Two purity theorems and the Grothendieck-Serre conjecture concerning principal $G$-bundles

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Abstract. The main results of the paper are two purity theorems for reductive group schemes over regular local rings containing a field. Using these two theorems a well-known Grothendieck-Serre conjecture on principal bundles is reduced to the simply-connected case. We point out that the mentioned reduction is one of the major steps in the proof of the conjecture that the author published in another work.

Bibliography: 25 titles.

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§ 1. Main results

Recall ([3], Exp. XIX, Definition 2.7) that an $R$-group scheme $G$ is called reductive (semi-simple, simple) if it is affine and smooth as an $R$-scheme and if, moreover, for each ring homomorphism $s: R \rightarrow \Omega(s)$ to an algebraically closed field $\Omega(s)$, its scalar extension $G_{\Omega(s)}$ is a connected and reductive (semi-simple, simple, respectively) algebraic group over $\Omega(s)$.

We stress that all the groups $G_{\Omega(s)}$ are connected. Observe also that the class of reductive group schemes contains the class of semi-simple group schemes which in turn contains the class of simple group schemes. This notion of a simple $R$-group scheme coincides with the notion of a simple semi-simple $R$-group scheme from Demazure-Grothendieck [3], Exp. XIX, Definition 2.7 and Exp. XXIV, §5.3. Throughout this paper $R$ denotes an integral noetherian domain and $G$ denotes a reductive $R$-group scheme, unless explicitly stated otherwise.

After the pioneering articles [2] and [21] on purity theorems for algebraic groups, various versions of purity theorems were proved in [1], [18], [25] and [13]. The most general result in the so-called constant case was given in [25], §3.3. The papers [18] and [25] contain results for the nonconstant case and they are proved even if the base field is finite. However they only consider specific examples of algebraic scheme morphisms $\mu: G \rightarrow C$ to a torus $C$. The following theorem covers all the mentioned results of this shape and it looks like the final one.

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Theorem 1.1. Let $k$ be a field. Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a $k$-smooth irreducible affine $k$-variety $X$. Let $K = k(X)$. Let

$$\mu: G \to C$$

be a smooth $\mathcal{O}$-morphism of reductive $\mathcal{O}$-group schemes, with a torus $C$. Suppose additionally that the kernel of $\mu$ is a reductive $\mathcal{O}$-group scheme. Then the sequence

$$C(\mathcal{O})/\mu(G(\mathcal{O})) \to C(K)/\mu(G(K)) \xrightarrow{\bigoplus_r \mu(G(K))} \prod_p C(K)/[C(\mathcal{O}_p) \cdot \mu(G(K))]$$

(1)

is exact, where $p$ runs over the height 1 primes of $\mathcal{O}$ and each $r_p$ is the natural map (the projection to the factor group).

If the field $k$ is infinite, this is Theorem 1.0.1 (Theorem A) in [13].

Let $k$, $\mathcal{O}$ and $K$ be as in Theorem 1.1. Let $G$ be a semi-simple $\mathcal{O}$-group scheme. Let $i: Z \to G$ be a closed subgroup scheme of the centre $\text{Cent}(G)$ of $G$. It is known that $Z$ is of multiplicative type. Let $G' = G/Z$ be the factor group and $\pi: G \to G'$ be the projection. It is known that $\pi$ is finite surjective and strictly flat. Thus the sequence of $\mathcal{O}$-group schemes

$$\{1\} \to Z \xrightarrow{i} G \xrightarrow{\pi} G' \to \{1\}$$

(2)

induces an exact sequence of group sheaves in the fppt-topology. Thus for every $\mathcal{O}$-algebra $R$ the sequence (2) gives rise to a boundary operator

$$\delta_{\pi,R}: G'(R) \to H^1_{\text{fppt}}(R, Z).$$

(3)

One can check that it is a group homomorphism (cf. [22], Ch. II, § 5.6, Corollary 2). Set

$$\mathcal{F}(R) = H^1_{\text{fppt}}(R, Z)/\text{Im}(\delta_{\pi,R}).$$

(4)

Theorem 1.2. Let $k$, $\mathcal{O}$ and $K$ be as in Theorem 1.1. Then the sequence

$$\frac{H^1_{\text{fppt}}(\mathcal{O}, Z)}{\text{Im}(\delta_{\pi,\mathcal{O}})} \to \frac{H^1_{\text{fppt}}(K, Z)}{\text{Im}(\delta_{\pi,K})} \xrightarrow{\bigoplus_p \text{res}_p} \frac{H^1_{\text{fppt}}(K, Z)}{H^1_{\text{fppt}}(\mathcal{O}_p, Z) + \text{Im}(\delta_{\pi,K})}$$

(5)

is exact, where $p$ runs over the height 1 primes of $\mathcal{O}$ and $\text{res}_p$ is the natural map (the projection to the factor group).

If the field $k$ is infinite, this is Theorem 1.0.2 in [13].

A well-known conjecture due to Serre and Grothendieck (see [22], Remarque, p. 31, [6], Remarque 3, pp. 26, 27, and [7], Remarque 1.11.a) asserts that, given a regular local ring $R$ and its field of fractions $K$ and given a reductive group scheme $G$ over $R$, the map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$$

induced by the inclusion of $R$ into $K$, has a trivial kernel. A survey paper [14] on the topic was published in the proceedings of the 2018 ICM. The Grothendieck-Serre conjecture on principal bundles over a semi-local regular ring containing a field was proved in [17]. That proof is based heavily on Theorem 1.3 stated below and proved in the present paper. We derive Theorem 1.3 from Theorems 1.1 and 1.2.
Theorem 1.3. Let $k$ be a field. Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a $k$-smooth irreducible affine $k$-variety $X$. Let $K = k(X)$. Assume that for all semi-simple simply connected reductive $\mathcal{O}$-group schemes $H$ the pointed set map

$$H^1_{\text{ét}}(\mathcal{O}, H) \to H^1_{\text{ét}}(k(X), H)$$

induced by the inclusion of $\mathcal{O}$ into its fraction field $k(X)$ has a trivial kernel. Then for any reductive $\mathcal{O}$-group scheme $G$ the pointed set map

$$H^1_{\text{ét}}(\mathcal{O}, G) \to H^1_{\text{ét}}(K, G)$$

induced by the inclusion of $\mathcal{O}$ into its fraction field $k(X)$ has a trivial kernel.

If the base field $k$ is infinite, then this theorem is proved in [13], Theorem 1.0.3. Theorem 1.0.3 in [13] is used in [5] for the proof of the conjecture in the case of regular local rings containing an infinite field.

Remark 1.4. The proof of the latter theorem is subdivided into two steps. First, given a semi-simple $\mathcal{O}$-group scheme $G$ we prove that the Grothendieck-Serre conjecture holds for an $\mathcal{O}$-group scheme $G$, provided it holds for its simply-connected cover $G^{sc}$ and all inner forms of that simply-connected $\mathcal{O}$-group scheme $G^{sc}$.

Second, given a reductive $\mathcal{O}$-group scheme $G$ we prove that the Grothendieck-Serre conjecture holds for $G$, provided it holds for the derived $\mathcal{O}$-group scheme $G_{\text{der}}$ of $G$ and for all inner forms of $G_{\text{der}}$.

The most basic obstacle in any attempt to prove Theorem 1.1 is as follows. Suppose the group $G$, the torus $C$ and the morphism $\mu$ are defined over the field $k$. Even under these additional assumptions we are not able to say that the functor $F(R) := C(R)/\mu(G(R))$ is a presheaf with transfers in the sense of Voevodsky [24] or in any other weaker sense. If this were the case, then Theorem 1.1 could be derived in a standard way. Why are we not able to check that the functor $F$ is a presheaf with transfers in a weak sense? Because to check this is more or less the same as to check that the morphism $\mu$ from Theorem 1.1 satisfies the norm principle at least for finite separable field extensions. However, the latter is a big open question, which is not even addressed in the present paper.

Here is our approach. We use transfers for the functor $R \mapsto C(R)$, but we do not use at all the norm principle for the homomorphism $\mu: G \to C$. Given an element $\xi \in C(K)$ such that its image $\bar{\xi} \in F(K)$ is an $\mathcal{O}$-unramified element, one can find a finite correspondence of the form

$$\mathbb{A}^1_{\mathcal{O}} \leftarrow \mathcal{X}' \xrightarrow{\mathbb{Q}_X} X$$

(see the diagram (10)) and use it in the ‘constant group’ case to write down a good candidate $\bar{\xi}_{\mathcal{O}} \in F(\mathcal{O})$ (see (17)) for a lift of the element $\bar{\xi}$ to $F(\mathcal{O})$. In general, since the group-scheme $G$ does not come from the ground field, we need to equate two of its pull-backs $\left(\text{pr}_{\mathcal{O}} \circ \sigma\right)^*(G)$ and $q_X^*(G)$ over $\mathcal{X}'$. Here $\text{pr}_{\mathcal{O}}: \mathbb{A}^1_{\mathcal{O}} \to \text{Spec}(\mathcal{O})$ is the projection. We need to do the same with the torus $C$. Due to these requirements our construction of the finite correspondence (6) and of a good candidate $\bar{\xi}_{\mathcal{O}} \in F(\mathcal{O})$ (see (18)) is quite involved.
It is especially involved in the case of a finite base field. Of course, we use Bertini-type results which are due to Poonen. Even this does not resolve all the difficulties. Nevertheless, there is a construction of the desired finite correspondence, as for the ‘constant group’ case, so for the general case. It is done below in Theorem 4.1.

The finite surjective morphism $\sigma$ of the $O$-schemes in Theorem 4.1 has the following property: for the corresponding fraction field extension $K(u) \subset \mathcal{K}$ the element

$$\zeta_u := N_{\mathcal{K}/K(u)}(q_X^*(\xi)) \in C(K(u))$$

is such that its image $\overline{\zeta}_u \in \mathcal{F}(K(u))$ is $K[u]$-unramified. The latter yields that the element $\overline{\zeta}_u \in \mathcal{F}(\mathcal{O})$ is indeed a lift of the element $\overline{\xi} \in \mathcal{F}(K)$. Details are given in §7.

It seems plausible to expect a purity theorem in the following context. Let $R$ be a regular local ring. Let $\mu: G \to C$ be a smooth morphism of reductive $R$-group schemes with an $R$-torus $C$. Let $\mathcal{F}$ be the covariant functor from the category of commutative $R$-algebras to the category of abelian groups given by $S \mapsto C(S)/\mu(G(S))$. Then $\mathcal{F}$ should satisfy purity for $R$.

The article is organized as follows. In §2 a theorem about equating certain group scheme morphisms is proved. In §§3 and 4 geometric results from [15] are used to prove stronger versions of some results from [16]. In §5 the construction of norm maps is recalled. In §6 groups of unramified elements and specialization maps are defined. A homotopy invariance theorem for the group of unramified elements is recalled (see Theorem 6.5). It is proved that in certain cases the norm map takes a specific unramified element to an unramified one (see Lemma 6.7). In §7 Theorems 1.1 and 1.2 are proved. Finally, in §8 Theorem 1.3 is proved.

Throughout the paper $k$ is a base field.

§2. Equating group scheme morphisms

Let $S$ be a regular semi-local irreducible scheme such that the residue fields at all its closed points are finite over $k$. Let $\mu_1: G_1 \to C_1$ and $\mu_2: G_2 \to C_2$ be two smooth $S$-group scheme morphisms with $S$-tori $C_1$ and $C_2$. Suppose that $G_1$ and $G_2$ are reductive $S$-group schemes which are forms of each other and suppose that $C_1$ and $C_2$ are forms of each other. Let $T \subset S$ be a connected nonempty closed subscheme of $S$, and $\varphi: G_1|_T \to G_2|_T$ and $\psi: C_1|_T \to C_2|_T$ be $T$-group scheme isomorphisms. By [16], Theorem 4.1, there exists a finite étale morphism $\overline{S} \xrightarrow{\pi} S$ with an irreducible scheme $\overline{S}$ and a section $\delta: T \to \overline{S}$ of $\pi$ over $T$ and $\overline{S}$-group scheme isomorphisms

$$\Phi: G_1,\overline{S} \to G_2,\overline{S} \quad \text{and} \quad \Psi: C_1,\overline{S} \to C_2,\overline{S}$$

such that $\delta^*(\Phi) = \varphi$ and $\delta^*(\Psi) = \psi$. For these $\overline{S}$, $T$, $\delta$, $\Phi$ and $\Psi$ the following result holds.
Theorem 2.1. Let $S$ be the above regular semi-local irreducible scheme. Let $T \subset S$ be the above connected nonempty closed subscheme of $S$. Suppose that the above morphisms $\varphi$ and $\psi$, $(\mu_1|_T)$ and $(\mu_2|_T)$ are such that the diagram

$$
\begin{array}{ccc}
G_1|_T & \xrightarrow{\varphi} & G_2|_T \\
\mu_1|_T & \downarrow & \downarrow \mu_2|_T \\
C_1|_T & \xrightarrow{\psi} & C_2|_T \\
\end{array}
$$

(7)

commutes. Then $\mu_{2,\bar{S}} \circ \Phi = \Psi \circ \mu_{1,\bar{S}} : G_{1,\bar{S}} \to C_{2,\bar{S}}$ as $\bar{S}$-group scheme morphisms.

Proof. Recall that $\mu_r$ can be naturally presented as a composition

$$G_r \xrightarrow{\text{can}_r} \text{Corad}(G_r) \xrightarrow{\pi_r} C_r.$$ 

Since $\text{can}_2, \bar{S} \circ \Phi = \text{Corad}(\Phi) \circ \text{can}_1, \bar{S}$, it remains to check that $\pi_{2,\bar{S}} \circ \text{Corad}(\Phi) = \Psi \circ \pi_{1,\bar{S}}$.

The equality $(\mu_2|_T) \circ \varphi = \psi \circ (\mu_1|_T)$ holds by the assumption of the Theorem. It yields the equality $(\pi_2|_T) \circ \text{Corad}(\varphi) = \psi \circ (\pi_1|_T)$. The equality $\pi_{2,\bar{S}} \circ \text{Corad}(\Phi) = \Psi \circ \pi_{1,\bar{S}}$ now follows from [16], Proposition 4.7, since $\bar{S}$ is irreducible.

This proves the theorem.

§ 3. Nice triples and group scheme morphisms

See [12], Definition 3.1, for the definition of a nice triple and see [12], Definition 3.2, for the definition of a morphism between nice triples. These definitions are reproduced in [15], Definitions 3.1 and 3.3. The notion of a special nice triple is given in [15], Definition 3.4. We need an extension of Theorem 3.9 in [15] and Theorem 3.9 in [16].

For this it is convenient to give two definitions in the following set up. Let $k$ be a field and $\Theta$ be the semi-local ring of finitely many closed points on a $k$-smooth irreducible affine $k$-variety $X$. Let $U = \text{Spec}(\Theta)$. Let $(\mathscr{X}, f, \Delta)$ be a special nice triple over $U$ and $G_{\mathscr{X}}$ be a reductive $\mathscr{X}$-group scheme, and let $G_U := \Delta^*(G_{\mathscr{X}})$ and $G_{\text{const}} := q^*_U(G_U)$. Let $\theta : (p' : \mathscr{X}' \to U, f', \Delta') \to (p : \mathscr{X} \to U, f, \Delta)$ be a morphism between nice triples over $U$ (see [15], Definition 3.3). The latter means that $\theta : \mathscr{X}' \to \mathscr{X}$ is an étale morphism of $U$-schemes such that $p' = p \circ \theta$, $\Delta = \theta \circ \Delta'$, $f' = \theta^*(f) \cdot h'$, where $h' \in \Gamma(\mathscr{X}', \Theta_{\mathscr{X}'}).$ The following definition is from [16], Definition 4.1.

Definition 3.1. We say that the above morphism $\theta$ equates the reductive $\mathscr{X}$-group schemes $G_{\mathscr{X}}$ and $G_{\text{const}}$ if there is an $\mathscr{X}'$-group scheme isomorphism $\Phi : \theta^*(G_{\text{const}}) \to \theta^*(G_{\mathscr{X}})$ with $(\Delta')^*(\Phi) = \text{id}_{G_U}$.

Further, let $C_{\mathscr{X}}$ be an $\mathscr{X}$-torus, $C_U := \Delta^*(C_{\mathscr{X}})$ and $C_{\text{const}} := p^*(C_U)$. Let $\mu_{\mathscr{X}} : G_{\mathscr{X}} \to C_{\mathscr{X}}$ be an $\mathscr{X}$-group scheme morphism which is smooth as a scheme morphism. Let $\mu_U = \Delta^*(\mu_{\mathscr{X}})$, and let $\mu_{const} : G_{\text{const}} \to C_{\text{const}}$ be the pull-back of $\mu_U$ to $\mathscr{X}$ by means of $p$. 

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Definition 3.2 (equating morphisms). We say that the morphism $\theta$ equates the reductive $\mathcal{X}$-group scheme morphisms $\mu_{\text{const}}$ and $\mu_\mathcal{X}$ if there are $\mathcal{X}'$-group scheme isomorphisms

$$
\Phi: \theta^*(G_{\text{const}}) \to \theta^*(G_\mathcal{X}) \quad \text{and} \quad \Psi: \theta^*(C_{\text{const}}) \to \theta^*(C_\mathcal{X})
$$

with $(\Delta')^*(\Phi) = \text{id}_{G_U}$, $(\Delta')^*(\Phi) = \text{id}_{G_U}$ and $\theta^*(\mu_\mathcal{X}) \circ \Phi = \Psi \circ \theta^*(\mu_{\text{const}})$. Clearly, if the morphism $\theta$ equates morphisms $\mu_{\text{const}}$ and $\mu_\mathcal{X}$, then it equates $G_{\text{const}}$ with $G_\mathcal{X}$ and $C_{\text{const}}$ with $C_\mathcal{X}$.

Remark 3.3. Let $\rho: (\mathcal{X}'', f'', \Delta'') \to (\mathcal{X}', f', \Delta')$ and $\theta: (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ be morphisms of nice triples over $U$. If $\theta$ equates $\mu_{\text{const}}$ with $\mu_\mathcal{X}$, then $\theta \circ \rho$ equates $\mu_{\text{const}}$ and $\mu_\mathcal{X}$ also.

Theorem 3.4. Let $U$ be as above in this section. Let $(p: \mathcal{X} \to U, f, \Delta)$ be a special nice triple over $U$. Let $G_\mathcal{X}$ be a reductive $\mathcal{X}$-group scheme and $G_U := \Delta^*(G_\mathcal{X})$ and $G_{\text{const}} := p^*(G_U)$. Let $C_\mathcal{X}$ be an $\mathcal{X}$-torus and $C_U := \Delta^*(C_\mathcal{X})$ and $C_{\text{const}} := p^*(C_U)$. Let $\mu_\mathcal{X}: G_\mathcal{X} \to C_\mathcal{X}$ be an $\mathcal{X}$-group scheme morphism which is smooth as a scheme morphism. Let $\mu_U = \Delta^*(\mu_\mathcal{X})$ and $\mu_{\text{const}}: G_{\text{const}} \to C_{\text{const}}$ be the pull-back of $\mu_U$ to $\mathcal{X}$ by means of $p$.

Then there exists a morphism $\theta'': (p'': \mathcal{X}'' \to U, f'', \Delta'') \to (p: \mathcal{X} \to U, f, \Delta)$ between nice triples over $U$ such that

(i) the morphism $\theta''$ equates the reductive $\mathcal{X}$-group scheme morphisms $\mu_{\text{const}}$ and $\mu_\mathcal{X}$;

(ii) the triple $(\mathcal{X}'', f'', \Delta'')$ is a special nice triple over $U$ subject to the conditions (1*) and (2*) from Definition 3.7 in [15].

Proof of Theorem 3.4. Let $U$ be as in the theorem. Let $(\mathcal{X}, f, \Delta)$ be the special nice triple over $U$ as in the theorem. By the definition of a nice triple there exists a finite surjective morphism $\Pi: \mathcal{X} \to \mathbf{A}^1 \times U$ of $U$-schemes. Construction 4.2 in [15] now gives us the data $(\mathcal{X}, \mathcal{Y}, S, T)$, where $\mathcal{X}, \mathcal{Y}$ and $T$ are closed subsets in $\mathcal{X}$, finite over $U$. In particular, they are semi-local. If $y_1, \ldots, y_n$ are all closed points of $\mathcal{Y}$, then $S = \text{Spec}(\mathcal{O}_{\mathcal{X}, y_1, \ldots, y_n})$. Recall that $T = \Delta(U)$.

Further, let $G_U = \Delta^*(G_\mathcal{X})$, $C_U = \Delta^*(C_\mathcal{X})$, $G_{\text{const}} = p^*(G_U)$ and $C_{\text{const}} = p^*(C_U)$ be as in the hypotheses of Theorem 3.4. Finally, let $\varphi: G_{\text{const}}|_T = G_U \to G_U = G_\mathcal{X}|_T$ and $\psi: C_{\text{const}}|_T = C_U \to C_U = C_\mathcal{X}|_T$ be the identity isomorphisms. Recall that by the definition of a nice triple $\mathcal{X}$ is $U$-smooth and irreducible, hence $S$ is regular and irreducible. By Theorem 4.1 in [16] and Theorem 2.1 there exists a finite étale morphism $\theta_0: S' \to S$, a section $\delta: T \to S'$ of $\theta_0$ over $T$ and isomorphisms $\Phi_0: \theta_0^*(G_{\text{const}}, S) \to \theta_0^*(G_\mathcal{X}|_S)$ and $\Psi_0: \theta_0^*(C_{\text{const}}, S) \to \theta_0^*(C_\mathcal{X}|_S)$ such that $\delta^*(\Phi_0) = \varphi$, $\delta^*(\Psi_0) = \psi$ and

$$
\theta_0^*(\mu_\mathcal{X}|_S) \circ \Phi_0 = \Psi_0 \circ \theta_0^*(\mu_{\text{const}}, S) : \theta_0^*(G_{\text{const}}, S) \to \theta_0^*(C_\mathcal{X}|_S),
$$

where the scheme $S'$ is irreducible. Consider now the diagram (6) from Construction 4.2 in [15]. We may and will now suppose that the neighbourhood $\mathcal{Y}'$ of the points $\{y_1, \ldots, y_n\}$ in that diagram is chosen so that there are $\mathcal{Y}'$-group scheme isomorphisms $\Phi: \theta^*(G_{\text{const}}, \mathcal{Y}') \to \theta^*(G_\mathcal{X}|_{\mathcal{Y}'})$ and $\Psi: \theta^*(C_{\text{const}}, \mathcal{Y}') \to \theta^*(C_\mathcal{X}|_{\mathcal{Y}'})$ with $\Phi|_{S'} = \Phi_0$ and $\Psi|_{S'} = \Psi_0$. Clearly, $\delta^*(\Phi) = \varphi$ and $\delta^*(\Psi) = \psi$. It is less clear, but
Theorem 4.1. Given a nonzero function following result is an extension of Theorem 6.1 in [16].

A morphism which is smooth as an is a diagram of the form

and let can:

Further we get

(i) the special nice triple \((a)\) a triple \(\mathcal{X}', f', \Delta'\);

(ii) that the morphism \(\theta: \mathcal{X}' \to \mathcal{X}\) and the section \(\delta: T \to \mathcal{X}'\) of \(\theta\) over \(T\) we get

1) a triple \((\mathcal{X}', f', \Delta')\);

2) the étale morphism of \(U\)-schemes \(\theta: \mathcal{X}' \to \mathcal{X}\).

To complete the proof of the theorem just apply Theorem 3.9 in [15] to the special nice triple \((p \circ \theta: \mathcal{X}' \to U, f', \Delta')\) and use Remark 3.3.

The theorem is proved.

§ 4. An extension of Theorem 6.1 from [16]

Let \(X\) be an affine \(k\)-smooth irreducible \(k\)-variety, and let \(x_1, x_2, \ldots, x_n\) be closed points in \(X\). Let \(\mathcal{O}\) be the semi-local ring \(\mathcal{O}_\mathcal{X}, \{x_1, x_2, \ldots, x_n\}\). Let \(U = \text{Spec}(\mathcal{O})\) and can: \(U \hookrightarrow X\) be the canonical embedding. Let \(G\) be a reductive \(X\)-group scheme and let \(G_U = \text{can}^*(G)\) be the pull-back of \(G\) to \(U\). Let \(C\) be an \(X\)-torus and let \(C_U = \text{can}^*(C)\) be the pull-back of \(C\) to \(U\). Let \(\mu: G \to C\) be an \(X\)-group scheme morphism which is smooth as an \(X\)-scheme morphism. Let \(\mu_U = \text{can}^*(\mu)\). The following result is an extension of Theorem 6.1 in [16].

Theorem 4.1. Given a nonzero function \(f \in k[X]\) vanishing at each point \(x_i\), there is a diagram of the form

\[
\begin{array}{ccc}
A^1 \times U & \xleftarrow{\sigma} & \mathcal{X}' \\
pr_U & & q_U \\
\downarrow & & \downarrow \Delta' \text{ can} \\
U & \xrightarrow{q_V} & X
\end{array}
\]

with an irreducible affine scheme \(\mathcal{X}'\), a smooth morphism \(q_U\), a finite surjective \(U\)-morphism \(\sigma\), an essentially smooth morphism \(q_X\) and a function \(f' \in q_X^*(f)k[\mathcal{X}']\), which enjoys the following properties:

(a) if \(\mathcal{X}'\) is the closed subscheme of \(\mathcal{X}'\) defined by the ideal \((f')\), then the morphism \(\sigma|_{\mathcal{X}'}: \mathcal{X}' \to A^1 \times U\) is a closed embedding and the morphism \(q_U|_{\mathcal{X}'}: \mathcal{X}' \to U\) is finite;

(a') \(q_U \circ \Delta' = \text{id}_U\) and \(q_X \circ \Delta' = \text{can}\) and \(\sigma \circ \Delta' = i_0\), where \(i_0\) is the zero section of the projection \(pr_U\);

(b) \(\sigma\) is étale in a neighbourhood of \(\mathcal{X}' \cup \Delta'(U)\);

(c) \(\sigma^{-1}(\sigma(\mathcal{X}')) = \mathcal{X}' \coprod \mathcal{X}''\) scheme theoretically for some closed subscheme \(\mathcal{X}''\) and \(\mathcal{X}'' \cap \Delta'(U) = \emptyset\);

(d) \(\mathcal{Z}_0 := \sigma^{-1}(\{0\} \times U) = \Delta'(U) \coprod \mathcal{Z}_0\) scheme theoretically for some closed subscheme \(\mathcal{Z}_0\) and \(\mathcal{Z}_0 \cap \mathcal{X}' = \emptyset\);

\[
\theta^*(\mu|_{\mathcal{X}'}) \circ \Phi = \Psi \circ \theta^*(\mu_{\text{const}, \mathcal{X}'}) : \theta^*(G_{\text{const}, \mathcal{X}'}) \to \theta^*(C_{\mathcal{X}'}).
\]

Applying the second part of Construction 4.2 in [15] and Proposition 4.3 in [15] to the finite étale morphism \(\theta: \mathcal{X}' \to \mathcal{X}\) and the section \(\delta: T \to \mathcal{X}'\) of \(\theta\) over \(T\) we get

\[
\begin{align*}
\sigma & : (a) \mathcal{X}' \to \mathcal{X}, \\
& \text{is smooth, } \theta \circ \sigma = \text{can}^*(\Phi), \\
& \Phi : G_U \to C_U, \\
& \text{and } \text{can}^*(\Phi) = \text{can}(\theta). \\
\end{align*}
\]
(e) for \( \mathcal{D}_1 := \sigma^{-1}(\{1\} \times U) \) one has \( \mathcal{D}_1 \cap \mathcal{X}' = \emptyset \);

(f) there is a monic polynomial \( h \in \mathcal{O}[t] \) with \( (h) = \text{Ker} \mathcal{O}[t] \xrightarrow{\sigma^*} k[\mathcal{X}'] \xrightarrow{\sim} k[\mathcal{X}']/(f') \), where the homomorphism 'bar' takes any \( g \in k[\mathcal{X}'] \) to \( \bar{g} \in k[\mathcal{X}']/(f') \);

(g) there are \( \mathcal{X}' \)-group scheme isomorphisms \( \Phi: q_U^*(G_U) \rightarrow q_X^*(G) \) and \( \Psi: q_U^*(C_U) \rightarrow q_X^*(C) \) with \( (\Delta')^*(\Phi) = \text{id}_{G_U}, (\Delta')^*(\Psi) = \text{id}_{C_U} \) and \( q_X^*(\mu) \circ \Phi = \Psi \circ q_U^*(\mu) \).

Remark 4.2. The triple \((q_U: \mathcal{X}' \rightarrow U, f', \Delta')\) is a nice triple over \( U \), since \( \sigma \) is a finite surjective \( U \)-morphism.

The morphism \( q_X \) is not equal to \( \text{can} \circ q_U \), since \( f' \in q_X^*(f)k[\mathcal{X}'] \) and the morphism \( q_U|_{\mathcal{X}'}: \mathcal{X}' = \{ f' = 0 \} \rightarrow U \) is finite.

We stress that usually the scheme \( \mathcal{X}' \) from diagram (10) does not coincide with the scheme \( \mathcal{X} \) in [16], Theorem 6.1.

Proof of Theorem 4.1. By [15], Proposition 3.6, one can shrink \( X \) such that \( x_1, x_2, \ldots, x_n \) are still in \( X \) and \( X \) is affine, and then construct a special nice triple \((p: \mathcal{X} \rightarrow U, \Delta, f)\) over \( U \) and an essentially smooth morphism \( p_X: \mathcal{X} \rightarrow X \) such that \( p_X \circ \Delta = \text{can}, f = p_X^*(f) \) and the set of closed points of \( \Delta(U) \) is contained in the set of closed points of \( \{ f = 0 \} \).

Set \( G_{\mathcal{X}} = p_X^*(G) \); then \( \Delta^*(G_{\mathcal{X}}) = \text{can}^*(G) \). Thus the \( U \)-group scheme \( G_U \) in Theorem 3.4 and the \( U \)-group scheme \( G_U \) defined just above Theorem 4.1 are the same. Set \( C_{\mathcal{X}} = p_X^*(C) \), then \( \Delta^*(C_{\mathcal{X}}) = \text{can}^*(C) \). Thus the \( U \)-group scheme \( C_U \) in Theorem 3.4 and the \( U \)-group scheme \( C_U \) defined just above Theorem 4.1 are the same.

Following the notation from Theorem 3.4 write \( G_{\text{const}} \) for \( p^*(G_U) \) and \( C_{\text{const}} \) for \( p^*(C_U) \). Let \( \mu_{\mathcal{X}} = p_X^*(\mu): G_{\mathcal{X}} \rightarrow C_{\mathcal{X}}, \mu_U = \Delta^*(\mu_{\mathcal{X}}) \) and let \( \mu_{\text{const}}: G_{\text{const}} \rightarrow C_{\text{const}} \) be the pull-back of \( \mu_U \) to \( \mathcal{X} \) by means of \( p \).

By Theorem 3.4 there exists a morphism \( \theta: (\mathcal{X}_{\text{new}}, f_{\text{new}}, \Delta_{\text{new}}) \rightarrow (\mathcal{X}, f, \Delta) \) between nice triples over \( U \) such that the triple \((p_{\text{new}}: \mathcal{X}_{\text{new}} \rightarrow U, f_{\text{new}}, \Delta_{\text{new}})\) is a special nice triple over \( U \) subject to conditions \((1^*)\) and \((2^*)\) from Definition 3.7 in [15]. Additionally, the morphism \( \theta \) equates the reductive \( \mathcal{X} \)-group scheme morphisms \( \mu_{\text{const}} \) and \( \mu_{\mathcal{X}} \). By Definition 3.2 the latter means that there are isomorphisms

\[
\Phi: \theta^*(G_{\text{const}}) \rightarrow \theta^*(G_{\mathcal{X}}) \quad \text{and} \quad \Psi: \theta^*(C_{\text{const}}) \rightarrow \theta^*(C_{\mathcal{X}})
\]

of \( \mathcal{X}_{\text{new}} \)-group schemes such that \( \Delta_{\text{new}}^*(\Phi) = \text{id}_{G_U}, \Delta_{\text{new}}^*(\Psi) = \text{id}_{C_U} \) and

\[
\theta^*(\mu_{\mathcal{X}}) \circ \Phi = \Psi \circ \theta^*(\mu_{\text{const}}).
\]

(11)

The triple \((\mathcal{X}_{\text{new}}, f_{\text{new}}, \Delta_{\text{new}})\) is a special nice triple over \( U \) subject to conditions \((1^*)\) and \((2^*)\) from Definition 3.7 in [15]. Thus by [15], Theorem 3.8, there is a finite surjective morphism \( A^1 \times U \xleftarrow{\sigma_{\text{new}}} \mathcal{X}_{\text{new}} \xrightarrow{p_{X,\text{new}}} X \) of the \( U \)-schemes satisfying conditions \((a)-(f)\) from that theorem. Hence one has a diagram of the form

\[
\begin{array}{ccc}
A^1 \times U & \xleftarrow{\sigma_{\text{new}}} & \mathcal{X}_{\text{new}} \\
p_{X,\text{new}} \downarrow & & \downarrow \Delta_{\text{new}} \circ \text{can} \\
U & \xrightarrow{p_U} & X
\end{array}
\]
with the irreducible affine scheme $\mathcal{X}_{\text{new}}$, the smooth morphism $p_{\text{new}} = p \circ \theta$, the finite surjective morphism $\sigma_{\text{new}}$, the essentially smooth morphism $p_{X,\text{new}} := p_X \circ \theta$ and the function $f_{\text{new}} \in (p_{X,\text{new}})^*(f)k[\mathcal{X}_{\text{new}}]$.

Put $\mathcal{X}' = \mathcal{X}_{\text{new}}$, $\sigma = \sigma_{\text{new}}$, $\Delta' = \Delta_{\text{new}}$, $qU = p_{\text{new}}$, $qX = p_{X,\text{new}}$ and $f' = f_{\text{new}}$. With this new notation diagram (12) becomes a diagram of the form (10) enjoying properties (a)–(f) from Theorem 4.1. The equality (11) shows that the isomorphisms $\Phi$ and $\Psi$ are subject to the condition (g).

This proves Theorem 4.1.

To formulate a consequence of Theorem 4.1 (see Corollary 4.3), note that using items (b) and (c) of Theorem 4.1 one can find an element $g \in I(\mathcal{X}')$ such that

1. $(f') + (g) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$;
2. $\ker((\Delta')^*(f) + (g)) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$;
3. $\sigma_g = \sigma|_{\mathcal{X}'}: \mathcal{X}' \to \mathbf{A}^1_U$ is étale.

Here is the corollary. It is proved in [15], Corollary 7.2.

**Corollary 4.3.** The function $f'$ in Theorem 4.1, the polynomial $h$ in item (f) of that theorem, the morphism $\sigma: \mathcal{X}' \to \mathbf{A}^1_U$ and the function $g \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ defined above enjoy the following properties:

i. the morphism $\sigma_g = \sigma|_{\mathcal{X}'}: \mathcal{X}' \to \mathbf{A}^1_U$ is étale;
ii. the data $(\mathcal{O}[t], \sigma_g^*: \mathcal{O}[t] \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g, h)$ satisfy the hypotheses of Proposition 2.6 in [1], that is, $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$ is a finitely generated $\mathcal{O}[t]$-algebra, the element $(\sigma_g)^*(h)$ is not a zero-divisor in $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$ and $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g/(h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g)$;
iii. $(\Delta'(U) \cup \mathcal{X}') \subseteq \mathcal{X}'$ and $\sigma_g \circ \Delta' = i_0: U \to \mathbf{A}^1_U$;
iv. $\mathcal{X}'_{gh} \subseteq \mathcal{X}'_{gh} \subseteq \mathcal{X}'_{ih} \subseteq \mathcal{X}'_{i(f)}$;
v. $(\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f'), h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) = (f') \cap I(\mathcal{X}')$ and $(f') + I(\mathcal{X}') = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$.

### § 5. Norms

In §§5 and 6 we prove results which will be used to prove Theorems 1.1, 1.2 and 1.3.

Let $k \subset K \subset L$ be field extensions and assume that $L$ is finite separable over $K$. Let $K^{\text{sep}}$ be a separable closure of $K$ and $\sigma_i: L \to K^{\text{sep}}$, $1 \leq i \leq n$, be the different embeddings of $L$ into $K^{\text{sep}}$. Let $C$ be a $k$-smooth commutative algebraic group scheme defined over $k$. One can define a norm map

$$\mathcal{N}_{L/K}: C(L) \to C(K)$$

by $\mathcal{N}_{L/K}(\alpha) = \prod_i C(\sigma_i)(\alpha) \in C(K^{\text{sep}})^{\Theta(K)} = C(K)$. In [13], following Suslin and Voevodsky (see [23], §6), we generalized this construction to finite flat ring extensions. Let $p: X \to Y$ be a finite flat morphism of affine schemes. Suppose that its rank is constant, and equal to $d$. Denote by $S^d(X/Y)$ the $d$th symmetric power of $X$ over $Y$.

Let $k$ be a field. Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a smooth affine irreducible $k$-variety $X$. Let $C$ be an affine smooth commutative $\mathcal{O}$-group scheme. Let $p: X \to Y$ be a finite flat morphism of affine $\mathcal{O}$-schemes
(of constant degree) and \( f : X \to C \) be any \( \mathcal{O} \)-morphism. In [13] the norm \( N_{X/Y}(f) \) of \( f \) is defined as the composite map

\[
Y \xrightarrow{N_{X/Y}} S^d(X/Y) \to S^d_{\mathcal{O}}(X) \xrightarrow{S^d_{\mathcal{O}}(f)} S^d_{\mathcal{O}}(C) \to C.
\]

(13)

Here we write ‘\( \times \)’ for the group law on \( C \). The norm maps \( N_{X/Y} \) satisfy the following conditions.

(i′) Base change: for any map \( f : Y' \to Y \) of affine schemes, putting \( X' = X \times_Y Y' \), we have a commutative diagram

\[
\begin{array}{ccc}
C(X) & \xrightarrow{(\text{id} \times f)^*} & C(X') \\
N_{X/Y} & \downarrow & \downarrow N_{X'/Y'} \\
C(Y) & \xrightarrow{f^*} & C(Y')
\end{array}
\]

(ii′) Multiplicativity: if \( X = X_1 \amalg X_2 \), then the following diagram commutes

\[
\begin{array}{ccc}
C(X) & \xrightarrow{(\text{id} \times f)^*} & C(X_1) \times C(X_2) \\
N_{X/Y} & \downarrow & \downarrow N_{X_1/Y}N_{X_2/Y} \\
C(Y) & \xrightarrow{\text{id}} & C(Y)
\end{array}
\]

(iii′) Normalization: if \( X = Y \) and the map \( X \to Y \) is the identity, then \( N_{X/Y} = \text{id}_{C(X)} \).

§ 6. Unramified elements

Let \( k \) be a field and \( \mathcal{O} \) be the semi-local ring of finitely many closed points on a \( k \)-smooth irreducible affine \( k \)-variety \( X \). Let \( K \) be the fraction field of \( \mathcal{O} \), that is, \( K = k(X) \). Let

\[ \mu : G \to C \]

be a smooth \( \mathcal{O} \)-morphism of reductive \( \mathcal{O} \)-group schemes, with a torus \( C \). Suppose additionally that the kernel of \( \mu \) is a reductive \( \mathcal{O} \)-group scheme. We work in this section with the category of commutative Noetherian \( \mathcal{O} \)-algebras. For a commutative \( \mathcal{O} \)-algebra \( S \) set

\[ \mathcal{F}(S) = C(S)/\mu(G(S)). \]

(14)

Let \( S \) be an \( \mathcal{O} \)-algebra which is a domain and let \( L \) be its fraction field. Define the subgroup of \( S \)-unramified elements of \( \mathcal{F}(L) \) as

\[ \mathcal{F}_{\text{nr},S}(L) = \bigcap_{p \in \text{Spec}(S)^{(1)}} \text{Im}[\mathcal{F}(S_p) \to \mathcal{F}(L)], \]

(15)

where \( \text{Spec}(S)^{(1)} \) is the set of height 1 prime ideals in \( S \). Obviously, the image of \( \mathcal{F}(S) \) in \( \mathcal{F}(L) \) is contained in \( \mathcal{F}_{\text{nr},S}(L) \). In most cases \( \mathcal{F}(S_p) \) injects into \( \mathcal{F}(L) \) and \( \mathcal{F}_{\text{nr},S}(L) \) is simply the intersection of all \( \mathcal{F}(S_p) \). For an element \( \alpha \in C(S) \) we write \( \overline{\alpha} \) for its image in \( \mathcal{F}(S) \). In this section we write \( \mathcal{F} \) for the functor (14).
Theorem 6.1 (see [9]). Let $S$ be an $O$-algebra which is a discrete valuation ring with fraction field $L$. Then the map $\mathcal{F}(S) \to \mathcal{F}(L)$ is injective.

Lemma 6.2. Let $\mu: G \to C$ be the above morphism of our reductive group schemes. Let $H = \ker(\mu)$. Then for an $O$-algebra $L$, where $L$ is a field, the boundary map

$$
\partial: C(L)/\mu(G(L)) \to H^1_{\text{ét}}(L, H)
$$

is injective.

Proof. For an $L$-rational point $t \in C$ set $H_t = \mu^{-1}(t)$. The action by left multiplication of $H$ on $H_t$ makes $H_t$ into a principal homogeneous $H$-space and, moreover, $\partial(t) \in H^1_{\text{ét}}(L, H)$ coincides with the isomorphism class of $H_t$. Now suppose that $s, t \in C(L)$ are such that $\partial(s) = \partial(t)$. This means that $H_t$ and $H_s$ are isomorphic as principal homogeneous $H$-spaces. We must check that for some $g \in G(L)$ one has $t = s\mu(g)$.

Let $L^{\text{sep}}$ be a separable closure of $L$. Let $\psi: H_s \to H_t$ be an isomorphism of principal homogeneous $H$-spaces. For any $r \in H_s(L^{\text{sep}})$ and $h \in H(L^{\text{sep}})$ one has

$$(hr)^{-1}\psi(hr) = r^{-1}h^{-1}h\psi(r) = r^{-1}\psi(r).$$

Thus, for any $\sigma \in \text{Gal}(L^{\text{sep}}/L)$ and any $r \in H_s(L^{\text{sep}})$ one has

$$r^{-1}\psi(r) = (r\sigma)^{-1}\psi(r\sigma) = (r^{-1}\psi(r))\sigma,$$

which means that the point $u = r^{-1}\psi(r)$ is a $\text{Gal}(L^{\text{sep}}/L)$-invariant point of $G(L^{\text{sep}})$. So $u \in G(L)$. The following relation shows that the morphism $\psi$ coincides with right multiplication by $u$. In fact, for any $r \in H_s(L^{\text{sep}})$ one has $\psi(r) = r\psi(r) = ru$. Since $\psi$ is right multiplication by $u$, one has $t = s\mu(u)$, which proves the lemma.

Let $k$, $O$ and $K$ be as above in this section. Let $\mathcal{H}$ be a field containing $K$ and $x: \mathcal{H}^* \to \mathbb{Z}$ be a discrete valuation vanishing on $K$. Let $A_x$ be the valuation ring of $x$. Clearly, $O \subset A_x$. Let $\hat{A}_x$ and $\hat{\mathcal{H}}_x$ be the completions of $A_x$ and $\mathcal{H}$ with respect to $x$. Let $i: \mathcal{H} \hookrightarrow \hat{\mathcal{H}}_x$ be the inclusion. By Theorem 6.1 the map $\mathcal{F}(\hat{A}_x) \to \mathcal{F}(\hat{\mathcal{H}}_x)$ is injective. We will identify $\mathcal{F}(\hat{A}_x)$ with its image under this map. Set

$$\mathcal{F}_x(\mathcal{H}) = i^{-1}\mathcal{F}(\hat{A}_x)).$$

The inclusion $A_x \hookrightarrow \mathcal{H}$ induces a map $\mathcal{F}(A_x) \to \mathcal{F}(\mathcal{H})$ which is injective by Theorem 6.1. So both groups $\mathcal{F}(A_x)$ and $\mathcal{F}_x(\mathcal{H})$ are subgroups of $\mathcal{F}(\mathcal{H})$. The following lemma shows that $\mathcal{F}_x(\mathcal{H})$ coincides with the subgroup of $\mathcal{F}(\mathcal{H})$ consisting of all elements unramified at $x$.

Lemma 6.3. $\mathcal{F}(A_x) = \mathcal{F}_x(\mathcal{H})$.

Proof. We only have to check the inclusion $\mathcal{F}_x(\mathcal{H}) \subseteq \mathcal{F}(A_x)$. Let $a_x \in \mathcal{F}_x(\mathcal{H})$ be an element. It determines elements $a \in \mathcal{F}(\mathcal{H})$ and $\hat{a} \in \mathcal{F}(\hat{A}_x)$, which coincide when regarded as elements of $\mathcal{F}(\hat{\mathcal{H}}_x)$. We denote this common element of $\mathcal{F}(\hat{\mathcal{H}}_x)$ by $\hat{a}_x$. Let $H = \ker(\mu)$ and let $\partial: C(\hat{\mathcal{H}}) \to H^1_{\text{ét}}(\hat{\mathcal{H}}, H)$ be the boundary map.

Let $\xi = \partial(a) \in H^1_{\text{ét}}(\hat{\mathcal{H}}, H)$, $\hat{\xi} = \partial(\hat{a}) \in H^1_{\text{ét}}(\hat{A}_x, H)$ and $\xi_x = \partial(\hat{a}_x) \in H^1_{\text{ét}}(\hat{\mathcal{H}}_x, H)$. Clearly, $\hat{\xi}$ and $\xi$ both coincide with $\hat{\xi}_x$ when regarded as elements of $H^1_{\text{ét}}(\hat{\mathcal{H}}_x, H)$. Thus one can glue $\xi$ and $\xi$ to get a $\xi_x \in H^1_{\text{ét}}(A_x, H)$ which maps to $\xi$ under the
map induced by the inclusion $A_x \hookrightarrow \mathcal{H}$ and maps to $\hat{\xi}$ under the map induced by the inclusion $A_x \hookrightarrow \hat{A}_x$.

Now we show that $\xi_x$ has the form $\partial(a'_x)$ for some $a'_x \in \mathcal{F}(A_x)$. In fact, observe that the image $\zeta$ of $\xi$ in $H^1_{\text{et}}(\mathcal{H}, \mathbb{G})$ is trivial. By [9], Theorem 4.2, and [8], Theorem 1.1, the map

$$H^1_{\text{et}}(A_x, \mathbb{G}) \to H^1_{\text{et}}(\mathcal{H}, \mathbb{G})$$

has a trivial kernel. Therefore, the image $\zeta_x$ of $\xi_x$ in $H^1_{\text{et}}(A_x, \mathbb{G})$ is trivial. Thus there exists an element $a'_x \in \mathcal{F}(A_x)$ with $\partial(a'_x) = \xi_x \in H^1_{\text{et}}(A_x, \mathbb{H})$.

We now prove that $a'_x$ coincides with $a_x$ in $\mathcal{F}_x(\mathcal{H})$. Since $\mathcal{F}(A_x)$ and $\mathcal{F}_x(\mathcal{H})$ are both subgroups of $\mathcal{F}(\mathcal{H})$, it suffices to show that $a'_x$ coincides with the element $a$ in $\mathcal{F}(\mathcal{H})$. By Lemma 6.2 the map

$$\mathcal{F}(\mathcal{H}) \xrightarrow{\partial} H^1_{\text{et}}(\mathcal{H}, \mathbb{H})$$

is injective. Thus it suffices to check that $\partial(a'_x) = \partial(a)$ in $H^1_{\text{et}}(\mathcal{H}, \mathbb{H})$. This is indeed the case because $\partial(a'_x) = \xi_x$ and $\partial(a) = \xi$, and $\xi_x$ coincides with $\xi$ when regarded over $\mathcal{H}$. We have proved that $a'_x$ coincides with $a_x$ in $\mathcal{F}_x(\mathcal{H})$. Thus the inclusion $\mathcal{F}_x(\mathcal{H}) \subseteq \mathcal{F}(A_x)$ is proved, whence the lemma.

For a regular domain $S$ with fraction field $\mathcal{H}$ and each height 1 prime $p$ in $S$ we construct a specialization map $s_p : \mathcal{F}_{\text{nr}, S}(\mathcal{H}) \to \mathcal{F}(K(p))$, where $K(p)$ is the residue field of $S$ at the prime $p$.

**Definition 6.4.** Let $\text{Ev}_p : \mathbb{C}(S_p) \to \mathbb{C}(K(p))$ and $\text{ev}_p : \mathcal{F}(S_p) \to \mathcal{F}(K(p))$ be the maps induced by the canonical $\mathcal{O}$-algebra homomorphism $S_p \to K(p)$. Define a homomorphism $s_p : \mathcal{F}_{\text{nr}, S}(\mathcal{H}) \to \mathcal{F}(K(p))$ by $s_p(\alpha) = \text{ev}_p(\widetilde{\alpha})$, where $\widetilde{\alpha}$ is a lift of $\alpha$ to $\mathcal{F}(S_p)$. Theorem 6.1 shows that the map $s_p$ is well defined. It is called the specialization map. The map $\text{ev}_p$ is called the evaluation map at the prime $p$.

Obviously for $\alpha \in \mathbb{C}(S_p)$ one has $s_p(\overline{\alpha}) = \text{Ev}_p(\overline{\alpha}) \in \mathcal{F}(K(p))$.

The following two results are Theorem 5.0.16 in [13] and Corollary 5.0.17 in [13], respectively.

**Theorem 6.5** (homotopy invariance). Let $K(t)$ be the rational function field in one variable over the field $K$ as above in this section. Define $\mathcal{F}_{\text{nr}, K(t)}(K(t))$ by formula (15). Then

$$\mathcal{F}(K) = \mathcal{F}_{\text{nr}, K(t)}(K(t)).$$

**Corollary 6.6.** Let $s_0, s_1 : \mathcal{F}_{\text{nr}, K(t)}(K(t)) \to \mathcal{F}(K)$ be the specialization maps at 0 and at 1 (at the primes $(t)$ and $(t-1)$). Then $s_0 = s_1$.

Let $k$, $\mathcal{O}$ and $K$ be as above in this section.

**Lemma 6.7.** Let $B \subseteq A$ be a finite extension of $K$-smooth algebras which are domains and each has dimension one. Let $0 \neq f \in A$ and let $h \in B \cap fA$ be such that the induced map $B/hB \to A/fA$ is an isomorphism. Suppose $hA = fA \cap J''$ for an ideal $J'' \subseteq A$ co-prime to the principal ideal $fA$.

Let $E$ and $F$ be the fields of fractions of $B$ and $A$, respectively. Let $\alpha \in \mathbb{C}(A_f)$ be such that $\overline{\alpha} \in \mathcal{F}(F)$ is $A$-unramified. Then, for $\beta = N_{F/E}(\alpha)$ the class $\overline{\beta} \in \mathcal{F}(E)$ is $B$-unramified.
Proof. The only primes at which $\overline{\beta}$ could be ramified are those which divide $hB$. Let $p$ be one of them. Check that $\overline{\beta}$ is unramified at $p$.

To do this we consider all primes $q_1, q_2, \ldots, q_n$ in $A$ lying over $p$. Let $q_1$ be the unique prime dividing $f$ and lying over $p$. Then

$$A \otimes_B \hat{B}_p = \hat{A}_{q_1} \times \prod_{i \neq 1} \hat{A}_{q_i}$$

with $\hat{A}_{q_1} = \hat{B}_p$. If $F$ and $E$ are the fields of fractions of $A$ and $B$, then

$$F \otimes_B \hat{B}_p = \hat{F}_{q_1} \times \prod_{i \neq 1} \hat{F}_{q_i}$$

and $\hat{F}_{q_1} = \hat{E}_p$. We will write $\hat{F}_i$ for $\hat{F}_{q_i}$ and $\hat{A}_i$ for $\hat{A}_{q_i}$. Let

$$\alpha \otimes 1 = (\alpha_1, \ldots, \alpha_n) \in C(\hat{F}_1) \times \cdots \times C(\hat{F}_n).$$

Clearly, for $i \geq 2$ one has $\alpha_i \in C(\hat{A}_i)$ and $\alpha_1 = \mu(\gamma_1)\alpha'$ with $\alpha' \in C(\hat{A}_1) = C(\hat{B}_p)$ and $\gamma_1 \in G(\hat{F}_1) = G(\hat{E}_p)$. Now $\beta \otimes 1 \in C(\hat{E}_p)$ coincides with the product

$$\alpha_1 N_{\hat{F}_2/\hat{E}_p}(\alpha_2) \cdots N_{\hat{F}_n/\hat{E}_p}(\alpha_n) = \mu(\gamma_1)[\alpha_1 N_{\hat{F}_2/\hat{E}_p}(\alpha_2) \cdots N_{\hat{F}_n/\hat{E}_p}(\alpha_n)]$$

Thus $\overline{\beta} \otimes 1 = \overline{\alpha_1 N_{\hat{F}_2/\hat{E}_p}(\alpha_2) \cdots N_{\hat{F}_n/\hat{E}_p}(\beta_n)} \in \mathcal{F}(\hat{B}_p)$. Let $i : E \hookrightarrow \hat{E}_p$ be the inclusion and $i_* : \mathcal{F}(E) \to \mathcal{F}(\hat{E}_p)$ be the induced map. Clearly, $i_*(\overline{\beta}) = \overline{\beta \otimes 1}$ in $\mathcal{F}(\hat{E}_p)$. Now Lemma 6.3 shows that the element $\overline{\beta} \in \mathcal{F}(E)$ belongs to $\mathcal{F}(B_p)$. Hence $\beta$ is $B$-unramified.

The lemma is proved.

§ 7. Proof of Theorems 1.1 and 1.2

Let $k$ be a field. Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a $k$-smooth irreducible affine $k$-variety $X$. Let $K = k(X)$. Let

$$\mu : G \to C$$

be a smooth $X$-morphism of reductive $X$-group schemes, with a torus $C$. Suppose additionally that the kernel of $\mu$ is a reductive $X$-group scheme. For each $X$-scheme $X'$ we write $\overline{C}(X')$ for $C(X')/\mu(C(X'))$ in this section.

**Definition 7.1.** Let $X'$ be an irreducible regular affine $X$-scheme. Let $a \in \overline{C}(k(X'))$. Its class $\overline{a} \in \overline{C}(k(X'))$ is called unramified at a height 1 prime ideal $p$ of $k[X']$ if $a$ is in the image of the group $\overline{C}(\mathcal{O}_{X', p})$. Let $M \subset k[X']$ be a multiplicative system. The class $\overline{a} \in \overline{C}(k(X'))$ is called $k[X']_M$-unramified if it is unramified at any height 1 prime ideal of $k[X']_M$. In particular, the class $\overline{a} \in \overline{C}(k(X'))$ is called $X'$-unramified if it is unramified at any height 1 prime ideal of $k[X']$. 
The following lemma is obvious.

**Lemma 7.2.** Let \( \varphi: Y \to X \) be a smooth morphism, where \( Y \) is an irreducible affine scheme. Then this morphism induces an obvious map

\[
\varphi^*: \overline{\mathcal{C}}(k(X)) \to \overline{\mathcal{C}}(k(Y)),
\]

which takes \( X \)-unramified elements to \( Y \)-unramified elements. If \( M \subset k[Y] \) is a multiplicative system, then the homomorphism \( \varphi^* \) takes \( X \)-unramified elements to \( k[Y]_M \)-unramified elements.

**Proof of Theorem 1.1.** The \( k \)-algebra \( O' \) is of the form \( O_{X,\{x_1,x_2,...,x_n\}} \), where \( X \) is a \( k \)-smooth irreducible affine variety. In general, \( G, C \) and \( \mu \) do not come from \( X \). However, clearing denominators, we may assume that \( G, C \) and \( \mu \) are defined over \( k \), \( G \) is reductive over \( X \), \( C \) is a torus and \( \mu \) is an \( X \)-group scheme morphism which is smooth. Let \( a_K \in \mathcal{C}(k(X)) \) be such that its class in \( \overline{\mathcal{C}}(K) \) is \( O' \)-unramified. Then there is a nonzero function \( f \in k[X] \) such that the element \( a_K \) is defined over \( X_f \), that is, there is an element \( a \in \mathcal{C}(k[X_f]) \) such that the image of \( a \) in \( \mathcal{C}(K) \) coincides with \( a_K \). Shrinking \( X \) we may assume further that \( f \) vanishes at each \( x_i \). Shrinking \( X \) further, we may and will assume also that \( \pi \in \overline{\mathcal{C}}(k[X_f]) \) is \( k[X] \)-unramified. One can choose these shrinkings of \( X \) so that the resulting scheme remains an affine neighbourhood of the points \( x_i \). By Theorem 4.1 there is a diagram of the form (10) together with a scheme \( \mathcal{X}' \), morphisms \( q_U, \Delta', \sigma, q_X \) and a function \( f' \in q_X^*(f)k[\mathcal{X}'] \), which enjoys properties (a)–(g) from that theorem. From now on and until the end of this proof we use the notation from Theorem 4.1. However, we write \( \mathcal{X} \) for \( \mathcal{X}' \), \( \Delta \) for \( \Delta' \) and \( f \) for \( f' \).

Assume first that \( \mu \) is ‘constant’, that is, there are a reductive group \( G_0 \), a torus \( C_0 \) over the field \( k \) and an algebraic \( k \)-group morphism \( \mu_0 \) and \( U \)-group scheme isomorphisms

\[
\Phi: G_{0,U} = G_0 \times_{\text{Spec}(k)} U \to G \quad \text{and} \quad \Psi: C_{0,U} = C_0 \times_{\text{Spec}(k)} U \to C
\]

such that \( \Psi \circ \mu_{0,U} = \mu \circ \Phi \).

The morphism \( \sigma \) in Theorem 4.1 is finite surjective and the schemes \( A^1_U \) and \( \mathcal{X} \) are regular and irreducible. Thus by a theorem of Grothendieck (see [4], Theorem 18.17) the morphism \( \sigma \) is flat and finite. Set \( \alpha := q_f^*(a) \in \mathcal{C}(\mathcal{X}_f) \), where \( q_f: \mathcal{X}_f \to X_f \) is the restriction of \( q_X \) to \( \mathcal{X}_f \) and set

\[
a_U := N_{\mathcal{X}_f/U}(\alpha|_{\mathcal{X}_f}) \cdot N_{\mathcal{X}_f/U}(\alpha|_{\mathcal{X}_f})^{-1} \in \mathcal{C}(U). \tag{17}
\]

**Claim 7.3.** Let

\[
\eta_U: \text{Spec}(k(X)) \to \text{Spec}(O') = U \quad \text{and} \quad \eta: \text{Spec}(k(X)) \to X_f
\]

be the generic points of \( U \) and \( X_f \), respectively. Then \( \eta^*_U(aU) = \eta^*(\alpha) \in \overline{\mathcal{C}}(k(X)) \).

Since \( \eta^*(\alpha) = a_K \), this claim completes the proof of Theorem 1.1 in the constant case. To prove the claim consider the scheme \( \mathcal{X} \) and its closed and open subschemes as \( U \)-schemes via the morphism \( q_U \). Set \( K = k(X) \). Taking the base change of \( \mathcal{X} \), \( A^1_U \) and \( \sigma \) via the morphism \( \eta_U: \text{Spec}(K) \to U \) we get a morphism of
the $K$-schemes $\mathbf{A}^{1}_K \overset{\sigma_K}{\leftarrow} \mathscr{A}_K$. Recall that the class $\overline{a} \in \overline{C}(X_f)$ is $X$-unramified. By Lemma 7.2 the class $\overline{a} \in \overline{C}((\mathscr{A}_f))$ is $\mathscr{A}$-unramified. Hence its image $\overline{a}_K$ in $\overline{C}(K(\mathcal{A}_K))$ is $\mathscr{A}_K$-unramified too.

Items (ii) and (v) of Corollary 4.3 and Lemma 6.7 show that for the element $\beta_t := N_{K(\mathcal{A}_K)/K}(A_{1,K}) (\alpha_K) \in C(K(t))$ the class $\overline{\beta}_t \in \overline{C}(K(t))$ is $\mathbf{A}^{1}_K$-unramified. By Theorem 6.5 the class $\overline{\beta}_t$ is constant, that is, it comes from the field $K$. By Corollary 6.6 its specializations at the $K$-points 0 and 1 of the affine line $\mathbf{A}^{1}_K$ coincide: $s_0(\overline{\beta}_t) = s_1(\overline{\beta}_t) \in \overline{C}(K)$.

Properties (c)–(e) and the equality $q_X \circ \Delta = \text{can}$ from Theorem 4.1 show that $\mathscr{A}_{1,K}, \mathscr{A}_{0,K}, \Delta(Spec(K)) \subset (\mathcal{A}_f)_K$. Thus there is a Zariski open neighbourhood $V$ of the $K$-points 0 and 1 in $\mathbf{A}^{1}_K$ such that $W := (\sigma_K)^{-1}(V) \subset (\mathcal{A}_f)_K$. Hence for $\beta := N_{W/V}(\alpha|_W)$ one has the equality $\beta = \beta_t$ in $C(K(t))$. Thus

$$\overline{\beta}(1) = s_1(\overline{\beta}_t) = s_0(\overline{\beta}_t) = \overline{\beta}(0) \in \overline{C}(K)$$

(see the remark at the end of Definition 6.4). By properties (i′)–(iii′) of the norm maps (see §5) one has

$$N_{\mathscr{A}_{1,K}/K}(\alpha|_{\mathscr{A}_{1,K}}) = \beta(1)$$

and

$$\beta(0) = N_{\mathscr{A}_{0,K}/K}(\alpha|_{\mathscr{A}_{0,K}}) = N_{\mathscr{A}_{0,K}/K}(\alpha|_{\mathscr{A}_{0,K}}) \cdot \Delta_{\mathcal{A}}(\alpha_K).$$

By the base change property of the norm maps one has the equality

$$\eta^*_U(a_U) = N_{\mathscr{A}_{1,K}/K}(\alpha|_{\mathscr{A}_{1,K}}) \cdot [N_{\mathscr{A}_{0,K}/K}(\alpha|_{\mathscr{A}_{0,K}})]^{-1}.$$  

Hence $\Delta_{\mathcal{A}}^*_K(\overline{\alpha}_K) = \eta^*_U(\overline{a}_U)$ in $\overline{C}(k(X))$. Finally, the composite map

$$\text{Spec}(K) \xrightarrow{\Delta_{\mathcal{A}}} (\mathcal{A}_f)_K \hookrightarrow \mathcal{A}_f \xrightarrow{q_f} X_f$$

coincides with the canonical map $\eta: \text{Spec}(K) \to X_f$. Hence $\Delta_{\mathcal{A}}^*(\overline{\alpha}_K) = \eta^*(\overline{a})$, which proves Claim 7.3. This proves Theorem 1.1 in the constant case.

In the general case there are two functors on the category of $\mathcal{A}$-schemes. Namely, $\overline{C}$ and $U \overline{C}$. If $r: \mathcal{Y} \to \mathcal{A}$ is a scheme morphism, then $\overline{C}(\mathcal{Y}) := C(\mathcal{Y})/\langle \mu(G(\mathcal{Y})) \rangle$, where $\mathcal{Y}$ is regarded as an $X$-scheme via the morphism $q_X \circ r$. On the other hand $U \overline{C}(\mathcal{Y}) := C_U(\mathcal{Y})/\langle \mu_U(G_U(\mathcal{Y})) \rangle$, where $\mathcal{Y}$ is regarded as a $U$-scheme via the morphism $q_U \circ r$. The $\mathcal{A}$-group scheme isomorphisms $\Phi$ and $\Psi$ from Theorem 4.1 induce a group isomorphism

$$\overline{\Psi}_{\mathcal{Y}}: U \overline{C}(\mathcal{Y}) \to \overline{C}(\mathcal{Y}),$$

which respects $\mathcal{A}$-scheme morphisms. Moreover, if the scheme $U$ is regarded as an $\mathcal{A}$-scheme via the morphism $\Delta$, then the isomorphism $\overline{\Psi}_U$ is the identity. And similarly for any $U$-scheme $g: W \to U$ regarded as an $\mathcal{A}$-scheme via the morphism $\Delta \circ g$, the isomorphism $\overline{\Psi}_W$ is the identity. Set $\alpha := q^*_f(a) \in C(\mathcal{A}_f)$, where $q_f: \mathcal{A}_f \to X_f$ is as above in this proof. Let $u \alpha \in C_U(\mathcal{A}_f)$ be a unique element such that $\Psi_{\mathcal{A}_f}(u \alpha) = \alpha$. Set

$$u a := N_{\mathcal{A}_{1,U}}((u \alpha)|_{\mathcal{A}_{1}}) \cdot N_{\mathcal{A}_{0,U}}((u \alpha)|_{\mathcal{A}_{0}})^{-1} \in C_U(U)$$

and $a_U := \Psi_U(u a) \in C(U)$.

We leave it to the reader to prove the following claim.
Claim 7.4. Let

\[ \eta_U: \text{Spec}(k(X)) \to \text{Spec}(\mathcal{O}) = U \quad \text{and} \quad \eta: \text{Spec}(k(X)) \to X_f \]

be as above in this proof. Then

\[ \eta_U^*(\alpha_U) = \eta^*(\alpha) \in \mathcal{C}(k(X)). \]

Since \( \eta^*(a) = a_K \), this claim completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Just repeat literally the proof of Theorem 10.0.30 in [13], replacing the reference to Theorem 1.0.1 in [13] with one to Theorem 1.1.

§ 8. Proof of Theorem 1.3

Proof of the semi-simple case of Theorem 1.3. Let \( \mathcal{O} \) and \( G \) be the same as in Theorem 1.3 and assume additionally that \( G \) is semi-simple. We need to prove that

\[ \ker[H^1_\text{ét}(\mathcal{O}, G) \to H^1_\text{ét}(K, G)] = \ast. \quad (19) \]

Let \( G^{\text{sc}} \) be the corresponding simply-connected semi-simple \( \mathcal{O} \)-group scheme.

Claim 8.1. Under the hypotheses of Theorem 1.3, for all semi-simple reductive \( \mathcal{O} \)-group schemes \( G \) the map \( H^1_\text{ét}(\mathcal{O}, G^{\text{sc}}) \to H^1_\text{ét}(K, G^{\text{sc}}) \) is injective.

In fact, let \( \xi, \zeta \in H^1_\text{ét}(\mathcal{O}, G^{\text{sc}}) \) be two elements such that their images \( \xi_K \) and \( \zeta_K \) in \( H^1_\text{ét}(K, G^{\text{sc}}) \) are equal. Let \( \xi G^{\text{sc}} \) and \( \zeta G^{\text{sc}} \) be the corresponding principal \( G^{\text{sc}} \)-bundles over \( \mathcal{O} \) and \( G^{\text{sc}}(\zeta) \) be the inner form of the \( \mathcal{O} \)-group scheme \( G^{\text{sc}} \) corresponding to \( \zeta \). The \( \mathcal{O} \)-scheme \( \text{Iso}(\xi G^{\text{sc}}, \zeta G^{\text{sc}}) \) is a principal \( G^{\text{sc}}(\zeta) \)-bundle over \( \mathcal{O} \), which is trivial over \( K \). Since \( G^{\text{sc}}(\zeta) \) is simply-connected semi-simple reductive over \( \mathcal{O} \), the \( \mathcal{O} \)-scheme \( \text{Iso}(\xi G^{\text{sc}}, \zeta G^{\text{sc}}) \) has an \( \mathcal{O} \)-point by the hypotheses of Theorem 1.3. Whence the claim.

To finish the proof of the semi-simple case of Theorem 1.3 it remains to repeat literally the arguments from the proof of the semi-simple case of Theorem 1.0.3 in [13], §11. This proves the semi-simple case of Theorem 1.3.

In turn, the semi-simple case of Theorem 1.3 has the following consequence, which is proved analogously to the proof of Claim 8.1.

Claim 8.2. Under the hypotheses of Theorem 1.3, for all semi-simple \( \mathcal{O} \)-group schemes \( G \) the map \( H^1_\text{ét}(\mathcal{O}, G) \to H^1_\text{ét}(K, G) \) is injective.

The end of the proof of Theorem 1.3. Just repeat literally the arguments from the corresponding part of the proof of Theorem 1.0.3 in [13], §11.

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Two purity theorems and the Grothendieck-Serre conjecture

Bibliography

[1] J.-L. Colliot-Thélène and M. Ojanguren, “Espaces principaux homogènes localement triviaux”, Inst. Hautes Études Sci. Publ. Math. 75:2 (1992), 97–122.

[2] J.-L. Colliot-Thélène, R. Parimala and R. Sridharan, “Un théorème de pureté locale”, C. R. Acad. Sci. Paris Sér. I Math. 309:14 (1989), 857–862.

[3] Schémas en groupes, vol. III: Structure des schémas en groupes réductifs, Séminaire de géométrie algébrique du Bois Marie 1962/64 (SGA 3), dirigé par M. Demazure, A. Grothendieck, Rev. reprint, Lecture Notes in Math., vol. 153, Springer-Verlag, Berlin–New York 1970, viii+529 pp.

[4] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry, Grad. Texts in Math., vol. 150, Springer-Verlag, New York 1995, xvi+785 pp.

[5] R. Fedorov and I. Panin, “A proof of the Grothendieck-Serre conjecture on principal bundles over regular local rings containing infinite fields”, Publ. Math. Inst. Hautes Études Sci. 122 (2015), 169–193; arXiv:1211.2678v2.

[6] A. Grothendieck, “Torsion homologique et sections rationnelles”, Séminaire C. Chevalley (2e année), vol. 3: Anneaux de Chow et applications, Secrétariat mathématique, Paris 1958, Exp. No. 5, 29 pp.

[7] A. Grothendieck, “Le group de Brauer. II. Théorie cohomologique”, Dix exposés sur la cohomologie de schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam 1968, pp. 67–87.

[8] N. Guo, “The Grothendieck-Serre conjecture over semilocal Dedekind rings”, Transform. Groups, Publ. online: 2020, 1–21.

[9] Y. A. Nisnevich, “Espaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind [Rationally trivial principal homogeneous spaces and arithmetic of reductive group schemes over Dedekind rings]”, C. R. Acad. Sci. Paris Sér. I Math. 299:1 (1984), 5–8.

[10] M. Ojanguren and I. Panin, “A purity theorem for the Witt group”, Ann. Sci. École Norm. Sup. (4) 32:1 (1999), 71–86.

[11] M. Ojanguren and I. Panin, “Rationally trivial Hermitian spaces are locally trivial”, Math. Z. 237:1 (2001), 181–198.

[12] I. Panin, A. Stavrova and N. Vavilov, “On Grothendieck-Serre’s conjecture concerning principal G-bundles over reductive group schemes: I”, Compos. Math. 151:3 (2015), 535–567.

[13] I. A. Panin, “On Grothendieck-Serre’s conjecture concerning principal G-bundles over reductive group schemes: II”, Izv. Ross. Akad. Nauk Ser. Mat. 80:4 (2016), 131–162; Izv. Math. 80:4 (2016), 759–790.

[14] I. Panin, “On Grothendieck-Serre conjecture concerning principal bundles”, Proceedings of the international congress of mathematicians, vol. 2 (Rio de Janeiro 2018), World Sci. Publ., Hackensack, NJ 2018, pp. 201–221.

[15] I. A. Panin, “Nice triples and moving lemmas for motivic spaces”, Izv. Ross. Akad. Nauk Ser. Mat. 83:4 (2019), 158–193; English transl. in Izv. Math. 83:4 (2019), 796–829.

[16] I. Panin, “Nice triples and the Grothendieck-Serre conjecture concerning principal G-bundles over reductive group schemes”, Duke Math. J. 168:2 (2019), 351–375.

[17] I. A. Panin, “Proof of the Grothendieck-Serre conjecture on principal bundles over regular local rings containing a field”, Izv. Ross. Akad. Nauk Ser. Mat. 84:4 (2020), 169–186; English transl. in Izv. Math. 84:4 (2020), 780–795.
[18] I. A. Panin and A. A. Suslin, “On a Grothendieck conjecture for Azumaya algebras”, Algebra i Analiz 9:4 (1997), 215–223; English transl. in St. Petersburg Math. J. 9:4 (1998), 851–858.

[19] B. Poonen, “Bertini theorems over finite fields”, Ann. of Math. (2) 160:3 (2004), 1099–1127.

[20] F. Charles and B. Poonen, “Bertini irreducibility theorems over finite fields”, J. Amer. Math. Soc. 29:1 (2016), 81–94; arXiv:1311.4960v1.

[21] M. Rost, “Durch Normengruppen definierte birationale Invarianten”, C. R. Acad. Sci. Paris Sér. I Math. 310:4 (1990), 189–192.

[22] J.-P. Serre, “Espaces fibrés algébriques”, Séminaire C. Chevalley (2e année), vol. 3: Anneaux de Chow et applications, Secrétariat mathématique, Paris 1958, Exp. No. 1, 37 pp.

[23] A. Suslin and V. Voevodsky, “Singular homology of abstract algebraic varieties”, Invent. Math. 123:1 (1996), 61–94.

[24] V. Voevodsky, “Cohomological theory of presheaves with transfers”, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ 2000, pp. 87–137.

[25] K. V. Zainoulline, “On Grothendieck’s conjecture about principal homogeneous spaces for some classical algebraic groups”, Algebra i Analiz 12:1 (2000), 150–184; English transl. in St. Petersburg Math. J. 12:1 (2001), 117–143.

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