Valid inequalities and global solution algorithm for Quadratically Constrained Quadratic Programs

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We consider the exact solution of problem \((P)\) that consists in minimizing a quadratic function subject to quadratic constraints. Starting from the classical convex relaxation that uses the McCormick’s envelopes, we introduce 12 inequalities that are derived from the ranges of the variables of \((P)\). We prove that these general Triangle inequalities cut feasible solutions of the McCormick’s envelopes. We then show how we can compute a convex relaxation \((P^*)\) which optimal value equals to the ”Shor’s plus RLT plus Triangle” semi-definite relaxation of \((P)\) that includes the new inequalities. We also propose a heuristic for solving this huge semi-definite program that serves as a separation algorithm. We then solve \((P)\) to global optimality with a branch-and-bound based on \((P^*)\). Moreover, as the new inequalities involved the lower and upper bounds on the original variables of \((P)\), their use in a branch-and-bound framework accelerates the whole process. We show on the unitbox instances that our method outperforms the compared solvers.

Key words: Quadratic Convex Relaxation, Valid inequalities, Global optimization, Semi-Definite programming, Lagrangian duality, sub-gradient algorithm, Quadratic Programming

1. Introduction

In this manuscript, we aim at the exact solution of Mixed-Integer Quadratically Constrained Programs. These are the class of optimization problems where the objective function to minimize and the constraints are all quadratic. Such a problem can be formulated as follows:

\[
(P) \begin{cases}
        \min f_0(x) \equiv \langle Q_0, xx^T \rangle + c^T_0 x \\
        \text{s.t. } f_r(x) \equiv \langle Q_r, xx^T \rangle + c^T_r x \leq b_r \quad r \in \mathcal{R} \\
        \ell_i \leq x_i \leq u_i \quad & i \in \mathcal{I} \\
        x_i \in \mathbb{R} \quad & i \in \mathcal{I}
    \end{cases}
\]

with \(\langle A, B \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}\), and where \(\mathcal{I} = \{1, \ldots, n\}, \mathcal{R} = \{1, \ldots, m\}, \forall r \in \{0\} \cup \mathcal{R}, (Q_r, c_r) \in \mathcal{S}_n \times \mathbb{R}^n\), with \(\mathcal{S}_n\) the set of \(n \times n\) symmetric matrices, \(b \in \mathbb{R}^m\), and \(u \in \mathbb{R}^n\).
Without loss of generality we suppose that $\ell \in \mathbb{R}^n_+$ and that the feasible domain of $(P)$ is non-empty. Problem $(P)$ trivially contains the case where there are quadratic equalities, since an equality can be replaced by two inequalities. It also contains the case of linear constraints since a linear equality is a quadratic constraint with a zero quadratic part.

Problem $(P)$ is a fundamental problem in global optimization. It arises in many applications, including facility location, production planning, multiperiod tankage quality problems in refinery processes, circle packing problems, euclidean distance geometry, or triangulation problems, see for instance Adhya et al. (1999), Aholt et al. (2012), Dorneich and Sahinidis (1995), Huy Hac (1982), Liu and Sahinidis (1997), Locatelli and Raber (2002), Sutou and Dai (2002), Xie and Sahinidis (2008).

In the specific case where matrices $Q_r$ are positive semi-definite, $(P)$ is a convex problem that is polynomially solvable. In this case, efficient solvers are available. However, in general matrices $Q_r$ are indefinite and problem $(P)$ is $\mathcal{NP}$-hard (Garey and Johnson (1979)). In this case, the development of suitable relaxations is required for exact solution algorithms. Indeed, global optimization methods for solving $(P)$ are classically based on a branch-and-bound framework in which a lower bound is computed by a certain relaxation scheme at each node of the search tree. The tightness of the relaxation and the efficiency in computing the lower bounds have a great impact on the behavior of such methods. Relaxation techniques for problems $(P)$ are mainly based on linearization, convex quadratic programming, or Semi-Definite Programming (SDP). Most of the proposed relaxations of the literature are either linear or quadratic and convex. To compute such a relaxation, the quadratic functions are reformulated as convex equivalent functions in an extended space of variables. More precisely, new variables $Y_{ij}$ are introduced for all $(i,j) \in I^2$, ($I^2$ is the cartesian product of a set $I$ by itself), that are meant to satisfy the equalities $Y_{ij} = x_i x_j$. The equivalent formulation is then solved by a branch-and-bound algorithm based on a relaxation of the later non-convex equalities, for instance by linear constraints (see for instance McCormick (1976), Sherali and Adams (2013), Yajima and Fujie (1998)). Software, implementing some of the methods described above, are available, see, for instance, Baron (Sahinidis and Tawarmalani (2010)), or GloMIQO (Misener and Floudas (2012, 2013), Misener et al. (2015)). Using semi-definite relaxations within branch-and-bound frameworks to solve $(P)$ was also widely studied (Anstreicher (2009), Chen and Burer (2012),
\textbf{Amélie Lambert}: \textit{Valid inequalities and global solution algorithm for QCQPs}.

A semi-definite relaxation of \((P)\) can be obtained by lifting \(x\) to a symmetric matrix \(X = xx^T\) where the later non-convex constraints are relaxed to \(X - xx^T \succeq 0\) (\(M \succeq 0\) means that \(M\) is positive semidefinite). This standard semi-definite relaxation is often referred to as "Shor’s" relaxation of \((P)\). In \textbf{Anstreicher (2009)}, the "Shor’s plus RLT" relaxation was introduced, where the McCormick’s envelopes where added to the latter relaxation. Method \textbf{MIQCR (Mixed Integer Quadratic Convex Reformulation)} \cite{Elloumi2019} also handles \((P)\). In this approach, a tight quadratic convex relaxation to \((P)\) is calculated using the "Shor’s plus RLT" relaxation. The original problem is then solved by a branch-and-bound based on the obtained relaxation.

In all these relaxations, the main common feature is that equalities \(Y_{ij} = x_i x_j\) are relaxed, and then forced by a branch-and-bound algorithm in order to come back to the original problem. The contributions of this paper are in order. In Section \textbf{2}, we start by the design of new families of valid inequalities for \((P)\). As the McCormick’s envelopes, they are derived from the ranges \([\ell_i, u_i]\) of each variable \(x_i\). After a complete description of these inequalities, we prove which of them cut feasible points of the McCormick’s envelopes. We then link them to the literature, showing that in the specific case were \(x \in [0,1]^n\), they amount to the well known Triangle inequalities introduced in \textbf{Padberg (1989)}. We thus call them General Triangle inequalities. Then in Section \textbf{3}, we use the General Triangles to build quadratic convex relaxations of \((P)\) sharper than the ones used in method \textbf{MIQCR}. We also prove that we can compute a quadratic convex relaxation of \((P)\) that has the same optimal value as the "Shor’s plus RLT plus Triangle" relaxation. Moreover, as the general upper and lower bounds \(\ell\) and \(u\) are involved within the new inequalities, the relaxation is again tighten in the course of the branch-and-bound accelerating the satisfaction of equalities \(Y_{ij} = x_i x_j\). Since, there exists a huge number of new inequalities, we propose in Section \textbf{4} to separate them within a bundle algorithm that heuristically solves the "Shor’s plus RLT plus Triangle" relaxation. Finally in Section \textbf{5}, we evaluate the new method, called \textbf{MIQCR-T}, on the instances of the literature and compare it to several solvers. Section \textbf{6} draws a conclusion.

\section{The General Triangle inequalities}

We start by building a convex relaxation of \((P)\) in an extended space of variables. As classically done, we introduce \(n(n+1)/2\) new variables \(Y_{ij}\) for all \((i,j) \in \mathcal{I}^2\) that represent
product $x_ix_j$. Then, we define sets $U = \{(i,j) \in \mathcal{I}^2 : i \leq j\}$ and $T = \{(i,j,t) : (i,j) \in U, t = 1, \ldots, 4\}$, and we recall McCormick’s envelopes $C = \{(x,Y) : h_{ij}^t(x,Y) \leq 0 \text{ } \forall (i,j,t) \in T\}$ with:

$$h_{ij}^t(x,Y) = \begin{cases} Y_{ij} - u_jx_i - \ell_i x_j + u_j \ell_i & t = 1 \\ Y_{ij} - u_i x_j - \ell_j x_i + u_i \ell_j & t = 2 \\ -Y_{ij} + u_j x_i + u_i x_j - u_i u_j & t = 3 \\ -Y_{ij} + \ell_j x_i + \ell_i x_j - \ell_i \ell_j & t = 4 \end{cases}$$

We also consider any set of positive semi-definite matrices $S_0, \ldots, S_m$. Then, $\forall r \in \{0\} \cup \mathcal{R}$, we formulate $f_r(x)$ as a sum of a quadratic function of the $x$ variables and a linear function of the $Y$ variables:

$$f_{r,S_r}(x,Y) = \langle S_r, xx^T \rangle + c_r^T x + \langle Q_r - S_r, Y \rangle.$$  

It holds that $f_{r,S_r}(x,Y)$ is equal to $f_r(x)$ if $Y_{ij} = x_i x_j$. By replacing the initial functions $f_r(x)$ by the convex functions $f_{r,S_r}(x,Y)$, and by relaxing the non-convex equalities $Y_{ij} = x_i x_j$ with the inequalities of set $C$, we obtain a family of quadratic convex relaxation of $(P)$:

$$\begin{align*}
\min_{(P_{S_0,\ldots,S_m})} & \langle S_0, xx^T \rangle + c_0^T x + \langle Q_0 - S_0, Y \rangle \\
\text{s.t.} & \langle S_r, xx^T \rangle + c_r^T x + \langle Q_r - S_r, Y \rangle \leq b_r & r \in \mathcal{R} \\
& h_{ij}^t(x,Y) \leq 0 & (i,j,t) \in T \\
& Y_{jj} = Y_{ij} & (i,j) \in U \end{align*}$$

where $U = \{(i,j) \in \mathcal{I}^2 : i < j\}$. Problem $(P_{S_0,\ldots,S_m})$ is parameterized by the set of matrices $S_0, \ldots, S_m$, we observe that taking $\forall r \in \{0\} \cup \mathcal{R}$, $S_r = 0_n$ the zero $n \times n$ matrices amounts to the standard linearization of $(P)$.

We now present new families of valid inequalities for $(P_{S_0,\ldots,S_m})$ that strengthen this relaxation. As the McCormick’s envelopes, they are derived from the ranges $[\ell_i, u_i]$ of each variable $x_i$. The idea is to consider $\forall (i,j,k) \in \mathcal{V} = \{(i,j,k) \in \mathcal{I}^3 : i < j < k\}$, three variables $x_i$, $x_j$ and $x_k$. Since these variables satisfy Constraints (2), we have $(u_i - x_i)(u_j - x_j)(u_k - x_k) \geq 0$, or equivalently:

$$u_k x_i x_j + u_j x_i x_k + u_i x_j x_k - u_i u_k x_j - u_j u_k x_i - u_i u_j x_k + u_i u_j u_k \geq x_i x_j x_k$$
using the McCormick inequality \( x_j x_k \geq \ell_j x_k + \ell_k x_j - \ell_j \ell_k \), we get:

\[
u_k x_i x_j + u_j x_i x_k + u_i x_j x_k - u_i u_k x_j - u_j u_k x_i + u_i u_j u_k \geq x_i (\ell_j x_k + \ell_k x_j - \ell_j \ell_k)
\]
or equivalently the new quadratic inequality:

\[
(\ell_k - u_k) x_i x_j + (\ell_j - u_j) x_i x_k - u_i u_k x_j + u_k x_i x_k + u_i u_k x_j + (u_j u_k - \ell_j \ell_k) x_i + u_i u_j x_k - u_i u_j u_k \leq 0
\]

that can be linearized by use of the variables \( Y \). These inequalities are obviously valid by construction. In the example above, we also could have chosen to substitute the product \( x_j x_k \) by its other McCormick envelope, i.e. \( x_j x_k \geq u_j x_k + u_j x_j - u_j u_k \), or, to substitute either the product \( x_i x_j \) or \( x_i x_k \) by one of its two McCormick estimators. Hence, considering all possible combinations for all \((i, j, k) \in \mathcal{V}\) we obtain 8 families of 6 inequalities. The question is now to determine which of these 48 inequalities, when they are linearized, are non redundant in \((P_{S_0,\ldots,S_m})\). We further prove in Propositions 1–8 that 12 out of 48 of these inequalities cut feasible solutions of \((P_{S_0,\ldots,S_m})\).

**Family 1** We consider \((u_i - x_i)(u_j - x_j)(u_k - x_k) \geq 0\), and we get:

\[
\Rightarrow (\ell_k - u_k) Y_{ij} + (\ell_j - u_j) Y_{ik} - u_i u_k x_j + (u_j u_k - \ell_j \ell_k) x_i + u_i u_j x_k - u_i u_j u_k \leq 0 \quad (T1)
\]
or symmetrically

\[
u_k x_i x_j + u_j x_i x_k + u_i x_j x_k - u_i u_k x_j - u_j u_k x_i + u_i u_j u_k \geq x_i x_j x_k \geq x_j (\ell_i x_k + \ell_k x_i - \ell_i \ell_k)
\]

\[
\Rightarrow (\ell_i - u_i) x_j x_k - x_i x_k + (\ell_i - u_i) x_j x_k + (u_k u_i - \ell_i \ell_k) x_j + u_j u_k x_i + u_i u_j x_k - u_i u_j u_k \leq 0
\]

\[
\Rightarrow (\ell_i - u_i) Y_{ij} - u_j Y_{ik} + (u_i u_k - \ell_i \ell_k) x_j + u_j u_k x_i + u_i u_j x_k - u_i u_j u_k \leq 0 \quad (T2)
\]
or symmetrically

\[
- u_k x_i x_j + u_j x_i x_k + u_i x_j x_k - u_i u_k x_j - u_j u_k x_i + u_i u_j u_k \geq x_i x_j x_k \geq x_k (\ell_j x_i + \ell_i x_j - \ell_i \ell_j)
\]

\[
\Rightarrow - u_k x_i x_j + (\ell_j - u_j) x_i x_k + (\ell_i - u_i) x_j x_k + u_i u_k x_j + u_j u_k x_i + (u_i u_j - \ell_i \ell_j) x_k - u_i u_j u_k \leq 0
\]

\[
\Rightarrow - u_k Y_{ij} + (\ell_j - u_j) Y_{ik} + (\ell_i - u_i) Y_{jk} + u_i u_k x_j + u_j u_k x_i + (u_i u_j - \ell_i \ell_j) x_k - u_i u_j u_k \leq 0 \quad (T3)
\]
or

\[
u_k x_i x_j + u_j x_i x_k + u_i x_j x_k - u_i u_k x_j - u_j u_k x_i - u_i u_j u_k \geq x_i x_j x_k \geq x_j (u_j x_k + u_k x_j - u_j u_k)
\]

\[
\Rightarrow u_i x_j x_k - u_i u_k x_j - u_i u_j x_k + u_i u_j u_k \geq 0
\]

\[
\Rightarrow - u_i Y_{jk} + u_i u_k x_j + u_i u_j x_k - u_i u_j u_k \leq 0 \quad (T4)
\]
or symmetrically

\[
- u_k x_i x_j + u_j x_i x_k + u_i x_j x_k - u_i u_k x_j - u_j u_k x_i - u_i u_j u_k \geq x_i x_j x_k \geq x_j (u_i x_k + u_k x_i -
Proposition 1. (T1)-(T3) cut feasible solutions of \((P_{s_0,\ldots,s_m})\), while (T4)-(T6) are redundant.

Proof.
(i) Consider the following solution \((x, Y)\) satisfying \(h_{ij}^l(x, Y) \leq 0\), for all \((i, j, t) \in T:\)

- \(x_i = \frac{u_i + \ell_i}{2}, x_j = \frac{u_j + \ell_j}{2},\) and \(x_k = \frac{u_k + \ell_k}{2}\)
- \(Y_{ij} = \frac{u_i \ell_j + u_j \ell_i}{2}, Y_{ik} = \frac{u_i \ell_k + u_k \ell_i}{2},\) and \(Y_{jk} = \frac{u_j \ell_k + u_k \ell_j}{2}\)

Using this solution in (T1), we get:

\[
\frac{\ell_i u_j \ell_k}{2} + u_i \ell_j \ell_k - \frac{\ell_i u_j \ell_k}{2} - u_i \ell_j u_k + \frac{u_i \ell_j \ell_k}{2} - \frac{\ell_i \ell_j \ell_k}{2} - u_i u_j \ell_k + \frac{u_i u_j u_k}{2} - \frac{\ell_i \ell_j \ell_k}{2} = \frac{1}{2} (\ell_i - u_i)(\ell_j - u_j)(\ell_k - u_k) \leq 0.
\]

This solution is thus cut off by the inequality (T1). By symmetry, the proof is similar for (T2) and (T3).

(ii) (T4) is equivalent to \(-u_i(Y_{jk} - u_k x_j - u_j x_k + u_j u_k) \leq 0\) that is redundant with (6).

By symmetry, the proof is similar for (T5) and (T6).

Family 2 We consider \((u_i - x_i)(u_j - x_j)(x_k - \ell_k) \geq 0\), and we get:

\[
\Rightarrow (u_j - \ell_j)Y_{ik} + (\ell_k - u_k)Y_{ij} + u_i Y_{jk} + (\ell_j u_k - \ell_k u_j)x_i - u_i u_j x_k + u_i u_j x_k \leq 0 \tag{T7}
\]

or symmetrically

\[
\Rightarrow u_j Y_{ik} + (\ell_k - u_k)Y_{ij} + (u_i - \ell_i)Y_{jk} - u_j \ell_k x_i + (\ell_i u_k - u_i \ell_k)x_j - u_i u_j x_k + u_i u_j \ell_k \leq 0 \tag{T8}
\]

or

\[
\Rightarrow (u_j - \ell_j)x_i x_k + \ell_k x_i x_j + u_i x_j x_k - u_j \ell_k x_i - u_i \ell_k x_j - u_i x_j x_k + u_i u_j \ell_k \leq x_i x_j x_k \leq x_i (u_k x_j + \ell_j x_k - \ell_j u_k)\]

\[
\Rightarrow (u_j - \ell_j)Y_{ik} + (\ell_k - u_k)Y_{ij} + u_i Y_{jk} + (\ell_j u_k - \ell_k u_j)x_i - u_i u_j x_k + u_i u_j x_k \leq 0 \tag{T9}
\]
or symmetrically
\[ u_j x_i x_k + \ell_k x_i x_j + u_i x_j x_k - u_j x_k x_i - u_i x_k x_j - u_i u_j x_k + u_i u_j \ell_k \leq x_i x_j x_k \leq x_k (u_j x_i + \ell_i x_j - \ell_i u_j) \]
\[ \Rightarrow \ell_k x_i x_j + (u_i - \ell_i) x_j x_i - u_i \ell_k x_i - u_i u_j x_k + (\ell_i - u_i) u_j x_k + u_i u_j \ell_k \leq 0 \]
\[ \Rightarrow \ell_k Y_{ij} + (u_i - \ell_i) Y_{jk} - u_j \ell_k x_i - u_i \ell_k x_j + (\ell_i - u_i) u_j x_k + u_i u_j \ell_k \leq 0 \quad (T10) \]
or
\[ u_j x_i x_k + \ell_k x_i x_j + u_i x_j x_k - u_j x_k x_i - u_i \ell_k x_i - u_i u_j x_k + u_i u_j \ell_k \leq x_i x_j x_k \leq x_j (u_i x_k + \ell_k x_j - u_i \ell_k) \]
\[ \Rightarrow u_i x_j x_k - u_i \ell_k x_j - u_i u_j x_k + u_i u_j \ell_k \leq 0 \quad (T11) \]
or symmetrically
\[ u_j x_i x_k + \ell_k x_i x_j + u_i x_j x_k - u_j x_k x_i - u_i \ell_k x_i - u_i u_j x_k + u_i u_j \ell_k \leq x_i x_j x_k \leq x_j (u_i x_k + \ell_k x_j - u_i \ell_k) \]
\[ \Rightarrow u_j x_i x_k - u_j \ell_k x_i - u_i u_j x_k + u_i u_j \ell_k \leq 0 \quad (T12) \]

**Proposition 2.** (T7) and (T8) cut feasible solutions of \((P_{S_0,...,S_m})\), while (T9)-(T12) are redundant.

**Proof.**

(i) Consider the following solution \((x, Y)\) satisfying \(h_{ij}^t(x, Y) \leq 0\), for all \((i, j, t) \in T:\n\[
\bullet \ x_i = \frac{w_i + \ell_i}{2}, \ x_j = \frac{w_j + \ell_j}{2}, \text{ and } x_k = \frac{w_k + \ell_k}{2} \\
\bullet \ Y_{ij} = \frac{u_i u_j + \ell_i \ell_j}{2}, \ Y_{ik} = \frac{u_i w_k + \ell_i \ell_k}{2}, \text{ and } Y_{jk} = \frac{u_j w_k + \ell_j \ell_k}{2} 
\]
Using this solution in (T7), we get:
\[
\frac{u_i u_j w_k + \ell_i \ell_j \ell_k}{2} - \frac{u_i w_k + \ell_i \ell_k}{2} + \frac{\ell_i \ell_j w_k - u_j \ell_k}{2} = \frac{1}{2} (u_j - u_i) (u_j - \ell_j) (\ell_k - u_k) \geq 0.
\]
This solution is thus cut off by the inequality (T7). By symmetry, the proof is similar for (T8).

(ii) (T9) is equivalent to \((u_j - \ell_j) (Y_{ik} - u_i x_k - \ell_k x_i + u_i \ell_k) + \ell_k (Y_{ij} - \ell_j x_i - u_i x_j + u_i \ell_j) \leq 0\) that is redundant with (5). By symmetry, the proof is similar for (T10).

(iii) (T11) is equivalent to \(u_i (Y_{jk} - \ell_k x_j - u_j x_k + u_j \ell_k) \leq 0\) that is redundant with (5). By symmetry, the proof is similar for (T12).

\[ \square \]

**Family 3** We consider \((u_i - x_i)(x_j - \ell_j)(u_k - x_k) \geq 0\) and we get:
\[
u_k x_i x_j + u_i x_j x_k + \ell_j x_i x_k - u_i u_k x_j - \ell_j u_k x_i - u_i \ell_j x_k + u_i \ell_j u_k \leq x_i x_j x_k \leq x_k (u_j x_i + \ell_j x_k - \ell_j u_k) \\
\Rightarrow u_i x_j x_k - u_i u_k x_j - u_i \ell_j x_k + u_i \ell_j u_k \leq 0 \\
\Rightarrow u_i Y_{jk} - u_i u_k x_j - u_i \ell_j x_k + u_i \ell_j u_k \leq 0 \quad (T13)\]
or symmetrically

\[ u_kx_ix_j + u_ix_jx_k + \ell_jx_ix_k - u_iu_kx_j - \ell_ju_kx_i - u_i\ell_jx_k + u_i\ell_ju_k \leq x_ix_jx_k \leq x_k(u_ix_j + \ell_jx_i - u_i\ell_j) \Rightarrow u_kx_ix_j - u_iu_kx_j - \ell_ju_kx_i + u_i\ell_ju_k \leq 0 \]

or

\[ u_kY_{ij} - u_iu_kx_j - \ell_ju_kx_i + u_i\ell_ju_k \leq 0 \quad (T14) \]

or symmetrically

\[ u_kx_ix_j + u_ix_jx_k + \ell_jx_ix_k - u_iu_kx_j - \ell_ju_kx_i - u_i\ell_jx_k + u_i\ell_ju_k \leq x_ix_jx_k \leq x_j(u_ix_k + \ell_kx_i - u_i\ell_k) \Rightarrow (u_k - \ell_k)x_ix_j + \ell_jx_ix_k + u_i(\ell_k - u_k)x_j - \ell_ju_kx_i - u_i\ell_jx_k + u_i\ell_ju_k \leq 0 \]

or symmetrically

\[ (u_k - \ell_k)Y_{ij} + \ell_jY_{ik} + u_i(\ell_k - u_k)x_j - \ell_ju_kx_i - u_i\ell_jx_k + u_i\ell_ju_k \leq 0 \quad (T15) \]

or symmetrically

\[ u_kY_{ij} + (u_i - \ell_i)Y_{jk} + (\ell_j - u_j)Y_{ik} - u_iu_kx_j + u_\ell_jx_j - \ell_ju_kx_i + u_i\ell_ju_k \leq 0 \quad (T16) \]

or symmetrically

\[ u_kY_{ij} + (u_i - \ell_i)Y_{jk} + (\ell_j - u_j)Y_{ik} - u_iu_kx_j - \ell_ju_kx_i + (\ell_i - u_i\ell_j)x_k + u_i\ell_ju_k \leq 0 \quad (T18) \]

**Proposition 3.** (T17) and (T18) cut feasible solutions of \((P_{S_0,...,S_m})\), while (T13)-(T16) are redundant.

**Proof.**

(i) Consider the following solution \((x, Y)\) satisfying \(h_{ij}^t(x,Y) \leq 0\), for all \((i,j,t) \in T:\)

- \(x_i = \frac{u_i + \ell_i}{2}, \quad x_j = \frac{u_j + \ell_j}{2}\), and \(x_k = \frac{u_k + \ell_k}{2}\)
- \(Y_{ij} = \frac{\ell_iu_j + u_iu_j}{2}, \quad Y_{ik} = \frac{u_k\ell_i + \ell_iu_k}{2}, \quad Y_{jk} = \frac{u_j\ell_k + \ell_ju_k}{2}\)

Using this solution in Constraints (T17), we get:

\[
\frac{u_iu_ju_k}{2} + \frac{\ell_iu_j\ell_k}{2} - \frac{u_i\ell_ju_k}{2} - \frac{u_\ell_j\ell_k}{2} + \frac{u_i\ell_j\ell_k}{2} - \frac{\ell_iu_j\ell_k}{2} - \frac{\ell_iu_k\ell_j}{2} - \frac{u_iu_j\ell_k}{2} = \frac{u_i}{2}(\ell_j - u_i)(u_j - \ell_j)(\ell_k - u_k) \geq 0.
\]

This solution is thus cut off by the inequality (T17). By symmetry, the proof is similar for (T18).

(ii) (T13) is equivalent to \((u_i - \ell_i)(Y_{jk} - u_kx_j - \ell_jx_k + \ell_ju_k) \leq 0\) that is redundant with (4). By symmetry, the proof is similar for (T14).

(iii) (T15) is equivalent to \((u_i - \ell_i)(Y_{jk} - u_kx_j - \ell_jx_k + \ell_ju_k) + \ell_j(Y_{ik} - \ell_iu_kx_i + \ell_iu_kx_i + \ell_iu_kx_i) \leq 0\) that is redundant with (4). By symmetry, the proof is similar for (T16).
Family 4 We consider \((x_i - \ell_i)(u_j - x_j)(u_k - x_k) \geq 0\) and we get:
\[
  u_k x_i x_j + \ell_i x_j x_k + u_j x_i x_k - \ell_i u_k x_j - u_j u_k x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq x_i x_j x_k \leq x_i (u_k x_j + \ell_j x_k - \ell_j u_k)
\]
\[
  \Rightarrow \ell_i x_j x_k + (u_j - \ell_j) x_i x_k - \ell_i u_k x_j + u_j (\ell_j - u_j) x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq 0
\]
\[
  \Rightarrow \ell_i Y_{jk} + (u_j - \ell_j) Y_{ik} - \ell_i u_k x_j + u_k (\ell_j - u_j) x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq 0 \quad (T19)
\]
or symmetrically
\[
  u_k x_i x_j + \ell_i x_j x_k + u_j x_i x_k - \ell_i u_k x_j - u_j u_k x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq x_i x_j x_k \leq x_i (u_j x_k + \ell_k x_j - u_j \ell_k)
\]
\[
  \Rightarrow (u_k - \ell_k) x_i x_j + \ell_i x_j x_k - \ell_i u_k x_j + u_j (\ell_k - u_k) x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq 0
\]
\[
  \Rightarrow (u_k - \ell_k) Y_{ij} + \ell_i Y_{jk} - \ell_i u_k x_j + u_j (\ell_k - u_k) x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq 0 \quad (T20)
\]
or
\[
  u_k x_i x_j + \ell_i x_j x_k + u_j x_i x_k - \ell_i u_k x_j - u_j u_k x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq x_i x_j x_k \leq x_j (u_k x_i + \ell_i x_k - \ell_i u_k)
\]
\[
  \Rightarrow u_j x_i x_k - u_j u_k x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq 0
\]
\[
  \Rightarrow u_j Y_{ik} - u_j u_k x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq 0 \quad (T21)
\]
or symmetrically
\[
  u_k x_i x_j + \ell_i x_j x_k + u_j x_i x_k - \ell_i u_k x_j - u_j u_k x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq x_i x_j x_k \leq x_k (u_j x_i + \ell_j x_j - \ell_j u_j)
\]
\[
  \Rightarrow u_k x_i x_j - \ell_i u_k x_j - u_j u_k x_i + \ell_i u_j u_k \leq 0
\]
\[
  \Rightarrow u_k Y_{ij} - \ell_i u_k x_j - u_j u_k x_i + \ell_i u_j u_k \leq 0 \quad (T22)
\]
or
\[
  u_k x_i x_j + \ell_i x_j x_k + u_j x_i x_k - \ell_i u_k x_j - u_j u_k x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq x_i x_j x_k \leq x_j (u_k x_i + \ell_k x_j - u_k \ell_k)
\]
\[
  \Rightarrow (u_k - \ell_k) Y_{ij} + (\ell_i - u_i) Y_{jk} + u_j Y_{ik} + (u_i \ell_k - u_i u_k) x_j - u_j u_k x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq 0 \quad (T23)
\]
or symmetrically
\[
  u_k x_i x_j + \ell_i x_j x_k + u_j x_i x_k - \ell_i u_k x_j - u_j u_k x_i - \ell_i u_j x_k + \ell_i u_j u_k \leq x_i x_j x_k \leq x_k (u_i x_j + \ell_j x_i - u_i \ell_j)
\]
\[
  \Rightarrow u_k Y_{ij} + (\ell_i - u_i) Y_{jk} + (u_j - \ell_j) Y_{ik} - \ell_i u_k x_j - u_j u_k x_i + (u_i \ell_j - \ell_i u_j) x_k + \ell_i u_j u_k \leq 0 \quad (T24)
\]

**Proposition 4.** \((T23)\) and \((T24)\) cut feasible solutions of \((P_{S_0,...,S_m})\), while \((T19)-(T22)\) are redundant.

**Proof.**

(i) Consider the following solution \((x, Y)\) satisfying \(h_{ij}^t(x, Y) \leq 0\), for all \((i, j, t) \in T\):

- \(x_i = \frac{u_i + \ell_i}{2},\ x_j = \frac{u_j + \ell_j}{2},\) and \(x_k = \frac{u_k + \ell_k}{2}\)
- \(Y_{ij} = \frac{\ell_i + u_i}{2} u_j + \frac{\ell_j + u_j}{2} u_i,\ Y_{ik} = \frac{\ell_i + u_i}{2} u_k + \frac{\ell_k + u_k}{2} u_i,\) and \(Y_{jk} = \frac{\ell_j + \ell_k}{2} u_i + \frac{\ell_i + \ell_k}{2} u_j + \frac{\ell_i + \ell_j}{2} u_k + \frac{\ell_i + \ell_j + \ell_k}{2} u_k - \frac{\ell_i + \ell_j}{2} u_k - \frac{\ell_i + \ell_k}{2} u_j - \frac{\ell_j + \ell_k}{2} u_i\)

Using this solution in Constraints \((T23)\), we get: \(\frac{u_i u_j u_k}{2} + \frac{\ell_i \ell_j u_k}{2} - \frac{u_i \ell_j u_k}{2} - \frac{\ell_i \ell_j \ell_k}{2} + \frac{\ell_i \ell_j \ell_k}{2} - \frac{\ell_i u_j u_k}{2} - \frac{\ell_j \ell_k u_i}{2} - \frac{\ell_i \ell_k u_j}{2} - \frac{\ell_j \ell_k u_i}{2} = \frac{1}{2} (\ell_i - u_i) (\ell_j - u_j) (u_k - \ell_k) \geq 0\). This solution is thus cut off by the inequality \((T23)\). By symmetry, the proof is similar for \((T24)\).
(ii) \((T_{10})\) is equivalent to \((u_j - \ell_j)(Y_{ik} - u_k x_i - \ell_i x_k + \ell_i u_k) + \ell_i (Y_{jk} - \ell_j x_k - u_k x_j + \ell_j u_k) \leq 0\) that is redundant with \([4]\). By symmetry, the proof is similar for \((T_{20})\).

(iii) \((T_{21})\) is equivalent to \(u_j (Y_{ik} - u_k x_i - \ell_i x_k + \ell_i u_k) \leq 0\) that is redundant with \([4]\). By symmetry, the proof is similar for \((T_{22})\).

\[
\square
\]

**Family 5** We consider \((u_i - x_i)(x_j - \ell_j)(x_k - \ell_k) \geq 0\) and we get:

\[
\begin{align*}
&u_i x_j x_k + \ell_j x_i x_k + \ell_k x_i x_j - u_i \ell_k x_j - \ell_j \ell_k x_i - u_i \ell_j \ell_k \geq x_i x_j x_k \geq x_i (\ell_j x_k + \ell_k x_j - \ell_j \ell_k) \\
\Rightarrow &-u_i x_j x_k + u_i \ell_k x_j + u_i \ell_j \ell_k \leq 0 \\
\Rightarrow &-u_i Y_{jk} + u_i \ell_k x_j + u_i \ell_j \ell_k - u_i \ell_j \ell_k \leq 0 \quad (T_{25})
\end{align*}
\]

or

\[
\begin{align*}
&u_i x_j x_k + \ell_j x_i x_k + \ell_k x_i x_j - u_i \ell_k x_j - \ell_j \ell_k x_i - u_i \ell_j \ell_k \geq x_i x_j x_k \geq x_j (u_i x_k + u_i x_j - u_i u_k) \\
\Rightarrow &-\ell_j x_i x_k + (u_k - \ell_k) x_i x_j + \ell_k u_i x_i + \ell_j \ell_k x_i + u_i \ell_j x_k - u_i \ell_j \ell_k \leq 0 \\
\Rightarrow &-\ell_j Y_{ik} + (u_k - \ell_k) Y_{ij} + (\ell_k - u_k) u_i x_j + \ell_j \ell_k x_i + u_i \ell_j x_k - u_i \ell_j \ell_k \leq 0 \quad (T_{26})
\end{align*}
\]

or symmetrically

\[
\begin{align*}
&u_i x_j x_k + \ell_j x_i x_k + \ell_k x_i x_j - u_i \ell_k x_j - \ell_j \ell_k x_i - u_i \ell_j \ell_k \geq x_i x_j x_k \geq x_k (u_j x_i + u_i x_j - u_i u_j) \\
\Rightarrow &u_j \ell_j x_i x_k - \ell_k x_i x_j + u_i \ell_k x_j + \ell_j \ell_k x_i + u_i \ell_j x_k - u_i \ell_j \ell_k \leq 0 \\
\Rightarrow &u_j \ell_j Y_{ik} - \ell_k Y_{ij} + u_i \ell_k x_j + \ell_j \ell_k x_i + u_i \ell_j x_k - u_i \ell_j \ell_k \leq 0 \quad (T_{27})
\end{align*}
\]

or symmetrically

\[
\begin{align*}
&u_i x_j x_k + \ell_j x_i x_k + \ell_k x_i x_j - u_i \ell_k x_j - \ell_j \ell_k x_i - u_i \ell_j \ell_k \geq x_i x_j x_k \geq x_j (\ell_i x_k + \ell_k x_i - \ell_i \ell_k) \\
\Rightarrow &u_j \ell_j x_i x_k - \ell_k x_i x_j + (u_i - \ell_i) \ell_k x_j + \ell_j \ell_k x_i + u_i \ell_j x_k - u_i \ell_j \ell_k \leq 0 \\
\Rightarrow &u_j \ell_j Y_{ik} - \ell_k Y_{ij} + (u_i - \ell_i) \ell_k x_j + \ell_j \ell_k x_i + u_i \ell_j x_k - u_i \ell_j \ell_k \leq 0 \quad (T_{28})
\end{align*}
\]

or symmetrically

\[
\begin{align*}
&u_i x_j x_k + \ell_j x_i x_k + \ell_k x_i x_j - u_i \ell_k x_j - \ell_j \ell_k x_i - u_i \ell_j \ell_k \geq x_i x_j x_k \geq x_k (\ell_i x_j + \ell_j x_i - \ell_i \ell_j) \\
\Rightarrow &u_j \ell_j x_i x_k - \ell_k x_i x_j + u_i \ell_k x_j + \ell_j \ell_k x_i + (u_i - \ell_i) \ell_j x_k - u_i \ell_j \ell_k \leq 0 \\
\Rightarrow &u_j \ell_j Y_{ik} - \ell_k Y_{ij} + u_i \ell_k x_j + \ell_j \ell_k x_i + (u_i - \ell_i) \ell_j x_k - u_i \ell_j \ell_k \leq 0 \quad (T_{29})
\end{align*}
\]

or

\[
\begin{align*}
&u_i x_j x_k + \ell_j x_i x_k + \ell_k x_i x_j - u_i \ell_k x_j - \ell_j \ell_k x_i - u_i \ell_j \ell_k \geq x_i x_j x_k \geq x_i (u_j x_k + u_k x_j - u_j u_k) \\
\Rightarrow &-u_i Y_{jk} + (u_j - \ell_j) Y_{ik} + (u_k - \ell_k) Y_{ij} + u_i \ell_k x_j + (\ell_j \ell_k - u_j u_k) x_i + u_i \ell_j x_k - u_i \ell_j \ell_k \leq 0 \quad (T_{30})
\end{align*}
\]

**Proposition 5.** \((T_{30})\) cuts feasible solutions of \((P_{S_0,...,S_m})\), while \((T_{25})-(T_{29})\) are redundant.
Proof.

(i) Consider the following solution \((x, Y)\) satisfying \(h^t_{ij}(x, Y) \leq 0\), for all \((i, j, t) \in T:\)

\[
x_i = \frac{u_i + \ell_i}{2}, \quad x_j = \frac{u_j + \ell_j}{2}, \quad \text{and} \quad x_k = \frac{u_k + \ell_k}{2}
\]

\[
Y_{ij} = \frac{u_iu_j + \ell_i\ell_j}{2}, \quad Y_{ik} = \frac{\ell_iu_k + u_k\ell_i}{2}, \quad \text{and} \quad Y_{jk} = \frac{u_j\ell_k + u_k\ell_j}{2}
\]

Using this solution in Constraints (T30), we get:

\[
\ell_iu_j\ell_k - \frac{u_iu_ju_k}{2} + \frac{u_i\ell_j\ell_k}{2} - \frac{u_iu_k\ell_j}{2} - \frac{u_j\ell_i\ell_k}{2} + \frac{u_ju_k\ell_i}{2} = \frac{1}{2}(\ell_i - u_i)(u_j - \ell_j)(\ell_k - u_k) \geq 0.
\]

This solution is thus cut off by the inequality (T30).

(ii) (T25) is equivalent to \(u_i(-Y_{jk} + \ell_k x_j + \ell_j x_k - \ell_j \ell_k) \leq 0\) that is redundant with (7).

(iii) (T26) is equivalent to \((u_k - \ell_k)(Y_{ij} - u_j x_j - \ell_j x_i + u_i \ell_j) + \ell_j(-Y_{ik} + u_i x_k + u_k x_i + u_i u_k) \leq 0\) that is redundant with (5) and (6). By symmetry, the proof is similar for (T27)-(T29).

\[\square\]

Family 6 We consider \((x_i - \ell_i)(u_j - x_j)(x_k - \ell_k) \geq 0\) and we get:

\[
\ell_i x_j x_k + u_j x_i x_k + \ell_k x_i x_j - \ell_i \ell_k x_j - u_j \ell_k x_i - \ell_i u_j x_k + \ell_i u_j \ell_k \geq x_i x_j x_k \geq x_i(\ell_j x_k + \ell_k x_j - \ell_j \ell_k)
\]

\[
\Rightarrow -\ell_i x_j x_k + (\ell_j - u_j)x_i x_k + \ell_i \ell_k x_j + (u_j - \ell_j)\ell_k x_i + \ell_i u_j x_k - \ell_i u_j \ell_k \leq 0
\]

\[
\Rightarrow -\ell_i Y_{jk} + (\ell_j - u_j)Y_{ik} + \ell_i \ell_k x_j + (u_j - \ell_j)\ell_k x_i + \ell_i u_j x_k - \ell_i u_j \ell_k \leq 0 \quad (T31)
\]

or symmetrically

\[
\ell_i x_j x_k + u_j x_i x_k + \ell_k x_i x_j - \ell_i \ell_k x_j - u_j \ell_k x_i - \ell_i u_j x_k + \ell_i u_j \ell_k \geq x_i x_j x_k \geq x_k(\ell_j x_i + \ell_i x_j - \ell_i \ell_j)
\]

\[
\Rightarrow (\ell_j - u_j)x_i x_k - \ell_k x_i x_j + \ell_i \ell_k x_j + u_j \ell_k x_i + \ell_i(u_j - \ell_j)x_k - \ell_i u_j \ell_k \leq 0
\]

\[
\Rightarrow (\ell_j - u_j)Y_{ik} - \ell_k Y_{ij} + \ell_i \ell_k x_j + u_j \ell_k x_i + \ell_i(u_j - \ell_j)x_k - \ell_i u_j \ell_k \leq 0 \quad (T32)
\]

or symmetrically

\[
\ell_i x_j x_k + u_j x_i x_k + \ell_k x_i x_j - \ell_i \ell_k x_j - u_j \ell_k x_i - \ell_i u_j x_k + \ell_i u_j \ell_k \geq x_i x_j x_k \geq x_i(u_k x_j + u_j x_k - u_j u_k)
\]

\[
-\ell_i x_j x_k + (u_k - \ell_k)x_i x_j + \ell_i \ell_k x_j + u_j(\ell_k - u_k)x_i + \ell_i u_j x_k - \ell_i u_j \ell_k \leq 0
\]

\[
\Rightarrow -\ell_i Y_{jk} + (u_k - \ell_k)Y_{ij} + \ell_i \ell_k x_j + u_j(\ell_k - u_k)x_i + \ell_i u_j x_k - \ell_i u_j \ell_k \leq 0 \quad (T33)
\]

or symmetrically

\[
\ell_i x_j x_k + u_j x_i x_k + \ell_k x_i x_j - \ell_i \ell_k x_j - u_j \ell_k x_i - \ell_i u_j x_k + \ell_i u_j \ell_k \geq x_i x_j x_k \geq x_j(\ell_k x_i + \ell_i x_k - \ell_i \ell_k)
\]

\[
\Rightarrow (u_i - \ell_i)x_j x_k - \ell_k x_i x_j + \ell_i \ell_k x_j + u_j(\ell_i - u_i)x_k + \ell_i u_j x_k - \ell_i u_j \ell_k \leq 0
\]

\[
\Rightarrow (u_i - \ell_i)Y_{jk} - \ell_k Y_{ij} + \ell_i \ell_k x_j + u_j(\ell_i - u_i)x_k + \ell_i u_j x_k - \ell_i u_j \ell_k \leq 0 \quad (T34)
\]

or \(\ell_i x_j x_k + u_j x_i x_k + \ell_k x_i x_j - \ell_i \ell_k x_j - u_j \ell_k x_i - \ell_i u_j x_k + \ell_i u_j \ell_k \geq x_i x_j x_k \geq x_j(\ell_k x_i + \ell_i x_k - \ell_i \ell_k)
\]

\[
\Rightarrow -u_j x_i x_k + u_j \ell_k x_i + \ell_i u_j x_k - \ell_i u_j \ell_k \leq 0
\]

\[
\Rightarrow -u_j Y_{ik} + u_j \ell_k x_i + \ell_i u_j x_k - \ell_i u_j \ell_k \leq 0 \quad (T35)
\]
or \( \ell_i x_j x_k + u_j x_i x_k + \ell_k x_i x_j - \ell_i \ell_k x_j - \ell_i u_j x_k + \ell_i u_j \ell_k \geq x_i x_j x_k \geq x_j (u_i x_k + u_k x_i - u_i u_k) \)

\[ \Rightarrow (u_i - \ell_i) Y_{jk} - u_j Y_{ik} + (u_k - \ell_k) Y_{ij} + (\ell_i \ell_k - u_i u_k) x_j + u_j \ell_k x_i + \ell_i u_j x_k - \ell_i u_j \ell_k \leq 0 \] (T36)

Proposition 6. (T36) cuts feasible solutions of \((P_{S_0,...,S_m})\), while (T31)-(T35) are redundant.

Proof.

(i) Consider the following solution \((x,Y)\) satisfying \( h_{ij}^i(x,Y) \leq 0 \), for all \((i,j,t) \in T:\)

- \( x_i = \frac{u_i + \ell_i}{2}, x_j = \frac{u_j + \ell_j}{2}, \) and \( x_k = \frac{u_k + \ell_k}{2} \)
- \( Y_{ij} = \frac{u_{ij} + \ell_{ij}}{2}, Y_{ik} = \frac{u_{ik} + \ell_{ik}}{2}, \) and \( Y_{jk} = \frac{u_{jk} + \ell_{jk}}{2} \)

Using this solution in Constraints (T36), we get:

\[ \frac{u_{ij} + \ell_{ij}}{2} - \frac{\ell_i u_k + \ell_k u_i}{2} + \frac{u_{ij} + \ell_{ij}}{2} - \frac{\ell_i u_k + \ell_k u_i}{2} - \frac{\ell_i \ell_k + u_i u_k}{2} + \frac{\ell_i \ell_k + u_i u_k}{2} = \frac{1}{2} (u_i - \ell_i)(\ell_j - u_j)(\ell_k - u_k) \geq 0 \]

This solution is thus cut off by the inequality (T36).

(ii) (T31) is equivalent to \((\ell_j - u_j)(Y_{ik} - \ell_k x_i - \ell_i x_k + \ell_i \ell_k) + \ell_i (-Y_{jk} + \ell_k x_j + \ell_j x_k - \ell_j \ell_k) \leq 0\) that is redundant with (5) and (7). By symmetry, the proof is similar for (T32)-(T34).

(iii) (T35) is equivalent to \(u_j(-Y_{ik} + \ell_k x_i + \ell_i x_k - \ell_i \ell_k) \leq 0\) that is redundant with (7).

\[ \square \]

Family 7 We consider \((x_i - \ell_i)(x_j - \ell_j)(u_k - x_k) \geq 0\) and we get:

\[ u_k x_i x_j + \ell_i x_j x_k + \ell_j x_i x_k - \ell_i u_k x_k - \ell_j u_k x_k + \ell_i \ell_j u_k x_k - x_i x_j x_k \geq x_j (\ell_i x_k + \ell_k x_j - \ell_j \ell_k) \]

\[ \Rightarrow (\ell_k - u_k) x_i x_j - \ell_i x_j x_k + \ell_i u_k x_k + \ell_j (u_k - \ell_k) x_i + \ell_i \ell_j x_k - \ell_i u_j \leq 0 \]

\[ \Rightarrow (\ell_k - u_k) Y_{ij} - \ell_i Y_{jk} + \ell_i u_k x_j + \ell_j (u_k - \ell_k) x_i + \ell_i \ell_j x_k - \ell_i u_j \leq 0 \] (T37)

or symmetrically

\[ u_k x_i x_j + \ell_i x_j x_k + \ell_j x_i x_k - \ell_i u_k x_k - \ell_j u_k x_k + \ell_i \ell_j u_k x_k \geq x_i x_j x_k \geq x_j (\ell_i x_k + \ell_k x_j - \ell_j \ell_k) \]

\[ \Rightarrow (\ell_k - u_k) x_i x_j - \ell_j x_i x_k + \ell_i (u_k - \ell_k) x_j + \ell_j u_k x_i + \ell_i \ell_j x_k - \ell_i u_j \leq 0 \]

\[ \Rightarrow (\ell_k - u_k) Y_{ij} - \ell_j Y_{ik} + \ell_i (u_k - \ell_k) x_j + \ell_j u_k x_i + \ell_i \ell_j x_k - \ell_i u_j \leq 0 \] (T38)

or

\[ u_k x_i x_j + \ell_i x_j x_k + \ell_j x_i x_k - \ell_i u_k x_k - \ell_j u_k x_k + \ell_i \ell_j u_k x_k \geq x_i x_j x_k \geq x_i (u_j x_k + u_k x_j - u_j u_k) \]

\[ \Rightarrow -\ell_i x_j x_k + (u_j - \ell_j) x_i x_k + \ell_i u_k x_j + u_k (\ell_j - u_j) x_i + \ell_i \ell_j x_k - \ell_i u_j \leq 0 \]

\[ \Rightarrow -\ell_i Y_{jk} + (u_j - \ell_j) Y_{ik} + \ell_i u_k x_j + u_k (\ell_j - u_j) x_i + \ell_i \ell_j x_k - \ell_i u_j \leq 0 \] (T39)

or symmetrically
Proposition 7. (T42) cuts feasible solutions of \( (P_{S_0,...,S_m}) \), while (T37)-(T41) are redundant.

Proof.

(i) Consider the following solution \((x, Y)\) satisfying \( h_{ij}^t(x, Y) \leq 0 \), for all \((i, j, t) \in T\):

- \( x_i = \frac{u_i + l_i}{2}, \; x_j = \frac{u_j + l_j}{2} \), and \( x_k = \frac{u_k + l_k}{2} \)
- \( Y_{ij} = \frac{l_i u_j + u_i l_j}{2}, \; Y_{ik} = \frac{l_i u_k + u_i l_k}{2}, \) and \( Y_{jk} = \frac{u_j u_k + l_j l_k}{2} \)

Using this solution in Constraints (T42), we get \( \frac{u_i l_j l_k}{2} - \frac{l_i u_j u_k}{2} + \frac{u_i u_j l_k}{2} + \frac{l_i u_j l_k}{2} - \frac{u_i u_j u_k}{2} - \frac{\ell_i l_j l_k}{2} + \frac{u_i u_j l_k}{2} + \frac{l_i u_j l_k}{2} = \frac{1}{2} (u_i - l_i)(l_j - u_j)(l_k - u_k) \geq 0 \). This solution is thus cut off by the inequality (T42).

(ii) (T37) is equivalent to \( (l_k - u_k)(Y_{ij} - l_i x_i - l_j x_j + l_i l_j) + l_i (u_j x_k + l_k x_j - l_j l_k) \leq 0 \) that is redundant with (5) and (7). By symmetry, the proof is similar for (T38)-(T40).

(iii) (T41) is equivalent to \( u_k (u_j x_k + l_i x_j + l_j x_i - l_i l_j) \leq 0 \) that is redundant with (7).

Family 8 We consider \((x_i - l_i)(x_j - l_j)(x_k - l_k) \geq 0 \) and we get:

\[
\ell_j x_j x_k + \ell_j x_i x_k + \ell_k x_i x_j - \ell_k x_j x_i - \ell_j x_k x_i - \ell_j x_k x_i \leq x_i x_j x_k \leq x_i (u_j x_k + l_k x_j - u_j l_k)
\]

\( \Rightarrow \ell_j x_j x_k + \ell_j x_i x_k - \ell_i x_i x_j - \ell_i x_i x_j + \ell_i x_j x_k + \ell_i x_j x_k \leq x_i x_j x_k \leq x_j (u_i x_k + l_k x_i - u_i l_k)
\]

\( \Rightarrow \ell_i Y_{jk} + (\ell_j - u_j)Y_{ik} - \ell_i l_j x_i + l_k (u_j - l_j) x_i - \ell_i l_j x_k + \ell_i l_j l_k \leq 0 \) (T43)

or symmetrically

\[
\ell_i x_j x_k + \ell_i x_i x_k + \ell_k x_i x_j - \ell_k x_j x_i - \ell_i x_k x_i - \ell_i x_k x_i \leq x_i x_j x_k \leq x_j (u_i x_k + l_k x_i - u_i l_k)
\]

\( \Rightarrow (\ell_i - u_i) x_j x_k + \ell_j x_i x_k + \ell_k (u_i - l_i) x_j - \ell_j l_k x_i - \ell_i l_j x_k + \ell_i l_j l_k \leq 0 \)
\[ (\ell_i - u_i)Y_{jk} + \ell_j Y_{ik} + \ell_k(u_i - \ell_i) x_j - \ell_j \ell_k x_i - \ell_i \ell_j x_k + \ell_i \ell_j \ell_k \leq 0 \quad (T44) \]

or symmetrically
\[ \ell_i x_j x_k + \ell_j x_i x_k + \ell_k x_j x_i - \ell_i \ell_k x_j - \ell_j \ell_k x_i - \ell_i \ell_j x_k + \ell_i \ell_j \ell_k \leq x_i x_j x_k \leq x_i (\ell_j x_k + u_k x_j - \ell_j u_k) \]
\[ \Rightarrow \ell_i x_j x_k + (\ell_k - u_k) x_i x_j + \ell_j (u_k - \ell_k) x_i - \ell_i \ell_j x_k + \ell_i \ell_j \ell_k \leq 0 \]
\[ \Rightarrow \ell_i Y_{jk} + (\ell_k - u_k) Y_{ij} + \ell_i(u_k - \ell_k) x_j - \ell_i \ell_k x_i - \ell_i \ell_j x_k + \ell_i \ell_j \ell_k \leq 0 \quad (T46) \]

or symmetrically
\[ \ell_i x_j x_k + \ell_j x_i x_k + \ell_k x_j x_i - \ell_i \ell_k x_j - \ell_j \ell_k x_i - \ell_i \ell_j x_k + \ell_i \ell_j \ell_k \leq x_i x_j x_k \leq x_j (\ell_i x_k + u_k x_i - \ell_i u_k) \]
\[ \Rightarrow \ell_j x_i x_k + (\ell_k - u_k) x_i x_j + \ell_i (u_k - \ell_k) x_j - \ell_j \ell_k x_i - \ell_i \ell_j x_k + \ell_i \ell_j \ell_k \leq 0 \]
\[ \Rightarrow \ell_j Y_{ik} + (\ell_k - u_k) Y_{ij} + \ell_i(u_k - \ell_k) x_j - \ell_j \ell_k x_i - \ell_i \ell_j x_k + \ell_i \ell_j \ell_k \leq 0 \quad (T47) \]

or symmetrically
\[ \ell_i x_j x_k + \ell_j x_i x_k + \ell_k x_j x_i - \ell_i \ell_k x_j - \ell_j \ell_k x_i - \ell_i \ell_j x_k + \ell_i \ell_j \ell_k \leq x_i x_j x_k \leq x_k (\ell_j x_i + u_i x_j - \ell_i u_j) \]
\[ \Rightarrow (\ell_i - u_i) x_j x_k + \ell_k x_i x_j - \ell_i \ell_k x_j - \ell_j \ell_k x_i + \ell_j (u_i - \ell_i) x_k + \ell_i \ell j \ell_k \leq 0 \]
\[ \Rightarrow (\ell_i - u_i) Y_{jk} + \ell_k Y_{ij} - \ell_i \ell_k x_j - \ell_j \ell_k x_i + \ell_j(u_i - \ell_i) x_k + \ell_i \ell j \ell_k \leq 0 \quad (T48) \]

**Proposition 8.** \((T44)-(T48)\) are redundant for \((P_{S_0,...,S_m})\).

**Proof.** \((T44)\) is equivalent to 
\[(\ell_j - u_j) (Y_{ik} - \ell_k x_i - \ell_i x_k + \ell_i \ell_k) + \ell_i (Y_{jk} - \ell_k x_j - u_j x_k + u_j \ell_k) \leq 0\]
that is redundant with \((4)\) and \((7)\). By symmetry, the proof is similar for \((T44)-(T48)\). 
\[\square\]

We now define the set \(T' = \{(i,j,k,t) : (i,j,k) \in V, t = 1, \ldots, 12\}\), and from Propositions \(18\) we introduce the set of general triangle inequalities,
\[ \mathcal{G} = \{(x,Y) : h_{ijk}^t(x,Y) \leq 0 \quad \forall (i,j,k,t) \in T'\} \]
with $h_{ijk}^o(x,Y)$:

$$\begin{align*}
(\ell_k - u_k)Y_{ij} + (\ell_j - u_j)Y_{ik} + u_iY_{jk} + u_iu_kx_j + (u_ju_k - \ell_j)x_i + u_iu_jx_k - u_iu_ju_k & \quad t = 1 \\
(\ell_k - u_k)Y_{ij} - u_jY_{ik} + (\ell_i - u_i)Y_{jk} + (u_iu_k - \ell_i)x_j + u_iu_kx_i + u_iu_jx_k - u_iu_ju_k & \quad t = 2 \\
-u_kY_{ij} + (\ell_j - u_j)Y_{ik} + (\ell_i - u_i)Y_{jk} + u_iu_kx_j + u_ju_kx_i + (u_iu_j - \ell_i)x_j - u_iu_ju_k & \quad t = 3 \\
(u_j - \ell_j)Y_{ik} + (\ell_k - u_k)Y_{ij} + u_iY_{jk} + (\ell_ju_k - u_j\ell_k)x_i - u_i\ell_kx_j - u_iu_jx_k + u_iu_j\ell_k & \quad t = 4 \\
u_jY_{ik} + (\ell_k - u_k)Y_{ij} + (u_i - \ell_i)Y_{jk} - u_j\ell_kx_i + (\ell_iu_k - u_i\ell_k)x_j - u_iu_jx_k + u_iu_j\ell_k0 & \quad t = 5 \\
(u_k - \ell_k)Y_{ij} + u_iY_{jk} + (\ell_j - u_j)Y_{ik} - u_iu_kx_j + (u_j\ell_k - \ell_ju_k)x_i - u_i\ell_jx_k + u_iu_j\ell_k & \quad t = 6 \\
u_kY_{ij} + (u_i - \ell_i)Y_{jk} + (\ell_j - u_j)Y_{ik} - u_iu_kx_j - \ell_ju_kx_i + (\ell_u - u_i\ell_k)x_j + u_iu_j\ell_k & \quad t = 7 \\
u_kY_{ij} + (\ell_i - u_i)Y_{jk} + (\ell_j - u_j)Y_{ik} - u_iu_kx_j - \ell_ju_kx_i + (\ell_iu_k - u_i\ell_k)x_i + u_iu_j\ell_k & \quad t = 8 \\
-u_kY_{jk} + (u_j - \ell_j)Y_{ik} + (u_k - \ell_k)Y_{ij} + u_i\ell_kx_j + (\ell_ju_k - u_j\ell_k)x_i + u_i\ell_jx_k - u_iu_j\ell_k & \quad t = 9 \\
u_kY_{jk} - u_iY_{ik} + (u_k - \ell_k)Y_{ij} + (u_i - \ell_i)Y_{jk} + (\ell_iu_k - u_i\ell_k)x_j + u_iu_jx_k - \ell_iu_j\ell_k & \quad t = 10 \\
-u_kY_{ij} + (u_i - \ell_i)Y_{jk} + (u_i - \ell_i)Y_{ik} + \ell_iu_kx_j + \ell_iu_kx_i + (\ell_ju_k - u_j\ell_k)x_i & \quad t = 11 \\
-u_kY_{ij} + (u_i - \ell_i)Y_{jk} + (u_i - \ell_i)Y_{ik} + \ell_iu_kx_j + \ell_iu_kx_i + (\ell_ju_k - u_j\ell_k)x_i - u_iu_j\ell_k0 & \quad t = 12 \\
& \quad t = 1, \ldots, 12
\end{align*}$$

We now state in Proposition 9 that these inequalities amounts to the Triangle inequalities, introduced in Padberg (1989), when for all $i \in \mathcal{I}$, $\ell_i = 0$ and $u_i = 1$.

**Proposition 9.** For all $(i,j,k,t) \in T'$, inequalities $h_{ijk}^o(x,Y) \leq 0$ are an extension of the Triangle inequalities to the case of general upper and lower bounds (i.e. $x_i \in [\ell_i, u_i]$).

**Proof.** By setting in inequalities $h_{ijk}^o(x,Y) \leq 0$ each lower and upper bound by values 0 and 1 respectively, we come back to the classical Triangle inequalities. \hfill \square

### 3. Computing a strengthened quadratic convex relaxation

By adding the general triangle inequalities to $(P_{S_0,\ldots,S_m})$, we obtain a strengthened family of quadratic convex relaxation to $(P)$:

$$\begin{align*}
\min & \quad \langle S_0, xx^T \rangle + c_0^T x + \langle Q_0 - S_0, Y \rangle \\
\text{s.t.} & \quad \langle S_r, xx^T \rangle + c_r^T x + \langle Q_r - S_r, Y \rangle \leq b_r \quad r = 1, \ldots, m \\
& \quad h_{ij}^1(x,Y) \leq 0 \quad (i,j,t) \in T \\
& \quad h_{ij}^o(x,Y) \leq 0 \quad (i,j,k,t) \in T' \\
& \quad Y_{jj} = Y_{ij} \quad (i,j) \in \mathcal{U}
\end{align*}$$
We then consider the problem of finding the best set of positive semi-definite matrices \( S_0, \ldots, S_m \), in the sense that the optimal solution value of \( (P_{S_0, \ldots, S_m}^\prime) \) is as large as possible. This amounts to solving the following problem \((\text{OPT})\):

\[
\text{(OPT)} \left\{ \begin{array}{c}
\max_{S_0, \ldots, S_m \succeq 0} v(P_{S_0, \ldots, S_m}^\prime)
\end{array} \right.
\]

where \( v(P_{S_0, \ldots, S_m}^\prime) \) is the optimal value of problem \( (P_{S_0, \ldots, S_m}^\prime) \). We further prove that \( v(\text{OPT}) \) is equal to the optimal value of the semi-definite relaxation of \( (P) \) called "Shor’s plus RLT plus Triangle".

**Theorem 1.** Let \((\text{SDP})\) be the "Shor’s plus RLT plus Triangle" semi-definite relaxation:

\[
(\text{SDP}) \left\{ \begin{array}{c}
\min f(X, x) \equiv \langle Q_0, X \rangle + c_0^T x \\
\text{s.t. } (X, x) \in \mathcal{W} \\
-h_{ij}^t(X, x) \equiv \langle M_{ij}^t, X \rangle + (v_{ij}^t)^T x + l_{ij}^t \leq 0, \quad (i, j, t) \in T \\
h_{ij}^t(X, x) \equiv \langle M_{ij}^t, X \rangle + (v_{ij}^t)^T x + l_{ij}^t \leq 0, \quad (i, j, k, t) \in T'
\end{array} \right.
\]

where \( T = \{(i, j, t) : (i, j) \in \mathcal{U}, t = 1, \ldots, 4\} \), and \( \forall (i, j, t) \in \mathcal{T} \), matrices \( M_{ij}^t \), vectors \( v_{ij}^t \), and scalars \( l_{ij}^t \) are the coefficients of constraints \( h_{ij}^t(X, x) \leq 0 \), \( \forall (i, j, k, t) \in T' \), matrices \( M_{ijk}^t \), vectors \( v_{ijk}^t \), and scalars \( l_{ijk}^t \) are the coefficients of constraints \( h_{ijk}^t(X, x) \leq 0 \), and \( \mathcal{W} \) is the following set:

\[
\mathcal{W} = (x, X) = \left\{ \begin{array}{c}
\langle Q_r, X \rangle + c_r^T x \leq b_r \\
X_{ii} - u_i x_i - \ell_i x_i + u_j \ell_i \leq 0 \quad i \in \mathcal{I}
\end{array} \right. \\
\left( \begin{array}{c}
1 \\
x^T \\
x \end{array} \right) \succeq 0 \\
x \in \mathbb{R}^n \quad X \in \mathcal{S}_n
\]

It holds that \( v(\text{OPT}) = v(\text{SDP}) \).

**Proof.**

\( \diamond \) To prove that \( v(\text{OPT}) \leq v(\text{SDP}) \), we show that \( v(P_{S_0, \ldots, S_m}^\prime) \leq v(\text{SDP}) \) for any \( \bar{S}_0, \ldots, \bar{S}_m \in \mathcal{S}_n^+ \), which in turn implies that \( v(\text{OPT}) \leq v(\text{SDP}) \) since the right hand side
is constant. For this, we show that, if \((\bar{x}, \bar{X})\) is feasible for \((SDP)\), then \((x, Y) := (\bar{x}, \bar{X})\) is i) feasible for \((P'_{\bar{S}_0, ..., \bar{S}_m})\) and ii) its objective value is less or equal than \(v(SDP)\). Since \((P'_{\bar{S}_0, ..., \bar{S}_m})\) is a minimization problem, \(v(P'_{\bar{S}_0, ..., \bar{S}_m}) \leq v(SDP)\) follows.

i) We prove that \((x, Y)\) is feasible to \((P'_{\bar{S}_0, ..., \bar{S}_m})\). Constraints of sets \(\mathcal{C}\) and \(\mathcal{G}\) are obviously satisfied. We now prove that Constraints \((I)\) are satisfied:

\[
\langle \bar{S}_r, xx^T \rangle + c_r^T x + \langle Q_r - \bar{S}_r, Y \rangle = \langle \bar{S}_r, \bar{x} \bar{x}^T \rangle + c_r^T \bar{x} + \langle Q_r, \bar{X} \rangle \leq b_r
\]

from Constraints \((20)\) and \((24)\), and since \(\bar{S}_r \geq 0\).

ii) Let us compare the objective values. For this, we prove that \((SDP)\) is the dual of \((OPT)\), where \(\Phi = \sum\limits_{(i,j)\subset \cup_{t=1}^{2\ell}} \phi_{ij}^T M_{ij}^t + \text{diag}(\varphi^1 - \varphi^2 - \varphi^3)\) and \(\Delta = \sum\limits_{(i,j,k)\in V_{\ell}} \delta_{ijk}^t M_{ijk}^t\) are the dual variables associated with Constraints \((4)\)–\((7)\), respectively, \(\varphi^1, \varphi^2, \varphi^3 \in \mathbb{R}^n\) are the dual variables associated to Constraints \((21)\)–\((23)\), respectively, and \(\delta_{ijk}^t\) are the dual variables associated with Constraints \((8)\)–\((19)\), respectively.

\diamond Let us secondly prove that \(v(OPT) \geq v(SDP)\) or equivalently \(v(OPT) \geq v(D)\) where \((D)\) is the dual of \((SDP)\):

\[
\begin{align*}
\max g(\alpha, \Phi, \Delta, \rho) = & -\sum_{r=1}^{m} \alpha_r b_r + \sum_{(i,j)\subset \cup_{t=1}^{2\ell}} \phi_{ij}^T + \sum_{(i,j,k)\in V_{\ell}} \delta_{ijk}^t - u^T \ell \varphi^1 - u^T \ell \varphi^2 - \ell^T \ell \varphi^3 - \rho \\
\text{s.t.} & \quad S = Q_0 + \sum_{r=1}^{m} \alpha_r Q_r + \Phi + \Delta \\
& \quad d = c_0 + \sum_{r=1}^{m} \alpha_r c_r + \sum_{(i,j)\subset \cup_{t=1}^{2\ell}} \phi_{ij}^T + \sum_{(i,j,k)\in V_{\ell}} \delta_{ijk}^t - \varphi^T(u + l) + 2 \varphi^T u + 2 \varphi^T \ell \\
& \quad \Phi = \sum_{(i,j)\subset \cup_{t=1}^{2\ell}} \phi_{ij}^T M_{ij}^t + \text{diag}(\varphi^1 - \varphi^2 - \varphi^3) \\
& \quad \Delta = \sum_{(i,j,k)\in V_{\ell}} \delta_{ijk}^t M_{ijk}^t \\
& \quad \left( \frac{1}{2} d^T S \right) \geq 0 \\
& \quad \alpha \in \mathbb{R}^m_+ \quad \varphi^1, \varphi^2, \varphi^3 \in \mathbb{R}^n_+ \quad \Phi \in \mathcal{S}_n \quad \phi_{ij}^t \geq 0 (i,j,t) \in \mathcal{T} \quad \Delta \in \mathcal{S}_n \quad \delta_{ijk}^t \geq 0 (i,j,k,t) \in \mathcal{T}'
\end{align*}
\]
Let \((\alpha^*, \Phi^*, \Delta^*, \rho^*)\) be an optimal solution to \((D)\), we build the following positive semi-definite matrices: \(\forall r \in \mathcal{R}, \bar{S}_r = 0_n\), and \(\bar{S}_0 = S^* = Q_0 + \sum_{r=1}^{m} \alpha^*_r Q_r + \Phi^* + \Delta^*\). By Constraint \((30)\), \(\bar{S}_0 \succeq 0\), and \((\bar{S}_0, \ldots, \bar{S}_m)\) forms a feasible solution to \((OPT)\). The objective value of this solution is equal to \(v(P'_0, \ldots, P'_m)\).

We now prove that \(v(P'_0, \ldots, P'_m) \geq v(D)\). For this, we prove that for any feasible solution \((\bar{x}, \bar{Y})\) to \((P'_0, \ldots, P'_m)\), the associated objective value is not smaller than \(g(\alpha^*, \Phi^*, \Delta^*, \rho^*)\).

Denote by \(\theta\) the difference between the objective values, i.e., \(\theta = (\bar{S}_0, \bar{x}^T) + c_0^T \bar{x} + \langle Q_0 - \bar{S}_0, \bar{Y}\rangle - g(\alpha^*, \Phi^*, \Delta^*, \rho^*)\). We below prove that \(\theta \geq 0\).

\[
\theta = (\bar{S}_0, \bar{x}^T) + c_0^T \bar{x} + \langle Q_0 - \bar{S}_0, \bar{Y}\rangle + \sum_{r=1}^{m} \alpha^*_r b_r - \sum_{(i,j) \in U} \sum_{t=1}^{4} \bar{\phi}^*_{ij} \ell^t_{ij} - \sum_{(i,j,k) \in V} \sum_{t=1}^{12} \bar{\delta}^*_{ijk} t_{ijk}^t - u^T \ell \varphi^1 + u^T w \varphi^2 + \ell^T \varphi^3 + \rho^*
\]
We end the proof by showing that \( \langle \bar{S}_0, \bar{x} x^T \rangle + d^T \bar{x} + \rho^* \geq 0 \). From Constraint (30), we know that for all \( x \in \mathbb{R}^n \), \( \left( \begin{array}{c} 1 \\ x \end{array} \right) \left( \begin{array}{c} \rho^* \\ \frac{1}{2} d \end{array} \right)^T \left( \begin{array}{c} 1 \\ \bar{S}_0 \end{array} \right) \left( \begin{array}{c} 1 \\ x \end{array} \right) \geq 0 \), which prove that \( \theta \geq 0 \). \( \square \)

From the proof of Theorem 1, we can characterize a set of optimal matrices \((S^*_0, \ldots, S^*_m)\).

**Corollary 1.** The following positive semi-definite matrices allow to build an optimal solution \((S^*_0, \ldots, S^*_m)\) of \((OPT)\):

i) \( \forall r = 1, \ldots, m, S^*_r = 0 \)

ii) \( S^*_0 = Q_0 + \sum_{r=1}^{m} \alpha^*_r Q_r + \Phi^* + \Delta^* \), where:

\( \diamond \alpha^* \) is the vector of optimal dual variables associated with Constraints (20),

\( \diamond \text{matrix } \Phi^* = \sum_{(i,j) \in U} \sum_{t=1}^{4} \delta^*_{ij} M^t_{ij} + \text{diag}(\varphi^1 - \varphi^2 - \varphi^3) \), where \( \varphi^1, \varphi^2, \varphi^3 \) are the vectors of optimal dual variables associated with Constraints (21)–(23), and \( \forall (i,j) \in \overline{T} \), \( \delta^*_{ij} \) is the optimal dual variables associated with Constraints (4)–(7), respectively.

\( \diamond \text{matrix } \Delta^* = \sum_{(i,j,k) \in V} \sum_{t=1}^{12} \delta^*_{ijk} M^t_{ijk} \), where \( \forall (i,j,k) \in T' \), \( \delta^*_{ijk} \) is the optimal dual variables associated with Constraints (8)–(19), respectively.

To sum up, we obtain the following quadratic convex relaxation to \((P)\):

\[
\begin{align*}
\text{(P*)} \quad \min_{f_0, S_0(x,Y)} & \quad \langle Q_0 + \sum_{r=1}^{m} \alpha^*_r Q_r + \Phi^* + \Delta^*, xx^T \rangle + c^T_0 x + \langle -\sum_{r=1}^{m} \alpha^*_r Q_r - \Phi^* - \Delta^*, Y \rangle \\
\text{s.t.} & \quad f_r(x,Y) = \langle Q_r, Y \rangle + c^T_r x \leq b_r & r \in \mathcal{R} \\
& \quad h^r_{ij}(x,Y) \leq 0 & (i,j) \in T \\
& \quad h''^r_{ij}(x,Y) \leq 0 & (i,j) \in \overline{T} \\
& \quad Y_{ij} = Y_{ij} & (i,j) \in U
\end{align*}
\]

As stated in Theorem 1, the optimal value of \((P*)\) is equal to the optimal value of \((SDP)\), and we now by Proposition 1–8 that this relaxation is tighter than the "shor plus RLT" relaxation. This sharp relaxation can then be used within a branch-and-bound algorithm to solve \((P)\) to global optimality.

**4. Using the dynamic bundle method for heuristically solving \((SDP)\) and for separating inequalities**

In this section we propose to separate the set of inequalities \(\mathcal{C} \cup \mathcal{G}\) by heuristically solving \((SDP)\), thanks to a dynamic bundle method. For this, we introduce a parameter \(p\) that
controls the number of considered constraints in \((SDP)\), and thus in the associated computed quadratic convex relaxation. Following the idea of Billionnet et al. (2017), we design sub-gradient algorithm within a Lagrangian duality framework using the callable Conic Bundle library of Helmberg (2011).

To describe the algorithm, we consider \((SDP)\) as a maximization problem, by changing the sign of its objective function. We then consider a partial Lagrangian dual of \((SDP)\) where we dualize the set of constraints \(C \cup G\). For this, with each constraint \(h_{ij}^t(x,X) \leq 0\), we associate a non-negative Lagrange multiplier \(\phi_{ij}^t\), and with each constraints \(h_{ijk}^t(x,X) \leq 0\) a non-negative Lagrange multiplier \(\delta_{ijk}^t\). We now consider the partial Lagrangian:

\[
L_{T,T'}(x,X,\phi,\delta) = -\langle Q_0, X \rangle - c_0^T x - \sum_{(i, j, t) \in T} \phi_{ij}^t h_{ij}^t(x,X) - \sum_{(i, j, k, t) \in T'} \delta_{ijk}^t h_{ijk}^t(x,X)
\]

and we obtain the dual functional

\[
g_{T,T'}(\phi, \delta) = \max_{(x, X) \in W} L_{T,T'}(x,X,\phi,\delta)
\]

By minimizing this dual functional we obtain the partial Lagrangian dual problem:

\[
(LD_{T,T'}) \left\{ \min_{\phi_{ij}^t \geq 0, (i, j, t) \in T} \min_{\delta_{ijk}^t \geq 0, (i, j, k, t) \in T'} g_{T,T'}(\phi, \delta) \right\}
\]

Problem \((LD_{T,T'})\) can then be solved with the bundle method for which a detailed description is available in Fischer et al. (2006). However, the number of elements in \(T \cup T'\) is \(4\binom{n}{2} + 12\binom{n}{3}\), and we are interested only in the subset of \(T \cup T'\) for which the constraints \(h_{ij}^t(x,X) \leq 0\) and \(h_{ijk}^t(x,X) \leq 0\) are active at the optimum. In order to preserve efficiency we adopt another idea from Fischer et al. (2006) that consists in dynamically adding and removing constraints in the course of the algorithm. Then, we now consider \(T \cup T' \subseteq T \cup T'\) and work with the function:

\[
g_{T,T'}(\phi, \delta) = \max_{(x, X) \in W} L_{T,T'}(x,X,\phi,\delta).
\]

Initially we set \(T \cup T' = \emptyset\) and after a first function evaluation we separate violated inequalities and add the elements to set \(T \cup T'\) accordingly. We keep on updating this set in course of the bundle iterations by removing elements with associated multiplier close to zero and separate newly violated constraints. Convergence for dynamic bundle methods
has been analyzed in detail in Belloni and Sagastizábal (2009), giving a positive answer for convergence properties in a rather general setting.

In our context, we know that any feasible dual solution to \((SDP)\) allows us to build a quadratic convex relaxation of \((P)\). Better this solution is, sharper is the associated bound at the root node of the branch-and-bound process. Another idea to reduce more the solution time of \((SDP)\) is to consider a parameter \(p\) that is an upper bound on the cardinality of \(\mathcal{T} \cup \mathcal{T}'\) (i.e. \(|\mathcal{T} \cup \mathcal{T}'| \leq p\)). In other words, \(p\) is the maximum number of constraints considered in the reduced problem. Introducing this parameter \(p\) leads to a dual heuristic that has two extreme cases:

- if \(p = 4 \binom{n}{2} + 12 \binom{n}{3}\), we solve \((SDP)\) and get the associated dual solution as described in Corollary 1.
- if \(p = 0\), we make a single iteration: we get the optimal solution of the reduced problem obtained from \((SDP)\) where we drop all constraints of sets \(\mathcal{C}\) and \(\mathcal{G}\), this amounts to solving the ”Shor’s plus diagonal RLT” semi-definite relaxation.

5. Computational results

In this section, we compare our algorithm \textsc{MIQCR-T} with GloMIQO (Misener et al. (2015)), Baron (Sahinidis and Tawarmalani (2010)), and the original \textsc{MIQCR} method (Elloumi and Lambert (2019)), on the 135 instances of quadratically constrained quadratic programs from Bao et al. (2009) called \textit{unitbox}.

Experimental environment

Our experiments were carried out on a server with 2 CPU Intel Xeon each of them having 12 cores and 2 threads of 2.5 GHz and 4 * 16 GB of RAM using a Linux operating system. For all algorithms, we use the multi-threading version of \textsc{Cplex} 12.7 with up to 48 threads. For methods \textsc{MIQCR} and \textsc{MIQCR-T}, we used the solver \textsc{csdp} (Borchers (1999)) together with the Conic Bundle library (Helmberg (2011)) for solving semi-definite programs, as described in Section 4. We used the C interface of the solver \textsc{Cplex}(IBM-ILOG (2017)) for solving the quadratic convex relaxations at each node of the search tree. For computing feasible local solutions, we use the local solver \textsc{Ipopt}(Wächter and Biegler (2006)). Parameter \(p\) is set to \(0.4 \cdot |\mathcal{C}|\) for \textsc{MIQCR}, and to \(0.04 \cdot |\mathcal{C} \cup \mathcal{G}|\) for \textsc{MIQCR-T}. 
Results for the unitbox instances

Each instance from Bao et al. (2009) consists in minimizing a quadratic function of \( n \) continuous variables in the interval \([0,1]\), subject to \( m \) quadratic inequalities. For the considered instances, \( n \) varies from 8 to 50, and \( m \) from 8 to 100. We set the time limit to 2 hours. In Figure 1, we present the performance profile of the CPU times for methods MIQCR-T, MIQCR, and the solvers GloMIQO, and Baron for the unitbox instances. We observe that MIQCR-T and MIQCR outperform the compared solvers in terms of CPU time and number of instances solved. In fact, BARON solve 109 instances, GloMIQO solves 110 instances, MIQCR solves 119 instances, and MIQCR-T solves 128 instances out of 135 within the time limit. Several additional remarks are in order: the initial gap is smaller for MIQCR-T than for MIQCR, since we pass from 1.63% to 1.18% on average on the 109 instances solved by both methods. Surprisingly, the CPU time for solving the semi-definite relaxation is divided by 2 on average for MIQCR-T in comparison to MIQCR, this is due to the sub-gradients considered in the course of the Conic Bundle algorithms that can be different for the two methods. Another consequence of the use of the new inequalities within the branch-and-bound process is the significant reduction of the number of nodes (by a factor 6.6). Hence, we can also see a reduction of CPU time for this phase, where we pass from 390 seconds to 85 seconds on average.

In Table 1, we present the detailed total CPU times for each method. Each line corresponds to one instance. We observe that GloMIQO or BARON are faster on most of the small and/or sparse instances, while MIQCR and MIQCR-T are faster on large and dense instances.
Figure 1  Performance profile of the total time for the *unitbox* instances with $n = 8$ to 50 with a time limit of 2 hours.
| Name   | MIQCR | MIQCR-T | BARON | GloMIQO | Name   | MIQCR | MIQCR-T | BARON | GloMIQO | Name   | MIQCR | MIQCR-T | BARON | GloMIQO |
|--------|-------|---------|-------|---------|--------|-------|---------|-------|---------|--------|-------|---------|-------|---------|
| 8.12.1.25 | 1 2 0 0 0 | 20.40.1.25 | 5 2 0 0 0 | 40.60.1.25 | 302 487 | 414 101 |
| 8.12.2.25 | 2 7 0 0 0 | 20.40.2.25 | 13 31 45 3 |
| 8.12.3.25 | 4 5 0 0 0 | 20.40.3.25 | 99 80 106 16 |
| 8.14.1.25 | 5 6 5 1 | 40.60.1.50 | 5516 190 282 1822 |
| 8.15.2.25 | 8 14 8 1 | 40.60.2.50 | 139 62 176 3616 |
| 8.16.2.25 | 1 3 0 0 0 | 20.40.3.25 | 323 61 546 - |
| 8.16.3.25 | 19 9 60 3 | 40.60.1.100 | 2913 - - - |
| 8.19.2.25 | 18 12 50 2 | 40.60.2.100 | 3133 449 - 2182 |
| 8.20.3.25 | 3 4 1 0 | 40.60.3.100 | 7203 2063 |
| 10.10.1.50 | 2 3 0 0 | 40.80.2.50 | 415 208 1184 1558 |
| 10.11.2.50 | 2 7 0 0 0 | 40.80.2.25 | 326 292 338 107 |
| 10.11.3.50 | 7 24 2 1 | 40.80.3.25 | 1248 181 203 86 |
| 10.11.4.50 | 10 19 2 0 | 40.80.2.50 | 190 139 112 85 |
| 10.11.5.50 | 12 6 0 0 | 40.80.2.50 | 154 51 207 4785 |
| 10.11.6.50 | 1 4 0 0 0 | 40.80.3.25 | 4086 1014 - - |
| 10.11.7.50 | 4 8 1 1 | 40.80.1.100 | 250 6957 741 |
| 10.11.8.50 | 4 8 0 1 | 40.80.2.100 | 951 286 - 654 |
| 10.11.9.50 | 19 21 8 1 | 40.80.3.100 | 411 211 2140 1958 |
| 10.11.10.50 | 38 26 36 3 | 45.8.2.50 | 161 100 108 57 |
| 10.11.11.50 | 60 17 20 12 | 45.8.2.100 | 1400 671 393 455 |
| 10.11.12.50 | 100 5 1 4 | 45.8.2.100 | 64 88 85 32 |
| 10.11.13.50 | 38 13 9 3 | 45.8.2.25 | 322 126 206 37 |
| 10.11.14.50 | 49 8 175 51 | 45.8.2.25 | 546 151 112 180 |
| 10.11.15.50 | 106 25 24 7 | 45.8.2.50 | 268 186 381 33 |
| 10.11.16.50 | 49 18 11 3 | 45.8.2.50 | 4710 547 - - |
| 10.11.17.50 | 64 17 38 4 | 45.8.2.50 | 1060 575 1420 2879 |
| 10.11.18.50 | 184 42 210 26 | 50.50.1.50 | - - 315 - |
| 10.11.19.50 | 755 85 360 61 | 50.50.2.50 | - - 2059 - |
| 10.11.20.50 | 101 27 92 7 | 50.50.2.50 | 2593 892 - |
| 10.11.21.50 | 833 180 188 323 | 50.50.1.100 | 2407 234 - |
| 10.11.22.50 | 302 50 54 279 | 50.50.2.100 | - - 2680 - |
| 10.11.23.50 | 57 15 9 7 | 50.50.2.100 | - - 3406 - |
| 10.11.24.50 | 231 43 537 29 | 50.75.1.50 | 3255 1225 - |
| 10.11.25.50 | 1602 230 5637 213 | 50.75.2.50 | - - 1233 - |
| 10.11.26.50 | 272 34 894 26 | 50.75.3.50 | 5126 632 - |
| 10.11.27.50 | 52 32 45 6 | 50.75.1.100 | - - - |
| 10.11.28.50 | 14 27 41 2 | 50.75.2.100 | - - - |
| 10.11.29.50 | 14 28 0 1 | 50.75.3.100 | - - - |
| 10.11.30.50 | 539 40 608 7 | 50.100.1.50 | - - - |
| 10.11.31.50 | 152 64 306 7034 | 50.100.2.50 | 823 324 - |
| 10.11.32.50 | 574 94 248 4 | 50.100.3.50 | 3337 1731 - |
| 10.11.33.50 | 65 42 235 14 | 50.100.1.100 | - - 3808 - |
| 10.11.34.50 | 297 72 2133 503 | 50.100.2.100 | - - - |
| 10.11.35.50 | 85 11 10 2 | 50.100.3.100 | - - - |

Table 1 Total CPU times of MIQCR, MIQCR-T, BARON 17.3.31 and GloMIQO 2 for the 135 unitbox instances. Time limit 2 hours.
6. Conclusion

We consider the general problem \((P)\) of minimizing a quadratic function subject to quadratic constraints where the variables are continuous. In this paper, we introduce 12 General Triangle inequalities and we prove that they cut feasible solutions of the McCormick envelopes. In fact, these inequalities are an extension of the Triangle inequalities to the case of general lower and upper bounds on the variables. Then, we show how we can compute a quadratic convex relaxation which optimal value is equal to the “Shor’s plus RLT plus Triangle” semi-definite relaxation \((SDP)\). Since there is a huge number of these inequalities, we then focus on selecting the \(p\) most violated ones. In particular, we separate them during the heuristic solution of \((SDP)\). We report computational results on 135 instances. These results show that the method allows us to solve 128 instances out of 135 within a time limit of 2 hours. From a general outlook, these new inequalities can be used in any branch-and-bound process based on the relaxation of the constraints \(Y = xx^T\). Indeed, since the upper and lower bounds \(\ell\) and \(u\) are involved within the general Triangles, the relaxation will be again tighten in the course of the algorithm.

References

Adhya N, Tawarmalani M, Sahinidis N (1999) A lagrangian approach to the pooling problem. Industrial & Engineering Chemistry Research 38(5):1956–1972, URL http://dx.doi.org/10.1021/ie980666q.

Aholt C, Agarwal S, Thomas R (2012) A QCQP Approach to Triangulation. Fitzgibbon A, Lazebnik S, Perona P, Sato Y, Schmid C, eds., Computer Vision – ECCV 2012, 654–667 (Berlin, Heidelberg: Springer Berlin Heidelberg), ISBN 978-3-642-33718-5.

Anstreicher KM (2009) Semidefinite programming versus the reformulation-linearization technique for non-convex quadratically constrained quadratic programming. Journal of Global Optimization 43(2):471–484, ISSN 1573-2916, URL http://dx.doi.org/10.1007/s10898-008-9372-0.

Bao X, Sahinidis N, Tawarmalani M (2009) Multiterm polyhedral relaxations for nonconvex, quadratically constrained quadratic programs. Optimization Methods Software 24(4-5):485–504, ISSN 1055-6788.

Belloni A, Sagastizábal C (2009) Dynamic bundle methods. Mathematical Programming A 120(2):289–311.

Billionnet A, Elloumi S, Lambert A, Wiegele A (2017) Using a Conic Bundle method to accelerate both phases of a Quadratic Convex Reformulation. INFORMS Journal on Computing 29(2):318–331, URL http://dx.doi.org/10.1287/ijoc.2016.0731.

Borchers B (1999) CSDP, A C Library for Semidefinite Programming. Optimization Methods and Software 11(1):613–623.
Burer S, Vandenbussche D (2008) A finite branch-and-bound algorithm for nonconvex quadratic programming via semidefinite relaxations. *Mathematical Programming A* 113(2):259–282, ISSN 1436-4646, URL [http://dx.doi.org/10.1007/s10107-006-0080-6](http://dx.doi.org/10.1007/s10107-006-0080-6).

Burer S, Vandenbussche D (2009) Globally solving box-constrained nonconvex quadratic programs with semidefinite-based finite branch-and-bound. *Comput Optim Appl* 43:181–195.

Chen J, Burer S (2012) Globally solving nonconvex quadratic programming problems via completely positive programming. *Mathematical Programming Computation* 4(1):33–52.

Dorneich M, Sahinidis N (1995) Global optimization algorithms for chip layout and compaction. *Engineering Optimization* 25(2):131–154, URL [http://dx.doi.org/10.1080/03052159508941259](http://dx.doi.org/10.1080/03052159508941259).

Elloumi S, Lambert A (2019) Global solution of non-convex quadratically constrained quadratic programs. *Optimization Methods and Software* 34(1):98–114, URL [http://dx.doi.org/10.1080/10556788.2017.1350675](http://dx.doi.org/10.1080/10556788.2017.1350675).

Fischer I, Gruber G, Rendl F, Sotirov R (2006) Computational experience with a bundle approach for semidefinite cutting plane relaxations of Max-Cut and equipartition. *Mathematical Programming B* 105(2-3):451–469.

Garey M, Johnson D (1979) Computers and Intractability: A guide to the theory of NP-Completness. *W.H. Freeman, San Francisco, CA*.

Helmberg C (2011) *Conic Bundle* v0.3.10. URL [http://www-user.tu-chemnitz.de/~helmberg/ConicBundle/](http://www-user.tu-chemnitz.de/~helmberg/ConicBundle/).

Huy Hao EP (1982) Quadratically constrained quadratic programming: Some applications and a method for solution. *Zeitschrift für Operations Research* 26(1):105–119, ISSN 1432-5217, URL [http://dx.doi.org/10.1007/BF01917102](http://dx.doi.org/10.1007/BF01917102).

IBM-ILOG (2017) IBM ILOG CPLEX 12.7 Reference Manual. [http://www-01.ibm.com/support/knowledgecenter/SSSA5P_12.7.0/ilog.odms.studio.help/Optimization_Studio/topics/CDS_home.html](http://www-01.ibm.com/support/knowledgecenter/SSSA5P_12.7.0/ilog.odms.studio.help/Optimization_Studio/topics/CDS_home.html).

Liu M, Sahinidis N (1997) Process planning in a fuzzy environment. *European Journal of Operational Research* 100(1):142 – 169, ISSN 0377-2217, URL [http://dx.doi.org/https://doi.org/10.1016/S0377-2217(96)00025-2](http://dx.doi.org/https://doi.org/10.1016/S0377-2217(96)00025-2).

Locatelli M, Raber U (2002) Packing equal circles in a square: a deterministic global optimization approach. *Discrete Applied Mathematics* 122(1):139 – 166, ISSN 0166-218X, URL [http://dx.doi.org/https://doi.org/10.1016/S0166-218X(01)00359-6](http://dx.doi.org/https://doi.org/10.1016/S0166-218X(01)00359-6).

McCormick G (1976) Computability of global solutions to factorable non-convex programs: Part i - convex underestimating problems. *Mathematical Programming A* 10(1):147–175.

Misener R, Floudas C (2012) Global optimization of mixed-integer quadratically-constrained quadratic programs (MIQCP) through piecewise-linear and edge-concave relaxations. *Mathematical Programming B* 136(1):155–182.
Amélie Lambert: Valid inequalities and global solution algorithm for QCQPs

Misener R, Floudas C (2013) GloMIQO: Global mixed-integer quadratic optimizer. Journal of Global Optimization 57(1):3–50, ISSN 0925-5001, URL http://dx.doi.org/10.1007/s10898-012-9874-7

Misener R, Smadbeck J, Floudas C (2015) Dynamically generated cutting planes for mixed-integer quadratically constrained quadratic programs and their incorporation into GloMIQO 2. Optimization Methods and Software 30(1):215–249.

Padberg M (1989) The boolean quadric polytope: some characteristics, facets and relatives. Mathematical programming 45(1):139–172.

Sahinidis N, Tawarmalani M (2010) Baron 9.0.4: Global optimization of mixed-integer nonlinear programs. User’s Manual URL http://www.gams.com/dd/docs/solvers/baron.pdf.

Sherali H, Adams W (2013) A reformulation-linearization technique for solving discrete and continuous nonconvex problems, volume 31 (Springer Science & Business Media).

Sutou A, Dai Y (2002) Global optimization approach to unequal sphere packing problems in 3d. Journal of Optimization Theory and Applications 114(3):671–694, ISSN 1573-2878, URL http://dx.doi.org/10.1023/A:1016083231326.

Vandenbussche D, Nemhauser G (2005a) A branch-and-cut algorithm for nonconvex quadratic programs with box constraints. Mathematical Programming A 102(3):259–275.

Vandenbussche D, Nemhauser G (2005b) A polyhedral study of nonconvex quadratic programs with box constraints. Mathematical Programming A 102(3):531–557.

Wächter A, Biegler L (2006) On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. Mathematical Programming A 106(1):25–57, ISSN 1436-4646, URL http://dx.doi.org/10.1007/s10107-004-0559-y.

Xie W, Sahinidis N (2008) A branch-and-bound algorithm for the continuous facility layout problem. Computers & Chemical Engineering 32(4):1016 – 1028, ISSN 0098-1354, URL http://dx.doi.org/https://doi.org/10.1016/j.compchemeng.2007.05.003.

Yajima Y, Fujie T (1998) A polyhedral approach for nonconvex quadratic programming problems with box constraints. Journal of Global Optimization 13(2):151–170.