SLICING, THREADING & PARAMETRIC MANIFOLDS

Stuart Boersma
Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA ¹
boersma@math.orst.edu

Tevian Dray
Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA
tevian@math.orst.edu

ABSTRACT

We present a unified treatment of the slicing (3+1) and threading (1+3) decompositions of spacetime in terms of foliations. It is well-known how to decompose the metric and connection in the slicing picture; this is at the heart of any initial-value problem in general relativity. We describe here the analogous problem in the threading picture, recovering the recent results of Perjés on parametric manifolds.

¹ Present address: Division of Mathematics and Computer Science, Alfred University, Alfred, NY 14802
1. Introduction

The main purpose of this paper is to present a unified treatment of the well-known slicing or ADM formalism with its less well-known dual formalism, which we call threading. In the slicing viewpoint, spacetime is foliated with spacelike hypersurfaces; in the threading viewpoint spacetime is foliated with timelike curves. Slicing and threading are equivalent in the special case where both foliations exist and are orthogonal to each other; the interesting case is when the curves are not hypersurface-orthogonal. The slicing viewpoint corresponds to a global time-synchronization, whereas the threading viewpoint treats a family of observers as fundamental. We show here how both slicing and threading can be obtained from a more general decomposition of the metric.

In the slicing viewpoint, one can regard tensor fields as time-dependent tensors on a particular hypersurface. One can then study the intrinsic and extrinsic geometry of the surface, for instance when formulating an initial value problem. This is not as simple from the threading viewpoint due to the likely absence of any hypersurfaces orthogonal to the given curves. One is thus naturally led to parametric manifolds, in which one considers time-dependent tensors on the manifold of orbits of the threading curves [1]. This is equivalent to using a projected geometric structure on a hypersurface of constant time, rather than the intrinsic geometry such a surface inherits from spacetime. This parametric viewpoint is closely related to Kaluza-Klein theories. Both can be viewed as projections of a higher dimensional geometry into a lower dimension. This perspective allows us to give the theory of parametric manifolds a solid mathematical foundation. Central to this discussion is a new, torsion-like quantity, the deficiency, which measures whether the given family of observers is hypersurface orthogonal.

We begin with a review of slicing in Section 2. Although this material is well-known, we give a detailed presentation of two different interpretations of slicing in order to set the stage for the completely analogous presentation of threading in Section 3. In Section 4 we discuss the general problem of decomposing the metric on a fibre bundle into metrics on the base space and on the fibres, and we then show how to recover both slicing and threading as special cases. An overview of parametric manifolds is given in Section 5, and a discussion of possible applications appears in Section 6.

2. Slicing

The (3+1)-decomposition of a 4-dimensional spacetime is the standard framework for formulating the dynamics of geometry (cf. [2]). There exist two standard approaches to such a splitting, both of which yield the standard definitions of lapse and shift, one being a construction process and the other a decomposition process. For the construction, one begins with 3-dimensional surfaces and attempts to “fill in” between these surfaces to construct a spacetime which admits the original 3-dimensional surfaces as a foliation of spacelike hypersurfaces. The spacetime metric is thus constructed out of the 3-metric of the hypersurfaces as well as additional bits of information. Alternatively, one could start with a spacetime which admits a 1-parameter family of spacelike hypersurfaces and then

\[\text{As discussed in more detail below, the threading metric corresponds to the distance between nearby observers, as measured orthogonally to the observers. This is in general different from the distance as measured on a hypersurface of constant time.}\]
decompose all of the original 4-dimensional geometrical information (e.g. tensor fields) into two pieces, one tangent to the surfaces and one normal to the surfaces. As we shall see, both approaches yield an equivalent (3+1)-interpretation of spacetime.

Throughout the next two sections we will be working with complete spacetimes foliated by spacelike hypersurfaces. We will further assume that the hypersurfaces are all diffeomorphic to each other. Thus, one may simplify the notation by working in extremely nice coordinate neighborhoods. Begin by introducing a global time function \( t \) which can be regarded as the parameter which labels the hypersurfaces. We will work in a neighborhood small enough so that the intersection of each hypersurface with the neighborhood admits coordinates \( \{ x^i \} = \{ x^1, x^2, x^3 \} \) on the hypersurface \( \Sigma_t \). Thus, \( p \in \Sigma_t \) can be given the coordinates \( (t, x^i) \).

To simplify the notation, we will use Greek letters as indices which take on all four spacetime dimensions with \( x^0 \equiv t \). Thus, \( \{ x^\alpha \} = \{ t, x^i \} \) and we can write the spacetime metric \( g \) in terms of its components given by \( ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \).

**a) The Construction**

Suppose one had a spacetime foliated by a 1-parameter family of spacelike hypersurfaces, \( \Sigma_t \). One would like to realize the 4-geometry of this spacetime as arising from the 3-geometries of these surfaces. Thus, one would like to “construct” the spacetime metric out of the spatial metrics of the surfaces. Of course, additional information must also be provided. Following the description in [2], let us assume that the 3-geometry of two infinitesimally close surfaces is known. Label these surfaces by \( \Sigma_t \) and \( \Sigma_{t+\Delta t} \). Each of these surfaces has an associated spatial metric, \( k_t \) and \( k_{t+\Delta t} \). At the risk of de-emphasizing the dependence on the coordinate \( t \), we will use the same notation to refer to both spatial metrics and write \( k_{ij} dx^i dx^j \) for the 3-metrics on both surfaces.

We now describe the 4-geometry that fills in between these slices. Given a point \( p_0 = (x^i, t) \in \Sigma_t \) and a nearby point \( q_0 = (x^i + \Delta x^i, t + \Delta t) \in \Sigma_{t+\Delta t} \), we are interested in calculating the coordinate distance \( d(p_0, q_0) \) between \( p_0 \) and \( q_0 \). By taking advantage of the existence of a metric in the slice \( \Sigma_t \), it seems most natural to use the Pythagorean Theorem of Lorentzian geometry and write

\[
d(p_0, q_0)^2 = d(p_0, p_1)^2 - d(p_1, q_0)^2
\]

where \( p_1 \in \Sigma_t \) is chosen so that \( d(p_1, q_0) \) is the orthogonal distance between the two hypersurfaces. See Figure 1.

![Figure 1: Calculating Distances](image)

It is now apparent that in order to fully construct a spacetime metric much information must still be specified. As we have no *a priori* knowledge of what it means to move...
orthogonally to the surfaces, the location of $p_1$ is not fully determined. As $\Delta t$ approaches zero, the point $q_0$ should approach $p_1$. However, there is no reason to assume that the coordinates of $p_1$ are $(x^i + \Delta x^i, t)$. Rather, the point $p_1$ could be “shifted” in any of the three spatial directions. Thus, we assign the coordinates $p_1 = (x^i + \Delta x^i + N^i \Delta t, t)$. The three functions $N^i$ depend on the coordinates of $\Sigma_t$ as well as the parameter $t$. As the functions $N^i$ describe how the nearby surfaces are shifted with respect to one another, they are commonly referred to as the components of the *shift vector*. The given metric $k_{ij}$ may now be used to measure $d(p_0, p_1)$.

The quantity $d(p_1, q_0)$ is still not determined. In order to fix this distance, one must know the relationship between the proper time from $\Sigma_t$ to $\Sigma_{t+\Delta t}$ and the arbitrary parameter $t$. Again, this distance may depend on the coordinates in $\Sigma_t$ as well as $t$. Define the *lapse function* $N$ by

$$d(p_1, q_0) = N(x^i, t) \Delta t.$$  

One may now describe the 4-geometry in terms of the lapse function and shift vector. Adding these newly defined quantities to Figure 1 yields the picture shown in Figure 2.

![Figure 2: Slicing Lapse Function and Shift Vector](image)

Using equation (1) and letting $\Delta t \to 0$ we see that the 4-geometry of the spacetime can be represented by the line element

$$ds^2 = k_{ij} (dx^i + N^i dt)(dx^j + N^j dt) - N^2 dt$$

$$= (N_i N^i - N^2) dt^2 + 2 N_i dt dx^i + k_{ij} dx^i dx^j$$

where we have used the 3-metric $k_{ij}$ to define $N_i = k_{ij} N^j$. As mentioned earlier, the functions $N^i$ are thought of as the component functions of a vector field “tangent” to each hypersurface. The *shift vector field* is defined by

$$N^i \frac{\partial}{\partial x^i}.$$  

Thus each hypersurface has a spatial metric, $k_{ij}$, a tangent vector field, $N^i \frac{\partial}{\partial x^i}$, and a function $N$. As we have seen, these three spatial quantities may be used to construct the 4-dimensional spacetime metric $g$.

In matrix notation, one can write the components of the spacetime metric tensor in terms of $N$, $N^i$, and $k_{ij}$ as follows:

$$(g_{\alpha\beta}) = \begin{pmatrix} - (N^2 - N_m N^m) & N_j \\ N_i & k_{ij} \end{pmatrix}$$
with inverse

\[
(g^{\alpha\beta}) = \begin{pmatrix}
-N^{-2} & N^{-2}N^j \\
N^{-2}N^i & k^{ij} - N^{-2}N^iN^j
\end{pmatrix}
\]

where \(k^{ij}\) is the inverse of \(k_{ij}\) defined by

\[k_{il}k^{lj} = \delta_i^j.\]

Our parameter \(t\) now takes the role of a spacetime coordinate whose coordinate vector field \(\frac{\partial}{\partial t}\) may be interpreted as representing the “flow of time” in the newly constructed spacetime. Since the coordinates \(x^i\) are constant along integral curves of \(\frac{\partial}{\partial t}\), one may think of the lapse and shift as the means of identifying points on different hypersurfaces. See Figure 3.

The shift vector field \(N^i \frac{\partial}{\partial x^i}\) was defined to account for the fact that there was no \(a\ priori\) knowledge of directions orthogonal to the surfaces \(\Sigma_t\). However, in terms of the shift vector and lapse function we may now easily describe a future pointing unit spacetime vector field normal to each surface. Call this normal vector field \(n\). Using \(\langle \cdot, \cdot \rangle\) to represent the 4-metric we just constructed, we observe that \(\langle Ndt, Ndt \rangle = -1\). Therefore, \(n\) is given by the metric-dual of the 1-form \(Ndt\). Explicitly,

\[
n = \frac{1}{N} \frac{\partial}{\partial t} - \frac{1}{N} N^i \frac{\partial}{\partial x^i}.\]  (3)

It is now apparent how the functions \(N^i\) describe the “shifting” of neighboring surfaces. The case of vanishing shift corresponds to the scenario where coordinate time is flowing orthogonally to the hypersurfaces (\(i.e.\) the directions of \(n\) and \(\frac{\partial}{\partial t}\) agree).

b) The Decomposition

As stated earlier, the above construction is simply an orthogonal splitting of spacetime geared towards an initial value formulation of spacetime. To see the obvious, begin as above with a spacetime which admits a foliation of spacelike hypersurfaces. If we let \(n^\alpha\) be the components of the future pointing unit vector field \(n\) normal to the hypersurfaces \(\Sigma_t\), the naturally induced metric on each hypersurface may be obtained from the projection tensor

\[
k_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta\]  (4)
where \( n_\alpha = g_{\alpha \beta} n^\beta \) (cf. [3]) For vector fields \( X = X^\alpha \frac{\partial}{\partial x^\alpha} \) and \( Y = Y^\beta \frac{\partial}{\partial x^\beta} \) tangent to \( \Sigma_t \),

\[
k_{\alpha \beta} X^\alpha Y^\beta = g_{\alpha \beta} X^\alpha Y^\beta = g_{ij} X^i Y^j.
\]

the functions \( k_{ij} = g_{ij} \) may thus be thought of as the components of a 3-dimensional metric on each hypersurface. One must not lose sight of the fact that the functions \( k_{ij} \) depend on the spacetime coordinate \( t \) (as do the \( g_{\alpha \beta} \)). We will refer to the functions \( k_{ij} \) as the components of the slicing metric.

The slicing metric on \( \Sigma_t \) is the naturally induced metric in the following sense: for each imbedding \( \iota_t : \Sigma_t \hookrightarrow \mathcal{M}, \ k = \iota_t^*(g) \), where \( \iota_t^* \) refers to the natural map on the cotangent spaces \( T^*\mathcal{M} \) and \( T^*\Sigma_t \). Thus, \( k \) is both the projection of \( g \) to \( \Sigma \) via (4) and the pullback of \( g \) to \( \Sigma \). Using \( \iota_{t*} \) to denote the natural map between the tangent spaces, we may work out the relationship explicitly. For tangent vector fields \( X \) and \( Y \) on \( \Sigma_t \) we can write

\[
X = X^i \frac{\partial}{\partial x^i}, \ Y = Y^i \frac{\partial}{\partial x^i}, \text{ and }
\]

\[
k_{ij} X^i Y^j = \iota_{t*}(g)_{ij} X^i Y^j = g_{\alpha \beta} (\iota_{t*}(X))^\alpha (\iota_{t*}(Y))^\beta = g_{ij} X^i Y^j.
\]

One may decompose the coordinate vector field \( \frac{\partial}{\partial t} \) into vector fields normal and tangent to each surface \( \Sigma_t \). Thus one has

\[
\frac{\partial}{\partial t} = N n + N^i \frac{\partial}{\partial x^i}.
\]

Equation (5) defines the slicing lapse function \( N \) and the slicing shift vector field \( N^i \frac{\partial}{\partial x^i} \) and can be seen to agree with the earlier definitions by comparing equation (5) with (3). In this scenario, the shift vector measures the tilting of \( \frac{\partial}{\partial t} \) away from the direction normal to the hypersurfaces.

Since \( N^i \frac{\partial}{\partial x^i} \) is tangent to each hypersurface, we use the slicing metric to define \( N_i = k_{ij} N^j \).

3. Threading

In the (3+1)-decomposition (or slicing) of spacetime, one has a foliation of spacetime by spacelike hypersurfaces labeled by a global time function \( t \). This time function together with the earlier definitions of the lapse function and shift vector gives one a way of identifying points on different hypersurfaces. In effect, one has, in addition to a foliation of spacetime by hypersurfaces, a congruence of curves given by the integral curves of the coordinate vector field \( \frac{\partial}{\partial t} \). While the spacelike nature of the hypersurfaces is an integral part of the standard (3+1)-decomposition, there are no similar causality conditions on the congruence of curves. Although we usually think of the parameter \( t \) as a local time coordinate, no formal causality restriction is necessary. When one adopts the dual ansatz of a foliation of spacetime by timelike curves together with a foliation of hypersurfaces (with
no causality conditions imposed upon them), one is led to consider a (1+3)-decomposition (or threading) of spacetime (see [4]).

In such a setting, the timelike congruence may be interpreted as the world-lines of a family of observers, while the hypersurfaces play the fundamental role of synchronizing the clocks of the different observers.

As in the last section, we will present the threading point of view from two different perspectives. First, we will address the issue of constructing a spacetime from a given family of curves. Second, we will illustrate the threading point of view by considering a certain decomposition of spacetime. One should notice the similarities between the slicing and threading points of view.

**a) The Construction**

In the previous section we saw how one would construct a spacetime metric from a 1-parameter family of 3-dimensional Riemannian manifolds. The resulting 4-metric was easily described in terms of the given metrics on the surfaces, the slicing lapse function, and the slicing shift vector field. Suppose one is instead given a family of timelike curves. How would one go about constructing a spacetime which realized the original family of curves as a congruence of timelike curves? We will proceed as we did in the case of slicing.

In the earlier (3+1)-construction we had a parameter which labeled each hypersurface. Let us assume we have parameters \( x^i, i = 1, 2, 3 \) which label each curve \( L_{x^i} \). On each curve suppose we have a coordinate \( t \) as well as a metric \( l \), which can thus be expressed as

\[
 l = -M^2 \, dt^2.
\]

We will interpret these curves as being world-lines of observers, and hence require that they be timelike. Consider the same measuring problem as before, that is, letting \( p_0 = (x^i, t) \in L_{x^i} \) and \( q_0 = (x^i + \Delta x^i, t + \Delta t) \in L_{x^i + \Delta x^i} \) we are interested in measuring the coordinate distance \( d(p_0, q_0) \) between \( p_0 \) and \( q_0 \). Since we are assuming we can measure distances in each curve \( L_{x^i} \), again use the Pythagorean Theorem to write

\[
 d(p_0, q_0)^2 = -d(p_0, q_1)^2 + d(q_1, q_0)^2.
\]

Here \( d(q_1, q_0) \) is meant to refer to the orthogonal distance between two nearby curves. See Figure 4.
Since we do not have any notion of traveling “orthogonally” to the curves $L_{x^i}$, the $t$-coordinate of $q_1$ is not determined. The position of $q_1$ along $L_{x^i}$ is affected by each of the $\Delta x^i$. Assign coordinates to $q_1$ by $q_1 = (x^i, t + \Delta t - M_i \Delta x^i)$. Again, the $M_i$ record the amount of “shifting” of $q_1$ with respect to nearby curves. That is, the $M_i$ may be thought of recording how $L_{x^i + \Delta x^i}$ has been shifted with respect to $L_{x^i}$ in the construction process. Therefore, we have

$$d(p_0, q_1) = M(\Delta t - M_i \Delta x^i).$$

The three functions $M_i$ and the function $M$ depend on the parameters $x^i$ as well as the coordinate $t$.

We now need to specify the relationship between the parameters $x^i$ and the proper coordinate distance between neighboring curves. We thus introduce a “spatial metric” of the form $h_{ij} \Delta x^i \Delta x^j$ which gives the distance between $L_{x^i}$ and $L_{x^i + \Delta x^i}$ for various choices of $\Delta x^i$. While we assume that $h_{ij} = h_{ji}$, the functions $h_{ij}$ may otherwise be chosen arbitrarily. We continue our construction of the 4-metric by assuming that this distance is precisely $d(q_1, q_0)$ (i.e. measured orthogonally). See Figure 5.

Thus, the Pythagorean Theorem implies that the spacetime metric may be written as

$$ds^2 = -M^2(dt - M_i dx^i)^2 + h_{ij} dx^i dx^j$$

$$= -M^2 dt^2 + 2M^2 M_i dx^i dt + (h_{ij} - M^2 M_i M_j) dx^i dx^j. \quad (6)$$

The component $M$ of the original metric along each curve is referred to as the threading lapse function. As we see in the above representation of the metric (equation (6)), the functions $M_i$ are most naturally associated with the 1-form $M_i dx^i$. The three functions $M_i$ are referred to as the components of the threading shift 1-form $M_i dx^i$. Finally, the functions $h_{ij}$ may be thought of as the components of a metric, the threading metric.

In the case of slicing, one thought of the slicing shift vector as a three dimensional spatial vector field on the surfaces $\Sigma_t$ and, hence, raised and lowered its indices with the slicing metric. In the present case of threading, one may again adopt the convention that the threading shift 1-form be treated as a three dimensional tensor. Under such a convention, the threading metric $h_{ij}$ may be used to raise and lower its indices.

The threading shift 1-form was defined in order to introduce some notion of traveling “orthogonally” to the curves $L_{x^i}$. The unit 1-form which annihilates the space of vectors orthogonal to the threading vector field may be written

$$m = -M(dt - M_i dx^i).$$
One should compare this equation with the analogous equation for slicing (equation (3)).

b) The Decomposition

The threading lapse function and shift 1-form field may also be viewed as arising from an orthogonal decomposition of spacetime. Analogous to the (3+1)-decomposition, the so-called (1+3)-decomposition attempts to decompose spacetime quantities into pieces orthogonal to the given congruence of curves, and pieces tangent to the congruence. As above, we will work in coordinates \((t, x^i)\) where \(t\) acts as a parameter along the integral curves of \(\frac{\partial}{\partial t}\) (the threading curves) and \(x^i\) are coordinates on each hypersurface \(\Sigma_{t_0} = \{t \equiv t_0\}\). As in [4], we will refer to \(\frac{\partial}{\partial t}\) as the threading vector field.

The normalization of the threading vector field is used to define the threading lapse function \(M\):

\[
\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle = -M^2,
\]

where \(\langle , \rangle\) refers to the spacetime metric \(g\). If one views the threading curves as the worldlines of a family of observers, the threading lapse function measures the rate of change of the observed proper time with respect to the coordinate time function \(t\).

In the slicing point of view one had to describe the discrepancy between \(\frac{\partial}{\partial t}\) and the direction normal to the hypersurfaces. Analogously, in the present scenario one wishes to measure the amount of tilting of the local rest spaces of the observers with respect to each coordinate direction \(\frac{\partial}{\partial x^i}\), keeping in mind that the local rest spaces of the observers need not constitute a hypersurface! Following the flavor of equation (5), we decompose the coordinate 1-form \(dt\) into a piece which annihilates the local rest spaces and a piece which is in the cotangent space of each surface. Letting \(m\) represent the metric dual of the unit vector field tangent to the threading curves, we have \(m(\frac{\partial}{\partial t}) = -M\), so that \(dt\) must have the form

\[
dt = -\frac{1}{M} m + M_i dx^i. \tag{7}
\]

The functions \(M_i\) thus defined are the components of the threading shift 1-form.

Using the above definitions of \(M\) and \(M_i\), the (1+3)-decomposition of the components of the 4-metric takes the following form:

\[
(g_{\alpha\beta}) = \begin{pmatrix}
-M^2 & M^2 M_j \\
M^2 M_i & h_{ij} - M^2 M_i M_j
\end{pmatrix}
\tag{8}
\]

with inverse

\[
(g^{\alpha\beta}) = \begin{pmatrix}
-(M^{-2} - M_m M^m) & M^3 \\
M^i & h^{ij}
\end{pmatrix}
\]

where we have defined \(M^i = h^{ij} M_j\) and where the functions \(h_{ij}\) defined by equation (8) are the components of the threading metric, with inverse \(h^{ij}\).

Although one may take equation (8) as the definition of \(h_{ij}\), historically it has a more familiar definition. For instance, in both [5] and [6] one is given a physical interpretation of the threading metric. In general relativity, to calculate the spatial distance between an
observer and an infinitesimally close event, one may direct a light signal from the observer to the event and back and calculate the “time” of propagation. One finds the spatial distance \( dl \) to be given by

\[
dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta
\]

where

\[
\gamma_{\alpha\beta} = \bar{g}_{\alpha\beta} - \frac{g_{0\alpha} g_{0\beta}}{g_{00}}.
\]

Note that \( \gamma_{00} = \gamma_{0\alpha} = \gamma_{\alpha0} \equiv 0 \) and that \( \gamma_{ij} \equiv h_{ij} \) \((i, j = 1, 2, 3)\) as defined earlier (in our adapted coordinate system). \(^3\)

One can see that the threading metric simply measures what Cattaneo [8] refers to as the *space norm* of any 4-vector. That is, \( h_{ij} \) measures the norm of the component perpendicular to the threading curves. Specifically, for any 4-vector \( V^\alpha \) write \( V^\alpha = V_\parallel^\alpha + V_\perp^\alpha \) where \( V_\parallel^\alpha \) is parallel to the threading curves and \( V_\perp^\alpha \) is perpendicular. Letting \( m^\alpha \) be the unit vector tangent to \( \frac{\partial}{\partial t} \) one has

\[
m_\alpha = g_{\alpha\beta} m^\beta = \frac{1}{\sqrt{-g_{00}}} g_{0\alpha}.
\]

Thus,

\[
g_{\alpha\beta} V_\perp^\alpha V_\perp^\beta = (g_{\alpha\beta} + m_\alpha m_\beta) V^\alpha V^\beta
\]

\[
= (g_{\alpha\beta} - \frac{g_{0\alpha} g_{0\beta}}{g_{00}}) V^\alpha V^\beta
\]

\[
= (g_{ij} + M^2 M_i M_j) V^i V^j
\]

\[
= h_{ij} V^i V^j.
\]

At this point one notices a crucial difference between the slicing and threading pictures of spacetime. When slicing spacetime with spacelike hypersurfaces, one defines the slicing metric which naturally lives on these hypersurfaces. While the threading metric arises in an analogous way, there exists no corresponding space (hypersurface) on which it naturally exists (since the local rest spaces of the observers may not be surface forming). One therefore constructs an abstract 3-manifold with the threading metric as its Riemannian metric. By identifying each threading curve with the point \((t_0, x^i)\) at which it pierces the slice \( \{t \equiv t_0\} \) one constructs the manifold of orbits, \( \Sigma_t \), of the threading. Even though \( \Sigma \) is diffeomorphic to the surfaces \( \Sigma_t \), \( \Sigma \) is not in general given the same geometry (metric) as any of the \( \Sigma_t \). One gives \( \Sigma \) the threading metric in an attempt to recapture some of the spacetime geometry associated with the local rest spaces. Thus, one may think of \( \Sigma \) as a smooth model for the collection of local rest spaces. \( \Sigma \) comes equipped with the coordinates \( x^i \) and the threading metric as a function of not only the points of \( \Sigma \), but

\(^3\) In [7], Einstein and Bergmann used a similar argument during their attempts to generalize Kaluza’s theory of electricity. Einstein and Bergmann, however, were working in a 5-dimensional space, so that in their formalism the 4-dimensional spacetime metric took the role of the threading metric in the above discussion.
also an additional parameter (the parameter along the original threading curves). Thus Σ has a 1-parameter family of Riemannian metrics! This is the beginning of the parametric manifold picture of spacetime.

4. Decomposition of Metrics on Fibre Bundles

The slicing and threading frameworks can be described as part of a single mathematical structure, a fibre bundle. The slicing and threading metrics, shifts, and lapse functions naturally arise when one examines the decomposition of a bundle metric in terms of metrics on the base space and the fibre space. By choosing the base space and fibre space correctly, one recovers either the threading or slicing framework.

As the presentations of the slicing and threading viewpoints mainly focused on the decomposition of the spacetime metric, that will be the main concern in this section as well. That is, given a metric on the total space, what conditions are necessary in order to construct metrics on the base space as well as the typical fibre. Since we are interested in this problem in order to better describe the slicing and threading decompositions, we are really only interested in local decompositions of the metric. Thus, we will be working in a single local trivialisation of the bundle.

Let $\mathcal{M}$ be a fibre bundle with base space $B$, fibre $F$, and continuous, surjective projection $\pi: \mathcal{M} \to B$. For the purposes of slicing and threading one should take $\mathcal{M}$ to be a spacetime and the collection of fibres $\{\pi^{-1}(x) : x \in B\}$ to represent the appropriate foliation (spacelike hypersurfaces for slicing or timelike curves for threading). Furthermore, we have $B = \mathcal{M}/F$ under the equivalence induced by the fibres (i.e. for $p, q \in \mathcal{M}$, $p \sim q \iff \pi(p) = \pi(q)$). $B$ is called the manifold of leaves, and is here the same as the manifold of orbits introduced earlier.

For the following we will work within a single local trivialisation with adapted coordinates. Let $U \subset B$ be a coordinate neighborhood with coordinates $x^i$ such that $\pi^{-1}(U) \approx U \times F$. Furthermore, within a coordinate neighborhood of $\pi^{-1}(U)$ we may use coordinates $(x^i, y^\alpha)$ where $y^\alpha$ are coordinates on $F$.

For any $p \in \pi^{-1}(U)$, there exists a natural subspace $V_p \subset T_p \mathcal{M}$ called the vertical subspace. $V_p$ is defined by

$$V_p = \{X \in T_p \mathcal{M} : \pi_*(X) = 0\}.$$ 

Complementing the notion of vertical, define a subspace $H_p \subset T_p \mathcal{M}$ so that $T_p \mathcal{M} = V_p \oplus H_p$ and call $H_p$ the horizontal subspace. Certainly there are many smooth choices for $H_p$. If $\mathcal{M}$ has a metric, $H_p$ may be chosen quite naturally to be the orthogonal complement to $V_p$.

Now, given a metric $g$ on $\mathcal{M}$ is there a natural choice for metrics $h$ and $k$ on $B$ and $F$ respectively? Not unless additional structure on $\mathcal{M}$ is given or we allow for the additional freedom of a parametric metric. Let us mention a few of these possibilities in greater detail.

For $X, Y \in T_p B$ there exist unique horizontal lifts of $X$ and $Y$ at each point $p \in \pi^{-1}(x)$. Call these lifted vectors $\hat{X}_p$ and $\hat{Y}_p$. It would seem natural to define $h(X, Y)$ in terms of these lifts. In order for $h$ to be well-defined there are various options depending on the additional structure one is willing to assume.

1. If $g$ were constant along each fibre, then $g(\hat{X}_p, \hat{Y}_p) = g(\hat{X}_q, \hat{Y}_q)$ for all $p, q \in \pi^{-1}(x)$. One could then define

$$h(X, Y) = g(\hat{X}_p, \hat{Y}_p)$$
for any \( p \in \pi^{-1}(x) \). Actually, we can loosen this restriction somewhat. We only need that \( g \) restricted to the horizontal subspace \( H_p \) is constant along each fibre. This is essentially Reinhart’s condition that \( g \) be bundle-like with respect to the given foliation [9].

2. If there were some preferred global section \( \sigma : B \to \mathcal{M} \) (e.g. \( F \) is a vector space) one could define

\[
h(X,Y) = g(\hat{X}_{\sigma(x)}, \hat{Y}_{\sigma(x)}).
\]

For instance, \( \sigma \) could refer to some initial hypersurface in an initial value formulation.

3. One could allow the metric on \( B \) to carry extra parameters, namely \( y^\alpha \), and define

\[
h(X,Y)_{|_{y^\alpha}} = g(\hat{X}_{(x^i, y^\alpha)}, \hat{Y}_{(x^i, y^\alpha)})
\]

and, hence, begin to consider \( B \) as a parametric manifold.

If \( h \) has been defined in one of the above situations, the component functions \( h_{ij} \) can be defined and computed. Suppose the horizontal direction is defined by the basis

\[
H_i = \frac{\partial}{\partial x^i} + \Gamma_i^\alpha \frac{\partial}{\partial y^\alpha}.
\]

We denote the horizontal lift of \( \frac{\partial}{\partial x^i} \) by \( H_i \), so that \( H_i = \hat{\frac{\partial}{\partial x^i}} \).

One may now define the components of \( h \) by

\[
h_{ij} = h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g\left(\hat{\frac{\partial}{\partial x^i}}, \hat{\frac{\partial}{\partial x^j}}\right) = g\left(\frac{\partial}{\partial x^i} + \Gamma_i^\alpha \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial x^j} + \Gamma_j^\beta \frac{\partial}{\partial y^\beta}\right) = g_{ij} + 2\Gamma_i^\alpha g_{j\alpha} + \Gamma_i^\alpha \Gamma_j^\beta g_{\alpha\beta}
\]

We are assuming \( h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \) is well defined, but the functions \( h_{ij} \) may be functions of \( y^\alpha \) as well as \( x^i \) (as in 3).

There are similar obstructions to defining a metric \( k \) on \( F \). Vectors \( Z,W \in T_y F \) can be naturally identified with vertical vectors (tangent to the fibres) of \( \mathcal{M} \). We have many natural embeddings of \( F \) into \( \mathcal{M} \). The problem is which fibre? As with \( h \), we need some additional structure which allows us a definition in the following sense:

\[
k(Z,W) = g(\hat{Z}, \hat{W})
\]

where \( \hat{Z} \) and \( \hat{W} \) represent some mapping of \( Z \) and \( W \) into vertical vectors of \( \mathcal{M} \). That is, we define \( k \) to be a pullback of \( g \). Since \( k \) depends on which imbedding of \( F \) we use, we can think of \( k \) as being parameterized by the coordinates \( x^i \) of \( B \). We therefore define the components of \( k \) by \( k_{\alpha\beta}|_{x^i} = g_{\alpha\beta}|_{x^i} \). That is, one can use the local trivialisations to pull back the metric \( g \) to a parametric metric on \( F \).
We can now write the original metric $g$ of $\mathcal{M}$ in terms of $h$ and $k$. We have:

\[
(g_{ab}) = \begin{pmatrix}
k_{\alpha\beta} & g_{\alpha j} \\
g_{i\beta} & h_{ij} - 2\Gamma_i^\alpha g_{\alpha j} - \Gamma_i^\alpha \Gamma_j^\beta k_{\alpha\beta}
\end{pmatrix}
\]

where

\[
g_{\alpha j} = g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^\alpha}\right) = g\left(\frac{\Gamma_j^\beta}{\partial y^\beta}, \frac{\partial}{\partial y^\alpha}\right) - \Gamma_j^\beta g_{\alpha\beta}
\]

\[
= g\left(H_j, \frac{\partial}{\partial y^\alpha}\right) - \Gamma_j^\beta g_{\alpha\beta}
\]

Since $\mathcal{M}$ has a metric, we may choose our notion of horizontal so that $H_i$ is orthogonal to $\frac{\partial}{\partial y^\alpha}$, in which case we have $g(H_j, \frac{\partial}{\partial y^\alpha}) = 0$ and $g_{\alpha j} = -\Gamma_j^\beta g_{\alpha\beta} = -\Gamma_j^\beta k_{\alpha\beta}$.

In this special case $g$ takes the following form:

\[
(g_{ab}) = \begin{pmatrix}
k_{\alpha\beta} & -\Gamma_j^\beta k_{\alpha\beta} \\
-\Gamma_i^\alpha k_{\alpha\beta} & h_{ij} + \Gamma_i^\alpha \Gamma_j^\beta g_{\alpha\beta}
\end{pmatrix}
\]

**Example 1. Threading.**

Take $F = \mathcal{R}$ with coordinate $y^0 = t$ and $B = \Sigma$ to be a 3-dimensional manifold with coordinates $x^i$ ($i = 1, 2, 3$) as before. In terms of the above decompositions, the spacetime metric is of the following form

\[
(g_{ab}) = \begin{pmatrix}
k_{00} & -\Gamma_j k_{00} \\
-\Gamma_i k_{00} & h_{ij} + \Gamma_i \Gamma_j k_{00}
\end{pmatrix}
\]

Since $k_{00}$ represents the squared norm of $\frac{\partial}{\partial t}$, according to previous notation $k_{00} = -M^2$. This decomposition is then precisely the same as the threading decomposition with $\Gamma_i = M_i$ and where $h_{ij}$ is the threading metric on the manifold of orbits $\Sigma$. Thus, the notion of horizontal is given by $H_i = \frac{\partial}{\partial x^i} + M_i \frac{\partial}{\partial t}$ which corresponds to the orthogonal subspace to $\frac{\partial}{\partial t}$.

**Example 2. Slicing**

By switching the roles of $F$ and $B$ in the above example, one has the original slicing story. Let $F = \Sigma$ with coordinates $y^\alpha = X^i$ ($i = 1, 2, 3$) and $B = \mathcal{R}$ with a single coordinate $x^0 = t$. One has:

\[
(g_{ab}) = \begin{pmatrix}
\Gamma^i k_{ij} & -\Gamma^j k_{ij} \\
-\Gamma^i k_{ij} & h_{00} + \Gamma^i \Gamma^j k_{ij}
\end{pmatrix}
\]
As before, $k_{ij}$ is the slicing metric, $\Gamma^i = -N^i$, and $h_{00} = -N^2$. Here the horizontal subspace is given by $H_0 = \frac{\partial}{\partial t} + N^i \frac{\partial}{\partial x^i}$, which is orthogonal to the hypersurfaces $\Sigma_t$.

Thus, both the slicing and threading viewpoints are special cases of a more general decomposition of metrics on fibre bundles. Furthermore, they are naturally dual to each other in the sense that the fibre and base space are interchanged.

5. Parametric Manifolds

Anyone familiar with the Kaluza-Klein theories of spacetime will notice a similarity between the threading (1+3) formalism and the standard Kaluza-Klein (1+4) framework. Kaluza attempted to describe ordinary 4-dimensional Einstein gravity and Maxwell electromagnetism by working in a 5-dimensional space [10]. Gravity and electromagnetism were then obtained by imposing a “cylindrical” condition on the fifth dimension. The resulting (1+4)-decomposition of a 5-dimensional space is reminiscent of the threading viewpoint. In place of the threading metric, Kaluza has the ordinary Einstein metric of spacetime, and taking the place of the threading shift is the electromagnetic vector potential.

In [7], Einstein and Bergmann reformulated Kaluza’s ideas and proceeded to replace the “cylindrical” condition imposed by Kaluza by a “periodic” assumption. In their resulting ansatz lies the beginning of a true parametric picture of spacetime (although still nestled in the comforts of a 5-dimensional space). The tensor analysis they described, translated into a 1+3 setting, turns out to be the same as the theory of parametric manifolds as presented recently by Perjés [1], based on earlier work by Zel’manov [11].

We recently showed [12] how to put this tensor analysis on a rigorous mathematical footing by generalizing the standard Gauss-Codazzi formalism for projections into an orthogonal hypersurface. We now present a summary of those results, stated explicitly in a threading framework in 1+3 dimensions. For further details, see [12].

Given a timelike vector field $A$ with norm $1/M$, i.e. $\langle A, A \rangle = -1/M^2$, introduce a parameter $t$ along the integral curves of $A$ such that $A = \frac{1}{M^2}\partial_t$. Now introduce (local) coordinates $x^i$ on the surfaces $\{t = \text{constant}\}$. This is precisely the threading decomposition of spacetime, adapted to the given family of timelike curves tangent to $A$ together with a particular choice of “parameter” $t$. In particular, the metric takes the form (8), where $M$ is to be identified with the threading lapse function, and the $M_i$ are the components of the threading shift 1-form.

The functions $h_{ij} = g_{ij} + M^2 M_i M_j$ are the components of the threading metric, and can also be thought of as the nonzero components of the tensor

\[ h_{\alpha\beta} = g_{\alpha\beta} + M^2 A_\alpha A_\beta \]

which is associated with the projection operator

\[ P_\alpha^\beta = h_\alpha^\beta = \delta_\alpha^\beta + M^2 A_\alpha A^\beta. \]

Any tensor can be projected orthogonally to $A$ by contracting all indices with $P$; we will denote this operation by the superscript $\perp$.

A parametric manifold consists of a generic $\{t = \text{constant}\}$ hypersurface — really the manifold of orbits of $A$ — together with 1-parameter families of “time-dependent”
projected tensors. All of the operations below are covariant in that they take projected tensors to projected tensors.

In what follows, we will assume that \(X, Y,\) and \(Z\) are orthogonal to \(A,\) so that e.g. \(X^\perp = X.\) We can define the **projected covariant derivative operator** \(D\) by

\[
D_X Y = (\nabla_X Y)^\perp.
\]

The spatial components of \(D_X Y\) turn out to be [12]

\[
(D_X Y)^i = X^j (Y^i*)^j + \Gamma^i_{kj} Y^k
\]

where we have introduced Perjés’ **starry derivative** notation, namely

\[
f^* = \partial_i f + M_i \partial f
\]

and where we have defined the symbol \(\Gamma^\alpha_{\nu\beta}\) by

\[
\Gamma^\alpha_{\nu\beta} = P^\alpha_{\gamma\delta} P^\delta_{\beta\mu} \Gamma_{\gamma\mu}. 
\]

We define the torsion \(T_D\) of \(D\) to be the projection of the torsion \(T\) of \(\nabla.\) It turns out that this yields the familiar coordinate expression [12]

\[
T_D^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}
\]

so that \(D\) is torsion free if \(\nabla\) is. However, the coordinate-free version of this equation does not quite take its usual form, and it is worth considering this carefully. We have [12]

\[
T_D(X, Y) = D_X Y - D_Y X - [X, Y]^\perp.
\]

Note that only the projected bracket appears here. If \(A\) is hypersurface-orthogonal, the bracket operation closes — this is just Frobenius’ theorem. It can now be shown that if \(\nabla\) is both torsion free and metric compatible, then \(D\) is the unique torsion free, metric compatible (projected) connection associated with the (parametric) metric \(h\) [13]. In this case, the connection symbols take the familiar form [1]

\[
\Gamma^i_{jk} = \frac{1}{2} h^{im} (h_{mj* k} + h_{mk* j} - h_{jk* m}). 
\]

However, \(A\) will not in general be hypersurface-orthogonal, and there will also be a parallel component. We are thus led to define the **deficiency** of \(D\) as [12]

\[
D(X, Y) = [X, Y]^\perp = [X, Y] - [X, Y]^\perp.
\]

In components we have

\[
([X, Y]^\perp)^i = X^m Y^i_m - Y^m X^i_m
\]
and $D(X, Y) = D_{ij} X^i Y^j \partial_t$ with $D_{ij} = M_{j^* i} - M_{i^* j}$. Note that the deficiency does not necessarily vanish even when $\nabla$ is torsion free.

There are now two natural candidates for the curvature tensor associated with $D$ [12]. One of these is the Zel’manov curvature used by Zel’manov [11] and Perjés [1], as well as (in a 5-dimensional setting) by Einstein and Bergmann [7], namely

$$\left[ \nabla^*_k \nabla^*_j - \nabla^*_j \nabla^*_k + (M_{j^* k} - M_{k^* j}) \frac{\partial}{\partial t} \right] X_i = Z^{r}_{ijk} X_r$$

with components

$$Z^l_{kij} = \Gamma^l_{kj} i - \Gamma^l_{ki} j + \Gamma^m_{ni} \Gamma^n_{jk} - \Gamma^m_{nj} \Gamma^n_{ik}.$$ 

Note that the deficiency appears here in much the same way that torsion would. The other candidate, derived by considering Gauss’ equation [12], is

$$R^l_{kij} = \Gamma^l_{kj} i - \Gamma^l_{ki} j + \Gamma^m_{ni} \Gamma^n_{kj} - \Gamma^m_{nj} \Gamma^n_{ki} + (M_{j^* i} - M_{i^* j}) h^{lm} (M^2 M_{m^* k} - M^2 M_{k^* m} + \partial_t h_{km})$$

This expression contains additional terms involving the threading lapse $M$, but possesses all the expected symmetries, which the Zel’manov curvature does not; both contain terms involving the deficiency. For a more detailed comparison of these two curvature tensors, see [12].

6. Discussion

The theory of parametric manifolds outlined above can also be expressed intrinsically [13]. The basic idea, as noted by Perjés, is to introduce a parametric structure on the manifold of orbits, which is essentially the threading shift 1-form. This leads to natural generalizations of Lie differentiation, exterior differentiation, and covariant differentiation, all based on a nonstandard action of vector fields on functions. For further details, see [13].

In the special case where the threading curves are integral curves of a Killing vector field, no physical fields depend on the parameter. In particular, the projected metric tensor $h$ on the manifold of orbits $\Sigma$ is now an ordinary metric tensor, and it follows immediately from (10) that the projected connection is precisely the Levi-Civita connection of $h$. Parametric manifolds thus generalize the formalism of Geroch [14] for spacetimes with (not necessarily hypersurface-orthogonal) Killing vectors.

Even in the non-Killing case, if the threading shift 1-form $\omega = M_i dx^i$ is independent of the parameter then “starry” partial differentiation defines a connection on the fibre bundle consisting of spacetime over the manifold of orbits. Parametric manifolds generalize this to something which is “almost” a connection on a fibre bundle. As hinted by Perjés [1],

---

4. The symmetries of this curvature tensor involve the deficiency in precisely the same way torsion would enter [12].

5. The missing property is invariance under the action of the group, which requires the notion of horizontal to be parameter independent.
it may thus be possible to use parametric manifolds to construct a generalized Yang-Mills theory in which exact symmetry under the gauge group is not required.

Since they correspond to what a given family of observers actually sees, parametric manifolds may provide a natural setting for initial-value problems. For instance, when considering the scalar field on a fixed spacetime background, formulating the initial-value problem requires one to decompose the spacetime Laplacian. Preliminary calculations indicate that, in the general, non-hypersurface-orthogonal setting, this decomposition is simplest using the parametric manifold approach [15]. The standard approach to Killing observers in the exterior of a Kerr black hole is to use the surfaces of constant time to slice the spacetime. This describes physics as seen by the ZAMOs (Zero Angular Momentum Observers) orthogonal to these surfaces. The parametric framework would describe instead the physics actually seen by the Killing observers, since a parametric manifold captures precisely the geometry of the instantaneous rest spaces of the given family of observers. Which approach to use depends on which questions one is asking.

For the initial value problem in general relativity itself, the situation is perhaps even more interesting. The usual constraints relating the extrinsic curvature and the 3-metric in the slicing approach are derived from the Gauss-Codazzi equations. As pointed out above, only one of the candidate parametric curvature tensors satisfies Gauss’ equation. This may be strong evidence in support of using this curvature tensor, and may lead to an initial value formulation of general relativity for appropriate data on the manifold of orbits of a given vector field. Work on this issue is continuing.

ACKNOWLEDGEMENTS

It is a pleasure to thank Bob Jantzen and Zoltan Perjés for their encouragement throughout this project, which builds on earlier results of theirs. This work forms part of a dissertation submitted to Oregon State University (by SB) in partial fulfillment of the requirements for the Ph.D. degree in mathematics, and was partially funded by NSF grant PHY-9208494.

REFERENCES
1. Z. Perjés, *The Parametric Manifold Picture of Space-Time*, Nuclear Physics **B403**, 809 (1993)

2. C. Misner, K. Thorne, J. Wheeler, *Gravitation*, W. H. Freeman and Company, San Fransisco, 1973

3. R. Wald, *General Relativity*, The University of Chicago Press, Chicago, 1984

4. R. Jantzen and P. Carini, *Understanding Spacetime Splittings and Their Relationships*, in *Classical Mechanics and Relativity: Relationship and Consistency*, ed. by G. Ferrarese, Bibliopolis, Naples, 185–241, 1991

5. L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley, Cambridge, 1951

6. C. Møller, *The Theory of Relativity*, Second Edition, Clarendon Press, Oxford, 1972

7. A. Einstein and P. Bergmann, Ann. of Math **39**, 683–701, (1938).
   *This article was reprinted in Introduction to Modern Kaluza-Klein Theories*, edited by T. Applequist, A. Chodos, and P.G.O. Freund, Addison-Welsey, Menlo Park, 1987.

8. C. Cattaneo, Nuovo Cimento, **10**, 318, (1958)

9. B. Reinhart, Ann. of Math., **69**, 119–131 (1959).

10. T. Kaluza, *On the Unity Problem of Physics*, reprinted in *Introduction to Modern Kaluza-Klein Theories*, edited by T. Applequist, A. Chodos, and P.G.O. Freund, Addison-Welsey, Menlo Park, 1987.

11. A. Zel’manov, Soviet Physics Doklady **1**, 227–230 (1956)

12. Stuart Boersma and Tevian Dray, *Parametric Manifolds I: Extrinsic Approach*, gr-qc/9407011, J. Math. Phys. (submitted).

13. Stuart Boersma and Tevian Dray, *Parametric Manifolds II: Intrinsic Approach*, gr-qc/9407012, J. Math. Phys. (submitted).

14. R. Geroch, *A Method for Generating Solutions of Einstein’s Equations*, J. Math. Phys. **12**, 918 (1971).

15. Stuart Boersma and Tevian Dray, *Parametric Manifolds and the Scalar Field*, (in preparation).