INCOMPLETE GAUSS SUMS MODULO PRIMES

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Abstract. We obtain a new bound for incomplete Gauss sums modulo primes. Our argument falls under the framework of Vinogradov’s method which we use to reduce the problem under consideration to bounding the number of solutions to two distinct systems of congruences. The first is related to Vinogradov’s mean value theorem, although the second does not appear to have been considered before. Our bound improves on current results in the range $N \geq q^{2k^{-1/2}+O(k^{-3/2})}$.

1. Introduction

The estimation of exponential sums of the form

$$\sum_{M<n\leq M+N} e^{2\pi i f(n)},$$

where $f$ is a polynomial of large degree, is a common problem in number theory with a wide range of arithmetic consequences. This problem has been considered by a number of previous authors, see [3, 8, 14, 16] for recent progress on the estimation of such sums. See also [5] and [9, Chapter 8] for a brief overview of current results and techniques.

Let $a$ and $q$ be integers with $(a, q) = 1$. In this paper we consider the problem of estimating sums of the form

$$(1) \quad \sum_{M<n\leq M+N} e_q(an^k),$$

where $e_q(z)$ is defined by $e^{2\pi i z/q}$.

A consideration of the sums (1) is motivated by Warings problem, which asks for the smallest integer $m$ such that all sufficiently large integers are the sum of at most $m$ $k$-th powers. The sums (1) arise when treating the minor arcs in an application of the Circle Method to
which we refer the reader to [15, 19]. In such applications, one uses a bound of the form

\[
\left| \sum_{M < n \leq M + N} e_q(an^k) \right| \leq N^{1-1/\rho},
\]

where \( \rho \) depends on the integer \( k \) and the size of \( N \) relative to \( q \). The order of \( \rho \) as a function of \( k \) is an important factor in applications and it is desirable for \( \rho \) to grow as slowly as possible for large \( k \).

For large \( k \), current methods for producing the sharpest bound for (2) reduce the problem to estimating the number of solutions to a certain system of equations known as Vinogradov’s mean value theorem, which we describe below. For these values of \( k \), the best known estimate for (2) relies on recent results of Bourgain, Demeter and Guth [4] (see for example [12, Lemma 2.1] and [5, Equation 4.5]) and may be stated as follows. For \( k \geq 3 \) and \( N \leq q \leq N^{k-1} \) we have

\[
\left| \sum_{M < n \leq M + N} e_q(an^k) \right| \leq N^{1-1/k(k-1)+o(1)},
\]

where \( q \) is an arbitrary integer and \((a, q) = 1\). More precisely, the bound (3) is sharpest known when \( N \) is small relative to \( q \). For example, when \( q \) is prime one may use the Weil bounds to show

\[
\left| \sum_{M < n \leq M + N} e_q(an^k) \right| \ll q^{1/2} \log q,
\]

which is better than (3) in the range \( N \geq q^{1/2+1/k^2+O(1/k^3)} \).

For integers \( k, r \) and \( V \) we let \( J_{r,k}(V) \) count the number of solutions to the system of equations

\[
v_1^j + \cdots + v_r^j = v_{r+1}^j + \cdots + v_{2r}^j, \quad j = 1, \ldots, k,
\]

with variables satisfying

\[1 \leq v_1, \ldots, v_{2r} \leq V.\]

Bounds for \( J_{r,k}(V) \) are usually referred to as Vinogradov’s mean value theorem and typically take the shape

\[
J_{r,k}(V) \leq (1 + V^{r-k(k+1)/2})V^{r+o(1)}.
\]

The main conjecture for \( J_{r,k}(V) \) is the statement that (6) holds for all integers \( r, k \) and \( V \). Significant progress concerning bounds for \( J_{r,k}(V) \) has been made by Wooley [17, 18, 20, 21] and in particular Wooley [22] has proven the main conjecture for \( J_{r,k}(V) \) in the case \( k = 3 \). More recently, Bourgain, Demeter and Guth [4] have proven
the main conjecture for $J_{r,k}(V)$ when $k > 3$. Combining the results of Wooley [22] for the case $k = 3$ with those of Bourgain, Demeter and Guth for the case $k > 3$, for any integers $r, k$ and $V$ we have

$$J_{r,k}(V) \leq (1 + V^{r-k(k+1)/2})V^{r+o(1)}.$$ (7)

In this paper we obtain a new bound for the sums (1) when $q$ is prime. Our argument falls under the framework of Vinogradov’s method which we use to reduce the problem to bounding two systems of congruences. The first concerns the number of solutions to the system

$$v_1^j + \cdots + v_r^j \equiv v_1^{j+1} + \cdots + v_r^{j+2} \mod q, \quad j = 1, \ldots, k,$$ (8)

which has been considered by Karatsuba [10] who attributes the problem to Korobov. We bound the number of solutions to this system by reducing to Vinogradov’s mean value theorem and applying results of Wooley [22] and Bourgain, Demeter and Guth [4]. The second system of congruences (see Lemma 5) does not appear to have been considered before and we use some ideas based on Mordell [13]. We also note that a related system of equations has been considered by Arkhipov and Karatsuba [1].

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2. Main Results

Our main result is as follows.

**Theorem 1.** Let $k \geq 3$ be an integer, $q$ a prime number, $a$ an integer with $(a, q) = 1$ and suppose $M$ and $N$ are integers with $N$ satisfying

$$N \leq q^{1/2+1/(k^{1/2}+1)}.$$ (9)

Then we have

$$\left| \sum_{M<n\leq M+N} e_q(an^k) \right| \leq \left( \frac{q^{1/(k-2)^{1/2}}}{N} \right)^{2/k(k+1)} Nq^{\rho(1)}.$$

We first note that the bound of Theorem 1 is nontrivial in the range

$$q^{(1+\epsilon)/(k-2)^{1/2}} \leq N,$$

and in this case we have a bound of the form

$$\left| \sum_{M<n\leq M+N} e_q(an^k) \right| \leq N^{1-\rho+o(1)}.$$
where
\[ \rho = \frac{2 \varepsilon}{(1 + \varepsilon)k(k + 1)}. \]
Comparing Theorem 1 with the bound (3), we see that Theorem 1 provides an improvement over (3) in the range
\[ N \geq q^{2(k-1)/(k-3)(k-2)^{1/2}} = q^{2k^{-1/2} + O(k^{-3/2})}. \]

3. Preliminary results

We first prove the following general inequality for systems of congruences.

Lemma 2. Suppose \( q \) is prime and \( t, m \) and \( \ell \) are integers. Let \( X \subseteq (\mathbb{Z}/q\mathbb{Z})^t \), be a set and \( f = \{f_i\}_{i=1}^m \) a sequence of functions on \( X \),
\[ f_i : X \to \mathbb{Z}/q\mathbb{Z}, \quad i = 1, \ldots, m. \]
Let \( \sigma = \{\sigma_i\}_{i=1}^{2\ell} \), be a sequence of numbers with each \( \sigma_i \not\equiv 0 \mod q \) and \( \lambda = \{\lambda_j\}_{j=1}^m \) a sequence of numbers with each \( \lambda_j \in \mathbb{Z}/q\mathbb{Z} \). We let \( I_{m,\ell}(f, X, \sigma, \lambda) \) denote the number of solutions to the system of equations
\[
\sigma_1 f_j(x_1) + \cdots + \sigma_{2\ell} f_j(x_{2\ell}) \equiv \lambda_j \mod q, \quad j = 1, \ldots, m, \\
\text{with variables } x_1, \ldots, x_{2\ell} \text{ satisfying } x_1, \ldots, x_{2\ell} \in X.
\]
In the special case that \( \sigma = \{(-1)^i\}_{i=1}^{2\ell} \) and each \( \lambda_j = 0 \) we write \( I_{m,\ell}(f, X, \{(-1)^i\}_{i=1}^{2\ell}, 0) = I_{m,\ell}(f, X) \).
For any \( X, f, \sigma \) and \( \lambda \) as above, we have
\[ I_{m,\ell}(f, X, \sigma, \lambda) \leq I_{m,\ell}(f, X). \]

Proof. Expanding the system (10) into additive characters, we see that \( I_{m,\ell}(f, X, \sigma, \lambda) \)
\[ = \frac{1}{q^m} \sum_{1\leq a_h \leq q} \prod_{1 \leq h \leq m} \left( \sum_{x_i \in X} e_q \left( \sigma_i \left( \sum_{j=1}^m a_j f_j(x_i) \right) \right) \right) \left( - \sum_{j=0}^m a_j \lambda_j \right), \]
so that writing
\[ f(a_1, \ldots, a_m) = \sum_{x \in X} e_q \left( \sum_{j=1}^m a_j f_j(x) \right), \]
the above implies
\[
I_{m,\ell}(f, X, \sigma, \lambda) = \frac{1}{q^m} \sum_{1 \leq a_h \leq q} \prod_{1 \leq h \leq m} f(\sigma_i a_1, \ldots, \sigma_i a_m) e_q \left( -\sum_{j=0}^{m} a_j \lambda_j \right),
\]
which we may bound by
\[
I_{m,\ell}(f, X, \sigma, \lambda) \leq \frac{1}{q^m} \sum_{1 \leq a_h \leq q} \prod_{1 \leq h \leq m} |f(\sigma_i a_1, \ldots, \sigma_i a_m)|.
\]
An application of Hölder’s inequality gives
\[
I_{m,\ell}(f, X, \sigma, \lambda) \leq \prod_{i=1}^{2\ell} \left( \frac{1}{q^m} \sum_{1 \leq a_h \leq q} |f(\sigma_i a_1, \ldots, \sigma_i a_m)|^{2\ell} \right)^{1/2\ell},
\]
and hence
\[
I_{m,\ell}(f, X, \sigma, \lambda) \leq \prod_{i=1}^{2\ell} \left( \frac{1}{q^m} \sum_{1 \leq a_h \leq q} |f(a_1, \ldots, a_m)|^{2\ell} \right)^{1/2\ell} = \frac{1}{q^m} \sum_{1 \leq a_h \leq q} |f(a_1, \ldots, a_m)|^{2\ell}.
\]
The result follows since the term
\[
\frac{1}{q^m} \sum_{1 \leq a_h \leq q} |f(a_1, \ldots, a_m)|^{2\ell},
\]
counts the number of solutions to the system of congruences
\[
\sum_{i=1}^{2\ell} (-1)^i f_j(x_i) \equiv 0, \quad j = 1, \ldots, m,
\]
with variables \(x_1, \ldots, x_{2\ell}\) satisfying
\[
x_1, \ldots, x_{2\ell} \in X.
\]
\[\square\]

The proof of the following uses the bound (7).

**Lemma 3.** For integers \(k, r, V\) and \(q\) we let \(J_{r,k}(V; q)\) count the number of solutions to the system of congruences
\[
v_1^j + \cdots + v_r^j \equiv v_{r+1}^j + \cdots + v_{2r}^j \mod q, \quad j = 1, \ldots, k,
\]
with variables satisfying

\[ 1 \leq v_1, \ldots, v_{2r} \leq V. \]

Let \( 1 \leq m < k \) be an integer and suppose \( V \) satisfies

\[ q^{1/m} \ll V < \frac{q^{1/m}}{r}. \]

If \( r \geq k(k+1)/2 \) we have

\[ J_{r,k}(V; q) \leq V^{2r-km(m-1)/2+o(1)}. \]

Proof. For integers \( \lambda_{m+1}, \ldots, \lambda_k \) we let \( J_{r,k}(V, \lambda_{m+1}, \ldots, \lambda_k) \) denote the number of solutions to the system of equations

\[ v_j^1 + \cdots + v_j^{2r} = v_{j+1}^r + \cdots + v_{2r}^r, \quad j = 1, \ldots, m, \]

\[ v_j^1 + \cdots + v_j^{2r} = v_{j+1}^r + \cdots + v_{2r}^r + \lambda_j, \quad j = m + 1, \ldots, k, \]

with variables \( v_1, \ldots, v_{2r} \) satisfying

\[ 1 \leq v_1, \ldots, v_{2r} \leq V. \]

Expressing \( J_{r,k}(V, \lambda_{m+1}, \ldots, \lambda_k) \) via additive characters we see that

\[ J_{r,k}(V, \lambda_{m+1}, \ldots, \lambda_k) = \int_0^1 \cdots \int_0^1 \left| \sum_{1 \leq v \leq V} e^{2\pi i (\alpha_1 v + \cdots + \alpha_k v^k)} \right|^{2r} e^{-2\pi i (\sum_{j=m+1}^k \alpha_j \lambda_j)} d\alpha_1 \cdots d\alpha_k, \]

and hence

\[ J_{r,k}(V, \lambda_{m+1}, \ldots, \lambda_k) \leq \int_0^1 \cdots \int_0^1 \left| \sum_{1 \leq v \leq V} e^{2\pi i (\alpha_1 v + \cdots + \alpha_k v^k)} \right|^{2r} d\alpha_1 \cdots d\alpha_k, \]

which implies

\[ J_{r,k}(V, \lambda_{m+1}, \ldots, \lambda_k) \leq J_{r,k}(V). \]

Using the assumption (12), we have

\[ J_{r,k}(V; q) = \sum_{|\lambda_j| \leq r V^j/q, \atop m+1 \leq j \leq k} J_{r,k}(V, \lambda_{m+1} q, \ldots, \lambda_k q), \]

and hence by (13)

\[ J_{r,k}(V; q) \ll \left( \prod_{j=m+1}^k \frac{V^j}{q} \right) J_{r,k}(V). \]
Since $r \geq k(k + 1)/2$, an application of (7) gives

$$J_{r,k}(V; q) \ll \left( \prod_{j=m+1}^{k} \frac{V^j}{q} \right) V^{2r-k(k+1)/2+o(1)},$$

which on recalling (12) simplifies to

$$J_{r,k}(V; q) \ll V^{2r-km+m(m-1)/2+o(1)}.$$

\[\square\]

For a proof of the following see [6]. See also [2, 7, 11] for related and more precise results.

**Lemma 4.** Let $q$ be prime and suppose $M, N$ and $U$ are integers with $N$ and $U$ satisfying

$$NU \leq q, \quad U \leq N.$$

The number of solutions to the congruence

$$n_1u_1 \equiv n_2u_2 \mod q,$$

with variables satisfying

$$M < n_1, n_2 \leq M + N, \quad 1 \leq u_1, u_2 \leq U,$$

is bounded by

$$O(NU \log q).$$

**Lemma 5.** Let $q$ be prime and $M, N$ and $U$ integers with $N$ and $U$ satisfying

$$NU \leq q, \quad U \leq N.$$

Let $\ell \geq 1$ be an integer and suppose $k$ is an integer satisfying

$$2\ell \leq k.$$

Let $I_{k,\ell}(N,U)$ denote the number of solutions to the system of congruences

(14) $$n_1^j u_1^{k-j} + \cdots + n_\ell^j u_\ell^{k-j} \equiv n_{\ell+1}^j u_{\ell+1}^{k-j} + \cdots + n_{2\ell}^j u_{2\ell}^{k-j} \mod q, \quad j = 0, \ldots, 2\ell - 1,$$

in variables $n_1, \ldots, n_{2\ell}, u_1, \ldots, u_{2\ell}$ satisfying

(15) $$M < n_1, \ldots, n_{2\ell} \leq M + N, \quad 1 \leq u_1, \ldots, u_{2\ell} \leq U.$$

Then we have

$$I_{k,\ell}(N,U) \ll (NU)^\ell (\log q)^{\ell-1}.$$
Proof. We fix an integer $k$ and first consider the case $\ell = 1$. Recalling that $I_{k,1}(N, U)$ counts the number of solutions to the congruences
\[ u_1^k \equiv u_2^k \mod q \quad \text{and} \quad u_1^{k-1} n_1 \equiv u_2^{k-1} n_2 \mod q, \]
in variables
\[ M \leq n_1, n_2 \leq M + N, \quad 1 \leq u_1, u_2 \leq U, \]
we fix a value of $u_1$ which determines $u_2$ with at most $k$ choices. Next we fix a value of $n_1$ which determines $n_2$ with at most 1 choice. This implies that
\[ I_{k,1}(N, U) \ll NU. \tag{16} \]

We next assume there exists some integer $m$ such that
\[ I_{k,m}(N, U) \gg (NU)^m (\log q)^{m-1}, \tag{17} \]
and out of all integers $m$ satisfying (17) suppose $\ell$ is the smallest, so that
\[ I_{k,\ell}(N, U) \gg (NU)^\ell (\log q)^{\ell-1}. \tag{18} \]
Considering (16), we see that $\ell \geq 2$.

Suppose the points $n = (n_1, \ldots, n_{2\ell})$ and $u = (u_1, \ldots, u_{2\ell})$ are a solution to (14) satisfying (15). Let
\[ H(u) \subseteq \mathbb{F}_q^{2\ell}, \]
denote the hyperplane
\[ H(u) = \{(z_1, \ldots, z_{2\ell}) \in \mathbb{F}_q^{2\ell} : z_1 u_1^k + \cdots + z_{2\ell} u_{2\ell}^k \equiv 0 \mod q \}. \]
Considering the system (14), we see that the $2\ell$ points
\[ p_j(n, u) = \left( \frac{n_1}{u_1} \right)^j, \ldots, \left( \frac{n_{2\ell}}{u_{2\ell}} \right)^j, \quad j = 0, \ldots, 2\ell - 1, \]
all lie on $H(u)$. Since $H(u)$ has dimension $2\ell - 1$, this implies there exists some nontrivial linear relation among the $p_j(n, u)$. In particular, there exists $\alpha_0, \ldots, \alpha_{2\ell-1} \in \mathbb{F}_q$, with at least one $\alpha_i \not\equiv 0 \mod q$ such that
\[ \alpha_0 p_0(n, u) + \cdots + \alpha_{2\ell-1} p_{2\ell-1}(n, u) \equiv 0 \mod q. \tag{19} \]
The equation (19) implies that the Vandermonde matrix of the points
\[ \frac{n_1}{u_1}, \ldots, \frac{n_{2\ell}}{u_{2\ell}}, \]
is singular mod $q$ and hence there exists integers $1 \leq r, s \leq 2\ell$ with $r \neq s$ such that
\begin{equation}
\frac{n_r}{u_r} \equiv \frac{n_s}{u_s} \mod q.
\end{equation}

Letting $I_{k,\ell}(N, U, r, s)$ denote the number of solutions to the system (14) with variables satisfying (15) subject to the further condition (20), since the pair $r, s$ can take at most $4\ell^2$ values, we see that
\begin{equation}
I_{k,\ell}(N, U) \ll I_{k,\ell}(N, U, r, s),
\end{equation}
for some pair $r, s$ with $r \neq s$. Considering $I_{k,\ell}(N, U, r, s)$, there exists some sequence of numbers $\sigma_i \in \{-1, 1\}$ for $1 \leq i \leq 2\ell$ such that
\begin{equation}
I_{k,\ell}(N, U, r, s) \leq \sum_{1 \leq i \leq 2\ell} \left| \sum_{i \neq r, s} \sigma_i n_i^j u_i^{k-j} \equiv \sigma_r n_r^j u_r^{k-j} + \sigma_s n_s^j u_s^{k-j} \mod q, \quad j = 0, \ldots, 2\ell - 1, \right|
\end{equation}
with variables satisfying (15) and (20).

For each fixed $n_r, n_s, u_r, u_s$, we let $I_0(n_r, n_s, u_r, u_s)$ count the number of solutions to the system (22) with variables satisfying
\begin{equation}
M < n_i \leq M + N, \quad 1 \leq u_i \leq U, \quad i = 1, \ldots, 2\ell, \quad i \neq r, s,
\end{equation}
so that the above implies
\begin{equation}
I_{k,\ell}(N, U) \ll \sum_{M < n_r, n_s \leq M + N} \left( \sum_{1 \leq u_r, u_s \leq U} I_0(n_r, n_s, u_r, u_s) \right).
\end{equation}

Letting $I'_0$ denote the number of solutions to the system of congruences
\begin{equation}
n_1^j u_1^{k-j} + \cdots + n_{\ell-1}^j u_{\ell-1}^{k-j} \equiv n_{\ell}^j u_{\ell}^{k-j} + \cdots + n_{2\ell-2}^j u_{2\ell-2}^{k-j} \mod q, \quad j = 0, \ldots, 2\ell - 1,
\end{equation}
in variables $n_1, \ldots, n_{2\ell-2}, u_1, \ldots, u_{2\ell-2}$ satisfying
\begin{equation}
M < n_1, \ldots, n_{2\ell-2} \leq M + N, \quad 1 \leq u_1, \ldots, u_{2\ell-2} \leq U,
\end{equation}
an application of Lemma 2 gives
\begin{equation}
I_0(n_r, n_s, u_r, u_s) \leq I'_0,
\end{equation}
for any $n_r, n_s, u_r$ and $u_s$. Considering only equations corresponding to
\begin{equation}
j = 0, \ldots, 2(\ell - 1) - 1,
\end{equation}
in the system (25), we see that
\begin{equation}
I'_0 \leq I_{k,\ell-1}(N, U),
\end{equation}
and hence by (27)

\[ I_0(n_r, n_s, u_r, u_s) \ll I_{k,\ell-1}(N, U). \]

Combining the above with (24) we get

\[
I_{k,\ell}(N, U) \ll \left( \sum_{M < n_r, n_s \leq M + N \atop 1 \leq u_r, u_s \leq U \atop n_r u_s \equiv n_s u_r \mod q} 1 \right) I_{k,\ell-1}(N, U).
\]

Since the term

\[
\sum_{M < n_r, n_s \leq M + N \atop 1 \leq u_r, u_s \leq U \atop n_r u_s \equiv n_s u_r \mod q} 1,
\]

counts the number of solutions to the congruence

\[ n_r u_s \equiv n_s u_r \mod q, \]

in variables \( n_r, n_s, u_r, u_s \) satisfying

\[ M < n_r, n_s \leq M + N, \quad 1 \leq u_r, u_s \leq U, \]

an application of Lemma 4 gives

\[
\sum_{M < n_r, n_s \leq M + N \atop 1 \leq u_r, u_s \leq U \atop n_r u_s \equiv n_s u_r \mod q} 1 \ll NU \log q,
\]

and hence

\[ I_{k,\ell}(N, U) \ll I_{k,\ell-1}(N, U)NU \log q. \]

Combining the above with (18), we see that

\[ I_{k,\ell-1}(N, U) \gg (NU)^{\ell-1}(\log q)^{\ell-2}, \]

contradicting the assumption that \( \ell \) is the smallest integer satisfying (17). \( \square \)

4. Proof of Theorem 1

We proceed by induction on \( N \) and fix an arbitrarily small \( \varepsilon \). Since the bound of Theorem 1 is trivial provided \( N \leq q^{1/(k-2)1/2} \), this forms the basis of the induction. We formulate our induction hypothesis as
follows. Let \( M \) be an arbitrary integer and suppose for all integers \( K \leq N - 1 \) we have
\[
\left| \sum_{M<n \leq M+K} e_q(an^k) \right| \leq \left( \frac{q^{1/(k-2)^{1/2}}}{K} \right)^{2/(k+1)} K q^\varepsilon,
\]
uniformly over \( M \). Using the above hypothesis, we aim to show
\[
\left| \sum_{M<n \leq M+N} e_q(an^k) \right| \leq \left( \frac{q^{1/(k-2)^{1/2}}}{N} \right)^{2/(k+1)} N q^\varepsilon.
\]
We define the integers \( \ell, m \) and \( r \) by
\[
\ell = \left\lfloor \frac{k}{2} \right\rfloor,
\]
\[
m = \left\lceil \sqrt{2\ell} \right\rceil,
\]
\[
r = \frac{k(k+1)}{2},
\]
and define the integers \( U \) and \( V \) by
\[
U = \left\lfloor \frac{rN}{2q^{1/m}} \right\rfloor, \quad V = \left\lceil \frac{q^{1/m}}{2r} \right\rceil.
\]
We first note that
\[
UV \leq \frac{N}{4}.
\]
Let \( 1 \leq u \leq U \) and \( 1 \leq v \leq V \) be integers and write
\[
\sum_{M<n \leq M+N} e_q(an^k) = \sum_{M-uv<n \leq M+N-uv} e_q(a(n+uv)^k)
\]
\[
= \sum_{M<n \leq M+N} e_q(a(n+uv)^k)
\]
\[
+ \sum_{M-uv<n \leq M} e_q(a(n+uv)^k) - \sum_{M+N-uv<n \leq M+N} e_q(a(n+uv)^k).
\]
Averaging the above over \( 1 \leq u \leq U \) and \( 1 \leq v \leq V \), using (33) and applying our induction hypothesis, we see that
\[
\left| \sum_{M<n \leq M+N} e_q(an^k) \right| \leq \frac{|W|}{UV} + \frac{1}{2} \left( \frac{q^{1/(k-2)^{1/2}}}{N} \right)^{2/(k+1)} N q^\varepsilon,
\]
where
\[ W = \sum_{1 \leq u \leq U} \sum_{1 \leq v \leq V} \sum_{M < n < M+N} e_q(a(n + uv)^k). \]

We have
\[ |W| \leq \sum_{1 \leq u \leq U} \sum_{1 \leq v \leq V} \sum_{M < n < M+N} e_q \left( a \sum_{j=0}^{k} \binom{k}{j} n^j u^{k-j} v^{k-j} \right), \]

hence by H"older’s inequality
\[ |W|^\ell \leq V^{\ell-1} \sum_{1 \leq u \leq U} \sum_{1 \leq v \leq V} \sum_{M < n < M+N} e_q \left( a \sum_{j=0}^{k-1} \binom{k}{j} (n_u^j u^{k-j} + \cdots + n_{u_\ell}^j u_{\ell}^{k-j}) v^{k-j} \right)^\ell, \]

so that for some complex numbers \( \theta_v \) with \( |\theta_v| = 1 \) we have
\[ |W|^\ell \leq V^{\ell-1} \sum_{1 \leq u_1 \leq U} \sum_{1 \leq \ell \leq k} \sum_{1 \leq v \leq V} |\theta_v| e_q \left( a \sum_{j=0}^{k-1} \binom{k}{j} (n_u^j u^{k-j} + \cdots + n_{u_\ell}^j u_{\ell}^{k-j}) v^{k-j} \right). \]

Let
\[ W_0 = \sum_{1 \leq u_1 \leq U} \sum_{M < n_1 \leq M+N} \sum_{1 \leq v \leq V} |\theta_v| e_q \left( a \sum_{j=0}^{k-1} \binom{k}{j} (n_u^j u^{k-j} + \cdots + n_{u_\ell}^j u_{\ell}^{k-j}) v^{k-j} \right), \]

so that
\[ (35) \quad |W|^\ell \ll V^{\ell-1} W_0. \]

Let \( I(\lambda_0, \ldots, \lambda_{k-1}) \) denote the number of solutions to the system of congruences
\[ \binom{k}{j} (n_u^j u^{k-j} + \cdots + n_{u_\ell}^j u_{\ell}^{k-j}) \equiv \lambda_j \mod q, \quad j = 0, \ldots, k-1, \]

in variables \( n_1, \ldots, n_\ell, u_1, \ldots, u_\ell \) satisfying
\[ M < n_1, \ldots, n_\ell \leq M + N, \quad 1 \leq u_1, \ldots, u_\ell \leq U. \]

The above implies that
\[ W_0 = \sum_{\lambda_0, \ldots, \lambda_{k-1}=0}^{q-1} I(\lambda_0, \ldots, \lambda_{k-1}) \sum_{1 \leq v \leq V} |\theta_v| e_q \left( a \sum_{j=0}^{k-1} \lambda_j v^{k-j} \right). \]
Two applications of Hölder’s inequality gives
\[
W_0^{2r} \leq \left( \sum_{\lambda_0, \ldots, \lambda_{k-1} = 0} I(\lambda_0, \ldots, \lambda_{k-1}) \right)^{2r-2} \left( \sum_{\lambda_0, \ldots, \lambda_{k-1} = 0} I(\lambda_0, \ldots, \lambda_{k-1}) \right)^{2r}
\]
(36)
\[
\times \left( \sum_{\lambda_0, \ldots, \lambda_{k-1} = 0} \left| \sum_{v = 1}^{V} \theta_v e_q \left( \sum_{j=0}^{k-1} \lambda_j v^{k-j} \right) \right| \right)^{2r}.
\]

The term
\[
\sum_{\lambda_0, \ldots, \lambda_{k-1} = 0} \left| \sum_{v = 1}^{V} \theta_v e_q \left( \sum_{j=0}^{k-1} \lambda_j v^{k-j} \right) \right|^{2r},
\]
is bounded by \(q^k\) times the number of solutions to the system of congruences
\[
v_1^j + \cdots + v_{2r}^j \equiv v_1^{j+1} + \cdots + v_{2r}^{j+1} \mod q, \quad j = 1, \ldots, k,
\]
in variables \(v_1, \ldots, v_{2r}\) satisfying
\[
1 \leq v_1, \ldots, v_{2r} \leq V.
\]

Recalling the choice of \(V\) in (32) and applying Lemma 3, we see that
\[
\sum_{\lambda_0, \ldots, \lambda_{k-1} = 0} \left| \sum_{v = 1}^{V} \theta_v e_q \left( \sum_{j=0}^{k-1} \lambda_j v^{k-j} \right) \right|^{2r} \leq q^k J_{r,k}(V; q) \leq q^k V^{2r-mk+m(m-1)/2+o(1)}.
\]

We have
\[
\sum_{\lambda_0, \ldots, \lambda_{k-1} = 0} I(\lambda_0, \ldots, \lambda_{k-1}) = (NU)^\ell,
\]
(38)
and the term
\[
\sum_{\lambda_0, \ldots, \lambda_{k-1} = 0} I(\lambda_0, \ldots, \lambda_{k-1})^2,
\]
is equal to the number of solutions to the system of congruences
\[
n_1^j u_1^{k-j} + \cdots + n_{2\ell}^j u_{2\ell}^{k-j} \equiv n_1^{j+1} u_{\ell+1}^{k-j+1} + \cdots + n_{2\ell}^{j+1} u_{2\ell}^{k-j+1} \mod q, \quad j = 0, \ldots, k - 1,
\]
in variables \(n_1, \ldots, n_{2\ell}, u_1, \ldots, u_{2\ell}\) satisfying
\[
M < n_1, \ldots, n_{2\ell} \leq M + N, \quad 1 \leq u_1, \ldots, u_{2\ell} \leq U.
\]
Recalling (29) and considering only equations corresponding to $j = 0, \ldots, 2\ell - 1$ in (39), we see that

$$
\sum_{\lambda_0, \ldots, \lambda_{k-1}=0}^{q-1} I(\lambda_0, \ldots, \lambda_{k-1})^2 \leq I_{k,\ell}(N, U),
$$

where $I_{k,\ell}(N, U)$ is defined as in Lemma 5. Recalling (9), we have

$$
(40) \quad \sum_{\lambda_0, \ldots, \lambda_{k-1}=0}^{q-1} I(\lambda_0, \ldots, \lambda_{k-1})^2 \leq I_{k,\ell}(N, U) \leq (NU)^{\ell+o(1)}.
$$

Combining (36) (37), (38) and (40) gives

$$
W_0^{2r} \leq q^{k+o(1)} (NU)^{\ell(2r-1)} V^{2r-mk+m(m-1)/2},
$$

and hence by (35)

$$
|W|^\ell \ll q^{h/2r+o(1)} V^{\ell-mk/2r+m(m-1)/4r} (NU)^{(1-1/2r)}.
$$

Combining the above with (34) gives

$$
\left| \sum_{M<n \leq M+N} e_q(an^k) \right| \leq \left( \frac{q^k}{V^{mk-m(m-1)/2}} \right)^{1/2r} N^{1-1/2r} q^{o(1)} U^{1/2r} + \frac{1}{2} \left( \frac{q^{1/(k-2)/2}}{N} \right)^{2/k(k+1)} N q^\varepsilon,
$$

which on recalling the choice of $U$ and $V$ in (32) the above simplifies to

$$
\left| \sum_{M<n \leq M+N} e_q(an^k) \right| \leq q^{((m-1)/2\ell+1/m)/2r+o(1)} N^{1-1/r} + \frac{1}{2} \left( \frac{q^{1/(k-2)/2}}{N} \right)^{2/k(k+1)} N q^\varepsilon.
$$

Recalling the choice of $\ell, m$ and $r$ in (29), (30) and (31) we get

$$
\left| \sum_{M<n \leq M+N} e_q(an^k) \right| \leq \left( \frac{q^{1/(k-2)/2}}{N} \right)^{2/k(k+1)} N q^{o(1)} + \frac{1}{2} \left( \frac{q^{1/(k-2)/2}}{N} \right)^{2/k(k+1)} N q^\varepsilon,
$$
and hence
\[ \left| \sum_{M < n \leq M + N} e_q(an^k) \right| \leq \left( \frac{q^{1/(k-2)^{1/2}}}{N^2} \right)^{2/k(k+1)} Nq^e, \]
by taking the term $o(1)$ in $q^{o(1)}$ to be sufficiently small.

References

[1] G. I. Arkhipov and A. A. Karatsuba, *A multidimensional analogue of Waring’s problem*, Sov. Math. Dokl., 36, 75–77, (1987).
[2] A. Ayyad, T. Cochrane and Z. Zheng, *The congruence $x_1x_2 \equiv x_3x_4 \pmod{p}$, the equation $x_1x_2 = x_3x_4$, and Mean Values of Character Sums*, J. Number Theory, 59, 398–413, (1996).
[3] E. Bombieri, *On Vinogradov’s mean value theorem and Weyl sums*, Automorphic forms and analytic number theory, (Montreal, PQ, 1989), Univ. Montreal, Montreal, QC, 7–24, (1990).
[4] J. Bourgain, C. Demeter and L. Guth, *Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three*, Ann. Math. (to appear).
[5] K. Ford, *Recent progress on the estimation of Weyl sums*, Proc. IV Intern. Conf. "Modern Problems of Number Theory and its Applications": Current Problems, Part II (Tula, 2001), Moscow State Univ., Moscow, 48–66, (2002).
[6] J. B. Friedlander and H. Iwaniec, *Estimates for character sums*, Proc. Amer. Math. Soc. 119, no. 2, 365–372, (1993).
[7] M. Z. Garaev and V.C. Garcia, *The equation $x_1x_2 = x_3x_4 + \lambda$ in fields of prime order and applications*, J. Number Theory, 128, 2520–2537, (2008).
[8] D. R. Heath-Brown, *A new $k$-th derivative estimate for exponential sums via Vinogradov’s mean value*, arXiv:1601.04493.
[9] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Colloquium Publications 53 (2004), American Math. Soc., Providence, RI.
[10] A. A. Karatsuba, *On systems of congruences*, Izv. Akad. Nauk SSSR Ser. Mat. 29, 935–944, (1965).
[11] B. Kerr *On the congruence $x_1x_2 \equiv x_3x_4 \pmod{q}$*, J. Number Theory, (to appear).
[12] A. V. Kumchev and T. D. Wooley *On the Waring-Goldbach problem for seventh and higher powers*, Monatsh. Math. (in press), 8pp.
[13] L. J. Mordell, *On a sum analogous to a Gauss’s sum*, Quart. J. Math. 3, 161–167, (1932).
[14] S. T. Parsell, *On the Bombieri-Korobov estimate for Weyl sums*, Acta. Arith., 138, 363–372, (2009).
[15] R. C. Vaughan, *The Hardy-Littlewood method*, Cambridge University Press, Cambridge, 1997.
[16] T. D. Wooley, *New estimates for Weyl sums*, Quart. J. Math. Oxford Ser. 46, 119–127, (1995).
[17] T. D. Wooley, *Vinogradov’s mean value theorem via efficient congruencing*, Annals of Math. (2) 175, no. 3, 265–273, (2012).
[18] T. D. Wooley, *Vinogradov’s mean value theorem via efficient congruencing, II*, Duke. Math. J. 162, no. 4, 673–730, (2013).
[19] T. D. Wooley, *Translation invariance, exponential sums and Waring’s problem*, arXiv:1401.7152v1. (2014).

[20] T. D. Wooley, *Multigrade efficient congruencing and Vinogradov’s mean value theorem*, arXiv:1310.8447.

[21] T. D. Wooley, *Approximating the Main Conjecture in Vinogradov’s Mean Value Theorem*, 2014, 52. pp. arXiv:1401.2932.

[22] T. D. Wooley, *The cubic case of the Main Conjecture in Vinogradov’s Mean Value Theorem*, Adv. Math. 294, 532–561, (2016).

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