ON INFINITE DETERMINANTS OF
FINITE POTENT ENDO MORPHISMS

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Abstract. The aim of this paper is to offer an algebraic definition of infinite
determinants of finite potent endomorphisms. It generalizes Grothendieck’s de-
terminal for finite rank endomorphisms on infinite-dimensional vector spaces,
and is equivalent to the classic analytic definitions. Moreover, the theory
can be interpreted as a multiplicative analogue to Tate’s formalism of ab-
stract residues in terms of traces of finite potent linear operators on infinite-
dimensional vector spaces, and allows us to relate Tate’s theory to the Segal-
Wilson pairing in the context of loop groups.

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1. Introduction

Infinite determinants have been studied since the second half of the last century.
In 1956, in [4] Grothendieck developed an algebraic method to compute det(1 + u),
where u is a finite rank endomorphism on an infinite-dimensional vector space.
When u is not of finite rank, there exist different approaches to define det(1 + u),
as far as we know all of them from an analytic point of view. Let us comment on
some of them.

In 1963, in [2] Dunford and Schwartz defined the infinite determinant det(1 + B)
in terms of the nonzero eigenvalues of a trace class operator B on a separable

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Hilbert space; also for trace class operators on Hilbert spaces, in [9] Simon defined (1976) the infinite determinant by:

$$\det_1(1 + B) = 1 + \sum_{r=1}^{\infty} \text{tr}(\Lambda^r A)$$

and showed that it satisfies the expected properties of a determinant. In 2001, in [3] Gohberg, Goldberg and Krupnik offered a generalization of determinants for trace potent operators on a separable complex Hilbert space.

In these notes we offer an algebraic construction of infinite determinants of finite potent endomorphisms, that generalizes Grothendieck’s determinant for finite rank endomorphisms (over infinite dimensional vector spaces). We show that there exists an equivalence between this algebraic definition and the classical analytic definitions commented above. The key point is to use Tate’s definition of traces of finite potent endomorphisms ([10]).

The algebraic construction of the determinant allows us to construct a central extension of groups whose associated cocycle can be interpreted as a multiplicative analogue of Tate’s abstract residue. Moreover, this cocycle shows a way to relate the theory of Segal and Wilson for loops groups ([5]) and Tate’s theory of abstract residues ([10]). Finally, a reciprocity law for this cocycle is stated in geometric terms and can be thought of as a multiplicative analogue of Tate’s theorem of residues. One could say that this self-contained theory could be thought of as an approach to a unified theory of local symbols (in characteristic zero).

The paper is organized as follows. In section 2 we briefly recall Tate’s definition of the trace of a finite potent endomorphism and its properties.

Section 3 is devoted to giving the algebraic construction of the infinite determinant of finite potent endomorphisms, to showing its basic properties and its relationship to the classical definitions in terms of the exterior algebra and the eigenvalues of the endomorphism.

Section 4 aims to give a definition of an exponential map for finite potent endomorphisms (over a field of characteristic zero), and to show the relationship between the infinite determinant (defined in the last section) and Tate’s trace via this exponential. We also include a subsection meant to extend the definition of the infinite determinant of a finite potent endomorphism to the case in which one has an infinite product of finite potent endomorphisms.

Finally, in section 5 we prove the existence of a central extension of groups and explicitly compute its associated cocycle. The proof shows that this cocycle is a multiplicative analogue of Tate’s abstract residue and offers us a way to see the equivalence between Segal and Wilson’s theory of determinants for loop groups and Tate’s theory. The properties of this cocycle (deduced directly from those of Tate’s residue) allow us to state a reciprocity law in geometric terms, that is no more than a multiplicative version of the theorem of residues given by Tate.

In Appendix A we show the equivalence between the algebraic construction given here and the classical analytic definitions.

The last Appendix, B is a quick overview of the theory of Segal and Wilson of determinants in the context of loop groups.

2. Preliminaries

Let $k$ be an arbitrary field, let $V$ be a $k$-vector space and let $\varphi$ be an endomorphism of $V$.

**Definition 2.1.** We say that $\varphi$ is finite potent if $\varphi^n V$ is finite dimensional for some $n$. 
If $\varphi$ is finite potent, a trace $\text{tr}_V(\varphi) \in k$ may be defined (see [10]), having the following properties:

(1) If $V$ is finite dimensional, then $\text{tr}_V(\varphi)$ is the ordinary trace.

(2) If $W$ is a subspace of $V$ such that $\varphi W \subset W$, then:
$$\text{tr}_V(\varphi) = \text{tr}_W(\varphi) + \text{tr}_{V/W}(\varphi).$$

(3) If $\varphi$ is nilpotent, then $\text{tr}_V(\varphi) = 0$.

(4) If $F$ is a “finite potent” subspace of $\text{End}(V)$ (i.e., if there exists an $n$ such that for any family of $n$ elements $\varphi_1, \ldots, \varphi_n \in F$, the space $\varphi_1 \ldots \varphi_n V$ is finite dimensional) then $\text{tr}_V: F \rightarrow k$ is $k$-linear.

(5) If $f: V' \rightarrow V$ and $g: V \rightarrow V'$ are $k$-linear and $fg$ is finite potent, then $gf$ is finite potent, and we have:
$$\text{tr}_V(fg) = \text{tr}_{V'}(gf).$$

Properties (1), (2) and (3) characterize traces, because if $W$ is a finite dimensional subspace of $V$ such that $\varphi W \subset W$ and $\varphi^n W \subset W$, for some $n$, then $\text{tr}_V(\varphi) = \text{tr}_W(\varphi|_W)$. And, since $\varphi$ is finite potent, we may take $W = \varphi^n V$ (for details see [10]).

**Remark 2.2.** M. Argerami, F. Szechtman and R. Tifenbach have recently shown in [1] that an endomorphism $\varphi$ is finite potent if and only if $V$ admits a $\varphi$-invariant decomposition $V = W_\varphi \oplus U_\varphi$ such that $\varphi|_{U_\varphi}$ is nilpotent, $W_\varphi$ is finite dimensional, and $\varphi|_{W_\varphi}: W_\varphi \rightarrow W_\varphi$ is an isomorphism. This decomposition is unique and one again has that $\text{tr}_V(\varphi) = \text{tr}_{W_\varphi}(\varphi|_{W_\varphi})$.

In this paper we shall call this decomposition the $\varphi$-invariant AST-decomposition of $V$. □

Let $M, N$ be two $k$-vector subspaces of $V$.

**Definition 2.3.** $M$ and $N$ are said to be commensurable if $M + N/M \cap N$ is a finite dimensional vector space over $k$. We write $M \sim N$ to denote commensurable subspaces ([10]).

If we set $M < N$ when $(M + N)/N$ is finite dimensional, it is clear that $M \sim N$ if and only if $M < N$ and $N < M$. Commensurability is an equivalent relation on the set of $k$-vector subspaces of $V$.

Let us fix a vector subspace $V_+ \subset V$ and let us define subspaces $E_i$, $E_0$, $E_1$, $E_2$ of $\text{End}_k(V)$ by:

$$\varphi \in E \iff \varphi(V_+) \subset V_+$$

$$\varphi \in E_1 \iff \varphi(V) \subset V_+$$

$$\varphi \in E_2 \iff \varphi(V_+) \subset (0)$$

$$\varphi \in E_0 \iff \varphi(V) \subset V_+ \text{ and } \varphi(V_+) \subset (0)$$

(2.1)

**Proposition 2.4.** [10][Prop.1] $E$ is a $k$-subalgebra of $\text{End}_k(V)$; the $E_i$ are two-sided ideals in $E$; the $E_i$ depend only on the $\sim$-equivalence class of $V_+$; we have $E_1 \cap E_2 = E_0$ and $E_1 + E_2 = E$; $E_0$ is finite potent.

Thus there is a $k$-linear map $\text{tr}_V: E_0 \rightarrow k$.

**Proposition 2.5.** [10][Prop. 2] Assume either $\varphi \in E_0$ and $\psi \in E$, or $\varphi \in E_1$ and $\psi \in E_2$. Thus, the commutator $[\varphi, \psi] = \varphi \psi - \psi \varphi \in E_0$ and has zero trace.

3. **Infinite determinants of finite potent endomorphisms.**

This section is devoted to defining infinite determinants of finite potent endomorphism over an arbitrary vector space.
3.A. Algebraic construction of infinite determinants: definition and basic properties.

Let \( k \) be an arbitrary field and let \( V \) be a \( k \)-vector space. Recall that an endomorphism \( \varphi \) of \( V \) is “finite potent” if \( \varphi^n V \) is finite dimensional for some \( n \).

For these endomorphisms, a determinant \( \det_V^k(1 + \varphi) \in k \) may be defined from the following properties:

1. If \( V \) is finite dimensional, then \( \det_V^k(1 + \varphi) \) is the ordinary determinant.
2. If \( W \) is a subspace of \( V \) such that \( \varphi W \subset W \), then:
   \[
   \det_V^k(1 + \varphi) = \det_W^k(1 + \varphi) \cdot \det_{V/W}^k(1 + \varphi). 
   \]
3. If \( \varphi \) is nilpotent, then \( \det_V^k(1 + \varphi) = 1 \).

Similar to Tate’s definition of traces for finite potent endomorphisms (see [10] and also subsection 2.2), where \( \det_{V_{\psi}} \) is nilpotent, \( W_{\varphi} \) is finite dimensional and \( \varphi^n V \subset W \), for some \( n \), then we have:

\[
\det_V^k(1 + \varphi) = \det_{W_{\varphi}}^k(1 + \varphi_{|_{W_{\varphi}}}). 
\]

And, since \( \varphi \) is finite potent, we may take \( W = \varphi^n V \).

Moreover, if we consider the \( \varphi \)-invariant AST-decomposition \( V = W_{\varphi} \oplus U_{\varphi} \) (see Remark 2.2), where \( \varphi_{|_{U_{\varphi}}} \) is nilpotent, \( W_{\varphi} \) is finite dimensional and \( \varphi_{|_{W_{\varphi}}} : W_{\varphi} \rightarrow W_{\varphi} \) is an isomorphism, we have again that:

\[
\det_V^k(1 + \varphi) = \det_{W_{\varphi}}^k(1 + \varphi_{|_{W_{\varphi}}}). 
\]

If follows from the definition of \( \det_V^k \) that:

**Lemma 3.1.** If \( \varphi \in \text{End}(V) \) is a finite potent endomorphism and \( V = V_1 \oplus V_2 \) is a decomposition of \( V \), where \( \varphi V_i \subset V_i \) for \( i \in \{1, 2\} \), one has that:

\[
\det_V^k(1 + \varphi) = \det_{V_1}^k(1 + \varphi_{|_{V_1}}) \cdot \det_{V_2}^k(1 + \varphi_{|_{V_2}}). 
\]

**Lemma 3.2.** If \( F \) is a finite potent subspace of \( \text{End}(V) \), then:

\[
\det_V^k(1 + \varphi) \cdot \det_V^k(1 + \psi) = \det_V^k[(1 + \varphi)(1 + \psi)] = \det_V^k[(1 + \psi)(1 + \varphi)],
\]

for all endomorphisms \( \varphi, \psi \in F \).

**Proof.** Recall from [10] that a subspace \( F \) of \( \text{End}(V) \) is called “finite potent” when there exists an \( n \) such that for any family of \( n \) elements, \( \varphi_1, \ldots, \varphi_n \in F \), the space \( \varphi_1 \ldots \varphi_n V \) is finite dimensional.

Thus, with this notation if \( \varphi, \psi \in F \), then \( < \varphi, \psi > := \varphi k[\varphi, \psi] + \psi k[\varphi, \psi] \) is also a finite potent subspace of \( \text{End}(V) \) such that the finite dimensional space \( W = < \varphi, \psi >^n V \) satisfies the following properties:

- \( W \) is invariant for \( \varphi \) and \( \psi \).
- \( \varphi^n V \subset W \) and \( \psi^n V \subset W \).
- \( \det_V^k(1 + \varphi) = \det_W^k(1 + \varphi) \) and \( \det_V^k(1 + \psi) = \det_W^k(1 + \psi) \).
- \( W \) is invariant for \( \varphi + \psi + \varphi \psi; (\varphi + \psi + \varphi \psi)^n V \subset W \) and \( \det_V^k[1 + \varphi + \psi + \varphi \psi] = \det_W^k[1 + (\varphi + \psi + \varphi \psi)_{|_{W}}] \).

Therefore:

\[
\det_V^k(1 + \varphi) \cdot \det_V^k(1 + \psi) = \det_W^k(1 + \varphi_{|_{W}}) \cdot \det_W^k(1 + \psi_{|_{W}}) =
\]

\[
= \det_W^k[(1 + \varphi_{|_{W}})(1 + \psi_{|_{W}})] =
\]

\[
= \det_W^k[1 + (\varphi + \psi + \varphi \psi)_{|_{W}}] =
\]

\[
= \det_V^k[(1 + \varphi)(1 + \psi)] = \det_V^k[(1 + \varphi)(1 + \varphi)]. \]

\( \square \)
Lemma 3.3. Given two finite potent endomorphisms \( \varphi, \psi \in \text{End}(V) \) such that the commutator \([\varphi, \psi] = \varphi \psi - \psi \varphi\) is of finite rank, one has that \( \varphi + \psi \) is a finite potent endomorphism.

Proof. Let \( N > 0 \) be a positive integer such that \( \varphi^N(V) \subseteq W_{\varphi} \) and \( \psi^N(V) \subseteq W_{\psi}, \)
\( W_{\varphi} \) and \( W_{\psi} \) being finite dimensional \( k \)-vector subspaces of \( V \). If \( W_{\varphi \psi} \) is a finite-dimensional subspaces of \( V \) such that \([\varphi \psi - \psi \varphi](V) \subseteq W_{\varphi \psi}\), it is clear that
\[
(\varphi + \psi)^{2N}(V) \subseteq \sum_{i=1}^{2N-1} \varphi^i(W_{\varphi}) + \sum_{j=1}^{2N-1} \psi^j(W_{\varphi}) + \sum_{h,k=1}^{2N-1} \varphi^h \psi^k(W_{\varphi \psi}),
\]
hence the statement is deduced. \( \square \)

Similarly, one can see that:

Lemma 3.4. Given two finite potent endomorphisms \( \varphi, \psi \in \text{End}(V) \) such that \( \varphi \psi - \psi \varphi \) is of finite rank, one has that \( < \varphi, \psi > := \varphi k[\varphi, \psi] + \psi k[\varphi, \psi] \) is a finite potent subspace of \( \text{End}(V) \).

Corollary 3.5. If \( \varphi, \psi \in \text{End}(V) \) are commuting finite potent endomorphisms, then \( < \varphi, \psi > \) is a finite potent subspace of \( \text{End}(V) \).

Proposition 3.6. If \( \varphi \) and \( \psi \) are finite potent endomorphisms such that \( (1 + \varphi)(1 + \psi) = 1 \), then \( < \varphi, \psi > \) is a finite potent subspace of \( \text{End}(V) \).

Proof. Since \( \varphi \psi = \psi \varphi \), the claim is a direct consequence of Corollary 3.5. \( \square \)

Proposition 3.7. Let \( \varphi \) be a finite potent endomorphism of \( V \). Then, \( \det_k^V(1+\varphi) \neq 0 \) if and only if \( 1 + \varphi \) is invertible.

Proof. Assume that \( 1 + \varphi \) is invertible and \( (1 + \varphi)\delta = \delta(1 + \varphi) = 1 \). Assume too that \( \varphi^n V \subseteq W \), with \( W \) a finite dimensional subspace of \( V \). Thus,
\[
\psi = \delta - 1 = -\varphi \delta = -\delta \varphi,
\]
and \( \psi^n V = \delta^n \varphi^n V \subseteq \delta^n W \), from which we deduce that \( \psi \) is a finite potent endomorphism such that \( \psi + \varphi + \psi \varphi = 0 \).

Therefore, it follows from Lemma 3.2 and Proposition 3.6 that
\[
\det_k^V(1+\varphi) \cdot \det_k^V(1+\psi) = \det_k^V 1 = 1,
\]
and, in particular, \( \det_k^V(1+\varphi) \neq 0 \).

On the other hand, if \( \det_k^V(1+\varphi) \neq 0 \) and \( V = W_{\varphi} \oplus U_{\varphi} \) is the \( \varphi \)-invariant AST-decomposition, one has that \( \det_k^V(1+\varphi|_{w_{\varphi}}) \neq 0 \) and \( 1 + \varphi|_{w_{\varphi}} \in \text{End}_k(W_{\varphi}) \) is invertible.

Thus, we can construct \( \delta \in \text{End}_k(V) \) such that \( (1 + \varphi)\delta = \delta(1 + \varphi) = 1 \) as follows:
\[
\delta(v) = \begin{cases} 
(1 + \varphi|_{w_{\varphi}})^{-1}(v) & \text{if } v \in W_{\varphi} \\
(1 + \sum_{i=1}^{s} (-1)^{i} \varphi^{i}|_{w_{\varphi}})(v) & \text{if } v \in U_{\varphi}
\end{cases}
\]
with \( \varphi|_{w_{\varphi}} = 0 \), hence the claim is proved. \( \square \)

Furthermore, when we consider an extension of fields we obtain the following result:
Proposition 3.8. If \( k \hookrightarrow K \) is a finite extension of fields, \( V \) is a \( K \)-vector space, and \( \varphi \in \text{End}_K(V) \) such that \( 1 + \varphi \) is invertible, then we have:

\[
\det_V^k(1 + \varphi) = N_{K/k}[\det_W^k(1 + \varphi)],
\]

\( N_{K/k} : K^\times \rightarrow k^\times \) being the norm of the extension \( k \hookrightarrow K \).

Proof. If \( \det_V^k(1 + \varphi) = \det_W^k(1 + \varphi|_W) \) for a finite-dimensional \( K \)-subspace \( W \subset V \), it is clear that \( \det_V^k(1 + \varphi) = \det_W^k(1 + \varphi|_W) \), and the claim therefore follows from the following well-known property of determinants on finite-dimensional vector spaces:

\[
\det_W^k(\phi) = N_{K/k}[\det_W^k(\phi)],
\]

where \( \phi \in \text{Aut}_K(W) \).

\( \square \)

Remark 3.9. In [3] A. Grothendieck developed an algebraic theory of determinants \( \det (1 + u) \) where \( u \) is a finite rank endomorphism on an infinite-dimensional vector space over a ground field \( k \) of characteristic zero. Obviously, each finite rank endomorphism is a finite potent endomorphism, and we should note that \( \det_V^k(1 + u) = \det(1 + u) \) for every finite rank linear operator \( u \in \text{End}(V) \). Hence, our definition of determinants of finite potent endomorphisms is a generalization of Grothendieck’s algebraic discussion of infinite determinants.

3.B. Infinite determinants and the exterior algebra.

For each positive integer \( r \), one has that an endomorphism \( \varphi \in \text{End}_k(V) \) induces an endomorphism \( \Lambda^r \varphi \in \text{End}_k(\Lambda^r V) \), defined as:

\[
[\Lambda^r \varphi](v_1 \wedge \cdots \wedge v_r) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_r),
\]

where \( \Lambda^r V \) is the component of degree \( r \) of the exterior algebra of \( V \).

If \( \varphi \) is finite potent, then \( \Lambda^r \varphi \) is also finite potent, and it is clear that:

Lemma 3.10. If \( \varphi \in \text{End}_k(V) \) is finite potent such that \( \varphi^n V \subset W \), then \( \Lambda^r \varphi \) is nilpotent for all \( r > \dim_k(W) \).

Moreover, an easy computation on finite dimensional vector spaces shows that:

Lemma 3.11. Let \( W \) be a finite dimensional \( k \)-vector space and let \( \phi \in \text{End}_k(W) \). One has that:

\[
\det(1 + \phi) = 1 + \sum_{r=1}^{\dim W} \text{tr}[\Lambda^r \phi],
\]

where \( \text{tr} \) denotes the ordinary trace.

Let \( \text{tr}_V \) be the trace defined in [10] (see also Section 2) for finite potent endomorphisms.

Proposition 3.12. If \( \varphi \in \text{End}_k(V) \) is finite potent, then:

\[
\det_V^k(1 + \varphi) = 1 + \sum_{r \geq 1} \text{tr}_{\Lambda^r V}[\Lambda^r \varphi].
\]

Proof. Since \( \varphi \) is finite potent, if \( \varphi^n V \subset W \) for some positive integer \( n \), then \( \text{tr}_V(\varphi) = \text{tr}_W(\varphi) \). Similarly, for all \( r \geq 1 \) one has that \( [\Lambda^r \varphi]^n V \subset \Lambda^r W \) for the same \( n \), and hence \( \text{tr}_{\Lambda^r V}[\Lambda^r \varphi] = \text{tr}_{\Lambda^r W}[\Lambda^r \varphi] \).

Thus, the claim is deduced immediately from the definition of the determinant \( \det_V^k(1 + \varphi) \) and the statement of Lemma 3.11 and the expression makes sense bearing in mind Lemma 3.10 (the trace of a nilpotent endomorphism is zero).

\( \square \)

Analogously, we have that:
Corollary 3.13. If \( \mu \in k \) and \( \varphi \in \text{End}_k(V) \) is finite potent, then:
\[
\det_k^V(1 + \mu \varphi) = 1 + \sum_{r \geq 1} \mu^r \text{tr}_{A^r V} [A^r \varphi].
\]

Lemma 3.14. For each finite potent endomorphism \( \varphi \) and for all automorphism \( \phi \in \text{Aut}_k(V) \), one has that:
\[
\det_k^V(1 + \phi \varphi \phi^{-1}) = \det_k^V(1 + \varphi).
\]

Proof. First, we should recall from [10] -see also Subsection (2)- that “if \( f : V' \to V \) and \( g : V \to V' \) are \( k \)-linear and \( fg \) is finite potent, then \( gf \) is finite potent, and \( \text{tr}_V(fg) = \text{tr}_{V'}(gf) \)”. Hence for each finite potent endomorphism \( \varphi \) and for all automorphism \( \phi \), since \( \varphi = \phi^{-1}(\phi \varphi) \) is finite potent, \( \phi \varphi \phi^{-1} \) is also finite potent and we have:
\[
\text{tr}_V(\phi \varphi \phi^{-1}) = \text{tr}_V(\varphi).
\]

Similarly, one has that:
\[
\text{tr}_{A^r V}[A^r(\phi \varphi \phi^{-1})] = \text{tr}_{A^r V}[(\Lambda^r \phi)(\Lambda^r \varphi)](\Lambda^r \varphi^{-1})] = \text{tr}_{A^r V}(\Lambda^r \varphi)
\]
for all \( r > 1 \), and it follows from Proposition 3.12 that:
\[
\det_k^V(1 + \phi \varphi \phi^{-1}) = 1 + \sum_{r \geq 1} \text{tr}_{A^r V}[A^r(\phi \varphi \phi^{-1})]
\]
\[
= 1 + \sum_{r \geq 1} \text{tr}_{A^r V}[A^r \varphi] = \det_k^V(1 + \varphi).
\]
\[\square\]

A direct consequence of the previous Lemma and of Proposition 3.12 is as follows:

Corollary 3.15. If \( F \subseteq \text{End}(V) \) is a finite potent subspace and \( \varphi, \psi \in F \), then \( \varphi \psi \) and \( \psi \varphi \) are finite potent endomorphisms satisfying the condition that:
\[
\sum_{r \geq 1} \text{tr}_{A^r V}[A^r(\varphi \psi)] = \sum_{r \geq 1} \text{tr}_{A^r V}[A^r(\psi \varphi)] = \sum_{i,j \geq 1} \left( \text{tr}_{A^i V}[A^i \varphi] \right) \cdot \left( \text{tr}_{A^j V}[A^j \psi] \right).
\]

As in the algebraic formalism of determinants of finite-dimensional vector spaces, to conclude this subsection we shall study the relationship between the scalar \( \det_k^V(1 + \varphi) \) and the “infinite wedge product” of the elements of a basis in an arbitrary \( k \)-vector space of infinite countable dimension \( \tilde{V} \).

Writing \( T^\infty \tilde{V} = \bigotimes_1^\infty \tilde{V} \) to denote the infinite countable tensor product, we have that \( \Lambda^\infty \tilde{V} = T^\infty \tilde{V}/J(\tilde{V}) \), where \( J(\tilde{V}) \) is the ideal generated by the elements \( \cdots \otimes \tilde{v} \otimes \cdots \otimes \tilde{v} \otimes \cdots \in T^\infty \tilde{V} \). It is clear that \( \Lambda^\infty \tilde{V} \) is also a \( k \)-vector space.

Given a basis \( \{\tilde{v}_1, \ldots, \tilde{v}_n, \ldots\} \), one has that:
\[
0 \neq \tilde{v}_1 \land \cdots \land \tilde{v}_n \land \cdots \in \Lambda^\infty \tilde{V}.
\]

Moreover, every endomorphism \( \psi \in \text{End}(\tilde{V}) \) induces a \( k \)-linear map defined by:
\[
\Lambda^\infty(\psi) : \Lambda^\infty \tilde{V} \longrightarrow \Lambda^\infty \tilde{V}
\]
\[
u_1 \land \cdots \land \nu_n \land \ldots \longmapsto \psi(\nu_1) \land \cdots \land \psi(\nu_n) \land \ldots
\]
for all \( \nu_1 \land \cdots \land \nu_n \land \cdots \in \Lambda^\infty \tilde{V} \). For each \( \psi, \phi \in \text{End}(\tilde{V}) \), one has that:
\[
\Lambda^\infty(\psi \circ \phi) = \Lambda^\infty(\psi) \circ \Lambda^\infty(\phi).
\]

Recall now from [5] that for every \( \phi \in \text{End}(V) \) possessing an annihilating polynomial of an arbitrary infinite-dimensional vector space \( V \) there exists a Jordan basis of \( V \) associated with \( \phi \).
Proposition 3.16. Let \( \tilde{V} \) be a k-vector space of infinite countable dimension, let \( \phi \in \text{End}(\tilde{V}) \) be a nilpotent endomorphism and let \( \{\tilde{w}_1, \ldots, \tilde{w}_n, \ldots\} \) be a Jordan basis of \( \tilde{V} \) associated with \( \phi \). One has that:

\[
\Lambda^\infty(1 + \phi)[\tilde{w}_1 \wedge \cdots \wedge \tilde{w}_n \wedge \ldots] = \tilde{w}_1 \wedge \cdots \wedge \tilde{w}_n \wedge \ldots.
\]

Proof. Since \( \phi \) is nilpotent, if \( \phi^N = 0 \), then a Jordan basis of \( \tilde{V} \) for \( \phi \) is:

\[
\bigcup_{i=1}^{\infty} \{\tilde{v}_i, \phi(\tilde{v}_i), \ldots, \phi^{\mu(i)}(\tilde{v}_i)\},
\]

with \( \mu(i) < N \) and \( \phi^{\mu(i)+1}(\tilde{v}_i) = 0 \).

Bearing in mind that:

\[
[1 + \phi](\tilde{v}_i) \wedge [1 + \phi](\phi(\tilde{v}_i)) \wedge \cdots \wedge [1 + \phi](\phi^{\mu(i)}(\tilde{v}_i)) = \tilde{v}_i \wedge \phi(\tilde{v}_i) \wedge \cdots \wedge \phi^{\mu(i)}(\tilde{v}_i)
\]

for all \( i \), we deduce the claim recurrently.

Let us now consider again a finite potent endomorphism \( \varphi \) of an arbitrary k-vector space \( \tilde{V} \) of infinite countable dimension with AST-decomposition \( \tilde{V} = \tilde{W}_\varphi \oplus \tilde{U}_\varphi \). Since \( \varphi \) admits an annihilating polynomial, then there exists a Jordan basis \( \{\tilde{w}_1^\varphi, \ldots, \tilde{w}_n^\varphi, \tilde{w}_{n+1}^\varphi, \ldots\} \), where \( \{\tilde{w}_1^\varphi, \ldots, \tilde{w}_n^\varphi\} \) is a Jordan basis of \( \tilde{W}_\varphi \) for \( \varphi \|_{\tilde{W}_\varphi} \), and \( \{\tilde{w}_{n+1}^\varphi, \ldots, \tilde{w}_{n+s}^\varphi, \ldots\} \) is a Jordan basis of the k-vector space \( \tilde{U}_\varphi \) for the nilpotent endomorphism \( \varphi|_{\tilde{U}_\varphi} \). Note that \( \tilde{U}_\varphi \) also has infinite countable dimension.

Thus, it follows from the definition of determinant \( \text{det}^k_V(1 + \varphi) \) and the statement of Proposition 3.16 that:

Proposition 3.17. With the previous notation, if \( \varphi \) is a finite potent endomorphism of an arbitrary k-vector space \( \tilde{V} \) of infinite countable dimension, one has that:

\[
\Lambda^\infty(1 + \varphi)[\tilde{w}_1^\varphi \wedge \cdots \wedge \tilde{w}_n^\varphi \wedge \tilde{w}_{n+1}^\varphi \wedge \cdots] = \text{det}^k_V(1 + \varphi) \cdot [\tilde{w}_1^\varphi \wedge \cdots \wedge \tilde{w}_n^\varphi \wedge \tilde{w}_{n+1}^\varphi \wedge \cdots].
\]

Accordingly, the relationship between the scalar \( \text{det}^k_V(1 + \varphi) \) and the “infinite wedge product” of the elements of an arbitrary basis of \( \tilde{V} \) is given by the following:

Theorem 3.18. If \( \varphi \) is a finite potent endomorphism of an arbitrary k-vector space \( \tilde{V} \) of infinite countable dimension and \( \{\tilde{v}_1, \ldots, \tilde{v}_n, \ldots\} \) is an arbitrary basis of \( \tilde{V} \), then:

\[
\Lambda^\infty(1 + \varphi)[\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_n \wedge \cdots] = \text{det}^k_V(1 + \varphi) \cdot [\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_n \wedge \cdots].
\]

Proof. Let \( \{\tilde{w}_1^\varphi, \ldots, \tilde{w}_n^\varphi, \tilde{w}_{n+1}^\varphi, \ldots\} \) be the above Jordan basis of \( \tilde{V} \). Considering the isomorphism:

\[
\phi: \tilde{V} \rightarrow \tilde{V}
\]

and bearing in mind Lemma 3.14 and Proposition 5.17 one has that:

\[
\Lambda^\infty(1 + \varphi)[\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_n \wedge \cdots] = \Lambda^\infty([1 + \varphi] \circ \phi^{-1})[\tilde{w}_1^\varphi \wedge \cdots \wedge \tilde{w}_n^\varphi \wedge \cdots] = (\Lambda^\infty(\phi^{-1}) \circ \Lambda^\infty(1 + [\phi \circ \phi^{-1}]))[\tilde{w}_1^\varphi \wedge \cdots \wedge \tilde{w}_n^\varphi \wedge \cdots] = \text{det}^k_V(1 + [\phi \circ \phi^{-1}]) \cdot \Lambda^\infty(\phi^{-1})[\tilde{w}_1^\varphi \wedge \cdots \wedge \tilde{w}_n^\varphi \wedge \cdots] = \text{det}^k_V(1 + \varphi) \cdot [\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_n \wedge \cdots].
\]

\( \square \)
3.C. Infinite determinant and eigenvalues.

Let \( \varphi \) be a finite potent endomorphism of a \( k \)-vector space \( V \) with AST-decomposition \( V = W_\varphi \oplus U_\varphi \). Let \( \lambda_1, \ldots, \lambda_N \) now be the nonzero eigenvalues of \( \varphi \) in the algebraic closure of \( k \) (repeated according their algebraic multiplicities), which coincide with the eigenvalues of the isomorphism \( \varphi|_{W_\varphi} \) (of a finite-dimensional vector space) in the algebraic closure of \( k \).

**Proposition 3.19.** If \( \lambda_1, \ldots, \lambda_N \) are all the nonzero eigenvalues of \( \varphi \), then one has that:

\[
\det_k^V (1 + \varphi) = \prod_{i=1}^N (1 + \lambda_i).
\]

**Proof.** If \( k \hookrightarrow L \) is an extension of the scalar field \( k \) that contains all eigenvalues of \( \varphi \), then:

\[
\det_k^{W_\varphi} (1 + \varphi|_{W_\varphi}) = \det_L^{W_\varphi \otimes_k L} [1 + (\varphi|_{W_\varphi} \otimes 1)] = \prod_{i=1}^N (1 + \lambda_i)
\]

and the claim is deduced. \( \square \)

If \( S_r(x_1, \ldots, x_N) \) is the elementary symmetric function given by:

\[
S_r(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_r \leq N} x_{i_1} \cdots x_{i_r},
\]

a direct consequence of Proposition 3.19 is:

**Corollary 3.20.** If \( \mu \in k \) and \( \lambda_1, \ldots, \lambda_N \) are all the nonzero eigenvalues of \( \varphi \), then one has that:

\[
\det_k^V (1 + \mu \varphi) = \sum_{i=0}^N \mu^i \cdot S_i(\lambda_1, \ldots, \lambda_N).
\]

**Remark 3.21.** If \( \lambda_1, \ldots, \lambda_N \) are again all the nonzero eigenvalues of a finite potent endomorphism \( \varphi \), note that from Corollary 3.20 and Corollary 3.13 we can deduce that:

\[
\text{tr}_{A^r V}[A^r \varphi] = S_r(\lambda_1, \ldots, \lambda_N).
\]

In particular, when \( r = 1 \) we have an algebraic version for finite potent endomorphisms of Lidskii’s Theorem from the equality:

\[
\text{tr}_V[\varphi] = \sum_{i=1}^N \lambda_i.
\]

4. Exponential map and infinite determinants.

4.A. Exponential map of finite potent endomorphisms.

Let \( k \) be a field of characteristic zero and let \( V \) be a \( k \)-vector space. Let \( k((z)) \) be the field of Laurent series and let us denote \( V_z := V \otimes_k k((z)) \).

**Proposition 4.1.** Let \( \varphi \in \text{End}_k(V) \) be a finite potent endomorphism. Then, there exists a well-defined exponential map:

\[
\exp_z: \text{End}_k(V) \to \text{Aut}_{k((z))}(V_z)
\]

\[
\varphi \mapsto \exp_z(\varphi) = 1 + z \varphi + \frac{z^2 \varphi^2}{2} + \frac{z^3 \varphi^3}{3!} + \cdots
\]
Moreover, the endomorphism of $V_z$:
\[ \tilde{\varphi} = \exp_z(\varphi) - 1 = z\varphi + \frac{z^2\varphi^2}{2} + \frac{z^3\varphi^3}{3!} + \cdots \]

is finite potent.

**Proof.** Since $\varphi$ is finite potent there exist a $\varphi$-invariant finite dimensional $k$-subspace $W$ of $V$ such that $\varphi^n(V) = W$. Let \( \{w_1, \ldots, w_r\} \) be a basis for $W$. Then, for any $v \in V$ we have:
\[ \exp_z(\varphi)(v) = v + z\varphi(v) + \cdots + \frac{1}{(n-1)!} \varphi^{n-1}(v) + s_1(z)w_1 + \cdots + s_r(z)w_r, \]
where $s_i(z)$ is well-defined. Let $\varphi$ be a basis for $V$ for all $v \in V$, and therefore $\exp_z$ is well-defined.

In particular, $\tilde{\varphi}$ is an endomorphism of $V_2$ that consists of a finite sum of commuting finite potent endomorphisms (the first $n - 1$ terms) plus a finite rank endomorphism of $W_z = W \otimes_k k((z))$. Accordingly, using Lemma 3.3 one concludes that $\tilde{\varphi}$ is finite potent. \( \square \)

**Proposition 4.2.** If $\varphi \in \text{End}_k(V)$ is finite potent, then:
\[ \det_{V_z}^{k((z))}(\exp_z(\varphi)) = \exp_z(\text{tr}_V(\varphi)) \in k((z))^\times. \]

**Proof.** By Proposition 4.1, the endomorphism $\tilde{\varphi} = \exp_z(\varphi) - 1$ is finite potent and therefore:
\[ \det_{V_z}^{k((z))}(\exp_z(\varphi)) = \det_{V_z}^{k((z))}(1 + \tilde{\varphi}) \]
is well-defined. Let $W$ be a finite $k$-subspace of $V$ such that $\varphi(W) \subset W$ and $\varphi^n(V) \subset W$, and let us denote $W_z = W \otimes_k k((z))$. One has that $(\varphi \otimes 1)(W_z) \subset W$ and $(\varphi \otimes 1)^n(W_z) \subset W_z$. Since the result is well-known to be true for finite dimensional vector spaces one finishes:
\[ \det_{V_z}^{k((z))}(\exp_z(\varphi)) = \det_{W_z}^{k((z))}(\exp_z(\varphi)_{|W_z}) = \det_{W_z}^{k((z))}(\exp_z(\varphi_{|W})) = \exp_z(\text{tr}_W(\varphi_{|W})) = \exp_z(\text{tr}_V(\varphi)) \]
\( \square \)

Consider the subspace $E_0$ of $\text{End}_k(V)$ of equation (2.1).

**Proposition 4.3.** If both $f$ and $g$ lie on $E_0$ then:
\[ \det_{V_z}^{k((z))}(\exp_z(f) \cdot \exp_z(g)) = \det_{V_z}^{k((z))}(\exp_z(f)) \cdot \det_{V_z}^{k((z))}(\exp_z(g)). \]

**Proof.** First of all, let us notice that the product of any two elements in $E_0$ is a finite rank endomorphism of $V$. Since $f, g \in E_0$, in particular they are finite potent endomorphisms and one has that $f^2(V)$ and $g^2(V)$ are finite dimensional subspaces of $V$. Moreover, $\langle f, g \rangle$ is a finite potent subspace of $\text{End}_k(V)$ (by Lemma 3.4) and $\langle f, g \rangle^>(V) = W$ is a finite dimensional subspace of $V$ such that:
- $f^2(V) \subseteq W$ and $f(W) \subseteq W$.
- $g^2(V) \subseteq W$ and $g(W) \subseteq W$.
- $fg(V) \subseteq W$ and $fg(W) \subseteq W$.
- $gf(V) \subseteq W$ and $gf(W) \subseteq W$.
- $[f, g](V) \subseteq W$ and $[f, g](W) \subseteq W$. 


Let us write $\exp_z(f) = 1 + \varphi$ and $\exp_z(g) = 1 + \psi$, where:

\[
\varphi = zf + \frac{z^2}{2} f^2 + \frac{z^3}{3!} f^3 + \cdots
\]
\[
\psi = zg + \frac{z^2}{2} g^2 + \frac{z^3}{3!} g^3 + \cdots
\]

are both finite potent endomorphisms of $\phi, \psi$. Moreover, one has that $\langle \phi, \psi \rangle$ is a finite rank endomorphism of $\phi, \psi$. Again applying Lemma 4.2 one concludes:

\[
\det_{V_z}^k((\exp_z(f) \cdot \exp_z(g))) = \det_{V_z}^k((1 + \varphi) \cdot (1 + \psi)) = \det_{V_z}^k(1 + \varphi) \cdot \det_{V_z}^k(1 + \psi) = \det_{V_z}^k((\exp_z(f)) \cdot \det_{V_z}^k((\exp_z(g))).
\]

\[\square\]

**Proposition 4.4.** If both $f$ and $g$ lie on $E_0$ then:

\[
\det_{V_z}^k((\exp_z(f + g))) = \det_{V_z}^k((\exp_z(f)) \cdot \det_{V_z}^k((\exp_z(g))).
\]

**Proof.** Since $E_0$ is a subspace one has that $f + g \in E_0$ and hence its exponential is well-defined. Therefore, using Proposition 4.2 and bearing in mind that $\text{tr}_V$ is $k$-linear on $E_0$ (see Proposition 2.4) one has:

\[
\det_{V_z}^k((\exp_z(f + g)) = \exp_z((\text{tr}_V(f + g)) = \exp_z((\text{tr}_V(f) + \text{tr}_V(g)) = \exp_z((\text{tr}_V(f)) \cdot \exp_z((\text{tr}_V(g)) = \det_{V_z}^k((\exp_z(f)) \cdot \det_{V_z}^k((\exp_z(g)).
\]

\[\square\]

**Corollary 4.5.** If both $f$ and $g$ lie on $E_0$ then:

\[
\det_{V_z}^k((\exp_z(f + g))) = \det_{V_z}^k((\exp_z(f)) \cdot \det_{V_z}^k((\exp_z(g))).
\]

**4.B. Determinant of an infinite product of exponentials.**

Let us denote the set $A_n = \prod_{i=1}^n \exp_{\varphi_i}(E_0)$ (for $n \geq 1$) and write:

\[
a_n = \{\exp_{\varphi_1}, \ldots, \exp_{\varphi_{n-1}}, \exp_{\varphi_n}\}.
\]

For each $n' \geq n$, let us consider the maps:

\[
\phi_{n'n}: A_{n'} \rightarrow A_n,
\]

\[
a_{n'} \mapsto a_n.
\]

Thus, $\{A_n, \phi_{n'n}\}$ is an inverse system of sets and $\lim_{n \rightarrow n} A_n = \prod_{i=1}^\infty \exp_{\varphi_i}(E_0)$.

**Remark 4.6.** Following Proposition 4.1 for every $\varphi \in E_0$ its exponential $\exp_{\varphi}(\varphi)$ has the shape:

\[
\exp_{\varphi}(\varphi) = 1 + \varphi + \frac{\varphi^2}{2} + \frac{\varphi^3}{3!} + \cdots
\]

where $\varphi = z^i \varphi + \frac{z^2}{2} \varphi^2 + \frac{z^3}{3!} \varphi^3 + \cdots$ is a finite potent endomorphism of $V_z$; that is, there exists a finite dimension subspace $W$ of $V$ and $m$ such that $\varphi^m(V_z) \subseteq W_z$, and moreover $W$ is such that $\varphi^m(V) \subseteq W$. Thus, its determinant is well-defined. \[\square\]
Proposition 4.7. There exists a well-defined map:
\[
\det_{V_z}^{k(z)}: A_n \to k(z)
\]
\[
a_n \mapsto \det_{V_z}^{k(z)}(a_n) = \prod_{i=1}^{n} \det_{V_z}^{k(z)}(\exp_{z_i}(\varphi_i))
\]

Proof. This follows from the fact that for each \( n \in \mathbb{N} \) there exists an inclusion:
\[
A_n = \prod_{i=1}^{n} \exp_{z_i}(E_0) \hookrightarrow \text{Aut}_{k(z)}(V_z)
\]
and hence, since Proposition 4.3 shows that:
\[
\prod_{i=1}^{n} \exp_{z_i}(\varphi_i) = \prod_{i=1}^{n} \det_{V_z}^{k(z)}(\exp_{z_i}(\varphi_i)),
\]
one concludes. \( \square \)

The aim now is to define the determinant for certain elements in the limit \( \prod_{i=1}^{\infty} \exp_{z_i}(E_0) \). Notice that one has:
\[
\prod_{i=1}^{\infty} \exp_{z_i}(\varphi_i) = 1 + \varphi_1 z + (\frac{1}{2} \varphi_1 + \varphi_2)z^2 + \cdots \in \text{Aut}_{k(z)}(V_z)
\]
and hence there exists a well-defined map:
\[
\lim_{\chi} A_n = \prod_{i=1}^{\infty} \exp_{z_i}(E_0) \hookrightarrow \text{Aut}_{k(z)}(V_z)
\]
\[
\{a_n\} = \{\exp_{z_1}(\varphi), \ldots, \exp_{z_{n-1}}(\varphi_{n-1}), \exp_{z_n}(\varphi), \ldots\} \to \prod_{i=1}^{\infty} \exp_{z_i}(\varphi_i).
\]

However, note too that even if we have:
\[
\prod_{i=1}^{\infty} \exp_{z_i}(\varphi_i) = 1 + \varphi_1 z + (\frac{1}{2} \varphi_1 + \varphi_2)z^2 + \cdots \in E_0 \otimes_k k((z)) \subseteq \text{Aut}_{k(z)}(V_z),
\]
that is, every coefficient of \( z^i \) is an element in \( E_0 \) (so in particular every coefficient is finite potent), we still don’t know whether a common finite dimensional subspace of \( V \) exists or not for all coefficients of \( z^i \). Therefore, we cannot directly define its determinant.

In order to solve this deficiency, it suffices to consider elements in the limit \( \{a_n\} \in \lim_{\chi} A_n \) that are compatible with \( \det_{V_z}^{k(z)}: A_n \to k((z)) \) (where \( k((z)) \) is endowed with the trivial inverse system); that is, elements \( \{a_n\} \in \lim_{\chi} A_n \) for which there exists \( m \in \mathbb{N} \) such that for each \( n \geq m \):
\[
\det_{V_z}^{k(z)}(a_n) = \det_{V_z}^{k(z)}(a_m).
\]

For these elements it makes sense to give the following definition.

Definition 4.8. Let \( \{a_n\} \in \lim_{\chi} A_n \) be compatible with \( \det_{V_z}^{k(z)}: A_n \to k((z)) \).

We define its determinant by:
\[
\det_{V_z}^{k(z)}(\{a_n\}) := \det_{V_z}^{k(z)}(a_m) \in k((z)).
\]
Therefore, using Proposition 4.3 we have:
\[
\det_{V_z}^{k(z)} \left( \prod_{i=1}^{n} \exp_{z_i}(\varphi_i) \right) := \det_{V_z}^{k(z)} \left( \prod_{i=1}^{m} \exp_{z_i}(\varphi_i) \right) = \prod_{i=1}^{m} \det_{V_z}^{k(z)} \left( \exp_{z_i}(\varphi_i) \right).
\]

Example 4.9. Let \( \{\varphi_i\}_{i \geq 1} \) be a family of elements in \( E_0 \) such that \( \text{tr}_V(\varphi_i) = 0 \) for all \( i \geq m \) for some \( m \in \mathbb{N} \). We have that:
- \( \det_{V_z}^{k(z)} \left( \exp_{z_i+1}(\varphi_i) \right) \) is well-defined for all \( i \geq 1 \) because \( \varphi_i \in E_0 \).
- \( \det_{V_z}^{k(z)} \left( \exp_{z_i+1}(\varphi_i) \right) = 1 \) for all \( i \geq m \), since \( \det_{V_z}^{k(z)} \left( \exp_{z_i+1}(\varphi_i) \right) = \exp_{z_i+1} \left( \text{tr}_V(\varphi_i) \right) \) (see Proposition 4.2) and \( \text{tr}_V(\varphi_i) = 0 \) for all \( i \geq m \).

Then, for all \( n \geq m \) we have:
\[
\det_{V_z}^{k(z)} \left( \prod_{i=1}^{n} \exp_{z_i+1}(\varphi_i) \right) = \prod_{i=1}^{m} \det_{V_z}^{k(z)} \left( \exp_{z_i+1}(\varphi_i) \right)
\]
and therefore the \( \varphi_i \)'s are compatible with \( \det_{V_z}^{k(z)} \) and we obtain:
\[
\det_{V_z}^{k(z)} \left( \prod_{i=1}^{\infty} \exp_{z_i+1}(\varphi_i) \right) = \prod_{i=1}^{m-1} \det_{V_z}^{k(z)} \left( \exp_{z_i+1}(\varphi_i) \right) \in k((z))
\]

5. Central extension of groups and Tate’s residue.

Using the theory of infinite determinants developed in the previous sections we are about to construct a central extension of groups by giving its associated cocycle explicitly. The importance of this extension lies in the fact that it can be viewed as the multiplicative analogue of Tate’s formalism of abstract residues in terms of traces of finite potent endomorphisms. Finally, a reciprocity law for this cocycle is given, which can be thought of as a multiplicative version of Tate’s theorem of residues.

5.A. Central extension of groups and Tate's residue.

Let us recall Tate’s definition of the “residue map”. Let \( K \) be a commutative \( k \)-algebra with unit, \( V \) a \( K \)-module and \( V_+ \) a \( k \)-subspace of \( V \) such that \( V_+ \subset V_+ \) for all \( f \in K \). With the notations \( E, E_0, E_1, E_2, V_1, V_2 \) of equation (2.1) this latter condition means that \( K \) operates on \( V \) through \( E \subset \text{End}_k(V) \), and in what follows we shall use the same letter, \( f \), to denote an element of \( K \) and its image in \( E \).

**Theorem 5.1.** (Definition of residue) [10] / Thm. 1/ In the situation just described, there exists a unique \( k \)-linear “residue map”:
\[
\text{Res}^V_{E, k} : \Omega^1_{K/k} \to k,
\]
such that for each pair of elements \( f \) and \( g \) in \( K \) we have:
\[
\text{Res}^V_{E, k}(fdg) = \text{tr}_V([f_1, g_1])
\]
for every pair of endomorphisms \( f_1 \) and \( g_1 \) in \( E \) satisfying the following conditions:

1. Both \( f \equiv f_1 \pmod{E_2} \) and \( g \equiv g_1 \pmod{E_2} \);
2. Either \( f_1 \in E_1 \) or \( g_1 \in E_1 \).

Note that given \( f \) and \( g \) in \( K \) it is always possible to find \( f_1 \) and \( g_1 \) in \( E \) satisfying (1) and (2) because \( E = E_1 \oplus E_2 \) (see Proposition 2.3). Thus, \([f, g_1] \in E_1 \) by (2) and \([f_1, g_1] = [f, g] = 0 \pmod{E_2} \) by (1). Hence, by Proposition 2.4 \([f_1, g_1] \in E_1 \cap E_2 = E_0 \) and therefore its trace is well-defined (recall that \( E_0 \) is finite potent).
Our next task consists of giving a multiplicative analogue of Tate’s residue. In order to do this, let us first recall the Zassenhaus formula:

\[(5.1) \quad \exp_z(f_1 + g_1) = \exp_z(f_1) \exp_z(g_1) \prod_{i \geq 1} \exp_{z^{i+1}} \left( \frac{-C_i(f_1, g_1)}{(i + 1)!} \right),\]

where:

\[C_1 = [f_1, g_1], \quad C_2 = 2[f_1, g_1] - [f_1, [f_1, g_1]], \quad C_3 = 3[[f_1, g_1], g_1] - 3[f_1, [f_1, g_1]], \quad g_1 + [f_1, [f_1, g_1]], \quad \ldots\]

Remark 5.2. It is worth noticing that if \([f_1, g_1] \in E_0\) then \(\operatorname{tr}_V(C_i) = 0\) for all \(i \geq 2\) by Proposition 2.3.

**Theorem 5.3.** Let \(f, g \in K\). The function:

\[c_{V_z} : K \times K \to k((z))^\times\]

\[(f, g) \mapsto c_{V_z}(f, g) := \det_{V_z}^{k((z))} \left( \exp_z(f_1) \exp_z(g_1) \exp_z(-(f_1 + g_1)) \right)\]

is a 2-cocycle of \(K\) with coefficients in \(k((z))^\times\), for every pair endomorphisms \(f_1\) and \(g_1\) in \(E\) satisfying:

- \(f \equiv f_1 \pmod{E_2}\) and \(g \equiv g_1 \pmod{E_2}\);
- Either \(f_1 \in E_1\) or \(g_1 \in E_1\).

In particular, there exists a central extension of groups:

\[1 \to k((z))^\times \to \overline{K}_{V_z} \to K \to 1.\]

**Proof.** Let us first check that \(c_{V_z}(f, g)\) is well-defined. If we denote:

\[e(f, g) = \exp_z(f_1) \exp_z(g_1) \exp_z(-(f_1 + g_1))\]

and make use of the Zassenhaus formula \((5.1)\), we have:

\[e(f, g) = \prod_{i \geq 1} \exp_{z^{i+1}} \left( \frac{-C_i(-g_1, -f_1)}{(i + 1)!} \right),\]

hence:

\[c_{V_z}(f, g) = \det_{V_z}^{k((z))} \left( \prod_{i \geq 1} \exp_{z^{i+1}} \left( \frac{-C_i(-g_1, -f_1)}{(i + 1)!} \right) \right).\]

Using Remark 5.2 and Example 4.9 (with \(m = 2\)) we may conclude that:

\[c_{V_z}(f, g) = \det_{V_z}^{k((z))} \left( \prod_{i \geq 1} \exp_{z^{i+1}} \left( \frac{-C_i(-g_1, -f_1)}{(i + 1)!} \right) \right) = \det_{V_z}^{k((z))} \left( \exp_z\left( \frac{1}{2}[f_1, g_1]\right) \right)\]

is well-defined since \([f_1, g_1] \in E_0\).

It is now clear that \(c_{V_z}\) can be regarded as a 2-cochain of the standard complex \(C^\bullet(K, k((z))^\times)\), and thus it suffices to check the cocycle condition:

\[c_{V_z}(f, g) \cdot c_{V_z}(f + g, h) = c_{V_z}(g, h) \cdot c_{V_z}(f, g + h)\]

for all \(f, g, h \in K\). However, this follows directly from Proposition 4.4 taking into account that:

\([f_1, g_1] + [f_1 + g_1, h_1] = [g_1, h_1] + [f_1, g_1 + h_1],\]

for every endomorphisms \(f_1, g_1\) and \(h_1\) in \(E_{V_z}\) satisfying:

- \(f \equiv f_1 \pmod{E_2}\), \(g \equiv g_1 \pmod{E_2}\) and \(h \equiv h_1 \pmod{E_2}\);
- At least two of \(f_1, g_1, h_1\) lie on \(E_1\).

□
Remark 5.4. If we write $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with respect to a fixed decomposition of $V = V_+ \oplus V_-$, we can assume that $f_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, the definition of the cocycle $c_{V_+}$ is inspired by Segal and Wilson’s cocycle of [8] Prop.3.6] (see also equation (1.1.1) in Appendix B).

Remark 5.5. Let us note that by Proposition 1.2 and Theorem 5.1 Theorem 5.3 states that:

$c_{V_+}(f, g) = \det_{V_+}^k(\exp_z^2(\frac{1}{2}[f_1, g_1])) = \exp_z^2(\frac{1}{2}\text{tr}_V([f_1, g_1])) = \exp_z^2(\frac{1}{2}\text{Res}_{V_+}^V(fdg))$.

Therefore, the cocycle $c_{V_+}$ is a multiplicative analogue for the abstract residue defined by Tate in [10].

Remark 5.6. Notice that the commutator of the central extension:

$$1 \to k((z)) \to \tilde{K}_{V_+} \to K \to 1.$$ 

is given by:

$$\{f, g\}_{V_+} = \exp_{z^2}(\text{tr}_V([f_1, g_1])), \quad \text{and therefore it has a similar shape to Segal and Wilson’s pairing given in [8] for loop groups (see Appendix B).}$$

Remark 5.7. It can be checked that:

$$\det_{V_+}^k(\exp_z^2(f_1)\exp_z(g_1)\exp_z(-f_1)\exp_z(-g_1)) = \exp_{z^2}(\text{tr}_V([f_1, g_1])), \quad \text{which generalizes the well-known formula:}$$

$$\det(\exp(A)\exp(B)\exp(-A)\exp(-B)) = \exp(\text{tr}[A, B])$$

for bounded operators $A$ and $B$ with trace-class commutator $[A, B]$ (see Appendix A for the analytic approach).

5.B. Properties of $c_{V_+}$ and reciprocity law.

Using Tate’s properties $(R_1), \ldots, (R_5)$ for the residue $\text{Res}_{V_+}^V$ (see [10]), we have the following:

$(C_1)$ If $A \sim A'$, then $c_A(f, g) = c_{A'}(f, g)$. That is, the cocycle does not depend on the commensurability class of $V_+$. In particular $\tilde{K}_A = \tilde{K}_{A'}$.

$(C_2)$ If $f(A) \subset A$ and $g(A) \subset A$, then $c_A(f, g) = 1$. In particular, the central extension $\tilde{K}_A$ is trivial.

$(C_3)$ $c_A(1, g) = 1$ for all $g \in K$.

$(C_4)$ If $g$ is invertible in $K$ and $h \in K$ is such that $h(A) \subset A$, then:

$$c_A(hg^{-1}, g) = \exp_{z^2}(\frac{1}{2}\text{tr}_{A^g\cap A}(h)) \cdot \exp_{z^2}(\frac{1}{2}\text{tr}_{A^g\cap A}(h)).$$

$(C_5)$ If $B$ is another $k$-subspace of $V$ such that $f(B) < B$ for all $f \in K$, then:

$$c_{A+B}(f, g) = c_A(f, g) \cdot c_B(f, g).$$

Using these properties and the Corollary of [10] Theorem 3] we have:

Corollary 5.8. (Reciprocity law) Let $X$ be a non-singular and complete curve over $k$ and let $K$ denote its function field. For a closed point $x \in X$, let us write $A_x = \hat{O}_{X, x}$ and $V_x = (\hat{O}_{X, x})_0$. Then:

$$\prod_{x \in C} c_{A_x}(f, g) = 1 \quad \text{for all } f, g \in K,$$

where the product is taken over all closed points $x$ of $X$.
Remark 5.9. Let us remark that this last section is an approach to a unified theory of local symbols over fields of characteristic zero. Using the algebraic definition of the determinant for finite potent endomorphisms developed along the paper, Theorem 5.3 reveals an equivalence between the theory of Segal and Wilson for loop groups (see Appendix B) and the theory of abstract residues given by Tate.

Further research will be done in order to extend this theory to characteristic \( p > 0 \) and modules over artinian local rings for the purpose of obtaining a unified theory of arithmetic symbols from infinite determinants.

Appendix A. Analytic approach.

Let \( H \) be a separable complex Hilbert space. Given an arbitrary finite potent linear operator \( \varphi : H \rightarrow H \), bearing in mind that it has an annihilating polynomial, it follows again from \([5]\) that \( H \) admits a Jordan basis for \( \varphi \), with arguments similar to those of M. Argerami, F. Szechtman and R. Tifenbach in \([1]\), one has that the Hilbert space \( H \) admits a \( \varphi \)-invariant decomposition \( H = W_\varphi \oplus U_\varphi \) such that \( \varphi|_{U_\varphi} \) is nilpotent, \( W_\varphi \) is finite dimensional, and \( \varphi|_{W_\varphi} : W_\varphi \rightarrow W_\varphi \) is an isomorphism.

Thus, since all the expressions referred to in the previous algebraic approach to infinite determinants consist of finite sums of well-defined analytic elements of \( \varphi \), the above definition and properties of \( \det_1(1 + \varphi) \in \mathbb{C} \) are valid within the analytic formalism of Hilbert spaces.

In fact, in \([9]\) B. Simon defined determinants of trace class operators \( B \) on a separable Hilbert space from the formula:

\[(A.1) \quad \det_1(1 + \mu B) = 1 + \sum_{n=1}^{\infty} \mu^n \text{tr}(\Lambda^n(B)),\]

and according to \([7]\) and \([9]\), for each trace class operator \( B \) one has that the infinite determinant \( \det_1(1 + B) \) satisfies the following properties:

- If \( A \) and \( B \) are trace class, then:
  \[
  \det_1(1 + A) \cdot \det_1(1 + B) = \det_1(1 + A + B + AB) = \det_1(1 + B) \cdot \det_1(1 + A).
  \]

- The operator \( 1 + B \) is invertible if and only if \( \det_1(1 + B) \neq 0 \).

- If \( U \) is unitary, then:
  \[
  \det_1(U^{-1}(1 + B)U) = \det_1(1 + U^{-1}BU) = \det_1(1 + B).
  \]

Corollary \([3,13]\) shows that our definition of determinants of finite potent endomorphisms coincides with expression \((A.1)\) when we replace the usual trace of linear operators of Hilbert spaces with Tate’s definition of traces of finite potent endomorphisms. The above properties of \( \det_1(1 + B) \) correspond to Lemma \([3,2]\) Proposition \([3,7]\) and Lemma \([3,14]\) (respectively) in the finite potent case.

Analogously, let \( \{\lambda_i(B)\}_{i=1}^{N(B)} \) be the listing of all nonzero eigenvalues of a trace class operator \( B \), counted up to algebraic multiplicity. In \([2]\) Dunford and Schwartz defined the infinite determinant \( \det_1(1 + \mu B) \) as the expression:

\[(A.2) \quad \det_1(1 + \mu B) = \prod_{i=1}^{N(B)} (1 + \mu \lambda_i(B)),\]

which coincides with the statement of Proposition \([3,19]\). This is another important equivalence between the classical results of well-known analytic treatments of infinite determinants and the above algebraic theory.
Furthermore, if \( \lambda_1, \ldots, \lambda_N \) are all the nonzero eigenvalues of a finite potent and trace class operator \( \varphi \), the algebraic and the analytic versions of the Lidskii’s Theorem imply that:

\[
\operatorname{tr}_V[\varphi] = \sum_{i=1}^{N} \lambda_i = \operatorname{tr}[\varphi],
\]

where \( \operatorname{tr}_V[\varphi] \) is the trace of \( \varphi \) as a finite potent endomorphism and \( \operatorname{tr}[\varphi] \) is the trace of \( \varphi \) as a linear operator of a Hilbert space.

Accordingly, from expression (A.2) and Proposition 3.19 we can also deduce that:

**Lemma A.1.** If \( H \) is a separable complex Hilbert space and \( \varphi: H \to H \) is a linear operator that is finite potent and of trace class, then:

\[
\det_C(V)(1 + \varphi) = \det_1(1 + \varphi).
\]

Therefore, the determinant \( \det_C(V)(1 + \varphi) \) is an extension to finite potent endomorphisms on arbitrary vector spaces of the most usual definition of infinite determinants of trace class operators on separable Hilbert spaces.

**Remark A.2.** Let \( H \) again be a separable complex Hilbert space and let \( \varphi: H \to H \) be a finite potent endomorphism. If \( \lambda_1, \ldots, \lambda_N \) are all the nonzero eigenvalues of \( \varphi \), since \( \operatorname{tr}_V(\varphi^n) = \lambda_1^n + \cdots + \lambda_N^n = p_n(\lambda_1, \ldots, \lambda_N) \), similar to trace class operators, it follows from the statement of Corollary 3.20 and from the properties of symmetric functions (see [6], Chapter 1) that:

- If \( \mu \in \mathbb{C}^\times \) and \( |\mu| \cdot \max |\lambda_i| < 1 \), then
  \[
  \det_C(H)(1 + \mu \varphi) = \exp(\operatorname{tr}_H[\ln(1 + \mu \varphi)]).
  \]

Although the linearity of Tate’s trace for arbitrary finite potent endomorphisms is still an open problem, note that

\[
\operatorname{tr}_H[\ln(1 + \mu \varphi)] = -\sum_{r=1}^{\infty} \frac{\mu^r}{r} \operatorname{tr}_H[(-\varphi)^r],
\]

because the \( \varphi \)-invariant AST-decomposition of \( V \) coincides with the \( \varphi^r \)-invariant AST-decomposition of \( V \) for all \( r \).

- If \( \alpha_0(\varphi) := 1, \alpha_1(\varphi) := \operatorname{tr}_H(\varphi) \) and

\[
\alpha_m(\varphi) = \begin{vmatrix}
  \operatorname{tr}_H(\varphi) & m-1 & 0 & \cdots & 0 & 0 \\
  \operatorname{tr}_H(\varphi^2) & \operatorname{tr}_H(\varphi) & m-2 & \cdots & 0 & 0 \\
  \operatorname{tr}_H(\varphi^3) & \operatorname{tr}_H(\varphi^2) & \operatorname{tr}_H(\varphi) & \ddots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
  \operatorname{tr}_H(\varphi^{m-1}) & \operatorname{tr}_H(\varphi^{m-2}) & \operatorname{tr}_H(\varphi^{m-3}) & \cdots & \operatorname{tr}_H(\varphi) & 1 \\
  \operatorname{tr}_H(\varphi^m) & \operatorname{tr}_H(\varphi^{m-1}) & \operatorname{tr}_H(\varphi^{m-2}) & \cdots & \operatorname{tr}_H(\varphi^2) & \operatorname{tr}_H(\varphi)
\end{vmatrix}
\]

for \( m \geq 2 \), similar to the “Plemelj-Smithies” formula, we have that:

\[
(A.3) \quad \det_C(H)(1 + \mu \varphi) = \sum_{m=0}^{\infty} \frac{\mu^m \alpha_m(\varphi)}{m!}.
\]

**Remark A.3.** It is important to emphasize that there is no relationship of inclusion between trace class and finite potent endomorphisms, as is deduced from the following example.
Let $H$ be a separable complex Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots, e_n, \ldots\}$. If we consider the endomorphisms $\phi_1, \phi_2 \in \text{End}_C H$ defined by:

$$
\phi_1(e_i) = \begin{cases} 
  e_i + e_{2i} & \text{if } i \text{ is odd} \\
  -e_{2i} - e_j & \text{if } i = 2j \text{ with } j \text{ odd} \\
  0 & \text{otherwise}
\end{cases},
$$

and $\phi_2(e_s) = e_{s+1}$ for all $s$, then we have that $\phi_1$ is a finite potent endomorphism of $H$ (in fact $\phi_1^2 = 0$), but it is not trace class, and $\phi_2$ is a trace class operator such that it is not finite potent. □

**Remark A.4.** If $\{\tilde{\lambda}_i\}_{i=1}^{N(\tilde{A})}$ are the eigenvalues of an operator $\tilde{A}$ of a separable Hilbert space $H$ (repeated again according to their algebraic multiplicities), the Carleman-Fredholm determinant is defined as:

$$
\det_2(1 + \tilde{A}) = \prod_{i=1}^{N(\tilde{A})} (1 + \tilde{\lambda}_i) \exp(-\tilde{\lambda}_i),
$$

and this product is known to converge for “Hilbert-Schmidt” operators.

Hence, this is a different method from the one described above for defining an infinite determinant of trace class operators.

**Remark A.5.** In 2001, in [3] I. Gohberg, S. Goldberg and N. Krupnik offered a generalized definition of determinants for trace-potent operators on a separable complex Hilbert space $H$. Indeed, a bounded linear operator $\bar{A}$ on $H$ is called “trace-potent” if there exists an integer $m > 1$ such that $\bar{A}^m$ is of trace class. Let $m$ be the smallest number for which $\bar{A}^m$ is of trace class; they defined a $m$-regularized determinant by the following equality:

$$
\tilde{\det}_m(1 - \mu \bar{A}) = \prod_{i=1}^{N(\bar{A})} (1 - \mu \bar{\lambda}_i) E_m(\mu \bar{\lambda}_i),
$$

where $\{\bar{\lambda}_i\}_{i=1}^{N(\bar{A})}$ are the eigenvalues of $\bar{A}$ (repeated again according to their algebraic multiplicities), $\mu \in \mathbb{C}$ and:

$$
E_m(\delta) = \exp \left( \sum_{j=1}^{m-1} \frac{j^m}{j} \delta \right).
$$

Note that if $\lambda = -1$ and $m = 2$, then $\tilde{\det}_2$ coincides with the Carleman-Fredholm determinant $\det_2$ referred to above.

Notice that each finite potent endomorphism of a Hilbert space is also trace-potent (Remark [3]), and we should note that our definition of determinant is different from that given in [3].

**Appendix B. The Segal-Wilson Pairing**

Let $H$ be a separable complex Hilbert space with a given decomposition $H = H_+ \oplus H_-$ as the direct sum of two infinite dimensional orthogonal closed subspaces. The Grassmannian $\text{Gr}(H)$ is the set of all closed subspaces $W$ of $H$ such that

- the orthogonal projection $\text{pr}: W \to H_+$ is a Fredholm operator (i.e. it has finite dimensional kernel and cokernel), and
- the orthogonal projection $\text{pr}: W \to H_-$ is a compact operator.

Let us write the operators $g \in \text{Gl}(H)$ in the block form

$$
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
with respect to the decomposition $H = H_+ \oplus H_-$. The restricted general linear group $\text{Gl}_{\text{res}}(H)$ is the closed subgroup of $\text{Gl}(H)$ consisting of operators $g$ whose off-diagonal blocks $b$ and $c$ are compact operators. The blocks $a$ and $d$ are therefore automatically Fredholm.

Any continuous non-vanishing function $f$ on $S^1$ defines an invertible multiplication operator, again written $f$, on $H$. Indeed, if $\Gamma$ denotes the group of continuous maps $S^1 \to \mathbb{C}^\times$, regarded as multiplication operators on $H$, then $\Gamma \subset \text{Gl}_{\text{res}}(H)$.

We now consider the subgroup $\Gamma_+$ of $\Gamma$ consisting of all-real analytic functions $f : S^1 \to \mathbb{C}^\times$ that extend to holomorphic functions $f : D_0 \to \mathbb{C}^\times$ in the disc $D_0 = \{ z \in \mathbb{C} : |z| \leq 1 \}$ satisfying $f(0) = 1$, and the subgroup $\Gamma_-$ of $\Gamma$ consisting of functions $f$, which extend to non-vanishing holomorphic functions in $D_\infty = \{ z \in \mathbb{C} \cup \infty : |z| \geq 1 \}$ satisfying $f(\infty) = 1$.

Let us recall from [9] that an operator of a Hilbert space has a determinant if and only if it differs from the identity by an operator of trace class.

In [8] G. Segal and G. Wilson offered the definition of a holomorphic line bundle $\text{Det}$ over $\text{Gr}$, as the bundle whose fiber over $W \in \text{Gr}(H)$ is an infinite expression $\lambda \cdot \omega_0 \wedge \omega_1 \wedge \omega_2 \wedge \ldots$ where $\{ \omega_i \}$ is what they called an “admissible basis” for $W$. If $\{ \omega_i \}$ and $\{ \omega'_i \}$ are two admissible basis of $W$, then the infinite matrix $t$ relating them is the kind that has a determinant, and it is possible to assert that:

$$\lambda \cdot \omega_0 \wedge \omega_1 \wedge \omega_2 \wedge \ldots = \lambda \cdot \det(t) \cdot \omega_0 \wedge \omega'_1 \wedge \omega'_2 \wedge \ldots$$

when $\omega_i = \sum t_{ij} \omega'_j$.

Let us now consider the subgroup $\text{Gl}_1(H)$ of $\text{Gl}_{\text{res}}(H)$ consisting of invertible operators $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Gl}_{\text{res}}(H)$ and where the blocks $b$ and $c$ are of trace class. If $\text{Gl}_1(H)^0$ is the identity component of $\text{Gl}_1(H)$, then the action of $\text{Gl}_1(H)^0$ on $\text{Det}$ does lift projectively to $\text{Det}$; that is: there exists a central extension of groups:

$$1 \to \mathbb{C}^\times \to \text{Gl}_1^\wedge \to \text{Gl}_1(H)^0 \to 1,$$

which acts on $\text{Det}$, covering the action of $\text{Gl}_1(H)^0$ on $\text{Gr}$.

If we consider the open set $\text{Gl}_1^{\text{reg}}$ of $\text{Gl}_1(H)^0$ where $a$ is invertible, there exists a section $s : \text{Gl}_1^{\text{reg}} \to \text{Gl}_1^\wedge$ of the projection $\text{Gl}_1^\wedge \to \text{Gl}_1(H)^0$ that induces a 2-cocycle $\langle \cdot, \cdot \rangle : \text{Gl}_1^{\text{reg}} \times \text{Gl}_1^{\text{reg}} \to \mathbb{C}^\times$, defined as:

\begin{equation}
\langle g_1, g_2 \rangle \mapsto \det(a_1 a_2 a_3^{-1}),
\end{equation}

where $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ and $g_3 = g_1 g_2$.

Accordingly, the elements of $\text{Gl}_1^{\text{reg}}$ act on $\text{Det}$ by means of the section $s$. However, $\text{Gl}_1^{\text{reg}}$ is not a group, and the map $s$ is not multiplicative.

Let us now consider the subgroup $\text{Gl}_1^+$ of $\text{Gl}_1^{\text{reg}}$ consisting of elements whose block decomposition has the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Thus, the restriction of $s$ to $\text{Gl}_1^+$ is an inclusion of groups $\text{Gl}_1^+ \hookrightarrow \text{Gl}_1^\wedge$ and one can regard $\text{Gl}_1^+$ as a group of automorphisms of the bundle $\text{Det}$. Similar remarks apply to the subgroup $\text{Gl}_1^-$ consisting of the elements of $\text{Gl}_1^{\text{reg}}$ whose block decomposition has the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. In particular, the subgroups $\Gamma_+$ and $\Gamma_-$ of the group of maps $S^1 \to \mathbb{C}^\times$ act on $\text{Det}$, for $\Gamma_{\pm} \subset \text{Gl}_1^\pm$.

Now, for every subgroup $G, H \subset \text{Gl}_1^{\text{reg}}$ such that the action of $G_1$ and $G_2$ commute with each other it is possible to define a map

\begin{equation}
\langle \cdot, \cdot \rangle^{\text{SW}}_{G, H} : G \times \tilde{H} \to \mathbb{C}^\times
\end{equation}

\begin{equation}
(g, \tilde{h}) \mapsto \det(a^{-1}a^{-1}a^{-1}),
\end{equation}

where $a$ is the admissible basis for $G$ and $\tilde{h}$ is the admissible basis for $H$.
where \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \) and \( \tilde{g} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \tilde{G} \). We have that the commutator has a determinant because, from the fact that \( g \) and \( \tilde{g} \) commute, it is equal to 

\[
1 - b\tilde{c}^{-1}\tilde{a}^{-1} + b\tilde{c}^{-1}a^{-1},
\]

and \( b, c, \tilde{b}, \text{ and } \tilde{c} \) are of trace class by the definition of \( \text{Gl}_{1}^{reg} \).

Hence this map is well-defined, and we shall call it “the Segal-Wilson pairing” associated with \( G_1 \) and \( G_2 \).

Thus, if \( g \in \Gamma_+ \) and \( \tilde{g} \in \Gamma_- \), with the above block decomposition, a computation shows that:

\[
(B.2) \quad (\tilde{g}, g)_{\text{SW}, \Gamma_+}^{\Gamma_-} = \det(\tilde{a}a^{-1}a^{-1}) = \exp(\text{trace}[\alpha, \tilde{\alpha}]),
\]

where \( g = \exp(f), \tilde{g} = \exp(f), \) and \( \alpha \) and \( \tilde{\alpha} \) are the \( H_+ \rightarrow H_+ \) blocks of \( f \) and \( \tilde{f} \) respectively. For details readers are referred to [8].

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