CASTELNUOVO-MUMFORD REGULARITY OF KOSZUL CYCLES AND
KOSZUL HOMOLOGIES

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Abstract. We extend to one dimensional quotients the result of A. Conca and S. Murai on the convexity of the regularity of Koszul cycles. By providing a relation between the regularity of Koszul cycles and Koszul homologies we prove a sharp regularity bound for the Koszul homologies of a homogeneous ideal in a polynomial ring under the same conditions.

1. Introduction

In order to attain better understanding of a projective variety classically there is a great interest to determine the equations of a projective variety and also the syzygies of its homogeneous ideal. In this regard, M. Green and R. Lazarsfeld defined the property $N_p$ which, roughly speaking, refers to the simplicity of syzygies of the homogeneous coordinate ring of a smooth projective variety embedded by a very ample line bundle. More precisely, let $R$ be a finitely generated graded $K$-algebra then we say that $R$ satisfies property $N_p$ if we have $\beta_{i,i+j}(R) = 0$ for all $1 < j$ and for all $1 \leq i \leq p$. The Green-Lazarsfeld index of $I$ denoted by $\text{index}(I)$ is the maximum of such $p$. In the case $X = \mathbb{P}^n$ with the line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$, M. Green proved that the coordinate ring of the image of Veronese embedding of degree $d$ of $X$ satisfies the property $N_d$. W. Bruns A. Conca and T. Römer [1] improved the lower bound of the Green-Lazarsfeld index of the Veronese subring $S^{(d)}$ to $d + 1$, their approach is based on investigation of the homological invariants of the Koszul cycles and Koszul homologies of $d$-th power of the maximal ideal.

By aforementioned motivation A. Conca and S. Murai studied the Castelnuovo-Mumford regularity of the Koszul cycles $Z_t(I, S)$ of a homogeneous ideal in a polynomial ring $S$. They proved that regularity of Koszul cycles $Z_t(I, S)$ as a function of $i$ is subadditive when $\dim S/I = 0$. We make a generalization showing that if $S$ is a polynomial ring over a field of characteristic 0 and $\dim S/I \leq 1$ then
\[ \text{reg}(Z_{s+t}(I, S)) \leq \text{reg}(Z_t(I, S)) + \text{reg}(Z_s(I, S)). \]

From the convexity of the regularity of Koszul cycles in dimension 0, A. Conca and S. Murai [3, Corollary 3.3] obtained a bound on the regularity of Koszul homologies. But inspired by the remarkable result of M.Chardin and P. Symonds [4] on the regularity of 2000 Mathematics Subject Classification. 13D02, 13D03.

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cycles and homologies of a general complex, first we determine the regularity of Koszul cycles by the regularity of the previous Koszul homologies. Let $S$ be a polynomial ring, $I$ be a homogeneous ideal of $S$. If $\dim S/I \leq 1$, then for all $0 < i < \mu(I)$

$$\text{reg}(Z_i(I, S)) = \max_{0 < j < n} \{\text{reg}(H_{i-j}(I, S)) + j + 1\}.$$  

Where $\mu(I)$ is the minimal number of generators of $I$.

As an application we state sharp bound for the regularity of Koszul homologies in dimension 1 which is a refinement of the result of A. Conca and S. Murai in dimension 0. Let $I$ be an ideal of $S$ and $\dim S/I \leq 1$, then we have the following inequalities between Koszul homologies of $I$ for all $i, j \geq 1$

$$\text{reg}(H_{i+j-1}(I, S)) \leq \max_{1 \leq \alpha, \beta \leq n-1} \{\text{reg}(H_{i-\alpha}(I, S)) + \text{reg}(H_{j-\beta}(I, S)) + \alpha + \beta\}.$$  

In the last section, we investigate the behavior of regularity of Koszul homologies of power of the maximal ideal. Thanks to the equivalence between Betti numbers of Veronese embedding and regularity of Koszul homologies, as an application we give a short proof for the theorem of M. Green in [5] on Green-Lazarsfeld index of Veronese embeddings.

2. Preliminaries

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ and $M$ be a finitely generated graded $S$-module. A minimal free resolution of $M$ is an exact sequence

$$0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to M \to 0,$$

where each $F_i$ is a graded $S$-free module of the form $F_i = \oplus S(-j)^{\beta_{i,j}(M)}$ such that the number of basis elements is minimal and each map is graded. The value $\beta_{i,j}(M)$ is called the $i$-th graded Betti numbers of $M$ of degree $j$. Note that the minimal free resolution of $M$ is unique up to isomorphism so the graded Betti numbers are uniquely determined.

**Definition 2.1.** Let $S$ be a polynomial ring and $I$ be a homogeneous ideal of $S$. We say that $I$ satisfies property $N_p$ if we have $\beta_{i,i+j}(I) = 0$ for all $j > 1$ and for all $i \leq p$. The Green-Lazarsfeld index of $I$ denoted by $\text{index}(I)$ is the maximum of such $p$.

**Remark 2.2.** We have $\text{index}(I) = \infty$ if and only if $I$ is generated in degree 2 and has a linear resolution.

Let $I = (f_1, \ldots, f_r)$ be a graded $S$-ideal minimally generated in degrees $d_1, \ldots, d_r$. Define $K(I, S) = \oplus K_t(I, S)$ as the Koszul complex associated to the $S$-linear map $\phi : F_0 = \oplus S(-d_i) \to S$ in which $\phi(e_i) = f_i$. Let $K(I, M) = K(I, S) \otimes M$ and denote $Z_t(I, M)$, $B_t(I, M)$ and $H_t(I, M)$ as the cycles, boundaries and homologies of $K(I, M)$ respectively at the homological position $t$. We use $Z_t(I)$, $B_t(I)$ and $H_t(I)$ whenever $M = S$. 
Remark 2.3. The Koszul complex does depend on the choice of the generators, but it is unique up to isomorphism if we choose minimal set of generators. Since we only deal with the case that the set of generators is minimal, we use $K(I)$ instead of $K(f_1, \ldots, f_r)$.

Let $m = (x_1, \ldots, x_n)$ be the graded maximal ideal of $S$, for a finitely generated module $M$ define the Čech complex as follows:

$$
C_m^\bullet(M) : 0 \to M \to \bigoplus_{1 \leq i \leq n} M_{x_i} \to \bigoplus_{1 \leq i, j \leq n} M_{x_ix_j} \to \cdots \to M_{x_1 \cdots x_n} \to 0
$$

The local cohomology modules of an $S$-module $M$ are the homologies of the Čech complex. It is a well known fact that each local cohomology module is artinian so we can speak about the last nonzero degree of each of them. We define

$$a^m_i(M) := \text{end}(H^i_m(M)) = \max\{j \mid H^i_m(M)_j \neq 0\},$$

then the Castelnuovo-Mumford regularity of module $M$ is defined as follows:

**Definition 2.4.** Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring, $m = (x_1, \ldots, x_n)$ be a unique graded maximal ideal of $S$ and $M$ be a finitely generated $S$-module, then

$$\text{reg}(M) = \max\{a^m_i(M) + i\}$$

**Veronese embedding.** Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring, for each $c$ consider $S$-subalgebra $V_S(c) = \oplus_{i \geq 0} S_{ic}$ where $S_d$ is a $k$-vector space spanned by all monomials of degree $c$ in $S$. We call $V_S(c)$ the $c$-th Veronese embedding of $S$. There is another interpretation of Veronese embedding as a coordinate ring of a variety, which is isomorphic to $k[t_1, \ldots, t_N]/J$ where $N = \binom{n+c-1}{n-1}$ and $J$ is a kernel of the map $\phi : k[t_1, \ldots, t_N] \to S$ by sending $t_i$’s to all monomials of degree $c$ in $S$. The variety correspond to $J$ is called Veronese Variety. We denote $\beta_{i,j}(k[t_1, \ldots, t_N]/J)$ by $\beta_{i,j}(V_S(c))$.

### 3. Regularity of Koszul cycles

In this section we will present a generalization of a result about convexity of regularity of Koszul cycles of A. Conca and S. Murai.

The following interesting property of Koszul cycles was first investigated in work of W. Bruns, A. Conca and T. Römer in [2]

#### Lemma 3.1. [2, Lemma 2.4]

Let $S$ be a polynomial ring, $I$ be a homogeneous ideal of $S$ and $M$ be a finitely generated graded $S$-module. Suppose that the element \( \left( \begin{array}{c} s + t \\ s \end{array} \right) \) is invertible in $S$. Then $Z_{s+t}(I, M)$ is a direct summand of $Z_s(I, Z_t(I, M))$.

The following lemma provides us to compare regularities of different terms of exact sequences and basically it plays the main role in the generalization of the result of A. Conca and S. Murai[3] on the convexity of regularity of Koszul cycles.
Lemma 3.2. Let $L : 0 \rightarrow L_4 \xrightarrow{d_4} L_3 \xrightarrow{d_2} L_2 \xrightarrow{d_2} L_1 \rightarrow 0$ be an exact sequence of finitely generated graded $S$-modules such that $L_1$ and $L_4$ are supported in dimension less than or equal 1, and depth $L_2 \geq 2$ then
\[
\text{reg}(L_3) = \max\{\text{reg}(L_4), \text{reg}(L_2), \text{reg}(L_1) - 1\},
\]
in particular $\text{reg}(L_2) \leq \text{reg}(L_3)$.

Proof. First we decompose the complex $L$ into the following short exact sequences
\[
0 \rightarrow L_4 \xrightarrow{d_4} L_3 \xrightarrow{\text{can}} \text{coker}(d_4) \rightarrow 0,
\]
\[
0 \rightarrow \text{coker}(d_4) \xrightarrow{d_4} L_2 \xrightarrow{d_2} L_1 \rightarrow 0.
\]
Given the above short exact sequences, one can obtain the following induced long exact sequences on local cohomology:
\[
\begin{align*}
\text{(I)} & \quad \cdots \rightarrow H^i_m(L_4) \rightarrow H^i_m(L_3) \rightarrow H^i_m(\text{coker}(d_4)) \rightarrow H^{i+1}_m(L_4) \rightarrow \cdots \\
\text{(II)} & \quad \cdots \rightarrow H^i_m(L_2) \rightarrow H^i_m(L_1) \rightarrow H^{i+1}_m(\text{coker}(d_4)) \rightarrow H^{i+1}_m(L_2) \rightarrow \cdots
\end{align*}
\]
$H^i_m(L_4) = 0$ for $i \geq 2$ as dim $L_4 \leq 1$ thus (I) gives
\[
H^2_m(L_3) \cong H^2_m(\text{coker}(d_4)).
\]
As dim $L_1 \leq 1$, by (II) we have
\[
H^i_m(\text{coker}(d_4)) \cong H^i_m(L_2) \forall i \geq 3.
\]
As depth $L_2 \geq 2$ and $H^i_m(L_1) = 0$ for $i = 0, 1$, by (II)
\[
H^0_m(L_1) \cong H^1_m(\text{coker}(d_4)) \text{ and } H^0_m(\text{coker}(d_4)) = 0.
\]
From the exact sequences (I) and (3.2) we get the following short exact sequences
\[
0 \rightarrow H^1_m(L_4) \rightarrow H^1_m(L_3) \rightarrow H^0_m(L_1) \rightarrow 0,
\]
also the exact sequences (II) and (3.1) give
\[
0 \rightarrow H^1_m(L_1) \rightarrow H^2_m(L_3) \rightarrow H^2_m(L_2) \rightarrow 0.
\]
As a result we have
\[
a^i_m(L_3) = \begin{cases} 
    a^0_m(L_4) & \text{if } i = 0 \\
    \max\{a^0_m(L_4), a^0_m(L_1)\} & \text{if } i = 1 \\
    \max\{a^1_m(L_2), a^1_m(L_1)\} & \text{if } i = 2 \\
    a^i_m(L_2) & \text{if } i \geq 3 
\end{cases}
\]
which proves the statement. \qed

Proposition 3.3. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring, let $M$ be a finitely generated graded $S$-module with depth $M \geq 2$ and let $I$ be a graded ideal of $S$ such that dim $S/I \leq 1$, then
\[
\text{reg}(Z_t(Z_s(I, M))) \leq \text{reg}(Z_t(I)) + \text{reg}(Z_s(I, M)).
\]
Proof. By definition one has the following exact sequences,

\[
\begin{align*}
(†) & \quad 0 \to Z_s(I, M) \to K_s(I, M) \xrightarrow{d_s} K_{s-1}(I, M) \\
(‡) & \quad 0 \to Z_t(I, Z_s(I, M)) \to K_t(I, Z_s(I, M)) \xrightarrow{d_t} K_{t-1}(I, Z_s(I, M))
\end{align*}
\]

Note that \(K_s(I, M)\) and \(K_{s-1}(I, M)\) are direct sums of copies of \(M\), (†) then implies that depth \(Z_s(I, M)\) \(\geq\) \(\min\{2, \text{depth } M\} = 2\). Using (‡), depth \(Z_t(I, Z_s(I, M))\) \(\geq\) \(\min\{2, \text{depth } Z_s(I, M)\} = 2\).

For the canonical map in [2, section 5]

\[u_{s,t} : Z_t(I) \otimes Z_s(I, M) \to Z_t(I, Z_s(I, M))\]

Proposition 5.1 in [2] gives an exact sequence,

\[0 \to \ker(u_{s,t}) \to Z_t(I) \otimes Z_s(I, M) \to Z_t(Z_s(I, M)) \to \text{Tor}_1^S(\frac{K_{s-1}(I, M)}{B_{s-1}(I, M)}, Z_t(I)) \to 0.\]

Notice that after localization at prime ideals not in the support of \(S/I\) all the Koszul cycles become direct sum of copies of \(M\) and the map \(u_{s,t}\) becomes an isomorphism. Therefore \(\text{Tor}_1^S(\frac{K_{s-1}(I, M)}{B_{s-1}(I, M)}, Z_t(I))\) and \(\ker(u_{s,t})\) are supported in \(S/I\), hence have a dimension at most 1.

Thus the conditions of Lemma 3.2 are fulfilled, and this lemma gives:

\[\text{reg}(Z_t(Z_s(I, M))) \leq \text{reg}(Z_t(I, M) \otimes Z_s(I, M)).\]

Notice that \(\text{Tor}_1^R(Z_t(I), Z_s(I, M))\) has Krull dimension at most 1 because \(Z_t(I)\) is free when we localize at prime ideals not in the support of \(S/I\), so we apply Corollary 3.1 in [6] to get

\[\text{reg}(Z_t(I, M) \otimes Z_s(I, M)) \leq \text{reg}(Z_t(I)) + \text{reg}(Z_s(I, M)).\]

As a result we get

\[\text{reg}(Z_t(Z_s(I, M))) \leq \text{reg}(Z_t(I)) + \text{reg}(Z_s(I, M)).\]

\[\square\]

**Theorem 3.4.** Let \(S = k[x_1, \ldots, x_n]\) and \(I\) be a graded ideal of \(S\), if \(\dim S/I \leq 1\) and characteristic of \(k\) is 0 or bigger than \(s + t\), then

\[\text{reg}(Z_{s+t}(I)) \leq \text{reg}(Z_t(I)) + \text{reg}(Z_s(I)).\]

Proof. The theorem follows from Proposition 3.3 and Lemma 3.1. \[\square\]
4. Regularity of Koszul Homologies

We start this section by a fact which is likely part of folklore but we did not find in the classical references.

**Proposition 4.1.** Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring and $I$ be an ideal of $S$ minimally generated by $f_1, \ldots, f_r$, then $Z_i(I) \subseteq \mathfrak{m}K_i(I)$ for all $i$.

**Proof.** Suppose it is not, then there exist $z \in Z_i(I)$ that is not in $\mathfrak{m}K_i(I)$. By symmetry we may assume it has the form:

$$z = e_1 \wedge \cdots \wedge e_i + \sum_{j > i} c_j e_1 \wedge \cdots \wedge e_{i-1} \wedge e_j + \text{terms without } e_1 \wedge \cdots \wedge e_{i-1}.$$

Since $\partial(z) = 0$ it follows that $(-1)^i f_1 + \sum_{j > i} (-1)^j c_j f_j = 0$, as it is the coefficient of $e_1 \wedge \cdots \wedge e_{i-1}$ in the expression of $\partial(z)$, which is a contradiction with the fact that $f_1, \ldots, f_r$ is a minimal set of generators for $I$. \hfill \Box

**Corollary 4.2.** Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring and $I = (f_1, \ldots, f_r)$ be a homogeneous ideal of $S$. Let $f_1, \ldots, f_r$ be a minimal generating set of $I$ and $\deg(f_i) = d_i$ where $d_1 \geq d_2 \geq \cdots \geq d_r$, then $\reg(Z_i(I)) > d_1 + \cdots + d_i$.

**Proof.** Fix a basis element $e_1 \wedge \cdots \wedge e_i \in K_i$. Since $K_\bullet(I)$ is a complex,

$$\partial(e_1 \wedge \cdots \wedge e_i \wedge e_{i+1}) = (-1)^{i+1} f_{i+1} e_1 \wedge \cdots \wedge e_i + \sum_{0 < j < i+1} (-1)^j f_j e_1 \wedge \cdots \wedge e_j \wedge \cdots \wedge e_{i+1} \in Z_i(I).$$

Therefore an element of the form $g e_1 \wedge \cdots \wedge e_i$ should appear as a summand in a minimal generating element of $Z_i(I)$. By Proposition 4.1, $g \in \mathfrak{m}$. So there exist minimal generator of degree at least $d_1 + \cdots + d_i + 1$. Hence $\reg(Z_i(I)) > d_1 + \cdots + d_i$. \hfill \Box

M. Chardin and P. Symonds state an approach in [4] for investigating the regularity of cycles of a general complex by the regularity of previous homologies. Here we determine a concrete relation between regularity of cycles and homologies of a koszul complex.

**Theorem 4.3.** Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring and $I$ be a homogeneous ideal of $S$ minimally generated by $f_1, \ldots, f_r$. If $\dim S/I \leq 1$, then for $0 < i < r$:

\begin{equation}
\reg(Z_i(I)) = \max_{0 < j < n}\{\reg(H_{i-j}(I)) + j + 1\}.
\end{equation}

**Proof.** Let $I = (f_1, \ldots, f_r)$ and $\deg(f_i) = d_i$ where $d_1 \geq d_2 \geq \cdots \geq d_r$. Let $K_\bullet^i(I)$ be the $i$-th truncated Koszul complex of $I$ as follows:

$$K_\bullet^i(I) : 0 \to Z_i(I) \xrightarrow{\partial_i^0} K_i(I) \xrightarrow{\partial_i} K_{i-1}(I) \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} K_0(I) \to 0.$$
and \(C^\bullet\) be the Čech complex. Consider double complex \(X = C^\bullet \otimes K^\bullet(I)\) where \(X_{p,q} = C^{\cdot-p} \otimes K^\bullet(I)_q\), and its associated spectral sequence. We first compute homology vertically and we get

\[
\begin{align*}
H^0_m(Z_i(I)) & \quad 0 \quad 0 \quad \cdots \quad 0 \\
H^1_m(Z_i(I)) & \quad 0 \quad 0 \quad \cdots \quad 0 \\
H^2_m(Z_i(I)) & \quad 0 \quad 0 \quad \cdots \quad 0 \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \\
H^n_m(Z_i(I)) & \quad H^p_m(K_i(I)) \quad H^n_m(K_{i-1}(I)) \quad \cdots \quad \cdots \quad \cdots \quad H^n_m(K_0(I)).
\end{align*}
\]

By continuing the process we have:

\[
E^\infty_{p,q} = E^2_{p,q} \begin{cases} 
H^p_m(Z_q(I)) & \text{if } q = i + 1, p < n \\
H^p_q(H^m_n(K^\bullet(I))) & \text{if } p = n, q \leq i \\
\ker(H^m_n(\partial^q)) & \text{if } (p, q) = (n, i + 1) \\
0 & \text{Otherwise.}
\end{cases}
\]

Notice that since \(a^m_n(K_j(I)) = d_1 + \cdots + d_j - n\), it follows that for all \(0 \leq q \leq i\) we have \(\text{end}(E^\infty_{n,q}) \leq \text{end}(E^1_{n,q}) = d_1 + \cdots + d_q - n\).

On the other hand, if we start taking homology horizontally we have \(E^2_{p,q} = H^p_m(H_q(I))\) for all \(p\) and \(q < i\) and \(E^2_{p,q} = 0\) for \(q = i, i+1\). Notice that \(\dim H_i(I) \leq \dim S/I \leq 1\), therefore spectral sequence collapses in the second page and we have:

\[
E^\infty_{p,q} = E^2_{p,q} \begin{cases} 
H^p_m(H_q(I)) & \text{if } p = 0, 1 \text{ and } q < i \\
0 & \text{otherwise.}
\end{cases}
\]

The comparison of two spectral sequences gives

\[
\begin{align*}
H^0_m(Z_i(I)) &= H^1_m(Z_i(I)) = 0 \\
\overset{2}{a}^m_m(Z_i(I)) &= a^{0}_m(H_{i-1}(I)) \\
\overset{j}{a}^m_m(Z_i(I)) &= \max\{\overset{1}{a}^m_m(H_{i-j+2}(I)), \overset{0}{a}^m_m(H_{i-j+1}(I))\}, \ \forall \ 2 < j < n
\end{align*}
\]

In addition, for the last local cohomology we have

\[
a^m_n(Z_i(I)) \leq \max\{\overset{1}{a}^m_m(H_{i-n+2}(I)), \overset{0}{a}^m_m(H_{i-n+1}(I)), d_1 + \cdots d_i - n\};
\]

furthermore

\[
a^m_n(Z_i(I)) = \max\{\overset{1}{a}^m_m(H_{i-n+2}(I)), \overset{0}{a}^m_m(H_{i-n+1}(I))\}
\]

if \(a^m_n(Z_i(I)) > d_1 + \cdots d_i - n\). By corollary 4.2, we can deduce that
\[ a_m^n(Z_i(I)) = \max\{a_m^1(H_{i-n+2}(I)), a_m^0(H_{i-n+1}(I))\} \text{ or } a_m^n(Z_i(I)) + n < \text{reg}(Z_i(I)) \]

In addition, the comparison of the two spectral sequences and Corollary 4.2 give
\[ a_m^1(H_{i-n+1}(I)) \leq \text{end}(E_{n,i-1}^\infty) \leq d_1 + \cdots + d_{i-1} - n < \text{reg}(Z_i(I)) - n. \]

As a result we have:
\[
\text{reg}(Z_i(I)) = \max_{0 \leq j \leq n} \{a_m^0(Z_i(I))) + j\},
\]
\[
= \max_{3 \leq j \leq n} \{a_m^0(H_i-1(I)) + 2, a_m^1(H_{i-j+2}(I)) + j, a_m^0(H_{i-j+1}(I)) + j, d_1 + \cdots + d_i\},
\]
\[
= \max_{2 \leq j \leq n} \{\text{reg}(H_{i-j+1}(I)) + j\}.
\]

\begin{remark}
From the proof of the Theorem 4.3, the following equality also holds
\[
\text{reg}(Z_i(I)) = \max_{j>0} \{\text{reg}(H_{i-j}(I)) + j + 1\}.
\]
\end{remark}

As a consequence of the Theorems 4.3 and 3.4 we give a regularity bound for Koszul homologies in dimension at most 1.

**Theorem 4.5.** Let \( S = k[x_1, \ldots, x_n] \) be a polynomial ring and \( I \) be a homogeneous ideal of \( S \). If \( \dim S/I \leq 1 \), then for all \( i, j \geq 1 \) we have the following regularity bound for the Koszul homologies of \( I \).

\[
(4.2) \quad \text{reg}(H_{i+j-1}(I)) \leq \max_{0 \leq \alpha, \beta < n} \{\text{reg}(H_{i-\alpha}(I)) + \text{reg}(H_{j-\beta}(I)) + \alpha + \beta\}.
\]

**Proof.** By Theorem 3.4 we have the following inequality for all \( i, j \)
\[
\text{reg}(Z_{i+j}(I)) \leq \text{reg}(Z_i(I)) + \text{reg}(Z_j(I)).
\]

By using Theorem 4.3 we have
\[
\text{reg}(H_{i+j-1}(I)) + 2 \leq \text{reg}(Z_{i+j}(I))
\]
\[
\leq \text{reg}(Z_i(I)) + \text{reg}(Z_j(I))
\]
\[
= \max_{0 \leq \alpha < n} \{\text{reg}(H_{i-\alpha}(I)) + \alpha + 1\} + \max_{0 \leq \beta < n} \{\text{reg}(H_{j-\beta}(I)) + \beta + 1\}
\]
\[
= \max_{0 \leq \alpha, \beta < n} \{\text{reg}(H_{i-\alpha}(I)) + \text{reg}(H_{j-\beta}(I)) + \alpha + \beta + 2\}.
\]
The following example shows the deviation degree of our bound comparing to the bound provided by A. Conca and S. Murai in dimension 0.

**Example 4.6.** Let $S = k[x, y, z]$ be a polynomial ring and $I = (x, y, z)^4$. We compare our bound for the regularity of $H_{12}(I)$ for different $i, j$ by the bound in the [3]. By using MACAULAY2 [7] one can see that the reg($H_{12}(I)$) = 57. For bounding regularity of $H_{12}(I)$ we should choose $i, j$ such that $i + j = 13$. By choosing $(i, j) = (1, 12)$ (respectively $(2, 11), (3, 10), (4, 9), (5, 8), (6, 7)$) the right hand side of 4.2 is 57 (respectively 58, 58, 59, 59, 58). On the other hand in the bound proposed by A. Conca and S. Murai the best possible estimate is 61.

**Corollary 4.7.** Let $S = k[x_1, \ldots, x_n]$ be a polynomial ideal and $I$ be an ideal of $S$. If $\dim S/I \leq 1$, then

$$\text{reg}(H_c(I)) \leq (c+1) \text{reg}(H_0(I)) + 2c.$$  

**Proof.** We prove by induction. For $c = 1$ by 4.5 we have

$$\text{reg}(H_1(I)) \leq \{ \text{reg}(H_0(I)) + 1 + \text{reg}(H_0(I)) + 1 \} = 2\text{reg}(H_0(I)) + 2.$$

Let reg($H_i(I)$) $\leq (i+1)\text{reg}(H_0(I)) + 2i$ for all $i \leq r$, by choosing $i = 1$ and $j = r + 1$ in 4.2 we have

$$\text{reg}(H_{r+1}(I)) \leq \max_{0 < \beta < n} \{ \text{reg}(H_0(I)) + \text{reg}(H_{r+1-\beta}(I)) + \beta + 1 \}$$

For all $0 < \beta < n$ we have

$$\text{reg}(H_{r+1-\beta}(I)) + \beta + 1 \leq (r - \beta + 2)\text{reg}(H_0(I)) + 2(r + 1 - \beta) + \beta + 1 \leq (r + 1)\text{reg}(H_0(I)) + 2(r + 1).$$

Therefore reg($H_{r+1}(I)$) $\leq (r + 2)\text{reg}(H_0(I)) + 2(r + 1)$. \qed

5. **Green-Lazarsfeld index of veronese embedding**

Let $X$ be a smooth projective variety with a very ample line bundle $L$ which sets up an embedding into projective space $\mathbb{P}^r$ where $r = h^0(X, L) - 1$. Let $S = \text{Sym} H^0(X, L)$ be the homogeneous coordinate ring of $\mathbb{P}^r$ and if we define $R := \oplus H^0(X, O(kL))$, then $R$ can be viewed as a finitely generated graded $S$-module. The syzygies of $R$ as an $S$-module is investigated by M. Green. Let $X$ be a curve of genus $p$ and let $L$ be a very ample line bundle on $X$. Green proved that if $\deg L = 2g + p + 1$ then the embedding defined by $L$ has property $N_p$. In the case of Veronese embedding of projective spaces $\varphi_c : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^N$ M. Green proved that the Veronese subring $S^{(c)} = \bigoplus_{i \in \mathbb{N}} S_{ic}$ satisfies property $N_c$ and then W. Bruns A. Conca and T. Römer extend the lower bound to $c + 1$.

G. Ottaviani and R. Paolletti [8] proved that the Veronese embedding $\varphi$ for $n \geq 2$ and $c \geq 3$ does not satisfy property $N_{3c-2}$. In zero characteristic therefore one can deduce that

$$c + 1 \leq \text{index}(S^{(c)}) \leq 3c - 3.$$
G. Ottaviani and R. Paolletti showed that if \( n = 3 \) then \( \text{index}(S^{(c)}) = 3c - 3 \) and they conjectured that the equality holds for arbitrary \( n \geq 3 \). Recently, T. Vu proved the conjecture of G. Ottaviani and R. Paolletti in the case \( c = 4 \) [9]. The following theorem provides an equivalence between the study of the syzygies of Veronese embedding and the study of Koszul homologies of powers of the maximal ideal.

**Theorem 5.1.** Let \( S = k[x_1, \ldots, x_n] \). For \( i \in \mathbb{N} \) and \( j \in \mathbb{Z} \) we have:

\[
\beta_{i,j}(V_S(c)) = \dim_K H_i(m^c, R)_{jc}
\]

*Proof.* See 4.1 in [2] for the proof. \( \square \)

Now we use our results on regularity of Koszul cycles and Koszul homologies to find a lower bound for the Green-Lazarsfeld index of Veronese embedding. In this regard, we are able to reproof the statement of M. Green in [5].

**Lemma 5.2.** Let \( S = k[x_1, \ldots, x_n] \) and \( V_{S}(c) \) be the \( c \)-th Veronese subring of \( S \), then

\[
\text{reg}(Z_{i+2}(m^c)) = \text{reg}(H_{i+1}(m^c)) + 2, \quad \text{for } i \leq \min\{\text{index}(V_{S}(c)), 2c\}
\]

*Proof.* By Theorem 4.3, it suffices to prove that the regularity of Koszul homologies are increasing as a function of \( i \). Since \( i \leq \text{index}(V_{S}(c)) \), by Theorem 5.1

\[
H_i(m^c)_{(i+2)c} = 0 \text{ and } \text{reg}(H_{i+1}(m^c)) \geq (i+2)c.
\]

As \( H_i(m^c)_{(i+2)c} = 0 \), \( \text{reg}(H_i(m^c)) < (i+2)c \) if and only if \( H_i(m^c) \) has no generator of degree greater than \( (i+2)c \). Hence it suffices to show that \( \text{reg}(Z_i(m^c)) \leq (i+2)c \). By 3.4 and 4.3 we have

\[
\text{reg}(Z_i(m^c)) \leq i(\text{reg}(Z_1(m^c))) = i(\text{reg}(H_0(m^c)) + 2) = i(c + 1)
\]

Since \( i \leq 2c \), then \( \text{reg}(Z_i(m^c)) \leq (i + 2)c \). \( \square \)

As a consequence of Theorem 5.1 we know that \( \text{reg}(H_i(m^c)) = (i + 1)c + r_i \). From the definition of Green-Lazarsfeld index one can see \( i \leq \text{index}(V_R(c)) \) if and only if \( r_i \leq c - 1 \). In order to find a bound for the index of Veronese embedding we can study the behavior of \( r_i \)'s. Notice that \( r_i \in \mathbb{Z} \) for instance \( r_0 = -1 \).

**Proposition 5.3.** With the above notations we have \( r_{i+1} \leq r_i + 1 \) for all \( i \leq \min\{\text{index}(V_{S}(c)) + 1, 2c + 1\} \).

*Proof.* By Theorem 3.4 we have a triangle inequality between the regularity of Koszul cycles, in particular for \( Z_1(m^c) \) and \( Z_i(m^c) \). By using Corollary 5.2 and Theorem 4.3

\[
\text{reg}(H_{i+1}(m^c)) + 2 = \text{reg}(Z_{i+2}(m^c)) \leq \text{reg}(Z_{i+1}(m^c)) + \text{reg}(Z_1(m^c)) = \text{reg}(H_i(m^c)) + 2 + \text{reg}(H_0(m^c)) + 2
\]

by the above notation we have that \( (i + 2)c + r_{i+1} \leq (i + 1)c + r_i + c + 1 \). In particular \( r_{i+1} \leq r_{i-1} + 1 \). \( \square \)
Corollary 5.4. The Green-Lazarsfeld index of Veronese embedding $V_S(c)$ is at least $c$.

Proof. As we mention above, for finding the Green-Lazarsfeld index of Veronese embedding we should control $r_i$'s. Proposition 5.3 shows that in each step, $r_i$'s can be increased only be one. Since $r_0 = -1$ so $r_c \leq c - 1$ that means $\text{index}(V_S(c)) \geq c$. □

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