Ladder proof of nonlocality without inequalities and without probabilities

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October 11, 2018

Abstract

The ladder proof of nonlocality without inequalities for two spin-1/2 particles proposed by Hardy [1] and Hardy et al. [2] works only for nonmaximally entangled states and goes through for 50% of pairs at the most. A similar ladder proof for two spin-1 particles in a maximally entangled state is presented. In its simplest form, the proof goes through for 17% of pairs. An extended version works for 100% of pairs. The proof can be extended to any maximally entangled state of two spin-s particles (with $s \geq 1$).

PACS numbers: 03.65.Bz

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1 Introduction

Recently, Hardy [1] and Hardy et al. [2] have presented a generalization of Hardy’s proof of Bell’s theorem without inequalities for two spin-$\frac{1}{2}$ particles [3]. Neither the original proof nor the ladder generalization and not even the generalizations for the case of two spin-$s$ particles proposed in [4, 5] work for maximally entangled states. The improvement in the ladder generalization of the two spin-$\frac{1}{2}$ particles case comes from the fact that, adding new observables, the probability for the proof to go through increases. It grows from 9% of pairs in the original proof with two alternative observables on each particle [3] to almost 50% of pairs when an infinite number of alternative observables are considered [1, 2].

In this paper, I present a similar ladder proof of Bell’s theorem without inequalities for two spin-1 particles prepared in the singlet state. In its simplest form, treated in section 3, the proof works for 17% of pairs. An extended version, considered in section 4, works for 100% of pairs and uses a finite number of alternative observables. In section 5, I explain how the proof can be extended to any other maximally entangled state of two spin-1 particles and to any maximally entangled state of two spin-$s$ particles (with $s \geq 1$). The advantages and disadvantages of the proposed ladder proof in order to design a real experiment to test Bell’s theorem are discussed in section 6. Finally, in section 7, some differences between ladder proofs of Bell’s theorem and the proofs of the so-called Kochen-Specker with locality theorem initially proposed by Heywood and Redhead [6] are remarked upon. In order to introduce some notations, I begin in section 2 with a brief review of the ladder proof by Hardy et al.
2 Ladder proof for two spin-$\frac{1}{2}$ particles

The scenario considered by Hardy et al. [1, 2] is the following: Two spin-$\frac{1}{2}$ particles, initially prepared in some specific quantum state, are confined to space-like separated regions of space-time. On the first particle only one measurement $\hat{A}_k$ chosen from a set $\{\hat{A}_j\}_{j=0}^K$ can be made. Each one of these potential measurements has the outcome $A_k$ or $A_k^\perp$. Similarly, on the second particle only one measurement $\hat{B}_k$ from the set $\{\hat{B}_j\}_{j=0}^K$ can be made to see whether it has the outcome $B_k$ or $B_k^\perp$. Hardy et al. show that there exist quantum states $|\eta\rangle$ and sets of measurements $\{\hat{A}_j\}_{j=0}^K$ and $\{\hat{B}_j\}_{j=0}^K$ with the following properties:

$$P_\eta(A_K, B_K) = P_K \neq 0,$$

(1)

$$P_\eta(B_{j-1}|A_j) = 1,$$

(2)

$$P_\eta(A_{j-1}|B_j) = 1,$$

(3)

$$P_\eta(A_0, B_0) = 0,$$

(4)

for $j = 1$ to $K$. From these properties we can build a ladder of inferences based on EPR condition for elements of reality (“If, without in any way disturbing a system, we can predict with certainty (i.e., with a probability equal to unity) the value of a physical quantity, then there exist an element of physical reality corresponding to this physical quantity” [5]). This ladder of inferences will conclude in a contradiction. We start from the proportion, $P_K$, of pairs in which we get $A_K$ and $B_K$, given by equation (1). Then, using EPR condition, we have the following inferences: since, according to (2) (3), $A_K$ ($B_K$) allows us to predict with certainty $B_{K-1}$ ($A_{K-1}$), then $B_{K-1}$ ($A_{K-1}$) was an element of reality. Similarly, since $A_{K-1}$ ($B_{K-1}$) allows us to predict with certainty $B_{K-2}$ ($A_{K-2}$), then $B_{K-2}$ ($A_{K-2}$) was an element
of reality, etc. Finally, since $A_1$ allows us to predict $B_0$ and $B_1$ allows us to predict $A_0$, $B_0$ and $A_0$ were both elements of reality. Therefore, in the state $|\eta\rangle$ we should have $A_0$ and $B_0$ for at least the proportion $P_K$ of pairs. But, according to (4), we should never get $A_0$ and $B_0$. So we reach a contradiction. In fact, this way of viewing the contradiction is not unique [1].

The whole reasoning can be summarized with the aid of some graphs. The graph $A_K \leftrightarrow_{P_K} B_K$ represents the statement (1): the outcomes $A_K$ and $B_K$ occur together with probability $P_K$. Graphs like $A_j \longleftrightarrow B_{j-1}$ represent statements like (2): if $A_j$ happens then we can predict $B_{j-1}$ with certainty. Analogously, graphs like $A_{j-1} \longleftrightarrow B_j$ represent statements like (3). Finally, the graph $A_0 \longleftrightarrow B_0$ represents the statement (4): the outcomes $A_0$ and $B_0$ never occur together. Using these graphs, the ladder proofs by Hardy et al. with two, three and $K + 1$ settings are represented in Figure 1. For the case of two alternative observables on each particle, the maximum value of $P_1$ is 9.0% [3]. Adding more observables this value grows. In case of three observables on each particle, the maximum value of $P_2$ is 17.5% [1, 2]. As $K$ tends to infinity $P_K$ tends to 50% [1, 2]. For details on these calculations the reader is referred to [1, 2, 3]. The name “ladder” comes from the fact that the proof uses a chain—of adjustable length—of predictions with certainty.

On the other hand, Clifton and Niemann [4] and, recently, Ghosh and Kar [5] have proposed generalizations for the case of two spin-$s$ particles (with $s \geq 1$) of Hardy’s original proof. Their generalizations are based on statements similar to (1)-(4) in which all the measurements $\hat{A}_j$ and $\hat{B}_j$ ($j = 0, 1$) correspond to nondegenerate operators (components of spin). In case of two spin-1 particles, Ghosh and Kar have found a maximum value for
$P_1$ of 13.2\%. Both Clifton and Niemann’s extension \[4\] and Ghosh and Kar’s extension \[5\] do not work for maximally entangled states.

3 Ladder proof for two spin-1 particles in the singlet state

As far as I know, no ladder proof for maximally entangled states exists. However, the Hilbert space corresponding to a system of two spin-$s$ particles, $\mathcal{H}_{2s+1} \otimes \mathcal{H}_{2s+1}$, with $s \geq 1$, is richer than the Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_2$ of two spin-$\frac{1}{2}$ particles. In particular, if $s \geq 1$, we can measure and predict the outcomes of local observables corresponding to degenerated operators. For instance, in case of two spin-1 particles, there are sets of three mutually compatible local observables which can be measured on the same run of the experiment. In this paper I exploit these facts to construct a ladder proof without inequalities of Bell’s theorem for maximally entangled states of two spin-1 particles.

The scenario is analogous to the one described in section 2, changing the two space-like separated spin-$\frac{1}{2}$ particles prepared in a nonmaximally entangled state by two space-like separated spin-1 particles prepared in the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{3}} (|\bar{h}\rangle |-\bar{h}\rangle + |\bar{h}\rangle |\bar{h}\rangle - |0\rangle |0\rangle) .$$

(5)

On the first particle a measurement of the square of the spin component in some direction $n_k$, $(S_1 \cdot n_k)^2$, chosen from a large but specific set of them can be made. Each one of these possible measurements has the outcome 0 or $\bar{h}^2$. 

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However, in the singlet state,

\[ P_\psi ([S_2 \cdot n_k]^2 = 0 \mid [S_1 \cdot n_k]^2 = 0) = 1, \quad (6) \]

and therefore, if the outcome of \((S_1 \cdot n_k)^2\) is 0 \((\hbar^2)\), we can predict with certainty that the outcome of \((S_2 \cdot n_k)^2\) will be 0 \((\hbar^2)\). Properties (6) and (7), and the corresponding properties obtained by interchanging particle 1 and 2, will be used in our proof in the same way as properties (2) and (3) in the ladder proof for two spin-\(\frac{1}{2}\) particles. To summarize this kind of inferences I shall continue to use the same kind of graphs as in the previous proof.

In addition, in case of two spin-1 particles in the singlet state, if the outcome of measuring \((S_1 \cdot n_k)^2\) is 0, then we can predict with certainty that the outcome of measuring \((S_2 \cdot n_j)^2\), in every direction \(n_j\) orthogonal to the direction \(n_k\), will be \(\hbar^2\). This occurs because

\[ P_\psi ([S_2 \cdot n_j]^2 = \hbar^2 \mid [S_1 \cdot n_k]^2 = 0) = 1, \quad \forall n_j \perp n_k, \quad (8) \]

and, consequently \((S_1 \cdot n_j)^2\) and \((S_2 \cdot n_k)^2\) cannot be both zero if \(n_j\) is orthogonal to \(n_k\), i.e.,

\[ P_\psi ([S_1 \cdot n_j]^2 = 0, [S_2 \cdot n_k]^2 = 0) = 0, \quad \forall n_j \perp n_k. \quad (9) \]

Property (8) can be used to predict with certainty the result of more than one measurement on the second particle from a single measurement on the first (or vice versa). New graphs must be introduced to reflect these new inferences. For instance, the graph \( A_k \overset{B_l}{\rightleftharpoons} B_m \) represents the case of two predictions: \(A_k\) could be \((S_1 \cdot n_k)^2 = 0\) and \(B_l\) and \(B_m\) could be \((S_2 \cdot n_l)^2 = \hbar^2\) and \((S_2 \cdot n_m)^2 = \hbar^2\), respectively, being \(n_l\) and \(n_m\) both orthogonal to \(n_k\).
Similarly, by interchanging the first and the second particles there are also graphs like \( A_l \rightarrow \bullet \rightarrow B_k \). Property (II) plays the same role as property (I) in the ladder proof by Hardy et al., therefore, to represent it I will use the same graph as in section 2.

Moreover, for a spin-1 particle the observables \((S_1 \cdot n_i)^2\), \((S_1 \cdot n_j)^2\), \((S_1 \cdot n_k)^2\) are compatible if \(n_i, n_j, n_k\) are mutually orthogonal directions. In fact, their values sum \(2h^2\)

\[
(S_1 \cdot n_i)^2 + (S_1 \cdot n_j)^2 + (S_1 \cdot n_k)^2 = 2h^2. \tag{10}
\]

Therefore, in the singlet state, if the outcome of measuring \((S_1 \cdot n_i)^2\) is \(h^2\) and the outcome of measuring \((S_1 \cdot n_j)^2\) is \(h^2\) then, using (I), we can predict with certainty that the outcome of measuring \((S_2 \cdot n_k)^2\) will be 0,

\[
P_\psi \left( |S_2 \cdot n_k|^2 = 0 \mid [S_1 \cdot n_i]^2 = h^2 \ & [S_1 \cdot n_j]^2 = h^2 \right) = 1. \tag{11}
\]

To represent these inferences I will use a new kind of graph: \( A_l \rightarrow \bullet \rightarrow B_k \). Or, interchanging the particles, \( A_k \rightarrow \bullet \rightarrow B_j \). To avoid confusion when different kinds of graph appear, note that the latter have thicker lines than the previous graphs.

### 3.1 First part: stepladder argument

The proof itself has two parts. In the first part, using a chain of predictions with certainty, I will show that no local realistic interpretation exists for the case in which the outcome of measuring \((S_1 \cdot i)^2\) on the first particle is 0 and the outcome of measuring \((S_2 \cdot a)^2\) on the second particle is also 0 when the directions \(i\) and \(a\) form an angle \(\phi\) bound between certain values. This chain of predictions is summarized in Figure 2 and will be explicitly explained in the following.
Let $A_4$ be the outcome $[S_1 \cdot (1, 0, 0)]^2 = 0$ and let $B_4$ be the outcome $[S_2 \cdot (\cos \phi, \sin \phi, 0)]^2 = 0$. In the singlet state,

$$P_\psi(A_4, B_4) = P_4 = \frac{1}{3} \cos^2 \phi. \quad (12)$$

Thus, if $\phi$ is not $\frac{\pi}{2}$, the probability $P_4$ is not zero. Let $A_3$ be $[S_1 \cdot (\tan \phi, -1, \cot \theta)]^2 = \hbar^2$ and let $A_2$ be $[S_1 \cdot (\tan \phi, -1, -\cot \theta)]^2 = \hbar^2$, where $\theta$ is not $\frac{\pi}{2}$). Then, in the singlet state,

$$P_\psi(A_3 | B_4) = 1, \quad (13)$$

$$P_\psi(A_2 | B_4) = 1. \quad (14)$$

Analogously, let $B_3$ be $[S_2 \cdot (0, \cos \theta, -\sin \theta)]^2 = \hbar^2$ and let $B_2$ be $[S_2 \cdot (0, \cos \theta, \sin \theta)]^2 = \hbar^2$. Then

$$P_\psi(B_3 | A_4) = 1, \quad (15)$$

$$P_\psi(B_2 | A_4) = 1. \quad (16)$$

Therefore, if $A_4$ and $B_4$ are found, then we can say that $A_2$, $A_3$, $B_2$ and $B_3$ were elements of reality in the sense of EPR.

Next, let $A_1$ be $[S_1 \cdot (0, \cos \theta, -\sin \theta)]^2 = \hbar^2$ and let $B_1$ be $[S_2 \cdot (\tan \phi, -1, \cot \theta)]^2 = \hbar^2$. In the singlet state,

$$P_\psi(A_1 | B_3) = 1, \quad (17)$$

and

$$P_\psi(B_1 | A_3) = 1. \quad (18)$$

Therefore, $A_1$ and $B_1$ were also elements of reality. Finally, let $A_0$ be $[S_1 \cdot (\cot \phi \csc^2 \theta, 1, -\cot \theta)]^2 = 0$ and let $B_0$ be $[S_2 \cdot (\cot \phi \csc^2 \theta, 1, \cot \theta)]^2 = 0$. Since in the singlet,

$$P_\psi(A_0 | B_1 \& B_2) = 1, \quad (19)$$
and
\[ P_\psi(B_0 \mid A_1 \& A_2) = 1, \]  \hspace{1cm} (20)
then we conclude that \( A_0 \) and \( B_0 \) were elements of reality and should be found in the singlet state at least with probability \( P_4 \). However, it is easy to see that \( A_0 \) and \( B_0 \) never happen together, i. e.,
\[ P_\psi(A_0, B_0) = 0, \]  \hspace{1cm} (21)
if
\[ \cot^2 \phi = \sin^2 \theta \cos(2\theta). \]  \hspace{1cm} (22)
Since the right-hand side of (22) is bound between \(-1\) and \(\frac{1}{3}\), then (22) is fulfilled if
\[ \arccos \left( \frac{1}{3} \right) \leq \phi \leq \arccos \left( -\frac{1}{3} \right), \]  \hspace{1cm} (23)
that is, if
\[ 70.5^\circ \leq \phi \leq 109.5^\circ. \]  \hspace{1cm} (24)
Therefore, the maximum value of \( P_4 \) is \(\frac{1}{27}\). In brief, if \( \phi \) satisfies (24), then no local realistic description is possible when both \( A_4 \) and \( B_4 \) occur. At the most, \( A_4 \) and \( B_4 \) occur for \(\frac{1}{27}\) of pairs. Strictly speaking, this is not (yet) a ladder argument—since it is not apparently extensible—but just a stepladder argument.

### 3.2 Second part: geometrical argument

The second part uses a particular geometrical situation to improve the conclusion of the previous stepladder argument. Let \( i, j, k \) be three mutually orthogonal vectors and let \( a, b, c \) be other three mutually orthogonal vectors.
Let us define
\[ \hat{a} = \hat{b} = \hat{c} = \phi_1, \]
\[ \hat{j} = \hat{k} = \hat{i} = \phi_2, \]
\[ \hat{k} = \hat{i} = \hat{j} = \phi_3. \]
These definitions allow us to easily implement the orthogonality relations between the members of each triad. These angles could be
\[ \phi_1 = \arccos \left( \frac{1 + \sqrt{3}}{3} \right) = 24.4^\circ, \]
\[ \phi_2 = \arccos \left( \frac{1}{3} \right) = 70.5^\circ, \]
\[ \phi_3 = \arccos \left( \frac{1 - \sqrt{3}}{3} \right) = 104.1^\circ. \]
This particular situation is represented in Figure 3. In that case, the angles \( \phi_2, \phi_3 \) satisfy (24) but the angle \( \phi_1 \) does not. It is easy to see that all 9 angles between each direction of one triad and all three directions of the other cannot satisfy (24).

Let us go back to physics. Suppose that, with the previous choice of angles, \( (\mathbf{S}_1 \cdot \hat{i})^2, (\mathbf{S}_1 \cdot \hat{j})^2, (\mathbf{S}_1 \cdot \hat{k})^2 \) are measured on the first particle and \( (\mathbf{S}_2 \cdot \hat{a})^2, (\mathbf{S}_2 \cdot \hat{b})^2, (\mathbf{S}_2 \cdot \hat{c})^2 \) are measured on the second. Since on each particle we will find one 0 and two \( \hat{h}^2 \), then there are 9 different possible results. On 6 of them, the previous stepladder argument works and therefore, in those cases, no local realistic interpretation exists. However, the stepladder argument can be eluded when \( (\mathbf{S}_1 \cdot \hat{i})^2 = (\mathbf{S}_2 \cdot \hat{a})^2 = 0 \), or when \( (\mathbf{S}_1 \cdot \hat{j})^2 = (\mathbf{S}_2 \cdot \hat{b})^2 = 0 \), or when \( (\mathbf{S}_1 \cdot \hat{k})^2 = (\mathbf{S}_2 \cdot \hat{c})^2 = 0 \). The probability for each of these three cases can be computed using (12) with the election for \( \phi_1 \) given in (28). As can be easily checked, the sum of these three probabilities is 0.829. For the
remaining 17.1% of pairs the previous ladder argument goes through. In fact, it can be proved that this is the maximum value for finding a contradiction, using definitions (25-27) and if $\phi_2$ and $\phi_3$ satisfy (24) and $\phi_1$ does not.

4 Ladder proof without inequalities and without probabilities

In this section I will describe a ladder extension of the previous proof. Here I will present a genuine ladder argument (i.e., with a variable number of steps) to provide a proof of Bell’s theorem without inequalities for two particles which works for 100% of pairs. The strategy will be the same as before. First I will develop the ladder argument and then I will use an additional geometrical argument to complete the proof.

4.1 First part: ladder argument

The ladder argument is completely analogous to the one presented in section 3, if longer. In fact, it would be too long to explicitly develop all the steps. It can be easily followed with the aid of Figure 4. To simplify the diagram, in Figure 4 I have sometimes substituted the graph $A_{4K} \rightarrow \rightarrow B_j$ with the graph $B_j$: i.e., this graph means that if $A_{4K}$ happens then we can predict with certainty $B_j$. The rest of the symbols mean the same as in section 3. Please follow the argument in Figure 4. Note that the basic step of the ladder is composed by 4 predictions on each particle. For instance, one basic step contains the predictions $A_4$ to $A_7$ and $B_4$ to $B_7$. Other basic step is the one which includes the predictions $A_{4(K-2)}$ to $A_{4K-5}$ and $B_{4(K-2)}$.
to $B_{4K-5}$. Note also that the initial step (the one which contains $A_{4(K-1)}$ to $A_{4K}$ and $B_{4(K-1)}$ to $B_{4K}$) and the final step (the one which contains $A_0$ to $A_3$ and $B_0$ to $B_3$) are both a little bit different from the basic steps in between. The coefficients $c_j$ are

$$c_1 = \sin \theta_1, \quad (31)$$

and, for $j \geq 2$, the coefficients can be obtained recursively using

$$c_{j+1} = c_j \cos (\theta_{j+1} - \theta_j), \quad (32)$$

or explicitly using

$$c_j = \sin \theta_1 \prod_{k=1}^{j-1} \cos (\theta_{k+1} - \theta_k). \quad (33)$$

Therefore, $P_\psi(A_0, B_0) = 0$ if

$$\cot^2 \phi = c_K^2 \left( \cos^2 \theta_K - \sin^2 \theta_K \right). \quad (34)$$

For $K = 2$, the right-hand side of (34) is bound between $-1$ and $\left(\frac{2+\sqrt{21}}{8}\right)^5$, then (34) is fulfilled if

$$59.5^\circ \leq \phi \leq 120.5^\circ. \quad (35)$$

Analogously, for $K = 3$, (34) is fulfilled if

$$55.2^\circ \leq \phi \leq 124.8^\circ. \quad (36)$$

In general, the right-hand side of (34) is bound between $-1$ and $\cos^{2K+1} \left(\frac{\pi}{2K+1}\right)$. Therefore, as $K$ tends to infinity, the right-hand side of (34) is bound between $-1$ and $1 - \epsilon$, with $\epsilon > 0$, and then (34) is fulfilled if

$$45^\circ < \phi < 135^\circ. \quad (37)$$
However, for our purposes, we will not need to consider an infinite number of observables. For the following geometrical argument, the particular case $K = 11$ will be of special interest. For $K = 11$, (34) is fulfilled if

$$48.08^\circ \leq \phi \leq 131.92^\circ.$$  \hfill (38)

### 4.2 Second part: geometrical argument

Let us change the particular geometrical situation considered in the second part of section 3. Maintaining the definitions (25-27), now let the relative angles between the mutually orthogonal vectors $i, j, k$ and the mutually orthogonal vectors $a, b, c$ be

$$\phi_1 = \phi_2 = \arccos \left( \frac{2}{3} \right) = 48.19^\circ,$$  \hfill (39)

$$\phi_3 = \arccos \left( -\frac{1}{3} \right) = 109.47^\circ.$$  \hfill (40)

This particular situation is represented in Figure 5. With this choice of angles, the 9 relative angles satisfy (38). Therefore, whatever the results of measuring $(S_1 \cdot i)^2$, $(S_1 \cdot j)^2$, $(S_1 \cdot k)^2$ on the first particle and $(S_2 \cdot a)^2$, $(S_2 \cdot b)^2$, $(S_2 \cdot c)^2$ on the second, we will always find the outcome 0 in one direction of the first particle that forms, with one direction of the second particle in which the outcome 0 has also been found, an angle $\phi$ satisfying (38). For this case, the ladder argument, when $K = 11$, gives a contradiction. Therefore, for 100% of pairs no local realistic interpretation exists.
5 Extension to any maximally entangled state

The above proof is based on some specific properties of the singlet state of two spin-1 particles. In this section I want to argue that similar proofs exist for any maximally entangled state of two spin-\(s\) particles (with \(s \geq 1\)). First we will see the case of two spin-1 particles and then the more general case of two spin-\(s\) particles. In case of two spin-1 particles, each maximally entangled state admits infinite Schmidt decompositions of the form

\[
|\Psi\rangle = \sum_{m=-1}^{1} c_m |S_1 \cdot n_j = m\hbar\rangle |S_2 \cdot n_k = m\hbar\rangle .
\] (41)

In fact, all maximally entangled states have their nonzero Schmidt coefficients \(c_m\) of the same absolute value \([8]\) (this is true in every Schmidt base since local unitary transformations can only change the Schmidt base vectors, not the Schmidt coefficients). On the other hand, for nonmaximally entangled states the Schmidt decomposition (41) is unique. The existence of infinite Schmidt decompositions implies that for every vector \(n_j\)

\[
P_\Psi \left( |S_2 \cdot n_k|^2 = 0 \bigg| |S_1 \cdot n_j|^2 = 0 \right) = 1 ,
\] (42)

and

\[
P_\Psi \left( |S_2 \cdot n_k|^2 = \hbar^2 \bigg| |S_1 \cdot n_j|^2 = \hbar^2 \right) = 1 .
\] (43)

These two properties would play the same role in the proof as properties (3) and (7) for the singlet state. In addition, for a spin-1 particle \((S_2 \cdot n_k)^2\) and \((S_2 \cdot n_l)^2\) cannot be both zero if \(n_l\) is orthogonal to \(n_k\). Therefore, \((S_1 \cdot n_j)^2\) and \((S_2 \cdot n_l)^2\) cannot be both zero if \(n_l\) is orthogonal to \(n_k\), i.e.,

\[
P_\Psi \left( |S_1 \cdot n_j|^2 = 0, |S_2 \cdot n_l|^2 = 0 \right) = 0 , \quad \forall n_l \perp n_k ,
\] (44)
These two properties would play the same role as, respectively, properties (9) and (8). Thus we can always find a set of inferences to build a ladder proof.

In fact, similar proofs exist for any maximally entangled state of two spin-$s$ particles (with $s \geq 1$). Each of these states has infinite Schmidt decompositions and therefore there exists a prediction with certainty between each local nondegenerate observable of one particle and other local nondegenerate observable of the other particle. On the other hand, there are local degenerate observables related with the previous nondegenerate observables, like $(S_1 \cdot n_j)^2$ is related with $S_1 \cdot n_j$, which form an orthogonal resolution of the identity of the Hilbert space of the corresponding particle. This resolution of the identity would play the same role as relation (10) plays in the previous proof.

6 On experiments

Until now we have been reasoning with thought experiments. In this section, I will mention some of the advantages and disadvantages of this ladder argument in a real experiment to test local realism. Real experiments based on the ladder proof proposed in this paper will share some common features with experiments based on Hardy’s argument [4, 10] or on its ladder extension [2]. For instance, since almost all the necessary experiments are measurements to confirm predictions with certainty, and since perfect certainties are hard to find in a laboratory (see the results of [2, 8, 11]), some inequalities must be derived to deal with the data [2, 8, 11]. With this analysis, real exper-
iments are not expected to elude the detection efficiency loophole [12], and therefore they will not provide more conclusive experimental tests against local realism than previous tests of Bell’s theorem [12].

On the other hand, the ladder proof proposed in this paper presents some advantages and disadvantages in relation to the ladder proof by Hardy et al. Pros: Maximally entangled states are easier to produce in a laboratory since some of them are associated to a conserved quantity of a physical system after its decay into two parts. Since my proof works for all the pairs, in principle, no postselection is needed. Only a finite number of observables are needed. Cons: I need at least a two-part three-level system. Each step of the ladder would require more experiments than in the case of Hardy et al. However, these experiments are always of the same kind. They consist on measuring the square of one spin component on one of the particles and the square of the same spin component or one orthogonal to it of the other particle. On the contrary, in the experiment by Hardy et al. the relative orientation of the polarizers changes in every step of the proof. More cons: The geometrical argument requires measuring a triad of the square of spin components in three mutually orthogonal directions (or the equivalent observables if a different physical system is considered). One way to do it is proposed in [13], however, it could be difficult to do this in practice.
7 Ladder proofs of Bell’s theorem versus proofs of the Kochen-Specker with locality theorem

In this section I clarify the differences between ladder proofs of Bell’s theorem and proofs of the so-called Kochen-Specker with locality (KSL) theorem. The KSL theorem shows that, for two spin-1 particles in the singlet state, there is no hidden variables theory that satisfies separability, locality and some additional assumptions. This result was first proved by Heywood and Redhead in 1983 [6] and then re-elaborated many times since [14]. Its proof is based on two points: First, on Kochen-Specker (KS) geometric proofs [13], which show that noncontextual values explaining all quantum predictions are impossible for certain sets of observables of a single spin-1 particle (usually these observables are squares of components of spin or other observables related to them, as in our ladder proof). Second, on EPR condition for elements of reality [7]. But this condition is used here in a different way than in the ladder proofs: it is used to justify why in the singlet state of two spin-1 particles the previously mentioned observables must have a predefined value. In contrast, ladder proofs use EPR condition to make predictions with certainty.

Indeed, no published proof of the KSL theorem can be used to construct a ladder proof of Bell’s theorem. To illustrate this point consider the following example. Consider a singlet state of two spin-1 particles and on each of them, consider the simplest known KS geometric proof in a three dimensional Hilbert space, due to Conway and Kochen [15]. All the directions used in the following explanation belong to this geometric proof. Suppose that we make
a measurement on the first particle and found that $[S_1 \cdot (1, 0, 0)]^2 = 0$. This implies that, for instance, $[S_2 \cdot (0, 1, -1)]^2 = \hbar^2$. Then, this would imply that one of $[S_1 \cdot (1, 1, 1)]^2$ or $[S_1 \cdot (-2, 1, 1)]^2$ must be 0 (and the other $\hbar^2$). But which one? To decide it, one would need to know the value in a direction orthogonal both to $(0, 1, -1)$ and $(1, 1, 1)$, or in a direction orthogonal to $(0, 1, -1)$ and $(-2, 1, 1)$, but such directions are not contained in the geometric proof by Conway and Kochen. Therefore, using this geometric proof one cannot decide which one must be 0. The same problem occurs sooner or later using every published geometric proof of the KS theorem and therefore occurs in every proof of the KSL theorem. In contrast, ladder proofs are based only on EPR inferences. Therefore, in a ladder proof one must be able to predict with certainty all the outcomes involved in the proof, except the two at the beginning ($A_K$ and $B_K$ in the ladder proof by Hardy et al., or $A_4K$ and $B_4K$ in the proof proposed in this paper).

8 Conclusions

Hardy’s argument [3] is “the best version of Bell’s theorem” [11] and possesses “the highest attainable degree of simplicity and physical insight” [10]. The recent ladder extension [1, 2] is an improvement in the sense that a greater proportion of the pairs is subject to a contradiction with local realism. However, it does not work for maximally entangled states. In this paper, I have presented a proof which fills the most important holes left by the ladder extension by Hardy et al.: the new proof works for maximally entangled states of two spin-$s$ particles (with $s \geq 1$), and the proportion of the pairs subject to a contradiction with local realism becomes 100%.
experimental implementation of this proof could be achieved with present
day technology, although in practice it would not provide more conclusive
results than previous tests of Bell’s theorem.

9 Acknowledgments

The author thanks Guillermo García Alcaine, Gonzalo García de Polavieja,
Lucien Hardy and Asher Peres for useful discussions and comments, José
Luis Cereceda for drawing my attention to Ref. 5, and Carlos Serra for
proofreading.
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Figure 1: Diagrams for the ladder proofs for two spin-$\frac{1}{2}$ particles by Hardy et al. Original proof by Hardy with two observables on each particle (a). Ladder proof with three observables (b) and ladder proof with $K + 1$ observables (c).
$[S_1 \cdot (1, 0, 0)]^2 = 0$
$[S_1 \cdot (\tan \phi, -1, \cot \theta)]^2 = h^2$
$[S_1 \cdot (\tan \phi, -1, -\cot \theta)]^2 = h^2$
$[S_1 \cdot (0, \cos \theta, -\sin \theta)]^2 = h^2$
$[S_1 \cdot (\cot \phi \csc^2 \theta, 1, -\cot \theta)]^2 = 0$

$[S_2 \cdot (\cos \phi, \sin \phi, 0)]^2 = 0$
$[S_2 \cdot (0, \cos \theta, -\sin \theta)]^2 = h^2$
$[S_2 \cdot (0, \cos \theta, \sin \theta)]^2 = h^2$
$[S_2 \cdot (\tan \phi, -1, \cot \theta)]^2 = h^2$
$[S_2 \cdot (\cot \phi \csc^2 \theta, 1, \cot \theta)]^2 = 0$

Figure 2: Diagram for the ladder proof for two spin-1 particles and 5 observables on each particle (center). Corresponding inferences on the first particle (left) and on the second particle (right).
Figure 3: Relative orientations between the three orthogonal directions $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of the first particle and the three orthogonal directions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of the second particle, corresponding to the geometrical argument of section 3.
Figure 4: Diagram for the chain of predictions between $4K + 1$ observables on each particle (center) used in section 4. Explicit predictions on the first particle (left) and on the second particle (right).
Figure 5: Relative orientations between the three orthogonal directions $i, j, k$ of the first particle and the three orthogonal directions $a, b, c$ of the second particle, corresponding to the geometrical argument of section 4.
Figure 3
Ladder proof...
A. Cabello
Figure 5
Ladder proof...
A. Cabello