On the Study of
Hyperbolic Triangles and Circles
by Hyperbolic Barycentric Coordinates
in
Relativistic Hyperbolic Geometry

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Abstract Barycentric coordinates are commonly used in Euclidean geometry. Following the adaptation of barycentric coordinates for use in hyperbolic geometry in recently published books on analytic hyperbolic geometry, known and novel results concerning triangles and circles in the hyperbolic geometry of Lobachevsky and Bolyai are discovered. Among the novel results are the hyperbolic counterparts of important theorems in Euclidean geometry. These are: (1) the Inscribed Gyroangle Theorem, (ii) the Gyrotangent-Gyrosecant Theorem, (iii) the Intersecting Gyrosecants Theorem, and (iv) the Intersecting Gyrochord Theorem. Here in gyrolanguage, the language of analytic hyperbolic geometry, we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and nonassociative algebra. Outstanding examples are gyrogroups and gyrovector spaces, and Einstein addition being both gyrocommutative and gyroassociative. The prefix “gyro” stems from “gyration”, which is the mathematical abstraction of the special relativistic effect known as “Thomas precession”.

1. Introduction

A barycenter in astronomy is the point between two objects where they balance each other. It is the center of gravity where two or more celestial bodies orbit each other. In 1827 Möbius published a book whose title, Der Barycentrische Calcul, translates as The Barycentric Calculus. The word barycentric means center of gravity, but the book is entirely geometrical and, hence, called by Jeremy Gray [5], Möbius’s Geometrical Mechanics. The 1827 Möbius book is best remembered for introducing a new system of coordinates, the barycentric coordinates. The historical contribution of Möbius’ barycentric coordinates to vector analysis is described in [1] pp. 48–50).

Commonly used as a tool in the study of Euclidean geometry, barycentric coordinates have been adapted for use as a tool in the study of the hyperbolic geometry of Lobachevsky and Bolyai as well, in several recently published books [20 22 25 26].
Relativistic hyperbolic geometry is a model of analytic hyperbolic geometry in which Einstein addition plays the role of vector addition. Einstein addition is a binary operation in the ball of vector spaces, which is neither commutative nor associative. However, Einstein addition is both gyrocommutative and gyroassociative, giving rise to gyrogroups and gyrovector spaces. The latter, in turn, form the algebraic setting for relativistic hyperbolic geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean geometry.

Relativistic hyperbolic geometry admits the notion of relativistic hyperbolic barycentric coordinates, just as Euclidean geometry admits the notion of Euclidean barycentric coordinates. Relativistic hyperbolic barycentric coordinates and classical Euclidean barycentric coordinates share remarkable analogies. In particular, they are both covariant. Indeed, Relativistic barycentric coordinate representations are covariant with respect to the Lorentz coordinate transformation group, just as classical, Euclidean barycentric coordinate representations are covariant with respect to the Galilean coordinate transformation group. The remarkable analogies suggest that hyperbolic barycentric coordinates can prove useful in the study of hyperbolic geometry, just as Euclidean barycentric coordinates prove useful in the study of Euclidean geometry.

Indeed, following the adaptation of Euclidean barycentric coordinates for use in hyperbolic geometry, where they are called gyrobarycentric coordinates, we employ here the technique of gyrobarycentric coordinates to rediscover and discover known and new results in hyperbolic geometry. An introduction to hyperbolic barycentric coordinates and their application in hyperbolic geometry is found in [29]. Some familiarity with relativistic hyperbolic geometry as studied in [29] is assumed. Relativistic hyperbolic geometry is studied extensively in [20, 22, 25, 26]; see also [19, 23, 30, 12, 13] and [16, 17, 18, 21, 24, 28].

Among the novel results in hyperbolic geometry that are discovered here are the following outstanding results:

1. The Inscribed Gyroangle Theorem, which is the hyperbolic counterpart of the well-known Inscribed Angle Theorem in Euclidean geometry (Sects. 9–10).
2. The Gyrotangent-Gyrosecant Theorem, which is the hyperbolic counterpart of the well-known Tangent-Secant Theorem in Euclidean geometry (Sect. 13).
3. The Intersecting Gyrosecants Theorem, which is the hyperbolic counterpart of the well-known Intersecting Secants Theorem in Euclidean geometry (Sect. 14).
4. The Intersecting Gyrochords Theorem, which is the hyperbolic counterpart of the well-known Intersecting Chords Theorem in Euclidean geometry (Sect. 26).

The prefix “gyro” that we extensively use stems from the term “gyration” [27], which is the mathematical abstraction of the special relativistic effect known as “Thomas precession”.

The use of nonassociative algebra and hyperbolic trigonometry (gyrotrigonometry) involves straightforward, but complicates calculations. Hence, computer algebra, like Mathematica, for algebraic manipulations is an indispensable tool in this work.

2. Einstein Addition

Our journey into the fascinating world of relativistic hyperbolic geometry begins in Einstein addition and passes through important novel theorems that capture remarkable analogies between Euclidean and hyperbolic geometry. Einstein addition, in turn, is the binary operation that stems from Einstein’s composition law of relativistically admissible velocities that he introduced in his 1905 paper [2] [3, p. 141] that founded the special theory of relativity.

Let \( c \) be an arbitrarily fixed positive constant and let \( \mathbb{R}^n = (\mathbb{R}^n, +, \cdot) \) be the Euclidean \( n \)-space, \( n = 1, 2, 3, \ldots \), equipped with the common vector addition, +, and inner product, \( \cdot \). The home of all \( n \)-dimensional Einsteinian velocities is the \( c \)-ball

\[
\mathbb{R}_c^n = \{ v \in \mathbb{R}^n : \| v \| < c \}
\]

It is the open ball of radius \( c \), centered at the origin of \( \mathbb{R}^n \), consisting of all vectors \( v \) in \( \mathbb{R}^n \) with norm smaller than \( c \).

Einstein addition and scalar multiplication play in the ball \( \mathbb{R}_c^n \) the role that vector addition and scalar multiplication play in the Euclidean \( n \)-space \( \mathbb{R}^n \).

**Definition 1.** Einstein addition is a binary operation, \( \oplus \), in the \( c \)-ball \( \mathbb{R}_c^n \) given by the equation, [19], [14, Eq. 2.9.2], [11, p. 55], [4],

\[
(2) \quad u \oplus v = \frac{1}{1 + \frac{u \cdot v}{c^2}} \left( u + \frac{1}{\gamma_u} v + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u} (u \cdot v) u \right)
\]

for all \( u, v \in \mathbb{R}_c^n \), where \( \gamma_u \) is the Lorentz gamma factor given by the equation

\[
(3) \quad \gamma_v = \frac{1}{\sqrt{1 - \frac{\| v \|^2}{c^2}}}
\]

where \( u \cdot v \) and \( \| v \| \) are the inner product and the norm in the ball, which the ball \( \mathbb{R}_c^n \) inherits from its space \( \mathbb{R}^n \).

A frequently used identity that follows immediately from (3) is

\[
(4) \quad \frac{v^2}{c^2} = \frac{\| v \|^2}{c^2} = \frac{\gamma_v^2 - 1}{\gamma_v^2}
\]

A nonempty set with a binary operation is called a groupoid so that, accordingly, the pair \( (\mathbb{R}_c^n, \oplus) \) is an Einstein groupoid.

In the Newtonian limit of large \( c \), \( c \rightarrow \infty \), the ball \( \mathbb{R}_c^n \) expands to the whole of its space \( \mathbb{R}^n \), as we see from (1), and Einstein addition \( \oplus \) in \( \mathbb{R}_c^n \) reduces to the ordinary vector addition \( + \) in \( \mathbb{R}^n \), as we see from (2) and (3).
In applications to velocity spaces, $\mathbb{R}^n = \mathbb{R}^3$ is the Euclidean 3-space, which is the space of all classical, Newtonian velocities, and $\mathbb{R}^n_c = \mathbb{R}^3_c \subset \mathbb{R}^3$ is the $c$-ball of $\mathbb{R}^3$ of all relativistically admissible, Einsteinian velocities. The constant $c$ represents in special relativity the vacuum speed of light. Since we are interested in geometry, we allow $n$ to be any positive integer and, sometimes, replace $c$ by $s$.

We naturally use the abbreviation $u \ominus v = u \oplus (\ominus v)$ for Einstein subtraction, so that, for instance, $v \ominus v = 0$, $\ominus v = 0 \ominus v = -v$. Einstein addition and subtraction satisfy the equations

\begin{equation}
\ominus (u \oplus v) = \ominus u \ominus v
\end{equation}

and

\begin{equation}
\ominus u \oplus (u \ominus v) = v
\end{equation}

for all $u, v$ in the ball $\mathbb{R}^n_c$, in full analogy with vector addition and subtraction in $\mathbb{R}^n$. Identity (5) is called the gyroautomorphic inverse property of Einstein addition, and Identity (6) is called the left cancellation law of Einstein addition. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (6) since, in general,

\begin{equation}
(u \ominus v) \ominus v \neq u
\end{equation}

However, this seemingly lack of a right cancellation law of Einstein addition is repaired, for instance, in [26 Sect. 1.9].

Finally, as demonstrated in [29] and in [13, 20, 22, 23, 25, 26], Einstein addition admits scalar multiplication, giving rise to Einstein gyrovector spaces. These, in turn, form the algebraic setting for relativistic model of hyperbolic geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean geometry. A brief description of the road from Einstein addition to gyrogroups and gyrovector spaces, necessary for a fruitful reading of this chapter, is found in [29].

3. GYROCIRCLES

Assuming familiarity with Einstein gyrovector spaces, as studied in [29], and particularly with the concepts of gyrobarcentric independence, gyroflats and hyperbolic span in [29 Def. 13], the gyrocircle definition follows.

**Definition 2. (Gyrocircle).** Let $S = \{A_1, A_2, A_3\}$ be a gyrobarcentrically independent set in an Einstein gyrovector space $(\mathbb{R}^n, \oplus, \otimes)$, $n \geq 2$, and let

\begin{equation}
A_3 = A_1 \oplus \text{Span}\{\ominus A_1 \oplus A_2, \ominus A_1 \oplus A_3\} \subset \mathbb{R}^n.
\end{equation}

The locus of a point in $A_3 \cap \mathbb{R}^n$ which is at a constant gyrdistance $r$ from a fixed point $O \in A_3 \cap \mathbb{R}^n$ is a gyrocircle $C(r, O)$ with gyrocenter $O$ and gyroradius $r$.

The gyrocircle $C(r, O)$ with gyroradius $r$, $0 < r < s$, and gyrocenter $O \in \mathbb{R}^2_s$ in an Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$ is the set of all points $P \in \mathbb{R}^2_s$ such that $\|\ominus P \oplus O\| = r$. It is given by the equation

\begin{equation}
C(r, O, \theta) = O \oplus \left(\frac{r \cos \theta}{r \sin \theta}\right),
\end{equation}

Figure 1. A sequence of gyrocircles with gyroradius $\frac{1}{3}$ in an
Einstein gyrovector plane $\mathbb{R}_s^2 = 1$ with gyrocenters approaching the
boundary of the open unit disc $\mathbb{R}_s^2 = 1$ is shown. The center of
the disc is conformal. Hence, the gyrocircle with gyrocenter at
the center of the disc coincides with a Euclidean circle. The Eu-
clidean circle is increasingly flattened as its gyrocenter approaches
the boundary of the disc.

$0 \leq \theta < 2\pi$. Indeed, by the left cancellation law we have

$$\|\ominus O \oplus C(r, O, \theta)\| = \left\| \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} \right\| = r.$$

A sequence of gyrocircles of gyroradius $\frac{1}{3}$ in an Einstein gyrovector plane $\mathbb{R}_s^2 = 1$
with gyrocenters approaching the boundary of the open unit disc $\mathbb{R}_s^2 = 1$ is shown in
Fig. 1. The center of the disc in Fig. 1 is conformal, as explained in [26, Sect. 6.2].
Accordingly, a gyrocircle with gyrocenter at the center of the disc is identical to
a Euclidean circle. This Euclidean circle is increasingly flattened in the Euclidean
sense when the gyrocircle gyrocenter approaches the boundary of the disc.

4. Gyrotriangle Circumgyrocenter

Definition 3. (Gyrotriangle Circumgyrocircle, Circumgyrocenter, Cir-
cumgyroradius). Let $A_1 A_2 A_3$ be a gyrotriangle in an Einstein gyrovector space
($\mathbb{R}_s^n, \oplus, \otimes$), $n \geq 2$, and let $A_3$ be the set

$$A_3 = A_1 \oplus \text{Span}\{ \ominus A_1 \oplus A_2, \ominus A_1 \oplus A_3 \} \subset \mathbb{R}^n.$$

The circumgyrocenter of the gyrotriangle is the point $O, O \in A_3 \cap \mathbb{R}_s^n$, equigyrodistant
from the three gyrotriangle vertices. The gyrodistance from $O$ to each vertex $A_k, k = 1, 2, 3,$ of the gyrotriangle is the gyrotriangle circumgyroradius, and the gyrocircle
with gyrocenter $O$ and gyroradius $r$ in $A_3 \cap \mathbb{R}_s^n$ is the gyrotriangle circumgyrocircle.
It should be noted that not every gyrotriangle in $\mathbb{R}^n_*$ possesses a circumgyrocenter.

Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_*, \oplus, \otimes)$ that possesses a circumgyrocircle, and let $O \in A_3 \cap \mathbb{R}^n_*$ be the circumgyrocenter of the gyrotriangle, as shown in Fig. 2. Then, $O$ possesses a gyrobarycentric representation,

\begin{equation}
O = \frac{m_1\gamma_{a_1}A_1 + m_2\gamma_{a_2}A_2 + m_3\gamma_{a_3}A_3}{m_1\gamma_{a_1} + m_2\gamma_{a_2} + m_3\gamma_{a_3}},
\end{equation}

with respect to the gyrobarycentrically independent set $S = \{A_1, A_2, A_3\}$. The gyrobarycentric coordinates $m_1, m_2$ and $m_3$ are to be determined in (20) below, in terms of gamma factors of the gyrotriangle sides and, alternatively in (34), in terms of the gyrotriangle gyroangles.
Following the *Gyrobarycentric representation Gyrocovariance Theorem*, [26, Theorem 4.6, pp. 90-91], with a left gyrotranslation by $X = \ominus A_1$, and using the index notation

$$a_{ij} = \ominus A_i \oplus A_j, \quad a_{ii} = \|a_{ij}\|, \quad \gamma_{ij} = \gamma_{a_{ij}},$$

noting that $a_{ij} = a_{ji}$, $\gamma_{ij} = \gamma_{ji}$, $a_{ii} = 0$, $a_{ij} = 0$ and $\gamma_{ii} = 1$, we have

$$\gamma_{\ominus A_1 \oplus O} = \frac{m_1\gamma_{\ominus A_1 \oplus A_1} + m_2\gamma_{\ominus A_1 \oplus A_2} + m_3\gamma_{\ominus A_1 \oplus A_3}}{m_O},$$

$$= \frac{m_1 + m_2\gamma_{12} + m_3\gamma_{13}}{m_O},$$

(14)

where by [26, Eq. (4.27), p. 90] and the *Gyrobarycentric representation Gyrocovariance Theorem*, [26, Theorem 4.6, pp. 90-91], the circumgyrocenter gyrobarycentric representation constant $m_O > 0$ with respect to the set of the gyrotriangle vertices is given by the equation

$$m_O^2 = m_1^2 + m_2^2 + m_3^2 + 2(m_1m_2\gamma_{12} + m_1m_3\gamma_{13} + m_2m_3\gamma_{23}).$$

Similarly, by the *Gyrobarycentric representation Gyrocovariance Theorem*, [26, Theorem 4.6, pp. 90-91], with left gyrotranslations by $X = \ominus A_1$, by $X = \ominus A_2$, and by $X = \ominus A_3$, we have, respectively,

$$\gamma_{\ominus A_2 \oplus O} = \frac{m_1\gamma_{12} + m_2\gamma_{23} + m_3\gamma_{13}}{m_O},$$

$$\gamma_{\ominus A_3 \oplus O} = \frac{m_1\gamma_{13} + m_2\gamma_{23} + m_3\gamma_{13}}{m_O}.$$

(16)

The condition that the circumgyrocenter $O$ is equigyrodistant from its gyrotriangle vertices $A_1, A_2,$ and $A_3$ implies

$$\gamma_{\ominus A_1 \oplus O} = \gamma_{\ominus A_2 \oplus O} = \gamma_{\ominus A_3 \oplus O}.$$

(17)

Equations (16) and (17), along with the normalization condition $m_1 + m_2 + m_3 = 1$, yield the following system of three equations for the three unknowns $m_1, m_2, m_3$,

$$m_1 + m_2 + m_3 = 1$$

$$m_1 + m_2\gamma_{12} + m_3\gamma_{13} = m_1\gamma_{13} + m_2\gamma_{23} + m_3$$

$$m_1\gamma_{12} + m_2 + m_3\gamma_{23} = m_1\gamma_{13} + m_2\gamma_{23} + m_3,$$

which can be written as the matrix equation,

$$\begin{pmatrix}
1 & 1 & 1 \\
1 - \gamma_{13} & \gamma_{12} - \gamma_{23} & \gamma_{13} - 1 \\
\gamma_{12} - \gamma_{13} & 1 - \gamma_{23} & \gamma_{23} - 1
\end{pmatrix}
\begin{pmatrix}
m_1 \\
m_2 \\
m_3
\end{pmatrix}
= \begin{pmatrix}1 \\ 0 \\ 0 \end{pmatrix}.$$
Solving (19) for the unknowns \(m_1, m_2,\) and \(m_3,\) we have
\[
m_1 = \frac{1}{D} (\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)(\gamma_{23} - 1)
\]
(20)
\[
m_2 = \frac{1}{D} (\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)(\gamma_{13} - 1)
\]
\[
m_3 = \frac{1}{D} (-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)(\gamma_{12} - 1),
\]
where \(D\) is the determinant of the \(3 \times 3\) matrix in (19),
\[
D = 2(\gamma_{12}\gamma_{13} + \gamma_{12}\gamma_{23} + \gamma_{13}\gamma_{23})
\]
\[\quad - (\gamma_{12}^2 - 1) - (\gamma_{13}^2 - 1) - (\gamma_{23}^2 - 1) - 2(\gamma_{12} + \gamma_{13} + \gamma_{23})
\]
\[\quad = 1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2 - 2(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1).
\]

Gyrotrigonometric substitutions into the extreme right-hand side of (21) from [26, Sect. 7.12] give the gyrotrigonometric representation of \(D,
\]
\[
D = \frac{16F^2}{\sin^2 \alpha_1 \sin^2 \alpha_2 \sin^2 \alpha_3} - \frac{16F}{\sin^2 \alpha_1 \sin^2 \alpha_2 \sin^2 \alpha_3} \sin^2 \frac{\delta}{2}
\]
\[\quad = \frac{16F}{\sin^2 \alpha_1 \sin^2 \alpha_2 \sin^2 \alpha_3} (F - \sin^2 \frac{\delta}{2})
\]
\[\quad = \frac{16F}{\sin^2 \alpha_1 \sin^2 \alpha_2 \sin^2 \alpha_3} \sin \frac{\delta}{2}
\]
\[\times \{\sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2}) - \sin \frac{\delta}{2}\},
\]
where \(\delta = \pi - \alpha_1 - \alpha_2 - \alpha_3\) is the defect of gyrotriangle \(A_1A_2A_3,\) and where
\[
F = F(\alpha_1, \alpha_2, \alpha_3) = \sin \frac{\delta}{2} \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2}).
\]

The extreme right-hand side of (22) is the product of two factors the first of which is positive. The second factor is positive if and only if the gyrotriangle circumgyrocenter \(O \in \mathbb{R}^n\) lies inside the ball \(\mathbb{R}^n_{+},\) as we will see in Theorem [4, p. 11]. This factor vanishes if and only if \(O\) lies on the boundary of the ball \(\mathbb{R}^n_{+},\) and it is negative if and only if \(O\) lies outside the closure of the ball \(\mathbb{R}^n,\) as we will see in Theorem [4].

The circumgyrocenter \(O\) of gyrotriangle \(A_1A_2A_3\) is given by (12), where the gyro-barycentric coordinates \(m_1, m_2,\) and \(m_3,\) are given by (20). Since in gyrobarcentric coordinates only ratios of coordinates are relevant, the gyrobarcentric coordinates, \(m_1, m_2,\) and \(m_3\) in (20) can be simplified by removing their common nonzero factor \(1/D.

Gyrobarcentric coordinates, \(m_1, m_2,\) and \(m_3,\) of the circumgyrocenter \(O\) of gyrotriangle \(A_1A_2A_3\) are thus given by the equations
\[
m'_1 = (\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)(\gamma_{23} - 1)
\]
(24)
\[
m'_2 = (\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)(\gamma_{13} - 1)
\]
\[
m'_3 = (-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)(\gamma_{12} - 1).
\]
Hence, by (15) along with the gyrobarycentric coordinates in (24), we have

\[
m^2_O = (1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2) \\
\times \{1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2 - 2(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1)\} \\
= \{(\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 - 2(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 1)\} \\
\times (1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2).
\]  

(25)

In order to emphasize comparative patterns that the gyrotriangle circumgyrocircle and the gyrotetrahedron circumgyrosphere possess, we present the first equation in (25) in the determinantal form

\[
m^2_O = D_3(D_3 - H_3),
\]

(26)

where \(D_3\) is the determinant

\[
D_3 = \begin{vmatrix} 1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & 1 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & 1 \end{vmatrix}
\]

(27)

and where

\[
H_3 = 2(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1).
\]

(28)

The gyrotriangle \(A_1A_2A_3\) in Fig. 2 possesses a circumgyrocenter if and only if \(m^2_O > 0\).

The factor \(D_3\) of \(m^2_O\) in (25) and in (26),

\[
D_3 = 1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2 = \begin{vmatrix} 1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & 1 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & 1 \end{vmatrix} > 0,
\]

(29)

is positive for any gyrotriangle \(A_1A_2A_3\) in an Einstein gyrovector space. Hence, as we see from (25), \(m^2_O > 0\) if and only if the points \(A_1, A_2,\) and \(A_3\) obey the circumgyrocircle existence condition

\[
(\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 > 2(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 1)
\]

or, equivalently, the circumgyrocircle existence condition

\[
4(\gamma_{12} - 1)(\gamma_{13} - 1) > (\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)^2,
\]

(30b)

or, equivalently, the circumgyrocircle existence condition

\[
D_3 > H_3.
\]

(30c)
Gamma factors of gyrotriangle side gyrolengths are related to its gyroangles by the AAA to SSS Conversion Law \[26, \text{Theorem 6.5, p. 137}\]

\[\gamma_{23} = \frac{\cos \alpha_1 + \cos \alpha_2 \cos \alpha_3}{\sin \alpha_2 \sin \alpha_3}\]
\[\gamma_{13} = \frac{\cos \alpha_2 + \cos \alpha_1 \cos \alpha_3}{\sin \alpha_1 \sin \alpha_3}\]
\[\gamma_{12} = \frac{\cos \alpha_3 + \cos \alpha_1 \cos \alpha_2}{\sin \alpha_1 \sin \alpha_2}.\]

(31)

Substituting these from (31) into (24) we obtain

\[m'_1 = F' \sin\left(-\frac{\alpha_1 + \alpha_2 + \alpha_3}{2}\right) \sin \alpha_1\]
\[m'_2 = F' \sin\left(\frac{\alpha_1 - \alpha_2 + \alpha_3}{2}\right) \sin \alpha_2\]
\[m'_3 = F' \sin\left(\frac{\alpha_1 + \alpha_2 - \alpha_3}{2}\right) \sin \alpha_3,\]

where the common factor \(F' = F'(\alpha_1, \alpha_2, \alpha_3)\) in (32) is given by the equation

\[F' = 2^3 \frac{\cos^2\left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{2}\right)}{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3} \cos\left(\frac{\alpha_1 - \alpha_2 + \alpha_3}{2}\right) \cos\left(\alpha_1 + \frac{\delta}{2}\right) \cos\left(\alpha_2 + \frac{\delta}{2}\right) \cos\left(\alpha_3 + \frac{\delta}{2}\right).\]

(33)

Since in gyrobarycentric coordinates only ratios of coordinates are relevant, the gyrobarycentric coordinates, \(m'_1, m'_2,\) and \(m'_3\) in (32) can be simplified by removing a common nonzero factor. Hence, convenient gyrobarycentric coordinates, \(m''_1, m''_2,\) and \(m''_3,\) of the circumgyrocenter \(O\) of gyrotriangle \(A_1A_2A_3,\) expressed in terms of the gyrotriangle gyroangles are given by the equations

\[m''_1 = -\sin\left(-\frac{\alpha_1 + \alpha_2 + \alpha_3}{2}\right) \sin \alpha_1 = \cos(\alpha_1 + \frac{\delta}{2}) \sin \alpha_1\]
\[m''_2 = -\sin\left(\frac{\alpha_1 - \alpha_2 + \alpha_3}{2}\right) \sin \alpha_2 = \cos(\alpha_2 + \frac{\delta}{2}) \sin \alpha_2\]
\[m''_3 = -\sin\left(\frac{\alpha_1 + \alpha_2 - \alpha_3}{2}\right) \sin \alpha_3 = \cos(\alpha_3 + \frac{\delta}{2}) \sin \alpha_3,\]

(34)

where \(\delta = \pi - \alpha_1 - \alpha_2 - \alpha_3\) is the defect of gyrotriangle \(A_1A_2A_3.\)

The circumgyrocenter \(O, (12),\) lies in the interior of its gyrotriangle \(A_1A_2A_3\) if and only if its gyrobarycentric coordinates are all positive or all negative. Hence, we see from the gyrobarycentric coordinates (31) of \(O\) that

1. the circumgyrocenter \(O\) lies in the interior of its gyrotriangle \(A_1A_2A_3\) if and only if the largest gyroangle of the gyrotriangle has measure less than the sum of the measures of the other two gyroangles. This result is known in hyperbolic geometry; see, for instance, \[6, \text{p. 132}\], where the result is proved synthetically. Similarly, we also see from the gyrobarycentric coordinates (31) of \(O\) in (12) that
2. the circumgyrocenter \(O\) lies in the interior of its gyrotriangle \(A_1A_2A_3\) if and only if all the three gyroangles \(\alpha_1 + \frac{\delta}{2}, \alpha_2 + \frac{\delta}{2}\) and \(\alpha_3 + \frac{\delta}{2}\) are acute.
Expressing Inequality (30) gyrotrigonometrically, by means of (31), it can be shown from (25) that
\[
\cos \frac{3\alpha_1 - \alpha_2 - \alpha_3}{2} + \cos \frac{-\alpha_1 + 3\alpha_2 - \alpha_3}{2} + \cos \frac{-\alpha_1 - \alpha_2 + 3\alpha_3}{2} > 3 \cos \frac{\alpha_1 + \alpha_2 + \alpha_3}{2}
\]
(35)
or, equivalently, if and only if
\[
\sin(2\alpha_1 + \frac{\delta}{2}) + \sin(2\alpha_2 + \frac{\delta}{2}) + \sin(2\alpha_3 + \frac{\delta}{2}) > 3 \sin \frac{\delta}{2}
\]
(36)
Inequality (36) is an elegant condition for the existence of a circumgyrocenter. In a different approach, we will discover below in (38) the circumgyrocenter existence condition in a different elegant form.

Gyrotrigonometric substitutions into the second equation in (25) from [26, Sect. 7.12] yield the following gyrotrigonometric expression for the constant \( m_O > 0 \) of the circumgyrocenter gyrobarycentric representation (12):
\[
m_O^2 = 4^2 F^2 \left\{ \frac{4^2 F^2}{\sin^2 \alpha_1 \sin^2 \alpha_2 \sin^2 \alpha_3} - \frac{4^2 F \sin^2 \frac{\delta}{2}}{\sin^2 \alpha_1 \sin^2 \alpha_2 \sin^2 \alpha_3} \right\}
\]
\[
= \frac{4^4 F^3}{\sin^4 \alpha_1 \sin^4 \alpha_2 \sin^4 \alpha_3} \left( F - \sin^2 \frac{\delta}{2} \right)
\]
\[
= \frac{4^4 F^3 \sin \frac{\delta}{2}}{\sin^4 \alpha_1 \sin^4 \alpha_2 \sin^4 \alpha_3} \left( \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2}) - \sin \frac{\delta}{2} \right)
\]
(37)
Hence, \( m_O^2 > 0 \) if and only if \( F > \sin^2(\delta/2) \) or, equivalently,
\[
\sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2}) > \sin \frac{\delta}{2}
\]
(38)
One may demonstrate directly that the two circumgyrocenter existence conditions (36) and (38) are equivalent.

Formalizing the main results of this section, we have the following theorem:

**Theorem 4. (Circumgyrocenter Theorem).** Let \( S = \{A_1, A_2, A_3\} \) be a gyrobarcentrically independent set of three points in an Einstein gyrovector space \((\mathbb{R}^n_\oplus, \oplus, \otimes)\). The circumgyrocenter \( O \in \mathbb{R}^n \) of gyrotriangle \( A_1A_2A_3 \), shown in Fig. 2, possesses the gyrobarcentric representation
\[
O = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}}
\]
(39)
with respect to the set \( S = \{A_1, A_2, A_3\} \), with gyrobarcentric coordinates \((m_1 : m_2 : m_3)\) given by
\[
m_1 = (\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)(\gamma_{23} - 1)
\]
\[
m_2 = (\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)(\gamma_{13} - 1)
\]
\[
m_3 = (-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)(\gamma_{12} - 1)
\]
(40)
or, equivalently, by the gyrotrigonometric gyrobarycentric coordinates
\[ m_1 = \cos(\alpha_1 + \frac{\delta}{2}) \sin \alpha_1 \]
\[ m_2 = \cos(\alpha_2 + \frac{\delta}{2}) \sin \alpha_2 \]
\[ m_3 = \cos(\alpha_3 + \frac{\delta}{2}) \sin \alpha_3. \]

The circumgyrocenter gyrobarycentric representation constant \( m_O \) with respect to the set \( S = \{A_1, A_2, A_3\} \) is an elegant product of two factors, given by the equation
\[ m_O^2 = \left(\gamma_{12} + \gamma_{13} + \gamma_{23} - 1\right) - 2\left(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 1\right) \]
\[ \times \left(1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2\right) = D_3(D_3 - H_3), \]
where \( D_3 \) and \( H_3 \) are given by (27) and (28).

The circumgyrocenter lies in the ball, \( O \in \mathbb{R}^n \), if and only if \( m_O^2 > 0 \) or, equivalently, if and only if one of the following mutually equivalent inequalities, each of which is a circumgyrocircle existence condition, is satisfied:
\[ \sin(2\alpha_1 + \frac{\delta}{2}) + \sin(2\alpha_2 + \frac{\delta}{2}) + \sin(2\alpha_3 + \frac{\delta}{2}) > 3 \sin \frac{\delta}{2} \]
\[ \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2}) > \sin \frac{\delta}{2} \]
\[ (\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 > 2(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 1) \]
\[ 4(\gamma_{12} - 1)(\gamma_{13} - 1) > (\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)^2. \]

5. Triangle Circumcenter

In this section the gyrotriangle circumgyrocenter in Fig. 2 p. 6 will be translated into its Euclidean counterpart in Fig. 3.

The gyrobarycentric representation (39) with gyrotrigonometric gyrobarycentric coordinates \( (m_1 : m_2 : m_3) \) given by (41) remains invariant in form under the Euclidean limit \( s \to \infty \), so that it is valid in Euclidean geometry as well, where \( \delta = 0 \). Hence, in the transition from hyperbolic geometry, where \( \delta > 0 \), to Euclidean geometry, where \( \delta = 0 \), the gyrobarycentric coordinates (41) reduce to the barycentric coordinates
\[ m_1 = \cos \alpha_1 \sin \alpha_1 = \frac{1}{2} \sin 2\alpha_1 \]
\[ m_2 = \cos \alpha_2 \sin \alpha_2 = \frac{1}{2} \sin 2\alpha_2 \quad (Euclidean \ Geometry) \]
\[ m_3 = \cos \alpha_3 \sin \alpha_3 = \frac{1}{2} \sin 2\alpha_3. \]

Hence, finally, a trigonometric barycentric representation of the circumcenter \( O \) of triangle \( A_1A_2A_3 \) in \( \mathbb{R}^n \), Fig. 3 with respect to the barycentrically independent set \( S = \{A_1, A_2, A_3\} \subset \mathbb{R}^n \) is given by Result (48) of the following corollary of Theorem 4 which recovers a well-known result in Euclidean geometry [7].
Figure 3. The circumcircle, and the circumcenter $O$, of triangle $A_1A_2A_3$ in a Euclidean vector space $\mathbb{R}^n$, $n = 2$, is shown along with its standard notation. Here $R = \| - A_1 + O \| = \| - A_2 + O \| = \| - A_3 + O \|$, where $R$ is the triangle circumradius and $O$ is the triangle circumcenter, given by its barycentric coordinate representation (48) with respect to the barycentrically independent set $S = \{A_1, A_2, A_3\}$. The hyperbolic counterpart of this figure is shown in Fig. 2, p. 6.

Corollary 5. Let $\alpha_k$, $k = 1, 2, 3$ and $O$ be the angles and circumcenter of a triangle $A_1A_2A_3$ in a Euclidean space $\mathbb{R}^n$. Then,

$$O = \frac{\sin 2\alpha_1 A_1 + \sin 2\alpha_2 A_2 + \sin 2\alpha_3 A_3}{\sin 2\alpha_1 + \sin 2\alpha_2 + \sin 2\alpha_3}.$$  

Theorem 4 and its Corollary 5 form an elegant example that illustrates the result that

(1) gyrotrigonometric gyrobarycentric coordinates of a point in an Einstein gyrovector space $\mathbb{R}^n$ survive unimpaired in Euclidean geometry, where they form

(2) trigonometric barycentric coordinates of a point in a corresponding Euclidean vector space $\mathbb{R}^n$.

The converse is, however, not valid since
(3) trigonometric barycentric coordinates of a point in a Euclidean vector space \( \mathbb{R}^n \) may embody the Euclidean condition that the triangle angle sum is \( \pi \), so that they need not survive in hyperbolic geometry.

6. GYROTriangle CIRCUMGYORADIUS

The circumgyroradius \( R \) of gyrotriangle \( A_1A_2A_3 \) with circumgyrocenter \( O \) in an Einstein gyrovector space \( (\mathbb{R}_e^n, \oplus, \otimes) \), shown in Figs. 2 and 3, is given by

\[
R = \| \ominus A_1 \oplus O \| = \| \ominus A_2 \oplus O \| = \| \ominus A_3 \oplus O \|.
\]

By (14), the circumgyroradius \( R \) satisfies the equation

\[
\gamma_R = \gamma_{\ominus A_1 \oplus O} = \gamma_{\ominus A_1 \oplus O} = \frac{m_1 + m_2 \gamma_{12} + m_3 \gamma_{13}}{m_O},
\]

where \( m_1, m_2 \) and \( m_3 \) are given by (40), and where \( m_O \) is given by (42).

Hence, following (50), (40) and (42), with the notation for \( D_3 \) and \( H_3 \) in (27) – (28), we have

\[
\gamma_R^2 = \frac{1 + 2 \gamma_{12} \gamma_{13} \gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{1 + 2 \gamma_{12} \gamma_{13} \gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2 - 2(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1)}
\]

\[
= \frac{2 \gamma_{12} \gamma_{13} \gamma_{23} - (\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 1)}{(\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 - 2(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 1)}
\]

\[
= \frac{D_3}{D_3 - H_3}.
\]

Interestingly, from (51) and (26) we obtain an elegant relationship between (i) the constant \( m_O \) of the gyrobarycentric representation of the circumgyrocenter \( O \) of a gyrotriangle \( A_1A_2A_3 \) with respect to the gyrotriangle vertices, and (ii) the gamma factor of the circumgyroradius \( R \),

\[
m_O \gamma_R = D_3.
\]

Following (51) we have, by (14),

\[
R^2 = s^2 \frac{\gamma_R^2 - 1}{\gamma_R^2} = 2s^2 \frac{(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1)}{1 + 2 \gamma_{12} \gamma_{13} \gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2} = s^2 \frac{H_3}{D_3}.
\]

Hence, finally, the circumgyroradius \( R \) of gyrotriangle \( A_1A_2A_3 \) in Figs. 2 and 3 is given by

\[
R = \sqrt{2s} \sqrt[\frac{2}{(\gamma_{12} + 1)(\gamma_{13} + 1)(\gamma_{23} + 1)}}
\]

\[
R = s \sqrt[\frac{2}{(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1)}] \cdot
\]

implying

\[
\sqrt[\frac{2}{(\gamma_{12} + 1)(\gamma_{13} + 1)(\gamma_{23} + 1)}} R = s \sqrt[\frac{2}{(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1)}] \cdot
\]
Identity (55) captures a remarkable analogy between the law of gyrosines and the law of sines. Indeed, following (55), the law of gyrosines \[26, \text{Theorem 6.9, p. 140}\] for gyrotriangle \(A_1A_2A_3\) in Fig. 4 is linked to the circumgyroradius \(R\) of the gyrotriangle by the equation

\[
\frac{\gamma_{23}a_{23}}{\sin \alpha_1} = \frac{\gamma_{13}a_{13}}{\sin \alpha_2} = \frac{\gamma_{12}a_{12}}{\sin \alpha_3} = \sqrt{\frac{(\gamma_{12}+1)(\gamma_{13}+1)(\gamma_{23}+1)}{2}} R,
\]

called the **extended law of gyrosines**.

Following the gamma-gyrotrigonometric identity \[26, \text{Eq. (7.154), p. 189}\], the extended law of gyrosines (56) can be written as

\[
\frac{\gamma_{23}a_{23}}{\sin \alpha_1} = \frac{\gamma_{13}a_{13}}{\sin \alpha_2} = \frac{\gamma_{12}a_{12}}{\sin \alpha_3} = 2 \frac{\sin (\alpha_1 + \frac{\delta}{2}) \sin (\alpha_2 + \frac{\delta}{2}) \sin (\alpha_3 + \frac{\delta}{2})}{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3} R,
\]

where \(\delta = \pi - (\alpha_1 + \alpha_2 + \alpha_3)\) is the defect of gyrotriangle \(A_1A_2A_3\).

In the Euclidean limit of large \(s\), \(s \to \infty\), gamma factors tend to 1 and gyrotriangle defects tend to 0. Hence, in that limit, the extended law of gyrosines (56)
tends to the well-known extended law of sines [9, p. 87],

\[ \frac{a_{23}}{\sin \alpha_1} = \frac{a_{13}}{\sin \alpha_2} = \frac{a_{12}}{\sin \alpha_3} = 2R \quad \text{(Euclidean Geometry)}. \]

Formalizing the results in (56) – (57) we have the following theorem:

**Theorem 6. (Extended Law of Gyrosines).** Let \( A_1A_2A_3 \) be a gyrotriangle in an Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\) with gyroangles \( \alpha_1, \alpha_2, \alpha_3 \), side-gyrolengths \( a_{23}, a_{13}, a_{12} \), and circumgyroradius \( R \), Fig. 4.

Then

\[ \frac{\gamma_{23}a_{23}}{\sin \alpha_1} = \frac{\gamma_{13}a_{13}}{\sin \alpha_2} = \frac{\gamma_{12}a_{12}}{\sin \alpha_3} = \sqrt{\frac{(\gamma_{12} + 1)(\gamma_{13} + 1)(\gamma_{23} + 1)}{2}} R \]

and

\[ \frac{\gamma_{23}a_{23}}{\sin \alpha_1} = \frac{\gamma_{13}a_{13}}{\sin \alpha_2} = \frac{\gamma_{12}a_{12}}{\sin \alpha_3} = 2 \left( \frac{\sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})}{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3} \right) R. \]

Interestingly, the gyrotriangle circumgyroradius \( R \) has an elegant representation in terms of its gyrotriangle gyroangles. Indeed, expressing the gamma factors in (53) in terms of the gyrotriangle gyroangles \( \alpha_k, k = 1, 2, 3 \), takes the gyrotrigonometric form

\[ R^2 = \frac{s^2}{\sin^2(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})} \]

\[ = \frac{1}{\sin^2(\alpha_1 + \frac{\delta}{2}) \sin^2(\alpha_2 + \frac{\delta}{2}) \sin^2(\alpha_3 + \frac{\delta}{2})} \]

\[ = \frac{F(\alpha_1, \alpha_2, \alpha_3)}{\sin^4(\alpha_1 + \frac{\delta}{2}) \sin^4(\alpha_2 + \frac{\delta}{2}) \sin^4(\alpha_3 + \frac{\delta}{2})} \]

in any Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\), where \( F = F(\alpha_1, \alpha_2, \alpha_3) \) is given by [26, Eq. (7.144), p. 187]. Equation (61) can be written conveniently as

\[ \frac{R}{s} = \sqrt{F} \]

Following (62), the first equation in [26, Eq. (7.143), p. 187] and [26, Eq. (7.150), p. 188] we have the equation

\[ \sin(\alpha_3 + \frac{\delta}{2}) = \frac{\gamma_{13}a_{12}}{R(\gamma_{12} + 1)}, \]

which will prove useful in (102), p. 25.

In the Euclidean limit, \( s \to \infty \), the gyrotriangle defect tends to 0, \( \delta \to 0 \), so that each side of (61) tends to 0.

An important relation that results from (61) is formalized in the following Theorem:
Theorem 7. Let $\alpha_k$, $k = 1, 2, 3$, and $R$ be the gyroangles and circumgyroradius of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s \oplus, \otimes)$. Then

$$s^2 \sin \frac{\delta}{2} = R^2 \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})$$

and

$$s\sqrt{F(\alpha_1, \alpha_2, \alpha_3)} = R \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})$$

where $F(\alpha_1, \alpha_2, \alpha_3)$ is given by [23, Eq. (7.144), p. 187].

Proof. Identity (64) follows immediately from (61), and Identity (65) follows from (64) and the definition of $F$. □

The Euclidean limit, $s \to \infty$, of the left-hand side of each of the equations (64) and (65) of Theorem 7 is indeterminate, being a limit of type $\infty \cdot 0$. In that limit, the gyrotriangle defect $\delta$ tends to zero, so that the right-hand side of (64) tends to $R^2 \sin \alpha_1 \sin \alpha_2 \sin \alpha_3$.

Theorem 7 gives rise to useful results, as indicated by the following theorem.

Theorem 8. Let $A_1A_2A_3$ be a gyrotriangle that possesses a circumgyroradius $R$ in an Einstein gyrovector space $(\mathbb{R}^n_s \oplus, \otimes)$, Fig. 2, p. 6. Then, in the gyrotriangle index notation (13),

$$a_{12} = \frac{2R \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})}{\cos \alpha_3 + \cos \alpha_1 \cos \alpha_2}$$

$$a_{13} = \frac{2R \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})}{\cos \alpha_2 + \cos \alpha_1 \cos \alpha_3}$$

$$a_{23} = \frac{2R \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})}{\cos \alpha_1 + \cos \alpha_2 \cos \alpha_3}$$

and

$$\gamma_{12}a_{12} = \frac{2R \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})}{\sin \alpha_1 \sin \alpha_2}$$

$$\gamma_{13}a_{13} = \frac{2R \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})}{\sin \alpha_1 \sin \alpha_3}$$

$$\gamma_{23}a_{23} = \frac{2R \sin(\alpha_1 + \frac{\delta}{2}) \sin(\alpha_2 + \frac{\delta}{2}) \sin(\alpha_3 + \frac{\delta}{2})}{\sin \alpha_2 \sin \alpha_3},$$

where $a_{ij} = \|\oplus A_i \oplus A_j\|$, $1 \leq i, j \leq 3$.

Proof. The identities in (66) and (67) follow from Identity (65) of Theorem 7 and from the AAA to SSS Conversion Law [26, Theorem 6.8, p. 140]. □

Comparing the first equation in each of (66) and (67) yields the gyrotriangle identity

$$\cos \alpha_3 + \cos \alpha_1 \cos \alpha_2 = \gamma_{12} \sin \alpha_1 \sin \alpha_2 \quad \text{(Hyperbolic Geometry)}$$
that any gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ obeys.

With $\delta = 0$, Theorem 8 specializes to the following corresponding theorem in Euclidean geometry:

**Theorem 9.** Let $A_1A_2A_3$ be a triangle with a circumradius $R$ in a Euclidean vector space $\mathbb{R}^n$, Fig. 8, p. 14. Then, in the triangle index notation,

\[
a_{12} = \frac{2R \sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{\cos \alpha_3 + \cos \alpha_1 \cos \alpha_2} \\
a_{13} = \frac{2R \sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{\cos \alpha_2 + \cos \alpha_1 \cos \alpha_3} \\
a_{23} = \frac{2R \sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{\cos \alpha_1 + \cos \alpha_2 \cos \alpha_3}
\]

and

\[
a_{12} = \frac{2R \sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{\sin \alpha_1 \sin \alpha_2} \\
a_{13} = \frac{2R \sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{\sin \alpha_1 \sin \alpha_3} \\
a_{23} = \frac{2R \sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{\sin \alpha_2 \sin \alpha_3},
\]

where $a_{ij} = \| - A_i + A_j \|$, $1 \leq i, j \leq 3$.

As expected from Identities (69) and (70) of Theorem 8, the triangle angles obey the triangle identity $\alpha_1 + \alpha_2 + \alpha_3 = \pi$, so that

\[
(71) \quad \cos \alpha_3 + \cos \alpha_1 \cos \alpha_2 = \sin \alpha_1 \sin \alpha_2 \quad \text{(Euclidean Geometry)}
\]

for any triangle $A_1A_2A_3$ in the Euclidean space $\mathbb{R}^n$. Indeed, the triangle identity (71) is the Euclidean counterpart of the gyrotriangle identity (68).

**7. Triangle Circumradius**

The circumradius $R_{\text{euc}}$ of triangle $A_1A_2A_3$ with circumcenter $O$ in a Euclidean space $\mathbb{R}^n$, shown in Fig. 17, p. 19, is given by

\[
(72) \quad R_{\text{euc}} = \| - A_1 + O \| = \| - A_2 + O \| = \| - A_3 + O \|. 
\]

We will determine the triangle circumradius in a Euclidean space $\mathbb{R}^n$ as the Euclidean limit of the circumgyroradius of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, that is, $R_{\text{euc}} = \lim_{s \to \infty} R$. Accordingly, let $A_1A_2A_3$ be a gyrotriangle that possesses a circumgyrocircle in the Einstein gyrovector space, and let $R$ be the gyrotriangle circumgyroradius.

Then, $R$ is given by (53), which can be written equivalently as

\[
(73) \quad R^2 = \frac{2s^2(\gamma_{23} - 1)}{D} 
\]
where
\[
D = 2 \left\{ 1 + \frac{\gamma_{13}}{\gamma_{12}} - 1 + \frac{\gamma_{23}}{\gamma_{12}} - 1 + (\gamma_{23} - 1) \right\} 
\]
(74)
\[
- \left\{ \frac{\gamma_{12}}{\gamma_{23}} - 1 + \frac{\gamma_{23}}{\gamma_{12}} - 1 + \frac{(\gamma_{23} - 1)^2}{(\gamma_{12} - 1)(\gamma_{13} - 1)} \right\},
\]
as one can check by straightforward algebra.

Expressing \( R \) by (73) – (74), rather than (53), enables its Euclidean limit to be determined manifestly.

Indeed, we have the following Lemma about Euclidean limits:

**Lemma 10.** Let \( A_i, A_j \in \mathbb{R}^n_s \subset \mathbb{R}^n \) be two distinct points in an Einstein gyrovector space \( (\mathbb{R}^n_s, \oplus, \otimes) \), and let
\[
a_{ij}^{\text{ein}} = \| \ominus A_i \oplus A_j \| 
\]
(75)
\[
\gamma_{ij} = \gamma \ominus A_i \oplus A_j,
\]
\[
a_{ij} = \| - A_i + A_j \|.
\]
Then,
\[
\lim_{s \to \infty} s^2 (\gamma_{ij} - 1) = \frac{1}{2} a_{ij}^2
\]
(76)
and
\[
\lim_{s \to \infty} s^2 (\gamma_{ij}^2 - 1) = a_{ij}^2.
\]
(77)

**Proof.** In the Euclidean limit, \( s \to \infty \), gamma factors tend to 1. Hence, by [4], p. 3
\[
\lim_{s \to \infty} s^2 (\gamma_{ij} - 1) = \lim_{s \to \infty} s^2 (\gamma_{ij} - 1) \times \frac{1}{2} \lim_{s \to \infty} (\gamma_{ij} + 1)
\]
(78)
\[
= \frac{1}{2} \lim_{s \to \infty} s^2 (\gamma_{ij}^2 - 1)
\]
\[
= \frac{1}{2} \lim_{s \to \infty} \gamma_{ij}^2 (a_{ij}^{\text{ein}})^2
\]
\[
= \frac{1}{2} a_{ij}^2,
\]
as desired. The proof of (77) is similar.

Following Lemma 10 we have the Euclidean limit
\[
\lim_{s \to \infty} D = 2 \left\{ 1 + \frac{a_{13}^2}{a_{12}^2} + \frac{a_{23}^2}{a_{12}^2} + 0 \right\} - \left\{ \frac{a_{12}^2}{a_{13}^2} + \frac{a_{23}^2}{a_{13}^2} + \frac{a_{23}^2}{a_{12}^2 a_{13}^2} \right\}.
\]
(79)
Hence, by (73), (78) – (79), and straightforward algebra,
\[
R_{\text{euc}}^2 := \lim_{s \to \infty} R^2 = \frac{a_{12}^2 a_{13}^2 a_{23}^2}{2(a_{12}^2 a_{13}^2 + a_{12}^2 a_{23}^2 + a_{13}^2 a_{23}^2) - (a_{12}^4 + a_{13}^4 + a_{23}^4)},
\]
(80)
where \( a_{ij} = \| - A_i + A_j \| \).

Formalizing the result of this section we obtain the following theorem.
Figure 5. Here $A_1$, $A_2$ and $A_3$ are arbitrarily selected three points of an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ that satisfy the circumgyrocircle existence condition (83). Accordingly, there exists a unique gyrocircle that passes through these points.

Figure 6. Here $A_1$, $A_2$ and $A_3$ are arbitrarily selected three points of an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ that do not satisfy the circumgyrocircle existence condition (83). Accordingly, there exists no gyrocircle that passes through these points.

**Theorem 11.** The circumradius $R$ of a triangle $A_1A_2A_3$ in a Euclidean space $\mathbb{R}^n$, $n \geq 2$, is given by the equation

$$R^2 = \frac{a_{12}^2a_{13}^2a_{23}^2}{2(a_{12}^2a_{13}^2 + a_{12}^2a_{23}^2 + a_{13}^2a_{23}^2) - (a_{12}^4 + a_{13}^4 + a_{23}^4)},$$

where

$$a_{ij} = \| - A_i + A_j \|,$$

$1 \leq i, j \leq 3$, are the sidelengths of the triangle.

8. **The Gyrocircle Through Three Points**

**Theorem 12.** (The Gyrocircle Through Three Points). Let $A_1$, $A_2$ and $A_3$ be any three points that form a gyrobarically independent set in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, shown in Figs. 5–6. There exists a unique gyrocircle that passes through these points if and only if gyrotriangle $A_1A_2A_3$ obeys the circumgyrocircle existence condition, (83).

$$(\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 > 2(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 1)$$
or, equivalently, if and only if gyrotriangle $A_1A_2A_3$ obeys the gyrotrigonometric circumgyrocircle existence condition, (83),

$$\sin(2\alpha_1 + \frac{\delta}{2}) + \sin(2\alpha_2 + \frac{\delta}{2}) + \sin(2\alpha_3 + \frac{\delta}{2}) > 3\sin\frac{\delta}{2},$$

where we use the index notation (13) for gyrotriangle $A_1A_2A_3$.

When a gyrocircle through the three points exists, it is the unique gyrocircle with gyrocenter $O$ given by, (39),

$$O = \frac{m_1\gamma_{A_1} A_1 + m_2\gamma_{A_2} A_2 + m_3\gamma_{A_3} A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}},$$

where

$$m_1 = (\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)(\gamma_{23} - 1)$$

$$m_2 = (\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)(\gamma_{13} - 1)$$

$$m_3 = (-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)(\gamma_{12} - 1)$$

and with gyroradius $R$ given by, (54),

$$R = \sqrt{2s} \sqrt{\frac{(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1)}{1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}} = s \sqrt{\frac{H_3}{D_3}}.$$

**Proof.** The gyrocircle in the theorem, if exists, is the circumgyrocircle of gyrotriangle $A_1A_2A_3$. The gyrocenter $O$ of the gyrocircle is, therefore, given by (85) – (86), as we see from Theorem 4, p. 11; and the gyroradius, $R$, of the gyrocircle is given by (54), p. 14.

Finally, the circumgyrocircle of gyrotriangle $A_1A_2A_3$ exists if and only if the gyrotriangle satisfies the circumgyrocircle existence condition (83), or, equivalently, (84), as explained in the paragraph of Inequality (30), p. 9, and in (35) – (36).

**Example 13.** If the three points $A_1$, $A_2$ and $A_3$ in Theorem 12 are not distinct, a gyrocircle through these points is not unique. Indeed, in this case we have

$$(\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 = 2(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 1),$$

as one can readily check, thus violating the circumgyrocircle existence condition (83).

**Example 14.** If the three points $A_1$, $A_2$ and $A_3$ in Theorem 12 are distinct and gyrocollinear, there is no gyrocircle through these points. Hence, in this case the circumgyrocircle existence condition (83) must be violated. Hence, these points must satisfy the inequality

$$(\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 \leq 2(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 1).$$

**Example 15.** Let the three points $A_1$, $A_2$ and $A_3$ in Theorem 12 be the vertices of an equilateral gyrotriangle with side gyrolengths $a$. Then, $\gamma_{12} = \gamma_{13} = \gamma_{23} = \gamma_a$, so that the circumgyrocircle existence condition (83) reduces to

$$\gamma_a > 1,$$
which is satisfied by any side gyrolength $a$, $0 < a < s$. Hence, by Theorem 12 any equilateral gyrotriangle in an Einstein gyrovector space possesses a circumgyrocircle. Moreover, the circumgyrocenter of an equilateral gyrotriangle lies on the interior of the gyrotriangle since the circumgyrocenter gyrobarycentric coordinates $m_k$, $k = 1, 2, 3$, in (86) are all positive.

More about the unique gyrocircle that passes through three given points of an Einstein gyrovector space is studied in Sect. 15.

A generalization of results in this section from the gyrotriangle circumgyrocircle to the gyrotetrahedron circumgyrosphere is presented in Chap. 10. Remarkably, the pattern that $D_3$ and $H_3$ exhibit in this section remains the same pattern in Chap. 10, exhibited by $D_4$ and $H_4$.

9. The Inscribed Gyroangle Theorem I

The Inscribed Gyroangle Theorem appears in two distinct, interesting versions, each of which reduces in the Euclidean limit of large $s$ to the well-known Inscribed Angle Theorem in Euclidean geometry. Version I is presented in this section, and Version II is presented in sect. 10.

Fig. 7 presents a gyrotriangle $A_1A_2A_3$ and its circumgyrocircle with gyrocenter $O$ at the gyrotriangle circumgyrocenter, given by (39), p. 11, and with gyroradius $R$, given by the gyrotriangle circumgyroradius (54), p. 14. The gamma factor $\gamma_R$ of $R$ is given by (51), p. 14.

A gyrocentral gyroangle of a gyrocircle is a gyroangle whose vertex is located at the gyrocenter of the gyrocircle. For instance, gyroangle $\angle A_1OA_2$ in Figs. 7 – 9 is gyrocentral.

An inscribed gyroangle of a gyrocircle is a gyroangle whose vertex is on the gyrocircle and whose sides each intersect the gyrocircle at another point. For instance, gyroangle $\angle A_1A_3A_2$ in Figs. 7 – 9 is inscribed. The inscribed gyroangle theorem gives a relation between inscribed gyroangles and a gyrocentral gyroangle of a gyrocircle that subtend on the same gyroarc of the gyrocircle in an Einstein gyrovector space.

**Theorem 16.** (The Inscribed Gyroangle Theorem I). Let $\theta$ be a gyroangle inscribed in a gyrocircle of gyroradius $R$, and let $2\phi$ be the gyrocentral gyroangle of the gyrocircle such that both $\theta$ and $2\phi$ subtend on the same gyroarc $A_1A_2$ on the gyrocircle, as shown in Fig. 7. Then, in the notation of Fig. 7 and in (13), p. 7,

$$
\sin \theta = \frac{2\gamma_R}{\sqrt{(\gamma_{13} + 1)(\gamma_{23} + 1)}} \sin \phi .
$$

**Proof.** Under the conditions of the theorem, as described in Fig. 7 let $M_{12}$ be the gyromidpoint of gyrosegment $A_1A_2$, implying

$$
\phi := \angle A_1OM_{12} = \angle A_2OM_{12} = \frac{1}{2} \angle A_1OA_2
$$

so that $2\phi$ is the gyrocentral gyroangle $\angle A_1OA_2$ shown in Fig. 7.
Figure 7. The Inscribed Gyroangle Theorem I. Gyroangle $\theta = \angle A_1A_3A_2$ is inscribed in a gyrocircle $C$ of gyroradius $R$ (the circum-gyroradius of gyrotriangle $A_1A_2A_3$) centered at $O$ in an Einstein gyrovector plane $(\mathbb{R}^2_1, \oplus, \otimes)$, and $\phi = \angle A_1OM_{12} = \angle A_2OM_{12}$, where $M_{12}$ is the gyromidpoint of the gyrosegment $A_1A_2$. Accordingly, $2\phi = \angle A_1OA_2$ is a gyrocentral gyroangle, and both $\theta$ and $2\phi$ subtend on the same gyroarc on the gyrocircle $C$. The relationship between $\theta$ and $\phi$, (91), is shown. In the Euclidean limit of large $s$, $s \to \infty$, gamma factors tend to 1 and, hence, the relationship between $\theta$ and $\phi$ in Euclidean geometry reduces to $\sin \theta = \sin \phi$ or, equivalently, $\theta = \phi$ (if $A_3$ and $O$ lie on the same side of gyrosegment $A_1A_2$) and $\theta = \pi - \phi$ (if $A_3$ and $O$ lie on opposite sides of gyrosegment $A_1A_2$; see Fig. [9]).

Furthermore, let

\[ a_{12} := \ominus A_1 \oplus A_2 \]

so that [26, p. 100],

\[ \ominus A_1 \oplus M_{12} = \frac{1}{2} \otimes a_{12} \]

and

\[ \frac{\gamma_{13}}{2} \otimes a_{12} \left( \frac{1}{2} \otimes a_{12} \right) = \frac{\gamma_{12}a_{12}}{\sqrt{2}\sqrt{1 + \gamma_{12}}} . \]
Taking magnitudes of both sides of (95) and noting the homogeneity property (V9) of Einstein gyrovector spaces [25, Sect. 2.7], we have

\begin{equation}
\gamma_{12}^{-1} (a_{12} \otimes a_{12}) = \frac{\gamma_{12} a_{12}}{\sqrt{2} \sqrt{1 + \gamma_{12}}}. \tag{96}
\end{equation}

Applying the extended law of gyrosines (56), p. 15, to gyrotriangle $A_1A_2A_3$ and its circumgyroradius $R$ in Fig. 7, we have

\begin{equation}
\frac{\gamma_{12} a_{12}}{\sin \theta} = \sqrt{\frac{(\gamma_{12} + 1)(\gamma_{12} + 1)(\gamma_{23} + 1)}{2}} R, \tag{97}
\end{equation}

implying

\begin{equation}
\sin \theta = \frac{\sqrt{2} \gamma_{12} a_{12}}{\sqrt{(1 + \gamma_{12})(1 + \gamma_{13})(1 + \gamma_{23})} R}. \tag{98}
\end{equation}

Applying the elementary gyrosine definition in gyrotrigonometry, illustrated in [26, Fig. 6.5, p. 147], to the right gyroangled gyrotriangle $A_1M_1O$ in Fig. 7, we obtain the first equation in (99),

\begin{equation}
\sin \phi = \frac{\gamma_{12} a_{12}}{\gamma R} = \frac{\gamma_{12} a_{12}}{\sqrt{2} \sqrt{1 + \gamma_{12}} \gamma R \hat{R}}. \tag{99}
\end{equation}

The second equation in (99) follows from (98).

Finally, the desired identity (101) follows by eliminating the factor $\gamma_{12} a_{12}$ between (98) and (99). \hfill \Box

10. The Inscribed Gyroangle Theorem II

**Theorem 17. (The Inscribed Gyroangle Theorem II).** Let $\theta$ be a gyroangle inscribed in a gyrocircle with gyrocenter $O$, and let $2\phi$ be the gyrocentral gyroangle of the gyrocircle such that both $\theta$ and $2\phi$ subtend on the same gyroarc $A_1A_2$ on the gyrocircle, as shown in Fig. 8. Furthermore, in the notation in Fig. 8, let $\delta_{A_1A_2A_3}$ be the defect of gyrotriangle $A_1A_2A_3$ and, similarly, let $\delta_{A_1A_2O}$ be the defect of gyrotriangle $A_1A_2O$. Then,

\begin{equation}
\sin(\theta + \frac{1}{2} \delta_{A_1A_2A_3}) = \sin(\phi + \frac{1}{2} \delta_{A_1A_2O}), \tag{100}
\end{equation}

that is, either

\begin{equation}
\theta + \frac{1}{2} \delta_{A_1A_2A_3} = \phi + \frac{1}{2} \delta_{A_1A_2O}, \tag{101a}
\end{equation}

as in Fig. 8 or

\begin{equation}
\theta + \frac{1}{2} \delta_{A_1A_2A_3} = \pi - (\phi + \frac{1}{2} \delta_{A_1A_2O}), \tag{101b}
\end{equation}

as in Fig. 9.
Figure 8. The Inscribed Gyroangle Theorem II. Unlike the Inscribed Gyroangle Theorem described in Fig. 7 here the relation between the inscribed gyroangle $\theta$ and the gyrocentral gyroangle $2\phi$ is expressed in terms of gyrotriangle defects. The latter vanish in Euclidean geometry, reducing the relation between $\theta$ and $\phi$ to the equation $\theta = \phi$.

Proof. In the gyrotriangle index notation (13), p. 7, with $\alpha_3 = \theta$ and $\delta = \delta_{A_1A_2A_3}$ to conform with the notation for gyrotriangle $A_1A_2A_3$ in Figs. 8 – 9, we have by (63), p. 10

$$\sin(\theta + \frac{1}{2}\delta_{A_1A_2A_3}) = \frac{\gamma_{12}a_{12}}{R(\gamma_{12} + 1)}.$$  

Here $\delta_{A_1A_2A_3}$ is the defect of gyrotriangle $A_1A_2A_3$ and $R$ is the circumgyroradius of gyrotriangle $A_1A_2A_3$ in Figs. 7 – 8. Accordingly,

$$R = \|\oplus O \ominus A_1\| = \|\oplus O \ominus A_2\|.$$  

The expression $\gamma_{12}a_{12}/(R(\gamma_{12} + 1))$ is expressed in (102) in terms of gyroangles of gyrotriangle $A_1A_2A_3$, shown in Figs. 8 – 9. We now wish to express it in terms of gyroangles of gyrotriangle $A_1A_2O$, also shown in Figs. 8 – 9. Accordingly, let

$$\epsilon = \angle A_1A_2O = \angle A_2A_1O$$

$$a_{12} = \|\oplus A_1 \ominus A_2\|$$

$$\gamma_{12} = \gamma_{a_{12}}$$
Figure 9. The Inscribed Gyroangle Theorem II. This figure is similar to Fig. 8 except that here the points $O$ and $A_3$ lie on opposite sides of the chord $A_1A_2$ of gyrocircle $C$ in the gyroplane through the points $A_1$, $A_2$ and $A_3$. As in Fig. 8, the relation between the inscribed gyroangle $\theta$ and the gyrocentral gyroangle $2\phi$ is expressed in terms of gyrotriangle defects. The latter vanish in Euclidean geometry, reducing the relation between $\theta$ and $\phi$ to the equation $\theta = \pi - \phi$.

be parameters of gyrotriangle $A_1A_2O$ and let $\delta = \delta_{A_1A_2O}$ be the defect of the gyrotriangle, as shown in Figs. 8-9.

Then,

$$\delta = \delta_{A_1A_2O} = \pi - 2\phi - 2\epsilon,$$

so that

$$\epsilon = \frac{\pi}{2} - (\phi + \frac{\delta}{2}),$$

implying

$$\epsilon + \frac{\delta}{2} = \frac{\pi}{2} - \phi,$$

so that

$$\sin(\epsilon + \frac{\delta}{2}) = \cos \phi.$$
Applying the first equation in [26, Eq. (7.143), p. 187] to gyrotriangle $A_1 A_2 O$ in Fig. 8, we have

\[(109) \quad \frac{1}{s} \gamma_{12} a_{12} = \frac{2\sqrt{F}}{\sin^2 \epsilon} \]

where, by [23], p. 8, and by (108),

\[(110) \quad F = \sin \frac{\delta}{2} \sin^2(\epsilon + \frac{\delta}{2}) \sin(2\phi + \frac{\delta}{2})
= \sin \frac{\delta}{2} \cos^2 \phi \sin(2\phi + \frac{\delta}{2}).\]

Applying [26, Eq. (7.150), p. 188] to gyrotriangle $A_1 A_2 O$ in Fig. 8, we have

\[(111) \quad \gamma_{12} + 1 = \frac{2 \sin^2(\epsilon + \frac{\delta}{2})}{\sin^2 \epsilon} \]

Following (109), (111) and (108) we have

\[(112) \quad \frac{1}{s} \gamma_{12} a_{12} + 1 = \frac{\sqrt{F}}{\sin^2(\epsilon + \frac{\delta}{2})} = \frac{\sqrt{F}}{\cos^2 \phi}. \]

We now turn to calculate $R/s$. Applying the AAA to SSS Conversion Law [26, Theorem 6.5, p. 137] to gyrotriangle $A_1 A_2 O$ in Fig. 8 we have

\[\gamma_R = \frac{\cos \epsilon + \cos \epsilon \cos 2\phi}{\sin \epsilon \sin 2\phi} = \frac{\cos \epsilon 1 + \cos 2\phi}{\sin \epsilon \sin 2\phi}\]

\[(113) \quad \gamma_R = \cot \epsilon \frac{2}{\sin 2\phi} \frac{1 + \cos 2\phi}{2} = \cot \epsilon \frac{\cos^2 \phi}{\sin \phi \cos \phi}
= \cot \epsilon \cot \phi.\]

By (4), p. 3, by (113) and by (110) we have

\[(114) \quad \frac{R^2}{s^2} = \frac{\gamma^2_R - 1}{\gamma^2_R} = \frac{\cot^2 \epsilon \cot^2 \phi - 1}{\cot^2 \epsilon \cot^2 \phi}
= \frac{\sin \frac{\delta}{2} \sin(2\phi + \frac{\delta}{2})}{\cos^2 \phi \sin^2(\phi + \frac{\delta}{2})} = \frac{F}{\cos^2 \phi \sin^2(\phi + \frac{\delta}{2})},\]

where $\delta = \delta_{A_1 A_2 O}$ is the defect of gyrotriangle $A_1 A_2 O$ given by (105), thus obtaining

\[(115) \quad \frac{R}{s} = \frac{\sqrt{F}}{\cos \phi \sin(\phi + \frac{\delta}{2})}. \]

Equations (112) and (115), along with the notation for $\delta$ in (105), imply

\[(116) \quad \frac{\gamma_{12} a_{12}}{R(\gamma_{12} + 1)} = \frac{1}{s} \frac{\gamma_{12} a_{12}}{\gamma_{12} + 1} = \frac{\sqrt{F}}{\cos \phi \sin(\phi + \frac{\delta}{2})} = \sin(\phi + \frac{\delta}{2}) = \sin(\phi + \frac{1}{2} \delta_{A_1 A_2 O}).\]
Finally, (102) and (116) imply
\[
\sin(\theta + \frac{1}{2}\delta_{A_1A_2A_3}) = \sin(\phi + \frac{1}{2}\delta_{A_1A_2O}),
\]
which, in turn, implies the results of the Theorem in (100) and in (101a) – (101b).

\[\square\]

**Remark 18.** In the proof of Theorem 17 we take for granted that the gyroline through the points \( A_1 \) and \( A_2 \) has two sides in the gyroplane through the points \( A_1, A_2 \) and \( A_3 \) so that in Fig. 9 the points \( A_3 \) and \( O \) lie on the same side of the gyroline while in Fig. 9 these points lie on opposite sides of the gyroline. The need for a careful study of the very intuitive idea that every line in a Euclidean plane has "two sides" was pointed out by Millman and Parker in [10, p. 63], resulting in what they call the Plane Separation Axiom (PSA). An analogous Gyroplane Separation Axiom (GPSA) for a gyroplane in an Einstein gyrovector space, is studied by Sönmez and Ungar in [15].

11. **Gyrocircle Gyrotangent Gyrolines**

Employing the Inscribed Gyroangle Theorem II, Theorem 17, p. 24 we prove the following theorem.

**Theorem 19.** Let \( C \) be a gyrocircle with gyrocenter \( O \) and let \( L_{PT} \) be a gyrotangent gyroline of \( C \) at a tangency point \( P \) in an Einstein gyrovector space \( (\mathbb{R}^n_s, \oplus, \otimes) \), shown in Fig. 10. Then, the gyrotangent gyroline \( L_{PT} \) is perpendicular to the gyroradius \( OP \) terminating at \( P \).

**Proof.** The proof is given by the following chain of equations, which are numbered for subsequent derivation. Let \( Q \) be a point close to \( P \) on gyrocircle \( C \), shown in Fig. 10. Then,

\[
\angle APT = \lim_{Q \to P} \angle APQ
\]

\[
\lim_{Q \to P} \left\{ \frac{1}{2}\angle AOP + \frac{1}{2}\delta_{AOP} - \frac{1}{2}\delta_{APQ} \right\}
\]

\[
\frac{1}{2}\angle AOP + \frac{1}{2}\delta_{AOP} - \frac{1}{2}\delta_{APQ}
\]

\[
\frac{\pi}{2}.
\]

Derivation of the numbered equalities in (118) follows:

1. This limit is clear from Fig. 10. Indeed, when \( Q \) approaches \( P \), the gyroline through \( P \) and \( Q \) approaches the gyrotangent gyroline through \( P \) and \( T \).
2. Follows from (1) by the Inscribed Gyroangle Theorem II, Theorem 17, p. 24 with \( \theta = \angle APQ \).
3. Follows from (2) by obvious limits as \( Q \to P \), noting that the defect of a degenerate gyrotriangle (that is, a gyrotriangle whose vertices are gyrocollinear) vanishes.
The circumgyrocircle of gyrotriangle $APQ$ with circumgyrocenter $O$ in an Einstein gyrovector plane. The circumgyrocircle is shown along with (i) its gyrodiometer $AP$ that extends its gyroradius $OP$, and (ii) its gyrotangent gyroline at the gyrotangency point $P$. When $Q$ approaches $P$, the gyroline through $P$ and $Q$ approaches the gyrotangent gyroline through $P$ and $T$. Shown are also an inscribed gyroangle $\theta$ and a gyrocentral gyroangle $2\phi$, which are related by the Inscribed Gyroangle Theorem II, Theorem 17, p. 24.

(4) Follows from (3) since $\angle AOP = \pi$.

12. SEMI-GYROCIRCLE GYROTRIANGLES

In the special case when the gyrocircle gyrochord $A_1A_2$ in Figs. 8–9 is the gyrocircle gyrodiometer, as shown in Fig. 11, we have $2\phi = \pi$ and $\delta_{A_1A_2O} = 0$ so that, by Theorem 17, (101a),

\[(119) \quad \theta + \frac{1}{2}\delta_{A_1A_2A_3} = \frac{\pi}{2}.\]

By definition, the gyrotriangular defect of gyrotriangle $A_1A_2A_3$ in Fig. 11 is given by the equation

\[(120) \quad \delta_{A_1A_2A_3} = \pi - (\alpha_1 + \alpha_2 + \theta).\]

Hence, by (119) – (120),

\[(121) \quad \theta = \alpha_1 + \alpha_2.\]
Figure 11. The Semi-Gyrocircle Gyrotriangle.

In Euclidean geometry triangle defects vanish, so that (119) reduces to the well known result according to which $\theta = \pi/2$ in Euclidean geometry.

13. THE GYROTANGENT–GYROSECANT THEOREM

A gyrotangent gyroline of a gyrocircle is a gyroline that intersects the gyrocircle in exactly one point. The point of contact is called the point of tangency.

A gyrosecant gyroline of a gyrocircle is a gyroline that intersects the gyrocircle in two different points. The gyrosegment that links these points is a gyrochord of the gyrocircle.

By Theorem 19, p. 28, the gyroradius of a gyrocircle drawn to the point of tangency of a gyrotangent gyroline of the gyrocircle is perpendicular to the gyrotangent gyroline. Accordingly, the gyrotangent gyrosegment $A_1 P$ of the gyrocircle in Fig. 12 with the tangency point $A_1$ is perpendicular to the gyroradius $OA_1$ drawn from the gyrocircle gyrocenter $O$ to the tangency point $A_1$. In full analogy with the well-known tangent–secant theorem of Euclidean geometry we present the gyrotangent–gyrosecant theorem of hyperbolic geometry.

Theorem 20. (The Gyrotangent–Gyrosecant Theorem, I). Let $A_1, A_2, A_3 \in \mathbb{R}^n$ be three distinct points that lie on a gyrocircle in an Einstein gyrovector space $\mathbb{R}^n = (\mathbb{R}^n, \oplus, \otimes)$ such that the gyrosecant gyroline of the gyrocircle through the points $A_2, A_3$ and the gyrotangent gyroline of the gyrocircle with the tangency point $A_1$ share a point $P$ not on the gyrocircle, as shown in Fig. 12.
Illustrating the Gyrotangent–Gyrosecant Theorem 20, a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_\oplus, \oplus, \otimes)$ is shown for $n = 2$, along with its circumgyrocircle and its circumgyrocenter $O$ and circumgyroradius $R = \|OA_1\|$. The gyrotangent gyrosegment $PA_1$ is tangent to the circumgyrocircle at the tangency point $A_1$. Gyrosecant gyrosegment $PA_3$ intersects the circumgyrocircle at the points $A_2$ and $A_3$. Gyrodistances $d_k$, $k = 1, 2, 3$ between various points that illustrate the gyrotangent–gyrosecant Theorem are also shown. Thus, in particular, $d_{23}$ is the gyrodistance between $A_2$ and $A_3$. Owing to the gyrotriangle equality $d_3 = d_2 \oplus d_{23}$, so that $d_{23}$ equals the gyrodifference between $d_3$ and $d_2$.

Furthermore, let

\[ d_1 = \|\ominus A_1 \oplus P\| \]
(122)
\[ d_2 = \|\ominus A_2 \oplus P\| \]
\[ d_3 = \|\ominus A_3 \oplus P\| \]

and

\[ d_{23} = \|\ominus A_2 \oplus A_3\| = d_3 \ominus d_2 \]
(123)

Then

\[ \gamma_{d_1}^2 d_1^2 = \frac{2}{\gamma_{d_3 \ominus d_2} + 1} \gamma_{d_2} d_2 \gamma_{d_3} d_3 \]
(124)
Proof. Using the gyrotriangle index notation [13], we have, in particular,
\[ \gamma_{23} = \gamma \ominus A_2 \oplus A_3 = \gamma \| \ominus A_2 \oplus A_3 \| \cdot \]
The points \( P, A_2 \) and \( A_3 \) are gyrocollinear, as shown in Fig. 12. Hence, by the gyrotriangle equality [20, Sect. 2.4],
\[ d_2 \oplus \| A_2 \oplus A_3 \| = d_3, \]
implying
\[ \| A_2 \oplus A_3 \| = d_3 \ominus d_2, \]
as stated in the Theorem, (123). Hence, by (125) and (127),
\[ \gamma_{23} = \gamma_{d_3 \ominus d_2}. \]

Let \( O \) be the circumgyrocenter of gyrotriangle \( A_1 A_2 A_3 \), as shown in Fig. 12. Gyrosegment \( PA_1 \) is tangent to the circumgyrocircle of the gyrotriangle at the tangency point \( A_1 \). Hence, gyrotriangle \( PA_1 O \) is right gyroangled, with \( \angle PA_1 O = \pi/2 \), so that, by the Einstein–Pythagoras Identity [26, Eq. (6.57), p. 144],
\[ \gamma_{\ominus A_1 \oplus O} = \gamma_{\ominus P \ominus O}. \]

The circumgyrocenter \( O \) of gyrotriangle \( A_1 A_2 A_3 \) in Fig. 12 possesses the gyrobaricentric representation
\[ O = \frac{m_{1,o} \gamma_{A_1} A_1 + m_{2,o} \gamma_{A_2} A_2 + m_{3,o} \gamma_{A_3} A_3}{m_{1,o} \gamma_{A_1} + m_{2,o} \gamma_{A_2} + m_{3,o} \gamma_{A_3}} \]
with respect to the gyrobaricentrically independent set \( \{ A_1, A_2, A_3 \} \) where, by (10), p. 11,
\[ m_{1,o} = (\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)(\gamma_{23} - 1) \]
\[ m_{2,o} = (\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)(\gamma_{13} - 1) \]
\[ m_{3,o} = (\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)(\gamma_{12} - 1). \]

Hence, by the Gyrobarycentric Representation Gyrocovariance Theorem [20, Theorem 4.6, pp. 90-91] with \( X = \ominus A_1 \), we have the following two equations (132) and (133). The first equation is
\[ \ominus A_1 \oplus O = \ominus A_1 \oplus \frac{m_{1,o} \gamma_{A_1} A_1 + m_{2,o} \gamma_{A_2} A_2 + m_{3,o} \gamma_{A_3} A_3}{m_{1,o} \gamma_{A_1} + m_{2,o} \gamma_{A_2} + m_{3,o} \gamma_{A_3}} \]
\[ = \frac{m_{1,o} \gamma_{A_1} A_1 + m_{2,o} \gamma_{A_2} A_2 + m_{3,o} \gamma_{A_3} A_3}{m_{1,o} \gamma_{A_1} + m_{2,o} \gamma_{A_2} + m_{3,o} \gamma_{A_3}} \]
\[ = \frac{m_{2,o} \gamma_{12} A_1 + m_{3,o} \gamma_{13} A_2}{m_{1,o} + m_{2,o} \gamma_{12} + m_{3,o} \gamma_{13}}. \]
The second equation is

\[ \gamma_R = \gamma_{\ominus A_1 \ominus O} = \frac{m_{1,O} + m_{2,O} \gamma_{12} + m_{3,O} \gamma_{13}}{m_O} \]

where \( R = \|\ominus A_1 \ominus O\| \) is the circumgyroradius of gyrotriangle \( A_1A_2A_3 \). Here, the constant \( m_O > 0 \) of the gyrobarycentric representation \((130)\) of \( O \) is given by the equation

\[ m_O^2 = m_{1,O}^2 + m_{2,O}^2 + m_{3,O}^2 + 2(m_{1,O}m_{2,O} \gamma_{12} + m_{1,O}m_{3,O} \gamma_{13} + m_{2,O}m_{3,O} \gamma_{23}) \]  

as we see from the gyrobarycentric representation constant associated with the Gyrobarycentric Representation Gyrocovariance Theorem \([26, \text{Theorem 4.6, pp. 90-91}]\). There will be no need to use the right-hand side of \((134)\) in the proof.

Let the point \( P \), shown in Fig. 12, be given by its gyrobarycentric representation

\[ P = \frac{m_2 \gamma_{A_2}A_2 + m_3 \gamma_{A_3}A_3}{m_2 \gamma_{A_2} + m_3 \gamma_{A_3}} \]

with respect to the gyrobarycentrically independent set \( \{A_1, A_2\} \), where the gyrobarycentric coordinates \( m_2 \) and \( m_3 \) are to be determined in \((136)\) below. The gyrobarycentric representation \((135)\) with respect to the set \( \{A_1, A_2\} \) exists since the point \( P \) lies on the gyroline that passes through the points \( A_2 \) and \( A_3 \).

By the Gyrobarycentric Representation Gyrocovariance Theorem \([26, \text{Theorem 4.6, pp. 90-91}]\) with \( X = \ominus A_1 \) we have

\[ \ominus A_1 \ominus P = \frac{m_2 \gamma_{12}a_{12} + m_3 \gamma_{13}a_{13}}{m_2 \gamma_{12} + m_3 \gamma_{13}} \]

and

\[ \gamma_{\ominus A_1 \ominus P} = \frac{m_2 \gamma_{12} + m_3 \gamma_{13}}{m_P} \]

Here, \( m_P > 0 \) is the constant of the gyrobarycentric representation \((135)\) of \( P \), given by the equation

\[ m_P^2 = m_2^2 + m_3^2 + 2m_2m_3 \gamma_{23} \]

as we see from the representation constant associated with the Gyrobarycentric Representation Gyrocovariance Theorem \([26, \text{Theorem 4.6, pp. 90-91}]\).

Similarly to \((136)\) and \((137)\), we have

\[ \ominus A_2 \ominus P = \frac{m_3 \gamma_{23}a_{23}}{m_2 + m_3 \gamma_{23}} \]

\[ \ominus A_3 \ominus P = \frac{m_2 \gamma_{23}a_{32}}{m_2 \gamma_{23} + m_3} \]
and

\[ \gamma_{\oplus A_1 \oplus P} = \frac{m_2 + m_3 \gamma_{23}}{m_p}, \]

\[ \gamma_{\oplus A_3 \oplus P} = \frac{m_2 \gamma_{23} + m_3}{m_p}. \]

Following (130) we have, by the Gyrobarycentric Representation Gyrcovariance Theorem [26, Theorem 4.6, pp. 90-91],

\[ \oplus P \oplus O = \frac{m_1,0 \gamma_{\oplus A_1 \oplus P} (\oplus P \oplus A_1) + m_2,0 \gamma_{\oplus A_2 \oplus P} (\oplus P \oplus A_2) + m_3,0 \gamma_{\oplus A_3 \oplus P} (\oplus P \oplus A_3)}{m_1,0 \gamma_{\oplus A_1 \oplus P} + m_2,0 \gamma_{\oplus A_2 \oplus P} + m_3,0 \gamma_{\oplus A_3 \oplus P}}, \]

noting that \( \gamma_{\oplus P \oplus A_k} = \gamma_{\oplus A_k \oplus P}, k = 1, 2, 3. \)

Hence, by a gamma factor identity of the Gyrobarycentric representation Gyrocovariance Theorem [26, Theorem 4.6, pp. 90-91], applied to (141), and by (137), (139), we have

\[ \gamma_{\oplus P \oplus O} = \frac{m_1,0 \gamma_{\oplus A_1 \oplus P} + m_2,0 \gamma_{\oplus A_2 \oplus P} + m_3,0 \gamma_{\oplus A_3 \oplus P}}{m_1,0 + m_2,0 + m_3,0} \]

\[ = \frac{m_1,0 (m_2 \gamma_{12} + m_3 \gamma_{13}) + m_2,0 (m_2 + m_3 \gamma_{23}) + m_3,0 (m_2 \gamma_{23} + m_3)}{m_1,0 m_p}. \]

Substituting (133), (137) and (142) into the Einstein–Pythagoras Identity (129), we obtain the equation

\[ m_1,0 (m_2 \gamma_{12} + m_3 \gamma_{13}) + m_2,0 (m_2 + m_3 \gamma_{23}) + m_3,0 (m_2 \gamma_{23} + m_3) \]

\[ = (m_2 \gamma_{12} + m_3 \gamma_{13}) (m_1,0 + m_2,0 \gamma_{12} + m_3,0 \gamma_{13}). \]

The latter, in turn, can be written as

\[ 0 = m_2 \{m_2,0 (\gamma_{12}^2 - 1) + m_3,0 (\gamma_{12} \gamma_{13} - \gamma_{23})\} \]

\[ + m_3 \{m_2,0 (\gamma_{12} \gamma_{13} - \gamma_{23}) + m_3,0 (\gamma_{13}^2 - 1)\}. \]

Solving (144) for \( m_2 \) and \( m_3 \) we obtain the equations

\[ m_2 = K \{m_2,0 (\gamma_{12} \gamma_{13} - \gamma_{23}) + m_3,0 (\gamma_{13}^2 - 1)\} \]

\[ m_3 = -K \{m_2,0 (\gamma_{12}^2 - 1) + m_3,0 (\gamma_{12} \gamma_{13} - \gamma_{23})\} \]

for any nonzero factor \( K. \)

Owing to the homogeneity of gyrobarycentric coordinates, the factor \( K \) is irrelevant. Hence, we select \( K = 1 \), obtaining the equations

\[ m_2 = m_2,0 (\gamma_{12} \gamma_{13} - \gamma_{23}) + m_3,0 (\gamma_{13}^2 - 1) \]

\[ m_3 = -m_2,0 (\gamma_{12}^2 - 1) - m_3,0 (\gamma_{12} \gamma_{13} - \gamma_{23}), \]

where \( m_{2,0} \) and \( m_{3,0} \) are given by (131).
Equations (146) for the gyrobarycentric coordinates \(m_2\) and \(m_3\) of the point \(P\) in (135) result from the Einstein–Pythagoras Identity (129). Hence, they insure that gyrosegments \(A_1P\) and \(A_1O\) are perpendicular to each other, as desired.

By the first equation in (122), and by (4), p. 3, we have

\[
\frac{1}{s^2}\gamma_i^2d_i^2 = \gamma_i^2 - 1 = \gamma_{\oplus A_1 \oplus P}^2 - 1,
\]

where the gamma factor \(\gamma_{\oplus A_1 \oplus P}\) is given by (137), (138) and (146).

Substituting (137) into (147), we have

\[
\frac{1}{s^2}\gamma_i^2d_i^2 = \gamma_{\oplus A_1 \oplus P}^2 - 1 = \gamma_i^2\oplus A_1 \oplus P - 1,
\]

where \(m_2\), \(m_3\) and \(m_2^p\) are given by (146) and (138).

Hence, by (148), (146) and by straightforward algebra,

\[
\frac{1}{s^2}\gamma_i^2d_i^2 = \left(\frac{(m_2\gamma_{12} + m_3\gamma_{13})^2 - m_2^2}{m_2^p}\right),
\]

where \(m_2\), \(m_3\) and \(m_2^p\) are given by (146) and (138).

By the second equation in (140), and by straightforward algebra,

\[
\frac{1}{s^2}\gamma_i^2d_i^2 = \gamma_i^2 - 1 = \gamma_{\oplus A_2 \oplus P}^2 - 1,
\]

where the gamma factor \(\gamma_{\oplus A_2 \oplus P}\) is given by the second equation in (140), and by (138) and (146).

Substituting the first equation in (140) into (151), we have

\[
\frac{1}{s^2}\gamma_i^2d_i^2 = \frac{1}{s^2D^2m_2^p}(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1),
\]

where

\[
D = \gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 2\gamma_{12}\gamma_{13}\gamma_{23}.
\]

Similarly to (147) and (151), by (122) and (4), p. 3, we have

\[
\frac{1}{s^2}\gamma_i^2d_i^2 = \frac{1}{s^2D^2m_2^p}(\gamma_{12} - 1)(\gamma_{23}^2 - 1),
\]

where \(m_2\), \(m_3\) and \(m_2^p\) are given by (146) and (138).

Similarly to (147) and (151), by (122) and (4), p. 3 we have

\[
\frac{1}{s^2}\gamma_i^2d_i^2 = \gamma_{13}^2 - 1 = \gamma_{\oplus A_3 \oplus P}^2 - 1,
\]

where the gamma factor \(\gamma_{\oplus A_3 \oplus P}\) is given by the second equation in (140), and by (138) and (146).

Substituting the second equation in (140) into (154), we have

\[
\frac{1}{s^2}\gamma_i^2d_i^2 = \frac{(m_2\gamma_{23} + m_3)^2 - m_2^2}{m_2^p},
\]

where \(m_2\), \(m_3\) and \(m_2^p\) are given by (146) and (138).
Hence, by (155), (146) and straightforward algebra,

\[ \frac{1}{s^2} \gamma_3 d_3^2 = \frac{1}{s^2 D^2 m_P^2} (\gamma_{13} - 1)(\gamma_{23} - 1). \]

Finally, following (149), (153) and (156) and straightforward algebra, noting (128), we have

\[ \frac{(\gamma_2 d_1 d_2)}{\gamma_2 d_2 d_3^2} = \left( \frac{2}{\gamma_{23} + 1} \right)^2 = \left( \frac{2}{\gamma_{d_3 d_2} + 1} \right)^2, \]

thus verifying the result (124) of the Theorem. \[ \square \]

In order to restate Theorem 20 in a way that emphasizes analogies with its Euclidean counterpart, we introduce the notation

\[ |AB| := \|A \oplus B\| = \|B \oplus A\| \quad \text{(Hyperbolic Geometry)} \]

for the gyrodistance between points A and B of an Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\). Accordingly, \(|AB|\) is the gyrolength of gyrosegment AB.

In order to emphasize analogies we use, ambiguously, the same notation in the context of Euclidean geometry as well, that is,

\[ |AB| := \|-A + B\| = \|-B + A\| \quad \text{(Euclidean Geometry)} \]

is the distance between points A and B of a Euclidean vector space \(\mathbb{R}^n\). Accordingly, \(|AB|\) is the length of segment AB. It should always be clear from the context whether \(|AB|\) represents the gyrolength of a gyrosegment in hyperbolic geometry, or the length of a segment in Euclidean geometry.

Using the notation in (158) – (159), we restate Theorem 20 as follows, noting that by (127) and (122),

\[ d_3 \otimes d_2 = |PA_3| \otimes |PA_2| = |A_2A_3|. \]

**Theorem 21. (The Gyrotangent–Gyrosecant Theorem, II).** If a gyrotangent of a gyrocircle from an external point P meets the gyrocircle at A1, and a gyrosecant from P meets the gyrocircle at A2 and A3, as shown in Fig. 12, then

\[ \gamma_{[PA_1]} |PA_1|^2 = \frac{2}{\gamma_{[A_2 A_3]} + 1} \gamma_{[PA_2]} |PA_2| \gamma_{[PA_3]} |PA_3|. \]

In the Euclidean limit, \(s \to \infty\), gyrolengths of gyrosegments tend to lengths of corresponding segments and gamma factors tend to 1. Hence, in that limit, the Gyrotangent–Gyrosecant Theorem 21 reduces to the following well-known Tangent–Secant Theorem of Euclidean geometry:

**Theorem 22. (The Tangent–Secant Theorem).** If a tangent of a circle from an external point P meets the circle at A1, and a secant from P meets the circle at A2 and A3, then

\[ |PA_1|^2 = |PA_2||PA_3|. \]
Figure 13. Illustrating the Intersecting Gyrosecants Theorem 23: two intersecting gyrosecants \( PA_3 \) and \( PB_3 \) of a gyrocircle are shown. They, respectively, intersect the gyrocircle at the points \( A_2, A_3 \) and at the points \( B_2, B_3 \).

14. THE INTERSECTING GYROSECANTS THEOREM

As an obvious corollary of the Gyrotangent–Gyrosecant Theorem 21, we have the following theorem for intersecting gyrosecants of a gyrocircle:

**Theorem 23. (The Intersecting Gyrosecants Theorem).** If two gyrosecants of a gyrocircle in an Einstein gyrovector space \( (\mathbb{R}^n, \oplus, \otimes) \), drawn to the gyrocircle from an external point \( P \), meet the gyrocircle at points \( A_2, A_3 \) and at points \( B_2, B_3 \), respectively, as shown in Fig. 13, then

\[
\frac{\gamma_{|PA_2|}|PA_2|\gamma_{|PA_3|}|PA_3|}{\gamma_{|A_2A_3|} + 1} = \frac{\gamma_{|PB_2|}|PB_2|\gamma_{|PB_3|}|PB_3|}{\gamma_{|B_2B_3|} + 1}.
\]

**Proof.** Let \( PA_1 \) be a gyrotangent gyrosegment of the gyrocircle drawn from \( P \) and meeting the gyrocircle at \( A_1 \), as shown in Fig. 13. Then, by Theorem 21, each of the two sides of (163) equals half the left-hand side of (161) thus verifying (163). □

In the Euclidean limit, \( s \to \infty \), gyrolengths of gyrosegments tend to lengths of corresponding segments and gamma factors tend to 1. Hence, in that limit, the Intersecting Gyrosecants Theorem 23 of hyperbolic geometry reduces to the following well-known Intersecting Secants Theorem of Euclidean geometry:

**Theorem 24. (The Intersecting Secants Theorem).** If two secants of a circle in a Euclidean vector space \( \mathbb{R}^n \), drawn to the circle from an external point \( P \), meet the circle at points \( A_2, A_3 \) and at points \( B_2, B_3 \), respectively, then

\[
|PA_2||PA_3| = |PB_2||PB_3|.
\]
15. **Gyrocircle Gyrobarycentric Representation**

Let $A_1A_2A_3$ be a gyrotriangle that possesses a circumgyrocircle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$. The circumgyrocenter $O$ of gyrotriangle $A_1A_2A_3$, shown in Figs. 13 and 14, possesses the gyrobarycentric representation

$$O = \frac{m_{1,0}\gamma_{A_1} A_1 + m_{2,0}\gamma_{A_2} A_2 + m_{3,0}\gamma_{A_3} A_3}{m_{1,0}\gamma_{A_1} + m_{2,0}\gamma_{A_2} + m_{3,0}\gamma_{A_3}}$$

with respect to the gyrobarycentrically independent set \{\$A_1, A_2, A_3\$\} of the reference gyrotriangle vertices where, by (40), p. 11,

$$m_{1,0} = (\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)(\gamma_{23} - 1)$$

$$m_{2,0} = (\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)(\gamma_{13} - 1)$$

$$m_{3,0} = (\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)(\gamma_{12} - 1),$$

as in (130) and (131).

The constant $m_0 > 0$ of the gyrobarycentric representation (165) of $O$ is given by (25)–(26), p. 9

$$m_0^2 = m_{1,0}^2 + m_{2,0}^2 + m_{3,0}^2$$

$$+ 2(m_{1,0}m_{2,0}\gamma_{12} + m_{1,0}m_{3,0}\gamma_{13} + m_{2,0}m_{3,0}\gamma_{23})$$

$$= D_3(D_3 - H_3).$$

With $A_3 = A_1 \oplus \text{Span}\{\ominus A_1 \oplus A_2, \ominus A_1 \oplus A_3\} \subset \mathbb{R}_s^n$, let $A$ be a generic point in $A_3 \cap \mathbb{R}_s^n$, given by its gyrobarycentric representation

$$A = \frac{m_{1}\gamma_{A_1} A_1 + m_{2}\gamma_{A_2} A_2 + m_{3}\gamma_{A_3} A_3}{m_{1}\gamma_{A_1} + m_{2}\gamma_{A_2} + m_{3}\gamma_{A_3}}$$

with respect to the gyrobarycentrically independent set $S = \{A_1, A_2, A_3\}$. A relationship between the gyrobarycentric coordinates $m_1, m_2$ and $m_3$ of $A$ is to be determined in (183) below by the condition that the point $A$ lies on the circumgyrocircle of gyrotriangle $A_1A_2A_3$, as shown in Fig. 14.

By the Gyrobarycentric Representation Gyrocovariance Theorem [26, Theorem 4.6, pp. 90-91] with $X = \ominus A$ we have

$$\ominus A \ominus O = \ominus A \ominus \frac{m_{1,0}\gamma_{A_1} A_1 + m_{2,0}\gamma_{A_2} A_2 + m_{3,0}\gamma_{A_3} A_3}{m_{1,0}\gamma_{A_1} + m_{2,0}\gamma_{A_2} + m_{3,0}\gamma_{A_3}}$$

$$= \frac{m_{1,0}\gamma_{A_1 \ominus A_1} (\ominus A \ominus A_1) + m_{2,0}\gamma_{A_2 \ominus A_2} (\ominus A \ominus A_2) + m_{3,0}\gamma_{A_3 \ominus A_3} (\ominus A \ominus A_3)}{m_{1,0}\gamma_{A_1} + m_{2,0}\gamma_{A_2} + m_{3,0}\gamma_{A_3}}$$

and

$$\gamma_d = \frac{m_{1,0}\gamma_{A_1 \ominus A} + m_{2,0}\gamma_{A_2 \ominus A} + m_{3,0}\gamma_{A_3 \ominus A}}{m_0}. $$
Figure 14. A generic point $A$ on the circumgyrocircle $C(A_1A_2A_3)$ of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_\oplus, \otimes)$. The point $A$ lies on the circumgyrocircle $C(A_1A_2A_3)$ if and only if it possesses the gyrobarcentric representation (168) with gyrobarcentric coordinates that satisfy (183).

where $d = \|\oplus A \oplus O\|$ is the gyrodistance from $A$ to $O$, and where the constant $m_O > 0$ of the gyrobarcentric representation (165) of $O$ is given by (167).

We will now calculate the gamma factors $\gamma_{\oplus A_k \oplus A}$, $k = 1, 2, 3$, that appear in (170).

Applying the Gyrobarcentric Representation Gyrocovariance Theorem [26, Theorem 4.6, pp. 90-91] with $X = \oplus A_1$ to the gyrobarcentric representation (168) of $A$, we have

(171)

\[
\ominus A_1 \oplus A = \ominus A_1 \oplus \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}}
\]

\[
= \frac{m_1 \gamma_{\oplus A_1 \oplus A_1} (\ominus A_1 \oplus A_1) + m_2 \gamma_{\oplus A_1 \oplus A_2} (\ominus A_1 \oplus A_2) + m_3 \gamma_{\oplus A_1 \oplus A_3} (\ominus A_1 \oplus A_3)}{m_1 \gamma_{\oplus A_1 \oplus A_1} + m_2 \gamma_{\oplus A_1 \oplus A_2} + m_3 \gamma_{\oplus A_1 \oplus A_3}}
\]

\[
= \frac{m_2 \gamma_{12} a_{12} + m_3 \gamma_{13} a_{13}}{m_1 + m_2 \gamma_{12} + m_3 \gamma_{13}}
\]

and

(172)

\[
\gamma_{\oplus A_1 \oplus A} = \frac{m_1 + m_2 \gamma_{12} + m_3 \gamma_{13}}{m_A}
\]
where $m_A > 0$ is the constant of the gyrobarycentric representation (168) of $A$, given by the equation

$$
\begin{align*}
\quad m_A^2 &= m_1^2 + m_2^2 + m_3^2 + 2(m_1m_2\gamma_{12} + m_1m_3\gamma_{13} + m_2m_3\gamma_{23}) \\
&= (m_1 + m_2 + m_3)^2 \\
&\quad + 2\{m_1m_2(\gamma_{12} - 1) + m_1m_3(\gamma_{13} - 1) + m_2m_3(\gamma_{23} - 1)\}. \\
\end{align*}
$$

(173a)

It proves useful to write (173a) as

$$
\begin{align*}
\quad m_A^2 &= (m_1 + m_2 + m_3)^2 \quad + \quad 2K(A; A_1, A_2, A_3),
\end{align*}
$$

(173b)

where

$$
\begin{align*}
\quad K &= K(A; A_1, A_2, A_3) = \sum_{i,j=1}^{3} m_im_j(\gamma_{ij} - 1),
\end{align*}
$$

(173c)

where $m_k, \ k = 1, 2, 3,$ are the gyrobarycentric coordinates of $A$ in the representation (168) of $A$ with respect to the vertices of gyrotriangle $A_1A_2A_3$, and $\gamma_{ij} = \|\oplus A_i \oplus A_j\|$.

Similarly to (171) – (172), we have

$$
\begin{align*}
\quad \ominus A_2 \oplus A &= \frac{m_1\gamma_{12} a_{21} + m_3\gamma_{23} a_{23}}{m_1\gamma_{12} + m_2 + m_3\gamma_{23}} \\
\quad \gamma_{\ominus A_2 \oplus A} &= \frac{m_1\gamma_{12} + m_2 + m_3\gamma_{23}}{m_A},
\end{align*}
$$

(174)

and

$$
\begin{align*}
\quad \ominus A_3 \oplus A &= \frac{m_1\gamma_{13} a_{31} + m_2\gamma_{23} a_{32}}{m_1\gamma_{13} + m_2\gamma_{23} + m_3} \\
\quad \gamma_{\ominus A_3 \oplus A} &= \frac{m_1\gamma_{13} + m_2\gamma_{23} + m_3}{m_A}.
\end{align*}
$$

(175)

Similarly to (171) – (172) and to (174) – (175), we have

$$
\begin{align*}
\quad \ominus A_3 \oplus A &= \frac{m_1\gamma_{13} a_{31} + m_2\gamma_{23} a_{32}}{m_1\gamma_{13} + m_2\gamma_{23} + m_3} \\
\quad \gamma_{\ominus A_3 \oplus A} &= \frac{m_1\gamma_{13} + m_2\gamma_{23} + m_3}{m_A},
\end{align*}
$$

(176)

and

$$
\begin{align*}
\quad \ominus A_3 \oplus A &= \frac{m_1\gamma_{13} a_{31} + m_2\gamma_{23} a_{32}}{m_1\gamma_{13} + m_2\gamma_{23} + m_3} \\
\quad \gamma_{\ominus A_3 \oplus A} &= \frac{m_1\gamma_{13} + m_2\gamma_{23} + m_3}{m_A}.
\end{align*}
$$

(177)

Substituting the gamma factors (172), (175) and (177), as well as (166) and (167), into (170), we obtain the elegant equation

$$
\begin{align*}
\quad \gamma_d^2 &= \frac{D_3}{D_3 - H_3} \left( \frac{m_1 + m_2 + m_3}{m_A} \right)^2,
\end{align*}
$$

(178)
where, as in  \((28) \rightarrow (29)\), p. 9,
\[
D_3 = 1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2 \\
= 2 \left\{ (\gamma_{12} - 1)(\gamma_{13} - 1) + (\gamma_{12} - 1)(\gamma_{23} - 1) + (\gamma_{13} - 1)(\gamma_{23} - 1) \right\} \\
- \left\{ (\gamma_{12} - 1)^2 + (\gamma_{13} - 1)^2 + (\gamma_{23} - 1)^2 \right\} \\
= \begin{vmatrix}
1 & \gamma_{12} & \gamma_{13} \\
\gamma_{12} & 1 & \gamma_{23} \\
\gamma_{13} & \gamma_{23} & 1 \\
\end{vmatrix} \\
(179a)
\]
and
\[
H_3 = 2(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1).
(179b)
\]

As expected, the denominator \(D_3 - H_3\) in \((178)\) is positive, by the circumgyrocircle existence condition \((30c)\), p. 9.

The point \(A\) in \((168)\) lies on the circumgyrocircle of gyrotriangle \(A_1A_2A_3\) (Fig. 14) if and only if
\[
\gamma_{d}^2 = \gamma_R^2.
(180)
\]
Inserting into \((180)\) \(\gamma_{d}^2\) from \((178)\) and \(\gamma_R^2\) from \((51)\), p. 14, we obtain the equation
\[
\frac{D_3}{D_3 - H_3} \left( \frac{m_1 + m_2 + m_3}{m_A} \right)^2 = \frac{D_3}{D_3 - H_3},
(181)
\]
implying
\[
m_A^2 = (m_1 + m_2 + m_3)^2.
(182)
\]
The latter, in turn, is valid if and only
\[
K(A; A_1, A_2, A_3) := \sum_{i,j=1}^{3} m_i m_j (\gamma_{ij} - 1) = 0,
(183)
\]
as we see from \((173)\). Equation \((183)\) expresses the circumgyrocircle condition in the sense that it provides a necessary and sufficient condition that the point \(A\) in \((168)\) lies on the circumgyrocircle \(C(A_1A_2A_3)\) of gyrotriangle \(A_1A_2A_3\).

A gyrotrigonometric version of the circumgyrocircle condition \((183)\) follows from \([26, Eq. (7.149), p. 188]\),
\[
m_1 m_2 \sin \alpha_3 \sin (\alpha_3 + \frac{\delta}{2}) + m_1 m_3 \sin \alpha_2 \sin (\alpha_2 + \frac{\delta}{2}) \\
+ m_2 m_3 \sin \alpha_1 \sin (\alpha_1 + \frac{\delta}{2}) = 0.
(184)
\]

In order to parametrize the points of the circumgyrocircle \(C(A_1A_2A_3)\) of a gyrotriangle \(A_1A_2A_3\) in an Einstein gyrovector space \(\mathbb{R}^n\) by a real parameter, we assume
Figure 15. Several points on the circumgyrocircle $C(A_1A_2A_3)$ of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n \geq 2$, determined by several values of the gyrocircle parameter $t$ in (200), $t = -8, -7, \ldots, -1, 0, 1, \ldots, 7, 8$, are presented. Clearly, the parameter value $t = 0$ corresponds to gyrobarycentric coordinates $m_1 = m_2 = 0, m_3 \neq 0$, as evidenced from (186). The latter, in turn, corresponds to the point $A = A_3$ on the circumgyrocircle, as shown here and as evidenced from (186). When $t \to \pm \infty$, corresponding points on the circumgyrocircle tend to $A_2$, as indicated here and as evidenced from (186), where $|m_1|, |m_3| \ll |m_2|$ for large $|t|$. The point $A_1$ on the circumgyrocircle corresponds to the parameter value $t = -(\gamma_{13} - 1)/(\gamma_{12} - 1)$.

$m_3 \neq 0$, so that the circumgyrocircle condition (183) can be written as

\begin{equation}
\frac{m_1 m_2}{m_3} (\gamma_{12} - 1) + \frac{m_1}{m_3} (\gamma_{13} - 1) + \frac{m_2}{m_3} (\gamma_{23} - 1) = 0.
\end{equation}

Selecting $m_2/m_3 = t$ as a parameter on the extended real line, $t \in \mathbb{R} \cup \{-\infty, \infty\}$, a system of parametric gyrobarycentric coordinates of the point $A$ in (185) with respect to the gyrobarycentrically independent set $\{A_1, A_2, A_3\}$ that obeys the circumgyrocircle condition (183) is obtained, given by

\begin{align}
m_1 &= -(\gamma_{23} - 1)t \\
m_2 &= (\gamma_{12} - 1)t^2 + (\gamma_{13} - 1)t = m_3t \\
m_3 &= (\gamma_{12} - 1)t + (\gamma_{13} - 1).
\end{align}
Several points $A$ of the circumgyrocircle $C(A_1A_2A_3)$ of a gyrotriangle $A_1A_2A_3$, which are given by (168) and (186) and which correspond to the circumgyrocircle parameter values $t = -8, -7, \ldots, -1, 0, 1, \ldots, 7, 8$, are shown in Fig. 15. The parameter values $t = -\infty$ and $t = \infty$ correspond jointly to the point $A_2$.

The two special values, $t = \pm\infty$, of the parameter $t$ are identified in the sense that they correspond jointly to the same point, $A_2$, on circumgyrocircle $C(A_1A_2A_3)$, as shown in Fig. 15. This identification suggests the replacement of the parameter $t \in \mathbb{R} \cup \{-\infty, \infty\}$ by the parameter $\theta$, $-\pi \leq \theta \leq \pi$, according to the bijective (one-to-one) relation

$$t = \tan \frac{\theta}{2},$$

for which a corresponding identification is built in naturally. Indeed, the two values $\theta = \pm\pi$ correspond to the two values $t = \pm\infty$ and, accordingly, $\theta = -\pi$ and $\theta = \pi$ are identified.
Inserting \( t \) from (187) in (186), and noting that \( \tan \frac{\theta}{2} = \sin \frac{\theta}{2}/\cos \frac{\theta}{2} \), and that gyrobaricentric coordinates are homogeneous, we obtain the following system of parametric gyrobaricentric coordinates of the point \( A \) in (168), p. 38 with respect to the set \( S \),

\[
\begin{align*}
    m_1 &= - (\gamma_{23} - 1) \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\
    m_2 &= (\gamma_{12} - 1) \sin \frac{\theta}{2} + (\gamma_{13} - 1) \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\
    m_3 &= (\gamma_{12} - 1) \sin \frac{\theta}{2} \cos \frac{\theta}{2} + (\gamma_{13} - 1) \cos \frac{\theta}{2}.
\end{align*}
\]

We now apply to (188) well-known trigonometric half-angle identities, noting that gyrobaricentric coordinates are homogeneous, and translate the parameter interval by \( \pi \), from the interval \([-\pi, \pi]\) to the interval \([0, 2\pi]\), obtaining the following elegant parametric gyrobaricentric coordinates of \( A \) in (168). p. 38.

\[
\begin{align*}
    m_1 &= - (\gamma_{23} - 1) \sin \theta \\
    m_2 &= (\gamma_{13} - 1) \sin \theta + (\gamma_{12} - 1)(1 - \cos \theta) \\
    m_3 &= (\gamma_{12} - 1) \sin \theta + (\gamma_{13} - 1)(1 + \cos \theta),
\end{align*}
\]

0 \leq \theta \leq 2\pi. The two values of the parameter \( \theta \), \( \theta = 0 \) and \( \theta = 2\pi \), are identified, as indicated in Fig. 16, where several points \( A \) on the circumference of circumgyrocircle \( C(A_1A_2A_3) \), which correspond to several values of the circumgyrocircle parameter \( \theta \), 0 \leq \theta \leq 2\pi, are presented.

16. **Gyrocircle Interior and Exterior Points**

We are now in the position to present a definition followed by two theorems that characterize the points of the circumgyrocircle of a given gyrotriangle, as well as interior and exterior points of the circumgyrocircle in Einstein gyrovector spaces.

**Definition 25. (Gyrocircle Interior and Exterior Points).** Let \( A_1A_2A_3 \) be a gyrotriangle that possesses a circumgyrocircle \( C(A_1A_2A_3) \) with circumgyrocenter \( O \) and circumgyroradius \( R \) in an Einstein gyrovector space \((\mathbb{R}^n_\oplus, \oplus, \otimes))\). Let \( P \) be a generic point in \( \mathbb{A}_3 \cap \mathbb{R}^n_\oplus \),

\[
P = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}} \in \mathbb{A}_3 \cap \mathbb{R}^n_\oplus,
\]

where, as in (8), p. 4

\[
\mathbb{A}_3 = A_1 \oplus \text{Span}\{ \ominus A_1 \oplus A_2, \ominus A_1 \oplus A_3 \} \subset \mathbb{R}^n,
\]

and let \( d = \| \ominus P \oplus O \| \) be the gyrodistance of \( P \) from \( O \). Then, \( P \) lies

(1) in the interior of \( C(A_1A_2A_3) \) if \( d < R \);
(2) in the exterior of \( C(A_1A_2A_3) \) if \( d > R \); and
(3) on \( C(A_1A_2A_3) \) if \( d = R \).
Theorem 26. (Point to Circumgyrocenter Gyrodistance). Let \(A_1A_2A_3\) be a gyrotriangle that possesses a circumgyrocircle \(C(A_1A_2A_3)\) with circumgyrocenter \(O\) and circumgyroradius \(R\) in an Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\), and let \(A \in A_3 \cap \mathbb{R}^n_s\) be a point given by its gyrobarycentric representation
\[
A = \frac{m_1\gamma A_1 A_1 + m_2\gamma A_2 A_2 + m_3\gamma A_3 A_3}{m_1\gamma A_1 + m_2\gamma A_2 + m_3\gamma A_3}
\]
with respect to the gyrobarycentrically independent set \(S = \{A_1, A_2, A_3\}\). Furthermore, let
\[
d = \|\Theta A \ominus O\|
\]
be the gyrodistance from \(A\) to \(O\).

Then,
\[
d = \sqrt{1 - \frac{2s^2K}{M^2\gamma_R^2 R^2}} R,
\]
where
\[
K = \sum_{i,j=1}^{3} m_i m_j (\gamma_{ij} - 1)
\]
and
\[
M = \sum_{k=1}^{3} m_k.
\]

Proof. Following (i) (178), p. 40, and (ii) (51), p. 14, and (iii) (173), p. 40, we have
\[
\gamma^2_d = \frac{D_3 M^2}{D_3 - H_3 m^2_A}
\]
(197)
\[
\gamma^2_R = \frac{D_3}{D_3 - H_3}
\]
\[
m^2_A = M^2 + 2K
\]
where \(D_3\) and \(H_3\) are given by (179).

Hence, by (197) and (1), p. 3, or (53), p. 44
\[
R^2 = s^2 \frac{\gamma^2_R - 1}{\gamma^2_R} = s^2 \frac{H_3}{D_3}
\]
and
\[
d^2 = s^2 \frac{\gamma^2_d - 1}{\gamma^2_d} = s^2 \frac{M^2 H_3 - 2K(D_3 - H_3)}{M^2 D_3},
\]
so that, by straightforward algebra,
\[
\frac{d^2}{R^2} = 1 - \frac{2s^2 K}{M^2 \gamma^2_R R^2},
\]

as desired.

\[ \square \]

**Theorem 27. (Gyrocircle Gyrobarycentric Representation).** Let \( A_1A_2A_3 \) be a gyrotriangle that possesses a circumgyrocircle \( C(A_1A_2A_3) \) with circumgyrocenter \( O \) and circumgyroradius \( R \) in an Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\), and let \( A \in A_3 \cap \mathbb{R}^n_s \) be a point given by its gyrobarycentric representation

\[
A = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}}
\]

with respect to the gyrobarycentrically independent set \( S = \{A_1, A_2, A_3\} \). Furthermore, let

\[
K(A; A_1, A_2, A_3) = m_1 m_2 (\gamma_{12} - 1) + m_1 m_3 (\gamma_{13} - 1) + m_2 m_3 (\gamma_{23} - 1),
\]

(202)

\[
T(A; A_1, A_2, A_3) = m_1 m_2 \sin \alpha_3 \sin (\alpha_3 + \frac{\delta}{2}) + m_1 m_3 \sin \alpha_2 \sin (\alpha_2 + \frac{\delta}{2}) + m_2 m_3 \sin \alpha_1 \sin (\alpha_1 + \frac{\delta}{2}),
\]

where \( \alpha_k, k = 1, 2, 3, \) is the vertex gyroangle of vertex \( A_k \) of gyrotriangle \( A_1A_2A_3 \), be two scalars associated with the point \( A \).

(1) The point \( A \) lies on the circumgyrocircle \( C(A_1A_2A_3) \) of gyrotriangle \( A_1A_2A_3 \) if and only if the gyrobarycentric coordinates \( m_1, m_2, m_3 \) of \( A \) in (201) satisfy the circumgyrocircle condition

(203a)

\[
K(A; A_1, A_2, A_3) = 0
\]

or, equivalently, the gyrotrigonometric circumgyrocircle condition

(203b)

\[
T(A; A_1, A_2, A_3) = 0.
\]

(2) The point \( A \) lies in the interior of circumgyrocircle \( C(A_1A_2A_3) \) if and only if

(204a)

\[
K(A; A_1, A_2, A_3) > 0
\]

or, equivalently,

(204b)

\[
T(A; A_1, A_2, A_3) > 0.
\]

(3) The point \( A \) lies in the exterior of circumgyrocircle \( C(A_1A_2A_3) \) if and only if

(205a)

\[
K(A; A_1, A_2, A_3) < 0
\]

or, equivalently,

(205b)

\[
T(A; A_1, A_2, A_3) < 0.
\]

Moreover, the circumgyrocircle \( C(A_1A_2A_3) \) is the locus of the point \( A \) in (201), with parametric gyrobarycentric coordinates \( m_k, k = 1, 2, 3, \) given by the parametric equations

(206)

\[
m_1 = -(\gamma_{23} - 1)t
\]

\[
m_2 = (\gamma_{12} - 1)t^2 + (\gamma_{13} - 1)t = m_3 t
\]

\[
m_3 = (\gamma_{12} - 1)t + (\gamma_{13} - 1)
\]
with the parameter $t$, $t \in \mathbb{R} \cup \{-\infty, \infty\}$, where the two parameter values $t = -\infty$ and $t = \infty$ are identified (Fig. 15), or, equivalently, by the parametric equations
\begin{align}
   m_1 &= -(\gamma_{23} - 1) \sin \theta \\
   m_2 &= (\gamma_{13} - 1) \sin \theta + (\gamma_{12} - 1)(1 - \cos \theta) \\
   m_3 &= (\gamma_{12} - 1) \sin \theta + (\gamma_{13} - 1)(1 + \cos \theta)
\end{align}
with the parameter $\theta$, $0 \leq \theta \leq 2\pi$, where the two parameter values $\theta = 0$ and $\theta = 2\pi$ are identified (Fig. 16).

(6) The circumgyrocenter, $O$, of gyrotriangle $A_1A_2A_3$ is given by (165) – (166), p. 38, and, gyrotrigonometrically, by (39) – (41), p. 12.

(7) The circumgyroradius, $R$, of gyrotriangle $A_1A_2A_3$ is given by (198).

Proof. The equivalence between $K$ and $T$ in the Theorem is proved in (183) – (184). The proof of each item of the Theorem follows.

(1) It follows from (194) that $K = K(A; A_1, A_2, A_3)$ vanishes if an only if $d = R$, that is, by Def. 25, if an only if the point $A$ lies on circumgyrocircle $C(A_1A_2A_3)$.

(2) It follows from (194) that $K = K(A; A_1, A_2, A_3) > 0$ if an only if $d < R$, that is, by Def. 25, if an only if the point $A$ lies in the interior of circumgyrocircle $C(A_1A_2A_3)$.

(3) It follows from (194) that $K = K(A; A_1, A_2, A_3) < 0$ if an only if $d > R$, that is, by Def. 25, if an only if the point $A$ lies in the exterior of circumgyrocircle $C(A_1A_2A_3)$.

(4) Item (4) follows from (186).

(5) Item (5) follows from (189).

(6) The proof of Item (6) is given in the derivation of (165) – (166), p. 38, and (39) – (41), p. 12.

(7) The proof of Item (7) is given in the derivation of (198).

The proof of the Theorem is thus complete. □

**Example 28.** Let $A$ be a point in an Einstein gyrovector space $\mathbb{R}^n_2$, $n \geq 2$, given by its gyrobarcentric representation (201) with respect to a gyrobarcentrically independent set $\{A_1, A_2, A_3\}$, with gyrobarcentric coordinates $m_1 = 0$, $m_2 = 0$ and $m_3 \neq 0$. Then, $A = A_3$, and the gyrobarcentric coordinates of $A$ satisfy the circumgyrocircle condition (203) of Theorem 27. Hence, by Theorem 27, the point $A_3$ lies on the circumgyrocircle of gyrotriangle $\{A_1, A_2, A_3\}$, as obviously expected.

**Example 29.** Let $A$ be a point in an Einstein gyrovector space $\mathbb{R}^n_2$, $n \geq 2$, given by its gyrobarcentric representation (201) with respect to a gyrobarcentrically independent set $\{A_1, A_2, A_3\}$, where gyrotriangle $A_1A_2A_3$ is equilateral. Then, $A$ lies on the circumgyrocircle of gyrotriangle $A_1A_2A_3$ if and only if the gyrobarcentric coordinates $m_1, m_2, m_3$ of $A$ in (201) satisfy the circumgyrocircle condition
\begin{equation}
   m_1 m_2 + m_1 m_3 + m_2 m_3 = 0.
\end{equation}
17. **Circle Barycentric Representation**

Lemma 10, p. 19 enables Theorem 27 to be reduced to its Euclidean counterpart, Theorem 31. As an immediate application of Lemma 10 we, therefore, note that

\[ \lim_{s \to \infty} 2s^2 K(A; A_1, A_2, A_3) = m_1m_2a_{12}^2 + m_1m_3a_{13}^2 + m_2m_3a_{23}^2 \]

where \( K = K(A; A_1, A_2, A_3) \) is given by (202), and \( a_{ij} = \| -A_i + A_j \| \).

**Definition 30. (Circle Interior and Exterior Points).** Let \( A_1A_2A_3 \) be a triangle and let \( C(A_1A_2A_3) \), \( O \) and \( R \) be, respectively, the circumcircle, circumcenter and circumradius of the triangle in a Euclidean space \( \mathbb{R}^n \). Let \( P \) be a generic point in \( \mathbb{A}_3^{euc} \),

\[ P = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \in \mathbb{A}_3^{euc}, \]

where

\[ \mathbb{A}_3^{euc} = A_1 + \text{Span}\{-A_1 + A_2, -A_1 + A_3\} \subset \mathbb{R}^n, \]

and let \( d = \| -P + O \| \) be the distance of \( P \) from \( O \). Then, \( P \) lies

1. in the interior of \( C(A_1A_2A_3) \) if \( d < R \);
2. in the exterior of \( C(A_1A_2A_3) \) if \( d > R \); and
3. on \( C(A_1A_2A_3) \) if \( d = R \).

**Theorem 31. (Point to Circumcenter Distance).** Let \( A_1A_2A_3 \) be a triangle and let \( C(A_1A_2A_3) \), \( O \) and \( R \) be, respectively, the circumcircle, circumcenter and circumradius of the triangle in a Euclidean space \( \mathbb{R}^n \), and let \( A \in \mathbb{A}_3^{euc} \) be a point given by its barycentric representation

\[ A = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \]

with respect to the barycentrically independent set \( S = \{A_1, A_2, A_3\} \). Furthermore, let

\[ d = \| -A + O \| \]

be the distance from \( A \) to \( O \).

Then,

\[ d = \sqrt{1 - \frac{K_{euc}}{M^2R^2}} R, \]

where

\[ K_{euc} = \sum_{i,j=1}^{3} m_im_ja_{ij}^2, \]

\( a_{ij} = \| -A_i + A_j \| \), and

\[ M = \sum_{k=1}^{3} m_k. \]
Figure 17. Illustrating Theorems 31 and 32. The circumcircle $C(A_1A_2A_3)$ of triangle $A_1A_2A_3$ in the Euclidean plane $\mathbb{R}^2$ is the locus of the point $A$ with barycentric representation (217), where its parametric barycentric coordinates are given by (222) or, equivalently, by (223). The radius of $C(A_1A_2A_3)$ is $R = \| -O + A_k \|$, $k = 1, 2, 3$, and the distance $d = \| -A + O \|$ from a generic point $A$ in the triangle plane to the center $O$ of $C(A_1A_2A_3)$ is given by (214). The latter provides an elegant condition in Theorem 32 that determines whether the point $A$ lies on the circumcircle $C(A_1A_2A_3)$, or in its interior or exterior.

**Proof.** Theorem 31 is the Euclidean counterpart of Theorem 26. The proof of Theorem 31 from Theorem 26 is immediate, noting (209), and noting that each equation in Theorem 31 is the Euclidean limit, $s \to \infty$, of a corresponding equation in Theorem 26. □

The Euclidean counterpart of the Gyrocircle Gyrobarycentric Representation Theorem 27 is the following elegant theorem, illustrated in Fig. 17.

**Theorem 32. (Circle Barycentric Representation).** Let $A, A \in A_1 + \text{Span}\{-A_1 + A_2, -A_1 + A_3\} \subset \mathbb{R}^n$, be a point given by its barycentric representation

$$A = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3}$$

(217)
with respect to the barycentrically independent set \( S = \{ A_1, A_2, A_3 \} \) in a Euclidean space \( \mathbb{R}^n \), \( n \geq 2 \), and let

\[
K_{\text{euc}}(A; A_1 A_2 A_3) = m_1 m_2 a_{12}^2 + m_1 m_3 a_{13}^2 + m_2 m_3 a_{23}^2
\]

\[
T_{\text{euc}}(A; A_1 A_2 A_3) = m_1 m_2 \sin^2 \alpha_3 + m_1 m_3 \sin^2 \alpha_2 + m_2 m_3 \sin^2 \alpha_3
\]

where \( \alpha_k, k = 1, 2, 3 \), is the vertex angle of vertex \( A_k \) of triangle \( A_1 A_2 A_3 \), be two scalars associated with \( A \).

(1) The point \( A \) lies on the circumcircle \( C(A_1 A_2 A_3) \) of triangle \( A_1 A_2 A_3 \) if and only if the barycentric coordinates \( m_1, m_2, m_3 \) of \( A \) in (217) satisfy the circumcircle condition

\[
K_{\text{euc}}(A; A_1, A_2, A_3) = 0
\]

or, equivalently, the trigonometric circumcircle condition

\[
T_{\text{euc}}(A; A_1, A_2, A_3) = 0.
\]

(2) The point \( A \) lies in the interior of circumcircle \( C(A_1 A_2 A_3) \) if and only if

\[
K_{\text{euc}}(A; A_1, A_2, A_3) > 0
\]

or, equivalently,

\[
T_{\text{euc}}(A; A_1, A_2, A_3) > 0.
\]

(3) The point \( A \) lies in the exterior of circumcircle \( C(A_1 A_2 A_3) \) if and only if

\[
K_{\text{euc}}(A; A_1, A_2, A_3) < 0
\]

or, equivalently,

\[
T_{\text{euc}}(A; A_1, A_2, A_3) < 0.
\]

Moreover, the circumcircle \( C(A_1 A_2 A_3) \) is the locus of the point \( A \) in (217), with parametric barycentric coordinates \( m_k, k = 1, 2, 3 \), given

(4) by the parametric equations

\[
m_1 = -t \sin^2 \alpha_1
\]

\[
m_2 = t \sin^2 \alpha_2 + t^2 \sin^2 \alpha_3 = m_3 t
\]

\[
m_3 = \sin^2 \alpha_2 + t \sin^2 \alpha_3
\]

with the parameter \( t \), \( t \in \mathbb{R} \cup \{-\infty, \infty\} \), where the two parameter values \( t = -\infty \) and \( t = \infty \) are identified, or, equivalently,

(5) by the parametric equations

\[
m_1 = -\sin^2 \alpha_1 \sin \theta
\]

\[
m_2 = \sin^2 \alpha_2 \sin \theta + \sin^2 \alpha_3 (1 - \cos \theta)
\]

\[
m_3 = \sin^2 \alpha_3 \sin \theta + \sin^2 \alpha_2 (1 + \cos \theta)
\]

with the parameter \( \theta \), \( 0 \leq \theta \leq 2\pi \), where the two parameter values \( \theta = 0 \) and \( \theta = 2\pi \) are identified.

(6) The circumcenter, \( O \), of triangle \( A_1 A_2 A_3 \) is given trigonometrically by (48), p. 13.

(7) The circumradius, \( R \), of triangle \( A_1 A_2 A_3 \) is given by (81), p. 20.
Proof. In the Euclidean limit, \( s \to \infty \), gamma factors tend to 1. Hence, in that limit, the gyrobarycentric representation of \( A \in \mathbb{R}^n \) in (201) tends to the corresponding barycentric representations of \( A \in \mathbb{R}^n \) in (217).

By Lemma 10 in the Euclidean limit the circumgyrocircle condition (203a) (where \( K \) is given by (202)) tends to (219a) (where \( K_{\text{euc}} \) is given by (218)).

In the Euclidean limit gyrotriangle defects vanish. Hence, in that limit, the circumgyrocircle condition (203b) (where \( T \) is given by (202)) tends to (219b) (where \( T_{\text{euc}} \) is given by (218)), as desired.

The proof of Inequalities (220) and (221) follows, similarly, from Inequalities (204) and (205).

Similarly, the proof of (222) – (223) follows from (206) – (207) and from Euclidean limits that result immediately from Lemma 10.

The proof of Item (6) is presented in the derivation of (48), p. 13, and the proof of Item (7) is presented in the derivation of (81), p. 20. □

18. Gyrocircle Gyrobarycentric Representation

Let \( P \in \mathcal{A}_3 \cap \mathbb{R}^n \) be a point given by its gyrobarycentric representation

\[
P = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}}
\]

with respect to a gyrobarycentrically independent set \( S = \{A_1, A_2, A_3\} \) in an Einstein gyrovector space \((\mathbb{R}^n, \oplus, \otimes)\). Additionally, let \( P' \in \mathbb{R}^n \) be a point that lies arbitrarily on the circumgyrocircle \( C(A_1 A_2 A_3) \) of gyrotriangle \( A_1 A_2 A_3 \), as shown in Figs. 18 – 19, given by its parametric gyrobarycentric representation

\[
P' = P'(t) = \frac{m_1' \gamma_{A_1} A_1 + m_2' \gamma_{A_2} A_2 + m_3' \gamma_{A_3} A_3}{m_1' \gamma_{A_1} + m_2' \gamma_{A_2} + m_3' \gamma_{A_3}}
\]

with respect to the set \( S \), where the parametric gyrobarycentric coordinates of \( P' \) are presented in (229) below.

Furthermore, let \( P'' \in \mathcal{A}_3 \cap \mathbb{R}^n \) be a point given by its gyrobarycentric representation

\[
P'' = \frac{m_1'' \gamma_{A_1} A_1 + m_2'' \gamma_{A_2} A_2 + m_3'' \gamma_{A_3} A_3}{m_1'' \gamma_{A_1} + m_2'' \gamma_{A_2} + m_3'' \gamma_{A_3}}
\]

with respect to the set \( S \), such that

1. \( P'' \) lies on the gyroline \( L_{P''} \) that passes through the points \( P \) and \( P' \); and
2. \( P'' \) and \( P' \), \( P'' \neq P' \), lie on the circumgyrocircle \( C(A_1 A_2 A_3) \) of gyrotriangle \( A_1 A_2 A_3 \), as shown in Figs. 18 – 19.

We will determine the gyrobarycentric coordinates of \( P'' \) in terms of those of \( P \) and \( P' \) and in terms of the reference gyrotriangle \( A_1 A_2 A_3 \) in (230) – (231) below.

Following the condition that \( P'' \) lies on circumgyrocircle \( C(A_1 A_2 A_3) \), by the Gyrocircle Gyrobarycentric Representation Theorem [27], p. 46, the gyrobarycentric...
Figure 18. The point $P'$ lies arbitrarily on the circumgyrocircle $C(A_1A_2A_3)$ of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$, and $P$, $P \neq P'$, is a point in $\mathbb{R}_s^2$. Both $P$ and $P'$ are given by their gyrobaricentric representation with respect to the gyrobaricentrically independent set $S = \{A_1, A_2, A_3\}$ of vertices of the reference gyrotriangle $A_1A_2A_3$. The point $P''$, $P'' \neq P'$, that lies on the intersection of the circumgyrocircle $C(A_1A_2A_3)$ and the gyroline $L_{PP'}$ through $P$ and $P'$ possesses a gyrobaricentric representation with respect to the set $S$, with gyrobaricentric coordinates that are determined by those of $P$ and $P'$ according to (237) – (238).

Coordinates $m''_k$ of $P''$ satisfy the circumgyrocircle condition (203). That is,

\begin{equation}
(227) \quad m''_1m''_2(\gamma_{12} - 1) + m''_1m''_3(\gamma_{13} - 1) + m''_2m''_3(\gamma_{23} - 1) = 0.
\end{equation}

We wish to express, in (230) below, the gyrobaricentric coordinates $m''_k$ of $P''$ in terms of the gyrobaricentric coordinates of $P$ and $P'$. 


Figure 19. In Fig. 18 the point $P$ lies in the exterior of the circumgyrocircle $C(A_1A_2A_3)$ of gyrotriangle $A_1A_2A_3$. Here, in contrast, the point $P$ lies in the interior of $C(A_1A_2A_3)$. As in Fig. 18 the point $P''$ lies on the intersection of circumgyrocircle $C(A_1A_2A_3)$ and the gyroline $L_{PP'}$ that passes through the points $P$ and $P'$. The gyrobaricentric coordinates of $P''$ are determined by those of $P$ and $P'$ according to (237) – (238). Both Figs. 18 and 19 illustrate Theorem 33, p. 54.

Since the point $P''$ lies on gyroline $L_{PP'}$, its gyrobaricentric coordinates $m_k''$, $k = 1, 2, 3$, are given parametrically by [20, Sect. 4.10], that is,

$$m_1'' = m_1(m_1' \gamma_{11} + m_2' \gamma_{12} + m_3' \gamma_{13}) - \left( \frac{m_1}{m_1'} m_2 \gamma_{12} + \frac{m_1}{m_1'} m_3 \gamma_{13} \right) t_0,$$

(228)  $$m_2'' = m_2(m_1' \gamma_{11} + m_2' \gamma_{12} + m_3' \gamma_{13}) + \left( \frac{m_1}{m_1'} m_2' \gamma_{11} - \frac{m_2}{m_2'} m_3' \gamma_{13} \right) t_0,$$

$$m_3'' = m_3(m_1' \gamma_{11} + m_2' \gamma_{12} + m_3' \gamma_{13}) + \left( \frac{m_1}{m_1'} m_3' \gamma_{11} + \frac{m_2}{m_2'} m_3' \gamma_{12} \right) t_0,$$

$\gamma_{11} = 1$, with the parameter $t_0 \in \mathbb{R}$. 
The point $P'$ lies on circumgyrocircle $C(A_1 A_2 A_3)$ of gyrotriangle $A_1 A_2 A_3$. Hence, by (206), the gyrobaricentric coordinates $m_k'$ of $P'$ can be parametrized as

$$
m'_k = - (\gamma_{23} - 1) t
$$

(229)

$$
m'_2 = (\gamma_{12} - 1) t^2 + (\gamma_{13} - 1) t = m'_3 t
$$

$$
m'_3 = (\gamma_{12} - 1) t + (\gamma_{13} - 1),
$$

with the parameter $t \in \mathbb{R} \cup \{-\infty, \infty\}$.

Inserting (229) in (228), and, successively, inserting the resulting gyrobaricentric coordinates $m_k'$ in the circumgyrocircle condition (227), we obtain a linear equation for the unknown $t_0$. The resulting solution, $T_0$ of $t_0$ is too involved and hence is not presented here. Inserting $t_0 = T_0$ in (228), we obtain the desired gyrobaricentric coordinates $m_k''$, $k = 1, 2, 3$, of $P''$. The resulting expressions of $m_k''$ are, initially, involved. Fortunately, however, these involved expressions can be factorized and, owing to the homogeneity of gyrobaricentric coordinates, nonzero common factors are irrelevant and, hence, can be omitted. The resulting gyrobaricentric coordinates $m_k''$ in the gyrobaricentric representation of $P''$ in (228) turn out to be simple and elegant. These are

$$
m''_1 = E_1 E_2
$$

(230)

$$
m''_2 = E_0 E_1 (\gamma_{13} - 1)
$$

$$
m''_3 = -E_0 E_2 (\gamma_{12} - 1),
$$

where

$$
E_0 = m_2 - m_3 t
$$

(231)

$$
E_1 = m_1 (\gamma_{13} - 1) + m_2 (\gamma_{23} - 1) + m_1 (\gamma_{12} - 1) t
$$

$$
E_2 = m_1 (\gamma_{13} - 1) + m_1 (\gamma_{12} - 1) t + m_3 (\gamma_{23} - 1) t.
$$

Interestingly, $E_0, E_1$ and $E_2$ are related by the equation

$$
E_1 - E_2 = E_0 (\gamma_{23} - 1).
$$

(232)

Formalizing, we obtain the following Theorem:

**Theorem 33. (The Gyroline Gyrocircle Intersection Theorem).**

1. Let $A_1 A_2 A_3$ be a gyrotriangle that possesses a circumgyrocircle in an Einstein gyrovector space $(\mathbb{R}_n^A, \oplus, \otimes)$, $n \geq 2$,
2. let $P \in A_3 \cap \mathbb{R}_n^A$ be a point given by its gyrobaricentric representation

$$
P = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}}
$$

(233)

with respect to the gyrobaricentrically independent set $S = \{A_1, A_2, A_3\}$ of the vertices of the reference gyrotriangle $A_1 A_2 A_3$,
3. let $P'$ be a point that lies arbitrarily on the circumgyrocircle $C(A_1 A_2 A_3)$ of gyrotriangle $A_1 A_2 A_3$, $P' \neq P$, and
4. let $P''$ be the unique point that lies simultaneously on the circumgyrocircle $C(A_1 A_2 A_3)$ and on the gyroline $L_{P P'}$ that passes through the points $P$ and $P'$ where, in general, $P'' \neq P'$, as shown in Figs. 18, 19. Finally,
(5) let $P'$ be parametrized by the parameter $t$,

$$t \in \mathbb{R} \cup \{-\infty, \infty\},$$

so that following the Gyrocircle Gyrobarycentric Representation Theorem 27, (206), p. 46, $P'$ possesses the gyrobarycentric representation

$$P' = P'(t) = \frac{m'_1 \gamma_{A_1} A_1 + m'_2 \gamma_{A_2} A_2 + m'_3 \gamma_{A_3} A_3}{m'_1 \gamma_{A_1} A_1 + m'_2 \gamma_{A_2} A_2 + m'_3 \gamma_{A_3}}$$

with respect to the set $S$, where

$$m'_1 = -(\gamma_{23} - 1)t$$

$$m'_2 = (\gamma_{12} - 1)t^2 + (\gamma_{13} - 1)t = m'_3$$

Then, the point $P''$ possesses the $t$-dependent gyrobarycentric representation

$$P'' = P''(t) = \frac{m''_1 \gamma_{A_1} A_1 + m''_2 \gamma_{A_2} A_2 + m''_3 \gamma_{A_3} A_3}{m''_1 \gamma_{A_1} A_1 + m''_2 \gamma_{A_2} A_2 + m''_3 \gamma_{A_3}}$$

with respect to the set $S$, where

$$m''_1 = E_1 E_2$$

$$m''_2 = E_0 E_1(\gamma_{13} - 1)$$

$$m''_3 = -E_0 E_2(\gamma_{12} - 1),$$

and where

$$E_0 = m_2 - m_3 t$$

$$E_1 = m_1(\gamma_{13} - 1) + m_2(\gamma_{23} - 1) + m_1(\gamma_{12} - 1)t$$

$$E_2 = m_1(\gamma_{13} - 1) + m_1(\gamma_{12} - 1)t + m_3(\gamma_{23} - 1)t.$$

19. Gyrocircle–Gyroline Tangency Points

**Theorem 34. (The Gyrocircle Gyrotangents Theorem).** Let $P \in A_3 \cap \mathbb{R}^n_*$ be a point given by its gyrobarycentric representation

$$P = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3}}$$

with respect to a gyrobarically independent set $S = \{A_1, A_2, A_3\}$ in an Einstein gyrovector plane $(\mathbb{R}^2_+, \oplus, \otimes)$, and let $C(A_1 A_2 A_3)$ be the circumgyrocircle of gyrotriangle $A_1 A_2 A_3$. Then the two tangency points $P_{\pm}$ of the gyrotangent gyrolines of circumgyrocircle $C(A_1 A_2 A_3)$ that pass through the point $P$, when exist as shown in Fig. 20, are given by their gyrobarically representations

$$P_{\pm} = \frac{m'_1 \gamma_{A_1} A_1 + m'_2 \gamma_{A_2} A_2 + m'_3 \gamma_{A_3} A_3}{m'_1 \gamma_{A_1} A_1 + m'_2 \gamma_{A_2} A_2 + m'_3 \gamma_{A_3}}.$$
with respect to $S$. The gyrobarycentric coordinates $m'_k$, $k = 1, 2, 3$, of $P_\pm$ are given in terms of the gyrobarycentric coordinates $m_k$ of $P$ and the sides of gyrotriangle $A_1A_2A_3$ by the equations

\begin{align*}
m'_1 &= F_0 F_1 (\gamma_{23} - 1) \\
m'_2 &= F_1 F_2 \\
m'_3 &= -F_0 F_2 (\gamma_{12} - 1),
\end{align*}

(242)

where

\begin{align*}
F_0 &= m_1 (\gamma_{12} - 1) + m_3 (\gamma_{23} - 1) \\
F_1 &= m_1 (\gamma_{12} - 1)(\gamma_{13} - 1) \pm \sqrt{-\Delta_1 \Delta_2} \\
F_2 &= -m_3 (\gamma_{13} - 1)(\gamma_{23} - 1) \pm \sqrt{-\Delta_1 \Delta_2},
\end{align*}

(243)

and where

\begin{align*}
\Delta_1 &= m_1 m_2 (\gamma_{12} - 1) + m_1 m_3 (\gamma_{13} - 1) + m_2 m_3 (\gamma_{23} - 1) \\
\Delta_2 &= (\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1) > 0.
\end{align*}

(244)

(1) Two distinct tangency points, $P_\pm$, exist if and only if $\Delta_1 < 0$ or, equivalently, if and only if the point $P$ lies in the exterior of the circumgyrocircle $C(A_1A_2A_3)$, as shown in Fig. 20.

(2) The two distinct tangency points degenerate to a single one, $P_\pm = P$ if and only if $\Delta_1 = 0$ or, equivalently, if and only if the point $P$ lies on the circumgyrocircle $C(A_1A_2A_3)$.

(3) There are no tangency points if and only if $\Delta_1 > 0$ or, equivalently, if and only if the point $P$ lies in the interior of the circumgyrocircle $C(A_1A_2A_3)$.

The constant $m_{P_\pm}$ of the gyrobarycentric representation (241) of $P_\pm$ is given by

\begin{align*}
m_{P_\pm} &= (\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1) E_1 \pm \sqrt{-\Delta_1 \Delta_2} E_2,
\end{align*}

(245)

where

\begin{align*}
E_1 &= (m_1 - m_2 + m_3) \{m_1 (\gamma_{12} - 1) + m_3 (\gamma_{23} - 1)\} - 2m_1 m_3 (\gamma_{13} - 1) \\
E_2 &= -m_1 (\gamma_{12} - 1)^2 + m_3 (\gamma_{23} - 1)^2 + m_1 (\gamma_{12} - 1)(\gamma_{13} - 1) \\
&\quad - m_3 (\gamma_{13} - 1)(\gamma_{23} - 1) - (m_1 - m_3)(\gamma_{12} - 1)(\gamma_{23} - 1).
\end{align*}

(246)

\textit{Proof.} The Gyrole Gyrocircle Intersection Theorem 33 enables gyrocircle–gyroline tangency points to be determined. Indeed, in the special case when the two points $P'$ and $P''$ in Theorem 33 in (235) and in (237), shown in Fig. 18, are coincident, the resulting point, $P_t := P' = P''$ is a point of tangency. There are two distinct ways for the points $P'$ and $P''$ to become coincident by approaching each other along the circumference of the circumgyrocircle $C(A_1A_2A_3)$ of gyrotriangle $A_1A_2A_3$. These give rise to two tangency points to the circumgyrocircle that lie on the two gyrotangent gyrolines that pass through the point $P$, shown in Fig. 20.
The two tangency points $P_\pm$ of the circumgyrocircle $C(A_1A_2A_3)$ of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector plane $(\mathbb{R}_2^s, \oplus, \otimes)$, which lie on the tangent lines that pass through a point $P$ that lies in the exterior of the circumgyrocircle, are shown. The Euclidean counterpart of this figure is presented in Fig. 21.

Indeed, if $P' = P''$ then it follows from the gyrobaricentric representations of $P'$ and $P''$ in (235) and (237) that

\[
\frac{m_2'}{m_1'} = \frac{m_2''}{m_1''}, \quad \frac{m_3'}{m_1'} = \frac{m_3''}{m_1''},
\]

thus obtaining two equations for the parameter $t$ that gives rise to the tangency points.
Inserting (236) and (238) – (239) in (247), we see that the two equations in (247) are equivalent to each other. Solving one of these equations for the unknown parameter value \( t \), we obtain

\[
(248) \quad t = \frac{-m_1(\gamma_{12} - 1)(\gamma_{13} - 1) \pm \sqrt{-\Delta_1 \Delta_2}}{\{m_1(\gamma_{12} - 1) + m_3(\gamma_{23} - 1)\}(\gamma_{12} - 1)},
\]

where

\[
(249) \quad \Delta_1 = m_1m_2(\gamma_{12} - 1) + m_1m_3(\gamma_{13} - 1) + m_2m_3(\gamma_{23} - 1)
\]

\[
\Delta_2 = (\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1) > 0.
\]

We see from the Gyrocircle Gyrobarycentric Representation Theorem [27, p. 46] that

1. \( \Delta_1 = 0 \) if and only if \( P \) lies on circumgyrocircle \( C(A_1A_2A_3) \);
2. \( \Delta_1 > 0 \) if and only if the point \( P \) in (233) lies in the interior of circumgyrocircle \( C(A_1A_2A_3) \); and
3. \( \Delta_1 < 0 \) if and only if the point \( P \) lies in the exterior of circumgyrocircle \( C(A_1A_2A_3) \).

We thus see from (248) that a value of the real parameter \( t \) that corresponds to a tangency point \( P_t, P_t := P' = P'' \), exists if and only if the point \( P \) lies on circumgyrocircle \( C(A_1A_2A_3) \) or in its exterior. Indeed, this result is expected since it is clear from Figs. [18, 19] that a gyrotangent gyroline to the circumgyrocircle that passes through the point \( P \) exists if and only if \( P \) lies in the exterior of the circumgyrocircle. Clearly, the two distinct tangency points \( P_\pm \) of circumgyrocircle \( C(A_1A_2A_3) \) degenerate to a single point, \( P_\pm = P \), if and only if the point \( P \) lies on the circumgyrocircle.

Inserting the parameter value of \( t \) from (248) into the gyrobarycentric coordinates \( m'_k, k = 1, 2, 3 \), of \( P' \) in (236), and omitting irrelevant nonzero common factors, we obtain the gyrobarycentric coordinates in (242) – (244), as desired.

By [26, Eq. (4.27), p. 90] and the Gyrobarycentric representation Gyrocovariance Theorem, [26, Theorem 4.6, pp. 90-91], the constant \( m_{P_\pm} \) of the gyrobarycentric representation (241) of \( P_\pm \) is given by the equation

\[
(250) \quad m_{P_\pm}^2 = (m'_1)^2 + (m'_2)^2 + (m'_3)^2 + 2(m'_1m'_2\gamma_{12} + m'_1m'_3\gamma_{13} + m'_2m'_3\gamma_{23}).
\]

The substitution of the gyrobarycentric coordinates \( m'_k \) from (242) into (250) yields (243), as desired.

\[ \square \]

20. Gyrocircle Gyrotangent Gyrolength

Let \( P_\pm \) denote collectively the two tangency points, \( P_+ \) and \( P_- \), of the gyrotangent gyrolines drawn from a point \( P \) to the circumgyrocircle \( C(A_1A_2A_3) \) of a gyrotriangle \( A_1A_2A_3 \) in an Einstein gyrovector space \( (\mathbb{R}^n, \oplus, \otimes) \), as shown in Fig. 20 for \( n = 2 \). Accordingly, let \( P \in \text{Span}\{\oplus A_1 \oplus A_2, \oplus A_1 \oplus A_3\} \subset \mathbb{R}^n \) be a point on the
gyroplane of gyrotriangle $A_1A_2A_3$ with gyrobarycentric representation

\[(251) \quad P = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2 + m_3\gamma_{A_3}A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}} \]

so that, following Theorem 34, (241), the tangency point $P_\pm$ possesses the gyrobarycentric representation

\[(252) \quad P_\pm = \frac{m'_1\gamma_{A_1}A_1 + m'_2\gamma_{A_2}A_2 + m'_3\gamma_{A_3}A_3}{m'_1\gamma_{A_1} + m'_2\gamma_{A_2} + m'_3\gamma_{A_3}} \]

with gyrobarycentric coordinates $m'_k$, $k = 1, 2, 3$, given by (242) – (244).

Hence, by [26, Sect. 4.9] with $N = 3$,

\[(253) \quad \gamma_{\Theta P \Theta P_\pm} = \frac{1}{m_pm_{P_\pm}} \left\{ (m_1m'_2 + m'_1m_2)\gamma_{12} + (m_1m'_3 + m'_1m_3)\gamma_{13} + (m_2m'_3 + m'_2m_3)\gamma_{23} + m_1m'_1 + m_2m'_2 + m_3m'_3 \right\} , \]

where by [26, Eq. (4.27), p. 90] and the Gyrobarycentric representation Gyro covariance Theorem, [26, Theorem 4.6, pp. 90-91], the constant $m_P$ of the gyrobarycentric representation (251) of $P$ is given by

\[(254) \quad m_P^2 = (m_1 + m_2 + m_3)^2 + 2(m_1m_2(\gamma_{12} - 1) + m_1m_3(\gamma_{13} - 1) + m_2m_3(\gamma_{23} - 1)) \]

and where $m_{P_\pm}$ is the constant of the gyrobarycentric representation (252) of $P_\pm$, given by (246).

Inserting $m'_k$, $k = 1, 2, 3$, from (242) – (244) into (253), we obtain the elegant equation

\[(255) \quad \gamma_{\Theta P \Theta P_\pm} = \frac{m_1 + m_2 + m_3}{m_P} . \]

The gyrolength of the gyrotangent gyrosegment $PP_\pm$ can readily be obtained from (255). Indeed, following (255) and (4), p. 8, we have

\[(256) \quad \|\Theta P \Theta P_\pm\| = s\sqrt{\gamma_{\Theta P \Theta P_\pm}^2 - 1} . \]

Unlike the gyrobarycentric coordinates $m'_k$ and the constant $m_{P_\pm}$ in (253), the gamma factor in (255) is free of the square root in (243) and (245). This implies the gyrodistance equality

\[(257) \quad \|\Theta P \Theta P_+\| = \|\Theta P \Theta P_-\| . \]

Following (254) – (256) we have

\[(258) \quad \|\Theta P \Theta P_\pm\| = s\sqrt{\gamma_{\Theta P \Theta P_\pm}^2 - 1} \]

\[= s\sqrt{-2(m_1m_2(\gamma_{12} - 1) + m_1m_3(\gamma_{13} - 1) + m_2m_3(\gamma_{23} - 1))} \]

\[\frac{m_1 + m_2 + m_3}{m_1 + m_2 + m_3} . \]
Note that by Theorem 27, p. 46, the radicand on the extreme right-hand side of (258) is positive (zero) if and only if the point $P$ lies in the exterior of circumgyrocircle $C(A_1A_2A_3)$ (if and only if the point $P$ lies on $C(A_1A_2A_3)$).

Formalizing results of this section, we obtain the following theorem.

**Theorem 35. (Gyrocircle Gyrotangent Gyrolength).**

1. Let $A_1A_2A_3$ be a gyrotriangle that possesses a circumgyrocircle $C(A_1A_2A_3)$ in an Einstein gyrovector space $(\mathbb{R}^n_\otimes, \otimes, \boxtimes)$,
2. let $P \in \text{Span}\{\ominus A_1 \oplus A_2, \ominus A_1 \oplus A_3\} \subset \mathbb{R}^n$ be a point that lies in the exterior of $C(A_1A_2A_3)$, given by its gyrobararycentric representation

$$P = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2 + m_3\gamma_{A_3}A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}}$$

with respect to the set $S = \{A_1, A_2, A_3\}$, and
3. let $P_\pm$ represent collectively the two tangency points, $P_+$ and $P_-$, of the two gyrotangent gyrolines drawn from $P$ to circumgyrocircle $C(A_1A_2A_3)$, as shown in Fig. 20.

Then, we have the following results:

1. The tangency points $P_\pm$ possess the gyrobararycentric representations

$$P_\pm = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2 + m_3\gamma_{A_3}A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}}$$

with gyrobararycentric coordinates $m'_k$, $k = 1, 2, 3$, given by (242) – (244).
2. The gyrolengths of the two gyroline gyrosegments $PP_+$ and $PP_-$ are equal,

$$\|\ominus P \oplus P_+\| = \|\ominus P \oplus P_-\|.$$

3. The gamma factor of each of the two gyrosegments $PP_+$ and $PP_-$ is given by the equations

$$\gamma_{\ominus P \oplus P_\pm} = \frac{m_1 + m_2 + m_3}{m_p},$$

where $m_p$ is the constant of the gyrobararycentric representation of $P$ with respect to $S$.
4. The gyrolength of each of the two gyrotangent gyrosegments $PP_+$ and $PP_-$ is given by

$$\|\ominus P \oplus P_\pm\| = s\sqrt{\frac{\gamma_{\ominus P \oplus P_\pm}^2 - 1}{\gamma_{\ominus P \oplus P_\pm}}}$$

$$= s\sqrt{\frac{-2\{m_1m_2(\gamma_{12} - 1) + m_1m_3(\gamma_{13} - 1) + m_2m_3(\gamma_{23} - 1)\}}{m_1 + m_2 + m_3}}.$$
Figure 21. Illustrating the Circle Tangents Theorem \[36\] The two tangency points \(P_\pm\) of the circumcircle \(C(A_1A_2A_3)\) of a triangle \(A_1A_2A_3\) in a Euclidean plane \(\mathbb{R}^2\), which lie on the tangent lines that pass through a point \(P\) that lies in the exterior of the circumcircle, are shown. This figure is the Euclidean counterpart of Fig. \[20\]

21. CIRCLE–LINE TANGENCY POINTS

The Euclidean counterpart of the Gyrocircle Gyrotangents Theorem \[34\] is the following elegant theorem, illustrated in Fig. \[21\]

**Theorem 36. (The Circle Tangents Theorem).** Let \(P \in \mathbb{R}^2\) be a point given by its barycentric representation

\[
P = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \tag{264}
\]

with respect to a barycentrically independent set \(S = \{A_1, A_2, A_3\}\) in a Euclidean plane \(\mathbb{R}^2\), and let \(C(A_1A_2A_3)\) be the circumcircle of triangle \(A_1A_2A_3\). Then the two tangency points \(P_\pm\) of the tangent lines of circumcircle \(C(A_1A_2A_3)\) that pass through the point \(P\), when exist, are given by their barycentric representations

\[
P_\pm = \frac{m'_1A_1 + m'_2A_2 + m'_3A_3}{m'_1 + m'_2 + m'_3}, \tag{265}
\]
shown in Fig. 21. The barycentric coordinates of \( P_\pm \) are given by the equations

\[
\begin{align*}
m'_1 &= F_0 F_1 \sin^2 \alpha_1 \\
m'_2 &= F_1 F_2 \\
m'_3 &= -F_0 F_2 \sin^2 \alpha_3,
\end{align*}
\]  
(266)

where

\[
\begin{align*}
F_0 &= m_1 \sin^2 \alpha_3 + m_3 \sin^2 \alpha_1 \\
F_1 &= \{m_1 \sin^2 \alpha_2 \sin^2 \alpha_3 \pm \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \sqrt{-\Delta}\} \sin^2 \alpha_1 \\
F_2 &= \{m_3 \sin^2 \alpha_1 \sin^2 \alpha_2 \pm \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \sqrt{-\Delta}\} \sin^2 \alpha_3,
\end{align*}
\]  
(267)

and where

\[
\Delta = m_1 m_2 \sin^2 \alpha_3 + m_2 m_3 \sin^2 \alpha_1.
\]  
(268)

(1) Two distinct tangency points, \( P_\pm \), exist if and only if \( \Delta < 0 \) or, equivalently, if and only if the point \( P \) lies in the exterior of the circumcircle \( C(A_1 A_2 A_3) \), as shown in Fig. 21.

(2) The two distinct tangency points degenerate to a single one, \( P_\pm = P \) if and only if \( \Delta = 0 \) or, equivalently, if and only if the point \( P \) lies on the circumcircle \( C(A_1 A_2 A_3) \).

(3) There are no tangency points if and only if \( \Delta > 0 \) or, equivalently, if and only if the point \( P \) lies in the interior of the circumcircle \( C(A_1 A_2 A_3) \).

(4) The lengths of the two tangent segments \( \| -P + P_\pm \| \) and \( \| -P + P_\mp \| \) are equal, given by

\[
\| -P + P_\pm \| = \frac{\sqrt{-\{m_1 m_2 a^2_{12} + m_1 m_3 a^2_{13} + m_2 m_3 a^2_{23}\}}}{m_1 + m_2 + m_3}.
\]  
(269)

Proof. We will show that each result of this theorem is the Euclidean limit of a corresponding result of its hyperbolic counterpart, Theorem 34. Noting that in the Euclidean limit, \( s \to \infty \), gamma factors tend to 1, the Euclidean limit of the gyrobarcentric coordinates \( m'_k, k = 1, 2, 3 \), in (241) – (244) is trivial. The resulting barycentric coordinates vanish, giving rise to an indeterminate barycentric representation. Thus, a straightforward application of the Euclidean limit to the gyrobarcentric coordinates \( m'_k \) in (241) – (244) in an attempt to recover the corresponding barycentric coordinates \( m'_k \) in (265) – (268) is destined to fail.

However, owing to their homogeneity, the gyrobarcentric coordinates \( m'_k \) in (241) – (244) can be written in a form that admits a Euclidean limit to a viable barycentric coordinate system. Specifically, the following equivalent form, in which each gyrobarcentric coordinate is multiplied by a common nonzero factor, is what
we need, in which we replace $m'_k$ by $m''_k$:

$$m''_1 = \frac{m'_1}{\gamma_{12} - 1} = \frac{F_0}{\gamma_{12} - 1} \frac{F_1}{(\gamma_{12} - 1)^2} \frac{\gamma_{23} - 1}{\gamma_{12} - 1}$$

$$m''_2 = \frac{m'_2}{(\gamma_{12} - 1)^4} = \frac{F_1}{(\gamma_{12} - 1)^2} \frac{F_2}{(\gamma_{12} - 1)^2}$$

$$m''_3 = \frac{m'_3}{(\gamma_{12} - 1)^4} = -\frac{F_0}{\gamma_{12} - 1} \frac{F_1}{\gamma_{12} - 1} \frac{\gamma_{12} - 1}{\gamma_{12} - 1}$$

where, by (243),

$$F_0 = m_1 + m_3 \frac{\gamma_{23} - 1}{\gamma_{12} - 1} := F'_0$$

$$F_1 = \frac{m_1 \gamma_{13} - 1}{\gamma_{12} - 1} \pm \left\{ \frac{-\Delta_1}{\gamma_{12} - 1} \frac{\Delta_2}{(\gamma_{12} - 1)^3} \right\}^{\frac{1}{2}} := F'_1$$

$$F_2 = \frac{-m_3 \gamma_{13} - 1}{\gamma_{12} - 1} \frac{\gamma_{23} - 1}{\gamma_{12} - 1} \pm \left\{ \frac{-\Delta_1}{\gamma_{12} - 1} \frac{\Delta_2}{(\gamma_{12} - 1)^3} \right\}^{\frac{1}{2}} := F'_2,$$

and where

$$\Delta_1 = \frac{m_1 m_2 + m_1 m_3 \gamma_{13} - 1}{\gamma_{12} - 1} + m_2 m_3 \frac{\gamma_{23} - 1}{\gamma_{12} - 1} := \Delta'_1$$

$$\Delta_2 = \frac{\gamma_{13} - 1 \gamma_{23} - 1}{\gamma_{12} - 1} := \Delta'_2.$$

Hence, by (270) – (272),

$$m''_1 = F'_0 F'_1 \frac{\gamma_{23} - 1}{\gamma_{12} - 1}$$

$$m''_2 = F'_1 F'_2$$

$$m''_3 = -F'_0 F'_2.$$

The gyrobarycentric coordinates $m''_k$, $k = 1, 2, 3$, in (270) – (273) appear in a form that admits a nontrivial Euclidean limit, $s \to \infty$, by means of Lemma 10. Indeed, one can readily show that the Euclidean limit of the gyrobarycentric coordinate system $m''_k$ in (273) turns out to be the barycentric coordinate system (266) – (268), in which we rename $m''_k$ as $m'_k$ ($k = 1, 2, 3$), and $F'_k$ as $F_k$ ($k = 0, 1, 2$).

Finally, it remains to prove (269). The length $\| -P + Pz \|$ of the tangent segment $PPz$, shown in Fig. 21, is the Euclidean limit, $s \to \infty$, of the gyrolength $\| \Theta P \otimes Pz \|$ of the gyrotangent gyrosegment $PPz$, shown in Fig. 20.

$$\lim_{s \to \infty} \| \Theta P \otimes Pz \| = \| -P + Pz \|.$$
Figure 22. Circumgyrocevians. A circumgyrocevian, $A_3Q_k$, $k = 2, 3, \ldots, 8$, is a gyrocevian $A_3P_k$ of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space, extended to the point $Q_k$ where it meets the gyrotriangle circumgyrocircle. The $k$th circumgyrocevian $A_3Q_k$ in this figure corresponds to the parameter value $t_1 = 1/k$ of the parameter $0 < t_1 < 1$ in (276). The point $A_1$ corresponds to $t_1 = 0$, and the point $A_2$ corresponds to $t_1 = 1$.

Indeed, by (274), (263) and Lemma 10, p. 19, we have

$$\| - P + P_\perp \| = \lim_{s \to \infty} \| \ominus P \oplus P_\perp \|$$

$$= \lim_{s \to \infty} \sqrt{-2s^2\{m_1m_2(\gamma_{12} - 1) + m_1m_3(\gamma_{13} - 1) + m_2m_3(\gamma_{23} - 1)\}}$$

$$= \sqrt{-\{m_1m_2a_{12}^2 + m_1m_3a_{13}^2 + m_2m_3a_{23}^2\}}$$

$$\frac{m_1 + m_2 + m_3}{m_1 + m_2 + m_3},$$

where $a_{ij} = \| - A_i + A_j \|$, as desired. \qed

22. Circumgyrocevians

Definition 37. (Circumgyrocevians). A circumgyrocevian of a gyrotriangle is a gyrocevian of the gyrotriangle extended to the point where it meets the gyrotriangle circumgyrocircle, as shown in Fig. 22.
The concept of the circumgyrocevian, together with its associated Theorem [38] p. 67, is employed in Sect. 23 and, consequently, proves useful in the study of the Intersecting Gyrochords Theorem [29] p. 75.

In this section we determine the circumgyrocevian associated with a given gyrocevian in an Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\).

Let \(P\) be a generic point of a gyrosegment \(A_1A_2\) in an Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\), as shown in Fig. 23. Then, \(P\) possesses the gyrobarycentric representation
\[
P = \frac{(1 - t_1)\gamma_{A_1}A_1 + t_1\gamma_{A_2}A_2}{(1 - t_1)\gamma_{A_1} + t_1\gamma_{A_2}}
\]
for any value of the parameter \(t_1, 0 \leq t_1 \leq 1\). The parameter \(t_1\) determines the location of the point \(P\) on gyrosegment \(A_1A_2\), as shown in Fig. 22. Thus, in particular, \(t_1 = 0\) gives \(P = A_1\), \(t_1 = 1\) gives \(P = A_2\), and \(t = 1/2\) gives the gyromidpoint \(M_{A_1A_2}\) of \(A_1\) and \(A_2\).

Applying the Gyrobarycentric Representation Gyrocovariance Theorem [26] Theorem 4.6, pp. 90-91] with \(X = \ominus A_1\), we have
\[
\ominus A_1 \oplus P = \ominus A_1 \oplus \frac{(1 - t_1)\gamma_{A_1}A_1 + t_1\gamma_{A_2}A_2}{(1 - t_1)\gamma_{A_1} + t_1\gamma_{A_2}}
\]
\[
= \frac{(1 - t_1)\gamma_{\ominus A_1 \oplus A_1}(\ominus A_1 \oplus A_1) + t_1\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2)}{(1 - t_1)\gamma_{\ominus A_1 \oplus A_1} + t_1\gamma_{\ominus A_1 \oplus A_2}}
\]
\[
= \frac{t_1\gamma_{12}a_{12}}{1 + (\gamma_{12} - 1)t_1},
\]
where we use the gyrotriangle index notation [13].

Let \(A_3P\) be the gyroray that emanates from the point \(A_3\) and passes through the point \(P\) in the Einstein gyrovector space, as shown in Fig. 23. Gyrolines and gyrorays in Einstein gyrovector spaces coincide with Euclidean segments, enabling methods of linear algebra to be applied for solving intersection problems for gyrolines and gyrorays.

Let us consider the gyroray
\[
(\ominus A_1 \oplus A_3)(\ominus A_1 \oplus P) = a_{13}(\ominus A_1 \oplus P)
\]
that emanates from \((\ominus A_1 \oplus A_3) = a_{13}\) and passes through the point \(\ominus A_1 \oplus P\). This ray is the left gyrotranslation by \(\ominus A_1\) of the gyroray \(A_3P\). By methods of linear algebra, the points \(A(t)\) of the gyroray \((278)\), parametrized by \(t, t \geq 0\), are given by
\[
A(t) = a_{13}(1 - t) + (\ominus A_1 \oplus P)t.
\]
Equivalently, by means of (277), (279) can be written as

\[ A(t) = a_{13}(1 - t) + \frac{t_1 \gamma_{12} a_{12}}{1 + (\gamma_{12} - 1) t_1} t \]

\[ = m_2 \gamma_{12} a_{12} + m_3 \gamma_{13} a_{13}, \]

where

\[ m_2 = \frac{t_1 t}{1 + (\gamma_{12} - 1) t_1} \]

\[ m_3 = \frac{1 - t}{\gamma_{13}}. \]

The set of points \( A(t), 0 \leq t < \infty \), is the set of all points of the left gyrotranslated gyroray \( (278) \) by \( \ominus A_1 \). Hence, the set of points \( A_1 \oplus A(t), 0 \leq t < \infty \), is the set of all points of the original gyroray

\[ A_3 P = \{ A_1 \oplus A(t) : 0 \leq t < \infty \}, \]

which is of interest as indicated in Figs. 22-24.

Noting that \( \ominus A_1 \oplus A_1 = 0 \) and \( \gamma_0 = 1 \), (280) can be written as

\[ A(t) = m_1 \gamma_{\ominus A_1 \oplus A_1} (\ominus A_1 \oplus A_1) + m_2 \gamma_{\ominus A_1 \oplus A_2} (\ominus A_1 \oplus A_2) + m_3 \gamma_{\ominus A_1 \oplus A_3} (\ominus A_1 \oplus A_3), \]

where \( m_1 \) is determined by the condition

\[ m_1 + m_2 \gamma_{12} + m_3 \gamma_{13} = 1 \]

that insures that (283) and (280) are identically equal.

Solving (283) for \( m_1 \), where \( m_2 \) and \( m_3 \) are given by (281), we have

\[ m_1 = \frac{(1 - t_1) t}{1 + (\gamma_{12} - 1) t_1}. \]

By means of the Gyrobarycentric Representation Gyrocovariance Theorem [26, Theorem 4.6, pp. 90-91] with \( X = \ominus A_1 \), (283) can be written as

\[ A(t) = A_1 \oplus \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}}, \]

where \( m_1, m_2 \) and \( m_3 \) are given by (283) and (281).

Following (286) we have, by a left cancellation,

\[ Q(t) := A_1 \oplus A(t) = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}}, \]

where the set of points \( Q(t) = A_1 \oplus A(t), 0 \leq t < \infty \), forms the gyroray \( A_3 P \), (282), shown in Fig. 23.

We have thus obtained in (287) the gyrobarycentric representation with respect to \( \{ A_1, A_2, A_3 \} \) of each point \( Q(t) \) of the gyroray \( A_3 P \). We now wish to determine the
unique point of the gyroray $A_3 P$, other than $A_3$, that lies on the circumgyrocircle of gyrotriangle $A_1 A_2 A_3$ in Fig. 23.

By the Gyrocircle Gyrobarycentric Representation Theorem [27] p. 46 a point $Q(t)$ of the gyroray $A_3 P$ in (287) lies on the circumgyrocircle of a gyrotriangle $A_1 A_2 A_3$ if and only if the gyrobarycentric coordinates $m_k$, $k = 1, 2, 3$, of $Q(t)$ in (287) satisfy the circumgyrocircle condition (203).

Accordingly, we substitute $m_1, m_2, m_3$ from (285) and (281) into (203), obtaining a linear equation for the unknown $t$, the solution $t = t_0$ of which is substituted into (285) and (281), obtaining gyrobarycentric coordinates $m_1, m_2, m_3$ for the point $Q = Q(t_0)$ that lies on the circumgyrocircle of gyrotriangle $A_1 A_2 A_3$, shown in Figs. 22, 24. Finally, by removing an irrelevant nonzero common factor of the gyrobarycentric coordinates, we obtain the gyrobarycentric coordinates of $Q$,

$$m_1 = \{\gamma_{13} - 1 + (\gamma_{23} - \gamma_{13})t_1\}(1 - t_1)$$

(288)

$$m_2 = \{\gamma_{13} - 1 + (\gamma_{23} - \gamma_{13})t_1\}t_1$$

$$m_3 = -(\gamma_{12} - 1)(1 - t_1)t_1,$$

where $t_1$, $0 < t_1 < 1$, is a parameter that determines by (270) the location of the point $P$ on the gyrochord $A_1 A_2$ of the circumgyrocircle of gyrotriangle $A_1 A_2 A_3$ in Fig. 23.

The point $Q$, given by (287) with gyrobarycentric coordinates given by (288) is thus the only point of gyroray $A_3 P$, other than $A_3$, that lies on the circumgyrocircle of gyrotriangle $A_1 A_2 A_3$ in Fig. 23. Formalizing, we thus obtain the following theorem:

**Theorem 38. (The Circumgyrocevian Theorem).** Let $A_3 P$ be a gyrocevian of a gyrotriangle $A_1 A_2 A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_\alpha, \oplus, \otimes)$, $n \geq 2$, and let $A_3 Q$ be its corresponding circumgyrocevian, as shown in Fig. 23. Furthermore, let

$$P = \frac{(1 - t_1)\gamma_{A_1} A_1 + t_1 \gamma_{A_2} A_2}{(1 - t_1)\gamma_{A_1} + t_1 \gamma_{A_2}}$$

(289)

be the gyrobarycentric representation of $P$ with respect to the gyrobarycentrally independent set $\{A_1, A_2\}$ where the location of $P$ on the circumgyrocircle gyrochord $A_1 A_2$ is determined by the parameter $t_1$.

$$0 \leq t_1 \leq 1.$$  

(290)

Then, the point $Q$ possesses the gyrobarycentric representation

$$Q = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}}$$

(291)

with respect to the gyrobarycentrally independent set $\{A_1, A_2, A_3\}$, where the gyrobarycentric coordinates $m_1, m_2, m_3$ are given by

$$m_1 = \{\gamma_{13} - 1 + (\gamma_{23} - \gamma_{13})t_1\}(1 - t_1)$$

(292)

$$m_2 = \{\gamma_{13} - 1 + (\gamma_{23} - \gamma_{13})t_1\}t_1$$

$$m_3 = -(\gamma_{12} - 1)(1 - t_1)t_1.$$
Figure 23. Illustrating the Circumgyrocevian Theorem 38, p. 67, and the Intersecting Gyrochords Theorem 42, p. 75. A circumgyrocevian, $A_3Q$, is shown. It is a gyrocevian $A_3P$ of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}_n^s, \oplus, \otimes)$, $n \geq 2$, extended to the point $Q$ where it meets the gyrotriangle circumgyrocircle. With $A_4 = Q$, this figure presents two gyrochords, $A_1A_2$ and $A_3A_4 = B_1B_2$, of a gyrocircle, intersecting at a point $P$ inside the gyrocircle. Here, the point $Q \in \mathbb{R}_n^s$ is determined by its gyrobarcentric representation (291) in Theorem 38. This figure is the hyperbolic counterpart of Fig. 24.

Example 39. For $t_1 = 0$ the gyrobarcentric coordinates in (292) specialize to $m_1 = \gamma_{13} - 1 > 0$ and $m_2 = m_3 = 0$ implying, by (291), $Q = A_1$.

Example 40. For $t_1 = 1$ the gyrobarcentric coordinates in (292) specialize to $m_2 = \gamma_{23} - 1 > 0$ and $m_1 = m_3 = 0$ implying, by (291), $Q = A_2$.

Exploring the Euclidean limit of the Circumgyrocevian Theorem 38 we obtain the following Circumcevian Theorem:

**Theorem 41. (The Circumcevian Theorem).** Let $A_3P$ be a Cevian of a triangle $A_1A_2A_3$ in a Euclidean space $\mathbb{R}^n$, $n \geq 2$, and let $A_3Q$ be its corresponding circumcevian, as shown in Fig. 24. Furthermore, let

\[
P = (1 - t_1)A_1 + t_1A_2
\]
Figure 24. Illustrating the Circumcevian Theorem [41] p. 68 and the Intersecting Chords Theorem [43] p. 76. A circular cevian, $A_3Q$, is shown. It is a cevian $A_3P$ of a triangle $A_1A_2A_3$ in a Euclidean space $\mathbb{R}^n$, $n \geq 2$, extended to the point $Q$ where it meets the triangle circumcircle. This figure presents two chords, $A_1A_2$ and $A_3Q$, of a circle, intersecting at a point $P$ inside the circle. Here, the point $Q \in \mathbb{R}^n$ is determined by its barycentric representation (295) in Theorem [41]. This figure is the Euclidean counterpart of Fig. 23.

be the barycentric representation of $P$ with respect to the barycentrically independent set $\{A_1, A_2\}$, so that the location of $P$ on the circumcircle chord $A_1A_2$ is determined by the parameter $t_1$,

$$0 \leq t_1 \leq 1.$$  \hspace{1cm} (294)

Then, the point $Q$ possesses the barycentric representation

$$Q = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3}$$  \hspace{1cm} (295)

with respect to the barycentrically independent set $\{A_1, A_2, A_3\}$, where the barycentric coordinates $m_1, m_2, m_3$ are given by

$$m_1 = \{\sin^2 \alpha_2 + t_1 \sin(\alpha_1 - \alpha_2) \sin(\alpha_3)\}(1 - t_1)$$  \hspace{1cm} (296)

$$m_2 = \{\sin^2 \alpha_2 + t_1 \sin(\alpha_1 - \alpha_2) \sin(\alpha_3)\}t_1$$

$$m_3 = -(1 - t_1)t_1 \sin^2 \alpha_3,$$
where $\alpha_k, k = 1, 2, 3$, are the corresponding angles of triangle $A_1A_2A_3$.

**Proof.** In the Euclidean limit, $s \to \infty$, the gyrobarycentric representation of $P \in \mathbb{R}^n_n$ in (289)–(290) reduces to the corresponding barycentric representations of $P \in \mathbb{R}^n$ in (293)–(294).

Obviously, in that Euclidean limit, the gyrobarycentric representation of $Q \in \mathbb{R}^n_n$ in (291) reduces to its corresponding barycentric representation of $Q \in \mathbb{R}^n$ in (295).

Hence, it remains to show that the gyrobarycentric coordinates $m_k, k = 1, 2, 3$, of $Q \in \mathbb{R}^n_n$ in (292) tend to their corresponding barycentric coordinates $m_k, k = 1, 2, 3$, of $Q \in \mathbb{R}^n$ in (296).

In the Euclidean limit, $s \to \infty$, gamma factors tend to 1. Hence, straightforward Euclidean limits of $m_k, k = 1, 2, 3$, in (292), as $s \to \infty$, give vanishing barycentric coordinates, $m_k = 0, k = 1, 2, 3$, resulting in an indeterminate barycentric representation of type $0/0$.

However, owing to the homogeneity of gyrobarycentric coordinates, the gyrobarycentric coordinates $m_k = 0, k = 1, 2, 3$, of $Q \in \mathbb{R}^n_n$ in (297) can be written in the following form, which admits the Euclidean limit as $s \to \infty$,

$$
m_1 = \left\{ \frac{\gamma_{13} - 1}{\gamma_{12} - 1} + \frac{\gamma_{23} - \gamma_{13}}{\gamma_{12} - 1} t_1 \right\} (1 - t_1),
$$

$$
m_2 = \left\{ \frac{\gamma_{13} - 1}{\gamma_{12} - 1} + \frac{\gamma_{23} - \gamma_{13}}{\gamma_{12} - 1} t_1 \right\} t_1,
$$

$$
m_3 = -(1 - t_1) t_1.
$$

Indeed, following Euclidean limits that can be derived from Lemma [10] we have the Euclidean limits

$$
\lim_{s \to \infty} \frac{\gamma_{13} - 1}{\gamma_{12} - 1} = \frac{\sin^2 \alpha_2}{\sin^2 \alpha_3},
$$

$$
\lim_{s \to \infty} \frac{\gamma_{23} - \gamma_{13}}{\gamma_{12} - 1} = \frac{\sin(\alpha_1 - \alpha_2)}{\sin \alpha_3}.
$$

Hence, in the Euclidean limit, $s \to \infty$, the gyrobarycentric coordinates $m_k, k = 1, 2, 3$, of $Q \in \mathbb{R}^n_n$ in (297) tend to the following barycentric coordinates of $Q \in \mathbb{R}^n$,

$$
m_1 = \left\{ \frac{\sin^2 \alpha_2}{\sin^2 \alpha_3} + \frac{\sin(\alpha_1 - \alpha_2)}{\sin \alpha_3} t_1 \right\} (1 - t_1),
$$

$$
m_2 = \left\{ \frac{\sin^2 \alpha_2}{\sin^2 \alpha_3} + \frac{\sin(\alpha_1 - \alpha_2)}{\sin \alpha_3} t_1 \right\} t_1,
$$

$$
m_3 = -(1 - t_1) t_1.
$$

Owing to their homogeneity, the barycentric coordinates (299) of $Q \in \mathbb{R}^n$ can be written as (296), as desired. □
23. Gyrodistances Related to the Gyrocevian

In order to prepare the stage for the Intersecting Gyrochords Theorem \[42\], p. 75 in this section we calculate the gyrodistance between \(A_k\) and the gyrocevian foot \(P\), \(k = 1, 2, 3\), of a gyrotriangle \(A_1A_2A_3\), shown in Fig. 23.

Let \(A_1A_2A_3\) be a gyrotriangle that possesses a circumgyrocircle in an Einstein gyrovector space \((\mathbb{R}^n, \oplus, \otimes)\), and let \(P\), given by \((289)\), be a generic point of the interior of chord \(A_1A_2\) of the circumgyrocircle, as shown in Fig. 23.

Applying the Gyrobarycentric Representation Gyrocovariance Theorem \[26\, \text{Theorem 4.6, pp. 90-91}\] to the gyrobarcentric representation \((289)\) of \(P\), using the gyrotriangle index notation \((13)\), p. 7, we have

\[
\Theta_{A_1 \oplus P} = \frac{t_1 \gamma_{12} a_{12}}{(1 - t_1) + t_1 \gamma_{12}}
\]

\[(300)\]

\[
\Theta_{A_2 \oplus P} = \frac{(1 - t_1) \gamma_{12} a_{21}}{(1 - t_1) \gamma_{12} + t_1}
\]

\[
\Theta_{A_3 \oplus P} = \frac{(1 - t_1) \gamma_{13} a_{31} + t_1 \gamma_{23} a_{23}}{(1 - t_1) \gamma_{13} + t_1 \gamma_{23}}
\]

and

\[
\gamma_{|A_1 P|} = \gamma_{\Theta_{A_1 \oplus P}} = \frac{(1 - t_1) + t_1 \gamma_{12}}{m_p}
\]

\[(301)\]

\[
\gamma_{|A_2 P|} = \gamma_{\Theta_{A_2 \oplus P}} = \frac{(1 - t_1) \gamma_{12} + t_1}{m_p}
\]

\[
\gamma_{|A_3 P|} = \gamma_{\Theta_{A_3 \oplus P}} = \frac{(1 - t_1) \gamma_{13} + t_1 \gamma_{23}}{m_p}
\]

where \(m_p > 0\) is the constant of the gyrobarcentric representation \((289)\) of \(P\) with respect to the set \(\{A_1, A_2\}\), given by

\[(302)\]

\[
m_p^2 = (1 - t_1)^2 + (1 - t_1) t_1 + 2(1 - t_1) t_1 = 1 + 2(\gamma_{12} - 1) (1 - t_1) t_1.
\]

By Identity \((3)\), p. 8 and \((301)\) and straightforward algebra, we have

\[
\frac{1}{s^2} \gamma^2_{|A_1 P|} |A_1 P|^2 = \gamma_{|A_1 P|}^2 - 1 = \frac{(\gamma_{12}^2 - 1) t_1^2}{1 + 2(\gamma_{12} - 1)(1 - t_1) t_1}
\]

\[(303)\]

\[
\frac{1}{s^2} \gamma^2_{|A_2 P|} |A_2 P|^2 = \gamma_{|A_2 P|}^2 - 1 = \frac{(\gamma_{12}^2 - 1)(1 - t_1)^2}{1 + 2(\gamma_{12} - 1)(1 - t_1) t_1}
\]

\[
\frac{1}{s^2} \gamma^2_{|A_3 P|} |A_3 P|^2 = \gamma_{|A_3 P|}^2 - 1 = \frac{\gamma_{13}(1 - t_1) + \gamma_{23} t_1}{1 + 2(\gamma_{12} - 1)(1 - t_1) t_1} - 1.
\]

The equations in \((303)\) prove useful in the proof of Theorem \[42\, \text{p. 75}\].

Finally, by means of Identity \((3)\), p. 8 it follows from \((303)\) that the squared gyrodistances \(|A_k P|^2 = ||\Theta_{A_k \oplus P}||^2\) between \(A_k\) and \(P\), \(k = 1, 2, 3\), are given by
the equations

\[
\frac{1}{s^2} \| \Theta A_1 \oplus P \|^2 = \frac{(\gamma_{12}^2 - 1)t_1^2}{(\gamma_{12} - 1)t_1 + 1}^2
\]  
(304)

\[
\frac{1}{s^2} \| \Theta A_2 \oplus P \|^2 = \frac{(\gamma_{12}^2 - 1)(1 - t_1)^2}{\gamma_{12}(1 - t_1) + t_1}^2
\]

\[
\frac{1}{s^2} \| \Theta A_3 \oplus P \|^2 = 1 - \frac{2(\gamma_{12} - 1)(1 - t_1)t_1 + 1}{\gamma_{13}(1 - t_1) + \gamma_{23}t_1}^2.
\]

It is clear from the first and the second equations in (304) that \(|A_1 P| = 0\) when \(t_1 = 0\) and \(|A_2 P| = 0\) when \(t_1 = 1\). Hence, \(P = A_1\) when \(t_1 = 0\) and \(P = A_2\) when \(t_1 = 1\), as indicated in Fig. 23.

24. A Gyrodistance Related to the Circumgyrocevian

In order to prepare the stage for the Intersecting Gyrochords Theorem 42, p. 75, in this section we calculate the gyrodistance \(|PQ| = \| \Theta P \oplus Q \|\) between the gyrocevian foot \(P\) and the circumgyrocevian foot \(Q\) of a gyrotriangle \(A_1 A_2 A_3\), shown in Fig. 23, p. 68.

By Theorem 38, the point \(Q\), shown in Fig. 23, is given by its gyrobarycentric representation, (291),

\[
Q = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2 + m_3 \gamma_{A_3} A_3}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2} + m_3 \gamma_{A_3}}
\]

with respect to the set \(\{A_1, A_2, A_3\}\), where, (292),

\[
m_1 = \{\gamma_{13} - 1 + (\gamma_{23} - \gamma_{13})t_1\}(1 - t_1)
\]

\[
m_2 = \{\gamma_{13} - 1 + (\gamma_{23} - \gamma_{13})t_1\}t_1
\]

\[
m_3 = -(\gamma_{12} - 1)(1 - t_1)t_1
\]

and the point \(P\) is given by its gyrobarycentric representation, (289),

\[
P = \frac{m_1' \gamma_{A_1} A_1 + m_2' \gamma_{A_2} A_2}{m_1' \gamma_{A_1} + m_2' \gamma_{A_2}},
\]

where

\[
m_1' = 1 - t_1
\]

\[
m_2' = t_1,
\]

\(0 < t_1 < 1\).
Hence, by [26] Sect. 4.9 with \( N = 3 \),

\[
\gamma_{\oplus P \oplus Q} = \frac{1}{m_P m_Q} \left\{ (m_1 m'_2 + m'_1 m_2)\gamma_{12} + (m_1 m'_3 + m'_1 m_3)\gamma_{13} \right\} + (m_2 m'_3 + m'_2 m_3)\gamma_{23} + m_1 m'_1 + m_2 m'_2 + m_3 m'_3 \right\}
\]

(307)

\[
= \frac{1}{m_P m_Q} \left\{ (m_1 m'_2 + m'_1 m_2)\gamma_{12} + m'_1 m_3 \gamma_{13} + m'_1 m_3 \gamma_{23} + m_1 m'_1 + m_2 m'_2 \right\}
\]

noting that \( m'_3 = 0 \) in the gyrobarycentric representation of \( P \) in (308).

As we see from the gyrobarycentric representation constant associated with the Gyrobarycentric Representation Gyrocovariance Theorem [26, Theorem 4.6, pp. 90-91], and by (305b) and (306b), \( m_Q > 0 \) and \( m_P > 0 \) are given by

\[
m_Q^2 = m_1^2 + m_2^2 + m_3^2 + 2(m_1 m_2 \gamma_{12} + m_1 m_3 \gamma_{13} + m_2 m_3 \gamma_{23})
\]

(308)

\[
m_P^2 = (m'_1)^2 + (m'_2)^2 + 2m'_1 m'_2 \gamma_{12}
\]

\[= 1 + 2(\gamma_{12} - 1)(1 - t_1),\]

noting that \( m_P^2 > 0 \) for all \( 0 < t_1 < 1 \). Also \( m_Q^2 > 0 \) for all \( 0 < t_1 < 1 \) according to the Circumgyrocenter Theorem [4] p. 11 since gyrotriangle \( A_1 A_2 A_3 \) possesses a circumgyrocircle on which the point \( Q \) lies, as shown in Fig. 23.

In (308), \( m_Q^2 \) is a perfect square in the sense that it appears in (308) as a squared polynomial function. Since \( m_Q > 0 \), we have

(309)

\[m_Q = (\gamma_{13} - 1)(1 - t_1) + (\gamma_{23} - 1)t_1 - (\gamma_{12} - 1)(1 - t_1)t_1 > 0,\]

where, indeed, \( m_Q > 0 \) for all \( 0 \leq t_1 \leq 1 \).

Substituting (305b) and (306b) into (307) and squaring, we obtain \( \gamma_{\oplus P \oplus Q}^2 \) and, hence, \( \gamma_{\oplus P \oplus Q}^2 - 1 \) expressed in terms of gamma factors and the parameter \( t_1 \), \( 0 < t_1 < 1 \),

(310)

\[\frac{1}{s^2} \gamma_{\oplus P \oplus Q}^2 |PQ|^2 = \gamma_{\oplus P \oplus Q}^2 - 1 \]

\[= \frac{\gamma_{12} - 1}{1} \left\{ (\gamma_{13} - 1) + 2(\gamma_{12} - 1)(1 - t_1)t_1 - (\gamma_{12} - 1)(1 - t_1)t_1 \right\} \left\{ (\gamma_{12} - 1)(1 - t_1) \right\} \left\{ (\gamma_{12} - 1)(1 - t_1) \right\} \]

\[= \frac{(\gamma_{12} - 1)^2 (\gamma_{13} - 1) + 2(\gamma_{12} - 1)(1 - t_1)t_1 - (\gamma_{12} - 1)(1 - t_1)t_1}{\left\{ (\gamma_{12} - 1)(1 - t_1) \right\} \left\{ (\gamma_{12} - 1)(1 - t_1) \right\}} \]

\[
\]

noting (41), p. 8. The extreme sides of (310) prove useful in the proof of Theorem [12] p. 25.

Substituting \( \gamma_{\oplus P \oplus Q}^2 \) from (310) into Identity, (41), p. 8

(311)

\[|PQ|^2 := ||\oplus P \oplus Q||^2 = s^2 \gamma_{\oplus P \oplus Q}^2 - 1,\]
In order to prepare the stage for the Intersecting Gyrochords Theorem 42, in this section we calculate the gyrodistance \( |A_3Q| = \|\triangle A_3\oplus Q\| \) between vertex \( A_3 \) and its opposing circumgyrocevian foot \( Q \) of a gyrotriangle \( A_1A_2A_3 \), shown in Fig. 23, p. 68.

Let \( A_3Q \) be a circumgyrocevian of a gyrotriangle \( A_1A_2A_3 \) in an Einstein gyrovector space \( (\mathbb{R}^n, \oplus, \otimes) \), as shown in Fig. 23, so that the gyrobarycentric representation of \( Q \) with respect to the gyrobarycentrically independent set \( \{A_1, A_2, A_3\} \) is given by (305).

Applying the Gyrobarycentric Representation Gyrocovariance Theorem [26, Theorem 4.6, pp. 90-91] with \( X = \triangle A_3 \) to the gyrobarycentric representation (305a) of \( Q \), and using the gyrotriangle index notation, (13), p. 7, we have

\[
\gamma_{\triangle A_3\oplus Q} = \frac{m_1\gamma_{\triangle A_3\oplus A_1} + m_2\gamma_{\triangle A_3\oplus A_2} + m_3\gamma_{\triangle A_3\oplus A_3}}{m_1\gamma_{\triangle A_3\oplus A_1} + m_2\gamma_{\triangle A_3\oplus A_2} + m_3\gamma_{\triangle A_3\oplus A_3}}
\]

and

\[
\gamma_{\triangle A_3\oplus Q} = \frac{m_1\gamma_{A_3} + m_2\gamma_{A_3} + m_3}{m_Q},
\]

where \( m_Q > 0 \) is the constant of the gyrobarycentric representation (305) of \( Q \), given by (309).

Substituting \( m_1, m_2, m_3 \) from (305a) and \( m_Q \) from (309) into (314), we obtain the equation

\[
\gamma_{\triangle A_3\oplus Q} = \frac{\gamma_{A_3}(\gamma_{A_3} - 1)(1 - t_1) + \gamma_{A_3}(\gamma_{A_3} - 1)t_1 - (\gamma_{A_3} - 1 + (\gamma_{A_3} - \gamma_{A_3})^2)(1 - t_1)t_1}{(\gamma_{A_3} - 1)(1 - t_1) + (\gamma_{A_3} - 1)t_1 - (\gamma_{A_3} - 1)(1 - t_1)t_1}.
\]

Equation (315) proves useful in the proof of Theorem 12 in Sect. 26.
Finally, the gyrolength $\|\ominus A_3Q\|$ of the circumgyrocevian $A_3Q$ of gyrotriangle $A_1A_2A_3$ in Fig. 23 is obtained from (315) by means of (4), p. 3.

(316) $\frac{1}{s^2}\|\ominus A_3Q\|^2 = \frac{\gamma_{\ominus A_3Q}^2 - 1}{\gamma_{\ominus A_3Q}^2} = \frac{\left[\gamma_{12}t_2^2 + \gamma_{23}t_1^2 - 1 - 2(\gamma_{12} - 1) + (\gamma_{23} - \gamma_{13})^2\right]t_2t_1}{\gamma_{13}(\gamma_{13} - 1)t_2 + \gamma_{23}(\gamma_{23} - 1)t_1 - \left(\gamma_{12} - 1 + (\gamma_{13} - \gamma_{23})^2\right)t_2t_1}$

where $t_2 = 1 - t_1$, $0 < t_1 < 1$.

26. The Intersecting Gyrochords Theorem

Following Sects. 23–25, we are now in the position to prove the Intersecting Gyrochords Theorem.

**Theorem 42. (The Intersecting Gyrochords Theorem).** If two gyrochords, $A_1A_2$ and $B_1B_2$, of a gyrocircle in an Einstein gyrovector space $(\mathbb{R}_n^s, \ominus, \otimes)$ intersect at a point $P$, as shown in Fig. 25, then

(317) $\frac{\gamma_{|PA_1|}|PA_1|\gamma_{|PA_2|}|PA_2|}{\gamma_{|A_1A_2|} + 1} = \frac{\gamma_{|PB_1|}|PB_1|\gamma_{|PB_2|}|PB_2|}{\gamma_{|B_1B_2|} + 1}$.

**Proof.** As a matter of notation we have

(318) $|A_1A_2| = \|\ominus A_1\ominus A_2\| = \gamma_{12}$

$|B_1B_2| = |A_3Q|$, where we use the notation in Fig. 23 in which, in particular, $B_1 = A_3$ and $B_2 = Q$.

Hence, noting that each side of (317) is positive, it can be written equivalently as

(319) $\left(\frac{\gamma_{|PA_1|}|PA_1|\gamma_{|PA_2|}|PA_2|}{\gamma_{12} + 1}\right)^2 = \left(\frac{\gamma_{|PA_3|}|PA_3|\gamma_{|PQ|}|PQ|}{\gamma_{|A_3Q|} + 1}\right)^2$,

where $|PA_1| = |A_1P|$ is the gyrolength $\|\ominus P\ominus A_1\|$ of gyrasegment $PA_1$, etc.

Substituting into (319)

(1) $\gamma_{|PA_k|}|PA_k|^2$, $k = 1, 2, 3$, from (303) in Sect. 23 and
(2) $\gamma_{|PQ|}|PQ|^2$ from (310) in Sect. 24 and
(3) $\gamma_{|A_3Q|}$ from (315) in Sect. 25.
we find that Identity (319) is valid, each side of which being equal to the right-hand side of each of the two equations in (320) below,

\[
\left( \frac{\gamma_{|PA_1|}|PA_1| \gamma_{|PA_2|}|PA_2|}{\gamma_{12} + 1} \right)^2 = s^4 \left( \frac{(\gamma_{12}^2 - 1)(1 - t_1)t_1}{2(\gamma_{12}^2 - 1)(1 - t_1)t_1 + 1} \right)^2.
\]

The two equations in (320) share the right-hand sides, so that the proof of (319) and, hence, of (317), is complete.

In the Euclidean limit, \( s \to \infty \), gyrolengths of gyrosegments tend to lengths of corresponding segments and gamma factors tend to 1. Hence, in that limit, the Intersecting Gyrochords Theorem 42 reduces to the following well-known Intersecting Chords Theorem of Euclidean geometry:

**Theorem 43. (The Intersecting Chords Theorem).** If two chords, \( A_1A_2 \) and \( B_1B_2 \), of a circle in a Euclidean space \( \mathbb{R}^n \) intersect at a point \( P \), as shown in Fig. 26, then

\[
|PA_1||PA_2| = |PB_1||PB_2|.
\]
Our ambiguous notation that emphasizes analogies between hyperbolic and Euclidean geometry should raise no confusion. Thus, in particular, one should note that

(1) in the Intersecting Gyrochords Theorem 42, \( |PA_1| \) is the gyrolength,

\[
|PA_1| = \| \odot P \odot A_1 \|, \tag{322}
\]

of gyrosegment \( PA_1 \), etc., while, in contrast,

(2) in the Intersecting Chords Theorem 43, \( |PA_1| \) is the length,

\[
|PA_1| = \| - P + A_1 \|, \tag{323}
\]

of segment \( PA_1 \), etc.

References

[1] Michael J. Crowe. *A history of vector analysis*. Dover Publications Inc., New York, 1994. The evolution of the idea of a vectorial system, Corrected reprint of the 1985 edition.

[2] Albert Einstein. *Zur Elektrodynamik Bewegter Körper* [on the electrodynamics of moving bodies] (We use the English translation in [4] or in [8], or in http://www.fourmilab.ch/etexts/einstein/specrel/www/). *Ann. Physik (Leipzig)*, 17:891–921, 1905.

[3] Albert Einstein. *Einstein’s Miraculous Years: Five Papers that Changed the Face of Physics*. Princeton, Princeton, NJ, 1998. Edited and introduced by John Stachel. Includes bibliographical references. Einstein’s dissertation on the determination of molecular dimensions – Einstein on Brownian motion – Einstein on the theory of relativity – Einstein’s early work on the quantum hypothesis. A new English translation of Einstein’s 1905 paper on pp. 123–160.

[4] Vladimir Fock. *The theory of space, time and gravitation*. The Macmillan Co., New York, 1964. Second revised edition. Translated from the Russian by N. Kemmer. A Pergamon Press Book.

[5] Jeremy Gray. M"obius’s geometrical Mechanics. 1993. In John Fauvel, Raymond Flood, and Robin Wilson, eds. M"obius and his band, Mathematics and astronomy in nineteenth-century Germany, The Clarendon Press Oxford University Press, New York, 1993, 78–103.

[6] Paul Joseph Kelly and Gordon Matthews. *The non-Euclidean, hyperbolic plane*. Springer-Verlag, New York, 1981. Its structure and consistency, Universitext.

[7] Clark Kimberling. Clark Kimberling’s Encyclopedia of Triangle Centers - ETC. 2012. http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[8] H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl. *The principle of relativity*. Dover Publications Inc., New York, N. Y., undated. With notes by A. Sommerfeld, Translated from the Russian by B. Jeffery, A collection of original memoirs on the special and general theory of relativity.

[9] Eli Maor. *Trigonometric delights*. Princeton University Press, Princeton, NJ, 1998.

[10] Richard S. Millman and George D. Parker. *Geometry*. Springer-Verlag, New York, second edition, 1991. A metric approach with models.

[11] C. Miller. *The theory of relativity*. Oxford, at the Clarendon Press, 1952.

[12] Themistocles M. Rassias. Book Review: *Analytic Hyperbolic Geometry and Albert Einstein’s Special Theory of Relativity*, by Abraham A. Ungar. *Nondinear Funct. Anal. Appl.*, 13(1):167–177, 2008.

[13] Themistocles M. Rassias. Book Review: *A gyrovector space approach to hyperbolic geometry*, by Abraham A. Ungar. *J. Geom. Symm. Phys.*, 18:93–106, 2010.

[14] Roman U. Szel and Helmut K. Urbantke. *Relativity, groups, particles*. Springer Physics. Springer-Verlag, Vienna, 2001. Special relativity and relativistic symmetry in field and particle physics, Revised and translated from the third German (1992) edition by Urbantke.

[15] Nilgün Sönumez and Abraham A. Ungar. The Einstein relativistic velocity model of hyperbolic geometry and its plane separation axiom. *Adv. Appl. Clifford Alg.*, 23:209–236, 2013.
[16] Abraham A. Ungar. Quasidirect product groups and the Lorentz transformation group. In Themistocles M. Rassias (ed.): Constantin Carathéodory: an international tribute, Vol. I, II, pages 1378–1392. World Sci. Publishing, Teaneck, NJ, 1991.

[17] Abraham A. Ungar. Gyrovector spaces in the service of hyperbolic geometry. In Themistocles M. Rassias (ed.): Mathematical analysis and applications, pages 305–360. Hadronic Press, Palm Harbor, FL, 2000.

[18] Abraham A. Ungar. Möbius transformations of the ball, Ahlfors’ rotation and gyrovector spaces. In Themistocles M. Rassias (ed.): Nonlinear analysis in geometry and topology, pages 241–287. Hadronic Press, Palm Harbor, FL, 2000.

[19] Abraham A. Ungar. Beyond the Einstein addition law and its gyroscopic Thomas precession: The theory of gyrogroups and gyrovector spaces, volume 117 of Fundamental Theories of Physics. Kluwer Academic Publishers Group, Dordrecht, 2001.

[20] Abraham A. Ungar. Analytic hyperbolic geometry: Mathematical foundations and applications. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[21] Abraham A. Ungar. The hyperbolic square and Möbius transformations. Banach J. Math. Anal., 1(1):101–116, 2007.

[22] Abraham A. Ungar. Analytic hyperbolic geometry and Albert Einstein’s special theory of relativity. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[23] Abraham A. Ungar. A gyrovector space approach to hyperbolic geometry. Morgan & Claypool Pub., San Rafael, California, 2009.

[24] Abraham A. Ungar. The hyperbolic triangle incenter. Dedicated to the 30th anniversary of Themistocles M. Rassias’ stability theorem. Nonlinear Funct. Anal. Appl., 14(5):817–841, 2009.

[25] Abraham A. Ungar. Barycentric calculus in Euclidean and hyperbolic geometry: A comparative introduction. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

[26] Abraham A. Ungar. Hyperbolic triangle centers: The special relativistic approach. Springer-Verlag, New York, 2010.

[27] Abraham A. Ungar. Gyrotations: the missing link between classical mechanics with its underlying Euclidean geometry and relativistic mechanics with its underlying hyperbolic geometry. In Essays in mathematics and its applications in honor of Stephen Smale’s 80th birthday, pages 463–504. Springer, Heidelberg, 2012. arXiv 1302.5678 (math-ph).

[28] Abraham A. Ungar. Möbius transformation and Einstein velocity addition in the hyperbolic geometry of Bolyai and Lobachevsky. In Nonlinear Analysis, volume 68 of Springer Optim. Appl., pages 721–770. Springer, New York, 2012. Stability, Approximation, and Inequalities. In honor of Themistocles M. Rassias on the occasion of his 60th birthday.

[29] Abraham A. Ungar. An introduction to hyperbolic barycentric coordinates and their applications. In Mathematics Without Boundaries: Surveys in Interdisciplinary Research, Springer Optim. Appl. Springer, New York, 2013 (in print). eds: Panos Pardalos and Themistocles M. Rassias.

[30] Scott Walter. Book Review: Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces, by Abraham A. Ungar. Found. Phys., 32(2):327–330, 2002.