FPTAS for mixed-integer polynomial optimization with a fixed number of variables

J. A. De Loera\textsuperscript{1}  R. Hemmecke\textsuperscript{2}  M. Köppe\textsuperscript{3}  R. Weismantel\textsuperscript{4}

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Abstract

We show the existence of an FPTAS for the problem of maximizing a non-negative polynomial over mixed-integer sets in convex polytopes, when the number of variables is fixed.

1 Introduction

A well-known result by H.W. Lenstra Jr. states that linear mixed integer programming problems with fixed number of variables can be solved in polynomial time on the input size \([10]\). It is a natural question to ask what is the computational complexity, when the number of variables is fixed, of the non-linear mixed integer problem

\[
\begin{align*}
\text{max} & \quad f(x_1, \ldots, x_{d_1}, z_1, \ldots, z_{d_2}) \\
\text{s.t.} & \quad Ax + Bz \leq b \\
& \quad x_i \in \mathbb{R} \quad \text{for } i = 1, \ldots, d_1, \\
& \quad z_i \in \mathbb{Z} \quad \text{for } i = 1, \ldots, d_2,
\end{align*}
\]

where \(f\) is a polynomial function of maximum total degree \(D\) with rational coefficients, and \(A \in \mathbb{Z}^{m \times d_1}, B \in \mathbb{Z}^{m \times d_2}, b \in \mathbb{Z}^m\) (here we assume that \(Ax + Bz \leq b\) describes a convex polytope, which we denote by \(P\)).

It was well-known that continuous polynomial optimization over polytopes, without fixed dimension, is NP-hard and that an FPTAS is not possible. Indeed the maxcut problem can be modeled as minimizing a quadratic form over the cube \([-1,1]^d\) \([9]\). More strongly, it turns out that, even for dimension two and total degree of \(f\) four, problem \([11]\) is an NP-hard problem too \([9]\). Thus the best we can hope for, even for fixed dimension, is an approximation result. This paper presents the best possible such result:

\textsuperscript{1}Address: University of California, Dept. of Mathematics, Davis CA 95616, USA; E-Mail Address: deloera@math.ucdavis.edu.
\textsuperscript{2}Address: Otto-von-Guericke-Universität Magdeburg, FMA/IMO, Universitätsplatz 2, 39106 Magdeburg, Germany; E-Mail Address: hemmecke@imo.math.uni-magdeburg.de.
\textsuperscript{3}Address: Otto-von-Guericke-Universität Magdeburg, FMA/IMO, Universitätsplatz 2, 39106 Magdeburg, Germany; E-Mail Address: mkoeppe@imo.math.uni-magdeburg.de.
\textsuperscript{4}Address: Otto-von-Guericke-Universität Magdeburg, FMA/IMO, Universitätsplatz 2, 39106 Magdeburg, Germany; E-Mail Address: weismant@imo.math.uni-magdeburg.de.
Theorem 1.1. Let the dimension \( d = d_1 + d_2 \) be fixed.

(a) There exists a fully polynomial time approximation scheme (FPTAS) for the optimization problem (1) for all polynomial functions \( f \in \mathbb{Q}[x_1, \ldots, x_{d_1}, z_1, \ldots, z_{d_2}] \) that are non-negative on the feasible region (1b–1d). (We assume the encoding length of \( f \) is at least as large as its maximum total degree.)

(b) Moreover, the restriction to non-negative polynomials is necessary, as there does not even exist a polynomial time approximation scheme (PTAS) for the maximization of arbitrary polynomials over mixed-integer sets in polytopes, even for fixed dimension \( d \geq 2 \).

The proof of Theorem 1.1 is presented in section 4. As we will see, Theorem 1.1 is a non-trivial consequence of the existence of FPTAS for the problem of maximizing a non-negative polynomial with integer coefficients over the lattice points of a convex rational polytope. That such FPTAS indeed exists was recently settled in our paper [6]. The knowledge of paper [6] is not necessary to understand this paper but, for convenience of the reader, we include a short summary in an appendix. Our arguments, however, are independent of which FPTAS is used in the integral case. Our results come to complement other approximation schemes investigated for continuous variables and fixed degree (see [4] and references therein).

One interesting property of our FPTAS is that it does not depend on dynamic programming, unlike most known FPTAS (see comments and references in the introduction of [11]). Instead our main approach is to use grid refinement in order to approximate the mixed-integer optimal value via auxiliary pure integer problems. One of the difficulties on constructing approximations is the fact that not every sequence of grids whose widths converge to zero leads to a convergent sequence of optimal solutions of grid optimization problems. This difficulty is addressed in section 2. In section 3 we develop techniques for bounding differences of polynomial function values. Finally, section 4 contains the proof of Theorem 1.1.

2 Grid approximation results

An important step in the development of an FPTAS for the mixed-integer optimization problem is the reduction of the mixed-integer problem (1) to an auxiliary optimization problem over a lattice \( \frac{1}{m} \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} \). To this end, we consider the grid problem

\[
\begin{align*}
\max & \quad f(x_1, \ldots, x_{d_1}, z_1, \ldots, z_{d_2}) \\
\text{s.t.} & \quad Ax + Bz \leq b \\
& \quad x_i \in \frac{1}{m} \mathbb{Z} \quad \text{for } i = 1, \ldots, d_1, \\
& \quad z_i \in \mathbb{Z} \quad \text{for } i = 1, \ldots, d_2.
\end{align*}
\]

(2)

We can solve this problem approximately using the integer FPTAS (Theorem A.3):

Corollary 2.1. For fixed dimension \( d = d_1 + d_2 \) there exists an algorithm with running time polynomial in \( \log m \), in the encoding length of \( f \) and of \( P \), in the maximum total degree \( D \) of \( f \), and in \( \frac{1}{\epsilon} \) for computing a feasible solution \( (x^m_m, z^m_m) \in P \cap \left( \frac{1}{m} \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} \right) \) to the grid problem (2), where \( f \) is non-negative on the feasible region, with

\[
f(x^m_m, z^m_m) \geq (1 - \epsilon)f(x^m, z^m),
\]

(3)
where \((x^m, z^m) \in P \cap (\frac{1}{m}Z^{d_1} \times Z^{d_2})\) is an optimal solution to (2).

**Proof.** We apply Theorem A.3 to the pure integer optimization problem:

\[
\begin{align*}
\max & \quad \tilde{f}(\tilde{x}, z) \\
\text{s.t.} & \quad A\tilde{x} + mBz \leq mb \\
& \quad \tilde{x}_i \in \mathbb{Z} \quad \text{for } i = 1, \ldots, d_1, \\
& \quad z_i \in \mathbb{Z} \quad \text{for } i = 1, \ldots, d_2,
\end{align*}
\]

where \(\tilde{f}(\tilde{x}, z) := m^D f(\frac{1}{m}\tilde{x}, z)\) is a polynomial function with integer coefficients. Clearly the binary encoding length of the coefficients of \(\tilde{f}\) increases by at most \([D \log m]\), compared to the coefficients of \(f\). Likewise, the encoding length of the coefficients of \(mB\) and \(mb\) increases by at most \([\log m]\). By Theorem 1.1 of [6], there exists an algorithm with running time polynomial in the encoding length of \(\tilde{f}\) and \(A\tilde{x} + mBz \leq mb\), the maximum total degree \(D\), and \(\frac{1}{r}\) for computing a feasible solution \((x^m, z^m) \in P \cap (\frac{1}{m}Z^{d_1} \times Z^{d_2})\) such that \(\tilde{f}(x^m, z^m) \geq (1 - \epsilon)\tilde{f}(x^m, z^m)\), which implies (3).

One might be tempted to think that for large-enough choice of \(m\), we immediately obtain an approximation to the mixed-integer optimum with arbitrary precision. However, this is not true, as the following example demonstrates.

**Example 2.2.** Consider the mixed-integer optimization problem

\[
\begin{align*}
\max & \quad 2z - x \\
\text{s.t.} & \quad z \leq 2x \\
& \quad z \leq 2(1 - x) \\
& \quad x \in \mathbb{R}_+, \quad z \in \{0, 1\},
\end{align*}
\]

whose feasible region consists of the point \((\frac{1}{2}, 1)\) and the segment \(\{ (x, 0) : x \in [0, 1]\}\). The unique optimal solution to (5) is \(x = \frac{1}{2}, z = 1\). Now consider the sequence of grid approximations of (5) where \(x \in \frac{1}{m}Z_+\). For even \(m\), the unique optimal solution to the grid approximation is \(x = \frac{1}{2}, z = 1\). However, for odd \(m\), the unique optimal solution is \(x = 0, z = 0\). Thus the full sequence of the optimal solutions to the grid approximations does not converge since it has two limit points.

However we can prove that it is possible to construct, in polynomial time, a subsequence of finer and finer grids that contain a lattice point \((x^*, z^*)\) that is arbitrarily close to the mixed-integer optimum \((x^*, z^*)\). This is the central statement of this section and a basic building block of the approximation result.

**Theorem 2.3 (Grid Approximation).** Let \(d_1\) be fixed. Let \(P = \{ (x, z) \in \mathbb{R}_+^{d_1 + d_2} : Ax + Bz \leq b \}\), where \(A \in \mathbb{Z}_{m \times d_1}, B \in \mathbb{Z}^{m \times d_2}\). Let \(M \in \mathbb{R}\) be given such that \(P \subseteq \{ (x, z) \in \mathbb{R}_+^{d_1 + d_2} : |x_i| \leq M \text{ for } i = 1, \ldots, d_1 \}\). There exists a polynomial-time algorithm to compute a number \(\Delta\) such that for every \((x^*, z^*) \in P \cap (\mathbb{R}_+^{d_1} \times \mathbb{Z}^{d_2})\) and \(\delta > 0\) the following property holds:

Every lattice \(\frac{1}{m}Z^{d_1} \times \mathbb{Z}^{d_2}\) for \(m = k\Delta\) and \(k \geq \frac{1}{\rho}(d_1 + 1)M\) contains a lattice point \((x, z^*) \in P \cap (\frac{1}{m}Z^{d_1} \times \mathbb{Z}^{d_2})\) with \(\|x - x^*\| \leq \delta\).
Theorem 2.3 follows directly from the next two lemmas.

Lemma 2.4 (Integral Scaling Lemma). Let \( P = \{ (x, z) \in \mathbb{R}^{d_1 + d_2} : Ax + Bz \leq b \} \), where \( A \in \mathbb{Z}^{m \times d_1} \), \( B \in \mathbb{Z}^{n \times d_2} \). For fixed \( d_1 \), there exists a polynomial time algorithm to compute a number \( \Delta \in \mathbb{Z}_{>0} \) such that for every \( z \in \mathbb{Z}^{d_2} \) the polyhedron 
\[
\Delta P_z = \{ \Delta x : (x, z) \in P \}
\]
is integral. In particular, the number \( \Delta \) has an encoding length that is bounded by a polynomial in the encoding length of \( P \).

Proof. Because the dimension \( d_1 \) is fixed, there exist only polynomially many simplex bases of the system \( Ax \leq b - Bz \), and they can be enumerated in polynomial time. The determinant of each simplex basis can be computed in polynomial time. Then \( \Delta \) can be chosen as the least common multiple of all these determinants.

Lemma 2.5. Let \( Q \subset \mathbb{R}^d \) be an integral polytope, i.e., all vertices have integer coordinates. Let \( M \in \mathbb{R} \) be such that \( Q \subseteq \{ x \in \mathbb{R}^d : |x_i| \leq M \text{ for } i = 1, \ldots, d \} \). Let \( x^* \in Q \) and let \( \delta > 0 \). Then every lattice \( \frac{1}{k} \mathbb{Z}^d \) for \( k \geq \frac{1}{\delta}(d + 1)M \) contains a lattice point \( x \in \mathbb{Q} \cap \frac{1}{k} \mathbb{Z}^d \) with \( \|x - x^*\|_\infty \leq \delta \).

Proof. By Carathéodory’s Theorem, there exist \( d + 1 \) vertices \( x^0, \ldots, x^d \in \mathbb{Z}^d \) of \( Q \) and convex multipliers \( \lambda_0, \ldots, \lambda_d \) such that \( x^* = \sum_{i=0}^d \lambda_i x^i \). Let \( \lambda'_i := \frac{1}{k} |k \lambda_i| \geq 0 \) for \( i = 1, \ldots, d \) and \( \lambda'_0 := 1 - \sum_{i=1}^d \lambda'_i \geq 0 \). Then \( x := \sum_{i=0}^d \lambda'_i x^i \in \mathbb{Q} \cap \frac{1}{k} \mathbb{Z}^d \), and we have 
\[
\|x - x^*\|_\infty \leq \sum_{i=0}^d (\lambda'_i - \lambda_i)\|x^i\|_\infty \leq (d + 1)\frac{1}{k}M \leq \delta.
\]

\( \square \)

3 Bounding techniques for polynomial functions

Using the results of Section 2 we are now able to approximate the mixed-integer optimal point by a point of a suitably fine lattice. The question arises how we can use the geometric distance of these two points to estimate the difference in objective function values. We prove Theorem 3.1 that provides us with a local Lipschitz constant for the polynomial to be maximized.

Lemma 3.1 (Local Lipschitz constant). Let \( f \) be a polynomial in \( d \) variables with maximum total degree \( D \). Let \( C \) denote the largest absolute value of a coefficient of \( f \). Then there exists a Lipschitz constant \( L \) such that \( |f(x) - f(y)| \leq L\|x - y\|_\infty \) for all \( |x_i|, |y_i| \leq M \). The constant \( L \) is \( O(D^{d+1}CM^D) \).

Proof. Using the usual multi-index notation, let \( f(x) = \sum_{\alpha \in \mathbb{D}} c_\alpha x^\alpha \). Let \( r = |\mathbb{D}| \) be the number of monomials of \( f \). Then we have 
\[
|f(x) - f(y)| \leq \sum_{\alpha \neq 0} |c_\alpha| \|x^\alpha - y^\alpha\|.
\]
We estimate all summands separately. Let $\alpha \neq 0$ be an exponent vector with $n := \sum_{i=1}^{d} \alpha_i \leq D$. Let

$$\alpha = \alpha^0 \geq \alpha^1 \geq \ldots \geq \alpha^n = 0$$

be a decreasing chain of exponent vectors with $\alpha^{i+1} - \alpha^i = e^i$ for $i = 1, \ldots, n$. Let $\beta^i := \alpha - \alpha^i$ for $i = 0, \ldots, n$. Then $x^\alpha - y^\alpha$ can be expressed as the “telescope sum”

$$x^\alpha - y^\alpha = x^{\alpha^0} y^{\beta^0} - x^{\alpha^1} y^{\beta^1} + x^{\alpha^2} y^{\beta^2} - x^{\alpha^3} y^{\beta^3} + \ldots - x^{\alpha^n} y^{\beta^n}$$

$$= \sum_{i=1}^{n} \left( x^{\alpha^{i-1}} y^{\beta^{i-1}} - x^{\alpha^i} y^{\beta^i} \right)$$

$$= \sum_{i=1}^{n} \left( (x_j^i - y_j^i)x^{\alpha^i} y^{\beta^{i-1}} \right).$$

Since $|x^{\alpha^i} y^{\beta^{i-1}}| \leq M^{n-1}$ and $n \leq D$, we obtain

$$|x^\alpha - y^\alpha| \leq D \cdot \|x - y\|_\infty \cdot M^{n-1},$$

thus

$$|f(x) - f(y)| \leq CrDM^{D-1}\|x - y\|_\infty.$$ 

Let $L := CrDM^{D-1}$. Since $r = O(D^d)$, we have $L = O(D^{d+1}CM^D)$. \qed

Moreover, in order to obtain an FPTAS, we need to put differences of function values in relation to the maximum function value. To do this, we need to deal with the special case of polynomials that are constant on the feasible region; here trivially every feasible solution is optimal. For non-constant polynomials, we can prove a lower bound on the maximum function value. The technique is to bound the difference of the minimum and the maximum function value on the mixed-integer set from below; if the polynomial is non-constant, this implies, for a non-negative polynomial, a lower bound on the maximum function value. We will need a simple fact about the roots of multivariate polynomials.

**Lemma 3.2.** Let $f \in \mathbb{Q}[x_1, \ldots, x_d]$ be a polynomial and let $D$ be the largest power of any variable that appears in $f$. Then $f = 0$ if and only if $f$ vanishes on the set $\{0, \ldots, D\}^d$.

**Proof.** This is a simple consequence of the Fundamental Theorem of Algebra. See, for instance, [3, Chapter 1, §1, Exercise 6 b]. \qed

**Lemma 3.3.** Let $f \in \mathbb{Q}[x_1, \ldots, x_d]$ be a polynomial with maximum total degree $D$. Let $Q \subseteq \mathbb{R}^d$ be an integral polytope of dimension $d' \leq d$. Let $k \geq D d'$. Then $f$ is constant on $Q$ if and only if $f$ is constant on $Q \cap \mathbb{Z}^d$.

**Proof.** Let $x^0 \in Q \cap \mathbb{Z}^d$ be an arbitrary vertex of $Q$. There exist vertices $x^1, \ldots, x^d \in Q \cap \mathbb{Z}^d$ such that the vectors $x^i - x^0, \ldots, x^{d'} - x^0 \in \mathbb{Z}^d$ are linearly independent. By convexity, $Q$ contains the parallelepiped

$$S := \left\{ x^0 + \sum_{i=1}^{d'} \lambda_i (x^i - x^0) : \lambda_i \in [0, \frac{1}{D}] \text{ for } i = 1, \ldots, d' \right\}.$$ 

5
We consider the set
\[ S_k = \frac{1}{k} \mathbb{Z}^d \cap S \supseteq \left\{ x^0 + \sum_{i=1}^{d'} \frac{1}{n_i} (x^i - x^0) : n_i \in \{0, 1, \ldots, D\} \text{ for } i = 1, \ldots, d' \right\}. \]

Now if there exists a \( c \in \mathbb{R} \) with \( f(x) = c \) for \( x \in Q \cap \frac{1}{k} \mathbb{Z}^d \), then all the points in \( S_k \) are roots of the polynomial \( f - c \), which has only maximum total degree \( D \). By Theorem 3.2 (after an affine transformation), \( f - c \) is zero on the affine hull of \( S_k \); hence \( f \) is constant on \( Q \).

**Theorem 3.4.** Let \( f \in \mathbb{Z}[x_1, \ldots, x_{d_1}, z_1, \ldots, z_{d_2}] \). Let \( P \) be a rational convex polytope, and let \( \Delta \) be the number from Theorem 2.4. Let \( m = k\Delta \) with \( k \geq D d_1, k \in \mathbb{Z} \). Then \( f \) is constant on the feasible region \( P \cap (\mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}) \) if and only if \( f \) is constant on \( P \cap (\frac{1}{m} \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}) \). If \( f \) is not constant, then
\[ |f(x_{\text{max}}, z_{\text{max}}) - f(x_{\text{min}}, z_{\text{min}})| \geq m^{-D}, \]
where \( (x_{\text{max}}, z_{\text{max}}) \) is an optimal solution to the maximization problem over the feasible region \( P \cap (\mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}) \) and \( (x_{\text{min}}, z_{\text{min}}) \) is an optimal solution to the minimization problem.

**Proof.** Let \( f \) be constant on \( P \cap (\frac{1}{m} \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}) \). For fixed integer part \( z \in \mathbb{Z}^{d_2} \), we consider the polytope \( \Delta P_z = \{ \Delta x : (x, z) \in P \} \), which is a slice of \( P \) scaled to become an integral polytope. By applying Theorem 3.3 with \( k = (D+1)d \) on every polytope \( \Delta P_z \), we obtain that \( f \) is constant on every slice \( P_z \). Because \( f \) is also constant on the set \( P \cap (\frac{1}{m} \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}) \), which contains a point of every non-empty slice \( P_z \), it follows that \( f \) is constant on \( P \).

If \( f \) is not constant, there exist \((x^1, z^1), (x^2, z^2) \in P \cap (\frac{1}{m} \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}) \) with \( f(x^1, z^1) \neq f(x^2, z^2) \). By the integrality of all coefficients of \( f \), we obtain the estimate
\[ |f(x^1, z^1) - f(x^2, z^2)| \geq m^{-D}. \]
Because \((x^1, z^1), (x^2, z^2) \) are both feasible solutions to the maximization problem and the minimization problem, this implies (6).

**4 Proof of Theorem 1.1**

Now we are in the position to prove the main result.

**Proof of Theorem 1.1.** Part (a). Let \((x^*, z^*) \) denote an optimal solution to the mixed-integer problem (11). Let \( \epsilon > 0 \). We show that, in time polynomial in the input length, the maximum total degree, and \( \frac{1}{\epsilon} \), we can compute a point \((x, z) \) that satisfies (11) such that
\[ |f(x, z) - f(x^*, z^*)| \leq \epsilon f(x^*, z^*). \]

First we note that we can restrict ourselves to the case of polynomials with integer coefficients, simply by multiplying \( f \) with the least common multiple of all denominators of the coefficients. We next establish a lower bound on \( f(x^*, z^*) \). To this end, let \( \Delta \) be the integer from Theorem 2.3, which can be computed in
polynomial time. By [Theorem 3.4] with \( m = Dd_1\Delta \), either \( f \) is constant on the feasible region, or

\[
f(x^*, z^*) \geq (Dd_1\Delta)^{-D},
\]

where \( D \) is the maximum total degree of \( f \). Now let

\[
\delta := \frac{\epsilon}{2(Dd_1\Delta)^D L(C, D, M)}
\]

and let

\[
m := \Delta \left[ \frac{2}{\epsilon} (Dd_1\Delta)^D L(C, D, M)(d_1 + 1)M \right],
\]

where \( L(C, D, M) \) is the Lipschitz constant from [Theorem 3.1]. Then we have \( m \geq \Delta \frac{1}{2}(d_1 + 1)M \), so by [Theorem 2.3], there is a point \( ([x^*_\delta], z^*) \in P \cap (\frac{1}{m}Z^{d_1} \times Z^{d_2}) \) with \( \| [x^*_\delta] - z^* \|_\infty \leq \delta \). Let \((x^m, z^m)\) denote an optimal solution to the grid problem (2). Because \(([x^*_\delta], z^*)\) is a feasible solution to the grid problem (2), we have

\[
f([x^*_\delta], z^*) \leq f(x^m, z^m) \leq f(x^*, z^*).
\]

Now we can estimate

\[
|f(x^*, z^*) - f(x^m, z^m)| \leq |f(x^*, z^*) - f([x^*_\delta], z^*)| \\
\leq L(C, D, M) \| x^* - [x^*_\delta] \|_\infty \\
\leq L(C, D, M) \delta \\
= \frac{\epsilon}{2} (Dd_1\Delta)^{-D} \\
\leq \frac{\epsilon}{2} f(x^*, z^*),
\]

where the last estimate is given by [3] in the case that \( f \) is not constant on the feasible region. On the other hand, if \( f \) is constant, the estimate [12] holds trivially.

By [Theorem 2.1], we can compute a point \((x^m_{\epsilon/2}, z^m_{\epsilon/2}) \in P \cap (\frac{1}{m}Z^{d_1} \times Z^{d_2}) \) such that

\[
(1 - \frac{\epsilon}{2}) f(x^m, z^m) \leq f(x^m_{\epsilon/2}, z^m_{\epsilon/2}) \leq f(x^m, z^m)
\]

in time polynomial in \( \log m \), the encoding length of \( f \) and \( P \), the maximum total degree \( D \), and \( 1/\epsilon \). Here \( \log m \) is bounded by a polynomial in \( \log M, D \) and \( \log C \), so we can compute \((x^m_{\epsilon/2}, z^m_{\epsilon/2}) \) in time polynomial in the input size, the maximum total degree \( D \), and \( 1/\epsilon \). Now we can estimate, using [13] and [12],

\[
f(x^*, z^*) - f(x^m_{\epsilon/2}, z^m_{\epsilon/2}) \leq f(x^*, z^*) - (1 - \frac{\epsilon}{2}) f(x^m, z^m) \\
= \frac{\epsilon}{2} f(x^*, z^*) + (1 - \frac{\epsilon}{2}) (f(x^*, z^*) - f(x^m, z^m)) \\
\leq \frac{\epsilon}{2} f(x^*, z^*) + \frac{\epsilon}{2} f(x^*, z^*) \\
= \epsilon f(x^*, z^*).
\]

Hence \( f(x^m_{\epsilon/2}, z^m_{\epsilon/2}) \geq (1 - \epsilon) f(x^*, z^*) \).

**Part (b).** Let the dimension \( d \geq 2 \) be fixed. We prove that there does not exist a PTAS for the maximization of arbitrary polynomials over mixed-integer sets of polytopes. We use the NP-complete problem AN1 on page 249 of [3]. This is to decide whether, given three positive integers \( a, b, c \), there exists a positive integer
\( x < c \) such that \( x^2 \equiv a \pmod{b} \). This problem is equivalent to asking whether the maximum of the quartic polynomial function \( f(x, y) = -(x^2 - a - by)^2 \) over the lattice points of the rectangle

\[
P = \left\{ (x, y) : 1 \leq x \leq c - 1, \quad \frac{1 - a}{b} \leq y \leq \frac{(c - 1)^2 - a}{b} \right\}
\]

is zero or not. If there existed a PTAS for the maximization of arbitrary polynomials over mixed-integer sets of polytopes, we could, for any fixed \( 0 < \epsilon < 1 \), compute in polynomial time a solution \((x_\epsilon, y_\epsilon) \in P \cap \mathbb{Z}^2\) with \( |f(x_\epsilon, y_\epsilon) - f(x^*, y^*)| \leq \epsilon |f(x^*, y^*)|\), where \((x^*, y^*)\) denotes an optimal solution. Thus, we have \( f(x_\epsilon, y_\epsilon) = 0 \) if and only if \( f(x^*, y^*) = 0 \); this means we could solve the problem AN1 in polynomial time. \qed
A Appendix: An FPTAS for the integer case

The first fully polynomial-time approximation scheme for the integer case appeared in our paper [6]. It is based on Alexander Barvinok’s theory for encoding all the lattice points of a polyhedron in terms of short rational functions [1, 2]. The set $P \cap \mathbb{Z}^d$ is represented by a Laurent polynomial $g_P(z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} z^\alpha$. From Barvinok’s theory this exponentially-large sum of monomials $g_P(z)$ can be written as a polynomial-size sum of rational functions (assuming the dimension $d$ is fixed) of the form:

$$g_P(z) = \sum_{i \in I} E_i \frac{z^{u_i}}{\prod_{j=1}^d (1 - z^{v_{ij}})},$$

(14)

where $I$ is a polynomial-size indexing set, and where $E_i \in \{1, -1\}$ and $u_i, v_{ij} \in \mathbb{Z}^d$ for all $i$ and $j$. There is a polynomial-time algorithm for computing this representation [1, 2, 5, 7].

By symbolically applying differential operators to the representation (14), we can compute a short rational function representation of the Laurent polynomial

$$g_{P,f}(z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} f(\alpha) z^\alpha.$$  

(15)

In fixed dimension, the size of the expressions occurring in the symbolic calculation can be bounded polynomially:

**Lemma A.1 (Lemma 3.1 of [6]).** Let the dimension $d$ be fixed. Let $g_P(z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} z^\alpha$ be the Barvinok representation of the generating function of $P \cap \mathbb{Z}^d$. Let $f \in \mathbb{Z}[x_1, \ldots, x_d]$ be a polynomial of maximum total degree $D$. We can compute, in time polynomial in $D$ and the input size, a Barvinok representation $g_{P,f}(z)$ for the generating function $\sum_{\alpha \in P \cap \mathbb{Z}^d} f(\alpha) z^\alpha$.

Now we present the algorithm to obtain bounds $U_k, L_k$ that reach the optimum. We make use of the elementary fact that, for a set $S = \{s_1, \ldots, s_r\}$ of non-negative real numbers,

$$\max\{s_i : s_i \in S\} = \lim_{k \to \infty} \sqrt[k]{\sum_{j=1}^r s_j^k}. \quad (16)$$

**Algorithm A.2 (Computation of bounds for the maximization problem).**

*Input:* A rational convex polytope $P \subset \mathbb{R}^d$, a polynomial objective $f \in \mathbb{Z}[x_1, \ldots, x_d]$ of maximum total degree $D$ that is non-negative over $P \cap \mathbb{Z}^d$.

*Output:* An increasing sequence of lower bounds $L_k$, and a decreasing sequence of upper bounds $U_k$ reaching the maximal function value $f^*$ of $f$ over $P \cap \mathbb{Z}^d$.

1. Compute a short rational function expression for the generating function $g_P(z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} z^\alpha$. Using residue techniques, compute $|P \cap \mathbb{Z}^d| = g_P(1)$ from $g_P(z)$.

2. From the rational function $g_P(z)$ compute the rational function representation of $g_{P,f_k}(z)$ of $\sum_{\alpha \in P \cap \mathbb{Z}^d} f^k(\alpha) z^\alpha$ in by Theorem A.1. Using residue techniques, compute

$$L_k := \sqrt[k]{g_{P,f_k}(1)/g_{P,f_0}(1)} \quad \text{and} \quad U_k := \sqrt[k]{g_{P,f_k}(1)}.$$
Theorem A.3 (FPTAS, Lemma 3.3 and Theorem 1.1 of [6]). Let the dimension $d$ be fixed. Let $P \subset \mathbb{R}^d$ be a rational convex polytope. Let $f$ be a polynomial with integer coefficients and maximum total degree $D$ that is non-negative on $P \cap \mathbb{Z}^d$.

(i) Theorem A.2 computes the bounds $L_k, U_k$ in time polynomial in $k$, the input size of $P$ and $f$, and the total degree $D$. The bounds satisfy the following inequality:

$$U_k - L_k \leq f^* \cdot \left( k \sqrt{|P \cap \mathbb{Z}^d|} - 1 \right).$$

(ii) For $k = (1 + 1/\epsilon) \log(|P \cap \mathbb{Z}^d|)$ (a number bounded by a polynomial in the input size), $L_k$ is a $(1 - \epsilon)$-approximation to the optimal value $f^*$ and it can be computed in time polynomial in the input size, the total degree $D$, and $1/\epsilon$. Similarly, $U_k$ gives a $(1 + \epsilon)$-approximation to $f^*$.

(iii) With the same complexity, by iterated bisection of $P$, we can also find a feasible solution $x_{\epsilon} \in P \cap \mathbb{Z}^d$ with

$$|f(x_{\epsilon}) - f^*| \leq \epsilon f^*.$$
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