Nonlinear magnetotransport theory and Hall induced resistance oscillations in graphene

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Abstract

The quantum oscillations of nonlinear magnetoresistance in graphene that occur in response to a dc current bias are investigated. We present a theoretical model for the nonlinear magnetotransport of graphene carriers. The model is based on the exact solution of the effective Dirac equation in crossed electric and magnetic fields, while the effects of randomly distributed impurities are perturbatively added. To compute the nonlinear current effects, we develop a covariant formulation of the migration center theory. The current is calculated for short- and large-range scatterers. The analysis of the differential resistivity in the large magnetic field region, shows that the extrema of the Shubnikov de Hass oscillations invert when the dc currents exceed a threshold value. These results are in good agreement with experimental observations. In the small magnetic field regime, corresponding to large filling factors, the existence of Hall induced resistance oscillations are predicted for ultra clean graphene samples. These oscillations originate from Landau–Zener tunneling between Landau levels, that are tilted by the strong electric Hall field.

Keywords: graphene, magnetotransport, Hall induced resistance oscillations

(Some figures may appear in colour only in the online journal)
resistance states were discovered \([9, 10]\) in 2DEG subjected to microwave irradiation and moderate magnetic fields. An analogous effect, Hall field-induced resistance oscillations (HIRO) and zero differential resistance states, have been observed in high mobility samples in response to a dc-current excitation \([11, 12]\).

Analogous phenomena to those described for conventional 2DEG in the previous paragraph, have been discovered in graphene, including the integer \([13, 14]\) and fractional quantum Hall effect \([15]\). However the non-linear magnetotransport effects in graphene are less studied. A recent experiment \([16]\) has analyzed the effect of a strong dc current bias on SdH oscillations in graphene. A phase inversion on the SdH oscillations of the differential resistance is observed with increasing bias, in which the oscillation maxima evolve into minima and vice versa. In the work of Wang and Lei \([17]\) the nonlinear effects on graphene magnetotransport are studied within the balance-equation approach.

In this paper we analyze a theoretical model for the nonlinear magnetotransport of graphene carriers. The model is based on the exact solution of the effective Dirac equation in crossed electric and magnetic fields, while the effects of randomly distributed impurities are perturbatively added. We take advantage of the relativistic-like structure of the model to compute the nonlinear current. With this purpose we develop a covariant formulation of the migration center theory. Based on this model the quantum oscillations of nonlinear magnetoresistance in graphene that occur in response to a dc current bias are investigated. The analysis of the differential resistivity in the large magnetic field region shows, in good agreement with experimental observations \([16]\) that the extrema of the Shubnikov de Hass oscillations invert when the dc currents exceeds a threshold value. In a small magnetic field, the existence of Hall-induced resistance oscillations are predicted for ultra clean graphene samples. These oscillations originate from Landau–Zener transitions between Landau levels, that are tilted by the strong electric Hall field. The formalism applies both for the cases of short-range and Coulomb randomly distributed scatterers. Although, most of the numerical results are presented for the simpler case of short-range delta interactions, at the end we compare the effects produced by the two types of impurity scatterers.

The outline of the paper is presented as follows. In section 2 the effective Dirac equation for charge carriers in graphene subject to crossed electric and magnetic fields, as well as randomly distributed impurities is described. We present the method, based on Lorentz-like transformations, that allows us to solve the problem. Section 3 describes the migration center theory, and the computation of the impurity assisted current. The details of the current controlled scheme that lead to the expression for the longitudinal resistivity are discussed in section 4. In section 5 we present the main results and discussion of the nonlinear effects in SdH oscillations as well as the Hall induced resistance oscillations in graphene. The formulation of the covariant migration center theory and details of the calculation of the impurity induced current are summarized in the appendixes.

2. Model

We consider the motion of charge carriers in an imperfect graphene sample, in the presence of an in-plane electric field and a magnetic field perpendicular to the graphene plane. The graphene defects are modeled by the impurity scattering potential \(V\). The goal of this work is to study the nonlinear response to a DC electrical current density \(J_{dc}\) of graphene placed in a magnetic field. In a typical experimental configuration the electric field is not explicitly controlled, instead the longitudinal current is fixed, \(J_l = J_{dc}\), while the transverse current cancels, \(J_t = 0\). Then, in general, the electric field has both longitudinal \((E_l)\) and transverse \((E_t)\) components, that have to be determined in terms of \(J_{dc}\). The calculations are simplified if we perform a rotation by an angle \(\theta = -\text{Arctan}(E_t/E_l)\) to a \(S_L\) frame with coordinates \(r' = (x', y', z')\), hence, the electric field points along the \(y'\) axis: \(E = (0, E_l)\). In section 2 the current density \(J'\) is calculated from the wave function solution that is obtained in the present section, an inverse rotation to the original \(S_L\) frame will lead to a set of two implicit equations for the unknown \(E_t\) and \(E_t\), that can be determined by a self-consistent iteration in terms of \(J_{dc}\).

The dynamic of the problem in the \(S_L\) frame is governed by a Dirac equation, that for convenience is written as

\[
\nu_{\nu} \Pi_{\nu} \Psi = \left[ \nu_{\nu} \sigma \cdot \Pi + V \right] \Psi, \tag{1}
\]

where \(\nu_{\nu}\) is the Fermi velocity and \(\sigma \equiv (\sigma_x, \sigma_y)\) are the Pauli matrices that act on the two-component spinor \(\Psi\); the pseudospin degrees of freedom are associated with the two sub-lattices. In the previous equation \(\Pi_{\nu} = p_{\nu} - c \varphi_{\nu}\), and \(\Pi = p - \frac{c}{E} A\), where the scalar potential corresponding to a constant electric field is given as \(\varphi = -E y'\), and the gage potential in the Landau gage is \(A = (0, \mu, 0)\). When necessary, we use a covariant notation with space-time 2 + 1-vectors, e.g.: \(x_i = (\nu \varphi_t, r', \mu)\), \(i = 0, 1, 2\). The momentum operator is \(p_{\nu} = \left( i \hbar \partial / \partial x_{\nu}, -i \hbar \nabla \right)\), \(\Pi_{\mu} = p_{\mu} = -\frac{c}{E} A_{\nu}\) and \(A_{\nu} = \left( -\frac{E}{\nu} y', -B y', 0 \right)\). The Dirac matrices are selected as \(\gamma_i = (\sigma_i, i \sigma_i, -i \sigma_i)\) \([18]\).

The impurity potential is decomposed in terms of its Fourier components

\[
V(\mathbf{r}) = e^{-\delta_{N}^{\text{th}} N} \sum_{i} \frac{d^2 q}{(2\pi)^2} \mathcal{V}(q) \exp[i \mathbf{q} \cdot (\mathbf{r} - \mathbf{r}_i)], \tag{2}
\]

where \(\mathbf{r}_i\) represents the impurity positions, that are chosen at random; \(N_i\) is the number of impurities; and the \(\delta\) parameter determines the rate at which the perturbation is turned on and off. The Fourier transform of the disorder potential, for screened Coulomb interactions, can be approximated by \([19]\)

\[
\mathcal{V}(q) = \frac{2\pi e^2}{e (q + 4 \alpha_s k_F)}, \tag{3}
\]

where \(k_F\) is the Fermi momentum, \(\alpha_s = e^2/\epsilon \hbar v_F\) is the effective fine structure constant and \(e\) is the material permittivity. We notice that in the strong screening limit: \(\alpha_s \gg 1\), the long-range interaction changes into a short-range interaction, the constant
value for \( V(q) \approx \frac{\hbar^2}{2k_F} \) leads to impurity potential, that is the sum of uncorrelated delta contributions:

\[
V(r) = e^{-i\delta(r)} \sqrt{\frac{2}{\hbar^2}} \sum_{q} \delta (r - r_q).
\]

(4)

The pseudo-relativistic structure of the Dirac equation was exploited by Lukose et al [20] in order to obtain an exact solution for the equation that represents a graphene charge carrier subjected to both magnetic and electric fields. The method requires a Lorentz-like transformation to an auxiliary reference frame \( S_L \) where the electric field vanishes. This is the case if the relative velocity of \( S_L \) with respect to the original frame \( S \) is selected as the drift velocity \( V_0 = c \mathbf{E} \times \mathbf{B}/B^2 \). Once the solution in \( S_L \) is known it is possible to boost back to \( S \) in order to obtain the appropriate eigenfunctions and eigenvalues of the problem. This is possible as long as the condition \( cE < v_s B \) is fulfilled.

We follow the procedure presented in the work of Lukose et al [20] and extend the method to take into account the impurity scattering effects. In order to cancel the electric field effect, a boost along the \( x' \) axis is performed, with a boost parameter \( \theta \) given by

\[
tanh \theta = \beta = \frac{\gamma_x}{\gamma_{x'0}}.
\]

(5)

Under the Lorentz boost, coordinates transform as covariant three-vectors: \( \tilde{x}_p = A_{p}^{x'} x'_p \); where the explicit components of \( A_x^{x'} \) can be read from the following relations:

\[
\begin{pmatrix}
\tilde{x}_0 \\
\tilde{x}_1 \\
\tilde{x}_2 
\end{pmatrix} = \begin{pmatrix}
\cosh \theta & \sinh \theta & 0 \\
\sinh \theta & \cosh \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
x_0 \\
x_1 \\
x_2
\end{pmatrix}.
\]

(6)

Whereas, the wave function transforms as a Dirac spinor

\[
\Psi(x') \rightarrow \tilde{\Psi}(\tilde{x}) = \exp \left[ -\frac{\theta}{2} \gamma_0 \right] \Psi(x').
\]

(7)

By re-expressing the Dirac equation (1) in terms of \( \tilde{\Psi}(\tilde{x}) \) with the aid of equations (7) and (6) one finds that \( \tilde{\Psi}(\tilde{x}) \) satisfies an equation with the same structure

\[
\gamma_x \tilde{\Gamma}_x \tilde{\Psi} = \left[ \gamma_x \mathbf{S} \cdot \tilde{\Pi} + \tilde{V} \right] \tilde{\Psi},
\]

(8)

with \( \tilde{\Pi}_x = \tilde{p}_x - i\tilde{A}_x \), but now the gauge potential is given as \( \tilde{A}_x = (0, -\tilde{B}y, 0) \), where \( \tilde{B} = B \sqrt{1-\beta^2} \). Hence in the boosted frame the charge carrier dynamics correspond to a problem with a vanishing electric field and a reduced magnetic field \( \tilde{B} \). Whereas the transformed impurity potential takes the form

\[
\tilde{V}(\tilde{r}, \tilde{t}) = \int \sum_{l}^{N} \frac{\partial^2}{\partial r^2} \tilde{V}(q) e^{-i(q \cdot \tilde{r} - \omega t)}. \]

(9)

Here \( \gamma \equiv \cosh \theta = 1/\sqrt{1-\beta^2} \), and the matrix \( \mathcal{M} \) is given by

\[
\mathcal{M} = \gamma \mathcal{M}_x \mathcal{M}_y \mathcal{M}_z = \gamma (1 + \beta_0 \mathcal{M}_x).
\]

Notice that in the \( S_L \) frame the impurity potential acquires an additional time-dependent term; the boost induces a superposition of oscillatory terms, each with a momentum dependent frequency \( \omega_{q\ell} = \beta \omega_{q\ell} \).

When the impurity potential is not included, equation (8) is exactly solvable. The eigenstates and spectra are given by the well-known relativistic Landau levels (LLs):

\[
\Psi_{n, \ell} = e^{i\ell} \left( \frac{\text{sgn}(n)\eta_{n\ell}}{\text{i} \hbar} \gamma_x \right) \left( \psi_{n\ell}(\tilde{r} - \tilde{Y}_n) \right),
\]

\[
\mathcal{E}_{n, \ell} = \hbar^2 v_F \sqrt{2n},
\]

(10)

here the Landau number \( n \) is an integer and the quantum number \( \ell \) corresponds to the translation symmetry along the \( x' \) direction. The function \( \psi_{n\ell} \) is the displaced harmonic oscillator eigenfunctions, centered at \( \tilde{Y}_n = i\tilde{\ell}_n \); and the magnetic length in the \( S_A \) system, \( \tilde{l}_n \), is related to the magnetic length \( l_n = \sqrt{\hbar v_F / \mathcal{E}_{n, \ell}} \) by the relation \( \tilde{l}_n = l_n \sqrt{1/\beta^2} \). Notice that the energy spectrum is degenerate with respect to \( \ell \). Applying the inverse boost to the solution in equation (10), yields the eigenfunctions and spectrum in the original \( S_L \) frame:

\[
\Psi_{n, \ell}(x, y) = e^{i\ell} e^{\frac{\theta}{2} \gamma_0} \left( \frac{\text{sgn}(n)\eta_{n\ell}}{\text{i} \hbar} \gamma_x \right) \left( \psi_{n\ell}(x - Y_n) \right),
\]

\[
\mathcal{E}_{n, \ell} = \hbar^2 v_F \sqrt{2n} \frac{\text{sgn}(n) \sqrt{2n_l} \beta_{n\ell}}{1-\beta^2},
\]

(11)

notice that, as expected, the electric field breaks the LL degeneracy, the Landau levels are tilted with respect to \( \ell_n \) with a slope given by \( \hbar v_F \beta \). The displaced harmonic oscillator wave function is now centered at \( Y_{n, \ell} \), given as

\[
Y_{n, \ell} = \frac{l_n^{1/2} \ell_n^2 + \text{sgn}(n) \sqrt{2n} \beta_{n\ell}}{1-\beta^2}.
\]

The wave function center position depends both on \( \ell_n \) and on the LL number \( n \). As a result, the LLs are mixed—indeed, the electric field mixes the particle and hole solutions—and consequently the eigenfunctions are not orthogonal.

In order to obtain the current within the linear response formalism it is convenient to use an orthogonal base, otherwise the algebra becomes rather cumbersome. To avoid this problem, and since the eigenfunctions in \( S_L \) are orthogonal, it is convenient to work out the current density \( J_\ell \) in the \( S_L \) frame and then apply a Lorentz transformation to obtain the current density \( J_\ell \) in the original \( S_L \) frame.

### 3. Current density in the nonlinear regime

We now turn to the calculation of the current density in the boosted \( S_A \) system where the dynamics is ruled by equation (8). To consider the dynamic of an electron in a magnetic field it is customary to use the migration center theory [21–23], where the electron coordinate \( r \) splits into the guiding center \( X = (X, Y) \) and the cyclotron or relative coordinate \( \eta \), i.e., \( r = X + \eta \), where \( \eta = \left( \frac{-c\Pi_x}{eB}, \frac{c\Pi_y}{eB} \right) \). With this decomposition the \( \eta_x \) and \( \eta_y \) coordinates become non-commutative, similarly for \( X \) and \( Y \):

\[
[\eta_x, \eta_y] = \frac{i\hbar}{eB}, [X, Y] = -\frac{i\hbar}{eB} [\eta_x, X] = 0.
\]

(12)
It is possible to develop a covariant extension of migration center theory using the Schwinger proper time formalism. In what follows we present the general aspects of the covariant migration center theory (details are worked out in appendix A). The covariant expression for the relative coordinate $\eta_\nu$ is selected as

$$\eta^\nu = \frac{1}{e} \frac{F^\mu \Pi_\mu}{F \cdot F},$$  \hspace{1cm} (13)

where the electromagnetic tensor $F_{\mu \nu}$ is constructed from the constant $E$ and $B$ fields as given in equation (A.6), and the quantity $F \cdot F = F_{\mu \nu} F^{\mu \nu} = B^2 - \frac{c^2}{v^2} \cong B^2$ is an invariant under the Lorentz transformation. In the $S_A$ frame the 2 + 1-vector $\eta^\nu$ is given by

$$\eta^\nu = \frac{c^2}{\beta} \left( - \beta \Pi_\nu, \frac{c \Pi_\nu}{B}, \frac{c \Pi_\nu}{B} + \beta \Pi_\nu \right),$$  \hspace{1cm} (14)

where we recall that the electric field points along the $y$-axis and $\beta = c/E v_B$. In the $S_A$ frame, the relative coordinate is given by $\eta^\nu = (c/E B)(0, -\Pi_\nu, \Pi_\nu)$. It is easily demonstrated that the relative coordinate transforms as a covariant 2 + 1-vector under the Lorentz transformations equations (6):

$$\eta^\nu_A = A^\nu_\nu \eta^\nu.$$

The three-vector guiding center coordinate is then defined as $X_g = x_\nu - n_\nu$. The guiding center velocity is obtained from the Heisenberg equation of motion (see the appendix A), in the boosted $S_A$ frame it reads:

$$\frac{d\vec{X}_g}{dr} = \kappa_v \left( v_e - \frac{c}{e B} \frac{\partial \tilde{V}}{\partial \tilde{B}}, \frac{c}{e B} \frac{\partial \tilde{V}}{\partial \tilde{B}} \right).$$  \hspace{1cm} (15)

It is a straightforward exercise to prove (see the appendix A) that the current density probability $J_\nu$ defined in terms of the guiding center velocity as $J_\nu = \Psi^\dagger \eta_\nu (d\vec{X}_g / dr) \Psi$, is also a Lorentz 2 + 1-vector.

We now turn to the calculation of the current density. The current density $J_\nu$ is computed from the impurity, thermal, and time average of the guiding center velocity

$$J_\nu = \frac{\epsilon}{\sqrt{T}} \int_{-\infty}^{\infty} dt \left( T \left[ \rho \dot{X}_\nu \right] \right).$$  \hspace{1cm} (16)

where $S$ is the sample area. In the $S_A$ frame the density matrix $\rho$ satisfies the von Neumann equation:

$$i \hbar \frac{\partial}{\partial t} \rho = [\nu v_s \sigma, \rho] + \frac{\partial}{\partial t} \rho.$$  \hspace{1cm} (17)

The Hamiltonian in equation (8) splits into the first part that is exactly solved, equation (10), and the impurity potential $\tilde{V}$, the latter is treated as a perturbation. The density matrix is also decomposed, perturbatively as $\rho = \rho_0 + \Delta \rho$, the equilibrium density matrix is $\rho_0 = \sum_n \int d\tilde{k}_n \rho_0 (\tilde{n}, \tilde{k}_n)$, and $f_\nu$ is the Fermi distribution function, and the eigenfunction $\tilde{L}_n (\tilde{k}_n)$ are explicitly given in equation (10). The solution for the equation of the density matrix to first order in $\Delta \rho$, and the corresponding matrix elements in the $\tilde{L}_n (\tilde{k}_n)$ base are presented in the appendix B. Combining the solution for the density matrix given in equation (B.2) and the expression for the guiding center velocity in equation (15), one can proceed to compute the current density $J_\nu$ according to the expression in equation (16). In the auxiliary reference frame $S_A$ the current space components get contributions that arise solely from impurity scattering. After a lengthy but straightforward evaluation the following result is worked out (details are presented in appendix B):

$$\mathbf{J}^{\text{imp}} (E) = \frac{e c v_\nu n_i}{B} \sum_{n,m} \int \frac{d^2 q}{2\pi} (e_i \times q) (f_n - f_m) G_{nm}(q, \nu) v_i (\mathbf{Q})(q, \nu) d^2 q,$$

where $n_i$ is the impurity density, the LL degeneracy per unit area is given by $n_i = 1/(2\pi L^2)$ and $e_i$ is an unitary vector in the direction of the magnetic field. In the transition matrix element $v_i (\mathbf{Q})(q, \nu)$, the potential is evaluated at the momentum $Q = (q_t, q_y, q_x - i \beta \delta) v_B$ and the function $\Theta_{nm}(\zeta)$ is expressed in terms of the matrix elements of the displacement operator $D_{nm}$ as given by the equations (B.4) and (B.5) in appendix B. Finally, the function $G_{nm}$ displays a Lorentzian distribution

$$G_{nm}(q, \nu) = \frac{\delta / \gamma}{(\tilde{E}_e - \tilde{E}_n - h v_B q, \nu)^2 + (\delta / \gamma)^2},$$  \hspace{1cm} (19)

we notice that the rate $\delta$ at which the impurity potential in equation (2) is turned on and off, appears in the previous equation as a broadening parameter for the Landau levels. A simple phenomenological prescription to simulate the disorder broadening effects on the LL, is dictated by simply retaining a finite value of $\delta$, that is expected to be related to the single quasiparticle’s lifetime $\tau$, as $\delta = 2\pi \tau$. In the $\delta \rightarrow 0$ limit, $G_{nm}$ embodies the energy conservation condition as discussed below. Due to the elastic scattering, the impurities transfer a momentum $q = \tilde{k}_e - \tilde{k}_n$, to the electrons. Recalling that the center of the oscillatory wave function is given as $Y = i \delta \tilde{k}_n$, it follows that the momentum transfer is equivalent to a hopping or shifting of the guiding center $\Delta Y = Y - Y' = i \delta q, \nu$. In the $\delta \rightarrow 0$ limit equation (19) gives the resonance condition $\tilde{E}_n = \tilde{E}_e + h v_B q, \nu$. If interpreted in the $S_A$ system, the inter-Landau transitions are induced by the time dependent impurity potential equation (9) that oscillates with the frequency $\omega_{0q} = v_B \beta q$. Alternatively, the results can be interpreted according to the displacement mechanism: using the momentum transfer $q = \tilde{k}_e - \tilde{k}_n$, and noticing that $\tilde{E}_n \rightarrow \gamma (\tilde{E}_e - \nu \hbar \tilde{k}_n)$, the resonance condition can be rewritten as an energy conserving condition in the $S_A$ frame $\tilde{E}_n = \tilde{E}_e + h v_B q, \nu$. The transition can be interpreted as a Landau-Zener tunneling between LLs, tilted by the electric field equation (11) [111].

When the Lorentzian distribution is considered in equation (18) the summation over the $n$ and $m$ indices does not converge. While it is customary to set a cut-off energy to avoid the convergence issue [24], we substitute for the Lorentzian distribution a Gaussian distribution, leading to a natural cut-off:

$$G_{nm}(q) \approx \frac{\gamma}{2 \sqrt{\pi} \delta} \exp \left[ - \frac{\gamma^2}{\delta} \left( \tilde{E}_e - \tilde{E}_n - q \nu \hbar \beta \right)^2 \right].$$
Finally, the thermal and impurity average of the time component of the current density in equation (16) is easily computed utilizing equations (15) and (10), yielding $\bar{J}_x = \bar{e}n_e v_F$, where $\bar{n}_e$ is the electron density in the boosted frame.

4. Current controlled scheme

We are now ready to obtain the current density $J'$ in the laboratory frame, from the corresponding current density $\bar{J}_x$ in the auxiliary frame. $\bar{J}_x$ was calculated in the previous section: the time component is simply given as $\bar{J}_x' = \bar{e}n_e v_F$; whereas the corresponding space components are obtained by computing the expressions contained in equation (18). It is easy to verify that $J'_{\text{imp}}$ cancels because of the angular integrations; hence, we only need to compute the component $J'_{\text{dc}}$. Applying the inverse Lorentz transformation the current $J_x$ in the $S_1$ frame is obtained as:

$$J_x = \left( e n_{\nu F}, e n_{\nu F} \frac{E_x}{B}, J_{\text{imp}} \right),$$  

where the electron density $n_e$ is related to the electron density in the boosted frame according to the relation $n_e = \gamma \bar{n}_e$. Recalling that the imposed dc-current points along the $x$-axis, we need to perform an additional rotation by an angle $\theta = \arctan(E_y/E_x)$ (see the beginning of section 2). Consequently this leads to the following conditions:

$$J_x = e n_{\nu F} \frac{E_x}{B} + \sin \theta J_{\text{imp}}^{\text{imp}}[E] \equiv I_{dc},$$

$$J_x = -e n_{\nu F} \frac{E_y}{B} + \cos \theta J_{\text{imp}}^{\text{imp}}[E] \equiv 0.$$  

The last equalities apply for the Hall configuration: the longitudinal current is fixed at the value $I_x = I_{dc}$, whereas the transverse current cancels. Equations (21) represent two implicit equations for the unknown $E_x$ and $E_y$. The equations can be solved following a self-consistent iteration method. However, it is easily verified that, for a typical Hall experiment, a correct approximate solution to the previous equation is obtained, assuming that the electric field is almost transverse with respect to the dc bias current: i.e. $E_x \ll E_y$ or $\theta \ll 1$. Retaining the leading terms in $\theta$, the explicit solution is worked out as:

$$E_x \approx E_{dc} = \frac{B}{e n_{\nu F}} I_{dc}, E_y \approx \frac{B}{e n_{\nu F}} \frac{J_{\text{imp}}^{\text{imp}}[E]}{I_{dc}}.$$  

(22)

The first equality sheds the well known result for the Hall electric field $E_{dc}$ induced by the dc bias in the magnetic field. The second equation gives the longitudinal electric field in terms of the impurity scattering current obtained in equation (18); the result allows us to calculate the longitudinal resistivity, $\rho_{xx}$, as well as the longitudinal resistivity differential $r_{xx}$, from the relations:

$$\rho_{xx} = \frac{E_x}{I_{dc}}, \quad r_{xx} = \frac{\partial E_x}{\partial I_{dc}}.$$  

(23)

It is interesting to notice that the $\beta$ parameter, equation (5), takes the form: $\beta_\| = \gamma I_{dc}/e n_{\nu F}$, so $\beta_\| \gamma$ can be understood as the rate between the current density and the charge carrier’s density moving at a top speed $v_F$. Clearly, the formalism breaks down when $I_{dc} > e n_{\nu F}$.

5. Results and discussion

Results are presented for the differential resistivity $\rho_{xx}$ both in the strong and weak magnetic field regimes. As explained in section 3, the displacement mechanism is the physical origin of the nonlinear effects: impurity scattering induces a spatial hopping of the charge carriers between tilted Landau levels. However, there is an additional effect that has to be considered. Recent experimental results [16] suggest that in graphene a current dc bias produces a strong electron heating effect. It was found that the charge carrier’s effective temperature depends linearly on the bias current as $T \sim T_0 + \alpha I_{dc}$. The parameter $\alpha$ was estimated as $\alpha = 3 \text{ K/(A m)}^{-1}$, for bias currents $2 \sim 20 \text{ A m}^{-1}$. For stronger bias currents the relation is no longer linear [25].
Except for the last graph, in all the other examples we shall consider short-range scatterers, with the impurity potential given by equation (4). The value for $V_0$ is obtained in the strong screening limit $\alpha r_s \gg 1$ of the Coulomb interaction in equation (3): $V(q) \approx V_0 = \pi \hbar v_F / 2 k_F$.

The present formalism requires two conditions that must be fulfilled. (i) First, we recall that at low energy scales the dynamic of quasiparticles in graphene is described by two independent $2 + 1$ dimensional Dirac equations; the equations remain decoupled as long as the magnetic field is not too strong. The scale of the energy separation between the $K$ and the $K'$ points is of order $h v_F / a$, ($a$ is the lattice spacing) that is required to be much larger than the scale of the energy eigenvalues in equation (10), leading to the condition $\mu (B/\text{Tesla}) \ll \hbar v_F / a^2$, here $\mu$ refers to the LL index. (ii) The impurity potential is treated as a perturbation. This condition is satisfied if the energy eigenvalues in equation (10) are much larger that the potential contribution in equation (2), the former contribution can be estimated as $V \sim n_c V_0 / \gamma = n_c \hbar v_F / 2 k_F$. For typical parameters used in this paper, the two restrictions for the magnetic field and the LL-index are estimated as: $10^{-3} \ll n_c (B/\text{Tesla}) / \gamma \ll 10^3$, conditions that are widely satisfied for the situations that are analyzed in the present work.

First, we consider the nonlinear effects on the Shubnikov–de-Hass (SdH) oscillations in the strong magnetic field region: $B \sim 5 - 10$ T. The parameters are selected corresponding to typical experiments in graphene samples: $v_F = 1 \times 10^6$ m s$^{-1}$, electron density $n_c = 3 \times 10^{12}$ cm$^{-2}$, and a base temperature $T_0 = 2$ K. The width of the LL depends on the broadening parameter $\delta$, that is calculated from a self-consistent Born approximation calculation as [26, 27]

$$\delta = \sqrt{2} \frac{\hbar v_F}{E_F}$$

In a typical graphene sample with a mobility $\mu = 10^4$ cm$^2$ V$^{-1}$ s$^{-1}$, the dimensionless parameter is estimated as $\delta \approx 0.14$. In figure 1 we present the longitudinal resistivity $\rho_{xx}(B)$ for various values of the dc current $J_{dc}$. As observed in the experiments [16], the amplitude of the SdH oscillations decreases with increasing temperature. Figure 2 displays the SdH oscillation for the differential resistivity as a function of $B$ and for the same values of $J_{dc}$. Besides the damping effect on the oscillation’s amplitude, it is observed that as $J_{dc}$ increases the maxima of the oscillation evolve into minima, and vice versa. These nonlinear effects on SdH oscillations originate mainly from the current heating effect. If we compare the results of $\rho_{xx}(J_{dc})$ without the heating effect ($\alpha = 0$), with the result of the linear response approximation $\rho_{xx}(0)$ we find essentially the same results, confirming the fact that the displacement mechanism has a negligible effect on the SdH oscillations.

We now explore the possibility of detecting nonlinear oscillations in the low magnetic region. The study of ultra clean semiconductor samples 2DEG has shown that dc current excitation leads to Hall field-induced resistance oscillations [11, 12, 28, 29]. In order to simulate a high mobility graphene sample the value of the LL width in equation (24) is reduced by an order of magnitude, by selecting $\delta \approx 0.014$; which would represent a sample with a high mobility $\mu = 10^5$ cm$^2$ V$^{-1}$ s$^{-1}$. Furthermore we consider an smaller value of the electron density $n_c = 1 \times 10^{11}$ cm$^{-2}$.

Figure 3 displays the differential resistivity $\rho_{xx}$ as a function of $B$. First we notice the results of the linear response solution, that produce a small monotonous decreasing curve. However, the two other curves obtained for $J_{dc} = 1.75$ A m$^{-1}$ and $J_{dc} = 3.5$ A m$^{-1}$, display strong differential magnetoresistance oscillations. The physical origin of the intense HIRO lies in the displacement mechanism, as explained below.

The oscillations are periodic in $1/B$ and are governed by the ratio $\varepsilon = \omega_H / \omega_C$, where the expression for the Hall frequency $\omega_H$ and a simple explanation of the oscillation pattern follows from the analysis of the structure of $J_{dc}^{\text{res}}(E_H)$ in equation (18). For this purpose we consider the limit $\delta \to 0$, in equation (19) or equation (20), that leads to the resonance condition

$$\frac{\Delta E}{\hbar \nu_F} = \sqrt{\frac{\Delta n}{\nu_F} (\sqrt{n} - \sqrt{m}) - \beta \Delta k} = 0,$$

here we are considering levels well above the Dirac point, so all the Landau numbers $n$ and $m$ are positive. The momentum
transfer $\Delta k$ is equivalent to hopping or shifting the guiding center $\Delta Y = Y - Y' = l_\text{q}^* q_z$. An estimation of the momentum transfer $\Delta k$ can be obtained from the overlap of the wave functions between the $n$ and $m$ LL. The overlap, obtained in section 3 is proportional to the function $D_{mn}(\zeta)$ in equation (B.4), where $\zeta = \frac{l_\text{q}^*}{\sqrt{2}} (q_x + iq_y)$. While this function is highly oscillatory, the envelope of the function can be approximated by $[30, 31]$

$$\sqrt{\frac{n!}{m!}} e^{-\zeta^2/2} L_n(-\zeta^2) \rightarrow \frac{e^{-\zeta^2/2}}{\sqrt{n! + m!}},$$

with a maximum at $\zeta_{\text{max}} = \sqrt{m + n}$. Hence, the maximum overlap corresponds to a momentum transfer of $\Delta k = \frac{\sqrt{2(m + n)}}{l_\text{q}^*}$. Inserting this result in equation (25) the following selection rule is obtained

$$\beta(\sqrt{n + m}) (\sqrt{n} + \sqrt{m}) = n - m.$$  

In HIRO we are interested in large LL; so we consider contributions from states that are close to the Fermi level, that is far away from the Dirac point, so $n_F \approx \pi l_\text{q}^2 n/2$. As we assume that $n_F \gg 1$ we write $m = n_F + m'$ and $n = n_F + n'$, with $(m', n') \ll n_F$. So the resonance condition (27) becomes

$$\sqrt{2} \beta n_F l_\text{q}^2 \approx N,$$

with $N = m' - n'$. We recall that, even though the linear spectrum in graphene implies zero rest mass, their cyclotron mass is not zero, but it is given as $[13] m^* = \frac{h}{\sqrt{2m_e \epsilon}}$. Hence, using the definition of the cyclotron frequency, $\omega_\text{c} \equiv \frac{eB}{m^* c}$, the resonance condition can be written as

$$\frac{2\pi}{\sqrt{n_e \epsilon}} \equiv \omega_\text{H} = N \omega_\text{c}.$$  

The HIRO oscillations are periodic in $1/B$, according to the condition (29) resistance maxima are expected at integer values of the dimensionless parameter $\epsilon = \omega_\text{H}/\omega_\text{c}$. The Hall frequency $\omega_\text{H}$ can be interpreted in terms of the energy

$$h\omega_\text{H} = e \frac{2}{R} E_\text{H}$$

associated with the Hall voltage drop across the cyclotron diameter: $E_\text{H}$ is the Hall field as given in equation (22) and $R$ is the cyclotron radius at the Fermi level. To confirm these observations, in figure 4 the differential resistivity is plotted as a function of $\epsilon$. Magnetoresistance oscillations are clearly observed up to the fourth resonance. The positions of the peak maxima appear in good agreement with the conditions in (29).

It was mentioned in section (3) that according to the interpretation in the $S_\text{At}$ system, the inter LL transitions are induced by the time dependent potential equation (9) that oscillates with the frequency $\omega_\text{q} = \sqrt{g \beta q_\text{c}}$. The transition matrix element peaks at the momentum transfer $q_\text{c} = q_\text{F}^* = 2 \sqrt{g^*} / l_\text{q}^*$. Hence we find that the leading oscillation frequency of the periodic potential, is precisely the Hall frequency $\omega_\text{H} \equiv \omega_\text{H}$, given in equation (29).

It is also interesting to observe that from equation (25) the hopping distance for a resonant transition between LL is expressed as

$$\Delta Y = \frac{\sqrt{2} \gamma n_j \sqrt{J_{\text{dc}}}}{J_{\text{dc}}} (\sqrt{n} - \sqrt{m}).$$  

(30)

that depends both on the magnetic field and $J_{\text{dc}}$. This equation can be interpreted as a selection rule that determines $\Delta Y$ for arbitrary transition between LLs. If we now consider the limit of large LL levels, the selection rule reduces to

$$\Delta Y = \frac{\gamma e \sqrt{\overline{n_f} \epsilon}}{\sqrt{2} \sqrt{J_{\text{dc}}}} N.$$  

(31)

which does not depend on $B$. This result is similar to the selection rule $\Delta Y = N (\epsilon n_j)(m' J_{\text{dc}})$ found by Yang et al [11] in their study of HIRO in 2DEG.

To acquire a deeper insight on the experimental parameters required to obtain the HIRO, in figure 5 the differential resistance is plotted as a function of $1/B$ for three values of the dimensionless broadening parameters $\delta_1 = 0.014$, $\delta_2 = 0.028$, and $\delta_3 = 0.042$, that correspond to a reduction of the mobility by...
a factor of 2 and 3. Clearly, the HIRO amplitudes are strongly suppressed as \( \delta \) increases. Although in the first case the HIROs are clearly identified, a small reduction of the mobility sweeps most of the oscillation. Hence, a well defined mobility threshold is expected. Producing ultra clean graphene samples with a mobility that exceeds the threshold value would probably lead to experimental observation of HIROs.

We now compare the effects on HIROs of short-range scatterers in the case of screened Coulomb scatterers. Figure 6 displays the numerical results for the differential resistivity as a function of \( \varepsilon \) for three values of the effective fine structure constant \( \alpha_g \). The first value (a) \((\alpha_g \rightarrow \infty)\) corresponds to short-range scatterers, whereas the two other cases are for screened Coulomb scatterers: \( \alpha_g = 3 \) is the value for graphene in vacuum, and \( \alpha_g = 1 \) is for a graphene sheet placed on a SiO\(_2\). The same Magnetoresistance oscillations pattern are observed in all the examples, with the same phases. Notice that the positions of the maxima in all cases agree with the condition in equation (29). The main difference between the two types of impurity scatterers is observed in the amplitude of the oscillations. We find that the differential resistivities evaluated with large-range impurity interactions are suppressed compared to the short-range case. The suppression increase as the value of \( \alpha_g \) is reduced and we approach the unscreened Coulomb limit.

6. Conclusion

We have presented a theoretical model for the nonlinear magnetotransport of graphene carriers. The response to a dc current bias is incorporated by the exact solution of the effective Dirac equation in crossed electric and magnetic fields, obtained by means of a Lorentz transformation to a system \( S_\Lambda \) in which the electric field vanishes [20]. We extend the method so the impurity scattering effects are taken into account, by a perturbative solution of the density matrix equation. The nonlinear current is computed by means of the migration center theory. The calculation is carried out within the \( S_\Lambda \) system, and is later transformed to the original \( S_1 \) frame. For this purpose we develop a covariant formulation of the migration center theory.

The analytical expression for the impurity assisted current equation (18), allows us to calculate the longitudinal resistivity and the longitudinal differential resistivity. Based on these results the quantum oscillations of nonlinear magnetoresistance that occur in response to a current bias are investigated. When applied to the analysis of the Shubnikov de Hass oscillations, it is observed that the strong \( J_{dc} \) produces a decay of the oscillations in the longitudinal resistivity figure 1. Whereas the differential resistivity plot, figure 2, shows a phase inversion in which the maxima of the oscillation evolve into minima. Both effects are in good agreement with the experimental observations [16]. It is demonstrated that these effects on SdH oscillations arise from the current heating effect, while the contribution from the displacement mechanism is negligible.

At the small magnetic field region, corresponding to large filling factors, magnetoresistance oscillation pattern emerge. Analogous to what has been observed in 2DEG, the magnetoresistance oscillations are induced by the intense electric field (HIRO). The nonlinear effects produce a strong magnification of the differential resistance, as well as an oscillatory behavior that is periodic in \( 1/B \). The oscillations are governed by the ratio \( \omega_{H}\omega_{HIRO} \). The origin of the HIRO lies on the displacement mechanism. The expression for the longitudinal current in equation (18) contains the main ingredients that explain the displacement mechanism. There are two possible interpretations, depending on which frame is visualized: (i) in \( S_\Lambda \), the relevant LL transitions are induced by the time dependent impurity potential, that oscillates with a distribution of frequencies \( \omega_q \) that peaks at the Hall frequency. Alternatively (ii) in \( S_1 \), the current oscillations appear as Landau–Zener tunneling transitions between tilted Landau levels. The hopping distance \( \Delta \gamma \) for the resonant transitions given in equation (31) satisfies \( \Delta \gamma = N \pi \omega_{HIRO} \), where \( N \) is an integer. It is argued that HIROs may be observable in ultra clean graphenes with mobilities above \( \mu = 10^6 \text{ cm}^2 \text{ V}^{-1} \text{ s}^{-1} \).

The formalism of this paper applies both for the cases of short-range and Coulomb randomly distributed scatterers. We find that the same magnetoresistance oscillations pattern for HIROs is observed independently of the nature of the scatterers, but the differential resistivities evaluated with large-range impurity interactions are suppressed compared to the short-range case. Finally we also mention that it will be interesting to look for the possibility of producing zero-differential resistance states, similar to those observed in 2DEG [11].

Appendix A. Covariant migration center theory

In this appendix, we work out the details of the covariant migration center formalism. First it is convenient to write the equation of motion in a manifestly covariant form

\[
\mathcal{H}\Psi \equiv (\sigma_\tau \gamma^0 \Pi_\tau + \gamma^0 V) \Psi = 0, \tag{A.1}
\]

where we recall that the Dirac matrices are select as \( \sigma_\tau = (\sigma_z, i\sigma_y, -i\sigma_x) \) [18]. The equation applies both in the \( S_1 \) frame (equation (1)), as well in the boosted frame \( S_\Lambda \) (equation (8)). To implement the Schwinger proper time method, suppose we seek the Green function \( G(x, x') \) solution of \( \mathcal{H}(x) G(x, x') = \delta^{(3)}(x - x') \). Hence, one considers \( \mathcal{H}(x) \) as a Hamiltonian that describes the proper time \( \tau \) evolution of the system

\[
\imath \hbar \frac{\partial \Psi}{\partial \tau} = \mathcal{H}\Psi. \tag{A.2}
\]

The Green function \( G(x, x') \) is obtained in terms of the unitary evolution operator \( U(x, x', \tau) \) by means of the relation

\[
G(x, x') = -\imath \int_{-\infty}^{\infty} U(x, x', \tau) \, d\tau. \tag{A.3}
\]

The evolution operator satisfies the equation

\[
\imath \hbar \frac{\partial U(x, x', \tau)}{\partial \tau} = \mathcal{H}(x) U(x, x', \tau). \tag{A.4}
\]

with the following boundary conditions: \( \lim_{\tau \to -\infty} U(x, x', \tau) = \delta^{(3)}(x, x') \) and \( \lim_{\tau \to +\infty} U(x, x', \tau) = 0 \). For the purposes of this paper, we are mainly interested in deducing covariant guiding center velocities. First we consider the Heisenberg equation of motion for the \( x^\mu \) coordinates, which gives
\[
\frac{dx^\nu}{dt} = i\hbar^{-1} [\mathcal{H}, x^\nu] = v_F \gamma x^\nu. \tag{A.5}
\]

The Dirac probability current is then written as \( j^\nu = \overline{\psi} \gamma^\nu \gamma^0 \psi = \psi^\dagger \gamma^\nu \gamma^0 \psi \). It is easily verified that the probability current transform, as a covariant 2 + 1-vector under the Lorentz-like transformations equations (6) and (7): \( \tilde{j}^\nu = \Delta^\nu j^\nu \). As explained in section 3 the electron coordinate \( x^\nu \) splits into the guiding center \( x^\alpha \) and the cyclotron or relative coordinate \( \eta^\nu \), i.e. \( x^\nu = x^\alpha + \eta^\nu \), where manifestly a covariant expression for the relative coordinate \( \eta_{\nu \lambda} \) is given in terms of the velocity operator \( \hat{\Pi}_\lambda \) and the electromagnetic tensor \( F_{\mu \nu} \) in equation (13); where \( F_{\mu \nu} \) is constructed from the constant fields: \( \frac{\mathbf{c}}{v_F} \mathbf{E} \) and \( \mathbf{B} \) as

\[
F_{\mu \nu} = \begin{pmatrix}
0 & c E_\nu & -c E_\mu & \gamma E_\mu \\
-c E_\nu & 0 & -B & 0 \\
-c E_\mu & B & 0 & 0 \\
\frac{c}{v_F} E_\nu & -\frac{c}{v_F} E_\mu & \frac{c}{v_F} & 0
\end{pmatrix} \tag{A.6}
\]

The quantity \( F_{\mu \nu} F^{\mu \nu} = F^2 \) is invariant under the Lorentz transformation. In the \( S_c \) frame the three-vector \( \eta^\nu \) is given by the expression in equation (14), whereas in the \( S_A \) frame it is given by usual result \( \eta^\nu = (c, eB) (0, -\hat{\Pi}_0, \hat{\Pi}_0) \). It can be readily demonstrated that \( \eta^\nu \) transforms as a covariant three-vector under the Lorentz transformation equations (6): \( \eta^\nu = M^\eta \eta^\nu \). With the aid of the Heisenberg equations of motion, and using the definition for \( \tilde{\eta}^\nu \), the relative proper velocity \( d \eta^\nu / d \tau \) can be obtained both in the \( S_c \) and in the \( S_A \) frames. In the boosted \( S_A \) frame we obtain:

\[
\frac{d\eta^\nu}{d\tau} = \left( 0, v_F \gamma - \frac{c}{v_F} \gamma_{\nu} \mathbf{e}_x \times \nabla \tilde{V} \right), \tag{A.7}
\]

where \( \mathbf{e}_x \) is a unitary vector in the direction of the magnetic field. On the other hand, \( d \eta^\nu / d \tau \) evaluated in the \( S_c \) yields the following result

\[
\frac{d\eta^\nu}{d\tau} = \left( -\beta \frac{d\eta^j}{d\tau}, \gamma_{\nu} \mathbf{e}_x \times (\nabla \tilde{V} - \mathbf{E}_j) \right), \tag{A.8}
\]

Notice that the temporal component \( d\eta^0 / d\tau \) is obtained in terms of spatial \( \alpha \)-component. It is a lengthy but straightforward exercise to demonstrate that the probability current defined in terms of the relative velocity \( \overline{\eta^0}(d\eta^0 / d\tau) \eta^0 \) transform as a covariant three-vector. Summarizing all these results, and taking into account that the relative coordinate is obtained as \( \eta^\nu = x^\nu - \eta^\nu \), it immediately follows that the current density probability \( j^\nu = \Psi^\dagger \gamma^\nu (dX^\nu / d\tau) \Psi \), is also a Lorentz three-vector. The guiding center velocity \( dX^\nu / d\tau \) in the \( S_c \) frame as given in equation (15) is used in the calculation of the impurity-induced current equation (18).

**Appendix B. Impurity-assisted transition rate**

In this appendix we present details concerning the derivation of the impurity-induced current in equation (18). In order to calculate the expectation value of the current density we need the time-dependent density matrix \( \rho(t) \). In the \( S_A \) frame the system dynamics is governed by the Dirac equation in (8). Accordingly, we assume that \( \rho(t) \) satisfies the von Neumann equation in (17). To the lowest order in the perturbation we linearize around the equilibrium density matrix \( \rho_0 = \sum \int df_n \langle n, \hat{k} \rangle \langle n, \hat{k} \rangle^\dagger \) where \( f_n \) is the Fermi distribution function. The first order deviation \( \Delta \rho = \rho - \rho_0 \) then obeys

\[
\frac{i \hbar}{\partial t} \frac{d\Delta \rho}{dt} = [H_0, \Delta \rho] + \left[ \tilde{V}, \rho_0 \right], \tag{B.1}
\]

where \( H_0 = v_F \mathbf{e} \cdot \hat{\Pi} \). The complete orthonormal set of eigenstates of \( H_0 \) represented by \( |\nu \rangle = |n, \hat{k} \rangle \) are explicitly given in equation (10). The expectation value of equation (B.1) can now be easily calculated in the \( |\nu \rangle \) base. Using the expression for the impurity potential given in equation (9), the differential equation for \( \langle \mu | \Delta \rho(t) | \nu \rangle \) can be solved with the initial condition \( \Delta \rho(t) = 0 \) as \( t \to -\infty \), giving the result

\[
\langle \mu | \Delta \rho(t) | \nu \rangle = \frac{1}{\gamma} (v^\dagger \delta_{\nu,\mu}) \int_i^\infty \frac{dq^2}{(2\pi)^2} \tilde{\xi}_{\nu} = \tilde{\xi}_{\mu} = v_F q \beta - i \delta / \gamma \exp(-i \mathbf{q} \mathbf{v}_F t), \tag{B.2}
\]

where \( \mu = (m, \hat{k}, \nu) \) and \( v = (n, \hat{k}, \nu) \), and \( M_{\mu \nu} \) denotes the matrix element \( M_{\mu \nu} = \langle \nu | M_{E}\Psi^\dagger \Psi | \mu \rangle \). The explicit contributions to \( M_{\mu \nu} \) can be calculated using the following result:

\[
\langle \mu | e^{i\mathbf{q} \mathbf{v}_F t} | \nu \rangle = \delta (\hat{k}_x - \hat{k}_x + q_y) D_{\mu \nu} (\xi), \tag{B.3}
\]

where \( D_{\mu \nu} \) are the matrix elements of the displacement operator given as

\[
D_{\mu \nu} (\xi) = \left\{ \begin{array}{ll}
\frac{N_1}{N_1^1} \frac{1}{\sqrt{2}} e^{-i \mathbf{q} \mathbf{v}_F t} L_{n', n} (\xi), \quad & \mathbf{q} \mathbf{v}_F t > 0, \\
\frac{N_1}{N_1^1} \frac{1}{\sqrt{2}} e^{i \mathbf{q} \mathbf{v}_F t} L_{n', n} (\xi), \quad & \mathbf{q} \mathbf{v}_F t < 0,
\end{array} \right. \tag{B.4}
\]

with \( \xi = \frac{l_B}{2} (q_x + iq_y) \). Including the matrix \( M \) in the calculation of \( M_{\mu \nu} \) we find \( M_{\mu \nu} = \delta (\hat{k}_x - \hat{k}_x + i \mathbf{q} \mathbf{v}_F t) \Theta_{\mu \nu} (\xi), \) where

\[
\Theta_{\mu \nu} = \left[ D_{\mu \nu \nu}^1 + \text{sgn}(nm) D_{\mu \nu \nu}^{1, -1} \right]
\]

\[
- \beta \text{sgn}(m) D_{\mu \nu \nu}^{1, -1} + \text{sgn}(n) D_{\mu \nu \nu}^{1, 1} \right]. \tag{B.5}
\]

Using the result for the density matrix given in equation (B.2) and the expression for the guiding center velocity in equation (15), we can proceed to compute the current density \( j^\nu \) according to the expression in equation (16). Once we take the average over the impurities the terms that have linear dependence on the impurity potential vanish, terms quadratic in \( V \) are evaluated using the following prescription for the average of randomly distributed impurities

\[
\frac{1}{S} \sum_i^{N_1} \sum_j^{N_2} \left[ i \left( q \cdot \hat{k}_x - q \cdot \hat{k}_x \right) \right] = (2\pi)^2 n_i \delta (q - \mathbf{q}), \tag{B.6}
\]

where \( S \) is the sample area and \( n_i = N_i / S \) is the impurity density. The evaluation of the density current requires to compute the sums over the complete set of quantum numbers \( \mu = (m, \hat{k}), \nu = (n, \hat{k}) \). The delta term in equation (B.3) allows for a trivial
evaluation of the sum over the $\tilde{k}_x$ numbers, the other sum yields
the LL degeneracy per unit area $(1/S) \sum_{\tilde{k}_x} = g_s = 1/(2\pi l^2)$.
Combining all the elements presented in the appendix leads to
a lengthy but straightforward evaluation of the impurity current
density as given in equation (18).

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