Neural Connectivity in M/EEG with Hidden Hermitian Gaussian Graphical Model

Deirel Paz-Linares a,b,1, Eduardo Gonzalez-Moreira a,c,2 and Pedro A. Valdés-Sosa a,b,2

a The Clinical Hospital of Chengdu Brain Science Institute, MOE Key Lab for Neuroinformation, University of Electronic Science and Technology of China, Chengdu, China; b Cuban Neuroscience Center, La Habana, Cuba; c Centro de Investigaciones de la Informática, Universidad Central “Marta Abreu” de las Villas

1 contributed equally to this research 2 senior and correspondence author

Abstract

Statistical analysis of functional images (signals) allows the assessment of neural connectivity patterns at mesoscopic scale. Scalp fields, recorded with Magneto/Electro Encephalography M/EEG, strongly correlate to the average effect of synaptic activity, i.e. Primary Current Density (PCD) at the Gray Matter space. Nevertheless, what is observed at the sensor space with either technique is a superposition of projected fields, from the whole Gray-Matter’s PCD. That is the reason for a major pitfall of M/EEG analysis methods: distorted reconstruction of source activity and its connectivity or “Leakage”. There has been a quest for the modelling to counterattack Leakage of both activation and connectivity. It has been proven that current methods produce incorrect connectomes, also related to incorrect connectivity modelling: they disregard either System Theory and Bayesian Formalism. We introduce a system theoretic formalism of M/EEG connectivity alongside its Bayesian inference: Hidden Hermitian Gaussian Graphical Model (H-HGGM). This is a source (state) GGM hidden by the M/EEG observation equation, i.e. equivalent to a frequency domain Linear State Space Model (LSSM). It is unknown due to the Type II Likelihood approximated representation: Expected Log-Likelihood (ELL). The essential contributions here are a theory for HGGM solvers, and its implementation for the subsequent Maximization step of the H-HGGM ELL. Its efficacy is demonstrated in high resolution EEG simulations and a Steady State Visual Evoked potential. Open source packages, to reproduce the results presented in this paper and to analyze external M/EEG databases, are freely available online.

M/EEG-connectivity | leakage | state-space-model | bayesian-formalism | gaussian-graphical-model

1 Introduction

Neural Connectivity underlies brain functional and behavioral aspects. Its description at the deepest level relies on synaptic events, which are not accessible by noninvasive techniques. Nevertheless, the PCD caused by spatially and temporally organized synaptic transmission can be indirectly observed, with high temporal resolution through scalp M/EEG signals (Lopes da Silva, 2013). The PCD pathways towards the scalp potentials are governed by the laws of electromagnetic field in media uniquely: Lead Field (LF) driven, which does not involve biological mechanisms (Valdés-Hernández et al. 2009).

M/EEG connectivity is studied by statistical dependencies of the time series at the space of Gray Matter generators. Doing so requires estimating the PCD in the first place. Electrophysiology Source Imaging (ESI) methods can tackle this by inverting the Lead Field forward equation. Lead Field ill-conditioning constitutes a major cause for the uncertainty of source activations. This has motivated the development of source penalization models, indistinctively used by the three large mathematical frameworks: Tikhonov Regularization, Bayesian Analysis and Beamformer Spatial Filtering.

A common issue of ESI methods was the “Activation Leakage” (AL): ghost activations present in the reconstructed source. It has been demonstrated that AL can be sorted with sparse models: such as L1 norm based penalization or combined L1/L2 norms (Vega-Hernández et al., 2008) or Linearly Constrained Minimum Variance (Van Veen et al., 1997).

Even when solving AL would improve the connectivity estimation, it does not prevent “Connectivity Leakage” (CL), in both MEG and EEG the time series of reconstructed sources are mixed. The estimated generator’s time series is contaminated with activity of the remaining ones: due to the mixing when they are projected forth (M/EEG signal generation) and back (ESI analysis) between source and sensor space (Van de Steen et al., 2016).

The situation worsens for EEG due to inhomogeneities of head volume conductance, that have a distortive effect on the Lead Field. Also, the high conductivity of the scalp tissue blurs the electric potential at the sensor space. It causes spurious correlations of the adjacent sensor time series, which are carried down to source space by ESI. Ergo, MEG has been the choice of leading connectome projects. In the interim, the wearability of EEG systems remains the main argument to center efforts on methods that might ameliorate this effect.

Recently, sparse GGMs of source times series has been proposed as the solution to CL (Colclough et al., 2016). This pursues de-mixed (sparse) estimation of the Source Partial Correlations (SPC) from the band limited Empirical Source Covariance (ESC) in Real domain.

This idea meets GGM theory, since the SPC represents the graph edges structure (Friedman et al., 2008). This method still incurs severe theoretical errors: it does not attain to model the effect of connectivity estimation (sparse SPC) back to the source time series (ESC). In addition, Real SPC models can only be related to zero-lag interactions within time domain models. These constitutes a contravention of both System Theory and
Bayesian Formalism. A consistent model would base on Source Partial Coherence: frequency domain representation of multiple-lag interactions (Faes et al., 2012; Baccalá and Sameshima, 2001). Its inference should be done considering the probabilistic conditioning between all levels in the Bayesian hierarchy. A deeper discussion on how the current models produce incorrect connectomes was given by (Pascual-Marqui et al., 2017; Bosch-Bayard and Biscay, 2018).

A physically plausible connectivity model can only be attained by using system theory, as with the Linear State Space Models (LSSM). A reliable inference only considering the three ontological levels: Observation and Neural dynamic and Neural connectivity. Doing so would allow modeling the probabilistic relationship between neural connections, neural dynamics and M/EEG signals.

These postulates are precisely those of the Bayesian information theory (MacKay, 2003). Bayesian hierarchy defines prior and posterior conditional probability maps: data (top level, M/EEG observations) $\leftrightarrow$ parameters (neural dynamics) $\leftrightarrow$ hyperparameters (bottom level, neural connectivity). There are two options for the inference of the hidden variables from Bayesian information viewpoint: direct estimation of the connectivity, after the marginalization of neural dynamic variables (which ultimately affects the inference of the latter), or the estimation of both with mutually dependent formulas.

Unfortunately, previous ESI methods based on Bayesian Formalism do not make use of adequate connectivity models. This is the case of univariate methods (using diagonal connectivity models) like Automatic Relevance Determination (ARD) (Neal, 1998), Structured Sparse Bayesian Learning (SSBL) (Wipf et al., 2009) and multivariate methods that model connectivity through Real Covariance (Friston et al., 2008).

An early frequency domain method based on system theory was the Variable Resolution Tomographic Analysis (VARETA) (Bosch-Bayard, et al., 2001). VARETA uses Bayesian inference: mutual estimation of neural states and connectivity, with the Expectation Maximization (EM) algorithm. Here, we formalize this in the context of system theory. We solve Leakage with sparse prior of VARETA’s SPC. The mathematical principle is the H-HGGM: a model that meets system theory and GGM connectivity methodology.

Significance

Accessing connectivity noninvasively is a key issue. M/EEG signals stand out for its purely electromagnetic link to neural events. Inverting M/EEG towards its source would reveal the connectivity at the mesoscopic scale (millimetric), something not accessible by current analysis methods. This is done here with a new frequency domain Bayesian formalism (H-HGGM). H-HGGM directly models the partial correlations (connectivity) and grounds in mesoscopic neural models (system theory). Severe GGM implementability issues are solved here: complex-variable and high-dimensionality.

2 Theory

A ubiquitous system theoretic representation of M/EEG time series is the LSSM. The state (source) is represented by the vector $\mathbf{v}(t)$, on the q-size discretized Gray Matter $\mathbf{G}$ and in time domain ($t \in \mathbb{T}$). Its dynamical regime is governed by an stochastic integral equation (State Equation), with a kernel that represents the directed connectivity at multiple time lags $\mathbf{K}_u(\tau)$ ($\tau \in \mathbb{T}$). Activity source space is transferred to M/EEG signal $\mathbf{v}(t)$, at the p-size Sensor Space $\mathbf{E}$, by the Lead Field $\mathbf{L}_{uv}(\tau)$ (Observation Equation). The Lead Field at multiple time lags represents the kernel of the Maxwell equation integral solution. In practice, an instantaneous effect operator $\mathbf{L}_{uv}$ is a useful simplification, but we keep this formulation for more generality of the framework.\(^2\)

The stochastic character of the state and observation equations is driven by additive noise processes at both levels: source $\mathbf{z}(t)$ and sensor $\mathbf{z}(t)$. See SI section C for further information about this model fundamend and frequency domain analysis. See also SI sections A and B for notation and nomenclature.

\begin{align}
\mathbf{v}(t) &= \int_0^t \mathbf{L}_{uv}(\tau) \mathbf{v}(t - \tau) d\tau + \mathbf{z}(t) \tag{1} \\
\mathbf{z}(t) &= \int_0^t \mathbf{K}_u(\tau) \mathbf{z}(t - \tau) d\tau + \mathbf{z}(t) \tag{2}
\end{align}

Fourier transforming (1) and (2) allows for a more compact representation in frequency domain ($\omega \in \mathbb{F}$), when $\mathbf{v}(t)$, $\mathbf{v}(t)$ and $\mathbf{z}(t)$ are square-integrable. The stochastic properties of the transformed “realizations” (of observations and states) are driven by a Hermitian Gaussian model ($\mathcal{N}^H$), it is direct if one assumes noise Gaussianity ($\mathcal{N}^H$): $\mathbf{z}(t) \sim \mathcal{N}^H_0 (\mathbf{z}(t)|0, \Sigma_{\mathbf{z}_z(t)})$ and $\mathbf{z}(t) \sim \mathcal{N}^H_0 (\mathbf{z}(t)|0, \Sigma_{\mathbf{z}_z(t)})$. This is strongly motivated by central limit theory (Rosenblatt, 1956). The model is compactly expressed by the Data Likelihood and Parameters Prior: Hierarchical model of both ontological levels.

\begin{align}
\mathbf{v}(\mathbf{v}) | \mathbf{v}(t), \Sigma_{\mathbf{z}_z(t)}(\tau) &\sim \mathcal{N}^H_0 (\mathbf{v}(\mathbf{v})| \mathbf{L}_{uv}(\mathbf{v}) \mathbf{v}(t), \Sigma_{\mathbf{z}_z(t)}) \tag{3} \\
\mathbf{z}(t) | \Sigma_{\mathbf{z}_z(t)}(\tau) &\sim \mathcal{N}^H_0 (\mathbf{z}(t)|0, \Sigma_{\mathbf{z}_z(t)}) \tag{4}
\end{align}

The signal spectral properties are stored in the Hermitian Covariance matrices of states/noise ($\Sigma_{\mathbf{z}_z(t)}(\tau)$). Due to equivalence between time and frequency domain system identification, its knowledge would allow retrieving the states

\(^2\)Observable quantities are denoted by Latin scripts and unobserved by Greek scripts. Check Appendices for mathematical notation and nomenclature.
and observations in time domain (Schoukens et al, 2004). The Hermitian matrix $\Sigma_u(v)$ encodes the Directed Transfer Function (DTF) $K_u(v)$ or connectivity (Kaminski and Blinowska, 1991). Its inverse: the SPC $\Theta_u(v) = \Sigma_u^{-1}(v)$, is regarded as an “undirected connectivity” measure. This is a consequence of the Spectral Factorization Theorem (Faes and Nollo, 2011).

$$\Theta_u(v) = \left( \mathbf{I}_q - K_u^*(v) \right) \Sigma_{\xi\xi}^{-1} \left( \mathbf{I}_q - K_u(v) \right) \quad (5)$$

Assuming in formula (5) an univariate biological noise model $\Sigma_{\xi\xi} = \text{diag}(\sigma_\xi^2)$, the following “necessary condition” can be verified: in relation to node’s directed and undirected connectivity.

“For the existence of undirected connectivity $i \leftrightarrow j$ ($\{\Theta_u(v)\}_{ij} \neq 0$) it must hold that either: There exists one of the directed connectivities $i \rightarrow j$ ($\{K_u(v)\}_{ij} \neq 0$) or $j \rightarrow i$ ($\{K_u(v)\}_{ji} \neq 0$). Or there exists directed connectivity from a third node $i \rightarrow k$ ($\{K_u(v)\}_{ik} \neq 0$) or $j \rightarrow k$ ($\{K_u(v)\}_{jk} \neq 0$).”

GGM theory in Real domain uses this argument, but only its extrapolation to complex variable (HGGM) is totally consistent to system theory. Determining the SPC $\Theta_u(v)$ would allow estimating the directionality encoded by the DTF $K_u(v)$, due to the uniqueness of the spectral factors in (5).

Here we will be limited to estimate the undirected connectivity or SPC $\Theta_u(v)$. This can be done by the analysis of the HGGM (prior) hidden by the Likelihood in formula (4): a consequence of the Observation Equation (1). It can be unhidden after computing the Expected Log-Likelihood (ELL): Local approximation to the Type II Log-Likelihood (T2L) Logarithm, inside the EM loop. This is done by integrating the model (3-4) over the parameters $\xi(v)$.

For realizations of the observations $V(v) = \{v_m(v)\}_{m \in \mathbb{M}}$ and states $I(v) = \{i_m(v)\}_{m \in \mathbb{M}}$ in the Fourier’s coefficient sample space $\mathbb{M}$ (size $m$), the derived Gibbs form of the DLL unfolds into two marginal Wishart models ($W^C$). It is expressed through two auxiliary quantities computed at the $k$-th Expectation step, for fixed values of the hyperparameters. These are specific of each ontological level: Effective Residual Empirical Covariance (ERE) $\Psi_u^{(k)}(v)$ and the Effective Source Empirical Covariance (ESC) $\Psi_u^{(k)}(v)$. This provides for a full interpretation of the EM algorithm in terms of GGM theory (Liu and Rubin, 1994): to unhide the H-HGGM. See Lemma 1 in Materials and Methods (SI section D and E).

$$V(v)|\Sigma_{\xi\xi} \sim W^C_P(\Psi_u^{(k)}(v)|m^{-1}\Theta_{\xi\xi}^{-1}, m) \quad (6)$$

$$V(v)|\Sigma_u(v) \sim W^C_P(\Psi_u^{(k)}(v)|m^{-1}\Theta_u^{-1}(v), m) \quad (7)$$

GGM solvers are not fully developed yet for complex variable. An important part here is also to develop a new procedure for HGGM estimation. At the neural level (connectivity) this will be done by leveraging a Complex Variable Local Quadratic Approximation (CV-LQA) of sparse Gibbs priors, see Lemma 2 and 3 in Materials and Methods (SI section F and G).

This is corollary of Andrews and Mallows Lemma (AML) that provides explicit formulas (Andrews and Mallows, 1974), able to tackle HGGM estimation in high dimensionality, see Lemma 4-5 in Materials and Methods (SI section H and I). The Gibbs prior distribution has exponential form with scale (regularization) parameter $\alpha_{\xi}$ and argument $\Pi(A_u \odot \Theta_u(v))$. Here $\Pi$ represents a given scalar function (penalty) and $A_u$ a selection matrix of connectivity that may encode neuro-anatomical information (SI section E).

For the M/EEG recordings level we assume that the Residuals Partial Coherence (RPC) is known but a scalar factor $\theta_{\xi\xi}^2$ to be estimated: $\theta_{\xi\xi}^2 = \theta_{\xi\xi}^2A_{\xi\xi}$. We use an exponential prior with scale parameter $\alpha_{\xi}$ for this factor, which provides an explicit solution. This formulation of the RPC formulation is interpretable in terms of experimental information: spurious EEG sensor connectivity due to scalp leakage currents can be encoded into $A_{\xi\xi}$ and instrumental noise inferior threshold encoded into $\alpha_{\xi}$.

$$\theta_{\xi\xi}^2|\alpha_{\xi} \sim \exp(\theta_{\xi\xi}^2|m\alpha_{\xi}) \quad (8)$$

$$\Theta_u(v) \sim \exp(\Pi(A_u \odot \Theta_u(v))|m\alpha_{\xi}) \quad (9)$$

The fitting of $\Pi$ and $\alpha_{\xi}$ biases the estimation of $\hat{\Theta}_u^{(k+1)}(v)$.

We solve this by implementing an unbiased estimator (desparsened) that provides for statistically justified limits to control the connectivity Leakage. This is an extrapolation to complex domain from the graphical LASSO real domain theory of (Jankova and Van De Geer, 2018), that we validate in simulations. It also holds for the hermitian Graphical LASSO ($\Pi = \Pi_{\text{H}}$), with fixed value of the regularization parameter $\alpha_{\xi} = \sqrt{m \log(q)}$ with $m \gg q$. See section J of SI.

$$\hat{\Theta}_u^{(k+1)}(v) = 2\hat{\Theta}_u^{(k+1)}(v) - \hat{\Theta}_u^{(k+1)}(v) \Psi_u^{(k)}(v) \hat{\Theta}_u^{(k+1)}(v) \quad (10)$$

For complex variable this limit is determined by Ryleigh tendency, in contrast to the Chi2 tendency for real variable, i.e. consequence of Jankova and Van De Geer (JVDG) theory. We implemented along other priors for the H-HGGM: based on naive (prior free) $\Pi = 0$ and ridge $\Pi = \|\cdot\|_2$ model. The naive case is precisely VARETA model: this is an extension of the LORETAs (Pascual-Marqui et al., 1994, 2002, 2006) with actual statistical fundament. Comparing these models in multiple scenarios of leakage demonstrates the robustness of the hidden hermitian graphical LASSO.
3 Results and Discussion

We validate the H-HGGM in three steps: 1-Statistical properties of the HGGM LASSO solution: Lemmas 2-5, JVDG unbiasing formula (10) and Rayleigh correction. 2-H-HGGM with the naïve, ridge and lasso penalization models, for two different Head Models: ideal EEG Lead Field (without the effect of volume conduction) and realistic SPM Human Lead Field. 3-H-HGGM connectivity analysis with the naïve, ridge and lasso penalization models SSVEP.

**HGGM (JVDG conditions and Rayleigh threshold)**

We generate 100 PC matrices (size 60x60), with block sparsity structure, and its corresponding empirical covariances for 600 samples of a Hermitian Gaussian generator ($m \gg q$). For each trial we compute the Hermitian graphical lasso solution (hggm-lasso PC) and the unbiased statistic proposed by JVDG theory (unbiased PC). From the latter we remove the values under threshold: obtained from the limit value of the Rayleigh distribution (Rayleigh corrected PC).

Figure 1 shows the retrieved sparsity pattern for a typical trial (upper row left), the likelihood evolution with iterations for all trials and for each of the models (upper row right) and histograms with the z-statistic at the null hypothesis subspace (bottom row). Computing the z-stat was done by scaling the unbiased PC using JVDG theoretical variances.

With Rayleigh threshold, from the originally dense unbiased PC, we obtain an improved sparse result in comparison to the lasso HGGM PC, as it is shown in the bidimensional maps of

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**Figure 1:** Experiment to evaluate JVDG conditions and statistical goodness of the proposed Hermitian graphical LASSO (hggm-lasso) solution. Top row (Left): Typical simulated PC and its estimators, “hggm-lasso” solution, “unbiased PC” derived from applying to the former unbiasing operation in formula (10), unbiased PC after thresholding (Rayleigh corrected PC). Top row (Right): Plots of the hggm-lasso likelihood function for all trials (100) and 30 iterations of the main loop. Bottom row: Histograms of the z-statistic computed across the 100 trials for null hypothesis cases.
Figure 1. The hermitian graphical LASSO (base of all estimated maps) shows a robust convergence pattern for any trail, according to the plots of the likelihood in Figure 1. The z-statistic histograms reflect the high coincidence with the theoretical Rayleigh distribution, for either modality (pdf and cdf).

These results demonstrate the hermitian graphical LASSO consistency to JVDG theory. The Rayleigh tendency of the unbiased PC absolute values is a natural extension of the real case Chi square tendency in the Real case. Furthermore, this analysis validates the solution given in Lemmas 2-5. The complexity of the proposed solution is bounded by the matrix square root of Formula (19): this is an optimal procedure in sight of GGM theory.

**H-HGGM (connectivity distortion by the Lead Field)**

We recreate EEG simulations considering two different Lead Field models: pseudo-LF and human-LF, see Figure 2 top row. The pseudo-LF was defined upon two concentric circles: pseudo-cortex and pseudo-scalp of bidimensional geometry. A basic electrostatic rule was implemented to construct the

Figure 2: Experiment to evaluate the volume conductance effect by using two different models. A pseudo lead field which does not regard any effect of volume conduction (left) and a realistic one which does regards it (right). The former was defined though a purely electrostatic rule for projecting the inner circle (pseudo cortex) dipolar elements onto the outer circle (pseudo cortex) sensor potential (top row left). The latter was computed by SPM software over a three layers conductance model (cortex, skull and scalp) extracted from a healthy human T1 image (top row right). The activity of approximately 20 sources was defined as samples (600) from a Hermitian Gaussian random generator, with 100 random binary SPCs (simulated SPC in bottom row) of which a single trial is shown. Then, for either model, the activity was projected to the sensor space and corrupted with noise (at source and sensor level).
projected pseudo-scalp fields, from normally oriented pseudo-cortical dipoles to 30 homogeneously distributed sensors. The human-LF was computed for a healthy subject T1 image, with SPM Boundary Element Method (BEM), from normally oriented cortical dipoles to 30 sensors in the 10-20 extended system (in which the SSVEP was also recorded).

We simulated distributed activity on approximately 20 generators. We followed similar procedure as with the previous simulation study: samples from a gaussian engine, with predefined SPC matrix, were projected to the sensor space via the LF. Additionally, we binarized the simulated SPCs for consistency to the binary classification measures (ROC analysis) used to evaluate the performance.

For the human model the active sources were picked randomly from areas spanning homogeneously all along the cortex (selecting one generator for area and avoiding deeper regions). For the pseudo model they were homogeneously distributed at the pseudo-cortex. We incorporated 7 dB noise at both sensor and generator space.

The results demonstrate the major shortcoming of EEG, distortion of the connectivity itself due to volume conduction, see Figure 2 bottom row. The reconstructed SPC deteriorates dramatically when moving from the ideal to realistic case. The H-HGGM model with LASSO penalization was the most robust to volume conduction effect, when compared to ridge and naive (VARETA) estimators. The simulation conditions isolate this effect from the possible distortion of source localization: we assumed knowledge about the active sources.

The binary classification performance across 100 simulation trials confirms what we illustrated in Figure 1, see Table 1. For the ideal case perfect classification was achieved, as shown by the ROC derived measures: AUC (total area under curve), SENS (sensitivity), SPEC (specificity), PREC (precision) and RECALL (F1 measure). Strong measures like PREC and RECALL evidence that, the large number of false positives, for ridge and naive estimates situated under optimal ROC threshold in all cases (trials). The Rayleigh threshold of the maps estimated with LASSO model were in total correspondence with (not higher than) the binary classification thresholds, due to the absence of false negative estimated values in the ideal case. This is the reason of its robustness under volume conduction effect (realistic lead field), as we illustrated with Figure 1 and Table 1 confirms. The three penalization models exhibited a robust convergence pattern for any trial of both simulations, see Figure 3 the likelihood evolution across 60 iterations.

| Measures/Penalty | AUC | SENS | SPEC | PREC | RECALL |
|------------------|-----|------|------|------|--------|
|                  |     |      |      |      |        |
| h-hgmm-lasso     |    |      |      |      |        |
| h-hgmm-ridge     |    |      |      |      |        |
| h-hgmm-naive     |    |      |      |      |        |

Table 1: Measures of connectivity binary classification: global AUC and partial measures computed for the optimal ROC threshold (SENS, SPEC, PREC and RECALL). The analysis covered both lead field simulations (pseudo and realistic) and h-hgmm retrieved connectivity with the proposed penalty models (lasso, ridge and naive).

Figure 3: Likelihood evolution along 60 iterations of the main loop and for all trials (100). It was computed for both lead field simulations pseudo (top row) and realistic (bottom row) and the proposed h-hgmm penalty models (lasso, ridge and naive).
**SSVEP neural connectivity with H-HGGM**

We analyze the connectivity of cortical sources, that are involved in visual processing a fixed frequency (4Hz) flickering light, i.e. Steady State Visual Evoked Potential (Bayram et al., 2011; Duru et al., 2011). This constitutes an advantageous experimental set up for the evaluation of our frequency domain system theory formalism. Cortical generators involved in this stimulus processing are expected to have multiple harmonic response of the input frequency (Müller-Putz et al., 2005), see top row left in Figure 4.

Either harmonic represents the brain response, caused by the nonlinear neural response in processing the stimulus. They carry on common signature of source activity and connectivity. Indeed, all harmonics can be regarded as replicas of the slowest harmonic under analysis: the phase shift between

![Power spectral density of task segments](image)

**Figure 4:** Connectivity analysis of single subject Steady State Visual Evoked Potential (4Hz stimulation frequency), harmonic components of occipital sensors are shown top row left. ENET-SSBL source reconstruction for three main harmonics reveals target areas of the stimulus (top row right), ranging from Occipital, Temporal and Frontal activations in both hemispheres. Second row shows the connectivity estimated with the three penalty models in block matrix plots, defining blocks in hierarchical (functional) order of the target areas (higher levels go down in the plot) and hemispherical organization (left hemisphere in the superior part). The estimation was restricted to the 10 better ranked active generators across the target areas to preserve JVDG condition ($m \gg q$). Third row shows roi-based average connectivity analysis reveals quantitatively different pathways retrieved with the three models. To the right the likelihood evolution along 60 iterations of the main loop for the three h-hggm penalty models (lasso, ridge and naive).
them is an “entire number” times $2\pi$ (full phase shift). Meanwhile analyzing the main harmonic only would rule out a large amount of information that is encoded by secondary harmonics.

For every experimental condition (flickering frequency) a single section was recorded. The aligned task segments spanned over 133 seconds approximately. The Fourier analysis with an acceptable frequency resolution (0.22Hz), reported (29 segments). This sample number is insufficient for cross-spectral estimation, in relation to number of sensors (30 recording points). To sort this we consider the band limited cross-spectra: for a narrow band of frequency components adjacent to three central harmonics. The band limited cross-spectra operates under the restriction that all of the frequency components have a unique cortical and connectivity signature. This is equivalent to consider analogous time domain LSSM for the dynamical regime of the band filtered signal. This reported a sample size of several hundred ($m = 435$) in computing the band limited cross-spectra.

To determine the source response to the stimulus (target) we use a special implementation of the Elastic Net Structured Sparse Bayesian Learning (ENET-SSBL) (Paz-Linares et al., 2017). This implementation uses similar assumptions to H-HGGM but with diagonal SPC structure. It allows us to screen out the source activity by thresholding a posterior distribution statistic given by the ratio of the posterior mean and posterior variances, see technical details in section K of SI. ENET-SSBL statistic reveals that the target areas extended over the Occipital (OL), Temporal (TL) and Frontal (FL) Lobe at both hemispheres, see Figure 4 (top row right). Previous studies on steady state evoked BOLD-fMRI responses have pointed out the participation and functional correlation of OL activations (stimulus spatial encoding) with FL processing at a higher level (Srinivasan et al., 2007).

The electrophysiology of similar experiments does not contravene this fact, but it also detaches the TL mediation in OL $\leftrightarrow$ FL communication. Source level spatio temporal analysis of grand-averaged Visual Evoked Potentials (VEP) have pointed out that FL activations are preceded by earlier and stronger TL activations. Something that was verified with two different SSBL methods: Elastic Net and Elitist LASSO (Paz-Linares et al., 2017). Also, in contradistinction with the results provided by other classical (Tikhonov Regularization) methods (Vega-Hernández et al., 2008). But in high correspondence with neurophysiological information (Boner and Price, 2013). The information flow at sensor space determined with different functional connectivity metrics is also in agreement with this (Miskovic and Keil, 2015). In this sense other studies fail in neglecting the TL participations (Li et al., 2015).

H-HGGM connectivity was computed for the most actively ranked generators across the target areas: according to the activity level up to 10 per area were selected. Those were the OL, TL and FL of both hemispheres: Left (L) and Right (R). See in Figure 4 second row the node wise connectivity (SPC) maps, and the qualitatively different results of the three penalization models (lasso, ridge and naive).

Roughly, the lasso model exhibited the higher sparsity level and relevancy of the estimates. Ridge and naive estimation were biased to a single connectivity block. Even when ridge performed much better, most of the connections were smaller and blurred. Nevertheless, all models revealed that the larger connectivity was OL-L $\leftrightarrow$ OL-R. This was expected since the same information is being processed by the visual cortex at both hemispheres.

To interpret the information flow between areas we analyze the average connectivity between regions of interest (ROIs), see Figure 4 third row. Lasso model connectivity was stronger for the OL $\leftrightarrow$ TL $\leftrightarrow$ FL hemispheric pathway. The neurophysiologically interpretation of this pattern relies on the mediation by TL in OL $\leftrightarrow$ FL communication. Ridge and naive models reinforced also the OL $\leftrightarrow$ FL hemispheric pathway due to its sensitivity to Leakage (this has been pinpointed as crosstalk). Lasso model also selected the interhemispheric pathway OL-R $\leftrightarrow$ TL-L and TL-R $\leftrightarrow$ FL-L, something more in correspondence with previous studies on sensor space connectivity of SSVEP (Yan and Gao, 2011; Zhang et al., 2015). Meanwhile ridge and naïve model reinforced totally different interhemispheric paths due to Leakage effect. That was also present between frontal areas: it has been demonstrated that not the FL-L $\leftrightarrow$ FL-R but OL $\leftrightarrow$ FL pathway carries on the mayor information flow (Li et al., 2015). In fact, that was the second highest connectivity guessed by ridge and naïve models. For lasso model estimation it was tremendously reduced, at a level that reflects some physiologically plausible statistical dependency.

4 Materials and Methods

An approximated expression to the T2L at the $k$-th EM step is given through ELL $Q(\Omega, \Omega^{(k)})$. The symbol $\Omega$ summarizes the hyperparameters for a single frequency component $v$, i.e. $\Omega = \{\Omega_{\xi \xi}, \Omega_{\xi q}(v)\}$. Formally, the model spans the entire frequency domain, for the sake of simplicity we hereinafter remove $v$ from the formulation.

\[
p(V|\Omega) \approx \exp\left(-Q(\Omega, \Omega^{(k)})\right)
\]

\[
Q(\Omega, \Omega^{(k)}) = \int_{\mathcal{F}} p(V, I|\Omega) p(I|\Omega^{(k)}) dI
\]

Lemma 1 (bimodal log-Wishart form of the ELL)

For the model in (3) and (4) the ELL $Q(\Omega, \Omega^{(k)})$ admits a decomposition into two sequentially independent factors of log-Wishart form, of the observation residuals and source:

\[
Q(\Omega, \Omega^{(k)}) = m \log|\Omega_{\xi \xi}| - m tr\left(\Omega_{\xi \xi}^{-1} \Psi_{\xi \xi}^{(k)}\right)...
\]
Lemma 2 (CV-LQA of AML corollary)

The measurable space with normalized Gibbs density (pdf)

\[ p(\mathbf{1}) \propto \exp(\Pi(A \otimes \mathbf{1})/n) \]

with penalization function of the complex LASSO model \( \Pi = \|\mathbf{1}\|_2 \), admits a hierarchical (conditional) representation of measurable spaces product: One of the conditional expectation \( \mathbf{1}|\Gamma \) with (unnormalized) Gaussian density and the random variable \( \Gamma \) with (normalized) Gamma pdf.

\[ \mathbf{1}|\Gamma \sim \Pi_{ij}^{q} N_i(||\Theta_{ij}||0,\Gamma_{ij}^2/m) \]

\[ \Gamma \sim \Pi_{ij}^{q} Ga(\Gamma_{ij}^2|1, \alpha^2 \Lambda_{ij}^2/2) \]

Lemma 3 (conavity of the CV-LQA)

The target function \( \mathcal{L}(\mathbf{1}, \Gamma) \) (minus-log-density of the measurable product space \( \mathbf{1} \times \Gamma \)) is strictly concave on the intercept of the region of positive definiteness of its arguments \( \{ \Theta \geq 0, \Gamma \geq 0 \} \) and the region comprehended by the set of inequalities:

\[ \{3m\Theta_{ij}\gamma_{ij}^2 - \gamma_{ij}^2 + m\alpha^2 \Lambda_{ij}^2 \geq 0\}^q_{ij=1} \]

(16)

Then, \( \mathcal{L}(\mathbf{1}, \Gamma) \) has a minimum within this region given by the intercept of the system of equations:

\[ -\mathbf{1}^{-1} + \mathbf{1} + \Theta \otimes \Gamma = 0_q \]

(17)

\[ -m\Theta_{ij}\theta_{ij}^2 + \theta_{ij}^2 + m\alpha^2 \Lambda_{ij}^2 \gamma_{ij}^2 = 0 \]  \[ q=1 \] 

(18)

Lemma 4 (standardization of the Wishart distribution)

Let \( \mathbf{1} \) and \( \mathbf{1} \) (referred as “Standard”) be \( q \times q \) Hermitian random matrices with complex Wishart density of \( m \) degrees of freedom and positive definite hermitic scale matrix \( \Sigma \) and \( \Sigma = (\Sigma^{-1} \otimes \Gamma)^{-1} \) (\( \Gamma \) is a \( q \times q \) positive definite matrix of positive weights), i.e. \( \mathbf{1} \sim W_q^c(\mathbf{1} | \Sigma, m) \) and \( \mathbf{1} \sim W_q^c(\mathbf{1} | \Sigma, m) \).

Then for \( \Phi \) (called “Version”) defined by the relationship to the Standard \( \Phi = (\Phi^{-1} \otimes \Gamma)^{-1} \) or \( \Phi = (\Phi^{-1} \otimes \Gamma)^{-1} \) (called “Unstandardization” of \( \Phi \)), it can be verified:

a) All entries \( (\Phi^{-1})_{ij} \) of the Version inverse \( \Phi^{-1} \) keep the Wishart density independency property: they are stochastically independent among them and from the set \( \{\Phi_{ij}(l)\} \) \( (l \neq i) \).

b) All entries \( (\Phi^{-1})_{ij} \) of the Version inverse \( \Phi^{-1} \) keep the Wishart marginal density: complex Inverse Gamma with parameter of shape \( m - q + 1 \) and scale \( (\Sigma^{-1})_{ij} \).

Lemma 5 (local graphical Ridge estimator)

Given the equivalence of Lemma 4 the graphical LASSO admits a graphical Ridge local representation: the pair given by the Likelihood \( W_q^c(\mathbf{1} | \mathbf{1}^{-1}, m) \) and prior \( \exp(\mathbf{1}|m) \). Furthermore, the graphical Ridge estimator is the solution of the Riccati matrix equation: \( \mathbf{1}^{-2} + \mathbf{1} \mathbf{1} - \mathbf{1} = 0_q \). That is the unique solution that shares the eigenspace with \( \mathbf{1} \), which is also positive definite and hermitic, expressed by the following matrix square root formula:

\[ \hat{\mathbf{1}} = -\frac{1}{2} \mathbf{1} + \frac{1}{2} \sqrt{\mathbf{1}^2 + 4\lambda \mathbf{1}} \]

(19)

5 Conclusions

Current Brain Connectivity methods are flawed due their lack of interpretability in terms of Systems Theory and Bayesian Formalism. We propose a new procedure that fills this gap: modelling functional connectivity by the Partial Correlations of a Hidden Hermitian Gaussian Graphical Model. Such quantity (SPC) is a bi-univocal representation of the neural system’s Directed Transfer Function. Its inference is leveraged by two elements: 1 - the marginal gaussian graphical models of residuals and states derived from the Type II Likelihood approximated representation (Lemma 1). 2 - the solution of the hermitian graphical LASSO (Lemma 2-5), alongside its statistical guarantees.

Doing so provided the link of two statistical modelling branches: Linear State Space Models and Gaussian Graphical Models. This strode for the solution to the Leakage problem: a fundamental limitation for M/EEG source space connectivity. Our solution to the hermitian graphical LASSO, also extendible to real variable, addresses the problem of sparse Partial Correlations in an optimal. The computational cost of our solution is bound by the matrix square root operation, this renders it a prospective approach for its entire field of applications. The accuracy of the implementation proposed here was tested in terms of the theoretically expected statistical tendency: the Rayleigh distribution (or complex circular extension of Jankova and Van Der Geer theoretical distribution). This is precisely the way to undertake Leakage control, with enough statistical guarantees, iteratively into the hidden hermitian gaussian graphical model scheme.

The analysis of the connectivity performance, under the effect of volume conduction, demonstrated the robustness of the hidden hermitian graphical LASSO. Different regularization models suffer a substantial distortion under this condition, as we illustrated with typical reconstructed connectivity maps and robust binary classification measures (precision and recall). This is the case with the VARETA model: the Bayesian generalization of the LORETAs. Also, the case of the hidden hermitian graphical Ridge: an efficient alternative to LASSO.
model that we tested here. Either hidden models (LASSO and Ridge) represent a potential assessment to state space inference in multiple scenarios (beyond M/EEG source connectivity). This is also constituting a natural extension to the theory of Hidden Markov Models.

We were able to extract a meaningful cortical connectivity pattern with the LASSO model in steady state visual evoked potentials, recorded with EEG at fixed light flicker (stimulus) frequency. We regard this experiment as the ideal scenario to validate frequency domain methods: in this case to explore the Leakage effect on the retrieved cortical connectivity at a single response frequency (in relation to the stimulus).

The connectivity study spanned across cortical areas that were first screened with a cross-spectral extension of ENET-SSBL, also a sparse source activity (not connectivity) method with statistical guarantees. Previous studies have proven these areas have a strong involvement in visual processing: a large scale network including the Occipital, Temporal and Frontal Lobes of both hemispheres.

Unfortunately, a large scale analysis as such has not been ever reported in similar experiments and at source space: for fMRI studies only Occipital <-> Frontal connection patterns were analyzed and for EEG it was take out only limited to Occipital intrinsic connections. Relevantly, previous studies of connectivity at sensor space have considered a large scale network, which agree with the cortical pathway revealed the sensor space connectivity analysis. These are not interpretable in terms of cortical connections, what they provide at most is a rough idea about the interareal information flow: something we use here as external verification.

These sensor space studies were corrupted by the effect of Leakage: the estimated sensor level network is denser than the original one at neural level. In part due to volume conduction but also because of the mixture of activity projected at the scalp.

The source space methods here used for comparison purpose (hidden graphical Ridge and VARETA), were also sensitive to Leakage and in contradiction with LASSO results. Great differences featured the recovered Occipital <-> Frontal pathway Frontal <-> Frontal interhemispheric communication. This is a typical example of Leakage, as critical reports point out, which provides an ideal scenario for the verification of the proposed theory of the hidden graphical LASSO.

This connectivity model will be further studied: in Human Connectome Project data-bases and on simulations to test both activation and connectivity Leakage. Something that would also involve testing its behavior along with that of the ENET-SSBL source screening. We offer freely an open source software package to reproduce automatically all the results presented in this manuscript: do it so by executing the script “h_hggm_simpack.m” from the following GitHub link:

https://github.com/CCC-members/MEEG_Source_Connectivity_SoftPack

An independent package is also offered for the general purpose of analyzing M/EEG databases, along with an example to help with the format of the inputting data. Do it so by executing the script “Principal.m” from the GitHub link:

https://github.com/CCC-members/BC-VARETA_Toolbox

6 Acknowledgements

This study was funded by the Grant No. 61673090 from the National Nature Science Foundation of China.

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SUPPLEMENTARY INFORMATION

Neural Connectivity in M/EEG with Hidden Hermitian Gaussian Graphical Model

Deirel Paz-Linares b,c, Eduardo Gonzalez-Moreira a,c,1 and Pedro A. Valdés-Sosa a,b,2

The Clinical Hospital of Chengdu Brain Science Institute, MOE Key Lab for Neuroinformation, University of Electronic Science and Technology of China, Chengdu, China; b Cuban Neuroscience Center, La Habana, Cuba; c Centro de Investigaciones de la Informática, Universidad Central "Marta Abreu" de las Villas

Pedro A. Valdés-Sosa
pedro.valdes@neuroinformatics-collaboratory.org

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A-Mathematical Notation

| Symbol | Description |
|--------|-------------|
| [A.1] | x, X, X | The following symbols denote respectively a vector (bold italic lowercase) a matrix (bold capital) a set (double struck capital). |
| [A.2] | x_m | Subscript indicating with lowercase script the m-th vector sample. |
| [A.3] | X_{ij}, (X)_{ij}, x_{i}(x)_{i} | Subscript indicating with lowercase the ij (i) element of a matrix X (vector x). |
| [A.4] | N_p(x|y, Z) | Normal distribution of a p size vector x with mean y and Covariance Matrix Z. |
| [A.5] | N^C_p(x|y, Z) | Circularly Symmetric Complex Normal distribution of a p size complex vector x with complex mean y and Complex Covariance Matrix Z. |
| [A.6] | exp(x|y) | Exponential distribution of the scalar x with parameter of shape y. |
| [A.7] | Ga(x|y, z) | Gamma distribution of the scalar x with parameters of shape y and rate z. |
| [A.8] | | Determinant of a matrix X. |
| [A.9] | tr(X) | Trace of a matrix X. |
| [A.10] | X^{-1} | Inverse of a matrix X. |
| [A.11] | X^T | Transpose of a matrix X. |
| [A.12] | X^† | Conjugate transpose of a matrix X. |
| [A.13] | X, x̃ | Estimator Parameters or Hyperparameters random matrix (X) or vector (x). |
| [A.14] | | Estimator of auxiliary magnitudes random matrix (X), dependent on Parameters or Hyperparameters estimators. |
| [A.15] | X^{(k)}, X^{(k)} | Updates at the k-th iteration of estimators. |
| [A.16] | \sum_{m=1}^{M} | Sum operator along index m. |
| [A.17] | \prod_{m=1}^{M} | Product operator along index m. |
| [A.18] | p(X) | Probability density function of a random variable X. |
| [A.19] | p(X|Y) | Conditional probability density function of a random variable X regarding the state of the variable Y. |
| B.20 | \( p(X,Y|Z) \) | Conditional joint probability density function of random variables \( X \) and \( Y \) regarding the state of the variable \( Z \). |
| B.21 | \( X|Y \sim p(X|Y) \) | Indicates that the variable \( X \) probability density function is conditioned to \( Y \). |
| B.22 | \( \|X\|_{i,A} \) \( \delta = 1,2 \) | L1 or L2 norm of the matrix \( X \) with weights or elementwise precisions defined by the mask matrix \( A \). |
| B.23 | \( I_p, 1_p, 0_p \) | Denotes respectively Identity, Ones and Ceros matrices of size \( p \). |
| B.24 | \( \odot \) \( \odot \) | Elementwise matrix product a division operators (Hadamard). |
| B.28 | \( \text{argmin}_X \{ f(X) \} \) \( \text{or} \) \( \text{argmax}_X \{ f(X) \} \) | Extreme values of the scalar function \( f \), correspondingly minimum or maximum, in the argument \( X \). |
| B.29 | \( \text{zeros}_X \{ f(X) \} \) | Zeros of the scalar function \( f \) in the argument \( X \). |

### B-Nomenclature

| B.1 | \( E \) | Scalp Sensors (Electrodes) Space. |
| B.2 | \( M \) | Random Samples space. |
| B.3 | \( G \) | Discretized Gray Matter (Generators) Space. |
| B.4 | \( p \) | Number of MEEG sensors at the scalp. |
| B.5 | \( m \) | Number of data samples obtained from MEEG single frequency bin Fourier coefficients from a number \( m \) of segments. |
| B.6 | \( q \) | Number of MEEG generators at the Cortex surface. |
| B.7 | \( \nu_m \) | Complex size MEEG data Fourier coefficients sample vector for a single frequency component \( \text{(observed variables or Data)} \). |
| B.8 | \( \xi_m \) | Complex size MEEG source’s Fourier coefficients sample vector for a single frequency component \( \text{(unobserved variables or parameters)} \). |
| B.9 | \( L \) | Lead Field matrix of \( n \times q \) size. |
| B.10 | \( \xi_m \) | Complex Fourier coefficients vector for a single frequency component from MEEG forward model residuals \( \text{(sensors’ noise)} \). |
| B.11 | \( \Sigma_u \) | Complex size Hermitian and positive semidefinite matrix of EEG/MEG sources’ Fourier coefficients \( \text{(unobserved variables or Parameters)} \) Covariance matrix. |
| B.12 | \( \Theta_u \) | Complex size Hermitian and positive semidefinite matrix of EEG/MEG sources’ Fourier coefficients \( \text{(unobserved variables or Parameters)} \) Inverse Covariance matrix. |
| B.13 | \( \Sigma \xi \) | Complex Hermitian and positive semidefinite matrix of EEG/MEG forward model residuals’ Fourier coefficients \( \text{(sensors’ noise)} \) Covariance matrix. |
| B.14 | \( \Lambda \) | Known Complex Hermitian and positive semidefinite matrix. |
| B.15 | \( \theta^2 \) | Positive nuisance level hyperparameter \( \sigma^2 \). |
| B.16 | \( \Xi \) | General variable defining the set of hyperparameters. |
| B.17 | \( Q(\Xi, \Xi) \) | Data expected log likelihood, obtained after the expectation operation of the data and parameters log joint conditional probability density function over the parameters accounting for the parameters posterior density function with estimated values of the hyperparameters. |
| B.18 | \( \Pi(\Theta_u, \Lambda) \) | Scalar general penalty function or exponent of the prior distribution Precision matrix \( \Theta_u \) parametrized in the regularization parameters or mask matrix \( \Lambda \). |
| B.19 | \( \lambda \) | Regularization parameters or tuning hyperparameters vector of the general penalty function. |
| B.20 | \( \bar{T}^{(R)} \) | MEEG Data to Source Transfer Function. |
| B.21 | \( \Sigma_u^{(R)} \) | Complex Hermitian and positive semidefinite matrix of MEEG source Fourier coefficients \( \text{(unobserved variables or parameters)} \) posterior Covariance matrix. |
| B.22 | \( \Sigma_u^{(R)} \) | Complex Hermitian and positive semidefinite matrix of MEEG sources’ Fourier coefficients \( \text{(unobserved variables or parameters)} \) empirical Covariance matrix. |
| B.23 | \( \Psi_u^{(R)} \) | Effective Sources Empirical Covariance \( \text{(ESEC)} \). It carries the information about sources correlations that will effectively influence the sources Covariance matrix estimator in the maximization step \( \text{(sources Graphical Model solution)} \), thus, it becomes the sources Covariance matrix estimator in the especial case of prior free model. |
| B.24 | \( \mathcal{S}_{pp} \) | Complex Hermitian matrix MEEG data Fourier coefficients Covariance matrix. |
C-Frequency Domain Transformation of the Linear State Model

The LSSM representation is settled in a more general context of State Space Model (SSM) like formulations, representing neural model that underlies MEEG signal generation. This is supported by previous experimental studies, evidencing the existence of deterministic dynamic regimes of neural populations (neural masses or equivalently Gray Matter generators) (Freeman, 1974). See an excellent review about these models and its biophysical basis in (Deco et al., 2008). Two SSM formulations can be accounted among the most stablished within the state of the art: Nonlinear Continuous Time models in either differential or integrodifferential form (Lopes da Silva et al., 1980; Valdes-Sosa 2004; David et al., 2003; Jirsa et al., 1997). Linear Discrete Autoregressive models (Faes et al., 2012; Galka et al., 2004; Pascual-Marqui et al., 2014; Baccalá and Sameshima, 2001; Babiloni et al., 2005).

Given its simplicity and generality, a ubiquitous formulation is that given by a dynamical integral representation of the source activity \( \tau (t) \), in the continuous time domain \( t \in \mathbb{T} \), i.e. State Equation. Thus, the associated LSSM builds upon the coupling of that with the Forward Model of the MEEG signal \( \nu (t) \) also in integral form, i.e. Observation Equation. See the expression (MEEG-LSSM) below, observable quantities are denoted by Latin scripts while the unobserved by Greek scripts. Check the mathematical notation and definition of variables across this document in sections A and B.

\[
\nu(t) = \int_0^t L_{\nu\nu}(\tau) \tau(t - \tau) d\tau + \xi(t) \quad \text{“Time Domain Observation Equation”} \\
\tau(t) = \int_0^t K_{\nu\tau}(\tau) \tau(t - \tau) d\tau + \xi(t) \quad \text{“Time Domain Neural State Equation”} 
\]

Above \( L_{\nu\nu}(\tau) \) represents the Source to Data Transfer Function (SDTO) of size \( p \times q \), i.e. Lead Field. Where \( p \) and \( q \) are respectively the number of sensors (at the \( p \)-size Sensor Space \( E \)) and the number of generators (at the \( q \)-size discretized Gray Matter Space \( G \)). The Lead Field is a result of the Maxwell Equations integration, expressed as temporal convolution \( (\tau \in \mathbb{T}) \), for a specific model of head additivity and numerical integration method for partial differential equations. The lowercases \( \xi(\tau) \) and \( \nu(\tau) \) represent respectively additive noise vectors of the MEEG signal and sources activity. The uppercase Kappa \( K_{\nu\tau}(\tau) \) denotes the sources directed connectivity matrix at multiple lags \( (\tau \in \mathbb{T}) \), of a mesoscopic neural mass model. At the mesoscopic scale, the connectivity between neural populations (masses) is expressed by unidirectional coupling strength coefficients. This represents the efficiency in neural communication, given by the spatio-temporal density of synaptic events. (Valdes-Sosa et al., 1999; Valdes-Sosa et al., 2005; Valdes-Sosa et al., 2006; Friston, 2009; Valdes-Sosa et al., 2009; Friston, 2011; Valdes-Sosa et al., 2011).

Strongly motivated by the central limit theory (Rosenblatt, 1956) and/or analytical tractability (Roweis and Ghahramani, 1999), the noise (residuals) are modeled as stationary gaussian processes. See the stochastic model below, built on Multivariate Real Gaussian distributions \( N^\mathbb{R} \), with time invariant parametrization in the covariance matrices of sensor \( \Sigma_{\xi\xi} \) and source \( \Sigma_{\xi\xi} \) noise.

\[
\xi(\tau) \sim N^\mathbb{R}(\xi(\eta) | 0, \Sigma_{\xi\xi}) \quad \text{“MEEG Residuals Process”} \\
\zeta(\tau) \sim N^\mathbb{R}(\zeta(\eta) | 0, \Sigma_{\xi\xi}) \quad \text{“Source Residuals Process”}
\]

The identification of the MEEG signal underlying model, when the periodicity of time series holds, is better suited by its representation in transformed spaces. The most interpretable and commonly used is the Fourier Transform, relevant discussion points about the use of time domain vs frequency domain system identification techniques are surveyed in (Ljung and Glover, 1981; Schoukens et al., 2004). In the frequency domain \( (\nu \in \mathbb{F}) \), the MEEG signal and source activity are respectively denoted by Fourier coefficients \( \nu_m(\nu) \) and \( \tau_m(\nu) \), that correspond to different time windows \( m \in \mathbb{M} \). Because of the Fourier Transform properties (linearity, convolution factorization), the model in (C-1) and (C-2) is expressed in the frequency domain as a linear system of equations. The spectral properties of the States are fully characterized by the Directed Transfer Function (DFT) \( K_{\nu\tau}(\nu) \), i.e. Fourier transform of \( K_{\nu\tau}(\tau) \). The DFT approach was introduced in the context of MEEG sensor connectivity analysis by (Kaminski and Blinowska, 1991).

\[
\nu_m(\nu) = L_{\nu\nu}(\nu) \tau_m(\nu) + \xi_m(\nu) \quad \text{“Frequency Domain Observation Equation”} \\
\tau_m(\nu) = K_{\nu\tau}(\nu) \tau_m(\nu) + \xi_m(\nu) \quad \text{“Frequency Domain Neural State Equation”}
\]

In formulas (C-5) and (C-6), the assumptions on the LSSM noise vectors stochastic properties predefine a frequency domain Bayesian model: For the MEEG signal (Data) \( \nu_m(\nu) \) and the source activity (Parameters) \( \tau_m(\nu) \) Fourier coefficients, categorized as independent random variables, consequently with the Bayesian formalism (MacKay, 2003). This can be demonstrated in virtue of the
stochastic invariance under the Fourier Transform of the gaussian process (B-3) and (B-4) and few algebraic transformations of equation (C-6). Thereby shaping the Data Likelihood and Parameters Prior, i.e. hierarchically conditioned CSG distribution $N^C$.

\[
\begin{align*}
\nu_m(v) | t_m(v), \Sigma_{\xi}(v) & \sim N^C \left( \nu_m(v) | L_{\nu t_m}(v), \Sigma_{\xi}(v) \right) \quad \text{"Data Likelihood"} \\
\iota_m(v) | \Sigma_u(v) & \sim N^C \left( \iota_m(v) | 0, \Sigma_u(v) \right) \quad \text{"Parameters Prior"}
\end{align*}
\]  

(C-7)  

(C-8)

Above, the former distribution is explicitly conditioned to the Parameters $\iota_m(v)$ and the MEEG noise covariance $\Sigma_{\xi}(v)$ (Hyperparameters). The latter is implicitly conditioned to the DTF $K_u(v)$ and source noise covariance $\Sigma_{\xi}(v)$, through the frequency dependent covariance matrix $\Sigma_u(v)$ (Hyperparameters). The frequency independency condition of the noise covariances is usually relaxed, i.e. $\Sigma_{\xi} \leftarrow \Sigma_{\xi}(v)$ and $\Sigma_{\xi} \leftarrow \Sigma_{\xi}(v)$. This relaxation still encompasses the LSSM frequency domain representation but enriches the model with generality of the noise dynamic model, e.g. non stationarity. Importantly, the DFT can be extracted by the decomposition into stable spectral factors of the covariance matrix inverse, i.e. precision matrix $\Theta_u(v) = \Sigma_u^{-1}(v)$. This is known as Spectral Factorization Theorem (Sayed and Kailath, 2001; Janashia et al, 2011; Jafarian and McWhirter et al, 2012; Faes and Nollo, 2011; Ephremidze et al, 2007).

\[
\Theta_u(v) = \left( L_q - K_u(v) \right) \Sigma_{\xi}(v) \left( L_q - K_u(v) \right)^{-1} \quad \text{"Spectral Factorization"}
\]

(C-9)

As a direct consequence of this parametric representation an additional category of random variables to be estimated (Hyperparameters) is introduced, i.e. Residuals (noise) Covariance (RC) and Source Covariance (SC). These are denoted by the Greek letter “Xi” $\Omega$. The symbol $\Omega$ summarizes the hyperparameters for a single frequency component $v$, i.e. $\Omega = \{ \Theta_{\xi}, \Theta_u(v) \}$. Formally, the model spans the entire frequency domain, for the sake of simplicity we hereinafter remove $v$ from the formulation.

\[
p \left( \iota_m(v), \Omega^{(k)} \right) \propto N^P \left( \nu_m(v) | L_{\nu t_m}, \tilde{\Sigma}_{\xi}(v) \right) N^C \left( \iota_m(v) | 0, \tilde{\Sigma}_{\xi}(v) \right)
\]

(D-1)

Analyzing the exponent in (D-1) is enough to find the structure of the posterior distribution:

\[
- (\nu_m - L_{\nu t_m})^t \Sigma_{\xi}^{-1}(v) (\nu_m - L_{\nu t_m}) - \iota_m^t \Sigma_{\xi}^{-1} \iota_m
\]

(D-2)

Reorganizing terms in [C.4]:

\[
- \iota_m^t \left( L_{\nu t_m} \Sigma_{\xi}^{-1} L_{\nu} + \Sigma_{\xi}^{-1} \right) L_{\nu} t_m + \nu_m^t \Sigma_{\xi}^{-1} L_{\nu} t_m + t_m^t L_{\nu} \Sigma_{\xi}^{-1} \nu_m
\]

(D-3)

Defining the auxiliary quantity Posterior Source Covariance (PSC) $\Sigma_u^{(k)} \leftarrow \left( L_{\nu t_m} \Sigma_{\xi}^{-1} L_{\nu} + \Sigma_{\xi}^{-1} \right)^{-1}$ and completing terms in (D-3):

\[
- \iota_m^t \Sigma_u^{(k)} \iota_m + \nu_m^t \Sigma_u^{(k)} \Sigma_u^{(k)} \Sigma_u^{(k)} \nu_m
\]

(D-4)
The second and third terms in (D-4) can be reorganized into the auxiliary quantity Data to Sources Transfer Function (DSTF) \( \hat{T}^{(k)} \leftarrow \Sigma_u L_u \psi \psi^{-1} \). Its left product with the Data samples \( v_m \) defines the source activity estimator \( \hat{i}^{(k)}_m \).

\[
\hat{i}^{(k)}_m \leftarrow \hat{T}^{(k)} v_m \tag{D-5}
\]

Substituting (D-5) in (D-4) and completing with the term \( \hat{i}^{(k)}_m \Sigma_u^{-1} \hat{i}^{(k)}_m \) we obtain the following expression for the exponent of the parameters posterior distribution.

\[
-(t_m - t^{(k)}_m)^T \Sigma_u^{-1} (t_m - t^{(k)}_m) + \hat{i}^{(k)}_m \Sigma_u^{-1} \hat{i}^{(k)}_m \tag{D-6}
\]

In virtue of (D-6) is clear that parameters posterior distribution constitutes a Circularly Symmetric Complex Gaussian, with posterior mean \( \hat{\mu}_m \) and posterior covariance matrix \( \hat{\Sigma}^{(k)}_u \).

\[
t_m | v_m, \hat{\Omega}^{(k)} \sim N_{\Sigma_u}^c(t_m | \hat{\mu}_m^{(k)}, \hat{\Sigma}^{(k)}_u) \tag{D-7}
\]

The Second Level of inference or Hyperparameter Posterior Analysis, formulated through the EM algorithm, constitutes an explicit and interpretable way to tackle the Hyperparameters estimation. This is done by iteratively maximizing its approximated representation of the intractable Type II likelihood (TL2) \( p((v_m)_m | \Omega) \), by the so-called Expected Log-Likelihood (ELL) \( q(\Omega, \hat{\Omega}^{(k)}) \). The latter is construed by the Expectation of the Data and Parameters Joint distribution given by the parameters Posterior distribution.

\[
Q \left( \Omega, \hat{\Omega}^{(k)} \right) = \sum_{m=1}^{M} \int \log(p(v_m, t_m | \Omega)) p(t_m | v_m, \hat{\Omega}^{(k)}) dt_m \tag{D-8}
\]

The Bayesian formalism, based on the HEGGM, can be reformulated regarding the Parameters as missing Data within the Complete Data defined as the pair \( \{v_m, t_m\} \). The complete Data Likelihood \( p(v_m, t_m | \Omega) \) can be factorized into the following expression:

\[
p(v_m, t_m | \Omega) = p(v_m | t_m, \Sigma_{\xi \xi})p(t_m | \Sigma_u) = N_{\Sigma_u}^c(v_m | L t_m, \Sigma_{\xi \xi}) N_{\Sigma_u}^c(t_m | 0, \Sigma_u) \tag{D-9}
\]

A single sample element of the Data Expected Log-Likelihood \( Q_m \left( \Omega, \hat{\Omega}^{(k)} \right) = E_{\hat{\Omega}^{(k)}}[\log p(v_m, t_m | \Omega) | v_m] \), in virtue of (D-9), can be expressed as follows.

\[
Q \left( \Omega, \hat{\Omega}^{(k)} \right) = \int N_{\Sigma_u}^c(t_m | \hat{\mu}_m^{(k)}, \hat{\Sigma}^{(k)}_u) \log \left( N_{\Sigma_u}^c(v_m | L u t_m, \Sigma_{\xi \xi}) N_{\Sigma_u}^c(t_m | 0, \Sigma_u) \right) dt_m \tag{D-10}
\]

Since the logarithm \( \log \left( N_{\Sigma_u}^c(v_m | L u t_m, \Sigma_{\xi \xi}) N_{\Sigma_u}^c(t_m | 0, \Sigma_u) \right) \) can be expressed as:

\[
\log |\Sigma_{\xi \xi}^{-1}| - (v_m - L_u t_m)^T \Sigma_{\xi \xi}^{-1} (v_m - L_u t_m) + \log |\Sigma_u^{-1}| - t_m^T \Sigma_u^{-1} t_m \tag{D-11}
\]

Using the trace properties on the second and fourth term in (D-11), and after some algebraic transformations we obtain:

\[
\log |\Sigma_{\xi \xi}^{-1}| - tr(\Sigma_{\xi \xi}^{-1} v_m v_m^T) + tr(\Sigma_{\xi \xi}^{-1} t_m L_u^T) + \log |\Sigma_u^{-1}| - t_m^T \Sigma_u^{-1} t_m \tag{D-12}
\]

From (D-12) we can deduce that the integral (D-8) can be rearranged as:

\[
Q_m \left( \Omega, \hat{\Omega}^{(k)} \right) = \log |\Sigma_{\xi \xi}^{-1}| - tr(\Sigma_{\xi \xi}^{-1} v_m v_m^T) + tr(\Sigma_{\xi \xi}^{-1} E_{\hat{\Omega}^{(k)}}(t_m | v_m) t_m^T L_u^T) + tr(\Sigma_{\xi \xi}^{-1} L_u E_{\hat{\Omega}^{(k)}}(t_m | v_m) v_m^T) - \\
tr(\Sigma_{\xi \xi}^{-1} L_u E_{\hat{\Omega}^{(k)}}(t_m | v_m) L_u^T) + \log |\Sigma_u^{-1}| - tr(\Sigma_u^{-1} E_{\hat{\Omega}^{(k)}}(t_m | v_m)) \tag{D-13}
\]

It can be easily checked that for the expectation terms inside [D.5] the following expressions hold:

\[
E_{\hat{\Omega}^{(k)}}(t_m | v_m) = \hat{t}^{(k)}_m \tag{D-14}
\]
\[ E_{\Omega^{(k)}}(t_m, t_m^t | v_m) = \Psi_{\Omega^{(k)}}^{(k)} + i_m^{(k)} t_m^+ \]  

(D-15)

Then, plugging (D-14) and (D-15) in equation (D-13):

\[ Q_m(\Omega, \Omega^{(k)}) = \log |\Sigma_{\xi}^{-1}| - tr(\Sigma_{\xi}^{-1} v_m v_m^t) + tr(\Sigma_{\xi}^{-1} v_m i_m^{(k)} L^T_{v}) + tr(\Sigma_{\xi}^{-1} L v_m i_m^{(k)} t_m^+ L^T_{v}) \]

\[ - tr(\Sigma_{\xi}^{-1} L v_m \Sigma_{\Omega}^{(k)} L^T_{v}) + \log |\Sigma_{\xi}^{-1}| - tr(\Sigma_{\xi}^{-1} (\Psi_{\Omega^{(k)}} + i_m^{(k)} t_m^+)) \]  

(D-16)

The previous analysis can be replicated for \( m \) samples of the Complete Data, defined as the set of independent pairs Samples \( \{v_m, t_m\}, m = 1 \ldots m \), to obtain the Data Expected Log-Likelihood. It is computed formulating the Complete Data Likelihood \( \prod_{m=1}^{m} p(\{v_m, t_m\} | \Omega) \) and the parameters posterior distribution \( \prod_{m=1}^{m} p(\{t_m | i_m^{(k)}, \hat{\Omega}^{(k)}\}) \), and in virtue of the linearity of the Integral in equation (D-8):

\[ Q(\Omega, \hat{\Omega}^{(k)}) = \sum_{m=1}^{m} Q_m(\Omega, \hat{\Omega}^{(k)}) \]  

(D-17)

The Expectation leads to a compact form, representing the two levels of the HEGGM hierarchical structure, i.e. MEEG noise (residuals) and source covariances. The ELL is through auxiliary quantities that can be interpreted as empirical covariances: Effective Empirical Residuals Covariance (EERC) matrix \( \Psi_{\xi}^{(k)} \) and Effective Empirical Source Covariance (EESC) matrix \( \Psi_{\Omega}^{(k)}(\omega) \).

The EESC is given by the additive combination of the auxiliary quantities PSC \( \Sigma_{\xi}^{(k)} \) and Empirical Source Covariance (ESC) \( \hat{S}_{\Omega}^{(k)} \). The ESC is the estimated Parameters empirical covariance \( \hat{S}_{\Omega}^{(k)} = \frac{1}{m} \sum_{m=1}^{m} i_m^{(k)} t_m^+ \). It can be expressed compactly by explicitly expressing \( i_m^{(k)} \) through the DSTF and effectuating empirical covariance of the Data, i.e. Empirical Data Covariance (EDC) \( \hat{S}_{vv} = \frac{1}{m} \sum_{m=1}^{m} v_m v_m^+ \). See in formulas below the expression of the EESC and ESC.

\[ \Psi_{\xi}^{(k)} = \Psi_{\xi}^{(k)} + \hat{S}_{\Omega}^{(k)} \]  

(D-18)

\[ \hat{S}_{\Omega}^{(k)} = \hat{T}_{v}^{(k)} S_{vv} \hat{T}_{v}^{(k)^T} \]  

(D-19)

The EREC \( \Psi_{\xi}^{(k)} \) depends on the EDC \( S_{vv} \) and the auxiliary quantities PSC \( \Sigma_{\xi}^{(k)} \) and DRTF \( \hat{T}_{v}^{(k)} \leftarrow I_p - L v_m i_m^{(k)} t_m^+ \) through the following expression.

\[ \Psi_{\xi}^{(k)} = \hat{T}_{v}^{(k)} S_{vv} \hat{T}_{v}^{(k)^T} + L v_m \Sigma_{\Omega}^{(k)} L^T_{v} \]  

(D-20)

Effectuating the sum (D-17) and rearranging the terms into (D-16) to conform the EERC and EESC we obtain the final expression of the Expected Log-Likelihood.

\[ Q(\Omega(\omega), \hat{\Omega}^{(k)}(\omega)) = m \log |\Sigma_{\xi}^{-1}(\omega)| - m tr(\Sigma_{\xi}^{-1}(\omega) \Psi_{\xi}^{(k)}(\omega)) \ldots \]

\[ + m \log |\Sigma_{\Omega}^{-1}(\omega)| - m tr(\Sigma_{\Omega}^{-1}(\omega) \Psi_{\Omega}^{(k)}(\omega)) \]  

“Data Expected Log-Likelihood”

(D-21)

Or equivalently formulated in terms of the Source Partial Correlations (SPC).

\[ Q(\Omega(\omega), \hat{\Omega}^{(k)}(\omega)) = m \log |\Theta_{\xi}(\omega)| - m tr(\Theta_{\xi}(\omega) \Psi_{\xi}^{(k)}(\omega)) \ldots \]

\[ + m \log |\Theta_{\Omega}(\omega)| - m tr(\Theta_{\Omega}(\omega) \Psi_{\Omega}^{(k)}(\omega)) \]  

“Data Expected Log-Likelihood”

(D-22)
E-Hyperparameter Posterior Analysis (maximization of the expected log-likelihood)

The Hyperparameters probabilistic posterior is thus expressed analytically by the combination of the Type II Likelihood approximated (iterated) representation and the Hyperparameter priors. The former builds on the exponentiation of the DELL of formula (D-22).

$$\Omega([v_m]_{m=1}^n, \bar{\Omega}^{(k)}) \sim e^{O(a_n^{(k)})}p(\Omega)$$  \hspace{1cm} \text{“Hyperparameters Posterior Distribution”} \hspace{1cm} (E-1)

$$\Theta_u \sim exp(\Pi(A_u \odot \Theta_u))|\mathcal{M}_u$$  \hspace{1cm} \text{“Source Partial Correlations Prior”} \hspace{1cm} (E-2)

$$\hat{\theta}^2 \sim \exp(\theta^2 | \mathcal{M}_u)$$  \hspace{1cm} \text{“Residual Partial Correlations Prior”} \hspace{1cm} (E-3)

The SPC estimator can be computed by maximizing the Hyperparameters Posterior distribution of formula (E-1) over $\Theta_u$. See Figure E-1 for the schematic representation of expectation computations and hyperparameters posterior maps:

$$\Theta_u^{(k+1)} \leftarrow \arg \max_{\Theta_u} \left\{ e^{0(a_n^{(k)})}p(\Theta_u) \right\}$$  \hspace{1cm} (E-4)

Equivalently, it can be done by direct differentiation after taking minus Logarithm in (E-4), by substituting formula (D-22) and the SPC Prior in formula (E-2):

$$\Theta_u^{(k+1)} \leftarrow \arg \min_{\Theta_u} \left\{ -\log |\Theta_u| + \text{tr} (\Theta_u \bar{\Psi}^{(k)}) + \alpha u_1(A_u \odot \Theta_u) \right\}$$  \hspace{1cm} (E-5)

The Nuisance Hyperparameter can be computed by maximizing the Posterior distribution in formula [E.1] over $\hat{\theta}^2$:

$$\hat{\theta}^2 \leftarrow \arg \max_{\theta^2} \left\{ e^{0(a_n^{(k)})}p(\theta^2) \right\}$$  \hspace{1cm} (E-6)

Equivalently, it can be done by direct differentiation after taking the Logarithm in (E-6) and substituting the Prior of formula (E-3):

$$\hat{\theta}^2 \leftarrow \text{zero} \hat{\theta}^2 \left\{ \frac{\partial}{\partial \theta^2} Q (\Omega, \bar{\Omega}^{(k)}) + \frac{\partial}{\partial \theta^2} \log p(\theta^2) \right\}$$  \hspace{1cm} (E-7)

Effectuating the derivative $\frac{\partial}{\partial \theta^2}$ of the first term inside (E-7) and in virtue of formula (D-22):

$$\frac{\partial}{\partial \theta^2} Q (\Omega, \bar{\Omega}^{(k)}) = -mp\theta^2 + \text{mtr} \left( A\hat{\xi} T^{(k)} S_{\nu \nu} T^{(k)} \dagger \right) \theta^2 + \text{mtr} \left( A\hat{\xi} L_{\nu \nu} \bar{\Sigma}^{(k)} L_{\nu \nu}^T \right) \theta^4$$  \hspace{1cm} (E-8)

Assuming $\alpha u_1 = ep$ and computing the derivative of the Nuisance Hyperparameter Log-Prior in the second term of [E.7]:

$$\frac{\partial}{\partial \theta^2} \log p(\theta^2) = ep$$  \hspace{1cm} (E-9)

Substituting (E-8) and (E-9) in formula (E-7) we obtain:

$$\hat{\theta}^2 \leftarrow \text{tr} \left( A\hat{\xi} T^{(k)} (\omega) \right) + \epsilon$$  \hspace{1cm} (E-10)
Figure E-1: More-Penrose diagram of the H-HGGM Posterior Analysis represented at a k-th iteration into the EM scheme and for single component in the Frequency Space $v \in \mathbb{F}$. The gray shapes represent different quantities categories: Observed (squares), Indirectly Observed (hexagons) and Variables/Estimators (circles). Accordingly, they lie on the Cartesian Product built of either Sensor Space $\mathbb{E}$ or Generator Space $\mathbb{G}$. The filled arrows represent variables generation by a specific distribution and the unfilled arrows the corresponding distribution parametrization. Inside the iterative scheme this posterior distribution is totally conditioned on the EECS matrix $\mathbf{X}_x^{(k)}$, $x = \{ \xi, \iota \}$. At every iteration, if we consider $\mathbf{X}_x^{(k)}$ as an indirectly observed Empirical Covariance, a complete analogy to a GCM can be found by assuming that $\mathbf{X}_x^{(k)}$ has Wishart Likelihood with $m$ degrees of freedom and scale matrix $\mathbf{m}^{-1} \mathbf{X}_{xx}^{-1}$:

$$
W_n^{(k)}(\mathbf{X}_x^{(k)} | m^{-1} \mathbf{X}_{xx}^{-1}, m) = |\mathbf{X}_x^{(k)}|^{(m-n)/2} |\mathbf{m}|^{n/2} e^{-m \text{tr}(\mathbf{X}_x^{(k)} \mathbf{X}_{xx}^{-1})}, n = \{p, q\}.
$$

F-Proof of Lemma 2 (Complex Variable Local Quadratic Approximation of Andrews and Mallows Lemma)

The hierarchical representation of the Gibbs Prior with LASSO exponent \( \Pi(A_u \circ \Theta_u) = \|\Theta\|_1 A \), can be built on corollaries of the Andrews and Mallows Lemma (Andrews and Mallows, 1974) for the extension of Real Laplace pdf to the Real/Complex matrix case, by considering unnormalized density functions or simply more general measurable spaces.

By Andrews and Mallows Lemma in the Real LASSO Gibbs pdf (Laplace), also for the Real/Complex case the integral representation holds \( e^{-|zl|} \propto \int N_l(|z||0, \tau) \text{Gamma}(\tau|1, a^2/2) \, d\tau. \) The measurable space in which the variable \( z\tau \) is defined has a normalized density function given by the Gaussian pdf \( p(z\tau) = N_l(|z||0, \tau), \) where its variance \( \tau \) has Gamma pdf \( p(\tau) = Ga(\tau|1, a^2/2). \) So, the measure in the product space of \( z \) and \( \tau \) has density represented as an unnormalized product of Gaussian and Gamma densities \( p(z, \tau) \propto N_l(|z||0, \tau) Ga(\tau|1, a^2/2). \) We call this the generalization of Andrews and Mallows Lemma for Real/Complex LASSO Gibbs pdf.

The GLASSO Gibbs pdf is expressed as \( p(\Theta) \propto \exp\left(\|\Theta\|_1 A\right) \), where there’s a priori independence between the Precision matrix elements, thus its pdf is factorizable as follows \( p(\Theta) \propto \prod_{i,j} \exp\left(\Theta_{ij}^2 \right) \). If we apply the generalization Andrews and Mallows Lemma in Real/Complex variable to the Precision matrix elements, by substituting \( \alpha = m^{1/2}A_{ij}, z = m^{1/2}\Theta_{ij} \) and \( \tau = \Gamma_{ij}^2 \) after some minor algebraic considerations we obtain \( p(\Theta_{ij}, \Gamma_{ij}) \propto N_l(0, \Gamma_{ij}^2/m)Ga(\Gamma_{ij}^2|1, m\alpha^2A_{ij}^2/2). \) For the measure in the product space of the Precision matrix \( \Theta \) and variances \( \Gamma \) matrix we can write the joint density.

\[
p(\Theta, \Gamma) = \prod_{i,j} N_l(0, \Gamma_{ij}^2/m)Ga(\Gamma_{ij}^2|1, m\alpha^2A_{ij}^2/2)
\]

Remark: In Lemma 2 we establish a statistical equivalence between the Gibbs Prior pdf with argument in the Complex LASSO and a hierarchical representation through a Second Level unnormalized density function \( p(\Theta|\Gamma) \), which is parametrized in a Third Level pdf. Remarkably, at this Second Level we are not using a pdf in the strict mathematical sense but a density function of a measurable space but a general measurable space. Setting up Sparse Models as General Penalty Function has been stablished in similar scenarios of Variable Selection, i.e. Graphical Models estimation (Jordan, 1998; Attias, 2000; Friedman et al, 2008; Mazumder et al. 2012; Wang, 2012; Wang, 2014; Schmidt, 2010; Hsieh, 2014; Witten et al., 2014; Zhang and Zou, 2014; Yuan and Zheng, 2017; Drton and Maathuis, 2017). Some of the most common Penalty Functions are referred into the family of Graphical LASSO Models, see Table 1 below.

| Table 1: Graphical LASSO family Penalty Functions Models |
|---------------------------------------------------|
| Graphical LASSO (GLASSO)             | Penalty function \( \Pi(\Theta_u, A) \) |
| Graphical Elastic Net (GENET)        | \( \|\Theta_u\|_{1,A} \) |
| Graphical Group Lasso (GGLASSO)      | \( \sum_{j=1}^q \|\Theta_u(x_j)\|_{2,A(x_j)} \) where \( x_j \subset \mathbb{G} \times \mathbb{G} ; j = 1 \cdots n \) |

G-Proof of Lemma 3 (Concavity of the Complex Variable Local Quadratic Approximation)

With the hierarchical representation of the Real/Complex LASSO, Lemma 2, we attain a modified Target Function of the Precisions matrix built on a Local Quadratic Approximation of the SSGM, i.e. combination of the EC Wishart Likelihood and the Precisions matrix univariate Gaussian Prior. Other terms are related to the normalization constant of the Precisions matrix Gaussian prior and the Precisions matrix variances (Weights) Gamma Prior.

To build the Target Function modified by AML, it is enough to formulate the posterior pdf of the PC and the PC variances as \( p(\Theta, \Gamma|\Psi) \propto p(\Psi|\Theta)p(\Theta|\Gamma)p(\Gamma), \) i.e. according to the distributions defined in formulas Lemma 1 and Lemma 2.

\[
p(\Theta, \Gamma|\Psi) \propto W_q^C(\Psi|m^{-1}\Theta^{-1}, m) \prod_{j=1}^q N_1(0, \Gamma_{ij}^2/m) p(\Gamma)
\]

We define the Target Function as the terms dependent on the Precisions matrix \( X \) and Precisions matrix variances \( \Gamma \) as \( L(\Theta, \Gamma) \propto \) 

\[-\log p(\Theta, \Gamma|\Psi) \], it is thus expressed as follows:
\[ \mathcal{L}(\Theta, \Gamma) = -m \log |\Theta| + m \text{tr}(\Theta \Psi) - \sum_{i=1}^{q} \log N_1(\theta_i, 0, \Gamma_i^2/m) - \log p(\Gamma) \] (G-2)

Given the definition of the univariate Gaussian pdf \( \log N_1(\theta_i, 0, \Gamma_i^2/m) \propto -\log \Gamma_i - m \theta_i \theta_i^T/2 \Gamma_i^2 \), the independence of the \( \Gamma_i \)'s and defining \( \|\Theta\|^2_{2,1,q} = \sum_{i=1}^{d} \theta_i \theta_i^T/\Gamma_i^2 \), we can write the target function as:

\[ \mathcal{L}(\Theta, \Gamma) = -m \log |\Theta| + m \text{tr}(\Theta \Psi) + \frac{m}{2} \|\Theta\|^2_{2,1,q} + \sum_{i=1}^{q} \log \Gamma_i - \sum_{i=1}^{q} \log p(\Gamma_i) \] (G-3)

The target function \( \mathcal{L}(\Theta, \Gamma) \) of arguments \( \Theta \) and \( \Gamma \) can be rearranged as a function of a vector argument \( \mathcal{L}(\vec{\Theta}, \vec{\Gamma}) \), where \( \vec{\Theta} = \text{vec}(\Theta) \) and \( \vec{\Gamma} = \text{vec}(\Gamma) \). The Hessian can be computed by a block array of the second derivatives over \( \vec{\Theta} \) and \( \vec{\Gamma} \):

\[
\begin{bmatrix}
\frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \vec{\Theta} \partial \vec{\Theta}} & \frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \vec{\Theta} \partial \vec{\Gamma}} \\
\frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \vec{\Gamma} \partial \vec{\Theta}} & \frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \vec{\Gamma} \partial \vec{\Gamma}}
\end{bmatrix}
\] (G-4)

The block Hessian matrix is Positive Definite if and only if its diagonal blocks are Positive Definite. Given the expression of \( \mathcal{L}(\Theta, \Gamma) \) in formula (G-3) we can deduce the structure of the diagonal blocks in (G-4) in the following expressions:

\[
\frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \vec{\Theta} \partial \vec{\Theta}} = \frac{\partial^2}{\partial \vec{\Theta} \partial \vec{\Theta}} \left( -m \log |\Theta| + m \text{tr}(\Theta \Psi) + \frac{m}{2} \sum_{i=1}^{q} \theta_i \theta_i^T/\Gamma_i^2 \right)
\] (G-5)

\[
\frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \vec{\Gamma} \partial \vec{\Theta}} = \frac{\partial^2}{\partial \vec{\Gamma} \partial \vec{\Theta}} \left( \frac{m}{2} \sum_{i=1}^{q} \theta_i \theta_i^T/\Gamma_i^2 + \sum_{i=1}^{q} \log \Gamma_i - \sum_{i=1}^{q} \log p(\Gamma_i) \right)
\] (G-6)

By considering the matrix differential properties the first block in expression (G-5) can be expressed as:

\[
\frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \vec{\Theta} \partial \vec{\Theta}} = m \Theta^{-1} \otimes \Theta^{-1} + \frac{m}{2} \text{diag} \left( \text{vec}(I_q \otimes \Gamma^2) \right)
\] (G-7)

In the region of Positive Definiteness of \( \Theta \) the Kronecker product \( \Theta^{-1} \otimes \Theta^{-1} \) is Positive Definite. Since the second term in (G1.13) is Positive Definite also the whole expression is Positive Definite, thus, the first block of the Hessian \( \frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \vec{\Theta} \partial \vec{\Theta}} \) is Positive Definite. To analyze the Positive Definiteness of \( \frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \vec{\Gamma} \partial \vec{\Theta}} \) it is enough to analyze the positivity of the expression \( \frac{\partial}{\partial \Gamma_i \partial \Gamma_j} \left( \frac{m \theta_i \theta_i^T}{2 \Gamma_i^2} + \log \Gamma_i - \log p(\Gamma_i) \right) \) for all \( i, j \). Substituting the term of the Gamma prior for the LASSO case, \( \log p(\Gamma_i) \propto -\frac{m^2 \lambda_i^2}{2 \Gamma_i^2} \), in the expression above we obtain:

\[
\frac{\partial^2 \mathcal{L}(\vec{\Theta})}{\partial \Gamma_i \partial \Gamma_j} = \frac{\partial}{\partial \Gamma_i \partial \Gamma_j} \left( \frac{m \theta_i \theta_i^T}{2 \Gamma_i^2} + \log \Gamma_i + \frac{m^2 \lambda_i^2}{2 \Gamma_i^2} \right) = 3m \theta_i \theta_i^T/\Gamma_i^2 - \frac{1}{\Gamma_i^2} + m \lambda_i^2 \] (G-8)

In virtue of (G-8), the Hessian is Positive Definite in the region of the product space \( \Theta \times \Gamma \) defined by the intercept of the region of Positive Definiteness of \( \Theta \) the region where the following set of inequalities \( \left\{ 3m \theta_i \theta_i^T/\Gamma_i^2 - \frac{1}{\Gamma_i^2} + m \lambda_i^2 > 0 \right\}_{i=1}^{q} \) hold. From this result the target function \( \mathcal{L}(\Theta, \Gamma) \) is strictly convex within this region. Then, applying derivatives of \( \mathcal{L}(\Theta, \Gamma) \) over \( \Theta \) and \( \Gamma \) we obtain:

---

3 See Proposition 3 of Lemma 4 in Section H to corroborate this.
\[
\frac{\partial}{\partial \theta} \left\{-m \log |\theta| + m \text{tr}(\theta \Sigma) + \frac{m}{2} \sum_{i,j=1}^{q} \frac{\theta_i \theta_j^\dagger}{r_{ij}} \right\} = -m(\theta^{-1})^T + m \Psi^T + m \theta^T \otimes \Gamma^2
\]  \hspace{1cm} \text{(G-9)}

\[
\frac{\partial}{\partial \Gamma_{ij}} \left\{m \theta_i \theta_j^\dagger + \log \Gamma_{ij} + \frac{m \alpha^2}{2} A_{ij}^2 \Gamma_{ij} \right\} = -m \frac{\theta_i \theta_j^\dagger}{r_{ij}} + \frac{1}{r_{ij}} + m \alpha^2 A_{ij}^2 \Gamma_{ij}
\]  \hspace{1cm} \text{(G-10)}

Equating (G-9) and (G-10) to zero and with some algebraic transformations we obtain the system of equations:

\[
-\theta^{-1} + \Psi + \theta \otimes \Gamma^2 = 0
\]  \hspace{1cm} \text{(G-11)}

\[
\left\{-m \frac{\theta_i \theta_j^\dagger}{r_{ij}} + \frac{1}{r_{ij}} + m \alpha^2 A_{ij}^2 = 0\right\}_{ij=1}^{q}
\]  \hspace{1cm} \text{(G-12)}

It can be checked that the point of the product space that satisfies the equation (G-12) also belongs to the region defined by the set of inequalities \(3m \frac{\theta_i \theta_j^\dagger}{r_{ij}} - \frac{1}{r_{ij}} + m \alpha^2 A_{ij}^2 > 0\). This can be checked by substituting into the inequality the term \(m \frac{\theta_i \theta_j^\dagger}{r_{ij}}\) given by (G-12), which leads to \(3 \left(\frac{1}{r_{ij}^2} + m \alpha^2 A_{ij}^2\right) - \frac{1}{r_{ij}^2} + m \alpha^2 A_{ij}^2 = \frac{2}{r_{ij}^2} + 4m \alpha^2 A_{ij}^2 \geq 0\).

For every element \(\Gamma_{ij}^2\) of the system of quadratic equations (G-12) it can be shown that it has a unique Real and Positive solution amongst the Discriminant formula Roots of Second Order Polynomials. This solution can be compactly expressed through the elementwise matrix operations \(\otimes\) (Hadamard product), \(\circ\) (Hadamard division), \(\text{abs}()\) (elementwise matrix absolute value), \(()^2\) (elementwise matrix Square Root) and \(1_q\) \((q \times q\) matrix of ones):

\[
\Gamma^2 = \left(-1_q + \left(1_q + 4\alpha^2 A^2 \otimes \text{abs}(\theta)^2\right)^{\frac{1}{2}}\right) \circ (2\alpha A^2)
\]  \hspace{1cm} \text{(G-13)}

Given the Positive Definiteness of \(A\) and \(\theta\), it can also be deduced that \(\Gamma\) is Positive Definite by following the steps in Proposition 3 of Lemma 4. \(\blacksquare\)

According to the hierarchical representation of the Real/Complex LASSO, through the Prior \(p(\theta|\Gamma)\) of equation (G-1), we can define a new random matrix through the Hadamard division scaling transformation \(\tilde{\theta} = \theta \otimes \Gamma\) (Standard Precision matrix), so that its Prior is a Gibbs pdf of the Squared L2 norm:

\[
\Theta \sim e^{-\frac{m}{2} \|\theta\|_2^2} 1_q \otimes \Gamma
\]  \hspace{1cm} \text{(G-14)}

\[
\tilde{\Theta} \propto e^{-\frac{m}{2} \|\theta\|_2^2}
\]  \hspace{1cm} \text{(G-15)}

Now we define a Standardization transformation on the ESEC \(\tilde{\Psi} = (\Psi^{-1} \otimes \Gamma)^{-1}\) that keeps the stochastic properties of the Wishart distribution (Likelihood) when conditioned to the Standard Precision matrix \(\tilde{\theta}\). See next section.

H-Proof of Lemma 4 (standardization of the Wishart distribution)

**Proposition 1**

If \(\Psi\) is a \((q \times q)\) Real/Complex Random matrix with Complex Wishart pdf of \(m\) degrees of freedom and positive definite scale matrix \(\Sigma\). Then the Random matrix obtained from the consecutive rows and columns permutation operations, denoted \(\Psi_{\text{row}}^{\text{row}}\) and \(\Psi_{\text{col}}^{\text{col}}\), respectively, has Wishart pdf of \(m\) degrees of freedom and positive definite scale matrix.

\[
\left(\Psi_{\text{col}}^{\text{col}} \Psi_{\text{row}}^{\text{row}} \Sigma^{-1}\right)^{-1}
\]
Proof of Proposition 1:

The Wishart pdf of of the posterior probability maps in Figure E-1 can be expressed as $W_q^C(\Psi|\Sigma, m) \propto |\Psi|^{(m-q)}|\Sigma^{-1}|^{-m} e^{-\text{tr}((\Sigma^{-1}\Psi))}$. These determinants and trace terms of the Wishart are invariant to consecutive rows and columns permutation operations $p_{f_i,j}^\text{col}, p_{i,j}^\text{row}$. So that:

$$|\Sigma^{-1}| = |p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|, |\Psi| = |p_{f_i,j}^\text{col}, p_{i,j}^\text{row}| \text{ and } \text{tr}(\Sigma^{-1}\Psi) = \text{tr}(p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Sigma^{-1}|p_{f_i,j}^\text{col}, p_{i,j}^\text{row}^\text{row}|\Psi|).$$

Thus the Wishart distribution can be expressed as a function of the Random matrix $p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Psi, W_q^C(\Psi|\Sigma, m) = W_q^C(p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Psi |(p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Sigma^{-1})^{-1}, m)$. Then, since the pdf is invariant to the organization of variables within the set $\{\Psi_{ij}; \ i, j = 1 \ldots q\}$, $p(\{|\Psi_{ij}; \ i, j = 1 \ldots q\}) = p(\Psi) = p(p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Psi |)$, it is clear that:

$$p(p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Psi |) \propto |\Psi|^{(m-q)} e^{-\text{tr}((\Sigma^{-1}\Psi))}$$

To find this conditional pdf it is enough to apply the consecutive permutation operation $p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Psi$ and then consider the conditional pdf of the element $(p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Psi |)_{11}$. In virtue of Proposition 1 the pdf of $(p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Psi |)_{11}$ is also Wishart. Thus, for $(p_{f_i,j}^\text{col}, p_{i,j}^\text{row}|\Psi |)_{11}$ a conditional pdf equivalent to the one in equation (F-1) can also be written. Without losing generality we can consider first the element $\Psi_{11}$ and then any result will also apply for all $\Psi_{ij}$:

$$p(\Psi_{11} | \{\Psi_{ij}\}) \propto |\Psi|^{(m-q)} e^{-\text{tr}((\Sigma^{-1}\Psi))}$$

Partitioning the random matrix $\Psi$ into the following block structure we can find a simplified expression of the determinant in the conditional pdf of equation (H-2):

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}$$

$$|\Psi| = |\Psi_{22}|(\Psi_{11} - \Psi_{12}^\text{col} \Psi_{22}^\text{row} \Psi_{21})$$

If we consider applying the same block structure as in expression (H-3) to $\Sigma^{-1}$ in the trace term if equation (H-2), completing the exponent with the term $\Psi_{12} \Psi_{22}^{-1} \Psi_{21}$ and by the determinant formula (H-4) we obtain:

$$p(\Psi_{11} | \{\Psi_{ij}\}) \propto (\Psi_{11} - \Psi_{12} \Psi_{22}^{-1} \Psi_{21})^{(m-q)} e^{-\Sigma^{-1}}(\Psi_{11}, \Psi_{12}, \Psi_{22}^{-1} \Psi_{21})$$

The argument $(\Psi_{11} - \Psi_{12} \Psi_{22}^{-1} \Psi_{21})$ in equation (H-5) above can be directly related to the element $(\Psi^{-1})_{11}$ of the inverse $\Psi^{-1}$ though the block inverse identity:

$$\Psi^{-1} = \begin{pmatrix} 1 - \Psi_{22}^{-1} \Psi_{21} & \Psi_{22}^{-1} \Psi_{21} \\ -\Psi_{22}^{-1} \Psi_{22} \Psi_{22}^{-1} & \Psi_{22}^{-1} + \Psi_{22}^{-1} \Psi_{21} \end{pmatrix}$$

$$(\Psi^{-1})_{11} = \frac{1}{(\Psi_{11} - \Psi_{12} \Psi_{22}^{-1} \Psi_{21})}$$
From this it is deduced that the random variable \((\Psi^{-1})_{11}\) is independent from \(\{\Psi_{11}\}\) and it has Complex Inverse Gamma pdf with shape parameter \((m - q)\) and scale parameter \((\Sigma^{-1})_{11}\):

\[
p((\Psi^{-1})_{11}) \propto ((\Psi^{-1})_{11})^{-(m-q)} e^{(\Sigma^{-1})_{11}}
\]

(H-8)

Now let's build the following auxiliary random matrix \(\Psi\) and scale matrix \(\Sigma\), based on scaling the elements \((\Phi^{-1})_{11}\) and \((\Sigma^{-1})_{11}\) respectively by a positive number \(\Gamma_{11}\):

\[
\Psi = \begin{pmatrix} (\Phi^{-1})_{11} & (\Phi^{-1})_{12} \\ (\Phi^{-1})_{21} & (\Phi^{-1})_{22} \end{pmatrix}^{-1}
\]

(H-9)

\[
\Sigma = \begin{pmatrix} (\Sigma^{-1})_{11} & (\Sigma^{-1})_{12} \\ (\Sigma^{-1})_{21} & (\Sigma^{-1})_{22} \end{pmatrix}^{-1}
\]

(H-10)

**Proposition 2**

The element \((\Phi^{-1})_{11}\) of the matrix \(\Phi\) has identical marginal pdf that the element \((\Psi^{-1})_{11}\) of the matrix \(\Psi\), when \(\Psi\) has Wishart pdf with \(m\) degrees of freedom and complex positive definite scale matrix \(\Sigma\).

**Proof of Proposition 2**

Using analogous representation of the conditional probability density of \(\Psi_{11}\) given \(\{\Psi_{11}\}\), as in formula (H-5), we can write

\[
p(\Psi_{11} | \{\Psi_{11}\}) \propto \Psi^{(m-q)} e^{-\ell(\Sigma^{-1}\Psi)}.
\]

Given the property of the Wishart pdf deduced before in equation (H-8) it is clear that the random variable \((\Phi^{-1})_{11}\) is independent from \(\{\Psi_{11}\}\) and it has Real/Complex Inverse Gamma pdf with shape parameter \((m - q)\) and scale parameter \((\Sigma^{-1})_{11}\). By construction of \(\Psi\) it holds that \((\Phi^{-1})_{11} = \Gamma_{11}(\Phi^{-1})_{11}\)' and \((\Phi^{-1})_{11}\) is also independent and it has Real/Complex Inverse Gamma pdf with shape parameter \((m - q)\) and scale parameter \(\Gamma_{11}((\Sigma^{-1})_{11} = (\Sigma^{-1})_{11}\).

In virtue of Proposition 1 we can iteratively apply the scaling operation described in equations (H-9) and (H-10) and get an standardized Wishart pdf of \(m\) degrees of freedom and positive definite scale matrix \(\Sigma = (\Sigma^{-1} \otimes \Gamma)^{-1}\), which argument is the complex random matrix \(\Psi = (\Phi^{-1} \otimes \Gamma)^{-1}\), i.e.

\[
p(\Psi) = W(\Psi, \Sigma, m).
\]

Which has identical marginal stochastic properties of the inverse \(\Psi^{-1}\).

To complete the proof, it is enough to show that the scale matrix \(\Sigma = (\Sigma^{-1} \otimes \Gamma)^{-1}\) of the Wishart pdf keeps being positive definite. This is a direct consequence of the fact that the scaling operation keeps positive definiteness property, as we can check from the following proposition which is a Corollary of Schur product theorem.

**Proposition 3**

If \(\Sigma\) and \(\Gamma\) are positive definite matrices then the matrix \(\Sigma = (\Sigma^{-1} \otimes \Gamma)^{-1}\) from the scaling operation is also positive definite.

**Proof of Proposition 3**

If \(\Sigma\) is positive definite so it is also the inverse \(\Sigma^{-1}\). Then, let's analyze the positive definiteness of the Hadamard scaling \(1_q \otimes \Gamma\). This Hadamard scaling can be expressed as an elementwise exponentiation \(1_q \otimes \Gamma = e^{-\log(\Gamma)}\), \(log(\Gamma)\) acts as an elementwise function. The logarithm can be expressed as an infinite Taylor series \(log(\Gamma) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} (\Gamma - 1_q)^{\ell}\). Without losing generality we can consider that the elements in \(\Gamma\) belong to the open interval \(\Gamma_{ij} \in (0,1), \forall i,j\). Given the assumption before, the odd terms in the Taylor series of \(-log(\Gamma)\) become positive, the series can be rearranged into \(-log(\Gamma) = \sum_{\ell=1}^{\infty} \frac{(1_q - 1_q)^{\ell}}{\ell}\). Finally, by substituting in the
elementwise exponential function we can express the Hadamard Scaling as the element wise product of exponentials $\mathbf{1}_q \odot \Gamma = \prod_{i=1}^{25} e^{\frac{(1q_i-T_i)^T}{\Gamma_i}}$. Here, given the positive definiteness of $\Gamma$ and since $\Gamma_i \in (0,1)$, $\forall i$, it holds that $(\mathbf{1}_q - \Gamma)$ is positive definite. Also, the elementwise exponentiation $(\mathbf{1}_q - \Gamma)^T$ and $e^{\frac{(1q_i-T_i)^T}{\Gamma_i}}$ and element wise product operation $\prod_{i=1}^{25}$, keep the positive definiteness. Thus, it is clear that $\Sigma^{-1} \odot \Gamma$ is positive definite so also it is $\mathbf{5}$. $\blacksquare$

Remark: By Lemma 4 we build a statistically equivalent model of the random matrix $\Psi$, the variable denominated ‘Version’ $\Phi$ defined through the Unstandardization of $\Psi$, in the sense that it keeps the same multivariate structure through the Wishart $pdf$, and its inverse $\Phi^{-1}$ keeps the same stochastic properties than $\Psi^{-1}$, i.e. all their elements are independent with identical marginal pdf (Drton et al., 2008). This result prescribes a Statistical equivalence between a Model of the Empirical Covariance $\Psi$ defined by a Wishart $pdf$, of $m$ degrees of freedom and positive definite scale matrix $(m\Theta)^{-1}$, with Prior of the Precision matrix $\Theta$ given by [G3.1], and Model of the Standard Empirical Covariance $\bar{\Psi}$ defined by a Wishart $pdf$, of $m$ degrees of freedom and positive definite scale matrix $(m\Theta)^{-1}$, with Prior of the Standard Precision matrix $\bar{\Theta}$ given by (G-15):

$$\bar{\Psi}|\sim W_q^c((m\Theta)^{-1}, m) \tag{H-11}$$

$$W_q^c((m\Theta)^{-1}, m) \propto |\Psi|^{m-q}|\Theta|^{-m} e^{-tr(\Theta\Psi)} \tag{H-12}$$

This Statistical Standardization is also consistent when the sample number $m$ tends to infinite, given the Inverse ESC tendency in probability to $\Theta$, i.e. $P(\Psi^{-1} \in B(\Theta)) \rightarrow 1$ as $m \rightarrow \infty$ for any open set $B(\Theta)$ containing $\Theta$ in the $q^2$-dimensional complex Euclidean space. A natural estimator of the Standard Precision matrix $\bar{\Theta}$ can be computed by maximum Likelihood through direct differentiation of equation (H-12) $\bar{\Theta} = \Psi^{-1}$, then, in agreement with the conditions and equivalence shown in Lemma 4:

$$\bar{\Theta} = \Psi \odot \Gamma \tag{H-13}$$

So, given the tendency in probability of $\Psi^{-1}$ it can also be directly deduced the tendency in probability of the Standard Precision matrix estimator, i.e. $P(\bar{\Theta} \in B(\Theta \odot \Gamma)) \rightarrow 1$ as $m \rightarrow \infty$ for any open set $B(\Theta \odot \Gamma)$ containing $\Theta \odot \Gamma$ in the $q^2$-dimensional complex Euclidean space. $\blacksquare$

I-Proof of Lemma 5 (local graphical Ridge estimator)

We attain a Standard formulation of the Precision matrix Posterior distribution, combining the equations of the Standard Wishart Likelihood and the Standard Prior:

$$\bar{\Theta}|\sim |\Theta|^{-m} e^{-mtr(\Theta\Psi)} e^{-\frac{m}{2}||\Theta||_2^2} \tag{I-1}$$

Applying minus Logarithm to the posterior distribution of formula (I-1) we obtain the Local Quadratic Approximation or Standard Target Function:

$$\bar{\Theta} = argmin_{\Theta} \{- \log |\Theta| + tr(\Theta\Psi) + \frac{1}{2} ||\Theta||_2^2 \} \tag{I-2}$$

Since (I-2) is a convex differentiable function the necessary and sufficient condition for a minimum is that its first matrix derivative over the argument $\Theta$ equals a matrix made of zeroes $0_q$, so it is direct that the minimization of (I-2) is reduced to solve the following matrix equation:

$$-\bar{\Theta}^{-1} + \Psi + \bar{\Theta} = 0_q \tag{I-3}$$
Remark: The expression (I-3) constitute a special case of the Riccati matrix equation. Positive definiteness and hermiticity of its solution $\hat{\Theta}$, as a natural property of the Standard Precision matrix estimator, is required. Also, it can be checked that the solution $\hat{\Theta}$ must share the same eigenspace with $\hat{\Psi}$, since from (I-3) two auxiliary Second Order matrix equations hold, i.e. $\hat{\Theta}^2 + \hat{\Theta}\hat{\Psi} - I_q = 0$ and $\hat{\Theta}^2 + \hat{\Phi}\hat{\Theta} - I_q = 0$, given the left and right multiplication by the Standard Precision matrix. This imply that any solution $\hat{\Theta}$ commute with $\hat{\Psi}$, thus they share the eigenspace. ■

Let’s show first that the proposed solution is positive definite. Given the positive definiteness and hermiticity of the complex matrix $\hat{\Psi}$ it admits a singular value decomposition $\hat{\Psi} = \hat{U}D\hat{U}^\dagger$ with real and positive singular values $\hat{D}$. Thus, the argument in the matrix square root of formula [G4.4] admits a singular value decomposition of the kind $\hat{D}^2 + 4I_q = 0(\hat{D}^2 + 4I_q)\hat{U}^\dagger$, where its singular values, given by $\hat{D}^2 + 4I_q$, are also real and positive. In consequence the square root term has also real and positive singular values given by the following singular value decomposition:

$$\sqrt{\hat{\Psi}^2 + 4I_q} = \hat{U} \sqrt{\hat{D}^2 + 4I_q} \hat{U}^\dagger$$

Finally, the singular value decomposition of the Standard Precision matrix estimator can be expressed as:

$$\hat{\Theta} = \hat{U} \left( \frac{1}{2} \sqrt{\hat{D}^2 + 4I_q} - \frac{1}{2} I_q \right) \hat{U}^\dagger$$

It is clear, by formula (I-5), that the singular values of the Standard Precision matrix estimator are real and positive numbers. The Standard Precision matrix estimator commutes with the Standard ESEC $\hat{\Psi}$ since they share the eigenspace, i.e. $\hat{\Psi}\hat{\Theta} = \hat{\Theta}\hat{\Psi}$. Given the singular value decomposition in equation (I-5) we can get that the singular value decomposition of $\hat{\Psi}\hat{\Theta}$ can be expressed as:

$$\hat{\Psi}\hat{\Theta} = \hat{U} \left( \hat{D} \left( \frac{1}{2} \sqrt{\hat{D}^2 + 4I_q} - \frac{1}{2} I_q \right) \right) \hat{U}^\dagger$$

The product of diagonal matrices always commutes, so we can directly check that (I-65) is also the singular value decomposition of $\hat{\Psi}\hat{\Theta}$.

To check that the proposed estimator satisfies the equation (I-3) it is enough to check that it also satisfies the pair of equations $\hat{\Theta}^2 = I_q - \hat{\Theta}^2$ and $\hat{\Phi}\hat{\Theta}^2 = I_q - \hat{\Theta}^2$. The left side in both equations are equal, i.e. $\hat{\Phi}\hat{\Theta}^2 = \hat{\Theta}^2$, and given by formula (I-6), so, lets evaluate the right side:

$$I_q - \hat{\Theta}^2 = I_q - \left( -\frac{1}{2} \hat{\Psi} + \frac{1}{2} \sqrt{\hat{\Psi}^2 + 4\lambda I_q} \right)^2$$

Effectuating the matrix square operation in (I-7) we obtain:

$$I_q - \hat{\Theta}^2 = I_q - \left( \frac{1}{2} \right)^2 \left( \hat{\Psi}^2 - \hat{\Psi} \sqrt{\hat{\Psi}^2 + 4I_q} - \sqrt{\hat{\Psi}^2 + 4I_q} \hat{\Psi} + \hat{\Psi}^2 + 4I_q \right)$$

From the singular value decomposition analysis above the matrices $\hat{\Psi}$ and $\sqrt{\hat{\Psi}^2 + 4I_q}$ share the same eigenspace and thus commute, so, rearranging (I-8) it can be obtained that:

$$I_q - \lambda \hat{\Theta}^2 = I_q - 2 \left( \frac{1}{2} \right)^2 \hat{\Psi}^2 + 2 \left( \frac{1}{2} \right)^2 \hat{\Psi} \sqrt{\hat{\Psi}^2 + 4I_q} - 4 \left( \frac{1}{2} \right)^2 I_q$$

From (I-9) and considering (I-8) it is direct that following identity holds:
\[ I_q - \tilde{\Theta}^2 = \frac{1}{2} \Psi^2 + \frac{1}{2} \Psi \sqrt{\Psi^2 + 4I_q} - \Psi \tilde{\Theta} \]  

(l-10)

Now we need to prove that the proposed estimator is the unique solution of equation (l-3) that commute with \( \tilde{\Psi} \). Let \( \hat{\Phi} \) be another solution that also commutes with \( \tilde{\Psi} \). Since the matrix \( \hat{\Phi} \) commutes with \( \tilde{\Psi} \), it has the same eigenspace, i.e., it admits a singular value decomposition of the kind \( \hat{\Phi} = \tilde{U} \tilde{\Sigma} \tilde{U}^T \). Also, since \( \hat{\Phi} \) satisfies equation (l-3), it can be checked that \( -\tilde{E}^{-1} + \hat{D} + \tilde{E} = 0 \). The solution of this equation is straightforward given diagonal matrices \( \tilde{E} \) and \( \hat{D} \), i.e., \( \hat{E} = \frac{1}{2} \sqrt{\hat{D}^2 + 4I_q} - \frac{1}{2} I_q \). It shows that \( \hat{\Phi} = \hat{\Theta} \) since they have identical eigenspace and eigenvalues.

The graphical Ridge estimator is thus expressed by the following matrix square root formula:

\[ \hat{\Theta} = -\frac{1}{2} \tilde{\Psi} + \frac{1}{2} \sqrt{\Psi^2 + 4I_q} \]  

(l-11)

**J-Connectivity estimator of the Local Quadratic Approximation**

As for choosing the Penalty Function and Regularization Parameter there is not ubiquitous rule. It is usually assumed that, for a given Penalty Function, fitting the Regularization Parameter by some Statistical Criteria by would suffice to rule out the ambiguity on the Variable Selection sparsity level (Resolution). This approach does not provide a Statistical guarantee, as discussed in (Jankova and Van De Geer, 2015, 2017), due the biasing introduced in the estimation by the Sparse Penalty in any case. For the typical Graphical LASSO, a solution was recently presented in (Jankova and Van De Geer, 2018) through an unbiased Precision Matrix estimator

\[ (\hat{\Theta}_u)^{k+1} \text{unbiased} = 2\hat{\Theta}_u^{(k+1)} - \hat{\Theta}_u^{(k)} \Psi_u^{(k)} \hat{\Theta}_u^{(k+1)} \]  

(J-1)

For the conditions \( \Pi(A_u \circ \Theta_u) = \| \Theta_u \|_1 A_u \) and \( \alpha = \sqrt{\log(q)/m} \), it is demonstrated, for the elements into the unbiased estimator

\[ \sigma_{ij}(\hat{\Theta}_u^{(k+1)}_{\text{unbiased}}) = (\hat{\Theta}_u^{(k+1)}_{\text{unbiased}})_{ij} (\hat{\Theta}_u^{(k+1)}_{\text{unbiased}})_{ji} + (\hat{\Theta}_u^{(k+1)}_{\text{unbiased}})_{ij} \]  

(J-2)

At every iteration of the outer cycle indexed \( k \)-th, of the Parameters \( \hat{\Theta}_m^{(k)} \) and Hyperparameter \( \hat{\Theta}_l^{(k)} \) estimators described in Section E, the unbiased Precision matrix estimator \( (\hat{\Theta}_u)^{k+1}_{\text{unbiased}} \) should be computed by effectuating an inner cycle indexed \( l \)-th, of the Local Quadratic Approximation formulas (G-11) and (G-12). If we denote \( \hat{\Theta}_u^{(k,l)} \) as the Local Quadratic Approximation Precision matrix estimator, the unbiased Precision matrix estimator is given by taking it to the limit:

\[ (\hat{\Theta}_u)^{k+1}_{\text{unbiased}} \leftarrow \lim_{l \to \infty} (2\hat{\Theta}_u^{(k,l)} - \hat{\Theta}_u^{(k)} \Psi_u^{(k)} \hat{\Theta}_u^{(k,l)}) \]  

(J-3)

Considering the unstandardization formula of the Precision matrix by its Standard estimator update \( \hat{\Theta}_u^{(k+1,l)} \) at the \( (l+1) \)-th iteration of the inner cycle, the Local Quadratic Approximation Precision matrix estimator update \( \hat{\Theta}_u^{(k,l+1)} \) is computed as:

\[ \hat{\Theta}_u^{(k,l+1)} = \hat{\Theta}_u^{(k,l)} \circ \hat{\Theta}_u^{(k,l+1)} \]  

(J-4)

By substituting in formula [H.4] the Standard estimator \( \hat{\Theta}_u^{(k,l+1)} \) given in formula (l-11) with Standard ESEC \( \Psi_u^{(k)} \circ \hat{\Theta}_u^{(k,l)} \) we obtain:
\[ \hat{\Theta}_u^{(k,l+1)} = \frac{1}{2\lambda} \hat{P}^{(k,l)} \odot \left( \sqrt{(\Psi_u^{(k)})^{-1} \odot \hat{P}^{(k,l)}} \right)^{-2} - 4\lambda \hat{q} - \left( \Psi_u^{(k)} - 1 \odot \hat{P}^{(k,l)} \right)^{-1} \] (I-5)

The solution to the Weights estimator updates at the \((l + 1)\)-th iteration of the inner cycle can be computed by (G-13), after substituting the Precision matrix estimator update \(\hat{\Theta}_u^{(k,l+1)}\).

\[ \hat{P}^{(k,l)} \leftarrow -1_q + \left( 1_q + 4(\lambda \sigma)^2 \odot \text{abs}(\hat{\Theta}_u^{(k,l)})^2 \right)^{\frac{1}{2}} \odot \left( 2^2(\lambda \sigma)^2 A \right) \] (I-6)

**K-Cross-spectral formulation of the Elastic Net Structured Sparse Bayesian Learning**

The results for complex LASSO of Section F can also be extended to the complex ENET, by the modification of Andrews and Mallows Lemma for Real ENET Gibbs pdf (Gaussian-Laplace). For the Real/Complex case the integral representation holds

\[ e^{-\frac{\left| a \right|^2}{4a^2}} \propto \int N_i(|z|,0,f(\tau)) TGamma \left( \tau \frac{1}{2}, 1, \left( \frac{a^2}{4a^2}, \infty \right) \right) d\tau \]

where the variance is defined as \(f(\tau) = \frac{1}{2a^2}(1 - \frac{a^2}{4a^2})\). The measurable space in which the variable \(z|\tau\) is defined has an unnormalized density function given by the Gaussian pdf \(p(z|\tau) = N_i(|z|,0,f(\tau))\) and its variance \(f(\tau)\) is dependent on the random variable \(x\) which has Truncated Gamma pdf \(p(\tau) = TGamma \left( \tau \frac{1}{2}, 1, \left( \frac{a^2}{4a^2}, \infty \right) \right)\). So, the measure in the space product of \(z\) and \(\tau\) is has density represented as an unnormalized product of Gaussian and Gamma densities \(p(z,\tau) \propto N_i(|z|,0,f(\tau)) TGamma \left( \tau \frac{1}{2}, 1, \left( \frac{a^2}{4a^2}, \infty \right) \right)\).

If for all parameter component we define the group penalization across samples \(z = \sqrt{\sum_{m=1}^{M} |(\tau_m)|^2}\) and \(f(\tau) = (\sigma_0^2)\). Then the transformed prior of the parameters is described analytically by the following prior distributions: \((\tau_m) \sim N_i(|(\tau_m)|,0,(\sigma_0^2)),\) \(\gamma_i \sim TGa \left( \gamma_i \frac{1}{2}, 1, \left( \frac{a^2}{4a^2}, \infty \right) \right)\). The full vector Bayesian model is as follows:

\[ v_m \sim N^C(v_m | L_u, \sigma_0^2 I) \quad \text{"Likelihood"} \quad (K-1) \]

\[ t_m \sim N_i(t_m | 0, \text{diag}(\sigma_0^2)) \quad \text{"Parameters prior"} \quad (K-2) \]

\[ \gamma \sim \prod_i TGa \left( \gamma_i \frac{1}{2}, 1, \left( \frac{a^2}{4a^2}, \infty \right) \right) \quad \text{"Hyperparameters prior"} \quad (K-3) \]

**Proposition K-1**

Let us define \(\mu_m = \hat{\Sigma}_u L_u^T (\sigma_0^2 I)^{\frac{1}{2}} v_m\) where \(\hat{\Sigma}_u = (L_u^T (\sigma_0^2 I)^{-1} L_u + \frac{1}{2} (\text{diag}(\sigma_0^2))^{-1})^{-1}\) then for the joint distribution of data and parameters, \(p(v_m, t_m) = N^C(v_m | L_u t_m, \sigma_0^2 I) N_i(t_m | 0, \text{diag}(\sigma_0^2))\) then the joint distribution of data and parameters, \(p(v_m, t_m) = N^C(v_m | L_u t_m, \sigma_0^2 I) N_i(t_m | 0, \text{diag}(\sigma_0^2))\), the following factorization holds:

\[ N^C(v_m | L_u t_m, \sigma_0^2 I) N_i(t_m | 0, \text{diag}(\sigma_0^2)) = |\pi \hat{\Sigma}_u N^C(t_m | \mu_m, \Sigma_u) N^C(v_m | L_u \mu_m, \sigma_0^2 I) N_i(t_m | 0, \text{diag}(\sigma_0^2)) \] (K-4)

**Proof of Proposition K-1**

Writing explicitly the distributions given in formula (K-4):

\[ N^C(v_m | L_u t_m, \sigma_0^2 I) N_i(t_m | 0, \text{diag}(\sigma_0^2)) = \frac{1}{\Gamma(\frac{1}{2})} e^{-\frac{1}{2} \left| v_m - L_u t_m \right|^2 (\sigma_0^2 I)} \frac{1}{2^{\frac{1}{2}} |\text{diag}(\sigma_0^2)|^{\frac{1}{2}}} e^{-\frac{1}{2} \left| t_m \right|^2 (\sigma_0^2 I)} \] (K-5)

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The form of the resultant distribution can be found by analyzing the terms that depend on the parameters (exponential argument) in formula (K-5):

$$-v_m^+ (\sigma^2 \xi)^{-1} v_m - t_m^+ L_w^+ (\sigma^2 \xi)^{-1} L_w t_m + v_m^+ (\sigma^2 \xi)^{-1} L_w v_m + t_m^+ L_w^+(\sigma^2 \xi)^{-1} v_m - \frac{1}{2} t_m^+ (\text{diag}(\sigma^2))^{-1} t_m \quad \text{(K-6)}$$

Reorganizing the exponent (K-6) in terms of $\widehat{\mu}_m$ and $\widehat{\Sigma}_u$ we obtain.

$$-(t_m - \widehat{\mu}_m)^+ \widehat{\Sigma}_u^{-1} (t_m - \widehat{\mu}_m) - v_m^+ (\sigma^2 \xi)^{-1} v_m + \widehat{\mu}_m^+ \widehat{\Sigma}_u^{-1} \widehat{\mu}_m \quad \text{(K-7)}$$

Completing terms (K-7) with the terms: $-v_m^+ (\sigma^2 \xi)^{-1} L_w \widehat{\mu}_m - \widehat{\mu}_m^+ L_w^+ (\sigma^2 \xi)^{-1} L_w v_m$ and $+ \widehat{\mu}_m^+ L_w^+(\sigma^2 \xi)^{-1} L_w \widehat{\mu}_m$ we obtain.

$$-(t_m - \widehat{\mu}_m)^+ \widehat{\Sigma}_u^{-1} (t_m - \widehat{\mu}_m) - (v_m - L_w \widehat{\mu}_m)^+ (\sigma^2 \xi)^{-1} (v_m - L_w \widehat{\mu}_m) - v_m^+ (\sigma^2 \xi)^{-1} L_w \widehat{\mu}_m - \widehat{\mu}_m^+ L_w^+(\sigma^2 \xi)^{-1} v_m + \widehat{\mu}_m^+ L_w^+(\sigma^2 \xi)^{-1} L_w \widehat{\mu}_m \quad \text{(K-8)}$$

In formula (K-8) the third and fourth term can be added with the sixth: due to $L_w^+(\sigma^2 \xi)^{-1} v_m = \widehat{\Sigma}_u^{-1} \widehat{\mu}_m$.

$$-(t_m - \widehat{\mu}_m)^+ \widehat{\Sigma}_u^{-1} (t_m - \widehat{\mu}_m) - (v_m - L_w \widehat{\mu}_m)^+ (\sigma^2 \xi)^{-1} (v_m - L_w \widehat{\mu}_m) - \widehat{\mu}_m^+ \widehat{\Sigma}_u^{-1} \widehat{\mu}_m + \widehat{\mu}_m^+ L_w^+(\sigma^2 \xi)^{-1} L_w \widehat{\mu}_m \quad \text{(K-9)}$$

Finally, rearranging terms in (K-9) we obtain.

$$-(t_m - \widehat{\mu}_m)^+ \widehat{\Sigma}_u^{-1} (t_m - \widehat{\mu}_m) - (v_m - L_w \widehat{\mu}_m)^+ (\sigma^2 \xi)^{-1} (v_m - L_w \widehat{\mu}_m) - \frac{1}{2} \widehat{\mu}_m^+ (\text{diag}(\sigma^2))^{-1} \widehat{\mu}_m \quad \text{(K-10)}$$

From (K-10) it holds that:

$$N^C(v_m | L_w t_m, \sigma^2 \xi) \prod N(\mu_m | \widehat{\mu}_m, \widehat{\Sigma}_u) N^C(v_m | L_w \widehat{\mu}_m, \sigma^2 \xi) \prod N(\widehat{\mu}_m | 0, \text{diag}(\sigma^2)) \equiv \text{(K-11)}$$

The estimation formulas can be derived by applying maximum posterior analysis to the hyperparameters, as in (Paz-Linares et al., 2017). Not the changes in notation used here with respect to the latter manuscript: $\Lambda := \sigma^2 \xi$, $\widehat{\Lambda} := \sigma^2 \xi$, $\beta := \sigma^2 \xi$, $a_1 := a_2$, $a_2 := a_1$, $r := \frac{a_1^2}{a_2}$, $\Sigma := \widehat{\Sigma}_u$, $S := q$. With starting hyperparameters $\alpha^{(0)} \xi$, $r^{(0)}$, $(\sigma^2 \xi)^{(0)}$ and $(\sigma^2 \xi)^{(0)}$ the estimators are as follows:

$$S_{vv} = \frac{1}{m} \sum_{m=1}^{m} v_m v_m^+ \quad \text{(K-12)}$$

$$\Sigma_u^{(k+1)} \leftarrow \left( (\sigma^2 \xi)^{-k} L_w^+ L_w + \frac{1}{2} (\text{diag}(\sigma^2 \xi)^{(k)})^{-1} \right)^{-1} \quad \text{(K-13)}$$

$$S_{\mu \mu}^{(k+1)} \leftarrow \frac{1}{m} \sum_{m=1}^{m} \widehat{\mu}_m (\widehat{\mu}_m)^+ \left( (\sigma^2 \xi)^{-k} \Sigma_u^{(k+1)} L_w^+ S_{vv} L_w \Sigma_u^{(k+1)} \right) \quad \text{(K-14)}$$

$$\sigma^2 \xi^{(k+1)} = \frac{1}{2\sigma^2 \xi} \left( \sigma^2 \xi \right)^{(k+1)} \quad \text{(K-15)}$$

Where $(\sigma^2 \xi)^{(k+1)} = \eta_i^{(k+1)}/(r^{(k)} + \eta_i^{(k+1)})$, for $i = 1: q$, and $\eta_i^{(k+1)}$ expressed as:
\[ \eta_{i}^{(k+1)} = -\frac{1}{4} + \frac{1}{16} + \left( \left( \frac{\epsilon_{\mu u}^{(k+1)}}{\bar{\epsilon}_{u}^{(k+1)}} \right)_{i} + \left( \bar{\epsilon}_{u}^{(k+1)} \right)_{i} \right) a_{2}^{(k)} r^{(k)}, \text{ for } i = 1: q \]  
\[ (K-16) \]

\[ a_{2}^{(k+1)} = \left( \frac{q}{2} \right) / \sum_{i=1}^{q} \left( \frac{\epsilon_{\mu u}^{(k+1)} + \bar{\epsilon}_{u}^{(k+1)}}{\bar{\epsilon}_{u}^{(k+1)}} \right) \]  
\[ (K-17) \]

\[ r^{(k)} = \text{argmin}|F(r)| \]  
\[ (K-18) \]

\[ F(r) = \sum_{i=1}^{q} \left( \frac{1}{1 - (\bar{\sigma}_{i}^{(k+1)})^{2}} \right) + v - \left( \frac{r - \frac{a}{2}}{r} - q(\pi r)^{-1/2} e^{-r} / \int_{0}^{\infty} \text{Ga}(x \mid \frac{1}{2}, 1) d x \right) \]  
\[ (K-19) \]

After convergence of the ENET-SSBL we can threshold the estimated source activity by constraining a biased statistic: this is due to the posterior distribution of source activity \( N^{C}(\bar{\mu}_{m}, \Sigma_{u}) \). In this distribution \( \bar{\mu}_{m} \) is the posterior mean and \( \Sigma_{u} \) the posterior covariance. The z-statistic for the analysis of variance has the following form:

\[ z_{\text{stat}} = \text{sqrt} \left( \text{diag} \left( \bar{\epsilon}_{u}^{(\omega)} / (\bar{\sigma}_{k}^{(\omega)}) \right) \right) \]  
\[ (K-20) \]

A plausible way to screen out the active sources is to extract the set of nodes \( J \) that return a value of the z-statistic greater than 1:

\[ J = \{ i : z_{\text{stat}}_{i} \geq 1 \} \]  
\[ (K-21) \]

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