Casimir densities for a conducting plate in de Sitter spacetime

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Abstract. Two-point functions for the electromagnetic field in background of \((D+1)\)-dimensional dS spacetime are evaluated assuming that the field is prepared in the Bunch-Davies vacuum state. By using these functions, the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor are investigated in the geometry of a conducting plate. The VEVs are explicitly decomposed into the boundary-free and plate-induced parts. For points outside of the plate the renormalization is needed for the first parts only. Because of the maximal symmetry of the background spacetime and of the Bunch-Davies vacuum state, the boundary-free parts do not depend on spacetime coordinates, whereas the plate-induced parts are functions of the proper distance of the observation point from the plate. The plate-induced part in the VEV of the energy-momentum tensor vanishes for \(D = 3\) which is a direct consequence of a conformal invariance of the electromagnetic field for this spatial dimension. For \(D > 3\), in addition to the diagonal components, the vacuum energy-momentum tensor has nonzero off-diagonal component which describes energy flux along the direction normal to the plate.

1. Introduction
De Sitter (dS) spacetime is among the most popular backgrounds in the gravitational physics. There are several reasons for this. First of all, dS spacetime is the maximally symmetric solution of Einstein’s equations with a positive cosmological constant and due to this maximal symmetry numerous physical problems are exactly solvable on its background. Further, dS spacetime is of key importance in the inflationary cosmology scenario [1], in accordance of which in the early stages of the cosmological expansion our Universe passed through a phase with the geometry well approximated by dS spacetime. During an inflationary epoch, quantum fluctuations in the inflaton field introduce inhomogeneities which play a central role in the generation of cosmic structures from inflation. Recent astronomical observations of high redshift supernovae, galaxy clusters and cosmic microwave background have indicated that at the present epoch the Universe is accelerating and can be approximated by a world with a positive cosmological constant [2]. If the Universe were to accelerate indefinitely, the Standard Cosmology would lead to an asymptotic dS Universe. It is therefore important to investigate physical effects in dS spacetime for understanding the early Universe and its future.

In particular, the investigation of quantum field theoretical effects in dS background has attracted a great deal of attention. The interaction of a fluctuating quantum field with the background gravitational field leads to the vacuum polarization. Another type of vacuum
polarization arises by the imposition of boundary conditions on the field operator. These conditions may arise because of the presence of boundaries having different physical nature: macroscopic bodies in QED, extended topological defects, horizons and branes in higher-dimensional models. Boundary conditions, imposed on the field operator, modify the zero-point modes of a quantized field and, as a result, the vacuum expectation values of physical observables are changed. In particular, forces arise acting on constraining boundaries. This is the well-known Casimir effect [3]. The nature of the Casimir forces depend on the quantum field, the type of spacetime manifold, the boundary geometry, and the specific boundary conditions imposed on the field.

In this paper, based on [4], we consider the combined effects of the background gravitational field and of boundaries on the properties of the electromagnetic vacuum. The background geometry is the dS spacetime and as a boundary we will consider a single plate on which the field obeys perfectly conducting boundary conditions. The Casimir effect in dS spacetime for a massive scalar field with general curvature coupling parameter obeying Robin boundary condition has been recently considered in [5, 6] and [7] for flat and spherical boundaries respectively (see also [8, 9] and [10] for special cases of conformally and minimally coupled massless fields). The case of the electromagnetic filed in background of Friedmann-Robertson-Walker cosmological models with power-law scale factors is discussed in [11]. By taking into account that the higher-dimensional models play an important role in high-energy physics, in particular, in string theories and in supergravity, we assume the general value of spatial dimension $D$.

2. Electromagnetic two-point functions

We consider the change in the properties of the electromagnetic vacuum induced by the presence of a perfectly conducting plate in the background of $(D + 1)$-dimensional dS spacetime. The geometry is described by the line element:

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_{i=1}^{D} (dz^i)^2 ,$$  \hspace{1cm} (1)

where the parameter $\alpha$ is related to the positive cosmological constant $\Lambda$ by the formula $\alpha^2 = D(D - 1)/(2\Lambda)$. In terms of conformal time $\tau$, defined as $\tau = -\alpha e^{t/\alpha}$, $-\infty < \tau < 0$, the metric tensor takes a conformally flat form:

$$g_{ik} = (\alpha/\tau)^2 \text{diag} (1, -1, ..., -1).$$  \hspace{1cm} (2)

We assume that the plate is placed at $z^D = 0$. On the plate the field obeys the boundary condition $n^\nu \tilde{F}_{\nu\cdot \nu\cdot D-1} = 0$, with the tensor $\tilde{F}_{\nu\cdot \nu\cdot D-1}$ dual to electromagnetic field tensor $F_{\nu\cdot \nu\cdot}$, and with $n^\mu$ being the normal to the plate. Here we consider the region $z^D > 0$.

All properties of the quantum vacuum are encoded in two-point functions. In the presence of the plate, the two-point function of the electromagnetic field tensor is decomposed as:

$$\langle F_l(x) F_m(x') \rangle = \langle F_l(x) F_m(x') \rangle_0 + \langle F_l(x) F_m(x') \rangle_b,$$  \hspace{1cm} (3)

where $\langle F_l(x) F_m(x') \rangle_0$ is the corresponding function in the boundary-free dS spacetime and the part $\langle F_l(x) F_m(x') \rangle_b$ is induced by the plate. Assuming that the electromagnetic field is prepared
in the Bunch-Davies vacuum state, for the boundary-free part of the two-point functions we have

\[
\langle F_{ml}(x) F_{0m}(x') \rangle_0 = \frac{(\eta l^2)}{2BD_\alpha^{D-3}} \left[ \left( \delta_{\text{lr}} \delta_{\text{mq}} - \delta_{\text{lm}} \delta_{\text{pq}} \right) \frac{\Delta z^p \Delta z^q}{2\eta l^2} \partial_z + (D - 1) \delta_{\text{lm}} \ G_D(z), \right.
\]

\[
\langle F_{pl}(x) F_{0m}(x') \rangle_0 = \frac{(\eta l^2)}{2BD_\alpha^{D-3}} \delta_{\text{pm}} \delta_{\text{lj}} \frac{\Delta z^q}{\eta l} \left[ 2 + (z - \frac{\eta + \eta'}{2\eta}) \partial_z \ F_D(z), \right.
\]

\[
\langle F_{pl}(x) F_{qm}(x') \rangle_0 = \frac{(\eta l^2)}{BD_\alpha^{D-3}} \delta_{\text{pm}} \delta_{\text{lj}} \frac{\Delta z^q}{\eta l^2} \partial_z + 2\delta_{\text{pq}} \delta_{\text{lm}} \ F_D(z),
\]

(4)

where \( l, m, p, q = 1, 2, \ldots, D \), \( z = (z^1, \ldots, z^D) \) and

\[
B_D = (4\pi)^{(D-1)/2} \Gamma \left( \frac{(D + 3)}{2} \right), \quad z = 1 + \frac{(\Delta \eta)^2 - |\Delta z|^2}{4\eta l^2}, \quad \Delta z = z - z'.
\]

In (4), we have defined the functions

\[
F_D(z) = \Gamma(D) F \left( D, 2; \frac{D + 3}{2}; z \right),
\]

\[
G_D(z) = 2\Gamma(D - 1) F \left( D - 1, 3; \frac{D + 3}{2}; z \right),
\]

(6)

with \( F(a, b, c; z) \) being the hypergeometric function. The expression for the two-point function \( \langle F_0m(x) F_{pl}(x') \rangle \) is obtained from the expression (4) for \( \langle F_{pl}(x) F_{0m}(x') \rangle \) by changing the sign and by the interchange \( \eta \leftrightarrow \eta' \).

The plate-induced part in (3) are related to the boundary-free functions by the formulas

\[
\langle F_{pl}(x) F_{qm}(x') \rangle_b = -\langle F_{pl}(x) F_{qm}(x'_-) \rangle_0,
\]

\[
\langle F_{pl}(x) F_{Dm}(x') \rangle_b = \langle F_{pl}(x) F_{Dm}(x'_-) \rangle_0,
\]

(7)

with \( p, l = 0, 1, \ldots, D \), \( q, m = 0, 1, \ldots, D - 1 \), and \( x'_{-} \) is the image of \( x' \) with respect to the plate: \( x'_{-} = (\tau', \bar{z}^1, \ldots, \bar{z}^{D-1}, -z^D) \).

For \( D = 3 \) for the functions (6) one has \( F_3(z) = G_3(z) = 2(z - 1)^{-2} \). In this case the two-point functions (4) are related to the corresponding functions in Minkowski spacetime by the standard conformal transformation. The latter property is a consequence of the fact that the electromagnetic field is conformally invariant for \( D = 3 \).

### 3. Casimir densities

First let us consider the VEV of the electric field squared. The boundary-induced part in this VEV is obtained from the two-point function \( \langle F_{0l}(x) F_{0m}(x') \rangle_b \) in the coincidence limit:

\[
\langle E^2 \rangle_b = \frac{D - 1}{2BD_\alpha^{D+1}} [2(1 - y) \partial_y - D + 2] G_D(y),
\]

(8)

where and in what follows we use the notation

\[
y = 1 - (z^D/\eta)^2.
\]

(9)

At proper distances from the plate much smaller than the dS curvature scale, \( z^D/\eta \ll 1 \), to the leading order we have

\[
\langle E^2 \rangle_b \approx \frac{3(D - 1) \Gamma((D + 1)/2)}{2(4\pi)^{(D-1)/2}(\alpha z^D/\eta)^{D+1}}.
\]

(10)
In this limit, the boundary-induced part dominates in the VEV of the electric field squared. At large distances from the plate one has the asymptotics:

\[
\langle E^2 \rangle_b \approx \frac{2^{3-D}}{(4\pi)^{D/2}(2\alpha)^{D+1}} \left( \frac{z^D}{\eta} \right)^{2(D-1)}, \quad D < 4,
\]

\[
\langle E^2 \rangle_b \approx \frac{(D-1)(8-D)}{2^4D} \frac{\Gamma(D/2-2)}{\Gamma(D/2)}, \quad D > 4.
\]

(11)

For \( D = 8 \) the leading term vanishes. For \( 3 \leq D \leq 8 \) the plate-induced part \( \langle E^2 \rangle_b \) is positive everywhere. For \( D \geq 9 \), it is positive near the plate and negative at large distances (see figure 1 below).

Another important characteristic of the vacuum state is the VEV of the energy-momentum tensor. It describes the local structure of the vacuum state and acts as the source of gravity in the quasiclassical Einstein equations. Therefore, the VEV of the energy-momentum tensor plays an important role in modelling self-consistent dynamics involving the gravitational field.

Having the two-point functions we can evaluate the VEV of the energy-momentum tensor by making use of the formula

\[
\langle T^\mu_\nu \rangle = -\frac{1}{4\pi} \left\langle F_\mu^\beta(x)F^\nu_\beta(x) \right\rangle + \frac{\delta^\nu_\mu}{16\pi} \left\langle F_\beta\gamma F_\gamma^\beta \right\rangle.
\]

(12)

Of course, the expression in the right-hand side is divergent and some regularization with the subsequent renormalization procedure is needed. The important thing here is that we have already decomposed the two-point functions. This allows us to have a similar decomposition of the vacuum energy-momentum tensor:

\[
\langle T^\mu_\nu \rangle = \langle T^\mu_\nu \rangle_0 + \langle T^\mu_\nu \rangle_b,
\]

(13)

with the boundary-free, \( \langle T^\mu_\nu \rangle_0 \), and boundary-induced, \( \langle T^\mu_\nu \rangle_b \), parts. For points outside the plate the local geometry is not changed by the presence of the plate and the divergences in \( \langle T^\mu_\nu \rangle \) and \( \langle T^\mu_\nu \rangle_0 \) are the same. Hence, with the decomposition (13), for the points outside the plate the renormalization of \( \langle T^\mu_\nu \rangle \) is reduced to the one for boundary-free dS spacetime. Because of the maximal symmetry of de Sitter spacetime and the Bunch-Davies vacuum state, the corresponding VEV of the energy-momentum tensor is proportional to the metric tensor: \( \langle T^\mu_\nu \rangle_0 = \text{const} \cdot \delta^\mu_\nu \).

Our main interest here is the boundary-induced part in the VEV. By using (3), (4) and (7), from (12) for the diagonal components of vacuum energy-momentum tensor one finds (no summation over \( l \))

\[
\langle T^0_0 \rangle_b = \frac{\alpha^{-D-1}}{A_D} \left\{ [2(1-y)\partial_y - D + 2] G(y) - [2(1-y)\partial_y + D - 4] F_D(y) \right\},
\]

\[
\langle T^l_l \rangle_b = \frac{\alpha^{-D-1}}{A_D} \frac{D-3}{D-1} \left\{ [2(1-y)\partial_y - (D-4)\frac{D-3}{D-1}] G(y) \right.
\]

\[
+ \left. [2(1-y)\partial_y + (D-4)\frac{D-1}{D-3} - 4] F_D(y) \right\}, \quad l = 1, \ldots, D-1,
\]

\[
\langle T^D_D \rangle_b = \frac{\alpha^{-D-1}}{A_D} [2(1-y)\partial_y - D] [F_D(y) - G_D(y)],
\]

(14)

with the notation

\[
A_D = (4\pi)^{(D+1)/2} (D + 1)\Gamma \left( \frac{D - 1}{2} \right).
\]

(15)
In addition to the diagonal components there is a non-zero off-diagonal component:

\[ \langle T_0^D \rangle_b = 4\alpha^{-D-1} \frac{xD}{\eta} \left[ (1 - y) \partial_y - 2 \right] F_D (y), \] (16)

which describes the energy flux along the direction normal to the plate. By using the expressions given above for the functions \( F_D (y) \) and \( G_D (y) \), we can see that the boundary-induced part in the VEV vanishes for \( D = 3 \). Again, this property is a consequence of the conformal invariance of the electromagnetic field in \( D = 3 \).

As it is seen from the expressions given above, the VEVs of the field squared and of the field vanishes for \( D = 3 \). At large proper distances from the plate, the stresses are isotropic. From (18) it follows that the decay of the stresses is oscillatory [5]. This effect is a qualitative new feature induced by the curvature of the Bunch-Davies vacuum state.

Simple expressions for the VEVs of the components of the energy-momentum tensor are obtained at small and large distances from the plate. At proper distances smaller than the dS curvature radius, \( z^D/\eta \ll 1 \), to the leading order one has (no summation over \( l = 0, \ldots, D - 1 \)):

\[ \langle T_l^D \rangle_b \approx -\frac{\eta}{z^D} \langle T_0^D \rangle_b \approx \frac{D - 1}{(z^D/\eta)^2} \langle T_D^D \rangle_b \approx -\frac{(D - 3)(D - 1)\Gamma((D + 1)/2)}{2(4\pi)^{(D+1)/2}(\alpha z^D/\eta)^{D+1}}. \] (17)

In this region, all diagonal components are negative for \( D \geq 4 \) and one has \( |\langle T_0^0 \rangle_b| \gg |\langle T_0^D \rangle_b| \). At large proper distances from the plate, compared with the dS curvature radius, \( z^D/\eta \gg 1 \), and for \( D > 4 \) for the leading terms in the corresponding asymptotic expansions we get (no summation over \( l = 1, \ldots, D \))

\[ \langle T_0^0 \rangle_b \approx \frac{D}{D - 4} \langle T_1^1 \rangle_b \approx -\frac{2^{D-4}D(D - 1)\Gamma(D/2 - 1)}{(4\pi)^{D/2+1}\alpha^{D+1}(z^D/\eta)^4}, \]
\[ \langle T_0^D \rangle_b \approx \frac{2^{D-2}(D - 1)\Gamma(D/2 - 1)}{(4\pi)^{D/2+1}\alpha^{D+1}(z^D/\eta)^5}. \] (18)

For \( D = 4 \) the asymptotic expressions for the components \( \langle T_0^0 \rangle_b \) and \( \langle T_0^D \rangle_b \) remain the same, whereas for the stresses one has (no summation over \( l = 1, \ldots, D \))

\[ \langle T_l^l \rangle_b = \frac{3\alpha^{-5} \ln(z^D/\eta)}{16\pi^3(z^D/\eta)^6}. \] (19)

As it is seen, at large distances the stresses are isotropic. From (18) it follows that the decay of the VEVs, as functions of the proper distance from the plate, at large distances is power-law. For massive scalar field with relatively large value of the mass (compared with the dS energy scale) the decay is oscillatory [5]. This effect is a qualitatively new feature induced by the curvature of the background spacetime.

On the left panel of figure 1 we have plotted the plate-induced part in the VEV of the electric field squared as a function of \( z^D/\eta \) for various values of \( D \) (numbers near the curves). Note that \( D = 9 \) corresponds to the critical dimension in string theories. On the right panel the boundary-induced parts in the components of the energy-momentum tensor, \( \langle T_l^l \rangle_b \), are plotted versus the distance from the plate. The numbers near the curves correspond to the value of the index \( l \). The dashed curve corresponds to the energy flux, \( \langle T_0^D \rangle_b \). As it is seen, the corresponding energy flux is directed from the plate.
Figure 1. The plate-induced parts in the VEVs of the field squared (left panel) and of the energy-momentum tensor (right panel) as functions of $z^D/\eta$. On the right panel the numbers near the curves are the corresponding values of $D$ and for the right panel $D = 9$.

4. Conclusion
Two-point function for the electromagnetic field tensor is evaluated in $(D + 1)$-dimensional de Sitter spacetime in the presence of a conducting plate. We have assumed that the field is prepared in the Bunch-Davies vacuum state. The VEVs of the field squared and of the energy-momentum tensor are decomposed into boundary-free and plate-induced parts. Because of the maximal symmetry of the background spacetime and of Bunch-Davies vacuum state, the boundary-free parts do not depend on spacetime coordinates. The diagonal components of plate-induced part in the vacuum energy-momentum tensor are negative and, hence, the effective pressures are positive. The VEV of the energy-momentum tensor contains a nonzero off-diagonal component which is positive and corresponds to the energy flux directed from the plate. The results obtained above can be used for the investigation of the vacuum characteristics in the geometry of two parallel plates, including the Casimir forces acting on the plates.

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