Abstract. Several results in previous works, strongly depend on the exponential tail of the linkages' distribution in our adhesive models. The purpose of this paper is to weaken this hypothesis and to allow more fat tails for large ages. From the biological point of view this means that we allow adhesions to be stronger, because linkages break less often.

Moreover, in our previous articles, the asymptotic expansion of adhesion site's position and the corresponding error estimates also used some fast decay properties of the kernel, we show, when the kernel is a given function of age but constant in time, how to overcome this problem and construct asymptotic expansions in a systematic way at any order with respect to a small parameter $\varepsilon$ representing the linkages' turnover.

Keywords. integral equation, renewal problem, asymptotic expansion, non-exponential distribution, comparison principle, Volterra equation, singular perturbation problem

MSC classification : 35B40, 45D05

1. Introduction

This article is part of a project related to the mathematical study of adhesion forces in the context of cell motility (see [1], [6], [7], [8] and [9]). Cell adhesion and migration play a crucial role in many biological phenomena such as embryonic development, inflammatory responses, wound healing and tumor metastasis.

The main motivation comes from the seminal papers [11, 13], where the authors built a complex and realistic model of the Lamellipodium. It is a cytoskeletal quasi-two-dimensional actin mesh, the whole structure propels the cell across a substrate. In these works, the authors represent the actin network through a set of 2D axi-symmetric equations whose solution models Lamellipodium’s position $x(s, t)$ evolving in time $t \in \mathbb{R}_+$ and with respect to a reference configuration $s \in (0, 1)$. Using gradient flow techniques, the authors obtain a system of non-linear equations:

\[
\begin{align*}
\kappa^B \partial_s^2(\eta \partial_s x) - \partial_s(\eta \lambda \partial_s x) + \eta A D_t x \quad & \text{bending} \\
+ \partial_s \left( \eta^2 \mu^M \arccos(\partial_s |x|) - \varphi_0 \right) \partial_s x & \text{unextensibility constraint} \\
+ \eta^2 \left( \mu^S D_t \varphi \right) x & \text{adhesion} \\
\end{align*}
\]

\[
|\partial_s x| = 1,
\]

supplemented by initial and boundary conditions that we omit for sake of conciseness.

The adhesion and stretching friction terms appearing in this force balance equation were obtained as formal limits of a microscopic description of adhesion mechanisms. For instance the first friction (boxed) term is obtained as the limit when $\varepsilon$, a small dimensionless parameter goes to zero in the expression:

\[
\frac{1}{\varepsilon} \int_{\mathbb{R}_+} (x_\varepsilon(s, t) - x_\varepsilon(s + \varepsilon a, t - \varepsilon a)) \varrho(s, a, t) da \rightarrow \mu_1(s, t) (\partial_t - \partial_s) x_0(s, t) =: \mu A D_t x
\]
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here the $a$ represents the age of a linkage established with the previous locations of the filament and the shift in the reference configuration $s + \varepsilon a$ comes from the (de)polymerization of the filaments inside the lamelipodium. The parameter $\varepsilon$ represents at the same time a characteristic age of the linkages, and the inverse of the stiffness of the elastic linkage under consideration. The elongation $(x(s, t) - x(s + \varepsilon a, t - \varepsilon a))/\varepsilon$, for a fixed $a$, is the linear elastic force exerted on $x(s, t)$ by a linkage established at $x(s + \varepsilon a, t - \varepsilon a)$ (see also [5, 12] for an extension to the non-axi-symmetric case).

Our initial goal was to give a rigorous justification of these formal computations but the non-linear space-dependent feature of the problem made it out of reach at that time. Instead we considered a simplified problem where only a single adhesion point $X_\varepsilon(t)$ was taken into account [6]. It is obtained solving a simplified force balance equation

$$
\begin{aligned}
\frac{1}{\varepsilon} \int_{\mathbb{R}_+} (X_\varepsilon(t) - X_\varepsilon(t - \varepsilon a)) \rho_\varepsilon(a, t) da &= f(t), \quad t > 0, \\
X_\varepsilon(t) &= X_p(t), \quad t \leq 0,
\end{aligned}
$$

where $f$ is an external force and the left hand side is a continuum of elastic spring forces with respect to past positions $X_\varepsilon(t - \varepsilon a)$. The density of linkages $\rho_\varepsilon$ is either a given function or it can be, for instance as in the original model [11], a solution of an age-structured problem:

$$
\begin{aligned}
(\varepsilon \partial_t + \partial_a + \zeta(a(t))) \rho_\varepsilon(a, t) &= 0, \quad (a, t) \in \mathbb{R}_+ \times (0, T), \\
\rho_\varepsilon(0, t) &= \beta(t) \left(1 - \int_{\mathbb{R}_+} \rho_\varepsilon(a, t) da\right), \quad (a, t) \in \{0\} \times (0, T), \\
\rho_\varepsilon(a, 0) &= \rho_1(a), \quad (a, t) \in \mathbb{R}_+ \times \{0\}.
\end{aligned}
$$

In the latter system, $\beta$ (resp. $\zeta \in \mathbb{R}_+$) is the kinetic on-rate (resp. off-rate) ( [6], [7], and [8]). Under the major assumption that the off-rate $\zeta$ admits a strictly positive lower bound

$$
0 < \zeta_{\min} \leq \zeta(a, t), \quad \text{a.e.} \quad (a, t) \in \mathbb{R}_+ \times \mathbb{R}_+
$$

in [6], the authors studied the asymptotic limit of the systems (1.1) and (1.2) when $\varepsilon$ goes to zero. They obtained rigorous convergence results, namely they showed that

$$
\|X_\varepsilon - X_0\|_{C^0([0, T])} + \|\rho_\varepsilon - \rho_0\|_{C^0((0, T]; L^1(\mathbb{R}_+))} \to 0,
$$

where the limit $X_0$ solves a simple force balance between the external load and a friction term,

$$
\begin{aligned}
\mu_{1, 0}(t) X_0' &= f(t), \quad t > 0, \\
X_0(t = 0) &= X_p(0), \quad t = 0,
\end{aligned}
$$

where $\mu_{1, 0}(t)$ denotes the first moment of $\rho_0$, i.e. $\mu_{1, 0}(t) := \int_0^\infty a \rho_0(a, t) da$ and $\rho_0$ satisfies the non-local problem:

$$
\begin{aligned}
\partial_a \rho_0 + \zeta(a(t)) \rho_0 &= 0, \quad t > 0, a > 0, \\
\rho_0(a = 0, t) &= \beta(t) \left(1 - \int_0^\infty \rho_0(t, \tilde{a}) d\tilde{a}\right), \quad t > 0.
\end{aligned}
$$
We underline that, if for instance \( f \) is constant or admits some limit for \( t \) large, the \( \varepsilon \) scaling provides some convergence results for long time asymptotics of (1.1) and (1.2) (see for more details [10, Theorem 4.4]).

In the previous setting, the convergence of \( \rho_\varepsilon \) towards \( \rho_0 \) stemmed form [6, Lemma 3.1 and Theorem 3.1, p. 495]. Using the very particular Lyapunov functional

\[
\mathcal{H}[u](t) := \left| \int_0^\infty u(a,t) \, da \right| + \int_0^\infty |u(a,t)| \, da, \tag{1.5}
\]

the authors obtained, setting \( \varphi(a,t) := \rho_\varepsilon(a,t) - \rho_0(a,t) \) that

\[
\varepsilon \frac{d}{dt} \mathcal{H}[\varphi](t) + \zeta_{\min} \mathcal{H}[\varphi](t) \leq o_\varepsilon(1) \tag{1.6}
\]

leading to exponential error estimates :

\[
\|\rho_\varepsilon(\cdot, t) - \rho_0(\cdot, t)\|_{L^1(\mathbb{R}^+)} \leq \exp(-t \zeta_{\min}/\varepsilon) \|\eta_1(\cdot) - \rho_0(\cdot, 0)\|_{L^1(\mathbb{R}^+)} + o_\varepsilon(1)
\]

where \( o_\varepsilon(1) \) means small when \( \varepsilon \) is small. The exponential nature of the initial layer is closely related to the strictly positive lower bound \( \zeta_{\min} \). This convergence result implies that, for \( \varepsilon \) small enough, \( \rho_\varepsilon \) is a perturbation of an exponentially decreasing profile in age. From the modelling point of view, the exponential tail of \( \rho_\varepsilon \) is of importance since it restricts the linkages to be of very short range in (1.1). On the mathematical side, if instead \( \zeta \) is non-negative but does not admit a lower bound this convergence result is to our knowledge an open question. Indeed, the Lyapunov functional then provides only stability; no convergence result is at hand since no lower-bound on the dissipation exists in (1.6). This work intends to fill this gap. Namely assuming that there exists a non-increasing positive function \( m \in L^1(\mathbb{R}^+) \), such that

\[
0 \leq -\frac{m'(a)}{m(a)} \leq \zeta(t,a) \leq \zeta_{\max}, \quad \text{a.e. } (a,t) \in \mathbb{R}^+ \times \mathbb{R}^+
\]

we show that it is again possible to establish convergence of \( \rho_\varepsilon \) towards \( \rho_0 \). For instance let’s examin some particular choices of \( m \). Choosing \( m(a) := \exp(-\zeta_{\min}a) \), one recovers the exponential case previously studied, whereas if \( m(a) := (1+a)^{-\sigma} \) for \( \sigma > 0 \), allows \( \zeta \) to tend to zero as \( a \) grows.

This new convergence result is made possible thanks to a higher order asymptotic expansion of \( \rho_\varepsilon \) which includes a first order far field term \( \rho_1 \) solution of :

\[
\begin{align*}
(\partial_a + \zeta(a,t)) \rho_1(a,t) &= -\partial_t \rho_0(a,t), & a > 0, t > 0, \\
\rho_1(0,t) &= -\beta(t) \int_{\mathbb{R}^+} \rho_1(a,t) \, da, & a = 0, t > 0,
\end{align*}
\]

and the initial layer (of order zero) \( \tau_0 \), solving :

\[
\begin{align*}
(\partial_t + \partial_a + \zeta(a,0)) \tau_0(a,t) &= 0, & a > 0, t > 0, \\
\tau_0(0,t) &= -\beta(0) \int_{\mathbb{R}^+} \tau_0(a,t) \, da, & a = 0, t > 0, \\
\tau_0(a,0) &= \rho_1(a) - \rho_0(a,0), & a > 0, t = 0.
\end{align*}
\]

We study finely the decay properties of this initial layer. First, we characterize in a new way the behavior of the trace \( \tau_0(0,t) \) in some weighted \( L^1(\mathbb{R}^+; (1+t)^\sigma) \)-space. As the
trace solves a convolution Volterra equation of the second kind, the concept of resolvent appears naturally [4, Chapters 2, 4, 7, 9] and its specific properties are used there. These results combined with Duhamel’s formula allow to show some stability of $r_0$ in other weighted $L^1$-spaces in age and in time (see Section 2.1.2 for more precise statements). By increasing the order of the asymptotic expansion, we can use the a priori estimates above, without dissipation, and still prove a convergence theorem. Indeed, for any $t > 0$, one has

$$\mathcal{H}[\rho_\varepsilon(\cdot, t) - \rho_0(\cdot, t) - \varepsilon \rho_1(\cdot, t) - v_0(\cdot, t/\varepsilon)] \lesssim o(1),$$

and the microscopic features of the initial layer (using again Section 2.1.2) allow then to recover:

$$\|\rho_\varepsilon - \rho_0\|_{L^1(\mathbb{R}_+ \times (0,T), (1+a))} \leq o(1).$$

In [6, Theorem 1.1], the convergence of $X_\varepsilon$ towards $X_0$ is also established through error estimates thanks to a comparison principle specific to non-negative Volterra non-convolution kernels [4, Chap. 9]. In the proof of this theorem, again a restrictive hypothesis appears on $\zeta$ [6, Assumption 1.1 (i) p. 487] as well. Indeed, at some point in the proof of [6, Theorem 1.1], one needs to control the magnitude of

$$A_\varepsilon[\rho_\varepsilon](t) := \frac{\int_{\mathbb{R}_+} a \varphi(a + t/\varepsilon, t) da}{\int_{\mathbb{R}_+} \varphi(a + t/\varepsilon, t) da}$$

of [6, Lemma 2.4, p. 491-492], this is possible if $\zeta(a + t/\varepsilon, t)$ is increasing for $a > a_0$ where $a_0 > 0$ is given, for a.e. fixed $t > 0$. Using weak-* convergence arguments from [7] on the elongation problem, we prove convergence of $X_\varepsilon$ without any restriction related to $\zeta$.

Nevertheless, if we still need to obtain error estimates on $X_\varepsilon$, in the case where $\varphi$ is a given non-negative kernel constant in time, we show, in a systematic way, how to construct an asymptotic expansion with respect to $\varepsilon$ of $X_\varepsilon$ and obtain error estimates. The novelty here is that we only use moments of $\varphi$, namely that for a fixed integer $N \geq 1$,

$$\int_{\mathbb{R}_+} (1+a)^{N+2} \varphi(a) da < +\infty.$$  

We construct a $N^{th}$-order asymptotic approximation

$$\overline{X}_\varepsilon,N = X_O(t) + X_I(\tau) + O(\varepsilon^N),$$

where $X_O$ (resp. $X_I$) represent a far field (resp. near field) Ansatz and $\tau = t/\varepsilon$ is the stretched variable. Then we prove that

$$\|X_\varepsilon - \overline{X}_\varepsilon,N\|_{C([0,T])} \lesssim \varepsilon^N.$$  

The key point is that we avoid the need of evaluating ratios such as $A_\varepsilon[\rho_\varepsilon](t)$ by a refined analysis of signed initial layers (see the proof of Theorem 5 and Remark 1 below).

The paper is structured as follows: in section 2, we analyze the asymptotic behavior of $(\rho_\varepsilon, X_\varepsilon)$ solutions of the weakly coupled problem (1.2)-(1.1) when $\varepsilon$ tends to 0. In section 3, we analyze the case where the kernel in (1.1) depends on the age variable only but does not solve any specific equation. Moreover, we construct the asymptotic expansion of $X_\varepsilon$ and show, under regularity hypotheses on the data, error estimates up to any order.
2. The weakly coupled case

We refer to [14] for a general framework for BV functions, our notations being coherent with this reference.

**Assumptions 1. Assume the kernel \( \rho_e \) solves (1.2) and that the data satisfies:**

a) the off-rate \( \zeta \) is in \( C([0,T];L^\infty(\mathbb{R}_+)) \) and there exists a non-increasing \( m \in L^1(\mathbb{R}_+;(1+a)^3) \) such that

\[
-\frac{m'(a)}{m(a)} \leq \zeta(a,t) \leq \zeta_{\max}, \quad \text{a.e. } a \in \mathbb{R}_+.
\]

b) the birth-rate \( \beta \in C(\mathbb{R}_+) \) is such that

\[
0 < \beta_{\min} \leq \beta(t) \leq \beta_{\max}.
\]

c) the initial condition \( \rho_1 \in BV(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+,(1+a)^3) \) is such that there exists \( C_{\rho_1} > 0 \) so that

\[
\rho_1(a) \leq C_{\rho_1} m(a), \quad \text{for a.e. } a \in \mathbb{R}_+.
\]

Whereas \( X_e \) solves (1.1) and the data satisfy

d) the source term \( f \) belongs to \( C^2(\mathbb{R}_+) \),

e) the past data \( X_p \in L^\infty(\mathbb{R}_-) \cap Lip(\mathbb{R}_-) \).

2.1. Asymptotic expansion of linkages

We introduce a first order asymptotic approximation of \( \rho_e \) solving (1.2) :

\[
\overline{\rho}_e(a,t) := \rho_0(a,t) - \tilde{r}_0(a,t) - \varepsilon \rho_1(a,t) \tag{2.1}
\]

where \( \rho_0 \) is the zeroth order macroscopic limit solving (1.4), \( \rho_1 \) is the first order macroscopic profile given when solving

\[
\begin{align}
(\partial_t + \partial_a + \zeta(a,0)) \rho_1(a,t) &= -\partial_t \rho_0(a,t), \quad a > 0, t > 0, \\
\rho_1(0,t) &= -\beta(t) \int_{\mathbb{R}_+} \rho_1(a,t) da, \quad a = 0, t > 0, 
\end{align} \tag{2.2}
\]

and \( \tilde{r}_0(a,t) := r_0(a,t/\varepsilon) \) is the (rescaled) initial layer \( r_0 \) solving

\[
\begin{align}
(\partial_t + \partial_a + \zeta(a,0)) r_0(a,t) &= 0, \quad a > 0, t > 0, \\
r_0(0,t) &= -\beta(0) \int_{\mathbb{R}_+} r_0(a,t) da, \quad a = 0, t > 0, \\
r_0(a,0) &= \rho_1(a) - \rho_0(a,0), \quad a > 0, t = 0. \tag{2.3}
\end{align}
\]

2.1.1. Decay properties of outer expansions

**Proposition 2.1.** Let Assumptions 1 hold, then there exists generic constants \( c_j > 0 \) and \( c_{j,1} > 0 \), such that \( \rho_0 \) (resp. \( \rho_1 \)) solution of (1.4) (resp. (2.2)) satisfies

\[
|\rho_j(a,t)| \leq c_j (1+a)^{2j} m(a), \quad \text{for } j \in \{0,1\}.
\]

In a generic way, for all \( j \in \{0,1\} \) and \( k \in \mathbb{N} \), if \( \zeta \in W^{k,\infty}(\mathbb{R}_+ \times \mathbb{R}_+) \) then :

\[
|\partial_{t^k_\zeta} \rho_j(a,t)| \leq c_{j,k} (1+a)^{2j+k} m(a).
\]

The proof uses equations (1.4) and (2.2) together with Assumptions 1 and is postponed to Appendix B.
2.1.2. A complete study of the initial layer

One considers the problem (2.3) and defines \( x(t) := \tau_0(0, t) \), then by using Duhamel’s principle (2.3) can be rewritten as

\[
\begin{cases}
x + k \ast x = b, \\
k(a) := \beta(0) \exp \left( - \int_0^a \zeta(\tau, 0) d\tau \right), \\
b(t) := -\beta(0) \int_t^\infty \tau_0(a - t, 0) \exp \left( - \int_a^{a-t} \zeta(\tau, 0) d\tau \right) da.
\end{cases}
\]  

(2.4)

As a consequence of the Paley-Wiener theorem [4, Theorem 4.1] and the fact that \( k \) is a decreasing function of \( a \), [4, p.264]:

**Theorem 1.** If \( k \) is a decreasing non-negative kernel such that \( k \in L^1(\mathbb{R}_+) \), then the resolvent associated to (2.3) satisfies : \( r + r \ast k = k \) and \( r \in L^1(\mathbb{R}_+) \). Moreover, the solution is unique, explicit and reads :

\[
x = b - r \ast b
\]

**Proposition 2.2.** Let Assumptions 1 hold. If moreover, \( \tau_0(\cdot, 0) \in L^1(\mathbb{R}_+, (1+a)^2) \) and there exists a constant \( C_{\tau_0} > 0 \) such that

\[
\tau_0(a, 0) \leq C_{\tau_0} m(a),
\]

then

i) \( x = \tau_0(0, \cdot) \in L^1_1(\mathbb{R}_+; (1+t)^2) \cap L^\infty(\mathbb{R}_+) \),

ii) \( \tau_0 \in L^1(\mathbb{R}_+ \times \mathbb{R}_+; (1+a)) \),

iii) there exists another constant \( c' > 0 \) such that

\[
\tau_0(a, t) \leq c' m(a).
\]

**Proof.** By standard fixed point arguments [6, Theorem 2.1, p. 497], there exists a unique solution \( \tau_0 \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)) \), Using [4, Theorem 2.3.5 p. 44], as \( b \in C(\mathbb{R}_+) \), so is \( x \). Using the Assumption 1.a) on \( \zeta \) and on the data, the source term \( b \) in (2.4) satisfies :

\[
|b(t)| \leq \beta \int_t^\infty \tau_0(a - t, 0) \frac{m(a)}{m(a - t)} da \leq \beta C_{\tau_0} \int_t^\infty m(a) da
\]

which is bounded since \( m \in L^1(\mathbb{R}_+) \) and it is also integrable because so is the first moment of \( m \). Writing then that

\[
x = b - r \ast b
\]

and since \( L^1 \) is an algebra for the convolution, \( x \in L^1(\mathbb{R}_+) \). Because, \( b \) is bounded and \( r \in L^1(\mathbb{R}_+) \), \( x \in L^\infty(\mathbb{R}_+) \) as well. This proves partially ii), the part concerning higher moment with respect to time is postponed to the end of the proof.

Then in a similar way, using Duhamel’s principle, one has :

\[
\tau_0(a, t) := \begin{cases}
\tau_0(0, t-a) \exp \left( - \int_0^{a} \zeta(\tilde{a}, 0) d\tilde{a} \right), & \text{if } t > a, \\
\tau_0(a-t, 0) \exp \left( - \int_a^{a-t} \zeta(\tilde{a}, 0) d\tilde{a} \right), & \text{otherwise ,}
\end{cases}
\]

so that

\[
|\tau_0(a, t)| \leq \begin{cases}
\|x\|_{L^\infty} \frac{m(a)}{m(0)}, & \text{if } t > a, \\
\frac{\tau_0(a-t, 0)}{m(a-t)} m(a) \leq C_{\tau_0} m(a), & \text{otherwise .}
\end{cases}
\]
which gives \( e' \) in the last estimates of the claim iii). Next we consider:

\[
\int_{\mathbb{R}_+} |\tau_0(a,t)| da \leq \int_0^t |\tau_0(0,t-a)| \exp \left( - \int_0^a \zeta(\tilde{a},0) d\tilde{a} \right) da \\
+ \int_t^\infty |\tau_0(a-t,0)| \exp \left( - \int_a^\infty \zeta(\tilde{a},0) d\tilde{a} \right) da \\
\leq \int_0^t |\tau_0(0,t-a)| \frac{m(a)}{m(0)} da + \int_t^\infty \tau_0(a-t,0) \frac{m(a)}{m(a-t)} da \\
= \frac{1}{m(0)} \left\{ (|\tau_0(0,\cdot)| \ast m)(t) + \int_{\mathbb{R}_+} |\tau_0(\tilde{a},0)| \frac{m(\tilde{a})}{m(\tilde{a}+t)} d\tilde{a} \right\} \\
= \frac{1}{m(0)} (|x| \ast m)(t) + C_{r_0} \int_t^\infty m(a) da
\]

Then using that \( L^1(\mathbb{R}_+) \) is an algebra for the convolution, and that \( m \in L^1(\mathbb{R}_+, (1+a)) \), one concludes that

\[
\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \tau_0(a,t) da dt < \frac{\|m\|_{L^1(\mathbb{R}_+, (1+a))}}{m(0)} \left( \|\tau_0(0,\cdot)\|_{L^1(\mathbb{R}_+)} + C_{r_0} m(0) \right)
\]

the same holds for the first moment as well. This proves ii).

Setting \( q_1(a,t) := t\tau_0(a,t) \), it solves

\[
\begin{align*}
(\partial_t + \partial_a + \zeta(a,0)) q_1(a,t) &= \tau_0(a,t) \\
q_1(0,t) &= -\beta(0) \int_{\mathbb{R}_+} q_1(\tilde{a},t) d\tilde{a} \\
q_1(a,0) &= 0
\end{align*}
\]

in the sense of characteristics [6, Lemma 2.1, p. 489]. Defining the trace \( y(t) := q_1(0,t) \), and using Duhamel’s principle, it can be shown that \( y \) solves:

\[
y(t) = -\beta(0) \int_0^t y(t-a) \exp \left( - \int_0^a \zeta(s,0) ds \right) da \\
- \beta(0) \int_0^t ax(t-a) \exp \left( - \int_0^a \zeta(s,0) ds \right) da \\
- \beta \int_t^\infty t\tau_0(a-t,0) \exp \left( - \int_{a-t}^a \zeta(s) ds \right) da =: -(k \ast y)(t) + b_x(t)
\]

i.e. \( y + k \ast y = b_x \). Assuming that \( \int_{\mathbb{R}_+} (1+a)^2 m(a) da < \infty \) shows that \( b_x \in L^1(\mathbb{R}_+) \). Then as above,

\[
\int_0^t |q_1(a,t)| da \leq \int_0^t (|q_1(0,t-a)| + a|x(t-a)|) \exp \left( - \int_0^a \zeta(s,0) ds \right) da \\
\leq C |q_1(0,\cdot)| \ast m + |x(0,\cdot)| \ast (\cdot) m(\cdot)
\]

together with

\[
\int_t^\infty |q_1(a,t)| da \leq t \int_t^\infty \tau_0(a-t,0) \exp \left( - \int_{a-t}^a \zeta(\tau,0) d\tau \right) da \leq tC_{r_0} \int_t^\infty m(a) da
\]
both left-hand sides are then $L^1_t(\mathbb{R}_+)$ functions in time, provided that $m \in L^1_a(\mathbb{R}_+,(1 + a)^2)$. For the next step, one works similarly and obtains that 
\[
\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} t^2 r_0(a,t) da dt < \infty
\]
since $m$ is in $L^1_a(\mathbb{R}_+,(1 + a)^2)$. This finishes the proof of i). \[ \square \]

**Corollary 1.** Under the Assumptions 1, one has $x(t) := r_0(0,t) \to 0$ when $t$ grows large and the first moment can be estimated as follows:
\[
\left| \int_{\mathbb{R}_+} (1 + a) |r_0(0,t)| da \right| \leq o(1) + c \int_{\mathbb{R}_+} (1 + a) m(a) da,
\]
where $o(1)$ denotes small when $t$ grows large.

**Proof.** Using Lyapunov’s functional (1.5), one has that 
\[
\int_{\mathbb{R}_+} |r_0(a,t)| da \leq \infty
\]
which, thanks to the boundary condition, provides that $x(t)$ is bounded on $\mathbb{R}_+$. This shows using Duhamel’s principle that $r_0 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$. Moreover, defining the discrete differences
\[
D^h_t r_0(a,t) := \frac{r_0(a,t+h) - r_0(a,t)}{h}
\]
solve the problem:
\[
\begin{cases}
(\partial_t + \partial_a + \zeta(a,0)) D^h_t r_0 = 0, & a > 0, t > 0, \\
D^h_t r_0(0,t) = -\beta(0) \int_{\mathbb{R}_+} D^h_t r_0(a,t) da, & a = 0, t > 0,
\end{cases}
\]
(2.5)

Thanks to Assumptions 1 and Proposition 2.1, one shows that $r_0(\cdot,0) := (\rho_f(\cdot) - \rho_0(\cdot,0)) \in BV(\mathbb{R}_+)$. Moreover, one has that
\[
|D^h_t x(t)| = |D^h_t r_0(0,t)| \leq \beta(0) \int_{\mathbb{R}_+} |D^h_t r_0(a,t)| da \leq \beta(0) \mathcal{H}[D^h_t r_0(a,t)]
\]
similarly as in [6, Lemma 3.1 p. 493, and Lemma 3.3, p. 495], we obtain that, for all $t > 0$, 
\[
\mathcal{H}[D^h_t r_0(\cdot,t)] \leq \mathcal{H}[D^h_t r_0(\cdot,0)].
\]
It suffices then to prove that the initial term $\mathcal{H}[D^h_t r_0(\cdot,0)]$ is bounded. Indeed, we have
\[
\mathcal{H}[D^h_t r_0(a,0)] = \int_{\mathbb{R}_+} \left| \frac{r_0(a,h) - r_0(a,0)}{h} \right| da + \left| \frac{\mu_{r_0}(h) - \mu_{r_0}(0)}{h} \right| = I_1 + I_2
\]
where $\mu_{r_0}(t) := \int_{\mathbb{R}_+} r_0(a,t) da$. For the first term, we split the integral in two parts
\[
I_1 = \int_0^h \left| \frac{r_0(a,h) - r_0(a,0)}{h} \right| da + \int_h^{\infty} \left| \frac{r_0(a,h) - r_0(a,0)}{h} \right| da
\]
\[
= \frac{1}{h} \int_0^h \left| r_0(0,h-a) \exp \left( - \int_0^a \zeta(\bar{a},0)d\bar{a} \right) - r_0(a,0) \right| da
\]
\[
+ \frac{1}{h} \int_h^{\infty} \left| r_0(a-h,0) \exp \left( - \int_{a-h}^a \zeta(\bar{a},0)d\bar{a} \right) - r_0(a,0) \right| da =: I_{1,1} + I_{1,2}
\]
where we used Duhamel’s principle. It easy to see that the first term

\[ I_{1,1} \lesssim \frac{1}{h} \int_0^h \left( |\beta(0)| \left\| \tau_0 \right\|_{L^\infty_t L^1_x} + \left\| \tau_0(a,0) \right\|_{L^\infty(\mathbb{R}_+)} \right) < \infty. \]

Concerning \( I_{1,2} \), one splits the integral adding and subtracting intermediate terms

\[ I_{1,2} \leq \frac{1}{h} \int_h^{+\infty} \left( \tau_0(a-h,0) - \tau_0(a,0) \right) \exp \left( - \int_{a-h}^a \zeta(\tilde{a},0) \tilde{a} \right) |\tau_0(a,0)| \, da \]

\[ + \frac{1}{h} \int_h^{+\infty} \left| \tau_0(a,0) \exp \left( - \int_{a-h}^a \zeta(\tilde{a},0) \tilde{a} \right) - \tau_0(a,0) \right| |\tau_0(a,0)| \, da \]

\[ \lesssim TV(\tau_0(\cdot,0)) + \zeta_{\text{max}} \left\| \tau_0(a,0) \right\|_{L^1(\mathbb{R}_+)} \]

where \( TV \) denotes total variation of \( \tau_0(\cdot,0) \) \[2\]. For the second term \( I_2 \) noting that

\[ |\partial_t \mu(\tau_0(t))| \leq (\zeta_{\text{max}} + \beta_{\text{max}}) \left\| \tau_0 \right\|_{L^\infty_t L^1_x}, \]

one obtains

\[ I_2 \leq (\zeta_{\text{max}} + \beta_{\text{max}}) \left\| \tau_0 \right\|_{L^\infty_t L^1_x}, \]

and finally, we obtain that \( \mathcal{H}[D^t \tau_0(a,t)] < \infty \), for all \( t > 0 \), which shows that \( x \in \text{Lip}(\mathbb{R}_+) \).

Since \( x \in L^1(\mathbb{R}_+) \) (see Proposition 2.2), this implies that \( \lim_{t \to +\infty} x(t) = 0 \). Now, we consider

\[ J(t) := \int_0^t (1+a) |\tau_0(a,t)| \, da \leq \int_0^t |x(t-a)| (1+a)m(a) \, da \]

\[ = \left( \int_0^{t/2} + \int_{t/2}^t \right) |x(t-a)| (1+a)m(a) \, da =: J_1 + J_2. \]

For every \( \delta > 0 \) there exists \( \eta_1 \) such that \( t > \eta_1 \), implying that

\[ \sup_{s \in (\frac{t}{2},t)} |x(s)| < \frac{\delta}{2 \int_{\mathbb{R}_+} (1+a)m(a) \, da} \]

which shows that \( J_1 < \delta/2 \). On the other hand, there exists \( \eta_2 \) such that \( t > \eta_2 \) which implies that

\[ J_2 \leq \left\| x \right\|_{L^\infty(\mathbb{R}_+)} \int_{t/2}^t (1+a)m(a) \, da < \delta/2, \]

by Lebesgue’s Theorem (since the integral of \( (1+a)m(a) \) is finite). These arguments show that \( J(t) \) vanishes when \( t \) grows large. On the other hand, from Proposition 2.2 iii) :

\[ \int_t^\infty (1+a) |\tau_0(a,t)| \, da \leq c' \int_t^\infty (1+a)m(a) \, da, \]

which is an initial layer. □
2.1.3. Error estimates for the linkage’s density  

We define the difference

$$\hat{\rho}_\varepsilon (a,t) := \rho_\varepsilon (a,t) - \bar{\rho}_\varepsilon (a,t)$$

where $\bar{\rho}_\varepsilon$ is defined by (2.1). We obtain:

**Theorem 2.** Under Assumptions 1, one has that

$$\mathcal{H} [\hat{\rho}_\varepsilon ] (t) \leq o_\varepsilon (1), \quad \text{a.e. } t \in \mathbb{R}_+$$

where $\mathcal{H} [u] (t) := \int_{\mathbb{R}_+} |u(a)| da + \left| \int_{\mathbb{R}_+} u(a) da \right|$ and $o_\varepsilon (1)$ means small when $\varepsilon$ is small.

**Proof.** We write the system satisfied by $\hat{\rho}_\varepsilon$:

$$\begin{align*}
(\varepsilon \partial_t + \partial_a + \zeta(a,t))\hat{\rho}_\varepsilon (a,t) &= -\varepsilon (\zeta(a,t) - \zeta(a,0)) \bar{\tau}_0 (a,t) - \varepsilon^2 \partial_t \rho_1 \\
\hat{\rho}_\varepsilon (0, t) &= -\beta(t) \int_{\mathbb{R}_+} \rho_\varepsilon (a,t) da - (\beta(t) - \beta(0)) \int_{\mathbb{R}_+} \bar{\tau}_0 (a,t) da \\
\hat{\rho}_\varepsilon (a,0) &= -\varepsilon \rho_1 (a,0)
\end{align*}$$

(2.7)

Then following the same steps as in [6, Lemma 3.1 and 3.3, p. 493-495], one has that

$$\varepsilon d \mathcal{H} [\hat{\rho}_\varepsilon ] (t) + \int_{\mathbb{R}_+} \zeta(a,t) \left\{ |\hat{\rho}_\varepsilon (a,t)| + \hat{\rho}_\varepsilon (a,t) \text{sgn} \left( \int_{\mathbb{R}_+} \hat{\rho}_\varepsilon (a,t) da \right) \right\} da
\leq 2\varepsilon^2 \int_{\mathbb{R}_+} |\partial_t \rho_1 (a,t)| da + 2 \int_{\mathbb{R}_+} |\zeta(a,t) - \zeta(a,0)| |\bar{\tau}_0 (a,t)| da
+ |\beta(t) - \beta(0)| \int_{\mathbb{R}_+} |\bar{\tau}_0 (a,t)| da$$

which after integration in time provides:

$$\mathcal{H} [\hat{\rho}_\varepsilon ] (t) \leq \mathcal{H} [\hat{\rho}_\varepsilon ] (0) + 2\varepsilon \int_0^t \int_{\mathbb{R}_+} |\partial_t \rho_1 (a,s)| da ds + 2 \int_0^t \int_{\mathbb{R}_+} |\zeta(a,\varepsilon \tilde{t}) - \zeta(a,0)| |\tau_0 (a,\tilde{t})| d\tilde{t} + 2 \int_0^t |\beta(\varepsilon \tilde{t}) - \beta(0)| \int_{\mathbb{R}_+} |\tau_0 (a,\tilde{t})| d\tilde{t}.$$

Now, here the crucial point is that, thanks to Lebesgue’s Theorem, the last two terms of the right-hand side do tend to zero as $\varepsilon$ goes to zero.

Thanks to the study of initial layer’s properties from Section 2.1.2, one can recover a sharper result on the zeroth order expansion of $\rho_\varepsilon$ estimating the contribution of higher order terms.

**Corollary 2.** Let Assumptions 1 hold, then one has

$$\int_{\mathbb{R}_+} (1+a) |\rho_\varepsilon (a,t) - \rho_0 (a,t)| da \leq o_\varepsilon (1) + \int_{\mathbb{R}_+} (1+a) m(a) da.$$

**Proof.** Considering the system solved by the difference $e(a,t) := \rho_\varepsilon (a,t) - \rho_0 (a,t)$

$$\begin{align*}
(\varepsilon \partial_t + \partial_a + \zeta(a,t))e(a,t) &= \varepsilon \partial_t \rho_0 (a,t), \quad a > 0, t > 0, \\
e(0,t) &= -\beta(t) \int_{\mathbb{R}_+} (\rho_\varepsilon (\tilde{a},t) - \rho_0 (\tilde{a},t) - \bar{\tau}_0 (\tilde{a},t)) da \\
-\beta(t) \int_{\mathbb{R}_+} \bar{\tau}_0 (a,t) da, & \quad a = 0, t > 0, \\
e(a,0) &= \rho_1 (a) - \rho_0 (a,0), \quad a > 0, t = 0.
\end{align*}$$

(2.8)
It satisfies (2.8) in the sense of characteristics [6, Lemma 2.1 and 3.1], namely

\[
\begin{aligned}
e(a,t) = e(0, t - \varepsilon a) \exp(- \int_{-a}^{0} \zeta(a + s, t + \varepsilon s) \, ds) + \\
&+ \varepsilon \int_{-a}^{0} \partial_t \rho_0(a + s, t + \varepsilon s) \exp(- \int_{s}^{0} \zeta(a + \tau, t + \varepsilon \tau) \, d\tau) \, ds, \quad a < t/\varepsilon, \\
&\quad e(a-t/\varepsilon,0) \exp(- \int_{-t/\varepsilon}^{0} \zeta(a + s, t + \varepsilon s) \, ds) + \\
&\quad + \varepsilon \int_{-t/\varepsilon}^{0} \partial_t \rho_0(a + s, t + \varepsilon s) \exp(- \int_{s}^{0} \zeta(a + \tau, t + \varepsilon \tau) \, d\tau) \, ds, \quad a > t/\varepsilon.
\end{aligned}
\]

(2.9)

One has that

\[
\int_{0}^{+\infty} (1+a)e(a,t)\, da = \int_{0}^{t/\varepsilon} (1+a)e(a,t)\, da + \int_{t/\varepsilon}^{+\infty} (1+a)e(a,t)\, da := I_1 + I_2.
\]

(2.10)

We treat each term separately because they correspond to two cases distinguished in (2.9):

\[
I_1(t) \leq \int_{0}^{t/\varepsilon} (1+a)|e(0, t - \varepsilon a)|\exp\left(- \int_{-a}^{0} \zeta(a + s, t + \varepsilon s) \, ds\right) \, da \\
+ \varepsilon \int_{0}^{t/\varepsilon} (1+a) \int_{-a}^{0} \partial_t \rho_0(a + s, t + \varepsilon s) \exp(- \int_{s}^{0} \zeta(a + \tau, t + \varepsilon \tau) \, d\tau) \, ds \, da.
\]

Using Theorem 2 and Corollary 1, one has that:

\[
|e(0, t - \varepsilon a)| \leq \alpha(1) + \int_{-a}^{\infty} m(a) \, da,
\]

which thanks to Proposition 2.1 gives that

\[
I_1(t) \leq \alpha(1) \int_{0}^{t/\varepsilon} (1+a)m(a) \, da + \int_{0}^{t/\varepsilon} (1+a)m(a) \int_{-a}^{\infty} m(\tilde{\alpha}) \, d\tilde{\alpha} \, da \\
+ \varepsilon c_{0,1} \int_{\mathbb{R}^+} (1+a)^3 m(a) \, da,
\]

using similar argument as in the proof of Corollary 1, one shows that

\[
\int_{0}^{t/\varepsilon} (1+a)m(a) \, da \int_{-a}^{\infty} m(\tilde{\alpha}) \, d\tilde{\alpha} \, da \leq \alpha(1),
\]

since \((1+a)m(a)\) is integrable and by Lebesgue’s Theorem \(\int_{t}^{\infty} m(a) \, da \to 0\) as \(t\) grows large. On the other hand,

\[
I_2 \leq \int_{t/\varepsilon}^{\infty} \big|e\big(a - \frac{t}{\varepsilon}, 0\big)\big| \exp\left(- \int_{-a}^{0} \zeta(a + s, t + \varepsilon s) \, ds\right) \, da \\
+ \varepsilon \int_{t/\varepsilon}^{\infty} (1+a) \int_{-a}^{0} |\partial_t \rho_0(a + s, t + \varepsilon s)| \exp\left(- \int_{s}^{0} \zeta(a + \tau, t + \varepsilon \tau) \, d\tau\right) \, ds \, da \\
\leq c \int_{t/\varepsilon}^{\infty} (1+a)m(a) \, da + \varepsilon \int_{\mathbb{R}^+} (1+a)^3 m(a) \, da.
\]
2.2. Convergence results for the position

**Theorem 3.**
Under Assumptions 1, if $\rho_\varepsilon$ is a solution of (1.2), $\rho_0$ solves (1.4), $X_\varepsilon$ is a solution of (1.1) and $X_0$ solves (3.5) then

$$\|X_\varepsilon - X_0\|_{C([0,T])} \leq o_\varepsilon(1).$$

**Proof.** Setting $u_\varepsilon(a,t):=(X_\varepsilon(t) - X_\varepsilon(t-\varepsilon a))/\varepsilon$ where $X_\varepsilon(t) = X_p(t)$ when $t < 0$, $u_\varepsilon$ solves in a weak sense [7]:

$$\begin{cases}
(\varepsilon T + \partial_n)u_\varepsilon = \partial_t X_\varepsilon = \frac{1}{\mu_{0,\varepsilon}} \left\{ \varepsilon \partial_t f + \int_{\mathbb{R}_+} u_\varepsilon(a,t) \zeta(a,t) \rho_\varepsilon(a,t) da \right\}, & a > 0, t > 0, \\
u_\varepsilon(0,t) = 0, & a = 0, t > 0, \\
u_\varepsilon(a,0) = u_I(a) := \frac{X_\varepsilon(0) - X_p(-\varepsilon a)}{\varepsilon}, & a > 0, t = 0,
\end{cases}$$

(2.11)

since problem (1.1) can be expressed in a integro-differential equation:

$$\mu_{0,\varepsilon}(t) \partial_t X_\varepsilon = \varepsilon \partial_t f + \int_{\mathbb{R}_+} \left( \frac{X_\varepsilon(t) - X_\varepsilon(t-\varepsilon a)}{\varepsilon} \right) \zeta(a,t) \rho_\varepsilon(a,t) da.$$

Following [7, Theorem 6.1], one has that

$$\int_{\mathbb{R}_+} |u_\varepsilon(a,t)| \rho_\varepsilon(a,t) da \leq \int_0^t |\partial_t f(\tau)| d\tau + |f(0)| + L X_p \mu_{1,\max} =: c_1$$

Moreover, one has also the bound:

$$\|\partial_t X_\varepsilon\|_{L^\infty(0,T)} \leq \frac{1}{\mu_{0,\min}} \left\{ \varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \zeta_{\max} c_1 \right\} =: c_2$$

which provides thanks to Ascoli-Arzella that there exists a converging sub-sequence $X_\varepsilon$ in $C([0,T])$. Moreover, it $t > \varepsilon a$, using Duhamel’s principle,

$$|u_\varepsilon(a,t)| \leq \frac{1}{\varepsilon} \int_{t-\varepsilon a}^t |\partial_t X_\varepsilon(\tau)| d\tau \leq c_2 a$$

whereas if $t < \varepsilon a$, by similar arguments,

$$|u_\varepsilon(a,t)| \leq \frac{t}{\varepsilon} c_2 + \left| \frac{f(0)}{\mu_{0,\min}} \right| + L X_p \frac{\mu_{1,\max}}{\mu_{0,\min}}$$

Thus, $u_\varepsilon(a,t)/(1+a) \in L^\infty(\mathbb{R}_+ \times (0,T))$ uniformly with respect to $\varepsilon$. These results provide that $u_\varepsilon$ weak-* converges in $L^\infty(\mathbb{R}_+ \times (0,T); (1+a)^{-1})$ to $u_0$ a weak solution of

$$\begin{cases}
\partial_n u_0 = \partial_t X_0 = \frac{1}{\mu_{0,0}} \int_{\mathbb{R}_+} u_0(a,t) \zeta(a,t) \rho_0(a,t) da \\
u_0(0,t) = 0
\end{cases}$$

(2.12)

which shows that $u_0(a,t) = a X_0(t)$. Since (1.1) reads as

$$A_\varepsilon := \int_0^T \int_{\mathbb{R}_+} u_\varepsilon(a,t) \rho_\varepsilon(a,t) da \varphi(t) dt = \int_0^T f(t) \varphi(t) dt, \quad \forall \varphi \in L^1(0,T),$$
ans since \( \partial_t X_\varepsilon \rightharpoonup \partial_t X_0 \) weak-* in \( L^\infty(0,T) \) together with
\[
\rho_\varepsilon \to \rho_0 \text{ in } L^1(\mathbb{R}_+ \times (0,T);(1+a)), \quad u_\varepsilon \rightharpoonup u_0 \text{ in } L^\infty(\mathbb{R}_+ \times (0,T);(1+a)^{-1})
\]
one concludes that
\[
A_\varepsilon \to \int_0^T \int_{\mathbb{R}_+} u_0(a,t)\rho_0(a,t)da\varphi(t)dt = \int_0^T \mu_{0,1}(t)\partial_t X_0(t)\varphi dt = \int_0^T f(t)\varphi(t)dt.
\]

\[\square\]

**3. The linkages’ density is a generic function constant in time**

In this section, we consider problem (1.1) with a given kernel \( \rho \) constant in time. It is not supposed to solve any particular problem, what only matters are its moments with respect to \( a \). Define the delay operator \( \mathcal{L}_\varepsilon \) as
\[
\mathcal{L}_\varepsilon[X](t) := \frac{1}{\varepsilon} \left\{ \mu_0 X(t) - \int_0^t X(t-\varepsilon a)\varrho(a)da \right\}.
\]
Then problem (1.1) can be rephrased as :
\[
\mathcal{L}_\varepsilon[X_\varepsilon](t) = f(t) + \frac{1}{\varepsilon} \int_0^\infty X_p(t-\varepsilon a)\varrho(a)da, \quad \forall t > 0
\]
and we aim at constructing an asymptotic expansion \( \mathbf{X}_\varepsilonN \) such that it satisfies
\[
\mathcal{L}_\varepsilon[\mathbf{X}_\varepsilonN] = f(t) + \frac{1}{\varepsilon} \int_0^\infty X_p(t-\varepsilon a)\varrho(a)da + O(\varepsilon^N).
\]
and we aim at showing that actually it is \( \varepsilon N+1 \) accurate i.e. \( \|X_\varepsilon - \mathbf{X}_\varepsilonN\|_{C([0,T])} \lesssim \varepsilon N+1 \).

Here, capital letters \( (X_i)_{i \in \mathbb{N}} \) denote the macroscopic correctors defined on \([0,T]\), they are independent on \( \varepsilon \), and the microscopic correctors \( (x_{i,j})_{(i,j) \in \mathbb{N}^2} \) or \( (w_k)_{k \in \mathbb{N}} \) are defined on \( \mathbb{R}_+ \). They are then renamed with a tilde when rescaled with respect to \( \varepsilon \) : \( \tilde{x}_{i,j}(t) := x_{i,j}(t/\varepsilon) \) for \( t \in (0,T) \) and \( (i,j) \in \mathbb{N}^2 \). The functions \( \tilde{x}_{i,j}(t) \) correct tails associated to
\[
\Xi_{j,k}(t) := \left( \frac{t}{\varepsilon} \right)^j \int_0^{\frac{t}{\varepsilon}} a^k \varrho(a)da =: \xi_{j,k}\left( \frac{t}{\varepsilon} \right),
\]
solving at the microscopic scale
\[
\mathcal{L}_1[x_{i,j}](t) := \mu_0 x_{i,j}(t) - \int_0^t x_{i,j}(t-a)\varrho(a)da =: \xi_{j,k}(t)
\]
while \( \tilde{w}_t \) take into account other tails related to kernel \( \varrho \) reading
\[
\mathcal{L}_1[w_t](t) = \mu_0 w_t(t) - \int_0^t w_t(t-a)\varrho(a)da = \int_t^\infty (t-a)^j \varrho(a)da.
\]
These tails occur when expanding the delay operator through Taylor series. We give ourselves some hypotheses on the data :

**Assumptions 2. Assume that :**

i) the source term is such that \( f \in C^N(\mathbb{R}_+) \).
ii) the past condition $X_p \in C^{N+1}(\mathbb{R}_+)$. 

iii) we assume that $\varrho$ is a measurable non-negative function, i.e. $\varrho: \mathbb{R}_+ \to \mathbb{R}_+$. Moreover, for all $a \in \mathbb{R}_+$, there exists $M \subset (a, \infty)$, $M$ compact and $|M| > 0$ such that $\varrho(a) > 0$ for almost every $\tilde{a} \in M$.

iv) defining the $k^{th}$ order moment of $\varrho$ as $\mu_k := \int_{\mathbb{R}_+} a^k \varrho(a) da$, for $k \in \mathbb{N}$, one assumes that $\mu_k < \infty$ for $k \in \{0, \ldots, N+2\}$.

3.1 Construction of the expansion In order to approximate the solution $X_\varepsilon$ of (1.1) within the framework defined using the previous hypotheses, we construct of the terms forming the $N^{th}$-order approximation of $X_\varepsilon$ solution of (1.1) for a fixed kernel as

$$X_{\varepsilon,N} := \sum_{i=0}^{N-1} \varepsilon^i X_i(t) + Y_N(t) + Z_N(t) + W_N(t),$$

(3.4)

where these terms are set later on.

**Proposition 3.1.** Let Assumptions 2 hold, let the sequence of functions $(X_i)_{i \in \{0, \ldots, N-1\}}$ be given and for all $i \in \{0, \ldots, N-1\}$ assume that $X_i \in W^{N+1, \infty}([0, T])$, then one has the expansion:

$$L_\varepsilon[X_i](t) = \sum_{k=1}^{N-i} \varepsilon^{k-1} (-1)^{k+1} X_i^{(k)}(t) (\mu_k - \Xi_{0,k}(t)) + \frac{1}{\varepsilon} \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da X_i(t)$$

$$+ \varepsilon^{-i} R_i^{N+1-i},$$

and the rest can be controlled:

$$|R_i^{N+1-i}| \leq \frac{1}{(N+1-i)!} \left\| X_i^{(N+1-i)} \right\|_\infty \mu_{N+1-i}$$

**Proof.** One writes:

$$L_\varepsilon[X_i](t) = \frac{1}{\varepsilon} \int_0^{\frac{t}{\varepsilon}} (X_i(t) - X_i(t - \varepsilon a)) \varrho(a) da + \frac{1}{\varepsilon} \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da X_i(t)$$

then using the Taylor expansion:

$$X_i(t - \varepsilon a) = \sum_{k=0}^{N-i} \varepsilon^k a^k (-1)^k X_i^{(k)}(t)$$

$$+ \frac{(-\varepsilon a)^{N+1-i}}{(N-i)!} \int_0^1 X_i^{(N+1-i)}(t - sa)(1-s)^{N-i} ds$$

so that the first term above becomes:

$$\frac{1}{\varepsilon} \int_0^{\frac{t}{\varepsilon}} (X_i(t) - X_i(t - \varepsilon a)) \varrho(a) da$$

$$= \sum_{k=1}^{N-i} \int_0^{\frac{t}{\varepsilon}} \varepsilon^{k-1} a^k \varrho(a) da (-1)^{k+1} X_i^{(k)}(t) + \varepsilon^{-i} R_i^{N+1-i}$$

$$= \sum_{k=1}^{N-i} \frac{(-1)^{k+1}}{k!} X_i^{(k)}(t) \left( \varepsilon^{k-1} \mu_k - \frac{1}{\varepsilon} \int_{\frac{t}{\varepsilon}}^{\infty} (\varepsilon a)^k \varrho(a) da \right) + \varepsilon^{-i} R_i^{N+1-i}$$
which provides the result. □

**Proposition 3.2** (Outer expansions).

*Under the same hypothesis as above, the zeroth-order macroscopic limit is given by*

\[
\mu_1 X_0^{(1)} = f, \quad (3.5)
\]

*and at any order* \( \ell \in \{1, \ldots, N\} \), *we have:*

\[
\mu_1 X_{\ell-1}^{(\ell)} = \sum_{k=2}^{\ell} (-1)^k \frac{k!}{k!} X_{\ell-k}^{(k)}. \quad (3.6)
\]

**Proof.** The result proved in Proposition 3.1 leads to:

\[
\mathcal{L}_\varepsilon \left[ \sum_{i=0}^{N-1} \varepsilon^i X_i \right] = \sum_{i=0}^{N-1} \varepsilon^i \sum_{k=1}^{N-i} \frac{(-1)^{k+1}}{k!} X_i^{(k)}(t) \left( \mu_k - \Xi_{0,k}(t) \right)
\]

\[
+ \sum_{i=0}^{N-1} \varepsilon^{i-1} \int_\frac{t}{\varepsilon}^\infty \rho(a) \, dX_i(t) + S_{N,0}, \quad (3.7)
\]

where we set \( S_{N,0} := \varepsilon^N \sum_{i=0}^{N-1} \mathcal{R}_{i}^{N+1-i} \) and

\[
|S_{N,0}| \leq \varepsilon^N \max_{i \in \{0, \ldots, N-1\}} \left\{ \mu_{N+1-i} \|X_i\|_{W^{N+1-i,\infty}(0,T)} \right\}.
\]

Considering the first sum gives:

\[
\sum_{i=0}^{N-1} \varepsilon^i \sum_{k=1}^{N-i} \frac{(-1)^{k+1}}{k!} X_i^{(k)}(t) \left( \mu_k - \Xi_{0,k}(t) \right)
\]

\[
= \sum_{k=1}^{N} \sum_{\ell=k}^{N} \frac{(-1)^{k+1}}{k!} X_{\ell-k}^{(k)}(t) \left( \mu_k - \Xi_{0,k}(t) \right)
\]

\[
= \sum_{\ell=1}^{N} \varepsilon^{\ell-1} \sum_{k=1}^{\ell} \frac{(-1)^{k+1}}{k!} X_{\ell-k}^{(k)}(t) \left( \mu_k - \Xi_{0,k}(t) \right)
\]

Separating powers of \( \varepsilon \) and considering that terms containing functions \( \Xi_{0,k} \) belong to the initial layer (these depend only on the microscopic variable \( t/\varepsilon \)) provides:

\[
\sum_{k=1}^{\ell} \frac{\mu_k}{k!} X_{\ell-k}^{(k)}(-1)^{k+1} = \begin{cases} 0 & \text{if } \ell \neq 1, \\ f(t) & \text{otherwise}, \end{cases} \quad (3.8)
\]

and by relating the lowest derivative with the highest index to the rest of the correctors, we establish macroscopic nested ODEs (3.5) and (3.6). □

**Remark 3.1.** The initial conditions of the macroscopic correctors \( X_i \) are to be defined later (cf Theorem 4).

**Proposition 3.3** (Inner expansion). It is threefold.

- The first part accounts for terms containing \( \Xi_{0,k} \) in the first sum of (3.7):

\[
Y_N(t) := \sum_{m=1}^{N} \varepsilon^m \sum_{q=1}^{m} \sum_{k=1}^{q} \frac{(-1)^{k+1}}{k!(m-q)!} X_{q-k}^{(k+m-q)}(0) \tilde{x}_{m,q,k}(t) \quad (3.9)
\]
where \( \tilde{x}_{j,k}(t) := x_{j,k}(t/\varepsilon) \) and the microscopic correctors solve:

\[
\mathcal{L}_1[x_{j,k}](t) = \xi_{j,k}(t) := t^j \int_t^\infty a^k g(a) da,
\]

(3.10)

and \( \mathcal{L}_1 \) is \( \mathcal{L}_\varepsilon \) defined for \( \varepsilon = 1 \).

- The second part corrects the second sum in (3.7) and reads:

\[
Z_N(t) := - \sum_{m=1}^N \varepsilon^m \sum_{q=0}^{m-1} \frac{1}{(m-q)!} X_q^{(m-q)}(0) \tilde{x}_{m-q,0}(t) - \sum_{i=0}^{N-1} \varepsilon^i X_i(0) \tilde{x}_{0,0}(t).
\]

(3.11)

and we underline that by uniqueness of the solution of (3.10), \( \tilde{x}_{0,0}(t) = 1 \).

- The last part concerns the remainders related to the past source term in (3.2)

\[
W_N(t) := \sum_{i=0}^N \varepsilon^i X_p^{(i)}(0) \tilde{w}_i(t),
\]

where \( \tilde{w}_i(t) := w_i(t/\varepsilon) \) and \( (w_\ell)_\ell \) solve for \( \ell \in \mathbb{N} \),

\[
\begin{cases}
\int_{\mathbb{R}^+} (w_\ell(t) - w_\ell(t-a)) g(a) da = 0, & t > 0, \\
w_\ell(t) = t^\ell, & t \leq 0.
\end{cases}
\]

(3.12)

Proof. First, we begin by constructing the first part of the initial layer \( Y_N \). We consider the second term in (3.7) and we use Taylor’s expansion:

\[
X_{\ell-k}^{(k)}(t) = \sum_{j=0}^{N-k} X_{\ell-k}^{(k+j)}(0) \frac{t^j}{j!} + \varepsilon^N T_{k,l}^N,
\]

where the integral rest reads:

\[
T_{k,l}^N := \frac{\varepsilon^{-k+1}}{(N-k)!} \left( \frac{t}{\varepsilon} \right)^{N-k+1} \int_0^1 (1-s)^{N-k} X_{\ell-k}^{(N-k+1)}(st) ds,
\]

which implies that

\[
\sum_{\ell=1}^N \varepsilon^{\ell-1} \sum_{k=1}^{\ell} \frac{(-1)^k}{k!} X_{\ell-k}^{(k)}(0) \Xi_{0,k}(t) = \sum_{\ell=1}^N \sum_{k=1}^{\ell} \frac{(-1)^k}{k!} \Xi_{0,k}(t) \left\{ \sum_{j=0}^{N-k} X_{\ell-k}^{(k+j)}(0) \frac{t^j}{j!} + \varepsilon^N T_{k,l}^N \right\} = \sum_{\ell=1}^N \sum_{k=1}^{N-k} \sum_{j=0}^{\ell-1} \frac{(-1)^k}{k! j!} \varepsilon^{\ell-j} X_{\ell-k}^{(k+j)}(0) \left( \frac{t}{\varepsilon} \right)^j \Xi_{0,k}(t) + S_{N,1} =: I + S_{N,1}
\]

(3.13)

where

\[
S_{N,1} := \sum_{\ell=1}^N \sum_{k=1}^{\ell} \frac{\varepsilon^{\ell+N-1}(-1)^k}{k!} \Xi_{0,k}(t) T_{k,l}^N,
\]

(3.14)
Since on \((t/\varepsilon, \infty), \ a > t/\varepsilon\) and since \(N - k + 1 \geq 0\):
\[
\left(\frac{t}{\varepsilon}\right)^j \Xi_{0,k}(t) \equiv \left(\frac{t}{\varepsilon}\right)^{N-k+1} \int_{\frac{t}{\varepsilon}}^{\infty} a^k \varrho(a) da \leq \int_{\frac{t}{\varepsilon}}^{\infty} a^{N+1} \varrho(a) da \leq \mu_{N+1}
\]
(3.15)
one has that:
\[
|S_{N,1}| \leq \varepsilon^N C\mu_{N+1} \sum_{\ell=1}^{N} \sum_{k=1}^{N} \varepsilon^{\ell-k} \leq C' \varepsilon^N \mu_{N+1}
\]
(3.16)
where the generic constants \(C\) and \(C'\) depend only on \(\max_{j \in \{0, \ldots, N-1\}} \|X_j\|_{W^{1,N}(0,T)}\).
The first triple sum in (3.13) can be decomposed thanks to Proposition A.1 as
\[
I := \sum_{m=1}^{N} \varepsilon^{m-1} \left( \sum_{k=1}^{q} \frac{(-1)^k}{k!(m-k)!} X_{q-k}(0) \Xi_{m-q,k}(t) \right)
\]
\[+ \sum_{m=1}^{N-1} \varepsilon^{m-1} \left( \sum_{q=m+1}^{N} \sum_{k=1}^{q} \frac{(-1)^k}{k!(m-k)!} X_{q-k}(0) \Xi_{m-q,k}(t) \right)
\]
\[=: I_1 + O(\varepsilon^N)
\]
where we recall the definition of \(\Xi_{j,k}\) in (3.3). In order to compensate \(I_1\), we define microscopic correctors \(\tilde{x}_{i,j}\) as (3.10) and set \(Y_N\) as in (3.9). Now, we need to correct the third term in (3.7), which we do with the same technique as above:
\[
\sum_{i=0}^{N-1} \varepsilon^{i-1} X_i(t) \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da
\]
\[= \sum_{i=0}^{N-1} \varepsilon^{i-1} \left\{ \sum_{j=0}^{N-i} \frac{t^j}{j!} X_i^{(j)}(0) + \frac{t^{N+1-i}}{(N-i)!} \int_0^1 X_i^{N+1-i}(st)(1-s)^{N-i} ds \right\} \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da
\]
\[= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \varepsilon^{i+j-1} \frac{1}{j!} \Xi_{j,0}(t) X_i^{(j)}(0) + S_{N,2}
\]
\[= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \varepsilon^{i+j-1} \Xi_{j,0}(t) X_i^{(j)}(0) + S_{N,2}
\]
\[= \sum_{i=0}^{N-1} \varepsilon^{i-1} \int_{\frac{t}{\varepsilon}}^{\infty} X_i^{N+1-i}(st)(1-s)^{N-i} ds \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da,
\]
(3.17)
and one has using the same argument as in (3.15):
\[
|S_{N,2}| \leq C\varepsilon^N \int_{R_+} (1+a)^{N+1} \varrho(a) da \sup_{i \in \{0, \ldots, N\}} \|X_i\|_{W^{N+1-i,\infty}(0,T)}
\]
(3.18)
It suffices then to add the correction $Z_N$ defined as in (3.11).
Lastly, the right hand side of (3.1) contains a multi-scale term related to the past positions $X_p$. As it depends on the fast variable $\frac{t}{\varepsilon}$, we Taylor-expand $X_p(t - \varepsilon a)$ around 0:

$$X_p(t - \varepsilon a) = \sum_{i=0}^{N} \varepsilon^i X_p^{(i)}(0) \frac{(t/\varepsilon - a)^i}{i!} + \varepsilon^{N+1} \frac{(t/\varepsilon - a)^{N+1}}{N!} \int_0^1 X_p^{(N+1)}(s(t - \varepsilon a))(1-s)^N ds,$$

leading to re-write the integral term in the rhs of (3.1) as

$$\frac{1}{\varepsilon} \int_{-\varepsilon}^{+\infty} X_p(t - \varepsilon a) g(a) da = \sum_{i=0}^{N} \varepsilon^{i-1} X_p^{(i)}(0) \int_{-\varepsilon}^{+\infty} \frac{(t/\varepsilon - a)^i}{i!} g(a) da + S_{N,3},$$

where

$$S_{N,3} := \varepsilon^N \int_{-\varepsilon}^{+\infty} \frac{(t/\varepsilon - a)^{N+1}}{N!} \int_0^1 X_p^{(N+1)}(s(t - \varepsilon a))(1-s)^N ds g(a) da, \quad (3.19)$$

and

$$|S_{N,3}| \leq \frac{\varepsilon^N}{(N+1)!} \|X_p^{(N+1)}\|_\infty \int_{-\varepsilon}^{+\infty} \left(\frac{t}{\varepsilon} - a\right)^{N+1} g(a) da \leq C \varepsilon^N \mu_{N+1} \quad (3.20)$$

This shows why the past microscopic correctors should reads as in the last part of the claim. □

3.2. Asymptotic behavior of initial layers

**Lemma 3.1.** Assume that the integers $j$ and $k$ are chosen s.t. $j+k \leq N$, moreover, if $\mu_{j+k+2} < \infty$, then one has :

$$x_{j,k}(0) = \begin{cases} \frac{\mu_k}{\mu_0}, & \text{if } j = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and $x_{j,k}(t) \to \mu_{j+1+k}/((j+1)\mu_1)$ when $t \to \infty$. The proof of this lemma strongly relies on the concept of resolvent introduced and deeply studied in [4, Chap. 2, 4, 7].

**Proof.** The resolvent associated to (3.10) [4, Theorem 2.3.1 and 2.3.5, p. 42-45], satisfies :

$$r(t) - (r * k)(t) = k(t),$$

where

$$k(a) := g(a)/\mu_0, \quad (r * k)(t) := \int_0^t r(t - \tau) k(\tau) d\tau \quad (3.21)$$

and it can be decomposed [4, Theorem 7.4.1, p.201] as

$$r(t) = \frac{\mu_0}{\mu_1} + \gamma(t),$$
where the function $\gamma \in L^1(\mathbb{R}_+)$. Moreover, the resolvent being defined the solution $x_{j,k}$ is explicit and reads:

$$x_{j,k} = \xi_{j,k} + \xi_{j,k} \star r = \xi_{j,k} + \xi_{j,k} \star (\mu_0 / \mu_1 + \gamma) = \xi_{j,k} + \xi_{j,k} \star \gamma + \frac{\mu_0}{\mu_1} \int_0^t \xi_{j,k}(s) ds.$$

Thus the leading term in $x_{j,k}$ when $t$ grows large is the last integral. Indeed

$$\frac{\mu_0}{\mu_1} \int_{\mathbb{R}_+} t^j \int_t^\infty a^k g(a) da dt = \frac{1}{(j+1) \mu_1} \int_{\mathbb{R}_+} (\int_0^a t^j dt) a^k g(a) da = \frac{\mu_{j+k+1}}{\mu_1} \int_{\mathbb{R}_+} t^j \int_0^\infty a^k g(a) da dt = \frac{1}{j+1} \mu_{j+k+1} \in L^1(\mathbb{R}_+).$$

which we define as the formal expression $x_{j,k}(t) = \mu_{j+k+1} / ((j+1) \mu_1)$ when $t \to \infty$.

**Lemma 3.2.** Under the same assumptions as in the previous Lemma, the microscopic functions $w_\ell$ are discontinuous at $t = 0$, for all $\ell \geq 0$:

$$w_\ell(0^+) = \frac{-\ell}{\mu_0}, \quad w_\ell(0^-) = 0,$$

and $w_\ell(t) \to ((-1)^\ell \mu_{\ell+1}) / ((\ell+1) \mu_1)$ when $t \to \infty$, provided that $\mu_{\ell+2} < \infty$.

**Proof.** Using (3.12), we can easily show the discontinuity of the correctors $w_\ell$ at $t = 0$. By the same arguments as proof of Lemma 3.1, one has that:

$$\lim_{t \to \infty} w_\ell(t) = \frac{1}{\mu_1} \int_{\mathbb{R}_+} (t-a)^\ell g(a) da = \frac{1}{\mu_1} \int_{\mathbb{R}_+} (\int_0^a (t-a)^\ell da) g(a) da = \frac{(-1)^\ell \mu_{\ell+1}}{(\ell+1) \mu_1},$$

which ends the proof.

**Lemma 3.3.** Under the previous hypotheses, the initial data can be estimated

$$\left| X_\varepsilon(0^+) - \tilde{X}_{\varepsilon,N}(0^+) \right| \lesssim \varepsilon^{N+1}.$$

**Proof.** By definition, one has:

$$\tilde{X}_{\varepsilon,N}(0^+) = \sum_{i=0}^{N-1} \varepsilon^i X_i(0) + Y_N(0) + Z_N(0) + W_N(0^+).$$

Next we analyze each of these terms. Firstly, one has thanks to

$$Y_N(0) = \frac{1}{\mu_0} \sum_{\ell=1}^N \varepsilon^\ell \sum_{k=1}^{\ell} \frac{(-1)^{k+1}}{k!} X_{\ell-k}(0) \mu_k = \frac{\varepsilon f(0)}{\mu_0}$$

where we used that $\bar{x}_{j,k}(0) = 0$ for all $j \neq 0$ and (3.8). By definition,

$$Z_N(0) = -\sum_{i=0}^{N-1} \varepsilon^i X_i(0) \bar{x}_{0,0}(0) = -\sum_{i=0}^{N-1} \varepsilon^i X_i(0),$$
and compensates the first terms of the sum (3.22), then

$$W_N(0) = \frac{1}{\mu_0} \sum_{i=0}^{N} \varepsilon^i X_p^{(i)}(0) (-1)^i \mu_i.$$  \hspace{1cm} (3.23)

Then one observes that if $X_\varepsilon$ solves (3.1), then when $t \downarrow 0^+$,

$$X_\varepsilon(0^+) = \varepsilon \frac{f(0)}{\mu_0} + \int_{\mathbb{R}_+} X_p(-\varepsilon a) \frac{\rho(a)}{\mu_0} da$$

and so an $N$th-order Taylor expansion of $X_p(-\varepsilon a)$ in 0 inside the last term coupled with (3.23) leaves an $O(\varepsilon^{N+1})$ rest. □

3.3. Matching inner and outer expansions  So far the initial conditions of the outer expansion are not defined. For this sake, we write the inner expansion’s limit when $t \to \infty$. This gives :

$$\lim_{t \to \infty} Y_N(t) = \sum_{m=1}^{N} \varepsilon^m \sum_{q=1}^{m} \sum_{k=1}^{q} (-1)^{k+1} \frac{(-1)^{m-q} (0)}{(m-q+1)!} X_{(k+m-q)}(0) \mu_{m-q+k+1} =: s_1,$$

and

$$\lim_{t \to \infty} Z_N(t) = - \sum_{m=1}^{N} \varepsilon^m \sum_{q=0}^{m-1} \frac{\mu_{m-q+1}}{(m-q+1)!} X_{q}(0)^{m-q} - \sum_{i=0}^{N-1} \varepsilon^i X_i(0)$$

and

$$\lim_{t \to \infty} W_N(t) = \sum_{i=0}^{N} \varepsilon^i X_p^{(i)}(0) \frac{(i+1)!}{\mu_{i+1}} (-1)^i =: s_3.$$

As we do not want the inner expansion to interfere with the outer expansion, we gather powers of $\varepsilon$ and define the initial conditions of the outer expansion so that

$$\lim_{t \to \infty} (Y_N(t) + Z_N(t) + W_N(t)) = 0$$

which then gives :

\textbf{Theorem 4.} The macroscopic Ansatz should be given the initial conditions : for $m = 0$, $X_0(0) = X_p(0)$ while for $m \in \{1, \ldots, N-1\}$

$$\mu_1 X_m(0) = (-1)^m X_p^{(m)}(0) \frac{\mu_{m+1}}{(m+1)!} - \sum_{q=0}^{m-1} X_q^{(m-q)}(0) \frac{\mu_{m-q+1}}{(m-q+1)!} + \sum_{q=1}^{m} \sum_{k=1}^{q} \frac{(-1)^{k+1}}{(m-q+1)!k!} X_{q-k}^{(k+m-q)}(0) \mu_{m-q+k+1}.$$  \hspace{1cm} (3.24)

3.4. Error estimates  In this section, we give error estimates between $X_\varepsilon$, the solution of (3.1) and the asymptotic expansion $\tilde{X}_{\varepsilon,N}$ given by (3.4). This result is based on a comparison principle [4, Chap. 9, Section 8] and the construction of a super solution $U_N$ such that $|X_\varepsilon(t) - \tilde{X}_{\varepsilon,N}(t)| \leq U_N(t)$ for all $t \in [0,T]$. Moreover, the super solution should satisfy $U_N \lesssim \varepsilon^N$ as well. In order to use such a principle, a strict control of the kernel in some norm is expected. The following lemma is thus required :
Lemma 3.4. Under the Assumptions 2, setting $K_\varepsilon(t,s) := \frac{1}{\varepsilon \mu_0} \hat{\varphi}(\frac{t}{\varepsilon})$ satisfies:

$$\|K_\varepsilon\|_{B^{\infty}(0,T)} := \text{esssup}_{t \in (0,T)} \int_0^t |K_\varepsilon(t,s)| ds < 1.$$  

We refer the reader to [4, Chap. 9, Sections 1 to 5] for a complete overview of non-convolution integral Volterra equations and the corresponding functional framework.

Proof. For almost every $t \in (0,T)$,

$$0 \leq \int_0^t |K_\varepsilon(\tilde{a},t)| \, d\tilde{a} = \frac{\int_0^T \varphi(a) \, da}{\int_{\mathbb{R}_+} \varphi(a) \, da} \leq \frac{\int_0^T \varphi(a) \, da}{\int_{\mathbb{R}_+} \varphi(a) \, da}$$  

But, by definition, for every fixed $\varepsilon$ there exists a compact set $M_\varepsilon \subset (T/\varepsilon, \infty)$ such that $\varphi(a) > 0$ for almost every $a \in M_\varepsilon$ so that

$$\int_{\mathbb{R}_+} \varphi(a) \, da = \int_{0}^{\infty} \varphi(a) \, da = \int_{M_\varepsilon} \varphi(a) \, da > 0.$$  

which ends the proof. \Box

Theorem 5. Suppose that Assumptions 2 holds, then:

$$\|X_\varepsilon - \overline{X}_{\varepsilon,N}\|_{C([0,T])} \lesssim \varepsilon^N.$$  

where $\overline{X}_{\varepsilon,N}$ is defined in (3.4) and $X_\varepsilon$ solves (3.1).

Proof. First, we consider the zero order approximation (i.e. $N = 1$). We denote $\hat{X}_1 := X_\varepsilon - \overline{X}_{\varepsilon,1}$, it solves:

$$\mathcal{L}_\varepsilon[\hat{X}_1] = S_1 := \sum_{i=1}^{3} S_{1,i},$$  

where $|S_1| \leq \varepsilon C_1$ (see estimates (3.16), (3.18) and (3.20) for $N = 1$). We construct a super-solution $U_1$ such that

$$\mathcal{L}_\varepsilon[\hat{X}_1] \leq \mathcal{L}_\varepsilon[U_1], \quad \text{and} \quad |\hat{X}_1(0)| \leq U_1(0).$$  

We set

$$U_1(t) := \varepsilon (c_1 + tc_2 - \varepsilon c_3 \hat{w}_1(t)),$$  

where $\hat{w}_1$ solves (3.12) with $\ell = 1$. As the resolvent associated to (3.12) is non-negative, applying the comparison principle [4, Proposition 9.8.1 and Lemma 9.8.2, p. 257], shows that $\hat{w}_1(t) \leq 0$ for $t > 0$. Then

$$\begin{align*}
\mathcal{L}_\varepsilon[U_1] &= c_1 \Xi_{0,0}(t) + \varepsilon c_2 \mu_1 + c_2 \int_{\frac{t}{\varepsilon}}^{\infty} (t - a) \varphi(a) \, da - \varepsilon c_3 \mathcal{L}_\varepsilon[\hat{w}_1] \\
&= c_1 \Xi_{0,0}(t) + \varepsilon c_2 \mu_1 + c_2 \int_{\frac{t}{\varepsilon}}^{\infty} \left(\frac{t}{\varepsilon} - a\right) \varphi(a) \, da - \varepsilon c_3 \int_{\frac{t}{\varepsilon}}^{\infty} \left(\frac{t}{\varepsilon} - a\right) \varphi(a) \, da \\
&\geq \varepsilon c_2 \mu_1.
\end{align*}$$  

The last inequality being true when $c_1 \geq 0$ and $c_2 = c_3$. Then, one tunes $c_2 \geq C_1/\mu_1$ so that

$$\mathcal{L}_\varepsilon[|\hat{X}_1|] \leq |S_1| \leq \varepsilon K_1 \leq \varepsilon \mu_1 c_2 \leq \mathcal{L}_\varepsilon[U_1],$$  

and the constant $c_1$ is chosen such that $|\dot{X}_1(0)| \leq \varepsilon c_1 \leq \varepsilon c_1 + \varepsilon^2 c_3 \frac{\mu_1}{\mu_0} = U_1(0)$. The result then follows using [4, Proposition 9.8.1 and Lemma 9.8.2].

More generally, for any $N$, one sets $U_N := \varepsilon^N (c_1 + c_2 t - \varepsilon c_3 \tilde{w}_1)$ and the result follows the same: choosing $c_1 := |\dot{X}_N(0)|$, $c_2 := K_N/\mu_1$ where $|S_N| \leq \varepsilon^N C_N$ and $c_3 = c_2$. □

**Remark 1.** We underline the novelty in the construction of the super-solution $U$ when compared with [6, Theorem 1.1]. Indeed, in this reference, terms $c_1\Xi_{0,0}(t)$ and $c_2 \int_\mathbb{R}^\infty (t-\varepsilon a)\varphi(a)da$ in (3.25) were compared so that the difference remain positive, and this translates into the control of the ratio $A_\varepsilon[\rho_e](t)$ presented in the introduction.

Here, instead, the negative tail is taken into account by the initial layer $-\varepsilon c_3 \tilde{w}_1$ and there is no need to relate the previous terms anymore. This in turn allows to relax hypotheses on the decay of $\varphi$.

### 4. Conclusion & perspectives

In this work we have obtain two-fold results. On one hand, we have weakened hypotheses on the death rate of the linkages and showed that still when the parameter $\varepsilon$ is small adhesive memory effects are close to friction.

Moreover, we made the previous statement more quantitative with respect to the size of the parameter $\varepsilon$ thanks to a general asymptotic expansion when $\varphi$ is constant in time.

To our knowledge, it is not possible yet to combine both parts of this article and construct asymptotic expansions (and error estimates) at any order $N \geq 2$, for the weakly coupled problem $(\rho_e, X_e)$ solving (1.2) and (1.1). The main reason for this is that at some point we use arguments based on Laplace transform of the resolvent (indirectly this leads to the asymptotic decomposition of the resolvent used in Lemmas 3.1 and 3.2 thanks to [4, Theorem 7.4.1]). When the density is time dependent, the microscopic correctors $x_{j,k}(s,t)$ an $w_{\ell}(s,t)$ depend on a macroscopic parameter $t$, ($s$ bien the local microscopic variable) (this is usual in the framework of quasi-periodic asymptotic expansions [3]) and so do the corresponding resolvents $r_{x_{j,k}}(s,t)$ and $r_{w_{\ell}}(s,t)$. Results provided by [4, Theorem 7.4.1 p. 201] do not allow for uniform estimates of the initial layers with respect to the macroscopic parameter $t$ and the analysis performed in Section 3 can’t be extended to time dependent kernels. This seems an interesting mathematical problem per se, since for the time being, asymptotic results established thanks to the Laplace transform, are to our knowledge, the sharpest possible under the weakest hypotheses.

**Appendix A. Some summations over integers.**

**Proposition A.1.** Assume that $N \geq 2$, and define the sum $S$ as follows

$$S := \sum_{i=1}^{N} \sum_{k=1}^{i} \sum_{j=0}^{N-k} a_{i,j,k}$$

where $a_{i,j,k}$ is a sequence of real numbers, then this sum is in fact equal to

$$S = \left( \sum_{m=1}^{N} \sum_{q=1}^{m} a_{m,1,1} \right) a_{q,1,1}$$

**Proof.** We distinguish between $i < N$ and $i = N$. In the first case, one has, by the following computation, that

$$s := \sum_{i=1}^{N-1} \sum_{k=1}^{i} \sum_{j=0}^{N-k} a_{i,j,k} = \sum_{i=1}^{N-1} \sum_{j=0}^{i} \sum_{k=1}^{i} a_{i,j,k} + \sum_{i=1}^{N-1} \sum_{j=0}^{N-i} \sum_{k=1}^{N-j} a_{i,j,k},$$
Fig. A.1. The index change from \((i,j)\) to \((m,n)\) (here as an example \(N = 5\)). It allows to separate the \(\varepsilon\)-scales as powers of \(m\). We look for a peculiar way of going through integer values of \((i,j)\) (crossed dots in the figure) although this does not correspond to all integer values of \((m,n)\).

because

\[
\begin{align*}
0 & \leq j \leq N - i - 1, \quad \min(i, N - j) = i \\
N - i & \leq j \leq N - 1, \quad \min(i, N - j) = N - j
\end{align*}
\]

One has that

\[
s = \sum_{i=1}^{N-1} \left( \sum_{j=0}^{N-1-i} \min(i, N-j) \right) \sum_{k=1}^{\min(i, N-j)} \sum_{j=0}^{N-1-i} \sum_{k=1}^{\min(i, N-j)} a_{i,j,k}.
\]

On the other hand, when \(i = N\), a simple check shows that

\[
s' := \sum_{k=1}^{N} \sum_{j=0}^{N-k} a_{N,j,k} = \sum_{k=1}^{N-1} \sum_{j=0}^{N-1-j} a_{N,j,k},
\]

and then one remarks simply that \(\min(i = N, N - j) = N - j\) which gathering the terms gives that

\[
S = s + s' = \sum_{i=1}^{N} \sum_{j=0}^{N-1-i} \sum_{k=1}^{\min(i, N-j)} a_{i,j,k}.
\]

The previous sum can be rewritten as:

\[
S = \left( \sum_{m=1}^{N} \sum_{n=3-m}^{m+1} + \sum_{m=N+1}^{2N-1} \sum_{n=m-2N+3}^{2N-m+1} \right) \sum_{k=1}^{\min(i(m,n), N-j(m,n))} a_{i(m,n), j(m,n), k} \chi_{\{i(m,n), j(m,n) \in \mathbb{N}^2\}}(m,n), \tag{A.2}
\]
where \( m := i + j \) and \( n := i - j + 1 \) and the inverse transform should provide integer values \( i(m, n) := (m + n - 1)/2 \) and \( j(m, n) := (m - n + 1)/2 \) (see Fig. A.1). This justifies the definition of the indicator function

\[
\chi_{\{i(m, n), j(m, n)\in\mathbb{N}^2\}}(m, n) := \begin{cases} 1, & \text{if } (i(m, n), j(m, n)) \in \mathbb{N}^2, \\ 0, & \text{otherwise}. \end{cases}
\]

When \( m > N \), one needs to bound the summation on \( n \) in an interval depending on \( m \) (see Fig. A.1). Indeed, when \( i = N \), we write:

\[
n = N - j + 1, \quad m = N + j, \quad \Rightarrow n = 2N - m + 1,
\]

while if \( j = N - 1 \),

\[
n = i - N + 2, \quad m = i + N - 1, \quad \Rightarrow n = m - 2N + 3,
\]

a simple check shows that \( n \leq 2N - 1 \Leftrightarrow m - 2N + 3 \leq 2N - m + 1 \).

Then since the indicator function \( \chi_{\{(i(m, n), j(m, n))\in\mathbb{N}^2\}}(m, n) \) in (A.2) is not zero when \( [n + m - 1]_2 = 0 \), there exists \( q \in \mathbb{Z} \) such that \( n + m - 1 = 2q \) or equivalently \( n = 1 + 2q - m \) so that the summation with respect to \( n \) can be replaced with a summation over \( q \). When \( n \in \{3 - m, m + 1\}, q \in \{1, m\} \), and similarly when \( n \in \{m - 2N + 3, 2N - m + 1\}, q \in \{m + 1 - N, N\} \). Moreover \( q = i \) and \( j = m - q \). Thus the previous sum becomes:

\[
S = \left( \sum_{m=1}^{N} \sum_{q=1}^{N} \sum_{k=1}^{\min(q, N-(m-q))} a_{q,m-q,k} \right) \sum_{m=1}^{N} \sum_{q=0}^{m-1} a_{q,m-q}.
\]

Since \( \min(q, q + N - m) = q + \min(0, N - m) = q \) as soon as \( m \leq N \) this gives the first part of the sum in (A.1). If \( m \in \{N + 1, \ldots, 2N\} \), then \( N - m \leq -1 \) and \( \min(q, q + N - m) = q + N - m \) which ends the proof. \( \square \)

**Proposition A.2.** In the same way as above

\[
S' := \sum_{i=0}^{N-1} \sum_{j=1}^{N-i} a_{i,j} = \sum_{m=1}^{N} \sum_{q=0}^{m-1} a_{q,m-q}.
\]

**Proof.** Again we perform the change of variables \( m = i + j \) and \( q = i \) and we proceed as above. \( \square \)

**Appendix B. Proof of Proposition 2.1.**

*Proof.** First, for \( j = 0 \), \( \rho_0 \) is explicitly given by

\[
\rho_0(a, t) = \rho_0(0, t) \exp \left( - \int_0^a \zeta(\bar{a}, t) d\bar{a} \right) \leq \rho_0(0, t) \frac{m(a)}{m(0)} \leq \frac{\beta_{\max} \zeta_{\max} m(a)}{\zeta_{\max} + \beta_{\min} m(0)}
\]

which gives \( c_0 \). Similarly, \( \partial_t \rho_0(a, t) \) it is explicit and reads:

\[
\partial_t \rho_0(a, t) = \partial_t \rho_0(0, t) \exp \left( - \int_0^a \zeta(\bar{a}, t) d\bar{a} \right) \left. - \int_0^a \exp \left( - \int_\tau^a \zeta(\bar{a}, t) d\bar{a} \right) \partial_t \zeta(\tau, t) \rho_0(\tau, t) d\tau, \right.
\]

where

\[
\partial_t \rho_0(0, t) = \frac{g(t)}{1 + \beta(t) \int_0^\infty \exp \left( - \int_\tau^a \zeta(\bar{a}, t) d\bar{a} \right) d\bar{a}}
\]
and
\[ g(t) = \beta'(t)(1 - \mu_0(t)) \int_0^{+\infty} \exp \left( - \int_0^a \zeta(\bar{a}, t) \, d\bar{a} \right) \, da \]
\[ - \int_0^{+\infty} \int_0^a \exp \left( - \int_0^\tau \zeta(\bar{a}, t) \, d\bar{a} \right) \partial_t \zeta(\tau, t) \rho_0(\tau, t) \, d\tau \, da. \]

So that
\[ |\partial_t \rho_0(a, t)| \leq |\partial_t \rho_0(0, t)| \frac{m(a)}{m(0)} + m(a) \|\zeta\|_{W^{1,\infty}} \int_0^a \rho_0(\tau, t) \frac{m(\tau)}{m(0)} \, d\tau \leq (k_1 + k_2 a) m(a) \]
\[ \leq c'(1 + a) m(a) \]
where
\[ k_1 := C \left( \frac{\zeta_{\max}}{\beta_{\max} + \zeta_{\max}}, \|\beta\|_{W^{1,\infty}}, m \|L^1(\mathbb{R}_+), \|\zeta\|_{W^{1,\infty}} \right), \quad k_2 = c_0 \frac{\|\zeta\|_{W^{1,\infty}}}{m(0)} \]
and \( c_{0,1} := \max(k_1, k_2) \). Now, for \( j = 1 \), \( \rho_1 \) can be given explicitly by
\[ \rho_1(a, t) = \rho_1(0, t) \exp \left( - \int_0^a \zeta(\bar{a}, t) \, d\bar{a} \right) - \int_0^a \exp \left( - \int_0^\tau \zeta(\bar{a}, t) \, d\bar{a} \right) \partial_t \rho_0(\tau, t) \, d\tau, \]
where
\[ \rho_1(0, t) = \frac{h(t)}{1 + \beta(t) \int_0^{+\infty} \exp \left( - \int_0^a \zeta(\bar{a}, t) \, d\bar{a} \right) \, da} \]
such that
\[ |h(t)| = \left| \int_{\mathbb{R}_+} \int_0^a \exp \left( - \int_0^\tau \zeta(\bar{a}, t) \, d\bar{a} \right) \partial_t \rho_0(\tau, t) \, d\tau \, da \right| \]
\[ \leq \int_{\mathbb{R}_+} \int_0^a \frac{m(a)}{m(\tau)} |\partial_t \rho_0(\tau, t)| \, d\tau \, da \leq c_{0,1} \int_{\mathbb{R}_+} (1 + a)^2 m(a) \, da, \]
and finally, we obtain that
\[ |\rho_1(a, t)| \leq k_1' \frac{m(a)}{m(0)} + \int_0^a c_{0,1} (1 + \tau) m(a) \, d\tau \leq \max(k_1', c_{0,1})(1 + a)^2 m(a) \]
where
\[ k_1' := C \left( \frac{\zeta_{\max}}{\beta_{\max} + \zeta_{\max}}, \int_{\mathbb{R}_+} (1 + a)^2 m(a) \, da \right). \]
Similarly, we can prove that
\[ |\partial_t \rho_1| \leq c_{1,1}(1 + a)^3 m(a), \]
and the generic way can be deduced by induction. \( \Box \)

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**Appendix D.**
The authors declare there is no conflict of interest.
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