CURRENT ALGEBRA AND
CONFORMAL FIELD THEORY
ON A FIGURE EIGHT

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ABSTRACT

We examine the dynamics of a free massless scalar field on a figure eight network. Upon requiring the scalar field to have a well defined value at the junction of the network, it is seen that the conserved currents of the theory satisfy Kirchhoff’s law, that is that the current flowing into the junction equals the current flowing out. We obtain the corresponding current algebra and show that, unlike on a circle, the left- and right-moving currents on the figure eight do not in general commute in quantum theory. Since a free scalar field theory on a one dimensional spatial manifold exhibits conformal symmetry, it is natural to ask whether an analogous symmetry can be defined for the figure eight. We find that, unlike in the case of a manifold, the action plus boundary conditions for the network are not invariant under separate conformal transformations associated with left- and right-movers. Instead, the system is, at best, invariant under only a single set of transformations. Its conserved current is also found to satisfy Kirchhoff’s law at the junction. We obtain the associated conserved charges, and show that they generate a Virasoro algebra. Its conformal anomaly (central charge) is computed for special values of the parameters characterizing the network.
1. Introduction

One dimensional networks are simple examples of topological spaces which are not manifolds. They can be physically realized in molecular systems, such as in the case of polymers, crystals and annulenes[1, 2]. Also, the manufacture and study of mesoscopic systems including networks is of current interest[3].

In the past, the theory of networks has been studied in the framework of quantum mechanics[1, 2, 4, 5, 6] [as contrasted with quantum field theory]. For the case of annulenes, the quantum mechanical particle represents itinerant \( \pi \) electrons which are free to propagate on the network. Recently, quantum mechanics was applied to the study of adiabatic transport phenomena[4], as well as the statistics of identical particles, on networks[3, 6]. Topology played a central role in these studies.

In this article, we shall explore some consequences of defining a field theory on a network. Here we choose a simple example of a field theory, consisting of a single massless scalar field, and a simple example of a network, the figure eight network. Physically, we can think of the figure eight as being made up of two superconducting loops of wire, with the scalar field representing the order parameter of the superconductor.

The dynamics of massless scalar fields on two dimensional manifolds (with circle as the spatial slice and the real line \( \mathbb{R}^1 \) accounting for time) have been well studied. Free massless scalar fields (which we shall also study here) are described by conformal field theories. They exhibit the affine \( U(1) \) Lie group [the centrally extended loop group \( \tilde{L}U(1) \) of \( U(1) \)] and the Virasoro group [7] as symmetries. One of the purposes of this investigation is to see what happens to these symmetries when the space-time domain is figure eight \( \times \mathbb{R}^1 \) (\( \mathbb{R}^1 \) again accounting for time).

The figure eight consists of two loops with one point in common, the junction. For purposes of generality, we shall allow the loops to have different lengths, \( \ell_1 \) denoting the
length of loop 1, and \(\ell_2\) denoting the length of loop 2. In addition to \(\ell_a\), four other parameters can be used to characterize a massless free scalar field theory on a general figure eight network. They correspond to the velocities of wave propagation \(v_a\), on loops \(a = 1\) and \(2\), along with the tension, \(T_a\), or energy per unit length associated with loop \(a\).

In general, the set of values for \(\{\ell_1, v_1, T_1\}\) may be different from \(\{\ell_2, v_2, T_2\}\). However, as we shall see in Section 2, the “physics” of the figure eight network depends on only four independent combinations of the parameters \(\ell_a\), \(v_a\) and \(T_a\).

In Section 2, we shall examine the classically conserved currents associated with a free scalar field theory on the figure eight. The boundary conditions on the fields at the junction are crucial in defining the theory. In this article, we shall primarily be concerned with scalar fields which have a well defined value at the junction, so that they do not possess any discontinuities. Physically, this is reasonable for a superconducting network (with the scalar field representing its order parameter), provided a potential does not exist across the junction. On the other hand, the associated currents need not be free of discontinuities. We find that the time-component of the current, or charge density, has a discontinuity at the junction when \(\frac{T_1}{v_1} \neq \frac{T_2}{v_2}\), while the space-component of the current must satisfy Kirchhoff’s law which states that the current flowing into the junction equals the current flowing out.

Section 3 examines the current algebra of the field theory of Section 2. As is well known, a quantum field is an operator valued distribution. The choice of the test function space for such distributions is an essential part of their definition. Distributions defined on different test function spaces, in general, are not equivalent. In this paper, the criterion we follow in order to define the test function spaces of our fields is that they lead to well defined Poisson brackets at the classical level. We think that this is a necessary condition in order to have a consistent quantization. We can show that, as a result of this choice of test function space, the left- and right-moving currents on the figure eight do not in
general commute in quantum theory. In contrast, the corresponding currents of a free massless scalar field on a manifold do of course commute.

In Section 4, we further study the classical currents for two special cases of the parameters \( \ell_a, v_a \) and \( T_a \) classifying the figure eight. For the first case (which we refer to as case \( a \)), \( \frac{T_1}{v_1} = \frac{T_2}{v_2} \) and there are no conditions on \( \ell_a \), while for the second case (which we refer to as case \( b \)) \( \frac{\ell_1}{T_1} = \frac{v_1}{v_2} = \frac{T_2}{T_2} \). The analysis of the currents simplifies for these cases, as we obtain certain periodic boundary conditions for the currents in case \( a \), and, even better, periodic currents in case \( b \). The current algebra for the latter case is easily expressible in terms of three sets of normal modes, and yields three \( U(1) \) current algebras upon quantization. Two sets of these modes are analogous to the left- and right-moving modes on a circle, while the remaining modes are unique to the figure eight. We then apply the Sugawara construction to these modes to obtain three classical Virasoro or Witt algebras with generators we denote by \( L^+_n, L^-_n \) and \( L^0_n \).

Normally, the existence of a Virasoro algebra indicates the presence of a conformal symmetry. We examine the question of conformal symmetry for the figure eight in Section 5. We show that, unlike a massless scalar field theory on a circle, the analogous theory on a figure eight is not invariant under separate left and right conformal transformations. Instead, the action plus boundary conditions are, at best, invariant only under a single set of transformations. The conserved current corresponding to the conformal symmetry transformation is shown to satisfy Kirchhoff’s law at the junction. This conformal symmetry exists provided \( \frac{\ell_1 v_2}{\ell_2 v_1} \) is rational. When \( \frac{\ell_1 v_2}{\ell_2 v_1} \) is not rational, there exists no analogue of conformal symmetry for the figure eight. For the former case, we find the associated conserved charges, and show that they generate the Virasoro algebra with zero central charge, which (as alluded to before) is also called the Witt algebra. If in addition to \( \frac{\ell_1 v_2}{\ell_2 v_1} \) being rational, the parameters satisfy the case \( b \) conditions \( \frac{\ell_1}{T_1} = \frac{v_1}{v_2} = \frac{T_2}{T_2} \), this algebra is spanned by \( L^+_n + L^-_n + \frac{1}{2} L^0_n \). Until this stage, our treatment is purely classical. The
quantum mechanical version of the above algebra, complete with the central extension, is commented on at the end of Section 5.

In Section 6, we show that, unlike on a circle, the left- and right-moving chiral currents of the classical theory cannot be independently quantized on the figure eight. By this we mean that the two chiral currents cannot be expanded in terms of two independent sets of bases such that their quantum analogues i) have a well defined action on the Fock space, and ii) provide a quantization of the currents which is unitarily equivalent to that derived from the eigenmodes of the one-particle Hamiltonian of the system.

In Appendix A of this paper, we sketch the possibility of having discontinuous boundary conditions for the scalar field at the junction. Boundary conditions, in general, are restricted only by the requirement that a certain differential operator acting on a Hilbert space of square integrable functions is self-adjoint, and there are such conditions admitting these discontinuities. In Appendix B, we write down the general solutions to the field equations on the figure eight consistent with the boundary conditions of Section 2, and carry out the eigenmode expansions of fields and currents for two special choices of the parameters of the figure eight corresponding to cases a and b.

2. The Singlevaluedness Condition and Kirchhoff’s Law

We first introduce a set of coordinates on the figure eight. Let $x$ be the spatial coordinate, with $0 \leq x \leq \ell_1 + \ell_2$, and let $t$ be time. We choose $x$ so that we are on loop 1 when $0 \leq x \leq \ell_1$ and we are on loop 2 when $\ell_1 \leq x \leq \ell_1 + \ell_2$. $x = 0 = \ell_1 = \ell_1 + \ell_2$ are all assumed to correspond to the same point, namely the junction (see Figure 1). Next, we introduce a complex scalar field $\Phi$ which is a function of $x$ and $t$. For the sake of simplicity, let us hold the magnitude of $\Phi(x, t)$ to be fixed at one, so that it just defines a single degree of freedom, a phase. If desired, we can justify this approximation by assuming
the presence of a symmetry breaking potential in the Lagrangian for the system, such as
$V(\Phi) = \lambda (\Phi^* \Phi - 1)^2$. Then $\Phi^* \Phi$ is frozen to 1 and we are left with just a phase $\chi$ defined by

$$\Phi(x, t) = e^{i\chi(x, t)}$$

in the limit $\lambda \to \infty$. For the dynamics of $\chi(x, t)$, we shall assume the free wave equation

$$\left[ \partial_x^2 - \frac{1}{v_a^2} \partial_t^2 \right] \chi(x, t) = 0$$

where $v_a$ represents the wave velocity on loop $a$.

Fig. 1. Figure 8 with its coordinates.

Rather than work with the spatial coordinate $x$, we find it more convenient to use another coordinate $\sigma$ where $0 \leq \sigma \leq 2\pi$. It is defined so that there is a two-to-one mapping from $\{x\}$ to $\{\sigma\}$. It is such that, a given value of $\sigma$ corresponds to a point on loop 1, and also to a point on loop 2. The relation between $x$ and $\sigma$ for points on loop 1
is
\[ x = \frac{\ell_1}{2\pi} \sigma , \]  
\tag{2.3}
while for loop 2, it is
\[ x = \frac{\ell_2}{2\pi} \sigma + \ell_1 . \]  
\tag{2.4}

With \( \sigma \) as the coordinate, it becomes necessary to distinguish the fields on the two loops of figure eight. For this purpose, we replace \( \Phi \) by a two component field \( \phi \) where \( \phi(\sigma,t) = (\phi_1(\sigma,t),\phi_2(\sigma,t)) \). \( \phi_a \) corresponds to the field \( \Phi \) evaluated on loop \( a \). More precisely, we define \( \phi_a \) by
\[ \phi_1(\sigma,t) = \Phi\left(\frac{\ell_1}{2\pi} \sigma,t\right) \quad \text{and} \quad \phi_2(\sigma,t) = \Phi\left(\frac{\ell_2}{2\pi} \sigma + \ell_1 , t\right) . \]  
\tag{2.5}

Since \( \Phi \) is a phase, so is \( \phi_a \) and we can write \( \phi_a(\sigma,t) = e^{i\chi_a(\sigma,t)} \). In terms of the degrees of freedom \( \chi_a \), the wave equation (2.2) becomes
\[ \left[ \frac{1}{\kappa_a^2} \partial^2_\sigma - \partial^2_t \right] \chi_a(\sigma,t) = 0 , \quad a = 1, 2 , \]  
\tag{2.6}
where \( \kappa_a = \frac{\ell_a}{2\pi v_a} \).

Eq. (2.6), by itself, is not sufficient to completely specify the dynamics of the system. It has to be supplemented with boundary conditions on the fields \( \chi_a \) at the junction. To show this, we first note that the substitution of \( \chi_a(\sigma,t) = e^{i\omega t} \bar{\chi}_a(\sigma) \) in eq. (2.6) leads to the eigenvalue equation
\[ \left[ H_a - \omega^2 \right] \bar{\chi}_a(\sigma) = 0 , \quad H_a = -\frac{1}{\kappa_a^2} \partial^2_\sigma . \]  
\tag{2.7}

The eigenfunctions of \( H_a \) will be interpreted as single particle wavefunctions in quantum theory. These eigenfunctions must form a complete set in the Hilbert space of square integrable functions of the figure eight for time evolution of the second quantized field theory to be unitary. This space consists of functions \( \bar{\chi} \equiv (\bar{\chi}_1, \bar{\chi}_2) \) with the inner product
\[ < \bar{\chi}, \bar{\psi} > = \sum_{a=1,2} \nu_a \kappa_a^2 \int d\sigma \ \bar{\chi}_a^*(\sigma) \bar{\psi}_a(\sigma) . \]  
\tag{2.8}
Here we have introduced a new parameter $\nu_a$, since in general, the coefficient of the measure $d\sigma$ need not be the same for the two loops.

Let us define the single particle Hamiltonian $H$ by $H\tilde{\chi} = (H_1\tilde{\chi}_1, H_2\tilde{\chi}_2)$. This is only a formal definition of $H$, since we have not specified its domain $\mathcal{D}(H)$. We can require that $\mathcal{D}(H)$ is so chosen that $H$ is self-adjoint, this condition being compatible with physical principles. There are an infinite number of options for $\mathcal{D}(H)$ consistent with this requirement. They correspond to different boundary conditions for the functions $\tilde{\chi} \in \mathcal{D}(H)$ at the junction, and lead to inequivalent definitions of the operator $H$. (We sketch the different possibilities in Appendix A.) These possibilities can be thought of as describing different junctions and the right one, in a given problem, must be chosen on physical grounds.

In this paper, we think of the figure eight as made of two superconducting wires, and of the field $\Phi$ as the order parameter. Then, if no potential is applied across the junction, $\Phi$ has to be continuous there. Therefore,

\[
\Phi(0,t) = \Phi(\ell_1,t) = \Phi(\ell_1 + \ell_2, t),
\]

or in terms of $\phi_a$,

\[
\phi_1(0,t) = \phi_1(2\pi,t) = \phi_2(0,t) = \phi_2(2\pi,t).
\]

Consequently, $\chi$ is allowed to have $2\pi$ discontinuities across the junction. These discontinuities, which represent winding modes of the field $\phi$ or $\chi$, are topologically stable under time evolution. A typical winding mode for $\chi_a$ is proportional to $\sigma$ for all $t$. It fulfills the wave equation (2.6). After subtracting such modes from $\chi_a$, we can regard $\chi$ too to be continuous across the junction. This requirement picks up a unique definition for $H$, namely that specified by the following domain:

\[
\mathcal{D}(H) = \left\{ \tilde{\chi} \mid \tilde{\chi}_1(0) = \tilde{\chi}_1(2\pi) = \tilde{\chi}_2(0) = \tilde{\chi}_2(2\pi), \sum_{a=1,2} \nu_a \partial_\sigma \tilde{\chi}_a |_{0}^{2\pi} = 0 \right\}.
\]
It is easy to verify that $H$ defined above is self-adjoint in the following manner: If $\tilde{\chi}$ is an arbitrary element of $\mathcal{D}(H)$, then the domain $\mathcal{D}(H^\dagger)$ of the adjoint $H^\dagger$ of $H$ consists of functions $\tilde{\psi}$ in the Hilbert space which fulfill

$$<\tilde{\psi}, H\tilde{\chi}> = <H\tilde{\psi}, \tilde{\chi}>.$$  \hspace{1cm} (2.12)

$H$ is self-adjoint if and only if $\mathcal{D}(H^\dagger) = \mathcal{D}(H)$. Now eq. (2.12) implies that

$$0 = -\sum_{a=1,2} \nu_a \left( \int d\sigma \tilde{\psi}_a^*(\sigma) \partial_\sigma^2 \tilde{\chi}_a(\sigma) \right) \bigg|_{0}^{2\pi} + \sum_{a=1,2} \nu_a \left( \int d\sigma \tilde{\psi}_a^*(\sigma) \tilde{\chi}_a(\sigma) \right) \bigg|_{0}^{2\pi}.$$  \hspace{1cm} (2.13)

Since the boundary values of $\tilde{\chi}$ and $\partial_\sigma \tilde{\chi}$ are arbitrary but for the conditions (2.11), we must have

$$\tilde{\psi}_1(0) = \tilde{\psi}_1(2\pi) = \tilde{\psi}_2(0) = \tilde{\psi}_2(2\pi) \quad \text{and} \quad \sum_{a=1,2} \nu_a \tilde{\psi}_a|_0^{2\pi} = 0.$$  

Hence $\tilde{\psi}$ is an element of $\mathcal{D}(H)$, or equivalently, the domain $\mathcal{D}(H^\dagger)$ of $H^\dagger$ is the same as $\mathcal{D}(H)$, proving that $H$ is self-adjoint.

The wave equation (2.2) and the boundary conditions (2.11) are obtainable from an action principle, the action $S$ being the sum of two terms:

$$S = S_1 + S_2, \quad S_a = \frac{\nu_a}{2} \int d\sigma dt \left( \kappa_a^2 |\partial_\sigma \phi_a|^2 - |\partial_t \phi_a|^2 \right).$$  \hspace{1cm} (2.14)

In the original coordinates $(x, t)$, the terms $S_1$ and $S_2$ can be written as

$$S_1 = \frac{T_1}{2} \int dt \int_0^{\ell_1} dx \left( \frac{1}{\nu_1^2} |\partial_t \Phi|^2 - |\partial_x \Phi|^2 \right),$$

$$S_2 = \frac{T_2}{2} \int dt \int_{\ell_1}^{\ell_2} dx \left( \frac{1}{\nu_2^2} |\partial_t \Phi|^2 - |\partial_x \Phi|^2 \right),$$  \hspace{1cm} (2.15)

$$T_a := \frac{\nu_a l_a}{2\pi}.$$  

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From (2.15), we see that $T_a$ can be interpreted as the “tension” in loop $a$. If the loops are made of different superconducting materials, there is no reason why $T_a$ should be identical.

To obtain the wave equation (2.6), we extremize (2.14) for variations of the field $\phi_a$ which vanish at the junction. If next we allow also variations of $\chi$ that are continuous at the junction, we recover in addition the boundary condition $\sum_{a=1,2} \nu_a \partial_\sigma \chi_a \big|_{0}^{2\pi} = 0$.

The solutions to the equations of motion (2.6) are of the form

$$\chi_a(\sigma, t) = \chi_a^+ (\sigma_a^+) + \chi_a^- (\sigma_a^-), \quad (2.16)$$

where $\sigma_a^\pm = \kappa_a \sigma \pm t$.

The equations of motion (2.6) can be recast in terms of current conservation laws. For this purpose, we define the time-components of the currents by

$$J_t^a = -\frac{i\nu_a \kappa_a}{2} (\phi_a^* \partial_t \phi_a - \phi_a \partial_t \phi_a^*) = \nu_a \kappa_a \partial_t \chi_a \quad \text{(no sum on } a \text{) ,} \quad (2.17)$$

and the space-components by

$$J_\sigma^a = -\frac{i\nu_a}{2} (\phi_a^* \partial_\sigma \phi_a - \phi_a \partial_\sigma \phi_a^*) = \nu_a \partial_\sigma \chi_a \quad \text{(no sum on } a \text{) .} \quad (2.18)$$

Then eqs. (2.6) imply that the currents are conserved:

$$\kappa_a \partial_t J_t^a - \partial_\sigma J_\sigma^a = 0 \quad (2.19)$$

From the solutions (2.16) to the equations of motion, we can form left- and right-moving combinations $J_\pm^a$ of currents. They are defined according to

$$J_\pm^a = J_\sigma^a \pm J_t^a \quad (2.20)$$

The solutions imply that $J_+^a$ and $J_-^a$ is each a function of just one variable:

$$J_+^a(\sigma_a^+) = 2\nu_a \kappa_a \frac{\partial \chi_a^+}{\partial \sigma_a^+} \quad \text{and} \quad J_-^a(\sigma_a^-) = 2\nu_a \kappa_a \frac{\partial \chi_a^-}{\partial \sigma_a^-} \quad (2.21)$$
Our choice for the coordinate $\sigma$ selects a particular orientation on the figure eight. The space-component of currents $J^a_\sigma$ will be regarded as positive (negative) when the current flows in the direction of increasing (decreasing) $\sigma$. Thus a positive $J^a_\sigma(0,t)$ corresponds to a current leaving the junction and flowing into loop $a$, and a positive $J^a_\sigma(2\pi,t)$ corresponds to a current entering the junction from loop $a$. The boundary condition for the space derivatives of $\chi$ is therefore just the Kirchhoff law for the currents, as it states that the total current flowing into the junction equals the total current flowing out of the junction:

$$\sum_{a=1,2} J^a_\sigma(0,t) = \sum_{a=1,2} J^a_\sigma(2\pi,t). \quad (2.22)$$

By taking the time derivative of this condition, we further have that

$$\frac{d}{dt} \left( \sum_{a=1,2} \nu_a \partial_\sigma \chi_a \right)|^{2\pi}_0 = 0 \text{ or } \left( \sum_{a=1,2} \frac{1}{\kappa_a} \partial_\sigma J^a_t \right)|^{2\pi}_0 = 0. \quad (2.23)$$

Boundary conditions exist also for the time-components $J^a_t$ of the currents. They are obtained by requiring that the boundary conditions (2.10) are preserved in time, that is $\partial_t \phi_1(0,t) = \partial_t \phi_1(2\pi,t) = \partial_t \phi_2(0,t) = \partial_t \phi_2(2\pi,t)$. This implies that the time component of the current satisfies

$$\frac{1}{\nu_1 \kappa_1} J^1_t(0,t) = \frac{1}{\nu_1 \kappa_1} J^1_t(2\pi,t) = \frac{1}{\nu_2 \kappa_2} J^2_t(0,t) = \frac{1}{\nu_2 \kappa_2} J^2_t(2\pi,t). \quad (2.24)$$

Thus the charge density is discontinuous at the junction when $\nu_1 \kappa_1 \neq \nu_2 \kappa_2$.

### 3. Current Algebra

In the Hamiltonian formulation of the theory, $\kappa_a J^a_t$ is canonically conjugate to the field $\chi_a$. We thus have the equal time Poisson brackets

$$\{ \chi_a(\sigma,t), J^b_t(\sigma',t) \} = \frac{1}{\kappa_a} \delta^b_a \delta(\sigma - \sigma'). \quad (3.1)$$
Upon using the definition (2.20) for the left and right moving currents $J^a_{\pm}$, we can also naively compute the Poisson brackets between the currents:

\[
\{ J^a_\pm(\sigma, t), J^b_\pm(\sigma', t) \} = \pm \frac{2\nu_a}{\kappa_a} \delta^b_a \partial_\sigma \delta(\sigma - \sigma') , \quad (3.2)
\]

\[
\{ J^a_\pm(\sigma, t), J^b_\pm(\sigma', t) \} = 0 . \quad (3.3)
\]

This resembles the current algebra for two scalar fields on a circle. However, the results (3.1-3) are only formal because we have not a) defined the delta function on a figure eight, and b) taken into account the boundary conditions on the fields and currents. Thus, for instance, the Poisson brackets which we have found cannot be valid in the limit where we approach the junction $\sigma = 0, 2\pi$. As a result, the application of (3.1-3) leads to incorrect Hamilton’s equations of motion for the time evolution of the system, as shown by the following: From the Hamiltonian

\[
\mathcal{H} = \sum_{a=1,2} \frac{1}{4\nu_a} \int_0^{2\pi} d\sigma \left[ (J^a_+(\sigma))^2 + (J^a_-\sigma) \right] , \quad (3.4)
\]

and (3.3), one would obtain the result that $\int_0^{2\pi} d\sigma (J^a_+(\sigma))^2$ is a constant of motion:

\[
\partial_t \int_0^{2\pi} d\sigma (J^a_+(\sigma))^2 = \left\{ \int_0^{2\pi} d\sigma (J^a_+(\sigma))^2 , \mathcal{H} \right\} = 0 . \quad (3.5)
\]

But this is incorrect because from current conservation (2.19) and the identity $\partial_\sigma J^a_\sigma = \kappa_a \partial_t J^a_\sigma$, we instead get

\[
\partial_t \int_0^{2\pi} d\sigma (J^a_+(\sigma))^2 = \frac{1}{\kappa_a} \left( J^a_+(\sigma) \right)^2 \bigg|_0^{2\pi} , \quad (3.6)
\]

where $\left( J^a_+(\sigma) \right)^2$ need not satisfy $2\pi$ periodic boundary conditions.

In order to account for the boundary conditions and obtain the correct Poisson bracket relations, let us introduce a set of “smearing” or “test” functions $\Lambda = (\Lambda^1_+, \Lambda^2_+, \Lambda^1_-, \Lambda^2_-)$ for the currents $J^a_{\pm}$. (We shall ignore the $t$ dependence.) Next we define the “smeared current” $\mathcal{J}(\Lambda)$ as follows:

\[
\mathcal{J}(\Lambda) = \sum_{a=1,2} \int d\sigma \left[ \Lambda^a_+(\sigma) J^a_+(\sigma) + \Lambda^a_-\sigma J^a_-(\sigma) \right]
\]
\[ \
\sum_{a=1,2} \int d\sigma \left[ (\Lambda_+^a + \Lambda_-^a) \nu_a \partial_{\sigma} \chi_a + (\Lambda_+^a - \Lambda_-^a) J_t^a \right]. \quad (3.7) \\
\]

In order to be able to define Poisson brackets involving the “smeared current” \( \mathcal{J}(\Lambda) \) consistently, we shall require that \( \mathcal{J}(\Lambda) \) is differentiable with respect to the phase space variables \( \chi_a \) and \( J_t^a \). From the definition (3.7), we see that differentiability is assured for variations in \( J_t^a \). But that is not, in general, true for variations \( \delta \chi_a \) in \( \chi_a \) as such variations will in general create boundary terms:

\[ \delta \mathcal{J}(\Lambda) = - \sum_{a=1,2} \nu_a \int d\sigma \partial_{\sigma} (\Lambda_+^a + \Lambda_-^a) \delta \chi_a + \sum_{a=1,2} \nu_a (\Lambda_+^a + \Lambda_-^a) \delta \chi_a \bigg|_{\sigma=0}^{\sigma=2\pi}. \quad (3.8) \]

If we assume continuity of the phase at the junction, so that \( \delta \chi_1(0,t) = \delta \chi_1(2\pi,t) = \delta \chi_2(0,t) = \delta \chi_2(2\pi,t) \), then the boundary term in eq. (3.8) can be made to vanish by requiring that

\[ \sum_{a=1,2} \nu_a (\Lambda_+^a + \Lambda_-^a) \bigg|_{\sigma=0}^{\sigma=2\pi} = 0. \quad (3.9) \]

We call \( \mathcal{T} \) the space of all test functions \( \Lambda \) satisfying eq. (3.9). It is our test function space for the currents. For \( \Lambda \in \mathcal{T} \), the variational derivatives of \( \mathcal{J}(\Lambda) \) with respect to \( \chi_a(\sigma) \) and \( J_t^a(\sigma) \) are given by

\[ \frac{\delta \mathcal{J}(\Lambda)}{\delta \chi_a(\sigma)} = -\nu_a \partial_{\sigma} \left[ \Lambda_+^a(\sigma) + \Lambda_-^a(\sigma) \right] \quad \text{and} \quad \frac{\delta \mathcal{J}(\Lambda)}{\delta J_t^a(\sigma)} = \Lambda_+^a(\sigma) - \Lambda_-^a(\sigma). \quad (3.10) \]

We can now compute the Poisson brackets between two smeared currents \( \mathcal{J}(\Lambda) \) and \( \mathcal{J}(\overline{\Lambda}) \), for \( \Lambda, \overline{\Lambda} \in \mathcal{T} \). Care is necessary in performing this computation as the \( \delta \) functions in (3.1) and (3.2) do not have all the usual properties. Perhaps the best way is to start with the basic definition

\[ \{ \mathcal{J}(\Lambda), \mathcal{J}(\overline{\Lambda}) \} = \sum_{a=1,2} \frac{\nu_a}{\kappa_a} \int_0^{2\pi} d\sigma \left( \frac{\delta \mathcal{J}(\Lambda)}{\delta \chi_a(\sigma)} \frac{\delta \mathcal{J}(\overline{\Lambda})}{\delta J_t^a(\sigma)} - (\Lambda \leftrightarrow \overline{\Lambda}) \right) \quad (3.11) \]

of the Poisson bracket and use (3.10). We then find,

\[ \{ \mathcal{J}(\Lambda), \mathcal{J}(\overline{\Lambda}) \} = \sum_{a=1,2} \frac{\nu_a}{\kappa_a} \int d\sigma (\Lambda_+^a \partial_{\sigma} \overline{\Lambda}_+^a - \overline{\Lambda}_+^a \partial_{\sigma} \Lambda_+^a) - \sum_{a=1,2} \frac{\nu_a}{\kappa_a} \int d\sigma (\Lambda_-^a \partial_{\sigma} \overline{\Lambda}_-^a - \overline{\Lambda}_-^a \partial_{\sigma} \Lambda_-^a) \]
\[
+ \sum_{a=1,2} \frac{\nu_a}{\kappa_a} (\Lambda^a_+ \overline{\Lambda}^a_+ - \overline{\Lambda}^a_- \Lambda^a_-) \bigg|_{\sigma=0}^{\sigma=2\pi}.
\] 

(3.12)

Of special interest is the boundary term in eq. (3.12). It is zero when the smearing functions are continuous at the junction, so that \(\Lambda^a_\pm(2\pi) = \Lambda^a_\pm(0)\) and \(\overline{\Lambda}^a_\pm(2\pi) = \overline{\Lambda}^a_\pm(0)\). In that case, we recover the result that the Poisson brackets between left and right moving currents are zero, which is consistent with eq. (3.3). On the other hand, eq. (3.12) shows that, contrary to eq. (3.3), there may be cases where the Poisson brackets between left and right moving currents do not vanish. This can happen, for instance, when one of the test functions \(\Lambda^a_+\) or \(\overline{\Lambda}^a_-\) is not continuous at the junction [but consistent with eq. (3.9)].

4. Periodic Boundary Conditions and Currents

We now examine the boundary conditions on the currents for two special cases of the parameters \(\kappa_a\) and \(\nu_a\). They are: a. \(\kappa_1 \nu_1 = \kappa_2 \nu_2\) (or equivalently, \(\frac{T_1}{\nu_1} = \frac{T_2}{\nu_2}\)), and b. \(\kappa_1 = \kappa_2 = \kappa\) and \(\nu_1 = \nu_2 = \nu\). As the case b parameters satisfy \(\kappa_1 \nu_1 = \kappa_2 \nu_2\), case b is actually a subcase of case a. (A detailed analysis of the solutions to the equations of motion (2.6) for both of these cases will be discussed in Appendix B.)

We first consider case a.

Case a \(\kappa_1 \nu_1 = \kappa_2 \nu_2\)

Here we can show that the currents \(J_\pm^a(\sigma, t)\) can be written as linear combinations of functions of \(\sigma\) and \(t\), where these functions satisfy either 2\(\pi\) or 4\(\pi\) periodic boundary conditions with regards to the spatial coordinate \(\sigma\).

Functions with 2\(\pi\) periodic boundary conditions are obtained by taking the sum of \(J^1_\pm\) and \(J^2_\pm\):

\[
J^\text{sum}_\pm := J^1_\pm + J^2_\pm, \quad J^\text{sum}_\pm(2\pi, t) = J^\text{sum}_\pm(0, t).
\] 

(4.1)

This result is due to the Kirchhoff law (2.22) and the boundary conditions (2.24), which reduce to \(J^1_\pm(0, t) = J^1_\pm(2\pi, t) = J^2_\pm(0, t) = J^2_\pm(2\pi, t)\) when we set \(\kappa_1 \nu_1 = \kappa_2 \nu_2\).
Functions with $4\pi$ periodic boundary conditions can be constructed by first taking the difference of $J^1_\pm$ and $J^2_\pm$:

$$J^{\text{dif}}_\pm := J^2_\pm - J^1_\pm .$$

Then eqs. (2.22) and (2.24) imply that

$$J^{\text{dif}}_+(0, t) = J^{\text{dif}}_-(0, t) \quad \text{and} \quad J^{\text{dif}}_+(2\pi, t) = J^{\text{dif}}_-(2\pi, t) . \tag{4.2}$$

To analyze these conditions it is helpful to introduce yet another function $K(s, t)$, which is defined on the spatial domain $\{ s; 0 \leq s \leq 4\pi \}$ as follows:

$$K(s, t) = \begin{cases} J^{\text{dif}}_+(s, t), & \text{if } 0 \leq s \leq 2\pi \\ J^{\text{dif}}_-(4\pi - s, t), & \text{if } 2\pi \leq s \leq 4\pi . \end{cases} \tag{4.3}$$

In view of eqs. (4.2), this function is continuous in $s$ and satisfies the $4\pi$ periodic boundary condition $K(0, t) = K(4\pi, t)$.

**Case b.** $\kappa_1 = \kappa_2 = \kappa$ and $\nu_1 = \nu_2 = \nu$

We discuss this case in the remainder of this Section.

In case b, the time evolution of the functions $J^{\text{sum}}_\pm(\sigma, t)$ and $K(s, t)$ can be given in a simple closed form, analogous to that found for the chiral currents on a circle. In fact, they can be expressed in terms of periodic functions of only one argument. We denote these functions by $f^{\text{sum}}_\pm$ and $f$. Then the result may be stated as follows:

$$J^{\text{sum}}_\pm(\sigma, t) = f^{\text{sum}}_\pm(\sigma \pm t/\kappa) ,$$

$$K(s, t) = f(s + t/\kappa) , \tag{4.4}$$

where

$$f^{\text{sum}}_\pm(x + 2\pi) = f^{\text{sum}}_\pm(x) , \quad f(x + 4\pi) = f(x) ; \quad -\infty < x < \infty . \tag{4.5}$$

To prove eqs. (4.4) and (4.5), we just recall that, as a consequence of eq. (2.16) and thanks to the condition $\kappa_1 = \kappa_2 = \kappa$, we can write the currents $J^a_\pm$ as functions of just a
single variable, the same for both loops as in eq. (2.21):

\[ J^a_\pm (\sigma, t) = f^a_\pm (\sigma \pm t/\kappa) . \]  

(4.6)

Upon substituting (4.6) into (4.1), we get

\[ f^\text{sum}_{\pm} (2\pi \pm t/\kappa) = f^\text{sum}_{\pm} (\pm t/\kappa) \]

where \( f^\text{sum}_{\pm} (x) \equiv (f^1_{\pm} + f^2_{\pm})(x) \). This is equivalent to the result \( f^\text{sum}_{\pm} (x + 2\pi) = f^\text{sum}_{\pm} (x) \).

As for the function \( K(s, t) \), upon substituting (4.6) into (4.3), we get

\[
K(s, t) = \begin{cases} 
(f^2_{\pm} - f^1_{\pm})(s + t/\kappa) & \text{if } 0 \leq s \leq 2\pi \\
(f^2_{\pm} - f^1_{\pm})(4\pi - s - t/\kappa) & \text{if } 2\pi \leq s \leq 4\pi 
\end{cases} .
\]  

(4.7)

But we have already proved, under case \( a \), that the function \( K(s, t) \) is continuous in \( s \) (in particular, at \( s = 2\pi \)) and satisfies \( 4\pi \) periodic conditions at all times. The former implies that

\[ f^\text{diff}_{\pm} (2\pi + t/\kappa) = f^\text{diff}_{\pm} (2\pi + t/\kappa) , \]

or that \( f^\text{diff}_{\pm} (x) = f^\text{diff}_{\pm} (x) \equiv f(x) \), while the latter implies that

\[ f^\text{diff}_{\pm} (t/\kappa) = f^\text{diff}_{\pm} (4\pi + t/\kappa) , \]

or that \( f(4\pi + x) = f(x) \). We have thus proved eqs. (4.4) and (4.5).

The periodicity of the currents allows us to make the Fourier expansions

\[
J^\text{sum}_{\pm} (\sigma, t) = \sum_{n=-\infty}^{\infty} \alpha^\pm_n (0) e^{-in(t/\kappa \pm \sigma)} ,
\]

(4.8)

\[
K(s, t) = \sum_{n=-\infty}^{\infty} \beta^2_n (0) e^{-i\frac{n}{2}(t/\kappa + s)} ,
\]

(4.9)

where \( n = 0, \pm1, \pm2, \ldots \) and \( \alpha^\pm_n (0) \equiv \alpha^\pm_n \) and \( \beta^2_n (0) \equiv \beta^2_n \) represent the values of the coefficients at time \( t = 0 \). The reality of the currents implies that \( \alpha_{-n} = (\alpha_n^\pm)^* \) and \( \beta_{-\frac{1}{2}} = (\beta_{\frac{1}{2}})^* \). From eqs. (4.8) and (4.9), we can obtain a basis for the test functions \( \Lambda = \)
$$(\Lambda^1_+ \Lambda^2_+, \Lambda^1_- \Lambda^2_-$$) appearing in the smeared currents (3.7). The test functions associated with the coefficients $\alpha_n^+, \alpha_n^-$ and $\beta_n^2$ are

$$
\Lambda^{(\alpha_n^+)} = -\frac{1}{2\pi} (e^{in\sigma}, e^{in\sigma}, 0, 0),
\Lambda^{(\alpha_n^-)} = \frac{1}{2\pi} (0, 0, e^{-in\sigma}, e^{-in\sigma}),
\Lambda^{(\beta_n^2)} = \frac{1}{4\pi} (e^{in\sigma/2}, -e^{in\sigma/2}, e^{-in\sigma/2}, -e^{-in\sigma/2}),
$$

(4.10)

respectively. These test functions satisfy the condition (3.9), and hence belong to the set $\mathcal{T}$. As $\alpha_n^+, \alpha_n^-$ and $\beta_n^2$ form a complete set of coefficients, the $\Lambda(X)$’s for $X = \alpha_n^+, \alpha_n^-$ and $\beta_n^2$ form a complete set of test functions spanning $\mathcal{T}$. The first two types of test functions $\Lambda^{(\alpha_{\pm}^n)}$ are associated with left- and right-moving modes, analogous to the modes on a circle, while the last type of test functions $\Lambda^{(\beta_n^2)}$ is unique to the figure eight.

The above $\Lambda^{(X)}$’s, are normalized to satisfy

$$
\mathcal{J}(\Lambda^{(X)}) = X.
$$

(4.11)

We can use this relation and (3.12) to compute the Poisson brackets of $\alpha_n^\pm$ and $\beta_n^2$. The nonzero brackets

$$
\{\alpha_m^{\pm}, \alpha_{-m}^\pm\} = -\frac{2in\nu}{\pi\kappa} \delta_{m,n}, \quad \{\beta_{\pm}^2, \beta_{-\pm}^2\} = -\frac{in\nu}{2\pi\kappa} \delta_{m,n},
$$

(4.12)

define three U(1) affine Lie algebras [7].

Of course, given the three classical affine U(1) algebras above, we can construct three classical Virasoro or Witt algebras, the generators being

$$
L_n^\pm = \frac{\pi\kappa}{4\nu} \sum_m \alpha_m^\pm \alpha_{-m}^\pm \quad \text{and} \quad L_n^0 = \frac{\pi\kappa}{\nu} \sum_m \beta_m^2 \beta_{-m}^2.
$$

(4.13)

From the Poisson brackets (4.12), it follows that

$$
\{L_n^\pm, L_m^\pm\} = -i(n-m)L_{n+m}^\pm \quad \text{and} \quad \{L_n^0, L_m^0\} = -i(n-m)L_{n+m}^0.
$$

(4.14)
Just as for conformal field theories on a circle, the $n = 0$ generators appear in the expression for the Hamiltonian since

$$H = \frac{1}{4\nu} \sum_{a=1,2} \int_0^{2\pi} d\sigma \left[ \left( J_a^+ (\sigma, t) \right)^2 + \left( J_a^- (\sigma, t) \right)^2 \right]$$

$$= \frac{1}{\kappa} \left( L_0^+ + L_0^- + \frac{1}{2} L_0^0 \right). \quad (4.15)$$

In quantum theory, we promote $\alpha_n^\pm$ and $\beta_n^2$ to operators, and replace the Poisson brackets of (4.12) by $-i$ times commutator brackets. The quantum operators act on a Fock space, and we assume, as usual, that $\alpha_n^\pm$ and $\beta_n^2$ for $n > 0$ annihilate its vacuum $|0\rangle$ and are destruction operators. The nonvacuum states of the Fock space are obtained by acting on $|0\rangle$ with $\alpha_n^\pm$ and $\beta_n^2$ for $n \leq 0$. The quantum version of the Virasoro generators are assumed to be normal ordered, with destruction operators appearing on the right. The classical Virasoro algebras are then modified by the standard central terms, with each algebra having central charge $c = 1$:

$$[L_n^\pm, L_m^\pm] = (n - m)L_{n+m}^\pm + \frac{1}{12} n(n^2 - 1)\delta_{n+m,0},$$

$$[L_n^0, L_m^0] = (n - m)L_{n+m}^0 + \frac{1}{12} n(n^2 - 1)\delta_{n+m,0}. \quad (4.16)$$

The eigenvalues of the Hamiltonian (4.15) are easily determined. If the vacuum is associated with zero energy, then by acting on $|0\rangle$ with $\alpha_n^\pm$ for $n > 0$, we obtain a state with energy equal to $\frac{n}{\kappa}$. By acting on $|0\rangle$ with $\beta_n^2$ for $n > 0$, we obtain a state with energy equal to $\frac{n}{2\kappa}$.

5. The Question of Conformal Symmetry

Normally, the existence of Virasoro algebras indicates that the system is conformally invariant. However, the notion of conformal invariance for fields defined on manifolds such as a circle and on networks are quite different. We will make this fact evident below.
The action (2.14) for fields on the figure eight for arbitrary $\kappa_a$ and $\nu_a$ can be written in the form

$$S = S_1 + S_2, \quad S_a = \nu_a\kappa_a \int d\sigma^+_a d\sigma^-_a \frac{\partial \chi_a}{\partial \sigma^+_a} \frac{\partial \chi_a}{\partial \sigma^-_a}, \quad \sigma^\pm_a = \kappa_a \sigma \pm t,$$

(5.1)

which by itself displays the usual conformal symmetries

$$\sigma^+_a \to \sigma^+_a + F^+_a(\sigma^+_a),$$

(5.2)

$$\sigma^-_a \to \sigma^-_a + F^-_a(\sigma^-_a).$$

(5.3)

However, once we impose the boundary conditions for the fields on a figure eight, the symmetry transformations (5.2) and (5.3) will not be independent. For infinitesimal $F^+_a$’s and $F^-_a$’s, the fields $\chi_a$ undergo the variations

$$\delta \chi_a = \frac{1}{2\nu_a\kappa_a} \left[ \left( F^+_a(\sigma^+_a) - F^-_a(\sigma^-_a) \right) J^a_t + \left( F^+_a(\sigma^+_a) + F^-_a(\sigma^-_a) \right) J^a_s \right].$$

(5.4)

Consistency with the boundary conditions (2.11) means that $\delta \chi_1(0, t) = \delta \chi_1(2\pi, t) = \delta \chi_2(0, t) = \delta \chi_2(2\pi, t)$. From the first term in brackets and the conditions (2.24), we then get

$$F^+_1(t) - F^-_1(-t) = F^+_1(2\pi\kappa_1 + t) - F^-_1(2\pi\kappa_1 - t)$$

$$= F^+_2(t) - F^-_2(-t) = F^+_2(2\pi\kappa_2 + t) - F^-_2(2\pi\kappa_2 - t).$$

(5.5)

From the second term in brackets, we get

$$0 = F^+_1(t) + F^-_1(-t) = F^+_1(2\pi\kappa_1 + t) + F^-_1(2\pi\kappa_1 - t)$$

$$= F^+_2(t) + F^-_2(-t) = F^+_2(2\pi\kappa_2 + t) + F^-_2(2\pi\kappa_2 - t).$$

(5.6)

Upon combining eqs. (5.5) and (5.6), we have

$$F^+_1(t) = F^+_2(t) = -F^-_1(-t) = -F^-_2(-t) \equiv F(t)$$

(5.7)
and
\[ F(t) = F(2\pi \kappa_1 + t) = F(2\pi \kappa_2 + t) = -F(2\pi \kappa_1 - t) = -F(2\pi \kappa_2 - t) . \] (5.8)

Eqs. (5.7) and (5.8) state that all \( F_a^\pm \) are given by just one independent function \( F \) which is odd in \( t \) and simultaneously \( 2\pi \kappa_1 \) periodic and \( 2\pi \kappa_2 \) periodic. This, of course, is possible only when \( \frac{\kappa_1}{\kappa_2} \) is rational (if the trivial case where \( F \) is the zero function is excluded). So a nontrivial analogue of conformal symmetry exists only in this case. We shall assume that \( \frac{\kappa_1}{\kappa_2} \) is rational in the rest of this Section. We note also that unlike the analogous field theory on a circle, there do not exist separate left and right conformal transformations. As a result, there do not exist two commuting sets of conformal generators on the figure eight, as there do on the circle.

We note that the transformations (5.2) and (5.3), along with the restrictions (5.7) and (5.8), preserve Kirchhoff’s law for the currents \( J_a^a = \nu_a \partial_\sigma \chi_a \). This follows from
\[
\delta J_a^a|_0 = (\partial_t F(t)) J_a^a|_0 + \frac{1}{\kappa_a} F(t) \partial_\sigma J_a^a|_0 ,
\]
\[
\delta J_a^a|_{2\pi} = (\partial_t F(t)) J_a^a|_{2\pi} + \frac{1}{\kappa_a} F(t) \partial_\sigma J_a^a|_{2\pi}
\]
and (2.23).

What are the generators of the transformation (5.2) and (5.3)? According to Noether’s theorem, for infinitesimal variations \( \delta \sigma_a^\pm \) which are such that the induced variations (5.4) of \( \chi_a \) leave the action (5.1) invariant, one has
\[
\sum_{a=1,2} \nu_a \kappa_a \int d\sigma_a^+ d\sigma_a^- \left\{ \partial_{\sigma_a^+} \left( \frac{\delta L_a}{\delta (\partial_{\sigma_a^+} \chi_a)} \delta \chi_a - L_a \delta \sigma_a^+ \right) + \partial_{\sigma_a^-} \left( \frac{\delta L_a}{\delta (\partial_{\sigma_a^-} \chi_a)} \delta \chi_a - L_a \delta \sigma_a^- \right) \right\} = 0 ,
\]
where
\[ L_a = \partial_{\sigma_a^+} \chi_a \partial_{\sigma_a^-} \chi_a , \]
and \( \partial_{\sigma_a^\pm} = \frac{\partial}{\partial \sigma_a^\pm} \). Upon substituting transformations (5.2-4), we have
\[
\sum_{a=1,2} \frac{1}{4 \nu_a \kappa_a} \int d\sigma_a^+ d\sigma_a^- \left\{ \partial_{\sigma_a^+} (F_a^- J_a^a) + \partial_{\sigma_a^-} (F_a^+ J_a^a) \right\} = 0 .
\]
This result can be written as a current conservation law. By changing variables from 
\((\sigma_+^a, \sigma_-^a)\) to \((\sigma, t)\), we get

\[
\int d\sigma dt \sum_{a=1,2} \left\{ \kappa_a \partial_t j_t^a - \partial_\sigma j_\sigma^a \right\} = 0 ,
\]
or

\[
\sum_{a=1,2} \left\{ \kappa_a \partial_t j_t^a - \partial_\sigma j_\sigma^a \right\} = 0 , \tag{5.9}
\]

where the currents \(j_t^a\) and \(j_\sigma^a\) are given by

\[
\begin{align*}
 j_t^a &= \frac{1}{8\nu_a\kappa_a} (F_a^- J_a^- J_a^a - F_a^+ J_a^+_a) , \\
 j_\sigma^a &= -\frac{1}{8\nu_a\kappa_a} (F_a^- J_a^- J_a^a + F_a^+ J_a^+_a) . \tag{5.10}
\end{align*}
\]

The conserved charge \(q(F)\) associated with these currents is a linear combination of \(\int_0^{2\pi} d\sigma j_t^1\) and \(\int_0^{2\pi} d\sigma j_t^2\) and can be obtained by integrating the time component of the Noether current in (5.10), the result being

\[
q(F) = \sum_{a=1,2} \kappa_a \int_0^{2\pi} d\sigma j_t^a
\]

\[
= -\sum_{a=1,2} \frac{1}{8\nu_a} \int_0^{2\pi} d\sigma \left( F(\sigma_a^-) J_a^a - F(\sigma_a^+) J_a^+_a + F(\sigma_a^+) J_a^- J_a^a + F(\sigma_a^-) J_a^- J_a^+_a \right) . \tag{5.11}
\]

The conservation law (5.9), by itself, does not guarantee that the conformal charges are conserved in time. We have

\[
\partial_t q(F) = \sum_{a=1,2} \kappa_a \int_0^{2\pi} d\sigma \partial_t j_t^a
\]

\[
= \sum_{a=1,2} \int_0^{2\pi} d\sigma \partial_\sigma j_\sigma^a \tag{5.12}
\]

\[
= \sum_{a=1,2} j_\sigma^a \bigg|_0^{2\pi} ,
\]

from which it follows that, in order for \(q(F)\) to be constant in time, the space component of the conformal current has to fulfill Kirchhoff’s law at the junction. It can be checked
that, thanks to the conditions (5.7) and (5.8) on $F$ and the boundary conditions (2.22) and (2.24) on $J^a_\sigma$ and $J^a_t$, this is indeed the case:

$$\sum_{a=1,2} j^a_\sigma \biggr|_0^{2\pi} = - \sum_{a=1,2} \frac{1}{8\nu_a \kappa_a} (F^{-}_a J^a_- + F^{+_a} J^{+_a}) \biggr|_0^{2\pi}$$

$$= - F(t) \sum_{a=1,2} \frac{1}{8\nu_a \kappa_a} (J^a_+ J^{+_a} - J^a_- J^{-}_a) \biggr|_0^{2\pi}$$

$$= - \frac{1}{2} F(t) \sum_{a=1,2} \frac{1}{\nu_a \kappa_a} J^a_\sigma J^a_t \biggr|_0^{2\pi}$$

$$= 0 .$$

(5.13)

One can show that $q(F)$ is differentiable with respect to variations in $\chi_a(\sigma)$ and the canonical momenta $\kappa_a J^a_t$. That is, that no boundary terms appear in the resulting variations of $q(F)$. Of course, this is obvious for variations in $J^a_t$. Concerning variations in $\chi_a$, upon substituting $\delta J^a_\pm = \nu_a \partial_\sigma \delta \chi_a$ into (5.11) we obtain the boundary term

$$- \frac{1}{4} \sum_{a=1,2} \left( F(-\sigma_\alpha) J^a_- + F(\sigma^+ a) J^a_+ \right) \delta \chi_a \biggr|_0^{\sigma=2\pi} .$$

However, if we again assume continuity of the phase at the junction, so that $\delta \chi_1(0, t) = \delta \chi_1(2\pi, t) = \delta \chi_2(0, t) = \delta \chi_2(2\pi, t)$, along with the result (5.8), this boundary term reduces to

$$- \frac{1}{2} \sum_{a=1,2} F(t) (J^a_- + J^a_+) \biggr|_0^{\sigma=2\pi} \delta \chi_a ,$$

which then vanishes by Kirchhoff’s law (2.22).

The variational derivatives of $q(F)$ with respect to $J^a_t$ and $\chi_a$ are given by

$$\frac{\delta q(F)}{\delta J^a_t(\sigma)} = \frac{1}{4\nu_a} \left( F(-\sigma^a_-) J^a_- - F(\sigma^a_+ J^a_+ \right)$$

and

$$\frac{\delta q(F)}{\delta \chi_a(\sigma)} = \frac{1}{4} \partial_\sigma \left( F(-\sigma^a_-) J^a_- + F(\sigma^a_+ J^a_+ \right)$$

(5.14)

Using eqs. (3.10) and (5.14), we can compute the Poisson brackets between a smeared current $J(\Lambda)$ and the conformal charge $q(F)$:

$$\{ J(\Lambda), q(F) \} = \sum_{a=1,2} \frac{1}{2\kappa_a} \int_0^{2\pi} d\sigma \left( F(\sigma^a_+) \partial_\sigma \Lambda^a_+ J^a_+ - F(-\sigma^a_-) \partial_\sigma \Lambda^a_- J^a_- \right)$$

22
\[ F(t) \sum_{a=1,2} \frac{1}{4\kappa_a} (\Lambda^a_+ - \Lambda^a_-) (J^a_+ + J^a_-) \bigg|_{\sigma=0}^{\sigma=2\pi} . \]  

(5.15)

For field theory on a circle, the Poisson bracket between a conformal generator and a current is still a current. This result does not seem to generalize to the figure eight. This is so firstly because of the boundary term in eq. (5.15). Further, the integral in eq. (5.15) cannot in general be replaced by a smeared current \( J(\overline{\Lambda}) \) for a test function \( \overline{\Lambda} \in \mathcal{T} \). This is because what stands for \( \overline{\Lambda} \) in the integral (5.15) does not satisfy the condition (3.9) and hence does not belong to the test function space \( \mathcal{T} \).

The Poisson brackets between two conformal charges \( q(F) \) and \( q(\overline{F}) \) defines the classical Virasoro or the Witt algebra. We get

\[ \{ q(F), q(\overline{F}) \} = q(\overline{F}) \]

where

\[ \overline{F}(\sigma) \equiv \frac{1}{2}(F\partial_\sigma F - F\partial_F)(\sigma) , \]

(5.16)

and we have used the conditions (5.7) and (5.8) to eliminate boundary terms. Eq. (5.16) is the standard relation defining the Witt algebra.

Let us rewrite (5.16) in terms of Fourier modes. Let the smallest period of the periodic function \( F \) be \( 2\pi\kappa \). Then in view of (5.8), \( \kappa_a \) has to be an integer multiple of \( \kappa \):

\[ \kappa_a = N_a \kappa , \quad N_a = \text{integer} . \]

We now define the Fourier components \( L_n \) of the conformal charge as follows:

\[ L_n = -2\kappa \, q(e^{in\sigma/\kappa}) . \]

(5.17)

The Poisson bracket of \( L_n \) with \( L_m \) is then a familiar one:

\[ \{ L_n , L_m \} = -i(n - m)L_{n+m} . \]

(5.18)
If we now specialize to the case $b$ where $\kappa_1 = \kappa_2 = \kappa$ and $\nu_1 = \nu_2 = \nu$, and apply the expansions (4.8) and (4.9), then $L_n$ can be written as

$$L_n = \frac{\kappa}{8\nu} e^{int/\kappa} \int_0^{2\pi} d\sigma \left\{ e^{i\sigma} \left( (J^\text{sum}_+)^2 + (J^\text{dif}_+)^2 \right) + e^{-i\sigma} \left( (J^\text{sum}_-)^2 + (J^\text{dif}_-)^2 \right) \right\}$$

$$= e^{int/\kappa} \left( L^+_n + L^-_n + \frac{1}{2} L^0_n \right), \quad (5.19)$$

where $L^+_n$, $L^-_n$ and $L^0_n$ were defined in eqs. (4.13). $L_n$ is thus the sum of three Virasoro generators which commute in quantum theory. In view of (4.15), we further obtain the result that the zero component $L_0$ of the algebra is the generator of time translations, that is that it is proportional to the Hamiltonian $\mathcal{H} = \frac{1}{\kappa} L_0$.

It is easy to verify (5.18) starting from the Poisson brackets (4.14).

So far our treatment of the figure eight has been purely classical. In quantum theory, we pick up an additional anomaly term in the Witt algebra defined by (5.18). If we regularize the theory so that the central terms for the algebra generated by $L^+_n$, $L^-_n$ and $L^0_n$ in quantum theory have the standard form as in eqs. (4.16), then the central term in the commutator $[L_n, L_m]$ will be

$$\frac{1}{24} n(8n^2 - 5) \delta_{n+m,0} \quad . \quad (5.20)$$

6. Absence of Chiral Currents in Quantum Theory

Here we show that the chiral currents $J^+_\sigma(\sigma)$ and $J^-_\sigma(\sigma)$ cannot be independently quantized on the figure eight. More precisely, the two chiral currents cannot be expanded in terms of two independent sets of bases, which i) when quantized have a well defined action on the Fock space, and ii) lead to the correct Poisson brackets between the chiral currents. We can state this claim in another way. Let us define left- and right-moving smeared classical currents, which we denote by $J_+(\Lambda)$ and $J_-(\Lambda)$ respectively, according
to

\[ J_+ (\Lambda) = \sum_{a=1,2} \int d\sigma \, \Lambda_a^+ (\sigma) J_a^+ (\sigma) \quad \text{and} \quad J_- (\Lambda) = \sum_{a=1,2} \int d\sigma \, \Lambda_a^- (\sigma) J_a^- (\sigma) \]  \quad (6.1)

Then there do not exist two subsets \( \mathcal{T}_+ \in \mathcal{T} \) and \( \mathcal{T}_- \in \mathcal{T} \) of test functions of the form

\[ \Lambda^{(A_n^+)} = (f_1^n, f_2^n, 0, 0) \]

\[ \Lambda^{(A_n^-)} = (0, 0, g_1^n, g_2^n) \]  \quad (6.2)

satisfying the properties of orthonormality and completeness,

\[ \sum_{a=1,2} \int_0^{2\pi} d\sigma \, f_a^n (\sigma) f_a^m (\sigma) = \delta_{n,m} , \]

\[ \sum_n f_a^n (\sigma) f_a^{* n} (\sigma') = \delta^a (\sigma - \sigma') , \]  \quad (6.3)

and

\[ \sum_{a=1,2} \int_0^{2\pi} d\sigma \, g_a^n (\sigma) g_a^m (\sigma) = \delta_{n,m} , \]

\[ \sum_n g_a^n (\sigma) g_a^{* n} (\sigma') = \delta^a (\sigma - \sigma') , \]  \quad (6.4)

such that i) the quantum operators \( A_n^+ \) and \( A_n^- \) corresponding respectively to \( J_+ (\Lambda^{(A_n^+)} \) and \( J_- (\Lambda^{(A_n^-)} \) have a well defined action on the Fock space, and ii) the chiral currents

\[ \hat{J}_+^a (\sigma) = \sum_n A_n^+ f_a^n (\sigma) , \quad \hat{J}_-^a (\sigma) = \sum_n A_n^- g_a^n (\sigma) \]  \quad (6.5)

give the correct Poisson brackets, eq. (3.12), for the corresponding classical observables.

[Here \( \delta^a \) denotes the \( \delta \) function corresponding to loop \( a \).]

For simplicity, we shall prove the result for case \( b \) defined by \( \kappa_1 = \kappa_2 = \kappa \) and \( \nu_1 = \nu_2 = \nu \). The proof can easily be generalized to any case. The mode expansions for the currents in case \( b \) are given in eqs. (4.8) and (4.9). (The mode expansions for the fields appear in Appendix B.)
The proof is by contradiction. We suppose that two complete and orthonormal sets of test functions, \( \{ \Lambda(A_n^+) \} \) and \( \{ \Lambda(A_n^-) \} \) satisfying the above conditions exist. Then, the condition of completeness implies that

\[
J^a_+(\sigma) = \sum_n J_+(\Lambda(A_n^+)) f^a_n(\sigma) ,
\]

\[
J^a_-(\sigma) = \sum_n J_-(\Lambda(A_n^-)) g^a_n(\sigma) ,
\]

for any classical currents \( J^a_+(\sigma) \) and \( J^a_-(\sigma) \). In quantum theory, we let \( A_n^+ (A_n^-) \) be the operators corresponding to \( J_+(\Lambda(A_n^+)) \) (\( J_-(\Lambda(A_n^-)) \)). Further, let \(|0>\) be the vacuum state in the Fock space on which the quantum operators \( A_n^+ \) and \( A_n^- \) can act. We first show that in order for \( A_n^+|0> \) to have finite norm, \( f^a_n \) must be continuous at the junction. Moreover, the limiting value of \( f^1_n \) at the junction must be the same as that of \( f^2_n \). Analogous results apply to the functions \( g^a_n \).

To proceed let us recall that the test function space \( T \) for the case \( \kappa_1 = \kappa_2 = \kappa \) and \( \nu_1 = \nu_2 = \nu \) is spanned by \( \Lambda(\alpha^\pm_n) \) and \( \Lambda(\beta^\pm_n) \) defined in eqs. (4.10). Therefore the left moving current \( \tilde{J}^a_+ \) has the expansion

\[
\tilde{J}^a_+(\sigma) = -\frac{1}{2} \sum_m \alpha^+_m e^{-im\sigma} - \frac{(-1)^a}{2} \sum_m \beta^+_m e^{-im\sigma/2} ,
\]

(6.7)

\( \alpha^+_m \) and \( \beta^+_m \) now being quantum operators. [For economy of notation, we will not introduce symbols for them distinct from those in (4.8) and (4.9).] Substituting it into the expression for \( J_+(\Lambda(A_n^+)) \), we obtain an expression for \( A_n^+ \) in terms of \( \alpha^+_n \) and \( \beta^+_n \):

\[
A^+_n = \sum_m \alpha^+_m N_{n,m} + \sum_m \beta^+_m M_{n,m} ,
\]

(6.8)

where

\[
N_{n,m} = -\frac{1}{2} \int d\sigma (f^1_n + f^2_n)^* e^{-im\sigma} \quad \text{and} \quad M_{n,m} = -\frac{1}{2} \int d\sigma (f^2_n - f^1_n)^* e^{-im\sigma/2} .
\]

(6.9)

If we now apply the quantum analogues

\[
[\alpha^+_m, \alpha^+_n] = \frac{2m\nu}{\pi\kappa} \delta_{m,n} , \quad [\beta^+_m, \beta^+_n] = \frac{m\nu}{2\pi\kappa} \delta_{m,n} ,
\]

(6.10)
of the Poisson brackets relations (4.12) and assume that \( \alpha_n^+ \) and \( \beta_n^- \) annihilate the vacuum when \( n > 0 \), we obtain the following expression for the squared norm of the state \( A_n^+|0> \):

\[
|A_n^+|0>|^2 = |(N_{0,0} \alpha_0^+ + M_{0,0} \beta_0)|0>|^2 + \sum_{m>0} \frac{2\nu}{\pi \kappa} \left( m|N_{n,m}|^2 + \frac{m}{4}|M_{n,m}|^2 \right) . \tag{6.11}
\]

By integrating by parts twice, we can rewrite \( N_{n,m} \) and \( M_{n,m} \) according to

\[
N_{n,m} = -\frac{i}{2m}\left(f_n^1 + f_n^2\right)^*|2\pi|_0 + O\left(\frac{1}{m^2}\right) ,
\]

\[
M_{n,m} = -(-1)^m \frac{i}{m}(f_n^2 - f_n^1)^*|2\pi + \frac{i}{m}(f_n^2 - f_n^1)^*|0 + O\left(\frac{1}{m^2}\right) .
\]

Substituting the above into eq. (6.11), we have

\[
|A_n^+|0>|^2 = |(N_{0,0} \alpha_0^+ + M_{0,0} \beta_0)|0>|^2 + \frac{\nu}{\pi \kappa} \sum_{m_{even}>0} \frac{1}{m} \left( |(f_n^1|2\pi|_0)^2 + |(f_n^2|2\pi|_0)^2 \right) + O\left(\frac{1}{m}\right) 
\]

\[
+ \frac{\nu}{2\pi \kappa} \sum_{m_{odd}>0} \frac{1}{m} \left( |(f_n^1 - f_n^2)|2\pi + (f_n^1 - f_n^2)|0|^2 \right) + O\left(\frac{1}{m}\right) . \tag{6.12}
\]

In order for the first summation in (6.12) to be convergent, we must require that \( f_n^1|2\pi|_0 = f_n^2|2\pi|_0 = 0 \), while in order for the second summation also to be convergent, we must in addition have \( f_n^1|0 = f_n^2|0 = 0 \). Thus the functions \( f_n^a \) for \( a = 1, 2 \) must have a unique value at the junction:

\[
f_n^1|2\pi = f_n^2|2\pi = f_n^1|0 = f_n^2|0 . \tag{6.13}
\]

The same argument can be applied to the test functions \( \Lambda(A_n^+) \) of the right moving currents, from which one finds that the functions \( g_n^a \), for \( a = 1, 2 \) must have a unique value at the junction:

\[
g_n^1|2\pi = g_n^2|2\pi = g_n^1|0 = g_n^2|0 . \tag{6.14}
\]

An immediate consequence of eqs. (6.13) and (6.14) is that the Poisson brackets of \( \mathcal{J}_+(\Lambda(A_n^+)) \) and \( \mathcal{J}_-(\Lambda(A_m^-)) \), and hence the corresponding commutators between \( A_n^+ \) and \( A_m^- \), vanish. In fact from eq. (3.12) it follows that

\[
\{\mathcal{J}_+(\Lambda(A_n^+)), \mathcal{J}_-(\Lambda(A_m^-))\} = \{\mathcal{J}(\Lambda(A_n^+)), \mathcal{J}(\Lambda(A_m^-))\}
\]
\[ = \frac{\nu}{k_a} \sum_{a=1,2} \sum_{\sigma=a^*, b^*} f_n^a g_m^b |_{\sigma=2\pi, g_m=0} = 0. \] 

From this and eqs. (6.5), we must also then conclude that the commutator between \( \tilde{J}_a^a(\sigma) \) and \( \tilde{J}_b^a(\sigma') \) vanishes.

But from eqs. (4.3), (4.8) and (4.9) we also have, for \( 0 \leq \sigma \leq 2\pi \),

\[
J^1_+(\sigma) = \frac{1}{2} (J^a_+(\sigma) - J^b_+(\sigma)) = \frac{1}{2} (J^a_+ - K(\sigma)) \\
= -\frac{1}{2} \sum_n \alpha_n e^{-in\sigma} + \frac{1}{2} \sum_n \beta_n e^{-i2\sigma} 
\]

and

\[
J^1_-(\sigma) = \frac{1}{2} (J^a_-(\sigma) - J^b_+(\sigma)) = \frac{1}{2} (J^a_- - K(4\pi - \sigma)) \\
= \frac{1}{2} \sum_n \alpha_n e^{-in\sigma} + \frac{1}{2} \sum_n \beta_n e^{i2\sigma}. 
\]

It follows that

\[
\{J^1_+(\sigma), J^1_-(\sigma')\} = \frac{1}{4} \sum_{n, m} \{\beta_n, \beta_m\} e^{-i2\sigma} e^{i2\sigma'} \\
= -\frac{1}{4} \sum_n \frac{in\nu}{2\pi k} e^{-i\frac{\sigma}{2}(\sigma + \sigma')} \neq 0. 
\]

A similar result holds on loop 2.

We thus see that if we try to quantise the chiral components of the currents separately, we get wrong commutation relations for them. This completes the proof.

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APPENDICES

A. Self-Adjoint Extensions

Here we examine boundary conditions more general than the ones we specified in Section 2 [cf. eq. (2.11)]. As we stated there, the boundary conditions must be so chosen that the operator $H$, defined formally in eq. (2.7), is self-adjoint. For this purpose, to start with, we can choose a domain $D_0$ such that the restriction $H_0$ of $H$ to it is symmetric. This means by definition that

$$<\tilde{\psi}^0, H_0 \tilde{\chi}^0 > = < H_0 \tilde{\psi}^0, \tilde{\chi}^0 >, \quad \forall \tilde{\chi}^0, \tilde{\psi}^0 \in D_0,$$

(A.1)

where the scalar product was defined in eq. (2.8). This equation is equivalent to

$$0 = \sum_{a=1,2} \nu_a \int d\sigma \tilde{\psi}^0_a(\sigma) \partial^2 \tilde{\chi}^0_a(\sigma) - \sum_{a=1,2} \nu_a \int d\sigma \partial^2 \tilde{\psi}^0_a(\sigma) \tilde{\chi}^0_a(\sigma)$$

$$= - \sum_{a=1,2} \nu_a \left( \partial_a \tilde{\psi}^0_a \tilde{\chi}^0_a - \tilde{\psi}^0_a \partial_a \tilde{\chi}^0_a \right) \bigg|_{0}^{2\pi} .$$

(A.2)

This condition is certainly fulfilled if $D_0$ is taken to be the set of functions which vanish at the junction together with their first derivatives:

$$D_0 \equiv \{ \tilde{\chi}^0 | \tilde{\chi}^0_1(0) = \tilde{\chi}^0_2(0) = \tilde{\chi}^0_1(2\pi) = \tilde{\chi}^0_2(2\pi) = \partial_a \tilde{\chi}^0_a(0) = \partial_a \tilde{\chi}^0_a(2\pi) = 0 \} .$$

The operator $H_0$ is not self-adjoint in view of the remark preceding eq. (2.13) since we can check that the domain $D_0^\dagger$ of its adjoint, $H_0^\dagger$, is larger than $D_0$. We recall that according to eq. (2.12), $D_0^\dagger$ is defined to be the set of all functions $\tilde{\psi}$ fulfilling

$$< \tilde{\psi}, H_0 \tilde{\chi}^0 > = < H_0^\dagger \tilde{\psi}, \tilde{\chi}^0 >, \quad \forall \tilde{\chi}^0 \in D_0.$$
This is equivalent to

\[ 0 = \sum_{a=1,2} \nu_a \int d\sigma \bar{\psi}_a^i(\sigma) \partial_\sigma^2 \chi_{a}^0(\sigma) - \sum_{a=1,2} \nu_a \int d\sigma \partial_\sigma^2 \bar{\psi}_a^i(\sigma) \chi_{a}^0(\sigma) \]

\[ = - \sum_{a=1,2} \nu_a (\partial_\sigma \bar{\psi}_a^0 \chi_{a}^0 - \bar{\psi}_a^0 \partial_\sigma \chi_{a}^0) \bigg|_0^{2\pi} . \]  

(A.4)

In order to satisfy this equation, neither \(\bar{\psi}\), nor its derivatives need vanish at the junction.

This means that, in order to make \(H_0\) self-adjoint, we have to extend it to a domain larger than \(D_0\). Whether this can be done and in how many ways, is determined by the deficiency index theorem, which we now briefly review [10].

The deficiency indices \(N_+\) and \(N_-\) of \(H_0\) are defined to be the number of linearly independent orthonormal eigenvectors \(\bar{\psi}_m^{(+)\, m}\) \(m = 1, \ldots N_+\) and \(\bar{\psi}_n^{(-)\, n}\) \(n = 1, \ldots N_-\) of \(H_0^\dagger\) in \(D_0^\dagger\) with eigenvalues \(+i\) and \(-i\) respectively:

\[ (H_0^\dagger \bar{\psi}_m^{(+)\, m}) = -\frac{1}{\kappa_a^2} \partial_\sigma^2 \bar{\psi}_m^{(+)\, m}(\sigma) = i \bar{\psi}_m^{(+)\, m}(\sigma) , \quad m = 1, \ldots N_+ , \quad \bar{\psi}_m^{(+)\, m} \in D_0^\dagger , \]

\[ (H_0^\dagger \bar{\psi}_n^{(-)\, n}) = -\frac{1}{\kappa_a^2} \partial_\sigma^2 \bar{\psi}_n^{(-)\, n}(\sigma) = -i \bar{\psi}_n^{(-)\, n}(\sigma) , \quad n = 1, \ldots N_- , \quad \bar{\psi}_n^{(-)\, n} \in D_0^\dagger , \]

\[ \langle \bar{\psi}_m^{(\epsilon)} , \bar{\psi}_n^{(\epsilon')} \rangle = \delta_{m,n} \delta_{\epsilon,\epsilon'} , \quad \epsilon, \epsilon' = \pm . \]  

(A.5)

According to the deficiency index theorem, \(H_0\) admits self-adjoint extensions if and only if \(N_+ = N_- = N\). With \(N_+ = N_- = N\), the self-adjoint extensions of \(H\) are in one-to-one correspondence with \(U(N)\) matrices \(\{g\}\). Their domains \(D_g\) are direct sums of \(D_0\) with the vector space spanned by the vectors \(\bar{\psi}_i = \bar{\psi}_i^{(+)\, i} + g_{ij} \bar{\psi}_j^{(-)\, j}\), \(1 \leq i \leq N\):

\[ D_g = D_0 \oplus \{\text{span} \left( \bar{\psi}_i^{(+)\, i} + g_{ij} \bar{\psi}_j^{(-)\, j} \right) \} , \quad g \in U(N) . \]  

(A.6)

It is easy to check that both of the deficiency indices of \(H_0\) are equal to 4, which implies the existence of a sixteen-fold infinity of self-adjoint extensions. It can be shown that the domains corresponding to any given choice of a matrix in \(U(4)\) can also be described in terms of boundary conditions involving the functions and their first derivatives at the
junction. Functions fulfilling a particular choice of these boundary conditions form a
domain $\mathcal{D}^{(h)}$, $h \in U(4)$. It can be shown that $\mathcal{D}^{(h)} = \mathcal{D}_g$ for some $g$. These boundary
conditions are such that if the surface term in eq. (A.4) is to vanish for all functions $\chi_a(\psi_a)$ in $\mathcal{D}^{(h)}$, then $\overline{\psi}_a(\chi_a)$ as well has to belong to $\mathcal{D}^{(h)}$.

The domain $\mathcal{D}(H)$ of (2.11) is $\mathcal{D}^{(h)}$ for a particular choice of $h$.

\section*{B. Mode Expansion}

Here we shall examine the general solutions of the field equations on a figure eight
consistent with the boundary conditions (2.10), (2.22) and (2.24), and carry out the
eigenmode expansions for two special choices of the parameters $\kappa_a$ and $\nu_a$, namely:

\begin{itemize}
  \item[a.] $\kappa_1 \nu_1 = \kappa_2 \nu_2$ \quad \text{and} \quad \kappa_1 = \kappa_2 = \kappa \quad \text{and} \quad \nu_1 = \nu_2 = \nu$ .
  \item[b.] $\kappa_1 \nu_1 = \kappa_2 \nu_2$ \quad \text{and} \quad \nu_1 = \nu_2 = \nu$ .
\end{itemize}

For case a, unlike in earlier Sections, we will in addition assume that $\frac{\omega}{\kappa_2}$ is irrational for reasons of simplicity. Our
aim is to find the basis of test functions $\Lambda$ for the currents $\mathcal{J}(\Lambda)$ for the two cases. For
case b, we show that our answer agrees with eq. (4.10).

The discussion which now follows is general and does not assume case a or b until
it is otherwise stated.

We first expand $\chi_a(\sigma, t)$ according to

$$
\chi_a(\sigma, t) = q + pt + N_a \sigma + \sum_n \chi_a^n(\sigma) e^{i \omega_n t} .
$$

(B.1)

$q$ and $p$ are constants corresponding to zero frequency modes, while $\chi_a^n(\sigma)$ denote the
oscillatory modes. The latter satisfy the equations

$$
\left[ H_a - \omega_n^2 \right] \chi_a^n(\sigma) = 0 , \quad H_a = -\frac{1}{\kappa_a^2} \partial_\sigma^2 .
$$

(B.2)

As in Section 2, we shall assume that $\chi_a^n$ are singlevalued at the junction, so that $\chi_1^n(0) = \chi_1^n(2\pi) = \chi_2^n(0) = \chi_2^n(2\pi)$. Since the phases $\chi_a(0, t)$ and $\chi_a(2\pi, t)$ can differ only by
$2\pi \times \text{integer}$, the constants $N_a$ must take on integer values. $N_a$ parametrize the “winding modes”.

For the solutions of eq. (B.2), we can take $\chi^n_a(\sigma) = A_{a,n} \cos k_{a,n} \sigma + B_{a,n} \sin k_{a,n} \sigma$ where $k_{a,n} = \kappa_a \omega_n ( > 0 )$ if $\chi^n_a(\sigma) \neq 0$, and the coefficients $A_{a,n}$ and $B_{a,n}$ are determined from the boundary conditions. [The value of $k_{a,n}$ is immaterial if $\chi^n_a(\sigma) = 0$. Also the case $k_{a,n} = -\kappa_a \omega_n ( < 0 )$ need not be separately considered as it can be brought back to the present form by letting $B_{a,n} \rightarrow -B_{a,n}$. ] For the latter, from the singlevaluedness conditions, we get

$$A_{1,n} = A_{2,n} = A_{1,n} \cos 2\pi k_{1,n} + B_{1,n} \sin 2\pi k_{1,n} = A_{2,n} \cos 2\pi k_{2,n} + B_{2,n} \sin 2\pi k_{2,n} .$$

(B.3)

In addition, the Kirchhoff law (2.22) gives

$$\sum_{a=1,2} \nu_a \kappa_a \left( A_{a,n} \sin 2\pi k_{a,n} + B_{a,n} (1 - \cos 2\pi k_{a,n}) \right) = 0 .$$

(B.4)

Eqs. (B.3) and (B.4) form a system of homogeneous linear equations for $A_{a,n}$ and $B_{a,n}$. Solutions for $A_{a,n}$ and $B_{a,n}$ exist provided the determinant of the associated matrix is zero, that is,

$$\nu_1 \kappa_1 (1 - \cos 2\pi k_{1,n}) \sin 2\pi k_{2,n} + \nu_2 \kappa_2 (1 - \cos 2\pi k_{2,n}) \sin 2\pi k_{1,n} = 0 .$$

(B.5)

Using this equation, we can classify five types of solutions for $k_{a,n}$, along with their corresponding eigenmodes ($\chi^n_1$, $\chi^n_2$). They are:

i) $k_{1,n} = n$ is a positive integer, and $(\chi^n_1(\sigma), \chi^n_2(\sigma)) = (\sin n\sigma, 0)$.

ii) $k_{2,n} = n$ is a positive integer, and $(\chi^n_1(\sigma), \chi^n_2(\sigma)) = (0, \sin n\sigma)$.

As $\chi^n_2$ ($\chi^n_1$) is zero in case i) (ii), the value of $k_{2,n}$ ($k_{1,n}$) in that case is immaterial.

iii) If $\frac{\kappa_1}{\kappa_2}$ is rational, we also have the solutions $k_{a,n} = n_a = \text{integer}$ where $\frac{n_1}{n_2} = \frac{\kappa_1}{\kappa_2}$, and $(\chi^n_1(\sigma), \chi^n_2(\sigma)) = (\cos n_1\sigma, \cos n_2\sigma)$.
iv) Both $2k_{1,n} = 2r_1$ and $2k_{2,n} = 2r_2$ are positive odd integers and
\[(\chi_1^n(\sigma), \chi_1^n(\sigma)) = (\nu_2 \kappa_2 \sin r_1 \sigma, -\nu_1 \kappa_1 \sin r_2 \sigma)\]. Just as for iii), these modes are possible only when $\frac{\kappa_1}{\kappa_2}$ is rational.

v) Neither $2k_{1,n}$ nor $2k_{2,n}$ are integers. Rather, $k_{a,n}$ are positive solutions of the transcendental equation
\[\nu_1 \kappa_1 \tan \pi k_{1,n} + \nu_2 \kappa_2 \tan \pi k_{2,n} = 0\]  \hspace{1cm} (B.6)

Then the corresponding eigenmodes are given by
\[\chi_a^n(\sigma) = \cos k_{a,n} \sigma + \tan \pi k_{a,n} \sin k_{a,n} \sigma\]  \hspace{1cm} (B.7)

Solutions i) and ii) correspond to independent oscillations on loops 1 and 2 respectively, and exist for arbitrary values of the independent parameters $\kappa_a$ and $\nu_a$. On the other hand, the presence of modes iii-v) depends on the values of $\kappa_a$ and $\nu_a$.

The solutions can be used to form an orthonormal basis with respect to the scalar product (2.8). If $\chi$ and $\psi$ are eigenmodes for distinct eigenvalues, so that $[H_a - \omega^2] \chi_a(\sigma) = [H_a - \omega'^2] \psi_a(\sigma) = 0$ and $\omega^2 \neq \omega'^2$, then $\chi$ and $\psi$ are of course orthogonal with respect to the scalar product. Orthogonal combinations of degenerate eigenmodes can be formed, and the modes can be normalized as well. The completeness of the eigenvectors $(\chi_1^n, \chi_2^n)$ follows from the result that the operator $H = (H_1, H_2)$ is self-adjoint, as was shown in Section 2.

We now expand the fields $\chi_a$ in terms of the eigenmodes i-v) for cases a and b. As case b is the simpler of the two, we begin with it.

**b.** $\kappa_1 = \kappa_2 = \kappa$ and $\nu_1 = \nu_2 = \nu$

The quantities $\ell_a$, $T_a$ and $v_a$ of Sections 1 and 2 correspond to length, tension and velocity respectively on loop $a$. This case requires that the ratio of these quantities for
the two loops must be the same: \( \ell_1 / \ell_2 = T_1 / T_2 = v_1 / v_2 \). In particular, if \( T_a \) and \( v_a \) are the same for both loops, then so must be the lengths \( \ell_a \).

For this case, the solutions for \( 2k_{a,n} \) can only be integers as follows from (B.5), \( \nu_a \kappa_a \) and \( k_{a,n} = \kappa_a \omega_n \) being independent of \( a \). Therefore type v) eigenmodes are not present in the expansion for \( \chi_a(\sigma, t) \). The expansion can be written as

\[
\chi_1(\sigma, t) = Q(t) + N_1 \sigma + \sum_{m=1}^{\infty} \left\{ a_{1,m}(t) \sqrt{2} \sin m\sigma + b_m(t) \cos m\sigma \right\} + \sum_{r=1/2}^{\infty} c_r(t) \sin r\sigma \quad \text{(B.8)}
\]

and

\[
\chi_2(\sigma, t) = Q(t) + N_2 \sigma + \sum_{m=1}^{\infty} \left\{ a_{2,m}(t) \sqrt{2} \sin m\sigma + b_m(t) \cos m\sigma \right\} - \sum_{r=1/2}^{\infty} c_r(t) \sin r\sigma , \quad \text{(B.9)}
\]

where \( m \) is a positive integer and \( 2r \) is a positive odd integer [so that \( r = \frac{1}{2}, \frac{3}{2}, \ldots \)]. \( a_{1,m}(t), a_{2,m}(t), b_m(t) \) and \( c_r(t) \) are real coefficients and they contain the \( t \) dependence of the oscillatory modes. \( Q(t) \) is the \( q + pt \) of (B.1) and denotes the zero frequency mode.

Upon substituting the expansions (B.8) and (B.9) into the action (2.14), we obtain, as expected, the action \( S \) and the Lagrangian \( L \) for an infinite number of harmonic oscillators:

\[
S = \int dt \, L ,
\]

where

\[
\frac{1}{\pi \nu} L = 2\kappa^2 \dot{Q}^2 - \left( N_1^2 + N_2^2 \right) \\
+ \sum_{m=1}^{\infty} \left( (\kappa^2 a_{1,m}^2 - m^2 a_{1,m}^2) + (\kappa^2 a_{2,m}^2 - m^2 a_{2,m}^2) + (\kappa^2 b_m^2 - m^2 b_m^2) \right) \\
+ \sum_{r=1/2}^{\infty} \left( \kappa^2 c_r^2 - r^2 c_r^2 \right) . \quad \text{(B.10)}
\]

The dot here denotes time differentiation.

In the Hamiltonian formalism, the momenta conjugate to \( Q, c_r, a_{a,m} \) and \( b_m \) are \( 4\pi \nu \kappa^2 \dot{Q}, 2\pi \nu \kappa^2 \dot{c_r}, 2\pi \nu \kappa^2 \dot{a}_{a,m} \) and \( 2\pi \nu \kappa^2 \dot{b}_m \) respectively, and the nonvanishing Poisson
brackets are given by

\[ \{ Q, \dot{Q} \} = \frac{1}{4\pi \nu \kappa^2}, \quad \{ a_{a,m}, \dot{a}_{b,n} \} = \frac{1}{2\pi \nu \kappa^2} \delta_{a,b} \delta_{m,n}, \]

\[ \{ b_m, \dot{b}_n \} = \frac{1}{2\pi \nu \kappa^2} \delta_{m,n}, \quad \{ c_r, \dot{c}_{r'} \} = \frac{1}{2\pi \nu \kappa^2} \delta_{r,r'}. \] (B.11)

Next we replace the real coefficients \( a_{a,m}, b_m, \text{ and } c_r \) along with their velocities by the complex variables \( \tilde{a}_{a,m}, \tilde{b}_m, \text{ and } \tilde{c}_r \) defined by

\[ \tilde{a}_{a,m} = \sqrt{\pi \nu \kappa} \left( \kappa \dot{a}_{a,m} - ima_{a,m} \right), \quad \tilde{b}_m = \sqrt{\pi \nu \kappa} \left( \kappa \dot{b}_m - imb_m \right) \] (B.12)

and

\[ \tilde{c}_r = \sqrt{\pi \nu \kappa} \left( \kappa \dot{c}_r - irc_r \right). \] (B.13)

Their nonzero Poisson brackets are all given by

\[ \{ \tilde{a}_{a,m}, \tilde{a}_{b,n}^* \} = -im \delta_{m,n} \delta_{a,b}, \quad \{ \tilde{b}_m, \tilde{b}_n^* \} = -im \delta_{m,n} \text{ and } \{ \tilde{c}_r, \tilde{c}_{r'}^* \} = -ir \delta_{r,r'}. \] (B.14)

Here, we allow the index \( m \) in \( \tilde{a}_{a,m} \) and \( \tilde{b}_m \) to be negative with \( \tilde{a}_{a,-m} = \tilde{a}_{a,m}^* \) and \( \tilde{b}_{-m} = \tilde{b}_m^* \).

We also allow the index \( r \) in \( \tilde{c}_r \) to be negative with \( \tilde{c}_{-r} = \tilde{c}_r^* \).

In terms of these variables, the Hamiltonian for the system is

\[ \mathcal{H} = 2\pi \nu \kappa^2 \dot{Q}^2 + \pi \nu (N_1^2 + N_2^2) + \frac{1}{2\kappa} \sum_{m \neq 0} \left( \tilde{a}_{1,-m} \tilde{a}_{1,m} + \tilde{a}_{2,-m} \tilde{a}_{2,m} + \tilde{b}_{-m} \tilde{b}_m \right) + \frac{1}{2\kappa} \sum_{r = -\infty}^{\infty} \tilde{c}_{-r} \tilde{c}_r. \] (B.15)

The currents \( J^a_\pm(\sigma) \) can be expanded according to

\[ \sqrt{\frac{\pi \kappa}{\nu}} J^a_\pm(\sigma) = \sqrt{\pi \nu \kappa} \left( N_a \pm \kappa \dot{Q} \right) \mp \frac{1}{2} \sum_{m=1}^{\infty} \left( i \sqrt{2} \tilde{a}_{a,\mp m} - \tilde{b}_{\pm m} \right) e^{im\sigma} + h.c. \]

\[ \pm \frac{(-1)^a}{2} \sum_{r = \frac{1}{2}}^{\infty} (i \tilde{c}_{\mp r} e^{ir\sigma} + h.c.), \] (B.16)

where \( h.c. \) denotes hermitean conjugate and time dependence has been suppressed.

If we take the sums and differences of \( J^1_\pm \) and \( J^2_\pm \), the resulting functions (\( J^\text{sum}_\pm \) and \( J^\text{diff}_\pm \) of Section 4) are \( 2\pi \) and \( 4\pi \) periodic respectively, that is they satisfy eqs. (4.4) and (4.5). The expansion given here for the currents must therefore be equivalent to
the one we wrote down in Section 4. Furthermore, the basis of test functions given in
that Section must be valid for this system. These expectations are readily verified. The
correspondence between the coefficients $\alpha_0^\pm, \beta_0^\pm$ defined in Section 4 and the coefficients
$\dot{Q}, N_a, \tilde{a}_{a,m}, \tilde{b}_m, \tilde{c}_r$ defined here is given by

$$
\alpha_0^\pm = \mp \nu (N_1 + N_2 \pm 2\kappa \dot{Q}), \quad \beta_0 = -\nu (N_2 - N_1),
$$

$$
\alpha_m^\pm = \mp \sqrt{\frac{\nu}{\pi \kappa}} \left( i \sqrt{2} \text{sgn}(m) (\tilde{a}_{1,m} + \tilde{a}_{2,m}) \pm \tilde{b}_m \right), \quad \beta_m = -i \text{sgn}(m) \sqrt{\frac{\nu}{2\pi \kappa}} (\tilde{a}_{2,m} - \tilde{a}_{1,m}),
$$

$$
\beta_r = i \text{sgn}(r) \sqrt{\frac{\nu}{\pi \kappa}} \tilde{c}_r, \quad m = \pm 1, \pm 2, \ldots, \quad r = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots, \quad (B.17)
$$

where $\text{sgn}(m) = \frac{m}{|m|}$.

**a. $\kappa_1 \nu_1 = \kappa_2 \nu_2$**

From the definitions of $\nu_a$ and $\kappa_a$, $\kappa_1 \nu_1 = \kappa_2 \nu_2$ corresponds to the situation where the
ratio of the tension to the velocity is the same for both loops of the figure eight. Thus,
for example, with the velocities and tensions identical on the two loops, we can still allow
for loops of different length. We will also assume that $\frac{\kappa_a}{\kappa_2}$ is irrational for simplicity, for
then the modes iii) and iv) are absent. When the velocities on the two loops are equal,
this condition implies that $\frac{\dot{a}_1}{\dot{a}_2}$ is irrational.

Unlike in case b, there now exist solutions of type v). This is because if $2k_{a,n}$ are
integers, then $\frac{2k_{1,n}}{2k_{2,n}} = \frac{\kappa_1}{\kappa_2}$ is rational, contrary to assumption. Eq. (B.6) which governs
type v) solutions reduces now to

$$
\tan \pi k_{1,n} + \tan \pi k_{2,n} = 0.
$$

It leads to $k_{1,n} + k_{2,n} = n$ where $n$ is a nonzero integer. Therefore for the eigenfrequencies
$\omega_n$ in $k_{a,n} = \kappa_a \omega_n$, we have,

$$
\omega_n = \frac{n}{\kappa_1 + \kappa_2}. \quad (B.18)
$$
The expansion of the fields $\chi_a$ can be written as

$$
\chi_a(\sigma, t) = Q(t) + N_a \sigma + \sqrt{2} \sum_{m=1}^{\infty} a_{a,m}(t) \sin m\sigma
+ \sqrt{2} \sum_{n=1}^{\infty} d_n(t) \left\{ \cos k_{a,n}\sigma + \tan \pi k_{a,n} \sin k_{a,n}\sigma \right\}.
$$

(B.19)

Upon substituting the expansions into the action (2.14), we obtain,

$$
S = \int dt \ L,
$$

$$
\frac{1}{\pi} L = \left( \sum_{a=1,2} \nu_a \kappa_a^2 \right) \dot{Q}^2 - \sum_{a=1,2} \nu_a N_a^2 + \sum_{m=1}^{\infty} \sum_{a=1,2} \nu_a (\kappa_a^2 a_{a,m}^2 - m^2 a_{a,m}^2)
+ \left( \sum_{a=1,2} \nu_a \kappa_a^2 \right) \sum_{n=1}^{\infty} \sec^2 \pi k_{1,n} \left( a_n^2 - \omega_n^2 d_n^2 \right),
$$

(B.20)

and the dot again denotes time differentiation.

In the Hamiltonian formalism, the momenta conjugate to $Q$, $a_{a,m}$ and $d_n$ are $2\pi (\sum_{a=1,2} \nu_a \kappa_a^2) \dot{Q}$, $2\pi \nu_a \kappa_a^2 \dot{a}_{a,m}$ and $2\pi (\sum_{a=1,2} \nu_a \kappa_a^2) \sec^2 \pi k_{1,n} \dot{d}_n$ respectively, and the nonvanishing Poisson brackets are all given by

$$
\{Q, \dot{Q}\} = \frac{1}{2\pi \left( \sum_{a=1,2} \nu_a \kappa_a^2 \right)},
$$

$$
\{a_{a,m}, \dot{a}_{b,n}\} = \frac{\delta_{a,b} \delta_{m,n}}{2\pi \nu_a \kappa_a^2}, \quad \{d_n, \dot{d}_m\} = \frac{\delta_{n,m}}{2\pi \left( \sum_{a=1,2} \nu_a \kappa_a^2 \right) \sec^2 \pi k_{1,n}}.
$$

(B.21)

Next we define the complex variables $\tilde{a}_{a,m}$ and $\tilde{d}_n$ according to

$$
\tilde{a}_{a,m} = \sqrt{\pi \nu_a \kappa_a} (\kappa_a \dot{a}_{a,m} - ima_{a,m}) \quad \text{and} \quad \tilde{d}_n = \frac{C_n}{\sqrt{2}} (\dot{d}_n - i\omega_n d_n),
$$

(B.22)

where

$$
C_n^2 = 2\pi (\kappa_1 + \kappa_2) \left( \sum_{b=1,2} \nu_b \kappa_b^2 \right) \sec^2 \pi k_{1,n}.
$$

(B.23)

Here we let $m, n$ to be negative with $\tilde{a}_{a,-m} = \tilde{a}_{a,m}^*$ and $\tilde{d}_{-n} = \tilde{d}_n^*$. The nonzero Poisson brackets involving (B.22) are

$$
\{\tilde{a}_{a,m}, \tilde{a}_{b,n}^*\} = -im\delta_{m,\nu} \delta_{a,b}, \quad \{\tilde{d}_n, \tilde{d}_m^*\} = -in\delta_{n,\nu}.
$$

(B.24)
In terms of these variables, the Hamiltonian for the system is

$$\mathcal{H} = \pi \left( \sum_{a=1,2} \nu_a \kappa_a^2 \right) \dot{Q}_2^2 + \pi \left( \sum_{a=1,2} \nu_a N_a^2 \right) + \sum_{a=1,2} \frac{1}{2 \kappa_a} \sum_{m \neq 0} \tilde{a}_{a,-m} \tilde{a}_{a,m} + \frac{1}{2(\kappa_1 + \kappa_2)} \sum_{n \neq 0} \tilde{d}_n \tilde{d}_n^* .$$  \hspace{1cm} (B.25)

The currents \( J_\pm^a(\sigma) \) are now given by

$$\frac{1}{\nu_a} J_\pm^a(\sigma) = N_a \pm \kappa_a \dot{Q} - \frac{i}{\sqrt{2 \pi \nu_a \kappa_a}} \sum_{m \neq 0} \tilde{a}_{a,m} e^{im\sigma} \pm \kappa_a \sum_{n \neq 0} \frac{\tilde{d}_{\pm n}}{C_n} \sec \pi \kappa_{a,n} e^{ik_{a,n}(\sigma - \pi)} .$$  \hspace{1cm} (B.26)

Unlike in case b, the sums and differences of \( J_1^\pm \) and \( J_2^\pm \) are not periodic, although they do satisfy the periodic boundary conditions (4.1) and (4.2). Thus the test function basis given in Section 4 can not be used here. A basis for the test functions \( \Lambda = (\Lambda_1^-, \Lambda_2^-, \Lambda_1^+, \Lambda_2^+) \) is instead obtained directly from eq. (B.26). The test functions associated with the constant modes \( N_1, N_2 \) and \( \dot{Q} \) are

$$\Lambda^{(N_1)} = \frac{1}{4 \pi \nu_1} (1, 0, 1, 0),$$

$$\Lambda^{(N_2)} = \frac{1}{4 \pi \nu_2} (0, 1, 0, 1)$$

$$\Lambda^{(\dot{Q})} = \frac{1}{8 \pi \nu_1 \kappa_1} (1, 1, -1, -1)$$  \hspace{1cm} (B.27)

respectively. They are normalized such that eq. (4.11) is satisfied. The test functions associated with the oscillating modes \( \tilde{a}_{1,m}, \tilde{a}_{2,m} \) and \( \tilde{d}_n \) are

$$\Lambda^{(\tilde{a}_{1,m})} = i \sqrt{\frac{\kappa_1}{8 \pi \nu_1}} (e^{im\sigma}, 0, e^{-im\sigma}, 0)$$

$$\Lambda^{(\tilde{a}_{2,m})} = i \sqrt{\frac{\kappa_2}{8 \pi \nu_2}} (0, e^{im\sigma}, 0, e^{-im\sigma}),$$  \hspace{1cm} (B.28)

$$\Lambda^{(\tilde{d}_n)} = \frac{C_n}{8 \pi} \left( \frac{e^{ik_{1,n}(\sigma - \pi)}}{\kappa_1 \nu_1 \sec \pi k_{1,n}}, \frac{e^{ik_{2,n}(\sigma - \pi)}}{\kappa_2 \nu_2 \sec \pi k_{2,n}}, -\frac{e^{-ik_{1,n}(\sigma - \pi)}}{\kappa_1 \nu_1 \sec \pi k_{1,n}}, -\frac{e^{-ik_{2,n}(\sigma - \pi)}}{\kappa_2 \nu_2 \sec \pi k_{2,n}} \right) .$$

They also satisfy eq. (4.11).
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