DISJOINTLY NON-SINGULAR OPERATORS ON BANACH LATTICES

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Abstract. An operator $T$ from a Banach lattice $E$ into a Banach space is disjointly non-singular (DN-S, for short) if no restriction of $T$ to a subspace generated by a disjoint sequence is strictly singular. We obtain several results for DN-S operators, including a perturbative characterization. For $E = L_p$ ($1 < p < \infty$) we improve the results, and we show that the DN-S operators have a different behavior in the cases $p = 2$ and $p \neq 2$. As an application we prove that the strongly embedded subspaces of $L_p$ form an open subset in the set of all closed subspaces.

1. Introduction

Inspired by the study of tauberian operators $T : L_1 \to Y$ using Banach lattice techniques in [5], we introduce the disjointly non-singular operators from a Banach lattice $E$ into a Banach space $Y$, denoted DN-S$(E, Y)$, and the dispersed subspaces of $E$. Note that $T : L_1 \to Y$ is tauberian if and only if $T \in$ DN-S$(L_1, Y)$.

We give several characterizations of the operators in DN-S$(E, Y)$ in terms of the action on the normalized disjoint sequences in $E$. We show that an operator $K : E \to Y$ is disjointly strictly singular [11] if and only if $M$ is dispersed when the restriction $K|_M$ is an isomorphism, that each $T \in$ DN-S$(E, Y)$ has dispersed kernel $N(T)$, and that $T$ in DN-S and $K$ disjointly strictly singular imply $T + K$ in DN-S. We prove a perturbative characterization: $T : E \to Y$ is in DN-S if and only if $N(T + K)$ is dispersed for every compact operator $K : E \to Y$, and we show that $T : E \to Y$ with closed range is in DN-S if and only if $N(T)$ is dispersed.

To study the operators in DN-S$(L_p, Y)$ ($1 < p < \infty$), we show that the dispersed subspaces of $L_p$ ($1 \leq p < \infty$) coincide with the strongly embedded subspaces [2] Definition 6.4.4], and that these subspaces form an open subset in the set of all closed subspaces of $L_p$ with respect to the gap metric. Since for $p \neq 2$ a closed subspace of $L_p$ is strongly embedded if and only if it does not contain copies of $\ell_p$ [2 Theorems 6.4.8 and 7.2.6], but this is not true for $p = 2$, these two cases have a different behavior. In fact, there are decompositions of $L_2$ as a direct sum of two strongly embedded subspaces [17, Theorem 8.22], but this is not true for $p \neq 2$. We show that these decompositions of $L_2$ are stable under small perturbations with respect to the gap metric. Moreover DN-S$(L_p)$ is stable under products for $p \neq 2$, but DN-S$(L_2)$ is not. We also prove that DN-S$(L_p, Y)$ is non-empty if and only if $Y$ contains a subspace isomorphic to $L_p$, that the disjointly strictly singular operators form the perturbation class of DN-S$(L_p, Y)$, and that there exist non-trivial examples of operators in DN-S$(L_p)$ for $1 < p < \infty$.

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Throughout the paper, $E$ and $F$ will be Banach lattices and $X$, $Y$ and $Z$ will be Banach spaces. We denote by $L(X,Y)$ the space of all (continuous linear) operators from $X$ into $Y$ and we write $L_p$ is the space $L_p(0,1)$ for $1 \leq p \leq \infty$, and we denote by $[x_n] \equiv [x_n : n \in \mathbb{N}]$ the closed subspace generated by a sequence $(x_n)$ in $X$.

An operator $T \in L(X,Y)$ is upper semi-Fredholm if it has closed range $R(T)$ and finite dimensional kernel $N(T)$. The operator $T$ is strictly singular if there is no closed infinite dimensional subspace $M$ of $X$ such that the restriction $T|_M$ is an isomorphism. The operator $T$ is $Z$-singular if there exists no subspace $M$ of $X$ isomorphic to $Z$ such that the restriction $T|_M$ is an isomorphism. The $\ell_1$-singular operators form an operator ideal and they are called Rosenthal operators by Pietsch in [16].

An operator $T \in L(X,Y)$ is called tauberian if the second conjugate $T^{**} \in L(X^{**}, Y^{**})$ satisfies $(T^{**})^{-1}(Y) = X$. Introduced by Kalton and Wilansky in [13], these operators are useful in several topics of Banach space theory like factorization of operators, equivalence between the KMP and the RNP, and refinements of James’ characterization of reflexive Banach spaces. We refer to [6] for information on tauberian operators.

2. Disjointly non-singular operators

We say that a sequence $(x_n)$ in $E$ is almost disjoint if there exists a normalized disjoint sequence $(y_n)$ in $E$ such that $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

**Definition 2.1.** A closed subspace $M$ of $E$ is said to be dispersed if it does not contain almost disjoint sequences.

Recall that each disjoint sequence of non-zero vectors in $E$ is an unconditional basic sequence.

**Lemma 2.2.** Let $M$ be a closed subspace of $X$ and let $(x_n)$ be a basic sequence in $X$.

1. If $M \cap [x_n]$ is finite dimensional, then $M \cap [x_n : n > n_0] = \{0\}$ for some $n_0 \in \mathbb{N}$.
2. If, additionally, $M + [x_n]$ is closed then so is $M + [x_n : n > n_0]$.

**Proof.** (1) Let $(x_n^*)$ be a sequence in $X^*$ such that $x_i^*(x_j) = \delta_{i,j}$ for $i, j \in \mathbb{N}$ and suppose that $\dim M \cap [x_n] = k$. If $0 \neq x \in M \cap [x_n]$ there exists $l \in \mathbb{N}$ such that $x_l^*(x) \neq 0$. Thus $x \notin M \cap [x_n : n > l]$, hence $\dim M \cap [x_n : n > l] < k$, and repeating the process we get the desired $n_0 \in \mathbb{N}$.

(2) Note that $M + [x_n : n > n_0]$ is the sum of $(M + [x_n]) \cap (\cap_{k=1}^{n_0} N(x_k^*))$ and a finite dimensional subspace. \hfill \Box

The argument in the proof of the following result will be used several times.

**Lemma 2.3** (Perturbation lemma). Let $M$ and $N$ be closed subspaces of $X$, and let $0 < \varepsilon < 1/2$. If $M + N$ is not closed, then there exists a compact operator $K \in L(X)$ with $\|K\| < \varepsilon$ and a closed infinite dimensional subspace $M_1$ of $M$ such that $(I-K)(M_1) \subset N$, $I - K$ is invertible and $(I-K)^{-1} = I + L$ with $\|L\| < 2\varepsilon$.

**Proof.** Since the sum $M + N$ is not closed, both subspaces $M$ and $N$ are infinite dimensional, and we can select a normalized sequence $(m_k)$ in $M$, and sequences $(n_k)$ in $N$ and $(x_k^*)$ in $X^*$ so that $x_k^*(m_j) = \delta_{i,j}$ and $\|x_k^*\| \|m_k - n_k\| < \varepsilon 2^{-k}$ for $i, j, k \in \mathbb{N}$. See the proof of [7 Proposition].
Thus $Kx = \sum_{k=1}^{\infty} x_k^* (x)(mk - nk)$ defines a compact operator $K \in L(X)$ such that $(I - K)m_i = n_i$ for each $i \in \mathbb{N}$ and $\|K\| \leq \sum_{k=1}^{\infty} \|x_k^*\|\|mk - nk\| < \varepsilon$. Hence $I - K$ is an invertible operator and we can take $M_1 = [m_k]$.

For the last part, apply $(I - K)^{-1} = I + \sum_{n=1}^{\infty} K^n$. \qed

Next we characterize the dispersed subspaces.

**Proposition 2.4.** A closed subspace $M$ of $E$ is dispersed if and only if, for every disjoint sequence of non-zero vectors $(x_n)$ in $E$, $M \cap [x_n]$ is finite dimensional and $M + [x_n]$ is closed.

**Proof.** Suppose that there is a disjoint sequence of non-zero vectors $(x_n)$ in $E$ such that $M \cap [x_n]$ is infinite dimensional, or $M \cap [x_n]$ finite dimensional and $M + [x_n]$ is not closed. In both cases we can construct a normalized block-basis $(y_k)$ of $(x_n)$ with $\lim_{k \to \infty} \text{dist}(y_k, M) = 0$. Since $(y_k)$ is a disjoint sequence, $M$ is not dispersed.

Conversely, suppose that for every disjoint sequence of non-zero vectors $(x_n)$ in $E$, $M \cap [x_n]$ is finite dimensional and $M + [x_n]$ is closed. Given a normalized disjoint sequence $(x_n)$ in $E$, by Lemma 2.2 we have $M \cap [x_n : n \geq n_0] = \{0\}$ for some $n_0$. Then $\lim_{n \to \infty} \text{dist}(x_n, M) > 0$; hence $M$ contains no almost disjoint sequence. \qed

The following class of operators was introduced in [11].

**Definition 2.5.** An operator $T \in L(E,Y)$ is said to be dispersively strictly singular if there is no disjoint sequence of non-zero vectors $(x_n)$ in $E$ such that the restriction of $T|_{[x_n]}$ is an isomorphism.

We denote $\text{DSS}(E,Y) := \{ T \in L(E,Y) : T \text{ is dispersively strictly singular} \}$.

For $1 \leq q < p < \infty$, the natural inclusion $L_p \to L_q$ is in $\text{DSS}$ [11], but it is not strictly singular because $\| \cdot \|_p$ and $\| \cdot \|_q$ are equivalent in the subspace generated by the Rademacher functions. Also, $\text{DSS}(E,Y)$ is a closed subspace of $L(E,Y)$ [10].

**Proposition 2.6.** For $T \in L(E,Y)$, the following assertions are equivalent:

1. $T$ is dispersively strictly singular.
2. Given a closed subspace $M$ of $E$, if $T|_M$ is an isomorphism then $M$ is dispersed.
3. For every disjoint sequence of non-zero vectors $(x_n)$ in $E$, $T|_{[x_n]}$ is strictly singular.

**Proof.** (1)$\Rightarrow$(2) Suppose that $M$ is a non-dispersed closed subspace of $E$ such that $T|_M$ is an isomorphism. Then there exists a normalized disjoint sequence $(x_k)$ in $E$ with $\lim_{k \to \infty} \text{dist}(x_k, M) = 0$. As in the proof of Lemma 2.3, we take a bounded sequence $(x_n)$ in $E$ such that $x_i^*(x_j) = \delta_{i,j}$ and, passing to a subsequence of $(x_k)$, we can find a sequence $(m_k)$ in $M$ so that $\sum_{k=1}^{\infty} \|x_k\| \cdot \|x_k - m_k\| < 1/2$. Thus $Kx = \sum_{k=1}^{\infty} x_k^* (x)(x_k - m_k)$ defines $K \in L(E)$ with $\|K\| < 1/2$. Hence $I - K$ is an isomorphism and $(I - K)x_k = m_k$.

Denoting $S = T(I - K)$, since $(I - K)([x_k]) = [m_k]$ and $T|_{[m_k]}$ is an isomorphism, $S|_{[x_k]}$ is also an isomorphism, thus $T|_{[x_k]} = S|_{[x_k]} + TK|_{[x_k]}$ is upper semi-Fredholm because $K$ is compact. Since $N(T|_{[x_k]})$ is finite dimensional, Lemma 2.2 implies that $T|_{[x_k : k \geq k_0]}$ is an isomorphism for some $k_0 \in \mathbb{N}$, hence $T$ is not disjointly strictly singular.

(2)$\Rightarrow$(3) Suppose that $(x_n)$ is a disjoint sequence of non-zero vectors in $E$ and $T|_{[x_n]}$ is not strictly singular. Then there exists an infinite dimensional closed subspace $M$ of
\[ x_n \] such that \( T|_M \) is an isomorphism, and \( M \) is not dispersed because there exists a normalized block basis \((y_k)\) of \((x_n)\) such that \( \text{dist}(y_k, M) < 2^{-k} \).

(3)\(\Rightarrow\)(1) Strictly singular operators are never isomorphisms on infinite dimensional subspaces. \( \square \)

**Definition 2.7.** We say that \( T \in L(E, Y) \) is disjointly non-singular if there is no disjoint sequence of non-zero vectors \((x_n)\) in \( E \) such that \( T|_{[x_n]} \) is strictly singular.

We denote \( \text{DN-S}(E,Y):= \{ T \in L(E,Y) : T \text{ is disjointly non-singular} \} \).

**Theorem 2.8.** For \( T \in L(E,Y) \), the following assertions are equivalent:

1. \( T \) is disjointly non-singular.
2. There is no disjoint sequence of non-zero vectors \((x_n)\) in \( E \) such that \( T|_{[x_n]} \) is compact.
3. For every disjoint sequence of non-zero vectors \((x_n)\) in \( E \), \( T|_{[x_n]} \) is upper semi-Fredholm.
4. For every normalized disjoint sequence \((x_n)\) in \( E \), \( \lim \inf_{n \to \infty} \|Tx_n\| > 0 \).

**Proof.** (1)\(\Rightarrow\)(2) is immediate, and for (3)\(\Rightarrow\)(4) observe that, if \( T|_{[x_n]} \) is upper semi-Fredholm, then Lemma 2.2 implies that \( T|_{[x_n:n>n_0]} \) is an isomorphism for some \( n_0 \).

(2)\(\Rightarrow\)(3) If \((x_n)\) is a disjoint sequence of non-zero vectors in \( E \) and \( T|_{[x_n]} \) is not upper semi-Fredholm, then there exists an infinite dimensional closed subspace \( M \) of \([x_n] \) such that \( T|_M \) is compact [1, Theorem 7.16]. Taking a normalized block basis \((y_k)\) of \((x_n)\) such that \( \text{dist}(y_k, M) < 2^{-k} \), the argument of Lemma 2.3 allows us to show that \( T|_{[y_k]} \) is compact.

(4)\(\Rightarrow\)(1) If \((x_n)\) is a disjoint sequence of non-zero vectors in \( E \) such that \( T|_{[x_n]} \) is strictly singular, then we can construct a normalized block basis \((y_k)\) of \((x_n)\) such that \( \lim_{k \to \infty} \|Ty_k\| = 0 \).

It was proved in [5, Theorem 2] that \( T \in L(L_1,Y) \) is tauberian if and only if it satisfies (4) in Theorem 2.8.

**Corollary 2.9.** Let \( T, K \in L(E,Y) \).

1. If \( T \in \text{DN-S} \) and \( K \in \text{DSS} \), then \( T + K \in \text{DN-S} \).
2. If \( T \in \text{DN-S} \), then \( N(T) \) is dispersed.

**Proof.** (1) is a consequence of Proposition 2.6, Theorem 2.8 and the fact that the class of upper semi-Fredholm operators is stable under the addition of strictly singular operators [1, Theorem 7.46].

(2) If \( T \in \text{DN-S} \) and \((x_n)\) is a disjoint sequence of non-zero vectors in \( E \) then \( T|_{[x_n]} \) is upper semi-Fredholm. Hence \([x_n] \cap N(T)\) is finite dimensional and \([x_n] + N(T)\) is closed, and the result follows from Proposition 2.4. \( \square \)

Next we give a perturbative characterization of disjointly non-singular operators. Similar characterizations can be found in [1, Theorem 7.16] for upper semi-Fredholm operators, and in [8] for tauberian operators.

**Theorem 2.10** (Perturbative characterization). An operator \( T \in L(E,Y) \) is disjointly non-singular if and only if \( N(T+K) \) is dispersed for every compact operator \( K \in L(E,Y) \).
Question 2. Suppose that the implication is valid for \(X\) by \(T \in \text{Theorem 2.10}\) imply the first part, and Proposition 2.6 implies the second part.

Proposition 2.12. Let \(X = Y\) if and only if \(N\).

Proof. We prove the countable case, since the proof of the general case is similar. Let \((x_n)\) be a 1-unconditional basis of \(E\) inducing its Banach lattice structure. A standard block-basis argument shows that for every closed infinite dimensional subspace \(M\) of \(E\) there is a normalized block basis \((y_k)\) such that \(\lim_{k \to \infty} \text{dist}(y_k, M) = 0\). Since \((y_k)\) is disjoint, we get that dispersed subspaces of \(E\) are finite dimensional. Thus the perturbative characterizations of the upper semi-Fredholm operators and DN-S(\(E, Y\)) ([11, Theorem 7.16] and Theorem 2.10) imply the first part, and Proposition 2.6 implies the second part. \(\square\)

For operators with closed range there is a simpler characterization of \(T \in \text{DN-S}\).

Proposition 2.12. Let \(T \in L(E, Y)\) with closed range. Then \(T\) is disjointly non-singular if and only if \(N(T)\) is dispersed.

Proof. The direct implication is proved in Corollary 2.9. For the converse one, suppose that \(T\) is not disjointly non-singular. By Theorem 2.8 there exists a normalized disjoint sequence \((x_n)\) in \(E\) such that \(\|Tx_n\| < 2^{-n}\). Since \(R(T)\) is closed there exists a constant \(C > 0\) so that \(\text{dist}(x_n, N(T)) < C \cdot 2^{-n}\), hence \(N(T)\) is not dispersed. \(\square\)

In the case \(\text{DN-S}(L_1, Y) \neq \emptyset\), it was proved in [5, Proposition 14] that \(K \in L(L_1, Y)\) is \(\ell_1\)-singular if and only if \(T + K \in \text{DN-S}\) for every \(T \in \text{DN-S}(L_1, Y)\). This means that \(\text{DSS}(L_1, Y)\) is the perturbation class of \(\text{DN-S}(L_1, Y)\). Moreover \(\text{DN-S}(L_1, Y)\) is an open subset of \(L(L_1, Y)\).

Question 1. Suppose that \(\text{DN-S}(E, Y)\) is non-empty.

(a) Is \(\text{DSS}(E, Y)\) the perturbation class of \(\text{DN-S}(E, Y)\)?

(b) Is \(\text{DN-S}(E, Y)\) an open subset of \(L(E, Y)\)?

In general, \(T \in L(X, Y)\) tauberian does not imply \(T^{**} \in L(X^{**}, Y^{**})\) tauberian [3], but the implication is valid for \(X = L_1\) [5, Corollary 9].

Also each operator \(T \in L(X, Y)\) induces an operator \(T^{co} \in L(X^{**}/X, Y^{**}/Y)\), defined by \(T^{co}(x^{**} + X) = T^{**}x^{**} + Y\), which is called the residuum operator. An operator \(T \in L(L_1, Y)\) is tauberian if an only if \(T^{co}\) is an isomorphism [5, Proposition 11].

Question 2. Suppose that \(E\) is non-reflexive and \(T \in \text{DN-S}(E, Y)\).

(a) Is \(T^{**} \in \text{DN-S}\)?

(b) Is \(T^{co}\) an isomorphism?
For information on the residuum operator $T^\infty$ we refer to [3] Section 3.1 and [9].

3. DISJOINTLY NON-SINGULAR OPERATORS ON $L_p$

Here we study the disjointly non-singular operators on $L_p$ ($1 < p < \infty$). The case $p = 1$ was studied in [5].

For $T \in L(E, Y)$ we consider the following quantity:

$$\beta(T) := \inf \{ \liminf_{n \to \infty} \|Tx_n\| : (x_n) \text{ normalized disjoint sequence in } E \},$$

and for a measurable function $f$ on $(0, 1)$ we denote $D(f) = \{ t \in (0, 1) : f(t) \neq 0 \}$.

**Proposition 3.1.** An operator $T \in L(L_p, Y)$ ($1 < p < \infty$) is disjointly non-singular if and only if $\beta(T) > 0$.

**Proof.** Clearly $\beta(T) > 0$ implies (4) in Theorem 2.8.

Conversely, suppose that $\beta(T) = 0$. Then for every $k \in \mathbb{N}$ there exists a normalized disjoint sequence $(f_n^k)_{n \in \mathbb{N}}$ in $L_p$ such that $\|Tf_n^k\| < 1/k$ for every $n \in \mathbb{N}$. We denote $D(k, n) = \{ t \in (0, 1) : f_n^k(t) \neq 0 \}$, and we take $n_1 = 1$ and $g_1 = f_1^1$. Since $\lim_{n \to \infty} \mu(D(k, n)) = 0$, we can select $n_2 > 1$ such that $\|g_1\chi_{D(2, n_2)}\|_p < 2^{-2}$, and take $g_2 = f_{n_2}^2$. Proceeding in this way we obtain $1 = n_1 < n_2 < \cdots$ in $\mathbb{N}$ so that $g_k = f_{n_k}^k$ satisfies $\|g_k\chi_{D(g_k)}\|_p < 2^{-k\cdot p}$ for $i < k$. We denote $A_k = D(g_k) \backslash \cup_{n > k} D(g_n)$.

Taking $f_k = g_k\chi_{A_k}$ we obtain a disjoint sequence $(f_k)$ satisfying

$$1 \geq \|f_k\|_p^p \geq 1 - \sum_{n > k} \|g_k\chi_{D(g_n)}\|_p^p \geq 1 - \sum_{n > k} 2^{-2k} \geq 1/2$$

and

$$\|Tf_k\| \leq \|Tg_k\| + \|T\| \sum_{n > k} \|g_k\chi_{D(g_n)}\|_p \leq 1/k + 2^{-kp}\|T\|.$$
(2) For $p > 2$, a closed infinite dimensional subspace of $L_p$ is strongly embedded if and only if it is isomorphic to $\ell_2$.

(3) For $p < 2$, the set of strongly embedded subspaces of $L_p$ include isomorphic copies of $L_q$ for every $q \in (p, 2]$.

(4) There exists an orthogonal decomposition $L_2 = M \oplus M^\perp$ with both $M$ and $M^\perp$ strongly embedded in $L_2$.

(5) For $p \neq 2$, we cannot write $L_p$ as the direct sum of two strongly embedded subspaces.

**Proof.** We refer to [2] Theorems 6.4.8 and 7.2.6] for (1), [2] Theorem 6.4.8] for (2), and [15] Corollary 2.f.5 for (3). Moreover, (4) is [17] Theorem 8.22, and (5) follows from the fact that containing no copy of $\ell_p$ is stable under direct sums.

For additional information on strongly embedded subspaces of $L_p$ we refer to [1], where they are called $\Lambda(p)$-spaces.

Next result is a special case of the Kadec-Pelczyński dichotomy as stated in [15] Proposition 1.c.8. We give an alternative proof.

**Proposition 3.5.** For $1 \leq p < \infty$, a closed subspace $M$ of $L_p$ is strongly embedded if and only if it is dispersed.

**Proof.** Suppose that $M$ is strongly embedded, and let $(f_n)$ be a disjoint sequence of non-zero vectors in $L_p$. For $p \neq 2$, since $M$ contains no copy of $\ell_p$ (Proposition 3.4) and $[f_n]$ is isomorphic to $\ell_p$, $M$ and $[f_n]$ are totally incomparable, hence $M \cap [f_n]$ is finite dimensional and $M + [f_n]$ is closed (see [7]). Thus $M$ is dispersed by Proposition 2.4.

For $p = 2$, it follows from Proposition 3.3 that $\| \cdot \|_2$ and $\| \cdot \|_1$ are equivalent on $M$. But, since every operator in $L(\ell_2, \ell_1)$ is compact [2] Theorem 2.1.4, the norms $\| \cdot \|_2$ and $\| \cdot \|_1$ are equivalent in no infinite dimensional subspace of $[f_n]$, hence $M \cap [f_n]$ is finite dimensional. Moreover, the argument in the proof of Lemma 2.3 allows us to show that $M + [f_n]$ is closed.

Conversely, if $M$ is not strongly embedded, arguing as in the proof of [2] Theorem 6.4.7], we get a normalized sequence $(g_n)$ in $M$ and a sequence of disjoint sets $(A_n)$ such that $\|g_n - g_n\chi_{A_n}\|_p \to 0$. Then $(g_n)$ is almost disjoint, hence $M$ is not dispersed. □

**Remark 3.6.** In $L_\infty$, there are dispersed subspaces which are not strongly embedded.

Indeed, strongly embedded subspaces of $L_\infty$ are finite dimensional because they are reflexive and, for $1 \leq p < \infty$, the natural map from $L_\infty$ to $L_p$ takes weakly convergent sequences into convergent sequences [2] Theorem 5.4.5]. See [18] Theorem 5.2 for an alternative argument. However, each closed subspace of $L_\infty$ containing no copy of $c_0$ is dispersed because it is totally incomparable with the subspace generated by a disjoint sequence of non-zero vectors. See the proof of Proposition 3.5.

The following result is essentially known, but we give a proof of it for convenience.

**Proposition 3.7.** Let $1 \leq p < \infty$, $p \neq 2$. An operator $T \in L(L_p, Y)$ is disjointly strictly singular if and only if it is $\ell_p$-singular.

**Proof.** By Proposition 2.3, $T$ is disjointly strictly singular if and only given a closed subspace $M$ of $L_p$, $T|_M$ isomorphism implies $M$ dispersed. Moreover, Propositions 3.4 and 3.5 show that $M$ is dispersed in $L_p$ ($1 \leq p < \infty$, $p \neq 2$) if and only if $M$ contains no copy of $\ell_p$. These two facts imply the result. □
Note that $T \in L(L_2, Y)$ is $\ell_2$-singular if and only it is strictly singular.

**Proposition 3.8.** Let $1 < p < \infty$, $p \neq 2$. For $T \in L(L_p, Y)$, the following assertions are equivalent:

1. $T$ is disjointly non-singular.
2. $T|_M$ is upper semi-Fredholm for every subspace $M$ of $L_p$ isomorphic to $\ell_p$.
3. $T|_M$ is compact for no subspace $M$ of $L_p$ isomorphic to $\ell_p$.

**Proof.** (2)$\Rightarrow$(3) The direct implication is trivial. Conversely, if there is a subspace $M$ of $L_p$ isomorphic to $\ell_p$ such that $T|_M$ is not upper semi-Fredholm, then $M$ contains a closed infinite dimensional subspace $N$ such that $T|_N$ is compact, and $N$ contains a subspace isomorphic to $\ell_p$.

(1)$\Rightarrow$(3) Suppose that there is a subspace $M$ of $L_p$ isomorphic to $\ell_p$ such that $T|_M$ is compact. By (1) in Proposition 3.4, $M$ is not strongly embedded; hence there is a normalized disjoint sequence $(f_n)$ in $L_p$ with $\lim_{n \to \infty} \text{dist}(f_n, M) = 0$ by Proposition 3.5. Passing to a subsequence, Lemma 2.3 allows us to show that $T|_{\{f_n\}}$ is compact, hence $T$ is not disjointly non-singular.

(2)$\Rightarrow$(1) It is a consequence of Theorem 2.8, because the closed subspace generated by a disjoint sequence of non-zero vectors in $L_p$ is isomorphic to $\ell_p$. \[\square\]

For $p = 2$, each of the assertions (2) and (3) in Proposition 3.8 is equivalent to $T$ being upper semi-Fredholm.

The following result shows some differences between the properties of DN-$S(L_2)$ and those of DN-$S(L_p)$ for $p \neq 2$.

**Proposition 3.9.** Let $1 < p < \infty$, $p \neq 2$.

1. $S, T \in \text{DN-}S(L_p)$ implies $ST \in \text{DN-}S(L_p)$.
2. There exist $S, T \in \text{DN-}S(L_2)$ such that $ST = 0$.

**Proof.** (1) It follows from Proposition 3.8, $T \in \text{DN-}S(L_p)$ if and only if $T|_M$ is upper semi-Fredholm for every subspace $M$ isomorphic to $\ell_p$.

(2) We consider the decomposition $L_2 = M \oplus M^\perp$ with both $M$ and $M^\perp$ strongly embedded given in Proposition 3.4. Let $S$ denote the orthogonal projection on $L_2$ onto $M$ and let $T = I - S$. Since both $S$ and $T$ have closed range and dispersed kernel, $S, T \in \text{DN-}S(L_2)$ by Proposition 2.12. \[\square\]

Next we show that the dispersed subspaces of $L_p$ form an open subset of the set of all closed subspaces with respect to the gap metric [14, Chapter IV].

Let $M$ and $N$ be closed subspaces of $X$, and let us denote $S_M = \{m \in M : \|m\| = 1\}$. The *gap between $M$ and $N$* is defined by $g(M, N) = \max\{\delta(M, N), \delta(N, M)\}$, where $\delta(M, N) = \sup_{m \in S_M} \text{dist}(m, N)$.

**Proposition 3.10.** Given a dispersed subspace $M$ of $L_p$ ($1 \leq p < \infty$), there exists $\varepsilon > 0$ such that if $M_1$ is a closed subspace of $L_p$ and $\delta(M_1, M) < \varepsilon$, then $M_1$ is dispersed.

**Proof.** By Proposition 2.12 and [5] (case $p = 1$), the quotient map $Q : L_p \to L_p/M$ is in DN-$S$, hence $\beta(Q) > 0$. Thus, for every normalized disjoint sequence $(x_n)$ in $L_p$,

$$\liminf_{n \to \infty} \|Qx_n\| = \liminf_{n \to \infty} \text{dist}(x_n, M) \geq \beta(Q) > 0.$$
By [14, Lemma IV.4.2], \((1 + \delta(M_1, M))\dist(x_n, M_1) \geq \dist(x_n, M) - \delta(M_1, M)\). Thus we can take \(\varepsilon = \beta(Q)/2\).

As a consequence, we obtain a stability result for the decompositions of \(L_2\) as a direct sum of strongly embedded subspaces, like the one given in Proposition 3.4.

**Corollary 3.11.** Let \(M\) and \(N\) be strongly embedded subspaces of \(L_2\) with \(L_2 = M \oplus N\). Then there exists \(\varepsilon > 0\) such that if \(M_1\) and \(N_1\) are closed subspaces of \(L_2\), \(g(M_1, M) < \varepsilon\) and \(g(N_1, N) < \varepsilon\) imply \(L_2 = M_1 \oplus N_1\).

**Proof.** Applying [14, Theorem IV.4.24] twice, we can find \(\varepsilon > 0\) such that \(g(M_1, M) < \varepsilon\) and \(g(N_1, N) < \varepsilon\) imply \(L_2 = M_1 \oplus N_1\). And, by Proposition 3.10, we can choose \(\varepsilon\) so that additionally \(M_1\) and \(N_1\) are strongly embedded. \(\square\)

There are some other differences between DN-S\((L_2)\) and DN-S\((L_p)\) for \(p \neq 2\).

**Proposition 3.12.** Let \(1 < p < 2\) and let \(M\) be a dispersed subspace of \(L_p\). Then the quotient map \(Q_M : L_p \to L_p/M\) satisfies \(\beta(Q_M) = 1\).

**Proof.** We showed in Proposition 3.5 that, for \(1 \leq p < \infty\), a closed subspace \(L_p\) is strongly embedded if and only if it is dispersed. Moreover, for \(1 < p < 2\), it was proved in [14, Theorem 2] that the unit ball \(B_M\) of a strongly embedded subspace of \(L_p\) is equi-integrable in \(L_p\); i.e., that for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(f \in B_M\) and \(A\) is a measurable subset of \((0, 1)\) with \(\mu(A) < \delta\) then \(\|f \cdot \chi_A\|_p < \varepsilon\).

If \(f \in L_p\) and \(\|f\|_p = 1\) then \(\|Q_Mf\| = \dist(f, M) = \inf_{g \in 3B_M} \|f - g\|_p\). Thus, given a normalized disjoint sequence \((f_n)\) in \(L_p\) and \(g \in 3B_M\), and denoting \(A_n = D(f_n)\), since \(\lim_{n \to \infty} \mu(A_n) = 0\), there exists \(n_\varepsilon\) so that

\[
\|f_n - g\|_p \geq \|f_n - g \cdot \chi_{A_n}\|_p \geq 1 - \varepsilon
\]

for \(n \geq n_\varepsilon\); hence \(\lim \inf_{n \to \infty} \|Q_Mf_n\| = 1\) and we get \(\beta(Q_M) = 1\). \(\square\)

**Remark 3.13.** Proposition 3.12 fails for \(p = 2\).

Indeed, if we consider the decomposition \(L_2 = M \oplus M^\perp\) with both \(M\) and \(M^\perp\) strongly embedded given in Proposition 3.4, then \(\beta(Q_M) > 0\) and \(\beta(Q_{M^\perp}) > 0\). Moreover, for \(f \in L_2\) we have \(\|f\|_2^2 = \dist(f, M)^2 + \dist(f, M^\perp)^2\), which implies \(\beta(Q_M)^2 + \beta(Q_{M^\perp})^2 \leq 1\). Hence \(0 < \beta(Q_M)\), \(\beta(Q_{M^\perp}) < 1\).

**Question 4.** Let \(M\) be a dispersed subspace of \(L_p\) \((2 < p < \infty)\). Is \(\beta(Q_M) = 1\)?

For \(2 < p < \infty\) there are strongly embedded subspaces of \(L_p\) whose unit ball is not equi-integrable in \(L_p\) [4]. So the argument in the proof of Proposition 3.12 is not valid.

Next we give further characterizations of DN-S\((L_p, Y)\) and some consequences.

**Theorem 3.14.** For \(T \in L(L_p, Y)\), \(1 < p < \infty\), the following assertions are equivalent:

1. \(T\) is disjointly non-singular.
2. For every normalized sequence \((f_n)\) in \(L_p\) with \(\lim_{n \to \infty} \mu(D(f_n)) = 0\), we have \(\lim \inf_{n \to \infty} \|Tf_n\| > 0\).
3. There exists \(r > 0\) such that \(f \in L_p\), \(\|f\|_p = 1\) and \(\mu(D(f)) \leq r\) imply \(\|Tf\| \geq r\).

**Proof.** (3) \(\Rightarrow\) (2) is clear and (2) \(\Rightarrow\) (1) follows from Theorem 2.8.
For the remaining implication, suppose that (3) fails. Then we can find a normalized sequence \((f_n)\) in \(L_p\) such that \(\mu(D(f_n)) < 1/n\) and \(\|Tf_n\| < 1/n\). Passing to a subsequence we can find a normalized disjoint sequence \((g_n)\) with \(\lim_{n \to \infty} \|f_n - g_n\| = 0\) and \(\lim_{n \to \infty} \|Tg_n\| = 0\), hence (1) fails. \(\square\)

**Corollary 3.15.** Let \(1 < p < \infty\). If \(DN-S(L_p,Y)\) is non-empty, then \(Y\) contains a subspace isomorphic to \(L_p\).

**Proof.** Let \(T \in DN-S(L_p,Y)\) and let \(r\) be as in Theorem 3.14. Note that we can assume that \(0 < r < 1\), hence the restriction \(T|_{L_p(0,r)}\) is an isomorphism. \(\square\)

Let us see that the perturbation classes problem (Question 1) has a positive answer for \(E = L_p\) (1 < \(p < \infty\)). The case \(p = 1\) was proved in [5].

**Theorem 3.16.** Let \(1 < p < \infty\) and suppose that \(DN-S(L_p,Y)\) is non-empty. An operator \(K \in L(L_p,Y)\) is in DSS if and only if \(T + K \in DN-S\) for every \(T \in DN-S(L_p,Y)\).

**Proof.** The direct implication is a consequence of Corollary 2.9.

The proof of the converse implication is similar to that of [5, Proposition 14] for the case \(p = 1\). We suppose that \(K \notin DSS\), and we will construct \(T \in DN-S(L_p,Y)\) such that \(T + K \notin DN-S(L_p,Y)\).

Since \(K \notin DSS\), there exists a disjoint sequence of non-zero vectors \((f_n)\) in \(L_p\) such that \(K||f_n||\) is an isomorphism. By [2, Proposition 6.4.1], \([f_n]\) is a complemented subspace of \(L_p\) isomorphic to \(\ell_p\) and \(N := K([f_n])\) is a subspace of \(Y\) isomorphic to \(\ell_p\). By Corollary 3.15 \(Y\) contains a subspace \(L\) isomorphic to \(L_p\).

Let \(M\) be a closed complement of \([f_n]\) in \(L_p\), and let \(U : M \to L\) be an isomorphic embedding. Considering the relative positions of the subspaces \(L\) and \(N\) inside \(Y\), we have three cases:

(a) \(L \cap N\) finite dimensional and \(L + N\) closed. By Lemma 2.2 we can assume that \(L \cap N = \{0\}\). Then \(T : L_p = M \oplus [f_n] \to Y\) defined by \(T|_M = U\) and \(T|[f_n] = -K||f_n||\) is an isomorphic embedding, hence \(T \in DN-S(L_p,Y)\), but \(T + K \notin DN-S\) because \((f_n) \subset N(T + K)\).

(b) \(L \cap N\) infinite dimensional. In this case \(L \cap N\) contains a subspace \(N_1\) isomorphic to \(\ell_p\) and complemented in \(L\) [2, Theorems 6.4.8 and 7.2.6], hence \(M_1 = (K|_M)^{-1}(N_1)\) is complemented in \(L_p\) and isomorphic to \(\ell_p\). Since \(L_p\) is primary [15, Theorem 2.4.11], the complements of \(N_1\) and \(M_1\) in \(L\) and \(L_p\) are isomorphic to \(L_p\), so we can construct \(T\) as in the previous case.

(c) \(L \cap N\) finite dimensional and \(L + N\) non-closed. An argument in [6, Proof of Theorem 4.3.5] provides a compact operator \(K_1 \in L(L_p,Y)\) such that \((K + K_1)||f_n||\) is an isomorphism and \(L \cap (K + K_1)([f_n])\) is infinite dimensional. The argument in case (b) gives \(T \in DN-S(L_p,Y)\) such that \(T + K + K_1 \notin DN-S\), hence \(T + K \notin DN-S\). \(\square\)

Each upper semi-Fredholm operator is tauberian, but no other examples of tauberian operators in \(L(L_1)\) were known (see [6, Section 4.1]) until, using probabilistic arguments, examples with infinite dimensional kernel or non-closed range were constructed in [12]. Next we show that it is much easier to find similar examples in \(L(L_p)\), 1 < \(p < \infty\).

**Example 3.17.** For 1 < \(p < \infty\), there exists a projection \(P \in L(L_p)\) onto the closed subspace generated by the Rademacher functions. See the proof of [2, Proposition 6.4.2].
The operator $I - P : L_p \to L_p$ has closed range and infinite dimensional kernel. In fact, $N(I - P)$ is the closed subspace generated by the Rademacher functions, which is strongly embedded \cite[Proposition 6.4.5]{Albiac}. Thus the operator $I - P$ is disjointly non-singular by Proposition \ref{proposition1}.

Also, it is not difficult to find a compact operator $K \in L(L_p)$ such that the range of $I - P + K$ is non-closed. Note that $I - P + K$ is disjointly non-singular.

References
\begin{enumerate}
\item P. Aiena. Fredholm and local spectral theory, with applications to multipliers. Kluwer, 2004.
\item F. Albiac, N. Kalton. Topics in Banach space theory. Springer, 2006.
\item T. Alvarez, M. González. Some examples of tauberian operators. Proc. Amer. Math. Soc. 111 (1991), 1023–1027.
\item S. V. Astashkin. $A(p)$-spaces. J. Funct. Anal. 266 (2014), 5174–5198.
\item M. González, A. Martínez-Abejón. Tauberian operators on $L_1(\mu)$ spaces. Studia Math. 125 (1997), 289–303.
\item M. González, A. Martínez-Abejón. Tauberian operators. Operator Theory: Advances and applications 194. Birkhäuser, 2010.
\item M. González, V.M. Onieva. On incomparrability of Banach spaces. Math. Z. 192 (1986), 581–585.
\item M. González, V.M. Onieva. Characterizations of tauberian operators and other semigroups of operators. Proc. Amer. Math. Soc. 108 (1990), 399–405.
\item M. González, E. Saksman, H.-O. Tylli. Representing non-weakly compact operators. Studia Math. 113 (1995), 289–303.
\item F. L. Hernández. Disjointly strictly-singular operators in Banach lattices. Acta Univ. Carolinae – Math. et Phys. 31 (1990), 35–40.
\item F. L. Hernández, B. Rodríguez-Salinas. On $\ell_p$ complemented copies in Orlicz spaces II. Israel J. Math. 68 (1989), 27–55.
\item W. B. Johnson, A. B. Nasser, G. Schechtman, T. Tkocz. Injective Tauberian operators on $L_1$ and operators with dense range on $\ell_\infty$. Canad. Math. Bull. 58 (2015), 276–280.
\item N. Kalton, A. Wilansky. Tauberian operators on Banach spaces. Proc. Amer. Math. Soc. 57 (1976), 251–255.
\item T. Kato. Perturbation theory for linear operators. Corrected printing of the 2nd. ed. 1980. Springer, 1995.
\item J. Lindenstrauss, L. Tzafriri. Classical Banach spaces II. Function spaces. Springer, 1979.
\item A. Pietsch. Operator ideals. North-Holland, 1980.
\item G. Pisier. Factorization of linear operators and geometry of Banach spaces. CBMS Reg. Conf. Series in Math., 60. Amer. Math. Soc., 1986.
\item W. Rudin. Functional analysis, 2nd. ed. McGraw-Hill, 1991.
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