Most of the theoretical results on the kinematic amplification of small-scale magnetic fluctuations by turbulence have been confined to the model of white-noise-like ($\delta$-correlated in time) advecting turbulent velocity field. In this work, the statistics of the passive magnetic field in the diffusion-free regime are considered for the case when the advecting flow is finite-time correlated. A new method is developed that allows one to systematically construct the correlation-time expansion for statistical characteristics of the field such as its probability density function or the complete set of its moments. The expansion is valid provided the velocity correlation time is smaller than the characteristic growth time of the magnetic fluctuations. This expansion is carried out up to first order in the general case of a $d$-dimensional arbitrarily compressible advecting flow. The growth rates for all moments of the magnetic-field strength are derived. The effect of the first-order corrections due to the finite correlation time is to reduce these growth rates. It is shown that introducing a finite correlation time leads to the loss of the small-scale statistical universality, which was present in the limit of the $\delta$-correlated velocity field. Namely, the shape of the velocity time-correlation profile and the large-scale spatial structure of the flow become important. The latter is a new effect, that implies, in particular, that the approximation of a locally-linear shear flow does not fully capture the effect of nonvanishing correlation time. Physical applications of this theory include the small-scale kinematic dynamo in the interstellar medium and protogalactic plasmas.

I. INTRODUCTION

The study of the statistics of magnetic fluctuations excited by a random Gaussian white-noise-like advecting velocity field, was pioneered by Kazantsev [1], and, in more recent times, has generated a considerable amount of research (see, e.g., Ref. [2] and references therein, as well as Refs. [3–10]). While much attention has concentrated on resistive dynamo problems, most often for very large magnetic Prandtl numbers, it is well known that the fundamental Zeldovich’s, or “stretch–twist–fold,” mechanism of the magnetic-energy amplification (the so-called “fast dynamo”) is active regardless of the presence of the resistive (diffusive) regularization [1]. If the initial seed magnetic field is concentrated on the scales of the same order as the characteristic scales of the advecting velocity, the stretching and folding of the magnetic-field lines by the random flow leads to an exponential growth of the magnetic fluctuations at scales that decrease exponentially fast, until the diffusive scales are reached [12,3,10]. This scenario is common in astrophysical applications such as the turbulence in the interstellar medium or in the protogalaxy where the Prandtl number ranges from $10^{14}$ to $10^{22}$, giving rise to 7 to 11 decades of small (subviscous) scales available to the magnetic fluctuations [3,13,10]. In fact, the initial diffusion-free regime may well be the only one practically important in such applications as far as the kinematic approximation is concerned, since the nonlinear saturation effects are likely to set in before the diffusion scales are reached [12]. On the fundamental physical level, the diffusion-free regime, in which the magnetic-field lines are fully frozen into the flow, exhibits most clearly the symmetry properties of the passive advection [8].

With a few notable exceptions (such as Refs. [15,16,4,7]), the dominant approach in the existing literature on the turbulent kinematic dynamo problem has been to study the statistics of passive magnetic fields advected by a flow $\mathbf{u}(t, \mathbf{x})$ whose two-time correlation function is approximated by a $\delta$ function, $\langle \mathbf{u}(t) \mathbf{u}(t') \rangle \propto \delta(t - t')$. This white-noise property of the velocity greatly simplifies matters: the evolution equations for such statistical quantities as the
correlation functions and probability density function of the magnetic field can be derived in closed form and yield themselves to exact solution.

In this paper, we relax the white-noise assumption and explore the effects that arise when a finite-time correlated velocity field is introduced. This immediately raises the level of difficulty associated with solving the statistical problem. Within the theoretical framework adopted here, the difficulty can be described in the following terms. In the zero-correlation-time approximation, one essentially has to deal with only one closed differential equation that fully determines the desired statistics. Allowing for a finite velocity correlation time leads to an infinite number of interlinked integro-differential equations involving time-history integrals. These equations form an infinite open hierarchy that formally constitutes the exact description of the problem (these matters are explained in more detail in Sec. II A). Solving this hierarchy in its entirety without additional assumptions appears to be an impossible task. The most obvious way to make progress is clearly to try a perturbative approach, i.e., to consider the kinematic dynamo problem with an advecting field whose correlation time is short, but finite. If the correlation time $\tau_c$ is assumed to be small, one can expect to be able to construct an expansion in the powers of $\tau_c$ (in what follows, we will frequently refer to it as the $\tau$ expansion) and calculate corrections to the growth rates of the moments of the magnetic field. This is the program that we undertake here.

We consider the one-point statistics of the passive magnetic field in the diffusion-free regime. In this context, the infinite hierarchy we have mentioned above interrelates the one-point probability density function (PDF) of the magnetic field and an infinite set of response functionals. These are averaged multiple functional derivatives of the magnetic field with respect to the velocity field and its gradients. We develop a functional expansion method that allows us to calculate successive terms in the $\tau$ expansion and derive in a closed form a Fokker–Planck equation for the one-point PDF of the magnetic field. We limit ourselves to advancing the expansion one order beyond the zero-correlation-time approximation. The result is a set of corrections to the growth rates of all moments of the magnetic field. These corrections are negative, so the growth rates are reduced.

The expansion is carried out assuming that the velocity correlation time is small and keeping the time integral of the velocity correlation function fixed. The latter constraint ensures that the dynamo growth rate remains finite when the correlation time vanishes. An alternative way, which is sometimes deemed preferable on physical grounds (see, e.g., Ref. [17]), is to fix the total energy of the velocity field. Since a $\delta$-correlated velocity field must necessarily possess infinite energy, fixing the energy at a finite value leads to vanishing of the growth rates when $\tau_c = 0$. The relative ordering of the terms in the expansion is, however, the same, regardless of what is kept fixed, so the technical side of the expansion method is unaffected.

Our expansion technique will be given detailed treatment in the body of this paper. Here, let us rather discuss the finite-correlation-time effects that can be distilled on the basis of our approach. As it turns out, a number of new interesting phenomena manifest themselves already at the level of the short-but-finite-correlation-time approximation.

In the case of the $\delta$-correlated advecting flow, the one-point statistics of the passive magnetic field are universal in the sense that they only depend on one small-scale property of the velocity: the time integral of the one-point correlation tensor of its gradients, $\int dt \langle \nabla \mathbf{u}(t) \nabla \mathbf{u}(0) \rangle$. The essential novelty in the case of finite correlation time is that this small-scale universality is lost on two accounts.

First, the $\tau$ expansion exhibits a sensitive dependence on the specific shape of the time-correlation profile of the velocity field (in recent literature, this was first explicitly pointed out by Boldyrev [18]; see also Refs. [19, 20]). Namely, multiple time integrals of products of velocity correlation functions enter the expressions for the expansion coefficients. Choosing different correlation profiles leads to order-one changes in the values of these coefficients. The root of this nonuniversality lies in the topology of the vertex-correction diagrams that contribute to the orders higher than the zeroth in the $\tau$ expansion (see Sec. II B).

Second, the first-order terms of the $\tau$ expansion feature a part that arises from the fourth-order derivatives of the velocity correlation function, i.e., from the second derivatives of the velocity field. In the one-point statistical approach, this is the first manifestation of the more general tendency that introducing finite correlation times brings into play the large-scale structure of the velocity field. A related effect is the loss of Galilean invariance due to the fact that the expansion terms also depend on the actual energy of the velocity field, i.e., on the rms value of the sweeping velocity. Indeed, now that the trajectories of the fluid elements have a “memory” of themselves, which extends approximately one $\tau_c$ back in time, we should naturally expect that there will appear an effective “correlation length” of the velocity (in what regards the one-point statistics of the fields it advects) approximately equal to $\mathbf{u} r_c$. Therefore, the one-point statistics of the passive fields now depend not only on the instantaneous velocity difference between two fluid particles that meet at a given time (i.e., the velocity gradient at a point), but also on the velocity that swept them into place and on the variation of the velocity gradient over the correlation length. This appearance of first-order corrections due to the second derivatives of the flow is a new effect, which indicates, in particular, that the customary approximation used in the Batchelor regime, where the advecting velocity is assumed to be locally linear, is only justified for the $\delta$-correlated-in-time advecting fields.

Such are the main qualitative consequences of introducing a finite-time-correlated velocity field into the kinematic
dynamo problem (or, in general, any passive-advection model). A few words are in order as to the quantitative impact of a finite correlation time on the dynamo action. As we have already mentioned, the effect of the first-order corrections is to reduce the growth rates of all moments of the magnetic field. Besides the nonuniversal dependence on the spatial and temporal structure of the velocity correlation function, the reduction depends in a universally calculable way on the usual set of parameters: the order of the moment, the dimension of space, and the degree of compressibility of the flow. The overall magnitude of this reductive effect is measured by the expansion parameter, which is of the order of \( \tau_c \gamma \), where \( \gamma \) is the growth rate of the magnetic energy. It is not hard to demonstrate (see Sec. III D) that \( \gamma d \sim (\tau_c / \tau_{eddy})^2 \), where \( \tau_{eddy} \) is the “eddy-turnover” time of the advecting turbulent velocity field and \( d \) is the dimension of space. In a standard Kolmogorov-type turbulence setting, one would, of course, expect any such approximation to be valid at best marginally, since \( \tau_c \sim \tau_{eddy} \). Astrophysical plasmas offer more variety in this respect, as their driving forces (typically supernova explosions) can, in fact, decorrelate faster than the turbulent eddies turn over [2]. In any event, the small-\( \tau_c \) expansion does not offer much more than qualitative, or, at best, semiquantitative, information about the way the dynamo action is modified by the finiteness of the correlation time. It is, of course, clear that introducing a finite correlation time cannot altogether suppress the fast-dynamo mechanism [16,7]. On the other hand, our conclusion that some reduction of the growth rate should be expected, is corroborated by numerical evidence [23,17,24] that suggests a reduction of about 40% to 50%. In fact, in Sec. IV we offer a semiquantitative evaluation of the finite-\( \tau_c \) correction to the growth rate which yields a reduction of approximately 40% in the three-dimensional case and for \( \tau_c \sim \tau_{eddy} \). Of course, this is at best just an indication of the well-behaved character of our expansion, rather than a truly solid quantitative confirmation of it.

The literature on the \( \tau \) expansion and finite-correlation-time effects is not extensive. Kliatskin and Tatarskii [23] were the first to propose the hierarchy of equations for the response functionals as a starting point for a method of successive approximations as applied to the description of waves propagating in a medium with random inhomogeneities. Vainshtein [24] applied this method to the mean-field kinematic dynamo theory. The Kliatskin–Tatarskii method and its relation to our functional expansion method are discussed at the end of Sec. II C. Van Kampen [19] and Terwiel [27] developed the so-called cumulant expansion method; van Kampen’s review article [20] also contains a good critical survey of other \( \tau \)-expansion schemes predating his work. His method was later applied in the kinematic-dynamo context by Knobloch [28] and Chandran [23]. Their treatment was Lagrangian and did not include any effects due to the explicit spatial dependence in the induction equation. Consequently, the nonuniversality of the \( \tau \) expansion with respect to the spatial structure of the velocity correlator was not captured. The van Kampen method is discussed in detail in Sec. III D. Parallel to our development of the functional expansion method, Boldyrev [18] proposed a \( \tau \)-expansion method that was based on the exact solution of the induction equation in the Lagrangian frame and offered a way to calculate the second moment of the magnetic field that elicited the nonuniversal character of the \( \tau \) expansion with respect to both temporal and spatial properties of the velocity correlation tensor. Molchanov, Ruzmaikin, and Sokoloff [10] considered the statistics of the kinematic dynamo in a renovating flow using the formalism of infinite products of random matrices. (See also Ref. [3] for the treatment of the kinematic mean-field dynamo in a renovating flow.) A version of their approach was later advanced by Gruzinov, Cowley, and Sudan [1]. Considerable progress was achieved in a nonperturbative way by Chertkov et al. [29], who studied the passive-scalar problem in two dimensions for arbitrary velocity correlation times. However, their method only works in the two-dimensional case.

Thus, while we now seem to have a fairly good understanding of the structure of the \( \tau \) expansion and such qualitative features as the loss of the small-scale universality, an adequate nonperturbative theory of the kinematic dynamo and passive advection in finite-time-correlated turbulent velocity fields remains an open problem.

This paper is organized in the following way. In Sec. I our functional expansion method is systematically developed on the example of the simplest available passive-advection problem: that of the Lagrangian passive vector in an incompressible flow. In this model, no explicit spatial dependence is present. In Sec. I A, Sec. I B, and Sec. I C we present a functional formalism that allows one to systematically construct successive terms in the \( \tau \)-expanded Fokker–Planck equation. The dependence of the expansion coefficients on the specific functional form of the velocity time correlation profile emerges. The expansion is carried out up to the first order in \( \tau_c \). In Sec. I D our method is compared with the van Kampen cumulant expansion method [19]. We ascertain that results obtained via the van Kampen method are consistent with ours. Finally, in Sec. I E we discuss the underlying structure of the \( \tau \) expansion in diagrammatic terms. In Sec. II, the general arbitrarily compressible space-dependent dynamo problem is solved with the aid of the functional expansion. At this level, the nonuniversality with respect to the spatial structure of the velocity correlations, as well as the loss of Galilean invariance, become evident. In Sec. II A we explain the emergence of an infinite hierarchy of equations for the characteristic function and various averaged response functionals of the magnetic field in the passive dynamo problem with finite-time-correlated advecting flow. The hierarchy is advanced up to the emergence of the second-order response functions. In Sec. II B we construct the \( \tau \) expansion up to first order in the correlation time, which leads to a closed equation for the characteristic function of the magnetic field. In Sec. II C we derive the Fokker–Planck equation for the one-point PDF of the magnetic-field strength valid to first order in the correlation time. The distribution is lognormal. In Sec. II D, we calculate the rates of growth of all
moments of the magnetic field with (negative) first-order corrections. Finally, in Sec. IV, we give a semi-quantitative argument that relates the expansion parameter to the ratio of the correlation and eddy-turnover times of the velocity field. We also evaluate the finite-τc reduction of the magnetic-energy growth rate in a model incompressible turbulence consisting of eddies of a fixed size. In Appendix A we provide the basic relations that allow one to express the results we have obtained in the configuration space in terms of the spectral characteristics of the velocity field. Some of the more cumbersome technical details of the τ expansion are exiled to Appendix B.

II. THE GAUSSIAN FUNCTIONAL EXPANSION FORMALISM

In this Section, we explain the Gaussian functional method for constructing the short-correlation-time expansion for passive advection problems. Working out such expansions for specific problems often involves a fair amount of algebra, which tends to obscure the otherwise transparent ideas behind them. In an attempt at the maximum possible clarity of exposition, we first consider a model that, while preserving most of the essential features of the passive-advection problems, offers much greater technical simplicity. Namely, let us consider the following stochastic equation in d dimensions:

\[ \partial_t B^i = \sigma^i_k B^k; \quad (1) \]

All the fields involved explicitly depend on time only. The specific initial distribution of \( B^i \) is not important for the derivation or the validity of the results below. Spatial isotropy is always assumed. The matrix field \( \sigma_k^i(t) \) is Gaussian with zero mean and a given two-point correlation tensor:

\[ \langle \sigma_k^i(t) \sigma^l_j(t') \rangle = T_k^{ij} \kappa(t - t'), \quad (2) \]

\[ T_k^{ij} = \delta^{ij} \delta_{kl} + a (\delta^i_k \delta^j_l + \delta^j_k \delta^i_l), \quad a = -1/(d + 1). \]

These equations can be interpreted to describe the evolution of a passive magnetic field in a Lagrangian frame, where the Lagrangian advecting velocity field is Gaussian and incompressible, and the tensor \( \sigma_k^i \) is its gradient matrix. In the more general context of the theory of passive advection, equations (1) and (2) model the stochastic dynamics of a vector connecting two Lagrangian tracer particles in an ideal fluid.

We assume that the temporal correlation function \( \kappa(t - t') \) of \( \sigma_k^i(t) \) has a characteristic width \( \tau_c \), i.e., the field \( \sigma_k^i(t) \) possesses a correlation time \( \tau_c \). Our task in this section is to construct an expansion of the statistics of \( B^i(t) \) in powers of \( \tau_c \), which is assumed to be small. The limit \( \tau_c \to 0 \) ought to be taken in such a way that the time integral of the correlation function is kept constant:

\[ \int_0^\infty d\tau \kappa(\tau) = \frac{\tilde{\kappa}}{2} = \text{const} \quad \text{and} \quad \tau_c \tilde{\kappa} \ll 1. \quad (3) \]

The white-noise limit of zero correlation time is realized by setting \( \kappa(\tau) = \tilde{\kappa} \delta(\tau) \).

The first step in our averaging scheme is to define the characteristic function of the field \( B^i(t) \),

\[ Z(t; \mu) = \langle \tilde{Z}(t; \mu) \rangle = \langle \exp[i \mu_i B^i(t)] \rangle. \quad (4) \]

Here and in what follows, the overtilde designates unaveraged random functions. Upon differentiating \( \tilde{Z}(t; \mu) \) with respect to time and making use of Eq. (1), we obtain a new stochastic equation:

\[ \partial_t \tilde{Z} = \sigma_k^i \mu_i \frac{\partial}{\partial \mu_k} \tilde{Z} = \tilde{\Lambda}_k^i \sigma_k^i \tilde{Z}, \quad (5) \]

where the auxiliary operator \( \tilde{\Lambda}_k^i \) has been introduced for the sake of notational compactness.

Our objective now is to learn how to obtain a closed equation for the averaged characteristic function \( Z(t; \mu) \), i.e., how to average Eq. (5) when \( \sigma_k^i \) has a nonzero correlation time. The inverse Fourier transform (with respect to \( \mu_i \)) of the resulting equation will be the Fokker–Planck equation for the PDF \( P(t; B) \) of the passive field \( B^i \).

A. The Hierarchy of Response Functions

We start the construction of the functional expansion by developing an exact formalism that describes the one-point statistics of the field \( B^i(t) \). Let us average both sides of Eq. (5) and “split” the mixed average that arises on the right-hand side with the aid of the well-known Furutsu–Novikov (or “Gaussian-integration”) formula [30,31]:

\[ \langle \sigma_k^i B^k \rangle = \int P(t; B) \langle \sigma_k^i \rangle dB^k. \]
\[
\frac{\partial}{\partial t} Z(t) = \hat{\Lambda}^k_k \langle \sigma_k^i(t) \hat{Z}(t) \rangle = \hat{\Lambda}^k_k T_{k\alpha} \int_0^t dt' \kappa(t-t') G^{\alpha \beta}_3(t|t'),
\]
where we have suppressed the \(\mu\)'s in the arguments, used the formula (4) for the second-order correlation tensor of \(\sigma_k^i(t)\), and introduced the averaged first-order response function:
\[
G^{\alpha \beta}_3(t|t') = \langle \hat{G}^{\alpha \beta}_3(t|t') \rangle = \left\langle \frac{\delta \hat{Z}(t)}{\delta \sigma^\beta_\alpha(t')} \right\rangle.
\]
This function is subject to the causality constraint: \(G^{\alpha \beta}_3(t|t') = 0\) if \(t' > t\) [hence the upper integration limit in Eq. (6)].
\[
\text{Integrating Eq. (5) from 0 to } t, \text{ taking the functional derivative } \delta/\delta \sigma^\beta_\alpha(t') \text{ of both sides, averaging, setting } t' = t, \text{ and taking causality into account, we get}
\]
\[
G^{\alpha \beta}_3(t) = \check{\Lambda}^\beta_\beta Z(t).
\]
We have thus obtained the equal-time form of \(G^{\alpha \beta}_3(t|t')\). In order to determine the response function at \(t' < t\), we simply take the functional derivative \(\delta/\delta \sigma^\beta_\alpha(t')\) of both sides of Eq. (5) and find that each element of the unaveraged tensor \(\hat{G}^{\alpha \beta}_3(t|t')\) satisfies an equation identical in form to Eq. (5). Upon averaging this, we obtain an evolution equation for \(G^{\alpha \beta}_3(t|t')\) subject to the initial condition (5) at \(t = t'\):
\[
\frac{\partial}{\partial t} G^{\alpha \beta}_3(t|t') = \hat{\Lambda}^\beta_\beta \langle \sigma^\beta_\alpha(t) \hat{G}^{\alpha \beta}_3(t|t') \rangle.
\]
This equation can now be handled in the same fashion as Eq. (5), the average on the right-hand side split via the Furutsu–Novikov formula in terms of the correlation tensor of \(\sigma^\beta_\alpha(t)\) and the appropriately defined second-order averaged response function \(G^{\alpha \beta}_{\beta_1 \beta_2}(t|t_1, t_2)\). At equal times, the latter can be expressed in terms of \(G^{\alpha \beta}_3(t|t')\) just as \(G^{\alpha \beta}_3(t|t)\) was expressed in terms of \(Z(t)\). At different times, we obtain the evolution equation for \(G^{\alpha \beta}_{\beta_1 \beta_2}(t|t_1, t_2)\) by taking the functional derivative of the equation for \(\hat{G}^{\alpha \beta}_3(t|t')\) and averaging.

An infinite linked hierarchy can be constructed by further iterating this procedure and introducing response functions of ascending orders. Let us give the general form of this hierarchy. Define the \(n\)-th order averaged response function:
\[
G^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}(t|t_1, \ldots, t_n) = \left\langle \frac{\delta \hat{Z}(t)}{\delta \sigma^\beta_\alpha(t_1) \ldots \delta \sigma^\beta_\alpha(t_n)} \right\rangle.
\]
This function has two essential properties: (i) it is causal: \(G^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}(t|t_1, \ldots, t_n) = 0\) if any \(t_i > t\); (ii) it remains invariant under all simultaneous permutations of the times \(t_1, \ldots, t_n\) and indices \(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n\), which correspond to changes of the order of functional differentiation in the definition (10). The \(n\)-th order response function satisfies the following recursive relations: if \(t_1, \ldots, t_n < t\),
\[
\frac{\partial}{\partial t} G^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}(t|t_1 \ldots t_n) = \hat{\Lambda}^k_k T_{k\alpha} \int_0^t dt' \kappa(t-t') G^{\alpha_1 \ldots \alpha_n+1}_{\beta_1 \ldots \beta_n+1}(t|t_1 \ldots t_{n+1});
\]
if, say, \(t_n = t\) and \(t_1, \ldots, t_{n-1} \leq t\),
\[
G^{\alpha_1 \ldots \alpha_n-1 \alpha_n}_{\beta_1 \ldots \beta_n-1 \beta_n}(t|t_1 \ldots t_{n-1}, t) = \check{\Lambda}^{\alpha_n} \hat{G}^{\alpha_1 \ldots \alpha_n-1}_{\beta_1 \ldots \beta_n-1}(t|t_1 \ldots t_{n-1}).
\]

The hierarchy is “forward” at different times and “backward” at equal times. The characteristic function \(Z(t)\) is formally treated as the zeroth-order response function.

**B. The White-Noise Approximation**

The white-noise approximation is obtained by setting \(\kappa(t-t') = \hat{\kappa} \delta(t-t')\). We are then left with just Eq. (5), where the time history integral reduces to \(\frac{\hat{\kappa}}{2} \hat{G}^{\alpha \beta}_3(t|t)\), which is substituted from Eq. (8). This produces a closed evolution equation for \(Z(t)\). The Fourier transform of it is the Fokker–Planck equation for the PDF of \(B^3\) at time \(t\) in the \(\delta\)-correlated regime:
\[
\frac{\partial}{\partial t} P(t) = \frac{\hat{\kappa}}{2} T_{k\alpha} \hat{\Lambda}^k_k \hat{\Lambda}^\beta_\beta P(t) = \frac{\hat{\kappa}}{2} \hat{L} P(t).
\]
In order to be not overly burdened by notation, we typically use the same symbol for denoting an operator in the Fourier space of the \( \mu \)'s and its analog in the configuration space of the \( B \)'s. This should lead to no confusion, as the context will always be clear. Thus,

\[
\hat{\Lambda}^k_l = \mu_i \frac{\partial}{\partial \mu_k} = - \frac{\partial}{\partial B^l} B^k = - \left( \delta^k_l + B^k \frac{\partial}{\partial B^l} \right). \tag{14}
\]

Due to isotropy, the probability density function \( P \) depends on the absolute value \( B = |B| \) only. The operator \( \hat{L} \) in Eq. (13) can therefore be written in the following isotropic form:

\[
\hat{L} = T^{i\beta}_{\kappa\alpha} \hat{\Lambda}^\alpha_{\beta} = \frac{d - 1}{d + 1} \left[ B^2 \frac{\partial^2}{\partial B^2} + (d + 1) B \frac{\partial}{\partial B} \right], \tag{15}
\]

which turns Eq. (13) into the familiar Fokker–Planck equation for the one-point PDF of the magnetic field in the kinematic \( \delta \)-correlated dynamo problem taken for the incompressible flow. The resulting distribution is lognormal and the moments of \( B \) satisfy

\[
\partial_t \langle B^n \rangle = \frac{1}{2} \frac{d - 1}{d + 1} n (n + d) \kappa \langle B^n \rangle. \tag{16}
\]

This is the expected outcome, because, as was shown in Ref. [8], the Lagrangian and Eulerian statistics are the same for the \( \delta \)-correlated incompressible flow.

Thus, the solution in the \( \delta \)-correlated limit is quite elementary. Things become much more complicated once the white-noise assumption is relaxed and a nonzero, however small, velocity correlation time is introduced.

### C. The Recursive Expansion

In order to construct an expansion in small correlation time, it is convenient to combine the equations (11) and (12) into one recursive integral relation that expresses the \( n \)th-order response function in terms of its immediate precursor and its immediate successor:

\[
G^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}(t|t_1 \ldots t_n) = \hat{\Lambda}^{\alpha_n}_{\beta_n} G^{\alpha_1 \ldots \alpha_{n-1}}_{\beta_1 \ldots \beta_{n-1}}(t_n|t_1 \ldots t_{n-1}) \\
+ \hat{\Lambda}^k_l T^{i\beta}_{\kappa\alpha} \int_{t_n}^{t} dt' \int_{t_n}^{t'} dt_{n+1} \kappa(t' - t_{n+1}) G^{\alpha_1 \ldots \alpha_{n+1}}_{\beta_1 \ldots \beta_{n+1}}(t'|t_1 \ldots t_{n+1}). \tag{17}
\]

The above relation is exact and valid for \( t_1, \ldots, t_{n-1} \leq t_n \leq t \). Due to the permutation symmetry of the response functions, this does not limit the generality. The desired expansion is constructed by repeated application of the formula (17).

Let us substitute the formula (17) with \( n = 1 \) for the first-order response function into the right-hand side of Eq. (1):

\[
\partial_t Z(t) = \hat{L} \int_0^t dt_1 \kappa(t - t_1) Z(t_1) \\
+ T^{i\beta}_{\kappa\alpha} T^{m\beta_2}_{\alpha_2} \hat{\Lambda}^k_l \hat{\Lambda}^n_m \int_0^t dt_1 \int_{t_1}^t dt' \int_0^{t'} dt_{2} \kappa(t - t_1) \kappa(t' - t_2) G^{\alpha_1 \alpha_2}_{\beta_1 \beta_2}(t'|t_1, t_2), \tag{18}
\]

where the operator \( \hat{L} \) is defined in (15). We now use the formula (17) to express the second-order response function on the right-hand side of the above equation: for \( t_2 > t_1 \), we have

\[
G^{\alpha_1 \alpha_2}_{\beta_1 \beta_2}(t'|t_1, t_2) = \hat{\Lambda}^{\alpha_2}_{\beta_2} G^{\alpha_1}_{\beta_1}(t_2|t_1) \\
+ \hat{\Lambda}^k_l T^{i\beta}_{\kappa\alpha} \int_{t_2}^{t'} dt'' \int_0^{t''} dt_3 \kappa(t'' - t_3) G^{\alpha_2 \alpha_3}_{\beta_2 \beta_3}(t''|t_1, t_2, t_3), \tag{19}
\]

while for \( t_2 < t_1 \) we flip the variables, \( t_1 \leftrightarrow t_2 \), to make sure that the first-order response function on the right-hand side do not vanish.
\[ G^{\alpha_1\alpha_2}(t'|t_1,t_2) = \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_2}(t_1|t_2) + \hat{T}_{\rho q_3}^{\beta_3} \int_{t_1}^{t'} dt'' \int_{0}^{t''} dt_3 \kappa(t'' - t_3)G^{\alpha_3\alpha_3}(t''|t_1,t_3). \]  

(20)

The recursion relation (17) is now applied to the first-order response functions in the formulas (19) and (20):

\[ G^{\alpha_1}_1(t_2|t_1) = \hat{\Lambda}_{\beta_1}^{\alpha_1} Z(t_1) + \hat{T}_{\rho q_3}^{\beta_3} \int_{t_1}^{t} dt'' \int_{0}^{t''} dt_3 \kappa(t'' - t_3)G^{\alpha_3\alpha_3}(t''|t_1,t_3), \]

(21)

\[ G^{\alpha_2}_2(t_2|t_1) = \hat{\Lambda}_{\beta_2}^{\alpha_2} Z(t_2) + \hat{T}_{\rho q_3}^{\beta_3} \int_{t_2}^{t} dt'' \int_{0}^{t''} dt_3 \kappa(t'' - t_3)G^{\alpha_3\alpha_3}(t''|t_2,t_3). \]

(22)

All this must be substituted into Eq. (13):

\[ \partial_t Z(t) = \hat{L} \int_{0}^{t} dt_1 \kappa(t - t_1)Z(t_1) + T_{k\alpha_1}^{\beta_1} T_{\alpha_2}^{m_2} \hat{\Lambda}_{k}^{m_2} \hat{\Lambda}_{\alpha_1}^{\beta_1} \int_{0}^{t} dt_1 \int_{0}^{t} dt_2 \kappa(t - t_1)\kappa(t' - t_2)Z(t_2) \]

\[ + R(t), \]

(23)

where the remainder \( R(t) \) contains the assembled terms that involve quintuple time integrals.

So far, all the manipulations we have carried out have been exact. It is now not hard to perceive the emerging contours of the small-\( \tau_c \) expansion. Since the time-correlation function \( \kappa(t - t') \) is a profile of width \( \sim \tau_c \), the area under which is constant and equal to \( \bar{c} \), the first in Eq. (30) is constant and equal to \( \bar{\tau} \).

Upon substituting this onto the right-hand side of Eq. (23) and again discarding all the terms of orders higher than \( \bar{\tau} \), we get

\[ \kappa = \frac{3}{\tau_c^2} \kappa(0), \]

(3)

This requirement is natural because it leads to finite dynamo growth rates in the limit of \( \tau_c \to \infty \). However, it is also acceptable to institute an alternative, arguably more physical, requirement that the zero correlation time [see Sec. II B, Eq. (16)]. Quantitatively, this means that \( \kappa(0) \), rather than \( \int_{0}^{\infty} dt \kappa(t) \), is kept fixed. Under this constraint, the terms that we have previously estimated to be of orders \( \kbar, \tau_c \kbar^2, \tau_c^2 \kbar^3 \), and hence, \( \bar{\kappa} \) being constant, to represent the zeroth, first, second, etc. orders of the \( \tau_c \) expansion, should now be reevaluated as follows. Since \( \bar{\kappa} \sim \tau_c \kappa(0) \), these terms are of orders \( \tau_c \kappa(0), \tau_c^2 \kappa(0)^2, \tau_c^3 \kappa(0)^3 \), etc., and therefore constitute the first, third, fifth, etc. orders of the \( \tau_c \) expansion. The shortcoming of this approach is that the dynamo growth rates vanish when \( \tau_c = 0 \), so formally there is no nontrivial zero-correlation-time limit. With \( \bar{\kappa} = \text{const} \), this problem was avoided because the energy was formally infinite when \( \tau_c = 0 \) (a \( \delta \)-correlated velocity field cannot have a finite energy). In any event, we see that, since the difference between keeping \( \bar{\kappa} \) and \( \kappa(0) \) constant does not affect the relative magnitudes of the terms in the expansion, our expansion scheme remains valid in both cases. Let us therefore proceed with our construction.

The dependence of the right-hand side of Eq. (24) on the “past” values of \( Z \) (i.e., on its values at times preceding \( t \)) can also be resolved in the framework of the small-\( \tau_c \) expansion. Formally integrating Eq. (24), we get, at times \( t_1 < t \),

\[ Z(t_1) = Z(t) - \hat{L} \int_{t_1}^{t} dt'_1 \int_{0}^{t'} dt_2 \kappa(t' - t_2)Z(t_2) + \cdots \]

(24)

Upon substituting this onto the right-hand side of Eq. (24) and again discarding all the terms of orders higher than the first in \( \tau_c \), we get
∂ₜZ(t) = \hat{L} \int_0^t dt_1 \kappa(t - t_1)Z(t)
- \hat{L}^2 \int_0^t dt_1 \int_0^{t_1} dt_1' \int_0^{t_1} dt_2 \kappa(t - t_1)\kappa(t' - t_2)Z(t)
+ (\hat{L}^2 - \hat{L}_1) \int_0^t dt_1 \int_0^{t_1} dt_1' \int_0^{t_1} dt_2 \kappa(t - t_1)\kappa(t' - t_2)Z(t)
+ (\hat{L}^2 - \hat{L}_2) \int_0^t dt_1 \int_0^{t_1} dt_1' \int_0^{t_1} dt_2 \kappa(t - t_1)\kappa(t' - t_2)Z(t).

We have introduced the following two operators:
\hat{L}_1 = \hat{\Lambda}^{\kappa\tau}_{\alpha\beta} \Lambda_{\beta_1} \Lambda_{\beta_2} T^{m\beta_2} \Lambda_{m\alpha} = \frac{d^2}{d + 1} \hat{L},
\hat{L}_2 = \hat{\Lambda}^{\kappa\tau}_{\alpha\beta} \Lambda_{\beta_1} \Lambda_{\beta_2} \hat{L} = 2d\hat{L},

where the square brackets denote commutators. We see that the terms in Eq. (25) that contain \hat{L}^2 cancel out, and only those terms remain that are due to the non-self-commuting nature of the operator \hat{\Lambda}^\kappa_{\beta_1}.

Finally, we inverse-Fourier transform Eq. (25) into the B space and take the long-time limit, \( t \gg \tau_c \). The following Fokker–Planck equation with constant coefficients results:
\partial_\tau P = \frac{\kappa}{2} \left[ 1 - \tau_c \kappa d \left( \frac{1}{2d + 1} K_1 + K_2 \right) \right] \hat{L} P,

where the coefficients,
\begin{align*}
K_1 &= \frac{4}{\tau_c k^2} \lim_{t \to \infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_1} dt_3 \kappa(t - t_1)\kappa(t_2 - t_3), \\
K_2 &= \frac{4}{\tau_c k^2} \lim_{t \to \infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_1} dt_3 \kappa(t - t_1)\kappa(t_2 - t_3),
\end{align*}

are constants that depend on the particular shape of the time-correlation function \( \kappa(t - t') \). Thus, the \( \tau \) expansion is nonuniversal in the sense that the specific choice of the functional form of the small-time regularization directly affects the values of the expansion coefficients (cf. Ref. [18]). As an example, let us give the values of the coefficients \( K_1 \) and \( K_2 \) for two popular choices of \( \kappa(t - t') \):
\begin{align*}
\kappa(t - t') &= \frac{\kappa}{2\tau_c} \exp[-|t - t'|/\tau_c] \quad \Rightarrow \quad K_1 = K_2 = 0.5 \\
\kappa(t - t') &= \frac{\kappa}{\sqrt{\pi} \tau_c} \exp[-(t - t')^2/\tau_c^2] \quad \Rightarrow \quad K_1 \approx 0.33, \quad K_2 \approx 0.23.
\end{align*}

In Sec. IV, we will apply the method we have presented above to the more realistic general compressible kinematic dynamo problem in the Eulerian frame.

Remark on the Kliatskin–Tatarskii method. The Gaussian hierarchy given by the equations (11) and (12) and based on repeated application of the Furutsu–Novikov formula was proposed by Kliatskin and Tatarskii [25] as a basis for constructing successive-approximation solutions of the problem of light propagation in a medium with randomly distributed inhomogeneities. Their method in its original form was carried over to the mean-field dynamo theory with finite-time-correlated velocity field by Vainshtein [26]. The method we have outlined in this section, while also based on the response-function hierarchy (11)–(12), differs substantially from that developed and applied by these authors. Their successive-approximation scheme consisted essentially in writing out the first \( n \) equations in the hierarchy (11)–(12) and then truncating it at the \( n \)th step by replacing \( \kappa(t - t_{n+1}) \) by \( \kappa \delta(t - t_{n+1}) \) in the equation for the \( n \)th-order response function. This gave a closed system of equations that could be solved. Carried out in the first order, such a procedure would correspond to setting \( \kappa(t' - t_2) = \kappa \delta(t' - t_2) \) in Eq. (18) and consequently \( \kappa(t_2 - t_3) = \kappa \delta(t_2 - t_3) \) in the expressions (26) and (30) for the coefficients \( K_1 \) and \( K_2 \). Such a substitution leads to \( K_1 = 0 \) and \( K_2 = (2/\tau_c \kappa) \int_0^{\infty} d\tau \kappa \kappa(\tau), \) which is incorrect. The reason for this discrepancy is that, in the time integrals involving multiple products of the correlation functions \( \kappa(t - t_1), \kappa(t_2 - t_3), \) etc., the latter cannot be approximated by \( \delta \) functions plus first-order corrections even in the small-\( \tau_c \) limit.
D. Comparison with the Van Kampen Cumulant Expansion Method

The evolution equation (3) for the “unaveraged characteristic function” \( \tilde{Z}(t; \mu) \) is a stochastic linear differential equation whose form agrees exactly with that of the general such equation considered by van Kampen \[19,20\] and simultaneously by Terwiel \[21\]:

\[
\partial_t \tilde{Z}(t) = \hat{A}(t) \tilde{Z}(t),
\]

where, in our case, \( \hat{A}(t) = \hat{\Lambda}_k^i \sigma_k^i(t) \). In his work, van Kampen developed a formalism that allowed one to construct successive terms in the short-correlation-time expansion of \( Z = \langle \tilde{Z} \rangle \) in terms of the cumulants of the operator \( \hat{A} \). Terwiel’s projection-operator method was shown by its author to be equivalent to that of van Kampen.

Let us see what happens if the passive-advection problem given by Eq. (1), or, equivalently, by Eq. (3) is subjected to van Kampen’s expansion algorithm. The latter proceeds as follows.

Start by writing the formal solution of Eq. (3) in terms of the time-ordered exponential:

\[
\tilde{Z}(t) = \left[ \exp \int_0^t dt' \hat{A}(t') \right] \tilde{Z}(0)
= \left[ 1 + \int_0^t dt_1 \hat{A}(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{A}(t_1)\hat{A}(t_2) + \cdots \right] \tilde{Z}(0).
\]

This solution is averaged assuming that the initial distribution of \( \tilde{Z} \) is independent of the statistics of \( \hat{A} \):

\[
Z(t) = \left[ 1 + \int_0^t dt_1 \int_0^{t_1} dt_2 \langle \hat{A}(t_1)\hat{A}(t_2) \rangle \right. \\
+ \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \langle \hat{A}(t_1)\hat{A}(t_2)\hat{A}(t_3)\hat{A}(t_4) \rangle + \cdots \left. \right] Z(0).
\]

Here all the odd-order averages have vanished (recall that \( \hat{A} = \hat{\Lambda}_k^i \sigma_k^i \)). The closed equation for \( Z(t) \) is now obtained as follows. First, the formal solution (35) is differentiated with respect to time:

\[
\partial_t Z(t) = \left[ \int_0^t dt_1 \langle \hat{A}(t)\hat{A}(t_1) \rangle \right. \\
+ \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \langle \hat{A}(t)\hat{A}(t_1)\hat{A}(t_2)\hat{A}(t_3) \rangle + \cdots \left. \right] Z(0).
\]

Second, \( Z(0) \) is expressed in terms of \( Z(t) \) by formally inverting the operator series on the right-hand side of Eq. (33), whereupon \( Z(0) \) is substituted into Eq. (36). Keeping only the terms that contain up to three time integrations, as we did in the previous section, we get

\[
\partial_t Z(t) = \left[ \int_0^t dt_1 \langle \hat{A}(t)\hat{A}(t_1) \rangle \right. \\
+ \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \langle \hat{A}(t)\hat{A}(t_1)\hat{A}(t_2)\hat{A}(t_3) \rangle \right. \\
\left. - \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \langle \hat{A}(t_1)\hat{A}(t_2) \rangle \langle \hat{A}(t_1)\hat{A}(t_2) \rangle + \cdots \right] Z(t).
\]

The quadruple average in the above expression splits into three products of second-order averages in the usual Gaussian way. Since

\[
\langle \hat{A}(t)\hat{A}(t_1) \rangle = \kappa(t - t_1) T_{kl}^{ij} \hat{\Lambda}_k^i \hat{\Lambda}_l^j = \kappa(t - t_1) \hat{L},
\]

we have

\[
\langle \hat{A}(t)\hat{A}(t_1)\hat{A}(t_2)\hat{A}(t_3) \rangle = \kappa(t - t_1)\kappa(t_2 - t_3) T_{kl}^{ij} T_{pq}^{mn} \hat{\Lambda}_k^i \hat{\Lambda}_l^j \hat{\Lambda}_p^m \hat{\Lambda}_q^n \\
+ \kappa(t - t_2)\kappa(t_1 - t_3) T_{kp}^{in} T_{lq}^{jm} \hat{\Lambda}_k^i \hat{\Lambda}_j^m \hat{\Lambda}_p^m \hat{\Lambda}_q^n
\]
\[ + \kappa(t - t_3)\kappa(t_1 - t_2) T_{kq}^{\text{im}} \hat{A}_k \hat{A}_l \hat{A}_m \hat{A}_n \]
\[ = \kappa(t - t_4)\kappa(t_2 - t_3) \hat{L}^2 \]
\[ + \kappa(t - t_4)\kappa(t_1 - t_3) \left( \hat{L}^2 - \hat{L}_1 \right) \]
\[ + \kappa(t - t_3)\kappa(t_1 - t_2) \left( \hat{L}^2 - \hat{L}_2 \right), \]
\[ \langle \hat{A}(t)\hat{A}(t_1) \rangle \langle \hat{A}(t_2)\hat{A}(t_3) \rangle = \kappa(t - t_1)\kappa(t_2 - t_3) \hat{L}^2. \]

The operators \( \hat{L}, \hat{L}_1, \) and \( \hat{L}_2 \) are the same as those in the previous section [see definitions (13), (26), and (27)]. The averages (38), (39), and (40) are now substituted into Eq. (37). The triple time integrals can be argued to represent the coefficients in Eq. (37) do not depend on time. The resulting Fokker–Planck equation for the PDF \( \hat{P}(t; \mu) \) is then
\[ \frac{\partial \hat{P}}{\partial t} = \frac{\kappa}{2} \left[ 1 - \tau_\kappa \int \frac{d}{2d + 1} C_1 + C_2 \right] \hat{L} \hat{P}, \]
where the coefficients \( C_0, C_1, \) and \( C_2, \) that depend on the shape function \( \kappa(t - t') \), are as follows:
\[ C_0 = \frac{4}{\tau_\kappa} \lim_{t' \to \infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_1} dt_3 \kappa(t - t_1)\kappa(t_2 - t_3), \]
\[ C_1 = \frac{4}{\tau_\kappa} \lim_{t' \to \infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_1} dt_3 \kappa(t - t_2)\kappa(t_1 - t_3), \]
\[ C_2 = \frac{4}{\tau_\kappa} \lim_{t' \to \infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_1} dt_3 \kappa(t - t_3)\kappa(t_1 - t_2). \]

By comparing the definition of \( C_0 \) with those of the coefficients \( K_1 \) and \( K_2 \) in Sec. [II.C] [see formulas (28) and (30)], we immediately establish that \( C_0 = K_1 + K_2 \). Furthermore, it is also not hard to ascertain that \( C_1 = K_1 \) and \( C_2 = K_2 \). Therefore, the last term in Eq. (11) vanishes, and the first-order Fokker–Planck equations (28) and (11) are identical. Thus, the results obtained via the van Kampen method are consistent with ours. Unlike the van Kampen method, however, our method does not involve any nontrivial operator algebra and is therefore better suited for a wide variety of applications. In particular, the stochastic equations containing spatial derivatives (such as the convective derivatives present in all Eulerian passive-advection problems) can be handled without much additional difficulty (this will be done in detail for the full kinematic dynamo problem in Sec. [II]).

**E. Discussion: The Vertex Corrections**

While the particular methods one employs to obtain the successive terms in the \( \tau \) expansion may vary and depend on one’s taste and the specific demands of the stochastic problem at hand, the underlying structure of the \( \tau \) expansion remains the same and is rooted in the common properties of all turbulence closure problems (see, e.g., Ref. [32]). As we have stated in general terms in the introduction to this paper, and as was clear from our construction of the response-function formalism in Sec. [II.A]–Sec. [II.C] or of van Kampen’s explicit series solution (35) in Sec. [II.D] averaged solutions of stochastic equations such as Eq. (1) can be represented in terms of infinite sums of multiple time-history integrals containing products of time-correlation functions \( \kappa(t_i - t_j) \) in the integrands. This summation can be visualized in terms of Feynman-style diagrams. The \( n \)-point diagrams represent the terms containing \( n \) time-history integrations. As an example, Fig. 1 lists the three possible four-point diagrams.

It was noted by Kazantsev [33] (see also Ref. [34]) that the white-noise approximation corresponds to the partial summation of all ladder-type diagrams such as the four-point one shown in Fig. 1(a). The distinctive property of these diagrams is that the pairs of points \( t_i, t_j \) at which the time-correlation functions in the integrands of the time-history integrals are taken, can be fused without interfering with each other. No essential information is therefore lost when the time-correlation functions \( \kappa(t_i - t_j) \) are approximated by \( \delta \) functions. However, in all orders of the \( \tau \) expansion but the zeroth, diagrams with more tangled topology appear: e.g., in the first order, these are the diagrams 1(b) and 1(c)]. Such diagrams are often referred to as the vertex corrections. Fusing points in these diagrams leads to the loss of terms that cannot be neglected [33]. This is the context in which the emerging nonuniversality with respect to the shape of the time-correlation profile should be viewed.

In this paper, we restrict our consideration to the first-order terms in the \( \tau \) expansion. The relevant diagrams are the four-point ones shown in Fig. 1. The diagrams 1(b) and 1(c) give rise to the coefficients \( C_1 \) and \( C_2, \) respectively
Upon changing variables $t_1 \leftrightarrow t_2$ in the diagram (b) and $t_1 \rightarrow t_2$, $t_2 \rightarrow t_3$, $t_3 \rightarrow t_1$ in the diagram (c), we see that these diagrams equally well correspond to the coefficients $K_1$ and $K_2$ [Eq. (28) and formulas (29) and (30)].

III. THE FUNCTIONAL EXPANSION FOR THE KINEMATIC DYNAMO IN A FINITE-TIME-CORRELATED VELOCITY FIELD

In this Section, we use the functional expansion method developed in Sec. II to construct the $\tau$ expansion for the general diffusion-free kinematic dynamo problem in the Eulerian frame with an arbitrarily compressible velocity field. Through the convective derivative, an explicit spatial dependence is now present in the problem. This leads to the appearance of the new effect advertised in the Introduction: while the zeroth-order terms in the expansion only depend on the one-point correlation properties of the velocity gradients, the first-order terms also depend on the energy of the advecting velocity field and on the one-point correlation function of its second derivatives. The former represents the loss of Galilean invariance, the latter the loss of the small-scale universality and the advent of the sensitive dependence of the statistics on the large-scale structure of the velocity correlations.

In this Section, all statistics are Eulerian. For the questions regarding the transformation of PDF’s of passive fields from the Eulerian to the Lagrangian frame, we address the reader to Ref. [8], as well as to Ref. [18], where the $\tau$ expansion is treated as a problem in stochastic calculus and Lagrangian statistics are discussed.

A. The Gaussian Hierarchy

The magnetic field passively advected by the velocity field $u^i(t, x)$ evolves according to the Hertz induction equation (formally in $d$ dimensions):

$$\partial_t B^i = -u^k B^i_{,k} + u^i_{,k} B^k - u^i_{,k} B^k,$$  \hspace{1cm} (45)

where $u^i_{,k} = \partial u^i / \partial x^k$, $B^i_{,k} = \partial B^i / \partial x^k$, and the Einstein summation convention is used throughout. Let the advecting velocity field $u^i(t, x)$ be a homogeneous and isotropic Gaussian random field whose statistics are defined by its second-order correlation tensor:

$$\langle u^i(t, x) u^j(t', x') \rangle = \kappa^{ij}(t - t', x - x'),$$  \hspace{1cm} (46)
where, as a function of the time separation \( t - t' \), the correlator \( \kappa^{ij} \) is assumed to have a finite width \( \tau_c \), which we will call the velocity correlation time. As we will only study the one-point statistics of the magnetic field, all relevant information about the velocity correlation properties is contained in the Taylor expansion of \( \kappa^{ij} \) around the origin:

\[
\kappa^{ij}(\tau, y) = \kappa_0(\tau) \delta^{ij} - \frac{1}{2} \kappa_2(\tau) [y^2 \delta^{ij} + 2ay^iy^j] \\
\quad + \frac{1}{4} \kappa_4(\tau) y^2[y^2 \delta^{ij} + 2by^iy^j] + \cdots,
\]

(47)
as \( y \to 0 \). Here \( a \) and \( b \) are the compressibility parameters. Between the purely incompressible and the purely irrotational cases, they vary in the intervals

\[
-\frac{1}{d+1} \leq a \leq 1, \quad -\frac{2}{d+3} \leq b \leq 2.
\]

(48)

We should like to mention here that the choice of the coefficients of the small-scale expansion (47) of the velocity correlation tensor is, strictly speaking, not entirely unconstrained. As \( \kappa^{ij}(\tau, y) \) is a correlation function, it must be an inverse Fourier transform of a proper correlation function in the Fourier space \([34]\). In Appendix \( \mathcal{A} \) we give the expressions for the coefficients of the expansion (47) in terms of the spectral characteristics of the velocity field. We further note that, while \( a \) and \( b \) can certainly be functions of \( \tau \), we will not overly shrink the limits of physical generality by assuming that they are either constant or slowly-varying functions of time, i.e. that they do not change appreciably over one correlation time.

In order to determine the one-point statistics of the magnetic field, we follow the standard procedure \([35]\) and introduce the characteristic function of \( B'(t, x) \) at an arbitrary fixed spatial point \( x \):

\[
Z(t; \mu) = \langle \tilde{Z}(t, x; \mu) \rangle = \langle \exp[i\mu_i B^i(t, x)] \rangle.
\]

(49)

As usual, the angle brackets denote ensemble averages and the overtilde marks unaveraged quantities. The function \( Z \) is the Fourier transform of the PDF of the vector elements \( B' \). Due to spatial homogeneity, \( Z \) does not depend on the point \( x \), where \( B'(t, x) \) is taken. Upon differentiating \( Z \) with respect to time and using Eq. (45), we get

\[
\partial_t \tilde{Z} = -u^k \tilde{Z}_{,k} + u^k_\mu \partial \tilde{Z} - u^k_\mu \partial \tilde{Z} = -u^k \tilde{Z}_{,k} + \hat{\Lambda}^k_i u^i_\mu \tilde{Z},
\]

(50)

where, for the sake of future convenience, we introduce the operator

\[
\hat{\Lambda}^k_i = \mu^j \partial \partial_{\mu^j} - \delta^k_i \mu^j \partial \partial_{\mu^j},
\]

(51)

which will turn up repeatedly in this calculation.

In order to obtain an evolution equation for the characteristic function \( Z(t; \mu) \) of the random magnetic field, we average both sides of Eq. (45). Since, due to the homogeneity of the problem, \( \langle u^k \tilde{Z}_{,k} \rangle = -\langle u^k \tilde{Z} \rangle \), we may write the equation for \( Z(t; \mu) \) in the following form:

\[
\partial_t Z(t) = (\delta^k_i + \hat{\Lambda}^k_i) \langle u^k_\mu(t, x) \tilde{Z}(t, x) \rangle
\]

\[
= -\langle \delta^k_i + \hat{\Lambda}^k_i \rangle \int_0^t dt_1 \int d^d\tau_1 \kappa_{,\tau_1}^{ij\beta} (t - t_1, x - x_1) G_{\beta_1}^{\alpha_1}(t, x | t_1, x_1),
\]

(52)

where the mixed average on the right-hand side has been “split” with the aid of the Furutsu–Novikov (“Gaussian-integration”) formula \([38, 39]\), and the \( \mu \) dependence in the arguments has been suppressed for the sake of notational compactness. We have introduced the first-order averaged response function of the following species:

\[
G_{\beta_1}^{\alpha_1}(t, x | t_1, x_1) = \langle G_{\beta_1}^{\alpha_1}(t, x | t_1, x_1) \rangle = \left\langle \frac{\delta \tilde{Z}(t, x)}{\delta u_{\alpha_1}^{\beta_1}(t_1, x_1)} \right\rangle.
\]

(53)

As a response function, \( G_{\beta_1}^{\alpha_1} \) satisfies the causality constraint: \( G_{\beta_1}^{\alpha_1}(t, x | t_1, x_1) = 0 \) for \( t_1 > t \). The same-time form of \( G_{\beta_1}^{\alpha_1} \) can be obtained in terms of the characteristic function \( \tilde{Z}(t) \); integrating Eq. (52) from 0 to \( t_1 \), taking the functional derivative \( \delta / \delta u_{\alpha_1}^{\beta_1}(t', x_1) \), averaging, setting \( t' = t_1 \), and taking causality into account, we get
\[ G_{\beta_1}^{\alpha_1}(t_1, x|t_1, x_1) = \delta(x - x_1) \hat{A}_{\beta_1}^{\alpha_1} Z(t_1). \] (54)

In order to find \( G_{\beta_1}^{\alpha_1}(t, x|t_1, x_1) \) at \( t > t_1 \), we take the functional derivative \( \delta/\delta u_\beta(t', x') \) of both sides of Eq. (54) and establish that each element of the unaveraged tensor \( \tilde{G}_{\beta_1}^{\alpha_1} \) satisfies an equation identical in form to Eq. (54):

\[ \partial_t \tilde{G}_{\beta_1}^{\alpha_1}(t, x|t_1, x_1) = -u^m(t, x) \tilde{G}_{\beta_1, m}^{\alpha_1}(t, x|t_1, x_1) + \hat{A}_{\beta_1}^m u^m(t, x) \tilde{G}_{\beta_1}^{\alpha_1}(t, x|t_1, x_1). \] (55)

Subscripts such as \( \cdot, m \) in the above equation mean, in accordance with the usual notation, the partial differentiation with respect to \( x^m \), viz., \( \partial/\partial x^m \).

We must now average Eq. (53) in its turn, to obtain an evolution equation for \( G_{\beta_1}^{\alpha_1}(t, x|t_1, x_1) \) at \( t > t_1 \). Using the initial condition (54), let us write this evolution equation in the integral form valid for all \( t \geq t_1 \):

\[ G_{\beta_1}^{\alpha_1}(t, x|t_1, x_1) = \delta(x - x_1) \hat{A}_{\beta_1}^{\alpha_1} Z(t_1) \]
\[ - \int_{t_1}^t dt' \int_0^{t'} dt_2 \int d^d x_2 \left[ \kappa^{m\beta_2}(t' - t_2, x - x_2) G_{\beta_1, \beta_2, m}^{\alpha_1}(t', x|t_1, x_1; t_2, x_2) \right. \]
\[ + \left. \kappa^{m\beta_2}(t' - t_2, x - x_2) G_{\beta_1, \beta_2}^{\alpha_1 \alpha_2}(t', x|t_1, x_1; t_2, x_2) \right], \] (56)

where the mixed averages have again been “split” by the Furutsu–Novikov formula, at the price of introducing two new second-order response functions:

\[ G_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}(t', x|t_1, x_1; t_2, x_2) = \frac{\delta^2 \tilde{Z}(t', x)}{\delta u_{\beta_1}(t_1, x_1) \delta u_{\beta_2}(t_2, x_2)}, \] (57)

\[ G_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}(t', x|t_1, x_1; t_2, x_2) = \frac{\delta^2 \tilde{Z}(t', x)}{\delta u_{\alpha_1}(t_1, x_1) \delta u_{\alpha_2}(t_2, x_2)}. \] (58)

In the same way that the equal-time first-order response function was expressed in terms of \( Z(t) \) [Eq. (54)], the second-order response functions at \( t' = t_1 \) or \( t' = t_2 \) can be expressed in terms of \( G_{\beta_2}^{\alpha_2}(t_1, x|t_2, x_2) \) or \( G_{\beta_1}^{\alpha_1}(t_2, x|t_1, x_1) \), respectively. Because of causality, the former representation would be valid provided \( t_1 \geq t_2 \), the latter in the opposite case \( t_2 \geq t_1 \). At other times, \( t_1, t_2 \leq t' \), the functions \( G_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} \) and \( G_{\beta_1}^{\alpha_1} G_{\beta_2}^{\alpha_2} \) satisfy integral equations analogous to Eq. (54), where third-order response functions make their appearance. An infinite open hierarchy can thus be obtained by further iterating this procedure and introducing response functions of ascending orders. This hierarchy constitutes the exact description of the statistics of the kinematic dynamo problem with arbitrary velocity correlation time.

**B. The \( \tau \) Expansion**

The expansion in small correlation time must be carried out in such a way that the time integral of the velocity correlator \( \kappa^{d}(\tau, y) \) remains constant. Since \( \kappa^{d}(\tau, y) \) has a finite (small) width \( \tau_\epsilon \), we can conclude that the double time integral on the right-hand side of Eq. (54) must be of first order in the correlation time \( \tau_\epsilon \). As we are only interested in constructing the \( \tau \) expansion up to first order, it is now sufficient to calculate the second-order response functions \( G_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} \) and \( G_{\beta_1}^{\alpha_1} G_{\beta_2}^{\alpha_2} \) with zeroth-order precision.

We have already mentioned that recursive relations completely analogous to the relation (54) can be derived for the second-order response functions. The latter are thereby expressed as their equal-time values plus double time integrals of the same sort as that which appeared on the right-hand side of Eq. (54). These time integrals are first order in the correlation time and can therefore be neglected. The equal-time values of the second-order response functions are obtained by formally integrating Eq. (54), taking functional derivatives of it, averaging, and using causality. The second-order response functions are thus expressed to zeroth order in terms of the first-order ones. These latter can by the same token be replaced by their equal-time values, which only contain the characteristic function \( Z(t) \). The resulting expressions, valid to zeroth order, must be substituted into the first-order term (the double time integral) in Eq. (54). All these manipulations, which require a fair amount of algebra, are relegated to Appendix B. Here we simply give the resulting expression for the first-order response function, valid to first order in \( \tau_\epsilon \):
$$G_{\beta_1}^{\alpha_1}(t, x|t_1, x_1) = \delta(x - x_1) \left[ \hat{\Lambda}^{\alpha_1}_{\beta_1} Z(t_1) \right.$$}

$$- \int_{t_1}^{t} dt' \int_{0}^{t_1} dt_2 \kappa_{\alpha_1\alpha_2}^{m,n}(t' - t_2, 0) (\delta_{m}^{n} + L_{m}^{n}) \hat{\Lambda}^{\alpha_1}_{\beta_1} \hat{\Lambda}^{\alpha_2}_{\beta_2} Z(t_2)$$

$$- \int_{t_1}^{t} dt' \int_{t_1}^{t} dt_2 \kappa_{\alpha_1\alpha_2}^{m,n}(t' - t_2, 0) (\delta_{m}^{n} + L_{m}^{n}) \hat{\Lambda}^{\alpha_2}_{\beta_2} \hat{\Lambda}^{\alpha_1}_{\beta_1} Z(t_1)$$

$$+ \int_{t_1}^{t} dt' \int_{0}^{t_1} dt_2 \kappa_{\alpha_1\alpha_2}^{m,n}(t' - t_2, 0) \hat{\Lambda}^{\alpha_2}_{\beta_2} Z(t_2) \right]$$

$$+ \frac{\partial^2 \delta(x - x_1)}{\partial x^m \partial x^n} \int_{t_1}^{t} dt' \int_{t_1}^{t} dt_2 \kappa_{mn}(t' - t_2, 0) \hat{\Lambda}^{\alpha_1}_{\beta_1} Z(t_1) + O(\tau_c^2).$$

(59)

This expression must now be substituted into the time-history integral on the right-hand side of Eq. (61). This gives a closed integro-diﬀerential equation for the characteristic function \(Z(t)\). However, the dependence on the past values of \(Z\) is spurious and can be resolved to first order in \(\tau_c\). Indeed, we can formally integrate Eq. (61) from \(t_1\) to \(t\) and, using the zeroth-order value of the first-order response function [the first term in the formula (59)], get

$$Z(t_1) = Z(t) + (\delta_{m}^{n} + \hat{\Lambda}^{n}_{m}) \int_{t_1}^{t} dt' \int_{0}^{t_1} dt_2 \kappa_{\alpha_1\alpha_2}^{m,n}(t' - t_2, 0) \hat{\Lambda}^{\alpha_2}_{\beta_2} Z(t_2) + O(\tau_c^2).$$

(60)

The double time integral in this equation is of first order in \(\tau_c\), as usual.

Upon assembling the equations (62), (63), and (60), we finally arrive at the following closed partial differential equation for \(Z(t)\):

$$\partial_t Z(t) = - \int_{0}^{t} dt_1 \kappa_{\alpha_1\alpha_2}(t - t_1, 0) (\delta_{i}^{k} + \hat{\Lambda}_{i}^{k}) \hat{\Lambda}^{\alpha_1}_{\beta_1} Z(t)$$

$$- \int_{0}^{t} dt_1 \int_{t_1}^{t} dt' \int_{0}^{t_1} dt_2 \kappa_{\alpha_1\alpha_2}(t' - t_2, 0) (\delta_{i}^{k} + \hat{\Lambda}_{i}^{k}) \left[ \hat{\Lambda}^{\alpha_1}_{\beta_1} \hat{\Lambda}^{\alpha_2}_{\beta_2} \right] Z(t)$$

$$- \int_{0}^{t} dt_1 \int_{t_1}^{t} dt' \int_{t_1}^{t} dt_2 \kappa_{\alpha_1\alpha_2}(t' - t_2, 0) (\delta_{i}^{k} + \hat{\Lambda}_{i}^{k}) \left[ \hat{\Lambda}^{\alpha_1}_{\beta_1} \hat{\Lambda}^{\alpha_2}_{\beta_2} \right] Z(t)$$

$$- \int_{0}^{t} dt_1 \int_{t_1}^{t} dt' \int_{0}^{t_1} dt_2 \kappa_{\alpha_1\alpha_2}(t - t_1, 0) (\delta_{i}^{k} + \hat{\Lambda}_{i}^{k}) \left[ \hat{\Lambda}^{\alpha_1}_{\beta_1} \hat{\Lambda}^{\alpha_2}_{\beta_2} \right] Z(t)$$

$$- \int_{0}^{t} dt_1 \int_{t_1}^{t} dt' \int_{t_1}^{t} dt_2 \kappa_{\alpha_1\alpha_2}(t - t_1, 0) (\delta_{i}^{k} + \hat{\Lambda}_{i}^{k}) \left[ \hat{\Lambda}^{\alpha_1}_{\beta_1} \hat{\Lambda}^{\alpha_2}_{\beta_2} \right] Z(t).$$

(61)

The square brackets denote commutators.

Note that, besides the second derivatives of the velocity correlation tensor, the first-order terms contain the fourth ones, as well as the undifferentiated tensor itself [in the last term in Eq. (61)]. The latter implies the loss of Galilean invariance, the former the loss of small-scale universality in the sense that the large-scale structure of the velocity correlation starts to play a role. This effect could not have been captured if the velocity field had been assumed to be purely a combination of the instantaneous velocity at a given point and a linear shear.

C. The Fokker–Planck Equation

In order to obtain the Fokker–Planck equation for the PDF of the magnetic field, we must inverse-Fourier transform Eq. (61) back to \(B\) dependence. The inverse Fourier transform of \(Z(t; \mu)\) is the one-point PDF \(P(t; \mathbf{B})\). We will continue using the symbol \(\hat{\Lambda}_{i}^{k}\) to denote the counterpart of the operator \(\Lambda_{i}^{k}\) in the \(\mathbf{B}\) space:

$$\hat{\Lambda}_{i}^{k} = (d - 1) \delta_{i}^{k} - B_{k} \frac{\partial}{\partial B_{i}} + \delta_{i}^{k} B_{l} \frac{\partial}{\partial B_{l}}. $$

(62)
Due to the isotropy of the problem, the PDF \( P(t; \mathbf{B}) \) will in fact be a scalar function of the field strength \( B \) only. Thus, all the operators that appear on the right-hand side of the \( \mathbf{B} \)-space counterpart of Eq. (61) must, after they are convolved with the velocity correlation tensors, be expressible in terms of \( B \). Let us use the Taylor expansion (47) of the velocity correlator to calculate the tensor convolutions in Eq. (61). We have

\[
\kappa^{ij}(\tau, 0) = \kappa_0(\tau) \delta^{ij},
\]

\[
\kappa^{ij}_2(\tau, 0) = -\kappa_2(\tau) \left[ \delta^{ij} \delta_{kl} + a \left( \delta^i_k \delta^j_l + \delta^j_k \delta^i_l \right) \right] = -\kappa_2(\tau) T^{ij}_{kl},
\]

\[
\kappa^{ij,klmm}(\tau, 0) = \kappa_4(\tau) \left[ 2(d + 2 + b) \delta^{ij} \delta_{kl} + (d + 4) b \left( \delta^i_k \delta^j_l + \delta^j_k \delta^i_l \right) \right]
\]

\[
= \kappa_4(\tau) U^{ij}_{kl}.
\]

A number of second-order differential operators (with respect to the velocity correlator to calculate the tensor convolutions in Eq. (61)). We have

\[
\hat{L} = T^{ij \beta}_{\alpha \gamma} \delta^k \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} = \frac{d-1}{d+1} \left( B \frac{\partial}{\partial B} + d \right) \left( 1 + \beta B \frac{\partial}{\partial B} + (d + 1) \beta \right);
\]

(66)

two operators appearing in the first-order terms result from the non-self-commuting nature of the operator \( \hat{\Lambda}^k \) [see the second and the third terms in Eq. (61)]:

\[
\hat{L}_1 = T^{ij \beta}_{\alpha \gamma} \tau^{m \beta}_{\alpha \gamma} \left( \delta^k \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right. \left. - \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right)
\]

\[
= \frac{d^2(d-1)}{(d+1)^2} \left( B \frac{\partial}{\partial B} + d \right) \left( 1 + \frac{\beta}{d^2} B \frac{\partial}{\partial B} + \frac{(d + 1)^2}{d^2} \beta \right);
\]

(67)

\[
\hat{L}_2 = T^{ij \beta}_{\alpha \gamma} \tau^{m \beta}_{\alpha \gamma} \left( \delta^k \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right. \left. - \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right) = T^{ij \beta}_{\alpha \gamma} \left( \delta^k \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right. \left. - \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right)
\]

\[
= \frac{2d(d-1)}{d+1} \left( B \frac{\partial}{\partial B} + d \right) \left( 1 + \frac{\beta}{d^2} B \frac{\partial}{\partial B} + \frac{(d + 1)^2}{d^2} \beta \right);
\]

(68)

and, finally, there are two other operators due to the presence of the convective term (i.e., explicit spatial dependence) in the induction equation [see the fourth and the fifth terms in Eq. (61)]:

\[
\hat{M}_1 = T^{ij \beta}_{\alpha \gamma} \tau^{m \beta}_{\alpha \gamma} \left( \delta^k \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right. \left. - \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right)
\]

\[
= \frac{d(d-1)}{d+1} \left( B \frac{\partial}{\partial B} + d \right) \left[ 1 + \frac{2\beta}{d^2} + \frac{d(d+1) - 1}{d^3} \beta \right]
\]

\[
\times \left( B \frac{\partial}{\partial B} + \frac{(d + 1)^2}{d^2} \beta \right);
\]

(69)

\[
\hat{M}_2 = U^{ij \beta}_{\alpha \gamma} \left( \delta^k \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right. \left. - \hat{\Lambda}^k \hat{\Lambda}^{\alpha \beta} \right)
\]

\[
= \frac{2(d-1)(d+4)}{d+3} \left( B \frac{\partial}{\partial B} + d \right)
\]

\[
\times \left[ 1 + \frac{4d + 2}{2d(d+4)} \zeta \right] \left( B \frac{\partial}{\partial B} + \frac{(d+2)(d+3)}{2d+4} \zeta \right).
\]

(70)

In all of the above, \( \beta = d[1 + (d + 1)a] \) and \( \zeta = d[2 + (d + 3)b] \) are compressibility parameters that vanish in the case of incompressible flow. In this latter case, the operators defined above simplify considerably:

\[
\hat{L}_1 = \frac{d^2}{d+1} \hat{L}, \quad \hat{L}_2 = 2d \hat{L},
\]

\[
\hat{M}_1 = \frac{d}{d+1} \hat{L}, \quad \hat{M}_2 = \frac{2(d+1)(d+4)}{d+3} \hat{L}.
\]

(71)

(72)

If we take the long-time limit, i.e., \( t \gg \tau_c \), the coefficients in Eq. (61) do not depend on time \( t \). We can now use the inverse Fourier transform of Eq. (61) taken in this limit and the isotropic operators listed above to assemble the Fokker–Planck equation for the PDF of the magnetic field. This equation contains the desired corrections that are of first order in the velocity correlation time \( \tau_c \) and represent the first available manifestation of the finite-correlation-time effects. We have

\[
\partial_t P = \frac{\kappa_2}{2} \left( \hat{L} - \frac{1}{2} \tau_c \kappa_2 \left[ K_1 (\hat{L}_1 + \hat{M}_1) + K_2 \hat{L}_2 + \hat{K}_2 \hat{M}_2 \right] \right) P,
\]

(73)
where the overall dimensional factor is

\[ \tilde{K}_2 = 2 \int_0^\infty d\tau \kappa_2(\tau), \]  

(74)

and the coefficients \[31\],

\[ K_1 = \frac{4}{\tau_c \tilde{K}_2} \lim_{t \to \infty} \int_0^t dt_1 \int_{t_1}^t dt_2 \int_{t_1}^{t_2} dt_3 \kappa_2(t - t_1) \kappa_2(t_2 - t_3), \]  

(75)

\[ K_2 = \frac{4}{\tau_c \tilde{K}_2} \lim_{t \to \infty} \int_0^t dt_1 \int_{t_1}^t dt_2 \int_{t_1}^{t_2} dt_3 \kappa_2(t - t_1) \kappa_2(t_2 - t_3), \]  

(76)

\[ \tilde{K}_2 = \frac{4}{\tau_c \tilde{K}_2} \lim_{t \to \infty} \int_0^t dt_1 \int_{t_1}^t dt_2 \int_{t_1}^{t_2} dt_3 \kappa_2(t - t_1) \kappa_0(t_2 - t_3), \]  

(77)

are constants that depend on the particular shapes of the time-correlation functions \( \kappa_0(\tau) \), \( \kappa_2(\tau) \), and \( \kappa_4(\tau) \). Such sensitive dependence is a new feature and represents a loss of universality with respect to the specific time-correlation profiles (cf. Ref. \[18\]). As we have pointed out in Sec. III B, the universality with respect to the functional form of the velocity correlator in space is also lost (this effect is incorporated into the coefficient \( \tilde{K}_2 \)).

Let us also list the much more compact form that the Fokker–Planck equation (73) assumes in the case of an incompressible velocity field:

\[ \partial_t P = \tilde{K}_2 \left[ 1 - \tau_c \tilde{K}_2 d \left( \frac{1}{2} K_1 + K_2 + \frac{(d + 1)(d + 4)}{d(d + 3)} \tilde{K}_2 \right) \right] \hat{L} P. \]  

(78)

Here it is especially manifest that the true expansion parameter in the problem is \( \tau_c \tilde{K}_2 d \). This is a general statement that holds regardless of the degree of compressibility, as can be readily verified by counting powers of \( d \) in the general expressions for the operators \( \hat{L}, \hat{L}_1, \hat{L}_2, \hat{M}_1, \) and \( \hat{M}_2 \) [formulas \[32\]–\[34\]].

It is evident that the distribution resulting from Eq. (73) is lognormal, which is a well-known fact in the kinematic-dynamo and passive-advection theory. Since we are interested in the quantitative description of the fast-dynamo effect, we will now proceed to calculate the growth rates of the moments of the magnetic field.

D. The Dynamo Growth Rates

The evolution of all moments of \( B \) can be determined from Eq. (73). The \( n \)th moment is calculated according to

\[ \langle B^n \rangle = \frac{2 \pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dB \, B^{n-d-1} P(t; B). \]  

(79)

Upon multiplying both sides of Eq. (73) by \( B^{d+n-1} \) and integrating over \( B \), we find that \( \langle B^n \rangle \) satisfies:

\[ \partial_t \langle B^n \rangle = \gamma(n) \langle B^n \rangle \]

\[ = \tilde{K}_2 \left\{ \Gamma(n) - \tau_c \tilde{K}_2 d \left[ K_1 \Gamma_1(n) + K_2 \Gamma_2(n) + \tilde{K}_2 \hat{\Gamma}_2(n) \right] \right\} \langle B^n \rangle, \]  

(80)

where the nondimensionalized zeroth-order growth rates are (cf. Ref. \[8\])

\[ \Gamma(n) = \frac{d - 1}{d+1} n [n + d + (n - 1) \beta], \]  

(81)

and the universal parts of the (negative) first-order corrections arising from the second- and fourth-order terms in the velocity correlator \[17\] are

\[ \Gamma_1(n) = \frac{1}{2} \frac{d - 1}{d+1} n \left[ (n + d) \left( 1 + \frac{2 \beta}{d^2} \right) + (n - 1) \frac{\beta^2}{d^2} \right], \]  

(82)

\[ \Gamma_2(n) = \frac{d - 1}{d+1} n (n + d) \left( 1 + \frac{\beta}{d^2} \right), \]  

(83)

\[ \hat{\Gamma}_2(n) = \frac{(d - 1)(d + 4)}{d(d + 3)} n \left[ n + d + \frac{n}{2} \left( \frac{d^2 + 4d + 2}{d(d + 4)} - 1 \right) \zeta \right]. \]  

(84)
We observe that, for $n = 0$, $\Gamma = \Gamma_1 = \Gamma_2 = \tilde{\Gamma}_2 = 0$. This simply means that both zeroth- and first-order terms in the $\tau$ expansion preserve the normalization of the PDF, i.e., our expansion is conservative, as it should be.

In the incompressible flow, the total growth rate of the $n$th moment can be written in a more compact form:

$$
\gamma(n) = \frac{\tilde{\kappa}_2}{2} \frac{d-1}{d+1} n(n+d) \left[ 1 - \tau_\kappa \tilde{\kappa}_2 d \left( \frac{1}{2} K_1 + K_2 + \frac{(d+1)(d+4)}{d(d+3)} \tilde{K}_2 \right) \right].
$$

We see that the corrections to the growth rates of the magnetic-field moments are negative, so the growth rates are reduced. The amount of reduction depends on a variety of factors including the dimension of space, the order of the moment, the degree of compressibility, the functional form of the velocity correlator in time and space, and, of course, the velocity correlation time. Let us note that our general results derived for an arbitrarily compressible velocity field reveal no qualitatively essential effect of compressibility on the behavior of the first-order finite-correlation-time corrections to the dynamo growth rates in the diffusion-free regime. Compressibility of the flow simply leads to additive (and positive) corrections to the incompressible values of $\Gamma(n)$, $\Gamma_1(n)$, $\Gamma_2(n)$, and $\tilde{\Gamma}_2(n)$. Quantitatively, these corrections may affect the exact conditions for the break-down of the first-order approximation. For more discussion of the compressibility effects in the kinematic dynamo (with a $\delta$-correlated velocity field), we address the reader to Refs. [3,4,13,14,10].

We remind the reader that here we have studied magnetic fluctuations in the diffusion-free regime and therefore dropped the term in the induction equation that is responsible for the resistive regularization. Such an approach is justified for plasmas with very large magnetic Prandtl numbers (e.g., the ISM or the protogalaxy) and applies to the initial stage of the small-scale dynamo that lasts for a time of order $t \sim \log \text{Pr}$ that elapses before the magnetic fluctuations reach resistive scales [14,10]. After that, or if the Prandtl number is of order unity or small (as is, e.g., the case for the Sun), resistive effects must be taken into account. In this case, the calculation of the moments of the magnetic field via the Fokker–Planck equation for its PDF as presented in this Section does not apply because of the closure problem associated with the diffusion term [the equations for $\tilde{Z}(t, x; \mu)$ and $Z(t; \mu)$ do not close]. However, the general $\tau$-expansion method proposed in this paper can, in principle, be applied to multipoint correlators of the magnetic field, for which treating the diffusive case presents no conceptual difficulty. One-point moments can then be obtained by fusing the points at which the multipoint correlators are taken (cf. Refs. [3,4,10]). Although it is the diffusive case that is studied in most numerical simulations, where $\text{Pr}$ rarely exceeds 100, it is not necessarily the most relevant one in the context of the (proto)galactic dynamo, for which $\text{Pr} \sim 10^{14} \div 10^{22}$. Indeed, as we already pointed out in the Introduction, the initial (proto)galactic seed field may well be strong enough for the kinematic approximation to break down while the dynamo is still in the diffusion-free stage [14]. If this is the case, the effect of magnetic diffusion must be studied in conjunction with nonlinear saturation of the magnetic fluctuations [17].

IV. A PHYSICAL EXAMPLE: THE ONE-EDDY MODEL

In real astrophysical environments, as the interstellar medium and the protogalactic plasmas, the magnetic fields are acted upon by a Kolmogorov-like turbulence with a fully developed inertial range about three decades wide ($\text{Re} \sim 10^6$). While the velocities of the turbulent eddies excited by the Kolmogorov cascade decrease with the scale of the eddy, the velocity gradients increase (see, e.g., Ref. [11]). Therefore, the dominant role in the process of amplification of the small-scale magnetic fluctuations is played by the smallest eddies. With this circumstance in mind, one often considers, for modeling purposes, a synthetic incompressible turbulent velocity field consisting of eddies all of which have the same fixed size but random isotropic orientation (for detailed discussions of the galactic and protogalactic dynamo, we refer the reader to Refs. [3,4,13,14,10]). In this Section, we will present a brief discussion of the implications of the $\tau$-expansion theory developed in Sec. [11] for such a model problem, which will henceforth be referred to as the one-eddy model.

The velocity field in the one-eddy model is specified as follows:

$$
u^i(t, \mathbf{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i \mathbf{k} \cdot \mathbf{x}} u^i(t, \mathbf{k}),$$

where the Fourier modes $u^i(t, \mathbf{k})$ are random variables that satisfy

$$
\langle u^i(t, \mathbf{k}) u^j(t', \mathbf{k}') \rangle = (2\pi)^d \delta(\mathbf{k} + \mathbf{k}') \left( \delta^{ij} - \frac{k_i k_j}{k^2} \right) \delta(k - k_0) \kappa(t - t').
$$

In this case, $\kappa_2(\tau) \propto \kappa(\tau)$, and, upon using the relations listed in Appendix [A], we get
\[ \kappa_0(\tau) = \frac{1}{k_0^2} \frac{(d-1)(d+2)}{d+1} \kappa_2(\tau), \]  
\[ \kappa_4(\tau) = \frac{k_0^2}{2(d+4)(d+1)} \kappa_2(\tau). \]  
(88)  
(89)

Let us specify a plausible velocity time-correlation profile:
\[ \kappa_2(\tau) = \frac{\bar{\kappa}_2}{2\tau_c} \exp \left( -\frac{\tau}{\tau_c} \right). \]  
(90)

For this correlation function, which corresponds, for example, to the well-known Ornstein–Uhlenbeck random process (see, e.g., Ref. [2]), the coefficients of the \( \tau \) expansion are \( K_1 = K_2 = 1/2 \). The relations (88) and (89) provide the value of \( K_2 \):
\[ \bar{K}_2 = \frac{(d-1)(d+2)(d+3)}{2(d+1)^2(d+4)} K_2. \]  
(91)

Let us define the “eddy-turnover” time \( \tau_{\text{eddy}} \sim (k_0 u)^{-1} \) of such a velocity field according to the following relation:
\[ \frac{1}{\tau_{\text{eddy}}^2} = k_0^2 \frac{1}{\tau_c} \int_{-\infty}^{+\infty} d\tau \kappa_{ii}(\tau, y = 0) = k_0^2 \frac{\bar{\kappa}_0}{\tau_c} = \frac{d(d-1)(d+2)}{d+1} \frac{\bar{\kappa}_2}{\tau_c}, \]  
(92)

where we have used Eq. (88) to express \( \bar{\kappa}_0 \) in terms of \( \bar{\kappa}_2 \). Note that the same expression is obtained if \( \tau_{\text{eddy}} \sim (\nabla u : \nabla u)^{-1/2} \) is formally defined in terms of the velocity gradients (without recourse to the one-eddy model):
\[ \frac{1}{\tau_{\text{eddy}}^2} = \frac{1}{\tau_c} \int_{-\infty}^{+\infty} d\tau |\kappa_{ij}^0(\tau, y = 0)| = \frac{d(d-1)(d+2)}{d+1} \frac{\bar{\kappa}_2}{\tau_c}. \]  
(93)

We recall that the zeroth-order growth rate \( \gamma_0 \) of the magnetic-fluctuation energy \( \langle B^2 \rangle \) is [see formula (83)]
\[ \gamma_0 = \frac{(d-1)(d+2)}{d+1} \bar{\kappa}_2. \]  
(94)

Formulas (92) and (93) then imply
\[ \left( \frac{\tau_c}{\tau_{\text{eddy}}} \right)^2 = \tau_c \gamma_0 d = \frac{(d-1)(d+2)}{d+1} \tau_c \bar{\kappa}_2 d. \]  
(95)

We have established a correspondence between the small parameter that has arisen in our expansion of the dynamo growth rates and the “physical” small parameter, which is the ratio of the correlation and eddy-turnover times. Of course, the above expression hinges on the definitions (92) or (93) of \( \tau_{\text{eddy}} \). A simple physical argument can be made in favor of these definitions and the resulting formula (95). Namely, let us observe that when \( \tau_c \sim \tau_{\text{eddy}} \) the eddy only stretches the magnetic field line in one of the \( d \) available directions during one turnover time, whence \( \tau_c \gamma \sim 1/d \). The same estimate follows from the formula (95).

Let us now evaluate the first-order correction to the growth rate of the magnetic energy. In the one-eddy model, one gets, upon using formulas (97) and (98) and taking \( K_1 = K_2 = 1/2 \) for the Ornstein–Uhlenbeck time-correlation profile (94),
\[ \gamma = \gamma(2) = \gamma_0 (1 - C_d \tau_c \gamma_0 d), \quad C_d = \frac{2d(d+1) - 1}{2d(d-1)(d+2)}. \]  
(96)

We note that in three dimensions, \( C_d = 23/60 \simeq 40\% \). When \( \tau_c \sim \tau_{\text{eddy}} \), we have \( \tau_c \gamma_0 d \sim 1 \), and the resulting growth-rate reduction of \( \sim 40\% \) is in a good qualitative agreement with the available numerical results [23-25]. Of course, as we have already stressed in the Introduction, our \( \tau \) expansion is not designed for the case of \( \tau_c \sim \tau_{\text{eddy}} \), so the fact that it gives a fairly reasonable prediction should not be considered as an adequate quantitative corroboration of our theory. At best, one might conclude that the first-order expansion is well behaved for not-too-small values of the expansion parameter.

Let us emphasize, however, that such a well-behaved expression has resulted from a number of essentially arbitrary (albeit physically reasonable) specifications of the parameters involved in the \( \tau \) expansion. One of the most physically
important points that we have tried to make in this work is, in fact, that the inclusion of finite-correlation-time
effects leads to nonuniversal statistics, so the quantitative predictions of the theory can and will change appreciably
if such factors as the shapes of the time-correlation profiles are changed. Namely, one would obtain expressions
of the form (96) with different values of the coefficient $C_d$. For sufficiently large values of $\tau_c\gamma_0 d$, the validity of the
expansion (96) will break down, and the expression in the brackets may even become negative. However, the following
heuristic argument can be envisioned in this context.

Let us recall that the finite-correlation-time effect was due to the presence of time-history integrals such as those
that appear in the equations (22), (24), and (29). The first-order corrections in the Fokker–Planck equation (23) arose
from systematically approximating the time evolution of the statistical quantities [response functions and characteristic
function $Z(t; \mu)$] that entered these time-history integrals. The corrected (“true”) value of $\gamma$ represents, in a rough
way, the rate at which these quantities change. It would appear then that a better estimate of $\gamma$ would be obtained
if $\gamma_0$ in the first-order term in the brackets in Eq. (96) were replaced with the corrected value $\gamma$. With this caveat,
we would find that

$$\gamma = \frac{\gamma_0}{1 + C_d \tau_c \gamma_0 d}.$$  

(97)

To first order, this formula is equally accurate as Eq. (96). However, it better represents the fact that, as $\tau_c \gamma_0 d$
increases, the corrected value of the growth rate should be expected to saturate (17). Of course, such considerations
cannot substitute for an adequate nonperturbative theory of the passive advection and kinematic dynamo in finite-
time-correlated flows, which remains an open problem.

ACKNOWLEDGMENTS

The authors would like thank S. A. Boldyrev for extensive and very fruitful discussions of the physics and the formal-
ism of the finite-time-correlated kinematic dynamo problem. Both the substance and the style of the presentation
have benefited from suggestions made by J. A. Krommes who read an earlier manuscript of this work. We are also
grateful to G. Falkovich, V. Lebedev, S. Cowley, and the anonymous referee for several useful comments.

This work was supported by the U. S. Department of Energy under Contract No. DE-AC02-76-CHO-3073.

APPENDIX A: SMALL-SCALE-EXPANSION COEFFICIENTS OF THE VELOCITY CORRELATION
TENSOR IN TERMS OF VELOCITY SPECTRA

In this Appendix, we list the basic formulas that relate the coefficients of the small-scale expansion (12) of the
velocity correlation tensor to the spectral characteristics of the velocity field. These relations allow one to apply the
results on the small-$\tau_c$ expansion obtained in Sec. IV to velocity fields that are specified in the Fourier, rather than
configuration, space. They also provide a set of consistency constraints that must be respected when the specific functional forms of $\kappa_0(\tau)$, $\kappa_2(\tau)$, and $\kappa_4(\tau)$ are chosen.

Let the advecting velocity field be given as a sum of spatial Fourier modes,

$$u^i(t, \mathbf{x}) = \int \frac{d^dk}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{x}} u^i(t, \mathbf{k}).$$  

(A1)

and let the Fourier coefficients $u^i(t, \mathbf{k})$ be random variables that satisfy

$$\langle u^i(t, \mathbf{k}) u^j(t', \mathbf{k}') \rangle = (2\pi)^d \delta(\mathbf{k} + \mathbf{k}') \left[ \kappa(k, t - t') \delta^{ij} + \tilde{\kappa}(k, t - t') \frac{k_i k_j}{k^2} \right].$$  

(A2)

For the incompressible flows, $\tilde{\kappa}(k, \tau) = -\kappa(k, \tau)$; for the irrotational ones, $\kappa(k, \tau) = 0$. The coefficients of the expansion (17) can then be expressed as follows:

$$\kappa_0(\tau) = \frac{1}{d} \int \frac{d^dk}{(2\pi)^d} \left[ d\kappa(k, \tau) + \tilde{\kappa}(k, \tau) \right],$$  

(A3)

$$\kappa_2(\tau) = \frac{1}{d(d + 2)} \int \frac{d^dk}{(2\pi)^d} k^2 \left[ (d + 2)\kappa(k, \tau) + \tilde{\kappa}(k, \tau) \right],$$  

(A4)
\[
\kappa_4(\tau) = \frac{1}{2d(d+2)(d+4)} \int \frac{d^d k}{(2\pi)^d} k^4 \left[ (d+4)\kappa(k, \tau) + \tilde{\kappa}(k, \tau) \right], \quad (A5)
\]

\[
a(\tau) = \kappa_2(\tau)^{-1} \frac{1}{d(d+2)} \int \frac{d^d k}{(2\pi)^d} k^2 \tilde{\kappa}(k, \tau), \quad (A6)
\]

\[
b(\tau) = \kappa_4(\tau)^{-1} \frac{1}{d(d+2)(d+4)} \int \frac{d^d k}{(2\pi)^d} k^4 \tilde{\kappa}(k, \tau), \quad (A7)
\]

where the \(k\)-space integrations of radial functions can, of course, be written more explicitly as

\[
\int \frac{d^d k}{(2\pi)^d} = \frac{S_d}{(2\pi)^d} \int_0^\infty dk k^{d-1}, \quad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (A8)
\]

The derivation of the above relations is straightforward and based on the expressions for the correlation functions of isotropic fields in configuration space in terms of their spectra. For the 3-D case, these expressions can be found in Ref. \[10\]. A detailed derivation of the formulas \[A3 \rightarrow A7\] for the \(d\)-dimensional case is also given in Appendix A of Ref. \[11\].

**Appendix B: Second-Order Response Functions**

In this Appendix, we provide the zeroth-order expressions for the second-order response functions that we used in Sec. \[III B\]. They are all derived in the same fashion: Eq. \[B5\] is formally integrated, functional derivatives of it are taken with respect to the velocity field \(u^i\) or its gradients \(u^i_k\) at the appropriate moments, the result is averaged, and the causality property of the response functions is used. Here we simply list the results.

When \(t_2 \geq t_1\), we have, to zeroth order in the correlation time \(\tau_c\),

\[
G^{\alpha_1}_{\beta_1, \beta_2}(t', t; x_1; x_2) = G^{\alpha_1}_{\beta_1, \beta_2}(t_2, x|t_1, x_1; t_2, x_2)
\]

\[
= -\delta(x - x_2)G^{\alpha_1}_{\beta_1, \beta_2}(t_2, x|t_1, x_1) + \frac{\partial}{\partial x^n} \delta(x - x_2) G^{\alpha_2}_{\beta_1, \beta_2}(t_2, x|t_1, x_1) + \delta(x - x_2) \check{\Lambda}_{\beta_2}^{\alpha_1 \alpha_2}(t_2, x|t_1, x_1), \quad (B1)
\]

where we have introduced the following notation: by definition,

\[
\frac{\partial \delta u^n(t, x)}{\partial \delta u^n_{\alpha_2}(t_2, x_2)} = \frac{\delta}{\delta u^n_{\alpha_2}(t_2, x_2)} \delta(t - t_2) \Delta^{\alpha_2}(x - x_2). \quad (B3)
\]

The function \(\Delta^{\alpha_2}(x - x_2)\) is nonrandom and has the following property, which will be all that we need to know about it:

\[
\frac{\partial}{\partial x^n} \Delta^{\alpha_2}(x - x_2) = \delta^{\alpha_2}_n \delta(x - x_2). \quad (B4)
\]

When \(t_1 > t_2\), the expressions \[B1\] and \[B2\] vanish by causality, so we have to flip the order of functional differentiation:

\[
G^{\alpha_1}_{\beta_1, \beta_2}(t', t; x_1; x_2) = G^{\alpha_1}_{\beta_2, \beta_1}(t_1, x|t_2, x_2; t_1, x_1)
\]

\[
= -\Delta^{\alpha_1}(x - x_1) G^{\alpha_1}_{\beta_2, \beta_1}(t_1, x|t_2, x_2) + \delta(x - x_1) \check{\Lambda}_{\beta_1}^{\alpha_1 
\alpha_2}(t_1, x|t_2, x_2), \quad (B5)
\]

\[
G^{\alpha_1}_{\beta_2, \beta_1}(t', t; x_1; x_2) = G^{\alpha_1}_{\beta_2, \beta_1}(t_1, x|t_2, x_2; t_1, x_1)
\]

\[
= -\Delta^{\alpha_1}(x - x_1) G^{\alpha_2}_{\beta_2, \beta_1}(t_1, x|t_2, x_2) + \delta(x - x_1) \check{\Lambda}_{\beta_1}^{\alpha_1 \alpha_2}(t_1, x|t_2, x_2). \quad (B6)
\]

In Eq. \[B5\], the following obvious notation was used:

\[
G^{\alpha_1}_{\beta_2, \beta_1}(t_1, x|t_2, x_2; t_1, x_1) = \left\langle \frac{\delta^2 \hat{Z}(t_1, x)}{\delta u^{\beta_2}(t_2, x_2) \delta u^{\alpha_1}(t_1, x_1)} \right\rangle. \quad (B7)
\]
and a new first-order response function appeared:

$$G_{\beta_2}(t_1, x|t_2, x_2) = \left\langle \frac{\delta \hat{Z}(t_1, x)}{\delta \hat{u}_{\beta_2}(t_2, x_2)} \right\rangle. \quad (B8)$$

The equal-time form of this function is

$$G_{\beta_2}(t_2, x|t_2, x_2) = \left[ \frac{\partial}{\partial x^n} \delta(x - x_2) \right] \hat{\Lambda}_{\beta_2}^n Z(t_2). \quad (B9)$$

The first-order response functions that appear in the formulas (B4), (B2), (B3), and (B4) can be written as their equal-time values (B1) and (B2) plus first-order terms. To zeroth order, we have therefore: for $t_2 \geq t_1$,

$$G_{\beta_1, \beta_2}^{\alpha_1}(t', x|t_1, x_1; t_2, x_2) = -\left[ \frac{\partial}{\partial x^n} \delta(x - x_1) \right] \delta(x - x_2) \hat{\Lambda}_{\beta_1}^{\alpha_1} Z(t_1) + \delta(x - x_1) \left[ \frac{\partial}{\partial x^n} \delta(x - x_2) \right] \hat{\Lambda}_{\beta_2}^{\alpha_1} Z(t_1), \quad (B10)$$

$$G_{\beta_1, \beta_2}^{\alpha_1, \alpha_2}(t', x|t_1, x_1; t_2, x_2) = -\left[ \frac{\partial}{\partial x^n} \delta(x - x_1) \right] \delta(x - x_2) \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_2} Z(t_1) + \delta(x - x_1) \delta(x - x_2) \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_2} Z(t_1); \quad (B11)$$

for $t_1 \geq t_2$,

$$G_{\beta_1, \beta_2}^{\alpha_1}(t', x|t_1, x_1; t_2, x_2) = -\Delta^{\alpha_1}(x - x_1) \left[ \frac{\partial^2}{\partial x^n \partial x^m} \delta(x - x_2) \right] \hat{\Lambda}_{\beta_1}^{\alpha_1} Z(t_2) + \delta(x - x_1) \left[ \frac{\partial}{\partial x^n} \delta(x - x_2) \right] \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_1} Z(t_2), \quad (B12)$$

$$G_{\beta_1, \beta_2}^{\alpha_1, \alpha_2}(t', x|t_1, x_1; t_2, x_2) = -\Delta^{\alpha_1}(x - x_1) \left[ \frac{\partial}{\partial x^n} \delta(x - x_2) \right] \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_2} Z(t_2) + \delta(x - x_1) \delta(x - x_2) \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_2} Z(t_2). \quad (B13)$$

These expressions must be substituted into Eq. (B14). The volume integrals with respect to $x_2$ can be done, taking into account the extremely useful fact that all odd spatial derivatives of the velocity correlator $\kappa_{ij}(\tau, y)$ vanish at the origin (at $y = 0$). The results are: for $t_2 \geq t_1$,

$$\int d^d x_2 \kappa_{m\beta_2}(t' - t_2, x - x_2) G_{\beta_1, \beta_2, m}^{\alpha_1}(t', x|t_1, x_1; t_2, x_2)$$

$$\quad = \left[ \frac{\partial^2}{\partial x^n \partial x^m} \delta(x - x_1) \right] \kappa_{m\beta_2}(t' - t_2, 0) \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_1} Z(t_1), \quad (B14)$$

$$\int d^d x_2 \kappa_{m\alpha_2}(t' - t_2, x - x_2) G_{\beta_1, \beta_2, m}^{\alpha_2}(t', x|t_1, x_1; t_2, x_2)$$

$$\quad = \delta(x - x_1) \kappa_{m\alpha_2}(t' - t_2, 0) \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_2} \hat{\Lambda}_{\beta_1}^{\alpha_1} Z(t_1); \quad (B15)$$

for $t_1 \geq t_2$,

$$\int d^d x_2 \kappa_{m\beta_2}(t' - t_2, x - x_2) G_{\beta_1, \beta_2, m}^{\alpha_1}(t', x|t_1, x_1; t_2, x_2)$$

$$\quad = \delta(x - x_1) \left[ -\kappa_{m\alpha_2}(t' - t_2, 0) + \kappa_{m\beta_2}(t' - t_2, 0) \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_1} Z(t_2), \quad (B16)$$

$$\int d^d x_2 \kappa_{m\beta_2}(t' - t_2, x - x_2) G_{\beta_1, \beta_2, m}^{\alpha_2}(t', x|t_1, x_1; t_2, x_2)$$

$$\quad = \delta(x - x_1) \kappa_{m\alpha_2}(t' - t_2, 0) \hat{\Lambda}_{\beta_1}^{\alpha_1} \hat{\Lambda}_{\beta_2}^{\alpha_2} Z(t_2). \quad (B17)$$

With the aid of these expressions and Eq. (B10), one obtains Eq. (B9) of Sec. II B.
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by $\tilde{\kappa}_2$). The coefficient $\tilde{K}_2$ would be proportional to $K_2$ if we assumed that the time-correlation properties of the velocity correlator $\kappa^{ij}(\tau,y)$ were embodied in a single scalar function, so that $\kappa_0(\tau) \propto \kappa_2(\tau)$ and $\kappa_4(\tau) \propto \kappa_2(\tau)$.

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