RESOLUTIONS IN FACTORIZATION CATEGORIES

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Abstract. Building upon ideas of Eisenbud, Buchweitz, Positselski, and others, we introduce the notion of a factorization category. We then develop some essential tools for working with factorization categories, including constructions of resolutions of factorizations from resolutions of their components and derived functors. Using these resolutions, we lift fully-faithfulness and equivalence statements from derived categories of Abelian categories to derived categories of factorizations. Some immediate geometric consequences include a realization of the derived category of a projective hypersurface as matrix factorizations over a noncommutative algebra and recover of a theorem of Baranovsky and Pecharich.

1. Introduction

Since their introduction by D. Eisenbud [Eis80], matrix factorizations have spread from commutative algebra into a wide range of fields. In theoretical physics, M. Kontsevich realized that matrix factorizations represent boundary conditions in Landau-Ginzburg models. In topology, matrix factorizations have been used to create knot and link invariants [KR08a, KR08b]. In algebraic geometry, deep statements tying the geometry of projective hypersurfaces to matrix factorizations of their defining polynomial have been proven by D. Orlov [Orl09]. In addition, through mirror symmetry, matrix factorizations allow access to the structure of Fukaya categories of symplectic manifolds, [Sei11, Efi12, AAEKO13, She11].

The original concept of matrix factorizations can be generalized in various ways, e.g., to the stable module category [Buc86], the category of singularities [Orl04], or, in another direction towards more general spaces [Pos09, EFT14, Orl12].

Much of the task of this paper is to repackage Positselski’s ideas towards a general theory of matrix factorizations for any Abelian category, in particular to derive functors of factorizations as one would functors of Abelian categories. To this end, we introduce the notion of a factorization category for a triple \((\mathcal{A}, \Phi, w)\) where \(\mathcal{A}\) is an Abelian category, \(\Phi : \mathcal{A} \to \mathcal{A}\) is an autoequivalence, and \(w : \text{Id} \to \Phi\) is a natural transformation. By appropriately altering \(\mathcal{A}\) and setting \(w = 0\), one fully recovers the usual construction of the derived category \(\mathcal{D}^b(\mathcal{A})\).

As factorization categories can rightly be viewed as a deformation of \(\Phi\)-twisted, two-periodic chain complexes over \(\mathcal{A}\), one should be able to build resolutions in a straightforward manner from resolutions of the components of a factorization. A key development of this paper is to provide a construction of such resolutions, see Theorems 3.8 and 3.11.

Now consider two triples, as above, \((\mathcal{A}, \Phi, w)\) and \((\mathcal{B}, \Psi, v)\), and an additive functor, \(\theta : \mathcal{A} \to \mathcal{B}\), such that

\[\theta \circ \Phi \cong \Psi \circ \theta\]

and

\[\theta(w_A) = v_{\theta(A)} : \theta(A) \to \theta(\Phi(A)) \cong \Psi(\theta(A)).\]

for all objects, \(A \in \mathcal{A}\). Furthermore, assume that \(\theta\) is left-exact, that \(\mathcal{A}\) has small coproducts and enough injectives, and that coproducts of injectives are injective. We can then use

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these resolutions to prove that if the right derived functor of $\theta$ is fully-faithful then so is a “right derived functor” associated to $\theta$ between the derived categories of factorizations. Moreover, if the right derived functor of $\theta$ is an equivalence induced by Abelian natural transformations, then so is a “right derived functor” associated to $\theta$ between the derived categories of factorizations. We can also use these resolutions to construct a spectral sequence computing the morphism spaces in the derived categories of factorizations whose $E_1$-page consists of Ext-groups between the components of this factorization in the underlying Abelian category.

From these results, we are able to lay much of the groundwork for handling these categories with the bulk of the machinery developed for derived categories. Moreover, one can deduce many results about factorization categories from results about the usual derived categories. Indeed, as a special application to geometry, we provide a derived equivalence between any smooth projective hypersurface and matrix factorizations of a noncommutative algebra. In addition, we recover the main result of [BP10]. For more results on standard algebro-geometric functors in the setting of factorization categories, see [Efi13 Appendix B].

Our work is also foundational to understanding derived categories of gauged Landau-Ginzburg models in algebraic geometry, such as in recent works on variations of GIT quotients and on Homological Projective Duality [Seg11 BFK12 BDFIK13]. Indeed, bootstrapping properties of functors from derived categories to factorization categories already appeared in [Seg11 BP10].

The reader is encouraged to compare the methods and results on derived functors for factorization categories in this article to the approach taken in [Efi13 Appendix A], see Remark 4.4.

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2. Basics

Let $\mathcal{A}$ be an Abelian category,

$$\Phi : \mathcal{A} \to \mathcal{A}$$

be an autoequivalence of $\mathcal{A}$, and

$$w : \text{Id}_\mathcal{A} \to \Phi$$

be a natural transformation from the identity functor to $\Phi$. We assume that

$$w_{\Phi(A)} = \Phi(w_A)$$

for all $A \in \mathcal{A}$.

Example 2.1. Let $X$ be a smooth algebraic variety and $G$ be an algebraic group acting on $X$. Consider a $G$-equivariant line bundle $\mathcal{L}$ on $X$. Set $\mathcal{A}$ to be the category of $G$-equivariant coherent sheaves on $X$, $\Phi$ to be tensoring with $\mathcal{L}$, and $w$ to be a section of $\mathcal{L}$. 


Remark 2.2. The above example is intended to be the category of $B$-branes on a gauged Landau-Ginzburg model. It is the one considered in [BFK11, BFK13, BFK12, BDFIK13].

Definition 2.3. A factorization of the triple, $(A, \Phi, w)$, consists of a pair of objects of $A$, $E^{-1}$ and $E^0$, and a pair of morphisms,

$$
\phi_E^{-1}: \Phi^{-1}(E^0) \to E^{-1} \\
\phi_E^0: E^{-1} \to E^0
$$

such that

$$
\phi_E^0 \circ \phi_E^{-1} = \Phi^{-1}(w_{E^0}) : \Phi^{-1}(E^0) \to E^0,
$$

$$
\Phi(\phi_E^{-1}) \circ \phi_E^0 = w_{E^{-1}} : E^{-1} \to \Phi(E^{-1}).
$$

We shall often simply denote the factorization, $(E^{-1}, E^0, \phi_E^{-1}, \phi_E^0)$, by $E$. The objects, $E^0$ and $E^{-1}$, are called the components of the factorization. We also set

$$
E^i := \begin{cases} 
\Phi^j(E^0) & \text{if } i = 2j \\
\Phi^j(E^{-1}) & \text{if } i = 2j - 1.
\end{cases}
$$

If $E$ is an object for which $w_E = 0$, then we shall denote the factorization $(0, E, 0, 0)$ simply by $E$.

A morphism of factorizations, $g : E \to F$, is a pair of morphisms in $A$,

$$
g^{-1}: E^{-1} \to F^{-1} \\
g^0: E^0 \to F^0,
$$

making the diagram,

$$
\begin{array}{ccc}
\Phi^{-1}(E^0) & \xrightarrow{\phi_E^{-1}} & E^{-1} & \xrightarrow{\phi_E^0} & E^0 \\
\Phi^{-1}(F^0) & \xrightarrow{\phi_F^{-1}} & F^{-1} & \xrightarrow{\phi_F^0} & F^0 \\
\end{array}
$$

commute.

We let $\text{Fact}(w)$ be the category of factorizations. If $\mathcal{E}$ is a full additive subcategory of $\mathcal{A}$ preserved by $\Phi$, we let $\text{Fact}(\mathcal{E}, w)$ be the full subcategory of $\text{Fact}(w)$ consisting of factorizations whose components lie in $\mathcal{E}$. The most common additive subcategories we will take are injective objects, where we will use the notation $\text{Fact}(\mathcal{I}nj w)$, and projective objects, where we will use the notation $\text{Fact}(\mathcal{P}roj w)$.

Lemma 2.4. The category, $\text{Fact}(w)$, is an Abelian category with component-wise kernels and cokernels of morphisms of factorizations.

Proof. We suppress the details.

There is a natural notion of translation, or shift, of a factorization.
Fact 2.10. Dg-categories come with natural notions of shifts and cones \cite[Definitions 3.5 and 3.7]{BLL04} given by universal properties. The shift and cone construction detailed above are shifts and cones in this universal sense. Hence, \Fact is a strongly pretriangulated dg-category, see \cite[Section 3]{BLL04}. As the homotopy category of a pretriangulated dg-category is naturally triangulated, Proposition 2.9 can be viewed as a corollary of the (strong) pretriangulation of \Fact.
Definition 2.11. Let
\[
\cdots \xrightarrow{g_s} E_s \xrightarrow{g_{s+1}} E_{s+1} \xrightarrow{g_{s+2}} \cdots \xrightarrow{g_t} E_t \xrightarrow{g_{t+1}} E_{t+1} \xrightarrow{g_{t+2}} \cdots
\]
be a complex of factorizations, i.e. a sequence of morphisms in \(\text{Fact}(w)\) satisfying
\[g_{i+1} \circ g_i = 0\]
for all \(i \in \mathbb{Z}\). We have two ways to totalize this complex of factorizations into new individual factorization.

The \(\bigoplus\)-totalization of Equation (2.1) is the factorization \(\text{tot}^{\bigoplus}(E_\bullet) = T\) whose components are given by the formula
\[
T = \bigoplus_{i \in \mathbb{Z}} E_i[-i]
\]
precisely
\[
T^{-1} = \bigoplus_{2i} \Phi^{-1}(E_{2i}^{-1}) \oplus \bigoplus_{2i+1} \Phi^{-1}(E_{2i+1}^{-1})
\]
\[
T^0 = \bigoplus_{2i} \Phi^{-1}(E_{2i}^{0}) \oplus \bigoplus_{2i+1} \Phi^{-1}(E_{2i+1}^{-1}).
\]

The morphisms \(\phi_T^{-1}, \phi_T^0\) defining \(T\) are determined uniquely by the conditions
\[
\phi_T^{-1}|_{\Phi^{-1}(E_{2i}^{-1})} = \Phi^{-1}(E_{2i}^{-1}) \oplus \Phi^{-1}(g_{2i+1}) : \Phi^{-1}(E_{2i}^{-1}) \to T^{-1}
\]
\[
\phi_T^{-1}|_{\Phi^{-1}(E_{2i+1}^{-1})} = -\Phi^{-1}(E_{2i+1}^{-1}) \oplus \Phi^{-1}(g_{2i+2}) : \Phi^{-1}(E_{2i+1}^{-1}) \to T^{-1}
\]
\[
\phi_T^0|_{\Phi^{-1}(E_{2i}^{0})} = -\Phi^{-1}(E_{2i}^{0}) \oplus \Phi^{-1}(E_{2i+1}^{-1}) : \Phi^{-1}(E_{2i}^{0}) \to T^0
\]
\[
\phi_T^0|_{\Phi^{-1}(E_{2i+1}^{0})} = -\Phi^{-1}(E_{2i+1}^{0}) \oplus \Phi^{-1}(E_{2i+2}^{-1}) : \Phi^{-1}(E_{2i+1}^{0}) \to T^0.
\]

The \(\prod\)-totalization of Equation (2.1) is the factorization \(\text{tot}^{\prod}(E_\bullet) = T\) whose components are given by the formula
\[
T = \prod_{i \in \mathbb{Z}} E_i[-i]
\]
\[
T^{-1} = \prod_{2i} \Phi^{-1}(E_{2i}^{-1}) \times \prod_{2i+1} \Phi^{-1}(E_{2i+1}^{-1})
\]
\[
T^0 = \prod_{2i} \Phi^{-1}(E_{2i}^{0}) \times \prod_{2i+1} \Phi^{-1}(E_{2i+1}^{-1}).
\]

The morphisms \(\phi_T^{-1}, \phi_T^0\) defining \(T\) are determined uniquely by the conditions that
\[
\pi_{2i}^{-1} \circ \phi_T^{-1} = (\Phi^{-1}(E_{2i}^{-1}) + \Phi^{-1}(g_{2i+1})) \circ (\Phi^{-1}(E_{2i}^{-1}) + \Phi^{-1}(g_{2i+1})) : \Phi^{-1}(T^0) \to \Phi^{-1}(E_{2i}^{-1})
\]
\[
\pi_{2i}^{-1} \circ \phi_T^0 = (\Phi^{-1}(E_{2i}^{0}) + \Phi^{-1}(g_{2i+1})) \circ (\Phi^{-1}(E_{2i}^{0}) + \Phi^{-1}(g_{2i+1})) : \Phi^{-1}(T^0) \to \Phi^{-1}(E_{2i}^{0})
\]
\[
\pi_{2i+1}^{-1} \circ \phi_T^{-1} = (\Phi^{-1}(E_{2i+1}^{-1}) + \Phi^{-1}(g_{2i+2})) \circ (\Phi^{-1}(E_{2i+1}^{-1}) + \Phi^{-1}(g_{2i+2})) : \Phi^{-1}(T^0) \to \Phi^{-1}(E_{2i+1}^{-1})
\]
\[
\pi_{2i+1}^{-1} \circ \phi_T^0 = (\Phi^{-1}(E_{2i+1}^{0}) + \Phi^{-1}(g_{2i+2})) \circ (\Phi^{-1}(E_{2i+1}^{0}) + \Phi^{-1}(g_{2i+2})) : \Phi^{-1}(T^0) \to \Phi^{-1}(E_{2i+1}^{0})
\]
where \(\pi_k^l\) denotes the projection onto the \(k\)-th component of \(T^l\).
If the complex from Equation (2.1) is bounded, then the $\bigoplus$-totalization and the $\prod$-totalization coincide. In this case, we call the result simply the totalization and denote it by $\text{tot}(E_\bullet)$.

Note that the two forms of totalization extend naturally to provide exact functors $\text{tot} \bigoplus, \text{tot} \prod : \text{Ch}(\text{Fact}(w)) \to \text{Fact}(w)$.

These definitions are due to Positselski, see [Pos09, EF14].

**Definition 2.12.** Let $E$ be a full additive subcategory of $A$ preserved by $\Phi$. A factorization is called $E$-totally acyclic if it lies in the smallest thick subcategory of $K(\text{Fact}(E, w))$ containing the totalizations of all bounded exact complexes from $\text{Fact}(E, w)$. We let $\text{Acycl}(E, w)$ denote the smallest thick subcategory of $K(\text{Fact}(E, w))$ consisting of $E$-totally acyclic factorizations. The absolute derived category of $E$-factorizations of $(E, A, \Phi, w)$ is the Verdier quotient,

$$D^\text{abs}(\text{Fact}(E, w)) := K(\text{Fact}(E, w))/\text{Acycl}(E, w).$$

A morphism in $\text{Fact}(E, w)$ which becomes an isomorphism in $D^\text{abs}(\text{Fact}(E, w))$ will be called a quasi-isomorphism, in analogy with the usual derived category. Similarly, two factorizations which are isomorphic in $D^\text{abs}(\text{Fact}(E, w))$ are called quasi-isomorphic. In the case where $E = A$, we will often omit $A$ from the notation.

**Definition 2.13.** Let $E$ be a full additive subcategory of $A$ preserved by $\Phi$. Assume that small coproducts exist in $E$ and that coproducts are exact. A factorization is called $E$-co-acyclic if it lies in the smallest thick subcategory of $K(\text{Fact}(E, w))$ containing the totalizations of all bounded exact complexes from $\text{Fact}(E, w)$ and closed under taking small coproducts. We let $\text{Co-acycl}(E, w)$ denote the thick subcategory of $K(\text{Fact}(E, w))$ consisting of $E$-co-acyclic factorizations. The co-derived category of $E$-factorizations of $(E, A, \Phi, w)$ is the Verdier quotient,

$$D^\text{co}(\text{Fact}(E, w)) := K(\text{Fact}(E, w))/\text{Co-acycl}(E, w).$$

A morphism in $\text{Fact}(E, w)$ which becomes an isomorphism in $D^\text{co}(\text{Fact}(E, w))$ will be called a co-quasi-isomorphism. Similarly, two factorizations which are isomorphic in $D^\text{co}(\text{Fact}(E, w))$ are called co-quasi-isomorphic. In the case where $E = A$, we will often omit $A$ from the notation.

**Definition 2.14.** Let $E$ be a full additive subcategory of $A$ preserved by $\Phi$. Assume that small products exist in $E$ and are exact. A factorization is called $E$ contra-acyclic if it lies in the smallest thick subcategory of $K(\text{Fact}(E, w))$ containing the totalizations of all bounded exact complexes from $\text{Fact}(E, w)$ and closed under taking small products. We let $\text{Ctr-acycl}(E, w)$ denote the thick subcategory of $K(\text{Fact}(E, w))$ consisting of acyclic factorizations. The contra-derived category of factorizations of $(E, A, \Phi, w)$ is the Verdier quotient,

$$D^\text{ctr}(\text{Fact}(E, w)) := K(\text{Fact}(E, w))/\text{Ctr-acycl}(E, w).$$

A morphism in $\text{Fact}(E, w)$ which becomes an isomorphism in $D^\text{ctr}(\text{Fact}(E, w))$ will be called a contra-quasi-isomorphism. Similarly, two factorizations which are isomorphic in $D^\text{ctr}(\text{Fact}(E, w))$ are called contra-quasi-isomorphic. In the case where $E = A$, we will often omit $A$ from the notation.

**Example 2.15.** Let $A$ be an Abelian category. Let $A^b$ be the category consisting of countably many objects $a_i \in A$ indexed by $\mathbb{Z}$ such that $a_i = 0$ for all but finitely many $i$. Let
Φ : \( A^b \rightarrow A^b \) be the autoequivalence which shifts the indexing i.e. \( \Phi(a)_i = a_{i-1} \). Let \( w = 0 \). Then \( \text{Fact}(0) \) is equal to \( \text{Ch}^b(\mathcal{A}) \), the category of bounded complexes in \( \mathcal{A} \).

Furthermore, \( \text{Acycl}(0) \) is nothing more than bounded acyclic complexes. Hence the usual bounded derived category of \( \mathcal{A} \) is nothing more than \( D^{\text{abs}}(\text{Fact} \ 0) \cong D^b(\mathcal{A}) \).

**Remark 2.16.** Let us attempt to provide some motivation for such definitions. Let us consider the derived category, \( D(\mathcal{A}) \). It is the localization of \( K(\mathcal{A}) \) at the class of quasi-isomorphisms. It can also be viewed as the Verdier quotient of \( K(\mathcal{A}) \) by acyclic complexes.

How does one make an acyclic complex? One way is to take an exact sequence of complexes, \( 0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_n \rightarrow 0 \), and totalize the complex to get an object of \( \text{Ch}(\mathcal{A}) \). This method of construction is fairly robust. Indeed, any finite acyclic complex is easily seen to be the totalization of an exact sequence of chain complexes. These are exactly the analogs of totally-acyclic factorizations. Thus, quotienting by totally-acyclic factorizations should be viewed as the analog of quotienting \( K(\mathcal{A}) \) by the thick subcategory of finite acyclic complexes.

To deal with unbounded complexes, we have to take some form of limit of totalizations of bounded exact complexes. Choice of direction of this limit naturally forces one to study infinite products or coproducts of bounded exact complexes. This connection motivates the definitions of co-acyclic and contra-acyclic complexes.

**Lemma 2.17.** Let

\[
\cdots \xrightarrow{g_s} E_s \xrightarrow{g_{s+1}} E_{s+1} \xrightarrow{g_{s+2}} \cdots \xrightarrow{g_t} E_t \xrightarrow{g_{t+1}} E_{t+1} \xrightarrow{g_{t+2}} \cdots
\]

be an exact complex over \( \text{Fact}(w) \). If \( \mathcal{A} \) possesses small coproducts and the complex \( E_* \) is bounded below, then \( \text{tot}^\otimes(E_*) \) is co-acyclic. If \( \mathcal{A} \) possesses small products and the complex \( E_* \) is bounded above, then \( \text{tot}^\prod(E_*) \) is contra-acyclic.

**Proof.** Assume that \( \mathcal{A} \) possesses small coproducts and that \( E_* \) bounded below. Note that shifting the \( E_* \) and applying any of the totalizations yields a shift of totalization. So we may assume that \( E_s = 0 \) for \( s < 0 \). Let \( C_s \) be the cokernel of \( g_s \) so that we have a bounded exact sequence

\[
0 \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{s-1} \rightarrow E_s \rightarrow C_s \rightarrow 0.
\]

Denote this complex by \( \tau_{\leq s} E_* \). Note that there is a natural chain map \( h_{s+1} : \tau_{\leq s} E_* \rightarrow \tau_{\leq s+1} E_* \).

One can check that \( E_* \) is isomorphic to the cokernel of the monomorphism

\[
\bigoplus_{s \geq 1} \tau_{\leq s} E_* \rightarrow \bigoplus_{s \geq 1} \tau_{\leq s} E_*
\]

determined by

\[
\tau_{\leq s} E_* \xrightarrow{\text{id}} \tau_{\leq s} E_* \xrightarrow{h_{s+1}} \tau_{\leq s} E_* \oplus \tau_{\leq s+1} E_*.
\]
Applying $\text{tot} \oplus$ yields an exact sequence of factorizations

$$0 \to \bigoplus_{s \geq 1} \text{tot}(\tau \leq s E \star) \to \bigoplus_{s \geq 1} \text{tot}(\tau \leq s E \star) \to \text{tot}(E \star) \to 0$$

showing that $\text{tot} \oplus (E \star)$ is co-acyclic. The proof of the other statement is completely analogous and therefore suppressed. $\square$

**Lemma 2.18.** The categories, $D^{\text{abs}}(\text{Fact} \mathcal{E}, w)$, $D^{\text{co}}(\text{Fact} \mathcal{E}, w)$, $D^{\text{ctr}}(\text{Fact} \mathcal{E}, w)$, with the shift and triangles inherited from $K(\text{Fact} \mathcal{E}, w)$, are triangulated categories.

**Proof.** Each of these categories is a Verdier quotient of a triangulated category by a thick triangulated subcategory hence triangulated $[\text{Ve77}, \S 3]$. $\square$

Next we demonstrate that, under some conditions, many of these categories coincide.

**Proposition 2.19.** Assume that $\mathcal{A}$ has small coproducts. Let $\mathcal{E}$ be an additive full subcategory of $\mathcal{A}$ satisfying the following conditions:

- $\mathcal{E}$ is closed under products.
- For any object $A \in \mathcal{A}$, there exists a monomorphism $A \to E$ with $E$ an object of $\mathcal{E}$.

Then,

$$K(\text{Fact} \mathcal{E}, w) \to K(\text{Fact} w) \to D^{\text{co}}(\text{Fact} w)$$

induces an essentially surjective functor

$$Q_{\mathcal{E}} : D^{\text{co}}(\text{Fact} \mathcal{E}, w) \to D^{\text{co}}(\text{Fact} w).$$

**Proof.** We first prove the following statement: for any morphism $h : F \to G$ in $K(\text{Fact} w)$, there exists a commutative diagram

$$\begin{array}{ccc}
F & \xrightarrow{h} & G \\
\downarrow & & \downarrow \\
E & \longrightarrow & E'
\end{array}$$

with vertical arrows being co-quasi-isomorphisms and $E$ and $E'$ being factorizations whose components lie in $\mathcal{E}$.

By assumption, we may choose objects of $\mathcal{E}$, $E^{-1}$ and $E^0$, and monomorphisms

$$F^{-1} f^{-1} \to E^{-1}$$

$$F^0 f^0 \to E^0.$$

Form the factorization, $G^-(E)$,

$$\Phi^{-1}(E^0) \oplus E^{-1} \xrightarrow{\left( \begin{array}{cc} 0 & 1_{E^{-1}} \\ w_{\phi^{-1}(E^0)} & 0 \end{array} \right)} E^{-1} \oplus E^0 \xrightarrow{\left( \begin{array}{cc} 0 & w_{E^{-1}} \\ 1_{E^0} & 0 \end{array} \right)} E^0 \oplus \Phi(E^{-1}).$$
The maps

\[
F^{-1} f^{-1} \oplus f_0 \circ \phi^{-1} F \rightarrow E^{-1} \oplus E^0 \\
F^0 f_0 \oplus \Phi(f^{-1}) \circ \phi_0 \oplus \Phi(E^{-1})
\]

give a monomorphism \( F \rightarrow G^{-}(E) \) in \( \text{Fact}(w) \). Thus, for any factorization \( F \), there exists a factorization with \( E \)-components which receives a monomorphism from \( F \).

Now, we can use the proceeding construction to get the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{h} & G \\
\downarrow & & \downarrow \\
E_0 & & 
\end{array}
\]

with \( E \) having components in \( \mathcal{E} \). Taking the pushout of this diagram,

\[
\begin{array}{ccc}
F & \xrightarrow{h} & G \\
\downarrow & & \downarrow \\
E_0 & \longrightarrow & \text{cok}(F \rightarrow G \oplus E_0)
\end{array}
\]

and applying the \( G^{-} \) construction to the pushout, we get a commutative diagram

\[
\begin{array}{ccc}
F & \xrightarrow{h} & G \\
\downarrow & & \downarrow \\
E_0 & \longrightarrow & E_0'
\end{array}
\]

with both \( E_0 \) and \( E_0' \) having components in \( \mathcal{E} \).

Taking cokernels and iterating the previous result, we can construct a commutative diagram
where the columns are exact sequences of factorizations and $E_s$ and $E'_s$ have components in $\mathcal{E}$ for all $s$. Taking $\bigoplus$-totalizations, we get the desired commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{h} & G \\
\downarrow & & \downarrow \\
E_0 & \longrightarrow & E'_0 \\
\downarrow & & \downarrow \\
E_1 & \longrightarrow & E'_1 \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
E_s & \longrightarrow & E'_s \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

by setting $E = \text{tot} \bigoplus(E_s)$ and $E' = \text{tot} \bigoplus(E'_s)$.

Restricting the previous construction to a single column immediately gives essential surjectivity of the natural functor

$$
Q_{\mathcal{E}} : D^{\text{co}}(\text{Fact } \mathcal{E}, w) \rightarrow D^{\text{co}}(\text{Fact } w).
$$

There is also the dual statement which we record separately.

**Proposition 2.20.** Assume that $\mathcal{A}$ has small products. Let $\mathcal{E}$ be an additive full subcategory of $\mathcal{A}$ preserved by $\Phi$ and satisfying the following conditions:

- $\mathcal{E}$ is closed under products.
- For any object $A \in \mathcal{A}$, there exists a epimorphism $E \to A$

with $E$ an object of $\mathcal{E}$.

Then, the composition

$$
K(\text{Fact } \mathcal{E}, w) \rightarrow K(\text{Fact } \mathcal{A}, w) \rightarrow D^{\text{ctr}}(\text{Fact } w)
$$

induces an essentially surjective functor

$$
Q_{\mathcal{E}} : D^{\text{ctr}}(\text{Fact } \mathcal{E}, w) \rightarrow D^{\text{ctr}}(\text{Fact } w).
$$
Proof. This proof is completely analogous to that of Proposition 2.19 and is therefore suppressed. □

Remark 2.21. In Section 3, we will see another method for producing injective or projective resolutions of factorizations. These will provide more control than those appearing in the arguments of the proof of Proposition 2.19.

Next we turn to fully-faithfulness. Fully-faithfulness in this generality is unclear. However, when \( E \) consists of injectives or projectives, see Corollary 2.25. In the case \( A \) is locally-Noetherian, one can boot-strap from this to general \( E \) in the co-derived case. We thank the referee for pointing this out. For our applications, we will only need fully-faithfulness in more specific setting.

Proposition 2.22. Let \( E \) be an additive full subcategory of \( A \) preserved by \( \Phi \) and satisfying the following conditions:

- For any object \( A \in A \), there exists a monomorphism
  
  \[ A \rightarrow E \]

  with \( E \) an object of \( E \).

- For each \( A \), there exists an \( N \) such that for any exact sequence
  
  \[ 0 \rightarrow A \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n, \]

  with each \( E_i \) lying in \( E \), the cokernel of the morphism \( E_{n-1} \rightarrow E_n \) lies in \( E \) whenever \( n \geq N \).

Or satisfying the following dual conditions:

- For any object \( A \in A \), there exists a epimorphism
  
  \[ E \rightarrow A \]

  with \( E \) an object of \( E \).

- For each \( A \), there exists an \( N \) such that for any exact sequence
  
  \[ E_n \rightarrow E_{n+1} \rightarrow \cdots \rightarrow E_0 \rightarrow A \rightarrow 0, \]

  with each \( E_i \) lying in \( E \), the kernel of the morphism \( E_n \rightarrow E_{n+1} \) lies in \( E \) whenever \( n \leq N \).

Under either set of assumptions, the composition

\[ K(\text{Fact} \, E, \, w) \rightarrow K(\text{Fact} \, (A, \, w)) \rightarrow D_{\text{abs}} \, (\text{Fact} \, w) \]

induces an equivalence

\[ Q_E : D_{\text{abs}} \, (\text{Fact} \, E, \, w) \rightarrow D_{\text{abs}} \, (\text{Fact} \, w). \]

Proof. The statement that the natural functor between bounded derived categories, \( D^b(E) \rightarrow D^b(A) \), is an equivalence is well-known and much easier since one can test whether a complex is acyclic by looking at its cohomology. Let \( \text{Fact}_{ab}(A, \, w) \) be \( \text{Fact} \, (A, \, w) \) with its Abelian category structure as given above. Then, totalization induces an exact functor

\[ \text{tot} : D^b \, \text{Fact}_{ab}(A, \, w) \rightarrow D_{\text{abs}} \, (\text{Fact} \, A, \, w). \]

In [Efi13, Proposition A.3.2], Efimov shows that \( \text{tot} \) realizes the Verdier quotient of \( D^b \, \text{Fact}_{ab}(A, \, w) \) by the thick subcategory generated by complexes of the form \( F[l] \) where \( F \in \text{Fact} \, (A, \, w) \) is
contractible. Note here shift is taken as a complex and not as a factorization. Since we have enough factorizations with $\mathcal{E}$-components, we have a natural equivalence

$$D^b\text{Fact}_{ab}(\mathcal{E}, w) \to D^b\text{Fact}_{ab}(\mathcal{A}, w)$$

where $\text{Fact}_{ab}(\mathcal{E}, w)$ is $\text{Fact}(\mathcal{E}, w)$ with the exact structure from its natural embedding in $\text{Fact}_{ab}(\mathcal{A}, w)$. (We keep the ab subscript so as not to conflict with the exact structure considered in [Efi13].) Totalization also induces a functor

$$\text{tot} : D^b\text{Fact}_{ab}(\mathcal{E}, w) \to D^\text{abs}(\text{Fact}\mathcal{E}, w).$$

which by arguments of [Efi13, Proposition A.3.2] is also a Verdier quotient by the thick subcategory of contractible factorizations, now with $\mathcal{E}$-components. Thus, we get a commutative diagram

$$\begin{array}{ccc}
D^b\text{Fact}_{ab}(\mathcal{E}, w) & \xrightarrow{\tau} & D^b\text{Fact}_{ab}(\mathcal{A}, w) \\
tot & & tot \\
D^\text{abs}(\text{Fact}\mathcal{E}, w) & \xrightarrow{Q_{\mathcal{E}}} & D^\text{abs}(\text{Fact}\mathcal{A}, w)
\end{array}$$

where $\tau$ is an equivalence. To check the bottom horizontal functor is an equivalence, one needs to check that $\tau$ induces an equivalence between the thick subcategory generated by contractible factorizations with $\mathcal{E}$-components in $D^b\text{Fact}_{ab}(\mathcal{E}, w)$ and thick subcategory generated by contractible factorizations in $D^b\text{Fact}_{ab}(\mathcal{A}, w)$. We already know it is fully-faithful so we only need essential surjectivity. It suffices to generate any contractible factorization using contractible factorizations with $\mathcal{E}$-components. Since any contractible factorization is a summand of the cone over its identity map, we can reduce to generating $\text{Cone}(1_F)$ for $F \in \text{Fact}(\mathcal{A}, w)$. We can find an exact sequence of factorizations

$$0 \to F \to E_0 \to E_1 \to \cdots \to E_t \to 0$$

and take cones over the identities to get an exact sequence of contractible factorizations

$$0 \to \text{cone}(1_F) \to \text{cone}(1_{E_0}) \to \cdots \to \text{cone}(1_{E_t}) \to 0.$$ 

Thus, $\text{cone}(1_F)$ is generated by contractible factorizations with components in $\mathcal{E}$ and we can conclude the functor $Q_{\mathcal{E}}$ is an equivalence. □

**Remark 2.23.** The conclusion of Proposition 2.22 in the setting of flat modules over a curved dg-ring appears in [PP12, Section 3.2], see also [Pos09], and one can rework the argument there to easily fit this general setting. However, the authors find the argument using [Efi13, Appendix I] prettier. It is also moves closer to an intrinsic characterization of acyclicity. We thank the referee for mentioning these references.

Following the analogy with derived categories of Abelian categories, one can realize the various derived categories of factorizations as homotopy categories of factorizations with injective or projective components.

**Lemma 2.24.** Let $I$ be an object of $\text{Fact}(w)$ with $I^{-1}, I^0$ injective objects of $\mathcal{A}$. Let $C$ be a co-acyclic factorization. Then,

$$\text{Hom}_{K(\text{Fact}w)}(C, I) = 0.$$
Let $P$ be an object of $\text{Fact}(w)$ with $P^{-1}, P^0$ projective objects of $\mathcal{A}$. Let $C$ be a contra-acyclic factorization. Then,

$$\text{Hom}_{K(\text{Fact}_w)}(P, C) = 0.$$ 

**Proof.** If $C_s, s \in S$ is a collection of objects left orthogonal to $I$, then $\bigoplus_{s \in S} C_s$ is also left orthogonal to $I$. We can reduce to checking that $I$ is right orthogonal to totalizations of exact sequences. Any exact sequence is an iterated sequence of totalizations of short exact sequences. Thus, it suffices to check that $I$ is left orthogonal to totalizations of short exact sequences.

Take a short exact sequence of factorizations,

$$0 \rightarrow E_1 \xrightarrow{g_1} E_2 \xrightarrow{g_2} E_3 \rightarrow 0.$$ 

Let $C$ be the totalization of this short exact sequence. Since $I$ has injective components, the sequence,

$$0 \rightarrow \text{Hom}_w^*(E_3, I) \rightarrow \text{Hom}_w^*(E_2, I) \rightarrow \text{Hom}_w^*(E_1, I) \rightarrow 0,$$

is an exact sequence of complexes. The complex $\text{Hom}_w^*(C, I)$ is the totalization of the exact sequence in Equation 2.2 and is therefore acyclic.

The proof for contra-acyclic and projective factorizations is completely analogous. □

In the case of factorizations with injective or projective components, we do not need to take any further quotients.

**Corollary 2.25.** If $\mathcal{A}$ has enough injectives and coproducts of injectives are injective, then the composition

$$Q_{\text{inj}} : K(\text{Fact}_w) \rightarrow K(\text{Fact}_w) \rightarrow D^{co}(\text{Fact}_w)$$

is an equivalence.

If $\mathcal{A}$ has enough projectives and products of projectives are projective, then the composition

$$Q_{\text{proj}} : K(\text{Fact}_w) \rightarrow K(\text{Fact}_w) \rightarrow D^{ctr}(\text{Fact}_w)$$

is an equivalence.

If every object of $\mathcal{A}$ admits a finite injective resolution, then the composition

$$Q_{\text{inj}} : K(\text{Fact}_w) \rightarrow K(\text{Fact}_w) \rightarrow D^{abs}(\text{Fact}_w)$$

is an equivalence.

If every object of $\mathcal{A}$ admits a finite projective resolution, then the composition

$$Q_{\text{proj}} : K(\text{Fact}_w) \rightarrow K(\text{Fact}_w) \rightarrow D^{abs}(\text{Fact}_w)$$

is an equivalence.

**Proof.** Lemma 2.24 shows that any co-acyclic or totally acyclic factorization with injective components is zero in the homotopy category and any contra-acyclic or totally-acyclic factorization with projective components is zero in the homotopy category. Thus, $Q_{\text{inj}}$ and $Q_{\text{proj}}$ are fully-faithful. Then Proposition 2.19 gives the first statement, Proposition 2.20 gives the second, and Proposition 2.22 gives the last two. □

Finally, we record a fact that allows one to reduce some arguments to factorizations with zero component morphisms.
Lemma 2.26. For any factorization $E = (E^{-1}, E^0, \phi^{-1}_E, \phi^0_E)$, there is an exact sequence in $\text{Fact}(w)$,
\[ 0 \to \ker \phi^0_E \xrightarrow{f} (E^{-1}, E^{-1}, w_{\phi^{-1}(E^{-1})}, 1_{E^{-1}}) \xrightarrow{g} E \xrightarrow{h} \text{coker} \phi^0_E \to 0. \]
This gives rise to an exact triangle in $D^{\text{abs}}(\text{Fact} w)$,
\[ \text{coker} \phi^0_E \to \ker \phi^0_E \to E. \]

Proof. The components of the morphisms $f, g, h$ are given by
\[ f^{-1} = 0, f^0 = i, g^{-1} = 1_{E^{-1}}, g^0 = \phi^0_E, h^{-1} = 0, h^0 = \pi \]
where is $i : \ker \phi^0_E \to E^{-1}$ is the inclusion and $\pi : E^0 \to \text{coker} \phi^0_E$ is the projection. It is straightforward to see that the sequences associated to each component are exact. \qed

Definition 2.27. Let $\mathcal{T}$ be a triangulated category. A full subcategory $\mathcal{S}$ is said to triangul-\textit{arly generate} $\mathcal{T}$ if the smallest triangulated full subcategory of $\mathcal{T}$ containing $\mathcal{S}$ is $\mathcal{T}$.

Remark 2.28. The usual notion of generation includes closure under formation of sum-mands [BV03]. Our language reflects the fact that only formation of cones is allowed.

Corollary 2.29. The categories $D^{\text{abs}}(\text{Fact} w), D^{\text{co}}(\text{Fact} w), D^{\text{ctr}}(\text{Fact} w)$ are each triangul-\textit{arly generated by factorizations of the form $(0, A, 0, 0)$ for $A \in \mathcal{A}$}.

Proof. This follows immediately from the exact triangle in Lemma 2.26 \qed

Remark 2.30. In fact, $D^{\text{abs}}(\text{Fact} w)$ is strongly triangul-\textit{arly generated by objects of the form $(0, A, 0, 0)$ for $A \in \mathcal{A}$ as we only need to take a single cone}. See [BV03] for a definition of strong generation.

3. Constructions of resolutions

In this section, we provide a useful method of replacing a factorization by a co-quasi-isomorphic factorization of injectives or by a contra-quasi-isomorphic factorization of projectives. We saw a few simple consequences of the existence of such replacements at the end of Section 2. In Section 4, we will present some more computationally-useful applications.

We first analyze a way to construct factorizations starting from complexes over $\mathcal{A}$. Assume we have two complexes of objects of $\mathcal{A}$
\[ \cdots \xrightarrow{d_{-1}} A_{-1} \xrightarrow{d_0} A_0 \xrightarrow{d_1} A_1 \xrightarrow{d_2} A_2 \xrightarrow{d_3} \cdots \]
\[ \cdots \xrightarrow{d_{-1}} A_{-1}^0 \xrightarrow{d_0^0} A_0^0 \xrightarrow{d_1^0} A_1^0 \xrightarrow{d_2^0} A_2^0 \xrightarrow{d_3^0} \cdots \]

If either of the complexes is infinite, we assume that $\mathcal{A}$ has small coproducts or small products. Define the following two objects of $\mathcal{A}$ by combining even and odd components of the two complexes:
\[ \text{tot}^\oplus (A_\bullet)^{-1} := \bigoplus_{2l} \Phi^{-l}(A_{2l+1}^{-1}) \oplus \bigoplus_{2l+1} \Phi^{-l-1}(A_{2l}^0) \]
\[ \text{tot}^\oplus (A_\bullet)^0 := \bigoplus_{2l} \Phi^{-l}(A_{2l}^0) \oplus \bigoplus_{2l+1} \Phi^{-l}(A_{2l+1}^{-1}). \]
Similarly, set
\[
\text{tot} \prod (A_\bullet)^{-1} := \prod_{2l} \Phi^{-l}(A_{2l}) \times \prod_{2l+1} \Phi^{-l-1}(A_{2l+1})
\]
\[
\text{tot} \prod (A_\bullet)^0 := \prod_{2l} \Phi^{-l}(A_{2l}) \times \prod_{2l+1} \Phi^{-l}(A_{2l+1})
\]

**Definition 3.1.** Assume that coproducts in \( \mathcal{A} \) preserve monomorphisms. We say the two complexes \((A_\bullet^{-1}, A_\bullet^0)\) are \(\bigoplus\)-foldable if there exists a factorization
\[
A = (\text{tot} \oplus (A_\bullet)^{-1}, \text{tot} \oplus (A_\bullet)^0, \phi_A^{-1}, \phi_A^0)
\]
such that
\[
\phi_{p,q}^{-1} = \phi_{p,q}^0 = 0 \text{ for } q > p + 1
\]
and
\[
\phi_{2l+1,2l+2}^{-1} = \Phi^{-l-1}(d_{2l+2}) : \Phi^{-l-1}(A_{2l+1}) \to \Phi^{-l-1}(A_{2l+2})
\]
\[
\phi_{2l,2l+1}^{-1} = -\Phi^{-l-1}(d_{2l+1}) : \Phi^{-l-1}(A_{2l}) \to \Phi^{-l-1}(A_{2l+1})
\]
\[
\phi_{2l+1,2l+2}^0 = -\Phi^{-l-1}(d_{2l+2}) : \Phi^{-l-1}(A_{2l+1}) \to \Phi^{-l-1}(A_{2l+2})
\]
\[
\phi_{2l,2l+1}^0 = \Phi^{-l}(d_{2l+1}) : \Phi^{-l}(A_{2l}) \to \Phi^{-l}(A_{2l+1})
\]
where
\[
\phi_{2l+1,2l+1}^{-1} : \Phi^{-l-1}(A_{2l+1}) \to \Phi^{-1}(A^0) \xrightarrow{\phi_{2l+1,2l+1}^{-1}} A^{-1} \to \Phi^{-j-1}(A_{2l+1})
\]
\[
\phi_{2l,2l}^{-1} : \Phi^{-l-1}(A_{2l}) \to \Phi^{-1}(A^0) \xrightarrow{\phi_{2l,2l}^{-1}} A^{-1} \to \Phi^{-j}(A_{2l})
\]
\[
\phi_{2l+1,2l+1}^0 : \Phi^{-l}(A_{2l+1}) \to \Phi^{-1}(A^0) \xrightarrow{\phi_{2l,2l+1}^0} A^{-1} \to \Phi^{-j}(A_{2l+1})
\]
\[
\phi_{2l,2l}^0 : \Phi^{-l}(A_{2l}) \to \Phi^{-1}(A^0) \xrightarrow{\phi_{2l,2l}^0} A^{-1} \to \Phi^{-j}(A_{2l})
\]

and
\[
\phi_{2l+1,2l+1}^0 : \Phi^{-l-1}(A_{2l+1}) \to A^{-1} \xrightarrow{\phi_{2l+1,2l+1}^0} A^0 \to \Phi^{-j}(A_{2l+1})
\]
\[
\phi_{2l,2l}^0 : \Phi^{-l}(A_{2l}) \to A^{-1} \xrightarrow{\phi_{2l,2l}^0} A^0 \to \Phi^{-j}(A_{2l})
\]
\[
\phi_{2l+1,2l+1}^0 : \Phi^{-l}(A_{2l+1}) \to A^{-1} \xrightarrow{\phi_{2l+1,2l+1}^0} A^0 \to \Phi^{-j}(A_{2l+1})
\]
\[
\phi_{2l,2l}^0 : \Phi^{-l}(A_{2l}) \to A^{-1} \xrightarrow{\phi_{2l,2l}^0} A^0 \to \Phi^{-j}(A_{2l})
\]

Any such factorization \( A \) will be called a \(\bigoplus\)-folding of \((A_\bullet^{-1}, A_\bullet^0)\), and, in general, a \(\bigoplus\)-folded factorization.

Assume that products in \( \mathcal{A} \) preserve epimorphisms. We say the two complexes \((A_\bullet^{-1}, A_\bullet^0)\) are \(\prod\)-foldable if there exists a factorization
\[
A = (\text{tot} \prod (A_\bullet)^{-1}, \text{tot} \prod (A_\bullet)^0, \phi_A^{-1}, \phi_A^0)
\]
such that
\[
\phi_{p,q}^{-1} = \phi_{p,q}^0 = 0 \text{ for } q > p + 1
\]
Remark 3.4. We provide a reformulation of notion of
and
Remark 3.3. As only finitely many terms in these sums will be nonzero, these equa-
tions are completely
above and
A
is a folding, requiring that
\[ \phi_A^0 \circ \phi_A^{-1} = w_{A^{-1}(A^0)} \]
\[ \Phi(\phi_A^{-1}) \circ \phi_A^0 = w_{A^{-1}} \]
is equivalent to requiring the following identities of the components of \( \phi_A^{-1} \) and \( \phi_A^0 \):
\[ \sum_{t \in \mathbb{Z}} \phi_{t,q}^0 \circ \phi_{p,t}^{-1} = \begin{cases} 0 & p \neq q \\
 & w \\
 & p = q \end{cases} \]
and
\[ \sum_{t \in \mathbb{Z}} \Phi(\phi_{t,q}^{-1}) \circ \phi_{p,t}^0 = \begin{cases} 0 & p \neq q \\
 & w \\
 & p = q. \end{cases} \]
As only finitely many terms in these sums will be nonzero, these equations are completely
unambiguous.

Remark 3.3. Note that if \( A \) folds \( (A_{\bullet}^{-1}, A_{\bullet}^0) \) then \( A[1] \) folds \( (A_{\bullet}^{-1}[1], A_{\bullet}^0[1]) \).

Remark 3.4. We provide a reformulation of notion of \( \bigoplus \)-folding as suggested by the referee; there is a similar version for \( \bigotimes \)-folding. We can make the following \( \mathbb{Z} \)-graded vector space
\[ C^m := \begin{cases} \Phi^n(\text{tot}(A_{\bullet}^0)) & m = 2n \\
\Phi^n(\text{tot}(A_{\bullet}^{-1})) & m = 2n - 1 \end{cases} \]
which becomes a complex with \( \delta^m : C^{m-1} \to C^m \) given by
\[ \delta^m := \begin{cases} \Phi^n(\text{tot}(d_{\bullet}^{-1})) & m = 2n \\
\Phi^n(\text{tot}(d_{\bullet}^0)) & m = 2n + 1. \end{cases} \]
The graded vector space \( C^\bullet \) admits a filtration
\[ F^p C^m := \begin{cases} \Phi^n(\text{tot}(\sigma_{\leq p}A_{\bullet}^0)) & m = 2n \\
\Phi^n(\text{tot}(\sigma_{\leq p}A_{\bullet}^{-1})) & m = 2n - 1 \end{cases} \]
where \( \sigma_{\leq p} \) is the brutal (sometimes called stupid) truncation of the complexes \( A_{\bullet}^{-1} \) and \( A_{\bullet}^0 \). The differential does not respect this filtration but instead satisfies
\[ \delta(F^p C) \subset F^{p+1} C. \]
One can define a $\bigoplus$-folding of $A_{\bullet}$ as a choice of $\alpha : C \to C[1]$ such that

$$\alpha^m = \begin{cases} 
\Phi^n(\alpha^0) & m = 2n \\
\Phi^n(\alpha^{-1}) & m = 2n - 1,
\end{cases} \tag{3.1}$$

and $\delta \circ \alpha + \alpha \circ \delta + \alpha^2 = w$. To understand the dictionary, we can look at the action of $\alpha$. If $\alpha$ is a factorization folding a pair of bounded exact complexes $(A_{-1}, A_{\bullet})$, then $A$ is totally-acyclic.

Assume that $A$ has small coproducts. If $A$ is a factorization $\bigoplus$-folding a pair of bounded exact complexes $(A_{-1}, A_{\bullet})$, then $A$ is co-acyclic.

Assume that $A$ has small products. If $A$ is a factorization $\prod$-folding a pair of bounded exact complexes $(A_{-1}, A_{\bullet})$, then $A$ is contra-acyclic.

Proof. For those used to derived categories, the idea is quite simple; the cone over the morphism behaves like the sum of the two good truncations of the resolutions, hence, like a complex with no cohomology. Morally, this complex is then split into short exact sequences.

Equation (3.1) says that $\alpha$ is $\Phi$-twisted two-periodic, (3.2) gives the vanishing required for a $\bigoplus$-folding, and (3.3) says that $\delta + \alpha$ endows $\text{tot} \bigoplus(A_{\bullet})$ with the structure of a factorization of $w$.

Lemma 3.5. If $A$ is a factorization folding a pair of bounded exact complexes $(A_{-1}, A_{\bullet})$, then $A$ is totally-acyclic.

Assume that $A$ has small coproducts. If $A$ is a factorization $\bigoplus$-folding a pair of bounded exact complexes $(A_{-1}, A_{\bullet})$, then $A$ is co-acyclic.

Assume that $A$ has small products. If $A$ is a factorization $\prod$-folding a pair of bounded exact complexes $(A_{-1}, A_{\bullet})$, then $A$ is contra-acyclic.

Proof. For those used to derived categories, the idea is quite simple; the cone over the morphism behaves like the sum of the two good truncations of the resolutions, hence, like a complex with no cohomology. Morally, this complex is then split into short exact sequences.

In the language of factorizations this amounts to constructing the complex as a colimit of totalizations, which is finite when the resolutions are finite.

After replacing $A_{-1}$ and $A_{\bullet}$ by a common appropriate shift, we may assume that both complexes vanish in negative degrees. Let $C_{0}^{-1} = A_{0}^{-1}$ and let $C_{j}^{-1}$ be the cokernel of $d_{j}^{-1} : A_{j-1}^{-1} \to A_{j}^{-1}$. Let $C_{0}^{0} = A_{0}^{0}$ and let $C_{j}^{0}$ be the cokernel of $d_{j}^{0} : A_{j-1}^{0} \to A_{j}^{0}$. From exactness, $C_{j}^{0}$ is the image of $d_{j+1}^{0}$ and $C_{j}^{-1}$ is the image of $d_{j+1}^{-1}$ and we have exact sequences,

$$0 \to C_{j-1}^{-1} \to A_{j-1}^{-1} \to C_{j}^{-1} \to 0$$

$$0 \to C_{j-1}^{0} \to A_{j-1}^{0} \to C_{j}^{0} \to 0.$$

Consider the subfactorization, $\tau_{\leq j} A$, of $A$ given by restricting the components to their good truncations. The factorization, $\tau_{\leq j} A$, has components,

$$\tau_{\leq j} A^0 = \bigoplus_{0 \leq 2t < j} \Phi^{-t}(A_{2t}^0) \oplus \bigoplus_{0 \leq 2t+1 < j} \Phi^{-t}(A_{2t+1}^{-1}) \oplus \begin{cases} 
\Phi^{-t}(C_{j-1}^{0}) & j = 2t \\
\Phi^{-t}(C_{j-1}^{-1}) & j = 2t + 1,
\end{cases}$$

$$\tau_{\leq j} A^{-1} = \bigoplus_{0 \leq 2t+1 < j} \Phi^{-t-1}(A_{2t+1}^0) \oplus \bigoplus_{0 \leq 2t < j} \Phi^{-t}(A_{2t}^{-1}) \oplus \begin{cases} 
\Phi^{-t}(C_{j-1}^{-1}) & j = 2t \\
\Phi^{-t-1}(C_{j-1}^{0}) & j = 2t + 1,
\end{cases}$$
and morphisms between components induced by those from $A$ using the inclusion $C_j^i \rightarrow A_{j+1}^i$.

Note that this is a well-defined factorization since $\phi_{p,q}^i = 0$ for $q > p + 1$ and $d_{j+1}^i$ vanishes on $C_j^i$. Let $S_j$ denote the factorization with components,

$$
S_j^0 = \begin{cases} 
\Phi^{-t+1}(C_{j-1}^i) \oplus \Phi^{-t}(C_{j-1}^i) & j = 2t \\
\Phi^{-t}(C_{j-1}^i) \oplus \Phi^{-t}(C_{j-1}^i) & j = 2t + 1
\end{cases}
$$

and morphisms

$$
\phi_{S_j}^0 = \begin{cases} 
0 & w_{\Phi^{-t}(C_{j-1}^i)} \\
\text{id}_{\Phi^{-t}(C_{j-1}^i)} & 0 \\
0 & w_{\Phi^{-t+1}(C_{j-1}^i)} \\
\text{id}_{\Phi^{-t+1}(C_{j-1}^i)} & 0
\end{cases}
$$

$$
\phi_{S_j}^{-1} = \begin{cases} 
0 & w_{\Phi^{-t}(C_{j-1}^i)} \\
\text{id}_{\Phi^{-t}(C_{j-1}^i)} & 0 \\
0 & w_{\Phi^{-t+1}(C_{j-1}^i)} \\
\text{id}_{\Phi^{-t+1}(C_{j-1}^i)} & 0
\end{cases}
$$

Note that $S_j$ is manifestly a null-homotopic factorization. There are short exact sequences,

$$
0 \rightarrow \tau_{\leq j} A \rightarrow \tau_{\leq j+1} A \rightarrow S_{j+1} \rightarrow 0,
$$

of factorizations.

Thus, in $\mathcal{D}^{\text{abs}}$ (Fact $w$), $\tau_{\leq j} A$ and $\tau_{\leq j+1} A$ are isomorphic for $j \geq 0$. If the resolutions are finite, we see that, since $\tau_{\leq j} A = 0$ for $j >> 0$, $A$ is totally-acyclic.

In general, the colimit of these morphisms is isomorphic to $A$. As we can write the colimit via the short exact sequence,

$$
0 \rightarrow \bigoplus_{j \geq 0} \tau_{\leq j} A \rightarrow \bigoplus_{j \geq 0} \tau_{\leq j} A \rightarrow \text{colim} \tau_{\leq j} A = A \rightarrow 0,
$$

we see that $A$ is co-acyclic in general.

The argument in the situation where $\mathcal{A}$ has small products is analogous and omitted. □

**Remark 3.6.** See also [Bec12, Section 3.2].

**Lemma 3.7.** Let $A$ be a $\bigoplus$-folding (respectively $\prod$-folding) of $(A_{-1}^i, A_0^i)$ and $B$ be a $\bigoplus$-folding (respectively $\prod$-folding) of $(B_{-1}^i, B_0^i)$. Let $\eta : A \rightarrow B$ be a morphism of factorizations.
Let
\[
\begin{align*}
\eta_{2l,2j}^{-1} & : \Phi^{-l}(A_{2l}^{-1}) \to \Phi^{-j}(B_{2j}^{-1}) \\
\eta_{2l,2j+1}^{-1} & : \Phi^{-l}(A_{2l}^{-1}) \to \Phi^{-j-1}(B_{2j+1}^{-1}) \\
\eta_{2l+1,2j}^{-1} & : \Phi^{-l-1}(A_{2l+1}^{-1}) \to \Phi^{-j}(B_{2j}^{-1}) \\
\eta_{2l+1,2j+1}^{-1} & : \Phi^{-l-1}(A_{2l+1}^{-1}) \to \Phi^{-j-1}(B_{2j+1}^{-1}) \\
\eta_{2l,2j+1}^{0} & : \Phi^{-l}(A_{2l}^{0}) \to \Phi^{-j}(B_{2j}^{0}) \\
\eta_{2l,2j+1}^{0} & : \Phi^{-l}(A_{2l}^{0}) \to \Phi^{-j}(B_{2j+1}^{0}) \\
\eta_{2l+1,2j}^{0} & : \Phi^{-l}(A_{2l+1}^{0}) \to \Phi^{-j}(B_{2j}^{0}) \\
\eta_{2l+1,2j+1}^{0} & : \Phi^{-l}(A_{2l+1}^{0}) \to \Phi^{-j}(B_{2j+1}^{0}) 
\end{align*}
\]
be the morphisms on the summands of the components of \(A\) and \(B\) determined by \(\eta\). Assume that
\[
\eta_{p,q}^{-1}, \eta_{p,q}^{0} = 0 \quad \text{for} \quad q > p
\]
and
\[
\tilde{\eta}^{-1} := \Phi^{p}(\eta_{2p,2p}^{-1}), \Phi^{p}(\eta_{2p+1,2p+1}^{-1}) : A_{p}^{-1} \to B_{p}^{-1} \\
\tilde{\eta}^{0} := \Phi^{p}(\eta_{2p,2p}^{0}), \Phi^{p+1}(\eta_{2p+1,2p+1}^{-1}) : A_{p}^{0} \to B_{p}^{0}
\]
are chain maps. Then, the cone over \(\eta\), \(\text{Cone}(\eta)\), folds the cones over \(\tilde{\eta}^{-1}, \tilde{\eta}^{0}\).

**Proof.** It is clear that the components of \(\text{Cone}(\eta)\) are of the correct form to fold the cones over \(\tilde{\eta}^{-1}, \tilde{\eta}^{0}\). We check the conditions on the morphisms. The morphisms in the factorization \(\text{Cone}(\eta)\) are given by
\[
\phi_{\text{Cone}(\eta)}^{-1} = \left( \begin{array}{c} \phi_{A[1]}^{-1} \\ \eta_{p}^{-1} \end{array} \right), \quad \phi_{\text{Cone}(\eta)}^{0} = \left( \begin{array}{c} \phi_{A[1]}^{0} \\ \eta_{p}^{0} \end{array} \right)
\]
The vanishing condition, Equation (3.4), together with the fact that \(A\) and \(B\) are \(\bigoplus\)-foldable (respectively \(\prod\)-foldable) implies that the terms \(\phi_{p,q}^{-1}, \phi_{p,q}^{0}\) vanish for \(q > p + 1\) in \(\text{Cone}(\eta)\) while
\[
\phi_{p,p+1}^{-1} = \left( \begin{array}{c} (\phi_{A[1]}^{-1})_{p,p+1} \\ \eta_{p,p}^{-1} \end{array} \right), \quad \phi_{p,p+1}^{0} = \left( \begin{array}{c} (\phi_{A[1]}^{0})_{p,p+1} \\ \eta_{p,p}^{0} \end{array} \right)
\]
We see that this is of the appropriate form. 

Now we assume that \(\mathcal{A}\) has enough injectives. Let \(E\) be a factorization. Choose injective resolutions
\[
\begin{array}{cccccccc}
0 & \longrightarrow & E^{-1} & \longrightarrow & I_{0}^{-1} & \longrightarrow & I_{1}^{-1} & \longrightarrow & \cdots \\
0 & \longrightarrow & E^{0} & \longrightarrow & I_{0}^{0} & \longrightarrow & I_{1}^{0} & \longrightarrow & \cdots 
\end{array}
\]
We may also choose lifts of \(\phi_{E}^{-1}\) and \(\phi_{E}^{0}\) to the specified injective resolutions. Such choices, of course, always exist. However, for certain applications, we will need to work with specific choices of such lifts. As such, it is useful to specify choices of lifts in advance,
Since $\phi_E^0 \circ \phi_E^{-1} = w$ and $\Phi(\phi_E^{-1}) \circ \phi_E^0 = w$, the compositions of the lifts to the injective resolutions are homotopic to $w$. It will also be useful to specify the homotopies beforehand.

\[
\begin{array}{c}
0 
\xrightarrow{\Phi^{-1}(E^0)} \Phi^{-1}(d_0^1) 
\xrightarrow{\Phi^{-1}(d_0^2)} \Phi^{-1}(d_1^0) 
\xrightarrow{\Phi^{-1}(d_1^1)} \Phi^{-1}(d_2^0) 
\xrightarrow{\Phi^{-1}(d_2^1)} \Phi^{-1}(d_3^0) 
\xrightarrow{\Phi^{-1}(d_3^1)} \ldots \nonumber
\end{array}
\]

\[
\begin{array}{c}
0 
\xrightarrow{E^{-1}} I_0^{-1} 
\xrightarrow{I_1^{-1}} I_2^{-1} 
\xrightarrow{I_3^{-1}} \ldots \nonumber
\end{array}
\]

\[
\begin{array}{c}
0 
\xrightarrow{E^0} I_0^0 
\xrightarrow{I_1^0} I_2^0 
\xrightarrow{I_3^0} \ldots \nonumber
\end{array}
\]

where $\beta_j^0 = w_{\Phi^{-1}(I_j^0)} - \phi_j^0 \circ \phi_j^{-1}$, and

\[
\begin{array}{c}
0 
\xrightarrow{E^{-1}} d_0^1 
\xrightarrow{I_0^{-1}} d_1^1 
\xrightarrow{I_1^{-1}} d_2^1 
\xrightarrow{I_2^{-1}} d_3^1 
\xrightarrow{I_3^{-1}} \ldots \nonumber
\end{array}
\]

\[
\begin{array}{c}
0 
\xrightarrow{E^{-1}} \Phi(E^{-1}) 
\xrightarrow{\Phi(d_0^1)} \Phi(I_0^0) 
\xrightarrow{\Phi(d_1^1)} \Phi(I_1^0) 
\xrightarrow{\Phi(d_2^1)} \Phi(I_2^0) 
\xrightarrow{\Phi(d_3^1)} \ldots \nonumber
\end{array}
\]

where $\beta_j^{-1} = w_{I_j^{-1}} - \Phi(\phi_j^{-1}) \circ \phi_j^0$.

Now, we state our construction of injective resolutions.

**Theorem 3.8.** Assume that small coproducts exist in $\mathcal{A}$ and that $\mathcal{A}$ has enough injective objects.

Let $E$ be an object of $\text{Fact}(w)$. Choose injective resolutions of its components, lifts of $\phi_E^{-1}$ and $\phi_E^0$ to these injective resolutions, and null-homotopies of the difference of $w$ and the compositions of the lifts as above.

There exists a $\bigoplus$-folding, $I = (\text{tot} \bigoplus (I^{-1}_*, \text{tot} \bigoplus (I_*^0, \phi_{I_*}^{-1}, \phi_{I_*}^0)), of I_*^{-1}$ and $I_*^0$ and a co-quasi-isomorphism, $d_0 : E \to I$, such that

- We have equalities

\[
\begin{align*}
\phi_{2l+1,2l+1}^{-1} & = \Phi^{-l-1}(d_{2l+1}^0) : \Phi^{-l-1}(I_{2l+1}^{-1}) \to \Phi^{-l-1}(I_{2l+1}^0) \\
\phi_{2l,2l}^{-1} & = \Phi^{-l}(d_{2l}^0) : \Phi^{-l-1}(I_{2l}^{-1}) \to \Phi^{-l}(I_{2l}^0) \\
\phi_{2l+1,2l+1}^0 & = \Phi^{-l}(d_{2l+1}^0) : \Phi^{-l-1}(I_{2l+1}^0) \to \Phi^{-l}(I_{2l+1}^{-1}) \\
\phi_{2l,2l}^0 & = \Phi^{-l}(d_{2l}^0) : \Phi^{-l}(I_{2l}^0) \to \Phi^{-l}(I_{2l}^{-1}) ,
\end{align*}
\]
and
\[
\phi_{2t+1,2t}^{-1} = \Phi^{-t-1}(h_{2t}^{-1}) : \Phi^{-t-1}(I_{2t+1}^{-1}) \to \Phi^{-t}(I_{2t}^{-1}) \\
\phi_{2t,2t-1}^{-1} = -\Phi^{-t}(h_{2t-1}^{0}) : \Phi^{-t-1}(I_{2t}^{0}) \to \Phi^{-t}(I_{2t-1}^{0}) \\
\phi_{2t+1,2t}^{0} = -\Phi^{-t}(h_{2t}^{0}) : \Phi^{-t-1}(I_{2t+1}^{0}) \to \Phi^{-t}(I_{2t}^{0}) \\
\phi_{2t,2t-1}^{0} = \Phi^{-t}(h_{2t-1}^{-1}) : \Phi^{-t-1}(I_{2t}^{-1}) \to \Phi^{-t+1}(I_{2t-1}^{-1}).
\]

- **d₀** is given by the compositions,
  \[
  E^{-1} \xrightarrow{d_{0}^{-1}} I_{0}^{-1} \to I^{-1} \\
  E^{0} \xrightarrow{d_{0}^{0}} I_{0}^{0} \to I^{0}.
  \]
- **d₀** is a quasi-isomorphism when both injective resolutions are finite.

**Proof.** We will construct \( \phi_{p,q}^{-1} \) and \( \phi_{p,q}^{0} \) such that
\[
\sum_{t \in \mathbb{Z}} \phi_{t,p+n}^{0} \circ \phi_{p,t}^{-1} = \begin{cases} 0 & n \neq 0 \\ w & n = 0 \end{cases} \tag{3.5}
\]
and
\[
\sum_{t \in \mathbb{Z}} \Phi(\phi_{t,p+n}^{-1}) \circ \phi_{p,t}^{0} = \begin{cases} 0 & n \neq 0 \\ w & n = 0 \end{cases} \tag{3.6}
\]
We will proceed by downward induction on \( n \). We begin by defining \( \phi_{p,q}^{-1} \) and \( \phi_{p,q}^{0} \) for \( p - 1 \leq q \) exactly as in the conclusions of the theorem. This satisfies the cases, \( n = 2, 1, 0 \), of Equations (3.5) and (3.6).

Now assume we have constructed \( \phi_{p,q}^{-1} \) and \( \phi_{p,q}^{0} \) for \( q \geq p - m \) satisfying Equations (3.5) and (3.6) for \( n \geq -m + 1 \). We need to construct \( \phi_{s,s-m-1}^{-1} \) and \( \phi_{s,s-m-1}^{0} \) such that
\[
\sum_{p-m \leq t \leq p} \phi_{t,p-m}^{0} \circ \phi_{p,t}^{-1} + \phi_{p-m-1,p-m}^{0} \circ \phi_{p,p-m-1}^{-1} + \phi_{p+1,p-m}^{0} \circ \phi_{p,p+1}^{-1} = 0 \tag{3.7}
\]
and
\[
\sum_{p-m \leq t \leq p} \Phi(\phi_{t,p-m}^{-1}) \circ \phi_{p,t}^{0} + \Phi(\phi_{p-m-1,p-m}^{-1}) \circ \phi_{p,p-m-1}^{0} + \Phi(\phi_{p+1,p-m}^{-1}) \circ \phi_{p,p+1}^{0} = 0. \tag{3.8}
\]
We will see that solving Equation (3.7) and (3.8) amounts to choosing a null-homotopy for an acyclic chain map between complexes of injectives. Solving Equation (3.7) for \( p \) even and Equation (3.8) for \( p \) odd is independent from solving Equation (3.7) for \( p \) odd and Equation (3.8) for \( p \) even. We will solve Equation (3.7) for \( p \) even and Equation (3.8) for \( p \) odd. The other case is completely analogous.

Assume that \( m = 2r \). The case of odd \( m \) follows analogously. Consider the chain complexes of injectives, \( (\Phi^{-u-1}(I_{v}^{0}), \Phi^{-u-1}(d_{v}^{0})) \) and \( (\Phi^{-u-r}(I_{v-m}^{0}), \Phi^{-u-r}(d_{v-m}^{0})) \). Each complex contains homology in a single degree, 0 for \( (\Phi^{-u-1}(I_{v}^{0}), \Phi^{-u-1}(d_{v}^{0})) \) and \( m \) for \( (\Phi^{-u-r}(I_{v-m}^{0}), \Phi^{-u-r}(d_{v-m}^{0})) \).
We claim that
\[ \psi \text{ (} \Phi^{-u} I^0_v, \Phi^{-u} (d^0_v) \text{)} \rightarrow (\Phi^{-u-r} I^0_{v-m}, \Phi^{-u-r} (d^0_{v-m}) \text{)} \text{ is a chain map.} \]
Let us assume the validity of this claim for the moment and continue. Since \( \psi \) must induce the trivial map on the homology of the complexes and the components of the complexes are injectives, there exists a null-homotopy,
\[ h_{u,v} : \Phi^{-u} I^0_v \rightarrow \Phi^{-u-r} I^0_{v-m-1}, \]
of \( \psi \). Let us draw the diagram for the homotopy. Recall that \( \phi_{-1} \) for \( 2q \) and \( \phi_{-2} \) for \( 2q+1 \).

We can rewrite the equations for the homotopy,
\[ \psi_{u,2q} = -h_{u,2q+1} \circ \Phi^{-u}(\phi_{2q,2q+1}) - \Phi^{-u}(\phi_{2q-1,2q-m}) \circ h_{u,2q}, \]
\[ \psi_{u,2q+1} = -h_{u,2q+2} \circ \Phi^{-u}(\phi_{2q+1,2q+2}) - \Phi^{-u+1}(\phi_{2q-1,2q-m+1}) \circ h_{u,2q+1}, \]
as
\[ \sum_{2q-m < t \leq 2q} \phi_{2q,m}^0 \circ \phi_{2q,t}^{-1} \circ \Phi^{-u}(h_{2q,2q+1}) \circ \phi_{2q,2q+1}^{-1} \circ \phi_{2q-1,2q-m}^{-1} \circ \Phi^{-u}(h_{u,2q}) = 0, \]
\[ \sum_{2q-m+1 < t \leq 2q+1} \Phi(\phi_{2q,m}^{-1} \circ \phi_{2q+1,2q+1}^{-1} \circ \Phi^{-u}(h_{u,2q+1}) \circ \phi_{2q+1,2q+2}^{-1} \circ \phi_{2q-1,2q-m+1}^{-1} \circ \Phi^{-u}(h_{u,2q+1}) = 0. \]

We then set
\[ \phi_{2q+1,2q-m}^0 := \Phi^{-u}(h_{u,2q+1}), \]
\[ \phi_{2q,2q-m}^{-1} := \Phi^{-u}(h_{u,2q}) \]
to solve Equation (3.7) for \( p = 2q \) and Equation (3.8) for \( p = 2q+1 \).

Thus, we have constructed \( \phi_{2q+1,2q-m}^0 \) and \( \phi_{2q,2q-m}^{-1} \) completing the induction step if we can show that \( \psi \) is a chain map. We check the commutativity of the square,
as the other squares are handled similarly. Commutativity of the above square is equivalent to the equality,

$$\left( \sum_{2q-m+1 \leq t \leq 2q+1} \Phi(\phi^{-1}_{t,2q-m+1}) \circ \phi^0_{2q+1,t} \right) \circ \phi^{-1}_{2q,2q+1} = \Phi(\phi^{-1}_{2q-m,2q+m+1}) \circ \left( \sum_{2q-m \leq t \leq 2q} \phi^0_{t,2q-m} \circ \phi^{-1}_{2q,t} \right).$$

(3.9)

From the induction hypothesis, for $2q - m + 1 \leq t \leq 2q + 1$, we have

$$\phi^0_{2q+1,t} \circ \phi^{-1}_{2q,2q+1} = - \sum_{t-1 \leq s \leq 2q} \phi^0_{s,t} \circ \phi^{-1}_{2q,s},$$

and, for $2q - m \leq t \leq 2q$,

$$\Phi(\phi^{-1}_{2q-m,2q+m+1}) \circ \phi^0_{t,2q-m} = - \sum_{2q-m-1 \leq s \leq t+1} \Phi(\phi^{-1}_{s,2q+m-1}) \circ \phi^0_{t,s}.$$

Thus, both sides of Equation (3.9) are equal to

$$- \sum_{2q-m-1 \leq s \leq t+1 \leq 2q} \Phi(\phi^{-1}_{s,2q+m-1}) \circ \phi^0_{t,s} \circ \phi^{-1}_{2q,t}.$$

This finishes the construction of the factorization, $(I^{-1}, I^0, \phi^{-1}_I, \phi^0_I)$.

We turn to checking that $d_0$, as defined in the conclusion of the theorem, is a morphism of factorizations. By the construction of $\phi^{-1}_I$ and $\phi^0_I$, to check that the bold squares in

$$\Phi^{-1}(E^0) \xrightarrow{\phi^{-1}_E} E^{-1} \xrightarrow{\phi^0_E} E^0$$

$$\Phi^{-1}(I^0_0) \xrightarrow{(d^{-1}_0, 0)} \Phi^{-1}(I^{-1}_0) \oplus \Phi^{-1}(I^0_1) \xrightarrow{(\phi^0_0, h_0, d^{-1}_1, \phi^{-1}_1)} I^0_0 \oplus I^{-1}_1 \oplus \Phi^{-1}(I^0_2)$$

commute, it suffices to show that the upper squares commute. This is immediate.

Finally, we demonstrate that the cone over $d_0$ is co-acyclic. The factorization $E$ folds the trivial complexes $(E^{-1}, E^0)$ while $I$ folds $(I^{-1}, I^0)$. The cone over $d_0$ folds the cones over the morphisms of the chain complexes $d_0^{-1} : E^{-1} \to I^{-1}$ and $d^0_0 : E^0 \to I^0$. Each of these complexes is bounded below and acyclic thus we may apply Lemma 3.5 to conclude that the
cone over $d_0$ is co-acylic and the cone over $d_0$ is acyclic if the chosen injective resolutions are bounded. □

**Remark 3.9.** The following was pointed out to us by the referee. Theorem 3.8 can be repackaged into a more elegant form. Using the reformulation of a $⨁$-folding from Remark 3.4, we are seeking a choice of $α : C → C[1]$ such that

$$α^m = \begin{cases} Φ^n(α^0) & m = 2n \\ Φ^n(α^{-1}) & m = 2n-1, \end{cases}$$

and $δ o α + α o δ + α^2 = w$

where $C$ is the $Φ$-twisted two-periodic extension of $\text{tot} Ω(\bigoplus I•)$ and $δ$ is the $Φ$-twisted two-periodic extension of $\text{tot} Ω(d•)$. Let

$$B^m := \text{Hom}_A(C^{-1}, C[m]^{-1}) \oplus \text{Hom}_A(C^0, C[m]^0).$$

The graded vector space $B = \bigoplus B^m$ has the structure of dg-algebra with differential $∂$ being the commutator with $δ$. The filtration $F^• C$ induces a decreasing filtration on $B$ given by

$$F^p B := \{ φ | φ(F^q C) ⊂ F^q-p B \text{ for all } q \}.$$

As we have specified the degree 0 and 1 pieces of $α$, with respect to this filtration, by choosing the lifts of $Φ_E$ and the homotopies, we have $α_2 ∈ B^1/F^2 B^1$ with

$$∂α_2 + α_2^2 - w ∈ F^1 B^2/F^2 B^2.$$

One can consider the following, more general problem. Assume we have dg-algebra $(B, ∂)$ with central element $w ∈ B^2$ and a decreasing filtration $F^• B$ satisfying

$$∂(F^p B) ⊂ F^{p-1} B,$$

$$F^p B · F^q B ⊂ F^{p+q} B,$$

$$\lim_p B/F^p B = B,$$

and such that the complex

$$\cdots \xrightarrow{∂} F^p B/F^{p+1} B \xrightarrow{∂} \cdots \xrightarrow{∂} F^1 B/F^2 B \xrightarrow{∂} F^0 B/F^1 B$$

(3.10)

is exact. If, for any $n ≥ 2$, there is a $\overline{α} ∈ B^1/F^n B^1$ satisfying

$$∂\overline{α} + α^2 - w ∈ F^{n-1} B^2/F^n B^2,$$

then there exists an $α ∈ B^1$ whose image in $B^1/F^n B^1$ is $\overline{α}$ and which solves

$$∂α + α^2 = w.$$

This more general question is proven via the same lifting order by order argument as in the proof of Theorem 3.8.

The additional condition, Equation 3.10 is satisfied when

$$B = \bigoplus_m \text{Hom}_A(C^{-1}, C[m]^{-1}) \oplus \text{Hom}_A(C^0, C[m]^0)$$

since the homology of the complex from Equation 3.10 in negative (cohomological) degrees computes the negative extensions of $Φ$-twists of $E^{-1}$ and $E^0$ with other $Φ$-twists of $E^{-1}$ and $E^0$, which must vanish. Compare with Lemma 3.13.
There is a special situation where the components, $\phi_{p,q}^{-1}$ and $\phi_{p,q}^0$, vanish for $q < p - 1$.

**Corollary 3.10.** Assume that

\[
\begin{align*}
    h_p^{-1} \circ \phi_{p+1}^{-1} &= \Phi(\phi_{p}^{-1}) \circ h_p^0 \\
    \Phi(h_p^0) \circ \phi_{p+1}^0 &= \Phi(\phi_{p}^0) \circ h_p^{-1} \\
    \Phi(h_{p-1}^0) \circ h_p^{-1} &= 0 \\
    \Phi(h_{p-1}^0) \circ h_p^0 &= 0.
\end{align*}
\]

Then, in the factorization constructed in Theorem 3.8, we may take

$$
\phi_{p,q}^{-1} = \phi_{p,q}^0 = 0
$$

for $q < p - 1$.

**Proof.** Under the hypotheses, we can take $\phi_{p,q}^{-1} = \phi_{p,q}^0 = 0$ for $q < p - 1$ and satisfy Equations (3.5) and (3.6) for all $n$. \qed

We also have the dual statement which we record in full detail for ease of future reference. Assume that $\mathcal{A}$ has enough projectives. Let $E$ be an object of Fact($\mathcal{w}$). Choose projective resolutions of its components and lifts of $\phi_{E}^1$ and $\phi_{E}^0$ to the those resolutions,

\[
\cdots \to \Phi^{-1}(d_{-2}^1) \to \Phi^{-1}(P_{-1}^0) \to \Phi^{-1}(d_{-1}^0) \to \Phi^{-1}(P_0^0) \to \Phi^{-1}(E^0) \to 0
\]

Also choose null-homotopies,

\[
\cdots \to \Phi^{-1}(d_{-3}^0) \to \Phi^{-1}(P_{-2}^0) \to \Phi^{-1}(d_{-2}^0) \to \Phi^{-1}(P_{-1}^0) \to \Phi^{-1}(d_{-1}^0) \to \Phi^{-1}(P_0^0) \to \Phi^{-1}(E^0) \to 0
\]

where $\beta_j^0 = w_{\Phi^{-1}(P_{j}^0)} - \phi_{j}^0 \circ \phi_{j}^{-1}$, and

\[
\cdots \to \Phi(P_{-2}^1) \to \Phi(P_{-1}^1) \to \Phi(P_0^1) \to \Phi(E^1) \to 0
\]

where $\beta_j^{-1} = w_{P_{j}^{-1}} - \Phi(\phi_{j}^{-1}) \circ \phi_{j}^0$. 


Theorem 3.11. Assume that small products exist in $\mathcal{A}$ and that $\mathcal{A}$ has enough projective objects.

Let $E$ be an object of $\text{Fact}(w)$. Choose projective resolutions of its components, lifts of $\phi_{E}^{-1}$ and $\phi_{E}^{0}$ to these projective resolutions, and null-homotopies of the difference of $w$ and the compositions of the lifts as above.

There exists a factorization, $P = (\text{tot} \prod (P_{\bullet})^{-1}, \text{tot} \prod (P_{\bullet})^{0}, \phi_{p}^{-1}, \phi_{p}^{0})$, $\prod$-folding $P_{\bullet}^{-1}$ and $P_{\bullet}^{0}$ and a contra-quasi-isomorphism, $d_{0} : P \to E$, such that

- The components of $\phi_{p,q}^{-1}$ and $\phi_{p,q}^{0}$ with $q = p, p-1$ are given by
  \[
  \Phi^{-l-1}(\phi_{2l+1}^{0}) = \phi_{2l+1,2l+1}^{-1} : \Phi^{-l-1}(P_{2l+1}^{-1}) \to \Phi^{-l-1}(P_{2l+1}^{0})
  \]
  \[
  \Phi^{-l}(\phi_{2l}^{-1}) = \phi_{2l,2l}^{-1} : \Phi^{-l-1}(P_{2l}^{-1}) \to \Phi^{-l}(P_{2l}^{0})
  \]
  \[
  \Phi^{-l}(\phi_{2l+1}^{0}) = \phi_{2l+1,2l+1}^{0} : \Phi^{-l-1}(P_{2l+1}^{0}) \to \Phi^{-l}(P_{2l+1}^{-1})
  \]
  \[
  \Phi^{-l}(\phi_{2l}^{0}) = \phi_{2l,2l}^{0} : \Phi^{-l}(P_{2l}^{-1}) \to \Phi^{-l}(P_{2l}^{0}).
  \]

and

\[
\Phi^{-l}(h_{2l}^{-1}) = \phi_{2l+1,2l}^{-1} : \Phi^{-l-1}(P_{2l+1}^{-1}) \to \Phi^{-l}(P_{2l}^{-1})
\]

- $d_{0}$ is given by the compositions,
  \[
P^{-1} d_{0}^{-1} P_{0}^{-1} \to E^{-1}
  \]
  \[
P^{0} d_{0} P_{0}^{0} \to E^{0}.
  \]

- $d_{0}$ is a quasi-isomorphism when both injective resolutions are finite.

Furthermore, if

\[
h_{p}^{-1} \circ \phi_{p+1}^{-1} = \Phi(\phi_{p}^{-1}) \circ h_{p}^{0}
\]
\[
\Phi(h_{p}^{0}) \circ \phi_{p+1}^{0} = \Phi(\phi_{p}^{0}) \circ h_{p}^{-1}
\]
\[
\Phi(h_{p-1}^{-1}) \circ h_{p}^{-1} = 0
\]
\[
\Phi(h_{p}^{0}) \circ h_{p}^{0} = 0.
\]

Then, we may take

\[
\phi_{p,q}^{-1} = \phi_{p,q}^{0} = 0
\]
for $q < p - 1$.

Proof. The statement is dual to those for Theorem 3.8 and Corollary 3.10. Therefore, we may replace $\mathcal{A}$ by its opposite category.

\[
\square
\]

Remark 3.12. The classical case of this construction is to let $R$ be a commutative Noetherian regular $k$-algebra, $\mathcal{A} = \text{mod } R$, $\Phi = \text{Id}$, and $w \in R$. We then consider an ideal $I$ containing $w$ and generated by a regular sequence $(x_{1}, \ldots, x_{n})$ so that we may write $w = \sum w_{i}x_{i}$. We may consider the factorization $(0, R/I, 0, 0)$. The Koszul complex on $(x_{1}, \ldots, x_{n})$ gives
a projective resolution of $R/I$ and contraction with $(w_1, \ldots, w_n)$ gives a homotopy $h$ such that $h^2 = 0$. The projective replacement

$$\bigoplus_{l=0}^{\lfloor \frac{n}{2} \rfloor} \bigoplus_{x_1, \ldots, x_n} 2^{l+1} \langle x_1, \ldots, x_n \rangle \otimes_k R, \bigoplus_{l=0}^{\lfloor \frac{n}{2} \rfloor} \bigoplus_{x_1, \ldots, x_n} 2^{l} \langle x_1, \ldots, x_n \rangle \otimes_k R, d + h, d + h$$

of $(0, R/I, 0, 0)$ is called the stabilization of $R/I$. This recovers Eisenbud’s original construction [Eis80].

As a first application, we give a spectral sequence for computing morphisms in $D^{\text{abs}}(\text{Fact } w)$.

**Lemma 3.13.** Let $E$ and $F$ be two factorizations of $w$. Assume that $\mathcal{A}$ has enough injectives and small coproducts, and assume that coproducts of injectives are injective.

There is a spectral sequence whose $E_1$-page is

$$E_1^{p,q} = \begin{cases} \text{Ext}_A^{2p+q-1}(E^{-1}, \Phi^{s}(F^0)) \oplus \text{Ext}_A^{2p+q}(E^0, \Phi^{s}(F^0)) & p = 2s \\ \text{Ext}_A^{2p+q-1}(E^{-1}, \Phi^{s}(F^{-1})) \oplus \text{Ext}_A^{2p+q}(E^0, \Phi^{s}(F^{-1})) & p = 2s + 1. \end{cases}$$

If the components of $F$ have finite injective resolutions, the spectral sequence strongly converges to $\bigoplus_r \text{Hom}(E, F[r])$ taken in $D^{\text{co}}(\text{Fact } w)$ or $D^{\text{abs}}(\text{Fact } w)$.

**Proof.** Choose finite injective resolutions of $F^{-1}$ and $F^0$,

$$0 \rightarrow F^{-1} \rightarrow I_0^{-1} \rightarrow I_1^{-1} \rightarrow \cdots$$

$$0 \rightarrow F^0 \rightarrow I_0^0 \rightarrow I_1^0 \rightarrow \cdots,$$

and use Theorem 3.8 to construct a co-quasi-isomorphic resolution, $I$, of $F$.

Filter the complex, $\text{Hom}_w^*(E, I)$, by

$$F^p\text{Hom}_w^n(E, I) := \{(g^{-1}, g^{0}) |$$

$$g^{-1}(E^{-1}) \subseteq \bigoplus_{2l \leq n+p-1} \Phi^{m-l}(I_{2l}^{-1}) \oplus \bigoplus_{2l+1 \leq n+p-1} \Phi^{m-l-1}(I_{2l+1}^{-1})$$

$$g^{0}(E^0) \subseteq \bigoplus_{2l \leq n+p} \Phi^{m-l}(I_{2l}^{0}) \oplus \bigoplus_{2l+1 \leq n+p} \Phi^{m-l-1}(I_{2l+1}^{0})$$

$$g^{-1}(E^{-1}) \subseteq \bigoplus_{2l \leq n+p-1} \Phi^{m-l}(I_{2l}^{-1}) \oplus \bigoplus_{2l+1 \leq n+p-1} \Phi^{m-l}(I_{2l+1}^{-1})$$

$$g^{0}(E^0) \subseteq \bigoplus_{2l \leq n+p} \Phi^{m-l+1}(I_{2l}^{1}) \oplus \bigoplus_{2l+1 \leq n+p} \Phi^{m-l}(I_{2l+1}^{1})$$

$$n = 2m$$

$$n = 2m + 1 \}.$$}

The associated graded complex is

$$\text{Gr}^p\text{Hom}_w^n(E, I) := \begin{cases} \text{Hom}_A(E^{-1}, \Phi^{s}(I_{p+n-1}^0)) \oplus \text{Hom}_A(E^0, \Phi^{s}(I_{p+n}^0)) & p = 2s \\ \text{Hom}_A(E^{-1}, \Phi^{s}(I_{p+n-1}^0)) \oplus \text{Hom}_A(E^0, \Phi^{s}(I_{p+n}^0)) & p = 2s + 1. \end{cases}$$

with differentials given by composition with the differentials in the complexes $I_{\bullet}^{-1}$ and $I_{\bullet}^0$. We set

$$E_0^{p,q} := \text{Gr}^p\text{Hom}_w^{p+q}(E, I)$$

to start our spectral sequence. The $E_1$-page is as above.

If we assume that the components of $F$ have injective resolutions of length $t$, then the spectral sequence degenerates at the $(t+1)$-st page.

**Lemma 3.14.** Let $E$ and $F$ be two factorizations of $w$. Assume that $\mathcal{A}$ has enough projectives and small products. Further, assume that products of projectives remain projective.
There is a spectral sequence whose $E_1$-page is

$$E_1^{p,q} = \begin{cases} \Ext_p^2(\Phi^{-s}(F^0)) \oplus \Ext_p^2(\Phi^{-s}(F^0)) & p = 2s \\ \Ext_{p+q}^2(\Phi^{-s}(F^{-1})) \oplus \Ext_{p+q}^2(\Phi^{-s}(F^{-1})) & p = 2s + 1. \end{cases}$$

If the components of $E$ have finite projective resolutions, the spectral sequence strongly converges to $\bigoplus_r \Hom_{D^\text{ct}(\text{Fact}_w)}(E, F^r).$

Proof. We may replace $\mathcal{A}$ by its opposite category and apply Lemma 3.13. □

4. Derived factored functors and some applications of the resolutions

Definition 4.1. Consider two triples, $(\mathcal{A}, \Phi, w)$ and $(\mathcal{B}, \Psi, v)$. An additive functor, $\theta : \mathcal{A} \to \text{Ch}^b(\mathcal{B})$, is called factored if there is a natural isomorphism

$$\epsilon_\theta : \theta \circ \Phi \cong \Psi \circ \theta$$

and

$$\theta(w_A) \circ \theta(A) \to \theta(\Phi(A)) \cong \Psi(\theta(A)).$$

for all objects, $A \in \mathcal{A}$. Here we extend $\Psi$ and $v$, in the obvious manner, to the Abelian category $\text{Ch}^b(\mathcal{B})$.

A factored functor induces a functor on factorization categories,

$$\theta^f : \text{Fact}(w) \to \text{Fact}(v)$$

$$E \mapsto \text{tot}(\theta(E^{-1}), \theta(E^0), \theta(\phi_E^{-1}), \theta(\phi_E^0))$$

where $\text{tot}$ is the totalization of the complex of factorizations $(\theta(E^{-1}), \theta(E^0), \theta(\phi_E^{-1}), \theta(\phi_E^0))$.

Definition 4.2. Let $\theta$ be a factored functor. A factorization is called $\theta^f$-co-adapted if $\theta$ is left-exact and its components are from a right-adapted class of objects for $\theta$. A factorization is called $\theta^f$-contra-adapted if $\theta$ is right-exact and its components are from a left-adapted class of objects for $\theta$.

As notation, we let $\mathcal{A}_{dp}^\theta$ be a class of (left or right) $\theta$-adapted objects in $\mathcal{A}$, viewed as a full subcategory. Specification will occur in instances of possible confusion.

We next define some notions of derived functors for factored functors.

Definition 4.3. Assume $\mathcal{A}$ has small coproducts and that $\theta$ is left exact and commutes with coproducts. Furthermore, assume that the natural functor

$$Q_{\mathcal{A}_{dp}^\theta} : D^{\text{co}}(\text{Fact} \mathcal{A}_{dp}^\theta, w) \to D^{\text{co}}(\text{Fact} w)$$

is an equivalence. Note that this is satisfied if $\mathcal{A}$ has enough injectives and coproducts of injectives remain injective by Corollary 2.25. We define the right co-derived factored functor of $\theta$ to be the composition

$$R^{\text{co}}\theta^f : D^{\text{co}}(\text{Fact} w) \xrightarrow{Q_{\mathcal{A}_{dp}^\theta}^{-1}} D^{\text{co}}(\text{Fact} \mathcal{A}_{dp}^\theta, w) \xrightarrow{\theta^f} D^{\text{co}}(\text{Fact} v).$$

Assume $\mathcal{A}$ has small products and that $\theta$ is right exact and commutes with products. Furthermore, assume that the natural functor

$$Q_{\mathcal{A}_{dp}^\theta} : D^{\text{ctr}}(\text{Fact} \mathcal{A}_{dp}^\theta, w) \to D^{\text{ctr}}(\text{Fact} w)$$
is an equivalence. Note that this is satisfied if \( \mathcal{A} \) has enough injectives and coproducts of injectives remain injective by Corollary 2.25. We define the left contra-derived factored functor of \( \theta \) to be the composition

\[
L^{\text{contra}} \theta^f : D^\text{ctr}(\text{Fact} \ A) \xrightarrow{Q^{-1}_{\mathcal{A}dp\theta}} D^\text{ctr}(\mathcal{A}dp\theta) \xrightarrow{\theta^f} D^\text{ctr}(\text{Fact} \ v).
\]

- Assume that \( \theta \) is left exact and the natural functor

\[
Q_{\mathcal{A}dp\theta} : D^\text{abs}(\text{Fact} \ A, w) \to D^\text{abs}(\text{Fact} \ A)
\]

is an equivalence. Note that this holds if the class of right-adapted objects in \( \mathcal{A} \) satisfies the appropriate set of hypotheses of Proposition 2.22. We define the right absolutely-derived factored functor of \( \theta \) to be the composition

\[
R^{\text{abs}} \theta^f : D^\text{abs}(\text{Fact} \ v) \xrightarrow{Q^{-1}_{\mathcal{A}dp\theta}} D^\text{abs}(\mathcal{A}dp\theta, w) \xrightarrow{\theta^f} D^\text{abs}(\text{Fact} \ v).
\]

- Assume that \( \theta \) is right exact and the natural functor

\[
Q_{\mathcal{A}dp\theta} : D^\text{abs}(\text{Fact} \ A, w) \to D^\text{abs}(\text{Fact} \ v)
\]

is an equivalence. Note that this holds if the class of right-adapted objects in \( \mathcal{A} \) satisfies the appropriate set of hypotheses of Proposition 2.22. We define the right absolutely-derived factored functor of \( \theta \) to be the composition

\[
L^{\text{abs}} \theta^f : D^\text{abs}(\text{Fact} \ v) \xrightarrow{Q^{-1}_{\mathcal{A}dp\theta}} D^\text{abs}(\mathcal{A}dp\theta, w) \xrightarrow{\theta^f} D^\text{abs}(\text{Fact} \ v).
\]

We shall often drop the superscript on the derived functors as context allows.

**Remark 4.4.** Another method for deriving functors on factorizations categories is presented in [Eh13]. This already made an appearance in the proof of Proposition 2.22. The general case of co-derived categories and a full set of details can be found in [Eh13, Appendix A]. It is clear that the two approaches yield isomorphic functors.

**Lemma 4.5.** Given two factored functors,

\[
\theta : \mathcal{A} \to \text{Ch}^b(\mathcal{B})
\]

and

\[
\gamma : \mathcal{B} \to \text{Ch}^b(\mathcal{C}),
\]

the functor

\[
\text{tot}(\gamma \circ \theta) : \mathcal{A} \to \text{Ch}^b(\mathcal{C})
\]

\[A \mapsto \text{tot}(\gamma(\theta(A)))\]

where \( \text{tot} \) denotes the total complex of a double complex over \( \mathcal{C} \), is factored.

**Proof.** This is immediate from the definitions and the way that totalization extends the autoequivalences and natural transformations. \( \square \)

**Corollary 4.6.** Consider three triples, \((\mathcal{A}, \Phi, w), (\mathcal{B}, \Psi, v)\) and \((\mathcal{C}, \Upsilon, u)\), and let

\[
\theta : \mathcal{A} \to \text{Ch}^b(\mathcal{B})
\]

\[
\gamma : \mathcal{B} \to \text{Ch}^b(\mathcal{C})
\]
be left exact factored functors. Suppose that there exists a class of right adapted objects for $\theta$ whose images form a class of right adapted objects for $\gamma$. Then, one has a natural isomorphism of derived factored functors
\[ R\gamma^f \circ R\theta^f \cong R(\text{tot}(\gamma \circ \theta))^f \]
whenever both sides exist.

If we replace the assumption that $\theta$ and $\gamma$ are left exact with the assumption that they are right exact and we suppose that there exists a class of left adapted objects for $\theta$ whose images form a class of left adapted objects for $\gamma$, then we have a natural isomorphism
\[ L\gamma^f \circ L\theta^f \cong L(\text{tot}(\gamma \circ \theta))^f \]
whenever both sides exist.

Proof. We can use $\theta$-adapted factorizations whose image under $\theta$ are $\gamma$-adapted. Plugging these in we are left with checking that
\[ \gamma^f \circ \theta^f \cong (\text{tot}(\gamma \circ \theta))^f. \]
The difference between the two sides is the order of totalization, which does not matter up to isomorphism. \qed

Definition 4.7. Let $\theta, \gamma : A \to \text{Ch}^b(B)$ be factored functors, factoring $w : \text{Id}_A \to \Phi$ and $v : \text{Id}_B \to \Psi$. A factored natural transformation is a natural transformation
\[ \eta : \theta \to \gamma \]
such that
\[ \Psi(\eta_A) = \eta\Phi(A) \]
for any $A \in A$.

Lemma 4.8. Let $\eta : \theta \to \gamma$ be a factored natural transformation. Then there is a natural transformation
\[ \eta^f : \theta^f \to \gamma^f \]
\[ \eta^f_E := (\eta_{E^{-1}}, \eta_{E^0}). \]

If $\theta$ and $\gamma$ are left exact and there exists a class of objects in $A$ that is simultaneously right $\theta$-adapted and right $\gamma$-adapted, then there is an induced natural transformations on derived factored functors
\[ R\eta^f : R\theta^f \to R\gamma^f \]
if both exist.

If $\theta$ and $\gamma$ are right exact and there exists a class of objects in $A$ that is simultaneously left $\theta$-adapted and left $\gamma$-adapted, then there is an induced natural transformations on derived factored functors
\[ L\eta^f : L\theta^f \to L\gamma^f \]
if both exist.

Proof. It suffices to check that we have a natural transformation
\[ \eta^f : \theta^f \to \gamma^f \]
\[ \eta^f_E := (\eta_{E^{-1}}, \eta_{E^0}) \]
as the natural transformation between the derived functors will come from restriction to jointly adapted factorizations.

One easily checks that the dashed arrows in the following diagram can be filled in with the natural morphisms

\[ \eta_{\Phi^{-1}(E^0)}, \eta_{E^{-1}}, \eta_{E^0}, \eta_{\Phi^{-1}(F^0)}, \eta_{F^{-1}}, \eta_{F^0}. \]

The requirement for the diagram to commute is equivalent to \( \eta \) being factored.

Commutativity of this diagram exactly means that these morphisms define a natural transformation with the components lying in \( \text{Ch}^b(B) \). Since \( \text{tot} \) is a functor, we may apply it to the entire commutative diagram, to obtain such a diagram in \( B \). The commutativity of the induced diagram in \( B \) is equivalent to verifying that these maps induce a natural transformation.

\[ \Box \]

**Definition 4.9.** Let \( \theta : A \to B \) be a left exact functor. If there exists a class of objects right adapted to \( \theta \), we have a derived functor

\[ \mathbf{R}\theta : D^+(A) \to D^+(B) \]

In the case, the codomain of \( \theta \) is chain complexes instead, \( \theta : A \to \text{Ch}^b(B) \), and assuming \( B \) admits countable coproducts, we have functor

\[ D^+(A) \to D^+(\text{Ch}^b(B)) \]

which we compose with totalization

\[ D^+(\text{Ch}^b(B)) \to D^+(B) \]

and denote by

\[ \mathbf{R}\theta : D^+(A) \to D^+(B). \]
An analogous definition exists when $\theta$ is right exact and $\mathcal{A}$ admits a class of objects left adapted to $\theta$

$$L\theta : D^-(\mathcal{A}) \to D^-(\mathcal{B}).$$

A similar variant exists for bounded derived categories if finite resolutions exists.

**Lemma 4.10.**

- Assume that $\mathcal{A}$ has enough injectives and that coproducts of injectives remain injective. Let

  $\theta, \gamma : \mathcal{A} \to \text{Ch}^b(\mathcal{B})$

be left exact and commute with coproducts and let

$$\eta : \theta \to \gamma$$

be a factored natural transformation which induces a natural isomorphism

$$R\theta \cong R\gamma : D^+(\mathcal{A}) \to D^+(\mathcal{B}),$$

then $R\eta^f$ induces a natural isomorphism,

$$R\theta^f \cong R\gamma^f : D^{\text{co}}(\text{Fact } w) \to D^{\text{co}}(\text{Fact } v).$$

- Assume that $\mathcal{A}$ has enough projectives and that products of projectives remain projectives. Let

  $\theta, \gamma : \mathcal{A} \to \text{Ch}^b(\mathcal{B})$

be right exact and commute with products and let

$$\eta : \theta \to \gamma$$

be a factored natural transformation which induces a natural isomorphism

$$L\theta \cong L\gamma : D^- (\mathcal{A}) \to D^- (\mathcal{B}),$$

then $L\eta^f$ induces a natural isomorphism,

$$L\theta^f \cong L\gamma^f : D^{\text{ctr}}(\text{Fact } w) \to D^{\text{ctr}}(\text{Fact } v).$$

- Assume that each object of $\mathcal{A}$ admits a finite injective resolution. Let

  $\theta, \gamma : \mathcal{A} \to \text{Ch}^b(\mathcal{B})$

be left exact and let

$$\eta : \theta \to \gamma$$

be a factored natural transformation which induces a natural isomorphism

$$R\theta \cong R\gamma : D^b(\mathcal{A}) \to D^b(\mathcal{B})$$

then $R\eta^f$ induces a natural isomorphism,

$$R\theta^f \cong R\gamma^f : D^{\text{abs}}(\text{Fact } w) \to D^{\text{abs}}(\text{Fact } v).$$

- Assume that each object of $\mathcal{A}$ admits a finite projective resolution. Let

  $\theta, \gamma : \mathcal{A} \to \text{Ch}^b(\mathcal{B})$

be right exact and let

$$\eta : \theta \to \gamma$$

be a factored natural transformation which induces a natural isomorphism

$$L\theta \cong L\gamma : D^b(\mathcal{A}) \to D^b(\mathcal{B})$$
then $L\eta^f$ induces a natural isomorphism,

$$L\theta^f \cong L\gamma^f : \text{D}^{\text{abs}}(\text{Fact } w) \to \text{D}^{\text{abs}}(\text{Fact } v)$$

**Proof.** We prove the first statement as the rest are extremely similar and follow from analogous arguments. By Theorem 3.8, we may restrict our attention to factorizations $I$ folding pairs of bounded below injective complexes

$$0 \to I_0^{-1} \to I_1^{-1} \to \cdots$$

Applying $\eta : \theta \to \gamma$ to each of these complexes yields a quasi-isomorphism by assumption. Thus, the two complexes $\text{Cone}(\eta^{-1}), \text{Cone}(\eta_{\bullet})$ are acyclic in the usual sense, they have no homology. Since the components of $\eta_{p,q}$ vanish for $p \neq q$, we can apply Lemma 3.7 to conclude that the cone over

$$\eta^f_I : \theta^I(I) \to \gamma^I(I)$$

folds the two complexes $\text{Cone}(\eta^{-1}), \text{Cone}(\eta_{\bullet})$. By Lemma 3.5, $\text{Cone}(\eta^f_I)$ is co-acyclic hence $\eta^f_I$ is a co-quasi-isomorphism.

We can also address fully-faithfulness.

**Lemma 4.11.** Let $\theta : \mathcal{A} \to \mathcal{B}$ be a left exact factored functor. Assume that each object of $\mathcal{B}$ admits a finite injective resolution. Let $E$ and $F$ be objects of $\text{Fact}(w)$ whose components have finite injective resolutions. Assume $R\theta$ exists. If the map,

$$R\theta : \text{Hom}_{\mathcal{D}^+(\mathcal{A})}(E^i, F^j[t]) \to \text{Hom}_{\mathcal{D}^+(\mathcal{B})}(R\theta(E^i), R\theta(F^j)[t]),$$

is an isomorphism for all $i, j, t \in \mathbb{Z}$, then the map,

$$R\theta^f : \text{Hom}_{\mathcal{D}^+(\text{Fact } w)}(E, F[t]) \to \text{Hom}_{\mathcal{D}^+(\text{Fact } v)}(R\theta^f(E), R\theta^f(F)[t]),$$

is an isomorphism for all $t \in \mathbb{Z}$.

The proof of Lemma 4.11 will be a direct result of studying a spectral sequence associated to a filtration on morphism complexes, $\text{Hom}^*$. Before presenting it, let us recall how one can compute the maps,

$$R\theta : \text{Hom}_{\mathcal{D}^+(\mathcal{A})}(A^i, A) \to \text{Hom}_{\mathcal{D}^+(\mathcal{B})}(R\theta(A^i), R\theta(A)),$$

on the ordinary derived categories.

Let $C, D$ be chain complexes from $\mathcal{A}$. We have the chain complex,

$$\text{Hom}^n_C(A, D) = \prod_{j-i=n} \text{Hom}_A(C^i, D^j),$$

with $d(\prod_i g^i : C^i \to D^{i+n}) := \prod_i (d^{i+n+1}_D \circ g^i - (-1)^n g^{i+1} \circ d^{i+1}_C)$.

First, we choose injective resolutions,

$$0 \to A' \xrightarrow{d'_0} I'_0 \xrightarrow{d'_1} I'_1 \xrightarrow{d'_2} \cdots$$

$$0 \to A \xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} \cdots.$$
Next, we construct a commutative diagram of bounded complexes,

\[
\begin{array}{cccccc}
\theta(I_0) & \theta(d_1) & \theta(I_1) & \theta(d_2) & \theta(I_2) & \theta(d_3) & \theta(I_3) & \theta(d_4) & \cdots \\
0 & d^0_{0,0} & J_0,0 & d^0_{1,0} & J_1,0 & d^0_{2,0} & J_2,0 & d^0_{3,0} & J_3,0 & d^0_{4,0} & \cdots \\
0 & d^1_{0,1} & J_0,1 & d^1_{1,1} & J_1,1 & d^1_{2,1} & J_2,1 & d^1_{3,1} & J_3,1 & d^1_{4,1} & \cdots \\
0 & d^2_{0,2} & J_0,2 & d^2_{1,2} & J_1,2 & d^2_{2,2} & J_2,2 & d^2_{3,2} & J_3,2 & d^2_{4,2} & \cdots \\
0 & d^3_{0,3} & J_0,3 & d^3_{1,3} & J_1,3 & d^3_{2,3} & J_2,3 & d^3_{3,3} & J_3,3 & \cdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\]

where the rows and columns are exact complexes of bounded complexes, all squares commute, and each \( J_{p,q} \) is injective. Form the associated total complex, \( J = (J_*, d^{\text{tot}}_{*}) \), where

\[
J_u = \bigoplus_{s+t=u} J_{s,t}
\]

with the differential,

\[
d^{\text{tot}}_{u+1} : J_u \to J_{u+1},
\]

being the product of the maps,

\[
J_{s,t} \xrightarrow{d^{s+1}_{s+1,t} \oplus (-1)^r d^{r}_{s,t+1}} J_{s+1,t} \oplus J_{s,t+1}.
\]

This comes with a map of chain complexes, \( \theta(I) \to J \).

The map,

\[
R\theta : \text{Hom}_{D^+(A')}(A', A[t]) \to \text{Hom}_{D^+(B')}(R\theta(A'), R\theta(A)[t]),
\]

is the cohomology in degree \( t \) of the map of chain complexes,

\[
\text{Hom}^*_A(I', I) \xrightarrow{\theta} \text{Hom}^*_B(\text{tot}(\theta(I'), \text{tot}(\theta(I)))) \to \text{Hom}^*_B(\text{tot}(\theta(I'), J)).
\]

With this recap fresh in our mind, let us proceed with the proof of Lemma 4.11.

Proof of Lemma 4.11. Choose finite injective resolutions of the components,

\[
0 \to E^{-1} \xrightarrow{d_0^{-1}} I_0^{-1} \xrightarrow{d_1^{-1}} I_1^{-1} \xrightarrow{d_2^{-1}} I_2^{-1} \xrightarrow{d_3^{-1}} \cdots
\]

\[
0 \to E^0 \xrightarrow{d_0^0} I_0^0 \xrightarrow{d_1^0} I_1^0 \xrightarrow{d_2^0} I_2^0 \xrightarrow{d_3^0} \cdots
\]

\[
0 \to F^{-1} \xrightarrow{d_0^{-1}} I_0^{-1} \xrightarrow{d_1^{-1}} I_1^{-1} \xrightarrow{d_2^{-1}} I_2^{-1} \xrightarrow{d_3^{-1}} \cdots
\]

\[
0 \to F^0 \xrightarrow{d_0^0} I_0^0 \xrightarrow{d_1^0} I_1^0 \xrightarrow{d_2^0} I_2^0 \xrightarrow{d_3^0} \cdots,
\]
and apply Theorem 3.8 to get resolutions of factorizations,

\[ E \to I^E \]
\[ F \to I^F. \]

Recall that the components of \( I^F \) are

\[
(I^F)^{-1} = \bigoplus_{2l} \Phi^{-l}(I_{2l}^{-1}) \oplus \bigoplus_{2l+1} \Phi^{-l-1}(I_{2l+1}^{0})
\]
\[
(I^F)^{0} = \bigoplus_{2l} \Phi^{-l}(I_{2l}^{0}) \oplus \bigoplus_{2l+1} \Phi^{-l-1}(I_{2l+1}^{-1}).
\]

Applying \( \theta^f \), we get the factorization, \( \theta^f(I^F) \), whose components are

\[
\theta^f(I^F)^{-1} = \bigoplus_{2l} \theta(\Phi^{-l}(I_{2l}^{-1})) \oplus \bigoplus_{2l+1} \theta(\Phi^{-l-1}(I_{2l+1}^{0}))
\]
\[
\theta^f(I^F)^{0} = \bigoplus_{2l} \theta(\Phi^{-l}(I_{2l}^{0})) \oplus \bigoplus_{2l+1} \theta(\Phi^{-l-1}(I_{2l+1}^{-1})).
\]

We want to replace \( \theta^f(I^F) \) by an injective factorization to compute \( R\theta^f \). We will apply Theorem 3.8, but, first, we need to choose injective resolutions of the components of \( \theta^f(I^F) \).

To do this, we first construct finite diagrams (which exist by assumption),

\[
\begin{array}{c}
\theta(I_{0}^{-1}) \xrightarrow{\theta(d_{0}^{F^{-1}})} \theta(I_{1}^{-1}) \xrightarrow{\theta(d_{2}^{F^{-1}})} \theta(I_{2}^{-1}) \xrightarrow{\theta(d_{3}^{F^{-1}})} \ldots \\
J_{0,0}^{F^{-1}} \xrightarrow{d_{0,0}^{F^{-1}}} J_{1,0}^{F^{-1}} \xrightarrow{d_{1,0}^{F^{-1}}} J_{2,0}^{F^{-1}} \xrightarrow{d_{2,0}^{F^{-1}}} J_{3,0}^{F^{-1}} \xrightarrow{d_{3,0}^{F^{-1}}} \ldots \\
J_{0,1}^{F^{-1}} \xrightarrow{d_{0,1}^{F^{-1}}} J_{1,1}^{F^{-1}} \xrightarrow{d_{1,1}^{F^{-1}}} J_{2,1}^{F^{-1}} \xrightarrow{d_{2,1}^{F^{-1}}} J_{3,1}^{F^{-1}} \xrightarrow{d_{3,1}^{F^{-1}}} \ldots \\
J_{0,2}^{F^{-1}} \xrightarrow{d_{0,2}^{F^{-1}}} J_{1,2}^{F^{-1}} \xrightarrow{d_{1,2}^{F^{-1}}} J_{2,2}^{F^{-1}} \xrightarrow{d_{2,2}^{F^{-1}}} \ldots \\
\end{array}
\]

and

\[
\begin{array}{c}
\theta(I_{0}^{0}) \xrightarrow{\theta(d_{0}^{F^{0}})} \theta(I_{1}^{0}) \xrightarrow{\theta(d_{2}^{F^{0}})} \theta(I_{2}^{0}) \xrightarrow{\theta(d_{3}^{F^{0}})} \ldots \\
J_{0,0}^{F^{0}} \xrightarrow{d_{0,0}^{F^{0}}} J_{1,0}^{F^{0}} \xrightarrow{d_{1,0}^{F^{0}}} J_{2,0}^{F^{0}} \xrightarrow{d_{2,0}^{F^{0}}} J_{3,0}^{F^{0}} \xrightarrow{d_{3,0}^{F^{0}}} \ldots \\
J_{0,1}^{F^{0}} \xrightarrow{d_{0,1}^{F^{0}}} J_{1,1}^{F^{0}} \xrightarrow{d_{1,1}^{F^{0}}} J_{2,1}^{F^{0}} \xrightarrow{d_{2,1}^{F^{0}}} J_{3,1}^{F^{0}} \xrightarrow{d_{3,1}^{F^{0}}} \ldots \\
J_{0,2}^{F^{0}} \xrightarrow{d_{0,2}^{F^{0}}} J_{1,2}^{F^{0}} \xrightarrow{d_{1,2}^{F^{0}}} J_{2,2}^{F^{0}} \xrightarrow{d_{2,2}^{F^{0}}} \ldots \\
\end{array}
\]
where the rows and columns are exact, all $J$’s are bounded complexes of injectives, and all squares commute.

Then, we use the injective resolutions,

$$0 \to \theta^f(I^F)^{-1} \oplus_{2l} \Psi^{-1}(d^{F-1}_{0,2l}) \oplus \oplus_{2l+1} \Psi^{-1}(d^{F-1}_{0,2l+1}) \oplus \oplus_{2l} \Psi^{-l-1}(J^{F-1}_{0,2l}) \oplus \oplus_{2l+1} \Psi^{-l-1}(J^{F-1}_{0,2l+1}) \to \cdots$$

and

$$0 \to \theta^f(I^F)^{0} \oplus_{2l} \Psi^{-1}(d^{F-1}_{0,2l}) \oplus \oplus_{2l+1} \Psi^{-1}(d^{F-1}_{0,2l+1}) \oplus \oplus_{2l} \Psi^{-l-1}(J^{F-1}_{0,2l}) \oplus \oplus_{2l+1} \Psi^{-l-1}(J^{F-1}_{0,2l+1}) \to \cdots$$

totalize and apply Theorem 3.8. Denote the resulting factorization by $J$. Note that the components of $J$ are

$$J^{-1} = \bigoplus_{s+t=2l} \Psi^{-l}(J^{F-1}_{s,t}) \oplus \bigoplus_{s+t=2l+1} \Psi^{-l-1}(J^{F-1}_{s,t})$$

$$J^{0} = \bigoplus_{s+t=2l} \Psi^{-l}(J^{F0}_{s,t}) \oplus \bigoplus_{s+t=2l+1} \Psi^{-l}(J^{F0}_{s,t}).$$
The chain complex, $\text{Hom}_v^*(\theta^J(I^E), J)$, admits a filtration,

$$\mathcal{F}^p \text{Hom}_v^n(\theta^J(I^E), J) := \{(g^{-1}, g^{0}) | \forall q$$

$$g^{-1} \left( \bigoplus_{2l \leq q-1} \Psi^{-l}(\theta(I_2^{-1})) \oplus \bigoplus_{u=2l+1 \leq q-1} \Psi^{-l-1}(\theta(I_u^{0})) \right)$$

$$\subseteq \bigoplus_{s+t=2l \leq q+p+n-1} \Psi^{-m-l}(J_{s,t}) \oplus \bigoplus_{s+t=2l+1 \leq q+p+n-1} \Psi^{-m-l-1}(J_{s,t})$$

$$g^{0} \left( \bigoplus_{2l \leq q} \Psi^{-l}(\theta(I_{2l}^{0})) \oplus \bigoplus_{2l+1 \leq q} \Psi^{-l}(\theta(I_{2l+1}^{0})) \right)$$

$$\subseteq \bigoplus_{s+t=2l \leq q+p+n} \Psi^{-m-l}(J_{s,t}^{0}) \oplus \bigoplus_{s+t=2l+1 \leq q+p+n} \Psi^{-m-l-1}(J_{s,t}^{0}) \right)$$

$$n = 2m$$

$$g^{-1} \left( \bigoplus_{2l \leq q-1} \Psi^{-l}(\theta(I_2^{-1})) \oplus \bigoplus_{2l+1 \leq q-1} \Psi^{-l-1}(\theta(I_2^{0})) \right)$$

$$\subseteq \bigoplus_{s+t=2l \leq q+p+n-1} \Psi^{-m-l}(J_{s,t}^{0}) \oplus \bigoplus_{s+t=2l+1 \leq q+p+n-1} \Psi^{-m-l}(J_{s,t}^{1})$$

$$g^{0} \left( \bigoplus_{2l \leq q} \Psi^{-l}(\theta(I_{2l}^{0})) \oplus \bigoplus_{2l+1 \leq q} \Psi^{-l}(\theta(I_{2l+1}^{0})) \right)$$

$$\subseteq \bigoplus_{s+t=2l \leq q+p+n} \Psi^{-m-l+1}(J_{s,t}^{1}) \oplus \bigoplus_{s+t=2l+1 \leq q+p+n} \Psi^{-m-l}(J_{s,t}^{1}) \right)$$

$$n = 2m + 1$$

After recombining even and odd parts, the associated graded complex is

$$\text{Gr}^p \text{Hom}_v^n(\theta^J(I^E), J) =$$

$$\begin{cases} 
\text{Hom}_B^{p+n}(\theta(I^E), \Psi^{-q}(J^F)) \oplus \text{Hom}_B^{p+n}(\theta(I^{E-1}), \Psi^{-q}(J^{F-1})) & p = 2q \\
\text{Hom}_B^{p+n}(\theta(I^E), \Psi^{-q}(J^{F-1})) \oplus \text{Hom}_B^{p+n}(\theta(I^{E-1}), \Psi^{-q}(J^{F}))) & p = 2q + 1 
\end{cases}$$

with the differential being the sum of the differentials on $\text{Hom}_B^*(\text{tot} \theta(I^E), \Psi^{-q}(J^F))$, $a, b \in \{-1, 0\}$.

There exists an analogous filtration on $\text{Hom}_v^*(I^E, I^F)$ whose associated graded complex is

$$\begin{cases} 
\text{Hom}_A^{p+n}(I^E, \Phi^{-q}(I^F)) \oplus \text{Hom}_A^{p+n}(I^{E-1}, \Phi^{-q}(I^{F-1})) & p = 2q \\
\text{Hom}_A^{p+n}(I^E, \Phi^{-q}(I^{F-1})) \oplus \text{Hom}_A^{p+n}(I^{E-1}, \Phi^{-q}(I^{F}))) & p = 2q + 1 
\end{cases}$$

These filtrations are compatible with the map,

$$\text{Hom}_w^*(I^E, I^F) \rightarrow \text{Hom}_v^*(\theta^J(I^E), J).$$

The map on the associated graded complexes,

$$\text{Gr}^p \text{Hom}_w^*(I^E, I^F) \rightarrow \text{Gr}^p \text{Hom}_v^*(\theta^J(I^E), J),$$

is exactly the sum of the maps,

$$\text{Hom}_A^*(I^E, \Phi^{-q}(I^F)) \rightarrow \text{Hom}_B^*(\theta(I^E), \Psi^{-q}(J^F)))$$

$$\text{Hom}_A^*(I^E, \Phi^{-q}(I^{F-1})) \rightarrow \text{Hom}_B^*(\theta(I^E), \Psi^{-q}(J^{F-1}))$$

$$\text{Hom}_A^*(I^{E-1}, \Phi^{-q}(I^F)) \rightarrow \text{Hom}_B^*(\theta(I^{E-1}), \Psi^{-q}(J^F)))$$

$$\text{Hom}_A^*(I^{E-1}, \Phi^{-q}(I^{F-1})) \rightarrow \text{Hom}_B^*(\theta(I^{E-1}), \Psi^{-q}(J^{F-1})))$$

which we have assumed to be quasi-isomorphisms. Since the terms in the $E_1$-page are cohomologies of the respective associated graded complexes, it follows that the corresponding map of spectral sequences is an isomorphism on the $E_2$-page. Since the injective resolutions are assumed to be finite, the spectral sequence degenerates and yields the desired statement.
Lemma 4.12. Let \( \theta : A \to B \) be a right exact factored functor. Assume that each object of \( B \) admits a finite projective resolution. Let \( E \) and \( F \) be objects of \( \text{Fact}(w) \) whose components admit finite projective resolutions. Assume \( L \theta \) exists. If the map

\[
L \theta : \text{Hom}_{D^c(A)}(E, F[t]) \to \text{Hom}_{D^c(B)}(L \theta E, L \theta F[t]),
\]

is an isomorphism for all \( t \in \mathbb{Z} \), then the map

\[
L \theta : \text{Hom}_{D^c(A)}(E^i, F^j[t]) \to \text{Hom}_{D^c(B)}(L \theta E^i, L \theta F^j[t]),
\]

is an isomorphism for all \( i, j, t \in \mathbb{Z} \).

Proof. The proof of this lemma is completely analogous to the proof of Lemma 4.11. \( \square \)

5. Geometric Applications

In this section we apply our results to smooth varieties/stacks. Recall that quasi-coherent sheaves on such spaces have finite injective resolutions.

5.1. Complete Intersections as matrix factorizations over noncommutative algebras. Let \( X \) be a smooth algebraic variety over \( k \), \( E \) be a finite rank locally-free sheaf on \( X \) and consider the associated geometric vector bundle

\[
V(E) := \text{Spec}(\text{Sym}^* E)
\]

together with the fiberwise scaling action by \( \mathbb{G}_m \). Suppose that \( V(E) \) admits a \( \mathbb{G}_m \)-equivariant tilting sheaf \( T \). Then \( A := \text{End}(T) \) is a \( \mathbb{Z} \)-graded algebra. Let \( \text{Mod}_Z A \) be the Abelian category of graded right modules over \( A \). Let \( (1) : \text{Mod}_Z A \to \text{Mod}_Z A \) be the autoequivalence given by shifting the grading of a module

\[
M(1)_j := M_{j+1}.
\]

Now, let \( s \in H^0(X, E) \) be a global section and \( Z \) be the zero locus of \( s \) and consider the equivariant line bundle \( \mathcal{O}(1) \) obtained by twisting \( \mathcal{O} \) by a character of weight 1. Then \( s \) gives a map \( T \to T \otimes \mathcal{O}(1) \) and hence an element of \( A \). We define a natural transformation

\[
v : \text{Id} \to (1)
\]

given by the action of this element of \( A \) on modules.

Theorem 5.1. Assume that the codimension of \( Z \) equals the rank of \( E \) and that \( A \) is (right) Noetherian. There is an equivalence of triangulated categories

\[
D^b(\text{coh} Z) \cong K(\text{Fact} \text{Proj}, v).
\]

Proof. The functor

\[
\text{Hom}(T, -) : \text{Qcoh}_{\mathbb{G}_m} V(E) \to \text{Mod}_Z A
\]

is a left exact factored functor which induces an equivalence

\[
D^b(\text{Qcoh}_{\mathbb{G}_m} V(E)) \to D^b(\text{Mod}_Z A).
\]

The inverse is also induced by a right exact factored functor

\[
(- \otimes T) : \text{Mod}_Z A \to \text{Qcoh}_{\mathbb{G}_m} V(E).
\]
Using the triples \((\text{Qcoh}_{\mathbb{G}_m} V(\mathcal{E}), (\mathcal{-} \otimes \mathcal{O}(1)), s)\) and \((\text{Mod}_Z(A), (1), v)\) we obtain the categories of factorizations (see Example 2.1). Applying Lemma 4.11 to \(\text{Hom}(T, -)\), we see that

\[
\mathcal{R}\text{Hom}(T, -)^f : \text{D}^{\text{abs}}(\text{Fact s}) \to \text{D}^{\text{abs}}(\text{Fact v})
\]

is fully-faithful. Similarly, applying Lemma 4.12 to \((- \otimes T)^f\) shows that

\[
\mathcal{L}(- \otimes T)^f : \text{D}^{\text{abs}}(\text{Fact v}) \to \text{D}^{\text{abs}}(\text{Fact s})
\]

is fully-faithful.

Consider the composition,

\[
\mathcal{R}\text{Hom}(T, -)^f \circ \mathcal{L}(- \otimes T)^f : \text{D}^{\text{abs}}(\text{Fact s}) \to \text{D}^{\text{abs}}(\text{Fact v}).
\]

Replacing \(T\) by a complex of injectives in \((- \otimes T)^f\) and any argument of \((- \otimes T)^f\) by one with projective components, we reduce to considering

\[
\text{Hom}(T, -)^f \circ (- \otimes T)^f : \text{K}(\text{Fact Proj}, v) \to \text{K}(\text{Fact Proj}, v)
\]

which is isomorphic to the identity by Lemma 4.10. Thus, \(\mathcal{R}\text{Hom}(T, -)^f\) is both fully-faithful and essentially surjective, hence an equivalence.

Since we have assumed that \(A\) is Noetherian, the following functors are well-defined

\[
\mathcal{R}\text{Hom}(T, -)^f : \text{D}^{\text{abs}}(\text{Fact coh}_{\mathbb{G}_m} V(\mathcal{E}), s) \to \text{D}^{\text{abs}}(\text{Fact mod}_Z A, v)
\]

\[
\mathcal{L}(- \otimes T)^f : \text{D}^{\text{abs}}(\text{Fact mod}_Z A, v) \to \text{D}^{\text{abs}}(\text{Fact coh}_{\mathbb{G}_m} V(\mathcal{E}), s).
\]

Using a variation on \([\text{EF}14, \text{Proposition 1.5.c}]\), we can identify the factorization category with coherent objects \(\text{D}^{\text{abs}}(\text{Fact coh}_{\mathbb{G}_m} V(\mathcal{E}), s)\) with the full subcategory of the factorization category \(\text{D}^{\text{abs}}(\text{Fact Qcoh}_{\mathbb{G}_m} V(\mathcal{E}), s)\) consisting of objects which are isomorphic to factorizations with coherent components. Similarly, via \([\text{Pos}09, \text{Theorem 3.11.1}]\), we identify the category \(\text{D}^{\text{abs}}(\text{Fact mod}_Z A, v)\) with the full subcategory of \(\text{D}^{\text{abs}}(\text{Fact Mod}_Z A, v)\) consisting of objects which are isomorphic to objects with finitely-generated components. The proof of this fact is analogous to the proof of Proposition 2.19.

Under these identifications, the equivalence \(\mathcal{R}\text{Hom}(T, -)^f\) restricts to an equivalence,

\[
\mathcal{R}\text{Hom}(T, -)^f : \text{D}^{\text{abs}}(\text{Fact coh}_{\mathbb{G}_m} V(\mathcal{E}), s) \to \text{D}^{\text{abs}}(\text{Fact mod}_Z A, v).
\]

Finally, we have equivalences

\[
\text{D}^b(\text{coh } Z) \cong \text{D}^{\text{abs}}(\text{Fact coh}_{\mathbb{G}_m} V(\mathcal{E}), s) \\
\cong \text{D}^{\text{abs}}(\text{Fact mod}_Z A, v) \\
\cong \text{K}(\text{Fact Proj}, v).
\]

The first line is precisely \([\text{Shi}12, \text{Theorem 3.4}]\) or equivalently \([\text{Isi}13, \text{Theorem 3.6}]\). The second line was established above. The final line is Corollary 2.25.

**Example 5.2.** Let \(X = \mathbb{P}^n\) and \(\mathcal{E} = \mathcal{O}(d)\). Then \(Z\) is a hypersurface defined by \(s\), a homogeneous polynomial of degree \(d\). Let \(\pi : V(\mathcal{O}(d)) \to \mathbb{P}^n\) be the projection. We may consider the tilting object

\[
T = \pi^*(\mathcal{O} \oplus \cdots \oplus \mathcal{O}(n)).
\]
Let $R := k[x_0, \ldots, x_n]$ and denote by $R_m$ homogeneous polynomials of degree $m$. Then one easily verifies that

$$A = \bigoplus_{0 \leq i, j \leq n} \bigoplus_{t=0}^{\infty} R_{td+i-j}$$

with the algebra structure given by

$$rr' = \begin{cases} rr' & \text{if } i' = j \\ 0 & \text{otherwise} \end{cases} \in R_{(t+t')d+i-j'}$$

where $r \in R_{td+i-j}, r' \in R_{t'd+i'-j'}$. Furthermore, an element of $R_{td+i-j}$ has degree $t$. For $0 \leq h \leq n$, our homogeneous polynomial $s$ gives an element $v_h$ in the summand $R_d$ corresponding to $i = j = h$. Then,

$$v := \sum_{h=1}^{n} v_h$$

is a central element of $A$ of degree 1. We may consider $v$ as a natural transformation $\text{Id} \rightarrow (1)$ where $(1)$ denotes the grading shift by 1. We get an equivalence

$$\text{D}^b(\text{coh } Z) \cong K(\text{Fact Proj}, v).$$

The right hand side is the same as graded matrix factorizations of $(A, v)$ as defined in [Orl09].

**Remark 5.3.** It may be of interest to compare this equivalence to the one found in [Orl09].

### 5.2. Integral transforms

In this section we use our results to recover a theorem of Baranovsky and Pecharich [BP10].

**Definition 5.4.** Let $X$ and $Y$ be smooth algebraic stacks with $P \in \text{D}($Qcoh $X \times Y)$. Denote the two projections by,

$$\pi_X : X \times Y \rightarrow X \quad \text{and} \quad \pi_Y : X \times Y \rightarrow Y.$$ 

The induced integral transform is the functor,

$$\Phi_P := R\pi_{Y*} \circ (- \otimes P) \circ \pi_X^* : \text{D}(\text{Qcoh } X) \rightarrow \text{D}(\text{Qcoh } Y)$$

The object $P$ is called the kernel of the transform $\Phi_P$.

Given two kernels $P \in \text{D}(\text{Qcoh } X \times Y), Q \in \text{D}(\text{Qcoh } Y \times Z)$ we define the convolution to be

$$P \ast Q := R\pi_{ZX*}(\pi_{XY}^* P \otimes \pi_{YZ}^* Q).$$

It is a standard fact that $\Phi_Q \circ \Phi_P$ is naturally isomorphic to $\Phi_{P \ast Q}$.

Let $w : X \rightarrow \mathbb{A}^1$ and $v : Y \rightarrow \mathbb{A}^1$ be morphisms.

**Definition 5.5.** A morphism $f : X \rightarrow Y$ is called factored with respect to $w, v$ if $w = v \circ f$.

**Lemma 5.6.** If $f : X \rightarrow Y$ is factored then $f_* : \text{Qcoh } X \rightarrow \text{Qcoh } Y$ and $f^* : \text{Qcoh } Y \rightarrow \text{Qcoh } X$ are factored with respect to the natural transformations $w : \text{Id}_{\text{Qcoh } X} \rightarrow \text{Id}_{\text{Qcoh } X}$ and $v : \text{Id}_{\text{Qcoh } Y} \rightarrow \text{Id}_{\text{Qcoh } Y}$.

**Proof.** This follows immediately from the definitions. \qed
Example 5.7. Let \( w : X \to \mathbb{A}^1 \) and \( v : Y \to \mathbb{A}^1 \) be morphisms. Then we get a morphism via composition \( X \times_{\mathbb{A}^1} Y \to X \to \mathbb{A}^1 \). Note this equals the composition \( X \times_{\mathbb{A}^1} Y \to Y \to \mathbb{A}^1 \) by definition of the fiber product. Then the natural projections from \( X \times_{\mathbb{A}^1} Y \) to \( X \) and \( Y \) are factored.

Definition 5.8. Let \( w : X \to \mathbb{A}^1 \) and \( v : Y \to \mathbb{A}^1 \) be morphisms. Let \( \pi_X, \pi_Y \) be the projections of \( X \times Y \) onto \( X, Y \) respectively and let \( P \in \text{Fact}(X \times Y, -w \boxplus v) \). We then get a functor
\[
- \otimes P : \text{Fact}(\text{Qcoh } X \times Y, w) \to \text{Fact}(\text{Qcoh } X \times Y, v)
\]
which is right exact, see e.g. \([\text{BFK11, Definition } 3.22]\). We denote the left (absolutely) derived functor as \((- \otimes P)\).

The factored integral transform is the functor
\[
\Phi_P := R\pi_Y^* \circ \left(- \otimes P\right) \circ L\pi_X^* : \text{D}^{\text{abs}}(\text{Fact } w) \to \text{D}^{\text{abs}}(\text{Fact } v).
\]
We are allowed to use absolute derived categories in place of co-derived categories as we have assumed that \( X \) and \( Y \) are smooth.

Remark 5.9. One simple way to make such a \( P \) is to take an object of \( \text{D}^b(\text{Qcoh } X \times \mathbb{A}^1 Y) \), view it as complex of factorizations of \(-w \boxplus v\) by pushing forward and taking zero component maps, and totalize it. Therefore, for such a \( P \), we will often use \( \Phi_P \) to denote the integral transform associated in this manner.

Proposition 5.10 (Base extension for factorizations). Consider a Cartesian square of factored morphisms
\[
\begin{array}{ccc}
Z & \xrightarrow{u'} & Y \\
\downarrow{v'} & & \downarrow{v} \\
X & \xrightarrow{u} & W
\end{array}
\]
Assume \( u \) is flat. Then there is a natural isomorphism between the composition of derived factored functors
\[
(u^*)^f \circ Rv_*^f \cong R(v'_*)^f \circ (u'^*)^f
\]
Proof. Recall that the usual statement of flat base change states that we have a natural morphism
\[
u^* \circ v_* \to v'_* \circ u^*.
\]
which induces a natural isomorphism
\[
u^* \circ Rv_* \cong Rv'_* \circ u'^*.
\]
Consequently, we see that the image of injectives under \( u'^* \) is \( v'_* \)-adapted. So,
\[
R(v'_* \circ u'^*) = Rv'_* \circ u'^*.
\]
Tautologically,
\[
R(u^* \circ v_*) = u^* \circ Rv_*.
\]
Thus, we are in exactly the situation of Lemma 4.10. \(\square\)

We will next need to prove a version of the projection formula.
Proposition 5.11 (Projection formula for factorizations). Let \( g : X \to Y \) be a factored morphism and \( P \in D^{ab}(\text{Qcoh} \, X, g^*v') \) for another potential \( v' : Y \to \mathbb{A}^1 \). There is a natural isomorphism between the composition of derived factored functors

\[
Rg^f_* \circ (P \otimes -) \circ L(g^*)^f \cong (Rg^f_* P \otimes -)^f.
\]

**Proof.** First, recall that the usual projection formula for \( g \) gives a factored natural transformation

\[
v_{E,F} : g_* (E \otimes g^*F) \to g_! E \otimes F
\]

which is an isomorphism whenever \( E \) is quasi-coherent and \( F \) is locally-free. Using the induced natural transformation on factorizations with \( E \) a factorization with injective components and \( F \) a locally-free factorization yields the desired projection formula. \( \square \)

**Remark 5.12.** We thank the referee for the general form of Proposition 5.11 and pointing that a similar fact for Grothendieck duality is proved in [Efi13, Theorem B.5].

Proposition 5.13. Let \( P \in D^{ab}(\text{Qcoh} \, X \times Y, -w \boxplus v) \), \( Q \in D^{ab}(\text{Qcoh} \, Y \times Z, -v \boxplus u) \) and assume that \( w : X \to \mathbb{A}^1 \) and \( v : Y \to \mathbb{A}^1 \) are flat. One has a natural isomorphism

\[
\Phi_Q^f \circ \Phi_P^f \cong \Phi_{P*Q}^f.
\]

where \( P \ast Q = R\pi_{XZ}^f(\pi_{YZ}^f Q \otimes \pi_{XY}^f P) \) for the projections from \( X \times Y \times Z \).

**Proof.** For notational simplicity, we assume \( E, P, \) and \( Q \) are factorizations with locally-free components. This is permissible by Proposition 2.22. We have natural isomorphisms

\[
\Phi_Q^f \circ \Phi_P^f (E) = R\pi_Z^f(Q \otimes \pi_Y^f (R\pi_Y^f (P \otimes \pi_X^f E)))
\]

\[
\cong R\pi_Z^f(Q \otimes R\pi_{YZ}^f \pi_{XY}^* P \otimes \pi_X^f E))
\]

\[
\cong R\pi_Z^f(R\pi_{YZ}^f (\pi_{YZ}^f Q \otimes \pi_{XY}^f P \otimes \pi_X^f E))
\]

\[
\cong R\pi_Z^f(R\pi_{XZ}^f (\pi_{XY}^f P \otimes \pi_X^f \pi_X^f E))
\]

\[
\cong R\pi_Z^f(R\pi_{XZ}^f (\pi_{YZ}^f Q \otimes \pi_{XY}^f P \otimes \pi_X^f E))
\]

\[
\cong R\pi_{XZ}^f (\pi_{YX}^f Q \otimes \pi_{XY}^f P \otimes \pi_X^f E))
\]

The first line is by definition. The second line is Proposition 5.10. The third line uses Proposition 5.11. The fourth line uses the fact that

\[
R\pi_Z^f \circ R\pi_{YZ}^f \cong R((\pi_Z \circ \pi_{YZ})^f)
\]

\[
\cong R((\pi_Z \circ \pi_{XZ})^f)
\]

\[
\cong R\pi_Z^f \circ R\pi_{XZ}^f
\]

where the first and third lines are Corollary 4.6 and the second line is just a natural isomorphism of functors at the Abelian level. The fifth line uses the isomorphism of functors at the Abelian level

\[
\pi_{XY}(- \otimes -) \cong \pi_{XY}^*(-) \otimes \pi_{XY}^*(-)
\]
which induces corresponding isomorphisms for factorizations. The sixth line uses the isomorphism of functors at the Abelian level

\[ \pi^*_{XZ} \circ \pi^*_X \cong \pi^*_{XY} \circ \pi^*_X \]

which induces corresponding isomorphisms for factorizations. The seventh line is Proposition 5.11 again. The rest is by definition. \(\square\)

**Lemma 5.14.** There is a natural isomorphism

\[ \Phi^f_{\mathcal{O}_\Delta} \cong \text{Id}. \]

**Proof.** Note that \(-w \boxplus w\) vanishes on \(\mathcal{O}_\Delta\) and so \((0, \mathcal{O}_\Delta, 0, 0)\) is a factorization. As in the above proof, we again let \(E\) be a factorization with locally-free components. We have natural isomorphisms

\[ R\pi_2^*(\mathcal{O}_\Delta \otimes \mathcal{O}_{X \times X} \pi_1^*E) \cong R\pi_2^*(\Delta^f \Delta^* \pi_1^*E) \cong R(\pi_2^* \circ \Delta^*)^f \pi_1^*E \cong E. \]

The first line is Proposition 5.11. The second line is from the isomorphism of functors \((\mathcal{O}_\Delta \otimes \mathcal{O}_{X \times X} \cong \Delta^* \circ \Delta^*\).

The third line is the isomorphism of functors

\[ \pi_{2*} \circ \Delta^* \cong \text{Id} \]

and

\[ \Delta^* \circ \pi_1^* \cong \text{Id}. \]

\(\square\)

As an application of these techniques, we recover a result of Baranovsky and Pecharich [BP10, Theorem 1.1].

**Theorem 5.15 (Baranovsky, Pecharich).** Let \(P \in \mathcal{D}(\text{Qcoh } X \times \mathbb{A}^1 Y)\) and assume \(\Phi_{i_! P}\) is an equivalence where \(i : X \times \mathbb{A}^1 Y \to X \times Y\) is the natural inclusion. Then \(\Phi^f_{i_! P}\) is an equivalence.

**Proof.** By general argument using the calculus of kernels, the functor \(\Phi_{i_! P}\) has a right adjoint \(\Phi_{i_* Q}\) where

\[ Q := R\mathcal{H}om(P, \mathcal{O}_{X \times \mathbb{A}^1 Y}) \otimes \pi^*_X \omega_X[\dim X]. \]

so that

\[ i_* Q \cong R\mathcal{H}om(i_* P, \mathcal{O}_{X \times Y}) \otimes \pi^*_X \omega_X[\dim X]. \]

Thus,

\[ \Phi_{i_* Q} \circ \Phi_{i_* P} \cong \Phi_{i_* Q i_* P} \cong \text{Id}. \]

More precisely, there exists a morphism \(\alpha : \mathcal{O}_\Delta \to i_* Q \ast i_* P\) lifting the unit of adjunction such that

\[ \Phi_\alpha : \Phi_{\mathcal{O}_\Delta} \to \Phi_{i_* Q i_* P} \]

is an isomorphism. This induces an isomorphism

\[ \Phi^f_\alpha : \Phi^f_{\mathcal{O}_\Delta} \to \Phi^f_{i_* Q i_* P} \]
given by applying $\alpha$ component-wise and folding. Now, from Proposition 5.13 there is natural isomorphism,

$$\Phi_{i_Q}^f \circ \Phi_{i_P}^f \cong \Phi_{i_Q+i_P}^f.$$ 

Finally, we get

$$\Phi_{i_\Delta}^f \cong \text{Id}$$

using Lemma 5.14. Using the same argument switching the roles of $P$ and $Q$, gives the result. \[\square\]

**Remark 5.16.** We thank the referee for providing the following perspective. Integral transforms give all continuous enhanced $\mathbb{Z}/2$-graded triangulated functors from $D^\text{co}(\text{Fact} \, w)$ to $D^\text{co}(\text{Fact} \, v)$ [LP11, Pre11]. This is very much analogous to the case of usual derived functors [To¨oe07]. From this point of view, Theorem 5.15 may be considered as part of the following 2-categorical structure. Take the 2-category $C_1$ with objects being pairs $(X, w: X \to \mathbb{A}^1)$, where $X$ is a smooth algebraic variety and $w$ is a function. The category of 1-morphisms from $(X, w)$ to $(Y, v)$ in $C_1$ is $D(\text{Qcoh} \, X \times_{\mathbb{A}^1} Y)$, with composition defined in the standard way. Further, take the 2-category $C_2$ with the same objects, such that the category of 1-morphisms from $(X, w)$ to $(Y, v)$ in $C_2$ is $D^\text{co}(\pi_Y^* v - \pi_X^* w)$. Then we have a 2-functor from $C_1$ to $C_2$ which is identity on objects, and such that on 1-morphisms it is just the direct image functor for the factored closed embedding $(X \times_{\mathbb{A}^1} Y, 0) \to (X \times Y, \pi_Y^* v - \pi_X^* w)$.

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