Envelop Dyadic Green’s Function for Uniaxial Metamaterials

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Based on the dyadic Green’s function (DGF) method, we present a formalism to study the propagation of electromagnetic fields with slowly varying amplitude (EMFSVA) in dispersive anisotropic media with two dyadic constitutive parameters, the dielectric permittivity and the magnetic permeability. We find the matrix elements of the envelope DGFs by applying the formalism for uniaxial anisotropic metamaterials. We present the relations for the velocity of the EMFSVA envelopes which agree with the known definition of the group velocity in dispersive media. We consider examples of propagation of the EMFSVA passing through active and passive media with the Lorentz and the Drude type dispersions, demonstrating beam focusing in hyperbolic media and superluminal propagation in media with inverted population. The results of this paper are applicable to the propagation of modulated electromagnetic fields and slowly varying amplitude fluctuations of such fields through dispersive and dissipative (or active) anisotropic metamaterials. The developed approach can be also used for the analysis of metamaterial-based waveguides, filters, and delay lines.

I. INTRODUCTION

Nowadays, there has been a growing interest in metamaterials (MMs) which are artificial composites used in various branches of science and engineering. After the pioneering works on the backward wave propagation in media \[1\]–\[4\], many scientists have focused on the appealing features of MMs such as the negative refractive index of active and passive MMs \[5\]–\[7\], the diffraction-unlimited imaging \[8\], and the remarkable control over the electromagnetic fields \[9\]. These features allowed for fabrication of flat lenses \[10\], invisible cloaking devices \[11\], and perfect electromagnetic absorbers \[12\]–\[13\]. MMs have been employed for the modeling of general relativity effects with artificial black holes \[14\] and to achieve frequency-agile or multi-band operation \[15\]–\[17\], to mediate the repulsive Casimir force \[18\], and to tune microwave propagation with light \[19\]. Another area in which MMs may have a broad impact that has recently attracted attention of science and technology is the near-field super-Planckian radiative heat transfer \[20\]–\[22\], with applications for thermophotovoltaics \[23\], \[24\].

In some situations, the unusual dispersive properties of the MMs result in superluminal or subluminal group velocities, \(V_g = \frac{\partial \omega}{\partial k}\), in which case this quantity can be higher \[25\] or extremely lower than the speed of light in a vacuum \[26\]. It can even become zero or negative \[27\]–\[29\]. These effects in MMs, as well as the negative refractive index, are well-accommodated within the framework of the causality principle \[30\]–\[33\]. In recent decades, superluminal and subluminal group velocities have attracted attention in nonlinear optics \[34\], quantum communication \[35\], photon controlling and storage \[36\]–\[38\], precision sensing \[39\], high-speed optical switching \[40\], \[41\], broadband electromagnetic devices and delay compensation circuits in ultra-high-speed communication systems \[42\], and in high resolution spectrometers \[43\].

In this article, based on the dyadic Green’s function (DGF) method, we present a self-consistent formalism to solve the Maxwell equations written for the electromagnetic fields with slowly varying amplitude (EMFSVA), when such fields propagate through a dispersive anisotropic medium with known dyadics of permittivity \((\vec{\varepsilon})\) and permeability \((\vec{\mu})\). Generalization to the case of bianisotropic media is as well possible. Here we focus on uniaxial anisotropic MMs, because they allow for a closed-form analysis. This formalism can be used, e.g. to study the propagation of electromagnetic fluctuations in super-Planckian radiative heat transfer systems \[22\], which is an actively developing topic in the context of MM applications. The uniaxial MMs are known for advantageous optical properties for sensing \[44\], nonlinear optical applications \[45\], and spontaneous emission control \[46\]. In this work, we examine our formalism on hyperbolic MMs and media with gain, in order to demonstrate the applicability of the method to the wave processes in such media, including the exotic superluminal processes in active media.

There is a significant body of literature on DGFs in anisotropic and bianisotropic media \[47\]–\[54\]. While in the most general case of bianisotropic medium there is no closed-form representation for the DGF, in the case of uniaxial magnetodielectrics, the time-harmonic DGF can be expressed through a pair of scalar electric and magnetic Green’s
functions. When interested in the EMFSVA in such media, the standard approach would be to start from such closed-form representations and expand the frequency-dependent parameters (such as propagation factors, etc.) in these relations into series around the carrier frequency \( \omega_0 \), with \( \Omega = \omega - \omega_0 \) being the small parameter. Although this would constitute a sound approach for the propagation of the EMFSVA in uniaxial media, in this article we develop a different method, which can be later extended to general bianisotropic media.

We start from introducing the slowly varying amplitudes (SVAs) of electromagnetic fields and write the Maxwell equations in terms of these quantities. With these equations, the dynamics of the EMFSVA can be studied directly, with material dispersion-related quantities like \( \frac{\partial \omega}{\partial \omega} \) and \( \frac{\partial \eta}{\partial \omega} \) appearing in these equations in a natural way. The envelope dyadic Green's function (EDGF) is then introduced as the solution of the corresponding dyadic equation with a point-like source. Note that, in contrast to the conventional DGF for the time-harmonic sources, the EDGF is a wavelet-type dyadic function that describes propagation of the amplitude fluctuations of quasi-monochromatic signals. These fluctuations propagate with a certain velocity that has the meaning of the group velocity. Despite being introduced in our formalism through a different way, this velocity agrees with the classical definition of the group velocity in dispersive media. Although not considered in this article, the presented formulation allows also for a direct finite-difference time-domain-based numerical solution of the dynamic equations for the EMFSVA, which can be useful, e.g. when studying propagation of wave packets in layered MMs.

In order to test our formalism, we consider the propagation of the EMFSVA through a hyperbolic medium and different method, which can be later extended to general bianisotropic media.

This article is organized as follows: In Sec.III we briefly set up the main tools to be used in our formalism for the EMFSVA; in Sec.IV, which employs the EDGF technique, the main formalism is presented; in Sec.V and VI we obtain the EDGF matrix elements for the uniaxial anisotropic MMs; in Secs. V and VI, we develop the paraxial approximation for the EDFG; in Sec.VII we study the above-mentioned effects associated with the propagation of the EMFSVA in active and passive media. Finally, we draw conclusions in Sec.VIII.

II. MAIN EMFSVA TOOLS AND DEFINITIONS

In order to investigate the propagation of the EMFSVA in a dispersive medium, we consider the Maxwell equations for the time-dependent electromagnetic fields and sources written as follows

\[
\nabla \cdot \mathbf{D} = \rho^E, \quad \nabla \cdot \mathbf{B} = \rho^M, \quad \nabla \times \mathbf{E} = -\partial_t \mathbf{B} - j^M, \quad \nabla \times \mathbf{H} = \partial_t \mathbf{D} + j^E, \tag{1}
\]

where the vectors \( \mathbf{E}, \mathbf{H}, \mathbf{D}, \) and \( \mathbf{B} \) are the electric field, the magnetic field, the electric displacement, and the magnetic flux density, respectively, and \( \partial_t = \frac{\partial}{\partial t} \). In Eq. (1), \( \rho^E(M) \) and \( j^E(M) \), the electric (magnetic) charge and current densities, respectively, are related by the continuity equation, \( \nabla \cdot j^E(M) + \partial_t \rho^E(M) = 0 \). The electromagnetic constitutive equation is given by

\[
\Xi = \tilde{\Xi} \cdot \mathbf{f}, \tag{2}
\]

where the dyadic integro-differential operator \( \tilde{\Xi} = \tilde{\Xi} (\mathbf{B}) \) represents the dispersive anisotropic permittivity (permeability) of the material, which relates \( \Xi = \mathbf{D} (\mathbf{B}) \) to \( \mathbf{f} = \mathbf{E} (\mathbf{H}) \).

The time-dependent electromagnetic field can be expanded into the monochromatic spectral components as follows

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_\omega (\omega) e^{-i\omega t} d\omega. \tag{3}
\]

On the other hand, considering the time-harmonic fields with SVAs, it is acceptable that the most spectral energy is concentrated in a narrow band \( \Delta \omega \) around \( \omega_0 \), the carrier frequency, with \( \Delta \omega \ll \omega_0 \). Thus, we can define the EMFSVA, \( f_m(t) \), as follows

\[
f(t) = \frac{1}{2} f_m(t) e^{-i\omega_0 t} + \text{c.c.}, \quad f_m(t) = \frac{1}{\pi} \int_{-\Delta \omega/2}^{+\Delta \omega/2} f_\omega (\omega_0 + \Omega) e^{-i\Omega t} d\Omega. \tag{4}
\]
where c.c. is the abbreviation for the complex conjugate of the previous term and \( \Omega = \omega - \omega_0 \). We also assume that \( \tilde{\zeta}_\omega \), the Fourier transform of \( \tilde{\zeta} \), has appreciably smooth behavior in the narrow band \( \Delta \omega \), which is a reasonable assumption for applications of our interest. So, expanding \( \tilde{\zeta}_\omega \) around \( \omega_0 \) and keeping only the two first terms, we obtain

\[
\tilde{\zeta}_\omega = \tilde{\zeta}_\omega|_{\omega_0} + (\partial_\omega \tilde{\zeta}_\omega)|_{\omega_0} \Omega,
\]

where \( \partial_\omega = \frac{d}{d\omega} \) and \( |_{\omega_0} \) denotes that the related quantity is computed at \( \omega_0 \). Using Eqs. (4)–(5), we obtain \( \partial_\omega \mathbf{\Xi} \) as follows

\[
\partial_\omega \mathbf{\Xi} = e^{-i\omega_0 t}(\partial_t - i\omega_0)\mathbf{\Xi}_{m}(t), \quad \mathbf{\Xi}_{m}(t) = \left(\tilde{\zeta}_\omega|_{\omega_0} + i(\partial_\omega \tilde{\zeta}_\omega)|_{\omega_0} \partial_t\right) \cdot \mathbf{f}_m(t) \equiv \tilde{\mathbf{\Xi}}_m \cdot \mathbf{f}_m(t),
\]

where the dyadic differential operator \( \tilde{\mathbf{\Xi}}_m \) is \( \tilde{\mathbf{\Xi}}_m = \tilde{\zeta}_\omega|_{\omega_0} + i(\partial_\omega \tilde{\zeta}_\omega)|_{\omega_0} \partial_t \) and we have to ignore the second and higher orders of \( \partial_t \) in expressions involving this operator. With having these tools at hand, in the next section we proceed to the Green’s function technique to solve the Maxwell equations written in terms of the EMFSVA.

### III. ENVELOPE DGF TECHNIQUE

Starting from Eq. (1) for the time-dependent electromagnetic fields and sources and using Eqs. (2)–(6), the Maxwell 6-vector equations for the EMFSVA assume the following operator form:

\[
\hat{O} \cdot \mathbf{F}_m = -i\mathbf{J}_{m}.
\]

In the above equation, \( \mathbf{F}_m \), the SVA of the 6-vector field, \( \mathbf{J}_{m} \), the 6-vector current density, and \( \hat{O} \), the electromagnetic operator, are given by

\[
\mathbf{F}_m \equiv \begin{bmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{bmatrix}, \quad \mathbf{J}_m \equiv \begin{bmatrix} \mathbf{I}^E_m \\ \mathbf{I}^H_m \end{bmatrix}, \quad \hat{O} \equiv \begin{bmatrix} (\partial_t - i\omega_0) \tilde{\mathbf{E}}_m - \nabla \times \tilde{\mathbf{I}} \\ \nabla \times \tilde{\mathbf{I}} \end{bmatrix},
\]

where \( \tilde{\mathbf{I}} \) is the identity dyadic. In Eq. (8), the differential operators \( (\partial_t - i\omega_0) \tilde{\mathbf{E}}_m \) are expanded as

\[
(\partial_t - i\omega_0) \tilde{\mathbf{E}}_m = -i\omega_0 \tilde{\mathbf{E}}_m|_{\omega_0} + \partial_\omega(\omega \tilde{\mathbf{E}}_m)|_{\omega_0} \partial_t,
\]

with the second derivative order term ignored, in line with the assumption of the SVA. We may introduce the envelope 6-dyadic Green’s function (EDGF), \( \overline{\mathbf{G}}(\mathbf{r}, t; \mathbf{r}', t') \), for Eq. (7) such that

\[
\hat{O} \cdot \overline{\mathbf{G}} = -i\delta(t - t')\delta(t - t') \tilde{\mathbf{I}},
\]

where \( \tilde{\mathbf{I}} \) is the identity 6-dyadic, and \( \delta(t) \), similar in role to the Dirac delta \( \delta(\cdot) \), emphasizes that we consider the slowly varying sources and fields. Respectively, \( \delta(t) \) has a finite duration and \( |\delta(t)| < \infty \). Eq. (10) means that the EDGF is obtained by inverting the operator \( \hat{O} \).

It is more convenient to work in the Fourier domain when the material parameters are uniform in both space and time. In this case, \( \overline{\mathbf{G}}(\mathbf{r}, t; \mathbf{r}', t') \equiv \overline{\mathbf{G}}(\mathbf{r} - \mathbf{r}', t - t') \), and we can define \( \overline{\mathbf{g}}(\mathbf{k}, \Omega) \), the Fourier transform of the EDGF, by the following relation

\[
\overline{\mathbf{G}}(\mathbf{R}, \tau) = \frac{1}{(2\pi)^4} \int d\mathbf{k} \int_{-\Delta\omega/2}^{+\Delta\omega/2} d\Omega \overline{\mathbf{g}}(\mathbf{k}, \Omega)e^{i(k\cdot R - \Omega \tau)},
\]

where \( \mathbf{R} = \mathbf{r} - \mathbf{r}' \) and \( \tau = t - t' \). Recalling Eq. (10) and using that, in the Fourier space, \( \partial_t = -i\Omega \) and \( \nabla = i\mathbf{k} \), where \( \mathbf{k} \) is the wave vector, we find that the 6-dyadic equation for \( \overline{\mathbf{g}} \) satisfies

\[
\hat{O} \cdot \overline{\mathbf{g}} = -\tilde{\mathbf{I}}, \quad \hat{O} \equiv \begin{bmatrix} D_\Omega \tilde{\mathbf{E}}_\omega \\ -\mathbf{k} \times \tilde{\mathbf{I}} \end{bmatrix}, \quad \begin{bmatrix} D_\Omega \tilde{\mathbf{E}}_\omega \\ -\mathbf{k} \times \tilde{\mathbf{I}} \end{bmatrix} = \begin{bmatrix} \mathbf{k} \times \tilde{\mathbf{I}} \\ D_\Omega \tilde{\mathbf{E}}_\omega \end{bmatrix},
\]

where \( D_\Omega = \frac{\partial^2}{\partial \omega^2} \).
where we have used operator-like notation for $D_{\Omega}$ such that
\[
D_{\Omega} \tilde{\zeta}_w = \omega [\tilde{\zeta}_w]_{\omega=\Delta \omega/2} + \Omega \partial_\omega [\omega \tilde{\zeta}_w]_{\omega=\Delta \omega/2},
\]
and the fact that $\delta(t) = (2\pi)^{-1} \int_{-\Delta \omega/2}^{+\Delta \omega/2} e^{-i\omega t} \, d\Omega$.

Knowing the dispersive constitutive parameters of the medium and inverting the operator $\hat{O}$ in Eq. (12), we can obtain $\hat{g}$ and, in turn, the EDGF. Afterward, for any given forms of the 6-vector source functions $J_m$ with the frequency spectra fitting the interval $\Omega \in \left[ -\frac{\omega}{2}, +\frac{\omega}{2} \right]$, the components of the 6-vector $F_m$ are obtained with the help of Kotelnikov’s theorem [58] (see Appendix A) as follows
\[
F_m(r, t) = \frac{2\pi}{\Delta \omega} \sum_{m=-\infty}^{+\infty} \int dr' G(r - r', t - t_m) \cdot J_m(r', t_m),
\]
where $t_m = \frac{2\pi m}{\Delta \omega}$. In the next section, we shall find the matrix elements of the EDGF for uniaxial anisotropic media.

IV. EDGF MATRIX ELEMENTS

Here, we apply the EDGF technique for reciprocal uniaxial media to obtain the matrix elements. For such media we can write
\[
\tilde{\zeta}_w = \zeta_0 \tilde{I}_1 + \zeta_\Omega \tilde{I}_2
\]
with $\tilde{I}_1 = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$ and $\tilde{I}_2 = \hat{\mathbf{z}}$ being the projection dyadics in the transverse and axial (the main axis) directions, respectively, along which the electromagnetic responses of the medium are different. In order to obtain the EDGF of the uniaxial medium, we impose this property of constitutive parameters to decompose the operator $\hat{O}$ in Eq. (12) as
\[
\hat{O} \approx \begin{bmatrix}
D_{\Omega} \tilde{\zeta}_1 & 0 & \mathbf{k}_c \times \tilde{\zeta}_1 & \mathbf{k}_c \times \tilde{\zeta}_2 \\
0 & D_{\Omega} \tilde{\zeta}_2 & \mathbf{k}_t \times \tilde{\zeta}_1 & 0 \\
-k_c \times \tilde{\zeta}_1 & -k_t \times \tilde{\zeta}_2 & D_{\Omega} \mu_1 \tilde{\zeta}_1 & 0 \\
-k_t \times \tilde{\zeta}_1 & 0 & D_{\Omega} \mu_2 \tilde{\zeta}_2 & 0
\end{bmatrix},
\]
where $\tilde{\zeta}_0$ is the null dyadic. When writing the matrix elements in Eq. (16), we keep in mind that $\mathbf{k} \times \tilde{I} = (\mathbf{k}_t + \mathbf{k}_c) \times (\tilde{I}_1 + \tilde{I}_2)$, where $\mathbf{k}_t = \tilde{I}_1 \cdot \mathbf{k}$ and $\mathbf{k}_c = \tilde{I}_2 \cdot \mathbf{k}$. Next, it is more convenient if we replace the second row with the third one and also the second column with the third one in the matrix representation of $\hat{O}$ in Eq. (16) so that the tangential and axial components of the constitutive parameters of the medium take place in separate blocks. When doing this, it is also necessary to respectively rearrange the dyadic components of $\tilde{\zeta}_0$ and $\tilde{I}$ in Eq. (12). After doing this and taking into account that $\mathbf{k}_t \times \tilde{I}_1 = \tilde{I}_2 \times \mathbf{k}_t$, the EDGF is expressed through the inverse of the reordered matrix of $\hat{O}$ as follows
\[
\tilde{\zeta}_w = \begin{bmatrix}
\bar{\zeta}_1 \\
\bar{\zeta}_2 \\
\bar{\zeta}_3 \\
\bar{\zeta}_4
\end{bmatrix} = -i \begin{bmatrix}
D_{\Omega} \tilde{\zeta}_1 & \mathbf{k}_c \times \tilde{\zeta}_1 & \mathbf{k}_c \times \tilde{\zeta}_2 & 0 \\
0 & D_{\Omega} \tilde{\zeta}_2 & \mathbf{k}_t \times \tilde{\zeta}_1 & 0 \\
-k_c \times \tilde{\zeta}_1 & -k_t \times \tilde{\zeta}_2 & D_{\Omega} \mu_1 \tilde{\zeta}_1 & 0 \\
-k_t \times \tilde{\zeta}_1 & 0 & D_{\Omega} \mu_2 \tilde{\zeta}_2 & 0
\end{bmatrix}^{-1}
\]
\[
\bar{\zeta}_3 = \frac{\bar{\zeta}_1 \times \mathbf{k}_t}{\bar{\zeta}_2} + \bar{\zeta}_2 \times \frac{\mathbf{k}_t \times \bar{\zeta}_1}{\bar{\zeta}_2}.
\]

Representation (17) allows for a straightforward splitting of the fields into a pair of orthogonal polarizations: The transverse-electric (TE or s-) polarization with vanishing axial component of the electric field, and the transverse-magnetic (TM or p-) polarization with vanishing axial component of the magnetic field. In order to perform such splitting, $\tilde{I}_1$ in Eq. (17) is expanded as $\tilde{I}_1 = \frac{\mathbf{k}_t \mathbf{k}_c - \mathbf{k}_c \mathbf{k}_t}{\mathbf{k}_c \mathbf{k}_c}$, where $\times$ denotes the dyadic double cross product: $\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \times \mathbf{d}$ and $k_c = |\mathbf{k}_c|$. Then, after a rather tedious but straightforward dyadic algebra, the matrix in Eq. (17) can be inverted and the following result obtained:
\[
\mathcal{g}^t = \left[ \frac{i t^p k \cdot k_\perp}{\mathcal{S}_m k_\perp^2} \mathcal{S}_e k_\perp^2 \right] + \left[ \frac{i t^p \cdot k \cdot k_\perp}{\mathcal{S}_m k_\perp^2} \mathcal{S}_e k_\perp^2 \right],
\]

(18)

\[
\mathcal{g}^z = \left( \mathcal{g}^T \right) = \left[ \frac{i t^z \cdot k \cdot k_\perp}{\mathcal{S}_m k_\perp^2} \mathcal{S}_e k_\perp^2 \right]
\]

(19)

\[
\mathcal{g}^{zz} = \left[ \begin{array}{cc}
\frac{\mathcal{S}_m k_\perp^2}{\mathcal{S}_e} & 0 \\
0 & \frac{\mathcal{S}_m k_\perp^2}{\mathcal{S}_e}
\end{array} \right]
\]

(20)

where the 12 independent non-vanishing components of \( \mathcal{g} \) are expressed as follows (here and thereafter we use shorthand notation \( \alpha[\beta] \) for respective selection of either \( \alpha \) or \( \beta \), where \( \alpha \) and \( \beta \) are isolated terms or groups of indices):

\[
\mathcal{S}^t_{e[mm]} = -i \frac{d^m[e]}{k^2_{\perp[e]} - k_z^2}, \quad \mathcal{S}^{tz}_{e[em]} = -i \frac{d^m[e]}{k^2_{\perp[e]} - k_z^2},
\]

(21)

\[
\mathcal{S}^{zz}_{e[em]} = -i k_z \frac{d^m[e]}{k^2_{\perp[e]} - k_z^2} \mathcal{S}^{zz}_{e[em]} = -i \frac{d^m[e]}{k^2_{\perp[e]} - k_z^2},
\]

(22)

where, from Eq. (13),

\[
d^m[e] = a^m[e] + b^m[e], \quad a^m[e] = \omega_0 \varepsilon_0 [\mu_1]_{\omega_0}, \quad b^m[e] = \partial_\omega (\omega \varepsilon_0 [\mu_1]_{\omega_0}),
\]

(23)

where the index \( l \) is either \( t \) or \( z \), and

\[
\kappa_{\parallel[e]} = \sqrt{d^m[e] \mathcal{S}^{zz}_{e[em]} - (d^m[e] / d^m_z[e]) k^2_z}, \quad \text{Im}(\kappa_{\parallel[e]}) \geq 0.
\]

(24)

As can be seen from Eqs. (21) and (22), there are two poles in \( k_z \) for each polarization: \( \pm \kappa_z \) for the TM case and, similarly, \( \pm \kappa_s \) for the TE one. Physically, the poles \( k_z = \pm \kappa_z \) correspond to the waves propagating in the halfspace \( z - z' > 0 \), and the poles with \( k_z = - \kappa_z \) correspond to the waves propagating at \( z - z' < 0 \).

Based on Eqs. (21) and (22), we can sort out the components of \( \mathcal{g} \) into two groups which correspond to the waves of \( p \)- and \( s \)-polarization

\[
\mathcal{g} = \mathcal{g}_p + \mathcal{g}_s,
\]

(25)

where \( \mathcal{g}_{\parallel} \) are formed by the corresponding terms in the numerators of Eqs. (21) and (22). Respectively, when taking the inverse Fourier transform of \( \mathcal{g} \) as given by Eq. (11) and considering the integral over \( dk_z \), we get two categories of residues

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \mathcal{g} e^{ik_z Z} = -\sum_{\gamma=p,s} \frac{i}{2\kappa_{\parallel}} \mathcal{A}_{\gamma}[\kappa_{\parallel} e^{i\kappa_{\parallel} Z}] = \sum_{\gamma=p,s} \mathcal{A}_{\gamma}[\kappa_{\parallel} e^{i\kappa_{\parallel} Z}],
\]

(26)

where \( Z = z - z' \), and \( \text{sgn}(Z) = \pm 1 \) is the sign of \( Z \).

Next, before taking the integral over \( d\Omega \), we recall the approximation of the SVA, and expand \( \mathcal{A}_{\gamma,\pm 1} \) and \( \kappa_{\gamma} \) around the point \( \Omega = 0 \) and ignore \( O(\Omega^2) \) terms so that

\[
\mathcal{A}_{\gamma,\pm 1} \approx \mathcal{A}_{\gamma,\pm 1} \bigg|_{\Omega=0} + \Omega \frac{\partial \mathcal{A}_{\gamma,\pm 1}}{\partial \Omega} \bigg|_{\Omega=0} = \mathcal{C}_{\gamma,\pm 1} + \Omega \mathcal{D}_{\gamma,\pm 1},
\]

(27)
\[ \kappa_\gamma = \kappa_\gamma|_{\Omega=0} + \Omega \frac{\partial \kappa_\gamma}{\partial \Omega}|_{\Omega=0} \equiv \kappa_0 + \Omega/V_\gamma^\prime, \]  

where the expressions for the components of \( C_{\gamma;1} = \overline{A}_{\gamma;1} \) and \( D_{\gamma;1} = \overline{A}_{\gamma;1} / \partial \Omega \) are given in Appendix B, and for the propagation factor \( \kappa_0^\prime = \kappa_\gamma|_{\Omega=0} \) and the complex group velocity \( V_\gamma^\prime = \left( \partial \kappa_\gamma / \partial \Omega \right)^{-1} \) we obtain

\[ \kappa_0^{[p]} = \sqrt{d_1^m a_m^m - d_z^m a_z^m k_z^2}, \]

\[ V_\gamma^{[p]} = \frac{2\kappa_0^{[p]}}{\left( d_1^m b_1^m + d_z^m b_z^m \right) - \left( d_1^m a_1^m - d_z^m a_z^m \right) k_z^2 / a_z^m k_z^2}, \]

for the \( p \)-[s] polarization. It should be noted that ignoring the second order terms in Eqs. (27) and (28) is reasonable for the frequency intervals of our interest, where the group velocity dispersion effects are relatively weak. In Appendix C, we discuss this with more detail and also present a closer look at the group velocity for a propagating envelope in an active and dispersive medium.

With these approximations at hand, when performing the integration over \( \Omega \) we obtain

\[ \frac{1}{2\pi} \int_{-\Delta \omega / 2}^{+\Delta \omega / 2} d\Omega \sum_{\gamma,p,s} \left( \overline{C}_{\gamma;\text{sgn}(Z)} + \Omega \overline{D}_{\gamma;\text{sgn}(Z)} \right) e^{i\kappa_0^\prime(2\Omega)\pm\Omega\tau_\gamma^\prime} = \frac{\Delta \omega}{2\pi} \sum_{\gamma,p,s} e^{i\kappa_0^\prime[Z]} \left[ \overline{C}_{\gamma;\text{sgn}(Z)} j_0 \left( \Delta \omega \tau_\gamma^\prime / 2 \right) - \frac{i \Delta \omega}{2} \overline{D}_{\gamma;\text{sgn}(Z)} j_1 \left( \Delta \omega \tau_\gamma^\prime / 2 \right) \right], \]

where \( \tau_\gamma^\prime = \tau - |Z|/V_\gamma^\prime \), and \( j_n(x) \) denotes the spherical Bessel function of the first kind and the \( n \)-th order. Therefore, from Eq. (11), we obtain the EDGF, \( \overline{G}(R, \tau) \), in the following form

\[ \overline{G} = \frac{\Delta \omega}{(2\pi)^3} \sum_{\gamma,p,s} \int d\mathbf{k} e^{i(k_z R + \kappa_0^\prime(2\mathbf{R}))} \times \left[ \overline{T}_{\gamma;\text{sgn}(Z)} j_0 \left( \Delta \omega \tau_\gamma^\prime / 2 \right) - \frac{i \Delta \omega}{2} \overline{D}_{\gamma;\text{sgn}(Z)} j_1 \left( \Delta \omega \tau_\gamma^\prime / 2 \right) \right], \]

where

\[ \overline{T}_{\gamma;\pm 1}(\chi) = \overline{C}_{\gamma;\pm 1} j_0(\chi) - \frac{i \Delta \omega}{2} \overline{D}_{\gamma;\pm 1} j_1(\chi). \]

In some special cases, the remaining integration over \( d\mathbf{k} \) can be performed analytically. In Sec. V, we consider such a special case of paraxial propagation. The representation (32) is most suitable for the calculation of fields of sources with a known spatial spectrum in the transverse plane, e.g. in near-field radiative heat transfer problems. Considering arbitrary source vectors and using Eqs. (14) and (32), we can investigate the propagation of the EMFSVA in any unbounded anisotropic dispersive media.

The present formalism can be extended to contain interface effects, which will make it applicable to investigate the propagation of the EMFSVA via multilayer media. In Sec. VII, we consider a special case of such media when the neighboring layers are (approximately) impedance matched.

V. EDGF FOR PARAXIAL PROPAGATION

Let us consider the case when the envelope propagation happens dominantly along the anisotropy axis. This case is typical for extremely anisotropic uniaxial MM in which \( |\epsilon_z| \gg |\epsilon_i| \) and (or) \( |\mu_z| \gg |\mu_i| \). Indeed, the propagation factor \( \kappa_0^\prime \) from Eq. (29) can be expressed as

\[ \kappa_0^{[p]} = \kappa_a \sqrt{1 - \frac{d_z^m k_z^2}{d_z^m k_z^2}}, \]

where \( \kappa_a = \sqrt{d_z^m k_z^2} \). When \( |d_z^m k_z^2| \ll |d_z^m k_z^2| \), we can expand \( \kappa_0^\prime \) as follows

\[ \kappa_0^{[p]} \approx \kappa_a - \frac{1}{2} \frac{d_z^m k_z^2}{\kappa_a}. \]
On the other hand, from Eqs. (21) and (22) it is seen that in this case the $\mathbf{g}_0$ component dominates over the $\mathbf{g}_{1r}$, $\mathbf{g}_{1z}$ and $\mathbf{g}_{2z}$ components, so that the EDGF is dominantly transverse which implies that the wave energy propagates dominantly along the $z$-axis.

Therefore, when calculating the integral over $d\boldsymbol{k}_t$ in Eq. (32), we will not make a big mistake if we evaluate the $\mathbf{T}_{t,n+1}$ term of Eq. (32) at $k_t \to 0$ while using the expansion (35) in the exponential terms. In this approximation, the integration over $d\boldsymbol{k}_t$ can be performed analytically, which results in the following expression for the $tt$-block of the paraxial EDGF:

$$
\mathbf{g}_{tt}^i(x' - x, y - y', \pm|z - z'|, t' - t) = \frac{\Delta \omega}{(2\pi)^2} \sum_{\gamma=p,s} \int_{d\xi} \mathbf{t}_{\gamma t} e^{i\xi (|z' - z| \pm (x - x')^2 + (y - y')^2)} 2\xi,
$$

(36)

where $\xi_{\gamma l} = a_{\gamma l}^0 |z - z'|$, and

$$
\mathbf{t}_{\gamma t} = \left[ \begin{array}{c} \frac{a_{\gamma p}^0 \sqrt{V_{\gamma t} \alpha}}{2a_{\gamma p}^0} + \frac{\sqrt{V_{\gamma t} \alpha}}{2a_{\gamma p}^0} \frac{\partial^2 \mathbf{V}_{\gamma t} \sqrt{V_{\gamma t} \alpha}}{2a_{\gamma p}^0} \frac{\Theta_{\gamma t}}{2} \end{array} \right] j_0(\tau) - i \Delta \omega \left[ \begin{array}{c} \frac{a_{\gamma p}^0 \sqrt{V_{\gamma t} \alpha}}{2a_{\gamma p}^0} + \frac{\sqrt{V_{\gamma t} \alpha}}{2a_{\gamma p}^0} \frac{\partial^2 \mathbf{V}_{\gamma t} \sqrt{V_{\gamma t} \alpha}}{2a_{\gamma p}^0} \frac{\Theta_{\gamma t}}{2} \end{array} \right] j_1(\tau),
$$

(37)

$$
\mathbf{t}_{\gamma s} = \left[ \begin{array}{c} \frac{a_{\gamma s}^0 \sqrt{V_{\gamma s} \alpha}}{2a_{\gamma s}^0} + \frac{\sqrt{V_{\gamma s} \alpha}}{2a_{\gamma s}^0} \frac{\partial^2 \mathbf{V}_{\gamma s} \sqrt{V_{\gamma s} \alpha}}{2a_{\gamma s}^0} \frac{\Theta_{\gamma s}}{2} \end{array} \right] j_0(\tau) - i \Delta \omega \left[ \begin{array}{c} \frac{a_{\gamma s}^0 \sqrt{V_{\gamma s} \alpha}}{2a_{\gamma s}^0} + \frac{\sqrt{V_{\gamma s} \alpha}}{2a_{\gamma s}^0} \frac{\partial^2 \mathbf{V}_{\gamma s} \sqrt{V_{\gamma s} \alpha}}{2a_{\gamma s}^0} \frac{\Theta_{\gamma s}}{2} \end{array} \right] j_1(\tau),
$$

(38)

with $V_{\gamma} = \hat{x} (\partial / \partial x) + \hat{y} (\partial / \partial y)$, $\tau = \Delta \omega \left( t - t' - |z - z'| / V_{\gamma}^0 \right)$, and $V_{\gamma}^0 = 2\kappa_{\gamma} / (a_{\gamma p}^0 + a_{\gamma s}^0 b_{\gamma}^0)$. The integral over $d\xi$ in Eq. (36) can be taken in a closed form after the dyadic differential operators $\mathbf{t}_{\gamma t}$ have acted on the exponential term. The integral representation (36) is especially handy when taking the convolution integral in Eq. (14) with the source currents having Gaussian profiles in the $xy$-plane.

VI. PROPAGATION THROUGH IMPEDANCE MATCHED MM LAYERS

For the sake of this section, we apply the developed formalism for multilayer uniaxial MMs in which layers are approximately impedance matched. The anisotropy axis is along the $z$-axis and is the same in all layers. The interfaces of the layers are perpendicular to the $z$-axis and are located at planes $z = z_l$, where $l$ is the layer index. Here, we are interested only in the paraxial propagation.

As the source of excitation, we consider oscillating surface electric current distributed in the plane $z = z'$ with some amplitude profile along the $x$-direction. The plane $z = z'$ happens to be inside one of the layers (which we will call the 0-th layer) located at $z \in (z_0, z_1)$. In the $y$-direction, the source current is uniform. The source current density is oscillating in time with the carrier frequency $\omega_0$ and has the SVA envelope of oscillations, $J_{m0}^E$, defined as

$$
J_{m0}^E = \sum_{\gamma=p,s} \int_{-\infty}^{\infty} j_{\gamma} \hat{u}_x e^{-\frac{x^2}{\sigma_x^2} - \frac{z^2}{\sigma_z^2}} \delta(z - z'),
$$

(39)

where $j_{\gamma} \hat{u}_x$ with $\gamma = p, s$, determines the initial vectorial amplitudes of the $s$- and $p$-polarized components of the surface current (here, $\hat{u}_x = \hat{x}$ and $\hat{u}_s = \hat{y}$), $\sigma_{\gamma l}$ defines the characteristic width of the amplitude profiles in $x$, separately for the two polarizations, and where we assume that $\sigma_{l0}$, the envelope duration in time, is such that $\Delta \omega / \sigma_{l0} \geq 1$. Under this condition, practically all source spectral power is concentrated within the frequency interval of width $\Delta \omega$. Thus, the SVA of the 6-vector source current density reads

$$
J_{m0}(x, z, t) = J_{00}(x, t) \delta(z - z'), \quad J_{00}(x, t) = \sum_{\gamma=p,s} \int_{-\infty}^{\infty} j_{\gamma} \hat{u}_x e^{-\frac{x^2}{\sigma_x^2} - \frac{z^2}{\sigma_z^2}}.
$$

(40)

Such a source creates the electromagnetic field in the 0-th layer which propagates in both $z > z'$ and $z < z'$ directions. When this field reaches the interface $z = z_1$ between the 0th and the next layer, it excites the fields in the next layer, and so on. If the characteristic wave impedances of the neighboring layers are mismatched, the reflected field will also appear, which can propagate to the other interface, be partially reflected again, etc.
We reserve the study of such multiple reflections in the EDGF context for a future work. Here, we assume that the neighboring layers are approximately impedance matched at the frequencies close to \( \omega_0 \), and the reflections may be neglected. Note that this does not mean that the layers must be made of the same materials, or that the materials must have the same dispersion. The impedance match condition for the case of the paraxial propagation considered in this section, means that the material parameters of the \( l \)-th and \((l+1)\)-th layer satisfy

\[
\frac{\varepsilon_l}{\mu_0} \mid_{\omega_0, l} \approx \frac{\varepsilon_l}{\mu_0} \mid_{\omega_0, l+1}.
\]

Under this assumption, the EMFSA created by the source \( S_0 \) in the 0-th layer can be obtained from Eqs. (40) and (36) and expressed in the following form, after evaluating all involved integrals:

\[
F_m(x, \pm|z-z'|, t; \vec{\varepsilon}_0, \vec{\mu}_0, J_0) = \sum_{\gamma=\pm} \sigma_\gamma e^{-i\kappa_0|z'-z| - \frac{\kappa_0^2}{2} t^2} \sum_{m=-\infty}^{\infty} \tilde{h}_{m\gamma}(\tau_n) \cdot J_0(0, \tau_n),
\]

where \( \tau_n = \frac{\Delta \omega}{2}(t_n - \frac{|z-z'|}{v_s}), \bar{\sigma}_{\gamma m} = \sqrt{\sigma_\gamma^2 + \frac{\varepsilon_0^m|\gamma-z'|}{\varepsilon_0^m v_s}}, \) and

\[
\tilde{h}_{m\gamma}(\tau) = \left[ -\frac{\varepsilon_0^m \kappa_0}{\kappa_0^2 + \kappa_0^2} \pm \frac{j \kappa_0}{\kappa_0^2 + \kappa_0^2} \right] j_0(\tau) - \frac{i \Delta \omega}{2} \left( \frac{b_m^m}{a^m} - \frac{b_l^m}{a_l^m} \right) \left[ \frac{a^m a^m}{a^m a^m} \right] \tilde{J}_m(\tau),
\]

\[
\tilde{h}_{m\gamma}(\tau) = \left[ -\frac{\varepsilon_0^m \kappa_0}{\kappa_0^2 + \kappa_0^2} \pm \frac{j \kappa_0}{\kappa_0^2 + \kappa_0^2} \right] J_0(\tau) - \frac{i \Delta \omega}{2} \left( \frac{b_m^m}{a^m} - \frac{b_l^m}{a_l^m} \right) \left[ \frac{a^m a^m}{a^m a^m} \right] \tilde{J}_l(\tau),
\]

where the parameters \( a^m_j, b^m_j \), etc. are expressed through the components of the material dyadics of the 0-th layer, \( \vec{\varepsilon}_0 \) and \( \vec{\mu}_0 \).

In order to find the fields in the next layer located at \( z \in (z_1, z_2) \) (the 1st layer), we introduce an equivalent Huygens source (a pair of electric and magnetic surface currents) placed at \( z = z_1 \) (where \( z_1 > z' \) defined as

\[
J_m(x, z_1, t) = J_1(x, t) \delta(z - z_1), \quad J_1(x, t) = \left[ \begin{array}{c} \vec{0} \\ \vec{L} \times \vec{I}_L \\ -\vec{L} \times \vec{I}_L \\ \vec{0} \end{array} \right] \cdot F_m(x, z_1 - z', t; \vec{\varepsilon}_0, \vec{\mu}_0, J_0).
\]

The field in the layer \( z \in (z_1, z_2) \) can be found from Eq. (42) as \( F_m(x, z-z_1, t; \vec{\varepsilon}_1, \vec{\mu}_1, J_1) \), where \( \vec{\varepsilon}_1 \) and \( \vec{\mu}_1 \) are the material dyadics of the 1st layer, from which the equivalent Huygens source \( J_2 \) at \( z = z_2 \) is expressed by Eq. (45) through \( F_m(x, z_2 - z_1, t; \vec{\varepsilon}_1, \vec{\mu}_1, J_1) \), etc. This procedure is repeated as many times as there are layers at \( z > z' \), after which the fields in the layers located at \( z < z' \) can be found in a completely analogous way.

**VII. NUMERICAL EXAMPLES**

A. Negative refraction and focusing by uniaxial MM with hyperbolic dispersion

When the transverse and longitudinal components of the permittivity dyadic of a MM have opposite signs, the isofrequency curves for the \( p \)-polarized waves are hyperbolas. It is known that the \( p \)-polarized light refracts negatively when impinging at the interface of a conventional material and such a hyperbolic MM \([53, 56]\). The following numerical example, we study the implications of this phenomenon on the paraxial propagation of the Gaussian envelopes considered in Sec. \( \text{VI} \). We shall confirm that the EDGF formalism correctly predicts focusing of a diverging \( p \)-polarized Gaussian beam by the hyperbolic MM.

At near-infrared frequencies, a hyperbolic MM can be realized, e.g. by embedding vertically aligned metallic nanowires into an isotropic dielectric host. For the sake of a numerical example, here we consider a MM formed by golden nanowires embedded into alumina substrate. By using the Maxwell-Garnett effective medium theory
Ex Ey (arb. u.)

| 0.00 | 0.00 |
|------|------|
| 0.05 | 0.05 |
| 0.10 | 0.10 |
| 0.15 | 0.15 |
| 0.20 | 0.20 |
| 0.25 | 0.25 |
| 0.30 | 0.30 |

\( \lambda \) follows the Drude dispersion model \([61]\)

follows the Drude dispersion model \([61]\)

\[
\varepsilon_{\text{eff},t} = \varepsilon_{\text{h}} \varepsilon_{\text{m}}(1 - f) + \varepsilon_{\text{h}}(1 - f)
\]

\[
\varepsilon_{\text{eff},z} = \varepsilon_{\text{m}} f + \varepsilon_{\text{h}}(1 - f)
\]

where \( \varepsilon_{\text{h}} \) and \( \varepsilon_{\text{m}} \) are the dielectric permittivities of the host material (Al\(_2\)O\(_3\) \([60]\)) and the plasmonic metal (Au), respectively, and \( f \) is the nanowires volume fraction. The relative permittivity of gold at near-infrared frequencies follows the Drude dispersion model \([61]\)

\[
\varepsilon_m(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\tau)}
\]

At the frequencies below the plasma frequency \( \omega_p \), \( \text{Re}(\varepsilon_m(\omega)) < 0 \), and one can achieve \( \text{Re}(\varepsilon_{\text{eff},z}(\omega)) < 0 \) with a proper choice of the nanowires volume fraction \( f \).

Let us consider a structure comprised of a hyperbolic MM sandwiched between two isotropic dielectrics with relative permittivity \( \varepsilon_d = 3.8 \) (e.g. aluminum nitride \([60]\)). The volume fraction of Au nanowires embedded into Al\(_2\)O\(_3\) substrate is \( f = 0.15 \). The carrier frequency is set to \( \hbar \omega_0 = 1.32 \) eV. An oscillating electric current source with this frequency and the amplitude profile given by Eq. \([39]\) with \( \sigma_p, z = \lambda_0/\sqrt{\varepsilon_d} \) (where \( \lambda_0 = 2\pi c/\omega_0 \)) and \( \sigma_t \to \infty \) is placed inside the first dielectric at \( z = 0 \). The hyperbolic MM layer is located at \( 10 < z/\lambda_0 < 30 \). At the frequency \( \omega_0 \), the relative transverse permittivity of this layer is \( \varepsilon_{\text{eff},t} = 3.8 + i6.5 \times 10^{-3} \) and the relative axial permittivity is \( \varepsilon_{\text{eff},z} = -3.8 + i0.22 \) [Eqs. \([46]\) and \([47]\)].

The results of the paraxial EDGF-based calculations for this structure for the \( p \)- and \( s \)-polarized source currents are displayed in Fig. \([1]\). As one can see, the initially diverging \( p \)-polarized Gaussian beam, after refracting negatively at the interface \( z/\lambda_0 = 10 \), is focused at \( z/\lambda_0 = 20 \), the middle point of the hyperbolic MM layer, and then it diverges again. After reaching the second interface at \( z/\lambda_0 = 30 \), the beam undergoes another negative refraction and is focused inside the second dielectric layer at the point \( z/\lambda_0 = 40 \). On the contrary, the \( s \)-polarized beam does not experience any refraction at the MM interfaces and simply diverges. Note that the scales on the \( x \)-axis and the \( z \)-axis in Fig. \([1]\) are different, so that the field profile along \( z \) is compressed in comparison with that along \( x \).

In order to explain how our EDGF formalism is able to reproduce these phenomena in the paraxial approximation,
let us consider the expression for the square of the effective beam width, $\tilde{\sigma}^2_{p(z)}$, at a point $z = z''$ inside the MM layer

$$
\tilde{\sigma}^2_{p(z=0)} = \tilde{\sigma}^2_{p(z=0)} + \frac{a_d^{[m]} z'' - z_1}{a_d^{[m]} \kappa_a} = a^2_{p[1]} + \frac{z_1}{\kappa_{a,d}} + \left( \frac{z_1}{\kappa_a} \right) - a^2_{p[1]}
$$

as follows from Eq. (42). Here, $z_1 = 10 \lambda_0$ is the coordinate of the first MM interface and $\kappa_{a,d} = 2\pi \sqrt{e_d/\lambda_0} \approx \kappa_a$ is the propagation factor in the dielectric layer. In the considered hyperbolic MM, $a_d^{[m]} / a_d^{[m]} \approx -1$, while $a_m^{[m]} / a_m^{[m]} = +1$. Therefore, when $|z'' - z_1| = |z_1|$, for the $p$-polarization, the propagation in the dielectric is compensated by the propagation in the MM and thus $\tilde{\sigma}^2_{p(z)} \approx \tilde{\sigma}^2_{p}',$ i.e., the $p$-polarized Gaussian beam is refocused at the middle of the MM layer.

### B. Negative group velocity and superluminality in active media

In this example, we apply our EDGF formalism to the EMFSVA propagation through a layered structure which is formed by an active medium sandwiched between two passive media. The active layer is the $^{132}$Xe gas with inverted population, and the passive media are air. The EMFSVA is created by an $s$-polarized surface electric current source located at $z = 0$ with the amplitude profile defined by Eq. (39) in which $\sigma_s \rightarrow \infty$ (i.e., only $k_1 = 0$ component is present). This source creates the EMFSVA propagating through the three media in the $z > 0$ direction. Because the relative permittivities and permeabilities of the layers are rather close to unity, the layers are well impedance matched and we can apply the theory of Sec. [VI]

The relative permittivity of the $^{132}$Xe gas with inverted population follows the Lorentzian dispersion with negative oscillator strength [27]:

$$
c_l^1 = c_l^2 = 1 - \frac{\eta \omega_p^2}{\omega_r^2 - \omega_0^2 - i\gamma \omega_0},
$$

where the parameter $\omega_p/2\pi = 0.42$ GHz accounts for both the magnitude of the oscillator strength and the atomic plasma frequency, $\eta = 0.9$ is the relative inversion, and $\omega_r/2\pi = 84$ THz and $\gamma/2\pi = 4.2$ MHz are the resonant frequency and the linewidth, respectively. We stay detuned from the resonant frequency and set $\omega_0 = \omega_r + \omega_p/3$. At this point, the group velocity in the active layer is $V_g \approx -0.97c$ and the assumptions of the EDGF approach hold (see Appendix C for details).
In Fig. 2 we depict the propagation of the EMFSVA envelope with \( \omega_p t_s = 80 \) through the three layers (separated by dashed vertical lines in the figure) at various moments of the normalized time: \( \omega_p t = 0, 150, 260, 300, 360, 500 \). The observed behavior agrees with that of Ref. [27]. As is seen from the top plots in Fig. 2 at \( \omega_p t = 0 \), the front edge (the precursor) of the Gaussian envelope penetrates into the second layer (the active layer). At \( \omega_p t = 150 \), the maximum of the original pulse passes forward and the field acquires a noticeable value at the interface of the first and the second layers and, at the same time, we can see that the amplified field in the active layer forms a sharp peak and penetrates into the third layer. We can see how the back propagating pulse is formed in the gain medium and how it interacts with the primary pulse which propagates forward in the first layer at the same interface at \( \omega_p t = 260, 300, \) and 360. Finally, at \( \omega_p t = 500 \), we can see that the main pulse has left the two layers, however, a small effect of its back edge is still present at the interface of the first and the second layers. It should be noted that at \( \omega_p t = 260 \) the maximum of the pulse exits the second layer earlier than it would do if it had traveled through an equal distance of air, i.e. it appears superluminal. This happens due to the action of the gain medium on the electromagnetic field with a Gaussian-shaped envelope which lacks a definite turn-on moment. Analogous results are reported in Ref. [27].

**VIII. CONCLUSIONS**

In this work, we have presented a theoretical formalism which is applicable for studying the propagation of amplitude fluctuations of the quasi-monochromatic electromagnetic field through anisotropic dispersive media. The developed formalism is aimed to be used in future works to model the dynamics of RHT in such media, in particular, in uniaxial MMs [22], however, it is equally applicable to the analysis of narrow-band signal propagation in these MMs.

Starting with the 6-vector Maxwell’s operator equation, we have formulated an EDGF-based method with which we have derived the envelope Green’s functions used to calculate the EMFSVA propagating through a dispersive medium with uniaxial dyadic constitutive parameters. We have obtained the matrix elements of the EDGF in the Fourier and the configuration spaces for the considered media. In the case of paraxial propagation, the EDGF for uniaxial media can be written in a closed form, resulting in a formulation analogous to the Gaussian beam-based paraxial approximation in optics. Finally, we have considered propagation of the EMFSVA through non-magnetic passive and active layered media. We simulated the propagation of the EMFSVA through such layered media by employing the effective Huygens sources at the interfaces of the neighboring layers.

We have demonstrated with numerical examples that the developed formalism correctly models negative refraction and focusing by hyperbolic MMs and is also applicable to exotic effects in optical media with inverted population, such as the negative group velocity and superluminality. The group velocity obtained in our formalism agrees with the standard definition known from the literature and results in similar behaviors. The considered examples confirm the applicability and validity of the EDGF approach developed in this paper.

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**APPENDIX A**

In its standard formulation, Kotelnikov’s theorem expresses the signal \( s(t) \) with a limited spectrum, \( \Omega \in \left[-\frac{\Delta \omega}{2}, \frac{\Delta \omega}{2}\right] \), through a set of the discrete samples, \( s(t_n) \):\n
\[
s(t) = \sum_{n=-\infty}^{+\infty} s(t_n) f_0 \left( \frac{\Delta \omega}{2} (t - t_n) \right), \quad t_n = \frac{2\pi n}{\Delta \omega},
\]

where \( f_0(x) = \sin(x)/x \). Using this theorem, the convolution integral of two functions, \( a(t) \) and \( s(t) \), both satisfying Kotelnikov’s spectral condition, can be written as follows:

\[
u(t) = \int_{-\infty}^{+\infty} a(t - t') s(t') dt' = \frac{2\pi}{\Delta \omega} \sum_{n=-\infty}^{+\infty} a(t - t_n) s(t_n).
\]
We have used this result when obtaining Eq. \((14)\).

**APPENDIX B**

Regarding Eq. \((27)\), the different blocks of \(\overline{\mathbf{C}_{\gamma, s\Omega}}\) and \(\overline{\mathbf{D}_{\gamma, s\Omega}} \ (\gamma' = p, s)\) matrices are given as follows

\[
\overline{\mathbf{C}_{\gamma, s\Omega}(Z)} = \begin{bmatrix}
\overline{C}^{tt}_{\gamma, s\Omega} \\
\overline{C}^{tz}_{\gamma, s\Omega} \\
\overline{C}^{zz}_{\gamma, s\Omega}
\end{bmatrix}, \quad \overline{\mathbf{D}_{\gamma, s\Omega}(Z)} = \begin{bmatrix}
\overline{D}^{tt}_{\gamma, s\Omega} \\
\overline{D}^{tz}_{\gamma, s\Omega} \\
\overline{D}^{zz}_{\gamma, s\Omega}
\end{bmatrix}.
\]  

(53)

Using Eqs. \((21)\) and \((22)\), recalling \(\kappa_{\gamma}^p \) and \(V_{\gamma}^p\) from Eqs. \((29)\) and \((30)\), respectively, we obtain the non-zero components of \(\overline{\mathbf{C}}\)’s and \(\overline{\mathbf{D}}\)’s as follows [the structure of these sub-blocks is the same as defined by Eqs. \((18)-(20)\)]:

\[
C^{tt,p[s]}_{em} = -\frac{a^m_{[e]} - k^2_a/d^m_{z}}{2\kappa_0}, \quad C^{tt,p[s]}_{mm[ee]} = -\frac{a^m_{[e]} - k^2_a/d^m_{z}}{2\kappa_0}, \quad C^{tt,m[p]}_{em} = \frac{-\text{sgn}(Z) k^2_a}{2\kappa_0},
\]  

(54)

\[
C^{tz,p[s]}_{em} = \frac{\text{sgn}(Z) k^2_a}{2\kappa_0}, \quad C^{tz,p[s]}_{mm[em]} = \frac{-a^m_{[e]} k^2_a}{2\kappa_0}, \quad C^{zz,p[s]}_{mm[zm]} = \frac{-a^m_{[e]} k^2_a}{2\kappa_0},
\]  

(55)

\[
D^{tt,p[s]}_{mm[mm]} = -b^m_{[e]} + k^2_a/b^m_{z} - k^2_a/d^m_{z} + V_{g}^{p[s]} a^m_{[e]} - k^2_a/d^m_{z}, \quad D^{tt,p[s]}_{mm[ee]} = -\frac{b^m_{[e]} + V_{g}^{p[s]} a^m_{[e]}}{2\kappa_0}, \quad D^{tt,m[p]}_{em} = 0,
\]  

(57)

\[
D^{tz,p[s]}_{mm[mm]} = -\frac{\text{sgn}(Z) k^2_a}{2\kappa_0}, \quad D^{tz,p[s]}_{mm[em]} = \frac{-a^m_{[e]} k^2_a}{2\kappa_0}, \quad D^{zz,p[s]}_{mm[zm]} = \frac{-a^m_{[e]} k^2_a}{2\kappa_0}, \quad D^{zz,m[p]}_{em} = 0,
\]  

(58)

\[
D^{zz,p[s]}_{mm[mm]} = \frac{k^2_a}{2\kappa_0} \left(2\frac{b^m_{[e]} - V_{g}^{p[s]} a^m_{[e]}}{d^m_{z} - k^2_a/d^m_{z}} + \frac{b^m_{[e]} - V_{g}^{p[s]} a^m_{[e]}}{d^m_{z} + k^2_a/d^m_{z}} \right),
\]  

(59)

\[
\overline{\mathbf{C}}^{tt,\gamma} = \begin{bmatrix}
\overline{C}^{tt}_{\gamma} \\
\overline{C}^{tz}_{\gamma} \\
\overline{C}^{zz}_{\gamma}
\end{bmatrix}, \quad \overline{\mathbf{D}}^{tt,\gamma} = \begin{bmatrix}
\overline{D}^{tt}_{\gamma} \\
\overline{D}^{tz}_{\gamma} \\
\overline{D}^{zz}_{\gamma}
\end{bmatrix}.
\]  

(60)

**APPENDIX C**

In order to check if it is reasonable to ignore the \(O(\Omega^2)\) term in Eq. \((28)\), we consider Eq. \((23)\) with an extra second-order term:

\[
d^l_{[m]} = a^m_{[e]} + b^m_{[e]} \Omega + \frac{c^m_{[e]} \Omega^2}{2},
\]  

(61)

where \(l = t \text{ or } z\), and substitute it into Eq. \((24)\). We define the smallness parameter \(\delta_{p[s]}\) as the ratio of the second-order \(O(\Omega^2)\) term [which is dropped in Eq. \((28)\)] to the first-order \(O(\Omega)\) term in the Taylor expansion of Eq. \((24)\) as follows

\[
\delta_{p[s]} = \frac{\Omega V_{g}^{p[s]} \delta^2 \kappa_{p[s]} \left| \frac{\partial \Omega^2}{\partial \Omega} \right|_{\Omega = 0}}{2},
\]  

(62)

where in the following calculations we replace \(\Omega\) with its maximum value \(\Omega = \Delta \omega / 2\). Obviously, \(\delta_{p[s]}\) depends on \(\omega_0\) and \(\Delta \omega\), in addition to the material properties. Considering \(k_t = 0\), when both polarizations are equivalent, we obtain

\[
\delta_\alpha = \frac{\Delta \omega}{4 \kappa_a} \left[ V_{g}^p \left( \frac{a^m_{[e]} b^m_{[e]} + c^m_{[e]}}{2} + b^m_{[e]} \right) - \frac{1}{V_{g}^p} \right].
\]  

(63)
In Fig. 3 we depict the normalized group velocity and the smallness parameter versus the normalized frequency shift (for the example of Sec. VII B). In this figure, $v_g = \text{Re}(V_g)$ and $\delta_a$ are measured by the scales on the left and the right, respectively. As can be seen from Fig. 3 $|\delta_a| \approx 0.2$ near the operational frequency $\omega_0 = \omega_r + \omega_p / 3$ from the example of Sec. VII B. This value can be considered sufficiently small. On the other hand, in the regions where the group velocity becomes extremely superluminal, we can see that $|\delta_a| \gg 1$, which indicates that in these regions the pulse propagation is very much affected by the group dispersion and the group velocity loses its physical meaning.

Figure 3. (Color online) The normalized group velocity $v_g / c$ (black solid curve) and the smallness parameter magnitude $|\delta_a|$ (blue dashed curve) as functions of the normalized frequency shift $(\omega_0 - \omega_r) / \omega_p$. The red arrow indicates the operating point corresponding to the pulse propagation shown in Fig. 2. The material parameters are the same as in Sec. VII B.
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