Abstract. In this paper, we introduce the concept of $f$-ideals and discuss its algebraic properties. In particular, we give the characterization of all the $f$-ideals of degree 2.

Key words : simplicial complex, height of an ideal, Primary Decomposition, $f$-vector.

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1. Introduction

The aim of this paper is to explore the algebraic and combinatorial properties of simplicial complexes. Let $S = k[x_1, ..., x_n]$ be a polynomial ring over an infinite field $k$. There is a natural bijection between a square-free monomial ideal and a simplicial complex:

$$\Delta \leftrightarrow I_N$$

Where $I_N$ is known as the Stanley Reisner ideal or non-face ideal of $\Delta$. This one to one correspondence has been discussed widely in the literature for instance in [1], [4], [5] and [7].

In [2], Faridi introduced another correspondence:

$$\Delta \leftrightarrow I_F$$

Where $I_F$ is the facet ideal of a given simplicial complex $\Delta$. In [2] and [3] Faridi has discussed and investigated some algebraic compatibilities of these two ideals for a given simplicial complex $\Delta$. Also when the simplicial complex is a tree (defined in [2]), its facet ideal posseses interesting algebraic and combinatorial properties discussed in [2] and [3].

Given a square free monomial ideal $I$, one can consider it as the facet ideal of a simplicial complex $\delta_F(I)$, and the Stanley-Reisner ideal of another $\delta_N(I)$. So for a square free monomial ideal $I$, one can explore some invariants of $\delta_F(I)$ and $\delta_N(I)$.

In this paper, We introduce the $f$-ideals and in Theorem 3.5 we give the characteriztion of all the $f$-ideals of degree 2. A monomial ideal is called $f$-ideal if both the simplicial complexes $\delta_F(I)$ and $\delta_N(I)$ have the same $f$-vector, where $f$-vector of
a $d$ dimensional simplicial complex $\Delta$ is the $(d+1)$-tuple:

$$f(\Delta) = (f_0, f_1, \ldots, f_d),$$

where $f_i$ is the number of faces of dimension $i$ of $\Delta$.

2. Basic combinatorics and algebra of simplicial complexes

This section is a review on the combinatorics and algebra associated to simplicial complexes discussed in [2] - [4] and [7].

**Definition 2.1.** A simplicial complex $\Delta$ over a set of vertices $V = \{x_1, x_2, \ldots, x_n\}$ is a collection of subsets of $V$, with the property that $\{x_i\} \in \Delta$ for all $i$, and if $F \in \Delta$ then all subsets of $F$ are also in $\Delta$ (including the empty set). An element of $\Delta$ is called a face of $\Delta$, and the dimension of a face $F$ of $\Delta$ is defined as $|F| - 1$, where $|F|$ is the number of vertices of $F$. The faces of dimension 0 and 1 are called vertices and edges, respectively, and $\dim \emptyset = -1$. The maximal faces of $\Delta$ under inclusion are called facets.

We denote simplicial complex $\Delta$ by a generating set of its facets $F_1, \ldots, F_q$ as

$$\Delta = \langle F_1, \ldots, F_q \rangle$$

Also, we denote the facet set by $\mathcal{F} = \{F_1, \ldots, F_q\}$. A simplicial complex with only one facet is called a simplex.

The following definitions lay the foundation of the dictionary between the combinatorial and algebraic properties of the simplicial complexes over the finite set of vertices $[n]$.

**Definition 2.2.** Let $\Delta$ be a simplicial complex over $n$ vertices $\{v_1, \ldots, v_n\}$. Let $k$ be a field, $x_1, \ldots, x_n$ be indeterminates, and $S$ be the polynomial ring $k[x_1, \ldots, x_n]$. Let $\mathcal{F}$ be the set of facets of $\Delta$.

- (a) We define $I_{\mathcal{F}}$ to be the ideal of $S$ generated by square-free monomials $x_i^{m_1} \cdots x_i^{m_s}$, where $\{v_i^{m_1}, \ldots, v_i^{m_s}\}$ is a facet of $\Delta$. We call $I_{\mathcal{F}}$ the facet ideal of $\Delta$.
- (b) We define $I_N$ to be the ideal of $S$ generated by square-free monomials $x_i^{m_1} \cdots x_i^{m_s}$, where $\{v_i^{m_1}, \ldots, v_i^{m_s}\}$ is not a face of $\Delta$. We call $I_N$ the non-face ideal or the Stanley-Reisner ideal of $\Delta$.

**Definition 2.3.** Let $I = (M_1, \ldots, M_q)$ be an ideal in a polynomial ring $k[x_1, \ldots, x_n]$, where $k$ is a field and $M_1, \ldots, M_q$ are square-free monomials in $x_1, \ldots, x_n$ that form a minimal set of generators for $I$.

- (a) We define $\delta_{\mathcal{F}}(I)$ to be the simplicial complex over a set of vertices $v_1, \ldots, v_n$ with facets $F_1, \ldots, F_q$, where for each $i$, $F_i = \{v_j \mid x_j^{m_i}, 1 \leq j \leq n\}$. We call $\delta_{\mathcal{F}}(I)$ the facet complex of $I$.
- (b) We define $\delta_N(I)$ to be the simplicial complex over a set of vertices $v_1, \ldots, v_n$, where $\{v_i^{m_1}, \ldots, v_i^{m_s}\}$ is a face of $\delta_N(I)$ if and only if $x_i^{m_1} \cdots x_i^{m_s} \notin I$. We call $\delta_N(I)$ the non-face complex or the Stanley-Reisner complex of $I$.

**Remark 2.4.** For given a square free monomial ideal $I$, one can construct $\delta_{\mathcal{F}}(I)$ by using the above definition (a). Where Faridi in [2], has given the construction of $\delta_N(I)$ by using the minimal vertex cover of $\delta_{\mathcal{F}}(I)$.
Example 2.5. Let \( I = (xy, yz) \subset k[x, y, z] \), then following are non-face complex and facet complex.

![Figure 1. Non-face and facet complex](image)

Definition 2.6. Let \( S = k[x_1, \ldots, x_n] \) be a polynomial ring, the Support of a monomial \( x^a = x_1^{a_1} \cdots x_n^{a_n} \) in \( S \) is given by \( \text{Supp}(x^a) = \{ x_i | a_i > 0 \} \). Similarly, let \( I = (g_1, \ldots, g_m) \subset S \) be a square-free monomial ideal then

\[
\text{Supp}(I) = \bigcup_{i=1}^{m} \text{Supp}(g_i)
\]

Remark 2.7. It is worth noting that for any square-free monomial ideal \( I \subset S \) the \( \delta_F(I) \) will be a simplicial complex on the vertex set \([s]\), where \( s = |\text{Supp}(I)| \). But \( \delta_N(I) \) will be a simplicial complex on \([n]\). So both \( \delta_F(I) \) and \( \delta_N(I) \) will have the same vertex set if and only if \( \text{Supp}(I) = \{x_1, \ldots, x_n\} \).

For example, for the ideal \( I = (x_2x_3, x_2x_4, x_3x_4) \) in \( S = k[x_1, x_2, x_3, x_4] \),

\[
\delta_F(I) = \langle \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\} \rangle
\]

is a simplicial complex on the vertex set \( \{v_2, v_3, v_4\} \). Whereas,

\[
\delta_N(I) = \langle \{v_1, v_4\}, \{v_1, v_3\}, \{v_1, v_2\} \rangle
\]

is the simplicial complex on the vertex set \( \{v_1, v_2, v_3, v_4\} \).

![Figure 2. facet and non-face complex](image)

Definition 2.8. Let \( I = (g_1, \ldots, g_m) \subset S \) be a square-free monomial ideal, the \( \text{deg}(I) \) is defined as:

\[
\text{deg}(I) = \text{Sup}\{\text{deg}(g_i)|i \in \{1, \ldots, m\}\}
\]
3. **f-ideals and classification of f-ideals of degree 2.**

Consider a polynomial ring $S = k[x_1, \ldots, x_n]$ over a field $k$. We say that a monomial ideal $I = (g_1, \ldots, g_m) \subset S$ where $g_1, \ldots, g_m$ are square-free monomials in $x_1, \ldots, x_n$ that form a minimal set of generators for $I$ is a pure square-free monomial ideal of degree $d$ if $\text{Supp}(I) = \{x_1, \ldots, x_n\}$ and all the monomials $g_i \in S_d$ for some $d > 0$, where $S_d$ is the graded component of $S$, or in other words all the $m_i$’s are of the same degree.

**Definition 3.1.** A square-free monomial ideal $I \subset S$ is said to be an $f$-ideal if and only if both $\delta_F(I)$ and $\delta_N(I)$ have the same $f$-vector.

There is a natural question to ask: characterize all the $f$-ideals in $S$. Here, we precisely give the characterization of $f$-ideals of degree 2. So it is still open to characterize all the $f$-ideals of degree $\geq 3$.

**Lemma 3.2.** For a pure square-free monomial ideal $I = (g_1, \ldots, g_m)$ in $S = k[x_1, \ldots, x_n]$ of degree $d$, the following equality holds:

$$\binom{n}{d} = f_{d-1}(\delta_F(I)) + f_{d-1}(\delta_N(I))$$

**Proof.** Let us take $I = (g_1, \ldots, g_m) \subset S$ be a square free monomial ideal of degree $d$, where $\{g_1, \ldots, g_m\}$ is the minimal set of generators for $I$ and $\text{deg}(g_i) = d$ for all $i \in \{1, \ldots, m\}$. So, corresponding to $I$ its facet simplicial complex $\delta_F(I)$ has

$$f_{d-1}(\delta_F(I)) = m.$$ 

As non-face complex $\delta_N(I)$ will have the $d-1$ dimensional face $\{v_{i1}, \ldots, v_{id}\}$ if and only if $x_{i1} \ldots x_{id} \notin I$ clear from the definition of $\delta_N(I)$. So $\delta_N(I)$ will have those $d-1$ dimensional faces which are not appearing in $\delta_F(I)$ because $I$ is a pure square-free monomial ideal of degree $d$. Also, for a simplicial complex on $n$ vertices the possible $d-1$ dimensional faces are $\binom{n}{d}$. Therefore, $f_{d-1}(\delta_N(I)) = \binom{n}{d} - f_{d-1}(\delta_F(I))$. □

**Remark 3.3.** For instance, in example 2.5 one can see that $I = (xy, yz) \subset k[x, y, z]$ is a pure square-free monomial ideal and;

$$\binom{3}{2} = f_1(\delta_N(I)) + f_1(\delta_F(I))$$

$$\binom{3}{2} = 1 + 2.$$ 

**Lemma 3.4.** Let $I$ be a square-free monomial ideal in $S$, $\dim(\delta_F(I)) = \dim(\delta_N(I))$

if and only if $\text{ht}(I) + \text{deg}(I) = n$.

**Proof.** From definition 2.3(a) it is clear that

$$\dim(\delta_F(I)) = \text{deg}(I) - 1.$$
Also, from Proposition 5.3.10 of [7] it is clear that,
\[ \dim(\delta_N(I)) = n - \text{ht}(I) - 1. \]
Which concludes the proof. \( \square \)

One dimensional simplicial complexes on the vertex set \([n]\) are the simple graphs. Also for a one dimensional simplicial complex the ideal \(I_F\) is same as the edge ideal of a graph, for details see [7].

Our main theorem is as follows:

**Theorem 3.5.** A pure square-free monomial ideal \(I = (g_1, \ldots, g_m) \subset S\) of degree 2 will be an \(f\)-ideal if and only if:

(i) \(I\) is unmixed with \(\text{ht}(I) = n - 2\),

(ii) \(\binom{n}{2} \equiv 0 \pmod{2}\) and

(iii) \(m = \frac{1}{2} \binom{n}{2}\)

**Proof.** Suppose \(I = (g_1, \ldots, g_m) \subset S\) is a pure square-free monomial ideal of degree 2 and let \(I\) be an \(f\)-ideal. So we have \(\dim(\delta_N(I)) = 1 = \dim(\delta_F(I))\) which by Lemma 3.4 implies \(\text{ht}(I) = n - 2\). As \(I\) is a pure square-free monomial ideal of degree 2, \(\delta_F(I)\) is a graph on the vertex set \([n]\) with no isolated vertex. So, since \(f(\delta_F(I)) = f(\delta_N(I))\), \(\delta_N(I)\) needs to be a graph on the same vertex set \([n]\) with no isolated vertex. As \(I_N = \cap F\) where the intersection is over all facets \(F\) of \(\delta_N(I)\) by [7] 5.3.10, which implies \(I\) is unmixed of height \(n - 2\).

As \(f_i\) denotes the number of \(i\)-dimensional faces, so
\[ f_1(\delta_F(I)) = m \]
where \(m\) is the number of monomial generators of \(I\). Also from Lemma 3.2
\[ f_1(\delta_N(I)) = \left( \binom{n}{2} \right) - m \]
As \(I\) is an \(f\)-ideal, so we have \(\binom{n}{2} = 2m \equiv 0 \pmod{2}\). Conversely, let us take the pure square-free monomial ideal \(I = (g_1, \ldots, g_m) \subset S\) of degree 2 satisfying the conditions (i), (ii) and (iii). The simplicial complexes \(\delta_F(I)\) and \(\delta_N(I)\) will have the same \(f\)-vector follows immediately from Lemma 3.2 and Corollary 3.4. \( \square \)

We conclude this paper with the following example.

**Example 3.6.** Let \(I = (x_1x_2, x_2x_3, x_3x_4) \subset S = k[x_1, x_2, x_3, x_4]\) be a pure square-free monomial ideal satisfying the conditions given in the above theorem, and \(\delta_N(I)\) and \(\delta_F(I)\) are as follows:

Clearly, both the simplicial complexes have the same \(f\)-vector \((4, 3)\). Hence \(I\) is an \(f\)-ideal of degree 2.
**Figure 3.** Non-face and facet complex

**References**

[1] W. Bruns, J. Herzog, *Cohen Macaulay rings*, Vol.39, Cambridge studies in advanced mathematics, revised edition, 1998.

[2] S. Faridi, *The facet ideal of a simplicial complex*, Manuscripta Mathematica, 109, (2002), 159-174.

[3] S. Faridi, *Simplicial Tree are sequentially Cohen-Macaulay*, Arxiv:math.AC/0702569.

[4] E. Miller, B. Sturmfels, *Combinatorial Commutative Algebra*, Springer-Verlag New York Inc. 2005.

[5] R. P. Stanley, *Combinatorics and commutative algebra*, Second edition. Progress in Mathematics, 41. Birkhuser Boston, MA, 1996, x+164 pp. ISBN: 0-8176-3836-9.

[6] R. P. Stanley, *Cohen-Macaulay Rings and constructible polytopes*, Bull. Amer. Math. Soc. 81(1975), 133-142.

[7] R. H. Villarreal, *Monomial algebras*, Dekker, New York, 2001.

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