Abstract

In this article, we prove exact estimates for the $W$-invariant Dunkl kernel and heat kernel, for the root system of type $A$ with arbitrary positive multiplicities. We apply the estimates of the $W$-invariant Dunkl heat kernel to compute sharp estimates for the Newton kernel and for the $s$-stable semigroups generated by a fractional power of the $W$-invariant Dunkl Laplacian.

Keywords: Dunkl kernel, root system, spherical function, heat kernel, Newton kernel, $s$-stable semigroup, fractional Dunkl Laplacian

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1 Introduction and notations

In Dunkl analysis for a root system $\Sigma$ on $\mathbb{R}^d$, a crucial role is played by the Dunkl kernel $E_k(X,Y)$ and by the Dunkl heat kernel $p_t(X,Y)$. Finding good estimates of the kernels $E_k$ and of the Dunkl heat kernel $p_t$ is a challenging and important subject, developed recently in [1, 11]. In this paper we prove exact estimates of both these kernels in the $W$-radial rational Dunkl case, for the root system $A_n$ with arbitrary positive multiplicities.

For a good introduction on rational Dunkl theory, the reader should consider the paper [1] or the book [9]. We provide here some details and notations on Dunkl analysis.

For every root $\alpha \in \Sigma$, let $\sigma_{\alpha}(X) = X - \frac{2(\alpha \cdot X)}{\langle \alpha, \alpha \rangle} \alpha$. The Weyl group $W$ associated to the root system is generated by the reflection maps $\sigma_{\alpha}$.

A function $k : \Sigma \to \mathbb{R}$ is called a multiplicity function if it is invariant under the action of $W$ on $\Sigma$.

Let $\partial_\xi$ be the derivative in the direction of $\xi \in \mathbb{R}^d$. The Dunkl operators indexed by $\xi$ are then given by

$$T_\xi(k)f(X) = \partial_\xi f(X) + \sum_{\alpha \in \Sigma_+} k(\alpha) \alpha(\xi) \frac{f(X) - f(\sigma_{\alpha}X)}{\langle \alpha, X \rangle}.$$ 

The $T_\xi$’s, $\xi \in \mathbb{R}^d$, form a commutative family.

For fixed $Y \in \mathbb{R}^d$, the Dunkl kernel $E_k(\cdot, \cdot)$ is then the only real-analytic solution to the system

$$T_\xi(k)|_X E_k(X,Y) = \langle \xi, Y \rangle E_k(X,Y), \ \forall \xi \in \mathbb{R}^d$$

with $E_k(0,Y) = 1$. In fact, $E_k$ extends to a holomorphic function on $\mathbb{C}^d \times \mathbb{C}^d$.

Its $W$-invariant version $E_k^W(X,\lambda)$ is called a Bessel function of Dunkl type (see [9, p. 57]) or a spherical function $\psi_\lambda(X)$ of type $\Sigma$ (refer to [17]). In this paper we use the latter terminology and notation. We have

$$\psi_\lambda(X) = E_k^W(X,\lambda) = \frac{1}{|W|} \sum_{w \in w} E_k(w \cdot X, \lambda)$$
and \( \psi_\lambda(X) \) is the only real-analytic solution of the system
\[
p(T_{e_1}, \ldots, T_{e_n})(k)|_X \psi_\lambda(X) = p(\lambda) \psi_\lambda(X), \quad \forall \lambda \in \mathbb{R}^d
\]
for every Weyl-invariant polynomial \( p \) (here \( e_1, \ldots, e_d \) represent the standard basis on \( \mathbb{R}^d \)).

Let \( \omega_k(X) := \prod_{\alpha \in \Sigma_+} |(\alpha, X)|^{2k(\alpha)} \) be the Dunkl weight function on \( \mathbb{R}^d \). Recall that the Dunkl transform of a \( W \)-invariant function \( f \) on \( \mathbb{R}^d \)
\[
\hat{f}(\lambda) := c_k^{-1} \int f(x) \psi_{-i\lambda}(X) \omega_k(X) dX, \quad \lambda \in \mathbb{R}^d,
\]
plays the role of the spherical Fourier transform in \( W \)-invariant Dunkl analysis. (here the constant \( c_k \) is the Macdonald–Mehta–Selberg integral.)

In [11, Conjecture 18], we made the following conjecture on the growth of spherical functions \( \psi_\lambda(x) \) of type \( \Sigma \). We use the Cartan algebra notation \( a = \mathbb{R}^d \). Then \( a^+ \) denotes the open positive Weyl chamber with respect to a system \( \Sigma^+ \) of positive roots.

**Conjecture 1.1.** If \( \lambda, X \in a^+ \), then
\[
\psi_\lambda(e^X) \asymp \frac{e^{\lambda(X)}}{\prod_{\alpha > 0} (1 + \alpha(X) \alpha(\lambda))^{k(\alpha)}},
\]
For the root system \( A_n \) on \( \mathbb{R}^d \), \( d \geq n \), and multiplicity \( k(\alpha) = k > 0 \), this becomes
\[
\psi_\lambda(e^X) \asymp \frac{e^{\lambda(X)}}{\prod_{i<j \leq n+1} (1 + (x_i - x_j)(\lambda_i - \lambda_j))^k}, \quad \lambda, X \in a^+
\]
(1.1)
(the underlying constants here only depend on \( k \)).

The notation \( f \asymp g \) in a domain \( D \) means that there exists \( C_1 > 0 \) and \( C_2 > 0 \) such that \( C_1 g(x) \leq f(x) \leq C_2 g(x) \) with \( C_1 \) and \( C_2 \) independent of \( x \in D \). Recall that for the root system \( A_n \) on \( \mathbb{R}^{n+1} \), the positive Weyl chamber is defined by \( a^+ = \{ X \in \mathbb{R}^{n+1} | x_1 > x_2 > \ldots > x_{n+1} \} \).

**Remark 1.2.** This conjecture includes the cases of the symmetric spaces of noncompact type \( \text{POS}_i(n, F) \), the positive definite matrices of determinant 1 over \( F \) where \( F = \mathbb{R} \) (the real numbers with \( k = 1/2 \)), \( F = \mathbb{C} \) (the complex numbers with \( k = 1 \)), \( F = \mathbb{H} \) (the quaternion numbers with \( k = 2 \) or \( F = \mathbb{O} \) (the Octonions with \( k = 4 \)) when \( n = 3 \). In [11], we proved the conjecture for the root system \( A_n \) in the complex case \( k = 1 \).

The main tool of the proof of Conjecture [11] for root systems \( A_n \) is the following iterative formula for the spherical functions of type \( A \), proven in [17]. Here we do not assume that the elements of the Lie algebra have trace 0. Here the Cartan subalgebra \( a \) for the root system \( A_n \) is isomorphic to \( \mathbb{R}^{n+1} \). For \( \lambda \in a = \mathbb{R}^{n+1} \) and \( X \in a^+ \), we have
\[
\psi_\lambda(e^X) = e^{\lambda(X)} \text{ if } n = 1 \quad \text{and}
\]
\[
\psi_\lambda(e^X) = \frac{\Gamma(k(n+1))}{\Gamma(k)^{n+1}} e^{\lambda(n+1)} \sum_{r=1}^{n+1} x_r \pi(X)^{1-2k} \int_{x_{n+1}}^{x_n} \cdots \int_{x_2}^{x_1} \psi_{\lambda_0}(e^Y)
\]
(1.2)
\[
\left[ \prod_{i=1}^{n} \left( \prod_{j=1}^{i} (x_j - y_j) \prod_{j=i+1}^{n+1} (y_i - x_j) \right) \right]^{k-1} \prod_{i<j \leq n} (y_i - y_j) dy_1 \cdots dy_n
\]

where \( \lambda_0(U) = \sum_{r=1}^{n} (\lambda_r - \lambda_{n+1}) u_k \) and \( \pi(X) = \prod_{i<j \leq n+1} (x_i - x_j) \).

**Remark 1.3.** Formula (1.2) concerns the action of the root system \( A_n \) on \( R^{n+1} \). If we assume \( \sum_{k=1}^{n+1} x_k = 0 = \sum_{k=1}^{n+1} \lambda_k \), we have then the action of the root system \( A_n \) on \( R^n \). We can also consider the action of \( A_n \) on any \( R^m \) with \( m \geq n \) by deciding on which \( n+1 \) entries \( x_k \), the roots act. These considerations do not affect the results of this article.

The Dunkl heat kernel \( p_t(X,Y) \) is given as

\[
p_t(X,Y) = \frac{1}{2^{\gamma+d/2} c_k} t^{\frac{d-\gamma}{2}} e^{-\frac{|X|^2-|Y|^2}{4t}} E_k \left( X, \frac{Y}{2t} \right).
\]  

(1.3)

where \( \gamma = \sum_{\alpha>0} k(\alpha) \). Establishing estimates of the Dunkl heat kernel is equivalent to estimating the Dunkl kernel as demonstrated by equation (1.3).

In [14, Lemma 4.5], it is shown that

\[
\int_{R^d} p_t(X,Y) \omega_k(Y) dY = 1
\]

\[
\Delta_k|_X p_t(X,Y) = \frac{\partial}{\partial t} p_t(X,Y)
\]

where the Dunkl Laplacian \( \Delta_k \) equals

\[
\Delta_k f(X) = \sum_{i=1}^{d} T_{e_i}^2 f(X)
\]

\[
= \Delta f(X) + 2 \sum_{\alpha \in \Sigma^+} k(\alpha) \left[ \frac{\langle \alpha, \nabla f(X) \rangle}{\langle \alpha, X \rangle} - \frac{f(X) - f(\sigma_{\alpha} X)}{\langle \alpha, X \rangle^2} \right].
\]

Here \( \Delta \) and \( \nabla \) denote the regular Laplacian and gradient.

The formula (1.3) remains true for the \( W \)-invariant kernels \( p_t^W \) and \( E^W \) and translates in a similar relationship between the spherical function \( \psi_\lambda \) and the heat kernel \( p_t^W(X,Y) \):

\[
p_t^W(X,Y) = \frac{1}{2^{\gamma+d/2} c_k} t^{\frac{d-\gamma}{2}} e^{-\frac{|X|^2-|Y|^2}{4t}} \psi_X \left( \frac{Y}{2t} \right).
\]  

(1.4)

In Section 2 we prove the Conjecture 1.1 for the root system \( A_n \), with an arbitrary multiplicity \( k > 0 \), i.e. we prove the formula (1.1) providing exact estimates for the spherical functions \( \psi_\lambda(X) \) in the two variables \( X, \lambda \) when \( \lambda \) is real.

In Section 3.1 we apply the sharp estimates (1.1) of the spherical functions \( \psi_\lambda(X) \) to the \( W \)-invariant Dunkl heat kernel \( p_t(X,Y) \) for the root system \( A_n \), with an arbitrary multiplicity \( k > 0 \). In the Theorem 3.1 we obtain sharp estimates of \( p_t(X,Y) \) in three variables \( t, X, Y \).

Next, in Sections 3.2 and 3.3 we apply the Theorem 3.1 to the \( W \)-invariant Dunkl Newton kernel and to the \( W \)-invariant s-stable semigroups, respectively. In all cases, we obtain sharp estimates.

3
2 Proof of the Conjecture in the case $A_n$.

We will assume from now on that $X \in a^+$ and $\lambda \in \overline{a}^+$. 

**Notation 2.1.** We will write $f(x) \lesssim g(x)$ ($f(x) \gtrsim g(x)$) for $x \in D$ if there exists a constant $C > 0$ independent of $x$ such that $f(x) \leq C g(x)$ ($f(x) \geq C g(x)$) for all $x \in D$. We will use the notation $M_k = (x_k + x_{k+1})/2$.

**Remark 2.2.** Suppose $i < j$. We will use repeatedly the fact that the functions $x/(1 + (\lambda_i - \lambda_j)x)$ and $(x/(1 + (\lambda_i - \lambda_j)x))^k$, $k \leq 1$ are increasing functions of $x$. A feature of our proofs will be the distinction between the cases $0 < k \leq 1$ and $k > 1$.

**Proposition 2.3.** Conjecture 1.1 is equivalent to

$$I(n) \propto \frac{\pi(X)^{2k-1}}{\prod_{i<j \leq n+1} ((1 + (\lambda_i - \lambda_j)(x_i - x_j))^k}$$

where

$$I(n) = \int_{x_{n+1}}^{x_n} \cdots \int_{x_2}^{x_1} e^{-\lambda \sum_{i=1}^{n} (\lambda_i - \lambda_{n+1})(x_i - x_1)} \left( \prod_{i \leq j \leq n} (x_i - y_j) \prod_{i < j \leq n+1} (y_i - x_j) \right)^{k-1} \prod_{i < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} dy_1 \ldots dy_n.$$

**Proof.** The integral $I(n)$ corresponds to a constant multiple of $e^{-\lambda(X)} \pi(X)^{2k-1} \psi_{\lambda}(e^X)$ in which we have replaced $\psi_{\lambda}(e^y)$ in (1.2) by its asymptotic expression conjectured in (1.1).

We start by two technical results.

**Lemma 2.4.** For $k > 0$ and $x \geq 0$, we have

$$\int_0^x u^{k-1} e^{-u} du \asymp \left( \frac{x}{1+x} \right)^k.$$

**Proof.** The result is clearly true if $0 \leq x < 1$ (use $e^{-1} \leq e^{-x} \leq 1$ and integrate). If $x \geq 1$ then

$$\int_0^1 u^{k-1} e^{-u} du \leq \int_0^x u^{k-1} e^{-u} du < \int_0^\infty u^{k-1} e^{-u} du$$

and the result follows.


Proposition 2.5. Assume that $\gamma = x_n - x_{n+1}$ is the largest positive root and let

$$I_1 = \int_{M_n}^{x_n} \ldots \int_{x_2}^{x_1} e^{-\sum_{i=1}^{n}(\lambda_i - \lambda_{n+1})(x_i - y_n)} \left( \prod_{i \leq j \leq n} (x_i - y_j) \prod_{i < j \leq n+1} (y_i - x_j) \right)^{k-1} \prod_{i < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))} dy_1 \ldots dy_n.$$ 

Then $I_1 \approx I^{(n)}$.

Proof. Let $I_2 = I^{(n)} - I_1$. In $I_1$ and $I_2$, consider only the corresponding integral in $y_n$, calling the resulting expressions $\tilde{I}_1$ and $\tilde{I}_2$. Observing that $y_n - x_{n+1} \succ \gamma$ for $y_n \in [x_n, M_n]$, we have

$$\tilde{I}_1 \approx \gamma^{k-1} \int_{M_n}^{x_n} e^{-(\lambda_n - \lambda_{n+1})} (x_n - y_n) \left( \prod_{i \leq n} (x_i - y_n) \right)^{k-1} \prod_{i < n} \frac{x_{i+1} - y_n}{(1 + (\lambda_i - \lambda_n) \gamma)^k} dy_n.$$

If $0 < k \leq 1$ then,

$$\tilde{I}_1 \gtrsim e^{-(\lambda_n - \lambda_{n+1}) \gamma/2} \gamma^{k-1} \int_{M_n}^{x_n} (x_1 - y_n)^{k-1} \prod_{i=2}^{n} (x_i - y_n)^{k-1} \prod_{i < n} \frac{x_{i+1} - y_n}{(1 + (\lambda_i - \lambda_n) \gamma)^k} dy_n$$

$$\gtrsim \gamma^{k-1} \prod_{i < n} (1 + (\lambda_i - \lambda_n) \gamma)^{k/2} \int_{M_n}^{x_n} (x_n - y_n)^{(n-1)k} (x_1 - y_n)^{k-1} dy_n$$

$$\gtrsim \gamma^{(n+1)k-1} \prod_{i < n} (1 + (\lambda_i - \lambda_n) \gamma)^{k}.$$

Indeed, if $n > 1$ then $(x_1 - y_n)^{k-1} \gtrsim \gamma^{k-1}$ and the rest can easily be integrated. If $n = 1$, then

$$\int_{M_n}^{x_n} (x_n - y_n)^{(n-1)k} (x_1 - y_n)^{k-1} dy_n = \int_{M_n}^{x_n} (x_n - y_n)^{k-1} dy_n \approx \gamma^k.$$ 

If $k > 1$, we have

$$\tilde{I}_1 \gtrsim \gamma^{k-1} e^{-(\lambda_n - \lambda_{n+1}) \gamma/2} \int_{M_n}^{x_n} \prod_{i=1}^{n-1} (x_i - y_n)^{k-1} \prod_{i < n} \frac{y_i - y_n}{(1 + (\lambda_i - \lambda_n)(y_i - y_n))^k} (x_n - y_n)^{k-1} dy_n$$

$$\gtrsim \gamma^{k-1} e^{-(\lambda_n - \lambda_{n+1}) \gamma/2} \int_{M_n}^{x_n} \left( \prod_{i < n} \frac{y_i - y_n}{1 + (\lambda_i - \lambda_n)(y_i - y_n)} \right)^k (x_n - y_n)^{k-1} dy_n.$$
\[ \gamma \geq \prod_{i<n} \frac{1}{1 + (\lambda_i - \lambda_n) \gamma} \frac{e^{-\frac{\gamma}{2}}}{(x_n - y_n)^k} \int_{x_n}^{x_{n-1}} e^{-\lambda_{n-1}} (x_n - y_n)^{k-1} dy_n \]

\[ \gamma \geq \prod_{i<n} \frac{1}{1 + (\lambda_i - \lambda_n) \gamma} \frac{e^{-\frac{\gamma}{2}}}{(x_n - y_n)^{n-1}} \int_{x_n}^{x_{n-1}} (y_n - x_n)^{k-1} dy_n \]

\[ \gamma \propto \prod_{i<n} \frac{1}{1 + (\lambda_i - \lambda_n) \gamma} \frac{e^{-\frac{\gamma}{2}}}{(x_n - y_n)^k} \]

On the other hand for \( k > 0 \), observing that \( x_i - y_n \propto \gamma \) and \( y_i - y_n \propto \gamma \) for \( y_n \in [M_n, x_{n+1}] \), we have

\[ I_2 \propto \prod_{i<n} \frac{\gamma}{(1 + (\lambda_i - \lambda_n) \gamma)} \int_{x_{n+1}}^{M_n} e^{-\lambda_{n-1}} (x_n - y_n) (y_n - x_{n+1})^{k-1} dy_n \]

\[ \propto \prod_{i<n} \frac{1}{1 + (\lambda_i - \lambda_n) \gamma} \frac{e^{-\frac{\gamma}{2}}}{(x_n - y_n)^{k-1}} \int_{x_{n+1}}^{y_{n+1}} (y_n - x_{n+1})^{k-1} dy_n \]

\[ \propto \prod_{i<n} \frac{1}{1 + (\lambda_i - \lambda_n) \gamma} \frac{e^{-\frac{\gamma}{2}}}{(x_n - y_n)^k} \leq I_1 \]

which allows us to conclude.

**Theorem 2.6.** Conjecture [1.1] holds for the root system \( A_n \), \( n \geq 1 \) with root multiplicity \( k > 0 \).

**Proof.** The result is proven using induction. Using Proposition 2.5 with \( n = 1 \), we have, using \( u = (\lambda_1 - \lambda_2) (x_1 - y_1) \) and Lemma 2.4

\[ I^{(1)} \propto \int_{M_1}^{x_1} e^{-(\lambda_1 - \lambda_2)} (x_1 - y_1)^{k-1} (y_1 - x_2)^{k-1} dy_1 \]

\[ \propto (x_1 - x_2)^{k-1} \int_{M_1}^{x_1} e^{-(\lambda_1 - \lambda_2)} (x_1 - y_1)^{k-1} dy_1 \]

\[ = (x_1 - x_2)^{k-1} (\lambda_1 - \lambda_2)^{-k} \int_{0}^{(\lambda_1 - \lambda_2) (x_1 - x_2)/2} e^{-u} u^{k-1} du \]

\[ \propto (x_1 - x_2)^{k-1} \left( \frac{(x_1 - x_2)/2}{1 + (\lambda_1 - \lambda_2) (x_1 - x_2)/2} \right)^k \]

which proves the formula [2.1] in the case \( n = 1 \).

Assume that the result holds for the root systems \( A_1, A_2, \ldots, A_{n-1} \). We will use Proposition 2.3 and will proceed by assuming, in turn for each \( m < n \), that \( \alpha_m = x_m - x_{m+1} \) is the largest root. We will discuss the case \( m = n \) at the end.

We will proceed as follows. As in the proof of Proposition 2.5, we will divide the integral in two parts \( I_1 \) and \( I_2 \), show that \( I_1 \) has the desired asymptotics and that \( I_2 \lesssim I_1 \).
Assume now that \( \alpha_m = x_m - x_{m+1}, 1 \leq m \leq n-1 \), is the largest root. Noting that \( x_i - y_j \leq \alpha_m, i \leq m, m < j \leq n, y_i - x_j \leq \alpha_m, i \leq m, j \geq m + 2, y_i - y_j \leq \alpha_m, i \leq m, m < j \leq n \), for \( y_m \in [M_m, x_m] \), we have

\[
I_1 = \int_{x_{n+1}}^{x_n} \cdots \int_{M_{m+1}}^{x_m} \int_{x_2}^{x_1} e^{-\sum_{i=1}^n (\lambda_i - \lambda_{n+1})(x_i - y_i)} \\
\left( \prod_{i \leq j \leq m} (x_i - y_j) \prod_{i < j \leq m+1} (y_i - x_j) \right)^{k-1}
\left( \prod_{m < i \leq j \leq n} (x_i - y_j) \prod_{m < i \leq j \leq n+1} (y_i - x_j) \right)^{k-1}
\left( \prod_{i \leq m < j \leq n} (x_i - y_j) \prod_{i \leq m < j \leq n+1} (y_i - x_j) \right)^{k-1}
\prod_{i \leq j \leq m} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} \prod_{m < i \leq j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k}
\prod_{i \leq m < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} \frac{\alpha_{m-1}}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k}
\int_{x_{n+1}}^{x_n} \cdots \int_{M_{m+1}}^{x_m} \int_{x_2}^{x_1} e^{-\sum_{i=1}^n (\lambda_i - \lambda_{n+1})(x_i - y_i)}
\left( \prod_{i \leq j \leq m} (x_i - y_j) \prod_{i < j \leq m+1} (y_i - x_j) \right)^{k-1}
\left( \prod_{m < i \leq j \leq n} (x_i - y_j) \prod_{m < i \leq j \leq n+1} (y_i - x_j) \right)^{k-1}
\prod_{i \leq m < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} \prod_{m < i \leq j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k}
dy_1 \cdots dy_n
\]

\[
= \frac{\alpha_m^{2m(n-m)(k-1) + m(n-m)}}{\prod_{i \leq m < j \leq n} (1 + (\lambda_i - \lambda_j)\alpha_m)^k} \int_{x_{n+1}}^{x_n} \cdots \int_{M_{m+1}}^{x_m} \int_{x_2}^{x_1} e^{-\sum_{i=1}^n (\lambda_i - \lambda_{n+1})(x_i - y_i)}
\left( \prod_{i \leq j \leq m} (x_i - y_j) \prod_{i < j \leq m+1} (y_i - x_j) \right)^{k-1}
\left( \prod_{m < i \leq j \leq n} (x_i - y_j) \prod_{m < i \leq j \leq n+1} (y_i - x_j) \right)^{k-1}
\prod_{i \leq m < j \leq n} \frac{y_i - y_j}{(1 + (\lambda_i - \lambda_j)(y_i - y_j))^k} dy_1 \cdots dy_m
\]
\[
\emph{\text{which has the desired asymptotics (we used Proposition 2.5 and the induction hypothesis on } A_n \text{ and on } A_{n-m}).}
\]

It remains to show that \( I_2 = \vec{I}^{(n)} - I_1 \leq I_1 \). As in the proof of Proposition 2.5, it suffices to show that \( \vec{I}_1 \geq \vec{I}_2 \) where \( \vec{I}_1 \) (respectively \( \vec{I}_2 \)) is the portion of \( I_1 \) (\( I_2 \)) integrated with respect to \( y_m \).

Now, since \( y_m - x_j \asymp \alpha_m, \; m < j \leq n + 1 \), and \( y_m - y_j \asymp \alpha_m, \; m < j \leq n \), when \( y_m \in [M_m, x_m] \), we have

\[
\vec{I}_1 \asymp \alpha_m^{(n+1-m)(k-1)+n-m} \int_{M_m}^{x_m} e^{-{(\lambda_{m+1} - \lambda_n)} (x_m - y_m)} \prod_{i \leq m} (x_i - y_m)^{k-1} \prod_{i < m} \frac{y_i - y_m}{(1 + (\lambda_i - \lambda_m)(y_i - y_m))^k} \, dy_m.
\]

If \( k > 1 \) then

\[
\vec{I}_1 \geq \alpha_m^{(n+1-m)(k-1)+n-m} e^{-(\lambda_{m+1} - \lambda_n) \alpha_m / 2} \int_{M_m}^{x_m} \prod_{i \leq m} (x_m - y_m)^{k-1} \prod_{i < m} \frac{x_m - y_m}{(1 + (\lambda_i - \lambda_m)(x_m - y_m))^k} \, dy_m \geq \alpha_m^{(n+1-m)(k-1)+n-m} e^{-(\lambda_{m+1} - \lambda_n) \alpha_m / 2} \int_{M_m}^{x_m} (x_m - y_m)^{m(k-1)+m-1} \, dy_m = \prod_{m < j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k \prod_{i < m} (1 + (\lambda_i - \lambda_m) \alpha_m)^k.
\]
If $0 < k \leq 1$ then
\[
\tilde{I}_1 \lesssim \frac{\alpha_m^{(n+1-m) (k-1)+n-m}}{\prod_{m<j \leq n} (1 + (\lambda_m - \lambda_j) \alpha_m)^k} e^{-\left(\lambda_m - \lambda_{n+1}\right) \alpha_m / 2} \int_{x_m}^{x_{m+1}} \prod_{i<m} \alpha_m^{k-1} (x_m - y_m)^{k-1} \prod_{i<m} (1 + (\lambda_i - \lambda_m) \alpha_m)^k \int_{M_m}^{x_m} (x_m - y_m)^{m-1+k} \, dy_m.
\]

On the other hand, since $x_i - y_m \asymp \alpha_m$, $i \leq m$, and $y_i - y_m \asymp \alpha_m$, $i < m$, when $y_m \in [x_{m+1}, M_m]$,
\[
\tilde{I}_2 \asymp \alpha_m^{(k-1) (n+1-m) (k-1)+n-m} \prod_{i<m} \alpha_m^{(n+1-m) (k-1)+n-m} e^{-\left(\lambda_m - \lambda_{n+1}\right) \alpha_m / 2} \int_{x_m}^{x_{m+1}} \prod_{i<m} (1 + (\lambda_i - \lambda_m) \alpha_m)^k \int_{x_{m+1}}^{x_m} (y_m - y_j)^{-1} \prod_{m<j \leq n+1} \frac{y_m - y_j}{(1 + (\lambda_m - \lambda_j) (y_m - y_j))^k} \, dy_m.
\]

If $k > 1$ then
\[
\tilde{I}_2 \lesssim \alpha_m^{(k-1) (n+1-m) (k-1)+n-m} \prod_{i<m} \alpha_m^{(n+1-m) (k-1)+n-m} e^{-\left(\lambda_m - \lambda_{n+1}\right) \alpha_m / 2} \int_{x_m}^{x_{m+1}} \prod_{m<j \leq n} (y_m - y_j)^{k-1} \prod_{m<j \leq n} \frac{y_m - y_j}{(1 + (\lambda_m - \lambda_j) (y_m - y_j))^k} \, dy_m.
\]

If $0 < k \leq 1$ then
\[
\tilde{I}_2 \lesssim \alpha_m^{(k-1) (n+1-m) (k-1)+n-m} \prod_{i<m} \alpha_m^{(n+1-m) (k-1)+n-m} e^{-\left(\lambda_m - \lambda_{n+1}\right) \alpha_m / 2} \int_{x_m}^{x_{m+1}} \prod_{m<j \leq n} (y_m - y_j)^{k-1} \prod_{m<j \leq n} \frac{y_m - y_j}{(1 + (\lambda_m - \lambda_j) (y_m - y_j))^k} \, dy_m.
\]
\[
\prod_{m<j\leq n} \frac{y_m - x_{j+1}}{(1 + (\lambda_m - \lambda_j)(y_m - x_{j+1}))^k} (y_m - x_{m+1})^{k-1} \, dy_m
\]

\[
= \alpha_m^{m(k-1)} \prod_{i<m} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_m) \alpha_m)^k} \,
\]

\[
e^{-.(\lambda_m-\lambda_{n+1}) \alpha_m/2} \int_{x_{m+1}}^{M_n} \prod_{m<j\leq n} \frac{y_m - x_{j+1}}{1 + (\lambda_m - \lambda_j)(y_m - x_{j+1})} \right)^k \, dy_m
\]

\[
\ll \alpha_m^{m(k-1)} \prod_{i<m} \frac{\alpha_m}{(1 + (\lambda_i - \lambda_m) \alpha_m)^k} e^{-.(\lambda_m-\lambda_{n+1}) \alpha_m/2} \alpha_m^k \]

\[
\left( \prod_{m<j\leq n} \frac{\alpha_m}{1 + (\lambda_m - \lambda_j) \alpha_m} \right)^k \lesssim \tilde{I}_i.
\]

By the structure of the root system \(A_n\), the case \(\alpha_n\) maximal is equivalent to the case \(\alpha_1\) maximal. Indeed, in formula (1.2), one does not assume that \(\lambda \in \overline{\mathbb{C}}\). We also know that \(\psi_\lambda(e^X)\) is invariant under permutation of its \(\lambda\) argument. Hence one can re-write (1.2) by exchanging \(\lambda_1\) and \(\lambda_{n+1}\),

\[
\psi_\lambda(e^X) = e^{\lambda(X)} \text{ if } n = 1 \text{ and } \\
\psi_\lambda(e^X) = \frac{\Gamma(k (n+1))}{\Gamma(k)^{n+1}} e^{\lambda_1 \sum_{i=1}^{n+1} x_i} \left( \prod_{i<j\leq n+1} (x_i - x_j) \right)^{1-2k} \int_{x_{n+1}}^{x_1} \cdots \int_{x_2}^{x_1} \psi_{\lambda_0}(e^Y)
\]

\[
\left[ \prod_{i=1}^{n} \left( \prod_{j=1}^{i} (x_j - y_j) \prod_{j=i+1}^{n+1} (y_i - x_j) \right) \right]^{k-1} \prod_{i<j\leq n} (y_i - y_j) \, dy_1 \cdots dy_n
\]

where \(\tilde{\lambda}_0(U) = \sum_{r=2}^{n+1} (\lambda_r - \lambda_1) u_r\). We used the fact that

\[
\psi_{[\lambda_{n+1}-\lambda_1, \ldots, \lambda_n-\lambda_1]}(e^Y) = \psi_{[\lambda_2-\lambda_1, \ldots, \lambda_n-\lambda_1, \lambda_{n+1}-\lambda_1]}(e^Y).
\]

Conjecture 1.1 is equivalent to

\[
J^{(n)} \propto \frac{\pi(X)^{2k-1}}{\prod_{i<j\leq n+1} (1 + (\lambda_i - \lambda_j)(x_i - x_j))^k}
\]

where

\[
J^{(n)} = \int_{x_{n+1}}^{x_n} \cdots \int_{x_2}^{x_1} e^{-\sum_{i=1}^{n+1} (\lambda_1 - \lambda_{i+1})(y_i - x_{i+1})} \left( \prod_{i<j\leq n} (x_i - y_j) \prod_{i<j\leq n+1} (y_i - x_j) \right)^{k-1} \prod_{i<j\leq n} \frac{y_i - y_j}{(1 + (\lambda_{i+1} - \lambda_{j+1})(y_i - y_j))^k} \, dy_1 \cdots dy_n.
\]
The term $J^{(n)}$ corresponds to a constant multiple of $e^{-\lambda(X)} \pi(X)^{2k-1} \psi_\lambda(e^X)$ in which we have replaced $\psi_\lambda(e^Y)$ in (12) by its asymptotic expression conjectured in (11). One then proves the case $\alpha_n$ maximal as one proves the case $\alpha_1$ maximal.

This concludes the proof of the estimate (11) for $X \in a^+$ (recall that the formula (12) holds for $X \in a^+$). The estimates that we find for $\psi_\lambda(e^X)$ extend to $X \in \overline{a^+}$ by continuity.

\[ \square \]

3 Applications

3.1 Estimates of the $W$-invariant Dunkl Heat Kernel

The following theorem establishes, for root systems $A_n$ and for any multiplicity $k > 0$, the estimates of the $W$-invariant Dunkl Heat Kernel conjectured in the Conjecture 18 of [11].

**Theorem 3.1.** For the root systems of type $A$, we have for $X, Y \in \overline{a^+}$

$$p_t^W(X, Y) \sim \frac{t^{-d/2} e^{-(X-Y)^2/(4t)}}{\prod_{\alpha > 0} (t + \alpha(X) \alpha(Y))^k}.$$

**Proof.** Consider the relation (14)

$$p_t^W(X, Y) = t^{-d/2-\gamma} e^{-(X-Y)^2/(4t)} \psi_X(Y/(2t))$$

with $\gamma = \sum_{\alpha > 0} k(\alpha) = k|\Sigma^+|$. From Theorem 2.6 we have

$$p_t^W(X, Y) \sim t^{-d/2-\gamma+|\Sigma^+|} e^{-(|X|^2+|Y|^2)/(4t)} \frac{e(X,Y/(2t))}{\prod_{\alpha > 0} (1 + \alpha(X) \alpha(Y/(2t)))^k}$$

$$= t^{-d/2-\gamma+|\Sigma^+|} e^{-(|X|^2+|Y|^2)/(4t)} \frac{e(X,Y/(2t))}{\prod_{\alpha > 0} (2t + \alpha(X) \alpha(Y))^k}$$

$$\approx \frac{t^{-d/2} e^{-(X-Y)^2/(4t)}}{\prod_{\alpha > 0} (t + \alpha(X) \alpha(Y))^k}.$$ 

\[ \square \]

3.2 Estimates of the $W$-invariant Dunkl Newton Kernel

The $W$-invariant Dunkl Newton kernel $N^W(X, Y)$ is the kernel of the inverse operator of the Dunkl Laplacian $\Delta^W$. It is the fundamental solution of the problem $\Delta^W u = f$ where $f$ is given and $|u(x)| \to 0$ as $x \to \infty$. It is defined by

$$N^W(X, Y) = \int_0^\infty p_t^W(X, Y) \, dt,$$

where $p_t^W(X, Y)$ is the heat kernel of $\Delta^W$.

In [10], we stated the following conjecture for the Weyl invariant Newton kernel for $d \geq 3$ and proved it for complex root systems.
**Conjecture 3.2.** For $X, Y \in \mathfrak{a}^+$ and $d \geq 3$, we have

$$N^W(X, Y) \approx \frac{1}{|X - Y|^{d-2} \prod_{\alpha \in \Sigma^+} |X - \sigma_\alpha Y|^{2k(\alpha)}}.$$ 

In this section, we prove the conjecture in the case of root systems of type $A$ and prove a similar result in the case $d = 2$.

The next three lemmas will be useful to derive sharp estimates for the Newton kernel.

**Lemma 3.3.** Suppose $k > 0$, $a \geq 0$, $b_i \geq 0$, $a + b_i > 0$, $i = 1, \ldots, m$ and $N > km - 1$. Then

$$J := \int_0^\infty \frac{u^N e^{-u} \, du}{\prod_{i=1}^m (a + b_i u)^k} \approx \frac{1}{\prod_{i=1}^m (a + b_i)^k}.$$ 

**Proof.** Note that $a + b_i u \leq (a + b_i) u$ whenever $u \geq 1$. Therefore, we have

$$J \geq \int_1^\infty \frac{u^N e^{-u} \, du}{\prod_{i=1}^m (a + b_i u)^k} \geq \frac{1}{\prod_{i=1}^m (a + b_i)^k} \int_1^\infty u^{N-k} e^{-u} \, du.$$ 

Let $\Lambda$ be the (possibly empty) set of indices where $a \leq b_i$.

$$J \lesssim \frac{1}{\prod_{i \in \Lambda}^m (a + b_i)^k} \max_{0 \leq m_0 \leq m} \int_0^\infty u^{N-m_0 k} e^{-u} \, du$$

(we understand an empty product to be equal to 1).

**Lemma 3.4.** Suppose $k > 0$, $a > 0$ and $b \geq 0$. Then

$$J := \int_0^\infty \frac{u^{k-1} e^{-u} \, du}{(a + b u)^k} \approx \frac{\ln(2 + b/a)}{(a + b)^k}.$$ 

**Proof.** If $0 \leq b \leq a$ then

$$\int_0^\infty \frac{u^{k-1} e^{-u} \, du}{(a + a u)^k} \leq J \leq \int_0^\infty \frac{u^{k-1} e^{-u} \, du}{a^k}$$

and the result holds. We now assume $a \leq b$. We then have

$$J \approx \int_0^a \frac{u^{k-1}}{a^k} \, du + \int_{a/b}^1 \frac{u^{k-1}}{(b u)^k} \, du + \int_1^\infty \frac{u^{k-1} e^{-u} \, du}{(b u)^k}$$

$$\approx \frac{1}{b^k} + \frac{1}{b^k} \int_{a/b}^1 u^{-1} \, du + \frac{1}{b^k} \int_1^\infty u^{-1} e^{-u} \, du$$

$$\approx \frac{1}{b^k} + \frac{1}{b^k} \ln(b/a) + \frac{1}{b^k} \approx \frac{\ln(2 + b/a)}{b^k}$$

which concludes the proof.

\[\square\]
Lemma 3.5. Suppose $k > 0$, $a \geq 0$ and $0 \leq b_1 \leq b_2 \leq b_3$ then

\[ J := \int_0^\infty \frac{u^{3k-1} e^{-u} du}{(a + b_1 u)^k (a + b_2 u)^k (a + b_3 u)^k} \preceq \frac{\ln(2 + b_1/a)}{(a + b_1)^k (a + b_2)^k (a + b_3)^k}. \]

Proof. If $b_1 \leq a$ then

\[ \int_1^\infty \frac{u^{3k-1} e^{-u} du}{(a + a u)^k ((a + b_2) u)^k ((a + b_3) u)^k} \leq J \leq \int_0^\infty \frac{u^{3k-1} e^{-u} du}{a^k (a + b_2 u)^k (a + b_3 u)^k} \]

and the result follows in this case using Lemma 3.3 for the upper bound.

If $a \leq b_1 \leq b_2 \leq b_3$ then

\[ J \gtrsim \int_{a/b_1}^2 \frac{u^{3k-1} du}{(b_1 u)^k (b_2 u)^k (b_3 u)^k} \gtrsim \frac{\ln(2 b_1/a)}{(b_1 b_2 b_3)^k} \]

while

\[ J \lesssim \int_0^{a/b_1} \frac{u^{3k-1} du}{a^k (b_2 u)^k (b_3 u)^k} + \int_{a/b_1}^2 \frac{u^{3k-1} du}{(b_1 u)^k (b_2 u)^k (b_3 u)^k} + \int_2^\infty \frac{u^{3k-1} e^{-u} du}{(b_1 u)^k (b_2 u)^k (b_3 u)^k} \]

\[ \gtrsim \frac{1}{b_1 b_2 b_3} + \frac{\ln(2 b_1/a)}{b_1 b_2 b_3} + \frac{1}{b_1 b_2 b_3} \]

and the result follows in this case.

Theorem 3.6. For the root system $A_n$ and $d \geq 3$, we have for $X, Y \in \mathbb{R}^n$

\[ N^W(X, Y) \preceq \frac{|X - Y|^{2-d}}{\prod_{\alpha > 0} |X - \sigma_\alpha Y|^{2k}}. \]

Proof. We have, using Theorem 3.1 and the change of variables $u = |X - Y|^2/(4t)$

\[ N^W(X, Y) = \int_0^\infty p_t^W(X, Y) dt \gtrsim \int_0^\infty t^{-d/2} e^{-|X - Y|^2/(4t)} \frac{dt}{\prod_{\alpha > 0} (t + \alpha(X) \alpha(Y))^k} \]

\[ \gtrsim |X - Y|^{-d} \int_0^\infty \frac{u^{d/2-2} e^{-u} du}{\prod_{\alpha > 0} (|X - Y|^2/(4u) + \alpha(X) \alpha(Y))^k} \]

\[ \gtrsim |X - Y|^{-d} \int_0^\infty \frac{u^{k|\Sigma^+|+d/2-2} e^{-u} du}{\prod_{\alpha > 0} (|X - Y|^2 + \alpha(X) \alpha(Y))^k} \]

\[ \gtrsim |X - Y|^{-d} \frac{1}{\prod_{\alpha > 0} (|X - Y|^2 + \alpha(X) \alpha(Y))^k} \gtrsim \frac{|X - Y|^{2-d}}{\prod_{\alpha > 0} |X - \sigma_\alpha Y|^{2k}} \]

(we have used Lemma 3.3 and the fact that $|X - \sigma_\alpha Y|^2 = |X - Y|^2 + 2 \alpha(X) \alpha(Y)$).

Proposition 3.7. If $d = 2$, the Newton kernel in the $A_1$ case satisfies

\[ N^W(X, Y) \preceq \frac{\ln(1 + |X - \sigma_\alpha Y|^2/|X - Y|^2)}{|X - \sigma_\alpha Y|^{2k}} \quad X, Y \in \mathbb{R}^1. \]
Here, it is important to recall that for \( X, Y \in \mathbb{a}^+ \), we have \(|X - \sigma_\alpha Y| \geq |X - Y|\) and therefore, the numerator of the last expression is at least \( \ln 2 \) for \( X \neq Y \). This remark also applies to the estimate in Proposition 3.8.

Proof. With computations similar as in the case \( d \geq 3 \), using Lemma 3.5.

\[
N^W(X, Y) = \int_0^\infty t^{-1} e^{-|X-Y|^2/(4t)} dt \times \int_0^\infty u^{k-1} e^{-u} du \times \frac{\ln(2 + \alpha(X) \alpha(Y))/|X - Y|^2}{(|X - Y|^2 + \alpha(X) \alpha(Y))^{k}}
\]

where \( \omega \) gives the minimum of \(|X - \sigma_\omega Y|\) for \( \omega \in \{\alpha, \beta\} \).

Proof. With computations similar as in the case \( d \geq 3 \), using Lemma 3.5.

\[
N^W(X, Y) \asymp \frac{\ln \left(1 + \frac{|X - \sigma_\omega Y|^{2k}}{|X - \sigma_\beta Y|^{2k}}\right)}{|X - \sigma_\alpha Y|^{2k} |X - \sigma_\beta Y|^{2k} |X - \sigma_{\alpha + \beta} Y|^{2k}}, \quad X, Y \in \mathbb{a}^+,
\]

where \( \omega \) gives the minimum of \(|X - \sigma_\omega Y|\) for \( \omega \in \{\alpha, \beta\} \).

Remark 3.9. In the Dunkl analysis, an important role is played by the intertwining operator \( V_k \), defined as a unique linear isomorphism on the space of polynomial functions on \( \mathbb{R}^d \) which intertwines the Dunkl operators with the usual partial derivatives:

\[
T_\xi V_k = V_k \partial_\xi \quad \text{for all } \xi \in \mathbb{R}^d
\]

and is normalized by \( V_k(1) = 1 \).
The following general formula for the Dunkl Newton kernel $N_k(x,y)$ involving the intertwining operator $V_k$ was proven in [7]:

$$N_k(x,y) = \frac{2^{2\gamma}((d-2)/2\gamma)}{|W| (d-2) w_d \pi(p)} V_k ((|y|^2 - 2 \langle y, y \rangle + |x|^2(2-d-2\gamma)/2) (Y))$$

(we are using a slightly different normalization of the operator $V_k$ than [9, 14], see [10] for details.)

Little is known explicitly on the intertwining operator. Theorem 3.6 and the formula $N_k(x,y) = \frac{1}{|W|} \sum_{w \in W} N_k(w x, y)$ imply the following explicit asymptotic formula.

**Corollary 3.10.** For the root system $\mathbb{A}_n$ and $d \geq 3$, we have for $x, y \in \mathfrak{a}^+$

$$V_k \left( \frac{|y|^2 - 2}{|W|} \sum_{w \in W} \langle w x, y \rangle + |x|^2(2-d-2\gamma)/2 \right) (y) \approx \frac{|x - y|^{2-d}}{\prod_{\alpha > 0} |x - \sigma_\alpha y|^{2k}}.$$

### 3.3 Heat semigroups for fractional powers of $\Delta_k^W$

Let $s \in (0, 2)$. The fractional powers $(-\Delta_k^W)^{s/2}$ of the $W$-invariant Dunkl Laplacian are the infinitesimal generators of important semigroups $(h_t^W(x,y))_{t \geq 0}$, called $W$-invariant Dunkl $s$-stable semigroups.

Fractional powers of the Dunkl Laplacian and related semigroups and processes were considered for $s = 1$ in [15, p.75], [16, Section 5] and for any $s \in (0, 2)$ in [3, 13]. Stable semigroups on Riemannian symmetric spaces of non-compact type were studied in [7, 12].

Like the heat semigroup $p_t^W(x,y)$, the densities $h_t^W(x,y)$ are to be considered with respect to the Dunkl weight function $\omega_k(y)$ on $\mathbb{R}^d$. We have

$$h_t^W(x,y) = \int_0^\infty p_t^W(x,y) \eta_t(u) du$$

where $\eta_t(u)$ is the density of the $s/2$-stable subordinator, i.e. of a positive Lévy process $(Y_t)_{t > 0}$ with the Laplace transform $\mathbb{E}(\exp(z Y_t)) = \exp(-t z^{s/2})$, $z > 0$ (see [2] for more details).

Denote by $h_t^{W d}(x,y)_{t \geq 0}$ the $s$-stable rotationally invariant semigroup on $\mathbb{R}^d$, with generator $(-\Delta)^{s/2}$. It is known ([3]) that

$$h_t^{W d}(x,y) \approx \min \left\{ \frac{1}{t^{d/s}}, \frac{t}{|X - Y|^{2d/s}} \right\} \approx \frac{t}{(t^{2/s} + |X - Y|^2)^{(d+s)/2}}. \quad (3.1)$$

**Remark 3.11.** It is useful to note that $\min \left\{ t^{-d/s}, t |X - Y|^{-(d+s)} \right\} = t^{-d/s}$ if and only if $t^{2/s} \geq |X - Y|^2$.

**Theorem 3.12.** Consider the $W$-invariant Dunkl Laplacian in the $\mathbb{A}_n$ case with multiplicity $k > 0$. Then for $x, y \in \mathfrak{a}^+$,

$$h_t^W(x,y) \approx \frac{h_t^{W d}(x,y)}{\prod_{\alpha > 0} (t^{2/s} + |X - \sigma_\alpha Y|^2)^k} \approx \frac{h_t^{W d}(x,y)}{\prod_{\alpha > 0} (t^{2/s} + |X - Y|^2 + \alpha(X)\alpha(Y))^k}.$$
Proof. The proof is inspired by the proof of [4, Theorem 3.1] providing estimates of stable semigroups on fractals.

Given Remark 3.11, it will make sense to consider the cases $t^{2/s} \geq |X - Y|^2$ and $t^{2/s} \leq |X - Y|^2$ separately. In the proof, $m$ will denote the number of positive roots.

We start by showing that our estimate is an upper bound with an appropriate constant. In [4, (14), page 168], it is shown that the subordinator density $\eta_t(u)$ satisfies

$$\eta_t(u) \leq C t u^{-1-s/2} e^{-t u^{-s/2}}.$$  \hfill (3.2)

Hence, using our estimates of the $W$-invariant Dunkl heat kernel in Theorem 3.1 and the change of variable $u = |X - Y|^2/(4w)$, we have

$$h_t^W(X,Y) \leq t \int_0^\infty \frac{u^{-d/2} e^{-|X-Y|^2/(4w)} u^{-1-s/2} e^{-t u^{-s/2}} du}{\prod_{\alpha > 0} (u + \alpha(X) \alpha(Y))^k}$$

$$\leq t |X - Y|^{-(d+s)} \int_0^\infty \frac{w^{(d+s)/2-1} e^{-w} dw}{\prod_{\alpha > 0} (|X - Y|^2/(4w) + \alpha(X) \alpha(Y))^k}$$

$$\leq t |X - Y|^{-(d+s)} \int_0^\infty \frac{w^{(d+s)/2+k m - 1} e^{-w} dw}{\prod_{\alpha > 0} (|X - Y|^2 + \alpha(X) \alpha(Y) w)^k}$$

$$\leq \frac{t |X - Y|^{-(d+s)}}{\prod_{\alpha > 0} (|X - Y|^2 + \alpha(X) \alpha(Y))^k}$$

with an application of Lemma 3.3 to get the last equivalence. This proves the upper bound in the case $|X - Y|^2 \geq t^{2/s}$.

We use Theorem 3.1 and the inequality (3.2) again with the change of variable $u = t^{2/s} w^{-2/s}$. Let $\Lambda$ be the set of $\alpha > 0$ such that $t^{2/s} \leq \alpha(X) \alpha(Y)$ with $m'$ the number of elements in $\Lambda$. We have

$$h_t^W(X,Y) \leq t \int_0^\infty \frac{u^{-d/2} u^{-1-s/2} e^{-t u^{-s/2}} du}{\prod_{\alpha > 0} (u + \alpha(X) \alpha(Y))^k}$$

$$\leq \frac{t}{d/s} \int_0^\infty \frac{w^{d/s} e^{-w} dw}{\prod_{\alpha > 0} (t^{2/s} w^{-2/s} + \alpha(X) \alpha(Y))^k}$$

$$\leq \frac{t}{d/s} \int_0^\infty \frac{w^{d/s+2 k m/s} e^{-w} dw}{\prod_{\alpha > 0} (t^{2/s} + \alpha(X) \alpha(Y) w^{2/s})^k}$$

$$\leq \frac{1}{\prod_{\alpha \in \Lambda} (t^{2/s})^k} \int_0^\infty \frac{w^{d/s+2 k m/s} e^{-w} dw}{\prod_{\alpha \in \Lambda} (\alpha(X) \alpha(Y) w^{2/s})^k}$$

$$\leq \frac{t}{d/s} \int_0^\infty \frac{w^{d/s+2 k (m-m'/s) / s} e^{-w} dw}{\prod_{\alpha > 0} (t^{2/s} + \alpha(X) \alpha(Y))^k}.$$}

This proves the upper bound in the case $|X - Y|^2 \leq t^{2/s}$.

Now we will justify the lower bound. Recall (see [4, (9,10), page 167] or [12, (9), page 89]) that for $u \geq t^{2/s}$, we have

$$\eta_t(u) \asymp t u^{-1-s/2}.$$ \hfill (3.3)
If $|X - Y|^2 \geq t^{2/s}$ then using formula (3.3), Theorem 3.1 and the change of variable $u = |X - Y|^2/(4w)$, we have

$$h_t^W(X,Y) \gtrsim t \int_{t^{2/s}}^{\infty} \frac{u^{-d/2} e^{-|X-Y|^2/(4w)} u^{-1-s/2}}{\prod_{\alpha > 0} (u + \alpha(X) \alpha(Y))^k} du$$

$$\gtrsim t |X - Y|^{-(d+s)} \int_0^{1/4} \frac{w^{(d+s)/2-1} e^{-w}}{\prod_{\alpha > 0} (|X - Y|^2/(4w) + \alpha(X) \alpha(Y))^k} dw$$

$$\gtrsim t |X - Y|^{-(d+s)} \int_1^{1/8} \frac{w^{(d+s)/2-1} e^{-w}}{\prod_{\alpha > 0} (|X - Y|^2 + \alpha(X) \alpha(Y))^k} dw$$

which proves the lower bound in that case.

Now assume $|X - Y|^2 \leq t^{2/s}$. We use formula (3.3) and Theorem 3.1. Then, since $-1/4 \leq -|X - Y|^2/(4t^{2/s}) \leq -|X - Y|^2/(4u) \leq 0$ for $u \geq t^{2/s}$, using the change of variable $u = t^{2/s} w$, we have

$$h_t^W(X,Y) \gtrsim t \int_{t^{2/s}}^{\infty} \frac{u^{-d/2} e^{-|X-Y|^2/(4w)} u^{-1-s/2}}{\prod_{\alpha > 0} (u + \alpha(X) \alpha(Y))^k} du$$

$$\gtrsim t \int_{t^{2/s}}^{\infty} \frac{u^{-(d+s)/2-1}}{\prod_{\alpha > 0} (u + \alpha(X) \alpha(Y))^k} du$$

$$\gtrsim t^{-d/s} \int_1^{\infty} \frac{w^{-d+s}/2-1} {\prod_{\alpha > 0} (t^{2/s} w + \alpha(X) \alpha(Y))^k} dw$$

$$\gtrsim t^{-d/s} \int_1^{\infty} \frac{w^{k/m} \prod_{\alpha > 0} (t^{2/s} + \alpha(X) \alpha(Y))^k} {t^{-d/s}} \prod_{\alpha > 0} (t^{2/s} + \alpha(X) \alpha(Y))^k$$

which proves the lower bound in that case.

Remark 3.13. The upper estimate in Theorem 3.12 may be deduced from our estimates of the $W$-invariant Dunkl heat kernel in Theorem 3.1 and from [8, Corollary 3.8 p. 11].

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