A generalization of primitive sets and a conjecture of Erdős

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Abstract: A set of integers greater than 1 is primitive if no element divides another. Erdős proved in 1935 that the sum of $1/(n \log n)$ for $n$ running over a primitive set $A$ is universally bounded over all choices for $A$. In 1988 he asked if this universal bound is attained by the set of prime numbers. We answer the Erdős question in the affirmative for 2-primitive sets. Here a set is 2-primitive if no element divides the product of 2 other elements.

Key words and phrases: primitive set, primitive sequence

1 Introduction and Statement of results

A set of integers greater than 1 is called primitive if no element divides any other. Erdős [4] showed that there is a constant $K$ such that for any primitive set $A$,

$$f(A) := \sum_{n \in A} \frac{1}{n \log n} \leq K.$$ 

Further, in 1988 he asked if $f(A)$ is maximized by the primes $A = \mathbb{P}$. This is now referred to as the Erdős conjecture for primitive sets:

For $A$ primitive, we have $f(A) \leq f(\mathbb{P}) = \sum_{p \in \mathbb{P}} \frac{1}{p \log p} =: C = 1.636616 \cdots$,

where $\mathbb{P}$ is the set of prime numbers. By a simple argument, the Erdős conjecture is equivalent to the assertion that $f(A) \leq f(\mathbb{P}(A))$ for any primitive set $A$, where $\mathbb{P}(A)$ denotes the set of primes dividing some member of $A$.

Recently, the second and third authors [9] proved that
Theorem 1. For any primitive set $A$, 

$$f(A) < e^\gamma = 1.781072\ldots$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. Further, if $A$ does not contain a multiple of 8, then 

$$f(A) \leq f(\mathcal{P}(A)) + 2.37 \times 10^{-7}.$$ 

One can strengthen the notion of primitivity as follows. We say that a set $A$ of integers greater than 1 with $|A| \geq k + 1$ is $k$-primitive if no element divides the product of $k$ distinct other elements. Note that $k$-primitive implies $j$-primitive for all $k \geq j \geq 1$.

In 1938, Erdős [6] first studied the maximal cardinality of 2-primitive sets in an interval. The first author together with Győri and Sárközy [3] extended it to all $k$ and it was subsequently improved in [2] and [10]. While the full conjecture remains out of reach, we prove the Erdős conjecture for 2-primitive sets (and hence $k$-primitive for all $k \geq 2$).

Theorem 2. For any 2-primitive set $A$, 

$$f(A) \leq f(\mathcal{P}(A)).$$

An immediate consequence is the following

Corollary 1. If $A$ is a primitive set with $f(A) > f(\mathcal{P}(A))$, then there exist three elements $a, b, c \in A$ with $a \mid bc$.

On the other hand, the primes are not optimal among primitive sets with respect to logarithmic density. Indeed, Erdős, Sárközy, and Szemerédi [8] obtained the best possible upper bound

$$\sum_{n \in A, \ n \leq x} \frac{1}{n} \leq \left(\frac{1}{\sqrt{2\pi}} + o(1)\right) \frac{\log x}{\sqrt{\log \log x}}$$

for any primitive set $A$, while Erdős [7] showed that

$$\sum_{n \in A', \ n \leq x} \frac{1}{n} \geq \left(\frac{1}{\sqrt{2\pi}} + o(1)\right) \frac{\log x}{\sqrt{\log \log x}}$$

where $A'$ is the set of positive integers $a \leq x$ with $\Omega(a) = \lfloor \log \log x \rfloor$. (Here, $\Omega(a)$ is the number of prime factors of $a$, counted with multiplicity.) By contrast, the primes satisfy

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + O(1).$$

Nevertheless, one may wonder if the primes still maximize the logarithmic density among 2-primitive sets. Indeed, we prove this to be the case.
**Proposition 1.** For all \( x \geq 2 \) and any 2-primitive set \( A \),

\[
\frac{\sum_{n \in A} 1}{n \leq x} \leq \frac{1}{\sum_{p \in \mathcal{P}(A)} p}.
\]

We use this to deduce Theorem 2.

**Proof of Theorem 2 given Proposition 1.** By Proposition 1, we have \( F(x) \geq 0 \) for all \( x \geq 2 \), where

\[
F(x) := \sum_{p \in \mathcal{P}(A)} \frac{1}{p} - \sum_{n \in A} \frac{1}{n}.
\]

Then by partial summation,

\[
\sum_{p \in \mathcal{P}(A)} \frac{1}{p \log p} - \sum_{n \in A} \frac{1}{n \log n} = \frac{F(x)}{\log x} + \int_{2}^{x} \frac{F(u)}{u \log^2 u} du \geq 0.
\]

Hence taking \( x \to \infty \) gives \( f(\mathcal{P}(A)) \geq f(A) \) as desired.

In light of Proposition 1, it is natural to ask if there exists an exponent \( \lambda < 1 \) for which

\[
\sum_{n \in A} \frac{1}{n^\lambda} \leq \sum_{p \in \mathcal{P}(A)} \frac{1}{p^\lambda}
\]

holds for all 2-primitive \( A, x \geq 2 \). Banks and Martin [1] settled the question in the setting of 1-primitive sets, proving (1.1) holds for all primitive \( A \) if and only if \( \lambda \geq \tau_1 := 1.1403659 \cdots \),

where \( t = \tau_1 \) is the unique real solution to the equation

\[
\sum_{p} p^{-t} = 1 + \left( 1 - \sum_{p} p^{-2t} \right)^{1/2}.
\]

The fact that \( \tau_1 \) is markedly larger than 1 gives some indication as to why the full Erdős conjecture remains open.

In the setting of 2-primitive sets, we extend the range of valid exponents \( \lambda \).

**Theorem 3.** For any \( \lambda \geq 0.7983, \ x \geq 2, \) and any 2-primitive set \( A \),

\[
\sum_{n \in A} \frac{1}{n^\lambda} \leq \sum_{p \in \mathcal{P}(A)} \frac{1}{p^\lambda}
\]
We remark it suffices to verify Theorem 3 with \( \lambda = 0.7983 \). Indeed, suppose that \( F_{\lambda}(x) \geq 0 \) for all \( x \geq 2 \), where

\[
F_\lambda(x) = \sum_{p \in \mathcal{P}(A), p \leq x} p^{-i} - \sum_{n \in A, n \leq x} n^{-i}.
\]

Then, by partial summation, for any \( t > \lambda \),

\[
F_t(x) = x^{\lambda - t}F_\lambda(x) + (t - \lambda) \int_2^x u^{\lambda - t - 1}F_\lambda(u) \, du \geq 0.
\]

Hence we may define the critical exponent \( \tau_2 \) for 2-primitive sets, as the infimum over all \( \lambda \) for which (1.2) holds. Thus, Theorem 3 implies that \( \tau_2 \leq 0.7983 \).

We also note that Theorem 3 with \( \lambda = 1 \) gives us Proposition 1. However, Theorem 3 does not hold for every positive value of \( \lambda \). Indeed, in [6], Erdős showed that there is a 2-primitive set \( A \) in \([1, x]\) of cardinality \( \pi(x) - \pi(x^{1/3}) + cx^{2/3}/(\log x)^2 \). It consists of primes in \((x^{1/3}, x]\) and a subset of \( \{p_1p_2p_3 : p_i \text{ are primes } \leq x^{1/3}\} \) where the triples \( \{p_1, p_2, p_3\} \) form a Steiner triple system. Thus, by the prime number theorem,

\[
\sum_{p \leq x} \frac{1}{p^\lambda} \geq \sum_{p \leq x} \frac{1}{p^{\lambda}} + \frac{cx^{2/3}}{(\log x)^2} \frac{1}{x^{\lambda}} > \sum_{p \leq x} \frac{1}{p^\lambda}
\]

when \( \lambda < 0.5 \) and \( x \) is sufficiently large. Hence the above argument and Theorem 3 together imply that the critical exponent lies in the interval

\[
0.5 \leq \tau_2 \leq 0.7983. \tag{1.3}
\]

In a sequel paper, we shall address the question of critical exponents for \( k \)-primitive sets, with \( k \geq 3 \).

### 2 Combinatorial Lemmas

Before proving Theorem 3, we need lemmas in counting the maximal number of elements in a \( k \)-primitive set.

We first recall the following famous result due to Erdős and Szekeres [5], whose proof we provide for completeness.

**Lemma 1** (Erdős–Szekeres). A sequence of \( (r - 1)(s - 1) + 1 \) real numbers has either a monotonic nondecreasing subsequence of length \( r \) or a monotonic nonincreasing subsequence of length \( s \).

**Proof.** Say the sequence is \( a_1, a_2, \ldots, a_n \), where \( n = (r - 1)(s - 1) + 1 \). For each \( a_i \) consider the ordered pair \( (b_i, c_i) \), where \( b_i \) is the length of the longest nondecreasing subsequence ending at \( a_i \) and \( c_i \) is the length of the longest nonincreasing subsequence ending at \( a_i \). Then no two pairs \( (b_i, c_i) \) and \( (b_j, c_j) \) can be equal, so for at least one choice of \( i \) we have \( b_i \geq r \) or \( c_i \geq s \).

We next bound the size of a \( k \)-primitive set based on the number of prime factors used to generate its elements.
Lemma 2. For $k \geq 2$, suppose $A$ is a $k$-primitive set and $T \subset A$ with $|\mathcal{P}(T)| = n$. If $n \leq k$, then $|T| \leq n$. If $n = k + 1$, then $|T| \leq n + 1$. Further, for $k = 2, n = 4$ we have $|T| \leq 19$.

Proof. We may assume that $|T| \geq n$. Let $\mathcal{P}(T) = \{q_1, \ldots, q_n\}$ and write each $t = \prod q_i^{e_i} \in T$ as an exponent vector $v = (e_1, \ldots, e_n)$. Define the notation $v \geq 0$ if $e_i \geq 0$ for all $i$, and define $v \preceq w$ if $w - v \preceq 0$. Take $\tilde{v}_1$ with maximal entry $e_1$ among $T$. Then take $\tilde{v}_2$ with maximal $e_2$ among the remaining 19 vectors, and similarly define $\tilde{v}_3, \ldots, \tilde{v}_n$. Thus, the chosen vectors are distinct.

Case $n \leq k$: If $|T| \geq n + 1$ then $T$ has some vector $\tilde{v} \neq \tilde{v}_i$ for all $i$. But then $\tilde{v} \preceq \tilde{v}_1 + \cdots + \tilde{v}_n$. This implies that $T$, and hence $A$, is not $n$-primitive, and since $n \leq k$, it implies that $A$ is not $k$-primitive, a contradiction. Hence we cannot have $|T| \geq n + 1$ when $n \leq k$.

Case $n = k + 1$: If $|T| \geq n + 2$ then $T$ has vectors $\tilde{w}_1 \neq \tilde{w}_2$ with $\tilde{w}_j \notin \{\tilde{v}_1, \ldots, \tilde{v}_n\}$ for $j = 1, 2$. Write $\tilde{w}_j = (f_1^{(j)}, \ldots, f_n^{(j)})$. By the pigeonhole principle, we may assume

$$f_i^{(1)} \leq f_i^{(2)}$$

for at least $n/2$ values of $i$, say $i = 1, \ldots, \lceil n/2 \rceil$. Thus, we deduce

$$\tilde{w}_1 \preceq \tilde{w}_2 + \tilde{v}_{\lceil n/2 \rceil} + \cdots + \tilde{v}_n$$

contradicting $T$ as $k$-primitive, since $1 + \lceil n/2 \rceil = 1 + \lceil (k + 1)/2 \rceil \leq k$.

Now say $k = 2, n = 4$. Suppose there are 20 members in $T$ with corresponding vectors

$$\tilde{w}_i := (e_{i,1}, e_{i,2}, e_{i,3}, e_{i,4})$$

for $1 \leq i \leq 20$.

Since $A$ is 2-primitive, so is $T$. Without loss of generality, say $\tilde{w}_{18}$ has maximal first coordinate, $\tilde{w}_{19} \neq \tilde{w}_{18}$ has maximal second coordinate among the remaining 19 vectors, and $\tilde{w}_{20}$ has maximal third coordinate among the remaining 18 vectors with $\tilde{w}_{20} \neq \tilde{w}_{18}, \tilde{w}_{19}$. Arrange the remaining 17 vectors in ascending order of their first coordinate (i.e., $e_{1,1} \leq e_{1,2} \leq \cdots \leq e_{17,1}$). By Lemma 1, there is a monotonic sequence of length 5 among the $e_{i,1}$’s. Without loss of generality, say $e_{1,2}, e_{2,2}, e_{3,2}, e_{4,2}, e_{5,2}$ form such a sequence.

Case 1: $e_{1,2} \leq e_{2,2} \leq e_{3,2} \leq e_{4,2} \leq e_{5,2}$. Consider the numbers $e_{i,3}$ for $i = 1, \ldots, 5$. By Lemma 1, there is a monotonic sequence of length 3 among the $e_{i,3}$’s, without loss of generality, say it is $e_{1,3}, e_{2,3}, e_{3,3}$. If $e_{1,3} \leq e_{2,3} \leq e_{3,3}$, this forces $e_{2,4} > e_{1,4} + e_{3,4}$ for otherwise $\tilde{w}_2 \preceq \tilde{w}_1 + \tilde{w}_3$, contradicting $T$ being 2-primitive. But this implies that $\tilde{w}_1 \preceq \tilde{w}_2$ which contradicts $T$ being primitive. Hence, we must have $e_{1,3} \geq e_{2,3} \geq e_{3,3}$. Again, this forces $e_{2,4} > e_{1,4} + e_{3,4}$, which in turn implies that $\tilde{w}_1 \preceq \tilde{w}_2 + \tilde{w}_{20}$, again a contradiction.

Case 2: $e_{1,2} \geq e_{2,2} \geq e_{3,2} \geq e_{4,2} \geq e_{5,2}$. By Lemma 1, there is a monotonic sequence of length 3 among the $e_{i,3}$’s, without loss of generality, say it is $e_{1,3}, e_{2,3}, e_{3,3}$. If $e_{1,3} \leq e_{2,3} \leq e_{3,3}$, then again this forces $e_{2,4} > e_{1,4} + e_{3,4}$. But then $\tilde{w}_1 \preceq \tilde{w}_2 + \tilde{w}_{19}$. Hence, we must have $e_{1,3} \geq e_{2,3} \geq e_{3,3}$. This forces $e_{2,4} > e_{1,4} + e_{3,4}$. But then $\tilde{w}_3 \preceq \tilde{w}_2 + \tilde{w}_{18}$, again a contradiction.

Therefore, there can be at most 19 members in $T$. \hfill \square

Remark 2.1. It is not clear if the number “19” in Lemma 2 is optimal. We will not need it here, but by similar methods one can prove that if $T$ is a 2-primitive set of positive integers with $|\mathcal{P}(T)| = n \geq 3$, then $|T| \leq 9^{2n-3}$.
3 Proof of Theorem 3

Let $A \subset (1, x]$ be a 2-primitive set. Let $0.79 \leq \lambda < 1$ be a parameter to be defined later. First, we partition $A$ into primes $S$ and composites $T$. Note $S$ and $\mathcal{P}(T)$ are disjoint since $A$ is primitive. For a prime $p$, define

$$T_p := \{t \in T : p \mid t\}.$$ 

If some prime $p \in \mathcal{P}(T)$ satisfies

$$\sum_{t \in T_p} \frac{1}{t^\lambda} \leq \frac{1}{p^\lambda},$$

then we replace the members of $T_p$ with the prime $p$ (i.e., redefine $A = (T \setminus T_p) \cup \{p\}$). This would make $\sum_{t \in T_p} \frac{1}{t^\lambda}$ at least as big while keeping $A$ 2-primitive. Repeat the process with each prime $p \in \mathcal{P}(T)$ until no such prime satisfies (3.1). If $T = \emptyset$ after doing this, then $A = S$ consists of primes so Proposition 1 follows. Otherwise $T \neq \emptyset$, so we may assume

$$\sum_{t \in T_p} \frac{1}{t^\lambda} > \frac{1}{p^\lambda} \quad \text{for all} \quad p \in \mathcal{P}(T).$$ (3.2)

Consider the set

$$D := \{t/p : t \in T, p \mid t\}$$ (3.3)

We record some useful properties of $T$ and $D$.

**Lemma 3.** Let $T$ be a 2-primitive set for which (3.2) holds and let $D$ be as in (3.3).

(i) For each $p \in \mathcal{P}(T)$, $T_p$ has at least 3 elements.

(ii) The map sending ordered pairs $(t, p)$ with $t \in T$ and $p \mid t$ to $t/p \in D$ is injective.

(iii) Each $t \in T$ has at least 3 prime factors (counted with multiplicity).

(iv) $D$ is a primitive set of composite numbers.

**Proof.** (i) For $p \in \mathcal{P}(T)$, (3.2) implies that

$$\sum_{t \in T_p} \frac{1}{t^\lambda} > 1 > 2^{-0.79} + 3^{-0.79},$$

Thus (i) follows, since $t/p \in \mathbb{Z}_{>1}$ for all $t \in T_p$.

(ii) If not, then $t_1/p_1 = t_2/p_2$ for some $t_1, t_2, p_1 \mid t_1$, and $p_2 \mid t_2$. If $t_1 \neq t_2$, by (i) there exists some $p_1 k \in T_{p_1}$ other than $t_1, t_2$. But then $t_1 = (t_1/p_1)p_1 = (t_2/p_2)p_1 \mid t_2(p_1 k)$, which contradicts $T$ as 2-primitive. Hence $t_1 = t_2$, which forces $p_1 = p_2$.

(iii) If not, say $t = pq$. Since $T_p, T_q$ each have at least 3 elements, there are some $pm$ and $qn$ other than $t \in T$. But then, $t = pq \mid (pm)(qn)$ which contradicts $T$ as 2-primitive. (This argument holds whether or not $p \neq q$.)
(iv) If not, then \((t/p) \mid (t_1/p_1)\) for some \(t, t_1 \in T, p \mid t, p_1 \mid t_1\), and \(t/p \neq t_1/p_1\). If \(p_1 = p\), then \(t \mid t_1\) which contradicts Theorem 3.1. And if \(p_1 \neq p\), then there is some \(p \in T_p\) other than \(t\) and \(t_1\). This implies \(t \mid t_1 \cdot pl\), and since \(t 
eq t_1\) (otherwise \(p = p_1\)), we have a contradiction to \(T\) being 2-primitive. Thus \(D\) is primitive, and also composite by (iii).

For Theorem 3, we must show
\[
\sum_{t \in T} \frac{1}{t^\lambda} - \sum_{p \in \mathbb{P}(T)} \frac{1}{p^\lambda} < 0. \tag{3.4}
\]
Suppose \(\mathbb{P}(T)\) consists of primes \(q_1 < q_2 < \cdots < q_r\). Let \(2 = p_1 < p_2 < \cdots < p_r\) be the first \(r\) primes in \(\mathbb{P}\). We are going to modify the set \(T\) by the following process. First, if each \(q_i = p_i\), we let \(T\) stand as it is. Otherwise, let \(i\) be the smallest index such that \(q_i > p_i\). Then \(q_j = p_j\) for all \(j < i\) and we have \(p_i \mid t\) for all \(t \in T\). Then replace each \(t \in T_{q_i}\) with \(p_i/q_i \cdot t\). This keeps \(T\) as 2-primitive, and by (3.2),
\[
0 < \sum_{t \in T_{q_i}} \frac{1}{t^\lambda} - \frac{1}{q_i^\lambda} < \frac{q_i^\lambda}{p_i^\lambda} \left( \sum_{t \in T_{q_i}} \frac{1}{t^\lambda} - \frac{1}{q_i^\lambda} \right) = \sum_{t \in T_{q_i}} \frac{1}{(p_i/q_i \cdot t)^\lambda} - \frac{1}{p_i^\lambda}.
\]
So replacing each \(t \in T_{q_i}\) with \(p_i/q_i \cdot t\) preserves (3.2). We repeat this process for each \(i\) with \(q_i > p_i\) and in the end we have \(\mathbb{P}(T) = \{p_1, p_2, \ldots, p_r\}\). By showing (3.4) for this \(T\) it would follow that (3.2) fails for some \(p_i\), and this contradiction would prove the theorem.

We have reduced Theorem 3 to the following.

**Theorem 3.1.** Suppose \(\lambda \geq 0.7983\) and \(T\) is a 2-primitive set of composite numbers satisfying (3.2) with \(\mathbb{P}(T) = \mathbb{P} \cap (1, Y]\) for some \(Y\). Then
\[
\sum_{t \in T} \frac{1}{t^\lambda} - \sum_{p \leq Y} \frac{1}{p^\lambda} < 0. \tag{3.5}
\]
Our goal now is to prove Theorem 3.1. For a parameter \(0 < \theta < 1\) to be chosen later, we define \(\lambda\) as
\[
\lambda = \tau (1 - \theta), \quad \text{where} \quad \tau = 1.140366. \tag{3.6}
\]
First consider those \(t \in T\) with greatest prime factor \(P(t) \geq t^\theta\). Then \(t^{1-\theta} \geq t/P(t)\) and so \(t^{-\lambda} \leq (t/P(t))^{-\lambda/(1-\theta)} = (t/P(t))^{-\tau}\). Hence
\[
\sum_{t \in T} t^{-\lambda} \leq \sum_{P(t) \geq t^\theta} \left( \frac{t}{P(t)} \right)^{-\tau} \leq \sum_{p \leq Y} p^{-\tau} \tag{3.7}
\]
by (1.1), since \(\{t/P(t) : t \in T\} \subset D\) is primitive by part (iii) of Lemma 3.

For a positive integer \(t\), we consider the following unique factorization
\[
t = m(t)M(t)
\]
into positive integers \(m(t) \leq M(t)\) with ratio \(M(t)/m(t)\) minimal. Let
\[
M(T) = \{m(t) : t \in T\} \cup \{M(t) : t \in T\}.
\]
We need two lemmas.
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Lemma 4. For any 2-primitive set $T$, consider the graph on the integers with edges $\{t,m(t)\}$ and $\{t,M(t)\}$ for $t \in T$, where if $m(t) = M(t)$, there is just one edge containing $t$. This graph contains a matching from $T$ into $M(T)$.

Proof. Let $t \in T$. If $m(t) \notin \{m(t'),M(t')\}$ for all other $t' \in T$, then we can match $t$ with $m(t)$. So assume $m(t) \in \{m(t'),M(t')\}$ for some other $t' \in T$. Then $M(t) \notin \{m(t''),M(t'')\}$ for all $t'' \in T$ with $t'' \neq t,t'$, since otherwise $t \mid t't''$, contradicting $T$ being 2-primitive.

If $m(t) < M(t)$, then 2-primitive implies $M(t) \notin \{m(t'),M(t')\}$ so we can match $t$ with $M(t)$.

Otherwise $m(t) = M(t)$, which means $t = m(t)^2$. Then $t' \neq t$ forces $m(t') < M(t')$, so we make define $m' = t'/m(t)$ (that is $m'$ is the singleton in $\{m(t'),M(t')\} \setminus \{m(t)\}$). We would like to match $t'$ with $m'$ instead of $m(t)$, freeing up $m(t)$ to be matched with $t$. So suppose this is blocked by some $t''$ different from $t'$ (and necessarily different from $t$) with $m'' \in \{m(t''),M(t'')\}$. But then $t' \mid t''$, a violation of 2-primitivity. Thus, the matching can be completed.

Lemma 5. Suppose $0 < \theta < 1/3$ and that $T$ is 2-primitive with $P(t) < t^{\theta}$ for each $t \in T$. Let $N(z) = |T \cap [2,z]|$. Then, with $q$ running over primes in the interval $I := [z^{(1+\theta)/4},z^{(1+\theta)/2}]$, we have

$$N(z) < z^{(1+\theta)/2} - \sum_{q \in I} \left\lfloor \frac{z^{1/2}}{q} \right\rfloor.$$

Proof. By Lemma 4, it suffices to bound $|M(T \cap [2,z])|$. We first show that $M(T \cap [2,z]) \subset [1,z^{(1+\theta)/2})$. Let $t \in T$ with $t \leq z$. Say $t = q_1q_2...q_r$ where the primes $q_i$ are written in nondecreasing order. Let $d = q_1q_2...q_i$ be maximal with $d \leq t^{(1-\theta)/2}$. Then $d' = dq_{i+1}$ satisfies $t^{(1-\theta)/2} < d' < t^{(1+\theta)/2}$. Also, $d'' = t/d'$ satisfies the same double inequality. Thus,

$$t^{(1-\theta)/2} < m(t) \leq M(t) < t^{(1+\theta)/2} \leq z^{(1+\theta)/2}.$$  

We further note that the members $m$ of $M(T \cap [2,z])$ satisfy $P(m) < z^\theta$, since $m$ divides some member of $T \cap [2,z]$ and every $t$ in that set has $P(t) < z^\theta$. In particular, $m$ is not divisible by any prime $q \geq z^\theta$. Note that if $\theta < 1/3$, then $\theta < (1+\theta)/4$. So, $m$ is not divisible by any prime in the interval $I$. Since no integer below $z^{(1+\theta)/2}$ is divisible by 2 primes from $I$, the lemma follows.

Set

$$T_p = \{t \in T : P(t) = p\},$$

so that $T_p \subset T_p$. We have the following variant of Lemma 5.

Lemma 6. For any 2-primitive set $T$ and prime $p$, let $N_p(z)$ denote the number of members $t$ of $T_p$ with $t \leq z$. With $q$ running over the primes in $I_p := \left(\max\{p,z^{1/4}\},z^{1/2}\right)$, we have

$$N_p(z) \leq z^{1/2} - \sum_{q \in I_p} \left\lfloor \frac{z^{1/2}}{q} \right\rfloor.$$

Proof. Note that if $T$ is 2-primitive, so too is $T_p/p = \{t/p : t \in T_p\}$. Thus, we may apply Lemma 4 to obtain a matching from $T_p/p$ into $M(T_p/p)$. The prime factors of each element $t/p \in T_p/p$ are at most $p$, so following the proof of Lemma 5, we have $m(t/p), M(t/p) \in \{t^{1/2}/p,t^{1/2}\}$. The lemma then follows in the same way as Lemma 5.  

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**Lemma 7.** For \( x \geq 2 \) we have

\[
\sum_{x^{1/2} < q < x \text{ prime}} \left\lfloor \frac{x}{q} \right\rfloor \geq \left( \log 2 - \frac{1.25}{\log x} - \frac{2.5}{(\log x)^2} \right)x.
\]

**Proof.** First suppose that \( x \geq 286^2 \). We have the sum is at least

\[
\sum_{x^{1/2} < q < x} \frac{x}{q} - \pi(x).
\]

From [11, (3.7)], we have that \( \pi(x) < 1.25x/\log x \) and from [11, (3.17)] that

\[
\sum_{q < x} \frac{1}{q} > \log \log x + B - \frac{1}{2(\log x)^2},
\]

where \( B \) is the Mertens constant. Further, from [11, (3.18)],

\[
\sum_{q \leq x^{1/2}} \frac{1}{q} < \log \log x^{1/2} + B + \frac{1}{2(\log x^{1/2})^2} = \log \log x - \log 2 + B + \frac{2}{(\log x)^2}.
\]

This proves the lemma in the range \( x \geq 286^2 \) and direct calculation shows that it holds in the wider range \( x \geq 2 \).

We shall find it useful to use the following asymptotically weaker estimates in small cases. The proof follows by checking values of \( x \leq 3213 \) after which Lemma 7 is stronger.

**Corollary 2.** For \( x \geq 185 \), we have \( \sum_{q \leq x^{1/2}} \left\lfloor x/q \right\rfloor > 0.5x. \) For \( x \geq 67 \), we have \( \sum_{q \in (x^{1/2}, x]} \left\lfloor x/q \right\rfloor > 0.45x. \)

Let

\[
\theta = 0.3, \quad \lambda = 0.7982562, \quad \nu = 1/\theta = 10/3.
\]

For each prime \( p \), let

\[
S_p = \sum_{t \in T : p(t) = p < t^\theta} \frac{1}{t^k}.
\]

With (3.7) it will suffice to prove Theorem 3.1 if we show under its hypotheses that for each \( Y \geq 2 \),

\[
\sum_{p \leq Y} S_p \leq \sum_{p \leq Y} \left( \frac{1}{p^4} - \frac{1}{p^2} \right).
\]
3.1 Small primes, \( Y \leq 37 \)

We are going to estimate \( S_p \) for various small primes \( p \). Take \( t \in T \) with \( P(t) < t^\theta \). If \( t \leq q^v \) for a prime \( q \), then \( P(t) < (q^v)^\theta = q \). If \( q = 3 \), we see there can be at most one such \( t \); that is, \( T \) can contain at most one power of 2. The values of \( t \leq 5^v \) are supported on \( \{2,3\} \), so by Lemma 2 with \( k = n = 2 \) we see that there are at most 2 such members of \( T \). Similarly, Lemma 2 with \( k = 2, n = 3 \) shows that \( T \) has at most 4 members below \( 7^v \), and with \( k = 2, n = 4, T \) has at most 19 members below \( 11^v \). Since members \( t \) of \( T \) with \( P(t) < t^\theta \) have at least \( \lceil v \rceil = 4 \) prime factors (counted with multiplicity), we have

\[
S_2 \leq \frac{1}{2^{4\lambda}} < 0.1093463, \\
S_2 + S_3 < 0.1093463 + \frac{2 - 1}{3^{\nu k}} < 0.1631052, \\
S_2 + S_3 + S_5 < 0.1631052 + \frac{4 - 2}{5^{\nu k}} < 0.1907220, \\
S_2 + S_3 + S_5 + S_7 < 0.1907220 + \frac{19 - 4}{7^{\nu k}} < 0.2753295. \tag{3.10}
\]

Computing \( \sum_{p \leq Y} (1/p^\lambda - 1/p^v) \) directly for \( Y = 2,3,5,7 \) gives lower bounds

\[
0.121399, 0.251741, 0.368904, 0.471733,
\]

respectively. Thus we observe \( \sum_{p \leq Y} S_p < \sum_{p \leq Y} (1/p^\lambda - 1/p^v) \), so by (3.9), Theorem 3.1 holds when \( Y = 2,3,5,7 \), respectively.

Now consider \( 11 \leq p \leq 37 \). By partial summation, we have the equality

\[
S_p = \int_{p^v}^{\infty} \frac{\lambda}{z^{1+k}} N_p(z) \, dz, \tag{3.11}
\]

noting that the integral converges, since \( N_p(z) \leq z^{(1+\theta)/2} \) by Lemma 5.

We use Lemmas 6 and 7 to get the upper estimates for \( N_p(z) \):

\[
N_p(z) \leq \lceil \sqrt{z} \rceil - \sum_{\max(p, z^{1/4}) < q \leq \sqrt{z}} \left\lfloor \frac{\sqrt{z}}{q} \right\rfloor, \tag{3.12}
\]

\[
N_p(z) \leq \sqrt{z} \left( 1 - \log 2 + \frac{2.5}{\log z} + \frac{10}{(\log z)^2} \right), \text{ when } p \leq z^{1/4}. \tag{3.13}
\]

We split the integral in (3.11) at \( p^4 \). In the first range when \( z < p^4 \), we bound the contribution to (3.11) by splitting up into intervals \( [m^2, (m+1)^2] \) and using (3.12), which gives

\[
S'_p := \int_{p^v}^{p^4} \frac{\lambda}{z^{1+k}} N_p(z) \, dz \leq \sum_{m_0 < m < p^2} \int_{m^2}^{(m+1)^2} \frac{\lambda}{z^{1+k}} N_p(z) \, dz + \int_{p^v}^{(m_0+1)^2} \frac{\lambda}{z^{1+k}} N_p(z) \, dz \leq \sum_{m_0 < m < p^2} \left( \frac{1}{m^{2\lambda}} - \frac{1}{(m+1)^{2\lambda}} \right) (m - \sum_{p < q \leq m} \left\lfloor \frac{m}{q} \right\rfloor) + \left( \frac{1}{p^{v\lambda}} - \frac{1}{(m_0+1)^{2\lambda}} \right) (m_0 - \sum_{p < q \leq p^{v/2}} \left\lfloor \frac{m_0}{q} \right\rfloor) \tag{3.14}
\]

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where \( m_0 = \lfloor p^{\nu/2} \rfloor \).

For the second range when \( z \geq p^4 \), we use (3.13) when \( z \geq 3213^2 \) and for smaller values of \( z \) we use Corollary 2. That is,

\[
S_p'' := \int_{p^4}^\infty \frac{\lambda}{z^{\lambda+1/2}} N_p(z) \, dz \leq \int_{3213^2}^\infty \frac{\lambda}{z^{\lambda+1/2}} \left( 1 - \log 2 + \frac{2.5}{\log z} + \frac{10}{(\log z)^2} \right) \, dz \\
+ 0.5 \int_{\max(p^4,185^2)}^{3213^2} \frac{\lambda}{z^{\lambda+1/2}} \, dz + 0.55 \int_{p^4}^{\max(p^4,185^2)} \frac{\lambda}{z^{1/2+\lambda}} \, dz.
\]

Denote the integrals

\[
f(y) := \int_y^\infty \frac{\lambda}{z^{\lambda+1/2}} \, dz \\
g(y) := \int_y^\infty \frac{\lambda}{z^{\lambda+1/2}} \left( 1 - \log 2 + \frac{2.5}{\log z} + \frac{10}{(\log z)^2} \right) \, dz.
\]

So we obtain

\[
S_p'' \leq (1 - \log 2)f(3213^2) + g(3213^2) \\
+ 0.5[f(\max(p^4,185^2)) - f(3213^2)] + 0.55[f(p^4) - f(\max(p^4,185^2))]
\]

Using the estimates in (3.14), (3.15), we bound \( S_p = S_p' + S_p'' \) by the following.

| \( p \) | \( S_p \leq \) | \( \Sigma_{q\leq p} S_q \leq \) | \( \Sigma_{q\leq p} (q^{-\lambda} - q^{-\tau}) \geq \) |
|---|---|---|---|
| 11 | 0.13259 | 0.40792 | 0.55427 |
| 13 | 0.11241 | 0.52033 | 0.62966 |
| 17 | 0.08382 | 0.60415 | 0.69432 |
| 19 | 0.07601 | 0.68016 | 0.75484 |
| 23 | 0.06194 | 0.74210 | 0.80868 |
| 29 | 0.04757 | 0.78967 | 0.85521 |
| 31 | 0.04501 | 0.83468 | 0.89978 |
| 37 | 0.03680 | 0.87148 | 0.93950 |

Note that the first entry in the third column is found by adding \( S_{11} \) to the estimate in (3.10). Since the entries in the fourth column exceed the entries in the third column, (3.9) implies Theorem 3.1 for \( Y \leq 37 \).
3.2 Large primes, \( Y \geq 41 \)

Now assume that \( Y \geq 41 \). We have via partial summation that

\[
\sum_{t \in T} \frac{1}{t^k} = \sum_{p \leq 7} S_p + \sum_{11 \leq p \leq 23} \int_{p^y}^{29^y} \frac{\lambda}{z^{1+\lambda}} N_p(z) \, dz + \int_{29^y}^{\infty} \frac{\lambda}{z^{1+\lambda}} N(z) \, dz.
\]

(As before, the last integral converges.) From (3.10) the \( S_p \) terms contribute at most 0.27533. Using Lemmas 5, 6, and 7, and Corollary 2, we similarly obtain

\[
\sum_{\frac{t}{p(t)} < t^\theta} \frac{1}{t^\lambda} < 0.27533 + 0.08455 + 0.06576 + 0.03756 + 0.02953 + 0.01487 + 0.45614 = 0.96374,
\]

where the second to the sixth terms correspond to the five finite integrals, and the last term is our estimate for the tail integral. We also note that

\[
\sum_{p \leq Y} \left( \frac{1}{p^\lambda} - \frac{1}{p^\tau} \right) \geq \sum_{p \leq 41} \left( \frac{1}{p^\lambda} - \frac{1}{p^\tau} \right) > 0.97661.
\]

Since this estimate exceeds the prior one, this gives Theorem 3.1 with \( \lambda = 0.7982562 \).

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