Experimental identification of multimode traveling waves in a coupled wave-tube

Yoav Vered and Izhak Bucher

Dynamics Laboratory, Faculty of Mechanical Engineering, Technion, Haifa 3200003, Israel

Keywords: model-based identification, traveling waves, model order selection, modal decomposing, acoustic waveguide, NDT of acoustic properties.

Abstract

An essential part of nondestructive testing and experimental modeling of waveguides is the decomposition of propagating wave patterns. The traveling wave ratio is a measure of partial reflections assisting in quantifying the pureness of a single traveling wave from a power flow perspective. This paper expands the notion of traveling wave ratio for multimode systems and outlines several schemes capable of decomposing the waves into its different traveling modes, while quantifying, individually, their traveling and standing proportions. A method to strike an optimal balance between increasing model order and maintaining low uncertainty is proposed. An experimental study performed on an acoustic wave tube, which utilizes the various methods while assessing their accuracy and performance is reported. The results described here emphasize the importance of including additional propagating modes. In addition, the results illustrate the capability of using the recursive multichannel least mean squares method for both a fast decomposition and as a basis to formulate closed-loop schemes controlling the wave's propagation patterns.

Electronic mail: syoavv@gmail.com ; syoavv@campus.technion.ac.il
1. Introduction

The analysis of continuum dynamics using a basis of traveling waves has been employed in various fields. A few notable examples are Acoustic [1], Elastic [2], Electromagnetic [3], Optics and Quantum mechanics [4]. While the simple model of propagation gives rise to a single propagating mode, a realistic model results in a several simultaneous propagating modes, multiple modes can exist at the entire frequency range [5,6] or only above the so-called cut-off frequency [7].

Acoustic wave tubes such as the impedance tube, standing wave tube, and pulse tube, play an important role in the conduction of nondestructive testing (NDT) of acoustic properties [8,9]. The wave tube is composed of a hollow elastic cylinder and is filled with the required fluid. Normally, as part of the analysis, the propagating pressure wave is decomposed into the forward and backward traveling wave under the assumptions of a single and planar propagating mode [10]. In reality, an acoustic wave tube which contains a compressible fluid surrounded by an elastic enclosure, gives rise to dispersion relation describing multiple propagation modes (referred to as multimode propagation [11]) at all frequencies [11,12]. While it is true for air-filled wave tubes [13], it has a larger effect on liquid-filled wave tubes [5]. Moreover, when considering air-filled tubes, it is common to neglect the propagation attenuation factor since it was found to be much smaller than the boundaries’ attenuation [13,14]; on the contrary, liquid-fluid wave tubes can give rise to non-negligible propagation attenuations which result in complex modal wavenumbers [15–17].

Each of the propagating mode branches of the dispersion relation is the outcome of a different energy conveying mechanism [18], whose contribution to the total transfer of energy can be quantified by the use of the modal power ratio [19]. The modal power ratio for fluid-filled acoustic waveguides was previously employed to characterized straight cylindrical acoustic waveguides [19], curved acoustic waveguides [20], and straight axisymmetric general cross-section acoustic waveguides [21].

The standing wave ratio (SWR) [22] and traveling wave ratio (TWR) [23,24] have been used to quantify the direction and purity of the propagating mode for the single-mode scenarios, and to identify the acoustic
impedance in a standing wave tube [25]. The power ratio and the TWR are equivalents [26], thus either of them can be used as an energy conveying measure. In a wave control application TWR is preferred over SWR because the former is always finite and thus more suitable for real-time control [23].

Traveling wave control schemes [27] are mostly developed on the basis of a single propagating mode in each propagation direction. The existing control schemes were developed and applied for many elastic and acoustic waveguides such as the cases of 1D beams [28–30], lumped flexible systems [31–33], systems governed by the 1D wave equation [23,33,34], and systems governed by the 2D wave equation[35–37]. As discussed, in [23] the open-looped model is highly sensitive to model uncertainties, therefore an experimental identification approach of the traveling wave is crucial for reasonable control. The decomposition of single traveling wave into its forward and backward components was done previously, both in a batch mode [23,29,38] and recursively [22,26], and is now a standard analysis tool.

In a previous work [39] the generalization of the TWR to the multimode scenario was used to control the TWR of the principal acoustic mode [11], by using a batch-based identification procedure. The present work develops a generalization of the TWR using the orthogonality relation defined by the Sturm–Liouville theory [40] and outlines tools to enable recursive and fast identification of each modal TWR. Four methods to identify the TWR in the multimode scenario are outlined: the first, a batch multichannel least-squares (MC-LS) method; the second, a recursive multichannel recursive-least-squares (MC-RLS) [41] method; the third a recursive multichannel least-mean-squares (MC-LMS) method [41]; the fourth, an extended recursive synchronous demodulation method [42]. While the first three methods are model-based and rely on the knowledge of the dispersion relation of all propagating modes; the fourth method relies on the dispersion relation of a single-mode but is limited by the number and distribution of sensors. The dispersion relation can be modeled analytically [5,12,16], numerically [43,44], or experimentally [13].

Least-squares (LS) is a conventional method and tool for the model-based identification technique [45], an extensive introduction to the method and its generalization can be found in [46], the recursive formulation of the LS problem as well as other model-based methods for the active control is also a
common knowledge nowadays, an extensive analysis of such method can be found in [47]. The number of propagating modes is not known a priori therefore, there is a need to avoid overfitting [46]. The overfitting can be avoided by employing an L-curved analysis [48,49] to choose the correct model-order (number of propagating modes).

Described in this work is a numerical and experimental study of state-of-the-art modeling techniques for recursive modal decomposition in an acoustic wave tube. The paper is organized as follows: Section 2 presents and discusses the fundamental work of Del Grosso's to introduce the concept of multimode propagation; Section 3.1 introduces the TWR based on the rigid wall assumption; Section 3.2 generalizes the TWR by employing the modal orthogonality based on the elastic acoustic wave tube model; Section 4 outlines methods to experimentally identify the modal TWR (MC-LS, MC-RLS, MC-LMS, synchronous demodulation), subsection 4.1 includes a discussion concerning the sensor amount and positions needed for the model-based methods; Section 5 presents a sensitivity-based analysis chart to avoid overfitting and choose the model order; Section 6 presents a numerical comparison between the methods; Section 7 presents the experimentally obtained results, subsection 7.1 describes the experimental air-filled acoustic wave tube, subsection 7.2 presents the experimentally sensitivity-based analysis, subsection 7.3 depicts the method comparison using experimentally obtained data, subsection 7.4 presents a comparison between the batch MC-LS and the recursive MC-LMS based on the modal TWR contour maps; The main conclusions of this paper are summarized in Section 8.
2. Motivation – multimode propagation in an acoustic wave tube

Fig. 1 – (a) Model of a coupled acoustic-elastic wave tube similar to the one used for NDT of acoustical properties. (b) A cross-section model of a coupled acoustic-elastic waveguide used for dispersion analysis.

Acoustic wave tubes, illustrated in Fig. 1a, are a common experimental system to conduct NDT of acoustic properties for different materials and different environments, a comprehensive review about their use and method of operation can be found in [10]. The use of a representative physical model composed of forward and backward traveling waves is the heart of the different evaluation methods. Most methods neglect the acoustic-elastic coupling by employing a rigid boundary condition model for the elastic tube. The rigid boundary model results in a single nondispersive planar propagating mode up to the first cut-off frequency [7]. On the other hand, models that incorporate the acoustic-elastic coupling, give rise to a rich, multimode dispersion relation, as is illustrate here for the case of a 3D axisymmetric model. The governing equations for the axisymmetric acoustic-elastic waveguide model at any cross-section are [6,11]

\[
\frac{1}{c_1} \frac{\partial^2}{\partial t^2} - \nabla^2 \Phi(r,\theta,z,t) = 0, \quad r \in (0,b),
\]

(1)

\[
\frac{1}{c_2} \frac{\partial^2}{\partial t^2} - \nabla^2 \vartheta(r,\theta,z,t) = 0, \quad r \in (b,d),
\]

(2)

\[
\frac{1}{c_3} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{1}{r^2} \psi_\theta(r,\theta,z,t) = 0, \quad r \in (b,d),
\]

(3)
where \( b \) and \( d \) denote the inner and outer radius of the waveguide (tube), \( \Phi \) is the acoustic velocity potential function [11], and \( \vartheta \) and \( \psi_\theta \) are the scalar and azimuthal elements of the vector elastic potential of the displacement field used in the Helmholtz decomposition \( \mathbf{u} = \nabla \vartheta + \nabla \times (0, \psi_\theta, 0) \) [6]. \( c_1 \) denotes the intrinsic speed of sound in the fluid, \( c_c = \sqrt{(2\mu + \lambda)/\rho_s} \) and \( \sqrt{\mu/\rho_s} \) denote the velocities of longitudinal and transverse waves in the solid, \( \rho_s \) denotes the solid density, and \( \lambda \) and \( \mu \) are the Lamé parameters of the solid medium. The axially symmetric potential solutions of Eq. (1)–(3) are [11, Eq. (23)]

\[
\Phi = \phi_{0m} J_0 \left( \sqrt{k_i^2 - k_{0m}^2} r \right) \exp \left( i \left( \omega t - k_{0m} z \right) \right),
\]

(4)

\[
\vartheta = \left( A_{0m} J_0 \left( \sqrt{k_c^2 - k_{0m}^2} r \right) + B_{0m} Y_0 \left( \sqrt{k_c^2 - k_{0m}^2} r \right) \right) \exp \left( i \left( \omega t - k_{0m} z \right) \right),
\]

(5)

\[
\psi_\theta = \left( C_{0m} J_1 \left( \sqrt{k_s^2 - k_{0m}^2} r \right) + D_{0m} Y_1 \left( \sqrt{k_s^2 - k_{0m}^2} r \right) \right) \exp \left( i \left( \omega t - k_{0m} z \right) \right),
\]

(6)

for angular frequency \( \omega \), modal wavenumber \( k_{0m} \), mediums wavenumbers \( k_i, k_c, \) and \( k_s \) which are defined by the ratio \( \omega/c_s \), where \( s \) takes either 1, \( c \), or \( s \). Also, \( J_n \) and \( Y_n \) are the \( n^{th} \) order Bessel function of the first kind and second kind, respectively. The constants \( \phi_{0m}, A_{0m}, B_{0m}, C_{0m}, \) and \( D_{0m} \) are obtained from the boundary conditions: fluid pressure equals to radial normal stress at the inner radii of the solid; fluid radial normal velocity equals that of the solid at the inner radii; zero solid radial shear stress at inner radii; zero solid radial shear stress at outer radii; zero solid radial normal stress at the outer radii.

By substituting the axially symmetric potential of Eq. (4)–(6), a homogenous linear system of equations can be written for the 5 constants, taking the matrix determinant yields the frequency-dependent dispersion equation in the modal wavenumber \( k_{0m} \) [5, Eq.(5)]. The characteristic equation’s real zeros are the axial wavenumbers of the propagating modes (their complex value counterparts are the wavenumbers of the evanescent modes). Fig. 2 presents the analytically computed dispersion curve of an air-filled wave tube which was used as part of this research.
Each branch (line) of the dispersion curve is associated with a different propagating mode and therefore acts as different energy conveying mechanism [21], it can be seen that at all frequencies at least two propagating modes exist, as the frequency gets higher additional modes begins to propagate, each of these frequencies is the cut–off frequency of the associated mode. The dispersion curves presented in Fig. 2 do not include all energy conveying mechanisms since the above mention model neglects non-axisymmetric modes that exist in reality. A full analytical model of the 3D waveguide dispersion relation can be found in [12], but its results do not shed new light beyond what was presented in the abovementioned axisymmetric model [11] for the purpose of the present work’s motivation.

In a previous paper [13] the experimental identification of the dispersion curves of the wave tube in use here was conducted using the two-actuator phase-perturbations (TAPP) method resulting in a strong agreement with both the axisymmetric analytical model of [11] (which is presented above) and the flexural beam deflection model of a circular cross-section Timoshenko's beam [6,50]. Therefore, the need to decompose the measured pressure field into all propagating modes, which affect the accuracy from an energy perspective, is crucial to the success of acoustical NDT methods.
3. Traveling wave ratio

The traveling wave ratio is a scalar measure of the propagating versus circulating energies ratio [26]. It serves as a basis for quantifying these proportions on a realistic, physical model as outlined below.

3.1 Background – single-mode propagation

When considering the case of an axisymmetric acoustic wave tube and modeling the elastic tube as a rigid boundary, the general solution of the fluid velocity potential of Eq. (4) holds, but the boundary condition in this case becomes

$$\frac{\partial \Phi}{\partial r} \bigg|_{r=b} = 0 \Rightarrow J_1 \left( \sqrt{k_1^2 - k_{0m}^2} b \right) = 0.$$  \hspace{1cm} (7)

Which defines the characteristic equation of the waveguide whose zeros are the modal wavenumbers, $k_{0m}$. Since $k_{0m} = k_1$ is a zero of the characteristic equation with a constant radial distribution ($J_0(0) = 1$). It follows that the principal acoustic mode, in this case, is a planar traveling wave. This is also the only real value solution until the first cut-off frequency, defined by

$$f_{c,2} = \frac{\xi_{0,2}}{2\pi b} c_1,$$  \hspace{1cm} (8)

where $\xi_{m,n}$ represents the $m^{th}$ zero of $J_n'$. Therefore, the pressure field can be expressed, up to this frequency as a one-dimensional wave by means of the D'Alembert solution [1]

$$p(z,t) = \text{Re} \left( P^+ \eta^+ (z,t) + P^- \eta^- (z,t) \right).$$  \hspace{1cm} (9)

Here $P^+$ and $P^-$ denote the forward and backward propagating wave amplitude, accordingly, and $\eta$ denotes the orthogonal basis of the sinusoidal traveling wave functions

$$\eta^+ (z,t) = \exp \left( i(\omega t \pm k z) \right).$$  \hspace{1cm} (10)

The dispersion relation in this case is linear $\omega = c_1 k$. Which is referred as a nondispersive wave where both the group and phase velocities are kept constants as function of angular frequency (wavenumber).

The Standing Wave Ratio [22] of the pressure wave is thus defined as
Since the SWR is unbounded when approaching a pure standing wave, a more usable quantification may be found in the Traveling Wave Ratio (TWR) [24]

\[
TWR = 1 - \left| \text{SWR}^{-1} \right| = 1 - \frac{\left| P^+ \right| - \left| P^- \right|}{\left| P^+ \right| + \left| P^- \right|}.
\]  

(12)

The TWR represents the ratio between the standing waves and the total maximal amplitude of the combined traveling and standing waves, which best suits traveling wave optimization procedures as TWR=0 means a pure traveling wave while TWR=1 a pure standing one. The traveling wave’s direction of propagation is defined by the sign of the numerator argument

\[
\text{dir} = \text{sign} \left( \left| P^+ \right| - \left| P^- \right| \right) = \pm 1,
\]

(13)
a positive sign means a forward traveling wave and vice-versa.

Thus, for the planar single-mode scenario, the TWR can and has been used to model the dynamic and control it [35]. In the more realistic case of multimode propagation, as presented in the motivation section, the presented definition of the TWR is not usable and a rigorous generalization is needed.

### 3.2 Traveling wave ratio generalization for a multimode scenario

In this subsection, the generalized, multimode case of the TWR, is presented. By considering the case of a multimode propagation that is predicted by the analytical solution of the elastic acoustic waveguide, several waves having different wavenumbers arise. In this case, the model of Eq. (9) is no longer adequate and the solution needs to be decomposed into a modal summation [11,38], via

\[
p(r, \theta, z,t) = \sum_{n=1}^{N} p_n(r, \theta, z,t),
\]

(14)

\[
p_n(r, \theta, z,t) = \text{Re} \left( g^+_n(r, \theta) \eta^+_n(z,t) + g^-_n(r, \theta) \eta^-_n(z,t) \right).
\]

(15)
Here $p_n$ represents the $n$th modal propagating mode, $z$ denotes the wave propagation direction as before, and $r$ and $\theta$ are the cross-section polar coordinates. Also, $N \in \mathbb{N}$ is the number of propagating modes which is known to be a finite number as discussed in Section 2. Each of the modal traveling wave coordinates, $\eta_n^\pm$, takes the form of Eq.(10), with the change of the single wavenumber, $k$, to the modal wavenumber, $k_n$. As discussed before the relation of each dispersion branch relates the modal wavenumber, $k_n$, to the angular frequency $\omega$ (the double indexing notation for the flexural order of the modes is omitted for brevity).

The wave amplitudes of Eq. (14), $g_n^\pm$, are functions of the cross-section coordinates as presented in Eq. (4)–(6), thus their norm will be used instead of the absolute value to compute the modal amplitudes, which helps in quantifying each mode’s relative contribution,

$$P_n^\pm = \|g_n^\pm(r, \theta)\|_2 = \sqrt{\int_0^L \int_0^{2\pi} |g_n^\pm(r, \theta)|^2 r dr d\theta}.$$  \hfill (16)

Since it is impractical to measure the integral, a numeric integration using a weighted sum should be used. The limiting case is that of a single sensor at each wave cross-section which can still be used if $g_n^\pm$ are either a priori experimentally measured or modeled analytically.

Using the orthogonality relation of the modal traveling wave functions, $\eta_n^\pm$ [40], and by projecting the pressure model of Eq. (14) onto a single-mode, one can define the modal traveling wave ratio, $TWR_n$, and modal traveling direction, $dir_n = \pm 1$, in a similar manner to the single propagating mode case as

$$TWR_n = 1 - \frac{|P_n^- - P_n^+|}{P_n^- + P_n^+}, \quad dir_n = \text{sign}(P_n^- - P_n^+).$$  \hfill (17)
Fig. 3 – (color online) Illustration of the decomposition procedure for the modal traveling wave ratio. The left pressure map shows the measured pressure which results from the combination of the two propagating modes shown in the right pressure maps, each map is characterized by the modal TWR and dir presented in Eq. (17). The local pressure level is color-coded, neutral (green) stands for zero, warm (red) for positive, cold (blue) for negative pressure fluctuations.

Fig. 3 illustrates the advantage of using wave decomposition and producing the modal TWR. The left pressure map, which is the simulated experiment raw measured one, does not provide a complete picture of the propagation phenomena. Therefore, by decomposing the two propagating modes (pressure maps on the right), it can be seen that the first mode (top) is a pure standing wave ($\text{TWR}_1 = 1$), while the second is a pure backward propagating traveling wave ($\text{TWR}_2 = 0$, $\text{dir}_2 = -1$). Thus, the modal TWR enables one to achieve a better understanding of the propagating phenomena.

4. Decomposition and identification of the modal traveling wave ratio

From the illustrated example shown in Fig. 3, it proves advantageous to be able to decompose propagating waves into the physical basis (modes) and quantify each one of them using the modal TWR. Several methods for the experimental modal traveling wave decomposition are outlined in this section.
The methods can be applied to the finite-length waveguide dealt with here in the multimode propagation scenario under the following assumptions: (i) S sensors are located along the propagation direction, z, their locations are denoted as z_s and they share the same position in each axial cross-section (r, \theta). (ii) All sensors are sampled simultaneously, where F_s and \Delta t = F_s^{-1} denote the sampling frequency and time accordingly. (iii) The dispersion relation of all modes are known, either from analytical models [5,11] or obtained experimentally [13]. (iv) Propagation attenuation is negligible compared to that at the boundaries [13].

The traveling wave basis sampled at location z_s and at time t_r = r\Delta t, r=0,1,… is thus

$$\eta_n(z_s, t_r) = \exp\left(i(\omega t_r + k_n(\omega) z_s)\right).$$  

(18)

By using a Cartesian representation for each wave amplitudes

$$P_n^+ = a_n + i b_n, \quad P_n^- = c_n + i d_n,$$  

(19)

each propagating mode can be written as a vector product

$$p_n(z_s, t_r) = \mathbf{h}_n^{r,s} w_n.$$

(20)

In which \mathbf{h} denotes the wave basis function vector evaluated at the s sensor and r time-step

$$\mathbf{h}_n^{r,s} = \begin{bmatrix} \cos(\omega t_r - k_n z_s) & \sin(\omega t_r - k_n z_s) & \cos(\omega t_r + k_n z_s) & \sin(\omega t_r + k_n z_s) \end{bmatrix},$$

(21)

and \mathbf{w}_n denotes the modal wave constant vector

$$\mathbf{w}_n = \begin{bmatrix} a_n & -b_n & c_n & -d_n \end{bmatrix}^T.$$  

(22)

Once \mathbf{w}_n is obtained, the corresponding modal TWR can be computed as

$$\text{TWR}_n = 1 - \frac{\sqrt{\mathbf{w}_n^2(1) + \mathbf{w}_n^2(2)} - \sqrt{\mathbf{w}_n^2(3) + \mathbf{w}_n^2(4)}}{\sqrt{\mathbf{w}_n^2(1) + \mathbf{w}_n^2(2)} + \sqrt{\mathbf{w}_n^2(3) + \mathbf{w}_n^2(4)}},$$

(23)

and the modal traveling wave direction, can be decided from

$$\text{dir}_n = \text{sign}\left(\sqrt{\mathbf{w}_n^2(1) + \mathbf{w}_n^2(2)} - \sqrt{\mathbf{w}_n^2(3) + \mathbf{w}_n^2(4)}\right).$$

(24)

Having identified each mode’s wave parameter vector is equivalent to the identification of its modal TWR.
Under assumption (iv), the model presented above is accurate when dissipation in the propagation can be negligible, i.e. when the fluid is modeled as a lossless media. This assumption may be used if the liquid viscosity is small compared to the boundary attenuation effects [13]. One may include attenuation effects by the use of a complex wavenumber [15,16] \( k_n = \gamma_n - i\alpha_n \), which will alter Eq. (21) to the following form

\[
\hat{h}_{r,s}^n = \left[ \begin{array}{cc}
\cos(\omega t_r - \gamma_n z_s) & \sin(\omega t_r - \gamma_n z_s) \\
\cos(\omega t_r + \gamma_n z_s) & \sin(\omega t_r + \gamma_n z_s)
\end{array} \right] e^{i\alpha_n z_s}. \tag{25}
\]

Thus assumption (iv) is not necessary, but from the analysis presented in [13] and as will be shown here, when the fluid under consideration is air, propagation related attenuation can be neglected without introducing noticeable inaccuracies and the assumption (iv) holds.

### 4.1 Multichannel least-squares method

The requirements on the experimental setup in terms of sensor placement and the model for the model-based methods are outlined in this section. Under the assumptions presented at the beginning of this section, by denoting the pressure measurement of a sensor located at the position \( z_s \) at a sampled time \( r \) as \( p_{r,s} \), given the time history of a batch at \( r=1,\ldots, R \) sampled times, and for the \( s=1,\ldots, S \) sensors, the modal coefficient vector can be identified by solving the following linear least-squares problem

\[
\min_{\mathbf{W}} \| \mathbf{Y} - \mathbf{AW} \|^2. \tag{26}
\]

Here \( \mathbf{Y} \) stands for the measured data vector measured from all sensors, \( \mathbf{A} \) is the model matrix of dimensions \( RS \times 4N \), composed of an assembly of \( \mathbf{h}_{r,s}^n \) for each sensor and mode, which is built on the basis of the physical modeling, and \( \mathbf{W} \) denotes the linear coefficient vector that defines the various waves amplitudes and modal TWR, their full description can be found in Appendix A.

A solution, \( \hat{\mathbf{W}} \) to the minimization problem Eq.(26) can be found by ([46])

\[
\hat{\mathbf{W}} = \mathbf{A}^\dagger \mathbf{Y}, \tag{27}
\]
where $A^+$ is the left pseudoinverse of $A$, which is computed by known decomposition methods \[46\]. The solution is unique as long as $A$ is uniquely left invertible, i.e. it has full column rank ($\text{rank}(A) = 4N$) \[46\]. The existence of a unique solution can be guaranteed given that a sufficient number of sensors exist

$$ S \geq 2N $$ \hfill (28)$$

as long as all sensors’ spatial spacing is not at a natural multiplication of any modal half-wavelength.

The latter can be proved by applying the angle sum trigonometric identities to Eq. (20) that can be written as

$$ p_n(z_s,t_r) = [\cos \omega t_r \sin \omega t_r] M_{ns} \Theta_n, $$ \hfill (29)$$

where $M_{ns}$ denotes the modal spatial matrix incorporating the sensors’ placement,

$$ M_{ns} = \begin{bmatrix} \cos k_n z_s & 0 & 0 & \sin k_n z_s \\ 0 & \cos k_n z_s & \sin k_n z_s & 0 \end{bmatrix}, $$ \hfill (30)$$

and the transformed amplitude vector $\Theta_n$ is defined by the following transformation

$$ \Theta_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} w_n. $$ \hfill (31)$$

Here, the elements of the transformed modal amplitude vector are denoted as

$$ \Theta_n = [A_n \quad B_n \quad B_{sn} \quad A_{sn}]^T. $$ \hfill (32)$$

The measured pressure at sensor $s$ at all given times can be derived by using the finite modal summation

$$ p(z_s,t) = [\cos \omega t \quad \sin \omega t] M_s \Theta, $$ \hfill (33)$$

where the matrix $M_s$ is composed of the modal spatial matrices and $\Theta$ is the $4N \times 1$ transformed modal amplitude vector of all modes. Multiplying Eq. (33) by the generalized left inverse temporal matrix gives

$$ M_s \Theta = \begin{bmatrix} P_n \cos (z_s, \omega) \\ P_n \sin (z_s, \omega) \end{bmatrix} \equiv P, $$ \hfill (34)$$
where $P_{\text{cos}}$ and $P_{\text{sin}}$ are the real value Fourier coefficients of the pressure signal measured at each sensor. Concatenating all sensors’ model and the real value Fourier coefficients in a similar manner leads to the augmented model

$$[\mathbf{M}]_{2S \times 4N}[\mathbf{\Theta}]_{4N \times 1} = [\mathbf{p}]_{2S \times 1}. \quad (35)$$

Thus, one can identify the Fourier coefficients and subsequently, the modal wave constants can be found. This decomposition is carried out employing the synchronous demodulation method which is presented below. In order for $\mathbf{M}$ to be uniquely left invertible, it has to have full column rank, which required that the row dimension will be larger or equal to that of the columns and that there are at least $4N$ linearly independent rows. These conditions are satisfied by Eq. (28) fulfilling the as long as the demand for sensor placement is fulfilled. From the linear transformations presented in Eq. (29) and Eq. (31), it follows that the model matrix $\mathbf{A}$ of Eq. (26) is uniquely left invertible under these conditions.

### 4.2 Multichannel recursive-least-squares method

In the cases were either batch measurement is not available, or that the inversion of the full model matrix $\mathbf{A}$ is computational unfeasible, the minimization problem of Eq.(26) can be solved by employing the MC-RLS method [41]. The MC-RLS method requires an initialization of the weight vector, $\mathbf{W}_0$, and the information matrix, $\mathbf{P}_0$, which is commonly chosen to be the identity matrix multiplied by a constant scalar, and a prechosen scalar forgetting factor $\lambda$ [41]. It is assumed that the new samples, $p_{r,s}$, are measured simultaneously from all sensors at times $t_r$. The modal wave amplitudes and TWR can be identified by using the traveling wave basis matrix

$$\mathbf{H}_r = \begin{bmatrix}
h_{r,1}^1 & \cdots & h_{r,1}^N \\
\vdots & \ddots & \vdots \\
h_{r,S}^1 & \cdots & h_{r,S}^N
\end{bmatrix}_{S \times 4N}, \quad (36)$$

as the multichannel filtered-reference matrix ($V(n)$ in [40, Eq. (20)]), and following the schematic given in [40, Eq. (23)-(25)] for the $r^{th}$ step, which can be found in Appendix B.
### 4.3 Multichannel least-mean-squares method

Although the RLS method is known to converge faster than the LMS method, in the case of multichannel the need to invert an $S \times S$ matrix arises at every time step of the RLS algorithm [41]. Thus, it is preferable to use the LMS method especially in the case of a high sampling rate on fixed-point hardware (FPGA for example). The MC-LMS method [41] requires initialization of the weight vector, $W_0$, and a prechosen, small, scalar constant step size $\mu$. The MC-LMS algorithm for the $r^{th}$ step schematic is given in [40, Eq. (11)–(13)], where again the traveling wave basis matrix $H_r$ is used instead of the filtered-reference matrix, and are outlined in Appendix C.

The MC-LMS method requires a significantly lower computational effort than the MC-RLS method and can, therefore, be implemented in real-time on a digital processor. The modal traveling wave amplitudes and modal TWR for each of the participating modes can be thus monitored in real-time.

### 4.4 Synchronous demodulation method

All three estimation methods described above result in an identification error if only a partial propagation model is used, i.e. some of the propagating modes are omitted (as shown in Section 5). To overcome this deficiency, the use of a modified synchronous detection method is proposed. The method uses a nonlinear filter of the following form; consider the signal presented in Eq. (14) under the cross-section sensor location assumption (i), it can be written by using an axial-dependent amplitude of temporal Cosine and Sine functions [22], as shown in Eq. (34)

$$ p(z_r,t_r) \equiv p_{s,r} = P_{\cos}(z_r)\cos(\omega t_r) + P_{\sin}(z_r)\sin(\omega t_r), $$

where $P_{\cos}$ and $P_{\sin}$ are in-phase and in-quadrature respectively. The synchronous demodulation [42] can be used to extract the in-phase and in-quadrature components of a signal with a known angular frequency by multiplying the signal with Cosine and Sine and filtering high harmonics

$$ \begin{bmatrix} P_{\cos}(z) \\ P_{\sin}(z) \end{bmatrix} = 2 \cdot \text{LPF} \left\{ \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix} p(z,t) \right\}, $$

$$ (38) $$
in which LPF stands for a real-time low-pass filter with no static gain. Equation (38) can be implemented directly every time a new measurement is acquired. The amplitudes $P_{\cos}$ and $P_{\sin}$ are both a linear combination of the modal complex amplitude and can be expanded in such a summation, as shown in Eq. (33). To extract a single element from the transformed modal amplitude vector, introduced in Eq. (32), spatial synchronous demodulation can be applied subsequently to the temporal synchronous demodulation

$$\begin{pmatrix} A_{cn} \\ B_{cn} \\ A_{sn} \\ B_{sn} \end{pmatrix} = 2 \text{LPF} \begin{pmatrix} P_{\cos} & 0 \\ P_{\sin} & 0 \\ 0 & P_{\cos} \\ 0 & P_{\sin} \end{pmatrix} \begin{pmatrix} \cos(k_n z) \\ \sin(k_n z) \end{pmatrix}. \quad (39)$$

The discrete nature of the LPF is enforced by the use of a finite number of samples and discrete locations of sensors. The modal traveling wave amplitudes and the modal TWR of the participating modes can be computed from $A_{cn}, B_{cn}, A_{sn},$ and $B_{sn}$ by using the transformation defined in Eq. (31).

Fig. 4 – Block diagram of the synchronous demodulation method, bold lines denote vectors and multiplication of vectors are defined element-wise. $z$ denotes the sensor position vector and the two long blocks represent the temporal and spatial low-pass filters. The outcome is the modal parameter vectors.
A schematic block diagram of the synchronous demodulation method is presented in Fig. 4. The measured pressure signals are passed through the temporal synchronous demodulation to identify the Fourier's coefficients. Then, the Fourier's coefficient vector is passed through the spatial synchronous demodulation to identify the modal wave amplitudes. Note the difference between the two LPF operations, while the temporal block represents 2S channels that are being updated simultaneously at sampling times, the spatial block represents 4 vectorized channels which are memoryless, meaning that the filter maximal order is equal to the number of sensors. As shown in Fig. 4 and the above discussion, the synchronous demodulation can be used to decompose and identify only the \( n^{th} \) mode modal amplitudes and modal TWR without considering additional modes. The downfall of the method lies in the need to use a spatial LPF which is limited by the number and spatial distribution of the sensors.

### 5. Sensitivity based analysis to choose the model-order

The number of modes to be included in the wave model of Eq. (14) is not known a priori. Therefore, a sensitivity analysis, [48], should be carried out to avoid overfitting [46]. A simulation was created, and on its basis, the proposed sensitivity analysis and its merits are discussed. The simulation was based on a simulated measurement signal (Eq. (14)), where the sum includes 3 propagating modes, and the simulated pressure wave is measured at 11 locations along the \( z \)-axis. A normally distributed measurement noise model was added to all sensors with a prechosen signal to noise ratio (SNR). The numerical values used in the sensitivity analysis simulation were chosen based on the experimental system properties and measurements and are provided in Table , in which the modes are ordered from the most to least dominant. Note the modal amplitude ratios between the dominant (first) mode and each of the other mode absolute amplitudes, which may intuitively lead to the (wrong) conclusion that the secondary modes may be neglected from the physical model.

The waves’ complex amplitudes (Eq. (16)) are identified using the MC-LS method where each time one additional mode was added to the model. The sensitivity of the parameters was analyzed using the
condition number of the model matrix $A$, which is an indicator of the sensitivity to small perturbations of the used model [46]; the estimated scaled residual norm

$$\hat{\sigma}^2 = \frac{\hat{r}^T \hat{r}}{RS - 4N}, \quad \hat{r} = Y - AW,$$

(40)

where $\hat{r}$ denotes the estimated residual and $\hat{W}$ is computed from Eq. (27); and the normalized amplitude error

$$e_n = \sqrt{\left(\frac{\hat{P}_n^+ - P_n^+}{P_n^+}\right)^2 + \left(\frac{\hat{P}_n^- - P_n^-}{P_n^-}\right)^2},$$

(41)

in which $\hat{P}_n^\pm$ are the estimated complex amplitudes of the $n^{th}$ mode computed by using $\hat{W}$, Eq. (22), and Eq. (19).

Fig. 5 – Simulation normalized residual histogram. Each subplot represents the residual when including the following number of modes in the estimation model: (a) – 1 mode, (b) – 2 modes, (c) – 3 modes, (d) – 4 modes.

Fig. 5 depicts the normalized residual histogram, $r/\text{max}(||Y||)$, when 1 to 4 modes are included in the fitted model. Note that the single-mode model histogram (a) differs from the other three, but still seems to have an uncorrelated zero mean nature. The differences between the other three histograms (a, b, and c) are not clear and do not contribute towards the effort of determining the suitable model order. The area
under all the histograms, which is proportional to the estimated residual norm, is of the same order of magnitude. Since the observation of noise level are not informative enough for the determination of the model order, the need to observe the change in another, measurable, parameter arises. The primary goal of the estimation is to correctly identify the wave complex amplitudes, in other words, one wishes to minimize the normalized amplitude error of a prechosen mode, which is not known a priori. Moreover, when analytical models are used, small perturbations of the models are to be expected. Therefore, if it were possible, one would wish to follow the change in the normalized amplitude error versus the sensitivity of the model to perturbations, which can be estimated by the condition number [46], as a function of the model order.

Fig. 6 – (color online) Sensitivity analysis simulation results, showing the model matrix’s condition number versus normalized amplitude error (Eq. (41)). Numbers represent the number of modes (Eq. (18)) included in the fitted model (Eq. (26)). Each subplot (a), (b) or (c) is related to the normalized amplitude error of the associated mode: (a) – dominant mode, (b) – second dominant mode, (c) – least dominant mode.

Fig. 6 shows the effect of adding each of the 4 modes to the model on the condition number and on the normalized estimation amplitude error of the particular mode. Clearly, as can be deduced from Fig.6a, fitting a single mode results in a relative error of 15% in the amplitude of this very mode. On the other hand, adding 2,3 and 4 modes to the model creates a clear preferred order were 3 modes seem to reduce
the fit error without making the model greatly uncertain, as reflected by the sudden increase in condition number upon increasing from 3 to 4 modes in the model.

The importance of observing both parameters: the model matrix's condition number and the normalized amplitude error parameters together is clarified. Upon adding a second mode, the fit error of the dominant mode is reduced by a factor of 3 to about 5%, while including 4 modes doubles the model matrix’s condition number thus the sensitivity, without a noticeable effect on the normalized error.

Fig. 6(a) shows the importance of including the correct amount of modes even in the case where only the dominant mode, is to be identified, although the dominant mode amplitude is four times larger than that of the second, neglecting additional modes results in a nonnegligible error in the identification.

When considering the more realistic case, in which the exact values are not known, and the normalized amplitude error cannot be calculated. A similar approach is proposed by observing the change in the estimated scaled residual norm versus the model matrix’s condition number, as a function of the model order which can be seen in Fig. 7.

![Fig. 7 – Sensitivity analysis simulation result, model matrix’s condition number versus estimated scaled residual norm (Eq. (40)). Numbers represent the number of modes (Eq. (18)) that are used in the model (Eq. (26)).](image-url)
Fig. 7 shows the L-shape-like behavior [48] of adding additional parameters to the LS estimation, from which the optimal number of modes to include can be determined. From this analysis, one can conclude to include 3 modes in the model, since it resides close to the L-curved "corner" [48,49]. In the current simulation, the 2 modes model also reside near the "corner", this is due to the large amplitude ratio between the dominant and the third modes (17/1) which is smaller than the chosen SNR. Still, from the analysis done based on this discussion and Fig. 7 L-curved behavior the 3-modes model is to be chosen in this case which exactly corresponds to the correct number of modes used in the simulated signal.

The sensitivity analysis done here applies to the three model-based least-squares methods introduced above, the statistical similarities between the LS, LMS, and RLS were shown previously in [45].

6. Simulated comparison between outlined methods

A numerical comparison between the 4 outlined methods was performed using the numerical parameters that are provided in Table . A model that simulated 3 modes in the acquired signal, already used in the previous section, is analyzed here, this time without added noise. Fig. 8 shows the results of the simulation-based identified modal TWR for the batch MC-LS method and for the three recursive methods. Both model-based recursive methods, MC-RLS and MC-LMS, converged to the batch LS solution for each of the 3 modes, which is accurate in the absence of noise. The synchronous demodulation was only able to correctly identify the modal TWR associated with the first and third modes. This is due to spatial filter design which includes a dynamical passband, up to the cutoff spatial frequency. For this reason, for the given sensor configuration, the spatial LPF was not able to reduce the magnitude of the double harmonics [42] of the second mode as required in Eq. (39) resulting in a biased estimation.
Fig. 8 – (color online) Modal TWR identification simulation results of the outlined methods. LS denotes the MC-LS solution. Each y-axis is bound between 0 to 1. $T$ denotes the simulated signal period time. Legend: dot (black) – MC-LS, solid (blue) – MC-RLS, dashed (red) – MC-LMS, square (orange) – synchronous demodulation.

Note that for the noiseless simulation the MC-RLS converges simultaneously to all modal TWR once sufficient data is collected. On the basis of the discussion presented in Sec. 4.1, one can show that the MC-RLS in the absence of noise convergence time is equal to the number of modes used in the model. The MC-LMS convergence rate on the other hand, is slower and differs between each of the modes. The convergence rate speed of the MC-LMS can be associated to the dominancy of the mode, the convergence rate is faster for the more dominant mode.

The difference in convergence time between the model-based methods arises from the nature of these methods, especially when noise is negligible. This is since the MC-LMS step-size is tuned at the initialization step for the worst-case scenario, while the MC-RLS step-size is being updated at each step. The MC-LMS convergence time is faster for more dominant modes, from the results shown in Fig. 8 the MC-LMS converges to the most dominant mode in under a cycle. Thus, if the goal is to accurately identify the dominant (first) mode TWR, the MC-LMS is preferred over the MC-RLS due to its ease of implementation and low computational cost.
7. Experimental study

The four methods outlined in Sec. 4 were analyzed and compared on an experimental PMMA air-filled wave tube. The comparison was conducted at an excitation frequency of 3000 Hz, for which the dispersion relation of five propagating modes was previously identified [13], the section opens with a description of the experimental system followed by a sensitivity analysis of the number of modes on the MC-LS method (similar to that of Sec. 5). Then the simulation discussed in Sec. 6 is done on the experimentally acquired signals. Finally, the experimental identified modal TWR results of the recursive MC-LMS method implemented on a field-programmable gate array (FPGA) in real-time are compared both qualitatively and quantitatively with these of the batch MC-LS method applied to signals measured using a NI© data-acquisition device.

7.1 Experimental system layout

Fig. 9 presents a photo of the experimental air-filled wave tube. The tube length is 2 m and the outer diameter is 0.09 m, it is made from PMMA. A 3”, 4 Ω, and two 60 W Dayton Audio© PC83-4 loudspeaker is connected to both ends of the wave tube. Eleven 9.7 mm diameter omnidirectional Adafruit© Max9814 microphones are placed along the tube’s inner surface at $\theta = 0$. The loudspeakers are driven by a current amplifier to eliminate the effect of electrical impedance coupling with the acoustic field.
Table presents all the experimental setup properties; the air density, $\rho_1$, and intrinsic speed of sound, $c_1$, were calculated for a temperature of 25 °C and pressure of 101325 Pa; the PMMA density, $\rho_s$, and elastic constants, $E$ and $v$, were estimated from a set of preliminary experiments.

The batch data used in Sec. 7.2-4 was sampled using a NI© PXIe-6358 data acquisition device and two BNC-2110 connector blocks with a total of 16 analog input channels and a 250 kHz sampling frequency. The recursive MC-LMS was implemented on a dSpace© DS 1005 computation node connected to two DS5203 FPGA boards with a total of 16 analog input channels and an internal clock of 100 MHz which also used to sample the identified modal TWR at a 10 kHz sampling frequency. The dispersion curves of the acoustic tube were identified using the two-actuator phase-perturbations (TAPP) method [13] using a similar setup to the one described here and the NI© data acquisition module.

**7.2 Multichannel least-squares sensitivity analysis**

In the dispersion curves identification phase [13], five propagating modes were identified. To avoid overfitting and choose the number of modes to include in the TWR identification phase, a sensitivity-based analysis procedure, similar to the one presented in Sec. 5, was carried out.

The last 100 cycles of the measured signal of each microphone were used in the sensitivity analysis, the analysis was done using the raw signals where the only filter which was used is a DC removal high-pass filter (windowed average) to account for the DC-biased voltage outputs of the microphones.

Fig. 10 depicts the experiment normalized residual histogram, $r/\text{max}(||Y||)$, when 1 to 4 modes are used in the fitted model. In this case, all 4 histograms have the same nature in principal the single-mode model histogram (a) has a higher peak and a wider distribution. As discussed in Sec. 5 it is clear from Fig. 10, the normalized residual by itself cannot provide a clear indication for the number of modes to include in the model.
Fig. 10 – Experiment normalized residual histogram. Each subplot represents the residual when including the following number of modes in the estimation model: (a) – 1 mode, (b) – 2 modes, (c) – 3 modes, (d) – 4 modes.

Fig. 11 – Sensitivity analysis experimental result, model matrix's condition number versus estimated scaled residual norm (Eq. (40)). Numbers represent the number of modes (Eq. (18)) that are used in the model (Eq. (26)).

Fig. 11 shows the sensitivity analysis introduced in Sec. 5 performed on the experimentally measured signals from the 11 microphones. The L-shape-like behavior [48,49] is noticeable, and the behavior seems to agree strongly with that of the simulation sensitivity analysis presented in Fig. 7. The resemblance
between the simulated and experimental results, Fig. 7, and Fig. 11, can be accounted for the accuracy of the physical parameters, these can be found in Table 1. The main difference between the simulated and experimentally measured signals is that in the simulated signal the "exact" parameter values and the noise model are known a priori.

It should be noted that in both the experimental result (Fig. 11) and the simulated one (Fig. 7), the addition of a fourth mode to the model caused a reduction in the estimated scaled residual norm. Still, the fourth mode’s addition has a significant effect on the model matrix’s condition number, it affects the sensitivity to perturbations greatly [46]. Using the L-shape "corner" location it is concluded that only the first 3 modes contribute to the acoustic wave-phenomena at this excitation frequency [48].

![Fig. 12 – (color online) Experimental measured signal (solid) at each sensor, and fitted signal (dot) using a 3 mode model as an input of the MC-LS method.](image)

Fig. 12 shows the experimentally measured signals of each microphone and the fitted signal based on the modal model of Eq. (20). The decomposition of the modes was done using a 3-mode model and the batch MC-LS method. Clearly, the curve-fitting seems adequate.

From this sensitivity analysis done on the experimentally measured signals, it was concluded that only 3 modes will be included in the model-based method when decomposing to propagating modes and identifying each mode modal TWR as presented in the following subsections.
7.3 Offline numerical comparison of the four methods based on experimentally measured data

To compare the outlined methods (Sec. 4), an offline recursive procedure was realized similarly to that presented in Sec. 6. The input data to the recursive procedure was obtained experimentally from the air-filled acoustic wave tube (Fig. 9), using the NI\textsuperscript{©} board with a similar configuration to the one detailed at the beginning of the previous subsection (Sec. 7.2). The numerical recursive procedure was run offline using Matlab\textsuperscript{©} where all model-based methods used the 3-mode model as was concluded from the experimentally obtained data sensitivity analysis.

Fig. 13 – (color online) Modal TWR identification experimental results of the 4 outlined methods. LS denotes the MC-LS solution. Each y-axis is bound between 0 to 1. $T$ denotes the simulated signal period.

Legend: dot (black) – MC-LS, solid (blue) – MC-RLS, dashed (red) – MC-LMS, square (orange) – synchronous demodulation.

Fig. 13 shows the three identified modal TWR, similarly to the simulation base analysis (Sec. 6), both the MC-RLS and the MC-LMS converged to the least-squares solution for all three modes, and the synchronous demodulation did not converge for the second mode modal TWR. The main difference from the experimentally based analysis (Fig. 13) and the simulated one (Fig. 8) is that the presence of noise has a significant effect on the convergence time of the MC-RLS, while its effect on the MC-LMS is less
obvious. Thus, when noise is nonnegligible, its magnitude affects the convergence rate advantage of the MC-RLS, diminishing its advantage, in particular when the computational effort is considered. The latter becomes especially important when implemented on an FPGA. For the MC-LMS the noise affects mainly the convergence of the weaker modes, but as can be seen in both Fig. 13 and Fig. 8 the MC-LMS converge to the dominant mode modal TWR in under a cycle. The synchronous demodulation’s behavior presented, as shown in Fig. 13, is similar to the one of the simulation-based analysis presented in Fig. 8, but the presence of noise and its effect on the LPF dynamic is clearly visible causing weak oscillations around the mean steady-state identified modal TWR.

7.4 Comparison between recursive multichannel least-mean-squares and batch multichannel least-squares methods

The recursive MC-LMS was implemented using the two dSpace© DS5203 FPGA boards. A comparison between the batch MC-LS and the recursive MC-LMS experimentally produced identified modal TWR contour maps [23,29], scanned for excitation frequency of 3000 Hz, was conducted. These maps show the effect of amplitude ratio and phase difference of the two loudspeakers (see Fig.9) on the TWR. Clearly, for specific combinations, the ration of traveling and standing waves varies and the TWR produces this information [23].

Fig. 14 depicts the modal TWR maps results for both methods, left column for the batch MC-LS and right column for the recursive MC-LMS, and for each of the three modes used in the model, top and bottom rows present the results for the most and least dominant modes accordingly. The blue (online) lines of each contour represent the traveling wave regions (TWR=0) while the red lines represent the standing wave region (TWR=1). Note that at the phase shift region of interest there are two traveling wave regions for each mode, one of them corresponds to the forward and the other to the backward traveling waves.
Fig. 14 – (color online) MC-LS and MC-LMS TWR\textsubscript{n} contour maps comparison at 3000 Hz versus loudspeakers amplitude ratio, $A_r$, and phase shift, $\Delta \varphi$, color represents the modal TWR\textsubscript{n} values: blue – 0, red – 1. (a), (c), and (e) – MC-LS results for the first, second, and third dominant modes. (b), (d), and (f) – MC-LMS results for the first, second, and third dominant modes.

From the modal TWR, the propagation pattern is clear, and if it is desired to convey energy in a chosen direction (TWR=0) or suppress it (TWR=1), it can be done by tuning one of the loudspeakers amplitude and phase relative to the other to achieve the desired modal TWR. The MC-LS obtained contour maps shows two anomalies, the first near $A_r = 0.9$ and $\Delta \varphi = 30^\circ$, the second at $A_r = 1.5$ and all phase shifts. These are due to the experimental nature of the identification process, the first may be accounted by some high-level correlated noise at the frequency of the experiments (external noise); the second anomaly is
more subtle and it may be associated with small variation in the resistivity of the sensors as a function of time.

When comparing the two methods one can note that the dominant (first) mode two maps (Fig. 14a and Fig. 14b) are qualitatively almost identical. When comparing the less dominant modes the MC-LMS modal TWR contours exhibit some distortions, but in principal, one identifies the different regions of the modal TWR (standing, traveling). This may be due to the numerical truncation caused by the finite number of bits in the FPGA realization, and due to the finite settling time given for the method at each amplitude ratio and phase shift. The modal TWR contours were also quantitively compared using the relative error norm

\[ \varepsilon_n \equiv \frac{1}{\#(A_r) \#(\Delta \phi)} \frac{\| TWR_{n,MC-LS} (A_r, \Delta \phi) - TWR_{n,MC-LMS} (A_r, \Delta \phi) \|}{\| TWR_{n,MC-LS} (A_r, \Delta \phi) \|}, \]  

(42)

where \( A_r \) and \( \Delta \phi \) denote the amplitude ratio and phase shift of the loudspeakers, \# denotes the number of elements scanned for each of them, and \( \| \|_f \) denotes the matrix Frobenius norm. The following results are obtained when applying Eq. (42) to the identified contours of Fig. 14:

\[ \varepsilon_1 = 0.08\%, \quad \varepsilon_2 = 0.64\%, \quad \varepsilon_3 = 0.83\% . \]  

(43)

The relative error norms of each mode show quantitively the same behavior of its qualitative counterpart discussed above, moreover, the ratios between the relative error norm of the dominant mode and each of the less dominant modes are smaller than their associated amplitude ratios. Although \( \varepsilon_2 \) and \( \varepsilon_3 \) are only one order of magnitude larger than \( \varepsilon_1 \) their contour map seems to have large distortion, but by using a smoothening filter (interpolation) the nature of the modal TWR of these maps can easily be retrieved as shown in Fig. 14. Thus, the MC-LMS may be used for the real-time identification of the modal TWR and modal amplitudes of all modes.

More sophisticated LMS algorithm can be used, the interested reader is referred to [47] chapter 3.4 for an extended overview of the many variations, but their implementation is beyond the scope of this current work which has shown the feasibility of using the MC-LMS for the real-time decomposition of the pressure
fields into its backward and forward traveling wave basis and discussed is advantages in the suggested form.

Fig. 15 – (color online) MC-LMS interpolated modal TWR maps, based on the modal TWR contour of Fig. 14, color represents the modal TWR, values: blue – 0, red – 1. (a) – second mode, (b) third mode.

8. Conclusions

It has been demonstrated that the modal TWR is an effective measure to assess the quality of modal decomposition in the case of an acoustical wave tube. It allows one to better understand the wave dynamics and it enables one to precisely control the desired propagation pattern by tuning the modal TWR using a secondary transducer. The prospect of using the MC-LMS for a real-time decomposition and identification of the modal amplitudes as well as the modal TWR was demonstrated experimentally with good agreement to simulation and models.

The importance of including all dominant modes as well as neglecting others was shown to improve the accuracy and reduce the sensitivity to model uncertainties of the identified modes. The number of modes one should include in the propagation model can be determined by using the proposed sensitivity analysis resembling the L-curve method. When employing methods that do not depend on the complete knowledge of the dispersion relation, i.e. the number of propagating modes and their wavenumbers, for example, the synchronous demodulation elaborate signal processing analysis and may be subject to sampling criteria.
Thus, the model-based methods are preferable in many cases, especially if one can rely on analytical models or combined analytical and experimental ones, for the derivation of the dispersion relation.

The MC-LMS has a slower convergence rate compared to the MC-RLS. Still, it is favorable in most cases since it is easy to implement and required a low computational effort. MC-LMS’ merit becomes more notable when noise is introduced to the system.

Hybrid methods like the synchronous demodulation – MC-LMS are a viable option, the first is effective to decompose the temporal harmonics by using time Fourier analysis the second to decompose the spatial harmonics by using a model-based approach.

Standardized methods to conduct NDT of acoustical properties which use two sensors and includes a single mode in their physical models are bound to have a biased estimation. It was shown that the model-based method requires at least two sensors for each additional mode in the model. Thus, based on the physical modeling and modal decomposition methods presented here, the knowledge of dispersion relation of the wave tube [13], and the use of a sufficient number of sensors, their accuracy can be enhanced by accounting for the additional modes.

**Appendix A – Equation (26) nomenclature full form**

\[
A = \begin{bmatrix}
    h_{1,1}^1 & \cdots & \cdots & \cdots & h_{1,1}^N \\
    \vdots & & & & \vdots \\
    h_{R,1}^1 & \cdots & \cdots & \cdots & h_{R,1}^N \\
    \vdots & & & & \vdots \\
    h_{1,S}^1 & \cdots & \cdots & \cdots & h_{1,S}^N \\
    \vdots & & & & \vdots \\
    h_{R,S}^1 & \cdots & \cdots & \cdots & h_{R,S}^N
\end{bmatrix}_{RS \times 4N}, \quad Y = \begin{bmatrix}
    p_{1,1} \\
    \vdots \\
    p_{R,1} \\
    \vdots \\
    p_{1,S} \\
    \vdots \\
    p_{R,S}
\end{bmatrix}_{RS \times 1}, \quad W = \begin{bmatrix}
    w_1 \\
    \vdots \\
    w_n \\
    \vdots \\
    w_N
\end{bmatrix}_{4N \times 1}
\]  

(A.1)
Appendix B – The multichannel recursive-least-squares algorithm [40 Eq. (23)–(25)]

1. Estimate error:
   \[ \alpha_r = p_r - H_r W_{r-1}, \]  
   \( (B.1) \)

2. Compute the RLS gain vector:
   \[ K_r = P_{r-1} H_r^T \left( \lambda I_S + H_r P_{r-1} H_r^T \right)^{-1}, \]  
   \( (B.2) \)

3. Update weight vector and information matrix:
   \[ W_r = W_{r-1} + K_r \alpha_r, \]  
   \( (B.3) \)
   \[ P_r = \frac{1}{\lambda} \left( I_{4N} - \lambda^{-1} K_r H_r \right) P_{r-1}, \]  
   \( (B.4) \)

where \( I_S \) and \( I_{4N} \) denote the \( S \times S \) and \( 4N \times 4N \) identity matrix.

Appendix C – The multichannel least-mean-squares algorithm [40 Eq. (11)–(13)]

1. Estimate the equation error:
   \[ \alpha_r = p_r - H_r W_{r-1}, \]  
   \( (C.1) \)

2. Update weight vector:
   \[ W_r = W_{r-1} + \beta \frac{H_r^T \alpha_r}{\|H_r\|^2}. \]  
   \( (C.2) \)
References

[1] A.D. Pierce, R.T Beyer, Acoustics: an introduction to its physical principles and applications, 1989.

[2] D.J. Mead, A general theory of harmonic wave propagation in linear periodic systems with multiple coupling, Journal of Sound and Vibration. 27 (1973) 235–260. https://doi.org/10.1016/0022-460X(73)90064-3.

[3] O.D. Jefimenko, Electricity and magnetism an introduction to the theory of electric and magnetic fields, Electret Scientific Company, 1989.

[4] C.W. Helstrom, VII Quantum detection theory, Progress in Optics. 10 (1972) 289–369. https://doi.org/10.1016/S0079-6638(08)70062-3.

[5] L.D. Lafleur, F.D. Shields, Low-frequency propagation modes in a liquid-filled elastic tube waveguide, Journal of the Acoustical Society of America. 97 (1995) 1435–1445. https://doi.org/10.1121/1.412981.

[6] J. Achenbach, Wave propagation in elastic solids, North-holland Publishing Company/ American elsevier, 1973. http://refhub.elsevier.com/S0888-3270(18)30800-8/h0005

[7] R.W. Morse, The velocity of compressional waves in rods of rectangular cross section, Journal of the Acoustical Society of America. (1950). https://doi.org/10.1121/1.1906592.

[8] ASTM International. E1050-12 Standard test method for impedance and absorption of acoustical materials using a tube, two microphones and a digital frequency analysis system. West Conshohocken, PA; ASTM International, 2012. https://doi.org/10.1520/E1050-12

[9] ASTM International. E2611-19 Standard test method for normal incidence determination of porous material acoustical properties based on the transfer matrix method. West Conshohocken, PA; ASTM International, 2019. https://doi.org/10.1520/E2611-19

[10] J. P. Dalmont, Acoustic impedance measurement, part I: A review, Journal of Sound and Vibration, 243.3 (2001) 427-439. https://doi.org/10.1006/jsvi.2000.3428
[11] DEL GROSSO VA, Analysis of multimode acoustic propagation in liquid cylinders with realistic boundary conditions. Application to sound speed and absorption measurements, Acustica. 24 (1971) 299–311.

[12] H. Sato, M. Lebedev, J. Akedo, Theoretical and experimental investigation of propagation of guide waves in cylindrical pipe filled with fluid, Japanese Journal of Applied Physics, Part 1: Regular Papers and Short Notes and Review Papers. 45 (2006) 4573–4576. https://doi.org/10.1143/JJAP.45.4573.

[13] Y. Vered, R. Gabai, I. Bucher, Waveguide dispersion curves identification at low-frequency using two actuators and phase perturbations, The Journal of the Acoustical Society of America. 146 (2019) 2443–2451. https://doi.org/10.1121/1.5128482.

[14] W. Sachse, Y.H. Pao, On the determination of phase and group velocities of dispersive waves in solids, Journal of Applied Physics. 49 (1978) 4320–4327. https://doi.org/10.1063/1.325484.

[15] M.G. Jones, T.L. Parrott, Evaluation of a multi-point method for determining acoustic impedance, Mechanical Systems and Signal Processing. 3 (1989) 15–35. https://doi.org/10.1016/0888-3270(89)90020-4.

[16] K. Baik, J. Jiang, T.G. Leighton, Acoustic attenuation, phase and group velocities in liquid-filled pipes: Theory, experiment, and examples of water and mercury, The Journal of the Acoustical Society of America. 128 (2010) 2610–2624. https://doi.org/10.1121/1.3495943.

[17] M. Oblak, M. Pircat, M. Boltežar, An impedance tube submerged in a liquid for the low-frequency transmission-loss measurement of a porous material, Applied Acoustics. 139 (2018) 203–212. https://doi.org/10.1016/j.apacoust.2018.04.014.

[18] R.S. Langley, On the modal density and energy flow characteristics of periodic structures, Journal of Sound and Vibration. 172 (1994) 491–511. https://doi.org/10.1006/jsvi.1994.1191.

[19] L. Feng, Acoustic properties of fluid-filled elastic pipes, Journal of Sound and Vibration. 176 (1994) 399–413. https://doi.org/10.1006/jsvi.1994.1384.
[20] A. Søe-Knudsen, S.V. Sorokin, Modelling of linear wave propagation in spatial fluid filled pipe systems consisting of elastic curved and straight elements, Journal of Sound and Vibration. 329 (2010) 5116–5146.  
https://doi.org/10.1016/j.jsv.2010.06.015.

[21] Y. Vered, I. Bucher, Tailoring phononic-like topologies for controlling the structural-acoustic coupling in fluid-filled cylinders, in: Proceedings of ISMA 2018 - International Conference on Noise and Vibration Engineering and USD 2018 - International Conference on Uncertainty in Structural Dynamics, 2018: pp. 3033–3045.

[22] I. Bucher, Estimating the ratio between travelling and standing vibration waves under non-stationary conditions, Journal of Sound and Vibration. 270 (2004) 341–359.  
https://doi.org/10.1016/S0022-460X(03)00539-X.

[23] R. Gabai, I. Bucher, Excitation and sensing of multiple vibrating traveling waves in one-dimensional structures, Journal of Sound and Vibration. 319 (2009) 406–425.  
https://doi.org/10.1016/j.jsv.2008.06.013.

[24] R. Gabai, D. Ilssar, R. Shaham, N. Cohen, I. Bucher, A rotational traveling wave based levitation device – Modelling, design, and control, Sensors and Actuators, A: Physical. 255 (2017) 34–45.  
https://doi.org/10.1016/j.sna.2016.12.016.

[25] R.A. Scott, An apparatus for accurate measurement of the acoustic impedance of sound-absorbing materials, Proceedings of the Physical Society. 58 (1946).  
https://doi.org/10.1088/0959-5309/58/3/303.

[26] I. Bucher, R. Gabai, H. Plat, A. Dolev, E. Setter, Experimental travelling waves identification in mechanical structures, Mathematics and Mechanics of Solids. 24 (2019) 152–167.  
https://doi.org/10.1177/1081286517732825.

[27] AH Von Flotow, Traveling wave control for large spacecraft structures, Journal of Guidance, Control, and Dynamics. 9 (1986) 462-468.  
https://doi.org/10.2514/3.20133.

[28] B.R. Mace, Active control of flexural vibrations, Journal of Sound and Vibration. 114 (1987) 253–270.  
https://doi.org/10.1016/S0022-460X(87)80152-9.

[29] A. Minikes, R. Gabay, I. Bucher, M. Feldman, On the sensing and tuning of progressive structural vibration waves, IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control. 52 (2005) 1565–1575.  
https://doi.org/10.1109/TUFFC.2005.1516029.
[30] F. Giraud, C. Giraud-Audine, M. Amberg, B. Lemaire-Semail, Vector control method applied to a traveling wave in a finite beam, IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control. 61 (2014) 147–158. https://doi.org/10.1109/TUFFC.2014.6689782.

[31] W.J. O’Connor, M. Zhu, Boundary-controlled travelling and standing waves in cascaded lumped systems, Mechanical Systems and Signal Processing. 39 (2013) 119–128. https://doi.org/10.1016/j.ymssp.2012.02.005.

[32] H. Habibi, W. O’Connor, Wave-based control of planar motion of beam-like mass–spring arrays, Wave Motion. 72 (2017) 317–330. https://doi.org/10.1016/j.wavemoti.2017.04.002.

[33] I. Peled, W.J. O’Connor, Y. Halevi, On the relationship between wave based control, absolute vibration suppression and input shaping, Mechanical Systems and Signal Processing. 39 (2013) 80–90. https://doi.org/10.1016/j.ymssp.2012.06.006.

[34] Y. Halevi, Control of flexible structures governed by the wave equation using infinite dimensional transfer functions, Journal of Dynamic Systems, Measurement, and Control. 127 (2005) 579–588. https://doi.org/10.1115/1.2098895.

[35] R. Gabai, I. Bucher, Spatial and temporal excitation to generate traveling waves in structures, Journal of Applied Mechanics, Transactions ASME. 77 (2010) 1–11. https://doi.org/10.1115/1.3176999.

[36] L. Sirota, Y. Halevi, Fractional order control of the two-dimensional wave equation, Automatica. 59 (2015) 152–163. https://doi.org/10.1016/j.automatica.2015.06.016.

[37] P.A. Musgrave, Patrick F and Malladi, VVN Sriram and Tarazaga, Generation of traveling waves in a 2D plate for future drag reduction manipulation, in: Special Topics in Structural Dynamics, Volume 6, Springer, 2016: pp. 129-138.

[38] M. Åbom, Modal decomposition in ducts based on transfer function measurements between microphone pairs, Journal of Sound and Vibration. 135 (1989) 95–114. https://doi.org/10.1016/0022-460X(89)90757-8.

[39] Y. Vered, R. Gabai, I. Bucher, Dispersion based reduced-order model identification and boundary impedance control in a weakly coupled impedance tube, Proceedings of Meetings on Acoustics. 39 (2019) 045005. https://doi.org/10.1121/2.0001215.
[40] P.M. Morse, H. Feshbach, E.L. Hill, Methods of theoretical physics, New York, McGraw-Hill, 1953.

[41] M. Bouchard, S. Quednau, Multichannel recursive-least-square algorithms and fast-transversal-filter algorithms for active noise control and sound reproduction systems, IEEE Transactions on Speech and Audio Processing. 8 (2000) 606–618. https://doi.org/10.1109/89.861382.

[42] M. Feldman, Hilbert transform in vibration analysis, Mechanical Systems and Signal Processing. 25 (2011) 735–802. https://doi.org/10.1016/j.ymssp.2010.07.018.

[43] L. Brillouin, Wave Propagation in Periodic Structures: Electric Filters and Crystal Lattices, Dover Publications, 1953.

[44] D. Duhamel, B.R. Mace, M.J. Brennan, Finite element analysis of the vibrations of waveguides and periodic structures, Journal of Sound and Vibration. 294 (2006) 205–220. https://doi.org/10.1016/j.jsv.2005.11.014.

[45] M.H. Hayes, Statistical digital signal processing and modeling, John Wiley & Sons, 2009.

[46] and G.S. Hansen, Per Christian, Victor Pereyra, Least squares data fitting with applications, Johns Hopkins University Press, Baltimore, 2012. https://doi.org/10.1353/book.21076.

[47] S.J. ELLIOTT, Signal Processing for Active Control, Elsevier, 2001. https://doi.org/10.1016/b978-0-12-237035-4.x5000-5.

[48] P.C. Hansen, Analysis of discrete ill-posed problems by means of the L-curve, SIAM Review. 34 (1992) 561–580. https://doi.org/10.1137/1034115.

[49] I. Bucher, S.G. Braun, Left eigenvectors: extraction from measurements and physical interpretation, Journal of Applied Mechanics. 64 (1997) 97–105. https://doi.org/10.1115/1.2787300.

[50] J.J. Jensen, On the shear coefficient in Timoshenko’s beam theory, Journal of Sound and Vibration. 87 (1983) 621–635. https://doi.org/10.1016/0022-460X(83)90511-4.
Table 1 – Numerical values used for the analysis of Sec. 5-7

| Property                  | Value, units                               |
|---------------------------|--------------------------------------------|
| Sampling frequency        | 250, kHz                                   |
| Excitation frequency      | 3000, Hz                                   |
| Number of cycles          | 100                                        |
| Random generator          | Rng('default')                             |
| Signal to Noise Ratio     | 5 (Sec. 5) / ∞ (Sec. 6)                    |
| Modal wavenumber          | (54.8189,12.1459,21.4010,16.3615), rad/m   |
| Forward modal amplitude   | (0.0392-0.0207i,0.0041-0.0038i ,0.0023-0.0021i,0) |
| Backward modal amplitude  | (0.0283+0.0444i,-0.0010+0.0148i,0.0011-0.0023i,0) |
| System Length             | 2.0122, m                                  |
| Sensor z position         | (0.3995,0.5016,0.6019,0.7019,0.8521,1.0031,… |
|                           | 1.1512,1.3003,1.4008,1.5015,1.6020), m      |
Table 2 – List of properties used in the experimental and the analytical modeling of the acoustic tube.

| Property                              | Symbol | Value, units          |
|---------------------------------------|--------|-----------------------|
| Air density                           | $\rho_1$ | 1.1839, kg m$^{-3}$ |
| Air intrinsic speed of sound          | $c_1$  | 346.13, m s$^{-1}$   |
| PMMA density                          | $\rho_s$ | 1220, kg m$^{-3}$    |
| PMMA Young’s modulus                  | $E$    | 3100, MPa             |
| PMMA Poisson’s ratio                  | $\nu$  | 0.38                  |
| Tube length                           | $L$    | 2.011, m              |
| Tube inner diameter                   | $a$    | 0.042, m              |
| Tube thickness                        | $h$    | 3, mm                 |
| Number of sensors                     | $S$    | 11                    |
| Sensors position                      | $z_s$  | See Table 1           |