INDEX THEORY AND DEFORMATIONS OF OPEN NONNEGATIVELY CURVED MANIFOLDS

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Abstract. We use an index-theoretic technique of Hitchin to show that the space of complete Riemannian metrics of nonnegative sectional curvature on certain open spin manifolds has nontrivial homotopy groups in infinitely many degrees. A new ingredient of independent interest is homotopy density of the subspace of metrics with cylindrical ends.

1. Introduction

A cheap way to deform a Riemannian metric $g$ on an $n$-manifold $M$ is to pull it back via diffeomorphisms of $M$. To simplify matters let us replace Diff($M$) by the subgroup of diffeomorphisms supported in a smoothly embedded $n$-disk in $M$. If $g$ enjoys some geometric property, one may ask whether the just described Diff($D^n, \partial$)-orbit map through $g$ is null-homotopic in the space of metrics with the property. The question was studied by N. Hitchin in [Hit74] and more recently in [CS13, CSS] in the setting when $M$ is a closed spin manifold and the property is invertability of the Dirac operator. These authors found order two elements in certain homotopy groups of Diff($D^n, \partial$) that remain nontrivial under the above orbit map in any Diff($M$)-invariant space of Riemannian metrics on $M$ with invertible Dirac operators. Nontriviality is detected by the $\alpha$-invariant, the $KO$-valued index of the Dirac operator.

Here we apply Hitchin’s method to (the compact-open smooth topology on) the space $R_{K \geq 0}(V)$ of complete metrics of nonnegative sectional curvature on an open connected manifold $V$. Any such $V$ is diffeomorphic to an open tubular neighborhood of a certain compact totally convex submanifold without boundary, called a soul, which is determined by the metric and the basepoint in $V$, see [CG72].

Given an element in a homotopy group of Diff($D^n, \partial$) with nontrivial $\alpha$-invariant, we push it to $R_{K \geq 0}(V)$ as in Hitchin’s method, assume it is null-homotopic, deform the homotopy so that the resulting family of metrics becomes cylindrical of positive scalar curvature on an end of $V$, double the metrics along a cylinder cross-section, and conclude that the homotopy on the double runs through metrics with invertible

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Dirac operators, which gives a contradiction. This strategy works under certain assumptions on $V$ resulting in the following theorem.

**Theorem 1.1.** Let $V$ be an open connected spin $n$-manifold with a complete metric $g$ of $K \geq 0$ whose soul is not flat. Suppose one of the following holds:

(i) $V$ is the product of $\mathbb{R}$ and a closed smooth manifold,

(ii) the normal sphere bundle to the soul of $(V, g)$ has no section, and for every metric in the path-component of $g$ in $\mathcal{R}_{K \geq 0}(V)$ the normal exponential map to the soul is a diffeomorphism.

If $m \geq n \geq 6$ and $m \equiv 0, 1 \pmod{8}$, then the homotopy group $\pi_{m-n}(\mathcal{R}_{K \geq 0}(V), g)$ has an element of order two.

Results of [GW01, GZ00, GZ11] and bundle-theoretic considerations provide a number of examples where Theorem 1.1 applies.

**Corollary 1.2.** Let $L$ be a closed spin positive-dimensional manifold of $K \geq 0$, and let $V$ be the total space of any of following bundles:

1. the trivial $\mathbb{R}$-bundle over $L$,
2. the tangent bundle to $S^k$ for even $k > 0$,
3. any nontrivial $\mathbb{R}^3$-bundle over such $S^4$, $S^5$, $S^7$,
4. any $\mathbb{R}^4$-bundle over $S^4$ with nonzero Euler class,
5. the tautological $\mathbb{R}$-bundle over $\mathbb{RP}^{2k}$ for odd $k$,
6. the tangent bundles to $\mathbb{CP}^k$ and $\mathbb{HP}^k$ for even $k > 0$,
7. the tautological quaternionic line bundle over $\mathbb{HP}^k$ with $k > 0$,
8. any nontrivial $\mathbb{R}^2$-bundle over $\mathbb{CP}^k$, $S^2 \times S^2$, or $\mathbb{CP}^k \# \pm \mathbb{CP}^k$ with the same 2nd Stiefel–Whitney as the base, where $k > 0$,
9. any non-spin $\mathbb{R}^3$-bundle over $\mathbb{CP}^2$ with no nowhere zero section,
10. the product of $L$ and any bundle in (2)–(9) above.

If $m \geq n = \dim(V) \geq 6$ and $m \equiv 0, 1 \pmod{8}$, then for any $g \in \mathcal{R}_{K \geq 0}(V)$ there is an order two element in $\pi_{m-n}(\mathcal{R}_{K \geq 0}(V), g)$.

A key step in the proof is to deform the metric to one with a cylindrical end. A Riemannian metric has a cylindrical end if the induced metric on the complement of a compact codimension zero submanifold is isometric to the product of a closed manifold and $[0, \infty)$. Let $\mathcal{C}_{K \geq 0}(V)$ denote the space of all metrics in $\mathcal{R}_{K \geq 0}(V)$ that have a cylindrical end. The two spaces of metrics coincide if $V$ has a metric with codimension one soul. L. Guijarro [Gui98] used a computation in [Kro79] to show that any open complete manifold $(V, g)$ of $K \geq 0$ admits a metric $h$ with a cylindrical end such that $g = h$ on a tubular neighborhood of the soul.
normal exponential map to the soul is a diffeomorphism, one can pick $h$ arbitrary close to $g$. A family version of this result holds if the normal bundle to a soul is sufficiently twisted:

**Theorem 1.3.** Let $V$ be an open $n$-manifold with a complete metric of $K \geq 0$ such that the normal sphere bundle to the soul of $(V, g)$ has no section and the normal exponential map to the soul is a diffeomorphism for every metric in $\mathcal{R}_{K \geq 0}(V)$. Then there is a homotopy $\rho_\varepsilon$ of self-maps of $\mathcal{R}_{K \geq 0}(V)$ such that $\rho_0$ is the identity and for all $g \in \mathcal{R}_{K \geq 0}(V)$ and $\varepsilon \in (0, 1]$ the metric $\rho_\varepsilon(g)$ lies in $\mathcal{C}_{K \geq 0}(V)$, and the $\rho_\varepsilon(g)$-distance from the soul to a cylindrical end is at most $1 + \frac{1}{\varepsilon}$.

Thus the homotopy $\rho_\varepsilon$ instantly pushes $\mathcal{R}_{K \geq 0}(V)$ into $\mathcal{C}_{K \geq 0}(V)$, in which case one says that the latter is *homotopy dense* in the former. That “the normal sphere bundle to the soul has no section” ensures that the soul is uniquely determined by the metric, and also depends continuously on the metric, see [BFK17, Theorem 2.1], which leads to continuity of $\rho_\varepsilon$.

Let us sketch the proof of Theorem 1.1. Fix an embedded $n$-disk in $M$ inside the $1$-neighborhood of the soul of $g$. Start with an order two element in the $k$th homotopy group of $\text{Diff}(D^n, \partial)$ that is detected by the $\alpha$-invariant. Represent the element by a map from the $k$-sphere, and push it to $\mathcal{R}_{K \geq 0}(V)$ via the $\text{Diff}(D^n, \partial)$-orbit map through $g$. Suppose arguing by contradiction that this singular $k$-sphere contracts in $\mathcal{R}_{K \geq 0}(V)$, i.e., extends to a map from $D^{k+1}$ giving a family of metrics $g_y$ parametrized by $y \in D^{k+1}$. Applying Theorem 1.3 we can arrange all metrics to have cylindrical ends that are within a definite distance to their souls. For $r \gg 1$ we get a family of $r$-neighborhoods $N_y^r$ of the souls of $g_y$ such that $\partial N_y^r$ lie in the cylindrical end of $N_y$, and in fact, $\partial N_y^r$ will be a cylinder cross-section, so that the induced metric on $\partial N_y^r$ has $K \geq 0$. Contractibility of the disk allows to find the ambient isotopies $\phi_y$ with compact support moving $N_y$ to $N_y$. A further ambient isotopy arranges that the pullback metrics $\phi_y^* g_y$ have the same product structure near $\partial N_y$, i.e., they are of the form $dr^2 + b_y$ where $r$ is the distance to $\partial N_y$. We refer to the cylindrical region outside $N_y$ as the “neck”. We wish to modify the metric on the neck to make it of positive scalar curvature. A result of C. Böhm and B. Wilking [BW07] implies that the Ricci flow on a closed non-flat manifold instantly turns a metric of $K \geq 0$ into a metric of positive scalar curvature. Applying Ricci flow to $b_y$ with running time depending of $r$ we get a family of metrics that agrees with $dr^2 + b_y$ for small $r$ and then has positive Ricci curvature on the fibers of $r$. Since isotopy implies concordance in the positive scalar curvature category, we can reparametrize the metric to make it of positive scalar curvature on the neck. Choosing the neck long enough we can arrange that its union $U_y$ with $N_y$ contains the support of every $\phi_y$. Pulling back the metrics via $\phi_y^{-1}$ we get a family of metrics on $U_y$ of nonnegative scalar curvature that is not identically zero. Near $\partial U_y$ the metrics are all equal, so they extend to the double of $U_y$. The result is a family of metrics parametrized by $y$ that has nonnegative scalar curvature that is not identically zero. It is easy to see that double is spin if and only if $V$ is spin,
so the double has invertible Dirac operator. Since the given singular $k$-sphere has nonzero $\alpha$-invariant, it cannot contract through such metrics, which is the desired contradiction.

Let us briefly review the previous works on topological properties of $R_{K \geq 0}(V)$. The results in [KPT05, BKS11, BKS15, Otta, Ottb, DKT, GAZ] give various examples where $R_{K \geq 0}(V)$, or even its Diff($V$)-quotient $\mathcal{M}_{K \geq 0}(V)$, has infinitely many path-components. The geometric ingredient is that metrics in the same path-component have ambiently isotopic (and hence diffeomorphic) souls. Examples where $R_{K \geq 0}(V)$ is shown to have finitely many nonzero rational higher homotopy groups are given in [BFK17]. By contrast, Theorem 1.1 yields order two elements of homotopy groups in infinitely many degrees. Results in [TW] consider the case when $V$ has a codimension one soul with a torus factor, and uses it to give examples where $\mathcal{M}_{K \geq 0}(V)$ has nontrivial rational homotopy and cohomology. Finally, [BH15, BH16, BB18] determines the homeomorphism type of $R_{K \geq 0}(\mathbb{R}^2)$, and also studies connectedness properties of $\mathcal{M}_{K \geq 0}(\mathbb{R}^2)$. There is a substantial recent literature on deformations of metrics subject to other curvature conditions, such as positive scalar curvature, and as a starting point the reader could consult [FT17].

**Convention:** in this paper smooth means $C^\infty$, and any set of diffeomorphisms, embeddings, submanifolds, or Riemannian metrics is equipped with the compact-open smooth topology.

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**Structure of the paper.** Section 2 builds on results in [GW01] related to the condition (ii) of Theorem 1.1, and needed for Corollary 1.2; in fact, the topic of Section 2 may be of independent interest. Section 3 contains a proof of Theorem 1.3. The other results stated in the introduction are proved in Section 4.

## 2. The normal exponential map to a soul is a diffeomorphism

We seek a topological condition on an open connected manifold $V$ ensuring that the normal exponential map $\exp^\perp: \nu_S \to V$ to the soul $S$ of each metric in $R_{K \geq 0}(V)$ is a diffeomorphism. A simple example of such condition is $H_{n-1}(V;\mathbb{Z}_2) \neq 0$, i.e., every soul have codimension one, which by the splitting theorem [CG72] ensures that $\exp^\perp$ is a diffeomorphism. Another such condition was discovered by L. Guijarro and G. Walschap in [GW01], and here we elaborate and extend their work.

By a standard argument the normal exponential map to a closed submanifold is a diffeomorphism if and only if it each unit vector that is normal to the submanifold exponentiates to a ray. Since $V$ is open, each $x \in S$ is a starting point of a ray, and since $S$ is a soul, the ray must be orthogonal to $S$ [CG72, Theorem 5.1(3)]. The parallel transport along $S$ preserves the set of rays orthogonal to $S$ [Wal88, Lemma 1.1], and in particular, the rays from $x$ are permuted by the corresponding normal
holonomy group $G_x$ of $\nu_S$ at $x$. Thus if $G_x$-action on the normal unit sphere at $x$ is transitive, then the normal exponential map is a diffeomorphism.

At this point let us switch to the following more general setting. Let $\xi$ be a smooth Euclidean vector bundle over a closed connected manifold $B$ (in fact, many results of this section also hold if $B$ is non-compact). Let $p: S(\xi) \to B$ be the unit sphere bundle. Equip $\xi$ with a metric connection, i.e., an (Ehresmann) connection whose parallel transport preserves the Euclidean inner product, and let $G$ be the holonomy group of the connection at $b \in B$. Then $G$ is a (possibly non-closed) subgroup in the isometry group of the sphere $F = p^{-1}(b)$, which is isomorphic to $O(k)$ where $k$ is the dimension of the fibers of $\xi$. The identity component $G_0$ of $G$ is a closed subgroup of $O(k)$, and the parallel transport defines a surjective homomorphism $\pi_1(B) \to G/G_0$, so $G/G_0$ is countable [KN63, Theorem 4.2 in Chapter II].

We say that the holonomy of $\xi$ is transitive if the $G$-action on $F$ is transitive.

Consider the following conditions on $\xi$:

(A) $\pi_q(p): \pi_q(S(\xi)) \to \pi_q(B)$ is not surjective for some $q \geq 1$,

(B) the holonomy of every metric connection on $\xi$ is transitive,

(C) $\xi$ does not split as a Whitney sum (of positive rank bundles),

(D) $p$ has no section.

The condition (B) is most relevant to this paper, while (D) is the easiest to check. In favorable situations (A)–(D) are all equivalent, and in general, the conditions help to isolate the cases where (B) hold. The following is contained in [GW01]:

**Theorem 2.1.** The implications (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D) hold.

**Proof.** The contrapositives of (B) $\Rightarrow$ (C) $\Rightarrow$ (D) are immediate: If $p$ has a section, then its span is a one-dimensional subbundle which together with its orthogonal complement gives a Whitney sum decomposition for $\xi$, and if $\xi$ is a nontrivial Whitney sum, then any connections on the summands gives rise to a connection on the Whitney sum whose holonomy violates (B). The implication (A) $\Rightarrow$ (B) is stated in [GW01, page 253] when $q \geq 2$ and $\xi$ is orientable but these assumptions are never used in the proof. $\square$

**Remark 2.2.** If (D) holds for a normal bundle to a soul, then the soul is uniquely determined by the metric [BFK17, Section 2]. The souls for different metrics need not even be diffeomorphic, but their normal sphere bundles are fiber homotopy equivalent, and more generally, if $\xi$, $\eta$ are smooth vector bundles over closed manifolds with the same total space, then their normal sphere bundles are fiber homotopy equivalent over the canonical homotopy equivalence of their zero sections [BKS11, Proposition 4.1]. Clearly, (A) and (D) are preserved under such fiber homotopy equivalences; I do not know if this holds for (B) or (C).
Lemma 2.3. The negation of each of the conditions (A)–(D) is inherited by the pullback via a smooth map $f$ of base manifolds, i.e., if $\xi$ does not satisfy the condition, then neither does $f^*\xi$.

Proof. Think of $f: B' \to B$ as the restriction of the deformation retraction $r$ of the mapping cylinder of $f$ onto $B$. If (A), (C), or (D) fails for $\xi$, then it does for the pullback $r^*\xi$, and hence for its restriction to $B'$. If (B) fails for a connection on $\xi$, consider the pullback connection on $f^*\xi$. For any two unit vectors in a fiber of $f^*\xi$ that are translates of each other along a loop $\gamma$, their images in $\xi$ are translates of each other along a loop $f \circ \gamma$. □

Example 2.4. For a (closed connected smooth) manifold $L$ consider the bundle $L \times \xi$ whose projection is the product of the identity map of $L$ and the projection of $\xi$. Then $\xi$ satisfies one of the conditions (A)–(D) if and only is so does $L \times \xi$, because the bundles are pullbacks of each other.

A partial converse of Lemma 2.3 for the condition (A) comes from the following observation: If both $f$ and the projection of the unit sphere bundle of $f^*\xi$ are $\pi_q$-surjective, then so is $p$. This is immediate from the map of the homotopy exact sequences of fiber bundles induced by $f$.

Corollary 2.5. If $f: B' \to B$ is a covering map and $k \geq 2$, then $\xi$ satisfies (A) if and only if so does $f^*\xi$. If $f: B' \to B$ is a torus bundle and $k \geq 3$, then $\xi$ satisfies (A) if and only if so does $f^*\xi$.

Proof. For both claims the “if” direction follows by Lemma 2.3. The converse of the first statement is true because $p$ is $\pi_1$-surjective as $k \geq 2$ so by assumption $p$ is not $\pi_q$-surjective for some $q \geq 2$, but $f$ is a $\pi_q$-isomorphism for all $q \geq 2$. Similarly, for the second statement from $k \geq 3$ we conclude that $p$ is surjective on $\pi_1$ and $\pi_2$, so by assumption $p$ is not $\pi_q$-surjective for some $q \geq 3$, but $f$ is a $\pi_q$-isomorphism for all $q \geq 3$. □

Example 2.6. If $V$ is an open complete manifold of $K \geq 0$, then any path-component of $R_{K \geq 0}(V)$ contains a metric whose pullback to a finite Galois cover splits as the Riemannian product of a torus and an open simply-connected complete manifold $N$ of $K \geq 0$, see [Wil00, Corollary 6.3]. Now Lemma 2.5, Example 2.4, and Remark 2.2 imply that the normal bundle to a soul of $V$ satisfies (A) if and only if so does the normal bundle to a soul of $N$. Thus in principle, verifying (A) for normal bundles to souls reduces to the case of simply-connected manifolds of $K \geq 0$. The caveat is that in some cases $\xi$ may be easier to understand than $N$.

Example 2.7. The bound $k \geq 3$ in Corollary 2.5 cannot be improved: $TS^2$ satisfies (A) while its pullback under the Hopf fibration $S^3 \to S^2$ does not.

Example 2.8. If $\xi$ is a nontrivial $\mathbb{R}$-bundle, then $\xi$ satisfies (A): the nontrivial two-fold-cover $p$ is not $\pi_1$-surjective. Of course, (D) fails for the trivial $\mathbb{R}$-bundle.
Remark 2.9. In many cases verifying (A) hinges on the following observation in [GW01]: the homotopy class of a map \( f : S^q \to B \) lies in the image of the \( \pi_q(p) : \pi_q(S(\xi)) \to \pi_q(B) \) if and only if the pullback of the sphere bundle \( p \) via \( f \) has a section. Since every sphere bundle with a section has zero Euler class, we conclude:

Condition (A) holds if the Euler class of \( \xi \) (with \( \mathbb{Z} \) or \( \mathbb{Z}_2 \) coefficients)

is nonzero on the image of the Hurewicz homomorphism.

Note that for an \((m - 1)\)-connected cell complex with \( m \geq 2 \) the Hurewicz homomorphism is an isomorphism in degree \( m \) and a surjection in degree \( m + 1 \) [Hat02, Theorem 4.37, and Exercise 23 in section 4.2]. In [GSW02, Theorem 1.4] one finds the following version of the above: (A) holds if the rational Euler class is nonzero, \( B \) is simply-connected and rationally \( \frac{k}{2} \)-connected.

Remark 2.10. If \( B \) is a sphere, then all the four conditions (A)–(D) are equivalent because if the projection of a sphere bundle over \( S^q \) is \( \pi_q \)-surjective, then it has a section, which is the contrapositive of \((D) \Rightarrow (A)\). More generally, by Example 2.4 we get: The conditions (A)–(D) are equivalent if \( \xi \) is the product of a closed smooth manifold \( L \) and a vector bundle bundle over a sphere.

Another natural way to obtain (B) is based on the fact that the structure group of \( \xi \) reduces to \( G \) [KN63, Theorem 7.1 in Chapter II], so the existence a connection with non-transitive potentially restricts \( \xi \), and here is how the idea can be exploited.

**Lemma 2.11.** \((D) \Rightarrow (B)\) for any \( \mathbb{R}^3 \)-bundle \( \xi \) with simply-connected \( B \).

**Proof.** Since \( B \) is simply-connected, the structure group of \( \xi \) lies in \( SO(3) \), and the holonomy group of any Euclidean metric connection on \( \xi \) is connected. If (B) fails for \( \xi \), then the holonomy group is a proper connected closed subgroup of \( SO(3) \), that is, the standard \( SO(2) \). Hence \( \xi \) has a section. \( \square \)

**Example 2.12.** By [GZ11, Theorem 1] every non-spin vector bundle over \( \mathbb{C}P^2 \) admits a complete metric of \( K \geq 0 \). Let us show that most non-spin \( \mathbb{R}^3 \)-bundles over \( \mathbb{C}P^2 \) satisfy (B). To this end recall that by [DW59] the set of isomorphism classes of non-spin \( \mathbb{R}^3 \)-bundles over \( \mathbb{C}P^2 \) is bijective to \( \{1 + 4d : d \in \mathbb{Z}\} \), where \( \xi \) is sento to the first Pontryagin class of \( \xi \) evaluated on the fundamental class of \( \mathbb{C}P^2 \). Such \( \xi \) has a section if and only if \( \xi \cong e^1 \oplus \eta \), the Whitney sum of a trivial \( \mathbb{R} \)-bundle and an \( \mathbb{R}^2 \)-bundle \( \eta \). This happens if and only if \( p_1(\xi) \) is the square of \( e(\eta) \), the Euler class of \( \eta \). Equivalently, \( 1 + 4d = k^2 \) for some integer \( k \). The letter happens if and only if \( d \) is the product of two consecutive integers. In summary, the non-spin \( \mathbb{R}^3 \)-bundle over \( \mathbb{C}P^2 \) that corresponds to \( 1 + 4d \) satisfies (B) if and only if \( d \) is not the product of two consecutive integers.

The following lemma illustrates yet another method of checking (B).
Lemma 2.13. Let \( L \) be a closed connected smooth manifold, and let \( \xi \) be a smooth Euclidean vector bundle over a closed manifold \( B \). If \( \xi \) shares the total space with \( L \times T\mathbb{C}P^n \) or \( L \times T\mathbb{H}P^n \) for an even positive \( n \), then \( \xi \) satisfies (B).

Proof. This improves on [GW01, Example 3] which deals with the case when \( L \) is a point and \( B \) is the zero section of \( T\mathbb{C}P^n \) or \( T\mathbb{H}P^n \). Consider the standard \( S^1 \)-action on \( TS^{2n+1} \) induced by the diagonal \( S^1 \)-action on \( \mathbb{C}^{n+1} \). The action preserves the \( \mathbb{R}^{2n} \)-subbundle \( \nu \) of \( TS^{2n+1} \) whose fibers are orthogonal to the \( S^1 \)-orbits. Let \( E(\nu) \) be the total space of \( \nu \), and \( S \subset L \times E(\nu) \) be the preimage of \( B \) under the orbit map \( L \times E(\nu) \to L \times T\mathbb{C}P^n \) that is the identity on the first factor. The circle bundle inclusion of \( S^1 \to S \to B \) into \( S^1 \to L \times E(\nu) \to L \times T\mathbb{C}P^n \) is the identity on the fibers and a homotopy equivalence on the base, so the inclusion \( S \to L \times E(\nu) \) is a homotopy equivalence (by the 5-lemma applied to the homotopy exact sequence of the bundles and the Whitehead theorem). Arguing by contradiction assume that \( \xi \) admits a connection with non-transitive holonomy. Let \( \nu_S \) be the normal bundle to \( S \) in \( L \times E(\nu) \). Since \( \nu_S \) is a pullback of \( \xi \), Lemma 2.3 shows that \( \nu_S \) admits a connection with non-transitive holonomy. Thus (A) fails for \( \nu_S \), and hence it also does for \( L \times \nu \), see [BKS11, Proposition 4.1]. Remark 2.10 implies that \( L \times \nu \) has a nowhere zero section, and hence so does \( \nu \). Thus \( S^{2n+1} \) has two orthonormal tangent vector fields which is impossible for even positive \( n \) [Hat02, Example 4L.5]. With obvious modifications the same proof works for \( \mathbb{H}P^n \).

Finally, we give several examples where (B) fails.

Example 2.14. The \( S^1 \) quotient of the Hopf fibration \( S^{4m+3} \to \mathbb{H}P^m \) is a smooth fiber bundle \( S^2 \to \mathbb{C}P^{2m+1} \to \mathbb{H}P^m \). Thus \( T\mathbb{C}P^{2m+1} \) splits as a Whitney sum if \( m > 0 \), and hence fails (C). I do not know if \( T\mathbb{H}P^{2m+1} \) satisfies (B) for \( m > 0 \).

Example 2.15. The Riemannian connection on the tangent bundle to a closed simply-connected irreducible compact symmetric space \( M \) has transitive holonomy if and only if \( M \) has rank one, see [Bes08, Sections 10.35, 10.79, 10.80]; thus (B) fails for \( TM \) if \( M \) has rank \( \geq 2 \). In Example 2.14 we saw that (B) can fail (for some other metric connection) even when the rank is one.

Example 2.16. (C) fails for the product of any two vector bundles with positive-dimensional fibers. If the factors have nonzero Euler classes, then \( \xi \) satisfies (D), and hence (D) \( \iff \) (C). For example, the tangent bundle to \( S^2 \times S^2 \) has no section (because the Euler characteristic is nonzero) but it splits as a Whitney sum.

Example 2.17. Let us show that (C) \( \iff \) (B). Let \( L \) denote any lens space of any (odd) dimension \( \geq 3 \) with \( \pi_1(L) \cong \mathbb{Z}_m \), see [Hat02, Example 2.43]. The universal coefficients theorem gives isomorphisms \( H^2(L; \mathbb{Z}) \cong H^2(L; \mathbb{Z}_m) \cong \mathbb{Z}_m \cong H^1(L; \mathbb{Z}_m) \). Denote generators of \( H^2(L; \mathbb{Z}) \) by \( \alpha \), \( \beta \), respectively. Let \( \xi \) be an oriented \( \mathbb{R}^2 \)-bundle over \( L \) with Euler class \( \alpha \). Since \( \alpha \) is torsion, \( \xi \) is flat [KT67, Theorem 6.1], and its holonomy group is an image of \( \mathbb{Z}_m \), and hence (B) fails. Let us show that if \( m = 4k \) with \( k \in \mathbb{N} \), then \( \xi \) satisfies (C). By [Hat02, Example 3.41] \( 2k \alpha \bmod m = \beta^2 \) and further reducing \( \bmod 2 \) gives \( 0 = \beta^2 \bmod 2 \).
Since $\beta$ is a generator, the cup-square of any element of $H^1(L;\mathbb{Z}_2)$ is a multiple of $\beta^2 (\text{mod } 2) = 0$. On the other hand, $\alpha (\text{mod } 2)$ generates a group isomorphic to $\mathbb{Z}_m \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$, hence $\alpha (\text{mod } 2)$ is not a square. Now $\alpha (\text{mod } 2) = w_2 (\xi)$ if $\xi = \gamma \oplus \gamma'$, the Whitney sum of two line bundles, then since $\xi$ is orientable we get $0 = w_1 (\xi) = w_1 (\gamma) + w_1 (\gamma')$, so $\gamma = \gamma'$ because an $\mathbb{R}$-bundle is determined by its $w_1$. The Whitney sum formula gives $0 \neq w_2 (\xi) = w_1 (\gamma)^2 = 0$, which is a contradiction.

**Lemma 2.18.** If $V$ is an open complete $n$-manifold of $K \geq 0$ with a flat soul and $H_{n-1} (V; \mathbb{Z}_2) = 0$, then every path-component of $R_{K \geq 0} (V)$ contains a metric for which the normal exponential map to soul is not a diffeomorphism and the normal holonomy group of the soul is finite.

**Proof.** By [Wi00, Corollary 6.3] any metric can be joined by a path in $R_{K \geq 0} (V)$ to a metric whose pullback to a finitely-sheeted Galois cover splits as a flat torus and the standard $\mathbb{R}^k$. Here $k \geq 2$ because the soul has codimension $\geq 2$. The action of the deck-transformation group $F$ on the $\mathbb{R}^k$-factor is orthogonal, and one can $F$-equivariantly deform $\mathbb{R}^k$ to a rotationally symmetric metric $h$ that has a cylindrical end and positive curvature near the origin (a point fixed by $F$). Then similarly to [GM69, page 77] one can $F$-equivariantly perturb $(\mathbb{R}^k, h)$ to a complete metric of $K \geq 0$ with no poles. Multiplying $h$ by the above flat torus, and quotienting by $F$ gives a metric with claimed properties. \qed

Recall that a vector bundle is *spin* if and only if its $w_1$, $w_2$ vanish, where $w_i$ is the $i$th Stiefel-Whitney class, and a manifold is *spin* if so is its tangent bundle. The following lemma lets us easily compute $w_i$ of the total space of $\xi$, or of the double of its unit disk bundle.

**Lemma 2.19.** Let $\xi$ be a smooth vector bundle over a closed manifold $B$, and let $S$ be the double of the unit disk bundle of $\xi$ along the boundary. Then $w_i (S) = 0$ if and only if $w_i (\xi \oplus TM) = 0$ if and only if $w_i$ of the total space of $\xi$ is zero.

**Proof.** The key point is that $S$ is the unit sphere bundle of $\xi \oplus \varepsilon^1$ where $\varepsilon^1$ is the trivial $\mathbb{R}$-bundle over $B$. Let $q: D \to B$ be the unit disk bundle of $\xi \oplus \varepsilon^1$ and set $q = q|_S: D \to B$, the sphere bundle projections. Then $TD = q^* (\xi \oplus TB)$, and since $q$ is homotopic to the identity, $w_i (\xi \oplus TB) = 0$ if and only if $w_i (\text{Int } D) = 0$ where $\text{Int } D$ is the total space of $\xi$. Since $TS \oplus \varepsilon^1 \cong TD|_S = q^* (\xi \oplus TB)$, the map $q^*$ takes $w_i (\xi \oplus TB)$ to $w_i (S)$. Since $\xi \oplus \varepsilon^1$ has a nowhere zero section, the $\mathbb{Z}_2$-Euler class of $q$ is zero, hence the $\mathbb{Z}_2$-Gysin sequence of $q$ gives injectivity of $q^*: H^1 (M; \mathbb{Z}_2) \to H^1 (S; \mathbb{Z}_2)$. Thus $w_i (\xi \oplus TB) = 0$ if and only if $w_i (S) = 0$. \qed

**Example 2.20.** The total space of the tautological line bundle over $\mathbb{R}P^n$ is spin if and only if $n \equiv 2 \pmod 4$, as easily follows from the fact that $T \mathbb{R}P^n$ is stably isomorphic to the Whitney sum of $n + 1$ copies of the tautological bundle.
In this section we prove Theorem 1.3. Since the normal sphere bundle to some (and hence any [BKS11, Proposition 4.1]) soul has no section, every metric $g \in \mathcal{R}_{K \geq 0}(V)$ has a unique soul $S = S_g$ which depends continuously on $g$ [BFK17, Theorem 2.1].

Since the normal exponential map to $S$ is a diffeomorphism, the distance function to the $S$ is smooth away from $S$. The function is also convex by Riccati comparison, see [EF92, Lemma 1.2]. In particular, the closed 1-neighborhood $D$ of $S$ is a totally convex, smooth codimension zero submanifold of $V$ with infinite normal injectivity radius. The distance function $d(\cdot, D)$ to $D$ is smooth away from $D$, and also convex, again, by Riccati comparison [Esc86, Lemma 3.4(a)], or alternatively, by concavity of $d(\cdot, \partial D)$ on $D$ established in [CG72, Theorem 1.10].

We are going to approximate $(V, g)$ by a family of convex hypersurfaces in $V \times \mathbb{R}$. To visualise the process think of the surface in $\mathbb{R}^3$ that is a rotationally symmetric smoothing of the “drinking glass”

$$\{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 \leq r^2\} \cup \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2 \text{ and } z \geq 0\}$$

which tends to the $xy$-plane as $r \to \infty$. Let $h: (-\infty, 0) \to [0, 1)$ be a surjective $C^\infty$ function such that

- $h|_{(-\infty, -1]} = 0$,
- the derivatives $h'$, $h''$ are positive on $(-1, 0)$,
- the inverse of $h|_{(-1, 0)}$ extends to a $C^\infty$ function $h: (0, \infty) \to (-1, 0]$ that vanishes on $[1, \infty)$.  

Fix $r > 1$. Let $N_r$ denote the closed $r$-neighborhood of $D$ in $V$. For $\rho \in [0, r)$ let $G_\rho \subset V \times \mathbb{R}$ be the graph of $h(d(\cdot, D) - r)$ over $\text{Int} N_\rho$, and let $\bar{G}_\rho$ be its closure. Thus $\bar{G}_r \setminus G_r = \partial N_r \times \{1\}$. The function $h(d(\cdot, D) - r)$ is convex because $d(\cdot, D) - r$ is convex and $h$ is non-decreasing convex. Hence the induced metric on $G_r$ has $K \geq 0$, see Lemma 3.6 below. The same holds for $\partial N_r \times [1, \infty)$ because $\partial N_r$ bounds $N_r$, a convex subset of a nonnegatively curved manifold. The union $H_r$ of $G_r$ and $\partial N_r \times [1, \infty)$ is clearly a topological submanifold of $V \times \mathbb{R}$ which is smooth away from $\partial N_r \times \{1\}$. Lemma 3.1 below shows that $H_r$ is smooth, and hence its induced metric also has $K \geq 0$.

To proceed we need some notations. Let $U$ denote the $g$-unit normal bundle to $\partial D$ so that $V \setminus D$ is identified with $U \times (0, \infty)$ via the normal exponential map $\exp^\perp$ of $D$. Let $f_r: U \times (0, \infty) \to V \times \mathbb{R}$ be the map given by $f_r(u, t) = (\exp^\perp(h_r(t)u), t)$ where $h_r := h + r$ takes values in $(r-1, r]$. We now verify smoothness of $H_r$.

**Lemma 3.1.** $f_r$ is a smooth embedding whose image is $H_r \setminus \bar{G}_{r-1}$.

**Proof.** Using that $\exp^\perp$ is a diffeomorphism and $U$ is compact one easily sees that $f_r$ is a smooth homeomorphism onto its image. Also $f_r$ is an immersion because if
q is given by \( q(\exp^t(v), t) = \left( \frac{v}{\|v\|}, t \right) \), where \( \exp^t(v) \in V \setminus D \) and \( \| \cdot \| \) is the norm induced by the Riemannian metric, then \( q \circ f_r \) is the identity. Thus \( f_r \) is a smooth embedding. For \( t \geq 1 \) we have \( f_r(u, t) = (\exp^t(ru), t) \), so \( f_r \) maps \( U \times [1, \infty) \) diffeomorphically onto \( \partial N_r \times [1, \infty) \). If \( 0 < t < 1 \), then \( h_r(t) \in (r-1, r) \), and \( x = \exp^t(h_r(t)u) \) satisfies \( d(x, D) = h_r(t) \), which can be rewritten as \( t = h(d(x, D) - r) \). Thus for each \( \rho \in (r-1, r) \) the map \( f_r \) takes \( U \times \{ h(\rho-r) \} \) onto \( \partial G_\rho \), and therefore, \( U \times (0, 1) \) onto \( G_r \setminus G_{r-1} \).

Next we pullback the metric on \( H_r \) to \( V \) in a way that is continuous in \( g \) and \( r \).

**Lemma 3.2.** There is a diffeomorphism \( s_r: V \to H_r \) which restricts to the identity on \( G_{r-1} \) and depends continuously on \( r \).

**Proof.** The idea is to map \( H_r \) diffeomorphically onto \( G_{r-\frac{1}{2}} \) by contracting along the geodesics orthogonal to \( \partial D \) so that the contraction is the identity on \( G_{r-\frac{1}{2}} \), and then project orthogonally onto \( N_{r-\frac{1}{2}} \), and finally stretch radially to \( V \). To do this continuously set \( a = h(-1/2) \) and \( b = h(-1/3) \). Let \( D_{a,b} \) be the diffeomorphism of \( U \times (0, \infty) \) onto \( U \times (0, b) \) given by \( D_{a,b}(u, t) = (u, d_{a,b}(t)) \) where \( d_{a,b} \) is as in Lemma 3.7 below. Then \( f_r \circ D_{a,b} \circ f_r^{-1} \) is a diffeomorphism of \( H_r \setminus G_{r-1} \) onto \( G_{r-\frac{1}{2}} \setminus G_{r-1} \) that is the identity on \( G_{r-\frac{1}{2}} \). Extend \( f_r \circ D_{a,b} \circ f_r^{-1} \) to a diffeomorphism \( H_r \to G_{r-\frac{1}{2}} \) that is the identity on \( G_{r-1} \). Postcompose the result with the coordinate projection \( V \times \mathbb{R} \to V \) and then with the self-diffeomorphism of \( V \) given by \( \exp^t(v) \to \exp^t \left( d_{a,b}(\|v\|) \frac{v}{\|v\|} \right) \). This gives a diffeomorphism \( H_r \to V \) whose inverse has desired properties. \( \square \)

Let \( g^{|r|} \) denote the the pullback of the induced metric on \( H_r \) via \( s_r \). For \( t \in [0, 1] \) define a self-map \( \rho_t \) of \( \mathcal{R}_{K \geq 0}(V) \) by setting \( \rho_t(g) := g^{\frac{1}{t}} \) for \( t > 0 \), and \( \rho_0(g) := g \).

**Lemma 3.3.** The map \( (t, g) \to \rho_t(g) \) is continuous.

**Proof.** Given a sequence \((t_i, g_i)\) converging to \((t, g)\) and a compact subset \( K \) of \( V \) we are to show that \( \rho_{t_i}(g_i) \) converges to \( \rho_t(g) \) on \( K \). If \( t > 0 \) this follows because by construction \( g^{|r|} \) depends continuously on \( g, r \). Suppose \( t = 0 \), so that \( \rho_t(g) = g \).

Since \( S_{g_i} \) converge to \( S_g \), the \( \left( \frac{1}{r_i} - 1 \right) \)-neighborhood of \( S_{g_i} \) contain \( K \) for all large \( i \), and hence over \( K \) we have \( g_i \left( \frac{1}{r_i} \right) = g_i \to g \) as \( i \to \infty \). \( \square \)

The distance in \( H_r \) between the soul and the cylindrical end \( \partial N_r \times [1, \infty) \) is realized by a geodesic that lies in a 2-dimensional half-flat \( \sigma \times \mathbb{R} \), where \( \sigma \) a ray in \( V \times \{0\} \) orthogonal to the soul. The portion of the geodesic in \( N_{r-1} \times \{0\} \) has length \( r - 1 \) and the remaining portion can be identified with the graph of \( h \) over \([-1, 0) \) whose length is at most 2. Setting \( r = \frac{1}{\varepsilon} \) gives the last assertion of Theorem 1.3. \( \square \)
Remark 3.4. If \((V,g)\) has a cylindrical end and \(r\) is sufficiently large, then \(g[r] = g\). More precisely, if \(r \in (0, r - 1)\) and \((V \setminus N_r, g)\) is isometric to the product of a closed manifold and \((0, \infty)\), then \(g[r] = g\) because \(H_r\) and \(V\) are both obtained by gluing \(N_r\) via the identity map of \(\partial N_r\) to the Riemannian product of \(\partial N_r\) and the image of a smooth proper embedding of \([0, \infty)\) into \(\mathbb{R} \times [0, \infty)\), and the diffeomorphism \(s_r\) of Lemma 3.2 identifies \(g\) and the induced metric on \(H_r\).

Remark 3.5. It follows by construction that the map \(g \to g[r]\) is \(\text{Diff} V\)-equivariant, i.e., if \(\phi \in \text{Diff} V\), then \((\phi^*g)[r] = \phi^*(g[r])\).

Finally, here are two elementary lemmas that were used above.

Lemma 3.6. If \(V\) is a Riemannian manifold of \(K \geq 0\) and \(f : V \to \mathbb{R}\) is convex and smooth, then the graph of \(f\) has nonnegative sectional curvature in the metric induced by Riemannian product \(V \times \mathbb{R}\).

Proof. By Gauss equation any smooth convex hypersurface in a manifold of \(K \geq 0\) has nonnegative sectional curvature. Since the graph of \(f\) is a level set of the function \(F(x, t) = f(x) - t\), it suffices to check that \(F\) is convex, or equivalently, that the Hessian of \(F\) is positive semidefinite. Fix normal coordinates centered at an arbitrary point of \(p \in V \times \mathbb{R}\), where the last coordinate is the projection on the \(\mathbb{R}\)-factor. Then the \(ij\) entry of \(\text{Hess} F\) at \(p\) equals \(\partial_i \partial_j f\) if \(i, j \leq \dim V\) and zero otherwise, so convexity of \(f\) implies convexity of \(F\).

Lemma 3.7. Given \(0 < a < b\) there is a \(C^\infty\) diffeomorphism \(d_{a,b} : (0, \infty) \to (0, b)\) which restricts to the identity on \((0, a]\), and varies continuously with \(a\) and \(b\) in the space of smooth self-maps of \((0, \infty)\).

Proof. Fix a bump function supported on \([-1, 1]\), apply an affine change of variable to produce a bump function with support \([a, \frac{a+b}{2}]\), integrate it from 0 to \(x\) and scale to get a nonnegative function that is zero on \((-\infty, a]\) and 1 on \([\frac{a+b}{2}, \infty)\). Multiply it on \((0, b)\) by \(\frac{2}{(b-x)^2}\) and denote the result by \(q_{a,b}\). The solution of \(f'' = q_{a,b}\), \(f(0) = 0\), \(f'(0) = 1\) satisfies \(f(x) = x\) for \(x \leq a\), and if \(\frac{a+b}{2} \leq x < b\), then \(f(x) = \frac{1}{b-x} + cx + d\) for some constants \(c\), \(d\) continuously depending on \(a\), \(b\). Now \(q_{a,b} \geq 0\) and \(f'(0) = 1\) gives \(f' \geq 1\), and \(d_{a,b} := f^{-1}\) has the desired properties.

4. Proof of main results

In this section we prove Theorem 1.1 and Corollary 1.2. Given \(r > 0\) and \(V\) as in Theorem 1.3 consider the set of pairs \((g, x)\) such that \(g \in \mathcal{R}_{K \geq 0}(V)\) and \(x \in D_{r,g}\), where \(D_{r,g}\) is the closed \(r\)-neighborhood of the (unique) soul of \(g\). Since \(D_{r,g}\) varies continuously with \(g\), the map \(\partial_r(g, x) = g\) is a smooth fiber bundle whose fiber over \(g\) is \(D_{r,g}\). The normal exponential map to the soul of \(g\) identifies \(D_{r,g}\) with \(D_{1,g}\), which gives rise to an isomorphism \(\partial_r \cong \partial_1\). While this fiber bundle is not really needed for the proof, it seems to illuminate our strategy.
Let $D_0(M)$ be the subspace of $\text{Diff } M$ consisting of compactly-supported diffeomorphisms that are isotopic to the identity. Let $R_{\text{scal} \geq 0}(M)$ be the space of complete metrics on $M$ of nonnegative curvature but not identically zero scalar curvature.

**Theorem 4.1.** Let $V$ be as in Theorem 1.1(ii) except that $V$ can be non-spin. Let $Y$ be a finite contractible CW complex $Y$ and $z \in Y$. Let $f: Y \to R_{K \geq 0}(V)$ be a continuous map. Let $X \subset Y$ be a subcomplex and let $\delta: X \to D_0(V)$ be a continuous map such that $f(x)$ is the pullback of $f(z)$ via $\delta(x)$ for all $x \in X$.

Let $D$ be a closed tubular neighborhood of the soul of $f(z)$ such that the support of every diffeomorphism in $\delta(X)$ lies in the interior of $D$. Let $\hat{D}$ be the double of $D$, and let $\hat{\delta}: X \to \text{Diff}(\hat{D})$ that extends $\delta$ by the identity outside $D$. Then there is a continuous map $\hat{f}: Y \to R_{\text{scal} \geq 0}(\hat{D})$ such that $\hat{f}(x)$ is the pullback of $\hat{f}(z)$ via $\hat{\delta}(x)$ for all $x \in X$.

**Proof.**

**Step 1: Push to metrics with cylindrical ends of definite size.** The homotopy $\rho_\varepsilon$ deforms $f$ into $C_{K \geq 0}(V)$, so after changing $f$ to a nearby homotopy map we can assume that each metric in the image of $f$ is a product outside the $(1 + \varepsilon)$-neighborhood of its soul. Also assume that $\varepsilon$ is so small that the interior of the neighborhood contains the support of every diffeomorphism in $\delta(X)$. Remark 3.5 ensures that after the homotopy the map $f|_X$ is still induced by $\delta$.

**Step 2: Preparing the neck topologically.** For $y \in Y$ set $g_y := f(y)$, let $S_y$ be the soul of $g_y$, let $D_{r.y}$ denote the $r$-neighborhood of $S_y$ in the metric $g_y$, and $\Sigma_{r.y} = \partial D_{r.y}$. Note that $D_{r,y}$ depends continuously of $y$, $r$.

Fix $\rho > 1 + \frac{1}{2}$ and set $D_y := D_{\rho.y}$, $\Sigma_y = \partial D_y$. Since $Y$ is contractible, the pullback of $\partial\rho$ via $f$ is trivial, which defines a continuous map $Y \to \text{Emb}(D_z, V)$ that sends $y$ to the diffeomorphism $D_z \to D_y$.

The map $D_0(V) \to \text{Emb}(D_z, V)$ that precomposes a diffeomorphism with the inclusion $D_z \to V$ is a fiber bundle over the path-component of the inclusion $[\text{Pal60}]$. Its fiber consists of diffeomorphisms in $D_0(V)$ that are the identity on $D_z$, and the space of such diffeomorphisms is contractible by the Alexander’s trick towards infinity. Thus the above map $Y \to \text{Emb}(D_z, V)$ lifts to a map $Y \to D_0(V)$, to be written as $y \to \phi_y$, such that $\phi_z$ is the identity. Thus $\phi_y(D_z) = D_y$.

For the metric $\phi_y^* g_y$ we consider the normal bundle $\nu_y$ to $\Sigma_z$ in $V$, the normal exponential map $e_y: \nu_y \to V$, and the unit normal vector field $U_v$ to $\Sigma_z$ pointing outside $D_z$. Let $L_y: \nu_z \to \nu_y$ be the (unique) linear isometry taking $U_z$ to $U_y$. Since $\phi_y^* g_y$ is a product metric outside $D_z$, the map $e_y \circ L_y \circ e_z^{-1}$ is a self-diffeomorphism of $V \setminus \text{Int} D_z$ which fixes every point of $\Sigma_z$. Restricting the composite $e_y \circ L_y \circ e_z^{-1}$ to the 1-neighborhood $N$ of $\Sigma_z$ in $(V \setminus \text{Int} D_z, g_z)$ gives a family of embeddings of $N$ into $V \setminus \text{Int} D_z$ that restrict to the identity on $\Sigma_z$. As in the previous paragraph the space of diffeomorphisms of $V \setminus \text{Int} D_z$ with compact support that are identity on $N$ is contractible, so by $[\text{Pal60}]$ the family of embeddings can be obtained by
restricting a continuous family of diffeomorphisms $\psi_y \in \mathcal{D}_0(V)$ such that $\psi_z$ is the identity.

An alternative way to extend isotopies in the above two paragraphs is to use contractibility of $Y$. In summary, the diffeomorphism $\iota_y := \phi_y \circ \psi_y$ has compact support, and maps $\Sigma_z$, $U_z$ to $\Sigma_y$, $U_y$, respectively. Also $\iota_z = \operatorname{id}$. By compactness of $Y$ there is a compact subset of $V$ outside which every diffeomorphisms $\iota_y$ equals the identity. By construction the $\iota^*_y g_y$-distance $r$ from a point of $N$ to $\Sigma_z$ in independent of $y$ (the induced metrics on the fibers of $r$ may depend on $y$).

**Step 3: Deforming metrics on the neck.** Let $b_y$ be the metric induced by $\iota^*_y g_y$ on $\Sigma_y$. Since $Y$ is contractible, there is a deformation retraction $q_s$: $Y \to Y$, $s \in [0, 1]$, where $q_1$ is the constant map to $z$ and $q_s$ is the identity for $s \in [0, \frac{1}{3}]$. Then $b_{q_s(y)}$ is a homotopy through maps $Y \to \mathcal{R}_{K \geq 0}(\Sigma)$ between $y \to b_y$ and the constant map to $b_z$. Applying Ricci flow for time $t$ with the initial metric $b_{q_s(y)}$ yields a map $(t, s, y) \to b^{s}_{q_{s}(y)}$ which is continuous by the smooth dependence of the Ricci flow under the initial data.

The normal sphere bundle to the soul is not flat, see Lemma 4.5 below. Hence a result in [BW07] stated in Lemma 4.3 below, shows that $b^t_{q_s(y)}$ has positive scalar curvature for all small positive $t$. Compactness of $Y$ gives $\tau > 0$ such that the flow exists and has positive scalar curvature for all $t \in [0, \tau]$, $y \in Y$, $s \in [0, 1]$.

Fix a continuous map $\mu: [0, 1] \to [0, \tau]$ with $\mu^{-1}(0) = [0, \frac{1}{3}]$ and $\mu^{-1}(\tau) = [\frac{1}{2}, 1]$.

Then $s \to b^s_{q_s(y)}$ is a homotopy from $y \to b_y$ to $y \to (b_z)^\tau$ such that $b^s_{q_s(y)}$ equals $b_y$, $b^s_z$ for $s$ in $[0, \frac{1}{3}]$, $[\frac{1}{2}, 1]$, respectively, and $b^s_{q_s(y)}$ has positive scalar curvature on $[\frac{1}{3}, 1]$.

By [PY99, Lemma 3] for all sufficiently large $a$ the metric $ds^2 + b^s_{q_{s/a}(y)}$ has positive scalar curvature on $\left(\frac{a}{13}, a\right] \times \Sigma_z$, and nonnegative sectional curvature on $\left[0, \frac{a}{3}\right) \times \Sigma_z$ where $b^s_{q_{s/a}(y)} = b_y$ for $s \in \left[0, \frac{a}{13}\right]$ and $b^s_{q_{s/a}(y)} = (b_z)^\tau$ for $s \in \left[\frac{a}{13}, a\right]$.

Let us pick $a$ so large that $a > 1$ and the union of the supports of $\iota_y$ over $y \in Y$ lies in the interior of $D_{\rho+a, z}$.

The metrics $\iota^*_y g_y$ on $D_{\rho+a, z}$ and $dr^2 + (b_{q_{s/a}(y)})^\mu(r/a)$ on $D_{\rho+a, z} \setminus D_z$ restrict to the metric $dr^2 + b_y$ on $D_{\rho + \frac{a}{13}, z} \setminus D_z$ where $r$ is the $g_z$-distance to $\Sigma_z$, so together they define a metric on $D_{\rho+a, z}$ which we denote $\bar{g}_y$.

**Step 3: Doubling.** The pullback of $\bar{g}_y$ via $\iota^{-1}_y$ is a metric on $D_{\rho+a, z}$ such that $\bar{g}_y = g_y$ on $D_0$ and $\bar{g}_y = dr^2 + (b_z)^\tau$ near $\Sigma_0+a, z$. If $\tilde{D}_{\rho+a, z}$ is the double of $D_{\rho+a, z}$ such that the doubling involution $j$ preserves every geodesic orthogonal to $\Sigma_{\rho+a, z}$, then $\bar{g}_y$ and $j^* \bar{g}_z$ form a smooth metric on $\tilde{D}_{\rho+a, z}$. Denote the metric by $\tilde{f}(y)$. Then the map $y \to \tilde{f}(y)$ is continuous, and for every $x \in X$ the restriction $\tilde{f}(x)|_{D_z}$
is the pullback of \( g_z = \hat{f}(z) \mid_{D_z} \) via \( \delta(x) \), while \( \hat{f}(x) = \hat{f}(z) \) on \( \hat{D}_{\rho+a,z} \setminus D_z \). Thus \( \hat{f} \) has desired properties.

**Remark 4.2.** The conclusion of Theorem 4.1 is true if the assumptions on \( V \) are replaced with “\( V \) is the product of \( \mathbb{R} \) and a closed non-flat manifold of \( K \geq 0 \)”. In this case the souls are precisely the fibers of the projection onto the \( \mathbb{R} \)-factor. Given a basepoint \( * \in V \) every metric \( g \in \mathcal{R}_{K \geq 0}(V) \) has a unique soul that contains \( * \), and the soul depends continuously on \( g \) (because the local \( \mathbb{R} \)-factor is uniquely determined by the metric, and the limit of local \( \mathbb{R} \)-factors is a local \( \mathbb{R} \)-factor). Since both ends of \( V \) are already cylindrical, Step 1 is unnecessary. The rest of the proof goes through without change.

**Lemma 4.3.** Let \( (M,g) \) be a closed manifold of \( K \geq 0 \) that is not flat. Then Ricci flow \( g_t \) with \( g_0 = g \) has positive scalar curvature for all small positive \( t \).

**Proof.** According to [BW07, Proposition 2.1 and Theorem A] for all small positive \( t \) the metric \( g_t \) has nonnegative Ricci curvature, which is positive if and only if \( \pi_1(M) \) is finite. The universal cover \( \tilde{M} \) of \( M \) splits as the product of a Euclidean space and a closed simply-connected manifold of \( K \geq 0 \), and the latter has dimension \( \geq 2 \) since \( M \) is not flat [CG72]. The pullback \( \tilde{g}_t \) of \( g_t \) to \( \tilde{M} \) is a solution of Ricci flow. The uniqueness of Ricci flow solutions with complete initial metrics of bounded curvature [CZ06] implies that the flow coincides with the product of Ricci flows on the factors. The flow leaves the Euclidean factor unchanged, and instantly turns the initial metric on the simply-connected factor to a metric of positive Ricci curvature, thanks to the above-mentioned result in [BW07]. In particular, \( \tilde{g}_t \) has positive scalar curvature, and hence so does \( g_t \). \( \square \)

**Proof of Theorem 1.1.** Fix a smoothly embedded \( n \)-disk in \( V \). Extending by the identity we think of \( \text{Diff}(D^n, \partial) \) as a subgroup of \( \mathcal{D}_0(V) \). Fix an order two element of \( \pi_k(\text{Diff}(D^n, \partial)) \) detected by the \( \alpha \)-invariant as in [CSS], and represent it by a continuous map \( \delta: S^k \to \mathcal{D}_0(V) \). Consider the map \( S^k \to \mathcal{R}_{K \geq 0}(V) \) that sends \( x \) to the pullback of \( g \) via \( \delta(x) \). Its homotopy class has order at most two. Arguing by contradiction assume the map extends to a map from \( D^{k+1} \). By assumption the soul is not flat, hence the normal sphere bundle to the soul is not a flat manifold, see Lemma 4.5 below. Also the double of any tubular neighborhood of the soul is spin because so is \( V \), see Lemma 2.19. Apply Theorem 4.1 or Remark 4.2 for \( (Y,X) = (D^{n+1}, S^k) \) to conclude that the map sending \( x \in S^k \) to the pullback of \( \tilde{g} \) via \( \tilde{\delta}(x) \) is null-homotopic. By [CSS] the \( \alpha \)-invariant of the map is nontrivial, which is a contradiction. \( \square \)

**Remark 4.4.** The proof of Theorem 1.1 fails when the soul is flat: if the flat soul has codimension one, then the double of the normal disk bundle to the soul is flat, hence the Dirac operator is not invertible, and if the flat soul has codimension \( \geq 2 \), its normal exponential map need not be a diffeomorphism by Lemma 2.18.
Lemma 4.5. Let $B$ be a soul of an open connected complete $n$-manifold of $K \geq 0$. Then its normal sphere bundle $E$ is flat if and only if $B$ is flat and $\dim(B) \geq n-2$.

Proof. By [CG72] a closed connected manifold of $K \geq 0$ has a finite cover diffeomorphic to the product of a simply-connected closed manifold and a torus. Any connected flat manifold is finitely covered by a torus. Write the normal sphere bundle to the soul as $S^k \to E \to B$ and refer to its homotopy exact sequence as HES. Here $\dim(E) = n-1 = k + \dim(B)$, and $E, B$ carry metrics of $K \geq 0$ (for $E$ this is due to [GW00]). If $B$ is flat and $k \leq 1$, then HES implies that $E$ is aspherical, and hence flat. Conversely, if $E$ is flat, then $\pi_1(E)$ is virtually-$\mathbb{Z}^{n-1}$. The portion $\pi_1(S^k) \to \pi_1(E) \to \pi_1(B) \to \pi_0(S^k)$ of HES implies that $\pi_1(B)$ is virtually-$\mathbb{Z}^m$ with $m \geq n-2$. Now $m \leq \dim(B)$ gives $n-2 \leq n-k-1$, i.e., $k \leq 1$. Then HES shows that $B$ is aspherical, and hence flat. □

Proof of Corollary 1.2. Let us first discuss why the manifolds in (1)-(10) admit complete metrics of $K \geq 0$. Having $K \geq 0$ is preserved under products, which covers (1) and (10). Another basic method is to realize the manifold as a base of Riemannian submersion from a complete manifold of $K \geq 0$, which is possible for tangent bundles of homogeneous spaces [GW09, Example 2.7.1] and for the tautological line bundles over projective spaces, e.g., as $\mathbb{C}P^n = (S^{2n+1} \times \mathbb{C})/S^1$, where in fact, by varying $S^1$-actions on $\mathbb{C}$ one gets all complex line bundles over $\mathbb{C}P^n$, see the proof of [BW02, Theorem 6.1]. This covers (2), (5)–(7), and the first example in (8). The other examples are due to K. Grove and W. Ziller: (3)–(4) can be found in Theorem B, Corollary 3.13, Proposition 3.14 in [GZ00], the remaining cases of (8) are treated in Theorem 4.5 and the remark that follows it in [GZ11], and (9) appears in [GZ11, Theorem 1].

Verifying that $V$ is spin is immediate from Lemma 2.19 and the Whitney sum formula for the Stiefel-Whitney class; for more details on (5) see Example 2.20.

Finally, let us check the assumptions (i) or (ii) of Theorem 1.1. The bundles in (1) satisfy (i) by Remark 4.2. The condition (B) of Section 2 implies (ii). By Example 2.4 we can assume that $L$ is a point. The bundles in (3) have no section (because every $\mathbb{R}$ or $\mathbb{R}^2$ bundle over $S^n$ with $n \geq 4$ is trivial), and hence by Lemma 2.11 they satisfy (B). By Remark 2.9 the bundles in (2), (4)–(5), (7)–(8) satisfies (A), and hence (B). By Lemma 2.13 the bundles in (6) satisfy (B), and (9) is covered in Example 2.12. Invoking Theorem 1.1 completes the proof. □

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