EXISTENCE OF MASS-CONSERVING WEAK SOLUTIONS TO THE SINGULAR COAGULATION EQUATION WITH MULTIPLE FRAGMENTATION

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Abstract. In this paper we study the continuous coagulation and multiple fragmentation equation for the mean-field description of a system of particles taking into account the combined effect of the coagulation and the fragmentation processes in which a system of particles growing by successive mergers to form a bigger one and a larger particle splits into a finite number of smaller pieces. We demonstrate the global existence of mass-conserving weak solutions for a wide class of coagulation rate, selection rate and breakage function. Here, both the breakage function and the coagulation rate may have algebraic singularity on both the coordinate axes. The proof of the existence result is based on a weak $L^1$ compactness method for two different suitable approximations to the original problem, namely, the conservative and non-conservative approximations. Moreover, the mass-conservation property of solutions is established for both approximations.

1. Introduction. We investigate the existence of mass-conserving weak solutions to the continuous coagulation and multiple fragmentation equation (CMFE). We first recall that the CMFE provides a mean-field description of a system of particles growing by successive mergers to form a larger one and a bigger particle splits into daughter particles. Each particle is fully identified by its volume (or size) $y \in \mathbb{R}_{>0} := (0, \infty)$. Denoting by $g(t, y) \geq 0$, the concentration of particles of volume $y \in \mathbb{R}_{>0}$ at time $t \geq 0$, the dynamics of $g$ is given by [21, 20, 12, 13, 11, 5, 15]

$$\frac{\partial g(t, y)}{\partial t} = \frac{1}{2} \int_0^y A(y - z, z)g(t, y - z)g(t, z)dz - \int_0^{\infty} A(y, z)g(t, y)g(t, z)dz$$
$$+ \int_y^{\infty} b(y|z)S(z)g(t, z)dz - S(y)g(t, y),$$

with the initial value

$$g(0, y) = g^{in}(y) \geq 0 \text{ a.e.}$$

Here the non-negative and symmetric function $A(y, z)$ represents the coagulation rate which describes the rate at which particles of volume $y$ unite with particles of volume $z$ to produce larger particles of volume $y + z$ whereas $b(y|z)$ is the breakage function.

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function which gives the contribution to the formation of particles of volume \( y \) from the breakage of particles of volume \( z \) and the selection rate \( S(y) \) represents the rate at which particles of volume \( y \) is selected to break. In addition, the breakage function is assumed to satisfy the following properties

\[
\int_0^z b(y|z)dy = N(z) \quad \forall z \in \mathbb{R}_{>0}, \quad \text{where } \sup_{z \in \mathbb{R}_{>0}} N(z) = N < \infty \quad \text{and} \quad b(y|z) = 0 \quad \forall y \geq z,
\]

and

\[
\int_0^z yb(y|z)dy = z, \quad \forall y \in (0, z).
\] (3)

In (3), \( N(z) \) stands for the total number of daughter particles obtained from the breakage of particles of volume \( z \) and is assumed its supremum \( N(\geq 2) \) to be a finite constant. The condition (4) ensures that the total volume (mass) in the system remains conserved during the fragmentation events.

The first term in (1) gives the production of particles of volume \( y \) after coalescing of particles of volumes \( z \) and \( y-z \) due to the coagulation process whereas the second term shows the disappearance of particles of volume \( y \) after combining with particles of volume \( z \). The third and fourth terms describe the gain and loss of particles of volume \( y \) due to the multiple fragmentation events, respectively.

From (4), it is clear that the total mass is conserved during the fragmentation process. Thus, we expect that the total mass will also conserve during both the coagulation and multiple fragmentation events. However, if the coagulation rate is very high compared to the fragmentation rate, the conservation of mass fails at a finite time due to the appearance of giant particles in the system. This process is called the gelation transition and the finite time at which this process occurs is known as the gelation time [8, 19].

Next, the total mass of particles for coagulation and multiple fragmentation equation can be defined as

\[
\mathcal{M}_1(t) = \mathcal{M}_1(g(t)) := \int_0^{\infty} yg(t, y)dy, \quad t \geq 0.
\] (5)

The well-posedness of weak solutions to the continuous CMFE with unbounded non-singular kernels have been investigated in many articles [7, 12, 13, 20, 21] and references therein. However, in [3, 5, 15, 22] the existence and uniqueness of solutions to the continuous CMFE with singular coagulation rates have been discussed. In particular, Norris [22] has studied the existence and uniqueness of solutions to the continuous Smoluchowski coagulation equation (SCE) locally in time when the coagulation rate \( A \) that satisfies \( A(y, z) \leq \psi_1(y)\psi_1(z) \), with \( \psi_1 : (0, \infty) \rightarrow [0, \infty) \) and \( \psi_1(ay) \leq a\psi_1(y) \) for all \( y \in (0, \infty) \), \( a \geq 1 \), where \( \psi_1 \) is a sub-linear function and the initial data \( g^{in} \in L^1((0, \infty); \psi_1(y)^2) \). Moreover, he has established the mass-conservation property of solutions for \( \epsilon > 0 \) such that \( \epsilon y \leq \psi_1(y) \). Later, Camejo and Warnecke [5] have discussed the existence of weak solutions to the continuous CMFE for the singular coagulation kernel, when the coagulation rate \( A_2 \) and the selection rate \( S_1 \), respectively, satisfy the following

\[
A_2(y, z) \leq k(1+y)^{\lambda}(1+z)^{\lambda}(yz)^{-\sigma}, \quad \text{for } \sigma \in [0, 1/2), \lambda - \sigma \in [0, 1) \text{ and } k > 0,
\]

and

\[
S_1(y) \leq k'y^\alpha \quad \text{where } \alpha \in (0, 1) \text{ and } k' > 0.
\]
Moreover, they have shown the uniqueness result for this kernel $A_2$ when $\lambda = 0$. Recently, a uniqueness of self-similar mass-conserving solutions to the SCE is established for the similar type of coagulation kernel, see [16]. In [15], Laurençot has proven an interesting result to show the existence of mass-conserving solutions to the continuous CMFE by considering the breakage function, $b(y|z) = (\nu + 2)\frac{y^{\nu + 1}}{z^{\nu + 1}}$, provided that $\nu \in (-2, -1]$. By taking on account of this breakage function, one can infer from (3) that an infinite number of particles are produced for $\nu = -1$ and on the other hand, for $\nu \in (-2, -1)$, infeasible number of particles are created. Furthermore, a uniqueness result is established for restricted coagulation rate. Later, we have investigated the existence of mass-conserving solutions to the continuous SCE having linear growth for large volumes and singularity for small volume particles whatever the approximations to the original problems, see [3]. In addition, we have relaxed the assumption on the initial data as in [22] to show the existence of solutions.

Since the general uniqueness result to (1)–(2) is not available for singular coagulation rate $A$, breakage function $b$, selection rate $S$ and initial data $g^{in}$ satisfying $(\Lambda_1)$–$(\Lambda_4)$ respectively, it is not confirmed whether the solution to (1)–(2) obtained by a non-conservative approximation is mass conserving or not ? In [10], Filbet and Laurençot have studied a finite volume scheme to discuss the gelation transition by using a non-conservative truncation. In addition, they have concluded that the loss of mass in the system decrease for a large domain. Hence, it is expected that when the upper limit of the truncated domain goes to infinity, then the mass conservation property holds for a non-conservative truncation. Later, in [9], they have established a mathematical proof of this numerical observation. A similar type of numerical observation for the coagulation-fragmentation equations (CFEs) by using a finite volume scheme has been discussed by Bourgade and Filbet in [4]. Recently, in [2], we have shown mathematically that a non-conservative coagulation and conservative fragmentation truncation for CFEs also gives the mass conserving solutions for certain classes of nonsingular unbounded coagulation and fragmentation kernels. A similar type result has been established to the continuous SCE by using both conservative and non-conservative approximations with singular rate in [3]. However, an analogous result has not been studied for the combined effect of coagulation and multiple fragmentation events. The main novelty of the present work is to investigate a similar result in [3] for the continuous CMFE by considering both conservative and non-conservative truncation of the coagulation and conservative form of multiple fragmentation to the original problem with singular rates. But unfortunately, it can easily be seen from $(\Lambda_1)$ that we have not covered the range of $\beta > 0$ which was in [3] due to the presence of algebraic singularity of the breakage function for the small volume particles. An important observation that $A$ and the coagulation rate in [3] are equivalent up to a positive constant. We have also generalized our previous result in [2] from CFEs with non-singular rates to CMFE with singular rates. Moreover, the mass-conserving solution is constructed to the CMFE (1)–(2) which was an open problem in [5] for the coagulation rate $A$ that satisfies $(\Lambda_1)$ whatever the approximations. In addition, $\alpha = 1$ is included in the selection rate $S_1$. The main motivation of the present work is from [2, 3, 5, 15].

Let us end the introductory section by describing the plan of the paper. In Section 2, we introduce some preliminary results, assumptions and statement of the main result i.e. Theorem 2.1. In Section 3, the existence and uniqueness of truncated solutions to (1)–(2) is shown by using both conservative and non-conservative
truncation. In addition, the existence of mass-conserving weak solutions is proved by using a weak $L^1$ compactness technique in this section.

2. Assumptions, preliminaries and statement of the main result. Before stating the main result of this paper, we first describe the class of functions $g^n$, $A$, $S$ and $b$. More precisely, we assume that the initial data $g^n$, $A$, $S$ and $b$ enjoy the following assumptions.

$$(A_1)\ A(y, z) \leq k_1 \frac{(1 + y + z)}{y^{2\beta}} \text{ for all } (y, z) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, \ k_1 \geq 0 \text{ and } \beta \in [0, 1/2).$$

$$(A_2) \text{ there exists a positive constant } c_1 > 2 \text{ (depending on } \nu \text{ and } \beta) \text{ such that }$$

$$\int_0^z y^{-2\beta} b(y|z)dy \leq c_1 z^{-2\beta},$$

where $b(y|z) = (\nu + 2) \frac{y^{\nu}}{z^{\nu + 2}}$, for $-1 < \nu \leq 0$ and $\beta \& \nu$ are related with the relation $2\beta - \nu < 1$.

Note: Throughout the paper, we assume $b(y|z) = (\nu + 2) \frac{y^{\nu}}{z^{\nu + 2}}$, for $-1 < \nu \leq 0$.

$$(A_3)\ S(y) \leq k_2 y^{1 + \nu}, \ \forall y \in \mathbb{R}_{>0} \text{ for } k_2 \geq 0 \text{ and there exists a } \gamma \in (1, 2) \text{ (depending on } \nu \& \beta) \text{ such that } \gamma(\nu - \beta) + 1 > 0, \text{ where } \nu \text{ is defined in } (A_2).$$

$$(A_4)\ g^n \in L^1_{-2\beta,1}(\mathbb{R}_{>0}).$$

**Remark 1.** One can easily be checked that our coagulation rate $A$ is covering the Smoluchowski coagulation kernel in Brownian motion [1], formation of bubbles in stochastic stirred froths [6] and Granulation kernel [14] in the existence result.

Now, we are in a position to state the main theorem of this paper.

**Theorem 2.1.** Consider a function $g^n$ satisfying $(A_4)$ and assume that the functions $A$, $b$ and $S$ enjoy assumptions $(A_1)-(A_3)$. Let $g_n$ be the solution to (19) for $n \geq 1$. Then there is a subsequence $(g_{n_k})$ of $(g_n)$ and a mass conserving solution $g$ to (1)–(2) such that

$$g_{n_k} \rightharpoonup g \text{ in } C([0, T]; L^1_{-2\beta,1}(\mathbb{R}_{>0})) \text{ for each } T > 0 \quad (6)$$

satisfying the following weak formulation

$$\int_0^\infty [g(t, y) - g^n(y)]\omega(y)dy = \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \tilde{\omega}(y, z)A(y, z)g(s, y)g(s, z)dzdyds$$

$$- \int_0^t \int_0^\infty H_\omega(y)S(y)g(s, y)dyds, \quad (7)$$

where

$$\tilde{\omega}(y, z) := \omega(y + z) - \omega(y) - \omega(z) \quad (8)$$

and

$$H_\omega(y) := \omega(y) - \int_0^y b(z|y)\omega(z)dz, \quad (9)$$

for every $t \in [0, T]$ and $\omega \in L^\infty(\mathbb{R}_{>0})$. 
Proof. This lemma can be easily proved in a similar way as given in [2, 3, 9, 15].

3. Existence of weak solutions. In this section, we construct a mass conserving solution relies on both the conservative and non-conservative approximations to (1)–(2). In general, we expect that a conservative approximation to (1)–(2) will give a mass-conserving solution whenever the rate of kernels are not high whereas a non-conservative approximation of coagulation and a conservative approximation of multiple fragmentation is appropriated to study the gelation transition. However, an obvious question arises whether this non-conservative approximation to coagulation and conservative approximation of multiple fragmentation will provide
a mass-conserving solution or not to (1)-(2)? The answer of this question is yes. In this paper, we have provided the proof of this question.

Next, we defined both conservative and non-conservation approximations to (1)-(2). For a given natural number $n \in \mathbb{N}$, we set

$$g_n^\in(y) = g_n(y)\chi_{(0,n)}(y),$$

(16)

for $\zeta \in \{0,1\}$,

$$A_n^{\zeta}(y,z) := A(y,z)\chi_{(1/n,n)}(y)\chi_{(1/n,n)}(z)\left[1 - \zeta + \zeta \chi_{(0,n)}(y + z)\right],$$

(17)

and

$$S_n(y) = S(y)\chi_{(0,n)}(y).$$

(18)

Using (16), (17) and (18), we can rewrite (1)-(2) as

$$\frac{\partial g_n(t,y)}{\partial t} = \frac{1}{2} \int_0^y A_n^{\zeta}(y - z,z)g_n(t,y - z)g_n(t,z)dz - \int_{y-n}^n A_n^{\zeta}(y,z)g_n(t,y)g_n(t,z)dz + \int_y^n b(y,z)S_n(z)g_n(t,z)dz - S_n(y)g_n(t,y),$$

(19)

with the truncated initial condition

$$g_n(0,y) = g_n^\in, \text{ for } y \in (0,n).$$

(20)

**Proposition 1.** Let $\zeta \in \{0,1\}$ and $n \geq 1$. Then, there exists a unique non-negative solution $g_n \in C^1([0,\infty);L^1(0,n))$ to (19)-(20). In addition, it satisfies

$$\int_0^n yg_n(t,y)dy = \int_0^n yg_n^\in(y)dy - (1 - \zeta) \int_0^t \int_0^n \int_{n-y}^n yA_n^{\zeta}(y,z)g_n(s,y)g_n(s,z)dzdyds$$

(21)

for $t \geq 0$.

**Proof.** The proof of the Proposition 1 is straightforward consequence of the Cauchy-Lipschitz theorem or Picard-Lindelöf theorem in the Banach space $L^1(0,n)$ due to the fact that the right-hand side of (19) is locally Lipschitz continuous in that space. Moreover, to complete the proof of this Proposition, one can follow similar steps as in [3, Proposition 4.1].

**Remark 2.** If $\zeta = 1$, then the last term on the right-hand side (21) vanishes and the total mass remains conserved at any time, thus this approximation is known as conservative approximation. On the other hand, for $\zeta = 0$, the corresponding approximation is called non-conservative approximation because in this case the total mass decreases with respect to time. In both cases, one can infer from (21) that

$$\int_0^n yg_n(t,y)dy \leq \int_0^n yg_n^\in(y)dy, \text{ for } t > 0.$$  

(22)

We now recall that, for $n \geq 1$, and $\omega \in L^\infty(\mathbb{R}_{>0})$, the solution $g_n$ to (19)-(20) satisfies the following weak formulation
\[ \int_0^t \left[ g_n(t,y) - g_n^m(y) \right] \omega(y) dy \]

\[ = \frac{1}{2} \int_0^t \int_0^n \int_0^n G_{\omega,n}(y,z) \chi_{(1/n, n)}(y) \chi_{(1/n, n)}(z) A(y, z) g_n(s, y) g_n(s, z) dz dy ds \]

\[ - \int_0^t \int_0^n H_\omega(z) S_n(z) g_n(s, z) dz ds, \quad (23) \]

where

\[ G_{\omega,n}(y, z) = \omega(y + z) \chi_{(0, n)}(y + z) - [\omega(y) + \omega(z)](1 - \zeta + \zeta \chi_{(0, n)}(y + z)) \quad (24) \]

and \( H_\omega \) is defined in (9).

Next our aim is to show that the family of solutions \( \{ g_n \}_{n \geq 1} \) is relatively compact in \( C([0, T], L_{2, 1}^1(\mathbb{R}^+)) \). For that purpose, we apply a weak \( L^1 \) compactness method which is used in the pioneering work of Stewart [23]. In the next lemma, we show the family of solutions \( \{ g_n \}_{n \geq 1} \) is uniformly bounded in \( L_{2, 1}^1(\mathbb{R}^+). \)

3.1. Uniform bound.

**Lemma 3.1.** Assume \( (\Lambda_1) - (\Lambda_4) \) hold. Let \( T > 0 \). Then there is a constant \( \mathcal{G}(T) \) depending on \( T \) such that

\[ \int_0^T (y^{-2\beta} + y) g_n(t,y) dy \leq \mathcal{G}(T) \quad \text{for all } t \in [0, T]. \]

**Proof.** Let \( \theta \in (0, 1) \). We set \( \omega(y) = (y + \theta)^{-2\beta} \chi_{(0, n)}(y) \), and inserting it into (23) to obtain

\[ \int_0^n (y + \theta)^{-2\beta} [g_n(t, y) - g_n^m(y)] dy \]

\[ = \frac{1}{2} \int_0^t \int_0^n \int_0^n G_{\omega,n}(y,z) \chi_{(1/n, n)}(y) \chi_{(1/n, n)}(z) A(y, z) g_n(s, y) g_n(s, z) dz dy ds \]

\[ - \int_0^t \int_0^n H_\omega(z) S_n(z) g_n(s, z) dz ds. \quad (25) \]

We simplify \( G_{\omega,n} \) and \( H_\omega \) separately. Next, in the first case, for \( y + z < n \), it follows from (24) that

\[ G_{\omega,n}(y, z) = \omega(y + z) \chi_{(0, n)}(y + z) - [\omega(y) + \omega(z)](1 - \zeta + \zeta \chi_{(0, n)}(y + z)) \]

\[ \leq (y + z + \theta)^{-2\beta} - (y + \theta)^{-2\beta} - (z + \theta)^{-2\beta} \leq 0. \]

On the other case, for \( y + z \geq n \), (24) yield

\[ G_{\omega,n}(y, z) = \omega(y + z) \chi_{(0, n)}(y + z) - [\omega(y) + \omega(z)](1 - \zeta + \zeta \chi_{(0, n)}(y + z)) \]

\[ = 0 - \{(y + \theta)^{-2\beta} + (z + \theta)^{-2\beta}\}(1 - \zeta) \leq 0. \]

Now, we estimate \( H_\omega(z) \), by using (9) and \( (\Lambda_2) \), as

\[ H_\omega(z) = (z + \theta)^{-2\beta} - \int_0^z (y + \theta)^{-2\beta} b(y|z) dy \]

\[ \geq (z + \theta)^{-2\beta} - \int_0^z y^{-2\beta} b(y|z) dy \geq -c_1 z^{-2\beta}. \quad (26) \]
Since $G_{\omega,n}^\xi$ is non-positive for both above cases, thus one can infer that the second integral on the right-hand side of (25) is non-positive. Then, by using (26), (A2), (A3), (A4) and (22) into (25), we evaluate

$$
\int_0^n (y + \theta)^{-2\beta} g_n(t, y) dy
\leq \int_0^n (y + \theta)^{-2\beta} g_n^\infty(y) dy + k_2 c_1 \int_0^n \int_0^n z^{-2\beta + \nu + 1} g_n(s, z) dz ds
\leq \int_0^n y^{-2\beta} g_n^\infty(y) dy + k_2 c_1 \int_0^n (z + \theta)^{-2\beta + \nu + 1} g_n(s, z) dz ds
\leq \int_0^\infty y^{-2\beta} g_n^\infty(y) dy + k_2 c_1 \left[ \int_0^t \int_0^n (z + \theta)^{-2\beta} g_n(s, z) dz ds + (1 + \theta)^{-2\beta + \nu + 1} \|g_n^\infty\|_{L_{2,\beta,1}(R_{>0})} \right].
$$

An application of Gronwall’s inequality gives

$$
\int_0^n (y + \theta)^{-2\beta} g_n(t, y) dy \leq G^*(T),
$$

where $G^*(T) := \|g_n^\infty\|_{L_{2,\beta,1}(R_{>0})}(1 + k_2 c_1 T(1 + \theta)^{-2\beta + \nu + 1})(1 + Te^{k_2 c_1 T})$. Then, consider $\theta \to 0$ to (27) and applying Fatou’s lemma, we have

$$
\int_0^n y^{-2\beta} g_n(t, y) dy \leq G^*(T).
$$

Finally, we estimate the following integral, by using (28) and (22), as

$$
\int_0^n (y^{-2\beta} + \gamma) g_n(t, y) dy \leq G(T),
$$

where $G(T) := G^*(T) + \int_0^\infty y g_n^\infty(y) dy$, for each $n$. This proves Lemma 3.1.  

In the coming lemma, we discuss the behaviour of $g_n$ for large volume particle $y$.

**Lemma 3.2.** Assume that the coagulation rate, breakage function, selection rate and initial data satisfy (A1)–(A4), respectively. Then for every $n \geq 1$ and for $T > 0$,

$$
\sup_{t \in [0,T]} \int_0^n \sigma_1(y) g_n(t, y) dy \leq \Theta(T),
$$

and

$$
(1 - \zeta) \int_0^T \int_0^n \int_{n-y}^n \sigma_1(y) \chi_{(1/n,n)}(y) \chi_{(1/n,n)}(z) \times A(y, z) g_n(s, y) g_n(s, z) dz dy ds \leq \Theta(T),
$$

and

$$
\sup_{t \in [0,T]} \int_0^T \int_0^n z \sigma_1^\prime(y) - \sigma_1(y) S_n(y) g_n(s, y) dy ds \leq \Theta(T),
$$

where $\Theta(T)$ (depending on $T$) is a positive constant and the $\sigma_1 \in C_{VP,\infty}$ satisfies (11) and (12).
Proof. We set $\omega(y) = \sigma_1(y)\chi_{(0,n)}(y)$, and inserting it into (23) to obtain

$$\int_0^t \sigma_1(y)g_n(t,y)dy = \frac{1}{2} \int_0^t \int_0^n \int_0^n G_{S_1,n}(y,z)\chi_{(1/n,n)}(y)\chi_{(1/n,n)}(z)A(y,z)g_n(s,y)g_n(s,z)dzdyds$$

$$+ \int_0^n \sigma_1(y)g^y_n(y)dy - \int_0^t \int_0^n H_{\sigma_1}(z)S_n(z)g_n(s,z)dzds,$$  

(30)

where

$$G_{S_1,n}(y,z) = \sigma_1(y + z)\chi_{(0,n)}(y + z) - [\sigma_1(y) + \sigma_1(z)](1 - \zeta + \zeta\chi_{(0,n)}(y + z))$$

and

$$H_{\sigma_1}(z) = \sigma_1(z) - \int_0^z \sigma_1(y)b(y|z)dy.$$

By using (12) and (16) into (30), we have

$$\int_0^n \sigma_1(y)g_n(t,y)dy \leq \Gamma_1 + \frac{1}{2} \int_0^t [P_n(s) + Q_n(s)]ds$$

$$- \int_0^t \int_0^n H_{\sigma_1}(z)S_n(z)g_n(s,z)dzds,$$  

(32)

where

$$P_n(s) = \int_0^n \int_0^{n-y} G_{S_1,n}(y,z)\chi_{(1/n,n)}(y)\chi_{(1/n,n)}(z)A(y,z)g_n(s,y)g_n(s,z)dzdy,$$

and

$$Q_n(s) = (1 - \zeta) \int_0^n \int_{n-y}^n G_{S_1,n}(y,z)\chi_{(1/n,n)}(y)\chi_{(1/n,n)}(z)$$

$$\times A(y,z)g_n(s,y)g_n(s,z)dzdy.$$  

Multiplying $A(y,z)$ with (31), we have

$$A(y,z)G_{S_1,n}(y,z) = A(y,z)[\sigma_1(y + z)\chi_{(0,n)}(y + z)$$

$$- [\sigma_1(y) + \sigma_1(z)](1 - \zeta + \zeta\chi_{(0,n)}(y + z)]].$$  

(33)

Next, we estimate $P_n(s)$, by using (33), (15), and (A1), as

$$P_n(s) = \int_0^n \int_0^{n-y} A(y,z)\chi_{(1/n,n)}(y)\chi_{(1/n,n)}(z)[\sigma_1(y + z) - \sigma_1(y) - \sigma_1(z)]$$

$$\times g_n(s,y)g_n(s,z)dzdy$$

$$\leq 2k_1 \int_0^n \int_0^{n-y} (1 + y + z) (yz)^\beta \frac{y\sigma_1(z) + z\sigma_1(y)}{y + z} g_n(s,y)g_n(s,z)dzdy$$

$$\leq 12k_1 \int_0^n \int_0^1 (yz)^{-\beta} \frac{y\sigma_1(z) + z\sigma_1(y)}{y + z} g_n(s,y)g_n(s,z)dzdy$$

$$+ 12k_1 \int_0^n \int_0^1 \frac{y}{(yz)^\beta} \frac{y\sigma_1(z) + z\sigma_1(y)}{y + z} g_n(s,y)g_n(s,z)dzdy$$

$$+ 4k_1 \int_0^n \int_0^1 \frac{y\sigma_1(z) + z\sigma_1(y)}{(yz)^\beta} g_n(s,y)g_n(s,z)dzdy.$$  

(34)
Let us estimate the first term on the right-hand side of (34), by using Lemma 3.1 and monotonicity of $\sigma_1$, as

$$12k_1 \int_0^1 \int_0^1 (yz)^{-\beta} \frac{y\sigma_1(z)}{y+z} g_n(s,y) g_n(s,z) dz dy \leq 12k_1 \sigma_1(1) G(T)^2.$$  \hspace{1cm} (35)

Again, by using Lemma 3.1 and monotonicity of $\sigma_1$, the second term on the right-hand side of (34) can be evaluated, as

$$12k_1 \int_1^n \int_0^1 \frac{y}{(yz)^{\beta}} \frac{y\sigma_1(z)}{y+z} g_n(s,y) g_n(s,z) dz dy \leq 12k_1 \sigma_1(1) \int_1^n \int_0^1 y z^{-2\beta} g_n(s,y) g_n(s,z) dz dy$$
$$+ 12k_1 \int_1^n \int_0^1 z^{-2\beta} \sigma_1(y) g_n(s,y) g_n(s,z) dz dy$$
$$\leq 12k_1 \sigma_1(1) G(T)^2 + 12k_1 G(T) \int_0^n \sigma_1(y) g_n(s,y) dy.$$ \hspace{1cm} (36)

Finally, we evaluate the last integral on the right-hand to (34), by applying Lemma 3.1, as

$$4k_1 \int_1^n \int_1^1 \frac{y\sigma_1(z)}{(yz)^{\beta}} g_n(s,y) g_n(s,z) dz dy \leq 8k_1 \int_1^n \int_1^1 \frac{y\sigma_1(z)}{(yz)^{\beta}} g_n(s,y) g_n(s,z) dz dy$$
$$\leq 8k_1 G(T) \int_0^n \sigma_1(y) g_n(s,y) dy.$$ \hspace{1cm} (37)

Inserting (35), (36) and (37) into (34), we obtain

$$P_n(s) \leq 24k_1 \sigma_1(1) G(T)^2 + 20k_1 G(T) \int_0^n \sigma_1(y) g_n(s,y) dy.$$ \hspace{1cm} (38)

If $y + z \geq n$, then

$$G_{\sigma_1}(y, z) = -\sigma_1(y) - \sigma_1(z).$$ \hspace{1cm} (39)

Using (39), $Q_n(s)$ can be rewritten as

$$Q_n(s) = -2k_1 (1 - \xi) \int_0^n \int_{n-y}^n \sigma_1(y) \chi_{(1/n, n)}(y) \chi_{(1/n, n)}(z) \times A(y, z) g_n(s, y) g_n(s, z) dz dy \leq 0.$$ \hspace{1cm} (40)

Since $\sigma_1$ is a non-decreasing convex function and its derivative is concave, then we estimate $H_{\sigma_1}(z)$, by using (4), as

$$H_{\sigma_1}(z) = \sigma_1(z) - \int_0^z \sigma_1(y) b(y|z) dy$$
$$= \int_0^z \left[ \frac{\sigma_1(z)}{z} y b(y|z) - \frac{\sigma_1(y)}{y} y b(y|z) \right] dy$$
$$= \int_0^z \left[ \frac{\sigma_1(z)}{z} - \frac{\sigma_1(y)}{y} \right] y b(y|z) dy \geq \int_0^z \left( \frac{\sigma_1(z)}{z} \right)' (z - y) y b(y|z) dy$$
$$= \frac{z \sigma_1'(z) - \sigma_1(z)}{z^2} \int_0^z (z - y) y b(y|z) dy = \frac{z \sigma_1'(z) - \sigma_1(z)}{\nu + 3}.$$ \hspace{1cm} (41)
Inserting (38), (40), and (41) into (32), we obtain
\[ \int_0^n \sigma_1(y)g_n(t,y)dy + \int_0^t \int_0^n \frac{y\sigma'_1(y) - \sigma_1(y)}{\nu + 3} S_n(y)g_n(s,y)dyds \]
\[ + k_1(1 - \zeta) \int_0^t \int_0^n \sigma_1(y)\chi_{1/n,n}(y)\chi_{1/n,n}(z) A(y,z)g_n(s,y)g_n(s,z)dzdyds \]
\[ \leq \Gamma_1 + 12k_1\sigma_1(1)G(T)^2T + 10k_1G(T) \int_0^t \int_0^n \sigma_1(y)g_n(s,y)dyds. \]

Then by Gronwall’s inequality, we get
\[ \int_0^n \sigma_1(y)g_n(t,y)dy + \int_0^t \int_0^n \frac{y\sigma'_1(y) - \sigma_1(y)}{\nu + 3} S_n(y)g_n(s,y)dyds \]
\[ + k_1(1 - \zeta) \int_0^t \int_0^n \sigma_1(y)\chi_{1/n,n}(y)\chi_{1/n,n}(z) A(y,z)g_n(s,y)g_n(s,z)dzdyds \]
\[ \leq \Theta(T), \]
where \( \Theta(T) := (\Gamma_1 + 12k_1\sigma_1(1)G(T)^2T)e^{10k_1G(T)T} \), which completes the proof of Lemma 3.2.

In order to apply a refined version of de la Vallée Poussin theorem [17] to show the equi-integrability condition for the family of solutions \( \{g_n\}_{n>1} \), we require the following lemma.

3.2. Equi-integrability.

**Lemma 3.3.** Assume \((\Lambda_1) - (\Lambda_4)\) hold. Let \(T > 0\). Then prove the following results.
(i) For \(n \geq R > 1\), there is a positive constant \(C(T,R)\) such that
\[ \sup_{\tau \in [0,T]} \int_0^R \sigma_2(y^{-\beta} g_n(t,y))dy \leq C(T,R), \]
where \(\sigma_2 \in C_{\nu,F,\infty}\) satisfies (11) and (12).
(ii) For every \(\epsilon > 0\), there is an \(R_\epsilon > 1\) (depending on \(\epsilon\)) such that
\[ \sup_{t \in [0,T]} \int_{R_\epsilon}^{\infty} g_n(t,y)dy \leq \epsilon. \]

**Proof.** We take \(h_n(t,y) := y^{-\beta} g_n(t,y)\) and \(n \geq R\). Next, by using the Leibniz’s integral rule and (19), we have
\[ \frac{d}{dt} \int_0^R \sigma_2(h_n(t,y))dy \]
\[ \leq \frac{1}{2} \int_0^R \int_0^y \sigma'_2(h_n(t,y))y^{-\beta} A_n(\beta,\nu) \chi_{1/n,n}(y-z)g_n(t,y-z)g_n(t,z)dzdy \]
\[ + \int_0^R \int_y^\infty \sigma'_2(h_n(t,y))y^{-\beta} b(y)S_n(z)g_n(t,z)dzdy. \quad (42) \]
Changing the order of integration by using Fubini’s theorem and simplifying it further, by substituting \( y - z = y' \) and \( z = z' \), as

\[
\frac{d}{dt} \int_0^R \sigma_2(h_n(t, y)) dy
\leq \frac{1}{2} \int_0^R \int_0^{R-z} \sigma_2'(h_n(t, y + z))(y + z)^{-\beta} A_n^\xi(y, z) g_n(t, y) g_n(t, z) dy dz
\]

\[
+ \int_0^R \int_0^z y^{-\beta} b(y|z) S_n(z) \sigma_2'(h_n(t, y)) g_n(t, z) dy dz
\]

\[
+ \int_R^n \int_0^R y^{-\beta} b(y|z) S_n(z) \sigma_2'(h_n(t, y)) g_n(t, z) dy dz.
\]

(43)

Now, we estimate each term on the right-hand side, individually. The first term on the right hand side of (43) can be evaluated, by using (A1), (14) and Lemma 3.1, as

\[
\frac{1}{2} \int_0^R \int_0^{R-z} \sigma_2'(h_n(t, y + z))(y + z)^{-\beta} A_n^\xi(y, z) g_n(t, y) g_n(t, z) dy dz
\leq \frac{1}{2} k_1 (1 + R) \int_0^R \int_0^{R-z} y^{-2\beta} z^{-\beta} \sigma_2'(h_n(t, y + z)) g_n(t, y) g_n(t, z) dy dz
\leq \frac{1}{2} k_1 (1 + R) \int_0^R \int_0^{R-z} y^{-2\beta} (\sigma_2(h_n(t, y + z)) + \sigma_2(h_n(t, y))) g_n(t, z) dy dz
\leq C_1(T, R) \int_0^R \sigma_2(h_n(t, z)) dz,
\]

(44)

where \( C_1(T, R) := k_1 (1 + R) G(T) \). Next, the second term can be estimated, by using (A3), (14), the definition of \( C_{V_p, \infty} \) and Lemma 3.1, as

\[
\int_0^R \int_0^z y^{-\beta} b(y|z) S_n(z) \sigma_2'(h_n(t, y)) g_n(t, z) dy dz
\leq k_2(\nu + 2) \int_0^R \int_0^z [\sigma_2(h_n(t, y)) + \sigma_2(y^{-\beta})] g_n(t, z) dy dz
\leq k_2(\nu + 2) \left[ G(T) \int_0^R \sigma_2(h_n(t, y)) dy + \int_0^R \int_0^z (y^{-\beta}) \gamma \sigma_2(y^{-\beta}) g_n(t, z) dy dz \right]
\leq k_2(\nu + 2) G(T) \int_0^R \sigma_2(h_n(t, y)) dy + k_2 \frac{(\nu + 2)}{\gamma \nu - \gamma \beta + 1} S_\gamma(\sigma_2) \int_0^R \sigma_2(y^{-\beta} + 1) g_n(t, z) dz
\leq k_2(\nu + 2) G(T) \int_0^R \sigma_2(h_n(t, y)) dy + k_2 \frac{(\nu + 2)}{\gamma \nu - \gamma \beta + 1} S_\gamma(\sigma_2) G(T).
\]

(45)

Finally, we estimate the third term, by using (A2), (A3), (14), the definition of \( C_{V_p, \infty} \) and Lemma 3.1, as

\[
\int_R^n \int_0^R y^{-\beta} b(y|z) S_n(z) \sigma_2'(h_n(t, y)) g_n(t, z) dy dz
\leq k_2(\nu + 2) \int_R^n \int_0^R y^{-\beta} y^\nu \sigma_2'(h_n(t, y)) g_n(t, z) dy dz
\leq k_2(\nu + 2) G(T) \left[ \int_0^R \sigma_2(h_n(t, y)) dy + \int_0^R \sigma_2(y^{-\beta}) dy \right]
\]
\[ \leq k_2(\nu + 2)G(T) \left[ \int_0^R \sigma_2(h_n(t, y))dy + \frac{S_\gamma(\sigma_2)}{\gamma\nu - \gamma\beta + 1} R^{\nu - \gamma\beta + 1} \right]. \quad (46) \]

Collecting all above estimates in (44), (45) and (46), and inserting them into (43), we have

\[
\frac{d}{dt} \int_0^R \sigma_2(h_n(t, y))dy \leq C(T, R) \int_0^R \sigma_2(h_n(t, z))dz + C_3(T, R),
\]

where \( C_2(T, R) := C_1(T, R) + 2k_2(\nu + 2)G(T) \) and \( C_3(T, R) := k_2(\frac{\nu + 2}{\gamma\nu - \gamma\beta + 1}) S_\gamma(\sigma_2) [G(T) + R^{\nu - \gamma\beta + 1}] \). Then applying the Gronwall’s inequality, we obtain

\[
\int_0^R \sigma_2(y^{-\beta} g_n(t, y))dy \leq C(T, R),
\]

where \( C(T, R) \) is a constant depending on \( T \) and \( R \). This completes the proof of the Lemma 3.3 (i). One can infer the second part of Lemma 3.3 by using (22) and (A4). This completes the proof of Lemma 3.3. \( \square \)

3.3. Equi-continuity w.r.t. time in weak sense.

**Lemma 3.4.** Assume (A1)–(A4) hold. For any \( T > 0 \) and \( R > 1 \), there is a positive constant \( C_5(T, R) \) depending on \( T \) and \( R \) such that

\[
\left| \int_0^T \Psi(y)[g_n(t, y) - g_n(s, y)]dy \right| \leq C_5(T, R)(t - s),
\]

for every \( n > 1 \), \( 0 \leq s \leq t \leq T \) and \( \Psi \in L^\infty(\mathbb{R}_>) \).

**Proof.** Let \( T > 0 \) and \( R > 1 \). For \( n > 1 \), \( 0 \leq s \leq t \leq T \) and \( \Psi \in L^\infty(\mathbb{R}_>) \), we evaluate the following integral as

\[
\int_0^T \Psi(y)[g_n(t, y) - g_n(s, y)]dy
\]

\[
\leq \left\| \Psi \right\|_{L^\infty(\mathbb{R}_>)} \int_s^t \int_0^R \int_0^y y^{-\beta} \frac{\partial g_n}{\partial t}(\tau, y)dyd\tau
\]

\[
\leq \left\| \Psi \right\|_{L^\infty(\mathbb{R}_>)} \int_s^t \left[ \frac{1}{2} \int_0^R \int_0^y y^{-\beta} A_n^\lambda(y - z, z)g_n(\tau, y - z)g_n(\tau, z)dzdy
\]

\[
+ \int_0^R \int_0^y y^{-\beta} A_n^\lambda(y, z)g_n(\tau, y)g_n(\tau, z)dzdy
\]

\[
+ \int_0^T \int_0^y y^{-\beta} b(y|z)S_n(y)g_n(\tau, z)dzdy + \int_0^R y^{-\beta} S_n(y)g_n(\tau, y)dy \right] d\tau. \quad (49)
\]

Next, we evaluate each integral on the right-hand side to (49) separately. First, we evaluate the first integral, by using Fubini’s theorem, (\( \Gamma_1 \)) and Lemma 3.1, as

\[
\frac{1}{2} \int_s^t \int_0^R \int_0^y y^{-\beta} A_n^\lambda(y - z, z)g_n(\tau, y - z)g_n(\tau, z)dzdyd\tau
\]

\[
\leq \frac{1}{2} k_1 \int_s^t \int_0^R \int_0^{R - z} (y + z)^{-\beta} \frac{(1 + y + z)}{(yz)^\beta} g_n(\tau, y)g_n(\tau, z)dydzd\tau
\]

\[
\leq \frac{1}{2} k_1 (1 + R) \int_s^t \int_0^R \int_0^y y^{-2\beta} z^{-\beta} g_n(\tau, y)g_n(\tau, z)dydzd\tau
\]

\[
\leq \frac{1}{2} k_1 (1 + R) G^2(T)(t - s). \quad (50)
\]
Similarly, by applying \((\Gamma_1)\) and Lemma 3.1, the second integral can be estimated, as

\[
\begin{align*}
\int_s^t \int_0^R \int_0^{n-\zeta y} y^{\beta} A_n^\zeta(y,z)g_n(\tau,y)g_n(\tau,z)dzdyd\tau \\
\leq k_1 \int_s^t \int_0^R \int_0^n y^{\beta} \frac{(1 + R + z)}{(yz)^\beta} g_n(\tau,y)g_n(\tau,z)dzdyd\tau \\
\leq k_1 G(T) \int_s^t \int_0^n (1 + R + z)z^{-\beta} g_n(\tau,z)dzd\tau \leq 2k_1(1 + R)G^2(T)(t - s). \quad (51)
\end{align*}
\]

We evaluate the third integral, by using Fubini’s theorem, \((\Gamma_2), (\Gamma_3)\), and Lemma 3.1, as

\[
\begin{align*}
\int_s^t \int_0^R \int_0^n y^{-\beta} b(y|z)S_n(z)g_n(\tau,z)dzdyd\tau \\
\leq \int_s^t \int_0^R \int_0^n y^{-\beta} b(y|z)S_n(z)g_n(\tau,z)dydzd\tau \\
\leq c_1 k_2 \int_s^t \int_0^n z^{1+\nu-\beta} g_n(\tau,z)dzd\tau \leq c_1 k_2 G(T)(t - s). \quad (52)
\end{align*}
\]

Finally, the last term can be estimated, by applying \((\Gamma_3)\), and Lemma 3.1, as

\[
\begin{align*}
\int_s^t \int_0^R y^{-\beta} S_n(y)g_n(\tau,y)dyd\tau \leq k_2 \int_s^t \int_0^R y^{1+\nu-\beta} g_n(\tau,y)dyd\tau \\
\leq k_2 G(T)(t - s). \quad (53)
\end{align*}
\]

Inserting (50), (51), (52) and (53) into (49), we have

\[
\int_0^R y^{-\beta} \Psi(y)g_n(t,y) - g_n(s,y)dy \leq C_5(T, R)(t - s), \quad (54)
\]

where

\[
C_5(T, R) = \|\Psi\|_{L^\infty(\mathbb{R}^+)} \left[ \frac{1}{2} k_1 (1 + R)G(T) + 2k_1(1 + R)G(T) + c_1 k_2 + k_2 \right] G(T).
\]

Now for arbitrary \(\epsilon > 0\), we evaluate the following integral, by using (54) and Lemma 3.3, as

\[
\begin{align*}
\left| \int_0^\infty y^{-\beta} \Psi(y)[g_n(t,y) - g_n(s,y)]dy \right| \\
\leq \left| \int_0^R y^{-\beta} \Psi(y)[g_n(t,y) - g_n(s,y)]dy \right| + \left| \int_0^\infty y^{-\beta} \Psi(y)[g_n(t,y) - g_n(s,y)]dy \right| \\
\leq C_5(T, R)(t - s) + 2\|\Psi\|_{L^\infty(\mathbb{R}^+)}\epsilon. \quad (55)
\end{align*}
\]

This completes the proof of Lemma 3.4. \(\square\)

We are now in a position to complete the proof of Theorem 2.1 in the next subsection.
3.4. **Convergence of integrals.** Proof of Theorem 2.1: From de la Vallée Poussin theorem, Lemma (3.1)–(3.3), and then using Dunford–Pettis theorem and a variant of the Arzelà-Ascoli theorem, see [24], we conclude that \((g_n)\) is relatively compact in \(C([0, T]; L^1_{-\beta}(\mathbb{R}_{>0}))\) for each \(T > 0\). There is thus a subsequence of \((g_n)\) (not relabeled) and a nonnegative function \(g \in C([0, T]; L^1_{-\beta}(\mathbb{R}_{>0}))\) such that
\[
g_n \rightarrow g \text{ in } C([0,T]; L^1(\mathbb{R}_{>0}) : y^{-\beta}dy) \tag{56}
\]
for each \(T > 0\).

Next, we can improve the weak convergence (56), by applying Lemma (3.1), (29) and (56), as
\[
g_n \rightarrow g \text{ in } C([0,T]; L^1(\mathbb{R}_{>0}) : (y^{-\beta} + y)dy). \tag{57}
\]

In order to show that \(g\) is actually a solution to (1)–(2) in the sense of (7), it remains to verify all the truncated integrals in (19) converges weakly to the original integrals in (1), respectively. This is now a standard procedure to prove this convergence of integrals, see [5, 3, 13, 18, 17, 15, 23]. Now, using the weak convergence (57) into (19), we have
\[
\int_0^\infty \omega(y)[g(t,y) - g^n(y)]dy = \lim_{n \rightarrow \infty} \int_0^n \omega(y)[g_n(t,y) - g^n_n(y)]dy
\]
\[
= \lim_{n \rightarrow \infty} \left\{ \frac{1}{2} \int_0^t \int_0^\infty \int_0^n C_{\omega,n}^\xi(y,z) \chi_{(1/n,n)}(y) \chi_{(1/n,n)}(z) A(y,z)g_n(s,y)g_n(s,z)dzdyds - \int_0^t \int_0^\infty H_{\omega}(z)S_n(z)g_n(s,z)dzds \right\}
\]
\[
= \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \tilde{\omega}(y,z)A(y,z)g(s,y)g(s,z)dzdyds - \int_0^t \int_0^\infty H_{\omega}(z)S(z)g(s,z)dzds,
\]
for every \(\omega \in L^\infty(\mathbb{R}_{>0})\). This confirms that \(g\) is a weak solution to (1)–(2) in the sense of (7).

Finally, to complete the proof of Theorem 2.1, it is required to show that \(g\) is a mass-conserving solution to (1)–(2). For the case of non-conservative one \((\zeta = 0)\), it can be easily proved similarly to [9, 2] and on the other hand, for conservative case \((\zeta = 1)\), we infer from (57) and (21), which completes the proof of Theorem 2.1.

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