CONTRACTING THE WEIERSTRASS LOCUS TO A POINT

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Abstract. We construct an open substack $U \subset \mathcal{M}_{g,1}$ with the complement of codimension $\geq 2$ and a morphism from $U$ to a weighted projective stack, which sends the Weierstrass locus $W \cap U$ to a point, and maps $\mathcal{M}_{g,1} \setminus W$ isomorphically to its image. The construction uses alternative birational models of $\mathcal{M}_{g,1}$ and $\mathcal{M}_{g,2}$ from [8].

Introduction

Let $W \subset \mathcal{M}_{g,1}$ denote the locus in the moduli stack of smooth one-pointed curves of genus $g$, consisting of $(C, p)$ such that $p$ is a Weierstrass point on $C$, i.e., $h^1(gp) \neq 0$. It is well known that $W$ is an irreducible divisor. In this paper we construct a rational map from $\mathcal{M}_{g,1}$ to a proper DM-stack with projective coarse moduli space, which contracts $W$ to a single point and maps $\mathcal{M}_{g,1} \setminus W$ isomorphically to its image (see Theorem A below). This is partly motivated by the question whether the class of the closure of $W$ in $\mathcal{M}_{g,1}$ generates an extremal ray (we do not solve this; however, see Prop. 2.4.6, Rem. 2.4.7 and the discussion below). Note that for small $g$ some pointed Brill-Noether divisors were shown to generate extremal rays in the effective cone of $\mathcal{M}_{g,1}$ in [9], [5] and [6].

The construction involves certain moduli stacks studied in [8]. Namely, in [8] we introduced and studied the moduli stack of curves with marked points $(C, p_1, \ldots, p_n)$, where $C$ is a reduced projective curve of arithmetic genus $g$, such that $h^1(a_1p_1 + \ldots + a_np_n) = 0$ for fixed integer weights $a_i \geq 0$ such that $a_1 + \ldots + a_n = g$ (we assume that the marked points are smooth and distinct). We denote this stack by $\mathcal{U}_{\text{ns}}^{g,n}(a_1, \ldots, a_n)$. We showed that $\mathcal{U}_{\text{ns}}^{g,n}(a_1, \ldots, a_n)$ can be realized as a quotient of an affine scheme by a torus action and studied the related GIT picture which leads to interesting projective birational models of $\mathcal{M}_{g,n}$. In particular, for $n = 1$ and $a_1 = g$ there is a unique nonempty GIT quotient stack $\overline{\mathcal{U}}_{\text{ns},1}(g)$, obtained from $\mathcal{U}^{\text{ns}}_{g,1}(g)$ by deleting one point corresponding to the most singular cuspidal curve. Furthermore, $\overline{\mathcal{U}}^{\text{ns}}_{g,1}(g)$ is a closed substack in a weighted projective space (see Sec. 1.1 for details).

We start by considering the natural rational map

$$\text{for}_2 : \mathcal{U}^{\text{ns}}_{g,2}(g-1,1) \longrightarrow \overline{\mathcal{U}}^{\text{ns}}_{g,1}(g)$$

(0.0.1)

given by forgetting the second marked point (more precisely, the map for$_2$ is regular on a certain open substack which is dense in the component corresponding to smoothable curves). Our main technical result is that (0.0.1) is regular on the open substack of $(C, p_1, p_2)$ such that $h^1((g+1)p_1) = 0$, and that the divisor, defined by the condition $h^1(gp_1) \neq 0$, gets contracted to a point (see Prop. 1.2.2). Furthermore, we show that this point has trivial group of automorphisms. We derive from this the following result.

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Theorem A. Assume that $g \geq 2$. The natural open embedding of stacks

\[ \mathcal{M}_{g,1} \setminus \mathcal{W} \hookrightarrow \overline{\mathcal{U}}_{g,1}^{\text{is}}(g) \]

extends to a regular morphism

\[ \phi = \phi_g : U \to \overline{\mathcal{U}}_{g,1}^{\text{is}}(g), \]

for some open substack $U \subset \mathcal{M}_{g,1}$ containing $\mathcal{M}_{g,1} \setminus \mathcal{W}$ and such that $\mathcal{M}_{g,1} \setminus U$ has codimension $\geq 2$ in $\mathcal{M}_{g,1}$. Furthermore, $\phi$ contracts $U \cap \mathcal{W}$ to a single point, which has no nontrivial automorphisms.

More precisely, the open substack $U$ in the above Theorem consists of $(C, p)$ such that $h^1((g+1)p) = 0$ and $h^0((g-1)p) = 1$.

We study the case $g = 2$ in more detail. In this case we get a more precise result involving a certain modular compactification of $\mathcal{M}_{2,1}$.

Recall that Smyth introduced in [10] the notion of an extremal assignment, which is a rule associating to each stable curve of given arithmetic genus some of its irreducible components (this rule should be stable under degenerations). For each extremal assignment $Z$, Smyth considered the moduli stack $\overline{\mathcal{M}}_{g,n}(Z)$ of $Z$-stable curves, i.e., pointed curves $C$ for which there exists a stable curve $C'$ and a map of pointed curves $C' \to C$, contracting precisely the components of $C'$, assigned by $Z$, in a certain controlled way. In this paper we consider only one extremal assignment which associates to every stable curve all of its unmarked components (see [10, Ex. 1.12]), so when we say $Z$-stable we always mean this particular extremal assignment.

We prove that the map $\phi_2$ extends to a regular morphism of stacks

\[ \phi_2 : \overline{\mathcal{M}}_{2,1}(Z) \to \overline{\mathcal{U}}_{2,1}^{\text{is}}(2) \]

contracting the closure of $\mathcal{W}$ to one point (see Theorem 2.4.5). Furthermore, we identify the point $\phi_2(\mathcal{W})$ explicitly as a certain cuspidal curve $C_0$ (see Definition 2.2.2), and show that $\phi_2$ induces an isomorphism of the complement of $\mathcal{W}$ to the complement of $\phi_2(\mathcal{W})$.

We also prove that the natural rational map of the coarse moduli spaces $\overline{\mathcal{M}}_{2,1} \dasharrow \overline{\mathcal{U}}_{2,1}^{\text{is}}(2)$ is a birational contraction with the exceptional divisors $\overline{W}$ and $\Delta_1$ (see Proposition 2.4.6). One can expect that the rational map $\overline{\mathcal{M}}_{g,1} \dasharrow \overline{\mathcal{U}}_{g,1}^{\text{is}}(g)$ is still a birational contraction for $g > 2$ (see Remark 2.4.7 for further discussion).

In addition, in Sec. 2.1 we obtain an isomorphism

\[ \overline{\mathcal{U}}_{2,1}^{\text{is}}(2) \simeq \mathbb{P}(2, 3, 4, 5, 6), \]

where the right-hand side is the weighted projective stack.

Conventions. In Sec. 2.1 we work over $\mathbb{Z}[1/6]$. Everywhere else we work over $\mathbb{C}$. By a curve we mean a connected reduced projective curve. By the genus of a curve we always mean arithmetic genus. For DM-stacks whose notation involves calligraphic letters $\mathcal{M}$, $\mathcal{U}$ and $\mathcal{W}$, we denote their coarse moduli spaces by replacing these letters by $M$, $U$ and $W$. 
1. Rational maps for $2$ and $\phi$

1.1. Moduli spaces of curves with non-special divisors. We start by recalling some results from [8] about the stacks $U_{g,n}^{ns}(a)$, where $a = (a_1, \ldots, a_n)$ and $a_i$ are non-negative integers with $a_1 + \ldots + a_n = g$. We denote by $\tilde{U}_{g,n}^{ns}(a)$ the $G_m^n$-torsor over $U_{g,n}^{ns}(a)$, corresponding to choices of nonzero tangent vectors at the marked points. It is proved in [8] that $\tilde{U}_{g,n}^{ns}(a)$ is an affine scheme of finite type. In this paper we only need the case when all $a_i$ are positive, so we assume this is the case.

The key result we will use is that for each $i = 1, \ldots, n$, and each $(C, p_1, \ldots, p_n, v_1, \ldots, v_n)$ in $\tilde{U}_{g,n}^{ns}(a)$ (where $v_i$ is a nonzero tangent vector at $p_i$), there is a canonical formal parameter $t_i$ on $C$ at $p_i$, such that $\langle v_i, dt_i \rangle = 1$, which is defined as follows. Given a formal parameter $t_i$, for each $m > a_i$ there is unique, up to adding a constant, rational function $f_i[-m] \in H^0(C, \mathcal{O}(mp_i + \sum_{j \neq i} a_j p_j))$ with the Laurent expansion in $t_i$ of the form

$$f_i[-m] = t_i^{-m} + \sum_{q \geq -a_i} \alpha_i[-m, q] t_i^q. \quad (1.1.1)$$

The canonical parameter is uniquely characterized by the condition that $\alpha_i[-m, -a_i] = 0$ for every $m > a_i$. Using these formal parameters we can consider for every pair $(i, j)$ and $m > a_i$ the expansion of $f_i[-m]$ at $p_j$:

$$f_i[-m] = \sum_{q \geq -a_j} \alpha_{ij}[-m, q] t_i^q$$

(note that $\alpha_i[-m, q] = \alpha_{ii}[-m, q]$). Now we can view the coefficients $\alpha_{ij}[-m, q]$ as functions on $\tilde{U}_{g,n}^{ns}(a)$, where we fix the ambiguity in adding a constant to $f_i[-m]$ by requiring that $\alpha_i[-m, 0] = 0$. It follows from the results of [8] that these functions are all expressed in terms of a finite number of them, which gives a closed embedding of $\tilde{U}_{g,n}^{ns}(a)$ into an affine space.

The rescaling of the tangent vectors $(v_i)$ defines an action of $G_m^n$ on $\tilde{U}_{g,n}^{ns}(a)$, so that the weight of the function $\alpha_{ij}[-m, q]$ is $me_i + qe_j$, where $(e_i)$ is the standard basis in the character lattice of $G_m^n$.

There is a special point in $\tilde{U}_{g,n}^{ns}(a)$ which is a unique point stable under the action of $G_m^n$: it is the point where all the functions $\alpha_{ij}[-m, q]$ vanish, i.e., it corresponds to the origin in the ambient affine space. The underlying curve is the union of $n$ rational cuspidal curves $C_{\text{cusp}}(a_i)$, glued transversally at the cusp. Here $C_{\text{cusp}}(a)$ is the projective curve with the affine part given by $\text{Spec}(k \cdot 1 + x^{a+1}[x])$, with one smooth point at infinity (see [8, Sec. 2.1]).

In [8] we also studied the GIT picture for the $G_m^n$-action on $\tilde{U}_{g,n}^{ns}(a)$. In general we have stability conditions depending on a character $\chi$ of $G_m^n$. In the case $n = 1$, i.e., for $\widetilde{U}_{g,1}^{ns}(g)$ there is a unique nonempty stability condition, so that the unique unstable point in $\widetilde{U}_{g,1}^{ns}(g)$ is the origin, i.e., the point corresponding to the curve $C_{\text{cusp}}(g)$. We denote this point by $[C_{\text{cusp}}(g)]$. Then the functions $\alpha_{ij}[-m, q]$ identify the corresponding GIT quotient stack,

$$\overline{U}_{g,1}^{ns} := (\widetilde{U}_{g,1}^{ns}(g) \setminus [C_{\text{cusp}}(g)])/G_m.$$
with a closed substack in the weighted projective stack.

For two collection of weights as above, \( a \) and \( a' \), we denote by \( \tilde{U}_{g,n}^{ns}(a, a') \) the intersection of the stacks \( \tilde{U}_{g,n}^{ns}(a) \) and \( \tilde{U}_{g,n}^{ns}(a') \). In other words, we impose both conditions, \( h^1(\sum a_ip_i) = \alpha \) and \( h^1(\sum a_ip_i) = \alpha' \), on the marked points.

1.2. The forgetful map. The rational map (0.0.1) corresponds to a regular morphism

\[
\text{for}_2 : \tilde{U}_{g,2}^{ns}((g - 1, 1), (g, 0)) \to \tilde{U}_{g,1}^{ns}(g)
\]

which is given as the composition of the open embedding \( \tilde{U}_{g,2}^{ns}((g - 1, 1), (g, 0)) \rightarrow \tilde{U}_{g,2}^{ns}(g, 0) \) followed by the forgetful map

\[
\text{for}_2 : \tilde{U}_{g,2}^{ns}(g, 0) \to \tilde{U}_{g,1}^{ns}(g)
\]
defined in [8, Thm. A]. The latter map sends \((C, p_1, p_2, v_1, v_2)\), with \( C \) irreducible, to \((C, p_1, v_1)\) (if \( C \) is reducible then it gets replaced by a certain curve \( \overline{C} \), such that \( C \to \overline{C} \) is contraction of the component containing \( p_2 \)).

Let \( Z \subset \tilde{U}_{g,2}^{ns}((g - 1, 1), (g, 0)) \) be the closed subscheme given as the preimage of the origin under (1.2.1). Then there is a regular morphism

\[
\tilde{U}_{g,2}^{ns}((g - 1, 1), (g, 0)) \setminus Z \to \tilde{U}_{g,1}^{ns}(g)
\]

induced by (1.2.1). Note that \( Z \) consists of \((C, p_1, p_2, v_1, v_2)\) such that \((C, p_1)\) is the cuspidal curve \( C_{\text{cusp}}(g) \) (with the marked point at infinity).

Let us denote by

\[
\tilde{U}_{g,2}^{ns}((g - 1, 1), (g + 1, 0)) \subset \tilde{U}_{g,2}^{ns}(g - 1, 1)
\]

the open subset given by the condition \( h^1((g + 1)p_1) = 0 \). Let also

\[
\tilde{W} \subset \tilde{U}_{g,2}^{ns}((g - 1, 1), (g + 1, 0))
\]

denote the closed locus given by the condition \( h^1(g, 0) \neq 0 \), so that

\[
\tilde{U}_{g,2}^{ns}((g - 1, 1), (g, 0)) = \tilde{U}_{g,2}^{ns}((g - 1, 1), (g + 1, 0)) \setminus \tilde{W}.
\]

Recall that we have sections \( f_1[-m] \in H^0(C, \mathcal{O}(mp_1 + p_2)) \), where \( C \) is the universal curve over \( \tilde{U}_{g,2}^{ns}(g - 1, 1) \), for \( m \geq g \), with expansions at \( p_1 \) of the form (1.1.1) (with \( i = 1 \)) with \( \alpha_1[-m, -g + 1] = \alpha_1[-m, 0] = 0 \).

**Lemma 1.2.1.** Let us set \( \alpha = \alpha_{12}[-g, -1], \beta = \alpha_{12}[-g - 1, -1] \). Then the open subset

\[
\tilde{U}_{g,2}^{ns}((g - 1, 1), (g, 0)) \subset \tilde{U}_{g,2}^{ns}(g - 1, 1)
\]

is given by the condition \( \alpha \neq 0 \). Similarly, the open subset

\[
\tilde{U}_{g,2}^{ns}((g - 1, 1), (g + 1, 0)) \subset \tilde{U}_{g,2}^{ns}(g - 1, 1)
\]

is the locus where either \( \alpha \neq 0 \) or \( \beta \neq 0 \).

**Proof.** Recall that the open subset \( \tilde{U}_{g,2}^{ns}((g - 1, 1), (g, 0)) \) is characterized by the condition \( h^1(gp_1) = 0 \). Since \( h^1(gp_1 + p_2) = 0 \), the long exact sequence of cohomology associated with the exact sequence of sheaves

\[
0 \to \mathcal{O}(gp_1) \to \mathcal{O}(gp_1 + p_2) \to \mathcal{O}(p_2)/\mathcal{O} \to 0
\]
shows that \( h^1(gp_1) \neq 0 \) precisely for those curves for which \( f_1[-g] \) is regular at \( p_2 \). But this is equivalent to the vanishing of \( \alpha \), since \( \alpha \) is the coefficient of \( t_2^{-1} \) in the expansion of \( f_1[-g] \) at \( p_2 \).

The case of \( \tilde{U}^{\text{ns}}_{g,2}((g-1,1),(g+1,0)) \) is similar: now we consider the exact sequence

\[
0 \to \mathcal{O}((g+1)p_1) \to \mathcal{O}((g+1)p_1+p_2) \to \mathcal{O}(p_2)/\mathcal{O} \to 0
\]

which shows that \( h^1((g+1)p_1) \neq 0 \) when both \( f_1[-g] \) and \( f_1[-g-1] \) are regular at \( p_2 \), i.e., both \( \alpha \) and \( \beta \) vanish.

The following Proposition is a crucial step in proving Theorem A.

**Proposition 1.2.2.** The subset \( Z \) is closed in \( \tilde{U}^{\text{ns}}_{g,2}((g-1,1),(g+1,0)) \), and we have \( Z \cap \tilde{W} = \emptyset \). There exists a regular morphism

\[
\tilde{\text{for}}_2 : \tilde{U}^{\text{ns}}_{g,2}((g-1,1),(g+1,0)) \setminus Z \to \tilde{U}^{\text{ns}}_{g,1}(g),
\]

extending the morphism (1.2.2) and sending \( \tilde{W} \) to a point. Furthermore, this point has no nontrivial automorphisms.

**Proof.** Let \( C' \) denote the universal curve over the open subset \( \tilde{U}^{\text{ns}}_{g,2}((g-1,1),(g,0)) \). To calculate explicitly the map (1.2.1), we need to find the sections \( f[-m] \in H^0(C',\mathcal{O}(mp_1)) \), for \( m \geq g + 1 \), and a modified formal parameter \( u \) at \( p_1 \), such that \( f[-m] \) would have expansions of the form

\[
f[-m] = u^{-m} + \alpha[-m, -g+1]u^{-g+1} + \alpha[-m, -g+2]u^{-g+2} + \ldots,
\]

where \( \alpha[-m, q] \) are some rational expressions of the coordinates on \( \tilde{U}^{\text{ns}}_{g,2}(g-1,1) \) with only powers of \( \alpha \) in the denominator.

As the first approximation let us set for \( m \geq g + 1 \),

\[
\tilde{f}[-m] = f_1[-m] - \frac{\alpha_{12}[-m,-1]}{\alpha} f_1[-g].
\]

The constant is chosen so that the poles at \( p_2 \) cancel out, so we have \( \tilde{f}[-m] \in H^0(\mathcal{O}(mp_1)) \), while the expansion of \( \tilde{f}[-m] \) at \( p_1 \) has form

\[
\tilde{f}[-m] = t_1^{-m} - \frac{\alpha_{12}[-m,-1]}{\alpha} t_1^{-g} + \ldots,
\]

where \( t_1 \) is the canonical parameter at \( p_1 \) on \( C' \).

Now we need to change the canonical parameter to \( u = t_1 + c_1 t_1^2 + \ldots \), and to add to each \( \tilde{f}[-m] \) a linear combination of \( \tilde{f}[-m'] \) with \( m' < m \), to get the expansions of the required form (1.2.4). We want to know only the highest order polar parts of the functions \( \alpha[-m, q] \), i.e., those with the highest power of \( \alpha \) (prescribed below) in the denominator, so we introduce the following filtration \( F_n \) on the space of formal Laurent series in \( t_1 \) with coefficients in \( R = \mathcal{O}(\tilde{U}^{\text{ns}}_{g,2}((g-1,1),(g,0))) \). By definition, a Laurent series belongs to \( F_n \) if it can be written in the form \( \sum_i a_i \alpha^{-i-n} t_1 \), where each \( a_i \) extends to a regular function on \( \tilde{U}^{\text{ns}}_{g,2}(g-1,1) \).
It will be enough for us to keep track only of \( f[-m] \mod F_{m-1} \). It is easy to see that the change of variables \( t_1 \mapsto t_1 + c_1 t_1^2 + c_2 t_1^3 + \ldots \), where for each \( i \), \( \alpha^i c_i \) extends to a regular function on \( \tilde{U}_{g,2}^{ns}(g-1,1) \), preserves the filtration \((F_n)\). Since to go from \( t_1 \) to \( u \) we will only use the changes of variables of this form, it suffices for us to know that

\[
\tilde{f}[-g-1] \equiv t_1^{-g-1} - \lambda t_1^{-g} \mod F_g,
\]  

(1.2.5)

where \( \lambda = \frac{a_{12}-g-1,-1}{a} = \frac{\beta}{\alpha} \), while

\[
\tilde{f}[-m] \equiv t_1^{-m} \mod F_{m-1} \text{ for } m > g+1.
\]  

(1.2.6)

We claim that there exist rational constants \((r_{m,j})\), \( 1 \leq j < m - g \), and \((r_i)\), \( i \geq 1 \), such that

\[
f[-m] \equiv \tilde{f}[-m] + \sum_{1 \leq j < m-g} r_{m,j} \lambda^j \tilde{f}[-m+j] \mod F_{m-1},
\]  

(1.2.7)

for each \( m \geq g + 1 \), and

\[
t_1 \equiv u + r_1 \lambda u^2 + r_2 \lambda^2 u^3 + \ldots \mod F_{-2}.
\]  

(1.2.8)

Namely, we prove by induction on \( n \geq 1 \) that (1.2.7) holds for all \( m \) with \( m \leq g + n \), and that the required relation between \( t_1 \) and \( u \) holds modulo \( t_1^n R[[t_1]] + F_{-2} \).

Let us recall the recursive construction of \((f[-g-n])\) and of formal parameters \( u_n \) such that \( u_n \equiv u \mod t_1^{n+1} R[[t_1]] \), where \( u \) is the canonical parameter (cf. \([4, \text{Lem. 4.1.3}]\)). For \( n = 1 \) we have \( f[-g-1] = \tilde{f}[-g-1] \) and \( u_1 = t_1 \). Assume \( f[-g-n'] \) are already defined for \( n' < n \) and \( u_{n-1} \equiv u \mod t_1^n R[[t_1]] \) is known, so that

\[
f[-g-n'] \equiv u_{n-1}^{-g-n'} \mod t_1^{-g+1} R[[t_1]] \text{ for } n' < n - 1, \text{ while}
\]

\[
f[-g-n+1] \equiv u_{n-1}^{-g-n+1} + c \cdot u_{n-1}^{-g} \mod t_1^{-g+1} R[[t_1]].
\]  

(1.2.9)

Then we set \( u_n = u_{n-1} + \frac{c}{g+n-1} u_{n-1}^n \), the expansion of \( f[-g-n+1] \) in \( u_n \) will take form

\[
f[-g-n+1] \equiv u_n^{-g-n+1} \mod t_1^{-g+1} R[[t_1]],
\]

and the expansions of \( f[-g-n'] \) for all \( n' < n - 1 \) in \( u_n \) will still have the correct form. Now, if the expansion of \( \tilde{f}[-g-n] \) in \( u_n \) has form

\[
\tilde{f}[-g-n] = u_n^{-g-n} + p_1 u_n^{-g-n+1} + \ldots + p_{n-1} u_n^{-g-1} + \ldots,
\]  

(1.2.10)

then we set

\[
f[-g-n] = \tilde{f}[-g-n] - p_1 f[-g-n+1] - \ldots - p_{n-1} f[-g-1].
\]  

(1.2.11)

The induction assumption implies that the function \( c \) in (1.2.9) has the leading polar term \( r \lambda^{n-1} \) for some \( r \in \mathbb{Q} \), so the change of variables from \( u_{n-1} \) to \( u_n \) is of the right form, as discussed above. It follows that

\[
t_1 \equiv u_n + s_1 \lambda u_n^2 + \ldots + s_{n-1} \lambda^{n-1} u_n^{n} \mod t_1^{n+1} R[[t_1]] + F_{-2}
\]

for some \( s_i \in \mathbb{Q} \). Now from (1.2.6) we get that

\[
\tilde{f}[-g-n] = (u_n + s_1 \lambda u_n^2 + \ldots + s_{n-1} \lambda^{n-1} u_n^n)^{-g-n} \mod t_1^{-g} R[[t_1]] + F_{g+n-1}.
\]
This implies that for \( i = 1, \ldots, n = 1 \), the leading polar term of the coefficient \( p_i \) in the expansion (1.2.10) is of the form \( a_i \lambda^i \), for \( a_i \in \mathbb{Q} \). Now (1.2.11) shows that (1.2.7) holds for \( m = g + n \). This finishes the proof of our claim.

Now combining (1.2.5)–(1.2.8), we get that for each \( m \geq g + 1 \) the expansion of \( f[-m] \) in the canonical parameter \( u \) has form

\[
f[-m] \equiv u^{-m} + \sum_{j \geq 1} s_{m,j} \lambda^{m-g+j} u^{-g+j} \mod F_{m-1},
\]

for some rational constants \((s_{m,j})\). In other words, the functions \( \alpha[-m, -g+j] \in R \), defining the map (1.2.1), have form

\[
\alpha[-m, -g+j] = s_{m,j} \lambda^{m-g+j} + \ldots
\]

where the omitted terms have smaller powers of \( \alpha \) in the denominator.

Finally, we need to know that not all \((s_{m,j})\) are zero, so let us compute \( s_{-g-1,-g+1} \) and \( s_{g-1,g+2} \) following the above procedure (we will need to look at two coordinates to prove that the point, which is the image of \( \mathcal{W} \), has no nontrivial automorphisms). Due to (1.2.5), the first change of variables is

\[
t_1 = u_2 - \frac{\lambda}{g+1} u_2^2 \mod F_{-2}.
\]

Then we get expansions

\[
f[-g-1] = \tilde{f}[-g-1] \equiv u_2^{-g-1} + \frac{2-g}{2(g+1)} \lambda^2 u_2^{-g+1} + \frac{-g^2 + g + 3}{3(g+1)^2} \lambda^3 u_2^{-g+2} \mod u_2^{-g+3} R[[u_2]] + F_g,
\]

\[
\tilde{f}[-g-2] \equiv u_2^{-g-2} + \frac{g+2}{g+1} \lambda u_2^{-g-1} + \frac{(g+2)(g+3)}{2(g+1)^2} \lambda^2 u_2^{-g} \mod u_2^{-g+1} R[[u_2]] + F_{g+1},
\]

\[
\tilde{f}[-g-3] \equiv u_2^{-g-3} + \frac{g+3}{g+1} \lambda u_2^{-g-2} + \frac{(g+3)(g+4)}{2(g+1)^2} \lambda^2 u_2^{-g-1} + \frac{(g+3)(g+4)(g+5)}{6(g+1)^3} \lambda^3 u_2^{-g} \mod u_2^{-g+1} R[[u_2]] + F_{g+2}.
\]

Hence, the coefficient of \( u_2^{-g} \) in \( f[-g-2] \mod F_{g+1} \) (which is the same as in \( \tilde{f}[-g-2] \mod F_{g+1} \)) is \( \frac{(g+2)(g+3)}{2(g+1)^2} \lambda^2 \). Thus, the second change of variables (defined so that the coefficient of \( u_3^{-g} \) in \( f[-g-2] \) is zero) is

\[
u_2 = u_3 + \frac{(g+2)(g+3)}{2(g+2)(g+1)^2} \lambda^2 u_3^2 \mod F_{-2},
\]

and we get the expansion

\[
f[-g-1] = u_3^{-g-1} - \frac{2g+1}{2(g+1)} \lambda^2 u_3^{-g+1} + \frac{-g^2 + g + 3}{3(g+1)^2} \lambda^3 u_3^{-g+2} \mod u_3^{-g+3} R[[u_3]] + F_g,
\]
which shows that
\[ s_{g+1,1} = \frac{2g + 1}{2(g+1)}. \]

Also, we see that the coefficient of \( u_3^g \) in the expansion of \( \tilde{f}[-g - 3] \mod F_{g+2} \) is equal to \(-\frac{(g+3)(g^2+3g-1)}{3(g+1)^3}\lambda^3\). This dictates that the next change of variables is
\[ u_3 = u_4 - \frac{(g^2 + 3g - 1)}{3(g + 1)^3}\lambda^3 u_4 \mod F_{-2}. \]

Finally, we get that the coefficient of \( u_4^{-g+2} \) in the expansion of \( f[-g - 1] \mod F_g \) is equal to
\[ \frac{-g^2 + g + 3}{3(g + 1)^2} \lambda^3 + \frac{(g^2 + 3g - 1)}{3(g + 1)^2} \lambda^3 = \frac{4g + 2}{3(g + 1)^2} \lambda^3, \]
and hence,
\[ s_{g+1,2} = \frac{4g + 2}{3(g + 1)^2}. \]

Now let us consider the modified map
\[ \alpha \cdot \tilde{\text{for}}_2 : \tilde{U}_{g,2}^{ns}(g-1,1), (g,0)) \to \tilde{U}_{g,1}^{ns}(g) : x \mapsto \alpha(x) \cdot \tilde{\text{for}}_2(x) \]
Since the weight of \( \alpha[-m, -g+j] \) is \( m - g + j \), the modified map sends \( x \) to the point in \( \tilde{U}_{g,1}^{ns}(g) \) with coordinates
\[ \alpha(x)^{m-g+j} \alpha[-m, -g+j](x) = s_{m,j} \beta(x)^{m-g+j} + \alpha(x) \cdot f_{m,j}(x), \tag{1.2.12} \]
where \( f_{m,j} \) are regular functions on \( \tilde{U}_{g,2}^{ns}(g-1,1) \). In particular, \( \alpha \cdot \tilde{\text{for}}_2 \) can be viewed as a regular map from \( \tilde{U}_{g,2}^{ns}(g-1,1) \).

Recall that by Lemma 1.2.1, the open subset \( \tilde{U}_{g,2}^{ns}(g-1,1), (g+1,0) \) is the locus where either \( \alpha \neq 0 \) or \( \beta \neq 0 \), and the locus \( \tilde{W} \) is given by \( \alpha = 0 \). Thus, (1.2.12) gives for \( x \in \tilde{W} \):
\[ \alpha \cdot \tilde{\text{for}}_2(x) = (s_{m,j} \beta(x)^{m-g+j}) = \beta(x) \cdot (s_{m,j}). \]
Furthermore, as we have seen above, the constants \( s_{g-1,1} \) and \( s_{g-1,2} \) are nonzero, so the corresponding coordinates in the above expression are also nonzero. Note also that the corresponding point of \( \tilde{U}_{g,1}^{ns}(g) \) is equal to \( (s_{m,j}) \), so it does not depend on \( x \).

Denoting by \( U_{\beta\neq0} \subset \tilde{U}_{g,2}^{ns}(g-1,1), (g+1,0) \) the open subset where \( \beta \neq 0 \), we get
\[ (\alpha \cdot \tilde{\text{for}}_2)^{-1}(0) \cap U_{\beta\neq0} = Z \cap U_{\beta\neq0}, \]
and so \( Z \cap U_{\beta\neq0} \) is closed in \( U_{\beta\neq0} \). Since, \( Z \) is closed in the open subset \( \alpha \neq 0 \), we derive that \( Z \) is closed in \( \tilde{U}_{g,2}^{ns}(g-1,1), (g+1,0) \).

We have a covering of \( \tilde{U}_{g,2}^{ns}(g-1,1), (g+1,0) \setminus Z \) by two open subsets: \( \tilde{U}_{g,2}^{ns}(g-1,1), (g,0) \setminus Z \) and \( U_{\beta\neq0} \). The required regular morphism (1.2.3) to \( \tilde{U}_{g,1}^{ns}(g) \) is induced by \( \tilde{\text{for}}_2 \) on \( \tilde{U}_{g,2}^{ns}(g-1,1), (g,0) \setminus Z \) and by \( \alpha \cdot \tilde{\text{for}}_2 \) on \( U_{\beta\neq0} \). As we have seen above, this morphism sends \( \tilde{W} \subset U_{\beta\neq0} \) to the point \( (s_{m,j}) \) of the weighted projective stack with two
Indeed, the only unstable point in $U$ since the condition $h_0(gp) = 1$ implies that $h_1((g + 1)p) = 0$ and $h_0((g - 1)p) = 1$.

Note also that we have an inclusion
\[ \mathcal{M}_{g,1} \setminus \mathcal{W} \subset U \]
since the condition $h_0(gp) = 1$ implies that $h_1((g + 1)p) = 0$ and $h_0((g - 1)p) = 1$.

Furthermore, the complement to $U$ is a proper closed subset in $\mathcal{W}$, so it has codimension $\geq 2$ in $\mathcal{M}_{g,1}$. In particular, $U \cap \mathcal{W}$ is dense in $\mathcal{W}$.

Note that we have a natural open inclusion
\[ \mathcal{M}_{g,1} \setminus \mathcal{W} \hookrightarrow \overline{\mathcal{U}}_{g,1}^{\text{ns}}(g). \] (1.3.1)
Indeed, the only unstable point in $\mathcal{U}_{g,1}^{\text{ns}}(g)$ corresponds to the singular curve $C_{\text{cusp}}(g)$. We are going to show that the above morphism extends to a regular morphism
\[ U \to \overline{\mathcal{U}}_{g,1}^{\text{ns}}(g), \]
such that $U \cap \mathcal{W}$ is mapped to a point.

Recall that by Proposition 1.2.2, we have a regular morphism
\[ \tilde{\phi}_2 : \tilde{\mathcal{U}}_{g,2}^{\text{ns}}((g - 1, 1), (g + 1, 0)) \setminus Z \to \overline{\mathcal{U}}_{g,1}^{\text{ns}}(g), \]
sending $\tilde{\mathcal{W}}$ to a point. Let $V \subset \tilde{\mathcal{U}}_{g,2}^{\text{ns}}((g - 1, 1), (g + 1, 0))$ be the open subset corresponding to smooth curves. Then $V \cap Z = \emptyset$ because for points of $Z$ the underlying curve is singular. Thus, the above morphism induces a regular morphism
\[ \tilde{\phi} : V \to \overline{\mathcal{U}}_{g,1}^{\text{ns}}(g), \] (1.3.2)
mapping $\tilde{\mathcal{W}} \cap V$ to a point.

Now we claim that the natural projection $V \to \mathcal{M}_{g,1}$ induces a smooth surjective morphism $V \to U$. Indeed, if $h_0((g - 1)p_1 + p_2) = 1$ then $h_0((g - 1)p_1) = 1$, so this projection factors through $U$. Conversely, if for $(C, p_1) \in \mathcal{M}_{g,1}$ one has $h_0((g - 1)p_1) = 1$ then for generic $p_2$ we will have $h_0((g - 1)p_1 + p_2) = 1$, hence the map $V \to U$ is surjective. It is smooth since $V$ is a $\mathbb{G}_m^2$-torsor over an open substack of a universal curve over $U$.

It remains to prove that the morphism (1.3.2) factors through a morphism $\phi : U \to \overline{\mathcal{U}}_{g,1}^{\text{ns}}(g)$ (it will then map $\mathcal{W} \cap U$ to a point, since (1.3.2) sends $\tilde{\mathcal{W}} \cap V$ to a point). Indeed, this is true if we restrict to the open subset $\mathcal{M}_{g,1} \setminus \mathcal{W}$, by the construction. Now let us set $T := V \times_U V$ and consider two morphisms
\[ f_1 = \tilde{\phi} \circ \pi_1, f_2 = \tilde{\phi} \circ \pi_2 : T \to \overline{\mathcal{U}}_{g,1}^{\text{ns}}(g), \]
where $\pi_1$ and $\pi_2$ are two projections to $V$. We know that these two maps agree on the open subset $\pi^{-1}(\mathcal{M}_{g,1} \setminus \mathcal{W})$, where $\pi$ is the projection $T = V \times_U V \to U$.

Note that the scheme $T$ parametrizes data $(C, p_1, p_2, p'_2, v_1, v_2, v'_2)$ such that $h_0((g - 1)p_1 + p_2) = h_0((g - 1)p_1 + p'_2) = 1$ and $h_1((g + 1)p_1) = 0$ (and $C$ smooth, $p_1 \neq p_2$, nonzero homogeneous coordinates, of weights 2 and 3. Hence, this point does not have nontrivial automorphisms. \[\text{□}\]
Let us work over Proposition 2.1.1. Let $U$ be the cartesian diagram

\[
\begin{array}{ccc}
T' & \longrightarrow & \overline{U}_{g,1}^{\text{ns}}(g) \\
\rho & \downarrow & \Delta \\
T & \longrightarrow & \overline{U}_{g,1}^{\text{ns}}(g) \times \overline{U}_{g,1}^{\text{ns}}(g)
\end{array}
\]

Since the stack $\overline{U}_{g,1}^{\text{ns}}(g)$ is separated, the vertical arrows are finite morphisms. Finally, we observe that a generic pointed curve $(C, p)$ in $\mathcal{M}_{g,1}$ does not have nontrivial automorphisms (note that in the case $g = 2$ this is true since we can take $p$ not to be a Weierstrass point). Hence, the preimages of points with trivial automorphisms in $\overline{U}_{g,1}^{\text{ns}}(g)$ under $f_1$ and $f_2$ are nonempty open subsets in $T$. Since $f_1$ and $f_2$ agree on a nonempty open subset, we deduce that there exists a nonempty open subset $W \subset T$ such that $\rho^{-1}(W) \to W$ is an isomorphism. Let $T'' \subset T'$ be an irreducible component of $T'$, containing $\rho^{-1}(W)$, with reduced scheme structure. Then $\rho|_{T''} : T'' \to T$ is a finite birational morphism. Since $T$ is smooth, we deduce that $\rho|_{T''}$ is an isomorphism. Hence, $\rho$ admits a section, and so we have $f_1 = f_2$, which means that the map (1.3.2) descends to a morphism from $U$. \hfill \Box

2. Curves of genus 2

2.1. Explicit identification of $\overline{U}^{\text{ns}}_{2,1}(2)$.

Proposition 2.1.1. Let us work over $\mathbb{Z}[1/6]$. One has an isomorphism of the moduli scheme $\overline{U}^{\text{ns}}_{2,1}(2)$ with the affine space $\mathbb{A}^5$ with coordinates $q_1, q_2, q_2, q_3, q_3$, so that the affine universal curve $C \setminus \{p\}$ is given by the following equations in the independent variables $f, h, k$:

\[
\begin{align*}
h^2 &= f k + q_1 h + 2q_2^2 + f(q_2 + q_2 f), \\
hk &= f(q_3 + q_3 f + f^2) - q_1 k + (q_2 + q_2 f)h + q_2(q_2 + q_2 f), \\
k^2 &= (q_3 + q_3 f + f^2)h + (q_2 + q_2 f)^2 - 2q_1(q_3 + q_3 f + f^2).
\end{align*}
\] (2.1.1)

The weights of the $\mathbb{G}_m$-action are:

\[
\deg(q_2) = 2, \quad \deg(q_3) = 3, \quad \deg(q_1) = 4, \quad \deg(q_2) = 5, \quad \deg(q_3) = 6.
\]

Hence, we get the identification of $\overline{U}^{\text{ns}}_{2,1}(2)$ with the weighted projective stack $\mathbb{P}(2, 3, 4, 5, 6)$.

Proof. This is proved using the same method as in [7, Thm. A] and [8, Thm. A]. Let $(C, p, v)$ be a point in $\overline{U}^{\text{ns}}_{2,1}(2)$. Since $h^1(2p) = 0$, we have $h^0(np) = n - 1$ for $n \geq 2$. Let $t$ be a formal parameter at $t$ compatible with the given tangent vector. We can find the elements $f \in H^0(C, \mathcal{O}(3p))$, $h \in H^0(C, \mathcal{O}(4p))$ and $k \in H^0(C, \mathcal{O}(5p))$ with the Laurent expansions

\[
\begin{align*}
f &= \frac{1}{t^3} + \ldots, \quad h &= \frac{1}{t^4} + \ldots, \quad k &= \frac{1}{t^5} + \ldots,
\end{align*}
\]

\[p_1 \neq p_2\). Thus, it is an open subset in a $\mathbb{G}_m^5$-torsor over the universal curve over $\mathcal{M}_{g,2}$ (via the projection to $(C, p_1, p_2, p_2)$), in particular, $T$ is smooth and irreducible.
where the omitted terms have poles of smaller order. Then the elements
\[ f^n, f^n h, f^n k, \quad \text{for } n \geq 0, \quad (2.1.2) \]
form a linear basis on \( H^0(C \setminus \{p\}, \mathcal{O}) \), so we can express \( h^2, hk \) and \( k^2 \) as their linear combinations. Taking into account the above Laurent expansion, we get relations of the form
\[
\begin{align*}
h^2 &= p_1(f)k + q_1(f)h + c_1(f), \\
hk &= p_2(f)k + q_2(f)h + c_2(f), \\
k^2 &= p_3(f)k + q_3(f)h + c_3(f),
\end{align*}
\]
where \( p_i, q_i, c_i \) are polynomials in \( f \) with the following restrictions:
\[
\begin{align*}
deg p_1 &= 1, \, \deg p_2 \leq 1, \, \deg p_3 \leq 1, \, \deg q_1 \leq 1, \, \deg q_2 \leq 1, \, \deg q_3 = 2, \\
\deg c_1 &= 2, \, \deg c_2 = 3, \, \deg c_3 \leq 3,
\end{align*}
\]
and the polynomials \( p_1, q_1 \) and \( c_2 \) are monic. Note that \( f \) is defined up to adding a constant, while \( h \) and \( k \) are defined up to the transformation
\[
(h, k) \mapsto (\tilde{h} = h + A(f), \tilde{k} = k + Bh + C(f)),
\]
where \( A \) and \( C \) are linear polynomials in \( f \) and \( B \) is a constant. It is easy to check that we can fix the ambiguity in the choice of \( h \) and \( k \) by requiring that \( p_3 = 0 \) and \( p_2 = -q_1 \) is a constant, i.e., does not have a linear term in \( f \). More precisely, we should set
\[
A = -\frac{q_1 + p_2}{3}, \quad B = \frac{1}{3}(q_1' - 2p_2'), \quad C = -\frac{p_3}{2} - \frac{B^2p_1}{2} - Bp_2 \quad (2.1.4)
\]
(here \( q_1' \) and \( p_2' \) are derivatives of the linear polynomials \( q_1 \) and \( p_2 \)). Note that here we use our assumption that \( 6 \) is invertible. Finally, we can fix the ambiguity in the choice of \( f \) by requiring that \( p_1(f) = f \).

Now the fact that the elements \((2.1.2)\) form a basis of \( H^0(C \setminus \{p\}, \mathcal{O}) \) is equivalent to the condition that the relations \((2.1.3)\) form a Gröbner basis in the ideal they generate (with respect to the degree reverse lexicographical order such that \( f < h < k \), \( \deg(f) = 3 \), \( \deg(h) = 4 \), \( \deg(k) = 5 \)). Applying the Buchberger’s Criterion (see [3, Thm. 15.8]) we compute that this condition is equivalent to the following expressions of \( c_1, c_2, c_3 \) in terms of the other variables (where in the second expression in each line we take into account the normalization \( p_3 = 0, \, p_2 = -q_1 \)):
\[
\begin{align*}
c_1 &= p_2^2 + p_1q_2 - q_1p_2 = 2q_1^2 + q_1q_2, \\
c_2 &= p_1q_3 - p_2q_2 = p_1q_3 + q_1q_2, \\
c_3 &= q_2^2 + p_2q_3 - q_1q_3 = q_2^2 - 2q_1q_3.
\end{align*}
\]
Thus, if we set
\[
q_2 = q_{2,0} + q_{2,1} f, \quad q_3 = q_{3,0} + q_{3,1} f + f^2,
\]
then we see that the constants \((q_1, q_{2,0}, q_{2,1}, q_{3,0}, q_{3,1})\) determine the curve \((C, p)\). The above process can be run in families and can be reversed (see the proofs of [7, Thm. A] and [8, Thm. A]), so this gives the required identification of our moduli space with \( \mathbb{A}^5 \). \( \Box \)
2.2. Special cuspidal curve $C_0$. Let $C_0$ denote the curve obtained from $\mathbb{P}^1$ by pinching the point 0 into a genus 2 cuspidal singular point, so that a regular function $f$ near 0 descends to $C_0$ if and only if the expansion of $f$ in the standard parameter $t$ has form
\begin{equation}
    f \equiv c_0 + c_2 \cdot t^2 \mod(t^4).
\end{equation}

Note that this condition depends on coordinates, i.e., the point $\infty \in C_0$ plays a special role. For example, the standard $\mathbb{G}_m$-action on $\mathbb{P}^1$, preserving 0 and $\infty$, descends to a $\mathbb{G}_m$-action on $C_0$. Also, note that $C_0 \setminus \{\infty\} = \text{Spec}(\mathbb{C}[t^2, t^5])$.

The next Lemma shows that if we equip $C_0$ with a smooth marked point $p \neq \infty$ then we get a point of $\overline{U}_{2,1}^{is}(2)$.

Lemma 2.2.1. Let $p \in C_0 \setminus \{0, \infty\}$. Then $h^0(C_0, \mathcal{O}(2p)) = 1$. On the other hand, for $p = \infty$ we have $h^0(C_0, \mathcal{O}(2p)) = 2$.

Proof. In the case $p \neq 0, \infty$ we can assume that $t(p) = 1$. Then $\mathcal{O}_{\mathbb{P}^1}(2p)$ is spanned by $1, \frac{1}{1-t}$ and $\frac{1}{(1-t)^2}$. Looking at the expansions at $t = 0$ we see that the only sections of $\mathcal{O}_{\mathbb{P}^1}(2p)$ satisfying (2.2.1) are constants.

In the case $p = \infty$ the functions $(1, t^2)$ give a basis of $H^0(C_0, \mathcal{O}(2p))$. $\square$

Definition 2.2.2. We denote by $[C_0]$ the point of $\overline{U}_{2,1}^{is}(2)$ corresponding to $(C_0, p)$, where $p \neq 0, \infty$.

2.3. Classification of singular irreducible curves of genus 2. Let $C$ be an irreducible curve of genus 2, and let $\rho : \tilde{C} \to C$ be the normalization. If $C$ is singular then the genus of $\tilde{C}$ is either 1 or 0.

If the genus of $\tilde{C}$ is 1 then $\text{coker}(\mathcal{O}_C \to \rho_* \mathcal{O}_{\tilde{C}})$ has length 1, so it is supported at one singular point $q \in C$. If $\rho^{-1}(q)$ contains two distinct points $q_1, q_2 \in C$ then $\rho$ factors through a morphism $C' \to C$, where $C'$ is the nodal curve obtained by gluing $q_1$ and $q_2$ on $\tilde{C}$. Since $C'$ has genus 2 we should have $C \simeq C'$. If $\rho^{-1}(q)$ is one point on $C$ then it is easy to see that $C$ has a simple cusp at $q$.

In the remaining case when $\tilde{C} = \mathbb{P}^1$ we have more possibilities. The length of the sheaf $\mathcal{F} := \text{coker}(\mathcal{O}_C \to \rho_* \mathcal{O}_{\tilde{C}})$ is now 2, so the support of $\mathcal{F}$ can consist of $\leq 2$ points.

Case I: support of $\mathcal{F}$ consists of two distinct points $q_1, q_2$. We have the following subcases.

Case Ia: $|\rho^{-1}(q_1)| > 1$ and $|\rho^{-1}(q_2)| > 1$. In this case the map $\rho$ factors through the nodal curve $C'$ obtained by gluing two pairs of distinct points in $\mathbb{P}^1$. Since the genus of $C'$ is 2, we should have $C \simeq C'$.

Case Ib: $|\rho^{-1}(q_1)| = 1$ and $|\rho^{-1}(q_2)| > 1$. In this case $\rho$ factors through the curve $C'$ obtained by gluing two pairs of distinct points in $\mathbb{P}^1$ and pinching one extra point to a simple cusp. Again, we have that the genus of $C'$ is 2, so $C \simeq C'$.

Case Ic: $|\rho^{-1}(q_1)| = |\rho^{-1}(q_2)| = 1$. In this case $C$ is obtained by pinching two points of $\mathbb{P}^1$ into simple cusps.

Case II: $\mathcal{F}$ is supported at one point $q$.

Case IIa: $|\rho^{-1}(q)| > 2$. In this case $\rho$ factors through the curve $C'$ obtained by gluing transversally 3 points on $\mathbb{P}^1$ into a single point (with the coordinate cross singularity). Since the genus of $C'$ is 2, we get $C \simeq C'$. 

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Case IIb: $|\rho^{-1}(q)| = 2$. Let $\rho^{-1}(q) = \{q_1, q_2\}$. Let $t$ be a generator of the maximal ideal $m_q \subset O_{C,q}$. Assume first that $t \in m_{q_1}^2$. Then $\rho$ factors through the curve $C'$ obtained from $\mathbb{P}^1$ by first pinching $q_1$ into a simple cusp and then gluing it transversally with the point $q_2$. Since $C'$ has genus 2, we have $C \simeq C'$. On the other hand, if $t$ maps to a generator of $m_{q_i}$ for $i = 1, 2$, then $\rho$ factors through the curve $C''$ obtained from $\mathbb{P}^1$ by gluing $q_1$ and $q_2$ into a tacnode singularity. Since such $C''$ has genus 2, we have $C \simeq C''$.

Case IIc: $|\rho^{-1}(q)| = 1$. In this case we can identify $C$ with $\mathbb{P}^1$ as a topological space, so that $O_C$ is a subsheaf of $O_{\mathbb{P}^1}$, which differs from it only at one point $q$, so that $m_{C,q} \subset m_{\mathbb{P}^1,q}^2$ is an embedding of codimension 1. We claim that there are two curves of this type, up to an isomorphism. If $m_{C,q} \subset m_{\mathbb{P}^1,q}^3$ then $m_{C,q} = m_{\mathbb{P}^1,q}^3$ and $C = C^{cusp}(2)$ (see Sec. 1.1).

Now assume that $m_{C,q} \not\subset m_{\mathbb{P}^1,q}^3$. Let $t$ be a formal parameter near $q$ on $\mathbb{P}^1$. Then $\hat{m}_{C,q}$ is a (non-unital) subalgebra in $t^2 \mathbb{C}[[t]]$ of codimension 1, and there exists an element $f \in \hat{m}_{C,q}$ such that $f \equiv t^2 \mod t^3 \mathbb{C}[[t]]$. Changing the formal parameter we can assume that $f = t^2$. There could not be an element $h \in \hat{m}_{C,q}$ such that $h \equiv t^3 \mod t^4 \mathbb{C}[[t]]$, since then we would have $\hat{m}_{C,q} = t^2 \mathbb{C}[[t]]$. Therefore,

$$\hat{m}_{C,q} = \mathbb{C} \cdot t^2 + t^4 \mathbb{C}[[t]].$$

Note that the subspace in the right-hand side depends only on $t \mod t^3 \mathbb{C}[[t]]$. Now we observe that any formal parameter at $q$, modulo $m_{\mathbb{P}^1,q}^3$, can be obtained from a unique regular function on $\mathbb{P}^1 \setminus \{p\}$, for some $p \neq q$. Using automorphisms of $\mathbb{P}^1$ we can make $q = 0$, $p = \infty$, so that $C$ is the curve $C_0$ defined before.

### 2.4. Comparison of stabilities for irreducible curves of genus 2.

**Proposition 2.4.1.** Let $C$ be an irreducible curve of genus 2, and let $p$ be a smooth point. Then $(C, p)$ $\mathcal{Z}$-stable if and only if $C$ is not of type IIc.

**Proof.** It is easy to see that a curve $C$ of type IIc is not $\mathcal{Z}$-stable. Indeed, if there is a contracting map $C' \rightarrow C$ then $C'$ would have a rational component with only two distinguished points, so it could not be stable. Assume now that $C$ is not of type IIc. If $(C, p)$ is nodal then it is stable (since $C$ is irreducible), hence it is $\mathcal{Z}$-stable.

Next, if $C$ is obtained by pinching a point on an irreducible nodal curve $E$ of genus 1 into a cusp, then there is a contraction $f : E \cup E' \rightarrow C$, where $E \cup E'$ is the stable curve with $E$ and $E'$ glued nodally at one point. Here the marked point is placed on $E$ and $f(E')$ is the cusp on $C$. This shows that $(C, p)$ is $\mathcal{Z}$-stable. Similarly, if $C$ is a rational curve with two cusps then there is a contraction to $C$ from $\mathbb{P}^1$ with two elliptic tails (that get contracted into cusps).

There remains two cases for $C$: IIa and IIb. In the case IIa we have a contraction to $C$ from the union of two $\mathbb{P}^1$’s, joined nodally at 2 points. In the case IIb there is a contraction to $C$ from the curve with an elliptic bridge. In other words, we consider the union $\mathbb{P}^1 \cup E$, where $E$ is an elliptic curve, $\mathbb{P}^1$ and $E$ are joined nodally at 2 points, so that there are no marked points on $E$. It is known (see [10, Ex. 2.5]) that there exists a contraction $\mathbb{P}^1 \cup E \rightarrow C$, mapping $E$ to the singular point, for both types of curves occurring in the case IIb.

**Corollary 2.4.2.** The stack $\overline{M}_{2,1}(\mathcal{Z})$ is smooth and irreducible.
Proof. The possible singular points that can appear in \( Z \)-stable curves of genus 2, other that nodes, are: a simple cusp, a tacnode, and a coordinate cross in 3-space. All of these have smooth versal deformation spaces and are smoothable, hence the assertion (see [11, Lem. 2.1]).

Using the classification from Sec. 2.3 we easily get the following codimension estimate.

Lemma 2.4.3. Away from a closed subset of codimension \( \geq 2 \), for every point \((C, p)\) in \( \overline{M}_{2,1}(Z) \) (resp., \( \overline{U}_{2,1}^{ss}(2) \)), \( C \) is either smooth, or a nodal curve with the normalization of genus 1.

Proof. Both \( \overline{M}_{2,1}(Z) \) and \( \overline{U}_{2,1}^{ss}(2) \) are irreducible of dimension 4. Now we just go through the strata described in Sec. 2.3 and see that they all have dimension \( \leq 2 \), except when \( C \) is either smooth or nodal with the normalization of genus 1.

We need one more simple observation.

Lemma 2.4.4. Let \( C \) be an irreducible curve of genus 2, and let \( p \in C \) be a smooth point. Then \( h^0(p) = 1 \) and \( h^1(3p) = 0 \).

Proof. First, if \( h^0(p) = 2 \) then we would get a degree 1 regular map \( C \rightarrow \mathbb{P}^1 \). Composing it with the normalization map \( \tilde{C} \rightarrow C \), we get that the normalization map is the inverse map \( \mathbb{P}^1 \rightarrow C \), which is impossible. Hence, \( h^0(p) = 1 \).

If \( h^1(2p) = 0 \) then we also have \( h^1(3p) = 0 \), so it is enough to consider the case \( h^1(2p) \neq 0 \), i.e., \( h^0(2p) = 2 \). Suppose that \( h^0(3p) = 3 \). Then we can choose \( f \in H^0(C, \mathcal{O}(2p)) \) and \( h \in H^0(C, \mathcal{O}(3p)) \) with the Laurent expansions \( f = \frac{1}{t^2} + \ldots \), \( h = \frac{1}{t^3} + \ldots \) at \( p \) (for some formal parameter \( t \) at \( p \)). Furthermore, there is a canonical choice of \( f \) and \( h \), such that the relation

\[
h^2 = f^3 + af + b
\]

holds for some constants \( a \) and \( b \). Then the algebra \( \mathcal{O}(C \setminus \{p\}) \) has the linear basis \((f^n), (hf^n)\), and is isomorphic to the algebra \( A = \mathbb{C}[h, f]/(h^2 - f^3 - af - b) \). Since \( C \) is irreducible, it is isomorphic to Proj of the Rees algebra of \( A \), which is a plane cubic, so we get that the arithmetic genus of \( C \) is equal to 1, which is a contradiction. This shows that \( h^0(3p) = 2 \), i.e., \( h^1(3p) = 0 \).

Theorem 2.4.5. Let \( \overline{W} \subset \overline{M}_{2,1}(Z) \) be the closure of the Weierstrass locus \( W \subset M_{2,1} \). Then \( \overline{W} \) coincides with the locus where \( h^1(2p) \neq 0 \). There is a regular morphism

\[
\phi_2 : \overline{M}_{2,1}(Z) \rightarrow \overline{U}_{2,1}^{ss}(2),
\]

such that \( \phi_2(\overline{W}) = [C_0] \) and \( \phi_2 \) induces an isomorphism

\[
\overline{M}_{2,1}(Z) \setminus \overline{W} \cong \overline{U}_{2,1}^{ss}(2) \setminus [C_0].
\]

Proof. First, we observe that every irreducible component of the locus \( h^1(2p) \neq 0 \) has codimension 1 in \( \overline{M}_{2,1}(Z) \) (recall that the latter stack is smooth and irreducible by Corollary 2.4.2). By Lemma 2.4.3, to see that this locus coincides with \( \overline{W} \), it is enough to see that the locus of \((C, p)\), such that \( C \) is nodal with normalization \( E \) of genus 1 and \( h^1(2p) \neq 0 \) has dimension 2 (and hence has codimension 2 in \( \overline{M}_{2,1}(Z) \)). But if \( C \) is obtained from \( E \)
by identifying points \( q_1 \neq q_2 \) then the condition that \( h^0(2p) = 2 \) implies the existence of a rational function on \( E \) with pole of order 2 at \( p \) and vanishing at both \( q_1 \) and \( q_2 \). In other words, we should have a linear equivalence \( 2p \sim q_1 + q_2 \). Thus, we have a finite number of choices for each \( (E, q_1, q_2) \), so the dimension is 2.

Next, let us denote by

\[ V^Z \subset \widetilde{U}_{2,2}^{\text{ns}}(1, 1) \]

the open substack consisting of \((C, p_1, p_2)\) such that \((C, p_1)\) is \(Z\)-stable (in particular, \(C\) is irreducible). Lemma 2.4.4 shows that every \((C, p)\) in \(\overline{M}_{2,1}(Z)\) satisfies \(h^0(p) = 1\) and \(h^1(3p) = 0\). This implies that

\[ V^Z \subset \widetilde{U}_{2,2}^{\text{ns}}((1, 1), (3, 0)) \]

and the projection \( V^Z \to \overline{M}_{2,1}(Z) \) is surjective. Furthermore, since the curve \([C^{\text{cusp}}(2)]\) is not \(Z\)-stable, we have the inclusion

\[ V^Z \subset \widetilde{U}_{2,2}^{\text{ns}}((1, 1), (3, 0)) \setminus Z. \]

Thus, the restriction of the map (2.1.3) gives us a regular morphism

\[ V^Z \to \overline{U}_{2,1}^{\text{ns}}(2), \tag{2.4.1} \]

contracting \(\overline{W} \) to a point.

Now, similarly to the proof of Theorem A we check that the morphism (2.4.1) factors through \(\overline{M}_{2,1}(Z)\). Note that to apply the same argument as in Theorem A we use the following facts: (i) \(\overline{M}_{2,1}(Z)\) is smooth (see Corollary 2.4.2); (ii) the projection \( V^Z \to \overline{M}_{2,1}(Z) \) is smooth (since \( p_2 \) varies in a smooth part of a curve); and (iii) \( V^Z \times_{\overline{M}_{2,1}(Z)} V^Z \) is irreducible, as a \(\mathbb{G}_m^3\)-torsor over the moduli stack of \((C, p_1, p_2, p')\) with \(C\) smoothable.

This gives us the required morphism \(\phi_2\) contracting to \(\overline{W} \) to some point in \(U_{2,1}^{\text{ns}}(2)\). On the other hand, by Proposition 2.4.1, the only point in \(U_{2,1}^{\text{ns}}(2)\), which is not \(Z\)-stable is \([C_0]\) (recall that by this we mean the pointed curve \((C_0, p)\), where \(p \neq 0, \infty\), see Lemma 2.2.1). Thus, the rational map \(\phi_2^{-1}\) is regular on \(U_{2,1}^{\text{ns}}(2) \setminus [C_0]\) (and sends \((C, p)\) to \((C, p)\)).

Also, the restriction of \(\phi_2\) to \(\overline{M}_{2,1}(Z) \setminus \overline{W}\), i.e., to the locus where \(h^0(2p) = 1\), is an open embedding sending \((C, p)\) to \((C, p)\). This implies that \(\phi_2(\overline{W}) = [C_0]\), and \(\phi_2\) induces an isomorphism of \(\overline{M}_{2,1}(Z) \setminus \overline{W}\) with \(U_{2,1}^{\text{ns}}(2) \setminus [C_0]\). \(\square\)

Let us consider the natural birational maps of the coarse moduli spaces

\[ \overline{M}_{2,1} \dashrightarrow \overline{M}_{2,1}(Z) \dashrightarrow U_{2,1}^{\text{ns}}(2). \]

Note that all these spaces are normal (for the last two this follows from Proposition 2.1.1 and Corollary 2.4.2). Note also that we only know that \(\overline{M}_{2,1}(Z)\) is a proper algebraic space.

Let \(\overline{W} \subset \overline{M}_{2,1}\) denote the closure of \(W\), and let \(\Delta_1 \subset \overline{M}_{2,1}\) be the boundary divisor, whose generic point corresponds to the union of two elliptic curves.

**Proposition 2.4.6.** The natural birational morphism \(f : \overline{M}_{2,1} \dashrightarrow \overline{M}_{2,1}(Z)\) (resp., \(g : \overline{M}_{2,1} \dashrightarrow U_{2,1}^{\text{ns}}(2)\)) is a birational contraction with the exceptional divisor \(\Delta_1\) (resp., exceptional divisors \(\Delta_1\) and \(\overline{W}\)).
Proof. Recall that to check that $f$ (resp., $g$) is a birational contraction we need to check that the exceptional locus $\text{Exc}(f^{-1})$ (resp., $\text{Exc}(g^{-1})$) has codimension $\geq 2$. But this immediately follows from Lemma 2.4.3. Next, the restriction of $f$ to the complement of $\Delta_1$ induces an isomorphism with the open subset in $\overline{M}_{2,1}(\mathcal{Z})$ consisting of $(C, p)$ with $C$ smooth or nodal, so we have an inclusion $\text{Exc}(f) \subset \Delta_1$. On the other hand, the generic point of $\Delta_1$ corresponds to the union of elliptic curves $E_1 \cup E_2$, with the marked point on $E_1$. Under the map $f$ this curve gets replaced by the cuspidal curve $\overline{E}_1$, so that we have a contraction $E_1 \cup E_2 \rightarrow \overline{E}_1$ sending the elliptic tail $E_2$ to the cusp. Since this map forgets the $j$-invariant of $E_2$, this means that $\Delta_1$ gets contracted by $f$. Now the fact that $\text{Exc}(g) = \Delta_1 \cup \overline{W}$ follows from Theorem 2.4.5. $\square$

Remark 2.4.7. Let $\overline{V}_{g,1}^{\text{ns}}(g) \subset \overline{U}_{g,1}^{\text{ns}}(g)$ be the irreducible component consisting of smoothable curves. Theorem A implies that the natural birational map

$$\overline{M}_{g,1} \dashrightarrow \overline{V}_{g,1}^{\text{ns}}(g)$$

contracts $\overline{W}$ to a point. Passing to the normalizations of the coarse moduli spaces we get the birational map $\phi : \overline{M}_{g,1} \dashrightarrow X$, where $X$ is a normal projective variety, contracting $\overline{W}$ to a point. It seems plausible that $\phi$ is a birational contraction (which would imply that $\overline{W}$ is an extremal divisor). To check this we would need to prove that $\text{Exc}(\phi^{-1})$ has codimension $\geq 2$. In other words, we would need to check that the locus in $\overline{V}_{g,1}^{\text{ns}}(g)$, consisting of unstable (i.e., non-nodal) curves, has codimension $\geq 2$. In the case $g = 2$ we have shown this in Lemma 2.4.3. Note that the fact that the class of $\overline{W}$ generates an extremal ray in $NE^1(\overline{M}_{g,1})$ is known for $g \leq 3$ and $g = 5$, by the works [9], [5] and [6].

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