ON THE DUALITY BETWEEN ROTATIONAL MINIMAL SURFACES AND MAXIMAL SURFACES

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Abstract. We investigate the duality between minimal surfaces in Euclidean space and maximal surfaces in Lorentz-Minkowski space in the family of rotational surfaces. We study if the dual surfaces of two congruent rotational minimal (or maximal) surfaces are congruent. We show that in the duality process by means of a one-parameter group of rotations, it appears the family of Bonnet minimal (maximal) surfaces and the Goursat transformations.

1. Introduction

There is a correspondence, known as duality, between minimal surfaces in Euclidean space $\mathbb{E}^3$ and maximal surfaces in Lorentz-Minkowski space $\mathbb{L}^3$. This correspondence assigns a maximal surface to each minimal surface and vice-versa and it was introduced by Calabi for minimal surfaces and maximal surfaces expressed as graphs on a simply-connected domain [5]. A similar correspondence between both families of surfaces also appeared on [8, 14] where the duality is now defined in terms of the isotropic curve that determines the surface and finally, it was proved in [11] that both methods are equivalent. We describe this correspondence in terms of complex analysis and the isotropic curve. If $X : \Omega \rightarrow \mathbb{E}^3$ is a conformal minimal surface defined on a simply-connected domain $\Omega$ of the complex plane $\mathbb{C}$ and $z$ is the conformal parameter, then the complex curve $\phi : \Omega \rightarrow \mathbb{C}^3$ defined by $\phi(z) = 2X_z = (\phi_1, \phi_2, \phi_3)$ is holomorphic and satisfies the isotropy relation $\langle \phi, \phi \rangle = \phi_1^2 + \phi_2^2 + \phi_3^2 = 0$. If we now define $\psi : \Omega \rightarrow \mathbb{C}^3$ by $(\psi_1, \psi_2, \psi_3) = (-i\phi_1, -i\phi_2, \phi_3)$, then $\psi_1^2 + \psi_2^2 - \psi_3^2 = 0$ and consequently this defines a maximal surface $X^\flat : \Omega \rightarrow \mathbb{L}^3$ by setting $X^\flat(z) = \Re \int^{z} \psi(z)dz$. This process has its converse: if $X : \Omega \rightarrow \mathbb{L}^3$ is a conformal maximal surface and $\psi(z) = 2X_z =$
(ψ₁, ψ₂, ψ₃), then the complex curve φ = (iψ₁, iψ₂, ψ₃) satisfies ⟨φ, φ⟩ = 0 and defines a minimal surface M♯ in E³ by means of X♯(z) = ℜ ∫ z φ(z) dz. Furthermore, and up to translations of the ambient space, we have M = (M♯)♭. If Min and Max denote the family of minimal surfaces of E³ and the maximal surfaces of L³, respectively, we have established two maps

♭ : Min → Max,  ♭ : Max → Min

with the property that ♭ ◦ ♬ and ♬ ◦ ♭ are the identities in Max and Min respectively. We say that M♭ (or M♯) is the dual surface of M (also named in the literature as twin surface [8, 12, 15]). This process of duality has been generalized in other ambient spaces: see for example, [1, 12, 19]

In this paper we are interested in a problem posed by Araujo and Leite in [3] that asks whether the dual surfaces of two congruent minimal (or maximal) surfaces are also congruent. We precise the terminology. We say that two surfaces M₁ and M₂ of E³ (or L³) are congruent, and we denote by M₁ ≃ M₂, if there is an orientation-preserving isometry of the ambient space taking one of the surface onto the other. Here we also suppose that this relation ≃ is up to an automorphism on M₁ and M₂ and up to dilations of the ambient space because dilations preserve the zero mean curvature property. In E³ the set of congruences preserving the orientation is SO(3) and in L³ is SO(2, 1). Then the problem can be formulated as follows:

Problem 1. If M₁ ≃ M₂ are two minimal surfaces of E³, does M♭₁ ≃ M♭₂ hold?

A similar question can be posed for maximal surfaces. Surprisingly the answer is ‘not in general’ and there are many congruent minimal surfaces whose dual surfaces are not congruent. The process to consider is the following. Let M ⊂ E³ be a minimal surface and the congruence class of M, namely, {T(M) : T ∈ SO(3)}, then compute {T(M)♭ : T ∈ SO(3)} and in this set we consider the equivalence relation by congruences. If M♭₁/ ≃ stands for the corresponding quotient space, we want to determine this set. Similarly, we use the notation M♯₁/ ≃. In [3] the authors study this problem in the the case that M is the Enneper surface, the Scherk surface and the catenoid.

In this paper we focus how the geometric properties of a minimal (or maximal) surface can transform to its dual surface and we pay our attention for rotational surfaces with the next question:

Problem 2. Is the dual surface rotational of a rotational minimal (or maximal) surface?
Let us observe that the maps $♭$ and $♯$ do not carry any geometrical information of the surface because in the definition of a dual surface only it is involved the complex coordinates of the surface. We point out that the class of rotational surfaces of $L^3$ is richer than in the Euclidean case because in $L^3$ there exist three types of rotational surfaces according the causal character of the axis of revolution. If we restrict to maximal surfaces, there are three types of non-congruent rotational maximal surfaces, named, elliptic catenoid, hyperbolic catenoid and parabolic catenoid when the rotational axis is timelike, spacelike or lightlike, respectively [10]. It was proved in [3] that for an Euclidean catenoid $M$, the quotient space $M♯/≃$ has the topology of the closed interval $[0, 1]$, obtaining a Bonnet maximal surface for $t < 1$ and the hyperbolic catenoid for $t = 1$. In Section 4 we recover this result by describing explicitly the dual surfaces of the rotational maximal surfaces of $L^3$. Then we take an Euclidean catenoid $C$ with axis $(0, 0, 1)$ and we consider the dual surfaces of the Euclidean catenoids obtained by rotation $C$ about a line orthogonal to the axis of $C$. In Section 5 we consider each one the three catenoids $C$ of $L^3$ and we deform by rotations about an axis with different causal character than the one of $C$. We will prove in Thms. 5.2, 5.3 and 5.4 that the dual surfaces belong to the Bonnet family of minimal surfaces up to a Goursat transformation or it is the Enneper surface.

2. Duality of minimal/maximal surfaces

Let $\mathbb{R}^3$ be the vector space where $(x, y, z)$ stands for the canonical coordinates. We endow $\mathbb{R}^3$ with the metric $ds^2 = dx^2 + dy^2 + \epsilon dz^2$, with $\epsilon = 1$ for the Euclidean metric and $\epsilon = -1$ for the Lorentzian metric, obtaining the Euclidean space $E^3$ ($\epsilon = 1$) and the Lorentz-Minkowski space $L^3$ ($\epsilon = -1$). Let $X : M \rightarrow (\mathbb{R}^3, ds^2)$ be a conformal immersion of a (connected oriented) surface $M$ with $X = X(z)$ and $z = u + iv \in \mathbb{C}$ stands for a conformal parameter. In case $\epsilon = -1$, we are assuming that the induced metric on $M$ via $X$ is Riemannian, that is, $(M, ds^2)$ is a spacelike surface in $L^3$. Suppose that the immersion has zero mean curvature at every point and we say that $M$ is a minimal surface ($\epsilon = 1$) or a maximal surface ($\epsilon = -1$). This is equivalent that the immersion $X$ is harmonic and this guarantees that the curve $\phi = \phi(z) : M \rightarrow \mathbb{C}^3$ defined by

$$\phi(z) = (\phi_1, \phi_2, \phi_3) = 2\frac{dX}{dz}$$

is holomorphic. Therefore $\phi$ satisfies $\phi_1^2 + \phi_2^2 + \epsilon \phi_3^2 = 0$, which means that $\phi$ lies on the complex null cone (resp. Lorentzian complex null cone) of $\mathbb{C}^3$ if $\epsilon = 1$ (resp. $\epsilon = -1$). The curve $\phi$ is called the isotropic curve of the immersion $X$. The induced metric on
the surface \( M \) reads as \( ds^2 = |\phi_1|^2 + |\phi_2|^2 + \epsilon|\phi_3|^2 \neq 0 \) and then the surface is obtained by \( X(z) = X(z_0) + \Re \int_{z_0}^{z} \phi(z)dz \) for any curve connecting a given point \( z_0 \in M \) and \( z \). The integral does not depend on the curve which is equivalent to \( \Re \int_{\gamma} \phi(z)dz = 0 \) for any closed curve \( \gamma \) in \( M \) and we say that \( \phi \) has no real periods. Letting

\[
g = \frac{\dot{\phi}_3}{\dot{\phi}_1 - i\dot{\phi}_2}, \quad \omega = (\phi_1 - i\phi_2)dz,
\]

the pair \((g, \omega)\) is called the Weierstrass representation of \( X \), where \( g \) is a meromorphic function on \( M \) and \( \omega \) is a holomorphic 1-form on \( M \). The parametrization \( X \) is now

\[
X(z) = X(z_0) + \Re \int_{z_0}^{z} \left( \frac{1}{2}(1 - \epsilon g^2)\omega, \frac{i}{2}(1 + \epsilon g^2)\omega, g\omega \right)
\]

and the metric is \( ds = |\omega|(1 + |g|^2)/2|dz| \). In order to distinguish minimal and maximal surfaces, we will denote by \( \phi \) the isotropic curve a minimal surface in \( \mathbb{E}^3 \) and by \( \psi \) for a maximal surface in \( \mathbb{L}^3 \). If the context is known, we do not explicit if \( M \) denotes a minimal or a maximal surface.

**Definition 2.1.**

1. Let \( M \) be a minimal conformal surface in \( \mathbb{E}^3 \) and let \( \phi \) be its isotropic curve. The dual surface of \( M \) is the maximal surface \( M^\flat \) of \( \mathbb{L}^3 \) whose isotropic curve is \( \psi = (-i\phi_1, -i\phi_2, \phi_3) \). Equivalently, if the Weierstrass representation of \( M \) is \((g, \omega)\), then the one of \( M^\flat \) is \((g^\flat, \omega^\flat) = (ig, -i\omega)\).

2. Let \( M \) be a maximal conformal surface in \( \mathbb{L}^3 \) and let \( \psi \) be its isotropic curve. The dual surface of \( M \) is the minimal surface \( M^\sharp \) of \( \mathbb{E}^3 \) whose isotropic curve is \( \phi = (i\psi_1, i\psi_2, \psi_3) \). Equivalently, if the Weierstrass representation of \( M \) is \((g, \omega)\), then the one of \( M^\sharp \) is \((g^\sharp, \omega^\sharp) = (-ig, i\omega)\).

**Remark 2.2.** If \((g, \omega)\) is the Weierstrass representation of a minimal (or maximal) surface \( M \), then \((ig, \omega/i)\) is the Weierstrass representation of a minimal (maximal) surface that is nothing \( M \) rotated \( \pi/2 \) about the \( z \)-axis. As the dual surface \( M^\flat \) is \((ig, -i\omega)\), then \((g, \omega)\) is \( M^\flat \) after a \( \pi/2 \)-rotation about the \( z \)-axis. We conclude that up to rotations of the ambient space, the Weierstrass representation of \( M^\flat \) coincides with the one of \( M \) (similarly for the dual surface of a maximal surface). As a consequence the duality process consists simply into to consider the same Weierstrass data with different parametrizations of the surface according to (2).

We immediately find that the third component of the dual surface is preserved up to vertical translations. We study the duality process under some transformations of the ambient space.
Proposition 2.3. (1) If \( T \) is a translation of the ambient space, then \( T(M)^\flat \simeq M^\flat \) and \( T(M)^\sharp \simeq M^\sharp \).

(2) If \( R_\theta \) is the rotation with respect to the axis \((0,0,1)\), then \( R_\theta(M)^\flat \simeq M^\flat \) and \( R_\theta(M)^\sharp \simeq M^\sharp \).

(3) If \( h_\lambda \) is a dilation of ratio \( \lambda > 0 \), then \( h_\lambda(M^\flat) \simeq M^\flat \) and \( h_\lambda(M^\sharp) \simeq M^\sharp \).

(4) If \( M_\theta \) denotes the associate surface of \( M \) corresponding to the parameter \( \theta \in \mathbb{R} \), then \( (M_\theta)^\flat \simeq (M^\flat)_\theta \) and \( (M_\theta)^\sharp \simeq (M^\sharp)_\theta \).

Proof. The three first properties are immediate by observing that if \((g,\omega)\) is the Weierstrass representation of \( M \), then the one of the translation of \( M \) is the same, the one of \( R_\theta(M) \) is \((e^{i\theta}g,e^{-i\theta}\omega)\) and the one of \( h_\lambda(M) \) is \((g,\lambda\omega)\). For the fourth item, recall that the associate surfaces of \( M \) are the minimal (resp. maximal) surfaces \( M_\theta \) whose isotropic curve is \( e^{i\theta}\phi \) (resp. \( e^{i\theta}\psi \)), \( \theta \in \mathbb{R} \). The surface \( M_{\pi/2} \) is called the adjoint of \( M \). Then the proof of item 4 follows now immediately. \( \square \)

As a consequence of this result, the answer to the question posed in Problem 1 is yes when the congruence is a translation, a rotation about the \( z \)-axis and also by a dilation of the ambient space.

Remark 2.4. The duality process can be also introduced by multiplying the coordinates of the isotropic curve by the unit imaginary number \( i \) and/or exchanging the order of the coordinates of the immersion. However all are equivalent up to congruences and associate surfaces. For example, for a minimal surface \( M \), the dual surface in [14] is given by the assignment \((\phi_1,\phi_2,-i\phi_3)\) as isotropic curve. If we calculate its adjoint surface, we multiply by \( i \), obtaining \((i\phi_1,i\phi_2,\phi_3)\) which is the surface \( M^\sharp \) of our definition. In [3], the isotropic curve of the dual surface of \( M \) is \((-i\phi_2,i\phi_1,\phi_3)\), which is a \( \pi/2 \)-rotation about the \( z \)-axis of \( M^\sharp \).

3. The Björling problem and rotational maximal surfaces

In this paper we need to know the isotropic curves that define a rotational minimal or maximal surface. For this purpose we utilize the Björling problem. The Björling problem consists into finding a surface with zero mean curvature containing a given real analytic curve \( \alpha \) called the core curve, and a prescribed unit analytic normal vector field \( V \) along \( \alpha \). In case that the ambient space is \( \mathbb{L}^3 \), the surface that we are looking for is spacelike and \( V \) must be a timelike vector field. The surface that solves the
problem parametrizes as

\[ X(u, v) = \Re \left( \alpha(z) - i \epsilon \int_{z_0}^{z} V(w) \times \alpha'(w) \, dw \right), \quad (3) \]

where \( z_0 \in I \) is fixed, \( z = u + iv \in \Omega \) and \( \times \) stands for the cross-product in each space, see [2, 17]. The solution defined in (3) considers the interval \( I \) as \( I \times \{0\} \subset \mathbb{C} \) and by analyticity, the functions \( \alpha \) and \( V \) have holomorphic extensions \( \alpha(z) \) and \( V(z) \) in a simply-connected domain \( \Omega \subset \mathbb{C} \) that contains \( I \times \{0\} \). The surface obtained, called the Björling surface, is unique under the condition that \( \alpha \) is the parameter curve \( v = 0 \) and its isotropic curve is \( X_z = \alpha'(z) - i \epsilon V(z) \times \alpha'(z) \).

It is usual in the literature to consider the rotational maximal surfaces of \( L^3 \) as surfaces obtained by rotating a planar curve and then imposing that the mean curvature vanishes on the surface. In contrast, we revisit the rotational maximal surfaces of \( L^3 \) as solutions of suitable Björling problems when the core curve is a circle of \( L^3 \). The discussion depends on the causal character of the rotational axis.

1. Timelike axis. Suppose that the axis is \((0,0,1)\). A rotational maximal surface with axis \((0,0,1)\) is the Björling surface when the core curve is the circle \( \alpha(t) = (\cos(t), \sin(t), 0) \) and the normal vector field along \( \alpha \) is \( V(t) = \sinh(a)n(t) + \cosh(a)b(t), \ a \in \mathbb{R}, \) where

\[ n(t) = (-\cos(t), -\sin(t), 0), \quad b(t) = (0,0,1). \]

If \( a = 0 \) the Björling surface is the plane of equation \( z = \text{constant} \). If \( a \neq 0 \), then (3) gives

\[ X(u, v) = \begin{pmatrix} \cos(u)(\cosh(a) \sinh(v) + \cosh(v)) \\ \sin(u)(\cosh(a) \sinh(v) + \cosh(v)) \\ -v \sinh(a) \end{pmatrix}. \quad (4) \]

The one-parametric group of rotations with axis \((0,0,1)\) is

\[ \left\{ R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\}. \]

Then it is immediate that \( R_\theta \cdot X(u, v) = X(u + \theta, v) \), proving that \( X(u, v) \) is rotational. The isotropic curve is

\[ \psi(z) = (-\sin(z) - i \cosh(a) \cos(z), \cos(z) - i \cosh(a) \sin(z), i \sinh(a)) \]
and the Weierstrass representation is
\[ g(z) = -\tanh(a/2)e^{iz}, \quad \omega = -\frac{i(1+e^a)^2}{2e^a}e^{-iz}dz. \]
This surface is called the elliptic catenoid.

(2) Spacelike axis. Suppose that the axis is \((1,0,0)\). The rotational maximal surface
is the Björling surface when the core curve is the spacelike hyperbola \(\alpha(t) = (0,\sinh(t),\cosh(t))\) and \(V(t) = \cosh(a)\mathbf{n}(t) + \sinh(a)\mathbf{b}(t)\), \(a \in \mathbb{R}\), where
\[
\mathbf{n}(t) = (0,\sinh(t),\cosh(t)), \quad \mathbf{b}(t) = (1,0,0).
\]
Expression (3) gives
\[
X(u,v) = \begin{pmatrix}
v \cosh(a) \\
\sinh(u)(\sinh(a)\sin(v) + \cos(v)) \\
\cosh(u)(\sinh(a)\sin(v) + \cos(v))
\end{pmatrix}.
\]
Since the one-parametric group of rotations with axis \((1,0,0)\) is
\[
\left\{ H_\theta = \begin{pmatrix} 1 & 0 & 0 \\
0 & \cosh(\theta) & \sinh(\theta) \\
0 & \sinh(\theta) & \cosh(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\},
\]
we find that \(H_\theta \cdot X(u,v) = X(u+\theta,v)\), proving that \(X(u,v)\) is rotational. This
surface is called the hyperbolic catenoid. The isotropic curve is
\[
\psi(z) = (-i\cosh(a),\cosh(z) - i\sinh(a)\sinh(z),\sinh(z) - i\sinh(a)\cosh(z))
\]
and the Weierstrass representation is
\[
g(z) = \frac{ie^{a+z} + e^a - e^z - i}{e^{a+z} + ie^a + ie^z + 1} \omega = -\frac{(e^{a+z} + ie^a + ie^z + 1)^2}{4e^{a+z}}dz.
\]

(3) Lightlike axis. Suppose that the axis is \((1,0,1)\). A rotational maximal surface
with axis \((1,0,1)\) is the solution of the Björling problem with \(\alpha(t) = (-1 + t^2/2,t,t^2/2)\) and \(V(t) = \sinh(a)e_2(t) + \cosh(a)e_3(t)\), \(a \in \mathbb{R}\), where \(e_2(t) = \mathbf{n}(t) - \mathbf{b}(t)\) and \(e_3(t) = \mathbf{n}(t) + \mathbf{b}(t)\) and
\[
\mathbf{n}(t) = \frac{1}{2}(1,0,1), \quad \mathbf{b}(t) = \frac{1}{2}(t^2 - 1,2t,t^2 + 1).
\]
Then the Björling surface given by (3) is
\[
X(u,v) = \begin{pmatrix}
e^{-a}(\frac{1}{6}v^3 - \frac{1}{2}u^2v) + \cosh(a)v + \frac{u^2}{2} - \frac{u^2}{2} \\
\quad u - e^{-a}uv \\
e^{-a}(\frac{1}{6}v^3 - \frac{1}{2}u^2v) + \sinh(a)v + \frac{u^2}{2} - \frac{u^2}{2}
\end{pmatrix} - \begin{pmatrix} 1 \\
0 \\
0 \end{pmatrix}.
\]
The one-parametric group of rigid motions with axis \((1, 0, 1)\) is

\[
P_\theta = \begin{pmatrix} 1 - \frac{\theta^2}{2} & \theta & \frac{\theta^2}{2} \\ -\theta & 1 & \frac{\theta^2}{2} \\ -\frac{\theta^2}{2} & \theta & 1 + \frac{\theta^2}{2} \end{pmatrix} : \theta \in \mathbb{R}
\]

and it is immediate that \(P_\theta \cdot X(u, v) = X(u + \theta, v)\) for all \(\theta\), proving that \(M\) is rotational. The isotropic curve of the surface \(X\) is

\[\psi(z) = (z + \frac{i}{2}(e^{-a}z^2 - 2 \cosh(a)), 1 + i e^{-a} z, z + \frac{i}{2}(e^{-a} z^2 - 2 \sinh(a)))\]

and the Weierstrass representation is

\[g(z) = \frac{e^a + iz - 1}{e^a + iz + 1}, \quad \omega = -\frac{i(e^a + iz + 1)^2}{2e^a}dz.
\]

This surface is called the parabolic catenoid.

Up to dilations and rigid motions, in each one of the three above cases, and independently of the value of the parameter \(a \in \mathbb{R}\) in the parametrizations (4), (5) and (6), the surface is the unique rotational maximal surface of \(L^3\). For example, and when the axis is timelike, if we multiply \(\omega\) by a real number, the surface changes by a dilation. Thus we suppose \(\omega = ie^{-iz}dz\). With the change \(-\tanh(a/2)e^{iz} \rightarrow z\), the Weierstrass representation is \(g(z) = z\) and \(\omega = -\tanh(a/2)dz/z^2\). After a dilation again, we conclude \(g(z) = z\) and \(\omega = dz/z^2\), where it does not appear the initial parameter \(a\). In the literature and following Kobayashi ([10]), the above three catenoids are also called as catenoid of first kind, catenoid of second kind for the surface and the Enneper surface of second kind respectively.

4. Duality of rotational minimal and maximal surfaces

In this section we address with Problem 2 and we ask if the dual surface of a rotational minimal (maximal) surface is, indeed, rotational. We begin calculating the dual surfaces of rotational maximal surfaces of \(L^3\), where the classification depends on the causal character of the rotational axis. In the next computations, we will suppose that the rotational axis is \((0, 0, 1)\) (timelike), \((1, 0, 0)\) (spacelike) and \((1, 0, 1)\) (lightlike) and we use the isotropic curves and the Weierstrass representations obtained in Sect. 3.

(1) Timelike axis. The Weierstrass representation of the elliptic catenoid \(C\) is \(g(z) = z\) and \(\omega = dz/z^2\) defined in \(M = \mathbb{C} - \{0\}\). Then the Weierstrass
representation of $C^\#$ is $g^\#(z) = -iz$ and $\omega^\# = idz/z^2$. Up to the automorphism $-iz \rightarrow z$, we have $g^\#(z) = z$ and $\omega^\# = dz/z^2$, which is the Weierstrass representation of the Euclidean catenoid of axis $(0,0,1)$: compare with Rem. 2.2.

(2) Spacelike axis. The Weierstrass representation of the hyperbolic catenoid $C$ is

$$g(z) = \frac{e^z - 1}{e^z + 1}, \quad \omega = -i\frac{(e^z + 1)^2}{2e^z}dz,$$

defined in $M = \mathbb{C}$. Then

$$g^\#(z) = \frac{e^z - 1}{e^z + 1}, \quad \omega^\# = \frac{(e^z + 1)^2}{2e^z}dz.$$

(7) An integration of (2) gives the parametrization of $C^\#$:

$$X^\#(u,v) = (u, -\cosh(u)\sin(v), \cos(v)\cosh(u)) - (0,0,1).$$

This surface is the Euclidean catenoid with respect to the axis of equation $y = 0, z = -1$.

(3) Lightlike axis. By taking $a = 0$ in (6), the Weierstrass representation of the parabolic catenoid $C$ is

$$g(z) = \frac{z}{z - 2i}, \quad \omega = \frac{i(z - 2i)^2}{2}dz.$$

Then $C^\#$ is given by

$$g^\#(z) = -\frac{iz}{z - 2i}, \quad \omega^\# = -\frac{(z - 2i)^2}{2}dz$$

and its parametrization by means of (2) is

$$X^\#(u,v) = \left(\frac{6(u - uv) - u^3 + 3uv^2}{6}, \frac{v^2 - u^2 - 2v}{2}, \frac{3(u^2 - v^2 - u^2v) + v^3}{6}\right).$$

We prove that $C^\#$ is the Enneper surface. First, the automorphism $z - 2i \rightarrow z$ and a dilation of $\mathbb{E}^3$ changes the Weierstrass representation (8) into

$$g^\#(z) = -i\frac{z + 2i}{z}, \quad \omega^\# = -z^2dz.$$

The isotropic curve $\psi^\#$ is now

$$\psi^\# = \left(i\frac{2z^2 + 2iz - 1}{2}, \frac{2iz - 1}{2}, z^2 + iz\right).$$
which has not real periods. The metric of $C^♯$ is

$$ds^♯ = \frac{1}{2}|\omega^♯|(1 + |g^♯(z)|^2) = \left(\frac{|z|^2}{2} + \frac{|z + 2i|^2}{2}\right)|dz|.$$

Because $ds^♯ \geq |z|^2/2|dz|$, then it is immediate that $ds^♯$ is complete. As the degree of $g^♯$ is 1, $C^♯$ has total finite curvature equal to $-4\pi$. Since the surface is not the catenoid, it is the Enneper surface by the classification of Osserman ([18]).

We explicit the above calculations.

**Proposition 4.1.** Let $M$ be a catenoid of $L^3$ with axis $(0, 0, 1)$ or $(1, 0, 0)$. Then its dual surface is the Euclidean catenoid with the same rotational axis. The dual surface of the parabolic catenoid of axis $(1, 0, 1)$ is the Enneper surface.

From this result, and reversing the dual process, the dual surfaces of the Euclidean catenoids of axis $(0, 0, 1)$ and $(1, 0, 0)$ are two non-congruent maximal surfaces in contrast to the existence of a deformation by congruences between the two initial Euclidean catenoids. Therefore we have a first example to give a negative answer to Problem 1. On the other hand, it also provides some answers to Problem 2: the dual surfaces of the elliptic and the hyperbolic catenoid are rotational, but the dual surface of the parabolic catenoid is not rotational.

We study the dual surfaces of the Euclidean catenoid under this deformation. We describe explicitly our setting. Consider $C$ the Euclidean catenoid with axis $(0, 0, 1)$ and let $\{G_t : t \in [0, \pi/2]\}$ be the one-parametric group of rotations about $(0, 1, 0)$. Then $G_0 = \text{id}$ and

$$G_t = \begin{pmatrix} \cos(t) & 0 & \sin(t) \\ 0 & 1 & 0 \\ -\sin(t) & 0 & \cos(t) \end{pmatrix}.$$

Thus $G_t$ is a rotation about the orthogonal line to the axis of $C$ passing through the origin that transforms the axis $(0, 0, 1)$ for $t = 0$ into $(1, 0, 0)$ when $t = \pi/2$. We rotate $C$ by means of $G_t$. Along this rotation of $C$ we obtain a one-parametric family of catenoids $\{C_t := G_t(C) : t \in [0, \pi/2]\}$ which, by Prop. 4.1, we know that the dual surfaces of the first and the last catenoid, namely, $C^♭_0$ and $C^♭_{\pi/2}$ are the elliptic catenoid and the hyperbolic catenoid, respectively. In particular, $C_0 \simeq C_{\pi/2}$ but $C^♭_0 \not\simeq C^♭_{\pi/2}$. In this context we ask which are the dual surfaces $C^♭_t$ during this process.
We compute the isotropic curves of $C_t$ and $C_♭_t$ and the corresponding Weierstrass representation. This can be done by solving the Björling problem where the core curve is $\alpha(s) = G_t(\cos(s), \sin(s), 0)$ and $V(s)$ is the unit normal vector of $\alpha$, that is, $V(s) = -\alpha''(s)$. The isotropic curves $\phi$ of $C_t$ and $\psi = \phi_♭$ of $C_♭_t$ are

$$\phi(z) = (-\cos(t) \sin(z) - i \sin(t), \cos(z), \sin(t) \sin(z) - i \cos(t)),$$

$$\psi(z) = (-\sin(t) + i \cos(t) \sin(z), -i \cos(z), \sin(t) \sin(z) - i \cos(t)).$$

We can obtain an explicit parametrization of $C_♭_t$ by integrating (2), obtaining

$$X_♭_t(u, v) = \begin{pmatrix} -u \sin(t) - \cos(t) \sin(u) \sinh(v) \\ \cos(u) \sinh(v) \\ -\sin(t) \cos(u) \cosh(v) + v \cos(t) + \sin(t) \end{pmatrix}.$$  \hfill (9)

A simple computation yields the Weierstrass representation of $C_♭_t$:

$$g_♭(z) = \frac{e^{it} + ie^{it} + e^{iz} - i}{-e^{it} + ie^{it} + e^{iz} - i}, \quad \omega_♭ = \frac{\left(-e^{i(t+z)} - ie^{it} + e^{iz} - i\right)^2}{4e^{i(t+z)}} dz,$$

respectively. With the change $z \to iz$ and after some transformations, we obtain

$$g_♭(z) = i \frac{\cos \frac{t}{2} e^z - \sin \frac{t}{2}}{\sin \frac{t}{2} e^z + \cos \frac{t}{2}}, \quad \omega_♭ = -\left(\frac{\sin \frac{t}{2} e^z + \cos \frac{t}{2}}{ie^z}\right)^2 dz.$$

Up to a multiplication and the division by $i$ in $g_♭$ and $\omega_♭$ (see Rem. 2.2), we have

$$g_♭(z) = \frac{\cos \frac{t}{2} e^z - \sin \frac{t}{2}}{\sin \frac{t}{2} e^z + \cos \frac{t}{2}}, \quad \omega_♭ = \left(\frac{\sin \frac{t}{2} e^z + \cos \frac{t}{2}}{e^z}\right)^2 dz. \hfill (10)$$

The maximal surfaces with this Weierstrass representation are the Bonnet maximal surfaces in $L^3$ obtained by Leite in [13]: compare the Weierstrass data (10) with the analogous surfaces in Euclidean space which will appear in (11) and the parametrization $X_♭_t$ in (9) with the parametrization of a Bonnet minimal surface of $E^3$ described in (12). With the change $e^z \to z$, the isotropic curve of $M_♭$ is

$$\psi = \left(\frac{z^2 + 1}{2z^2}, \frac{i \cos(t)(1 - z^2) - 2iz \sin(t)}{2z^2}, \frac{2z \cos(t) + \sin(t)(1 - z^2)}{2z^2}\right)$$

defined in $M = \mathbb{C} - \{0\}$. We point out that the Weierstrass representation of $M_♭$ has real periods for $\psi_2$, except at $t = 0, \pi/2$, which means that the surface is singly periodic.
Theorem 4.2. Let \( \{C_t : t \in [0, \pi/2]\} \) be the one-parametric family of Euclidean catenoids where the parameter \( t \) indicates the angle that makes the rotation axis with the \( z \)-axis. Then the dual surfaces \( C_t^\flat \) belong to the family of Bonnet surfaces in \( L^3 \) starting from the elliptic catenoid for \( t = 0 \) until the hyperbolic catenoid for \( t = \pi/2 \).

Thus \( C_t \simeq C_s \) but \( C_t^\flat \not\simeq C_s^\flat \) for any \( s, t \in [0, \pi/2] \). From the Lorentzian viewpoint, the axis of \( C_t \) is timelike if \( t \in [0, \pi/4) \), spacelike if \( t \in (\pi/4, \pi/2] \) and lightlike if \( t = \pi/4 \). Then we observe that when the axis \( L \) is \((0, 0, 1)\) and \((1, 0, 0)\), the dual surface of the Euclidean catenoid of axis \( L \) is a catenoid in \( L^3 \) with the same axis. However, when \( t = \pi/4 \), the surface \( C_{\pi/4}^\flat \) is not the parabolic catenoid, but a surface in the Bonnet family of maximal surfaces. In the next pictures, we show the dual surfaces of the catenoids \( C_t \) when they are close to the elliptic catenoid (Figure 1) and close to the hyperbolic catenoid (Figure 2).

**Figure 1.** The surfaces \( C_t^\flat \) for \( t = 0 \) (left), \( t = 0.1 \) (middle) and \( t = 0.3 \) (right)

**Figure 2.** The surfaces \( C_t^\flat \) for \( t = 1.1 \) (left), \( t = 1.3 \) (middle) and \( t = \pi/2 \) (right)
Proposition 2.3 allows to consider the dual surfaces of a helicoid. In both ambient spaces, minimal and maximal ruled surfaces are called helicoids. It was proved in [16] that an associate surface of a catenoid of \( L^3 \) with axis \( L \) is characterized to be a maximal surface invariant by a one-parameter group of helicoidal motions. Since the adjoint surface of the Euclidean catenoid is the helicoid (with the same axis) and the adjoint of the Enneper surface coincides with itself up to a reparametrization and a rotation, Propositions 2.3 and 4.1 conclude:

**Corollary 4.3.** The dual surface of a helicoid of \( L^3 \) with axis \((0,0,1)\) or \((1,0,0)\) is a helicoidal surface of \( E^3 \) with the same axis. The dual surface of the helicoid of axis \((1,0,1)\) is the Enneper surface.

### 5. Deformations of the Lorentzian catenoid and duality

In this section we deform a catenoid \( C \subset L^3 \) by a one-parameter group of rotations of \( L^3 \) and we ask which are their dual surfaces. Now we have three types of catenoid and we can deform under the three types of rotations of \( L^3 \). We separate in cases depending on the type of the catenoid \( C \). On the other hand, by Prop. 2.3, the rotations \( R_t \) about the axis \((0,0,1)\) satisfy \( R_t(M) \# \simeq M\# \), and thus we discard this type of rotations because they do not provide new minimal surfaces. As a conclusion, the rotations that we consider are the hyperbolic rotations \( H_t \) about \((1,0,0)\) and the parabolic rotations \( P_t \) about \((1,0,1)\), see Sect. 3. We also do not use rotations with the same axis of the given catenoid because they do not change the initial catenoid.

#### 5.1. Elliptic catenoid.

Consider an elliptic catenoid \( C \) with rotation axis \((0,0,1)\). Here \( C \) is determined by the Weierstrass representation \( g(z) = e^z \) and \( \omega = e^{-z}dz \) in \( M = \mathbb{C} \) and the isotropic curve is \( \psi(z) = (\cosh(z), -i \sinh(z), 1) \).

(1) Hyperbolic rotations. We consider the family of surfaces \( \{H_t(C)\} : t \in \mathbb{R} \} \).

Then the isotropic curve of \( H_t(C) \) is

\[
H_t(\psi) = (\cosh(z), -i \sinh(z) \cosh(t) + \sinh(t), \cosh(t) - i \sinh(z) \sinh(t)).
\]

The isotropic curve of the dual surface \( H_t(C)\# \) is

\[
H_t(\psi)\# = (i \cosh(z), \sinh(z) \cosh(t) + i \sinh(t), \cosh(t) - i \sinh(z) \sinh(t))
\]

and its Weierstrass representation is

\[
g^\#(z) = -ie^z \cosh \frac{t}{2} + i \sinh \frac{t}{2}, \quad \omega^\# = i e^{-z} \left( \cosh \frac{t}{2} - ie^z \sinh \frac{t}{2} \right)^2 dz.
\]

(11)
We can integrate explicitly the parametrization of the surface by means of $H_2(\psi)$, obtaining

$$Y_t(u,v) = \begin{pmatrix} -\cosh(u) \sin(v), \\
\cosh(t) \cosh(u) \cos(v) - v \sinh(t) - \cosh(t), \\
\sin(v) \sinh(u) \sinh(t) + u \cosh(t) \end{pmatrix}.$$  \(12\)

See Figure 3, left. As it is known by Prop. 4.1, for $t = 0$ we obtain the Euclidean catenoid of axis $(0,0,1)$. For $t \neq 0$, the parametrization $Y_t$ in (12) coincides with the parametrization of the family of minimal surfaces discovered by Bonnet in [4], see [17, §175]. Besides the plane, the catenoid and the Enneper surface, the Bonnet surfaces are the only nonplanar minimal surfaces with planar lines of curvature (see [6] from a different point of view).

**Remark 5.1.** The Weierstrass representation of the Bonnet minimal surfaces in Euclidean space $E^3$ is, up to congruences and dilations, $g(z) = e^z + \lambda$ and $\omega = e^{-z} dz$ defined in $M = \mathbb{C}$, where $\lambda \in (0, \infty)$. We denote this surface by $B(\lambda)$. After the change $e^z \rightarrow z$, it is not difficult to see that if $\lambda \neq 0$, the isotropic curve $\phi$ has real periods in the second coordinate $\phi_2$ which means that the surface is singly periodic. The Weierstrass representation of the Bonnet maximal surfaces coincide with the Euclidean case ([3]).

(2) Parabolic rotations. We consider the family of surfaces $\{P_t(C)^\sharp : t \in \mathbb{R}\}$ obtaining that the isotropic curve is

$$P_t(\psi)^\sharp = \begin{pmatrix} i(t(t - 2i \sinh(z)) - (t^2 - 2) \cosh(z)) / 2 \\
it(1 - \cosh(z)) + \sinh(z) \\
(t^2 - t(t \cosh(z) + 2i \sinh(z)) + 2) / 2 \end{pmatrix}$$

and its Weierstrass representation is

$$g^\sharp(z) = -i \frac{(t + 2i)e^z - t}{te^z + 2i - t}, \quad \omega^\sharp = -i \frac{(te^z + 2i - t)^2}{4e^z} dz.$$  \(13\)

We find an explicit parametrization of the surface by means of (2), yielding

$$Z_t(u,v) = \begin{pmatrix} t^2 \cosh(u) \sin(v) / 2 - t^2 v / 2 + t \cosh(u) \cos(v) - t - \cosh(u) \sin(v) \\
\cosh(u)(t \sin(v) + \cos(v)) - tv - 1 \\
-t^2 \sinh(u) \cos(v) / 2 + t^2 u / 2 + t \sinh(u) \sin(v) + u \end{pmatrix}.$$  \(14\)

See Figure 3, right. We prove that the surface $P_t(C)^\sharp$ is a Bonnet surface after a suitable Goursat transformation. Here we recall a Goursat transformation of a minimal surface. If $\phi$ is the isotropic curve of a minimal surface $M \subset E^3$, ...
a Goursat transformation of $M$ is the minimal surface whose isotropic curve is $A\phi$, where $A$ is an element of the complex rotation group $O(3, \mathbb{C})$ ([7]). In terms of the conformal parameter, a Goursat transformation is characterized by a change of the Gauss map under a Möbius transformation $T \in \text{Aut}(\mathbb{C} \cup \{\infty\})$ that preserves the Hopf differential ([9, p. 206]). In particular, the Weierstrass representation $(g, \omega)$ of $M$ changes into \(\{T(g), \omega/T'(g)\}\) of $T(M)$.

Returning with the surface $Z_t(u,v)$ described in (14), we are able to find a Möbius transformation $T(z) = \left(\frac{az + b}{cz + d}\right)$ such that $T(e^z + \lambda) = g^\sharp(z)$, where $\lambda \in (0, \infty)$ is the Bonnet parameter. For this, and in view of the Gauss map $g^\sharp(z)$, we fix $a = 2 - it$ and $c = t$. Because $\lambda$ is a positive real number, it is not difficult to find that $b$ and $d$ are given by $b = \mu - \frac{1}{2}i(\mu - 2)t$, $d = \frac{1}{2}(\mu - 2)t + 2i$ and $\mu < 0$ is a real parameter. The Bonnet parameter is $\lambda = -\mu/2$. This proves definitely that if $B(\lambda)$ is the Bonnet minimal surface for the above value of $\lambda$, then $T(B(\lambda)) = P_t(C)^\sharp$.

In (13) we do the change $e^z \rightarrow z$ and after a $\pi/2$-rotation about the $z$-axis and reflection about the $xy$-plane, the Weierstrass representation of $P_t(C)^\sharp$ is

$$g^\sharp(z) = \frac{(t + 2i)z - t}{tz + 2i - t}, \quad \omega^\sharp = \frac{(tz + 2i - t)^2}{4z^2}dz.$$  

The isotropic curve is

$$\phi = \left(\begin{array}{c}
-1 - it + 2iz + (1 - it)z^2 \\
-2i + 2t + it^2 - 2it^2z + i(-2 + 2it + t^2)z^2 \\
t^2 - 2it - (2t^2 + 4)z + (t^2 + 2it)z^2 \\
4z^2
\end{array}\right).$$

Thus the surface $P_t(C)^\sharp, t \neq 0$, has real periods along the only non-trivial homological curve of $M$ and this says that $P_t(C)^\sharp$ is singly periodic.

We summarize the result in the next theorem:

**Theorem 5.2.** Consider $C \subset \mathbb{L}^3$ the elliptic catenoid of axis $(0, 0, 1)$. The dual surface of $C$ by the hyperbolic rotation $H_t$ about $(1, 0, 0)$ is a Bonnet minimal surface. The dual surface of $C$ by a parabolic rotation $P_t$ about $(1, 0, 1)$ is a Goursat transformation of a Bonnet minimal surface.
5.2. Hyperbolic catenoid. Let $C$ be the hyperbolic catenoid of axis $(1,0,0)$ whose Weierstrass representation is $g(z) = i e^{z-1} e^{z+1}$ and $\omega = -i \frac{(e^z+1)^2}{2e^z} dz$ defined in $M = \mathbb{C}$. If $\psi$ is the isotropic curve of $C$, then the isotropic curve of $P_t(C)$ is

$$P_t(\psi) = \frac{1}{2} \begin{pmatrix} 2 - t^2 + i t (t \sinh(z) + 2 \cosh(z)) \\ 2 i (\cosh(z) + t(\sinh(z) + i)) \\ (t^2 + 2) \sinh(z) + it^2 + 2t \cosh(z) \end{pmatrix}$$

and the Weierstrass representation is

$$g^a(z) = \frac{(1 + t + i) e^z - 1 - i + it}{(1 + i + it) e^z + 1 + it}, \quad \omega^a = \frac{((1 - i + t)e^z + 1 - i + it)^2}{4ie^z} dz.$$

By integrating $P_t(\psi)$, the parametrization of $P_t(C)$ is

$$W_t(u,v) = \begin{pmatrix} -t^2 u/2 - t \sin(v)(t \sinh(u) + 2 \cosh(u))/2 + u \\ - \sin(v)(t \sinh(u) + \cosh(u)) - tu \\ ((t^2 + 2) \cosh(u) \cos(v) - (v + 1)t^2 + 2t \sinh(u) \cos(v) - 2)/2 \end{pmatrix}.$$ 

We prove that this surface is the Goursat transformation of a Bonnet minimal surface. The computations are similar as in Th. 5.2. In view of $g^a(z)$, we choose

$$T(z) = \frac{az + b}{cz + d}, \quad a = 1 + t + i, \quad c = 1 + i + it.$$
Then let
\[ b = \mu + \frac{(t^2 + \mu)}{t + 1}i, \quad d = \frac{2 + \mu - t^2}{1 + t} + (2 + \mu)i \]
and \( \mu \) is a real parameter. The parameter \( \lambda \) of the Bonnet minimal surface is \( \lambda = -(1 + \mu)/(1 + t) \) and the Weierstrass representation of the minimal surface \( T(\mathcal{B}(\lambda)) \) is \((g^\sharp(z), \omega^\sharp)\). This proves that \( T(\mathcal{B}(\lambda)) = P_t(C)^\sharp \).

With the change \( e^z \to z \), and up to a \( \pi/2 \)-rotation about the \( z \)-axis, dilations and a reflection with respect to the \( xy \)-plane, we have
\[
g^\sharp(z) = \frac{(t + 1 + i)z - 1 - i + it}{(1 + t - i)z + 1 - i + it}, \quad \omega^\sharp = \frac{((1 + i - i - i + it)^2}{4z^2}dz
\]
and
\[
P_t(\psi)^\sharp = \begin{pmatrix}
-i + it + 2tz - (i + it)z^2 \\
(2t + t^2)z + (2 + 2t + t^2)z^2
\end{pmatrix}.
\]

Let us observe that \( P_t(\psi)^\sharp \) is defined in \( \mathbb{C} - \{0\} \) and has real periods in the third coordinate. This means that the surface is singly periodic. As a conclusion:

**Theorem 5.3.** Consider \( C \) the hyperbolic catenoid of axis \((1, 0, 0)\). The dual surface of \( C \) by a parabolic rotation \( P_t \) about \((1, 0, 1)\) is a Goursat transformation of a Bonnet minimal surface.

5.3. **Parabolic catenoid.** We consider the parabolic catenoid \( C \) with axis \((1, 0, 1)\) and whose Weierstrass representation is \( g(z) = z/(z - 2i) \) and \( \omega = i(z - 2i)^2/2dz \) defined in \( M = \mathbb{C} \). The isotropic curve of \( C \) is \( \psi(z) = (iz^2/2 + z - i, 1 + iz, z + iz^2/2) \).

We deform \( C \) by the hyperbolic rotations \( H_t \) about the axis \((1, 0, 0)\) and we compute the dual surfaces of \( H_t(C) \). Then the isotropic curve of \( H_t(C)^\sharp \) is
\[
H_t(\psi)^\sharp = \begin{pmatrix}
1 + iz - z^2/2 \\
i \cosh t + (- \cosh t + i \sinh t)z - \sinh t \cdot z^2/2
\end{pmatrix}.
\]

The Weierstrass representation is
\[
g^\sharp(z) = \frac{(\sinh \frac{t}{2} - i \cosh \frac{t}{2})z - 2i \sinh \frac{t}{2}}{(\cosh \frac{t}{2} - i \sinh \frac{t}{2})z - 2i \cosh \frac{t}{2}}, \quad \omega^\sharp = -\frac{1}{2} \left( (\cosh \frac{t}{2} - i \sinh \frac{t}{2})z - 2i \cosh \frac{t}{2} \right)^2 dz.
\]
The parametrization of $H_t(C^\sharp)$ is

$$X_t(u,v) = \frac{1}{6} \begin{pmatrix} -u(u^2 - 3v^2 + 6v - 6) \\ -u \sinh(t)(u^2 - 3v^2 + 6v) - 3 \cosh(t)(u^2 - v^2 + 2v) \\ \cosh(t)(-3u^2v + 3u^2 + v^3 - 3v^2) - (6uv - 6u) \sinh(t) \end{pmatrix}. $$

By (15), the isotropic curve $H_t(\psi)^2$ has not real periods. We show that $H_t(C)^\sharp$ is a complete surface. Then it is not difficult to see that the induced metric $ds^\sharp$ on $H_t(C)^\sharp$ satisfies

$$ds^\sharp = \frac{1}{2}(|\omega^\sharp| + |\omega^\sharp||g^\sharp|^2) \geq \frac{1}{2}|\omega^\sharp| \geq (a|z|^2 + b|z| + c)|dz|$$

for certain numbers $a,b,c \in \mathbb{R}$, $a > 0$, related with $\cosh(t/2)$ and $\sinh(t/2)$. Then it is immediate that if $\gamma$ is a divergent path on $M = \mathbb{C}$, that is, if $\gamma$ is a path on $\mathbb{C}$ that has $\infty$ as an end point, then the length of $\gamma$ is $\infty$. Therefore $H_t(C)^\sharp$ is a complete surface and because the degree of its Gauss map $g^\sharp$ is 1, then the total curvature is $-4\pi$. The classification of Osserman proves that $H_t(C)^\sharp$ is the Enneper surface ([18]). Since this argument holds for any value of $t$, we conclude that the quotient space of $\{H_t(C)^\sharp : t \in \mathbb{R}\}$ by congruences has only one element.

**Theorem 5.4.** Consider $C$ the parabolic catenoid of axis $(1, 0, 1)$. The dual surface of $C$ by the hyperbolic rotations $H_t$ of axis $(1, 0, 0)$ is the Enneper surface.

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