A new graph perspective on max-min fairness in Gaussian parallel channels

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Abstract

In this work we utilize a novel view of graph theory to characterize the notion of max-min fairness in Gaussian parallel channels. We show that the max-min fair performance behaves as a version of the Lovasz function in terms of a graph induced by the channel sharing. This constructive result leads also to the specification of some easily computable fair policies.

1. INTRODUCTION

The allocation of resources (power, bandwidth, time) to multiple users sharing the parallel channels poses serious practical problems and still needs a deeper understanding [1], [2]. In particular, the optimum interrelations between the combinatorics of channel sharing and the (real-valued) power allocation are still not characterized. Also the essential questions such as “what is the power/time/bandwidth function describing it” remain unanswered so far.

In this work we make a step towards answers to the above questions for the max-min fair performance, with regards to a general performance function, in parallel channels shared by multiple users. We state optimistic and pessimistic bounds on the user performance (Sections 4, 5, respectively). The bounds give rise to insights into the problem structure: The essence of our results is that the fair performance behaves as a specialized version of the Lovasz function [3] of the channel sharing topology and is influenced by the set of allowable power allocations only through a certain gap/distance. The constructive proofs of the bounds lead furthermore to algorithmic fair policies achieving the proved bounds on user performance (Section 6). In this paper we present results without proofs, and refer for full proofs, extensive discussion and examples to [4].

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2. MODEL

Let \( \mathbf{x}_k = (x_{k1}, \ldots, x_{kN}) \) \( \in \mathbb{C}^N \) be a random vector grouping the independent zero-mean symbols of user \( k \in \mathcal{K} \), \( \mathcal{K} = \{1, \ldots, K\} \), transmitted equidistantly at distance \( T_t \) over the channels \( n \in \mathcal{N} \) in the parallel channels ensemble \( \mathcal{N} = \{1, \ldots, N\} \). The signal of user \( k \in \mathcal{K} \) received over the parallel channels is \( \mathbf{y}_k = (y_{k1}, \ldots, y_{kN}) \), with \( y_{kn} = h_{kn}x_{kn} + n_{kn}, \) \( n \in \mathcal{N} \), where \( h_{k} = (h_{k1}, \ldots, h_{kN}) \in \mathbb{C}^N \) collects the path coefficients between the transmitter and receiver of user \( k \in \mathcal{K} \) on channels \( n \in \mathcal{N} \) and where \( \mathbf{n}_k = (n_{k1}, \ldots, n_{kN}) \in \mathbb{C}^N \) is a random zero-mean iid Gaussian noise vector with \( \sigma_n^2 = E(|n_{kn}|^2) > 0 \).

The transmit power allocation to users and channels is represented by a matrix \( \mathbf{P} = (p_1, \ldots, p_K)' \in \mathbb{R}^K_{++} \times \mathbb{R}^{K \times N}_+ \), where vector \( p_k = (p_{k1}, \ldots, p_{kN}) \) is such that \( p_{kn} = E(|x_{kn}|^2) \).

Let \( \mathbf{A} = (a_1, \ldots, a_K)' \in \mathbb{R}^K_{++} \times \mathbb{R}^{K \times N}_+ \) denote the sharing matrix of the parallel channels among users such that \( a_k = (a_{k1}, \ldots, a_{kN}) \) collects the relative time fractions assigned to user \( k \in \mathcal{K} \) for the exclusive access to channels \( n \in \mathcal{N} \). The set of all sharing matrices takes the form

\[
\mathcal{A} (\mathbf{r}) = \{ \mathbf{A} \in \mathbb{R}^{K \times N}_+ : \| \mathbf{a}_k \|_1 \leq r_k, \left( \sum_{k \in \mathcal{K}} a_{kn} \right) \leq 1 \},
\]

where a predefined \( \mathbf{r} = (r_1, \ldots, r_K) \in \mathbb{R}^K_+ \) is such that \( r_k/N \) represents the fraction of the parallel channels which is assigned to user \( k \in \mathcal{K} \) over time (under \( \mathbf{r} = \mathbf{1}/N \), any user is assigned an equal \( \frac{1}{N} \)-fraction of the parallel channels ensemble over time).

Given a sharing matrix \( \mathbf{A} \in \mathcal{A} (\mathbf{r}) \) under use we assume an arbitrary set \( \mathcal{P} (\mathbf{A}) \) of allowed power allocations, requiring merely that \( \mathcal{P} (\mathbf{A}) \supseteq (S_1 (0) \cap \mathbb{R}^{K \times N}_+) \) for some \( \epsilon = \epsilon (\mathbf{A}) > 0 \) (i.e., all power allocations in sufficiently small ball around \( \mathbf{0} \) are allowable). Assuming the partition of transmission into frames of duration \( T_t \), we can consider either energy constraints per frame or
transmit power constraints at any time in a frame. For instance, the set

\[ \mathcal{P}(A) = \{ P \in \mathbb{R}_+^{K \times N} : \sum_{k \in \mathcal{K}} T(a_k, p_k) \leq E, \ A \in \mathcal{A}(r), \} \]

corresponds to the limitation of the aggregate energy budget (per frame) of all users by \( E > 0 \), while \( \mathcal{P}(A) = \mathcal{P} = \{ P \in \mathbb{R}_+^{K \times N} : \| p_k \|_1 \leq P_k, k \in \mathcal{K} \} \) mirrors the constraints \( P_k > 0 \) on power of each user \( k \in \mathcal{K} \) at any time in a frame. Note that in the latter case of instantaneous power constraints, the set of allowable is independent of the sharing matrix.

For any user \( k \in \mathcal{K} \) we assume a general nonnegative performance/QoS function \( p \mapsto f_k(p) \in \mathbb{R}_+^N \), \( p \in \mathbb{R}_+^N \). Hereby \( f_k(p) = (f_{k1}(p_1), \ldots, f_{kN}(p_N)) \), where map \( p \mapsto f_{kn}(p), \ p \geq 0 \), expresses the performance of user \( k \in \mathcal{K} \) on channel \( n \in \mathcal{N} \) as a function of power allocated to channel \( n \in \mathcal{N} \), when the user accesses this channel exclusively throughout. To avoid technical queerness we assume Fréchet-differentiability and increasingness for infinitesimally small power allocations, i.e. \( \frac{\partial}{\partial p_k} f_k(p) > 0 \), \( k \in \mathcal{K} \), for \( p \in S_{\mathcal{K}}(0) \cap \mathbb{R}_+^N \).

We refer to the inner product \( \langle a_k, f_k(p_k) \rangle \) as the (aggregate) user performance/QoS of user \( k \in \mathcal{K} \) under policy \( (A, P) \in \mathcal{A}(r) \times \mathcal{P}(A) \) and denote the user-specific requirements with respect to user QoS by \( \gamma_k, k \in \mathcal{K} \). The focus of this work is on the maximum attainable relative user performance of the worst-case user accessing the parallel channels, which can be expressed as

\[
\max_{(A, P) \in \mathcal{A}(r) \times \mathcal{P}(A)} \min_{k \in \mathcal{K}} \langle a_k, f_k(p_k) \rangle \gamma_k
\]

and is called the (max-min) fair performance [1], [2]. Two useful examples of QoS are the following (see also [3]).

1. **Capacity:** Under Maximum Likelihood receivers, \( f_{kn}(p) = \log(1 + \frac{p_k b_{kn}^2}{\sigma_k^2}) \) represents the information capacity achievable by user \( k \in \mathcal{K} \) when accessing the channel \( n \in \mathcal{N} \). Then, the function \( \overline{c}(a_k, f_k(p_k)) \) corresponds to the achievable (under Gaussian codebook) number of reliably decoded bits/nats per frame under policy \( (A, P) \).

2. **Symbol decoding reliability:** If user \( k \in \mathcal{K} \) accesses channel \( n \in \mathcal{N} \) and uses uncoded constant-envelope modulation, then the achieved probability of error-free symbol decoding is \( f_{kn}(p) = 1 - Q(\sqrt{\frac{p_k b_{kn}^2}{\sigma_k^2}}) \), with \( Q \) as the Marcum Q-function, \( M \) as the constellation size, and some \( c > 0 \) [5]. Then, \( \overline{c}(a_k, f_k(p_k)) \) expresses the aggregate (over channels \( n \in \mathcal{N} \)) average number of error-free decoded symbols in a frame under policy \( (A, P) \).

### 3. THE SHARING GRAPH

Our key concept is a sharing graph induced by a sharing matrix (a graph \( G = (\mathcal{K}, \mathcal{E}) \) has the vertex set \( \mathcal{K} \) and the set of edges, i.e. adjacent vertex pairs, \( \mathcal{E} \)). The proposed definition of a sharing graph is a version of orthonormal graph representation in [3].

**Definition 1** For \( N \geq K \) and any \( A \in \mathcal{A}(r) \), a corresponding sharing graph \( G = G(A) \) is such that \( G = (\mathcal{K}, \mathcal{E}) \) where \( (k, l) \in \mathcal{E}, k \neq l \), if \( \langle a_k, a_l \rangle > 0 \).

That is, any two vertices \( k, l \in \mathcal{K}, k \neq l \), of the sharing graph are adjacent if some of the parallel channels are shared by users \( k, l \), where a channel is said to be shared by some two users if both users access this channel exclusively some fraction of time. We group the sharing matrices which induce a sharing graph \( G \) in the set

\[
\mathcal{A}(G, r) = \{ A \in \mathcal{A}(r) : G = G(A) \}, \quad r \in \mathbb{R}_+^K.
\]

Note furthermore that, for any fixed \( A \in \mathcal{A}(r) \), an induced sharing graph \( G = G(A) \) is in general not unique. The construction of a sharing graph is illustrated for exemplary parallel channels in Fig. 1.

A special role in our considerations is played by cycles, where a cycle of length \( M \) in \( G = (\mathcal{K}, \mathcal{E}) \) is a sequence of distinct graph vertices \( k_i \in \mathcal{K}, 1 \leq i \leq M \), such that \( (k_i, k_{i+1}) \in \mathcal{E}, 1 \leq i \leq M - 1 \) and \( (k_M, k_1) \in \mathcal{E} \) [6]. In a sharing graph, a cycle of length, say, \( M \) corresponds to a chain of \( M \) users accessing the parallel channels such that any pair of subsequent users shares some channel and the last user shares a channel with the first user. In this context, a specific role is played by so-called \( M \)-partite graphs. Such graphs contain edges only between some \( M \) disjoint vertex subsets (and no
edge between two vertices within the same vertex subset) and thus, cannot contain any cycle longer than \(M\). Evidently, \(M\)-partite sharing graphs are induced by sharing policies which distinguish \(M\) classes of users with the property that users within one class are not allowed, or not able, to share any channels over time.

The first key sharing graph function which we make use of is the Lovasz function, which was used originally, in the celebrated approach to the problem of graph capacity in [3]. The (weighted) Lovasz function of a graph \(G = (\mathcal{K}, \mathcal{E})\) can be expressed as the map

\[
(G, v) \mapsto \theta^0(G, v) = \min_{B \in \mathcal{B}(G, v)} \lambda_{\text{max}}(B), \quad v \in \mathbb{R}^K_+,
\]

where, with \(\mathcal{S}^K\) as the set of symmetric matrices in \(\mathbb{R}^{K \times K}\),

\[
\mathcal{B}^0(G, v) = \{B = (b_{kl}) \in \mathcal{S}^K : b_{kl} = (v_k v_l)^{\frac{1}{2}}/(k, l) \notin \mathcal{E} \text{ or } k = l\}. \tag{2}
\]

For some graph classes, e.g. for perfect graphs, the value \(\theta^0(G, 1)\) was shown to be equal to the capacity of graph \(G\) [3]. From the algebraic framework of coding theory we know also the concept of a Delsarte number, or bound, which is a generalization of the Lovasz function of the form \(G, v) \mapsto \theta^i(G, v) = \min_{B \in \mathcal{B}(G, v)} \lambda_{\text{max}}(B), \quad v \in \mathbb{R}^K_+, \quad i = 1, 2, 3\), for any graph \(G = (\mathcal{K}, \mathcal{E})\),

\[
\mathcal{B}^i(G, v) = \{B = (b_{kl}) \in \mathcal{S}^K : b_{kl} \geq (v_k v_l)^{\frac{1}{2}}/(k, l) \notin \mathcal{E} \text{ or } k = l\}. \tag{3}
\]

The Delsarte bound corresponds to an upper bound on the cardinality of a so-called \(\mathcal{M}\)-clique, \(\mathcal{M} \subset \{1, \ldots, M\}\), in an association scheme with \(M\) associate classes (see [4] for references). For the purposes of this work, we define two further sets of the type (2), (3) as

\[
\mathcal{B}^2(G, v) = \{B = (b_{kl}) \in \mathcal{S}^K : b_{kl} = (v_k v_l)^{\frac{1}{2}}/(k, l) \notin \mathcal{E} \text{ or } k = l,
\]

\[
b_{kl} \leq (v_k v_l)^{\frac{1}{2}}, \quad k, l \in \mathcal{K}\} \tag{4}
\]

and

\[
\mathcal{B}^3(G, v) = \{B = (b_{kl}) \in \mathcal{S}^K : b_{kl} = (v_k v_l)^{\frac{1}{2}}/(k, l) \notin \mathcal{E} \text{ or } k = l,
\]

\[
v^\dagger v^\frac{1}{2} B + \lambda_{\text{max}}(B) I \in \mathbb{P}^K\}. \tag{5}
\]

Hereby, \(\mathbb{P}^K\) denotes the class of completely positive matrices in \(\mathbb{R}^{K \times K}\), i.e. those symmetric matrices which are nonnegative and nonnegatively factorizable in the sense that for any \(C \in \mathbb{P}^K\) we can write \(C = V^\dagger V\) for some \(V \in \mathbb{R}^{(\phi(C)) \times K}\), where the minimum number \(\phi(C)\) is referred to as the cp-rank of \(C\) [6], [4]. For any given graph \(G = (\mathcal{K}, \mathcal{E})\), the related functions in the spirit of \(\theta^0\) and \(\theta^1\) take then the form \((G, v) \mapsto \theta^i(G, v) = \min_{B \in \mathcal{B}^i(G, v)} \lambda_{\text{max}}(B), \quad v \in \mathbb{R}^K_+, \quad i = 2, 3\).

By (2)-(5) it is readily seen that, given a graph \(G = (\mathcal{K}, \mathcal{E})\) and any \(v \in \mathbb{R}^K_+\), we have \(\theta^2(G, v) \geq \theta^2(G, v) \geq \theta^0(G, v) \geq \theta^1(G, v)\).

4. OPTIMISTIC BOUNDS

We first discuss two upper bounds on the fair performance in parallel channels. According to our model, an upper bound represents an optimistic case, i.e. a value of fair performance no worse than the true one. In the next Section we combine these bounds with analogous pessimistic ones to obtain our central conclusions.

Consider the following characterization of fair performance under a fixed sharing graph.

**Proposition 1** Given \(N \geq K\), any \(G = (\mathcal{K}, \mathcal{E})\), and \(r \in \mathbb{R}^K_+\), we have

\[
\max_{(A, P) \in \mathcal{A}(G, r) \times \mathcal{P}(A)} \min_{k \in \mathcal{K}} \frac{\langle a_k, f_k(p_k) \rangle^2}{\gamma_k^2} \leq \min_{f \in \mathcal{F}(G, r)} \frac{\langle f, f \rangle}{\theta^i(G, w)},
\]

\(i = 0, 1, 2, \text{ where } w\) is such that

\[
w_k = \frac{\gamma_k^2}{\theta^2_k}, \quad k \in \mathcal{K}, \tag{6}
\]

and where we defined

\[
\mathcal{F}(G, r) = \{f \in \mathbb{R}^N_+ : \langle a_k, f_k(p_k) \rangle \geq \langle a_k, f_k(p_k) \rangle, \quad k \in \mathcal{K}, \quad \text{for some } A \in \mathcal{A}(G, r)\},
\]

with \((\hat{A}, \hat{P}) = \arg \max_{(A, P) \in \mathcal{A}(G, r) \times \mathcal{P}(A)} \min_{k \in \mathcal{K}} \frac{\langle a_k, f_k(p_k) \rangle^2}{\gamma_k^2}\).

It is readily seen that \(\mathcal{F}(G, r)\) is the set of values of \((\mathbb{R}^N_+\text{-valued})\) performance functions which are equal for any user accessing the parallel channels and, for some sharing matrix which induces the same sharing graph as \(A\), attain user performance no worse than under policy \((A, P)\). Thus, \(\mathcal{F}(G, r)\) is a set of QoS function values which in (some sense) dominate the QoS of the fair policy \((\hat{A}, \hat{P})\) under fixed sharing topology \(G\).

We can also formulate the following loosened, but more illustrative, bound based on Proposition 1.

**Corollary 1** Given \(N \geq K\), any \(G = (\mathcal{K}, \mathcal{E})\) and \(r \in \mathbb{R}^K_+\), we have

\[
\max_{(A, P) \in \mathcal{A}(G, r) \times \mathcal{P}(A)} \min_{k \in \mathcal{K}} \frac{\langle a_k, f_k(p_k) \rangle^2}{\gamma_k^2} \leq \min_{f \in \mathcal{F}(G, r)} \frac{\langle f, f \rangle}{\theta^i(G, w)},
\]

\(i = 0, 1, 2, \text{ where } w\) is such that (6) and where, with \(\hat{A}\) defined as in Proposition 1,

\[
\mathcal{F}(G, r) = \{f \in \mathbb{R}^N_+ : \|f\|_1 \geq \|f_k(p_k)\|_1, \quad k \in \mathcal{K}, \quad P \in \mathcal{P}(\hat{A})\}.
\]
Here, the set $\tilde{F}(G, r)$ includes all QoS function values, equal for all users, which are in the sum over all channels superior to any QoS function value achieved by an allowable power allocation under the sharing policy $\tilde{A}$. Thus, $\tilde{F}(G, r)$ can be seen as a hull of any user dimension of the feasible QoS/performance set of parallel channels, which we define here in analogy to the (equivalently, $A_k$) of the lower bounds make strong use of the notion of min fair performance in parallel channels. The proofs no worse. These bounds are analogs of the optimistic sense that the fair performance is guaranteed to be given $A$ defined as in Proposition 1.

Moreover, given a particular

$$B = \arg\min_{B \in \mathbb{R}^N} \lambda_{\text{max}}(B),$$

(9)

this further implies

$$\max_{\mathbb{V} \in \mathbb{R}^N} \left\{ \frac{(a_k, f_k(p_k))^2}{\gamma_k^2} \right\} \geq \frac{\mu(RV) \max_{f \in \tilde{F}(G, r)} \langle f, f \rangle}{\theta^2(G, w)}$$

for some $P \in \mathcal{P}(A)$.

Proposition 2: Given any $G = (K, E)$ and $r \in \mathbb{R}^K_{++}$, we have

$$\max_{(\mathcal{A}, P) \in \mathcal{A}(G, r) \times \mathcal{P}(\mathcal{A})} \min_{k \in K} \frac{(a_k, f_k(p_k))^2}{\gamma_k^2} \geq \frac{\mu(RV) \max_{f \in \tilde{F}(G, r)} \langle f, f \rangle}{\theta^2(G, w)},$$

(10)

where $RV$ in the last expression is any nonnegative factor of the particular matrix

$$R(\lambda_{\text{max}}^{-1}(B)(w^I \cdot w^I - B) + I)R$$

such that (9). The set $\tilde{F}(G, r)$ for is dimension of the feasible QoS set (7) of the parallel channels. On the other hand, recall that the hull $\tilde{F}(G, r)$ as the set of all (vector forms of) $\tilde{A}_1 \leq \tilde{A}_2$. Here, $\tilde{A}_1$ is the maximum smallest geometric mean of pairs of diagonal entries among scalings from $X(C, r, c)$.

The lower bounds on max-min fair performance presented below correspond to pessimistic values, in the sense that the fair performance is guaranteed to be no worse. These bounds are analogs of the optimistic bounds from Section 4, and together embrace the max-min fair performance in parallel channels. The proofs of the lower bounds make strong use of the notion of an $(r, c)$-scaling of a nonnegative matrix $C \in \mathbb{R}^K \times N$, which is $C = \text{diag}(C)$, for $c \in \mathbb{R}^N_{++}$; it is defined as a matrix pair $(X, Y) \in \mathbb{R}^K \times \mathbb{R}^K \times N$ such that $X = \text{diag}(X)$, $Y = \text{diag}(Y)$ and $XYC = r$, $1^T XCY = c$. [9]. This gives rise to the definition of

$$X(C, r, c) = \{ x \in \mathbb{R}^K : y = X1 : (X, Y) \in (r, c) \}$$

(8)

as the set of all (vector forms of) $(r, c)$-scalings of a given matrix $C$ which yield row an column sum vectors no larger than $(r, c)$. Further, we use the map

$$\mu(C) = \max_{(x, y) \in X(C, r, 1)} \min_{n \in N} (x_k y_n)^2$$

as a kind of metric (see also [7]) of such entire class of $(r, c)$-scalings which do not exceed $(r, 1)$: $\mu(C)$ represents the maximum smallest geometric mean of pairs of diagonal entries among scalings from $X(C, r, 1)$.

Consider the following main lower bound characterization (for other tighter bounds we refer to [4]).
the QoS functions set of parallel channels, that is, in terms of the structure of the feasible performance interval (10) of candidate fair performance values $\theta^2(G, w)$ and normalized function values $\theta^3(G, w)/\mu(RV)$ for the sharing graph $G$; the normalization is here by the metric $\mu$ of the corresponding factor of (11) such that $\lambda_{\max}(B) = \theta^3(G, w)$. Thus, the tightest pessimistic bound is obtained for a matrix $B \in B^3(G, w)$ and a nonnegative factor $RV$ of (11) which provide the minimum normalized eigenvalue $\lambda_{\max}(B)/\mu(RV)$. Similarly, the outer bounds in (10) are tightest for a factor $RV$ of the particular matrix (11), such that (9), which maximizes metric $\mu$.

5.1. THE ROLE OF CYCLES

As the proposed upper bounds depend on the sharing graph $G$ via the spectral properties of the set $B^3(G, w)$ and the lower bounds are governed via the eigenvalues achievable within the smaller set $B^2(G, w)$, we obtain an undesired asymmetry in the proposed bounds (10). By the recent results on completely positive graphs we can, however, unify the dependence on the sharing graph for a large class of sharing topologies (see [6] and references therein).

**Proposition 3** Let $G = (\mathcal{K}, \mathcal{E})$ be any sharing graph with either $K \leq 4$ or with no odd cycles longer than 4. Then, the bounds from Proposition 2 and in (10) are satisfied with

$$B^2(G, w) = B^3(G, w),$$

and thus, $\theta^3(G, w) = \theta^2(G, w)$.

Proposition 3 implies that whenever the parallel channels are accessed by no more than $K = 4$ users, the value of the function $\theta^2$ assumed for the sharing graph $G$ and vector $w$ is a sufficient characterization of the sharing policy for enclosing the fair performance according to (10). Similarly, the value $\theta^2(G, w)$, for the given sharing graph $G$, is a sufficient description of the channel sharing for the proposed fair performance characterization (10) when there is no odd chain of more than $K = 4$ users such that any two subsequent users share some channel and the last user shares a channel with the first one (this makes up a cycle in the sharing graph). In particular, we have such property when the users accessing the parallel channels can be partitioned into no more than $M = 4$ groups such that no pair of users within one group is allowed, or able, to share a channel; for instance due to certain constraints on traffic class processing or hardware, as illustrated by examples in [4]. The channel sharing is represented in such case by $M$-partite sharing graphs, discussed in Section 3.

6. SOME FAIR POLICIES
Figure 3: The user QoS under policy \((\bar{A}, \bar{P})\) from Algorithm 1 (dashed line) and fair performance given by (1) (solid line), under sum-power constraint of all users and capacity as QoS function. A randomly picked instance of parallel channels with \(K = 4\) and \(N = 6\) was simulated; the randomly picked graphs \(G(j) = (K, E(j))\), \(1 \leq j \leq 100\), are such that edges occur independently with probability 0.5.

The proofs of Proposition 2 and the related lower bounds in [4] are constructive: They contain algorithmic specifications of certain parallel channels policies in the case when the sharing topology/graph is predefined, e.g. due to regulations of traffic/signal processing, hardware constraints, etc. [4]. Such policies can be seen as (sub-) fair in the sense that they ensure user performance of any user be no worse than the respective bounds proven. We present here one of the corresponding algorithms and refer to [4] for further algorithms and their detail derivation (we restrict us to the case of power constraints at any time in a frame). The algorithm uses arbitrary small scalars \(\alpha, \delta > 0\), the given sharing graph \(G = (K, E)\) and \(P\) as input.

**Algorithm 1**

1. Compute a sharing matrix \(\bar{A}\) and vector \(\bar{f}\) as

\[
(\bar{A}, \sqrt{\bar{f}}) = \arg \max_{(A, c) \in K} \frac{(a_k, c)}{\gamma_k}
\]

subject to

\[
\left\{ \begin{array}{l}
(A, c) \in A(r) \times \mathbb{R}^N_+ \\
(a_k, a_l) \leq 0, \quad (k, l) \notin E \\
(c, c) \leq 1,
\end{array} \right.
\]

by any bilinear programming method.

2. Compute a power allocation \(P\) from \(f_k(\bar{p}_k) = \alpha \bar{f}, k \in K\).

3. If \(P \in P\) then set \(\alpha \mapsto \alpha + \delta\) and go to step 2, otherwise stop.

By the algorithm, the solution of the original nonlinear and nonconvex program (1) is replaced by fundamental algebraic operations and the solution of a simple bilinear program, for which efficient local and global solution approaches exist (see e.g. [8]).

**Corollary 2** Given \(G = (K, E)\) with no odd cycles longer than 4, the policy \((\bar{A}, \bar{P})\) from Algorithm 1 achieves the worst user QoS \(\min_{k \in K} \frac{(a_k, x_k)}{\gamma_k}\) which is at most

\[
\min_{f \in \mathcal{F}(r)} f - \min_{k \in K} x_k \max_{f \in \mathcal{F}(r)} f \leq \frac{\theta(f, G, w)}{\theta^2(G, w)} \cdot (x, 1) \in \mathcal{X}(RV, r, 1),
\]

away from the fair performance under predefined \(G\).

Fig. 3 provides an exemplary comparison of the policy from Algorithm 1 and the fair policy solving (1). We observe a particular loss of about 20% to the fair performance. Such loss can be shown to decrease further if the differences between the user channel vectors \(h_k, k \in K\), and between the variance ensembles \(\sigma_{kn}, n \in N\), of users \(k \in K\) diminish.

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