Born-Infeld gravity coupled to Born-Infeld electrodynamics

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We investigate spherically symmetric, static spacetimes in Eddington-inspired Born-Infeld gravity coupled to Born-Infeld electrodynamics. The two constants, $b^2$ and $\kappa$ which parametrise the Born-Infeld structures in the electrodynamics (matter) and gravity sectors, characterise the features of our analytical solutions. Black holes or naked singularities are found to arise, depending on the values of $b^2$ and $\kappa$, as well as charge and mass. Several such solutions are classified and understood through the analysis of the associated metric functions. Interestingly, for a particular relation between these two parameters, $b^2 = 1/\kappa, \kappa > 0$, we obtain a solution resembling the well-known Reissner-Nordström line element, albeit some modifications. Using this particular solution as the background spacetime, we study null geodesics for Born-Infeld photons and also, gravitational lensing. Among interesting features we note (i) an increase in the radius of the photon sphere with increasing $\kappa$ and (ii) a net positive contribution in the leading order correction term involving $\kappa$, in the weak lensing formula for the deflection angle. In summary, our work is the first attempt towards figuring out how Born-Infeld structures in both the matter and gravity sectors can influence the nature and character of resulting gravitational fields.

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I. INTRODUCTION

General Relativity (GR) is largely successful as a classical theory of gravity. It has been tested through experiments and observations to very high precision, in vacuum. However, it is not entirely certain that GR is the correct theory of gravity inside a matter distribution or, in the strong field regime. Moreover, it possesses many unsolved puzzles, both theoretical and observational. This has prompted many researchers to actively pursue various modified theories of gravity. For example, one motivation for constructing modified theories of gravity is to look for the possibility of avoiding the singularity problem in GR, e.g. the occurrence of a big bang in cosmology or black holes in a astrophysical context. In a classical metric theory of gravity, under very general and physical assumptions, these singularities are inevitable, as proved through the Hawking–Penrose theorems [1]. However, it is possible to obtain non-singular geometries as solutions in an alternative theory. For example, non-singular Friedman–Robertson–Walker (FRW) type cosmologies do exist in modified gravity theories. One can also argue that the singularity in the classical theory is expected to be resolved in quantum gravity. Unfortunately, though specific examples exist in the context of loop quantum cosmology we are still far away from any general statements which may be possible when we have a fully consistent quantum description of gravity.

In this article, we investigate one such alternate theory which is inspired from Eddington’s alternative formulation of GR. In this formulation, the $\sqrt{-g} R$ (Ricci scalar) in the Einstein-Hilbert action is replaced by $\sqrt{-\text{det}(R)}$ and the connection is chosen as the basic variable, instead of the metric. Thus, Eddington’s formulation is essentially an affine formulation [2]. In vacuum, it is equivalent to GR and it describes de Sitter gravity. However, coupling of matter was never addressed in this formulation until very recently.

We are also aware of Born–Infeld electrodynamics [3] wherein the divergence in the electric field at the location of the charge/current is removed by using an action different from that of Maxwell. The new field theory due to Born and Infeld includes the determinantal structures in the action, inspired by Eddington’s work. The virtue of this nonlinear electrodynamics is in removing the singularity of the electric field, inspired Deser and Gibbons to construct a Born-Infeld gravity theory in the metric formulation [4]. Later, Vollick [5] introduced the Palatini formulation in Born-Infeld gravity and worked on various related aspects. He also introduced a non-trivial and somewhat artificial way of coupling matter in such a theory [6, 7]. More recently, Banados and Ferreira [8] have come up with a formulation where the matter coupling is different and simpler compared to Vollick’s proposal. We will focus here on the theory proposed in [8] and refer to it as Eddington-inspired Born–Infeld (EiBI) gravity, for obvious reasons. Note that the EiBI theory has the feature that it reduces to GR, in vacuum.

Interestingly, EiBI theory also falls within the class of bimetric theories of gravity (also called bi-gravity). The current bimetric theories have their origin in the seminal work of Isham, Salam and Strathdee [9]. Several articles have appeared in the last few years on various aspects of such bi-gravity theories. In [10], the authors have pointed out that the EiBI field equations can also be derived from an equivalent bi-gravity action. This action is closely related to a recently discovered family of unitary massive gravity theories which are built as bi-gravity theories. Several others have contributed in this direction, in various ways [11, 12].

Let us now briefly recall Eddington–inspired Born–Infeld gravity. The central feature here is the existence of a physical metric which couples to matter and another auxiliary metric which is not used for matter couplings. One needs to solve for both metrics through the field equations. The action for the theory developed in [8], is given as:

$$S_{BI}(g, \Gamma, \Psi) = \frac{1}{8\pi\kappa} \int d^4x \left[ \sqrt{-g} \left( g_{\mu\nu} + \kappa R_{\mu\nu} \right) - \lambda \sqrt{-g} \right] + S_M(g, \Psi)$$

(1)

where $\Lambda = \frac{\lambda - 1}{\kappa}$. We assume $G = 1$ and $c = 1$ throughout in the paper. Variation w.r.t $\Gamma$, done using the auxiliary metric $q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}(q)$ gives

$$q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}(q)$$

(2)

Variation w.r.t $g_{\mu\nu}$ gives

$$\sqrt{-q} q^{\mu\nu} = \lambda \sqrt{-g} g^{\mu\nu} - 8\pi\kappa \sqrt{-g} T^{\mu\nu}$$

(3)

In order to obtain solutions, we need to assume a $g_{\mu\nu}$ and a $q_{\mu\nu}$ with unknown functions, as well as a matter stress energy ($T^{\mu\nu}$). Thereafter, we write down the field equations and obtain solutions using some additional assumptions about the metric functions and the stress energy.

Quite some work on various fronts have been carried out on various aspects of this theory, in the last few years. Astrophysical scenarios have been discussed in [13–21] while cosmology has been dealt with in [10, 22–29]. Among other topics, a domain wall brane in a higher dimensional generalisation, has been analysed in [30]. Generic features of the paradigm of matter-gravity couplings have been analysed in [31]. Some interesting cosmological and circularly
symmetric solutions in $2 + 1$ dimensions are shown in [32]. Constraints on the EiBI parameter $\kappa$ are obtained from studies on compact stars [13, 17], tests in the solar system [33], astrophysical and cosmological observations [18], and nuclear physics studies [34]. In [35], a major problem related to surface singularities has been noticed in the context of stellar physics. However, gravitational backreaction has been sugested as a cure to this problem in [36].

Not much has however been done on black hole physics, or, broadly on the spherically symmetric, static solutions in this theory. It may be noted that the vacuum spherically symmetric static solution is trivially same as the Schwarzschild de Sitter black hole. But, the electrovacuum solutions are expected to deviate from the usual Reissner-Nordström solution in GR. This has been shown in [37, 38] where the authors consider EiBI gravity coupled to a Maxwell electric field of a localized charge. They obtain the resulting spacetime geometries, and study its properties. The basic features of such spacetimes includes a singularity at the location of the charge which may or may not be covered by an event horizon. The strength of the electric field remains nonsingular as in Born-Infeld electrodynamics. However, this may not be the only solution because, in EiBI gravity, the matter coupling is nonlinear. The authors of [39] have shown, in a different framework, that the central singularity could be replaced by a wormhole supported by the electric field.

In our present work, we couple EiBI gravity with Born-Infeld electrodynamics. Thus, we have Born-Infeld structures in both the gravity and matter sectors. We obtain some new classes of spherically symmetric static spacetimes and study their properties. The solutions we present include black holes and naked singularities. Earlier, a lot of work has indeed been done by considering nonlinear electrodynamics coupled to GR [40–47]. Some of them are motivated by string theory since Born-Infeld structures naturally arise in the low energy limit of open string theory [48, 49]. Our article is sectioned as follows. In Section II, we derive the form of the stress-energy tensor assuming an ansatz for the physical metric ($g_{\mu\nu}$). We obtain the general expressions for the metric functions by solving the EiBI field equations with an ansatz for the auxiliary metric ($q_{\mu\nu}$). In Section III, we list the solutions by scanning across regions in the parameter space ($b^2$, $\kappa$), where $b^2$ and $\kappa$ are the parameters in the electromagnetic and gravity sectors, respectively. In Section IV, we analyze, in detail, a particular class of solutions of the Reissner-Nordström-type for which $b^2 = \frac{4\kappa}{q}$ ($\kappa > 0$), i.e. when the two parameters are the same. Here we also discuss null geodesics and gravitational lensing. Finally, in Section V, we summarize our results and indicate possibilities for future work.

## II. SPHERICALLY SYMMETRIC STATIC SPACETIME DUE TO A CHARGED MASS

In curved spacetime, the Lagrangian density for the Born-Infeld electromagnetic field theory is given by [2],

$$\mathcal{L} = \frac{b^2}{4\pi} \sqrt{-g} \left[ 1 - \sqrt{1 + \frac{F}{b^2} - \frac{G}{b^4}} \right]$$  \hspace{1cm} (4)

where, $F = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}$ and $G = \frac{1}{4} F_{\mu\sigma} G^{\mu\sigma}$ are two scalar quantities constructed from the components of the electromagnetic field tensor ($F_{\mu\nu}$) and the dual field tensor ($G_{\mu\nu}$). Here, $b$ sets an upper limit on the electromagnetic field and, when $b \to \infty$, Maxwell’s theory is recovered. The resulting energy-momentum tensor has the following general expression:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = -\frac{b^2}{4\pi} g_{\mu\nu} \left[ \sqrt{1 + \frac{F}{b^2} - \frac{G}{b^4}} - 1 \right] - \frac{b^2}{b^4} \left[ \frac{F_{\mu\sigma} F^{\sigma\nu} - G_{\mu\nu}}{1 + \frac{F}{b^2} - \frac{G}{b^4}} \right]$$  \hspace{1cm} (5)

In our work here we restrict ourselves to an electrostatic scenario (i.e. $A_\mu \equiv \{\phi, 0, 0, 0\}$), in the background of a spherically symmetric, static spacetime. We assume the line element to be of the form,

$$ds^2 = -U(r)e^{2\psi(r)} dt^2 + U(r)e^{2\nu(r)} dr^2 + V(r)r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$  \hspace{1cm} (6)

With this assumption, solving the equations of motion for the electromagnetic field, we obtain

$$\frac{d\phi}{dr} = -\frac{q U e^{\nu + \psi}}{\sqrt{V^2 r^4 + q^2 / b^2}}$$  \hspace{1cm} (7)
Here, \( q \) is the electric charge. In the Minkowski limit of Eq. (3), the Eq. (7) resembles the known Born-Infeld electric field due to a point charge. The non-zero components of the energy-momentum tensor are given as:

\[
T_{tt} = \frac{b^2 U}{4\pi} \left( \frac{\sqrt{V^2 r^4 + q^2 b^2}}{V r^2} - 1 \right) e^{2\nu},
\]

\[
T_{rr} = -\frac{b^2 U}{4\pi} \left( \frac{\sqrt{V^2 r^4 + q^2 b^2}}{V r^2} - 1 \right) e^{2\nu},
\]

\[
T_{\theta\theta} = \frac{b^2}{4\pi} \left( 1 - \frac{V r^2}{\sqrt{V^2 r^4 + q^2 b^2}} \right) V r^2,
\]

and, \( T_{\phi\phi} = \frac{b^2}{4\pi} \left( 1 - \frac{V r^2}{\sqrt{V^2 r^4 + q^2 b^2}} \right) V r^2 \sin^2 \theta. \) (8)

The energy-momentum tensor is not traceless (i.e. \( g^{\mu\nu} T_{\mu\nu} \neq 0 \) for any \( q/b \neq 0 \)). The auxiliary line element is assumed to be of the form

\[
ds_q^2 = -e^{2\psi} dt^2 + e^{2\nu} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)
\] (9)

It worth noting here that our choice of the physical and auxiliary line elements are opposite to the standard choices in [8, 37–39]. While making this choice, we have exploited two facts: (i) the absence of differential expressions in the field equations (2) which opens up the scope of algebraic manipulations, and (ii) the specific forms of the \( tt \) and \( rr \) components of the stress-energy tensor. This gauge choice simplifies the situation to quite an extent. Using Eqs. (8), (11), and (12) in the field equations obtained from the ‘\( g^4 \)’-variation [Eq. (2)], and after some algebra, we obtain the expressions for \( U \) and \( V \). We get a quadratic equation for \( V \) as,

\[
(1 - 4kb^2) V^2 - 2(1 - 2kb^2) V + 1 - \frac{4\kappa^2 q^2 b^2}{r^4} = 0
\] (10)

There are two possible solutions for \( V \) from Eq. (10) and consequently, two solutions for \( U \). We choose the solution for \( U \) and \( V \) such that \( U \rightarrow 1, V \rightarrow 1 \) for large \( r \) and there, the physical spacetime is that found in the GR regime. Therefore, we have,

\[
V(r) = \frac{1 - 2kb^2 \left( 1 + \sqrt{1 + \frac{q^2 b^2}{r^4} - \frac{4\kappa q^2}{r^4}} \right)}{1 - 4kb^2}
\] (11)

\[
U(r) = \frac{(1 - 2kb^2) \sqrt{1 + \frac{q^2 b^2}{r^4} - \frac{4\kappa q^2}{r^4}} - 2kb^2}{(1 - 4kb^2) \sqrt{1 + \frac{q^2 b^2}{r^4} - \frac{4\kappa q^2}{r^4}}}
\] (12)

From the field equations obtained via ‘\( \Gamma \)’-variation [Eq. (2)], we get,

\[
k e^{-2\psi} R_{tt}(q) - U = -1, \quad (13)
\]

\[
k e^{-2\nu} R_{rr}(q) + U = 1, \quad (14)
\]

and \( V r^2 + \kappa \left[ -\frac{d}{dr}(re^{-2\nu}) + 1 \right] = r^2, \) (15)

where, \( R_{tt}(q) = e^{-2\nu+2\psi} \left( \frac{2\nu'}{r^2} - \nu' \psi' + \psi'^2 + \psi'' \right) \) and \( R_{rr}(q) = \frac{2\nu'}{r^2} + \nu' \psi' - \psi'^2 - \psi'' \). Here, primes denote derivatives with respect to \( r \). From Eq. (13) and Eq. (14), we get \( \psi' + \nu' = 0 \). So, without loss of any generality, we can assume \( \nu = -\psi \). Thus, from the Eq. (15), we arrive at,

\[
e^{2\psi(r)} = 1 - \frac{r^2}{3\kappa} + \frac{1}{kr} \int V(r) r^2 dr + \frac{C}{r}
\] (16)

where, \( C \) is a constant of integration. For, \( q = 0 \) (i.e. vacuum), \( U = V = 1 \) [Eqs. (12), (11)], for any \( \kappa \) and the solution should be the Schwarzschild spacetime. Therefore, \( C \) is identified as related to the total mass, i.e., \( C = -2M \).
III. CLASSIFICATION OF SPACETIMES

In this section, we obtain different classes of solutions by scanning the $b^2 - \kappa$ parameter space. For this purpose, we introduce the definition $b^2 = \frac{\alpha}{\kappa}$. We note that, if $\kappa > 0$, then $\alpha > 0$, and if $\kappa < 0$ then $\alpha < 0$. The characteristics/form of the solution changes over different ranges of $\alpha$. These are (i) $\alpha \to \infty$, (ii) $\infty > \alpha > 1$, (iii) $\alpha \to 1$, (iv) $-\infty < \alpha < 1$ (but $\neq 0$), and (v) $\alpha \to -\infty$. We discuss the $\alpha = 1$ case elaborately, as a special case, in a separate section. The remaining classes of solutions are analysed here. Firstly, we note that $\alpha \to \pm \infty$ cases are actually those belonging to the Maxwell limit, i.e., a coupling of EiB gravity with a Maxwellian electric field. This has already been studied earlier [37–39]. One can verify that the solution for this case (particularly for $\alpha \to \infty$), after rewriting in the Schwarzschild gauge, takes exactly the same form as that of [37]. Next, we obtain the solutions for the cases (ii) and (iv) below:

First we rewrite $U$, $V$ as,

$$U(r) = \frac{2 - \alpha}{2(1 - \alpha)} - \frac{\alpha}{2(1 - \alpha)} \sqrt{1 + \frac{4\kappa q^2(1 - \alpha)}{\alpha r^4}},$$

$$V(r) = \frac{2 - \alpha}{2(1 - \alpha)} - \frac{\alpha}{2(1 - \alpha)} \sqrt{1 + \frac{4\kappa q^2(1 - \alpha)}{\alpha r^4}}.$$  

(17) (18)

Using, the Eq. (13), we compute the integral ($\int Vr^2 dr$) (see the Appendix) and use it in the Eq. (16).

$$(ii) \infty > \alpha > 1 \text{ case:}$$

Using Eq. (13), we get,

$$e^{2\psi} = 1 + \frac{\alpha r^2}{6\kappa(\alpha - 1)} \left[ \frac{1}{\kappa q^2(1 - \alpha) - 1} \right] + \frac{\alpha^{1/4}(4q^2)^{3/4}}{3\kappa^{1/4}(\alpha - 1)^{3/4}} \frac{\text{EllipticF} \left[ \text{arcsin} \left( \frac{(4q^2(\alpha - 1))^{1/4}}{\alpha^{1/4}r} \right) \right]}{-1}.$$

$$\frac{2M}{r}$$

(19)

Thus we find all the metric functions. Here, the radius-squared of the 2-sphere at each value of the radial coordinate $r$ is given by $R^2(r) = V(r)r^2$. We note that, for $1 < \alpha \leq 2$; at $r = \frac{(\alpha q^2)^{1/4}}{\kappa}$, the radius-square becomes zero. But, for $\alpha > 2$, there is a non-zero minimum value of $R^2$, i.e., $R^2(r_0) = \frac{(\alpha - 2)^2}{2\kappa q^2}$, where $r_0 = \left[ 4\kappa q^2(1 - 1/\alpha) \right]^{1/4}$. This means, for $1 < \alpha \leq 2$, the spacetime is due to a point charge ($q$) of total mass $M$, but, for $\alpha > 2$, it is distributed over a spherical shell in space. One can verify that, for a very small value of $\kappa$, the solution converges to the known Reissner-Nordström solution. The reason for this is that, as $\kappa$ becomes very small with respect to the value of $\alpha$, $b^2$ becomes large, which then is the Maxwell limit for the electric field and the GR limit in the gravity sector. Further, we have examined the singularities of the spacetime and found that curvature scalar $(\mathcal{R})$ diverges at the position of the charge. Moreover, the existence of the horizon depends upon the values of the parameters $\alpha, \kappa, q$, and $M$. Some of these features are demonstrated in Fig. 1 and Fig. 2, where the plots of the metric functions in the Schwarzschild coordinates are shown.

In the Schwarzschild gauge, the line element [eq. (10)] becomes

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

where, $g_{tt} = U(r)e^{2\psi(r)}$, $g_{rr} = \frac{V(r)}{U(r)}e^{-2\psi(r)}$, and, $r = R \left[ 1 - \frac{\alpha}{2} + \frac{\alpha}{2} \sqrt{1 + \frac{4\kappa q^2}{\alpha R^4}} \right]^{1/2}$.

(iv) $-\infty < \alpha < 1$ case:

In this case, using Eq. (17) we get,

$$e^{2\psi} = 1 - \frac{2M}{r} - \frac{\alpha r^2}{6\kappa(1 - \alpha)} \left[ \frac{1}{\kappa q^2(1 - \alpha) - 1} \right] + \frac{4q^2}{3\kappa^2} \text{Hypergeometric2F1} \left[ \begin{array}{c} 1, 1 \ 4, 2, 4 \\ 1 - \frac{4\kappa q^2(1 - \alpha)}{\alpha r^4} \end{array} \right]$$

(20) (21)

Here, for $0 < \alpha < 1$, the $R^2(r_0) = 0$, for $r_0 = (\alpha q^2)^{1/4}$. But, for $\alpha < 0$ (i.e., for $\kappa < 0$), there is a minimum value of the radius-square, which is $R^2(r_0) = \left( \frac{\alpha q^2}{1 - \alpha} \right)^{1/2}$, for $r_0 = 0$. Therefore, for $\alpha < 0$ or $\kappa < 0$, the charge ($q$) and the
FIG. 1. The plots of the metric functions $g_{tt}$ and $g_{RR}$ are shown as a function of $R$ in the Schwarzschild gauge [eq. (20)]. The value of the parameter $\alpha$ is chosen in the range $1 < \alpha < 2$. In the left column, the plots are specified for $q = M = 1$ and these are compared with the corresponding Reissner-Nordström solution (dashed curve). The character of the spacetime, for this particular choice of parameter values, is that of a naked singularity. In the plots of the right column, $q/M = 0.9 < 1$. The spacetime represents a black hole and has a single horizon. In the last plot, the vertical red coloured dashed line indicates that the metric function $g_{RR}$ diverges to infinity for the corresponding parameter values. The same interpretations are applied in the subsequent plots of the metric functions.

FIG. 2. The plots of the metric functions for $\alpha = 2.5 > 2$. 

mass ($M$) are distributed over a 2-sphere, but, for $1 > \alpha > 0$, it is a point charge. This solution also reduces to the Reissner-Nordström solution for very small $\kappa$ ($\kappa \to \pm 0$). Here also, the spacetime is singular at the location of the charge and it may be a black hole or naked singularity depending upon the parameter values. The metric functions for this class of solutions are plotted in the Schwarzschild coordinates [eq. (20)] for some parameter values and these are shown in Fig. 3 and Fig. 4.

Additionally, we note that, in the limits– $\alpha \to \pm 0$ and $\kappa \to \pm 0$, but a finite $b^2$, it reduces to a geonic black hole.
solution. The geonic black hole is a solution in GR coupled to a Born-Infeld electric field due to a point charge. In a geonic black hole scenario, a distant observer associates a total mass which comprises $M$ (the black hole mass) and a pure electromagnetic mass stored as the self energy in the electromagnetic field. If $M$ is zero, the spacetime becomes regular everywhere.

IV. THE $\alpha = 1$ CASE

For $\alpha = 1$, the quadratic equation [Eq. (10)] becomes linear in $V$. Hence, the metric functions $V$, $U$ and $e^{2\psi}$ [Eqs. (14), (15), (16)] become

$$V = 1 - \frac{\kappa q^2}{r^4}, \quad U = 1 + \frac{\kappa q^2}{r^4}, \quad \text{and} \quad e^{2\psi} = 1 - \frac{2M}{r} + \frac{q^2}{r^2}$$

Then, the physical line element becomes,

$$ds^2 = - \left(1 + \frac{\kappa q^2}{r^4}\right) \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \frac{\left(1 + \frac{\kappa q^2}{r^4}\right)}{\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)} \left(1 - \frac{\kappa q^2}{r^4}\right) r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Here, the radius-square: $R^2(r) = r^2 - \frac{\kappa q^2}{r^4}$, becomes zero at $r = (\kappa q^2)^{1/4}$. So, this describes a point charge $q$ with the total mass $M$. This line element [Eq. (23)] clearly shows that, for $\kappa = 0$, it becomes the Reissner-Nordström solution. The auxiliary line element is also the same as Reissner-Nordström.

The source for such a spacetime has the following form of the stress-energy tensor in the framework of EiBI gravity:

$$T^\mu_\nu = \text{diag.} \left\{ \frac{-q^2}{8\pi(r^4 - \kappa q^2)}, \frac{-q^2}{8\pi(r^4 - \kappa q^2)}, \frac{-q^2}{8\pi(r^4 - \kappa q^2)}, \frac{-q^2}{8\pi(r^4 - \kappa q^2)} \right\}.$$

One may verify that the stress energy satisfies the weak/null energy conditions.

In the Schwarzschild gauge, the line element takes the form

$$ds^2 = -f(R)dt^2 + \frac{2R^2}{(R^2 + \sqrt{R^4 + 4\kappa q^2})} dR^2 + R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where, the redshift function, $f(R) = \left[1 + \frac{4\kappa q^2}{(R^2 + \sqrt{R^4 + 4\kappa q^2})^2}\right] \left[1 - \frac{2\sqrt{M}}{\sqrt{R^4 + \sqrt{R^4 + 4\kappa q^2}}} + \frac{2q^2}{R^2 + \sqrt{R^4 + 4\kappa q^2}}\right]$. At the horizon-radius, $R_{hz}$, $f(R_{hz}) = 0$. Then, the expression for the horizon-radius is given by,

$$R^2_{hz} = \frac{\left(M \pm \sqrt{M^2 - q^2}\right)^4 - \kappa q^2}{\left(M \pm \sqrt{M^2 - q^2}\right)^2}$$

Here, we note that horizon does not exist for $q > M$, similar to the Reissner-Nordström case. For $q = M$, we have $R^2_{hz} = q^2 - \kappa$. So, unlike the Reissner-Nordström extremal solution, for each value of $q = M$, there is an upper limit on $\kappa$ for the existence of the horizon, (see the top panel of the Fig. [E]). From the bottom panel of the Fig. [E] we note that, for $q < M$, for sufficiently small values of $\kappa$ there may be double horizon. For a range of higher values of $\kappa$ a single horizon exists, and for even more high values of $\kappa$ the horizon vanishes. However, in the Reissner-Nordström counterpart, we always have the double horizon.

One can verify that, in general, all the scalar quantities, like the Ricci scalar ($R = g^{ij} R_{ij}$), $R_{ij} R^{ij}$, and the Kretschmann scalar ($R_{ijkl} R^{ijkl}$), diverge at the location of the point charge, i.e. at $r = (\kappa q^2)^{1/4}$, or, $R = 0$. However, for a special choice, $q = \sqrt{R}/3$ and $M = 2\sqrt{R}/3\sqrt{3}$, the Ricci scalar becomes regular everywhere. But, the spacetime still remains singular at the location of the point charge, for such a choice, since the Kretschmann scalar and $R_{ij} R^{ij}$ diverge. Fig. [E] demonstrates the singularity for two different value of the parameters. So, the spacetime is eventually singular, but may or may not be covered by horizon. However, we show that the strength of the electric field at the location of the charge remains finite. For this, we consider a static observer having the four velocity, $u^\mu = \{e^{-\psi}/\sqrt{U}, 0, 0, 0\}$. The electric field vector is defined as $E_\mu = u^\nu F_{\nu\mu}$. So, the square of the strength of the electric field is $E^2 = E^\mu E_\mu = \frac{q^2}{R^2(r_0^2 + q^2)/r^2}$. At the location of the charge, $E^2 = \frac{q^2}{R(r_0 + q^2)/r^2} = b^2 = 1/4\kappa$ (in this case,
FIG. 5. (Top panel) the parameter space: \( q^2 - \kappa \) showing the existence/non-existence of horizon, for \( q = M \). (Bottom panel) the same plot for \( q \neq M \) case.

\( R(r_0) = 0 \). The energy density observed by such a static observer is \( \rho_{\text{obs}} = u^\mu u^\nu T_{\mu\nu} = \frac{b^2}{4\pi} \left( \frac{\sqrt{V^2 + q^2/r^2}}{Vr^2} - 1 \right) \).

In this case, \( \rho_{\text{obs}} = \frac{q^2}{8\pi(r^4 - \kappa q^2)} \). It diverges at the location of the point charge (\( r = r_0 = (\kappa q^2)^{1/4} \)). We note that the non-singular feature of the strength of the electric field remains even in the solution of EiBI gravity coupled to Maxwell’s electrodynamics \([8, 37]\) and this is due to the Born-Infeld like structure of the gravity sector. However, this does not ensure the nonsingularity of the spacetime. In \([8, 37]\), the charge is not point-like, but distributed over a spherical surface. \( \rho_{\text{obs}} \) remains finite, though the scalar invariants diverge at this surface. But, in our solution, both the energy density and the curvature scalar are singular. A non-linear correlation between the gravity and the physical matter sector in the EiBI theory is thus revealed.

**Null geodesics**

Let us now turn to the null geodesics in this spacetime. On the equatorial plane (\( \theta = \pi/2 \)), for null geodesics, we have, from Eq. (23):

\[
\frac{\dot{R}^2}{2} \left( 1 - \frac{\kappa q^2}{r^4} \right) + \frac{L^2}{2R^2(r)} \left( 1 + \frac{\kappa q^2}{r^4} \right) \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) = E^2/2
\]

(27)

where, \( E = \left( 1 + \frac{\kappa q^2}{r^4} \right) \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) \dot{t} \), \( L = R^2(r) \dot{\phi} \) are two conserved quantities and \( \dot{t}, \dot{\phi}, \dot{R} \) are defined as \( \dot{t} = \frac{dt}{d\lambda}, \dot{\phi} = \frac{d\phi}{d\lambda}, \dot{R} = \frac{dR}{d\lambda} \). \( \lambda \) is the affine parameter. So, from Eq. (27) we identify the effective potential for the null
FIG. 6. (Top panel) the ricci scalar \( \mathcal{R} \) is plotted as a function of \( r \) for the parameter values, \( q = M = 0.4, \kappa = 1.0 \). The plot shows that \( \mathcal{R} \) diverges at the location of the charge, i.e. \( r = (\kappa q^2)^{1/4} \) or, \( \mathcal{R} = 0 \). (Bottom panel) the same plot for a \( q \neq M \) case. Here, \( \mathcal{R} \) diverges to negative values.

geodesics as

\[
V_{\text{eff}}(r) = \frac{L^2}{2H^2(r)} \left( 1 + \frac{\kappa q^2}{r^4} \right) \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right)
\]

(28)

The abovementioned effective potential is useful to study the propagation of gravitational signals in the current scenario. We discuss photon propagation in the following subsection.

The propagation of photons

Photons (lightlike particles) in Born-Infeld electromagnetism (BI photon), no longer propagate along the null geodesics of the background spacetime geometry. This is generally true for nonlinear electrodynamics \[51\]. Instead, the BI photon trajectories are determined by the null geodesics of the so-called effective geometry, given by \[51\ [52\],

\[
g^{\mu\nu}_{\text{eff}} = \left( 1 + \frac{1}{b^2} F \right) g^{\mu\nu} + \frac{1}{b^2} F^\mu_\sigma F^{\sigma\nu}
\]

(29)

where, \( F = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \) and \( g_{\mu\nu} \) defines the metric of the background spacetime. Such an effective geometry is induced due to nonlinear self-interaction of the photon. Here, in Eq. \[29\], we assume the same \( F_{\mu\nu} \) which generates background
spacetime geometry. Then, the effective line element becomes

\[
\text{d} s^2_{\text{eff}} = - \left( 1 + \frac{\kappa q^2}{r^4} \right) \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) \text{d}t^2 + \frac{1 + \frac{\kappa q^2}{r^4}}{\left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right)^2} \text{d}r^2 + \frac{1 + \frac{\kappa q^2}{r^4}}{r^2} \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right)
\]

(30)

Since our interest is to study the null geodesics of the effective geometry, we take out the conformal factor \((1 + \frac{\kappa q^2}{r^4})\) and rewrite the effective line element as,

\[
\text{d} s^2_{\text{eff}} = - \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) \text{d}t^2 + \frac{\text{d}r^2}{\left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right)^2} + \frac{1 + \frac{\kappa q^2}{r^4}}{r^2} \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right)
\]

(31)

The effective geometry becomes exactly the same as the Reissner-Nordström solution when \(\kappa = 0\), which is expected. The analysis of the squared area radius in the effective geometry, \(R_{ph}^2 = r^2 \left( \frac{r^4 + \kappa q^2}{r^4 - \kappa q^2} \right)\), shows that the area radius \(R_{ph}\) becomes infinity both at \(r = r_0 = (\kappa q^2)^{1/4}\) and \(r \to \infty\), but has a minimum value \(R_{th}^2 = \frac{\sqrt{\kappa q}}{(\sqrt{2} - 2)^{1/2}} \left( \frac{\sqrt{2} - 1}{\sqrt{2} - 2} \right)\) at \(r_{th} = \left( \frac{\kappa q^2}{(\sqrt{2} - 2)^2} \right)^{1/4}\) (similar to the throat radius square of a wormhole). This implies that for each \(R_{ph} \in [R_{th}, \infty]\), there exists a \(r \in [r_{th}, \infty]\) and a \(r \in [r_0, r_{th}]\). But these two sets of ranges of \(r\) describe two different geometries. We are interested in studying the propagation of photons coming from and going towards asymptotically flat regions of the given spacetime. Hence, we focus on \(r_{th} \leq r < \infty\), which describes an asymptotically flat geometry.

Now, for the null geodesics, we have

\[
\frac{1}{2} \dot{r}^2 + \frac{L^2}{2r^2} \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) \left( \frac{r^4 - \kappa q^2}{r^4 + \kappa q^2} \right) = \frac{E^2}{2}
\]

(32)

where, \(E = \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) \dot{t}\) and \(L = R_{ph}^2(r) \dot{\phi}\) are two conserved quantities. So, the effective potential for the BI photons become

\[
V_{\text{eff}}(r) = \frac{L^2}{2r^2} \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) \left( \frac{r^4 - \kappa q^2}{r^4 + \kappa q^2} \right)
\]

(33)

We plot the effective potential for different values of \(K = \kappa/M^2\) (a dimensionless parameter) and extract the essential features. A few plots are shown in the top panel of the Fig. 7. In these plots, the effective potential is rescaled as \(v_{\text{eff}} = 2M^2 V_{\text{eff}}/L^2\) and is plotted as a function of the dimensionless quantity \(R_{ph}/M\). In these parametric plots, the parameter \(\dot{r}\) value ranges from \(r_{th}\) to infinity. The maxima of the effective potential corresponds to the unstable circular orbits for photons. The corresponding spherical surface is called the ‘photon sphere’ [50]. In the Reissner-Nordström solution the radius of the photon sphere depends on \(q\) and \(M\). In our solution, we note that, it depends additionally on \(\kappa\). As an example, we consider the case for \(q/M = 1\). Then, in the Reissner-Nordström solution, the radius of this photon sphere is \(R_{\text{ph}} = 2M = 2q\) (see Fig. 7(a)). In our solution, we have a different picture. Here, from the figures of the top panel of Fig. 7 we see that as \(\kappa\) is increased the radius of the photon sphere increases. This feature becomes more clear when we look at the plot of deflection angles \((\alpha)\) in the bottom panel of Fig. 7. Near photon sphere, the deflection angle becomes infinitely large, indicating multiple full rounds of the light around the lensing object. Consequently, a sequence of a large number of highly demagnified relativistic images may arise [53, 54].

Analysing the plots of the effective potential for greater values of \(K\), one can easily verify that the radius of the photon sphere \((R_{\text{ph}})\) becomes closer and closer to the throat radius \((R_{th})\) as \(K\)-value increases.

**Strong deflection:** In this effective geometry, the exact expression for the deflection angle [50] becomes,

\[
\alpha = 2|\phi(\infty) - \phi(R_{tp})| - \pi
\]

\[
= 2 \int_{r_{tp}}^{\infty} \frac{\left( r^4 - \kappa q^2 \right)}{r^2 (r^4 + \kappa q^2)} \left[ \frac{1}{r_{tp}^2} \left( \frac{r_{tp}^4 - \kappa q^2}{r_{tp}^4 + \kappa q^2} \right) \left( 1 - \frac{2M}{r_{tp}} + \frac{q^2}{r_{tp}^2} \right) - \frac{1}{r^2} \left( \frac{r^4 - \kappa q^2}{r^4 + \kappa q^2} \right) \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) \right]^{-1/2} dr
\]

(34)
With the conserved quantities, a spherically symmetric static line element, \( ds \) compare this result with the Reissner-Nordström case. We obtain the weak deflection angle perturbatively \([57]\). For \( M \) radius of the photon sphere increases from the value \( 2 \) (Fig. 7(f)), we see that the nature of plot do not change. However, as the \( Q \) increases rapidly (>> \( b \)), where, \( K \) of Fig. 7. Fig. 7(d) (for the Eq. (36) becomes, \( R_{ph}^{2} = \frac{r_{ph}^{2} + \kappa q^{2}}{r_{ph}^{2} - \kappa q^{2}} \). Here, \( R_{tp} \) indicates the turning point, \( i.e. \frac{dR_{ph}}{d\phi} |_{R_{tp}} = \frac{R_{ph}}{\phi} |_{R_{tp}} = 0 \). The deflection angle is plotted as a function of the dimensionless parameter \( R_{tp}/M \) and the figures are shown in the bottom panel of Fig. 7. Fig. 7(d) (for \( K = 0, Q = 1 \)) shows that, in the extremal Reissner-Nordström case, the deflection angle increases rapidly (>> 2\( \pi \)) near the photon sphere (\( R_{tp} = R_{ps} = 2M \)). From the next two figures (Fig. 7(e) and Fig. 7(f)), we see that the nature of plot do not change. However, as the \( K \) value increases, it is clearly seen that radius of the photon sphere increases from the value 2\( M \).

**Weak deflection**: Let us now look for the leading order contribution of \( \kappa \) in the weak-deflection angle formula. We compare this result with the Reissner-Nordström case. We obtain the weak deflection angle perturbatively \([57]\). For a spherically symmetric static line element, \( ds^{2} = -A(r)dt^{2} + B(r)dr^{2} + r^{2}C(r)d\Omega^{2} \), the radial geodesic equation is

\[
2B\ddot{r} + B'\dot{r}^{2} + A'\dot{r}^{2} - \dot{r}^{2}C' \dot{\phi}^{2} = 0
\]

(35)

With the conserved quantities, \( E = A\dot{t}, L = Cr^{2}\dot{\phi} \), and the new variable \( u = \frac{1}{r} \), for the null geodesics, the Eq. (35) becomes \([57]\)

\[
\frac{d^{2}u}{d\phi^{2}} + \frac{C}{B}u = -\frac{1}{2}u^{2} \frac{d}{du} \left( \frac{C}{B} \right) + \frac{1}{2b_{0}^{2}} \frac{d}{du} \left( \frac{C^{2}}{AB} \right)
\]

(36)

where, \( b_{0} = L/E \) is the impact parameter. In the Reissner-Nordström case, \( (A = B^{-1} = 1 - 2Mu + q^{2}u^{2}, C = 1) \), the Eq. (36) becomes,

\[
\frac{d^{2}u}{d\phi^{2}} + u = 3Mu^{2} - 2q^{2}u^{3}
\]

(37)
In our case, $A = B^{-1} = (1 - 2M u + q^2 u^2)$, and $C = (1 + \kappa q^2 u^4)/(1 - \kappa q^2 u^4)$. We assume small $\kappa (\kappa q^2 u^4 << 1)$ and write the Eq. (36) keeping only the first order terms in $\kappa$,

$$\frac{d^2 u}{d\phi^2} + u = 3M u^2 - \left(2 - \frac{8\kappa}{q^2}ight) q^2 u^3 - 6\kappa q^2 u^5 + 14\kappa q^2 M u^6 - 8\kappa q^4 u^7$$

We set $\epsilon = M u_N$, where $u_N = 1/b_0$, i.e. the inverse of the Newtonian distance of closest approach. Further, we define a dimensionless variable $\xi = u/u_N$. Thereafter, we write the Eq. (38) to second order in $\epsilon$,

$$\frac{d^2 \xi}{d\phi^2} + \xi \approx 3\epsilon \xi^2 - \epsilon^2 \left(2 - 8\kappa u_N^2\right) \frac{q^2}{M^2} \epsilon^3$$

Writing $\xi$ as $\xi = \xi_0 + \epsilon \xi_1 + \epsilon^2 \xi_2 + \ldots$, and substituting them in the Eq. (39) we find the following set of equations, by collecting terms of different order:

$$\frac{d^2 \xi_0}{d\phi^2} + \xi_0 = 0$$

$$\frac{d^2 \xi_1}{d\phi^2} + \xi_1 - 3\xi_0^2 = 0$$

$$\frac{d^2 \xi_2}{d\phi^2} + \xi_2 - 2\xi_0 \xi_1 + 2 \left(1 - 4\kappa u_N^2\right) \frac{q^2}{M^2} \xi_0^2 = 0$$

Solving the equations (40), (41), and (42) we get the approximate inverse radial distance,

$$u \approx u_N \cos \phi + M u_N^2 \left(\frac{3}{2} - \frac{1}{2} \cos(2\phi)\right) + \frac{3M^2 u_N^3}{16} \left(20 - 4(1 - 4\kappa u_N^2) \frac{q^2}{M^2}\right) \phi \sin \phi + \left(1 + \frac{q^2}{3M^2(1 - 4\kappa u_N^2)}\right) \cos(3\phi)$$

Assuming, $u \to 0$ at $\phi = \pi/2 + \delta$, we solve for $\delta$ using the Eq. (43). For a small deflection angle, we find

$$\delta \approx 2M u_N + 3\pi M^2 u_N^2 \left[5 - (1 - 4\kappa u_N^2) \frac{q^2}{M^2}\right] + O(M^3 u_N^3)$$

The deflection angle is $\Delta \phi = 2\delta$. At the distance of closest approach, $r_{tp} = 1/u(0)$. So, converting from $u_N$ to $u(0)$ and substituting $u(0) = 1/r_{tp}$, we finally obtain the approximate weak deflection angle

$$\Delta \phi \approx \frac{4M}{r_{tp}} + \frac{M^2}{r_{tp}^2} \left(\frac{15}{4} \pi - 4\right) - \frac{3\pi}{4} \left(1 - \frac{4\kappa}{r_{tp}^2}\right) \frac{q^2}{r_{tp}^2}$$

In this formula, the term involving $\kappa$ is the leading order contribution to weak deflection. If $\kappa = 0$, the expression becomes the same as in the Reissner-Nordström case [55]. By extending the calculation up to the fourth order perturbation (i.e. $O(\epsilon^4)$), one can verify that the sign of the leading order $\kappa$-involving term that appears in the fourth order correction ($1/r_{tp}^2$-term) in the deflection angle, is positive. This term involving $\kappa$ increases the net deflection angle. As the value of $\kappa$ is increased, the deflection angle also increases further. Earlier, we have seen that, with the increase of $\kappa$, the radius of photon-sphere increases. Thus, the study of both strong and weak deflection seems to suggest an overall attractive effect of $\kappa$.

V. CONCLUSIONS

In this article, we have derived new classes of spherically symmetric static solutions in EiBI gravity coupled to Born-Infeld electrodynamics. The spacetimes demonstrate the effect of coupling of a non-linear electrodynamics (the Born-Infeld electrodynamics) to a modified theory of gravity (the EiBI gravity). All solutions are analytical. We summarize our results below, point wise:

(i) In our work here we have used a non-standard gauge to find the spacetimes in EiBI gravity. The use of the standard ansätze for the physical and auxiliary line elements as employed in [8, 37–39], led to a complicated set of coupled differential equations. We note that our non-standard choice is far more useful insofar as solution construction
is concerned. Similar unconventional gauge choices may be used in other cases with the hope of simplifying the set of equations we need to solve.

(ii) We obtain two classes of spacetimes in section III. All the known results for different electrically charged, static scenarios can be obtained by taking different limits for \( \alpha = 4\kappa b^2 \) and \( \kappa \). In general, the spacetimes we obtain are singular at the location of the charge, where the charge can be localized about a point or distributed over a spherical shell (2-sphere). The singularity may or may not be covered by an event horizon, i.e. it may be black hole or a naked singularity. However, all solutions converge to the Reissner-Nordström solution at large distances.

(iii) Although the results are analytical, the expressions for the metric functions are complicated. However, a special choice, \( \alpha = 1 \) \((\kappa > 0)\), leads to a simple solution that looks similar to the Reissner-Nordström. We analyze this special solution in detail in Section IV. We find that this spacetime also possesses the same generic features mentioned earlier in (ii) above. Subsequently, we investigate the null geodesics in the effective geometry for BI photons as well as gravitational lensing. We find that as \( \kappa \) is increased, the radius of the photon sphere increases. One may explore strong gravitational lensing further using the approach in [59]. We also find the leading order contribution of \( \kappa \) in the expression for the weak deflection angle. The overall sign of this term is positive but it depends on \( \kappa \).

In summary, using non-standard ansatze for the physical and auxiliary metrics, we are able to obtain analytical solutions in Born-Infeld gravity coupled to Born-Infeld electrodynamics. A particularly simple solution for the case when the two parameters \( \kappa \) and \( b^2 \) are related is demonstrated and then used to study null geodesics and light deflection. It is possible that there are other non-singular solutions as well which are non-singular from both the gravitational and electrodynamic perspectives. For this it is essential to scan the \((\kappa, b^2, q_M)\) parameter space more thoroughly. We hope to work on this and communicate our results in the near future.

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Appendix: Integral results

Here, we compute the integrals \( \int V(r)r^2 dr \), in detail, which have been used in the section III to obtain the analytical expressions of two different classes of spacetimes [Eq. (19) and Eq. (21)]. These integrals are non-trivial and may not be computed correctly otherwise (e.g. using Mathematica).

\( (a) \infty > \alpha > 1 \) case:

\[
\int Vr^2 dr = -\frac{(2 - \alpha)r^3}{6(\alpha - 1)} + \frac{\alpha}{2(\alpha - 1)} \int \sqrt{r^4 - \frac{4\kappa q^2(\alpha - 1)}{\alpha}} dr \quad \text{(A.1)}
\]

The last term on the R.H.S. of the Eq. A.1 is a non-trivial integral which we evaluate below:

\[
\int \sqrt{r^4 - \beta} dr = r\sqrt{r^4 - \beta} - \int \frac{2r^4 dr}{\sqrt{r^4 - \beta}} \quad \text{(Using partial integration)}
\]

\[
= r\sqrt{r^4 - \beta} - 2 \int \sqrt{r^4 - \beta} dx - 2\beta \int \frac{dr}{\sqrt{r^4 - \beta}}
\]

\[
= \frac{1}{3} r\sqrt{r^4 - \beta} - \frac{2\beta^{3/4}}{3} \int \frac{dy}{\sqrt{y^4 - 1}} \quad (r = \beta^{1/4}y)
\]

Now,

\[
\int \frac{dy}{\sqrt{y^4 - 1}} = -\int \frac{du}{\sqrt{1 - u^4}} \quad (u = 1/y)
\]

\[
= -\text{EllipticF}[\arcsin(1/y), -1] + C_{ext}
\]

So,

\[
\int \sqrt{r^4 - \beta} dr = \frac{1}{3} r\sqrt{r^4 - \beta} + \frac{2\beta^{3/4}}{3} \left(\text{EllipticF}[\arcsin(\beta^{1/4}/r), -1] - C_{ext}\right) \quad \text{(A.2)}
\]
where, \( \beta = 4kq^2(\alpha-1) \) (\( \beta > 0 \)) and \( C_{ext} \) is an additional constant term appearing due to the fact that Elliptic integrals are defined with the zero lower limit and hence, these are not the so-called indefinite integrals. The Eq. (A.1) becomes,

\[
\int Vr^2 dr = \frac{r^3}{3} + \frac{\alpha r^3}{6(\alpha - 1)} \left[ \sqrt{1 - \frac{4kq^2(\alpha - 1)}{\alpha^4}} - 1 \right] + \frac{\alpha^{1/4}(4kq^2)^{3/4}}{3(\alpha - 1)^{1/4}} \left( \text{EllipticF} \left[ \arcsin \left( \frac{(4kq^2(\alpha - 1))^{1/4}}{\alpha^{1/4}r} \right), 1 \right] - C_{ext} \right)
\]

(A.3)

Now, to fix up the extra constant \( C_{ext} \), we use the fact that, in the limit of \( \alpha \to \infty \) and \( \kappa \to 0 \), the full expression of \( e^{2\psi} \) must take form of the Reissner-Nordström solution (i.e. \( e^{2\psi} \to 1 - \frac{2M}{r} + \frac{\kappa^2}{r^2} \)). Under this constraint, \( C_{ext} = 0 \).

(b) \(-\infty < \alpha < 1 \) case:

\[
\int Vr^2 dr = -\frac{(2-\alpha)r^3}{6(\alpha - 1)} + \frac{\alpha}{2(\alpha - 1)} \int \sqrt{r^4 + \gamma} dr
\]

(A.4)

where, \( \gamma = \frac{4kq^2(1-\alpha)}{\alpha} \) (\( \gamma > 0 \)). We evaluate the last integral on R.H.S. of the Eq. (A.4) below:

\[
\int \sqrt{r^4 + \gamma} dr = \frac{1}{3} \sqrt{r^4 + \gamma} + \frac{2\gamma^{3/4}}{3} \int \frac{dy}{\sqrt{y^4 + 1}}, \quad (r = \gamma^{1/4}y)
\]

Now,

\[
\int \frac{dy}{\sqrt{y^4 + 1}} = -i\text{EllipticF} \left[ i\text{ArcSinh}(\sqrt{r}), -1 \right] + C'_{ext}
\]

So,

\[
\int \sqrt{r^4 + \gamma} dr = \frac{1}{3} \sqrt{r^4 + \gamma} - \frac{2\gamma^{3/4}}{3} \left[ \sqrt{i\text{EllipticF} \left[ i\text{ArcSinh}(\sqrt{r}(\gamma^{1/4(1-\alpha)})), -1 \right] - C'_{ext}} \right]
\]

(A.5)

So, the Eq. (A.4) becomes

\[
\int Vr^2 dr = \frac{r^3}{3} + \frac{\alpha r^3}{6(\alpha - 1)} \left[ \sqrt{1 + \frac{4kq^2(1-\alpha)}{\alpha^4}} - 1 \right] + \frac{\alpha^{1/4}(4kq^2)^{3/4}}{3(1-\alpha)^{1/4}} \left( \sqrt{i\text{EllipticF} \left[ i\text{arcsinh} \left( \frac{\sqrt{\alpha^{1/4}r}}{(4kq^2(1-\alpha))^{1/4}} \right), 1 \right] - C'_{ext}} \right)
\]

(A.6)

Similar to the earlier case, we fix up the extra constant \( C'_{ext} \), by taking the limit of \( \alpha \to -\infty \) and \( \kappa \to -0 \). Then also, the full expression of \( e^{2\psi} \) takes the form of the Reissner-Nordström solution. Under this constraint, \( C'_{ext} = -\int_0^\infty \frac{dy}{\sqrt{y^4 + 1}} = -\frac{\Gamma^2(1/4)}{4\sqrt{\pi}} \). Eq. (A.6) can be rewritten further as,

\[
\int Vr^2 dr = \frac{r^3}{3} + \frac{\alpha r^3}{6(\alpha - 1)} \left[ \sqrt{1 + \frac{4kq^2(1-\alpha)}{\alpha^4}} - 1 \right] + \frac{4kq^2}{3r} \text{Hypergeometric2F1} \left[ \frac{1}{4}, \frac{1}{2}; \frac{5}{4}, \frac{4kq^2(1-\alpha)}{\alpha r^4} \right]
\]

(A.7)

where, we use

\[
\sqrt{i\text{EllipticF} \left[ i\text{arcsinh} \left( \sqrt{y} \right), -1 \right] + \frac{\Gamma^2(1/4)}{4\sqrt{\pi}} = \int_0^\infty \frac{dz}{\sqrt{z^4 + 1}} = \frac{1}{y} \text{Hypergeometric2F1} \left[ \frac{1}{4}, \frac{1}{2}; \frac{5}{4}, \frac{1}{y^4} \right].
\]

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