Private and Robust Distributed Nonconvex Optimization via Polynomial Approximation

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Abstract—There has been work that exploits polynomial approximation to solve distributed nonconvex optimization problems involving univariate objectives. This idea facilitates arbitrarily precise global optimization without requiring local evaluations of gradients at every iteration. Nonetheless, there remains a gap between existing guarantees and practical requirements, e.g., privacy preservation and robustness to network imperfections. To fill this gap and keep the above strengths, we propose a private and robust Chebyshev-proxy-based distributed optimization algorithm (PR-CPOA). Specifically, to ensure both the accuracy of solutions and the privacy of local objectives, we design a new privacy-preserving mechanism. This mechanism leverages the randomness in blockwise insertions of perturbed vector states and, hence, provides a stronger privacy guarantee in the scope of $(\alpha, \beta)$-data-privacy. Furthermore, to gain robustness against network imperfections, we use the push-sum consensus protocol as a backbone and discuss its specific enhancements. Thanks to the purely consensus-type iterations, we avoid the privacy-accuracy tradeoff and are spared from selecting proper step sizes in different settings. We rigorously analyze the accuracy, privacy, and complexity of the proposed algorithm. We show that the advantages brought by introducing polynomial approximation are maintained when all the above requirements exist.

Index Terms—Chebyshev polynomial approximation, data privacy, distributed optimization, privacy preservation, robustness.

NOMENCLATURE

Symbol | Definition
---|---
$c_j$ | $j$th Chebyshev coefficient.
$f_i(x)$ | Local objective function of $f_i(x)$.
$p_i^{(m_i)}(x)$ | Polynomial approximation of degree $m_i$ for agent $i$.
$p_i^0$ | Coefficient vector of $p_i^{(m_i)}(x)$ for agent $i$.
$m$ | Highest of the degrees of local approximations.
$\alpha$ | Estimation accuracy.
$\beta$ | Maximum disclosure probability.
$\Theta$ | Domain of a random variable $\theta_i(k)$.
$g_{\theta_i(k)}(y)$ | Probability density function of a random variable $\theta_i(k)$.
$I_t^i$ | Information set used by the adversaries to estimate agent $i$’s local objective at time $t$.

I. INTRODUCTION

DISTRIBUTED optimization enables multiple agents in a network to optimize the average of local objective functions in a collaborative manner. This global aim is achieved by exploiting local computations and communication between neighboring agents. Such a distributed architecture is preferable in various applications related to networked systems, e.g., distributed learning [2], energy management [3], and resource allocation [4]. In these applications, the need to improve efficiency, scalability, and robustness and protect privacy motivates the development of distributed strategies, which serve as plausible alternatives to their centralized counterparts.

A. Motivations

Considerable effort has been devoted to designing the efficient primal [5], [6] and dual-based [7] distributed optimization algorithms and extending them to meet diverse practical requirements, including privacy preservation [4], [8], [9], [10], [11]; time-varying directed communication [12], [13]; and asynchronous computations to allow lack of coordination [13], [14]; delays; and packet drops [15]. These extensions mainly focus on convex problems, and critical issues including privacy–accuracy tradeoff [9] and network scaling [12] are explored. Despite their wide applicability, the above algorithms only ensure convergence to stationary points for nonconvex problems, and their loads of locally evaluating gradients or function values increase linearly with the number of iterations. These issues...
motivate the study of [16], where polynomial approximations are used to substitute for univariate nonconvex local objectives, and a gradient-free, consensus-type iteration rule is adopted to exchange vectors of coefficients of local approximations. These designs help to achieve arbitrarily precise global optimization and reduce the costs of communication and local evaluations. Nonetheless, there are two issues that limit the practical values of the Chebyshev-proxy and consensus-based algorithm (CPCA) in [16]. First, it is not privacy-preserving due to the potential leakage of sensitive local objectives. This issue is critical because such leakage can cause the disclosure of secret local patterns. For instance, if local demand functions are revealed, then users’ personal details (e.g., daily schedules) are at risk of being inferred [8]. The above leakage stems from consensus-type iterations, where vectors of coefficients of local approximations are directly exchanged. Once the adversaries obtain the exact initial vector of a target agent, they can recover a close estimate of its local objective. Hence, how to preserve the privacy of local objectives and quantify protection results is well worth consideration. Second, it only handles optimization over static undirected networks with perfect communication. Given that issues including time-varying directed links, lack of coordination, and packet drops are common in applications, it is meaningful to investigate their effects and find countermeasures to gain robustness. In this article, we aim to demonstrate that the introduction of polynomial approximation into distributed optimization allows enhancements to meet practical needs of privacy and robustness while maintaining advantages in solution accuracy and complexity.

B. Contributions

We propose a private and robust Chebyshev-proxy-based distributed optimization algorithm (PR-CPOA). The key idea is to construct Chebyshev polynomial approximations (i.e., proxies) for objectives, employ consensus-type iterations with a privacy-preserving and robust mechanism to exchange coefficient vectors of local proxies, and solve an approximate problem by optimizing the recovered global proxy. The main contributions are summarized as follows.

1) We propose PR-CPOA to solve distributed optimization problems with nonconvex univariate objectives and convex constraint sets, pursuing privacy preservation and robustness against network imperfections. We demonstrate that it maintains the advantages of CPCA in obtaining \( \epsilon \)-globally optimal solutions for any arbitrarily small given precision \( \epsilon \) and being distributed terminable.

2) We incorporate a new privacy-preserving mechanism to prevent sensitive local objective functions from being disclosed. This mechanism exploits two types of randomness. One lies in the obfuscation of local states with zero-sum random noises. The other is the randomness in the blockwise insertions of perturbed vector states to render their dimensions uncertain to the adversaries.

3) We prove that in the scope of \( (\alpha, \beta) \)-data-privacy [17], a stronger privacy guarantee is obtained compared to the design where existing algorithms (e.g., [18] and [19]) are directly extended to handle vector states. Moreover, we demonstrate that the solution accuracy and privacy of local objectives are simultaneously ensured. These guarantees are in contrast with differentially private distributed convex optimization algorithms [4], [8], [9], [10].

4) We address the robustness issue in the face of various network imperfections. We employ the push-sum average consensus protocol [20] as a backbone to handle time-varying directed graphs and discuss its asynchronous extensions. We analyze the relationship between the accuracy of iterations and that of the obtained solutions, thus, verifying that PR-CPOA remains effective against the above network imperfections. Thanks to the linear, consensus-type iterations, we are spared from selecting appropriate step sizes in different settings.

Compared to the conference version [1], we 1) explore the design and analysis of the privacy-preserving mechanism for approximation-based distributed optimization; 2) investigate the strategies to deal with diverse network imperfections; and 3) provide rigorous analysis of the accuracy and complexities.

C. Organization

The rest of this article is organized as follows. Section II describes the problem of interest and gives some preliminaries. Section III presents the algorithm PR-CPOA. Section IV analyzes the accuracy, privacy, and complexity of the proposed algorithm. Numerical evaluations are performed in Section V, followed by the review of related work in Section VI. Finally, Section VII concludes this article.

II. PROBLEM DESCRIPTION AND PRELIMINARIES

Consider a network system of \( N \) agents. Each agent \( i \) owns a univariate local objective \( f_i(x) : X_i \rightarrow \mathbb{R} \) and a local constraint set \( X_i \subset \mathbb{R} \). The network at time \( t (t \in \mathbb{N}) \) is described as a directed graph \( G^t = (V, \mathcal{E}^t) \), where \( V \) is the set of agents, and \( \mathcal{E}^t \subset V \times V \) is the set of edges. Note that \( (i, j) \in \mathcal{E}^t \) if and only if (iff) agent \( j \) can receive messages from agent \( i \) at time \( t \). The script \( k \) in parentheses denotes the index of components in a vector. We summarize important notations in the Nomenclature for ease of reference.

A. Problem Description

We aim to solve the following constrained problem:

\[
\min_x \ f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x) \\
\text{s.t. } \ x \in X = \bigcap_{i=1}^{N} X_i \tag{1}
\]

in a distributed, private, and robust manner. The global aim of optimization will be achieved through local communication and computations. Meanwhile, we will address practical requirements including preservation of the privacy of local objectives and robustness to time-varying directed communication and asynchrony. Some basic assumptions are as follows.
**Assumption 1:** The local objective \( f_i(x) \) is Lipschitz continuous on \( X_i \).

**Assumption 2:** The local constraint set \( X_i \) is a closed, bounded, and convex set.

Assumptions 1 and 2 are satisfied by problems of practical interest and are extensively made by the related literature (e.g., [2], [21], and the references therein).

**Assumption 3:** \( \{g_i^j\} \) is \( B \)-strongly connected, i.e., there exists a positive integer \( B \), such that for any \( k \in \mathbb{N} \), the graph \( (V, E_{(k+1)}B−1) \) is strongly connected.

Assumption 3 states that the union graph is strongly connected for a time window of length \( B \). It is weaker than that requiring connectivity every time and is sufficient for information flow from one agent to networks to another [12].

Problem (1) involves nonconvex objective functions and convex constraint sets. Under Assumption 2, the set \( X_i \) is a closed interval for any \( i \in V \). Hence, let \( X_i = [a_i, b_i] \), where \( a_i, b_i \in \mathbb{R} \). As a result, the intersection set \( X \) is \( [a, b] \), where \( a = \max_{i \in V} a_i \), \( b = \min_{i \in V} b_i \).

### B. Preliminaries

**Consensus protocols:** Let \( N_i^{\text{in},t} = \{j|(j, i) \in E^t_i\} \) and \( N_i^{\text{out},t} = \{j|(i, j) \in E^t_i\} \) be the sets of agent \( i \)'s in-neighbors and out-neighbors, respectively, and \( d_i^{\text{out},t} = |N_i^{\text{out},t}| \) (i.e., the cardinality of \( N_i^{\text{out},t} \)) is its out-degree. Suppose that agent \( i \) owns a local variable \( x_i \). There are two consensus protocols, i.e., maximum consensus and average consensus, that allow agents to reach global agreement via local information exchange. The maximum consensus protocol is

\[
x_{i}^{t+1} = \max_{j \in N_i^{\text{in},t}} x_{j}^{t}.
\]

It can be proven that with (2), all \( x_i \) converge to \( \max_{i \in V} x_i \) in \( T (\leq (N − 1)B) \) iterations.

The push-sum average consensus protocol [20] is

\[
x_{i}^{t+1} = \sum_{j \in N_i^{\text{in},t}} a_{ij} x_{j}^{t}, \quad y_{i}^{t+1} = \sum_{j \in N_i^{\text{in},t}} a_{ij} y_{j}^{t}
\]

where \( y_i \) is initialized to be 1 for all \( i \in V \). The key to the convergence of (3) lies in constructing a column stochastic weight matrix \( A^t \triangleq (a_{ij})_{N \times N} \). A feasible choice is to set \( a_{ij} \) as \( 1/d_{ij}^{\text{out},t} \) if \( j \in N_i^{\text{in},t} \), and as 0 otherwise.

In the implementation, every agent \( j \) transmits \( x_{j}^{t}/d_{ij}^{\text{out},t} \) and \( y_{j}^{t}/d_{ij}^{\text{out},t} \) to its out-neighbors. With (3), the ratio \( z_k \triangleq x_k/y_k \) converges geometrically to the average of all the initial values \( \bar{x} = 1/N \sum_{i=1}^{N} x_i \) [20].

**Chebyshev polynomial approximation** focuses on using truncated Chebyshev series to approximate functions, thus, facilitating numerical analysis. These series (i.e., approximations) are efficiently computed by interpolation. The degree \( m \) Chebyshev interpolant \( p^m(x) \) corresponding to a Lipschitz continuous function \( g(x) \) defined on \([a, b]\) is

\[
p^m(x) = \sum_{j=0}^{m} c_j T_j \left( \frac{2x - (a + b)}{b - a} \right), \quad x \in [a, b]
\]

where \( c_j \) is the Chebyshev coefficient, and \( T_j(\cdot) \) is the \( j \)th Chebyshev polynomial defined on \([-1, 1]\) and satisfies \( |T_j(x')| \leq 1, \forall x' \in [-1, 1] \). As \( m \) increases, \( p^m(x) \) converges uniformly to \( g(x) \) on the entire interval [22]. In practice, \( p^m(x) \) with a moderate degree \( m \) generally suffices to be a rather close approximation of \( g(x) \) [22]. The dependence of \( m \) on the smoothness of \( g(x) \) and the specified precision \( \epsilon \) is quantified in Section IV-D.

### C. Models of Adversaries of Privacy

We consider honest-but-curious adversaries [23]. These adversaries are agents that faithfully follow the specified protocol but intend to infer information about the target agent \( i \) based on the received data. They are linked with malicious or Byzantine adversaries [24], [25] sending manipulated information, in the sense that their existence will jeopardize the well-functioning of the network system. However, the difference is that here the goal is to prevent sensitive local information from being leaked instead of resilience to manipulation.

We are concerned with the issue of privacy disclosure arising in the consensus iterations of PR-CPOA. At time \( t \), the information exchanged during iterations is

\[
I_{i}^{\text{own},t} = \{a_{ii}; x_{i}^{t}\}, \quad I_{i}^{\text{in},t} = \{a_{ij}; x_{j}^{t} | j \in N_{i}^{\text{in},t}\}
\]

i.e., the information sets of the states and weights of agent \( i \) and those transmitted from \( N_i^{\text{in},t} \) to agent \( i \), respectively. The following assumption characterizes the abilities of adversaries.

**Assumption 4:** At every time \( t \), for the target agent \( i \), the adversaries can always access \( I_{i}^{\text{in},t} \) but can only obtain \( I_{i}^{\text{own},t} \) with a probability whose upper bound is \( p \in (0, 1) \).

**Remark 1:** We assume the constant access of \( I_{i}^{\text{own},t} \) to include the scenario where some out-neighbors are adversaries and can, therefore, always receive the information transmitted by agent \( i \), as in [17] and [19]. For the assumption of \( I_{i}^{\text{in},t} \), the rationality is that the switching nature of time-varying networks can inhibit the persistent and perfect access to \( I_{i}^{\text{in},t} \). In practice, this setting holds if, at time \( t \), there exists a trustworthy agent whose link with agent \( i \) occurs with a probability not less than \( 1 - p \), or the adversaries are mobile and contact agent \( i \) to gather \( I_{i}^{\text{in},t} \) with a probability not more than \( p \).

### D. Privacy Requirement

We consider preserving the privacy of agent \( i \)'s local objective \( f_i(x) \), where \( i = 1, \ldots, N \). In CPCA, local communication happens in its consensus iterations, where agents directly exchange and update their local variables \( p_i^0 \) that lie in a set \( \mathcal{P} \subset \mathbb{R}^{m+1} \), i.e., \( p_i^0 \subset \mathcal{P} \). These variables are the coefficient vectors of approximations \( p_i^{(m)}(x) \) for \( f_i(x) \). Hence, \( p_i^0 \) is the sensitive information of \( f_i(x) \), and its privacy should be preserved. To characterize the privacy effect, we use \((\alpha, \beta)\)-data-privacy [17].

Let \( \bar{p}_i \) be any estimation of \( p_i^0 \) in \( \mathcal{P} \) based on the available information set \( \mathcal{I} \). The definition of \((\alpha, \beta)\)-data-privacy, where \( \alpha \geq 0 \) and \( \beta \in [0, 1] \), is given as follows.

**Definition 1:** A distributed algorithm achieves \((\alpha, \beta)\)-data-privacy for \( p_i^0 \in \mathcal{P} \) with a given \( \mathcal{I} \) if

\[
\max_{\bar{p}_i \in \mathcal{P}} \left\{ \|\bar{p}_i - p_i^0\|_1 \leq \alpha |\mathcal{I}| \right\} = \beta.
\]
In (5), \( \alpha \) and \( \beta \) are parameters that indicate the estimation accuracy and the maximum disclosure probability of \( \tilde{p}^0_i \), respectively. When \( \alpha \) is specified, a smaller \( \beta \) corresponds to a higher degree of privacy preservation.

**Remark 2:** (Connection with differential privacy (DP) [4])

DP emphasizes indistinguishability (i.e., a single substitution in the input will lead to a similar output). In some problems of network systems, however, we may further care about the exact privacy degree of the noise-adding mechanism in the face of estimation. Some adversaries may estimate the true local value, and the probability that such an estimation is close can be difficult to quantify through DP. Through the lens of data privacy [17], we can explicitly characterize the relationship between estimation accuracy and disclosure probability.

### III. DESIGN OF PR-CPOA

We present the proposed PR-CPOA algorithm. It consists of three stages and is illustrated in Fig. 1.

#### A. Construction of Local Chebyshev Approximations

In this stage, every agent computes a polynomial approximation \( p_{m_i}(x) \) of degree \( m_i \) for \( f_i(x) \) on \( X = [a, b] \), s.t.

\[
| f_i(x) - p_{m_i}(x) | \leq \epsilon_1 \quad \forall x \in [a, b] \tag{6}
\]

where \( \epsilon_1 > 0 \) is a specified tolerance. This goal is achieved via the adaptive Chebyshev interpolation method [26]. In this method, the degree of the interpolant is systematically increased until a certain stopping criterion is satisfied. First, agent \( i \) sets \( m_i = 2 \) and evaluates \( f_i(x) \) at the set \( S_{m_i} \Delta \{ x_0, \ldots, x_{m_i} \} \) of \( m_i + 1 \) points by

\[
x_k = \frac{b - a}{2} \cos \left( \frac{k \pi}{m_i} \right) + \frac{a + b}{2}, \quad f_k = f_i(x_k) \tag{7}
\]

where \( k = 0, 1, \ldots, m_i \). Then, it calculates the Chebyshev coefficients of the interpolant of degree \( m_i \) by

\[
c_j = \frac{1}{m_i} \left( f_0 + f_{m_i} \cos(j \pi) \right) + \frac{2}{m_i} \sum_{k=1}^{m_i-1} f_k \cos \left( \frac{jk \pi}{m_i} \right) \tag{8}
\]

where \( j = 0, 1, \ldots, m_i \) [26]. At every iteration, the degree \( m_i \) is doubled until the stopping criterion

\[
\max_{x_k \in (S_{2m_i} \setminus S_{m_i})} | f_i(x_k) - p_{m_i}(x_k) | \leq \epsilon_1 \tag{9}
\]

is met, where \( S_{2m_i} \setminus S_{m_i} \) is the set difference of \( S_{2m_i} \) and \( S_{m_i} \), and \( p_{m_i}(x) \) takes the form of (4) with \( \{ c_j \} \) being its coefficients. Since \( S_{m_i} \subset S_{2m_i} \), the evaluations of \( f_i(x) \) are constantly reused. The intersection \( X = [a, b] \) of local constraint sets is known by running some numbers of max/min consensus iterations as (2) beforehand.

#### B. Privacy-Preserving Information Dissemination

Every agent now owns a local variable \( p_i^0 \in \mathcal{P} \subset \mathbb{R}^{m_i+1} \), which is the vector of coefficients of \( p_i^{(m_j)}(x) \). In this stage, the goal is to enable agents to converge to the average\(^\dagger\) \( \tilde{p} = 1/N \sum_{i=1}^{N} p_i^0 \) of their initial values via a distributed mechanism, and the privacy of these initial values is preserved.

We propose a privacy-preserving scheme to achieve this goal. The key ideas are as follows:

1) adding random noises to \( p_i^0 \) to mask the true values;
2) inserting the components of the perturbed initial states block by block, thus, making their dimensions uncertain to the adversaries;
3) subtracting the noises separately to guarantee the convergence to the exact average.

**1) Step 1: Addition of random noises**

First, every agent \( i \) generates a noise vector \( \theta_i \in \Theta^{m_i+1} \), whose components are independent random variables in the domain \( \Theta \). Then, it adds \( \theta_i \) to its initial state \( p_i^0 \) to form a perturbed state \( \tilde{p}_i^0 \), i.e., \( \tilde{p}_i^0 = p_i^0 + \theta_i \).

**2) Step 2: Blockwise insertions of perturbed states**

Agents exchange and update their local variables \( x_i^l \) and \( y_i^l \) based on the push-sum consensus protocol [20]. The initial value of \( y_i^0 \) is all set as 1 for all \( i \in V \). Instead of directly setting the initial value of \( x_i^l \) as \( \tilde{p}_i^0 \), every agent \( i \) will gradually extend \( x_i^l \) with different blocks of \( \tilde{p}_i^0 \) in the first \( K_i \) iterations. Let \( (d_1^i, \ldots, d_{K_i}^i) \) be drawn from the multinomial distribution \( \text{mult}(m_i + 1, 1/K_i \cdot 1_{K_i}) \), where \( 1_{K_i} \) denotes an all-ones vector of size \( K_i \). Then, \( (d_1^i, \ldots, d_{K_i}^i) \) denotes the numbers of components of \( \tilde{p}_i^0 \) that are inserted into \( x_i^l \) at every iteration. Let \( l_0^i = 0 \) and \( l_t^i = \sum_{k=1}^{t} d_k^i \), where \( t = 1, \ldots, K_i \). At the \( t \)th iteration, the \( (l_t^i - 1) \)th to \( l_t^i \)th components of \( x_i^l \) and \( \tilde{p}_i^0 \) are added together to form the corresponding components of \( x_i^{l+1} \).

The remaining components of \( x_i^{l+1} \) and \( x_i^l \) are the same. Specifically

\[
x_i^{l+1}(k) = \begin{cases} x_i^l(k) + \tilde{p}_i^0(k), & \text{for } k = l_t^{i-1} + 1, \ldots, l_t^i \\ x_i^l(k), & \text{else} \end{cases} \tag{10}
\]

where \( x_i^l(k) \) and \( \tilde{p}_i^0(k) \) denote the \( k \)th components of \( x_i^l \) and \( \tilde{p}_i^0 \), respectively. If the corresponding \( x_i^l(k) \) is null (i.e.,
The bound $U$ can be obtained via the technique in [28] to estimate $N$ and the prior knowledge of $B$. Specifically, there are two auxiliary variables, i.e., $r_i^t$ and $s_i^t$. They are initialized as $\tilde{p}_1^{K_2} = x_i^{K_2}/y_i^{K_2}$ and updated together with $x_i^t$ and $y_i^t$ by the following max/min consensus protocols:

$$r_i^{t+1}(k) = \max_{j \in \mathcal{N}^m_{i,t}} r_j^{t}(k), \quad s_i^{t+1}(k) = \min_{j \in \mathcal{N}^m_{i,t}} s_j^{t}(k) \quad \forall k.$$  \hfill (14)

With (14), $r_i^{t}(k)$ and $s_i^{t}(k)$ are guaranteed to converge in finite time to $\max_{j \in \mathcal{V}} p_i^{K_2}(k)$ and $\min_{j \in \mathcal{V}} p_i^{K_2}(k)$, respectively. The required number of iterations for convergence is less than $(N - 1)B$, and is, therefore, less than $U$. Hence, at time no later than $t = K_2 + U$, all the local variables $x_i^t$, $y_i^t$, $r_i^t$, and $s_i^t$ become $(m + 1)$-dimensional vectors, where $m \triangleq \max_{i \in \mathcal{V}} m_i$ is the maximum degree of all the local approximations. The variables $r_i^t$ and $s_i^t$ are reinitialized as $\tilde{p}_i^t$ every $U$ iteration to allow the continual dissemination of the recent information on $p_i^t$. When the stopping criterion

$$\|r_i^K - s_i^K\|_\infty \leq \delta \triangleq \frac{\epsilon_3}{m + 1}$$  \hfill (15)

is satisfied at the $K_i$th iteration, agents terminate the iterations and set $p_i^{K_i} = x_i^K/y_i^K$.

### C. Polynomial Optimization by Solving SDPs

In this stage, agents locally optimize the polynomial proxy $p_i^t(x)$ recovered from $p_i^t$ on $X$ to obtain $\epsilon$-optimal solutions of problem (1). This problem is transformed into a semidefinite program (SDP) based on the sum-of-squares decomposition of nonnegative polynomials [29]. We refer the reader to [16] for details on the transformed problem.

The transformed problems are SDPs and can, therefore, be efficiently solved via the primal-dual interior-point method [30]. The iterations of this method are terminated when $0 \leq f^*_p - p^* \leq \epsilon_3$, where $f^*_p$ is the obtained estimate of the optimal value $p^*$ of $p_i^t(x)$ on $X$, and $\epsilon_3 > 0$ is the specified precision. The optimal point $x^*_p$ of $p_i^t(x)$ on $X$ can then be calculated by the complementary slackness condition [29].

The full details of the proposed algorithm are summarized as Algorithm 1. We set all the precision used in three stages, i.e., $\epsilon_1, \epsilon_2,$ and $\epsilon_3$, as $\epsilon/3$. Their sum equals $\epsilon$, thus, helping to ensure the reach of $\epsilon$-optimality.

### IV. PERFORMANCE ANALYSIS

#### A. Accuracy

The following lemma guarantees the accuracy of the privacy-preserving iterations of the proposed algorithm.

**Lemma 1:** If Assumptions 3 and 5 hold, when (15) is satisfied, we have

$$\max_{i \in \mathcal{V}} \|p_i^K - \tilde{p}\|_\infty \leq \delta = \frac{\epsilon_2}{m + 1}. \hfill (16)$$

**Proof:** The proof relies on the investigation of consensus iterations and is referred to in our online report [31]. \hfill \Box

In the following theorem, we characterize the distance between the obtained solution $f^*_p$ and the optimal value $f^*$ of problem (1), and the distance between the optimal point $x^*_p$ of

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**Fig. 2.** Illustration of blockwise insertions.
Algorithm 1: PR-CPOA.

Input: $f_i(x), X_i = \{a_i, b_i\}, U, K_1, K_2, \alpha, \epsilon$
Output: $f_o^*, x_o^*$
1: Initialize: $a_i^0 = a_i, b_i^0 = b_i, m_i = 2$.
2: for each agent $i \in \mathcal{V}$ do
3: for $t = 0, \ldots, U - 1$ do
4: $a_i^{t+1} = \max_{j \in \mathcal{N}_i^{m_i}} a_i^j, b_i^{t+1} = \min_{j \in \mathcal{N}_i^{m_i}} b_i^j$.
5: end for
6: Set $a = a_i^t, b = b_i^t$.
7: Calculate $\{x_j\}$ and $\{f_j\}$ by (7).
8: Calculate $\{c_k\}$ by (8).
9: If (9) is satisfied (where $\epsilon_1 = \frac{\epsilon}{2}$), go to step 10.
Otherwise, set $m_i \leftarrow 2m_i$ and go to step 7.
10: Set $\hat{p}^0_i = p_i^0 + \theta_i, x_0^0 = \text{null}, y_0^0 = 1, (d_1^0, \ldots, d_{K_1}^0)$ drawn from multi$(m_i + 1, 1/K_1 \cdot 1_{K_1}), l = 1, \epsilon_2 = \frac{\epsilon}{2}$.
11: for $t = 0, 1, \ldots, d$ do
12: if $t \leq K_1$ then
13: Extend $x_i^t$ to form $x_i^{t+1}$ by (10).
14: Update $x_i^{t+1} \forall k$ and $y_i^{t+1}$ by (11).
15: else if $K_1 + 1 \leq t \leq K_2$ then
16: for each component $k = 1, \ldots, m_i$ do
17: Update $x_i^{t+1}(k) \forall k$ and $y_i^{t+1}$ by (11), or additionally by (13) if subtractions need to be performed.
18: end for
19: else
20: if $t = IU$ then
21: if $\|x_i^t - s_i^t\|_\infty \leq \epsilon_2/(m + 1)$ then
22: $p_i^0 = x_i^t/y_i^t, \text{ break}$
23: end if
24: $r_i^t = s_i^t = x_i^t/y_i^t, l \leftarrow l + 1$.
25: end if
26: Update $x_i^{t+1}, y_i^{t+1}, r_i^{t+1}, s_i^{t+1}$ by (3) and (14).
27: end if
28: end for
29: end for
30: end for

$p_i^K(x)$ on $X$ (i.e., the returned solution) and the optimal point $x_i^*$ of problem (1).2

Theorem 2: Suppose that Assumptions 1–5 hold. PR-CPOA ensures that every agent obtains $\epsilon$-optimal solutions $f_i^*$ for problem (1), i.e., $\|f_i^* - f^*\| \leq \epsilon$, where $\epsilon > 0$.

Moreover

$|x_i^* - x_i^e| \leq \text{diam}(S), S = \{x \in X | f(x) \leq f(x_i^e) + \frac{\epsilon}{2} \epsilon\}.$

Proof: Please refer to our online report [31].

In Theorem 2, $\epsilon$ is any arbitrarily small specified precision, and diam$(S)$ is the diameter of $S$, i.e., the maximum distance

between any two points in $S$. It implies that for a given accuracy requirement, we can control the updates of the inner stages to ensure overall accuracy. The acquisition of $\epsilon$-optimal solutions benefits from the use of polynomial approximation and the characteristics of univariate objectives.

B. Data-Privacy

We investigate the performance of PR-CPOA in preserving the privacy of $p_i^0$.

We first define the information set $I_i^t$ used by the adversaries at time $t$ for estimation. Let

$\mathcal{I}_i^t = \mathcal{I}_i^{\text{own},t} \cup \mathcal{I}_i^{\text{in},t}$.

$\mathcal{I}_i^{\text{own},t} = \bigcup_{s=1}^{t} \mathcal{I}_i^{\text{own},s} = \bigcup_{s=1}^{t} \{a_{i,s}, x_{i,s}^*\}$

$\mathcal{I}_i^{\text{in},t} = \bigcup_{s \in \mathcal{S}_t} (a_{ij}^s, x_j^s) | j \in N_i^{\text{in},s}).$ (17)

The set $\mathcal{S}_t$ contains those numbers of iterations $s (s \leq t)$ when the adversaries have obtained the full knowledge of $\mathcal{I}_i^{\text{in},s}$. Note that $\mathcal{I}_i^t$ consists of all the available information on the states and weights owned by and transmitted to agent $i$ up to the $t$th iteration. Consider a random variable $X : \Omega \rightarrow \mathbb{R}$ whose distribution and any other relevant information are unknown. In this case, the probability that the adversaries can generate an accurate estimation $\hat{X}$ is small, i.e.,

$$\text{Pr} \{ |\hat{X} - X| \leq \alpha \} \leq \gamma \ll p_{\max} \int_{\Omega}^{\alpha} g_{\Theta,k}(y)dy \quad \forall k$$

(18)

where $\Theta$ is the domain of the $k$th component $\theta_i(k)$ of the noise vector $\theta_i$, and $g_{\Theta,k}(y)$ is the probability density function of $\theta_i(k).$3

Recall that we aim to preserve the privacy of $p_i^0 \in \mathbb{R}^{m_i + 1}$. Thanks to the blockwise insertions in (10), the adversaries are unaware of the exact size $m_i + 1$ of $p_i^0$. They do know the maximum size $m + 1$, however, based on the received $p_i^0$. Hence, the estimation of $p_i^0$ consists of two parts, i.e., to estimate its components $p_i^0(k)$, where $k = 1, \ldots, m_i + 1$, and to infer that $p_i^0(k)$ is null for $k = m_i + 2, \ldots, m + 1$. Let $\alpha$ and $\alpha_k$ be the estimation accuracy of $p_i^0$ and each component $p_i^0(k)$, respectively, s.t.

$$\sum_{k=1}^{m_i+1} \alpha_k = \alpha, \quad \alpha_k \in [0, \alpha] \quad \forall k = 1, \ldots, m_i + 1. \quad (19)$$

Moreover, since $m_i$ is unknown and varies with $\epsilon_1$ and $f_i(x)$, it is viewed as a random variable by the adversaries. Let $F_{m_i|X_i^e}(\cdot)$ be the cumulative distribution function of $m_i$ given $I_i^t$. The following theorem characterizes the effects of privacy preservation of PR-CPOA.

3To illustrate, let $L_\Theta$ be the total length of $\Omega$. The sensible policy for the adversaries is to uniformly generate an estimation $\hat{X}$ from $\Omega$. Hence, $\gamma = 2\alpha/L_\Theta$. We can ensure (18) by choosing $g_{\Theta,k}(y)$ such that $\exists \delta : \delta [\hat{v} - \alpha, \hat{v} + \alpha], \forall y \in \Theta : g_{\Theta,k}(y) \geq 1/pL_\Theta$, which is not difficult to satisfy if $\Omega$ is a large domain (i.e., $L_\Theta$ is large).
**Theorem 3:** If Assumptions 3 and 4 hold, given $I^k_1$, PR-CPOA achieves $(\alpha, \beta)$-data-privacy for $p^k_0$, where $\alpha$ satisfies (19)

$$\beta = \prod_{k=1}^{m+1} \beta_k \cdot \prod_{k=m+2}^{m+1} F_{m_k}(k-2)$$

$$\beta_k = (1 - p^{K_2-K_1+1}) h_i(\alpha_k) + p^{K_2-K_1+1}$$

$$h_i(\alpha_k) = p \max_{v \in \Theta} \int_{v-\alpha_k}^{v+\alpha_k} g_0(k)(y)dy + \gamma.$$

**Proof:** Please see Appendix A. \(\square\)

Theorem 3 states that PR-CPOA preserves the privacy of $p^0_k$ and characterizes such effects through $(\alpha, \beta)$-data-privacy. It clarifies the relationship between estimation accuracy and disclosure probability. We note that

$$\|\hat{p}_i - p^0_i\|_1 = \sum_{k=1}^{m+1} |\hat{p}_i(k) - p^0_i(k)| \leq \sum_{k=1}^{m+1} \alpha_k = \alpha.$$

Hence, we sequentially consider the relationship between the estimation accuracy $\alpha_k$ and the maximum disclosure probability $\beta_k$ of each component $p^0_i(k)$, and then synthesize them to obtain the result concerning $p^0_k$. The interpretation of $\beta$ in (20) is as follows. Note that $\beta \in (0, 1)$ is the product of a set of bounds $\beta_k \in (0, 1)$ for all components of $p^0_i(k)$ and the probabilities of correctly identifying null components [see (33)]. The bounds $\beta_k$ are derived via the law of total probability. The probabilities of the correct decision on null components are obtained based on whether the index $k$ exceeds $m_i + 1$ (i.e., the dimension of $p^0_i$).

If we directly extend existing algorithms [17], [18], [19] to handle vector states $p^0_i$, then $\beta$ will at least equal $\prod_k \beta_k$. In contrast, the design of blockwise insertions causes adversaries to additionally identify null components, thus, further reducing the disclosure probability of $p^0_i$. In addition, such a benefit does not cause an increase in communication complexity.

From (20), we know that for $p^0_i$ of a larger size (i.e., with larger $m_i$), $\beta$ will generally be smaller, which implies a higher degree of privacy preservation. In addition, $\beta$ increases with $\alpha_k$ but decreases with $K_2 - K_1$. These relationships support the intuitions that less accurate estimations can be acquired with higher probabilities, and more room for randomness leads to lower probabilities of privacy disclosure.

**C. Further Discussions on Privacy and Robustness**

1) **Privacy Guarantee:** Although the characterization of data privacy in Theorem 3 is different from differential privacy [4], [32], which emphasizes indistinguishability, they are closely related. The links lie in the use of noise-adding mechanisms to pursue privacy and the similar intuition that large noises ensure strong privacy preservation but degrade the utility of data. We now discuss the privacy effects of differential privacy. We define the database $D$ and the randomized query output $\mathcal{M}(D)$ until time $t$ as the set of initial states and the information set $I^t_1$ of the adversaries [see (17)], respectively, i.e.,

$$D = \{p^0_i|\forall i \in V\}, \mathcal{M}(D) = I^t_1.$$

Based on [32], in our setting, a privacy-preserving consensus protocol is $\epsilon$-differentially private if

$$\Pr \{\mathcal{M}(D) \in O\} \leq e^\epsilon \Pr \{\mathcal{M}(D') \in O\}$$

holds for any $O \subseteq \text{range} \mathcal{M}$ and $\sigma$-adjacent $D, D'$ satisfying

$$\|p^0_i - (p^0_i')\|_1 \leq \{\sigma, \text{ if } i = i_0, \text{ 0, if } i \neq i_0$$

for all $i \in V$, where $i_0$ is some element in $V$. Note that we have used correlated noises [see (12)] to pursue the proximity of $p^0_i$ to the exact average $p$ (see Lemma 1), thus, ensuring the accuracy of the obtained solutions (see Theorem 3). Based upon the impossibility result of simultaneously achieving exact average consensus and differential privacy [32], we conclude that our algorithm is not $\epsilon$-differentially private. To pursue differential privacy at the cost of losing certain solution accuracy, we can add uncorrelated noises (e.g., independent Laplace noises) to the transmitted states at every iteration.

2) **Asynchrony:** We discuss the robustness issue from the angle of asynchronous paradigms.

They handle uncoordinated computations and imperfect communication, e.g., transmission delays and packet drops. The design of consensus-type information dissemination in Algorithm 1 is synchronous. Its extension to cope with asynchrony is feasible and can benefit from the extensive research on asynchronous consensus protocols, including those allowing for random activations (e.g., gossip algorithms [20]), delays [33], packet drops, and all these issues [15]. The aforementioned asynchronous protocols converge deterministically to the average of initial values. If they are incorporated into PR-CPOA, by Lemma 1 and Theorem 2, the accuracy of the obtained solutions is still guaranteed, although the proof will be relatively more involved. In addition, since the iterations of Algorithm 1 are consensus-based and gradient-free, there is no need to select varying step sizes in different circumstances of asynchrony.

**D. Complexity**

We present a lemma about the dependence of the degree $m_i$ of the local approximation $p^k{i}{m_i}(x)$ on the specified tolerance $\epsilon_i$ and the smoothness of the local objective $g_i(x)$.

**Lemma 4 (16):** If $f_i(x)$ and its derivatives through $f^{(v-1)}(x)$ are absolutely continuous and $f^{(v)}(x)$ is of bounded variation on $X_i$, then $m_i \sim O(\epsilon_i^{-1})$. If $f_i(x)$ is analytic on $X_i$, then $m_i \sim O(\ln \frac{1}{\epsilon_i})$.

Lemma 4 suggests that for functions that are smooth to some extent, polynomial approximations of moderate degrees (e.g., of the order of $10^1 \sim 10^2$) can serve as rather accurate representations [22]. The following theorem describes the complexities of PR-CPOA in terms of the maximum degree of local proxies $m$ and the solution accuracy $\epsilon$, which equal $\max_{i \in V} m_i$ and $3\epsilon$, respectively. We measure the computational complexity via the order of flops\(^4\) and use $F_0$ to denote the cost of flops in one evaluation of $f_i(x)$.

\(^4\)A flop is defined as one addition, subtraction, multiplication, or division of two floating-point numbers [30].

\(^5\)This cost depends on $f_i(x)$ [30] and, hence, is not explicitly specified.
TABLE I

COMPARISONS OF PR-CPOA AND OTHER TYPICAL DISTRIBUTED OPTIMIZATION ALGORITHMS

| Algorithms       | Nonconvex Objectives | Networks | Privacy Guarantee | Asynchrony | Accuracy Guarantee | Complexities |
|------------------|----------------------|----------|-------------------|------------|-------------------|--------------|
| ASY-SONATA [15]  | ✓                    | ✓        | ✓                 | ✓          | ✓                 | $\text{scvx}^1$, linear $\text{ncvx}^1$: $O\left(\frac{1}{\epsilon}\right)^2$ |
| Algorithm in [9]$^3$ | ✓                    | ✓        | DP                | ✓          | ✓                 | trade-off$^2$ |
| FS protocol [34] | ✓                    | ✓        | ✓                 | ✓          | ✓                 | $\text{scvx}: O\left(\frac{1}{\epsilon}\right)$ |
| PR-CPOA          | ✓                    | ✓        | ✓                 | ✓          | ✓                 | $0^\text{th}$-ord. oracle: $O(m)$ Commn.: $O\left(\log \frac{\epsilon}{\delta}\right)^2$ (Theorem 5) |

$^1$ “scvx” and “ncvx” refer to strongly-convex and nonconvex objective functions, respectively.
$^2$ The complexities of inter-agent communication and evaluations of gradients (i.e., queries of the first-order oracle) are $O\left(\frac{1}{\epsilon}\right)$.
$^3$ This strategy of function perturbation can be combined with any distributed convex constrained optimization algorithms.
$^4$ There is a trade-off between accuracy and privacy.
$^5$ Detailed discussions are provided in Sec. IV-C.

**Theorem 5:** PR-CPOA ensures that every agent obtains $\epsilon$-optimal solutions for problem (1) with $O(m)$ evaluations of local objective functions, $O\left(\log \frac{m^3}{\epsilon}\right)$ rounds of interagent communication, $O\left(\sqrt{m} \log \frac{1}{\epsilon}\right)$ iterations of primal-dual interior-point methods, and $O(m \cdot \max(m^{3.5}, \log \frac{1}{\epsilon}, F_0))$ flops.

**Proof:** Please see Appendix B.

We compare PR-CPOA with typical distributed optimization algorithms in Table I. These algorithms exhibit sublinear convergence for nonconvex problems, and the complexities of evaluations and communication are $O\left(\frac{1}{\epsilon}\right)$. In comparison, PR-CPOA is more efficient in the complexities of function evaluations and communication rounds.

**Remark 3:** Although PR-CPOA involves the exchange of $m$-dimensional vectors, its total transmission costs in communication can be acceptable given 1) the decreased rounds of communication (see Theorem 5 and Table I) and 2) the typically moderate degrees of approximations in numerical practice (see [22] and also the discussion below Lemma 4).

**E. Applications and Extensions**

In this article, we consider problems with univariate objectives to highlight the advantages brought by exploiting polynomial approximation, e.g., achieving efficient optimization of nonconvex problems and allowing for enhancement to be private and robust when diverse practical needs exist. We discuss the relevant applications and multivariate extensions.

1) **Application scenario:** Some multivariate distributed optimization problems are naturally separable or can be transformed into a separable form through a change of coordinates. Consider the regularized facility location problem [30], [35]

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{N} w_i \|x - u_i\|_1 + \tilde{g}(x). \tag{22}$$

In problem (22), $\mathbb{R}^n \subset \mathbb{R}^n$ is a constraint set, $u_i \in \mathbb{R}^n$ is the location of agent $i$, $w_i$ is a nonnegative weight, and $\tilde{g}(x) \triangleq \sum_{k=1}^{n} \kappa \log(1 + |x(k)|/\nu)$ is the log-sum penalty, where $\kappa, \nu > 0$ are parameters. Note that $\tilde{g}(x)$ is one of the common surrogates of the $\ell_0$ norm to promote sparsity, and it is a nonconvex Lipschitz continuous regularizer [35].

Problem (22) can be solved by considering a set of univariate subproblems as follows:

$$\min_{x(k) \in X} \sum_{i=1}^{N} (w_i|x(k) - u_i(k)| + \kappa \log(1 + |x(k)|/\nu))$$

where $k = 1, \ldots, n$. The proposed algorithm applies to the above problem involving nonconvex Lipschitz objectives.

The key idea of exploiting function approximation is also of interest to address other problems of data analytics in network systems. We can use function approximation to represent local features and perform consensus to obtain the global features (e.g., the average, product, or other functions of local features), thus, facilitating operations of statistics or estimation [16].

2) **Multivariate extension:** Compared to the univariate setup, the differences of the multivariate extension mainly lie in the stages of initialization and optimization of approximations. Specifically, let $L_2(X)$ be the set of square-integrable functions over $X \subset \mathbb{R}^n$, and $f_i(x) \in L_2(X)$ be a general local objective. Then, there exists an orthonormal basis $\{h_k(x)\}_{k \in \mathbb{N}_+}$ (e.g., orthonormalization of Taylor polynomials) and an arbitrarily precise approximation $f_i(x) = \sum_{k=1}^{m} c_k h_k(x)$ for $f_i(x)$, where $\{c_k\}_{k=1}^{m}$ is the set of coefficients. Afterward, agents exchange and update their local vectors that store these coefficients (as in Section III-B) and acquire an approximation for the global objective. Finally, they locally optimize this approximation via the tools for polynomial optimization or for finding stationary points of general nonconvex functions [36]. The main challenge of this extension lies in systematically calculating $f_i(x)$ of an appropriate degree to strictly satisfy the accuracy requirement. It calls for further investigation and careful analysis and is still part of our ongoing work. Nonetheless, the idea of introducing approximation still: 1) offers a new perspective of representing possibly complicated objectives by simple coefficient vectors of approximations, thus allowing the dissemination of the global objective in the network; 2) naturally allows using random blockwise insertions to pursue a strong privacy.
guarantee in data privacy; 3) is promising to obtain approximate globally optimal solutions in a distributed and asymptotic manner.

V. NUMERICAL EVALUATIONS

Consider a network with \( N = 20 \) agents. At each time \( t \), besides itself, every agent \( i \) has two out-neighbors. One belongs to a fixed cycle, and the other is chosen uniformly at random. Hence, \( \{G^t\} \) is 1-strongly connected. We set the local constraint sets as the same interval \( X = [-1, 1] \) and generate the local objective function \( f_i(x) \) of agent \( i \) by

\[
f_i(x) = \frac{a_i}{1 + e^{-x}} + b_i \log(1 + x^2)
\]

where \( a_i \sim N(10, 2) \) and \( b_i \sim N(5, 1) \) are independent and identically distributed samples drawn from normal distributions. It follows that \( f_i(x) \) is nonconvex and Lipschitz continuous on \( X \). We use the Chebfun toolbox [22] to construct Chebyshev polynomial approximations \( p_i(x) \) for the local objective function \( f_i(x) \).

For PR-CPOA, we set \( K_1 = 10, K_2 = 20 \), generate i.i.d. random noises \( \theta_i(k) \) from the uniform distribution \( U(-1, 1) \), and randomly select \( L \) from the discrete uniform distribution \( U\{1, K_2 - K_1\} \) to satisfy (12). The convergence of PR-CPOA is shown in Fig. 3(a). The markers on the blue line indicate how many iterations of information dissemination are performed when certain precisions \( \epsilon \) are specified. The markers on the orange line represent the actual objective errors \( |f_i^* - f_i^*| \) when those iterations are completed. The relationship between \( \log \epsilon \) and \( t \) is roughly linear. This phenomenon results from the linear convergence of consensus-type information dissemination. For comparison, we implement DiaDSP [10], where the step size \( \alpha \) and the parameters \( d \) and \( q \) used in the Laplace distribution are set as 0.05, 1/3, and 0.95, respectively. We run 100 experiments and plot the average curve. We observe that the solution accuracy of DiaDSP is sacrificed to some extent due to its differentially private mechanism.

The effects of privacy preservation are presented in Fig. 3(b). This figure demonstrates the relationships between the estimation accuracy \( \alpha_k \) and the maximum disclosure probability \( \beta_k \) for a single component \( p_1(k) \) when different types of noises \( \theta_i(k) \) are used. These relationships are explicitly characterized by (21) in Appendix A. In the experiment, we set \( K_1 = 10, K_2 = 20, p = 0.8, \) and \( \gamma = 10^{-5} \). We consider three types of noises that satisfy uniform, normal, and Laplace distributions. The mean and variance of these noises are 0 and 1, respectively. We observe that \( \beta_k \) increases with \( \alpha_k \), which confirms the intuition that a less accurate estimate can be obtained with a higher probability. We also notice that uniformly distributed noises yield the smallest \( \beta_k \) and, thus, the most effective preservation of \( p_1(k) \). This observation supports the conclusion in [17]. Note that the maximum disclosure probability \( \beta \) of \( p_1 \) is the product of all \( \beta_k \), where \( k = 1, \ldots, m_i + 1 \) [see (20) in Section IV]. The degrees \( m_i \) of local approximations vary from 20 to 40 when the specified precision \( \epsilon = 10^{-10} \). Hence, in this case, \( \beta \) is an extremely small number, given \( \alpha_k \) and \( \beta_k \) in the figure.

The robustness of the consensus-type iterations is shown in Fig. 3(c). We consider cases where the time-varying links between agents suffer from different rates of failure, which may result from packet drops or delays. We observe that the iterations still converge, thus, ensuring the solution accuracy of the proposed algorithm. Nonetheless, the convergence rates tend to be slower as the link failure rates increase.

VI. RELATED WORK

Various distributed optimization algorithms can generally be categorized as primal methods [5, 6, 14] and dual-based methods [7]. The core idea of the primal methods is to combine consensus with gradient-based optimization algorithms, thus, achieving consensusal convergence in the primal domain. The intuition of the dual-based methods is to formulate the consensus requirement as equality constraints, and then solve the dual problems or carry on primal-dual updates in a distributed manner. For convex problems, distributed algorithms guarantee convergence to globally optimal points; for nonconvex problems, the convergence to first-order stationary solutions is ensured [2, 21, 37].

The aforementioned work bridges the gap in convergence behaviors between distributed and centralized optimization algorithms. To deploy these algorithms into applications, some
practical issues need to be addressed. These issues include privacy preservation and robustness to allow time-varying, directed communication, and asynchronous computations.

The privacy concerns of distributed algorithms have received growing attention. Exchanging exact data can lead to the disclosure of sensitive local objective functions, constraints, and states [4]. To tackle this problem, a typical approach based on message perturbation is to add random noises to the transmitted data during iterations. The perturbation of the critical data (e.g., states [17], [18], [19], [32], [38], gradients [4], [8], step-sizes [39], and functions [9], [34]) limits its utility for yielding sensible estimations. Some work uses uncorrelated Laplacian or Gaussian noises and develops differentially private consensus [32], [40] and distributed optimization algorithms [4], [8], [9], [10], [11], [41]. The differentially private mechanism offers strong privacy guarantees even against adversaries that own arbitrarily much auxiliary information. Nonetheless, it causes the tradeoff between privacy and accuracy [9], [32]. Other work, thus, turns to correlated noises and fulfills exact average consensus [17], [18], [19] and optimization [34]. There are also methods that utilize state decomposition to achieve complete indistinguishability, provided noncolluding neighbors or private interaction weights exist [23]. Another typical approach is to apply cryptographic techniques [42], [43]. These methods are suitable if trusted agents or shared keys/secrets exist and the extra computation and communication burdens induced by encryption and decryption are acceptable.

Another widely investigated issue is robustness. Time-varying directed communication inhibits the construction of doubly stochastic weight matrices, which are crucial for convergence over undirected graphs. Push-sum-based [2], [12] and push-pull-based algorithms [13] overcome this challenge. The former combines the push-sum consensus [20] with gradient-based methods. The latter uses a row stochastic matrix and a column stochastic matrix to mix solution estimates and gradient trackers, respectively. Algorithms that handle transmission delays include [3], [15], with the idea of locally fused the delayed information as soon as it arrives. To achieve asynchronous computations, gossip-type algorithms [13], [14] and those allowing delays and packet drops [15] are developed.

Different from the existing work, we exploit polynomial approximation and introduce effective mechanisms to meet practical requirements on privacy and robustness. We show that the proposed algorithm achieves efficient distributed optimization of general nonconvex objective functions, and the issues of privacy–accuracy tradeoff and step-size selections are avoided.

VII. CONCLUSION

We proposed the PR-CPOA to solve a class of constrained distributed nonconvex optimization problems, considering privacy preservation and robustness to network imperfections. We achieved exact convergence and effective preservation of the privacy of local objectives by incorporating a new privacy-preserving mechanism for consensus-type iterations. This mechanism utilized the randomness in blockwise insertions of perturbed data, and the privacy degree was explicitly characterized through \((\alpha, \beta)\)-data-privacy. We ensured robustness by using the push-sum average consensus protocol as a basis, and we discussed its extensions to maintain the performance when diverse network imperfections exist. We proved that the major benefits brought by the use of polynomial approximation were preserved, and the above demanding requirements were satisfied in the meantime. Future directions include handling noisy evaluations of local objectives and quantized communication.

APPENDIX

A. Proof of Theorem 3

We first consider the estimation of \(p^0_i(k)\), where \(k = 1, \ldots, m_k + 1\). Suppose that at the \(t_k\)th iteration, agent \(i\) inserts the perturbed state \(\vec{p}^0_i(k)\) by (10). The estimation \(\hat{p}_i(k)\) of \(p^0_i(k)\) can be calculated at three types of time: 1) before \(t_k\), 2) at \(t_k\), and 3) after \(t_k\). We discuss these cases as follows.

Case 1: At time \(t < t_k\), \(p^0_i(k)\) has not been inserted yet. What the adversaries have collected are either null values or combinations of the perturbed states of agent \(i\)’s neighbors. Since there is not any available information on \(p^0_i(k)\) that serves as a basis for estimation, by (18), we have

\[
\Pr \{ |\hat{p}_i(k) - p^0_i(k)| \leq \alpha_k |I^k_i \} \leq \gamma.
\]

Case 2: At time \(t = t_k\), \(p^0_i(k)\) is inserted. By Assumption 4, the probability that the adversaries acquire the full knowledge of \(p^0_i(k)\) is not more than \(p\). If this is the case, based on (10) and (11), they can easily calculate \(\vec{p}^0_i(k)\) by

\[
\vec{p}^0_i(k) = x_{i}^{t_k + 1} - \sum_{j \in N_k} a_{ij}^{-1} x_{j}^{(t_k - 1) + 1} + (k).
\]

Note that \(\vec{p}^0_i(k) = p^0_i(k) + \theta_i(k)\). Hence, after an estimation \(\hat{\theta}_i(k)\) of \(\theta_i(k)\) is obtained, \(\hat{p}_i(k)\) is calculated by \(\hat{p}_i(k) = \vec{p}^0_i(k) - \hat{\theta}_i(k)\). Therefore, for any estimation \(\hat{p}_i(k)\), we have

\[
\Pr \{ |\hat{p}_i(k) - p^0_i(k)| \leq \alpha_k |I^k_i \} = \Pr \{ |\hat{\theta}_i(k) - \theta_i(k)| \leq \alpha_k |I^k_i \} = \Pr \{ \theta_i(k) \in [\hat{\theta}_i(k) - \alpha_k, \hat{\theta}_i(k) + \alpha_k] |I^k_i \} = \int_{\hat{\theta}_i(k) - \alpha_k}^{\hat{\theta}_i(k) + \alpha_k} g_{\theta_i(k)}(y)dy \leq \max_{\nu \in \Theta} \int_{\nu - \alpha_k}^{\nu + \alpha_k} g_{\theta_i(k)}(y)dy \tag{24}
\]

where \(\hat{\theta}_i(k) \in \Theta\). However, if the adversaries can only access part of \(I^k_i\), they are unable to calculate \(x_{i}^{t_k + 1}\) by (11) and then recover \(\vec{p}^0_i(k)\) by (23). Note that

\[
x_{i}^{t_k + 1} = x_{i}^{t_k} + \theta_i(k) = x_{i}^{t_k} + \theta_i(k) + p^0_i(k).
\]

Hence, in this case, they need to obtain an estimation \(\hat{\theta}_i(k)\) of \(x_{i}^{t_k} + \theta_i(k)\), and then calculate \(\hat{p}_i(k)\) by \(\hat{p}_i(k) = x_{i}^{t_k + 1} - \hat{\theta}_i(k)\). According to (11), \(x_{i}^{t_k + 1}\) is a linear combination of the states \(x_{j}^{(t_k - 1) + 1}\) for \(j \in N_k^{m_k} \cdot x_{k}^{t_k + 1}\) and \(\theta_i(k)\), where \(l \in V\). Note that
the adversaries only have partial knowledge of $I^ {m,t_k-1}$ and know part of these states. Hence, there exist certain independent random variables, i.e., $\theta_i(k)$, of which the adversaries do not own any prior or relevant knowledge. As a result, by (18), it is hard to estimate $x^k_i(k)$ with high precision. It follows that:
\[
\Pr \left\{ |\hat{\theta}_i(k) - \theta^0_i(k)| \leq \alpha_k |I^t_i| \right\} = \Pr \left\{ |\eta_i(k) - (x^k_i(k) + \theta_i(k))| \leq \alpha_k |I^t_i| \right\} \\
\leq \Pr \left\{ |\eta_i(k) - x^k_i(k) \in [\theta_i(k) - \alpha_k, \theta_i(k) + \alpha_k]|I^t_i, \theta_i(k) \right\} \\
\leq \gamma. \quad (25)
\]
Combining (24) and (25), for any estimation $\hat{\theta}_i(k)$ of $p^0_i(k)$, we have
\[
\max_{\hat{\theta}_i(k)} \Pr \left\{ |\hat{\theta}_i(k) - \theta^0_i(k)| \leq \alpha_k |I^t_i| \right\} \\
\leq \max_{\theta \in \Theta} \int_{\nu - \alpha_k}^{\nu + \alpha_k} g_{\eta_i(k)}(y) dy + \gamma \triangleq h_i(\alpha_k). \quad (26)
\]
**Case 3:** At time $t > t_k$, the adversaries can estimate $p^0_i(k)$ either by the same rule that is adopted at time $t = t_k$ or by the new rule based on the new information. In the former case, we still obtain (26). We now discuss the latter case in detail. We first consider the time $t_k + 1$. Note that
\[
x^k_i{t_k+1} = \frac{x^k_i{t_k+1} - x^k_i{t_k+1}}{a^k_{i{t_k}}} \\
= x^k_i{t_k+1} + \frac{1}{a^k_{i{t_k}}} \left( \sum_{j \in N^k_i \setminus \{i\}} a^k_{ij} x^k_j - \tau_i, t_k+1(k) \right) \\
= p^0_i(k) + \theta_i(k) + x^k_i(k) + \frac{1}{a^k_{i{t_k}}} \left( \sum_{j \in N^k_{i \setminus \{t_k\}}} a^k_{ij} x^k_j - \tau_i, t_k+1(k) \right) \\
= p^0_i(k) + \theta_i(k) + \theta'_i(k) \quad (27)
\]
where $\tau_i, t_k+1(k) = \zeta_i(k)$ if noises are subtracted at time $t$, and $\tau_i, t(k) = 0$ otherwise. If the full knowledge of $I^m_{i,t}$ is available, the adversaries can only collect all the $x^k_j +$ for $j \in N^m_i$, but also accurately infer $\tau_i, t_k+1(k)$ by
\[
\tau_i, t_k+1(k) = \sum_{j \in N^m_{i \setminus \{t_k\}}} a^k_{ij} x^k_j - x^k_t(k) + (k). \quad (28)
\]
Hence, $\theta'_i(k)$ is a deterministic constant. In this case, by using (27), we still have
\[
\Pr \left\{ |\hat{\theta}_i(k) - p^0_i(k)| \leq \alpha_k |I^t_i| \right\} = \Pr \left\{ |\hat{\theta}_i(k) - \theta_i(k)| \leq \alpha_k |I^t_i| \right\} \\
\]
Next, we analyze the disclosure probability of $\theta_i(k)$ given $I^t_i$. The newly available information, i.e., the subtracted noise $\zeta_i(k)$, allows for another means of inferring $\theta_i(k)$. We now show that the resulting disclosure probability is rather small when $L_i$ is drawn from an unknown distribution. Note that
\[
\Pr \left\{ |\hat{\theta}_i(k) - p^0_i(k)| \leq \alpha_k |I^t_i| \right\} = \Pr \left\{ |\hat{\theta}_i(k) - \theta_i(k)| \leq \alpha_k |I^t_i| \right\} \\
= \Pr \left\{ |\hat{\theta}_i(k) - \theta_i(k) - \alpha_k|I^t_i| \right\} \\
\leq \gamma. \quad (28)
\]
where $L_i$ is any estimation of $L_i$, and the last inequality follows from (18). Thus, the disclosure probability will not exceed the upper bound in (24), i.e.,
\[
\Pr \left\{ |\hat{\theta}_i(k) - p^0_i(k)| \leq \alpha_k |I^t_i| \right\} = \Pr \left\{ |\hat{\theta}_i(k) - \theta_i(k)| \leq \alpha_k |I^t_i| \right\} \\
\leq \max_{\nu \in \Theta} \int_{\nu - \alpha_k}^{\nu + \alpha_k} g_{\theta_i(k)}(y) dy. \quad (29)
\]
If the full knowledge of $I^m_{i,t}$ is unavailable, then $\theta'_i(k)$ contains those independent random variables whose relevant information is unknown to the adversaries. Specifically, if $t_k + 1 \leq K_1$, then those variables refer to certain added noises $\theta_i(k)$ that are included in $x^k_i(k)$, where $l \in \mathcal{V}$. Otherwise, those variables refer to certain subtracted noises $\zeta_i(k)$ for some $l \in \mathcal{V}$. Thus, it follows from (18) that:
\[
\Pr \left\{ |\hat{\theta}_i(k) - p^0_i(k)| \leq \alpha_k |I^t_i| \right\} \leq \gamma. \quad (30)
\]
Combining (29) and (30), for any estimation $\hat{\theta}_i(k)$ of $p^0_i(k)$
\[
\max_{\hat{\theta}_i(k)} \Pr \left\{ |\hat{\theta}_i(k) - p^0_i(k)| \leq \alpha_k |I^t_i| \right\} \\
\leq \max_{\nu \in \Theta} \int_{\nu - \alpha_k}^{\nu + \alpha_k} g_{\theta_i(k)}(y) dy + \gamma = h_i(\alpha_k). \quad (31)
\]
A similar analysis can be performed for other arbitrary $t \geq t_k + 1, t \in \mathbb{N}$. However, for $t \geq K_2$, there exists an extreme case where the adversaries successfully obtain the full knowledge of $I^m_{i,t}$ at time $t = t_k - 1$ and also from time $t = K_1$ to time $t = K_2$. In this case, they can not only calculate $p^0_i(k)$ by (23), but also acquire $\tau_i, t(k)$ and perfectly infer $\theta_i(k)$ by $\theta_i(k) = \sum_{t=K_1}^{K_2} \tau_i, t(k)$. Hence, the exact value of $p^0_i(k)$ can be obtained, and
\[
\Pr \left\{ |\hat{\theta}_i(k) - p^0_i(k)| \leq \alpha_k |I^t_i| \right\} = \Pr \left\{ |\hat{\theta}_i(k) - \theta_i(k)| \leq \alpha_k |I^t_i| \right\} = 1. \quad (32)
\]
The probability that such an extreme case happens is not more than $p^{K_2-K_1+1}$. Thus, for any $k = 1, \ldots, m_t + 1$ and $t \in \mathbb{N}$
\[
\max_{\hat{\theta}_i(k)} \Pr \left\{ |\hat{\theta}_i(k) - p^0_i(k)| \leq \alpha_k |I^t_i| \right\} \leq \beta_k \quad (32)
\]
where $\beta_k$ is given by (21). Since $h_i(\alpha_k) \leq p + \gamma < 1, \beta_k$ is larger than the right-hand side of (26).
Finally, we consider the inference on whether $p_i^0(k)$ is null for $k = m_i + 2, \ldots, m + 1$, i.e., whether $p_i^0$ is an $(m_i + 1)$-dimensional vector. Note that there is no action of insertions or subtractions corresponding to the aforementioned components. Hence, the adversaries will not find any inconsistency between $x_i^{(k+1)t}$ and $\sum_{j \in \mathcal{S}_i} x_j^{(k+1)t}$, where $t \in \mathcal{S}_i$. Let this event be denoted by $A$. Once it occurs, the adversaries need to decide between the following two hypotheses, i.e., $H_0 : p_i^0(k)$ is null, and $H_1 : p_i^0(k)$ is a nonzero number. Based on the algorithmic design, we have $Pr\{A|H_0\} = 1$, $Pr\{A|H_1\} = (1 - p)^{L_1+1}$. It follows from the maximum likelihood rule that the adversaries will always choose $H_0$ when $A$ occurs. The probability that they successfully decide that $p_i^0(k)$ is null for $k = m_i + 2, \ldots, m + 1$ equals

$$Pr\{m_i + 1 \leq k - 1\} = F_{m_i+2}(k - 2).$$

Combining (32) and (33), we arrive at (20).

### B. Proof of Theorem 5

Note that the evaluations of local objective functions (i.e., queries of the zeroth-order oracle) are only performed in the stage of initialization, and the primal-dual interior-point method [30] is used to solve the reformulated SDP in the stage of polynomial optimization. By referring to the proof of [16, Th. 5], we know that for every agent, the orders of evaluations of local objective functions and primal-dual iterations are of $O(m)$ and $O(\sqrt{m \log \delta})$, respectively. Also, the orders of the required flops of these two stages are of $O(m \cdot \max(m, F_0))$ and $O(m^{4.5} \log \delta)$, respectively.

In the stage of information dissemination, the blockwise insertions of vectors and the subtractions of noises are completed in finite time, i.e., in $K_2$ iterations. Since the consensus-type protocol converges geometrically, the order of the total number of iterations (i.e., interagent communication) is of

$$K_2 + O\left(\log \frac{4}{\delta}\right) = O\left(\log \frac{1}{\delta}\right) = O\left(\log \frac{m}{\delta}\right),$$

where the required precision $\delta$ is given by (15). The order of flops needed in this stage is of $O(m \log \frac{m}{\delta})$. The results in the theorem follow from the above analysis.

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