On the Relation of Symplectic Algebraic Cobordism to Hermitian $K$-Theory

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To the memory of great mathematician I. R. Shafarevich

Abstract—We reconstruct hermitian $K$-theory via algebraic symplectic cobordism. In the motivic stable homotopy category $\text{SH}(S)$, there is a unique morphism $\varphi : \text{MSp} \to \text{BO}$ of commutative ring $T$-spectra which sends the Thom class $\text{th}^{\text{MSp}}$ to the Thom class $\text{th}^{\text{BO}}$. Using $\varphi$ we construct an isomorphism of bigraded ring cohomology theories on the category $\text{SmOp}/S$, $\varphi : \text{MSp}^{*,*}(X,U) \otimes_{\text{MSp}^{*,*,2*}(\text{pt})} \text{BO}^{*,2*}(\text{pt}) \cong \text{BO}^{*,*}(X,U)$. The result is an algebraic version of the theorem of Conner and Floyd reconstructing real $K$-theory using symplectic cobordism. Rewriting the bigrading as $\text{MSp}^{p,q} = \text{MSp}^{[q]}_{2q-p}$, we have an isomorphism $\varphi : \text{MSp}^{[q]}_{2q-p}(X,U) \otimes_{\text{MSp}^{[q]}_{2q-p}(\text{pt})} \text{KO}^{[q]}_{2q}(\text{pt}) \cong \text{KO}^{[q]}_{q}(X,U)$, where the $\text{KO}^{[q]}_{q}(X,U)$ are Schlichting’s hermitian $K$-theory groups.

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1. A MOTIVIC VERSION OF A THEOREM BY CONNER AND FLOYD

Our main result relates symplectic algebraic cobordism to hermitian $K$-theory. It is an algebraic version of the theorem of Conner and Floyd [2, Theorem 10.2] reconstructing real $K$-theory using symplectic cobordism. The algebraic version of the reconstruction of complex $K$-theory using unitary cobordism was presented in [6].

In [8] the present authors constructed a commutative ring $T$-spectrum $\text{BO}$ representing hermitian $K$-theory in the stable homotopy category $\text{SH}(S)$ for any regular Noetherian separated base scheme $S$ of finite Krull dimension without residue fields of characteristic 2. (These restrictions allowed us to use particularly strong results of M. Schlichting [12].) The $T$-spectrum $\text{BO}$ has a standard family of Thom classes for special linear vector bundles and hence for symplectic bundles. The symplectic Thom classes can all be derived from a single class $\text{th}^{\text{BO}} \in \text{BO}^{4,2}(\text{Th}_U) = \text{BO}^{4,2}(\text{MSp}_2)$, the symplectic Thom orientation. For a general discussion of symplectically oriented commutative ring $T$-spectra, see Section 4.

In [7, Definition 6.1] we constructed the commutative monoid $\text{MSp}$ of algebraic symplectic cobordism in the category of symmetric $T^2$-spectra. The reason for constructing the commutative monoid $\text{MSp}$ in the category of symmetric $T^2$-spectra is that the unit map naturally arises in the symmetric $T^2$-spectra but not in the symmetric $T$-spectra. The categories of symmetric $T$-spectra and of symmetric $T^2$-spectra are both symmetrical monoidal, and their homotopy categories are equivalent symmetric monoidal categories by [7, Theorem 3.2]. Thus $\text{MSp}$ defines a commutative monoid in $\text{SH}(S)$.

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The canonical map \( u_2: \Sigma^\infty_+ \text{MSP}_2(-2) \to \text{MSP} \) described in (2.1) gives the symplectic Thom orientation \( \text{th} \text{MSP} \in \text{MSP}_{4,2}(\text{MSP}_2) \) (see Example 4.2). The pair \((\text{MSP}, \text{th} \text{MSP})\) is the universal symplectically oriented commutative ring \( T \)-spectrum by Theorem 4.5.

Therefore, there is a unique morphism \( \varphi: \text{MSP} \to \text{BO} \) of commutative monoids in \( \text{SH}(S) \) with \( \varphi(\text{th} \text{MSP}) = \text{th} \text{BO} \). Our notation is that for a motivic space \( Y \) and a bigraded cohomology theory we write \( A^{*,*}(Y) = \bigoplus_{p,q} A^{p,q}(Y) \) and \( A^{*,2*}(Y) = \bigoplus_{i} A^{4i,2i}(Y) \). A motivic space \( Y \) is small if \( \text{Hom}_{\text{SH}}(\Sigma^\infty_+Y, -) \) commutes with arbitrary coproducts. Our main result is the following theorem.

**Theorem 1.1.** Let \( S \) be a regular Noetherian separated scheme of finite Krull dimension with \( \frac{1}{2} \in \Gamma(S, \mathcal{O}_S) \). For all small pointed motivic spaces \( Y \) over \( S \) the map

\[
\varphi: \text{MSP}^{*,*}(Y) \otimes_{\text{MSP}^{*,*}(\text{pt})} \text{BO}^{4*,2*}(\text{pt}) \to \text{BO}^{*,*}(Y)
\]

induced by \( \varphi \) is an isomorphism.

This has as a consequence the result mentioned in the abstract. For a pair \((X, U)\) consisting of a smooth \( S \)-scheme of finite type \( X \) and an open subscheme \( U \), there is a quotient pointed motivic space \( X_+/U_+ \). We define \( \text{MSp}^{*,*}(X, U) = \text{MSp}^{*,*}(X_+/U_+) \) and \( \text{BO}^{*,*}(X, U) = \text{BO}^{*,*}(X_+/U_+) \).

There are natural isomorphisms \( \text{BO}^{*,q}(X, U) = \text{KO}_{2q-2}(X, U) \) with the hermitian \( K \)-theory of \( X \) with supports in \( X - U \) as defined by Schlichting [11]. The weight \( q \) is the degree of the shift in the duality used for the symmetric bilinear forms on the chain complexes of vector bundles.

For a field \( k \) of characteristic different from 2, the ring \( \text{BO}^{4*,2*}(k) \) is not large. For all \( i \) one has \( \text{BO}^{8i,4i}(k) \cong \text{GW}(k) \) and \( \text{BO}^{8i+4,4i+2}(k) \cong \mathbb{Z} \). All members of \( \text{BO}^{0,0}(k) \) therefore come from composing endomorphisms in \( \text{SH}(k) \) of the sphere \( T \)-spectrum \( 1 = \Sigma^\infty_+ \text{pt} \). The unit \( e: 1 \to \text{BO} \) of the monoid. (See Morel [3, Theorem 4.36] and Cazanave [1] for calculations of the endomorphisms of the sphere \( T \)-spectrum.) Consequently, \( \varphi^{0,0}: \text{MSp}^{0,0}(k) \to \text{BO}^{0,0}(k) \) is surjective. We do not know what happens in other bidegrees.

This is the fourth paper in a series on symplectically oriented motivic cohomology theories. All depend on the quaternionic projective bundle theorem proven in the first paper [9].

2. PRELIMINARIES

Let \( S \) be a Noetherian separated scheme of finite Krull dimension. We will be dealing with hermitian \( K \)-theory, and we prefer to avoid the subtleties of negative \( K \)-theory, so we will assume that \( S \) is regular and that \( \frac{1}{2} \in \Gamma(S, \mathcal{O}_S) \). Let \( \mathcal{S}m/S \) be the category of smooth \( S \)-schemes of finite type. Following [5], we consider the category \( \mathcal{S}m\mathcal{O}p/S \) whose objects are pairs \((X, U)\) with \( X \in \mathcal{S}m/S \) and \( U \subseteq X \) an open subscheme and whose arrows \( f: (X, U) \to (X', U') \) are morphisms \( f: X \to X' \) of \( S \)-schemes with \( f(U) \subseteq U' \). Note that all \( X \) in \( \mathcal{S}m/S \) have an ample family of line bundles.

A motivic space over \( S \) is a simplicial presheaf on \( \mathcal{S}m/S \). We will often write \( \text{pt} \) for the base scheme regarded as a motivic space over itself. Inverting the motivic weak equivalences in the category of pointed motivic spaces gives the pointed motivic unstable homotopy category \( \text{H}_*(S) \).

Let \( T = A^1/(A^1 - 0) \) be the Morel–Voevodsky object. A \( T \)-spectrum \( M \) is a sequence of pointed motivic spaces \((M_0, M_1, M_2, \ldots)\) equipped with structural maps \( \sigma_n: M_n \to T \to M_{n+1} \). Inverting the stable motivic weak equivalences gives the motivic stable homotopy category \( \text{SH}(S) \). A pointed motivic space \( X \) has a \( T \)-suspension spectrum \( \Sigma^\infty_+ X \). For any \( T \)-spectrum \( M \) there are canonical maps of spectra

\[
u_n: \Sigma^\infty_+ M_n(-n) \to M.
\]

Both \( \text{H}_*(S) \) and \( \text{SH}(S) \) are equipped with closed symmetric monoidal structures, and the functor \( \Sigma^\infty_+: \text{H}_*(S) \to \text{SH}(S) \) is a strict symmetric monoidal functor. The symmetric monoidal structure \((\wedge, 1_S = \Sigma^\infty_+ \text{pt}+)\) on the homotopy category \( \text{SH}(S) \) can be constructed on the model category level using symmetric \( T \)-spectra.

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Any $T$-spectrum $A$ defines a cohomology theory on the category of pointed motivic spaces. Namely, for a pointed space $(X, x)$ one sets

$$A^{p,q}(X, x) = \text{Hom}_{H_2(S)}(\Sigma_T^\infty(X, x), \Sigma^{p,q}(A))$$

and

$$A^{*,*}(X, x) = \bigoplus_{p,q \in \mathbb{Z}} A^{p,q}(X, x).$$

We write (somewhat inconsistently)

$$A^{4i,2i}(X, x) = \bigoplus_{i \in \mathbb{Z}} A^{4i,2i}(X, x).$$

For an unpointed space $X$ we set $A^{p,q}(X) = A^{p,q}(X_+, +)$, with $A^{*,*}(X)$ and $A^{4i,2i}(X)$ defined accordingly. We will not always write the pointings explicitly.

Each $Y \in \mathcal{S}m/S$ defines an unpointed motivic space $\text{Hom}_{\mathcal{S}m/S}(-, Y)$, which is constant in the simplicial direction. So we regard smooth $S$-schemes as motivic spaces and set $A^{p,q}(Y) = A^{p,q}(Y_+, +)$. Given a monomorphism $U \hookrightarrow Y$ of smooth $S$-schemes, we write $A^{p,q}(Y, U) = A^{p,q}(Y_+/U_+, U_+/U_+)$. A commutative ring $T$-spectrum is a commutative monoid $(A, \mu, e)$ in $(\mathcal{S}H(S), \wedge, 1)$. We will use the following version of the definition. Let $A$ be the trivial symplectic bundle on $\mathbb{Z}$, with pullback map $f_E \colon E \to E$ one has $f_E^* \text{th}(E, \phi) = \text{th}(f_E^* E, f_E^* \phi)$;

(3) for the rank 2 trivial symplectic bundle $H$ over pt the map

$$- \times \text{th}(H) : A^{*,*}(X) \to A^{*+4,*,+2}(X \times \mathbb{A}^2, X \times (\mathbb{A}^2 - 0))$$

is an isomorphism for all $X \in \mathcal{S}m/S$.

The Borel class of $(E, \phi)$ is $b_1(E, \phi) = -z^* \text{th}(E, \phi) \in A^{4,2}(X)$ where $z : X \to E$ is the zero section.
The sign in the Borel class is simply conventional. It is chosen so that if $A^{*,*}$ is an oriented cohomology theory with an additive formal group law, then the Chern and Borel classes satisfy the traditional formula $b_i(E, \phi) = (-1)^ic_{2i}(E)$.

From the Mayer–Vietoris sequence one sees that for any rank 2 symplectic bundle $(E, \phi)$ over $X \in \mathcal{S}m/S$,

$$\cup \text{th}(E, \phi): A^{*,*}(X) \xrightarrow{\cong} A^{*+4,*+2}(E, E - X)$$

is an isomorphism.

The quaternionic Grassmannian $\text{HGr}(r, n) = \text{HGr}(r, \mathbb{H}^n)$ is defined as the open subscheme of $\text{Gr}(2r, 2n) = \text{Gr}(2r, \mathbb{H}^n)$ parametrizing those subspaces of dimension $2r$ of the fibers of $\mathbb{H}^n$ on which the symplectic form of $\mathbb{H}^n$ is nondegenerate (see [9] for details). We write $\mathcal{U}_{\text{HGr}(r,n)}$ for the restriction to $\text{HGr}(r,n)$ of the tautological rank $2r$ subbundle of $\text{Gr}(2r, 2n)$. The symplectic form of $\mathbb{H}^n$ restricts to a symplectic form on $\mathcal{U}_{\text{HGr}(r,n)}$, which we denote by $\phi_{\text{HGr}(r,n)}$. The pair $(\mathcal{U}_{\text{HGr}(r,n)}, \phi_{\text{HGr}(r,n)})$ is the tautological symplectic subbundle of rank $2r$ on $\text{HGr}(r,n)$.

More generally, given a symplectic bundle $(E, \phi)$ of rank $2n$ over $X$, the quaternionic Grassmannian bundle $\text{HGr}(r, E, \phi)$ over $X$ is the open subscheme of the Grassmannian bundle $\text{Gr}(2r, E)$ over $X$ parametrizing those subspaces of dimension $2r$ of the fibers of $E$ on which $\phi$ is nondegenerate.

For $r = 1$ we have quaternionic projective spaces and bundles $\text{HP}^n = \text{HGr}(1, n + 1)$ and $\text{HP}(E, \phi) = \text{HGr}(1, E, \phi)$.

The quaternionic projective bundle theorem is proven in [9] using the symplectic Thom structure and not any other version of a symplectic orientation. It is proven first for trivial bundles.

**Theorem 3.2** [9, Theorem 8.1]. Let $(A, \mu, e)$ be a commutative ring $T$-spectrum with a symplectic Thom structure on $A^{*,*}$. Let $(\mathcal{U}_{\text{HP}^n}, \phi_{\text{HP}^n})$ be the tautological rank 2 symplectic subbundle over $\text{HP}^n$ and $t = b_1(\mathcal{U}_{\text{HP}^n}, \phi_{\text{HP}^n}) \in A^{4,2}(\text{HP}^n)$ its Borel class. Then for any $X$ in $\mathcal{S}m/S$ we have an isomorphism of bigraded rings

$$A^{*,*}(\text{HP}^n \times X) \cong A^{*,*}(X)[t]/(t^{n+1}).$$

A Mayer–Vietoris argument gives a more general theorem [9, Theorem 8.2].

**Theorem 3.3** (quaternionic projective bundle theorem). Let $(A, \mu, e)$ be a commutative ring $T$-spectrum with a symplectic Thom structure on $A^{*,*}$. Let $(E, \phi)$ be a symplectic bundle of rank $2n$ over $X$, let $(\mathcal{U}, \phi|_{\mathcal{U}})$ be the tautological rank 2 symplectic subbundle over the quaternionic projective bundle $\text{HP}(E, \phi)$, and let $t = b_1(\mathcal{U}, \phi|_{\mathcal{U}})$ be its Borel class. Then we have an isomorphism of bigraded $A^{*,*}(X)$-modules

$$(1, t, \ldots, t^{n-1}): A^{*,*}(X) \oplus A^{*,*}(X) \oplus \cdots \oplus A^{*,*}(X) \rightarrow A^{*,*}(\text{HP}(E, \phi)).$$

**Definition 3.4.** Under the hypotheses of Theorem 3.3 there are unique elements $b_i(E, \phi) \in A^{4i,2i}(X)$ for $i = 1, 2, \ldots, n$ such that

$$t^n - b_1(E, \phi) \cup t^{n-1} + b_2(E, \phi) \cup t^{n-2} - \cdots + (-1)^{n-1}b_{n-1}(E, \phi) = 0.$$ 

The classes $b_i(E, \phi)$ are called the Borel classes of $(E, \phi)$ with respect to the symplectic Thom structure of the cohomology theory $(A, \partial)$. For $i > n$ one sets $b_i(E, \phi) = 0$, and one sets $b_0(E, \phi) = 1$.

**Corollary 3.5.** The Borel classes of a trivial symplectic bundle vanish: $b_i(\text{H}^n) = 0$ for $i > 0$.

The Cartan sum formula holds for Borel classes [9, Theorem 10.5]. In particular, the following theorem is valid.

**Theorem 3.6.** Let $(A, \mu, e)$ be a commutative ring $T$-spectrum with a symplectic Thom structure on $A^{*,*}$. Let $(E, \phi)$ and $(F, \psi)$ be symplectic bundles over $X$. Then we have

$$b_1((E, \phi) \oplus (F, \psi)) = b_1(E, \phi) + b_1(F, \psi).$$

(3.1)
We also have the following result [9, Proposition 8.5].

**Proposition 3.7.** Suppose that \((E, \phi)\) is a symplectic bundle over \(X\) with a totally isotropic subbundle \(L \subset E\). Then for all \(i\) we have

\[
b_i(E, \phi) = b_i\left(\left(L^\perp/L, \overline{\phi}\right) \oplus \left(L \oplus L^\vee, \begin{pmatrix} 0 & 1_L \varepsilon \varepsilon \varepsilon \\ -1_L & 0 \end{pmatrix}\right)\right).
\]

This is because there is an \(A^1\)-deformation between the two symplectic bundles.

**Definition 3.8.** The Grothendieck–Witt group of symplectic bundles \(GW^-(X)\) is the abelian group of formal differences \([E, \phi] - [F, \psi]\) of symplectic vector bundles over \(X\) modulo the following three relations:

1. for an isomorphism \(u: (E, \phi) \cong (E_1, \phi_1)\) one has \([E, \phi] = [E_1, \phi_1]\);
2. for an orthogonal direct sum one has \([E, \phi] + [E_1, \phi_1] = [E, \phi] + [E_1, \phi_1]\);
3. if \((E, \phi)\) is a symplectic bundle over \(X\) with a totally isotropic subbundle \(L \subset E\), then we have \([E, \phi] = [L^\perp/L, \overline{\phi}] + [L \oplus L^\vee, \begin{pmatrix} 0 & 1_L \varepsilon \varepsilon \varepsilon \\ -1_L & 0 \end{pmatrix}]\).

The Grothendieck–Witt group of orthogonal bundles \(GW^+(X)\) is defined analogously.

**Theorem 3.9.** Let \((A, \mu, e)\) be a commutative ring \(T\)-spectrum with a symplectic Thom structure on \(A^{*,*}\). Then the rule \((E, \phi) \mapsto b_1(E, \phi)\) gives a well-defined additive map

\[
b_1: GW^-(X) \to A^{4,2}(X),
\]

which is functorial in \(X\).

In [10] Schlichting constructed hermitian \(K\)-theory spaces for exact categories. This gives hermitian \(K\)-theory spaces \(KO(X)\) and \(KSp(X)\) for orthogonal and symplectic bundles on schemes. Their \(\pi_0\) are \(GW^+(X)\) and \(GW^-(X)\), respectively. In [11] he constructed hermitian \(K\)-theory spaces \(KO^{[n]}(X, U)\) for complexes of vector bundles on \(X\) acyclic on the open subscheme \(U\) equipped with a nondegenerate symmetric bilinear form for the duality shifted by \(m\). For an even integer \(2n\) an orthogonal bundle \((U, \psi)\) gives a chain complex \(U[2n]\) equipped with a nondegenerate symmetric bilinear form \(\psi[4n]: U[2n] \otimes \mathcal{O}_X U[2n] \to \mathcal{O}_X[4n]\) in the symmetric monoidal category \(D^b(VBX)\). For an odd integer \(2n + 1\) a symplectic bundle \((E, \phi)\) gives a chain complex \(E[2n + 1]\) equipped with a nondegenerate symmetric bilinear form \(\phi[4n + 2]: E[2n + 1] \otimes \mathcal{O}_X E[2n + 1] \to \mathcal{O}_X[4n + 2]\). These functors induce homotopy equivalences of spaces \(KO(X) \to KO^{[4n]}(X)\) and \(KSp(X) \to KO^{[4n+2]}(X)\) (see [11, Proposition 6]).

The simplicial presheaves \(X \mapsto KO^{[n]}(X)\) are pointed motivic spaces. Dévissage gives scheme-wise weak equivalences

\[
KO^{[n]}(X) \to KO^{[n+1]}(X \times A^1, X \times (A^1 - 0)),
\]

which are adjoint to maps \(KO^{[n]} \wedge T \to KO^{[n+1]}\). These are the structural maps of a \(T\)-spectrum \((KO^{[0]}, KO^{[1]}, KO^{[2]}, \ldots)\) for which our \(BO\) is a fibrant replacement (see [8, §§ 7, 8]). One has \(KO^{[n]}_i(X, U) = BO^{4n-2i}(X_+/U_+)\) for all \(i \geq 0\) and \(n\). So \(BO^{4n,2n}(X_+/U_+)\) is the Grothendieck–Witt group for the usual duality shifted by \(n\) of symmetric chain complexes of vector bundles on \(X\) which are acyclic on \(U\).

**Definition 3.10.** The right isomorphisms are

\[
\text{unsign.trans}_{4n}: GW^+(X) \xrightarrow{\cong} KO_0^{[4n]}(X) = BO^{8n,4n}(X),
\]

\[
[U, \psi] \mapsto [U[2n], \psi[4n]]
\]
and
\[ \text{sign.trans}_{4n+2} : \GW^-(X) \xrightarrow{\cong} \KO_0^{4n+2}(X) = \BO^{8n+4.4n+2}(X), \]
\[ [E, \phi] \mapsto -[E[2n+1], \phi[4n+2]]. \]

The sign in \text{sign.trans}_{4n+2} is chosen so that it commutes with the forgetful maps to \( K_0(X) \), where we have \([E] = -[E[2n+1]].\) Most authors of papers on Witt groups do not use this sign because Witt groups do not have forgetful maps to \( K_0(X) \).

**Definition 3.11.** The periodicity elements \( \beta_8 \in \BO^{8.4}(pt) \) and \( \beta_8^{-1} \in \BO^{-8.4}(pt) \) correspond to the unit 1 = \([\mathcal{O}_X, 1] \in \GW^+(X)\) under the isomorphisms \( \BO^{8.4}(pt) \cong \GW^+(pt) \cong \BO^{-8.4}(pt) \) of Definition 3.10.

We have the composition
\[ \overline{b}_1^A : \BO^{4.2}(X) \xleftarrow{\text{sign.trans}_2} \GW^-(X) \xrightarrow{\text{sign.trans}_1} A^{4.2}(X). \]

The Thom classes for hermitian \( K \)-theory are constructed by the same method that Nenashev used for Witt groups [4, Sect. 2]. Suppose we have an \( \SL_n \)-bundle \( (E, \lambda) \) consisting of a vector bundle \( \pi : E \to X \) of rank \( n \) and an isomorphism \( \lambda \) : \( \mathcal{O}_X \cong \det E \) of line bundles. The pullback \( \pi^*E = E \oplus E \to E \) has a canonical section \( \Delta_E \), the diagonal. There is a Koszul complex
\[ K(E) = (0 \to \Lambda^n \pi^*E^\vee \to \Lambda^{n-1} \pi^*E^\vee \to \ldots \to \Lambda \pi^*E^\vee \to \pi^*E^\vee \to \mathcal{O}_E \to 0) \]
in which each boundary map is the contraction with \( \Delta_E \). It is a locally free resolution of the coherent sheaf \( z \mathcal{O}_X \) on \( E \), with \( z \) the zero section of \( E \). There is a canonical isomorphism \( \Theta(E, \lambda) : K(E) \to K(E)^\vee[n] \) induced by \( \lambda \) which is symmetric for the shifted duality.

**Definition 3.12.** In the standard special linear Thom structure on \( \BO \), the Thom class of the special linear bundle \( (E, \lambda) \) of rank \( n \) is
\[ \text{th}^{BO}(E, \lambda) = [K(E), \Theta(E, \lambda)] \in \KO_0^n(E, E-X) = \BO^{2n,n}(E, E-X). \]

In the standard symplectic Thom structure on \( \BO \), the Thom class of the symplectic bundle \( (E, \phi) \) of rank 2\( r \) is
\[ \text{th}^{BO}(E, \phi) = \text{th}^{BO}(E, \lambda_\phi) \in \BO^{4r,2r}(E, E-X) \]
for \( \lambda_\phi = (\text{Pf} \phi)^{-1}, \) where \( \text{Pf} \phi \in \Gamma(X, \det E^\vee) \) denotes the Pfaffian of \( \phi \in \Gamma(X, \Lambda^2 E^\vee) \).

The corresponding first Borel class of a rank 2 symplectic bundle is therefore
\[ b_1^{BO}(E, \phi) = -[K(E), \Theta(E, \lambda_\phi)]|_X \in \BO^{4.2}(X). \]

A short calculation shows that this is the class which corresponds to \([E, \phi] - [H] \in \GW^-(X)\) under the isomorphism \( \text{sign.trans}_2 \). The symplectic splitting principle [9, Theorem 10.2] and Theorem 3.6 now give the next proposition.

**Proposition 3.13.** Let \((E, \phi)\) be a symplectic bundle of rank 2\( r \) on \( X \). Then \( b_1^{BO}(E, \phi) \in \BO^{4.2}(X) \) is the class corresponding to \([E, \phi] - r[H] \in \GW^-(X)\) under the isomorphism \( \text{sign.trans}_2 \).

Let \( X = \bigsqcup X_i \), where \( X_i \) are the connected components of \( X \). We consider the elements and functions
\[ 1_{X_i} \in \BO^{0,0}(X), \quad \text{rk}_{X_i} : \BO^{4.2}(X) \to \mathbb{Z}, \quad h \in \BO^{4.2}(pt). \]

The first is the central idempotent which is the image of the unit \( 1_{X_i} \in \BO^{0,0}(X_i) \). The second is the rank function on the Grothendieck–Witt group \( \KO_0^{[2]}(X) \) of bounded chain complexes of vector bundles. The third is the class corresponding to \([H] \in \GW^-(pt)\) under the right isomorphism \( \text{sign.trans}_2 : \GW^-(pt) \cong \BO^{4.2}(pt) \).

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Let \( \overline{b}_1^{BO} \colon BO^{4,2}(X) \to BO^{4,2}(X) \) be the map of (3.2).

**Corollary 3.14.** For all \( \alpha \in BO^{4,2}(X) \) we have

\[
\alpha = \overline{b}_1^{BO}(\alpha) + h \prod_i \frac{1}{2}(\text{rk}_X, \alpha) 1_X.
\]

4. SYMPLECTICALLY ORIENTED COMMUTATIVE RING T-SPECTRA

In this section, we recall the notion of a symplectically oriented commutative ring \( T \)-spectrum, recall the construction of the symplectic cobordism \( T \)-spectrum \( MSp \), and formulate the universality theorem for the \( T \)-spectrum \( MSp \).

Embed \( H^{\otimes n} \) in \( H^{\otimes \infty} \) as the direct sum of the first \( n \) summands. The ensuing filtration \( H \subset H^{\otimes 2} \subset H^{\otimes 3} \subset \ldots \) for each \( r \) gives rise to a direct system of schemes

\[
pt = HGr(r, r) \hookrightarrow HGr(r, r + 1) \hookrightarrow HGr(r, r + 2) \hookrightarrow \ldots.
\]

The ind-scheme and motivic space

\[
BSp_{2r} = HGr(r, \infty) = \colim_{n \geq r} HGr(r, n)
\]

are pointed by \( h_r : pt = HGr(r, r) \hookrightarrow BSp_{2r} \). Each \( HGr(r, n) \) has a tautological symplectic subbundle \((U_{HGr(r,n)}, \phi_{HGr(r,n)})\), and their colimit is an ind-scheme \( \mathcal{U}_{BSp_{2r}} \), which is a vector bundle over the ind-scheme \( BSp_{2r} \). It has a Thom space \( TH\mathcal{U}_{BSp_{2r}} \) just as for ordinary schemes. We write

\[
MSp_{2r} = Th \mathcal{U}_{BSp_{2r}} = Th \mathcal{U}_{HGr(r, \infty)} = \colim_{n \geq r} Th \mathcal{U}_{HGr(r,n)}.
\]

We refer the reader to [7, Sect. 6] for the complete construction of \( MSp \) as a commutative monoid in the category of symmetric \( T^{\wedge 2} \)-spectra. The unit comes from the pointings \( h_r : pt \hookrightarrow BSp_{2r} \), which induce canonical inclusions of Thom spaces

\[
e_r : T^{\wedge 2r} \hookrightarrow MSp_{2r}.
\]

Let \((A, \mu, e)\) be a commutative ring \( T \)-spectrum. The unit of the monoid defines the unit element \( 1_A \in A^{0,0}(pt) \). Applying the \( T \)-suspension isomorphism twice gives an element \( \Sigma^2 T 1_A \in A^{4,2}(T^{\wedge 2}) = A^{4,2}(Th A^2) \).

**Definition 4.1.** A symplectic Thom orientation on a commutative ring \( T \)-spectrum \((A, \mu, e)\) is an element \( th \in A^{4,2}(MSp_2) = A^{4,2}(Th \mathcal{U}_{HP^{\infty}}) \) with \( th|_{T^{\wedge 2}} = \Sigma^2 T 1_A \in A^{4,2}(T^{\wedge 2}) \).

The element \( th \) should be regarded as the symplectic Thom class of the tautological quaternionic line bundle \( \mathcal{U}_{HP^{\infty}} \) over \( HP^{\infty} \).

**Example 4.2.** The standard symplectic Thom orientation on algebraic symplectic cobordism is the element \( th^{MSp} = u_2 \in MSp^{4,2}(MSp_2) \) corresponding to the canonical map \( u_2 : \Sigma^\infty T MSp_2(-2) \to MSp \) described in (2.1).

The main theorem of [7] gives seven other structures containing the same information as a symplectic Thom orientation. In particular, the following theorem holds.

**Theorem 4.3** [7, Theorem 10.2]. Let \((A, \mu, e)\) be a commutative monoid in \( SH(S) \). There is a canonical bijection between the sets of

(a) symplectic Thom structures on the ring cohomology theory \( A^{*,*} \) such that for the trivial rank 2 symplectic bundle \( H \) over \( pt \) we have \( th(H) = \Sigma^2 T 1_A \) in \( A^{4,2}(T^{\wedge 2}) \) and

(a) symplectic Thom orientations on \((A, \mu, e)\).
Thus a symplectic Thom orientation determines Thom and Borel classes for all symplectic bundles.

**Lemma 4.4.** In the standard special linear and symplectic Thom structures on $BO$ we have $\theta(A^1, 1) = \Sigma T^1 BO$ and $\theta(H) = \Sigma^2 T^1 BO$.

**Proof.** The structural maps $KO[n] \wedge T \to KO^{[n+1]}$ of the spectrum are by definition [8, §8] adjoint to maps $KO[n] \to Hom_r(T, KO^{[n+1]})$ which are fibrant replacements of maps of simplicial presheaves
\[
\left(- \otimes (K(\emptyset), \Theta(\emptyset, 1))\right)_\wedge : KO^n(-) \to KO^{[n+1]}(- \wedge T),
\]
which act on the homotopy groups as $- \cup [K(\emptyset), \Theta(\emptyset, 1)] = - \cup \theta(A^1, 1)$. Therefore, we have $\Sigma T^1 BO = \theta(A^1, 1)$. It then follows that we have $\theta(H) = \theta(A^1, 1)^{\infty, 2} = \Sigma^2 T^1 BO$. \qed

The standard symplectic Thom structure on $BO$ thus satisfies the normalization condition of Theorem 4.3. It corresponds to the standard symplectic Thom orientation on hermitian $K$-theory $\theta^{BO} \in BO^{1,2}(MSp_2)$. It is given by the formulas of Definition 3.12 for the tautological subbundle $(E, \phi) = (1_{HP^\infty}, \phi_{HP^\infty})$ on $HP^\infty = BSp_2$.

A symplectically oriented commutative ring $T$-spectrum is a pair $(A, \vartheta)$ with $A$ a commutative monoid in $SH(S)$ and $\vartheta$ a symplectic Thom orientation on $A$. We could write the associated Thom and Borel classes as $\theta^g(E, \phi)$ and $b_i^g(E, \phi)$.

A morphism of symplectically oriented commutative ring $T$-spectra $\varphi: (A, \vartheta) \to (B, \varpi)$ is a morphism $\varphi: A \to B$ of commutative monoids in $SH(S)$ with $\varphi(\vartheta) = \varpi$. For such a $\varphi$ one has $\varphi(\theta^g(E, \phi)) = \theta^g(E, \phi)$ and $\varphi(b_i^g(E, \phi)) = b_i^g(E, \phi)$ for all symplectic bundles.

**Theorem 4.5** (universality of $MSp$). Let $(A, \mu, e)$ be a commutative monoid in $SH(S)$. The assignment $\varphi \mapsto \varphi(\theta^{MSp})$ gives a bijection between the sets of

- $(\varepsilon)$ morphisms $\varphi: (MSp, \mu_{MSp}, e_{MSp}) \to (A, \mu, e)$ of commutative monoids in $SH(S)$ and
- $(\alpha)$ symplectic Thom orientations on $(A, \mu, e)$.

This is [7, Theorems 12.3, 13.2]. Thus $(MSp, \theta^{MSp})$ is the universal symplectically oriented commutative ring $T$-spectrum.

Let $\varphi: (A, \vartheta) \to (B, \varpi)$ be a morphism of symplectically oriented commutative ring $T$-spectra. For a space $X$ the isomorphisms $X \wedge pt_+ \cong X \cong pt_+ \wedge X$ make $A^{*,*}(X)$ into a two-sided module over the ring $A^{*,*}(pt)$ and into a bigraded commutative algebra over the commutative ring $A^{4*,2*}(pt)$. The morphism $\varphi$ induces morphisms of graded rings
\[
\overline{\varphi}_X: A^{*,*}(X) \otimes A^{4*,2*}(pt) B^{4*,2*}(pt) \to B^{*,*}(X),
\]
\[
\overline{\varphi}_X: A^{4*,2*}(X) \otimes A^{4*,2*}(pt) B^{4*,2*}(pt) \to B^{4*,2*}(X),
\]
which are natural in $X$ in the obvious sense.

**Theorem 4.6** (weak quaternionic cellularity of $MSp_{2r}$). Let $\varphi: (A, \vartheta) \to (B, \varpi)$ be a morphism of symplectically oriented commutative ring $T$-spectra. Then for all $r$ the natural morphism of graded rings
\[
\overline{\varphi}_{MSp_{2r}}: A^{4*,2*}(MSp_{2r}) \otimes A^{4*,2*}(pt) B^{4*,2*}(pt) \to B^{4*,2*}(MSp_{2r})
\]
is an isomorphism.
Proof. Let $t_1, \ldots, t_r$ be independent indeterminates with $t_i$ of bidegree $(4i, 2i)$. By [7, Theorems 9.1–9.3] there is a commutative diagram of isomorphisms

\[
\begin{array}{c}
A^{*,*}(pt)[[t_1, \ldots, t_r]]_{\text{hom}} \xrightarrow{t_i \mapsto b_{i}^{\phi}(U_{BSp_{2r}}, \phi_{BSp_{2r}})} A^{*,*}(BSp_{2r}) \\
\times t_r \xrightarrow{\cong} t_r A^{*,*}(pt)[[t_1, \ldots, t_r]]_{\text{hom}} \xrightarrow{\cong} A^{*+4r,*+2r}(MSP_{2r})
\end{array}
\]

The notation on the left refers to homogeneous formal power series. There is a similar diagram for $(B, \varpi)$. The maps $\varphi: A^{*,*} \to B^{*,*}$ commute with the maps of the two diagrams because $\varphi$ sends the Thom and Borel classes of $(A, \vartheta)$ onto the Thom and Borel classes of $(B, \varpi)$. The morphism $\varphi_{MSp_{2r}}$ is an isomorphism because

\[
t_r A^{4*,2*}(pt)[[t_1, \ldots, t_r]]_{\text{hom}} \otimes A^{4*,2*}(pt) B^{4*,2*}(pt) \to t_r B^{4*,2*}(pt)[[t_1, \ldots, t_r]]_{\text{hom}}
\]

is an isomorphism. □

5. WHERE THE CLASS $b_1$ TAKES THE PLACE OF HONOR

We suppose that $(U, u) \to (BO, \text{th} BO)$ is a morphism of symplectically oriented commutative ring $T$-spectra. We set

\[
\begin{align*}
\overline{U}^{*,*}(X) &= U^{*,*}(X) \otimes_{U^{4*,2*}(pt)} BO^{4*,2*}(pt), \\
\overline{U}^{4*,2*}(X) &= U^{4*,2*}(X) \otimes_{U^{4*,2*}(pt)} BO^{4*,2*}(pt)
\end{align*}
\]

and write $\overline{\tau}_X$ for the morphisms of $(4.1)$.

**Theorem 5.1.** Let $(U, u) \to (BO, \text{th} BO)$ be a morphism of symplectically oriented commutative ring $T$-spectra. Suppose there exists an $N$ such that for all $n \geq N$ the maps $\overline{\tau}_{U_{2n}}: \overline{U}^{4i,2i}(U_{2n}) \to BO^{4i,2i}(U_{2n})$ are isomorphisms for all $i$. Then for all small pointed motivic spaces $X$ and all $(p,q)$ the homomorphism $\overline{\tau}_X: \overline{U}^{p,q}(X) \to BO^{p,q}(X)$ is an isomorphism.

Before turning to the theorem itself, we prove a series of lemmas. The first three demonstrate the significance of the first Borel class for this problem.

**Lemma 5.2.** The functorial map $\overline{\tau}_X: \overline{U}^{4,2}(X) \to BO^{4,2}(X)$ has a section $s_X$ which is functorial in $X$.

**Proof.** Write $HGr = \text{colim}_r HGr(r, \infty)$. According to [8, Theorem 10.1, (11.1)], there is an isomorphism à la Morel–Voevodsky $\tau: (Z \times HGr, (0, x_0)) \cong KSp$ in $H_*(S)$ such that the restrictions are

\[
\tau|_{(i)\times HGr(n,2n)} = [U_{HGr(n,2n)}, \phi_{HGr(n,2n)}] + (i - n)[H]
\]

in $KSp_0(HGr(n,2n)) = GW^-(HGr(n,2n))$. Composing with the isomorphisms in $H_*(S)$,

\[
(Z \times HGr, (0, x_0)) \xrightarrow{\tau} KSp \xrightarrow{\text{trans}_1} KO^{[2]} \xrightarrow{-1} KO^{[2]},
\]

where the $\text{trans}_1$ comes from the translation functor $(\mathcal{F}, \phi) \mapsto (\mathcal{F}[1], \phi[2])$ and the $-1$ is the inverse operation of the $H$-space structure on $KO^{[2]}$, we obtain an element

\[
\tau_2 \in KO^{[2]}_0(Z \times HGr, (0, x_0)) = BO^{4,2}(Z \times HGr, (0, x_0))
\]

corresponding to the composition. By Corollary 3.14 we have

\[
\tau_2|_{(i)\times HGr(n,2n)} = b_1(U_{HGr(n,2n)}, \phi_{HGr(n,2n)}) + i\hbar.
\]
For any symplectically oriented cohomology theory $A^{*,*}$ we have [8, (9.3)]

$$A^{*,*}(\mathbb{Z} \times \text{HGr}) = (A^{*,*}(\text{pt})[[b_1, b_2, b_3, \ldots]]^\text{hom}) \times \mathbb{Z}.$$ 

For such a theory let

$$\frac{1}{2} \text{rk}^A = (i 1_{\text{HGr}})_{i \in \mathbb{Z}} \in A^{0,0}(i 1_{\text{HGr}}) \times \mathbb{Z}, \quad b_1^A = (b_1)_{i \in \mathbb{Z}} \in A^{4,2}(\mathbb{Z} \times \text{HGr}).$$

Then $\tau_2 = b_1^A BO + \frac{1}{2} \text{rk}^A h$. Consider the element

$$s = b_1^U \otimes 1_{BO} + \frac{1}{2} \text{rk}^U h \in \mathbb{U}_{i,2}^{4,2}(\mathbb{Z} \times \text{HGr}).$$

Clearly one has $\overline{\tau}(s) = \tau_2$. By the Yoneda lemma, the element $s$ may be regarded as a morphism of functors

$$\text{Hom}_{H_\bullet(S)}(-, \mathbb{Z} \times \text{HGr}) \to \mathbb{U}_{i,2}^{4,2}(-),$$

i.e., a morphism of presheaves on $H_\bullet(S)$. The composite map

$$\text{Hom}_{H_\bullet(S)}(-, \mathbb{Z} \times \text{HGr}) \xrightarrow{s} \mathbb{U}_{i,2}^{4,2}(-) \to \text{BO}^{4,2}(-)$$

coincides with a functor transformation given by the adjoint $\Sigma^\infty_2(\mathbb{Z} \times \text{HGr})(-2) \to \text{BO}$ of the motivic weak equivalence $\tau_2 : \mathbb{Z} \times \text{HGr} \to \text{KO}^{[2]}$. Thus for every pointed motivic space $X$ the map

$$s_X : \text{BO}^{4,2}(X) = \text{Hom}_{H_\bullet(S)}(X, \text{KO}^{[2]}) = \text{Hom}_{H_\bullet(S)}(X, \mathbb{Z} \times \text{HGr}) \xrightarrow{s} \mathbb{U}_{i,2}^{4,2}(X)$$

is a section of the map $\overline{\tau}_X : \mathbb{U}_{i,2}^{4,2}(X) \to \text{BO}^{4,2}(X)$ which is natural in $X$. □

**Lemma 5.3.** For any integer $i$ the functorial map

$$\overline{\tau}_X : \mathbb{U}_{i,2}^{4,2}(X) \to \text{BO}_{8i+4,4i+2}(X)$$

has a section $t_X$ which is functorial in $X$.

**Proof.** We have $\text{BO}^{8i+4,4i+2} = \text{BO}_{4,2}^{[2]}[\beta_8, \beta_8^{-1}]$ for the periodicity element $\beta_8 \in \text{BO}^{8,4}(\text{pt})$ of Definition 3.11. So any element of $\text{BO}^{8i+4,4i+2}(X)$ may be written uniquely in the form $a \cup \beta_8^i$ with $a \in \text{BO}_{4,2}(X)$ and $i \in \mathbb{Z}$. We define

$$t_X(a \cup \beta_8^i) = s_X(a) \cup (1_U \otimes \beta_8^i) \in \mathbb{U}_{8i+4,4i+2}(X).$$

Then $t_X$ is a section of $\overline{\tau}_X$ which is natural in $X$. □

**Lemma 5.4.** If $X$ is a small pointed motivic space and $i$ is an integer, then for any $\alpha \in \mathbb{U}_{i,2}^{4,2}(X)$ there exists an $n \geq 0$ with

$$(t_X \wedge T^{\wedge 2n} \circ \overline{\tau}_X \wedge T^{\wedge 2n})(\Sigma^2 n) = \Sigma^2 n \alpha.$$ 

**Proof.** We may assume that $\alpha = a \otimes b$ with $a \in \mathbb{U}_{4d,2d}(X)$ and $b \in \text{BO}_{4i-4d,2i-2d}(\text{pt})$. For a small motivic space $X$ there is a canonical isomorphism [13, Theorem 5.2]

$$\mathbb{U}_{4d,2d}(X) = \text{colim}_m \text{Hom}_{H_\bullet(S)}(X \wedge T^{\wedge m}, \text{U}_{2d+m}).$$

This isomorphism implies that there exists an integer $n \geq 0$ such that $\Sigma^2 n a = f^*[u_{2d+2n}]$ for an appropriate map $f : X \wedge T^{\wedge 2n} \to \text{U}_{2d+2n}$ in $H_\bullet(S)$, where $u_{2d+2n}$ is as defined in (2.1). We may assume that $d + n \geq N$ and that $n + i$ is odd.
We have \([u_{2d+2n}] \otimes b \in U^{4n+4i, 2n+2i}(U_{2d+2n})\). By hypothesis
\[
\varphi_{U_{2d+2n}} : U^{4n+4i, 2n+2i}(U_{2d+2n}) \to BO^{4n+4i, 2n+2i}(U_{2d+2n})
\]
is an isomorphism. So its section \(t_{U_{2d+2n}}\) is the inverse isomorphism. Hence we have
\[
(t_{U_{2d+2n}} \circ \varphi_{U_{2d+2n}})([u_{2d+2n}] \otimes b) = [u_{2d+2n}] \otimes b.
\]
Then by the functoriality of \(U\), \(t\), and \(\varphi\) we have
\[
\Sigma_T^{2n} \alpha = f^*[u_{2d+2n}] \otimes b = f^* \circ t_{U_{2d+2n}} \circ \varphi_{U_{2d+2n}}([u_{2d+2n}] \otimes b)
\]
\[
= t_{X \wedge T \wedge 2n} \circ \varphi_{X \wedge T \wedge 2n}(\Sigma_T^{2n} \alpha). \quad \square
\]

**Lemma 5.5.** Suppose that for some \((p, q)\) the homomorphism \(\varphi_X : \U^{p,q}(X) \to BO^{p,q}(X)\) is an isomorphism for all small pointed motivic spaces \(X\). Then the same holds for \((p - 1, q)\) and \((p - 1, q - 1)\).

**Proof.** For \((p - 1, q)\) this is because the suspension \(\Sigma_{S^1}\) induces isomorphisms \(U^{p-1,q}(X) \cong U^{p,q}(X \wedge S^1)\) and similar isomorphisms for \(U\) and \(BO\), and these are compatible with \(\varphi\) and \(\varphi\). For \((p - 1, q - 1)\) use the suspension \(\Sigma_{Sm}\). \(\square\)

**Proof of Theorem 5.1.** First suppose \((p, q) = (8i + 4, 4i + 2)\) for some \(i\). Then for any small motivic space \(X\) the map \(\varphi_X : \U^{8i+4,4i+2}(X) \to BO^{8i+4,4i+2}(X)\) is surjective because it has the section \(t_X\) of Lemma 5.3. To show it is injective, we suppose \(\alpha\) is in its kernel. The suspension \(\Sigma_T\) is compatible with \(\varphi\) and \(\varphi\), so we have \(\varphi_{X \wedge T \wedge 2n}(\Sigma_T^{2n} \alpha) = \Sigma_T^{2n} \varphi_X(\alpha) = 0\). By Lemma 5.4 we therefore also have \(\Sigma_T^{2n} \alpha = 0\). But \(\Sigma_T^{2n}\) induces an isomorphism of cohomology groups. So we have \(\alpha = 0\). Thus \(\varphi_X : \U^{p,q}(X) \to BO^{p,q}(X)\) is an isomorphism for all small motivic spaces \(X\) for \((p, q) = (8i + 4, 4i + 2)\).

The result for other values of \((p, q)\) follows from Lemma 5.5 and a numerical argument. \(\square\)

6. LAST DETAILS

**Proof of Theorem 1.1.** By the universality of the symplectically oriented commutative ring \(T\)-spectrum \((\MSp, \text{th}^{\MSp})\) (Theorem 4.5), there is a unique morphism \(\varphi : \MSp \to BO\) in \(\text{SH}(S)\) of commutative ring \(T\)-spectra with \(\varphi(\text{th}^{\MSp}) = \text{th}^{BO}\). It induces the morphisms of (4.1):
\[
\varphi_X : \MSp^{*,*}(X) \otimes_{\MSp^{4*,2r}(pt)} BO^{4*,2r}(pt) \to BO^{*,*}(X),
\]
\[
\varphi_X : \MSp^{4*,2r}(X) \otimes_{\MSp^{4*,2r}(pt)} BO^{4*,2r}(pt) \to BO^{4*,2r}(X).
\]
The second morphism, with the bidegrees \((4i, 2i)\) only, is an isomorphism for \(X = \MSp_{2r}\) for all \(r\) by Theorem 4.6. So all the hypotheses of Theorem 5.1 hold with \((U, u) = (\MSp, \text{th}^{\MSp})\). The conclusions of Theorem 5.1 imply Theorem 1.1. \(\square\)

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REFERENCES

1. C. Cazanave, “Algebraic homotopy classes of rational functions,” Ann. Sci. Éc. Norm. Supér., Sér. 4, 45 (4), 511–534 (2012).
2. P. E. Conner and E. E. Floyd, The Relation of Cobordism to K-Theories (Springer, Berlin, 1966), Lect. Notes Math. 28.
3. F. Morel, $\mathbb{A}^1$-Algebraic Topology over a Field (Springer, Berlin, 2012), Lect. Notes Math. 2052.
4. A. Nenashev, “Gysin maps in Balmer–Witt theory,” J. Pure Appl. Algebra 211 (1), 203–221 (2007).
5. I. Panin, “Oriented cohomology theories of algebraic varieties. II (after I. Panin and A. Smirnov),” Homology, Homotopy Appl. 11 (1), 349–405 (2009).
6. I. Panin, K. Pimenov, and O. Röndigs, “On the relation of Voevodsky’s algebraic cobordism to Quillen’s $K$-theory,” Invent. Math. 175 (2), 435–451 (2009).
7. I. Panin and C. Walter, “On the algebraic cobordism spectra $\text{MSL}$ and $\text{MSp}$,” arXiv:1011.0651 [math.AG].
8. I. Panin and C. Walter, “On the motivic commutative ring spectrum $BO$,” St. Petersbg. Math. J. 30 (6), 933–972 (2019) [repr. from Algebra Anal. 30 (6), 43–96 (2018)].
9. I. Panin and C. Walter, “Quaternionic Grassmannians and Borel classes in algebraic geometry,” arXiv:1011.0649 [math.AG].
10. M. Schlichting, “Hermitian $K$-theory of exact categories,” J. K-Theory 5 (1), 105–165 (2010).
11. M. Schlichting, “The Mayer–Vietoris principle for Grothendieck–Witt groups of schemes,” Invent. Math. 179 (2), 349–433 (2010).
12. M. Schlichting, “Hermitian $K$-theory, derived equivalences and Karoubi’s fundamental theorem,” J. Pure Appl. Algebra 221 (7), 1729–1844 (2017).
13. V. Voevodsky, “$\mathbb{A}^1$-homotopy theory,” Doc. Math., Extra Vol. ICM Berlin 1998, I, 579–604 (1998).

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