WELL-POSEDNESS FOR TWO-DIMENSIONAL STEADY SUPERSONIC EULER FLOWS PAST A LIPSCHITZ WEDGE

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Abstract. For a supersonic Euler flow past a straight wedge whose vertex angle is less than the extreme angle, there exists a shock-front emanating from the wedge vertex, and the shock-front is usually strong especially when the vertex angle of the wedge is large. In this paper, we establish the $L^1$ well-posedness for two-dimensional steady supersonic Euler flows past a Lipschitz wedge whose boundary slope function has small total variation, when the total variation of the incoming flow is sufficiently small. In this case, the Lipschitz wedge perturbs the flow and the waves reflect after interacting with the strong shock-front or the wedge boundary. We first obtain the existence of solutions in $BV$ when the incoming flow has small total variation by the wave front tracking method and then study the $L^1$ stability of the solutions. In particular, we incorporate the nonlinear waves generated from the wedge boundary to develop a Lyapunov functional between two solutions, which is equivalent to the $L^1$ norm, and prove that the functional decreases in the flow direction. Then the $L^1$ stability is established, so is the uniqueness of the solutions by the wave front tracking method. Finally, we show the uniqueness of solutions in a broader class, i.e. the class of viscosity solutions.

1. Introduction

For the Cauchy problem of a strictly hyperbolic system of conservation laws:

(1.1) \[ U_t + F(U)_x = 0, \quad U \in \mathbb{R}^n, \]

(1.2) \[ U|_{t=0} = U(x), \]

whose each characteristic field is either linearly degenerate or genuinely nonlinear, the existence of weak solutions to (1.1)–(1.2) with small total variation was first proved by Glimm [19] by a probabilistic algorithm, the Glimm scheme; and a deterministic version of the Glimm scheme was developed by Liu [28]. Alternative methods for constructing solutions of the Cauchy problem were first introduced in [16] [18], based on wave front tracking. For the scalar equation, $F$ is approximated by piecewise linear functions $F_\nu$ in Dafermos [10] so that the approximate solutions are piecewise constants and all the interactions are determined by solving the Riemann problem. The method was generalized to the $2 \times 2$ case in DiPerna [18] in...
which piecewise constant approximate solutions are constructed so that the wave interactions can be determined by only solving the Riemann problem. Bressan [3] developed the wave front tracking method for \( n \times n \) systems by overcoming the difficulty that the procedure used in [18] may yield an infinite number of discontinuities in finite time when \( n > 2 \); and the wave front tracking method was further simplified later in Baiti-Jenssen [1]. Also see Bressan [5], Dafermos [17], Holden-Risebro [21], and LeFloch [22] for further references.

Within the class of initial data \( \bar{U} \in L^1 \cap BV (R; R^n) \) with suitably small total variation, it was established that problem (1.1)–(1.2) is well-posed in \( L^1 (R; R^n) \) for the solutions generated by the wave front tracking algorithm. In particular, in Bressan-Colombo [6], Bressan-Crasta-Piccoli [8], and Bressan-Liu-Yang [9], it was proved that the entropy solutions of (1.1)–(1.2) constitute a semigroup which is Lipschitz continuous with respect to time and initial data. Lewicka-Trivisa [26] obtained the \( L^1 \) well-posedness of solutions generated by the wave front tracking method for the Cauchy problem (1.1)–(1.2) with the initial data \( \bar{U} \) being a small perturbation of a fixed Riemann problem \((U_-, U_+)\) containing two large shocks, under the necessary stability condition (cf. [7, 26, 24]; also see [23, 25]). The \( L^1 \) well-posedness in the class of viscosity solutions for the Cauchy problem has been also established (cf. [4, 2, 14] and the references therein).

In this paper, we are concerned with the \( L^1 \) well-posedness of a physical nonlinear problem of initial-boundary value type, which governs two-dimensional steady supersonic Euler flows past a curved wedge. More specifically, the two-dimensional steady supersonic Euler flows are generally governed by

\[
\begin{align*}
(\rho u)_x + (\rho v)_y &= 0, \\
(\rho u^2 + p)_x + (\rho u v)_y &= 0, \\
(\rho uv)_x + (\rho v^2 + p)_y &= 0, \\
(\rho (E + p/\rho))_x + (\rho (E + p/\rho))_y &= 0,
\end{align*}
\]

where \((u, v)\) is the velocity, \(\rho\) the density, \(p\) the scalar pressure, and \(E = \frac{1}{2}(u^2 + v^2) + c(\rho, p)\) the total energy with \(c\) the internal energy (a given function of \((\rho, p)\) defined through thermodynamical relationships). The other two thermodynamic variables are the temperature \(\theta\) and the entropy \(S\). For an ideal gas,

\[
p = R\rho \theta, \quad e = c_v \theta, \quad \gamma = 1 + \frac{R}{c_v} > 1,
\]

and

\[
p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \quad e = \frac{\kappa}{\gamma - 1} \rho^{\gamma-1} e^{S/c_v} = \frac{R \theta}{\gamma - 1},
\]

where \(R, \kappa, \) and \(c_v\) are all positive constants.

If the flow is isentropic, i.e. \(S = \text{const.}\), then the pressure \(p\) is a function of the density \(\rho\), \(p = p(\rho)\), and the flow is governed by the following simpler isentropic
Euler equations:

\[
\begin{align*}
\left\{ \begin{array}{l}
(\rho u)_x + (\rho v)_y &= 0, \\
(\rho u^2 + p)_x + (\rho uv)_y &= 0, \\
(\rho uv)_x + (p\rho^2 + p)_y &= 0.
\end{array} \right.
\]

For polytropic isentropic gases, by scaling, the pressure-density relationship can be expressed as

\[
p(\rho) = \rho^\gamma, \quad \gamma > 1.
\]

For the isothermal flow, \(\gamma = 1\). The quantity

\[
c = \sqrt{p(\rho, S)}
\]

is defined as the sonic speed and, for polytropic gases, \(c = \sqrt{\gamma p/\rho}\).

System (1.3) or (1.6) governing a supersonic flow (i.e., \(u^2 + v^2 > c^2\)) has all real eigenvalues and is hyperbolic, while system (1.3) or (1.6) governing a subsonic flow (i.e., \(u^2 + v^2 < c^2\)) has complex eigenvalues and is elliptic-hyperbolic mixed and composite.

The study of two-dimensional steady supersonic flows past a wedge can date back to the 1940s (cf. Courant-Friedrichs [15]). Local solutions around the wedge vertex were first constructed in Gu [20], Li [27], Schaeffer [29], and the references cited therein. Global potential solutions were constructed in various different setups in [11, 12, 13, 31] when the wedge vertex angle is less than the critical angle.

For the full Euler equations, when a wedge is straight and the wedge vertex angle is less than the critical angle, there exists a supersonic shock-front emanating from the wedge vertex so that the constant states on both sides of the shock-front are supersonic; the critical angle condition is necessary and sufficient for the existence of the supersonic shock (cf. Courant-Friedrichs [15]). When the incoming flow is uniform, Chen-Zhang-Zhu [10] first established the existence of global supersonic Euler flows, especially the nonlinear structural stability of the strong shock-front emanating from the wedge vertex under the BV perturbation of the wedge boundary. In this paper, we first show the existence of solutions to the above problem when the incoming flow is a BV perturbation of the uniform flow by the wave front tracking method, and then we establish the \(L^1\)-stability of entropy solutions generated by this method. Based on these, we establish estimates on the uniformly Lipschitz semigroup \(\mathcal{P}\) generated by the wave front tracking limit and prove the uniqueness of solutions by means of local integral estimates within a broader class of solutions, i.e. the class of viscosity solutions.

One of the main new ingredients in our approach here is to develop techniques to handle the boundary difficulty, in comparison with the earlier works on the Cauchy problem. For the \(L^1\) stability of solutions of the Cauchy problem, the decrease of the Lyapunov functional was achieved by essentially using the cancellation of
the distances on both sides of waves. However, for our wedge problem that is the problem of initial-boundary value type, there is no such cancellation near the boundary, since only one-side is possible near the wedge boundary. In order to overcome this difficulty, we employ the exact feature of the boundary condition to obtain an estimate and refine the functional based on this estimate. In particular, since the flow of two solutions near the boundary must be parallel, we identify the relation between these two states, which is desirable for redesigning the functional to ensure the decreasing of the functional in the flow direction.

For concreteness, as in Chen-Zhang-Zhu [10], we will analyze the problem in the region below the lower side $\Gamma$ of the wedge for the Euler flows for $U = (u, v, p, \rho)$ governed by system (1.3) and $U = (u, v, \rho)$ by (1.6); the case above the wedge can be handled in the same fashion. Then we have

(i) There exists a Lipschitz function $g \in Lip(\mathbb{R}_+)$ with $g' \in BV(\mathbb{R}_+)$, $g'(0+) = 0$, and $g(0) = 0$ such that

$$\Omega := \{(x, y) : y < g(x), x \geq 0\}, \quad \Gamma := \{(x, y) : y = g(x), x \geq 0\},$$

and $n(x\pm) = \frac{(-g'(x\pm), 1)}{\sqrt{(g'(x\pm))^2 + 1}}$ is the outer normal vector to $\Gamma$ at the point $x\pm$ (see Fig. 1):

![Figure 1. Supersonic flow past a curved wedge](image)

(ii) The upstream flow $U = \bar{U}(y) = (\bar{u}, \bar{v}, \bar{p}, \bar{\rho})(y)$ at $x = 0$ satisfies

$$\bar{u}(y) > 0, \quad \bar{v}(y) > 0, \quad \bar{u}(y)^2 + \bar{v}(y)^2 > \bar{c}(y)^2 := \gamma \bar{p}(y)/\bar{\rho}(y),$$

and

$$0 < \arctan(\bar{v}(y)/\bar{u}(y)) < \omega_{crit},$$

where $\omega_{crit}$ is the critical vertex angle so that there is a supersonic shock-front emanating from the wedge vertex.

With this setup, the wedge problem can be formulated into the following problem of initial-boundary value type for system (1.3) or (1.6):
Cauchy Condition:

\[ U|_{x=0} = \overline{U}(y); \]

Boundary Condition:

\[ (u, v) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \]

This paper is organized as follows. In Section 2, we discuss the basic properties for the adiabatic Euler equations and related nonlinear waves. In Section 3, we discuss the wave front tracking method and define the interaction potential \( Q \), and then we prove the existence of entropy solutions to the initial-boundary value problem. In Section 4, we construct the Lyapunov functional \( \Phi \) between two solutions to incorporate the nonlinear waves generated by the wedge boundary vertices, which is equivalent to the \( L^1 \) distance between these two solutions. In Section 5, we prove the decrease of \( \Phi \) in the flow direction, which implies the \( L^1 \) stability of the solutions. In Section 6, we prove the existence of the semigroup generated by the wave front tracking method and establish some estimates on the uniformly Lipschitz semigroup \( S \) generated by the wave front tracking limit. In Section 7, we prove the uniqueness of solutions by means of local integral estimates within a broader class of solutions.

2. Euler Equations and Nonlinear Waves

In this section, we review some basic properties of the adiabatic Euler equations and related nonlinear waves, which will be used in the subsequent sections. The Euler equations for steady supersonic flows can be written in the following conservation form:

\[ W(U)_x + H(U)_y = 0, \quad U = (u, v, p, \rho)^\top, \]

with

\[ W(U) = (\rho u, \rho u^2+p, \rho uv, \rho u(h+\frac{u^2+v^2}{2}))^\top, \quad H(U) = (\rho v, \rho uv, \rho v^2+p, \rho v(h+\frac{u^2+v^2}{2}))^\top, \]

and \( h = \frac{\gamma p}{(\gamma-1)p} \). The eigenvalues of system (2.1) are

\[ l_j = \frac{uv + (-1)^j c\sqrt{u^2+v^2-c^2}}{u^2-c^2}, \quad j = 1, 4; \quad l_i = v/u, \quad i = 2, 3, \]

where \( c^2 = \frac{\gamma p}{\rho} \). If the flow is supersonic (i.e. \( u^2 + v^2 > c^2 \)), system (2.1) is hyperbolic; and, in particular, when \( u > c \), system (2.1) has the four corresponding linearly independent eigenvectors:

\[ r_j = \kappa_j(-l_j, 1, \rho(l_j u - v), \rho(l_j u - v)/c^2)^\top, \quad j = 1, 4, \]

\[ r_2 = (u, v, 0, 0)^\top, \quad r_3 = (0, 0, 0, \rho)^\top, \]
where \( \kappa_j \) are chosen so that \( r_j \cdot \nabla l_j = 1 \) since the \( j \)-th-characteristic fields are genuinely nonlinear, \( j = 1, 4 \). Note that the second and third characteristic fields are always linearly degenerate: \( r_j \cdot \nabla \lambda_j = 0, j = 2, 3 \).

**Definition 2.1** (Entropy Solutions). A BV function \( U = U(x, y) \) is called an entropy solution of the initial-boundary value problem (2.1) and (1.8)–(1.9) provided that

(i) \( U \) is a weak solution of (2.1) and satisfies

\[
U|_{x=0} = \bar{U}(y), \quad (u, v) \cdot n|_{y=g(x)} = 0
\]

in the trace sense;

(ii) \( U \) satisfies the entropy inequality, i.e. the steady Clausius inequality:

\[
(\rho u S)_x + (\rho v S)_y \geq 0
\]

in the sense of distributions in \( \Omega \) including the wedge boundary.

We now discuss the wave curves in the phase space. The contact Hugoniot curves \( C_i(U_0) \) through \( U_0 \) are

\[
C_i(U_0) : \quad p = p_0, \quad w = v/u = v_0/u_0, \quad i = 2, 3,
\]

which describe compressible vortex sheets. We remark that, although the two contact discontinuities coincide as a single vortex sheet in the physical \( xy \)-plane, it requires two independent parameters to describe them in the phase space \( U = (u, v, p, \rho) \) since there are two linearly independent eigenvectors corresponding to the repeated eigenvalues \( \lambda_2 = \lambda_3 = v/u \) of the two linearly degenerate fields.

Moreover, the rarefaction wave curves \( R_j(U_0) \) in the phase space through \( U_0 \) are

\[
R_j(U_0) : \quad dp = c^2 dp, \quad du = -\lambda_j dv, \quad \rho(\lambda_j u - v)dv = dp \quad \text{for} \quad \rho < \rho_0, \quad j = 1, 4.
\]

The Rankine-Hugoniot conditions for (2.1) are

\[
s[W(u)] = [H(u)],
\]

where \( s \) is the propagation speed of the discontinuity. Then

\[
s = s_j := \frac{u_0 v_0 + (-1)^j \sigma_0 \sqrt{u_0^2 + v_0^2 - c_0^2}}{u_0^2 - c_0^2}, \quad \sigma = \sigma_i = v_0/u_0, i = 2, 3,
\]

where \( c_0^2 = \frac{\gamma}{\gamma - 1} \frac{\rho_0}{u_0^2} \) and \( b_0 = \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} \frac{\rho_0}{u_0^2} \).

Plugging \( s_i, i = 2, 3 \), into (2.7), we obtain the \( i \)-th-contact Hugoniot curves \( C_i(U_0), i = 2, 3 \), as defined in (2.5); while plugging \( s_j, j = 1, 4 \), into (2.7), we obtain the \( j \)-th-Hugoniot curve \( S_j(U_0) \) through \( U_0 \):

\[
S_j(U_0) : \quad [p] = \frac{c_0^2}{b_0} [\rho], \quad [u] = -s_j [v], \quad \rho_0(s_j v_0 - u_0) [v] = [p] \quad \text{for} \quad \rho > \rho_0, \quad j = 1, 4.
\]
The half curves of $S_j(U_0)$ for $\rho > \rho_0$, denoted by $S^+_j(U_0)$, $j = 1, 4$, in the phase space are called the shock curves on which any state with $U_0$ forms a shock in the $x-y$ plane satisfying the entropy condition as explain in Lemma 2.1 below.

Note that $S^+_j(U_0)$ contacts with $R^-_j(U_0)$ at $U_0$ up to second-order.

As indicated in [10], we have

**Lemma 2.1.** If $U$ is a piecewise smooth solution, then, on the shock wave, the entropy inequality in Definition 2.1 is equivalent to any of the following:

(i) The physical entropy condition: the density increases across the shock in the flow direction,

\begin{equation}
\rho_{\text{front}} < \rho_{\text{back}};
\end{equation}

(ii) The Lax entropy condition: on the $j^{th}$-shock with the shock speed $\sigma_j$,

\begin{equation}
\lambda_j(\text{back}) < s_j < \lambda_j(\text{front}), \quad j = 1, 4,
\end{equation}

\begin{equation}
s_1 < \lambda_{2,3}(\text{back}), \quad \lambda_{2,3}(\text{front}) < s_4.
\end{equation}

The following properties and related estimates of wave interactions in Lemmas 2.2–2.8 have been obtained in Chen-Zhang-Zhu [10]. We list them below for subsequent use in this paper.

### 2.1. Riemann problems and Riemann solutions.

We start with Riemann problems and their solutions.

**Lateral Riemann problem.** The simplest case of problem (2.1) and (1.8)–(1.9) is $g \equiv 0$. It can be shown that, if $g \equiv 0$, then problem (2.1) admits an entropy solution that consists of a constant state $U_-$ and a constant state $U_+$, with $U_+ = (u_+, 0, p_+, \rho_+)$ and $u_+ > c_+ > 0$ in the subdomain of $\Omega$ separated by a straight shock emanating from the vertex. That is to say that the state ahead of the shock-front is $U_-$, while the state behind the shock-front is $U_+$. When the angle between the flow direction of the front state and the wedge boundary at a boundary vertex is larger than $\pi$, then an entropy solution contains a rarefaction wave that separates the front state from the back state.

**Riemann problem involving only weak waves.** Consider the following initial value problem:

\begin{equation}
\begin{cases}
W(U)_x + H(U)_y = 0, \\
U|_{x=x_0} = U_b = \begin{cases}
U_a, & y > y_0, \\
U_b, & y < y_0,
\end{cases}
\end{cases}
\end{equation}

where $U_b$ and $U_a$ are constant states.

**Lemma 2.2.** There exists $\varepsilon > 0$ such that, for any states $U_a, U_b \in O_\varepsilon(U_+)$ or $U_a, U_b \in O_\varepsilon(U_-)$, problem (2.11) admits a unique admissible solution consisting of four elementary waves.
**Riemann problem involving a strong 1-shock.** For simplicity, we use notation \( \{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) to denote the solution to the Riemann problem, where \( \alpha_i \) is the strength of the \( i \)th elementary wave. For any \( U \in S_1(U_-) \), we also use \( \{U_-, U\} = (\sigma, 0, 0, 0) \) to denote the 1-shock that connects \( U_- \) and \( U \) with speed \( \sigma \).

Then we have

**Lemma 2.3.** Let \( \{U_-, U_+\} = (\sigma_0, 0, 0, 0), \rho_+ > \rho_-, \) and \( \gamma > 1. \) Then
\[
\sigma_0 < 0, \quad u_+ < u_- < (1 + 1/\gamma)u_+,
\]

and
\[
\det(\nabla U H(U_+) - \sigma_0 \nabla U W(U_+)) > 0.
\]

Furthermore, there exists a neighborhood \( O_4(U_+) \) of \( U_+ \) and a neighborhood \( O_4(U_-) \) of \( U_- \) such that \( U_0 \in O_4(U_-) \) and the shock polar \( S_1(U_0) \cap O_4(U_+) \) can be parameterized by the shock speed \( \sigma \) as \( \sigma \to G(U_0, \sigma) \) with \( G \in C^2 \) near \( (U_-, \sigma_0) \) and \( G(U_-, \sigma_0) = U_+ \).

**2.2. Estimates on wave interactions and reflections.** We have

**Lemma 2.4** (Estimates on weak wave interactions). Suppose that \( U_b, U_m, U_a \in O_4(U_+) \), or \( U_b, U_m, U_a \in O_4(U_-) \), are three states with \( \{U_b, U_m\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \) \( \{U_m, U_a\} = (\beta_1, \beta_2, \beta_3, \beta_4) \), and \( \{U_b, U_a\} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \). Then
\[
\gamma_1 = \alpha_1 + \beta_1 + O(1)\triangle(\alpha, \beta),
\]
where \( \triangle(\alpha, \beta) = (|\alpha_4| + |\alpha_3| + |\alpha_2|)|\beta_1| + |\alpha_4|(|\beta_2| + |\beta_3|) + \sum_{j=1,4} \triangle_j(\alpha, \beta) \) with
\[
\triangle_j(\alpha, \beta) = \begin{cases}
0, & \alpha_j \geq 0, \beta_j \geq 0, \\
|\alpha_j| |\beta_j|, & \text{otherwise}.
\end{cases}
\]

Denote \( \{C_k(a_k, b_k)\}_{k=0}^{\infty} \) the points \( \{(a_k, b_k)\}_{k=0}^{\infty} \) in the \( xy \)-plane with \( a_{k+1} > a_k \). Set
\[
\omega_{k,k+1} = \arctan\left(\frac{b_{k+1} - b_k}{a_{k+1} - a_k}\right), \quad \omega_k = \omega_{k,k+1} - \omega_{k-1,k}, \quad \omega_{-1,0} = 0,
\]
\[
\Omega_{k+1} = \{(x, y) : x \in [a_k, a_{k+1}), y < b_k + (x - a_k) \tan(\omega_{k,k+1})\},
\]
\[
\Gamma_{k+1} = \{(x, y) : x \in [a_k, a_{k+1}), y = b_k + (x - a_k) \tan(\omega_{k,k+1})\},
\]
and the outer normal vector to \( \Gamma_k \):
\[
n_{k+1} = \frac{(b_{k+1} - b_k, a_{k+1} - a_k)}{\sqrt{(b_{k+1} - b_k)^2 + (a_{k+1} - a_k)^2}} = (-\sin(\omega_{k,k+1}), \cos(\omega_{k,k+1})).
\]

Then we consider the initial-boundary value problem with \( U \) a constant state:
\[
\begin{cases}
\text{(2.1)} & \text{in } \Omega_{k+1}, \\
U|_{x=a_k} = U, \\
(u, v) \cdot n_{k+1} = 0 & \text{on } \Gamma_{k+1}.
\end{cases}
\]
Lemma 2.5 (Estimate on the boundary perturbation of the strong shock). For \( \varepsilon > 0 \) sufficiently small, there exists \( \bar{\varepsilon} = \bar{\varepsilon}(\varepsilon) < \varepsilon \) so that \( G(O_{\bar{\varepsilon}}(\sigma_0)) \subset O_{\varepsilon}(U_+) \) and, when \( |\sigma_k| < \varepsilon \), the equation \( G(\sigma) \cdot (n_k, 0, 0) = 0 \) admits a unique solution \( \sigma_k \in O_{\varepsilon}(\sigma_0) \). Moreover, we have
\[
\sigma_{k+1} = \sigma_k + K_{bs} \theta_k,
\]
where \( |K_{bs}| \) is bounded.

Lemma 2.6 (Estimate on the boundary perturbation of weak waves). Let \( U_k = (u_k, v_k, p_k, \rho_k) \) be the state near the boundary with \( (u_k, v_k) \cdot n_k = 0 \). Then there exists \( U_{k+1} \) such that \( \{U_k, U_{k+1}\} = (\delta_1, 0, 0, 0) \) and \( (u_{k+1}, v_{k+1}) \cdot n_{k+1} = 0 \). Furthermore,
\[
\delta_1 = K_{b0} \theta_k,
\]
where \( K_{b0} \) is bounded.

Lemma 2.7 (Estimates on the reflection of weak waves on the boundary). Let \( \{U_b, U_a\} = (0, \alpha_2, \alpha_3, \alpha_4) \) and \( (u_k, v_k) \cdot n_k = 0 \). Then there exists \( U_{k+1} \) such that \( \{U_b, U_{k+1}\} = (\delta_1, 0, 0, 0) \) and \( (u_{k+1}, v_{k+1}) \cdot n_{k+1} = 0 \). Furthermore,
\[
\delta_1 = K_{b4} \alpha_4 + K_{b3} \alpha_3 + K_{b2} \alpha_2,
\]
where \( K_{b4}, K_{b3}, K_{b2}, \) and \( K_{b0} \) are \( C^2 \)-functions of \( (\alpha_4, \alpha_3, \alpha_2, \beta_1, \theta_k; U_b) \) satisfying
\[
K_{b4}(\{\sigma_k = \alpha_4 = \alpha_3 = \alpha_2 = \beta_1 = 0, U_b = U_+\}) = 1,
K_{b2}(\{\sigma_k = \alpha_4 = \alpha_3 = \alpha_2 = \beta_1 = 0, U_b = U_+\}) = K_{b3}(\{\sigma_k = \alpha_4 = \alpha_3 = \alpha_2 = \beta_1 = 0, U_b = U_+\}) = 0.
\]

Lemma 2.8 (Estimates on the interaction between the strong shock and weak waves from above). Let \( U_m, U_a \in O_{\varepsilon}(U_+) \) with \( G(U_b, \sigma), U_m \} = (0, 0, 0, 0) \) and \( \{U_m, U_a\} = (\beta_1, 0, 0, 0) \). Then there exists a unique \( (\sigma', \delta_2, \delta_3, \delta_4) \) such that the Riemann problem \( \ref{2.11} \) with \( U_b \in O_{\varepsilon}(U_-) \) admits an admissible solution consisting of a strong 1-shock, two contact discontinuities of strengths \( \delta_2 \) and \( \delta_3 \), and a weak 4-wave of strength \( \delta_4 \):
\[
\{U_b, U_a\} = (\sigma', \delta_2, \delta_3, \delta_4).
\]

Moreover,
\[
\sigma' = \sigma + K_{s1} \beta_1, \quad \delta_2 = K_{s2} \beta_1, \quad \delta_3 = K_{s3} \beta_1, \quad \delta_4 = K_{s4} \beta_1,
\]
where \( |K_{s4}| < 1 \), and \( |K_{s1}| + |K_{s2}| + |K_{s3}| \) is bounded. Furthermore, we have
\[
|\frac{l_{\sigma_0}^{1+} - \sigma_0}{l_{\sigma_0}^{1+} - \sigma_0}| = \frac{|\sigma_0 u_- Q - u_+ l_{\sigma_0}^{1+} P|}{|\sigma_0 u_- Q + u_+ l_{\sigma_0}^{1+} P|} < 1.
\]

Lemma 2.9 (Estimates on the interaction between the strong shock and weak waves from below). Let \( U_m, U_b \in O_{\varepsilon}(U_-) \) and \( U_a \in O_{\varepsilon}(U_+) \) with
\[
\{U_b, U_m\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \{U_m, U_a\} = (\sigma, \beta_2, \beta_3, \beta_4).
\]
Moreover, the Riemann solvers.

Then there exists a unique \((\sigma', \delta_2, \delta_3, \delta_4)\) such that the Riemann problem admits an admissible solution consisting of a strong 1-shock, two contact discontinuities of strengths \(\delta_2\) and \(\delta_3\), and a weak 4-wave of strength \(\delta_4\):

\[
\{U_b, U_a\} = (\sigma', \delta_2, \delta_3, \delta_4).
\]

Moreover,

\[
\sigma' = \sigma + \sum_{i=1}^{4} K_{1i} \alpha_i + O(1) \Delta, \quad \delta_2 = \beta_2 + \sum_{i=1}^{4} K_{2i} \alpha_i + O(1) \Delta,
\]

\[
\delta_3 = \beta_3 + \sum_{i=1}^{4} K_{3i} \alpha_i + O(1) \Delta, \quad \delta_4 = \beta_4 + \sum_{i=1}^{4} K_{4i} \alpha_i + O(1) \Delta,
\]

where \(|K_{ij}|, i, j = 1, \ldots, 4\), are bounded and \(\Delta = \sum_{i=1,2,3,4,j=2,3,4} |\alpha_i \beta_j|\). Furthermore, we can write the estimates in a more precise fashion:

\[
\sigma' = \sigma + \sum_{i=1}^{4} \overline{K}_{1i} \alpha_i, \quad \delta_2 = \beta_2 + \sum_{i=1}^{4} \overline{K}_{2i} \alpha_i, \quad \delta_3 = \beta_3 + \sum_{i=1}^{4} \overline{K}_{3i} \alpha_i, \quad \delta_4 = \beta_4 + \sum_{i=1}^{4} \overline{K}_{4i} \alpha_i,
\]

where \(\sum_{i,j=1}^{4} \overline{|K_{ij}|} \leq M\) for some \(M > 0\).

Proof. We first consider the interaction between \((U_b, U_m) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) and \((U_m, G(U_m, \sigma)) = (\sigma, 0, 0, 0)\) to find that the solution is the perturbation of the unperturbed states of the strong shock. From Lemma 2.3, we know that \(U = G(U_0, \sigma)\) near \((U_-, \sigma_0)\) with \(G \in C^2\), which implies that \(U = (u, v, p, \rho)\) depends continuously on the state \(U_0 = (u_0, v_0, p_0, \rho_0)\). The perturbation near \(U_-\) is equivalent to \(\sum_{i=1}^{4} O(1) \alpha_i\). Then the interaction estimate between \((U_b, G(U_m, \sigma))\) and \((G(U_m, \sigma), U_a)\) follows from Lemma 2.4.

\(\square\)

3. Wave front tracking method and existence of entropy solutions

The basic idea of the wave front tracking method is to construct approximate solutions within a class of piecewise constant functions: First, approximate the initial data by a piecewise constant function and solve the resulting Riemann problems and replace the rarefaction waves by step functions with many small discontinuities; then track the outgoing fronts until the first time when two waves interact which are determined by a new Riemann problem; and finally design a simplified Riemann solver so that the number of wave fronts is finite for all \(x \geq 0\) in the flow direction.

3.1. The Riemann solvers. As mentioned in Section 2, the solution to the Riemann problem \((U_b, U_a)\) is a self-similar solution given by at most five states separated by shocks or rarefaction waves. More precisely, there exists \(C^2\) curves \(\alpha \rightarrow \psi(\alpha)(u)\) parameterized by arc length such that

\[
U_b = \psi_4(\alpha_4) \circ \ldots \circ \psi_1(\alpha_1)(U_a)
\]
for some $\alpha = (\alpha_1, \ldots, \alpha_4)$ and $U_i = \psi_i(\alpha_1) \circ \ldots \circ \psi_1(\alpha_1)(U_0)$. When $\alpha_i$ is positive (negative), states $U_{i-1}$ and $U_i$ are separated by an $i$-rarefaction ($i$-shock) wave, so we call $\alpha_i$ the strength of the $i$-wave.

For given initial data $U$, let $U(\varepsilon, \varepsilon > 0)$ be a sequence of piecewise constant functions approximating $U$ in the $L^1$ norm, and the wedge boundary is also approximated as in Section 2. Let $N_\varepsilon$ be the total number of discontinuities in the function $U$ and the tangential angle functions of the wedge boundary. Choose a parameter $\delta_\varepsilon > 0$ controlling the maximum strength of rarefaction fronts, and $\hat{\lambda}$ (strictly larger than all the characteristic speeds of (2.1)) that is the speed of non-physical waves generated whenever the simplified method is used. The strength of the non-physical waves is the error due to the simplified Riemann solver.

A. Accurate Riemann solver: The accurate Riemann solver is just the solution to the Riemann problem (as in Section 2), except every rarefaction wave $(w, R_i(w)(\alpha))$ is approximated by a piecewise constant rarefaction fan.

B. Simplified Riemann solver: For the weak waves, it is exactly the same as in [1]. When a weak wave interacts with the large shock, the simplified Riemann solver is that we ignore the strength of the weak wave, keep the strength of the strong shock, and put the error in the non-physical wave as follows:

Case 1 (A weak wave $(U_-, U_1)$ hits the large shock $(U_1, U_2)$ from below): We solve the Riemann problem $(U_-, U_+)$ in the following way:

$$
\begin{align*}
\begin{cases}
U_- & \text{for } y/x < \Lambda(U_1, U_+), \\
U_1 & \text{for } \Lambda(U_1, U_+) < y/x < \hat{\lambda}, \\
U_+ & \text{for } y/x > \hat{\lambda},
\end{cases}
\end{align*}
$$

where $\Lambda(U_1, U_+)$ is the speed of the strong shock;

Case 2 (A weak wave $(U_2, U_+)$ hits the large shock $(U_-, U_2)$ from above): We solve the Riemann problem $(U_-, U_+)$ in the following way:

$$
\begin{align*}
\begin{cases}
U_- & \text{for } y/x < \Lambda(U_-, U_2), \\
U_2 & \text{for } \Lambda(U_-, U_2) < y/x < \hat{\lambda}, \\
U_+ & \text{for } y/x > \hat{\lambda},
\end{cases}
\end{align*}
$$

where $\Lambda(U_-, U_2)$ is the speed of the strong shock.

3.2. The algorithm to construct the approximate solutions. Given $\varepsilon$, we construct the approximate solution $U^\varepsilon(x, y)$ as follows. When $x = 0$, all the Riemann problems in $U^\varepsilon$ are solved by the accurate Riemann solver. By slightly perturbing the speed of a wave, we can assume that, at any time, we have at most one collision involving only two incoming fronts. Let $\mu_\varepsilon$ be a fixed small parameter with $\mu_\varepsilon \to 0$, as $\varepsilon \to 0$, which will be specified later. For simplicity of notation, we will drop the index $i$ in $\alpha_i$ and do not distinguish between $\alpha_i$ and $\alpha$ when there is no ambiguity from now on; the same for $\beta$; also we use the same notation $\alpha$ as a wave and its strength as before.
Case 1 (There is a collision between two weak waves with strengths $\alpha$ and $\beta$ at some $x > 0$, respectively): The Riemann problem generated by this interaction is solved as follows:

- If $|\alpha \beta| > \mu \varepsilon$ and the two waves are physical, then we use the accurate solver;
- If $|\alpha \beta| < \mu \varepsilon$ and the two waves are physical, or one wave is non-physical, then we use the simplified Riemann solver.

Case 2 (There is a collision between the large shock and one weak wave $\alpha$ at some $x > 0$): The Riemann problem generated by this interaction is solved as follows:

- If $|\alpha| > \mu \varepsilon$ and the weak wave is physical, then we use the accurate solver;
- If $|\alpha| < \mu \varepsilon$ and the weak wave is physical, or this wave is non-physical, then we use the simplified Riemann solver.

Case 3 (The wave hits the boundary or the boundary perturbs the flow): We use the accurate Riemann solver to solve the lateral Riemann problem.

3.3. Glimm’s functional and interaction potential. We now develop the Glimm-type functional and interaction potential for the initial-boundary value problem by carefully incorporating additional nonlinear waves generated from the wedge boundary vortices.

**Definition 3.1** (Approaching waves). (i) We say that two weak fronts $\alpha$ and $\beta$, located at points $x_\alpha < x_\beta$ and belonging to the characteristic families $i_\alpha, i_\beta \in \{1, \ldots, 4\}$ respectively, approach each other if the following two conditions hold:

- $x_\alpha$ and $x_\beta$ both lay in one of the two intervals into which $\mathbb{R}$ is partitioned by the location of the large 1-shock, i.e. the waves both belong to $\Omega^-$ or $\Omega^+$;
- Either $i_\alpha = i_\beta$ and one of them is a shock, or $i_\alpha > i_\beta$.

In this case we write $(\alpha, \beta) \in A$.

(ii) We say that a weak wave $\alpha$ of the characteristic family $i_\alpha$ is approaching the large 1-shock if either $\alpha \in \Omega^-$ and $i_\alpha \in \{1, 2, 3, 4\}$, or $\alpha \in \Omega^+$ and $i_\alpha = 1$. We then write $\alpha \in A_1$.

(iii) We say that a weak wave $\alpha$ of the characteristic family $i_\alpha$ is approaching the boundary if $\alpha \in \Omega^+$ and $i_\alpha = 4$. We then write $\alpha \in A_b$.

For a weak wave $\alpha$ of $i$-family, we define its weighted strength as

$$b_\alpha = \begin{cases} 
\alpha & \text{if } \alpha \in \Omega^+,
\kappa_{-\alpha} & \text{if } \alpha \in \Omega^-,
\end{cases}$$

where $\kappa_- = 2 \max_{1 \leq i \leq 4, 2 \leq j \leq 4} \{K_{ij}\}$ for coefficients $K_{ij}$ in Lemma 2.9.
Definition 3.2. The wave interaction potential $Q(x)$ is

$$
Q(x) = C^* \sum_{(\alpha, \beta) \in A} |b_\alpha b_\beta| + K^* \sum_{\alpha \in A_1} |b_\alpha| + \sum_{\beta \in A_b} |b_\beta| + \overline{K}_{\alpha_0} \sum_{\alpha_k > x} |\alpha_k|
$$

(3.2)

where $K^* \in (K_4, 1)$ and $\overline{K}_{\alpha_0} > K_{\alpha_0}$.

Definition 3.3. The total (weighted) strength of weak waves in $U^\varepsilon(x, \cdot)$ is defined by

$$
V(x) = \sum_\alpha |b_\alpha|.
$$

The Glimm-type functional is defined by

$$
\mathcal{F}(x) = V(x) + \kappa Q(x) + |U^*(x) - U_0^+| + |U_0^-(x) - U_0^-|,
$$

(3.3)

where $\kappa > 0$ is a constant to be specified later, the vectors $U^*(x)$ and $U_0(x)$ are the above and below states of the large shock respectively at “time” $x$, and $U_0^+$ and $U_0^-$ are the right and left states of the large shock at $x = 0$, respectively.

Note that $V$, $Q$, and $\mathcal{F}$ are constant between any pair of subsequent interaction times. On the other hand, we can show that, across an interaction “time” $x$, both $Q$ and $\mathcal{F}$ decrease.

Proposition 3.1. If $TV(\tilde{U}(\cdot)) + TV(g'(\cdot))$ is sufficiently small, then, for any $x > 0$, $V(x)$ is sufficiently small and $\mathcal{F}(U^\varepsilon(x, \cdot))$ is uniformly bounded in $\varepsilon > 0$.

Proof. Define

$$
\Delta \mathcal{F}(x) = \mathcal{F}(x^+) - \mathcal{F}(x^-),
$$

where $x^+$ and $x^-$ are the “times” right after and right before the interaction “time”, respectively.

Case 1 (Weak waves $\alpha$ and $\beta$ interact): Then $U^*(x)$ and $U_0(x)$ do not change across this interaction time. Thus,

$$
\mathcal{F}(x^+) - \mathcal{F}(x^-) = V(x^+) - V(x^-) + \kappa (Q(x^+) - Q(x^-))
$$

$$
\leq M_1 |b_\alpha b_\beta| + \kappa (-C^* |b_\alpha b_\beta| + C^* |b_\alpha b_\beta| V(x^-) + M_0 |b_\alpha b_\beta|),
$$

for some constants $M_0$ and $M_1$ independent of $\varepsilon$.

Case 2 (Weak wave $\alpha$ of 4-family interacts with the boundary):

$$
\Delta \mathcal{F}(x) = K_{b_4} \alpha - \alpha + \kappa \left(C^* K_{b_4} V(x^-) \alpha + K^* K_{b_4} \alpha - \alpha\right).
$$

Case 3 (New 1-wave $\alpha$ produced by the boundary):

$$
\Delta \mathcal{F}(x) = K_{\alpha_0} \phi_k + \kappa \left(C^* K_{\alpha_0} \phi_k V(x^-) + K^* K_{\alpha_0} \phi_k - \overline{K}_{\alpha_0} \phi_k\right).
$$

The next two cases when $U^*(x)$ and $U_0(x)$ change across this interaction “time”. Then
Case 4 (Weak wave $\alpha$ of $i$-family interacts with the strong shock from below):

$$
\Delta \mathcal{F}(x) = V(x^+) - V(x^-) + |U^*(x^+) - U^*(x^-)| \\
+ |U_\ast(x^+) - U_\ast(x^-)| + K(Q(x^+) - Q(x^-)) \\
= \sum_{j=1,2,3,4} K_{ji} \varepsilon_\alpha - b_\alpha + K(C^* \sum_{j=2,3,4} K_{ji} V(x^-) - b_\alpha + K_{44} \alpha);
$$

Case 5 (Weak wave $\alpha$ of 1-family interacts with the strong shock from above):

$$
\Delta \mathcal{F}(x) = \sum_{j=1,2,3,4} K_{si} \alpha - \alpha + K(C^* \sum_{j=2,3,4} K_{sj} V(x^-) - \alpha - K^* \alpha + K_{44} \alpha).
$$

In these cases, $K_{44} < K^* < 1$, $b_\alpha \geq 2 \max\{K_{ji}\} |\alpha|$ due to the choice of the weight $k_-$, and $C^* > M_0 > 0$ is a constant that is not small.

We now prove $V(x) \ll 1$ for all $x > 0$.

Case 1 ($x > 0$ is the first interaction “time” $x_1$): Since $V(x^-_1) = V(0) \leq TV(U(\cdot)) \ll 1$ and $\sum_{k=0}^{\infty} \phi_k \leq TV(g(\cdot)) \ll 1$ for all Cases 1–5, we find that, when $\kappa$ is larger enough and $\mu_\varepsilon$ is sufficiently small,

$$
\Delta \mathcal{F}(x_1) \leq 0, \quad i.e. \quad \mathcal{F}(x_1^+) \leq \mathcal{F}(x_1^-) = \mathcal{F}(0).
$$

Therefore,

$$
V(x_1^+) \leq \mathcal{F}(x_1^+) \leq \mathcal{F}(0) \leq V(0) + \kappa Q(0) \\
= V(0) + \kappa (C^* V^2(0) + V(0) + \tilde{K}_{60} \sum_{x=0}^{\infty} \phi_k) \\
\leq C(V(0) + \sum_{x=0}^{\infty} \phi_k) \ll 1.
$$

Case 2 ($V(x^-_k) \ll 1$ and $\mathcal{F}(x_k^+) \leq \mathcal{F}(x_k^-)$ for any $k < n$): Then, for the next interaction “time” $x_n$, similarly to Case 1, we also have

$$
\Delta \mathcal{F}(x_n) \leq 0, \quad i.e. \quad \mathcal{F}(x_n^+) \leq \mathcal{F}(x_n^-) = \mathcal{F}(x_n^{*-1}).
$$

Thus, we have

$$
V(x_n^+) + |U^*(x_n^+) - U_0^+| + |U_\ast(x_n^+) - U_0^-| \\
\leq \mathcal{F}(x_n^+) \leq \mathcal{F}(x_n^-) = \mathcal{F}(x_n^{-1}) \leq \ldots \leq \mathcal{F}(0) = V(0) + \kappa Q(0) \\
= V(0) + \kappa (C^* V^2(0) + V(0) + \tilde{K}_{60} \sum_{x=0}^{\infty} \phi_k) \\
\leq C(V(0) + \sum_{x=0}^{\infty} \phi_k) \ll 1.
$$

Therefore, $V(x) \ll 1$ is proved for all $x$, since $C$ is independent of $x$. Then

$$
(3.4) \quad TV\{U(x, \cdot)\} \approx V(x) + |U^*(x) - U_0^+| + |U_\ast(x) - U_0^-| + |\sigma_0| = O(1).
$$
Lemma 3.2. For any sufficiently small fixed ε > 0, the number of wave fronts in $U^\varepsilon(x,y)$ is finite so that the approximate solutions $U^\varepsilon(x,y)$ are defined for all $x$.

Proof. Recall that the total interaction potential $Q(x)$ is constant except decreasing when it crosses an interaction time. From Cases 1–5 in Proposition 3.1, we have known that $V(t) \ll 1$. Therefore, we can find some $c \in (0,1)$ so that

$$\Delta Q(x) = Q(x^+) - Q(x^-) \leq \begin{cases} 
-c|b_\alpha b_\beta| & \text{if both waves } \alpha \text{ and } \beta \text{ are weak}, \\
-c|b_\alpha| & \text{if weak wave } \alpha \text{ hits the strong shock or the boundary}, \\
-c|\theta_k| & \text{if the angle of the boundary changes}.
\end{cases}$$

The following argument is similar to that in [1]: $Q$ decreases for each case and $Q(0)$ is bounded; When the interaction potential between the incoming waves is greater than $\mu \varepsilon$, $Q$ decreases by at least $c\mu \varepsilon$ in these interactions, by the bound in (3.5); Following the wave front tracking, new physical waves can be only generated by this kind of interactions, which implies that the number of the waves is finite; Since non-physical waves are produced only when physical waves interact, the number of non-physical waves is also finite; and, since two waves can interact only once, the number of interactions is also finite. Therefore, the approximate solutions are defined for all $x > 0$. □

Similar to [1], we have following lemma.

Lemma 3.3. The total strength of all non-physical waves at any $x$ is of the order $O(1)(\delta \varepsilon + \mu \varepsilon)$.

Following the framework of the wave front tracking in [3,1] and Lemmas 3.1–3.2, we obtain the existence of global entropy solutions to (1.3) and (1.8)–(1.9).

Theorem 3.1. If $TV(\bar{U}()) + TV(g'(\cdot))$ is sufficiently small, then there exists a global entropy solution in $BV$ of problem (1.3) and (1.8)–(1.9) of initial-boundary value type in the sense of Definition 2.1.

4. The Lyapunov functional

We now follow the approach of [9,20,30] to construct the Lyapunov functional $\Phi(U, V)$ by incorporating additional new waves generated from the wedge boundary vortices, which is equivalent to the $L^1$-distance:

$$C_1^{-1}\|U(x, \cdot) - V(x, \cdot)\|_{L^1} \leq \Phi(U, V) \leq C_2\|U(x, \cdot) - V(x, \cdot)\|_{L^1},$$

$$\Phi(U(x_2, \cdot), V(x_2, \cdot)) - \Phi(U(x_1, \cdot), V(x_1, \cdot)) \leq C_3\varepsilon(x_2 - x_1)$$

for any $x_2 > x_1 > 0$, for some constants $C_i$, $i = 1, 2, 3$, where $U$ and $V$ are two approximate solutions obtained by the wave front tracking, and the small parameter $\varepsilon$ controls the following three types of errors:
• Errors in the approximation of initial data and boundary;
• Errors in the speeds of shock and rarefaction fronts;
• The maximum strength of rarefaction fronts;
• The total strength of all non-physical waves.

When \( x \) is fixed, for each \( y \), the connection \( U(y) \) with \( V(y) \) always moves along Hugoniot curves \( S_1, C_2, C_3, \) and \( S_4 \) in the phase space. We call \( p_i(y) \) the strength of the \( i \)-th discontinuity wave, which is determined by \( U(y) \) and \( V(y) \) as follows:

- If \( U(y) \) and \( V(y) \) are both in \( \Omega_- \) or in \( \Omega_+ \), then start from \( U(y) \) moving along Hugoniot curves and end at \( V(y) \);
- If \( U(y) \) is in \( \Omega_- \), \( V(y) \) is in \( \Omega_+ \), then also start from \( U(y) \) moving along Hugoniot curves and end at \( V(y) \);
- If \( V(y) \) is in \( \Omega_- \) and \( U(y) \) is in \( \Omega_+ \), then start from \( V(y) \) moving along Hugoniot curves and end at \( U(y) \).

Now we define the weighted \( L^1 \) strength:

\[
(4.1) \quad q_i(y) = \begin{cases} 
  c^b_ip_i(y) & \text{if } U(y) \text{ and } V(y) \text{ are both in } \Omega_- , \\
  c^m_ip_i(y) & \text{if } U(y) \text{ and } V(y) \text{ are in different domains,} \\
  c^p_ip_i(y) & \text{if } U(y) \text{ and } V(y) \text{ are both in } \Omega_+ .
\end{cases}
\]

where the constants \( c^b_i, c^m_i, \) and \( c^p_i \) are to be determined later. Then we define the Lyapunov functional:

\[
(4.2) \quad \Phi(U, V) = \sum_{i=1}^{4} \int_{-\infty}^{g(x)} |q_i(y)|W_i(y)dy,
\]

with

\[
(4.3) \quad W_i(y) = 1 + \kappa_1A_i(y) + \kappa_2(Q(U) + Q(V)).
\]

Here \( \kappa_1 \) and \( \kappa_2 \) are two constants to be defined later, \( Q \) is the total wave interaction potential defined in Definition 3.2. \( A_i(y) \) is the total strength of waves in \( U \) and \( V \) which approach the \( i \)-wave \( q_i(y) \) defined by

\[
(4.4) \quad A_i(y) = B_i(y) + D_i(y) + \begin{cases} 
  C_i(y) & \text{if } q_i(y) \text{ is small,} \\
  F_i(y) & \text{if } i = 1 \text{ and } q_1(y) = B \text{ is large,}
\end{cases}
\]
where the “small” or “large” describes the waves that connect the states in the same or in the distinct domains $\Omega^-$ and $\Omega^+$, respectively, and

\[
B_i(y) = \left( \sum_{\alpha \in J(U)} + \sum_{\alpha \in J(V)} \right) |\alpha|,
\]

\[
C_i(y) = \begin{cases} 
\left( \sum_{a < y, k_{\alpha} = i} + \sum_{a > y, k_{\alpha} = i} \right) |\alpha| & \text{if } q_i(y) < 0, \\
\left( \sum_{a < y, k_{\alpha} = i} + \sum_{a > y, k_{\alpha} = i} \right) |\alpha| & \text{if } q_i(y) > 0,
\end{cases}
\]

\[
F_i(y) = \left( \sum_{\alpha \in J(U)} + \sum_{\alpha \in J(V)} \right) |\alpha|,
\]

For fixed $x$, $J = J(U) \cup J(V)$ is the set of all weak waves in $U$ and $V$, $\alpha$ is the strength of wave $\alpha \in J$, located at point $y_{\alpha}$ and belonging to the characteristic family $k_{\alpha}$.

\[
D_i(y) = \begin{array}{c|c|c|c}
U, V \text{ are both in } \Omega_- & U, V \text{ are in different domains} & U, V \text{ are both in } \Omega_+ \\
D_1(y) & B & 0 & B \\
D_2(y) & B & B & 0 \\
D_3(y) & B & B & B \\
\end{array}
\]

Since, for any $U(x, \cdot), V(x, \cdot) \in BV \cap L^1$ and $TV(\bar{U}(\cdot)) + TV(\bar{V}(\cdot)) + TV(g'(\cdot))$ is sufficiently small, we have

\[
C_0^{-1} \|U(x, \cdot) - V(x, \cdot)\|_{L^1} \leq \sum_{i=1}^{4} \int_{-\infty}^{g(x)} |q_i(y)| dy \leq C_0 \|U(x, \cdot) - V(x, \cdot)\|_{L^1},
\]

\[
1 \leq W_i(y) \leq C_0, \quad i = 1, 2, 3, 4,
\]

for some constant $C_0$ independent of $x$ and $\varepsilon$. Therefore, for any $x \geq 0$,

\[
(4.5) \quad C_1^{-1} \|U(x, \cdot) - V(x, \cdot)\|_{L^1} \leq \Phi(U, V) \leq C_2 \|U(x, \cdot) - V(x, \cdot)\|_{L^1},
\]

where $C_1$ and $C_2$ depend only on $TV(\bar{U}(\cdot)) + TV(\bar{V}(\cdot)) + TV(g'(\cdot))$ and the strength of the strong shock, which are independent of $x$.

Now we examine how the Lyapunov functional $\Phi$ evolves in the flow direction $x > 0$. Denote $\lambda_i$ the speed of the $i$–wave $q_i(x)$ (along the Hugoniot curve in the phase space). At a time $x$ which is not the interaction time of the waves either in
or
\[
d\frac{dx}{\Phi(U(x), V(x))} = \sum_{\alpha \in J} \sum_{i=1}^{4} \left( |q_i(y_\alpha^-)| W_i(y_\alpha^-) - |q_i(y_\alpha^+)| W_i(y_\alpha^+) \right) \dot{y}_\alpha + \sum_{i=1}^{4} |q_i(b)| W_i(b) \dot{y}_b
\]
\[
= \sum_{\alpha \in J} \sum_{i=1}^{4} \left( |q_i(y_\alpha^-)| W_i(y_\alpha^-)(\dot{y}_\alpha - \lambda_i(y_\alpha^-)) - |q_i(y_\alpha^+)| W_i(y_\alpha^+)(\dot{y}_\alpha - \lambda_i(y_\alpha^+)) \right)
\]
\[
+ \sum_{i=1}^{4} |q_i(b)| W_i(b)(\dot{y}_b - \lambda_i(b)),
\]
where \( \dot{y}_\alpha \) is the speed of the discontinuity at wave \( \alpha \in J \), \( b = g(x) \) stands for the points on the boundary, and \( \dot{y}_b \) is the slope of the boundary. Define
\[
E_{\alpha,i} = |q_i^+| W_i^+(\lambda_i^+ - \dot{y}_\alpha) - |q_i^-| W_i^-(\lambda_i^- - \dot{y}_\alpha),
\]
\[
E_{b,i} = |q_i(b)| W_i(b)(\dot{y}_b - \lambda_i(b)),
\]
where \( q_i^\pm = q_i(y_\alpha \pm) \), \( W_i^\pm = W_i(y_\alpha \pm) \), and \( \lambda_i^\pm = \lambda_i(y_\alpha \pm) \). Then
\[
\frac{d}{dx} \Phi(U(x), V(x)) = \sum_{\alpha \in J} \sum_{i=1}^{4} E_{\alpha,i} + \sum_{i=1}^{4} E_{b,i}.
\]
Our main goal is to establish the following bounds:

1. \[
\sum_{i=1}^{4} E_{b,i} \leq 0 \quad \text{near the boundary},
\]
2. \[
\sum_{i=1}^{4} E_{\alpha,i} \leq 0 \quad \text{when } \alpha \text{ is a strong shock wave in } J,
\]
3. \[
\sum_{i=1}^{4} E_{\alpha,i} \leq O(1)|\alpha| \quad \text{when } \alpha \text{ is a non-physical wave in } J,
\]
4. \[
\sum_{i=1}^{4} E_{\alpha,i} \leq O(1)\varepsilon|\alpha| \quad \text{when } \alpha \text{ is a weak wave in } J.
\]

From (4.9)–(4.12), we have
\[
\frac{d}{dx} \Phi(U(x), V(x)) \leq O(1)\varepsilon.
\]
If the constant \( \kappa_2 \) in the Lyapunov functional is chosen large enough, by the Glimm interaction estimates, all weight functions \( W_i(y) \) decrease at each time where two fronts of \( U \) or two fronts of \( V \) interact. By the self-similar property of the Riemann solutions, \( \Phi \) decreases at this time. Integrating (4.13) over interv al \([0, x]\), we obtain
\[
\Phi(U(x), V(x)) \leq \Phi(U(0), V(0)) + O(1)\varepsilon x.
\]
In Section 5, we prove (4.13)–(4.14).
5. Estimates for the $L^1$ Stability

For the case that the weak wave $\alpha \in J := J(U) \cup J(V)$ appears when $U$ and $V$ both in $\Omega_-$ or $\Omega_+$ and for the case of the non-physical waves in $J$, estimates (4.11)-(4.12) can be obtained if $|B/\sigma_0|$ is small enough and $\kappa_1$ is large enough, by following [9].

Therefore, in this section, we focus the other cases. Cases 1–3 below are all related to the strong shock and depend on the wave jump $\alpha$ in $U$ or $V$; and, by carefully adjusting the coefficients $c_i$ and especially relying on estimate (2.15), we can obtain desirable results, which is similar to the Cauchy problem discussed in [26]. Case 4 is the case near the boundary, which is different from those for the Cauchy problem.

Case 1 (Cross the first strong shock $\alpha$ in $U$ or $V$): For this case,

$$E_1 = BW_1^+(\lambda_1^+ - \hat{y}_\alpha) - |q_1^-|W_1^- (\lambda_1^- - \hat{y}_\alpha)$$

$$\leq O(1)B \sum_{i=1}^{4} |q_i^-| - \kappa_1 B|q_i^-| |\lambda_i^- - \hat{y}_\alpha|,$$

and

$$\sum_{i=2}^{4} E_i = \sum_{i=2}^{4} \left( |q_i^-| (\lambda_i^- - \hat{y}_\alpha) (W_i^+ - W_i^-) + W_i^+ (|q_i^-| (\lambda_i^+ - \hat{y}_\alpha) - |q_i^-| (\lambda_i^- - \hat{y}_\alpha)) \right)$$

$$\leq 4 \sum_{i=2}^{4} \kappa_1 B|q_i^-||\lambda_i^+ - \hat{y}_\alpha| - \frac{3}{4} \sum_{i=2}^{4} \kappa_1 B|q_i^-||\lambda_i^- - \hat{y}_\alpha|.$$

Therefore, when $\kappa_1$ is large enough, we have

$$\sum_{i=1}^{4} E_i \leq \sum_{i=2}^{4} \kappa_1 B|q_i^+||\lambda_i^+ - \hat{y}_\alpha| - \sum_{i=1}^{4} \frac{1}{2} \kappa_1 B|q_i^-||\lambda_i^- - \hat{y}_\alpha|.$$
For $i = 2, 3, 4,$

$$E_i = |q_i^+| (W_i^+ - W_i^-) (\lambda_i^+ - \hat{\gamma}_o) + W_i^+ (|q_i^+| (\lambda_i^+ - \hat{\gamma}_o) - |q_i^-| (\lambda_i^- - \hat{\gamma}_o))$$

$$\leq \kappa_1 |q_i^+| |\alpha| |\lambda_i^+ - \hat{\gamma}_o| + \kappa_1 B (|q_i^+| - |q_i^-|) (\lambda_i^+ - \hat{\gamma}_o) + |q_i^-| (\lambda_i^+ - \lambda_i^-))$$

$$\leq \kappa_1 |q_i^+| |\sigma| |\lambda_i^+ - \hat{\gamma}_o| + \kappa_1 B (|q_i^+| - |q_i^-|) (\lambda_i^+ - \hat{\gamma}_o) + O(1) |q_i^-| |\alpha|).$$

Then we have

$$\sum_{i=1}^{4} E_i \leq \kappa_1 O(1)( - |\alpha| + |\alpha| \sum_{k \neq 1} (|q_k^+| + |q_k^-|) + \sum_{k \neq 1} (|q_k^+| - |q_k^-|)) + O(1) |\alpha|.$$}

Since $||q_k^+| - |q_k^-|| \leq |q_k^+ - q_k^-| \leq O(1)|\alpha|$ when $k \neq 1,$ we can obtain $\sum_{i=1}^{4} E_i \leq 0$ if all the weights $c_i^m$ are sufficiently small and $\kappa_1$ is large enough.

Case 3 (Cross the second strong shock $\alpha$ in $U$ or $V$): For this case, by Lemma 2.3 we have

$$p_4 = p_4^+ + K_{ss} p_4^+.$$}

Since we have (2.1.4) the following lemma can be easily obtained.

**Lemma 5.1.** There exist $c_1^a,$ $c_2^a,$ and $\gamma$ such that

$$\frac{c_2^a}{c_1^a} < 1,$$

$$\frac{c_2^a}{c_1^a} |\lambda_i^+ - \sigma| < |\lambda_i^+ - \sigma| < \gamma < 1.$$}

With Lemma 5.1 then we estimate $E_i$:}

$$E_1 = -BW_1^- (\lambda_i^- - \hat{\gamma}_o) + |q_i^+| W_1^+ (\lambda_i^+ - \hat{\gamma}_o)$$

$$\leq O(1) B |q_i^+| - \kappa_1 B |q_i^+| |\lambda_i^+ - \hat{\gamma}_o|$$

$$= O(1) B |q_i^+| - \kappa_1 B c_i^a |p_i^+| |\lambda_i^+ - \hat{\gamma}_o|,$$

and, for $i = 2, 3,$

$$E_i = |q_i^-| (\lambda_i^- - \hat{\gamma}_o) (W_i^- + W_i^+) + W_i^+ (|q_i^+| (\lambda_i^+ - \hat{\gamma}_o) - |q_i^-| (\lambda_i^- - \hat{\gamma}_o))$$

$$\leq -\kappa_1 B |q_i^-| (\lambda_i^- - \hat{\gamma}_o) + O(1) |q_i^-|$$

$$\leq -\kappa_1 B |q_i^-| (\lambda_i^- - \hat{\gamma}_o) + O(1) (|q_i^-| + |q_i^+|).$$}

By 5.1 and (5.3),

$$E_4 = |q_4| (\lambda_4^- - \hat{\gamma}_o) (W_4^- + W_4^+) + W_4^+ (|q_4^+| (\lambda_4^+ - \hat{\gamma}_o) - |q_4^-| (\lambda_4^- - \hat{\gamma}_o))$$

$$\leq \kappa_1 B \left( c_1^a |p_4| (\lambda_4^- - \hat{\gamma}_o) + c_2^a K_{ss} |p_4^+| (\lambda_4^+ - \hat{\gamma}_o) - c_2^a |p_4^-| (\lambda_4^- - \hat{\gamma}_o) \right)$$

$$\leq \kappa_1 B \left( c_1^a |p_4| (\lambda_4^- - \hat{\gamma}_o) + \gamma c_1^a |p_4^+| |\lambda_4^- - \hat{\gamma}_o| - c_2^a |p_4^-| (\lambda_4^- - \hat{\gamma}_o) \right).$$
From above, if we choose \( c_4^2 \) is small enough relatively to \( c_i^p \) and choose \( k_1 \) is large enough, then we obtain

\[
\sum_{i=1}^{4} E_i &= -(1 - \gamma) \kappa_1 B |q_i^+| |\lambda_i^+ - \dot{y}_a| + O(1) |q_i^+| \\
&+ \kappa_1 B (c_4^2 |p_4^+| (\lambda_4^+ - \dot{y}_a) - c_4^p |p_4^-| (\lambda_4^- - \dot{y}_a)) \\
&+ \sum_{i=2}^{3} (-\kappa_1 B |q_i^-| (\lambda_i^- - \dot{y}_a) + O(1) \cdot |q_i^-|) \leq 0.
\]

Case 4 (near the boundary): For the previous cases, all the desire results depend on the cancellation between the two sides of a wave in \( J \). However, it is not the case near the boundary since there is only one side near the boundary. Then we exploit the exclusive property of the boundary condition \( \lambda_4 \): the flows of \( U \) and \( V \) are tangent to the boundary, which implies that they must be parallel to each other. Then we solve the Riemann problem determined by \( U(b) \) and \( V(b) \).

**Proposition 5.2.** Let \( U(b) = (\dot{u}, \dot{v}, \ddot{p}, \ddot{\rho}) \) and \( V(b) = (\hat{u}, \hat{v}, \hat{p}, \hat{\rho}) \) be both in \( O(U_+) \), \( v_1/u_1 = v_2/u_2 = \ddot{y}_b \), and \( \ddot{v}, \ddot{\hat{v}} \approx 0 \). Let \( p_i(b) \) and \( \lambda_i \) be the strength and speed of the \( i \)th shock in the Riemann problem determined by \( U(b) \) and \( V(b) \). Then

\[
\begin{align*}
|\dot{y}_b - \lambda_i| &\sim |p_i(b)|, \quad i = 2, 3, \\
|p_1(b)| &\leq |p_4(b)| + O(1)|p_2(b)||\lambda_2 - \ddot{y}_b| + |p_1(b)|O(1)|\ddot{y}_b|, \\
|p_4(b)| &\leq O(1)|p_1(b)|.
\end{align*}
\]

**Proof.** We divide the proof into two cases.

Case 1. \( p_1(b) = 0 = p_4(b) \) which corresponds the case \( \ddot{p} = \hat{p} \): Starting from \( U_b \), go along the curves of the second and third families to reach \( V_b \). These two families are the contact Hugoniot curves, and \( \lambda_2 \) and \( \lambda_3 \) are constant along the curves. Since \( \lambda_{2,3} = v/u \), \( \mathbf{r}_2 = (1, v/u, 0, 0)^\top \), and \( \mathbf{r}_3 = (0, 0, 0, \rho)^\top \), \( v/u \) keeps unchanged as the initial value \( v(U_b)/u(U_b) \), i.e. \( \ddot{y}_b \) in this process. Therefore, \( \lambda_{2,3} = \ddot{y}_b \), i.e.,

\[\ddot{y}_b - \lambda_{2,3} = 0.\]

Case 2. \( p_1(b) \neq 0 \) which corresponds to \( \ddot{p}_1 \neq \hat{p}_4 \). Starting from \( U(b) \), go along the first Hugoniot curve to reach \( U_1 \), then possibly along the second curve to reach \( U_2 \), the third curve to reach \( U_3 \), and the fourth Hugoniot curve to reach \( V(b) \).

We project \((u, v, p, \rho)\) onto the \( u - v \) plane to see the relation among \( p_1(b) \), \( p_2(b) \), \( p_3(b) \), and \( p_4(b) \) more clearly. Denote \( \mathbf{r}_1|_u \) the projection of \( \mathbf{r}_1 \) onto the \( u \) axis, \( \mathbf{r}_2|_{(u,v)} \) the projection of \( \mathbf{r}_2 \) onto the \( u - v \) plane; and the others are defined similarly. At \( U_+ \), we have

\[
\mathbf{r}_1|_u = -\mathbf{r}_4|_u, \quad \mathbf{r}_1|_v = \mathbf{r}_4|_v, \quad \mathbf{r}_1|_{(p,\rho)} = -\mathbf{r}_4|_{(p,\rho)}, \quad \mathbf{r}_2 = \mathbf{r}_2|_{(u,v)}, \quad \mathbf{r}_3 = \mathbf{r}_3|_{(u,v)} = 0.
\]

The first observation is \( p_4(b) \neq 0 \). Since \( \mathbf{r}_1|_{(u,v)} = k_1(-\lambda_1, 1)^\top \), the characteristic speed is finite and \( \ddot{y}_b \approx 0 \), so we always have \( \frac{1}{\lambda_i} > \ddot{y}_b \) near \( U_+ \), i.e. the
derivative $dv/du$ along the 1st curve is always larger than $\dot{y}_6$ in the $u-v$ plane. Therefore, $v(U_1)/u(U_1) \neq v(U_b)/u(U_b)$. On the other hand, we have $v(U_1)/u(U_1) = v(U_2)/u(U_2) = v(U_3)/u(U_3)$ and $v(U_6)/u(U_6) = v(U_b)/u(U_b)$. Therefore, $v(U_1)/u(U_1) = v(U_2)/u(U_2) = v(U_3)/u(U_3) \neq v(V_b)/u(V_b)$. To reach $V_b$, there must be some distance along the 4th Hugoniot curve. Thus, $p_4 \neq 0$.

On the $u-v$ plane, we define the signed length of $(U_1-U_b)_{(u,v)}$ and $(V_b-U_3)_{(u,v)}$ by $l_1$ and $l_4$ as follows:

$$l_1 = \begin{cases} \| (U_1-U_b)_{(u,v)} \| & \text{if } p_1 > 0, \\ -\| (U_1-U_b)_{(u,v)} \| & \text{if } p_1 < 0; \end{cases}$$

and

$$l_4 = \begin{cases} \| (V_b-U_3)_{(u,v)} \| & \text{if } p_4 > 0, \\ -\| (V_b-U_3)_{(u,v)} \| & \text{if } p_4 < 0. \end{cases}$$

The second observation is

$$|\lambda_2 - \dot{y}_b| = O(1)|l_1| = O(1)|p_1(b)|,$$

and, since $\lambda_2 = v(U_1)/u(U_1) = v(U_2)/u(U_2) = \lambda_3$, we also have

$$|\lambda_3 - \dot{y}_b| = O(1)|p_1(b)|.$$

The third observation is

$$-l_4 = p_2(b) \cdot O(1)(\lambda_2 - \dot{y}_b) + \bar{l},$$

where $\bar{l} = l_1 \cos \theta_1 = l_4 \cos \theta_2$, $\theta_1$ is the angle between $(1, \dot{y}_b)$ and $r_1_{(u,v)}$, $\theta_2$ is the angle between $r_1_{(u,v)}$ and $(1, \dot{y}_b)$, $\theta_1 = \theta_2 + 2\beta$ for $\beta = \arctan \dot{y}_b$, and

$$\bar{l} = l_1 \frac{\cos \theta_2}{\cos \theta_1} = l_4 \frac{\cos (\theta_1 - 2\beta)}{\cos \theta_1} = l_1 \frac{\cos \theta_1 \cos (2\beta) + \sin \theta_1 \sin (2\beta)}{\cos \theta_1}.$$

Therefore, we have

$$-l_4 = O(1)p_2(b)(\lambda_2 - \dot{y}_b) + l_4(1 + O(1)\dot{y}_b).$$

Since $r_1_{(u,p)} = -r_4_{(u,p)}$ and $r_1_{(v)} = r_4_{(v)}$ at $U_+$, we have

$$l_1 = \frac{l_4}{p_1}.$$

Then we obtain

$$-p_4(b) = O(1)p_2(b)(\lambda_2 - \dot{y}_b) + p_1(b)(1 + O(1)\dot{y}_b).$$

Therefore, from (5.7), we obtain

$$|p_1(b)| \leq |p_4(b)| + O(1)|p_2(b)||\lambda_2 - \dot{y}_b| + |p_1(b)||O(1)|\dot{y}_b|,$$

and

$$|p_4(b)| = O(1)|p_1(b)|.$$

□
By Proposition 6.2,

\[ E_{b,1} = |q_1(b)|W_1(b)(\hat{y}_b - \lambda_1) = c_1^p|p_1(b)|\kappa_1B(-\lambda_1) + O(1)|p_1(b)| \]
\[ \leq c_1^a|p_4(b)|\kappa_1B(-\lambda_1) + O(1)|p_1(b)| , \]
\[ E_{b,i} = |q_i(b)|W_i(b)(\hat{y}_b - \lambda_i) = c_i^p|p_i(b)|O(1)(\hat{y}_b - \lambda_i) = O(1)|p_1(b)| , \quad i = 2, 3, \]
\[ E_{b,4} = |q_4(b)|W_4(b)(\hat{y}_b - \lambda_4) = c_4^a|p_4(b)|\kappa_1B\lambda_1 + O(1)|p_4(b)| \]
\[ = c_4^a|p_4(b)|\kappa_1B\lambda_1 + O(1)|p_1(b)| . \]

From Lemma 5.1, we can find \( c_1^a \) and \( c_4^a \) such that \( c_1^a < c_4^a \). Then, when \( \kappa_1 \) is large enough, we conclude

\[ \sum_{i=1}^{4} E_{b,i} = (c_1^a - c_4^a)|p_4(b)|\kappa_1B(-\lambda_1) + O(1)|p_1(b)| \]
\[ \leq (c_1^a - c_4^a)O(1)|p_1(b)|\kappa_1B(-\lambda_1) + O(1)|p_1(b)| \leq 0. \]

6. Semigroup

We now prove the existence of the semigroup generated by the wave front tracking method.

**Proposition 6.1.** If \( TV(\bar{U}(\cdot)) + TV(g(\cdot)) \) is sufficiently small, then, the map \( \bar{U}(\cdot), x) \mapsto U^\varepsilon(x, \cdot) := S^\varepsilon\bar{U}(\cdot) \) produced by the wave front tracking method is a uniformly Lipschitz continuous semigroup with the following properties:

(i) \( S_0^\varepsilon\bar{U} = \bar{U} \), \( S_1^\varepsilon S_1^\varepsilon\bar{U} = S_{1,1}^\varepsilon\bar{U} \);
(ii) \( \|S_1^\varepsilon\bar{U} - S_{1,1}^\varepsilon\bar{U}\|_{L^1} \leq C\|\bar{U} - \bar{V}\|_{L^1} + C\varepsilon x \).

**Proof.** Property (i) is obvious since \( S^\varepsilon \) is produced by the wave front tracking method. Then we see property (ii).

Let \( \{U^\varepsilon\} \) and \( \{V^\varepsilon\} \) be the front tracking \( \varepsilon \)-approximate solutions of \eqref{eq:1} and \eqref{eq:2} with initial data functions \( \bar{U}(\cdot) \) and \( \bar{V}(\cdot) \), respectively. By \eqref{eq:1} and \eqref{eq:2}, we obtain that, for any \( x \geq 0 \),

\[ \|U^\varepsilon(x) - V^\varepsilon(x)\|_{L^1} \leq C_1 \Phi(U^\varepsilon(x), V^\varepsilon(x)) \]
\[ \leq C_1 \Phi(U^\varepsilon(0), V^\varepsilon(0)) + C_1 O(1)\varepsilon x \]
\[ \leq C_1 C_2\|\bar{U} - \bar{V}\|_{L^1} + C_1 O(1)\varepsilon x \]

This establishes the Lipschitz continuity of the \( \varepsilon \)-semigroup. \( \square \)

**Definition 6.1.** Given \( \delta_0 > 0 \), define the domain \( \mathbb{D} \) as the closure of the set consisting of the points \( U : \mathbb{R} \mapsto \mathbb{R}^4 \) such that there exists one point \( y^i \in \mathbb{R} \) so that \( U - \bar{U} \in L^1(\mathbb{R}, \mathbb{R}^4) \) and \( TV(U - \bar{U}) \leq \delta_0 \), where

\[ \bar{U}(y) = \begin{cases} U_-, & y < y^i, \\ U_+, & y^i \leq y \leq \text{boundary} . \end{cases} \]
Remark 6.1. For a solution $U(x,y)$ to the initial-boundary value problem of (1.3) and (1.8)–(1.9), if, for any fixed $x \geq 0$, $U_x(y) = U(x,y) \in \mathbb{D}$, then $y^i = g(0) = 0$ when $x = 0$, but $y^i < g(x)$ when $x > 0$ since there is a strong shock.

The semigroup defined by the wave front tracking method is set up in the following theorem.

Theorem 6.1. If $TV(\bar{U}(\cdot)) + TV(g'(\cdot))$ is sufficiently small, then $S^\varepsilon$ defined by the wave front tracking method is a Cauchy sequence in the $L^1$ sense. Let $S_x(\bar{U}) = \lim_{\varepsilon \to 0} S^\varepsilon_x(\bar{U})$. There exists a constant $L$ such that $S : [0, \infty) \times \mathbb{D} \to \mathbb{D}$ is a uniformly Lipschitz continuous semigroup with following properties:

(i) $S_0\bar{U} = \bar{U}$, $S_xS_y\bar{U} = S_{x+y}\bar{U}$;

(ii) $\|S_x\bar{U} - S_y \bar{V}\|_{L^1} \leq L\|\bar{U} - \bar{V}\|_{L^1}$;

(iii) Each trajectory $x \mapsto S_x\bar{U}$ yields an entropy solution to the initial-boundary problem (1.3) and (1.8)–(1.9);

(iv) If $\bar{U} \in \mathbb{D}$ is piecewise constant, then, for $x > 0$ sufficiently small, the function $U(x,\cdot) = S_x\bar{U}$ coincides with the solution of (1.3) and (1.8)–(1.9) obtained by piecing together the standard Riemann solutions and the lateral Riemann solutions.

Corollary 6.1. If $TV(\bar{U}(\cdot)) + TV(g'(\cdot))$ is sufficiently small, the entropy solution to the initial-boundary problem (1.3) and (1.8)–(1.9) produced by the wave front tracking method is unique.

To prove Theorem 6.1 we need the following lemma which can be found in [6].

Lemma 6.2. Let $S : [0, \infty) \times \mathbb{D} \to \mathbb{D}$ be a globally Lipschitz semigroup. Let $X > 0$, $\bar{V} \in \mathbb{D}$, and $V : [0, X] \to \mathbb{D}$ be a continuous map whose values are piecewise constant in the $(x,y)$–plane, with jumps occurring along finitely many polygonal lines. Let $L$ be the Lipschitz constant of the semigroup. Then

$$||V(X) - S_X\bar{V}||_{L^1} \leq L\left(\|V(0) - \bar{V}\|_{L^1} + \int_0^X \lim_{h \to 0^+} \frac{||V(x + h) - S_hV(x)||_{L^1}}{h} \, dx \right).$$

Proof of Theorem 6.1. It is quite similar to the proof in [6]. The difference is that the front tracking method in [4] is to use the cut-off function in the order of $\sqrt{\varepsilon}$, while the front tracking method here is to employ the simplified Riemann solver rather than the accurate Riemann solver when the interaction term is less than $\varepsilon$.

By Lemma 6.2

$$||S^\varepsilon_x\bar{V}_n - S^\varepsilon_x\bar{V}_m||_{L^1} \leq L\left(\|\bar{V}_n - \bar{V}_m\|_{L^1} + \int_0^X \lim_{h \to 0^+} \frac{||S^\varepsilon_h(S^\varepsilon_x\bar{V}_m) - S^\varepsilon_x\bar{V}_m||_{L^1}}{h} \, dx \right).$$
Let $\epsilon_m > \epsilon_n$. Then the $\epsilon_m$–approximate solution and $\epsilon_n$–approximate solution only differ when the interaction term of two weak waves or the strength of the wave interacting with the strong wave is in $[\epsilon_n, \epsilon_m]$. Suppose that there are $N + 1$ such interactions. For each weak wave interaction between $\alpha$ and $\beta$,

$$|\alpha\beta| = \epsilon_m,$$

and

either $\alpha \geq \sqrt{\epsilon_m}$ or $\beta \geq \sqrt{\epsilon_m}$.

Let $\alpha$ be large. Then

$$\lim_{h \to 0^+} \frac{\|S^n_h(S^n_x V_n) - S^m_h(S^m_x V_m)\|_{L^1}}{h} = \sum_{i=1}^{N} O(1)\epsilon_m + O(1)\epsilon_m$$

$$= \sum_{i=1}^{N} O(1)\sqrt{\epsilon_m}\alpha + O(1)\epsilon_m$$

$$= O(1)\sqrt{\epsilon_m} TV(\bar{V}(\cdot)) = O(1)\sqrt{\epsilon_m}.$$ 

Therefore, $S^n_x V_n$ is a Cauchy sequence, which converges in the $L^1$ sense. Hence, the map $S : [0, \infty) \times D \to D$ as the limit of the approximate solutions produced by the wave front tracking method is well-defined.

Next, we prove (i) to (iv). Facts (i), (ii), and (iv) are obvious since $S$ is the limit of $S^\varepsilon$ produced by the wave front tracking method. It is similar to prove (iii) as [6], the only difference is that the wave front tracking method we employ here is slightly different. Finally, we can see that the entropy solution satisfies the boundary condition due to the construction of our approximate solutions. This completes the proof of Theorem 6.1.

7. **Uniqueness of Entropy solutions in a broader class**

In this section, we first prove the semigroup $S$ defined by the wave front tracking method is the only standard Riemann semigroup (SRS) which is defined as Definition [7.1]. That is, the semigroup defined by the wave front tracking method is the canonical trajectory of the standard Riemann semigroup (SRS). Then we prove that the uniqueness of entropy solutions in a broader class, i.e. the class of viscosity solutions as defined in [4]. The main point is to prove that, in the class of viscosity solutions, the entropy solution is unique, which coincides with the trajectory produced by the wave front tracking method.

**Definition 7.1.** We say that problem (1.3) and (1.8)–(1.9) admits a standard Riemann semigroup (SRS) if, for some $\delta_0$, there exists a continuous mapping:

$R : [0, \infty) \times D \to D$ and a constant $L$ with the following properties:

(i) $R_0 \bar{U} = \bar{U}$, $R_{x_1} R_{x_2} \bar{U} = R_{x_1 + x_2} \bar{U}$;

(ii) $\|R \bar{U} - R \bar{V}\|_{L^1} \leq L \|ar{U} - \bar{V}\|_{L^1}$;
If \( \bar{U} \in \mathbb{D} \) is piecewise constant, then, for \( x > 0 \) sufficiently small, the function \( U(x, \cdot) = R_x \bar{U} \) coincides with the solution of (1.8) and (1.9) obtained by piecing together the standard Riemann solutions and the lateral Riemann solutions.

**Theorem 7.1.** Let problem (1.3) and (1.8)–(1.9) admits a standard Riemann semi-group \( R : [0, \infty) \times \mathbb{D} \mapsto \mathbb{D} \). Let \( S \) be the semigroup generated by the wave front tracking method, i.e. \( S_x(\bar{U}) = \lim_{\varepsilon \to 0} S_x^\varepsilon(\bar{U}) \). If \( \bar{U} \in \mathbb{D} \), then \( R_x \bar{U} = S_x \bar{U} \) for all \( x \geq 0 \).

The proof of the theorem is similar to the proof in [4] by using Lemma 6.2 and the fact that, locally in \( x \) direction, the wave front tracking method and the standard Riemann semigroup (SRS) both have the structure of the Riemann solutions.

As in [4], there are two types of local approximate parametrices for (1.3): One is derived from the self-similar solution of a Riemann problem, and the other is obtained by “freezing” the coefficients of the corresponding quasilinear hyperbolic system in a neighborhood of a given point.

Let \( U : [0, \infty) \times \mathbb{R} \mapsto \mathbb{R}^4 \) be a function. Fix any point \((\tau, \xi)\) in the domain of \( U \). If \( U(\tau, \cdot) \in \mathbb{D} \), then the bound on the total variation implies the existence of the limits

\[
U^- = \lim_{y \to \xi^-} U(\tau, y), \quad U^+ = \lim_{y \to \xi^+} U(\tau, y).
\]

Denote by \( \omega = \omega(x, y) \) the corresponding solution of the Riemann problem with \( U^- \) and \( U^+ \) and by \( \hat{\lambda} \) a upper bound for all characteristic speeds, i.e.,

\[
(7.1) \quad \sup_U |\lambda_i(U)| < \hat{\lambda}, \quad i = 1, 2, 3, 4.
\]

For \( x > \tau \), define the function

\[
W^\#(U, \tau, \xi)(x, y) = \begin{cases} 
    w(x - \tau, y - \xi) & \text{if } |y - \xi| \leq \hat{\lambda}(x - \tau), \\
    U(\tau, y) & \text{if } |y - \xi| > \hat{\lambda}(x - \tau).
\end{cases}
\]

Set \( \tilde{A} = DW(U(\tau, \xi)) \) and \( \tilde{B} = DH(U(\tau, \xi)) \) the Jacobian matrices computed at the point \( U(\tau, \xi) \). For \( x > \tau \), define \( W^b(U, \tau, \xi) \) as the solution of the linear Cauchy problem with constant coefficients

\[
(7.3) \quad \tilde{A} V_x + \tilde{B} V_y = 0, \quad V(\tau, y) = U(\tau, y).
\]

Then the functions \( W^\# \) and \( W^b \) depend on the values \( U(\tau, \xi) \) and \( U(\tau, \xi \pm) \). Next, we define viscosity solutions which have the same local characterization as \( W^\# \) and \( W^b \).

**Definition 7.2.** Let \( U : [0, X] \mapsto \mathbb{D} \) be continuous with respect to the \( L^1 \) norm. We say that \( U \) is a viscosity solution of system (1.3) and (1.8)–(1.9) if there exist
constants \( C \) and \( \hat{\lambda} \) satisfying (7.1) such that, at each point \((\tau, \xi) \in [0, X] \times \mathbb{R}\), when \( \rho \) and \( \varepsilon \) are sufficiently small,

\[
\frac{1}{\varepsilon} \iint_{\xi - \rho + \varepsilon} U(\tau + \varepsilon, y) - W_{(U, \tau, \xi)}^\#(x, y) \, dx \leq C \text{TV}\{U(\tau) : (\xi - \rho, \xi) \cup (\xi, \xi + \rho)\},
\]

(7.4)

\[
\frac{1}{\varepsilon} \iint_{\xi - \rho + \varepsilon} U(\tau + \varepsilon, y) - W_{(U, \tau, \xi)}^b(x, y) \, dx \leq C (\text{TV}\{U(\tau) : (\xi - \rho, \xi + \rho)\})^2.
\]

(7.5)

Lemma 7.1. Let \((a, b)\) be a (possibly unbounded) open interval and let \( R \) be the standard Riemann semigroup (SRS). If \( \hat{U}, \hat{V} \in \mathbb{D} \), then, for all \( x \geq 0 \),

\[
\int_a^b |R_x \hat{U}(y) - R_x \hat{V}(y)| \, dy \leq L \int_a^b |\hat{U}(y) - \hat{V}(y)| \, dy.
\]

(7.6)

Proof. If two initial data functions \( \hat{U} \) and \( \hat{V} \in \mathbb{D} \) coincide on \((a, b)\), due to the finite speed of propagation, then the semigroup generated by the wave front tracking method has \( S_x \hat{U}(y) = S_x \hat{V}(y) \) for \( y \in (a + \lambda_x, b - \lambda_x) \). By Theorem 7.1, \( R_x \hat{U}(y) = S_x \hat{U}(y) = S_x \hat{V}(y) = R_x \hat{V}(y) \) for all \( y \in (a + \lambda_x, b - \lambda_x) \). Next, for \( \hat{U}, \hat{V} \in \mathbb{D} \), define

\[
\hat{U} = \hat{U}_x \chi_{[a, b]} + U_- \chi(-\infty, a) + U_+ \chi(b, g(x)),
\]

where \( \chi \) is the characteristic function, and \( U_- \) and \( U_+ \) are the states in the definition of \( \mathbb{D} \). It follows that \( \hat{U} \in \mathbb{D} \). Similarly, \( \hat{V} = \hat{V}_x \chi_{[a, b]} + U_- \chi(-\infty, a) + U_+ \chi(b, g(x)) \in \mathbb{D} \).

The uniform Lipschitz continuity of the semigroup \( R \) implies

\[
\int_a^b |R_x \hat{U}(y) - R_x \hat{V}(y)| \, dy = \int_a^b |R_x \hat{U}(y) - R_x \hat{V}(y)| \, dy \leq \|R_x \hat{U}(y) - R_x \hat{V}(y)\| \leq L \|\hat{U} - \hat{V}\| = L \int_a^b |\hat{U} - \hat{V}| \, dy.
\]

\[\square\]

Theorem 7.2. Assume that problem (1.3) and (1.5)–(1.9) admits a standard Riemann semigroup \( R \). Then a continuous map \( U : [0, X] \mapsto \mathbb{D} \) is a viscosity solution of (1.3) and (1.5)–(1.9) if and only if

\[
U(x, \cdot) = R_x \hat{U} \quad \text{for any } x \in [0, X].
\]

(7.7)

Corollary 7.1. For system (1.3) and (1.5)–(1.9), the entropy solution is unique in the class of the viscosity solutions, which coincides with the trajectory \( S_x \hat{U} \) generated by the wave front tracking method, i.e. a continuous map \( U : [0, X] \mapsto \mathbb{D} \) is a viscosity solution if and only if

\[
U(x, \cdot) = S_x \hat{U} \quad \text{for any } x \in [0, X].
\]

(7.8)
The proof is similar to the argument in \[4\]. The only difference is that there is a strong shock in our case; however, we can still carry out the proof as long as the convergence of the wave front tracking method is achieved which has been proved in Section 3.

**Remark 7.1.** For the potential flow, isentropic or isothermal Euler flow \[14\], which are the simpler cases as the $L^1$ stability problem as concerned, we obtain the same results as the full Euler equations.

**Acknowledgments.** The research of Gui-Qiang Chen was supported in part by the National Science Foundation under Grants DMS-0505473, DMS-0426172, DMS-0244473, and an Alexandre von Humboldt Foundation Fellowship. The research of Tian-Hong Li was supported in part by the National Science Foundation under Grants DMS-0244383 and DMS-0244473. The authors would like to thank Professor Tai-Ping Liu for helpful discussion.

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