A Noise Model for Multilayer Graded-Bandgap Avalanche Photodiodes

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Abstract

Multilayer graded-bandgap avalanche photodiodes (APDs) are the future deterministic photomultipliers, owing to their deterministic amplification with twofold stepwise gain via impact ionization when operated in the staircase regime. Yet, the stepwise impact ionization irregularities worsen as the number of steps increases. These irregularities in impact ionization are the major source of noise in these APDs. These solid-state devices could replace conventional silicon photomultiplier tubes if they are carefully studied and designed. A noise model for multistep staircase APDs, considering equal stepwise ionization probabilities is previously reported. However, we derive a generalized noise model for multilayer graded-bandgap APDs, applicable for all operating biases, which include the sub-threshold, staircase, and tunneling breakdown regimes. Moreover, the previous noise model’s expression for the total excess noise factor in terms of ionization probabilities of the multistep staircase APD follows Friis’s total noise factor. However, we demonstrate that our derived expression matches Bangera’s correction to Friis’s total noise factor.

1 Introduction

Photomultiplier tubes (PMTs) [1] are most often used for deterministic photomultiplication, owing to their extremely high gain and low noise [2]. PMTs consist of a photocathode for the emission of photoelectrons (primary electrons), a focusing grid, an array of metallic dynodes which enable amplification by emitting Poisson-distributed secondary electrons, and an anode for collection of electrons at the output, all enclosed in a vacuum glass tube [1]. These PMTs have several applications in photon counting [1, 3, 4], spectroscopy [1, 5, 6], high energy physics [1, 7, 8], radiation measurement [1], electron microscopy [1, 9], and so on. Some biomedical applications of PMTs include positron emission tomography, in-vitro assay, computed radiography, and gamma scintigraphy [1, 10, 11]. However, these devices are large in size (approximately a few cm), fragile, expensive, and operated at high voltages of greater than 1kV in a vacuum [2].

Although the conventional avalanche photodiodes (APDs) are alternate solid-state devices for PMT applications such as photon counting [12–17], spectroscopy [18–20], biomedical [21, 22], and so on; they produce a nonlinear output with high noise [23–26]. Further attempts were made to reduce the excess noise and improve the gain in conventional APDs by fabricating quantum dot resonant tunneling diode based APDs [13, 27], waveguide-integrated Germanium APDs [28], nanowire APDs [29, 30], separate absorption-multiplication (SAM) APDs using tunable direct bandgap digital alloys such as AlInAsSb [31–33], and so forth.

Recently, multilayer graded-bandgap APDs operated in their staircase operating regime have been the solid-state analogue of PMTs with deterministic amplification [24, 26, 34–39]. The conduction band profile in the energy-position band-diagram of these APDs appears similar to a series of steps when the applied bias ranges within the staircase regime [35, 37–41]. Here, the function of the steps is similar to the metallic dynodes in a PMT. Thus, these APDs operated in the staircase regime may be referred to as multistep staircase APDs. Moreover, these solid-state devices have the advantages of micro-size, are low-cost, and are operated at low voltages. However, the irregularities in the stepwise impact ionization worsen as the number of steps increases [24, 26, 39, 41–43]. These solid-state devices could be a replacement for conventional silicon PMTs if they are carefully designed.

This article first briefly discusses the existing theory and noise model of the conventional p-i-n APDs
[24, 26, 44]. We then propose a new noise model for a multilayer graded-bandgap APD. Owing to the irregularities in the stepwise impact ionization, we have derived a generalized noise model for a non-ideal n-layer graded-bandgap APD. The previous noise models [35, 41, 43, 45] for an n-step staircase APD provide the expression for the excess noise factor in terms of the number of steps when the stepwise impact ionizations are equal. However, our generalized model is applicable for all operating biases, which include the sub-threshold, staircase operating, and tunnelling breakdown regimes. Further, we present our model’s simplified noise expression for n-step staircase APDs. For validation, we compare the noise power ratios (NPRs) and the excess noise factors determined using our model with the previous models [35, 41, 43, 45], when the stepwise ionization probabilities are equal. Then we discuss the inter-dependent variations of ionization probabilities, NPRs, noise current ratios (NCRs), excess noise factors, and staircase gains of 1-step, 2-step, and 3-step staircase APDs.

2 Theory and Noise Model

2.1 Conventional p-i-n APD

The theory and noise model of a conventional p-i-n APD is well known in the literature [24, 26, 44]. Let the input photocurrent of a conventional p-i-n APD be represented as,

\[ i_{ph} = \sum_{\alpha=1}^{N_0} h(t - t_{\alpha}) \]  

(1)

Where \( \alpha \) is the injected photoelectron count; \( N_0 \) is the total number of charges generated by the incident radiation or the total injected photoelectrons; \( h(t - t_{\alpha}) \) is the pulse function that defines the input photocurrent corresponding to the injected photoelectron generated by the pulse of radiation incident at time \( t_{\alpha} \), such that, \( \int h(t)\,dt = q \), where \( q \) is the electron charge.

Then, the time-dependent output current of the conventional p-i-n APD, neglecting the dark current will be,

\[ i(t) = \sum_{\alpha=1}^{N_0} M_C h(t - t_{\alpha} - t_o) \]  

(2)

Where \( t_o \) is the time taken for the photoelectron to reach the output, and \( M_C \) is the avalanche gain of the conventional p-i-n APD for each pulse of charge. Here, \( M_C = 1 \) when the device is operated in the unity gain mode regime such that the incident photon count is equal to the number of photoelectrons generated and collected at the output, which implies that no carrier multiplication happens [26].

The noise current spectral intensity (A Hz\(^{-1}\)) of this conventional APD [26, 44], neglecting the dark current is given by,

\[ S_C^i(f) = 2q(M_C^2)I_0 = 2q(M_C)^2F(M_C)I_0 \]  

(3)

Where \( q \) is the electron charge which is a constant, \( F(M_C) \equiv \frac{(M_C^2)}{(M_C^2)} \) is the excess noise factor of the conventional p-i-n APD, \( I_0 = \langle N_0 \rangle |H(f)| \) is the unity gain photocurrent.

The spectral noise current (A\(\sqrt{\text{Hz}}\)) is defined as the square-root of the noise current spectral intensity [26, 44]. Therefore, the spectral noise current of this conventional APD is given by,

\[ \sigma_C^i(f) = \sqrt{2q(M_C^2)I_0} = \sqrt{2q(M_C)^2F(M_C)I_0} \]  

(4)

The noise power spectral density (W Hz\(^{-1}\)) is defined as the product of the noise current spectral intensity and the noise figure analyser’s AC load resistance \( (R_L) \) [26]. For this conventional APD, the noise power spectral density is given by,

\[ S_C^L(f) = 2q(M_C^2)I_0R_L = 2q(M_C)^2F(M_C)I_0R_L \]  

(5)

2.2 A generalized noise model for an n-layer graded-bandgap APD

The block diagram shown in Fig. 1 elaborates the amplification of the photo-generated electrons (photoelectrons) in an n-layer graded-bandgap APD. Here, the first amplifier with gain \( M_0 \) corresponds to the photoelectrons, and \( M_0 = 1 \) indicates that each incident photon generates only one photoelectron and no carrier multiplication happens in the absorption region of the APD. Moreover, the band structure of an n-layer graded-bandgap APD appears discontinuous due to the heterojunction interfaces, forming step-like structures when biased, especially in the staircase regime [35, 38]. This article refers to these junction discontinuities as steps, even when the devices are unbiased. Therefore, the junction/step gains are represented as \( M_x \) corresponding to the gains at junction/step ‘x’. From the block diagram, it is clear that the total step gain (total multiplication gain) of the n-layer graded-bandgap APD can be defined as the fraction of the total number of electrons at the output of step ‘n’ \( (N_n) \) and the total number of input photoelectrons \( (N_0) \). This is also equal to the product of
The noise power spectral density of this \( n \)-layer graded-bandgap APD is,

\[
S_N^L(f) = 2q\langle M_0^2 \rangle \langle M_S^2 \rangle I_0 R_L
= 2q\langle M_0 \rangle^2 F(M_0)\langle M_S \rangle^2 F(M_S)I_0 R_L
\tag{10}
\]

Further, this article defines the noise power ratio of the \( n \)-layer graded-bandgap APD (NPR\(_S_n\)) as the ratio of the noise power spectral density of the \( n \)-layer graded-bandgap APD to the noise power spectral density of the conventional p-i-n APD. When, \( M_0 = M_C \), we get, \( F(M_0) = F(M_C) \). Thus,

\[
\text{NPR}_{S_n} = \frac{S_N^L(f)}{S_N^C(f)} = \frac{(M_0)^2 F(M_0)\langle M_S \rangle^2 F(M_S)}{\langle M_C \rangle^2 F(M_C)}
\tag{11}
\]

Similarly, we define the noise current ratio of the \( n \)-layer graded-bandgap APD (NCR\(_S_n\)) as the ratio of the spectral noise current of the \( n \)-layer graded-bandgap APD to the spectral noise current of the conventional p-i-n APD. This may also be defined as the square-root of the noise power ratio. Therefore,

\[
\text{NCR}_{S_n} = \frac{\sigma_L^S(f)}{\sigma_C^S(f)} = \sqrt{\frac{(M_0)^2 F(M_0)\langle M_S \rangle^2 F(M_S)}{\langle M_C \rangle^2 F(M_C)}}
\tag{12}
\]

In the new noise model proposed by us, let \( X_x \) be a random variable for multiplication at step ‘\( x \)’ that defines the number of extra electrons generated by a single electron at the input of each step, such that

all the step gains. Therefore, the total step gain is represented as,

\[
M_S = \frac{N_0}{N_0} = M_1M_2M_3...M_n = \prod_{x=1}^{n} M_x \tag{6}
\]

If \( h(t-t_\alpha) \) is a pulse-shaped function that represents input photocurrent corresponding to the injected photoelectron generated by the pulse of radiation incident at time \( t = t_\alpha \), then, \( h(t-t_\alpha - t_x) \) is the pulse-shaped function shifted by a time instant ‘\( t_x \)’. In comparison to the conventional p-i-n APD, the time-dependent output current of an \( n \)-layer graded-bandgap APD, neglecting the dark current is given by,

\[
i(t) = \sum_{n=1}^{N_0} M_SM_0 \delta(t-t_n) * h(t-t_\alpha)
= \sum_{n=1}^{N_0} M_SM_0 h(t-t_\alpha - t_n)
= \sum_{n=1}^{N_0} \left( \prod_{x=1}^{n} M_x \right) M_0 h(t-t_\alpha - t_n) \tag{7}
\]

Where ‘\( n \)’ is the total number of layers in the \( n \)-layer graded-bandgap APD, \( \delta(t-t_n)^*h(t-t_\alpha) \) represents the time-shifted pulse-shaped function and ‘\( * \)’ indicates convolution operator.

The noise current spectral intensity of this \( n \)-layer graded-bandgap APD is,

\[
S_N^L(f) = 2q\langle M_0^2 \rangle \langle M_S^2 \rangle I_0
= 2q\langle M_0 \rangle^2 F(M_0)\langle M_S \rangle^2 F(M_S)I_0 \tag{8}
\]

Where \( F(M_0) \) is the excess noise factor corresponding to the gain \( M_0 \) and \( F(M_S) \) is the excess noise factor corresponding to the gain \( M_S \) of the \( n \)-layer graded-bandgap APD.

The spectral noise current of this \( n \)-layer graded-bandgap APD is given by,

\[
\sigma_L^S(f) = \sqrt{2q\langle M_0^2 \rangle \langle M_S^2 \rangle I_0}
= \sqrt{2q\langle M_0 \rangle^2 F(M_0)\langle M_S \rangle^2 F(M_S)I_0} \tag{9}
\]
gain may be represented as,

$$X \sim \begin{cases} 
0 & 1 - p_{x1} - p_{x2} - \ldots - p_{xm} \\
1 & p_{x1} \\
2 & p_{x2} \\
\vdots & \vdots \\
m & p_{xm}
\end{cases} \quad \text{[i.e., No Ionization at step ‘x’]}
\quad \Rightarrow \quad X_x = 0
$$

$$\text{[i.e., Ionization at step ‘x’ \(\forall\) it generates 1 e\textsuperscript{-}s]}
\quad X \sim \begin{cases} 
0 & 1 - p_{x1} - p_{x2} - \ldots - p_{xm} \\
1 & p_{x1} \\
2 & p_{x2} \\
\vdots & \vdots \\
m & p_{xm}
\end{cases} \quad \text{[i.e., Ionization at step ‘x’ \(\forall\) it generates 2 e\textsuperscript{-}s]}
\quad \text{[i.e., Ionization at step ‘x’ \(\forall\) it generates ‘m’ e\textsuperscript{-}s]}
\quad \text{(13)}
$$

Then, the generalized form of the time-dependent current at the output of the step ‘n’, in terms of the random variable for multiplication ‘\(X_x\)’ of an \(n\)-layer graded-bandgap APD, can be written as,

$$i(t) = \sum_{\alpha=1}^{N_0} \left[ \prod_{x=1}^{n} (1 + X_x) \right] M_0 h(t - t_\alpha - t_n)
= \sum_{\alpha=1}^{N_0} \left\{ \left[ \prod_{x=1}^{n} (1 + p_{x1} + 2p_{x2} + \ldots + np_{xm}) \right] \times M_0 h(t - t_\alpha - t_n) \right\}
\quad \text{(14)}$$

Where the generalized equation for the total step gain may be represented as,

$$M_S = \prod_{x=1}^{n} M_x = \prod_{x=1}^{n} (1 + X_x)
= \prod_{x=1}^{n} (1 + p_{x1} + 2p_{x2} + \ldots + np_{xm})
\quad \text{(15)}$$

Similarly, the generalized equation for the noise power ratio and noise current ratio of the \(n\)-layer graded-bandgap APD may, respectively, be formulated as

$$\text{NPR}^{\text{new}}_{S_n} = \left\langle \prod_{x=1}^{n} M_x^2 \right\rangle = \left\langle \prod_{x=1}^{n} (1 + X_x)^2 \right\rangle
= \left\langle \prod_{x=1}^{n} (1 + p_{x1} + 2p_{x2} + \ldots + np_{xm})^2 \right\rangle
\quad \text{NPR}^{\text{new}}_{S_n} = \left\langle \prod_{x=1}^{n} M_x^2 \right\rangle = \left\langle \prod_{x=1}^{n} (1 + X_x)^2 \right\rangle
= \left\langle \prod_{x=1}^{n} (1 + p_{x1} + 2p_{x2} + \ldots + np_{xm})^2 \right\rangle
\quad \text{(16)}$$

and

$$\text{NCR}^{\text{new}}_{S_n} = \sqrt{\prod_{x=1}^{n} M_x^2} = \sqrt{\prod_{x=1}^{n} (1 + X_x)^2}
= \sqrt{\prod_{x=1}^{n} (1 + p_{x1} + 2p_{x2} + \ldots + np_{xm})^2}
\quad \text{(17)}$$

Here, since the \(X_x\) is considered a random variable ranging from 0 to \(m\) with different probabilities, this is a generalized noise model for a practical \(n\)-layer graded-bandgap APD applicable for all operating biases, which includes the sub-threshold, staircase operating, and tunneling breakdown regimes.

3 Results and Discussion

3.1 Staircase operation of an \(n\)-layer graded-bandgap APD

An \(n\)-layer graded-bandgap APD operated in its staircase operation regime is called a multistep staircase APD or an \(n\)-step staircase APD. Considering that each incident photon generates only one photoelectron and no carrier multiplication happens in the absorption region of the APD, the photo-generated electron gain is expected to be \(M_0 = 1\). Thus, the excess noise factor corresponding to \(M_0\) will be \(F(M_0) = 1\). Therefore, the time-dependent output current of the \(n\)-step staircase APD will be,

$$i(t) = \sum_{\alpha=1}^{N_0} M_S h(t - t_\alpha - t_n)
= \sum_{\alpha=1}^{N_0} \left[ \prod_{x=1}^{n} M_x \right] h(t - t_\alpha - t_n)
\quad \text{(18)}$$

The noise power spectral density of this \(n\)-step staircase APD is,

$$S^P_S(f) = 2q\langle M_S^2 \rangle^2 F(M_S) I_0 R_L
= 2q\langle M_S^2 \rangle^2 I_0 R_L
\quad \text{(19)}$$

Fig. 2 depicts the energy-position band-diagram of an ideal multistep staircase APD. Therefore, for an ideal \(n\)-step staircase APD, an electron at the input of each step will contribute to ionization by generating only one extra free electron. Thus, the gain at each step will be \(M_x = (1 + X_x) = 2\), where ‘\(x\)’ is the step number; and the total gain of the staircase APD will be a deterministic gain equal to \(M_S = (1 + X_x)^n = 2^n\). Here, the \(X_x\) will always be equal to 1, \(i.e.,\) the probability \((p_x)\) of the event \(X_x = 1\) is ‘one.’
Therefore, the noise power spectral density of such an \( n \)-step staircase APD with equal stepwise ionization probability \( p_x = 1 \) at all steps such that all input electrons at every step generates only one extra electron is,

\[
S^P_S(f) = 2q \left\langle \prod_{x=1}^{n} (1 + X_x)^2 \right\rangle I_0 R_L = 2q(2)^{2n} I_0 R_L
\]  

(20)

Further, comparing the equations (19) and (20), the noise power ratio of the \( n \)-step staircase APD in terms of \( X_x \) is formulated as,

\[
\text{NPR}_{S,n}^{\text{new}} = \left\langle \prod_{x=1}^{n} (1 + X_x)^2 \right\rangle = \left\langle M^2_S \right\rangle = \left\langle \prod_{x=1}^{n} (1 + X_x)^2 \right\rangle
\]  

(21)

However, in the case of a practical \( n \)-step staircase APD, an electron at the input of step ‘\( x \)’ would generate one or more extra electrons when ionized, or the input electrons may not ionize [35, 38, 39, 41, 43, 45]. Furthermore, all the steps may not have equal ionization probabilities, owing to the non-identically manufactured APD heterojunctions, the device operating bias, and so on [39]. Thus, \( X_x \) indicated in equation (13) would provide a generalized solution for all the irregularities included. Moreover, from the literature [35,38,39,41,43,45], it is known that the probability of input electrons generating more than two extra electrons at each step is quite low when the devices are operated in the staircase operating regime. Therefore, simplifying the equations and neglecting the ionization events with lower probabilities at step ‘\( x \)’, we consider that an electron at the input of step ‘\( x \)’ would generate only one extra electron when ionized and no extra electrons if not ionized. Then, \( X_x \) could be considered as a random variable with just two possibilities 0 or 1 with probabilities \( (1 - p_x) \) or \( p_x \), respectively. Therefore,

\[
X_x \sim \begin{cases} 0 & \text{[i.e., No Ionization at step ‘} x \text{’} \implies X_x = 0] \\ 1 & \text{[i.e., Ionization at step ‘} x \text{’ \( \geq 1 \) e\text{– generated}]} \\
\end{cases}
\]  

The simplified time-dependent current of a staircase APD at the output of step ‘\( n \)’ will be,

\[
i(t) = \sum_{\alpha=1}^{N_0} \left[ \prod_{i=1}^{n} (1 + X_x) \right] \delta(t - t_n) * h(t - t_\alpha)
\]  

(23)

Moreover, the noise power ratio of the proposed new noise model is,

\[
\text{NPR}_{S,n}^{\text{new}} = \left\langle \prod_{x=1}^{n} (1 + X_x)^2 \right\rangle
\]  

(24)

Equation (24) may be simplified and rewritten as follows,

\[
\text{NPR}_{S,n}^{\text{new}} = 1 + \sum_{i=1}^{n} (3)^i \left[ \sum_{j=1}^{n} \sum_{j_2=j+1}^{n} \sum_{j_2=j_1+1}^{n} \cdots \frac{p_{j_1} p_{j_2} \ldots p_{j_i}}{(1 - p_{j_1})(1 - p_{j_2}) \cdots (1 - p_{j_i})} \right]
\]  

(25)

The derivation and proof of the above equations (24) and (25) are included in Appendix A.

3.2 A comparison with the previous noise models for staircase APDs

From one of the previous noise models [35,41,43], it is reported that if ‘\( \delta \)’ is the fraction of electrons at the input of each step that do not impact-ionize, then the total step gain of the \( n \)-step staircase APD is \( M_\text{prev}^S = (2 - \delta)^n \) and its excess noise factor is
Another literature on the noise model \cite{39,45} reports that if \( p' \) is the ionization probability at each step, then the excess noise factor \( F(M_S) \) is,

\[
F(M_S, \delta)_{\text{prev}} = 1 + \sum_{x=1}^{n} \left( \frac{1}{(2 - \delta)^x} \frac{\delta(1 - \delta)}{2 - \delta} \right) 
= 1 + \frac{\delta(1 - (2 - \delta)^{-n})}{2 - \delta} \tag{26}
\]

From the literature, it is known that Bangera’s excess noise factors \( \text{NPR}_n^{\text{prev}} \) in the range \( 0 < p < 1 \), the \( \text{NPR}_n^{\text{prev}} < \text{NPR}_n^{\text{new}} \). Since the expressions for the staircase gain (total step gain of a staircase APD) in the previous and new noise models remain the same, the difference in the \( \text{NPR} \) is attributed to the differences in the total excess noise factors in the two models, as shown in Fig. 3b. Here, the excess noise factors \( F(M_S) \) of the multistep staircase APDs with steps \( n = 1, 2, 3 \) can be considered as the total noise factors of cascade networks with the total number of stages \( n = 1, 2, 3 \); with no externally added stage-wise noise. Thus, if the ionization probability is equal to one, then the stage-wise noise factors must be equal to one. However, suppose the ionization probabilities are less than one. In that case, there will be irregularities in the stage-wise multiplication factor, leading to the addition of internally generated stage-wise noise that is dependent on the ionization probabilities of that stage. These irregularities in ionization thus contribute to the stage-wise excess noise factors. Thus, in the case of staircase APDs with no externally added stage-wise noise, the stepwise noise factors \( F_x(M_S) \) must be equal for all steps \( x' \), with \( F_x(M_S) > 1 \) if \( p \neq 1, 0 \).

From Fig. 3b, the previous model’s excess noise factors \( F(M_S)_{\text{prev}} \) of the multistep staircase APDs with the total number of steps/stages \( n \) are in accordance with Friis’s total noise factor formula for cascade networks \cite{46,47} given by,

\[
F_{T_n}^{\text{Friis}} = F_1^{\text{Friis}} + \sum_{x=2}^{n} \left( \frac{F_1^{\text{Friis}} - 1}{\prod_{y=1}^{x-1} M_y} \right) \tag{31}
\]

An illustration of how the excess noise factors obtained using the previous noise models follow Friis’s total noise factor formula for cascade networks is included in Appendix B. However, our new model’s excess noise factors \( F(M_S)_{\text{new}} \) of the multistep staircase APDs agree with Bangera’s total noise factor of \( n \)-stage cascade networks \cite{48} given by equation (32) and is illustrated in Appendix C.

\[
F_{T_n}^{\text{Cor}} = \prod_{x=1}^{n} F_x^{\text{Cor}} \tag{32}
\]

From the literature, it is known that Bangera’s expression for the total noise factor of an \( n \)-stage cascade network in terms of the stage-wise noise factors is a correction to the corresponding Friis’s formula \cite{48}. Thus, our new noise model for graded-bandgap \( n \)-step staircase APDs is an improved noise model.

For better understanding and visualization of our proposed new noise model, the variation of the noise current ratio \( \text{NCR}_n \) and staircase gain \( M_S \) of the
n-step staircase APD with the ionization probabilities is shown in Fig. 3c. We have also plotted the variation of the \( \text{NPR}_{R_n}, \text{NCR}_{R_n}, \) and \( F(M_S) \) versus \( M_S \) of the \( n \)-step staircase APD, for steps 1, 2, and 3, shown in Fig. 4a, 4b, and 4c, respectively.

4 Conclusion

We conclude that our generalized noise model for multilayer graded-bandgap APDs is applicable to estimate various noise expressions for all operating biases, which includes the sub-threshold, staircase operating, and tunnelling breakdown regimes. To better understand our noise model, a detailed discussion of our proposed noise model for an \( n \)-layer graded-bandgap APD in its staircase operating regime is presented and compared with the two previous noise models. From the discussion, the expressions for the staircase gain of the multistep staircase APD obtained using all three models are equivalent. Moreover, comparing the previous noise models, both models provided equivalent noise expressions. However, comparing our proposed and previous noise models [35,39,41,43,45], values of the noise power ratio and excess noise factors of the \( n \)-step staircase APDs obtained using our model are comparatively greater when \( n > 1 \) and \( p \neq 0, 1 \). Nevertheless, our noise model and the expression for total excess noise factors of the multistep staircase APDs agree with Bangera’s total noise factor of \( n \)-stage cascade networks [48]. Since, Bangera’s total noise factor expression for \( n \)-stage cascade networks [48] is a correction to Friis’s formula [46,47], especially for multistage networks with \( n \geq 2 \); we conclude that our new noise model for \( n \)-step staircase APDs is an improved noise model.
A.2 Solution for a 2-step staircase APD

\[(1 + X_1)^2 (1 + X_2)^2 \sim \begin{cases} 1 & (1 - p_1)(1 - p_2) \\ 4 & p_1(1 - p_2) \\ 4 & (1 - p_1)p_2 \\ 16 & p_1 p_2 \end{cases} \]

Then, the \( \text{NPR}_{S_n}^{\text{new}} \) of a 2-step staircase APD is,

\[
\text{NPR}_{S_2}^{\text{new}} = \left( (1 + X_1)^2 (1 + X_2)^2 \right) = (1 - p_1)(1 - p_2) + 4p_1(1 - p_2) + 4(1 - p_1)p_2 + 16p_1p_2 \\
= (1 - p_1)(1 - p_2) \left\{ 1 + (2^2) \left[ \frac{p_1}{(1 - p_1)} + \frac{p_2}{(1 - p_2)} \right] \right\} \\
+ (2^2)^2 \left[ \frac{p_1}{(1 - p_1)} \frac{p_2}{(1 - p_2)} \right] = 1 + 3\{p_1 + p_2\} + 9\{p_1p_2\} 
\]

If ionization probabilities at both the steps are equal \( i.e., p_1 = p_2 = p \), then \( \text{NPR}_{S_n}^{\text{new}} \) of the staircase APD with number of steps \( n = 2 \) may be rewritten as,

\[
\text{NPR}_{S_2}^{\text{new}} = 1 + ^2 C_1(3p) + ^2 C_2(9p^2) = \sum_{i=0}^{2} 2C_i(3p)^i 
\]

A.3 Solution for a 3-step staircase APD

\[
\begin{align*}
(1 + X_1)^2 (1 + X_2)^2 (1 + X_3)^2 & \sim \begin{cases} 1 & (1 - p_1)(1 - p_2)(1 - p_3) \\ 4 & p_1(1 - p_2)(1 - p_3) \\ 4 & (1 - p_1)p_2(1 - p_3) \\ 4 & (1 - p_1)(1 - p_2)p_3 \\ 16 & p_1 p_2(1 - p_3) \\ 16 & p_1(1 - p_2)p_3 \\ 16 & (1 - p_1)p_2 p_3 \\ 64 & p_1 p_2 p_3 \end{cases} \\
\end{align*}
\]

Then, the \( \text{NPR}_{S_n}^{\text{new}} \) of a 3-step staircase APD is,
\[ \text{NPR}_{S_3}^{\text{new}} = (1 + X_1)^2(1 + X_2)^2(1 + X_3)^2 \]
\[ = (1 - p_1)(1 - p_2)(1 - p_3) + 4p_1(1 - p_2)(1 - p_3) + 4(1 - p_1)p_2(1 - p_3) + 4(1 - p_1)(1 - p_2)p_3 + 16p_1p_2(1 - p_3) + 16p_1(1 - p_2)p_3 + 16(1 - p_1)p_2p_3 + 64p_1p_2p_3 \]
\[ = (1 - p_1)(1 - p_2)(1 - p_3) \left\{ 1 + (2^2) \left[ \frac{p_1}{1 - p_1} + \frac{p_2}{1 - p_2} + \frac{p_3}{1 - p_3} \right] \right\} + \left[ (2^2)^2 \left[ \frac{p_1}{1 - p_1} \right] + \left( \frac{p_2}{1 - p_2} \right) \right] + \left( \frac{p_3}{1 - p_3} \right) \left\{ \frac{p_1p_2p_3}{(1 - p_2)(1 - p_3)} \right\} \]
\[ = 1 + 3(p_1 + p_2 + p_3) + 9(p_1p_2 + p_1p_3 + p_2p_3) + 27(p_1p_2p_3) \]

If ionization probabilities at all the steps are equal \( i.e., p_1 = p_2 = p_3 = p \), then NPR_{S_3}^{\text{new}} of the staircase APD with number of steps \( n = 3 \) may be rewritten as,

\[ \text{NPR}_{S_3}^{\text{new}} = 1 + 3C_1(3p) + 3C_2(9p^2) + 3C_3(27p^3) \]
\[ = \sum_{i=0}^{3} 3C_i(3p)^i \]

**A.4 Solution for an \( n \)-step staircase APD**

\[ \prod_{x=1}^{n} (1 + X_x)^2 \sim \begin{cases} 1 & \prod_{x=1}^{n} (1 - p_x) \\ 4 & p_1 \prod_{x:y\neq 1}(1 - p_x) \\ 4 & p_2 \prod_{x:y\neq 2}(1 - p_x) \\ \vdots & \vdots \\ 4 & p_n \prod_{x:y\neq n}(1 - p_x) \\ 16 & p_1p_2 \prod_{x:y\neq 1,2}(1 - p_x) \\ 16 & p_1p_3 \prod_{x:y\neq 1,3}(1 - p_x) \\ \vdots & \vdots \\ (2^2)^n & \prod_{x=1}^{n} p_x \end{cases} \]

Then, the \( \text{NPR}_{S_n}^{\text{new}} \) of an \( n \)-step staircase APD is,

\[ \text{NPR}_{S_n}^{\text{new}} = \prod_{x=1}^{n} (1 + X_x)^2 \]
\[ = \prod_{x=1}^{n} (1 + p_x) \left\{ 1 + 2^2 \left[ \sum_{j=1}^{n-1} \frac{p_{j_1}}{1 - p_{j_1}} \right] + (2^2)^2 \left[ \sum_{j_1=1}^{n-2} \frac{p_{j_1}}{1 - p_{j_1}} \right] + \left( \sum_{j_2=1, j_1+1}^{n-1} \frac{p_{j_2}}{1 - p_{j_2}} \right) \right\} + \left( \sum_{j_2=j_1+1}^{n} \frac{p_{j_2}}{1 - p_{j_2}} \right) \]
\[ = \prod_{x=1}^{n} (1 + p_x) \left\{ 1 + \sum_{i=1}^{n-1} (2^2)^i \left[ \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} \ldots \frac{p_{j_i}}{1 - p_{j_i}} \right] + \sum_{j_i=j_{i-1}+1}^{n} p_{j_1}p_{j_2}\ldots p_{j_i} \right\} \]

Solving the above expression, \( \text{NPR}_{S_n}^{\text{new}} \) may also be formulated as,

\[ \text{NPR}_{S_n}^{\text{new}} = 1 + \sum_{i=1}^{n} (3)^i \left[ \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} \ldots \frac{p_{j_i}}{1 - p_{j_i}} \right] \]

If ionization probabilities at all the steps are equal \( i.e., p_1 = p_2 = p_3 = \ldots = p_n = p \), then \( \text{NPR}_{S_n}^{\text{new}} \) of the staircase APD with number of steps \( n \) may be rewritten as,

\[ \text{NPR}_{S_n}^{\text{new}} = 1 + \sum_{i=1}^{n} nC_i(3p)^i = \sum_{i=0}^{n} nC_i(3p)^i \]

**B Illustrations: Previous noise models follow Friis’s total noise factor formula for \( n \)-stage cascade networks**

Friis’s total noise factor formula for cascade networks [46, 47] is given by,

\[ F_{1n}^{\text{Friis}} = F_1^{\text{Friis}} + \sum_{x=2}^{n} \left( F_x^{\text{Friis}} - \frac{1}{\prod_{y=1}^{x-1} M_y} \right) \]
Moreover, the Friis noise factor at the \( x \)-th stage of a cascade network is,

\[
F_{x}^{\text{Friis}} = 1 + \frac{N_{a(x)}}{N_{1}M_{x}}
\]

Where \( N_{a(x)} \) is an externally added noise at the output of the \( x \)-th stage or \( x \)-th step. However, the above expression for \( F_{x}^{\text{Friis}} \) is valid only when all the multiplication probabilities are unity (or the multiplication gain is deterministic). Here, if \( N_{a(x)} = 0 \), then \( F_{x}^{\text{Friis}} = 1 \) for all stages or steps. However, if the multiplication probabilities are less than unity and non-zero, \( F_{x}^{\text{Friis}} \) will be greater than 1.

In the case of solid state devices such as staircase APDs, since there is no external noise added at every step (i.e., \( N_{a(x)} = 0 \)) and if all the stepwise ionization or multiplication probabilities of an \( n \)-step staircase APD are equal i.e., \( p_{1} = p_{2} = ... = p_{n} = p \); we could consider that all its stepwise excess noise factors will also be equal.

According to previous models [35, 39, 41, 43, 45], if ‘\( p \)’ is the ionization probability at each step, then the excess noise factor \( F(M_{S})^{\text{prev}} \) of an \( n \)-step staircase APD is given by,

\[
F(M_{S})^{\text{prev}} = 1 + \frac{(1-p)(1-(1+p)^{-n})}{(1+p)} = F_{T_{1}}^{\text{prev}}
\]

The total gain of the \( n \)-step staircase APD in terms of ‘\( p \)’ will be \( M_{S}^{\text{prev}} = (1+p)^{n} \).

**Illustration 1:** If \( p = 0.3 \) and \( n = 1 \), then substituting the values of ‘\( p \)’ and ‘\( n \)’ we get the total excess noise factor of a 1-step staircase APD as \( F_{T_{1}}^{\text{prev}} = 1.12426 \). Since this corresponds to a staircase APD with only one step, it could be considered that the stepwise excess noise factor at step one of a 1-step staircase APD is \( F_{1} = F_{T_{1}}^{\text{prev}} = 1.12426 = F_{T_{1}}^{\text{Friis}} \).

**Illustration 2:** Similarly, if \( p = 0.3 \) and \( n = 2 \), the total excess noise factor of a 2-step staircase APD will be \( F_{T_{2}}^{\text{prev}} = 1.12426 \). Since we have considered the same values of ‘\( p \)’ as in the previous illustration, we get \( F_{2} = F_{1} = 1.12426 \). Therefore, substituting \( F_{2} = F_{1} = 1.12426 \) in Friis’s equation, we get \( F_{T_{2}}^{\text{Friis}} = F_{1} + \frac{(F_{2}-1)}{M_{1}} = F_{1} + \frac{(F_{2}-1)}{(1+p)} = 1.12426 \).

**Illustration 3:** Similarly, if \( p = 0.3 \) and \( n = 3 \), the total excess noise factor of a 3-step staircase APD will be \( F_{T_{3}}^{\text{prev}} = 1.293372 \). Since we have considered the same values of ‘\( p \)’ as in the previous illustration, we get \( F_{3} = F_{2} = F_{1} = 1.12426 \). Therefore, substituting \( F_{3} = F_{2} = F_{1} = 1.12426 \) in Friis’s equation, we get \( F_{T_{3}}^{\text{Friis}} = F_{1} + \frac{(F_{3}-1)}{M_{1}} + \frac{(F_{3}-1)}{M_{1}M_{2}} = F_{1} + \frac{(F_{3}-1)}{(1+p)(1+p)} = 1.293372 \).

Thus, we conclude that the previous noise models are in accordance with Friis’s total noise factor formula for cascade networks.

**C Illustrations: Our noise model follows Bangera’s total noise factor expression for \( n \)-stage cascade networks**

Bangera’s total noise factor of \( n \)-stage cascade networks [48] given by,

\[
F_{T_{n}}^{\text{Cor}} = \prod_{x=1}^{n} F_{x}^{\text{Cor}}
\]

Moreover, Bangera’s formula for the stage-wise noise factor at the \( x \)-th stage or \( x \)-th step \( (F_{x}^{\text{Cor}}) \) is,

\[
F_{x}^{\text{Cor}} = 1 + \frac{N_{a(x)}}{N_{1} \prod_{j=1}^{x} M_{j} + \sum_{k=1}^{(n-1)} \{N_{a(k)} \prod_{l=k+1}^{n} M_{l} \}}
\]

Where \( N_{a(x)} \) is an externally added noise at the output of the \( x \)-th stage or \( x \)-th step. Similar to Friis conditions, the above expression for \( F_{x}^{\text{Cor}} \) is valid only when all the multiplication probabilities are unity (or the multiplication gain is deterministic). Here, if \( N_{a(x)} = 0 \), then \( F_{x}^{\text{Cor}} = 1 \) for all stages or steps. However, if the multiplication probabilities are less than unity and non-zero, \( F_{x}^{\text{Cor}} \) will be greater than 1.

Again, in the case of solid state devices such as staircase APDs, since there is no external noise added at every step (i.e., \( N_{a(x)} = 0 \)) and if all the stepwise ionization or multiplication probabilities of an \( n \)-step staircase APD are equal i.e., \( p_{1} = p_{2} = ... = p_{n} = p \); we could consider that all its stepwise excess noise factors will also be equal.

According to our new model, if ‘\( p \)’ is the ionization probability at each step, then the excess noise factor \( F(M_{S})^{\text{new}} \) of an \( n \)-step staircase APD is given by,

\[
F(M_{S})^{\text{new}} = \frac{\sum_{i=0}^{n} nC_{i}(3p)^{i}}{(1+p)^{2n}} = F_{T_{n}}^{\text{new}}
\]

The total gain of the \( n \)-step staircase APD in terms of ‘\( p \)’ will be \( M_{S}^{\text{new}} = (1+p)^{n} \).
Illustration 1: If $p = 0.3$ and $n = 1$, then substituting the values of ‘$p$’ and ‘$n$’ we get the total excess noise factor of a 1-step staircase APD as $F_{T_{1}}^{\text{new}} = 1.12426$. Since this corresponds to a staircase APD with only one step, it could be considered that the stepwise excess noise factor at step one of a 1-step staircase APD is $F_1 = F_{T_{1}}^{\text{new}} = 1.12426 = F_{T_{1}}^{\text{Cor}}$. This value matches the total excess noise factor of a 1-step staircase APD obtained using the previous models.

Illustration 2: Similarly, if $p = 0.3$ and $n = 2$, the total excess noise factor of a 2-step staircase APD will be $F_{T_{2}}^{\text{new}} = 1.26396$. Since we have considered the same values of ‘$p$’ as in the previous illustration, we get $F_2 = F_1 = 1.12426$. Therefore, substituting $F_2 = F_1 = 1.12426$ in Bangera’s equation, we get $F_{T_{2}}^{\text{Cor}} = F_1 F_2 = 1.26396$.

Illustration 3: Similarly, if $p = 0.3$ and $n = 3$, the total excess noise factor of a 3-step staircase APD will be $F_{T_{3}}^{\text{new}} = 1.42102$. Since we have considered the same values of ‘$p$’ as in the previous illustration, we get $F_3 = F_2 = F_1 = 1.12426$. Therefore, substituting $F_3 = F_2 = F_1 = 1.12426$ in Bangera’s equation, we get $F_{T_{3}}^{\text{Cor}} = F_1 F_2 F_3 = 1.42102$.

From the above illustrations, when $n \geq 2$, the total excess noise factors of $n$-step staircase APDs obtained using our model are greater than the corresponding total excess noise factors obtained using the previous models. Moreover, we conclude that our noise model follows Bangera’s total noise factor expression for cascade networks.

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