Modelling and Dynamic Analysis of an Unbalanced and Cracked Cardan Shaft for Vehicle Propeller Shaft Systems

Bernard Xavier Tchomeni * and Alfayo Alugongo *

Industrial Engineering, Operations Management & Mechanical Engineering, Vaal University of Technology, Vanderbijlpark Campus Private Bag X021, Andries Potgieter Blvd, Vanderbijlpark 1911, South Africa

* Correspondence: bernardt@vut.ac.za (B.X.T.); alfayoa@vut.ac.za (A.A.)

Abstract: The vibrational behaviour of misaligned rotating machinery is described and analysed in this paper. The model, constructed based on the equations of vehicle dynamics, considered the dynamic excitation of a single Hooke’s joint. The system adopted the breathing functions from a recent publication to approximate the actual breathing mechanism of a cracked driveshaft. The study aimed to understand the transmission of a nonlinear signal from the unbalanced and cracked driveshaft to an unbalanced driven shaft via a Hooke’s joint. The governing equation of the system was established based on the energy principle and the Lagrangian approach. The instantaneous frequency (IF) identification of the cracked driveshaft was extracted based on the synchrosqueezing wavelet technique. To correlate the results, the nonlinear synchrosqueezing wavelet transforms combined with the classical waves techniques were experimentally used in various scenarios for dynamic analysis of the Cardan shaft system. The variations in the dynamic response in the form of a rising trend of higher harmonics of rotational frequency and increased level of sub-harmonic peaks in both shafts were presented as significant crack indicators. The synchrosqueezing response showed breathing crack excitation played a crucial role in the mixed faults response and caused divergence of the vibration amplitudes in the rotor’s deflections. The simulation and test results demonstrated that the driveshaft damage features impacted the transfer motion to the driven shaft and the Hooke’s joint coupling was the principal source of instability in the system. The proposed model offers new perspectives on vibration monitoring and enhancement analysis to cover complex Cardan shaft systems.

Keywords: Cardan shaft; breathing crack; Hooke’s joint; misalignment; perturbation function; transient stiffness

1. Introduction

In practice, Cardan joints are required to transmit rotational forces (torque) between two offset shafts. Generally, a vehicle is mainly composed of a transmission shaft, axle, half shafts, and wheels. The axle is connected to a sub-frame through mounts and the propeller shaft is linked to the axle by a Cardan joint. Due to its simplicity and low cost, its applications can be found in automobiles, marine propulsion systems, and petrochemical facilities. In vehicle transmission such as 4WD, Cardan shafts are positioned between the engine and vehicle wheel and are connected using a joint. The Cardan joint easily produces vibration and roaring noise in the interior of a four-wheel-drive and also a rear-wheel-drive vehicle, especially for vehicles running under high torque conditions [1]. Thus, an intensive study of the lateral and torsional vibration caused by the Cardan joint is essential for the design and layout of the automotive driveline.

The Hooke’s joint (universal, Cardan, or U-joint) is a widely studied mechanical connection commonly used in various types of industrial drivelines [2–6]. The maintenance of such a system, unfortunately, is often arduous due to its inaccessibility, function, and size, making its mechanism one of the most commonly misunderstood issues a vehicle operator.
may experience. Cardan joint has a wide range of applications in the automobile industry for transmitting rotary motion. However, when it comes to mechanical failures, faults such as unbalance and crack may contribute to premature wear of the joints if not early detected. Unfortunately, in modern machinery, due to high rotation and periodical fluctuation speed, the shafts system connected by a Hooke’s joints is often challenging, making their analysis one of the most complex and most overlooked. The industry is continuously seeking scientific innovations and techniques to improve the lifespan of rotating machinery and equipment, as well as a new technology for monitoring and maintaining rotor systems. These systems composed of rigid or flexible misaligned shafts connected by joints, transmit the rotational motion to the driven shafts. In most published studies, the Cardan shaft systems are considered with a maximum of two degrees of freedom (DOF) lumped-mass models consisting of torsionally elastic, massless shafts. However, for reliable dynamic system analysis and prediction of the instability characteristics of these shaft systems, studies of a complex system cannot be expected to be successful using such approximate low-dimensional models.

Many studies have investigated the dynamic characteristics of complex Cardan joint and misalignment in the transmission system. From a single to multiple DOF, a more realistic model of a multi-shaft system consisting of n shafts interconnected via Hooke’s joints which results in the n-DOF system has been modelled for the offset shaft system [7–10]. The mathematical model consisted of a set of coupled, linear partial differential equations with time-dependent coefficients. These proposed model systems demonstrated via numerical simulations that the coupled rotor system might display chaotic behaviour under specific conditions. Unlike the numerical studies mentioned, researchers have investigated numerically and validated experimentally the dynamic stability and vibration analysis of torsional vibrations of a shaft system connected via a Hooke’s joint. Their studies have concentrated mostly on the behaviour of mechanisms with a Hooke’s joint [11–16]. However, couplings are key components to ensure the effectiveness of the transfer motion and power in various rotating machines systems, in many industries.

It is estimated that misaligned shafts in rotating machines cause more than one-third of all rotor system failures, up to half the cost of downtime and, as a result, misaligned joint and couplings [17]. Such tiny misalignments tend to grow into more significant machine problems as the error often trickles down to affect many other parts of the plant, leading to high vibrations, unbalance and drive shaft crank which is affecting many industries in terms of work stoppages. Thus, the identification of the faults in such rotor systems plays an integral role in the performance of equipment and machinery [18]. Despite the voluminous research works conducted on the propeller shaft, there are only a few well-researched on the torsional vibration of connected shaft systems involving more than one parametric excitation fault. Only a few researchers recently have investigated the dynamic characteristics of the universal joint, misalignment, and crack of the drive or driven shafts in the transmission system using algorithms involving time-frequency representation. Considering the elasticity and energy of the system, Alugongo [19,20] proposed a two-axis model for analysing the torsional vibration of a propeller shaft with a crack-induced excitation using wavelet transform techniques. Tchomeni and Alugongo [21] reconsidered the same problem as Alugongo, developed a six DOF Cardan shaft system based on Lagrangian transformation and carried out a parametric study of the coupled lateral and torsional vibration of an unbalanced Cardan shaft system. Their numerical and experimental investigations included the perturbation parameter characteristic of Hooke’s joint distortion and unbalanced parameter in the various speed of rotors. Hu et al., [22] diagnosed numerically and validate experimentally the Cardan shaft unbalance and misalignment using acceleration signal and dual-tree complex wavelet packet transformation method to extract the relevant features for the detection of shaft misalignment.

To our best knowledge, the dynamic analysis of a partially cracked Cardan shaft model, which considers the lateral and torsional deflections of both unbalanced drive and the driven shaft, has not been analysed yet. Regarded as a multi-body system consisting
of two identical elastic shafts modelled with two discrete rotating discs at its mid-span; the model also has taken into account the local flexibility induced by a breathing crack on the driveshaft and the dynamic excitation of the perturbation function representing the Hooke’s joint disturbance. The differential equations of the seven DOF model of a system coupled with a Hooke’s joint is constructed from the equations of vehicle dynamics and vibration theory. The impact of a Cardan joint disturbance, shaft-unbalance, and transverse breathing crack fluctuation are the main parameters requiring the time–frequency techniques for effective faults feature extractions. These reliable methods based on the synchrosqueezing wavelet technique (SWT) can extract any instantaneous frequency (IFs) resonance in parameter space and present the results in the form of energy concentration graphs. To provide correlation, the simulation results (parametric resonances) are compared with the experimental results in various scenarios.

The remainder of this paper is organised as follows: In Section 2, the modelling of the Cardan shaft with shafts unbalance is established. The derivation of the parametric excitation such as the breathing crack functions is described. Then, the multi-scale signal processing tools for nonlinear crack detection is briefly described in Section 3 as well as the experimental set-up on a rotor test rig to perform real driveshaft crack detection is proposed. In Section 4, the numerical findings and the experiments are presented. An experiment was conducted on an RK4 rotor test rig to validate the effectiveness of applying the nonlinear synchrosqueezing wavelet transforms (NWSST) method to the feature extraction of IF for an unbalanced and cracked rotor system. Finally, a discussion and several conclusions are drawn in Section 5.

2. Mathematical Model Descriptions

The nonlinear equations of the Cardan shaft linked with a single Hooke’s joint were derived based on several assumptions. Figure 1 represents the modelled Cardan shaft system composed of drive and driven shafts connected by a single Cardan joint. The sub-assembly in Figure 1 is the assembly in a deformed frame. The model at its end was developed based on a simple Jeffcott’s approach with the following assumptions: (1) each shaft is uniform, balanced, and symmetric across its axis; (2) the power transmission system is only loaded torsionally, and the lateral oscillations are significant enough to be considered; (3) the joint mass is negligible comparing to the shaft-disc mass and is conceived as ideal, without any clearance, depreciation, and friction; (4) the input angular velocity is inconstant; (5) the U parts attached to the middle shaft ends are in phase; (6) the intermediate shaft is available in most vehicles; however, the choice not to consider it was based on the lower operating angle between the two offset shafts. Due to the nonlinear vibrations of the driveline in both torsional and lateral directions, the joint angle was considered not constant during operation. The system considered was characterised by the system’s kinetic energy. The two discs were of masses $M_1$ and $M_2$ and carry eccentric masses, $m_{1u}$ and $m_{2u}$, respectively. We assumed that $m_{1u}$ and $m_{2u}$ remained in the mid-span of the rotor disc during vibration. Gyroscopic effects due to discs’ spinning were neglected. The flexural stiffness of the shaft was considered relatively small compared to bearings. Self-aligning bearings were used to ensure that the bearing took up the bending mode shape of the shafts at the supports. Transfer of vibration from the rest of the chassis to the Cardan shaft was negligibly small in comparison with the torsional vibration of the Cardan shaft itself. This assumption allowed us to ignore and self-align bearing moments due to the mode of the shaft. Only linear viscous damping effects of the bearings were considered.

A Hooke’s joint connects the two shafts, and each behaves as an Euler–Bernoulli beam represented by the first mode. Therefore, the model is valid below the second lateral critical speed. The coordinate frame used to develop the model is shown in Figure 1. The system’s degree of freedom was lumped at the centres of the inertias, $J_M$, $J_{D1}$, and $J_{D2}$. $J_M$ and $J_{D2}$ were coupled by a light torsional spring while avoiding the risk of mass matrix singularity of the equation of motion. Figure 1, shows the orientation of the masses on the two shafts viewed in the mid-span planes perpendicular to each shaft axis. It was considered that, at a
time $t$, the centres of the discs are at the points $x_1, y_1$ and $x_2, y_2$, i.e., the rotating coordinates attached to the respective centres of the discs.

![Figure 1](image-url)

**Figure 1.** Illustration of a vehicle with two propeller shafts interconnected by a single Hooke’s joint.

### 2.1. The Lagrange Expression

Two inertial reference frames, $X_1 Y_1 Z_1$ and $X_2 Y_2 Z_2$ (Figure 1) were adopted for the global representation of the lumped mass system. $X_1 Y_1 Z_1$ is fixed to the motor with $Z_1$, coincident with the motor output shaft axis, whereas $X_2 Y_2 Z_2$ is attached to the left bearing of the secondary shaft, such that $Z_2$ is parallel to the central axis of the bearing as shown in Figure 1. The vectors $R_{e1}$ and $R_{e2}$ represent the global position of $m_{u1}$ and $m_{u2}$, respectively. The pairs of vectors $R_{e1}, \psi_1$, and $R_{e2}, \psi_2$ represent the centres of the rotor masses $M_1$ and $M_2$, respectively. The kinetic energy of the system can be expressed as:

$$G_{e1} = \frac{1}{2} J_M \dot{\theta}_1^2 + \frac{1}{2} J_{D1} (\theta_1 + \psi_1)^2 + \frac{1}{2} M_1 \left( X_1^2 + Y_1^2 \right) + \frac{1}{2} m_{u1} \dot{R}_{e1}^2 R_{e1}$$

$$G_{e2} = \frac{1}{2} J_{D2} (\theta_2 + \psi_2)^2 + \frac{1}{2} M_2 \left( X_2^2 + Y_2^2 \right) + \frac{1}{2} m_{u2} \dot{R}_{e2}^2 R_{e2}$$

where individual inertias and their net displacements are as follows: $J_M$—inertia of motor output shaft, which undergoes only rigid-body rotation $\theta_1$; $J_{D1}$—elastic deformation of spring, $k_1$, i.e., $\psi_1$ superposed on the flexible rotation of $J_M$; $J_{D2}$—elastic deformation of the spring $k_2$, i.e., $\psi_2$ superposed on the elastic rotation of the shaft two, i.e., $\theta_2$; $R_1$ and $R_2$—the velocity vectors of $m_{u1}$ and $m_{u2}$, respectively. The vectors $R_{e1}$ and $R_{e2}$ can be expressed as [23]:

$$R_{e1} = [A(\theta_1)] [A(\psi_1)] e_1$$

$$R_{e2} = [A(\theta_2)] [A(\psi_2)] e_2$$

where, $e_1, \gamma_1$ and $e_2, \gamma_2$ represent locations of $m_{u1}$ and $m_{u2}$ in the disc’s body coordinate systems $x_1, y_1$ and $x_2, y_2$. The angles $\theta_1$ and $\psi_1$ are the motor shaft elastic rotation and the torsional deformation angle of the primary shaft measured with respect to the motor coordinate system, respectively. The angles, $\theta_2$ and $\psi_2$ are the secondary bearing elastic rotation and the torsional deformation angle measured with respect to the bearing coordinate system, respectively. The matrices have the following significance: $[A(\theta_1)]$—rotational transformation from the motor coordinate system $x_1^m, y_1^m$ to the inertial reference frame $X_1, Y_1; [A(\psi_1)]$—rotational transformation from the disc coordinate systems $x_1, y_1$ to the motor coordinate system $x_1^m, y_1^m; [A(\theta_2)]$—rotational transformation of the secondary shaft-bearing coordinate system $x_2^b, y_2^b$ to the reference frame $X_2, Y_2; [A(\psi_2)]$—rotational
transformational matrices from the disc coordinate system $x_2, y_2$ to the secondary shaft-bearing coordinate system $x_2', y_2'$. The matrices $[A(\theta_1)]$ and $[A(\psi_1)]$ are given as:

$$[A(\theta_1)] = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix}, \text{ and } [A(\psi_1)] = \begin{bmatrix} 1 & -\psi_1 \\ \psi_1 & 1 \end{bmatrix}$$  \hspace{1cm} (5)

and analytical expressions for $[A(\theta_2)]$ and $[A(\psi_2)]$ are developed as follows:

$$[A(\theta_2)] = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{bmatrix}, \text{ and } [A(\psi_2)] = \begin{bmatrix} 1 & -\psi_2 \\ \psi_2 & 1 \end{bmatrix}$$  \hspace{1cm} (6)

Differentiating $R_{c1}$ and $R_{c2}$ with respect to time yields:

$$\dot{R}_{c1} = \dot{\theta}_1 [A_{\theta_1}(\theta_1)] [A(\psi_1)] e_1 + \psi_1 [A(\theta_1)] [A_{\psi_1}(\psi_1)] e_1$$  \hspace{1cm} (7)

$$\dot{R}_{c2} = \dot{\theta}_2 [A_{\theta_2}(\theta_2)] [A(\psi_2)] e_2 + \psi_2 [A(\theta_2)] [A_{\psi_2}(\psi_2)] e_2$$  \hspace{1cm} (8)

Let the net displacement of $f_{D2}$ be expressed as:

$$\theta_2 = (\theta_1 + \psi_1) - \mu(\theta_1 + \psi_1) \text{ and } \psi_2 = \psi_1 - \mu \psi_1$$  \hspace{1cm} (9)

where $\mu$ is a perturbation function of the rigid and elastic body motion ($\theta_1$ and $\psi_1$), characteristic of a disturbance by a Hooke’s joint. For a finite range of $\mu$, the product $\mu \times (\theta_1 + \psi_1)$ will be finite and periodically convergent. $\mu$ used in this paper has been partially developed in [24] and is extended here in Appendix A.1:

$$\mu = \frac{\alpha \tan(\theta_1 + \psi_1)}{(\theta_1 + \psi_1) (\gamma \tan^2(\theta_1 + \psi_1) + 1)}$$  \hspace{1cm} (10)

where $\alpha = 1 - \gamma$ and $\gamma = \cos \beta$. $\theta_2$ was determined by eliminating $\psi_1$ in Equation (9), after which:

$$\theta_2 = \theta_1 - \frac{an}{(\gamma n^2 + 1)}$$  \hspace{1cm} (11)

where $n = \tan \theta_1$. The first derivative of Equation (11) with respect to time gives the relationship of angular velocity between input and output shaft of the Cardan joint as follows:

$$\dot{\theta}_2 = \dot{\theta}_1 (1 - g) \text{ where } g = \frac{an}{\gamma n^2 + 1} \left( \frac{n^2 + 1}{n} - \frac{2\gamma n(n^2 + 1)}{\gamma n^2 + 1} \right)$$  \hspace{1cm} (12)

Differentiating $R_{c1}$ and $R_{c2}$ with respect to time gives:

$$\dot{R}_{c1} = \dot{\theta}_1 [A_{\theta_1}(\theta_1)] [A(\psi_1)] e_1 + \psi_1 [A(\theta_1)] [A_{\psi_1}(\psi_1)] e_1$$  \hspace{1cm} (13)

$$\dot{R}_{c2} = \dot{\theta}_2 [A_{\theta_2}(\theta_2)] [A(\psi_2)] e_2 + \psi_2 [A(\theta_2)] [A_{\psi_2}(\psi_2)] e_2$$  \hspace{1cm} (14)

where $[A_{\theta_1}(\theta_1)] = d[A(\theta_1)]/d\theta_1$, $[A_{\psi_1}(\psi_1)] = d[A(\psi_1)]/d\psi_1$, $[A_{\theta_2}(\theta_2)] = d[A(\theta_2)]/d\theta_2$, $[A_{\psi_2}(\psi_2)] = d[A(\psi_2)]/d\psi_2$.

Upon substituting for $[A_{\theta_1}(\theta_1)]$ and $[A_{\psi_1}(\psi_1)]$ in Equations (14) through (15), $R_{c_1}$ and $\dot{R}_{c_2}$ can be presented by:

$$R_{c_1} = \begin{cases} X_1 - \theta_1 \epsilon_2 a_1 - \theta_2 \epsilon_2 \epsilon_1 - \psi_1 \eta_1 \\ Y_1 + \theta_1 \epsilon_2 a_1 - \theta_2 \epsilon_2 \epsilon_1 + \psi_1 \eta_1 \end{cases} \text{ and } \dot{R}_{c_2} = \begin{cases} \dot{X}_2 - \dot{\theta}_2 \epsilon_2 a_2 - \dot{\theta}_2 \epsilon_2 \epsilon_2 - \dot{\psi}_2 \eta_2 \\ \dot{Y}_2 + \dot{\theta}_2 \epsilon_2 a_2 - \dot{\theta}_2 \epsilon_2 \epsilon_2 + \dot{\psi}_2 \eta_2 \end{cases}$$  \hspace{1cm} (15)

where

$$\begin{align*}
\alpha_1 &= \sin \theta_1 + \psi_1 \cos \theta_1; \quad \delta_1 = \cos \theta_1 - \psi_1 \sin \theta_1 \\
\eta_1 &= \epsilon_2 \sin \theta_1 + \epsilon_1 \cos \theta_1; \quad \eta_1 = \epsilon_2 \cos \theta_1 - \epsilon_1 \sin \theta_1
\end{align*}$$  \hspace{1cm} (16)

and

$$\begin{align*}
\alpha_2 &= \sin \theta_2 + \psi_2 \cos \theta_2; \quad \delta_2 = \cos \theta_2 - \psi_2 \sin \theta_2 \\
\eta_2 &= \epsilon_2 \sin \theta_2 + \epsilon_2 \cos \theta_2; \quad \mu_2 = \epsilon_2 \cos \theta_2 - \epsilon_2 \sin \theta_2
\end{align*}$$  \hspace{1cm} (17)
The pairs $e_{x1}, e_{y1}$ and $e_{x2}, e_{y2}$ are components of $e_1$ and $e_2$ in $x_1, y_1$ and $x_2, y_2$ coordinate. Substituting Equations (11)–(17) in Equations (1) and (2), in terms of the system model in Figure 1 the kinetic energy $G_{r1}$ of the primary shaft is given as:

$$G_{r1} = \frac{1}{2} I_{x1} \dot{e}_{x1}^2 + \frac{1}{2} I_{y1} \dot{e}_{y1}^2 + m_{1x} e_{x1}^2 \ddot{e}_{x1} + m_{1y} e_{y1}^2 \ddot{e}_{y1} + \frac{1}{2} m_{1z} e_{z1}^2 \ddot{e}_{z1} \left( 1 + \psi_1^2 \right)$$

$$+ \frac{1}{2} \left( M_1 + m_{1a} \right) \left( X_1 + Y_1 \right)^2 + m_{1w} Y_1 \left( e_{x1} \cos \theta_1 - e_{y1} \sin \theta_1 \right)$$

$$- m_{1a} X_1 \dot{e}_{x1} \left[ \left( e_{x1} - \psi_1 \epsilon_{y1} \right) \sin \theta_1 + \left( e_{y1} \psi_1 + e_{y1} \right) \cos \theta_1 \right]$$

$$+ m_{1a} Y_1 \dot{e}_{x1} \left[ \left( e_{x1} - \psi_1 \epsilon_{y1} \right) \cos \theta_1 - \left( e_{y1} \psi_1 + e_{y1} \right) \sin \theta_1 \right] - m_{1a} X_1 \dot{e}_{y1} \left( e_{x1} \sin \theta_1 + e_{y1} \cos \theta_1 \right)$$

The kinetic energy $G_{r2}$ of the secondary shaft is constituted as:

$$G_{r2} = \frac{1}{2} I_{x2} \dot{e}_{x2}^2 + \frac{1}{2} I_{y2} \dot{e}_{y2}^2 \left( 1 + \psi_2^2 \right) + \frac{1}{2} m_{2x} e_{x2}^2 \ddot{e}_{x2} + m_{2y} e_{y2}^2 \ddot{e}_{y2} + \frac{1}{2} m_{2z} e_{z2}^2 \ddot{e}_{z2}$$

$$- m_{2a} X_2 \dot{e}_{x2} \left[ \left( e_{x2} \psi_2 + e_{y2} \right) \cos \theta_2 + \left( e_{x2} \psi_2 - e_{y2} \right) \sin \theta_2 \right] +$$

$$+ m_{2a} Y_2 \dot{e}_{x2} \left[ \left( e_{x2} \psi_2 - e_{y2} \psi_2 \right) \cos \theta_2 - \left( e_{x2} \psi_2 + e_{y2} \psi_2 \right) \sin \theta_2 \right] + \frac{1}{2} \left( M_2 + m_{2a} \right) \left[ X_2^2 + Y_2^2 \right]$$

The kinetic energy of the system comprises the kinetic energy of the primary shaft $G_{r1}$ and secondary shaft $G_{r2}$:

$$G_{\text{Total}} = G_{r1} + G_{r2}$$

2.2. The Potential Energy Expression

The potential energy comprises strain energy of bending and torsion and is expressed as:

$$V = \frac{1}{2} K_{x1} X_1^2 + \frac{1}{2} K_{y1} Y_1^2 + \frac{1}{2} K_{x2} X_2^2 + \frac{1}{2} K_{y2} Y_2^2 + \frac{1}{2} K_{t1} \psi_1^2 + \frac{1}{2} K_{t2} \left( (1 - \mu) \psi_1 - \psi_2 \right)^2$$

where $K_{x1}, K_{y1}, K_{x2}, K_{y2}, K_{t1}$, and $K_{t2}$ are the stiffness coefficients associated with the degrees of freedom of the system.

2.3. The Rayleigh Dissipation Function

Assuming a case of viscous modal damping, Rayleigh’s dissipation function has been expressed as:

$$D = \frac{1}{2} C_{x1} X_1^2 + \frac{1}{2} C_{y1} Y_1^2 + \frac{1}{2} C_{x2} X_2^2 + \frac{1}{2} C_{y2} Y_2^2 + \frac{1}{2} C_{t1} \psi_1^2 + \frac{1}{2} C_{t2} \left( \mu - 1 \right) \psi_1 + \psi_2 \right)^2$$

where, $C_{x1}, C_{y1}, C_{x2}, C_{y2}, C_{t1}$, and $C_{t2}$ are the respective degrees of freedom damping coefficient.

2.4. The Equation of Motion of the Coupled Cardan Shaft

Substituting the expression of kinetic and potential energy and Rayleigh’s dissipation function into Lagrange’s formulation with respect to $\{q\}$, the vector of generalised coordinates

$$\frac{d}{dt} \left( \frac{\partial G}{\partial \dot{q}} \right) + \frac{\partial V}{\partial q} + \frac{\partial }{\partial q} - \frac{\partial G}{\partial q} = T_{\text{q}\dot{q}} \quad q = \theta_1, \psi_1, \psi_2, X_1, Y_1, X_2, Y_2$$

yielding the system vibration equation in the form:

$$[M] \{\ddot{q}\} + [C] \{\dot{q}\} + [K] \{q\} = \{NL_q\} + \{F_q\} + \{C_q\}$$

$[M], [C],$ and $[K]$ are respectively the mass matrix, the damping matrix, and stiffness matrix. The terms on the right hand are obtained and defined in Section 3.1. Elements of
the respective matrices and vectors have been determined based on Equations (20)–(22), and their final forms are:

\[
\begin{align*}
    m_{\theta_1} &= (J_{11} + J + m_{44} \varepsilon_1^2(1 + \psi_1^2)) + (J_{22} + m_{24} \varepsilon_2^2(1 + \psi_2^2))(g - 1)^2; \\
    m_{\theta_2} &= J_{12} + m_{44} \varepsilon_2^2; \\
    m_{\theta_1} &= -m_0 \left[ (\varepsilon_{11} + \varepsilon_{11} \psi_1) \cos \theta_1 + (\varepsilon_{11} - \varepsilon_{11} \psi_1) \sin \theta_1 \right]; \\
    m_{\theta_2} &= -m_0 \left[ (\varepsilon_{21} \psi_2) \cos \theta_2 - (\varepsilon_{22} \psi_2) \sin \theta_2 \right]; \\
    m_{\psi_1} &= m_{11}(\varepsilon_{11} + \varepsilon_{11} \psi_1) \cos \theta_1 + (\varepsilon_{11} - \varepsilon_{11} \psi_1) \sin \theta_1; \\
    m_{\psi_2} &= m_{11}(\varepsilon_{22} \psi_2) \sin \theta_2 - (\varepsilon_{22} \psi_2) \cos \theta_2; \\
    m_{X_1} &= m_{12}(\varepsilon_{11} \varepsilon_{11} \psi_1) \cos \theta_1 - (\varepsilon_{11} \varepsilon_{11} \psi_1) \sin \theta_1; \\
    m_{X_2} &= m_{12}(\varepsilon_{11} \varepsilon_{11} \psi_1) \sin \theta_1 + (\varepsilon_{11} \varepsilon_{11} \psi_1) \cos \theta_1; \\
    m_{Y_1} &= m_{12}(\varepsilon_{22} \varepsilon_{22} \psi_2) \cos \theta_2 + (\varepsilon_{22} \varepsilon_{22} \psi_2) \sin \theta_2; \\
    m_{Y_2} &= m_{12}(\varepsilon_{22} \varepsilon_{22} \psi_2) \sin \theta_2 - (\varepsilon_{22} \varepsilon_{22} \psi_2) \cos \theta_2.
\end{align*}
\]

(25)

Based on partial Equation (22) and following Equation (23), the damping of the rotor system can be expressed by differentiating Rayleigh’s dissipation function with respect to each generalised coordinate. The damping elements of the rotor system can be described by:

\[
\begin{align*}
    C_{\theta_1} &= C_{T_2} \left( \frac{\varepsilon_{11}(g - 1)}{\varepsilon_{11} + \varepsilon_{11} \mu} \right)^2 \psi_1 C_{\theta_1}; \\
    C_{\theta_2} &= C_{T_2} \left( \frac{\varepsilon_{22}(g - 1)}{\varepsilon_{22} + \varepsilon_{22} \mu} \right)^2 \psi_1 C_{\theta_2}; \\
    C_{\psi_1} &= C_{T_1} \left( \frac{\varepsilon_{11}(g - 1)}{\varepsilon_{11} + \varepsilon_{11} \mu} \right)^2 \psi_1 C_{\psi_1}; \\
    C_{\psi_2} &= C_{T_1} \left( \frac{\varepsilon_{22}(g - 1)}{\varepsilon_{22} + \varepsilon_{22} \mu} \right)^2 \psi_1 C_{\psi_2}; \\
    C_{X_1} &= C_{X_2} \left( \frac{\varepsilon_{11}(g - 1)}{\varepsilon_{11} + \varepsilon_{11} \mu} \right)^2 \psi_1 C_{X_1}; \\
    C_{X_2} &= C_{X_2} \left( \frac{\varepsilon_{22}(g - 1)}{\varepsilon_{22} + \varepsilon_{22} \mu} \right)^2 \psi_1 C_{X_2}.
\end{align*}
\]

(32)

Following the approaches of the previous section and based on Equation (21), the transient stiffness matrix \([K]\) of the rotor system obtained by differentiating the potential energy can be expressed as:

\[
\begin{align*}
    K_{\theta_1} &= K_{T_1} - m_0 \varepsilon_1^2 \psi_1^2 + K_{T_2}(\mu - 1)^2; \\
    K_{\theta_2} &= K_{T_1}(\mu - 1); \\
    K_{\psi_1} &= K_{T_1}(\varepsilon_{11} \psi_1); \\
    K_{\psi_2} &= K_{T_2}(\varepsilon_{22} \psi_2).
\end{align*}
\]

(33)

In general, the negative torsional stiffness has no physical meaning in itself. The torsional stiffness \(K_{T_1}\) and \(K_{T_2}\) are always higher than the actual softening dissipative energy of the connecting input shafts \(m_1 \varepsilon_1^2 \psi_1^2\) and \(m_2 \varepsilon_2^2 \psi_2^2\), which are subtracted in the expression of the torsional stiffness and, consequently, the term \(K_{\theta_2}\) is positive. Thus, the coupling between lateral and torsional vibrations enhances when the torsional stiffness of the driven shaft augments and the coupling weakens when the softening effect due to the driveshaft speed increases.

### 2.5. Introduction of a Breathing Crack into the Model

The breathing mechanism presented in [25] was adopted to a driveshaft to analyse the influence of the breathing crack during transmission of motion through the Hooke’s joint. The detailed calculation process of the area moments of inertia of the cracked shaft cross-section and the area moments of inertia values can be found in [26], and its derivation is not repeated in this paper. The coordinates were built on the cross-section at the crack location (Figure 2). The neutral axis shift, the time-varying moment of inertia of the cross-section at the crack location was approximated by the Fourier series expansion.

The time-varying area moments of inertia \(I_{X_1}(t), I_{Y_1}(t),\) and \(I_{X_1 \gamma_1}(t)\) about the centroidal \(X_1\) and \(Y_1\) axes during the driveshaft rotation were obtained by considering the
cross-sectional area $A_1$ of the cracked driveshaft about the fixed $\overline{X}_1$ and $Y_1$ axes and were given as:

$$I_{\overline{X}_1}(t) \cong I - h_1(t) I_{11}$$  \hspace{1cm} (34)$$

$$I_{\overline{Y}_1}(t) \cong I - h_1(t) I_{11} + h_2(t) I_{22}$$  \hspace{1cm} (35)$$

and

$$I_{\overline{X}_1\overline{Y}_1}(t) \cong - \left( \frac{I_{\overline{X}_1}(t) - I_{\overline{Y}_1}(t)}{2} + \frac{A_1 e^2}{2} \right) \times \sum_{n=1}^{p} \frac{2 \phi_2 \sin(\pi - n \phi_2)}{\pi^2 - n^2 \phi_2^2} \sin(n \Omega t)$$  \hspace{1cm} (36)$$

where $I = \pi R^4 / 4$ is the cross-sectional area moment of inertia for the fully closed crack.

![Schematic diagram and assembly of the interior of a driveshaft structure. (b) Deformed configurations of shaft discs 1 and 2.](image)

**Figure 2.** (a) Schematic diagram and assembly of the interior of a driveshaft structure. (b) Deformed configurations of shaft discs 1 and 2.

The breathing functions $h_1(t)$ and $h_2(t)$ are defined by [18]:

$$h_1(t) = \left( \cos\left(\frac{\Omega t}{2}\right) \right)^m = \frac{1}{2^m} \left( \begin{array}{c} m \\ m/2 \end{array} \right) + 2 \sum_{n=0}^{(m/2)-1} \left( \begin{array}{c} m \\ n \end{array} \right) \cos\left( (m - 2n) \frac{\Omega t}{2} \right)$$  \hspace{1cm} (37)$$

$$h_2(t) = \frac{1}{\pi} \left[ -\frac{\phi_1 + \phi_2}{2} + \frac{2}{\phi_2 - \phi_1} \times \sum_{i=1}^{p} \cos(i \phi_2) \cos(i \phi_1) \cos(\Omega t) \right]$$  \hspace{1cm} (38)$$
where \( i, n, \) and \( m \) are positive even numbers that control the deflection and shape of the breathing functions. As a result, the complete stiffness matrix \( [K_{\text{stiffness}}] \) of the driveshaft with a breathing crack can be approximated by:

\[
[K_{\text{cracked}}] = \begin{bmatrix}
K_{X_1 X_1}(t) & K_{X_1 Y_1}(t) \\
K_{Y_1 X_1}(t) & K_{Y_1 Y_1}(t)
\end{bmatrix} \approx \frac{48E}{I^3} \begin{bmatrix}
\hat{I}_{X_1}(t) & \hat{I}_{Y_1}(t) \\
\hat{I}_{X_1}(t) & \hat{I}_{Y_1}(t)
\end{bmatrix}
\]

(39)

where \( E \) is the elastic modulus of elasticity, \( L \) is the length of the rotor. Since the cross-section of the cracked driveshaft is symmetric about the \( Y_1 \) axis, so that \( I_{X_1 Y_1} = 0 \), the cross-section \( I_{X_1 Y_1}(t) = \hat{I}_{Y_1}(t) = 0 \). For the non-dimensional crack depth \( \varepsilon_{1c} = h/R = 0.1 \), the time-varying stiffness, which represents the breathing mechanism of the transverse crack, and the torsional stiffness during the rotating process are shown in Figure 3.

Figure 3a shows self-excited oscillations when cracks occurred on the driveshaft. The cracked shaft stiffness components oscillated periodically with an oscillation period equal to the rotating shaft period. The torsional stiffness of the driveshaft was significantly excited up at the start-up and then self-oscillated into harmonic and periodic motion before converging to stability. Compared with the time-varying stiffness in the \( X_1 \) and \( Y_1 \) directions, the amplitude of the torsional stiffness was much smaller; therefore, we experimentally neglected its effect.

Upon introducing the breathing crack functions into the governing equation, the system equation of motion was found to be:

\[
\begin{bmatrix}
m_{\psi_1} \psi_1 & m_{\psi_1 \psi_2} & m_{\psi_1 X_1} & m_{\psi_1 Y_1} & m_{\psi_2 \psi_2} & m_{\psi_2 X_1} & m_{\psi_2 Y_1} \\
m_{\psi_2 \psi_1} & m_{\psi_1 \psi_2} & m_{\psi_1 X_1} & m_{\psi_1 Y_1} & m_{\psi_2 \psi_2} & m_{\psi_2 X_1} & m_{\psi_2 Y_1} \\
m_{\psi_1 X_1} & m_{\psi_1 Y_1} & 0 & 0 & 0 & 0 & 0 \\
m_{\psi_2 X_1} & m_{\psi_2 Y_1} & 0 & 0 & 0 & 0 & 0 \\
m_{\psi_2 X_1} & m_{\psi_2 Y_1} & 0 & 0 & 0 & 0 & 0 \\
m_{\psi_2 X_1} & m_{\psi_2 Y_1} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
X_1 \\
Y_1 \\
X_2 \\
Y_2
\end{bmatrix}
= \begin{bmatrix}
\hat{I}_{X_1} \\
\hat{I}_{Y_1}
\end{bmatrix}
\]

(40)

The terms \( \{Q_{\text{NL}}\} \) represent the influence of the Hooke’s joints on the coupling of the elastic Cardan shaft for each generalized coordinate expressed as:

\[
Q_{\text{NL}} = \frac{dV}{\theta_1} = Q_{X_1 \psi_1} = \frac{dV}{\psi_1} = \frac{K_{T_1} (\Gamma - \mu) \psi_1}{\theta_1 + \psi_1} = (\mu - 1) \psi_1 + \psi_2
\]

\[
Q_{\text{NL}} = Q_{X_2 \psi_1} = Q_{X_2 \psi_2} = Q_{X_1 \psi_2} = Q_{Y_1 \psi_2} = 0
\]

where \( \Gamma = \frac{(\gamma - 1)N}{2\pi N^2 - \frac{2\pi N}{N^2}} \).
\[ \xi_{t_1} = \xi_{t_1} = \frac{C_{x_1}}{K_{x_1}x_1} ; \quad \text{and} \quad \xi_{t_2} = \xi_{t_2} = \frac{C_{x_2}}{K_{x_2}x_2} \]

\[ \omega_{t_1} = \omega_{t_1} = \sqrt{\frac{K_{x_1}x_1}{M_{x_1}+m_{x_1}}}; \quad \omega_{t_2} = \omega_{t_2} = \sqrt{\frac{K_{x_2}x_2}{M_{x_2}+m_{x_2}}} \]  

(43)

The dynamic behaviour of the coupled lateral and torsional vibration of the driveline was then investigated numerically by integrating the non-linear state-space equations of motion (40). However, the perturbation function \( \mu(\dot{\theta}_1 + \dot{\psi}_1) \), characteristic of disturbance of the Hooke’s joint, and fluctuating mode manifested by the cracked driveshaft necessitated the use of sensitive time-frequency tools to extract the feature of the breathing crack, generally hidden in the weak components.

3. The Proposed Features Extraction Method

As a refined version of the synchrosqueezing transform (SST), the nonlinear wavelet synchrosqueezing transform (NWSST), squeezes the energy in a partial frequency window, which can provide both excellent time and frequency resolution for signals with weak fluctuation for IFs estimation. In this section, the brief principle of IF is reviewed first, to introduce better the advantages and effectiveness of NWSST for extraction of the feature of a connected Cardan shaft.

3.1. The Adopted Instantaneous Frequency-Shift Operation

As a powerful tool for analysis of the nonlinear signal, the wavelet synchrosqueezing technique has been successfully used in the signal analysis of faulted rotor systems [27–29]. Moreover, a general application of Hilbert transform (HT) and IF widely used in [30–32] is briefly presented in this subsection. The collected signal from Equation (40) can be instantly characterized by IF which can be extracted by Hilbert transform to obtain an analytic signal. A typical time-varying signal is expressed as a sum of N intrinsic mode functions and residual in the form

\[ x(t) = \sum_{i=1}^{N} x_n(t) + r(t) \]  

(44)

where \( x_n(t) = A_n(t)e^{i\phi_n(t)} \), the amplitude \( A_n(t) > 0 \), and phase \( \phi(t) > 0 \). The Hilbert transform of the analytical signal \( z_n(t) \) is given by:

\[ z_n(t) = x_n(t) + iy_n(t) = A_n(t)e^{i\phi_n(t)} \]  

(45)

whereas the Hilbert transform of \( x_n(t) \) is defined as

\[ y_n(t) = \frac{P}{\pi} \int_{-\infty}^{+\infty} x_n(\tau)/t - \tau d\tau \]  

(46)
in which $P$ represents the Cauchy principal value. The obtained signal $z_n(t)$ permits the amplitude $A_n(t)$ and phase $\phi_n(t)$ to be uniquely defined. The original signal is recovered by

$$z_n^*(t) = A_n(t)e^{2\pi i \phi_n(t)}$$  \hspace{1cm} (47)

In this step, $z_n(t)$ is replaced with $z_n^*(t)$ to perform the candidate IF calculation. The instantaneous angular frequency of the component $x_n(t)$ is the first derivative of its phase

$$\omega_n(t) = \left\{ \frac{d \phi_n}{dt} \right\}$$  \hspace{1cm} (48)

and the actual IF of the original signal $s(t)$ can be calculated by the actual IF of the original signal $z_n^*(t)$ can be calculated by $IF_n(t) = \phi_n(t)/2\pi$.

Since the vibration signals generated by a cracked driveshaft system present strong nonlinear features, discretization of the frequency resolution for such signals with weakly fluctuating IFs estimation can become more complex. To resolve the issue, NWSST was proposed in [28,29], which can provide both excellent time and frequency resolutions for signals with highly oscillating IFs in a partial frequency region where the shifted IFs are located. After the IF-shift operation, the NWSST consists of the partial zoom synchrosqueezing reconstruction.

### 3.2. The Adopted Nonlinear Wavelet Synchrosqueezing and Its Reconstruction

After the completion of the signal conversion frequency, the next step was to use the NWSST which mathematically can be described as a post-processing method of the wavelet transform (WT) to extract the fault information contained in the time-frequency plot. As a singular reallocation method, synchrosqueezing is based on continuous wavelet transform (CWT) and aims to refine the WT coefficient $W(t, \omega)$ by transferring its value to different points $(t', \omega')$, from which IF lines and curves can be extracted. For a chosen mother wavelet function, the CWT of the HT signal $z_n(t)$ is defined by:

$$W_x(a, b) = \int_{-\infty}^{+\infty} z_n(t) \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) dt$$  \hspace{1cm} (49)

where $a$ is the scale factor, $b$ the dilation factor, and $\bar{\psi}\left((t-b)/a\right)$ represents the complex conjugate of $\psi\left((t-b)/a\right)$. The mapping between scale factor $a$ and signal frequency $\omega$ makes it easier to visualise wavelet coefficients in the time–frequency plane. The Fourier transform of the decayed mother wave function $\psi$ is approximately equal to zero in the negative frequencies: $\bar{\psi}(\xi) = 0$ for $\xi < 0$ and is concentrated around $a = \omega_0/\omega$. Its IF is suggested to be preliminary evaluated by taking derivatives of wavelet coefficients. The formula for its computation is as follows:

$$\omega_x(a, b) = \begin{cases} -j\partial_b W_x(a, b)/W_x(a, b) & \text{if } |W_x(a, b)| > 0 \\ \infty & \text{if } |W_x(a, b)| < 0 \end{cases}$$  \hspace{1cm} (50)

With the map built by Equation (48), in the next synchrosqueezing step, the frequency variable $\omega$ and scale factor $a$ were computed only at discrete points $a_i$, with $(\Delta a) = a_i - a_{i-1}$, and its synchrosqueezing value was similarly determined only at the centres $\omega_\delta$ of closed intervals $[\omega_s - 1/2\Delta\omega, \omega_s + 1/2\Delta\omega]$, with $\Delta\omega = \omega_\delta - \omega_{\delta-1}$. After obtaining the candidate IF, $\omega_x(a, b)$, the reassignment operation of $W_x(a, b)$, could be performed by summing these different contributions; the synchrosqueezing wavelet transform of $x(t)$ was obtained as:

$$T_x(\omega_x, b) = \int_{\delta(b)} W_x(a, b)a^{-3/2}\delta(\omega(a, b) - \omega) da$$  \hspace{1cm} (51)

The partial zoom of nonlinear synchrosqueezing of the time–frequency representation (TFR) can be mathematically described as constructing a time–frequency band-pass filter of
the Hilbert transform $H(\omega, b)$ for denoising. The partial zoom nonlinear synchrosqueezing TFR of the signal $z_n^*(t)$ similar to Equation (49) was obtained from

$$T_{zz}(\omega_n^*, b) = \int_{A(b)} W_z^*(a, b)a^{-3/2}d(\omega_n^*(a) - \omega^*)da$$  \hspace{1cm} (52)

where $\omega_n^*(a, b) - \omega^*$ is the partial discrete frequency sequence of the partial zoom nonlinear synchrosqueezing TFR. The corresponding elements of matrices $T_{zz}(\omega_n^*, b)$ were then multiplied by $H(\omega, b)$ to retain the components from the IF $f_i$, and the fixed frequency bandwidth is 2D for time–frequency filtering as described in Equation (53).

$$T_{zz}^*(\omega_n^*, b) = T_{zz}(\omega_n^*, b) \times H(\omega_n^*, b)$$  \hspace{1cm} (53)

where the coefficients of the filter $H(\omega, b)$ in this zone are 1 as proposed [15,33], while the other time-frequency coefficients outside it are 0 for denoising, i.e., $H(\omega, b) = \begin{cases} 1 & (\omega, b) \in f_i \pm d \\ 0 & (\omega, b) \notin f_i \pm d \end{cases}$

The flow chart in Figure 4 describes the generic framework of the proposed method for multi-faults diagnostics of the coupled rotor system:

![Flowchart of the proposed method](image)

**Figure 4.** Flowchart of the proposed method.

### 3.3. Description of the Test Cardan Shaft System

To validate the proposed model, an experiment was conducted on an RK4 rotor test rig (RK4 Rotor Kit) benchmark, as shown in Figure 5. Measurement of the displacement due to misalignment of a Cardan shaft are discussed in this section. For this reason, specially designed Hooke’s joints mimicking the misalignment of the Cardan shaft were constructed and equipped in the experimental set-up shown in Figure 5c. The set mainly consisted of the driveshaft with a created breathing crack and a driven shaft connected by a miniaturized Hooke’s joint, which consisted of three essential parts: two yokes (1a and 1b), a cross, and two central and identical discs (Figure 5e). Self-aligning bearings were used to ensure that the bearing would take up the bending mode shape of the shafts at the supports, including a motor and rotor. There were displacement sensors (Bently 3300 XL 8 mm with sensitivity 7.87 V/mm) and a tachometer that produced one pulse per revolution to provide a reference for the angle position of the unbalance. Four eddy current sensors were connected to the data acquisition device to capture and display the experimental time histories for the whirl twin-rotor. To measure the relative displacement of the cracked driveshaft, two mutually perpendicular proximity probes were mounted
at ±90° from the true vertical and oriented in two planes perpendicular to the shaft. The probes were staggered in the probe block probe (Figure 5d) at the vicinity of the crack location. An asset condition monitoring Scout 100 from the Ascent 2011 Contest was used to collect the vibration data of the input cracked rotor system and to improve transient response performance. The input voltage of the Bently speed controller varied from 0 to 5 V, corresponding to the input motion to the driveshaft of a rotational speed from 0 to 10,000 RPM.

The problem of joint flexibility when the driveshaft is damaged is sometimes a limiting factor in achieving excellent performance. The critical part of preparation for the experiment was the extraction of the breathing crack mechanism on the driveshaft by two perpendicular probes connected to a data acquisition system. For this purpose, the sampling frequency was 12,800 Hz, and the nominal rotational frequency of the rotor was 20 Hz. An unbalanced excitation force, rotating with each disc-shaft at a frequency equal to the rotors spin speed, was considered. Then the real breathing crack was generated by adding cyclic loading discs to the driveshaft at the slit location. The detailed parameters of the completed rotors system used for both simulation and experiment are given in Table 1.

**Figure 5.** The experimental setup consisting of (a) a connected shaft with a crack and Hooke’s joint; (b–d) modified Rotor-Kit 4 components: 1–3—data acquisition control; 4—motor; 5—probe sensors; 6—driveshaft-crack; 7—disc 1; 8—flexible coupling; 9—self-aligning bearing; 10—tachometer; 11—disc 2; 12—output shaft 2; (e) 1a–1b yoke; 2a cross shaft; 3a sleeve bearing.
Table 1. Rotor system R-K 4 parameters.

| Shaft Parameters                          | Value & Units | Shaft Parameters                          | Values and Units           |
|------------------------------------------|---------------|------------------------------------------|-----------------------------|
| Length of the shaft (L)                  | 570 mm        | Shaft stiffness $K_0$                     | $7.35 \times 10^5 \text{Nm}^{-1}$ |
| Shaft Diameter (D)                       | 10 mm         | Shaft torsional stiffness $K_T$          | $8.2 \times 10^4 \text{Nm}^{-1}$ |
| Density of the material ($\varphi$)      | 7800 kg m$^{-3}$ | Damper $c$                             | 200 Ns/m                   |
| Modulus of elasticity (E)                | $2.11 \times 10^{11} \text{Pa}$ | Discs Values and Units | |
| Viscous damping ratio                    | $0.8 \times 10^{-6} \text{s}^{-1}$ | Mass (m)                             | 16.45 kg                   |
| Lateral modes damping ratio $\xi_L$      | 0.0287        | Eccentricity mass ($m_u$)                | 0.25 kg                     |
| Torsional mode damping ratio $\xi_{T1}$  | 0.0088        | Moment of inertia, $J_M$                  | 10.610 kg m$^2$            |
| Torsional mode damping ratio $\xi_{T2}$  | 0.01234       | Disc moment of inertia, $J_{D1,2}$       | 0.1861 kg m$^2$            |

4. Simulation and Experiment on the Cardan Shaft under Unbalanced Condition

4.1. Simulation Response of an Unbalanced Coupled Rotor

In this section, a theoretical analysis of an unbalanced reaction is described, as well as an experimental dynamic analysis performed at a maximum driveshaft rotational speed of 1500 rpm in a start period of 5 s. The dynamic response of the rotor system under crack effects was numerically determined by the fourth-order Runge–Kutta algorithm with an integration time step of 0.001 s. To make sure that the condition of the experiment would match simulation parameters, the torsional deflection angle was very small compared to the rotor’s lateral-torsional deflection; i.e., no torsional flexibilities accounted for ($\psi = 0$). The resulting rotor velocity profile at a maximum rotational speed of 1500 rpm in a start period of 5 s is shown in Figure A2 (Appendix A). The vibration signal of the unbalanced driveshaft due to the constant angular speed was analysed and compared with that at the angular speed fluctuation of the driven shaft. The fast Fourier transform (FFT) spectrum and the lateral deflection of the system were used to extract the feature of the connected Cardan shaft (Figure 5).

Figure 6a–d provides baseline information on the time history and FFT of the system for small shafts misalignment. From the spectrum comparison of both shafts in Figure 6a,b, the first observation is the presence of a second peak (27.83 Hz) at almost half of the driveshaft calculated critical speed (550.42 rpm), which characterises the effect of the unbalance mass. The driveshaft peaks at the first critical speed differ from the driven shaft only in the magnitude representing the attenuation of vibration due to the Hooke’s joint presence. The same observation is displayed in Figure 6c,d. The obtained simulated response was further compared with the experiment response to check its accuracy.

4.2. Simulation Response and Fault Diagnosis of a Cracked Cardan Shaft

In order to extract the features of the driveshaft with a breathing crack and analyse the transmission motion to the unbalanced driven shaft, the breathing shaft stiffness $\epsilon_{1c} = 0.3$ was allowed, and an initial unbalance with $m_{1u} = 0.25$ kg was adopted. The steady-state response of the coupled rotor system with a breathing crack and an unbalance is shown for the vertical deflection and the FFT spectrum (Figure 7). From the analysis below (Figure 7c), it can be concluded that the identification of initial unbalances from the FFT spectrum response contains the information on synthetic unbalance, which is composed of initial unbalanced drive and driven shaft at its first critical speed (56.83 Hz).

The steady-state response of the coupled rotor system with a breathing crack and unbalance (Figure 7) permitted to extract the highly oscillating peaks induced by the
transverse crack successfully. An increase in amplitudes of vibration was observed for cracked driveshaft with the occurrence of a resonance point at the critical driveshaft speed of $\omega$ to $5\omega$ amplitudes (red ellipses in Figure 7a). Figure 7a shows the lateral deflection of the driveshaft, which indicates that the maximum deflection at the resonance point was increased by including the breathing crack into the unbalanced model. The driven shaft response is shown in Figure 7b in terms of lateral deflection time history; there was a main resonance point corresponding to the ordinary unbalance and the other ones developed as a result of the Hooke’s joint parametric excitation between the two shafts. It was observed that the motion transmitted through the Hooke’s joint caused consistent fluctuation, as observed in the lateral deflection response of the secondary shaft with the appearance of some resonance points at the rotational speed of $\Omega = \omega/2$, $\Omega = \omega$, and $\Omega = n\omega$ (Figure 7b). The input shaft underwent substantial vibrations, which could be easily captured by the rotational speed, which happened at $\Omega = n\omega$. This produced synchronous vibrations similar to those caused by a cracked shaft at lower amplitudes compared to the primary shaft response. These results were confirmed by analysis of the frequency spectrum of both shafts (Figure 7c). It was noted that the influence of the cracked shaft was important at a high speed, as a variety of frequency components was observed (presence of $1 \times$, $2 \times$, $\ldots$, $6 \times$); then, it faded to regain its dominance, contrary to what is commonly known in previous research [25,26]. A closer zoom on the frequency spectrum of the second shaft (in black colour) showed that it FFT peaks decreased considerably in amplitude compared to the driveshaft frequency. The fluctuation amplitude of the cracked driveshaft increased continually with time while the fluctuation period of the driven shaft decreased with time (Figure 7c). The prediction of the simulated results outlined above was further compared with results from the experimental rig, presented in the next section.
Figure 7. Simulation response of lateral vibration of the system with a crack: (a,b) vertical deflection of shafts 1 and 2 and (c) Fast Fourier Transform of shafts 1 and 2.

4.3. Experimental Response of an Unbalanced Coupled Rotor

In this experiment, the eccentric mass was added at 10 mm from the axis of rotation of the discs; the two extra masses of 250 g each were attached to the disc located at the angle of 0° (Figure 5c). Then the driveshaft was driven to a predetermined speed at the range of 600–650 rpm, suitable for capturing the signal. The unbalanced responses of both rotors were extracted and compared to the simulation for the model’s validation purpose.

Figure 8 displays the waveform and FFT of the dynamic response of the rotor system when the unbalance mass is attached to each disc. The original signals obtained from a dynamic experiment presented qualitatively the same observation of the simulated results with a specific remark on the driveshaft (630.5 rpm) different from the driven shaft (630.5 rpm). It can be seen that the disc-shaft 2 is likely to be excited by the disc-shaft 1 through the Hooke’s joint to synchronise resonance when the extended end has an eccentric mass. Quantitatively, the rotors are not excited in the same way according to the plans due to the non-considered factors (the gyroscopic phenomenon due to the joint). Frequency analysis of the second shaft confirmed that the increase in unbalance in phase had the effect of accentuating the intensity of the sub-synchronous frequencies. The frequency of appearance corresponded to the flexible shaft mode excited by the joint. This qualitative observation correlated well with the frequency phenomena displayed in Figure 6a,b. The illustration in the time domain of the lateral deflection indicated that the fluctuation due to eccentric mass was significant and perceptible in experimental results (Figure 8a,b) and validated the simulation response (Figure 6c,d). In both cases, the waveform pattern of vibration of the periodic drive and driven shaft during the passage through the first critical speed was observed; the amplitude of vibration increased with time until reaching the
maximum peak and decreased gradually afterwards. The secondary shaft responses were discarded for an unbalanced case because it showed almost a similar pattern to the primary shaft vibration and exhibited envelope changes due to Hooke’s joint conditions.

4.4. Experimental Response and Fault Diagnosis of a Cracked Cardan Shaft

Experiments using a four-bearing spin bench supporting two connected shafts (one of them exhibited a transverse breathing crack) were conducted to validate the simulation presented in Sections 4.1 and 4.2. The reduced diameter of the driveshaft (Section 3.3) has been designed so that at the reduced area, the crack would not propagate while the driveshaft is operating (Figure 5a) and at least one critical speed is in the 0–2000 rpm rotating range.

Following the measurement of the high stresses in the cracked driveshaft, an assessment of the possibility of identifying a transverse crack by vibration monitoring was conducted and an estimate was made of the depth of the crack that could be detected as presented in Figure 9.

![Figure 8](image_url)

**Figure 8.** Experimental response of the lateral-torsional Cardan shaft: (a,b) lateral deflection of shaft 1 and shaft 2, and (c,d) frequency spectrum of the drive and driven shafts.

The dynamic response obtained experimentally during start-up of a secondary shaft disc M_2 was clearly affected by the breathing crack but only at low frequencies. Time-domain response contained many tighten harmonics (Figure 9b); however, it may be seen that the amplitude of output time response of secondary shaft-disc M_2 was lower compared with the case shown in Figure 9a. The crack on the primary shaft M_1 gradually affected the disturbance of the shaft-disc M_2 and the appearance frequency value of the harmonics of the second shaft increased, which corresponded to a reduction of the maximum periods of the movement due to the crack of the main shaft. During the primary shaft deflection, a split at resonance occurred as a result of the crack impact (Figure 9a). For the multi-fault condition, the frequency response (extracted from the signals in Figures 8c and 9c,d) provided useful information about the resonance frequency of the rotor at the vicinity of its first critical speed. The behaviour of the cracked system during this period was highly non-linear, and standard features of a cracked shaft manifested by the occurrence
of a vibration component of sup-harmonic peaks at low amplitude were observed. The intensity of the sub-synchronous frequencies increased significantly, and its characteristics were found on the second shaft as well, thus destabilising the system. This observation has been recently reported for a coupled lateral and rigid-rotor rotation with neglected torsional deflection of the shaft [21]. Extraction of features of the secondary shaft M2 contained harmonics of the main resonance phenomenon in a low-frequency amplitude, which can be used as a good indicator of the existence of cracks. However, the contribution of the crack is always present, the parametric nonlinearity of the Hooke’s joint affects the total non-linear response and more particularly on the super-harmonic orders $n \times$, which are directly proportional to the intensity of the rotor amplitudes. The evolution of orders $2 \times$ and $3 \times$ as a function of the speed of rotation of the rotor clearly showed peaks at the passages of sub-harmonic resonances (Figure 9c,d). This observation constitutes an essential point for the prediction of the rotor crack.

![Figure 9](image_url)

**Figure 9.** Experimental response of lateral-torsional coupled Cardan shaft with a crack: (a,b) vertical deflection of drive and driven shaft and (c,d) frequency spectrum of shaft 1 and shaft 2.
In conclusion, the use of super-harmonic order tracking makes it possible to identify the presence of a crack regardless of the intensity of the unbalance compared with the non-linear effects due to the presence of a rotating crack. Knowing, therefore, that a crack strongly influences the dynamic behaviour of a rotor, and that noise influences the analysis of the data collected, a more in-depth analysis is necessary because of the coupling joint and the speed ratio existing between the two rotors. The experimental synchrosqueezing technique, therefore, modifies the non-linear dynamic response of the cracked rotor in order to rule on the possibility of quickly detecting the presence of a crack from the influences of the energy concentration in the two lateral directions. As described in the next section, the response of the Cardan shaft system accounting for masses unbalances and transverse crack was experimentally conducted to characterise the highly oscillated feature using the IF and NWSST extracted features.

4.5. Experimental Verification and Fault Diagnosis Using Synchrosqueezing Technique

The transient fluctuation induced by the oscillating driveshaft with a transverse crack has been discussed experimentally in a previous Section 4.4.

To enhance the time and frequency resolution of the FFT, the synchrosqueezing transform methods were applied to extract the amplitude and instantaneous frequency of the signal (Figures 10–13).

One of the remaining problems is the search of an adequate non-destructive diagnostic technique to identify the parameters of cracks by relating to the modifications of the modal behaviour of the considered structure (frequencies and specific deformities). Thus, all the experimentally collected results and data were transported and associated with a program developed in the MATLAB to extract the IF estimation and the NWSST response of the system. The estimated IF and the NWSST representation of the basic harmonic is presented in Figures 10–13. The results firstly indicated that the IF and NWSST methods are the least sensitive to noise. When the crack effect occurred, both the extracted IF of the response in the lateral direction of the shafts oscillated periodically near $t = 55$ ms (Figure 10c,d), which is the fundamental harmonic frequency. However, the structure of the estimated IF and the NWSST representation were not as organized as the obtained numerical results. The fluctuation of the flexible Hooke’s joint exerted a significant influence on the system dynamics (Figures 11 and 13). The excitations of the driveshaft at frequencies of $2\omega, \ldots, 6\omega$ were very small in amplitude, whereas the excitations of the driven shaft accompanied by frequencies of $4\omega, \ldots, 8\omega$ showed conspicuous magnitudes (Figure 11c,d) compared with those in Figure 13c,d. The magnitudes of the unbalanced driven shaft excitations were, therefore, independent of the joint angle but highly dependent on the effects of the model parameters, such as rotational speed ratio, mass eccentricity, and inter-shaft connection. A comparison of these experimental results with the simulation has shown good agreement with results obtained in Section 5.2. Consequently, the effectiveness of the application of the NWSST method on the denoised signal facilitated the extraction of the highly oscillated IF for the cracked and interconnected rotor system. The existence of a Hooke’s joint on an unbalanced and cracked driveshaft gave a complex, driven shaft signal. Moreover, the oscillation feature of the IF also validated that the highly oscillated IF could qualitatively reveal the fluctuating rule of the transient stiffness for a cracked rotor system. Because the oscillation of the driven shaft is directly influenced by the flexible Hooke’s joint and the cracked driveshaft, the oscillated IF and the NWSST may be applied to qualitatively describe the features of the crack. Therefore, the oscillation feature of the IF can be considered as an index to reflect the highly oscillated features of the instantaneous stiffness for a cracked rotor system, as well as to monitor the occurrence of the crack, and the nonlinear coupling of factors needs to be always considered and explored in power transmission.

In summary, the qualitative analysis results indicated that the Hooke’s joint, when it is combined with the time-varying stiffness induced by transverse crack, significantly affects the system response. From the TFR, the experimental results validated that noise can be
decreased effectively, and the flexible joint is the original cause of the frequency-modulated feature of the dynamic response for a cracked rotor system. Therefore, the highly oscillated features of IF and the fluctuation of the NWSST can characterize the time-varying stiffness qualitatively and can serve as a monitoring index to detect a crack in rotating shafts.

Figure 10. Experimental instantaneous frequency estimation of (a) unbalanced drivshaft, (b) unbalanced driven shaft.

Figure 11. Experimental instantaneous frequency of (a) cracked unbalanced drivshaft, (b) unbalanced driven shaft.

Figure 12. 3D experimental NWSST of (a) unbalanced driveshift, (b) unbalanced driven shaft.
showed conspicuous magnitudes (Figure 11c,d) whereas the excitations of the driven shaft ac-

In this paper, the vibration behaviour of a large class of machines such as ground vehicle propeller shaft connected by the Cardan joint or other inter-coupled machines with primary and secondary rotary shaft inertia is described by this simplified equivalent configuration. Numerical and experimental methods study a model for analysing the coupled lateral and torsional vibrations of unbalanced and cracked driveshaft systems inter-connected with a Hooke’s joint. The application of the classical techniques and the time-frequency techniques for monitoring methods of the cracked driveshaft, including the nonlinear perturbation of the flexible Hooke’s joint under several operational conditions, were theoretically analysed and verified through measurements. The results are summarized as follows:

The nonlinear dynamic model constructed from the equations of vehicle dynamics driveline, including 7 DOFs, is established by the Lagrangian formalism. The perturbation factor of the flexible coupling, the breathing crack mechanism and the second-order lateral and torsional vibrations of the coupled shaft system are simulated numerically.
The concept of transmission of motion through a Hooke’s joint should also take into consideration the perturbation generated by the input shaft as the study revealed that the weakness of the cracked driveshaft contributes to the high disturbance of the secondary shaft. This implies that the mixed mode of the rigid and disturbing Hooke’s joint noise should be considered when selecting the Time-Frequency techniques.

The numerical and experimental responses validate that noise and vibration are effectively significant when the intersection angle of the Cardan joint and the torsional and bending vibration of connected components increased. Experimentally, this phenomenon also means that the vibration amplitude transmission of the driveshaft throughout the Hooke’s joint decreases considerably, and thus the flexible coupling fluctuation has as well good damping performance properties which were proved using the IF estimation method.

The applications of the nonlinear synchrosqueezing wavelet transform (NWSST) and IFs, which directly deals with non-stationary response signals, are in good agreement with each other and exhibit the same changing trend as well.

The results simulated by the model ultimately seemed quantitatively close to those obtained by experimentation on a test bench. However, the bench used generated a lot of noise, and the correlation was difficult. In the future, it could be considered to validate the model from another fairly robust test bench in order, on the one hand, to reduce the possible inaccuracies due to the experimental dimension, and on the other hand, to be able to check the validity of taking the support into account.

**Author Contributions:** The topic of this research was conceptualized by B.X.T. and A.A.; formal analysis was performed by B.X.T.; the first version manuscript was prepared by B.X.T.; and this version of the manuscript was read and approved by B.X.T. and A.A., who also substantially contributed to it. Both authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Acknowledgments:** This work is based on the research supported in part by the Vaal University of Technology (VUT), South Africa, through the Department of Higher Education and Training University Capacity Development Grant.

**Conflicts of Interest:** The authors declare no conflict of interest.

**Nomenclature**

| Symbol | Description |
|--------|-------------|
| $\beta$ | Angle between the shafts |
| $\mu$ | Perturbation parameter |
| $\mathbf{K}_e$ | Velocity vector of the unbalanced mass |
| $e_{(j)}$ | Unbalance eccentricity |
| $\delta$ | Local crack depth |
| $a$ | Inner radius of the rotor |
| $2b$ | Crack width |
| $L$ | Length of the two shafts |
| $Q$ | Coriolis forces and total rotor kinetic energy |
| $G_{Total}$ | Total rotor kinetic energy |
| $A_1$ | Uncracked cross-sectional area |
| $A_c$ | Cracked cross-sectional area |
| $C_e$ | Centroid location |
| $I$ | Moment of inertia of the rotor section |
| $k_{crack}$ | Cracked shaft stiffness |
| $x^m, y^m$ | Motor coordinate system |
| $x, y$ | Rotating coordinate system |
| $\theta_1; \theta_2$ | Angular displacement of the drive and driven shaft |
| $\psi_1; \psi_2$ | Shafts torsional deformation angle |
| $\zeta_l; \zeta_T$ | Lateral and torsional modes damping ratio |
| $V, D$ | System elastic strain energy and Rayleigh’s dissipation |
| $J_i (i = 1, 2, 3)$ | Moment of inertia of rotor shaft about inertia coordinate |
Appendix A

Terms used in the derived equations of motion.

Appendix A.1. Perturbation Function between Shaft Input and Output

One of the critical parameters governing the rotors system is a mixed and fluctuating mode characteristic of Hooke’s joint disturbance. Its analysis was needed to assess the influence of variations of the input variables (input angle, breathing crack, and fluid properties) on the response and guide the selection of the critical rotor parameters.

Let the net displacement of the driven shaft be expressed as:

\[ \theta_2 = (\theta_1 + \psi_1) - \mu(\theta_1 + \psi_1) \]  

(A1)

where \( \mu \) is a small perturbation parameter that depends on the input angular speed \( \theta \) and the torsional deflection angle \( \psi_1 \). Consistent with the assumptions and considerations in Section 2.1., the kinematic relationship between the output \( \theta_2 \) and input \( \theta_1 \) is given by:

\[ \tan \theta_2 = \cos \beta \tan \theta_1 \]  

(A2)

where \( \beta \) is the angle of inclination of the secondary shaft. Combining Equations (A1) and (A2) yields:

\[ \tan[(\theta_1 + \psi_1) - \mu(\theta_1 + \psi_1)] = \cos \beta \tan \theta_1 \]  

(A3)

and allowing for infinitesimal small-angle approximation on \( \mu(\theta + \psi_1) \) leads to:

\[ \gamma \tan(\theta_1 + \psi_1) = \frac{\tan[(\theta_1 + \psi_1) - \mu(\theta_1 + \psi_1)]}{1 + \mu(\theta_1 + \psi_1) \tan(\theta_1 + \psi_1)} \]  

(A4)

Making \( \mu \) the subject in Equation (A4) gives:

\[ \mu = \frac{(1 - \gamma) \tan(\theta_1 + \psi_1)}{(\theta_1 + \psi_1)(\gamma \tan^2(\theta_1 + \psi_1) + 1)} \]  

(A5)

where \( \gamma = \cos \beta \), within a finite range of \( (\theta + \psi_1) \), the dependent product \( \mu(\theta + \psi_1) \) will be finite, and periodically convergent, as shown in the figures below. In the presented model, governed by Equation (A5), Figure A1 shows that when the angle between the two shafts is less than 10°, there is not much fluctuation in perturbation function and the driveshaft speed cannot be assumed constant as shown in Figure A2.

Appendix A.2. Driveshaft Speed Profil during Operation

The driveshaft speed profile used to drive the system to a constant maximum rotational speed of 1500 rpm in a starting period is shown in Figure A2.
Appendix B

Derivation of the governing equations by applying the Lagrangian formalism in Equation (22), a system of seven order inertia equations was obtained with nonlinear excitation correspondingly to the coupling elastic deformations across the Hooke’s joint in the following form:

\[ \text{NL}_{\omega_1} = -m_{1u}((c_{1u} \cos \theta_1 - c_{1u} \sin \theta_1)X_1 \psi_1 - m_{1u}((e_{1u1} + e_{1u1} \psi_1) \cos \theta_1 - (e_{1u1} + e_{1u1} \psi_1) \sin \theta_1)X_1 \psi_1 \\
-m_{1u}((c_{1u} \sin \theta_1 + e_{1u} \cos \theta_1) Y_1 \psi_1 - m_{1u}((e_{1u} + e_{1u} \psi_1) \cos \theta_1 + (e_{1u} + e_{1u} \psi_1) \sin \theta_1)Y_1 \psi_1 \\
+2m_{1u}e_{1u}^2 \psi_1 \psi_1 \beta_1 + 2m_{2u}e_{2u} \psi_2 \psi_2 (g-1)^2 + m_{2u}((g-1)(e_{2u} \cos \theta_2 - e_{2u} \sin \theta_2)X_2 \psi_2 \\
+ m_{2u}((g-1)(e_{2u} \sin \theta_2 + e_{2u} \sin \theta_2)Y_2 \psi_2 \\
-2m_{2u}((g-1)((e_{2u} - e_{2u} \psi_2) \cos \theta_2 - (e_{2u} + e_{2u} \psi_2) \sin \theta_2)X_2 \psi_2 \\
-2m_{2u}((g-1)((e_{2u} - e_{2u} \psi_2) \cos \theta_2 + (e_{2u} + e_{2u} \psi_2) \sin \theta_2)Y_2 \psi_2) \right) \]  \tag{A6}

\[ F_{\omega_1} = (g-1)(f_{d2} + m_{2u}e_{2u}^2(1 + \psi_2^2))g_1 \beta_1 - (f_{d2} + m_{2u}e_{2u}^2)g_1 \psi_2 \\
+ m_{2u}((e_{2u} + e_{2u} \psi_2) \cos \theta_2 + (e_{2u} - e_{2u} \psi_2) \sin \theta_2)Y_2 \beta_1 \\
- m_{2u}((e_{2u} - e_{2u} \psi_2) \cos \theta_2 - (e_{2u} + e_{2u} \psi_2) \sin \theta_2)X_2 \beta_1 \]  \tag{A7}

\[ \text{NL}_{X_1} = -2m_{1u}((e_{1u} \cos \theta_1 - e_{1u} \sin \theta_1) \psi_1 \beta_1 \]  \tag{A8}

\[ C_{X_1} = -m_{1u}e_{1u}^2((e_{1u1} + e_{1u1} \psi_1) \cos \theta_1 - (e_{1u1} + e_{1u1} \psi_1) \sin \theta_1) \tag{A9} \]

\[ \text{NL}_{Y_1} = -2m_{1u}((e_{1u} \cos \theta_1 + e_{1u} \sin \theta_1) \psi_1 \beta_1 \]  \tag{A10}

\[ C_{Y_1} = -m_{1u}((e_{1u} + e_{1u} \psi_1) \cos \theta_1 + (e_{1u} + e_{1u} \psi_1) \sin \theta_1) \beta_1 \psi_1 t^2 \]  \tag{A11}

\[ \text{NL}_{\psi_2} = m_{2u}((g-1)((e_{2u} \cos \theta_2 - e_{2u} \sin \theta_2)X_2 \beta_1 + (e_{2u} \cos \theta_2 + e_{2u} \sin \theta_2)Y_2 \beta_1) \]  \tag{A12}

\[ F_{\psi_2} = -(f_{d2} + m_{2u}e_{2u}^2)g_1 \beta_1 \]  \tag{A13}

\[ \text{NL}_{X_2} = 2m_{2u}((g-1)((e_{2u} \cos \theta_2 - e_{2u} \sin \theta_2) \beta_1 \psi_2 \]  \tag{A14}
\[ F_{x_2} = m_2 a \left[ (c_{2y_2} + c_{2y_2}y_2) \cos \theta_2 + (c_{2y_2} - c_{2y_2}y_2) \sin \theta_2 \right] \dot{g}_2 \dot{\theta}_1 + m_2 a \left[ (c_{2y_2} + c_{2y_2}y_2) \cos \theta_2 + (c_{2y_2} - c_{2y_2}y_2) \sin \theta_2 \right] \dot{g}_1 \dot{\theta}_1 \]  
\[ C_{x_2} = m_2 a \left( g - 1 \right) \left[ (c_{2y_2}y_2 - c_{2y_2}y_2) \cos \theta_2 + (c_{2y_2} + c_{2y_2}y_2) \sin \theta_2 \right] \dot{g}_2 \dot{\theta}_1 \]  
\[ N_{y_2} = 2m_2 a \left( g - 1 \right) \left( c_{2y_2} \cos \theta_2 + c_{2y_2} \sin \theta_2 \right) \dot{\theta}_2 \dot{\psi}_2 \]  
\[ F_{x_2} = -m_2 a \left[ (c_{2y_2} - c_{2y_2}y_2) \cos \theta_2 - (c_{2y_2} + c_{2y_2}y_2) \sin \theta_2 \right] \dot{g}_2 \dot{\theta}_1 \]  
\[ C_{x_2} = m_2 a \left( g - 1 \right)^2 \left[ (c_{2y_2} + c_{2y_2}y_2) \cos \theta_2 + (c_{2y_2} - c_{2y_2}y_2) \sin \theta_2 \right] \dot{g}_2 \dot{\theta}_1 \]  
\[ \frac{\partial \bar{r}}{\partial \theta_1} = \bar{r}_1 = -\frac{2n}{\gamma n^2 + 1} \left[ (3 - \gamma) \gamma n^2 + 3 \gamma - 1 \right] \]  
\[ \frac{\partial \bar{r}}{\partial \psi_1} = \bar{r}_1 = \frac{a(n^2 + 3) \gamma n^2 + 1}{\gamma n^2 + 1} \left[ (3 - \gamma) \gamma n^2 + 3 \gamma - 1 \right] \]  
\[ \frac{\partial \mu}{\partial \bar{r}} = \frac{\partial \mu}{\partial \bar{r}} + \left( \frac{\Gamma - \mu}{\bar{r}_1 + \psi_1} \right) \]  
\[ \frac{\partial \mu}{\partial \psi} = \frac{\partial \mu}{\partial \psi} \]  
\[ \frac{\partial \mu}{\partial \psi} = \frac{\partial \mu}{\partial \psi} \]  

For brevity, the expression \( g_1, g_2 \) in the nonlinear terms have been compactly written as:

\[ \frac{\partial \bar{r}}{\partial \theta_1} = \frac{\partial \bar{r}}{\partial \theta_1} = -\frac{2n}{\gamma n^2 + 1} \left[ (3 - \gamma) \gamma n^2 + 3 \gamma - 1 \right] \]  
\[ \frac{\partial \bar{r}}{\partial \psi_1} = \frac{\partial \bar{r}}{\partial \psi_1} = \frac{a(n^2 + 3) \gamma n^2 + 1}{\gamma n^2 + 1} \left[ (3 - \gamma) \gamma n^2 + 3 \gamma - 1 \right] \]

References

1. Xia, Y.; Pang, J.; Yang, L.; Zhao, Q.; Yang, X. Nonlinear numerical and experimental study on the second-order torsional and lateral vibration of driveline system connected by Cardan joint. J. Vib. Control 2019, 25, 540–551. [CrossRef]
2. Asokanthan, S.F.; Hwang, M.-C. Torsional Instabilities in a System Incorporating a Hookes Joint. J. Vib. Acoust. 1996, 118, 368–374. [CrossRef]
3. Asokanthan, S.F.; Wang, X.H. Characterization of torsional instabilities in a Hookes joint driven system via maximal Lyapunov exponents. J. Sound Vib. 1996, 194, 83–91. [CrossRef]
4. Asokanthan, S.F.; Meehan, P.A. Non-linear vibration of a torsional system driven by a Hookes’ joint. J. Sound Vib. 2000, 233, 297–310. [CrossRef]
5. Bulut, G.; Parlar, Z. Dynamic stability of a shaft system connected through a Hookes’ joint. Mech. Mach. Theory 2011, 46, 1689–1695. [CrossRef]
6. Savković, M.; Gašić, M.; Petrović, D.; Zdravković, N.; Pljakić, R. Analysis of the drive shaft fracture of the bucket wheel excavator. Eng. Fail. Anal. 2012, 20, 105–117. [CrossRef]
7. Bulut, G. Dynamic stability analysis of torsional vibrations of a shaft system connected by a Hookes’ joint through a continuous system model. J. Sound Vib. 2014, 333, 3691–3701. [CrossRef]
8. Mazzei, A.J.; Scott, R.A. Effects of internal viscous damping on the stability of a rotating shaft driven through a universal joint. J. Sound Vib. 2003, 265, 863–885. [CrossRef]
9. Glavardanov, V.B.; Maretic, R.B.; Grahovac, N.M. Buckling of a twisted and compressed rod supported by Cardan joints. Eur. J. Mech. A Solids 2009, 28, 131–140. [CrossRef]
10. Asokanthan, S.F.; Wang, X.H.; Baik, S.H. Torsional vibration control of a Hookes’ joint driven flexible shaft system. In Asme 2014 International Mechanical Engineering Congress and Exposition; American Society of Mechanical Engineers Digital Collection: New York, NY, USA, 2014.
11. Browne, M.; Palazzolo, A. Super harmonic nonlinear lateral vibrations of a segmented driveline incorporating a tuned damper excited by non-constant velocity joints. J. Sound Vib. 2009, 323, 334–351. [CrossRef]
12. Patel, T.H.; Darpe, A.K. Experimental investigations on vibration response of misaligned rotors. Mech. Syst. Signal Process. 2009, 23, 2236–2252. [CrossRef]
13. Vesali, F.; Rezvani, M.A.; Kashi, M. Dynamics of universal joints, its failures and some propositions for practically improving its performance and life expectancy. J. Mech. Sci. Technol. 2012, 26, 2439–2449. [CrossRef]
14. Yao, Z.; Jingrui, Z. The imaging stability enhancement of optical payload using multiple vibration isolation platforms. J. Vib. Control 2015, 21, 1848–1865. [CrossRef]
15. Tchomeni, B.X.; Sozinando, D.F.; Alugongo, A. Influences of Hydrodynamic Forces on the Identification of the Rotor-Stator-Rubbing Fault in a Rotating Machinery. Int. J. Rotating Mach. 2020, 2020, 8816191. [CrossRef]
16. Dutta, S.; Choi, S.B. Control of a shimmy vibration in vehicle steering system using a magneto-rheological damper. J. Vib. Control 2018, 24, 797–807. [CrossRef]
17. Sudhakar, G.N.D.S.; Sekhar, A.S. Coupling misalignment in rotating machines: Modelling, effects and monitoring. Noise Vib. World 2009, 40, 17–39. [CrossRef]
18. Jayaswal, P.; Wadhwni, A.K.; Mulchandani, K.B. Machine fault signature analysis. Int. J. Rotating Mach. 2008, 2008, 583982. [CrossRef]
19. Alugongo, A.A. A nonlinear torsional vibration model of a propeller shaft with a crack-induced parametric excitation and a Hookes joint type of kinematic constraint. In IEEE Africon’11; IEEE: Piscataway, NJ, USA, 2011; pp. 1–7.
20. Alugongo, A.A. Parametric excitation and wavelet transform analysis of a ground vehicle propeller shaft. J. Vib. Control 2014, 20, 280–289. [CrossRef]
21. Tchomeni, B.X.; Alugongo, A. Theoretical and experimental analysis of an unbalanced and cracked cardan shaft in the vicinity of the critical speed. Math. Models Eng. 2020, 6, 34–49. [CrossRef]
22. Hu, Y.; Zhang, B.; Tan, A.C. Acceleration signal with DTCWPT and novel optimize SNR index for diagnosis of misaligned cardan shaft in high-speed train. *Mech. Syst. Signal Process.* 2020, 140, 106723. [CrossRef]

23. Al-bedoor, B.O. Transient torsional and lateral vibrations of unbalanced rotors with rotor-to-stator rubbing. *J. Sound Vib.* 2000, 229, 627–645. [CrossRef]

24. Alugongo, A.A. Parametric vibration of a cardan shaft and sensitivity analysis. In Proceedings of the World Congress on Engineering and Computer Science, London, UK, 4–6 July 2018; Volume 2.

25. Al-Shudeifat, M.A.; Butcher, E.A. New breathing functions for the transverse breathing crack of the cracked rotor system: Approach for critical and subcritical harmonic analysis. *J. Sound Vib.* 2011, 330, 526–544. [CrossRef]

26. Guo, C.; Al-Shudeifat, M.A.; Yan, J.; Bergman, L.A.; McFarland, D.M.; Butcher, E.A. Application of empirical mode decomposition to a Jeffcott rotor with a breathing crack. *J. Sound Vib.* 2013, 332, 3881–3892. [CrossRef]

27. Jiang, Q.; Suter, B.W. Instantaneous frequency estimation based on synchro-squeezing wavelet transform. *Signal Process.* 2017, 138, 167–181. [CrossRef]

28. Wang, S.; Chen, X.; Wang, Y.; Cai, G.; Ding, B.; Zhang, X. Nonlinear squeezing time–frequency transform for weak signal detection. *Signal Process.* 2015, 113, 195–210. [CrossRef]

29. Chen, H.; Lu, L.; Xu, D.; Kang, J.; Chen, X. The synchrosqueezing algorithm based on generalized S-transform for high-precision time-frequency analysis. *Appl. Sci.* 2017, 7, 769. [CrossRef]

30. Hazra, B.; Sadhu, A.; Narasimhan, S. Fault detection of gearboxes using synchro-squeezing transform. *J. Vib. Control* 2017, 23, 3108–3127. [CrossRef]

31. Hu, H.Y.; Bao, E.; Kang, J. Rotor complex fault vibration analysis based on frequency heterodyne EMD method. *Adv. Mater. Res.* 2011, 199, 845–849. [CrossRef]

32. Wang, Z.C.; Ren, W.X.; Chen, G. Time-frequency analysis and applications in time-varying/nonlinear structural systems: A state-of-the-art review. *Adv. Struct. Eng.* 2018, 21, 1562–1584. [CrossRef]

33. Barkat, B. Instantaneous frequency estimation of nonlinear frequency-modulated signals in the presence of multiplicative and additive noise. *Signal Process.* 2001, 49, 2214–2222. [CrossRef]