On a conjecture of Pisier on the analyticity of semigroups

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Abstract

We show that the analyticity of semigroups \((T_t)_{t \geq 0}\) of selfadjoint contractive Fourier multipliers on \(L^p\)-spaces of compact abelian groups is preserved by the tensorisation of the identity operator of a Banach space for a large class of K-convex Banach spaces, answering partially a conjecture of Pisier. We also give versions of this result for some semigroups of Schur multipliers and Fourier multipliers on noncommutative \(L^p\)-spaces. Finally, we give a precise description of semigroups of Schur multipliers to which the result of this paper can be applied.

1 Introduction

In the early eighties, in a famous paper on the geometry of Banach spaces, Pisier [Pis3] showed that a Banach space \(X\) does not contain \(\ell_1^n\)'s uniformly if and only if the tensorisation \(P \otimes \text{Id}_X\) of the Rademacher projection

\[
P: \quad L^2(\Omega) \rightarrow L^2(\Omega)
\]

\[
f \mapsto \sum_{k=1}^{\infty} \left( \int_{\Omega} f \varepsilon_k \right) \varepsilon_k
\]

induces a bounded operator on the Bochner space \(L^2(\Omega, X)\) where \(\Omega\) is a probability space and where \(\varepsilon_1, \varepsilon_2, \ldots\) is a sequence of independant random variables with \(P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}\). Such a Banach space \(X\) is called K-convex. The heart of his proof relies on the fact, proved by himself in his article, that if \(X\) is a K-convex Banach space then any \(w^*\)-continuous semigroup \((T_t)_{t \geq 0}\) of positive unital selfadjoint Fourier multipliers on a locally compact abelian group \(G\) induces a strongly continuous bounded analytic semigroup \((T_t \otimes \text{Id}_X)_{t \geq 0}\) of contractions on the Bochner space \(L^p(G, X)\) where \(1 < p < \infty\). In 1981, in the seminars [Pis1] and [Pis2] which announced the results of his paper, he stated several natural questions raised by his work. In particular, he conjectured that the same property holds for any \(w^*\)-continuous semigroup \((T_t)_{t \geq 0}\) of selfadjoint contractive operators on a space \(L^\infty(\Omega)\) where \(\Omega\) is a measure space (see also the recent preprint [Xu] Problem 11] for a more general question). Note that it is well known that such a semigroup induces a strongly continuous bounded analytic semigroup of contractions on the associated \(L^p\)-space \(L^p(\Omega)\) and the conjecture says that the property of analyticity is preserved by the tensorisation of the identity \(\text{Id}_X\) of a K-convex Banach space \(X\).
Using operator space theory (see [ER], [Pau] and [Pis7]), a quantised theory of Banach spaces, we are able to give the following partial answer to this purely Banach spaces question. First of all, let us recall that an operator space $E$ is $\OK$-convex if the vector valued Schatten space $S^p(E)$ is $K$-convex for some (equivalently all) $1 < p < \infty$. It means that the Rademacher projection $P$ is completely bounded. This notion was introduced by [JuP] and is the noncommutative version of the property of $K$-convexity. Our main result is the following theorem.

**Theorem 1.1** Suppose that $G$ is a compact abelian group. Let $(T_t)_{t \geq 0}$ be a $w^*$-continuous semigroup of selfadjoint contractive Fourier multipliers on $L^{\infty}(G)$. Let $X$ be a $K$-convex Banach space isomorphic to a Banach space $\mathcal{E}$ which admits an $\OK$-convex operator space structure. Consider $1 < p < \infty$. Then $(T_t)_{t \geq 0}$ induces a strongly continuous bounded analytic semigroup $(T_t \otimes 1)_{t \geq 0}$ of contractions on the Bochner space $L^p(G,X)$.

This result can be used, by example, in the case where the Banach space $X$ is an $L^p$-space or a Schatten space $S^q$ with $1 < q < \infty$. Our methods also give a result for some $w^*$-continuous semigroups of Schur multipliers and a generalization for semigroups of Fourier multipliers on amenable discrete groups.

The paper is organized as follows. Section 2 gives a brief presentation of vector valued noncommutative $L^p$-spaces, Fourier multipliers on group von Neumann algebras and Schur multipliers. We introduce here some notions which are relevant to our paper. The next section contains a proof of Theorem 1.1. Finally, in Section 4, we describe the semigroups of Schur multipliers to which the results of this paper can be applied. This result is of independent interest.

## 2 Preliminaries

The readers are referred to [ER], [Pau] and [Pis7] for details on operator spaces and completely bounded maps and to the survey [PX] for noncommutative $L^p$-spaces and the references therein.

If $T : E \to F$ is a completely bounded map between two operator spaces $E$ and $F$, we denote by $\|T\|_{\cb,E \to F}$ its completely bounded norm.

The theory of vector valued noncommutative $L^p$-spaces was initiated by Pisier [Pis5] for the case where the underlying von Neumann algebra is hyperfinite and equipped with a faithful normal semifinite trace. Suppose $1 \leq p < \infty$. Under these assumptions, for any operator space $E$, we can define by complex interpolation

$$L^p(M,E) = (M \otimes_{\min} E, L^1(M) \widehat{\otimes} E)^p,$$

where $\otimes_{\min}$ and $\widehat{\otimes}$ denote the injective and the projective tensor product of operator spaces.

If $M = B(\ell_2^n)$ is the algebra of bounded operators on the Hilbert space $\ell_2^n$ equipped with its canonical trace, we obtain the vector valued Schatten space $S^p_\ell(E) = L^p(\ell_2^n, E)$. Sometimes, we will use the notation $S^p(B(\ell_2^n) \otimes E)$ for the space $L^p(B(\ell_2^n) \otimes E)$.

Note the following vector valued extension property of completely positive maps between noncommutative $L^p$-spaces, see [Pis4].

**Proposition 2.1** Suppose $1 < p < \infty$. Let $M$ and $N$ be hyperfinite von Neumann algebras equipped with normal faithful semifinite traces and let $E$ be an operator space. Let $T : M \to N$ be a trace preserving unital normal completely positive map. Then the operator $T \otimes 1_E$ extends to a bounded operator from $L^p(M, E)$ into $L^p(N, E)$ and we have

$$\|T \otimes 1_E\|_{L^p(M, E) \to L^p(N, E)} \leq \|T\|_{L^p(M) \to L^p(N)}.$$
In particular, this result applies to canonical conditional expectations. Recall that a linear map
\( T : L^p(M) \to L^p(M) \) defined on a noncommutative \( L^p \)-space where \( 1 < p < \infty \) is said to be
regular [Pis4] if for any operator space \( E \) the linear map \( T \otimes \text{Id}_E : L^p(M, E) \to L^p(M, E) \) is bounded. The above proposition give an example of regular map.

Suppose that \( G \) is a discrete group. We denote by \( \epsilon_g \) the neutral element of \( G \). For
\( f \in \ell^2(G) \), we write \( L_f \) for the left convolution by \( f \) acting on \( \ell^2(G) \) by:

\[
L_f(h)(g') = \sum_{g \in G} f(g)h(g^{-1}g')
\]

for any \( h \in \ell^2(G) \) and any \( g' \in G \). Let \( \text{VN}(G) \) be the von Neumann algebra generated by
\( \{L_f\}_{f \in \ell^1(G)} \). This \( \text{VN}(G) \) is called the group von Neumann algebra of \( G \) and is equal to the
von Neumann algebra generated by \( \{\lambda_g\}_{g \in G} \), where \( \lambda_g \) is the left translation acting on \( \ell^2(G) \)
defined by \( \lambda_g(h)(g') = h(g^{-1}g') \). It is an finite algebra with normalized trace given by

\[
\tau_G(x) = \langle \epsilon_{\epsilon_G}, x(\epsilon_{\epsilon_G}) \rangle_{\ell^2_G}
\]

where \( (\epsilon_g)_{g \in G} \) is the canonical basis of \( \ell^2_G \) and \( x \in \text{VN}(G) \).

For a compact abelian group \( G \), the Fourier transform of \( f \in \ell^1(G) \) is defined on the dual
group \( \hat{G} \) of \( G \) by

\[
\hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}dx
\]

for \( \gamma \in \hat{G} \). In this case, \( (\text{VN}(G), \tau_G) \) is equivalent as a von Neumann algebra to \( L^\infty(\hat{G}) \) with
the usual integration on the dual group \( \hat{G} \) of \( G \) under the mapping

\[
\begin{align*}
\text{VN}(G) & \quad \longrightarrow \quad L^\infty(\hat{G}) \\
L_f & \quad \longmapsto \quad \hat{f}.
\end{align*}
\]

Moreover, it is well known that \( G \) is discrete if and only if the dual group \( \hat{G} \) is compact.

Let \( G \) be a discrete group. Recall that the von Neumann algebra \( \text{VN}(G) \) is hyperfinite if and
only if \( G \) is amenable [SS, Theorem 3.8.2]. A Fourier multiplier is a normal linear map
\( T : \text{VN}(G) \to \text{VN}(G) \) such that there exists a complex function \( \varphi : G \to \mathbb{C} \) such that for any
\( g \in G \) we have \( T(\lambda_g) = \varphi_g\lambda_g \). In this case, we denote \( T \) by

\[
M_\varphi : \quad \text{VN}(G) \quad \longrightarrow \quad \text{VN}(G) \\
\lambda_g & \quad \longmapsto \quad \varphi_g\lambda_g.
\]

By [DCH, Corollary 1.8], if the discrete group \( G \) is amenable, every contractive Fourier multiplier
\( M_\varphi : \text{VN}(G) \to \text{VN}(G) \) is completely contractive.

If \( I \) is an index set and if \( E \) is a vector space, we write \( \mathcal{M}_I \) for the space of the \( I \times I \) matrices
with entries in \( \mathbb{C} \) and \( \mathcal{M}_I(E) \) for the space of the \( I \times I \) matrices with entries in \( E \).

Let \( A = [a_{ij}]_{i,j \in I} \) be a matrix of \( \mathcal{M}_I \). By definition, the Schur multiplier on \( B(\ell^2_I) \) associated
with this matrix is the unbounded linear operator \( M_A \) whose domain is the space of all \( B = [b_{ij}]_{i,j \in I} \) of \( B(\ell^2_I) \) such that \( [a_{ij}b_{ij}]_{i,j \in I} \) belongs to \( B(\ell^2_I) \), and whose action on \( B = [b_{ij}]_{i,j \in I} \) is given by \( M_A(B) = [a_{ij}b_{ij}]_{i,j \in I} \). For all \( i, j \in I \), the matrix \( e_{ij} \) belongs to \( D(M_A) \), hence
\( M_A \) is densely defined for the weak* topology. Suppose \( 1 \leq p < \infty \). If for any \( B \in \mathcal{S}^p_I \), we have \( B \in D(M_A) \) and the matrix \( M_A(B) \) represents an element of \( \mathcal{S}^p_I \), by the closed graph theorem, the matrix \( A \) of \( \mathcal{M}_I \) defines a bounded Schur multiplier \( M_A : \mathcal{S}^p_I \to \mathcal{S}^p_I \). We have a
similar statement for bounded Schur multipliers on \( B(\ell^2_I) \). Recall that every contractive Schur
multiplier $M_A: B(\ell^2_1) \to B(\ell^2_1)$ is completely contractive (see [Pau]). We say that a matrix $A$ of $M_I$ induces a completely positive Schur multiplier $M_A: B(\ell^2_1) \to B(\ell^2_1)$ if and only if for any finite set $F \subset I$ the matrix $[a_{i,j}]_{i,j \in F}$ is positive (see [Pau]).

Let $M$ be a von Neumann algebra equipped with a semifinite normal faithful trace $\tau$. Suppose that $T: M \to M$ is a normal contraction. We say that $T$ is selfadjoint if for all $x, y \in M \cap L^1(M)$ we have

$$\tau(T(x)y^*) = \tau(x(T(y))^*).$$

In this case, it is not hard to show that the restriction $T|_{M \cap L^1(M)}$ extends to a contraction $T: L^1(M) \to L^1(M)$. By complex interpolation, for any $1 \leq p \leq \infty$, we obtain a contractive map $T: L^p(M) \to L^p(M)$. Moreover, the operator $T: L^2(M) \to L^2(M)$ is selfadjoint. If $T: M \to M$ is actually a normal selfadjoint complete contraction, it is easy to see that the map $T: L^p(M) \to L^p(M)$ is completely contractive for any $1 \leq p \leq \infty$. It is not difficult to show that a contractive Fourier multiplier $M_\varphi: VN(G) \to VN(G)$ is selfadjoint if and only if $\varphi: G \to \mathbb{C}$ is a real function. Finally, one can prove that a contractive Schur multiplier $M_\varphi$ is selfadjoint if and only if $\varphi$ is a real function.

### 3 Analyticity of semigroups on vector valued $L^p$-spaces

Let $X$ be a Banach space. A strongly continuous semigroup $(T_t)_{t \geq 0}$ is called bounded analytic if there exist $0 < \theta < \frac{\pi}{2}$ and a bounded holomorphic extension

$$\Sigma_\theta \to B(X), \quad z \mapsto T_z,$$

where $\Sigma_\theta = \{ z \in \mathbb{C}^* : |\text{Arg}(z)| < \theta \}$ denotes the open sector of angle $2\theta$ around the positive real axis $\mathbb{R}_+$. See [EN] and [Haa] for more information on this notion. We need the following theorem which is a corollary [Pis3, Lemma 4] of a result of Beurling [Beu] (see also [Fac] and [Hin]).

**Theorem 3.1** Let $X$ be a Banach space. Let $(T_t)_{t \geq 0}$ be a strongly continuous semigroup of contractions on $X$. Suppose that there exists some integer $n \geq 1$ such that for any $t > 0$

$$\| (I - T_t)^n \|_{X \to X} < 2^n.$$

Then the semigroup $(T_t)_{t \geq 0}$ is bounded analytic.

Moreover, we recall the following lemma, see [DJT, Lemma 13.12] and [Pis3, Lemma 1.5].

**Lemma 3.2** Suppose that $X$ is a $K$-convex Banach space. Then there exist a real number $0 < \rho < 2$ and an integer $n \geq 1$ such that if $P_1, \ldots, P_n$ is any finite collection of mutually commuting norm one projections on $X$, then

$$\left\| \prod_{1 \leq k \leq n} (I - P_k) \right\|_{X \to X} \leq \rho^n.$$

We will use the useful next ‘absorption Lemma’ which is a variant of [Arh2, Proposition 3.4]. The proof is left to the reader.
Lemma 3.3 Suppose $1 \leq p < \infty$. Let $E$ be an operator space. For any positive integer $n \geq 1$ and any matrix $x \in M_n(E)$ finitely supported on $I \times I$, we have

\[
\left\| \sum_{i,j \in I} e_{ij} \otimes e_{ij} \otimes \cdots \otimes e_{ij} \otimes x_{ij} \right\|_{\mathcal{S}^p(B(\ell^2_n)\otimes E)} = \left\| \sum_{i,j \in I} e_{ij} \otimes x_{ij} \right\|_{\mathcal{S}^p(E)}.
\]

Moreover, for any regular Schur multiplier $M_M: \mathcal{S}^p \to \mathcal{S}^p$ and any positive integer $n \geq 1$ we have

\[
\left\| M_M \otimes Id_E \right\|_{\mathcal{S}^p(E) \to \mathcal{S}^p(E)} \leq \left\| M_M \otimes Id_E \right\|_{\mathcal{S}^p(B(\ell^2_n)\otimes E) \to \mathcal{S}^p(B(\ell^2_n)\otimes E)}.
\]

We also need the following transfer results of Neuwirth and Ricard [NR] between Fourier multipliers and Schur multipliers. Let $G$ be a discrete amenable group. If $\varphi: G \to \mathbb{C}$ is a complex function, we denote by $\hat{\varphi}$ the Schur multiplier defined by the symbol $\hat{\varphi} \in \ell^\infty(G \times G)$ defined by $\hat{\varphi}(g, h) = \varphi(gh^{-1})$ where $g, h \in G$. Then, for any $1 \leq p \leq \infty$, we have

\[
\left\| M_\varphi \otimes Id_E \right\|_{L^p(VN(G), E) \to L^p(VN(G), E)} \leq \left\| M_\hat{\varphi} \otimes Id_E \right\|_{\mathcal{S}^p(G) \to \mathcal{S}^p(G)}
\]

and

\[
\left\| M_\varphi \otimes Id_E \right\|_{cb, L^p(VN(G), E) \to L^p(VN(G), E)} = \left\| M_\hat{\varphi} \otimes Id_E \right\|_{cb, \mathcal{S}^p(\mathcal{G}) \to \mathcal{S}^p(\mathcal{G})}.
\]

Using the identification \ref{identification}, we see that Theorem 3.4 is a particular case of the following more general result, which improves a part of [Arh1, Theorem 5.1].

**Theorem 3.4** Suppose that $G$ is an amenable discrete group. Let $(T_t)_{t \geq 0}$ be a $w^*$-continuous semigroup of selfadjoint contractive Fourier multipliers on the group von Neumann algebra $VN(G)$. Suppose that $E$ is an OK-convex operator space. Consider $1 < p < \infty$. Then $(T_t)_{t \geq 0}$ induces a strongly continuous bounded analytic semigroup $(T_t \otimes Id_E)_{t \geq 0}$ of contractions on the noncommutative vector valued $L^p$-space $L^p(VN(G), E)$.

**Proof:** We consider the associated semigroup $(\hat{T}_t)_{t \geq 0}$ of selfadjoint contractive Schur multipliers on the space $B(\ell^2(G))$. Using \ref{identification}, we see that

\[
\left\| \hat{T}_t \right\|_{\mathcal{S}^p(\mathcal{G}) \to \mathcal{S}^p(\mathcal{G})} \leq \left\| T_t \right\|_{cb, \mathcal{S}^p(\mathcal{G}) \to \mathcal{S}^p(\mathcal{G})} = \left\| T_t \right\|_{cb, L^p(VN(G), E) \to L^p(VN(G), E)} \leq 1.
\]

In the sequel, we denote by $(\hat{T}_t)^\circ$ the Schur multiplier defined by the adjoint matrix of the matrix of the Schur multiplier $\hat{T}_t$. As the proof of [Arh1, Corollary 4.3], for any $t \geq 0$, there exists Schur multipliers $S_{1,t}$ and $S_{2,t}$ on $B(\ell^2(G))$ such that

\[
W_t = \begin{bmatrix} S_{1,t} & \hat{T}_t \\ (\hat{T}_t)^\circ & S_{2,t} \end{bmatrix},
\]

is a completely positive unital self-adjoint Schur multiplier on $B(\ell^2_{(1,2)} \times G)$. Note that, for any $t \geq 0$, we have

\[
(W_t)^2 = \begin{bmatrix} S_{1,t} \hat{T}_t & \hat{T}_t \hat{T}_t \\ (\hat{T}_t)^\circ S_{1,t} & S_{2,t} \end{bmatrix}^2 = \begin{bmatrix} S_{1,t} \hat{T}_t & (\hat{T}_t)^\circ S_{1,t} \\ S_{2,t} \hat{T}_t & S_{2,t} \end{bmatrix}^2 = \begin{bmatrix} S_{3,t} \hat{T}_t & \hat{T}_t \hat{T}_t \\ (\hat{T}_t)^\circ S_{3,t} & S_{4,t} \end{bmatrix}.
\]
By [Rin] (see also [JMX, Section 10.B] for some related information), for any \( t \geq 0 \), the Schur multiplier \( (W_\pi^t)^2 \) admits a Rota dilation

\[
(W_\pi^t)^2 = Q_E \pi, \quad k \geq 1
\]

where \( \pi : M_2(B(\ell_2^2)) \to M \) is a normal unital faithful *-representation into a hyperfinite von Neumann algebra (equipped with a normalized trace) which preserve the traces, where \( Q : M \to M_2(B(\ell_2^2)) \) is the conditional expectation associated with \( \pi \) and where the \( E_k \)'s are conditional expectations onto von Neumann subalgebras of \( M \). In particular, we have

\[
(W_\pi^t)^2 = Q_E \pi.
\]

We infer that

\[
Id_{M_2(B(\ell_2^2))} - (W_\pi^t)^2 = Q(Id_M - E_1) \pi.
\]

Let \( n \) be a positive integer. We deduce that

\[
(3.6) \quad \left( Id_{M_2(B(\ell_2^2))} - (W_\pi^t)^2 \right)^{\otimes n} = Q(Id_M - E_1)^{\otimes n} \pi.
\]

For any integer \( 1 \leq k \leq n \), we let

\[
\Pi_k = Id_{L^p(M)} \otimes \cdots \otimes Id_{L^p(M)} \otimes E_1 \otimes Id_{L^p(M)} \otimes \cdots \otimes Id_{L^p(M)}.
\]

By Proposition 2.1 we deduce that the \( \Pi_k \otimes Id_E \)'s are well-defined and form a family of mutually commuting contractive projections on the Banach space \( L^p(M^{\otimes n}, E) \). Moreover, by [Arh2, Proposition 3.5], the latter space is \( K \)-convex. Now, we choose the integer \( n \geq 1 \) and \( 0 < \rho < 2 \) as in Lemma 3.2. Hence, we obtain that

\[
(3.7) \quad \left\| \prod_{1 \leq k \leq n} \left( Id_{L^p(M^{\otimes n}, E)} - (\Pi_k \otimes Id_E) \right) \right\|_{L^p(M^{\otimes n}, E) \to L^p(M^{\otimes n}, E)} \leq \rho^n.
\]

Furthermore, it is easy to see that

\[
(3.8) \quad (Id_{L^p(M)} - E_1)^{\otimes n} \otimes Id_E = \prod_{1 \leq k \leq n} \left( Id_{L^p(M^{\otimes n}, E)} - (\Pi_k \otimes Id_E) \right).
\]

Now, using (3.5), (3.2), (3.6) and (3.7) we obtain that

\[
\left\| \left( Id_{S^p_G} - T_\pi^t \right)^{\otimes n} \otimes Id_E \right\|_{S^p_G(E) \to S^p_G(E)} \leq \left\| \left( Id_{S^p_G} - (W_\pi^t)^2 \right)^{\otimes n} \otimes Id_E \right\|_{S^p_G(E) \to S^p_G(E)} \leq \rho^n.
\]
Remark 3.6
Wo does not know if any K-convex Banach space $E$ space
The description of self-adjoint contractive Schur multipliers on
Semigroups of contractive selfadjoint Schur multipliers
similar question for other Banach spaces properties (UMD, cotype...).

Theorem 4.1
rediscovered by many authors, see [Pis6, Chapter 5] for more information. Here, we give a
Theorem 3.5
continuous version of this result which precisely describes the semigroups of Schur multipliers
on the vector valued Schatten space $S^p_{G}(E)$.

Theorem 3.5
Let $(T_t)_{t \geq 0}$ be a $w^*$-continuous semigroup of selfadjoint contractive Schur multipliers on $B(\ell^2_1)$. Suppose that $E$ is an OK-convex operator space. Consider $1 < p < \infty$. Then $(T_t)_{t \geq 0}$ induces a strongly continuous bounded analytic semigroup $(T_t \otimes I_d)_{t \geq 0}$ of contractions on the vector valued Schatten space $S^p_{G}(E)$.

Remark 3.6
Wo does not know if any K-convex Banach space $X$ is isomorphic to a Banach space $E$ admitting an operator space structure such that the Banach space $S^p(E)$ is K-convex for $1 < p < \infty$ (i.e. E is OK-convex). Moreover, it would be also interesting to examine a similar question for other Banach spaces properties (UMD, cotype...).

4 Semigroups of contractive selfadjoint Schur multipliers

The description of self-adjoint contractive Schur multipliers on $B(\ell^2_1)$ (and more generally of contractive Schur multipliers) is well-known and essentially goes back to Grothendieck and was rediscovered by many authors, see [Pis6, Chapter 5] for more information. Here, we give a continuous version of this result which precisely describes the semigroups of Schur multipliers of Theorem 3.5.

Theorem 4.1
Suppose that $A$ is a matrix of $M_I$. For any $t \geq 0$, let $T_t$ be the unbounded Schur multipliers on $B(\ell^2_1)$ associated with the matrix

\[ e^{-ta_{ij}} \] \quad i,j \in I.

The semigroup $(T_t)_{t \geq 0}$ extends to a semigroup of selfadjoint contractive Schur multipliers $T_t: B(\ell^2_1) \to B(\ell^2_1)$ if and only if there exists a Hilbert space $H$ and two families $(\alpha_i)_{i \in I}$ and $(\beta_j)_{j \in I}$ of elements of $H$ such that $a_{ij} = \|\alpha_i - \beta_j\|_H^2$ for any $i,j \in I$.

In this case, the Hilbert space may be chosen as a real Hilbert space and moreover, $(T_t)_{t \geq 0}$ is a $w^*$-continuous semigroup.

Proof: First, suppose that the semigroup $(T_t)_{t \geq 0}$ extends to a semigroup of selfadjoint contractive Schur multipliers $T_t: B(\ell^2_1) \to B(\ell^2_1)$. In particular, for any integer $n \geq 1$, the Schur multiplier $T_n: B(\ell^2_1) \to B(\ell^2_1)$ is contractive and selfadjoint. Thus, as the proof of [Arh2, Corollary 4.3], we can find matrices $S_{1,n}, S_{2,n} \in M_I$ such that the block matrix

\[
\begin{bmatrix}
S_{1,n} & e^{-\frac{\alpha}{n}} \\
e^{-\frac{\beta}{n}} & S_{2,n}
\end{bmatrix}
\]

defines a unital selfadjoint completely positive Schur multiplier on $B(\ell^2_{\{1,2\} \times I})$. This matrix identifies to a matrix $[b_{n,k,i,j}]_{(k,i) \in \{1,2\} \times I, (m,j) \in \{1,2\} \times I} \in M_{\{1,2\} \times I}$ such that for any $i,j \in I$

\[
b_{n,1,i,1,j} = (S_{1,n})_{i,j}, \quad b_{n,1,i,2,j} = e^{-\frac{\alpha}{n}}, \quad b_{n,1,2,i,j} = e^{-\frac{\beta}{n}}, \quad b_{n,1,2,j} = (S_{2,n})_{i,j}.
\]
Then it is obvious that the map
\[
(\{1, 2\} \times I) \times (\{1, 2\} \times I) \rightarrow \mathbb{R}
\]
\[
((k, i), (m, j)) \mapsto n(1 - b_{n,k,i,m,j})
\]
is a real-valued negative definite kernel which vanishes on the diagonal of \((\{1, 2\} \times I) \times (\{1, 2\} \times I)\).

By [BCR, Proposition 3.2, Chapter 3], there exist a real Hilbert space \(H_n\) and a map \(\xi^n : \{1, 2\} \times I \rightarrow H_n\) such that
\[
\|\xi^n_{k,i} - \xi^n_{m,j}\|_{H_n} = n(1 - b_{n,k,i,m,j}), \quad k, m \in \{1, 2\}, \ i, j \in I.
\]

We can assume that \(\xi^n_{2,1} = 0\) for any integer \(n \geq 1\). For any \(a \in \mathbb{R}\), we have \(\frac{1 - e^{-\alpha/n}}{n} \xrightarrow{n \to 0} a\).

Hence, for any \(i, j \in I\), we see that
\[
\|\xi^n_{i} - \xi^n_{j}\|_{H_n} = n(1 - b_{n,1,i,2,j}) = n(1 - e^{-a_{ij}/n}) \xrightarrow{n \to \infty} a_{ij}.
\]

Since \(\xi^n_{2,1} = 0\), we infer that \(\|\xi^n_{i} - \xi^n_{j}\|_{H_n}\) is a bounded sequence for each \(i \in I\). Note that for any \(i, j \in I\) we have
\[
\|\xi^n_{i} - \xi^n_{j}\|_{H_n} \leq \|\xi^n_{i} - \xi^n_{2,1}\|_{H_n} + \|\xi^n_{1,1}\|_{H_n}.
\]

Thus, for any \(j \in I\), we deduce that \(\|\xi^n_{i} - \xi^n_{j}\|_{H_n}\) is also a bounded sequence.

Now, consider the ultraproduct \(H = \prod_{\mathcal{U}} H_n\) of the Hilbert spaces \(H_n\) with respect to some ultrafilter \(\mathcal{U}\) on \(\mathbb{N}\) refining the Fréchet filter. For any \(k \in \{1, 2\}\) and any \(i \in I\), let \(\xi_{k,i}^n\) denote the vector of \(H\) corresponding to the sequence \(\xi^n_{k,i}\), i.e. the equivalence class of this sequence.

The above computations give
\[
(4.2) \quad \|\xi_{i} - \xi_{j}\|_{H} = a_{ij}, \quad i, j \in I.
\]

For any \(i, j \in I\) we let \(\alpha_i = \xi_{1,i}\) and \(\beta_j = \xi_{2,j}\). Then Equation (4.2) becomes
\[
\|\alpha_i - \beta_j\|_{H} = a_{ij}, \quad i, j \in I.
\]

Conversely, suppose that there exists a Hilbert space \(H\) and two families \((\alpha_i)_{i \in I}\) and \((\beta_j)_{j \in I}\) of elements of \(H\) such that for any \(t \geq 0\) the Schur multiplier \(T_t : B(\ell^2_1) \rightarrow B(\ell^2_1)\) is associated with the matrix
\[
A_t = \left[ e^{-t\|\alpha_i - \beta_j\|_{H}^2} \right]_{i,j \in I}.
\]

Now, for any \(t \geq 0\), we define the following matrices of \(M_I\)
\[
B_t = \left[ e^{-t\|\alpha_i - \alpha_j\|_{H}^2} \right]_{i,j \in I}, \quad C_t = \left[ e^{-t\|\beta_i - \beta_j\|_{H}^2} \right]_{i,j \in I} \quad \text{and} \quad D_t = \left[ e^{-t\|\alpha_i - \beta_j\|_{H}^2} \right]_{i,j \in I}.
\]

For any \(i \in I\) and any \(n \in \{1, 2\}\), we define the vector \(\gamma_{(n,i)}\) of \(H\) by
\[
\gamma_{(n,i)} = \begin{cases}
\alpha_i & \text{if } n = 1 \text{ and } i \in I \\
\beta_i & \text{if } n = 2 \text{ and } i \in I.
\end{cases}
\]

Now, by the identification \(\mathbb{M}_2(M_I) \simeq \mathbb{M}_{\{1,2\} \times I}\), the block matrix \(\begin{bmatrix} B_t & A_t \\ D_t & C_t \end{bmatrix}\) of \(\mathbb{M}_2(M_I)\) can be identified with the matrix
\[
(4.3) \quad \left[ e^{-t\|\gamma_{(n,i)} - \gamma_{(m,j)}\|_{H}^2} \right]_{(n,i) \in \{1,2\} \times I, (m,j) \in \{1,2\} \times I}.
\]
of $M_{1,2} \times I$. Using [Arh1, Proposition 5.4], we deduce that, for any $t \geq 0$, the Schur multiplier associated with the matrix $A_3$ is contractive on $B(\ell^2_{1,2} \times I)$. We deduce that $T_1: B(\ell^2_1) \to B(\ell^2_1)$ is also contractive. An alternative proof of this implication can be obtained in adapting the proof of [JMX, Proposition 8.17].

Finally, it is easy to see that $(T_t)_{t \geq 0}$ is a $w^*$-continuous semigroup. □

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