A CONSTRUCTION OF THE QUANTUM STEENROD SQUARES AND THEIR ALGEBRAIC RELATIONS

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Abstract. We construct a quantum deformation of the Steenrod square construction on closed monotone symplectic manifolds, based on the work of Fukaya, Betz and Cohen. We prove quantum versions of the Cartan and Adem relations. We compute the quantum Steenrod squares for all \( \mathbb{CP}^n \) and give the means of computation for all toric varieties. As an application, we also describe two examples of blowups along a subvariety, in which a quantum correction of the Steenrod square on the blowup is determined by the classical Steenrod square on the subvariety. The relationship between the quantum Steenrod square and Seidel’s equivariant pair-of-pants product will be explored in a further paper.

1. Introduction

The Steenrod squares are additive homomorphisms

\[ Sq^i : H^n(M) \to H^{n+i}(M) \]

using \( \mathbb{Z}/2 \) coefficients, which generalize the squaring operation on cohomology with respect to the cup product, \( x \mapsto x \cup x \).

Because we use \( \mathbb{Z}/2 \) coefficients and \((x + y)^2 = x^2 + y^2 \) modulo 2, the \( Sq^i \) are additive and hence well defined.

These operations determine the Steenrod square,

\[ Sq : H^*(M) \to H^*(M)[h] \]

where

\[ Sq(x) = \sum Sq^{|x|-i}(x) h^i. \]

Here \( h \) is a formal variable in degree 1 that represents the generator of

\[ H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[h]. \]

The Steenrod square satisfies the Cartan relation

\[ S(q(x \cup y) = S(q(x) \cup S(q(y)) \]

which, for example, allows one to inductively compute the Steenrod squares for \( \mathbb{CP}^n \) (which we will review in example 2.3).

The Steenrod squares are cohomology operations, historically constructed first on \( H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2) \) for Eilenberg-MacLane spaces \( K(\mathbb{Z}/2, n) \). This paper describes two different constructions: with Morse homology and with intersections of cycles. The first is based on the definition for Floer theory by Seidel in [10], with origins in

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the flowlines construction mentioned by Betz in [3], and extended by Betz, Cohen and Norbury in [2, 5].

We then construct a quantum Steenrod square on the quantum cohomology of a closed monotone symplectic manifold \((M, \omega)\).

Recall that the quantum cohomology \(QH^*(M, \omega)\) is \(H^*(M)[[t]]\) as a vector space using a graded formal variable \(t\) of degree 2, but the cup product is deformed by quantum contributions from counting 3-pointed genus zero Gromov-Witten invariants, i.e. counting certain \(J\)-holomorphic spheres in \(M\) for \(J\) an almost complex structure on \(M\) compatible with \(\omega\). We often abbreviate by \(T = t^N\) for \(N\) the minimal Chern number. A full description is given in Section 2.3.

The quantum Steenrod square will be a degree doubling operation

\[
QS : QH^*(M) = H^*(M)[[t]] \to H^*(M)[[t]][h] = QH^*(M)[h]
\]

involving additive homomorphisms \(QS_{i,j}\) with

\[
QS(x) = \sum_{i,j \geq 0} QS_{i,j}(x) h^iT^j.
\]

A construction of the quantum Steenrod square was first suggested by Fukaya in [6] based on Betz and Cohen’s definition of the classical Steenrod square in [2]. Our construction (of the same operations) is slightly different, and can be viewed as a Morse theory analogue of the work by Seidel in Floer theory in [10]. The first goal of this paper is to solve an open problem posed by Fukaya in [6, Problem 2.11] as to whether the Adem and Cartan relations hold for quantum Steenrod squares and, if not, what their quantised versions should be.

**Example 1.1.** \(M = \mathbb{P}^1\). Let \(x\) be the generator of \(H^2(M)\). Recall that the quantum product is \(x \ast x = T\), where \(T = t^2\) has degree 4. Then

\[
QS(x \ast x) = QS(T) = T^2.
\]

For degree reasons \(QS(x) = xh^2 + T\). Thus

\[
QS(x) \ast QS(x) = (xh^2 + T) \ast (xh^2 + T) = Th^4 + T^2.
\]

This shows that the immediate generalisation of the Cartan relation fails, i.e. that \(QS : QH^*(M) \to QH^*(M)[h]\) is not a ring homomorphism.

The reason the more obvious generalisation of the Cartan relation fails is because the moduli space \(\overline{M}_{0,5}\) of genus zero stable curves with 5 marked points has non-trivial \(\mathbb{Z}/2\)-equivariant cohomology, under the \(\mathbb{Z}/2\) action that transposes marked points via \((12)(34)\). More precisely, the two configurations in Figure 5 that determine \(QS(x \ast y)\) and \(QS(x) \ast QS(y)\) are not connected by a \(\mathbb{Z}/2\)-invariant path in \(\overline{M}_{0,5}\). We will explain this in detail in Section 5.

We will prove however that a quantum deformation of the Cartan relation holds:

**Theorem 1.2 (Quantum Cartan relation).**

\[
QS(x \ast y) = QS(x) \ast QS(y) + \sum_{i,j} q_{i,j}(W_0 \times D^{i-2,+})(x, y)h^i
\]

where the notation \(q_{i,j}\) will be defined precisely in Section 6.
The correction term is an operation determined by the homology class
\[ W_0 \times D^{i,+} \subset M_{0.5} \times \mathbb{Z}/2 S^{\infty} \]
where \( W_0 \subset M_{0.5} \cong Bl_{((0,0),(1,1),C^{\infty})}(\mathbb{C}P^1 \times \mathbb{C}P^1) \) is the exceptional divisor over \((0,0)\) (compare Figure 5).

Using Theorem 1.2 we can calculate the quantum Steenrod squares for all Fano toric varieties:

**Theorem 1.3.** Let \( M \) be a Fano toric manifold. For \( b, x \in H^*(M) \) and \(|x| = 2\),

\[ q_{i,j}(W_0 \times D^{i,+})(b, x) = \sum_{j \geq 1} \sum_{k=1}^j \sum_{c_1(\mu) = 2kN} \#(PD(x) \cap \mu) \cdot (QS_{b,-i+2,j-k}(b) \ast_{\mu,k} x) \cdot h^i T^j \]

summing over \( \mu \in H_2(M) \), where \( \ast_{\mu,k} \) denotes the coefficient of \( T^k \) in the quantum product, using spheres representing \( \mu \). \( N \) is the minimal Chern number, and \(|T| = 2N\).

For example, if \( M = \mathbb{C}P^n \) then setting \( b = x^i \) for \( x \in H^2(\mathbb{C}P^n) \) being the generator, we have:

\[ q_{4i,2-2n,1}(W_0 \times D^{4i-2n,+})(x^i, x) = \binom{i}{n-i} T, \text{ else } q_{i,j} = 0 \]

hence

\[ QS(x^i) = \sum_{j=0}^i \left( \binom{n/2+1}{j} \sum_{k=0}^{n-1} \left( \binom{n-k}{k} \cdot \binom{i-(n+1-k)}{j-k} \right) \right) x^{i+j} h^{2(i-j)} \]

for \( x^p \) the \( p \)-th quantum power of \( x \). Omitting the inner summation would give the classical Steenrod square.

**Corollary 1.4.** Let \( M \) be a Fano toric manifold. Then \( QS \) is determined by \( QH^*(M) \).

Recall that the classical Adem relations are relations between compositions of the \( Sq^i \). Namely, for all \( p, q > 0 \) such that \( q < 2p \),

\[ Sq^p Sq^q = \sum_{s=0}^{[q/2]} \binom{p - s - 1}{q - 2s} Sq^{p+q-s} Sq^s \]

where \([i/2]\) is the integer part of \( i/2\).

**Definition 1.5.** Define \( QS_{a,b} \) by

\[ QS(xT^i) = \sum_{s=1}^{s=1} QS_{a,b}(xT^i) \cdot h^{i-2N_{i-a}} \text{ where } QS_{a,b}(xT^i) \in T^{b+i}H^*(M), \]

for any \( x \in H^*(M) \).

**Example 1.6.** Let \( M = \mathbb{C}P^2 \), so \( 2N = 6 \). Then \( QS_{2^{-2N,1}} \circ QS_{2^{-0N,0}}(x) = T \), but

\[ \sum_{s=0}^{s=1} \binom{1-s}{2-2s} QS_{2^{-2N,1}} \circ QS_{s^{-2N,0}}(x) = 0 \]
for all $i, j$. This demonstrates that the immediate generalisation of the Adem relations fails.

**Corollary 1.7 (Quantum Adem Relations).** For $p, q > 0$, $q < 2p$, $\alpha \in QH^*(M)$,

$$
\sum_{b,d} \left( QS^{q,b} \circ QS^{p,d}(\alpha) - \sum_{s=0}^{q/2} \left( \frac{p-s-1}{q-2s} \right) QS^{p+q-s,b} \circ QS^{s,d}(\alpha) \right) = T \cdot Q(\alpha) \quad (3)
$$

for the correction term

$$
T \cdot Q(\alpha) = qD_8 \left( (gm_1 + g^2m_1) \otimes \Psi(e^{[\alpha]+p-q\sigma_2[\alpha]-p}) \right)(\alpha) - \sum_{s=0}^{[q/2]} \left( \frac{q-s-1}{q-2s} \right) qD_8 \left( (gm_1 + g^2m_1) \times \Psi(e^{[\alpha]+2s-p-q\sigma_2[\alpha]-s}) \right)(\alpha).
$$

The $qD_8$ is an operation determined by homology classes in $\overline{M_{0,5}} \times D_8 ED_8$. Here $m_1 \in \overline{M_{0,5}}$, $e^2\sigma_2 \in H^*(BD_8)$, $\Psi$ is the universal coefficients isomorphism and $g = (123) \in S_3$ such that the cosets of $D_8$ in $S_4$ are $D_8, gD_8, g^2D_8$.

The quantum Steenrod squares can also be calculated in the case of $Bl_Y(\CP^3)$ and $Bl_Y(\CP^1 \times \CP^1 \times \CP^1)$ for $Y$ respectively the intersection of two quadrics and the intersection of two linear hypersurfaces. The set up here is similar to Blaier [4]. We will prove:

**Theorem 1.8.**

$$
QS_{1,1} = id : H^3(Bl_Y(\CP^3)) \to H^3(Bl_Y(\CP^3)) \quad (4)
$$

and

$$
QS_{1,1} = id : H^3(Bl_Y(\CP^1 \times \CP^1 \times \CP^1)) \to H^3(Bl_Y(\CP^1 \times \CP^1 \times \CP^1)). \quad (5)
$$

Observe that $QS_{1,1}$ are quantum correction terms to the classical Steenrod square on the blowup. Nevertheless, these are determined by lifts of contributions to the classical Steenrod square on $Y$.

There is a PSS isomorphism $\psi : QH^*(M) \to HF^*(M, H)$ for closed monotone symplectic manifolds. Likewise an isomorphism exists for certain open symplectic manifolds (Liouville manifolds) in the exact case for “small” Hamiltonians (Hamiltonians radial at infinity with small slope). In all cases this isomorphism generalises to give an isomorphism between equivariant quantum cohomology and equivariant Floer cohomology that intertwines the quantum Steenrod square and the equivariant pair-of-pants defined in [10].

Symplectic cohomology is a direct limit of the groups $HF^*(M, H)$ as $H$ becomes “large” at $\infty$. This yeilds a canonical unital ring homomorphism $c^* : QH^*(M) \to SH^*(M)$, although this is not typically an isomorphism. There is an equivariant version of the $c^*$ map, and this also intertwines the quantum Steenrod square and the equivariant pair-of-pants.

Analogues of the quantum Cartan relation also hold for the equivariant pair-of-pants in both Floer and symplectic cohomology.

This work will appear in a forthcoming paper.
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2. Preliminaries

Henceforth we always work with coefficients in $\mathbb{Z}/2$. For example $H^*(M)$ means $H^*(M;\mathbb{Z}/2)$.

2.1. Equivariant Cohomology. We follow section 2 of [10].

Definition 2.1 (Equivariant cohomology of a chain complex). Let $(C^\bullet,d)$ be a cochain complex over $\mathbb{Z}/2$. Suppose $(C^\bullet,d)$ has a chain involution $\iota$, so $\iota : C^\bullet \to C^\bullet$ is a chain map with $\iota^2 = id_{C^\bullet}$.

Let $h$ be a formal variable in grading 1. The equivariant chain complex is $(C^\bullet \mathbb{Z}/2, \delta) = (C^\bullet[[h]], d + h(id_{C^\bullet} + \iota))$.

Define $H^\ast \mathbb{Z}/2(C) := H^*(C^\bullet \mathbb{Z}/2, \delta)$, the equivariant cohomology of $(C,d,\iota)$.

Definition 2.2 (Equivariant Cohomology of a manifold). Let $N$ be a topological space with a continuous involution $\iota : N \to N$. Let $C = C^\ast(N)$ be the singular cochain complex of the topological space $N$. There is a $\mathbb{Z}/2$ action on $C^\ast(N)$ induced by $\iota$. As in Definition 2.1, define the equivariant cohomology of $N$ as $H^\ast \mathbb{Z}/2(N) := H^\ast (C^\ast(N))$.

The important examples of this will be $M^2$ with the involution swapping the factors and $M$ with the trivial involution.

Remark. There is another description of $H^\ast \mathbb{Z}/2(N)$ for a manifold $N$ with a continuous involution. Recall that $EZ/2$ is the classifying space of $\mathbb{Z}/2$: a contractible space with a free $\mathbb{Z}/2$ action, for example $EZ/2 = S^\infty$ with the involution being the antipodal map. Then

$$H^\ast \mathbb{Z}/2(N) := H^\ast (N \times_{\mathbb{Z}/2} EZ/2).$$

This definition is the same as the previous one.

If we let $N = \{pt\}$ then we obtain $pt \times_{\mathbb{Z}/2} S^\infty = S^\infty / (\mathbb{Z}/2) = \mathbb{R}P^\infty$.

Hence

$$H^\ast \mathbb{Z}/2(pt) = H^\ast (\mathbb{R}P^\infty) = \mathbb{Z}/2[h].$$

2.2. The Steenrod Squares. For a reference, see [7, Section 4.L].

The Steenrod square operations $\{Sq^i\}$ are the unique collection of additive homomorphisms such that:

1. $Sq^i : H^n(M) \to H^{n+i}(M)$ for each $n \geq 0$ and topological space $M$,
2. Each $Sq^i$ is natural in $M$,
3. $Sq^0$ is the identity,
4. $Sq^n$ acts as the cup square on $H^n$, so $Sq^{|x|}(x) = x \cup x$,
5. If $n > |x|$ or $n < 0$ then $Sq^n(x) = 0$,
6. (Cartan relation) For each $n$, $Sq^n(x \cup y) = \sum_{i+j=n} Sq^i(x) \cup Sq^j(y)$. 
$|x|$ is the cohomological grading of $x \in H^*(M)$. Recall that we use $\mathbb{Z}/2$ coefficients to ensure additivity: $(x + y) \cup (x + y) = x \cup x + y \cup y$ modulo 2.

These $Sq^i$ define a single operator $Sq : H^n(M) \to (H^*(M)[h])^{2n}$, where $Sq^i$ is the coefficient of $h^{n-i}$, so $Sq(x) = \sum Sq^{|x| - i}(x) \cdot h^i$.

The cup product on $H^n(M)[h]$ is $(a \cdot h^i) \cup (b \cdot h^j) = (a \cup b) \cdot h^{i+j}$, so the Cartan relation becomes $Sq(x \cup y) = Sq(x) \cup Sq(y)$ and thus $Sq$ is a unital ring homomorphism.

These axioms imply uniqueness, but one also needs to prove existence.

**Example 2.3** (The classical Steenrod square for $\mathbb{CP}^n$).

$$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$$

where $|x| = 2$.

$Sq^0(x) = x$ and $Sq^2(x) = x^2$ using axioms 3 and 4, and these are all of the nonzero terms by axiom 5. Hence $Sq(x) = xh^2 + x^2$. By the Cartan relation (axiom 6),

$$Sq(x^i) = Sq(x)^i = (xh^2 + x^2)^i = x^i \sum_{j=0}^{i} \binom{i}{j} x^j h^{2(i-j)}.$$ 

Looking at the coefficient of $h^{2i-k}$, $Sq^k(x) = 0$ for $k$ odd and $Sq^{2j}(x) = \binom{j}{i} x^{i+j}$.

### 2.3. The Quantum Cup Product

For more details on the quantum cup product, see chapter 8 of [8]. Throughout, $PD$ refers to the Poincaré duality operation over $\mathbb{Z}/2$ coefficients for closed manifolds.

**Definition 2.4.** A symplectic manifold $M$ is monotone if there exists a constant $\lambda > 0$ such that every $J$-holomorphic sphere $u : S^2 \to M$ satisfies

$$c_1(u_*([S^2])) = \lambda \cdot E(u)$$

where $E(u) = \int_{S^2} u^*\omega \geq 0$ is the symplectic energy of $u$ and $c_1 = c_1(TM, J)$.

Let $(M, \omega)$ be a monotone symplectic manifold of dimension $n$, with a fixed almost complex structure $J$ compatible with $\omega$.

As an abelian group, $QH^*(M) = H^*(M)[[t]]$ where $t$ is a formal variable of degree 2. Let $T = t^n$ for $N \geq 0$ the minimal Chern number of $M$ determined by $c_1(\pi_2(M)) = NZ$. Throughout we assume $\lambda = N$.

Pick a basis $B$ for $H^*(M)$ and a dual basis with respect to the nondegenerate cup product pairing $(e, f) \mapsto \langle e \cup f, [M] \rangle$. Then one can form a dual basis $B^\vee$ with respect to this pairing. Let $\alpha^\vee \in H^{n-|\alpha|}(M)$ denote the dual of the cohomology class $\alpha \in H^{2|\alpha|}(M)$. Our operations on cohomology will not depend on this choice of basis, although they may affect the chain level description.

Given $A \in H_2(M)$, let $\mathcal{M}_A(J)$ be the moduli space of $J$-holomorphic spheres $u : S^2 \to M$ such that $u_*([S^2]) = A$, up to reparametrisation by $PSL(2, \mathbb{C})$. For a generic choice of $J$, this moduli space is a smooth manifold with

$$\dim \mathcal{M}_A(J) = 2c_1(A) + \dim(M).$$

For each $z \in S^2$, there is an evaluation map $ev_{A,z} : \mathcal{M}_A(J) \to M$ with $ev_{A,z}(u) = u(z)$.

Pick three distinct points, $z_1, z_2, z_3 \in S^2$. We use $0, 1, \infty$ throughout.
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**Definition 2.5 (Quantum Product).** For \( \alpha, \beta \in H^*(M) \subset QH^*(M) \),

\[
\alpha \ast \beta = \sum_{j \in \mathbb{Z}, \gamma \in B: |\gamma| = |\beta| + |\alpha| - 2jN} n(\gamma, \alpha, \beta, j) \cdot \gamma \cdot T^j,
\]

\[
n(\gamma, \alpha, \beta, j) = \sum_{A \in H_2(M), c_1(A) = jN} \# ev_{A,z_1}^{-1}(PD(\alpha)) \cap ev_{A,z_2}^{-1}(PD(\beta)) \cap ev_{A,z_3}^{-1}(PD(\gamma'))
\]

for \( \# \) denoting the mod 2 count of a 0-dimensional compact manifold.

Extending \( \mathbb{Z}/2[|t|] \)-linearly defines \( \ast \) on \( QH^*(M) \).

In concrete terms we count the number of \( J \)-holomorphic spheres in \( M \) intersecting the cycles \( PD(\alpha), PD(\beta) \) and \( PD(\gamma') \). The degree condition ensures the intersection is a 0-dimensional compact submanifold of \( M_A(J) \) (for generic \( J \), the evaluation maps satisfy the necessary transversality conditions, \[8\]).

Notice that \( \ast \) is compatible with the grading, using \( |T| = 2N \).

If \( A = 0 \) (so \( E(u) = 0 \) and \( u \) is constant), the intersection is a point and this recovers the classical intersection product.

### 3. Two Constructions of the Steenrod Squares

The first construction will be based on that given in \[10\], \[6\] and \[2\]. The second construction is new.

In this section \( \Gamma \) is the Y-shaped graph with incoming edge \( e_1 \) and outgoing edges \( e_2 \) and \( e_3 \). Let \( e_1 \) be parametrised by \((-\infty, 0]\) and \( e_2, e_3 \) by \([0, \infty)\).

Recall the Morse theoretic cup product: given a manifold \( M \) and a Morse function \( f \), pick three generic perturbations \( f^i_s \) for \( s \in (-\infty, 0] \) and \( f^2_s, f^3_s \) for \( s \in [0, \infty) \), transverse at 0 (genericity ensures the moduli space below is cut out transversely). There is an \( R > 0 \) such that \( f^i_s = f \) if \( |s| \geq R \).

Denote by \( \text{Crit}_k(f) \) the critical points of \( f \) of Morse index \( k \). Write \( |x| \) for the Morse index of \( x \in \text{Crit}(f) \).

**Definition 3.1 (Morse cup product).** Let \( a_2, a_3 \) be critical points of \( f \), with Morse indices \( |a_2|, |a_3| \) and let \( k = |a_2| + |a_3| \), then

\[
a_2 \cdot a_3 := \sum_{a_1 \in \text{Crit}_k(f)} n_{a_1,a_2,a_3} a_1
\]

where \( n_{a_1,a_2,a_3} \) is the number of solutions in the 0 dimensional moduli space \( \mathcal{M}(f^i_s, a_1, a_2, a_3) \) of continuous maps \( u : \Gamma \to M \), smooth on the edges, such that:

1. \( d(u|_{e_1})/ds = -\nabla f^i_s \),
2. \( u|_{e_1}(x) \to a_1 \) as \( x \to -\infty \),
3. \( u|_{e_i}(x) \to a_i \) as \( x \to \infty \) for \( i = 2, 3 \).

If \( a_2 = a_3 \) then there is a \( \mathbb{Z}/2 \) symmetry in maps \( u : \Gamma \to M \) with conditions as above, which one exhibits to define Steenrod squares as follows.
3.1. Morse Steenrod square. We refine the choice of \( f^i_s \) by picking a collection of smooth functions \( f^i_{v,s} : M \to \mathbb{R} \), smoothly parametrised by \( v \in S^\infty \) and \( s \in (-\infty,0] \) for \( i = 1 \), respectively \( s \in [0,\infty) \) for \( i = 2,3 \), satisfying the following conditions:

1. \( f^2_{v,s} = f^3_{v,s} \), and \( f^2_{v,s} \) is generic near \( s = 0 \) so that the moduli space below is cut out transversely.

2. There is an \( R > 0 \) such that \( f^i_{v,s} = f \) for \( |s| \geq R \).

3. \( f^1_{v,s} = f^1_{v,s} \).

For each \( v \in S^\infty \) we define \( M'_v(a_1,a_2,a_3) \) to be the set of pairs \( (u : \Gamma \to M,v) \) such that:

- \( d(u|_{e_i})/ds = -\nabla f^i_{v,s} \).
- \( u|_{e_1}(x) \to a_1 \) as \( s \to -\infty \) and \( u|_{e_i}(x) \to a_i \) for \( i = 2,3 \) as \( s \to \infty \).

In this section we assume \( a_2 = a_3 \) as in figure 1 and we abbreviate \( M'_v(a_1,a_2) := M'_v(a_1,a_2,a_2) \).

Choose a nested sequence \( S^0 \subset S^1 \subset \ldots \subset S^\infty \) of equators in \( S^\infty \) that exhaust \( S^\infty \) and are preserved under the involution \( v \mapsto -v \).

Let \( M'_i(a_1,a_2) = \bigcup_{v \in S^i} M'_v(a_1,a_2) \), topologised so that the projection to \( S^i \) is continuous for all \( i \).

Let \( r : \Gamma \to \Gamma \) be the reflection that swaps \( e_2 \) and \( e_3 \) (preserving parametrisations) and fixes \( e_1 \). There is a free \( \mathbb{Z}/2 \) action on the moduli space \( M'_i(a_1,a_2) \),

\[ (u,v) \mapsto (u \circ r, -v). \]

Let \( M_i(a_1,a_2) = M'_i(a_1,a_2)/(\mathbb{Z}/2) \), the quotient by the \( \mathbb{Z}/2 \) action.

Consider the natural projection to \( S^i \). Over generic \( v \in S^i \) there is a moduli space of degree \( |a_1| - 2|a_2| \), so the dimension of the moduli space is

\[ \dim M_i(a_1,a_2) = |a_1| - 2|a_2| + i. \]

**Definition 3.2** (The Morse Steenrod Square). Let \( a \in H^{|a|}(M) \).
Define $Sq : H^{|a|}(M) \to H_{\mathbb{Z}/2}^{2|a|}(M)$ by

$$Sq(a) = \sum_{i \in \mathbb{Z}, \ b \in \text{Crit}_{2|a| - i}(f)} n_{a,b,i} \cdot b \cdot h^i$$

where $n_{a,b,i} = \# M_i(b,a)$ for $\#$ the number of points modulo 2.

Define $Sq^i(a) \in H^{|a|+i}(M)$ to be the coefficient of $h^{|a|-i}$

One can show that this is another definition of the Steenrod square by proving that it is equivalent to the definition in [5]. We now verify the axioms.

**Lemma 3.3.** $Sq$ satisfies the axioms from section 2.2.

**Sketch proof.** Some properties will use the definition in the next section.

**Axiom 1 and 2:** these are immediate from the definition of $Sq^i$ and naturality is true for the same reason as for the Morse cup product.

**Axiom 4:** $Sq^{|x|}(x) = x^2$ as this counts the moduli space $M^0(y, x)$ for $|y| = 2|x|$, and the definition is the same as the Morse cup product.

**Axiom 5 (1):** $Sq^i(x) = 0$ for $i > |x|$ as only non-negative powers of $h$ are allowed: $\mathbb{R}P^\infty$ has no cohomology in negative degrees so $\mathcal{M}_{|x| - i}(\cdot, x) = \emptyset$.

**Axiom 5 (2):** $f_{v,s}$ is a perturbation of $f$. Assume the perturbation is small. For generic $f$ there is no $-\nabla f$ flowline from $b$ to $a$ if $|b| < |a|$.

As $f_{v,s}$ is close to $f$, this means that generically for any $v$ there is no ‘flowline’ from $b$ to $a$ that has gradient $-\nabla f$ for $s < 0$ and $-\nabla f_{v,s}$ for $s > 0$.

Hence $Sq^i(x) = 0$ for $i < 0$.

We prove axiom 3 in subsection 3.4 and axiom 6 in subsection 3.2. \hfill $\square$

**Remark.** It is not straightforward to prove $Sq^0 = id$ without a specific choice of Morse functions. We prove it in section 3.3 using a different approach.

### 3.2. The Cartan Relation.

Let $T$ be a family of graphs as in figure 2 parametrised by $t \in (0, \infty)$. Edges $e_2, e_5$ are parametrised by $[0, t]$. Compactify $T$ by adding the 0 and $\infty$ as in the figure, so the compactification $T^c \cong [0, 1]$.

Use edge labels as given in Figure 2. Fix a Morse function $f$ on $M$. We always parametrise the edges $p$ with the appropriate coordinates $s$.

Pick 5 perturbations of $f$ corresponding to the 5 tree edges in $t = 0 \in T^c$ in figure 2. These are $f^p_{v,s,0}$ for $p$ the edge label, $s \in \mathbb{R}^\pm$ and $v \in S^\infty$. For asymptotics, $f^1 = f$ for all $s, v$ and $f^3, f^4, f^6, f^7 = f$ for $s$ near $+\infty$.

We ensure that $f^3_{v,s,0} = f^4_{-v,s,0}$ and $f^6_{v,s,0} = f^7_{v,s,0}$ for all $v, s$. We choose $f^p$ along with $S_0$ such that $f^p_{v,s,0} = f$ for $s \geq S_0$.

Choose 7 perturbations of $f$ labelled $f^p_{v,s,t}$ for $p = 1, \ldots, 7$ corresponding to the edge labels in Figure 2 where $t \in T^C$, $v \in S^\infty$ and $s \in \mathbb{R}^+$ for $p = 3, 4, 6, 7$, $s \in \mathbb{R}^-$ for $p = 1$ and $s \in [0, t]$ for $p = 2, 5$. Choose $f^1$ to be independent of $s, v, t$ in this case. Choose Morse functions $f^2_{s,2}, f^5_{s,2}$ for $s \in [0, 2]$ such that $f^s = f$ for $s > 1$ and $p = 2, 5$.

The $f^p$ must be chosen generic at each vertex of $\Gamma$, to ensure transversality of moduli spaces. The $f^p_{v,s,t}$ satisfy the following conditions:

1. $f^p_{v,s,t} = f^p_{v,s,0}$ as picked previously for $p = 1, 3, 4, 6, 7$.
2. $f^3_{v,s,t} = f^4_{-v,s,t}$ and $f^6_{v,s,t} = f^7_{-v,s,t}$ for all $s, t$.
(3) $f^2, f^5$ are independent of $v$.

(4) For $t \geq 2$, $f^p_{s,t} = f^p_{s,2}$ for $s \leq 2$, and $f^p_{s,t} = f$ for $s \geq 2$.

Fix $i \in \mathbb{N}$, $x, y \in \text{Crit}(f)$.

Let $T \rightarrow T^c$ consist of pairs $\left(|t|, t\right)$ where $t \in T^c \cong [0, 1]$ and $|t|$ is the metric tree represented by $t$ as a topological space (with outer edges semi-infinite and inner edges of length $t$).

For $z \in \text{Crit}^{2|a|+2|b|-i}(f)$ consider the space $\mathcal{M}_1(x, y, z)$ of pairs $(u, v)$ with $u : T \rightarrow M$ a map and $v \in S^{||x|+|y|-i}$, such that $u$ satisfies:

$$\partial_s u_{s,t} = -\nabla f^p_{v,s,t}$$

along edge $p$, with asymptotic conditions $(z, x, x, y, y)$ on the exterior edges $(1, 3, 4, 6, 7)$.

For generic $t \in T^c$ there is a 0-dimensional subset of pairs $(u(\cdot), t, v)$ satisfying the condition. So $\mathcal{M}_1(x, y, z)$ is 1-dimensional.

$\mathcal{M}_1(x, y, z)$ has a free $\mathbb{Z}/2$ action, $(u, v) \mapsto (u \circ \bar{\pi}, -v)$ for $\bar{\pi}$ acting on $T$ by the permutation of edges $(34)(67)$. Let $\mathcal{M}_{1,\pi}(x, y, z) = \mathcal{M}_1(x, y, z)/(\mathbb{Z}/2)$, which is still 1-dimensional.
Figure 3. Tree labelling for $\mathcal{M}_2$.

We also define a moduli space $\mathcal{M}_2(x,y,z)$ by choosing another 7 Morse functions, labelled $f^p_{v,s,t}$ as above, but now with the conditions:

1. $f^p_{v,s,t} = f^q_{v,s,t}$ for $(p,q) = (3,4),(6,7),(2,5)$
2. $f^p_{v,s,t}$ is independent of $v,t$ for large enough $t$, for $p = 1,3,4,6,7$
3. For large enough $t$ and $s \in [1,t]$, $f^2_{v,s,t} = f^5_{v,s,t} = f$

In defining equations for pairs $(u,v) \in \mathcal{M}_2(x,y,z)$, use the edge labellings in figure 3, i.e. the edge labels 4 and 6 from figure 2 have been swapped. For each edge label the equations and asymptotic conditions are the same as in the $\mathcal{M}_1$ case. Further, there is a free $\mathbb{Z}/2$ action on $\mathcal{M}_2(x,y,z)$ similarly to $\mathcal{M}_1$ but with edge permutation $(25)(36)(47)$. Quotienting gives $\mathcal{M}_{2,\pi}(x,y,z)$.

Remark. Some of the conditions require a choice of $f^p_{v,s,t}$ that becomes independent of $v$ for large enough $t$. We can do this with moduli spaces remaining transverse because we keep the $v$ dependence on some of the other $f^q$.

Specifically, there is a tuple $(f^1_{v,s,t},...,f^7_{v,s,t})$ for each $v \in S^\infty$, along with associated $\mathbb{Z}/2$ action $\iota$. Our choices ensure that $\iota : v \mapsto -v$ always acts freely on tuples, hence the union of tuples over all $v \in S^\infty$ is an $\mathbb{E}\mathbb{Z}/2 \subset (C^\infty(M))^7$.

Each pair $(f^i_{v,s,t},f^j_{v,s,t})$ for $(i,j) = (2,5),(3,4),(6,7)$ also defines an $\mathbb{E}\mathbb{Z}/2$. So as long as $\iota$ acts freely on some pair, the set of these tuples is an $\mathbb{E}\mathbb{Z}/2$. This is the fact that if $E,F$ are both an $\mathbb{E}\mathbb{Z}/2$ then $E \times F$ is an $\mathbb{E}\mathbb{Z}/2$.

Theorem 3.4 (The Cartan Relation).

$$Sq^i(x \cup y) = \sum_{j+k=i} Sq^j(x) \cup Sq^k(y)$$

Proof. $\#\partial_1 \mathcal{M}_1(x,y,z) = 0$, the boundary of a 1 dimensional manifold.

The boundary at $t = \infty$ is the count of the contribution of $z$ in

$$\sum_{j+k=i} Sq^j(x) \cup Sq^k(y)$$
(Figure 4 and Lemma 3.5 below).

The boundary at $t = 0$ for $M_{2,\pi}(x, y, z)$ is the same as for $M_{1,\pi}$. Hence, the boundary count for $t = \infty$ for $M_1$ and $M_2$ is the same in cohomology. The count for $M_{2,\pi}$ is the count of the contribution of $z$ in $Sq^i(x \cup y)$.

**Lemma 3.5.** Summing over all choices of $w_1, w_2$, counting $[(u, v)]$ satisfying configurations as shown in Figure 4, gives $\sum_{j+k=i} Sq^j(x) \cup Sq^k(y)$.

**Proof.** $|w_1| + |w_2| = |z| = |x| + |y| + i$. Hence if $|w_1| = |x| + j$ and $|w_2| = |y| + k$ then $j + k = i$. Throughout fix $w_1, w_2$ both for the configuration and as outputs of $Sq^j(x), Sq^k(y)$ respectively.

Restrict attention to the upper right-hand Y-shaped graph of Figure 4. If the $v$ parameter space was $\mathbb{RP}^{||x||-j}$, then counting $[(u, v)]$ satisfying the configuration conditions would be exactly the coefficient of $w_1 \cdot h^{||x||-j}$ in $Sq^j(x)$, which we denote $n_x$. Letting $v$ vary in $\mathbb{RP}^{||x||+||y||-i}$, we call the set of such pairs $\mathcal{U}_x$.

Similarly for the lower right-hand branch there is a count of $n_y$, the coefficient of $w_2 \cdot h^{||y||-k}$ in $Sq^k(y)$, for each $\mathbb{RP}^{||y||-k} \subset \mathbb{RP}^{||x||+||y||-i}$. Define similarly $\mathcal{U}_y$.

The contribution to $z \cdot h^{||x||+||y||-i}$ in $w_1 \cup w_2$ from $Sq^j(x) \cup Sq^k(y)$ is $n_x \cdot n_y \cdot n_z$ for $n_z$ the coefficient of $z$ in $w_1 \cup w_2$, i.e. Figure 4.

Let

$$\pi_x : \mathcal{U}_x \to \mathbb{RP}^{||x||+||y||-i}$$

**Figure 4.** Flowline configurations for $Sq(x) \cup Sq(y)$. 
be the second projection. This represents a cycle \( u_x := \pi_x \circ [u_x] \in H|_j (\mathbb{R} P^{|x| - j}) \).
Its intersection with a generic \( \mathbb{R} P^{|x| - j} \) implies that \( u_x = n_x \cdot [\mathbb{R} P^{|y| - k}] \). Similarly \( u_y = n_y \cdot [\mathbb{R} P^{|z| - l}] \).

The count of all solutions \([u, v]\) satisfying the configuration in Figure 4 is
\[ n_z \cdot |u_x \cap u_y| = n_z \cdot n_x \cdot n_y. \]

3.3. Steenrod Squares via intersections of cycles. Recall there are nested equators \( S^i \subset S^\infty \), invariant under the antipodal action.

Let \( a \in H^{|a|}(M) \). Let \( \mathcal{B} \) be a basis of \( H^*(M) \).

Henceforth use the notation \( a = PD(a) \in H_{\dim M - |a|}(M) \) for both the homology class and the underlying cycle. Similarly \( \beta = PD(b) \).

There is an involution
\[ \iota = id_M \times (-id_{S^\infty}): M \times S^\infty \rightarrow M \times S^\infty \]

Choose a sequence of chains \((\alpha_i)_{i=0}^\infty\) in \( M \times S^\infty \) such that:
1. For \( \pi_2 \) the second projection, \( \pi_2(\alpha_i) \) is the equatorial \( S^i \subset S^\infty \).
2. For \( \pi_1 \) the first projection, for any \( v \in S^i \) then \( \alpha_v := \pi_1(\alpha_i \cap \pi_2^{-1}(\{v\})) \) is a representative of \( \alpha \) in \( M \).
3. The restriction \( \alpha_i|_{M \times S^j} = \alpha_j \) for \( j \leq i \).
4. For \( b \in \mathcal{B} \), the triple intersection
\[ \alpha_i \cap (\beta^j \times S^i) \cap \iota(\alpha_i) \]

is transverse in \( M \times S^i \).

The chains \( \alpha_i \) cannot be a standard representative of \( \alpha \times S^i \), i.e. \( \{(p, v) | p \in \mathcal{A}, v \in S^i\} \) for some representative \( \mathcal{A} \subset M \) of \( \alpha \), because then it would not be transverse to \( \iota \alpha_i \). In the next section we construct a family of admissible choices. \( B^\infty \times S^i \) can and will be such a standard representative.

All triple intersections in (6) come in pairs related by the free involution \( \iota \). Provided \( |b| = 2|a| - i \), define \( \mathcal{M}_i'(b, a) \) as the triple intersection (6): this moduli space is 0 dimensional.

Quotienting by the action by \( \iota \) defines
\[ \mathcal{M}_i(b, a) = \mathcal{M}_i'(b, a)/(\mathbb{Z}/2). \]

This moduli space is compact and 0 dimensional, so is a collection of points.

**Definition 3.6** (Steenrod Square). Define
\[ Sq(a) = \sum_{i \in \mathbb{Z}, \ b \in \mathcal{B}, \ |b| = 2|a| - i} \# \mathcal{M}_i(b, a)bh^i \]

**Remark.** The Morse case, as in subsection 3.1, is the same as definition 3.6 using the isomorphism \( HM^*(M, f) \cong H^*(M) \) that intertwines the Morse product and the cup product.

Recall we assume \( f_{v, s} = f \) for \(|s| \geq R\).

In the Morse case there are critical points \( a_1, a_2 \). Say \( a_2 \) has a stable manifold \( W \) under \( f \). By definition, \( W \) is the set of points \( p \) such that the flow of \( p \) under \(-\nabla f\) from time \( R \) to \( \infty \) is asymptotically \( a_2 \). For each \( v \in S^i \), every \( w \in W \) is
mapped bijectively to another point in $M$ by backwards integrating for the vector field $-\nabla f_{2,x,v}$ from $R$ to $0$: call this backwards operation $\phi_v : W \to M$.

This manifold $\phi_v(W)$ is homotopy equivalent to the stable manifold $W$. If $a_2$ represents a cohomology class then $W$, hence $\phi_v(W)$, represents $PD(a_2)$, viewed as a pseudocycle in $M$.

In the Morse case we count the number of maps $u : \Gamma \to M$ with $u$ restricting to be a flowline on the edges. $\bigcup_{v \in S^i} \phi_v(W) \times \{v\}$ represents $PD(a_2) \times S^i$, and

$$\bigcup_{v \in S^i} \phi_v(W) \times \{v\} = \iota\left(\bigcup_{v \in S^i} \phi_v(W) \times \{v\}\right)$$

represents $\iota(PD(a_2) \times S^i)$.

Further, these representatives of $PD(a_2) \times S^i$ satisfy all of the conditions at the beginning of this section, so the Morse definition is a special case of Definition 3.6.

In general $a_2 \in \text{Crit}(f)$ is not a cycle, but some linear combination of critical points is: the union of the stable manifolds represents (as a pseudocycle) the Poincaré dual of the sum.

### 3.4. Properties of the Steenrod Square.

**Lemma 3.7.** $Sq(x + y) = Sq(x) + Sq(y)$ for all $x, y \in H^*(M)$

**Proof.** Fix a choice of $X_v$, for $X = PD(x)$, and likewise for $Y_v$ (using the notation from Section 3.3 (2)).

Define $PD(x + y)_v = X_v \cup Y_v$, representing the sum of the cycles.

Let $c_{z,x+y}$ be the coefficient of $z \cdot h^i$ in $Sq(x + y)$,

$$c_{z,x+y} = \# \bigcup_{v \in D^{i,+}} Z^v \cap (X_v \cup Y_v) \cap (X_{-v} \cup Y_{-v})$$

for $Z = PD(z)$ the output. Likewise $c_{z,x}$ is the coefficient of $z \cdot h^i$ in $Sq(x)$.

Expanding the right hand side of equation (7) gives

$$c_{z,x+y} = c_{z,x} + c_{z,y} + \# \bigcup_{v \in D^{i,+}} Z^v \cap X_v \cap Y_{-v} + \# \bigcup_{v \in D^{i,+}} Z^v \cap Y_v \cap X_{-v}$$

The last two terms combine to $\# \bigcup_{v \in S^i} Z \cap X_v \cap Y_{-v}$.

For $i \neq 0$, suppose first that generically $X \cap Y = \emptyset$. Then as $X_v$ is a perturbation of $X$ and $Y_v$ is a perturbation of $Y$ for each $V$, we may assume that $X_v \cap Y_{-v} = \emptyset$ for all $v$. Otherwise generically $X \cap Y = W$, a cycle. Hence there is a choice of $X_v, Y_v$ so that $X_v \cap Y_{-v} \simeq W$. Genericity implies that this solution is nonisolated, i.e. given any neighbourhood $v \in U$ there is $v' \in U$ such that $X_{v'} \cap Y_{-v'} \simeq W$. So isolated solutions do not exist for dimension reasons.

For $i = 0$, $Sq$ is the cup product square in which case additivity holds. \qed

As promised in section 3.1 we now check axiom 3 from section 2.2

**Lemma 3.8.** $Sq^0(PD(pt)) = PD(pt)$.

**Proof.** Let $n = \dim(M)$. Write $a = PD(pt)$.

Construct a representative of $\{pt\} \times S^n$ in $M \times S^n$:
pt \subset M$ has trivialisable normal bundle, so the disc subbundle $D(pt)$ of the normal bundle $N(pt)$ embeds into $M$ as a small disc around $pt$.

Let $S^n, D^n \subset \mathbb{R}^{n+1}$, where $D^n$ is the $n$-disc with the $n+1$st coordinate 0.

There is a natural flattening map $\phi' : S^n \to D^n$, where $\phi'$ is projection of $S^n$ onto the first $n$ coordinates. $\phi'$ is a double cover except on the equator, which bijects onto $\partial D^n$.

There is a diffeomorphism $D^n \cong D(pt) \subset M$. Composing $\phi'$ with this diffeomorphism defines $\phi : S^n \to M$.

$\phi$ is homotopic to a constant map hence $\bigsqcup_{v \in S^n} (\phi(v), v)$, the graph of $\phi$ in $M \times S^n$, is a representative of $\{pt\} \times S^n$.

To calculate the coefficient of $a$ in $Sq^0(a)$, count the number of solutions $\phi(v) = \phi(-v)$ modulo $\mathbb{Z}/2$. The solution set is $\{(0, \ldots, 0, \pm 1)\} \subset S^n \subset \mathbb{R}^{n+1}$. Taking this modulo the $\mathbb{Z}/2$ action gives $Sq^0(a) = a + \ldots$.

$\{a\}$ generates $H^n(M)$ so we are done.  

An easy generalisation of the above proof shows:

**Lemma 3.9.** $Sq^0 = id$. 

**Corollary 3.10.** Let $A$ be a closed submanifold of $M$, with trivialisable normal bundle. Then $Sq^i(PD(A)) = 0$ for $i \neq 0$.

**Proof.** Use the embedding $e : A \times D^{n-\dim(A)} \to M$ by inclusion of the unit disc bundle of $A$, which exists because $A$ has trivialisable normal bundle, to define $A_v$ for $v \in S^i$ for $i > 0$.

**Remark.** More generally, for any immersed submanifold $A$, $Sq^i(PD(A))$ gives the Stiefel-Whitney class $w_i(A)$.

4. **Quantum Steenrod Square via Morse theory**

Let $M$ be a closed monotone symplectic manifold. The definition of the quantum Steenrod square uses a $Y$-shaped graph as with the Morse Steenrod square, but now allows for a $J$-holomorphic sphere at the trivalent vertex in the $Y$-shaped graph in the definition. This is a $J$-holomorphic sphere with 2 + 1 marked points, and 2 incoming and 1 outgoing Morse flowlines from the respective marked points.

Make a choice of $f_{S^p}$ as in subsection 3.1 for $p = 1, 2, 3$.

Let $N$ be the minimal Chern number of $M$. Fix $a, b \in H^*(M)$ with

$$|b| - 2|a| + 2jN = 0.$$

Let $M_{i,j}(b, a)$ be the moduli space of pairs $(u, v)$, for $v \in S^i$ and $u$ a $J$-holomorphic map $u : S^2 \to M$ of Chern number $2jN$, such that the $-\nabla f_{S^p}$ flowline from $u(0)$ converges to $b$ as $s \to -\infty$ and the $-\nabla f_{S^p}$ flowline from $u(1), u(\infty)$ converge to $a$ as $s \to \infty$ for $p = 2, 3$.

There is a $\mathbb{Z}/2$ action on this moduli space as follows:

$$\iota_M(u, v) = (u \circ \mu, -v)$$

where $\mu$ is the unique Möbius map in $PSL(2, \mathbb{C})$ swapping 1 and $\infty$ and fixing 0. Let

$$M_{i,j}(a, b) = M_{i,j}(a, b)/\iota_M.$$
**Definition 4.1** (Quantum Steenrod Square version 1). Pick a basis $B$ of $H^*(M)$. Let $a \in H^*(M)$. For each $i, j$, let

$$QS_{i,j}(a) = \sum_{b \in B : |b| + i + 2jN = 2|a|} \#M_{i,j}(b, a) \cdot b,$$

$$QS(a) = \sum_{i,j} QS_{i,j}(a) \cdot h^i T^j.$$

Extend to a general element of $QH^*(M)$ by $QS(at^j) = QS(a)t^{2j}$, and so that $QS$ is an additive homomorphism.

**Remark.**

1. For $a \in H^*(M)$,

$$QS_{i,0}(a) = Sq^{|a| - i}(a)$$

as it counts constant spheres. Further,

$$\sum_{j \geq 0} QS_{0,j}(a) T^j = a * a$$

is the usual quantum product.

2. Observe that we do not need to use an almost complex structure $J_v$ parametrised by $S^\infty$. See the remark before theorem 3.4. A free $\mathbb{Z}/2$-action is only needed somewhere in the setup: here it is on the Morse flowlines via $f_v$. However, we could use a domain dependent $J_v$ in the definition.

4.1. **Quantum Steenrod Squares via intersections of cycles.** Let $a \in H^{|a|}(M)$, and we pick a basis $B$ of $H^*(M)$. As usual $\alpha = PD(a), \beta = PD(b)$ for $b \in B$.

We define a moduli space and evaluation maps analogously to subsection 2.3: given $j \geq 0$ consider $M_j(J) \times S^i$ consisting of pairs $(u, v)$ where $u$ is a $J$-holomorphic map of energy $j$ and $v \in S^i$. The evaluation maps are $ev_q \times id_{S^i} : M_j(J) \times S^i \to M \times S^i$, which we abusively call $ev_q$, for $q \in \mathbb{CP}^1$.

Choose a sequence of $(\alpha_i)_{i=0}^\infty$ in $M \times S^\infty$ as in subsection 3.3, satisfying conditions 1, 2 and 3 but we modify 4:

4. For a choice of generators $b$ of the cohomology of $M$, the triple intersection

$$M'_{i,j}(b, a) := ev_1^{-1}(\alpha_i) \cap ev_0^{-1}(\beta^\nu \times S^i) \cap ev_\infty^{-1}(\tau(\alpha_i))$$

(8)

is transverse in $M_j(J) \times S^i$, where $\tau = \text{id} \times -\text{id}$.

All such intersections come in pairs, related by $\iota_M : (u, v) \mapsto (u \circ \mu, -v)$ as in the previous section. Define $M_{i,j}(b, a) = M'_{i,j}(b, a)/\iota_M$.

Given some $i, j, N$ such that $|b| = 2|a| - i - 2jN$, where $N$ is the minimal Chern number of $M$ and $b$ a generator of $H^{|b|}(M)$, count the number of intersections as in equation (8) in the 0-dimensional components of $M_{i,j}(b, a)$.

**Definition 4.2** (Quantum Steenrod Square Version 2). For $a \in H^*(M)$ define $QS : QH^*(M) \to QH^*(M)[h]$ such that

$$QS(a) := \sum_{i,j \in \mathbb{Z}_{\geq 0}, b \in B, |b| = 2|a| - i - 2jN} \#M_{i,j}(b, a) \cdot bT^j h^i$$
with $QS$ a linear homomorphism. Then extend $QS$ linearly to $QH^*$ by requiring that $QS(ata^k) = QS(a)t^{2k}$. Also define $QS_{i,j}(a)$ as previously.

As in the classical case this is the same as Definition 4.1.

4.2. Quantum Stiefel Whitney Class. For a manifold $M$, the classical Stiefel-Whitney class of $TM$, $w(TM)$, is constructed as in [5] Section 5.3, using a certain graph operation. We will not go into details.

Let $x = PD$\{pt\}. If $a = n \cdot x \cdot h^i + \sum_{y \in B - \{x\}} n_{i,y} \cdot y \cdot h^j(y) \in H^*_Z(M)$ for $n, n_{i,y}$ coefficients, then $\langle a, [M] \rangle := nh^i$. A gluing theorem implies that:

**Lemma 4.3.** $w(TM) = \sum_{y \in B} S(q(y)) \cdot \langle S(q(y^\vee)), [M] \rangle$.

Now let $M$ be a closed monotone symplectic manifold.

**Definition 4.4** (Quantum Stiefel-Witney Class). The Quantum Stiefel-Witney class is $w_Q(TM) := \sum_{y \in B} QS(y) \langle QS(y^\vee), [M] \rangle$

It follows from this definition and a grading argument that:

**Lemma 4.5.** If the minimal Chern number $N > \frac{1}{2} \dim M$ then $w_Q(TM) = w(TM)$.

**Corollary 4.6.** Let $M = \mathbb{CP}^n$. Then $w_Q(TM) = w(TM)$.

**Proof.** The minimal Chern number for $\mathbb{CP}^n$ is $N = n + 1 > n$. Now apply Lemma 4.5.

5. The Quantum Cartan relation

We continue the discussion from example 1.1. Consider the space $M_{0,5}^\#$ of 5 distinct marked points on the 2-sphere, and let

$$M_{0,5} = M_{0,5}^\#/PSL(2, \mathbb{C})$$

where the Möbius group $G = PSL(2, \mathbb{C})$ acts diagonally on the 5 marked points.

There are two different representations of $M_{0,5}$ that will be useful:

1. $\{(z_0, z_1, z_2, z_3, z_4)\}/G$ of five distinct points modulo the action of $G$, reparametrising Möbius maps.
2. $\{(0, 1, \infty, z_3, z_4)\}$ with $z_3, z_4$ distinct from each other and from 0, 1, $\infty$.

The former gives a simpler definition of the compactification, but the latter is more useful when describing homology classes.

Letting $z_3, z_4$ vary in the description 2 shows that

$$M_{0,5} \cong ((\mathbb{CP}^1 - \{0, 1, \infty\}) \times (\mathbb{CP}^1 - \{0, 1, \infty\})) - \Delta$$

for $\Delta$ the diagonal.

By compactifying, adding nodal curves as required (there are 10 copies of $\mathbb{CP}^1 - \{0, 1, \infty\}$ and 15 points to add), there is a space

$$\overline{M}_{0,5} \simeq Bl_{\{(0,0),(1,1),(\infty,\infty)\}}(\mathbb{CP}^1 \times \mathbb{CP}^1).$$
We model $\mathcal{M}_{0,5}$ as $(\mathbb{CP}^1 \times \mathbb{CP}^1)\#3(\mathbb{CP}^2)$, which means:

$$H^*(\mathcal{M}_{0,5}) = \mathbb{F}_2[\delta_1, \delta_2, w_0, w_1, w_\infty]/(\delta_1^2, \delta_2^2, w_i^3, w_i^2 - \delta_1 \delta_2 \text{ for } i = 0, 1, \infty)$$

where $w_i$ corresponds to the exceptional divisor at $(i, i)$ and $\delta_1, \delta_2$ correspond to the spheres $\mathbb{CP}^1 \times \{pt\}$ and $\{pt\} \times \mathbb{CP}^1$ respectively: thus all the generators have degree 2. A good treatment of this is [9], Section D.7.

Henceforth $W_i = PD(w_i)$ and $\Delta_i = PD(\delta_i)$.

Let $X, Y, Z$ be homology classes with Poincaré duals $x, y, z$ respectively.

Given $m = (z_0, z_1, z_2, z_3, z_4) \in \mathcal{M}^\#_{0,5}$, there is a natural $\mathbb{Z}/2$ action

$$\iota : (z_0, z_1, z_2, z_3, z_4) \mapsto (z_0, z_2, z_1, z_5, z_3).$$

Then $\iota \times -\text{id}$ defines a free diagonal $\mathbb{Z}/2$ action on $\mathcal{M}_{0,5} \times S^i$. Define

$$P_i := (\mathcal{M}_{0,5} \times S^i)/(\iota \times -\text{id})$$

and

$$P_i^\# := (\mathcal{M}^\#_{0,5} \times S^i)/(\iota \times -\text{id}).$$

Pick representatives $X_v, Y_v$ varying smoothly with $v \in S^\infty$ as previously. $\mathcal{M}_{i,j}(x, y, z)$ consists of equivalence classes $[(u, m, v)]$ for $u : m \to M$ a smooth J-holomorphic curve representing a homology class of Chern number $jN$, $m \in \mathcal{M}^\#_{0,5}$ and $v \in S^i$, where $u$ satisfies $u(z_0) \in Z^t$, $u(z_1) \in X_v$, $u(z_2) \in X_{-v}$, $u(z_3) \in Y_v$ and $u(z_4) \in Y_{-v}$.

The equivalence class is taken after quotienting by two actions:

1. the $\mathbb{Z}/2$ action $\text{id} \times \iota \times -\text{id}$,
   $$(u, (z_0, z_1, z_2, z_3, z_4), v) \mapsto (u, (z_0, z_2, z_1, z_4, z_3), -v)$$

2. reparametrisation via all Möbius maps $g$ on the first two factors,
   $$(u, (z_0, z_1, z_2, z_3, z_4), v) \mapsto (u \circ g^{-1}, (g \circ z_0, g \circ z_1, g \circ z_2, g \circ z_3, g \circ z_4), v)$$

There is a natural map

$$\pi : \mathcal{M}_{i,j}(x, y, z) \to P_i$$

obtained by ‘projecting’ onto its last two factors.

**Remark.** $\mathcal{M}_{i,j}(x, y, z)$ is an inverse image under the evaluation map

$$ev : \mathcal{M}_{j}(J) \times_G P_i^\# \to M \times (M^4 \times \mathbb{Z}/2 S^i)$$

where $\mathcal{M}(A, J)$ is the set of all J-holomorphic maps $u : \mathbb{CP}^1 \to M$ of energy $j$. Specifically,

$$\mathcal{M}_{i,j}(x, y, z) = ev^{-1}(Z^t \times \bigcup_{v \in \mathbb{RP}^i} X_v \times X_{-v} \times Y_v \times Y_{-v}).$$

This allows us to calculate $\dim \mathcal{M}_{i,j}(x, y, z)$.

Let $Q$ be a closed submanifold of the parameter space $P_i$: $Q$ represents a cycle in $H_*(P_i)$.

$$\dim \pi^{-1}(Q) = |z| - 2|x| - 2|y| + \dim(Q) + 2jN. \tag{9}$$

In particular, for $Q = P_i$, using $\dim(P_i) = 4 + i$,

$$\dim \mathcal{M}_{i,j}(x, y, z) = \dim \pi^{-1}(P_i) = |z| - 2|x| - 2|y| + i + 4 + 2jN.$$
Definition 5.1. Let $Q$ be a cycle in $P_i$ with $i,j$ fixed. Define
\[ q_{i,j}(Q)(x,y) = \sum_{z: (9) = 0} \#(\pi^{-1}(Q)) z T^j \]
where the sum is taken over a basis of $z$ for $H_{|z|}(M)$ such that equation (9) is 0.

Explicitly, this defines a bilinear map
\[ q_{i,j}(Q): QH^k(M) \otimes QH^l(M) \to QH^{k+l-\dim(Q)}(M). \]

Remark. More formally, we work with $\rho = P D(Q) \in H^*(P_i)$. Consider the cohomology class $\pi^*\rho \in H^*(M_{i,j}(x,y,z))$. It is this cohomology class that defines the operation. However, for intuition we use homology.

In the following, use a cell decomposition for $S^i$ with cells $D^{i,\pm}$ in degree $i$, corresponding to the two hemispheres of dimension $i$. For $d$ the boundary map on cellular chains, $d(D^{i,\pm}) = D^{i-1,+,+} + D^{i-1,-,-}$.

The class of cases we consider are $q = \dim(Q) = i$. If $m_1, m_2 \in \overline{M}_{0,5}$ are as given in figure 5, then $m_1$ and $m_2$ are invariants under the $\mathbb{Z}/2$ action on $\overline{M}_{0,5}$. Hence $\{m_1\} \times D^{i,+}$ and $\{m_2\} \times D^{i,+}$ are well defined cycles in $\overline{M}_{0,5} \times_{\mathbb{Z}/2} S^i$. For $p = 1, 2$ call these
\[ Q_p := \{m_p\} \times D^{i,+}. \]

For $i > 0$, the chain “$\{pt\} \times D^{i,++}$” is only a cycle when $pt$ is a fixed point of the $\mathbb{Z}/2$ action on $\overline{M}_{0,5}$. The space of fixed points $(\overline{M}_{0,5})_{\mathbb{Z}/2}$ is the disjoint union of a sphere containing $m_2$ and the single point $m_1$. To see this, consider a general element of $M_{0,5}$: after reparametrising we represent this element by $(0, 1, \infty, z_3, z_4)$. Such an element is fixed by $\mathbb{Z}/2$ exactly when $z_4 = \frac{z_3}{z_3-1}$, which determined a 2-family of points. When we add the elements in $\overline{M}_{0,5}$ fixed by $\mathbb{Z}/2$, we get the compactification of the 2-family to a 2-sphere, and an isolated point.

Lemma 5.2.
\[ \sum_{i,j} q_{i,j}(Q_{1i})(x,y)h^i = Q S(x) \ast Q S(y) \quad \text{and} \quad \sum_{i,j} q_{i,j}(Q_{2i})(x,y)h^i = Q S(x \ast y). \]
Proof. We prove for each fixed $i,j$ that

$$q_{i,j}(Q_i^1)(x,y) = [QS(x) * QS(y)]_{i,j} T^j.$$  

To do this we proceed as in Lemma 3.5 using 1-dimensional moduli spaces, the ends of which count e.g.

$$\sum_{i,j} q_{i,j}(Q_i^1)(x,y) \cdot h^i$$

and $QS(x) * QS(y)$ respectively. The setups mimic those in Lemma 3.5 with a $J$-holomorphic sphere at every trivalent vertex and the appropriate nodal curve at the valence 5 vertex at the end of the moduli space.

Let $A^i = Q_i^1 - Q_i^2$. Then, by the above Theorem 1.2 holds.

Let $W_0$ be the exceptional $\mathbb{CP}^1$ divisor in $Bl_{(0,0)}(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

The group $H_{rq}(P_i)$, for $r > 2, r \neq i + 2$, is generated by the cycles:

$$\{m_1\} \times D^{r,+}, \quad W_0 \times D^{r-2,+}, \quad \overline{M}_{0,5} \times D^{r-4,+}.$$

For $r = i + 2$, we need additionally $(W_1 + W_\infty) \times D^{i,+}$ and $(\Delta_1 + \Delta_2) \times D^{i,+}$. For $r \leq 2$, we need additionally $A \times D^{0,+}$ for any closed $A$, e.g.

$$A = \Delta_i, \; W_1, \; W_\infty.$$

$W_0$ consists of sphere configurations as in Figure 6 removing reparametrization freedom by fixing the positions of $z_0, z_1, z_2, z_3, z_4$ and letting $z$ vary. There are three special curves, when the free point $z$ “collides” with each of the fixed points, giving rise to a bubbling configuration as shown in the Figure.

To compute $A^i$ we intersect chains, observing the following:

1. For the intersection theory in degree $r$, we may assume that $r = i$, so we work with $\overline{M}_{0,5} \times_{Z/2} S^i$ which is of dimension $i + 4$.

2. So the elements of dual dimension to $\{m_p\} \times D^{i,+} \in H^i(\overline{M}_{0,5} \times_{Z/2} S^i)$ are in degree 4, and (for $i > 2$) these are generated by $\{m_1\} \times D^{4,+}, W_0 \times D^{2,+}$ and $\overline{M}_{0,5} \times D^{0,+}$.

3. Hence if two homology classes in degree $i$ have the same intersections with the three degree 4 elements listed above, then they must be the same by Poincaré duality.

Lemma 5.3.

$$\{m_1\} \times D^{i,+} = \{m_2\} \times D^{i,+} + [W_0 \times D^{i-2,+}]$$

in $H_i(P_i)$.

Proof. We tabulate the intersections of cycles (except for the $i = 2$ case):

|               | $\{m_1\} \times D^{i,+}$ | $\{m_2\} \times D^{i,+}$ | $W_0 \times D^{i-2,+}$ |
|---------------|---------------------------|---------------------------|-------------------------|
| $\{m_1\} \times D^{4,+}$ | 1                         | 0                         | 1                       |
| $W_0 \times D^{2,+}$       | 1                         | 0                         | 1                       |
| $\overline{M}_{0,5} \times D^{0,+}$ | 1                         | 1                         | 0                       |

For the bottom row, the first two entries follow because of intersections in $\overline{M}_{0,5}$. The third entry occurs because $D^{i-2,+} \cap D^{0,+} = \emptyset$ in $S^i$.

For the middle column, we may choose our representative of $m_2$ to be distinct from our choice of $m_1$ and $W_0$, giving the entries of 0.
For the rest, we need to pick appropriate transversely intersecting chain representatives. For $i = 2$, the cohomology group in degree 4 has more generators to consider, meaning we must add more rows. However these extra generators have trivial intersection with each of the columns, and so all the extra rows will be 0. \[\square\]

**Corollary 5.4** (Quantum Cartan relation).

\[QS(x \ast y) = QS(x) \ast QS(y) + q(W_0)(x, y)\]

where

\[q(W_0) := \sum_{i,j} q_{i,j}(W_0 \times D^{i-2, +}) h^i\]

and the $q_{i,j}$ are as in Definition 5.1

*Proof.* By lemma 5.3, $q_{i,j}(\{m_1\} \times D^{i-+}) = q_{i,j}(\{m_2\} \times D^{i+}) + q_{i,j}(W_0 \times D^{i-2, +})$.

Now multiply by $h^i$ and sum over all $i, j$ and apply Lemma 5.2 \[\square\]
We will verify Corollary 5.4 in the case of $\mathbb{C}P^1$.

**Example 5.5 ($\mathbb{C}P^1$).** Let $x$ be the generator of 2-dimensional cohomology. We verify that

$$[QS(x) * QS(x)]_{i,j} = QS_{i,j}(x * x) + q_{i,j}(W_0 \times D^{i-2,+})(x,x).$$

This is trivial except for $(i,j) = (4,1)$.

$$[QS(x) * QS(x)]_{4,1} = [T^2 + h^4 T]_{4,1} = h^4 T$$

$$QS_{4,1}(x * x) = QS_{4,1}(T) = [T^2]_{4,1} = 0$$

so it remains to calculate $q_{4,1}(W_0 \times D^{2,+})(x,x)$.

Pick representatives of $PD(x) \times S^2$ as follows: let $\phi : S^2 \to D^2$ be the “flattening map” of the sphere, e.g. if $S^2 \subset \mathbb{R}^3$ it is projection onto $\mathbb{R}^2 \subset \mathbb{R}^3$.

Pick two disjoint discs in $\mathbb{C}P^1 = S^2$, call them $D$ and $D'$, and pick maps $\eta$ and $\eta'$ identifying $D^2$ with $D, D'$ respectively. Let $\psi = \eta \circ \phi$, and likewise $\psi'$. Then two representatives of $PD(x) = \{pt\}$, are $\psi(v)$ and $\psi'(v)$ where $v$ varies in $S^2$.

Recall that elements of a standard choice for $W_0$ are as in figure 3

The sphere $S^2$ has Chern number $N = 2$ so only one of the J-holomorphic spheres may be non-constant, and has degree 1.

The sphere with three marked points must be trivial (if the other sphere were constant, then the solution cannot be rigid as $z_4$ can vary freely), and the other sphere of $u$ has degree 1. Then $u(z_1) = \psi(v)$ and $u(z_2) = \psi(-v)$ meet at the unique point on the sphere where $\psi(v) = \psi(-v)$. Hence there is one solution, and this solution gives the correction term $h^4 T$.

**Lemma 5.6.** $q(W)$ only contains terms of the form $aT^p h^q$ for $p > 0, q > 0$.

**Proof.** The $p = 0$ case corresponds to J-holomorphic maps that are constant. For such spheres, if there is a solution for some $m \in W_0$ there is a solution for every element of $W_0$. So there are no isolated solutions.

For $q = 0$, the $h^q$ terms correspond to $q(W \times D^{q-2,+})$ which vanishes for $q < 2$. Specifically $S^\infty$ has no cells of negative dimension. □

Remark (Quantum Cartan in Classical Case). Lemma 5.6 gives a sanity check that in the classical case, $Sq(x) \cup Sq(y) = Sq(x \cup y)$.

6. Computing the Quantum Steenrod Square for toric varieties

6.1. Quantum Steenrod squares for $\mathbb{C}P^n$. Let $x^i$ generate $H^{2i}(\mathbb{C}P^n)$. By the quantum Cartan relation, Corollary 5.4

$$QS(x^{i+1}) = QS(x^i) * QS(x) + q(W)(x^i, x)$$

We can iteratively construct $QS(x^{i+1})$ as long as we know $q(W)(x^i, x)$. For degree reasons, $QS(x) = x h^2 + x * x$.

**Lemma 6.1.** For $2i < n$, $q(W)(x^i, x) = 0$. For $n \leq 2i \leq 2n$,

$$q(W)(x^i, x) = \binom{i}{n-i} Th^{4i+2-2n}.$$
Proof. Observe that $q(W)(x^i, x)$ has degree $4i + 4$, so by lemma 5.6 for $i = 1, ..., n$ we deduce that:

$$q(W)(x^i, x) = \sum_{j=0}^{n-i} m_j x^{i+j-n} h^{2(i+1)-2j}$$

$$= m_{n-i} x^0 h^{4i+2-2n} + m_{n-i-1} x^1 h^{4i-2n} + \cdots + m_{i-1} x^{2i-n} h^2$$

where $m_j$ are coefficients and the degrees are $|x| = 2$, so $|x^i| + |x| = 2i + 2$ and $|T| = 2(n + 1)$. Equation (10) follows for grading reasons.

We claim that $m_j$, the coefficient of $x^{i+j-n} h^{2(i+1)-2j}$, is the number of (unparametised) $J$-holomorphic spheres that intersect both $\mathbb{C}P^{i+j-n}$ and $PD(Sq^2(x^i))$. This follows by three observations:

i ) Counting the component of $x^{i+j-n} h^{2(i+1)-2j}$ is the same as counting setups as in figure 7 for $v \in D^{2i-2j,+}$ (recall $D^{2i-2j,+}$ corresponds to $h^{2i-2j+2}$ terms when defining $q(W)$ in corollary 5.4).

Only $T^1$ appears, so one of the holomorphic bubbles has degree 0 and the other degree 1. For the solutions to be rigid the sphere with the two marked points $z_1, z_2$ must be constant, giving the setup in figure 7.

ii ) The sphere is a $\mathbb{C}P^1$ and $\mathbb{C}P^1 \cap \mathbb{C}P^{n-1} = \{pt_v\}$ for each $v$. The sphere cannot be contained in $\mathbb{C}P^{n-1}$ because if it were then under a generic setup it could not satisfy all of the other intersection conditions.

iii ) The intersection of $\mathbb{C}P^{n-i}$ and $\mathbb{C}P^{n-i}$, taken over all $v \in D^{2i-2j,+}$, is $PD(Sq^2(x^i))$. The output of $Sq(x^i)$ corresponding to $D^{2i-2j,+}$ is

$$Sq_{q(i-j)}(x^i) = Sq^2(x^i) = \binom{i}{j} x^{i+j}$$

with $PD(x^{i+j}) = \mathbb{C}P^{n-(i+j)}$, for $i + j \leq n$. So $m_j$ is:

$$\binom{i}{j} \cdot (\#\text{lines intersecting } \mathbb{C}P^{i+j-n} \text{ and } \mathbb{C}P^{n-(i+j)})$$

As $i + j - n \geq 0$ and $n - (i + j) \geq 0$ the only nontrivial term arises for $j = n - i$, in which case there is precisely one line. \hfill \Box

Theorem 6.2. For all $i \geq 0$,

$$QS(x^i) = \sum_{j=0}^{i} \binom{i}{j} + \sum_{k=0}^{\lfloor n/2 \rfloor + 1} \binom{n-k}{k} \cdot \binom{i-(n+1-k)}{j-k} x^{i+j} h^{2(i-j)},$$

where $x^{i+j}$ is the $(i+j)$-th quantum power of $x$.

Recall $QS(x^i) = Sq(x^i) + ...$ where by example 2.3

$$Sq(x^i) = \sum_{j=0}^{n-i} \binom{i}{j} x^{i+j} h^{2i-2j}.$$
Proof of theorem 6.2. Since $T = x^{n+1}$, we can express the square as:

$$QS(x^i) = \sum_{j=0}^{i} l_j^i x^{i+j} h^{2i-2j}$$

$$= l_0^i x^i h^{2i} + l_1^i x^{i+1} h^{2i-2} + \cdots + l_i^i x^{2i} h^0$$

for some $l_j^i \in \mathbb{Z}/2$.

By the Quantum Cartan relation, Corollary 5.4 and Lemma 6.1, the coefficients $l_j^i$ satisfy $l_j^{i+1} = l_j^i + \delta_{j,0}$ for $j \neq n - i$ and $l_{n-i}^{i+1} = l_{n-i-1}^i + \delta_{i,n-i} + \binom{i}{n-i}$ (the latter comes from the quantum correction).

Using a Pascal Triangle and the iterative formula for the $l_j^i$, one can write down the closed form solution.

In particular, truncating the sum in equation (11) to $j \leq n - i$ recovers the classical Steenrod square formula for $\mathbb{CP}^n$ from example 2.3. This is because if $j \leq n - i$ then every term in the second summation in equation (11) vanishes because either

- $j - k < 0$
- or $i - (n + 1 - k) = k + i - n - 1 \leq j + i - n - 1 \leq -1 < 0$

Explicit examples:

$\mathbb{CP}^1 : q(W)(x, x) = \binom{1}{i-1} T h^{4+2-2}$
\[ QS(x) = xh^2 + T \]
\[ QS(T) = (xh^2 + T)^2 + Th^4 = T^2. \]
\[ \mathbb{CP}^2 : q(W)(x, x) = (\frac{1}{2})Th^{1+2-4} \]
\[ QS(x) = xh^2 + x^2 \]
\[ QS(x^2) = (xh^2 + x^2)^2 + Th^2 = x^2h^2 + Th^2 + xT. \]
\[ \mathbb{CP}^3 : q(W)(x, x) = (\frac{1}{3})Th^{1+2-6} = 0 \] and \[ q(W)(x^2, x) = (\frac{2}{3})Th^{8+2-6} = 0 \]
\[ QS(x) = xh^2 + x^2 \]
\[ QS(x^2) = (xh^2 + x^2)^2 = x^2h^4 + T \]
\[ QS(x^3) = (xh^2 + x^2)(x^2h^2 + T) = x^3h^6 + Th^4 + xTh^2 + x^2T. \]

6.2. **Fano Toric Varieties.** Let \( M \) be a compact monotone toric manifold, with \( b \in H^{|b|}(M) \) and \( x \in H^2(M) \). Then analogously to \( \mathbb{CP}^n \), one proves Theorem 1.3.

**Proof of Theorem 1.3.** Consider setups as in figure 8.

We consider related setups, which we call reduced setups, where we neglect the cycle \( X_v \) and the marked point \( z_4 \) corresponding to it. We still have a dimension 0 setup: removing \( z_4 \) removes 2 dimensions, and removing the intersection with \( X_v \) adds 2 dimensions.

Setups and reduced setups are in a 1 to \#(\( X \cap \mu \)) correspondence. Strictly, for every reduced setup (so a pair consisting of a J-holomorphic map with evaluation conditions and some \( v_0 \in D^{i,^+} \)), there are exactly \#(\( X_{v_0} \cap \mu \)) setups. We must note that a generic point in \( S^2 \) is an injective point of \( u \), and hence fixing an intersection point in the image of \( u \) fixes the free point \( z_4 \) in \( W_0 \). A generic choice of \( X_v \) ensures \#(\( X_{v_0} \cap \mu \)) = \#(\( X \cap \mu \)) for each \( v_0 \).

Hence it is sufficient to prove that reduced setups count

\[
\sum_{2i=0}^{\frac{|b|}{2}} \sum_{j \geq 1} \sum_{k=1}^{j} \sum_{\mu \in H_2(M) : E(\mu) = k} (QS_{2i,j-k}(b) \ast_{\mu, k} x) \cdot h^{|b| - 2i + 2j}. 
\]

However, considering reduced setups alone one may choose \( X_{-v} \) to be independent of \( v \in D^{i,^+} \). Hence the result follows immediately from the definitions of \( QS \) and the quantum product.

As the cohomology of a toric variety \( M \) is generated by \( H^2(M) \), iterated application of the above formula yields a general solution, i.e. one can calculate \( QS(x_{p_1}, x_{p_2}, ..., x_{p_r}) \) assuming the base cases \( QS(x_{p_i}) \) for a basis \{\( x_{p} \)\} of \( H^2(M) \) are known. For degree reasons \( QS(x_p) = x_p \ast x_p + x_p \cdot h^2 \).

**Corollary 6.3.** For \( M \) a compact, monotone toric variety, \( QS \) is determined completely by \( QH^*(M) \).

**Proof.** The base case \(|x| = 2\), \( QS(x) = xh^2 + x \ast x \) is determined by \( QH^*(M) \).

Given \( a \in H^*(M) \) for \( |a| > 2 \), write \( a = b \ast x \) for \( x \in H^2(M) \).

\( QS(a) = QS(b) \ast QS(x) + q(W)(b, x) \). By induction \( QS(b) \) and \( QS(x) \) are determined by \( QH^*(M) \), hence so is \( QS(b) \ast QS(x) \).

By theorem 1.3 \( q(W)(b, x) \) is determined by \( QH^*(M) \).
Figure 8. Configurations for $q(W)(b,x_p)$ for toric varieties, where $X = PD(x)$, $B = PD(b)$ and $z$ is the output. Energy $k$ means $E(u) = c_1(u_*[S^2])/2N = k$.

Example 6.4 ($\mathbb{CP}^1 \times \mathbb{CP}^1$). We let $x, y$ be the generators of $H^2(\mathbb{CP}^1 \times \mathbb{CP}^1)$, with $PD(x) = \{pt\} \times \mathbb{CP}^1$ and $PD(y) = \mathbb{CP}^1 \times \{pt'\}$ . Here $q(W_0 \times D^{i-2+})(x,y) = 0$ hence $QS(x) \ast QS(y) = QS(x \ast y)$. Indeed by equation [2],

$$q(W_0 \times D^{i-2+})(x,y) = \sum_{2i=0}^{2} \sum_{j \geq 1} \sum_{k=1}^{j} Q_{2i,j-k}(x) *_{\mu,k} y h^{4-2i} T^j.$$ 

Working from definitions, $QS(x) = xh^2 + T$.

$\alpha \ast_k y \neq 0 \implies k = 1, \alpha = y$. There are no $i, j, k$ such that $Q_{2i,j-k}(x) = y$. Hence the sum on the right hand side is 0.

7. The Quantum Adem Relations

7.1. Classical Adem Relations. We begin with a discussion of the group cohomology of $S_4$ and $D_8$. This will involve adding details to the argument alluded to by Cohen-Norbury to prove the classical relations in [5, Section 5.2].

It is proved in [1] Sections IV.1, VI.1] that

$$H^\ast(BD_8) = \mathbb{Z}/2[e, \sigma_1, \sigma_2]/(e\sigma_1)$$

where $e, \sigma_1$ are of degree 1 and $\sigma_2$ is of degree 2, and

$$H^\ast(BS_4) = \mathbb{Z}/2[n_1, n_2, c_3]/(n_1c_3),$$

where again subscripts denote the degree of the elements.

Considering $D_8 = \langle(12), (34), (13)(24)\rangle \subset S_4$ there are subgroups

$$\mathbb{Z}/2 = \langle(13)(24)\rangle, \quad \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle(12), (34)\rangle.$$ 

Then

$$H^\ast(B\mathbb{Z}/2) = \mathbb{Z}/2[e], \quad H^\ast(B(\mathbb{Z}/2 \times \mathbb{Z}/2)) = \mathbb{Z}/2[x, y].$$

Consider the commutative diagram [12] induced by the various inclusion maps of groups. As in [1], one shows that:
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\[ H^*(B\mathbb{Z}/2) \]
\[ \xymatrix{ & H^*(BD_8) \ar[dl]_{i_1} \ar[dr]^{j_1} \ar[r]_{\pi^*} \ar[d] & H^*(BS_4) \ar[dl]_{i_2} \ar[dr]^{j_2} \ar[d] & \cr H^*(B(\mathbb{Z}/2 \times \mathbb{Z}/2)) & & & \}
\]

\( i_1(e) = e \)
\( i_2(\sigma_1) = x + y \quad i_2(\sigma_2) = xy \)
\( j_1(n_2) = e^2 \)
\( j_2(n_1) = x + y \quad j_2(n_2) = xy \)
All other generators map to 0 via the \( i, j \) maps. From this, and the fact that \( \pi^* \) is injective, we deduce that
\( \pi^*(n_1) = \sigma_1 \quad \pi^*(n_2) = \sigma_2 + e^2 \quad \pi^*(c_3) = e\sigma_2. \)

By Cohen-Norbury, [5], there is a commutative diagram, namely diagram (13), where \( qq \) satisfies
\[ qq(\alpha) = \sum_{p,q} Sq^q \circ Sq^p(\alpha)e^{[\alpha]+p-q}e^{|\alpha|-p}. \]

\[ H^*(M) \xrightarrow{\pi^*} H^*(M) \otimes H^*(BS_4) \]
\[ \xrightarrow{\text{id}_{H^*(M)} \otimes \pi^*} \]
\[ H^*(M) \xrightarrow{qq} H^*(M) \otimes H^*(BD_8) \]

**Fact 1.** By theorem 19 (Invariance) in [5], the diagram (13) commutes. This implies that the image of \( qq \) lies in the image of \( id_{H^*(M)} \times \pi^* \). Hence there are constraints on the image. Specifically, \( e^{2l}\sigma_2^j \) may only appear in \( qq(\alpha) \) if it arises from some \((e\sigma_2)^{2k}(e^2 + \sigma_2)^{i+j-3k} \) for \( k = 0, 1, \ldots \), with coefficient \( \binom{i+j-3k}{i-3k} \).

This will be proven as a special case of Lemma 7.8.

**Lemma 7.1.** For any \( s, m, \)
\[ \binom{3s + m}{s + m} = \sum_{l=0}^{\infty} \binom{m + l - 1}{2l} \binom{3s + m}{s - l} \]
modulo 2.

**Proof.** We prove this by induction. Let \( c(m, s) = \binom{m+3s}{m+s} \). Then modulo 2,
\[ c(m+2, s) = c(m, s) + c(m+3, s-1). \]
Define \( S(m, s) = \sum_l \binom{m+3s-m}{2l} \binom{3s+m}{s-l} \). Check that \( S(m, s) = c(m, s) \) for \( s = 0, 1 \) and \( m = 1, 2 \). These are the base cases.

Hence if \( S(m+2, s) = S(m, s) + S(m+3s, s-1) \) for all \( m, s \) then the lemma holds by induction. This is an exercise in binomial coefficient algebra modulo \( \mathbb{Z}/2 \). \qed
**Theorem 7.2** (Classical Adem Relations). Given $\alpha \in H^*(M)$, $q, p > 0$, $q < 2p$,

$$Sq^q Sq^p(\alpha) = \sum_{k=0}^{[q/2]} \binom{p-k-1}{q-2k} Sq^{p+q-k} Sq^k(\alpha).$$

**Proof.** Suppose $q$ is even. Let $l = |\alpha| - p$, $m = p - q/2$, $n = q/2 - k$, thus

$$\binom{p-k-1}{q-2k} = \binom{m+n-1}{2n}.$$  

Assume $l = 2r$. The cases for $q$ or $l$ odd are similar. Throughout let $\text{cff}(E)$ be the coefficient of $E$ in $qq(\alpha)$. By definition of $qq$

$$Sq^q \circ Sq^p(\alpha) = \text{cff}(e^{l+2m}\sigma_2^l) \quad \text{and} \quad Sq^{p+1-k} \circ Sq^k(\alpha) = \text{cff}(e^{l-2n}\sigma_2^{l+m-n}).$$

By Fact 1,

$$\text{cff}(e^{l+2m}\sigma_2^l) = \sum_i \binom{3r+m-3i}{r+m-i} \cdot \text{cff}((e\sigma_2)^{2i}(e^2 + \sigma_2)^{3r+m-3i})$$

and

$$\text{cff}(e^{l-2n}\sigma_2^{l+m+n}) = \sum_i \binom{3r+m-3i}{r-n-i} \cdot \text{cff}((e\sigma_2)^{2i}(e^2 + \sigma_2)^{3r+m-3i}).$$

The claim now follows since by lemma 7.1

$$\binom{3r+m-3i}{r+m-i} = \sum_{n=0}^{\infty} \binom{m+n-1}{2n} \binom{3r+m-3i}{r-n-i}.$$  

The terms with $n > q/2$ will not appear in the final statement because $n > q/2$ implies $k < 0$, and $Sq^k(\alpha) = 0$ for $k < 0$. $\Box$

### 7.2. Quantum Adem Relations

Recall in Section 5 we constructed homomorphisms $q(W)$ for $W \in H^*_{\mathbb{Z}/2}(\overline{M}_{0,5})$. We will define a similar construction for elements of $H^*_{D_8}(\overline{M}_{0,5})$ and $H^*_S(\overline{M}_{0,5})$, where

$$D_8 = \langle (12), (34), (13)(24) \rangle \subset S_4$$

acts by permutations on the indices of $[z_0, z_1, z_2, z_3, z_4] \in \mathcal{M}_{0,5}$.

We will abbreviate $P_{D_8} = \overline{M}_{0,5} \times_{D_8} ES_4$ and $P_{S_4} = \overline{M}_{0,5} \times_{S_4} ES_4$.

We note that for any $M$ with $H^*(M)$ finitely generated in all degrees, there is a map $\Psi : H^*(M) \rightarrow H_*(M)$ which is an isomorphism via universal coefficients: e.g. by picking a dual basis under the pairing $\langle \alpha, a \rangle \mapsto \alpha(a)$. $P_{D_8}$ and $P_{S_4}$ satisfy this finite generation condition, which one can see using the Cartan Leray spectral sequence.

Given $\alpha \in H^*(M)$, let $A = PD(\alpha)$. Choose $A_v$ for $v \in ES_4$, with invariance and genericity conditions: importantly, $A_v = A_{(23)v} = A_{(24)v}$.

**Definition 7.3.** For $d \in H^*_D(\overline{M}_{0,5})$, we pick a representative $D$ of $\Psi(d) \in H_*(P_{D_8})$. Then

$$q_{D_8}(D)(\alpha) := \sum_{z \in H^*(M), j \geq 0} n_{z,\alpha, b,j} \cdot z \cdot T^j$$
where \(n_{z,a,b,j}\) counts the number of \(D_8\) equivalence classes of triples \((m,u,v)\) with \([m,v]\in D\subset P_{D_8}\) and \(u : m\to M\) a \(J\)-holomorphic curve of energy \(j\) such that \(u(z_0)\in PD(z')\), \(u(z_1)\in A_v\), \(u(z_2)\in A_{(12)}v\), \(u(z_3)\in A_{(13)}v\), \(u(z_4)\in A_{(14)}v\).

On cohomology the operation will be independent of the representative \(D\) of \(\Psi(d)\).

The definition of \(q_{S_4}(D)\) is analogous.

We pick a basis \(\mathcal{B}\) for \(H^*(BS_4)\).

As in the case of the quantum Cartan relation, we would like to consider cycles in \(H_*(PS_4)\) parametrised by \(\mathcal{B}\). Compare this to the proof of the quantum Cartan relations, where \(\{m_1\} \times D^{i+} \in H_*(P_{\mathbb{Z}/2})\) was parametrized by \(D^{i+} \in H_*(B\mathbb{Z}/2) = H_*(\mathbb{R}P^{\infty})\).

Further, we will show later that

\[
Q\mathcal{S} \circ Q\mathcal{S}(\alpha) = \sum_{i,j} q_{D_8}([m_1] \otimes \Psi(e^i \sigma^j))(\alpha) \cdot e^i \sigma^j. \tag{15}
\]

Hence, ideally we would like the basis of \(H_*(PS_4)\) to be of the form \(\{m_1\} \otimes B\) for \(B \in \mathcal{B}\). However, this will not work because \(m_1\) is not \(S_4\)-invariant. However, the cycle \(m_1 + gm_1 + g^2m_1 \in H_*(M_{0,5})\) is \(S_4\) invariant for \(g = (123)\) generating the cosets of \(D_8\) in \(S_4\) (note that \(gm_1 = m_2\)).

**Definition 7.4.**

\[
q_{S_4} : H^*(M) \to QH^*(M) \otimes H^*(BS_4), \quad q_{S_4}(\alpha) = \sum_{b \in \mathcal{B}} q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b))(\alpha) \cdot b.
\]

Recall that there are quotient maps \(\pi_* : H_*(BD_8) \to H_*(BS_4)\) and \(\pi^* : H^*(BS_4) \to H^*(BD_8)\). There are also

\[
i_* : H_*(BS_4) \to H_*(BD_8), \quad i_*(D) = D + gD + g^2D
\]

and

\[
i^* : H^*(BD_8) \to H^*(BS_4), \quad i^*(d) = d + gd + g^2d.
\]

As we work over \(\mathbb{Z}/2\) we see that \(\pi_* \circ i_* = id\) and \(i^* \circ \pi^* = id\), which also shows that \(\pi^*\) is injective.

As \(\pi^*\) is injective, \(\pi^*\mathcal{B}\) is linearly independent in \(H^*(BD_8)\). We extend this to a basis \(\tilde{\mathcal{B}} = \pi^*\mathcal{B} \cup \mathcal{B}'\).

**Definition 7.5.**

\[
qq : H^*(M) \to QH^*(M) \otimes H^*(BD_8), \quad qq(\alpha) := \sum_{b \in \mathcal{B}} q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b))(\alpha) \cdot \hat{b}.
\]

**Lemma 7.6.**

\[
q_{S_4}((m_1 + gm_1 + g^2m_1) \otimes \pi_* \Psi(b)) = q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b)).
\]

**Proof.** This is immediate from the definitions. A representative of \((m_1 + gm_1 + g^2m_1) \otimes \pi_* \Psi(b)\) is in \(3 : 1\) correspondence with a representative of \((m_1 + gm_1 + g^2m_1) \otimes \Psi(b)\). Quotienting by \(S_4\), and as we work over \(\mathbb{Z}/2\) the count of the number of solutions is the same. \(\square\)
Lemma 7.7. For $b' \in \mathcal{B}' = \hat{\mathcal{B}} - \pi^*\mathcal{B}$.

$$q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b')) = 0.$$  

Proof. By Lemma 7.6

$$q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b')) = q_{S_4}((m_1 + gm_1 + g^2m_1) \otimes \pi_*\Psi(b')).$$

If $B' = \Psi(b')$ then for all $b \in B$,

$$\langle b, \pi_*B' \rangle = \langle \pi^*b, B' \rangle = \langle \pi^*b, \Psi(b') \rangle = 0$$

by definition of the dualising isomorphism $\Psi$. Hence $\pi_*\Psi(b') = 0$. \hfill \Box

This implies that

$$qq(\alpha) := \sum_{\pi^*b \in \pi^*\mathcal{B}} q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(\pi^*b))(\alpha) \cdot \pi^*b. \quad (16)$$

Lemma 7.8. The following diagram commutes:

$$\begin{array}{ccc}
H^*(M) & \xrightarrow{q_{S_4}} & QH^*(M) \otimes H^*(BS_4) \\
\downarrow & & \downarrow \text{id}_{H^*(M) \otimes \pi^*} \\
H^*(M) & \xrightarrow{qq} & QH^*(M) \otimes H^*(BD_8)
\end{array} \quad (17)$$

Proof. 

$$(id \otimes \pi^*)q_{S_4}(\alpha) = \sum_{\pi^*b \in \pi^*\mathcal{B}} q_{S_4}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b))(\alpha) \cdot \pi^*b.$$ 

$$qq(\alpha) = \sum_{\pi^*b \in \pi^*\mathcal{B}} q_{S_4}((m_1 + gm_1 + g^2m_1) \otimes \pi_*\Psi(\pi^*b))(\alpha) \cdot \pi^*b$$

using Equation (16) and Lemma 7.6.

For $b \in \mathcal{B}$, let $D = \Psi(\pi^*b)$. Then $\langle \pi^*b, D \rangle = 1$ and $\langle \hat{b}, D \rangle = 0$ for all $\hat{b} \in \hat{\mathcal{B}} - \pi^*b$, specifically for $\hat{b} = \pi^*d$ with $d \in \mathcal{B} - b$. Hence $\langle d, \pi_*D \rangle = 0$ for $d \in \mathcal{B} - b$ and $\langle b, \pi_*D \rangle = 1$, so $\pi_*\Psi(\pi^*b) = \Psi(b)$ by definition of $\Psi$. \hfill \Box

Observe that using monomials in $n_1, n_2, c_3$ for our basis $\mathcal{B}$ of $H^*(BS_4)$ (see the beginning of Section 7.1), $\pi^*\mathcal{B}$ consists of products of $e^2 + \sigma_2$, $e\sigma_2$ and $\sigma_1$. We may partition the basis $\mathcal{B} = \mathcal{B}_0 \sqcup \mathcal{B}_1$, where each basis element (which are all monomials) in $\mathcal{B}_1$ contains $n_1^k$ for some $k > 0$, whereas $\mathcal{B}_0$ is generated by $n_2, c_3$. Then $\pi^*\mathcal{B}_0$ extends to a linear basis for the subalgebra of $H^*(BD_8)$ that do not involve $\sigma_1$. This allows us to work only with this subalgebra generated by $\pi^*\mathcal{B}_0$. We also change our basis of $H^*(BD_8)$ to a standard basis consisting of elements of the form $e^j\sigma_2^k\sigma_1^k$ (again, see Section 7.1).

Let $q_{(a,b)}(\alpha) := q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(e^a\sigma_2^b))$, the coefficient of $e^a\sigma_2^b$ in $qq(\alpha).

Theorem 7.9. For $p, q > 0, q < 2p$:

$$q_{(|\alpha|+p-|\alpha|-p)}(\alpha) = \sum_{s=0}^{q/2} \binom{p-s-1}{q-2s} q_{(|\alpha|+2s-p-|\alpha|-s)}(\alpha).$$
Theorem 7.2. The theorem follows by Lemma 7.8 and the combinatorial argument in Theorem 7.2.

Proof. The theorem follows by Lemma 7.8 and the combinatorial argument in Theorem 7.2.

To relate this to a composition of quantum Steenrod squares:

Proof of Corollary 1.7. To show that Equation (15) holds, we use a similar idea to proving the Cartan relation. Specifically, we use an interpolating moduli space as in Figure 9, here using Morse flowlines where for all $v$:

1. $f_{(23)v,t} = f_{(24)v,t} = f_{v,t}$ for all $t$.
2. $g_{(12)v,t} = g_{(34)v,t} = g_{v,t}$ for all $t$.

We see that the $t = 0$ boundary is

$$\sum_{i,j} q_{D_8} (\{ m_1 \} \otimes \Psi(e^i \sigma^j)) (\alpha) \cdot e^i \sigma^j.$$ 

The $t = \infty$ boundary is split into regions I, II, III.

Consider the contractible space $S^\infty \times (S^\infty \times S^\infty)$, along with an action of $D_8 = \langle r, a \mid r^2 = a^4 = 1, r \cdot a = a^3 \cdot r \rangle$ as follows:

- $r \cdot (z, (z_1, z_2)) = (-z, (z_2, z_1))$.
- $a \cdot (z, (z_1, z_2)) = (z, (-z_2, z_1))$.

This is a free action of $D_8$ on the contractible space $S^\infty \times (S^\infty \times S^\infty)$, hence it is an $ED_8$. If $D_8^{i+}$ is the upper $i$-dimensional hemisphere as usual, then $D_8^{i+} \times D_8^{j+} \times D_8^{k+}$ represents $\Psi(e^i \sigma^j \sigma^k)$. Using this model, and restricting attention to each of the regions I for $(QS_i \circ \ldots) \circ QS_j \circ \ldots$ and II, III for $(\ldots \circ QS_j \circ \ldots)$ respectively, we see that the $t = \infty$ boundary is $QS \circ QS(\alpha)$.

Observe that in the classical case, i.e. constant spheres, the operations

$$q_{D_8} (gm_1 \times \Psi b) = q_{D_8} (g^2 m_1 \times \Psi b),$$

and so the quantum case reduces to the classical case, since the solutions corresponding to those two operations cancel in pairs mod 2.

Remark. $q_{(j,0)}(\alpha)$ is the coefficient of $h^j$ in $QS(\alpha) \circ QS(\alpha)$. This is one of the correction terms that can be computed, e.g. the $p = |\alpha|$ term in Corollary 1.7.
8. QS FOR BLOW-UPS

We will demonstrate calculations of $S_{2,1,1}$ in two cases. The setup in both cases will be similar to the setup in [4, Section 8], where Blaier computes the quantum Massey product.

8.1. $\mathbb{CP}^3$. Fix two generic quadric hypersurfaces in $X = \mathbb{CP}^3$. Let $Y$ be their intersection. $Y$ is an elliptic curve, hence a torus. We let $M = Bl_Y X$.

Recall that there is a $\mathbb{CP}^1$-bundle $\pi : E \to Y$ over the torus and an inclusion $i : E \to M$ of the exceptional divisor $E$. Specifically $E$ is the projectivisation of the normal bundle of $Y \subset X$.

Consider the 3-disc bundle $DY \to Y$ such that $E \hookrightarrow \to DY$ is an inclusion of the subbundle $E$, with the maps of fibres being inclusion of the boundary $S^2 \hookrightarrow D^3$.

One can use the Mayer-Vietoris sequence, by observing that $M \cup_E DY \simeq X$, to write down the long exact sequence of a blow-up,

$$\ldots \to H_*(E) \to H_*(M) \oplus H_*(Y) \to H_*(X) \to H_{*-1}(E) \to \ldots$$ (18)

One can also use the homological Gysin sequence in this case to get an exact sequence:

$$\ldots \to H_*(E) \to H_*(Y) \to H_{*-3}(Y) \xrightarrow{\phi} H_{*-1}(E) \to \ldots$$ (19)

Putting these together, we can prove that:

$$H_2(M) \cong H_2(X) \oplus H_2(E)/H_2(Y)$$

$$i_* : H_3(E) \xrightarrow{\cong} H_3(M)$$

$$\phi : H_1(Y) \xrightarrow{\cong} H_3(E).$$

In particular $\dim H_3 E = 2$.

Moreover, the two generators of $H_2(M)$ are: the spheres lifted from $\mathbb{P}^3$ to $M$ and the spheres representing a fibre $\pi^{-1}$ over a point $y \in Y$.

We calculate the first Chern class of $M$.

Observe that we may embed $j : M \to \mathbb{P}^3 \times \mathbb{P}^1$ as a complex hypersurface of degree $(2, 1)$, where this degree is defined to be the standard definition of algebraic degree on each factor.

Then functorality and the Whitney sum formula imply that

$$c_1(TM) + c_1(vM) = c_1(T(\mathbb{P}^3 \times \mathbb{P}^1)|_M)$$

where $vM$ is the normal bundle of $M$ in $\mathbb{P}^3 \times \mathbb{P}^1$.

It is known that $c_1(T(\mathbb{P}^3 \times \mathbb{P}^1)|_M) = (4, 1)$, and $c_1(vM) = (2, 1)$ because it is the same as the degree (here we note that the Euler class can be reinterpreted as $j^*PD([M])$, the self intersection of $M$ with itself, where $[M] \in H_6(\mathbb{P}^3 \times \mathbb{P}^1)$). Hence $c_1(TM) = (2, 1)$.

Therefore, when calculating $QS_{1,1}$ we only need to consider the spheres in the fibre class of $M$ as these are the only $J$-holomorphic spheres of Chern number 1, which are confined to be in $E$.

$QS_{1,1} : H^3(M) \to H^3(M)$. We will show that $QS_{1,1}|_{H^3(M)} = id$. 
Figure 10. Configurations for $QS_{1,1}$ on $M$ and $QS_{1,2}$ on $E$.

First we show that we may reduce to $QS_{1,2}$: $H^3(E) \to H^1(E)$. We use $i_*$ on cohomology to mean $PD \circ i_* \circ PD^{-1}$.

**Lemma 8.1.** Fix $a \in H^3(M)$. Let $A = PD(a)$.

$$QS_{1,2} \circ i^* (a) = i_* \circ QS_{1,1} (a)$$

for $a \in H^3(M)$, i.e. (20) commutes:

$$
\begin{array}{ccc}
H^3(M) & \xrightarrow{QS_{1,1}} & H^3(M) \\
\downarrow{i^*} & & \downarrow{i_*} \\
H^3(E) & \xrightarrow{QS_{1,2}} & H^1(E)
\end{array}
$$

**Proof.** Consider inputs $A_v = PD(a)_v$ for $v \in S^1$ and output $B = PD(b)$ for $a, b \in H^3(M)$. In this instance one can show that

$$i^* B^\vee = (i_*^{-1} B)^\vee$$

for an appropriate choice of basis.

Figure [I] counts setups for $QS_{1,1}(a)$ and Figure [II] counts setups for $i_* QS_{1,2}(i^* a) \in H^1(E)$, using Equation (21). Given a type I setup, taking the intersection of everything with $E$, the $J$-holomorphic spheres are fibres and hence contained in $E$ and unaffected by this intersection. However their Chern number changes from 1 to 2. Further $a = PD(A)$ is replaced by $i^* a = PD(E \cap A)$ and $b^\vee$ is replaced by $i^* b^\vee$. This is a type II setup. Given a type II setup, and using that $i^*$ is an isomorphism, we obtain a type I setup. 

Note the change in degree of 2, which comes from the fact that $i_*$ changes cohomological degree by 2.

We now use the Morse theoretic definition of the quantum square. $E = Y \times \mathbb{P}^1$ so we may pick the Morse function on $E$ to be $f + g$ where $f : Y \to \mathbb{R}$ and $g : \mathbb{P}^1 \to \mathbb{R}$, such that $g$ has two critical points of index 0, 2, which we call $a_0, a_2$, and $f$ has critical points $b_0, b_1, b'_1, b_2$ (whose indices are the subscripts).
**Lemma 8.2.** $QS_{1,2} = \pi^* \circ Sq_1 \circ \pi_*$. 

*Proof.* Given a setup to calculate $QS_{1,2}$, Figure [10][II], project down to $Y$ to get a setup for calculating $Sq$: a fibre sphere in $E$ lives above a point $y \in Y$, hence if three flowlines meet a fibre sphere then this projects down to three flowlines coinciding at a point $y \in Y$. Input elements of $H^3(E)$ correspond to $(b_1, a_2)$ or $(b'_1, a_2)$, which project down to $b_1$ or $b'_1$ respectively. Output elements of $H^1(E)$ correspond to $(b_1, a_0)$ or $(b'_1, a_0)$.

For the opposite direction, consider a configuration for $Sq$ in $Y$ with the three flowlines meeting at $y$. Input elements of the form $b_1$ lift to $(b_1, a_2)$. The flowline $l$ incoming to $b_1$ in $Y$ lifts uniquely to an incoming flowline to $(b_1, a_2)$ in $E$ because $a_2$ is the maximum of $g$, and hence we may flow back uniquely along $(-\nabla f, -\nabla g)$ (subject to the condition that the flowline projected to $Y$ is $l$). See Figure [II].

Likewise the output critical point $b_1$ lifts to $(b_1, a_0)$, with the output flowline on $Y$ lifting uniquely to $E$.

Moreover, because this setup lifts from $Y$ we know that the three flowlines will all intersect $\pi^{-1}y$. Because this is a blow-up, we also know where each lifted flowline intersects $\pi^{-1}y$, as this is determined by the gradient of $f+g$ on each of the flowlines at $s = 0$. Hence there is a unique $J$-holomorphic sphere that fits into the lifted setup, giving a unique configuration on $E$ corresponding to the configuration on $Y$. \qed 

*Proof of Equation (4) in Theorem 1.8.* $Sq : H^1(Y) \rightarrow H^1(Y)$ is the identity for degree reasons. Lemmas 8.1 and 8.2 imply that Diagram (22) commutes.

$$
\begin{array}{ccc}
H^3(M) & \xrightarrow{QS_{1,1}} & H^3(M) \\
\downarrow i^* & & \downarrow i_* \\
H^3(E) & \xrightarrow{QS_{1,2}} & H^1(E) \\
\downarrow \pi_* & & \downarrow \pi^* \\
H^1(Y) & \xrightarrow{Sq_1} & H^1(Y)
\end{array}
$$

$Sq_1 = id$. In Morse cohomology,

$$
\pi_*^{-1}(b_1) = (b_1, a_0) \text{ and } \pi^*(b_1) = (b_1, a_2)
$$
so $Q S_{1,2}(b_1,a_0) = (b_1,a_2)$.

$H^i(E)$ is generated by $\{(b_1,a_{i-1}),(b'_1,a_{i-1})\}$ for $i = 1,3$.

Identify the cycles $B_1 \in H_1(E)$ to $(b_1,a_2) \in H^3(E)$ and $B_3 \in H_3(E)$ to $(b_1,a_0) \in H^3(E)$, using Poincaré duality and the isomorphism between Morse and classical cohomology (likewise $B'_1$ for $(b'_1,a_{3-i})$). Note that $B'_3 = B'_1$ and so on. In this notation $Q S_{1,2}(B_1) = B_3$.

$B_3 \cap B_3 = \emptyset$ so $i_* B_3 \cap i_* B_3 = \emptyset$ generically. As $H_3(M)$ is generated by $i_* B_3$ and $i_* B'_3$, this implies

$$(i_* B_3)^\vee = i_* B'_3.$$ 

By Equation (23),

$$i_* \circ Q S_{1,2} \circ i^*(i_* B_3)^\vee = i_* \circ Q S_{1,2} \circ i^*(i_* B'_3)).$$

By Equation (21),

$$i_* \circ Q S_{1,2} \circ i^*(i_* B_3)^\vee = i_* \circ Q S_{1,2} B'_3 = i_* B'_3.$$

From this and similar calculations, and Diagram (22), $i_* \circ Q S_{1,2} \circ i^* = \text{id}$. 

8.2. $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. Now let $X = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, with $Y \subset X$ a torus defined by the intersection of two linear hypersurfaces, and $M = B_{Y,X}$.

Using a similar method to the $\mathbb{CP}^3$ case, we can show that the Chern class of $M$ is $(1,1,1,1)$.

Hence, when calculating $Q S_{1,1} : H^3(M) \to H^3(M)$ there are contributions from the fibre direction plus those from $J$-holomorphic spheres in $X$ that have been lifted to $M$.

The fibrewise contributions are calculated in exactly the same way as for $\mathbb{CP}^3$, so we turn our attention to the spheres lifted from $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

Proof of Equation [5] in Theorem 1.8 Suppose the defining linear equations for $Y$ are $P_1(x,y,z)$ and $P_2(x,y,z)$ in local coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. Fixing $x,y$, there is at most one solution $z$ such that $P_1(x,y,z) = P_2(x,y,z) = 0$.

Hence, for any $S = \{pt\} \times \{pt'\} \times \mathbb{CP}^1$ a lift of a $J$-holomorphic curve of $X$ to $M$, we have that $A \cap Y$ is at most one point, hence there is no rigid solution to Figure 12. Here, the $A_v$ represents $A \in H_3(M)$ for each $v$. We recall that as we count degree 1 spheres, we do not need a transverse intersection $A_v \cap A_{-v}$, but rather that $A_v, A_{-v}$ meet the evaluation maps transversely. As such, we can choose $A_v = D_v \times \mathbb{CP}^1$ for $D_v$ representing an element $D \in H_1(Y)$, where $D_v$ is used in the calculation of $S q_1 (P D(D))$. We must verify that $S \subset Y$ is not possible, and this follows because there is no degree 1 holomorphic map $\mathbb{CP}^1 \to Y$. This ensures that $P_1^{-1}(0) \cap P_2^{-1}(0) \cap S$ is either empty or a single point. 

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