Runge-Lenz Vector, Accidental $SU(2)$ Symmetry, and Unusual Multiplets for Motion on a Cone

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Abstract

We consider a particle moving on a cone and bound to its tip by $1/r$ or harmonic oscillator potentials. When the deficit angle of the cone divided by $2\pi$ is a rational number, all bound classical orbits are closed. Correspondingly, the quantum system has accidental degeneracies in the discrete energy spectrum. An accidental $SU(2)$ symmetry is generated by the rotations around the tip of the cone as well as by a Runge-Lenz vector. Remarkably, some of the corresponding multiplets have fractional “spin” and unusual degeneracies.

1 Introduction

It is well-known that $1/r$ and harmonic oscillator potentials are exceptional because, in addition to rotation invariance, they have accidental dynamical symmetries. At the classical level the accidental symmetries imply that all bound orbits are closed, while at the quantum level they give rise to additional degeneracies in the discrete energy spectrum. In particular, the $SO(d)$ rotational symmetry of the $d$-dimensional $1/r$ potential (the Coulomb potential for $d = 3$) is enlarged to the accidental symmetry $SO(d + 1)$. The additional conserved quantities form the components of the Runge-Lenz vector. Similarly, the $d$-dimensional harmonic oscillator has an $SO(d)$ rotational symmetry which is contained as a subgroup in an accidental $SU(d)$ symmetry.
It has been shown by Bertrand in 1873 that the $1/r$ and $r^2$ potentials are the only spherically symmetric scalar potentials in Euclidean space for which all bound orbits are closed [1]. Still, there exist a number of other systems with accidental symmetries involving vector potentials or non-Euclidean spaces. For example, a free particle confined to the surface of the $d$-dimensional hyper-sphere $S^d$ moves along a great circle (which obviously is closed). Indeed the rotational $SO(d+1)$ symmetry of this system corresponds to the accidental symmetry of the $1/r$ potential. Already in 1935 Fock has realized that the hydrogen atom possesses “hyper-spherical” symmetry [2]. Based on this work, Bargmann [3] has shown that the generators of the accidental symmetry are the components of the Runge-Lenz vector [4]

$$\vec{R} = \frac{1}{2M} \left( \vec{p} \times \vec{L} - \vec{L} \times \vec{p} \right) - \kappa \vec{e}_r.$$  

(1.1)

Here $\vec{p}$ and $\vec{L}$ are the momentum and angular momentum of a particle of mass $M$, $\kappa$ is the strength of the $1/r$ potential, and $\vec{e}_r$ is the radial unit-vector. An example of an accidental symmetry involving a vector potential is cyclotron motion [5, 6]. In all these cases, there is a deep connection between the fact that all bound classical orbits are closed and additional degeneracies in the discrete energy spectrum of the corresponding quantum system. The subject of accidental symmetry has been reviewed, for example, by McIntosh [7].

In order to further investigate the phenomenon of accidental symmetries, in this paper we study a particle confined to the surface of a cone. A cone is obtained from the plane by removing a wedge of deficit angle $\delta$ and gluing the open ends back together. As a consequence, the polar angle $\chi$ no longer extends from 0 to $2\pi$, but only to $2\pi - \delta$. The geometry of the cone is illustrated in figure 1. It is convenient to rescale the polar angle such that it again covers the full interval, i.e.

$$\varphi = \frac{\chi}{s} \in [0, 2\pi],$$  

(1.2)

with the scale factor

$$s = 1 - \frac{\delta}{2\pi}. $$  

(1.3)

The kinetic energy of a particle of mass $M$ then takes the form

$$T = \frac{M}{2} \left( \dot{r}^2 + r^2 \dot{\varphi}^2 \right) = \frac{M}{2} \left( \dot{r}^2 + r^2 s^2 \dot{\varphi}^2 \right).$$  

(1.4)

The radial component of the momentum $p_r$ is canonically conjugate to $r$, i.e.

$$p_r = \frac{\partial T}{\partial \dot{r}} = M \dot{r}. $$  

(1.5)

Similarly, the canonically conjugate momentum corresponding to the rescaled angle $\varphi$ is given by the angular momentum

$$L = \frac{\partial T}{\partial \dot{\varphi}} = M r^2 s^2 \dot{\varphi}, $$  

(1.6)
Figure 1: A cone is obtained by cutting a wedge of deficit angle $\delta$ out of the 2-dimensional plane, and by gluing the open ends back together. Points on the cone are described by the distance $r$ from the tip and an angle $\chi$ which varies between $0$ and $2\pi - \delta$. Unlike the cone in the figure, the actual cone considered in this work extends over the whole range $r \in (0, \infty)$.

such that

$$T = \frac{1}{2M} \left( \frac{p_r^2}{r^2} + \frac{L^2}{r^2 \sin^2 \theta} \right).$$

As usual, upon canonical quantization (and using natural units in which $\hbar = 1$) the angular momentum conjugate to the rescaled angle $\varphi$ is represented by the operator

$$L = -i \partial_{\varphi}.$$  

(1.8)

The issues of domains of operators in a corresponding Hilbert space as well as of Hermiticity and self-adjointness play a certain role in this paper. For some mathematical background we refer to [8, 9]. Let us begin to address these issues in the context of the operator $L$. First of all, the Hilbert space $\mathcal{H} = L_2((0, \infty) \times (0, 2\pi); r)$ for a particle moving on a cone consists of the square-integrable functions $\Psi(r, \varphi)$ with $r \in (0, \infty)$, $\varphi \in [0, 2\pi]$, and with the norm $\langle \Psi | \Psi \rangle < \infty$, which is induced by the scalar product

$$\langle \Phi | \Psi \rangle = \int_0^\infty dr \int_0^{2\pi} d\varphi \, \Phi(r, \varphi)^* \Psi(r, \varphi).$$

(1.9)

It should be noted that functions in the Hilbert space need not be continuous or differentiable, in particular, they need not be periodic. In order to completely define a quantum mechanical operator $O$, one must identify the domain $\mathcal{D}[O] \subset \mathcal{H}$ of wave functions on which the operator acts. Since the self-adjoint operator $L$ is unbounded in $\mathcal{H}$, its domain is dense in $\mathcal{H}$ but not equal to $\mathcal{H}$. In order to be able to act on it with $L$, a wave function $\Psi$ must be differentiable at least once with respect to $\varphi$.  

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An operator $O$ is said to be Hermitean (symmetric in mathematical parlance) if

$$\langle O\Phi|\Psi \rangle = \langle \Phi|O\Psi \rangle, \quad (1.10)$$

for all wave functions $\Phi, \Psi \in \mathcal{D}[O]$. In order to investigate the question of Hermiticity of $L$, we perform a partial integration and obtain

$$\langle O\Phi|\Psi \rangle = \int_0^\infty dr \int_0^{2\pi} d\varphi \left[ -i \partial_\varphi \Phi(r, \varphi) \right]^* \Psi(r, \varphi)$$

$$= \int_0^\infty dr \int_0^{2\pi} d\varphi \Phi(r, \varphi)^* \left[ -i \partial_\varphi \Psi(r, \varphi) \right]$$

$$+ i \int_0^\infty dr \int_0^{2\pi} d\varphi \Phi(r, \varphi)^* \Psi(r, \varphi) \big|_{\varphi=2\pi}$$

$$= \langle \Phi|O\Psi \rangle + i \int_0^\infty dr \Phi(r, \varphi)^* \Psi(r, \varphi) \big|_{\varphi=2\pi}. \quad (1.11)$$

Thus, the operator $L$ is Hermitean if

$$\Phi(r, \varphi)^* \Psi(r, \varphi) \big|_{\varphi=2\pi} = 0. \quad (1.12)$$

Let us now address the issue of self-adjointness versus Hermiticity. In particular, self-adjointness (but not Hermiticity alone) guarantees a real-valued spectrum. An operator $O$ is self-adjoint (i.e. $O = O^\dagger$) if it is Hermitean and the domain of its adjoint $O^\dagger$ coincides with the domain of $O$, i.e. $\mathcal{D}[O^\dagger] = \mathcal{D}[O]$. The domain $\mathcal{D}[O^\dagger]$ consists of all functions $\xi \in \mathcal{H}$ for which there exists a function $\eta \in \mathcal{H}$ such that

$$\langle \eta|\Psi \rangle = \langle \xi|O\Psi \rangle, \quad (1.13)$$

for all $\Psi \in \mathcal{D}[O]$. For $\xi \in \mathcal{D}[O^\dagger]$ one then has $O^\dagger \xi = \eta$. For example, let us consider the operator $L$ in the domain of differentiable functions $\Psi \subset \mathcal{H}$ (with $L\Psi \in \mathcal{H}$), which obey the boundary condition $\Psi(r,0) = \Psi(r,2\pi) = 0$. In this domain $L$ acts as a Hermitean operator because the condition of eq.(1.12) is indeed satisfied. However, the functions in the domain of $L^\dagger$ need not obey this condition (i.e. the domain of $L^\dagger$ is larger than the one of $L$) and thus $L$ restricted to the above domain is not self-adjoint. However, there is a family of self-adjoint extensions. To see this, let us extend the domain of $L$ to differentiable functions obeying

$$\Psi(r,2\pi) = z\Psi(r,0), \quad z \in \mathbb{C}. \quad (1.14)$$

Then the condition of eq.(1.12) implies

$$\Phi(r,2\pi)^*\Psi(r,2\pi) - \Phi(r,0)^*\Psi(r,0) = [\Phi(r,2\pi)^*z - \Phi(r,0)^*] \Psi(r,0) = 0, \quad (1.15)$$

such that

$$\Phi(r,2\pi) = \frac{1}{z^*} \Phi(r,0). \quad (1.16)$$
In order to be self-adjoint (i.e. to have $D[L^\dagger] = D[L]$) the functions $\Phi \in D[L^\dagger]$ must obey the same condition as $\Psi \in D[L]$, which implies $z = 1/z^* = \exp(i\theta)$. The angle $\theta$ characterizes a one-parameter family of self-adjoint extensions of the operator $L$ to the domain of differentiable functions obeying the boundary condition

$$\Psi(r, 2\pi) = \exp(i\theta)\Psi(r, 0). \quad (1.17)$$

Since the coordinates $\varphi = 0$ and $\varphi = 2\pi$ describe the same physical point on the cone, the requirement of single-valuedness of the physical wave function restricts us to $\theta = 0$. Hence, for wave functions on the cone the domain $D[L] \in \mathcal{H}$ consists of the periodic differentiable functions $\Psi$ (with $L\Psi \in \mathcal{H}$) which obey

$$\Psi(r, 2\pi) = \Psi(r, 0). \quad (1.18)$$

The operator for the kinetic energy takes the form

$$T = -\frac{1}{2M} \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2 s^2} \partial_\varphi^2 \right). \quad (1.19)$$

Since $\partial_\varphi = s \partial_\chi$, this operator seems to be identically the same as the standard one operating on wave functions on the plane. However, in order to completely define $T$, one must again identify the domain $D[T]$ of wave functions on which it acts. First of all, these functions should again obey eq.(1.18), i.e. they must be periodic in the rescaled angle $\varphi$ (not in the original polar angle $\chi$ of the full plane). Separating the angular dependence

$$\Psi(r, \varphi) = \psi(r) \exp(im\varphi), \quad (1.20)$$

eq(1.18) leads to $m \in \mathbb{Z}$ as well as to the kinetic energy

$$T = -\frac{1}{2M} \left( \partial_r^2 + \frac{1}{r} \partial_r \right) + \frac{m^2}{2Mr^2 s^2}. \quad (1.21)$$

Effectively, a positive deficit angle $\delta$ (i.e. $s < 1$) leads to an enhancement of the centrifugal barrier, while a negative deficit angle ($s > 1$) leads to its reduction.

Let us now address the issues of Hermiticity and self-adjointness of $T$. First of all, the radial wave functions belong to the radial Hilbert space $\mathcal{H}_r = L_2((0, \infty); r)$. It is well-known that the operator $-i\partial_r$ is not even Hermitean in $\mathcal{H}_r$. Indeed, the Hermitean conjugate of $\partial_r$ is

$$\partial_r^\dagger = -\partial_r - \frac{1}{r}. \quad (1.22)$$
This follows from
\[
\langle \phi | \partial_r \psi \rangle = \int_0^\infty dr \, r\phi(r)^* \partial_r \psi(r)
\]
\[
= -\int_0^\infty dr \, \partial_r \left[ r\phi(r)^* \psi(r) + r\phi(r)^* \psi(r) \right]_0^\infty
\]
\[
= -\int_0^\infty dr \, \left[ r\partial_r \phi(r)^* + \phi(r)^* \right] \psi(r) + r\phi(r)^* \psi(r) |_0^\infty
\]
\[
= \int_0^\infty dr \, \left[ -\partial_r \phi(r)^* - \frac{1}{r} \phi(r)^* \right] \psi(r) + r\phi(r)^* \psi(r) |_0^\infty
\]
\[
= \langle \partial^\dagger_r \phi | \psi \rangle + r\phi(r)^* \psi(r) |_0^\infty. \tag{1.23}
\]

Since the partial integration should not lead to boundary terms, we must require
\[
r\phi(r)^* \psi(r) |_0^\infty = 0.
\]
It is natural to define the operator
\[
D_r = -i \left( \partial_r + \frac{1}{2r} \right) = -i \frac{1}{\sqrt{r}} \partial_r \sqrt{r}. \tag{1.24}
\]

In the domain \( D[D_r] \) of differentiable functions \( \psi(r) \) (with \( D_r \psi \in \mathcal{H}_r \)) obeying \( \psi(0) = 0 \), the operator \( D_r \) is indeed Hermitean because formally
\[
D_r^\dagger = i \left( \partial^\dagger_r + \frac{1}{2r} \right) = i \left( -\partial_r - \frac{1}{r} + \frac{1}{2r} \right) = -i \left( \partial_r + \frac{1}{2r} \right). \tag{1.25}
\]

However, it does not represent a proper physical observable because it is not self-adjoint. It is interesting to note that
\[
D^2_r = -\left( \partial_r + \frac{1}{2r} \right)^2 = -\partial^2_r - \frac{1}{r} \partial_r + \frac{1}{4r^2}, \tag{1.26}
\]

which is closely related to the kinetic energy operator \( T \), possesses a family of self-adjoint extensions. Eq.\((1.22)\) also seems to readily imply Hermiticity of the kinetic energy operator \( T \) because, at least formally,
\[
\left( \partial^2_r + \frac{1}{r} \partial_r \right)^\dagger = \partial^2_r + \frac{1}{r} \partial_r \partial^\dagger_r = \left( \partial_r + \frac{1}{r} \right)^2 - \left( \partial_r + \frac{1}{r} \right) \frac{1}{r}
\]
\[
= \partial^2_r + \frac{2}{r} \partial_r - \frac{1}{r} \partial_r = \partial^2_r + \frac{1}{r} \partial_r. \tag{1.27}
\]
However, the issue is again more subtle because one should consider

\[ \langle \phi | \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \psi \rangle = \int_0^\infty dr \ r \phi(r)^* \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \psi(r) \]

\[ = - \int_0^\infty dr \ \partial_r \left[ r \phi(r)^* \right] \partial_r \psi(r) + r \phi(r)^* \partial_r \psi(r) \bigg|_0^\infty \]

\[ - \int_0^\infty dr \ \partial_r \phi(r)^* \psi(r) + \phi(r)^* \psi(r) \bigg|_0^\infty \]

\[ = \int_0^\infty dr \ \partial_r^2 \left[ r \phi(r)^* \right] \psi(r) - \partial_r \left[ r \phi(r)^* \right] \psi(r) \bigg|_0^\infty \]

\[ - \int_0^\infty dr \ \partial_r \phi(r)^* \psi(r) + [r \phi(r)^* \partial_r \psi(r) + \phi(r)^* \psi(r)]_0^\infty \]

\[ = \int_0^\infty dr \ r \left[ \left( \partial_r^2 + \frac{2}{r} \partial_r - \frac{1}{r} \partial_r \right) \phi(r)^* \right] \psi(r) \]

\[ + [r \phi(r)^* \partial_r \psi(r) - r \partial_r \phi(r)^* \psi(r)]_0^\infty \]

\[ = \langle \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \phi | \psi \rangle + [r \phi(r)^* \partial_r \psi(r) - r \partial_r \phi(r)^* \psi(r)]_0^\infty . \]  

(1.28)

Hence, in order to ensure the Hermiticity of \( T \), one must impose the condition

\[ [r \phi(r)^* \partial_r \psi(r) - r \partial_r \phi(r)^* \psi(r)]_0^\infty = 0, \]  

(1.29)

which admits a one-parameter family of self-adjoint extensions. The self-adjoint extensions of \( T \) have been studied in [10]. It turns out that the tip of the cone is a singular point that may be endowed with non-trivial physical properties. These properties are described by a real-valued parameter that defines a family of self-adjoint extensions. Physically speaking, the different self-adjoint extensions correspond to properly renormalized \( \delta \)-function potentials of different strengths located at the tip of the cone. In this paper, we limit ourselves to the case without \( \delta \)-function potentials, which corresponds to the so-called Friedrichs extension [3] characterized by the boundary condition

\[ \lim_{r \to 0} r \partial_r \psi(r) = 0. \]  

(1.30)

If we impose this condition on \( \psi \in \mathcal{D}[T] \) and also want to satisfy eq.(1.29), the function \( \phi \in \mathcal{D}[T^+] \) must also obey eq.(1.30). As a result, \( \mathcal{D}[T^+] = \mathcal{D}[T] \), such that \( T = T^+ \) is indeed self-adjoint.

While the cone is as flat as the plane, its singular tip and its deficit angle \( \delta \) have drastic effects on the dynamics. In the following, we will consider a particle moving on a cone and bound to its tip by a \( 1/r \) or \( r^2 \) potential. Interestingly, when the deficit angle divided by \( 2\pi \) (or equivalently \( s \)) is a rational number, all bound classical orbits are again closed and once more there are additional degeneracies in the discrete spectrum of the Hamilton operator \( H \). Just like in the plane, the \( 1/r \)
and $r^2$ potentials on a cone have accidental $SU(2)$ symmetries. However, unlike in the plane, the corresponding multiplets may now have fractional “spin” and unusual degeneracies. This unusual behavior arises because, in this case, the Runge-Lenz vector $\vec{R}$ — although Hermitean in its appropriate domain $D[\vec{R}]$ — does not act as a Hermitean operator in the domain $D[H]$ of the Hamiltonian and thus does not represent a proper physical observable.

It should be noted that motion on a cone may not be an entirely academic problem. First, space-times with a conical singularity arise in the study of cosmic strings. Indeed, $1/r$ and $r^2$ potentials have already been considered in this context [11, 12], however, without discussing accidental symmetries. Furthermore, graphene — a single sheet of graphite, i.e. a honeycomb of carbon hexagons — can be bent to form cones by adding or removing a wedge of carbon atoms and by replacing one hexagon by a carbon hepta- or pentagon [13, 14]. While the low-energy degrees of freedom in graphene are massless Dirac fermions, in this paper we limit ourselves to studying the Schrödinger equation for motion on a cone.

The rest of the paper is organized as follows. In section 2 we discuss the $1/r$ and in section 3 we discuss the $r^2$ potential on a cone. In both cases, we construct the Runge-Lenz vector which, together with the angular momentum, generates the accidental $SU(2)$ symmetry. We also relate the Hamilton operator to the corresponding Casimir operator and we discuss the unusual multiplets realized in the discrete spectrum. Section 4 contains our conclusions.

## 2 The $1/r$ Potential on a Cone

In this section we consider a particle on the surface of a cone bound to its tip by a $1/r$ potential

$$V(r) = -\frac{\kappa}{r}. \quad (2.1)$$

The corresponding total energy is thus given by

$$H = T + V = \frac{1}{2M} \left( p_r^2 + \frac{L^2}{r^2 s^2} \right) - \frac{\kappa}{r}. \quad (2.2)$$

### 2.1 Classical Solutions

It is straightforward to find the most general bound solution of the classical equations of motion and one obtains the classical orbit

$$\frac{1}{r} = \frac{M_\kappa s^2}{L^2} \left[ 1 + e \cos(s(\varphi - \varphi_0)) \right], \quad (2.3)$$
with the eccentricity given by
\[ e = \sqrt{1 + \frac{2EL^2}{MK^2s^2}}, \quad (2.4) \]
where \( E < 0 \) is the energy and \( L \) is the angular momentum. The radial component of the momentum takes the form
\[ p_r = \frac{MKs}{L}e \sin(s(\varphi - \varphi_0)). \quad (2.5) \]
The angle \( \varphi_0 \) determines the direction of the perihelion. Obviously, the classical orbit is closed as long as \( s = p/q \) is a rational number (with \( p, q \in \mathbb{N} \) not sharing a common divisor). In that case, after \( q \) revolutions around the tip of the cone, both \( r \) and \( p_r \) return to their initial values. Some examples of classical orbits are shown in figure 2.

![Figure 2: Examples of bound classical orbits for the 1/r potential with s = 3 (left), s = 1/2 (middle), and s = 1 (right). The latter case represents a standard Kepler ellipse. The orbits are shown in the x-y-plane with \((x,y) = r(\cos \varphi, \sin \varphi)\) where \( \varphi = \chi/s \in [0,2\pi] \) is the rescaled polar angle.](image)

### 2.2 Semi-classical Bohr-Sommerfeld Quantization

Let us consider Bohr-Sommerfeld quantization. The quantization condition for the angular momentum takes the form
\[ \oint d\varphi \ L = 2\pi L = 2\pi m, \quad (2.6) \]
such that \( L = m \in \mathbb{Z} \). Similarly, the quantization condition for the radial motion is given by
\[ \oint dr \ p_r = 2\pi \left( n_r + \frac{1}{2} \right), \quad n_r \in \{0,1,2,...\}. \quad (2.7) \]
The factor \(1/2\), which is sometimes not taken into account in Bohr-Sommerfeld quantization, arises for librations but is absent for rotations. Using eqs. (2.3), (2.4), and (2.5) and integrating over the period \(2\pi/s\) it is straightforward to obtain

\[
\oint dr p_r = \int_0^{2\pi/s} d\varphi \frac{|L| e^2 \sin^2(s(\varphi - \varphi_0))}{(1 + e \cos(s(\varphi - \varphi_0)))^2} = 2\pi \left(\sqrt{-\frac{M\kappa^2}{2E}} - \frac{|L|}{s}\right),
\]

which leads to

\[
E = -\frac{M\kappa^2}{2\left(n_r + \frac{|m|}{s} + \frac{1}{2}\right)^2}.
\]

It will turn out that this result is exact and not just limited to the semi-classical regime.

### 2.3 Solution of the Schrödinger Equation

The radial Schrödinger equation takes the form

\[
\left[-\frac{1}{2M} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) + \frac{m^2}{2Mr^2s^2} - \frac{\kappa}{r}\right] \psi(r) = E \psi(r),
\]

which is straightforward to solve. Using \(\alpha = \sqrt{8M|E|}\), one finds [11]

\[
\psi_{n_r,m}(r) = A \exp\left(-\frac{\alpha r}{2}\right)(\alpha r)^{|m|/s} {_1F_1}(-n_r, \frac{2|m|}{s} + 1, \alpha r),
\]

where \(_1F_1\) is a confluent hyper-geometric function. The corresponding quantized energy values are given by eq. (2.9). Now \(n_r\) is the number of nodes of the radial wave function and \(m \in \mathbb{Z}\) is the angular momentum quantum number. Parity symmetry together with the \(SO(2)\) rotational symmetry ensures the degeneracy of states with quantum numbers \(m\) and \(-m\). It is interesting to note that there are additional accidental degeneracies in the spectrum when the scale factor \(s\) is a rational number. Remarkably, this is just the condition under which all classical orbits are closed. It is worth mentioning that there are other mathematical solutions of the Schrödinger equations which, however, do not qualify as physical wave functions because they diverge at the origin. These solutions will play a certain role later, when we discuss the accidental symmetry multiplets.

For example, let us consider the case \(s = \frac{1}{2}\) corresponding to the deficit angle \(\delta = \pi\). In that case, a wave function without nodes (i.e. with \(n_r = 0\)) and with angular momentum \(m = \pm 1\) is degenerate in energy with a wave function with two nodes \((n_r = 2)\) and with \(m = 0\). As another example, let us consider \(s = 2\) which corresponds to the negative deficit angle \(\delta = -2\pi\). In this case, one builds a “cone” by cutting two planes open and gluing them together in the same way as the
double-layered Riemann surface of the complex square root. This effectively lowers the centrifugal barrier by a factor of $s^2 = 4$. In this case, a wave function without nodes ($n_r = 0$) and with $m = \pm 2$ is degenerate with a wave function with one node ($n_r = 1$) and with $m = 0$. Similarly, for $s = n \in \mathbb{N}$, one glues $n$ cut planes to a “cone” in the same way as the multi-layered Riemann surface of the complex $n$-th root. Now, a wave function without nodes ($n_r = 0$) and with $m = \pm n$ is degenerate with a wave function with one node ($n_r = 1$) and with $m = 0$. Some features of the energy spectrum are illustrated in figure 3.

Figure 3: The $1/r$ potential (solid curve) together with an effective potential including the centrifugal barrier with $m = \pm 1$ (dashed curve) for $s = 3$. The energies of the ground state and the first three excited states are indicated by horizontal lines. The numbers besides the lines specify the degree of degeneracy. The ground state (with $n_r = 0, m = 0$) is non-degenerate, while the first and second excited states (with $n_r = 0, m = \pm 1$ and $n_r = 0, m = \pm 2$, respectively) are two-fold degenerate due to parity symmetry. The third excited level has an accidental three-fold degeneracy and consists of the states with $n_r = 0, m = \pm 3$ and $n_r = 1, m = 0$.

### 2.4 Runge-Lenz Vector and $SU(2)$ Algebra

The fact that, for rational $s$, all classical orbits are closed and the discrete spectrum of the Hamiltonian has accidental degeneracies suggests that there is a hidden conserved quantity generating a corresponding accidental symmetry. From the plane (with $s = 1$) we are familiar with the Runge-Lenz vector and an accidental $SU(2)$ symmetry. Indeed a Runge-Lenz vector can be constructed for any $s$. However, it should be pointed out that, due to the conical geometry, the resulting object
no longer transforms as a proper vector. We still continue to refer to it as the “Runge-Lenz vector”.

At the classical level, we can use eqs. (2.3) and (2.5) to write

\[\kappa e \cos(s(\varphi - \varphi_0)) = \kappa e \left[ \cos(s\varphi) \cos(s\varphi_0) + \sin(s\varphi) \sin(s\varphi_0) \right] = \frac{L^2}{M r s^2} - \kappa,\]
\[\kappa e \sin(s(\varphi - \varphi_0)) = \kappa e \left[ \sin(s\varphi) \cos(s\varphi_0) - \cos(s\varphi) \sin(s\varphi_0) \right] = \frac{p_r L}{M s},\] (2.12)

such that

\[R_x = \kappa e \cos(s\varphi_0) = \left( \frac{L^2}{M r s^2} - \kappa \right) \cos(s\varphi) + \frac{p_r L}{M s} \sin(s\varphi),\]
\[R_y = \kappa e \sin(s\varphi_0) = \left( \frac{L^2}{M r s^2} - \kappa \right) \sin(s\varphi) - \frac{p_r L}{M s} \cos(s\varphi),\] (2.13)

are indeed independent of time. Furthermore, for \(s = 1\), \(R_x\) and \(R_y\) are just the components of the familiar Runge-Lenz vector. It should be noted that, for non-integer values of \(s\), the quantities \(R_x\) and \(R_y\) are not conserved quantities in the usual sense. In particular, they are not single-valued functions of the coordinates \(x = r \cos \varphi\) and \(y = r \sin \varphi\), but depend on the angle \(\varphi\) itself. As a consequence, the values of \(R_x\) and \(R_y\) depend on the history of the motion, i.e. on the number of revolutions around the tip of the cone. However, quantities that are “conserved” only because they refer back to the initial conditions, do not qualify as proper physical constants of motion. To further clarify this issue, it is useful to construct the complex variables

\[R_\pm = R_x \pm iR_y = \left( \frac{L^2}{M r s^2} - \kappa \mp \frac{p_r L}{M s} \right) \exp(\pm is\varphi).\] (2.14)

For rational values \(s = p/q\) (with \(p, q \in \mathbb{N}\)) the quantities

\[R_\pm^q = \left( \frac{L^2}{M r s^2} - \kappa \mp \frac{p_r L}{M s} \right)^q \exp(\pm ip\varphi)\] (2.15)

are single-valued functions of \(x = r \cos \varphi\) and \(y = r \sin \varphi\), and hence qualify as proper conserved quantities. Remarkably, exactly for rational values of \(s\) all bound classical orbits are closed.

The length of the Runge-Lenz vector is given by

\[R^2 = R_x^2 + R_y^2 = \left( \frac{L^2}{M r s^2} - \kappa \right)^2 + \left( \frac{p_r L}{M s} \right)^2 = 2 \frac{L^2}{M s^2} \left( \frac{p_r^2}{2M} + \frac{L^2}{2Mr^2s^2} - \frac{\kappa}{r} \right) + \kappa^2 = \frac{2HL^2}{Ms^2} + \kappa^2.\] (2.16)
In the quantum mechanical treatment it will turn out to be useful to introduce the rescaled variables

\[ \tilde{R}_x = \sqrt{-\frac{M}{2H}} R_x, \quad \tilde{R}_y = \sqrt{-\frac{M}{2H}} R_y, \quad \tilde{L} = \frac{L}{s}, \]  

(2.17)

which makes sense for bound orbits with negative energy. We then obtain

\[ C = \tilde{R}_x^2 + \tilde{R}_y^2 + \tilde{L}^2 = -\frac{M\kappa^2}{2H} \Rightarrow H = -\frac{M\kappa^2}{2C}. \]  

(2.18)

It will turn out that the quantum analogue of \( C \) is the Casimir operator of an accidental \( SU(2) \) symmetry.

At the quantum level, the components of the Runge-Lenz vector turn into operators

\[ R_x = -\frac{1}{Mr^2} \cos(s\varphi)\partial_x^2 + \frac{1}{2Mr} \sin(s\varphi)\partial_x - \kappa \cos(s\varphi), \]
\[ R_y = -\frac{1}{Mr^2} \sin(s\varphi)\partial_y^2 - \frac{1}{2Mr} \cos(s\varphi)\partial_y - \kappa \sin(s\varphi) \]
\[ + \frac{1}{Ms} \cos(s\varphi)\partial_r\partial_x - \frac{1}{2M} \sin(s\varphi)\partial_r, \]  

(2.19)

It is straightforward (but somewhat tedious) to show that not only the angular momentum \( L \), but also both components of the Runge-Lenz vector commute with the Hamiltonian, i.e.

\[ [R_x, H] = [R_y, H] = [L, H] = 0, \]  

(2.20)

and that these operators obey the algebra

\[ [R_x, R_y] = -i\frac{2HL}{Ms}, \quad [R_x, L] = -isR_y, \quad [R_y, L] = isR_x. \]  

(2.21)

Applying the rescaling of eq. (2.17), this leads to

\[ [\tilde{R}_x, \tilde{R}_y] = i\tilde{L}, \quad [\tilde{R}_y, \tilde{L}] = i\tilde{R}_x, \quad [\tilde{L}, \tilde{R}_x] = i\tilde{R}_y. \]  

(2.22)

Hence, \( \tilde{R}_x, \tilde{R}_y, \) and \( \tilde{L} \) generate an \( SU(2) \) algebra.

### 2.5 Casimir Operator

It is straightforward to construct the Casimir operator of the \( SU(2) \) algebra and one obtains

\[ C = \tilde{R}_x^2 + \tilde{R}_y^2 + \tilde{L}^2 = -\frac{M\kappa^2}{2H} - \frac{1}{4}, \]  

(2.23)
such that
\[ H = -\frac{M\kappa^2}{2\left(C + \frac{1}{4}\right)} = -\frac{M\kappa^2}{2\left(S + \frac{1}{2}\right)^2}. \] (2.24)

In the last step we have used the fact that the eigenvalue of the Casimir operator of an \( SU(2) \) symmetry is naturally represented by \( S(S + 1) \) such that
\[ C + \frac{1}{4} = S(S + 1) + \frac{1}{4} = \left(S + \frac{1}{2}\right)^2. \] (2.25)

By comparison with eq.(2.9) for the energy spectrum, we thus identify
\[ S = n_r + \frac{|m|}{s}. \] (2.26)

This result is puzzling, because for \( 2|m|/s \notin \mathbb{N} \) the abstract spin \( S \) is not an integer or a half-integer. For a general scale factor \( s \) (or equivalently for general deficit angle \( \delta \)), the abstract spin is, in fact, continuous. Even for general rational \( s \), for which all bound classical orbits are closed and there are accidental degeneracies in the discrete spectrum of the Hamiltonian, the spin \( S \) is not just an integer or a half-integer.

### 2.6 Domains of Operators and Hermiticity

In order to better understand the puzzling result that the Casimir “spin” may not be an integer or half-integer, let us address the questions of Hermiticity and of the domains of the various operators. As we have discussed in the introduction, once it is endowed with an appropriate extension, the Hermitean kinetic energy operator \( T \) becomes self-adjoint and thus qualifies as a physical observable. The same is true for the full Hamiltonian including the potential. In this case, we assume the standard Friedrichs extension \([8]\), which implies that there is no \( \delta \)-function potential located at the tip of the cone.

Using \( \partial_r^\dagger = -\partial_r - 1/r \) as well as \( \partial_\varphi^\dagger = -\partial_\varphi \), it is straightforward to show that, at least formally, \( \tilde{R}_x^\dagger = \tilde{R}_x \) and \( \tilde{R}_y^\dagger = \tilde{R}_y \), which implies \( \tilde{R}_z^\dagger = \tilde{R}_z \). However, as we have seen in the introduction, Hermiticity also requires appropriate boundary conditions, which restrict the domains of the corresponding operators. It is interesting to note that, using \( s\varphi = \chi \), the operators \( \tilde{R}_x \) and \( \tilde{R}_y \) of eq.(2.19) formally agree with the components of the standard Runge-Lenz vector for the plane from eq.(1.1). The Runge-Lenz vector for the plane is a Hermitean and even self-adjoint operator acting in a domain \( \mathcal{D}[\tilde{R}] \) that contains the domain of the Hamiltonian. This domain contains smooth functions which are \( 2\pi \)-periodic in the polar angle \( \chi \) of the plane. The operators \( \tilde{R}_x \) and \( \tilde{R}_y \), on the other hand, act on the Hilbert space of square-integrable wave functions on the cone. In this case, the domain of the Hamiltonian \( \mathcal{D}[H] \) contains smooth functions which are \( 2\pi \)-periodic in the rescaled angle \( \varphi \) and
obey the boundary condition of eq. (1.30). While \( \tilde{R}_x \) and \( \tilde{R}_y \) on the cone are still Hermitean in their appropriate domain, in contrast to the case of the plane, they are not Hermitean in the domain \( \mathcal{D}[H] \) of the Hamiltonian. In particular, for \( s \neq 1 \) the operators \( \tilde{R}_x \) and \( \tilde{R}_y \) map \( 2\pi \)-periodic physical wave functions onto functions outside \( \mathcal{D}[H] \), because they contain multiplications with the \( 2\pi/s \)-periodic functions \( \cos(s\varphi) \) and \( \sin(s\varphi) \). Proper symmetry generators should map wave functions from the domain of the Hamiltonian back into \( \mathcal{D}[H] \). Hence, for \( s \notin \mathbb{N} \), the operators \( \tilde{R}_x \) and \( \tilde{R}_y \) do not represent proper symmetry generators.

It is interesting to consider the case of rational \( s = p/q \) with \( p, q \in \mathbb{N} \). In this case, a single application of

\[
\tilde{R}_\pm = \tilde{R}_x \pm i\tilde{R}_y
\]

may take us out of the domain of the Hamiltonian, but a \( q \)-fold application of these operators brings us back into \( \mathcal{D}[H] \). Indeed, just as for rational \( s \) the classical object \( R^2 \) represents a proper physical conserved quantity, \( \tilde{R}_\pm^q \) (but not \( \tilde{R}_\pm \) itself) qualifies as a proper symmetry generator. The case of integer \( s = n \) is also interesting, because in that case \( \cos(s\varphi) \) and \( \sin(s\varphi) \) are indeed \( 2\pi \)-periodic. Hence, by acting with \( \tilde{R}_\pm \) we might expect to stay within \( \mathcal{D}[H] \), although for \( n \geq 3 \) the abstract spin \( S = n_r + |m|/s = n_r + |m|/n \) is still quantized in unusual fractional units. However, as we will see below, another subtlety arises because \( \tilde{R}_\pm \) may turn a physical wave function that is regular at the origin (and thus obeys the boundary condition of eq. (1.30)) into a singular one. This further limits the domain of the operators \( \tilde{R}_\pm \).

The unusual (not properly quantized) value of the Casimir spin can be traced back to the mathematical fact that the Runge-Lenz vector — although Hermitean in its appropriate domain — does not act as a Hermitean operator in the domain of the Hamiltonian. Hence, in retrospect the \( SU(2) \) commutation relations of eq. (2.22) are rather formal. In fact, they are satisfied for functions \( \Psi(r, \varphi) \) with \( \varphi \in \mathbb{R} \), but not for the periodic functions in \( \mathcal{D}[H] \) for which \( \varphi \in [0, 2\pi] \). This is another indication that the accidental “\( SU(2) \)” symmetry of eq. (2.22) is rather unusual.

### 2.7 Unusual Multiplets

How can we further understand the puzzling result that the Casimir spin \( S \) is not always quantized in integer or half-integer units? It seems that we found new unusual representations for something as well understood as an \( SU(2) \) algebra. Acting with \( \tilde{R}_\pm \) on a \( 2\pi \)-periodic wave function

\[
\langle r, \varphi|n_r, m \rangle = \psi_{n_r, m}(r) \exp(im\varphi)
\]

one changes both \( n_r \in \mathbb{N} \) and \( m \in \mathbb{Z} \). For \( m > 0 \) one obtains

\[
\tilde{R}_+|n_r, m \rangle \propto |n_r - 1, m + s \rangle, \quad \tilde{R}_-|n_r, m \rangle \propto |n_r + 1, m - s \rangle,
\]

(2.29)
and for $m < 0$ one finds
\[
\tilde{R}_+|n_r, m\rangle \propto |n_r + 1, m + s\rangle, \quad \tilde{R}_-|n_r, m\rangle \propto |n_r - 1, m - s\rangle. \tag{2.30}
\]

Finally, for $m = 0$ we have
\[
\tilde{R}_+|n_r, 0\rangle \propto |n_r - 1, s\rangle, \quad \tilde{R}_-|n_r, 0\rangle \propto |n_r - 1, -s\rangle. \tag{2.31}
\]

These relations follow from the $SU(2)$ algebra which implies that $\tilde{R}_\pm$ are raising and lowering operators for $\tilde{L} = L/s$. Hence, by acting with $\tilde{R}_\pm$ the eigenvalue $m$ of $L$ is shifted by $\pm s$. Using the fact that the eigenvalue of the Casimir operator, which is determined by $S = n_r + |m|/s$, does not change under applications of $\tilde{R}_\pm$, one immediately obtains the effects of $\tilde{R}_\pm$ on the radial quantum number $n_r$.

Eqs. (2.29), (2.30), and (2.31) also follow directly by applying the explicit forms of $\tilde{R}_\pm$ or with $R_\pm = R_x \pm iR_y$ from eq. (2.19) to the wave functions of eq. (2.11).

Again, by using the wave functions of eq. (2.11) one can show that
\[
\tilde{R}_+^{n_r+1}|n_r, m \geq 0\rangle \propto \tilde{R}_+|0, m + n_rs \geq 0\rangle = 0,
\]
\[
\tilde{R}_-^{n_r+1}|n_r, m \leq 0\rangle \propto \tilde{R}_-|0, m - n_rs \leq 0\rangle = 0. \tag{2.32}
\]

Hence, depending on the sign of $m$, by acting $n_r + 1$ times either with $\tilde{R}_+$ or with $\tilde{R}_-$ we reach zero, and thus the multiplet naturally terminates. This allows us to confirm the value of the Casimir spin $S = n_r + |m|/s$ by evaluating
\[
C|0, m + n_rs \geq 0\rangle = \left[ \frac{1}{2}(\tilde{R}_+\tilde{R}_- + \tilde{R}_-\tilde{R}_+) + \tilde{L}^2 \right]|0, m + n_rs \geq 0\rangle = \left[ \frac{1}{2}(\tilde{R}_+\tilde{R}_- + 2\tilde{R}_-\tilde{R}_+) + \tilde{L}^2 \right]|0, m + n_rs \geq 0\rangle = (\tilde{L}^2)|0, m + n_rs \geq 0\rangle = \left( \frac{m}{s} + n_r \right)\left( \frac{m}{s} + n_r + 1 \right)|0, m + n_rs \geq 0\rangle = S(S + 1)|0, m + n_rs \geq 0\rangle,
\]
\[
C|0, m - n_rs \leq 0\rangle = \left[ \frac{1}{2}(\tilde{R}_+\tilde{R}_- + \tilde{R}_-\tilde{R}_+) + \tilde{L}^2 \right]|0, m - n_rs \leq 0\rangle = \left[ \frac{1}{2}(\tilde{R}_+\tilde{R}_- + 2[\tilde{R}_-, \tilde{R}_-]) + \tilde{L}^2 \right]|0, m - n_rs \leq 0\rangle = (-\tilde{L}^2)|0, m - n_rs \leq 0\rangle = \left( -\frac{m}{s} + n_r \right)\left( -\frac{m}{s} + n_r + 1 \right)|0, m - n_rs \leq 0\rangle = S(S + 1)|0, m - n_rs \leq 0\rangle. \tag{2.33}
\]

The multiplet of degenerate states with the same value of $S$ can now be obtained by $n$ repeated applications of either $\tilde{R}_+$ or $\tilde{R}_-$. It is important to note that, if $s$ is not
an integer, \( m \pm ns \) may also not be an integer and thus the corresponding state may be outside \( \mathcal{D}[H] \). Despite this, its radial wave function is still defined by eq.(2.11) and it still solves the radial Schrödinger equation.

Let us first consider the generic case of irrational \( s \). In that case, the classical orbits are not closed, there are no accidental degeneracies in the discrete spectrum of the Hamiltonian, and the Casimir spin \( S = n_r + |m|/s \) is irrational. Acting with \( \tilde{R}_\pm \) on the \( 2\pi \)-periodic wave function \( |n_r, m \in \mathbb{Z} \rangle \) an arbitrary number of times, one generates functions which are not \( 2\pi \)-periodic and thus outside \( \mathcal{D}[H] \). As a consequence of parity symmetry, for \( m \neq 0 \) the two levels with the quantum numbers \( m \) and \(-m\) are still degenerate. However, that two-fold degeneracy is not accidental.

Next, let us discuss the case of rational \( s = p/q \) in which all classical orbits are closed and there are accidental degeneracies in the discrete spectrum of the Hamiltonian. First, we consider the case \( 2|m|/s = 2|m|q/p \in \mathbb{N} \) for which the Casimir spin \( S \) is an integer or a half-integer. Only in that case, the set of degenerate wave functions terminates on both ends, i.e.

\[
\tilde{R}_+^{2S+1}|0, m + n_r s \geq 0 \rangle = 0, \quad \tilde{R}_-^{2S+1}|0, m - n_r s \leq 0 \rangle = 0. \tag{2.34}
\]

This follows by applying the operators of eq.(2.19) to the wave function of eq.(2.11), which is a somewhat tedious procedure that requires using non-trivial properties of the confluent hyper-geometric functions. Our experience with \( SU(2) \) algebras would suggest that there are \( 2S + 1 \) degenerate states. However, we should not forget that a single application of the raising and lowering operators \( \tilde{R}_\pm \) may take us outside \( \mathcal{D}[H] \), and only \( q \) applications of \( \tilde{R}_\pm \) take us back into \( \mathcal{D}[H] \). It is straightforward (but not very illuminating) to count the number of states inside \( \mathcal{D}[H] \). In any case, for \( s \neq 1 \) this number is smaller than the naively expected \( 2S + 1 \). When \( sS \in \mathbb{N} \), the degeneracy is given by \( g = [2S/q] + 1 \), where \([2S/q]\) denotes the nearest integer below \( 2S/q \).

As we will see now, the multiplets are even more unusual in the case of rational \( s = p/q \) with the Casimir spin \( S \) neither being an integer nor a half-integer. In that case, the set of degenerate wave functions only terminates on one end, but not on the other. In particular, while still \( \tilde{R}_+|0, m + n_r s \geq 0 \rangle = 0 \), \( \tilde{R}_-|0, m - n_r s \geq 0 \rangle \) does not vanish, even for arbitrarily large \( n \). Since an infinite number of values \( m + (n_r - n)s \) will be integers, one might then think that the multiplet of degenerate states inside \( \mathcal{D}[H] \) should contain an infinite number of states. Interestingly, this is not the case for a rather unusual reason. For \( S \) neither being an integer nor a half-integer, the states \( \tilde{R}_-|0, m + n_r s \geq 0 \rangle \) with \( m + (n_r - n)s < 0 \) are outside \( \mathcal{D}[H] \) because the corresponding wave function is singular at the origin. This again follows from applying the operators of eq.(2.19) to the wave function of eq.(2.11). Although they do not qualify as physical states, the divergent wave functions still are mathematical solutions of the Schrödinger differential equation which take the
The singularity of the wave function may or may not make the wave function non-normalizable. Even if it remains normalizable, the corresponding singular wave function does not belong to $D[H]$ because it does not obey the boundary condition of eq.(1.30). For $S$ neither being an integer nor a half-integer, the states with positive and negative $m$ have the same energy as a consequence of parity symmetry, but they are not related to one another by applications of the raising and lowering operators $\tilde{R}_\pm$. Remarkably, in this case, by acting with a symmetry generator $\tilde{R}_x$ or $\tilde{R}_y$ on a wave function inside $D[H]$, one may generate a physically unacceptable wave function outside $D[H]$. A sequence of physical and unphysical wave functions is illustrated in figure 4.

![Graphs showing wave functions](image)

Figure 4: A sequence of wave functions for the $1/r$ potential with $s = 3$ obtained from repeated applications of $\tilde{R}_-$. The quantum numbers are $n_r = 0$, $m = 4$ (left), $n_r = 1$, $m = 4 - s = 1$ (middle), and $n_r = 2$, $m = 4 - 2s = -2$ (right). The third state in the sequence is outside the domain of the Hamiltonian because the corresponding wave function does not obey the boundary condition of eq.(1.30) and the state is thus unphysical.

To summarize, for $s \neq 1$ different types of unusual multiplets arise. First, even for integer or half-integer $S = n_r + |m|/s$, the degeneracy of the physical multiplet is not $2S + 1$ because $m \pm ns$ may not be an integer in which case the corresponding wave function is not $2\pi$-periodic. When $S = n_r + |m|/s$ is neither an integer nor a half-integer, there is an infinite number of degenerate solutions of the Schrödinger equation. However, only a finite number of them obeys the boundary condition of eq.(1.30) and thus belongs to $D[H]$. 

\[
\psi(r) = A \exp\left(-\frac{\alpha r}{2}\right)(\alpha r)^{-|m|/s} F_1(-n_r, -\frac{2|m|}{s} + 1, \alpha r). \tag{2.35}
\]
3 The $r^2$ Potential on a Cone

Let us now turn to the problem of a particle moving on a cone and bound to its tip by a harmonic oscillator potential

$$V(r) = \frac{1}{2} M \omega^2 r^2. \quad (3.1)$$

The Hamiltonian is then given by

$$H = T + V = \frac{1}{2M} \left( p_r^2 + \frac{L^2}{r^2 s^2} \right) + \frac{1}{2} M \omega^2 r^2. \quad (3.2)$$

### 3.1 Classical Solutions

Using the corresponding classical equations of motion one obtains the classical orbits

$$\frac{1}{r^2} = \frac{M E s^2}{L^2} \left[ 1 + f \cos(2s(\varphi - \varphi_0)) \right], \quad (3.3)$$

with $E$ and $L$ again denoting energy and angular momentum and with

$$f = \sqrt{1 - \frac{\omega^2 L^2}{E^2 s^2}}. \quad (3.4)$$

The radial component of the momentum is given by

$$\frac{p_r}{r} = \frac{M E s}{L} f \sin(2s(\varphi - \varphi_0)). \quad (3.5)$$

All classical orbits are closed as long as $2s = p/q$ is a rational number (with $p, q \in \mathbb{N}$ again not sharing a common divisor). Some examples of classical orbits are shown in figure 5.

### 3.2 Semi-classical Bohr-Sommerfeld Quantization

As in the case of the $1/r$ potential, the semi-classical quantization condition for the angular momentum is again given by $L = m \in \mathbb{Z}$. For the harmonic oscillator the quantization condition for the radial motion takes the form

$$\oint dr \, p_r = \int_0^{\pi/s} d\varphi \frac{|L| f^2 \sin^2(2s(\varphi - \varphi_0))}{(1 + f \cos(2s(\varphi - \varphi_0))^2} = \pi \left( \frac{E}{E - \frac{|L|}{s}} \right) = 2\pi \left( n_r + \frac{1}{2} \right), \quad (3.6)$$

such that

$$E = \omega \left( 2n_r + \frac{|m|}{s} + 1 \right). \quad (3.7)$$

Again, it will turn out that the semi-classical result exactly reproduces the one of the full quantum theory.
Figure 5: Examples of bound classical orbits for the $r^2$ potential with $s = 3$ (left), $s = \frac{1}{2}$ (middle), and $s = 1$ (right). The latter case represents an elliptic orbit of the standard harmonic oscillator. The orbits are shown in the $x$-$y$-plane with $(x, y) = r(\cos \varphi, \sin \varphi)$ where $\varphi = \chi/s \in [0, 2\pi]$ is the rescaled polar angle.

### 3.3 Solution of the Schrödinger Equation

For the particle on the cone with harmonic oscillator potential the radial Schrödinger equation takes the form

$$
\left[ -\frac{1}{2M} \left( \partial_r^2 + \frac{1}{r} \partial_r \right) + \frac{m^2}{2Mr^2s^2} + \frac{1}{2} M\omega^2 r^2 \right] \psi(r) = E \psi(r). \tag{3.8}
$$

In this case, the solution is given by [11]

$$
\psi_{n,r,m}(r) = A \exp\left( -\frac{\alpha^2 r^2}{2} \right) (\alpha r)^{|m|/s} {}_1F_1\left( -n_r, \frac{|m|}{s} + 1, \alpha^2 r^2 \right), \quad \alpha = \sqrt{M\omega}. \tag{3.9}
$$

The corresponding quantized energy values are given by eq.(3.7). There are accidental degeneracies if $2s = p/q$ is a rational number, which thus again arise exactly when all classical orbits are closed. Some features of the energy spectrum are illustrated in figure 6.

### 3.4 Runge-Lenz Vector

The accidental degeneracies for rational $s$ again point to the existence of a conserved Runge-Lenz vector. At the classical level, we can use eqs.(3.3) and (3.5) to write

$$
Ef \cos(2s(\varphi - \varphi_0)) = Ef \left[ \cos(2s\varphi) \cos(2s\varphi_0) + \sin(2s\varphi) \sin(2s\varphi_0) \right] = \frac{L^2}{Mr^2s} - H
$$

$$
Ef \sin(2s(\varphi - \varphi_0)) = Ef \left[ \sin(2s\varphi) \cos(2s\varphi_0) - \cos(2s\varphi) \sin(2s\varphi_0) \right] = \frac{p_v L}{M rs}, \tag{3.10}
$$
Figure 6: The $r^2$ potential (solid curve) together with an effective potential including the centrifugal barrier with $m = \pm 1$ (dashed curve) for $s = \frac{1}{2}$. The energies of the ground state and the first two excited states are indicated by horizontal lines. The numbers besides the lines specify the degree of degeneracy. The ground state (with $n_r = 0, m = 0$) is non-degenerate, while the first excited level (consisting of the states with $n_r = 0, m = \pm 1$ and $n_r = 1, m = 0$) and the second excited level (consisting of the states with $n_r = 0, m = \pm 2$, $n_r = 1, m = \pm 1$, and $n_r = 2, m = 0$) are accidentally three-fold, respectively five-fold degenerate.

\[ R_x = E f \cos(2s\varphi_0) = \left( \frac{L^2}{Mr^2s^2} - H \right) \cos(2s\varphi) + \frac{p_r L}{Mr} \sin(2s\varphi), \]
\[ R_y = E f \sin(2s\varphi_0) = \left( \frac{L^2}{Mr^2s^2} - H \right) \sin(2s\varphi) - \frac{p_r L}{Mr} \cos(2s\varphi). \] (3.11)

It should again be pointed out that $R_x$ and $R_y$ are proper conserved quantities only if $2s$ is an integer. Otherwise the Runge-Lenz vector is not a $2\pi$-periodic function of the angle $\varphi$, and its value depends on the number of revolutions of the particle around the tip of the cone. As before, it is useful to introduce the complex quantities

\[ R_\pm = R_x \pm iR_y = \left( \frac{L^2}{Mr^2s^2} - H \mp i \frac{p_r L}{Mr} \right) \exp(\pm 2is\varphi). \] (3.12)

For rational values $2s = p/q$ (with $p, q \in \mathbb{N}$) the quantities

\[ R^q_\pm = \left( \frac{L^2}{Mr^2s^2} - H \mp i \frac{p_r L}{Mr} \right)^q \exp(\pm ip\varphi) \] (3.13)

are again single-valued functions of $x = r \cos \varphi$ and $y = r \sin \varphi$, and are hence proper conserved quantities.
For the harmonic oscillator, the length of the Runge-Lenz vector is given by
\[ R^2 = R_x^2 + R_y^2 = \left( \frac{L^2}{Mr^2s^2} - H \right)^2 + \left( \frac{p_r L}{Mr} \right)^2 \]
\[ = \left( \frac{p_r^2}{2M} - \frac{L^2}{2Mr^2s^2} + \frac{1}{2} M \omega^2 r^2 \right)^2 + \left( \frac{p_r L}{Mr} \right)^2 = H^2 - \left( \frac{\omega}{s} \right)^2. \tag{3.14} \]

As in the case of the $1/r$ potential, it is useful to introduce rescaled variables which now take the form
\[ \tilde{R}_x = \frac{1}{2\omega} R_x, \quad \tilde{R}_y = \frac{1}{2\omega} R_y, \quad \tilde{L} = \frac{L}{2s}. \tag{3.15} \]

We thus obtain
\[ C = \tilde{R}_x^2 + \tilde{R}_y^2 + \tilde{L}^2 = \left( \frac{H}{2\omega} \right)^2 \Rightarrow H = 2\omega \sqrt{C}. \tag{3.16} \]

Once again, it will turn out that the quantum analogue of $C$ is the Casimir operator of an accidental $SU(2)$ symmetry.

At the quantum level the Runge-Lenz vector now takes the form
\[
R_x = \frac{1}{2M} \cos(2s \varphi) \partial_r^2 - \frac{1}{2Mr^2s^2} \cos(2s \varphi) \partial_r^2 + \frac{1}{Mr^2s} \sin(2s \varphi) \partial_\varphi \\
- \frac{1}{2} M \omega^2 r^2 \cos(2s \varphi) - \frac{1}{Mr} \sin(2s \varphi) \partial_r \partial_\varphi - \frac{1}{2Mr} \cos(2s \varphi) \partial_r, \\
R_y = \frac{1}{2M} \sin(2s \varphi) \partial_r^2 - \frac{1}{2Mr^2s^2} \sin(2s \varphi) \partial_r^2 - \frac{1}{Mr^2s} \cos(2s \varphi) \partial_\varphi \\
- \frac{1}{2} M \omega^2 r^2 \sin(2s \varphi) + \frac{1}{Mr} \cos(2s \varphi) \partial_r \partial_\varphi - \frac{1}{2Mr} \sin(2s \varphi) \partial_r. \tag{3.17} \]

One can show that the Runge-Lenz vector as well as the angular momentum $L$ commute with the Hamiltonian, and that these operators obey the algebra
\[ [R_x, R_y] = 2i\omega \frac{L}{s}, \quad [R_x, L] = -2isR_y, \quad [R_y, L] = 2isR_x. \tag{3.18} \]

Applying the rescaling of eq.(3.15), this leads to
\[ [\tilde{R}_x, \tilde{R}_y] = i\tilde{L}, \quad [\tilde{R}_y, \tilde{L}] = i\tilde{R}_x, \quad [\tilde{L}, \tilde{R}_x] = i\tilde{R}_y, \quad [\tilde{L}, \tilde{R}_y] = i\tilde{R}_x. \tag{3.19} \]

which again represents an $SU(2)$ algebra.

### 3.5 Casimir Operator

The Casimir operator for the harmonic oscillator on the cone takes the form
\[ C = \tilde{R}_x^2 + \tilde{R}_y^2 + \tilde{L}^2 = \left( \frac{H}{2\omega} \right)^2 - \frac{1}{4}, \tag{3.20} \]
which implies

\[ H = 2\omega \sqrt{C + \frac{1}{4}} = 2\omega \left( S + \frac{1}{2} \right). \]  

(3.21)

Comparing with eq. (3.7) for the energy spectrum, we now identify

\[ S = n_r + \frac{|m|}{2s}. \]  

(3.22)

### 3.6 Unusual Multiplets

Let us now consider the unusual multiplets in case of the harmonic oscillator. The discussion is similar to the one of the $1/r$ potential and will thus not be repeated in all details. For $m > 0$ one now obtains

\[ \tilde{R}_+ |n_r, m\rangle \propto |n_r - 1, m + 2s\rangle, \quad \tilde{R}_- |n_r, m\rangle \propto |n_r + 1, m - 2s\rangle, \]  

(3.23)

and for $m < 0$ one finds

\[ \tilde{R}_+ |n_r, m\rangle \propto |n_r + 1, m + 2s\rangle, \quad \tilde{R}_- |n_r, m\rangle \propto |n_r - 1, m - 2s\rangle, \]  

(3.24)

while, for $m = 0$ we have

\[ \tilde{R}_+ |n_r, 0\rangle \propto |n_r - 1, 2s\rangle, \quad \tilde{R}_- |n_r, 0\rangle \propto |n_r - 1, -2s\rangle. \]  

(3.25)

As before, these relations follow from the $SU(2)$ algebra which now implies that $\tilde{R}_\pm$ are raising and lowering operators for $\tilde{L} = L/2s$. Hence, by acting with $\tilde{R}_\pm$ the eigenvalue $m$ of $L$ is now shifted by $\pm 2s$.

One now confirms the value of the Casimir spin $S = n_r + |m|/2s$ by evaluating

\[ C|0, m + 2n_r s \geq 0\rangle = (\tilde{L} + \tilde{L}^2)|0, m + 2n_r s \geq 0\rangle \]
\[ = (m + n_r) (m + n_r + 1) |0, m + 2n_r s \geq 0\rangle \]
\[ = S(S + 1)|0, m + 2n_r s \geq 0\rangle, \]

\[ C|0, m - 2n_r s \leq 0\rangle = (-\tilde{L} + \tilde{L}^2)|0, m - 2n_r s \leq 0\rangle \]
\[ = (-m + n_r) (-m + n_r + 1) |0, m - 2n_r s \leq 0\rangle \]
\[ = S(S + 1)|0, m - 2n_r s \leq 0\rangle. \]  

(3.26)

The multiplet of degenerate states with the same value of $S$ is again obtained by repeated applications by $\tilde{R}_+$ or $\tilde{R}_-$.

As in the case of the $1/r$ potential, for $s \neq 1$ different types of unusual multiplets arise. Again, even for integer or half-integer $S = n_r + |m|/2s$, the degeneracy of the physical multiplet is not $2S + 1$ because $m \pm 2ns$ may not be an integer in which
case the corresponding wave function is not \(2\pi\)-periodic. When \(S = n_r + |m|/2s\) is neither an integer nor a half-integer, there is again an infinite number of degenerate solutions of the Schrödinger equation. However, once more, only a finite number of them obeys the boundary condition of eq.(1.30) and thus belongs to \(D[H]\). A sequence of physical and unphysical wave functions is illustrated in figure 7.

Figure 7: A sequence of wave functions for the \(r^2\) potential with \(s = 3\) obtained from repeated applications of \(\tilde{R}_-\). The quantum numbers are \(n_r = 0\), \(m = 7\) (left), \(n_r = 1\), \(m = 7 - 2s = 1\) (middle), and \(n_r = 2\), \(m = 7 - 4s = -5\) (right). The third state in the sequence is outside the domain of the Hamiltonian because the corresponding wave function does not obey the boundary condition of eq.(1.30) and the state is thus unphysical.

4 Conclusions

We have considered the physics of a particle confined to the surface of a cone with deficit angle \(\delta\) and bound to its tip by a \(1/r\) or an \(r^2\) potential. In both cases, for rational \(s = 1 - \delta/2\pi\), all bound classical orbits are closed and there are accidental degeneracies in the discrete energy spectrum of the quantum system. There is an accidental \(SU(2)\) symmetry generated by the Runge-Lenz vector and by the angular momentum. However, the Runge-Lenz vector is not necessarily a physical operator. For example, by acting with the Runge-Lenz vector on a physical state one may generate an unphysical wave function outside the domain of the Hamiltonian. As a result, the representations of the accidental \(SU(2)\) symmetry are larger than the multiplets of degenerate physical states. In particular, some physical states are contained in multiplets with an unusual value of the Casimir spin \(S\) which is neither an integer nor a half-integer. Still, the fractional value of the spin yields the correct value of the quantized energy.

The particle on a cone provides us with an interesting physical system in which
symmetries manifest themselves in a very unusual manner. Although the Hamiltonian commutes with the generators of an $SU(2)$ symmetry, the multiplets of degenerate states do not always correspond to integer or half-integer Casimir spin. This is because the application of the symmetry generators may lead us out of the domain of the Hamiltonian. Only the states with square-integrable single-valued $2\pi$-periodic wave functions belong to the physical spectrum, and all other members of the corresponding “$SU(2)$” representation must be discarded. Mathematically speaking, the symmetry generators — although Hermitean in their respective domain — do not act as Hermitean operators in the domain of the Hamiltonian.

In contrast to many other quantum mechanics problems, in order to understand motion on a cone it was necessary to address mathematical issues such as the domains of operators as well as Hermiticity versus self-adjointness. Still, we have not elaborated on some questions related to different possible self-adjoint extensions of the Hamiltonian. For the particle on the cone, such issues seem worth investigating. In this work, we have limited ourselves to the standard Friedrichs extension of the Hamiltonian. Alternative self-adjoint extensions correspond to an additional $\delta$-function potential located at the tip of the cone. This will modify the problem in an interesting way. In particular, we expect that, in the presence of an additional $\delta$-function potential, the accidental degeneracy will be partly lifted. However, since the $\delta$-function only affects states with $m = 0$, some accidental degeneracy will remain. The particle on the cone provides us with another example for the deep connection between the closedness of all bound classical orbits and accidental degeneracies in the discrete spectrum of the Hamiltonian. Even if the classical system has various quantum analogues (because there are different possible self-adjoint extensions) some accidental degeneracy still persists. It is also remarkable that, like in other cases with accidental symmetries, for the particle on the cone semi-classical Bohr-Sommerfeld quantization provides the exact quantum energy spectrum.

We are unaware of another system for which a similarly unusual symmetry behavior has been observed. It is interesting to ask if symmetry can manifest itself in this unusual manner also in other quantum systems. For example, cones of graphene may provide a motivation to study accidental degeneracies of the Dirac equation on a cone. Also higher-dimensional spaces with conical singularities may be worth investigating. In any case, we hope that we have convinced the reader that motion on a cone provides an illuminating example for a rather unusual manifestation of symmetry in quantum mechanics.

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