Triangle colorings require at least seven colors

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Abstract

We show that if a coloring of the plane has the properties that any two points at distance one are colored differently and the plane is partitioned into uniformly colored triangles (under certain conditions), then it requires at least seven colors. This is also true for a coloring using uniformly colored convex polygons if it has a point bordering at least four polygons.

1 Introduction

The chromatic number of the plane is the minimum number of colors needed to color all points of \( \mathbb{R}^2 \) in such a way that any two points at distance one are colored differently. See Soifer [5] for a detailed history of this problem. Although the problem was proposed by Nelson in the 1950s, the best lower bound has only recently been proven by De Grey [2], who showed that at least five colors are needed to color the plane. On the other hand, the plane can be colored with seven colors by tiling the plane with hexagons or with squares (see Figure 1), so the chromatic number of the plane is 5, 6, or 7.

It is natural to ask what the chromatic number is for colorings with certain restrictions. For instance, before De Grey’s work, Falconer [3] proved that five colors are needed for colorings that only use measurable sets. Woodall [7] and Townsend [6] proved that certain map-type colorings require at least six colors. See Soifer [5, Chapters 8, 9, and 24] for more details about the chromatic number of restricted colorings.

Because the upper bound for the general problem follows from restricted colorings that only use polygons, it is natural to ask for the chromatic number of such colorings. Coulson [1] proved that all polygon colorings require at least six colors, under the condition that all polygons are convex and have an area greater than some positive constant. Furthermore, Moustakis [4] showed that at least seven colors are needed if the plane is tiled with congruent squares under certain conditions, which matches the square coloring in Figure 1b.

In this paper, we continue the study of restricted colorings that use polygons. In particular, we look at colorings that use triangles (see Figure 7 for an example), and we prove...
that at least seven colors are needed for any triangle coloring of the plane. We do this by showing that if any convex polygon coloring has a point that borders at least four polygons, then at least seven colors are needed. This, together with the observation that any triangle coloring has at least one point that borders at least four triangles, implies that triangle colorings require at least seven colors.

2 Definitions and theorem

We begin by defining the colorings using polygons that we study.

**Definition 2.1.** A *convex polygon coloring* is a mapping from $\mathbb{R}^2$ to a finite set of colors, such that any two points a unit distance apart are colored differently, and the plane is partitioned into *regions*, *borders*, and *vertices* with the following properties. Vertices are points, borders are line segments that connect vertices, and regions are uniformly colored convex polygons that are enclosed by borders and vertices. Regions that share a border or a vertex are colored differently, and the coloring is locally finite in the sense that any disk intersects finitely many regions, borders, and vertices.

Note that all borders and vertices may be colored arbitrarily, meaning that they may be colored independently of the regions that they border. The arguments in this paper are not affected by how the border points and vertices are colored.

The *degree* of a vertex is the number of borders that are connected to it. Observe that for any convex polygon coloring, the degree of any vertex is at least 3. We say that a color *occurs* at a border point or vertex $P$ if a region bordering $P$ contains that color. If $n$ colors occur at a vertex, then we call it an *$n$-colored vertex*. Since in our definition regions that share a vertex must be colored differently, if a vertex has degree $n$ then it is *$n$-colored*.

We now state the first of our two main theorems, which applies to all convex polygon colorings.

**Theorem 2.2.** If a convex polygon coloring of the plane contains a vertex of degree at least 4, then at least 7 colors are required.

We define a *triangle coloring* of the plane as a convex polygon coloring in which all the regions are triangles. We now state our second main theorem that applies to all triangle colorings, which is a consequence of Theorem 2.2.

**Theorem 2.3.** Any triangle coloring of the plane requires at least 7 colors.

3 Definitions and lemmas

In this section, we introduce some definitions and lemmas that we use in our proof. These all refer to a convex polygon coloring with a fixed vertex $O$ and the circle $C_O$ of unit radius centered at $O$.

3.1 Crossings

The proof relies on considering certain border points and vertices on $C_O$ which we call crossings.

**Definition 3.1.** A *crossing* is either a point on $C_O$ that lies on a border not tangent to $C_O$ or a vertex on $C_O$ connected to at least one border that has points inside $C_O$.
(a) Points $P_1$ and $P_2$ are crossings

(b) Points $P_3$ and $P_4$ are non-crossings

Figure 2: Examples of crossings and non-crossings on $C_O$

Note that a point on $C_O$ which lies on a border that is tangent to $C_O$ is not a crossing. Also, a vertex on $C_O$ connected to borders that do not have any points inside $C_O$ is not a crossing (see Figure 2b). We call these points non-crossings.

We now record the obvious fact that $C_O$ must contain at least one crossing. It would not be difficult to prove that there are at least six crossings on $C_O$ for any convex polygon coloring of the plane, but this is not needed in our proof.

**Lemma 3.2.** The circle $C_O$ must contain at least one crossing.

**Proof.** Assume that there are no crossings on $C_O$. This implies that all but finitely many points on $C_O$ lie within the same region, with non-crossings being the only possible exceptions. Since the convex polygon coloring is locally finite, there must be two points on $C_O$ at distance one that are not exceptions. Thus, these two points are in the same region and must be colored the same. However, this is a contradiction since points at distance one from each other must be colored differently.

Now we state the following lemma about points that are not crossings on $C_O$.

**Lemma 3.3.** If a point $P$ on $C_O$ is not a crossing, then there is a region that $P$ lies in or borders that cannot be colored any of the colors that occur at $O$.

**Proof.** If $P$ lies within a region, consider a small circular neighborhood of $P$. Within each of the regions bordering $O$, there is a point a unit distance away from a point in the neighborhood of $P$. Each of these points must be colored differently, so the region $P$ lies in cannot be colored any of the colors that occur at $O$. On the other hand, if $P$ does not lie within a region and is a non-crossing, then we can apply the previous argument to a point on $C_O$ in a region that $P$ borders.

### 3.2 Point types and special arcs

The following definitions and lemmas all refer to a convex polygon coloring with a fixed 4-colored vertex $O$ and the circle $C_O$ of unit radius centered at $O$.

We define an **inside neighborhood** of a point $P$ on $C_O$ as an open semi-circular neighborhood that lies on the side closer to $O$ of the tangent line to $C_O$ at $P$. Similarly, we define an **outside neighborhood** of a point $P$ on $C_O$ as an open semi-circular neighborhood that lies on the side away from $O$ of the tangent line to $C_O$ at $P$. We say that if any points within a
neighboring neighborhood of a point cannot be colored any of $n$ colors, then those $n$ colors are excluded in that neighborhood.

**Definition 3.4.** An *inward point* is a point $P$ on $C_O$ that has an inside neighborhood in which all 4 colors occurring at $O$ are excluded (see Figure 3a).

**Definition 3.5.** An *outward point* is a point $P$ on $C_O$ that has an outside neighborhood in which all 4 colors occurring at $O$ are excluded (see Figure 3b).

**Definition 3.6.** An *alternative point* is a point $P$ on $C_O$ that has an outside neighborhood in which 3 of the 4 colors occurring at $O$ are excluded (see Figure 3c).

We can determine whether a point $P$ is inward, outward, or alternative as follows. Take the tangent line at $O$ to the unit circle $C_P$ centered at $P$ and count the borders on each side of the tangent. Adding one to this number usually gives the number of excluded colors on the corresponding side of $P$. The only exception to this heuristic is when two borders lie on one side of the tangent line and the other two borders lie on the side closer to $P$ (see Figure 4a for an example).

![Figure 3: The shaded areas are the inside and outside neighborhoods of $P$.](image)

With these definitions, we can show that these are the only kinds of points on $C_O$.

**Lemma 3.7.** Any point on $C_O$ is an inward, outward, or alternative point.

**Proof.** Let $P$ be a point on $C_O$ and $C_P$ be the unit circle centered at $P$. The circle $C_P$ passes through $O$. We examine whether the borders connected to $O$ within a neighborhood of $O$ lie inside or outside of $C_P$. There are four possibilities for how many of these borders lie inside and outside of $C_P$: all four outside, three outside and one inside, two outside and two inside, and one outside and three inside. It is not possible for all four borders to lie inside of $C_P$ because then two of the borders form an angle that is greater than $\pi$, which contradicts the coloring being convex.

In the first case of four borders outside of $C_P$ and the second case of three borders outside of $C_P$ and one border inside of $C_P$, 4 regions are intersected by any unit circle centered at a point within a sufficiently small inside neighborhood of $P$. Thus, the 4 colors occurring at $O$ are excluded in an inside neighborhood of $P$, which makes $P$ an inward point.

In the third case of two borders outside of $C_P$ and two borders inside of $C_P$, 3 regions are intersected by any unit circle centered at a point within a sufficiently small outside neighborhood of $P$. Thus, 3 of the 4 colors occurring at $O$ are excluded in an outside neighborhood of $P$, which makes $P$ an alternative point.
Lastly, in the case of three borders inside of $C_P$ and one border outside of $C_P$, 4 regions are intersected by any unit circle centered at a point within a sufficiently small outside neighborhood of $P$. Thus, the 4 colors occurring at $O$ are excluded in an outside neighborhood of $P$, which makes $P$ an outward point.

Since these are the only possible cases, all points on $C_O$ must be inward, outward, or alternative points.

With the understanding that all points on $C_O$ come in three different types, we can define arcs that consist of points of the same type.

**Definition 3.8.** A maximal arc of $C_O$ consisting of only inward points is an inward arc. Similarly, a maximal arc consisting of only outward points is an outward arc. Lastly, a maximal arc consisting of only alternative points in which the same three colors are excluded in outside neighborhoods of each point is called an alternative arc.

Note that an arc could be a single point of $C_O$ and endpoints may or may not be included in an arc.

With this definition, we can introduce the following lemma, which determines the maximum number of alternative arcs on $C_O$.

**Lemma 3.9.** The circle $C_O$ has at most four alternative arcs on it.

**Proof.** A point $P$ is an alternative point if and only if a unit circle $C_P$ centered at $P$ intersects two opposite regions bordering $O$. This is because within a neighborhood of $O$ two borders connected to $O$ must lie inside $C_P$ and two borders must lie outside $C_P$. There are two cases that are considered for the borders connected to $O$. One is a special case where only two borders are collinear and all other borders lie on one side of the pair, and the other case is where the borders are oriented in any other way.

We first consider the case of one pair of collinear borders. In this case, there are two alternative arcs and one alternative arc consisting of one point such that the unit circle centered at any point on each arc intersects two opposite regions bordering $O$.

We next consider the general case. In this case, there are four alternative arcs because each pair of opposite regions forces two alternative arcs.

Therefore, there are at most four alternative arcs on $C_O$. 

![Figure 4: The dark arcs are alternative arcs and the dotted circles are examples of $C_P$.](image)

Note that the only case in which the number of alternative arcs on $C_O$ is fewer than four occurs when only two borders are collinear with all other borders on one side of the pair as shown in Figure 4a.
3.3 Lemmas about crossings

The following lemmas all refer to a convex polygon coloring with a fixed 4-colored vertex $O$ and the circle $C_O$ of unit radius centered at $O$.

**Lemma 3.10.** The regions that border a crossing $P$ on $C_O$ must be colored a fifth and sixth color that do not occur at $O$.

*Proof.* Because the coloring is locally finite, there must be two regions that border $P$. By Lemma 3.3, these regions cannot be colored any of the colors which occur at $O$. Since both regions cannot be colored the same fifth color by our definition, one region must be colored a fifth color and the other must be colored a sixth color. \[\square\]

Using Lemma 3.10, we can now introduce lemmas that prove that certain crossings necessitate seven colors for the coloring.

**Lemma 3.11.** If a crossing $P$ on $C_O$ is an inward or outward point, then at least seven colors are needed for the coloring.

*Proof.* Let $Q$ be a point at distance one from $P$ on $C_O$. Since a fifth and sixth color occur at $P$ by Lemma 3.10, if $Q$ is not a crossing, then the region $Q$ lies in or borders must be colored a seventh color by Lemma 3.3.  

Hence, we consider the case where $Q$ is a crossing (see Figure 5a). Since a fifth and sixth color occur at $Q$ by Lemma 3.10, we consider the points on $C_Q$ within a neighborhood of $P$. Regardless of whether $P$ is an inward or outward point, there are points on $C_Q$ that lie inside a region and also in an inside or outside neighborhood of $P$. Since these points cannot be colored any of the six colors that occur at $O$ and $Q$ by Lemma 3.3, at least seven colors are needed. \[\square\]

In addition to considering the case where crossings are inward or outward points, we also introduce a lemma that considers the case of crossings that are alternative points.

**Lemma 3.12.** If two crossings lie at distance one on the same alternative arc, then at least seven colors are needed for the coloring.

*Proof.* Let $P$ and $Q$ be the crossings and let $C_P$ and $C_Q$ be the unit circles centered at $P$ and $Q$ respectively. Without loss of generality, if the points on $C_Q$ within an outside neighborhood of $P$ lie within a region that $C_Q$ passes through, then at least seven colors are needed by Lemma 3.3 since a fifth and sixth color occur at $Q$ by Lemma 3.10.  

Therefore, we consider the case in which points on $C_P$ and $C_Q$ within outside neighborhoods at $P$ and $Q$ lie within regions that $C_O$ does not pass through. Without loss of generality, consider a region bordering $Q$ that only $C_P$ passes through. If this region is colored a seventh color, then we are done.  

Suppose that this region is not colored a seventh color. Since $Q$ is an alternative point, this region can only be colored one color not excluded from the colors occurring at $O$ (see Figure 5b). By Lemma 3.3 a region which borders $P$ and contains points on $C_Q$ cannot be colored any of the six colors occurring at $Q$ and $O$. Thus, at least seven colors are needed. \[\square\]
4 Proof of Theorem 2.2

Let \( \alpha \) be a convex polygon coloring of the plane and suppose it contains a vertex \( O \) of degree 5 or more. By Lemma 3.2 there is at least one crossing \( P \) on the unit circle \( C_O \) centered at \( O \). The regions that \( C_O \) passes through which border \( P \) cannot be colored any five colors that occur at \( O \) by Lemma 3.3. Therefore, by our definition, the bordering regions must be colored using a sixth and seventh color.

Hence, we consider the case where all vertices in \( \alpha \) have degree at most 4, and we let vertex \( O \) be a vertex of degree 4. Let \( P \) be a crossing on \( C_O \) and \( Q \) be a point at distance one clockwise from \( P \) on \( C_O \). If \( Q \) is not a crossing, then by Lemma 3.3 the region \( Q \) lies in \( C_O \) or borders must be colored a seventh color. Therefore, we can assume that \( Q \) is a crossing. Similarly, if we repeat the same reasoning clockwise starting from point \( Q \), we can assume that all points on the inscribed hexagon on \( C_O \) that contains \( P \) and \( Q \) are crossings.

If any of the points on the hexagon are inward or outward points, then by Lemma 3.11 at least seven colors are needed for \( \alpha \). Hence, we can assume all points on the hexagon are alternative points. By Lemma 3.9 there are at most four alternative arcs on \( C_O \). Therefore, by the pigeonhole principle, there must be two adjacent points on the inscribed hexagon that lie on the same alternative arc, so by Lemma 3.12 more than six colors are needed for \( \alpha \).

Hence, if a polygon coloring has at least one vertex of degree at least 4, then at least seven colors are needed.

5 Triangle colorings

In this section, we introduce an observation about triangle colorings that proves that Theorem 2.3 is a consequence of Theorem 2.2. Though this is a basic fact, we provide a proof to keep this paper self-contained.

**Proposition 5.1.** Any triangle coloring has a vertex of degree 4 or more.

**Proof.** Consider a triangle coloring of the plane. Assume that all vertices on the plane have degree 3. Let \( \triangle ABC \) be the triangle with greatest angle of any triangle on the plane. Without loss of generality, let \( \angle ABC \) be the angle with the greatest angle measure.

Consider the border that is connected to \( C \), not connected to \( A \) or \( B \), and let \( D \) be the vertex closest to \( C \) on it. Since no angle between any adjacent borders can be greater than
Figure 6: Triangle ABC

π because the coloring is convex and all vertices have degree 3, $D$ must lie on or between the lines $\overrightarrow{AC}$ and $\overrightarrow{BC}$ (see Figure 6). The angle $\angle BCD$ must be at least the sum of $\angle BAC$ and $\angle ABC$ by the exterior angle theorem. However, since $\angle BCD$ must be in a triangle on the plane, this is a contradiction because $\angle BCD$ is larger than $\angle ABC$, which is assumed to be the biggest angle of any triangle on the plane. Therefore, there must be at least one vertex of degree 4 on the plane.

Hence, by Theorem 2.2, at least 7 colors are needed to color the plane for any triangle coloring, which proves Theorem 2.3.

6 Discussion and open problems

Since Theorem 2.2 requires the polygon coloring to have a vertex with degree at least 4, the only possible colorings that are left to be examined are those with only vertices of degree 3. This type of coloring can not be analyzed using the same method.

For instance, let $O$ be a vertex of degree 3 and $C_O$ be the unit circle centered at $O$. By Lemma 3.2 there is at least one crossing on $C_O$. Let $P$ be the crossing and $Q$ be a point one unit away on $C_O$. In the proof of Theorem 2.2 we can assume $Q$ is a crossing because otherwise seven colors are needed immediately.

However, in a polygon coloring with vertices that only have degree 3, $Q$ does not have to be a crossing because if $Q$ lies in a region, then it does not immediately imply that seven colors needed. Therefore, each crossing on $C_O$ can be a unit apart from non-crossings and points within regions.

This leads us to ask for the fewest number of colors needed for polygon colorings in which all vertices have degree 3 (see Figure 1 for examples). If it is shown that at least seven colors are needed for this case, then the chromatic number of convex polygon colorings is 7 as a consequence of Theorem 2.2.

Question 6.1. What is the minimum number of colors needed for convex polygon colorings with only vertices of degree 3?

Though Theorem 2.3 shows that triangle colorings require at least seven colors, this does not imply that the chromatic number of these colorings is 7. The best coloring that we have found for triangle colorings (see Figure 7) sets the upper bound for the chromatic number of triangle colorings to 8.

Since this coloring uses more than seven colors, naturally one asks what the chromatic number is for triangle colorings. We conjecture that the chromatic number for such colorings is 8.
Figure 7: Each triangle has unit side length and every vertex and border point is given the color of the region directly above it.

**Question 6.2.** *What is the chromatic number of triangle colorings of the plane?*

Finally, we note that it may be possible to extend our approach to polygon-like colorings that use differentiable curves instead of line segments. In such a coloring, we can approximate the curved borders connected at vertices by the tangent lines of those curves, which allows us to use most of the arguments for colorings with straight line borders. It would thus follow that a coloring with regions that are curved triangles also requires seven colors.

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