ON THE RESIDUAL FINITENESS AND OTHER PROPERTIES
OF (RELATIVE) ONE-RELATOR GROUPS

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Abstract. A relative one-relator presentation has the form \( \mathcal{P} = \langle x, H; R \rangle \)
where \( x \) is a set, \( H \) is a group, and \( R \) is a word on \( x^{\pm 1} \cup H \). We show that if
the word on \( x^{\pm 1} \) obtained from \( R \) by deleting all the terms from \( H \) has what
we call the unique max-min property, then the group defined by \( \mathcal{P} \) is residually
finite if and only if \( H \) is residually finite (Theorem 1). We apply this to obtain
new results concerning the residual finiteness of (ordinary) one-relator groups
(Theorem 4). We also obtain results concerning the conjugacy problem for one-
relator groups (Theorem 5), and results concerning the relative asphericity of
presentations of the form \( \mathcal{P} \) (Theorem 6).

1. Introduction

The question of when one-relator groups are residually finite is still open.
In the torsion-free case there are well-known examples of groups which are not
residually finite, namely the Baumslag-Solitar/Meskin groups [4], [15]:
\[ G = \langle x; U^{-1}V^lUV^m \rangle, \]
where \( U, V \) do not generate a cyclic subgroup of the free group on \( x \), and \(|l| \neq |m|, |l|, |m| > 1\). On the other hand, there are some examples which are known to be
residually finite. For instance, it was shown in [3] that if
\[ W = UV^{-1}, \]
where \( U, V \) are positive words on an alphabet \( x \) and the exponent sum of \( x \) in
\( UV^{-1} \) is 0 for each \( x \in x \), or if
\[ W = [U, V], \]
where \( U, V \) are (not necessarily positive) words on \( x \) such that no letter \( x \in x \)
appears in both \( U \) and \( V \), then \( G = \langle x; W \rangle \) is residually finite.

In the torsion case there is the well-known open question:

Question 1 ([2], [5, Question OR1]). Is every one-relator group with torsion residually finite?
Question 1 is known to be true when \( G = \langle x; W^n \rangle \) where \( W \) is a positive word and \( n > 1 \) [9] (see also [19]). In [20], Wise obtains further related results, summed up by his “Quasi-Theorem 1.3”: If \( W \) is sufficiently positive, and \( W^n \) is sufficiently small cancellation, then \( G \) is residually finite.

A related open question is:

**Question 2** ([5, Question OR6], [11, Question 8.68]). If a torsion-free one-relator group \( G_1 = \langle x; W \rangle \) is residually finite, then is \( G_n = \langle x; W^n \rangle \) also residually finite for \( n > 1 \)?

(Of course, if Question 1 is true, then Question 2 is trivially true.)

It was shown in [1] that Question 2 holds true when \( W \) has the form (1) or (2).

Here, amongst other things, we tackle Question 2 by considering relative presentations.

A relative presentation has the form

\[ P = \langle x, H; r \rangle, \]

where \( H \) is a group and \( r \) is a set of expressions of the form

\[ R = x_i^{\varepsilon_1} h_1 x_i^{\varepsilon_2} \ldots x_i^{\varepsilon_r} h_r \ (r > 0, x_i \in x, \varepsilon_i = \pm 1, h_i \in H, 1 \leq i \leq r). \]

The word

\[ W = x_i^{\varepsilon_1} x_i^{\varepsilon_2} \ldots x_i^{\varepsilon_r} \ (r > 0, x_i \in x, \varepsilon_i = \pm 1, 1 \leq i \leq r) \]

is called the \( x \)-skeleton of \( R \). We do not require that the \( x \)-skeleton be reduced or cyclically reduced. The group \( G = G(P) \) defined by \( P \) is the quotient of \( H \ast F \) (where \( F \) is the free group on \( x \)) by the normal closure of the elements of \( H \ast F \) represented by the expressions \( R \in r \). The composition of the canonical imbedding \( H \rightarrow H \ast F \) with the quotient map \( H \ast F \rightarrow G \) is called the natural homomorphism, denoted by \( \nu : H \rightarrow G \) (or simply \( H \rightarrow G \)).

As is normal, we will often abuse notation and write \( G = \langle x, H; r \rangle \), or \( G \cong \langle x, H; r \rangle \).

When \( r \) consists of a single element \( R \), then we have the one-relator relative presentation

\[ P = \langle x, H; R \rangle. \]

Heuristically, \( G = G(P) \) should be governed by the shape of the \( x \)-skeleton of \( R \) and the algebraic properties of \( H \).

Here we introduce the unique max-min property for the shape of \( W \). (Words of the form (1) are a very special case.) For a group \( H \), denote by \( \mathcal{M}_H \) the class of one-relator relative presentations of the form (5), where \( W \) has the unique max-min property.

**Theorem 1.** If \( P \) is in \( \mathcal{M}_H \), then

(i) the natural homomorphism \( H \rightarrow G(P) \) is injective;

(ii) \( G(P) \) is residually finite if and only if \( H \) is residually finite.

We can deduce from this

**Theorem 2** (Substitution theorem). Let \( K \) be a one-relator group given by an ordinary presentation \( \langle y, z; S(y, z) \rangle \), and let \( P = \langle x, H; R \rangle \) be an \( \mathcal{M}_H \)-presentation. Then the group given by the relative presentation \( \langle x, y, H; S(y, R) \rangle \) is residually finite if and only if \( H \) and \( K \) are residually finite.
We can give the proof of this straightaway. Consider the $\mathcal{M}_{H \ast K}$-presentation $P = \langle x, H \ast K; Rz^{-1} \rangle$. By Theorem 1, $L = G(P)$ is residually finite if and only if $H \ast K$ is residually finite, which is equivalent to requiring that both $H$ and $K$ are residually finite (using results discussed in [12], p. 417). Now note that

$$L \cong \langle x, y, z, H; S(y, z), Rz^{-1} \rangle \cong \langle x, y, H; S(y, R) \rangle.$$ 

In particular, taking $K$ to be defined by $\langle z; zn \rangle$ ($n > 1$) we have

**Theorem 3.** If $G = \langle x, H; R \rangle$ is a residually finite $\mathcal{M}_H$-group, then the group $G_n = \langle x, H; R^n \rangle$ ($n > 1$) is also residually finite.

Now take $H$ to be a free group $\Phi$. Then $\mathcal{M}_\Phi$-groups are one-relator groups. Since $\Phi$ is residually finite ([12], p. 116 or p. 417), we obtain the following theorem concerning the residual finiteness of one-relator groups.

**Theorem 4.** Every $\mathcal{M}_\Phi$-group $G = \langle x, \Phi; R \rangle$ is a residually finite one-relator group. Moreover, if $K = \langle y, z; S(y, z) \rangle$ is a one-relator group, then the one-relator group $\bar{K} = \langle x, y, \Phi; S(y, R) \rangle$ is residually finite if and only if $K$ is residually finite. In particular, $G_n = \langle x, \Phi; R^n \rangle$ ($n > 1$) is residually finite.

The solution of the conjugacy problem for one-relator groups with torsion has been solved by B. B. Newman [16]. However, for the torsion-free case the problem is still open [5, Question O5].

**Theorem 5.** Every $\mathcal{M}_\Phi$-group ($\Phi$ a finitely generated free group) has a solvable conjugacy problem. Also, such groups have a solvable power conjugacy problem.

(Two elements $c, d$ of a group are said to be *power conjugate* if some power of $c$ is conjugate to some power of $d$.)

Other aspects of relative presentations (and in particular, one-relator relative presentations) have been studied intensively, particularly asphericity. Recall [6] that a relative presentation $\mathcal{P}$ is *aspherical* (more accurately, diagrammatically aspherical) if every spherical picture over $\mathcal{P}$ contains a dipole. Under a weaker condition on shape (the *unique min property*, or equivalently the *unique max property*) we can prove

**Theorem 6.** Let $\mathcal{P}$ be a relative presentation as in (5), where $W$ has the unique min property. Then $\mathcal{P}$ is aspherical.

It then follows from [6] (see Corollary 1 of Theorem 1.1, Theorem 1.3, and Theorem 1.4) that for the group $G = G(\mathcal{P})$ we have

(i) the natural homomorphism $H \to G$ is injective;
(ii) every finite subgroup of $G$ is contained in a conjugate of $H$;
(iii) for any left $\mathbb{Z}G$-module $A$ and any right $\mathbb{Z}G$-module $B$,

$$H^n(G, A) \cong H^n(H, A),$$

$$H_n(G, B) \cong H_n(H, B)$$

for all $n \geq 3$. 

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2. Max-min property

Let \( x \) be an alphabet. A \textit{weight function} on \( x \) is a function
\[ \theta : x \rightarrow \mathbb{Z} \]
such that \( \text{Im} \, \theta \) generates the additive group \( \mathbb{Z} \) (that is, \( \text{gcd}\{\theta(x) : x \in x\} = 1 \)). A \textit{strict} weight function is one for which \( \theta(x) \neq 0 \) for all \( x \in x \).

Let \( W \) be a word on \( x \) as in (4). Given a weight function \( \theta \), we then have the function
\[ \phi = \phi^\theta_W : \{0,1,2,\ldots,r\} \rightarrow \mathbb{Z}, \]
\[ \phi(j) = \sum_{i=0}^{j} \varepsilon_i \theta(x_i), \]
where \( \phi(0) = 0 \) since the empty sum is taken to be 0. We will say that the weight function is \textit{admissible} for \( W \) if \( \phi(r) = 0 \).

For visual purposes, it is useful to extend \( \phi \) to a piecewise linear function \( \phi : [0, r] \rightarrow \mathbb{R} \), so that the graph of \( \phi \) in the interval \([j−1, j]\) is the straight line segment joining the points \((j−1, \phi(j−1)), (j, \phi(j)) (0 < j \leq r)\). We will informally refer to this graph as “the graph of \( W' \)” (with respect to \( \theta \)).

A word \( W \) as in (4) will be said to have the \textit{unique max-min property} if for some admissible strict weight function \( \theta \), the graph of \( W \) has a unique maximum and a unique minimum. To be precise, we require that, for some admissible strict weight function and some \( k,l \in \{1,2,\ldots,r\} \), we have \( \phi(j) < \phi(k) \) for all \( j \in \{1,2,\ldots,r\} \) and \( \phi(j) > \phi(l) \) for all \( j \in \{1,2,\ldots,r\} \) \( \{l\} \). We also require that \( x_k \neq x_{k+1} \) and \( x_l \neq x_{l+1} \) (subscripts modulo \( r \)). This amounts to requiring that \( W \) is “reduced at the unique maximum and minimum”; that is, \( x_k^{\varepsilon_k} \neq x_{k+1}^{\varepsilon_{k+1}}, x_l^{\varepsilon_l} \neq x_{l+1}^{\varepsilon_{l+1}} \) (subscripts modulo \( r \)). For at the maximum and minimum we must have either \( x_j \neq x_{j+1} \), or \( x_j = x_{j+1} \) and \( \varepsilon_j = -\varepsilon_{j+1} \) (\( j = k,l \)). If the two letters occurring at the unique maximum are not disjoint from the two letters occurring at the unique minimum (i.e. \( \{x_k, x_{k+1}\} \cap \{x_l, x_{l+1}\} \) is not empty), then we will say that \( W \) has the \textit{strong} unique max-min property.

A word \( W \) as in (4) will be said to have the \textit{unique min property} if for some strict weight function \( \theta \), the graph of \( W \) has a unique minimum (but not necessarily a unique maximum). The \textit{unique max property} is defined similarly, but is not really of interest because replacing \( \theta \) by \( -\theta \) will convert this property to the unique min property.

We let \( M^1_H \) (respectively \( S^1_H \)) denote the subclass of \( M_H \) consisting of relative presentations of the form (5) for which \( W \) has the unique max-min property (respectively, the strong unique max-min property) with respect to the weight function
\[ 1 : x \rightarrow \mathbb{Z} \quad x \mapsto 1 \, (x \in x). \]

Lemma 1. Every \( M_H \)-group can be embedded into an \( M^1_H \)-group.

Proof. Let \( G = (x,H; R) \) with \( R \) as in (3), and suppose \( W = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_r^{\varepsilon_r} \) has the unique max-min property with respect to some strict weight function \( \theta : x \rightarrow \mathbb{Z} \). We can assume \( \theta(x) > 0 \) for all \( x \). For if \( \theta(x) < 0 \), then we can replace \( x \) by \( x^{-1} \).

Let \[ y = \{ y : y \in x, \theta(y) > 1 \}, \]
and let
\[ \hat{x} = (x - y) \cup \{ y_1, y_2, \ldots, y_{\theta(y)} : y \in y \}. \]

Let \( \hat{G} = \langle \hat{x}, H; \hat{R} \rangle \), where \( \hat{R} \) is obtained from \( R \) by replacing each occurrence of \( y^{\pm 1} \)
by \((y_1y_2 \ldots y_{\theta(y)})^{\pm 1} (y \in y) \). It is easy to see that the \( \hat{x} \)-skeleton \( \hat{W} \) of \( \hat{R} \) has the
unique max-min property with respect to \( 1 : \hat{x} \rightarrow \mathbb{Z} \). (The graph of \( \hat{W} \) is obtained
from that of \( W \) by stretching along the horizontal axis.) Moreover, \( G \) is embedded
into \( \hat{G} \), for we have the retraction \( \rho \) with section \( \mu \):
\[
\frac{\hat{G}}{\mu} \quad \rho \mu = \text{id}_G,
\]
\[
\rho : x \mapsto x (x \in x - y), \quad y_1 \mapsto y, \quad y_i \mapsto 1 (y \in y, 1 < i \leq \theta(y)), \quad h \mapsto h (h \in H),
\]
\[
\mu : x \mapsto x (x \in x - y), \quad y \mapsto y_1y_2 \ldots y_{\theta(y)} (y \in y), \quad h \mapsto h (h \in H). \quad \square
\]

**Lemma 2.** Every \( M^1_H \)-group can be embedded into an \( S^1_H \)-group.

*Proof.** Let \( G = \langle x, H; \hat{R} \rangle \), where the \( x \)-skeleton \( W \) of \( R \) has the unique max-
min property with respect to the constant function \( 1 : x \rightarrow \mathbb{Z} \). Suppose the
letters occurring at the unique maximum are \( a, b \), and those occurring at the unique
minimum are \( c, d \). We can assume that \( \{ a, b \} \cap \{ c, d \} \) is empty; otherwise, there is
nothing to prove. Let \( y = x - \{ a, b, c, d \} \), and introduce a new alphabet
\[ \hat{x} = \{ a, b, c, d, e \} \cup \{ y_1, y_2 : y \in y \}. \]

Let \( \hat{R} \) be obtained from \( R \) as follows. For each \( y \in y \), replace all occurrences of \( y^{\pm 1} \)
by \((y_1y_2)^{\pm 1} \), and replace all occurrences of \( a^{\pm 1} \) (respectively, \( b^{\pm 1} \), \( c^{\pm 1} \), \( d^{\pm 1} \))
by \((ea)^{\pm 1} \) (respectively, \((be)^{\pm 1}, (ec)^{\pm 1}, (de)^{\pm 1} \)). Let \( \hat{G} = \langle \hat{x}, H; \hat{R} \rangle \), and let \( \hat{W} \) be the
word obtained from \( \hat{R} \) by deleting all terms from \( H \). The graph of \( \hat{W} \) under the
weight function \( 1 : \hat{x} \rightarrow \mathbb{Z} \) is the graph of \( W \) magnified by a factor of 2, and \( e \) occurs
at the unique maximum and the unique minimum. Moreover, \( G \) is embedded into
\( \hat{G} \) for we have the retraction \( \rho \) with section \( \mu \):
\[
\frac{\hat{G}}{\mu} \quad \rho \mu = \text{id}_G,
\]
\[
\rho : z \mapsto z (z \in \{ a, b, c, d \}), \quad e \mapsto 1, \quad y_1 \mapsto y, \quad y_2 \mapsto 1 (y \in y), \quad h \mapsto h (h \in H),
\]
\[
\mu : a \mapsto ea, \quad b \mapsto be, \quad c \mapsto ec, \quad d \mapsto de, \quad y \mapsto y_1y_2 (y \in y), \quad h \mapsto h (h \in H). \quad \square
\]

**Remark 1.** Note that in both the above proofs, we have \( \mu \nu = \hat{\nu} \), where \( \nu : H \rightarrow G \),
\( \hat{\nu} : H \rightarrow \hat{G} \) are the natural homomorphims. Thus if \( \hat{\nu} \) is injective, then so is \( \nu \).

**Remark 2.** Note also from the proof of the above two lemmas, we get that every
\( M_H \)-group is a retract of an \( S_H \)-group.

**Remark 3.** The referee has brought to my attention the work of K. S. Brown [8],
which is concerned with whether a homomorphism \( \chi \) from a one-relator group
\( B = \langle x : W \rangle \) \(| |x| \geq 2, W \) as in (4) and cyclically reduced) onto \( Z \) has a finitely
generated kernel. Such a homomorphism is induced by a weight function \( \theta \) which
is admissible for \( W \). However, since \( \theta \) need not be strict, it is necessary to interpret
the max-min property more widely. Thus the unique maximum could be a plateau:
and \( \phi(j) < \phi(k) \) for all \( j \in \{ k, k + 1 \} \). Similarly, the unique minimum
could be a reverse plateau. Then according to Brown [8], as restated in Theorem 2.2
of [13], ker $\chi$ is finitely generated if and only if $|x| = 2$, and $W$ has the unique maximin property in the above sense with respect to the corresponding weight function. In our work we could also allow non-strict weight functions. However, for the most part this can be avoided. For example, if the unique maximum is a plateau with $x_k \neq x_{k+2}$, then we could transform it to a genuine maximum by deleting $x_{k+1}$ from $x$ and replacing $H$ by $H \ast \langle x_{k+1} \rangle$. However, if the unique maximum is a plateau with $x_k = x_{k+2}$, then some of our arguments need to be modified, which we leave as an exercise for the reader.

3. A CONSTRUCTION

By a 2-complex of groups we mean a connected graph of groups (in the sense of Serre [18]) with trivial edge groups, together with a set of closed paths which we call defining paths. (These are essentially the generalized complexes defined in §1 of [10], where more detail can be found. Note however, that in [10] a 2-cell $c(\alpha)$ consists of all cyclic permutations of $\alpha^{\pm 1}$ for each one of our defining paths $\alpha$. We specifically do not add these extra paths. This makes no significant difference.)

Let $P$ be as in (5), and let $\theta$ be an admissible weight function for $W$. There is then an induced epimorphism

$$
\psi : G \to \mathbb{Z} \quad x \mapsto \theta(x) \ (x \in x), \ h \mapsto 0 \ (h \in H).
$$

We can construct a 2-complex of groups

$$
\tilde{P} = \langle \Gamma, H_n \ (n \in \mathbb{Z}) ; \ (n, R) \ (n \in \mathbb{Z}) \rangle
$$

whose fundamental group is isomorphic to the kernel $K$ of $\psi$. The underlying graph $\Gamma$ has vertex set $\mathbb{Z}$, edges $(n, x^\varepsilon) \ (n \in \mathbb{Z}, x \in x, \varepsilon = \pm 1)$, and initial, terminal and inversion functions $i, \tau, \nu$ given by $i(n, x^\varepsilon) = n, \tau(n, x^\varepsilon) = n + \varepsilon \theta(x), (n, x^\varepsilon)^{-1} = (n + \varepsilon \theta(x), x^{-\varepsilon})$. The vertex groups are copies $H_n = \{(n, h) : h \in H\}$ of $H$ (with the obvious multiplication $(n, h)(n, h') = (n, hh')$). We extend $i, \tau, \nu$ to the elements of the vertex groups by defining $i(n, h) = n = \tau(n, h), (n, h)^{-1} = (n, h^{-1})$, where $h^{-1}$ is the inverse of $h$ in $H$. We extend $\theta$ to $x^{\pm 1} \cup H$ by defining $\theta(x^{-1}) = -\theta(x)$ ($x \in x$), $\theta(h) = 0 \ (h \in H)$. Then for any sequence $\alpha = z_1 z_2 \ldots z_q$ with $z_i \in x^{\pm 1} \cup H$ and any vertex $n \in \Gamma$, we have a path $(n, \alpha)$ in the graph of groups starting at $n$, where

$$(n, \alpha) = (n, z_1)(n + \theta(z_1), z_2)(n + \theta(z_1) + \theta(z_2), z_3) \ldots
$$

$$(n + \theta(z_1) + \theta(z_2) + \ldots + \theta(z_{q-1}), z_q).
$$

In particular we have the (closed) paths $(n, R)$.

There is an obvious action of $\mathbb{Z}$ on the above graph of groups, with $i \in \mathbb{Z}$ acting on vertices by $i \cdot n = i + n \ (n \in \mathbb{Z})$, and on the edges and vertex groups by $i \cdot (n, z) = (i + n, z) \ (n \in \mathbb{Z}, z \in x^{\pm 1} \cup H)$. This action of course extends to paths. Thus $(i, \alpha) = i(0, \alpha)$. In particular, $(i, R) = i(0, R)$, so $\mathbb{Z}$ acts on $\tilde{P}$.

If we regard $P$ as a 2-complex of groups with a single vertex $o$, edges $x^\varepsilon \ (x \in x, \varepsilon = \pm 1)$, vertex group $H$, and defining path $R$, then we have a mapping of 2-complexes of groups

$$
\rho : \tilde{P} \to P,
$$

$$
n \mapsto o, \ (n, x^\varepsilon) \mapsto x^\varepsilon, \ (n, h) \mapsto h, \ (n, R) \mapsto R
$$

$(n \in \mathbb{Z}, x \in x, \varepsilon = \pm 1, h \in H)$. This induces a homomorphism $\rho_* : \pi_1(\tilde{P}, 0) \to \pi_1(P, o) = G$.
which is injective, and \( \text{Im} \rho_s = K \). This can easily be proved by adapting the standard arguments of covering space theory for ordinary 2-complexes (see for example [17], pp. 157-159) to this relative situation.

4. PROOF OF THEOREM 1

Since residual finiteness is closed under taking subgroups, it follows from Lemmas 1 and 2 and Remark 1 at the end of §2 that it suffices to prove Theorem 1 for \( S_H \)-groups.

We will make use of the following results:

(a) A free product \( F \ast B \), where \( F \) is a free group, is residually finite if and only if \( B \) is residually finite;

(b) An infinite cyclic extension of a finitely generated group \( L \) is residually finite if and only if \( L \) is residually finite.

(The first of these follows from results on p. 417 of [12]; the second is a special case of Theorem 7, p. 29 of [14].)

We can assume \( x \) is finite. For if not let \( x' \) be the set of letters occurring in \( R \). Then \( G \) is isomorphic to \( G' \ast \Psi \) where \( G' \cong (\langle x' \rangle, H; R) \), and \( \Psi \) is the free group on \( x - x' \). So by (a) above, it is enough to work with \( G' \).

Let \( G \) be defined by an \( S_H \) presentation as in (5), with \( e \in x \) occurring at both the unique maximum and the unique minimum of the graph of \( W \) under the weight function \( \theta = 1 \). We denote the maximum and minimum values of \( \phi_W \) by \( M, m \), respectively. Note that \( m \leq 0 \leq M \) and \( m < M \).

We first deal with the trivial case when \( M - m = 1 \). Then up to cyclic permutation and inversion, \( R = eha^{-1}h' \), where \( a \in x - \{e\} \), \( h, h' \in H \). Thus \( G = \Phi \ast H \), where \( \Phi \) is the free group on \( x - \{e\} \), so the theorem holds by (a) above.

Now suppose \( M - m > 1 \). Let \( f \in x - \{e\} \).

We have the epimorphism

\[
\psi : G \to Z \quad x \mapsto 1 \quad (x \in x), \quad h \mapsto 0 \quad (h \in H).
\]

Also, we have the homomorphism

\[
\eta : Z \to G \quad 1 \mapsto f.
\]

Then \( \psi \eta = \text{id}_Z \), so \( G \) is a semidirect product \( K \rtimes \mathbb{Z} \), where \( K = \ker \psi \), and with the action of \( n \in \mathbb{Z} \) on \( K \) being induced by conjugation by \( f^n \).

The fundamental group of \( \tilde{\mathcal{P}} \) (at the vertex 0), as in §3, is isomorphic to \( K \).

We will obtain a relative presentation for \( K \) by collapsing a maximal tree.

The edges \( (n, f)^{\pm 1} \) form a maximal tree \( T \) in \( \Gamma \). Let \( R_n \) be the word on

\[
\{(i, x) : i \in \mathbb{Z}, x \in x, x \neq f\} \cup (\bigcup_{i \in \mathbb{Z}} H_i) \) obtained from \( (n, R) \) by deleting all edges from \( T \) which occur in \( (n, R) \) and replacing all terms \( (i, x^{-1}) \) by \( (i - 1, x)^{-1} \) \((i \in \mathbb{Z}, x \in x, x \neq f) \). Then

\[
\mathcal{Q} = \{(n, x) : (n \in \mathbb{Z}, x \in x, x \neq f) \ast_{n \in \mathbb{Z}} H_n ; R_n \quad (n \in \mathbb{Z})\}
\]

is a relative presentation for \( K \). Moreover, since the edges in \( T \) constitute an orbit under the action of \( \mathbb{Z} \) on our graph of groups, the action of \( \mathbb{Z} \) on \( K \) is given by the automorphism

\[
\mu : (n, x) \mapsto (n + 1, x) \quad (x \in x, x \neq f), \quad (n, h) \mapsto (n + 1, h) \quad (h \in H)
\]

\((n \in \mathbb{Z})\).
Now consider the HNN-extension $\overline{K}$ of $K$ given by the relative presentation

$$\overline{Q} = \langle (n, x) \ (n \in \mathbb{Z}, x \in \mathbf{x}, x \neq f), s; R_n \ (n \in \mathbb{Z}) \rangle$$

$$s(n, x)s^{-1} = (n + 1, x) \ (n \in \mathbb{Z}, x \in \mathbf{x}, x \neq e, f),$$

$$s(n, h)s^{-1} = (n + 1, h) \ (n \in \mathbb{Z}, h \in H)).$$

The automorphism $\mu$ of $K$ can be extended to an automorphism $\overline{\mu}$ of $\overline{K}$ by defining $\overline{\mu}(s) = s$. Then $G = \overline{K} \rtimes \mu \mathbb{Z}$ can be embedded into $\overline{G} = \overline{K} \rtimes \overline{\mu} \mathbb{Z}$.

By our assumption, up to cyclic permutation and inversion, $(0, R)$ will have the form

$$(M - 1, e)(M, h)(M - 1, a)^{-1}\gamma_0((m, b)^{-1}(m, h')(m, e))^\varepsilon\delta_0,$$

where $h, h' \in H, \varepsilon = \pm 1, a, b \in \mathbf{x} - \{e\}$, and each term $(i, z)$ occurring in the paths $\gamma_0, \delta_0$ is such that both its initial and terminal vertices lie in the range $m + 1, m + 2, \ldots, M - 1$. Then

$$R_0 = (M - 1, e)\alpha_0(m, e)^\varepsilon\beta_0,$$

where $\alpha_0, \beta_0$ do not contain any occurrence of $(i, e)^{\pm 1}$ with $i \leq m$ or $i \geq M - 1$. More generally, for $n \in \mathbb{Z}$

$$R_n = (n + M - 1, e)\alpha_n(n + m, e)^\varepsilon\beta_n,$$

where $\alpha_n, \beta_n$ do not contain any occurrence of $(i, e)^{\pm 1}$ with $i \leq n + m$ or $i \geq n + M - 1$.

Let $F_0$ be the free group on

$$(\mathbf{x} - \{e, f\}) \cup \{s, (m + 1, e), (m + 2, e), \ldots, (M - 1, e)\}.$$

Then there is a homomorphism

$$\overline{K} \to H \ast F_0$$

defined as follows:

$$s \mapsto s,$$

$$(n, x) \mapsto s^nxs^{-n} \ (x \in \mathbf{x}, x \neq e, f, n \in \mathbb{Z}),$$

$$(n, h) \mapsto s^nhs^{-n} \ (h \in H, n \in \mathbb{Z}),$$

$$(i, e) \mapsto (i, e) \ (m + 1 \leq i \leq M - 1),$$

and inductively, for $k = 0, 1, 2, \ldots$,

$$(k + M, e) \mapsto \beta_{k+1}^{-1}(k + 1 + m, e)^{-\varepsilon}\alpha_{k+1}^{-1},$$

$$(-k + m, e) \mapsto (\beta_{-k}(-k + M - 1, e)\alpha_{-k})^{-\varepsilon}.$$

This homomorphism is actually an isomorphism. The inverse is defined by

$$x \mapsto (0, x) \ (x \in \mathbf{x}, x \neq e, f),$$

$$h \mapsto (0, h) \ (h \in H),$$

$$(i, e) \mapsto (i, e) \ m + 1 \leq i \leq M - 1,$$

$$s \mapsto s.$$

Thus $\overline{G}$ is an infinite cyclic extension of the group $F_0 \ast H$.

**Remark 4.** Note that by sending $s$ to the generator $1 \in \mathbb{Z} \subseteq G = \overline{K} \rtimes \mu \mathbb{Z}$, we obtain a retraction of $\overline{G}$ onto $G$ (with section induced by the inclusion of $K$ into $\overline{K}$).

We can now complete the proof.
Clearly, the natural homomorphism from \( H \) into \( \overline{G} \) is injective (and is thus injective into \( G \)). Hence if \( H \) is not residually finite, then neither is \( G \). It remains to show that if \( H \) is residually finite, then so is \( \overline{G} \) (and thus \( G \)).

**Case 1.** If \( H \) is finitely generated, then the result holds straightaway by (a) and (b) above.

**Case 2.** Suppose that \( H \) is not finitely generated. For any homomorphism \( \theta \) from \( H \) to a group \( H_\theta \) we obtain an induced homomorphism from \( \overline{G} = (F_0 \ast H) \rtimes \mathbb{T} \mathbb{Z} \) to \( \overline{G}_\theta = (F_0 \ast H_\theta) \rtimes \mathbb{T} \mathbb{Z} \) which acts as \( \theta \) on \( H \) and acts as the identity on \( F_0 \) and \( \mathbb{T} \mathbb{Z} \). Let \( g = (w_0 h_1 \ldots h_q w_q)n \) be a non-trivial element of \( \overline{G} \), where \( q \geq 0, h_1 \ldots h_q \in H - \{1\}, w_1, \ldots, w_q-1 \in F_0 - \{1\}, w_0, w_q \in F_0, n \in \mathbb{Z} \), and if \( q \) is 0, then either \( n \neq 0 \) or \( w_0 \) is non-trivial. Since residually finite groups are fully residually finite, there is a homomorphism \( \tau \) from \( H \) onto a finite group \( H_\tau \) such that \( \tau(h_i) \neq 1 \) \((i = 1, \ldots, q)\). So the image of \( g \) in \( \overline{G}_\tau = (F_0 \ast H_\tau) \rtimes \mathbb{T} \mathbb{Z} \) is non-trivial, and then Case 1 applies.

5. Proof of Theorem 5

**Lemma 3.** Let \( C \) be a group which is a retract of a group \( B \). If \( B \) has solvable conjugacy (or power conjugacy) problem, then so does \( C \).

**Proof.** By assumption we have maps \( \xymatrix{ B \ar@<1ex>[rr]^\rho \ar@<-1ex>[rr]_\mu & & C \} \), \( \rho \mu = \text{id}_C \). Clearly if \( c, d \in C \) are conjugate (respectively, power conjugate) in \( C \), then \( \mu(c), \mu(d) \) are conjugate (respectively, power conjugate) in \( B \). Conversely if there exists \( b \in B \) such that \( b \mu(c)b^{-1} = \mu(d) \) (respectively, \( b \mu(c)^{\epsilon}b^{-1} = \mu(d)^{\epsilon} \)), then \( \rho(b)c \rho(b)^{-1} = d \) (respectively, \( \rho(b)c^{\epsilon} \rho(b)^{-1} = d^{\epsilon} \)). Thus the result follows. \( \square \)

Now it is shown in [7] that infinite cyclic extensions of finitely generated free groups have a solvable conjugacy, and power conjugacy; problem. By Remarks 2 and 4, every \( \mathcal{M}_G \)-group is a retract of such a group.

6. Proof of Theorem 6

We will assume familiarity with the terminology in §3 of [6].

As in Lemma 1, we can assume that \( \theta(x) > 0 \) for all \( x \). We can extend \( \theta \) to any word \( U = y_1^{\epsilon_1} y_2^{\epsilon_2} \ldots y_s^{\epsilon_s} \) \((s > 0, \epsilon_i \in \mathbb{X}, \epsilon_i = \pm 1, 1 \leq i \leq s)\) by \( \theta(U) = \sum_{i=0}^{s} \epsilon_i \theta(y_i) \).

Let \( \mathbb{P} \) be a based connected spherical picture (with at least one disc) over \( \mathcal{P} \), with global basepoint \( O \), and basepoint \( O_\Delta \) for each disc \( \Delta \). (Note that since \( R \) is not periodic, there will be just one basepoint for each disc.) We will also choose, for each region \( R \), a point \( O_R \) in the interior of \( R \).

We can relabel \( \mathbb{P} \) to obtain a picture \( \overline{\mathbb{P}} \) over \( \overline{\mathcal{P}} \) as follows:

(a) For each region \( R \), choose a tranverse path \( \gamma_R \) from \( O \) to \( O_R \), and let \( U_R \) (a word on \( \mathbb{X} \)) be the label on the path \( \gamma_R \). Then the potential \( q(R) \) of \( R \) is \( \theta(U_R) \).

(b) For an arc tranversely labelled \( x \in \mathbb{X} \) say, relabel it by \((q(R), x)\) where \( R \) is the region where the tranverse arrow on the arc begins.

(c) For a corner of a disc, with label \( h \in H \) say, relabel the corner by \((q, h)\), where \( q \) is the potential of the region in which the corner occurs.

For a disc \( \Delta \), let \( q_\Delta \) be the potential of the region containing \( O_\Delta \). Then in the relabelled picture, \( \Delta \) will be labelled by the path \((q_\Delta, R)\).
Let $\Theta$ be a minimal disc, that is, a disc such that $q_{\Theta} \leq q_{\Delta}$ for all discs $\Delta$. Let $m$ be the minimum value of $\phi_{\Theta}^\theta$, and let $e$ be one of the two distinct letters occurring at the unique minimum. Then in the path $(0,R)$ there is a unique edge labelled $(m,e)$. Now $\Theta$ is labelled by $(q_{\Theta},R)$ in $\tilde{P}$, and thus there is a unique edge labelled $(m+q_{\Theta},e)$ incident with $\Theta$. This arc must intersect another disc $\Theta'$, which must also be labelled by $(q_{\Theta},R)$, but with the opposite orientation. Thus we obtain a dipole in $\tilde{P}$ where $\Theta, \Theta'$ are the discs of the dipole. Reverting to $P$, this dipole in $\tilde{P}$ gives rise to a dipole in $P$.

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