INVARINANCE AND INNER FRACTALS IN POLYNOMIAL AND TRANSCENDENTAL FRACTALS

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Abstract. A lot of formal and informal recreational study took place in the fields of Meromorphic Maps, since Mandelbrot popularized the map $z \leftarrow z^2 + c$. An immediate generalization of the Mandelbrot $z \leftarrow z^n + c$ also known as the Multibrot family were also studied. In the current paper, general truncated polynomial maps of the form $z \leftarrow \sum_{p \geq 2} a_p x^p + c$ are studied. Two fundamental properties of these polynomial maps are hereby presented. One of them is the existence of shape preserving transformations on fractal images, and another one is the existence of embedded Multibrot fractals inside a polynomial fractal. Any transform expression with transcendental terms also shows embedded Multibrot fractals, due to Taylor series expansion possible on the transcendental functions. We present a method by which existence of embedded fractals can be predicted. A gallery of images is presented alongside to showcase the findings.

1. Introduction

Mandelbrot popularized the map $T(z) : z \leftarrow z^2 + c$ which bears his name ([1], [5]). The Mandelbrot set $M$, comprise of points ‘$z$’, such that $T(z)$, repeated infinite time, upon itself, remains bounded. The iteration is defined as in [1] :-

1. Initialize current point $c = P(x, y)$ and $z_0 = c$.
2. Use the Transform map $T : z \leftarrow z^2 + c$ to generate $z_1$.
3. Repeat the procedure (2), to get $z_2, z_3, ..., z_l$ until $|z_l| > k$ or $l = N$ is the maximum number of iteration.
4. If iterations expired, the original point $c \in M$.
5. Else $c \notin M$.
6. Take another point in $(x, y)$ plane, and start again from step (1), till all the points in the plane are exhausted.

When $T(z)$ is applied on itself once it can be said $T^2(z)$. Making things general, an ‘$x$’ times repetition of $T(z)$ upon itself would be termed as $T^x(z)$.

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Dedicated to Benoit B. Mandelbrot.

Dedicated to our parents and children, without their presence we are nothing.
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Then, Formally:
\[ M(k) = \{ z : \lim_{x \to \infty} |T^x(z)| \leq k \} \] (1.1)

A plot of the set yields outstanding visual imagery, and patterns.

1.1. **Escape Time Fractals.** A generic way to produce any set \( \Phi \) of this kind would be then to have a transform function \( F(z) : z \leftarrow F(z) \) such as to:-

\[ \Phi(k) = \{ z : \lim_{x \to \infty} |F^x(z)| \leq k \} \] (1.2)

Clearly then, \( k \) is a very important parameter. However the removal of \( k \) can be easily done with:-

\[ \Phi = \{ z : \lim_{x \to \infty} |F^x(z)| < \infty \} \] (1.3)

For in case of Mandelbrot set \( M \) it can be easily shown that any point \( |z| > 2 \) would escape to infinity, that means:-

\[ M(z^2 + c, 2) = M(z^2 + c, k) ; 2 < k \leq \infty \]

so for \( M \), \( k \) is actually omitted.

The images generated by equation (1.3) are called “Escape Time Fractals”, also known as “orbits” fractals.

1.2. **Our Methodology.** We have modified an open source ‘Mandelbrot Set Viewer’ computer program, and incorporated the ability to type in mathematical expressions for the function \( F(z) \) in equation (1.3). The source code is available [here](#).

There is an expression handler in the program that can handle many standard mathematical functions. Given the expression, the program would evaluate the expression dynamically, to generate the ‘escape time fractal’ for the function \( F(z) \).

The program is capable of:-

1. zoom-in and zoom-out.
2. modifying the radii of convergence ‘\( k \)’, to be expressed as a power of ‘\( e \)’.
3. modifying number of iterations to perform as ‘infinity’.
4. saving the image as ‘png’ format image file, that is how illustrations of this paper were generated.
5. plotting the set in Colors, to be discussed more in next.

1.2.1. **Plotting the Set using Software.** From a computational point of view, there is no way to perform the iteration of equation (1.3) up-to infinity. Hence, a maximum ‘\( N \)’ number of iteration is performed. Using the custom software we can manipulate ‘\( N \)’, \( N \in [0, 99999] \). If a point \( z \) exceeds the limit \( k \) at iteration number \( m \), then a ratio \( x = m/N \) is defined. Clearly \( x = 1 \) means, the function remained bounded till the ‘\( N \)’th iteration. Based upon the value of \( x \), the color of the point is chosen, among a fixed set of colors. Though a point might show \( x = 1.0 \), that does not imply that, it would remain so for an increased \( N \). Also, another point
having current \( x < 0 \) might get into the set, for an increased \( k \). The coloring can be formally defined as a mapping:

\[
C(x) : \mathbb{R} \rightarrow \mathbb{N}
\]

The custom application which produced the illustrations of present paper is capable of plotting in any integer number of colors, and can handle any mathematical expression and function. The color codes are dependent on the color map used. The program uses equation (1.5) for coloring.

\[
C(x) : \left\lceil px \right\rceil ; \forall x \in (0, 1)
\]

In equation (1.5) ‘\( p \)’ is the total number of colors in the color map. The default value for color maps used is \( p = 256 \), and the default color is Gray Scale. The user of the software can choose from a list of color maps, or can create his own color map.

1.2.2. Unique Findings from the Experiments. As the software can evaluate almost any mathematical expressions to generate the fractals of type equation (1.3), we noticed certain interesting findings.

The most interesting phenomenon we noticed is the existence of ‘embedded’ fractals while using Transcendental expressions, that is fractals, which are created by expressions with transcendental functions in it. The first observation was ‘\( \tan(z)^2 + c \)’ fractal (figure 8 on page 12) contains ‘\( z^2 + c \)’ or Mandelbrot fractals inside it (figure 18 on page 19).

There are Mandelbrot fractals inside ‘\( (\tan(z))^{-2} + c \)’ fractal. Figure 19 on page 19 illustrates a \( (\cotan(z))^2 + c \) fractal, and figure 20 on page 20 depicts the embedded Mandelbrot fractals inside it.

The second observation was, that the Mandelbrot set is capable of ‘Shape Preserving Transformations’, like rotation, scaling, and translation, just by multiplying the \( z^2 \) term by different (complex) factors. We have established a ‘generic’ way to transform polynomial fractals without a \( z^1 \) term.

1.3. Generalization of the Transformation Function. Equation (1.1) can be further generalized into:

\[
F : z \leftarrow z^n + c ; n \in \mathbb{N}
\]

where \( \mathbb{N} \) is the set of Integers.

Further generalization can be done by:

\[
F : z \leftarrow z^n + c ; n \in \mathbb{R}
\]

where \( \mathbb{R} \) is the set of real numbers. The resulting sets are known as “Multibrots” family. In particular, [3] and [4] consider these maps in details.

Figure 1 on page 4 shows a “Multibrot” fractal with \( n = 4.0 \).
2. Invariance: Shape Preserving Transforms on Fractal Shapes

We know that the only shape preserving transforms are:-

1. Translation
2. Rotation
3. Scaling

In this section we review the integer multibrot map (1.6), under the influence of shape preserving transformations. Our observation is that Standard shape preserving transformations works on the function family of ‘Multibrots’ with positive ‘n’ (equation (1.6), \( n > 0 \)). We note that the family with \( n \in \mathbb{Q} \) does not preserve shape under rotation, simply because the transform cease to become a function:

\[
z \leftarrow z^{\frac{2}{a}} + c \implies z \leftarrow b\sqrt{z^a} + c
\]

The expression \( b\sqrt{z^a} \) is not a function, and therefore no 1-1 relationship between the old point, and mapped point can be established.

2.1. Shape Preserving Transformation Theorems. We show that the point which is now current transformed ‘\( t \)z’, has the same trajectory of the earlier ‘z’. So, if ‘\( z \)’ was included in the set, so would be ‘\( t \)z’ with modification in \( k \)’s value, and, if ‘\( z \)’ was not included in the set, so would not be ‘\( t \)z’. That would establish the transformation properties.

Theorem 2.1. A variable transformation of the form \( z : z - a \) translates a Multibrot Map.

Translation can be achieved by :-

\[
T_t(z) : z \leftarrow (z - a)^n + c ; a(x, y) \in \mathbb{C} ; n > 0
\]

The result is the set image shifted co-ordinates to \( a = (x, y) \).

One point to be noted here is that as the radii of convergence is \( k \), if the point \( a \) resides further than \( k \), then we need to increase the radii appropriately, for seeing the whole set’s image.
Proof of the Translation Theorem. To prove this we note that all points which were mapped as \( z \) according to equation (1.6), are now mapped as \( z + a \) in the new mapping with equation (2.1). That would mean that if \( z \in M \) for equation (1.6), then \( z + a \in M' \) for equation (2.1), and vice versa. We take a point \( t^c = c + a \) and hence \( t^z_0 = c + a \).

Clearly at 1st level the original transformation can be written as:-

\[
z_1 = z_0^n + c = c^n + c \tag{2.2}
\]

Compare with 1st level transformation using equation (2.1):

\[
t^z_1 = (t^z_0 - a)^n + t^c = (t^c - a)^n + t^c = (c + a - a)^n + c + a = c^n + c + a
\]

where \( t^z \) and \( t^c \) signifies the transformed \( z, c \).

So we have established that:-

\[
t^z_1 = z_1 + a \tag{2.3}
\]

Now, let it be true for some \( l \geq 1 \). Then we have:-

\[
t^z_l = z_l + a
\]

We have:-

\[
z_{l+1} = z_l^n + c \tag{2.4}
\]

and

\[
t^z_{l+1} = (t^z_l - a)^n + t^c
\]

Replacing the value of \( t^z_l \) we have:-

\[
t^z_{l+1} = (z_l + a - a)^n + t^c = (z_l^n + c) + a
\]

hence:-

\[
t^z_{l+1} = z_{l+1} + a \tag{2.5}
\]

Using equation (2.3) as basis and then having equation (2.5), implies that using principle of mathematical induction, we can say all points which were mapped as \( z \) according to equation (1.6), are now mapped as \( z + a \) in the new mapping with equation (2.1). That establishes the translational property. \( \square \)

Theorem 2.2. Transformation \( z : e^{i\theta} z \) rotates a Multibrot Map.

A “clockwise” rotation of angle \( \rho = \theta/(n-1) \) can be achieved by:-

\[
T_r(z) : z \leftarrow e^{i\theta} z^n + c \tag{2.6}
\]

Proof of the Rotation Theorem. To prove that we note again: \( z_0 = c \) and \( t^z_0 = ce^{\left(\frac{-i\theta}{n-1}\right)} \).

We have:-

\[
t^z_1 = e^{i\theta}(t^z_0^n) + t^c = c^n e^{\left(\frac{i\theta}{n-1}\right)} + c e^{\left(\frac{-i\theta}{n-1}\right)} = e^{\left(\frac{-i\theta}{n-1}\right)} (c^n + c)
\]

Hence, using equation (2.2)

\[
t^z_1 = e^{\left(\frac{-i\theta}{n-1}\right)} z_1 \tag{2.7}
\]
Assume now that for some \( l \geq 1 \)

\[ t_{z_l} = e^{\frac{-i\theta}{n-1}} z_l \]

Clearly we would have:

\[ t_{z_{l+1}} = e^{i\theta} (t_{z_l}^n) + t \cdot c = e^{\frac{-i\theta}{n-1}} z_l^n + c = e^{(\frac{-i\theta}{n-1}) (z_l^n + c)} \]

Then, using equation (2.4) we get:

\[ t_{z_{l+1}} = e^{\frac{-i\theta}{n-1}} z_{l+1} \quad (2.8) \]

Using equation (2.7) as basis and having equation (2.8) implies that, using principle of mathematical induction, we can say all points which were mapped as \( z \) according to equation (1.6), are now mapped as \( e^{(\frac{-i\theta}{n-1})} z \) in the new mapping with equation (2.6). That establishes the rotational property.

□

Note that for “Multibrots” with \( n < 0 \) this formula holds true. As the term \( \rho < 0 \) \( \forall n < 0 \), this rotation would be in “anti-clock-wise” direction. For \( n > 0 \), this rotation is in “clockwise” direction. Also note that, for “Multibrots”with fractional \( n > 0 \), rotation leads to altogether different type of images. They are not invariant under rotation.

**Theorem 2.3. Transformation** \( z : az \) **scales a Multibrot Map**

A Scaling of factor \( \sigma \) can be achieved by :-

\[ T_s(z) : z \leftarrow az^n + c ; \sigma = a^{(\frac{1}{\rho-n})} ; a \in \mathbb{R}_+ ; n > 0 \quad (2.9) \]

The resulting figure is a scaled version of the original Map, by a factor of \( \sigma \).

**Proof of the Scaling Theorem.** To show this, we start with \( z_0 = c \) and \( t \cdot z_0 = ca^{(\frac{1}{\rho-n})} \) as initial points.

Clearly then:-

\[ t_{z_1} = a(t_{z_0}^n) + t \cdot c = a(ca^{(\frac{1}{\rho-n})})^n + ca^{(\frac{1}{\rho-n})} = a^{(\frac{1}{\rho-n})}(c^n + c) \]

hence, using equation (2.2)

\[ t_{z_1} = a^{(\frac{1}{\rho-n})} z_1 \quad (2.10) \]

Now, assume it is true for some \( l \geq 1 \)

\[ t_{z_l} = a^{(\frac{1}{\rho-n})} z_l \]

Clearly we can say:-

\[ t_{z_{l+1}} = a(t_{z_l}^n) + t \cdot c = a(a^{(\frac{1}{\rho-n})} z_l)^n + ca^{(\frac{1}{\rho-n})} = a^{(\frac{1}{\rho-n})}(z_l^n + c) \]
Then, using equation (2.4) we get:-

\[ t_{z_{l+1}} = a^{(\frac{1}{n})} z_{l+1} \]  \hspace{1cm} (2.11)

Using equation (2.10) as basis and having equation (2.11) implies that, using principle of mathematical induction, we can say all points which were mapped as \( z \) according to equation (1.6), are now mapped as \( a^{(\frac{1}{n})} z \) in the new mapping with equation (2.9). That establishes the scaling property.

□

Note that, “Multibrots” with fractional \( n > 0 \) are invariant under scaling.

2.2. **Generic Shape Preserving Transformation.** The most generic shape preserving transformation would be, then

\[ T(z) : z \leftarrow a(z - b)^n + c ; \quad a = |a|e^{i\theta}, b(x, y) \in \mathbb{C} \]  \hspace{1cm} (2.12)

The transformation of equation (2.12) translates the origin by \( b = (x, y) \) amount, then rotates the map by \( \theta/(n - 1) \), and then scales down the map by \( |a|^{(\frac{1}{n-1})} \).

2.2.1. **Shape Preserving Transformations For Polynomial Fractals.** Polynomials are important due to Taylors Series expansion available for any infinitely differentiable function. The generic formula for a polynomial transform is:-

\[ T(z) = \sum_{j=0}^{n} a_j z^j + c = P(z) + c \]  \hspace{1cm} (2.13)

We note that while equation (2.1) can be immediately applied to a polynomial, due to variations in power transforms like rotation: equation (2.6) and scaling: equation (2.9) can not be used on equation (2.13). Hence, only with the translation part, we can redefine equation (2.12) for polynomials as:-

\[ T(z) : z \leftarrow P(z - b) + c ; \quad b(x, y) \in \mathbb{C} \]  \hspace{1cm} (2.14)

2.2.2. **Rotation and Scaling of Truncated Polynomial Fractals.** We can not consider polynomial with a linear \( z^1 \) term because then the results from rotation (2.6) and scaling (2.9) becomes undefined with \( n = 1 \).

**Definition 2.1. Left Truncated Polynomial.**

A Left Truncated polynomial is defined as:-

\[ P(z, l) = \sum_{k=l} a_k z^k ; \quad l \geq 2 \]  \hspace{1cm} (2.15)

We define a truncated polynomial is a polynomial with at least no linear term, (and possibly constant term) to be defined formally in a later section, with equation (2.15). We can drop the requirement of not having a constant term, because in that case \( z^0 \), and the equations (2.6) and (2.9) remains defined.
We have found out from equations (2.6) and (2.9) that they affect different powers of \( z \) differently. But assume, we still want to find a way to rotate, and scale a polynomial fractal image appropriately. For notational convenience, we treat \( P(z) \) as a vector, with components \( \chi_j = a_j z^j \); \( j \neq 1 \).

\[
P = [a_j z^j]
\]

We note that, to generate an uniform scaling, \( \sigma_i \neq \sigma_j \).

\[
s = [\sigma_j] \quad (2.16)
\]

We note that, to generate an uniform rotation, \( \rho_i \neq \rho_j \).

\[
r = [\rho_j] \quad (2.17)
\]

It is easy to see that the for the ‘effect’ of them all should be equal to some predefined value ‘\( u \)’,

\[
u = \sigma_j^{-1}
\]

rewriting, we get:-

\[
\sigma_j = u^{1-j}
\]

In the same way, to get a fixed angle of rotation ‘\( \theta \)’ :-

\[
\rho_j = e^{i(j-1)\theta}
\]

Hence, a properly scaled and rotated original Polynomial fractal \( P + c \) transform, with translation, would have components:-

\[
\sigma\rho P = [^T \chi_j] = [u^{1-j} a_j e^{i(j-1)\theta} (z - b)^j]
\]

Using equation (2.18) we can modify and shift any ‘non-infinite’ \(( j \neq \infty \) ) truncated \(( j \neq 1 \) ) polynomial fractal, such a way that the original shape is preserved. So, the set image remains ‘invariant’ under equation (2.18).

Note that equation (2.18) can handle \( n < 0 \), but not the scaling part, as ‘\( k \)’ needed to be adjusted.

2.2.3. Demonstration of Invariance. We take the equation of :

\[
T(z) = z^2 + z^3 + c \quad (2.19)
\]

The image of equation (2.19) can be seen in figure 9 on page 16. The image of this can be rotated clockwise \( \pi/2 \) radian, by multiplying the \( \chi_2 \) by \( i \) and \( \chi_3 \) by \( e^{i\pi} \). Hence,

\[
1/2 T(z) = i(z - 1)^2 - (z - 1)^3 + c \quad (2.20)
\]
Equation (2.20) is a clockwise $\pi/2$ rotated, and right shifted by 1, image of the equation (2.19), which is clearly established by the figure 12 on page 17. To further establish the result, we present the second equation:,

$$1 \cdot \frac{\pi}{4} T(z) = e^{i \frac{\pi}{4}} (z - 1)^2 + i(z - 1)^3 + c$$

Equation (2.21) is a clockwise $\pi/4$ rotated, and right shifted by 1, image of the equation (2.19), which is clearly established by the figure 13 on page 17.

2.3. Self Similarity. With the previous set of transforms in mind, we focus on what is observed self similarity in the Multibrot fractals.

The tool we seek is the magnify a small section of the fractal shape, and finding how it should look. We note that this has to deal with the transform scaling ( theorem 2.3 ). But that won’t be all.

The point with respect to which we need to magnify or zoom first needs to be made the origin of the co-ordinate, and then magnified, or scaled. That would make the transform looks like translation followed by scaling. But that is not so.

We established that the direction of the co-ordinate system matters for fractals, as in Multibrot with fractional powers does not show remain same under an induced co-ordinate rotation. Therefore, the transform should also take care of aligning the co ordinates properly, via rotation.

Definition 2.2. The Zoom Transform.

Zoom of a fractal image can be done using :

$$z \leftarrow s e^{i \phi} f(z - a) + c$$

(2.22)

where $s$ is a positive constant factor, $a = (x_n, y_n)$ a complex number to have the co-ordinate origin shifted to, and $\phi$ denotes the rotation angle.

Conjecture 2.1. Existence of Self Similar Embedded Fractals.

A fractal of the type :

$$z \leftarrow f(z) + c$$

has quasi self similar embedded fractals under magnification iff it is invariant under the equation (2.22) or the zoom transform.

3. Generic Analytic Family and Embedded Fractals

It is to be noted here that all geometric transcendental functions can be expanded in Taylor series, and hence a transform of the form:

$$T : z \leftarrow G(z) + c$$

where $G(z)$ is a geometric transcendental function, qualifies as an infinite polynomial map.
We have noticed an important property of such maps, that they have embedded Multibrot fractals inside small regions within them. Some of them we have already shared in the introduction section. We furnish some more examples here:

1. The fractal of ‘$\cos(z) - 1 + c$’ (figure 2 at page 10) contains Mandelbrot fractals inside (figure 3 at page 10).
2. The fractal of ‘$\sin(z^2) + c$’ (figure 4 at page 10) has Mandelbrots inside (figure 5 at page 11), and looks like a deformed eaten away Mandelbrot image overall.
(3) Increasing the power, ‘$\sin(z^4) + c$’ is (figure 6 at page 11) virtually same as a Multibrot with $n = 4$ (figure 4 at page 4), and has Multibrots $n = 4$ inside it (figure 7 at page 11).
(4) What more, the fractal of ‘$\frac{6(z - \sin(z))}{c}$’ shows up just like a Multibrot of $n = 3$ (figure 8 at page 12).

These observations lead us to think that for those maps, the original transform function can probably be approximated by the Multibrot type function. This is the topic of discussion in this section.
3.1. Asymptotic Bounds : Bachmann Landau notation. What we informally seek is that in the neighborhood of a point, can we replace a function by another much simpler function, because, they behave approximately the same way. This concept is already available as asymptotic bounds proposed by Bachmann \[6\], and reintroduced by Landau \[7\], for real functions, with a parameter \(x\).

Definition 3.1. Big Theta : \(\Theta\).

Let there exist functions \(f(x), g(x)\) and constants \(k_1 > 0, k_2 > 0\) such that:

\[ k_1 f(x) \leq g(x) \leq k_2 f(x) ; \ x \to x_{\text{crit}} \]

then,

\[ g(x) \in \Theta(f(x)) \] (3.1)

The equation (3.1) is the Bachmann-Landau ‘\(\Theta\)’ notation, mostly used in computer science (Knuth \[8\]) for getting approximate bounds for functions (runtime analysis of algorithms). With simple modification, complex analytic functions can be included in \(\Theta\) notation.

Definition 3.2. Complex Generalization of Big Theta.

Iff for \(F(z), G(z)\) and constants \(k_1, k_2 \in (0, k)\) at neighborhood of a point \(z_c \in \mathbb{C}\):

\[ k_1 |F(z)| \leq |G(z)| \leq k_2 |F(z)| ; \ z \to z_c \] (3.2)

then, we define:-

\[ G(z_c) \in \Theta(F(z_c)) \] (3.3)

As both \(F(z), G(z)\) are analytic, then, there are infinite points in the neighborhood of \(z_c\) where equation (3.2) will be true. On that neighborhood, \(|F(z)|\) will approximate \(|G(z)|\) with arbitrary accuracy.

Theorem 3.1. There is a \(\Theta\) relationship for Complex Polynomials.

Let

\[ f(z) = \sum_{k=l}^{u} a_k z^k ; \ a_k \in \mathbb{R} \]

Then, if this holds:

\[ \lim_{z \to z_c} \frac{|z|^k}{|z|^m} = 0 \]
then
\[ \lim_{z \to z_c} f(z) \in \Theta(z^m) \]

In particular:-
\[ \forall |z| \to \infty \implies f(z) \in \Theta(z^n) \]

and,
\[ \forall |z| \to 0 \implies f(z) \in \Theta(z^l) \]

Proof of the theorem 3.1. We rewrite the polynomial as:-
\[ f(z) = z^m \sum_{k=l}^{u} a_k \frac{z^k}{z^m} \]

Taking mods we get:-
\[ |f(z)| = |z|^m | \sum_{k=l}^{u} a_k \frac{|z|^k}{|z|^m} | \]

which immediately gives the result. \(\Box\)

Theorem 3.2. The relation \(\Theta\) is true over iterations over polynomials.
Let \(f(z)\) be a polynomial function with
\[ \lim_{z \to z_c} f(z) \in \Theta(g(z)); \]
where \(g(z) = z^m\) Then,
\[ \lim_{z \to z_c} f^n(z) \in \Theta(g^n(z)) \]

Proof of the theorem 3.2. We rewrite the polynomial as:-
\[ f(z) = z^m \sum_{k=l}^{u} a_k \frac{z^k}{z^m} \]

and then the \(f^n(z)\) becomes:-
\[ f^n(z) = z^{mn} \sum_{k=ln}^{um} b_k \frac{z^k}{z^{mn}} \]

Taking mod, the result follows immediately. \(\Box\)
3.2. **The Principle of Dominance.** What we informally seek, is when we “zoom in” to a polynomial fractal, what “inner” fractal type we may find. If we find a fractal of type $f_c(z) = f(z) + c$ at a point $z_d$ inside a transformation function $T_c(z) = T(z) + c$, then we say that the function $f_c(z)$ must have “dominated” the function $T_c(z)$ at point $z_d$.

**Theorem 3.3.** \( \Theta \) can be used to find escape time fractals.

Let the transform function $z \leftarrow T(z)$ be:-

$$T = f(z) + c$$

\( f(z) \) being a polynomial with \( \lim_{z \to z_d} f(z) \in \Omega(g) \). Let then:-

$$S = \{ z : \lim_{n \to \infty} |f^n(z)| < \infty \}$$

and

$$S_\Theta = \{ z : \lim_{n \to \infty} |g^n(z)| < \infty \}$$

so that:-

$$S = S_\Theta$$

**Proof of the theorem 3.3.** We note that from theorem (3.2) :-

$$\lim_{z \to z_d} f^n \in \Theta(g^n)$$

This would imply that:-

$$|f^n| \leq K_u|g^n|$$

which would imply that:-

$$S \subseteq S_\Theta$$

also

$$|f^n| \geq K_l|g^n|$$

which would imply that:-

$$S_\Theta \subseteq S$$

combining we get $S_\Theta = S$. \( \square \)

**Theorem 3.4.** Dominance : Relation $f \in \Omega(g)$ guarantees a ‘sub-fractal’.

Let $\lim_{z \to z_d} f \in \Omega(g)$. Then, there for the escape time fractal generated from the transform function:-

$$z \leftarrow f(z) + c$$

there will be a embedded sub fractal of the form:-

$$z \leftarrow g(z) + c$$

In general for $f(z) = P(z, l)$, a truncated polynomial (definition 2.1), there would be sub fractals of type $z^l + c$. 
Theorem 3.3 makes proof of this theorem a triviality. However, another proof can be furnished from the perspective of scaling transform. We demonstrate the proof here.

**Proof of the theorem 3.4.** We assume that:

\[ |l g(z)| \leq |f(z)| \leq u |g(z)| \] with \( u, l > 0 \)

If that is the case, then we know that the fractal:

\[ z \leftarrow l g(z) + c \]

would be bigger in size than the fractal:

\[ z \leftarrow u g(z) + c \]

if the fractal \( z \leftarrow g(z) + c \) exists due to the scaling theorem (2.9).

That would mean that the fractal \( z \leftarrow f(z) + c \) is to be bound between the two fractals of the same shape, which is only possible, iff the shape of the fractal \( z \leftarrow f(z) + c \) is similar to that of \( z \leftarrow g(z) + c \), which proves the theorem. □

### 3.3. Implications of Dominance

The theorem (3.4) suggests, that there will be Multibrots found in the vicinity of truncated polynomials. In this section we elaborate what observations we have made, and how those observations fit into the property of dominance. We demonstrate simple truncated polynomial fractals.

#### 3.3.1. Demonstration of Dominance

To illustrate the result, we exhibit two transformation functions. The first transformation function:

\[ T_{2,3}(z) : z^2 + z^3 + c \]

has Mandelbrot \((z^2 + c)\) fractals inside it. Figure 9 on page 16 shows the \( T_{2,3}(z) \), and inner \((z^2 + c)\) in the next image.

The second transformation function:

\[ T_{4,7,-10}(z) : z^4 + z^7 - z^{10} + c \]

has Multibrot \((z^4 + c)\) fractals inside it. Figure 14 on page 17 shows the \( T_{4,7,-10}(z) \), and inner \((z^4 + c)\) in the next image.

These two observations shows validity of the dominance property.

#### 3.3.2. Implication for Functions Expandable with Taylor Series

The most important result of Dominance lies with the Taylor Series expansion. For a function \( f(z) \) which is capable of expansion by Taylor Series, “multibrot” type fractals can be generated by removing lower powers, or multiplying or dividing the function by \( z^k \).

We note down the expansion of \( \sin(z) \):

\[ \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \] (3.4)
Figure 9. The Polynomial Fractal of $z^2 + z^3 + c$

Figure 10. $z^2 + z^3 + c$ zoomed, showing existence of $z^2 + c$ fractal inside

Figure 11. $z^2 + z^3 + c$, zoomed, showing clearly $z^2 + c$ fractals

and the expansion of $\cos(z)$ :-

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...$$

(3.5)
Figure 12. Image of \((z - 1)^2 - (z - 1)^3 + c\), showing the invariance under translation and rotation. Cross reference with Figure 9 on page 16.

Figure 13. Image of \(e^{(i\pi/4)}(z - 1)^2 + i(z - 1)^3 + c\), Cross reference with Figure 12 on page 17.

Figure 14. The Polynomial Fractal of \(z^4 + z^7 - z^{10} + c\), notice the similarity with figure 1 on page 4.

In this subsection for brevity we have used the informal \(G \sim F\) to denote \(G \in \Theta_z(F)\).

Now clearly then due to dominance effect:-
Figure 15. The Polynomial Fractal of \( z^4 + z^7 - z^{10} + c \), zoomed, showing how the borders look like a Multibrot with \( n = 4 \)

Figure 16. \( z^4 + z^7 - z^{10} + c \) zoomed, showing \( z^4 + c \) fractals inside, compare with figure : \[ \text{on page 4} \]

(1) \[ \cos(z) - 1 + c \sim \frac{z^2}{2} + c \rightarrow z^2 + c \]

(2) \[ 6(\sin(z) - z + c) \sim z^3 + c \]

(3) \[ \sin(z^n) + c \sim z^n + c \]

(4) \[ z^k \sin(z^n) + c \sim z^{n+k} + c \quad ; \quad k \in \mathbb{R} \]

(5) And finally we explain, the initial \( \tan(z) \) fractals, which served as the starting point of this paper, in the next subsection.
3.4. The \( \tan(z) \) Fractals. Figure 17 on page 19 shows a \( (\tan(z))^2 + c \) fractal, and figure 18 on page 19 shows embedded Mandelbrots inside it.

\[
\tan(z) = z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots
\]

hence,

\[
(\tan(z))^n + c \sim z^n + c
\]

Now we proceed to explain the observation of \( (\tan(z))^{-2} + c \). Figure 19 on page 19 shows a \( (\cotan(z))^2 + c \) fractal, and figure 20 on page 20 shows embedded Mandelbrots inside it.
Figure 20. \((\cotan(z))^2 + c\) zoomed, showing Mandelbrots inside

\[
\frac{1}{\tan(z)} = \cotan(z) = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \ldots \quad (3.7)
\]

We note down that equation (3.7) contains power of \(z^{-1}\), and hence, we can not directly use the theorem of existence of embedded Multibrots.

Let us define:

\[
g(a, b, z) = \frac{a}{z} - \frac{z}{b} ; \quad a, b \in \mathbb{C} \quad (3.8)
\]

and

\[
T_g(a, b, n) : z \leftarrow (g(a, b, z))^n + c \quad (3.9)
\]

A fractal of type \(T_g(a, b, n)\) is shown in figure 19 on page 19.

We state without proof:

\[
(cotan(z))^n + c \sim T_g(1, 3, n) ; \quad |z| < 1
\]

and

\[
(cotan(z))^n + c \sim z^n + c ; \quad |z| > 1
\]

**Theorem 3.5. Existence of Multibrot inside Fractal of \(T_g(a, b, n)\).**

Fractal of equation family \(T_g(a, b, n)\) (3.9) would contain Multibrots inside.
**Proof of theorem** (3.5). We note that when $z \to 0$, the equation (3.8) gets unbounded.

Assume $k$ is large enough so that:

$$k > a - \frac{1}{b}$$

Now, when $z \to 1$ but $z \neq 1$ and $|z| < 1$, we have:

$$z = 1 - \epsilon ; \epsilon \in \mathbb{C} ; \epsilon \to 0 ; \epsilon \neq 0$$

(3.10)

In that case:

$$\lim_{z \to 1} g(a, b, z) = \lim_{\epsilon \to 0} (\frac{a}{1 - \epsilon} - \frac{1 - \epsilon}{b})$$

That approximates into :-

$$\lim_{\epsilon \to 0} (a(1 + \epsilon + \epsilon^2 + \epsilon^3...) - \frac{1 - \epsilon}{b}) = a(1 + \epsilon) - \frac{1 - \epsilon}{b}$$

Ignoring the higher powers of $\epsilon$ we get:-

$$g(a, b, z) = 2a - z(a + \frac{1}{b}) ; z \to 1 ; z \neq 1$$

(3.11)

And hence equation (3.9) can be written as:

$$T_g(a, b, n) : z \leftarrow (g(a, b, z))^n + c \sim (2a - z(a + \frac{1}{b}))^n + c$$

(3.12)

But, equation (3.12) is a ‘shape preserving transform’ of equation (2.12). Hence, the fractal of equation (3.12) is expected to have Multibrot fractals inside.

A zoom of fractal of type $T_g(2, 5, 3)$ is shown in figure 22 on page 21, revealing the Multibrot with $n = 3$ inside it.

Hence we complete the explanations of the observations what we have stated in the beginning of the current section.
4. SUMMARY AND FUTURE WORK AND THANKS

This paper started as a celebration of what Mandelbrot’s work has been. We started working on creating a software tool to produce ‘only’ Mandelbrot set in color, and ended up creating a software tool capable of evaluating any Mathematical expression, to generate escape time fractals, in any color. While experimenting with this tool, we came up with heather to unknown, unexplained phenomenon. The phenomenon of shape preserving transform, and the phenomenon of embedded inner fractal images, were so fascinating that we tried to explain them, so as to generate a preliminary theoretical basis. The usage of transformation vector to preserve shape and transform any finite truncated polynomial fractal is proven. The dominance phenomenon we borrowed, and reused from Computer Science, and shown that it is a very powerful mechanism to predict existence of embedded inner fractals. While dominance principle shows promise, the principle is fairly new at least in the fractal geometry domain. Further theoretical research work is needed to take the preliminary concepts discussed in this paper, into mainstream fractal geometry.

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