NUMBER OF COMPONENTS OF
THE NULLCONE

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Abstract. For every pair \((G, V)\) where \(G\) is a connected simple
linear algebraic group and \(V\) is a simple algebraic \(G\)-module with
a free algebra of invariants, the number of irreducible components
of the nullcone of unstable vectors in \(V\) is found.

1. We fix as the base field an algebraically closed field \(k\) of character-
istic zero. Below the standard notation and terminology of the theory
of algebraic groups and invariant theory [23] are used freely.

Consider a finite dimensional vector space \(V\) over the field \(k\) and a
connected semisimple algebraic subgroup \(G\) of the group \(GL(V)\). Let
\(\pi_{G,V}: V \rightarrow V//G\) be the categorical quotient for the action of \(G\) on \(V\),
i.e., \(V//G\) is the irreducible affine algebraic variety with the coordinate
algebra \(k[V]^G\) and the morphism \(\pi_{G,V}\) is determined by the identity
embedding \(k[V]^G \hookrightarrow k[V]\). Denote by \(\mathcal{N}_{G,V}\) the nullcone of the G
module \(V\), i.e., the fiber \(\pi_{G,V}^{-1}(\pi_{G,V}(0))\) of the morphism \(\pi_{G,V}\). A point
of the space \(V\) lies in \(\mathcal{N}_{G,V}\) if and only if its \(G\)-orbit is nilpotent, i.e.,
contains in its closure the zero of the space \(V\) (see [23, 5.1]).

This article owes its origin to the following A. Joseph’s question [8]:
may it happen that the nullcone \(\mathcal{N}_{G,V}\) is reducible if the group \(G\) is sim-
ple, its natural action on \(V\) is irreducible, and the algebra of invariants
\(k[V]^G\) is free?

Pairs \((G, V)\) with a free algebra of invariants \(k[V]^G\) have been studied intensively in the 70s of the last century (see [23], [17] and
the literature cited there). Under the assumptions of simplicity of the group
\(G\) and irreducibility of its action on \(V\) they are completely classified
and constitute a remarkable class which admits a number of other important
caracterizations.

In Theorem 3 proved below we find the number of irreducible compo-

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2. Up to conjugacy in $GL(V)$, the group $G$ is uniquely determined as the image of a representation $\tilde{G} \to GL(V)$ of its universal covering group $\tilde{G}$. The equivalence class on this representation, if it is irreducible, is uniquely determined by its highest weight $\lambda$ (with respect to a fixed maximal torus and a Borel subgroup of the group $\tilde{G}$ containing this torus). With this in mind, we shall write $G = (R, \lambda)$, where $R$ is the type of the root system of the group $G$. Note that $(R, \lambda) = (R, \lambda^*)$, where $\lambda^*$ is the highest weight of the dual representation. We denote by $\varpi_1, \ldots, \varpi_r$ the fundamental weights of the group $\tilde{G}$ numbered as in Bourbaki [4]. If $R = A_r, B_r, C_r, D_r$, then we assume that, respectively, $r \geq 1, 3, 2, 4$.

The following theorem is proved in [10]:

**Theorem 1.** All connected nontrivial simple algebraic subgroups $G$ of the group $GL(V)$ that act on $V$ irreducibly and have a free algebra of invariants $k[V]^G$, are exhausted by the following list:

(i) (adjoint groups):

$$(A_r, \varpi_1 + \varpi_r); (B_r, \varpi_2); (D_r, \varpi_2); (C_r, 2\varpi_1);$$
$$\quad (E_6, \varpi_2), (E_7, \varpi_1); (E_8, \varpi_8); (F_4, \varpi_1); (G_2, \varpi_2)$$

(ii) (isotropy groups of symmetric spaces):

$$(B_r, \varpi_1); (D_r, \varpi_1); (A_3, \varpi_2); (A_1, 2\varpi_1);$$
$$\quad (B_r, 2\varpi_1); (D_r, 2\varpi_1); (A_3, 2\varpi_2); (C_2, 2\varpi_1); (A_1, 4\varpi_1);$$
$$\quad (C_r, \varpi_2); (A_7, \varpi_4); (B_4, \varpi_4); (C_4, \varpi_4); (D_8, \varpi_8); (F_4, \varpi_4);$$

(iii) (groups $G$ with $k[V]^G = k$):

$$(A_r, \varpi_1); (A_r, \varpi_2), r \geq 4 \text{ even}; (C_r, \varpi_1); (D_5, \varpi_5);$$

(iv) (groups $G$ with $\text{tr deg} k[V]^G = 1$ not included in (i) and (ii)):

$$(A_r, 2\varpi_1), r \geq 2; (A_r, \varpi_2), r \geq 5 \text{ odd};$$
$$\quad (A_1, 3\varpi_1); (A_5, \varpi_3); (A_6, \varpi_3); (A_7, \varpi_3);$$
$$\quad (B_3, \varpi_3); (B_5, \varpi_5); (C_3, \varpi_3); (D_6, \varpi_6); (D_7, \varpi_7);$$
$$\quad (G_2, \varpi_1); (E_6, \varpi_1); (E_7, \varpi_7);$$

(v) (other groups):

$$(A_2, 3\varpi_1); (A_8, \varpi_3); (B_6, \varpi_6).$$

**Remark 1.** There are no repeated groups inside each of these five lists (i)–(v). The unique group included in two different lists (namely, in (i) and (ii)) is $(A_1, 2\varpi_1)$. The groups $G$ with $\text{tr deg} k[V]^G = 1$ included in
at least one of the lists (i), (ii) are \((B_r, \varpi_1), (D_r, \varpi_1), (A_3, \varpi_2), (C_2, \varpi_2), (A_1, 2\varpi_1), (B_4, \varpi_4)\) and only these groups.

3. Recall from [23, 3.8, 8.8], [17, Chap. 5, §1, 11], [18] that an algebraic subvariety \(S\) in \(V\) is called a Chevalley section with the Weyl group \(W(S) := N(S)/Z(S)\), where \(N(S) := \{g \in G \mid g \cdot S = S\}\) and \(Z(S) := \{g \in G \mid g \cdot s = s \quad \forall s \in S\}\), if the homomorphism of \(k\)-algebras \(k[V]^G \to k[S]^W(S), f \mapsto f|_S\), is an isomorphism. A linear subvariety in \(V\) that is a Chevalley section with trivial Weyl group (i.e., a linear subvariety intersecting every fiber of the morphism \(\pi_{G,V}\) at a single point) is called a Weierstrass section. A linear subspace in \(V\) that is a Chevalley section with a finite Weyl group is called a Cartan subspace.

Recall also (see [23, Thm. 3.3 and Cor. 4 of Thm. 2.3]) that semisimplicity of the group \(G\) implies the equality

\[
m_{G,V} := \max_{v \in V} \dim G \cdot v = \dim V - \dim V/G. \tag{1}
\]

Consider the following properties:

- (FA) \(k[V]^G\) is a free \(k\)-algebra;
- (FM) \(k[V]\) is a free \(k[V]^G\)-module;
- (ED) all fibers of the morphism \(\pi_{G,V}\) have the same dimension;
- (ED\(_0\)) \(\dim N_{G,V} = m_{G,V}\) (see (1));
- (FO) every fiber of the morphism \(\pi_{G,V}\) contains only finitely many \(G\)-orbits;
- (FO\(_0\)) \(N_{G,V}\) contains only finitely many \(G\)-orbits;
- (NS) \(G\)-stabilizers of points in general position in \(V\) are nontrivial;
- (CS) there is a Cartan subspace in \(V\);
- (WS) there is a Weierstrass section in \(V\).

The following implications between them hold true:

\[
(FM) \iff (FA) \& (ED) \quad \text{(see [17, p. 127, Thm. 1])};
(ED\(_0\)) \iff (ED) \iff (FO\(_0\)) \quad \text{(see [17, p. 128, Thm. 3, Cor.])};
(FO\(_0\)) \iff (FO) \quad \text{(see [23, Cor. 3 of Prop. 5.1])};
(CS) \implies (FM) \iff (WS) \quad \text{(see [17, p. 133, Thm. 7])}.
\]

**Theorem 2.** For the connected simple algebraic subgroups \(G\) in \(GL(V)\), acting on \(V\) irreducibly, all nine properties (FA), (FM), (ED), (ED\(_0\)), (FO), (FO\(_0\)), (NS), (CS), and (WS) are equivalent\(^1\).

\(^1\)In [13, p. 207, Thm.], the property (NS) is replaced by the property that the \(G\)-stabilizer of every point of \(V\) is nontrivial. It is a mistake: for instance, the \(SL_2\)-module of binary forms in \(x\) and \(y\) of degree 3 has the property (FA), but the \(SL_2\)-stabilizer of the form \(x^2y\) is trivial.
Proof. The complete list of the groups $G$ having the property (FA) is obtained in [10]; the one having the property (ED) is obtained in [16], [17, p. 141, Thm. 8] and, in the same papers, that having the property (FM); the one having the property (FO) is obtained in [11]. The results of papers [3], [2], [14], [15] yield the complete list of the groups $G$ having the property (NS). Matching the obtained lists proves the equivalence of the properties (FA), (FM), (ED), (FO), and (NS) (see [23, Thm. 8.8] and [17, p. 127, Thm. 1]). It is proved in [17, p. 142, Thm. 9] that each of the properties (CS) and (WS) is equivalent to the property (ED). □

Remark 2. The conditions of simplicity of the group $G$ and irreducibility of its action on $V$ in Theorem 2 are essential, see [18].

4. Now we turn to finding the number of irreducible components of the nullcone $\mathcal{N}_{G,V}$.

Lemma 1. If $\dim V/\!\!/G \leq 1$, then the nullcone $\mathcal{N}_{G,V}$ is irreducible. If $\dim V/\!\!/G = 0$, then it contains an open dense $G$-orbit.

Proof. The equality $\dim V/\!\!/G = 0$ means that $\dim V/\!\!/G$ is a single point. By the definition of the nullcone, the latter condition is equivalent to the equality $\mathcal{N}_{G,V} = V$. In particular, in this case the nullcone $\mathcal{N}_{G,V}$ is irreducible. On the other hand, in view of (1), the equality $\dim V/\!\!/G = 0$ is equivalent to that $V$ contains a $G$-orbit of dimension $\dim V$, i.e., an open and dense orbit.

In view of smoothness of $V$, the algebraic variety $V/\!\!/G$ is normal (see [23, Thm. 3.16]). Let $\dim V/\!\!/G = 1$. It follows from rationality of the algebraic variety $V$, dominance of the morphism $\pi_{G,V}$, and Lüroth’s theorem that the curve $V/\!\!/G$ is rational. Being normal, it is smooth. Hence $V/\!\!/G$ is isomorphic to an open subset of the affine line. Since every invertible element of the algebra $k[V]$ is a constant, the algebra $k[V]^G$ has the same property. Hence the curve $V/\!\!/G$ is isomorphic to the affine line, and therefore, there is a polynomial $f \in k[V]^G$ such that $f(0) = 0$ and $k[V]^G = k[f]$. Since the group $G$ is connected and has no nontrivial characters, the polynomial $f$ is irreducible (see [23, Thm. 3.17]). Since $\mathcal{N}_{G,V} = \{v \in V \mid f(v) = 0\}$, this implies irreducibility of the nullcone $\mathcal{N}_{G,V}$.

Theorem 3. The nullcone $\mathcal{N}_{G,V}$ of the connected nontrivial simple algebraic group $G \subseteq \text{GL}(V)$ acting irreducibly on $V$ and having the equivalent properties listed in Theorem 2 is reducible if and only if $G$ is contained in the following list:

$$(D_r, 2\varpi_1), (A_3, 2\varpi_2), (A_7, \varpi_4).$$

(2)
For every group $G$ from list (2), the number of irreducible components of the nullcone $N_{G,V}$ is equal to 2.

**Proof.** From Theorem 2 we obtain the following interpretation of the number of irreducible components of the nullcone $N_{G,V}$. Using (1) and the fiber dimension theorem (see [7, Chap. II, §3]), we infer that dimension of every irreducible component of the nullcone $N_{G,V}$ is at least $m_{G,V}$. This and the property $(ED_0)$ imply that dimension of every irreducible component of the nullcone $N_{G,V}$ is equal to $m_{G,V}$. But in view of the property $(FO_0)$ every irreducible component of the nullcone $N_{G,V}$ is the closure of some $G$-orbit. Hence the number of irreducible components of the nullcone $N_{G,V}$ is equal to the number of $m_{G,V}$-dimensional nilpotent $G$-orbits in $V$.

Now we shall use Theorem 1 and find, for every group $G$ listed in it, the number of irreducible components of the nullcone $N_{G,V}$.

1. If the group $G$ is adjoint, then according to [11, Cor. 5.5], the nullcone $N_{G,V}$ is irreducible. This conclusion covers all the groups $G$ from list (i) of Theorem 1.

2. In view of Lemma 1, the nullcone $N_{G,V}$ is irreducible for all the groups $G$ from lists (iii) and (iv) of Theorem 1 and also for the groups with $\text{trdeg}_k k[V]^G = 1$ mentioned in Remark 1.

3. Consider all the groups $G$ from list (v) of Theorem 1.

   (3a) The orbits of the group $(A_2, 3\varpi_1)$ are the orbits of the natural action of the group $\text{SL}_3$ on the space of cubic forms in three variables. According to [23, 5.4, Example 2$^\circ$], the Hilbert–Mumford criterion implies the existence of a linear subspace $L$ in $V$ such that $N_{G,V} = G \cdot L$. Hence the nullcone $N_{G,V}$ is irreducible.

   (3b) The orbits of the group $(A_8, \varpi_3)$ are the orbits of the natural action of the group $\text{SL}_9$ on the space of 3-vectors $\bigwedge^3 k^9$. The classification of them is obtained in [24]; it shows (see [24, Table 6, dim $S = 0$]) that in this case there is a unique nilpotent orbit of dimension $m_{G,V} = 80$. Hence the nullcone $N_{G,V}$ is irreducible.

   (3c) The orbits of the group $(B_6, \varpi_6)$ are the orbits of the natural action of the group $\text{Spin}_{13}$ on the space of spinor representation. The classification of them is obtained in [6]; it shows (see [6, Thm. 1(3)]) that in this case there is a unique nilpotent orbit of dimension $m_{G,V} = 62$, and hence the nullcone $N_{G,V}$ is irreducible.

4. Let us now consider all the groups $G$ from the remaining list (ii) of Theorem 1. By virtue of the Lefschetz principle, we may (and shall) assume that $k = \mathbb{C}$. All these groups are obtained by means of the following general construction.
Consider a semisimple complex Lie algebra \( \mathfrak{h} \), its adjoint group \( \text{Ad} \mathfrak{h} \), and an involution \( \theta \in \text{Aut} \mathfrak{h} \). The decomposition
\[
\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{where } \mathfrak{k} := \{ x \in \mathfrak{h} \mid \theta(x) = x \}, \quad \mathfrak{p} := \{ x \in \mathfrak{h} \mid \theta(x) = -x \}.
\]
is a \( \mathbb{Z}_2 \)-grading of the Lie algebra \( \mathfrak{h} \), and \( \mathfrak{k} \) is its proper reductive subalgebra (see [25]). Let \( K \) be the connected algebraic subgroup of \( \text{Ad} \mathfrak{h} \) with the Lie algebra \( \mathfrak{k} \). The subspace \( \mathfrak{p} \) is invariant with respect to the restriction to \( K \) of the natural action of the group \( \text{Ad} \mathfrak{h} \) on \( \mathfrak{h} \). The action of \( K \) on \( \mathfrak{p} \) arising this way determines a homomorphism \( \iota : K \to \text{GL}(\mathfrak{p}) \).

For every group from list (ii) of Theorem 1, there is a pair \((\mathfrak{h}, \theta)\) such that \( V = \mathfrak{p} \) and \( G = \iota(K) \).

Next, we use the following facts (see [12], [5], [25], [22]).

In \( \mathfrak{h} \), there is a \( \theta \)-stable real form \( \mathfrak{r} \) of the Lie algebra \( \mathfrak{h} \), such that \( \mathfrak{r} = (\mathfrak{r} \cap \mathfrak{k}) \oplus (\mathfrak{r} \cap \mathfrak{p}) \) is its Cartan decomposition (thereby \( \mathfrak{r} \cap \mathfrak{k} \) is a compact real form of the Lie algebra \( \mathfrak{k} \)). The semisimple real Lie algebra \( \mathfrak{r} \) is noncompact and the juxtaposition \( \mathfrak{r} \rightsquigarrow \theta \) determines a bijections between the noncompact real forms of the Lie algebra \( \mathfrak{h} \), considered up to an isomorphism, and the involutions in \( \text{Aut} \mathfrak{h} \), considered up to conjugation. By means of this bijection and described construction, every group \( G \) from list (ii) of Theorem 1 is determined by some noncompact semisimple real Lie algebra \( \mathfrak{s} \); we say that \( G \) and \( \mathfrak{s} \) correspond each other.

The nullcone \( \mathcal{N}_{K,\mathfrak{p}} \) for the action of \( K \) on \( \mathfrak{p} \) contains only finitely many \( K \)-orbits, therefore, every its irreducible component contains an open dense \( K \)-orbit; the latter is called \emph{principal} nilpotent \( K \)-orbit and its dimension is equal to the maximum of dimensions of \( K \)-orbits in \( \mathfrak{p} \).

Let \( \sigma : \mathfrak{h} \to \mathfrak{h}, \ x + iy \mapsto x - iy, \ x, y \in \mathfrak{r} \). Denote by \( \mathcal{N}_r \) the set of nilpotent elements of the Lie algebra \( \mathfrak{r} \). In every nonzero \( K \)-orbit \( \mathcal{O} \subset \mathcal{N}_{K,\mathfrak{p}} \), there is an element \( e \) such that \( \{ e, f := -\sigma(e), h := [e, f] \} \) is an \( \mathfrak{sl}_2 \)-triple (i.e., \( [h, e] = 2e \) and \( [h, f] = -2f \)). Then the element \( (i/2)(e+f-h) \) lies in \( \mathcal{N}_r \), its \( \text{Ad} \mathfrak{r} \)-orbit \( \mathcal{O}' \) does not depend on the choice of \( e \), the equality \( 2 \dim_{C} \mathcal{O} = \dim_{R} \mathcal{O}' \) holds, and the map \( \mathcal{O} \mapsto \mathcal{O}' \) is a bijection between the set of nonzero \( K \)-orbits in \( \mathcal{N}_{K,\mathfrak{p}} \) and the set of nonzero \( \text{Ad} \mathfrak{r} \)-orbits in \( \mathcal{N}_r \).

A nilpotent element of a real semisimple Lie algebra \( \mathfrak{s} \) is called \emph{compact} if the reductive Levi factor of its centralizer in \( \mathfrak{s} \) is a compact Lie algebra, [22]. For all simple real Lie algebras \( \mathfrak{s} \) and their compact elements \( x \), the orbits \( \text{Ad} \mathfrak{s} \cdot x \) are classified (and their dimensions are found) in [22]. If, in the above notation, the elements of an \( \text{Ad} \mathfrak{r} \)-orbit \( \mathcal{O}' \) are compact, then the \( K \)-orbit \( \mathcal{O} \) is called \((−1)\)-\emph{distinguished}, [19]. All principal nilpotent \( K \)-orbits are \((−1)\)-distinguished, [21].
It follows from the aforesaid that the number of irreducible components of the nullcone $N_{K,p}$ is equal to the number of $(-1)$-distinguished $K$-orbits of maximal dimension in $p$, and also to the number of orbits $(\text{Ad} \tau) \cdot x$ of maximal dimension, where $x$ is a compact element in $\tau$.

This reduces the problem to pointing out for every group $G$ from list (ii) of Theorem 1 the simple real Lie algebra $s$ corresponding to it, and then to finding the number of orbits $(\text{Ad} s) \cdot x$, where $x$ is a compact element of $s$, such that their dimension is maximal.

Now we shall perform this for every group from list (ii) of Theorem 1, except those from Remark 1 that have already been considered above.

(4a) Let $G$ be one of the groups $(B_r, 2 \varpi_1), (D_r, 2 \varpi_1), (A_3, 2 \varpi_2), (C_2, 2 \varpi_1), (A_1, 4 \varpi_1)$. Therefore, $\mathfrak{t} = \mathfrak{so}_n$, where, respectively, $n = 2r + 1$ (with $r \geq 3$), $2r$ (with $r \geq 4$), 6, 5, 3. Hence the maximal compact subalgebra in $s$ is $\mathfrak{so}_{n,0}$ (see [25], [5], [22, Table 1]). In this case, $s$ is a real form of the Lie algebra $\mathfrak{sl}_n$ (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and Table 8 in Reference Chapter of [25] that $s = \mathfrak{sl}_n(\mathbb{R})$. According to [22, Thm. 8], the number of orbits $(\text{Ad} s) \cdot x$, where $x$ is a nonzero compact element of $s$, is equal to 1 if $n$ is odd, and to 2 if $n$ is even, and in the case of even $n$ both of these orbits have the same dimension. Therefore, the nullcone $N_{G,V}$ is irreducible for odd $n$ and has exactly two irreducible components for even $n$.

(4b) Let $G = (C_r, \varpi_2)$. Therefore, $\mathfrak{t} = \mathfrak{sp}_{2r}$, so the maximal compact subalgebra in $s$ is $\mathfrak{sp}_{r,0}$ (see [25], [5], [22, Table 1]). In this case, $s$ is a real form of the Lie algebra $\mathfrak{sl}_{2r}$ (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and Table 8 in Reference Chapter of [25] that $s = \mathfrak{sl}_{r}(\mathbb{H})$. According to [22, Thm. 8], the number of orbits $(\text{Ad} s) \cdot x$, where $x$ is a nonzero compact element of $s$, is equal to 1. Therefore, the nullcone $N_{G,V}$ is irreducible.

(4c) Let $G = (A_7, \varpi_4)$. Then $\mathfrak{t} = \mathfrak{su}_8$ (see [25], [5], [22, Table 1]). In this case, $s$ is a real form of the Lie algebra $E_7$ (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and [22, Table 5] that, using E. Cartan’s notation, $s = E_7(7)$. According to [22, Table 12], for this $s$, the number of $(-1)$-distinguished $K$-orbits of maximal dimension ($= 63$) in $N_{K,p}$ is equal to 2. Therefore, the number of irreducible components of the nullcone $N_{G,V}$ is equal to 2 as well.

(4d) Let $G = (C_4, \varpi_4)$. Therefore, $\mathfrak{t} = \mathfrak{sp}_{8}$, and hence the maximal compact subalgebra in $s$ is $\mathfrak{sp}_{4,0}$ (see [25], [5], [22, Table 1]). In this case, $s$ is a real form of the Lie algebra $E_6$ (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from
this and [22, Table 5] that $s = E_{6(6)}$. According to [22, Table 7], for this $s$, there is a unique $(-1)$-distinguished $K$-orbit of maximal dimension ($= 36$) in $N_{K,p}$. Therefore, the nullcone $N_{G,V}$ is irreducible.

(4e) Let $G = (D_8, \varpi_8)$. Therefore, $\mathfrak{k} = \mathfrak{so}_{16}$, so the maximal compact subalgebra in $\mathfrak{s}$ is $\mathfrak{so}_{16,0}$ (see [25], [5], [22, Table 1]). In this case, $\mathfrak{s}$ is a real form of the Lie algebra $E_8$ (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and [22, Table 5] that $s = E_{8(8)}$. According to [22, Table 14], for this $s$, there is a unique $(-1)$-distinguished $K$-orbit of maximal dimension ($= 129$) in $N_{K,p}$. Hence the nullcone $N_{G,V}$ is irreducible.

(4f) Let $G = (F_4, \varpi_4)$. Therefore, $\mathfrak{k} = \mathfrak{f}_4$, so the maximal compact subalgebra in $\mathfrak{s}$ is $\mathfrak{f}_4(-52)$ (see [22, Sect. 5]). In this case, $\mathfrak{s}$ is a real form of the Lie algebra $E_8$ (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and [22, Table 5] that $s = E_{6(-26)}$. According to [22, Table 9], for this $s$, there is a unique $(-1)$-distinguished $K$-orbit of maximal dimension ($= 24$) in $N_{K,p}$. Hence in this case the nullcone $N_{G,V}$ is irreducible.  

□

Remark 3. In [20] is obtained an algorithm that employs only elementary geometric operations (the orthogonal projection of a finite system of points onto a linear subspace and taking its convex hull) and, starting from the system of weights of the $G$-module $V$ and the system of roots of the group $G$, finds a finite set of linear subspaces $L$ in $V$ such that the irreducible components of maximal dimension of the nullcone $N_{G,V}$ are the varieties $G \cdot L$. In particular, if the property (ED$_0$) holds (see above the list of properties after formula (1)), this algorithm describes all the irreducible components of the nullcone $N_{G,V}$. For instance, this is so for every pair $(G, V)$ from Theorem 1. The computer implementation of this algorithm is obtained in [1].

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