v_n \leq v \), then using the fact that \( (-\Delta)^sv_n \geq 0 \), it follows that
\[
\int_{\Omega} (-\Delta)^sv_n(v-v_n)\,dx \geq 0.
\]
Hence
\[
\int_{\Omega} (-\Delta)^sv_n\,dx \geq \int_{\Omega} (-\Delta)^sv_nv\,dx.
\]
Now, using Young’s inequality, we obtain that
\[
\int_{\Omega} \int_{\Omega} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy \leq \int_{\Omega} \int_{\Omega} \frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}\,dx\,dy.
\]
Thus
\[
\limsup_{n \to \infty} \|v_n\| \leq \|v\|.
\]
Since
\[
\limsup_{n \to \infty} \|v_n - v\|^2 = \limsup_{n \to \infty} (\|v_n\|^2 + \|v\|^2 - 2\langle v_n, v \rangle) \leq 2\|v\|^2 - 2 \limsup_{n \to \infty} \langle v_n, v \rangle,
\]
taking into consideration that \( v_n \rightharpoonup v \) in \( H^s \), we get
\[
\limsup_{n \to \infty} \|v_n - v\|^2 = 0.
\]
As a consequence, \( v_n \to v \) strongly in \( H^s \). 

3. **Proof of Theorem 1.1.** In this section we prove Theorem 1.1. We split the proof into several auxiliary Lemmas. Let us begin by proving an existence result.

**Lemma 3.1.** Assume that \( 0 < q < 1 < p \), then problem \((P_\lambda)\) has a nontrivial bounded solution at least for \( \lambda > 0 \) small.
Proof. The main idea is to show that for $\lambda$ small, the problem $(P_\lambda)$ has a comparable bounded sub and supersolution. Let $\mathcal{V}$ be the unique positive solution to the problem

\[
\begin{cases}
(-\Delta)^s \mathcal{V} = 1 & \text{in } \Omega, \\
\mathcal{V} > 0 & \text{in } \Omega, \\
\mathcal{B}_s \mathcal{V} = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Notice that the existence of $\mathcal{V}$ follows by using the Lax-Milgram theorem in the space $H^s$, however the positivity of $\mathcal{V}$ follows from [8]. It is clear that $\mathcal{V} \in C^{\alpha}(\bar{\Omega})$ for some $\alpha < 1$. Let $C = \|\mathcal{V}\|_\infty$, it is not difficult to show the existence of $\lambda^* > 0$ such that for all $\lambda < \lambda^*$, the inequality

\[ M \geq \lambda M^q C^q + M^p C^p, \]

has a solution $M > 0$. Fix $\lambda, M$ as above and define $v_1 = MV$, then $v_1$ solves

\[
\begin{cases}
(-\Delta)^s v_1 = M \geq \lambda v_1^q + v_1^p & \text{in } \Omega, \\
v_1 > 0 & \text{in } \Omega, \\
\mathcal{B}_s v_1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Thus $v_1$ is a supersolution to problem $(P_\lambda)$.

We consider now the following problem

\[
\begin{cases}
(-\Delta)^s z = z^q & \text{in } \Omega, \\
z > 0 & \text{in } \Omega, \\
\mathcal{B}_s z = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Since $q \in (0,1)$, then setting

\[ M = \min \left\{ \frac{1}{2} \|w\|^2 - \frac{\lambda}{q+1} \int_\Omega w^{q+1} dx, \ w \in H^s \right\}, \]

it follows that $M$ is achieved by a minimizer $z$. It is clear that $z \geq 0$, then by Proposition 5 and Lemma 2.7, it follows that $z > 0$ and it is unique. In particular, $z$ is the solution to problem (19). By Theorem 2.5, it holds that $z \in L^\infty(\Omega)$.

Now setting $z_\lambda = \lambda^{-\frac{1}{q-1}} z$, then $z_\lambda$ is a solution to

\[
\begin{cases}
(-\Delta)^s z_\lambda = \lambda z_\lambda^q & \text{in } \Omega, \\
\mathcal{B}_s z_\lambda = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

By the comparison result in Lemma 2.7, it holds that $z_\lambda \leq v_1$. It is clear that $z_\lambda$ is a subsolution to problem $(P_\lambda)$. Hence a monotonicity argument allows us to get the existence of a solution $u_\lambda$ to problem $(P_\lambda)$ with $z_\lambda \leq u_\lambda \leq v_1$.

\[ \square \]

Lemma 3.2. Let $\Lambda$ be defined by

\[ \Lambda = \sup \{ \lambda > 0 : \text{ problem } (P_\lambda) \text{ has a solution } \} . \]

Then $0 < \Lambda < \infty$.

Proof. By Lemma 3.1, we reach that $\Lambda > 0$.

We show now that $\Lambda < \infty$. Let $\lambda$ be such that problem $(P_\lambda)$ has a solution $\bar{u}_\lambda$.

By the comparison principle in Lemma 2.7, we get $z_\lambda \leq \bar{u}_\lambda$ where $z_\lambda$ is the unique positive solution to problem (20). Let $\phi \in H^s$, then using Picone’s inequality we obtain that

\[ \int \int_{D_{\Omega}} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+2s}} \ dx \ dy \]
\[
\geq \int_\Omega \partial^2 (-\Delta) u_{\lambda} \, dx \geq \int_\Omega \partial^2 (\lambda u_{\lambda}^{p-1} - u_{\lambda}^{p-1}) \, dx
\]
\[
\geq \int_\Omega \lambda z^{p-1} \partial^2 \phi \, dx \geq \lambda^{\frac{1}{p-1}} \int_\Omega \lambda z^{p-1} \partial^2 \phi \, dx.
\]
Hence
\[
\lambda^{\frac{1}{p-1}} \leq \inf_{\phi \in H^s} \left( \int \int_{\Omega} \left( \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}} \right) \, dx \, dy \right) \int \Omega \lambda z^{p-1} \partial^2 \phi \, dx = \Lambda^*.
\]
Consequently, \( \Lambda \leq (\Lambda^*)^{\frac{p-2}{p-1}} < \infty \). This gives point (2) in Theorem 1.1. \( \square \)

We show now that for all \( 0 < \lambda < \Lambda \), problem \((P_\lambda)\) has a solution. This will be a consequence of the following lemma.

**Lemma 3.3.** Let
\[
S = \{ \lambda > 0 : \text{problem } (P_\lambda) \text{ has a solution} \}.
\] (21)
Then \( S \) is an interval.

**Proof.** Notice that \( S \neq \emptyset \), thanks to Lemma 3.1. Let \( \lambda_1 \in S \) be fixed, we have just to prove that for all \( 0 < \lambda_2 < \lambda_1 \), problem \((P_{\lambda_2})\) has a non trivial solution.

Since \( \lambda_1 \in S \), then we get the existence of \( u_1 \in H^s \) such that \( u_1 \) solves \((P_{\lambda_2})\). It is clear that \( u_1 \) is a supersolution to problem \((P_{\lambda_2})\). Recall that \( z \) is the unique solution to problem \((19)\). Setting \( z_2 = \lambda_2^\frac{1}{p-1} z \), then \( z_2 \) solves
\[
\begin{cases}
(-\Delta)^s z_2 = \lambda_2 z_2^p & \text{in } \Omega, \\
B_s z_2 = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]
By the comparison principle in Lemma 2.7, it holds that \( z_2 \leq u_1 \).

Since \( z_2 \) is a subsolution to problem \((P_{\lambda_2})\), then using a monotonicity argument we get the existence of \( u_2 \in H^s \) such that \( u_2 \leq u_1 \) and \( u_2 \) solves problem \((P_{\lambda_2})\). Thus \( \lambda_2 \in S \) and the result follows. \( \square \)

We now prove that \((P_\lambda)\) possesses a minimal solution and we give some energy properties of such solutions.

**Lemma 3.4.** For all \( 0 < \lambda < \Lambda \), problem \((P_\lambda)\) has a minimal solution \( u_\lambda \) such that \( J_\lambda(u_\lambda) < 0 \). Moreover the family \( u_\lambda \) of minimal solutions is increasing with respect to \( \lambda \).

**Proof.** Suppose that \((P_\lambda)\) has a solution \( v_\lambda \) for a given \( \lambda \in S \). Define the sequence \( v_n \) by \( v_0 = z_\lambda \),
\[
\begin{cases}
(-\Delta)^s v_n = \lambda v_n^{p-1} + v_n^{p-1} & \text{in } \Omega, \\
v_n \geq 0 & \text{in } \Omega, \\
B_s v_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\] (22)
where \( z_\lambda \) is the unique solution to problem \((20)\). By the comparison result in Lemma 2.7, we have that \( z_\lambda \leq ... \leq v_{n-1} \leq v_n \leq v_\lambda \) and then, by Proposition 5, it follows that \( z_\lambda < v_n < v_\lambda \).

So, using \( v_n \) as a test function in \((22)\), we get \( \|v_n\| \leq \|v_\lambda\| \). Hence there exists \( u_\lambda \in H^s \) such that \( v_n \rightharpoonup u_\lambda \). Accordingly, since \((-\Delta)^s v_n \geq 0\), using Lemma 2.8, we conclude that \( v_n \rightarrow u_\lambda \) strongly in \( H^s \) and \( u_\lambda \leq v_\lambda \). This shows that \( u_\lambda \) is a minimal solution.
Then, by Lemma 2.7 and Proposition 5, we obtain the monotonicity of the family \( \{ u_\lambda, \lambda \in (0, \Lambda) \} \).

Henceforth, given \( \lambda \in (0, \Lambda) \), we use the notation \( u_\lambda \) for the minimal solution. Let us define \( a(x) = \lambda q u_\lambda^{q-1} + p u_\lambda^{p-1} \) and let \( \mu_1 \) be the first eigenvalue of the following problem

\[
\begin{cases}
( -\Delta )^s \phi - a(x) \phi &= \mu_1 \phi \quad \text{in } \Omega, \\
\phi &> 0 \quad \text{in } \Omega, \\
B_s \phi &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\] (23)

Using closely the same argument as in the proof of Lemma 3.5 in [3], we can prove that

\[ \mu_1 \geq 0. \] (24)

It is clear that (24) is equivalent to

\[ \| \phi \|^2 \geq \int_\Omega a(x) \phi^2 \, dx \quad \forall \phi \in H^s. \] (25)

Since \( u_\lambda \) is a solution to \((P_\lambda)\), testing the equation against \( u_\lambda \) itself, we find that

\[ \| u_\lambda \|^2 = \lambda \| u_\lambda \|_{q+1}^{q+1} + \| u_\lambda \|_{p+1}^{p+1}. \] (26)

By (25), it follows that

\[ \| u_\lambda \|^2 - \lambda q \| u_\lambda \|_{q+1}^{q+1} - p \| u_\lambda \|_{p+1}^{p+1} \geq 0. \] (27)

By inserting these relations into (4), we obtain that \( J_\lambda(u_\lambda) < 0 \), as desired.

This gives point (1) in Theorem 1.1. Thus, to complete the proof of Theorem 1.1, we can now focus on the proof of point (3). To this end, we have the following result:

**Lemma 3.5.** Problem \((P_\lambda)\) has at least one solution if \( \lambda = \Lambda \).

**Proof.** Let \( \{ \lambda_n \} \) be a sequence such that \( \lambda_n \nearrow \Lambda \). We denote by \( u_n \equiv u_{\lambda_n} \) the minimal solution to problem \((P_{\lambda_n})\), then the sequence \( \{ u_n \} \) is increasing in \( n \). Since \( J_{\lambda_n}(u_n) < 0 \), we get

\[
0 > J_\lambda(u_n) - \frac{1}{p+1} J'_\lambda(u_n)
\]

\[
\geq (\frac{1}{2} - \frac{1}{p+1}) \| u_n \|^2 + \lambda \left( \frac{1}{p+1} - \frac{1}{q+1} \right) \| u_n \|_{q+1}^{q+1}
\]

\[
\geq (\frac{1}{2} - \frac{1}{p+1}) \| u_n \|^2 - \lambda \left( \frac{1}{q+1} - \frac{1}{p+1} \right) \| u_n \|_{q+1}^{q+1}.
\]

Then, it follows that \( \{ u_n \} \) is bounded in \( H^s \). Accordingly, we have that \( u_n \rightharpoonup u^* \) in \( H^s \), for some \( u^* \in H^s \). Since \( \{ u_n \} \) is increasing in \( n \), using the fact that \(( -\Delta )^s u_n \geq 0\), recalling Lemma 2.8, we conclude that \( u_n \rightarrow u^* \) strongly in \( H^s \). As a consequence, \( u^* \) is a solution of \((P_\lambda)\) for \( \lambda = \Lambda \).

**Remark 1.** If \( p \leq 2^*_s - 1 \) then using Theorem 2.5, we can easily prove that \( u^* \in L^\infty(\Omega) \), that means that \( u^* \) is a regular extremal solution.

In view of Lemma 3.5, we obtain point (3) of Theorem 1.1. The proof of Theorem 1.1 is thus complete.
4. Proof of Theorem 1.2. In this section we prove the existence of a second positive solution to \((P_\lambda)\).

Since \(p < \frac{N+2}{2s} \), we observe that problem \((P_\lambda)\) has a variational structure, indeed it is the Euler-Lagrange equation of the energy functional in (4). We note that \(J_\lambda\) is well defined, it is differentiable on \(H^s\) and for any \(\varphi \in H^s\),

\[
(J'_\lambda(u), \varphi) = \langle u, \varphi \rangle - \lambda \int_\Omega |u|^q \varphi dx - \int_\Omega |u|^p \varphi dx.
\]

Thus critical points of the functional \(J_\lambda\) are solutions to \((P_\lambda)\).

To prove Theorem 1.2, we will use a mountain pass-type argument. The proof goes as follows. As in the local case, we can prove that the problem has a second positive solution for \(\lambda\) small. This follows using the mountain pass theorem. For this purpose it is essential to have a first solution which is a local minimum in \(H^s\).

Let

\[
f_\lambda(r) = \begin{cases} 
\lambda r^q + r^p, & \text{if } r \geq 0, \\
0, & \text{if } r < 0,
\end{cases}
\]

and

\[
F_\lambda(u) = \int_0^u f_\lambda(r) \, dr.
\]

We define the functional \(J_\lambda(u) = \frac{1}{2} \|u\|^2 - \int_\Omega F_\lambda(u).
\)

Critical points of \(J_\lambda\) correspond to solutions of \((P_\lambda)\). Define the set

\[A = \{\lambda > 0 : J_\lambda \text{ has a local minimum } u_{0,\lambda}\}.
\]

It is clear that if \(\lambda \in A\) and \(w_\lambda\) is a minimum of \(J_\lambda\) in \(H^s\), then \(v = 0\) is a local minimum of the functional

\[
J_\lambda(v) = \frac{1}{2} \|v\|^2 - \int_\Omega G_\lambda(v) \, dx,
\]

where

\[
G_\lambda(v) = \int_0^v g_\lambda(r) \, dr
\]

and

\[
g_\lambda(r) = \begin{cases} 
\lambda ((u_{0,\lambda}(x) + r)^q - u_{0,\lambda}(x)^q) + (u_{0,\lambda}(x) + r)^p - u_{0,\lambda}(x)^p, & \text{if } r \geq 0, \\
0, & \text{if } r < 0,
\end{cases}
\]

We can see that \(J_\lambda\) possesses the mountain pass geometry. Thus, let \(v_0 \in H^s\) be such that \(J_\lambda(v_0) < 0\) and define

\[\Gamma = \{\gamma : [0, 1] \to H^s : \gamma(0) = 0, \gamma(1) = v_0\} \text{ and } c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda (\gamma(t))\).
\]

We have that \(c \geq 0\) and since \(p < 2^*_s - 1\), then \(J_\lambda\) satisfies the Palais-Smale condition. If \(c > 0\), then using the Ambrosetti-Rabinowitz theorem we reach a non trivial critical point. If \(c = 0\), then we use the Ghoussoub-Preiss Theorem, see [21].

As a consequence if we start with a local minimum of the functional \(J_\lambda\), then we obtain a second critical point of \(J_\lambda\), and hence a second solution to \((P_\lambda)\).

Next, to show that problem \((P_\lambda)\) has a second solution for all \(\lambda \in (0, \Lambda)\), we follow some arguments similar to those developed by Alama in [1] taking into consideration the nonlocal nature of the operator.
We prove first, using a variational formulation of the Perron’s method, that the functional has a constrained minimum and then that this minimum is a local minimum in the whole $\mathbb{H}^n$. To this end, we use a truncation technique and some energy estimates.

Fix $\lambda_0 \in (0, \Lambda)$ and let $\lambda < \bar{\lambda} < \Lambda$. Define $u_0, \bar{u}$ to be the minimal solutions to problem $(P_\lambda)$ with $\lambda = \lambda_0$ and $\lambda = \bar{\lambda}$ respectively. By definition we obtain that $u_0 < \bar{u}$. Let us define

$$M = \{ u \in \mathbb{H}^n : 0 \leq u \leq \bar{u} \}.$$ 

It is clear that $u_0 \in M$ and that $M$ is a convex closed subset of $\mathbb{H}^n$. Since $J_{\lambda_0}$ is bounded from below in $M$ and lower semi-continuous, then we get the existence of $\vartheta \in M$ such that

$$J_{\lambda_0}(\vartheta) = \inf_{u \in M} J_{\lambda_0}(u).$$

Let $v$ be the unique solution to

$$\begin{cases}
(\Delta)u = \lambda_0 u^q & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
B_u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}$$

We have that $J_{\lambda_0}(v) < 0$, and then $\vartheta \neq 0$. As in Theorem 2.4 in [32], page 17, we conclude that $\vartheta$ is a solution to problem $(P_\lambda)$.

If $\vartheta \neq u_0$, then the proof of Theorem 1.2 is complete. Accordingly, we can assume that $\vartheta = u_0$. We show that

$$\vartheta$$

is a local minimum of $J_{\lambda_0}$. \hfill (29)

For this, we argue by contradiction, and we assume that $\vartheta$ is not a local minimum of $J_{\lambda_0}$. Then there exists a sequence $\{v_n\} \subset \mathbb{H}^n$ such that $\|v_n - \vartheta\|_{\mathbb{H}^s} \to 0$ as $n \to \infty$ and

$$J_{\lambda_0}(v_n) < J_{\lambda_0}(\vartheta).$$

We define $w_n = (v_n - \bar{u})_+$ and $u_n = \max\{0, \min\{v_n, \bar{u}\}\}$. It is clear that $u_n \in M$ and

$$u_n(x) = \begin{cases}
0 & \text{if } v_n(x) \leq 0, \\
v_n(x) & \text{if } 0 \leq v_n(x) \leq \bar{u}(x), \\
\bar{u}(x) & \text{if } \bar{u}(x) \leq v_n(x).
\end{cases}$$

Thus $u_n = v_n^+ - w_n$. Let $T_n = \{ x \in \Omega : u_n(x) = v_n(x) \}$ and $S_n = \text{supp } w_n \cap \Omega$. Notice that $\text{supp } v_n^+ \cap \Omega = T_n \cup S_n$. We claim that

$$|S_n| \to 0 \text{ as } n \to \infty.$$

To this end, let $\epsilon > 0$,

$$E_n = \{ x \in \Omega : v_n(x) \geq \bar{u}(x) > \vartheta(x) + \delta \}$$

and

$$F_n = \{ x \in \Omega : v_n(x) \geq \bar{u}(x) \text{ and } \bar{u}(x) \leq \vartheta(x) + \delta \},$$

where $\delta$ has to be suitably chosen. Since

$$0 = \| \{ x \in \Omega : \bar{u}(x) < \vartheta(x) \} \| = \left\| \bigcap_{j=1}^{\infty} \left\{ x \in \Omega : \bar{u}(x) \leq \vartheta(x) + \frac{1}{j} \right\} \right\|$$

and

$$\lim_{j \to \infty} \left\| \left\{ x \in \Omega : \bar{u}(x) \leq \vartheta(x) + \frac{1}{j} \right\} \right\|,$$
then we get the existence of a suitable $\delta_0 = \frac{1}{n_0}$ such that if $\delta < \delta_0$, then
\[ |\{ x \in \Omega : \bar{u}(x) \leq \vartheta(x) + \delta \}| \leq \frac{\epsilon}{2}. \]
Thus $|F_n| \leq \frac{\epsilon}{2}$. Since $|u_n - v_0|_{L^2(\Omega)} \to 0$ as $n \to \infty$, we get that for $\eta = \frac{\delta^2}{2}$, if $n \geq n_0$, we have that
\[ \frac{\delta^2 \epsilon}{2} \geq \int_{\Omega} |v_n - \vartheta|^2 dx \geq \int_{E_n} |v_n - \vartheta|^2 dx \geq \delta^2 |E_n|. \]
Hence $|E_n| \leq \frac{\varepsilon}{2}$. Since $S_n \subset F_n \cup E_n$, we conclude that $|S_n| \leq \epsilon$ for $n \leq n_0$ and then the claim in (31) follows.

Now we define
\[ H(u) = \lambda_0 \frac{u^q+1}{q+1} + \frac{u^{p+1}}{p+1}. \]
Using the fact that
\[ \|v_n\|^2 \geq \|v_n^+\|^2 + \|v_n^-\|^2, \]
we obtain that
\[ J_{\lambda_0}(v_n) = \frac{1}{2} \|v_n\|^2 - \int_{\Omega} H(v_n) dx \]
\[ \geq \frac{1}{2} \|v_n^+\|^2 - \int_{\Omega} H(v_n) dx + \frac{1}{2} \|v_n^-\|^2 \]
\[ = \frac{1}{2} \|v_n^+\|^2 - \int_{\Gamma_n} H(u_n) dx - \int_{S_n} H(v_n) dx + \frac{1}{2} \|v_n^-\|^2 \]
\[ = \frac{1}{2} \|v_n^+\|^2 - \int_{\Gamma_n} H(u_n) dx - \int_{S_n} H(w_n + \bar{u}) dx + \frac{1}{2} \|v_n^-\|^2 \]
\[ = J_{\lambda_0}(u_n) + \frac{1}{2} \left(\|v_n^+\|^2 - \|u_n\|^2\right) + \frac{1}{2} \|v_n^-\|^2 - \int_{S_n} \left( H(w_n + \bar{u}) - H(\bar{u}) \right) dx, \]
where we have used the fact that
\[ \int_{\Omega} H(u_n) dx = \int_{\Gamma_n} H(u_n) dx + \int_{S_n} H(\bar{u}) dx. \]
Also, since $v_n^+ = u_n + w_n$, then
\[ \frac{1}{2} \left(\|v_n^+\|^2 - \|u_n\|^2\right) = \frac{1}{2} \|w_n\|^2 + \langle u_n, w_n \rangle. \]
Using that
\[ \{ w_n \neq 0 \} = \{ u_n = \bar{u} \}, \]
we see that
\[ \langle u_n, w_n \rangle \geq \int_{\Omega} (-\Delta)^s \bar{u} w_n dx \geq \lambda \int_{S_n} \bar{u}^q w_n dx + \int_{S_n} \bar{u}^p w_n dx. \]
Therefore, recalling that $\bar{u}$ is a supersolution to problem $(P_\lambda)$ for $\lambda = \lambda_0$, we conclude that
\[ J_{\lambda_0}(v_n) \geq J_{\lambda_0}(\bar{u}) + \frac{1}{2} \|w_n\|^2 + \frac{1}{2} \|v_n^-\|^2 \]
\[ - \int_{S_n} \left\{ H(w_n + \bar{u}) - H(\bar{u}) - \lambda_0 \bar{u}^q w_n - \bar{u}^p w_n \right\} dx. \]
Taking into account that
\[ 0 \leq \frac{1}{q+1}(w_n + \bar{u})^{q+1} - \frac{1}{q+1}\bar{u}^{q+1} - \bar{u}^q w_n \leq \frac{q}{2} \frac{w_n^2}{\bar{u}^{q-1}}, \]
and using the Picone inequality in Theorem 2.6, we find that
\[ \lambda \int_{\Omega} \frac{w_n^2}{\bar{u}^{q-1}} dx \leq \int_{\Omega} \frac{w_n^2}{\bar{u}} (-\Delta)^q \bar{u} \leq ||w_n||^2. \]

Then, we obtain that
\[ \lambda_0 \int_{\Omega} \left\{ \frac{1}{p+1} (w_n + \bar{u})^{p+1} - \frac{1}{p+1} \bar{u}^{p+1} - \bar{u}^p w_n \right\} dx \leq \frac{q}{2} \int_{\Omega} \frac{w_n^2}{\bar{u}^{q-1}} dx \leq \frac{q}{2} ||w_n||^2. \]

Moreover, since \( 2 \leq p + 1, \)
\[ 0 \leq \frac{1}{p+1}(w_n + \bar{u})^{p+1} - \frac{1}{p+1}\bar{u}^{p+1} - \bar{u}^p w_n \leq \frac{p}{2} w_n^2 (w_n + \bar{u})^{p-1} \leq C (\bar{u}^{p-1} w_n^2 + w_n^{p+1}). \]

Hence, using the Sobolev inequality and the fact that \(|S_n| \to 0\) as \( n \to \infty, \) we reach that
\[ \int_{\Omega} \left\{ \frac{1}{p+1} (w_n + \bar{u})^{p+1} - \frac{1}{p+1} \bar{u}^{p+1} - \bar{u}^p w_n \right\} dx \leq o(1)||w_n||^2. \]

Hence
\[ J_{\lambda_0}(v_n) \geq J_{\lambda_0}(\emptyset) + \frac{1}{2} ||w_n||^2 (1 - q - o(1)) + \frac{1}{2} ||v_n^-||^2 \]
\[ \geq J_{\lambda_0}(\emptyset) + \frac{1}{2} ||w_n||^2 (1 - q - o(1)) + o(1). \]

So we get that
\[ 0 > J_{\lambda_0}(v_n) - J_{\lambda_0}(\emptyset) \geq \frac{1}{2} ||w_n||^2 (1 - q - o(1)) + \frac{1}{2} ||v_n^-||^2. \]

Since \( q < 1, \) we conclude that \( w_n = v_n^- = 0 \) for \( n \) large, so \( v_n \in M \) and then
\[ J_{\lambda_0}(v_n) \geq J_{\lambda_0}(\emptyset), \]
which is in contradiction with (30).

This completes the proof of (29). From this, we have that \( \emptyset \) is a local minimum for \( J_{\lambda_0}, \) and \( J_{\lambda_0} \) has \( u = 0 \) as a local minimum and then \( \hat{J}_{\lambda_0} \) has a nontrivial critical point \( \hat{u}. \) As a consequence, \( u = \emptyset + \hat{u} \) is a solution, different from \( \emptyset, \) of problem (P\(_{\lambda}\)). This concludes the proof of Theorem 1.2.

**Remark 2.** If we consider the odd symmetric version of problem (P\(_{\lambda}\)), namely,
\[
\begin{cases}
(-\Delta)^s u = \lambda |u|^{q-1} u + |u|^{p-1} u & \text{in } \Omega, \\
B_s u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
(32)
the associated functional
\[ I_{\lambda}(u) = \frac{1}{2} ||u||^2 - \frac{\lambda}{q+1} ||u||^{q+1}_{q+1} - \frac{1}{p} ||u||^{p+1}_{p+1} \]
is even. Then, for \( p < \frac{N+2s}{N+2q}, \) by using the Lusternik-Schnirelman min-max argument, it is possible to prove that problem (32) has infinitely many solutions with negative energy, see [3] and [6], and following closely the arguments in [4], [3] the same holds for solutions with positive energy.
Acknowledgements. The authors would like to express their gratitude to the anonymous referee for his/her comments and suggestions that improve the last version of the manuscript. Part of this work was carried out while the second author was visiting the Weierstraß-Institut für Angewandte Analysis und Stochastik in Berlin. He thanks the institute for the warm hospitality.

REFERENCES

[1] S. Alama, Semilinear elliptic equation with sublinear indefinite nonlinearities, *Adv. Differential Equation*, 4 (1999), 813–842.

[2] A. Ambrosetti, Critical points and nonlinear variational problems, *Mem. Soc. Math. France (N.S.)*, 49 (1992), 1–139.

[3] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.*, 122 (1994), 519–543.

[4] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, 14 (1973), 349–381.

[5] D. Applebaum, *Lévy Processes and Stochastic Calculus*, 2nd edition, Cambridge Studies in Advanced Mathematics, vol. 116, Cambridge University Press, 2009.

[6] J. G. Azorero and I. Peral, Multiplicity of solutions for elliptic problems with critical exponent or with a non-symmetric term, *Trans. Am. Math. Soc.*, 323 (1991), 877–895.

[7] B. Barrios, E. Colorado, R. Servadei and F. Soria, A critical fractional equation with concave-convex power nonlinearities, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32 (2015), 875–900.

[8] B. Barrios and M. Medina, Strong maximum principles for fractional elliptic and parabolic problems with mixed boundary conditions, *arXiv:1607.01505*.

[9] B. Barrios, M. Medina and I. Peral, Some remarks on the solvability of non-local elliptic problems with the Hardy potential, *Commun. Contemp. Math.*, 16 (2014), 1350046, 29 pp.

[10] H. Brezis and S. Kamin, Sublinear elliptic equations in *R*^N*, *Manuscripta Math.*, 74 (1992), 87–106.

[11] C. Bucur and M. Medina, A fractional elliptic problem in *R*^N* with critical growth and convex nonlinearities, *arXiv:1609.01911*.

[12] C. Bucur and E. Valdinoci, *Nonlocal Diffusion and Applications*, Lecture Notes of the Unione Matematica Italiana, 20. Springer; Unione Matematica Italiana, Bologna, 2016.

[13] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations*, 32 (2007), 1245–1260.

[14] L. Caffarelli and L. Silvestre, Regularity results for nonlocal equations by approximation, *Arch. Ration. Mech. Anal.*, 200 (2011), 59–88.

[15] E. Colorado and I. Peral, Semilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions, *J. Funct. Anal.*, 199 (2003), 468–507.

[16] M. Cozzi, *Qualitative Properties of Solutions of Nonlinear Anisotropic PDEs in Local and Nonlocal Settings*, PhD thesis, 2015.

[17] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, 136 (2012), 521–573.

[18] S. Dipierro, M. Medina, I. Peral and E. Valdinoci, Bifurcation results for a fractional elliptic equation with critica exponent in *R*^N*, *Manuscripta Math.*, 153 (2017), no. 1-2, 183–230.

[19] S. Dipierro, M. Medina and E. Valdinoci, Fractional elliptic problems with critical growth in the whole of *R*^N*, *Appunts. Scuola Normale Superiore di Pisa (Nuova Serie)* [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, 2017.

[20] S. Dipierro, X. Ros-Oton and E. Valdinoci, Nonlocal problems with Neumann boundary conditions, *Rev. Mat. Iberoam.*, 33 (2017), 377–416.

[21] N. Ghoussoub and D. Preiss, A general mountain pass principle for locating and classifying critical points, *Ann. Inst. H. Poincaré Anal. Nonlinéaire*, 6 (1989), 321–330.

[22] M. Grossi and F. Pacella, Positive solutions of nonlinear elliptic equations with critical Sobolev exponent and mixed boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A.*, 116 (1990), 23–43.

[23] N. S. Landkof, *Foundations of Modern Potential Theory*, Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180, Springer-Verlag.
[24] T. Leonori, I. Peral, A. Primo and F. Soria, Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations, Discrete Contin. Dyn. Syst., 35 (2015), 6031–6068.

[25] A. C. Ponce, Elliptic PDEs, Measures and Capacities, Tracts in Mathematics 23, European Mathematical Society (EMS), Zurich, 2016.

[26] X. Ros-Oton, Nonlocal elliptic equations in bounded domains: A survey, Publ. Mat., 60 (2016), 3–26.

[27] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, J. Math. Pures Appl., 101 (2014), 275–302.

[28] R. Servadei and E. Valdinoci, Mountain pass solutions for non-local elliptic operators, J. Math. Anal. Appl, 389 (2012), 887–898.

[29] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, Discrete Contin. Dyn. Syst., 33 (2013), 2105–2137.

[30] R. Servadei and E. Valdinoci, Weak and viscosity solutions of the fractional Laplace equation, Publ. Mat., 58 (2014), 133–154.

[31] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble), 15 (1965), 189–258.

[32] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Ergeb. Math. Grenzgeb. (3), Springer-Verlag, Berlin Heidelberg, 1990.

Received November 2016; revised October 2017.

E-mail address: boumediene.abdellaoui@inv.uam.es
E-mail address: dieb.d@yahoo.fr
E-mail address: enrico@mat.uniroma3.it