A NOTE ON MINKOWSKI FORMULA OF CONFORMAL KILLING-YANO 2-FORM

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ABSTRACT. We study the Minkowski formula of conformal Killing-Yano two-forms in a spacetime of constant curvature. We establish the spacetime Alexandrov theorem with a free boundary.

1. Introduction

The Minkowski formula states that for a smooth closed hypersurface \( X : \Sigma \to \mathbb{R}^n \),

\[
(n - k) \int_{\Sigma} \sigma_{k-1} d\mu = k \int_{\Sigma} \sigma_k \langle X, \nu \rangle.
\]

Here \( \sigma_k \) is the \( k \)-th elementary symmetric functions of principal curvatures of \( \Sigma \). It has found itself many applications in Riemannian geometry for example a proof of the celebrated Alexandrov theorem which says that an closed embedded hypersurface of constant mean curvature must be an sphere. The same ideas of proof lead to a free boundary generalization due to Wang-Xia [WX19] establishing the rigidity of spherical caps in balls of space forms. Both closed and the free boundary settings made use of a specially chosen conformal Killing vector field. Tachibana introduced the conformal Killing-Yano two-form as a generalization of conformal Killing vector field.

**Definition 1** (Tachibana [Tac69]). A two-form \( Q \) on an \((n+1)\)-dimensional spacetime is called a conformal Killing-Yano 2-form if for every vector field \( X, Y \) and \( Z \) the following identity holds

\[
(\nabla_X Q)(Y, Z) + (\nabla_Y Q)(X, Z) = [2 \langle X, Y \rangle \langle \xi, Z \rangle - \langle X, Z \rangle \langle \xi, Y \rangle - \langle Y, Z \rangle \langle \xi, X \rangle]
\]

where \( \xi = \frac{1}{n} \text{div} Q \). We call \( \xi \) the associated 1-form of \( Q \).

In physics literature, these two forms are usually termed as *hidden symmetry* and can give information about the spacetime. See for example [LD06] and the references therein. Besides its physical significance, mathematically the conformal Killing-Yano two-forms are also interesting. In particular, they also allow a Minkowski type formula. Chen, Wang, Yau [CWW19] expressed quasilocal masses using this Minkowski formula. The authors of [WWZ17] established a spacetime version of the Alexandrov theorem for codimension two spacelike hypersurfaces via the Minkowski formula.

In this work, we are going to extend results in [WWZ17] where they used only conformal Killing-Yano two-forms \( rdr \wedge dt \). First we state the spacetime CMC condition with free boundary.
Definition 2. We say that $\Sigma^2$ in a spacetime $\mathbb{R}^{3,1}$ is CMC with free boundary if $\Sigma$ admits a null normal vector field $L$ with $\langle \vec{H}, L \rangle$ is constant, $(DL)^\perp = 0$ and $\Sigma$ meets the de Sitter sphere $S^{2,1}$ orthogonally.

Of course, one can allow arbitrary spacetime and boundary in the above definition. One interesting problem related to such surfaces is the uniqueness problem of a topological disk (cf. [FS15]). Without the free boundary condition, similar questions can be asked for two-spheres in $3 + 1$ dimensional de Sitter sphere (cf. [Che69]). One can also ask whether a spacelike graph over $\mathbb{R}^2$ in $\mathbb{R}^{3,1}$ with $\langle \vec{H}, L \rangle = 0$ and $(DL)^\perp = 0$ is linear which is analogous to the Bernstein problem for minimal graphs.

We generalize the spacetime Alexandrov theorem to the free boundary settings via establishing a spacetime Heintz-Karcher inequality. We state here the theorem in the Minkowski spacetime.

Theorem 1. Let $\Sigma$ be a codimension two, future incoming null embedded submanifold in the $(3 + 1)$-dimensional Minkowski spacetime with free boundary on the de Sitter sphere $S^{2,1}$. If $\Sigma$ lies in a half spacetime, and there exists a null vector field $L$ such that along $\Sigma$ that $\langle \vec{H}, L \rangle$ is a positive constant and $(DL)^\perp = 0$. Then $\Sigma$ lies in a shear free null hypersurface.

The theorem is a direct corollary from Theorem 4 and similar proofs as in [WWZ17, Theorem 3.14]. The article is organized as follows:

In Section 2 we collect basics of spacetime of constant curvature and the conformal Killing-Yano two-forms they admit. In Section 3 we prove a spacetime Heintz-Karcher inequality with a free boundary leading to a free boundary, spacetime Alexandrov theorem. We mention briefly the generalization to higher order curvatures.

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2. Conformal Killing-Yano 2-form on spacetime of constant curvature

A spacetime of dimension $3 + 1$ can only admit 20 conformal Killing-Yano two-forms. Actually, if a spacetime admits all twenty of them, then the spacetime has to be a spacetime of constant curvature. Note that for similar statements are also true for conformal Killing vector fields. In Minkowski, de Sitter and anti-de Sitter spacetime, these two forms are found explicitly. See the works by Jezierski and Lukasik [JL06, Jez08]. Now we collect some basics of these spacetimes and the conformal Killing-Yano two forms that live on them.

2.1. Minkowski spacetime. Let $(x^0, x^1, x^2, x^3)$ be the standard coordinates of the Minkowski space $\mathbb{R}^{3,1}$, define

$$\mathcal{D} = -x^0 dx^0 + x^1 dx^1 + x^2 dx^2 + x^3 dx^3,$$

$$\mathcal{T}_0 = -dx^0,$$

$$\mathcal{T}_i = dx^i,$$

$$\mathcal{L}_{0i} = -x^0 dx^i + x^i dx^0.$$
The conformal Killing-Yano 2-forms on Minkowski spacetime $\mathbb{R}^{3,1}$ are

\[(1) \quad \mathcal{T}_\mu \wedge \mathcal{T}_\nu, \mathcal{D} \wedge \mathcal{T}_\mu, *(\mathcal{D} \wedge \mathcal{T}_\mu) \text{ and } \mathcal{D} \wedge \mathcal{L}_{\mu\nu} + \frac{1}{2}(\mathcal{D} \wedge \mathcal{D}) \mathcal{T}_\mu \wedge \mathcal{T}_\nu,\]

where $*$ is the Hodge star operator and $\mu, \nu$ range from 0 to 3. See [JL06] for a calculation. Note that all are still conformal Killing-Yano 2-forms on $\mathbb{R}^{n,1}$ except $*(\mathcal{D} \wedge \mathcal{T}_\mu)$.

We remark that the last one in (1) can be used to prove formulas relating the center of mass (See [MT16]) and a Brown-York type quasi-local quantity by following similar procedures in [CWWY19].

2.2. Anti-de Sitter spacetime. We recall some basics of four-dimensional anti-de Sitter spacetime. The anti-de Sitter spacetime $\text{adS}^{3,1}$ is defined to be the set in $\mathbb{R}^{3,2}$

\[-(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 = -1\]

with metric induced from $\eta = -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2 - (dy^4)^2$. We will use coordinates of the Poincaré ball model by setting $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, $y^0 = \frac{1+x^2}{1-r^2}$, $y^4 = \frac{1-x^2}{1-r^2}$, and $y^1 = \frac{2x^1}{1-r^2}$. The metric of $\text{adS}^{3,1}$ is then

\[-(\frac{1+r^2}{1-r^2})^2 dt^2 + \frac{4\sum (dx^i)^2}{(1-r^2)^2}.\]

It is shown in [Jez08] that the conformal Killing-Yano 2-forms in four-dimensional anti-de Sitter spacetime are

\[dy^0 \wedge dy^i, dy^0 \wedge dy^4, dy^i \wedge dy^4, dy^i \wedge dy^j\]

and their Hodge duals with respect to the anti-de Sitter metric. We fix the frame $\theta^i = \frac{2x^i}{1-r^2} dx^i$ and $\theta^0 = \frac{1-x^2}{1-r^2} dt$. Let $\omega = \frac{2x^i}{1-r^2} dr$, then the length of $\omega$ is one. We have

\[dy^i = \theta^i + y^i r \omega, dy^4 = \cos t \theta^0 + \frac{2r}{1-r^2} \omega \sin t.\]

Note that $y^4$ and $y^i$ are static potentials, that is $\nabla_i dy^\mu = y^\mu \theta^i$ and $\nabla_\mu dy^\mu = -y^\mu \theta^0$ for each $\mu = 0, 1, \ldots, 4$. Here $\nabla_\mu$ denotes covariant derivative with respect to the vector field $(\theta^\mu)^i$. Then it is easy to obtain that

\[\text{div}(dy^i \wedge dy^4) = 3(y^i dy^4 - y^4 dy^i).\]

Note that $y^i dy^4 - y^4 dy^i$ is a Killing 1-form. Using the properties of Hodge operators, we find that $\text{div}(*(dy^2 \wedge dy^3))$ vanishes.

We remark that the 2-form $dy^i \wedge dy^4$ can be used similarly as in [CWWY15] to recover a formula relating the integrals of Ricci tensor and Brown-York type mass vector of an asymptotically hyperbolic manifold. These formulas are overlooked by the authors of [CWWY15]. The original proof is due to [MTX17].

2.3. de Sitter spacetime. The case with de Sitter spacetime is similar to the anti-de Sitter case (See [Jez08]). We consider here the $3 + 1$ dimensional case i.e. $\text{S}^{3,1}$. The de Sitter spacetime is the subset

\[y_0^2 + y_1^2 + y_2^2 + y_3^2 - y_4^2 = 1\]
in \( \mathbb{R}^{4,1} \) with the metric inherited from the standard Lorentz metric of \( \mathbb{R}^{4,1} \). We use the coordinate change
\[
\begin{align*}
y^0 &= \frac{1-r^2}{1+r^2} \cosh t, \\
y^i &= \frac{2x^i}{1+r^2}, \\
y^4 &= \frac{1-r^2}{1+r^2} \sinh t,
\end{align*}
\]
where \( r = \sqrt{\sum_{i=1}^{3}(x^i)^2} < 1 \). Now the metric of the de Sitter spacetime \( \mathbb{S}^{3,1} \) takes the form
\[
\eta = -(\frac{1-r^2}{1+r^2})^2 dt^2 + \frac{4}{(1+r^2)^2}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2].
\]
It is shown in [Jez08] that the conformal Killing-Yano 2-forms in four-dimensional de Sitter spacetime are
\[
dy^0 \wedge dy^i, \ dy^0 \wedge dy^4, \ dy^i \wedge dy^4, \ dy^i \wedge dy^j
\]
and their Hodge duals with respect to the de Sitter metric. We fix the frame \( \theta^i = \frac{2x^i}{1+r^2} dx^i \) and \( \theta^0 = \frac{1-r^2}{1+r^2} dt \). Let \( \omega = \frac{2r}{1+r^2} dr \), then the length of \( \omega \) is one. We have
\[
dy^i = \theta^i - y^i r \omega, \ dy^4 = \cosh \theta^0 - \frac{2r}{1+r^2} \omega \sinh t.
\]
Note that \( y^4 \) and \( y^i \) are static potentials, that is \( \nabla_i dy^\mu = -y^\mu \theta^i \) and \( \nabla_0 dy^\mu = y^\mu \theta^0 \) for each \( \mu = 0, 1, \ldots, 4 \). Here \( \nabla_\mu \) denotes covariant derivative with respect to the vector field \( (\theta^\mu)^2 \). Then it is easy to obtain that
\[
\text{div}(dy^i \wedge dy^4) = -3(y^i dy^4 - y^4 dy^i).
\]
Note that \( y^i dy^4 - y^4 dy^i \) is a Killing 1-form. We found also easily that \( \text{div}(\ast(dy^2 \wedge dy^3)) \) vanishes.

3. Spacetime Alexandrov theorem with free boundary

We start by proving a Minkowski formula for a codimension two spacelike hypersurface in \( \mathbb{R}^{3,1} \) with boundary meeting orthogonally with the de Sitter sphere. The result is related to mean curvature only, the generalization to higher order curvatures is quite straightforward.

The Minkowski spacetime is used as a prototype. First, we fix a conformal Killing-Yano 2-form in Minkowski spacetime \( \mathbb{R}^{3,1} \)
\[
Q = \mathcal{D} \wedge \mathcal{L}_{\theta i} + \frac{1}{2}[1 + (\mathcal{D}, \mathcal{D})] e^0 \wedge e^i.
\]
The associated 1-form is \( \xi := \frac{1}{n} \text{div} Q = \mathcal{L}_{\theta i} \) since

**Lemma 1.** The divergence of the 2-form \( Q = \mathcal{D} \wedge \mathcal{L}_{\theta i} + \frac{1}{2}[1 + (\mathcal{D}, \mathcal{D})] dx^0 \wedge dx^i \) is given by \( \text{div} Q = 3 \mathcal{L}_{\theta i} \).

Define the Minkowski unit ball
\[
\mathcal{B}^{3,1} = \{ x \in \mathbb{R}^{3,1} : (x, x) \leq 1 \}.
\]
The boundary of \( \mathcal{B}^{3,1} \) is the de Sitter sphere \( \mathbb{S}^{2,1} \). It is easy to check that \( \mathcal{D} \wedge Q \) is zero along \( \partial \mathcal{B}^{3,1} \), so \( Q \) has no components normal to \( \mathbb{S}^{2,1} \).
Theorem 2. Let $\Sigma$ be an immersed oriented spacelike codimension two submanifolds of the Minkowski spacetime $\mathbb{R}^{3,1}$, $\partial \Sigma$ lies in the de Sitter sphere $S^{2,1}$ and $\Sigma$ meets $S^{2,1}$ orthogonally. For any null vector field $\underline{L}$ of $\Sigma$, we have

$$\int_{\Sigma} [(n-1)\langle \underline{\xi}, \underline{L} \rangle + Q(\underline{H}, \underline{L}) + Q(\partial_a, (D^a \underline{L})^\perp)]d\mu = 0.$$  

Proof. Define $Q = Q(\partial_a, \underline{L})du^a$ on $\Sigma$ and the proof is almost the same as Theorem 2.2 of [WWZ17]. We include their proof for convenience. Let $\chi = \langle D_a \underline{L}, \partial_b \rangle$. Consider the 1-form $Q = Q(\partial_a, \underline{L})du^a$, we have

$$\text{div } Q = \nabla_a Q^a - Q(\nabla_a \partial_a, \underline{L})$$

$$= (D^a Q)(\partial_a, \underline{L}) + Q(\underline{H}, \underline{L}) + Q(\partial_a, D^a \underline{L})$$

$$= (n-1)\langle \underline{\xi}, \underline{L} \rangle + Q(\underline{H}, \underline{L}) + \sum_{ab} Q^{ab} + Q(\partial_a, (D^a \underline{L})^\perp)$$

$$= (n-1)\langle \underline{\xi}, \underline{L} \rangle + Q(\underline{H}, \underline{L}) + Q(\partial_a, (D^a \underline{L})^\perp).$$

Integration by parts and noting that $Q$ has no components normal to the de Sitter sphere. \hfill $\square$

3.1. A monotonicity formula. Let $\Sigma$ be a spacelike submanifold of codimension two in a spacetime $(S^{3,1}, g)$ which admits a Killing-Yano two form $Q$. Here, $S$ is either one of the four dimensional Minkowski, de Sitter and anti de Sitter spacetime. We require that $Q$ has no normal component normal to a support hypersurface $S$. Suppose that $\langle \underline{H}, \underline{L} \rangle \neq 0$, we define the following functional

$$F(\Sigma, [\underline{L}]) = (n-1) \int_{\Sigma} \frac{\langle \underline{\xi}, \underline{L} \rangle}{\langle \underline{H}, \underline{L} \rangle} d\mu - \frac{1}{2} \int_{\Sigma} Q(L, L)d\mu.$$  

Note $F$ is invariant under the change $L \to aL$ and $\underline{L} \to \frac{1}{a} \underline{L}$.

Let $\chi$ and $\chi$ be respectively the second fundamental form with respect to $L$, $\underline{L}$; let $C_0$ denote the future incoming null hypersurface of $\Sigma$. $C_0$ is obtained by taking the collection of all null geodesics emanating from $\Sigma$ with initial velocity $\underline{L}$. We then extend it to a future directed null vector field along $C_0$. Consider the evolution of $\Sigma$ along $C_0$ by a family of immersions $F : \Sigma \times [0, T) \to C_0$, satisfying

$$\frac{\partial F}{\partial s}(x, s) = \varphi(x, s)\underline{L},$$

$$F(x, 0) = F_0(x),$$

$$\Sigma \perp S$$

for some positive function $\varphi(x, s)$.

We have the following monotonicity property of the flow $\varphi$.

Theorem 3. Suppose that $\langle \underline{H}, \underline{L} \rangle > 0$ for some null vector field $\underline{L}$. Then $F(F(\Sigma, s), [\underline{L}])$ is monotone decreasing under the flow.

Proof. See Theorem 3.2 of [WWZ17]. We only have to use the extra fact that $Q$ has no components normal to the de Sitter space as in the proof of Theorem 2. \hfill $\square$

The monotonicity property leads to a spacetime Heintz-Karcher inequality. More, specifically, if under certain flow $\varphi$, the surface $\Sigma$ with $\langle \underline{H}, \underline{L} \rangle > 0$ flows into a submanifold of the time slice $\{x^0 = 0\}$ at $s = T$ and for $\Sigma$

$$F(\Sigma, [\underline{L}]) \geq 0.$$
holds provided $\varphi(\Sigma, T) \subset \{x^0 = 0\}$ and $\mathcal{F}(\varphi(\Sigma, T), [L]) \geq 0$.

**Lemma 2.** For any $\Sigma \subset \{x^0 = 0\}$, $\mathcal{F}(\Sigma, [L]) \geq 0$ reduces to
\begin{equation}
(n - 1) \int_{\Sigma} \langle \xi, \nu \rangle d\mu \geq \int_{\Sigma} \langle X_{\partial_i}, \nu \rangle d\mu.
\end{equation}

**Proof.** We have $\nu = \partial_i - e_n$ where $e_n$ is a unit normal. So $(\tilde{H}, \nu) = H$ where $H$ is the mean curvature of $\Sigma$ in $B^n$. We have that $\xi = L_{\partial_i} = x^i dx^0$, so $\langle \xi, L \rangle = x^i$.

Also,
\[Q(L, L) = Q(\partial_i + \nu, \partial_i - \nu) = 2\langle X_{\partial_i}, \nu \rangle,
\]

where $X_a = (X, a)X + \frac{1}{2}(|X|^2 + 1)a$ where $a = a^i \partial_i$ is a constant vector in $\mathbb{R}^n$. It easily leads to (2).

Note that this is precisely an inequality proven already by Wang-Xia [WX19 (5.5)] with the assumption that $\Sigma$ has positive mean curvature and lies in a half ball.

Combining with their result, we have

**Theorem 4** (spacetime Heintz-Karcher inequality). *If there exists a flow $\varphi$ of a hypersurface $\Sigma$ with $(\tilde{H}, L) > 0$ for some null vector field $L$ and a free boundary on $S^{2,1}$ which flows $\Sigma$ into the half unit ball of the slice $\{x^0 = 0\}$, then we have the inequality
\begin{equation}
\int_{\Sigma} \frac{\langle \xi, L \rangle}{\langle \tilde{H}, L \rangle} d\mu \geq \frac{1}{2(n - 1)} \int_{\Sigma} Q(L, L) d\mu.
\end{equation}

Equality occurs if and only if $\Sigma$ lies in a shear free null hypersurface with free boundary on $S^{n-1,1}$.*

**Proof.** Let $\Sigma_t = \varphi_t(\Sigma)$, then for each $t > 0$, the equality holds. Suppose that $\Sigma_T \subset \{x^0 = 0\}$ for some $T > 0$. So $\Sigma_T$ has to be a spherical cap orthogonal to the unit sphere in $\mathbb{R}^n$ according to [WX19]. In particular, under the flow $\varphi$, $\Sigma_t$ foliates a shear free null hypersurface $S$ with free boundary.

3.2. **Anti-de Sitter case.** Theorems 2, 3 and 4 work well in the case with $\partial \Sigma = \emptyset$. The same proof also adapts in the anti-de Sitter and de Sitter settings. We use the notations in Section 2.2. For simplicity, we set $i$ to be 1, we use the 2-forms $dy^1 \wedge dy^4$ and $*(dy^2 \wedge dy^3)$ only. Note that the Hodge star operator commutes with the covariant derivative. Using this, we see easily that $\text{div}(*(dy^2 \wedge dy^3))$ vanishes.

We use the 2-form
\[Q = dy^1 \wedge dy^4 + l * (dy^2 \wedge dy^3)
\]

where $l > 0$ is a positive constant. We define the surface $B^{3,1}$ to be the surface with distance less than $d$ from the point $t = 0$, $r = 0$ where $\cosh d = l$. If $Y_1, Y_2 \in \lambda S^{3,1}$ (using the embedding into $\mathbb{R}^{3,2}$) are two points which can be connected via a spacelike geodesic, then the distance from $Y_1$ to $Y_2$ is $\cosh d = -\eta(Y_1, Y_2) > 0$. The boundary $S = \partial B^{3,1}$ is a timelike hypersurface of dimension three of constant distance from the point $t = 0$, $r = 0$ and it is unibital hence null geodesics intrinsic to $S$ are also null geodesic in $\text{ad}S^{3,1}$. It is the analog of de Sitter sphere which is of constant distance to the origin in Minkowski spacetime. It is a tedious task to check that $Q$ has no component normal to $S$. We state here the spacetime Heintz-Karcher inequality and leave the spacetime Alexandrov theorem to the reader.
Theorem 5. (spacetime Heintz-Karcher inequality in $B^{3,1}$) If there exists a flow $\varphi$ of a hypersurface $\Sigma$ with $\langle \tilde{H}, L \rangle > 0$ for some null vector field $L$ and a free boundary on $S$ which flows $\Sigma$ into the half geodesic ball of the slice $\{t = 0\}$, then we have the inequality

$$ \int_{\Sigma} \frac{\langle \xi, L \rangle \langle \tilde{H}, L \rangle}{\langle H, L \rangle} d\mu \geq \frac{n}{2(n-1)} \int_{\Sigma} Q(L, L) d\mu. $$

Equality occurs if and only if $\Sigma$ lies in a shear free null hypersurface.

Proof. The proof is the same with Theorem 4. We only have to verify when $t = 0$ the inequality holds. Let $\nu$ be the unit normal of $\Sigma$ in the $\{t = 0\}$ slice. Indeed, when $t = 0$, $\xi = y^1 dy^4$ and

$$ L = e_0 - \nu = \frac{y^2}{1+r^2} \partial_1 - \nu, $$

so

$$ \langle \xi, L \rangle = y^1 = \frac{2x^j}{1+r^2}. $$

We turn to $Q(L, L)$. We have

$$(dy^1 \wedge dy^4)(L, L) = 2dy^1(\nu)$$

and

$$(dy^1)^2 = \frac{1}{2} \partial_1 + (x^1 x^j \partial_j - \frac{1}{2} r^2 \partial_1).$$

And

$$ *(dy^2 \wedge dy^3) = -\theta^1 \wedge \theta^0 + y^2(x^1 \theta^0 - x^2 \theta^1) \wedge \theta^0 + y^3(x^1 \theta^3 - x^3 \theta^1) \wedge \theta^0,$$

so the 1-form $*(dy^2 \wedge dy^3)(\cdot, e_0)$ is dual to $-\frac{1}{2} \partial_1 + (x^1 x^j \partial_j - \frac{1}{2} r^2 \partial_1)$. As usual, $Q(L, L) = 2Q(\nu, e_0)$. Thus,

$$ Q(L, L) = 2(X_{\partial_1}, \nu), $$

where $X_a = (1 + l) \left[ x^k a_k x^j \partial_j - \frac{1}{4}(l^2 + \frac{1}{4})a \right]$ with $a = a^j \partial_j$ being a constant vector in $\mathbb{R}^n$. Letting $l = \frac{1+R_2}{1-R_2}$, [WX19] reduces to also [WX19].

Remark 1. It is easy to check that the higher dimensional analog of $*(dy^2 \wedge dy^3)$ in the $n$-dimensional anti-de Sitter spacetime

$$ adS^n = \{- (y^0)^2 + (y^1)^2 + \cdots + (y^n)^2 - (y^{n+1})^2 = 1\} $$

is

$$ -e^1 \wedge e^0 + \sum_{i \neq 1} y^i (x^1 e^i - x^i e^1) \wedge e^0. $$

3.3. de Sitter case. We calculate below the quantities needed for a theorem parallel to Theorem 3. We follow similar notations and omit the the statements or details. Generalizing to higher dimension is also straightforward. The conformal Killing-Yano 2-form is

$$ Q = dy^k \wedge dy^1 + l \ast (dy^3 \wedge dy^2) $$

and its associated 1-form

$$ \xi = \text{div} Q = 3y^1 dy^4 - y^4 dy^1.$$
Notice the order of the superscripts. Within the slice \( \{ t = 0 \} \), we have that \( \xi = y^1 \mathrm{d} y^4 \) and \( L = e_0 - \nu = \frac{1}{\left( 1 + r^2 \right)^2} \partial_t - \nu \) and

\[
\langle \xi, L \rangle = \frac{2y^1}{1 + r^2},
\]

We turn to \( Q(L, L) \). We have

\[
(dy^4 \wedge dy^1)(L, L) = -2dy^1(\nu).
\]

Note that

\[
A := -(dy^1)^4 = -\frac{1}{2} \partial_t - (\frac{1}{r^2} \partial_1 - x^1 x^j \partial_j)
\]

and

\[
*(dy^3 \wedge dy^2) = \theta^1 \wedge \theta^0 + y^2(x^1 \theta^2 - x^2 \theta^1) \wedge \theta^0 + y^3(x^1 \theta^3 - x^3 \theta^1) \wedge \theta^0,
\]

so the 1-form \( *(dy^3 \wedge dy^2)(\cdot, e_0) \) is dual to

\[
B := \frac{1}{r} \partial_t - \frac{1}{r^2} \partial_1 + x^1 x^j \partial_j.
\]

\[
A + lB
\]

is then

\[
X_{\partial_t} := (1 + l) \left[ x^1 x^j \partial_j + \frac{1}{2} \left( 1 + \frac{1}{r^2} - r^2 \right) \partial_1 \right].
\]

Therefore \( Q(L, L) = 2(X_{\partial_t}, \nu) \). Setting \( \frac{1}{r^2} = |x|^2 \) with \( 0 < \frac{1}{r} < 1 \) recovers the form of [WX19]. We have not given the support hypersurface of the boundary yet. To this end, we fix a point \( O = \{ t = 0, r = 0 \} \), let \( S \) be the hypersurface in \( S^{3:1} \) be a hypersurface of constant distance \( d \) from the point \( O \) where \( \cos d = l \). It is fairly easy to check that \( Q \) has no components to the hypersurface \( S \).

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