Abstract
We analyse the structure of the set of solutions to the following class of boundary value problems

\[-\text{div}(A(x)Du) = c_\lambda(x)u + (M(x)Du, Du) + h(x), \quad u \in H^1_0(\Omega) \cup L^\infty(\Omega) \]  

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded domain with boundary $\partial \Omega$ of class $C^{1,D}$. We assume that $c, h \in L^p(\Omega)$ for some $p > n$, where $c^+ \geq 0$ are such that $c_\lambda(x) := \lambda c^+(x) - c^-(x)$ for a parameter $\lambda \in \mathbb{R}$, $A(x)$ is a uniformly positive bounded measurable matrix and $M(x)$ is a positive bounded matrix. Under suitable assumptions, we describe the continuum of solutions to problem $(P_\lambda)$ and also its bifurcation points proving existence and uniqueness results in the coercive case ($\lambda \leq 0$) and multiplicity results in the non-coercive case ($\lambda > 0$).

Keywords: Quasilinear elliptic equations, quadratic growth on the gradient, sub and super solutions.

1. Introduction
We consider the following class of boundary value problems

\[ \left\{ \begin{array}{l}
-\text{div}(A(x)Du) = c_\lambda(x)u + (M(x)Du, Du) + h(x) \\
u \in H^1_0(\Omega) \cup L^\infty(\Omega)
\end{array} \right. \]  

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded domain with boundary $\partial \Omega$ of class $C^{1,D}$ and $c, h \in L^p(\Omega)$ for some $p > n$, with $c^+$ and $c^-$ nonnegative functions such that $c_\lambda(x) := \lambda c^+(x) - c^-(x)$ for a parameter $\lambda \in \mathbb{R}$. Furthermore, $A(x)$ is a uniformly positive bounded measurable matrix, i.e. $\theta I_n \leq (a_{ij}(x)) \leq \theta^{-1} I_n$, $\theta$ is a positive constant, and $I_n$ is the identity matrix; and that $M(x)$ is an positive matrix such that

\[ 0 < \mu_1 I_n \leq M(x) \leq \mu_2 I_n \quad \text{in } \Omega, \]  

for some positive constants $\mu_1$ and $\mu_2$.

Following the ideas in [6], we assume the additional assumption

\[ \left\{ \begin{array}{l}
|\Omega_{c^+}| > 0, \quad \text{where } \Omega_{c^+} := \text{supp}(c^+) \\
\text{There exists } \epsilon > 0 \text{ such that } c^- = 0 \quad \text{in } \{x \in \Omega : d(x, \Omega_{c^+}) < \epsilon\}.
\end{array} \right. \]  

(A)
This hypothesis means that we are dealing with the “hard” noncoercive case, when the zero order coefficient is not negative and the uniqueness of solution is expected to fail. For a definition of $\text{supp}(f)$ when $f \in L^p(\Omega)$ for some $p \geq 1$, we refer [3, Proposition 4.17].

Depending on the parameter $\lambda \in \mathbb{R}$ we study the existence and multiplicity of solutions of $(P_\lambda)$. We recall that $u$ is a weak Sobolev (super, sub) solution of $(P_\lambda)$, if $u$ satisfies

$$
\int_\Omega A(x) Du D\varphi (\geq, \leq) = \int_\Omega c_\lambda(x) u \varphi + \int_\Omega \varphi (M(x) Du, Du) + \int_\Omega h(x) \varphi,
$$

for each nonnegative $\varphi \in C_0^\infty(\Omega)$. Such definitions are going to be essential for our arguments throughout this paper.

The class of problems $(P_\lambda)$ is more challenging and delicate to study due to the quadratic dependence in the gradient, which gives to the gradient term the same order as the Laplacian, with respect to dilations. We refer to [5, 6, 11] for a review of the large literature on this topic. The study of the coercive case, i.e. $c \leq 0$, was initiated by Boccardo, Murat and Puel in the 80’s, and the uniqueness of solution for this case was proved in [2]. On the other hand, the noncoercive case have remained unexplored until very recently. We refer a particular case considered by Jeanjean and Sirakov, where they studied a problem directly connected to $(P_\lambda)$, see [9].

In order to state our main results, we will denote by $\gamma_1 > 0$ the first eigenvalue of the linear problem, which in our case means that the problem

$$
\begin{cases}
-\text{div}(A(x) D \varphi) = c_\gamma_1(x) \varphi & \text{in } \Omega \\
\varphi > 0 & \text{in } \Omega \\
\varphi = 0 & \text{on } \partial\Omega
\end{cases}
$$

has a solution. In this case when $h(x) \not\geq 0$ the problem $(P_\lambda)$ does not have a solution $u$ with $c^+(x) u \not\geq 0$ when $\lambda = \gamma_1$, neither nonnegative solutions when $\lambda \geq \gamma_1$, see [1, Lemma 6.1] for more details.

In this paper we aim to contribuit to the literature providing a description of the set of solutions to $(P_\lambda)$. In what follows a continuum means a closed and connected set and the above assumptions on the coefficients of the equation are assumed to hold. More precisely, defining

$$
\Sigma := \{(\lambda, u) \in \mathbb{R} \times C(\Omega) : u \text{ solves } (P_\lambda)\},
$$

we will show that it is possible to obtain a description of the set $\Sigma$. Following the strategy developed in [6], as expressed in the next two theorems, we show the existence of a continuum of solutions of $(P_\lambda)$, for the case that the coercive problem $(P_0)$ with $\lambda = 0$ has a solution. The suitable conditions on the coefficients which ensure the existence of such a solution can be found for instance in [1, 6].

**Theorem 1.1.** Suppose that $(P_0)$ has a solution $u_0$ with $c^+(x) u_0 \not\geq 0$. Then

(i) For all $\lambda \leq 0$, $(P_\lambda)$ has a unique solution $u_\lambda$ which satisfies $u_0 - \|u_0\|_\infty \leq u_\lambda \leq u_0$;

(ii) There exists a continuum $\mathcal{C} \subset \Sigma$ such that the projection of $\mathcal{C}$ on the $\lambda$-axis is an unbounded interval $(-\infty, \overline{X})$ for some $\overline{X} \in (0, +\infty)$ and $\mathcal{C}$ bifurcates from infinity to the right of the axis $\lambda = 0$;
Illustration of Theorem 1.1

(iii) There exists \( \lambda_0 \in (0, \bar{\lambda}] \) such that, for all \( \lambda \in (0, \lambda_0) \), \((P_\lambda)\) has at least two solutions with \( u_i \geq u_0 \) for \( i = 1, 2 \).

**Theorem 1.2.** Suppose that \((P_0)\) has a solution \( u_0 \leq 0 \) with \( c^+(x)u_0 \nless 0 \). Then

(i) For \( \lambda \leq 0 \), \((P_\lambda)\) has a unique nonpositive solution \( u_\lambda \) and this solution satisfies \( u_\lambda + \|u_0\|_\infty \geq u_\lambda \geq u_0 \);

(ii) There exists a continuum \( C \subset \Sigma \) such that its projection of \( C^+ \) on the \( \lambda \)-axis is \([0, +\infty)\);

(iii) For \( \lambda > 0 \), every non-positive solution of \((P_\lambda)\) satisfies \( u_\lambda \nless u_0 \). Furthermore, if \( 0 < \lambda_1 < \lambda_2 \) we have \( u_{\lambda_2,1} \leq u_{\lambda_1,2} \).

Note that Theorems 1.1 and 1.2 require \((P_0)\) to have a solution, thus we are in a situation such that a branch of solutions starts from \((0, u_0)\). Our next results consider the alternative situation when there exists a supersolution of \((P_\lambda)\) for some \( \lambda_0 > 0 \).

**Theorem 1.3.** Assume that \((P_0)\) does not have a solution \( u_0 \leq 0 \) and that there exist \( \lambda_0 > 0 \) and \( \beta_0 \leq 0 \) a supersolution of \((P_\lambda)\). Then there exists \( 0 < \underline{\lambda} \leq \lambda_0 \) such that

(i) for every \( \lambda \in (\underline{\lambda}, \infty) \), \((P_\lambda)\) has at least two solutions with \( u_{\lambda,1} \leq 0 \) and \( u_{\lambda,2} \leq u_{\lambda,1} \). Moreover, if \( \lambda_1 < \lambda_2 \), we have \( u_{\lambda_1,1} \gg u_{\lambda_2,1} \);

(ii) \((P_\lambda)\) has a unique solution \( u_{\lambda} \leq 0 \);

(iii) for \( \lambda < \bar{\lambda} \), \((P_\lambda)\) has no solution \( u \leq 0 \).

Furthermore, for every \( \lambda < 0 \) problem \((P_\lambda)\) has at most one nonpositive solution \( u_\lambda \), there exists an unbounded continuum \( C \subset \Sigma \) and \( \lambda = 0 \) is a bifurcation point from infinity.

In the proof of Theorem 1.2 we define the auxiliary problem \((P_{\lambda,k})\), whose solutions are supersolutions of \((P_\lambda)\). In particular, from Theorem 1.1 and Lemma 4.3 we are able to deduce the following corollary, which concerns to the case \( h \nless 0 \), and in which we can see the achievement of two of the above theorems simultaneously.
Illustration of Theorem 1.2

Illustration of Theorem 1.3

Corollary 1.4. Assume that $h \leq 0$. For all $\tilde{\lambda} > \gamma_1$ where $\gamma_1 > 0$ is the first eigenvalue $(P_{\gamma_1})$, there exists $\tilde{k} > 0$ such that, for all $k \in (0, \tilde{k}]$,

(i) there exists $\lambda_1 \in (0, \gamma_1)$ such that
   
   (a) for all $\lambda \in (0, \lambda_1)$, $(P_{\lambda,k})$ has at least two positive solutions;
   (b) for $\lambda = \lambda_1$, $(P_{\lambda,k})$ has exactly one positive solution;
   (c) for $\lambda > \lambda_1$, $(P_{\lambda,k})$ has no non-negative solution;

(ii) for $\lambda = \gamma_1$ $(P_{\gamma_1,k})$ has no solution;

(iii) there exists $\lambda_2 \in (\gamma_1, \tilde{\lambda}]$ such that

(a) for $\lambda > \lambda_2$, $(P_{\lambda,k})$ has at least two solutions with $u_{\lambda,1} \ll 0$ and $\min u_{\lambda,2} < 0$;
(b) for $\lambda = \lambda_2$, $(P_{\lambda,k})$ has a unique non-positive solution;
(c) $\lambda < \lambda_2$, $(P_{\lambda,k})$ has no non-positive solution.

On the particular, but important case $h(x) \equiv 0$ we have the following result. Further considerations about the cases when $h(x)$ has a sign are given in Section 4.

Theorem 1.5. Assume that $h(x) \equiv 0$ and $\gamma_1 > 0$ is the first eigenvalue of $(P_{\gamma_1})$. Then

(i) for all $\lambda \in (0, \gamma_1)$, the problem

\[- \text{div}(A(x)Du) = c_\lambda(x)u + (M(x)Du, Du) \quad (P_{h \equiv 0})\]

has at least two solutions $u_{\lambda,1} \equiv 0$ and $u_{\lambda,2} \geq 0$;

(ii) for $\lambda = \gamma_1$, $(P_{h \equiv 0})$ has only the trivial solution;

(iii) for $\lambda > \gamma_1$, $(P_{h \equiv 0})$ has at least two solutions $u_{\lambda,1} \equiv 0$ and $u_{\lambda,2} \leq 0$;

(iv) for all $\lambda \leq 0$, $(P_{h \equiv 0})$ has a unique solution $u_{\lambda} \equiv 0$;

(v) There exists a continuum $C \subset \Sigma$ that bifurcates from infinity to the right of the axis $\lambda = 0$ and whose projection on the $\lambda$-axis is an unbounded interval $(0, +\infty)$. 

This paper is organized as follows. Section 2 presents auxiliary results, which are fundamental for the construction of our arguments. In Section 3 we derive a priori bounds for the solutions of problem $(P_\lambda)$. Finally, Section 4 we prove our main results.

2. Auxiliary Results

The Strong Maximum Principle is extremely important in our approach. As stated below, it guarantees that a nonnegative supersolution of an elliptic equation in a domain cannot vanish inside the domain, unless it vanishes identically.

**Theorem 2.1 (Strong Maximum Principle - SMP).** Let $\Omega \subset \mathbb{R}^n$ be a domain. If $u$ satisfies

\[
\begin{cases}
- \text{div}(A(x)Du) - \mu_1|Du|^2 - c_\lambda(x)u - h(x) & \geq 0 \quad \text{in } \Omega \\
u & \geq 0 \quad \text{in } \Omega 
\end{cases}
\]

then either $u > 0$ in $\Omega$ or $u \equiv 0$ in $\Omega$.

The SMP is an immediate consequence of the well known Interior Weak Harnack Inequality - IWHI, for more details we refer to [6] and also [7, Theorem 3.5, Theorem 8.18].

The next theorem is a generalization of the IWHI up to the boundary. Such a result is the core of our arguments in order to describe the solutions of the class of problems $(P_\lambda)$. Its proof can be found in [12, Theorem 4.7], see also [13, Theorem 1.1] for a more general version.

**Theorem 2.2 (Boundary Weak Harnack Inequality, BWHI).** Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded domain with boundary $\partial \Omega$ of class $C^{1,\alpha}$, $\|a_{ij}\|_{C^0(\Omega)} \leq \vartheta^{-1}$, $i,j = 1,\ldots,n$ and the coefficients $b,c \in L^p(\Omega)$ for some $p > n$. Then, there exist constants $\varepsilon = \varepsilon(n,p,\vartheta) > 0$ and $C = C(n,p,\vartheta) > 0$ depending also on $R > 0$ and the $C^{1,\alpha}$-representation of the boundary, such that for each nonnegative solution of

\[
- \text{div}(A(x)Du) + b(x)Du + c(x)u \geq f \quad \text{in } \Omega,
\]

it follows that

\[
\inf_{B'_R} \frac{u}{d} \geq C \left( \int_{B'_R} \left( \frac{u}{d} \right)^\varepsilon \right)^{1/\varepsilon} - C\|f\|_{L^p(B'_{2R})}.
\]
Before stating the next auxiliary result, we denote by $C(\Omega)$ the real Banach space of continuous functions defined over $\Omega$ and let $T: \mathbb{R} \times C(\Omega) \to C(\Omega)$ be a completely continuous map, i.e. it is continuous and maps bounded sets to relatively compact sets. For the purposes of this paper, we consider the problem

$$u \in C(\Omega); \quad \Phi(\lambda, u) := u - T(\lambda, u) = 0, \quad (2.1)$$

of finding the zeroes of $\Phi(\lambda, u) := u - T(\lambda, u)$, for each fixed $\lambda \in \mathbb{R}$. Let $\lambda_0 \in \mathbb{R}$ be arbitrary but fixed and assume that $u_{\lambda_0}$ is an isolated solution of $\Phi(\lambda_0, u)$, then the degree $\deg(\Phi(\lambda_0, .), B(u_{\lambda_0}, r), 0)$ is well defined and it is constant for $r > 0$ small enough. Thus, it is possible to define the index

$$i(\Phi(\lambda_0, .), u_{\lambda_0}) := \lim_{r \to 0} \deg(\Phi(\lambda_0, .), B(u_{\lambda_0}, r), 0).$$

Now we are able to enunciate the following theorem, which was proved in [1, Theorem 2.2].

**Theorem 2.3.** If $(2.1)$ has a unique solution $u_{\lambda_0}$, and $i(\Phi(\lambda_0, .), u_{\lambda_0}) \neq 0$ then $\Sigma$ possesses two unbounded components $C^+, C^-$ in $[\lambda_0, +\infty] \times C(\Omega)$ and $[-\infty, \lambda_0] \times C(\Omega)$ respectively which meet at $(\lambda_0, u_{\lambda_0})$.

We recall that a strict subsolution of $(P_\lambda)$ is a subsolution $\alpha$ such that for every solution $u$ of $(P_\lambda)$ satisfying $\alpha \leq u$ it implies that $\alpha \ll u$. As well as a strict supersolution of $(P_\lambda)$ is a supersolution $\beta$ such that $u \ll \beta$, for every solution $u$ of $(P_\lambda)$ satisfying $u \leq \beta$.

In order to consider the situation where $(P_{\lambda_0})$ has a supersolution, we need the following formulation of the anti-maximum principle. This result was established in [8], under slightly smoother data, but its proof may be directly extended under our assumptions.

**Lemma 2.4.** Let $\overline{c}, \overline{h}, \overline{d} \in L^p(\Omega)$ with $p > n$ and assume $\overline{h} \geq 0$. We denote by $\overline{\gamma}_1 > 0$ the first eigenvalue of

$$-\operatorname{div}(A(x)Du) + \overline{d}(x)u = \overline{c}\gamma_1(x)u, \quad u \in H^1_0(\Omega).$$

Then there exists $\varepsilon_0 > 0$ such that, for all $\lambda \in (\overline{\gamma}_1, \overline{\gamma}_1 + \varepsilon_0)$, the solution $v \in H^1_0(\Omega)$ of

$$-\operatorname{div}(A(x)Dv) + \overline{d}(x)v = \overline{c}\lambda(x)v + \overline{h}(x), \quad \text{satisfies } v \ll 0.$$

3. A priori Bound

This section is devoted to the derivation of some a priori bounds results for the solutions of $(P_\lambda)$. Most of our results hold true under more general assumptions than $(A)$. Firstly, we obtain the following essential upper bound on the supersolutions of $(P_\lambda)$, which shows that any unbounded continuum of solutions of $(P_\lambda)$, for $\lambda > 0$ in a bounded interval, can only bifurcate to the right of $\lambda = 0$.

**Theorem 3.1 (A priori Upper Bound).** Under the stated assumptions of problem $(P_\lambda)$, including hypothesis $(A)$, for any $\Lambda_2 > \Lambda_1 > 0$, there exists a constant $M > 0$ such that, for each $\lambda \in [\Lambda_1, \Lambda_2]$, any solution of $(P_\lambda)$ satisfies $\sup \overline{u} \leq M$.

Let us point out that if $\lambda = 0$ or $c^+ \equiv 0$ i.e. $|\Omega_{c^+}| = 0$ the problem $(P_\lambda)$ reduces to $(P_0)$ which is independent of $\lambda$, and has a solution. In fact, in [1, 4] the authors give sufficient
conditions to ensure the existence of a solution of \((P_0)\). Such a solution is unique and so, automatically we have an a priori bound for this particular case.

For the general case, in order to prove Theorem 3.1, firstly we show that an a priori bound to solution of \((P_\lambda)\) depends only on controlling the solution on \(\Omega_{c+}\). By compactness, it is equivalent to study what happens around any fixed point \(\pi \in \overline{\Omega}_{c+}\).

**Lemma 3.2.** Assume the hypotheses of \((P_\lambda)\), there exists a constant \(M > 0\) such that, for any \(\lambda \in \mathbb{R}\), any solution \(u\) of the problem \((P_\lambda)\) satisfies

\[
- \sup_{\Omega_{c+}} u^- - M \leq u \leq \sup_{\Omega_{c+}} u^+ + M.
\]

**Proof.** In case problem \((P_\lambda)\) has no solution for any \(\lambda \in \mathbb{R}\), there is nothing to prove. Hence, we assume the existence of \(\tilde{\lambda} \in \mathbb{R}\) such that \((P_\lambda)\) has a solution \(\tilde{u}\). We shall prove the result for \(M := 2\|\tilde{u}\|_{\infty}\). Let \(u\) be an arbitrary solution of \((P_\lambda)\). Setting \(D := \Omega \setminus \overline{\Omega}_{c+}\) and \(v = u - \sup_{\partial D} u^+\), we have

\[
- \text{div}(A(x)Dv) = -c^-(x)u + (M(x)Dv, Dv) + h(x) - c^-(x)\sup_{\partial D} u^+
\]

\[
\leq -c^-(x)u + (M(x)Dv, Dv) + h(x) \text{ in } D.
\]

Since \(v \leq 0\) on \(\partial D\), the function \(v\) is a subsolution of \((P_0)\). On the other hand, setting \(\tilde{v} = \tilde{u} + \|\tilde{u}\|_{\infty}\) we obtain

\[
- \text{div}(A(x)D\tilde{v}) = -c^-(x)\tilde{v} + (M(x)D\tilde{v}, D\tilde{v}) + h(x) + c^-(x)\|\tilde{u}\|_{\infty}
\]

\[
\geq -c^-(x)\tilde{v} + (M(x)D\tilde{v}, D\tilde{v}) + h(x) \text{ in } D,
\]

and thus, as \(\tilde{v} \geq 0\) on \(\partial D\), the function \(\tilde{v}\) is a supersolution of \((P_0)\). By standard regularity results, see for instance [2, Lemma 2.1], we get \(u, \tilde{u} \in H^1(\Omega) \cap W_{loc}^{1,n}(\Omega) \cap C(\overline{\Omega})\) and hence, \(v, \tilde{v} \in H^1(\Omega) \cap W_{loc}^{1,n}(\Omega) \cap C(\overline{\Omega})\) and the right-hand sides of the above inequalities are \(L^n\) functions. Therefore, we are able to apply the Comparison Principle, and conclude that \(v \leq \tilde{v}\) in \(D\), namely, \(u \leq \tilde{u} + \|\tilde{u}\|_{\infty} + \sup_{\partial D} u^+\) in \(D\). Hence \(u \leq M + \sup_{\partial D} u^+\) in \(\Omega\). For the other inequality, we now define \(v := u + \sup_{\partial D} u^+\) and hence obtain that \(v \geq 0\) on \(\partial D\), and also that \(v\) is a supersolution of \((P_0)\). Furthermore, defining \(\tilde{v} = \tilde{u} - \|\tilde{u}\|_{\infty}\), we have that \(\tilde{v} \leq 0\) on \(\partial D\) and also that \(\tilde{v}\) is a subsolution of \((P_0)\). As previously, we have that \(v, \tilde{v} \in H^1(\Omega) \cap W_{loc}^{1,n}(\Omega) \cap C(\overline{\Omega})\), and applying again the Comparison Principle we get \(\tilde{v} \leq v\) in \(D\). Namely, \(\tilde{u} - \|\tilde{u}\|_{\infty} u \leq \sup_{\partial D} u^-\) in \(D\). Therefore, it yields \(u \geq -\sup_{\partial D} u^- - M\) in \(\Omega_{c+}\), ending the proof.  

Now, let \(u \in H^1_0(\Omega) \cap L^\infty(\Omega)\) be a solution of \((P_\lambda)\). We introduce the exponential change of variable

\[
w_i(x) := \frac{1}{\nu_i} (e^{\nu_i u(x)} - 1) \quad \text{and} \quad g_i(x) := \frac{1}{\nu_i} \ln(1 + \nu_i s), \quad i = 1, 2 \quad (3.1)
\]

where \(\nu_1 := \mu_1 \vartheta\) and \(\nu_2 := \mu_2 \vartheta^{-1}\) for \(\mu_1, \mu_2\) given in (1.1) and \(\vartheta\) given in the definition of the matrix \(A(x)\). The following change of variables lemma follows straightway from an algebraic computation and it is going to be useful for proving our results.
Lemma 3.3 (Exponential change). Let $u$ be a weak solution of problem

$$-\text{div}(A(x)Du) = f(x), \quad f \in L^p(\Omega).$$

For $m > 0$ we define $v := \frac{e^{mu} - 1}{m}$ and $w := \frac{1 - e^{-mu}}{m}$. Then $Dv = (1 + mw)Du$, $Dw = (1 - mw)Du$, and for each $\vartheta > 0$ we have,

$$-\text{div}(A(x)Du) - \vartheta^{-1}m|Du|^2 \leq -\text{div}(A(x)Du) - \vartheta m|Du|^2,$$

$$-\text{div}(A(x)Du) + \vartheta m|Du|^2 \leq -\text{div}(A(x)Du) - \vartheta^{-1}m|Du|^2,$$

and $\{u = 0\} = \{v = 0\}$ and $\{u > 0\} = \{v > 0\}$. Therefore if $u$ is a weak supersolution of

$$-\text{div}(A(x)Du) \geq \mu_1|Du|^2 + c_\lambda(x)u + h(x), \quad (3.2)$$

and for $m = \mu_1\vartheta$, $v$ is a weak supersolution of

$$-\text{div}(A(x)Dv) \geq h(x)(1 + mv) + \frac{c_\lambda(x)}{m}(1 + mv)\ln(1 + mv).$$

By Lemma 3.3 we have,

$$-\text{div}(A(x)Dw_i) = (1 + \nu_iw_i) \left[ c_\lambda(x)g_i(w_i) + h(x) + (|M(x) - \nu_iA(x)|Du, Du) \right]. \quad (3.3)$$

Note that the last term is negative for $i = 1$ and positive for $i = 2$. Using (3.3) we shall obtain a uniform a priori upper bound on $u$ in a neighborhood of any fixed point $\bar{x} \in \overline{\Omega}_{r+}$. We consider the two cases $\bar{x} \in \overline{\Omega}_{r+} \cap \Omega$ and $\bar{x} \in \overline{\Omega}_{r+} \cap \partial\Omega$ separately.

Lemma 3.4. Assume that (A) holds and that $\bar{x} \in \overline{\Omega}_{r+} \cap \Omega$. For each $\Lambda_2 > \Lambda_1 > 0$, there exist $M_1 > 0$ and $R > 0$ such that, for any $\lambda \in [\Lambda_1, \Lambda_2]$, any solution $u$ of $(P_\lambda)$ satisfies

$$\sup_{B_R(\bar{x})} u \leq M_1.$$

Proof. Under the assumption (A) we can find a $R > 0$ such that $M(x) \geq \mu_1I_n > 0$, $c^- \equiv 0$ in $B_{4R}(\bar{x})$ and $c^+ \equiv 0$ in $B_R(\bar{x})$. Observe that from (3.3) for $i = 1$, we get

$$-\text{div}(A(x)Dw_1) \geq (1 + \nu_1w_1)[|c^+(x)g_1(w_1) + h^+(x)| - h^-(x) - \nu_1h^-(x)w_1]
+ (1 + \nu_1w_1)(\mu_1 - \vartheta^{-1}\lambda)\nu_1||Du||^2.$$

Therefore, in $B_{4R}(\bar{x})$ it yields,

$$-\text{div}(A(x)Dw_1) + \nu_1h^-(x)w_1 \geq (1 + \nu_1w_1)[|c^+(x)g_1(w_1) + h^+(x)| - h^-(x)]. \quad (3.4)$$

Let $z_0$ be the solution of

$$-\text{div}(A(x)Dz_0) + \nu_1h^-(x)z_0 = -\Lambda_2c^+(x)\nu_1^{-1}, \quad z_0 \in H_0^1(B_{4R}(\bar{x})). \quad (3.5)$$

By classical regularity, see [10, Theorem III-14.1], $z_0 \in C(B_{4R}(\bar{x}))$ and there exists a positive constant $\overline{C} = \overline{C}(\bar{x}, \nu_1, \Lambda_2, p, R, ||h^-||_{L^p(B_{4R})}, ||c^+||_{L^p(B_{4R})})$ such that $z_0 \geq -\overline{C}$ in $B_{4R}$. Further, by the Weak Maximum Principle we know that $z_0 \leq 0$. Since

$$\min_{(-\frac{\pi}{2}, \infty)} (1 + \nu_is)g_i(s) = -\nu_1^{-1}, \quad \text{setting} \quad v_1 := w_1 - z_0 + \frac{1}{\nu_1} \quad \text{it satisfies,}$$
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\[-\text{div}(A(x) Dv_1) + \nu_1 h^-(x) v_1 \geq (1 + \nu_1 w_1) [\lambda c^+(x) g_1(w_1) + h^+(x)] + \Lambda_2 c^+(x) \frac{e^{-1}}{\nu_1} \]
\[\geq (1 + \nu_1 w_1) (\Lambda_2 - \Lambda_2) [c^+(x) g_1^-(w_1)] + (1 + \nu_1 w_1 \Lambda_1 c^+(x) g_1^-(w_1)] \]
\[\geq \frac{\Lambda_1 c^+(x)}{\nu_1} (1 + \nu_1 w_1) \ln(1 + \nu_1 w_1) \]
\[= f(x, v_1), \text{ in } B_{4R}(\overline{\Omega}), \]

(3.6)

where

\[f: \Omega \times \mathbb{R} \rightarrow \mathbb{R} (x, s) \rightarrow \text{ in } B_{4R}(\overline{\Omega}), \]

is a superlinear function in the variable \(s\). Since \(w_1 > -1/\nu_1\) we have \(v_1 > 0\) in \(B_{4R}(\overline{\Omega})\).

On the other hand, for \(i = 2\), in view of (1.1) and \(w_2 > -1/\nu_2\), by (3.3) in a similar way we conclude that \(w_2\) satisfies

\[-\text{div}(A(x) Dv_2) + \nu_1 h^-(x) w_2 \leq (1 + \nu_2 w_2) [\lambda c^+(x) g_2(w_2) + h^+(x)] + \Lambda_2 c^+(x) \frac{e^{-1}}{\nu_2} \]
\[\leq (1 + \nu_2 w_2) \left( \Lambda_2 c^+(x) \frac{e^{-1}}{\nu_2} (1 + \nu_2 w_2) + h^+(x) \right) \]
\[=: g(x, w_2), \text{ in } B_{4R}(\overline{\Omega}), \]

(3.7)

where \(g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) satisfies

\[g(x, s) \leq a_0 [1 + (\nu_2 s)^{\alpha + 1}], \text{ for each } \alpha > 0, \text{ where } a_0(x) \in L^p(\Omega). \]

(3.8)

In fact, in order to obtain (3.8), let \(c_\alpha > 0\) be a constant such that

\[\ln(1 + x) \leq (1 + x)^\alpha + c_\alpha, \text{ for all } x \geq 0.\]

Then,

\[g(x, w_2) \leq \left(1 + \nu_2 w_2\right) \left( \Lambda_2 c^+(x) \frac{e^{-1}}{\nu_2} (1 + \nu_2 w_2)^\alpha + \Lambda_2 c^+(x) + h^+(x) \right) \]
\[\leq \left(1 + \nu_2 w_2\right)^{\alpha + 1} \left( \Lambda_2 c^+(x) (1 + c_\alpha) + h^+(x) \right) \leq \left(1 + (\nu_2 w_2)^{\alpha + 1}\right)a_0(x).\]

In addition, we note that \(\left(1 + \nu_2 w_2\right)^{\frac{\alpha}{\beta}} = (e^{\nu_2 u})^{\frac{\alpha}{\beta}} = (e^{\nu_2 u}) = 1 + \nu_1 w_1 = \nu_1 \nu_1 + z_0\),

which means that \(w_2 = \xi(\nu_1 + z_0)\), where \(\xi(\nu_1 + z_0) = [\nu_1 e^{\nu_2 u} - 1]^{\nu_1} \) is an increasing function satisfying

\[\lim_{s \rightarrow \infty} \frac{\xi(s)}{s^\beta} = \lim_{s \rightarrow \infty} \left( \frac{\nu_1}{\nu_2 s^{\nu_2/\nu_1}} - 1 \right) = \lim_{s \rightarrow \infty} \frac{\nu_1^{\nu_2/\nu_1} - s^{\nu_2/\nu_1}}{\nu_2} = \frac{\nu_1^{\nu_2/\nu_1}}{\nu_2} < \infty, \text{ for } \beta = \nu_2/\nu_1.\]

(3.9)

Thus, we are in position to apply the following theorem, which under our assumptions is a straightforward generalization of [14, Theorem 2]. In fact, as a consequence of Theorem 2.2 the Theorems 3-6 stated in [14] are valid under our assumptions on the domain and on the coefficients of \((P_\lambda)\). Hence, it remains to observe that the other generalizations on the hypotheses of Theorem 5.3, in comparison with [14, Theorem 2], are natural, in view of [14, Remark 4]. As a matter of completeness, we state here our adapted version.

**Theorem 3.5.** Let \(\Omega \subset \mathbb{R}^n, n \geq 2\) be a bounded domain with boundary \(\partial \Omega\) satisfying the interior \(C^1, D\)-paraboloid condition and \((P_\lambda)\) be a uniformly elliptic operator under our
First observe that both \( (A) \) including hypothesis Theorem 3.7 (A priori lower bound) \( \lambda \) details.
by applying a topological approach relying on the derivation of a priori bounds, see [6] for proof follows the same lines as the proof of the Interior Weak Harnack inequality-IWHI,
quantities such that
\[ 1 \] into (3.4) in \( \Omega \) assumptions allow us to find \( \Omega \)
This proof is very similar to the previous one, we only need to observe that our Proof.
In view of the Lemmas 3.4 and 3.6 we have the existence of a uniform Proof of Theorem 3.1.
Assume that \( u \) a priori upper bound on \( \Lambda \) holds for \( u \).
Lemma 3.6.
Lemma 3.4 we deduce (3.6),(3.7) and then we are able to apply Theorem 3.5 getting an of \( H \) depend on the concerned quantities we have
\begin{align*}
x(v(x) + z_0) & \leq C d(x) \text{ in } \Omega \text{ and hence } v(x) \leq C.
\end{align*}
In view of (3.6) and (3.7) we are able to apply Theorem 3.5 for \( u \equiv 0 \text{ in } \Omega \), as in Lemma 3.4, we get
\begin{align*}
& r < \frac{n+1}{n-1} + \left( \frac{1}{\beta} - 1 \right) \frac{2}{n-1}.
\end{align*}
Then for some \( C \) depending on the concerned quantities we have
\begin{align*}
x(v(x) + z_0) & \leq C d(x) \text{ in } \Omega \text{ and hence } v(x) \leq C.
\end{align*}
Lemma 3.6. Assume that (A) holds and that \( \pi \in \overline{\Omega}_{c^+} \cap \partial \Omega \). For each \( \Lambda_2 > \Lambda_1 > 0 \), there exist \( R > 0 \) and \( M_2 > 0 \) such that, for any \( \lambda \in [\Lambda_1, \Lambda_2] \), any solution \( u \) of (\( P_\lambda \)) satisfies
\begin{align*}
\sup_{B_R(\pi) \cap \Omega} u \leq M_2.
\end{align*}
Proof. This proof is very similar to the previous one, we only need to observe that our assumptions allow us to find \( \Omega_1 \subset \Omega \) with \( \partial \Omega_1 \) of class \( C^{1,\beta} \) such that \( B_{2R}(\pi) \cap \Omega \subset \Omega_1 \) and \( M(x) \geq \mu_1 I_n > 0 \) \( c^-(x) \equiv 0 \) and \( c^+(x) \geq 0 \) in \( \Omega_1 \). Hence, for \( i = 1 \) note that (3.3) turn into (3.4) in \( \Omega_1 \) instead of \( B_{4R}(\pi) \). Then, if \( z_0 \) is the solution of (3.5) in \( H_0^1(\Omega_1) \) instead of \( H_0^1(B_{4R}(\pi)) \), as in Lemma 3.4, we get \( z_0 \in C(\overline{\Omega}_1) \) and \( c^+ > 0 \) depending on the usual quantities such that \( -c^+ \leq z_0 \leq 0 \) in \( \Omega_1 \). In addition, defining \( v_1 \) as before, we observe that \( v_1 \) satisfies the equation (3.6) in \( \Omega_1 \) and \( v_1 > 0 \) on \( \overline{\Omega}_1 \). Therefore, arguing exactly as Lemma 3.4 we deduce (3.6),(3.7) and then we are able to apply Theorem 3.5 getting an upper bound to \( u \) in \( \Omega_1 \).
Proof of Theorem 3.1. In view of the Lemmas 3.4 and 3.6 we have the existence of a uniform a priori upper bound on \( u \) in a neighborhood of any fixed point \( \pi \in \overline{\Omega}_{c^+} \). Then this proof follows the same lines as the proof of the Interior Weak Harnack inequality-IWHI, by applying a topological approach relying on the derivation of a priori bounds, see [6] for details.
We will now see that the solutions of problem (\( P_\lambda \)) are bounded from below, even when \( \lambda \rightarrow 0 \), \( \lambda > 0 \).

Theorem 3.7 (A priori lower bound). Under the standing assumptions on problem (\( P_\lambda \)), including hypothesis (A), let \( \Lambda_2 > 0 \). Then every supersolution \( u \) of (\( P_\lambda \)) satisfies
\begin{align*}
\|u^-\|_{L^\infty} & \leq C \text{ for all } \lambda \in [0, \Lambda_2] \text{, } \text{where } C = C(n, p, \nu_1, \Omega, \Lambda_2, \|c\|_{L^p(\Omega)}, \|h^-\|_{L^p(\Omega)}).
\end{align*}
Proof. First observe that both \( U_1 = -u \) and \( U_2 = 0 \) are subsolutions of
\begin{align*}
-\text{div}(A(x)DU) & \leq c_\lambda U - (M(x)DU, DU) + h^-(x) \text{ in } \Omega.
\end{align*}
Then, these functions are also subsolutions of
\[
\begin{cases}
- \text{div}(A(x)DU) + \mu_1|Du|^2 \leq c_\lambda U + h^{-}(x) & \text{in } \Omega \\
U \leq 0 & \text{on } \partial \Omega
\end{cases}
\]
and so is \( U := u^- = \max\{U_1, U_2\} \), as the maximum of subsolutions. Moreover \( U \geq 0 \) in \( \Omega \) and \( U = 0 \) on \( \partial \Omega \). We make the following exponential change of variables \( w := \frac{1 - e^{-\nu_1 U}}{\nu_1} \).

From Lemma 3.3,
\[- \text{div}(A(x)Dw) \leq (1 - \nu_1 w) \left[ c_\lambda(x)U + h^{-}(x) \right],\]
hence, we know that \( w \) is a weak solution of
\[
\begin{cases}
- \text{div}(A(x)Dw) + \nu_1 h^{-}(x)w \leq h^{-}(x) + \frac{c_\lambda(x)}{\nu_1} \ln(1 - \nu_1 w)(1 - \nu_1 w) & \text{in } \Omega \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(Q\(\lambda\))

Now set \( w_1 := \frac{1 - e^{-\nu_1 U}}{\nu_1} \), where \( u_1 \) is some fixed supersolution of \((P\(\lambda\)), \lambda \geq 0\), Note that if there was not such supersolution, we had nothing to prove. Then, from the above conclusions, \( w_1 \in [0, 1/\nu_1) \) is a solution of \((Q\(\lambda\))\).

Define
\[
\overline{w} := \sup A, \quad \text{where } A := \{ w : w \text{ is a solution of } (Q\(\lambda\)) ; 0 \leq w < 1/\nu_1 \text{ in } \Omega \}.
\]

Observe that \( A \neq \emptyset \), since \( w_1 \in A \), and also that \( w_1 \leq \overline{w} \leq 1/\nu_1 \) in \( \Omega \). Further, as a supremum of subsolutions, \( \overline{w} \) is a weak solution of \((Q\(\lambda\))\), and also that \( \overline{w} = 0 \) on \( \partial \Omega \). Then,
\[
f(x) := h^{-}(x) + \frac{c_\lambda(x)}{\nu_1} \ln(1 - \nu_1 \overline{w})(1 - \nu_1 \overline{w}) \in L_+^p(\Omega)
\]
where
\[
\|f^+\|_{L^p(\Omega)} \leq \|h^{-}\|_{L^p(\Omega)} + \frac{1}{\nu_1} \left( A_2\|c^+\|_{L^p(\Omega)} + \|c^-\|_{L^p(\Omega)} \right) C_0,
\]
since \( A(\overline{w}) := \|\ln(1 - \nu_1 \overline{w})\|_{L^p(\Omega)} \leq C_0 \). Therefore, by the Boundary Lipschitz bound,
\[
\overline{w} \leq C \delta^{1-n/p} \|f^+\|_{L^p(\Omega)} d(x) \rightarrow 0 \text{ as } x \rightarrow \partial \Omega,
\]
and so \( \overline{w} \neq 1/\nu_1 \). Note also that the function \( \overline{w} \) may be equal to \( 1/\nu_1 \) at some interior points. In order to complete the proof, we argue by contradiction. Assume that there is a sequence of supersolutions \( u_k \) of \((P\(\lambda\))\) in \( \Omega \) with unbounded negative parts, then there would exist a subsequence such that
\[
u_k^{-1}(x) = \|u_k\|_{L^\infty} \rightarrow +\infty, \quad x_k \in \overline{\Omega}, \quad x_k \rightarrow x_0 \in \overline{\Omega} \text{ as } k \rightarrow \infty,
\]
with \( x_k \in \Omega \) for large \( k \), since \( u_k \geq 0 \) on \( \partial \Omega \). Then, the respective sequence \( (w_k(x_k)) \) satisfies
\[
w_k(x_k) = \frac{1 - e^{-\nu_1 \overline{w}(x_k)}}{\nu_1} \rightarrow \frac{1}{\nu_1}, \quad w_k \in A.
\]
Hence, for every \( \epsilon > 0 \), there exists some \( k_0 \in \mathbb{N} \) such that
\[
\frac{1}{\nu_1} - \epsilon \leq w_k(x_k) \leq \overline{w}(x_k) \leq \frac{1}{\nu_1}, \quad \text{for all } k \geq k_0.
\]
Thus, \( \overline{w}(x_0) \geq \lim_{k \to \infty} \overline{w}(x_k) = \lim_{k \to \infty} \overline{w}(x_k) = 1/\nu_1 \), and \( x_0 \in \Omega \), since \( \overline{w} = 0 \) on \( \partial \Omega \).
Moreover, \( \overline{w}(x_0) = 1/\nu_1 \). Finally, set \( z := 1 - \nu_1 \overline{w} \) and observe that
\[
\text{div}(A(x)Dz) = -\nu_1 \text{div}(A(x)D\overline{w}) \leq \nu_1 (1 - \nu_1 \overline{w}) \left[ \frac{c_\lambda(x)}{\nu_1} \left| \ln(1 - \nu_1 \overline{w}) \right| + h^-(x) \right] = c_\lambda(x) |\ln z| z + \nu_1 h^-(x)z.
\]

Then \( z \) is a supersolution of
\[
\begin{cases}
-\text{div}(A(x)Dz) + \nu_1 h^-(x)z & \geq -c_\lambda(x) |\ln z| z \quad \text{in } \Omega \\
z(x_0) = 0 \quad \text{and } z & \geq 0 \quad \text{in } \Omega.
\end{cases}
\]

But this contradicts the nonlinear version of the SMP, see for instance [11, Lemma 5.3], and its extension in [15], which says that \( z \equiv 0 \) or \( z > 0 \) in \( \Omega \).

\( \square \)

4. Main Results

This section is devoted to the proof of our main results. We start by proving a lemma, which is going be useful in order to deal with degree arguments.

**Lemma 4.1.** Under assumption (A) for every \( \lambda > 0 \), there exists a strict subsolution \( v_\lambda \) of \( (P_\lambda) \) such that, every supersolution \( \beta \) of \( (P_\lambda) \) satisfies \( v_\lambda \leq \beta \).

**Proof.** Let \( C > 0 \) be given by Theorem 3.7 and \( M \) be given by Theorem 3.1 such that for every supersolution \( \beta \) of
\[
\begin{cases}
-\text{div}(A(x)Du) = c_\lambda(x)u + (M(x)Du, Du) - h^-(x) - 1 & \text{in } \Omega \\
u & = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]
we have \( \beta \geq -C \). Let \( k > C \) and consider \( \alpha_k \) the solution of
\[
\begin{cases}
-\text{div}(A(x)Dv) + c^-(x)v & = -\lambda c^+(x) - h^-(x) - 1 & \text{in } \Omega \\
v & = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

As \( -\lambda c^+(x) - h^-(x) - 1 < 0 \) we have \( \alpha_k \ll 0 \) by the SMP and the Hopf lemma.

We claim that every supersolution \( \beta \) of \( (P_\lambda) \) satisfies \( \beta \geq \alpha_k \). In fact, taking regular supersolutions \( \beta_1, \ldots, \beta_l \) of \( (P_\lambda) \) such that \( \beta = \min \{ \beta_j : 1 \leq j \leq l \} \) and setting \( w = \beta_j - \alpha_k \) for some \( 1 \leq k \leq l \) we have
\[
\begin{cases}
-\text{div}(A(x)Dw) + c^-(x)w & \geq \lambda c^+(x)(\beta_j + k) + \mu_1 |D\beta_j|^2 \geq 0 & \text{in } \Omega \\
w & = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

Hence, by the maximum principle \( w \geq 0 \) i.e. \( \beta_j \geq \alpha_k \) and this proves the claim.

Now, consider the problem
\[
-\text{div}(A(x)Dv) = c_\lambda(x)T_k(v) + (M(x)Dv, Dv) - h^-(x) - 1, \quad \text{(4.1)}
\]
where \( T_k(v) = \begin{cases} -k, & \text{if } v \leq -k, \\ v, & \text{if } v > -k. \end{cases} \)
Furthermore, we claim that, in fact, \( v \) is a supersolution of (4.1) and \( \alpha_k \) is a subsolution of (4.1). Note that \( -\lambda c^+(x) = \lambda c^+(x)T_k(\alpha_k) \), \( c^-(x)k = -c^-(x)T_k(\alpha_k) \) and hence, by the standard method of sub and supersolutions, (4.1) has a minimal solution \( v_k \) with \( \alpha_k \leq v_k \leq \beta \). Furthermore, we also note that every supersolution \( \tilde{\beta} \) of (4.1) has a minimal solution \( v \) such that \( c \leq \tilde{\beta} \), \( \beta \geq \alpha_k \) and by the construction of (4.1), every supersolution \( \tilde{\beta} \) of (4.1) is also a supersolution of (4.1), then the minimality of \( v_k \) implies that \( v_k \leq \tilde{\beta} \).

Now we observe that \( v_k \) is a subsolution of (4.1), since \( v_k \geq C > -k \) and it satisfies

\[
-\text{div}(A(x)Dv_k) = c_\lambda(x)T_k(v_k) + (M(x)Dv_k, Dv_k) - h^-(x) - 1 \\
\leq c_\lambda(x)v_k + (M(x)Dv_k, Dv_k) + h(x).
\]

Furthermore, we claim that, in fact, \( v_k \) is strict subsolution of (4.1). In order to see this, let \( u \) be a solution of (4.1) with \( u \geq v_k \). Then, \( w = u - v_k \) satisfies

\[
-\text{div}(A(x)Dw) \geq c_\lambda(x)u + (M(x)Du, Du) + h(x) - c_\lambda(x)v_k - (M(x)Dv_k, Dv_k) + h^-(x) + 1 \\
= c_\lambda(x)w + (M(x)[Du + Dv_k], Dw) + h^+(x) + 1,
\]

which means that

\[
\begin{align*}
-\text{div}(A(x)Dw) - (M(x)[Du + Dv_k], Dw) & \geq c_\lambda(x)w + h^+(x) + 1 & \text{in } \Omega \\
w & = 0 & \text{on } \partial\Omega.
\end{align*}
\]

Therefore, by the maximum principle, we deduce that \( w \gg 0 \), namely, \( \lambda \gg v_k \). \( \Box \)

By adapting [6, Lemma 5.1] to our setting, we obtain the following auxiliary result, which is going to be useful for proving Theorem 1.1.

**Lemma 4.2.** Under the assumptions of Theorem 1.1, assume that (4.1) has a solution \( u_0 \) such that \( c^+(x)u_0 \geq 0 \), \( c^-(x) \equiv 0 \). Then, there exists \( \bar{\lambda} \in (0, \infty) \) such that, for \( \lambda \geq \bar{\lambda} \), the problem (4.1) has no solution \( u \) with \( u \geq u_0 \) in \( \Omega \).

**Proof.** Let \( \varphi_1 > 0 \) the first eigenfunction of (4.1). If (4.1) has a solution \( u \) with \( u \geq u_0 \), multiplying (4.1) by \( \varphi_1 \) and integrating we obtain

\[
\int_\Omega c_\gamma(x)u\varphi_1 = \int_\Omega A(x)D\varphi_1 Du = \int_\Omega c_\lambda(x)u\varphi_1 dx + \int_\Omega (M(x)Du, \varphi_1 Du) + \int_\Omega h(x)\varphi_1 dx,
\]

and hence \( \lambda \geq \bar{\lambda} \). As \( u \geq u_0 \), we have

\[
0 \geq (\lambda - \gamma_1) \int_\Omega c^+(x)u\varphi_1 dx + \mu_1 \int_\Omega |Du|^2 dx + \int_\Omega h(x)\varphi_1 dx \\
\geq (\lambda - \gamma_1) \int_\Omega c^+(x)u_0 \varphi_1 dx + \mu_1 \int_\Omega |Du|^2 dx + \int_\Omega h(x)\varphi_1 dx,
\]

which gives a contradiction for \( \lambda \) large enough. \( \Box \)

In view of the previous results, we now are able to prove Theorem 1.1.
Proof of Theorem 1.1. Applying all previous results and adopting strategies presented in [1, 2, 6] we give the proof of Theorem 1.1 treating separately the cases \( \lambda \leq 0 \) and \( \lambda > 0 \).

Case (i): \( \lambda \leq 0 \). This case has been studied in previous works. We briefly recall the following argument. If \( (P_0) \) has a solution \( u_0 \), then \( u_0 \) is a supersolution of \( (P_\lambda) \). By applying Lemma 3.2 and the arguments found in [1], we obtain the existence of a solution \( u_\lambda \) of \( (P_\lambda) \) for any \( \lambda < 0 \). We observe that the uniqueness of solutions for \( \lambda \leq 0 \) is ensured by [1, Proposition 4.1]. On the other hand, for \( \lambda \leq 0 \), we have \( c_\lambda(x) = \lambda c^+(x) - c^-(x) \leq -c^- \) so by applying the Comparison Principle, we get \( u_\lambda \leq u_0 \). Also by Lemma 3.2, setting \( v = u_0 - \|u_0\|_\infty \), we see that \( v_0 \) a subsolution of \( (P_\lambda) \) for \( \lambda < 0 \), so again by the Comparison Principle we get \( u_0 - \|u_0\|_\infty \leq u_\lambda \).

Case (ii): \( \lambda > 0 \). With the aim of showing the existence of a continuum of solution of \( (P_\lambda) \), for \( \lambda \geq 0 \) we introduce the auxiliary problem

\[-\text{div}(A(x)Du) + u = [c_\lambda(x) + 1][(u - u_0)^+ + u_0] + (M(x)Du, Du) + h(x). \tag{P_\lambda}\]

As in the case of \( (P_\lambda) \), any solution of \( (P_\lambda) \) belongs to \( C^{0,\tau}(\Omega) \) for some \( \tau > 0 \). Moreover, observe that \( u \) is a solution of \( (P_\lambda) \) if and only if it is a fixed point of the operator \( T_\lambda : C(\Omega) \to C(\Omega) : v \to u \) with \( u \) the solution of

\[-\text{div}(A(x)Du) + u - (M(x)Du, Du) = [c_\lambda(x) + 1][(v - u_0)^+ + u_0] + h(x). \]

Applying [1, Lemma 5.2], we see that \( T_\lambda \) is completely continuous. Now, we denote \( \Sigma := \{ (\lambda, u) \in \mathbb{R} \times C(\Omega), u \text{ solves } (P_\lambda) \} \) and we split the rest of the proof into three steps.

Step 1: If \( u \) is a solution of \( (P_\lambda) \), then \( u \geq u_0 \) and hence, it is a solution of \( (P_\lambda) \). Observe that \((u - u_0)^+ + u_0 - u \geq 0 \) and \( \lambda c^+(x)[(u - u_0)^+ + u_0] \geq \lambda c^+(x)u_0 \geq 0 \). Hence, we deduce that a solution \( u \) of \( (P_\lambda) \) is a supersolution of

\[-\text{div}(A(x)Du) = [c_\lambda(x) + 1][(u - u_0)^+ + u_0] + (M(x)Du, Du) + h(x). \tag{4.2}\]

Since \( u_0 \) is a solution of \( (P_\lambda) \), it implies that \( u_0 \) solves \( (4.2) \). Thus, applying again the Comparison Principle we get \( u \geq u_0 \).

Step 2: \( u_0 \) is the unique solution to \( (P_0) \) as well as to the problem \( (P_0) \) and \( i(I - T_0, u_0) = 1 \).

For \( \lambda = 0 \), if \( u \) is a solution of \( (4.2) \), then by Step 1, \( u \geq u_0 \) and \( u \) solves \( (P_\lambda) \). From Case (i), we conclude that \( u = u_0 \). In order to prove that \( i(I - T_0, u_0) = 1 \), we consider the operator \( S_t \) defined by \( S_t : C(\Omega) \to C(\Omega) \), given by \( S_t(v) = tT_0v = u \), where \( u \) is the solution of

\[-\text{div}(A(x)Du) + u = (M(x)Du, Du) + th(x) + t([-c^- + 1][u_0 + (v - u_0)^+ - (v - u_0 - 1)^+]). \]

First, note that the complete continuity of \( T_\lambda \) follows from the fact that every solution \( u \) of \( (P_\lambda) \) is \( C^\alpha \) up to the boundary, and hence there exists \( R > 0 \) such that for all \( t \in [0, 1] \) and all \( v \in C(\Omega) \), it follows that \( \|S_tv\|_L^\infty < R \). Then, \( I - S_t \) does not vanish on \( \partial B_R(0) \) and \( \text{deg}(I - T_0, B_\varepsilon(0)) = \text{deg}(I - S_t, B_\varepsilon(0)) = \text{deg}(I - S_0, B_\varepsilon(0)) = \text{deg}(I, B_\varepsilon(0)) = 1 \). Therefore, \( T_0 \) has a fixed point \( u_0 \), which is a solution of \( (P_0) \). Applying the degree’s properties, for all \( \varepsilon > 0 \) small enough, it follows that \( \overline{\text{deg}}(I - T_0, B_\varepsilon(0)) = \overline{\text{deg}}(I - T_0, B_\varepsilon(0)) = 1 \). Thus, for \( \varepsilon < 1 \), we conclude that \( i(I - T_0, u_0) = \lim_{\varepsilon \to 0} \overline{\text{deg}}(I - T_0, B_\varepsilon(0)) = 1 \).
Step 3: Existence and behavior of the continuum. Proceeding as in [5, Theorem 1.2], we are able to apply Theorem 2.3 gives us a continuum $C = C^+ \cup C^- \subset \Sigma$ such that $C^+ = C \cap ([0, \infty) \times C(\Omega))$ and $C^- = C \cap ((-\infty, 0] \times C(\Omega))$ are unbounded in $\mathbb{R}^+ \times C(\Omega)$.

By Step 1, we get that if $u \in C^+$, then $u \geq u_0$ and it is a solution of $(P_\lambda)$. Thus, applying Lemma 4.2 we infer that the projection of $C^+$ on $\lambda$-axis is $[0, \overline{\lambda}]$, a bounded interval. A consequence of Case (i) is that none of $\lambda \in (-\infty, 0]$ is a bifurcation point of infinity of $(P_\lambda)$, and then, we deduce that the projection of $C^-$ on $\lambda$-axis is $(-\infty, 0]$. Hence,

$$\text{Proj}_{\mathbb{R}} C = \text{Proj}_{\mathbb{R}} C^- \cup \text{Proj}_{\mathbb{R}} C^+ = (-\infty, \overline{\lambda}], \quad \text{for some } \overline{\lambda} > 0.$$ 

Finally, by Theorem 3.1 for any $0 < \Lambda_1 < \Lambda_2$ there exists a priori bound for the solutions of $(P_\lambda)$, for all $\lambda \in [\Lambda_1, \Lambda_2]$, then the projection of $C \cap ([\Lambda_1, \Lambda_2] \times C(\Omega))$ on $C(\Omega)$ is bounded. Since the component $C^+$ is unbounded in $\mathbb{R}^+ \times C(\Omega)$, its projection on the $C(\Omega)$ axis must be unbounded. By Case (i), the projection $C^-$ on $C(\Omega)$ is bounded. Hence,

$$\text{Proj}_{C(\Omega)} C = \text{Proj}_{C(\Omega)} C^- \cup \text{Proj}_{C(\Omega)} C^+ = [0, +\infty).$$

Therefore, we deduce that $C$ must emanate from infinity on the right of axis $\lambda = 0$.

Now, we prove our multiplicity results in (iii). Since $C$ contains $(0, u_0)$, with $u_0$ being the unique solution of $(P_0)$, from Case (ii) we know that $C$ also emanates from infinity on the right of axis $\lambda = 0$ and then, we conclude that there exists $\lambda_0 \in (0, \overline{\lambda})$ such that the problems $(P_{\lambda_0})$ and $(P_{\lambda})$ have at least two solutions satisfying $u \geq u_0$ for $\lambda \in (0, \lambda_0)$. Hence, the quantity

$$\overline{\lambda} := \sup\{\mu > 0 : \forall \lambda \in (0, \mu), (P_\lambda) \text{ has at least two solutions}\} \quad \text{is well defined.}$$

We claim that for all $\lambda \in (0, \overline{\lambda})$, the problem $(P_\lambda)$ has at least two solutions with $u_{\lambda, 1} \ll u_{\lambda, 2}$. Let us consider the strict subsolution $\alpha_{\lambda}$ given by Lemma 4.1. As $\alpha_{\lambda} \leq u$ for all $u$ solution of $(P_{\lambda})$, we can choose $u_{\lambda, 1}$ as the minimal solution with $u_{\lambda, 1} \geq \alpha$. Hence we have $u_{\lambda, 1} \ll u_{\lambda, 2}$, otherwise there would exist a solution $u$ with $\alpha \leq u \leq \min\{u_{\lambda, 1}, u_{\lambda, 2}\}$, which contradicts the minimality of $u_{\lambda, 1}$. Observe that, the function $\beta = (u_{\lambda, 1} + u_{\lambda, 2})/2$ is a supersolution of $(P_\lambda)$ which is not a solution. Now, for each $\xi \in \mathbb{R}^n$, we can define the function $\varphi(\xi) := (M(x)\xi, \xi)$ and observe that by assumption (A) we have $D^2(\varphi) > 0$, therefore $\varphi$ is convex. Then, we obtain

$$-\text{div}(A(x)D\beta) = -\frac{1}{2} \text{div}(A(x)Du_{\lambda, 1}) - \frac{1}{2} \text{div}(A(x)Du_{\lambda, 2})$$

$$= c_\lambda(x)\beta + \frac{1}{2}(M(x)Du_{\lambda, 1}, Du_{\lambda, 1}) + \frac{1}{2}(M(x)Du_{\lambda, 2}, Du_{\lambda, 2}) + h(x)$$

$$\geq c_\lambda(x)\beta + \varphi\left(\frac{Du_{\lambda, 1}}{2} + \frac{Du_{\lambda, 2}}{2}\right) + h(x) = c_\lambda(x)\beta + (M(x)D\beta, D\beta) + h(x).$$

Let us prove that $\beta$ is a strict supersolution of $(P_\lambda)$. Consider a solution $u$ of $(P_\lambda)$ with $u \leq \beta$. Then $v := \beta - u$ satisfies

$$-\text{div}(A(x)Dv) \geq c_\lambda(x)\beta + (M(x)D\beta, D\beta) + h(x) - (M(x)Du, Du) - c_\lambda u - h(x)$$

$$= (M(x)[D\beta + Du], Du) + c_\lambda v,$$

and hence

$$-\text{div}(A(x)Dv) - (M(x)D\beta + Du, Dv) + c^-\lambda v \geq \lambda c^+(x)v \geq 0.$$
By Theorem 2.1 we deduce that either \( v \gg 0 \) or \( v \equiv 0 \). If \( v \equiv 0 \), then \( \beta = u \) is solution, which contradicts the construction of \( \beta \). Then we have \( \beta \gg u \). As \( u_{\lambda,1} \ll \beta \lesssim u_{\lambda,2} \) we deduce that, \( u_{\lambda,1} \ll \beta \not\lesssim u_{\lambda,2} \) and hence we have \( u_{\lambda,1} \ll u_{\lambda,2} \). We finish the proof claiming that if \( \lambda < \infty \), the solution \( u_{\lambda} \) of \((P_{\lambda})\) is unique.

In order to prove that \((P_{\lambda})\) has at least one solution, take \( \{\lambda_n\} \subset (0, \overline{\lambda}) \) such that \( \lambda_n \to \overline{\lambda} \) and by the regularity result [2, Lemma 2.1] let \( \{u_n\} \subset H^1(\omega) \cap W^{1,\alpha}_0(\Omega) \cap C(\overline{\Omega}) \) be a sequence of corresponding solutions. By Theorem 3.1, there exists \( M > 0 \) such that \( ||u_n||_{L^\infty} < M \) for all \( n \in \mathbb{N} \), and hence by the \( C^{1,\alpha} \) global estimates we get \( ||u_n||_{C^{1,\alpha}(\overline{\Omega})} \leq C \). Hence, up to a subsequence, \( u_n \to u \) in \( C^1_0(\Omega) \). From this strong convergence we easily observe that \( u \) is a solution of \((P_{\lambda})\). Now we proof the uniqueness of the solution of \((P_{\lambda})\).

Let us assume by contradiction that we have two distinct solutions, \( u_1 \) and \( u_2 \) of \((P_{\lambda})\), we prove that \( \beta = (u_1 + u_2)/2 \) is a strict supersolution of \((P_{\lambda})\). Let us consider the strict subsolution \( \alpha \ll \beta \) of \((P_{\lambda})\) given by Lemma 4.1, and look at the set,

\[
\mathcal{S} = \{ u \in C^1_0(\overline{\Omega}); \alpha < u < \beta, ||u||_{C^1_0} < R \}
\]

for some \( R > C > 0 \). Again, by the \( C^{1,\alpha} \) estimates,

\[
||u||_{C^{1,\alpha}} \leq C \text{ for all } u \text{ solution of } (P_{\lambda}), \lambda \in [\overline{\lambda}, \overline{\lambda} + 1]
\]

such that \( \text{deg}(I - T_{\lambda}, \mathcal{S}) = 1 \). Now we prove the existence of \( \varepsilon > 0 \) such that

\[
\text{deg}(I - T_{\lambda}, \mathcal{S}) = 1, \text{ for all } \lambda \in [\overline{\lambda}, \overline{\lambda} + \varepsilon].
\]

We will verify that there exists some \( \varepsilon \in (0, 1) \) such that there is no fixed points of \( T_{\lambda} \) on the boundary of \( \mathcal{S} \) for all \( \lambda \) in the preceding interval. Indeed, if this was not the case, there would exists a sequence \( \lambda_k \to \overline{\lambda} \) with the respective solutions \( u_k \) of \((P_{\lambda_k})\) belonging to \( \mathcal{S} \). Say \( \lambda_k \in [\overline{\lambda}, \overline{\lambda} + 1] \) for \( k \geq k_0 \). Then, since \( \alpha < u_k < \beta \) in \( \Omega \), by (4.3) we must have \( u_k \in \partial \mathcal{S} \) for \( k \geq k_0 \), which means that for each such \( k \),

\[
\max_{\overline{\Omega}}(\alpha - u_k) = 0 \text{ or } \min_{\overline{\Omega}}(u_k - \beta) = 0.
\]

By (4.3) and the compact inclusion \( C^\alpha(\overline{\Omega}) \subset C(\Omega) \), \( u_k \to u \) in \( \Omega \) for some \( u \in C(\Omega) \), up to subsequences. Then, \( u \) is a solution of \((P_{\lambda})\) and by taking the limit as \( k \to +\infty \) in the corresponding inequalities for \( u_k \), it follows that \( \alpha \leq \beta \) in \( \Omega \). Thus, \( \alpha \ll u \ll \beta \) in \( \Omega \), since \( \alpha \) and \( \beta \) are strict. Passing (4.5) to the limit, we obtain that \( u(x) = \alpha(x) \) or \( u(x) = \beta(x) \) for \( x \in \overline{\Omega} \), which contradicts the definition of \( \alpha \ll u \ll \beta \). Hence, for obtaining (4.4) it is just necessary to apply the homotopy invariance in \( \lambda \) in the interval \( [\overline{\lambda}, \overline{\lambda} + \varepsilon] \). With (4.4) in hand, we repeat exactly the same argument done in (iii) to obtain the existence of a second solution \( u_{\lambda,2} \) of \((P_{\lambda})\) for all \( \lambda \in [\overline{\lambda}, \overline{\lambda} + \varepsilon] \), which contradicts the definition of \( \lambda \). \( \square \)

In order to prove Theorem 1.2, we start by constructing an auxiliary problem \((P_{\lambda,k})\), for which we can assume that there is no solution for large \( k \). This is a typical but essential argument that allows us to find a second solution via degree theory, by homotopy invariance in \( k \). Fix \( \Lambda_2 > 0 \) and recall that Theorem 3.7 gives us an a priori lower uniform bound \( C_0 \) such that \( u \geq -C_0 \), for every weak supersolution \( u \) of \((P_{\lambda})\), for all \( \lambda \in [0, \Lambda_2] \). Consider, the problem

\[
\begin{align*}
- \text{div}(A(x)Du) &= c_\lambda(x)u + (M(x)Du, Du) + h(x) + k\overline{c}(x) \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{align*}
\]

\((P_{\lambda,k})\)
for $k \geq 0$, $\lambda \in [0, \Lambda_2]$ and $\bar{c}$ being defined as
\[
\bar{c}(x):= \bar{c}_{\Lambda_2}(x) = h^-(x) + \Lambda_2 C_0 c^+(x) + \tilde{M} c^-(x) + B c^+(x) \tag{4.6}
\]
with $B = \gamma_1/\nu_1$, where $\gamma_1 = \gamma_1^+ > 0$ is the first eigenvalue with weight $c$, associated to the eigenfunction $\varphi_1 \in W^{2,p}(\Omega)$, given by $(P_{\gamma_1})$. Note that every solution of $(P_{\lambda,k})$ is also a supersolution of $(P_{\lambda})$ since $k\bar{c}(x) \geq 0$. From this and (4.6) we have for all $k \geq 1$ that
\[
c_{\lambda}(x)u + h(x) + k\bar{c}(x) \geq -\Lambda_2 C_0 c^+(x) - \tilde{M} c^-(x) - h^-(x) + \bar{c}(x) = B c^+(x) \geq 0.
\]

We now derive some results about the solutions of problem $(P_{\lambda,k})$.

**Lemma 4.3.** Under assumption (A), assume that $(P_0)$ has a solution $u_0 \leq 0$ with $c^+(x)u_0 \leq 0$. Then for each fixed $\Lambda_2 > 0$ and $\lambda \in [0, \Lambda_2]$, there exists $k \geq 0$ such that

(i) For all $k > 1$, the problem $(P_{\lambda,k})$ has no solutions;

(ii) For all $k \in (0, 1)$, $(P_{\lambda,k})$ has at least two solutions $u_{\lambda,1} \ll u_{\lambda,2}$;

(iii) For $k = 1$, and $h \leq 0$ the problem $(P_{\lambda,k})$ has exactly one solution.

**Proof.** We proceed in several steps.

**Step 1:** For $k > 0$ small, $(P_{\lambda,k})$ admits a solution. Let $\lambda > \gamma_1$ and $\varepsilon_0 > 0$ be given by Lemma 2.4 corresponding to $\tau = c(x)$, $\bar{d} = \nu_2 h^-(x)$, $\bar{b} = \nu_2 \bar{c}(x) + k^{-1} \nu_2 h^+(x)$, and choose $\lambda_0 \in \{\gamma_1 + \epsilon_0, \gamma_1 + (\lambda - \gamma_1)/2\}$. Then the problem
\[
-\text{div}(A(x)Du) + \nu_2 h^-(x)u = c_{\lambda_0} u + \nu_2 \bar{c}(x) + \frac{1}{k} \nu_2 h^+(x)
\]
has a solution $u \ll 0$. Also taking $\delta > 0$ small enough we have that
\[
\lambda_0 s \geq (1 + \lambda s) \ln(1 + \lambda s) \quad \text{for all} \quad s \in [-\delta, 0].
\]
Thus defining $\bar{\beta}_k = k\lambda^{-1} u$ for $k > 0$ small enough, it follows that $\bar{\beta}_k \in [-\delta, 0]$ and satisfies
\[
-\text{div}(A(x)D\bar{\beta}_k) = c_{\lambda_0} \bar{\beta}_k + \nu_2 k \bar{c}(x) - \nu_2 \frac{k}{\lambda} h^+(x)u + \frac{1}{\lambda} \nu_2 h^+(x), \quad \text{and hence}
\]
\[
-\text{div}(A(x)D\bar{\beta}_k) + \nu_2 h^-(x)\bar{\beta}_k = c_{\lambda_0}(x)\bar{\beta}_k + \nu_2 \frac{k}{\lambda} \bar{c}(x) + \frac{1}{\lambda} \nu_2 h^+(x).
\]
Hence for $\beta_k$ being defined by $\beta_k = \nu_2^{-1} \ln(1 + \lambda \bar{\beta}_k)$, we have
\[
-\text{div}(A(x)D\beta_k) = -\frac{\lambda}{\nu_2} \text{div}(A(x)D\bar{\beta}_k) - \frac{\lambda}{\nu_2} \left(\frac{\lambda}{\nu_2} \left[A(x)D\bar{\beta}_k, D \left[\left(1 + \lambda \bar{\beta}_k\right)^{-1}\right]\right] - c_{\lambda}(x)\beta_k + \frac{k\bar{c}(x) + h^+(x) - \lambda h^-(x)\bar{\beta}_k}{1 + \lambda \bar{\beta}_k} + \frac{\lambda^2}{\nu_2(1 + \lambda \bar{\beta}_k)^2} (A(x)D\bar{\beta}_k, D\bar{\beta}_k)
\]
\[
\geq c_{\lambda}(x)\beta_k + \frac{k\bar{c}(x) + h^+(x) - \lambda h^-(x)\bar{\beta}_k}{1 + \lambda \bar{\beta}_k} + \frac{\lambda^2}{\nu_2(1 + \lambda \bar{\beta}_k)^2} (A(x)D\bar{\beta}_k, D\bar{\beta}_k)
\]
\[
\geq c_{\lambda}(x)\beta_k + \frac{k\bar{c}(x) + h^+(x) - \lambda h^-(x)\bar{\beta}_k}{1 + \lambda \bar{\beta}_k} + \frac{\lambda^2}{\nu_2(1 + \lambda \bar{\beta}_k)^2} (A(x)D\bar{\beta}_k, D\bar{\beta}_k).
\]
Therefore, we conclude that

\[
\begin{aligned}
&\begin{cases}
-\text{div}(A(x)D\beta_k) \geq c_\lambda(x)\beta_k + k\tilde{c}(x) + h(x) + (M(x)D\beta_k, D\beta_k) & \text{in } \Omega \\
\beta_k = 0 & \text{on } \partial\Omega
\end{cases}
\end{aligned}
\]

has a supersolution \( \beta_k \) with \( \beta_k \ll 0 \) and that \((P_{\lambda,k})\) has at least one solution, by following the proof of Theorem 1.1.

**Step 2:** For \( k > 1 \) the problem \((P_{\lambda,k})\) has no solution. First we observe that every solution of \((P_{\lambda,k})\) for \( \lambda \in [0, \Lambda_2] \) is positive in \( \Omega \). In fact, we observe that

\[
\begin{aligned}
&\begin{cases}
-\text{div}(A(x)Du) \geq (M(x)Du, Du) + Bc^+(x) \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\end{aligned}
\]

and this implies that \( u > 0 \) in \( \Omega \) by the SMP. In order to obtain a contradiction, assume that \( u \) is a solution of \((P_{\lambda,k})\) in \( \Omega \). Let \( \varphi \in C_0^\infty(\Omega) \) such that \( \varphi^2 \gg 0 \). Then using \( \varphi^2 \) as a test function, by Theorem 3.7 we obtain

\[
\begin{aligned}
\int \frac{1}{\mu_1}|D\varphi|^2 \geq 2 \int (\varphi Du, D\varphi) - \mu_1 \int |Du|^2 \varphi^2 \geq 2 \int (\varphi Du, D\varphi) - \int (M(x)Du, \varphi^2 Du) \\
\geq -\Lambda_2 C_0 \int c^+(x)\varphi^2 - M \int c^-(x)\varphi^2 - \int h^-(x)\varphi^2 + \int k\tilde{c}(x)\varphi^2
\end{aligned}
\]

which is a contradiction for \( k > 1 \) large enough.

**Step 3:** For \( k = 1 \) \((P_{\lambda,k})\) has a unique solution and for \( k \in (0, 1) \) the problem \((P_{\lambda,k})\) has a strict supersolution. By Step 1 and 2 we have \( 1 = \sup \{ \lambda > 0 ; (P_{\lambda,k}) \text{ has at least one solution} \} \).

Let \( k \in (0, 1) \) and \( \tilde{k} \in (k, 1) \) be such that \((P_{\lambda,\tilde{k}})\) has a solution \( \tilde{\beta} \). Then \( \beta = k\tilde{k}^{-1}\tilde{\beta} \) is a supersolution of \((P_{\lambda,k})\). Now, as in (iii) of the proof of Theorem 1.1 we can prove that \( \beta \) is a strict supersolution of \((P_{\lambda,k})\) and also derive the existence of the second solution \( u_{\lambda,2} \) with \( u_{\lambda,1} \ll u_{\lambda,2} \).

**Lemma 4.4.** Under assumption (A), assume that \((P_0)\) has a solution \( u_0 \leq 0 \) with \( c^+(x)u_0 \ll 0 \). Then, for all \( \lambda \geq 0 \), problem \((P_\lambda)\) has at most one solution \( u \leq 0 \).

**Proof.** The proof is divided in several steps.

**Step 1:** If \( u \) is a subsolution of \((P_\lambda)\) with \( u \leq 0 \), then \( u \ll 0 \). In fact, \( u \) is a subsolution of \((P_0)\) and by the Comparison Principle, we have \( u \leq u_0 \). In addition for \( w = u_0 - u \) we have

\[
\begin{aligned}
-\text{div}(A(x)Dw) \geq -c^-(x)u_0 + (M(x)Du_0, Du_0) - c_\lambda(x)u - (M(x)Du, Du) \\
= (M(x)Du + Dw, Dw) - c^-(x)w - \lambda c^+(x)u,
\end{aligned}
\]

and hence, we get

\[
\begin{aligned}
&\begin{cases}
-\text{div}(A(x)Dw) - (M(x)Du + Dw, Dw) - c^-(x)w \gg 0 & \text{in } \Omega \\
w = 0 & \text{on } \partial\Omega
\end{cases}
\end{aligned}
\]

This implies that \( w \gg 0 \) i.e. \( u \ll u_0 \leq 0 \).

**Step 2:** If we have two solutions \( u_1, u_2 \leq 0 \) of \((P_\lambda)\), then such solutions are ordered as \( \tilde{u}_1 \ll \tilde{u}_2 \leq u_0 \). By Step 1, we have \( u_1, u_2 \ll u_0 \). In case \( u_1 \) and \( u_2 \) are not ordered, as \( u_0 \) is a supersolution of \((P_\lambda)\), applying [5, Theorem 2.1] there exists a solution \( u_3 \) of \((P_\lambda)\) with
max\{ u_1, u_2 \} \leq u_3 \leq u_0. This proves Step 2 by choosing \( \hat{u}_1 = u_1 \) and \( \hat{u}_2 = u_3 \).

**Step 3:** There exists at most one nonpositive solution to \((P_\lambda)\). Let us assume by contradiction that we have two ordered nonpositive solutions, we can suppose \( u_1 < u_2 < 0 \). As \( |u_2| \gg 0 \) the set \( \{ \epsilon > 0, u_2 - u_1 < \epsilon|u_2| \} \) is nonempty. Defining

\[
\hat{\epsilon} := \min\{ \epsilon > 0, u_2 - u_1 \leq \epsilon|u_2| \}\quad \text{and setting} \quad w_\epsilon := \frac{(1 + \hat{\epsilon})u_2 - u_1}{\hat{\epsilon}},
\]

we can use the convexity of the function \( \varphi(\xi) := (M(x)\xi, \xi) \) for each \( \xi \in \mathbb{R}^n \) and write \( u_2 = \hat{\epsilon}(1 + \hat{\epsilon})^{-1}w_\epsilon + (1 + \hat{\epsilon})^{-1}u_1 \), then we obtain

\[
(M(x)Du_2, Du_2) = \varphi\left(\frac{\hat{\epsilon}}{1 + \hat{\epsilon}} Dw_\epsilon + \frac{1}{1 + \hat{\epsilon}} Du_1\right) \leq \frac{\hat{\epsilon}}{1 + \hat{\epsilon}} \varphi(Dw_\epsilon) + \frac{1}{1 + \hat{\epsilon}} \varphi(Du_1) = \frac{1}{1 + \hat{\epsilon}} \left[ \hat{\epsilon}(M(x)Dw_\epsilon, Dw_\epsilon) + (M(x)Du_1, Du_1) \right], \quad \text{and hence}
\]

\[
\frac{1 + \hat{\epsilon}}{\hat{\epsilon}} (M(x)Du_2, Du_2) \leq (M(x)Dw_\epsilon, Dw_\epsilon) + \frac{1}{\hat{\epsilon}} (M(x)Du_1, Du_1).
\]

Thus, it yields

\[
- \text{div}(A(x)Dw_\epsilon) \leq \frac{1 + \hat{\epsilon}}{\hat{\epsilon}} \left[ c_\lambda(x)u_2 + (M(x)Du_1, Du_1) + h(x) \right] - \frac{1}{\hat{\epsilon}} \left[ c_\lambda(x)u_1 + (M(x)Du_1, Du_1) + h(x) \right] \leq c_\lambda(x)w_\epsilon + (M(x)Dw_\epsilon, Dw_\epsilon) + h(x).
\]

Applying again the Comparison Principle, we get \( w_\epsilon \preceq u_2 \leq 0 \), which is a contradiction due to the definition of \( \hat{\epsilon} \).

Finally, we have all the necessary tools to prove Theorem 1.2.

**Proof of Theorem 1.2.** We treat separately the case \( \lambda \leq 0 \) and \( \lambda > 0 \).

**Case (i):** \( \lambda \leq 0 \). As in the proof of Theorem 1.1 we can apply [6, Theorem 1.2]. Moreover, observe that \( u_0 \) is a subsolution of \((P_\lambda)\). Hence we conclude that \( u_\lambda \geq u_0 \) applying the Comparison Principle. By [1, Proposition 4.1] the problem \((P_\lambda)\) has at most one solution and by Lemma 3.2 the function \( v = u_0 + \|u_0\|_\infty \) is a supersolution of \((P_\lambda)\) when \( \lambda < 0 \). Then, the Comparison Principle implies that \( u_0 + \|u_0\|_\infty \geq u_\lambda \).

**Case (ii):** \( \lambda > 0 \). With the aim of showing the existence of a continuum of solution of \((P_\lambda)\), for \( \lambda \geq 0 \) we introduce the auxiliary problem

\[
- \text{div}(A(x)Du) + u = [c_\lambda(x) + 1]|u_0 - (u - u_0)^-| + (M(x)Du, Du) + h(x). \quad (P_\lambda)
\]

As in the case of \((P_\lambda)\), any solution of \((P_\lambda)\) belong to \( C^{0,\tau}(\Omega) \) for some \( \tau > 0 \). Moreover observe that \( u \) is a solution of \((P_\lambda)\) if and only if it is a fixed point of the operator \( \tilde{T}_\lambda \) defined by \( \tilde{T}_\lambda : C(\overline{\Omega}) \to C(\overline{\Omega}) : v \to u \), where \( u \) is the solution of

\[
- \text{div}(A(x)Du) + u - (M(x)Du, Du) = [c_\lambda(x) + 1]|u_0 - (v - u_0)^-| + h(x).
\]

Applying the same argument to \( \tilde{T}_\lambda \) as the one used in the proof of Theorem 1.1, we see that \( \tilde{T}_\lambda \) is completely continuous, and we split the rest of the proof into three steps.
Step 1: If \( u \) is a solution of \((P_\lambda)\) then \( u \leq u_0 \) and it is a solution of \((P_0)\). Observe that \( u_0 - u - (u - u_0)^- \leq 0 \). Moreover, we also have \( \lambda c^+(x) |u_0 - u - (u - u_0)^+| \leq \lambda c^+(x) u_0 \leq 0 \). Hence we deduce that a solution \( u \) of \((P_\lambda)\) is a subsolution of

\[
- \text{div}(A(x) Du) = -c^-(x)[u_0 - (u - u_0)^-] + (M(x) Du, Du) + h(x).
\]

(4.7)

Since \( u_0 \) is a solution of \((P_0)\), it implies that \( u_0 \) solves \((4.7)\). Then applying again the Comparison Principle we get \( u \leq u_0 \).

Step 2: \( u_0 \) is the unique solution to \((P_0)\) as well as to the problem \((P_\lambda)\) and \( i(I - \hat{T}_0, u_0) = 1 \).

For \( \lambda = 0 \), if \( u \) is a solution to \((4.7)\), then by Step 1, \( u \leq u_0 \) and \( u \) solves \((P_0)\). From (i) we conclude that \( u = u_0 \). In order to prove that \( i(I - \hat{T}_0, u_0) = 1 \), we consider the operator \( \hat{S}_t : C(\bar{\Omega}) \to C(\bar{\Omega}) \) given by \( \hat{S}_t(v) := t\hat{T}_0 v = u \) where \( u \) is the solution of

\[
- \text{div}(A(x) Du) + u = (M(x) Du, Du) + th(x) + t[-c^-((x) + 1)[u_0 - (v - u_0)^- - (v - u_0 + 1)^-].
\]

By the complete continuity of \( \hat{T} \) and also the fact that every solution \( u \) of \((P_\lambda)\) is \( C^\alpha \) up to the boundary, there exists \( R > 0 \) such that for all \( t \in [0,1] \) and all \( v \in C(\bar{\Omega}) \), it follows that \( ||\hat{S}_t v||_{C^\alpha} < R \). Then \( I - S_t \) does not vanish on \( \partial B_R(0) \) and

\[
\text{deg}(I - \hat{T}_0, B_R(0)) = \text{deg}(I - S_1, B_R(0)) = \text{deg}(I - S_0, B_R(0)) = \text{deg}(I, B_R(0)) = 1.
\]

Therefore, \( \hat{T}_0 \) has only a fixed point \( u_0 \), which is a solution of \((P_0)\). Therefore, arguing as in Step 2 of Theorem 1.1 we conclude this step.

Step 3: Existence and behavior of the continuum. It follows as well as the Step 3 of Theorem 1.1.

For the multiplicity results in (iii), we observe that by Step 1, we get the existence of a first solution \( u_{\lambda,1} \leq u_0 \). To prove that \( u_0 \) is a strict supersolution of \((P_\lambda)\), we argue as in the proof of Theorem 1.1, and by Lemma 4.1 \((P_\lambda)\) has a strict subsolution \( \alpha \) with \( \alpha \leq u_0 \). Then, by [5, Theorem 2.1], there exists \( R > 0 \) such that \( u_{\lambda,1} \in S \), where

\[
S = \{ u \in C^1_0(\Omega); \alpha < u < u_0 \text{ in } \Omega, ||u||_{C^1_0} < R \}.
\]

Now, fixing \( \lambda > 0 \) and setting \( \Lambda_2 = 2\lambda \), we replace \( h \) by \( h + k\hat{c} \) in the problem \((P_{\lambda,k})\), and then Theorem 3.1 gives us an \( L^\infty \) a priori bound for solutions of \((P_{\lambda,k})\) for every \( k \in [0,1] \). This provides, by the \( C^{1,\alpha} \) global estimates, an a priori bound for solutions in \( C^1_0(\Omega) \), i.e. \( ||u||_{C^1_0(\Omega)} < R_0 \) for every solution \( u \) of \((P_{\lambda,k})\), for all \( k \in [0,1] \), where \( R_0 > R \) also depends on \( \lambda \). Hence, by the homotopy invariance of the degree, and the fact that, for \( k > 1 \), \((P_{\lambda,k})\) has no solution we have

\[
\text{deg}(I - \hat{T}_{\lambda,k}, B_{R_0}(0)) = ... = 0,
\]

where \( \hat{T}_{\lambda,k} \) is the operator \( \hat{T}_{\lambda} \) in which we replace \( h(x) \) by \( h(x) + k\hat{c} \), note that \( \hat{T}_{\lambda,k} \) is clearly still completely continuous. But then, by the excision property of the degree,

\[
\text{deg}(I - \hat{T}_{\lambda,k}, B_{R_0} \setminus S(0)) = ... = -1
\]

and the existence of a second solution \( u_{\lambda,2} \in B_{R_0} \setminus S \) is derived. By Lemma 4.4 we have \( u_{\lambda,2} > 0 \).
For finishing, we claim that for fixed $\lambda_1 < \lambda_2$ we have $u_{\lambda_2, 1} \ll u_{\lambda_1, 1}$. In fact, note that

$$c_{\lambda_1}(x)u_{\lambda_1, 1} = \lambda_1 c^+(x)u_{\lambda_1, 1} - c^-(x)u_{\lambda_1, 1} \geq \lambda_2 c^+(x)u_{\lambda_1, 1} - c^-(x)u_{\lambda_1, 1} = c_{\lambda_2}(x)u_{\lambda_1, 1},$$

since $u_{\lambda_1, 1} < 0$. Then, $u_{\lambda_1, 1}$ is a strict supersolution of $(P_{\lambda_2})$, which is not a solution and, in particular, $u_{\lambda_1, 1} \neq u_{\lambda_2, 1}$. As in the proof of [11, Claim 6.16], we observe that $u_{\lambda_2, 1}$ is the minimal solution of $(P_{\lambda_2})$. In fact, recall that $\xi = \xi_{\lambda_2}$, given by Lemma 4.1, is such that $\xi \leq u$ for every strictly supersolution of $(P_{\lambda_2})$ and in particular $\xi \leq u_{\lambda_1, 1}$. Remember also that $u_{\lambda_2, 1}$ is the minimal weak solution such that $u_{\lambda_2, 1} \geq \xi$ in $\Omega$. Now, if there was a $x_0 \in \Omega$ such that $u_{\lambda_2, 1}(x_0) > u_{\lambda_1, 1}(x_0)$, by defining $\eta := \min \{u_{\lambda_1, 1}, u_{\lambda_2, 1}\}$, as the minimum of the strict supersolutions of $(P_{\lambda_2})$ not less than $\xi$, we have $\xi \leq \eta$ in $\Omega$. Thus, applying again [5, Theorem 2.1] we get a solution $u$ of $(P_{\lambda_2})$ such that $\xi \leq u \leq \eta \ll u_{\lambda_2, 1}$ in $\Omega$, which contradicts the minimality of $u_{\lambda_2, 1}$. Finally, this ends the proof of Theorem 1.2. \(\square\)

In what follows we prove Theorem 1.3, considering the alternative situation when there exists a supersolution of $(P_{\lambda})$ for some $\lambda_0 > 0$.

**Proof of Theorem 1.3.** For proving (i), we first observe that if $(P_{\lambda})$ has a supersolution $\beta_{\lambda} \leq 0$, then $\beta_{\lambda}$ satisfies also $c^+(x)\beta_{\lambda} \leq 0$, otherwise, it is also a supersolution of $(P_0)$, which contradicts the assumption (A). Let us define

$$\underline{\lambda} = \inf \{\lambda \geq 0; (P_{\lambda}) \text{ has a supersolution } \beta_{\lambda} \leq 0 \text{ with } c^+(x)\beta_{\lambda} \leq 0\}.$$

Given $\lambda > \underline{\lambda}$, by the definition of $\underline{\lambda}$ there exists $\tilde{\lambda} \in [\underline{\lambda}, \lambda)$, such that $(P_{\tilde{\lambda}})$ has a supersolution $\beta_{\tilde{\lambda}} \leq 0$ with $c^+(x)\beta_{\tilde{\lambda}} \leq 0$. Note that

$$c_{\tilde{\lambda}}(x)\beta_{\tilde{\lambda}} = \lambda c^+(x)\beta_{\tilde{\lambda}} - c^-(x)\beta_{\tilde{\lambda}} \geq \lambda c^+(x)\beta_{\tilde{\lambda}} - c^-(x)\beta_{\tilde{\lambda}} = c_{\lambda}(x)\beta_{\tilde{\lambda}}.$$

Then, $\beta_{\tilde{\lambda}}$ is a supersolution of $(P_{\lambda})$, which is not a solution and hence, as in the proof of Theorem 1.2(iii), it is a strict supersolution of $(P_{\lambda})$. By Lemma 4.1, $(P_{\lambda})$ has a strict subsolution $\alpha_{\lambda} \leq \beta_{\lambda}$ and $\alpha_{\lambda} \leq 0$ for all solutions $u$ of $(P_{\lambda})$. As in Step 2 of the proof of Theorem 1.2, there exists $R > 0$ such that $\deg(I - \tilde{T}_{\lambda}, S) = 1$ with

$$S = \{u \in C_0^1(\Omega), \alpha \ll u \ll \beta_{\tilde{\lambda}}, \|u\|_{C^1, \Omega} \leq R\},$$

and hence the existence of the first solution $u_{\lambda, 1} \ll 0$ is derived. To obtain a second solution $u_{\lambda, 2}$ satisfying $u_{\lambda, 1} \ll u_{\lambda, 2}$ and $u_{\lambda, 2} > \beta_{\lambda}$ we repeat the argument in the proof of the Theorem 1.2(iii). By Lemma 4.4 in this case we have $u_{\lambda, 2} > u_{\lambda}$, Finally, arguing as at the ending of the proof of Theorem 1.2, we prove that if $\lambda_1 < \lambda_2$ we have $u_{\lambda_1, 1} \gg u_{\lambda_2, 1}$.

For proving that $(P_{\lambda})$ has a nonpositive solution, let $\{\lambda_n\} \subset (\underline{\lambda}, \infty)$ be a decreasing sequence such that $\lambda_n \to \underline{\lambda}$. By the regularity result proved in [2, Lemma 2.1], we know that there exists a sequence of corresponding solutions $\{u_n\} \subset H^1(\omega) \cap W^{1,n}_0(\Omega) \cap C(\overline{\Omega})$ with $u_n \leq u_{n+1} \leq 0$. As $\{u_n\}$ is increasing and bounded above, by Theorem 3.1, there exists $M > 0$ such that $\|u_n\|_{L^\infty} \ll M$ for all $n \in \mathbb{N}$, and hence by the $C^{1,n}$ global estimates, we get $\|u_n\|_{C^{1,n}}(\overline{\Omega}) \leq C$. Then, up to a subsequence, $u_n \to u$ in $C_0^1(\overline{\Omega})$. From this strong convergence we easily conclude that $u$ is a solution of $(P_{\lambda})$ with $u \leq 0$.

Now we prove the uniqueness of the nonpositive solution of $(P_{\lambda})$. Let us assume by contradiction that we have two distinct solutions, $u_1$ and $u_2$ of $(P_{\lambda})$, then as in the Step 3 of the proof of Theorem 1.2, we prove that $\beta = (u_1 + u_2)/2$ is a strict supersolution of
(\(P_\lambda\)). Let us consider the strict subsolution \(\alpha \ll \beta\) of \((P_\lambda)\) given by Lemma 4.1, and define the set, \(\mathcal{S} = \{u \in C^1_0(\Omega); \alpha \ll u \ll \beta, ||u||_{C^1_0} < R\}\) for some \(R > C > 0\). Again, by the \(C^{1, \alpha}\) estimates, we have that

\[
||u||_{C^{1, \alpha}} \leq C \text{ for all } u \text{ solution of } (P_\lambda), \lambda \in [\lambda - 1, \lambda]
\]

(4.8)
such that \(\text{deg}(I - \hat{T}_\lambda, \mathcal{S}) = 1\). Now we prove the existence of \(\varepsilon > 0\) such that

\[
\text{deg}(I - \hat{T}_\lambda, \mathcal{S}) = 1, \text{ for all } \lambda \in [\lambda - \varepsilon, \lambda].
\]

(4.9)

We argue as at the end of proof of Theorem 1.1, in order to verify that there exists some \(\varepsilon \in (0, 1)\) such that there is no fixed points of \(T_\lambda\) on the boundary of \(\mathcal{S}\) for all \(\lambda\) in the preceding interval.

Hence for obtaining (4.9) it is sufficient to apply the homotopy invariance in \(\lambda\) in the interval \([\lambda - \varepsilon, \lambda]\). Next, with (4.9) at hand, we repeat exactly the same argument done in the proof of (i) to obtain the existence of a second solution \(u_{\lambda,2}\) of \((P_\lambda)\), for all \(\lambda \in [\lambda - \varepsilon, \lambda]\). But this, finally, contradicts the definition of \(\lambda\) completing the proof of (ii). By the definition of \(\lambda\) and since \(\beta\) is a strict supersolution of \((P_\lambda)\) we infer that (iii) holds.

Finally, in order to describe the behaviour of the solutions for \(\lambda \to 0^+\), observe that in Lemma 3.2 we have proved that \(\limsup_{\lambda \to 0^-} ||u_\lambda||_{\infty} < 2 ||u_3||_{\infty}\) for all \(\lambda \leq \tilde{\lambda} < 0\). In particular, if \(C_0 := \limsup_{\lambda \to 0^-} ||u_\lambda||_{\infty} < \infty\), then there exists a sequence \(\lambda_n \to 0^-\) such that \(C_0 = \limsup_{n \to \infty} ||u_{\lambda_n}||_{\infty} < \infty\). Hence, for every sequence \(\lambda_n \to 0^-\) we deduce by the above inequality that \(\limsup_{\lambda \to 0^-} ||u_\lambda||_{\infty} \leq 2C_0\), which implies that \(\limsup_{\lambda \to 0^-} ||u_\lambda||_{\infty} < \infty\). Therefore, we have either \(\lim_{\lambda \to 0^-} ||u_\lambda||_{\infty} = \infty\) or \(\limsup_{\lambda \to 0^-} ||u_\lambda||_{\infty} < \infty\). By assumption we know that \((P_0)\) does not have a solution \(u_0 \leq 0\), then the first case holds, finishing the proof.

\(\square\)

**Proof of Corollary 1.4.** First observe that \((P_{\gamma_1, k})\) has no solution. In fact, if we assume by contradiction that \(u\) is a solution of \((P_{\gamma_1, k})\), using \(\varphi_1 > 0\) the first eigenfunction of \((P_{\gamma_1})\) as a test function in \((P_{\gamma_1, k})\), we have

\[
\int c_{\gamma_1}(x) u \varphi_1 = \int A(x) Du D\varphi_1 = \int c_{\lambda}(x) u \varphi_1 + \int \varphi_1 (M(x) Du, Du) + \int (h(x) + k\tilde{c}(x)) \varphi_1
\]

and \((\gamma_1 - \lambda) \int c^+(x) u \varphi_1 \leq - \int |h(x)| \varphi_1 < 0\), which is a contradiction for \(\lambda = \gamma_1\).

Hence also, for all \(\lambda > 0\) problem \((P_\lambda)\) has no solution with \(c^+ (x) u \equiv 0\), otherwise \(u\) would be a solution to \((P_\lambda)\) for every \(\lambda \in \mathbb{R}\), which contradicts the nonexistence of a solution for \(\lambda = \gamma_1\). By Step 3 of the proof of Lemma 4.3 there exists \(\tilde{k} > 0\) such that, for all \(k \in (0, \tilde{k}]\), the problem \((P_{\gamma_1, k})\) has a strict supersolution \(\beta_0\) with \(\beta_0 \ll 0\). The existence of \(\lambda_2 \geq \gamma_1\) as in (iii) can then be deduced from Theorem 1.3. By [1, Theorem 1.1], decreasing \(\tilde{k}\) if necessary, we know that for all \(k \in (0, \tilde{k}]\), the problem \((P_{0, k})\) has a solution \(u_0 \geq 0\). Hence the existence of \(\lambda_1\) as in (i) can be deduced from Theorem 1.1. 

\(\square\)

Before proving Theorem 1.5, we observe that particular cases of Theorems 1.1 and 1.2 are given when \(h(x) \geq 0\) and \(h(x) \leq 0\). Indeed, if \(h \geq 0\) holds, then \(u_0\) is a supersolution of

\[
\begin{cases}
- \text{div}(A(x)Du_0) & \geq c_{\lambda}(x)u_0 + (M(x)Du_0, Du_0) + h(x) \geq 0 & \text{in } \Omega \\
u_0 = 0 & \text{on } \partial \Omega
\end{cases}
\]
Hence for $\beta$

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Then applying the SMP we obtain

\[ u \]

and so $v_0 = \nu_2^{-1}(e^{\nu_2 u_0} - 1)$ is a subsolution of

\[ -\text{div}(A(x)Du_0) \leq c_\lambda(x)u_0 + (M(x)Du_0, Du_0) + h(x) \leq (M(x)Du_0, Du_0) \]

and so $v_0 = \nu_2^{-1}(e^{\nu_2 u_0} - 1)$ is a subsolution of

\[ -\text{div}(A(x)Du_0) \leq [1 + \nu_2 v][-\text{div}(A(x)Du_0) - \mu_2 |Du_0|^2] \]

\[ \leq [1 + \nu_2 v]((M(x)Du_0, Du_0) - \mu_2 |Du_0|^2) \leq 0 \text{ in } \Omega. \]

Again, by the SMP and the Hopf Lemma we get $v_0 \equiv 0$ and therefore $u_0 \equiv 0$.

**Proof of Theorem 1.5.** For proving (i), firstly we note that for all $\lambda \in \mathbb{R}$, $u \equiv 0$ is a solution of $(P_{h=0})$. In order to prove that for all $\lambda \in (0, \gamma_1)$ problem $(P_{h=0})$ has a second solution $u_{\lambda, 2} \not\equiv 0$, we claim that problem $(P_{h=0})$ has a supersolution $\beta \not\equiv 0$. In fact, taking $\lambda < \gamma_1$ and $\varepsilon > 0$ such that $\lambda \varepsilon^{-1}(1 + \nu_2 v_1) \ln(1 + \nu_2 v_1) \leq \gamma_1 v_1$ for all $v_1 \in [0, \varepsilon]$, we consider the function $\bar{\beta} = \varepsilon \varphi_1$, where $\varphi_1$ denotes the first eigenfunction of $(P_{\gamma_1})$ with $\|\varphi_1\|_{L^\infty} = 1$ and

\[
\begin{aligned}
-\text{div}(A(x)D\bar{\beta}) &= c_\gamma(x)\beta \geq c_\lambda(x)\frac{(1 + \nu_2 \bar{\beta})\ln(1 + \nu_2 \bar{\beta})}{\nu_2}, & \text{in } \Omega \\
\bar{\beta} &= 0 & \text{on } \partial\Omega.
\end{aligned}
\]

Hence for $\beta$ being defined by $\beta := \nu_2^{-1}\ln(1 + \nu_2 \bar{\beta})$, we have

\[
-\text{div}(A(x)D\beta) = -\frac{\text{div}(A(x)D\bar{\beta})}{(1 + \nu_2 \bar{\beta})^2} - \left(A(x)D\bar{\beta}, D\left[(1 + \nu_2 \bar{\beta})^{-1}\right]\right)
\]

\[
\geq c_\lambda(x)\beta + \frac{\nu_2}{(1 + \nu_2 \bar{\beta})^2}(A(x)D\bar{\beta}, D\bar{\beta}) \geq c_\lambda(x)\beta + \nu_2 \bar{\beta} \frac{|D\bar{\beta}|^2}{(1 + \nu_2 \bar{\beta})^2}
\]

\[
= c_\lambda(x)\beta + \mu_2 |D\beta|^2 \geq c_\lambda(x)\beta + (M(x)D\beta, D\beta)
\]

and thus,

\[
\begin{aligned}
-\text{div}(A(x)D\beta) &\geq c_\lambda(x)\beta + (M(x)D\beta, D\beta) & \text{in } \Omega \\
\beta &\equiv 0 & \text{on } \partial\Omega.
\end{aligned}
\]

By the Comparison Principle, we have that $\beta \geq 0$ is a strict supersolution of $(P_{h=0})$. Since we know that every solution $u$ of problem $(P_{h=0})$ satisfies $u \geq 0$, by Lemma 4.1 problem $(P_{h=0})$ has a strict subsolution $\alpha \not\equiv 0$. Therefore, we conclude that problem $(P_{h=0})$ has at least two solutions by following the proof of Theorem 1.1 with the solution $u_{\lambda, 1} \equiv 0$.

For proving (ii), suppose by contradiction that $u \not\equiv 0$ is another solution to problem $(P_{h=0})$ and use $\varphi_1 > 0$ the first eigenfunction of $(P_{\gamma_1})$ as a test function in $(P_{h=0})$. Then,

\[
\int c_{\gamma_1}(x)u\varphi_1 = \int A(x)DuD\varphi_1 = \int c_\lambda(x)u\varphi_1 + \int (M(x)Du, Du)\varphi_1
\]

\[
(\gamma_1 - \lambda)\int c^+(x)u\varphi_1 = \int (M(x)Du, Du)\varphi_1 \geq \mu_1 \int |Du|^2\varphi_1 > 0,
\]

which provides a contradiction for $\lambda = \gamma_1$. 


In case $\lambda > \gamma_1$, for showing that problem $(P_{h=0})$ has a second solution $u_{\lambda,2}$, let $\lambda_0 \in (\gamma_1, \lambda]$ be such that by Lemma 2.4, the problem
\[
- \text{div}(A(x)Du) = c_{\lambda_0}(x)u + 1,
\]
has a solution $u \ll 0$. Then, for $\varepsilon > 0$ small enough, the function $\beta_0 = \varepsilon u$ satisfies
\[
- \text{div}(A(x)D\beta_0) = c_{\lambda_0}(x)\varepsilon u + \varepsilon \geq c_{\lambda_0}(x)\beta_0 + \varepsilon^2 \mu_2 |Du|^2 \geq c_{\lambda_0}(x)\beta_0 + (M(x)D\beta_0, D\beta_0)
\]
and the problem $(P_{h=0})$ has a supersolution $\beta_0$ with $\beta_0 \leq 0$ and $c^+(x)\beta_0 \leq 0$. Therefore, (iii) follows by Theorem 1.3 with $u_{\lambda,2} \equiv 0$.

With the aim of showing the existence of a continuum of solution to problem $(P_{h=0})$, we define the operator $T_{\lambda} := \{ T_{\lambda,\lambda} \}$, if $\lambda \leq \gamma_1$, where $T_{\lambda}$ for $\lambda \leq \gamma_1$ is defined in (ii) of the proof of Theorem 1.1 and the operator $T_{\lambda}$ for $\lambda \geq \gamma_1$ is defined in (ii) of the proof of Theorem 1.2 in both cases with $h \equiv 0$.

Observe that the case $\lambda \in (-\infty, \gamma_1]$ can be proved as in the proof of Theorem 1.1(ii). Note that, if $u$ is a solution of $(T_{\lambda})$, then $u \geq u_{\gamma_1}$ and hence it is a solution of $(P_{h=0})$. On the other hand, the case $\lambda \in [\gamma_1, +\infty)$ follows the same lines of the proof of Theorem 1.2(ii). If $u$ is a solution of $(P_{\lambda})$, then $u \leq u_{\gamma_1}$ and hence it is a solution of $(P_{h=0})$. Furthermore, since $u_{\gamma_1} \equiv 0$ is the unique solution of the problem $(P_{h=0})$ for $\lambda = \gamma_1$, then $i(I - T_{\gamma_1}, u_{\gamma_1}) = 1$.

Applying Theorem 2.3 with $\gamma_1 > 0$, we obtain a continuum $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^- \subset \Sigma$ such that $\mathcal{C}^+ := \mathcal{C} \cap ([\gamma_1, +\infty) \times C(\Omega))$ and $\mathcal{C}^- := \mathcal{C} \cap ((-\infty, \gamma_1] \times C(\Omega))$ are unbounded in $\mathbb{R}^+ \times C(\Omega)$. By Step 1, we get that if $u \in \mathcal{C}^-$, then $u \geq u_{\gamma_1}$ and it is a solution of $(P_{h=0})$. Thus, from (iv) we infer that the projection of $\mathcal{C}^-$ on $\lambda$-axis is $(0, \gamma_1]$, a bounded interval, and then we deduce that the projection of $\mathcal{C}^+$ on $\lambda$-axis is $[\gamma_1, +\infty)$. Hence,
\[
\text{Proj}_\mathbb{R} \mathcal{C} = \text{Proj}_\mathbb{R} \mathcal{C}^- \cup \text{Proj}_\mathbb{R} \mathcal{C}^+ = (0, +\infty).
\]
Finally, by Theorem 3.1 for any $0 < \Lambda_1 < \Lambda_2 < \gamma_1$ there is a priori bound for the solution of $(P_{h=0})$, for all $\lambda \in [\Lambda_1, \Lambda_2]$. Then, we have also a $C^\alpha$ a priori bound for these solutions i.e. the projection of $\mathcal{C} \cap ([\Lambda_1, \Lambda_2] \times C(\Omega))$ on $C(\Omega)$ is bounded. Since the component $\mathcal{C}^-$ is unbounded in $\mathbb{R}^- \times C(\Omega)$, its projection on the $C(\Omega)$ axis must be unbounded. Therefore, we deduce that $\mathcal{C}$ must emanate from infinity on the right of axis $\lambda = 0$.

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Acknowledgment: The author Mayra Soares would like to thank the financial support received by the postdoctoral fellowship from DGAPA-Unam.

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