THE COEFFICIENTS OF THE PERIOD POLYNOMIALS

SERBAN BARCANESCU

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Abstract

A general description of the Viète coefficients of the gaussian period polynomials is given, in terms of certain symmetric representations of the subgroups and the corresponding quotient groups of the multiplicative group \( \mathbb{F}_p^* \) of a finite prime field of characteristics \( p \), an odd prime number. The known values of these coefficients are recovered by this technique and further results of general nature are presented.

(Key words: gaussian symbols, gaussian periods, symmetric modules, k-sets, difference vectors, sliding classes).

I. GAUSS PERIODS

Let \( \mathbb{F}_p \) be the prime finite field of characteristics \( p \) (an odd prime number) and let \( g \) be a fixed primitive root modulo \( p \) (i.e. a generator of the cyclic multiplicative group \( \mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\} \)).

I.1 THE GAUSS SYMBOL

For non empty subsets \( M_1, M_2, ..., M_n \subseteq \mathbb{F}_p \) (\( n \geq 1 \)) we define the “Gauss symbol” by:

\[
\{M_1, M_2, ..., M_n\} = \# \{(x_1, x_2, ..., x_n) | x_j \in M_j \text{ for } j = 1, 2, ..., n \text{ and } x_1 + x_2 + ... + x_n = 0 \}.
\]

The following properties are immediate:

(i) \( \{M_1, M_2, ..., M_n\} = \{M_{\pi(1)}, M_{\pi(2)}, ..., M_{\pi(n)}\} \) for any permutation \( \pi \) of \( \{1, 2, ..., n\} \).

(ii) \( \{M_1 \sqcup M_1', M_2, ..., M_n\} = \{M_1' + M_2 + ... + M_n\} + \{M_1', M_2, ..., M_n\} \), where \( \sqcup \) stands for disjoint union.

(iii) \( \{\lambda M_1, \lambda M_2, ..., \lambda M_n\} = \{M_1, M_2, ..., M_n\} \) for any scalar \( \lambda \in \mathbb{F}_p^* \), where \( \lambda M = \{\lambda x | x \in M\} \).

When \( M_1 = \{x\} \), we write \( \{x, M_2, ..., M_n\} \) instead of \( \{\{x\}, M_2, ..., M_n\} \).

( In [3] slightly different notations are used for the same notion) .
I.2 SUBGROUPS OF $\mathbf{F}_p^*$

We fix $d \geq 1$ a divisor of $p - 1$ and denote by:

$$p - 1 = dm$$

($m \in \mathbb{N}$) the resulting factorization of $p - 1$.

Let $C_0 = \{x^d \mid x \in \mathbf{F}_p^*\} = \{1, g^d, g^{2d}, \ldots, g^{(m-1)d}\}$ be the unique subgroup of order $m$ (and index $d$) of $\mathbf{F}_p^*$, defining the partition into classes (mod $C_0$):

$$\mathbf{F}_p^* = C_0 \sqcup C_1 \sqcup \ldots \sqcup C_{d-1},$$

where $C_s = \{g^{jd+s} \mid j = 0, 1, \ldots, m - 1\}$ for $s = 0, 1, 2, \ldots, d - 1$.

If $\Gamma_h$ denotes the abstract cyclic group of order $h$, we have the models:

$$\Gamma_m \cong C_0 \text{ and } \Gamma_d \cong \mathbf{F}_p^*/C_0$$

We have the following simple property concerning the sign repartition on the classes mod $C_0$:

**Proposition 1**

(i) For odd $d$: $(-1) \in C_0$

(ii) For even $d$: $(-1) \in C_0$ for even $m$ and $(-1) \in C_{\frac{d}{2}}$ for odd $m$.

**Proof.**

Let $(-1) \in C_s$ for some $s \pmod{d}$. The multiplication by a non zero element is bijective on $\mathbf{F}_p^*$; $(-1)$ has period 2 as an element of this group and $C_aC_b = C_{a+b}$ for all $a, b \pmod{d}$, therefore:

$$-C_s = C_{2s} \implies C_s = -(-C_s) = -C_{2s} = C_{3s} \implies 3s \equiv s \pmod{d} \implies 2s \equiv 0 \pmod{d},$$

so (i) results.

To see that (ii) holds, observe first that $-1 = g^{\frac{p-1}{2}}$ giving $s \equiv \frac{p-1}{2} \equiv \frac{dm}{2} \pmod{d}$ (so $s = 0$ for $d$ odd (necessary $m$ is even)). For $d$ even we have: either $s \equiv \frac{d}{2} \equiv \frac{dm}{2} \pmod{d}$ so $m$ should be odd, or $s \equiv \frac{dm}{2} \equiv 0 \pmod{2 \frac{d}{2}}$ so $m$ should be even.

**Corollary**

For any $s \pmod{d}$:

(i) $-C_s = C_s$ for odd $d$

(ii) $-C_s = C_s$ for even $d$ and even $m$ and $-C_s = C_{s+\frac{d}{2}}$ for even $d$ and odd $m$.

**Remark.**

Although very simple in the above situation, the sign repartition is not a trivial fact on a prime finite field. For instance, with respect to the canonical halbsystem of the positive residues modulo $p$ the sign repartition is an easy problem for $p \equiv 1 \pmod{4}$, but a difficult one for $p \equiv 3 \pmod{4}$: in this case it is equivalent to the determination of the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$, see [1].
1.3 THE GAUSS PERIODS

We preserve the notations above. Let $\zeta$ be a fixed complex root of unity of order $p$. The complex numbers:

$$\eta_j = \sum_{x \in C_j} \zeta^x \text{ for } j = 0, 1, ..., d - 1$$

are called "the Gauss $d$-periods". Since the complex conjugate of $\zeta$ is $\zeta^{-1}$, the Corollary to Proposition 1 shows that the $d$-periods are actually real numbers for odd $d$ or for even $d$ and $m$.

The periods $\eta_0, \eta_1, ..., \eta_{d-1}$ constitute an integral basis (for $p^k$, $k \geq 2$ only a rational basis) of the subfield of degree $d$ over $\mathbb{Q}$ of the cyclotomic field $\mathbb{Q}(\zeta)$.

As such, they satisfy a separable, irreducible equation over $\mathbb{Z}$:

$$P_d(X) = \prod_{j=0}^{d-1} (X - \eta_j) = X^d + \sum_{k=1}^{d} (-1)^k a_k(p, d) X^{d-k}$$

whose Viète coefficients $a_k = a_k(p, d)$ are the integer numbers given by:

$$a_k = \sum_{S \in \left( \begin{array}{c} [d] \\ [k] \end{array} \right)} \eta_S \text{ for } k = 1, 2, ..., d$$

where $\left( \begin{array}{c} [d] \\ [k] \end{array} \right)$ denotes the set of all $k$-element subsets of $[d] = \{0, 1, ..., d-1\}$ and $\eta_S = \prod_{s \in S} \eta_s$.

Let us remark that $a_1(p, d) = -1$ for all $p, d$ because $\sum_{x=1}^{p-1} \zeta^x = \sum_{j=0}^{d-1} \eta_j = -1$.

In this work we display a general formula for the computation of the Viète coefficients $a_k(p, d)$. This formula covers the previously known cases ($d = 2, 3, 4$), it is easily applied to find the general $a_2(p, k)$ and $a_3(p, k)$ and, by conveniently developing the supporting combinatorics, indicates a general algorithm with interesting number theoretic and perhaps geometric connections.

In order to begin the investigation, let us (incorrectly, for the moment) write in condensed form:

$$\eta_j = \zeta^{C_j} \text{ for } j = 0, 1, ..., d - 1 \text{ and consequently } \eta_S = \zeta^{\sum_{s \in S} C_s}$$

for a $k$-element subset $S$ of $[d]$.

In contrast to the usual writing of a set of elements: $\{x, y, z, ...\}$, we shall use the notation $\|x, y, z, ...\|$ for a list (multiset) of elements, i.e. taking into account the multiplicities of the elements.

With this convention, we write a generic term of the coefficient $a_k$ as:

$$\eta_S = \zeta^{\|\sum_{s \in S} C_s\|}$$

where it naturally appears the tableau:

$$T(S) = \|\theta_j^S\| \text{ with } S \in \left( \begin{array}{c} [d] \\ [k] \end{array} \right) \text{ and } J = (j_1, ..., j_k) \in (\mathbb{Z}/m\mathbb{Z})^k$$

(1)
whose entries are:

$$
\theta^S_J = g^{d_{j_1} + s_1} + g^{d_{j_2} + s_2} + ... + g^{d_{j_k} + s_k} \text{ for } S = \{s_1, ..., s_k\}.
$$

The condensed writing:

$$
\eta_S = \zeta^{T(S)}
$$

actually means the sum development of the product $\cap_{s \in S} \eta_s$, i.e.

$$
\sum_J \zeta^{\theta^S_J} = \sum_{(j_1, ..., j_k)} \zeta^{g^{d_{j_1} + s_1}} \zeta^{g^{d_{j_2} + s_2}} ... \zeta^{g^{d_{j_k} + s_k}},
$$

where we keep track of the individual factors, without effectively replacing the actual value of their exponents in $F_p$.

The tableau $T(S)$ defined above has $m^k$ entries (which may be computed as elements of $F_p$), each indexed by a sequence $J = (j_1, j_2, ..., j_k)$ (because for any integer $t$ we have: $g^{d(tm+j)}+s = g^{dj+s}$ since $g^{tdm} = g^{t(p−1)} = 1$).

**Remark.**

One may conveniently consider the tableau $T(S)$ as a generalized matrix-like object. Namely, we indentify the index set $(\mathbb{Z}/m\mathbb{Z})^k$ with the integral $k$-cube $[m] \times [m] \times ... \times [m] \subset \mathbb{N}^k$, $(m) = \{0, 1, ..., m − 1\}$ (addition inside the cube being considered modulo $m$ -see the proof of Prop.2 below) and put the value $\theta^S_J$ on the point $J = (j_1, ..., j_k)$ of the cube. The resulting function is the tableau associated to the $k$-element set $S$. ■

The individual tableaux of the type $T(S)$ will be investigated in the next section.

The main objects of study in this paper are the sets

$$
TAB(p, d, k) = \{T(S)|S \in \binom{[d]}{[k]} \} \quad k = 1, 2, ..., d
$$

When $p$ is fixed, the notation $TAB(d, k)$ will be used instead of the above one. We investigate the properties of these sets beginning with section III.

**II THE $C_0$-MODULE $T(S)$**

Throughout this section we fix a $k$-element subset $S = \{0 \leq s_1 < s_2 < ... < s_k \leq d − 1\}$ of $[d]$ and consider the tableau $T(S) = [\theta^S_J]$, as defined in (1).

The cyclic group $C_0 = \langle g^d \rangle = \{1, g^d, g^{2d}, ..., g^{(m−1)d}\}$ naturally acts on $T(S)$ by the multiplication law of the field $F_p$. Algebraically, the action is defined by:

$$
(\forall) \lambda (\text{mod } m) : (g^{\lambda d}, \theta^S_J) = g^{\lambda d} \theta^S_J
$$

Since $g^d g^{dj+s} = g^{d(j+1)+s}$, this action may also be combinatorially described the following way:

$$
(\forall) \lambda (\text{mod } m) : (g^{\lambda d}, \theta^S_J) = \theta^S_{J+\lambda[1]} \quad [1] = [1, 1, ..., 1] (k\text{times}) \text{ and } [\lambda] = [\lambda, ..., \lambda]
$$
By separating the first coordinate in each multi-index \( J \) we may write:

\[
J = (i, \bar{J}), \; i = 0, 1, \ldots, m - 1 \text{ with } \bar{J} \in (\mathbb{Z}/m\mathbb{Z})^{k-1}
\]

so the following sub-tableaux do naturally appear:

\[
T_i(S) = \left\| \theta^S_{(i, \bar{J})} \right\| \; i = 0, 1, \ldots, m - 1
\]

giving the partition:

\[
T(S) = T_0(S) \sqcup T_1(S) \sqcup \ldots \sqcup T_{m-1}(S).
\]

**Proposition 2.**

(i) For \( i=0,1,\ldots,m-1 \) : \( \#T_i(S) = m^{k-1} \).

(ii) Each set \( T_i(S) \) is a transversal (i.e. a complete and independent set of representatives) to the orbits of the action of \( C_0 \) on \( T(S) \).

(iii) The elements of the transversal \( T_0(S) \) are indexing the orbits of the action of \( C_0 \) on \( T(S) \). Each orbit is either \( \|0,0,\ldots,0\| \) (m positions) or one of \( C_0, C_1, \ldots, C_{d-1} \).

**Proof.**

(i) is a direct consequence of the definition of the sub-tableaux \( T_i(S) \).

(ii) We have \( g^d T_i(S) = g^d \cdot \theta^S_{(i, \bar{J})} \) because the multiplication by \( g^d \) is injective and \( \bar{J} + [1] \) and \( \bar{J} \) simultaneously cover all of \( (\mathbb{Z}/m\mathbb{Z})^{k-1} \) (here we denote also by \([1]\) the list of \((k-1)\) positions equal to 1: in order to avoid cumbersome notation, we implicitly adapt to the situation considered the length of such vectors).

If \( \theta^S_{(i, \bar{J})} \equiv \theta^S_{(i, \bar{L})} \pmod{C_0} \) for some \( \bar{J}, \bar{L} \in (\mathbb{Z}/m\mathbb{Z})^{k-1} \) and fixed \( i \in \{0, 1, \ldots, m-1\} \) then there exists \( \lambda \pmod{m} \) such that

\[
(i, \bar{J}) = (i, \bar{L}) + \lambda [1] \implies i = i + \lambda \pmod{m} \implies \bar{J} = \bar{L}.
\]

Therefore the entries of \( T_i(S) \) cannot be congruent modulo \( C_0 \).

(iii) The first assertion results from (ii). For the second one, let \( x = g^S_{(0, \bar{J})} \in T_0(S) \). Then \( x \in F_p = \{0\} \sqcup C_0 \sqcup C_1 \sqcup \ldots \sqcup C_{d-1} \), so there are two possible cases:

1. \( x = 0 \implies C_0, x = \{0, 0, \ldots, 0\}, \text{ m positions} \)
2. \( x \in C_j \implies C_0, x = C_j \).

So the orbits are of the enounced form.■

**Corollary**

Let \( T_0(S) = Z(S) \sqcup A_0(S) \sqcup \ldots \sqcup A_{d-1}(S) \) where \( Z(S) = \|x \in T_0(S) | x = 0\| \) and \( A_j(S) = \|x \in T_0(S) | x \in C_j\| \) for \( j = 0, 1, \ldots, d-1 \).

The structure of the \( C_0 \)-module \( T(S) \) is:

\[
T(S) = C_0, Z(S) \sqcup C_0, A_0(S) \sqcup \ldots \sqcup C_0, A_{d-1}(S)
\]

where \( C_0, A = \sqcup_{a \in A} C_0, a \).■
Considering the multiplicities of the elements in the lists above, namely:

\[ z(S) = \#Z(S) \quad \text{and} \quad \mu_j(S) = \#A_j(S) \]

the structure of the \( C_0 \) module \( T(S) \) as described in the above Corollary may also be written:

\[ T(S) = \sqcup z(S) [0] \sqcup (\sqcup_1^{\mu_0} C_0) \sqcup (\sqcup_1^{\mu_1} C_1) \sqcup ... \sqcup (\sqcup_1^{\mu_{d-1}} C_{d-1}) \]  \hspace{1cm} (5)

where \([0]\) is the list of \( m \) entries each equal to 0.

Directly from the definition of \( Z(S) \) and the definition of the subtableau \( T_0(S) \), writing \( S = \{0 \leq s_1 < s_2 < ... < s_k \leq d-1\} \) we have:

\[ z(S) = \{g^{s_1}, C_{s_2}, C_{s_3}, ..., C_{s_k}\} \] \hspace{1cm} (6)

In particular, for \( s_1 = 0 \):

\[ z(S) = \{1, C_{s_2}, ..., C_{s_{d-1}}\} \]

**Proposition 3.**

With the above notations, for any fixed \( k \)-subset \( S \) of \([d]\):

\[ \eta_S = mz(S) + \mu_0 \eta_0 + \mu_1 \eta_1 + ... + \mu_{d-1} \eta_{d-1} \]  \hspace{1cm} (7)

where

\[ z(S) + \mu_0 + \mu_1 + ... + \mu_{d-1} = m^{k-1} \]  \hspace{1cm} (8)

**Proof.**

As we have seen in Section I: \( \eta_S = \zeta_{T(S)} \) and the decomposition of \( T(S) \) into \( C_0 \) orbits as given by (5) directly implies (7). Passing to cardinals in the Corollary to Proposition 2 gives (8). \( \blacksquare \)

### III. THE \( \mathbf{F}_p^*/C_0 \) MODULE \( TAB(p, d, k) \)

In the previous section we were concerned with the individual \( C_0 \) modules \( T(S) \), each associated to a \( k \)-element subset \( S \) of \([d]\). We now gather them in a combinatorial variety:

\[ TAB(d, k) = \{T(S)|S \in \binom{[d]}{[k]}\} \]

for \( k \) fixed in \( \{1, 2, ..., d\} \).

Let us consider the cyclic group \( \Gamma_d \) realized as \( \mathbf{F}_p^*/C_0 = \{1, \bar{g}, \bar{g}^2, ..., \bar{g}^{d-1}\} \), \( \bar{g} = g.C_0 = g(mod C_0) \). This group naturally acts on \( TAB(k) \) via the multiplication in \( \mathbf{F}_p \) of each entry of a given tableau \( T(S) \) with a representative of an element of \( \mathbf{F}_p^*/C_0 \).

Precisely, for any \( \nu(mod d) \) and any tableau \( T(S) = \|\theta_j^S\|_J \), the algebraic description of the action is:

\[ (\bar{g}^\nu, \|\theta_j^S\|_J) = \|\bar{g}^\nu \cdot \theta_j^S\|_J \]  \hspace{1cm} (9)
The action is well-defined because if \( \nu' = \nu + hd \), \((h \mod m)\) we have 
\[
g'^{\nu} \theta^{g^\nu}_j = g^{\nu} \theta^{g^\nu}_{j+h[1]} \quad ([1] = [1,1,...,1], k \text{ positions}) \text{ and the indices } J_j, J + h[1]
\]
simultaneously cover the index set \((\mathbb{Z}/m\mathbb{Z})^k\).

Since for \( \nu \) and \( s \mod d \) we have:
\[
g^{\nu} g^{id+s} = g^{id+(\nu+s)}, \quad (\nu + s) \mod d
\]
the action \((9)\) may be combinatorially described as:
\[
(g^{\nu}, T(S)) = T(S + \nu [1])
\]  \hspace{1cm} (10)

where \([1] = [1,1,...,1]\) \((k \text{ positions})\), the \(k\) - element set \( S + \nu [1] \) being the translation with \( \nu \) of \( S \), taken modulo \( d \) \((\text{in order to obtain the result as a subset of } [d])\).

We will now give an alternative combinatorial description of the \( \mathbb{F}_p^*/C_0 \simeq \Gamma_d \)-module \( TAB(k) \). Namely, let \( \mathbb{Z}/d\mathbb{Z} \simeq \Gamma_d \) be a new model of the abstract cyclic group of order \( d \) and let:
\[
M(d,k) = \{ \{p_1,\ldots,p_k\} | p_i \in \mathbb{Z}/d\mathbb{Z} \text{ for } i = 1,2,\ldots,k \} = \{S(\mod d)|S \in \left( \begin{array}{c} [d] \\ [k] \end{array} \right) \} \text{ (simply denoted by } M(k) \text{ for fixed } d \} \text{ with the structure of a } \mathbb{Z}/d\mathbb{Z}-\text{module given by:}
\]
\[
(\nu(\mod d), S(\mod d)) = S + \nu [1] (\mod d)
\]  \hspace{1cm} (11)

**Proposition 4.**

With the above notations and definitions the \( \Gamma_d \) - modules \( TAB(d,k) \) and \( M(d,k) \) are isomorphic.

**Proof.**

The two models of \( \Gamma_d : \mathbb{F}_p^*/C_0 \) and \( \mathbb{Z}/d\mathbb{Z} \) are isomorphic by \( g^{\nu} \rightarrow \nu(\mod d) \) and \( T(S) \rightarrow S(\mod d) \) is a bijection between \( TAB(k) \) and \( M(k) \). The formula (10) and (11) show that these correspondences actually define an isomorphism of \( \Gamma_d \)-modules. \(\blacksquare\)

Therefore we shall investigate the structure of the \( \Gamma_d \) - module \( M(d,k) \) and automatically translate the results in \( TAB(d,k) \). We will identify a \( k \)-element subset \( S = \{0 \leq s_1 < s_2 < \ldots < s_k \leq d-1\} \) of \([d]\) with its \( \mod d \) reduction \( S(\mod d) \), taking care to consider its translates \( \{S + \nu [1] | \nu(\mod d)\} \) also modulo \( d \). The elements of \( S(\mod d) \), as the ones of \( S \), will usually be written in their increasing order of magnitude.

For a divisor \( d' \) of \( d \) we denote the image via the canonical epimorphism
\[
\Gamma_d \simeq \mathbb{Z}/d\mathbb{Z} \rightarrow \Gamma_{d'} \simeq \mathbb{Z}/d'\mathbb{Z} : x(\mod d) \rightarrow x(\mod d') \text{ of a set } S (\text{considered as subset of } [d]) \text{ by } S(\mod d')(\text{considered as subset of } [0,1,...,d']) \text{.}
\]

Also, we will freely use the already introduced convention to automatically adapt the length of the list \([1] = [1,1,...,1]\) to a given particular situation, using the single notation \([1]\). When necessary, we put \([1]_n\) to indicate that the list has precisely \( n \) entries equal to 1.
We begin with the following structural result.

**Proposition 5.**

(i) For any \( S = S \mod d \in M(d,k) \) there exists an unique divisor \( e \) of \( \gcd(d,k) \) such that, putting \( d = ed' \) and \( k = ek' \), there exists a \( k \) set \( S^* \in M(d,k') \) whose reduction modulo \( d' \) preserves the cardinality and such that

\[
S = S^* \cup S^* + \tilde{d} \left[ 1 \right]_{k'} \cup S^* + 2\tilde{d} \left[ 1 \right]_{k'} \cup \ldots \cup S^* + (e - 1)\tilde{d} \left[ 1 \right]_{k'}.
\]

The canonical selection of \( S^* \), making it unique, is : \( S^* \subset \left[ d' \right] \) (considered as the initial segment of \([d]\)).

(ii) For each common divisor \( e \) of \( d \) and \( k \), with \( d = ed' \) and \( k = ek' \) there exists a \( k \) set \( S \in M(d,k) \) having the decomposition described in (i).

**Proof.**

Let \( S \) be \( \{s_1 < s_2 < \ldots < s_k\}(\mod d) \) where the elements \( s_i \) are minimal representatives, i.e. \( s_i \in [d] \). By (11) the \( \Gamma_d \)-orbit of \( S \) is :

\[
\Gamma_dS = \{S, S + [1]_k, S + 2[1]_k, \ldots, S + (d - 1)[1]_k \} \text{ for an integer } d' \geq 1,
\]

minimal with the property that:

\[(*): S + d \left[ 1 \right]_k = S.\]

Because \( S + d \left[ 1 \right]_k = S \) and \( d' \) is minimal, it follows that \( d' \) divides \( d \), so there is an \( e \geq 1 \) such that: \( d = ed' \) (the number \( e \) is the order of the stabilizer subgroup in \( \Gamma_k \) of \( S \) and \( d' \) is the length of the \( \Gamma_d \)-orbit of \( S \)).

The condition (11) means that there exists a permutation \( \pi \) on \( k \) symbols, such that:

\[s_i + d' = s_{\pi(i)}, i = 1, 2, \ldots, k.\]

Let \( \pi = \gamma_1 \gamma_2 \ldots \gamma_{k'} \) (\( k' \leq k \)) be the unique decomposition into disjoint cycles of \( \pi \) and (modulo renumbering the cycles) let \( \{r_1 < r_2 < \ldots < r_{k'}\} \) be the fixed transversal of the cycles \( \gamma_1, \ldots, \gamma_{k'} \) such that each \( r_i \) is the minimal element (in the usual order relation on \([d]\)) within the cycle \( \gamma_i \). Let \( e_i \) be the order (in the symmetric group on \( k \) symbols) of the cycle \( \gamma_i, i = 1, 2, \ldots, k' \). Then \( \gamma_i|S = \{\gamma_i^n(r_i) \mid n \in \mathbb{Z}\} = \{r_i, \gamma_i(r_i), \ldots, \gamma_i^{e_i - 1}(r_i)\} = \{r_i, r_i + d', r_i + 2d', \ldots, r_i + (e_i - 1)d'\} \) for \( i = 1, 2, \ldots, k' \). Since \( \gamma_i^e = id' \) (the identical permutation) we have \( e \equiv 0 \pmod{e_i} \) for all \( i \)'s.

Since \( r_i + e_i d' = r_i \) it follows that \( e_i d' \equiv 0 \pmod{d} \) i.e. \( e_i d' \equiv 0 \pmod{ed'} \) for all \( i \)'s. It results:

\[e = e_1 = e_2 = \ldots = e_{k'}.
\]

Denoting by \( e \) the common value of the orders of the cycles \( \gamma_1, \ldots, \gamma_{k'} \) we already have \( k = ek' \).

The \( k \) set \( S^* = \{r_1 < r_2 < \ldots < r_{k'}\} \) has the required property (since no two of the \( r_i \)'s are congruent modulo \( d' \), belonging to different cycles) and it
is unique because of the minimality of the \( r_i \)'s, which gives \( r_i \in \{0, 1, 2, ..., d' \} \) for all \( i \)'s.

(ii) We take any \( k' \)-subset \( U = \{u_1, u_2, ..., u_{k'} \} \) of \([d]\) for which the reduction modulo \( d' \) preserves the cardinality i.e. \( i \neq j \Rightarrow u_i \) non congruent to \( u_j \) modulo \( d' \). Let \( r_i \) be the minimal element ( in the natural order on \([d]\)) in the set \( \{u_i, u_i + d', u_i + 2d', ..., u_i + (e - 1)d' \} \), put the \( r_i \)'s in their ascending order of magnitude and consider the set \( S^* = \{r_1, ..., r_{k'} \} \). The union of the arithmetic progressions each of length \( e \) and ratio \( d' \) beginning with each of the \( r_i \)'s constitutes the required set \( S \).

Remarks.

(a) In the above proof, any set of representatives for the cycles \( \gamma_1, ..., \gamma_{k'} \) produces a \( k' \)-set in \([d]\) giving a decomposition of \( S \) as the one in the enounce. Such a set is a realization modulo \( d \) of the canonical minimal set \( S^* \), i.e. its elements are two-by-two non congruent modulo \( d' \). Any such realization would do, but we fix the minimal one because, from \( r_{k'} < d' \) it follows \( r_{k'} < r_1 + d' \) implying the following description of the set \( S \):

\[
S = \{ S^* < S^* + d' [1]_{k'} < ... < S^* + (e - 1)d' [1]_{k'} \}.
\]

(b) In the setting of Proposition 5, for any \( i \in \{1, 2, ..., k' \} \) the set \( \{r_i, r_i + 1, ..., r_i + (d' - 1)\} \) is a complete and independent set of representatives for the residues modulo \( d' \) and, simultaneously, a complete and independent set of representatives for the elements (which are sets) in the \( \Gamma_d \)-orbit of \( S \). Therefore, since between these representatives one should be \( \equiv 0 \) modulo \( d' \), it follows:

in every \( \Gamma_d \)-orbit in \( M(d,k) \) there is a representative \( \{0 = s_1 < s_2 < ... < s_k \} \).

The result (i) in Proposition 5 says that for each element \( S \in M(d,k) \) there exists a divisor \( e | gcd(d,k) \) such that \( S \) decomposes as an union of \( k' \) arithmetic progressions, each of length \( e \) and of the same ratio \( d' \), their initial terms being the elements of an uniquely determined \( k' \)-subset \( S^* \) of \([d]\). When \( e = 1 \) the assertion says that \( S \) is a \( - \)-subset of \([d]\) whose \( \Gamma_d \)-orbit has maximal length \( d \). In particular, for \( gcd(d,k) = 1 \), \( M(d,k) \) decomposes into disjoint orbits of the same length \( d \).

The result (ii) shows that there exists a well defined surjection:

\[
\varphi : M(d,k) \rightarrow Div(gcd(d,k))
\]

given by \( S \rightarrow e \) (here \( Div(n) \) denotes the set of all divisors of the natural number \( n \)). We denote by \( M_e(d,k) \) the \( \varphi \)-preimage of the divisor \( e \in Div(gcd(d,k)) \), so it results the partition:

\[
M(d,k) = \cup_{e | gcd(d,k)} M_e(d,k)
\] (12)
(here “|” stands for the divisibility relation). Obviously, \( M_e(d, k) \) consists of the elements \( S \in M(d, k) \) having as stabilizer the unique subgroup of order \( e \) in \( \Gamma_d \). With these notations we have the:

**Proposition 6.**

For every divisor \( e \in \text{Div}(\gcd(d, k)) \) the set \( M_e(d, k) \) is a \( \Gamma_d \)-submodule of \( M(d, k) \).

**Proof.**

For \( S \in M_e(d, k) \) any element in the \( \Gamma_d \)-orbit of \( S \) has the same stabilizer, therefore the entire orbit is contained in \( M_e(d, k) \), \( \Gamma_d \) being abelian (combinatorially, using the above notations, we see that:
\[
(S + [1]_k)^+ = (S^* + [1]_{k'}) \pmod{d'}
\]
so each element in the orbit of \( S \) has the same structure as \( S \), therefore it belongs to \( M_e(d, k) \).

The result in Proposition 6 shows that every set \( M_e(d, k) \) is a disjoint union of \( \Gamma_d \)-orbits. We define:

\[ T_e(d, k) = \text{a fixed transversal of the } \Gamma_d \text{-orbits partitioning } M_e(d, k). (\#) \]

In this context it is clear that \( M_1(d, k) \) is a disjoint union of “complete” orbits, i.e. of orbits of maximal length \( d \) and every \( M_e(d, k) \) is a disjoint union of orbits of length \( d' = d/e \), for every common divisor \( e \) of \( d \) and \( k \). In particular, for \( \delta = \gcd(d, k) \), the set \( M_\delta(d, k) \) consists of the orbits of minimal length \( d/\delta \). For \( k = d \) we have the unique total \( d \)-set \( [d] = \{0, 1, 2, ..., d-1\} \) having an unique \( \Gamma_d \)-orbit of length 1, namely \( \{[d]\} \). We do also obtain the following result.

**Corollary.**

Let \( \gcd(k, d) = 1 \). Then \( d \) divides the binomial coefficient \( \binom{d}{k} \) and:

\[ \#T_1(d, k) = 1 \binom{d}{k}. \]

**Proof.**

For \( \gcd(k, d) = 1 \) we have a single common divisor \( e = 1 \), so \( d' = d \) and \( k' = k \) and we pass to cardinalities in the decomposition into complete \( \Gamma_d \)-orbits of the set \( M_1(d, k) = M(d, k) \), obtaining the conclusion.

In general, there exists the relation:

\[
\binom{d}{k} = \sum_{e|\gcd(d, k)} \frac{d}{e} \#(T_e(d, k))
\]

as one can see using the definition (\#) and passing to cardinalities in (12).

From this relation, putting \( \delta = \gcd(d, k) \) and \( \overrightarrow{d} = \frac{d}{\delta} \), \( \overrightarrow{k} = \frac{k}{\delta} \) using the Möbius inversion on the lattice \( \text{Div}(\delta) \) we obtain:
\[
\#T_d(d, k) = \sum_{e \mid d} \mu\left(\frac{\delta}{e}\right) \left(\frac{ed}{ek}\right),
\]

where \(\mu\) stands for the usual arithmetic Möbius function.

In the extreme case \(k = d\) we have \(\#T_d(d, d) = 1\).

For a fixed divisor \(e \mid \gcd(d, k)\) let us remark that \(M_e(d, k)\) actually is a \(\Gamma'_d\)-module, when we take as model for \(\Gamma'_d\) the quotient \(\Gamma_d/\Gamma_e = (\mathbb{Z}/d\mathbb{Z})/(d'(\mathbb{Z}/d\mathbb{Z}))\), since the stabilizer of each orbit (of length \(d'\)) in the decomposition in \(\Gamma_d\)-orbits of \(M_e(d, k)\) is the cyclic group \(\Gamma_e \simeq d'\mathbb{Z}/d\mathbb{Z}\). Writing \(k = ek'\), we see that \(M_1(d', k')\) is also a \(\Gamma'_d\)-module, this time with \(\mathbb{Z}/d'\mathbb{Z}\) as model for \(\Gamma'_d\).

Let us consider the function:

\[\psi : M_e(d, k) \rightarrow M_1(d', k')\]
given by : \(S \rightarrow S^* (\mod d')\) (in the setting of Proposition 5).

**Proposition 7.**

The function \(\psi\) is an isomorphism of \(\Gamma'_d\)-modules.

**Proof.**

For any \(S \in M_e(d, k)\) and \(j \in \{0, 1, ..., d'-1\}\) we have \((S + j[1], k)^* = (S^* + j[1], k')(\mod d')\) so, by Proposition 5, the \(\Gamma'_d\)-orbit of \(S\) is taken bijectively and with compatibility with the actions of the models of \(\Gamma'_d\) into the \(\Gamma'_d\)-orbit of \(S^* (\mod d')\). 

As an immediate consequence we have the formula :

\[\#M_e(d, k) = \#M_1\left(\frac{d'}{e}, \frac{k'}{e}\right),\]

( here \(\frac{d'}{e}\) and \(\frac{k'}{e}\) are not necessarily coprime, unless \(e = \gcd(d, k)\)).

**IV. THE VIETE COEFFICIENTS OF THE PERIOD POLYNOMIALS**

Using Proposition 4 we translate back into \(T_A B(d, k)\) the results obtained above for \(M(d, k)\). With the notations established in Section II let us consider the \(C_0\) - module structure on the tableau \(T(S)\) given by (5), i.e. :

\[T(S) = \sqcup_{1}^{\left[S\right]} [0] \sqcup (\sqcup_{1}^{\mu_0} C_0) \sqcup (\sqcup_{1}^{\mu_1} C_1) \sqcup ... \sqcup (\sqcup_{1}^{\mu_{d-1}} C_{d-1})\]

where \(\sqcup_{1}^{\left[S\right]} [0] = \mathbb{Z}(S)\) is the multiset of the entries equal to 0 in \(T_0(S)\) and \(\mu_i\) is the number of entries belonging to \(C_i\) in \(T_0(S)\), \(i = 0, 1, ..., d - 1\).

We look now at the evolution of the \(C_0\)-module structure within the \(\Gamma'_d\)-orbit of \(S\). The first fact is described in the following:

**Proposition 8.**
For any \( \nu(\text{mod } d) \):

(i) \( Z(S + \nu [1]_k) = Z(S) \)

(ii) \( T(S + \nu [1]_k) = Z(S) \cup (\cup \nu^n C_\nu) \cup (\cup \nu C_{\nu+1}) \cup \ldots \cup (\cup \nu^{d-1} C_{\nu+d-1}) \) (indices modulo \( d \))

**Proof.** From (9) and (10), the translation with \( \nu [1]_k \) comes to the multiplication with \( g^\nu \) of the entries of \( T_0(S) \), which is bijective on the entries equal to 0 (see also (iii) of I.1).

(ii) results from (i) and \( g^\nu C_i = C_{\nu+i} \) (indices modulo \( d \)), \( i = 0, 1, \ldots, d - 1. \)

For an entire \( \Gamma_d \)-orbit we can now compute the corresponding value of the sum of the products of gaussian periods (with the notations established in I):

**Proposition 9.**

Let \( e \) be a divisor of \( \gcd(d,k) \) (\( d = ed \) and \( k = ek \)), let \( S \in M_e(d,k) \) and \( T(S) \) the tableau associated with \( S \). Let \( \Gamma_dT(S) \) be its orbit under the action of \( \Gamma_d \). Then:

\[
\sum_{S' \in \Gamma_dS} \eta_{S'} = \frac{1}{e} [pz(S) - m^{k-1}] \tag{13}
\]

**Proof.**

Because of Proposition 4: \( \Gamma_dT(S) = \{ T(S), T(S + [1]_k), \ldots, T(S + (d' - 1)[1]_k) \} \), i.e. the \( \Gamma_d \)-orbit of \( T(S) \) is indexed by the \( \Gamma_d \)-orbit of \( S \). From (i) Proposition 8 and from (7), (8) Proposition 3 we have:

\[
z(S) + e.(\mu_0(S) + \mu_1(S) + \ldots + \mu_{d'-1}(S)) = m^{k-1} \tag{8'}
\]

and successively:

\[
\eta_S = mz(S) + \mu_0(S)(\eta_0 + \eta_d + \ldots + \eta_{(e-1)d'}) + \ldots + \mu_{d'-1}(S)(\eta_{d'-1} + \eta_{2d'-1} + \ldots + \eta_{(e-1)d'-1})
\]

\[
\eta_{S+[1]_k} = mz(S) + \mu_0(S)(\eta_1 + \eta_{d'+1} + \ldots + \eta_{(e-1)d'+1}) + \ldots + \mu_{d'-1}(S)(\eta_{d'} + \eta_{2d'} + \ldots + \eta_{(e-1)d'})
\]

\[
\ldots
\]

\[
\eta_{S+(d'-1)[1]_k} = mz(S) + \mu_0(S)(\eta_{d'-1} + \eta_{2d'-1} + \ldots + \eta_{(e-1)d} + \eta_{d' -1}) + \ldots + \mu_{d'-1}(S)(\eta_{2d'-2} + \ldots + \eta_{(e-1)d'+d'-2})
\]

(remark that \( (e - 1)d' + d' - n = d - n \)).

Adding these equalities, we obtain:

\[
\sum_{S' \in \Gamma_dS} \eta_{S'} = d' mz(S) + \mu_0(S)(\sum_{j=0}^{d-1} \eta_j) + \ldots + \mu_{d'-1}(S)(\sum_{j=0}^{d-1} \eta_j) = (\text{because } \sum_{j=0}^{d-1} \eta_j = -1)
\]
\[ = d' m z(S) - (\mu_0(S) + \ldots + \mu_{d'-1}(S)) = (\text{using } (8')) = d' m z(S) - \frac{1}{e}(m^{k-1} - z(S)) = \frac{1}{e}[(ed m + 1)z(S) - m^{k-1}] \text{ and the result follows because } ed m + 1 = dm + 1 = p. \]

With the notations and definitions above we are now in position to formulate the

**Theorem 1.**

Let \( p \) be an odd prime number and \( p - 1 = dm \), \( d \geq 2 \). For any \( k \in \{1, 2, \ldots, d\} \) the \( k \)-th Viète coefficient of the period polynomial of degree \( d \) is:

\[
    a_k(p, d) = \sum_{e \mid \gcd(d, k)} \frac{1}{e} [p(\sum_{S \in T_e(d, k)} z(S)) - \#T_e(d, k).m^{k-1}] \tag{14}
\]

where \( T_e(d, k) \) is a transversal to the \( \Gamma_d \)-orbits in the decomposition of the \( \Gamma_d \)-module \( M_e(d, k) \).

**Proof.**

Considering the decomposition (12) of \( M(d, k) \), the decomposition into \( \Gamma_d \)-orbits of every component \( M_e(d, k) \) and the definition \((\#)\) of \( T_e(d, k) \) the formula (14) results directly from (13), Proposition 9.

**Remarks.**

(a) Because of (ii), Proposition 8 the expression given in the Theorem 1 for the general Viète coefficient does not depend upon the particular transversals chosen. Also, according to III, remark (b), Proposition 5, we can always fix a representative of any \( \Gamma_d \)-orbit to be a \( k \)-set beginning with 0. This we will do in the sequel.

(b) The general form for the Viète coefficients given in (14) makes the actual computation of these numbers depend upon:

(i) the determination of the transversals \( T_e(d, k) \) to the \( \Gamma_d \)-orbits partitioning \( M_e(d, k) \)
(ii) the computation of their cardinality
(iii) the computation of the gaussian symbols \( z(S) \), one for each representative \( S \).

An approach to (i) will be done below. Indication for combinatorial solutions to (ii) are already given in III above.

The problem (iii) is very difficult in general: its nature is neither algebraic nor combinatorial. It is connected to some deep unsolved problems about the properties of the particular prime number \( p \), for instance:

for any \( x \in \mathbb{F}_p^* = C_0 \cup C_1 \cup \ldots \cup C_{d-1} \) let \( i(x) \) be the unique index such that \( x \in C_i(x) \); determine \( i(x + 1) \).

In particular cases, for small values of \( p \) and of the parameters \( d, k \) it can be solved by brute force. Also, in certain cases, the computation of adequate gaussian symbols allows remarkable conclusions about the structure of some elliptic curves over finite fields, see [3].
(c) The extreme Viète coefficients are easily determined by the above formula (see also I.3), namely:

\[ a_1(p, d) = -1 \text{ and } a_d(p, d) = \frac{1}{d}(p, z(\{0, 1, ..., d - 1\}) - m^{d-1}). \]

(here \(z(\{0, 1, ..., d - 1\})\) is the gaussian symbol \(\{1, C_1, C_2, ..., C_{d-1}\}\).)

V. THE TRANSVERSAL \(\mathcal{T}_e(d, k)\)

We will now give a new interpretation of the sets in \(M(d, k)\) leading to a simplified combinatorial description of the transversals \(\mathcal{T}_e(d, k)\), \(e(\gcd(d, k))\).

Let \(S = \{0 \leq s_1 < s_2 < \ldots < s_k \leq d - 1\}\) be a \(k\) - subset of \([d] = \{0, 1, ..., d-1\}\) (considered as a set of representatives for the elements in \(\mathbb{Z}/d\mathbb{Z}\)). We associate with \(S\) the "difference vector" \(\delta\) :

\[ \delta(S) = [\delta_1(S), \delta_2(S), ..., \delta_{k-1}(S)], \quad \delta_j(S) = s_{j+1} - s_j, \quad j = 1, 2, ..., k - 1 \]

and complete it with "the positioning entry" (value)

\[ \pi(S) = d - s_k + s_1. \]

Obviously \(S = \{s_1, s_1 + \delta_1(S), ..., s_1 + \delta_1(S) + \delta_2(S) + ... + \delta_{k-1}(S)\}\) (compactely written as \(S = \{s_1|\delta(S)\}\)) , i.e. \(S\) is uniquely determined by its first element an its associated difference vector. For \(s_1 = 0\) , \(\delta(S)\) alone characterizes \(S\) and in this case the positioning entry is \(\pi(S) = d - s_k = d - (\delta_1(S) + \delta_2(S) + ... + \delta_{k-1}(S)) = d - |\delta(S)|\), where for a vector \(\delta\) by \(|\delta|\) we denote the sum of its components. Since \(s_k < d\) we also have in this case \(|\delta(S)| < d\) and \(\pi(S) \geq 1\).

The maximum possible value of a difference is obtained for \(S = \{0, 1, 2, ..., k - 2, d - 1\}\) therefore it is \(d - k + 1\), so:

\[ 1 \leq \delta_j(S) \leq d - k + 1, \quad j = 1, 2, ..., k - 1. \]

Thus , we see that the difference vectors of the \(k\) - subsets of \([d]\) are in fact those functions:

\[ \delta : \{1, 2, ..., k - 1\} \rightarrow \{1, 2, ..., d - k + 1\} \text{ satisfying } |\delta| = \sum_{1 \leq x \leq k - 1} \delta(x) \leq d - 1. \]

Now , let us consider the evolution of the difference vectors within the \(\Gamma_d\) - orbit of \(S\) . Remembering that inside any orbit there exists a representative whose first element is 0 , we begin with :

**Proposition 10.**

Let \(S = \{0|\delta(S)\}\) be a representative of its \(\Gamma_d\)-orbit , with positioning value \(\pi(S) = d - |\delta(S)|\). Then:

\[ S \equiv \{1|\delta(S)\} \equiv \{2|\delta(S)\} \equiv ... \equiv \{\pi(S) - 1|\delta(S)\} \pmod{\Gamma_d}. \]

**Proof.**
The translation with \([1]_k\) does not change the difference vector of a set in the \(\Gamma_d\)-orbit of \(S\) until the first time \(d \equiv 0 \pmod{d}\) is reached, i.e. exactly after 
\[
d - s_k = d - |\delta(S)| = \pi(S)\text{ steps. So the last step preserving the difference vector is precisely }\pi(S) - 1.\]

The result in Proposition 10 shows that , after translating \(S\) exactly \(\pi(S)\) times , in the \(\Gamma_d\)-orbit of \(S\) the following element is reached:

\[
S^{(1)} = \{0[\pi(S), \delta_1(S), \delta_2(S), ..., \delta_{k-2}(S)]\}
\]
whose difference vector is therefore :

\[
\delta_1(S^{(1)}) = \pi(S), \delta_2(S^{(1)}) = \delta_1(S), ..., \delta_{k-1}(S^{(1)}) = \delta_{k-2}(S)
\]
and whose positioning value is :

\[
\pi(S^{(1)}) = d - |\delta(S^{(1)})| = d - (d - \delta_{k-1}(S)) = \delta_{k-1}(S).
\]

We continue the same procedure with \(S^{(1)}\) instead of \(S\) and reach after exactly \(\pi(S^{(1)}) = \delta_{k-1}(S)\) translations the following element in the orbit of \(S\):

\[
S^{(2)} = \{0[\delta_{k-1}(S), \pi(S), \delta_1(S), ..., \delta_{k-3}(S)]\}
\]
whose difference vector is :

\[
\delta_1(S^{(2)}) = \delta_{k-1}(S), \delta_2(S^{(2)}) = \pi(S), \delta_3(S^{(2)}) = \delta_1(S), ..., \delta_{k-1}(S^{(2)}) = \delta_{k-3}(S)
\]
and whose positioning value is :

\[
\pi(S^{(2)}) = d - |\delta(S^{(2)})| = d - (d - \delta_{k-2}(S)) = \delta_{k-2}(S).
\]

Continuing the translations we reach after \(j \leq k - 1\) steps the following element in the \(\Gamma_d\)-orbit of \(S\):

\[
S^{(j)} = \{0[\delta_{k-j+1}(S), \delta_{k-j+2}(S), ..., \delta_{k-1}(S), \pi(S), \delta_1(S), \delta_2(S), ..., \delta_{k-j-1}(S)]\} \tag{15}
\]
whose difference vector is :

\[
\delta_n(S^{(j)}) = \delta_{k-j+n}(S) \text{ for } n = 1, 2, ..., j-1, \delta_j(S^{(j)}) = \pi(S) \text{ and } \delta_m(S^{(j)}) = \delta_{m-j}(S) \text{ for } m = j + 1, ..., k - 1 \tag{16}
\]
and whose positioning value is :

\[
\pi(S^{(j)}) = \delta_{k-j}(S).
\]

The described procedure has either \(k\) steps or is periodic with period a divisor of \(k\). In the first case, in the \(\Gamma_d\)-orbit of \(S\) we have the pivotal elements \(S = S^{(0)}, S^{(1)}, S^{(2)}, ..., S^{(k-1)}\) defined by (15) ( with \(S^{(k)} = S\) , implying the notation \(\delta_0(S) = \delta_k(S) = \pi(S)\) and , by the above description, the entire orbit is structured as follows:
Proposition 11.

Let us suppose that the k pivotal elements $S^{(0)}, S^{(1)},..., S^{(k-1)}$ are all distinct. Then the $\Gamma_d$- orbit of $S = S^{(0)}$ consists of k blocks:

$$\Gamma_d.S = \{B_0(S), B_1(S),..., B_{k-1}(S)\}$$

identified by:

$$B_j(S) = \{[0|\delta(S^{(j)})], [1|\delta(S^{(j)})],..., [\pi(S^{(j)}) - 1|\delta(S^{(j)})]\} \quad j = 0, 1, ..., k-1$$

and $S^{(j)}$ defined by (15). ■

This presentation of the orbit of $S$ shows the role of the pivotal elements: they constitute a transversal to the decomposing blocks $B_0, B_1,..., B_{k-1}$. Inside each block $B_j$ the elements have the same difference vector (16).

Since $\pi(S^{(0)}) + \pi(S^{(1)}) + ... + \pi(S^{(k-1)}) = \pi(S) + \delta_{k-1}(S) + \delta_{k-2}(S) + ... + \delta_1(S) = d$, the situation considered in Proposition 11 appears precisely when the $\Gamma_d$-orbits are complete, i.e. of maximal length $d$. In the previous notations this means: $S \in M_1(d,k)$. Below, we present the general case, of the orbits having length $d' = \frac{d}{e}$ for the admissible divisors $e$ of $d$.

The decomposition presented in Proposition 11 may be described combinatorially as follows.

Let $\delta = [\delta_1, \delta_2,..., \delta_{k-1}]$ be a “difference vector” i.e. a function $\delta: \{1, 2,..., k-1\} \rightarrow \{1, 2,..., d-k+1\}$ satisfying:

$$|\delta| = \delta_1 + \delta_2 + ... + \delta_{k-1} \leq d-1$$

and let $\pi(\delta) = d - |\delta|$ be its “positioning value” (this value determines and is determined by the embedding of $Im(\delta)$ as a subset of $[d]$).

Definition 1.

The set of vectors

$$SC(\delta) = \{\delta = \delta^{(0)}, \delta^{(1)},..., \delta^{(k-1)}\}$$

where $\delta^{(j)} = [\delta_{k-j+1}, \delta_{k-j+2},..., \delta_{k-1}, \pi(\delta), \delta_1, \delta_2,..., \delta_{k-j-1}], j = 0, 1, 2,..., k-1$ is called “the sliding class” of $\delta$ .

In the above definition, let us remark that $\pi(\delta^{(j)}) = \delta_{k-j}$ for every $j$.

In this setting, the Proposition 11 becomes:

Proposition 11'.

For any $S \in M_1(d,k)$ the transversal to the blocks $B_0, B_1,..., B_{k-1}$ partitioning the $\Gamma_d$-orbit of $S$ is given by the sliding class of the difference vector $\delta(S)$. ■

(In this enounce “is given” means the bijection $S^{(j)} \rightarrow \delta(S^{(j)})$ between the pivotal sets and their difference vectors, since each set $S^{(j)}$ has 0 as its first element).
The blocks $B_0, B_1, ..., B_{k-1}$ actually define a partition of the orbit of $S$, therefore Proposition 11 allows only $k$ operations in order to define the entire $d$-element orbit of $S$. Applying the same procedure to each $\Gamma_d$-orbit in the decomposition into orbits of $M_1(d, k)$ it results the following:

**Corollary.**

Any transversal $T_1(d, k)$ to the $\Gamma_d$-orbits partitioning $M_1(d, k)$ is in bijection with any transversal $T^1(d, k)$ to the sliding classes of the difference vectors $\delta(S)$ for $S \in T_1(d, k)$. ■

Thus, because the difference vectors are purely combinatorial objects (arrangements with repetitions), this Corollary translates the problem of determining the transversals $T_1(d, k)$ to the $\Gamma_d$-orbits partitioning $M_1(d, k)$ into the combinatorial problem of determining the transversals $T^1(d, k)$ to the sliding classes of the corresponding difference vectors. For small values of $d$ this combinatorial problem is easily solvable, as we shall see below. For $k = 2, 3$ and any $d$ the direct computation is also feasible, indicating both a possible algorithm and the complexity of the computations.

Now, let $e$ be a divisor of $gcd(d, k)$ with $d = ed'$, $k = ek'$ and let $S \in M_e(d, k)$. By III, Proposition 5 (see also Remark (a) after the proof of Proposition 5) we may write:

$$S = \{S^* < S^* + d' [1]_{k'} < ... < S^* + (e - 1)d' [1]_{k'}\}$$

where $S^* = \{0 = r_1 < r_2 < ... < r_{k'}\}$, $r_j \in \{0, 1, ..., d'\}$ for $j = 2, 3, ..., k'$ (as a representative of its $\Gamma_d$-orbit, $S$ has 0 as its first element).

This representation shows that the difference vector of $S$ is:

$$\delta(S) = [\delta(S^*), w, \delta(S^*), w, ..., \delta(S^*)] \quad (e \text{ copies of } \delta(S^*))$$

where $w = d' - r_{k'} + r_1 = \pi(S^*)$. To determine the positioning vector of $\delta(S)$ we first remark that: $|\delta(S)| = e.\delta(S^*)+(e-1)w = e(\delta(S^*)+w)-w = ed' - w = d-w$

therefore $\pi(S) = d - |\delta(S)| = d - (d - w) = w = \pi(S^*)$.

By using the same procedure as in the case $e = 1$ treated above, by remarking that $S(j)^* = S^{*(j)}$, the following result is directly obtained:

**Proposition 12.**

With the above notations the $\Gamma_d$-orbit of $S$ has length $d'$ and decomposes into $k'$ blocks $B_0, B_1, ..., B_{k'}$ identified by:

$$B_j = \{[0|\delta(S^{(j)})], [1|\delta(S^{(j)})], ..., \{\pi(S^{(j)}* - 1|\delta(S^{(j)})]\} \quad j = 0, 1, ..., k' - 1,$$

with $S^{(j)}$ defined by (15) with $k$ replaced by $k'$ and $d$ replaced by $d'$. ■

As above, the immediate consequence is:

**Proposition 12'.**

For any $S \in M_e(d, k)$ the transversal to the blocks $B_0, B_1, ..., B_{k'-1}$ partitioning the $\Gamma_d$-orbit of $S$ is given by the sliding class of the difference vector $\delta(S^*)$. ■
Corollary.
Any transversal $T_e(d, k)$ to the $\Gamma_d$-orbits partitioning $M_e(d, k)$ is in bijection to any transversal $T^e(d, k)$ to the sliding classes of the difference vectors $\delta(S^e)$ for $S \in T_e(d, k)$. ■

VI. THE PERIOD POLYNOMIALS FOR $d = 2, 3, 4$

VI.1 $d = 2$ (QUADRATIC RESTS)
In this case $F_p^* = C_0 \sqcup C_1$, $p - 1 = 2m$, $\#C_0 = \#C_1 = m$, the subgroup $C_0$ consists of the quadrats and the residual class $C_1$ consists of the non-quadrats in $F_p^*$. Using the established notations $a_k(p, d)$ for the Viète coefficients of the gaussian period polynomials, we have $a_1(p, d) = -1$ in all cases, therefore:

$$a_1(p, 2) = -1$$

For $k = 2$ we may directly use remark (c) after the Theorem1. In order to illustrate the theory so far developed we proceed differently. Namely, using the notations established in Section V, we have:

$$e = \gcd((d, k)) = \gcd((2, 2)) = 2$$

and

$$k - 1 = 1, d - 1 = 1$$

so there exists a single difference vector $\delta$ of length 1 and of modulus (i.e. the sum of all components) $|\delta| \leq 1$ namely: $\delta = [1]$ with positioning value $\pi(\delta) = d - |\delta| = 2 - 1 = 1$, giving a single $\Gamma_2$-orbit of the set $S = \{0|1\} = \{0, 1\}$. The unique Gauss symbol is:

$$z(\{0, 1\}) = \{1C_1\} = \begin{cases} 0, & -1 \notin C_1 \\ 1, & -1 \in C_1 \end{cases} = \begin{cases} 0, & p \equiv 1 \pmod{4} \\ 1, & p \equiv 3 \pmod{4} \end{cases}.$$  

Theorem 1 gives the value of the coefficient $a_2(p, 2)$:

$$a_2(p, 2) = \frac{1}{2} [pz(01) - m] = \begin{cases} \frac{-m}{2}, & p \equiv 1 \pmod{4} \\ \frac{p - m}{2}, & p \equiv 3 \pmod{4} \end{cases}.$$  

Using the standard notation: $p^* = (-1)^{\frac{p+1}{2}}p$ we obtain in all cases:

$$a_2(p, 2) = \frac{1 - p^*}{4}.$$  

Therefore, the (well known) equation of the gaussian 2-periods is:

$$X^2 + X + \frac{1 - p^*}{4} = 0$$
Remarks.

(a) Knowing the equation of the gaussian d-periods does not give any information about the actual value of the periods \( \eta_0, \eta_1, \ldots, \eta_{d-1} \) : the periods are given only modulo a permutation on \( d \) symbols. Although \( \eta_j \) is precisely defined as \( \sum_{x \in C_j} \zeta^x \), solving the period equation does not tell which root actually is \( \eta_j \).

For \( d = 2 \) this is connected to the famous problem of the determination of the sign of the gaussian sum \( \eta_0 - \eta_1 \).

(b) The equation for the gaussian 2-periods depends only on \( p \). No parameter enters its coefficients.

VI.2 \( d = 3 \) (CUBIC RESTS)

In this case \( \mathbb{F}_p^* = C_0 \sqcup C_1 \sqcup C_2 \) where \( C_0 \) consists of the cubes mod \( p \) with residual classes \( C_1, C_2 \) associated to the residues 1, 2 (mod 3) and \( p - 1 = 3m \) (so \( p \) should be \( \equiv 1 \) (mod 3)). As in the general case:

\[
a_1(p,3) = -1.
\]

For \( k = 2 \) we have \( k - 1 = 1, d - 1 = 2, d - k + 1 = 2 \) and \( \gcd(d,k) = \gcd(3,2) = 1 \) so difference vectors of length \( k - 1 = 1 \) and modulus (i.e. the sum of all entries) \( \leq d - 1 = 2 \) there are only two:

\[
\delta = [1] \quad \text{and} \quad \delta' = [2]
\]

The sliding class of \( \delta = [1] \) (whose positioning value is \( \pi(\delta) = 3 - 1 = 2 \)) is therefore:

\[
SC([1]) = \{[1],[2]\}
\]

whose representative \([1]\) produces the single representative of the \( \Gamma_3 \)-orbit on \( M_1(3,2) : S = \{0|1\} = \{0,1\} \).

Theorem 1 implies the following value for the second Viète coefficient:

\[
a_2(p,3) = p z(\{0,1\}) - m
\]

The gaussian symbol is \( z(\{0,1\}) = \{1C_1\} = 0 \) because, by Proposition 1 \( d = 3 \) is odd so \(-1 \in C_0 \), i.e. \(-1 \notin C_1 \).

The second coefficient is, finally:

\[
a_2(p,3) = -m = \frac{1 - p}{3}.
\]

For \( k = 3 \) we may apply Remark (c) to the Theorem 1 to obtain directly:

\[
a_3(p,3) = \frac{1}{3} p z(\{0,1,2\}) - m^2
\]

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the gaussian symbol being: \(z(\{0, 1, 2\}) = \{1C_1 C_2\} = \#([1 + C_1] \cap C_2)\) since again by Proposition 1 \(-C_2 = C_2\) in this case. Therefore the equation of the gaussian 3-periods is the following:

\[X^3 + X^2 - \frac{p-1}{3} X - \frac{1}{3} \alpha - \left(\frac{p-1}{3}\right)^2 = 0\]

This polynomial depends upon the unique parameter \(\alpha = z(\{0, 1, 2\})\), which is the gaussian symbol computed as \(\#([1 + C_1] \cap C_2)\).

G. Myerson in [2] gives the following expression for the polynomial having the gaussian 3-periods as roots:

\[X^3 + X^2 - \frac{p-1}{3} X - \frac{1}{27} [p.(c + 3) - 1]\]

where \(p \equiv 1 (mod\ 3) \implies 4p = c^2 + 27b^2\) for integers \(c, b\) such that \(c\) is uniquely determined by the condition \(c \equiv 1 \pmod{3}\) and \(b\) uniquely determined modulo its sign.

By comparing the two expressions for the 3-periods polynomials, it follows that the gaussian symbol \(\alpha\) and the parameter \(c\) are connected by the relation:

\[9\alpha = p + 1 + c.\]

Remarkably, this value is precisely the number of the \(\mathbb{F}_p\)-rational points on the projective plane curve \(X^3 + Y^3 + Z^3 = 0\), cf [3].

VI.3 \(d = 4\) (BIQUADRATIC RESTS).

In this case \(\mathbb{F}_p^* = C_0 \sqcup C_1 \sqcup C_2 \sqcup C_3\) with \(C_0\) consisting of the biquadrats in \(\mathbb{F}_p^*\) and the residual classes associated to the residues 1, 2, 3 modulo 4. We have \(p - 1 = 4m\) implying \(p \equiv 1 (mod\ 4)\). As in the general case:

\[a_1(p, 4) = -1.\]

For \(k = 2\) the difference vectors should have the length \(k - 1 = 1\) and the modulus \(\leq d - 1 = 4 - 1 = 3\) therefore the only possible such vectors are:

\[[1], [2], [3].\]

The positioning value of \([1]\) is 4-1=3 and the one of \([3]\) is 4-3=1, while the positioning value of \([2]\) is 4-2=2 therefore we have the following sliding classes:

\[SC([1]) = \{[1], [3]\}(represented\ by\ [1]) \quad SC([2]) = \{[2]\}(represented\ by\ [2]).\]

Now, \(gcd(4, 2) = 2\) so we have the decomposition:

\[M(4, 2) = M_1(4, 2) \sqcup M_2(4, 2)\]

Applying the Proposition 12 and its Corollary a transversal to the unique \(\Gamma_4\)-orbit in \(M_1(4, 2)\) is \(T_1(4, 2) = \{\{0, 1\}\}\) and a transversal to the unique orbit in \(M_2(4, 2)\) is \(T_2(4, 2) = \{\{0, 2\}\}\).

Theorem 1 gives the following value for the second Viète coefficient:
Proposition 1: The third Viète coefficient is $\gamma_3$. 

In order to determine the gaussian symbols we may apply the Proposition 1 to the case $d = 4$ obtaining two cases:

(i) either $-1 \in C_0 \Rightarrow -1 \notin C_1$ and $-1 \notin C_2 \Rightarrow z(01) = z(02) = 0 \Rightarrow a_2(p, 4) = \frac{-3m}{2} = \frac{3(1-p)}{2}$. But $-1 \in C_0$ means $-1 = \frac{p}{2} = g^{4n}$ for some $n \equiv 1 \pmod{8}$ therefore this case appears for $p \equiv 1 \pmod{8}$. Obviously $m$ should be even in this case.

(ii) or $-1 \notin C_0 \Rightarrow -1 \notin C_1 \Rightarrow z(01) = 0$ and $z(02) = 1 \Rightarrow a_2(p, 4) = p + \frac{1}{2}(p - m) = m + \frac{3}{2}(4m + 1 - m) = \frac{m+1}{2} = \frac{k+3}{8}$.

But $-1 \in C_2$ means $-1 = g^{4n+2} = g^{4n+2}$ for some $n \equiv 0 \pmod{8}$ therefore this case appears for $p \equiv 5 \pmod{8}$ (and $m = \frac{k-1}{2}$ odd). It results: 

$$a_2(p, 4) = \begin{cases} \frac{3(1-p)}{2} & p \equiv 1 \pmod{8} \\ \frac{m+1}{2} & p \equiv 5 \pmod{8} \end{cases}.$$ 

For $k = 3$ the difference vectors have length $k - 1 = 2 - 1 = 2$, maximum value $d - k + 1 = 4 - 3 + 1 = 2$ and modulus $d - 1 = 3$. There are only three such vectors, namely:

$$[11], 
[12], 
[21]$$

whose respective positioning values are: 2, 1, 1. So we have the unique sliding class:

$$SC([11]) = \{[11], [12], [21]\}$$

with transversal $T^1(4, 3) = \{[11]\}$, giving the unique representative of the $\Gamma_4$-orbit $M(4, 3) = M_1(4, 3)$ namely:

$$S = \{0|[11]\} = \{0, 1, 2\} \text{ (because } gcd(4, 3) = 1 \text{ the decomposition of } M(4, 3) \text{ into orbits consists of a unique orbit of maximal length 4). By Theorem 1 the third Viète coefficient is:}$$

$$a_3(p, 4) = pz([0, 1, 2]) - m^2$$

the gaussian symbol being $z([0, 1, 2]) = \{1C_1C_2\} = \#(1 + C_1) \cap (-C_2)$. By Proposition 1: $-C_2 = C_2$ for $p \equiv 1 \pmod{8}$ and $-C_2 = C_0$ for $p \equiv 5 \pmod{8}$.

For $k = 4$ we may directly apply Remark (c) after Theorem 1 and obtain:

$$a_4(p, 4) = \frac{1}{4}\{pz([0, 1, 2, 3]) - m^3\}$$

where the gaussian symbol is: $z([0, 1, 2, 3]) = \#(1 + C_1) \cap (-C_2)$. Here for the first time it appears the general phenomenon, namely that the sum $-C_2 - C_3$ actually is a multiset (i.e. a set with multiplicities attached to its elements) and the cardinality of the intersection also counts the multiplicities of the common elements. By Proposition 1 and its Corollary we see that:
for \( p \equiv 1 \pmod{8} \) \( \Rightarrow -1 \in C_0 \Rightarrow -C_2 = C_2 \) and \( -C_3 = C_3 \)

\( (m \text{ is even in this case}) \), respectively:

for \( p \equiv 5 \pmod{8} \) \( \Rightarrow -1 \in C_2 \Rightarrow -C_2 = C_0 \) and \( -C_3 = C_3 \)

\( (m \text{ is odd in this case}) \).

We may now write the equation of the gaussian 4-periods as follows:

A. For \( p \equiv 1 \pmod{8} \) (\( m \text{ even} \)):

\[
X^4 + X^3 + \frac{3m}{2}X^2 - [p.z(012) - m^2].X + \frac{1}{4}[p.z(012) - m^3] = 0
\]

(with \( z(012) \) instead of \( z(\{0,1,2\}) \) etc.) where \( m = \frac{p-1}{4} \) and the gaussian symbols are computed by:

\[ \alpha = z(012) = \#\{1 + C_1 \cap C_2 \} \] (intersection as sets) and \( \beta = z(0123) = \#\{1 + C_1 \cap [C_2 + C_3] \} \) (intersection as multisets).

B. For \( p \equiv 5 \pmod{8} \) (\( m \text{ odd} \)):

\[
X^4 + X^3 + \frac{m+1}{2}X^2 - [p.z(012) - m^2].X + \frac{1}{4}[p.z(012) - m^3] = 0
\]

where \( m = \frac{p-1}{4} \) and the gaussian symbols are computed by:

\[ \alpha = z(012) = \#\{1 + C_1 \cap C_0 \} \] (intersection as sets) and \( \beta = z(0123) = \#\{1 + C_1 \cap [C_0 + C_1] \} \) (intersection as multisets).

In both cases the equation for the gaussian 4-periods depends upon two parameters \( \alpha \) and \( \beta \) which are defined as the gaussian symbols \( z(012) \) and \( z(0123) \) respectively.

G. Myerson in \[ 2 \] gives the following expressions for the polynomials having the gaussian 4-periods as roots:

we have \( p \equiv 1 \pmod{4} \) \( \Rightarrow p = s^2 + 4t^2 \) for integers \( s, t \) such that \( s \) is uniquely determined by the condition \( s \equiv 1 \pmod{4} \) and \( t \) uniquely determined modulo sign and then:

A’. For \( p \equiv 1 \pmod{8} \) (\( m \text{ even} \)) the equation is:

\[
X^4 + X^3 - \frac{3(p-1)}{8}X^2 + \frac{1}{16}[(2s-3)p+1]X + \frac{1}{256}[p^2 - (4s^2 - 8s + 6)p + 1] = 0
\]

B’. For \( p \equiv 5 \pmod{8} \) (\( m \text{ odd} \)) the equation is:

\[
X^4 + X^3 + \frac{1}{8}(p+3)X^2 + \frac{1}{16}[(2s+1)p+1]X + \frac{1}{256}[9p^2 - (4s^2 - 8s - 2)p + 1] = 0.
\]
Remarkably, a comparison between the coefficients of the biquadratic polynomials equations $A'$, $B'$ with the coefficients of the biquadratic equations $A$, $B$ directly gives the expressions of the Gaussian symbols (computable only for each $p$ separately and for reasonable small values of $p$) in terms of the representation of $p$ by the quadratic form $U^2 + 4V^2$:

\[
\begin{align*}
(i) \alpha &= z(012) = \frac{1}{16}(p + 1 - 2s) \quad \text{for} \quad p \equiv 1 \pmod{8} \\
(ii) \alpha &= z(012) = \frac{1}{16}(p - 3 - 2s) \quad \text{for} \quad p \equiv 5 \pmod{8}
\end{align*}
\]

respectively:

\[
\begin{align*}
(i) \beta &= z(0123) = \frac{1}{64}[p^2 - 2p - (4s^2 - 8s + 3)] \quad \text{for} \quad p \equiv 1 \pmod{8} \\
(ii) \beta &= z(0123) = \frac{1}{64}[p^2 + 6p - (4s^2 - 8s - 5)] \quad \text{for} \quad p \equiv 5 \pmod{8}
\end{align*}
\]

Eliminating $s$ between (i) and (i)', respectively between (ii) and (ii)' we find:

\[
\begin{align*}
2\beta &= \alpha(p - 2 - 8\alpha) \quad \text{for} \quad p \equiv 1 \pmod{8} \\
4\beta &= (p - 1) - 2\alpha(p - 5) - 4\alpha^2 \quad \text{for} \quad p \equiv 5 \pmod{8}
\end{align*}
\]

These formulae show that the period equations $A$ and $B$ actually depend upon the single parameter $\alpha = z(012)$.

VII. THE COEFFICIENTS $a_2(p, d)$ and $a_3(p, d)$

We now determine the simplest non trivial Viète coefficients using the above developed combinatorics. The notations and definitions introduced up to now will be used throughout. In particular we have the notation $p - 1 = dm$.

VII.1 THE COEFFICIENT $a_2(p, d)$.

In this case $k = 2$ so we work with difference vectors $\delta$ of length $2 - 1 = 1$ only, with maximum value $d - k + 1 = d - 1$ and modulus $|\delta| \leq d - 1$. Therefore the difference vectors are:

\[ \delta : [1], [2], \ldots, [d - 1] \]

having the positioning values respectively:

\[ \pi(\delta) : \quad d - 1, \quad d - 2, \quad \ldots, \quad 1. \]

The sliding classes are:

\[ SC([j]) = \{[j], [d - j]\} \quad j = 1, 2, \ldots, \left\lfloor \frac{d}{2} \right\rfloor. \]

Thus, we must consider separately the following two cases:

\( (1) \ d \equiv 1 \pmod{2} \)

In this case we have \( \frac{d - 1}{2} \) sliding classes represented by the difference vectors \([1], [2], \ldots, \left\lfloor \frac{d - 1}{2} \right\rfloor \). The Theorem 1 produces the value:

\[ a_2(p, d) = p\left(\sum_{j=1}^{\frac{d-1}{2}} z(0j)\right) - \frac{d - 1}{2} m \]
Because \(d\) is odd Proposition 1 shows that \(-1 \in C_0 \implies z(01) = z(02) = \ldots = z(\frac{d-1}{2}) = 0\) such that:

\[
a_2(p, d) = -\frac{(d - 1)m}{2} = -\frac{(d - 1)(p - 1)}{2d}.
\]

(2) \(d \equiv 0(\text{mod } 2)\)

In this case, proceeding as above, we see that there are \(d^2 - 1\) sliding classes of cardinality 2 represented by \([1], [2], \ldots, [\frac{d}{2} - 1]\) and a single class of cardinality 1 represented by \([\frac{d}{2}]\) (with positioning value \(\frac{d}{2}\)). The Theorem 1 produces the value:

\[
a_2(p, d) = \left[p\left(\sum_{j=1}^{\frac{d}{2}-1} z(0j)) - \left(\frac{d}{2} - 1\right)m\right] + \frac{1}{2}[pz(\frac{d}{2}) - m]
\]

Here, by Proposition 1, there appear two possibilities:

(i) \(-1 \in C_0 \implies z(0j) = 0\) for \(j = 1, 2, \ldots, \frac{d}{2} - 1, \frac{d}{2}\) (this case appears for even \(m\)) so it follows:

\[
a_2(p, d) = -\frac{d - 1}{2}m - \frac{1}{2}m = -\frac{(d - 1)}{2}m.
\]

(ii) \(-1 \in C_{\frac{d}{2}} \implies z(0j) = 0\) for \(j = 1, 2, \ldots, \frac{d}{2} - 1\) and \(z(\frac{d}{2}) = 1\) (this case appears for odd \(m\)) so it follows:

\[
a_2(p, d) = \frac{d - 1}{2}m + \frac{1}{2}[p - m] = \left(\because p = dm + 1\right) = \frac{1}{2}(m + 1) = \frac{p - 1 + d}{2d}.
\]

We put the above discussion under the form of

**Proposition 13.**

The general second Viète coefficient of the period equation is:

\[
a_2(p, d) = \begin{cases} 
\frac{(d-1)(p-1)}{p+d-1}, & \text{for } d \equiv 1(\text{mod } 2) \text{ or } d \equiv 0(\text{mod } 2) \text{ and } m \text{ even} \\
\frac{p-1+d}{2d}, & \text{for } d \equiv 0(\text{mod } 2) \text{ and } m \text{ odd}
\end{cases}
\]

It is clear that the values listed in Proposition 13 are in accordance with \(a_2(p, 2), a_2(p, 3), a_2(p, 4)\) computed in Section VI.

**VII.2 THE COEFFICIENT \(a_3(p, d)\)**

In this case \(k = 3\) so we work with difference vectors of length \(k - 1 = 2\), maximum value \(d - k + 1 = d - 2\) and modulus \(\leq d - 1\). Therefore the difference vectors are:

\[
\delta = [ij] \text{ with } i, j \in \{1, 2, \ldots, d - 2\} \text{ and } i + j \leq d - 1.
\]

We display these vectors in the following triangle, named (Tr1) hereafter:
This triangle contains all the difference vectors we are considering, has \(j\) elements on the \(j\)th line (consisting of the vectors of modulus \(j + 1\)) for \(j = 1, 2, ..., d - 2\) and a total of \(\binom{d - 1}{2}\) entries.

A difference vector \(\delta = [ij]\) has positioning value \(\pi(\delta) = d - i - j\) and sliding class:

\[
SC(\delta) = \{[ij], [\pi(\delta)i], [\pi(\delta)]\}.
\]

We see that each sliding class has three elements, except for the case \(i = j = \frac{1}{3}d\) (which is possible for \(d \equiv 0(\text{mod } 3)\) only), when the sliding class reduces to the single element \(\{[d:3:d:3]\}\). The possible situations are distinguished by \(d(\text{mod } 3)\):

(i) \(d \equiv 1\) or \(d \equiv 2\) (mod 3) \(\Rightarrow\) there are \(\frac{1}{3}\left(\frac{d - 1}{2}\right)\) sliding classes, each having 3 elements

(ii) \(d \equiv 0(\text{mod } 3)\) \(\Rightarrow\) there are \(\frac{1}{3}\left(\frac{d - 1}{2}\right) - 1\) 3-element classes and one class having one element, namely \(\{[d:3:d:3]\}\).

We must compute a transversal to the sliding classes in order to apply Theorem 1. We proceed as follows.

Let us fix \(i = 1\) and consider the sliding classes of the difference vectors \(\delta = [1j]\) for \(j = 1, 2, ..., d - 3\):

\[
SC([1j]) = \{[1j], [(d - j - 1)1], [j(d - j - 1)]\}
\]

The locations in the triangle (Tr1) of the elements of all such sliding class cover the sides of the “exterior” triangle with extremal vertices \([11]\), \([(d-2)1]\), \([1(d-2)]\). Therefore we obtain the representatives for these classes:
We eliminate from (T_r1) the exterior sides and obtain a smaller triangle, named (T_r2) hereafter.

Let us fix \( i = 2 \) and consider the sliding classes of the difference vectors \( \delta = [2j] \) for \( j = 2, 3, ..., d - 5 \):

\[
SC([2j]) = \{[2j], [(d - j - 2)2], [j(d - j - 2)]\}
\]

The locations in (T_r2) of the elements of all such sliding classes cover the "exterior" triangle with extremal vertices \( [22], [(d - 4)2], [2(d - 4)] \). Therefore we obtain the representatives for these classes:

\[
(\text{REP}2) = [22] [23] [24] ... [2(d - 5)] (d - 6 \text{ elements})
\]

We eliminate from (T_r2) the exterior sides and obtain a new triangle, named (T_r3) hereafter.

Let us fix \( i = 3 \) and consider the sliding classes of the difference vectors \( \delta = [3j] \) for \( j = 3, 4, ..., d - 7 \):

\[
SC([3j]) = \{[3j], [(d - j - 3)3], [j(d - j - 3)]\}
\]

The locations in (T_r3) of the elements of all such sliding classes cover the "exterior" triangle with extremal vertices \( [33], [(d - 6)3], [3(d - 6)] \). Therefore we obtain the representatives for these classes:

\[
(\text{REP}3) = [33] [34] ... [3(d - 7)] (d - 9 \text{ elements})
\]

We continue the procedure by induction.

At step \( n \) we obtain the representatives for the corresponding sliding classes:

\[
(\text{REP}_n) = [nn], [n(n + 1)], ... [n(2n - 1)] (d - 3n \text{ elements})
\]

To specify the final step we must distinguish the \( (\text{mod} \ 3) \) residue of \( d \).

\[\text{(a) THE CASE } \quad d = 1(\text{mod} \ 3)\]

In this case we have \( d = 3n + 1 \) so the last step in the above procedure is \( n = \frac{d - 1}{3} \) with:

\[
(\text{REP} \ \frac{d - 1}{3}) = [\frac{d - 1}{3}, \frac{d - 1}{3}]
\]

The union of all these representatives (REPl) for \( l = 1, 2, 3, ..., \frac{d - 1}{3} \) produces the corresponding transversal to the \( \Gamma_3 \) orbits on \( M_3(p, d) \) and the following sum of the intervening gaussian symbols:

\[
A_d(1 \text{ mod } 3) = z(012) + z(013) + ... + z(01(d - 2)) + z(024) + z(025) + ... + z(02(d - 3)) + ... + z(0\frac{d - 1}{3} 2\frac{2d - 1}{3}).
\]

\[\text{(b) THE CASE } \quad d = 2(\text{mod} \ 3)\]

In this case we have \( d = 3n + 2 \) so the last step in the above procedure is \( n = \frac{d - 2}{3} \) and:

\[
(\text{REP} \ \frac{d - 2}{3}) = [\frac{d - 2}{3}, \frac{d - 2}{3}, \frac{d - 2}{3}]
\]

The union of all these representatives (REPl) for \( l = 1, 2, 3, ..., \frac{d - 2}{3} \) produces the corresponding transversal to the \( \Gamma_3 \) orbits on \( M_3(p, d) \) and the following sum of the intervening gaussian symbols:

\[\text{26}\]
\[ A_d(2 \mod 3) = z(012) + z(013) + \ldots + z(01(d-2)) + z(024) + z(025) + \ldots + z(02(d-3)) + \ldots + z(0 \frac{d-2}{3} \frac{2d-1}{3}) + \]

(c) THE CASE \( d \equiv 0 \mod 3 \)

In this case we have (cf.(ii) above) \( d = 3n \) so the above procedure gives as last step for the 3-elements classes:

\[ \text{(REP d)} = \{\frac{d-3}{3} \frac{d-3}{3} d \}, \{\frac{d-3}{3} \frac{d-3}{3} \frac{d-3}{3} \}, \text{and an unique representative for the 1-element class: } \{\frac{d}{3}\} \].

The union of all these representatives ((REPl) for \( l = 1, 2, \ldots, \frac{d-3}{3} \)) produces the corresponding transversal to the \( \Gamma_3 \) - orbits of maximal length \( d \) in \( M_1(d, 3) \) and the following sum of the intervening gaussian symbols:

\[ A_d(0 \mod 3) = z(012) + z(013) + \ldots + z(01(d-2)) + z(024) + z(025) + \ldots + z(0 \frac{d-2}{3} \frac{2d-1}{3}) + \ldots + z(0 \frac{d-3}{3} \frac{2d-3}{3}) + z(0 \frac{d-3}{3} \frac{2d}{3}). \]

For the unique orbit of length \( \frac{d}{3} \) the unique representative \( \{\frac{d}{3}\} \) produces the gaussian symbol:

\[ B_d(0 \mod 3) = z(0 \frac{2d}{3}). \]

Applying the Theorem 1 and using the notations introduced above at (a),(b),(c) we obtain:

**Proposition 14.**

The third general Viète coefficient of the period equation is:

\[ a_3(p,d) = \begin{cases} 
  p.A_d(1 \mod 3) - \frac{1}{3} \left( \frac{d-1}{2} \right) m^2 & , d \equiv 1 \mod 3 \\
  p.A_d(2 \mod 3) - \frac{1}{3} \left( \frac{d-1}{2} \right) m^2 & , d \equiv 2 \mod 3 \\
  p.A_d(0 \mod 3) - \frac{1}{3} \left( \frac{d-1}{2} \right) - 1 \right] m^2 + \frac{1}{3}[p.B_d(0 \mod 3) - m^2] & , d \equiv 0 \mod 3 
\end{cases} \]

where \( m = \frac{p-1}{3} \).

**Remarks.**

(i) The anterior values of \( a_3(p, 3) \) and \( a_3(p, 4) \) may trivially be recovered from the formulae in Proposition 14.

For \( d = 5 \) we have \( A_5(2 \mod 3) = z(012) + z(013) \) therefore \( a_3(p, 5) = p[z(012) + z(013)] - 2m^2 \).

Ulterior values are: \( A_6(0 \mod 3) = z(012) + z(013) + z(014) \) and \( B_6(0 \mod 3) = z(024) \) therefore:

\[ a_3(p, 6) = p[z(012) + z(013) + z(014)] - 3m^2 + \frac{1}{3}[p.z(024) - m^2]. \]
(ii) For $k = 3$ the gaussian symbols are all of the form:

$$z(0jl) = \{1C_jC_l\} = \#([1 + C_j] \cap (-C_l))$$

The Corollary to Proposition 1 show that $-C_l = C_l$ for: (odd $d$ and all $m$) or (even $d$ and even $m$) and $-C_l = C_{l+\frac{d}{2}}$ for (even $d$ and odd $m$). The actual values of the gaussian symbols strongly depends upon the properties of the prime number $p$. ■

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Institute of Mathematics of the Romanian Academy , Calea Grivitei 21 , Bucharest , ROMANIA