1. Introduction

We prove the unpolarized Shafarevich conjecture for K3 surfaces: the set of isomorphism classes of K3 surfaces over a fixed number field $K$ with good reduction away from a fixed and finite set of places $S$ is finite. Our proof is based on the theorems of Faltings and André, as well as the Kuga-Satake construction.

1.1. Finiteness theorems. Faltings [Fal86, Thm 6] proved the (polarized) Shafarevich conjecture for abelian varieties over number fields:

Theorem 1.1.1 (Faltings). Let $K$ be a number field, $S$ be a fixed finite set of places of $K$, and $n, d \in \mathbb{N}$. Then the set

$$\mathcal{A}_{n,d}(\mathfrak{O}_{K,S}) := \left\{ \frac{A/K}{\sim_K} \mid (A, \lambda) \text{ is a polarized abelian variety which has dimension } n, \right.$$  
$$\text{admits good reduction outside } S, \quad \text{and } \deg \lambda = d. \right\}$$

is finite.

Polarized abelian varieties of any degree exist and are parametrized by a countably infinite set of moduli spaces. This leaves open the question of finiteness across polarizations of all degrees $d$. More precisely, is the union

$$(1) \quad \text{Shaf}_{AV}(K, S, n) := \bigcup_d \mathcal{A}_{n,d}(\mathfrak{O}_{K,S})$$

finite? In [Zar85], Zarhin proved that the answer is yes:

Theorem 1.1.2 (Zarhin). The set of isomorphism classes of abelian varieties over $K$, of dimension $n$, and having good reduction away from $S$ is finite; i.e. the set (1) is finite.
The unpolarized Shafarevich conjecture for K3 surfaces

Using the Kuga-Satake map (Section 3) André [And96, Thm. 1.3.1] derived the analogue of Faltings’ theorem for polarized K3 surfaces:

**Theorem 1.1.3** (Andre). Let $K$ be a number field, $S$ a fixed finite set of its places, $d \in \mathbb{N}$. The set

$$\mathcal{F}_d(K, S) := \left\{ X/K \bigg| (X, \lambda) \text{ a K3 surface which admits good reduction outside } S \right\} \sim \mathcal{K}$$

is finite.

**Remark 1.1.4.** By *good reduction* of a K3 surface $X$ (resp. polarized K3 surface $(X, \lambda)$) at a place $\nu \in K$ we mean that there exists a K3 space $\mathcal{X}$ (resp. polarized K3 space $(\mathcal{X}, \lambda)$) over $\mathcal{O}_{K,\nu}$ (see Defs. 2.1.1 and 2.3.2) with generic fiber $X$ (resp. $(X, \lambda)$).

In this paper we prove the unpolarized Shafarevich conjecture. This is the analogue of Zarhin’s unpolarized finiteness theorem for K3 surfaces.

**Theorem 1.1.5** (Main theorem). Let $K$ be a number field and $S$ a fixed finite set of places of $K$. The set

$$\text{Shaf}_{K3}(K, S) := \bigcup_d \mathcal{F}_d(K, S)$$

of isomorphism classes of K3 surfaces defined over $K$ with good reduction away from $S$ is finite.

2. K3 surfaces

2.1. Definitions. We review the basic theory and definitions for K3 surfaces, closely following [Riz06].

**Definition 2.1.1.** [Riz06, 1.1.1, 1.1.5]

1. Let $k$ be a field. A non-singular proper surface $X/k$ is a K3 surface if $\Omega^2_{X/k} \simeq \mathcal{O}_{X/k}$ and $H^1(X, \mathcal{O}_{X/k}) = 0$.
2. Let $S$ be a scheme. A K3 scheme over $S$ is a scheme $X$, with a proper smooth morphism $\pi: X \to S$ whose geometric fibers are K3 surfaces.
3. Let $S$ be a scheme. A K3 space is an algebraic space $X$ with a proper smooth morphism $\pi: X \to S$ and an etale cover (of schemes) $S' \to S$ such that $\pi': X' = X \times_S S' \to S'$ is a K3 scheme.

To prove the main theorem, we will study the moduli spaces $\mathcal{F}_{d, \mathbb{Z}}$ (see 2.3) of polarized K3 surfaces as well as some invariants of the second cohomology of the K3 surfaces. First we review properties of $H^2$ and the Picard lattice.

**Definition 2.1.2.** An $R$-lattice is a free $R$-module with a symmetric bilinear pairing. Maps of $R$-lattices are the maps of $R$-modules compatible with the pairing. We mean $\mathbb{Z}$-lattice when $R$ is omitted.

2.2. Cohomology of K3 surfaces. Let $X/k$ be a K3 surface. When $k \subseteq \mathbb{C}$, the Betti numbers of $X(\mathbb{C})$ are computed by Noether’s formula: $(b_0, b_1, b_2, b_3, b_4) = (1, 0, 22, 0, 1)$. There is a perfect pairing given by the intersection product $\cup$ on $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22}$, which has been computed to be

$$(H^2(X, \mathbb{Z}), \cup) = - (\mathbb{E}_8^2 \oplus \mathbb{H}_2^2)$$

see [Sha96, 12.3]. By the $-$ sign, we mean negating the pairing, i.e. $-\mathbb{H}_2$ is the rank 2 hyperbolic lattice with pairing given by \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}. We will refer to $\mathbb{E}_8^2 \oplus \mathbb{H}_2^2$ as the K3 lattice and denote it by $L_{K3}$. 


When $X/k$ is a proper scheme over a field, the relative Picard functor $\text{Pic}_{X/k}$ is representable by a scheme, locally of finite type over $k$ [BLR90, 8.2, Thm 3]. When $X/k$ is a K3 surface, this scheme is separated, smooth, 0-dimensional over $k$, and the torsion subgroup $\text{Pic}_{X/k}^c$ is trivial [Riz06, 3.1.2].

**Definition 2.2.1.** Let $X/k$ be a K3 surface. We define the *Picard lattice* of $X$, denoted $\text{Pic}_X$, to be the abelian group $\text{Pic}_{X/k}(k)$. The pairing is given by the intersection form.

Note that $\text{Pic}_X$ contains the absolute Picard group $\text{Pic}(X)$ as a finite index subgroup [And96, 2.3.1]. For any abelian group $A$, we denote by $A_\ell$ the base change $A \otimes \mathbb{Z}_\ell$, e.g. we have $(\text{Pic}_X)_\ell = \text{Pic}_X \otimes \mathbb{Z}_\ell$.

For each $\ell$, the $\ell$-adic chern map defines an embedding of lattices

$$c_\ell: (\text{Pic}_X)_\ell \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(X_K, \mathbb{Z}_\ell(1))^{\text{Gal}(\overline{K}/K)}$$

If $k \subseteq \mathbb{C}$, then the analytic chern map defines an embedding of lattices

$$c: \text{Pic}_X \to H^2(X(\mathbb{C}), \mathbb{Z}(1)) \cap H^{1,1}(X(\mathbb{C}), \mathbb{C})$$

The Tate conjecture computes the image of the $\ell$-adic chern maps in the case when $k$ is a number field. Although the first proof in the literature is due to André (see [And96]), we give a citation to a paper with a more precise statement.

**Theorem 2.2.2 (The Tate Conjecture for K3 surfaces [Tat94, 5.6]).** Let $X/K$ be a K3 surface over a number field. Then the map

$$c_\ell \otimes \mathbb{Q}_\ell: (\text{Pic}_X)_\ell \otimes \mathbb{Z}_\ell \otimes \mathbb{Q}_\ell \to H^2_{\text{ét}}(X_K, \mathbb{Q}_\ell(1))^{\text{Gal}(\overline{K}/K)}$$

is surjective (and is therefore an isomorphism).

**Lemma 2.2.3.** The images of the Picard lattice $\text{Pic}_X$ in $H^2(X(\mathbb{C}), \mathbb{Z}(1))$ and $H^2_{\text{ét}}(X_K, \mathbb{Z}(1))$ under the respective chern maps are saturated.

**Proof.** The saturation of $\text{Pic}_X = \text{Pic}_{X/K}(K)$ in $H^2(X(\mathbb{C}), \mathbb{Z}(1))$ follows from the fact that $\text{Pic}_{X/K}(K)$ is saturated in $H^2_{\text{ét}}(X_K, \mathbb{Z}(1))$ and the equalities

$$\text{Pic}_{X/K}(K) = \text{Pic}_{X/K}^{\text{Gal}(\overline{K}/K)} = \text{Pic}(X_{\overline{\mathbb{Q}}})^{\text{Gal}(\overline{K}/K)} = \text{Pic}_{X_{\overline{\mathbb{Q}}}/\overline{\mathbb{Q}}}^{\text{Gal}(\overline{K}/K)}.$$

All the equalities in the above line result from the fact $\text{Pic}_{X/K}$ and $\text{Pic}_{X/K}^{\text{Gal}(\overline{K}/K)}$ are representable functors. The last term is clearly saturated in $\text{Pic}_{X_{\overline{\mathbb{Q}}}/\overline{\mathbb{Q}}}^{\text{Gal}(\overline{K}/K)}$. □

By the Artin comparison theorems, we have an isomorphism $H^2(X(\mathbb{C}), \mathbb{Z}(1)) \otimes \mathbb{Z} = H^2_{\text{ét}}(X, \mathbb{Z}(1))$ as $\mathbb{Z}$-lattices. This isomorphism is compatible with the chern maps and intersection forms in that it restricts to isomorphisms of $\mathbb{Z}_\ell$-lattices $(c(\text{Pic}_X))_\ell \simeq (c_\ell(\text{Pic}_X))_\ell$ for all $\ell$.

2.3. **Moduli of K3 surfaces.** We recall the construction and basic properties of moduli spaces of K3 surfaces following [Riz06].

**Definition 2.3.1.** [Riz06, 3.2]

1. Let $k$ be a field. A *polarization* on a K3 surface $X/k$ is a global section $\lambda \in \text{Pic}_{X/k}(k)$ which over $\overline{k}$ is the class of an ample line bundle. A polarization $\lambda$ is *primitive* if it is not a non-trivial multiple of another line bundle over $\overline{k}$.

2. Let $S$ be a scheme and let $\pi: \mathcal{X} \to S$ be a K3 space. A polarization on $\mathcal{X}$ is a global section $\lambda \in \text{Pic}_{\mathcal{X}/S}(S)$ such that for every geometric point $\overline{s}$ of $S$, the section $\lambda_{\overline{s}} \in \text{Pic}_{X_{\overline{s}}/\mathbb{Q}(\overline{s})}(\kappa(\overline{s}))$ is a polarization of $X_{\overline{s}}$. A polarization $\lambda$ on $\mathcal{X}/S$ is *primitive* if $\lambda_{\overline{s}}$ is primitive for every geometric point $\overline{s}$ of $S$. 
Definition 2.3.2. [Riz06, 4.3] Fix $d$ a natural number and let $R$ be a $\mathbb{Z}$-algebra. Let $\mathcal{F}_{d,R}$ be the moduli of degree $d$-primitively polarized $K3$ surfaces defined by:

$$\mathcal{F}_{d,R}(S) = \left\{ (\pi : \mathcal{X} \to S, \lambda) \middle| S \text{ is an } R\text{-scheme, } \pi : \mathcal{X} \to S \text{ is a K3 space, } \text{and } \lambda \text{ is a primitive degree } d \text{ polarization on } \mathcal{X} \right\}$$

Proposition 2.3.3 ([Riz06, 4.3.3]). The functor $\mathcal{F}_{d,\mathbb{Z}}$ is a separated Deligne-Mumford stack of finite type over $\mathbb{Z}$.

To represent this moduli problem with schemes, we will need level structures.

Definition 2.3.4. Let $\mathcal{L}$ be a lattice over $\mathbb{Z}$. The discriminant kernel of $\mathcal{L}$ is the maximal open compact subgroup of $SO_{\mathcal{L}}(\mathbb{Z})$ which acts trivially on $(\mathcal{L}_\mathbb{Z})^\vee/\mathcal{L}_\mathbb{Z} = \mathcal{L}^\vee/\mathcal{L}$.

For a polarized K3 surface $(X, \lambda)$ over $K$, there is an explicit description of the discriminant kernel of the primitive lattice. Recall the K3 lattice $\mathcal{L}_{K3} = E_8^3 \oplus \mathbb{H}_2^3$.

We fix a direct summand $\mathbb{H}_2 \subset \mathcal{L}_{K3}$ and a symplectic basis $e, f$ for $\mathbb{H}_2$ (i.e. $e^2 = f^2 = 0$ and $ef = fe = 1$). Let $v_d = e - df$, a primitive vector of degree $2d$. The Betti realization $(\mathcal{L}(X, Z), v_d)$ of the primitive cohomology is isomorphic to $v_d^\perp \subset \mathcal{L}_{K3}$. Let $\mathcal{L}_d$ be the $\mathbb{Z}$-lattice:

$$\mathcal{L}_d = E_8^d \oplus \mathbb{H}_2^d \oplus (2d) \cong v_d^\perp$$

Let $SO_{\mathcal{L}_{K3}}$ denote the $\mathbb{Z}$-algebraic group of isometries of $\mathcal{L}_{K3}$. The discriminant kernel of $\mathcal{L}(X, Z)$ can be identified with the subgroup of $g \in SO_{\mathcal{L}_{K3}}(\mathbb{Z})$ which fixes $v_d$.

Definition 2.3.5. Let $R$ be a $\mathbb{Z}$-algebra. Let $I_d$ be the sheaf on the étale site of a K3 space $\pi : \mathcal{X} \to S$ with sections given by

$$I_d(S) = \left\{ \alpha : \mathcal{L}_{K3} \otimes \mathbb{Z} \cong R_\mathbb{Z}(\mathcal{X}, \mathcal{L}_d(1)) \right\} \quad \alpha \text{ is an isometry such that } \alpha(v_d) = c_2(\lambda)$$

Here $c_2$ is the product of the $\ell$-adic chern maps $c_j$. Let $K_d$ be a compact open subgroup of the discriminant kernel of $\mathcal{L}_d$. Precomposition defines a $K_d$ action on $I_d(S)$. A $K_d$-level structure on a K3 space $(\pi : \mathcal{X} \to S, \lambda)$ is a section $\alpha \in H^0_{\mathbb{Z}}(S, I_d/K_d)$.

Definition 2.3.6. Let $d > 0$ and $K_d$ be a compact open subset of the discriminant kernel of $\mathcal{L}_d$. The moduli functor $\mathcal{F}_{d,K_d,R}$ of degree $d$-primitively polarized K3 surfaces with level $K_d$ structure is defined by

$$\mathcal{F}_{d,K_d,R}(S) = \left\{ (\pi : \mathcal{X} \to S, \lambda) \middle| S \text{ is an } R\text{-scheme, } \pi : \mathcal{X} \to S \text{ is a K3 space, } \text{and } \lambda \text{ is a degree } d \text{ polarization on } \mathcal{X} \text{ and } \alpha \text{ is a } K_d\text{-level structure on } \mathcal{X} \right\}$$

Proposition 2.3.7. [Riz10, 2.4.3] For a torsion free compact open $\mathbb{K}_d$ of the discriminant kernel, the stack $\mathcal{F}_{d,K_d}$ is representable by a quasi-projective scheme over $\mathbb{Q}$.

Proposition 2.3.8. [MP15, 3.11] For a torsion free compact open $\mathbb{K}_d$ of the discriminant kernel, the stack $\mathcal{F}_{d,K_d,\mathbb{Z}(p)}$ is representable by an algebraic space over $\mathbb{Z}(p)$. For $p \nmid d$, the space $\mathcal{F}_{d,K_d,\mathbb{Z}(p)}$ is smooth over $\mathbb{Z}(p)$.

3. A uniform Kuga-Satake map

We will use a Kuga-Satake construction to associate abelian varieties to K3 surfaces in a uniform way. The classical Kuga-Satake map [KS67] associates a complex abelian variety $A_X^{KS}$ to a complex polarized K3 surface $X$, such that there is an isomorphism of $\mathbb{Z}$-Hodge structures [Del72, 5.7]

$$C^+(\mathcal{L}(X, Z)(1)) \cong \text{End}_C(H^1(A_X^{KS}, \mathbb{Z}))$$

(compatible with the Hodge filtration after tensoring with $\mathbb{C}$)
where $C$ (resp. $C^+$) is the Clifford (resp. even Clifford) algebra associated to the lattice $P^2(X, \mathbb{Z})(1)$.

This construction was shown to be compatible in families over $\text{Spec}(\mathbb{Z}[\frac{1}{d}])$; the case of $\mathbb{Z}[\frac{1}{d}]$ is due to [Riz05], and that of $\mathbb{Z}[\frac{1}{2}]$ to [MP15]. In this section we use this construction to define a variation of the classical Kuga-Satake map over $\mathbb{Z}[\frac{1}{2}]$ in order to prove our main theorem.

**Remark.** In [And96] and [Del72], each $\mathcal{F}_d$ is embedded in a different moduli space of abelian varieties (the degree of the polarization of the abelian variety increases as a function of $d$). The goal of this construction is to embed $\mathcal{F}_d$ in the same moduli space (of abelian varieties) as $d$ varies.

### 3.1. Orthogonal Shimura varieties

We recall the theory of Shimura varieties for certain orthogonal groups. For more details and general theory of Shimura varieties, see [Del79], [Mil05], and [Moo98].

Let $(\mathcal{L}, \mathcal{Q})$ be a $\mathbb{Z}$-lattice of rank $n + 2$ and signature $(n+2, -2)$. Let $G_L := SO(\mathcal{L}, \mathcal{Q})$ be the algebraic group of automorphisms over $\mathbb{Z}$, and let $G_L := (G_L)_\mathbb{Z}$. Let $\Omega$ be the space of negative definite planes in $\mathcal{L} \otimes \mathbb{R}$. Then $\Omega$ is identified with the space of weight zero Hodge structures on $\mathcal{L} \otimes \mathbb{C}$ as follows (for details see [MP15, 4] or [Riz10, 2]). For $h \in \Omega$ with basis $(e_h, f_h)$, the corresponding decomposition is given by

$$(\mathcal{L} \otimes \mathbb{C})_{h,p}^\mathbb{c} = \begin{cases} 
\langle e_h + if_h \rangle & (p, q) = (-1, 1); \\
h_h^\mathbb{c} = \langle e_h, f_h \rangle^\perp & (p, q) = (0, 0); \\
\langle e_h - if_h \rangle & (p, q) = (1, -1)
\end{cases}$$

The pair $(G_L, \Omega)$ is a Shimura datum of weight 0. In our application we have $n \geq 1$ so that the group $G_L$ is split over $\mathbb{Q}$ and the pair $(G_L, \Omega)$ has reflex field $E(\mathcal{L}, \Omega) = \mathbb{Q}$. For a compact open $\mathbb{K} \subset G_L(A_f)$, let $\text{Sh}_\mathbb{K}(G_L, \Omega)$ be the canonical model associated to $(G_L, \Omega)$ and $\mathbb{K}$ over $\mathbb{Q}$. We will assume $\mathbb{K}$ is torsion free so that the canonical model is a quasi-projective scheme.

**Theorem 3.1.1 ([MP15, 4.6]).** Assume $(\mathcal{L}, \mathcal{Q})$ has cyclic discriminant of order $d$ and let $\mathbb{K}_d$ be a subgroup of the discriminant kernel. For $p > 2$, $\text{Sh}_{\mathbb{K}_d}(G_L, \Omega)$ admits an integral canonical model $\mathcal{F}_{\mathbb{K}_d}(\mathcal{L})(p)/\mathbb{Z}(p)$.

By the torsion freeness of $\mathbb{K}_d$, the functor $\text{Sh}_{\mathbb{K}_d}(G_L, \Omega)$ (resp. $\mathcal{F}_{\mathbb{K}_d}(\mathcal{L})(p)$) is a quasi-projective scheme over $\mathbb{Q}$ (resp. quasi-projective algebraic space over $\mathbb{Z}(p)$). We refer to [MP16, 4.3] and [Moo98, 3.3] for the definition of integral canonical models.

### 3.2. The Torelli Theorem

We now apply the definitions above to construct period domains for K3 surfaces and to state the Torelli theorem.

Let $(X, \lambda)$ be a degree $2d$-primitively polarized K3 surface over a field $k \subset \mathbb{C}$. Recall the notation of 2.3. We fix an isomorphism $H^2(X, \mathbb{Z}, \cup) \simeq \mathcal{L}_{K3} = \mathbb{H}_2 \oplus \mathbb{H}_2^*$ and identify the two lattices (here $\cup$ is the negative of the natural intersection product, this negation does not affect any arguments). Let $ch(\lambda) \in \mathcal{L}_{K3}$ be the chern class of the polarization under this identification. We fix the summand $\mathbb{H}_2 \subset \mathcal{L}_{K3}$, a basis $e, f \in \mathbb{H}_2$ and $e_d$ as in 2.3. Then there is an isometry $\phi \in SO(\mathcal{L}_{K3})(\mathbb{Z})$ so that $\phi(ch(\lambda)) = e - df$ [BBD85, Exp. IX, §1, Prop. 1] and (automatically) restricting to an isometry $\phi(\lambda^\perp) = (e - df)^\perp \subset \mathcal{L}_{K3}$.

Let $\mathcal{L}_d$ be the $\mathbb{Z}$-lattice defined in 2.3, which has signature $(19+, 2-)$, and denote the associated $\mathbb{Q}$-algebraic group of automorphisms of $\mathcal{L}_d$ by $G_{\mathcal{L}_d}$. Let $\Omega_d$ be the space of negative definite planes in $\mathcal{L}_d \otimes \mathbb{R}$, and let $\mathbb{K}_d \subset G_{\mathcal{L}_d}(A_f)$ be a compact open subgroup of the discriminant kernel (see 2.3.4).

We denote the Shimura variety associated to $(G_{\mathcal{L}_d}, \Omega_d)$ over $\mathbb{Q}$ with $\mathbb{K}_d$-level structure by $\text{Sh}_{\mathbb{K}_d}(G_{\mathcal{L}_d}, \Omega_d)$. Since $\mathcal{L}_d$ has cyclic discriminant $\mathbb{Z}/2d\mathbb{Z}$, we have an integral model $\mathcal{F}_{\mathbb{K}_d}(\mathcal{L}_d)(p)/\mathbb{Z}(p)$ of $\text{Sh}_{\mathbb{K}_d}(G_{\mathcal{L}_d}, \Omega_d)$ for each $p > 2$ by 3.1.1.
Proposition 3.2.1 (The Torelli theorem). Let \( p > 2 \). Let \( \mathbb{K}_d \subset G_{L_d}(\hat{\mathbb{Z}}) \) be a compact open subgroup of the discriminant kernel. Assume \( \mathbb{K}_d \) is sufficiently small so that \( \mathbb{K}_d \) is torsion free, and that \( \mathbb{K}_d^2 \) coincides with the discriminant kernel away from 2. Then there is an open immersion

\[ i_{\mathbb{K}_d}: \mathcal{F}_{d, \mathbb{K}_d, \mathcal{Z}(p)} \to \mathcal{H}_{\mathbb{K}}(\mathcal{L}_d)(p) \]

Proof. This theorem is due to Piatetski Shapiro-Shafarevich over \( \mathbb{C} \) in [PSS71], Rizov over \( \mathbb{Q} \) in [Riz10, 3.9.1], and Madapusi-Pera over \( \mathbb{Z}(p), p > 2 \) in [MP15, 5.15]. \( \square \)

3.3. Embeddings of orthogonal Shimura varieties. We will use the following unimodular \( \mathbb{Z} \)-lattice to construct the uniform Kuga-Satake construction

\[ \mathcal{L} = \mathbb{E}_8^d \oplus \mathbb{H}_2^d \oplus (1)^5 \]

Let \( \mathcal{G}_L = \text{SO}(\mathcal{L}) \) be the \( \mathbb{Z} \)-algebraic group of isometries of \( \mathcal{L} \), and let \( L = L_\mathbb{Q} \).

Let \( \mathbb{K} \subset G_L(\mathbb{A}_f) \) a torsion free compact open contained in the discriminant kernel. By 3.1.1, the Shimura variety \( \text{Sh}_G(G_L, \Omega) \) admits an integral canonical model \( \mathcal{H}_\mathbb{K}(L)(p) \) over \( \mathbb{Z}(p), p > 2 \). We will construct embeddings \( i_d: \mathcal{F}_{d, \mathbb{K}_d, \mathcal{Z}(p)} \to \mathcal{H}_\mathbb{K}(L)(p) \).

Lemma 3.3.1. For any \( d \in \mathbb{N} \) there exists a primitive embedding of lattices

\[ i_d: \mathcal{L}_d = \mathbb{E}_8^d \oplus \mathbb{H}_2^d \oplus (2d) \hookrightarrow \mathbb{E}_8^d \oplus \mathbb{H}_2^d \oplus (1)^5 = \mathcal{L} \]

Proof. It suffices to construct a primitive embedding of \( (2d) \) in \( (1)^5 \). We can express \( 2d = 1 + z^2 + w^2 + v^2 + u^2 \) by Lagrange’s theorem. Hence if \( v \) is a generator of \( (2d) \), \( v \mapsto (1, z, w, v, u) \) is a primitive metric embedding of lattices. \( \square \)

Let \( \Lambda_d = \mathcal{L}_d^+ \subset \mathcal{L} \). We define the closed subgroup scheme \( \mathcal{G}_{\mathcal{L}_d} \) of \( \mathcal{G}_\mathcal{L} \) by its points as follows. For a \( \mathbb{Z} \)-algebra \( R \),

\[ \mathcal{G}_{\mathcal{L}_d}(R) := \{ g \in \mathcal{G}_{\mathcal{L}}(R) : g(\Lambda_d)_R = \text{Id} \} \]

Note that \( (\mathcal{G}_{\mathcal{L}_d})_\mathbb{Q} \) is isomorphic to \( G_{L_d} \) from Section 2. The restriction map \( r_d: \mathcal{G}_{\mathcal{L}_d} \to \text{SO}(\mathcal{L}_d) \) is defined in the obvious way by their values on the \( R \) points.

Proposition 3.3.2 ([MP16, 6.1]). The inclusions in 3.3.1 define maps of Hodge structures \( i_d: \Omega_d \to \Omega \), so that we have an embedding of Shimura data \( i_d: (G_{L_d}, \Omega_d) \hookrightarrow (G_L, \Omega) \). Let \( \mathbb{K} \subset G_L(\mathbb{A}_f) \) be a fixed compact open. Then for any \( \mathbb{K}_d \) with \( i_d(\mathbb{K}_d) \subset \mathbb{K} \), we have a finite and unramified map \( i_{\mathbb{K}_d}: \text{Sh}_{\mathbb{K}_d}(G_{L_d}, \Omega_d) \to \text{Sh}_{\mathbb{K}_d}(G_L, \Omega) \).

The integral canonical model \( \mathcal{H}_{\mathbb{K}_d}(L_d)(p) \) satisfies the extension property by definition. See [MP16, 4.2, 4.3] for the definition of extension property.

Corollary 3.3.3. For each \( p > 2 \), the map \( i_d \) in 3.3.2 extends to a map of integral models \( \mathcal{H}_{\mathbb{K}_d}(L_d)(p) \to \mathcal{H}_{\mathbb{K}_d}(L)(p) \) over \( \mathbb{Z}(p) \).

3.4. A uniform Kuga-Satake morphism. We continue the notation from the previous section, in particular \( \mathcal{L} = \mathbb{E}_8^d \oplus \mathbb{H}_2^d \oplus (1)^5 \) and \( G_L = \text{SO}(\mathcal{L})_\mathbb{Q} \). The goal of this section is to construct a family of abelian varieties over \( \text{Sh}_{\mathbb{K}}(G_L, \Omega) \), i.e., a Kuga-Satake morphism for \( \text{Sh}_{\mathbb{K}}(G_L, \Omega) \). The inclusions \( i_d: G_{L_d} \to G_L \) will then induce Kuga-Satake morphisms for \( \text{Sh}_{\mathbb{K}_d}(G_{L_d}, \Omega_d) \) and \( \mathcal{F}_{d, \mathbb{K}_d, \mathcal{Z}(p)} \).

Let \( \mathcal{C}(\mathcal{L}) \) (resp. \( \mathcal{C}^+(\mathcal{L}) \)) be the Clifford algebra (resp. even Clifford algebra) of \( \mathcal{L} \). Let \( \mathcal{G}\text{Spin}_{\mathcal{L}} \) be the spin group associated to \( \mathcal{L} \). This is the \( \mathbb{Z} \)-algebraic group which has \( R \) points

\[ \mathcal{G}\text{Spin}_{\mathcal{L}}(R) = \{ g \in C_R^+(\mathcal{L}_R) : g \mathcal{L}_R g^{-1} = \mathcal{L}_R \} \]

where \( C_R^+(\mathcal{L}_R) \) is the even Clifford algebra of \( \mathcal{L}_R \). We refer to [Del72, 3] for details on Clifford algebras and Spin groups.
We denote $C(\mathcal{L})$ by $\mathcal{W}$ when we consider it as its own left module. The natural inclusion $i: GSpin \rightarrow C(\mathcal{L})^*$ defines a faithful representation $GSpin_{\mathcal{L}} \rightarrow G\mathcal{L}(\mathcal{W})$. Let $i: C(\mathcal{L}) \rightarrow C(\mathcal{L})$ be the canonical anti-involution of $C(\mathcal{L})$ (defined on a basis of $C(\mathcal{L})$ by $e_1 \cdots e_k \mapsto e_k \cdots e_1$ where $\{e_i\}_{i+2}$ is a basis of $\mathcal{L}$). Let $a \in C(\mathcal{L})$ be a nonzero fixed point of $i$. We have a bilinear pairing on $\mathcal{W}$ defined by

$$\phi_a: \mathcal{W} \otimes \mathcal{W} \rightarrow \mathbb{Z} \quad \phi_a(x, y) := \text{Tr}(i(x)y).$$

The pairing $\phi_a$ is nondegenerate, alternating and invariant (up to the norm character) under the action of $GSpin$ [Riz10, 5.5]). Thus the map $GSpin \rightarrow G\mathcal{L}(\mathcal{W})$ factors through $GSp := GSp(\mathcal{W}, \phi_a)$. We denote $GSp_{\mathcal{Q}}$ by $GSp$ and the space of Lagrangian subspaces of $(\mathcal{W}_K, \phi_a)$ by $\mathcal{O}'$.

**Lemma 3.4.1** ([MP16, 3.6]). In the notation above:

1. We can choose $a \in C(\mathcal{L})$ so that the embedding $GSpin_{\mathcal{L}} \rightarrow GSp(\mathcal{W}, \phi_a)$ induces an embedding of Shimura data $i^{KS}: (GSpin_{\mathcal{L}}, \mathcal{O}) \rightarrow (GSp, \mathcal{O})$.

2. Let $K$ be a compact open subset of the discriminant kernel, and let $K' \subset GSp(A_f)$ be a compact open containing $K$. Then $i^{KS}$ induces a map of canonical models over $\mathbb{Q}$

$$i^{KS}_K: Sh_K(GSpin_{\mathcal{L}}, \mathcal{O}) \rightarrow Sh_K(GSp, \mathcal{O}')$$

By explicit computation we have

**Lemma 3.4.2** ([Riz05, p.102]). This map descends to a map (which we also denote by $i^{KS}_K$)

$$i^{KS}_K: Sh_K(G_L, \mathcal{O}) \rightarrow Sh_K(GSp, \mathcal{O}')$$

**Definition 3.4.3.** Over $Sh_K(GSp, \mathcal{O}')$, we have the universal abelian variety $(\mathcal{A}, \lambda, \alpha)$. We define the Kuga-Satake abelian variety $A^{KS}_{Sh_K} \rightarrow Sh_K(G_L, \mathcal{O})$ to be the pullback of $\mathcal{A}$ under $i^{KS}_K$.

**Definition 3.4.4.** Recall the embedding $i_\ell: \mathcal{L}_d \rightarrow \mathcal{L}$ (3.3.1) and its induced map $i_d: Sh_{\mathcal{L}_d}(G_{L_d}, \mathcal{O}_d) \rightarrow Sh_{\mathcal{L}}(G_L, \mathcal{O})$ (3.3.2). Let $s_d: T \rightarrow Sh_{\mathcal{L}_d}(G_{L_d}, \mathcal{O}_d)$ be a map corresponding to the polarized K3 surface with level structure $(X_{s_d}, \lambda_{s_d}, \alpha_{s_d})$ over $T$. We define the uniform Kuga-Satake abelian variety (suppressing notation for polarization and level structure) $A^{KS}_{T}$ associated to $s_d$ to be the pullback of $A_{Sh_{\mathcal{L}_d}}$ under $s_d \circ i_d \circ i^{KS}_K$.

3.5. Cohomology. The goal of this section is to establish Galois-equivariant $\ell$-adic realizations of the classical cohomological relation $P^2(X) \leftrightarrow \text{End}(A^{KS}_{X})$. We retain the notation of the previous section.

Let $f: A^{KS}_{Sh_K} \rightarrow Sh_K(G_L, \mathcal{O})$ be the Kuga-Satake abelian variety constructed (in the third column of the diagram) above. The relative étale cohomology $R^1f_*\mathbb{Z}_{\ell}$ defines a lisse $\mathbb{Z}_{\ell}$-sheaf of rank $2^{\text{rank}(L)} = 2^{25}$ on $Sh_K(G_L, \mathcal{O})$.

**Lemma 3.5.1** ([MP15, 4.2]). Let $Sh_K(G_L, \mathcal{O})$ and $Sh_{\mathcal{L}_d}(G_{L_d}, \mathcal{O}_d)$ be the Shimura varieties over $\mathbb{Q}$ constructed in the previous section. For all $\ell$, we have lisse $\mathbb{Z}_{\ell}$-sheaves $(\mathcal{L}_{\mathcal{L}})^{\text{shf}}$ and $(\mathcal{L}_{\mathcal{L}_d, \mathcal{O}_d})^{\text{shf}}$ on $Sh_K(G_L, \mathcal{O})$ and $Sh_{\mathcal{L}_d}(G_{L_d}, \mathcal{O}_d)$ respectively.

**Definition 3.5.2.** We define $\tilde{F}_d$ to be the degree two étale cover of $F_d$ given by adding a trivialization $\det(L_d) \otimes \mathbb{Z}_2 \simeq P^2_{\mathbb{Z}_2}(X, \mathbb{Z}_2)$ to the data of $\mathcal{F}_d$. We use the same notation for the degree two étale cover for all the other K3 moduli functors.
Remark 3.5.3. The cover $\hat{F}_d \to F_d$ has a (non-canonical) section, and we may lift $S$ points of the latter to $S$ points of the former.

Lemma 3.5.4 ([MP15, 5.6.1]). Let $\hat{j}_{d, K_d, Q} : F_{d, K_d, Q} \to \text{Sh}_{K_d}(G_{L_d}, \Omega_d)$ be the Torelli map (see 3.2.1). For all $\ell$, we have an isomorphism of lisse $\mathbb{Z}_\ell$-sheaves
\[ j^*_{d, K_d, Q}((L_d, \mathbb{Z}_\ell)^{sh,f}) \cong P^2_{\ell, K}(X, \mathbb{Z}_\ell(1)). \]

Lemma 3.5.5. If $L$ is unimodular, then $(L_{\mathbb{Z}})^{sh,f}$ is a sub-$\mathbb{Z}_d$-sheaf of $\text{End}_{C(\mathbb{C})}(R^1f_*\mathbb{Z}(\mathbb{Z}_d))$. 

Proof. The statement needed here is found in [MP15, 4.2], which refers to [MP16, 3.3-3.12] for a more detailed proof. The proof in loc. cit. outlines the construction for $(L_{\mathbb{Q}})^{sh,f}$ associated to $L_{\mathbb{Q}}$. However all but one step go through for a $\mathbb{Z}$-lattice $L$. The only place where $\mathbb{Q}$-coefficients are needed is the use of Lemma 1.4 in loc. cit. 3.12. This lemma uses division by $\text{disc}(L)$ to construct a projector $\pi : \text{End}(C(\mathbb{C}))_{\mathbb{Q}} \to L_{\mathbb{Q}}$, however under the additional assumption that $L$ is unimodular, it holds without tensoring with $\mathbb{Q}$.

Lemma 3.5.6 ([MP15, 4.9], [MP16, 7.15]). Recall the embedding $i_d : L_d \to L$, and the induced map of Shimura varieties $i_d : \text{Sh}_{K_d}(G_{L_d}, \Omega_d) \to \text{Sh}(G_L, \Omega)$. 

1. We have a metric embedding $(L_d)^{sh,f} \hookrightarrow i^*_d((L)^{sh,f})$.
2. The sublocal system $((L_d)^{sh,f})^\perp \subset i^*_d((L)^{sh,f})$ is trivial.

Lemma 3.5.7 ([MP15, 5.6.(3)]). Let $s : K \to \text{Sh}_{K_d}(G_{L_d}, \Omega_d)$ be a $K$-point and let $(X_s, \lambda_s, \alpha_s)$ be the corresponding tuple of $K$3 surface, polarization, and level structure. We have a Galois equivariant isomorphism $((L_d)^{sh,f})^s \cong P^2_d(X, \mathbb{Z}_d(1))$.

Combining the above lemmas, we have:

Proposition 3.5.8. Let $s : K \to \text{Sh}_{K_d}(G_{L_d}, \Omega_d)$ be a $K$-point. Let $(X_s, \lambda_s, \alpha_s)$ be the corresponding tuple of $K$3 surface, polarization, and level structure. Then $P^2_d(X, \mathbb{Z}(1)) \cong s^*((L_d)^{sh,f})$. The $\mathbb{Z}$-lattice $P^2_d(X, \mathbb{Z}(1))$ is isometrically and $\text{Gal}(\overline{K}/K)$-equivariantly embedded in $i^*_d((L)^{sh,f})^s$. Its orthogonal complement $P^2_d(X, \mathbb{Z}(1))^\perp \subset ((L)^{sh,f})^s$ is trivial as a $\text{Gal}(\overline{K}/K)$ module.

4. Proof of theorem

In this section we prove the main theorem:

Theorem 4.1.1. Let $K$ be a fixed number field and $S$ a fixed set of places of $K$. The set of isomorphism classes of $K$3 surfaces defined over $K$ with good reduction away from $S$ is finite. We denote this set by $\text{Shaf}_{K3}(K, S)$.

First we prove that the minimal degree of a polarization on a $K$3 surface $X$ is a function of its Picard lattice $\text{Pic}_X$ as a lattice.

Proposition 4.1.2. Fix a number field $K$ and let $N \subset L_{K3}$ be a primitive sublattice of the $K$3 lattice with signature $((\text{rank}(N) - 1)+, 1-)$. Let
\[ C_N := \left\{ v \in N \mid v^2 > 0 \text{ and } \forall w \text{ s.t. } w^2 = -2, w \cdot v \neq 0 \right\} \]
Let $D_N := \inf_{v \in C_N} v^2$. Then $C_N$ is nonempty, so that $D_N < \infty$. Any $K$3 surface $X/K$ with Picard lattice $\text{Pic}_X \simeq N$ admits a polarization of degree $D_N$.

Proof. The proof is the same as [LMS14, 2.3.2] and mostly due to Ogus, the only difference is that we are working with $\text{Pic}_X = \text{Pic}_{X/K}(K)$ instead of the Picard group $\text{Pic}(X)$. A polarization $\lambda \in \text{Pic}_X$ is ample iff a multiple $\lambda^n \in \text{Pic}(X)$ is ample. Ampleness in $\text{Pic}(X)$ is checked by the Nakai-Moishezon-Kleiman (NMM) criterion [Huy14, Sec 8, Thm 1.2]. By adjunction, effective $K$-rational integral curves $C \subset X$ have $|C|^2 \geq -2$. 

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Since $X$ is minimal it suffices to check the NMK criterion with respect to $-2$ curves. Let $w \in N$ be any $-2$-class (not necessarily effective).

Reflection across $w$ is an isometry. We denote the group generated by these reflections and $-id$ by $R_N$. Let

$$V_N := N \otimes \mathbb{R} - \bigcup_{w^2 = -2} w^\perp$$

be the open subset (see [Ogu83, 1.10]) of $N \otimes \mathbb{R}$ where intersection with any integral $-2$-class is non-zero. By [Ogu83, 1.10], $V_N$ is a union of cones. The group $R_N$ acts transitively on the set of topologically connected components of $V_N$, and each component has non-empty intersection with the lattice $N$. Since one of the components of $V_N$ is the ample cone, the transitivity of the isometry group $R_N$ implies the result. □

This implies the more or less straightforward corollary:

**Corollary 4.1.3.** Fix $K, S$ as in 4.1.1, and additionally fix a finite set of lattices $\{N_i\}_{i \in \mathbb{N}}$. The subset of $K3$ surfaces $X \in \text{Shaf}_{K3}(K, S)$ which have Picard lattice $\text{Pic}_X \simeq N_i$ for some $i$ is finite.

**Proof.** The proposition 4.1.2 shows that a K3 surface $X$ with Picard lattice $\text{Pic}_X \simeq N_i$ admits a polarization whose degree is a function of the lattice $N_i$. Hence if $\text{Pic}_X \simeq N_i$, we have that $X$ admits some polarization of bounded degree. André’s theorem shows that the set of K3’s with good reduction outside $S$ and having a polarization of bounded degree is finite. □

We return to the proof of 4.1.1. We will need to make a base change to get level structures to use the Kuga-Satake morphism. We will prove the finiteness of the set

$$(4) \quad \{X' \in \text{Shaf}_{K3}(K', S') \mid X' = X \otimes K'/K \text{ for some } X \in \text{Shaf}_{K3}(K, S)\}$$

where $K'/K$ is any Galois extension and where $S'$ is the set of all places of $K'$ over $S$.

The next lemma shows that it is enough to prove the finiteness of $(4)$.

**Lemma 4.1.4.** Fix $K, S$ and let $K'/K$ be any finite Galois extension. The finiteness of $(4)$ implies Theorem 4.1.1.

**Proof.** Let $\Gamma = \text{Gal}(K'/K)$, $X/K$ be an element of $\text{Shaf}_{K3}(K, S)$, and set $X' := X_{K'}$. We have that $\Gamma$ is a finite group acting on $\text{Pic}_X = \text{Pic}_{X/K}(K')$ by isometries of the lattice. The fixed sublattice of this action is $\text{Pic}_X = \text{Pic}_{X/K}(K)$. The set of conjugacy classes of finite subgroups of the $\mathbb{Z}$-points of any linear algebraic group, e.g. $\text{SO}(\text{Pic}_{X'})$, is finite [Bor63, 5a]. Thus the set of conjugacy classes of images of $\text{Gal}(K'/K)$ in $\text{SO}(\text{Pic}_{X'})$ is finite, hence the set of possible Picard lattices $\{\text{Pic}_X \mid X \in \text{Shaf}_{K3}(K, S)\}$ is finite. Now apply 4.1.3. □

Now we specify the extension $K'/K$. Let $k_d$ be the discriminant kernel of $L_d$. The subgroup $k_d(4) \subset k_d$ defines an étale cover $\text{Sh}_{k_d(4)}(G_{L,d}, \Omega) \to \text{Sh}_{k_d}(G_{L,d}, \Omega)$, which extends to an étale cover of the integral models away from 2. Since $k_d(4)$ is torsion free, $\text{Sh}_{k_d(4)}(G_{L,d}, \Omega)$ is a quasi-projective scheme over $\mathbb{Q}$. Let $K'/\mathbb{Q}$ be the maximal extension of degree at most $\#(\text{SO}_{\mathbb{Z}, K3}(\mathbb{Z}/4\mathbb{Z}))$, and unramified away from 2. This is a finite extension by Hermite-Minkowski. Moreover the $K'$ base extension $(X', \lambda')/K'$ of a polarized K3 surface $(X, \lambda)/K$ has level $k_d(4)$ structure and is represented by a map of schemes $\alpha : \text{Spec}(K') \to \text{Sh}_{k_d(4)}(G_{L,d}, \Omega)$. We will prove the finiteness of $(4)$. Using Lemma 4.1.4, we assume from now on that $K = K'$.

**Lemma 4.1.5** ([Cas78, Ch. 9, Thm. 1.1]). The set of isomorphism classes of lattices of bounded rank and discriminant is finite.
The rest of the proof consists of bounding this discriminant using the given hypotheses of 4.1.1, that \( K \) and \( S \) are fixed. We will do this by applying a Kuga-Satake construction to Faltings’ finiteness theorem for abelian varieties.

**Proposition 4.1.6** (Faltings). The Shimura variety \( \text{Sh}_K(GSp, \Omega') \) has finitely many \( K \) points with good reduction outside of \( S \).

**Proof.** The variety \( \text{Sh}_K(GSp, \Omega') \) parametrizes tuples \((A, \lambda, \alpha)\). By the (polarized) Shafarevich conjecture (cf. [Fal86], [Mil86, 18.1]), we have that this set is finite. For any fixed \( K' \), the set of \( \mathcal{K}' \)-level structures over \( K \) is finite.

We denote the finite set of abelian varieties from the above proposition by \((A_i, \lambda_i, \alpha_i)\), \( i \leq n \).

Let \((X, \lambda, K)\) be a \( K \)-point \( s \to \mathcal{F}_d, K_d \) for any \( d \). The Torelli map 3.2.1 and the embeddings 3.3.2 and 3.4.1 define a point \( s \to \text{Sh}_K(GSp, \Omega') \) corresponding to the uniform Kuga-Satake abelian variety \((A^{uKS}_s, \lambda^{uKS}_s, \alpha^{uKS}_s)\).

**Proposition 4.1.7.** Let \( s: \text{Spec}(K) \to \mathcal{F}_{d_0}, K_{d_0} \) be the \( K \)-point corresponding to a \( K3 \) surface \((X_s, \lambda_s, \alpha_s)\) over \( K \) such that \( X_s \) has good reduction away from \( S \). Then its associated uniform Kuga-Satake abelian variety \( A^{uKS}_s \) has good reduction away from \( S \).

**Proof.** Let \( \nu \notin S \) be a place of \( K \) over \( p \) and where \((X, \lambda, \alpha)\) has good reduction, i.e. there is a local model \((X, \lambda)/\mathcal{O}_{K, \nu}\). Let \( s \to \mathcal{F}_{d_0}, K_{d_0} \) be the \( \mathcal{O}_{K, \nu} \)-point representing this model. Composition with \( \mathcal{F}_{d_0}, K_{d_0} \to \mathcal{F}(K)(p) \) defines a polarized abelian variety \((A_s, \lambda_s, \alpha_s)\) which is a model over \( \mathcal{O}_{K, \nu} \). Here \( \mathcal{F}(K)(\Omega')_{(p)} \) is the local model of \( \text{Sh}_K(GSp, \Omega') \).

**Corollary 4.1.8.** For any \( d \), let \( s \in \text{Sh}_K(GSp, \Omega_d) \) be the point corresponding to a polarized \( K3 \) surface with good reduction away from \( S \). Then \((A^{uKS}_s, \lambda^{uKS}_s, \alpha^{uKS}_s)\) is isomorphic to \((A_i, \lambda_i, \alpha_i)\) for some \( i \).

**Remark 4.1.9.** Since \(|\text{disc}(\text{Pic}_X)|_\ell = |\text{disc}(c_{\ell}(\text{Pic}_X))|_\ell\) we can compute the \( \ell \)-adic valuation of \( \text{disc}(\text{Pic}_X) \) from its image in \( \ell \)-adic cohomology. Doing so for all \( \ell \) will allow us to bound \( \text{disc}(\text{Pic}_X) \).

**Definition 4.1.10.** Let \( X/K \) be a surface over a number field \( K \). The lattice of transcendental cycles \( T(X) \) is the orthogonal complement of the image of \( c: \text{Pic}_X \to H^2(X(\mathbb{C}), \mathbb{Z}(1)) \)

\[
T(X) := \text{Pic}_X^+ \subset H^2(X(\mathbb{C}), \mathbb{Z}(1)).
\]

**Proposition 4.1.11.** Let \( K, S \) be as above. There is a finite set \( \rho_i: \text{Gal}(\overline{K}/K) \to \text{GL}(M_i) \) of \( \text{Gal}(\overline{K}/K) \) representations with coefficients in \( \mathbb{Z} \) such that for any \( X/K \) satisfying the hypotheses of 4.1.1, we have an isometry

\[
T(X) \otimes \mathbb{Z} \simeq M_i
\]

of \( \text{Gal}(\overline{K}/K) \) representations for some \( i \).

**Proof.** Let \((A_i, \lambda_i, \alpha_i)\) be the finite set of polarized abelian varieties from 4.1.6. For a fixed \( A_i \), the set of \( \phi: C(L) \to \text{End}(H^1(A_i, \mathbb{Z})) \) is finite by [Lan13, 1.3.3.7]. Since \((L_i)_{\mathbb{Z}} \subset \text{End}_{C(L)}(H^1_{et}(A_i, \mathbb{Z}))\) is determined by \( \phi \), it suffices to show that each \((L_i)_{\mathbb{Z}}\)
determines finitely many Galois representations $M_i$ in the list above. In fact, each $(\mathcal{L}_i)_{\mathbb{Z}}$ determines one Galois representation
\[ M_i = \left( ((\mathcal{L}_i)_{\mathbb{Z}})^{\text{Gal}(\mathbb{K}/\mathbb{K})} \right)^{\perp} \]
where the orthogonal complement is taken in the lattice $(\mathcal{L}_i)_{\mathbb{Z}}$.

By Lemmas 4.1.8 and 3.5.8, for any point $(X, \lambda, \alpha)$ of $\text{Sh}_{\mathbb{K}}(G_{Ld}, \Omega_d)$, we have Galois equivariant, metric embeddings
\[ i_d: P^2_d(X, \hat{\mathbb{Z}}(1)) \to (\mathcal{L}_i)_{\mathbb{Z}} \to \text{End}_{C(\mathcal{L})}(H^1_{\text{et}}(A_i, \hat{\mathbb{Z}})) \]
where $(\mathcal{L}_i)_{\mathbb{Z}}$ is the $\text{Gal}(\mathbb{K}/\mathbb{K})$-submodule of $\text{End}_{C(\mathcal{L})}(H^1_{\text{et}}(A_i, \hat{\mathbb{Z}}))$ defined by 3.5.5. The Tate conjecture for K3 surfaces over number fields shows that the transcendental lattice $T(X)_{\mathbb{Z}}$ is isomorphic to $(P^2_d(X, \hat{\mathbb{Z}}(1)))^{\text{Gal}(\mathbb{K}/\mathbb{K})}$ (the orthogonal complement taken in $P^2_d(X, \hat{\mathbb{Z}}(1))$).

We have that $P^2_d(X, \hat{\mathbb{Z}}(1)) \perp (\mathcal{L}_i)_{\mathbb{Z}}$ is trivial as a $\text{Gal}(\mathbb{K}/\mathbb{K})$ representation by 3.5.8. Hence $T(X) \otimes \hat{\mathbb{Z}} \simeq \left( ((\mathcal{L}_i)_{\mathbb{Z}})^{\text{Gal}(\mathbb{K}/\mathbb{K})} \right)^{\perp}$. \hfill \qed

**Corollary 4.1.12.** For $K,S$ as above, there is an upper bound $D$ such that for any $X$, we have $\text{disc}(T(X)) \leq D$. This bound is determined by the set $(A_i, \lambda_i, \alpha_i)$, $i \leq n$ so that $D$ depends only on $K$ and $S$.

**Proof.** By the proposition above we have $T(X) \otimes \hat{\mathbb{Z}} \simeq M_i$ as a $\hat{\mathbb{Z}}$-lattice, in particular $\text{disc}(T(X)) = \text{disc}(M_i)$. Since the set of $M_i$ is finite, the conclusion follows. \hfill \qed

**Corollary 4.1.13.** For $X$ satisfying the hypotheses of Theorem 4.1.1, the discriminant of $\text{Pic}_X$ is bounded as a function depending only on $K$ and $S$.

**Proof.** For two mutually orthogonal and saturated sublattices $N$ and $N'$, whose sum is finite index in a unimodular lattice $M$, we have
\[ \text{disc}(N) = \left| \frac{M}{N + N'} \right| = \text{disc}(N'). \]
This applies to $\text{Pic}_X$ and $T(X)$ as sublattices of $H^2(X(\mathbb{C}), \mathbb{Z}(1))$. Hence the discriminant of $\text{Pic}_X$ is equal to the discriminant of $T(X)$, which is constrained to a finite set by 4.1.12. \hfill \qed

**Corollary 4.1.14.** The set of isomorphism classes of lattices $\text{Pic}_X$ for $X$ satisfies the hypotheses of 4.1.1 is finite. The main theorem 4.1.1 now follows from 4.1.3.

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