The Phase Diagram of the $N = 2$ Kazakov-Migdal Model

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Abstract

We have determined the phase diagram of the simplest version of a lattice model introduced in the recent work of Kazakov and Migdal. If $m_0$ and $\lambda$ are the bare mass and self coupling of the scalar field in the model respectively, we find a line of first order phase transitions in the $(m_0, \lambda)$ plane ending in a critical point where $\lambda$ is nonzero. Kazakov and Migdal speculate that their model of scalar field theory could induce QCD. Our work indicates that for $N = 2$ there is no continuum limit for the Kazakov-Migdal model except at the critical end point. Whether or not a nontrivial continuum limit exists in the vicinity of the critical point requires careful study of the renormalization group properties of the model.
Recently Kazakov and Migdal (KM) \[1\] came up with an intriguing idea: consider the following model, defined on a d-dimensional hypercubic lattice with action

$$ S = N \sum_x \left[ V(\Phi(x)) - tr \sum_{\mu=1}^d \Phi(x)U_\mu(x)\Phi(x + \mu)U_\mu^\dagger(x) \right], $$

(1)

where $\Phi(x)$ is a traceless scalar field in the adjoint representation of $SU(N)$, covariantly coupled to the gauge fields $U_\mu(x)$ defined on the links of the lattice. Kazakov and Migdal speculate that perhaps, when the parameters of the scalar potential $V$ are tuned appropriately, the model Eq. (1) will have a non-trivial continuum limit. Moreover the resulting continuum theory might actually be equivalent to QCD.

If the above were true this apparent reformulation of QCD should not be considered as merely an interesting curiosity. The potential of this speculation lies in the fact that Eq. (1) in many ways is much simpler than conventional lattice QCD. As a matter of fact, based on the work of KM and of Migdal \[2\] it looks as though in the limit of large $N$ the model might be analytically tractable and that the degrees of freedom associated with the eigenvalues of $\Phi$ could actually be the long sought “master field” of large $N$ QCD.

In order to see how such a scenario could possibly be true and also to motivate the present study, let us examine Eq. (1) in more detail. Integrating out the scalar fields in the partition function will give rise to an induced action for the gauge fields which because of the gauge invariance of Eq. (1), will be given as an (infinite) sum over Wilson loops on the lattice. Moreover, because the scalar field is in the adjoint representation, the Wilson loops will also be. Concretely, for $V(\phi) = \frac{m^2}{2} tr\phi^2$, the $\phi$ integral is Gaussian and one obtains the induced action\[1\]:

$$ S_{\text{ind}} = - \sum_\Gamma \frac{2^{l(\Gamma)-1}||trU(\Gamma)||^2}{l[\Gamma]m_0^{2l(\Gamma)}}, $$

(2)

where $l(\Gamma)$ is the length of the path $\Gamma$. Note that Eq. (2) inherits from Eq. (1) a local center symmetry $U_\mu(x) \rightarrow Z_\mu(x)U_\mu(x)$, a fact recently discussed by Kogan, Semenoff and Weiss \[3\]. We will return to this point again later on.

How could the theory defined by Eq. (2) possibly be related to QCD? To begin with the induced action in Eq. (2) is as good a lattice action as any other in the sense that in the naive continuum limit, it produces a term proportional to $F^2_{\mu\nu}$. If a continuum limit indeed exists for this model, we expect that only this term will survive based on the familiar renormalization group argument. The model is unusual only because the basic building blocks of the action are arbitrary loops (in the adjoint representation). In the weak smooth gauge field limit this model has a critical point at $m_0^2 = 2d$ as can be seen easily from Eq. (1) where the lowest order action becomes an action for a free scalar field. Taking fluctuations into account the critical point is expected to shift to a different value $m_c^2$. Using the definition $m^2 = m_0^2 - m_c^2$ KM \[1\] calculated the coefficient of the $F^2_{\mu\nu}$ term and obtained the correspondence

$$ \frac{1}{g_0^2} \rightarrow - \frac{N}{96\pi^2}ln(m^2 a^2), $$

(3)
in the smooth gauge fields limit.

KM [1] and Migdal [2] pointed out that the model in Eq. (1) can be easier dealt with analytically and numerically if one first integrates out the gauge fields. In the large $N$ limit, the resulting effective action for the master field is best rewritten as a two-matrix model, which can be treated using modern matrix model technology. KM suggest that the master field gives nontrivial scaling laws for physical quantities (for example, glueball mass $M_g$)

$$M_g^2 = (m^2)\gamma,$$

(4)

where the value of $\gamma$ is different from the usual trivial scalar theory ($\gamma = 1$ with possible logarithmic corrections). KM have given the following argument for the value of $\gamma$: if the continuum theory of Eq. (1) is to “induce” QCD correctly, we must view the scalar as a heavy “constituent” field whose mass acts as an effective UV cutoff for the gauge field in the continuum. This immediately gives the usual relation between a physical quantity, the cutoff and the bare coupling

$$ln\frac{m^2}{M_g^2} = \frac{48\pi^2}{11Ng_0^2}.$$

(5)

Combining this expression with that of Eq. (3) one gets the scaling relation for $M_g$ and find $\gamma = 23/22$ (This is the value in the smooth gauge field limit. When the gauge field is treated nonperturbatively $\gamma$ value can be quite larger than one [2]).

Based on the above discussions we expect that in order for Eq. (1) to correctly induce QCD, we need two things, a critical point and nontrivial scaling of physical quantities. In the present letter we follow KM’s suggestion and study the simplest version of the model, $SU(2)$ ($N = 1$ is trivial). Certainly the possibility of Eq. (1) inducing QCD does not crucially depend on $N$ being large. Integrating out the gauge fields one obtains the following partition function [1]

$$Z = \int_{\phi>0} [d\phi] \exp \left\{ \sum_x [ln\phi^2 - 2V(\phi)] + \sum_{<xy>} ln \frac{sinh(4\phi(x)\phi(y))}{\phi(x)\phi(y)} \right\},$$

(6)

where $<xy>$ denotes the sum over nearest neighbors of x. The field $\phi(x)$ in Eq. (6) is the (positive) eigenvalue of the matrix $\Phi$ and the restriction to $\phi(x) > 0$ expresses the fact that it is a “radial” variable. According to what was said before, we must find a critical point close to which a continuum limit can be constructed. It is well known that the continuum limit of a scalar model with polynomial interaction terms is trivial with vanishing renormalized coupling [4]. In principle the continuum limit of Eq. (6) could avoid the triviality problem due to the presence of the nonpolynomial interaction terms (if we could integrate out the angular variables in an $O(N)$ model exactly, however, the resulting effective action for the radial variables would be also nonpolynomial). Whether or not this is the case and Eq. (6) allows for nontrivial scaling is numerically a much more difficult question and remains to be answered in the future.

We restrict ourselves to a scalar potential of the form $V(\phi) = \frac{m^2}{2}tr\phi^2 + \frac{\lambda}{4}tr\phi^4$. A nonzero $\lambda$ is necessary to stabilize the system and allows numerical simulation in the small $m_0^2$ region.
In order to get an idea of what the phase diagram looks like in the \((\lambda, m_0)\) plane, we first use a simple mean field theory (MFT), i.e. we set \(\phi(x) = e^u\) and assume \(u(x) = \text{constant}\). We then analyzed the resulting potential looking for minima and phase transitions. As it turns out the qualitative features of the phase diagram from the MFT effective potential analysis are in excellent agreement with Monte Carlo simulations. In Fig. 1, we show a typical plot of the MFT potential \(V_{\text{eff}}(u)\) at a rather small value of \(\lambda = 0.01\). There obviously is a first order phase transition at \(m_0^2 = m_c^2 = 7.65\), which is not very far away from the smooth gauge field limit \(m_0^2 = 8\). For this particular value of \(\lambda\) we show a comparison of MFT and Monte Carlo for \(\phi\)-field vacuum expectation value in Fig. 2. All our Monte Carlo simulations were done on \(8^4\) lattice using a single-hit Metropolis algorithm on the field \(u\). The errors in Fig. 2 were obtained by blocking averages and \(<\phi>\) values are the result of \(5K\) sweeps through the lattice at each point. Clearly, the agreement between MFT and Monte Carlo is very good. The transition is obviously first order with a very clear hysteresis in \(<\phi>\) (we also have obtained clear two state signals at \(m_0^2\)). Interestingly then, our first conclusion is that in the \(\lambda \to 0\) limit no continuum limit exists.

The appearance of first order phase transitions does not come as a surprise. The simple \(SU(2)\) lattice theory in the adjoint representation \((SU(2)/Z_2 = SO(3))\) has action \(S = \beta_A \sum_D |TrU_D|^2\) and possesses a first order phase transition at some critical value of \(\beta_A\) \([5]\). Presumably this transition is related to the \(Z_2\) invariance of the action \([3]\). The question now is whether a first order transition can be avoided by tuning \(\lambda\). The answer is yes: In MFT the gap in \(u\) (or \(<\phi>\)) decreases rapidly as \(\lambda\) is increased. For \(\lambda > 2.57\) the gap disappears all together and there is no phase transition anymore. Thus the phases of strong and weak \(m_0^2\) are analytically connected. The point \((\lambda = 2.57, m_0^2 = 4.515)\) is a critical point in MFT: \(V_{\text{eff}}''(\phi)\) vanishes at the minimum of the potential. In Fig. 3 we show how this MFT scenario is confirmed by Monte Carlo simulations. The location of the critical endpoint by Monte Carlo is not very precise. It is taken at the value of \(\lambda\) where the hysteresis effect in \(<\phi>\) disappears and the critical value of \(m_0^2\) is determined by the position of the peak of \(\phi\)-field susceptibility

\[
\chi = \frac{1}{V} \sum_{x,y} [ < \phi(x)\phi(y) > - < \phi(x) > < \phi(y) > ] .
\] (7)

In conclusion, we have found a line of first order phase transitions in the \((\lambda, m_0^2)\) plane of the \(SU(2)\) KM model. There is no continuum limit at \(\lambda \to 0\). The first order phase transition line terminates in a critical point at \(\lambda > 0\), in the vicinity of which one may attempt to construct a continuum theory. The precise nature of this continuum theory remains to be determined. This is a challenge for further research.

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Figure Caption

Figure 1: The MFT effective potential $V_{\text{eff}}(u)$ at $\lambda = 0.01$. The two minima become degenerate at $m_0^2 = 7.65$ which agrees with the observed critical point for the first order phase transition in Monte Carlo simulation.

Figure 2: Comparison of the mean field theory and Monte Carlo simulation result for $\phi$-field vacuum expectation value. The solid lines are the MFT predictions and Monte Carlo data points are indicated by *. The errors are smaller than the size of the symbols.

Figure 3: The phase diagram. The solid line is the mean field theory prediction and the simulation results are indicated by diamonds. The end point of the first order phase transition line is marked by * for MFT and square for Monte Carlo estimate. At the end point the theory becomes critical.