Mean-performance of sharp restart: II. Inequality roadmap

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Abstract
Restarting a deterministic process always impedes its completion. However, it is known that restarting a random process can lead to an opposite outcome—expediting completion. Hence, the effect of restart is contingent on the underlying statistical heterogeneity of the process’ completion times. To quantify this heterogeneity we introduce a novel approach to restart research: the methodology of inequality indices, which is widely applied in economics and in the social sciences to measure income and wealth disparities. Utilizing this approach we establish an ‘inequality roadmap’ for the mean-performance of sharp restart: a whole new set of universal inequality criteria that determine when restart with sharp timers (i.e. with fixed deterministic timers) impedes/expedites mean completion. The criteria are based on key Lorenz-curve inequality indices including Bonferroni, Gini, and Pietra. From a practical perspective, the criteria offer researchers highly useful tools to tackle the common real-world situation in which only partial information of the completion-time statistics is available. From a theoretical perspective, the criteria yield—with unprecedented precision and resolution—a powerful and overarching take-home-message: restart impedes/expedites mean completion when the underlying statistical heterogeneity is low/high, respectively. As sharp restart can match the mean-performance of any other restart protocol, the results established here apply to restart research at large.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Following [1], this paper is the second in a duo of works exploring the mean-performance of sharp restart. Sharp restart is an algorithm whose input is the random duration of a general task to be accomplished—e.g. a completion time or a first-passage time of a given stochastic process [2–6]. As long as the task is not accomplished, the algorithm restarts the underlying stochastic process periodically, using a ‘sharp’ timer—i.e. a fixed deterministic timer. The algorithm’s output is the random duration it takes to accomplish the task ‘under restart’. The motivation for exploring sharp restart, together with a rather rich literature survey (including [7–27]), was described in detail in part one of the duo [1].

Following the main approach in restart research, the analysis presented in both parts of the duo focus on mean performance: comparing the output’s mean to the input’s mean. Based on a formula expressing the output’s mean in terms of the input’s distribution function, a comprehensive statistical analysis of the mean-performance of sharp restart was established in part one [1]. In this context we termed sharp restart ‘beneficial’ when the output’s mean is smaller than the input’s mean—i.e. when task completion is expedited on average. Conversely, we termed sharp restart ‘detrimental’ when the output’s mean is larger than the input’s mean—i.e. when task completion is impeded on average.

The aforementioned formula (which was utilized in part one [1]) can be readily used to check, per any given timer, if sharp restart is beneficial or detrimental. Yet, in order to do so, one requires full information regarding the input's statistical distribution (i.e. the task’s completion-time statistics). In practice, such full information is seldom available. Indeed, in practice, the information available constitutes a finite sample of input observations. Thus, when facing reality, the mean-performance analysis of sharp restart calls out for tools and criteria that are based on partial information.

This paper answers the call, and does so via a rather surprising approach: socioeconomic inequality indices. This approach is widely applied in economics and in the social sciences to quantify wealth and income disparities [28–31]. More generally, socioeconomic inequality indices can be used to gauge the statistical heterogeneity of non-negative random quantities at large [32–34]. Here we apply socioeconomic inequality indices to gauge the input’s statistical heterogeneity. Doing so we establish universal inequality-based criteria that: (i) determine the very existence of timers with which sharp restart is beneficial/detrimental; and (ii) determine if sharp restart with specific timers—e.g. the input’s mean and the input’s median—is beneficial/detrimental. These universal criteria are highly practical, as they only require partial information: low order moments of the input, which can be estimated easily and reliably from finite samples of input observations (i.e. observations of the task’s completion times).

From a practical perspective, the universal inequality-based criteria established here provide a novel set of tools that facilitate informed restart decisions that are based on partial information. From a theoretical perspective, the universal inequality-based criteria established here provide a fundamental linkage between restart and socioeconomic inequality. In a nutshell, this linkage is described, informally, by the following profound insight: restart is detrimental when the input’s statistical heterogeneity is low; restart is beneficial when the input’s statistical heterogeneity is high; and the separation between low and high statistical heterogeneity is via precise thresholds. Both practically and theoretically, the results presented in this paper are a significant advancement to the multidisciplinary field restart research.
The remainder of this paper is organized as follows. Section 2 sets the stage: it recaps sharp restart and relevant results from [1]; and it reviews the notion of socioeconomic inequality indices, as well as the very foundation of many socioeconomic inequality indices—the Lorentz curve. Sections 3–5 present the universal inequality-based criteria for the mean-performance of sharp restart: Gini and Bonferroni existence criteria (section 3); Pietra and Vdiam criteria for, respectively, mean and median timers (section 4); ‘vertical’ and ‘horizontal’ criteria for arbitrary timers (section 5). Section 6 concludes with an ‘inequality roadmap’ for the mean-performance of sharp restart: a summary of the key results established here.

A note about notation: along the paper $E[\xi]$ denotes the expectation of a (non-negative) random variable $\xi$; and IID is acronym for independent and identically distributed (random variables).

2. Setting the stage

This section sets the stage for a comprehensive mean-performance analysis of sharp restart—to be based on socioeconomic inequality indices—that will be carried out in the next sections. Subsections 2.1 and 2.2 recap material that was presented in [1]. Subsection 2.3 and 2.4 review, respectively, the socioeconomic notions of inequality indices and the Lorenz curve.

2.1. Sharp restart

Sharp restart admits the following algorithmic description. There is a general task with completion time $T$, a positive-valued random variable. To this task a three-steps algorithm, with a positive deterministic timer $\tau$, is applied. Step I: initiate simultaneously the task and the timer. Step II: if the task is accomplished up to the timer’s expiration—i.e. if $T \leq \tau$—then stop upon task completion. Step III: if the task is not accomplished up to the timer’s expiration—i.e. if $T > \tau$—then, as the timer expires, go back to step I and start afresh (independently of the past).

The sharp-restart algorithm generates an iterative process of independent and statistically identical task-completion trials. This process halts during its first successful trial, and we denote by $T_R$ its halting time. Namely, $T_R$ is the overall time it takes—when the sharp-restart algorithm is applied—to complete the task. The sharp-restart algorithm is a non-linear mapping whose input is the random variable $T$, whose output is the random variable $T_R$, and whose parameter is the deterministic timer $\tau$.

This paper shall use the following notation regarding the input’s statistics: distribution function, $F(t) = \Pr(T \leq t)$ ($t \geq 0$); survival function, $\bar{F}(t) = \Pr(T > t)$ ($t \geq 0$); density function, $f(t) = F'(t) = -\bar{F}'(t)$ ($t > 0$); and mean, $\mu = E[T] = \int_0^\infty tf(t)\,dt$. The input’s density function is henceforth considered to be positive-valued over the positive half-line: $f(t) > 0$ for all $t > 0$.

Throughout this paper $M(\tau) = E[T_R]$ shall denote the output’s mean; this notation underscores the fact that the output’s mean is a function of the timer $\tau$, the parameter of the sharp-restart algorithm. In terms of the input’s distribution and survival functions, the output’s mean is given by [1, 35, 36]:

$$M(\tau) = \frac{1}{F(\tau)} \int_0^\tau \bar{F}(t)\,dt. \tag{1}$$

1 This is merely a technical assumption, which is introduced in order to assure that all positive timers $\tau$ are admissible. In general, admissible timers are in the range $t_{\text{low}} < \tau < \infty$, where $t_{\text{low}}$ is the lower bound of the support of the input’s density function: $t_{\text{low}} = \inf\{t > 0 \mid f(t) > 0\}$. 
As the input’s mean is the integral of the input’s survival function, \( \mu = \int_0^\infty \bar{F}(t) \, dt \), equation (1) implies that: in the limit \( \tau \to \infty \), the output’s mean coincides with the input’s mean, \( \lim_{\tau \to \infty} M(\tau) = \mu \).

Examining the sharp-restart algorithm from a mean-performance perspective, it is key to determine if the application of the algorithm will expedite task-completion, or if it will impede task-completion. To that end the following terminology shall be used [1]:

- **Sharp restart with timer \( \tau \) is beneficial** if it improves mean-performance, \( M(\tau) < \mu \).
- **Sharp restart with timer \( \tau \) is detrimental** if it worsens mean-performance, \( M(\tau) > \mu \).

Equation (1) implies that the output’s mean is always finite: \( M(\tau) < \infty \) for all timers \( \tau \). Thus, if the input’s mean is infinite, \( \mu = \infty \), then the application of the sharp-restart algorithm is highly beneficial—as it reduces the input’s infinite mean to the output’s finite mean: \( M(\tau) < \mu = \infty \). Having resolved the case of infinite-mean inputs, we henceforth set the focus on the case of positive-mean inputs, \( 0 < \mu < \infty \).

Equation (1) can be readily used in order to determine if sharp restart with timer \( \tau \) is beneficial or detrimental. However, the construction of equation (1) requires full information of the input’s statistical distribution. In reality such full information is rarely available. Rather, the information available is usually partial, and it is often a finite sample of input observations. So, while in theory equation (1) is a ‘master tool’ for the mean-performance of sharp restart, in practice equation (1) is of limited applicability.

As practitioner deal with real and simulated data, they need efficient tools that are based on partial information of the input’s statistical distribution. To date, the only such tool in the restart-research arsenal is the, so called, ‘CV criteria’. This tool is recapped below.

### 2.2. CV criteria

The variance of the input \( T \) is the input’s mean square deviation from its mean, \( \sigma^2 = \mathbb{E}[T - \mu]^2 \). The input’s standard deviation \( \sigma \) is the square root of its variance. And, the input’s coefficient of variation (CV) is its ‘noise-to-signal’ ratio: the ratio of its standard deviation to its mean, \( \sigma / \mu \). The input’s CV, \( \sigma / \mu \), is a ‘normalized’ version of its standard deviation; this CV is the most popular standardized gauge of the input’s statistical heterogeneity.

A key result established in [1] is based on the input’s CV. This key result yields the following pair of CV criteria:

- **If the CV is smaller than one**—which is equivalent to \( \sigma < \mu \)—then there exist timers \( \tau \) for which sharp restart is detrimental.
- **If the CV is larger than one**—which is equivalent to \( \sigma > \mu \)—then there exist timers \( \tau \) for which sharp restart is beneficial.

Similar, yet not identical, CV criteria hold with regard to the introduction of exponential restart [36–43]. While in sharp restart the periods between consecutive restart epochs are deterministic, in exponential restart these periods are drawn (independently) from a given exponential distribution.

The CV criteria stem from the following formula:

\[
\int_0^\infty [M(\tau) - \mu] F(\tau) \, d\tau = \frac{1}{2} \left( \mu^2 - \sigma^2 \right).
\] (2)

Namely, equation (2) manifests how the interplay between the input’s standard deviation and mean—\( \sigma \) versus \( \mu \)—affects the output’s mean \( M(\tau) \). Equivalently, equation (2) manifests the
effect of the input’s CV on the output’s mean \( M(\tau) \). The derivation of equation (2) is detailed in the methods.

The CV criteria have the following threshold form. The input’s CV is compared to the threshold level 1. To determine the existence of timers for which the application of the sharp-restart algorithm is detrimental or beneficial, one needs to check if the input’s CV is below or above the threshold level, respectively. Evidently, in the CV criteria, one can replace the CV by its square, \( \sigma^2/\mu^2 \).

As noted above, equation (1) requires full information of the input’s statistical distribution. In sharp contrast to equation (1), the CV criteria require partial information of the input’s statistical distribution: the input’s first two moments. These moments are easily and reliably estimated from finite samples of input observations. Hence the CV criteria are a highly practical tool for the mean-performance of sharp restart.

The CV gauges statistical heterogeneity via an Euclidean-geometry perspective [33]. We now turn to review inequality indices—which are quantitative measures that gauge statistical heterogeneity via a socioeconomic-inequality perspective.

2.3. Inequality indices

Consider a human society comprising of members with non-negative wealth values. Such a society has two socioeconomic extremes: perfect equality and perfect inequality. In a perfectly equal society all members share a common positive wealth value. In a perfectly unequal society 0% of the members possess 100% of the overall wealth.

An inequality index \( \mathcal{I} \) is a measure of the society’s socioeconomic inequality: the larger the index—the more socioeconomically unequal the society. More specifically, an inequality index \( \mathcal{I} \) takes values in the unit interval, \( 0 \leq \mathcal{I} \leq 1 \), and it has three basic properties [28–31].

(I) The index meets its zero lower bound \( \mathcal{I} = 0 \) if and only if the society is perfectly equal. (II) If the society is perfectly unequal then the index meets its unit upper bound \( \mathcal{I} = 1 \). (III) The index is invariant with respect to the specific currency via which wealth is measured.

While devised to gauge socioeconomic disparity, the application of inequality indices is not confined to this field alone. Indeed, an inequality index \( \mathcal{I} \) can be used to measure the inherent ‘socioeconomic inequality’ of any given non-negative random variable with a finite mean [33, 34]. To that end, deem the random variable under consideration to represent the wealth of a randomly-sampled member of a virtual society. Then, the socioeconomic inequality of the random variable is that of its corresponding virtual society. Here we take the random variable under consideration to be the input \( T \) of the sharp-restart algorithm, and we will apply inequality indices to gauge the input’s statistical heterogeneity.

An illustrative example of an inequality index of the input is what shall be henceforth referred to as the CV index. This inequality index was introduced in [44], and it is a special case of the input’s Rényi spectra [34, 44, 45]. In terms of the input’s CV, \( \sigma/\mu \), the input’s CV index admits the following representation [44]:

\[
\mathcal{I}_{CV} = 1 - \frac{1}{1 + (\sigma/\mu)^2}.
\]  

The input’s CV index \( \mathcal{I}_{CV} \) is a monotone increasing function of the input’s CV. This index meets its zero lower bound if and only if the input’s CV vanishes, i.e.—for a given positive-mean input—if and only if the input’s standard deviation vanishes, \( \sigma = 0 \). And, this index

\[2\] In the socioeconomic extreme of perfect inequality the society’s population is infinitely large [33].
meets its unit upper bound if and only if the input’s CV diverges, i.e.—for a given positive-mean input—if and only if the input’s standard deviation diverges, \(\sigma = \infty\). For a more detailed account of the CV index and its properties, see [34].

In terms of the input’s CV index \(I_{CV}\), the pair of the CV criteria of subsection 2.2 can be re-formulated as the following pair of CV-index criteria:

- If the input’s CV index is smaller than half, \(I_{CV} < \frac{1}{2}\), then there exist timers \(\tau\) for which sharp restart is detrimental.
- If the input’s CV index is larger than half, \(I_{CV} > \frac{1}{2}\), then there exist timers \(\tau\) for which sharp restart is beneficial.

The threshold form of the CV criteria is induced to the CV-index criteria. Specifically, the input’s CV index \(I_{CV}\)—the input’s ‘CV inequality’—is compared to the threshold level \(\frac{1}{2}\). To determine the existence of timers for which the application of the sharp-restart algorithm is detrimental or beneficial, one needs to check if the input’s ‘CV inequality’ \(I_{CV}\) is below or above the threshold level \(\frac{1}{2}\), respectively.

The CV-index criteria expose a connection between sharp restart and the measurement of socioeconomic inequality. In the following sections we shall show that this connection is profound and broad, and that it extends far beyond the CV index. To establish this deep connection we will use an additional object that quantifies socioeconomic inequality: the Lorenz curve, which is the bedrock of many inequality indices. The Lorenz curve is reviewed below.

### 2.4. The Lorenz curve

As in subsection 2.3, consider a human society comprising of members with non-negative wealth values. The distribution of wealth among the society’s members is quantified by the society’s Lorenz curve \(y = L(x)\) \((0 \leq x, y \leq 1)\) [46–49]. Specifically, the Lorenz curve \(y = L(x)\) has the following socioeconomic meaning: the low (poor) \(100x\%\) of the society members possess \(100y\%\) of the society’s overall wealth.

The Lorenz curve \(y = L(x)\) resides in the unit square \((0 \leq x, y \leq 1)\), and it has three basic properties [46–49]. (I) It ‘starts’ at the square’s bottom-left corner, \(L(0) = 0\); and it ‘ends’ at the square’s top-right corner, \(L(1) = 1\). (II) It is monotone increasing and concave. (III) It is invariant with respect to the currency via which wealth is measured. The first two properties imply that the Lorenz curve \(y = L(x)\) is bounded from above by the square’s diagonal line \(y = x\) (see figure 1).

As noted in subsection 2.3, the human society under consideration has two socioeconomic extremes: perfect equality and perfect inequality. Recall that in a perfectly equal society all members share a common positive wealth value. In the space of Lorenz curves the perfect-equality socioeconomic extreme is characterized by the diagonal line \(y = x\) \((0 \leq x, y \leq 1)\). Consequently, the deviation of the Lorenz curve \(y = L(x)\) from the diagonal line \(y = x\) can serve as a geometric gauge of the society’s socioeconomic inequality: the deviation of the society from the perfect-equality socioeconomic extreme.

Evidently, there are many different ways of measuring the deviation of the Lorenz curve \(y = L(x)\) from the diagonal line \(y = x\). In turn, these different ways yield different inequality indices. An illustrative example is described as follows. For a fixed number \(q\) (where \(0 < q < 1\)), consider the vertical line \(x = q\) of the unit square. The vertical distance—along the vertical line \(x = q\)—between the Lorenz curve \(y = L(x)\) and the diagonal line \(y = x\) is: \(q - L(q)\) (see figure 1). This vertical distance takes values in the range \([0, q]\). Consequently, the corresponding ‘normalized’ vertical distance \([q - L(q)]/q\) takes values in the unit interval...
Figure 1. Lorenz-curve illustration. The Lorenz curve $y = L(x)$ is depicted in blue, and the diagonal line $y = x$ is depicted in orange; the Lorenz curve and the diagonal line reside in the unit square ($0 \leq x, y \leq 1$). The line segment depicted in solid black is the vertical distance—along the vertical line $x = q$, depicted in dashed black—between the Lorenz curve and the diagonal line.

$[0, 1]$. It is straightforward to check that the normalized vertical distance $[q - L(q)]/q$ meets the three inequality-index properties that were postulated in subsection 2.3.

Any non-negative valued random variable, with a positive mean, has a corresponding Lorenz curve [32–34]. Indeed, as with inequality indices: deem the random variable under consideration to represent the wealth of a randomly-sampled member of a virtual society. Then, the Lorenz curve of the random variable is that of its corresponding virtual society. As above, we take the random variable under consideration to be the input $T$ of the sharp-restart algorithm. Thus, henceforth, $y = L(x)$ shall manifest the input’s Lorenz curve.

2.5. Outlook

Armed with the notion of inequality indices and with the Lorenz curve, we are now all set to construct an ‘inequality roadmap’ for the mean-performance of sharp restart. In the following sections we will use a host of inequality indices—all based on the Lorenz curve—to gauge the input’s statistical heterogeneity. Doing so we will establish a whole new set of universal inequality-based criteria that determine when sharp restart is beneficial, and when it is detrimental. Moreover, we will show that all the novel criteria share a common ‘bedrock structure’ that is described by the following take-home-message: high statistical heterogeneity implies that sharp restart is beneficial; and low statistical heterogeneity implies that sharp restart is detrimental. Using precise threshold levels, the novel criteria will articulate the take-home-message via exact and explicit mathematical formulations.

The new set of universal inequality-based criteria is a significant advancement—both practically and theoretically—to the multidisciplinary field restart research. As noted above, the only tool that is based on partial information of the input’s statistical distribution is, to date, the CV criteria. From the practical perspective, the novel criteria offer additional partial-information tools which: on the one hand are as easy-to-use as the CV criteria; and, on the other hand, yield results that are unattainable by the CV criteria. From the theoretical perspective, the novel criteria provide a profound and deep insight that is summarized by the above take-home-message.
3. Gini and Bonferroni indices

In this section we establish connections between sharp restart on the one hand, and the Gini and Bonferroni inequality indices on the other hand. Specifically, we shall show that these two inequality indices of the input $T$ yield existence criteria that are similar to the CV-index criteria of subsection 2.3, and that have a similar threshold form.

3.1. Gini-index criteria

Perhaps the best-known and most popular socioeconomic inequality index is the Gini index [50–56]. This subsection addresses the Gini index $I_{\text{Gini}}$ of the input $T$.

In terms of the input’s Lorenz curve, the input’s Gini index $I_{\text{Gini}}$ is twice the area captured between the Lorenz curve $y = L(x)$ and the diagonal line $y = x$ [54]. Namely,

$$I_{\text{Gini}} = 2 \int_0^1 [q - L(q)] \, dq. \tag{4}$$

The integral appearing on the right-hand side of equation (4) is the average of the vertical distances $[q - L(q)]$ ($0 < q < 1$) between the Lorenz curve and the diagonal line (see figure 1); these vertical distances were noted in subsection 2.4.

The Gini-index representation of equation (4) exhibits no clear connection to equation (1), the output’s mean. To make the connection more apparent we use the following, alternative, Gini-index representation [34]:

$$I_{\text{Gini}} = 1 - \frac{1}{\mu} E \left[ \min \{T_1, T_2\} \right], \tag{5}$$

where $T_1$ and $T_2$ are IID copies of the input $T$. The Gini-index representation of equation (5) is based on the disparity between two means: the mean of the minimum $\min \{T_1, T_2\}$ versus the input’s mean $\mu$. The smaller the disparity between the two means—the closer is the Gini index to its zero lower bound. Conversely, the larger the disparity between the two means—the closer is the Gini index to its unit upper bound.

In addition to the Gini-index representation of equation (5), we also use the following representation of the output’s mean:

$$M(\tau) = \frac{1}{F(\tau)} E \left[ \min \{T, \tau\} \right]. \tag{6}$$

The derivation of equation (6) is detailed in the methods. Equations (5) and (6) incorporate similar terms—the term $E \left[ \min \{T_1, T_2\} \right]$ in the former, and the term $E \left[ \min \{T, \tau\} \right]$ in the latter. This similarity suggests that a connection between the input’s Gini index $I_{\text{Gini}}$ and the output’s mean $M(\tau)$ may exist. We shall now establish such a connection indeed.

Denote by $f_{\max}(t)$ ($t \geq 0$) the density function of the random variable $\max \{T_1, T_2\}$ where, as in equation (5), $T_1$ and $T_2$ are IID copies of the input $T$. With this density function at hand, the following formula is presented:

$$\int_0^\infty \left[ \frac{M(\tau) - \mu}{\mu} \right] f_{\max}(\tau) \, d\tau = 1 - 2I_{\text{Gini}}. \tag{7}$$

The derivation of equation (7) is detailed in the methods.

Equation (7) manifests the effect of the input’s Gini index $I_{\text{Gini}}$ on the output’s mean $M(\tau)$. Indeed, equation (7) yields the following pair of Gini-index criteria:
• If the input’s Gini index is smaller than half, $\mathcal{I}_{\text{Gini}} < \frac{1}{2}$, then there exist timers $\tau$ for which sharp restart is detrimental.

• If the input’s Gini index is larger than half, $\mathcal{I}_{\text{Gini}} > \frac{1}{2}$, then there exist timers $\tau$ for which sharp restart is beneficial.

These Gini-index criteria have a threshold form that is identical to that of the CV-index criteria of subsection 2.3. Specifically, the input’s Gini index $\mathcal{I}_{\text{Gini}}$—the input’s ‘Gini inequality’—is compared to the threshold level $\frac{1}{2}$. To determine the existence of timers for which the application of the sharp-restart algorithm is detrimental or beneficial, one needs to check if the input’s ‘Gini inequality’ $\mathcal{I}_{\text{Gini}}$ is below or above the threshold level $\frac{1}{2}$, respectively.

On the one hand, $\mathcal{I}_{\text{CV}}$ and $\mathcal{I}_{\text{Gini}}$ are markedly different inequality indices of the input $T$. On the other hand, these very different inequality indices yield very similar existence results regarding timers $\tau$ for which sharp restart is detrimental or beneficial. We shall elaborate on the relation between the CV-index criteria and the Gini-index criteria in the discussion at the end of this section.

3.2. Bonferroni-index criteria

Not as popular as the Gini index, yet no less profound, is the Bonferroni index [57–63]. This subsection addresses the Bonferroni index $\mathcal{I}_{\text{Bonf}}$ of the input $T$.

In terms of the input’s Lorenz curve, the input’s Bonferroni index $\mathcal{I}_{\text{Bonf}}$ is the average of the normalized vertical distances $[q - L(q)]/q$ ($0 < q < 1$) between the Lorenz curve and the diagonal line [63]. Namely,

$$\mathcal{I}_{\text{Bonf}} = \int_0^1 \frac{q - L(q)}{q} dq.$$  

(8)

As noted in subsection 2.4, for any fixed number $q$, the normalized vertical distance $[q - L(q)]/q$ is an inequality index. Hence, the Bonferroni index $\mathcal{I}_{\text{Bonf}}$ is an average of inequality indices.

The Bonferroni-index representation of equation (8) exhibits no clear connection to equation (1), the output’s mean. To make the connection more apparent we use $\phi(\tau) = \mathbb{E}[T|T \leq \tau]$—the input’s conditional mean, given the information that the input is no larger than the timer. In terms of the input’s density and distribution functions, this conditional mean is given by:

$$\phi(\tau) = \int_0^\tau \frac{f(t)}{F(\tau)} dt = \frac{1}{F(\tau)} \int_0^\tau tf(t) dt.$$  

(9)

Evidently, this conditional mean is no larger than the input’s mean: $\phi(\tau) \leq \mu$.

In terms of the conditional mean $\phi(\tau)$, the input’s Bonferroni index $\mathcal{I}_{\text{Bonf}}$ admits the following representation [63]:

$$\mathcal{I}_{\text{Bonf}} = 1 - \frac{1}{\mu} \int_0^\infty \phi(\tau) f(\tau) d\tau.$$  

(10)

The integral appearing on the right-hand side of equation (10) is a weighted average of the conditional mean $\phi(\tau)$, where the averaging is with respect to the input’s density function. The Bonferroni-index representation of equation (10) is based on the disparity between two terms: the weighted average of the conditional mean $\int_0^\infty \phi(\tau) f(\tau) d\tau$ versus the input’s mean $\mu$. The smaller the disparity between the two terms—the closer is the Bonferroni index to its
zero lower bound. Conversely, the larger the disparity between the two terms—the closer is the Bonferroni index to its unit upper bound.

In terms of the conditional mean $\phi(\tau)$, the output’s mean of equation (1) admits the following representation:

$$M(\tau) = \phi(\tau) + \frac{\tau}{\int F(\tau)}.$$  \hspace{1cm} (11)

The derivation of equation (11) is detailed in the methods. Both equations (10) and (11) incorporate the term $\phi(\tau)$. This commonality suggests that a connection between the input’s Bonferroni index $I_{\text{Bonf}}$ and the output’s mean $M(\tau)$ may exist. We shall now establish such a connection indeed.

Introduce the value

$$\nu = \int_{0}^{\infty} tf(t) F(t) \, dt = \int_{0}^{\infty} \ln \frac{1}{F(t)} \, dt.$$  \hspace{1cm} (12)

With the value $\nu$ at hand, the following formula is presented:

$$\int_{0}^{\infty} \left[ \frac{M(\tau) - \mu}{\mu} \right] f(\tau) \, d\tau = \frac{\nu - \mu}{\mu} - I_{\text{Bonf}}.$$  \hspace{1cm} (13)

The derivation of equation (13) is detailed in the methods.

Equation (13) manifests the effect of the input’s Bonferroni index $I_{\text{Bonf}}$ on the output’s mean $M(\tau)$. Indeed, setting the threshold level $l_{\text{Bonf}} = \frac{\nu - \mu}{\mu}$, equation (13) yields the following pair of Bonferroni-index criteria:

- If the input’s Bonferroni index is smaller than its threshold level, $I_{\text{Bonf}} < l_{\text{Bonf}}$, then there exist timers $\tau$ for which sharp restart is detrimental.
- If the input’s Bonferroni index is larger than its threshold level, $I_{\text{Bonf}} > l_{\text{Bonf}}$, then there exist timers $\tau$ for which sharp restart is beneficial.

The Bonferroni-index criteria have a threshold form that is similar to the threshold form of the CV-index criteria of subsection 2.3, and to the threshold form of the Gini-index criteria of subsection 3.1. Specifically, the input’s Bonferroni index $I_{\text{Bonf}}$—the input’s ‘Bonferroni inequality’—is compared to the threshold level $l_{\text{Bonf}} = \frac{\nu - \mu}{\mu}$. To determine the existence of timers for which the application of the sharp-restart algorithm is detrimental or beneficial, one needs to check if the input’s ‘Bonferroni inequality’ $I_{\text{Bonf}}$ is below or above the threshold level $l_{\text{Bonf}}$, respectively.

3.3. Discussion

We conclude this section with remarks regarding the resemblance and the interplay between the CV criteria of subsection 2.2, and the Gini-index criteria of subsection 3.1. The remarks are based on the deviation $T_1 - T_2$ between two IID copies, $T_1$ and $T_2$, of the input.

It is straightforward to observe that the mean square deviation (MSD) between the two IID copies is twice the input’s variance: $\mathbb{E}[(T_1 - T_2)^2] = 2\sigma^2$. Consequently, in terms of the MSD, the input’s squared CV admits the following representation:

$$\frac{\sigma^2}{\mu^2} = \frac{\mathbb{E}[(T_1 - T_2)^2]}{2\mu^2}.$$  \hspace{1cm} (14)
An alternative to the aforementioned MSD is the mean absolute deviation (MAD) between the two IID copies, $E[|T_1 - T_2|]$ [64]. In [33] it is shown that, from a planar-geometry perspective, the square root of the MSD is analogous to aerial distance, and the MAD is analogous to walking distance. In terms of the MAD, the input’s Gini index admits the following representation [34]:

$$\mathcal{F}_{\text{Gini}} = \frac{E[|T_1 - T_2|]}{2\mu}.$$  \hfill (15)

Equations (14) and (15) highlight the resemblance between the input’s squared CV, $\sigma^2/\mu^2$, and the input’s Gini index, $\mathcal{F}_{\text{Gini}}$. Both the CV and the Gini index use partial information of the input’s statistical distribution: the CV uses the input’s mean and the MSD of equation (14); and the Gini index uses the input’s mean and the MAD of equation (15). The input’s mean, as well as the MSD of equation (14) and the MAD of equation (15), are easily and reliably estimated from finite samples of input observations. Hence, the Gini-index criteria are as practical a tool as the CV criteria. As we shall now argue, the CV criteria and the Gini-index criteria are complementary tools.

Jensen’s inequality implies that the squared MAD is no larger than the MSD, $E[|T_1 - T_2|^2] \leq E[|T_1 - T_2|^2]$. Consequently, equations (14) and (15) imply the following relation between the input’s Gini index and the input’s CV:

$$\sqrt{2} \cdot \mathcal{F}_{\text{Gini}} \leq \frac{\sigma}{\mu}.$$ \hfill (16)

In turn, equation (16) implies that the interplay between the CV criteria and the Gini-index criteria comprises four different scenarios (see figure 2): ‘DD’, ‘BB’, ‘DB’ and ‘BD’.

In the ‘DD’ and ‘BB’ scenarios the Gini and CV criteria are in accord—both asserting the existence of timers for which sharp restart is either detrimental (‘DD’) or beneficial (‘BB’). In the ‘DB’ and ‘BD’ scenarios the Gini and CV criteria are in disaccord—one criterion asserts the existence of timers for which sharp restart is detrimental, and another criterion asserts the existence of timers for which sharp restart is beneficial. The ‘DB’ and ‘BD’ scenarios underscore the fact that sharp restart can be detrimental for some timers, and beneficial for other timers [1].

The ‘DB’ and ‘BD’ scenarios also underscore the fact that the CV and Gini criteria are complementary results that can yield complementary answers. For example, consider the information that the CV index is below its threshold, from which the CV criteria infer that there exists a detrimental sharp-restart timer. Note that this information does not necessarily imply that a beneficial sharp-restart timer does not exist—but rather that the CV criteria cannot affirm the existence of such a timer. Turning to the Gini criteria, we may very well obtain an answer that affirms the existence of a beneficial sharp-restart timer—an answer that the CV criteria is unable to provide. So, jointly, the CV and the Gini criteria can yield answers that are more informative than the answers provided by each of these criteria alone.

A demonstration of a scenario in which the Gini and CV criteria are in dis-accord is illustrated by the real-world example of lognormal task-completion times (figure 3). Specifically, for the lognormal example, figure 3 shows that there is a set of parameters in which the ‘DB’ scenario holds. Lognormal random variables are of major importance across the sciences [65–69], and as task-completion times they are widely observed in call centers [70–72]. We note that the queueing processes that underpin call centers are intricate and complex; yet, on a system level, empirical evidence asserts that these processes generate lognormal service times.

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Figure 2. A Gini–CV ‘phase diagram’. The horizontal axis represents the value of the input’s Gini index $I_{\text{Gini}}$, and the vertical axis represents the value of the input’s $\text{CV} = \sigma/\mu$. The line $\sqrt{2}I_{\text{Gini}} = \sigma/\mu$ is depicted in blue; according to equation (16), feasible pairs of Gini–CV values $(I_{\text{Gini}}, \sigma/\mu)$ do not reside below this line. The Gini–CV ‘phase diagram’ comprises four different regions—each manifesting a different scenario. The trapezoid region ‘DD’ ($0 \leq I_{\text{Gini}} < 1/2$ and $\sqrt{2}I_{\text{Gini}} \leq \sigma/\mu < 1$): in this region both the Gini-index criteria and the CV criteria assert the existence of timers for which sharp restart is detrimental. The polygonal region ‘BB’ ($1/2 < I_{\text{Gini}} \leq 1$, and the intersection of $1 < \sigma/\mu < \infty$ and $\sqrt{2}I_{\text{Gini}} \leq \sigma/\mu < \infty$): in this region both the Gini-index criteria and the CV criteria assert the existence of timers for which sharp restart is beneficial. The rectangular region ‘DB’ ($0 \leq I_{\text{Gini}} < 1/2$ and $1 < \sigma/\mu < \infty$): in this region a Gini-index criterion asserts the existence of timers for which sharp restart is detrimental, and a CV criterion asserts the existence of timers for which sharp restart is beneficial. The triangular region ‘BD’ ($1/2 < I_{\text{Gini}} < 1$ and $\sqrt{2}I_{\text{Gini}} \leq \sigma/\mu < 1$): in this region a Gini-index criterion asserts the existence of timers for which sharp restart is beneficial, and a CV criterion asserts the existence of timers for which sharp restart is detrimental.

4. Mean and median timers

The inequality-indices criteria established so far, in sections 2 and 3, were existence results. Namely, these criteria determined the existence of timers $\tau$ for which the application of sharp restart is detrimental or beneficial. However, these criteria do not address particular timers $\tau$. Indeed, these criteria are incapable of determining if sharp restart with a particular timer $\tau$ is detrimental or beneficial. This section begins to address—via socioeconomic inequality indices—particular timers $\tau$. Specifically, this section shall address sharp restart with two specific and ‘natural’ timers: a timer whose value is the input’s mean $\mu$, and a timer whose value is the input’s median $m$.

Along this section the following representation of the output’s mean of equation (1) shall be used:

$$M(\tau) = \frac{\mu + \tau - E[|T - \tau|]}{2F(\tau)}.$$  

(17)

The derivation of equation (17) is detailed in the methods. The term $E[|T - \tau|]$ appearing in equation (17) is the MAD [64] between the input $T$ and the timer $\tau$.

3 Specifically, the median $m$ is the unique positive value at which the input’s distribution function and survival function intersect: $F(m) = \frac{1}{2} = F(m)$. As the input’s density function is considered to be positive-valued over the positive half-line, the input’s median is well defined indeed.
Figure 3. Lognormal example of the ‘DB’ scenario. In this example we consider the input $T$ to be lognormal: $\ln(T)$ is a normal random variable with an arbitrary mean, and with standard deviation $d$. The squared CV of this input is $\exp(d^2) - 1$, and the Gini index of this input is $\text{erf}(d/2)$ [73] (here $\text{erf}(z)$ denotes the Gauss error function). As functions of the positive standard-deviation parameter $d$: the squared CV is depicted in blue, and twice the Gini index is depicted in orange. The CV is larger than one if and only if $d > \sqrt{\ln(2)}$, and the Gini index is smaller than half if and only if $d < 2\text{erf}^{-1} \left( \frac{1}{2} \right)$. Hence, the scenario ‘DB’ holds when the standard-deviation parameter is in the range $\sqrt{\ln(2)} < d < 2\text{erf}^{-1} \left( \frac{1}{2} \right)$, which is indicated in the figure.

4.1. Mean timer

In this subsection we consider sharp restart with the particular timer $\tau = \mu$, the input’s mean. Namely, a timer that equals the mean of the task’s completion time $T$. As shall be shown below, sharp restart with this particular timer is intimately related to an inequality index of the input that is termed Pietra index [74–79].

From a functional-analysis perspective, equation (4) implies that the input’s Gini index $\mathcal{J}_{\text{Gini}}$ is based on the ‘$L_1$ distance’ between the Lorenz curve $y = L(x)$ and the diagonal line $y = x$. Switching from the $L_1$ distance to the ‘$L_\infty$ distance’ yields the input’s Pietra index [77]: $\mathcal{J}_{\text{Pietra}}$ is the maximal vertical distance between the input’s Lorenz curve $y = L(x)$ and the diagonal line $y = x$. Namely,

$$\mathcal{J}_{\text{Pietra}} = \max_{0 \leq q \leq 1} \left[ q - L(q) \right].$$ (18)

The Pietra-index representation of equation (18) exhibits no clear connection to equation (17), the output’s mean. To make the connection more apparent we use the following, alternative, Pietra-index representation [34]:

$$\mathcal{J}_{\text{Pietra}} = \frac{1}{2\mu} \mathbb{E} \left[ |T - \mu| \right].$$ (19)

The Pietra-index representation of equation (19) is based on the term $\mathbb{E} \left[ |T - \mu| \right]$, the MAD between the input $T$ and its mean $\mu$ [80]. The smaller this MAD—the closer is the Pietra index to its zero lower bound. Conversely, the larger this MAD—the closer is the Pietra index to its unit upper bound.

Setting $\tau = \mu$ in equation (17), and using equation (19), yields the formula

$$\frac{M(\mu) - \mu}{\mu} = \frac{\mathcal{F}(\mu) - \mathcal{J}_{\text{Pietra}}}{\mathcal{F}(\mu)}. $$ (20)
Equation (20) manifests, for the particular timer $\tau = \mu$, the effect of the input’s Pietra index $I_{\text{Pietra}}$ on the output’s mean $M(\mu)$. Indeed, set the threshold level $I_{\text{Pietra}} = F(\mu)$, the probability that the input exceeds its mean. Then, equation (20) yields the following pair of Pietra-index criteria:

- Sharp restart with the timer $\tau = \mu$ is detrimental if and only if the input’s Pietra index is smaller than its threshold level, $I_{\text{Pietra}} < l_{\text{Pietra}}$.
- Sharp restart with the timer $\tau = \mu$ is beneficial if and only if the input’s Pietra index is larger than its threshold level, $I_{\text{Pietra}} > l_{\text{Pietra}}$.

Jensen’s inequality implies that the squared Pietra-index MAD is no larger than the input’s variance, i.e. $\mathbb{E} \left[ (T - \mu)^2 \right] \leq \mathbb{E} \left[ (T - \mu)^2 \right] = \sigma^2$. Consequently, twice the input’s Pietra index is no larger than the input’s CV, $2I_{\text{Pietra}} \leq 2\mu$. Also, the input’s Pietra index is no larger than the input’s Gini index, $I_{\text{Pietra}} \leq I_{\text{Gini}}$ [81].

These Pietra–CV and Pietra–Gini relations—combined, respectively, with the CV criteria of subsection 2.2, and with the Gini-index criteria of subsection 3.1—yield the following Pietra-index corollary: if the input’s Pietra index is larger than half, $I_{\text{Pietra}} > \frac{1}{2}$, then there exist timers $\tau$ for which sharp restart is beneficial. In case the input’s median is no-larger than the input’s mean, $m \leq \mu$, the Pietra-index criteria ‘upgrade’ the Pietra-index corollary as follows: if the input’s Pietra index is larger than half, $I_{\text{Pietra}} > \frac{1}{2}$, then sharp restart with the timer $\tau = \mu$ is beneficial.

Last, with regard to being larger than the level $\frac{1}{2}$, we note the following resemblances between the CV, Gini, and Pietra indices. CV index: $I_{\text{CV}} > \frac{1}{2}$ is equivalent to $\mathbb{E} \left[ (T - \mu)^2 \right] > \mu^2$. Gini index: $I_{\text{Gini}} > \frac{1}{2}$ is equivalent to $\mathbb{E} \left[ (T_1 - T_2)^2 \right] > \mu$. Pietra index: $I_{\text{Pietra}} > \frac{1}{2}$ is equivalent to $\mathbb{E} \left[ (T - \mu)^2 \right] > \mu$.

4.2. Median timer

In this subsection we consider sharp restart with the particular timer $\tau = m$, the input’s median. Namely, a timer that equals the median of the task’s completion time $T$. As shall be shown below, sharp restart with this particular timer is intimately related to an inequality index of the input that is termed vertical-diameter index (Vdiam index) [81, 82].

The description of the Vdiam index uses the Lorenz curve $y = L(x)$ and its corresponding complementary Lorenz curve, $y = \bar{L}(x)$ ($0 \leq x, y \leq 1$) [34]. The socioeconomic meaning of the complementary Lorenz curve is: the top (rich) 100x% of the society members possess 100y% of the society’s overall wealth. The complementary Lorenz curve $y = \bar{L}(x)$ has the same properties as the Lorenz curve $y = L(x)$, albeit one: it is concave rather than convex. Consequently, in the unit square, the complementary Lorenz curve $y = \bar{L}(x)$ is bounded from below by the diagonal line $y = x$.

As described above, the input’s Pietra index $I_{\text{Pietra}}$ is the ‘$L_\infty$ distance’ between the input’s Lorenz curve $y = L(x)$ and the diagonal line $y = x$. Switching from the ‘$L_\infty$ distance’ between the Lorenz curve and the diagonal line, to the ‘$L_\infty$ distance’ between the Lorenz curve and the complementary Lorenz curve, yields the input’s Vdiam index [81, 82]: $I_{\text{Vdiam}}$ is the maximal vertical distance between the input’s Lorenz curve $y = L(x)$ and the complementary Lorenz curve $y = \bar{L}(x)$. It turns out that $I_{\text{Vdiam}}$ is the normalized vertical distance—a long the vertical line $x = \frac{1}{2}$—between the input’s Lorenz curve $y = L(x)$ and the diagonal line $y = x$ [81, 82]. Namely,

$$I_{\text{Vdiam}} = 1 - 2L\left(\frac{1}{2}\right). \quad (21)$$
The Vdiam-index representation of equation \((21)\) exhibits no clear connection to equation \((17)\), the output’s mean. To make the connection more apparent we use the following, alternative, Vdiam-index representation [34]:

\[
I_{\text{Vdiam}} = \frac{1}{\mu} \mathbb{E} \left[ \left| T - m \right| \right].
\]  
\[(22)\]

The Vdiam-index representation of equation \((22)\) is based on the term \(\mathbb{E} \left[ \left| T - m \right| \right]\), the MAD between the input \(T\) and its median \(m\). The smaller this MAD—the closer is the Vdiam index to its zero lower bound. Conversely, the larger this MAD—the closer is the Vdiam index to its unit upper bound.

Setting \(\tau = m\) in equation \((17)\), and using equation \((22)\), yields the formula

\[
M(m) - \frac{\mu}{\mu} = \frac{m}{\mu} - I_{\text{Vdiam}}.
\]  
\[(23)\]

Equation \((23)\) manifests, for the particular timer \(\tau = m\), the effect of the input’s Vdiam index \(I_{\text{Vdiam}}\) on the output’s mean \(M(m)\). Indeed, set the threshold level \(l_{\text{Vdiam}} = \frac{\mu}{\mu}\), the ratio of the input’s median to the input’s mean. Then, equation \((23)\) yields the following pair of Vdiam-index criteria:

- **Sharp restart with the timer \(\tau = m\) is detrimental if and only if the input’s Vdiam index is smaller than its threshold level, \(I_{\text{Vdiam}} < l_{\text{Vdiam}}\).**
- **Sharp restart with the timer \(\tau = m\) is beneficial if and only if the input’s Vdiam index is larger than its threshold level, \(I_{\text{Vdiam}} > l_{\text{Vdiam}}\).**

### 4.3. Discussion

The Pietra-index criteria of subsection 4.1 and the Vdiam-index criteria of subsection 4.1 are highly practical tools for determining the mean-performance of sharp restart. As the CV criteria and the Gini-index criteria, the Pietra-index criteria and the Vdiam-index criteria both use partial information of the input’s statistical distribution. Indeed, the Pietra index uses the input’s mean and the MAD of equation \((19)\); and the Vdiam index uses the input’s mean, the input’s median, and the MAD of equation \((22)\). The input’s mean and median, as well as the corresponding MADs, are easily and reliably estimated from finite samples of input observations.

We conclude this section with remarks regarding the input’s mean and median, the CV criteria, and the Vdiam-index criteria. The remarks are based on an optimization-problem perspective.

The ‘\(L_p\) distance’ between the input \(T\) and the timer \(\tau\) is

\[
D_p(\tau) = \left\{ \mathbb{E} \left[ \left| T - \tau \right|^p \right] \right\}^{1/p}.
\]
\[(24)\]

The parameter \(p\) of the \(L_p\) distance takes values in the range \(p \geq 1\), and its most notable values are \(p = 1\) and \(p = 2\). Specifically, the \(L_1\) distance is the aforementioned MAD \(\mathbb{E} \left[ \left| T - \tau \right| \right]\) between the input \(T\) and the timer \(\tau\). And, the \(L_2\) distance is the square root of the MSD \(\mathbb{E} \left[ \left| T - \tau \right|^2 \right]\) between the input \(T\) and the timer \(\tau\).

The inputs’ mean and median are intimately related, respectively, to the \(L_2\) and \(L_1\) distances. Indeed, consider the optimization problem \(\min_{0 < \tau < \infty} D_p(\tau)\). This optimization problem seeks the timer \(\tau\) that is closest—in the \(L_p\) distance—to the input \(T\). For the \(L_2\) distance the minimum is attained at the input’s mean, and the minimum value is the input’s standard deviation: \(\arg \min_{0 < \tau < \infty} D_2(\tau) = \mu\) and \(\min_{0 < \tau < \infty} D_2(\tau) = \sigma\). For the \(L_1\) distance the minimum is
attained at the input’s median, and the minimum value is the MAD between the input and its median: \( \arg\min_{0<\tau<\infty} D_1(\tau) = m \) and \( \min_{0<\tau<\infty} D_1(\tau) = E[|T-m|] \).

The CV criteria of subsection 2.2 assert that—in order to determine the existence of timers \( \tau \) for which the sharp restart algorithm is detrimental or beneficial—one has to compare the input’s standard deviation \( \sigma \) to the input’s mean \( \mu \). In other words, the CV criteria compare the minimal value \( \min_{0<\tau<\infty} D_2(\tau) \) to the minimizing point \( \arg\min_{0<\tau<\infty} D_2(\tau) \).

Equation (23) can be re-written as follows:

\[
M(m) - \mu = m - E[|T-m|].
\]

Consequently, the Vdiam-index criteria of subsection 4.2—regarding restart with the particular timer \( \tau = m \)—can be re-formulated as follows: sharp restart is detrimental if and only if \( E[|T-m|] < m \), and is beneficial if and only if \( E[|T-m|] > m \). In other words, the Vdiam-index criteria compare the minimal value \( \min_{0<\tau<\infty} D_1(\tau) \) to the minimizing point \( \arg\min_{0<\tau<\infty} D_1(\tau) \).

Thus, from the perspective of the optimization problem \( \min_{0<\tau<\infty} D_p(\tau) \), there is an analogy between the CV criteria of subsection 2.2 and the Vdiam-index criteria of subsection 4.2. Indeed, as pointed out above, both these criteria compare the minimal value \( \min_{0<\tau<\infty} D_p(\tau) \) to the minimizing point \( \arg\min_{0<\tau<\infty} D_p(\tau) \). Notably, the CV criteria and the Vdiam-index criteria are based on rather ‘neat’ formulae—equations (2) and (25), respectively.

5. General timers

The previous section addressed two particular timers: \( \tau = \mu \), where \( \mu \) is the input’s mean; and \( \tau = m \), where \( m \) is the input’s median. To each of these timers a specific inequality index of the input was matched, and using these inequality indices it was determined when sharp restart (with these timers) is detrimental or beneficial. In this section, we show that the connection between sharp restart and statistical inequality extends from the particular timers \( \tau = \mu \) and \( \tau = m \) to general sharp-restart timers \( 0 < \tau < \infty \).

5.1. Sampling

As noted in subsection 2.3, in order to measure the inherent ‘socioeconomic inequality’ of the input \( T \), this random variable was deemed to be the wealth of a randomly-sampled member of a virtual society. Now, with respect to this virtual society, consider also a randomly-sampled dollar. Namely, sample at random a single dollar from the society’s overall wealth, and set \( T_{dol} \) to be the wealth of the society member to whom the randomly-sampled dollar belongs [34].

A temporal description of the random variable \( T_{dol} \) is as follows. Perform, repeatedly and independently, \( n \) rounds of the task under consideration (whose completion time is the random variable \( T \)). Specifically: starting at time \( t = 0 \), perform the task for the first round; then, upon the first completion, start performing the task for the second round; then, upon the second completion, start performing the task for the third round; and continue so on and so forth for \( n \) consecutive rounds. Denoting by \( \{T_1, \ldots, T_n\} \) the durations of the tasks—these durations being IID copies of the input \( T \)—we obtain that: \( T_1 \) is the completion time of the first round, \( T_1 + T_2 \) is the completion time of the second round, \( \ldots \), and \( T_1 + \cdots + T_n \) is the completion time of the last round. Now, place an observer at a random time epoch along the temporal interval \([0, T_1 + \cdots + T_n]\). This random placement of the observer implies that: the observer samples the task that is performed at round \( k \) with probability \( T_k/(T_1 + \cdots + T_n) \). In the limit \( n \to \infty \), the duration of the task that is sampled by the observer converges, in law, to the random variable \( T_{dol} \).
As the input $T$, also $T_{dol}$ is a positive-valued random variable. In terms of the input’s density function and mean, the density function of the random variable $T_{dol}$ is $f_{dol} = \frac{1}{\mu} f(t)$ ($t > 0$) [34]. In turn, the distribution and survival functions of the random variable $T_{dol}$ are, respectively:

$$ F_{dol}(t) = \frac{1}{\mu} \int_0^t s f(s) \, ds \quad (t \geq 0), $$

and

$$ \bar{F}_{dol}(t) = \frac{1}{\mu} \int_t^{\infty} s f(s) \, ds \quad (t \geq 0). $$

The input’s Lorenz curve $y = L(x)$ couples together the input’s distribution function $F(t)$, and the distribution function $F_{dol}(t)$ of the random variable $T_{dol}$. Indeed, the socioeconomic definitions of the Lorenz curve $y = L(x)$, and of the distribution functions $F(t)$ and $F_{dol}(t)$, imply that [81, 82]:

$$ L[F(t)] = F_{dol}(t) \quad (t \geq 0). $$

An alternative formulation of equation (28) is:

$$ F(t) = L^{-1}[F_{dol}(t)] \quad (t \geq 0), $$

where $x = L^{-1}(y)$ denotes the inverse function of the input’s Lorenz curve $y = L(x)$.

### 5.2. Vertical-index criteria

Subsection 2.4 introduced, for a fixed number $q$ (where $0 < q < 1$), the inequality index $[q - L(q)]/q$: the ‘normalized’ vertical distance—along the vertical line $x = q$—between the Lorenz curve $y = L(x)$ and the diagonal line $y = x$. With respect to the input’s distribution function $F(t)$, set $q$ to be the quantile corresponding to the timer $\tau$, i.e. $q = F(\tau)$. Equation (28) implies that

$$ \frac{q - L(q)}{q} = 1 - \frac{F_{dol}(\tau)}{F(\tau)}. \quad (30) $$

The quantity appearing in equation (30) is an inequality index with an underpinning vertical Lorenz-curve geometric meaning. This quantity is henceforth termed the input’s vertical index, and is denoted $I_v(\tau)$.

Introduce the threshold level

$$ l_v(\tau) = \frac{\tau \bar{F}(\tau)}{\mu F(\tau)}. \quad (31) $$

With the vertical index $I_v(\tau)$ of equation (30) at hand, as well as the threshold level $l_v(\tau)$ of equation (31), the following formula is presented:

$$ M(\tau) - \mu \frac{\tau \bar{F}(\tau)}{\mu F(\tau)} = l_v(\tau) - I_v(\tau). \quad (32) $$

The derivation of equation (32) is detailed in the methods.

Equation (32) manifests the effect of the input’s vertical index $I_v(\tau)$ on the output’s mean $M(\tau)$. Indeed, equation (32) yields the following pair of vertical-index criteria:
• Sharp restart with timer $\tau$ is detrimental if and only if the input’s vertical index is smaller than its threshold level, $S_V(\tau) < l_V(\tau)$.

• Sharp restart with timer $\tau$ is beneficial if and only if the input’s vertical index is larger than its threshold level, $S_V(\tau) > l_V(\tau)$.

In particular, setting $\tau = \mu$ in the vertical-index criteria yields the Pietra-index criteria of subsection 4.1. And, setting $\tau = m$ in the vertical-index criteria yields the Vdiam-index criteria of subsection 4.2.

5.3. Horizontal-index criteria

The vertical-index criteria of the previous subsection are based on the vertical distance between the Lorenz curve $y = L(x)$ and the diagonal line $y = x$. This subsection shifts from the vertical-distance perspective to a horizontal-distance perspective.

For a fixed number $q$ (where $0 < q < 1$), consider the horizontal line $y = q$ of the unit square. The horizontal distance—along the horizontal line $y = q$—between the Lorenz curve $y = L(x)$ and the diagonal line $y = x$ is: $L^{-1}(q) - q$. This horizontal distance takes values in the range $[0, 1 - q]$. Consequently, the ‘normalized’ horizontal distance $[L^{-1}(q) - q] / (1 - q)$ takes values in the unit interval $[0, 1]$. It is straightforward to check that the normalized horizontal distance $[L^{-1}(q) - q] / (1 - q)$ meets the three inequality-index properties that were postulated in subsection 2.3.

With respect to the distribution function $F_{\text{dol}}(t)$ of the random variable $T_{\text{dol}}$, set $q$ to be the quantile corresponding to the timer $\tau$, i.e. $q = F_{\text{dol}}(\tau)$. Equation (29) implies that

$$\frac{L^{-1}(q) - q}{1 - q} = 1 - \frac{\bar{F}(\tau)}{\bar{F}_{\text{dol}}(\tau)}. \quad (33)$$

The quantity appearing in equation (33) is an inequality index with an underpinning horizontal Lorenz-curve geometric meaning. This quantity is henceforth termed the input’s horizontal index, and is denoted $S_H(\tau)$.

Introduce the threshold level

$$l_H(\tau) = \frac{\tau}{\tau + \mu}. \quad (34)$$

With the horizontal index $S_H(\tau)$ of equation (33) at hand, as well as the threshold level $l_H(\tau)$ of equation (34), we present the following formula:

$$\frac{M(\tau) - \mu}{\mu} = \frac{\mu + \mu \bar{F}_{\text{dol}}(\tau)}{\mu \bar{F}(\tau)} \cdot \left[ l_H(\tau) - S_H(\tau) \right]. \quad (35)$$

The derivation of equation (35) is detailed in the methods.

Equation (35) manifests the effect of the input’s horizontal index $S_H(\tau)$ on the output’s mean $M(\tau)$. Indeed, equation (35) yields the following pair of horizontal-index criteria:

• Sharp restart with timer $\tau$ is detrimental if and only if the input’s horizontal index is smaller than its threshold level, $S_H(\tau) < l_H(\tau)$.

• Sharp restart with timer $\tau$ is beneficial if and only if the input’s horizontal index is larger than its threshold level, $S_H(\tau) > l_H(\tau)$.

As in equation (29) above, $x = L^{-1}(y)$ denotes the inverse function of the input’s Lorenz curve $y = L(x)$.
In particular, setting $\tau = \mu$ in the horizontal-index criteria yields the Pietra-index criteria of subsection 4.1.

5.4. Discussion

We conclude this section with remarks regarding the vertical-index criteria of subsection 5.2 and the horizontal-index criteria subsection 5.3. Also, using the horizontal-index criteria, we shall elaborate on a particular timer: $\tau = m_{dol}$, the median of the random variable $T_{dol}$.

In this section, given a positive timer $\tau$, two inequality indices of the input $T$ were matched to the timer: the vertical index $\mathcal{I}_V(\tau)$ of equation (30), and the horizontal index $\mathcal{I}_H(\tau)$ of equation (33). The vertical index $\mathcal{I}_V(\tau)$ quantifies the disparity, at the time point $t = \tau$, between the distribution functions of the random variables $T$ and $T_{dol}$. Similarly, the horizontal index $\mathcal{I}_H(\tau)$ quantifies the disparity, at the time point $t = \tau$, between the survival functions of the random variables $T$ and $T_{dol}$.

Accompanying the input’s vertical and horizontal indices, $\mathcal{I}_V(\tau)$ and $\mathcal{I}_H(\tau)$, are corresponding threshold levels: $l_V(\tau)$ of equation (31), and $l_H(\tau)$ of equation (34). The comparisons between the vertical and horizontal indices and their corresponding threshold levels determines if sharp restart with the timer $\tau$ is detrimental or beneficial. While these comparisons are equivalent, they provide two different geometric Lorenz-curve perspectives—vertical and horizontal.

The Pietra index of subsection 4.1 emanated from maximizing the vertical distance between the Lorenz curve $y = L(x)$ and the diagonal line $y = x$. Analogously, the Vdiam index of subsection 4.1 emanated from maximizing the vertical distance between the Lorenz curve $y = L(x)$ and the complementary Lorenz curve $y = \bar{L}(x)$. Shifting from the vertical perspective to the horizontal perspective has the following effects. As in the vertical perspective, maximizing the horizontal distance between the Lorenz curve $y = L(x)$ and the diagonal line $y = x$ yields the Pietra index [81, 82]. However, maximizing the horizontal distance between the Lorenz curve $y = L(x)$ and the complementary Lorenz curve $y = \bar{L}(x)$ yields an inequality index of the input that is termed horizontal-diameter index [81, 82].

It turns out that the input’s Hdiam index $\mathcal{I}_{Hdiam}$ is the normalized horizontal distance—along the horizontal line $y = \frac{1}{2}$—between the input’s Lorenz curve $y = L(x)$ and the diagonal line $y = x$ [81, 82]. Namely,

$$\mathcal{I}_{Hdiam} = 2L^{-1} \left( \frac{1}{2} \right) - 1.$$  \hspace{1cm} (36)

This index is intimately related to the median $m_{dol}$ of the random variable $T_{dol}$. Indeed, equations (33) and (36) imply that $\mathcal{I}_{Hdiam} = \mathcal{I}_H(m_{dol})$.

The Hdiam index has several representations. One of the input’s Hdiam-index representations is [34]:

$$\mathcal{I}_{Hdiam} = \frac{1}{m_{dol}} \mathbb{E} \left[ |T - m_{dol}| \right].$$ \hspace{1cm} (37)

The Hdiam-index representation of equation (37) is based on the term $\mathbb{E} \left[ |T - m_{dol}| \right]$, the MAD between the input $T$ and the median $m_{dol}$ (which is the median of the random variable $T_{dol}$, not of the input $T$). The smaller this MAD—the closer is the Vdiam index to its zero lower bound. Conversely, the larger this MAD—the closer is the Vdiam index to its unit upper bound.

Setting $\tau = m_{dol}$ in equation (35), and using equation (34), yields the formula

$$\frac{M(m_{dol}) - \mu}{\mu} = \frac{m_{dol}}{m_{dol} + \mu} - \mathcal{I}_{Hdiam}.$$ \hspace{1cm} (38)
Equation (38) manifests, for the particular timer $\tau = m_{\text{dol}}$, the effect of the input’s Hdiam index $I_{\text{Hdiam}}$ on the output’s mean $M(m_{\text{dol}})$. Indeed, set the threshold level $l_{\text{Hdiam}} = \frac{m_{\text{dol}}}{m_{\text{dol}}+\mu}$. Then, equation (38) yields the following pair of Hdiam-index criteria:

- Sharp restart with the timer $\tau = m_{\text{dol}}$ is detrimental if and only if the input’s Hdiam index is smaller than its threshold level, $I_{\text{Hdiam}} < l_{\text{Hdiam}}$.
- Sharp restart with the timer $\tau = m_{\text{dol}}$ is beneficial if and only if the input’s Hdiam index is larger than its threshold level, $I_{\text{Hdiam}} > l_{\text{Hdiam}}$.

6. Conclusion

Following up on [1], in this paper we continued exploring the effect of sharp restart on mean performance. Using a positive deterministic timer $\tau$, sharp restart is applied to a general task whose completion time is a positive random variable $T$. Namely, initiating the task at time $t = 0$, the task is restarted—as long as it is not accomplished—at the fixed time epochs $t = \tau, 2\tau, 3\tau, \ldots$. The mean completion time of the task ‘under restart’, $M(\tau)$, is compared to the task’s mean completion time, $E[T] = \mu$. This comparison determines if sharp restart improves mean performance, $M(\tau) < \mu$, or if it worsens mean performance, $M(\tau) > \mu$.

The analysis presented in this paper established that inequality indices of the random variable $T$ hold a treasure trove of information regarding the effect of sharp restart on mean performance. We showed that three inequality indices—CV, Gini, and Bonferroni—determine the very existence of timers with which sharp restart improves/worsens mean performance. Given a specific timer $\tau$, we further showed that there are inequality indices that relate to this timer, and that these inequality indices determine if sharp restart with the specific timer $\tau$ improves/worsens mean performance.

The novel results established in this paper provide a detailed ‘inequality roadmap’ for sharp restart: eight pairs of universal inequality criteria that determine the effect of sharp restart on mean performance. The underpinning inequality indices are summarized in table 1, and the criteria are summarized in table 2. Each pair of criteria comprises a specific inequality index $\mathcal{I}$, and a corresponding threshold level $l$. All eight pairs of criteria share in common the following threshold pattern. If the index is larger than its corresponding threshold level then mean performance improves: $\mathcal{I} > l \Rightarrow M(\tau) < \mu$. And, if the index is smaller than its corresponding threshold level then mean performance worsens: $\mathcal{I} < l \Rightarrow M(\tau) > \mu$.

The first two pairs of inequality criteria—CV and Gini—have a fixed threshold level $(l = \frac{1}{4})$ that is independent of the task’s completion time. Conversely, the latter six pairs of inequality criteria have threshold levels that depend on the task’s completion time. For the latter six pairs of inequality criteria, the common threshold pattern admits an alternative interpretation that is described as follows. Given the value of the inequality index $\mathcal{I}$, there is a critical ‘mean value’ $\mu_c$ that is based on $\mathcal{I}$. If the task’s mean completion time is larger than the critical value then mean performance improves: $\mu > \mu_c \Rightarrow M(\tau) < \mu$. And, if the task’s mean completion time is smaller than the critical value then mean performance worsens: $\mu < \mu_c \Rightarrow M(\tau) > \mu$. The critical values $\mu_c$ are specified in the right column of table 2.

A natural question that arises is: why need eight different pairs of universal inequality criteria? The answer to this question is: partial information. Indeed, if full information of the statistical distribution of the task’s completion time $T$ is known then equation (1) can be used in order to determine the effect of sharp restart on mean performance. However, in practice, only partial information of the statistical distribution of $T$ is known—and in this common real-world situation the universal inequality criteria come into play. Each pair of criteria exploit a
Table 1. The eight inequality indices that underpin, respectively, the eight pairs of universal inequality criteria for sharp restart. These inequality indices have various equivalent representations [34], two of which are presented in the table: representations based on mean square/absolute deviations (MSD/MAD) of the task’s completion time $T$; and representations based on the Lorenz curve $L(\cdot)$ of the task’s completion time $T$ (see section 2.4 for the details). Remarks regarding specific inequality indices are the following. CV and Gini: $T_1$ and $T_2$ are two IID copies of the random variable $T$. Vdiam: $m$ is the median of the random variable $T$. Hdiam: $m_{dol}$ is the median of the random variable $T_{dol}$ (see section 5.1 for the details). Vertical: $q = F(\tau)$, where $F(\cdot)$ is the distribution function of the random variable $T$. Horizontal: $q = F_{dol}(\tau)$, where $F_{dol}(\cdot)$ is the distribution function of the random variable $T_{dol}$ (see section 5.1 for the details).

| Inequality index | MSD/MAD representation | Lorenz representation |
|------------------|------------------------|----------------------|
| CV: $\mathcal{I}_{CV} = \frac{1}{2\sigma^2} E [\{T_1 - T_2\}^2]$ | $-\int_0^1 q \cdot L(q) dq$ |
| Gini: $\mathcal{I}_{Gini} = \frac{1}{\sigma^2} E [\{T_1 - T_2\}]$ | $2 \int_0^1 q \cdot L(q) dq$ |
| Bonferroni: $\mathcal{I}_{Bonf} = -\int_0^1 q \cdot L(q) dq$ |
| Pietra: $\mathcal{I}_{Pietra} = \frac{1}{\sigma^2} E [\{T - \mu\}]$ | $\max_{0 \leq q \leq 1} [q - L(q)]$ |
| Vdiam: $\mathcal{I}_{Vdiam} = \frac{1}{\mu} E [\{T - m\}]$ | $1 - 2L(\frac{1}{2})$ |
| Hdiam: $\mathcal{I}_{Hdiam} = \frac{1}{m_{dol}} E [\{T - m_{dol}\}]$ | $2L^{-1}(\frac{1}{2}) - 1$ |
| Vertical: $\mathcal{I}_V(\tau) = -\frac{q \cdot L(q)}{q}$ |
| Horizontal: $\mathcal{I}_H(\tau) = -\frac{L^{-1}(q) \cdot q}{1-q}$ |

different type of partial information, and hence: the partial information that is available points out to the pair of criteria that can be used.

The inequality indices that underpin the inequality criteria have various equivalent representations [34]. As specified in table 1, one such representation is based on mean square/absolute deviations (MSD/MAD) of the task’s completion time $T$; these deviations can be easily and reliably estimated from empirical data. Thus, the application of the inequality criteria whose underpinning inequality indices admit a MSD/MAD representation is highly practical. Indeed, for example: scientists from various disciplines are well accustomed to estimating the CV, and estimating the Gini index is common practice in economics and in the social sciences; consequently, the use of the CV and Gini criteria should be straightforward to all.

Given a task, a ‘natural’ sharp-restart timer to consider is the mean of the task’s completion time, $\tau = \mu$. (Recall that the inequality index that corresponds to the timer $\tau = \mu$ is the Pietra
Table 2. Eight pairs of universal inequality criteria that determine the mean performance of sharp restart. For each pair of criteria, the table’s columns specify: the inequality index \( I \) on which the criteria are based; the threshold level \( l \) that corresponds to the inequality index; and the timer parameters \( \tau \) for which the criteria apply. Comparing the inequality index \( I \) to the threshold level \( l \), the criteria assert that: if \( I > l \) then mean performance improves; and if \( I < l \) then mean performance worsens. For each pair of criteria—excluding the CV and Gini criteria—the table’s right column further specifies the critical ‘mean value’ \( \mu_c \) that is based on the inequality-index value \( I \). Comparing the task’s mean completion time \( \mu \) to the critical ‘mean value’ \( \mu_c \), the criteria assert that: if \( \mu > \mu_c \) then mean performance improves; and if \( \mu < \mu_c \) then mean performance worsens. Remarks regarding specific inequality indices are the following. Bonferroni: the value \( \nu \) is given by equation (12). Pietra and Vertical: \( F(\cdot) \) and \( \bar{F}(\cdot) \) are, respectively, the distribution function and the survival function of the task’s completion time \( T \).

| Inequality index \( I \) | Threshold level \( l \) | Timer Parameters | Critical value \( \mu_c \) |
|--------------------------|-------------------------|------------------|--------------------------|
| \( I_{CV} \)             | \( \frac{1}{2} \)      | Existence        | —                        |
| \( I_{Gini} \)           | \( \frac{1}{3} \)      | Existence        | —                        |
| \( I_{Bonf} \)           | \( \frac{\mu-\bar{\mu}}{\mu} \) | Existence | \( \frac{\mu}{1+I_{Bonf}} \) |
| \( I_{Pietra} \)         | \( F(\mu) \)           | \( \tau = \mu \) | \( \bar{F}^{-1}(I_{Pietra}) \) |
| \( I_{Vdiam} \)          | \( \frac{m}{\bar{v}} \) | \( \tau = m \)   | \( \frac{m}{I_{Vdiam}} \) |
| \( I_{Hdiam} \)          | \( \frac{m \mu}{\mu_0 + \mu} \) | \( \tau = m\mu_0 \) | \( \frac{m \mu}{I_{Hdiam}} \) |
| \( I_{V} \)              | \( \frac{\bar{v}}{\mu} \) | \( 0 < \tau < \infty \) | \( \frac{\bar{v}}{F_{V}(\tau)} \) |
| \( I_{H} \)              | \( \frac{\bar{v}}{\mu} \) | \( 0 < \tau < \infty \) | \( \frac{\bar{v}}{F_{H}(\tau)} \) |

Index.) Utilizing relations between the CV, Gini, and Pietra inequality indices, the two following universal Pietra corollaries were also established. If the Pietra index is larger than the level \( \frac{1}{2} \) then there exist timers with which sharp restart is beneficial. Moreover, if—in addition to the Pietra index being larger than the level \( \frac{1}{2} \)—the median of \( T \) is no-larger than its mean, \( m \leq \mu \), then: sharp restart with the specific timer \( \tau = \mu \) is beneficial. (We note that, for positive random variables, the scenario ‘median \( \leq \) mean’ is prevalent [83].) As the Pietra index admits a MAD representation, the Pietra criteria and corollaries are highly practical and easy to use.

In conclusion, this paper unveiled profound connections between two seemingly unrelated topics: the measurement of socioeconomic inequality on the one hand, and the mean performance of sharp restart on the other hand. These connections have major implications, practically and theoretically alike. Practically, the paper established a whole new set of universal criteria that are highly applicable tools in the common real-world situation of partial information. Theoretically, the paper yielded—with unprecedented mathematical precision and resolution—a powerful and overarching take-home-message: restart improves mean performance when the underlying statistical heterogeneity is high; and it worsens mean performance when the underlying statistical heterogeneity is low. As sharp restart can match the mean-performance of any other restart protocol [36, 84], the results presented here are relevant to the multidisciplinary field of restart research at large.
7. Methods

7.1. Derivation of equation (2)

Introduce the function

\[ f_{\text{res}}(t) = \frac{1}{\mu} F(t) \]  

(\( t \geq 0 \)). As the integral of the input’s survival function \( \bar{F}(t) \) is the input’s mean, \( \mu = \int_0^{\infty} \bar{F}(t) \, dt \), the function \( f_{\text{res}}(t) \) is a probability density: it is non-negative, \( f_{\text{res}}(t) \geq 0 \); and it has a unit integral, \( \int_0^{\infty} f_{\text{res}}(t) \, dt = 1 \). In effect, \( f_{\text{res}}(t) \) is the density function of the input’s ‘residual lifetime’ [85]. The corresponding distribution and survival functions are \( F_{\text{res}}(t) = \int_0^{t} f_{\text{res}}(s) \, ds \) and \( \bar{F}_{\text{res}}(t) = \int_t^{\infty} f_{\text{res}}(s) \, ds \). Moreover, the corresponding mean is

\[ \mu_{\text{res}} = \int_0^{\infty} F_{\text{res}}(t) \, dt = \frac{1}{\mu} \int_0^{\infty} t \bar{F}(t) \, dt = \frac{1}{2\mu} \int_0^{\infty} t^2 \bar{f}(t) \, dt \]  

(40)

In equation (40) we used integration by parts, and the following representation of the input’s variance:

\[ \sigma^2 = \mathbb{E}[T^2] - \mathbb{E}[T]^2. \]

Dividing both sides of equation (1) by \( \mu \), and using the distribution function \( F_{\text{res}}(t) \), we have

\[ \frac{M(\tau)}{\mu} = \frac{F_{\text{res}}(\tau)}{F(\tau)}. \]  

(41)

Equation (41), together with the coupling between distribution and survival functions, implies that

\[ \frac{M(\tau) - \mu}{\mu} = \frac{M(\tau)}{\mu} - 1 = \frac{F_{\text{res}}(\tau)}{F(\tau)} - 1 = \frac{F(\tau) - F_{\text{res}}(\tau)}{F(\tau)} = \frac{\bar{F}(\tau) - \bar{F}_{\text{res}}(\tau)}{F(\tau)}. \]  

(42)

In turn, cross multiplying the right-hand side and the left-hand side of equation (42) we obtain

\[ [M(\tau) - \mu] F(\tau) = \mu \left[ \bar{F}(\tau) - \bar{F}_{\text{res}}(\tau) \right]. \]  

(43)

Integrating both sides of equation (43), and using equation (40), yields equation (2):

\[ \int_0^{\infty} [M(\tau) - \mu] F(\tau) \, d\tau = \mu \int_0^{\infty} \left[ \bar{F}(\tau) - \bar{F}_{\text{res}}(\tau) \right] \, d\tau \]

\[ = \mu \left[ \int_0^{\infty} \bar{F}(\tau) \, d\tau - \int_0^{\infty} \bar{F}_{\text{res}}(\tau) \, d\tau \right] = \mu \left( \mu - \mu_{\text{res}} \right) \]

\[ = \mu^2 - \frac{1}{2} (\sigma^2 + \mu^2) = \frac{1}{2} \left( \mu^2 - \sigma^2 \right). \]  

(44)

7.2. Derivation of equations (6) and (7)

The output of the sharp-restart algorithm admits the following stochastic representation [1]:

\[ T_R = \min \{ T, \tau \} + T_R' \cdot I \{ T > \tau \}. \]  

(45)
where $T'_R$ is an IID copy of the output $T_R$, and where $I \{ T > \tau \}$ is the indicator of the event \{ $T > \tau$ \} (i.e. the indicator is 1 if the event occurs, and it is 0 if the event does not occur). Applying expectation to both sides of equation (45) yields

$$E[T_R] = E\left[ \min \{ T, \tau \} \right] + E\left[ T'_R \cdot I \{ T > \tau \} \right]. \quad (46)$$

As $T'_R$ is an IID copy of $T_R$, we have

$$E\left[ T'_R \cdot I \{ T > \tau \} \right] = E[T'_R] \cdot E\left[ I \{ T > \tau \} \right] = E[T'_R] \cdot \Pr(T > \tau) = E[T_R] \cdot \bar{F}(\tau). \quad (47)$$

And, substituting equation (47) in equation (46) we have

$$E[T_R] = E\left[ \min \{ T, \tau \} \right] + E[T_R] \cdot \bar{F}(\tau). \quad (48)$$

In turn, equation (48) implies that

$$E[T_R] \cdot F(\tau) = E\left[ \min \{ T, \tau \} \right], \quad (49)$$

and equation (49) yields equation (6).

In what follows $T_1$ and $T_2$ are IID copies of the input $T$. The distribution function of the random variable $\max \{ T_1, T_2 \}$ is $F(t)$ ($t \geq 0$), and consequently its density function is: $f_{\max}(t) = 2F(t)f(t)$ ($t > 0$). Multiplying both sides of equation (49) by the term $f(\tau)$, and using the density function $f_{\max}(t)$, we have

$$M(\tau) f_{\max}(\tau) = 2E\left[ \min \{ T, \tau \} \right] f(\tau). \quad (50)$$

Integrating equation (50) yields

$$\int_0^\infty M(\tau) f_{\max}(\tau) d\tau = 2 \int_0^\infty E\left[ \min \{ T, \tau \} \right] f(\tau) d\tau$$

$$= 2 \int_0^\infty \left( \int_0^\tau \min \{ t, \tau \} f(t) dt \right) f(\tau) d\tau$$

$$= 2 \int_0^\infty \int_0^\tau \min \{ t, \tau \} f(t) f(\tau) dt d\tau$$

$$= 2E\left[ \min \{ T_1, T_2 \} \right]. \quad (51)$$

As $f_{\max}(t)$ is a density function, we obtain equation (7):

$$\int_0^\infty \left[ M(\tau) - \frac{\mu}{\mu} \right] f_{\max}(\tau) d\tau = \int_0^\infty \left[ \frac{M(\tau)}{\mu} - 1 \right] f_{\max}(\tau) d\tau$$

$$= \frac{1}{\mu} \int_0^\infty M(\tau) f_{\max}(\tau) d\tau - \int_0^\infty f_{\max}(\tau) d\tau$$

$$= \frac{2}{\mu} E\left[ \min \{ T_1, T_2 \} \right] - 1 = 2(1 - G_{\text{Gini}}) - 1$$

$$= 1 - 2G_{\text{Gini}}. \quad (52)$$

In equation (52) we used equation (5).
7.3. Derivation of equations (11) and (13)

Integration by parts implies that
\[
\int_0^\tau \bar{F}(t) \, dt = \int_0^\tau tf(t) \, dt + \tau \bar{F}(\tau).
\] (53)

Combined together, equations (1), (53) and (9) yields equation (11):
\[
M(\tau) = \frac{1}{F(\tau)} \int_0^\tau F(t) \, dt = \frac{1}{F(\tau)} \int_0^\tau tf(t) \, dt + \tau \frac{\bar{F}(\tau)}{F(\tau)} = \phi(\tau) + \tau \frac{\bar{F}(\tau)}{F(\tau)}.
\] (54)

Note that we can write equation (54) in the following form
\[
M(\tau) = \phi(\tau) + \tau \frac{1}{F(\tau)} - \tau.
\] (55)

Multiplying both sides of equation (55) by the term \(f(\tau)\) yields
\[
M(\tau) f(\tau) = \phi(\tau) f(\tau) + \tau \frac{f(\tau)}{F(\tau)} - \tau f(\tau).
\] (56)

Integrating equation (56), and using equation (12), we have
\[
\int_0^\infty M(\tau) f(\tau) \, d\tau = \int_0^\infty \phi(\tau) f(\tau) \, d\tau + \int_0^\infty \tau \frac{f(\tau)}{F(\tau)} \, d\tau - \int_0^\infty \tau f(\tau) \, d\tau = \int_0^\infty \phi(\tau) f(\tau) \, d\tau + \nu - \mu.
\] (57)

As \(f(t)\) is a density function, we obtain equation (13):
\[
\int_0^\infty \left( \frac{M(\tau) - \mu}{\mu} \right) f(\tau) \, d\tau = \int_0^\infty \left[ \frac{M(\tau)}{\mu} - 1 \right] f(\tau) \, d\tau = \frac{1}{\mu} \int_0^\infty M(\tau) f(\tau) \, d\tau - \int_0^\infty f(\tau) \, d\tau = \frac{1}{\mu} \int_0^\infty \phi(\tau) f(\tau) \, d\tau + \nu - \mu = \frac{1}{\mu} \int_0^\infty \phi(\tau) f(\tau) \, d\tau + \frac{\nu}{\mu} - 2 = (1 - \mathcal{I}_{\text{Bonf}}) + \frac{\nu}{\mu} - 2 \equiv \frac{\nu - \mu}{\mu} - \mathcal{I}_{\text{Bonf}}.
\] (58)

In equation (58) we used equation (10).
7.4. Derivation of equation (17)

Note that

\[ \max \{T, \tau\} + \min \{T, \tau\} = T + \tau, \]

(59)

and

\[ \max \{T, \tau\} - \min \{T, \tau\} = |T - \tau|. \]

(60)

Subtracting equation (60) from equation (59) yields

\[ 2 \min \{T, \tau\} = T + \tau - |T - \tau|. \]

(61)

In turn, applying expectation to both sides of equation (61) we have

\[
2E \left[ \min \{T, \tau\} \right] = E \left[ T + \tau - |T - \tau| \right] \\
= E[T] + E[\tau] - E[|T - \tau|] \\
= \mu + \tau - E[|T - \tau|]. 
\]

(62)

Combining together equations (6) and (62) yields equation (17):

\[
M(\tau) = \frac{E \left[ \min \{T, \tau\} \right]}{F(\tau)} = \frac{\mu + \tau - E[|T - \tau|]}{2F(\tau)}. 
\]

(63)

7.5. Derivation of equations (32) and (35)

Combining together equations (54) and (26) we have

\[
M(\tau) = \frac{1}{F(\tau)} \int_0^\tau tf(t) \, dt + \frac{\tau F(\tau)}{F(\tau)} = \frac{\mu}{F(\tau)} F_d(\tau) + \frac{\tau F(\tau)}{F(\tau)}. 
\]

(64)

Equation (64) yields equation (32):

\[
\frac{M(\tau) - \mu}{\mu} = \frac{1}{\mu} M(\tau) - 1 = \frac{1}{\mu} \left[ \frac{F_d(\tau)}{F(\tau)} + \frac{\tau}{F(\tau)} \right] - 1 \\
= \frac{\tau F(\tau)}{\mu F(\tau)} \left[ 1 - \frac{F_d(\tau)}{F(\tau)} \right]. 
\]

(65)

Equation (65) implies that

\[
[M(\tau) - \mu] F(\tau) = \tau [1 - F(\tau)] - \mu [F(\tau) - F_d(\tau)] \\
= \tau \bar{F}(\tau) - \mu [\bar{F}_d(\tau) - \bar{F}(\tau)] = (\tau + \mu) \bar{F}(\tau) - \mu \bar{F}_d(\tau). 
\]

(66)

Dividing the left-hand and the right-hand sides of equation (66) by the term \((\tau + \mu)\bar{F}_d(\tau)\) yields

\[
\frac{M(\tau) - \mu}{\mu} \left[ \frac{\mu}{\tau + \mu} F_d(\tau) \right] = \frac{M(\tau) - \mu}{\tau + \mu} \left[ \frac{\tau + \mu}{F_d(\tau)} \right] \\
= \frac{1}{\tau + \mu} \left[ \frac{(\tau + \mu) \bar{F}(\tau) - \mu \bar{F}_d(\tau)}{\bar{F}_d(\tau)} \right]. 
\]
\[
\frac{\bar{F}(\tau)}{\bar{F}_{dol}(\tau)} = \frac{\mu}{\tau + \mu} \left[ 1 - \frac{\mu}{\tau + \mu} \right] - \left[ 1 - \frac{\bar{F}(\tau)}{\bar{F}_{dol}(\tau)} \right] \left[ 1 - \frac{\mu}{\tau + \mu} \right] - \left[ 1 - \frac{\bar{F}(\tau)}{\bar{F}_{dol}(\tau)} \right] \frac{\mu}{\tau + \mu} \left[ 1 - \frac{\mu}{\tau + \mu} \right] \right]. \tag{67}
\]

Utilizing the horizontal index \( J_H(\tau) \) of equation (33), as well as the threshold level \( l_H(\tau) \) of equation (34), we obtain

\[
M(\tau) = \frac{\mu}{\tau + \mu} \left[ \frac{\mu}{\tau + \mu} \bar{F}(\tau) \right] = [J_H(\tau) - J_H(\tau)] \tag{68}
\]

from which equation (35) follows directly.

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**Data availability statement**

No new data were created or analysed in this study.

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