Three "quantum" models of competition and cooperation in interacting biological populations and social groups

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In present paper we propose the consistent statistical approach which appropriate for a number of models describing both behavior of biological populations and various social groups interacting with each other. The approach proposed based on the ideas of quantum theory of open systems (QTOS) and allows one to account explicitly both discreteness of a system variables and their fluctuations near mean values. Therefore this approach can be applied also for the description of small populations where standard dynamical methods are failed. We study in detail three typical models of interaction between populations and groups: 1) antagonistic struggle between two populations 2) cooperation (or, more precisely, obligatory mutualism) between two species 3) the formation of coalition between two feeble groups in their conflict with third one that is more powerful. The models considered in a sense are mutually complementary and include the most types of interaction between populations and groups. Besides this method can be generalized on the case of more complex models in statistical physics and also in ecology, sociology and other "soft" sciences.

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I. INTRODUCTION

Among various classical open systems of interest in physics and also in so-called "soft" sciences such as ecology, sociology, economics and so on, the important role is played by the systems whose states in accordance with the sense of the problem are specified by the set of integers determined by a concrete problem.

In statistical physics $n_i$ may represent occupation numbers of various cells in phase space of the nonideal gas, in ecology - numbers of individuals in distinct populations living in the area and interacting with each other, in economics the number of different companies operating on the market at the same time and so on. As far as states of such systems change with time they are described as a rule by a set of autonomous differential equations of the next form:

$$\frac{dn_i}{dt} = F_i\{n_\alpha\}, \quad (1)$$

where $F_i\{n_\alpha\}$ - some nonlinear state depending functions determined by a concrete problem.

Obviously, that notation of Eq. (1) assumes that all variables $n_i$ - are continuous quantities. In the case when all $n_i \gg 1$ such "smooth" approximation of discrete system is quite reasonable. But for small occupation numbers when $n_i \gtrsim 1$, the dynamical approach of Eq. (1) becomes inapplicable and it is necessary to take into account both as discreteness of variables $n_i$ and their possible fluctuations near mean values. The statistical method proposed in the present paper has undoubted advantage compared with dynamic approach Eq. (1), since it completely free from mentioned restrictions.

The method proposed is based on application of the quantum Lindblad master equation (LME) for density matrix (DM) evolution of a open quantum system. The LME in general case has the next form (1), (2):

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_i [R_i\rho, R_i^\dagger] + h.c., \quad (2)$$

where $H$ is some hermitian operator, describing the inner dynamics of open system and $\{R_i\}$ are a set of nonhermitian operators that simulate different types of interaction of open system in question with its environment).

Although Eq. (2) has obviously quantum origin, nevertheless, under the certain conditions which we will discuss more detail later in this paper, it may be applied also to the study of many classical open systems, for example for the systems with discrete variables. The main advantage of the LME compared with other similar master equations lies in the fact that the number and form of operators $R_i$ entering in Eq. (2) for many concrete problems can be defined from simple heuristic reasons. As soon as such choice is made the consistent mathematical framework for the description of the problem is at our disposal.

Since the method proposed is to some extent heuristic, we demonstrate its reasonableness and effectiveness on the examples of different typical models which equally can be applied to ecology for the description of behavior of interacting populations and in sociology for studying dynamics of cooperating or competing groups. Although in the present paper we specially restricted our choice only the simplest examples, which admit the complete qualitative analysis, it is clear that the method proposed can be generalized on more complex cases without any difficulties of principle. The rest of the paper is organized...
as follows. In Sect.2 we present the main features of approach proposed, and discuss all conditions and restrictions for its application. In Sect.3 we study the model of the struggle for existence between two antagonistic biological populations in the case when total size of both populations remains invariable. In Sect.4 we consider the model of cooperation (or more precisely the model of obligatory mutualism) between two distinct populations or social groups. In Sect.5 we study somewhat more complicated model of three groups interaction in situation when two feeble groups join together to confront successfully more powerful rival. In conclusion we are summing up all results obtained in the paper.

Now let us turn to the concrete presentation of the paper.

II. DESCRIPTION OF THE METHOD

As we already note the mathematical framework of the method proposed is the Lindblad master equation (2) that is used in QTOS for the description of open quantum Markov systems. Although this equation has undoubtedly quantum origin under certain conditions it can be applied also for statistical description of some classical open systems.

This significant point should be explained more detail. First of all note that density matrix (DM) of quantum system apart from quantum correlations containes also exhaustive information about its classical correlations. We believe that information about classical correlations is recorded mainly in diagonal elements of DM. Let us assume now that the open system of interest is such that the relevant LME implies closed and complete set of equations connecting only diagonal elements of its DM. Clearly, in this case the LME could serve as appropriate framework for the statistical description of classical analog of corresponding quantum system. Although similar situation is realized for several classes of open classical systems, but in the present paper we are interested in only special such class, namely classical systems, whose states can be specified by integer variables \( n_i \). Let us describe this class more explicitly. Our first and main assumption is that "hamiltonian term" in the r.h.s. of the LME (2) is missing.

Besides we demand that all operators \( R_i \) in Eq. (2) have monomial form that is: \( R_i = C_i \prod_{\alpha,\beta} (a_\alpha^+)^{k_{i,\alpha}} (a_\beta)^{l_{i,\beta}} \) (in general case \( C_i \) can be considered as functions of occupation numbers , but in what follows they are assumed to be constants). In such situation one can directly verify that all equations for diagonal elements of \( \rho(n_1,\ldots,n_N) \), (where \( n_i = a_i^+ a_i \) and \( a_i, a_i^+ \) are usual Bose-operators, that is \( [a_i, a_j^+] = \delta_{jk}, \)) form complete set of differential-difference equations, which give self-consistent description of the system. It should be note also that with assumptions made the LME (2) actually reduced to the specific form of Pauli master equation (PME) (3), namely:

\[
\frac{d\rho_n}{dt} = \sum_{n_1} (W_{nn_1}\rho_{n_1} - W_{n_1n}\rho_n).
\]

We could suggest also that for the "good enough" form of transition probabilities \( W_{nn_1} \), entering in the Eq. (4) one can find an appropriate set of operators \( R_i \) of monomial form such that the LME (2) (without "hamiltonian term") and the PME (3) actually will coincide. Now to complete the description of the method we need to specify how one must choose a set of operators \( \{ R_i \} \) for a concrete problem of interest. The quality of the method proposed in our opinion lies exactly in this point because this choice can be done according to simple heuristic considerations. Namely, we believe the number and form of a set \( \{ R_i \} \) are entirely determined by the condition what types of transitions \( W_{nn_1} \), one want to take into account for the concrete problem. The best way to illustrate all features of the method proposed is to apply it to study various concrete models. Let us now turn to this matter.

III. ANTAGONISTIC STRUGGLE BETWEEN TWO COMPETING POPULATIONS

As the first example that illustrates all features of the method proposed we consider the model of antagonistic opposition between two populations or social groups struggling with each other for certain resources or some other preferences. In most naked form such struggle is realized when two tribes or kins of cannibals living side by side in the area in a literal sense eat each other (but with unequal voracity). Assume for the simplicity that the total number of individuals in both populations remains constant in this struggle. As regards to sociology it could be for example the struggle between two political parties competing at the parliament elections when the total number of vacant seats is fixed. It is clear that in such situation the benefit for one group necessarily means the failure for the other and vice versa. Using the language of game theory, one can say that we consider statistical model of two person game with zero sum (but from nonstandard point of view). Let us turn now to the explicit mathematical formulation of the model. We assume that our model of antagonistic struggle can be properly described by the help of the LME with two operators: \( R_1 = \sqrt{a_1^+ a_2} \) and \( R_2 = \sqrt{a_2^+ a_1} \) (where operator \( R_1 \) corresponds to events when first population or group benefits and the second fails, and \( R_2 \) respectively to the opposite one, coefficients \( a \) and \( b \) reflect corresponding competitiveness of both groups). After these principal assumptions remainder of our analysis is completely rigorous and can be represented as the sequence of three consecutive steps.

Step1. We write down the LME for the DM of the system, that characterize total correlations between two
groups of individuals with antagonistic interaction. The

\[ \frac{dp}{dt} = a (a_1^+ a_2 \rho a_2^+ a_1 - a_2^+ a_1 a_1^+ \rho a_2) + b (a_2^+ a_1 \rho a_1^+ a_2 - a_1^+ a_2 \rho a_2^+ a_1). \] (4)

One can easily see that Eq. (4) implies the closed set of equations for diagonal elements of DM that describes

\[ \frac{d\rho_{n_1 n_2}}{dt} = a \left[ n_1 \pi_{2 n_2} \rho_{n_1 n_2} - n_2 \pi_{n_1} \rho_{n_1 n_2} \right] + b \left[ n_2 \pi_{n_2} \rho_{n_1 n_2} - n_1 \pi_{2 n_1} \rho_{n_1 n_2} \right]. \] (5)

(note that we use the notation \( \pi \equiv n + 1 \), and \( \nu \equiv n - 1 \) to reduce the length of formula [5]).

Step2. It is convinient to represent the difference-differential system of equations in the form of equivalent differential equation. Such representation can be obtained with the help of the generation function for the distribution \( \rho_{n_1 n_2} \). According to definition, the generating function (GF) \( G(u, v, t) \) is a consequence of two reasons: 1) the linearity of Eq. (6) and 2) the existence of conservation law, namely, one can easily see that total number \( G(1, 1, t) \equiv 1 \). All moments of distribution \( \rho_{n_1 n_2} \) can be found by differentiating \( G(u, v, t) \). For example:

\[ \pi_1 = \frac{\partial G}{\partial u} \bigg|_{u=v=1}, \pi_1 = \frac{\partial G}{\partial v} \bigg|_{v=u=1} \text{and so on.} \]

One can see that Eq. (4) implies the next equivalent equation for GF \( G(u, v, t) \):

\[ \frac{\partial G}{\partial t} = a (u - v) \frac{\partial^2 (u G)}{\partial u \partial v} + b (v - u) \frac{\partial^2 (v G)}{\partial u \partial v}. \] (6)

Let us show that Eq. (5) admits an exhaustive qualitative analysis because its general solution can be represented in the form of the decomposition:

\[ G(u, v, t) = \sum_N K_N G_N (u, v, t), \] (7)

where \( G_N (u, v, t) = A_0 u^N + A_1 u^{N-1} v + ... A_N v^N \) is normalized homogeneous polynomial of degree \( N \) in variables \( u \) and \( v \); and \( \{K_N\} \) is a set of constants satisfying the condition : \( \sum_N K_N = 1 \). The capability of decomposition Eq. (7) is a consequence of two reasons: 1) the linearity of Eq. (6) and 2) the existence of conservation law, namely, one can easily see that total number of individuals in both populations \( N = \pi_1 + \pi_2 \) remains constant.

Thus we conclude that for the study of general dynamics of Eq. (i) enough to consider it in each of the subspaces with fixed \( N \), where this dynamics can be reduced to the simple system of linear equations for the coefficients \( A_i \) of homogeneous polynomial \( G_N \):

\[ \frac{dA_i}{dt} = L_{ik} A_k. \] (8)

In the Eq. (8) \( L_{ik} \) is some nonhermitian \( N + 1 \times N + 1 \) matrix which can be easily calculated from Eq. (9) for any concrete \( N \). For example in the simplest case \( N = 2 \) matrix \( L_{ik} \) has the form:

\[ \begin{pmatrix} -2b & 2a & 0 \\ 2b & -(2a + 2b) & 2a \\ 0 & 2b & -2a \end{pmatrix}. \]

Having in hands decomposition (4) we are able to answer on the main question: how the time evolution of arbitrary initial distribution \( G_0 (u, v) = G_N (u, v, t = 0) \) is happened? To answer this question let us begin with the end and find the stationary solutions of Eq. (8). Equating the r.h.s. of (8) to zero we obtain that:

\[ G^{st} (u, v) = \frac{A(u) + B(v)}{au - bv}. \] (9)

To obtain required polynomial form of \( G (u, v, t) \) we must choose \( A_N (u) \) as \( C_N (au)^{N+1} \) and \( B (v) \) as \( -C_N (bv)^{N+1} \) after that Eq. (10) leads to the next result:

\[ G_N^{st} (u, v) = C_N \left[(au)^N + (au)^{N-1} (bv) + ... (bv)^N\right], \] (10)

( where \( C_N = a^N + a^{N-1} b + ... b^N \) is normalization constant). With the help of Eq. (10) for the GF one can find all moments that is all statistical characteristics of the system of interest in its stationary states. In this connection we would like to mention one curious result.Let \( N \) is total (and conserved) number of individual in both populations and assumed that \( a \geq b \). Then with the help of (10) one can show that when \( t \) tends to infinity, the fraction \( \pi_{n_1} \equiv n_1 \) tends to zero, that is less voracious population gradually disappears.

This conclusion is completely coincides with result obtained from dynamical approach of Eq. (i), since for
$N \gg 1$, the role of discreteness of variables $n_i$ and their fluctuations becomes negligible. On the other hand the expression (10) is valid also in the case of small populations where dynamical approach is not applicable. For example when $N = 2$ GF is equal to: $G_2 = \frac{a^2v^2 + abu + b^2v^2}{a^2 + ab + b^2}$ and we can find that $\overline{n}_1 = \frac{2a^2 + ab + b^2}{a^2 + ab + b^2}$; $\overline{n}_2 = \frac{2b^2 + ab + b^2}{a^2 + ab + b^2}$. Going over to the study of time evolution of arbitrary initial distribution $G_0(u,v)$ we propose the following prescription: using Eq. (10) one must expand $G_0$ in homogeneous polynomials $G_N$, thus determining a set of $K_N$, and after that we can immediately write down the definitive result, namely $G(u,v,\infty) = \sum_N K_N G_N^t (u,v)$. In what follows we will call states with GF $G_N$ the pure states.

Note also that using the Euler theorem about homogeneous functions: $u \frac{\partial G}{\partial u} + v \frac{\partial G}{\partial v} = N G_N$, one can easily obtain for the pure states two simple relations:

\[ \overline{n}_1^2 + \overline{n}_1 \overline{n}_2 = \overline{n}_1 \] and \[ \overline{n}_2^2 + \overline{n}_1 \overline{n}_2 = \overline{n}_2. \] (11)

If we define by standard way the variance of the number of individuals in every population as $\sigma_i = \overline{n}_i^2 - (\overline{n}_i)^2$ (where $i=1,2$) and correlation coefficient $k = \frac{n_{12} - \overline{n}_1 \overline{n}_2}{\sqrt{n_1 \sigma_1 \sqrt{n_2 \sigma_2}}}$ then the Eq. (11) implies that $\sigma_1 = \sigma_2 \equiv \sigma = \overline{n}_2 - \overline{n}_1 \overline{n}_2$ and hence for any values $a$ and $b$ correlation coefficient $k$ in any pure state is equal to $-1$. This result exactly justifies the name for considered model as antagonistic model of interaction between two populations or groups.

### IV. COOPERATION BETWEEN TWO POPULATIONS OR SOCIAL GROUPS

In this part we consider the model of interaction between two groups of individuals in which both populations either together get the benefit in the struggle for existence or under adverse circumstances are failing together. In biology such kind of interaction is known as symbiosis (or more precisely as obligatory mutualism). As the simplest example of such interaction we can specify the alliance between hermit- crab and actinium, that is sea coral.[4] Actinium attached to the shell in which crab lives, and then moves with it and eats the remains of its food. On the other hand actinium protects the crab from its enemies with special cells located in its tentacles. Thus neither actinium neither hermit -crab cannot successfully exist without each other. As regards to sociology the number of distinct groups that share common goals and (or) values and cooperates with each other to achieve them are truly unlimited. To formulate the mathematical model describing the case of two groups obligatory cooperation we start again from Lindblad equation (2), but now to accordance with the sense of the problem as a collection of operators $R_k$ we choose two operators: $R_1 = \sqrt{\frac{\sigma}{2}} a_1^+ a_2^+$ and $R_2 = \sqrt{\frac{\sigma}{2}} a_1^+ a_2^-$. The first operator describes the events in which both groups get the benefit while the second one is responsible for the common failure. The required Lindblad equation for diagonal elements of density matrix of the system $\rho_{n_1 n_2}$ in this case takes the form:

\[ \frac{\partial \rho_{n_1 n_2}}{\partial t} = a \left[ n_1 n_2 \rho_{n_1 n_2} - \overline{n}_1 \cdot n_2 \cdot \rho_{n_1 n_2} \right] + b \left[ n_1 \cdot n_2 \cdot \rho_{n_1 n_2} - n_1 n_2 \rho_{n_1 n_2} \right]. \] (12)

(In the Eq. (12) as before we used the notation $\overline{n}_i = n_i + 1$ and $n_i = n_i - 1$). It is easy to verify that difference-differential equation (12) by introducing the generating function $G(u,v,t) = \sum_{n_1 n_2} \rho_{n_1 n_2} u^{n_1} v^{n_2}$ can be transformed to the equivalent form of single differential equation, namely:

\[ \frac{\partial G}{\partial t} = (1 - uv) \frac{\partial^2}{\partial u \partial v} \left[ (b - auv) G \right]. \] (13)

Note that Eq. (13) as well as Eq. (10) admits the exhaustive qualitative analysis, due to linearity and the presence of conservation law. We begin our analysis of Eq. (13) with finding its stationary solutions. Equating the r.h.s. of Eq. (13) to zero, we obtain that a set of stationary solutions of Eq. (13) can be represented in the form:

\[ G_{st} (u,v) = \frac{A(u) + B(v)}{b - auv}, \] (14)

where functions $A(u)$ and $B(v)$ yet to be determined. As the second step let us write down equations of motion for the occupation numbers mean values of both populations. By differentiating Eq. (13) with respect to $u$
and \(v\) respectively and then putting \(u = v = 1\) we obtain:

\[
\frac{\partial \mu_1}{\partial t} = \frac{\partial \mu_2}{\partial t} = a(n_1 + n_2 + 1) + (a - b)n_1n_2.
\]

(15)

In what follows we consider the case \(b \geq a\). The Eq. (15) implies that \(n_1 - n_2 = N\) is the integral of motion for Eq. (13). Further we will consider \(N\) as a fixed integer (positive, zero or negative). Let us prove now that the dynamics of Eq. (13) can be considered as superposition of independent dynamics for each \(N\) respectively. Indeed, let \(N\) is some positive integer and take as generating function the normalized function of the form:

\[G_N(u, v, t) = u^Nf(uv, t)\] (where \(f(1, t) = 1\)). It is easy to see that for such \(GF\) \(\mu_1 - \mu_2 = N\) for arbitrary \(f(x, t) \equiv f(uv, t)\). Substituting this expression for \(GF\) in Eq. (13) after simple algebra we obtain the closed equation of evolution for the function \(f(x, t)\):

\[
\frac{\partial f(x, t)}{\partial t} = (1 - x) \left[ (N + 1) \frac{\partial}{\partial x} (b - ax) f + x \frac{\partial^2}{\partial x^2} (b - ax) f \right].
\]

(16)

Let us prove now that \(f(x, t)\) tends to \(\frac{C}{b - ax}\) when \(t\) tends to infinity. For this purpose we introduce auxiliary function \(h(x, t) = (b - ax) f(x, t)\). Then Eq. (13) implies the next equation for function \(h(x, t)\):

\[
\frac{\partial h}{\partial t} = (1 - x) (b - ax) \left[ (N + 1) \frac{\partial h}{\partial x} + x \frac{\partial^2 h}{\partial x^2} \right] = - \frac{(1 - x) (b - ax) \delta F[h(x)]}{\delta h} x^N.
\]

(17)

where \(F[h(x)]\) is the functional of \(h(x)\) which has the form:

\[
F[h(x)] = \frac{1}{2} \int_0^x N + 1 \frac{\partial h}{\partial x} dx;
\]

The Eq. (17) implies that \(\frac{\partial F}{\partial h} \mid_{x=1} = 0\) and hence when \(t\) tends to infinity functional \(F[h(x, t)]\) tends to zero and therefore \(h(x, t)\) tends to certain constant value (remind that we assume \(b \geq a\)). QED.

It is clear that all above arguments remain valid also in the case when integer \(N \leq 0\), if one takes as generating function:

\[G_N = v^Nf(uv, t).\]

Thus we reach the required conclusion: the general solution of Eq. (13) admits the decomposition:

\[G(u, v, t) = \sum_{N=\infty}^{N=\infty} K_N G_N(u, v, t),\]

where \(K_N G_N(u, v, t)\) is the distribution function of the form \(u^N f_0(uv)\) such GF we again connect with certain state (transformation to the distribution \(u^N f_0(uv)\)). Similar result is valid for the initial GF component of the form \(u^N f_0(uv)\). Thus qualitative description of evolution for arbitrary initial \(GF\) is completed. In conclusion of this part we present two notable results relating to evolution of pure states in this model. Result 1. Let \(G_0 = u^N f(uv)\). We have just proved that at any time \(G(u, v, t)\) has the same form and hence satisfies to the equation:

\[
\frac{\partial G}{\partial t} = - v \frac{\partial G}{\partial t} = NG.
\]

(18)

The Eq. (18) implies relations which are similar to the relations (11) for foregoing model, namely:

\[
\frac{\mu_1 - \mu_2}{\mu_1 - \mu_2} = N, \quad \frac{\mu_1 - \mu_2}{\mu_1 - \mu_2} = N\mu_1, \quad \mu_2 - \mu_1 \mu_2 = N\mu_2,
\]

(19)

from which we are able to obtain expressions for variations of statistical distributions of \(n_1\) and \(n_2\), namely:

\[
s_1 = n_1^2 - \langle n_1 \rangle^2 = \sigma_2 = n_2^2 - \langle n_2 \rangle^2 = n_1 n_2 - \langle n_1 \rangle \langle n_2 \rangle\]

and hence correlation coefficient \(k\) between two populations 1 and 2 is:

\[k_{12} = \frac{n_1 n_2 - \langle n_1 \rangle \langle n_2 \rangle}{\sqrt{n_1^2 - \langle n_1 \rangle^2} \sqrt{n_2^2 - \langle n_2 \rangle^2}} = 1.\]

Thus we come to the conclusion that all pure states in this model are maximally correlated.

Result 2. Let us again assume that initial pure state has \(GF\) of the form:

\[G_0 = u^N f_0(uv)\]. Then mean occupation numbers of two populations are:

\[\mu_1 = N + \frac{\partial f_0}{\partial v} \mid_{x=1}\] and \(\mu_2 = \frac{\partial f_0}{\partial u} \mid_{x=1}\).

When \(t\) tends to infinity then \(G\) tends to \(G^t = \frac{(b-a)u^N}{\mu_1 - \mu_2}\) and occupation numbers in stationary state are:

\[\mu_1 = N + \frac{\alpha}{b-a} \quad \text{and} \quad \mu_2 = \frac{\alpha}{b-a}\].

Let \(\frac{\partial f_0}{\partial u} < 1\) then initial number \(\mu_2\) is very small, but in situation when ratio \(\propto \frac{\alpha}{b-a}\) tends to one, the number of individuals in both populations increases with time without any bound. However, the difference between occupation numbers \(\mu_1\) and \(\mu_2\) remains constant as before.
V. THE FORMATION OF COALITION BETWEEN TWO POPULATIONS OR SOCIAL GROUPS IN THE STRUGGLE AGAINST COMMON RIVAL

In this part we consider more complex model of interaction between three distinct populations which is, in a sense, the superposition of two foregoing models.

We will focus on the situation when two feeble groups of individuals enter into an alliance to resist together the common rival. Thus in this model on the one hand there is a cooperation between two feeble groups but on the other hand there is mutual antagonism in relation to common enemy. Unfortunately, I find it difficult to bring vivid examples of such coalition in biology (although they are apparently exist). On the other hand it is clearly that in social, political and military conflicts there are the great number of forming such coalitions. However, I include this example in the present paper mainly to demonstrate how one can construct more and more complex models based on simple ones.

Let us now turn to the explicit mathematical formulation of this model. As before we are starting from the Lindblad equation [2] but this time we choose as the set of \( R_i \) the next two operators: \( R_1 = \sqrt{\frac{1}{2}a_1^+ a_2 a_3} \) and \( R_2 = \sqrt{\frac{1}{2}a_1 a_2^+ a_3^+} \). We believe that operator \( R_1 \) describes events when more stronger population with index 1 wins in the conflict and respectively the coalition formed from two populations with indices 2 and 3 fail, while the operator \( R_2 \) takes into account the contrary events. The Lindblad equation for the diagonal elements of density matrix \( \rho_{n_1 n_2 n_3} \) in this case takes the form:

\[
\frac{\partial \rho_{n_1 n_2 n_3}}{\partial t} = a \left[ n_1 \cdot \bar{n}_2 \cdot \bar{n}_3 \rho_{n_1 n_2 n_3} \right] - n_1 \cdot n_2 \cdot n_3 \rho_{n_1 n_2 n_3} \\
+ b \left[ \bar{n}_1 \cdot n_2 \cdot n_3 \rho_{n_1 n_2 n_3} \right] - n_1 \cdot \bar{n}_2 \cdot \bar{n}_3 \rho_{n_1 n_2 n_3} \quad \text{(20)}
\]

(we again use in Eq. [20] the notation: \( \bar{n}_i = n_i + 1 \) and \( n_i = n_i - 1 \).

As before we convert the system of difference-differential equations Eq. [20] to equivalent differential equation for generation function: \( G(u, v, w, t) = \sum_{n_1, n_2, n_3} \rho_{n_1 n_2 n_3} u^{n_1} v^{n_2} w^{n_3} \). Required equation for GF in this model reads as:

\[
\frac{\partial G}{\partial t} = (u - v w) \frac{\partial^3}{\partial u \partial v \partial w} \left[ (au - b w) G \right]. \quad \text{(21)}
\]

\[
\begin{align*}
\frac{\partial n_1}{\partial t} &= a n_2 n_3 (n_1 + 1) - b n_1 (n_2 + 1) (n_3 + 1) \\
\frac{\partial n_2}{\partial t} &= \frac{\partial n_3}{\partial t} = -a n_2 n_3 (n_1 + 1) + b n_1 (n_2 + 1) (n_3 + 1) \quad \text{(22)}
\end{align*}
\]

One can see that Eq. [22] imply the existence of two integrals of motion, that can be written as: 1) \( \frac{\partial n_1}{\partial t} + \frac{\partial n_2}{\partial t} = N + p \) and 2) \( \frac{\partial n_1}{\partial t} + \frac{\partial n_2}{\partial t} = N \). We will consider further only the case when \( N \) and \( p \) are positive integers. The presence of these integrals allows one to give complete qualitative description of evolution [21] for any initial generating function \( G_0 (u, v, w, t = 0) \). To realize this intention we firstly find the stationary solutions of Eq. [21]. Equating its r.h.s to zero we obtain required result:

\[
G^{st} (u, v, w, \,) = \frac{A (u, v) + B (u, w) + C (v, w)}{au - b w}, \quad \text{(23)}
\]

(4) where functions \( A (u, v), B (u, w), C (v, w) \) must be determined in such a way to obtain for \( G^{st} \) some polynomial expansion. It is easy to see that after relevant choice we can rewrite Eq. [23] in the required polynomial form:

\[
G^{st} (u, v, w) = [K (v) + L (w)] \left[ (au)^N + (au)^{N-1} (bw) + \ldots (bw)^N \right], \quad \text{(24)}
\]
(where polynomials \( K(v), L(w) \) are determined by initial \( \text{GF} \)). Now we will seek a complete set of pure states with self-closed dynamics, on which arbitrary generating function can be expanded. To this end let us consider the generating function in the form: \( G_p = v^p f(u, vw, t) \). Substituting this expression in Eq. (21) after simple algebra we obtain the closed evolution equation for the function \( f(u, x, t) \equiv f(u, vw, t) \), namely:

\[
\frac{\partial f}{\partial t} = (u - x) \frac{\partial}{\partial u} \left\{ x \frac{d^2}{dx^2} [(au - bx) f] + (p + 1) \frac{d}{dx} [(au - bx) f] \right\}.
\]

(25)

It is easy to see that the general solution of Eq. (25) can be represented as some linear superposition of homogeneous polynomials of \( N \) degree in variables \( x \) and \( u \), that is \( f(u, x, t) = \sum_n P_n(u, x, t) \).

Besides it is clear, if we take the generation function in the form \( G_p = v^p g(u, vw) \) we come to the similar result. Now we are able to claim that the general solution of Eq. (21) can be expanded on these two classes of pure states. We can write this decisive for the further study of this model result as:

\[
G(u, v, w, t) = \sum_{N, p} \left[ v^p P_N(u, vw) + w^p R_N(u, vw) \right].
\]

(26)

(\( \text{in Eq. (26) summation is over all integers } N \text{ and } p, \) \( \text{and } P_N(u, x), R_N(u, x) \text{ are homogeneous polynomials of } N \text{ degree in } u \text{ and } x \equiv vw \text{ depending on } p \text{ as well}. \))

The result obtained allows one to restrict the study of the Eq. (21) to the case of pure states evolution only. So, let us consider the pure state which continually has the form:

\[
G_{pN}(u, v, w, t) = v^p R_N(u, vw, t) \text{ (where } R_N \text{ is homogeneous polynomial of } N \text{ degree)}. \]

We specify here only two notable properties of such states in this model.

1) Cooperating groups being in pure states are fully correlated, while conflicting groups are fully anticorrelated. To prove this result note that pure states in question satisfy to the equations:

\[
u \frac{\partial G_{pN}}{\partial u} + w \frac{\partial G_{pN}}{\partial w} = NG_{pN},
\]

(27)

and

\[
u \frac{\partial G_{pN}}{\partial v} - w \frac{\partial G_{pN}}{\partial w} = pG_{pN}.
\]

(28)

The Eq. (27) implies following relations: a) \( \overline{\mu_1 \mu_1} = N \), and b) \( \overline{\mu_1 \mu_1} + \overline{\mu_1} = N \), from which one can find the variance of first group population, namely:

\[
\sigma_1 = \overline{\mu_1^2} - (\overline{\mu_1})^2 = \overline{\mu_1} - (\overline{\mu_1})^2 = \overline{\mu_1 \mu_1} - (\overline{\mu_1})^2 - \overline{\mu_1} \overline{\mu_1}.
\]

(\( \text{Same result one can obtain for the variance } \sigma_3 \text{ and hence correlation coefficient } k_{13} = \overline{\mu_1 \mu_3} \overline{\mu_1 \mu_1} \)).

Between groups 1 and 3 is equal to -1. Similar algebra results in that \( k_{12} = -1 \), and \( k_{23} = 1 \). QED. Let us consider now another important question: how the properties of stationary pure state \( G^{st}_{pN} \) depend on coefficients \( a \) and \( b \) and also on values of integers \( N \) and \( p \). We intend to analyze this problem in all details elsewhere and in this paper, as an illustration, consider only single special case when integer \( N \) tends to infinity while integer \( p \) remains finite. So, let stationary GF is:

\[
G^{st}_{pN} = v^p \left[ (au)^N + (au)^{N-1} (bw) + \ldots (bw)^N \right].
\]

(29)

We want to understand how in this stationary state ratio of mean occupation numbers in populations 1 and 3 depend on the coefficients \( a \) and \( b \). With the help of GF from Eq. (21) the required ratio \( \overline{\mu_1 \mu_3} \) can be represented in the form:

\[
\overline{\mu_1 \mu_3} = N a^N + (N - 1) a^{N-1} b + \ldots + a b^{N-1} \overline{N a^N + (N - 1) a^{N-1} b + \ldots + a b^{N-1}} = \frac{N a^N + (N - 1) a^{N-1} + \ldots + a b^{N-1}}{N + (N - 1) a + \ldots + a b^{N-1}} = \frac{x \frac{d}{dx} [Ln f_N(x)]}{N - x \frac{d}{dx} [Ln f(x)]}.
\]

(30)

(where we use the notation: \( x = \frac{b}{a} \) and \( f_N(x) = 1 + x + \ldots + x^N = \frac{x^{N+1}}{x - 1} - 1 \)). Let \( N \) tends to infinity. Using the Eq. (30) it is easy to see that \( \overline{\mu_1 \mu_3} \) tends to \( (\frac{x-1}{x}) N \) when \( x > 1 \), and \( \overline{\mu_1 \mu_3} \) tends to \( \frac{x}{(1-x)N} \) when \( x < 1 \). Thus in the case when \( x > 1 \) we get: \( \overline{\mu_1 \mu_3} \rightarrow p \) and \( \overline{\mu_1 \mu_3} \rightarrow \frac{x}{(1-x)N} \), while in the case \( x < 1 \) we get: \( \overline{\mu_1 \mu_3} \rightarrow \frac{x}{(1-x)N} \rightarrow N + (p - 1) \frac{x}{(1-x)N} \) and \( \overline{\mu_1 \mu_3} \rightarrow N \). In other words if \( N \gg 1 \) and \( x > 1 \) the first population becomes dominating, contrary in the case \( x < 1 \) coalition dominates. On the other hand when \( N \approx p \) the behavior of the model becomes considerably more complex and
requires a separate study.

Let us briefly summing up the main results of present paper. Based on ideas of QTOS we propose the consistent approach for statistical description of open classical systems with integer variables. We proved that for broad class of open systems possessing specific restrictions on the form of their interaction with environment the LME actually reduced to the Pauli master equation for diagonal elements of density matrix. This fact gives one the reason to use the LME for quantitative study of different problems relating both to statistical physics and to various "soft" sciences such as ecology, sociology, economics and so on.

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