Geodesic deviation equation in Bianchi cosmologies

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Abstract. We present the Geodesic Deviation Equation (GDE) for the Friedmann Robertson Walker (FRW) universe and we compare it with the equation for Bianchi type I model. We justify consider this cosmological model due to the recent importance the Bianchi Models have as alternative models in cosmology. The main property of these models, solutions of Einstein Field Equations (EFE) is that they are homogeneous as the FRW model but they are not isotropic. We can see this because they have a non-null Weyl tensor, which is zero for FRW model. We study some consequences of this Weyl tensor in the GDE.

1. Introduction
Bianchi cosmological models are homogeneous solutions of EFE. These models generalize FRW model, although they are homogeneous, as FRW one, generally they are anisotropic. Standard Cosmological Model has survived almost all observational tests [1], however Bianchi cosmologies haven’t been discarded. Today the Universe is highly isotropic, but in early times it couldn’t been so, and it is an open question how anisotropies died. Standard Model fails to explain quadrupole anomaly [2], obtained from Cosmic Microwave Background data. This anomaly have a plausible explanation with Bianchi models, for that reason these models are considered today [2]. Thus we considered the idea of studying Geodesic Deviation Equation (GDE) in some of these models.

2. 1+3 Covariant description
We assume a fluid description for the cosmological model. The 4-velocity $u^\alpha$ of preferred worldlines is normalized, $u_\alpha u^\alpha = -1$. Given this 4-velocity, we define the projection tensor $h_{\alpha\beta}$, the expansion scalar $\Theta$, the shear $\sigma_{\alpha\beta}$ and the vorticity $\omega_{\alpha\beta}$ [3], [4]. In FRW case $\sigma_{\alpha\beta} = \omega_{\alpha\beta} = 0$. These quantities are related with Weyl tensor. In FRW case Weyl tensor is zero, so it is conformally flat, but it is not the case in Bianchi models. In the 1 + 3 covariant description we can split the Weyl tensor into an electric $E_{\alpha\beta}$ and a magnetic part $H_{\alpha\beta}$ [4], [3].

3. Deviation Equation
In a space-time torsion free, if $V^\alpha$ is the normalized tangent vector field of a fiducial geodesic, parameterized by an affine parameter $\nu$, with $V_\alpha V^\alpha = \varepsilon$, with $\varepsilon = +1, 0, -1$ if the geodesics are spacelike, null or timelike respectively; if $\eta^\alpha$ is the deviation vector for the congruence, then the evolution of the separation or the deviation vector $\eta^\alpha$ evolves according to GDE [5]:

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\[
\frac{\delta^2 \eta^\alpha}{\delta y^2} = - R^\alpha_{\beta\gamma\delta} V^\beta \eta^\gamma V^\delta
\]

where \( \frac{\delta T^\alpha_{\beta\gamma\delta}}{\delta y^y} = V^\gamma \nabla^\gamma T^\alpha_{\beta\gamma\delta} \) is the covariant derivative along the geodesic.

Now, let’s see how is this equation in Bianchi type I model, given by the metric\[6\], \[7\]:

\[
ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2
\]

If we define \( \tau(t) := a(t)b(t)c(t) \), from the energy-momentum tensor conservation we get an analogous expression to a known Friedmann equation \[6\], \[7\]:

\[
\frac{\ddot{\tau}}{\tau} = \frac{3\kappa}{2} (\mu - p)
\]

where \( \mu \) is the Energy density and \( p \) is the pressure. As we see in the GDE, there is a difference if space-time has a nonzero Weyl tensor. Ellis and van Elst in \[5\] consider this equation for FRW space-time.

For Bianchi type I model vorticity and Magnetic part of Weyl tensor are equal to zero, \( \omega_{\alpha\beta} = 0, H_{\alpha\beta} = 0 \). The Electric and shear tensors are non-null and diagonal. For that reason, if we define \( F = E_{\gamma\beta} V^\gamma \eta^\beta \) and \( E = -V_\alpha u^\alpha \), if we suppose a perfect fluid, \( q^\alpha = 0, \pi_{\alpha\beta} = 0 \) we have that GDE for this space is:

\[
R^\alpha_{\beta\gamma\delta} V^\beta \eta^\gamma V^\delta = \left[ \frac{1}{3} (\mu + \Lambda) + \frac{1}{2} (\mu + p) E^2 \right] \eta^\alpha + F (2w^\alpha + \eta^\alpha - V^\alpha) + E^\alpha \eta^\gamma (\varepsilon + 2E^2)
\]

where \( \Lambda \) is the cosmological constant. The first part of this expression is the force term representing the perfect part of the fluid \[4\] and only gives isotropic deviation. In FRW case we only have this term and reflects spatial isotropy of space-time, deviation vector only changes in magnitude but not in direction. The other terms change the direction of \( \eta^\alpha \).

If we suppose an equation of state \( p = w\epsilon, (w = 0 \text{ for dust}, w = \frac{1}{3} \text{ for radiation}) \), then, for \( \Lambda = 0 \) we get\[6\]:

\[
\tau = At^{\frac{2}{1+w}}
\]

Given \( \tau \) as a function of \( t \) we can get \( a(t), b(t) \) and \( c(t) \). The electric Weyl tensor tends to zero. It is diagonal and its components decay. For this model the vorticity and the Magnetic part of Weyl tensor are null. The EFE imply that \( q^\alpha = 0 \). This means we have a fluid without viscosity. However, we can have an imperfect fluid because the model allows to have an anisotropic pressure \( \pi_{\alpha\beta} \). But this anisotropic pressure tensor must be diagonal, following EFE’s. Given \( \tau \) as a function of \( t \) we have expressions for \( a(t), b(t) \) and \( c(t) \). The evolution of the three scale factors for \( w = 0 \) is shown in Figure 1 and for \( w = \frac{1}{3} \) in figure 2. We can see that in radiation case anisotropy is greater than in matter dominated case, but in both cases anisotropy dies.

The \( \sigma \) scalar defined by \( \sigma^2 = \frac{1}{2} \sigma_{\alpha\beta} \sigma_{\alpha\beta} \) also measure the degree of anisotropy in the model. In Figure 4 we illustrate the evolution of \( \sigma^2 \).
Figure 1. Scale factors for \( w = 0, A = \frac{3}{4} \).

Figure 2. Scale factors for \( w = \frac{1}{3}, A = \frac{3}{4} \).

Figure 3. Electric Weyl tensor for \( A = \frac{3}{4}, w = \frac{1}{3} \).

4. Conclusions

The evolution of the Electric Weyl tensor show us that this model tends to isotropy as \( t \) grows, the model isotropizes. Also the evolution of shear scalar. In FRW only the magnitude \( \eta \) will change along a geodesic, while its spatial orientation remain fixed [5]. So, in early times the deviation is anisotropic, the orientation is important, for example, along the \( z \) axis deviation will grow, while along \( x \) and \( y \) axis the effect of Weyl Tensor is to contract the deviation, because there are the other terms that are also present in FRW. Here we considered a perfect fluid, \( \pi_{\alpha\beta} = 0 \). Along \( x \) axis, the effect of anisotropy is minimal.

Figure 4. Electric Weyl tensor for \( A = \frac{3}{4}, w = \frac{1}{3} \).

Figure 5. Shear scalar evolution for \( A = \frac{3}{4} \).

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