ON THE HERMITE-HADAMARD INEQUALITIES FOR INTERVAL-VALUED
CO-ORDINATED CONVEX FUNCTIONS

DAFANG ZHAO, MUHAMMAD AAMIR ALI, AND GHULAM MURTAZA

ABSTRACT. In this paper, we establish Hermite-Hadamard inequality for interval-valued convex function on the co-ordinates on the rectangle from the plane. We also present Hermite-Hadamard inequality for the product of interval-valued convex functions on co-ordinates. Some examples are also given to clarify our new results.

1. Introduction

The Hermite–Hadamard inequality discovered by C. Hermite and J. Hadamard, (see [8], [17, pp. 137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that, if \( f : I \to \mathbb{R} \) is a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \), then

\[
\begin{align*}
    f\left(\frac{a+b}{2}\right) & \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\end{align*}
\]

Both inequalities in (1.1) hold in the reversed direction if \( f \) is concave. We note that Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. Hermite–Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied.

In [7], Dragomir established the following similar inequality of Hadamard type for the co-ordinated convex functions.

**Theorem 1.** Let \( f : \Delta = [a,b] \times [c,d] \to \mathbb{R} \) is convex on co-ordinates \( \Delta \). Then following inequalities holds:

\[
\begin{align*}
    f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) dx + \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy \right] \\
    & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \\
    & \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x,c)dx + \frac{1}{b-a} \int_a^b f(x,d)dx \\
    & \quad + \frac{1}{c-d} \int_c^d f(a,y)dy + \frac{1}{d-c} \int_c^d f(b,y)dy \right] \\
    & \leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}.
\end{align*}
\]

For more results related to (1.2) we refer ([1], [10], [16]) and references therein.

On the other hand, interval analysis is a particular case of set–valued analysis which is the study of sets in the spirit of mathematical analysis and general topology. It was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real–world phenomena. An old example of interval enclosure is Archimede’s method which is related to compute of the circumference of a circle. In 1966, the first book related to interval analysis
was given by Moore who is known as the first user of intervals in computational mathematics, see [12]. After his book, several scientists started to investigate theory and application of interval arithmetic. Nowadays, because of its applications, interval analysis is a useful tool in various area which are interested intensely in uncertain data. You can see applications in computer graphics, experimental and computational physics, error analysis, robotics and many others.

What’s more, several important inequalities (Hermite–Hadamard, Ostrowski, etc.) have been studied for the interval–valued functions in recent years. In [3, 4], Chalco–Cano et al. obtained Ostrowski type inequalities for interval–valued functions by using Hukuhara derivative for interval–valued functions. In [18], Román–Flores et al. established Minkowski and Beckenbach’s inequalities for interval–valued functions. For the others, please see [5, 6, 9, 18, 19]. However, inequalities were studied for more general set–valued maps. For example, in [20], Sadowska gave the Hermite–Hadamard inequality. For the other studies, you can see [11, 14].

2. Preliminaries and Known Results

In this section we recalling some basics definitions, results, notions and properties, which are used throughout the paper. We denote \( \mathbb{R}_+^2 \) the family of all positive intervals of \( \mathbb{R} \). The Hausdorff distance between \( [\underline{X}, \overline{X}] \) and \( [\underline{Y}, \overline{Y}] \) is defined as
\[
d([\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}]) = \max \{ |\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}| \}.
\]
The \((\mathbb{R}_+, d)\) is a complete metric space. For more details and basic notations on interval-valued functions see ( [13], [21]).

It is remarkable that Moore [12] introduced the Riemann integral for the interval–valued functions. The set of all Riemann integrable interval–valued functions and real–valued functions on \([a, b]\) are denoted by \( \mathcal{IR}_{([a, b])} \) and \( \mathcal{R}_{([a, b])} \), respectively. The following theorem gives relation between \((IR)\)–integrable and Riemann integrable \((R\)–integrable) (see [13], pp. 131):

**Theorem 2.** Let \( F : [a, b] \rightarrow \mathbb{R}_+ \) be an interval–valued function such that \( F(t) = [\underline{E}(t), \overline{E}(t)] \). \( F \in \mathcal{IR}_{([a, b])} \) if and only if \( \underline{E}(t), \overline{E}(t) \in \mathcal{R}_{([a, b])} \) and
\[
\left( IR \right) \int_a^b F(t)dt = \left( R \right) \int_a^b \underline{E}(t)dt, \left( R \right) \int_a^b \overline{E}(t)dt.
\]

In [21, 23], Zhao et al. introduced a kind of convex interval–valued function as follows:

**Definition 1.** Let \( h : [c, d] \rightarrow \mathbb{R} \) be a non–negative function, \((0, 1) \subseteq [c, d] \) and \( h \neq 0 \). We say that \( F : [a, b] \rightarrow \mathbb{R}_+ \) is a \( h\)–convex interval–valued function, if for all \( x, y \in [a, b] \) and \( t \in (0, 1) \), we have
\[
h(t)F(x) + h(1-t)F(y) \subseteq F(tx + (1-t)y).
\]
With \( SX(h, [a, b], \mathbb{R}_+^2) \) will show the set of all \( h\)–convex interval–valued functions.

The usual notion of convex interval–valued function corresponds to relation (2.1) with \( h(t) = t \), see [20]. Also, if we take \( h(t) = t^* \) in (2.1), then Definition 1 gives the other convex interval–valued function defined by Breckner, see [2]. Otherwise, Zhao et al. obtained the following Hermite–Hadamard inequality for interval–valued functions by using \( h\)–convex:

**Theorem 3.** [21] Let \( F : [a, b] \rightarrow \mathbb{R}_+ \) be an interval–valued function such that \( F(t) = [\underline{F}(t), \overline{F}(t)] \) and \( F \in \mathcal{IR}_{([a, b])}, h : [0, 1] \rightarrow \mathbb{R} \) be a non–negative function and \( h \left( \frac{1}{2} \right) \neq 0 \). If \( F \in SX(h, [a, b], \mathbb{R}_+^2) \), then
\[
\frac{1}{2h \left( \frac{1}{2} \right)} F \left( \frac{a + b}{2} \right) \supseteq \frac{1}{b - a} (IR) \int_a^b F(x)dx \supseteq [F(a) + F(b)] \frac{1}{2}\int_0^1 h(t)dt.
\]

**Remark 1.** (i) If \( h(t) = t \), then (2.2) reduces to the following result:
\[
F \left( \frac{a + b}{2} \right) \supseteq \frac{1}{b - a} (IR) \int_a^b F(x)dx \supseteq F(a) + F(b) \frac{1}{2},
\]
which is obtained by [20].

(ii) If \( h(t) = t^* \), then (2.2) reduces to the following result:

\[
2^{n-1} F \left( \frac{a+b}{2} \right) \geq \frac{1}{b-a} (IR) \int_a^b F(x)dx \geq \frac{F(a) + F(b)}{s+1},
\]

which is obtained by [15].

**Theorem 4.** Let \( F, G : [a, b] \to \mathbb{R}_+^\times \) be two interval-valued functions such that \( F(t) = [F(t), \mathcal{F}(t)] \) and \( G(t) = [G(t), \mathcal{G}(t)] \), where \( F, G \in IR_{(a,b)}, h_1, h_2 : [0, 1] \to \mathbb{R} \) are two non-negative functions and \( h_1 \left( \frac{1}{2} \right) h_2 \left( \frac{1}{2} \right) \neq 0 \). If \( F, G \in SX(h, [a, b], \mathbb{R}_+^\times) \), then

\[
\frac{1}{2h_1 \left( \frac{1}{2} \right) h_2 \left( \frac{1}{2} \right)} F \left( \frac{a+b}{2} \right) G \left( \frac{a+b}{2} \right) \geq \frac{1}{b-a} (IR) \int_a^b F(x)G(x)dx + \int_0^1 h_1(t)h_2(1-t)dt + \int_0^1 h_1(t)h_2(t)dt
\]

and

\[
\frac{1}{b-a} (IR) \int_a^b F(x)G(x)dx \geq M(a, b) \int_0^1 h_1(t)h_2(t)dt + N(a, b) \int_0^1 h_1(t)h_2(1-t)dt,
\]

where

\[
M(a, b) = F(a)G(a) + F(b)G(b) \text{ and } N(a, b) = F(a)G(b) + F(b)G(a).
\]

**Remark 2.** If \( h(t) = t \), the (2.4) reduces to the following result:

\[
\frac{1}{b-a} (IR) \int_a^b F(x)G(x)dx \geq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b).
\]

**Remark 3.** If \( h(t) = t \), then (2.5) reduces to the following result:

\[
2 F \left( \frac{a+b}{2} \right) G \left( \frac{a+b}{2} \right) \geq \frac{1}{b-a} (IR) \int_a^b F(x)G(x)dx + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b).
\]

3. **Interval-valued double integral**

A set of numbers \( \{t_{i-1}, \xi_i, t_i\}^m_{i=1} \) is called tagged partition \( P_1 \) of \([a, b]\) if

\[
P_1 : a = t_0 < t_1 < \ldots < t_n = b
\]

and if \( t_{i-1} \leq \xi_i \leq t_i \) for all \( i = 1, 2, 3, \ldots, m \). Moreover if we have \( \Delta t_i = t_i - t_{i-1} \), then \( P_1 \) is said to be \( \delta \)-fine if \( \Delta t_i < \delta \) for all \( i \). Let \( \mathcal{P}(\delta, [a,b]) \) denote the set of all \( \delta \)-fine partitions of \([a,b]\). If \( \{t_{i-1}, \xi_i, t_i\}^m_{i=1} \) is a \( \delta \)-fine \( P_1 \) of \([a, b]\) and if \( \{s_{j-1}, \eta_j, s_j\}^n_{j=1} \) is \( \delta \)-fine \( P_2 \) of \([c, d]\), then rectangles

\[
\Delta_{i,j} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]
\]

partition the rectangle \( \Delta = [a, b] \times [c, d] \) and the points \((\xi_i, \eta_j)\) are inside the rectangles \([t_{i-1}, t_i] \times [s_{j-1}, s_j]\). Further, by \( \mathcal{P}(\delta, \Delta) \) we denote the set of all \( \delta \)-fine partitions \( P \) of \( \Delta \) with \( P_1 \times P_2 \), where \( P_1 \in \mathcal{P}(\delta, [a,b]) \) and \( P_2 \in \mathcal{P}(\delta, [c,d]) \). Let \( \Delta A_{i,j} \) be the area of rectangle \( \Delta_{i,j} \). In each rectangle \( \Delta_{i,j} \), where \( 1 \leq i \leq m, 1 \leq j \leq n \), choose arbitrary \((\xi_i, \eta_j)\) and get

\[
S(F, P, \delta, \Delta) = \sum_{i=1}^m \sum_{j=1}^n F(\xi_i, \eta_j) \Delta A_{i,j}.
\]

We call \( S(F, P, \delta, \Delta) \) is integral sum of \( F \) associated with \( P \in \mathcal{P}(\delta, \Delta) \).

Now we recall the concept of interval-valued double integral given by Zhao et al. in [22].
Theorem 5. [22] Let $F : \Delta \to \mathbb{R}_I$. Then $F$ is called ID-integrable on $\Delta$ with ID-integral $U = \langle \text{ID} \rangle \int_{\Delta} F(t,s)dA$, if for any $\epsilon > 0$ there exist $\delta > 0$ such that

$$d(S(F,P,\delta,\Delta)) < \epsilon$$

for any $P \in \mathcal{P}(\delta,\Delta)$. The collection of all ID-integrable functions on $\Delta$ will be denoted by $\mathcal{ID}(\Delta)$.

Theorem 6. [22] Let $\Delta = [a,b] \times [c,d]$. If $F : \Delta \to \mathbb{R}_I$ is ID-integrable on $\Delta$, then we have

$$\langle \text{ID} \rangle \int_{\Delta} F(s,t)dA = \langle I \rangle \int_{a}^{b} \langle I \rangle \int_{c}^{d} F(s,t)dsdt.$$

Example 1. Let $F : \Delta = [0,1] \times [1,2] \to \mathbb{R}_I^+$ be defined by

$$F(s,t) = [st, s + t],$$

then $F(s,t)$ is integrable on $\Delta$ and $\langle \text{ID} \rangle \int_{\Delta} F(t,s)dA = \left[ \frac{3}{4}, 2 \right]$.

4. Main Results

In this section, we define interval-valued co-ordinated convex function and prove some inequalities of Hermite-Hadamard type by using our new definition. Throughout this section we will use $\Delta = [a,b] \times [c,d]$, where $a < b$ and $c < d$, $a,b,c,d \in \mathbb{R}$.

Definition 2. A function $F : \Delta \to \mathbb{R}_I^+$ is said to be interval-valued co-ordinated convex function, if the following inequality holds:

$$F(tx + (1-t)y, su + (1-s)w)$$

$$\geq tsF(x,u) + t(1-s)F(x,w) + s(1-t)F(y,u) + (1-s)(1-t)F(y,w),$$

for all $(x,y),(u,w) \in \Delta$ and $s,t \in [0,1]$.

Lemma 1. A function $F : \Delta \to \mathbb{R}_I^+$ is interval-valued convex on co-ordinates if and only if there exists two functions $F_x : [c,d] \to \mathbb{R}_I^+$, $F_x(w) = F(x,w)$ and $F_y : [a,b] \to \mathbb{R}_I^+$, $F_y(u) = F(y,u)$ are interval-valued convex.

The proof of this lemma follows immediately by the definition of interval-valued co-ordinated convex function.

It is easy to proof that an interval-valued convex function is interval-valued co-ordinated convex but the converse may not be true. For this we can see the following example.

Example 2. An interval-valued function $F : [0,1]^2 \to \mathbb{R}_I^+$ defined as $F(x,y) = [xy, (6 - e^x)(6 - e^y)]$ is interval-valued convex on co-ordinates but it is not interval-valued convex on $[0,1]^2$.

Proposition 1. If $F, G : \Delta \to \mathbb{R}_I^+$ are two interval-valued co-ordinated convex functions on $\Delta$ and $\alpha \geq 0$, then $F + G$ and $\alpha F$ are interval-valued co-ordinated convex functions.

Proposition 2. If $F, G : \Delta \to \mathbb{R}_I^+$ are two interval-valued co-ordinated convex functions on $\Delta$, then $(FG)$ is interval-valued co-ordinated convex function on $\Delta$.

Proof. Since $F$ and $G$ are interval-valued co-ordinated convex functions, we have

$$F(tx + (1-t)y, su + (1-s)w)$$

$$\geq tsF(x,u) + t(1-s)F(x,w) + s(1-t)F(y,u) + (1-s)(1-t)F(y,w)$$

and

$$G(tx + (1-t)y, su + (1-s)w)$$

$$\geq tsG(x,u) + t(1-s)G(x,w) + s(1-t)G(y,u) + (1-s)(1-t)G(y,w).$$
Multiplying (4.1) and (4.2), we have
\[
F(tx + (1-t)y, su + (1-s)w)G(tx + (1-t)y, su + (1-s)w)
\geq [tsF(x, u) + t(1-s)F(x, w) + s(1-t)F(y, u) + (1-s)(1-t)F(y, w)]
\times [tsG(x, u) + t(1-s)G(x, w) + s(1-t)G(y, u) + (1-s)(1-t)G(y, w)]
= stF(x, u)G(x, w) + t(1-s)F(x, w)G(x, w)
+(1-t)sF(y, u)G(y, u) + (1-t)(1-s)F(y, w)G(y, w)
\]
and therefore \(FG\) is interval-valued co-ordinated convex function.

In what follows, without causing confusion, we will delete notations of \((R), (IR)\) and \((ID)\). We start with the following Theorem.

**Theorem 7.** If \(F: \Delta \to \mathbb{R}^+_I\) is interval-valued co-ordinated convex function on \(\Delta\) such that \(F(t) = [F(t), F(t)]\), then following inequalities holds:

\[
\begin{align*}
F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\geq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b F\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d F\left(\frac{a+b}{2}, y\right) dy\right] \\
&\geq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) dy dx \\
&\geq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b F(x, c) dx + \frac{1}{b-a} \int_a^b F(x, d) dx \\
&\quad + \frac{1}{d-c} \int_c^d F(a, y) dy + \frac{1}{d-c} \int_c^d F(b, y) dy\right] \\
&\geq \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{4}.
\end{align*}
\]

**Proof.** Since \(F\) is interval-valued co-ordinated convex function on co-ordinates \(\Delta\), then \(F_x : [c, d] \to \mathbb{R}^+_I\), \(F_x(y) = F(x, y)\) is interval-valued convex function on \([c, d]\) and for all \(x \in [a, b]\), From inequality (2.3), we have

\[
F_x\left(\frac{c+d}{2}\right) \geq \frac{1}{d-c} \int_c^d F_x(y) dy \geq \frac{F_x(c) + F_x(d)}{2},
\]

that can be written as

\[
F\left(\frac{x, c+d}{2}\right) \geq \frac{1}{d-c} \int_c^d F(x, y) dy \geq \frac{F(x, c) + F(x, d)}{2}.
\]

Integrating (4.4) with respect to \(x\) over \([a, b]\) and dividing both sides by \((b-a)\), we have

\[
\begin{align*}
\frac{1}{b-a} \int_a^b F\left(x, \frac{c+d}{2}\right) dx &\geq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) dy dx \\
&\geq \frac{1}{2(b-a)} \left[ \int_a^b F(x, c) dx + \int_a^b F(x, d) dx\right].
\end{align*}
\]

Similarly, \(F_y = [a, b] \to \mathbb{R}^+_I\), \(F_y(x) = F(x, y)\) is interval-valued convex function on \([a, b]\) and \(y \in [c, d]\), we have

\[
\begin{align*}
\frac{1}{d-c} \int_c^d F\left(\frac{a+b}{2}, y\right) dy &\geq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) dy dx \\
&\geq \frac{1}{2(d-c)} \left[ \int_c^d F(a, y) dy + \int_c^d F(b, y) dy\right].
\end{align*}
\]

By adding (4.5) and (4.6) and using Theorem 2, we have second and third inequality in (4.3).
We also have from (2.3),

\[
F \left( \frac{a + b + c + d}{2}, \frac{c + d}{2} \right) \geq \frac{1}{b - a} \int_a^b F \left( x, \frac{c + d}{2} \right) dx
\]

(4.7)

\[
F \left( \frac{a + b + c + d}{2}, \frac{c + d}{2} \right) \geq \frac{1}{d - c} \int_c^d F \left( \frac{a + b}{2}, y \right) dy.
\]

(4.8)

By adding (4.7) and (4.8) and using Theorem (2), we have first inequality in (4.3).

At the end, again from (2.2) and Theorem (2), we have

\[
\frac{1}{b - a} \int_a^b F(x, c)dx \geq \frac{F(a, c) + F(b, c)}{2},
\]

\[
\frac{1}{b - a} \int_a^b F(x, d)dx \geq \frac{F(a, d) + F(b, d)}{2},
\]

\[
\frac{1}{d - c} \int_c^d F(a, y)dy \geq \frac{F(a, c) + F(a, d)}{2},
\]

\[
\frac{1}{d - c} \int_c^d F(b, y)dy \geq \frac{F(b, c) + F(b, d)}{2},
\]

and proof is completed. \(\square\)

**Example 3.** Suppose that \([a, b] = [0, 1]\) and \([c, d] = [1, 2]\). Let \(F : [a, b] \times [c, d] \rightarrow \mathbb{R}^+_I\) be given as \(F(x, y) = [xy, 4xy]\), for all \(x \in [a, b]\) and \(y \in [c, d]\). We have

\[
F \left( \frac{a + b + c + d}{2}, \frac{c + d}{2} \right) = \left[ \frac{3}{4}, 3 \right],
\]

\[
\frac{1}{2} \left[ \frac{1}{b - a} \int_a^b F \left( x, \frac{c + d}{2} \right) dx + \frac{1}{d - c} \int_c^d F \left( \frac{a + b}{2}, y \right) dy \right] = \left[ \frac{3}{4}, 3 \right].
\]

\[
\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d F(x, y)dydx = \left[ \frac{3}{4}, 3 \right],
\]

\[
\frac{1}{4} \left[ \frac{1}{b - a} \int_a^b F(x, c)dx + \frac{1}{b - a} \int_a^b F(x, d)dx \right.
\]

\[
+ \frac{1}{d - c} \int_c^d F(a, y)dy + \frac{1}{d - c} \int_c^d F(b, y)dy \right] = \left[ \frac{3}{4}, 3 \right],
\]

\[
\frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{4} = \left[ \frac{3}{4}, 3 \right].
\]

Hence \(\left[ \frac{3}{4}, 3 \right] \supseteq \left[ \frac{3}{4}, 3 \right] \supseteq \left[ \frac{3}{4}, 3 \right] \supseteq \left[ \frac{3}{4}, 3 \right] \).

**Remark 4.** If \(\overline{F} = F\) in Theorem 7, then Theorem 7 reduces to Theorem 1.

**Theorem 8.** If \(F, G : \Delta \rightarrow \mathbb{R}^+_I\) are two interval–valued co-ordinated convex functions such that \(F(t) = \left[ F(t), F(t) \right]\) and \(G(t) = \left[ G(t), G(t) \right]\), then the following inequality holds:

\[
\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d F(x, y)G(x, y)dydx \geq \frac{1}{9} P(a, b, c, d) + \frac{1}{18} M(a, b, c, d) + \frac{1}{36} N(a, b, c, d),
\]

(4.9)

where

\[
P(a, b, c, d) = F(a, c)G(a, c) + F(a, d)G(a, d) + F(b, d)G(b, d),
\]

\[
M(a, b, c, d) = F(a, c)G(a, d) + F(a, d)G(a, c) + F(b, c)G(b, d) + F(b, d)G(b, c) + F(b, c)G(b, c) + F(b, d)G(b, d),
\]

\[
N(a, b, c, d) = F(b, c)G(a, d) + F(a, d)G(b, c) + F(b, d)G(a, c) + F(a, c)G(b, d).
\]
Proof. Since $F$ and $G$ are interval-valued co-ordinates convex functions on $\Delta$, therefore $F(x) : [a, b] \rightarrow \mathbb{R}_+^+$, $F_y(x) = F(x, y)$, $G(x) : [c, d] \rightarrow \mathbb{R}_+^+$, $G(x, y) = G(x, y)$ and $F_y(x) : [a, b] \rightarrow \mathbb{R}_+^+$, $F_y(x) = F(x, y)$, $G_y : [a, b] \rightarrow \mathbb{R}_+^+$, $G_y(x) = G(x, y)$ are interval-valued convex functions on $[c, d]$ and $[a, b]$ respectively for all $x \in [a, b], y \in [c, d]$.

Now from inequality (2.6), we have

$$\frac{1}{d-c} \int_c^d F(x)G_x(y)dy \geq \frac{1}{3}[F_x(c)G_x(c) + F_x(d)G_x(d)]$$

that can be written as

$$\frac{1}{d-c} \int_c^d F(x)G(x,y)dy \geq \frac{1}{3}[F(x,c)G(x,c) + F(x,d)G(x,d)]$$

Integrating the above inequality with respect to $x$ over $[a, b]$ and and dividing both sides by $b-a$, we have

$$(4.10)\quad \frac{1}{b-a}(d-c) \int_a^b \int_c^d F(x,y)G(x,y)dydx$$

$$\geq \frac{1}{3(b-a)} \int_a^b [F(x,c)G(x,c) + F(x,d)G(x,d)]dx$$

$$\quad + \frac{1}{6(b-a)} \int_a^b [F(x,c)G(x,d) + F(x,d)G(x,c)]dx.$$  

Now using inequality (2.6) to each integral on right hand side of (4.10), we have

$$(4.11)\quad \frac{1}{b-a} \int_a^b F(x,c)G(x,c)dx \geq \frac{1}{3}[F(a,c)G(a,c) + F(b,c)G(b,c)]$$

$$\quad + \frac{1}{6}[F(a,c)G(b,c) + F(b,c)G(a,c)],$$

$$(4.12)\quad \frac{1}{b-a} \int_a^b F(x,d)G(x,d)dx \geq \frac{1}{3}[F(a,d)G(a,d) + F(b,d)G(b,d)]$$

$$\quad + \frac{1}{6}[F(a,d)G(b,d) + F(b,d)G(a,d)],$$

$$(4.13)\quad \frac{1}{b-a} \int_a^b F(x,c)G(x,d)dx \geq \frac{1}{3}[F(a,c)G(a,d) + F(b,c)G(b,d)]$$

$$\quad + \frac{1}{6}[F(a,c)G(b,d) + F(b,c)G(a,d)],$$

$$(4.14)\quad \frac{1}{b-a} \int_a^b F(x,d)G(x,c)dx \geq \frac{1}{3}[F(a,d)G(a,c) + F(b,d)G(b,c)]$$

$$\quad + \frac{1}{6}[F(a,d)G(b,c) + F(b,d)G(a,c)].$$

Substituting (4.11)-(4.14) in (4.10), we have our desired inequality (4.9). Similarly we can find same inequality by using $F_y(x)G_y(x)$ on $[a, b]$. \qed

Remark 5. If $F = F_y$ in Theorem 8, then Theorem 8 reduces to ( [10, Theorem 4]).
Theorem 9. If \( F, G : \Delta \to \mathbb{R}^+_I \) are two interval-valued co-ordinated convex functions such that 
\[
F(t) = [F(t), \overline{F(t)}] \quad \text{and} \quad G(t) = \left[ G(t), \overline{G(t)} \right],
\]
then we have the following inequality:

\[
4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) G\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \geq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x,y)G(x,y)dydx
\]

\[
+ \frac{5}{36} P(a,b,c,d) + \frac{7}{36} M(a,b,c,d) + \frac{2}{9} N(a,b,c,d),
\]

where \( P(a,b,c,d), M(a,b,c,d) \) and \( N(a,b,c,d) \) are defined in Theorem 8.

Proof. Since \( F \) and \( G \) are interval-valued co-ordinated convex functions, from (2.7) we have

\[
2F\left(\frac{a+b}{2}\right) G\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a} \int_a^b F\left(x, \frac{c+d}{2}\right) G\left(x, \frac{c+d}{2}\right) dx
\]

\[
+ \frac{1}{6} \left[ F\left(a, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) \right]
\]

\[
+ \frac{1}{3} \left[ F\left(a, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) \right],
\]

and

\[
2F\left(\frac{a+b}{2}\right) G\left(\frac{a+b}{2}\right) \geq \frac{1}{d-c} \int_c^d F\left(\frac{a+b}{2}, y\right) G\left(\frac{a+b}{2}, y\right) dy
\]

\[
+ \frac{1}{6} \left[ F\left(\frac{a+b}{2}, c\right) G\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right) G\left(\frac{a+b}{2}, d\right) \right]
\]

\[
+ \frac{1}{3} \left[ F\left(\frac{a+b}{2}, c\right) G\left(\frac{a+b}{2}, d\right) + F\left(\frac{a+b}{2}, d\right) G\left(\frac{a+b}{2}, c\right) \right].
\]

Adding (4.16), (4.18) and multiplying on both sides of the resultnat one by 2, we get

\[
8 \left(\frac{a+b}{2}\right) G\left(\frac{a+b}{2}\right) \geq \frac{2}{b-a} \int_a^b F\left(x, \frac{c+d}{2}\right) G\left(x, \frac{c+d}{2}\right) dx
\]

\[
+ \frac{2}{d-c} \int_c^d F\left(\frac{a+b}{2}, y\right) G\left(\frac{a+b}{2}, y\right) dy
\]

\[
+ \frac{1}{6} \left[ 2F\left(a, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) + 2F\left(b, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) \right]
\]

\[
+ \frac{1}{6} \left[ 2F\left(\frac{a+b}{2}, c\right) G\left(\frac{a+b}{2}, c\right) + 2F\left(\frac{a+b}{2}, d\right) G\left(\frac{a+b}{2}, d\right) \right]
\]

\[
+ \frac{1}{3} \left[ F\left(a, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) \right]
\]

\[
+ \frac{1}{3} \left[ F\left(\frac{a+b}{2}, c\right) G\left(\frac{a+b}{2}, d\right) + F\left(\frac{a+b}{2}, d\right) G\left(\frac{a+b}{2}, c\right) \right].
\]
Now from (2.7), we have

\[
2F\left(a, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) \\
\supseteq \frac{1}{d-c} \int_{c}^{d} F(a, y) G(a, y) dy \\
+ \frac{1}{6} \left[F(a, c) G(a, c) + F(a, d) G(a, d)\right] \\
+ \frac{1}{3} \left[F(a, c) G(a, d) + F(a, d) G(a, c)\right],
\]

(4.19)

\[
2F\left(b, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) \\
\supseteq \frac{1}{d-c} \int_{c}^{d} F(b, y) G(b, y) dy \\
+ \frac{1}{6} \left[F(b, c) G(b, c) + F(b, d) G(b, d)\right] \\
+ \frac{1}{3} \left[F(b, c) G(b, d) + F(b, d) G(b, c)\right],
\]

(4.20)

\[
2F\left(a + \frac{b}{2}, c\right) G\left(a + \frac{b}{2}, c\right) \\
\supseteq \frac{1}{b-a} \int_{a}^{b} F(x, c) G(x, c) dx \\
+ \frac{1}{6} \left[F(a, c) G(a, c) + F(b, c) G(b, c)\right] \\
+ \frac{1}{3} \left[F(a, c) G(b, c) + F(b, c) G(a, c)\right],
\]

(4.21)

\[
2F\left(a + \frac{b}{2}, d\right) G\left(a + \frac{b}{2}, d\right) \\
\supseteq \frac{1}{b-a} \int_{a}^{b} F(x, d) G(x, d) dx \\
+ \frac{1}{6} \left[F(a, d) G(a, d) + F(b, d) G(b, d)\right] \\
+ \frac{1}{3} \left[F(a, d) G(b, d) + F(b, d) G(a, d)\right],
\]

(4.22)

\[
2F\left(a, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) \\
\supseteq \frac{1}{d-c} \int_{c}^{d} F(a, y) G(b, y) dy \\
+ \frac{1}{6} \left[F(a, c) G(b, c) + F(a, d) G(b, d)\right] \\
+ \frac{1}{3} \left[F(a, c) G(b, d) + F(a, d) G(b, c)\right],
\]

(4.23)
Using (4.19)-(4.26) in (4.18), we have

\[
8 \left( \frac{a + b}{2} \right) G \left( \frac{a + b}{2} \right) \supseteq \frac{2}{b - a} \int_a^b F \left( x, \frac{c + d}{2} \right) G \left( x, \frac{c + d}{2} \right) dx \\
+ \frac{2}{d - c} \int_c^d F \left( \frac{a + b}{2}, y \right) G \left( \frac{a + b}{2}, y \right) dy \\
+ \frac{1}{6(d - c)} \int_c^d F(a, y)G(a, y)dy + \frac{1}{6(d - c)} \int_c^d F(b, y)G(b, y) \\
+ \frac{1}{6(b - a)} \int_a^b F(x, c)G(x, c)dx + \frac{1}{6(b - a)} \int_a^b F(x, d)G(x, d)dx \\
+ \frac{1}{3(d - c)} \int_c^d F(a, y)G(b, y)dy + \frac{1}{3(d - c)} \int_c^d F(b, y)G(a, y)dy \\
+ \frac{1}{3(b - a)} \int_a^b F(x, c)G(x, d)dx + \frac{1}{3(b - a)} \int_a^b F(x, d)G(x, c)dx \\
+ \frac{1}{18} P(a, b, c, d) + \frac{1}{9} M(a, b, c, d) + \frac{2}{9} N(a, b, c, d).
\]
Again from (2.7), we have following relations

\[(4.28)\]
\[
\frac{2}{d-c} \int_c^d F(\frac{a+b}{2}, y) G(\frac{a+b}{2}, y) \, dy \\
\geq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) G(x, y) \, dy \, dx \\
+ \frac{1}{6(d-c)} \int_c^d F(a, y) G(a, y) \, dy + \frac{1}{6(d-c)} \int_c^d F(b, y) G(b, y) \, dy \\
+ \frac{1}{3(d-c)} \int_c^d F(a, y) G(b, y) \, dy + \frac{1}{3(d-c)} \int_c^d F(b, y) G(a, y) \, dy,
\]

\[(4.29)\]
\[
\frac{2}{b-a} \int_a^b F(\frac{x, c+d}{2}) G(\frac{x, c+d}{2}) \, dx \\
\geq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) G(x, y) \, dy \, dx \\
+ \frac{1}{6(b-a)} \int_a^b F(x, c) G(x, c) \, dx + \frac{1}{6(b-a)} \int_a^b F(x, d) G(x, d) \, dx \\
+ \frac{1}{3(b-a)} \int_a^b F(x, c) G(x, d) \, dx + \frac{1}{3(b-a)} \int_a^b F(x, d) G(x, c) \, dx.
\]

Using (4.28) and (4.29) in (4.27), we have

\[(4.30)\]
\[
8 \left(\frac{a+b}{2}\right) G\left(\frac{a+b}{2}\right) \\
\geq \frac{2}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) G(x, y) \, dy \, dx \\
+ \frac{1}{3(d-c)} \int_c^d F(a, y) G(a, y) \, dy + \frac{1}{3(d-c)} \int_c^d F(b, y) G(b, y) \, dy \\
+ \frac{1}{3(b-a)} \int_a^b F(x, c) G(x, c) \, dx + \frac{2}{3(b-a)} \int_a^b F(x, d) G(x, d) \, dx \\
+ \frac{2}{3(d-c)} \int_c^d F(a, y) G(b, y) \, dy + \frac{2}{3(d-c)} \int_c^d F(b, y) G(a, y) \, dy \\
+ \frac{2}{3(b-a)} \int_a^b F(x, c) G(x, d) \, dx + \frac{2}{3(b-a)} \int_a^b F(x, d) G(x, c) \, dx \\
+ \frac{1}{18} F(a, b, c, d) + \frac{1}{9} M(a, b, c, d) + \frac{2}{9} N(a, b, c, d)
\]

and by using (2.6) on each integral in (4.30), we have our required result. \(\square\)

**Remark 6.** If \(\mathcal{F} = \overline{\mathcal{F}}\) in Theorem 9, then Theorem 9 reduces to the ( [10, Theorem 5]).

**Conclusion**

In this article, interval-valued co-ordinated convex function and double integral for the interval-valued functions are introduced and establish some new inequalities of Hermite-Hadamard type. Our inequalities are the extend some previously obtained results.

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**College of Science, Hohai University, Nanjing, P. R. China and School of Mathematics and Statistics, Hubei Normal University, Huangshi, P. R. China**

E-mail address: dafangzhao@163.com

**Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, 210023, China**

E-mail address: mahr.muhammad.aamir@gmail.com

**Department of Mathematics, Government College University Faisalabad, Pakistan**

E-mail address: gmizami@gmail.com