On the chiral and deconfinement phase transitions in parity-conserving $QED_3$ at finite temperature

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Abstract

We present some results about the interplay between the chiral and deconfinement phase transitions in parity-conserving $QED_3$ (with $N$ flavours of massless 4 component fermions) at finite temperature. Following Grignani et al. [11,12], confinement is discussed in terms of an effective Sine-Gordon theory for the timelike component of the gauge field $A_0$. But whereas in [11,12] the fermion mass $m$ is a Lagrangian parameter, we consider the $m=0$ case and ask whether an effective S-G theory can again be derived with $m$ replaced by the dynamically generated mass $\Sigma$ which appears below $T_{ch}$, the critical temperature for the chiral phase transition. The fermion and gauge sectors are strongly interdependent, but as a first approximation we decouple them by taking $\Sigma$ to be a constant, depending only on the constant part $\tilde{A}_0$ of the gauge field. We argue that the existence of a low-temperature confining phase may be associated with the generation of $\Sigma$; and that, analogously, the vanishing of $\Sigma$ for $T > T_{ch}$ drives the system to its deconfining phase. The effect of the gauge field dynamics on mass generation is also indicated.

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I. INTRODUCTION

Quantum electrodynamics in (2+1)-dimensions (QED$_3$) is an interesting theoretical laboratory in which to explore, in a simpler environment, some of the fundamental features of more complicated four-dimensional gauge theories, such as chiral symmetry breaking and confinement; it may also be of direct physical relevance in some condensed matter systems [1]. In this paper, we shall be concerned with N-flavour QED$_3$ at finite temperature, in its parity invariant (four-component) form. Our aim is two-fold: to cast some light on the nature of the phase transition associated with the dynamical generation of fermion mass (the “chiral phase transition”); and to argue for a dynamical interplay between the chiral phase transition and the confinement-deconfinement one.

Chiral symmetry breaking (“csb”), or equivalently dynamical fermion mass generation, in QED$_3$ at $T \neq 0$ has been extensively studied using Schwinger-Dyson equations, in the large $N$ limit [1–6]. Though the approximations are admittedly not fully controlled, all calculations show that a fermion mass is dynamically generated for $T < T_{ch}$, thereby spontaneously breaking the chiral symmetry. But the Coleman-Mermin-Wagner theorem [7,8] forbids the occurrence of an order parameter transforming non-trivially under chiral transformations in this two-dimensional system. One presumes, therefore, that the transition at $T = T_{ch}$ is of the BKT type [9,10]. However, to our knowledge there has been no explicit demonstration of this – for example, by mapping (or otherwise relating) the theory to one with a known BKT transition, which could then be identified with the csb transition in QED$_3$.

On the other hand, in a quite independent development, Grignani et al. [11,12] have shown that, when chiral symmetry is explicitly broken by a fermion mass $m$ in the Lagrangian, the relevant effective theory (at least in a certain region of parameter space) has a BKT transition at a temperature $T_c$, which the authors interpret as a confinement-deconfinement phase transition.

It is natural to ask, first, whether the analysis of [11,12] can be extended to the case in which the fermion mass is dynamically generated, rather than appearing explicitly in the Lagrangian; and secondly, whether the resulting effective theory leads us any nearer to an understanding of the chiral phase transition, and of any possible connection between the two transitions. The purpose of this paper is to make a first attempt at answering these questions.

It will be helpful to recall the argument of Grignani et al. [11,12]. According to them, confinement in QED$_3$ at $T \neq 0$ can be probed by an Abelian analogue of the Polyakov loop operator, namely

$$P_\tilde{e}(\vec{x}) = e^{i \tilde{e} \int_0^\beta A_0(r,\vec{x})dr},$$

where $\beta = 1/T$ ($k_B = 1$), and $\tilde{e} \neq \text{integer} \times e_f$ where $e_f$ is the charge of the dynamical fermions in the theory. The Euclidean path integral for the partition function is invariant under gauge transformations $A'_\mu = A_\mu + \partial_\mu \chi$, where

$$\chi(\beta, \vec{x}) = \chi(0, \vec{x}) + 2\pi n/e_f, \quad n \in \mathbb{Z}.$$  

The group of all gauge transformations modulo those which are strictly periodic is $\mathbb{Z}$, the additive group of integers; this $\mathbb{Z}$-symmetry is a global symmetry. Under a $\mathbb{Z}$-transformation,
\[ P(t) \rightarrow P(t) e^{2\pi i \hat{e}_f / e_f}. \] (3)

If Z-symmetry is unbroken, then \( \langle P(t) \rangle = 0 \) and the system confines \( \hat{e} \) charges; if Z-symmetry is spontaneously broken (as it can be, with an order parameter, being discrete), then \( \langle P(t) \rangle \neq 0 \) and the \( \hat{e} \) charges are deconfined.

As noted in [11,12], it is possible to choose a gauge in which \( A_0 \) is independent of \( \tau \). In this gauge, \[ P(t) = e^{i \hat{e} \beta A_0(t)} \] (4) and Z-transformations are \[ e_f \beta A_0(t) \rightarrow e_f \beta A_0(t) + 2\pi n, \] (5) under which \( P(t) \) transforms as in (3). The effective field theory for \( P(t) \) is then the effective action for the static field \( A_0(t) \), which is obtained by integrating out the other degrees of freedom from the path integral. The resulting effective action is in general non-local and non-polynomial, but it must still obey the Z-symmetry periodicity:

\[ S_{\text{eff}}[e_f \beta A_0(t)] = S_{\text{eff}}[e_f \beta A_0(t) + 2\pi n]. \] (6)

A local approximation can be obtained by expanding \( S_{\text{eff}} \) in powers of derivatives of \( A_0(t) \) divided by the (Lagrangian) fermion mass \( m \). In such an expansion, the effective potential is obtained from the fermion determinant in the presence of a constant background field \( A_0 \) – and it has to be periodic as in (6). It is, one might think, very likely that such a potential should be essentially of “Sine-Gordon” type, and indeed this is what the authors of [11,12] were able to show, provided that

\[ m \gg T, e_f^2 / 12\pi \] (7)

(see also section II.A below). In this regime, the critical behaviour of the effective theory for \( A_0(t) \) is that of the 2-D Sine-Gordon model, which has a line of critical points along which a BKT transition occurs, separating confining and deconfining regions of parameter space. The critical temperature is

\[ T_c = \frac{e_f^2 / 8\pi}{1 + e_f^2 / 12\pi m + \cdots} \] (8)

up to one-loop order. Thus the confinement-deconfinement transition has been shown to be of BKT type, by demonstrating that the relevant effective field theory has this critical behaviour.

Consider now how dynamical (fermion) mass generation (dmg) might fit into this picture. Our basic idea is to see if it might be consistent for the dynamically generated mass \( \Sigma \) to play the role of the Lagrangian mass \( m \) in the analysis of [11,12]. To investigate dmg, we shall use Schwinger-Dyson (SD) equations as previously, but here with one important difference: we wish to safeguard the crucial invariance (6). We shall therefore split \( A_0 \) into its slowly varying (\( \tilde{A}_0 \)) and fluctuating (\( a_0 \)) parts, and - as a first approximation - solve the SD equations for the fermion and \( a_0 \)-field propagators in the presence of the uniform \( \tilde{A}_0 \)
which, at this stage, plays the role of an external parameter, and which may be “large”. The fermion self-energy equation will produce a dynamically generated mass $\Sigma$ for every set of external parameters (in an appropriate region of parameter space) and for each $\tilde{A}_0$. The low momentum components of $A_0$ are then treated as in [11,12] with $m$ replaced by $\Sigma$; for the analysis to go through as before we shall need to find that $\Sigma$ does not depend too strongly on $\tilde{A}_0$, at least in some parameter region. This picture will only be consistent if (c.f. (7) ) the generated $\Sigma$ satisfies the condition

$$\Sigma \gg \frac{T, e^2}{12\pi}$$

with, of course, $T < T_{ch}$. In this case, we can tentatively associate the generation of fermion mass with the occurrence of a confining phase.

Of course, the fermion and gauge dynamics of $a_0$ are coupled: the vacuum polarisation function will be the bubble diagram evaluated in the $A_0$-background, and using the dynamically generated mass $\Sigma$, which also depends on $\tilde{A}_0$. We shall see that in the low-temperature confined phase, where $A_0$ is fluctuating about the trivial minimum of the Sine-Gordon potential, the properties of the gauge field propagator differ little from those in the normal ($\tilde{A}_0 = 0$) case. However, for $A_0$ configurations fluctuating near a position of unstable equilibrium for the S-G potential – which may be associated with the onset of the deconfining phase transition – we find that an instability develops in the vacuum polarisation, which threatens to destroy the non-zero gap solution and so restore chiral symmetry. There is an indication here, therefore, for an intimate connection between the chiral and the confining transitions. Unfortunately, our method does not allow us to explore the behaviour close to the phase transitions in any detail.

The paper is organised as follows. In Section II we define the model and discuss the formalism used to study confinement and csb simultaneously. In Section III we calculate the vacuum polarisation in the presence of $A_0$. In Section IV we write down the self-consistent gap equation in the presence of $A_0$, and present numerical results together with our conclusions. An Appendix discusses the error due to neglecting the dynamics of the space components of the gauge field.

**II. TOWARDS A UNIFIED TREATMENT OF THE CHIRAL AND DECONFINEMENT PHASE TRANSITIONS**

In this work we shall be concerned with a theory whose generating functional of complete Green’s functions $Z[j_\mu; \bar{\eta}, \eta]$, in the imaginary time (Matsubara) formalism, is defined by

$$Z[j_\mu; \bar{\eta}, \eta] = \int D A_\mu D \bar{\psi} D \psi \exp \left\{ -S + \int d^3 x [j_\mu(x) A_\mu(x) + \bar{\eta}(x) \psi(x) + \bar{\psi}(x) \eta(x)] \right\}$$

where $\int d^3 x \cdots \equiv \int_0^\beta d\tau \int d^3 x \cdots$, $\beta = \frac{1}{T}$, and $S$ denotes the Euclidean action

$$S = \int d^3 x \mathcal{L} \quad \mathcal{L} = \mathcal{L}_G + \mathcal{L}_F.$$

The gauge field Euclidean Lagrangian, $\mathcal{L}_G$, is assumed to be of the standard Maxwell type,
\[ \mathcal{L}_G = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \]  

(12)

where we have omitted, for the time being, the gauge fixing. The fermionic field \( \psi \) denotes \( N \) flavours \( \psi_a \), each one minimally coupled to the gauge field \( A_\mu \). The corresponding Euclidean Lagrangian, \( \mathcal{L}_F \), is then

\[ \mathcal{L}_F = \sum_{a=1}^{N} \bar{\psi}_a(\tau, x) \left( \partial \tau + ie_N A + m \right) \psi_a(\tau, x) \]  

(13)

where the coupling constant \( e_N = \frac{e}{\sqrt{N}} \) (replacing the \( e_f \) of Section I) has the proper \( N \)-dependence to ensure that, in the large-\( N \) limit keeping \( e \) fixed, all Feynman diagrams are finite. The Lagrangian mass \( m \) may or may not be present. We use the notation \( \tau \) and \( \vec{x} \) to refer to the time and space coordinates, respectively. The \( \gamma \)-matrices are in a \( 4 \times 4 \) (reducible) representation of the Dirac algebra in \( 2 + 1 \) dimensions:

\[ \gamma_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix} \]  

(14)

where \( \mu = 0, 1, 2 \), and

\[ \sigma_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]  

(15)

are the usual Pauli matrices. This representation allows for the introduction of a parity-conserving mass term for the four-component fermions. This corresponds, in the two-component formalism, to mass terms with opposite signs for each fermion field within a given flavour. We shall here use a description that allows us to study the possibility of both chiral and deconfinement phase transitions simultaneously. Note that the latter transition, as presented in [11,12], is considered in terms of an effective action for \( A_0 \), the time component of the gauge field. This effective action is derived in the static-\( A_0 \) gauge, performing a derivative expansion. The usual way to study the chiral phase transition is, on the other hand, to work with truncated Schwinger-Dyson equations (SDE’s), an approximation which is justified in the large-\( N \) limit. It is possible, as we shall see, to find a non-empty region in parameter space where the simplifying assumptions made for the study of each phase transition are indeed compatible.

We shall briefly review here the derivation of the effective action for the field \( A_0 \), for the case of massive fermions, following [11,12]. This effective action is the interesting object to study when dealing with the deconfinement phase transition. We will afterwards extend this procedure, in order to include the non-perturbative dynamics responsible for the chiral phase transition. The fermion mass, rather than being a Lagrangian parameter, will then be dynamically generated, and determined by a gap equation. Following references [11,12] (with trivial changes due to the fact that we now have \( N \) fermionic flavours) the effective action \( S_{\text{eff}}[A_0] \) for a system where the fermions do have a Lagrangian mass \( m \), is obtained by first using a gauge transformation such that \( A_0 \) is \( \tau \)-independent \(^1\), and then integrating out all the remaining fields, i.e.,

\(^1\)\( A_0 \) itself cannot be set equal to zero, because the time coordinate is compact.
\[
e^{-S_{eff}[A_0]} = e^{-\frac{1}{2} \int d^3x(\partial_\mu A_\mu)^2} \int \mathcal{D}A \exp \left\{ - \int d^3x \left[ \frac{1}{2} (\partial_\mu A_\mu)^2 + \frac{1}{4} F_{jk}^2 \right] \right\} \times e^{-\Gamma_N[A_0,A_j]} \tag{16}\]

where the functional \( \Gamma_N[A_0,A_j] \) is the result of integrating out the fermionic fields,

\[
\Gamma_N[A_0,A_j] = \int \Pi_{a=1}^N \mathcal{D}\bar{\psi}_a \mathcal{D}\psi_a e^{-\int d^3x \sum_{a=1}^N \bar{\psi}_a [\mathcal{D} + i e_N \gamma_0 A_0 + i e_N \gamma_j A_j + m] \psi_a}
\]

\[
= \left\{ \det[\mathcal{D} + i e_N \gamma_0 A_0(x) + i e_N \gamma_j A_j(\tau,x) + m] \right\}^N. \tag{17}\]

Of course, we have \( \Gamma_N[A_0,A_j] = N \Gamma[A_0,A_j] \), where \( \Gamma[A_0,A_j] \) denotes the contribution corresponding to the integration of just one fermionic flavour. Following [11,12], \( \Gamma[A_0,A_j] \) may be evaluated by using a derivative expansion for the external gauge field, taking into account the fact that \( A_0 \) may have 'large' values, i.e., its constant component cannot be assumed to be small, since that would spuriously break the invariance under shifts in \( A_0 \). To use an approximation which preserves this periodicity is crucial for the study of the deconfinement phase transition. Thus we let

\[
A_\mu(x) = \tilde{A}_\mu(x) + a_\mu(x) \tag{18}\]

where \( \tilde{A}_0 \) is the large, almost constant piece, \( \tilde{A}_j = 0 \), and \( a_\mu(x) \) denotes the fluctuating part of \( A_\mu(x) \). As \( \tilde{A}_0 \) cannot be assumed to be small, the expansion of the determinant must include that object exactly:

\[
\det[\mathcal{D} + i e_N \gamma_0 A_0(x) + i e_N \gamma_j A_j(\tau,x) + m] = \det[\mathcal{D} + i e_N \gamma_0 \tilde{A}_0 + m] \times \det \left[ 1 + i e_N (\mathcal{D} + i e_N \gamma_0 \tilde{A}_0 + m)^{-1} \mathcal{D}(\tau,\vec{x}) \right], \tag{19}\]

where, for the last factor in (19), we can attempt a small-\( a_\mu \) expansion. In a first approximation, and to be consistent with the approximation usually invoked in the SDE's approach to the study of the chiral phase transition, we shall ignore the dynamics of \( a_j \), the space components of the gauge field. A discussion about the error due to this approximation is postponed to the Appendix. Then, we only need the fermionic determinant as a function of the time component of the gauge field. For one flavour, we have

\[
e^{-\Gamma[A_0]} = \det[\mathcal{D} + i e_N \gamma_0 A_0(\vec{x}) + m] \]

\[
= \det[\mathcal{D} + i e_N \gamma_0 \tilde{A}_0 + m] \det \left[ 1 + i e_N (\mathcal{D} + i e_N \gamma_0 \tilde{A}_0 + m)^{-1} \gamma_0 a_0(\vec{x}) \right]. \tag{20}\]

The \( a_0 \)-independent factor in (19), involving no derivatives of the gauge field, plays the role of an effective potential, and is given by

\[
\det[\mathcal{D} + i e_N \gamma_0 \tilde{A}_0 + m] \equiv e^{\int d^2x V(m, \frac{e_N \tilde{A}_0}{T})} \tag{21}\]

with

\[
V(m, \frac{e_N \tilde{A}_0}{T}) = \frac{1}{Vol.} \log \det [(\mathcal{D} \partial_0 - e_N \tilde{A}_0)^2 - \nabla^2 + m^2]. \tag{22}\]

6
Regarding the second factor, when expanded for low momentum, it produces a wave function renormalization of the kinetic term for $A_0$. Putting together these two pieces, and including the factor $N$, we see that the effective action for $A_0$ becomes

$$S_{\text{eff}}[A_0] = \int d^2 x \left( Z(m, e_N \tilde{A}_0/T) \frac{1}{2T} \tilde{\nabla} A_0 \cdot \tilde{\nabla} A_0 - N V(m, e_N A_0/T) \right),$$  \hspace{1cm} (23)

where

$$N \Pi_{00}(0, \vec{k}^2) = N \Pi_{00}(0, 0) + \vec{k}^2 (Z - 1) + \ldots$$  \hspace{1cm} (24)

where $\Pi_{00}(n, \vec{k}^2)$ denotes the 00 component of the vacuum polarization tensor in the presence of the constant gauge field $\tilde{A}_0$.

To relate this form of the effective action to a Sine-Gordon description, one performs a harmonic expansion of the effective potential:

$$V(m, e_N A_0/T) = -\frac{T^2}{\pi} \sum_{n=1}^{\infty} (-1)^n e^{-nm/T} \left( 1 + \frac{nm}{T} \right) \frac{\cos(ne_N A_0/T)}{n^3}.$$  \hspace{1cm} (25)

This effective potential may be further simplified in some particular regimes. For example, when $m >> T$, $T/m$ and $e^2/(12\pi m)$ are small and $e_N / T$ is arbitrary, so that $Z \approx 1$ and the higher harmonics in the effective potential are exponentially small perturbations; one may then keep only the leading term,

$$V(m, e_N A_0/T) \sim \frac{Tm}{\pi} e^{-m/T} \cos(e_N A_0/T),$$  \hspace{1cm} (26)

which is a Sine-Gordon potential. Under these conditions, one has an effective description which corresponds to the Sine-Gordon theory for the scalar field $A_0$:

$$S_{\text{eff}}^{(m>>T)}[A_0] = \int d^2 x \left\{ \frac{1}{2T} \tilde{\nabla} A_0 \cdot \tilde{\nabla} A_0 - N \frac{Tm}{\pi} e^{-m/T} \cos(e_N A_0/T) \right\},$$  \hspace{1cm} (27)

which has a BKT phase transition. The "$\beta$" parameter of the Sine-Gordon model is now $N$ dependent, and given by $\beta = \frac{e}{\sqrt{NT}}$. Under the assumption (7) (with $e_f$ replaced by $e_N$), the critical temperature is given by (8) with $e_f$ replaced by $e_N$.

It is important to remark that, in this analysis, the Lagrangian mass $m$ has been regarded as a free parameter. We shall now argue that, when $m$ is zero, one can - under certain conditions - arrive at an effective theory of the same form as (27), but with $m$ replaced by $\Sigma$, the dynamically induced gap.

The effective action (27) has been obtained by performing a derivative expansion. In consequence, when the $A_0$ field is regarded as dynamical, the effects due to its large-momentum modes are either neglected, or improperly taken into account. This is, of course, not an issue when one is dealing with the deconfinement phase transition, but it will have an important influence for the particular case of massless fermions. In this case the mass parameter used to perform the low-momentum expansion is proportional to the temperature, and one is then missing modes of the gauge field which may indeed be responsible for the generation of a dynamical mass for the fermions. In other words, the procedure of using the low-momentum effective action for $A_0$ for the case of massless fermions would
not be accurate, since one still has to include virtual $a_0$ corrections which contribute to the
generation of a mass for the fermions. If a non-vanishing dynamical mass for the fermion is
generated, the momentum expansion procedure can be made consistent, but one should be
careful to include in that dynamical mass the dependence on the temperature and on the
parameters of the model.

We shall now give a formal derivation to justify this procedure. The definition of the
effective action for the large field $A_0$ should now take into account the existence of the high
momentum modes for $A_\mu$. The natural way to proceed is to decompose the gauge field into
its large and small components, as in (18), where only up to two derivatives of $\tilde{A}$ will be
kept, while $a_\mu$ is assumed to take care of the higher derivatives. We shall, to simplify the
notation, get rid of the tilde, and $A$ will denote just the large component of the gauge field.

Then, $S_{\text{eff}}$ is defined by

$$\exp\{-S_{\text{eff}}[A_0]\} = \int \mathcal{D}a_\mu \Pi_\alpha \mathcal{D}\bar{\psi}_\alpha \mathcal{D}\psi_\alpha \exp\{-S[A_0 + a_0, a_j, \bar{\psi}, \psi]\}$$

(28)

where $S$ is the Euclidean action, as defined in ([1]). More explicitly,

$$S[A_0 + a_0, a_j, \bar{\psi}, \psi] = \int d^3x \left[ \frac{1}{4} F_{\mu\nu}(A + a) F_{\mu\nu}(A + a) + \sum_b \bar{\psi}_b(\partial + ieN(\bar{A} + \phi))\psi_b \right].$$

(29)

It is crucial to realise that $A$ and $a$ are assumed to have complementary supports in mo-
mentum space. Thus the mixed quadratic term for the gauge field in (29) vanishes

$$\int d^3x \frac{1}{2} F_{\mu\nu}(A) F_{\mu\nu}(a) = 0,$$

as can be easily verified by Fourier transformation.

Then we see that

$$\exp\{-S_{\text{eff}}[A_0]\} = \exp\{-\frac{1}{2} \int d^3x (\partial_j A_0)^2 - W[A_0]\}$$

(31)

where

$$e^{-W[A_0]} = \int \mathcal{D}a_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\{-\int d^3x \left[ \frac{1}{4} F_{\mu\nu}(a) F_{\mu\nu}(a) + \sum_{b=1}^N \bar{\psi}_b(\phi + ieN(\bar{A} + \phi))\psi_b \right] \}. $$

(32)

Note that $W[A_0]$ has a simple interpretation: it is the free energy corresponding to parity
conserving $QED_3$ in the presence of an $A_0$ background. The leading term in a large $N$
expansion is

$$W[A_0] = -\text{Tr}[\ln S_F^{-1}]$$

(33)

where $S_F$ denotes the full fermion propagator, for the theory defined by (32). This propa-
gator is the result of solving the S-D equations in the $A_0$ background.

In principle, since $A_0$ depends on $x$ so will $\Sigma$, but we shall make the assumption that the
induced mass is not sensitive to the gradient of $A_0$, but only to the constant part $\bar{A}_0$. This
assumption will only be reliable in the confined phase, where the topological excitations
of the S-G theory are absent, and $A_0$ has only small fluctuations around one of the stable minima of the effective potential. With $\Sigma$ assumed constant, $S_F^{-1}$ will have the form

$$S_F^{-1} = \vartheta + i e_N \gamma_0 A_0 + \Sigma ;$$

(34)
evaluation of the trace in (33) may then proceed as in the $m \neq 0$ case, via Equations (20) - (27) above. But an important question to answer will be whether, for at least some part of the solution-space for $\Sigma$, conditions (9) do in fact hold, allowing the passage from (25) to (26).

The low-momentum effective action for the gauge field is then constructed with the 1PI functions derived (see Sections III and IV) from the self-consistent SD equations, with $\tilde{A}_0$ appearing as an external field. As the low-momentum component of the gauge field has not yet been integrated, one has to treat its dynamics separately; this is done with the S-G description, as in [11,12], which does indeed allow for a BKT transition. The phase transition for the deconfinement transition should then be studied taking into account the fact that $\Sigma$ is now not an independent parameter (as was $m$), but rather a function of $T$, $e$, $N$ and $\tilde{A}_0$. In this connection, we shall have to check that the dependence of $\Sigma$ on $\tilde{A}_0$ is, in fact, not so rapid as to change the effective potential in (27) significantly away from the S-G form.

III. THE VACUUM POLARISATION FUNCTION $\Pi_{00}$ IN THE PRESENCE OF $\tilde{A}_0$

We derive here the vacuum polarisation function in the one-loop (large-$N$) approximation, keeping the full dependence on the ‘large’ (uniform) piece of the time-like component of the gauge field, $A_0$.

In principle, the coupled S-D equations for the 1PI functions should be solved self-consistently, having made the usual truncations. However, experience shows [13] that, in dynamical mass calculations, it makes very little difference whether a fermion mass is included in $\Pi_{00}$ or not. We shall present the calculation for massless fermions, indicating at the end the changes needed to deal with massive ones.

For one single four-component fermionic flavour (for $N$ flavours we only have to multiply the result by $N$), the object we consider is the quadratic form $\Gamma^{(2)}[A_0]$, defined by

$$\Gamma^{(2)}[A_0] = \frac{1}{2\beta} \int \frac{d^2 k}{(2\pi)^2} \tilde{a}_0(0,-k)\Pi_{00}(0,k)\tilde{a}_0(0,k)$$

(35)

where $\tilde{a}_\mu(n,k)$ denotes the Fourier transformed of $a_\mu$, namely

$$a_\mu(\tau,x) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} e^{i(\alpha_n \tau + k \cdot x)} \tilde{a}_\mu(n,k)$$

(36)

with $\alpha_n = \frac{2\pi n}{\beta}$, the bosonic Matsubara frequency. From the definition of the fermionic determinant, we see that the vacuum polarization tensor is given by
\[
\Pi_{\mu\nu}(n, k) = -\frac{e_N^2}{\beta} \sum_{l=-\infty}^{l=+\infty} \int_0^1 \frac{d^2 p}{(2\pi)^2} \text{tr} \left[ \frac{1}{i(p+\beta k) + i\gamma_0\omega_l + n} \frac{1}{i(p+\beta k) + i\gamma_0\omega_l - n} \right]
\]  
(37)

where we adopted the notation \( \tilde{\omega}_l = \omega_l + e_N\tilde{A}_0 \), where \( \omega_l = (2l + 1)^2 \) is the fermionic Matsubara frequency. Note that the Dirac slash is now defined for the two spacelike indices in the ‘dimensionally reduced’ theory. The ‘tr’ denotes Dirac trace, over the four component spinor space. After evaluating this Dirac trace, and performing the angular integration for the spatial momentum, \( F(k) \equiv \Pi_{00}(0, k) \), the zero frequency part of \( \Pi_{00} \) may be written as follows

\[
F(k) = e_N^2 \int_0^1 d\alpha \int_0^{\pi} \frac{dp}{\pi} \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \frac{p^2 - \alpha(1-\alpha)k^2 - \tilde{\omega}_n^2}{p^2 + \alpha(1-\alpha)k^2 + \tilde{\omega}_n^2},
\]  
(38)

where \( \alpha \) is a Feynman parameter. To calculate \( F(k) \), we found it convenient to split that function into \( F_{T=0}(k) \) (its \( T = 0 \) limit) and \( \bar{F}(k) = F(k) - F_{T=0}(k) \). By performing the sum over the discrete frequencies and the integration over the modulus of the spatial momentum, the subtracted function \( \bar{F}(k) \) may be expressed as an integral over \( \alpha \):

\[
\bar{F}(k) = \frac{e_N^2}{\pi\beta} \int_0^1 d\alpha \ln\left[ 1 + 2\cos(e_N\beta\tilde{A}_0)e^{-\beta\sqrt{\alpha(1-\alpha)k}} + e^{-2\beta\sqrt{\alpha(1-\alpha)k}} \right].
\]  
(39)

The zero-temperature value of \( F \) is well-known [3], and given by \( \alpha_N k/8 \).

It does not seem possible to evaluate (39) in closed form. However, since \( \Pi_{00} \) appears inside an integral in the equation for the dynamically generated fermion mass (Eq. (15) below), it would make the solution of (15) much easier if an accurate analytic approximation were available for (39). A similar problem was encountered in [3], and there a convenient approximation was found which preserved correctly the limits \( \beta k \to 0 \) and \( \beta k \to \infty \) (see Eq.(9) of [3]). Following the same procedure here, we approximate \( \Pi_{00} \) by

\[
\Pi_{00}(0, k) \approx \frac{e_N^2}{8\beta} \left[ \beta k + \frac{8\ln[2 + 2\cos(e_N\beta\tilde{A}_0)]}{\pi} \exp\left\{ -\frac{\pi\beta k}{8\ln[2 + 2\cos(e_N\beta\tilde{A}_0)]} \right\} \right]
\]  
(40)

which incorporates

\[
\lim_{\beta k \to \infty} \Pi_{00}(0, k) = \frac{e_N^2}{8}
\]  
(41)

and

\[
\lim_{\beta k \to 0} \Pi_{00}(0, k) = \frac{2e_N^2}{\pi\beta} \ln \left\{ 2 \left| \cos \left( \frac{e_N\beta\tilde{A}_0}{2} \right) \right| \right\}.
\]  
(42)

and where we have now written the result explicitly for \( N \neq 1 \) flavours (recall that \( e_N = e/\sqrt{N} \), and \( e^2 \) is held fixed as \( N \) varies). Expression (42) is the “thermal mass”, in the presence of \( \tilde{A}_0 \). It is clear that when \( e_N\beta\tilde{A}_0 > \frac{2\pi}{3} \) the thermal mass goes negative. We shall discuss this further in Section IV. For the moment, note that this value is in any case approaching near to the “unstable vacuum” of the S-G potential at \( e_N\beta\tilde{A}_0 = \pi \), which lies beyond the region of validity of our approach, based as it is on perturbing around one of
the stable minima of the S-G potential. For \(e_N \beta \tilde{A}_0 \sim 1.5\), we find that the approximation (40) is always within 10% of the exact formula (39).

It is not hard to show that, when \(m \neq 0\), the zero frequency part values of \(\Pi_{00}\) are given by a function \(F(k)\) defined by

\[
F(k) = e_N^2 \int_0^1 d\alpha \int_0^\infty \frac{dp^2}{\pi} \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \frac{p^2 - \alpha(1-\alpha)k^2 - \tilde{\omega}_n^2 + m^2}{[p^2 + \alpha(1-\alpha)k^2 + \tilde{\omega}_n^2 + m^2]^2},
\]

(43)

An analogous procedure to the one followed in the massless fermion case allows us to write the subtracted function \(\bar{F}\) as

\[
\bar{F}(k) = \frac{e_N^2}{\pi \beta} \left(1 - m \frac{\partial}{\partial m}\right) \int_0^1 d\alpha \ln[1 + 2 \cos(e_N \beta \tilde{A}_0)e^{-\beta \sqrt{\alpha(1-\alpha)k^2 + m^2}} + e^{-2\beta \sqrt{\alpha(1-\alpha)k^2 + m^2}}].
\]

(44)

As for the massless case, an approximate expression for \(\bar{F}(k)\) may also be obtained, but in our calculations we have not used the massive form of \(\Pi_{00}\). In a fully self-consistent treatment, \(m\) would be replaced by \(\Sigma\) in (44).

**IV. DYNAMICAL MASS GENERATION IN THE PRESENCE OF \(\tilde{A}_0\)**

In this first exploration of the interplay between csb and confinement in QED at \(T \neq 0\), we shall adopt the simplest approach to the S-D equation for \(\Sigma\) (the “gap equation”) which was taken in the early paper by Dorey and Mavromatos [2]. We shall assume that \(\Sigma\) is momentum-independent, that fermion wave function renormalization may be neglected, and that only the instantaneous-exchange part of the kernel is retained (see [3,4] for evidence that relaxing these assumptions will not change the qualitative conclusions very much). The new feature of our calculation of \(\Sigma\), as for \(\Pi_{00}\), is the inclusion of the \(\tilde{A}_0\) background.

The fermion propagator has the form \((ip + i\gamma_0 \tilde{\omega}_n)^{-1}\) where \(\tilde{\omega}_n = \omega_n + e_N \tilde{A}_0\) and \(\omega_n\) is the fermionic Matsubara frequency as before; the gap equation is then

\[
1 = \frac{e^2}{4\pi N} \int \frac{k dk}{[k^2 + \Pi_{00}(k, \beta, \tilde{A}_0)]\sqrt{k^2 + \Sigma^2}} \left[\frac{\sinh \beta \sqrt{k^2 + \Sigma^2}}{\cosh \beta \sqrt{k^2 + \Sigma^2} + \cos(e\beta \tilde{A}_0/\sqrt{N})}\right]
\]

(45)

where \(\Pi_{00}\) is the \(\mu = \nu = 0\) component of the vacuum polarization tensor \(\Pi_{\mu\nu}\) as calculated in the preceding Section, Eq. (10), and \(e^2\) is fixed as \(N\) varies. Eq.(45) is to be compared with Eq.(20) of [2], to which it reduces when \(\tilde{A}_0 \rightarrow 0\) (note that the latter equation is in error by a factor of \(\frac{1}{2}\) on the right hand side: “\(2N\pi\)” should read “\(4N\pi\)”, so that the \(N = 2\) results of [4] correspond to our \(N = 1\) results, etc.; note also that our \(\Pi_{00}\) is equal to \(k^2 \times \Pi_{00}\) of [2], but is the same as \(\Pi_0\) of [4]). The upper limit of integration in (45) is taken to be unity, as in [2].

It is worth noting that the momentum integral in the S-D equation for \(\Sigma\) should really have an IR cutoff \(\epsilon\), given by a small fraction of \(\Sigma\), since the gauge field corresponds to the fluctuating part of \(A_0\). This should not make any difference to the solution of the
equation, for all the cases where there is a non-vanishing $\Sigma$, since there the IR behaviour of the integrand is smooth. We have verified numerically that indeed the solution to (45) is insensitive to the introduction of a small infrared cutoff.

On the other hand, when the only stable solution is $\Sigma = 0$, the IR cutoff would make a difference, but then the separation into low and high momentum modes collapses. Again, this would correspond to the physics near to the transition, where our procedure is not applicable.

We turn now to a discussion of the numerical solutions of (17), and their implications. Figure 1 shows $s = \Sigma/\pi t$ as a function of $t = \pi/\beta$ for various values of $b = \beta/\pi$ for the case $N = 1$. For $b = 0$, the curve reproduces the known results of [2]. For $b \neq 0$, a mass continues to be generated; indeed, at a given temperature the mass increases with $b$ (see Figure 2 where $s$ is shown as a function of $b$ for fixed $t$). However, for $b \neq 0$ we cannot probe temperatures below $\frac{\beta_0}{2\pi}$, since below this value the thermal mass goes negative, as remarked after (12). In fact, from (10) it is clear that when $t = \frac{3\pi}{2\pi}$ (i.e. $\pi e\beta A_0 = \frac{2\pi}{3}$), $\Pi_{00}$ reduces to $\frac{e^{2k}}{\pi}$, the zero temperature value. This is the reason why the curves all tend to $s(t = 0)$ as $t$ tends to $t = \frac{3\pi}{2\pi}$.

In reality, the calculation should probably only be accepted for $t \geq \frac{b}{\pi}$ (i.e. $\pi e\beta A_0 \leq \pi$), since we must remember that, at $e\beta A_0 = \pi$, the Sine-Gordon potential in (27) has a point of unstable equilibrium, while our mass calculation is understood as being performed around a stable minimum of the potential. Inspection of Figure 1 shows that the condition $t \geq \frac{b}{\pi}$ is met for the curves corresponding to $b = 0.01$ and $0.02$, but only for part of the $b = 0.03$ curve, and for less of the $b = 0.04$ curve.

Figure 2 shows $s$ as a function of $b$ for various values of $t$, also for the case $N = 1$. As before, the curves tend to the zero-temperature value as $b$ tends to $\frac{2\pi}{3} t$. For the values of $t$ and $b$ shown, the more conservative limit $b \leq \pi t$ is always satisfied. The critical temperature for $b = 0$ is near $t_{ch} = 0.0137$, which is why the curve for this value of $t$ “divides” the curves in Figure 2.

We must now consider whether these results support our approach \textit{a posteriori}. First and most importantly, we need to check whether conditions (13) hold. Inspection of Figure 1 shows that indeed the condition $\Sigma >> T$ (i.e. $s >> t$) is met for small $T$, say $T \leq 0.01$. The condition $s >> \frac{1}{12\pi N}$ is less satisfactorily met, but it is not flagrantly violated. Secondly, we see from Figure 2 that, while there is certainly a dependence of $\Sigma$ on $A_0$, it does not seem to be sufficiently rapid to call into question the effective S-G description of the deconfinement transition.

So far we have discussed the case $N = 1$, and we must now remember that some approximations we have made depend on the large-$N$ limit. However, it is well known (see for example [3] and references therein) that in the model considered here there is a critical value of $N$, $N_c(T)$, above which chiral symmetry is restored. $N_c$ is typically in the region 1-3. We shall therefore consider the case $N = 2$ for definiteness. While $N = 2$ is hardly a large value, there is evidence to suggest that the inclusion of terms which are higher order in $1/N$ will not alter the qualitative conclusions [4].

In Figure 3 we present, for $N = 2$, the curves of $s$ as a function of $t$, for two values of $b$. We see that both the critical temperature and the value of $s$ at $t = 0$ are reduced, but still there is an important fraction of parameter space where $s >> t$. As for the condition $s >> \frac{1}{12\pi N}$, it is satisfied to the same extent as for the $N = 1$ case, since going from $N = 1$ to $N = 2$ reduces both sides of that inequality by (roughly) a factor 2.
We may conclude that we have demonstrated the feasibility of our approach, in which the dynamically generated $\Sigma$ replaces (at least in a limited region of parameter space) the Lagrangian mass $m$ of [11,12]. We may therefore argue that in the $m = 0$ case, dynamical generation of $\Sigma$ is associated with a low-temperature confining phase, via the emergence of an effective S-G theory. There is also an indication of a (reciprocal) effect of the gauge field dynamics on mass generation. As noted after Eq. (42), when $e_N\beta\tilde{A}_0 > (2\pi/3)$ the thermal mass goes negative, which would imply a singularity in (43) due to the ensuing pole in the gauge field propagator in the Euclidean region. We may interpret this as indicating that, as the gauge field fluctuates towards the value $e_N\beta\tilde{A}_0 = \pi$, which is an unstable stationary point of the S-G potential, the stability of the gap equation is lost, and mass generation ceases. Interestingly, apparently similar special temperatures occur in a study of the Gross-Neveu model with an imaginary chemical potential [15]. Clearly it would be desirable to consider more fully the coupled dynamics of the $A_0$ and fermion sectors, including an $x$-dependence for $\Sigma$. Nevertheless, as a first orientation, our procedure seems to be reasonably justified.

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APPENDIX : INCLUSION OF THE SPATIAL COMPONENTS OF THE GAUGE FIELD.

In the SDE’s approach, one usually neglects the dynamics of the spatial components of the gauge field. This produces an error in the result of the self-consistent gap equation. In the evaluation of the effective action for $A_0$, on the other hand, the spatial components of the vacuum polarization tensor yield a wave function renormalization for the spatial components of $A_\mu$.

Let us call $\delta S_{\text{eff}}$ the contribution arising from the integration over the spatial modes $\vec{a}$ in (10). It is quite straightforward to see that

$$e^{-\delta S_{\text{eff}}} = \int D\vec{a} \exp \left\{- \sum_{n=-\infty}^{+\infty} \frac{1}{4} \int \frac{d^2k}{(2\pi)^2} \tilde{F}_{jk}(-n,-k)\tilde{F}_{jk}(n,k) \right\}$$

$$\times \exp \left\{- \sum_{n=-\infty}^{+\infty} \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \tilde{a}_i(-n,-k)\Pi_{ij}(n,k)(n,k)\tilde{a}_j(n,k) \right\}$$

(46)

where the tilde denotes Fourier transform. In the static $A_0$ gauge there is a remaining gauge invariance under space dependent gauge transformations, which implies that the function $\Pi_{ij}$ is spatially transverse:

$$\Pi_{ij}(n,k) = \Pi(n,k)(-\Delta \delta_{ij} + \partial_i \partial_j)$$

(47)

Fixing these gauge field components to be spatially transverse,

$$e^{-\delta S_{\text{eff}}} = \int D\vec{a} \exp \left\{- \sum_{n=-\infty}^{+\infty} \frac{1}{4} \int \frac{d^2k}{(2\pi)^2} \tilde{F}_{jk}(-n,-k)[1 + \Pi(n,k)]\tilde{F}_{jk}(n,k) \right\}.$$ 

(48)

Then we may write for the correction to the effective action

$$e^{-\delta S_{\text{eff}}} = \Pi_{n,k}[1 + \Pi(n,k)]^{-1}$$

(49)

or, equivalently, that there is a correction to the effective potential $V$, namely, $V \rightarrow V + \delta V$, with

$$\delta V = -\sum_n \int \frac{d^2k}{(2\pi)^2} \ln[1 + \Pi(n,k)] .$$

(50)

In spite of the fact that this contribution exists, it is negligible in the large $N$ limit (note the absence of the factor $N$ which appears when integrating fermions). Moreover, even for only one fermionic flavour, we checked that $\delta V$ is exponentially suppressed in comparison with $V$. 

14
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Figure 1

Figure 2
FIGURE CAPTIONS

- **Figure 1:** Curves of $s = \frac{\sum e^2}{e^2}$ as a function of $t = \frac{T}{e^2}$ for various values of $b = \frac{\tilde{A}_0}{e}$, for the case $N = 1$.

- **Figure 2:** Curves of $s = \frac{\sum e^2}{e^2}$ as a function of $b = \frac{\tilde{A}_0}{e}$ for various values of $t = \frac{T}{e^2}$, for the case $N = 1$.

- **Figure 3:** Curves of $s = \frac{\sum e^2}{e^2}$ as a function of $t = \frac{T}{e^2}$ for two values of $b = \frac{\tilde{A}_0}{e}$, for the case $N = 2$. 

