THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF THE BRANCH CURVE OF $T \times T$ IN $\mathbb{C}^2$

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Abstract. This paper is the second in a series of three papers concerning the surface $T \times T$, where $T$ is a complex torus. We compute the fundamental group of the branch curve of the surface in $\mathbb{C}^2$, using the van Kampen Theorem and the braid monodromy factorization of the curve.

1. Background

The concentration on the fundamental group of a complement of a branch curve of an algebraic surface $X$ with respect to a generic projection onto $\mathbb{CP}^2$, leads us to the computation of $\pi_1(X_{\text{Gal}})$ the fundamental group of the Galois cover of $X$ with respect to this generic projection. Galois covers are surfaces of a general type.

Bogomolov conjectured that the Galois covers corresponding to generic projections of algebraic surfaces to $\mathbb{CP}^2$ have infinite fundamental groups.

In [7] we justify Bogomolov’s conjecture by proving that $\pi_1(T \times T)_{\text{Gal}}$ is an infinite group.

In order to compute $\pi_1(T \times T)_{\text{Gal}}$, we have to encode the braid monodromy factorization of the branch curve $S$ of $T \times T$. Then we have to apply the van Kampen Theorem on the factors in the factorization in order to get relations for $\pi_1(\mathbb{C}^2 - S, \ast)$ the fundamental group of the complement of $S$ in $\mathbb{C}^2$.

The fundamental group of the Galois cover $X_{\text{Gal}}$ is known to be a quotient of a certain subgroup of the fundamental group of the complement of $S$.

We recall shortly the computations from [6].

Let $X = T \times T$ be an algebraic surface (where $T$ is a complex torus) embedded in $\mathbb{CP}^5$, and $f : X \rightarrow \mathbb{CP}^2$ be a generic projection. We degenerate $X$ to a union of 18 planes $X_0([6, Section 3])$. We numerate the lines and vertices as shown in Figure 1.

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We have a generic projection $f_0 : X_0 \to \mathbb{C}P^2$. We get a degenerated branch curve $S_0$ which is a line arrangement and compounds nine 6-points. We regenerate each 6-point separately.

We concentrate for example in a regeneration in a neighbourhood of $V_2$. We consider the local numeration of lines meeting at $V_2$ ([6, Figure 6]). First, the diagonal lines 4 and 5 become conics which are tangent to the lines 2,3 and 1,6 respectively, see Figure 2.
Now, concentrating in a neighbourhood of the left 4-point (the intersection point of 1,2,3,6), the two horizontal lines become a hyperbola. Each one of the two vertical lines is replaced by two parallel lines, which are tangent to the hyperbola, see Figure 3.

Finally, the hyperbola is doubled. Each one of the tangent points is replaced by three cusps ([6, Theorem 13]). Naturally, it occurs that each node is replaced by two or four ones.

We end up with a regenerated cuspidal curve, which has a degree of 12.

We do so to each one of the 6-points and get regenerated curves $S_i, 1 \leq i \leq 9$. The union of these curves is the regenerated branch curve $S$. $\deg S_0 = 27$, thus $\deg S = 54$. The reason is that each intersection point $q_j$ of the curve $S_0$ with the typical fiber was replaced by two close points $\{q_j, q'_j\}$.

In order to define a $g$-base for the fundamental group of the complement of $S$ in $\mathbb{C}^2$, we present the following situation (following Figure 4 to understand the below notions).

$S$ is an algebraic curve in $\mathbb{C}^2$, $54 = \deg S$. $\pi: \mathbb{C}^2 \to \mathbb{C}$ a generic projection on the first coordinate. $K(x) = \{y \mid (x,y) \in S\}$ is the projection to the $y$-axis of $\pi^{-1}(x) \cap S$. Let $N = \{x \mid \#K(x) < 54\}$ and $M' = \{x \in S \mid \pi|_x \text{ is not étale at } x\}$ such that $\pi(M') = N$. Assume $\#(\pi^{-1}(x) \cap M') = 1, \forall x \in N$. Let $E$ (resp. $D$) be a closed disk on $x$-axis (resp.
\(y\)-axis), such that \(M' \subset E \times D, N \subset \text{Int } (E)\). We choose \(u \in \partial E, x \ll u \ \forall x \in N\).

\[C_u = \{q_1, q_1', \ldots, q_{27}, q_{27}'\}.\]

\[\text{Figure 4.}\]

We now specify a standard set of generators for the fundamental group \(\pi_1(\mathbb{C}^2 - S, M)\), where \(M\) is a point outside \(S\).

Write \(S \cap C_u = \{q_1, \ldots, q_{27}\}\). Let \(\gamma_j\) be paths from \(M\) to \(q_j \ \forall j\), such that the \(\gamma_j\) do not meet each other in any point except \(M\). Let \(\eta_j\) be a small oriented circle around \(q_j\). Let \(\gamma_j'\) be the part of \(\gamma_j\) outside \(\eta_j\), and take \(\Gamma_j = \gamma_j'\eta_j(\gamma_j')^{-1}\). In the same way we specify generators \(\Gamma_j'\). The set \(\{\Gamma_j, \Gamma_j'\}_{j=1}^{27}\) freely generates \(\pi_1(\mathbb{C}_u - S, M)\) \([40]\). Such a set is called a g-base for \(\pi_1(\mathbb{C}_u - S, M)\).

By Lemma \([12]\) we have a surjection \(\pi_1(\mathbb{C}_u - S, M) \xrightarrow{\nu} \pi_1(\mathbb{C}^2 - S, M) \rightarrow 0\), so the set \(\{\nu(\Gamma_j)\}\) generates \(\pi_1(\mathbb{C}^2 - S, M)\). By abuse of notation, we shall denote \(\nu(\Gamma_j)\) by \(\Gamma_j\). A presentation for \(\pi_1(\mathbb{C}^2 - S, M)\) is obtained by the van Kampen Theorem, from a list of braids in \(B_{54}[\mathbb{C}_u, \mathbb{C}_u \cap S]\).

The group \(\pi_1(\mathbb{C}^2 - S, M)\) acts on the points in \(C_u\). This leads to a permutation representation \(\psi : \pi_1(\mathbb{C}^2 - S, M) \rightarrow S_{18}\), 18 is the number of planes in \(X_0\).

Let \(\langle \Gamma_j^2, \Gamma_j' \rangle\) denote the normal subgroup generated by \(\Gamma_j^2, \Gamma_j'\).

**Definition 1.** Define

\[
\tilde{\pi}_1 = \frac{\pi_1(\mathbb{C}^2 - S, M)}{\langle \Gamma_j^2, \Gamma_j' \rangle}.
\]
Since $f$ is stable, $\Gamma_j$ and $\Gamma'_j$, induce a transposition in $S_{18}$ so that $\langle \Gamma_j^2 \rangle \subseteq \ker \psi$. The map $\tilde{\pi}_1 \to S_{18}$ is also denoted $\psi$.

By the isomorphism theorems, we have an exact sequence

(1) \[ 1 \to \ker \psi \to \tilde{\pi}_1 \xrightarrow{\psi} S_{18} \to 1. \]

**Definition 2.** Consider the fibered product

\[ X \times_f \cdots \times_f X = \{(x_1, \cdots, x_n) \in X^n : f(x_1) = \cdots = f(x_n)\}, \]

and the diagonal

\[ \Delta = \{(x_1, \cdots, x_n) \in X \times_f \cdots \times_f X \mid x_i = x_j \text{ for some } i \neq j\}. \]

The surface $X_{Gal}$ is the Galois cover of $X$ with respect to the generic projection $f : X \to \mathbb{C}\mathbb{P}^2$.

That is the Zariski closure of the complement of $\Delta$:

\[ X_{Gal} = \overline{X \times_f \cdots \times_f X - \Delta}. \]

Let $X_{Gal}^{Aff}$ be the part of $X_{Gal}$ lying over $\mathbb{C}^2 (\subseteq \mathbb{C}\mathbb{P}^2)$. There is a surjective map $X_{Gal}^{Aff} \to X_{Gal}$.

**Theorem 3.** [16, Section 0.3]

$\pi_1(X_{Gal}^{Aff})$ is isomorphic to the kernel of $\psi : \tilde{\pi}_1 \to S_{18}$.

We denote $\mathcal{A} = \pi_1(X_{Gal}^{Aff})$.

The exact sequence (1) gets the form

(2) \[ 1 \to \mathcal{A} = \ker \psi \to \tilde{\pi}_1 = \left\langle \frac{\pi_1(\mathbb{C}^2 - S, M)}{\langle \Gamma_j^2, \Gamma'_j \rangle} \right\rangle \xrightarrow{\psi} S_{18} \to 1. \]

The plan is to use the van Kampen Theorem and the braid monodromy technique to obtain a presentation of $\pi_1(\mathbb{C}^2 - S, M)$ and by adding the relations $\Gamma_j^2 = \Gamma'_j = 1$ to get a presentation of $\tilde{\pi}_1$. Then we use the Reidemeister Schreier method to obtain a presentation of $\mathcal{A}$ (see [7]).
1.1. Braid monodromy.

Recall that \( S \) is the branch curve of \( X, \deg S = 54 \). Recall the above \( N, M', K(x), u, \mathbb{C}_u, E, D \).
Let \( B_{54}[D, \mathbb{C}_u] \) be the braid group, and \( H_1, \ldots, H_{53} \) be its frame. Let \( M \in \partial D \) and \( \pi_1(\mathbb{C}^2 - S, M) \) is the fundamental group of the complement of \( S \), with a \( g \)-base \( \Gamma_1, \ldots, \Gamma_{27} \).

The braid monodromy of \( S \) is a map \( \varphi : \pi_1(E - N, u) \to B_{54}[D, \mathbb{C}_u] \) defined as follows: every loop in \( E - N \) starting at \( u \) has liftings to a system of 54 paths in \( (E - N) \times D \) starting at \( q_1, \ldots, q_{27} \). Projecting them to \( D \) we get 54 paths in \( D \) defining a motion \( \{q_1(t), \ldots, q_{27}(t)\} \) of 54 points in \( D \) starting and ending at \( \mathbb{C}_u, 0 \leq t \leq 1 \). This motion defines a braid in \( B_{54}[D, \mathbb{C}_u] \).

**Theorem 4. The Artin Theorem**

*Let \( S \) be a curve and let \( \delta_1, \ldots, \delta_q \) be a \( g \)-base of \( \pi_1(E - N, u) \). Assume that the singularities of \( S \) are cusps, nodes and branch points. Let \( \varphi : \pi_1(E - N, u) \to B_{54}[D, \mathbb{C}_u] \) be the braid monodromy. Then for all \( i \), there exist a halftwist \( V_i \in B_{54}[D, \mathbb{C}_u] \) and \( r_i \in \mathbb{Z} \), such that \( \varphi(\delta_i) = V_i^{r_i} \) and \( r_i \) depends on the type of the singularity: \( r_i = 1, 2, 3 \), for branch point, node, cusp respectively.*

**Proposition 5. [17, Proposition VI. 2.1]**

*Let \( S \) be the regenerated branch curve of degree 54 in \( \mathbb{CP}^2 \). Let \( \pi, u, D, E, \mathbb{C}_u \) be as above. Let \( \varphi \) be the braid monodromy of \( S \) with respect to \( \pi, u \). Let \( \delta_1, \ldots, \delta_q \) be a \( g \)-base of \( \pi_1(E - N, u) \). Then \( \prod_{i=1}^{9} \varphi(\delta_i) = \Delta^2_{54} \).

\[ 2. \Delta^2_{54} = \prod_{i=1}^{9} C_i H_{V_i} \]

2.1. \( C_i \).

There are some lines in \( X_0 \) which do not meet \( x \), but when projecting them to \( \mathbb{CP}^2 \), they may intersect. These intersections \( C_i, i = 1, \ldots, 9 \) are called parasitic intersections.

Recall from [6, Section 5]:

\[ \tilde{C}_1 = \prod_{t=1,2,4,6,13,22} D_t, \tilde{C}_2 = \prod_{t=3,7,9,14,17} D_t, \tilde{C}_3 = \prod_{t=5,10,18,23} D_t, \]
\[ \tilde{C}_4 = \prod_{t=8,11,15,19} D_t, \tilde{C}_5 = \prod_{t=12,20,24} D_t, \tilde{C}_6 = \prod_{t=16,25} D_t, \tilde{C}_7 = \prod_{t=21,26} D_t, \tilde{C}_8 = D_{27}, \tilde{C}_9 = Id. \]

\( D_t, 1 \leq t \leq 27 \), are regenerated. We use the Complex Conjugation [6, Subsection 7.2] and obtain the following results (denoted as above):
expressions such as $\rho$, $H$ regenerate in their neighbourhood. That we can consider any relation we need for calculations, but setting $2.2$. We computed in detail $H$ in \cite{4} an Invariance Property to each $D$.

During the regeneration, each $\tilde{C}_i$ is regenerated to $C_i$, $1 \leq i \leq 9$.

Each $C_i$ is now a product of the certain regenerated $D_t$ (as shown for $\tilde{C}_i$).

2.2. $H_{V_i}$.

Recall that the regenerated branch curve compounds nine 6-points $V_i$, $i = 1, \cdots, 9$. We regenerate in their neighbourhood. $H_{V_i}$ are the resulting regenerated braid monodromies. We computed in detail $H_{V_i}$ for $i = 1, 4, 7$ in \cite[Sections 8, 9, 10]{6}. In this section we present the resulting $H_{V_i}$ and the paths which correspond to their different factors. We have proved in \cite{4} an Invariance Property to each $H_{V_i}$, therefore in the following tables, one can see expressions such as $\rho^j_4 \rho^{i-1}_3 z_{44} \rho^{-i}_3$ where $\rho^j_4 = Z^{j}_{44}$ and $\rho^{i}_4 = Z^{i}_{33}$, $i, j \in \mathbb{Z}$. The meaning is that we can consider any relation we need for calculations, but setting $i = j = \pm 1$ is enough in order to derive all other possibilities.

There are two tables which correspond to each $H_{V_i}$. The first table presents the paths which correspond to the factors in $H_{V_i}$ and some complex conjugates, the braids themselves and...
the exponent of the braids according to the Emil Artin Theorem. The second table presents the same ones but for the factors of the form $\hat{F}_1 \cdot (\hat{F}_1)^{p^{-1}}$. 

\[
H_{V_1} = (Z_{4i_1}^2)_{i=22',55',6,6'} \cdot Z_{33',4}^5 \cdot (Z_{4i}^2)_{i=22',55',6,6'} \cdot Z_{11',4}^3 \cdot (Z_{4i}^4) \cdot Z_{33',4}^2 \cdot Z_{55',6}^3 \cdot (Z_{3i}^2)_{i=22',33',4}. 
\]

\[
Z_{22',6}^3 \cdot (Z_{66'}^2)_{i=11',33'} \cdot Z_{22',33'}^2 \cdot Z_{22',6}^{Z_{66'}} \cdot (Z_{66'}) \cdot Z_{33',4}^2 \cdot Z_{55',6}^2 \cdot (\hat{F}_1(\hat{F}_1)^{-1})Z_{33'}^2Z_{55'}^2 \cdot Z_{33',4}^2. 
\]
|   | The paths/complex conjugates | The braids | The exponent of braids |
|---|-------------------------------|------------|-----------------------|
| (1) | ![Diagram](image1.png) | $\rho_4^i \rho_m^i \rho_m^i \rho_4^i \rho_4^{-j}$ | 2 |
|     |                               | $m = 2, 5$ |                       |
|     |                               | $\rho_6^i \rho_4^i \rho_4^{-i} \rho_6^{-i}$ |                       |
| (2) | ![Diagram](image2.png) | $\rho_4^i \rho_3^i \rho_3^i \rho_4^{-j}$ | 3 |
| (3) | ![Diagram](image3.png) | $\rho_4^i \rho_m^i \rho_m^i \rho_4^{-j}$ | 2 |
|     |                               | $m = 2, 5$ |                       |
|     |                               | $\rho_6^i \rho_4^i \rho_4^{-i} \rho_6^{-i}$ |                       |
| (4) | ![Diagram](image4.png) | $\rho_4^i \rho_4^i \rho_4^{-i} \rho_4^{-j}$ | 3 |
| (5) | ![Diagram](image5.png) | $\tilde{z}_{44'}$ | 1 |
| (6) | ![Diagram](image6.png) | $\rho_6^i \rho_5^i \rho_5^i \rho_6^{-i}$ | 3 |
| (7) | ![Diagram](image7.png) | $\rho_6^i \rho_m^i \rho_m^i \rho_6^{-j}$ | 2 |
|     |                               | $m = 1, 3$ |                       |
| (8) | ![Diagram](image8.png) | $\rho_6^i \rho_2^i \rho_2^i \rho_6^{-i}$ | 3 |
| (9) | ![Diagram](image9.png) | $\rho_6^i \rho_m^i \rho_m^i \rho_6^{-j}$ | 2 |
|     |                               | $m = 1, 3$ |                       |
| (10) | ![Diagram](image10.png) | $\tilde{z}_{66'}$ | 1 |
\[
(F_1 \cdot (F_1)^{Z_{\gamma 3}^{-1} Z_{\gamma 5}^{-1}} Z_{\gamma 5' 6}^{2} Z_{\gamma 3' 4}^{2})
\]

| Diagram | Symbol | Count |
|---------|--------|-------|
| ![Diagram 1](https://via.placeholder.com/150) | $\rho_{5}^{i} \rho_{5}^{j} \tilde{z}_{23} \rho_{2}^{i} \tilde{\rho}_{3}^{j}$ | 3 |
| ![Diagram 2](https://via.placeholder.com/150) | $\rho_{5}^{i} \rho_{5}^{i} \tilde{z}_{35} \rho_{3}^{j} \rho_{5}^{i}$ | 2 |
| ![Diagram 3](https://via.placeholder.com/150) | $\rho_{5}^{i} \rho_{5}^{i} \tilde{z}_{35} \rho_{3}^{j} \rho_{5}^{i}$ | 2 |
| ![Diagram 4](https://via.placeholder.com/150) | $\rho_{5}^{i} \rho_{5}^{j} \tilde{z}_{25} \rho_{2}^{i} \tilde{\rho}_{5}^{j}$ | 3 |
| ![Diagram 5](https://via.placeholder.com/150) | $z_{12}$ | 1 |
| ![Diagram 6](https://via.placeholder.com/150) | $z_{1' 2}$ | 1 |
| ![Diagram 7](https://via.placeholder.com/150) | $z_{1' 2}^{\rho^{-1}}$ | 1 |
| ![Diagram 8](https://via.placeholder.com/150) | $z_{12}^{\rho^{-1}}$ | 1 |
\[
H_{V_2} = \mathbb{Z}_{33'}^3 \cdot (\mathbb{Z}_{22'}^{3})_{i=11',5',66'} \cdot \mathbb{Z}_{22'}^{2} \cdot (\mathbb{Z}_{22'}^{3})_{i=11',5',66'} \cdot (\mathbb{Z}_{22'}^{2})_{i=11',5',66'} \cdot (\mathbb{Z}_{22'}^{2})_{i=11',5',66'} \cdot (\mathbb{Z}_{22'}^{2})_{i=11',5',66'} \cdot (\mathbb{Z}_{22'}^{2})_{i=11',5',66'} \cdot (\mathbb{Z}_{22'}^{2})_{i=11',5',66'}.
\]

The table below shows the paths/complex conjugates, the braids, and the exponent of braids for each case.

| The paths/complex conjugates | The braids | The exponent of braids |
|-----------------------------|------------|-----------------------|
| (1)                         | \( p_4^j p_3^j z_{334} p_3^{-j} \rho_4^{-j} \) | 3                     |
| (2)                         | \( p_4^j p_3^j \tilde{z}_{m4} \rho_m^{-j} \rho_4^{-j} \) \( m = 1, 6 \) | 2                     |
| (3)                         | \( p_4^j p_5^j \tilde{z}_{224} \rho_2^{-j} \rho_4^{-j} \) | 3                     |
| (4)                         | \( p_4^j p_5^j \tilde{z}_{m4} \rho_m^{-j} \rho_4^{-j} \) \( m = 1, 6 \) | 2                     |
| (5)                         | \( \tilde{z}_{44'} \) | 1                     |
| (6)                         | \( \rho_6^j \rho_5^j \tilde{z}_{56} \rho_5^{-j} \rho_6^{-j} \) | 3                     |
| (7)                         | \( \rho_5^j \rho_3^j \tilde{z}_{35} \rho_3^{-j} \rho_5^{-j} \) | 2                     |
| (8)                         | \( \rho_5^j \rho_3^j \tilde{z}_{25} \rho_2^{-j} \rho_5^{-j} \) | 2                     |
| (9)                         | \( \rho_5^j \rho_1^j \tilde{z}_{15} \rho_1^{-j} \rho_5^{-j} \) | 3                     |
| (10)                       | \( \rho_5^j \rho_3^j \tilde{z}_{35} \rho_3^{-j} \rho_5^{-j} \) | 2                     |
| (11)                       | \( \rho_5^j \rho_2^j \tilde{z}_{25} \rho_2^{-j} \rho_5^{-j} \) | 2                     |
| (12)                       | \( \tilde{z}_{55'} \) | 1                     |
\[
(\hat{F}_1 \cdot (\hat{F}_1)^{-1}) z_{33'}^{-1} z_{66'}^{-1} z_{33'}^{-2} z_{33'}^{-4}
\]
\[
H_{V_3} = (Z^2_{3k})_{i=22',5,5',6'} \cdot Z^3_{3',44'} \cdot Z^3_{1',3} \cdot (Z^2_{2'})_{i=22',5,5',6'} \cdot (Z_{3'})^2 \cdot (Z_{3'})^2 \cdot (Z_{3'})^2 \cdot (Z_{3'})^2 \cdot \left( \hat{F}_1 (Z^2_{1',3} Z_{66'}) \right)^2 \cdot (Z_{66'})^2.
\]
\[
(\tilde{F}_1 \cdot (\tilde{F}_1)^{-1} \cdot \gamma_{90'})^2 \gamma_{12} \gamma_{12'} \gamma_{12''}
\]
\[
H_{V_4} = (Z_{2,1}^2)_{i=31',55',6,6'} \cdot Z_{11',2}^3 \cdot Z_{2,44'}^3 \cdot (Z_{2,1'}^2)_{i=33',55',6,6'} \cdot (Z_{22}^2)_{i=31',2} \cdot Z_{2,44'}^2 \cdot Z_{2,33'}^2 \cdot (Z_{60}^2)_{i=33',6} \cdot Z_{55',6}^2.
\]

| The paths/complex conjugates | The braids | The exponent of braids |
|-----------------------------|------------|-----------------------|
| (1)                         | \( \rho_m^i \rho_2^i \bar{z}_{2m} \rho_2^{-i} \rho_m^{-j} \) | 2                     |
| (2)                         | \( \rho_2^i \rho_1^i \bar{z}_{12} \rho_1^{-i} \rho_2^{-j} \) | 3                     |
| (3)                         | \( \rho_4^i \rho_2^i \bar{z}_{24} \rho_2^{-i} \rho_4^{-j} \) | 3                     |
| (4)                         | \( \rho_m^i \rho_2^i \bar{z}_{2m} \rho_2^{-i} \rho_m^{-j} \) | 2                     |
| (5)                         | \( \bar{z}_{22} \) | 1                     |
| (6)                         | \( \bar{z}_{66} \) | 1                     |
| (7)                         | \( \rho_6^i \rho_m^i \bar{z}_{m6} \rho_m^{-i} \rho_6^{-j} \) | 2                     |
| (8)                         | \( \rho_6^i \rho_3^i \bar{z}_{36} \rho_3^{-i} \rho_6^{-j} \) | 3                     |
| (9)                         | \( \rho_6^i \rho_m^i \bar{z}_{m6} \rho_m^{-i} \rho_6^{-j} \) | 2                     |
| (10)                        | \( \rho_6^i \rho_5^i \bar{z}_{56} \rho_5^{-i} \rho_6^{-j} \) | 3                     |
\[(\hat{F}_1 \cdot (\hat{F}_1)^{Z_{11}^{-1}Z_{55}^{-1}})^{Z_{11}^{2}Z_{55}^{-2}}\]

| \[\rho_i^j \rho_i^j \hat{z}_i\rho_i^{-i} \rho_i^{-j}\] | 3 |
|-----------------------------------------------|---|
| \[\rho_i^j \rho_i^j \hat{z}_4\rho_i^{-i} \rho_i^{-j}\] | 3 |
| \[z_{34}^r\] | 1 |
| \[z_{34}^r\] | 1 |
| \[z_{34}^r\] | 1 |
| \[z_{34}^r\] | 1 |
| \[\rho_i^j \rho_i^j \hat{z}_5\rho_i^{-i} \rho_i^{-j}\] | 2 |
| \[\rho_i^j \rho_i^j \hat{z}_5\rho_i^{-i} \rho_i^{-j}\] | 2 |
$H_{V_5} = Z_{2v,2z}^2 \cdot (Z_{6v}) Z_{6v,6}^6 Z_{55'}^2 \cdot Z_{33',6}^2 \cdot \tilde{Z}_{44',6}^3 \cdot (Z_{33'}^6)^{Z_{55'}^2} \cdot (Z_{33',6}^2)^{Z_{55'}^2} \cdot \tilde{Z}_{44',6}^2 \cdot (Z_{33',6}^2)^{Z_{55'}^2} \cdot (Z_{33',6}^2)^{Z_{55'}^2} \cdot \tilde{Z}_{44',6}^2$.

$(\tilde{F}_1(F_1) Z_{2v,2z}^2 Z_{55'}^2 \cdot Z_{6v,6}^6 Z_{55'}^2 \cdot Z_{33',6}^2 \cdot (Z_{33',6}^2)^{Z_{55'}^2} \cdot \tilde{Z}_{44',6}^2 \cdot (Z_{33',6}^2)^{Z_{55'}^2} \cdot \tilde{Z}_{44',6}^2)_{i=44',55',6}$.

| The paths/complex conjugates | The braids | The exponent of braids |
|----------------------------|------------|-----------------------|
| (1) ![Diagram](image1)      | $\rho_2^j \rho_1^j \tilde{z}_{v2}^j \rho_1^i \rho_2^{-j}$ | 3                     |
| (2) ![Diagram](image2)      | $\tilde{z}_{6v}$ | 1                     |
| (3) ![Diagram](image3)      | $\rho_6^j \rho_3^j \tilde{z}_{v3}^j \rho_3^{-i} \rho_6^{-j}$ | 2                     |
| (4) ![Diagram](image4)      | $\rho_6^j \rho_4^j \tilde{z}_{v4}^j \rho_4^{-i} \rho_6^{-j}$ | 3                     |
| (5) ![Diagram](image5)      | $\rho_6^j \rho_3^j \tilde{z}_{v3}^j \rho_3^{-i} \rho_6^{-j}$ | 2                     |
| (6) ![Diagram](image6)      | $\rho_6^j \rho_2^j \tilde{z}_{v2}^j \rho_2^{-i} \rho_6^{-j}$ | 2                     |
| (7) ![Diagram](image7)      | $\rho_6^j \rho_2^j \tilde{z}_{v2}^j \rho_2^{-i} \rho_6^{-j}$ | 2                     |
| (8) ![Diagram](image8)      | $\rho_6^j \rho_5^j \tilde{z}_{v5}^j \rho_5^{-i} \rho_6^{-j}$ | 3                     |
| (9) ![Diagram](image9)      | $\rho_m^j \rho_1^m \tilde{z}_{v1}^m \rho_1^{-i} \rho_m^{-j}$ | 2                     |
| (10) ![Diagram](image10)    | $\rho_3^j \rho_1^j \tilde{z}_{v3}^j \rho_1^{-i} \rho_3^{-j}$ | 3                     |
| (11) ![Diagram](image11)    | $\rho_m^j \rho_1^m \tilde{z}_{v1}^m \rho_1^{-i} \rho_m^{-j}$ | 2                     |
| (12) ![Diagram](image12)    | $\tilde{z}_{11v}$ | 1                     |
\[
(\tilde{F}_1 \cdot (\tilde{F}_1)^{Z_{22'}^{-1}Z_{55'}^{-1}})Z_{133'}^{2'}Z_{122'}^{2'}
\]

| 1 | 1' | 2 | 2' | 3 | 4 | 5 | 5' | 6 | 6' |
|---|---|---|---|---|---|---|---|---|---|
| \(\rho_5\rho_2\tilde{z}_{25}\rho_2^{-i}\rho_5^{-i}\) | 3 |

| 1 | 1' | 2 | 2' | 3 | 4 | 5 | 5' | 6 | 6' |
|---|---|---|---|---|---|---|---|---|---|
| \(\rho_5\rho_4\tilde{z}_{45}\rho_4^{-i}\rho_5^{-i}\) | 3 |

| 1 | 1' | 2 | 2' | 3 | 4 | 5 | 5' | 6 | 6' |
|---|---|---|---|---|---|---|---|---|---|
| \(z_{34'}\) | 1 |

| 1 | 1' | 2 | 2' | 3 | 4 | 5 | 5' | 6 | 6' |
|---|---|---|---|---|---|---|---|---|---|
| \(z_{34'}\) | 1 |

| 1 | 1' | 2 | 2' | 3 | 4 | 5 | 5' | 6 | 6' |
|---|---|---|---|---|---|---|---|---|---|
| \(z_{34'}^{-1}\) | 1 |

| 1 | 1' | 2 | 2' | 3 | 4 | 5 | 5' | 6 | 6' |
|---|---|---|---|---|---|---|---|---|---|
| \(z_{34'}^{-1}\) | 1 |

| 1 | 1' | 2 | 2' | 3 | 4 | 5 | 5' | 6 | 6' |
|---|---|---|---|---|---|---|---|---|---|
| \(\rho_5\rho_2\tilde{z}_{25}\rho_2^{-i}\rho_5^{-i}\) | 2 |

| 1 | 1' | 2 | 2' | 3 | 4 | 5 | 5' | 6 | 6' |
|---|---|---|---|---|---|---|---|---|---|
| \(\rho_5\rho_2\tilde{z}_{25}\rho_2^{-i}\rho_5^{-i}\) | 2 |
THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF THE BRANCH CURVE OF $T \times T$ IN $\mathbb{C}^2$  

\[
H_{V_6} = Z^2_{1,22'} \cdot (Z^2_{1,1'})^{Z^2_{1,22'}} \cdot (Z^3_{1,44'})^{Z^2_{1,22'}} \cdot (Z^2_{1,1'})^{Z^2_{1,44'}} \cdot (Z_{11'}) \cdot (Z_{11'})^{Z^2_{1,44'}} \cdot (Z_{55}) \cdot (Z^2_{1,1'})^{Z^2_{1,22'}} \cdot (Z_{55})^{Z^2_{1,22'}}.
\]

| The paths/complex conjugates | The braids | The exponent of braids |
|-----------------------------|------------|-----------------------|
| (1)                         | $\rho_2^j \rho_1^i \bar{z}_1 \rho_1^i \rho_2^{-j}$ | 3                     |
| (2)                         | $\rho_m^j \rho_1^i \bar{z}_1 \rho_1^i \rho_m^{-j}$ | 2                     |
|                             | $\rho_5^j \rho_1^i \bar{z}_1 \rho_1^i \rho_5^{-j}$ |                       |
| (3)                         | $\rho_4^j \rho_1^i \bar{z}_1 \rho_1^i \rho_4^{-j}$ | 3                     |
| (4)                         | $\rho_m^j \rho_1^i \bar{z}_1 \rho_1^i \rho_m^{-j}$ | 2                     |
|                             | $\rho_5^j \rho_1^i \bar{z}_1 \rho_1^i \rho_5^{-j}$ |                       |
| (5)                         | $\bar{z}_{11'}$ | 1                     |
| (6)                         | $\bar{z}_{55'}$ | 1                     |
| (7)                         | $\rho_5^j \rho_1^i \bar{z}_1 \rho_1^i \rho_5^{-j}$ | 2                     |
|                             | $m = 2, 4$   |                       |
| (8)                         | $\rho_5^j \rho_3 \bar{z}_3 \rho_3^{-i} \rho_5^{-j}$ | 3                     |
| (9)                         | $\rho_5^j \rho_1^i \bar{z}_1 \rho_1^i \rho_5^{-j}$ | 2                     |
|                             | $m = 2, 4$   |                       |
| (10)                        | $\rho_4^j \rho_5^i \bar{z}_5 \rho_5^i \rho_6^{-j}$ | 3                     |
\[
\left( \hat{F}_1(\hat{F}_1) Z'_{22} Z'_{66} \right)^{-2} Z'_{22} Z'_{66}^{-2}
\]

| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| --- | --- |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
| \( \rho_{6} \rho_{6} Z'_{26} \rho_{2} Z'_{26} \rho_{2} \) | 2 |
THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF THE BRANCH CURVE OF $T \times T$ IN $\mathbb{C}^2$:

$$H_{V_7} = (Z^2_{2,1})_{i=33',4',66'} \cdot Z^3_{11',2} \cdot (Z^3_{2,55'})^2_{2',33'} \cdot (Z^2_{2,1})_{i=33',4',66'} \cdot (Z^2_{2,1})_{i=33',4',66'} \cdot Z^3_{33',4'} \cdot (Z^2_{4',i})_{i=11',55'} \cdot Z^3_{4',66'} \cdot (Z^2_{4',i})_{i=11',55'} \cdot (Z^2_{4',i})_{i=11',55'} \cdot (Z^2_{4',66'})^{22}_{33',4'} \cdot (\hat{F}_1(F_1)^{Z^2_{11',33'}}_{Z^2_{4',66'}})^{Z^2_{11',33'}}_{Z^2_{4',66'}}.$$
\[
\left( \hat{F}_1 \cdot (\hat{F}_1) \right) Z_{11'} Z_{33'} Z_{56'} Z_{66'} Z_{11'} Z_{22'} Z_{33'} Z_{56'}
\]
\[ H_{V_5} = Z_{11',2}^3 \cdot (Z_{22'}^2 z_{11',66'})^{Z_{6,66'}^{i,2}} \cdot (Z_{33'}^2 z_{55',55'})^{Z_{55',66'}^{i,2}} \cdot (Z_{31'}^2 z_{31',44'}^{i,2}) . \]

| The paths/complex conjugates | The braids | The exponent of braids |
|-------------------------------|------------|-----------------------|
| (1)                           | \( \rho_2^j \rho_1^i \zeta_{12} \rho_1^{-j} \rho_2^{-i} \) | 3                     |
| (2)                           | \( \rho_m^j \rho_2^i \rho_2^{-i} \) \( m = 4, 5 \) | 2                     |
| (3)                           | \( \rho_6^j \rho_2^i \zeta_{26} \rho_6^{-i} \rho_6^{-j} \) | 3                     |
| (4)                           | \( \rho_m^j \rho_3^i \rho_3^{-i} \) \( m = 4, 5 \) | 2                     |
| (5)                           | \( \zeta_{22'}^j \) | 1                     |
| (6)                           | \( \rho_4^j \rho_3^i \zeta_{44} \rho_4^{-i} \rho_4^{-j} \) | 3                     |
| (7)                           | \( \rho_m^j \rho_3^i \zeta_{33} \rho_3^{-i} \) \( m = 1, 6 \) | 2                     |
| (8)                           | \( \rho_5^j \rho_3^i \zeta_{55} \rho_5^{-i} \rho_5^{-j} \) | 3                     |
| (9)                           | \( \rho_m^j \rho_3^i \zeta_{33} \rho_3^{-i} \) \( m = 1, 6 \) | 2                     |
| (10)                          | \( \zeta_{333}^j \) | 1                     |
\[
(\hat{F}_1 \cdot (\hat{F}_1)^{Z_{-1}^{2}Z_{44}^{1}})^{Z_{1,1,3}^{2}Z_{3,44}^{2}Z_{1,2}^{2}}
\]

| 2  | \(\rho_4\rho_1\hat{Z}_{i4}^{4}\rho_4^{-1}\rho_4^{i}\) |
| 3  | \(\rho_5\rho_4\hat{Z}_{i5}^{4}\rho_4^{-1}\rho_5^{j}\) |
| 2  | \(\rho_4\rho_1\hat{Z}_{i4}^{4}\rho_4^{-1}\rho_4^{i}\) |
| 3  | \(\rho_5\rho_1\hat{Z}_{i5}^{4}\rho_1^{-1}\rho_5^{j}\) |
| 1  | \(Z_{56}^{4}\) |
| 1  | \(Z_{56}^{5}\) |
| 1  | \(Z_{56}^{6}\) |
| 1  | \(Z_{56}^{7}\) |

\[
H_{V_9} = Z_{1,22}^{3} \cdot (Z_{2i}^{2}Z_{1,22}^{2})_{i=3,44,66} \cdot (Z_{1,55}^{3})^{Z_{1,22}^{2}} \cdot (Z_{1,44}^{3})^{Z_{1,22}^{2}} \cdot (Z_{11}^{3})^{Z_{1,66}^{2}Z_{1,55}^{2}Z_{1,22}^{2}} \cdot Z_{3,44}^{3} \cdot (Z_{3i}^{3})_{i=22,55} \cdot (Z_{3i}^{3})_{i=22,55} \cdot (Z_{3,66}^{3})^{Z_{3,44}^{2}} \cdot (Z_{3,44}^{3})^{Z_{3,44}^{2}} \cdot (F_1 (F_1)^{Z_{22}^{1}Z_{44}^{1}})^{Z_{22}^{2}Z_{3,44}^{2}Z_{1,22}^{2}}.
\]
|   | The paths/complex conjugates | The braids | The exponent of braids |
|---|-----------------------------|------------|-----------------------|
| (1) | ![paths](image1) | $\rho_2^j \rho_1^i z_1^j \rho_2^{-j}$ | 3 |
| (2) | ![paths](image2) | $\rho_m^j \rho_1^i z_1^j \rho_1^{-j} \rho_m^{-i}$ $m = 4, 6$ $\rho_3^i \rho_1^i z_1^j \rho_1^{-i} \rho_3^{-i}$ | 2 |
| (3) | ![paths](image3) | $\rho_2^j \rho_1^i z_1^j \rho_2^{-j}$ | 3 |
| (4) | ![paths](image4) | $\rho_m^j \rho_1^i z_1^j \rho_1^{-j} \rho_m^{-i}$ $m = 4, 6$ $\rho_3^i \rho_1^i z_1^j \rho_1^{-i} \rho_3^{-i}$ | 2 |
| (5) | ![paths](image5) | $\tilde{z}_{11'}$ | 1 |
| (6) | ![paths](image6) | $\rho_4^j \rho_3^i z_3^j \rho_3^{-j}$ | 3 |
| (7) | ![paths](image7) | $\rho_m^j \rho_3^i z_3^j \rho_3^{-j} \rho_m^{-i}$ $m = 2, 5$ | 2 |
| (8) | ![paths](image8) | $\rho_m^j \rho_3^i z_3^j \rho_3^{-j} \rho_m^{-i}$ $m = 2, 5$ | 2 |
| (9) | ![paths](image9) | $\rho_6^j \rho_3^i z_3^j \rho_3^{-j}$ | 3 |
| (10) | ![paths](image10) | $\tilde{z}_{33'}$ | 1 |
2.3. Properties of $\Delta^2_{54}$.

Now we are checking if the braids we have found are the only ones. For that, we assume that
$\Delta^2_{54} = \prod_{i=1}^{9} C_i H_{V_i} \prod b_i$ be the braid monodromy factorization. $b_i$ are factors corresponding to
singularities that are not covered by $\prod_{i=1}^{9} C_i H_{V_i}$, and each $b_i$ is of the form $Y^{t_i}_i$, $Y_i$ is a positive
halftwist and $0 \leq t_i \leq 3$. We compute degrees of all factors involved. By [6, Theorem 8],
$\deg \Delta^2_{54} = 54 \cdot 53 = 2862$.

On the other hand, $\prod_{i=1}^{9} C_i$ compounds 864 factors, so $\deg \prod_{i=1}^{9} C_i = 864 \cdot 2 = 1728$. In
the computations of each $H_{V_i}, 1 \leq i \leq 9$, $\deg \tilde{F}_1 = \deg (\tilde{F}_1)^{\rho^{-1}} = 24$, so $\deg \tilde{F}_1 (\tilde{F}_1)^{\rho^{-1}} = 48$. The factors outside $\tilde{F}_1 (\tilde{F}_1)^{\rho^{-1}}$ in each $H_{V_i}$ are: 20 degree two factors, 12 degree three factors, 2 degree one factor. The sum of the factors’ degrees resulting from each $H_{V_i}$ is

\[
(\tilde{F}_1 \cdot (\tilde{F}_1)^{Z_{22}^{-1} Z_{44}^{-1}})^{Z_{22}^2 Z_{33}^2 Z_{44}^2 Z_{11}^{*22} Z_{11}^{*22}}
\]

| $\rho_4^i \rho_2^j \tilde{z}_{24} \rho_2^{-i} \rho_4^{-1}$ | 2 |
| $\rho_5^i \rho_4^j \tilde{z}_{24} \rho_2^{-i} \rho_5^{-j}$ | 3 |
| $\rho_4^j \rho_2^i \tilde{z}_{24} \rho_2^{-i} \rho_4^{-i}$ | 2 |
| $\rho_5^j \rho_2^i \tilde{z}_{24} \rho_2^{-i} \rho_5^{-j}$ | 3 |
| $z_{56}$ | 1 |
| $z_{5/6}$ | 1 |
| $z_{6/5}$ | 1 |
| $z_{6/5}^{-1}$ | 1 |
| $z_{6/5}^{-1}$ | 1 |
20 \cdot 2 + 12 \cdot 3 + 2 \cdot 1 = 78. \text{ Thus } \deg H_{V_i} = 78 + 48 = 126 \text{ for } 1 \leq i \leq 9, \text{ and therefore } \\
\deg \prod_{i=1}^{9} H_{V_i} = 126 \cdot 9 = 1134. \text{ Finally } \deg \prod_{i=1}^{9} C_i H_{V_i} = 1134 + 1728 = 2862.

Therefore, \deg \prod_{i=1}^{9} b_i = 2862 \cdot 2862^{-1} = 1. \text{ Since } \forall i, b_i \text{ is a positive exponent of a positive halftwist, we get } b_i = 1 \forall i.

Finally, \Delta_2^{54} = \prod_{i=1}^{9} C_i H_{V_i}.

In the following lemma we prove the Invariance Property for \( C_i \).

**Lemma 6. Complex Invariance of \( C_i \).** For every \( i, i=1, \cdots, 9 \) and \( \forall m_j \in \mathbb{Z}, 1 \leq j \leq 27 \), \( C_i \) is invariant under \( \prod_{j=1}^{27} Z_{m_j}^{m_j} \).

**Proof.** We apply Invariance Rule II and Invariance Remark (iv) [6, Subsection 7.3] on each of the factors of \( C_i \) of the form \( Y_{ii'}^{2} \).

We can summarize the properties of \( \Delta_2^{54} \) in the following theorems.

**Theorem 7. Invariance Theorem for \( \Delta_2^{54} \).** Let \( \rho = \prod_{j=1}^{27} \rho_{m_j} \) for \( \rho_{m_j} = Z_{ij}^{m_j} \) and \( 1 \leq j \leq 27 \). Then \( \Delta_2^{54} \) is invariant under \( \rho \) for every \( m_j \in \mathbb{Z} \).

**Proof.** By Invariance Properties of \( H_{V_i} \) ([4 , Sections 3.1 - 3.9]) and by Lemma 6. \( \square \)

**Theorem 8. Complex Conjugation Theorem.** \( \Delta_2^{54} \) is invariant under complex conjugation.

**Proof.** The same proof from Lemma 19, [22]. \( \square \)

### 2.4. Consequences from the Invariance Theorems.

Recall that \( S^{(0)} = S \) is the regenerated branch curve. Every factor of a braid monodromy factorization \( \Delta_2^{54} \) induces a relation on \( \pi_1(\mathbb{C}^2 - S, M) \). The Invariance Properties are an essential addition to the van Kampen Theorem, since we get more relations in \( \pi_1(\mathbb{C}^2 - S, M) \).

**Theorem 9.** [6, Section 3.10]. Let \( S, \varphi, B_{54} \) be as above. If a sub-factorization \( \prod_{i=s}^{r} Z_{i} \) of \( \Delta_2^{54} \) is invariant under any element \( h \), and \( \prod_{i=s}^{r} Z_{i} \) induces a relation \( \Gamma_{i_1} \cdot \cdots \cdot \Gamma_{i_t} \) on \( \pi_1(\mathbb{C}^2 - S, M) \) via the van Kampen method, then \( (\Gamma_{i_1})_h \cdot \cdots \cdot (\Gamma_{i_t})_h \) is also a relation.
Corollary 10. If \( R \) is any relation in \( \pi_1(C^2 - S, M) \), then \( R_\rho \) is also a relation in \( \pi_1(C^2 - S, M) \), where \( R_\rho \) is the relation induced from \( R \) by replacing \( \Gamma_j \) and \( \Gamma_{j'} \) with \( (\Gamma_j)^{\rho_m_j} \) and \( (\Gamma_{j'})^{\rho_m_{j'}} \) respectively for \( 1 \leq j \leq 27 \), where \( \rho = \prod_{j=1}^{27} \rho_m_j \) and \( \rho_m_j = Z_{jj'}^{m_j} \).

Proof. By Theorem 7. □

Corollary 11. \( \pi_1(C^2 - S, M) \) satisfies all the relations induced in \( R(\Delta_{54}^2) \), all the relations induced from \( (R(\Delta_{54}^2))_\rho \) and all relations induced from the complex conjugation of \( \Delta_{54}^2 \).

Proof. By Theorems 7 and 8. □

3. \( \pi_1(C^2 - S, M) \) AND \( \tilde{\pi}_1 \)

3.1. The van Kampen Theorem.

The van Kampen Theorem induces a finite presentation of the fundamental group of complements of curves by meaning of generators and relations.

We obtained the regenerated braid monodromy factorization \( \Delta_{54}^2 = \prod_{i=1}^{9} C_i H_{V_i} \). We have to apply the van Kampen Theorem on the paths, which correspond to the factors in \( \Delta_{54}^2 \). We take any path from \( k \) to \( \ell \), cut it in \( M \), then towards \( k \) along the path, around \( k \) and coming back the same way. Consider \( A \) as an element of the fundamental group. Do the same to \( \ell \) to obtain \( B \). So \( A, B \) are conjugations of \( \Gamma_k \) and \( \Gamma_\ell \) respectively, see Figure 5.

![Figure 5](image)

**Figure 5.**

Lemma 12. \[40\] \( \{A, B\} \) can be extended to a \( g \)-base of \( \pi_1(C_u - S, M) \) denoted by \( \{\Gamma_j, \Gamma_{j'}\}_{j=1}^{27} \). Moreover, there is an epimorphism \( \pi_1(C_u - S, M) \to \pi_1(C^2 - S, M) \).

Theorem 13. van Kampen for cuspidal curves. Let \( S \) be the regenerated cuspidal branch curve, \( u, M, \varphi, A, B \) defined as above. Let \( \{\delta_i\} \) be a \( g \)-base of \( \pi_1(E - N, u) \). Let
\[ \varphi(\delta_i) = V_i^{\nu_i}, V_i \text{ be a halftwist, } \nu_i = 1, 2, 3. \]

Let \( \{ \Gamma_j, \Gamma'_j \}_{i=1}^{27} \) be a \( g \)-base for \( \pi_1(\mathbb{C}_u - S, M) \).

Then: \( \pi_1(\mathbb{C}^2 - S, M) \) is generated by the images of \( \Gamma_j, \Gamma'_j \) in \( \pi_1(\mathbb{C}^2 - S, M) \) (denoted also by \( \Gamma_j, \Gamma'_j \)) and we get a complete set of relations from those induced from \( \varphi(\delta_i) = V_i^{\nu_i} \), as follows (when \( A, B \) are expressed in terms of \( \{ \Gamma_j, \Gamma'_j \} \)):

(a). \( A = B \), when \( \nu_i = 1 \);

(b). \([A, B] = 1\), when \( \nu_i = 2 \);

(c). \( < A, B > = ABABAB = 1 \), when \( \nu_i = 3 \).

Recall that

\[ \tilde{\pi}_1 = \frac{\pi_1(\mathbb{C}^2 - S, M)}{\langle \Gamma_j^2, \Gamma'_j \rangle}. \]

Let us apply the theorem on three examples, taken from the table corresponding to \( H_{V_2} \).

We follow Figures 6, 7, 8.

**Figure 6.**

Following Figure 6, \( A \) and \( B \) are conjugations of \( \Gamma_5 \) and \( \Gamma'_5 \) respectively. All factors in the conjugations are with a positive exponent since \( \Gamma_j = \Gamma_j^{-1} \) and \( \Gamma'_j = \Gamma'_j^{-1} \) in \( \tilde{\pi}_1 \).

We construct \( B \) by proceeding from \( M \) towards \( 5' \) above \( 6' \) and \( 6 \) encircling \( 5' \) counterclockwise and proceeding back above \( 6 \) and \( 6' \). Therefore \( B = \Gamma_6 \Gamma_6 \Gamma_7 \Gamma_6 \Gamma_6' = (\Gamma_5')^{\Gamma_6 \Gamma_6'}. \) In a similar way, \( A = \Gamma_5 \Gamma_2 \Gamma_1 \Gamma_5 \Gamma_1 \Gamma_1 \Gamma_2 \Gamma_2 \Gamma_5 = (\Gamma_5)^{\Gamma_1 \Gamma_1 \Gamma_2 \Gamma_2 \Gamma_5}. \)

This path is related to a braid which is induced from a branch point. Therefore by Theorem \[ A = B. \] The derived relation is: \( (\Gamma_5)^{\Gamma_1 \Gamma_1 \Gamma_2 \Gamma_2 \Gamma_5} = (\Gamma_5')^{\Gamma_6 \Gamma_6'}. \)
Following the same technique in Figure 7 we get three relations according to three paths. These paths represent the three braids derived from three cusps. The relations are: $\langle \Gamma_2', \Gamma_4 \rangle = 1$, $\langle \Gamma_2, \Gamma_4 \rangle = 1$, $\langle (\Gamma_2')^{\text{rev}}, \Gamma_4 \rangle = 1$

Following the same technique in Figure 8, we get two relations corresponding to the two paths. These paths represent the two braids derived from two nodes. The relations are: $[\Gamma_5,(\Gamma_2)^{\text{rev}}_{1}] = 1$ and $[\Gamma_5,(\Gamma_2')^{\text{rev}}_{1}] = 1$.

Since we followed the process on the point $V_2$, we continue and present the list of relations for $V_2$. We get an infinite number of relations by Corollary 10. We present partially the infinite list for $V_2$, following the list of paths and braids involved in $H_{V_2}$ (Subsection 2.2). Recall that the generators involved in this certain case are $\{\Gamma_j, \Gamma_{j'}\}_{i=1,3,7,9,14,17}$ (see [4,
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(See Figure 6) which are numerated locally as $\{\Gamma_j, \Gamma_{j'}\}_{i=1,2,3,4,5,6}$. Moreover, we add the relations $\Gamma_j^2 = \Gamma_{j'}^2 = 1$.

3.2. Relations $H_{V_2}$.

The following relations are induced from the factors in $H_{V_2}$ and their paths in the first table.

Relation[1] : $< \Gamma_3, \Gamma_4 > = 1$
Relation[2] : $< \Gamma_3, \Gamma_{4'} > = 1$
 Relation[3] : $< \Gamma_{3'}, \Gamma_4 > = 1$
 Relation[4] : $< \Gamma_{3'}, \Gamma_{4'} > = 1$
 Relation[5] : $[\Gamma_1, \Gamma_4^{(\Gamma_2 \cdot \Gamma_{2'})}] = 1$
 Relation[6] : $[\Gamma_1, \Gamma_{4'}^{(\Gamma_2 \cdot \Gamma_{2'})}] = 1$
 Relation[7] : $[\Gamma_{1'}, \Gamma_4^{(\Gamma_2 \cdot \Gamma_{2'})}] = 1$
 Relation[8] : $[\Gamma_{1'}, \Gamma_{4'}^{(\Gamma_2 \cdot \Gamma_{2'})}] = 1$
 Relation[9] : $[\Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_5] = 1$
 Relation[10] : $[\Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{5'}] = 1$
 Relation[11] : $[\Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_5] = 1$
 Relation[12] : $[\Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{5'}] = 1$
 Relation[13] : $[\Gamma_4^{(\Gamma_{4'} \cdot \Gamma_2)}, \Gamma_5] = 1$
 Relation[14] : $[\Gamma_4^{(\Gamma_{4'} \cdot \Gamma_2)}, \Gamma_{5'}] = 1$
 Relation[15] : $[\Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_6] = 1$
 Relation[16] : $[\Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{6'}] = 1$
 Relation[17] : $[\Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_6] = 1$
 Relation[18] : $[\Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{6'}] = 1$
 Relation[19] : $< \Gamma_2, \Gamma_4 > = 1$
 Relation[20] : $< \Gamma_2, \Gamma_{4'} > = 1$
 Relation[21] : $< \Gamma_{2'}, \Gamma_4 > = 1$
 Relation[22] : $< \Gamma_{2'}, \Gamma_{4'} > = 1$
 Relation[23] : $[\Gamma_1, \Gamma_4] = 1$
 Relation[24] : $[\Gamma_1, \Gamma_{4'}] = 1$
 Relation[25] : $[\Gamma_{1'}, \Gamma_4] = 1$
 Relation[26] : $[\Gamma_{1'}, \Gamma_{4'}] = 1$
Relation[27] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_5^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[28] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_5^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[29] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_5^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[30] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_5^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[31] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_5^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[32] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_5^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[33] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_6^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[34] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_6^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[35] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_6^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[36] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}, \Gamma_6^{(\Gamma_4 \Gamma_5)} = 1\]
Relation[37] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)} = \Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2)}\]
Relation[38] : \[< \Gamma_5, \Gamma_6 >= 1\]
Relation[39] : \[< \Gamma_5, \Gamma_6 >= 1\]
Relation[40] : \[< \Gamma_5, \Gamma_6 >= 1\]
Relation[41] : \[< \Gamma_5, \Gamma_6 >= 1\]
Relation[42] : \[\Gamma_3^{(\Gamma_3)}, \Gamma_5^{(\Gamma_3)} = 1\]
Relation[43] : \[\Gamma_3^{(\Gamma_3)}, \Gamma_5^{(\Gamma_3)} = 1\]
Relation[44] : \[\Gamma_3^{(\Gamma_3)}, \Gamma_5^{(\Gamma_3)} = 1\]
Relation[45] : \[\Gamma_3^{(\Gamma_3)}, \Gamma_5^{(\Gamma_3)} = 1\]
Relation[46] : \[\Gamma_2^{(\Gamma_3)}, \Gamma_5^{(\Gamma_3)} = 1\]
Relation[47] : \[\Gamma_2^{(\Gamma_3)}, \Gamma_5^{(\Gamma_3)} = 1\]
Relation[48] : \[\Gamma_2^{(\Gamma_3)}, \Gamma_5^{(\Gamma_3)} = 1\]
Relation[49] : \[\Gamma_2^{(\Gamma_3)}, \Gamma_5^{(\Gamma_3)} = 1\]
Relation[50] : \[< \Gamma_1, \Gamma_5 >= 1\]
Relation[51] : \[< \Gamma_1, \Gamma_5 >= 1\]
Relation[52] : \[< \Gamma_1, \Gamma_5 >= 1\]
Relation[53] : \[< \Gamma_1, \Gamma_5 >= 1\]
Relation[54] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2 \Gamma_5)} = 1\]
Relation[55] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2 \Gamma_5)} = 1\]
Relation[56] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2 \Gamma_5)} = 1\]
Relation[57] : \[\Gamma_4^{(\Gamma_3 \Gamma_3 \Gamma_2 \Gamma_5)} = 1\]
Relation [58]: \([\Gamma_2^{(\Gamma'_2 \cdot \Gamma_1)}, \Gamma_5] = 1\)

Relation [59]: \([\Gamma_2^{(\Gamma'_2 \cdot \Gamma_1)}, \Gamma_5'] = 1\)

Relation [60]: \([\Gamma_2^{(\Gamma'_2 \cdot \Gamma_1)}, \Gamma_5] = 1\)

Relation [61]: \([\Gamma_2^{(\Gamma'_2 \cdot \Gamma_1)}, \Gamma_5'] = 1\)

Relation [62]: \(\Gamma_5^{(\Gamma_1 \cdot \Gamma'_1 \cdot \Gamma_2 \cdot \Gamma_2' \cdot \Gamma_5')} = \Gamma_5^{(\Gamma'_6 \cdot \Gamma'_7)}\)

Relation [63]: \(\Gamma_1 \cdot \Gamma_1 = 1\)

Relation [64]: \(\Gamma_1' \cdot \Gamma_1' = 1\)

Relation [65]: \(\Gamma_2 \cdot \Gamma_2 = 1\)

Relation [66]: \(\Gamma_2' \cdot \Gamma_2' = 1\)

Relation [67]: \(\Gamma_3 \cdot \Gamma_3 = 1\)

Relation [68]: \(\Gamma_3' \cdot \Gamma_3' = 1\)

Relation [69]: \(\Gamma_4 \cdot \Gamma_4 = 1\)

Relation [70]: \(\Gamma_4' \cdot \Gamma_4' = 1\)

Relation [71]: \(\Gamma_5 \cdot \Gamma_5 = 1\)

Relation [72]: \(\Gamma_5' \cdot \Gamma_5' = 1\)

Relation [73]: \(\Gamma_6 \cdot \Gamma_6 = 1\)

Relation [74]: \(\Gamma_6' \cdot \Gamma_6' = 1\).

3.3. Relations \(\hat{F}_1(\hat{F}_1)^{-1}\).

\(\hat{F}_1(\hat{F}_1)^{-1}\) are conjugated by \(\mathbb{Z}_{5,66',33',4}^{-2}\). Therefore, the corresponding elements \(A, B\) in this case are conjugations of the corresponding \(\Gamma_j, \Gamma'_j\) by \(\mathbb{Z}_{5,66',33',4}^{-2}\), \(1 \leq j \leq 6\).

We denote the conjugated \(\Gamma_j\) and \(\Gamma'_j\) by \(\tilde{\Gamma}_j\) and \(\tilde{\Gamma}'_j\) respectively.

\(\tilde{\Gamma}_j = \Gamma_j\) for \(j = 1, 1', 2, 2', 4'\).

We can view the other ones in the following figures.

3.3. Relations \(\hat{F}_1(\hat{F}_1)^{-1}\).

\(\hat{F}_1(\hat{F}_1)^{-1}\) are conjugated by \(\mathbb{Z}_{5,66',33',4}^{-2}\). Therefore, the corresponding elements \(A, B\) in this case are conjugations of the corresponding \(\Gamma_j, \Gamma'_j\) by \(\mathbb{Z}_{5,66',33',4}^{-2}\), \(1 \leq j \leq 6\).

We denote the conjugated \(\Gamma_j\) and \(\Gamma'_j\) by \(\tilde{\Gamma}_j\) and \(\tilde{\Gamma}'_j\) respectively.

\(\tilde{\Gamma}_j = \Gamma_j\) for \(j = 1, 1', 2, 2', 4'\).

We can view the other ones in the following figures.

\[
\tilde{\Gamma}_3, \tilde{\Gamma}'_3, \tilde{\Gamma}_4:
\]

\[
\tilde{\Gamma}_3 = \Gamma_3^{\Gamma_4}
\]

\[
\tilde{\Gamma}'_3 = \Gamma_3^{\Gamma_4'}
\]

\[
\tilde{\Gamma}_4 = \Gamma_4^{(\Gamma_3 \cdot \Gamma_3' \cdot \Gamma_4)}
\]
\[
\tilde{\Gamma}_5 = \Gamma_5^{(\Gamma_5', \Gamma_6', \Gamma_5)}.
\]

\[
\tilde{\Gamma}_5' = \Gamma_5^{(\Gamma_5', \Gamma_6', \Gamma_5', \Gamma_6, \Gamma_5)}.
\]

\[
\tilde{\Gamma}_6, \tilde{\Gamma}_6'.
\]

\[
\tilde{\Gamma}_6 = \Gamma_6^{\Gamma_5}
\]
\[
\tilde{\Gamma}_6' = \Gamma_6^{\Gamma_5'}
\]

The following relations are induced from the factors in \((\hat{F}_1(\hat{F}_1)^{-1})Z_{5,66}^{-2}Z_{33',4}^{2}\) and their paths in the second table of \(H_{v_2}\).

Relation[1] : \(\tilde{\Gamma}_2, \tilde{\Gamma}_3 = 1\)

Relation[2] : \(\tilde{\Gamma}_2, \tilde{\Gamma}_3' = 1\)

Relation[3] : \(\tilde{\Gamma}_2', \tilde{\Gamma}_3 = 1\)

Relation[4] : \(\tilde{\Gamma}_2', \tilde{\Gamma}_3' = 1\)

Relation[5] : \(\tilde{\Gamma}_2, \tilde{\Gamma}_3^3 = 1\)

Relation[6] : \(\tilde{\Gamma}_2', \tilde{\Gamma}_3^3' = 1\)

Relation[7] : \([\tilde{\Gamma}_3, \tilde{\Gamma}_6] = 1\)

Relation[8] : \([\tilde{\Gamma}_3^3, \tilde{\Gamma}_6] = 1\)

Relation[9] : \([\tilde{\Gamma}_3', \tilde{\Gamma}_6'] = 1\)

Relation[10] : \([\tilde{\Gamma}_3^{(\Gamma_2', \Gamma_2')}, \tilde{\Gamma}_6] = 1\)

Relation[11] : \([\tilde{\Gamma}_3^{(\Gamma_2', \Gamma_2')}, \tilde{\Gamma}_6] = 1\)

Relation[12] : \([\tilde{\Gamma}_3^{(\Gamma_2', \Gamma_2')}, \tilde{\Gamma}_6] = 1\)

Relation[13] : \(\tilde{\Gamma}_2^3, \tilde{\Gamma}_6 = 1\)

Relation[14] : \(\tilde{\Gamma}_2^3', \tilde{\Gamma}_6' = 1\)

Relation[15] : \(\tilde{\Gamma}_2^3', \tilde{\Gamma}_6 = 1\)
3.4. Relations $C_i$.

In the same way as above, we derive relations from the braids $C_i$. The list appears in [4, Section 4.11] in a global numeration of the generators.

4. Results

In this paper we presented the list of braids and the braid monodromy factorization $\Delta^2_{54}$ corresponding to the branch curve $S$ of $T \times T$. Recall that $S$ compounds nine curves, each one of them is treated in [4]. The computations in detail appear in [4, Chapter3].

We quoted the van Kampen Method and Theorems and we used them to obtain relations for the fundamental group $\pi_1(C^2 - S, M)$. We presented some relations, the complete list appears in [4, Chapter4]. Setting $\Gamma_j = \Gamma_{j'} = 1$, we can get a presentation for the group $\tilde{\pi}_1$.

Through this paper we concentrated on the 6-point $V_2$ and showed complete computations concerning this point. As quoted above, all other computations appear in [4, Chapters 3 and 4].

In [7] we compute the fundamental group $\pi_1((T \times T)_{Gal})$ of the Galois cover of $T \times T$ with respect to a generic projection to $\mathbb{CP}^2$. Recall that the fundamental group $\pi_1((T \times T)_{Aff_{Gal}})$ is the kernel of the surjection $\psi : \tilde{\pi}_1 \rightarrow S_{18}$.

The Galois cover is a surface of a general type. In [7] we verify the Bogomolov Conjecture. Bogomolov conjectured that if a surface of a general type has a positive index, then it has an infinite fundamental group.

Let $X$ be a surface, $f : X \rightarrow \mathbb{CP}^2$ is a generic projection, $S \subset \mathbb{CP}^2$ its branch curve, $m = \text{deg}S, d = \# \text{ nodes in } S, \rho = \# \text{ of cusps in } S, \mu = \# \text{ of tangency points in } S$ for a
generic projection to \( \mathbb{CP}^1 \), \( n = \text{deg} f \). Then \( C_1^2(X_{Gal}) = \frac{n!}{4}(m-6)^2 \), \( C_2(X_{Gal}) = n!(\frac{m^2}{2} - \frac{3m}{2} + 3 - \frac{3d}{4} - \frac{4\rho}{3}) \) and the index \( \tau = \frac{1}{3}(C_1^2(X_{Gal}) - 2C_2(X_{Gal})) \), see [16, pp. 603-604].

We compute the index of \((T \times T)_{Gal}\).

By the above computations, \( n = 18 \), \( \mu = 54 \), \( \rho = 216 \), \( d = 1080 \), \( m = 54 \). \( C_1^2(X_{Gal}) = 576 \cdot 18! \) and \( C_2(X_{Gal}) = 282 \cdot 18! \). Therefore \( \tau(X_{Gal}) = 4 \cdot 18! \). The index is positive and the group we derive in [7] is infinite.

5. Notations

\((A)_B = B^{-1}AB = A^B.\)

\(X\) an algebraic surface, \( X \subseteq \mathbb{CP}^n.\)

\(X_0\) a degenerated object of a surface \( X \), \( X_0 \subseteq \mathbb{CP}^N.\)

\(S\) an algebraic curve defined over \( \mathbb{R} \), \( S \subseteq \mathbb{C}^2.\)

\(E(\text{ resp.} D)\) be a closed disk on the \( x\)-axis (resp. \( y\)-axis) with the center on the real part of the \( x\)-axis (resp. \( y\)-axis), such that \( \{\text{singularities of } \pi_1\} \subseteq E \ast (D - \partial D).\)

\(\pi : S \rightarrow E.\)

\(K(x) = \pi^{-1}(x).\)

\(N = \{x \in E : \#K(x) < n\}.\)

\(u\) real number such that \( x << u \ \forall x \in N.\)

\(C_u = \pi^{-1}(u).\)

\(\rho_j = Z_{jj}^r.\)

\(\varphi = \text{the braid monodromy of an algebraic curve } S \text{ in } M.\)

\(B_p[D, K] = \text{the braid group.}\)

\(\Delta_p^2 = (H_1 \cdots H_{p-1})^p.\)

\(\pi_1(C^2 - S, M) = \text{the fundamental group of a complement of a branch curve } S.\)

\(\tilde{\pi}_1 = \frac{\pi_1(C^2 - S, M)}{(\Gamma_j, \tilde{\Gamma}_j)}.\)

\(T = \text{complex torus.}\)

\(\tilde{z}_ij(\text{ resp. } \tilde{\bar{z}}_ij) = \text{a path from } q_i \text{ to } q_j \text{ below(resp. above) the real line.}\)

The corresponding halftwists are: \(H(\tilde{z}_ij) = \bar{Z}_{ij} \); \(H(\tilde{\bar{z}}_ij) = Z_{ij}.\)

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