f-Frequently hypercyclic \( C_0 \)-semigroups on complex sectors

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Abstract
We analyze \( f \)-frequently hypercyclic, \( q \)-frequently hypercyclic \((q > 1)\), and frequently hypercyclic \( C_0 \)-semigroups \((q = 1)\) defined on complex sectors, with values in separable infinite-dimensional Fréchet spaces. Some structural results are given for a general class of \( \mathcal{F} \)-frequently hypercyclic \( C_0 \)-semigroups, as well. We investigate generalized frequently hypercyclic translation semigroups and generalized frequently hypercyclic semigroups induced by semiflows on weighted function spaces. Several illustrative examples are presented.

Keywords \( C_0 \)-Semigroups on complex sectors · \( \mathcal{F} \)-Frequent hypercyclicity · \( f \)-Frequent hypercyclicity · Translation semigroups and semigroups induced by semiflows · Fréchet spaces

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1 Introduction and preliminaries

The main aim of this paper is to investigate the classes of $\mathcal{F}$-frequently hypercyclic $C_0$-semigroups and $f$-frequently hypercyclic $C_0$-semigroups on a complex sector $\Delta$, in the setting of a separable infinite-dimensional Fréchet space. Here, we assume that $\Delta \in \{[0, \infty), \mathbb{R}, \mathbb{C}\} \text{ or } \Delta = \Delta(\alpha)$ for an appropriate angle $\alpha \in (0, \pi/2]$, where $\Delta(\alpha) = \{ re^{i\theta} : r \geq 0, \theta \in [-\alpha, \alpha]\}$. Further on, $\mathcal{F} \in P(P(\Delta))$, where $P(\Delta)$ denotes the power set of $\Delta$, $\mathcal{F} \neq \emptyset$ and $f : [0, \infty) \to [1, \infty)$ is an increasing mapping.

The notion of a frequently hypercyclic $C_0$-semigroup $(T(t))_{t \in \Delta}$ on a separable Fréchet space $E$ seems to be not considered elsewhere even in the case that $\Delta = [0, \infty)$. We have been forced to reconsider and slightly extend several known results from [25,26] to $C_0$-semigroups in Fréchet spaces. Moreover, the notion of a frequently hypercyclic $C_0$-semigroup on a complex sector has not yet been defined in the existing literature, even in the setting of separable Banach spaces. A great deal of our work is devoted to the study of the much more general class of $f$-frequently hypercyclic $C_0$-semigroups on complex sectors, which is the central object of our investigations.

The notion of a frequently hypercyclic linear continuous operator $T$ on a separable Fréchet space $E$ was introduced in [1] and analyzed by many authors; see [1,5,12,13]. We refer to [25,26] for the analysis on Banach spaces and more specifically on weighted function spaces: hypercyclic and chaotic translation $C_0$-semigroups on complex sectors were investigated in [7,8].

The paper is organized as follows. We recall lower and upper densities in Sect. 1.1. The basic facts about weighted function spaces and translation semigroups and semigroups induced by semiflows are given in Sect. 1.2. Conjugacy lemma for $\mathcal{F}$-frequently hypercyclic $C_0$-semigroups on complex sectors is stated in Lemma 3. In Propositions 1–2, we generalize [26, Proposition 2.1] for $\mathcal{F}$-frequently hypercyclic $C_0$-semigroups and $f$-frequently hypercyclic $C_0$-semigroups on complex sectors. In Proposition 3, we show that for any hypercyclic $C_0$-semigroup $(T(t))_{t \in \Delta}$, one can always find an increasing mapping $f : [0, \infty) \to [1, \infty)$ satisfying that $(T(t))_{t \in \Delta}$ is $f$-frequently hypercyclic. In Theorem 1, case (a), which is commonly used in applications, we formulate sufficient conditions for $f$-frequent hypercyclicity of a single operator $T(t_0)$, where $t_0 \in \Delta \setminus \{0\}$, as well as for the whole semigroup $(T(t))_{t \in \Delta}$ and its restriction to the ray $R$ connecting the origin and $t_0$. The frequent hypercyclicity criterion for $C_0$-semigroups [26, Theorem 2.2] cannot be satisfactorily reformulated for $q$-frequent hypercyclicity, provided that $q > 1$. Theorem 1, case (b), is an extension of the case (a) for a sequence of single operator of the considered $C_0$-semigroup; this theorem provides an important tool for proving the existence of a frequently hypercyclic semigroup $(T(t))_{t \in \Delta(\pi/4)}$ without any (frequently) hypercyclic single operator; see also Examples I in Sect. 2.1, in which we reconsider several known examples from [7,8] and [19, Chapter 3]. Essentially, our first main contribution is Theorem 2, where we formulate and prove an $f$-frequent hypercyclicity criterion for $C_0$-semigroups on complex sectors by following an approach which does not use the notion of Pettis integrability. In Theorem 3, we state an important extension of [26, Proposition 2.1] for $f$-frequently hypercyclic semigroups with the index set $\Delta = [0, \infty)$. 

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The third section is devoted to the study of \( f \)-frequently hypercyclic translation semigroups and \( f \)-frequently hypercyclic semigroups induced by semiflows on certain weighted function spaces. The method presented in the proof of Theorem 4, in which we provide sufficient conditions for \( f \)-frequent hypercyclicity of translation semigroups on complex sectors, is repeated after that several times with minor modifications. Note that obtaining necessary and sufficient conditions is a non-trivial problem, even for \( q \)-hypercyclicity. These results can be slightly improved by considering analysis of the sequences of operators of the considered \( C_0 \)-semigroups (see Theorem 5). As explained in Remark 2, these results are applicable for \( q \)-frequent hypercyclicity, where \( q > 1 \), and not for the usual frequent hypercyclicity. In Theorem 6 and Proposition 6, we slightly extend the assertions of [7, Theorem 6] and [25, Proposition 3.4], respectively, concerning frequently hypercyclic \( C_0 \)-semigroups. In our examinations of \( C_0 \)-semigroups induced by semiflows, we essentially use the same methods as for translation \( C_0 \)-semigroups, so that we essentially shorten the proofs and only explain how these proofs can be deduced. It is worth noting that frequently hypercyclic weighted composition \( C_0 \)-semigroups with the index set \( \Delta = [0, \infty) \) have been considered in [16], where the author gave a certain conditions under which these semigroups satisfy the frequent hypercyclicity criterion for \( C_0 \)-semigroups iff they are chaotic (an application has been made to the linear von Foerster–Lasota equation in the spaces \( L^p[0, 1] \) and \( W^{1,p}_*(0, 1) \)). Concerning the abstract first-order differential equations with a non-vanishing zero-order term, whose solutions are governed by weighted composition \( C_0 \)-semigroups, we would like to note that the assertions of [20, Theorem 23, Theorem 25] hold for the general \( \mathcal{F} \)-hypercyclicity and its subnotions; in other words, our main structural results established here for the abstract, first-order differential equations without a non-vanishing zero-order term can be almost technically modified for the abstract first-order differential equations containing a non-vanishing zero-order term. Because of that, we will skip all related details concerning this question here. Chaotic and frequently hypercyclic weighted composition \( C_0 \)-semigroups on complex sectors deserve a special attention and these classes of \( C_0 \)-semigroups will be further analyzed in our follow-up research [24]. In Sect. 3, we also present a great number of illustrative examples, applications, and open problems (see Sects. 3.1 and 3.4–3.6).

1.1 Notation and definitions

We use the standard notation throughout the paper. For any \( s \in \mathbb{R} \), set \( \lfloor s \rfloor := \sup \{ l \in \mathbb{Z} : s \geq l \} \) and \( \lceil s \rceil := \inf \{ l \in \mathbb{Z} : s \leq l \} \). We denote by \( E \) a separable Fréchet space over the field \( K \in \{ \mathbb{R}, \mathbb{C} \} \); we assume that the topology of \( E \) is induced by the fundamental system \( (p_n)_{n \in \mathbb{N}} \) of increasing seminorms. If \( Y \) is also a Fréchet space, over the same field of scalars as \( E \), then \( L(E, Y) \) is the space of all continuous linear mappings from \( E \) into \( Y \). The translation invariant metric \( d : E \times E \to [0, \infty) \), defined by:

\[
d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)}, \quad x, y \in E,
\]
satisfies, among many other properties, the following ones: \( d(x + u, y + v) \leq d(x, y) + d(u, v) \) and \( d(cx, cy) \leq (|c| + 1)d(x, y) \), \( c \in \mathbb{K}, x, y, u, v \in E \). Set \( L(x, \epsilon) := \{ y \in E : d(x, y) < \epsilon \} \) and \( L_n(x, \epsilon) := \{ y \in E : p_n(x - y) < \epsilon \} \) \((n \in \mathbb{N}, \epsilon > 0)\). Denote by \( E^* \) the dual space of \( E \). For a linear operator \( A \) on \( E \), we denote by \( D(A) \), \( R(A) \) and \( N(A) \) its domain, range, and kernel, respectively. Set \( D_\infty(A) := \bigcap_{k \in \mathbb{N}} D(A^k) \). We refer to [27, pp. 99–102] and references cited therein for the properties of the Bochner integration of Fréchet space valued functions. Let us recall that a collection of series \( \sum_{n=1}^{\infty} x_{n,k}, k \in J \) is called unconditionally convergent, uniformly in \( k \in J \) iff for any \( \epsilon > 0 \), there is an integer \( N \in \mathbb{N} \), such that for any finite set \( F \subseteq [N, \infty) \cap \mathbb{N} \) and for every \( k \in J \), we have \( \sum_{n \in F} x_{n,k} \in L(0, \epsilon) \) (cf. [4, 15, 30] and references cited therein). Set \( \Delta_\delta := \{ t \in \Delta : |t| \leq \delta \} \) and \( B(t, \delta) := \{ x \in \mathbb{K} : |x - t| \leq \delta \} \). We refer to [16, 19] for the basic material related to \( C_0 - \) semigroups on Fréchet spaces. Within our framework, any semigroup \( (T(t))_{t \in \Delta} \) is locally equicontinuous.

**Definition 1** ([22]) Let \((T_n)_{n \in \mathbb{N}} \) be a sequence in \( L(X, Y) \) and \( T \in L(X, Y) \). Let \( x \in E \). Suppose that \( \mathcal{F} \in P(P(\mathbb{N})) \) and \( \mathcal{F} \neq \emptyset \). Then, it is said that:

(i) \( x \) is an \( \mathcal{F} \)-hypercyclic element of the sequence \((T_n)_{n \in \mathbb{N}} \) iff \( x \in \bigcap_{n \in \mathbb{N}} D(T_n) \) and for each open non-empty subset \( V \) of \( Y \), we have that:

\[
S(x, V) := \{ n \in \mathbb{N} : T_n x \in V \} \in \mathcal{F} ;
\]

\((T_n)_{n \in \mathbb{N}} \) is said to be \( \mathcal{F} \)-hypercyclic iff there exists an \( \mathcal{F} \)-hypercyclic element of \((T_n)_{n \in \mathbb{N}} \);

(ii) \( T \) is \( \mathcal{F} \)-hypercyclic iff the sequence \((T^n)_{n \in \mathbb{N}} \) is \( \mathcal{F} \)-hypercyclic; \( x \in D_\infty(T) \) is said to be an \( \mathcal{F} \)-hypercyclic element of \( T \) iff \( x \) is an \( \mathcal{F} \)-hypercyclic element of the sequence \((T^n)_{n \in \mathbb{N}} \).

**Definition 2** ([22]) Suppose that \( q \in [1, \infty) \), \( A \subseteq \mathbb{N} \) and \((m_n)\) is an increasing sequence in \([1, \infty)\). Then, we define:

(i) The lower \( q \)-density of \( A \), denoted by \( d_q(A) : \)

\[
d_q(A) := \liminf_{n \to \infty} \frac{|A \cap [1, n^q]|}{n} .
\]

(ii) The lower \((m_n)\)-density of \( A \), denoted by \( d_{m_n}(A) : \)

\[
d_{m_n}(A) := \liminf_{n \to \infty} \frac{|A \cap [1, m_n]|}{n} .
\]

Assume that \( q \in [1, \infty) \) and that \((m_n)\) is an increasing sequence in \([1, \infty)\). Consider the notion introduced in Definition 1 with: (i) \( \mathcal{F} = \{ A \subseteq \mathbb{N} : d(A) := d_1(A) > 0 \} \), (ii) \( \mathcal{F} = \{ A \subseteq \mathbb{N} : d_q(A) > 0 \} \), (iii) \( \mathcal{F} = \{ A \subseteq \mathbb{N} : d_{m_n}(A) > 0 \} \); Then, we say that \((T_n)_{n \in \mathbb{N}} \) or \((T)\) is frequently hypercyclic, \( q \)-frequently hypercyclic and \( l-(m_n) \)-hypercyclic, respectively. In this case, \( x \) is called a frequently hypercyclic vector, \( q \)-frequently hypercyclic vector, and \( l-(m_n) \)-hypercyclic vector of \((T_n)_{n \in \mathbb{N}} \) \((T)\), respectively. Here, we use the prefix “\( l-\)” to emphasize the difference between
the concept of l-$(m_n)$-hypercyclicity, which is employed in this paper, and the concept of $(m_n)$-hypercyclicity, introduced in [2,22]. For the sake of simplicity of notation, we will drop this prefix in our considerations of continuous analogues of l-$(m_n)$-hypercyclicity.

In the sequel, $m(\cdot)$ denote the Lebesgue measure; $\Omega \neq \emptyset$ is open subset of $\mathbb{R}^n$.

**Definition 3** ([22]) Suppose that $q \in [1, \infty)$, $A \subseteq \Delta$ and $f : [0, \infty) \to [1, \infty)$ is an increasing mapping. Then, we define:

(i) The lower $qc$-density of $A$, denoted by $d_{qc}(A)$:

$$d_{qc}(A) := \lim \inf_{t \to \infty} \frac{m(A \cap \Delta_{tq})}{t}.$$

(ii) The lower $f$-density of $A$, denoted by $d_f(A)$:

$$d_f(A) := \lim \inf_{t \to \infty} \frac{m(A \cap \Delta_{f(t)})}{t}.$$

1.2 Weighted function spaces, translation semigroups, and semigroups induced by semiflows

A measurable function $\rho : \Delta \to (0, \infty)$ is said to be an admissible weight function iff there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$, such that $\rho(t) \leq Me^{\omega|t'|}\rho(t+t')$, $t, t' \in \Delta$. For such a function $\rho(\cdot)$, we introduce the following Banach spaces:

$$L^p_\rho(\Delta, K) := \{ u : \Delta \to K ; u(\cdot) \text{ is measurable and } ||u||_p < \infty \},$$

where $p \in [1, \infty)$, $||u||_p := \left( \int_{\Delta} |u(t)|^p \rho(t) \, dt \right)^{1/p}$, and:

$$C_{0,\rho}(\Delta, K) := \{ u : \Delta \to K ; u(\cdot) \text{ is continuous and } \lim_{t \to \infty} u(t)\rho(t) = 0 \},$$

with $||u|| := \sup_{t \in \Delta} |u(t)\rho(t)|$. The spaces $L^p_\rho(\Omega, K)$ and $C_{0,\rho}(\Omega, K)$ are defined in the same way. It is well known that $C_c(\Omega, K)$, the subspace of $C(\Omega, K)$, consisting of all compactly supported functions, is dense in $L^p_\rho(\Omega, K)$ and $C_{0,\rho}(\Omega, K)$. If the field $K$ is clearly determined, then we also write $C_c(\Omega)$ for $C_c(\Omega, K)$.

Assume that $\Delta' \in \{ [0, \infty), \mathbb{R} \}$ or $\Delta' = \Delta(\alpha')$ for an appropriate angle $\alpha' \in (0, \frac{\pi}{2}]$, and $\Delta' \subseteq \Delta$. Then, the translation semigroup $(T(t))_{t \in \Delta'}$, given by:

$$(T(t)f)(x) := f(x + t), \quad x \in \Delta, \ t \in \Delta',$$

is strongly continuous on $L^p_\rho(\Delta, K)$ and $C_{0,\rho}(\Delta, K)$.

Hypercyclic and chaotic $C_0$-semigroups induced by semiflows have been analyzed for the first time in [16]. We will use the following definition:
\textbf{Definition 4} ([16,19]; see also [23]) A continuous mapping $\varphi : \Delta \times \Omega \rightarrow \Omega$ is called a semiflow if $\varphi(0, x) = x$, $x \in \Omega$, $\varphi(t+s, x) = \varphi(t, \varphi(s, x))$, $t, s \in \Delta$, $x \in \Omega$ and $x \mapsto \varphi(t, x)$ is injective for all $t \in \Delta$.

Denote by $\varphi(t, \cdot)^{-1}$ the inverse mapping of $\varphi(t, \cdot)$, i.e., $y = \varphi(t, x)^{-1}$ iff $x = \varphi(y, t)$, $t \in \Delta$.

Given $t \in \Delta$, a semiflow $\varphi : \Delta \times \Omega \rightarrow \Omega$ and a function $f : \Omega \rightarrow \mathbb{K}$, we define $T_\varphi(t) f : \Omega \rightarrow \mathbb{K}$ by $(T_\varphi(t) f)(x) := f(\varphi(t, x))$, $x \in \Omega$. Then, $T_\varphi(0) f = f$, $T_\varphi(t) T_\varphi(s) f = T_\varphi(s) T_\varphi(t) f = T_\varphi(t+s) f$, $t, s \in \Delta$ and Brouwer’s theorem implies $C_c(\Omega) \subseteq T_\varphi(t) C_c(\Omega)$. Furthermore, we have the following lemma.

\textbf{Lemma 1} ([19])

(i) Suppose that $\varphi : \Delta \times \Omega \rightarrow \Omega$ is a semiflow and $\varphi(t, \cdot)$ is a locally Lipschitz continuous function for all $t \in \Delta$. Then, (b) implies (a), where:

(a) $(T_\varphi(t))_{t \in \Delta}$ is a $C_0$-semigroup in $L^p_{\rho_1}(\Omega)$ and

(b)  

$$\exists M, \omega \in \mathbb{R} : \rho_1(x) \leq M e^{\omega t} |\rho_1(\varphi(t, x))| \det D\varphi(t, x)| \text{ a.e. } t \in \Delta, x \in \Omega.$$

(1)

If, additionally, $\varphi(t, \cdot)^{-1}$ is locally Lipschitz continuous for every $t \in \Delta$, then (a) and (b) are equivalent.

(ii) Let $\varphi : \Delta \times \Omega \rightarrow \Omega$ be a semiflow. Then, $(T_\varphi(t))_{t \in \Delta}$ is a $C_0$-semigroup in $C_{0, \rho}(\Omega)$ iff the following holds:

(a)  

$$\exists M, \omega \in \mathbb{R} : \rho(x) \leq M e^{\omega t} |\rho(\varphi(t, x))|, t \in \Delta, x \in \Omega;$$

(b) for $K \subseteq \Omega$ and $\forall \delta > 0$, $\forall t \in \Delta$:

$$\varphi(t, \cdot)^{-1}(K) \cap \{x \in \Omega : \rho(x) \geq \delta\} \text{ is a compact subset of } \Omega.$$

(2)

\section{2 Generalized frequent hypercyclicity for $C_0$-semigroups on complex sectors}

We start our work by proposing the following general definition:

\textbf{Definition 5} Assume that $(T(t))_{t \in \Delta}$ is a $C_0$-semigroup on $E$, and $x \in E$. Let $\mathcal{F} \in P(P(\Delta))$ and $\mathcal{F} \neq \emptyset$. Then, it is said that $x$ is an $\mathcal{F}$-hypercyclic element (vector) of $(T(t))_{t \in \Delta}$ iff for each open non-empty subset $V$ of $E$, we have:

$$S(x, V) := \{t \in \Delta : T(t)x \in V\} \in \mathcal{F}.$$ 

It is said that $(T(t))_{t \in \Delta}$ is $\mathcal{F}$-hypercyclic iff there exists an $\mathcal{F}$-hypercyclic element of $(T(t))_{t \in \Delta}$, while $(T(t))_{t \in \Delta}$ is said to be hypercyclic iff it is hypercyclic with $\mathcal{F}$ being the collection of all non-empty subsets of $P(\Delta)$. 

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In the next lemma, we generalize Theorem 3.1.2 (i) of [19], with a completely different proof (see also [20, Theorem 2(i))].

**Lemma 2** Suppose that $(T(t))_{t \in \Delta}$ is a hypercyclic $C_0$-semigroup, $x \in E$ is a hypercyclic element of $(T(t))_{t \in \Delta}$ and $s > 0$. Then, the set $\{T(t)x : t \in \Delta \setminus \Delta_s\}$ is dense in $E$.

**Proof** Suppose that $x^* \in E^*$ and $\langle x^*, T(t)x \rangle = 0$ for $|t| \geq s$. Let $t_0 \in \Delta$ and $|t_0| < s$. Then, there exists a sufficiently large real number $k > 0$, such that $\|kT(t_0)x\| > \max_{t \in \Delta_s} \|T(t)x\|$. By this inequality and the fact that $x$ is a hypercyclic element of $(T(t))_{t \in \Delta}$, it readily follows that there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $\Delta \setminus \Delta_s$, such that $\lim_{n \to \infty} T(t_n)x = kT(t_0)x$, so that $0 = \lim_{n \to \infty} \langle x^*, T(t_n)x \rangle = \langle x^*, kT(t_0)x \rangle$. Therefore, $0 = \langle x^*, T(t_0)x \rangle$. Since $t_0$ was arbitrary, we get that $0 = \langle x^*, T(t)x \rangle$ for all $t \in \Delta$. This implies that $x^* = 0$, since $x^*$ equals zero on a dense set in $E$. This completes the proof. \hfill \Box

In the sequel, we will always denote by $f$ an increasing function $f : [0, \infty) \to [1, \infty)$. Also, $(t_n)_{n \in \mathbb{N}}$ will always denote a sequence in $\Delta \setminus \{0\}$ and $m_k = f(k), k \in \mathbb{N}$. We will use the following special cases of Definition 5 (compared with discrete case, we have made a choice to add the word “frequently” to denote new terms):

**Definition 6** Let $q \in [1, \infty)$, and $(T(t))_{t \in \Delta}$ be a $C_0$-semigroup. Then, it is said that:

(i) $(T(t))_{t \in \Delta}$ is $q$-frequently hypercyclic iff there exists $x \in E$ ($q$-frequently hypercyclic vector of $(T(t))_{t \in \Delta}$), such that for each open non-empty subset $V$ of $E$, we have $d_{qc}(\{t \in \Delta : T(t)x \in V\}) > 0$;

(ii) $(T(t))_{t \in \Delta}$ is $f$-frequently hypercyclic iff there exists $x \in E$ ($f$-frequently hypercyclic vector of $(T(t))_{t \in \Delta}$), such that for each open non-empty subset $V$ of $E$, we have $d_{f}(\{t \in \Delta : T(t)x \in V\}) > 0$.

Recall that, in the case that $q \geq 1$ and $f(t) := t^q + 1, t \geq 0$, an $f$-frequently hypercyclic $C_0$-semigroup $(T(t))_{t \in \Delta}$ is also said to be $q$-frequently hypercyclic, while the usual frequent hypercyclicity is obtained by plugging $q = 1$.

Concerning the notion introduced in Definition 5 (and therefore, in Definition 6), we have the following conjugacy lemma; the proof is very simple and, therefore, omitted.

**Lemma 3** Assume that $(T(t))_{t \in \Delta}$ is a $C_0$-semigroup on $E$, and $x \in E$. Let $\mathcal{F} \in P(P(\Delta))$ and $\mathcal{F} \neq \emptyset$. Suppose that $F$ is a separable infinite-dimensional Fréchet space over the same field of scalars as $E$, and $(S(t))_{t \in \Delta}$ is a $C_0$-semigroup on $F$. If $\Psi : E \to F$ is a continuous mapping with dense range, $x$ is an $\mathcal{F}$-hypercyclic vector of $(T(t))_{t \in \Delta}$ and $\Psi \circ T(t) = S(t) \circ \Psi$ for all $t \geq 0$, then $\Psi x$ is an $\mathcal{F}$-hypercyclic vector of $(S(t))_{t \in \Delta}$.

Suppose now that $\mathcal{F} \in P(P(\mathbb{N}))$ and $\mathcal{F} \neq \emptyset$. If $\mathcal{F}$ satisfies

(I): $A \in \mathcal{F}$ and $A \subseteq B$ imply $B \in \mathcal{F}$,

then it is said that $\mathcal{F}$ is a Furstenberg family; a proper Furstenberg family $\mathcal{F}$ is any Furstenberg family satisfying that $\emptyset \notin \mathcal{F}$ (see [11] for more details).

Since the $C_0$-semigroup $(T(t))_{t \in \Delta}$ under our consideration is locally equicontinuous, we can argue as in the proof of [26, Proposition 2.1], where $\Delta = [0, \infty)$, to deduce the following result:

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Proposition 1 Let $\mathcal{F}$ be a Furstenberg family and $(T(t))_{t \in \Delta}$ be a $C_0$-semigroup on $E$. If $x \in E$ is an $\mathcal{F}$-hypercyclic element of a sequence $(T(t_n))_{n \in \mathbb{N}}$, then $x$ is an $\mathcal{F}'$-hypercyclic element of $(T(t))_{t \in \Delta}$, where:

$$\mathcal{F}' = \left\{ B \subseteq \Delta : (\exists A \in \mathcal{F}) \left( \exists \delta_0 > 0 \right) \bigcup_{k \in A} [B(t_k, \delta_0) \cap \Delta] \subseteq B \right\}.$$ 

Furthermore, if there exists a number $t_0 \in \Delta \setminus \{0\}$, such that $t_n = n t_0$ for all $n \in \mathbb{N}$, then $x$ is an $\mathcal{F}''$-hypercyclic element of $(T(\epsilon^{i \text{arg}(t_0)}))_{t \geq 0}$, where:

$$\mathcal{F}'' = \left\{ B \subseteq [0, \infty) : (\exists A \in \mathcal{F}) \left( \exists \delta_0 > 0 \right) \bigcup_{k \in A} [k|t_0|, k|t_0| + \delta_0] \subseteq B \right\}.$$ 

For the sake of completeness, we give a sketch of the proof of the next proposition proved in [26] (Proposition 2.1) for $\Delta = [0, \infty)$. (Recall that $m_k = f(k)$.)

Proposition 2 Suppose that $(T(t))_{t \in \Delta}$ is a $C_0$-semigroup on $E$, and that $(t_n)_{n \in \mathbb{N}} \subseteq \Delta \setminus \{0\}$ is a given sequence, such that there exists $\delta > 0$ and for $m \neq n$ holds:

$$|t_n - t_m| \geq \delta. \quad (3)$$ 

Suppose that $g : [1, \infty) \rightarrow [1, \infty)$ is an increasing mapping and $|t_n| \leq g(n)$ for all $n \in \mathbb{N}$. If $x \in E$ is an $l$-$(m_n)$-hypercyclic element of sequence $(T(t_n))_{n \in \mathbb{N}}$, then $x$ is a $(g \circ f)$-frequently hypercyclic element of $(T(t))_{t \in \Delta}$. Furthermore:

(i) If $x$ is a frequently hypercyclic element of sequence $(T(t_n))_{n \in \mathbb{N}}$ ($f(t) = t + 1, t \geq 0$), then $x$ is a $g$-frequently hypercyclic element of $(T(t))_{t \in \Delta}$.

(ii) Let $q \geq 1$ and $x$ be a $q$-frequently hypercyclic element of sequence $(T(t_n))_{n \in \mathbb{N}}$ ($f(t) = t^q + 1, t \geq 0$). Then, $x$ is a $(g \circ (\cdot + 1))$-frequently hypercyclic element of $(T(t))_{t \in \Delta}$.

(iii) If there exists $t_0 \in \Delta \setminus \{0\}$, such that $t_n = n t_0$, $n \in \mathbb{N}$ and $x \in E$ is a frequently hypercyclic ($q$-frequently hypercyclic, $l - (m_n)$-hypercyclic) element of sequence $(T(t_n))_{n \in \mathbb{N}}$, then $x$ is a frequently hypercyclic ($q$-frequently hypercyclic, $f$-hypercyclic) element of $(T(t))_{t \in \Delta}$ and $(T(\epsilon^{i \text{arg}(t_0)}))_{t \geq 0}$.

Proof Let $x \in E$ be an $l$-$(m_n)$-hypercyclic element of sequence $(T(t_n))_{n \in \mathbb{N}}$. To prove that $x$ is an $f$-frequently hypercyclic element of $(T(t))_{t \in \Delta}$, let an open non-empty set $V$ of $E$ be given. The prescribed assumptions yield the existence of $c > 0$ and a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$, such that the interval $[1, m_{n_k}]$ contains at least $\lfloor cn_k \rfloor$ elements of set $\{ n \in \mathbb{N} : T(t_n)x \in V \}$, say $l_1, \ldots, l_{\lfloor cn_k \rfloor}$. Hence, $T(t_{l_1})x \in V, \ldots, T(t_{l_{\lfloor cn_k \rfloor}})x \in V, |l_j| \leq g(m_{n_k}) = (g \circ f)(n_k)$ for $1 \leq j \leq \lfloor cn_k \rfloor$ and $|l_j - l_{j'}| \geq \delta$, provided $l_j \neq l_{j'}$ and $1 \leq j, j' \leq \lfloor cn_k \rfloor$. Arguing similarly as in the proof of [26, Proposition 2.1], the strong continuity and the local equicontinuity of $(T(t))_{t \in \Delta}$ together imply the existence of a positive real number $\delta \in (0, 1)$, such that $T(t)x \in V$ for any $t \in \bigcup_{1 \leq j \leq \lfloor cn_k \rfloor} B(t_j, \delta)$. This implies the first conclusion. The parts (i)–(ii) simply follow from this fact, while the proof of (iii) can be deduced using (i)–(ii) and Proposition 1.
We continue by stating the following existence type result (a similar statement can be formulated for a single operator, as well):

**Proposition 3** Assume that \((T(t))_{t \in \Delta}\) is a hypercyclic \(C_0\)-semigroup. Then, there exists an increasing mapping \(f : [0, \infty) \to [1, \infty)\), such that \((T(t))_{t \in \Delta}\) is \(f\)-frequently hypercyclic and, for every open non-empty subset \(V\) of \(E\), we have:

\[
d_f(\{t \in \Delta : T(t)x \in V\}) = +\infty.
\] (4)

**Proof** Let \((O_n)_{n \in \mathbb{N}}\) be a base of the topology on \(E\), where \(O_n \neq \emptyset\) for all \(n \in \mathbb{N}\). Assume that \(x \in E\) is a hypercyclic vector of \((T(t))_{t \in \Delta}\). Using Lemma 2 as well as the hypercyclicity and strong continuity of \((T(t))_{t \in \Delta}\), we can prove that for each \(n \in \mathbb{N}\), there exists a sequence \((t_{n,k})_{k \in \mathbb{N}}\) in \(\Delta\), such that \(T(t_{n,k})x \in O_n\) for all \(k \in \mathbb{N}\) as well as that \(|t_{n,k} - t_{n',k'}| \geq 1\) for all \((n, k), (n', k') \in \mathbb{N}^2\), such that \((n, k) \neq (n', k')\). Define \(F : \mathbb{N}^2 \to \mathbb{N}\) by:

\[
F(n, k) := \frac{(n + k - 2)(n + k - 1)}{2} + n, \ (n, k) \in \mathbb{N}^2.
\]

Then, \(F(\cdot, \cdot)\) is a bijection together with its inverse mapping \(F^{-1} : \mathbb{N} \to \mathbb{N}^2\). We define \(f_0 : [0, \infty) \to [1, \infty)\) by \(f_0(x) := f_0(\lceil x \rceil)\) for all \(x \geq 0\), where the sequence \((f_0(N))_{N \in \mathbb{N}_0}\) is defined inductively as follows: Set \(f_0(0) := 1\). For any \(N \in \mathbb{N}\), we define \(f_0(N)\), such that \(f_0(N) > f_0(N - 1) + 1 + \max_{1 \leq j \leq N, 1 \leq k \leq N^2} F(n, k)\). It is clear that there exists an increasing mapping \(g : [1, \infty) \to [1, \infty)\), such that \(|t_{F^{-1}(N)}| \leq g(N)\) for all \(N \in \mathbb{N}\). Set \(f := g \circ f_0\). We will prove that \((T(t))_{t \in \Delta}\) is \(f\)-frequently hypercyclic and that (4) holds by applying Proposition 2 with the sequence \((T(t_N))_{N \in \mathbb{N}}\), where \(t_N := t_{F^{-1}(N)}\) for all \(N \in \mathbb{N}\). Evidently, condition (3) holds with \(\delta = 1\) and \(|t_N| \leq g(N)\) for all \(N \in \mathbb{N}\). Let \(V\) be an open non-empty subset \(E\) of \(E\). It is sufficient to prove that \(x\) is an \(l-(m_N)\)-hypercyclic vector of the sequence \((T(t_N))_{N \in \mathbb{N}}\), as well as that for each \(n \in \mathbb{N}\), there exists a strictly increasing sequence \((N^k_n)_{k \in \mathbb{N}}\) of positive integers, such that the interval \([1, f(N^k_n)]\) contains at least \((N^k_n)^2\) positive integers \(m \in \mathbb{N}\), such that \(F^{-1}(m) = (n, k)\) for some \(k \in \mathbb{N}\). However, this follows with the sequence \((N^k_n = F(n, k))_{k \in \mathbb{N}}\), using the fact that \(f(N) > F(n, N^2)\) for \(1 \leq n \leq N\).

The following result (case (a)) is a simple consequence of [21, Theorem 2.5], for \(\Delta = [0, \infty)\). The proof for general region \(\Delta\) is similar.

**Theorem 1** Assume that \((T(t))_{t \in \Delta}\) is a \(C_0\)-semigroup on \(E\). Further on, assume that there exists a sequence of mappings \(S_n : E_0 \to E, n \in \mathbb{N}\), where \(E_0\) is dense in \(E\), such that the series \(\sum_{n=1}^{\infty} S_{[m_n]}y\) converges unconditionally for every \(y \in E_0\). We consider the following cases:

(a) \(\tau_k = t_{0 | m_k|}, k \in \mathbb{N}\) for some \(t_0 \in \Delta \setminus [0]\). (Recall, once more, that \(m_k = f(k)\).)

(b) \(\tau_k = t_{[m_k]}, k \in \mathbb{N}\), \((t_n)_{n \in \mathbb{N}}\) is a sequence in \(\Delta \setminus [0]\), such that (3) holds for some \(\delta > 0\). In this case, we assume that there exists \(g : [1, \infty) \to [1, \infty)\), such that \(|t_n| \leq g(n)\), \(n \in \mathbb{N}\).
Assume that for \( \tau_k \) defined as in (a) and (b) the following three conditions hold, for every \( y \in E_0 \) (condition (iii) is essential):

(i) The series \( \sum_{n=1}^{k} T(\tau_k)S_{[m_k-n]}y, k \in \mathbb{N} \) converges unconditionally, uniformly in \( k \in \mathbb{N} \).

(ii) The series \( \sum_{n=1}^{\infty} T(\tau_k)S_{[m_k+n]}y \) converges unconditionally, uniformly in \( k \in \mathbb{N} \).

(iii) \( \lim_{n \to \infty} T(\tau_k)S_{\lfloor m_n \rfloor}y = y, y \in E_0. \)

Then:

In case (a), \( (T(t))_{t \in \Delta} \) is \( f \)-frequently hypercyclic, \( (T(te^{i\arctg(t_0)})_{t \geq 0} \) is \( f \)-frequently hypercyclic, and the operator \( T(t_0) \) is \( l-(m_k) \)-frequently hypercyclic.

In case (b), the sequence \( (T_n := T(t_n))_{n \in \mathbb{N}} \) is \( l-(m_k) \)-frequently hypercyclic and \( (T(t))_{t \in \Delta} \) is \( (g \circ f) \)-frequently hypercyclic.

**Proof** By [22, Theorem 3.1], we easily infer in case (b) that the sequence \( (T_n)_{n \in \mathbb{N}} \) is \( l-(m_k) \)-frequently hypercyclic. Then, the final statement follows from Proposition 2.

Note that in the case (a), by setting \( f(t) := t^q + 1, t \geq 0 (q \geq 1) \), we obtain a sufficient condition for the \( q \)-frequent hypercyclicity of \( (T(t))_{t \in \Delta} \), \( (T(te^{i\arctg(t_0)})_{t \geq 0} \) and \( T(t_0) \).

Now, we give an \( f \)-frequent hypercyclicity criterion for \( C_0 \)-semigroups on complex sector (with the stated notation for \( f \) and \( m_k = f(k) \)).

**Theorem 2** Assume that \( (T(t))_{t \in \Delta} \) is a \( C_0 \)-semigroup on \( E \). Set \( A_{k,n}^f := \lfloor m_k+n \rfloor - \lfloor m_k \rfloor - 1, [m_k+n] - [m_k] \rangle (k, n \in \mathbb{N}) \) and \( B_{k,n}^f := \lfloor m_k \rfloor - \lfloor m_k - n \rfloor, \lfloor m_k \rfloor - \lfloor m_k - n \rfloor + 1 \rangle (k, n \in \mathbb{N}, k \geq n) \). Suppose that there exists a dense subset \( E_0 \) of \( E \), such that \( T(t)E_0 \subseteq E_0, t \in \Delta \) and mappings \( S(t) : E_0 \to E_0 \) (\( \Delta \)), such that:

(i) \( T(t)S(t)x = x, x \in E_0, t \in \Delta \) and \( S(r)T(t)x = T(t)S(r)x = S(r-t)x, x \in E_0 \) if \( r, t \in \Delta \setminus [0], \arctg(r) = \arctg(t) \) and \( |r| > |t| \).

(ii) For every \( t_0 \in \Delta \setminus [0] \) and \( s \geq 0 \), the sequence \( \sum_{n=1}^{k} \int_{x+B_{k,n}^f} T(tt_0e^{i\arctg(t_0)})x dt, n \in \mathbb{N} \) converges unconditionally, uniformly in \( k \in \mathbb{N} \), for every \( x \in E_0 \).

(iii) For every \( t_0 \in \Delta \setminus [0] \), the mapping \( t \mapsto S(tt_0e^{i\arctg(t_0)})x, t \in A_{k,n}^f \) is Bochner integrable for \( k, n \in \mathbb{N}, x \in E_0 \), and the series \( \sum_{n=1}^{k} \int_{A_{k,n}^f} S(tt_0e^{i\arctg(t_0)})x dt \) converges unconditionally for \( x \in E_0 \), uniformly in \( k \in \mathbb{N} \).

Then, \( (T(t))_{t \in \Delta} \) is \( f \)-frequently hypercyclic and, for every \( t_0 \in \Delta \setminus [0] \), \( (T(te^{i\arctg(t_0)})\) is \( f \)-frequently hypercyclic and the operator \( T(t_0) \) is \( l-(m_k) \)-frequently hypercyclic.

**Proof** We will include the most relevant details of proof, which is similar to that of [26, Proposition 2.8]. Let a number \( t_0 \in \Delta \setminus [0] \) be fixed. First, we will prove that the range of operator \( I - T(t_0) \) is dense in \( E \). Assume that \( x^* \in E^* \) and \( \langle x^*, x - T(t_0)x \rangle = 0, x \in E \). Using the semigroup property, this inductively implies:

\[
\langle x^*, x - T(nt_0)x \rangle = 0, \quad x \in E, \ n \in \mathbb{N}. \tag{5}
\]
Fix $s \geq 0$. The continuity of $x^*$ and unconditional convergence of the series:
\[ \sum_{n=1}^{k} \int_{s+B_{k/n}^i}^{s+B_{k/n}^i} T(t_0 e^{it_0 \phi}) \, dt, \]
uniformly in $k \in \mathbb{N}$, imply that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that:
\[ \left| \int_{s+B_{k/n}^i}^{s+B_{k/n}^i} \langle x^*, T(r)x \rangle \, dr \right| \leq \epsilon, \quad k \in \mathbb{N}, k \geq N. \]

Due to (5), we have:
\[ \int_{s+B_{k/n}^i}^{s+B_{k/n}^i} \langle x^*, T(r)x \rangle \, dr = \int_{0}^{s} \langle x^*, T(r)x \rangle \, dr = 0, \quad x \in E, \]
so that $\int_{0}^{s} \langle x^*, T(r)x \rangle \, dr = 0$, $x \in E$, $s \geq 0$, which yields $x^* = 0$. Since the integral generator of the semigroup $(T(t e^{i \phi}))_{t \geq 0}$ is dense in $E$ (see, e.g., [18]), we can proceed as in the proof of [26, Theorem 2.2] to deduce that the set $E'_0 := \{ \int_{0}^{s} T(t)x \, dt : x \in E_0 \}$ is dense in $E$. For any $x \in E_0$, we have $y = \int_{0}^{s} T(t)x \, dt \in E'_0$ and, similarly as in the proof of [26, Proposition 2.8], the condition (i) and an elementary argumentation show that:
\[ \sum_{n=1}^{k} T(t_0 [m_k]) S[m_k-n] \cdot y = t_0 e^{i \phi} \sum_{n=1}^{k} \int_{[m_k] - [m_k-n]}^{[m_k] - [m_k-n]+1} T(t_0 e^{i \phi}) \, dt, \]
\[ \sum_{n=1}^{\infty} T(t_0 [m_k]) S[m_k+n] \cdot y = t_0 e^{i \phi} \sum_{n=1}^{\infty} \int_{[m_k-n]}^{[m_k-n]-[m_k]} S(t_0 e^{i \phi}) \, dt \]
and
\[ \sum_{n=1}^{\infty} S[m_n] \cdot y = t_0 e^{i \phi} \sum_{n=1}^{\infty} \int_{[m_n]}^{[m_n]-1} S(t_0 e^{i \phi}) \, dt. \]
The result now follows by applying Theorem 1, case (a).

We prove now the extension of [26, Proposition 2.1] for $f$-frequently hypercyclic semigroups on Fréchet spaces (we refer to [6] for the ideas for the proof).

**Theorem 3** Let $(T(t))_{t \geq 0}$ be a $C_0$-semigroup on $E$. Suppose that:
\[ \liminf_{k \to \infty} \frac{m_k}{k} > 0. \]  
(6)

Then, the following statements are equivalent:

(i) $(T(t))_{t \geq 0}$ is $f$-frequently hypercyclic;
(ii) for every $t > 0$, the operator $T(t)$ is $l$-$(m_k)$-frequently hypercyclic;
(iii) there exists $t > 0$, such that the operator $T(t)$ is $l$-$(m_k)$-frequently hypercyclic.
Proof The implication (ii) $\Rightarrow$ (iii) is trivial, while the implication (iii) $\Rightarrow$ (i) follows from an application of Proposition 2. Let a number $t_0 > 0$ be given and let (i) hold. All that we need to show is that the operator $T(t_0)$ is $l$-(m_k)-frequently hypercyclic. Without loss of generality, we may assume that $R(f) \subseteq \mathbb{N}$. Let $x \in E$ be an $f$-frequently hypercyclic vector of $(T(t))_{t \geq 0}$. Fix $k \in \mathbb{N}, y \in E$ and two open-balanced neighborhoods of the origin $U$ and $U'$ in $E$, such that $U' + U' \subseteq U$. Then, due to [6, Theorem 2.3], $T(t_0 j/k)x$ is a hypercyclic vector of $T(t_0)$ for $j = 0, 1, 2, \ldots, k - 1$, which means that there exist positive integers $n_j$, such that $T(n_j t_0 + t_0(j/k))x \in y + U'$ for $j = 0, 1, 2, \ldots, k - 1$. On the other hand, $(T(t))_{t \geq 0}$ is strongly continuous, so that there exists an open-balanced neighborhood of the origin $V$ in $E$, such that $T(s)(V) \subseteq U'$, for $s \leq N_0 = \max\{n_j : j = 0, 1, 2, \ldots, k - 1\} + \max(1, t_0)$. By our assumption, there are numbers $C > 0$ and $N_1 \in \mathbb{N}$, such that $m(\{t \in [0, m_N] : T(t)x - x \in V\}) \geq CN$, for every $N \geq N_1$. For every $N \in \mathbb{N}$, we define $L := \{t \in [0, m_N] : T_l x - x \in V\}$. Furthermore, for every $j \in \mathbb{N}$, we define:

$$I_j := \bigcup_{n \in \mathbb{N}, m \in \mathbb{N}_0} \left[ t_0 m_n + t_0 s + \frac{j}{k} t_0, t_0 m_n + t_0 s + \frac{j-1}{k} t_0 \right], \quad L_j := I_j \cap L$$

and the mappings $f_j : [0, \infty) \to [0, \infty)$ by $f_j(t) := t + n_k j t_0 + \frac{k-j-1}{k} t_0$. Then, it can be easily seen that $f_j(t) \in I_{k-1}$ and $T(f_j(t))x \subseteq U$ for every $t \in L_j$. Hence, we have:

$$\begin{align*}
m\left( \left\{ t \in [0, m_N + N_0 + 1] : T_l x - y \in U \text{ and } t \in I_{k-1} \right\} \right) \\
\geq m\left( \bigcup_{j=0}^{k-1} f_j(L_j) \right) \geq \sum_{j=0}^{k-1} \frac{m(f_j(L_j))}{k} \\
= \sum_{j=0}^{k-1} \frac{m(L_j)}{k} \geq \frac{m_N - m_1}{k}.
\end{align*}$$

Using (6) and this estimate, it readily follows that:

$$\liminf_{N \to \infty} \frac{m(\{m_N \geq t \geq 0 : T(t)x - y \in V, \ t \in I_{k-1}\})}{N} > 0.$$
real number $t_2 \in [0, m_{N_l}]$, such that $m_{n_2} + s_2 \neq m_{n_1} + s_1$, $T(t_2)x - y \in V$ and $t_2 \in [t_0 m_{n_2} + t_0 s_2 + \frac{k}{k-1} t_0, t_0 m_{n_2} + t_0 s_2 + t_0)$; otherwise, the Lebesgue measure of set $m(m_{N_l} \geq t \geq 0 : T(tx) - y \in V$, $t \in I_{k-1})$ cannot be greater than $t_0/k$. Set $t'_2 := t_0 (m_{n_2} + s_2 + 1)$. Then, $t'_2 = t'_1$ and $T(t'_2)x \in y + U$, as well. Proceeding in such a way, we may construct $[c'N_l]$ positive numbers $t'_1, \ldots, t'_1 \in [0, m_{N_l}]$, such that $T(t'_j)x \in y + U$ for every $j = 1, \ldots, [c'N_l]$, where $c' > 0$ is a certain positive constant. This implies that $x$ is an $l-(m_k)$-frequently hypercyclic vector of $T(t_0)$, finishing the proof of theorem. \hfill \Box

Note that our result is optimal, since the estimate (6) is necessary for the operator $T(1)$ to be $l-(m_k)$-frequently hypercyclic; it is also clear that (6) holds iff

$$\liminf_{x \to +\infty} \frac{f(x)}{x} \geq 0,$$

which is a necessary condition for $(T(t))_{t \geq 0}$ to be $f$-frequently hypercyclic. By the proofs of Proposition 2 and Theorem 3, we may conclude that, for a given element $x \in E$, we have that $x$ is an $f$-frequently hypercyclic vector of $(T(t))_{t \geq 0}$ iff $x$ is an $l-(m_k)$-frequently hypercyclic vector of $T(t)$ for all $t > 0$ iff $x$ is an $l-(m_k)$-frequently hypercyclic vector of $T(t)$ for some $t > 0$.

Using this fact as well as the method proposed in the proof of Proposition 3, we may deduce the following:

**Proposition 4** Suppose that $(T(t))_{t \geq 0}$ is a hypercyclic $C_0$-semigroup on $E$. Then, there exist $x \in E$ and a strictly increasing sequence $(m_n)_{n \in \mathbb{N}}$ of positive integers, such that for each open non-empty subset $V$ of $E$ and for each $t > 0$, one has $d_{m_n} (\{ k \in \mathbb{N} : T(kt) \in V \}) = +\infty$.

**Remark 1** Since the assertions of [26, Theorem 4.3, Corollary 4.4] remain true in the setting of Fréchet spaces (see e.g. [29, pp. 281–282, 291]) and since the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Fréchet space is densely defined, the proof of [26, Theorem 2.2] can be repeated verbatim to see that this statement holds true for $C_0$-semigroups in infinite-dimensional separable Fréchet spaces. The same holds with the statements of [26, Propositions 2.7 and 2.8].

### 2.1 Examples I

We reconsider several illustrative examples proposed in [7,8,19].

(i) ([7, Example 2]) Let $\Delta := \Delta(\pi/4)$, let

$$\rho(x + iy) := \begin{cases} 1, & x + y \geq \sqrt{x - y}, \\ e^{x+y-\sqrt{x-y}}, & \text{otherwise}, \end{cases}$$

and let $E := L^2_{\rho}(\Delta, \mathbb{R})$. Then, we know that the translation semigroups $(T(t))_{t \in \Delta}$ and $(T(te^{-i\pi/4}))_{t \geq 0}$ are chaotic as well as that any period of $(T(t))_{t \in \Delta}$ lies on the boundary of $\Delta$. We will prove that the translation semigroups $(T(t))_{t \in \Delta}$ and $(T(te^{-i\pi/4}))_{t \geq 0}$ are frequently hypercyclic as well as that any semigroup $(T(te^{i\varphi}))_{t \geq 0}$, where $-\pi/4 < \varphi \leq \pi/4$, is not hypercyclic.
To see that \((T(te^{-iπ/4}))_{t≥0}\) is frequently hypercyclic, we will apply the frequent hypercyclicity criterion for \(C₀\)-semigroups [26, Theorem 2.2]. Let \(E₀\) denote a set consisting of all continuous functions \(f : Δ → ℝ\) with compact support; then \(E₀\) is dense in \(E\) (terminology “\(f\)-frequent hypercyclicity” is a little bit misleading, since \(f\) also denotes a function in the underlying function space, but the use of symbol \(f\) will be clear from the context henceforth). Define \(S(t)f : Δ → ℝ\) by \((S(t)f)(x + iy) := f(x + iy - te^{-iπ/4})\), \(x + iy \in te^{-iπ/4} + \text{supp}(f)\) and \((S(t)f)(x) := 0\), otherwise \((f ∈ E₀, n ∈ ℕ)\). Then, \(T(t)S(t)f = f, f ∈ E₀\) and \(T(t)S(r)f = S(r - t)f, f ∈ E₀, r > t > 0\), so that it suffices to show that the mappings \(t ↦ T(t)f, t ≥ 0\), and \(t ↦ S(t)f, t ≥ 0\) are Bochner integrable. This is clear for the first mapping, because, for every \(f ∈ E₀\), there exists \(t₀ > 0\), such that \(T(t)f = 0\) for all \(t ≥ t₀\). To see that the mapping \(t ↦ S(t)f, t ≥ 0\) is Bochner integrable for any \(f ∈ E₀\) with \(\text{supp}(f) = K\), observe first that:

\[
\int_0^∞ \|S(t)f\| \, dt \leq ∫_0^∞ \left(∫_K ρ(x + iy + te^{-iπ/4}) \, dx \, dy\right)^{1/p} \, dt.
\]

It is clear that there exist \(t₀ > 0\) and \(c₀ > 0\), such that \(x + y < \sqrt{x - y + t^2}\) for all \(t ≥ t₀\). Hence:

\[
\int_0^∞ \|S(t)f\| \, dt \leq ∫_0^∞ ∫_0^{t₀} \left(∫_K ρ(x + iy + te^{-iπ/4}) \, dx \, dy\right)^{1/p} \, dt
\]

\[
+ ∫_0^∞ ∫_0^{t₀} \left(∫_K ρ(x + iy + te^{-iπ/4}) \, dx \, dy\right)^{1/p} \, dt
\]

\[
≤ c₀\|f\|_∞ + c₀\|f\|_∞ ∫_0^∞ \left(∫_K e^{-\sqrt{t^2 + 2}} \, dx \, dy\right)^{1/p} \, dt < ∞,
\]

proving the claim. The frequent hypercyclicity of \((T(t))_{t∈Δ}\) follows by applying [26, Proposition 2.1] and Proposition 2. If \(-π/4 < ϕ ≤ π/4\), then there exists a finite constant \(t_ϕ > 0\), such that:

\[
t(\cos ϕ + \sin ϕ) ≥ \sqrt{t(\cos ϕ - \sin ϕ)}, \quad t ≥ t_ϕ,
\]

so that

\[
ρ(te^{iϕ}) = 1, \quad t ≥ t_ϕ.
\]

(7)

Assume that the semigroup \((T(te^{iϕ}))_{t≥0}\) is hypercyclic on \(E\). Repeating literally the argumentation given in the first part of proof of [10, Theorem 4.7], with appealing to [8, Lemma 4.2] in place of [10, Lemma 4.2], we obtain that there exists an increasing sequence \((tₙ)_{n∈ℕ}\) of positive reals tending to infinity, such that \(\lim_{n→∞} ρ(tₙe^{iϕ}) = 0\). This is in contradiction with (7).
(ii) ([8, Example 4.17]) Let $\Delta := \Delta(\pi/2)$, let $\rho(x + iy) := e^{-x + |y|}$ for $x + iy \in \Delta$, and let $E := L_p(\rho, \mathbb{R})$. Applying the frequent hypercyclicity criterion for $C_0$-semigroups, it can be easily seen that for each $\varphi \in (-\pi/4, \pi/4)$ the semigroup $(T(\rho e^{i\varphi}))_{t \geq 0}$ is frequently hypercyclic on $E$, so that the frequent hypercyclicity of $(T(t))_{t \in \Delta}$ follows again by applying [26, Proposition 2.1] and Proposition 2. If $|\varphi| \geq \pi/4$, then the semigroup $(T(\rho e^{i\varphi}))_{t \geq 0}$ cannot be hypercyclic, which can be established by the arguments given in the part (i) and the fact that there thus not exist a sequence $(t_n)_{n \in \mathbb{N}}$ of positive reals tending to infinity, such that $\lim_{n \to \infty} \rho(t_ne^{i\varphi}) = 0$.

(iii) We will show that there exists a frequently hypercyclic semigroup with the index set $\Delta(\pi/4)$ and without any (frequently) hypercyclic single operator.

This example is based on [8, Example 4.14]. Let $\Delta := \Delta(\pi/4)$, $\zeta \geq 1$, let

$$\rho(x + iy) := \begin{cases} 
\frac{(x+y+1)}{(x-y+1)}^{2\zeta}, & x + y + 1 \geq \sqrt{x-y+1}, \\
\frac{1}{x+y+1}^{2\zeta}, & x + y + 1 < \sqrt{x-y+1},
\end{cases}$$

and let $E := L_p^\rho(\Delta, \mathbb{K})$. Then, $\rho(\cdot)$ is an admissible weight function and, arguing as in the aforementioned example, we can prove that $(T(t))_{t \in \Delta}$ is hypercyclic and that there is no number $t_0 \in \Delta \setminus \{0\}$, such that $T(t_0)$ is hypercyclic.

We apply Theorem 1, case (b) with $E_0$ being the space of continuous compactly supported functions, the sequence $T_n = T(t_n)$ with:

$$t_n := n + i \frac{-3 - 2n + \sqrt{8n + 9}}{2}, \quad n \in \mathbb{N},$$

and mappings $S_n f : \Delta \to \mathbb{K}$ given by $(S_n f)(x) := f(x - t_n), x \in t_n + \text{supp}(f)$ and $(S_n f)(x) := 0$, otherwise $(f \in E_0, n \in \mathbb{N})$. To do that, let us observe first that the condition of Theorem 1, holds with a certain positive number $\delta > 0$ and an increasing mapping $g(\cdot)$ with $g(n) \sim n\sqrt{2}$ as $n \to \infty$. Fix now a function $f \in E_0$ with $\text{supp}(f) = K$ and observe first that for each $x + iy \in \Delta$, we have the following equivalence relation: $x + y + 1 \geq \sqrt{x-y+1}$ iff $y \leq \frac{-3 - 2x + \sqrt{8x + 9}}{2}$. Consider first the case $p = 1$. Then, $(T_{t_k} S_{n+k} f)(x) = f(x - t_{k+n} + t_k), x \in t_{k+n} - t_k + \text{supp}(f)$ and $(T_{t_k} S_{n+k} f)(x) = 0$, otherwise $(k, n \in \mathbb{N})$. Furthermore, it can be easily seen that:

$$\sum_{n=1}^{\infty} \|T_{t_n} S_{n+k} f\| \leq \|f\|_\infty \sum_{n=1}^{\infty} \int_{t_{n+k} - t_k + K} \rho(x + iy) \, dx \, dy$$

$$\leq \|f\|_\infty \sum_{n=1}^{\infty} \int_{t_{n+k} - t_k + K} \left(\frac{1}{x + y + 1}\right)^{2\zeta} \, dx \, dy.$$
\[
= \| f \|_\infty \sum_{n=1}^{\infty} \int_k \left( \frac{1}{x + y + 1 + \sqrt{8(n + k) + 9} - \sqrt{8k + 9}} \right)^{2\zeta} dx \, dy
\]

so that \( \sum_{n=1}^{\infty} T(t_k)S_{k+n}f \) converges absolutely, uniformly in \( k \in \mathbb{N} \). Similarly, \( \sum_{n=1}^{k} T(t_k)S_{k-n}f \) converges absolutely, uniformly in \( k \in \mathbb{N} \), and \( \sum_{n=1}^{\infty} S_n f \) converges absolutely. Since \( T(t_k)S_k f = f \), the claimed assertion follows. Suppose now that \( p > 1 \). Then, \( E \) does not contain the space \( c_0 \) and it suffices to prove that for each functional \( \psi \in L^p_p(\Delta, \mathbb{K}) \), where \( p' \in (1, \infty) \) satisfies \( 1/p + 1/p' = 1 \), we have that the series \( \sum_{n=1}^{k} |\langle \psi, T(t_k)S_{k-n}f \rangle| \), \( \sum_{n=1}^{\infty} |\langle \psi, T(t_k)S_{k+n}f \rangle| \) and \( \sum_{n=1}^{\infty} |\langle \psi, S_n f \rangle| \) unconditionally converge, the first two of them uniformly in \( k \in \mathbb{N} \). This can be deduced as in the final part of proof of [26, Proposition 3.3]. Observe that \((T(t))_{t \in \Delta}\) is \( q \)-frequently hypercyclic for any \( q > 1 \), provided that \( \zeta = 1 \), and that it is not clear whether \((T(t))_{t \in \Delta}\) is frequently hypercyclic in this case.

Finally, we note, without going into full details, that the trick used above in combination with the analysis contained in [8, Example 4.15] enables one to simply construct an example of translation semigroup

\[(T(t))_{t \in \Delta(\pi/4)}\] satisfying the following conditions: For any ray \( R \), the semigroup \((T(t))_{t \in R}\) is both topologically mixing and frequently hypercyclic, but the whole semigroup is \((T(t))_{t \in \Delta(\pi/4)}\) which is frequently hypercyclic and not topologically mixing (see [8] for the notion).

(iv) Let \( \Delta = \mathbb{C} \) or \( \Delta = \Delta(\alpha) \) for some \( \alpha \in (0, \frac{\pi}{4}] \), let \( m \in \mathbb{N}_0 \), and let \( C^m(\Delta, \mathbb{K}) \) denote the vector space consisting of all functions \( \varphi : \Delta \to \mathbb{K} \) that are \( m \) times continuously differentiable in \( \Delta^\circ \) (the interior of \( \Delta \)) and whose partial derivatives \( D^\alpha \varphi \) can be extended continuously throughout \( \Delta \). Define \( C^\infty(\Delta, \mathbb{K}) := \bigcap_{m \in \mathbb{N}} C^m(\Delta, \mathbb{K}) \). The Fréchet topology on \( C^m(\Delta, \mathbb{K}) \), resp. \( C^\infty(\Delta, \mathbb{K}) \), is induced by the following system of increasing seminorms:

\[
p_n(f) := \sup_{\tau \in \Delta_n} \sup_{|\alpha| \leq m} |D^\alpha f(\tau)|, \quad f \in C^m(\Delta, \mathbb{K}), \text{ resp.}
\]

\[
p_n(f) := \sup_{\tau \in \Delta_n} \sup_{|\alpha| \leq n} |D^\alpha f(\tau)|, \quad f \in C^\infty(\Delta, \mathbb{K}), \quad n \in \mathbb{N}.
\]

Let \( E := C^m(\Delta, \mathbb{K}) \) for some \( m \in \mathbb{N}_0 \) or \( E := C^\infty(\Delta, \mathbb{K}) \). Based on the consideration carried out in [19, Example 3.1.29], we know that the translation semigroup \((T(t))_{t \in \Delta}\) is a locally equicontinuous \( C_0 \)-semigroup in \( E \), and we already know that \((T(t))_{t \in \Delta}\) is both chaotic and topologically mixing. An easy application of Theorem 1, case (a) shows that \((T(t))_{t \in \Delta}, (T(t))_{t \in R} \) and any single operator \( T(t_0) \) are frequently hypercyclic, as well (\( R \subseteq \Delta \) is a ray, \( t_0 \in \Delta \setminus \{0\} \)).
3 Generalized frequent hypercyclicity for translation semigroups and semigroups induced by semiflows

We will continue to use the same notation as in Sect. 1.2. Define \( \tilde{\rho} : \mathbb{C} \rightarrow [0, \infty) \) by \( \tilde{\rho}(x) := \rho(x), x \in \Delta \) and \( \tilde{\rho}(x) := 0, \) otherwise. In the sequel, we use the next condition: for every \( k \in \mathbb{N} \) and \( \delta > 0, \) there exist only a finite number \( c_{k, \delta} \in \mathbb{N} \) of tuples \( (n_1, n_2) \in \mathbb{N} \mathbb{N}_0 \), such that \( n_1 \neq n_2 \) and:

\[
\left| \lfloor m_{k+n_1} \rfloor - \lfloor m_{k+n_2} \rfloor \right| \leq \delta, \text{ and } \left| \lfloor m_{k-n_1} \rfloor - \lfloor m_{k-n_2} \rfloor \right| \leq \delta, \tag{8}
\]

for \( k \geq \max(n_1, n_2) \), as well as the sequence \((c_{k, \delta})_{k \in \mathbb{N}}\) is bounded for every fixed \( \delta > 0. \)

**Theorem 4** Assume that \( 1 \leq p < \infty, E := L^p_{0}(\Delta, \mathbb{K}), \) resp., \( E := C_{0, p}(\Delta, \mathbb{K}) \) and \( t_0 \in \Delta' \setminus \{0\}. \) Assume that the following conditions hold for any compact subset \( K \) of \( \Delta: \)

(i) \( \sum_{n=1}^{k} \int_{K} \tilde{\rho}(x-t_0(\lfloor m_k \rfloor - \lfloor m_{k-n} \rfloor)) \, dx \) converges unconditionally, uniformly in \( k \in \mathbb{N}, \) resp., \( \forall \epsilon > 0, \exists N \in \mathbb{N}, \) such that for any finite set \( F \subseteq \lfloor N, \infty \rfloor \cap \mathbb{N} \) and \( \forall k \in \mathbb{N}, \) we have:

\[
\sup_{n \in F, k \geq n, x \in K} \tilde{\rho}(x - t_0(\lfloor m_k \rfloor - \lfloor m_{k-n} \rfloor)) \in (0, \epsilon).
\]

(ii) \( \sum_{n=1}^{\infty} \int_{K} \tilde{\rho}(x-t_0(\lfloor m_{k+n} \rfloor - \lfloor m_k \rfloor)) \, dx \) converges unconditionally, uniformly in \( k \in \mathbb{N}, \) resp., \( \forall \epsilon > 0, \exists N \in \mathbb{N}, \) such that for any finite set \( F \subseteq \lfloor N, \infty \rfloor \cap \mathbb{N} \) and \( \forall k \in \mathbb{N}, \) we have:

\[
\sup_{n \in F, x \in K} \tilde{\rho}(x - t_0(\lfloor m_{k+n} \rfloor - \lfloor m_k \rfloor)) \in (0, \epsilon).
\]

(iii) \( \sum_{n=1}^{\infty} \int_{K} \rho(x + t_0 \lfloor m_n \rfloor) \, dx \) converges unconditionally, resp., \( \forall \epsilon > 0, \exists N \in \mathbb{N}, \) such that for any finite set \( F \subseteq \lfloor N, \infty \rfloor \cap \mathbb{N}, \) we have:

\[
\sup_{n \in F, x \in K} \rho(x + t_0 \lfloor m_n \rfloor) \in (0, \epsilon).
\]

Then, \((T(t))_{t \in \Delta'}\) is \( f \)-frequently hypercyclic, \((T(te^{\text{arg}(t_0)}))_{t \geq 0}\) is \( f \)-frequently hypercyclic, and the operator \( T(t_0) \) is \( l-(m_k) \)-frequently hypercyclic.

In particular, if \( f(t) = t^q + 1, t \geq 0, q > 1 \) and \( \tilde{\rho}(t) = (1 + |t|)^{-s}, t \geq 0, \) where \( qs > 1, \) then (8) holds as well as (i)–(iii). Thus, \((T(t))_{t \in \Delta'}\) satisfies all the conclusions given above.

**Proof** We will prove the first assertion (for \( L^p_{0}(\Delta, \mathbb{K}) \)). Without loss of generality, we may assume that \( \Delta' = \Delta. \) First, we will prove the particular case. Assume that \( k \in \mathbb{N} \) and \( \delta > 0 \) are given as well as that \( \lfloor |m_{k+n_1}|-|m_{k+n_2}|-l \rfloor \leq \delta, \) for some positive integers \( n_1, n_2 \in \mathbb{N}, \) such that \( n_1 < n_2. \) Then, \( |(k+n_1)^q -(k+n_2)^q| \leq \delta + 2. \) By
the mean value theorem, $|(k + n_1)^q - (k + n_2)^q| \leq q|n_1 - n_2|(k + n_1)^{q-1}$, we get $q(k + n_1)^{q-1} \leq \delta + 2$. This implies that:

$$n_1 \leq \left(\frac{\delta + 2}{q}\right)^{1/(q-1)} - k \text{ and } n_2 \leq \left(\delta + 2 + \left[\left(\frac{\delta + 2}{q}\right)^{1/(q-1)}\right]\right)^{1/q} - k.$$ 

Hence, there exist only a finite number $c_{k,\delta}' \in \mathbb{N}$ of tuples $(n_1, n_2) \in \mathbb{N}_0^2$, such that $n_1 \neq n_2$, the first inequality in (8) holds and the sequence $(c_{k,\delta}')_{k\in\mathbb{N}}$ is bounded for every fixed $\delta > 0$. Similarly, the assumption $|[m_k - n_1] - [m_k - n_2]| \leq \delta$ for $k \geq \max(n_1, n_2)$ implies:

$$[n_1, n_2] \subseteq \left[k - \left(\delta + 2 + \left[\left(\frac{\delta + 2}{q}\right)^{1/(q-1)}\right]\right)^{1/q}, k - \left(\frac{\delta + 2}{q}\right)^{1/(q-1)}\right],$$

on account of which there exist only a finite number $c_{k,\delta}'' \in \mathbb{N}$ of tuples $(n_1, n_2) \in \mathbb{N}_0^2$, such that $n_1 \neq n_2$, the second inequality in (8) holds and the sequence $(c_{k,\delta}'')_{k\in\mathbb{N}}$ is bounded for every $\delta > 0$. Note that $n_1$, $n_2$ are integers and that the interval appearing on the right hand side of (9) contains only a finite number of integers, which is independent of $k \in \mathbb{N}$. Suppose now that $qs > 1$. Since $t_0 \neq 0$, elementary inequalities give:

$$\frac{1}{(1 + |x - t_0([k^q] - [(k - n)^q]))^s} \sim \frac{1}{|t_0|^s([k^q] - [(k - n)^q])^s} \leq \frac{1}{|t_0|^s n^{qs}},$$

(since $(a + b)^q \geq a^q + b^q$), uniformly in $x \in K$. Thus, (i) holds; the proofs of (ii) and (iii) follow in the same way. Thus, in this case, $(T(t))_{t \in \Delta'}$ satisfies all the assumptions of the main part of the theorem which will be proved now.

The main result is a consequence of Theorem 1, case (a), with $E_0$ being the set consisting of all continuous functions with compact support and a mapping $S_{[m_n]} f : \Delta \rightarrow \mathbb{K}$ given by $(S_{[m_n]} f)(x) := f(x - t_0([m_n]), x \in [m_n] + \text{supp}(f)$, $(S_{[m_n]} f)(x) := 0$ otherwise ($f \in E_0$, $n \in \mathbb{N}$). Then, $T(t_0[m_n])S_{[m_n]} f = f$ for all $f \in E_0$, $n \in \mathbb{N}$ and the conditions (i)–(iii) enable one to see that the requirements of Theorem 1, case (a) are satisfied. Strictly speaking, let a function $f \in E_0$ be given and let $\text{supp}(f) = K \subseteq \Delta$. Set $\delta := \max\{|t| : t \in K\}/|t_0|$. To see that $\sum_{n=1}^{K} T(t_0[m_k])S_{[m_k-n]} f$ converges unconditionally, uniformly in $k \in \mathbb{N}$, observe first that:

$$\left\| \sum_{n=1}^{K} T(t_0[m_k])S_{[m_k-n]} f \right\| = \left( \int_{\Delta} \left( \sum_{n=1}^{K} f(x + t_0([m_k] - [m_k-n])) 1_{K-t_0([m_k]-[m_k-n])}(x) \right)^p \rho(x) \, dx \right)^{1/p}.$$
For any fixed number \( x \in \Delta \), the sum:

\[
\sum_{n=1}^{k} f(x + t_0(|m_k| - |m_{k-n}|)) \chi_{K-t_0(|m_k| - |m_{k-n}|)}(x)
\]

consists of at most \( c_{k,\delta} \) summands, since the assumption \( x \in K - t_0(|m_k| - |m_{k-n}|) \) and \( x \in K - t_0(|m_k| - |m_{k-n_2}|) \) for some integers \( n_1, n_2 \in [1, k] \) implies \( |m_{k+n_1}| - |m_{k+n_2}| \leq \delta \). Hence:

\[
\left\| \sum_{n=1}^{k} T(t_0|m_k|)S|m_{k-n}|f \right\| \\
\leq c_{k,\delta}^{(p-1)/p} \left( \int_{\Delta} \sum_{n=1}^{k} \left| f(x + t_0(|m_k| - |m_{k-n}|)) \right| \chi_{K-t_0(|m_k| - |m_{k-n}|)}(x) \right)^{1/p} \\
\cdot \chi_{K-t_0(|m_k| - |m_{k-n}|)}(x) \right)^{1/p}
\]

Similarly, the series \( \sum_{n=1}^{\infty} T(t_0|m_k|)S|m_{k+n}|f \) converges unconditionally, uniformly in \( k \in \mathbb{N} \), due to (ii) and the monotone convergence theorem:

\[
\left\| \sum_{n=1}^{\infty} T(t_0|m_k|)S|m_{k+n}|f \right\| \\
\leq c_{k,\delta}^{(p-1)/p} \left( \int_{\Delta} \sum_{n=1}^{\infty} \left| f(x + t_0(|m_k| - |m_{k-n}|)) \right| \chi_{K-t_0(|m_k| - |m_{k-n}|)}(x) \right)^{1/p} \\
\cdot \chi_{K-t_0(|m_k| - |m_{k-n}|)}(x) \right)^{1/p}
\]
\[
\begin{align*}
&\leq c_{k,\delta}^{(p-1)/p} \left( \sum_{n=1}^{\infty} \int_K |f(x)|^p \hat{\rho}(x - t_0([m_{k+n}] - [m_k])) \, dx \right)^{1/p} \\
&\leq c_{k,\delta}^{(p-1)/p} \|f\|_{\infty} \left( \sum_{n=1}^{\infty} \int_K \hat{\rho}(x - t_0([m_{k+n}] - [m_k])) \, dx \right)^{1/p}.
\end{align*}
\]

Since (8) holds, for any compact set \( K \) of \( \Delta \) there exists a finite number \( n_K \in \mathbb{N} \), such that any point \( x \in \Delta \) is contained in at most \( n_K \) sets of the form \( t_0([m_n] + K) \). Using this condition and (iii), we can similarly verify that the series \( \sum_{n=1}^{\infty} S_{[m_n]} f \) converges unconditionally.

Now, we prove the assertion for \( E := C_{0,\rho}(\Delta, \mathbb{K}) \).

The proof is similar to the first part and the basic differences are given below (the particular case is the same). Again, we may assume that \( \Delta' = \Delta \). We can apply Theorem 1, case (a), with \( E_0 \) being the set consisting of all continuous functions with compact support and the mappings \( S_{[m_n]} \), \( n \in \mathbb{N} \), defined as in the proof of the first part. Then, \( T(t_0([m_n])), S_{[m_n]} f = f \) for all \( f \in E_0 \), \( n \in \mathbb{N} \) and (i)–(iii) implies the validity of conditions necessary for applying Theorem 1, case (a). Strictly speaking, suppose that \( f \in E_0 \) and \( \text{supp}(f) = K \subseteq \Delta \). Set \( \delta := \max\{2|t| : t \in K\}/|t_0| \). To see that \( \sum_{n=1}^{\infty} T(t_0([m_n]), S_{[m_{k-n}]} f \) converges unconditionally, uniformly in \( k \in \mathbb{N} \), observe first that for each finite set \( F \subseteq \mathbb{N} \) and each fixed number \( x \in \Delta \), the sum

\[
\sum_{1 \leq n \leq k, n \in F} f(x + t_0([m_k] - [m_{k-n}]))\chi_{K - t_0([m_k] - [m_{k-n}])}(x)
\]

consists of at most \( c_{k,\delta} \) summands, and therefore:

\[
\left| \sum_{1 \leq n \leq k, n \in F} f(x + t_0([m_k] - [m_{k-n}]))\chi_{K - t_0([m_k] - [m_{k-n}])}(x) \right| \leq c_{k,\delta} \sup_{1 \leq n \leq k, n \in F} \|f(x + t_0([m_k] - [m_{k-n}]))\| \hat{\rho}(x).
\]

This implies that:

\[
\left\| \sum_{1 \leq n \leq k, n \in F} T(t_0([m_k])), S_{[m_{k-n}]} f \right\|
\leq c_{k,\delta} \sup_{x \in \Delta} \left| \sum_{1 \leq n \leq k, n \in F} f(x + t_0([m_k] - [m_{k-n}]))\chi_{K - t_0([m_k] - [m_{k-n}])}(x) \right| \hat{\rho}(x)
\leq c_{k,\delta} \sup_{1 \leq n \leq k, n \in F, x \in \Delta} \|f(x + t_0([m_k] - [m_{k-n}]))\| \hat{\rho}(x)
\leq c_{k,\delta} \|f\|_{\infty} \sup_{1 \leq n \leq k, n \in F, x \in K} \hat{\rho}(x - t_0([m_k] - [m_{k-n}])).
\]
Using this estimate and condition (i), we get that

\[ \sum_{n=1}^{k} T(t_0 |m_{k}|)S_{m_{k-n}} f \]

converges unconditionally, uniformly in \( k \in \mathbb{N} \). Similarly, we can prove that the series \( \sum_{n=1}^{\infty} T(t_0 |m_{k}|)S_{m_{k+n}} f \) converges unconditionally, uniformly in \( k \in \mathbb{N} \), and that the series \( \sum_{n=1}^{\infty} S_{m_{n}} f \) converges unconditionally.

\[ \square \]

**Remark 2** Condition (8) does not hold for frequent hypercyclicity \( (q = 1) \).

**Remark 3** Let \( f : [0, \infty) \to [1, \infty) \) be twice continuously differentiable satisfying additionally that \( f''(x) \geq 0 \) for all \( x \geq 0 \), as well as \( \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} f'(x) = +\infty \). Arguing in the same manner, we can prove that (8) holds for \( f \)-frequent hypercyclicity, i.e., for any numbers \( k \in \mathbb{N} \) and \( \delta > 0 \) given in advance, there exist only finite number \( c_{k,\delta} \in \mathbb{N} \) of tuples \( (n_1, n_2) \in \mathbb{N}_0^2 \), such that \( n_1 \neq n_2 \), (8) holds and the sequence \( (c_{k,\delta})_{k \in \mathbb{N}} \) is bounded for every \( \delta > 0 \). By a careful analysis, one can find a corresponding \( \hat{\rho}(\cdot) \), so that conditions (i)–(iii) automatically hold. We will not consider these cases.

We can similarly prove the following results continuing our previous analysis from Theorem 1, case (b). It is only worth to note that we need the condition (10) below for proving that for any compact set \( K \) of \( \Delta \), there exists a finite number \( n_K \in \mathbb{N} \), such that any point \( x \in \Delta \) is contained at most \( n_K \) sets of the form \( t_{(m_{n})} + K \).

**Theorem 5** Assume that \( 1 \leq p < \infty \), \( E := L^p_{\rho}(\Delta, \mathbb{K}) \), resp., \( E := C_{0,\rho}(\Delta, \mathbb{K}) \), \( (t_n)_{n \in \mathbb{N}} \) is a sequence in \( \Delta' \setminus \{0\} \), \( \delta' > 0 \), that (3) holds (with \( \delta \) replaced therein with \( \delta' \)). Furthermore, assume that there exists an increasing mapping \( g : [1, \infty) \to [1, \infty) \), such that \( |t_n| \leq g(n) \), \( n \in \mathbb{N} \) and that (8) holds. Assume that:

\[ \lim_{n \to \infty} |t_{(m_{n+1})} - t_{(m_{n})}| = +\infty \quad (10) \]

as well as that the following conditions hold for any compact subset \( K \) of \( \Delta \):

(i) \( \sum_{n=1}^{k} \int_{K} \tilde{\rho}(x - t_{(m_{k})} + t_{(m_{k-n})}) \, dx, \ k \in \mathbb{N} \) converges unconditionally, uniformly in \( k \in \mathbb{N} \), resp., \( \forall \epsilon > 0 \ \exists N \in \mathbb{N} \), such that for any finite set \( F \subseteq [N, \infty) \cap \mathbb{N} \) and \( \forall k \in \mathbb{N} \):

\[ \sup_{n \in F, k \geq n, x \in K} \tilde{\rho}(x - t_{(m_{k})} + t_{(m_{k-n})}) \in (0, \epsilon). \]

(ii) \( \sum_{n=1}^{\infty} \int_{K} \tilde{\rho}(x - t_{(m_{k+n})} + t_{(m_{k})}) \, dx \) converges unconditionally, uniformly in \( k \in \mathbb{N} \), resp., \( \forall \epsilon > 0 \ \exists N \in \mathbb{N} \), such that for any finite set \( F \subseteq [N, \infty) \cap \mathbb{N} \) and \( \forall k \in \mathbb{N} \):

\[ \sup_{n \in F, x \in K} \tilde{\rho}(x - t_{(m_{k+n})} + t_{(m_{k})}) \in (0, \epsilon). \]
(iii) \( \sum_{n=1}^{\infty} \int_{K} \rho(x + t_{\lfloor mn \rfloor}) \, dx \) converges unconditionally, resp., \( \forall \epsilon > 0 \exists N \in \mathbb{N} \), such that for any finite set \( F \subseteq [N, \infty) : \cap N \)
\[
\sup_{n \in F, x \in K} \rho(x + t_{\lfloor mn \rfloor}) \in (0, \epsilon).
\]

Then, \((T(t))_{t \in \Delta'}\) is \((g \circ f)\)-frequently hypercyclic and the sequence of operators \((T_n := T(t_n))_{n \in \mathbb{N}}\) is \(l-(m_k)\)-frequently hypercyclic.

Keeping in mind [8, Lemma 4.2], it is almost straightforward to extend the assertions of [26, Propositions 3.6 and 3.8] for \(f\)-frequently hypercyclic \(C_0\)-semigroups on complex sectors which do have at least one single \(l-(m_n)\)-hypercyclic operator (the situation is not so clear for \(C_0\)-semigroups defined on complex sector \(\Delta \neq [0, \infty)\) without any single \(l-(m_n)\)-hypercyclic operator):

**Proposition 5** Assume that \(1 \leq p < \infty\), \(E := L^p_\rho(\Delta, \mathbb{K})\), resp., \(E := C_{0,\rho}(\Delta, \mathbb{K}), t_0 \in \Delta' \setminus \{0\}\). Let the operator \(T(t_0)\) be \(l-(m_n)\)-hypercyclic. Then, for each \(\epsilon > 0\), there exists a strictly increasing sequence \((n_k)_{k \in \mathbb{N}}\) in \(\mathbb{N}\), such that the set \(\{n_k : k \in \mathbb{N}\}\) has a positive lower \((m_n)\)-density and that, for every \(i \in \mathbb{N}\):

\[
\sum_{k > i} \rho((n_k - n_i)e^{i\arg(t_0)}) < \epsilon, \quad \text{resp.,} \quad \rho((n_k - n_i)e^{i\arg(t_0)}) < \epsilon.
\]

In particular, if \(f(t) = t + 1\) for all \(t \geq 0\), then the mapping \(t \mapsto \rho(te^{i\arg(t_0)}), t \geq 0\) is bounded.

### 3.1 Examples II

(i) Let \(\Delta = [0, \infty)\) or \(\Delta = \mathbb{R}\), and let \(\rho(x) := (1 + |x|)^{-1}, x \in \Delta\). Then, \(\rho(\cdot)\) is an admissible weight function and, due to [26, Theorem 3.8, Theorem 3.9], the translation semigroup \((T(t))_{t \geq 0}\) is not frequently hypercyclic on \(L^p_\rho(\Delta, \mathbb{K})\), for any \(p \geq 1\). Suppose now that \(q > 1, 1 \leq p < \infty\) and \(f(t) := t^q + 1, t \geq 0\). Then, we can apply Theorem 4 to see that \((T(t))_{t \geq 0}\) is \(q\)-frequently hypercyclic on \(L_\rho^p(\Delta, \mathbb{K})\). Further on, due to [26, Proposition 2.7], any single operator \(T(t_0)\) of a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) satisfying the requirements of frequent hypercyclicity criterion for \(C_0\)-semigroups needs to be chaotic \((t_0 > 0)\). Here, we would like to point out that, due to [9, Corollary 4.8], the translation \(q\)-frequently hypercyclic semigroup \((T(t))_{t \geq 0}\) constructed in this way, with \(\Delta = [0, \infty),\) is not chaotic and that, for every \(t_0 > 0\), the single operator \(T(t_0)\) is not chaotic.

(ii) ([31, Example 1]; see also [26, Example 3.7] and [19, Example 3.1.28(ii)]) Let \(p \in [1, \infty), \Delta = \Delta' := [0, \infty)\) and \(\rho(x) := e^{-(x+1)\cos(\ln(x+1))+1}, x \geq 0\). Assume first that \(E := L^p_\rho([0, \infty), \mathbb{K})\). Then, the translation semigroup \((T(t))_{t \geq 0}\) is hypercyclic, because \(\lim_{t \to \infty} \rho(x) = 0\), and not chaotic, because \(\int_0^{\infty} \rho(x) \, dx = +\infty\); by [25, Theorem 3], it follows that \((T(t))_{t \geq 0}\) is not frequently hypercyclic. Let \(\zeta = \pi/2 + \theta\), where \(\theta\) is an infinitesimally small positive number. We will prove that \((T(t))_{t \geq 0}\) is \(f\)-frequently hypercyclic with \(f(x) := e^{(2x+1)\pi + \zeta},\)
x ≥ 0. By Remark 3, we have that for given numbers k ∈ N and δ > 0, there exist only finite number ck,δ ∈ N of tuples (n1, n2) ∈ N2, such that n1 ≠ n2, (8) holds as well as that the sequence (ck,δk∈N is bounded for every fixed number δ > 0. Let t0 = 1 and let K = [a, b] ⊆ [0, ∞) be a compact set. First of all, it can be easily seen that the series \( \sum_{n=1}^{\infty} \int_{K} \rho(x - (\lfloor m_k \rfloor - \lfloor m_{k-n} \rfloor)) dx \) and \( \sum_{n=1}^{\infty} \int_{K} \rho(x - (\lfloor m_k \rfloor - \lfloor m_{k-n} \rfloor)) dx \) converge unconditionally, uniformly in k ∈ N, because the sums are finite and equal zero for \( k \geq k_0 \), where \( k_0 \in N \) depends only on K. The series \( \sum_{n=1}^{\infty} \int_{a}^{b} \rho(x + \lfloor m_n \rfloor) dx = \sum_{n=1}^{\infty} \int_{a+\lfloor m_n \rfloor}^{b+\lfloor m_n \rfloor} \rho(x) dx \)
\[ = \sum_{n=1}^{\infty} \int_{a+\lfloor e^{n+2n+1} \pi \rfloor}^{b+\lfloor e^{n+2n+1} \pi \rfloor} \rho(x) dx \leq c_{\xi,K} \sum_{n=1}^{\infty} e^{-2n+1} \pi. \]
Hence, an application of Theorem 4 shows that \( (T(t))_{t \geq 0} \) is \( f \)-frequently hypercyclic, as claimed. Suppose now that \( E := C_{0,\rho}([0, \infty), \mathbb{K}) \). Then, \( (T(t))_{t \geq 0} \) is hypercyclic, not chaotic; moreover, \( (T(t))_{t \geq 0} \) cannot be frequently hypercyclic by [26, Proposition 3.8]. Arguing as above, we may conclude that \( (T(t))_{t \geq 0} \) is \( f \)-frequently hypercyclic with the same choice of function \( f(\cdot) \). Let us finally note that it is not clear how we can apply Proposition 5 here to see that \( (T(t))_{t \geq 0} \) is not \( g \)-frequently hypercyclic if \( g(\cdot) \) is of subexponential growth.

### 3.2 On frequently hypercyclic translation semigroups on complex sectors

Our main aim is to state the following slight extension of [7, Theorem 6]:

**Theorem 6** Assume that 1 ≤ \( p < \infty \), \( \Delta = \Delta(\alpha) \) for some \( \alpha \in (0, \pi/2) \) or \( \Delta = \mathbb{C} \) and \( E = L^p_{\rho}(\Delta, \mathbb{K}) \). Then, the following statements are equivalent:

(i) The translation semigroup \( (T(t))_{t \in \Delta} \) is chaotic.

(ii) There exists a ray \( R \subseteq \Delta \) starting at zero, such that, for every \( m \in \mathbb{N} \), we have:
\[ \int_{F_{R,m}} \rho(s) ds < \infty, \text{ where } F_{R,m} := \left\{ t \in \Delta : d(t, R) = \inf_{s \in R} |t - s| < m \right\}. \]

(iii) There exists a ray \( R \subseteq \Delta \) starting at zero, such that the translation semigroup \( (T(t))_{t \in R} \) is frequently hypercyclic on \( E \).

Any of these statements implies that \( (T(t))_{t \in \Delta} \) is frequently hypercyclic on \( E \).

**Proof** The equivalence of (i) and (ii) has been proved in [7, Theorem 6]. Assume that (ii) holds with the ray \( R := \{ te^{i\varphi} : t \geq 0 \} \subseteq \Delta \) and some angle \( \varphi \in [0, 2\pi) \). The
implication (ii) ⇒ (iii) follows by applying the frequent hypercyclicity criterion for $C_0$-semigroups with the mapping $S(t) : E_0 \to E$ defined by $(S(t)f)(x) := f(x - te^{i\varphi})$, $x \in te^{i\varphi} + \text{supp}(f)$ and $(S(t)f)(x) := 0$, otherwise ($f \in E_0$, $t > 0$); see the proof of [26, Proposition 3.3], with appealing to [8, Lemma 4.2] in place of [10, Lemma 4.2]. Here, it is only worth noting that, in the case $p = 1$, the Bochner integrability of mapping $t \mapsto S(te^{i\varphi})f$, $t \geq 0$ ($f \in E_0$) follows essentially from the corresponding part of the proof of [26, Proposition 3.3] and the fact that $\int_0^2 \rho(te^{i\varphi}) \, dt$ is convergent, which follows from the next calculation involving [10, Lemma 4.2] and the last estimate from the proof of [7, Theorem 6]:

$$\int_0^\infty \rho(te^{i\varphi}) \, dt = \sum_{k=0}^\infty \int_k^{k+1} \rho(te^{i\varphi}) \, dt \leq \text{Const.} \sum_{k=0}^\infty \rho(kte^{i\varphi}) \leq \text{Const.} \int_{F_{R,m}} \rho(s) \, ds;$$

the Pettis integrability of mapping $t \mapsto S(te^{i\varphi})f$, $t \geq 0$ ($f \in E_0$) for $p > 1$ follows much easier, by showing functional. Arguing as in the proof of [25, Theorem 3.8], it can be simply shown that the validity of (iii) in the case $\Delta = C$ implies that the backward shift operator $B$ is frequently hypercyclic on the Banach space $l_p^\varphi := \{(\alpha_k)_{k \in \mathbb{Z}} \mid \| (\alpha_k)_{k \in \mathbb{Z}} \| := (\sum_{k \in \mathbb{Z}} |\alpha_k|^p v_k)^{1/p} < \infty \}$, where $v_k := \rho(ke^{i\varphi})$ for $k \in \mathbb{Z}$ (just use [8, Lemma 4.2] and replace the segment $[k, k+1]$ in the proof with the region $ke^{i\varphi} + \Delta_1$). By [3, Theorem 12.3], we get that $\sum_{k=0}^\infty \rho(ke^{i\varphi}) < \infty$ and (i) follows by applying [7, Theorem 4]. The proof of implication (iii) ⇒ (i), in the case that $\Delta = \Delta(\alpha)$ for some $\alpha \in (0, \pi/2]$, is similar and, therefore, omitted. \hfill \Box

A simple modification of the proof of [25, Proposition 3.4] yields the following:

**Proposition 6** Assume that $E := C_{0, \rho}(\Delta, K) and \lim_{t \in \Delta', |t| \to \infty} \rho(t) = 0$. Then, $(T(t))_{t \in \Delta'}$ is frequently hypercyclic and any single operator $T(t_0)$ is frequently hypercyclic $(t_0 \in \Delta' \setminus \{0\})$.

As pointed out in [25, Remark 3.5], the condition $\lim_{t \in \Delta', |t| \to \infty} \rho(t) = 0$ is only sufficient but not necessary for $(T(t))_{t \in \Delta'}$ to be frequently hypercyclic ($\Delta' = [0, \infty)$, $\Delta = \mathbb{R}$). In the case that $\Delta' = \Delta = \Delta(\pi/4)$, a simple counterexample can be obtained by considering the translation semigroup from [8, Example 4.15], with the pivot space being $C_{0, \rho}(\Delta, K)$.

### 3.3 On semigroups and semiflows

In this subsection, we turn our attention to $C_0$-semigroups induced by semiflows. For $L_{p_1}^\varphi$ setting, we define $\tilde{\rho}_1 : C \to [0, \infty)$ by $\tilde{\rho}_1(x) := \rho_1(x)$, $x \in \Delta$ and $\tilde{\rho}_1(x) := 0$; otherwise, for $C_{0, \rho}$ setting, the notion of $\tilde{\rho}$ is understood as before. The following result is very similar to Theorem 4. For this purpose, in next two theorems, we impose the following condition: $\forall k \in \mathbb{N} \forall K \subset \subset \Omega \exists e_{k,K}$ of tuples $(n_1, n_2) \in \mathbb{N}^2_0$, such that: $n_1 \neq n_2$.
\[ \varphi(t_0[m_{k+n}], K) \cap \varphi(t_0[m_{k+n+1}], K) \neq \emptyset \tag{11} \]
\[ \varphi(t_0[m_{k-n}], K) \cap \varphi(t_0[m_{k-n+1}], K) \neq \emptyset, \text{ provided } k \geq \max(n_1, n_2), \tag{12} \]
as well as that the sequence \((c_k, K)_{k \in \mathbb{N}}\) is bounded for every fixed compact \(K \subseteq \Omega\).

**Theorem 7** Let \(\varphi : \Delta \times \Omega \to \Omega\) be a semiflow. Suppose that \(\varphi(t, \cdot)\) is a locally Lipschitz continuous function for all \(t \in \Delta\) and the condition (i)-(b), resp., (ii)-(b) of Lemma 1 holds. Further on, suppose that \(E := L_{\rho_1}^p(\Omega, \mathbb{K})\), resp., \(E := C_0, \rho(\Omega, \mathbb{K})\) and let there exist a number \(t_0 \in \Delta \setminus \{0\}\), such that the following conditions hold for any compact subset \(K \subseteq \Omega\):

(a) \(\sum_{n=1}^k \int_K |\det D\varphi(t_0([m_k] - [m_{k-n}]), x)| \tilde{\rho}_1(\varphi(t_0([m_k] - [m_{k-n}]), x)^{-1}) dx, \]
\[ k \in \mathbb{N} \text{ converges, uniformly in } k \in \mathbb{N}, \text{ resp., } \forall \epsilon > 0 \exists N \in \mathbb{N}, \text{ such that for any finite set } F \subseteq [N, \infty) \cap \mathbb{N} \text{ and } \forall k \in \mathbb{N}: \]
\[ \sup_{n \in F, k \geq n} \tilde{\rho}(\varphi(t_0([m_k] - [m_{k-n}]), x)^{-1}) \in (0, \epsilon). \]

(b) \(\sum_{n=1}^\infty \int_K |\det D\varphi(t_0([m_{k+n}] - [m_k]), x)| \tilde{\rho}_1(\varphi(t_0([m_{k+n}] - [m_k]), x)^{-1}) dx \text{ converges unconditionally, uniformly in } k \in \mathbb{N}, \text{ resp., } \forall \epsilon > 0 \exists N \in \mathbb{N}, \text{ such that for any finite set } F \subseteq [N, \infty) \cap \mathbb{N} \text{ and } \forall k \in \mathbb{N}: \]
\[ \sup_{1 \leq n \leq k, n \in F} \tilde{\rho}(\varphi(t_0([m_{k+n}] - [m_k]), x)^{-1}) \in (0, \epsilon). \]

(c) The series \(\sum_{n=1}^\infty \int_K |\det D\varphi(t_0[m_n], x)|^{-1} \rho_1(\varphi(t_0[m_n], x)) dx \text{ converges unconditionally, resp., } \forall \epsilon > 0 \exists N \in \mathbb{N}, \text{ such that for any finite set } F \subseteq [N, \infty) \cap \mathbb{N}: \]
\[ \sup_{n \in F, x \in K} \rho(\varphi(t_0[m_n], x)) \in (0, \epsilon). \]

Then, \((T_\varphi(t))_{t \in \Delta}\) is \(f\)-frequently hypercyclic, \((T(te^{i\arg(t_0)})_{t \geq 0}\) is \(f\)-frequently hypercyclic, and the operator \(T(t_0)\) is \(l-(m_k)\)-frequently hypercyclic.

**Proof** The proofs of the theorem in both cases \((E := L_{\rho_1}^p(\Omega, \mathbb{K})\) and \(E := C_0, \rho(\Omega, \mathbb{K})\)) can be deduced by repeating almost literally the arguments given in the proof of Theorem 4, with appropriate technical modifications. Set \(E_0 := C_c(\Omega, \mathbb{K})\) and \(S_{[m_n]} f : \Omega \to \mathbb{K}\) by \((S_{[m_n]} f)(x) := f(\varphi(t_0[m_n], x)^{-1}), x \in \varphi(t_0[m_n], \text{supp}(f))\) and \((S_{[m_n]} f)(x) := 0\), otherwise \((f \in E_0, n \in \mathbb{N})\). Then, for each \(f \in E_0\), one has \(T_\varphi(t_0[m_n])S_{[m_n]} f = f, n \in \mathbb{N}\) and:
\[ T_\varphi(t_0[m_k])S_{[m_{k+n}] f} = f(\varphi(t_0[m_{k+n}], \varphi(t_0[m_k], x)^{-1})^{-1} \chi_{\varphi(t_0[m_k], \varphi(t_0[m_{k+n}], \text{supp}(f)))^{-1}, n, k \in \mathbb{N}.} \]
Due to the conditions (a)–(c) and chain rule, the requirements of Theorem 1, case (a) are satisfied and Theorem 7 for the space \(L_{\rho_1}^p(\Omega, \mathbb{K})\) immediately follows.
The remaining part of proof of Theorem 7 for the space $E := C_{0, \rho}(\Omega, \mathbb{K})$ is much easier and can be deduced along the same lines. \[ \square \]

We can also formulate the following analogue of Theorem 7 for the semigroup $(T_\phi(t))_{t \in \Delta'}$ induced by the semiflow $\phi(t, \cdot)$, as it has been done in Theorem 5.

**Theorem 8** Let $\phi : \Delta \times \Omega \to \Omega$ be a semiflow. Assume that $(t_n)_{n \in \mathbb{N}}$ is a sequence in $\Delta \setminus \{0\}$, that (3) holds (with $\delta$ replaced by $\delta'$, $\delta' > 0$), and that an increasing mapping $g : [1, \infty) \to [1, \infty)$ satisfies $|t_n| \leq g(n)$, $n \in \mathbb{N}$. Further on, suppose that $\phi(t, \cdot)$ is a locally Lipschitz continuous function for all $t \in \Delta$ and the condition (i)-(b), resp., (ii)-(b) of Lemma 1 holds. Suppose, further, that $E := L^p_{\rho_1}(\Omega, \mathbb{K})$, resp., $E := C_{0, \rho}(\Omega, \mathbb{K})$. Let the following conditions hold for any compact subset $K$ of $\Omega$:

(a) $\sum_{n=1}^{\infty} \int_K |\det D\phi(t_{[m_k]} - t_{[m_{k-n}]} , x)|\rho_1(\phi(t_{[m_k]} - t_{[m_{k-n}]} , x)^{-1}) dx$, $k \in \mathbb{N}$ converges unconditionally, uniformly in $k \in \mathbb{N}$, resp., $\forall \epsilon > 0 \exists N \in \mathbb{N}$, such that for any finite set $F \subseteq [N, \infty) \cap \mathbb{N}$ and $\forall k \in \mathbb{N}$:

$$\sup_{n \in F, k \geq n, x \in K} \rho(\phi(t_{[m_k]} - t_{[m_{k-n}]}, x)^{-1}) \in (0, \epsilon).$$

(b) $\sum_{n=1}^{\infty} \int_K |\det D\phi(t_{[m_{k+n}]} - t_{[m_k]} , x)|\rho_1(\phi(t_{[m_{k+n}]} - t_{[m_k]} , x)^{-1}) dx$ converges unconditionally, uniformly in $k \in \mathbb{N}$, resp., $\forall \epsilon > 0 \exists N \in \mathbb{N}$, such that for any finite set $F \subseteq [N, \infty) \cap \mathbb{N}$ and $\forall k \in \mathbb{N}$:

$$\sup_{1 \leq n \leq k, n \in F, x \in K} \rho(\phi(t_{[m_{k+n}]} - t_{[m_k]} , x)^{-1}) \in (0, \epsilon).$$

(c) $\sum_{n=1}^{\infty} \int_K |\det D\phi(t_{[m_n]} , x)|^{-1}\rho_1(\phi(t_{[m_n]} , x)) dx$ converges unconditionally, resp., $\forall \epsilon > 0 \exists N \in \mathbb{N}$, such that for any finite set $F \subseteq [N, \infty) \cap \mathbb{N}$:

$$\sup_{n \in F, x \in K} \rho(\phi(t_{[m_n]} , x)) \in (0, \epsilon).$$

Then, $(T_\phi(t))_{t \in \Delta'}$ is $(g \circ f)$-frequently hypercyclic and the sequence of operators $(T_{\phi,n} := T_\phi(t_n))_{n \in \mathbb{N}}$ is $l$-$(m_k)$-frequently hypercyclic.

### 3.4 Examples III

(i) (see [16, Example 3.20]) Let $p \geq 1$, $\Omega := (0, 1)$, $\Delta = \Delta' := [0, \infty)$, $E := L^p_{\rho_1}(\Omega, \mathbb{K})$ and

$$\varphi(t, x) := \frac{x}{x + (1-x)e^{-t}}, \quad t \geq 0, \quad x \in (0, 1).$$

Then, $\varphi(t, \cdot)$ is a continuously differentiable function for all $t \geq 0$, $D\varphi(t, x) = e^{-t}(x + (1-x)e^{-t})^{-2}$ for all $t \geq 0$ and $x \in (0, 1)$, $\varphi(t, x)^{-1} = xe^{-t}(xe^{-t} + 1 - x)^{-1}$ for all $t \geq 0$ and $x \in (0, 1)$, $D\varphi(t, x)^{-1} = e^{-t}(xe^{-t} + 1 - x)^{-2}$ for all...
\( t \geq 0 \) and \( x \in (0, 1) \), and the condition (i)-(b) of Lemma 1(i) holds iff there exist \( M \geq 1 \) and \( \omega \geq 0 \), such that:

\[
\rho_1(x)^t(x + (1 - x)e^{-t})^2 \leq M e^{\omega t}\rho_1\left(x(x + (1 - x)e^{-t})^{-1}\right), \quad t > 0, \ a.e. \ x \in (0, 1).
\]

It can be easily seen that the last estimate holds for \( \rho_1(x) := 1/x, \ x \in (0, 1) \), with \( M = 1 \) and \( \omega = 3 \), so that \((T_\varphi(t))_{t \geq 0}\) is a \( C_0\)-semigroup on \( E \). Using \cite[Theorem 2, Remark 3]{17}, it readily follows that \((T_\varphi(t))_{t \geq 0}\) is neither frequently hypercyclic nor chaotic. With the help of Theorem 7, we will prove that there exist only \( c_k \in \mathbb{N} \) and a compact \( K = [a, b] \subseteq (0, 1) \) and that the sequence \((c_k, K) \subseteq (0, 1) \) be given. Assume that \( x \in \varphi\left([k + n_1]^q + 1, K\right) \cap \varphi\left([k + n_2]^q + 1, K\right) \). This implies the existence of numbers \( x_1, x_2 \in [a, b] \), such that:

\[
\frac{x_1}{x_1 + (1 - x_1)e^{-\left([k + n_1]^q + 1\right) + 1}} = \frac{x_2}{x_2 + (1 - x_2)e^{-\left([k + n_2]^q + 1\right) + 1}}.
\]

This yields:

\[
\frac{x_1(1 - x_2)}{x_2(1 - x_1)} = e^{[k + n_1]^q + 1 - [k + n_2]^q + 1},
\]

so that there exists \( \delta > 0 \), such that \( (k + n_1)^q - (k + n_2)^q \leq \delta \). Arguing as in Remark 2, we get that there exist only \( c_{k, K}^p \) tuples \((n_1, n_2) \in \mathbb{N}^2_0\), such that \( n_1 \neq n_2 \) and (11) hold, as well as that the sequence \((c_{k, K}^p)_{k \in \mathbb{N}}\) is bounded. We can similarly prove that there exist only \( c_{k, K}^{p'} \) tuples \((n_1, n_2) \in \mathbb{N}^2_0\), such that \( n_1 \neq n_2 \) and (12) hold as well as that the sequence \((c_{k, K}^{p'})_{k \in \mathbb{N}}\) is bounded. Hence, the requirements necessary for applying Theorem 7 are fulfilled. Conditions (a)–(c) also hold and we will verify this only for the condition (b), i.e., we will prove that the series \( \sum_{n=1}^{\infty} \int_a^b |D\varphi\left([k + n]^q\right) - [k^q], x\right| \mu_1\left(\varphi\left([k + n]^q\right) - [k^q], x\right)^{-1}\ dx \)

converges unconditionally, uniformly in \( k \in \mathbb{N} \). This follows from the following calculation:

\[
\sum_{n=1}^{\infty} \int_a^b \left|D\varphi\left([k + n]^q\right) - [k^q], x\right| \mu_1\left(\varphi\left([k + n]^q\right) - [k^q], x\right)^{-1}\ dx
\]

\[
= \sum_{n=1}^{\infty} \int_a^b e^{-2([k + n]^q) - [k^q]} \ dx
\]

\[
= \sum_{n=1}^{\infty} \int_a^b \frac{x e^{-2([k + n]^q) - [k^q]}}{(x e^{-2([k + n]^q) - [k^q]} + 1 - x)(x + (1 - x)e^{-2([k + n]^q) - [k^q]})} \ dx
\]

\[
= \sum_{n=1}^{\infty} \int_a^b \frac{x}{(x e^{-2([k + n]^q) - [k^q]} + 1 - x)(e^{[k + n]^q} - [k^q] x + 1 - x)} \ dx
\]

\[
\leq \frac{b(b - a)}{(1 - b)a} \sum_{n=1}^{\infty} e^{-([k + n]^q) - [k^q]}.
\]
ii) Let \( p \in [1, \infty), \alpha \in (0, \pi/2], \) let \( \Delta = [0, \infty) \) or \( \Delta = \Delta(\alpha), \) and let \( \Omega := \Delta^c. \)

Suppose that \( F : \Omega \to \Omega \) is a continuously differentiable bijective mapping together with its inverse mapping \( F^{-1} : \Omega \to \Omega. \) Define \( \varphi(t, x) := F^{-1}(t + F(x)), t \in \Delta, x \in \Omega. \) Then, it can be simply shown that \( \varphi(\cdot, \cdot) \) is a semiflow. Let \( E := L^p_\Omega(\Delta, \mathbb{R}), t_0 \in \Delta \setminus \{0\}, q > 1 \) and \( f(s) := s^q + 1, s \geq 0. \) The condition (1) is fulfilled iff:

\[
\exists M, \omega \in \mathbb{R} \forall t \in \Delta : \rho_1(x) \leq M e^{\omega |t|} \rho_1\left(F^{-1}(t + F(x))\right) \left| \det DF^{-1}(t + F(x)) \right|
\]

(13)

for a.e. \( x \in \Omega, \) when \( (T_\varphi(t))_{t \in \Delta} \) is a \( C_0 \)-semigroup on \( E. \) It is clear that:

\[
\varphi(t, x)^{-1} = y \iff y = F^{-1}(F(x) - t) \quad \text{for} \quad t \in \Delta, \ x, y \in \Omega.
\]

(14)

Arguing similarly as in Remark 2 and Example II(i), it can be simply shown that conditions (11)–(12) hold true, as well as that the series in the formulations of conditions (a)–(b) of Theorem 7 converge unconditionally, uniformly in \( k \in \mathbb{N}, \) since, for a given compact set \( K \subseteq \Omega, \) the sums in their definitions are finite and equal zero for any sufficiently large number \( k \geq k_0(K), \) \( (T_\varphi(t))_{t \in \Delta} \) will be \( f \)-frequently hypercyclic, \( (T(te^{i \arg(t_0)}))_{t \geq 0} \) will be \( f \)-frequently hypercyclic and the operator \( T(t_0) \) will be \( l-(m_k) \)-frequently hypercyclic if the series in the formulation of condition (c) of Theorem 7 converges unconditionally, i.e., if the series:

\[
\sum_{n=1}^{\infty} \int_K \left| \det DF^{-1}(t_0[n^q] + F(x)) \right|^{-1} \rho_1\left(F^{-1}(t_0[n^q] + F(x))\right) dx
\]

(15)

converges unconditionally. Let us examine some concrete cases in which the conditions (13) and (15) hold true with the sector \( \Delta = \Delta(\alpha):\)

(a) Let \( c > 0 \) and \( F(x + iy) := c(x \pm iy), x + iy \in \Omega. \) If \( \rho_1(\cdot) \) is an admissible weight function on \( \Delta \) and \( \sum_{n=1}^{\infty} \rho_1(t_0n^q/c) < \infty \) for the choice of sign +, resp., \( \sum_{n=1}^{\infty} \rho_1(\bar{t}_0n^q/c) < \infty \) for the choice of sign –. Then, it can be easily seen that (13)–(15) are valid. An application of Theorem 8 can be also made provided that the conditions \( \sum_{n=1}^{\infty} \rho_1(t/n/c) < \infty \) or \( \sum_{n=1}^{\infty} \rho_1(\bar{t}/n/c) < \infty \) hold for an appropriate sequence \( (t_n)_{n \in \mathbb{N}} \) in \( \Delta' \setminus \{0\}. \)

(b) Let \( F(x + iy) := (x + iy)^a e^{i\alpha(1-a)}, x + iy \in \Delta \) for some number \( a \in (0, 1). \) Then, \( \varphi(t, x + iy) = e^{i\alpha \frac{2a}{a-1}} (t + (x + iy)^a e^{i\alpha(1-a)})^{1/a} \) for \( t, x + iy \in \Delta \) and:

\[
\left| \det DF^{-1}(t + F(x)) \right| \sim \text{Const.} \left| t + (x + iy)^a \right|^{\frac{2a-2}{a}} |x + iy|^{2a-2}, \quad t \in \Delta, \ x + iy \in \Delta.
\]
Using this estimate, it can be simply verified that the conditions (13) and (15) hold provided that $\zeta > 0$, $\rho_1(x + iy) := (1 + |x + iy|)^{-\zeta}$, $x + iy \in \Delta$ and $q(2 - \frac{2}{a}) - \frac{q}{a} \zeta < -1$.

(iii) ([17]) Let $p \in [1, \infty]$ and let $E := L^p[0, 1]$. Assume that $h \in C[0, 1]$ and $(h(x) - \Re h(0))/x \in L^1[0, 1]$. ($\Re$ is real part.) It is well known that the $C_0$-semigroup $(T(t))_{t \geq 0}$ governing the solutions of (linear) von Foerster–Lasota equation:

$$u_t(t, x) = -xu_x(t, x) + h(x)u(t, x), \quad t \geq 0, \ x \in (0, 1); \ u(0, x) = u_0(x),$$

(16)
is hypercyclic iff it is chaotic iff it is frequently hypercyclic iff $\Re(h(0)) \geq (-1)/p$. Let $f : [0, \infty) \to [1, \infty)$ be an increasing mapping, such that $\lim_{t \to \infty} f(x)/x > 0$. Then, frequent hypercyclicity implies $f$-frequent hypercyclicity which further implies hypercyclicity, so that any of above conditions is necessary and sufficient for $f$-frequent hypercyclicity of solutions to (16). This example particularly shows that finding the necessary and sufficient conditions for $f$-frequent hypercyclicity of $C_0$-semigroups induced by semiflows is far from being a trivial problem.

In contrast to Theorems 7, 1, case (a) and [26, Theorem 2.2] can be always applied for proving frequent hypercyclicity of $C_0$-semigroups induced by semiflows.

### 3.5 Examples IV

(i) ([19, Example 3.1.28(v)]) Assume that $\Delta := [0, \infty)$, $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$, $p > 0$, $q \in \mathbb{R}$ and:

$$\varphi(t, x, y) := e^{pt}(x \cos qt - y \sin qt, x \sin qt + y \cos qt), \ t \geq 0, \ (x, y) \in \Omega.$$  

Then, $\varphi : \Delta \times \Omega \to \Omega$ is a semiflow and we already know that $(T_\varphi(t))_{t \geq 0}$ is topologically mixing and chaotic in the Fréchet space $C(\Omega, \mathbb{K})$. Using Theorem 1, case (a), it can be simply shown that $(T(t))_{t \geq 0}$ is frequently hypercyclic, as well.

(ii) ([16, Example 5.5]) Let $\Delta := [0, \infty)$, $E := L^p_{\rho_1}(\mathbb{R}^n)$, $\rho_1(x) := 1/(1 + |x|^2)$ $(x \in \mathbb{R}^n)$ and $\varphi(t, x) := e^{t}x$ $(t \geq 0, x \in \mathbb{R}^n)$. We already know that the $C_0$-semigroup $(T_\varphi(t))_{t \geq 0}$ is chaotic. Using [26, Theorem 2.2], it can be simply proved that $(T_\varphi(t))_{t \geq 0}$ is frequently hypercyclic, as well. Strongly continuous (weighted composition) semigroups considered in [16, Example 3.20, Example 4.13] are also frequently hypercyclic, which can be shown by applying [17, Theorem 2].

(iii) ([19, Example 3.1.41(i)]) Assume that $p \in [1, \infty)$, $\alpha \in (0, \frac{1}{p})$, $\Delta \in \{[0, \infty), \Delta(\alpha)\}$, $\Omega = (1, \infty)$ and $0 < \alpha_1 \leq 1$. Define $\varphi : \Delta \times \Omega \to \Omega$ and $\rho_1 : \Omega \to (0, \infty)$ through:

$$\varphi(t, x) := (\Re t + x^{\alpha_1})^{1/\alpha_1} \text{ and } \rho_1(x) := e^{-x^{\alpha_1}}, \ t \in \Delta, \ x \in \Omega.$$  

(17)

It is straightforward to verify that $\varphi(\cdot, \cdot)$ is a semiflow and $(T_\varphi(t))_{t \in \Delta}$ is a $C_0$-semigroup on $L^p_{\rho_1}(\Omega, \mathbb{K})$. Employing [26, Theorem 2.2], we can simply prove...
that \((T_\varphi(t))_{t \in R}\) is frequently hypercyclic for any ray \(R \subseteq \Delta\), so that \((T_\varphi(t))_{t \in \Delta}\) is frequently hypercyclic, as well.

### 3.6 Open problems

We propose two open problems motivated by reading the paper [13] by K.-G. Grosse-Erdmann. Let us recall that for every operator \(T \in L(E)\), such that \(T^{-1} \in L(E)\), the hypercyclicity of \(T\) implies the hypercyclicity of \(T^{-1}\); see, e.g., [12]. The corresponding question for frequent hypercyclicity, pointed out in [13, Problem 9], was recently solved in [28]. In [10, Theorem 2.5], W. Desch, W. Schappacher, and G. F. Webb have proved that for any \(C_0\)-group of linear operators \((T(t))_{t \in \mathbb{R}}\) on a separable Banach space \(E\), the following assertions are equivalent?

\[
\begin{align*}
(i) & \quad \text{the semigroup } (T(t))_{t \geq 0} \text{ is hypercyclic;} \\
(ii) & \quad \text{the semigroup } (T(-t))_{t \geq 0} \text{ is hypercyclic;} \\
(iii) & \quad \text{there exists some } x \in E, \text{ such that both sets } \{T(t)x : t \geq 0\} \text{ and } \{T(-t)x : t \geq 0\} \\
& \quad \text{are dense in } E.
\end{align*}
\]

A similar assertion holds in separable Fréchet spaces (see, e.g., [20, Theorem 2(iv), Theorem 4(ii)] and the proof of [10, Theorem 2.5]) and, based on the above discussion, we would like to propose the following problems:

**Problem 1** Let \((T(t))_{t \in \mathbb{R}}\) be a \(C_0\)-group of linear operators \((T(t))_{t \in \mathbb{R}}\) on a separable Fréchet space \(E\). Is it true that the following assertions are equivalent?

\[
\begin{align*}
(i) & \quad \text{the semigroup } (T(t))_{t \geq 0} \text{ is frequently hypercyclic;} \\
(ii) & \quad \text{the semigroup } (T(-t))_{t \geq 0} \text{ is frequently hypercyclic;} \\
(iii) & \quad \text{there exists some } x \in E, \text{ such that both sets } \{T(t)x : t \geq 0\} \text{ and } \{T(-t)x : t \geq 0\} \\
& \quad \text{are } f\text{-frequently hypercyclic vectors for } (T(t))_{t \geq 0} \text{ and } \{T(-t)x : t \geq 0\} \text{ are } f\text{-frequently hypercyclic vectors for } (T(t))_{t \geq 0} ?
\end{align*}
\]

**Problem 2** Let \((T(t))_{t \in \mathbb{R}}\) be a \(C_0\)-group of linear operators \((T(t))_{t \in \mathbb{R}}\) on a separable Fréchet space \(E\). Profile the class of increasing functions \(f : [0, \infty) \to [1, \infty)\) for which the following assertions are equivalent:

\[
\begin{align*}
(i) & \quad \text{the semigroup } (T(t))_{t \geq 0} \text{ is } f\text{-frequently hypercyclic;} \\
(ii) & \quad \text{the semigroup } (T(-t))_{t \geq 0} \text{ is } f\text{-frequently hypercyclic;} \\
(iii) & \quad \text{there exists some } x \in E, \text{ such that both } f\text{-frequently hypercyclic vector for } (T(t))_{t \geq 0} \text{ and } f\text{-frequently hypercyclic vector for } (T(-t))_{t \geq 0}.
\end{align*}
\]

The notion of \(f\)-frequent hypercyclicity is still very unexplored and we can propose a great number of other problems for \(C_0\)-semigroups and single operators in Fréchet spaces.

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