On some control problems of dynamic of reactor

A V Baskakov and N P Volkov

National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), 31, Kashirskoe shosse, 115409 Moscow, Russia

e-mail: npvolkov@mephi.ru

Abstract. The paper analyzes controllability of the transient processes in some problems of nuclear reactor dynamics. In this case, the mathematical model of nuclear reactor dynamics is described by a system of integro-differential equations consisting of the non-stationary anisotropic multi-velocity kinetic equation of neutron transport and the balance equation of delayed neutrons. The paper defines the formulation of the linear problem on control of transient processes in nuclear reactors with application of spatially distributed actions on internal neutron sources, and the formulation of the nonlinear problems on control of transient processes with application of spatially distributed actions on the neutron absorption coefficient and the neutron scattering indicatrix. The required control actions depend on the spatial and velocity coordinates. The theorems on existence and uniqueness of these control actions are proved in the paper. To do this, the control problems mentioned above are reduced to equivalent systems of integral equations. Existence and uniqueness of the solution for this system of integral equations is proved by the method of successive approximations, which makes it possible to construct an iterative scheme for numerical analyses of transient processes in a given nuclear reactor with application of the developed mathematical model. Sufficient conditions for controllability of transient processes are also obtained. In conclusion, a connection is made between the control problems and the observation problems, which, by to the given information, allow us to reconstruct either the function of internal neutron sources, or the neutron absorption coefficient, or the neutron scattering indicatrix.

1. Introduction

The needs for reliable, easily controllable and safe nuclear power facilities permanently increase in the recent time. This growth represents an additional motivation for further developing mathematical theory of nuclear reactors, which is a relatively young and rapidly progressing theory. Some scientific publications [1-12] on mathematical theory of nuclear reactors confirm this fact. When developing the theory, nuclear specialists and technicians are trying to create such models of nuclear reactors whose reliable operation should be maximally independent on human factor effects. The problems related with mathematical modeling of physical processes occurring in nuclear reactors appear on the agenda. These problems should be resolved both for the processes of the reactor designing and for the control processes of the reactor operation.
2. Definition of the linear control problem

In this section we considered the mathematical model that was used to describe reactor dynamics without accounting for temperature feedbacks in Ref. [4]. The mathematical model is based on system of the following integro-differential equations:

1) non-stationary anisotropic multi-velocity equation of neutron transport:

\[
\frac{\partial u(x,v,t)}{\partial t} + (v, \nabla_x)u(x,v,t) + \Sigma(x,v,t)u(x,v,t) + \sum_{k=1}^{N} z_k R_k(x,v,t) + \int_{V} J(x,v,v',t)u(x,v',t)dv' + F(x,v,t),
\]

(1)

2) balance equation of delayed neutrons:

\[
\frac{\partial R_k(x,v,t)}{\partial t} = -z_k R_k(x,v,t) + \int_{V} J_{k}(x,v,v',t)u(x,v',t)dv' \quad \forall \ k = \overline{1,N},
\]

(2)

where \((x,v,t) \in D = G \times V \times [0,T]\).

In this mathematical model the function \(u(x,v,t)\) defines distribution of the neutrons flying through spatial point \(x \in G\) with velocity \(v \in V\) at time moment \(t \in [0,T]\). The functions \(\Sigma(x,v,t)\), \(J(x,v,v',t)\) and \(F(x,v,t)\) characterize properties of the medium where some mass transport processes take place in. The function \(\Sigma(x,v,t)\) is a neutron absorption coefficient, the function \(J(x,v,v',t)\) is a neutron scattering indicatrix, and the function \(F(x,v,t)\) is an intensity of neutron source. Kernels of the scattering integrals \(J_k(x,v,v',t)\) characterize distribution of secondary neutrons. The function \(R_k(x,v,t)\) characterizes distribution of nuclides – emitters of delayed neutrons in the k-th energy group. The strictly convex region \(G\) is a domain of spatial coordinates. The closed set \(V\) in the spherical layer \(\{0 < v_s \leq |v| \leq v_f < \infty\}\) is a set of velocities \(v\) with which the emitted particles can move.

It seems natural to assume that the transient process takes place without accounting for external neutron sources, i.e. an external neutron radiation is not able to pass through walls of the reactor vessel. This assumption can be mathematically written as the following boundary condition:

\[
u(x,v,t) = 0, \quad (x,v,t) \in \partial G \times V.\]

(3)

where \(\partial G = \{(x,v) \in \partial G \times V : (v,n_x) < 0\}\), \(n_x\) is an outer normal to the border \(\partial G\) of the domain \(G\) in spatial point \(x\).

The following initial conditions can be also set:

\[
u(x,v,0) = \varphi(x,v), \quad (x,v) \in \overline{G} \times V, \quad R_k(x,v,0) = R_{k0}(x,v), \quad \forall \ k = \overline{1, N}, \quad (x,v) \in \overline{G} \times V.
\]

(4)

(5)

The linear control problem can be defined by setting the following final condition:

\[
u(x,v,t_f) = \psi(x,v), \quad (x,v) \in \overline{G} \times V,
\]

(6)

where time \(t_f \in [0,T]\).

The control problem 1. It is required to find sufficient conditions on initial data under which the reactor could be converted from the initial state (4) to the target state (6) for the time interval \(t_f \in [0,T]\) by using the time-independent part \(f(x,v)\) of the function of neutron sources:

\[
F(x,v,t) = f(x,v)g(x,v,t),
\]

(7)
where $f$ is an acceptable control action (distributed stationary control), $g_i$ is an a priori known function (the corrective function).

The control problem 1 is a linear problem from the standpoint of searching for the spatially distributed control action $f(x, v)$. The control problem 1 by its definition is identical to the inverse problem described in Ref. 12.

3. Solution of the linear control problem

To resolve the problem, we used the following functional spaces introduced in Ref. 5:

1) $H_{\infty}(D)$ is a space of the functions $u$, which belong to the class $L_{\infty}(D)$ together with their generalized derivatives $u_t$ and $(v, \nabla_x)u$ on the set $D$ and possess the traces $u|_{\Gamma_-}$ on the set $\Gamma_- = \gamma_x \times (0, T)$ from $L_{\infty}(\Gamma_-)$, i.e.

$$H_{\infty}(D) = \left\{ u \in L_{\infty}(D) : \frac{\partial u}{\partial t} + (v, \nabla_x)u \in L_{\infty}(D), u|_{\Gamma_-} \in L_{\infty}(\Gamma_-) \right\},$$

which is a Banach space with respect to the following norm:

$$\|u\|_{H_{\infty}} = \left\| u \right\|_{L_{\infty},D} + \left\| \frac{\partial u}{\partial t} \right\|_{L_{\infty},D} + \left\| (v, \nabla_x)u \right\|_{L_{\infty},D} + \left\| u|_{\Gamma_-} \right\|_{L_{\infty},\Gamma_-} < \infty.$$  

2) $L'_{\infty}(D)$ is a space of the functions $R_k$, which belong to the class $L_{\infty}(D)$ together with their generalized derivatives $\frac{\partial R_k}{\partial t}$ on the set $D$, i.e.

$$L'_{\infty}(D) = \left\{ R_k \in L_{\infty}(D) : \frac{\partial R_k}{\partial t} \in L_{\infty}(D) \right\},$$

which is a Banach space with respect to the following norm:

$$\|R_k\|_{L'_{\infty}} = \left\| R_k \right\|_{L_{\infty},D} + \left\| \frac{\partial R_k}{\partial t} \right\|_{L_{\infty},D} < \infty.$$  

Where $\| \cdot \|_{L_{\infty},D}$ is a norm in the space $L_{\infty}(D)$.

The solution of the linear control problem 1 is presented in the following theorem.

**Theorem 1.** Let us assume that $\Sigma \in L'_{\infty}(D)$; $J \in L'_{\infty}(D \times V)$; $\varphi, (v, \nabla_x)\varphi \in L_{\infty}(G \times V)$; $\psi, (v, \nabla_x)\psi \in L_{\infty}(G \times V)$; $g_i \in L_{\infty}(\Gamma_-) \in L_{\infty}(\Gamma_-)$, $g \in L_{\infty}(D)$,

$$|g_i(x, v, l)| \geq g_o > 0; \quad J_k \in L'_{\infty}(D \times V); \quad R_{k0} (x, v) \in L_{\infty}(G \times V) \quad \forall k = 1, N;$$

and the following concordance conditions are satisfied.

$$\varphi(x, v) = 0 \quad \psi(x, v) = 0 \quad (x, v) \in \gamma_-.$$  

Then the reactor can be converted from the initial state (4) to the target state (6) for the time interval $t_1$ by using the only possible stationary spatially distributed control action $f(x, v)$ that belongs to the space $L_{\infty}(G \times V)$. In this case, the state function $u(x, v, t)$ belongs to the class $H_{\infty}(D)$. 

---

3
Proof. To prove the theorem, we used the method proposed in Refs. 7-9] for kinetic equation of neutron transport.

By integrating on the characteristic, the system of equations (1)-(8) can be equivalently reduced to the following system of integral equations:

\[ u(y + v \xi, v, t) = \Phi(y, v, \xi, t) + \]
\[ + \int \left[ (Pu)(y + v \theta, v, \theta - \xi + t) + f(y + v \theta, v)g_i(y + v \theta, v, \theta - \xi + t) \right] d\theta, \]
\[ f(y + v \xi, v) = [g_i(y + v \xi, v, t_i)]^{-1} \left\{ \frac{\partial}{\partial \xi} \psi(y + v \xi, v) - (P \psi)(y + v \xi, v, t_i) + \right. \]
\[ + \int \left[ \frac{\partial}{\partial t} (Pu)(y + v \theta, v, \theta - \xi + t_i) + f(y + v \theta, v) \frac{\partial}{\partial t} g_i(y + v \theta, v, \theta - \xi + t_i) \right] d\theta - \]
\[ - \sum_{k=1}^{N} z_k \left[ R_{k0}(y + v \xi, v) \exp\{ -z_i t_i \} + z_k \int R_{k0}(y + v \theta, v) \exp\{ -z_i (\theta - \xi + t_i) \} d\theta \right] \]

where \( y + v \xi = x \) is a characteristic direct line with the direction vector \( v, \xi \) is a parameter from the range \( \xi_-, \xi_+ \), while \( y + v \xi_\pm \in \partial G \), and \( (y + v \xi, v) \in \gamma_+ \);

\[ \Phi(y, v, \xi, t) = \varphi(y + v(\xi - t), v) + \sum_{k=1}^{N} z_k \int R_{k0}(y + v \theta, v) \exp\{ -z_i (\theta - \xi + t) \} d\theta, \]
\[ \eta = \xi - t \text{ at } 0 \leq t \leq \xi - \xi_-, \xi \in [\xi_-, \xi_+] \text{ and } \]
\[ \Phi(y, v, \xi, t) = \sum_{k=1}^{N} z_k \int R_{k0}(y + v \theta, v) \exp\{ -z_i (\theta - \xi + t) \} d\theta, \]
\[ \eta = \xi_- \]
at \( \xi - \xi_- < t \leq T, \xi \in [\xi_-, \xi_+] \);

\( (Pu)(y + v \xi, v, t) = \sum_{k=1}^{N} z_k \int \exp\{ -z_i (t - \tau) \} \int J_k(y + v \xi, v', v', t) u(y + v \xi, v, v', t) dv' d\tau + \]
\[ + \int J(y + v \xi, v, v', t) u(y + v \xi, v, v', t) dv' - \Sigma(y + v \xi, v, v', t) u(y + v \xi, v, v', t) \]

Hence, the control problem 1 has the sole generalized solution \( f \in L_\infty(G \times V) \) and \( u \in H_\infty(D) \), only if the system of integral equations (9), (10) can be unambiguously solved in the mentioned functional class. This is possible under the following condition: \( v_0^d \text{diam } G \leq a_1 \), where \( a_1 \) is a constant depending on the time interval \( t_1 \) and on the norm of the operator \( P \).

Farther, we studied a principal capability to resolve the system of integral equations (9), (10). This system can be re-written in the following operator form:
\[ \{ u, f \} = A \{ u, f \} = \{ A_1 \{ u, f \}, A_2 \{ u, f \} \} \]

where \( A_j, A_2 \) are the operators with the domains \( H_\alpha(D) \times L_\alpha(G \times V) \), which can act on the functions \( \{ u, f \} \) in the accordance to the rules defined by the right parts of integral equations (9) and (10), respectively. In this case, the operator \( A \) can continuously reflect the Banach space \( H_\alpha(D) \times L_\alpha(G \times V) \) into itself. Since the image \( A_1 \{ u, f \} \) has the generalized derivatives belonging to the space \( L_\alpha(D) \), its trace is an element of the space \( L_\alpha(\Gamma) \). The image \( A_2 \{ u, f \} \) belongs to the space \( L_\alpha(G \times V) \).

Further, we studied compressibility of the operator \( A \) and revealed that its second degree is a contraction operator under the following condition: \( \nu_0 \delta G \leq \lambda_2 \), where \( \lambda_2 \) is a constant depending on the norm of the functions \( \Sigma, J, J_k, \varphi, \psi, g, f \) in the respective spaces.

Hence, according to the Banach principle for immovable point, under the condition \( \nu_0 \delta G \leq \min \{ \lambda_1, \lambda_2 \} \), where \( \{ u, f \} \) is the sole solution for the system of integral equations (8), (9) in the space \( H_\alpha(D) \times L_\alpha(G \times V) \). Hence, the linear control problem 1 has the sole solution \( \{ u, f \} \) under the found constraints on initial data, i.e. we obtained sufficient conditions for controllability of transient processes for the given mathematical model of the reactor dynamics.

Thus, the theorem 1 was completely proved.

**Remark 1.** In the theorem 1 the conditions \( \Sigma \in L'_\alpha(D), J \in L'_\alpha(D \times V), g \in L'_\alpha(D), J_k \in L'_\alpha(D \times V) \) \( \forall \ k = \overline{1, N} \) can be replaced by the following new conditions: \( \Sigma, (v, \nabla_x) \Sigma \in L_\alpha(D), J, (v, \nabla_x)J \in L_\alpha(D \times V), g \in L_\alpha(D), J_k, (v, \nabla_x)J_k \in L_\alpha(D \times V) \) \( \forall \ k = \overline{1, N} \), respectively. Under these new conditions the conclusions derived from the theorem 1 remain to be correct.

4. **Non-linear control problems**

**The control problem 2.** Is it possible to convert nuclear reactor from the initial state (4) to the target state (6) for the time interval \( t_f \in (0, T) \) by using the time-independent part \( \sigma(x, v) \) of the neutron absorption coefficient:

\[ \Sigma(x, v, t) = \sigma(x, v)g_2(x, v, t), \]  \( \text{(11)} \)

where \( \sigma \) is an acceptable control action (distributed stationary control), and \( g_2 \) is an a priori known function (the corrective function)?

The control problem 2 is a non-linear problem from the standpoint of searching for the spatially distributed control action \( \sigma(x, v) \).

Positive response to the question asked above was obtained in the following theorem.

**Theorem 2.** Let us assume that \( J \in L'_\alpha(D \times V) \); \( F \in L'_\alpha(D) \); \( \varphi, (v, \nabla_x) \varphi \in L_\alpha(G \times V) \), \( \psi, (v, \nabla_x) \psi \in L_\alpha(G \times V) \). \( \varphi \big| \gamma_\nu - \varphi \big| \gamma_\nu, \psi \big| \gamma_\nu - \psi \big| \gamma_\nu \); \( g_2 \in L'_\alpha(D) \), \( |g_2(x, v, t)| \psi(x, v) \geq g_2 > 0 \); \( J_k \in L_\alpha(D \times V) \); \( R_k(x, v) \in L_\alpha(G \times V) \) \( \forall \ k = \overline{1, N} \); and the following concordance conditions are satisfied:

\[ \varphi(x, v) = 0 \quad \psi(x, v) = 0 \quad (x, v) \in \gamma_- \]  \( \text{(12)} \)
Then the reactor can be converted from the initial state (4) to the target state (6) for the time interval $t_f$ by using the only possible stationary spatially distributed control action $\sigma(x,v)$ that belongs to a compact set $\Omega$ from the space $L_\infty(G \times V)$. In this case, the state function $u(x,v,t)$ belongs to the closed limited set $U$ from the class $H_\infty(D)$. 

Proof. The two sought functions $\{u, \sigma\}$ can be found in the space $H_\infty(D) \times L_\infty(G \times V)$ on the following set

$$U \times \Omega = \{\{u, \sigma\} : u \in H_\infty(D), \sigma \in L_\infty(G \times V), \|u\|_{H_\infty} \leq k_1, \|\sigma\|_{L_\infty} \leq k_2\},$$

where constants $k_1, k_2$ depend on the norms of initial data.

Like the method we used in proving the theorem 1, we reduced the system of equations (1)-(6), (12) to the equivalent system of the second kind integral equations for the two sought functions $\{u, \sigma\}$. This system can be written in the following operator form:

$$\{u, \sigma\} = B\{u, \sigma\} = B\{u, \sigma\}, B\{u, \sigma\}.$$ 

In this case, the second degree of the operator $B$ is a contraction operator for all elements of the set $U \times \Omega$, under condition of the following inequality: $\nu_3 \|diam G \| \leq b$, where $b$ is a constant that depends on initial data of the control problem 2.

This means that the derived system of integral equations has the sole solution $\{u, \sigma\}$ in the set $U \times \Omega$. Hence, there is the only one control action $\sigma(x,v)$, which is able to convert nuclear reactor from the initial state (4) to the target state (6) for the time interval $t_f$. So, the control problem 2 is an unambiguously resolvable problem. Thus, the theorem 2 was completely proved.

Remark 2. The conclusions derived from the theorem 2 remain to the same if the following conditions $J \in L_\infty(D \times V)$, $F \in L_\infty(D)$, $g_1 \in L_\infty(D)$, $J_k \in L_\infty(D \times V) \forall k = I, N$ are replaced by the following new conditions $J$, $(v, \nabla_x)J \in L_\infty(D \times V)$, $F$, $(v, \nabla_x)F \in L_\infty(D)$, $g_2$, $(v, \nabla_x)g_2 \in L_\infty(D)$, $J_k$, $(v, \nabla_x)J_k \in L_\infty(D \times V) \forall k = I, N$, respectively.

Now we considered one else type of non-linear control problems. In these control problems the time-independent part $j(x,v)$ of the kernel in the scattering integral

$$J(x,v,v',t) = j(x,v)g_1(x,v,v',t),$$

is used as a spatially distributed control action, where $j$ is an acceptable control action (distributed stationary control), $g_1$ is an a priori known function (the corrective function).

The control problem 3. Is it possible to convert nuclear reactor from the initial state (4) to the target state (6) for the time interval $t_f \in (0, T]$ by using the time-independent part $j(x,v)$ of the neutron scattering indicatrix $J(x,v,v',t)$?

Positive response to the question asked above was obtained by proving the following theorem.

Theorem 3. Let us assume that $\Sigma \in L_\infty(D)$; $F \in L_\infty(D)$; $\varphi$, $(v, \nabla_x)\varphi \in L_\infty(G \times V)$, $\psi$, $(v, \nabla_x)\psi \in L_\infty(G \times V)$, $\varphi|_{x_0}$, $\psi|_{x_0} \in L_\infty(\gamma_0)$; $g_1 \in L_\infty(D)$,
\[ |g_j(x,v,t_j)\psi(x,v)| \geq g_{a}\mathbf{v} > 0 ; \quad J_k \in L_{a}(D \times V) ; \quad R_{k}(x,v) \in L_{a}(G \times V) \]

\[ \forall \ k = \overline{1,N} ; \text{ and the following concordance conditions are satisfied.} \]

\[ \varphi(x,v) = 0 \quad \psi(x,v) = 0 \quad (x,v) \in \mathcal{G} \quad (14) \]

Then the reactor can be converted from the initial state (4) to the target state (6) for the time interval \( t_1 \) by using the only possible stationary spatially distributed control action \( f(x,v) \) that belongs to a compact set \( \Omega \) from the space \( L_{a}(G \times V) \). In this case, the state function \( u(x,v,t) \) belongs to the closed limited set \( U \) from the class \( H_{a}(D) \).

The theorem 3 can be proved in quite the same manner as we used to prove the theorem 2.

**Remark 3.** In the theorem 3 the conditions \( \Sigma \in L_{a}(D) \), \( F \in L_{a}(D) \), \( g_j \in L_{a}(D) \), \( J_k \in L_{a}(D \times V) \) \( \forall \ k = \overline{1,N} \) can be replaced by the following new conditions:

\( \Sigma \), \( (v,\nabla_x)\Sigma \in L_{a}(D) \), \( F \), \( (v,\nabla_x)F \in L_{a}(D) \), \( g_j \), \( (v,\nabla_x)g_j \in L_{a}(D) \).

\( J_k \), \( (v,\nabla_x)J_k \in L_{a}(D \times V) \) \( \forall \ k = \overline{1,N} \), respectively. Under these new conditions the conclusions derived from the theorem 3 remain to be correct.

5. **Conclusion**

So, the present paper analyzed one linear and two non-linear problems related with controllability of transient processes in the objects which can be described by equations of nuclear reactor dynamics. The methods proposed in the paper make it possible to construct the spatially distributed control actions, namely the solutions of the defined control problems, as well as to determine the respective phase states of the objects under investigation.

It should be noted here that, as the proofs of the theorems mentioned above are based on the method of successive approximations, this fact allows us to constructively build solutions for three control problems 1, 2, 3. The proposed methods open the possibilities to construct numerical algorithms for determination of the required functions.

In conclusion, we would like to note that the control problems we considered in the paper may be interpreted as the problems of final observation, i.e. as the problems related with restoration of the spatially distributed control actions in the forms (7), (11), (13) and with determination of the respective phase states from the results obtained during observation of the objects under investigation.

The work was performed and the paper was prepared for publishing within the frames of the Program on Competitiveness Enhancement of National Research Nuclear University MEPhI (Moscow Engineering Physics Institute); contract No. 02.a03.21.0005,27.08.2013.

**References**

[1]  Marchuk G I Methods for calculating nuclear reactors. Moscow, 1961, pp 667 (In Russian).
[2]  Albertoni S and Montagnini B On the Spectrum of Neutron Transport Equation in Finite Bodies J. Math. Analys. Appl., 1966, v 13, p 19.
[3]  Akcasu Z., Lellouche G S and Shotkin L M Mathematical Methods in Nuclear Reactor Dynamics Academic Press, New York – London, 1971, pp 465.
[4]  Lewins J Nuclear Reactor Kinetics and Control Jefery Lewins. Pergamon Press, 1978, pp 264.
[5]  Kryanev A V and Shihkov S B Problems of nuclear reactor mathematical theory (Non-linear analysis). Moscow, Energoatomizdat, 1983, pp 280. (In Russian).
[6]  Germogenova T A Local properties of solutions of the transport equation. Moscow, Nauka, 1986, pp 272. (In Russian).
[7] Kuznetsov Yu A and Morozov S F System of integro-differential equations for nuclear reactor kinetics. Differential equations, 1974, v 10, N. 8, pp 1491–1503. (In Russian).

[8] Prilepko A I and Volkov N P Inverse problems of finding parameters of a nonstationary kinetic transport equation from supplementary information on traces of the unknown function. Differential equations, 1988. v 24, N 1, pp 136-146. (In Russian).

[9] Volkov N P On some inverse problems for time-dependent transport equation. Ill-posed problems in natural sciences. Moscow: TVP-VSP, 1992, pp 431-438.

[10] Volkov N P Solvability of certain inverse problems for the non-stationary kinetic transport equation. Computational mathematics and mathematical physics. Pleiades Publishing, 2016, v 56, N 9, pp 1598-1603.

[11] Baskakov A V and Volkov N P On some conditions for the solvability of control problems for the non-stationary kinetic neutron transport equation. Electronic journal “MGOU Vestnik” (Series on physics and mathematics), 2013, N 1, pp 1-10. (In Russian).

[12] Volkov N P Linear Inverse Problem of Dynamics of the Reactor. Journal of Physics: Conference Series, 2017, v 788, N 1, pp 1-4.