Prevalence of Delay Embeddings with a Fixed Observation Function

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Abstract

Let \( x_{j+1} = \phi(x_j), \ x_j \in \mathbb{R}^d \), be a dynamical system with \( \phi \) being a diffeomorphism. Although the state vector \( x_j \) is often unobservable, the dynamics can be recovered from the delay vector \( (o(x_1), \ldots, o(x_D)) \), where \( o \) is the scalar-valued observation function and \( D \) is the embedding dimension. The delay map is an embedding for generic \( o \), and more strongly, the embedding property is prevalent. We consider the situation where the observation function is fixed at \( o = \pi_1 \), with \( \pi_1 \) being the projection to the first coordinate. However, we allow polynomial perturbations to be applied directly to the diffeomorphism \( \phi \), thus mimicking the way dynamical systems are parametrized. We prove that the delay map is an embedding with probability one with respect to the perturbations. Our proof introduces a new technique for proving prevalence using the concept of Lebesgue points.

1 Introduction

Let \( x_{j+1} = \phi(x_j) \) be a dynamical system. If \( o \) is a scalar valued observation function, the delay map is given by
\[
F_0(x_1) = (o(x_1), \ldots, o(x_D)).
\]
The question of when \( F_0 \) is an embedding was considered by Aeyels [1] and Takens [10]. Suppose that \( x_j \in \mathbb{R}^n \) but with the dynamics confined to an invariant submanifold of dimension \( d \leq n \). Alternatively, we may assume \( x_j \in \mathbb{m} \), where \( \mathbb{m} \) is a manifold of dimension \( d \). Based on an analogy to Whitney embedding [2], we may expect \( F_0 \) to be an embedding for generic \( o \) for embedding dimension \( D \geq 2d + 1 \). Here genericity is with respect to the space of functions \( o \) under a \( C^r \) topology with \( r \geq 2 \) [2].

Sauer et al [9] introduced a new point of view, supported by deep ideas, into the theory of delay embeddings. If \( x \in \mathbb{R}^n \) and \( \alpha \in \mathbb{Z}_0^n \) is a multi-index, denote the monomial \( x^\alpha \) by \( p_\alpha(x) \). Instead of assuming the observation function \( o \) to be any \( C^r \) function, Sauer et al take the observation function to be the sum of some fixed function \( o^* \) and a finite linear combination of the monomials \( p_\alpha(x) \). Proofs of genericity rely on “bump” functions or \( C^\infty \) functions with compact support. Although the device of bump functions is of much utility in differential topology [2], bump functions hardly ever arise in applications. In contrast, physical
models often use polynomials. Thus, limiting the perturbations to a finite linear combination of polynomials is a welcome shift in point of view.

A property is generic in a Baire space if it holds for a countable intersection of open and dense sets. A generic set is always dense but it may be of probability zero (in a reasonable sense). For example, generic subsets of $[0,1]$ of probability zero may be constructed easily. Thus, it may be questioned whether the concept of genericity captures the notion of what is typical in applications.

Sauer et al [9] answered that question by introducing the notion of prevalence. To say that delay embeddings are prevalent is equivalent to saying that the delay map is an embedding for almost every linear combination of polynomials. If probabilities are defined by normalizing the Lebesgue measure, we may say that the delay map is an embedding with probability one.

Suppose $x_k = \phi^k(x_1)$ and $y_k = \phi^k(y_1)$. For $F_0$ to be an injection, we must have $F_0(x_1) \neq F_0(y_1)$ whenever $x_1 \neq y_1$. A major difficulty in the proof of injectivity arises in handling points $x_1 \neq y_1$ but with overlapping orbits. For example, we may have $y_1 = x_2$ or $y_1 = x_3$. Related difficulties arise in handling periodic points and in the proof of immersivity (an embedding must be injective as well as immersive). Sauer et al [9] introduced several key ideas for handling these difficulties. However, there is a minor gap in their proof. In section 4, we fix that gap and show that earlier mathematical treatments have serious deficiencies. Therefore, proofs prior to Sauer et al cannot be accepted.

The proof of Sauer et al [9] is quite informal. We give a more formally precise development of their ideas in sections 2 and 3. Later, we consider the case where the observation map is fixed at $o = \pi_1$, with $\pi_1$ being the projection to the first coordinate and with polynomial perturbations applied directly to $\phi$. Ideas essential for the new developments are interspersed in sections 2 and 3. Sauer et al include a filtering step applied to the delay map in their main theorems. In addition to mathematical informality, the filtering step makes the essential ideas difficult to grasp and verify. Thus, the filtering step is omitted in section 4, where we derive their main results in a modified form.

From section 5 onwards, we treat the case where $o = \pi_1$ and $\phi$ itself is perturbed by polynomials. There are two main motivations for considering this case. First, from a purely aesthetic point of view, it is desirable to make the theory of delay embeddings depend upon the dynamics and not the observation function. Second, the setting with $o = \pi_1$ is pertinent to applications. For example, the most natural way to extract a time series from a fluid flow is to simply record the fluid velocity at a fixed point [10].

The main technical novelty in our approach is related to the concept of Lebesgue points. Our delay embedding theorem for the $o = \pi_1$ case requires $D \geq 4d + 2$, although our earlier work [4] suggests $D \geq 2d + 1$. In the concluding section, we express the hope that the technique of Lebesgue points may prove useful in obtaining prevalence versions of some classical results in dynamical systems theory. In that regard, we mention the extensions of delay embedding theory to PDE by Robinson [6, 7]. A more complete account of other mathematical investigations in embedding theory may be found in the introduction to our earlier work [4].

2 Transfer of volume

A key idea in the work of Sauer et al [9] is to transfer volumes from embedding space to parameter space. For an example of what we mean by transfer of volume, suppose $A$ is a
square matrix. Then a volume equal to $\mathbf{v}$ in the range is transferred to $\mathbf{v}/\det A$ in the domain.

Suppose $G : \mathbb{R}^{D_{\alpha}} \times \mathbb{R}^n \to \mathbb{R}^D$ is a $C^r$ function with $r \geq 2$. Here $\mathbb{R}^{D_{\alpha}}$ is the space of parameters and we will denote a point in parameter space by $(c_{\alpha})$ or $c_{\alpha}$, with the understanding that $(c_{\alpha})$ (or $c_{\alpha}$) is a column vector. The transfer of volume is carried out with fixed $\mathbf{j} \in \mathbb{R}^n$. Thus, the dependence of $G(c_{\alpha}, \mathbf{j})$ on $\mathbf{j}$, which will be nonlinear, does not come up in the transfer of volume argument. When the map $\phi$ is fixed and only the observation function is parametrized, $G$ is linear in the parameters $c_{\alpha}$. The embedding space is $\mathbb{R}^D$ and the dimension $D$ of this space is of much importance. The rank of $G$ is mainly constrained by $D$ because $D_{\alpha} \gg D$, and the rank determines how much volume (or how little, with lesser the better) is transferred from embedding space to parameter space.

In the following lemma and later we refer to $\mu(B_1 \cap B_2)/\mu(B_2)$, where $\mu(\cdot)$ is the Lebesgue measure, as the probability of $B_1$ relative to $B_2$ (both sets are assumed to be measurable). Measure will always refer to Lebesgue measure. The following lemma transfers the volume of a ball of radius $L\epsilon$ in $\mathbb{R}^D$ to parameter space. All norms in this paper are spectral or $L^2$ norms.

**Lemma 1** ([9]). Let $g(c_{\alpha}) = A(c_{\alpha}) + g_0$ be a linear (affine) map from $\mathbb{R}^{D_{\alpha}}$ to $\mathbb{R}^D$, with $A$ being a $D \times D_{\alpha}$ matrix. Suppose the first $r$ singular values of $A$ are at least as great as $\sigma > 0$. Then the measure of the set

$$\left\{c_{\alpha}\left|\|A(c_{\alpha}) + g_0\| \leq L\epsilon\right\} \cap \left\{c_{\alpha}\left|\|c_{\alpha}\| \leq a\right\}\right.$$  

is less than or equal to

$$2^{D_{\alpha}}L^r\epsilon^r a^{D_{\alpha} - r}/\sigma^r,$$  

and the probability of $\|A(c_{\alpha}) + g_0\| \leq L\epsilon$ relative to $\|c_{\alpha}\| \leq a$ is less than or equal to

$$D_{\alpha}!L^r\epsilon^r/\sigma^r a^r.$$  

**Proof.** Suppose $u_1, \ldots, u_{D_{\alpha}}$ are the right singular vectors, $v_1, \ldots, v_D$ are the left singular vectors, and $\sigma_1, \ldots, \sigma_{D_{\alpha}}$ the singular values of $A$. (see [11]). Let $(c_{\alpha}) = \sum_{i=1}^{D_{\alpha}} c_{\alpha} u_i$ and $g_0 = \sum_{i=1}^{D_{\alpha}} g_0 v_i$.

For $i = 1, \ldots, r$, $\|A(c_{\alpha}) + g_0\| \leq L\epsilon$ implies that $|\sigma_i c_{\alpha} + g_0| \leq L\epsilon$ and therefore $|c_{\alpha} + g_0/\sigma_i| \leq L\epsilon/\sigma_i \leq L\epsilon/\sigma$. Thus, the coefficient $c_{\alpha}$ must lie in an interval of measure less than $2L\epsilon/\sigma$ for $i = 1, \ldots, r$.

For $i = r + 1, \ldots, D_{\alpha}$, $\|c_{\alpha}\| \leq a$ implies that $c_{\alpha}$ must vary inside the interval $[-a, a]$, whose length is $2a$.

Therefore, the volume of the set (2.1) is bounded above by $(2L\epsilon/\sigma)^r (2a)^{D_{\alpha} - r}$, which simplifies to (2.2).

For the statement about the probability of $\|A(c_{\alpha}) + g_0\| \leq L\epsilon$ relative to $\|c_{\alpha}\| \leq a$, we divide (2.2) by $\gamma a^{D_{\alpha}}$, where $\gamma$ is the volume of the unit sphere in $\mathbb{R}^{D_{\alpha}}$, to obtain

$$2^{D_{\alpha}}L^r\epsilon^r/\gamma\sigma^r a^r.$$  

The proof is completed using $\gamma = \pi^{D_{\alpha}/2}/\Gamma(D_{\alpha}/2 + 1) \geq 2^{D_{\alpha}}/D_{\alpha}!$. \qed
Lemma 1 shows how a volume $\|g(c_\alpha)\| \leq L\varepsilon$ in embedding space is transferred to a probability relative to $\|c_\alpha\| \leq a$ in parameter space. The transferred probability is proportional to $\varepsilon^r$, and therefore, as the rank $r$ increases, the probability becomes smaller.

To obtain prevalence with the observation function fixed and the map parametrized, we will rely on the following nonlinear transfer of volume lemma. When the previous Lemma 1 is applied, $L$ will be a Lipshitz constant. When the following lemma is applied, $L$ will be a Lipshitz constant as well as a bound on the quadratic remainder term in a Taylor series.

**Lemma 2.** Suppose $g : \mathbb{R}^{D_a} \to \mathbb{R}^D$ is a $C^2$ function, with the Taylor series $g(c_\alpha) = g_0 + A(c_\alpha) + h(c_\alpha)$. We assume that both $g(\cdot)$ and $h(\cdot)$ are defined for $\|c_\alpha\| \leq a$ and that $\|h(c_\alpha)\| \leq L \|c_\alpha\|^2$. We also assume that the first $r$ singular values of $A$ are at least as great as $\sigma > 0$. Then the probability of $\|g(c_\alpha)\| \leq L \varepsilon$ relative to $\|c_\alpha\| \leq \varepsilon^{1/2}$ is less than or equal to

$$D_\alpha!^2 L^r \varepsilon^{1/2} / \sigma^r$$

for $0 < \varepsilon^{1/2} \leq a$.

**Proof.** If $\varepsilon^{1/2} \leq a$ and $\|c_\alpha\| \leq \varepsilon^{1/2}$, then $\|h(c_\alpha)\| \leq L \varepsilon$. Therefore, $\|A(c_\alpha) + g_0\| \leq 2L \varepsilon$. The proof is completed by applying the previous lemma with $L \leftarrow 2L$ and $a \leftarrow \varepsilon^{1/2}$. \qed

Applications of Lemmas 1 and 2 will require us to get a handle on singular values. We will turn to that in the next section. Before doing so, we recapitulate an elegant argument of Sauer et al. This argument, although elementary, gives a good idea of the general approach when the observation function is parametrized.

Suppose $K$ is a smooth sub-manifold or even a fractal set of box counting dimension $d$ and with compact closure that is a subset of $\mathbb{R}^n$. Let the embedding dimension be $D > 2d$. If $d \in \mathbb{Z}^+$, we can take $D = 2d + 1$ as in Whitney’s embedding theorem. The following assumptions are made about the constant $C_K$:

**Assumption** about $C_K$ (1): The set $K$ can be covered with $C_K/\varepsilon^d$ $\varepsilon$-balls for any $\varepsilon > 0$.

**Assumption** about $C_K$ (2): The set $K \times K$ can be covered with $C_K/\varepsilon^{2d}$ $\varepsilon$-balls for any $\varepsilon > 0$.

All balls are spherical.

A linear map from $\mathbb{R}^n$ to $\mathbb{R}^D$ can be written as $F_\alpha(x) = \sum_{\alpha \in I} c_\alpha m_\alpha x$, where $I$ is the index set $(i, j)$, $1 \leq i \leq D$ and $1 \leq j \leq n$, and $m_\alpha$ is the matrix with 1 in the $i, j$th position if $\alpha = (i, j)$ and zero everywhere else. Here $D_\alpha = nD$. We use $c_\alpha$ both to refer to an entry of the vector $(c_\alpha)$ as in the definition of $F_\alpha$ and to the vector as a whole as in $\|c_\alpha\|$. The slight ambiguity, which is resolved from context, is highly convenient. In most instances, $c_\alpha$ refers to the vector as a whole.

Define $G_\alpha(x, y) = F_\alpha(x) - F_\alpha(y)$. Assume $\|c_\alpha\| \leq a_0$. By compactness of the ball $\|c_\alpha\| \leq a_0$, we may assume the Lipshitz constant of $G_\alpha(x, y)$ (with respect to $x, y$) to be bounded above by $L$. Define $K(\delta)$ to be the set of all points $(x, y) \in K \times K$ satisfying $\|x - y\| \geq \delta > 0$. Cover $K(\delta)$ using $C_K/\varepsilon^{2d}$ balls. Suppose $G_\alpha(x, y) = 0$ for some $(x, y) \in K(\delta)$. Then by the Lipshitz bound, we must have $\|G_\alpha(x, y)\| \leq L \varepsilon$ for $(x, y)$ that is a center of one of the $C_K/\varepsilon^{2d}$ covering $K(\delta)$.

The rest of the argument hinges on transferring the volume $\|G_\alpha(x, y)\| \leq L \varepsilon$ to parameter space. To do so, write $G_\alpha(x, y)$ in the form

$$\left( m_{1,1}(x - y), \ m_{1,2}(x - y), \ldots \right)(c_\alpha)$$
and observe that every column in the resulting matrix is in $\mathbb{R}^D$ and is all zeros except for a single entry equal to $\pi_ix - \pi_iy$, where $\pi_i$ denotes the projection to the $i$th coordinate, for some $i \in \{1, \ldots, n\}$. The first $D$ singular values of that matrix are all equal to $\|x - y\| \geq \delta$. Thus, we may transfer volumes using Lemma 1 and assert that the probability of $G_\alpha(x, y) = 0$ for some $(x, y) \in \mathcal{K}(\delta)$ relative to $\|c_\alpha\| \leq a_0$ is at most

$$\frac{C_K}{\varepsilon^{2d} \times \frac{(nD)!L^D e^D}{\delta^D a_0^D}}.$$  

By taking the limit $\varepsilon \to 0$ and because $D > 2d$, it follows that $G_\alpha(x, y) = 0$ for some $(x, y) \in \mathcal{K}(\delta)$ only for a set of $c_\alpha$ of probability zero relative to the ball $\|c_\alpha\| \leq a_0$. By taking the union of the probability zero sets with $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$, we may conclude that $G_\alpha(x, y) = 0$ for some $(x, y) \in K \times K$, $x \neq y$, only for a set of $c_\alpha$ of probability zero relative to $\|c_\alpha\| \leq a_0$. Equivalently, $x \to F_\alpha(x)$ is injective for $x \in K$ with probability one relative to the ball $\|c_\alpha\| \leq a_0$ in parameter space.

The argument derives its power by simply refining the cover of $\mathcal{K}(\delta)$ by using smaller and smaller $\varepsilon$-balls. If $dF_\alpha(x, v)$ is the tangent map at $x$ applied to the tangent vector $v$, then $dF_\alpha(x, v) = F_\alpha(v)$ because of the linearity of $F_\alpha(x)$ in $x$. If $K$ is a submanifold then $T_zK$ is the unit tangent bundle consisting of points $(x, v)$ with $\|v\| = 1$. Injectivity may be proved by considering $dF_\alpha(x, v)$ instead of $G_\alpha(x, y)$, with Lemma 4 invoked with $\sigma \leftarrow 1$.

### 3 Rank lemmas

In proving a version of the Whitney embedding theorem, the argument of Sauer et al. [9] reviewed above writes $G_\alpha(x, y) = M.c_\alpha$ and relies on explicit knowledge of singular values of $M$. In general, singular values of $M$ cannot be obtained so explicitly. Instead, the approach is to first argue that $M$ has rank $D$ or greater for every $(x, y) \in \mathcal{K}(\delta)$ and then observe that

$$\sigma_\delta = \min_{(x, y) \in \mathcal{K}(\delta)} \sigma_D(M) > 0$$

because the $D$th singular value $\sigma_D(M)$ is continuous in $x, y$ and $\mathcal{K}(\delta)$ is compact. The argument may then be completed by applying Lemma 4 with $\sigma \leftarrow \sigma_\delta$.

To support such an argument, we give a few rank lemmas in this section. The first two lemmas are from [3]. Rank lemmas of this type are known in multivariate approximation theory [3], although they are buried inside more sophisticated results.

Suppose $z \in \mathbb{R}^d$. As noted already, the projection to the $i$th coordinate is denoted by $\pi_i$. If $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$, is a multi-index, then $z^\alpha = \prod_{i=1}^d (\pi_i z)^{\alpha_i}$ as usual and $|\alpha| = \sum_{i=1}^d |\alpha_i|$. In later arguments, it is essential to take the gradient of $z^\alpha$ with respect to $z$. For notational convenience, we always denote $z^\alpha$ by $p_\alpha(z)$. The index set $I_{D^+}$ is the set of all $\alpha$ such that $|\alpha| \leq D^+$. By elementary combinatorics, the cardinality of $I_{D^+}$ is $(D^+ D^+)$.

Suppose $z_1, z_2, \ldots, z_{D'}$ are distinct points in $\mathbb{R}^d$. Then

$$\begin{pmatrix} p_\alpha(z_1) \\ \vdots \\ p_\alpha(z_{D'}) \end{pmatrix}$$

(3.1)
denotes the multivariate Vandermonde matrix with the column index \( \alpha \in I_{D^+} \) for some \( D^+ \).
The dimension of the matrix is \( D' \times |I_{D^+}| \), where \(|I_{D^+}|\) is the cardinality of \( I_{D^+} \).

**Lemma 3** ([9]). For \( \alpha \in I_{D^+} \) and \( D^+ \geq D' - 1 \), the rank of the Vandermonde matrix (3.1) is equal to the number of its rows.

**Proof.** Following [9], let \( Q \) be a \( d \times d \) orthogonal matrix drawn from the Haar measure. If \( z_1 \) and \( z_2 \) are distinct, then \( \pi_i z_1 \neq \pi_i z_2 \) for any \( i \) for \( Q \) outside a set of measure 0. Therefore, we can find a \( Q \) such that \( \pi_1 Q z_1, \ldots, \pi_1 Q z_{D'} \) are distinct. We may interpolate arbitrary values at \( z_j \) using a univariate polynomial \( p(\pi_1 Q z) \) of degree \( D' - 1 \). Because we can write

\[
p(\pi_1 Q z) = \begin{pmatrix} 
    p_\alpha(z_1) \\
    \vdots \\
    p_\alpha(z_{D'})
\end{pmatrix} (c_\alpha)
\]

for a suitable choice of \( c_\alpha \), it follows that the rank of (3.1) is equal to the number of its rows. \( \square \)

Let

\[
\begin{pmatrix}
    \nabla p_\alpha(z_1) \\
    \vdots \\
    \nabla p_\alpha(z_{D'})
\end{pmatrix}
\]

be the multivariate incomplete Hermite matrix at \( z_1, \ldots, z_{D'} \) and with \( \alpha \in I_{D^+} \).

**Lemma 4** ([9]). The rank of the incomplete Hermite matrix (3.2) is equal to the number of its rows if \( D^+ \geq D' \).

**Proof.** Arguing as in the previous lemma, we may assume \( \pi_i Q z_1, \ldots, \pi_i Q z_{D'} \) to be distinct for \( i = 1, \ldots, d \). Following [9] and assuming \( Q \) to be the identity without loss of generality, we may then find a polynomial \( p_i(\pi_i z) \) of degree \( D' - 1 \) that interpolates the \( i \)th component of the prescribed gradients at \( z_1, \ldots, z_{D'} \). We may then obtain the prescribed gradients from \( p(z) = \int p_1 d\pi_1 z + \cdots + \int p_d d\pi_d z \). Thus, a linear combination of the columns of (3.2) can produce any prescribed gradients. \( \square \)

To obtain prevalence results with a fixed observation function, Lemmas 3 and 4 need to be combined into another lemma. Therefore, let

\[
\begin{pmatrix}
    p_\alpha(z_1) \\
    \vdots \\
    p_\alpha(z_{D'}) \\
    \nabla p_\alpha(z_1) \\
    \vdots \\
    \nabla p_\alpha(z_{D'})
\end{pmatrix}
\]

be the multivariate Hermite matrix at \( z_1, \ldots, z_{D'} \) and with \( \alpha \in I_{D^+} \).

**Lemma 5.** The rank of the Hermite matrix (3.3) is equal to the number of its rows if \( D^+ \geq 2D' - 1 \).
Proof. Suppose function values as well as gradients are prescribed at $z_1, \ldots, z_{D'}$. We may obtain the prescribed gradients at $z_2, \ldots, z_{D'}$ as in the previous proof in the form $p(z) = \int p_2 \, d\pi_2 z + \cdots + \int p_d \, d\pi_d z$. To obtain suitable function values as well as the $\pi_1$ component of the gradients, we may take the polynomial $p(z) + q(\pi_1 z)$ with $q$ being a suitable univariate Hermite interpolant of degree $2D' - 1$.

A matrix $M$ is said to be circulant if its subsequent rows are obtained by rotating the first row. If the number of columns is $n$ and the first row is $a_1, \ldots, a_n$, the second row must be $a_n, a_1, \ldots, a_{n-1}$. The following lemma about circulant matrices will be used in the next section to refine the discussion of [9].

Lemma 6. Let $M$ be an $m \times D'$ circulant matrix whose first row is $1, 0^{j_1}, -1, 0^{j_2}$, where $0^{j_1}$ is $0$ repeated $j_1$ times. The rank of $M$ is equal to $m$ if $m \leq \lceil D'/2 \rceil$.

Proof. We must have $j_1 + j_2 = D' - 2$. Either $j_1$ or $j_2$ must be less than or equal to $(D' - 2)/2$. Because they are both integers, either $j_1$ or $j_2$ must be $\leq \lfloor D'/2 \rfloor$. Without loss of generality, we assume $j_1 \leq \lfloor D'/2 \rfloor - 1$. As the rows are rotated, the $-1$ appears in column $j_1 + k + 1$ for $k = 1, \ldots, m$. The columns do not wrap around because

$$j_1 + m + 1 \leq \lfloor D'/2 \rfloor - 1 + \lfloor D'/2 \rfloor + 1 \leq D'.$$

All those columns are linearly independent.

The final rank lemma is obvious from elementary linear algebra. We state it explicitly because it is invoked often and has a key position in the framework of [9]. For the most part, the lemma is invoked silently.

Lemma 7. If the rank of the matrix $B$ is equal to the number of its rows, the rank of the product $AB$ is equal to the rank of $A$.

4 Review of Sauer et al [9]

In this section, we review the main results and proofs of [9]. Our aim is twofold. The review helps us prepare the ground for our results about prevalence with a fixed observation map. Second, we point out and fix an error in [9], while presenting the proof with greater formal precision and completeness. The error in [9] is a minor one relative to the depth of ideas found in that paper. We also point out errors and gaps in earlier mathematical treatments that are much more serious.

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism that is at least $C^2$. We adopt the following convention:

Convention about $x, y$: If $x_1$ is a point in $\mathbb{R}^n$, then $x_2 = \phi(x_1), x_3 = \phi(x_2)$, and so on. Similarly, $y_2 = \phi(y_1), y_3 = \phi(y_2)$, and so on. It must be noted that this convention does not apply to $z$. For example, $z_1, \ldots, z_{D'}$ are any distinct points in Lemma [3].

The observation function is assumed to be the (at least twice continuously differentiable) function $o : \mathbb{R}^n \to \mathbb{R}$, which maps every state vector to a real number. If the state vector
the corresponding delay vector is

$$F_0(x_1) = \begin{pmatrix} o(x_1) \\ \vdots \\ o(x_D) \end{pmatrix},$$

where $D$ will be referred to as the embedding dimension.

Let $K \subset \mathbb{R}^n$ be a possibly fractal set of box counting dimension $d$. The set $K$ is assumed to be compact. The delay mapping $F_0$ restricted to $K$ may not be injective. To examine the injectivity more generally, we perturb the observation function to

$$o(x) + \sum_{\alpha \in \mathcal{I}_{2D-1}} c_{\alpha} p_{\alpha}(x)$$

and examine injectivity in the ball $||c_{\alpha}|| \leq a_0$ with $a_0 > 0$ and fixed. The perturbed delay vector becomes

$$F_{\alpha}(x) = F_0(x) + \begin{pmatrix} p_{\alpha}(x_1) \\ \vdots \\ p_{\alpha}(x_D) \end{pmatrix} \cdot (c_{\alpha}),$$

with $\alpha$ ranging over $\mathcal{I}_{2D-1}$. We use $F_{\alpha}$ instead of $F_{c_{\alpha}}$ to denote the delay vector for simplicity and without risk of confusion. The two assumptions about $C_K$ made in the previous section are carried forward.

**Theorem 8 ([9]).** If $D > 2d$ and $\phi$ has finitely many periodic points $x$ of periods less than $2D$, the delay mapping $x \rightarrow F_{\alpha}(x)$ is injective for $x \in K$ for a set of $c_{\alpha}$ of probability 1 relative to $||c_{\alpha}|| \leq a_0$.

Theorem 8 is less general than corresponding statements in [9]. Our aim is to exhibit techniques while forsaking generality. The manner in which more general statements can be obtained is discussed later.

**Proof.** Define $G_{\alpha}(x_1, y_1) = F_{\alpha}(x_1) - F_{\alpha}(y_1)$. We then have $G_{\alpha}(x_1, y_1) = F_0(x_1) - F_0(y_1) + \mathcal{M}(c_{\alpha})$, where

$$\mathcal{M} = \mathcal{J} \mathcal{V}, \quad \mathcal{J} = \begin{pmatrix} 1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & -1 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} p_{\alpha}(x_1) \\ \vdots \\ p_{\alpha}(x_D) \\ p_{\alpha}(y_1) \\ \vdots \\ p_{\alpha}(y_D) \end{pmatrix}.$$

Here $\mathcal{J}$ is $D \times 2D$ and $\mathcal{V}$ is $2D \times D_{\alpha}$, where $D_{\alpha}$ is the cardinality of $\mathcal{I}_{2D-1}$. The proof turns on the determination of the rank of $\mathcal{M}$. If $x_i$ and $y_i$, $1 \leq i \leq D$, are $2D$ distinct points, we may apply Lemmas 3 and 4 and immediately conclude that the rank of $\mathcal{M}$ is $D$. However, if not all points are distinct, the rank of $\mathcal{V}$ is obviously not equal to the number of rows. Several cases need to be considered to determine the rank of $\mathcal{M}$.
Case 1: both \( x_1 \) and \( y_1 \) are periodic of period less than \( 2D \) with \( x_1 \neq y_1 \). The set of such pairs \((x_1, y_1)\) is finite (by assumption) and will be denoted by \( \mathcal{K}_1 \). There are two subcases.

Case 1.1: \( x_1 \) and \( y_1 \) lie on distinct orbits. If so \( \mathcal{M} \) can be written in a compressed form as \( \mathcal{M} = J_c \mathcal{V}_c \) with

\[
J_c = \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \quad \mathcal{V}_c = \begin{pmatrix} p_\alpha(x_1) \\ \vdots \\ p_\alpha(x_p) \\ p_\alpha(y_1) \\ \vdots \\ p_\alpha(y_q) \end{pmatrix},
\]

where \( p, q \) are the periods of \( x_1, y_1 \) (or \( D \) if the periods are greater than \( D \)), respectively. Further, \( C_1 \) is a \( D \times p \) circulant matrix with first row \( 1, 0, \ldots \) and \( C_2 \) is a \( D \times q \) circulant matrix with first row \(-1, 0, \ldots\). The rank of \( \mathcal{V}_c \) is equal to the number of its rows by Lemma \( \Box \) and \( J_c \) is nonzero. Therefore, we may assert that the rank of \( \mathcal{M} \) is 1 or greater.

Case 1.2: \( x_1 \) and \( y_1 \) lie on the same periodic orbit. In this case, we may write

\[
J_c = C_1, \quad \mathcal{V}_c = \begin{pmatrix} p_\alpha(x_1) \\ \vdots \\ p_\alpha(x_p) \end{pmatrix},
\]

where \( p \) is the period of \( x_1 \), \( C_1 \) is a \( D \times p \) circulant matrix whose first row is of the form \( 1, 0, \ldots, 0, -1, 0, \ldots, 0 \). Again, we conclude that the rank of \( \mathcal{M} \) is greater than 1.

Suppose \( G_\alpha(x_1, y_1) = 0 \) for some \((x_1, y_1) \in \mathcal{K}_1\). Then \( \mathcal{M} c_\alpha = 0 \) and \( c_\alpha \) must lie on a hyperplane of co-dimension 1 or greater. Because \( \mathcal{K}_1 \) is finite, we may assert \( G_\alpha(x_1, y_1) \neq 0 \) for all \((x_1, y_1) \in \mathcal{K}_1\) with probability 1 relative to the ball \( ||c_\alpha|| \leq a_0 \). Case 1 is now complete.

Case 2: Define \( \mathcal{K}_2(\delta) \) to be the set of all \((x_1, y_1) \in \mathcal{K} \times \mathcal{K} \) satisfying

1. \( ||x_1 - y_1|| \geq \delta \),
2. \( \text{dist}((x_1, y_1), \mathcal{K}_1) \geq \delta \). All distances in this paper use the \( L_2 \) or spectral norm.

The matrix \( \mathcal{M} \) has a rank equal to \( D \) for each point in \( \mathcal{K}_2(\delta) \), as we will prove by breaking up case 2 into subcases.

Case 2.1: \( x_1, \ldots, x_D, y_1, \ldots, y_D \) are \( 2D \) distinct points. In this case, \( \mathcal{M} = J \mathcal{V} \) has rank equal to \( D \) as noted at the beginning of the proof.

Case 2.2: \( x_1, \ldots, x_D \) are distinct, \( y_1, \ldots, y_D \) are distinct, and neither \( x_1 \) nor \( y_1 \) is a periodic point of period less than \( 2D \), but \( y_1 = x_j \) or \( x_1 = y_j \) for \( j \in \{2, \ldots, D\} \). Without loss of generality, we assume \( y_1 = x_j \).

In this case, the compressed form is \( \mathcal{M} = J_c \mathcal{V}_c \) with

\[
J_c = (C_1), \quad \mathcal{V}_c = \begin{pmatrix} p_\alpha(x_1) \\ \vdots \\ p_\alpha(x_D+j-1) \end{pmatrix},
\]

where \( C_1 \) is \( D \times (D + j - 1) \) circulant matrix with first row equal to \( 1, 0^{j-2}, -1, 0^{D-1} \). The \(-1\) does not wrap around and the rank of \( C_1 \) and therefore of \( \mathcal{M} \) is \( D \).
Case 2.3: $x_1$ periodic of period less than $2D$ and $y_1$ not so (or vice versa, which may be ignored without loss of generality). In this case, the compressed form is $\mathcal{M} = J_c \mathcal{V}_c$ with

$$J_c = (C_1, C_2), \quad \mathcal{V}_c = \begin{pmatrix} p_\alpha(x_1) \\ \vdots \\ p_\alpha(x_p) \\ p_\alpha(y_1) \\ \vdots \\ p_\alpha(y_D) \end{pmatrix},$$

where $p$ is the period of $x_1$, $C_1$ is a $D \times p$ circulant matrix with first row 1, 0, ..., and $C_2$ is a $D \times D$ circulant matrix with first row $-1, 0, \ldots$. The column rank of $C_2$ is equal to $D$ and therefore the rank of $\mathcal{M}$ is also $D$.

We can now complete case 2 as follows. Suppose $G_\alpha(x_1, y_1) = 0$ for some $(x_1, y_1) \in \mathcal{K}_2(\delta)$. By assumption (2) about $C_K$, cover $\mathcal{K}_2(\delta)$ with $C_K/\epsilon^{2d}$ or fewer $\epsilon$-balls for $\epsilon > 0$. At this point, we introduce an assumption about $L$:

**Assumption** about $L$ (1): The Lipschitz constant of $G_\alpha(x_1, y_1)$ with respect to $(x_1, y_1) \in K \times K$ and with $||c_\alpha|| \leq a_0$ is bounded by $L$. The existence of $L$ is a consequence of the compactness of $K \times K$, the compactness of $||c_\alpha|| \leq a_0$, and the differentiability assumption about the observation function $o$ and the diffeomorphism $\phi$.

It then follows that if $G_\alpha(x_1, y_1) = 0$ at some point $(x_1, y_1) \in \mathcal{K}_2(\delta)$, then $||G_\alpha(x_1, y_1)|| = ||\mathcal{M}(c_\alpha)|| \leq L \epsilon$ at the center of one of the $\epsilon$-balls covering $\mathcal{K}_2(\delta)$. Define

$$\sigma_\delta = \min_{(x_1, y_1) \in \mathcal{K}_2(\delta)} \sigma_D(\mathcal{M}).$$

By compactness of $\mathcal{K}_2(\delta)$, $\sigma_\delta$ exists and is positive. By the transfer of volume Lemma II which is applied with $r \leftarrow D$, the probability of $||G_\alpha(x_1, y_1)|| \leq L \epsilon$ relative to the ball $||c_\alpha|| \leq a_0$ at a point $(x_1, y_1) \in \mathcal{K}_2(\delta)$ is upper bounded by

$$\frac{D_\alpha ! L^D \epsilon^D}{\sigma_D^D a_0^D}.$$  

Because $\mathcal{K}_2(\delta)$ can be covered with $C_K/\epsilon^{2d}$ or fewer $\epsilon$-balls, the probability that $G_\alpha(x_1, y_1) = 0$ for some $(x_1, y_1) \in \mathcal{K}_2(\delta)$ is upper bounded by

$$\frac{C_K \epsilon^{2d}}{\epsilon^{2d}} \cdot \frac{D_\alpha ! L^D \epsilon^D}{\sigma_D^D a_0^D}.$$  

Because $D > 2d$ and by taking $\epsilon \to 0$, we conclude that the probability of $G_\alpha(x_1, y_1) = 0$ for some $(x_1, y_1) \in \mathcal{K}_2(\delta)$ relative to $||c_\alpha|| \leq a_0$ is one. Case 2 is now complete.

To complete the proof of injectivity, take the union of the measure zero sets in case 2 with $\delta = 1, \frac{1}{2}, \frac{1}{2^2}, \ldots$ and the measure zero set in case 1. Outside of that measure 0 subset of the ball $||c_\alpha|| \leq a_0$, we have $G_\alpha(x_1, y_1) \neq 0$ for $(x_1, y_1) \in K \times K$ and $x_1 \neq y_1$. \qed
The ideas in the proof presented above are from [9], although our presentation is more precise and formally complete. Theorem 8 makes an assumption on periodic points of period \( < 2D \) and not \( \leq D \) as in [9]. To see why the more stringent assumption is needed, we turn to [9, p. 611, case 3]. The case “\( x \) and \( y \) are not both periodic with period \( \leq w \)” is considered (\( w \) is \( D \) in our notation) and it is stated that \( J_{xy} \) (which is \( J_c \) in our notation) is triangular of rank \( D \). Unfortunately, that statement is not correct.

To understand why that statement is not true, assume \( D = 6 \). Suppose \( x_1 \) is a periodic point of period \( 8 > D \) and that \( y_1 = x_5 \). Then \( J_c \) will be a \( 6 \times 8 \) circulant matrix which looks as follows:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

Evidently, the rank of this matrix is \( 4 < D \).

The easiest way to fix the minor error is to assume the number of periodic points of period \( < 2D \) to be finite as we have done. However, Sauer et al [9] place conditions on the box counting dimension of the set of periodic points of period \( p \). The conditions involving quantities such as rank(\( BC_{pq}^w \)) are not easy to interpret and it is unclear what they mean. The basic idea of assuming a bound on the box counting dimension of periodic points of a certain period is a sound one. It can be developed fully using Lemma 6 about the rank of circulant matrices and variations of that lemma. We have not done so for two reasons. The proof becomes a great deal more complicated, and at this point having a clear and complete account of the main ideas appears more important than a slightly more general theorem. Additionally, if the box counting dimension of the set of periodic points is greater than 1, then 1 will be a characteristic multiplier that is repeated more than once, which is excluded in the immersivity theorem.

The gaps in [10] and [1] are much less minor. In [10], it is assumed that the delay map is an embedding in some neighborhood of the periodic points. The proof of that assumption is unlikely to be as straightforward as assumed. Even granting that assumption, the argument for transversality [10, p. 371] appears incomplete. In particular, it does not consider the possibility that perturbing the delay map of \( x \) may also perturb the delay map of \( x' \), for example, when \( x' = \phi(x) \) and the orbits of \( x, x' \) overlap. There are yet other aspects of the proof we were not able to verify. For example, [10, p. 370, case iii] seems to require \( x \) to be close to a periodic point and \( x' \) to be away from a periodic point. It is then asserted that \( x, \ldots, \phi^{2m}(x), x', \ldots, \phi^{2m}(x') \) are distinct. How could that be true if \( x \) is a fixed point? How is the possibility \( x' = \phi(x) \) handled?

The gaps in [1] also occur in handling overlaps of orbits and periodic points. The main argument [1, p. 598] entirely ignores the possibility that orbits of \( x^* \) and \( y^* \) may overlap. Further, it is suggested that difficulties associated with fixed points can be handled by adjusting the delays but no details are provided about carrying out that suggestion.

Going back to the work of Sauer et al [9], a point in our proof of Theorem 8 is worth calling to attention. In the proof, \( K_2(\delta) \) is covered with \( \epsilon \)-balls and it is assumed that every ball center is in \( K_2(\delta) \). It is not sufficient to start with any cover of \( K \times K \) because a ball center can be arbitrarily close to the diagonal or to a pair of periodic points and \( \sigma_D(M) \) may become
arbitrarily small.

If we say that a certain compact set s is covered by a certain number of ε-balls, it is assumed that each ball has a center that lies in s. That assumption comes up repeatedly in the proof of immersivity, which we now turn to. Once again all the ideas are from [9]. Here K is assumed to be a smooth, closed, and compact submanifold of dimension d and T₁K denotes its unit tangent bundle. If x ∈ K and v is tangent to K at x, then (x, v) ∈ T₁K if and only if ||v|| = 1.

**Theorem 9.** [9] If D > 2d − 1, K is invariant under ϕ, ϕ has finitely many points x ∈ K of period less than D, and all characteristic multipliers of each of those points are distinct, then x → Fα(x) is immersive over K with probability 1 relative to the ball ||cα|| ≤ a₀.

**Proof.** If x → Fα(x) and v is a tangent vector to K at x, then we denote the vector that v is mapped to by dFα(x, v). The following convention about v is an extension of the convention about x, y explained earlier.

**Convention** about v: If v₁ is tangent to K at x₁, then v₂ = ∂φ/∂x₁ | v₁, v₃ = ∂φ/∂x₂ | v₂, and so on. Because ϕ is a diffeomorphism, vᵢ are all nonzero like v₁.

We write dFα(x₁, v₁) = dF₀(x₁, v₁) + N(cα), where

\[
\mathcal{N} = J\mathcal{H}, \quad J = \begin{pmatrix} v₁^T & \cdots & v_D^T \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \nabla p₁(x₁) \\ \vdots \\ \nabla p_D(x_D) \end{pmatrix}.
\]

The proof will turn on the rank of \(\mathcal{N} = J\mathcal{H}\). If x₁, ..., x_D are distinct, the rank of \(\mathcal{N}\) is D because the rank of \(\mathcal{H}\) is equal to the number of its rows by Lemma 4 and the rank of J is obviously D.

To study the rank of \(\mathcal{N}\), it is useful to define the following disjoint sets of T₁K.

- \(\mathcal{K}₁\) is the set of all (x₁, v₁) such that x₁ is a periodic point of period less than D and v₁ is an eigenvector of the periodic point x₁. By eigenvector of a periodic point, we mean an eigenvector of the corresponding monodromy matrix.

- \(\mathcal{K}₂(δ)\) is the set of all (x₁, v₁) such that x₁ is a periodic point of period less than D and v₁ is a linear combination of two eigenvectors of x₁. It is also required that

  \[\text{dist}((x₁, v₁), \mathcal{K}₁) ≥ δ.\]

We will denote \(\mathcal{K}₂(0)\), where this last condition is not operative, by \(\mathcal{K}₂\). Evidently, \(\mathcal{K}₁\) is a subset of \(\mathcal{K}₂\).

- In general, \(\mathcal{K}_r(δ)\), where r = 2, ..., d, is defined as the set of (x₁, v₁) ∈ T₁K such that x₁ is a periodic point of period D or less and v₁ is a linear combination of r eigenvectors of the periodic point x₁. It is also required that

  \[\text{dist}((x₁, v₁), \mathcal{K}_{r⁻¹}) ≥ δ.\]

We will denote \(\mathcal{K}_r(0)\), where this last condition is not operative, by \(\mathcal{K}_r\). Evidently, \(\mathcal{K}_{r⁻１}\) is a subset of \(\mathcal{K}_r\).

This sequence of cases stops at r = d and does not go up to r = n because we are only interested in those eigenvectors of the periodic point x₁ that are also tangent to K. The assumption about the invariance of K is used here.
The rank of $H$ has full rank, the $\lambda_i$ about where $\lambda_i$ each ($x_k$ because the Vandermonde matrix obtained, and $d$ Assumption about map $\phi$. Assume $v_t = u_1 + \cdots + u_r$, where $u_i$ are eigenvectors at the periodic point $x_1$. Assume $v_2 = v_1 + \cdots + v_r$, where $v_i$ are eigenvectors at point $x_2$ obtained by pushing $u_i$ along with the map $\phi$. Likewise, if $x_1$ is of period $p$, assume that $v_p = w_1 + \cdots + w_r$.

Then the compressed form of $\mathcal{N}$ is $J_c = J_c H_c$ with

$$J_c = \begin{pmatrix}
    u_1^T + \cdots + u_r^T & v_1^T + \cdots + v_r^T \\
    \lambda_1 u_1^T + \cdots + \lambda_r u_r^T & \vdots & \vdots \\
    \lambda_1^2 u_1^T + \cdots + \lambda_r^2 u_r^T & \vdots & \vdots \\
    \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots \\
\end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_r$ are characteristic multipliers and the pattern is repeated until $D$ rows are obtained, and

$$H_c = \begin{pmatrix}
    \nabla p_0(x_1) \\
    \vdots \\
    \nabla p_0(x_p)
\end{pmatrix}.$$

The rank of $H_c$ is equal to the number of its rows by Lemma 4. The rank of $J_c$ is $\min(rp, D)$ because the Vandermonde matrix

$$\begin{pmatrix}
    1 & 1 & \cdots & 1 \\
    \lambda_1 & \lambda_2 & \cdots & \lambda_r \\
    \vdots & \vdots & \cdots & \vdots \\
    \lambda_1^{-1} & \lambda_2^{-1} & \cdots & \lambda_r^{-1}
\end{pmatrix}$$

has full rank, the $\lambda_i$ being distinct by assumption. Therefore, the rank of $\mathcal{N}$ is $r$ or greater for each $(x_t, y_1) \in \mathcal{K}_r(\delta)$.

To complete the proof, we note that $\mathcal{K}_r$ is of dimension $r - 1$ for $r = 1, \ldots, d$ and that $T_1K$ is of dimension $2d - 1$. A new assumption about $C_K$ is useful.

**Assumption** about $C_K$ (3): It is assumed that $\mathcal{K}_r$ can be covered with $C_K/\epsilon^{r-1}$ $\epsilon$-balls for $r = 1, \ldots, d$. It is assumed that $T_1K$ and therefore $\mathcal{K}_D(\delta)$ can be covered with $C_K/\epsilon^{2d-1}$ $\epsilon$-balls.

We also extend the assumption about the Lipshitz bound $L$.

**Assumption** about $L$ (2): It is assumed that the Lipshitz constant of $dF_0$ with respect to $(x_t, y_1) \in T_1K$ for $||c_0|| \leq a_0$ is upper bounded by $L$. This assumption too may be verified using compactness like the first assumption about $L$. 

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The proof may now be completed easily. Suppose \( dF_\alpha(x_1,v_1) = 0 \) for some \((x_1,v_1) \in K_r(\delta)\). Then \( ||dF_\alpha(x_1,v_1)|| \leq L \) at the center of one of the \( C_K/\epsilon^{-1} \) balls covering \( K_r(\delta) \). By the transfer of volume Lemma 1, the probability of such an event is upper bounded by

\[
\frac{C_K}{\epsilon^{-1}} \times \frac{D_\alpha! L^r \epsilon^r}{\sigma_0^r \sigma_{a_0}},
\]

where \( \sigma_\delta = \min \sigma_r(N) \) over \((x_1,v_1) \in K_r(\delta)\). The probability evidently goes to 0 as \( \epsilon \to 0 \) leaving us with a measure zero set of \( c_\alpha \) where \( F_\alpha \) is not immersive at some point in \( K_r(\delta) \) for \( r = 2, \ldots, d \). The sets \( K_1 \) and \( K_D(\delta) \) are handled similarly.

Theorem 9 assumes \( K \) to be a closed and compact submanifold. That assumption implies \( K_D(\delta) \) to be compact. If \( K_D(\delta) \) is compact, we may conclude that \( \min \sigma_D(N) \) over \((x_1,v_1) \in K_D(\delta) \) exists and is positive. The assumptions on \( K \) can be reduced. However, the technicalities that arise (see [2]) are extraneous to the main ideas in this paper.

5 Perturbing the dynamical system

Let \( \phi : \mathbb{R}^d \to \mathbb{R}^d \) be a diffeomorphism, which is as before but with \( n = d \). Let \( \psi(x) \) denote \( \frac{\partial \phi}{\partial x} \). The vector in \( \mathbb{R}^d \) with first component 1 and the others zero is denoted by \( e_1 \). The perturbed dynamical system is

\[
\phi_\alpha(x) = \phi(x) + e_1 (p_\alpha(x)) (c_\alpha)
\]

with \( \alpha \in I_{2D-1} \), where \( D \) is the embedding dimension. It may be noted we are only perturbing the first coordinate of \( \phi \). Because the observation function will be assumed to be \( o = \pi_1 \), it is enough to perturb only the first coordinate.

The delay vector under \( \phi \) is

\[
F_0(x_1) = \begin{pmatrix}
\pi_1 x_1 \\
\vdots \\
\pi_1 x_D
\end{pmatrix}.
\]

**Convention** about \( \tilde{x} \): It is assumed that \( \tilde{x}_1 = x_1 \). Thereafter, it is assumed that \( \tilde{x}_2 = \phi_\alpha(\tilde{x}_1) \), \( \tilde{x}_3 = \phi_\alpha(\tilde{x}_2) \), and so on.

The delay vector under \( \phi_\alpha \) is therefore

\[
F_\alpha(x_1) = \begin{pmatrix}
\pi_1 \tilde{x}_1 \\
\vdots \\
\pi_1 \tilde{x}_D
\end{pmatrix}.
\]

It is worthy of notice that \( \phi_\alpha \) perturbs only the first component of \( \phi \). Because the delay vector is built up using \( \pi_1 \), \( \phi_\alpha \) must perturb the first component. If not, the perturbation may not propagate to the delay vector at all. It turns out that perturbing only the first component is also sufficient to obtain a prevalence theorem.
Our first task is to express \( F_\alpha \) as a perturbation of \( F_0 \). That can be done by simply iterating the definition of \( \phi_\alpha \):

\[
\begin{align*}
\tilde{x}_1 &= x_1 \\
\tilde{x}_2 &= x_2 + e_1 (p_\alpha(x_1)) (c_\alpha) \\
\tilde{x}_3 &= x_3 + e_1 (p_\alpha(x_2)) (c_\alpha) + \psi(x_2)e_1 (p_\alpha(x_1)) (c_\alpha) + \mathcal{O}(c_\alpha^2).
\end{align*}
\]

Above and later, \( \mathcal{O}(c_\alpha^2) \) is the same as \( \mathcal{O} \left( ||c_\alpha||^2 \right) \). By following the pattern, we obtain

\[
\tilde{x}_j = x_j + e_1 (p_\alpha(x_{j-1})) (c_\alpha) + \rho_{j-1}(x_2, \ldots, x_{j-1}, p_\alpha(x_1), \ldots, p_\alpha(x_{j-2}))(c_\alpha) + \mathcal{O}(c_\alpha^2) \quad (5.1)
\]
for \( j = 2, \ldots, D \). Here it is important to note that \( \rho_{j-1} \) is linear in \( p_\alpha(x_1), \ldots, p_\alpha(x_{j-2}) \). For brevity, we will rewrite (5.1) as

\[
\tilde{x}_j = x_j + e_1 (p_\alpha(x_{j-1})) (c_\alpha) + \rho_{j-1}(c_\alpha) + \mathcal{O}(c_\alpha^2). \quad (5.2)
\]

We then get

\[
F_\alpha(x_1) = F_0(x_1) + \begin{pmatrix} 0 \\ V(x_1) \end{pmatrix} (c_\alpha) + \mathcal{O}(c_\alpha^2),
\]
with the matrix \( V(x_1) \) defined by

\[
V(x_1) = \begin{pmatrix} p_\alpha(x_1) \\ p_\alpha(x_2) + \pi_1 \rho_2 \\ \vdots \\ p_\alpha(x_D) + \pi_1 \rho_{D-1} \end{pmatrix}.
\]

The next lemma is about the rank of \( V(x_1) \).

**Lemma 10.** If \( x_1, \ldots, x_{D-1} \) are distinct, the rank of \( V(x_1) \) is equal to the number of its rows.

**Proof.** Suppose we consider

\[
\mathcal{V} = \begin{pmatrix} p_\alpha(x_1) \\ p_\alpha(x_2) \\ \vdots \\ p_\alpha(x_{D-1}) \end{pmatrix}.
\]

The rank lemma tells us that the rank of \( \mathcal{V} \) is equal to the number of its rows.

Now to produce a vector \( (a_1, \ldots, a_{D-1})^T \) in the range of \( V(x_1) \), we proceed as follows. Define

\[
\begin{align*}
a_1' &= a_1 \\
a_2' &= a_2 - \rho(x_1, a_1') \\
a_3' &= a_3 - \rho(x_1, x_2, a_1', a_2')
\end{align*}
\]
and so on. Because of the linearity of \( \rho \) in \( a_1 \), the vector \( (c_\alpha) \) that satisfies \( \mathcal{V}(c_\alpha) = (a_1', \ldots, a_{D-1}')^T \) also satisfies \( V(x_1)(c_\alpha) = (a_1, \ldots, a_{D-1})^T \). \( \square \)
The next lemma is similar. Part (c) of the following lemma is more general than Lemma 10 because we allow \( D^+ > D \).

**Lemma 11.** The following matrices have rank equal to the number of rows:

a. \[
\begin{pmatrix}
V(x_1) \\
V(y_1)
\end{pmatrix}
\]
assuming \( x_1, \ldots, x_{D-1}, y_1, \ldots, y_{D-1} \) to be distinct.

b. \[
\begin{pmatrix}
V(x_1) \\
m_k
\end{pmatrix}
\]
where \( m_k \) is the first \( k \) rows of \( V(y_1) \), assuming \( x_1, \ldots, x_{D-1}, y_1, \ldots, y_k \) to be distinct.

c. \[
\begin{pmatrix}
p_\alpha(x_1) \\
p_\alpha(x_2) + \pi_1 \rho_2 \\
\vdots \\
p_\alpha(x_{D+1}) + \pi_1 \rho_{D+1}
\end{pmatrix}
\]
assuming \( x_1, \ldots, x_{D+1} \) are distinct and \( D^+ \leq 2D \).

**Proof.** Similar to the previous proof.

Our second task in this section is to obtain \( dF_\alpha(x_1, v_1) \) as a perturbation of \( dF_0(x_1, v_1) \). It is helpful to introduce another convention:

**Convention** about \( w \): \( w_1 = v_1 \), \( w_2 \) is obtained as \( \frac{\partial \phi_\alpha}{\partial x} \bigg|_{\tilde{x}_1} \), \( w_3 \) is obtained as \( \frac{\partial \phi_\alpha}{\partial x} \bigg|_{\tilde{x}_2} \), and so on.

Thus, in effect we need to obtain perturbative expansions of \( w_i \). To do so, let us first note that

\[
\frac{\partial \phi_\alpha}{\partial x} = \psi(x) + e_1 \left( \nabla p_\alpha(x)^T \right) (c_\alpha).
\]

We substitute the above equation into the iteration that defines \( w_i \) and obtain

\[
\begin{align*}
w_1 &= v_1 \\
w_2 &= \psi(\tilde{x}_1)w_1 + e_1 \left( v_1^T \nabla p_\alpha(x_1) \right) (c_\alpha) \\
w_3 &= \psi(\tilde{x}_2)w_2 + e_1 \left( v_2^T \nabla p_\alpha(x_2) \right) (c_\alpha) + e_2 + \mathcal{O}(c_\alpha^2)
\end{align*}
\]

and so on. If we now use (5.1) to substitute for \( \tilde{x}_j \), we obtain

\[
w_j = v_j + e_1 \left( v_{j-1}^T \nabla p_\alpha(x_{j-1}) \right) (c_\alpha) + e_{j-1}(c_\alpha) + \mathcal{O}(c_\alpha^2),
\]

where \( e_{j-1} \) is linear in

\[
p_\alpha(x_1), \ldots, p_\alpha(x_{j-2}), \nabla p_\alpha(x_1), \ldots, \nabla p_\alpha(x_{j-2}).
\]

We may then write

\[
dF_\alpha(x_1, v_1) = \begin{pmatrix}
\pi_1 w_1 \\
\vdots \\
\pi_1 w_D
\end{pmatrix}
\]

\[
dF_0(x_1, v_1) + \begin{pmatrix}
0 \\
H(x_1, v_1)
\end{pmatrix} (c_\alpha) + \mathcal{O}(c_\alpha^2),
\]

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where

\[
H(x_1, v_1) = \begin{pmatrix}
v_T^1 \nabla p_\alpha(x_1) \\
v_T^2 \nabla p_\alpha(x_2) + \pi_1 q_2 \\
v_T^3 \nabla p_\alpha(x_3) + \pi_1 q_3 \\
\vdots \\
v_{D-1}^T \nabla p_\alpha(x_{D-1}) + \pi_1 q_{D-1}
\end{pmatrix}.
\]

The second task for this section concludes with a lemma about the rank of \(H(x_1, v_1)\).

**Lemma 12.** If \(x_1, \ldots, x_{D-1}\) are distinct, the rank of \(H(x_1, v_1)\) is equal to the number of its rows.

**Proof.** The proof is similar to that of Lemma 10. First consider

\[
\begin{pmatrix}
p_\alpha(x_1) \\
\vdots \\
p_\alpha(x_{D-1}) \\
v_T^1 \nabla p_\alpha(x_1) \\
\vdots \\
v_{D-1}^T \nabla p_\alpha(x_{D-1})
\end{pmatrix}.
\]

By Lemma 5, the rank of this matrix is equal to the number of its rows. Suppose we want to find \((c_\alpha)\) such that \(H(x_1, v_1)(c_\alpha)\) equals a specified vector \((a_1, \ldots, a_{D-1})^T\). To do so, we find a vector \((c_\alpha)\) such that the matrix displayed above applied to \(c_\alpha\) is equal to

\[
\begin{pmatrix}
0 \\
\vdots \\
0 \\
a_1' \\
\vdots \\
a_{D-1}'
\end{pmatrix},
\]

where \(a_1' = a_1, a_2' = a_2 - r_2\), where \(r_2\) is \(\pi q_2\) evaluated by replacing \(p_\alpha(x_1)\) by 0 and \(v_T^1 \nabla p_\alpha(x_1)\) by \(a_1'\), and so on. \(\square\)

The third and final task of this section is to track the perturbation of fixed points when the map \(\phi\) is perturbed to \(\phi_\alpha\).

**Lemma 13.** Suppose \(z_0 = \phi(z_0)\) and \(\psi(z_0)\) has no eigenvalue equal to 1. Under \(\phi \to \phi_\alpha\), the fixed point \(z_0\) perturbs to

\[
z_0(c_\alpha) = z_0 + (I - \psi(z_0))^{-1} e_1(p_\alpha(z_0))(c_\alpha) + \mathcal{O}(c_\alpha^2).
\]

**Proof.** The function \(z_0(c_\alpha)\) exists by the implicit function theorem. To obtain the expansion given in the lemma, start with

\[
\phi(z_0) + e_1(p_\alpha(z_0))(c_\alpha) = z_0
\]

differentiate with respect to \(c_\alpha\) and obtain \(\frac{\partial \phi}{\partial c_\alpha}\) at \(c_\alpha = 0\) using implicit differentiation. \(\square\)
6 The setting for injectivity and immersivity theorems

In the case where \( \phi \) is fixed and only the observation function \( o \) is perturbed, injectivity and immersivity are proved with respect to the ball \( ||c_\alpha|| \leq a_0 \), where \( a_0 > 0 \) can be anything. Such a thing is plainly impossible when \( \phi \) is perturbed to \( \phi_\alpha \). Under a perturbation, the map may even fail to be well defined or might blow-up in finite time. Therefore, we have to specify the setting for injectivity and immersivity theorems more carefully.

We will assume that \( K \) is a compact sphere in \( \mathbb{R}^d \) centered at the origin. The map \( \phi_\alpha \) will be proved to be injective and immersive over \( K \). It is assumed that \( K^+ \) is a compact sphere bigger than \( K \) and containing \( K \). If \( x_1 \in K \), it is assumed that \( x_1, \ldots, x_D \) all remain in \( K^+ \) for all \( ||c_\alpha|| \leq a_0 \). Further assumptions are enumerated below:

1. \( \phi_\alpha : \mathbb{R}^d \to \mathbb{R}^d \) is assumed to be a diffeomorphism (for \( ||c_\alpha|| \leq a_0 \)), that is \( C^3 \) or better.
2. The map \( \phi_\alpha \) has exactly \( m \) fixed points and those will be denoted by \( \xi_1(c_\alpha), \ldots, \xi_m(c_\alpha) \).
3. The map \( \phi_\alpha \) has no other periodic points of period less than 2D.
4. All the fixed points are hyperbolic and \( \pi_1 \xi_i(c_\alpha) \neq \pi_1 \xi_j(c_\alpha) \) if \( i \neq j \). This assumption is made with the intention of simplifying the proof so as to bring out the main techniques with greater clarity. Here we are essentially assuming injectivity between fixed points.
5. We will also assume that \( dF_\alpha \) is immersive at each fixed point for the same reason.

Now we will recall a few basic facts about Lebesgue points. A point \( a \in \mathbb{R}^n \) is a Lebesgue point of a measurable set \( A \subset \mathbb{R}^n \) if

\[
\lim_{\epsilon \to 0} \frac{\mu(A \cap \{u ||u - a|| < \epsilon\})}{\mu(\{u ||u - a|| < \epsilon\})} = 1.
\]

We will need the following basic lemma.

**Lemma 14.** If every point of the measurable set \( B \) is a Lebesgue point of the measurable set \( A \), then \( \mu(B - A) = 0 \).

**Proof.** Almost every point of \( A \) is a Lebesgue point of \( A \). Similarly, almost every point of \( A^c \), the complement of \( A \), is a Lebesgue point of \( A^c \). If \( a \) is a Lebesgue point of \( A^c \),

\[
\lim_{\epsilon \to 0} \frac{\mu(A \cap \{u ||u - a|| < \epsilon\})}{\mu(\{u ||u - a|| < \epsilon\})} = 0.
\]

The lemma follows from these observations. \( \Box \)

Lemma 14 will be crucial to our proof that \( \phi_\alpha \) is an embedding with probability 1 relative to \( ||c_\alpha|| < a_0 \). In the case where \( \phi \) is fixed and only the observation function is perturbed, the proofs of injectivity and immersivity consider the ball \( ||c_\alpha|| \leq a_0 \) all at once. Such a thing is not possible here. Instead, we have to pick \( c_\alpha^* \) satisfying \( ||c_\alpha^*|| < a_0 \) and localize around it and that is where Lemma 14 comes in.

In order to localize around \( c_\alpha^* \), we adopt new notation that is centered at \( c_\alpha^* \). The re-centered diffeomorphism \( \phi(x) + e_1(p_\alpha(x))(c_\alpha^*) \) is denoted by \( \Phi(x) \). Similarly, \( \Psi \) denotes \( \psi(x) + \)
e_1(\nabla p_{\alpha}(x))(c^*_\alpha). When we localize around c^*_\alpha, \Phi_{\alpha}(x) will denote \Phi(x) + e_1(p_{\alpha}(x))(c_{\alpha}). The fixed point \xi_j(c^*_\alpha) is denoted \Sigma_j. The fixed point \xi_j(c^*_\alpha + c_{\alpha}) is denoted \Sigma_j(c_{\alpha}).

**Convention** about \(x, y\) updated: \(x_1, x_2, \ldots\) are iterates of \(x_1\) under \(\Phi\). Similarly, \(y_1, y_2, \ldots\) are iterates of \(y_1\) under \(\Phi\).

**Convention** about \(\tilde{x}\) updated: \(\tilde{x}_1 = x_1\) and \(\tilde{x}_1, \tilde{x}_2, \ldots\) are iterates of \(x_1\) under \(\Phi_{\alpha}\).

**Convention** about \(v\) updated: we assume \((x_1, v_1) \in T_1K\) and \(v_2, v_3, \ldots\) are obtained by iterating \(d\Phi\).

All the lemmas of the previous section continue to hold after re-centering. The delay vector \(F_{\alpha}(x)\) defined in the previous section will be denoted by \(F_0(x)\) if \(c_{\alpha}\) is replaced by \(c^*_\alpha\). Similarly, if \(c_{\alpha}\) is replaced by \(c^*_\alpha + c_{\alpha}\) in the definition of \(F_{\alpha}(x)\), we will denote the re-centered delay vector by \(F_{\alpha}(x)\).

We may write

\[
F_{\alpha}(x_1) = F_0(x_1) + \begin{pmatrix} 0 \\ \Psi(x_1) \end{pmatrix}(c_{\alpha}) + O(c_{\alpha}^2),
\]

with the definition of \(\Psi(x_1)\) being the same as that of \(\Psi(x_1)\) but with \(\psi\) replaced by \(\Psi\). Likewise,

\[
dF_{\alpha}(x_1, v_1) = dF_0(x_1, v_1) + \begin{pmatrix} 0 \\ H(x_1, v_1) \end{pmatrix}(c_{\alpha}) + O(c_{\alpha}^2),
\]

with a similar alteration of the definition of \(H(x_1, v_1)\) to get \(H(x_1, v_1)\).

Finally, we note that the centered analogue of \(G_{\alpha}(x_1, y_1) = F_{\alpha}(x_1) - F_{\alpha}(y_1)\) is \(G_{\alpha}(x_1, y_1) = F_{\alpha}(x_1) - F_{\alpha}(y_1)\).

### 7 Proof of injectivity

In this section, our purpose is to prove that \(F_{\alpha}(x_1)\), defined in section 5, is injective for \(x_1 \in K\). The assumptions about \(C_K\) and \(L\) are carried forward from earlier sections, although the third assumption about \(C_K\) is not necessary in its entirety. Further assumptions will be stated as the need arises. Let us define \(\Delta\) is the minimum distance between fixed points of \(F_{\alpha}\) in \(K\) for \(\|c_{\alpha}\| \leq a_0\).

Let us define \(A_{1, \delta}\) to be the set of \(c_{\alpha}\) satisfying

1. \(\|c_{\alpha}\| < a_0\)
2. \(G_{\alpha}(\xi_j(\alpha), x_1) \neq 0\) for \(j \in \{1, \ldots, m\}\) and \(x_1 \in K\) with \(\|x_1 - \xi_j(\alpha)\| \geq 3\delta\) for each \(j \in \{1, \ldots, m\}\).

In this section and the next, we always assume \(\delta < \Delta / 3\).

**Lemma 15.** If \(D \geq 2d + 2\), every point of \(\|c_{\alpha}\| < a_0\) is a Lebesgue point of \(A_{1, \delta}\) and therefore the probability of \(A_{1, \delta}\) relative to the open ball \(\|c_{\alpha}\| < 1\) is \(1\).

**Proof.** Pick \(c^*_\alpha\) satisfying \(\|c^*_\alpha\| < a_0\). We will use an argument centered at \(c^*_\alpha\) to show that \(c^*_\alpha\) is a Lebesgue point of \(A_{1, \delta}\).

Pick \(a_1 > 0\) so small that \(\|\Sigma_j(\alpha) - \Sigma_j\| < \delta\) for \(\|c_{\alpha}\| \leq a_1\). Define \(K_{1, \delta}\) as the set of \(x_1 \in K\) such that \(\|x_1 - \Sigma_j\| \geq 2\delta\) for each \(j \in \{1, \ldots, m\}\).
Let us look at $G_\alpha(\Sigma_j(c_\alpha), x_1)$. Using Lemma 13 and the definition of $\nabla(x_1)$, we get

$$G_\alpha(\Sigma_j(c_\alpha), x_1) = G_0(\Sigma_j, x_1) + M(c_\alpha) + O(c_\alpha^2) \quad (7.1)$$

with $M = JV$ and

$$J = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ \vdots & \vdots \\ -1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} \nabla(x_1) \\ \pi_1(I - \Psi(\Sigma_j))^{-1}e_1p_\alpha(\Sigma_j) \end{pmatrix}.$$ 

There are two cases here. Suppose $\pi_1(I - \Psi(\Sigma_j))^{-1}e_1$ is nonzero. Then by Lemma 11 (b), the rank of $V$ is equal to the number of its rows. Therefore, the rank $JV$ is $D$. If in fact the corner entry $\pi_1(I - \Psi(\Sigma_j))^{-1}e_1$ is zero, we can drop the last column and first row of $J$ and conclude that the rank of $JV$ is $D - 1$. In either case, the rank of $M$ is $D - 1$ or greater.

Define $\sigma_\delta = \min \sigma_{D-1}(M)$, where the minimum is over $x_1 \in K_{1,\delta}$ and $||c_\alpha|| \leq a_1$. Cover $K_{1,\delta}$ with $C_K/\epsilon^d$ $\epsilon$-balls.

**Assumption about $L$ (3):** In (7.1), the $O(c_\alpha^2)$ term is upper bounded by $L||c_\alpha||^2$. Like the earlier assumptions about $L$, this assumption too is a direct consequence of compactness.

The earlier assumptions used $L$ as a bound on Lipschitz constants. Here $L$ is used as a bound on the Taylor series remainder.

Now suppose $G_\alpha(\Sigma_j(c_\alpha), x_1) = 0$ for some $j \in \{1, \ldots, m\}$ and some $x_1 \in K_{1,\delta}$. Because the Lipschitz constant of $G_\alpha(\Sigma_j(c_\alpha), x_1)$ with respect to $x_1$ is bounded by $L$, we must have $||G_\alpha(\Sigma_j(c_\alpha), x_1)|| \leq L\epsilon$ at an $x_1$ that is at the center of one the balls covering $K_{1,\delta}$.

Applying the nonlinear transfer of volume Lemma 2 with $r \leftarrow D - 1$ and $\sigma \leftarrow \sigma_\delta$, we find that the probability of $||G_\alpha(\Sigma_j(c_\alpha), x_1)|| \leq L\epsilon$ relative to $||c_\alpha|| \leq \epsilon^{1/2} < a_1$ is upper bounded by

$$D_\alpha^{2D-1}L^{D-1}\epsilon^{(D-1)/2}/\sigma_\delta^{D-1}.$$ 

Because the number of fixed points is $m$ and the number balls covering $K_{1,\delta}$ is $C_K/\epsilon^d$, the probability of $G_\alpha(\Sigma_j(c_\alpha), x_1) = 0$ for some $j \in \{1, \ldots, m\}$ and some $x_1 \in K_{1,\delta}$ relative to $||c_\alpha|| \leq \epsilon^{1/2}$ is upper bounded by

$$m \times \frac{C_K}{\epsilon^d} \times \frac{D_\alpha^{2D-1}L^{D-1}\epsilon^{(D-1)/2}}{\sigma_\delta^{D-1}}.$$ 

Evidently, the probability goes to zero as $\epsilon \to 0$ if $D \geq 2d + 2$. Thus, we have shown that $c_\alpha^*$ is a Lebesgue point of $A_{1,\delta}$ proving the lemma. 

Now define $A_{2,\delta}$ to be the set of $c_\alpha$ satisfying

1. $||c_\alpha|| < a_0$
2. $G_\alpha(x_1, \phi_\alpha(x_1)) \neq 0$ for $x_1 \in K$ with $||x_1 - \xi_j(\alpha)|| \geq 3\delta$ for each $j \in \{1, \ldots, m\}$.

**Lemma 16.** If $D \geq 2d + 1$, every point of $||c_\alpha|| < a_0$ is a Lebesgue point of $A_{2,\delta}$ and therefore the probability of $A_{2,\delta}$ relative to $||c_\alpha|| < a_0$ is 1.
Proof. As before, we pick $c_α^*$ satisfying $||c_α^*|| < a_0$ and will give an argument centered at $c_α^*$ to show that $c_α^*$ is a Lebesgue point of $A_{1,δ}$. As before, pick $a_1 > 0$ so small that $||Σ_j(α) - Σ_j|| < δ$ for $||c_α|| ≤ a_1$. As before, define $K_{1,δ}$ as the set of $x_1 ∈ K$ such that $||x_1 - Σ_j|| ≥ 2δ$ for each $j ∈ \{1, \ldots, m\}$.

Using (5.2), we get

$$G_α(\bar{x}_1, \bar{x}_2) = \begin{pmatrix} π_1x_1 - π_1x_2 & \vdots & π_1x_D - π_1x_{D+1} \end{pmatrix} + M(c_α) + O(c_α^2)$$

(7.2)

with $M = JV$ and

$$J = \begin{pmatrix} -1 & 1 & -1 & \vdots & -1 & 1 \\ 1 & -1 & 1 & \vdots & 1 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} p_α(x_1) & p_α(x_2) + π_1ρ2 & \vdots & p_α(x_D) + π_1ρD \end{pmatrix}.$$

By Lemma 11 (c), the rank of $V$ is equal to the number of its rows. Therefore, the rank of $M = JV$ is equal to $D$.

Define $σ_δ = \min σ_D(M)$, where the minimum is over $x_1 ∈ K_{1,δ}$ and $||c_α|| ≤ a_1$. Cover $K_{1,δ}$ with $C_K/ε^d$ $ε$-balls.

**Assumption** about $L (4)$: In (7.2), the $O(c_α^2)$ term is upper bounded by $L ||c_α^2||$. The first two assumptions about $L$ are both obtained from upper bounds on the derivative of $F_α(x)$ or $F_α(x)$ with respect to $x$. This assumption as well as the preceding one are obtained from upper bounds on the second derivative. In all cases, the assumptions are direct consequences of the compactness of $K$ and the ball $||c_α|| ≤ a_0$.

If $G_α(\bar{x}_1, \bar{x}_2) = 0$ for some $x_1 ∈ K_{1,δ}$, we must have $||G_α(\bar{x}_1, \bar{x}_2)|| ≤ Lε$ for some $x_1$ that is the center of one of the balls covering $K_{1,δ}$. Using the nonlinear transfer of volume Lemma 2, we find the probability of $G_α(\bar{x}_1, \bar{x}_2) = 0$ for some $x_1 ∈ K_{1,δ}$ relative to the ball $||c_α|| ≤ ε^{1/2} < a_1$ to be upper bounded by

$$\frac{C_K}{ε^d} × \frac{D_α^{1/2}L^Dε^{D/2}}{σ_δ^D}.$$

The limit of this probability as $ε → 0$ is zero. It follows that $c_α^*$ is a Lebesgue point of $A_{2,δ}$ completing the proof of this lemma.

Lemma 16 allows us to conclude that the delay vectors of $x_1$ and $φ_α(x_1)$ do not coincide typically if $x_1$ is a little removed from the fixed points of $φ_α$. More generally, we need to argue that the delay vectors of $x_1$ and $φ_α^{k-1}(x)$ do not coincide for $k = 3, \ldots, D$. To make that argument, we define $A_{k,δ}$ to be the set of $c_α$ satisfying

1. $||c_α|| ≤ a_0$
2. $G_α(x_1, φ_α^{k-1}(x_1)) ≠ 0$ for $x_1 ∈ K$ with $||x_1 - ξ_δ|| ≥ 3δ$ for each $j ∈ \{1, \ldots, m\}$ for $k = 2, \ldots, D$.

**Lemma 17.** For $D ≥ 2d + 1$ and $k = 2, \ldots, D$, every point of $||c_α|| < a_0$ is a Lebesgue point of $A_{k,δ}$ and therefore the probability of $A_{k,δ}$ relative to the ball $||c_α|| < a_0$ is 1.
Proof. The proof is almost identical to that of the previous lemma, which is a special case. The only significant difference occurs in the definition of $V$. In the general case,

$$V = \begin{pmatrix}
    p_\alpha(x_1) \\
p_\alpha(x_2) + \pi_1 p_2 \\
    \vdots \\
p_\alpha(x_{D+k-2}) + \pi_1 p_{D+k-2}
\end{pmatrix}.$$ 

Note that Lemma [11](c) still applies, implying the rank of $V$ to be equal to the number of its rows, because $D + k - 2 \leq 2D$. 

The final lemma of this section pertains to the set $A_{xy}(\delta)$. It is defined as the set of all $c_\alpha$ such that $||c_\alpha|| < a_0$ and $G_\alpha(x_1, y_1) \neq 0$ provided

1. $x_1, y_1 \in K$
2. $||x_1 - y_1|| \geq \delta$ (which excludes the diagonal of $K \times K$)
3. $||x_1 - \xi_j(\alpha)|| \geq 3\delta$ and $||y_1 - \xi_j(\alpha)|| \geq 3\delta$ for $j \in \{1, \ldots, m\}$ (so that both $x_1$ and $y_1$ stay away from fixed points)
4. $||x_1 - \phi^{-k}(y_1)|| \geq 2\delta$ and $||y_1 - \phi^{-k}(x_1)|| \geq 2\delta$ for $k = 2, \ldots, D$ (so that $x_1$ does not come too close to the iterates of $y_1$ and vice versa).

**Lemma 18.** For $D \geq 4d + 2$, every point of $||c_\alpha|| < a_0$ is a Lebesgue point of $A_{xy, \delta}$ and therefore the probability of $A_{xy, \delta}$ relative to the ball $||c_\alpha|| < a_0$ is 1.

**Proof.** Again the argument begins by centering at some $c_\alpha^*$ satisfying $||c_\alpha^*|| < a_0$. However, the conditions on $a_1$ this time are different. The radius $a_1$ must be so small that for $||c_\alpha|| \leq a_1$ the following conditions are satisfied:

1. $||\Sigma_j(\alpha) - \Sigma_j|| < \delta$
2. For any $x_1 \in K$, $||\tilde{x}_j - x_j|| \leq \delta$ for $j = 1, \ldots, D$.

The set $K_{xy, \delta}$ is defined as the set of $(x_1, y_1) \in K \times K$ satisfying the following conditions:

1. $||x_1 - \Sigma_j|| \geq 2\delta$ and $||y_1 - \Sigma_j|| \geq 2\delta$ for $j \in \{1, \ldots, m\}$
2. $||x_1 - y_1|| \geq \delta$
3. $||x_1 - y_j|| \geq \delta$ and $||y_1 - x_j|| \geq \delta$ for $j \in \{2, \ldots, m\}$.

We have

$$G_\alpha(x_1, y_1) = G_\alpha(x_1, y_1) + M(c_\alpha) + O(c_\alpha^2). \quad (7.3)$$ 

The top row of $M$ is zero. The rest of the $D - 1$ rows below are given by $JY$

$$J = \begin{pmatrix}
    1 & \cdots & -1 \\
    \cdots & 1 & \cdots \\
    1 & \cdots & 1
\end{pmatrix}, \quad Y = \begin{pmatrix}
    \mathcal{V}(x_1) \\
    \mathcal{V}(y_1)
\end{pmatrix}.$$ 

By Lemma [11] the rank of $Y$ is equal to the number of its rows. Therefore the ranks of $JY$ and $M$ are both equal to $D - 1$.

Define $\sigma_D = \min \sigma_{D-1}(M)$, where the minimum is over $(x_1, y_1) \in K_{xy, \delta}$ and $||c_\alpha|| \leq a_1$. Cover $K_{xy}$ with $C_K/\epsilon^{2d}$ balls.

**Assumption** about $L$ (5): The $O(c_\alpha^2)$ term in (7.3) is upper bounded by $L ||c_\alpha||^2$. 

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Suppose $G_\alpha(x_1, y_1) = 0$ for some $(x_1, y_1) \in K_{xy, \delta}$. Then we must have $\|G_\alpha(x_1, y_1)\| \leq L\epsilon$ for an $(x_1, y_1)$ that is at the center of one of the balls covering $K_{xy, \delta}$. Applying the nonlinear transfer of volume Lemma 2 we find the probability of $G_\alpha(x_1, y_1) = 0$ for some $(x_1, y_1) \in K_{xy, \delta}$ relative to the ball $\|c_\alpha\| \leq \epsilon^{1/2} < a_1$ to be upper bounded by

$$C_K \epsilon^{2d} \times \frac{D_{\alpha} \gamma^{D-1} L^{D-1} \epsilon^{D-1}}{\sigma_D^D}.$$ 

If $D \geq 4d + 2$, the limit of this probability as $\epsilon \to 0$ is 0. Therefore, every $c_\alpha^*$ satisfying $\|c_\alpha^*\| < a_0$ is a Lebesgue point of $A_{xy, \delta}$, which completes the proof of the lemma.

We are now prepared to state and prove the main theorem of this section.

**Theorem 19.** Assuming $a_0$ and $\phi_\alpha$ satisfy the conditions laid down in section 6 and $D \geq 4d + 2$, the delay mapping $F_\alpha$ is injective on the set $K$ with probability one relative to the ball $\|c_\alpha\| < a_0$.

**Proof.** The proof follows from Lemmas 15, 17, and 18 by taking the limit $\delta \to 0$ through a countable sequence.

## 8 Proof of immersivity

All the main techniques have been demonstrated in the proof of injectivity of the delay mapping $F_\alpha$. The assumption in section 6 that $dF_\alpha$ is immersive at all fixed points in $K$ simplifies the proof of immersivity considerably.

Define $A_{T, \delta}$ as the set of all $c_\alpha$ satisfying $\|c_\alpha\| < a_0$ and $F_\alpha$ is immersive at all $x_1 \in K$ satisfying $\|x_1 - \xi_j(\alpha)\| \geq 3\delta$ for $j \in \{1, \ldots, m\}$. In other words, we are requiring $dF_\alpha(x_1, v_1) \neq 0$ if $(x_1, v_1) \in T_1K$ and $x_1$ is removed from each periodic point by at least $3\delta$.

**Lemma 20.** For $D \geq 4d$, every point of $\|c_\alpha\| < a_0$ is a Lebesgue point of $A_{T, \delta}$ and therefore the probability of $A_{T, \delta}$ relative to $\|c_\alpha\| < a_0$ is 1.

**Proof.** We center at $c_\alpha^*$ satisfying $\|c_\alpha^*\| < a_0$ as before. Again as before, we assume $a_1$ to be so small that $\|\Sigma_j(c_\alpha) - \Sigma_j\| < \delta$ for $\|c_\alpha\| \leq a_1$.

Define $K_{T, \delta}$ to be the set of all $(x_1, v_1) \in T_1K$ satisfying $\|x_1 - \Sigma_j\| \geq 2\delta$ for $j \in \{1, \ldots, m\}$. Then

$$dF_\alpha(x_1, v_1) = dF_0(x_1, v_1) + \mathcal{N}(c_\alpha) + O(c_\alpha^2) \quad (8.1)$$

with

$$\mathcal{N} = \begin{pmatrix} 0 \\ \mathbb{H}(x_1, v_1) \end{pmatrix}.$$

By Lemma 12 the rank of $\mathcal{N}$ is $D - 1$.

Define $\sigma_D = \min \sigma_{D-1}(\mathcal{N})$, where the minimum is taken over $(x_1, v_1) \in K_{T, \delta}$ and $\|c_\alpha\| \leq a_1$. Cover $K_{T, \delta}$ with $C_K/\epsilon^{2d-1} \epsilon$-balls.

**Assumption** about $L$ (5): In (8.1), the $O(c_\alpha^2)$ term is upper bounded by $L \|c_\alpha\|^2$. Here, we are effectively assuming a bound on the third derivative of $F_\alpha(x_1)$ with respect to $x_1$ over the compact sets $x_1 \in T_1K$ and $\|c_\alpha\| \leq a_0$. 

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If $dF_\alpha(x_1, v_1) = 0$ for some $(x_1, v_1) \in K_{T, \delta}$, then we must have $||dF_\alpha(x_1, v_1)||$ for some $(x_1, v_1)$ that is at the center of one of the $\epsilon$-balls covering $K_{T, \delta}$. The nonlinear transfer of volume lemma implies that the probability of $dF_\alpha(x_1, v_1) = 0$ for some $(x_1, v_1) \in K_{T, \delta}$ relative to $||c_\alpha|| \leq \epsilon^{1/2} < a_1$ is upper bounded by

$$
\frac{C_K}{\epsilon^{2d-1}} \times \frac{D_\alpha^{12D-1}L^{D-1}\epsilon^{D-1}}{\sigma^{D-1}}.
$$

If $D \geq 4d$, this probability goes to zero as $\epsilon \to 0$. Therefore, every $||c_\alpha^*|| < a_0$ is a Lebesgue point of $\mathcal{A}_{T, \delta}$, proving the lemma.

We are now prepared to state and prove the immersivity theorem.

**Theorem 21.** Suppose $a_0$ and $\phi_\alpha$ satisfy the assumptions laid down in section 6 and suppose $D \geq 2d$. The delay map $F_\alpha$ is then immersive at every point of $K$ with probability 1 relative to the ball $||c_\alpha|| < a_0$.

**Proof.** The proof follows by taking $\delta \to 0$ through a countable sequence in the previous Lemma and using the assumption made in section 6 about immersivity at fixed points.

9 Discussion

The delay map may be viewed in light of the Whitney embedding theorem [2]. However, it has some characteristics of its own. One of these is the possibility that orbits of two distinct points can overlap. There are other distinctive characteristics related to periodic orbits and eigenvectors.

In this article, we showed how to prove that the delay map is an embedding using the concept of Lebesgue points. For the delay map $F_\alpha(x)$ with $o = \pi_1$ to be an embedding with probability 1 relative to the ball $||c_\alpha|| < 1$, we require the embedding dimension to satisfy $D \geq 4d + 2$.

We conjecture that the delay mapping is an embedding for $D \geq 2d + 1$. The more restrictive $4d + 2$ requirement comes in when applying the nonlinear transfer of volume lemma. The extra dimensions are used to absorb the effect of the nonlinear term. Some evidence for this conjecture may be found in our earlier work [4].

In our opinion, it would be desirable to obtain prevalence versions of classical theorems such as the Kupka-Smale theorem [5]. The differential topology proofs rely heavily on the bump function and genericity is weaker than almost sureness in probability. It is hoped that the technique based on Lebesgue points introduced here will be useful in that regard.

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