PRIMORDIAL GRAVITATIONAL WAVES AND RESCATTERED ELECTROMAGNETIC RADIATION IN THE COSMIC MICROWAVE BACKGROUND

DONG-HOON KIM¹ AND SASCHA TRIPPE²,³

¹ Basic Science Research Institute, Ewha Womans University, Seoul 03760, Korea; k113130@gmail.com
² Department of Physics and Astronomy, Seoul National University, Seoul 08826, Korea; trippe@astro.snu.ac.kr

ABSTRACT

Understanding the interaction of primordial gravitational waves (GWs) with the Cosmic Microwave Background (CMB) plasma is important for observational cosmology. In this article, we provide an analysis of an apparently as-yet-overlooked effect. We consider a single free electric charge and suppose that it can be agitated by primordial GWs propagating through the CMB plasma, resulting in periodic, regular motion along particular directions. Light reflected by the charge will be partially polarized, and this will imprint a characteristic pattern on the CMB. We study this effect by considering a simple model in which anisotropic incident electromagnetic (EM) radiation is rescattered by a charge sitting in spacetime perturbed by GWs, and becomes polarized. As the charge is driven to move along particular directions, we calculate its dipole moment to determine the leading-order rescattered EM radiation. The Stokes parameters of the rescattered radiation exhibit a net linear polarization. We investigate how this polarization effect can be schematically represented out of the Stokes parameters. We work out the representations of gradient modes (E-modes) and curl modes (B-modes) to produce polarization maps. Although the polarization effect results from GWs, we find that its representations, the E- and B-modes, do not practically reflect the GW properties such as strain amplitude, frequency, and polarization states.

Key words: cosmic background radiation – gravitational waves – polarization

1. INTRODUCTION

Ever since its experimental detection (Penzias & Wilson 1965) and subsequent interpretation as a relic of the earliest epoch of the universe (Dicke et al. 1965), studies of the Cosmic Microwave Background (CMB) have been crucial for constraining cosmological models. Present-day space-based measurements of the angular power spectrum of the CMB temperature distribution on the sky are able to constrain the parameters included in the ΛCDM model of cosmology (Bahcall et al. 1999) with statistical uncertainties down to a few percent (Ade et al. 2014b). Of special importance is the fact that state-of-the-art cosmological observations can, at least in principle, place constraints on the large family of cosmic inflation models (Martin et al. 2014), which are a cornerstone of hot big-bang cosmology.

A key prediction made by models of inflation is the occurrence of primordial gravitational waves (GWs). Even though not observable directly (currently), they should be detectable indirectly via a characteristic linear polarization of the CMB (Polnarov 1985). CMB polarization can be decomposed into contributions from gradient modes (E-modes) and curl modes (B-modes), with the latter ones being excited by either tensor perturbations, i.e., propagation of primordial GWs through the plasma emitting the CMB, or gravitational lensing by foreground matter (Kamionkowski et al. 1997; Zaldarriaga & Seljak 1997). Polarimetric observations (Tripp et al. 2014) of the CMB at radio frequencies around 100 GHz have been numerous, with different classes of observations aimed at different polarization modes (E or B) and angular scales. A first observation of E-mode polarization was achieved by the Degree Angular Scale Interferometer in 2002 (Kovac et al. 2002), B-mode polarization due to gravitational lensing was first detected by the South Pole Telescope in 2013 (Hanson et al. 2013) and has since been observed by POLARBEAR (Ade et al. 2014c, 2015), ACTPol (Naess et al. 2014), and BICEP1 (Barkats et al. 2014). A detection of B-mode polarization caused by primordial GWs was claimed by the BICEP2 collaboration in 2014 (Ade et al. 2014a) but was shown to be likely caused by interstellar dust soon thereafter Flaugher et al. (2014).

The aforementioned studies aim at comparisons of theoretical polarization—specifically B-mode—patterns to observational ones in order to constrain cosmological quantities. In a nutshell, the polarization patterns studied so far originate as follows: GWs propagating through the CMB plasma cause local quadrupole anisotropies in the radiation field; this in turn causes characteristic polarization patterns in the light scattered at CMB plasma electrons. However, at least a priori one should also expect polarization immediately from interactions between GWs and individual electric charges. An electric charge located in the path of propagation of a GW will be forced into an oscillatory motion by the wave. This should lead to characteristic linear polarization of light scattered by the charge. We herewith present what appears to be the first quantitative analysis of this effect. We work out the representations of gradient modes (E-modes) and curl modes (B-modes) to produce maps of the polarization patterns resulting from scattering at a single electric charge. Throughout this work, we adopt the negative metric signature (−, +, , , ) and the Minkowski metric is given by \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1, 1) \).

2. RESCATTERED EM RADIATION AND POLARIZATION IN THE CMB

2.1. EM Radiation Rescattered by a Charge in Perturbed Spacetime

Consider an electric charge \( q \) with mass \( m \) in the CMB plasma. If the charge encounters incident light, it reradiates (outgoing) reflected light. If, however, the charge is set in
oscillating motion by a GW, the geometry of reflection will be given by the GW: as the charge is driven to move along particular directions, like the GW polarizations $h_+$ or $h_\times$, it will reanimate the reflected light along the same directions.

The electromagnetic (EM) radiation rescattered by the charge, as measured by a distant observer, can be obtained by solving the Maxwell equations

$$\Box A_\mu = -\frac{4\pi}{c} j_\mu, \quad (1)$$

where $\Box \equiv -\partial^2/c^2 \partial t^2 + \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ denotes the d’Alembertian, with $c$ representing the speed of light, and $A_\mu$ represents the EM vector potential produced by the charge and $j_\mu$, the charge’s current density. A solution to the Maxwell’s equations can be expressed to the leading order as

$$A_\mu \sim \frac{Q_\mu(t_R)}{r}, \quad (2)$$

where $r$ is the distance from the charge to the observer and $t_R \equiv t - r/c$ denotes the retarded time, and $Q_\mu$ is the charge-dipole moment defined as

$$Q_\mu = q X_\mu, \quad (3)$$

for the charge $q$ at position $X_\mu$, and the overdot denotes differentiation with respect to time $t$.

Our rescattered EM radiation can be computed using Equations (2) and (3), with the charge’s trajectory $X_\mu(t)$ determined in the spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (4)$$

where $\eta_{\mu\nu}$ is the flat background and $h_{\mu\nu}$ describes the perturbations by GWs. As the charge is moving in curved spacetime, to treat the general relativistic effects of the charge’s motion we employ a semi-relativistic approximation (Ruffini & Sasaki 1981), in which the charge’s trajectory $X_\mu(t)$ is identified with a geodesic in the spacetime $g_{\mu\nu}$. For a weakly perturbed spacetime, in which $|h_{\mu\nu}| \ll 1$, one may split the dipole moment,

$$Q^\mu = q X^\mu_{(\flat)} + q X^\mu, \quad (5)$$

and the vector potential accordingly,

$$A^\mu = A^\mu_{(\flat)} + A^\mu, \quad (6)$$

where the perturbation $A^\mu$ to the potential $A^\mu_{(\flat)}$ results from the deviation $X^\mu$ from the trajectory $X^\mu_{(\flat)}$, which is caused by the perturbation of the flat background $h_{\mu\nu}$.

Now, we prescribe an anisotropic incident monochromatic radiation field on a charge as shown in Figure 1 by means of the two waves,

$$E_{\text{it}(\text{in})} = E_i \{ e_y \exp[i(Kx - \Omega t + \varphi_1(t))] + e_z \exp[i(Ky - \Omega t + \varphi_2(t))], (7)$$

where $E_i$ and $E_{\text{it}(\text{in})}$ are the amplitudes, and $\Omega$ and $K$ are the angular temporal frequency and angular spatial frequency with $\Omega = cK$, and $\varphi_1(t)$, $\varphi_2(t)$, and $\varphi_{\text{II}}(t)$ are the phase modulation functions, which are assumed to vary on a timescale much slower than the period of the waves, i.e., $|\varphi_1(t)|$, $|\varphi_2(t)|$, $|\varphi_{\text{II}}(t)| \ll \Omega$. Here both $E_{\text{it}(\text{in})}$ and $E_{\text{it}(\text{in})}$ are unpolarized: as the relative phases $\varphi_1(t) - \varphi_2(t)$ and $\varphi_{\text{II}}(t) - \varphi_{\text{II}}(t)$ fluctuate in time, each wave will not remain in a single polarization state and hence is unpolarized. That is, the Stokes parameters for each wave must exhibit

$$\begin{bmatrix} I_{\text{it}(\text{in})} \\ Q_{\text{it}(\text{in})} \\ U_{\text{it}(\text{in})} \\ V_{\text{it}(\text{in})} \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which can be verified via Equations (40)–(43) (see Appendix A).

2.1.1. Case 1. For GWs Propagating along the Line of Sight

As mentioned above, to calculate the EM radiation rescattered by a charge moving in curved spacetime, we determine the charge’s trajectory $X_\mu(t)$, which is indeed identified with a geodesic worldline that the charge follows in the spacetime $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ represents GWs (Ruffini & Sasaki 1981). For a weakly perturbed spacetime, in which $|h_{\mu\nu}| \ll 1$, one can write down the geodesic equation of motion for the charge in linearized gravity as follows.

Figure 1. Anisotropic incident EM radiation is rescattered by a charge sitting on spacetime ripples (i.e., GWs) and becomes polarized. The GWs propagate along the line of sight while being polarized in a plane perpendicular to the line of sight, and the rescattered (outgoing) EM radiation is induced by the GWs. The directions of propagation and polarization of the GWs are fitted into the observer’s reference frame, with propagation along the $z$-axis (coinciding with the line of sight) and polarization along the $x$-axis.

Credit: CAPMAP website, http://quiet.uchicago.edu/capmap; reproduced with some modifications.
 where the indices \(i, j, \ldots\) refer to the spatial coordinates \((x, y, z)\), and the overdot “\(\dot{}\)" represents differentiation with respect to time \(t\), and the comma before the subscripts indicates partial differentiation with respect to the subscript that follows the comma. In this equation the GWs \(h_{ij}\) are given by

\[
h_{ij} = h(e_i^1 \otimes e_j^2 - e_i^2 \otimes e_j^1) \exp \left[ i \omega \left( \frac{z}{c} - t \right) \right],
\]

where \(h\) denotes the strain amplitude and \(\omega\) the frequency, \(e_i^{1(2)}\) refers to the \((x/y)\)-component of the unit vector \(e_i\), and \(+ \) and \(\times\) represent the two polarization states prescribed by the tensors \(e_i^1 \otimes e_j^2 - e_i^2 \otimes e_j^1\) and \(e_i^1 \otimes e_j^1 + e_i^2 \otimes e_j^2\), respectively. On the right-hand side of Equation (10) the EM field is prescribed by means of Equations (7) and (8)) as

\[
E_i = e_i^2 E_i \exp \left[ i \left( \Omega \left( \frac{z}{c} - t \right) + \varphi_1 (t) \right) \right] + e_i^1 E_i \exp \left[ i \left( \Omega \left( \frac{z}{c} - t \right) + \varphi_\Pi (t) \right) \right],
\]

which describes the outgoing field reflected by the change: the charge is driven to move along the \(x\)-axis and the \(y\)-axis by GWs \(h_x\) or \(h_y\), and hence radiaates the outgoing reflected light most easily by moving along the same directions. The incident and outgoing EM radiation fields and the GWs are illustrated in Figure 1: the GWs propagate along the line of sight while being polarized in a plane perpendicular to the line of sight, and the rescattered (outgoing) EM radiation is induced by the GWs. Note here that the directions of propagation and polarization of the GWs are fitted into the observer’s reference frame; with propagation along the \(z\)-axis (coinciding with the line of sight) and polarization along the \(x\)-axis and the \(y\)-axis.

Now, solving the geodesic Equation (10) via perturbation, we obtain the following results (see Appendix B for a detailed derivation):

For \(h_+\),

\[
x = v_{ox} - h v_{ox} \left( 1 - \frac{v_{ox}}{c} \right) \exp \left[ i \omega \left( \frac{z}{c} - t \right) \right] + i \frac{q E_i}{m \omega} \exp \left[ i \left( \Omega \left( \frac{z}{c} - t \right) + \varphi_\Pi (t) \right) \right] - h \left( 1 - \frac{v_{ox}}{c (\Omega + \omega)} \right) \exp \left[ i \left( \Omega + \omega \right) \left( \frac{z}{c} - t \right) + \varphi_\Pi (t) \right] \right] + \mathcal{O}_{\Pi[0]} + \mathcal{O}_{\Pi[\Phi]}, \tag{14}\]

\[
y = v_{oy} + h v_{oy} \left( 1 - \frac{v_{oy}}{c} \right) \exp \left[ i \omega \left( \frac{z}{c} - t \right) \right] + i \frac{q E_i}{m \omega} \exp \left[ i \left( \Omega \left( \frac{z}{c} - t \right) + \varphi_1 (t) \right) \right] - h \left( 1 - \frac{v_{oy}}{c (\Omega + \omega)} \right) \exp \left[ i \left( \Omega + \omega \right) \left( \frac{z}{c} - t \right) + \varphi_1 (t) \right] \right] + \mathcal{O}_{\Pi[0]} + \mathcal{O}_{\Pi[\Phi]}, \tag{15}\]

\[
z = v_{oz} - \frac{h v_{oz}^2 - v_{oz}^2}{2 c} \exp \left[ i \omega \left( \frac{z}{c} - t \right) \right] - i \frac{h q \omega}{m c \Omega (\Omega + \omega)} \left\{ E_i v_{oz} \exp \left[ i \left( \Omega + \omega \right) \left( \frac{z}{c} - t \right) + \varphi_\Pi (t) \right] \right. \right] \left( \varphi_1 (t) \right) \right] + \mathcal{O}_{\Pi[0]} + \mathcal{O}_{\Pi[\Phi]}, \tag{16}\]

For \(h_x\),

\[
x = v_{ox} - h v_{ox} \left( 1 - \frac{v_{ox}}{c} \right) \exp \left[ i \omega \left( \frac{z}{c} - t \right) \right] + i \frac{q E_i}{m \omega} \exp \left[ i \left( \Omega \left( \frac{z}{c} - t \right) + \varphi_\Pi (t) \right) \right] - h \frac{E_i}{E_i} \left( 1 - \frac{v_{ox}}{c (\Omega + \omega)} \right) \exp \left[ i \left( \Omega + \omega \right) \left( \frac{z}{c} - t \right) + \varphi_1 (t) \right] \right] + \mathcal{O}_{\Pi[0]} + \mathcal{O}_{\Pi[\Phi]}, \tag{17}\]
\[
\dot{y} = v_{oy} - hv_{ox} \left( 1 - \frac{v_{oz}}{c} \right) \exp \left[ i \omega \left( \frac{z}{c} - t \right) \right] + \frac{qE_1}{m \omega} \exp \left[ i \left( \Omega \left( \frac{z}{c} - t \right) + \varphi_1 (t) \right) \right] - \frac{h E_1}{E_q} \left( 1 - \frac{v_{ox} \omega}{c (\Omega + \omega)} \right) \exp \left[ i \left( \Omega + \omega \right) \left( \frac{z}{c} - t \right) + \varphi_\Pi (t) \right) \right] + O_{LII[0]} + O_{LII[1]} \right],
\]

\[
\dot{z} = v_{oz} - \frac{hv_{ox} v_{oz}}{c} \exp \left[ i \omega \left( \frac{z}{c} - t \right) \right] - \frac{i h q \omega}{mc \Omega (\Omega + \omega)} E_v \exp \left[ i \left( \Omega + \omega \right) \left( \frac{z}{c} - t \right) + \varphi_1 (t) \right] \]
\[
+ E_{Hv} \exp \left[ i \left( \Omega + \omega \right) \left( \frac{z}{c} - t \right) + \varphi_\Pi (t) \right] \right] + \frac{hq^2 E_1}{mc^2 \omega^2} \exp \left[ i \left( 2 \Omega + \omega \right) \left( \frac{z}{c} - t \right) + \varphi_1 (t) + \varphi_\Pi (t) \right] \right] + O_{LII[0]} + O_{LII[1]} \right],
\]

where \((v_{ox}, v_{oy}, v_{oz})\) represents the charge’s constant drifting velocity, and \(O_{[0]} = O(\varphi_1 / \Omega, \varphi_1^2 / \Omega^2, \varphi_1 \varphi_\Pi / \Omega^3, \varphi_\Pi / \Omega^3, \ldots), \)
\(O_{[1]} \equiv O(\varphi_1 / \Omega, \varphi_1^2 / \Omega^2, \varphi_1 \varphi_\Pi / \Omega^3, \varphi_\Pi / \Omega^3, \ldots), \)
\(O_{LII[0]} \equiv O(\varphi_1 / \Omega, \varphi_1^2 / \Omega^2, \varphi_1 \varphi_\Pi / \Omega^3, \varphi_\Pi / \Omega^3, \ldots), \)
\(O_{LII[1]} \equiv O(\varphi_1 / \Omega, \varphi_1^2 / \Omega^2, \varphi_1 \varphi_\Pi / \Omega^3, \varphi_\Pi / \Omega^3, \ldots), \)
\(O_{[0]} \sim h \times O_{[1]}, \)
\(O_{[1]} \sim h \times O_{[0]}, \)
\(O_{LII[0]} \sim h \times O_{LII[1]} \)
are the errors generated by the approximate solution for \(X^i = (x, y, z)\): the subscripts \([0]\) and \([1]\)
denote that the errors are generated from the unperturbed part and the perturbed part of the solution, respectively. Above, however, we have assumed that the phase modulation functions \(\varphi_1 (t)\) and \(\varphi_\Pi (t)\) vary on a timescale much slower than the period of the wave, namely, \(|\varphi_1|, |\varphi_\Pi| \ll \Omega\) (and further \(|\varphi_1|, |\varphi_\Pi| \ll \Omega^2, |\varphi_1|, |\varphi_\Pi| \ll \Omega^3, \ldots\), and therefore all these errors can be regarded as sufficiently small. A more detailed discussion of the errors is given in Appendix B.

The rescattered EM potentials are calculated by substituting Equations (14)–(19) into Equation (2). They are trivially obtained from
\[
A_{x(\text{scat})} = \frac{q \tilde{q}_1 (t_R)}{r},
\]
\[
A_{y(\text{scat})} = \frac{q \tilde{q}_1 (t_R)}{r},
\]
\[
A_{z(\text{scat})} = \frac{q \tilde{q}_1 (t_R)}{r},
\]

obtained from
\[
E_{x(\text{scat})} = \frac{\partial_A x (\text{scat})}{r} = \frac{q \tilde{q}_1 (t_R)}{r},
\]
\[
E_{y(\text{scat})} = \frac{\partial_A y (\text{scat})}{r} = \frac{q \tilde{q}_1 (t_R)}{r},
\]
\[
E_{z(\text{scat})} = \frac{\partial_A z (\text{scat})}{r} = \frac{q \tilde{q}_1 (t_R)}{r},
\]

together with Equations (14)–(16) for \(h_+\) and Equations (17)–(19) for \(h_-\).

Finally, the rescattered EM fields are calculated by means of
\[
E_{ab} = \nabla_a A_b - \nabla_b A_a \]
and Equations (20)–(22). They are
\[
E_{x(\text{scat})} = \frac{q \tilde{q}_1 (t_R)}{r},
\]
\[
E_{y(\text{scat})} = \frac{q \tilde{q}_1 (t_R)}{r},
\]
\[
E_{z(\text{scat})} = \frac{q \tilde{q}_1 (t_R)}{r},
\]

For \(h_+\),
\[
E_{x(\text{scat})} = \frac{i h q \omega (1 - \frac{v_{oz}}{c})}{r} \exp \left[ i \omega \left( \frac{z}{c} - t_R \right) \right] + \frac{q^2 E_1}{mr} \exp \left[ i \left( \Omega \left( \frac{z}{c} - t_R \right) + \varphi_\Pi (t_R) \right) \right] - h \left( 1 + \left( 1 - \frac{v_{ox} \omega}{c} \right) \exp \left[ i \left( \Omega + \omega \right) \right] \right] \times \left( \frac{z}{c} - t_R + \varphi_\Pi (t_R) \right) \right] + O_{[1]},
\]
\[
E_{y(\text{scat})} = -i h q \omega (1 - \frac{v_{oz}}{c}) \exp \left[ i \omega \left( \frac{z}{c} - t_R \right) \right] + \frac{q^2 E_1}{mr} \exp \left[ i \left( \Omega \left( \frac{z}{c} - t_R \right) + \varphi_1 (t_R) \right) \right] + h \left( 1 + \left( 1 - \frac{v_{ox} \omega}{c} \right) \exp \left[ i \left( \Omega + \omega \right) \right] \right] \times \left( \frac{z}{c} - t_R + \varphi_1 (t_R) \right) \right] + O_{[1]},
\]
\[
E_{z(\text{scat})} = -\frac{h q^2 \omega}{2 \epsilon c r} \exp \left[ i \omega \left( \frac{z}{c} - t_R \right) \right] + \frac{h q^2 \omega}{mc \epsilon \Omega \omega} \exp \left[ i \left( \Omega + \omega \right) \left( \frac{z}{c} - t_R \right) \right] + \varphi_\Pi (t_R) \right] \right] - E_t \exp \left[ i \left( \Omega + \omega \right) \left( \frac{z}{c} - t_R \right) \right] \right] \times \left( \frac{z}{c} - t_R + \varphi_1 (t_R) \right) \right] \right] + O_{[1]},
\]

(23)
(24)
(25)
For \( h_\epsilon \),

\[
E_{x(\text{scan})} = i \frac{\hbar q v_o}{r} \left( 1 - \frac{v_o}{c} \right) \omega \exp \left[ i \left( \frac{z}{c} - t_R \right) \right] + \frac{q^2 E_{\text{scat}}}{m r} \left\{ \exp \left[ i \left( \frac{z}{c} - t_R \right) + \varphi_{\Pi}(t_R) \right] \right. \\
- h \frac{E_{\text{scat}}}{E_{\text{scat}}} \left[ 1 + \left( 1 - \frac{v_o}{c} \right) \omega \right] \exp \left[ i ((\Omega + \omega)) \right] \times \left( \frac{z}{c} - t_R \right) + \varphi_{\Pi}(t_R) \right\} + O_{\text{LII}[a]}, \tag{29}
\]

\[
E_{y(\text{scan})} = i \frac{\hbar q v_o}{r} \left( 1 - \frac{v_o}{c} \right) \omega \exp \left[ i \left( \frac{z}{c} - t_R \right) \right] + \frac{q^2 E_{\text{scat}}}{m r} \left\{ \exp \left[ i \left( \frac{z}{c} - t_R \right) + \varphi_{\Pi}(t_R) \right] \right. \\
- h \frac{E_{\text{scat}}}{E_{\text{scat}}} \left[ 1 + \left( 1 - \frac{v_o}{c} \right) \omega \right] \exp \left[ i ((\Omega + \omega)) \right] \times \left( \frac{z}{c} - t_R \right) + \varphi_{\Pi}(t_R) \right\} + O_{\text{LII}[a]}, \tag{30}
\]

\[
R(\phi, \theta, \psi) = R_3(\psi) R_2(\theta) R_1(\phi) = \begin{bmatrix} \cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi \\
- \sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi \\
\sin \theta \sin \phi \end{bmatrix}, \tag{33}
\]

\[
E_{z(\text{scan})} = i \frac{\hbar q v_o v_o}{c r} \omega \exp \left[ i \left( \frac{z}{c} - t_R \right) \right] - \frac{\hbar q^2}{m c \Omega} \frac{E_{\text{scat}}}{E_{\text{scat}}} \exp \left[ i (\Omega + \omega) \left( \frac{z}{c} - t_R \right) \right] + \varphi_{\Pi}(t_R) \right\} + \frac{\hbar q^2 E_{\text{scat}}}{m^2 c r \omega} \exp \left[ i (2 \Omega + \omega) \left( \frac{z}{c} - t_R \right) \right] + \varphi_{\Pi}(t_R) \right\} + O_{\text{LII}[a]}, \tag{31}
\]

Here we note that the errors from the unperturbed part, \( O_{[0]} \), \( O_{[I]} \), and \( O_{\text{LII}[0]} \), are absent and only the errors from the perturbed part, \( O_{[II]} \), \( O_{\text{LII}[II]} \), and \( O_{\text{LII}[II]} \) are present. This is because \( E_{\text{scat}}^i \sim \dot{X}^i \) due to Equations (23)–(25), where the unperturbed part of a solution for \( \dot{X}^i = (x, y, z) \) is trivially obtained from the right-hand side of the geodesic Equation (10), which is equivalent to Equation (13) apart from the factor; thereby not generating \( O_{[II]} \). Later in Section 2.2, we will disregard any contributions from the errors \( O_{[II]} \), \( O_{\text{LII}[II]} \), and \( O_{\text{LII}[II]} \) in computing Stokes parameters. A more detailed discussion of the errors is given in Appendix B.

### 2.1.2. Case 2. For GWs Propagating along an Arbitrary Direction

Our physical quantities for Case 1 above were expressed in the observer’s reference frame associated with Cartesian coordinates \((x, y, z)\): the re scattered EM radiation was induced by GWs that propagate along polarization along the \( x \)-axis and the \( y \)-axis. This does not imply that radiation can only be observed along the \( z \)-axis: we now discuss the case that line of sight and \( z \)-axis do not coincide. A new reference frame can be defined by rotating the Cartesian frame through Euler angles \( \{\phi, \theta, \psi\} \) following the convention by Goldstein (1980):

\[
x' = R(\phi, \theta, \psi)x, \tag{32}
\]

where \( x' = (x', y', z') \) refer to the new frame while \( x = (x, y, z) \) is the original frame, and

\[
R(\phi, \theta, \psi) = R_3(\psi) R_2(\theta) R_1(\phi) = \begin{bmatrix} \cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi \\
- \sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi \\
\sin \theta \sin \phi \end{bmatrix}, \tag{33}
\]

with

\[
R_1(\phi) = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ - \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}, \quad R_3(\psi) = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ - \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{34}
\]

and \( \{\phi, \theta\} \) denote the direction angles in spherical coordinates, defined with respect to the Cartesian coordinates \((x, y, z)\), and \( \psi \) denotes the polarization-ellipse angle (Pai et al. 2001).

Let us now consider a more general situation in which GWs propagate along a general direction through the CMB. One may let the propagation direction of our GWs coincide with the \( z' \)-axis in the new frame \((x', y', z')\), which is obtained by rotating the original frame \((x, y, z)\) via Equation (32) above. Then a charge \( q \) in the CMB would be driven to move along the \( x' \)-axis and the \( y' \)-axis by GWs \( h_\pm \) or \( h_\mp \) prescribed in the new frame. Therefore, the geodesic Equation (10), together with Equations (11)–(13) should be rewritten in the coordinates \((t, x', y', z')\). Solving the geodesic equation in the new frame, one should obtain the same results as (14)–(19) in Case I.
above, except that the solutions are now expressed in $(t, x', y', z')$ rather than $(t, x, y, z)$. The same argument goes for the rescattered EM fields (26)--(31). In Figure 2 are illustrated the incident and outgoing EM radiation fields and the GWs with respect to the new coordinate frame, namely, the source frame $(x', y', z')$: the GWs propagate along the $z'$-axis while being polarized along the $x'$-axis and the $y'$-axis, and the rescattered (outgoing) EM radiation induced by the GWs should be expressed in the same frame. Here, concerning the relationship between the source frame $(x', y', z')$ and the observer’s frame $(x, y, z)$, we note the two cases: (i) the propagation direction of the GWs does not coincide with the line of sight, i.e., $e^\prime = e_\circ$, (ii) the propagation direction of the GWs coincides with the line of sight, i.e., $e^\prime = e_\circ$.

Now, the inverse transformation to Equation (32),

$$ x = R^{-1}(\phi, \theta, \psi) x' $$

is given by $R^T(\phi, \theta, \psi)$, the transpose of the matrix $R(\phi, \theta, \psi)$ (Goldstein 1980):

$$ R^{-1}(\phi, \theta, \psi) = R^T(\phi, \theta, \psi) $$

where the quantity $(E_x^{(\text{scat})}, E_y^{(\text{scat})}, E_z^{(\text{scat})})$ is to be taken from Equations (26)--(31), and the values $\{\phi, \theta, \psi\}$ are to be specified for the configuration of the source frame $(x', y', z')$ relative to the observer’s frame $(x, y, z)$. For convenience in Sections 2.2 and 3.1, we represent the transformation matrix by

$$ R^{-1}(\phi, \theta, \psi) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} $$

(38)

where $a_{ij} (i, j = 1, 2, 3)$ can be directly read off from Equation (36).

However, as in the case (i) of Figure 2, when the propagation direction of the GWs does not coincide with the line of sight, i.e., $e^\prime = e_\circ$, the rescattered EM radiation induced by the GWs cannot actually be detected by an observer who is very far away from the source; for $r \gg$ source size in Equations (26)--(31). That is, only in the case (ii) of Figure 2, namely, when the propagation direction of the GWs coincides with the line of sight, i.e., $e^\prime = e_\circ$, will the rescattered EM radiation be detectable by a distant observer. This will let us fix the direction angles $\{\phi, \theta\} = \{0, 0\}$ and reduce $R^{-1}(\phi, \theta, \psi)$ to

$$ R^{-1}(0, 0, \psi) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

(39)

which simply represents the rotation through the polarization-ellipse angle $\psi$.

### 2.2. Stokes Parameters for the Rescattered EM Radiation

In the previous subsection we have modeled a situation in which an anisotropic incident radiation is rescattered by a charge being agitated by GWs and the rescattered radiation is indeed induced by the GWs. The charge is set in motion along particular directions by the GWs $h_+$ or $h_\times$, and this in fact opens polarization channels for the rescattered radiation. For example, a charge being driven to move along the $x$-axis and the $y$-axis by GWs $h_+$ or $h_\times$ will reradiate the reflected light most easily by moving along the same directions. Now, if the incident unpolarized radiation is more intense along one axis than the other, namely, $E_1 \neq E_2$ in Equations (7) and (8), it causes the charge to oscillate more strongly along one axis than the other, i.e., along the $x$-axis [y-axis] than the $y$-axis [x-axis], for rescattering (see Figures 1 and 2). Then the radiation emitted by this accelerating charge, which propagates along the third axis, namely, the $z$-axis, has a net polarization. Therefore, the following can be said of the CMB polarization induced by primordial GWs: as primordial GWs agitate a charge in the CMB, setting it in periodic, regular motion, the EM radiation rescattered by the charge is partially polarized and this will imprint a characteristic pattern on the light of the CMB. A net linear polarization is expected from the rescattered radiation, and it can be shown by determining the Stokes parameters. For...
2.2.1. Case 1. For GWs Propagating along the Line of Sight

For the rescattered EM radiation given by (26)–(31), the Stokes parameters can be calculated in a straightforward manner:

\[ I_{\text{(case 1)}} = \langle |E_x(\text{scat})|^2 \rangle + \langle |E_y(\text{scat})|^2 \rangle, \]

\[ Q_{\text{(case 1)}} = \langle |E_x(\text{scat})|^2 \rangle - \langle |E_y(\text{scat})|^2 \rangle, \]

\[ U_{\text{(case 1)}} = \langle E_x(\text{scat})E_x^*(\text{scat}) \rangle + \langle E_y(\text{scat})E_y^*(\text{scat}) \rangle, \]

\[ V_{\text{(case 1)}} = i\langle E_x(\text{scat})E_y^*(\text{scat}) \rangle - \langle E_x(\text{scat})E_y^*(\text{scat}) \rangle, \]

where \( \langle \rangle \) represents the time average of the enclosed quantity and \( * \) denotes the complex conjugate. After some calculations, these parameters turn out to exhibit linear polarization as expected:

\[ I_{\text{(case 1)}}^{\pm} = \frac{q^4(E_1^2 + E_2^2)}{m^2r^2} \left[ 1 + h^2 \left( 1 + \left( 1 - \frac{v_\infty}{c} \right) \frac{\omega}{\Omega} \right)^2 \right] \]

\[ + \frac{h^2q^2(v_\infty^2 + v_o^2)}{r^2} \left( 1 - \frac{v_\infty}{c} \right) \frac{\omega}{\Omega} \]

\[ Q_{\text{(case 1)}}^{\pm} = \frac{q^4(E_0^2 - E_1^2)}{m^2r^2} \left[ 1 \pm h^2 \left( 1 + \left( 1 - \frac{v_\infty}{c} \right) \frac{\omega}{\Omega} \right)^2 \right] \]

\[ \pm \frac{h^2q^2(v_\infty^2 - v_o^2)}{r^2} \left( 1 - \frac{v_\infty}{c} \right) \frac{\omega}{\Omega}, \]

\[ U_{\text{(case 1)}}^{\pm} = 0, \]

\[ V_{\text{(case 1)}}^{\pm} = 0, \]

where the labels + and \( \times \) denote the two polarization states of GWs, and the ± signs on the right-hand side of Equation (45) follow the order of \( ^{\pm} \). In these calculations, we have disregarded any contributions from the errors \( O_{[\Omega]}, O_{[\delta]}, \) and \( O_{[\Pi]} \); they would be \( O_{[\Omega]}^2 \sim h^2 \times O_{[\delta]}^2 \) and thus too small to consider.

Figure 3 shows the expected spectrum of the characteristic GW amplitude over a broad interval of frequencies (The Cardiff Gravitational Physics Group Tutorial 2016). The values of amplitude in this spectrum have been estimated for present time observations, and thus our GW amplitude \( h \) in Equations (44) and (45) should be distinguished from the values of amplitude for the regime of the CMB anisotropies in the spectrum: our \( h \) should be larger than the characteristic amplitude of primordial GWs \( \sim 10^{-10} \) as shown in Figure 3. Nevertheless, due to the assumption of linearized gravity as made in Section 2.1, the order of \( h^2 \) should still be regarded as negligibly small, and hence the terms led by \( h^2 \) in Equations (44) and (45) may be cast off. Taking into consideration the redshift effect from cosmic expansion, our GW frequency \( \omega \) in Equations (44) and (45) should be \( \sim 10^3 \) times higher than the characteristic frequency for the regime of the CMB anisotropies in the spectrum shown in Figure 3. However, our \( \omega \) should still be negligibly small (in particular, compared to our EM wave frequency \( \Omega \)), and hence the terms with \( \omega/\Omega \) and \( \omega^2 \) in Equations (44) and (45) may be cast off, too. Therefore, no profiles of our GWs would indeed remain practically in the Stokes parameters (44)–(47). Henceforth we may express the parameters simply in the flat spacetime limit:

\[ I_{\text{(case 1)}} = \frac{q^4(E_1^2 + E_2^2)}{m^2r^2}, \]

\[ Q_{\text{(case 1)}} = \frac{q^4(E_0^2 - E_1^2)}{m^2r^2}, \]

\[ U_{\text{(case 1)}} = 0, \]

\[ V_{\text{(case 1)}} = 0, \]

where the expressions are now independent of the polarization states and therefore the signs \( ^{\pm} \) have been dropped.

2.2.2. Case 2. For GWs Propagating Along an Arbitrary Direction

As mentioned in the previous subsection, our rescattered EM radiation \( (E_x(\text{scat}), E_y(\text{scat}), E_z(\text{scat})) \) in the source frame takes the same values as Equations (26)–(31), except that it is expressed in the coordinates \( (x', y', z') \). However, to compute the Stokes parameters in the observer’s frame \( (x, y, z) \), the quantity \( (E_x(\text{scat}), E_y(\text{scat}), E_z(\text{scat})) \) should be projected into that frame (see Figure 2 for comparison of the observer’s frame \( (x, y, z) \) and the source frame \( (x', y', z') \)). Inserting Equations (37) and (38) into Equations (40)–(43), after rearranging terms, we have:

\[ I_{\text{(case 2)}} = (a_{11}^2 + a_{22}^2) \langle |E_x(\text{scat})|^2 \rangle \]

\[ + (a_{12}^2 + a_{21}^2) \langle |E_y(\text{scat})|^2 \rangle \]

\[ + (a_{11}a_{12} + a_{22}a_{21}) \langle E_x(\text{scat})E_y^*(\text{scat}) \rangle + \langle E_x(\text{scat})E_y^*(\text{scat}) \rangle, \]
$$Q_{\text{(case 2)}} = (a_{11}^2 - a_{22}^2)(|E_{x'}(\text{scat})|^2)$$
$$+ (a_{12}^2 - a_{21}^2)(|E_{y'}(\text{scat})|^2)$$
$$+ (a_{11}a_{12} - a_{21}a_{22})(E_{x'}(\text{scat})E_{y'}^*(\text{scat}))$$
$$+ (E_{x'}^*(\text{scat})E_{y'}(\text{scat})), \quad (53)$$

$$U_{\text{(case 2)}} = 2a_{11}a_{21}(|E_{x'}(\text{scat})|^2)$$
$$+ 2a_{12}a_{22}(|E_{y'}(\text{scat})|^2)$$
$$+ (a_{11}a_{22} + a_{12}a_{21})(E_{x'}(\text{scat})E_{y'}^*(\text{scat}))$$
$$+ (E_{x'}^*(\text{scat})E_{y'}(\text{scat})), \quad (54)$$

$$L = \frac{1}{2} \begin{bmatrix}
1 + \cos^2 \theta & \sin^2 \theta \cos (2\psi) & 0 \\
-2 \cos \theta \sin (2\phi) & (1 + \cos^2 \theta) \cos (2\phi) \cos (2\psi) & -2 \cos \theta \sin (2\phi) \cos (2\psi) \\
\sin^2 \theta \sin (2\phi) & 2 \cos \theta \cos (2\phi) \sin (2\psi) & 2 \cos \theta \cos (2\phi) \cos (2\psi) \\
\sin^2 \theta \sin (2\phi) & 0 & 0 \\
0 & 0 & 2 \cos \theta 
\end{bmatrix} \dagger (56)$$

where \( a_{ij} \) is read from Equation (36), and any terms associated with \( E_{x'}(\text{scat}) \) have been disregarded in the flat spacetime limit: the time average \( \langle \rangle \) of these terms would scale as \( h^2 \), which is negligibly small and beyond our measurement limit.

Now, from Equations (40)–(43) we may write

$$\langle |E_{x'}(\text{scat})|^2 \rangle = \frac{1}{2} (U_{\text{(case 1)}} - Q_{\text{(case 1)}}), \quad (56)$$

$$\langle |E_{y'}(\text{scat})|^2 \rangle = \frac{1}{2} (U_{\text{(case 1)}} - Q_{\text{(case 1)}}), \quad (57)$$

$$\langle E_{x'}(\text{scat})E_{y'}^*(\text{scat}) \rangle = \frac{1}{2} (U_{\text{(case 1)}} - iV_{\text{(case 1)}}), \quad (58)$$

$$\langle E_{x'}^*(\text{scat})E_{y'}(\text{scat}) \rangle = \frac{1}{2} (U_{\text{(case 1)}} + iV_{\text{(case 1)}}), \quad (59)$$

where the Stokes parameters are defined in the source frame \((x', y', z')\); however, their values are equal to those in Equations (48)–(51) for Case 1. Substituting Equations (56)–(59) into Equations (52)–(55), we establish the following relation:

$$L \begin{bmatrix}
U_{\text{(case 2)}} \\
Q_{\text{(case 2)}} \\
V_{\text{(case 2)}} \\
U'_{\text{(case 1)}} \\
Q'_{\text{(case 1)}} \\
V'_{\text{(case 1)}} 
\end{bmatrix} = \begin{bmatrix}
I_{\text{(case 2)}} \\
Q'_{\text{(case 1)}} \\
V_{\text{(case 2)}} \\
U'_{\text{(case 1)}} \\
Q'_{\text{(case 1)}} \\
V'_{\text{(case 1)}} 
\end{bmatrix}, \quad (60)$$

with the values \( \{U_{\text{(case 1)}}, Q_{\text{(case 1)}}, U'_{\text{(case 1)}}, V_{\text{(case 1)}}\} \) to be taken from Equations (48)–(51). By means of Equations (36) and (38) we can write out Equation (61):

$$L = \frac{1}{2} \begin{bmatrix}
1 + \cos^2 \theta & \sin^2 \theta \cos (2\psi) & 0 \\
-2 \cos \theta \sin (2\phi) & (1 + \cos^2 \theta) \cos (2\phi) \cos (2\psi) & -2 \cos \theta \sin (2\phi) \cos (2\psi) \\
\sin^2 \theta \sin (2\phi) & 2 \cos \theta \cos (2\phi) \sin (2\psi) & 2 \cos \theta \cos (2\phi) \cos (2\psi) \\
\sin^2 \theta \sin (2\phi) & 0 & 0 \\
0 & 0 & 2 \cos \theta 
\end{bmatrix} \dagger (62)$$

By Equations (60) and (62) together with (48)–(51), we can write down explicitly,

$$I_{\text{(case 2)}} = \frac{1 + \cos^2 \theta}{2} \frac{q^4(E_{11}^2 + E_{11}^2)}{m^2r^2}$$
$$+ \frac{\sin^2 \theta \cos (2\psi)}{2} \frac{q^4(E_{11}^2 - E_{11}^2)}{m^2r^2} \quad (63)$$

$$Q_{\text{(case 2)}} = \frac{\sin^2 \theta \cos (2\phi)}{2} \frac{q^4(E_{11}^2 + E_{12}^2)}{m^2r^2}$$
$$+ \frac{[1 - \cos \theta \sin (2\phi) \sin (2\psi)]}{2} \frac{q^4(E_{11}^2 - E_{11}^2)}{m^2r^2} \quad (64)$$

$$U_{\text{(case 2)}} = \frac{\sin^2 \theta \sin (2\phi)}{2} \frac{q^4(E_{11}^2 + E_{12}^2)}{m^2r^2}$$
$$+ \frac{[1 - \cos \theta \sin (2\phi) \sin (2\psi)]}{2} \frac{q^4(E_{11}^2 - E_{11}^2)}{m^2r^2} \quad (65)$$
These represent the Stokes parameters as computed by an observer as the rescattered EM radiation \( (E_x'\text{scat}, E_y'\text{scat}, E_z'\text{scat}) \) in the source frame \( (x', y', z') \) is projected into the observer’s frame \( (x, y, z) \).

However, the argument presented at the end of Section 2.1.2 suggests that the only physically meaningful transformation from Case 1 to Case 2 would be the rotation through the polarization-ellipse angle \( \psi \) while fixing the direction angles \( \{\phi, \theta\} = \{0, 0\} \); namely, keeping the propagation direction of GWs aligned with the line of sight, i.e., \( e_z' = e_z \) (see Figure 2). Then Equations (63)–(66) will reduce to

\[
\begin{align*}
I_{(\text{case } 2)} &= \frac{q^4(E_1^2 + E_\parallel^2)}{m^2r^2}, \\
Q_{(\text{case } 2)} &= \cos(2\psi)\frac{q^4(E_\parallel^2 - E_1^2)}{m^2r^2}, \\
U_{(\text{case } 2)} &= \sin(2\psi)\frac{q^4(E_\parallel^2 - E_1^2)}{m^2r^2}, \\
V_{(\text{case } 2)} &= 0.
\end{align*}
\]

In comparison to Equations (48)–(51) for Case 1, this implies that we have the linear polarization components along the axes rotated from the \( x \)-axis and the \( y \)-axis by \( \psi \).

\[
M = L (a_{ij} \leftrightarrow a_{ji}; \psi = 0)
\]

\[
= \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where \( a_{ij} \) identified from Equations (36) and (38). Here Equation (71) means that the Stokes parameters \( \{I', Q', U', V'\} \) (in the spherical coordinate system) are equal to the values of the Stokes parameters \( \{I, Q, U, V\} \) (in the Cartesian coordinate system) as measured relative to \( (e_0, e_\phi) \): the Stokes parameters \( \{I', Q', U', V'\} \) are functions of spherical polar angles \( \{\phi, \theta\} \) while the Stokes parameters \( \{I, Q, U, V\} \) are constant evaluated values in the Cartesian frame, given by Equations (48)–(51) or (67)–(70). Hereafter, for notational convenience, we drop the sign ‘ from \( \{I', Q', U', V'\} \) on the left-hand side of Equation (71) but put the subscript \( c \) (to stand for “constant”) in \( \{I, Q, U, V\} \) on the right-hand side. Then we may rewrite Equation (71) as

\[
\begin{bmatrix}
I' \\
Q' \\
U' \\
V'
\end{bmatrix} = \begin{bmatrix}
1 + \cos^2 \theta & \sin^2 \theta \cos(2\phi) & \sin^2 \theta \sin(2\phi) & 0 \\
\sin^2 \theta & (1 + \cos^2 \theta) \cos(2\phi) & (1 + \cos^2 \theta) \sin(2\phi) & 0 \\
0 & -2 \cos \theta \sin(2\phi) & 2 \cos \theta \cos(2\phi) & 0 \\
0 & 0 & 0 & 2 \cos \theta
\end{bmatrix} \begin{bmatrix}
I_c \\
Q_c \\
U_c \\
V_c
\end{bmatrix}. 
\]

3. CMB POLARIZATION OBSERVABLES

3.1. The Stokes Parameters Redefined on a Sphere and Polarization Patterns

In the previous section we have computed the Stokes parameters for the EM radiation scattered by a charge being shaken by GWs propagating in the CMB. As shown by Equations (48)–(51) and (67)–(70), the Stokes parameters exhibit a net linear polarization as desired. However, to analyze the CMB polarization on the sky, it is natural to redefine the Stokes parameters on a sphere such that they can be decomposed into spherical harmonics; to be precise, spin-weighted spherical harmonics as shall be seen later. Below we describe how this is achieved.

First, recall from the previous section that the Stokes parameters (48)–(51) and (67)–(70) have been computed with respect to \( (e_\phi, e_\phi) \) as the rescattered EM radiation propagates along \( e_\phi \) in the Cartesian coordinate system \( (x, y, z) \). Now, to redefine the Stokes parameters on a sphere, one should measure them relative to \( (e_\phi, e_\phi) \) as the rescattered EM radiation propagates along \( e_\phi \) in the spherical coordinate system \( (r, \theta, \phi) \). This can be achieved via Equation (32); by rotating the Cartesian frame through the Euler angles \( \{\phi, \theta\} \) (direction angles) while fixing the Euler angle \( \psi = 0 \) (polarization-ellipse angle). This will render the inverse transformation to Equation (37), which can be equivalently obtained by switching \( a_{ij} \leftrightarrow a_{ji} \) (i.e., transpose) and fixing \( \psi = 0 \) in Equations (36) and (38). Further, we find that the inverse transformation to (60) can be obtained by applying the same argument to Equation (61). That is,

\[
\begin{bmatrix}
I' \\
Q' \\
U' \\
V'
\end{bmatrix} = M \begin{bmatrix}
I \\
Q \\
U \\
V
\end{bmatrix}, 
\]

where

\[
M = \begin{bmatrix}
1 + \cos^2 \theta & \sin^2 \theta \cos(2\phi) & \sin^2 \theta \sin(2\phi) & 0 \\
\sin^2 \theta & (1 + \cos^2 \theta) \cos(2\phi) & (1 + \cos^2 \theta) \sin(2\phi) & 0 \\
0 & -2 \cos \theta \sin(2\phi) & 2 \cos \theta \cos(2\phi) & 0 \\
0 & 0 & 0 & 2 \cos \theta
\end{bmatrix} \bigg|_{\psi=0}. 
\]
where \( I_c, Q_c, U_c, V_c \) refer to Equations (48)–(51) or (67)–(70), and the factor 1/2 from the matrix (72) has been readjusted to 3/4 for normalization (Chandrasekhar 1960; Halonen et al. 2013); such that the scattered intensity integrated over all directions, \( 0 \leq \phi < 2\pi, \ 0 \leq \theta < \pi \) equals the intensity of the incoming radiation.

To study the linear polarization of the CMB, we examine only \( Q \) and \( U \) among the Stokes parameters. From Equation (73) the combinations of the two parameters turn out to be

\[
Q \pm iU = \frac{3}{4}I_c \sin^2 \theta + Q_c \left[ \frac{3}{4} (1 + \cos^2 \theta) \cos (2\phi) \right. \\
\left. \pm \frac{3}{2} \cos \theta \sin (2\phi) \right] \\
+ U_c \left[ \frac{3}{4} (1 + \cos^2 \theta) \sin (2\phi) \right. \\
\left. \pm \frac{3}{2} \cos \theta \cos (2\phi) \right].
\]

(74)

Or we can put this in the representation

\[
Q \pm iU = \frac{6\pi}{5} I_c \pm 3Y_2^0 (\theta, \phi) \\
+ \frac{9\pi}{5} (Q_c - iU_c) \pm 3Y_2^0 (\theta, \phi) \\
+ \frac{9\pi}{5} (Q_c + iU_c) \pm 3Y_2^0 (\theta, \phi).
\]

(75)

where instances of \( \pm sY_\ell^m \) represent the spin-weighted spherical harmonics of spin \( \pm s \) (Newman & Penrose 1966). This clearly shows that \( Q \pm iU \) are spin \( \pm 2 \) quantities.

However, from Equations (60) and (62) we can express \( \{I_c, Q_c, U_c, V_c\} \) in the most general case for observation:

\[
I_c \equiv I_o, \quad Q_c \equiv \cos (2\psi_o)Q_o - \sin (2\psi_o)U_o, \quad U_c \equiv \sin (2\psi_o)Q_o + \cos (2\psi_o)U_o, \quad V_c \equiv V_o,
\]

(76)

(77)

(78)

(79)

where \( \psi_o \) denotes the value of polarization-ellipse angle, by which the source frame \((x', y', z')\) is rotated from the observer’s frame \((x, y, z)\) with respect to the \(z\)-axis while keeping \( e'_z = e_z \) (see Figure 2 for comparison of the two frames), and

\[
I_o \equiv I_{\text{case 1}} = \frac{q^4 (E_i^2 + E_i^2)}{m^2 r^2}, \quad Q_o \equiv Q_{\text{case 1}} = \frac{q^4 (E_i^2 - E_i^2)}{m^2 r^2},
\]

(80)

(81)

\[
U_o \equiv U_{\text{case 1}} = 0, \quad V_o \equiv V_{\text{case 1}} = 0.
\]

(82)

(83)

by Equations (48)–(51). In the limit \( \psi_o \to 0 \), we have \( \{I_o, Q_o, U_o, V_o\} \to \{I_c, Q_c, U_c, V_c\} \). Inserting Equations (76)–
(79) into Equation (74), we may rewrite

\[
Q \pm i\bar{U} = \frac{3}{4} I_0 \sin^2 \theta \\
+ \frac{3}{4} \tilde{Q}_0 \left[ (1 + \cos^2 \theta) \cos (2(\phi - \psi_0)) \right. \\
\left. \pm i2 \cos \theta \sin (2(\phi - \psi_0)) \right].
\]

(84)

Here the term \( \frac{3}{4} I_0 \sin^2 \theta \) refers to the total intensity of radiation and does not concern polarization. Therefore, we may remove this and redefine

\[
\tilde{Q} \pm i\bar{U} = \frac{3}{4} \tilde{Q}_0 \left[ (1 + \cos^2 \theta) \cos (2(\phi - \psi_0)) \right. \\
\left. \pm i2 \cos \theta \sin (2(\phi - \psi_0)) \right].
\]

(85)

Or we can represent this using the spin-weighted spherical harmonics,

\[
\tilde{Q} \pm i\bar{U} = \sqrt{\frac{9\pi}{5}} \tilde{Q}_0 \left[ \pm \frac{1}{2} Y_{2,0}^2 (\theta, \phi - \psi_0) \right. \\
\left. \pm \frac{1}{2} Y_{2,-2}^2 (\theta, \phi - \psi_0) \right].
\]

(86)

By means of Equations (85) or (86), we can illustrate polarization patterns; namely, “electric” E-mode and “magnetic” B-mode patterns (Hu \\& White 1997a). The polarization amplitude and angle clockwise from the north pole (at \( \theta = 0 \)) are defined to represent E-modes, and B-modes can be represented by rotating the E-modes by 45°:

\[
\text{Amplitude} = \sqrt{Q^2 + \bar{U}^2} \\
= \frac{3}{4} |\tilde{Q}_0| \left[ ((1 + \cos^2 \theta) \cos (2(\phi - \psi_0)))^2 \right. \\
\left. + [-2 \cos \theta \sin (2(\phi - \psi_0))]^2 \right]^{1/2},
\]

(87)

\[
\text{Angle(E-modes)} = \frac{1}{2} \arctan \left( \frac{\bar{U}}{Q} \right) \\
= \frac{1}{2} \arctan \left( \frac{-2 \cos \theta \tan (2(\phi - \psi_0))}{1 + \cos^2 \theta} \right).
\]

(88)

\[
\text{Angle(B-modes)} = \text{Angle(E-modes)} + \frac{\pi}{4},
\]

(89)

with \( \tilde{Q}_0 \) given by Equation (81). Regarding the definitions of E-modes/B-modes, we followed Hu \\& White (1997a). Equations (88) and (89) refer to the angles for E-modes/B-modes derived from the “spin \pm 2” quantity as given by Equation (86); this should be distinguished from the later case of Equations (103) and (104), which refer to the angles for E-modes/B-modes derived from the “spin 0” quantity as given by Equation (101).

**Graphic representation 1.** The E-mode and B-mode patterns based on Equations (87)–(89) above are graphically represented in Figure 4; given the polarization-ellipse angle \( \psi_0 \), which determines the orientation of the source frame \((x', y', z')\) with respect to the observer’s or observers’ frame \((x, y, z)\) (see Figure 2 for comparison of the two frames). The patterns may be characterized as follows.

**i. Figures 4(a) (E-modes) and 4(b) (B-modes) with \( \psi_0 = 0 \):**

The source frame \((x', y', z')\) coincides with the observer’s or observers’ frame \((x, y, z)\), and therefore the rescattered EM radiation is received by the observer with its full intensity and the maximum polarization effect. The E-modes and B-modes exhibit regular gradient-like and curl-like patterns, respectively.

**ii. Figures 4(c) (E-modes) and 4(d) (B-modes) with \( \psi_0 = \pi/4 \):**

The source frame \((x', y', z')\) is rotated by \( \pi/4 \) from the observer’s frame \((x, y, z)\) with respect to the z-axis. The E-modes and B-modes exhibit the same patterns as in case (i) except that the both patterns are now shifted along \( \phi \) by \( \psi_0 = \pi/4 \); therefore, they are physically equivalent to case (i).

**3.2. Plane Wave Modulation of Polarization Patterns**

The E-mode and B-mode patterns of polarization in the previous subsection represent the local signature from scattering over the sphere. However, in real-world observations the polarization patterns on the sky are not simply the local signature from scattering but are modulated by plane wave fluctuations on the last scattering surface. The plane wave modulation changes the amplitude, sign, and angular structure of the polarization but not its nature: that is, it does not mix \( Q \) and \( \bar{U} \) (Hu \\& White 1997a).

The quantities given by Equations (85) or (86) have a direct association with physical observables, namely, Stokes parameters, and from them we can illustrate the polarization patterns as was done at the end of the previous Subsection. However, as can be seen from Equation (86), these are spin \( \pm 2 \) quantities, and it would be desirable to derive rotationally invariant spin 0 quantities out of them (Kamionkowski et al. 1997; Zaldarriaga \\& Seljak 1997; Kim 2011). Now, we define

\[
\tilde{Q} \pm i\bar{U} \equiv (\tilde{Q} \pm i\bar{U}) \exp (ik \cdot r) \\
= (\tilde{Q} \pm i\bar{U}) \exp (ip \cos \theta) ; \rho = kr,
\]

(90)

where \( k = ke \), and \( r = re \), with \( k \) denoting the wave number and \( r \) the comoving distance to the last scattering surface, and \( \exp (ik \cdot r) = \exp (ikr \cos \theta) = \exp (ip \cos \theta) \) represents the plane wave projected into the spherical sky (Hu \\& White 1997b). From this we can construct the following scalar quantities (Kamionkowski et al. 1997; Zaldarriaga \\& Seljak 1997; Kim 2011):

\[
E \equiv -\frac{1}{2} \left[ \partial^2 (\tilde{Q} + i\bar{U}) + \partial^2 (\tilde{Q} - i\bar{U}) \right],
\]

(91)

\[
B \equiv \frac{i}{2} \left[ \partial^2 (\tilde{Q} + i\bar{U}) - \partial^2 (\tilde{Q} - i\bar{U}) \right],
\]

(92)

where the spin raising and lowering operators are defined as

\[
\partial f(\theta, \phi) \\
= - (\sin \theta)^2 \left( \frac{\partial}{\partial \theta} + i \csc \theta \frac{\partial}{\partial \phi} \right) (\sin \theta)^{-1} f(\theta, \phi),
\]

(93)

\[
\partial f(\theta, \phi) \\
= - (\sin \theta)^{-1} \left( \frac{\partial}{\partial \theta} - i \csc \theta \frac{\partial}{\partial \phi} \right) (\sin \theta)^{1} f(\theta, \phi),
\]

(94)
for an arbitrary spin $s$ quantity $q_f(s)$. Substituting Equation (85) into Equation (90), and subsequently Equation (95) into Equations (91) and (92), we obtain

$$E = \frac{3}{4} Q_o \sin^2 \theta \left[ \rho^2 (1 + \cos^2 \theta) - 12 - 8i\rho \cos \theta \right] \times \cos (2(\phi - \psi_o)) \exp (i\rho \cos \theta),$$

(95)

$$B = \frac{3}{2} Q_o \sin^2 \theta (-\rho^2 \cos \theta + 4i\rho) \sin \times (2(\phi - \psi_o)) \exp (i\rho \cos \theta).$$

(96)

Here we note that the spin 0 quantity $\exp (i\rho \cos \theta)$ is a common phase factor for the complex quantities $E$ and $B$. With this factor suppressed, the relative phase between the two complex quantities still remains the same. Then we may define

$$E \equiv E \exp (-i\rho \cos \theta),$$

(97)

$$B \equiv B \exp (-i\rho \cos \theta),$$

(98)

such that their combinations are

$$E + iB = (E + iB) \exp (-i\rho \cos \theta)$$

$$= -\overline{\theta}^2 (\bar{Q} + i\bar{U}) \exp (-i\rho \cos \theta),$$

(99)

$$E - iB = (E - iB) \exp (-i\rho \cos \theta)$$

$$= -\overline{\theta}^2 (\bar{Q} - i\bar{U}) \exp (-i\rho \cos \theta),$$

(100)

by means of Equations (91) and (92). By Equations (95) and (96) these read

$$E + iB = \frac{3}{4} Q_o \sin^2 \theta \left[ \rho^2 (1 + \cos^2 \theta) - 12 \right] \times \cos (2(\phi - \psi_o)) \pm 8i\rho \sin (2(\phi - \psi_o))$$

$$+ i\rho \cos \theta \left[ -8 \rho \cos (2(\phi - \psi_o)) \right. + 2\rho^2 \sin (2(\phi - \psi_o)) \right].$$

(101)

Here we note that $E - iB$ can be obtained by changing $\phi - \psi_o \rightarrow -\phi - \psi_o$ in $E + iB$. This means that $E - iB$ is the reflection of $E + iB$ about $\phi = \psi_o$. Therefore, $E + iB$ and $E - iB$ lead to physically equivalent representations: they produce patterns that are simply mirror images of each other.\footnote{The same argument applies to $\bar{Q} + i\bar{U}$ and $\bar{Q} - i\bar{U}$ due to Equation (85).}

We choose $E + iB$ for our representation of polarization patterns. Similarly as in the previous subsection, one can illustrate the E-mode and B-mode polarization patterns by means of Equation (101). We represent E-modes by the polarization amplitude and angle clockwise from the North Pole (at $\theta = 0$), and B-modes by rotating the E-modes by $45^\circ$.\footnote{The same argument applies to $\bar{Q} + i\bar{U}$ and $\bar{Q} - i\bar{U}$ due to Equation (85).}

$$\text{Amplitude} = \sqrt{\Re(E + iB)^2 + \Im(E + iB)^2},$$

(102)

$$\text{Angle(E-modes)} = \frac{1}{2} \arctan \left( \frac{\Im(E + iB)}{\Re(E + iB)} \right).$$

(103)
\[
\text{Angle (B-modes)} = \text{Angle (E-modes)} + \frac{\pi}{4}, \quad (104)
\]

where

\[
\Re(E + iB) = \frac{3}{4} Q_\alpha \sin^2 \theta \left\{ [\rho^2 (1 + \cos^2 \theta) - 12] \times \cos(2(\phi - \psi_\alpha)) - 8 \rho \sin(2(\phi - \psi_\alpha)) \right\}, \quad (105)
\]

\[
\Im(E + iB) = -\frac{3}{2} Q_\alpha \sin^2 \theta \cos \theta \left[ 4 \rho \cos(2(\phi - \psi_\alpha)) + \rho^2 \sin(2(\phi - \psi_\alpha)) \right], \quad (106)
\]

with \(Q_\alpha\) given by Equation (81).

In particular, for \(\rho \gg 1\), which corresponds to small angular scales, the representations of E-modes and B-modes tend toward

\[
\text{Amplitude} = \frac{3}{4} |Q_\alpha| \rho^2 \sin^2 \theta \times \left[ [(1 + \cos^2 \theta) \cos(2(\phi - \psi_\alpha))]^2 + [-2 \cos \theta \sin(2(\phi - \psi_\alpha))]^2 \right]^{1/2}, \quad (107)
\]

\[
\text{Angle (E-modes)} = \frac{1}{2} \arctan \left( \frac{-2 \cos \theta \tan(2(\phi - \psi_\alpha))}{1 + \cos^2 \theta} \right). \quad (108)
\]

\[
\text{Angle (B-modes)} = \text{Angle (E-modes)} + \frac{\pi}{4}. \quad (109)
\]

We note that these scale to the spin 2 representations given by Equations (87)–(89), apart from the factor \(\rho^2 \sin^2 \theta\) in the amplitude (Chiang & Komatsu 2011).

**Graphic representation 2.** The E-mode and B-mode patterns based on Equations (102)–(104) above are graphically represented in Figures 5–7; with the dimensionless parameter \(\rho\), which defines the plane wave projection into the spherical sky (i.e., \(\exp(ik \cdot r) = \exp(ikr \cos \theta) = \exp(i\rho \cos \theta)\)) and with the polarization-ellipse angle \(\psi_\alpha\), which determines the orientation of the source frame \((x', y', z')\) with respect to the observer’s frame \((x, y, z)\) (see Figure 2 for comparison of the two frames). Given different values of \(\rho\) and \(\psi_\alpha\), the patterns may be characterized as follows.

1. **In Figure 5**
   i'. Figures 5(a) (E-modes) and 5(b) (B-modes) with \(\rho = 5\), \(\psi_\alpha = 0\): The source frame \((x', y', z')\) coincides with the observer’s frame \((x, y, z)\), and therefore the rescattered EM radiation is received by the observer with its full intensity and the maximum polarization effect. The E-modes and B-modes exhibit gradient-like and curl-like patterns but with broken symmetric and anti-symmetric images in each quadrant for \(\phi\), respectively. This is caused by the plane wave modulation at long wavelengths or at small comoving distances to the last
scattering surface, due to Equations (103) and (104) with \( \rho = 2\pi r/\lambda \).

ii'. Figures 5(c) (E-modes) and 5(d) (B-modes) with \( \rho = 5 \), \( \psi_0 = \pi/4 \): The source frame \((x', y', z')\) is rotated by \( \pi/4 \) from the observer’s frame \((x, y, z)\) with respect to the \(z\)-axis. The E-modes and B-modes exhibit the same patterns as in case (i’ii) except that the both patterns are now shifted along \( \phi \) by \( \psi_0 = \pi/4 \); therefore, they are physically equivalent to case (i’i).

2. In Figure 6

i”’. Figures 6(a) (E-modes) and 6(b) (B-modes) with \( \rho = 25 \), \( \psi_0 = 0 \): The source frame \((x', y', z')\) coincides with the observer’s frame \((x, y, z)\), and therefore the rescattered EM radiation is received by the observer with its full intensity and the maximum polarization effect. The E-modes and B-modes exhibit gradient-like and curl-like patterns but with broken symmetric and anti-symmetric images in each quadrant for \( \phi \), respectively; however, less broken than in case (i’ii’). This is caused by the plane wave modulation at medium wavelengths or at medium comoving distances to the last scattering surface, due to Equations (103) and (104) with \( \rho = 2\pi r/\lambda \).

ii”’. Figures 6(c) (E-modes) and 6(d) (B-modes) with \( \rho = 25 \), \( \psi_0 = \pi/4 \): The source frame \((x', y', z')\) is rotated by \( \pi/4 \) from the observer’s frame \((x, y, z)\) with respect to the \(z\)-axis. The E-modes and B-modes exhibit the same patterns as in case (i”ii’) except that the both patterns are now shifted along \( \phi \) by \( \psi_0 = \pi/4 \); therefore, they are physically equivalent to case (i”ii’).

3. In Figure 7

i””. Figures 7(a) (E-modes) and 7(b) (B-modes) with \( \rho = 125 \), \( \psi_0 = 0 \): The source frame \((x', y', z')\) coincides with the observer’s frame \((x, y, z)\), and therefore the rescattered EM radiation is received by the observer with its full intensity and the maximum polarization effect. The E-modes and B-modes exhibit regular gradient-like and curl-like patterns with proper symmetric and anti-symmetric images in each quadrant for \( \phi \), respectively. This is caused by the plane wave modulation at short wavelengths or at large comoving distances to the last scattering surface, due to Equations (103) and (104) with \( \rho = 2\pi r/\lambda \).

ii””. Figures 7(c) (E-modes) and 7(d) (B-modes) with \( \rho = 125 \), \( \psi_0 = \pi/4 \): The source frame \((x', y', z')\) is rotated by \( \pi/4 \) from the observer’s frame \((x, y, z)\) with respect to the \(z\)-axis. The E-modes and B-modes exhibit the same patterns as in case (i”ii’) except that the both patterns are now shifted along \( \phi \) by \( \psi_0 = \pi/4 \); therefore, they are physically equivalent to case (i”ii’).

Cases (i’) and (i”ii’) from Figures 5 and 6 show that the broken symmetric and anti-symmetric images in the E-mode and B-mode patterns are due to the plane wave modulation at long
to medium wavelengths or at small to medium comoving distances to the last scattering surface, which corresponds to small to medium $\rho$. However, case (ii) from Figure 7 shows that the E-mode and B-mode patterns take proper symmetric and anti-symmetric images, respectively, with the plane wave modulation at short wavelengths or at large comoving distances to the last scattering surface, which corresponds to large $\rho$: in fact, these patterns should be equivalent to the ones to be produced via Equations (107)–(109) in the limit $\rho \gg 1$. While the plane wave modulation affects the symmetric and anti-symmetric properties of the E-modes and B-modes, depending on the size of $\rho$, it does not change the periodicity of the patterns in $\phi$ (i.e., $\pi/2$): in both the E-mode and B-mode patterns, four identical images are produced over $0 \leq \phi < 2\pi$, one in each of the four quadrants.

4. DISCUSSION AND CONCLUSIONS

We present an analysis of an apparently as-yet overlooked effect in the interaction between (primordial) GWs and individual electric charges. We computed the Stokes parameters for the EM radiation rescattered by a charge that is forced into oscillatory motion by GWs propagating through the CMB plasma: as shown by Equations (48)–(51) and (67)–(70) in Section 2.2, the Stokes parameters exhibit a net linear polarization—as expected. The GWs open polarization channels for the rescattered radiation by setting a charge in motion along particular directions. In other words, the GWs provide for polarization in the photon-electron scattering by agitating a charge along particular directions. Taking the $O(h)$ terms into consideration, the resulting contribution to our Stokes parameters will be as shown in Equations (44)–(47), which is $O(h^2)$. In our analysis, due to the assumption of linearized gravity in Section 2.1, $O(h^2)$ would be considered as very small, and therefore the terms of this order may be disregarded for the final expressions of the Stokes parameters as in Equations (48)–(51). However, until we are able to estimate how large or small the strain $h$ is in the CMB regime (unlike its estimates for present time observations as in Figure 3), it is not clear whether the negligence of $O(h^2)$ will make any significant difference (or not) in our final E-mode/B-mode maps. Our analysis is presently limited to modeling single-electron scattering; accordingly, we are not yet able to provide a reliable estimate of the amount of polarized flux to be expected. Evidently, reliable amplitude estimates will be crucial for attempts to observe the effect we describe in this study; current B-mode polarization studies have sensitivities of about 0.3 $\mu$K/$\sqrt{\text{s}}$ (in CMB temperature units, using BICEP2 as benchmark: Ade et al. 2014a).

If our measurement were indeed sensitive enough to tell the values of the strain $h$ and the frequency $\omega$ of the GWs in the CMB regime, then we should rather express our Stokes parameters using Equations (44)–(47). This way, we would be able to see the properties of the GWs directly from our E-mode/B-mode polarization maps through the Stokes parameters. In particular, the second parameter $Q$ would take different values, depending on the GW polarization states, $+$ and $\times$, while the first parameter $I$ would be the same regardless of the states. This implies that the linear polarization of our CMB radiation would be quantified differently, depending on the GW polarization states. That is, $Q_+ = Q_0 + O(h^2)$ and $Q_\times = Q_0 - O(h^2)$ while $I_+ = I_0 = I_\times + O(h^2)$, and therefore we find the ratios, $Q_+/I_+ = Q_\times/I_\times$. This means that the GW-induced polarization should show this distinctive feature, which will distinguish itself from other degenerate cases, where polarization could possibly have been induced from non-GW sources.

In order to analyze the CMB polarization on the sky, we have redefined the Stokes parameters on a sphere such that their representations can be expressed as spin-weighted spherical harmonics; namely, spin $\pm 2$ quantities. The representations yield the two polarization patterns, “electric” E-mode and “magnetic” B-mode patterns (Hu & White 1997a). They are constructed by means of Equations (87)–(89), and graphically represented in Figure 4 (see Graphic representation 1) in Section 3.1. While these patterns represent the local signature from scattering over the sphere, in real-world observations, the polarization patterns on the sky are not simply the local signature from scattering but are modulated by plane wave fluctuations on the last scattering surface (Hu & White 1997a). Thus we have modified the representations of the Stokes parameters such that they contain the properties of the plane wave projected into the spherical sky. The modified representations are spin 0 quantities, and they also yield the E-mode and B-mode polarization patterns (Kamionkowski et al. 1997; Zaldarriaga & Seljak 1997; Kim 2011). They are constructed by means of Equations (102)–(104), and graphically represented in Figures 5–7 (see Graphic representation 2) in Section 3.2; with different conditions of the plane wave projection into the spherical sky.

Overall, GWs can generate the gradient- and curl-like E- and B-modes as expected, even though notable differences between Graphic representations 1 and 2 are observed due to the plane wave modulation, parameterized by $\rho$, which defines the plane wave projection into the spherical sky. With small to medium $\rho$, which corresponds to the plane wave modulation at long to medium wavelengths or at small to medium comoving distances to the last scattering surface, the E-modes and B-modes show broken symmetric and anti-symmetric patterns, respectively in Graphic representation 2. However, with large $\rho$, which corresponds to the plane wave modulation at short wavelengths or at large comoving distances to the last scattering surface, the E-modes and B-modes take proper symmetric and anti-symmetric patterns. In fact, in the limit $\rho \gg 1$, the E-modes and B-modes in Graphic representation 2 scale to their counterparts in Graphic representation 1, apart from the factor $\rho^2 \sin^2 \theta$ in the amplitude Chiang & Komatsu (2011). In both Graphic representations 1 and 2, however, the periodicity of the patterns in $\phi$ for the E-modes and B-modes remains the same, i.e., $\pi/2$.

In this work we have restricted our attention to the case of a single charged particle under the influence of GWs passing through the CMB. We have investigated how the EM radiation is rescattered by the charge being agitated by the GWs and how the rescattered radiation becomes polarized in the CMB. More realistic astronomical scenarios, however, would involve multiple charged particles interacting with each other. At the same time, as we assume that our universe was expanding during the epoch of reionization, the interaction between the particles would be significantly affected by the scale factor of the expanding universe. Therefore, to handle a situation with the interaction properly, the motion of the particles should be described in the spacetime geometry, which is prescribed by the perturbed Friedmann–Robertson–Walker metric rather than by the perturbed flat spacetime metric. Unlike the case of a
single particle, dynamic evolution of the multiple particles in this setup should be treated statistically, e.g., using the Boltzmann equation (Hu & White 1997b; Kamionkowski et al. 1997; Zaldarriaga & Seljak 1997). Likewise, astrophysically realistic CMB polarization maps—which we do not provide here—result from convolution of the patterns caused by single-particle interaction with the density distribution of the CMB plasma. Furthermore, we do not yet present power spectra in this study. Following Hu & White (1997b), we can determine the $\ell$-mode expressions for the power spectra out of Equations (86) and (90)–(92) by means of the Clebsch–Gordan relation. With these we will be able to estimate the E-mode/B-mode contributions in a quantitative manner as shown by Figure 11(b) in Hu & White (1997a). We plan to address these open issues in a follow-up study.

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APPENDIX A

NON-POLARIZED ANISOTROPIC INCIDENT RADIATION

In Section 2.1 we prescribe an anisotropic incident monochromatic radiation field on a charge via the two waves,

$$E_{(m)} = E_{I} \{ e_{I} \exp[i(Kx - \Omega t + \varphi_{I}(t))] + e_{II} \exp[i(Ky - \Omega t + \varphi_{II}(t))] \},$$

(110)

$$E_{II(m)} = E_{II} \{ e_{II} \exp[i(Ky - \Omega t + \varphi_{II}(t))] + e_{I} \exp[i(Kx - \Omega t + \varphi_{I}(t))] \},$$

(111)

where $E_{I}$ and $E_{II}$ ($E_{I} = E_{II}$) are the amplitudes, and $\Omega$ and $K$ are the angular temporal frequency and angular spatial frequency with $\Omega = cK$, and $\varphi_{I}(t)$, $\varphi_{II}(t)$, and $\varphi_{II}(t)$ are the phase modulation functions, which are assumed to vary on a timescale much slower than the period of the waves, i.e., $|\varphi_{I}|, |\varphi_{II}|, |\varphi_{II}| \ll \Omega$.

With this prescription, $E_{(m)}$ and $E_{II(m)}$ are unpolared. This is due to the relative phases $\varphi_{I}(t) - \varphi_{I}(t)$ and $\varphi_{II}(t) - \varphi_{II}(t)$, which fluctuate in time and will not let each wave remain in a single polarization state. This can be shown formally by computing the Stokes parameters out of the waves (110) and (111). By Equations (40)–(43) we calculate

$$I_{II} = \langle |E_{II(m)}|^2 \rangle = \langle |E_{II(z)}|^2 \rangle,$$

(112)

$$Q_{II} = \langle |E_{II(m)}|^{*} \rangle - \langle |E_{II(z)|}^{*} \rangle,$$

(113)

$$U_{II} = \langle |E_{II(m)}|^{*} \rangle + \langle |E_{II(z)|}^{*} \rangle,$$

(114)

$$V_{II} = i \langle |E_{II(m)}|^{*} \rangle - \langle |E_{II(z)|}^{*} \rangle,$$

(115)

and

$$I_{II} = \langle |E_{II(m)}|^2 \rangle + \langle |E_{II(z)}|^2 \rangle,$$

(116)

$$Q_{II} = \langle |E_{II(m)}|^{*} \rangle - \langle |E_{II(z)|}^{*} \rangle,$$

(117)

$$U_{II} = \langle |E_{II(m)}|^{*} \rangle + \langle |E_{II(z)|}^{*} \rangle,$$

(118)

$$V_{II} = i \langle |E_{II(m)}|^{*} \rangle - \langle |E_{II(z)|}^{*} \rangle.$$
hand side the EM field is given by

\[ E_i = e_i^2 E_i \exp \left( \frac{i}{\Omega} \left( \frac{\hat{z}}{c} - t \right) + \phi_1(t) \right) \]

\[ + e_i^1 E_{i\Pi} \exp \left( i \left( \frac{\hat{z}}{c} - t \right) + \phi_{\Pi}(t) \right) \] (128)

which describes the outgoing field reflected by the charge.

The geodesic Equation (125) can be solved via perturbation. First, the unperturbed part of a solution for \( X^i = (\hat{x}, \hat{y}, \hat{z}) \), namely, \( \hat{X}^i_{[0]} = (\hat{x}_{[0]}, \hat{y}_{[0]}, \hat{z}_{[0]}) \), is obtained as follows. Plugging \( \hat{X}^i \) into Equation (125), and keeping only the unperturbed terms, we have

\[ \hat{X}^i_{[0]} = \frac{q}{m} \eta^a E_k. \] (129)

Inserting Equation (128) into Equation (129), and integrating this over \( t \),

\[ \hat{X}^i_{[0]} = v_0^i + \frac{q}{m} \eta^a \int_0^t dt' E_k \]

\[ = v_0^i + \frac{qE_{\text{scat}}}{m} \int_0^t dt' \exp \left[ i \left( \Omega \left( \frac{\hat{z}}{c} - t' \right) + \phi_1(t') \right) \right], \] (130)

where \( E_{\text{scat}} = (E_\text{scat}, E_{\text{scat}}, 0) \) and \( \phi_1(t) = (\phi_{\Pi}(t), \phi(t), 0) \). Now, we take the time-dependent part from the integral in Equation (130) and define

\[ \mathcal{I}(t) \equiv \int_0^t dt' e^{-i[(\hat{x} - \hat{y}'(t'))]}, \] (131)

where \( \phi(t) \) refers to either \( \phi_1(t) \) or \( \phi_\Pi(t) \). Using integration by parts repeatedly, we arrive at

\[ \mathcal{I}(t) = e^{-i[(\hat{x} - \phi(t'))]} \left[ 1 + \phi(t') + \left( \frac{\phi(t)}{\Omega} \right)^2 - \frac{\phi(t)}{\Omega^2} \right] \]

\[ + \left( \frac{\phi(t)}{\Omega} \right)^3 - \frac{3}{2} \left( \frac{\phi(t)}{\Omega} \right)^2 - \frac{\phi(t)}{\Omega^3} \] \[ + \ldots \]

\[ = e^{-i[(\hat{x} - \phi(t'))]} \]

\[ + \mathcal{O}_0(\phi/\Omega, \phi/\Omega^2, \phi/\Omega^3, \phi/\Omega^3, \ldots) \] (132)

where \( \phi/\Omega \sim \varepsilon, \phi/\Omega^2 \sim \varepsilon^2, \phi/\Omega^3 \sim \varepsilon^3 \), etc. with \( \varepsilon \ll 1 \), from the assumption that \( \phi(t) \) varies on a timescale much slower than the period of the wave, i.e., \( |\phi| \ll \Omega, |\phi| \ll \Omega^2, |\phi| \ll \Omega^3 \), etc. This leads to

\[ \hat{X}^i_{[0]} = v_0^i + \frac{qE_{\text{scat}}}{m} \exp \left[ i \left( \Omega \left( \frac{\hat{z}}{c} - t \right) + \phi_1(t) \right) \right] \]

\[ + \mathcal{O}_0(\phi/\Omega, \phi/\Omega^2, \phi/\Omega^3, \phi/\Omega^3, \ldots). \] (133)

A first-order perturbation of \( X^i \), namely, \( \hat{X}^i_{[1]} \), is obtained by recycling the unperturbed part, i.e., Equation (133) into Equation (125). Plugging \( \hat{X}^i = \hat{X}^i_{[0]} + \hat{X}^i_{[1]} \) into Equation (125), and keeping only first-order terms in \( h \),

\[ \hat{X}^i_{[1]} = -\frac{q}{m} h_\hat{E}_k - \eta^a h_{jk,l} \hat{X}^j_{[1]} \hat{X}^k_{[0]} - \frac{1}{2} \eta^a \]

\[ \times \left( h_{jk,l} + h_{kl,j} - h_{jl,k} \right) \hat{X}^j_{[1]} \hat{X}^k_{[0]}, \] (134)

Integrating this over \( t \),

\[ \hat{X}^i_{[1]} = -\frac{q}{m} \int_0^t dt' h_\hat{E}_k - \eta^a \int_0^t dt' h_{jk,l} \hat{X}^j_{[1]} \hat{X}^k_{[0]} - \frac{1}{2} \eta^a \]

\[ \times \left( h_{jk,l} + h_{kl,j} - h_{jl,k} \right) \hat{X}^j_{[1]} \hat{X}^k_{[0]}, \] (135)

where the integrands are specified by Equations (126)–(128) and (133). Here each integral has the same form as Equation (131) and hence can be approximated in the same manner as in Equation (132), thereby generating the error \( \mathcal{O}_0(\phi/\Omega, \phi/\Omega^2, \phi/\Omega^3, \phi/\Omega^3, \ldots) \). Combining the result from Equation (135) with Equation (133), we obtain the first-order perturbation solution, i.e., \( \hat{X} = \hat{X}^i_{[0]} + \hat{X}^i_{[1]} \), which has the total error \( \mathcal{O}_0(\phi/\Omega, \phi/\Omega^2, \phi/\Omega^3, \phi/\Omega^3, \phi/\Omega^3, \ldots) + \mathcal{O}_0(h/\Omega, \phi/\Omega^2, \phi/\Omega^3, \phi/\Omega^3, \phi/\Omega^3, \ldots) \), as presented in Equations (14)–(19).

The full first-order perturbation solution for \( X^i \), i.e., \( X^i = \hat{X}^i_{[0]} + \hat{X}^i_{[1]} \), can be obtained by combining Equations (129) and (134). The unperturbed part \( \hat{X}^i_{[0]} \) is trivially obtained from the right-hand side of (129), which is equivalent to \( E_i \) (128) apart from the factor; thereby not generating \( \mathcal{O}_0 \). However, the perturbed part \( \hat{X}^i_{[1]} \) is obtained by recycling the unperturbed part \( \hat{X}^i_{[1]} \) into (134); thereby generating the error \( \mathcal{O}_0 h/\Omega \). Hence the total error from \( \hat{X} = \hat{X}^i_{[0]} + \hat{X}^i_{[1]} \) is \( \mathcal{O}_0 \) only. The rescattered EM radiation is obtained by

\[ E_{\text{scat}} = \frac{q}{r} \hat{X}^i_{[0]} + \frac{q}{r} \hat{X}^i_{[1]}, \] (136)

due to Equations (23)–(25), and therefore it has the error \( \mathcal{O}_0 \) only, as shown by Equations (26)–(31).

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