Nature of Electric and Magnetic Fields; How the Fields Transform

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In this paper the proofs are given that the electric and magnetic fields are properly defined vectors on the four-dimensional (4D) spacetime (the 4-vectors in the usual notation) and not the usual 3D fields. They are the 4D geometric quantities (GQs). Furthermore, the proofs are presented that under the mathematically correct Lorentz transformations (LT), e.g., the electric field vector transforms as any other vector transforms, i.e., again to the electric field vector; there is no mixing with the magnetic field vector \( B \), as in the usual transformations of the 3D fields. Different derivations of these usual transformations of the 3D fields, including those from some well-known textbooks, are discussed and objected. This formulation with the 4D GQs is in a true agreement, independent of the chosen inertial reference frame and of the chosen system of coordinates in it, with experiments in electromagnetism, e.g., the motional emf. It is not the case with the usual 3D formulation which agrees with experiments only if the standard basis is used and for \( \gamma \approx 1 \).

In our living arena, the four-dimensional (4D) spacetime, physical laws, e.g., the Lorentz force law, are geometric, coordinate-free relationships between the 4D geometric, coordinate-free quantities.

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1. Introduction

It is generally accepted that the electric and magnetic fields are the 3D vectors and that their transformations, e.g., equations (11.148) and (11.149) in [1], are the mathematically correct Lorentz transformations (LT) of these fields. In this paper the transformations of the 3D fields \( E \) and \( B \) will be called the “apparent” transformations (AT). The name is explained below. According to the mentioned AT, the transformed 3D vector \( E' \) is expressed by the mixture of the 3D vectors \( E \) and \( B \), equation (11.149) in [1]. In the usual covariant approaches, e.g., [1], the AT for the components of \( E \) and \( B \) are derived assuming that for the observers in an inertial frame, the \( S \) frame, these components are identified with the six independent components \( F^{\alpha\beta} \) of the electromagnetic field tensor. These identifications are

\[
E_i = F^{i0}, \quad B_i = (1/2)\varepsilon_{ijk}F_{kj}
\]  

(1)
(the indices $i, j, k, \ldots = 1, 2, 3$), equation (11.137) in [1], e.g., $E_x = E_1 = F^{10}$. The components of the 3D fields $E$ and $B$ are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric $\varepsilon$ tensor too. The super- and subscripts are used only on the components of the 4D quantities. The 3D $E$ and $B$ are geometric quantities in the 3D space and they are constructed from these six independent components of $F^\mu\nu$ and the unit 3D vectors $i, j, k$, e.g., $\mathbf{E} = F^{10}i + F^{20}j + F^{30}k$. Observe that $F^{\alpha\beta}$ is not a tensor since $F^{\alpha\beta}$ are only components implicitly taken in the standard basis. The components are coordinate quantities and they do not contain the whole information about the physical quantity, since a basis of the spacetime is not included. Then, it is supposed that the same identification of the components as in equation (1) holds for a relatively moving inertial frame $S'$, i.e., for the transformed components $E'_i$ and $B'_i$

$$E'_i = F'^{i0}, \quad B'_i = (1/2c)\varepsilon_{ijk}F'_{kj}. \quad (2)$$

The same remark about the (generic) subscripts holds also here. The components $F^{\alpha\beta}$ transform under the LT as, e.g.,

$$F'^{10} = F^{10}, \quad F'^{20} = \gamma(F^{20} - \beta F^{21}), \quad F'^{30} = \gamma(F^{30} - \beta F^{31}), \quad (3)$$

which yields (by equations (1) and (2)) that

$$E'_1 = E_1, \quad E'_2 = \gamma(E_2 - \beta cB_3), \quad E'_3 = \gamma(E_3 + \beta cB_2), \quad (4)$$

what is equation (11.148) in [1]. Thus, in the usual covariant approaches, e.g., [1], the AT of the components of $E$ and $B$ are derived assuming that they transform under the LT as the components of $F^{\alpha\beta}$ transform.

However, there are several objections to the mathematical correctness of such a procedure. Some of them are the following:

1) As seen, e.g., from section 3.1 in [2], such an identification of the components of $E$ and $B$ with the components of $F^{\alpha\beta}$ is synchronization dependent and, particularly, it is meaningless in the “radio,” “r” synchronization, i.e., in the $\{r_\mu\}$ basis, see [3] and below.

2) The 3D vectors $E$, $B$ and $E'$, $B'$ are constructed in both frames in the same way, i.e., multiplying the components, e.g., $E_{x,y,z}$ and $E'_{x,y,z}$ by the unit 3D vectors $i, j, k$ and $i', j', k'$, respectively. This procedure gives the AT of the 3D vectors $E$ and $B$, equation (11.149) in [1]. But, as seen from (3), the components $F^{\alpha\beta}$ are multiplied by the bivector basis $\gamma_\alpha \wedge \gamma_\beta$ and not by the unit 3D vectors. In the 4D spacetime the unit 3D vectors are ill-defined algebraic quantities and there are no LT, or some other transformations, that transform the unit 3D vectors $i, j, k$ into the unit 3D vectors $i', j', k'$.

In [4], in section 12.3.2 under the title “How the Fields Transform,” the AT, equations (12.109), are derived using the Lorentz contraction and the 3D fields. But, as shown, e.g., in [3] and [5], the Lorentz contraction is ill-defined in the 4D spacetime; it is synchronization dependent and consequently it is not an intrinsic relativistic effect. The LT have nothing in common with the Lorentz
contraction; the LT cannot connect two spatial lengths that are simultaneously determined for relatively moving inertial observers. The Lorentz contracted length and the rest length are two different quantities and they are not related by the LT. Rohrlich [6] named such transformations (Lorentz contraction) that do not refer to the same quantity - the “apparent” transformations, whereas the transformations which refer to the same 4D quantity as the “true” transformations, e.g., the LT. It is visible from (4), (33) and (36) that the transformations of the components of $E$ and $B$ do not refer to the same quantity and therefore they are also the AT and not the true transformations, i.e., the LT.

In [3] and [5] instead of the Lorentz contraction and the time dilation the 4D geometric quantities (GQs) are used, the position 4-vector, the distance 4-vector between two events and the spacetime length. In [5] it is shown that all well-known experiments that test special relativity, e.g., the “muon” experiment, the Michelson-Morley type experiments, the Kennedy-Thorndike type experiments and the Ives-Stilwell type experiments are in a complete agreement, independently of the chosen synchronization, with the 4D geometric approach, whereas it is not the case with Einstein’s approach with the Lorentz contraction and the time dilation if the “r” synchronization is used.

In this paper, in section 2, the geometric algebra formalism, the standard basis and the $\{r_{\mu}\}$ basis with the “r” synchronization are briefly discussed. In sections 2.1, some additional objections to the derivations of the AT are presented. In sections 3.1 and 3.3 it is proved in a mathematically correct way that in the 4D spacetime the electric and magnetic fields are not the usual 3D fields $E$ and $B$ but that they are properly defined vectors on the 4D spacetime, the 4D vectors $E$ and $B$. In the whole text $E$, $B$ will be simply called - vectors - or the 4D vectors, whereas the usual $E$, $B$ will be called the 3D vectors. In sections 4.1 and 4.2 the proofs are given that the AT of the 3D fields are not the mathematically correct LT, because the LT are properly defined on the 4D spacetime and cannot transform the 3D quantities. The LT transform the electric field vector in the same way as any other vector transforms, i.e., again to the electric field vector. Sections 3.1, 3.3, 4.1 and 4.2 are the central sections and they contain the most important results that are obtained in this paper. In sections 5.1 and 5.2, for the reader’s convenience, the derivations of the AT and the LT are compared using matrices. In section 6, the derivation of the AT from the textbook by Blandford and Thorne (BT) [7] is discussed and objected. In [7], in contrast to, e.g., [1, 4], a geometric viewpoint is adopted; the physical laws are stated as geometric, coordinate-free relationships between the geometric, coordinate-free quantities. Particularly, in section 1.10 in [7], it is discussed the nature of electric and magnetic fields and they are considered to be the 4D fields. But, nevertheless, BT also derived the AT of the 3D vectors $E$ and $B$, their equation (1.113), and not the correct LT of the 4D fields, equations (29), (30) and (32) here. They have not noticed that under the LT the electric field 4D vector must transform as any other 4D vector transforms. In section 7.1, it is discussed the derivation of the AT from the paper by Klajn and Smolić (KS) [8]. KS [8] use the tensor formalism with the abstract index notation, but in section 3 in [8] they made almost the same mistakes as in BT [7]. In section
7.2 Similar shortcomings in the treatment of the angular momentums and spin that are made in section 4 in [8] are discussed and objected. In section 8, the mathematically correct definitions with the 4D GQs of the orbital angular momentums and spins are discussed. In section 9, the electromagnetic field of a point charge in uniform motion is investigated and it is explicitly shown that 1) the primary quantity is the bivector $F$ (equations (76) and (77)) and 2) that the observer dependent 4D vectors $E$ and $B$, equation (82), correctly describe both the electric and magnetic fields for all relatively moving inertial observers and for all bases chosen by them. In section 10, a brief discussion is presented of the comparison with the experiments on the motional emf. It is shown that the theory with the 4D quantities and their LT, equations (29), (30) and (32) here, is in agreement with the principle of relativity, equations (91) and (92), whereas it is not the case with the usual approach with the 3D quantities and their AT, equations (87) - (90). In section 11, the discussion of the obtained results is presented and the conclusions are given.

2. The geometric algebra formalism. The $\{r_\mu\}$ basis with the “r” synchronization

Here, we shall also deal either with the abstract, coordinate-free 4D GQs, or with their representations in some basis, the 4D coordinate-based geometric quantities (CBGQs) comprising both components and a basis, e.g., the position vector, $x = x^\nu r_\nu$. The coordinate-free 4D GQs will be called the abstract quantities (AQs). An independent physical reality is attributed to the 4D GQs and not, as usual, to the 3D quantities. Every 4D CBGQ is invariant under the passive LT. The invariance of a 4D CBGQ under the passive LT reflects the fact that such 4D GQ represents the same physical quantity for relatively moving inertial observers. We shall use the geometric algebra formalism. The geometric (Clifford) product of two multivectors $A$ and $B$ is written by simply juxtaposing multivectors $AB$. For vectors $a$ and $b$ the geometric product $ab$ decomposes as $ab = a \cdot b + a \wedge b$, where the inner product $a \cdot b$ is $a \cdot b \equiv (1/2)(ab + ba)$ and the outer (or exterior) product $a \wedge b$ is $a \wedge b \equiv (1/2)(ab - ba)$. For the reader’s convenience, all equations will be written with the CBGQs in the standard basis. Therefore, the knowledge of the geometric algebra is not required for the understanding of this presentation. The standard basis $\{\gamma_\mu\}$ is a right-handed orthonormal frame of vectors in the Minkowski spacetime $M^4$ with $\gamma_0$ in the forward light cone, $\gamma_0^2 = 1$ and $\gamma_k^2 = -1$ ($k = 1, 2, 3$). The $\gamma_\mu$ generate by multiplication a complete basis for the spacetime algebra: $1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \gamma_5, \gamma_5$ ($2^4 = 16$ independent elements). $\gamma_5$ is the right-handed unit pseudoscalar, $\gamma_5 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$. Any multivector can be expressed as a linear combination of these 16 basis elements of the spacetime algebra. The $\{\gamma_\mu\}$ basis corresponds to Einstein’s system of coordinates in which the Einstein synchronization of distant clocks [9] and Cartesian space coordinates $x^i$ are used in the chosen inertial frame of reference. Here, we shall also introduce another basis, the $\{r_\mu\}$ basis with the “r” synchronization. The “r” synchronization is commonly used in everyday
life. If the observers who are at different distances from the studio clock set their clocks by the announcement from the studio then they have synchronized their clocks with the studio clock according to the “r” synchronization.

The unit vectors in the \( \{ \gamma_\mu \} \) basis and the \( \{ r_\mu \} \) basis are connected as

\[
r_0 = \gamma_0, \quad r_i = \gamma_0 + \gamma_i.
\]

Hence, the components \( g_{\mu\nu} \) of the metric tensor are \( g_{ii, r} = 0 \), and all other components are \( = 1 \). Obviously it is completely different than in the \( \{ \gamma_\mu \} \) basis, i.e. than the Minkowski metric, which, here, is chosen to be \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \). (Note that in [3] and [5] the Minkowski metric is \( g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \).) Then, according to equation (4) from [3], one can use \( g_{\mu\nu, r} \) to find the transformation matrix \( R_{\mu}^{\nu} \) that connects the components in the \( \{ \gamma_\mu \} \) and the \( \{ r_\mu \} \) bases. The only components that are different from zero are

\[
R_{\mu}^{\mu} = -R_{0}^{0} = 1.
\]

The inverse matrix \( (R_{\mu}^{\nu})^{-1} \) connects the “old” basis, \( \{ \gamma_\mu \} \), with the “new” one, \( \{ r_\mu \} \). The components of any vector are connected in the same way as the components of the position vector \( x \) are connected, i.e., as

\[
x_r^0 = x^0 - x^1 - x^2 - x^3, \quad x_r^i = x^i.
\]

This reveals that in the \( \{ r_\mu \} \) basis the space \( r \) and the time \( t \) cannot be separated; the “3+1 split” of the spacetime into space + time is impossible. Note that there is the zeroth component of \( x \) in the \( \{ r_\mu \} \) basis, \( x_r^0 \neq 0 \), even if in the standard basis \( x^0 = 0 \), but the spatial components \( x^i \neq 0 \). This means that in the 4D spacetime only the position vector \( x, x = x_\mu \gamma_\mu = x_\mu r_\mu, \) is properly defined quantity. In general, the position in the 3D space \( r \) and the time \( t \) have not an independent reality in the 4D spacetime. Although the Einstein and the “r” synchronizations are completely different they are equally well physical and relativistically correct synchronizations. Every synchronization is only a convention and physics must not depend on conventions. An important consequence of the result that in the 4D spacetime \( r \) and \( t \) are not well-defined is presented in section 4 in [10]. There, it is shown that only the world parity \( W, W x = -x \), is well defined in the 4D spacetime and not the usual \( T \) and \( P \) inversions. We remark that in order to treat different bases on an equal footing the general transformation matrix \( T_{\mu}^{\nu} \) is presented in [3], equation (4), that connects the \( \{ \gamma_\mu \} \) basis and some other basis, e.g., the \( \{ r_\mu \} \) basis, in the same reference frame. That matrix \( T_{\mu}^{\nu} \) is expressed in terms of the basis components of the metric tensor and for the connection with the \( \{ r_\mu \} \) basis it is given by equation (5). It is worth mentioning that in equation (1) in [3] it is derived such form of the LT, which is independent of the chosen system of coordinates, including different synchronizations.

2.1. Other objections to the derivations of the AT

3) As already mentioned above (the objection 1)) the identification of the components of \( E \) and \( B \) with the components of \( F^{\alpha\beta} \), (1), is synchronization dependent. If the components \( F^{\alpha\beta} \) of \( F \) are transformed by the transformation
matrix \( R^\mu_\nu \) to the \( \{ r_\mu \} \) basis, then it is obtained that, e.g.,
\[
F_{\mu}^{10} = F^{10} - F^{12} - F^{13}.
\] (7)

Hence, as shown in [3], [10], [2], in the \( \{ r_\mu \} \) basis the identification \( E_1 = F^{10}_r \)
as in (1), yields that the component \( E_1 \) is expressed as the combination of \( E_i \)
and \( B_i \) components from the \( \{ \gamma_\mu \} \) basis
\[
E_1 = F^{10}_r, \quad E_1 = E_1 + cB_3 - cB_2.
\] (8)

This means that if the “\( r \)” synchronization is used then it is not possible to make the usual identifications (1) and (2).

4) As discussed in the next section, in the 4D geometric approach the primary quantity for the whole electromagnetism is a physically measurable quantity, the bivector field \( F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu \), where \( \gamma_\mu \wedge \gamma_\nu \) is the bivector basis and the basis components \( F^{\mu\nu} \) are determined as \( F^{\mu\nu} = \gamma_\mu \cdot (\gamma_\nu \cdot F) = (\gamma_\nu \wedge \gamma_\mu) \cdot F. \)

In the same way as for any other CBGQ it holds that bivector \( F \) is the same 4D quantity for relatively moving inertial observers and for all bases chosen by them, e.g.,
\[
F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = (1/2)F^{\mu\nu}_r r_\mu \wedge r_\nu = (1/2)F^{\mu\nu}_r' \gamma'_\mu \wedge \gamma'_\nu = (1/2)F^{\mu\nu}_r' \gamma'_\mu \wedge \gamma'_\nu, \] (9)

where the primed quantities in both bases \( \{ \gamma_\mu \} \) and \( \{ r_\mu \} \) are the Lorentz transforms of the unprimed ones. For the \( \{ r_\mu \} \) basis and the LT in that basis see [3]. Only the whole \( F \) from (9) is a mathematically correctly defined quantity and it does have a definite physical reality. The components \( F^{0i} \), or \( F^{ij} \) (implicitly determined in the standard basis \( \{ \gamma_\mu \} \), if taken alone, are not properly defined physical quantities in the 4D spacetime. The transformations of these components, e.g., equation (3), which are extracted from the LT of the whole properly defined physical quantity \( F = (1/2)F^{\alpha\beta}\gamma_\alpha \wedge \gamma_\beta \), are not the relativistically correct LT and actually they have nothing to do with the LT. They do not refer to the same 4D quantity for relatively moving observers. Hence, the determination of \( E \) and \( B \) by the components \( F^{0i} \) and \( F^{ij} \), respectively, as the quantities that do not depend on the 4-velocity of the observer is not mathematically and relativistically correct. In contrast to it, the determination of vectors \( E \) and \( B \) relative to the observer by the decomposition of \( F \), i.e., by equations (18) and (19) with coordinate-free quantities, or (20) and (21) with the CBGQs is mathematically and relativistically correct. Every antisymmetric tensor of the second rank (as a geometric quantity) can be decomposed into two vectors and a unit timelike vector, in this case, \( v/c. \) This proves in another way that the usual identification of the components of \( E \) and \( B \) with the components of \( F^{\alpha\beta} \), (1), cannot have a definite physical sense; the components are coordinate quantities and they are only a part of the representation in some basis of an abstract, coordinate-free bivector \( F. \)

5) In addition, it is worth mentioning that in the usual covariant approaches, e.g., [1], the components \( F^{\alpha\beta} \) are defined in terms of a 4-vector potential \( A^\alpha = \)
(Φ, A), equation (11.132) in [1], as $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, equation (11.136) in [1]. The 3D fields $E$ and $B$ are determined in terms of the potentials by equation (11.134) in [1], which, together with equation (11.136) in [1], leads to equation (11.137) in [1] in which, as already stated, the components $F^{\alpha\beta}$ are expressed in terms of the components of the 3D vectors $E$ and $B$. According to that procedure from [1] the 4-vector potential $A^\alpha$ (gauge dependent and thus unmeasurable quantity) is considered to be the primary quantity which determines the measurable quantities, the electric and magnetic fields and also $F^{\alpha\beta}$. Observe that, contrary to the assertions from [1], $A^\alpha$ is not a 4D vector. $A^\alpha$ are only components implicitly taken in the standard basis of the 4D vector $A = A^\mu \gamma^\mu$. In the 4D spacetime only the whole 4D potential $A = A^\mu \gamma^\mu = A^\mu r^\mu$ is a well-defined quantity, whereas it is not the case with the usual scalar potential Φ and the 3D vector potential $A$ in which the components $A^{x,y,z}$ are multiplied by the unit 3D vectors $i$, $j$, $k$ and not by the properly defined unit 4D vectors $\gamma^\mu$.

3. The proofs that the electric and magnetic fields are properly defined vectors on the 4D spacetime and not the usual 3D fields

3.1. Oziewicz’s proof

There is a simple but very strong and completely correct mathematical argument, which is stated by Oziewicz, e.g., in [11]:

What is essential for the number of components of a vector field is the number of variables on which that vector field depends, i.e., the dimension of its domain. In general, the dimension of a vector field that is defined on a n-dimensional space is equal - n. The electric and magnetic fields are defined on a 4D space, i.e., the spacetime. They are always functions of the position vector $x$. This means that they are not the usual 3D fields, but they are properly defined vectors on the 4D spacetime, $E(x)$ and $B(x)$. In any basis they have four components some of which can be zero. This is a fundamental argument and it cannot be disputed in any way. It is very surprising that this argument is not applied in physics much earlier.

The mentioned argument holds in the same measure for the polarization vector $P(x)$ and the magnetization vector $M(x)$, which are discussed in detail in [12, 13, 2]. In [12] the electromagnetic field equations for moving media are presented, whereas in [13] the constitutive relations and the magnetoelectric effect for moving media are investigated from the geometric point of view. $P(x)$ and $M(x)$ are also properly defined vectors on the 4D spacetime and not the 3D vectors as usually considered, e.g., in [1, 4]. Note that in the 4D spacetime we always have to deal with correctly defined vectors $E(x)$, $B(x)$, $P(x)$, $M(x)$, etc. even in the usual static case, i.e., if the usual 3D fields $E(r)$, $B(r)$ do not explicitly depend on the time $t$. The reason is that if in the 4D spacetime the standard basis is used then the LT cannot transform the spatial coordinates from
one frame only to spatial coordinates in a relatively moving inertial frame of reference. What is static case for one inertial observer is not more static case for relatively moving inertial observer, but a time dependent case. Furthermore, if an observer uses the “t” synchronization and not the standard Einstein’s synchronization, then, as seen from (4), the space and time are not separated and the usual 3D vector \( r \) is meaningless. If the principle of relativity has to be satisfied and the physics must be the same for all inertial observers and for \( \{ \gamma_\mu \}, \{ r_\mu \}, \{ \gamma'_\mu \}, \) etc. bases which they use, then the properly defined quantity is the position vector \( x \),

\[
x = x'^\nu \gamma_\nu = x'^\nu r_\nu = x'^\nu r'_\nu ,
\]

and not \( r \) and \( t \). Consequently, in the 4D spacetime, e.g., the electric field is properly defined as the vector \( E(x) \) for which the relation (37) given below holds.

### 3.2. Briefly about the \( F \) formulation

In [14] an axiomatic geometric formulation of electromagnetism with only one axiom, the field equation for the bivector field \( F \), equation (4) in [14], is constructed. There, it is shown that the bivector \( F = F(x) \), which represent the electromagnetic field, can be taken as the primary quantity for the whole electromagnetism. It yields a complete description of the electromagnetic field and, in fact, there is no need to introduce either the field vectors or the potentials. If the field equation for \( F \) is written with AQs it becomes

\[
\partial \cdot F + \partial \wedge F = j/\varepsilon_0 c ,
\]

where the source of the field is the charge-current density vector \( j(x) \) (equation (4) in [14]). If \( j(x) \) is the sole source of \( F \) then the general solution for \( F \) with AQs is given by equation (8) in [14]. Particularly, the general expression for \( F \) for an arbitrary motion of a charge is given by equation (10) in [14] with AQs and as a CBGQ in the \( \{ \gamma_\mu \} \) basis by equation (11) in [14]. \( F \) of point charge in uniform motion as an AQ is given by equation (12) in [14], i.e., equation (76) here. The components in the standard basis \( F_{\alpha\beta} \) from that equation (11) in [14] are the same as the usual result from Chapter 14 in [1]. If the equation for \( F \) (11) is written with CBGQs in the \( \{ \gamma_\mu \} \) basis it becomes equation (5) in [14],

\[
\partial_\alpha F^{\alpha\beta} \gamma_\beta - \partial_\alpha \ast F^{\alpha\beta} \gamma_\beta \gamma_\gamma = (1/\varepsilon_0 c) j^\beta \gamma_\beta ,
\]

where the usual dual tensor (components) is \( \ast F^{\alpha\beta} = (1/2) \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \). From that equation one easily finds the usual covariant form (only the basis components of the 4D geometric quantities in the \( \{ \gamma_\mu \} \) basis) of the field equations as equation (6) in [14],

\[
\partial_\alpha F^{\alpha\beta} = j^\beta /\varepsilon_0 c , \quad \partial_\alpha \ast F^{\alpha\beta} = 0 .
\]

These two equations for the components in the standard basis \( F^{\alpha\beta} \) are the equations (11.141) and (11.142) in [1].
In the same paper, [14], it is also shown that this formulation with the $F$ field is in a complete agreement with the Trouton-Noble experiment, i.e., in the approach with $F$ as a 4D GQ there is no Trouton-Noble paradox. It is clearly visible from [14] and this short presentation that, in principle, the components $F^\alpha\beta$ of the electromagnetic field tensor, i.e., of the bivector $F$ here and in [14], have nothing to do with the components of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$. Only the whole $F$ has an independent physical reality; it is a physically measurable quantity by the Lorentz force density, $K_{(j)} = F \cdot j/c$, equation (27) in [14], or, for a charge $q$ by the Lorentz force

$$K_L = (q/c) F \cdot u,$$

where $u$ is the 4D velocity vector of a charge $q$ (it is defined to be the tangent to its world line).

It is worth noting that the expression for the Lorentz force density, $K_{(j)} = F \cdot j/c$, is directly derived from the field equation for $F$ [11]. Similarly, in [14], the coordinate-free expressions for the stress-energy vector $T(n)$ (equations (37) and (38)), the energy density $U$ (scalar, equation (39)), the Poynting vector $S$ (equation (40)), the momentum density vector $g$ (equation (42)), the angular momentum density $M$ (bivector, equation (43)), the local charge conservation law (equation (48)) and the local energy-momentum conservation law (equations (49) and (50)) are all directly derived from that field equation (11). In that axiomatic geometric formulation from [14] $T(n)$ is the most important quantity for the momentum and energy of the electromagnetic field,

$$T(n) = -\varepsilon_0/2 \left[ (F \cdot F)n + 2(F \cdot n) \cdot F \right],$$

equation (37) in [14]. $T(n)$ is a vector-valued linear function on the tangent space at each spacetime point $x$ describing the flow of energy-momentum through a hypersurface with normal $n = n(x)$. It can be expressed by $U$ and $S$ as in equation (41) in [14],

$$T(n) = U n + (1/c) S, \quad U = -\varepsilon_0/2 \left[ (F \cdot F) + 2(F \cdot n)^2 \right],$$
$$S = -\varepsilon_0 c \left[ (F \cdot n) \cdot F - (F \cdot n)^2 n \right]$$

(16)

Observe that $T(n)$ as a whole quantity, i.e., the combination of $U$ and $S$ from [16] enters into a fundamental physical law, the local energy-momentum conservation law

$$\partial \cdot T(n) = 0$$

(17)

for the free fields, equation (49) in [14]. This means, as stated in [14], that only $T(n)$, as a whole quantity, does have a physically correct interpretation. In [14] this viewpoint is nicely illustrated considering an apparent paradox in the usual 3D formulation in which the Poynting vector $S$ is interpreted as an energy flux due to the propagation of fields. If such an interpretation of $S$ is adopted then there is a paradox for the case of an uniformly accelerated charge, e.g., section
In that case, \( S = 0 \) (there is no energy flow) but at the same time \( U \neq 0 \) (there is an energy density) for the field points on the axis of motion. The obvious question is how the fields propagate along the axis of motion to give that \( U \neq 0 \). In the formulation with 4D GQs the important quantity is \( T(n) \) and not \( S \) and \( U \) taken separately. \( T(n) \) is \( \neq 0 \) everywhere on the axis of motion and the local energy-momentum conservation law \( (17) \) holds everywhere.

3.3. Proof by the use of the decomposition of \( F \)

In contrast to the usual covariant approach, which deals with the identification of components \( (1) \) and \( (2) \), it is possible to construct in a mathematically correct way the 4D vectors of the electric and magnetic fields using the decomposition of \( F \). There is a mathematical theorem according to which any antisymmetric tensor of the second rank can be decomposed into two space-like vectors and the unit time-like vector. For the proof of that theorem in geometric terms see, e.g., [15].

If that theorem is applied to the bivector \( F \) then it is obtained that

\[
F = E \wedge v/c + (IcB) \cdot v/c,
\]

where the electric and magnetic fields are represented by vectors \( E(x) \) and \( B(x) \), see, e.g., [14]. The unit pseudoscalar \( I \) is defined algebraically without introducing any reference frame. If \( I \) is represented in the \( \{\gamma_\mu\} \) basis it becomes \( I = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_5 \). The vector \( v \) in the decomposition \( (18) \) is interpreted as the velocity vector of the observers who measure \( E \) and \( B \) fields. Then \( E(x) \) and \( B(x) \) are defined with respect to \( v \), i.e., with respect to the observer, as

\[
E = F \cdot v/c, \quad B = -(1/c)I(F \wedge v/c).
\]

It also holds that \( E \cdot v = B \cdot v = 0 \); both \( E \) and \( B \) are space-like vectors. If the decomposition \( (18) \) is written with the CBGQs in the \( \{\gamma_\mu\} \) basis it becomes

\[
F = (1/2)F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad F^{\mu\nu} = (1/c)(E^\mu v^\nu - E^\nu v^\mu) + \varepsilon^{\mu\nu\alpha\beta}v_\alpha B_\beta,
\]

where \( \gamma_\mu \wedge \gamma_\nu \) is the bivector basis. If the equations for \( E \) and \( B \) \( (19) \) are written with the CBGQs in the \( \{\gamma_\mu\} \) basis they become

\[
E = E^{\mu} \gamma_\mu = (1/c)F^{\mu\nu} v_\nu \gamma_\mu, \quad B = B^{\mu} \gamma_\mu = (1/2c^2)\epsilon^{\mu\nu\alpha\beta}F_{\nu\alpha} v_\beta \gamma_\mu.
\]

All these relations, \( (18) - (21) \) are the mathematically correct definitions. They are first reported (only components implicitly taken in the standard basis) by Minkowski in section 11.6 in [16].

Let us introduce the \( \gamma_0 \) - frame; the frame of “fiducial” observers for which \( v = c\gamma_0 \) and in which the standard basis is chosen. Therefore, in the \( \gamma_0 \)-frame, e.g., \( E \) becomes \( E = F \cdot \gamma_0 \). It can be shown that in the \( \gamma_0 \) - frame \( E \cdot \gamma_0 = B \cdot \gamma_0 = 0 \), which means that \( E \) and \( B \) are orthogonal to \( \gamma_0 \); they refer to the
3D subspace orthogonal to the specific timelike direction $\gamma_0$. If $E$ and $B$ are written as CBGQs in the standard basis they become

$$
E = E^\mu \gamma_\mu = 0 \gamma_0 + F^{0i} \gamma_i,
$$

$$
B = B^\mu \gamma_\mu = 0 \gamma_0 + \frac{1}{2c} \varepsilon^{0ijk} F_{kj} \gamma_i.
$$

Note that $\gamma_0 = (\gamma_0)^\mu \gamma_\mu$ with $(\gamma_0)^\mu = (1, 0, 0, 0)$. Hence, in the $\gamma_0$-frame the temporal components of $E$ and $B$ are zero and only the spatial components remain

$$
E^0 = B^0 = 0, \quad E^i = F^{i0}, \quad B^i = \frac{1}{2c} \varepsilon^{0ijk} F_{kj}.
$$

It is visible from (22) and (23) that $E^i$ and $B^i$ are the same as the components of the 3D $E$ and $B$, equation (1), i.e., the same as in equation (11.137) in [1]. However, there are very important differences between the identifications (1) and equations (22) and (23). The components of $E$ and $B$ in (1) are not the spatial components of the 4D quantities. They transform according to the AT (4). The antisymmetric $\varepsilon$ tensor in (1) and (2) is a third-rank antisymmetric tensor. On the other hand, the components of $E$ and $B$ in (22) and (23) are the spatial components of the 4D geometric quantities that are taken in the standard basis. They transform according to the LT that are given below, equation (30). The antisymmetric $\varepsilon$ tensor in (22) and (23) is a fourth-rank antisymmetric tensor. Furthermore, it is shown above, equations (1) and (2), that the identifications (1) and (2) do not hold in the $\{r_\mu\}$ basis. But, the relations (21) hold for any chosen basis, including the $\{r_\mu\}$ basis, e.g.,

$$
E = E^\nu r_\nu = E^\nu r_\nu = (1/c) F^{\nu\rho} r_\nu v_\rho r_\mu.
$$

This can be easily checked using the above mentioned matrix $R^\mu_\nu$. Thus, for the components of vector $E$ it also holds that

$$
E^0_r = E^0 - E^1 - E^2 - E^3, \quad E^i_r = E^i.
$$

From these relations it follows that there is the zeroth component of $E$ in the $\{r_\mu\}$ basis, $E^0_r \neq 0$, even if it is $= 0$ in the standard basis, $E^0 = 0$, but the spatial components $E^i \neq 0$. This again shows that the components taken alone are not physical. The whole consideration presented here explicitly reveals that in the 4D spacetime the usual identifications (1) and (2) are not mathematically correct and that

the electric field $E$ is a vector (4D vector); it is an inner product of a bivector $F$ and the velocity vector $v$ of the observer who measures fields.

It is worth mentioning that in the 4D spacetime the mathematically correct relations (18) - (21) are already firmly theoretically founded and they are known to many physicists. The recent example is in [17]; it is only the electric part (the magnetic part is zero there). Similarly, in the component form these relations are presented, e.g., in [18] and in the basis-free form with the abstract 4D quantities in [7, 8, 15] and in, e.g., [19]. But, it has to be noted that from all of them only Oziewicz, see [11] and references to his papers in it, exclusively deals with the
abstract, basis-free 4D quantities. He correctly considers from the outset that in the 4D spacetime such quantities are physical quantities and not the usual 3D quantities. All others, starting with Minkowski [16], are not consistent in the use of the 4D electric and magnetic fields. They use together the 4D fields and the usual 3D fields $\mathbf{E}$ and $\mathbf{B}$ considering that the 3D fields are physically measurable quantities and that their AT are the correct LT. Minkowski [16] introduced only in section 11.6 the 4D fields and their LT. In other sections he also dealt with the 3D fields and their AT.

4. The proofs that under the mathematically correct LT the electric field vector transforms as any other vector transforms, i.e., again to the electric field vector

As proved in section 2 the electric field is properly defined vector on the 4D spacetime and the same holds for the magnetic field. Hence, under the LT, e.g., the electric field vector must transform as any other vector transforms, i.e., again to the electric field vector; there is no mixing with the magnetic field vector $\mathbf{B}$. In [20] the same result is obtained for the electric field as a bivector and for the magnetic field as well. This will be explicitly shown both for the active LT in 4.1 and for the passive LT in 4.2.

4.1. Proof with the coordinate-free quantities, AQs, and the active LT

Regarding the correct LT let us start from the definition with the coordinate-free quantities $E = c^{-1}F \cdot v$ and with the active LT. Mathematically, as noticed by Oziewicz [11], an active LT must act on all tensor fields from which the vector field $E$ is composed, including an observer’s time-like vector field. This means that the mathematically correct active LT of $E = c^{-1}F \cdot v$ are $E' = c^{-1}F' \cdot v'$; both $F$ and $v$ are transformed. It was first discovered by Minkowski in section 11.6 in [16] but with components implicitly taken in the standard basis and reinvented and generalized in terms of 4D GQs in [21-26] and [20], see also section 5 in [2]. As explicitly shown, e.g., in [26], in the geometric algebra formalism any multivector $N$ transforms by the active LT in the same way, i.e., as $N \rightarrow N' = RN\tilde{R}$, where $R$ is given by equation (10) in [26] (equation (39) in [2]); for boosts in an arbitrary direction the rotor $R$ is

$$R = (1 + \gamma + \gamma_0\beta)/(2(1 + \gamma))^{1/2},$$

(26)

where $\gamma = (1 - \beta^2)^{-1/2}$, the vector $\beta = \beta n$, $\beta$ on the r.h.s. of that equation is the scalar velocity in units of $c$ and $n$ is not the basis vector but any unit space-like vector orthogonal to $\gamma_0$. The reverse $\tilde{R}$ is defined by the operation of reversion according to which $AB = \tilde{B}\tilde{A}$, for any multivectors $A$ and $B$, see section 3 in [26] (section 5 in [2]). Hence, the vector $E = c^{-1}F \cdot v$ transforms by the mathematically correct active LT $R$ into

$$E' = RE\tilde{R} = c^{-1}R(F \cdot v)\tilde{R} = c^{-1}(RF\tilde{R}) \cdot (Rv\tilde{R}) = c^{-1}F'v'.$$

(27)
If \( v = c \gamma_0 \) is taken in the expression for \( E \) then \( E \) becomes \( E = F \cdot \gamma_0 \) and it transforms as in [16], i.e., that both \( F \) and \( \gamma_0 \) are transformed by the LT.

\[
E = F \cdot \gamma_0 \rightarrow E' = R(F \cdot \gamma_0)\tilde{R} = (RF\tilde{R}) \cdot (R\gamma_0\tilde{R}).
\]  

(28)

Hence, the explicit form for \( E' \) with the abstract, coordinate-free quantities is given by equation (13) in [26],

\[
E' = E + \gamma(E \cdot \beta)\{\gamma_0 - (\gamma/(1 + \gamma))\beta\},
\]  

(29)

In (29) \( \beta \) is a vector. In the standard basis and for boosts in the direction \( x^1 \) the components of that \( E' \) are

\[
E'^\mu = (E'^0 = -\beta\gamma E^1, \ E'^1 = \gamma E^1, \ E'^{2,3} = E^{2,3}).
\]  

(30)

Under the active LT the electric field vector \( E = F \cdot \gamma_0 \) (as a CBGQ \( E = E^\mu \gamma_\mu = 0\gamma_0 + F^\mu \gamma_\mu \)) is transformed into a new electric field vector \( E' \). Note that under the active LT the components are changed, (30), but the basis remains unchanged,

\[
E'^\mu \gamma_\mu = -\beta\gamma E^1\gamma_0 + \gamma E^1\gamma_1 + E^2\gamma_2 + E^3\gamma_3,
\]  

(31)

see equation (14) in [26] (equation (43) in [2]), i.e., equation (54) below. The components \( E^\mu \) transform by the LT again to the components \( E'^\mu \) and there is no mixing with \( B^\mu \). In general, the LT of the components \( E^\mu \) (in the \( \{\gamma_\mu\} \) basis) of \( E = E^\mu \gamma_\mu \) are given as

\[
E'^0 = \gamma(E^0 - \beta E^1), \ E'^1 = \gamma(E^1 - \beta E^0), \ E'^{2,3} = E^{2,3},
\]  

(32)

for a boost along the \( x^1 \) axis, i.e., the same LT as for any other 4D vector.

On the other hand, if in \( E = F \cdot \gamma_0 \) only \( F \) is transformed by the active LT and not \( \gamma_0 \), which is not a mathematically correct procedure, then the components of that \( E'_F \) will be denoted as \( E'^\mu_F \) and they are

\[
E'^\mu_F = (E'^0_F = 0, \ E'^1_F = E^1, \ E'^2_F = \gamma(E^2 - c\beta B^3), \ E'^3_F = \gamma(E^3 + c\beta B^3)),
\]  

(33)

see equation (17) in [26] (equation (46) in [2]), i.e., (48) below. The transformations of the spatial components (taken in the standard basis) of \( E \) are exactly the same as the transformations of \( E_{x,y,z} \) from equation (11.148) in [1], i.e., as in equation (14). However, from \( E = F \cdot \gamma_0 \) it follows that the components of \( E \) are \( E^\mu = (E^0 = 0, \ E^1, \ E^2, \ E^3) \). Hence, if only \( F \) is transformed by the LT then the temporal components of both \( E \) and \( E'_F \) are zero, \( E^0 = E'^0_F = 0 \), which explicitly reveals that such transformations are not the mathematically correct LT; the LT cannot transform \( E^0 = 0 \) again to \( E'^0_F = 0 \). This proves that the transformations (30) in which both \( F \) and \( \gamma_0 \) are transformed are the correct LT.

4.2. Proof with CBGQs and the passive LT
If $E$ is written as a CBGQ, i.e., as in (21), then we have to use the passive LT. For example, in the $\gamma_0$-frame $E$ is given as

$$E = E^\mu \gamma_\mu = [(1/c)^{F^i_0 v_0}] \gamma_i = 0 \gamma_0 + E^i \gamma_i$$

(34)

For boosts in the $\gamma_1$ direction and if both $F^i_0$ and $v_0$ are transformed by the LT then, as for any other CBGQ, it holds that

$$E = E^\mu \gamma_\mu = [(1/c)^{F^\mu_\nu v'_\nu}] \gamma'_\mu = E'^\mu \gamma'_\mu,$$

(35)

where, again, the components $E'^\mu$ are the same as in (30), see [24]. On the other hand, if only $F^i_0$ is transformed but not $v_0$ the transformed components $E'_F$ are again the same as in (4) and the same objections as in section 4.1 hold also here. In addition, it can be easily checked that

$$E'^\mu \gamma'_\mu \neq E^\mu \gamma_\mu,$$

(36)

which additionally proves that the transformations in which only $F$ is transformed are not the relativistically correct LT. In that way it is also proved that the transformations given by equations (11.148) ([23] here) and (11.149) from [1] are not the LT but, as called here, the mathematically incorrect AT that do not refer to the same quantity. As can be seen from the above discussion if $E$ is written as a CBGQ then, as for any other 4D CBGQ, it holds that

$$E = E^\nu \gamma_\nu = E'^\nu \gamma'_\nu = E^\nu r_\nu = E'^\nu r'_\nu.$$  

(37)

Here, as in [4], the primed quantities in both bases $\{\gamma_\mu\}$ and $\{r_\mu\}$ are the Lorentz transforms of the unprimed ones.

4.3. A short discussion of the field equations with vectors $E$ and $B$

If the decomposition of $F$ from (20) is introduced into (12) then the field equation (38) is obtained

$$\partial_\alpha (\delta^{\alpha\beta} \mu_\nu E^\mu \nu + \epsilon^{\alpha\beta\mu\nu} v_\mu c B_\nu) - (j^\beta / \varepsilon_0) \gamma_\beta + \partial_\alpha (\delta^{\alpha\beta} \mu_\nu v_\mu c B_\nu + \epsilon^{\alpha\beta\mu\nu} v_\mu E_\nu) \gamma_\beta \gamma_\beta = 0,$$

(38)

where $E^\alpha$ and $B^\alpha$ are the basis components in the standard basis of the 4D vectors $E$ and $B$, $\delta^{\alpha\beta} \mu_\nu = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu$ and $\gamma_\beta$ is the pseudoscalar in the $\{\gamma_\mu\}$ basis. This is equation (40) in [23], but there it is written using some unspecified basis $\{e_\mu\}$. The first part in (38) comes from $\partial \cdot F = j/\varepsilon_0 c$ and the second one (the source-free part) comes from $\partial \wedge F = 0$. As discussed in detail in [23] equation (38) is the relativistically correct, manifestly covariant field equation that generalizes the usual Maxwell equations with the 3D fields $E$ and $B$. It, (38), can be compared with the usual formulation with the 3D quantities going to the $\gamma_0$-frame in which $v = c \gamma_0$ and equation (23) holds. This yields that equation (38) becomes
\[
(\partial_k E^k - j^0/c\varepsilon_0)\gamma_0 + (-\partial_0 E^i + c\varepsilon^{ijk}\partial_j B_k - j^i/c\varepsilon_0)\gamma_i +
(-c\partial_k B^k)\gamma_5\gamma_0 + (c\partial_0 B^i + \varepsilon^{ijk}\partial_j E_k)\gamma_5\gamma_i = 0. \tag{39}
\]

The equation (39) contains all four usual Maxwell equations in the component form. The first part (with \(\gamma_\alpha\)) in (39) contains two Maxwell equations in the component form, the Gauss law for the electric field (the first bracket, with \(\gamma_0\)) and the Ampère-Maxwell law (the second bracket, with \(\gamma_i\)). The second part (with \(\gamma_5\gamma_\alpha\)) contains the component form of another two Maxwell equations, the Gauss law for the magnetic field (with \(\gamma_5\gamma_0\)) and Faraday’s law (with \(\gamma_5\gamma_i\)).

Observe that the component form of the Maxwell equations with the 3D \(E\) and \(B\)

\[
\partial_k E_k - j^0/c\varepsilon_0 = 0, \quad -\partial_0 E_i + c\varepsilon_{ikj}\partial_j B_k - j^i/c\varepsilon_0 = 0,
\]

\[
\partial_k B_k = 0, \quad c\partial_0 B_i + \varepsilon_{ikj}\partial_j E_k = 0 \tag{40}
\]

is obtained from the covariant Maxwell equations (13) using the usual identifications of six independent components of \(F^{\mu\nu}\) with three components \(E_i\) and three components \(B_i\) as in (1) and also in (2). But, as shown above, such an identification is meaningless in the \(\{r_\mu\}\) basis, which means that Maxwell equations (40) do not hold in the \(\{r_\mu\}\) basis. Moreover, the components of the 3D fields from (40) transform according to the AT (4) and not according to mathematically correct LT (29) - (32), which causes, as explicitly shown in [23], that equations (40) are not covariant under the LT. On the other hand, contrary to the formulation of the electromagnetism with \(E\) and \(B\),

the formulation with the 4D fields \(E\) and \(B\), i.e., with equation (38), is correct not only in the \(\gamma_0\) - frame with the standard basis \(\{\gamma_\mu\}\) but in all other relatively moving frames and it holds for any permissible choice of coordinates, i.e., bases.

This consideration reveals that the 4D fields \(E\) and \(B\) that transform like in (29) - (32) and the field equation (38) do not have the same physical interpretation as the usual 3D fields \(E\) and \(B\) and the usual Maxwell equations (40) except in the \(\gamma_0\) - frame with the \(\{\gamma_\mu\}\) basis in which \(E^0 = B^0 = 0\).

Here, it is at place a remark about the \(\gamma_0\) - frame. The dependence of the relations (21) and the field equation (38) on \(v\) reflects the arbitrariness in the selection of the \(\gamma_0\) - frame, but at the same time this arbitrariness makes that equations (21) and (38) are independent of that choice. The \(\gamma_0\) - frame can be selected at our disposal depending on the considered problem which proves that we don’t have a kind of “preferred” frame theory. Some examples will be discussed in sections 9 and 10.

4.4. The generalization of the field equation for \(F\) (11) to a magnetized and polarized moving medium

The generalization of the field equation for \(F\) (11) to a magnetized and polarized moving medium with the generalized magnetization-polarization bivector \(M(x)\)
is presented in [12]. That generalization is obtained simply replacing $F$ by $F + \mathcal{M}/\varepsilon_0$, which yields the primary equations for the electromagnetism in moving media

$$\partial (\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c; \quad \partial \cdot (\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c, \quad \partial \wedge F = 0,$$

(41)
equation (7) in [12]. $j^{(C)}$ is the conduction current density of the free charges and $j^{(M)} = -c \partial \cdot \mathcal{M}$ is the magnetization-polarization current density of the bound charges. The total current density vector $j$ is $j = j^{(C)} + j^{(M)}$. If written with the CBGQs in the standard basis that equation becomes

$$\partial_\alpha (\varepsilon_0 F^{\alpha\beta} + \mathcal{M}^{\alpha\beta}) \gamma_\beta - \partial_\alpha (\varepsilon_0 * F^{\alpha\beta}) \gamma_5 \gamma_\beta = c^{-1} j^{(C)\beta} \gamma_\beta,$$

(42)what is equation (8) in [12]. Observe that if in equation for $F$ (11) $j = j^{(C)} + j^{(M)}$ is the total current density then (11), i.e., (12), holds unchanged in moving medium as well.

In the same way as in (20) the generalized magnetization-polarization bivector $\mathcal{M}(x)$ can be decomposed into two vectors, the polarization vector $P(x)$ and the magnetization vector $M(x)$ and the unit time-like vector $u/c$, equation (21) in [12],

$$\mathcal{M} = P \wedge u/c + (MI) \cdot u/c^2,$$

(43)or, with the CBGQs in the $\{\gamma_\mu\}$ basis, equation (22) in [12],

$$\mathcal{M} = (1/2)\mathcal{M}^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad \mathcal{M}^{\mu\nu} = (1/c)(P^{\mu} u^{\nu} - P^{\nu} u^{\mu}) + (1/c^2)\varepsilon^{\mu\nu\alpha\beta} M_\alpha u_\beta. \quad (44)$$

The vector $u$ is identified with bulk velocity vector of the medium in spacetime. Hence, as in (21), equation (24) in [12],

$$P = (1/c)\mathcal{M}^{\mu\nu} u_\nu \gamma_\mu, \quad M = (1/2)\varepsilon^{\mu\nu\alpha\beta} \mathcal{M}_{\alpha\nu} u_\beta \gamma_\mu, \quad (45)$$

with $P^{\mu} u_\mu = M^{\mu} u_\mu = 0$, only three components of $P$ and three components of $M$ are independent since $\mathcal{M}$ is antisymmetric. Inserting the decompositions of $F(x)$ (20) and $\mathcal{M}(x)$ (44) into the field equation (42) one finds equation (29) in [12],

$$\partial_\alpha \{\varepsilon_0 [\delta^{\alpha\beta}_{\mu\nu} E^{\mu} v^{\nu} + \varepsilon^{\alpha\beta\mu\nu} v_\mu B_\nu] + [\delta^{\alpha\beta}_{\mu\nu} P^{\mu} u^{\nu} + (1/c)\varepsilon^{\alpha\beta\mu\nu} M_{\mu} u_\nu] \} \gamma_\beta = j^{(C)\beta} \gamma_\beta,$$

(46)where $\delta^{\alpha\beta}_{\mu\nu} = \delta^\alpha_{\mu} \delta^\beta_{\nu} - \delta^\alpha_{\nu} \delta^\beta_{\mu}$. This is the part of the equation (42) with sources, whereas another part, the equation without sources, equation (30) in [12], becomes

$$\partial_\alpha \{c \delta^{\alpha\beta}_{\mu\nu} B^{\mu} v^{\nu} + \varepsilon^{\alpha\beta\mu\nu} E_\mu v_\nu \} \gamma_5 \gamma_\beta = 0.$$

(47)The equations (46) and (47) are the fundamental equations for moving media and they replace all usual Maxwell’s equations (with 3D vectors) for moving media.

As stated in [12], in contrast to all usual formulations of the field equations for
moving media, the equation (46) contains two different velocity vectors, $v$ - the velocity of the observers and $u$ - the velocity of the moving medium, which come from the decompositions of $F$ and $\mathcal{M}$, equations (20) and (44), respectively. It is shown in [12] that, in the same way as for vacuum, the field equations (46) and (47) with the 4D fields are not equivalent to the usual Maxwell’s equations (with 3D vectors) for moving media because the AT of the 3D fields are not the mathematically correct LT.

Furthermore, in the same way as for vacuum, i.e., as in [14], one can derive from (11) the stress-energy vector $T(n)$ for a moving medium simply replacing $F$ by $F + \mathcal{M}/\epsilon_0$ in equations (26), (37-47) in [14], i.e., in equations (15), (16) here. The expression for $T(n), T(n) = \bar{U}n + (1/c)S$, will remain unchanged, but the energy density $U$ and the Poynting vector $S$ will change according to the described replacement. This will be important in the discussion of Abraham-Minkowski controversy.

5. The comparison of the derivations of the AT and the LT using matrices (the components in the standard basis)

5.1. The electric and magnetic fields as vectors

For the reader’s convenience the same results as in sections 3 - 3.3 can be obtained explicitly using the matrices. We write the relation $E^\mu = c^{-1}F^{\mu\nu}v_\nu$ in the $\gamma_0$ - frame, i.e., for $v = c\gamma_0$. From the matrix for $F^{\mu\nu}$ and $v_\nu = (c, 0, 0, 0)$ one finds $E^\mu = (0, F^{10} = E^1, F^{20} = E^2, F^{30} = E^3)$.

Then, for the AT only $F^{\mu\nu}$ is transformed by the LT but not the velocity of the observer $v = c\gamma_0$. The Lorentz transformed $F^{\mu\nu}$ is (symbolically) $F' = AF\tilde{A}$; here $A, F, ..$ denote matrices. This relation can be written with components as $F'^{\mu\nu} = A^{\mu}_{\rho}F^{\rho\sigma}A^{\sigma}_{\nu}$. The matrix $A$ is the boost in the direction $x^1$ (in the standard basis) and it is written in equation (54). $A$ is also given by equation (11.98) in [1] (with only $\beta_1 \neq 0$) and $\tilde{A}$ is obtained transposing $A$. The transformed components $E'^{\mu}_F$ are obtained as $E'^{\mu}_F = c^{-1}F'^{\mu\nu}v_\nu$, or explicitly with matrices as

$$
\begin{bmatrix}
0 & -F'^{10} & -F'^{20} & -F'^{30} \\
E^{1} & 0 & -F'^{21} & -F'^{31} \\
\gamma(E^2 - \beta cB^3) & \gamma(-\beta E^2 + cB^3) & 0 & -F'^{32} \\
\gamma(E^3 + \beta cB^2) & \gamma(-\beta E^3 - cB^2) & cB^1 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
E^1 \\
\gamma(E^2 - \beta cB^3) \\
\gamma(E^3 + \beta cB^2)
\end{bmatrix},
$$

(48)

where the first matrix is the Lorentz transformed $F'^{\mu\nu}$, i.e., $F'^{\mu\nu}$, and the second matrix is $c^{-1}v^\mu = \gamma^\mu_0$. The components $E'^{\mu}_F$ are already written in equation (43). As seen from (48) the transformed zeroth component $E'^{0}_F$ is again = 0, which shows, as previously stated, that such transformations cannot be the mathematically correct LT; the LT cannot transform the 4D vector with $E^0 = 0$ into the 4D vector with $E'^{0}_F = 0$. Furthermore, it can be simply checked using
that for the CBGQs holds
\[ E^\nu_{\mu} \neq E^\nu_{\gamma \mu}, \]  
(49)

where \( E^\nu_{\mu} \) is from (48). This is the same as in (48), i.e., it additionally proves that \( E^\nu_{\mu} \) is not obtained by the mathematically correct LT from \( E^\mu \).

Under the mathematically correct LT both \( F^{\mu \nu} \) and the velocity of the observer \( v = c \gamma_0 \) are transformed. Then (symbolically)
\[ E = c^{-1} F \cdot v \rightarrow E' = c^{-1} F' \cdot v' = c^{-1} (A F A \gamma)(A^{-1} v) = A(c^{-1} F v) = A E, \]  
(50)

where, here, \( E, F, v, A, F', \ldots \) denote matrices. Hence, \( E^\nu_{\mu} \) can be written as
\[ E^\nu_{\mu} = c^{-1} F^{\mu \nu} v'_\nu = c^{-1} (A^\rho_\mu F^{\sigma \tau} A^\tau_\nu)(A^{-1})^\rho_\alpha v_\alpha = A^\rho_\mu (c^{-1} F^{\rho \tau} v_\tau). \]  
(51)

Using the explicit matrices \( c^{-1} A^{-1} v \) is given as
\[ c^{-1} A^{-1} v = c^{-1} \begin{bmatrix} \gamma & \beta \gamma & 0 & 0 \\ \beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma \\ \beta \gamma \\ 0 \\ 0 \end{bmatrix} \]  
(52)

and \( E^\nu_{\mu} \) is \( E^\nu_{\mu} = c^{-1} F^{\mu \nu} v'_\nu \), i.e.,
\[ \begin{bmatrix} E^1 \\ E^2 \\ E^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -E^1 \\ -F^2 0' \\ -F^3 0' \end{bmatrix} \begin{bmatrix} 0 \\ -E^1 \\ -F^2 1' \\ -F^3 1' \end{bmatrix} = \begin{bmatrix} -\beta \gamma E^1 \\ \gamma E^1 \\ E^2 \\ E^3 \end{bmatrix}, \]  
(53)

where again the first matrix is \( F^{\mu \nu} \), as in (48), but the second matrix is the Lorentz transformed 4-velocity of the observer, i.e., it is given by equation (52).

Observe that the same result for \( E^\nu_{\mu} \) is obtained from \( E^\nu_{\mu} = A^\rho_\mu E^\nu_\rho \),
\[ E^\nu_{\mu} = A^\rho_\mu E^\nu_\rho = \begin{bmatrix} \gamma & -\beta \gamma & 0 & 0 \\ -\beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ E^1 \\ E^2 \\ E^3 \end{bmatrix} = \begin{bmatrix} -\beta \gamma E^1 \\ \gamma E^1 \\ E^2 \\ E^3 \end{bmatrix}. \]  
(54)

The components \( E^\nu_{\mu} \) are the same as in (30). This result clearly shows that the transformations in which both \( F \) and the velocity of the observer \( v \) are transformed are the mathematically correct LT; under such LT the electric field 4D vector transforms again only to the electric field 4D vector as any other 4D vector transforms.

As an additional proof of that result it can be simply checked using (51) that for the CBGQs \( E^\nu_{\gamma \nu}, E^\nu_{\gamma' \nu}, \ldots \) again holds the relation (37), \( E = E^\nu_{\gamma \nu} = E^\nu_{\gamma' \nu} = E^\nu_{\gamma' \nu} = E^\nu_{\gamma' \nu}, \) as for any other CBGQ.

5.2. The electric and magnetic fields as bivectors
In [20] the same result about the fundamental difference between the AT and the correct LT is obtained representing the electric and magnetic fields by bivectors. The representation by bivectors is used, e.g., in [27, 28] and they derived the AT in which the components of the transformed electric field bivector are expressed by the combination of components of the electric and magnetic field bivectors like in [4]. In the $\gamma_0$-frame the electric field bivector $E_H$ is determined from the electromagnetic field bivector, equation (2) in [20], $E_H = (F \cdot \gamma_0)\gamma_0 = (1/2)(F - \gamma_0 F\gamma_0)$. In section 5 in [20] the derivation of the AT from [27, 28] is presented. The space-time split is made and accordingly the space-space components are zero for the matrix of the electric field bivector $(E_H)^{\mu\nu}$, equation (5) in [20], i.e., $(E_H)^{00} = E^0 = E^0$, $(E_H)^{ij} = 0$. Then, in [27, 28], the same is supposed to hold for the electric field bivector that is transformed by the AT, equations (18) and (19) in [20]. The transformed electric field bivector $E'_{H,at}$ is not obtained in the way in which all other multivectors transform, but it is obtained that only $F$ is transformed whereas $\gamma_0$ is not transformed, equation (16) in [20], $E'_{H,at} = (1/2)[F' - \gamma_0 F'\gamma_0] = (F' \cdot \gamma_0)\gamma_0$. This is the treatment from [27, 28]. They have not noticed that such transformations cannot be the correct LT because the LT cannot transform the matrix (5) in [20] in which the space-space components are zero to the matrix (18) in [20] in which again the space-space components are zero. The space-time split is not a Lorentz covariant procedure. In section 4 in [20] the derivation of the correct LT is presented. If the matrix (5) in [20], $(E_H)^{\mu\nu}$, is transformed in the way in which the matrix of any other bivector transforms under the LT, equation (13) in [20], then the matrix (12) in [20], $(E'_{H})^{\mu\nu}$, is obtained in which the space-space components are different from zero and the components $(E_H)^{\mu\nu}$ transform under the LT again to the components $(E'_{H})^{\mu\nu}$; there is no mixing with the components of the matrix of the magnetic field bivector. In general, as shown in [22, 23] the electric and magnetic fields can be represented by different algebraic objects; vectors, bivectors or their combination.

The correct LT always transform the 4D algebraic object representing the electric field only to the electric field; there is no mixing with the magnetic field.

6. The derivations of the AT of $E$ and $B$ in BT [7]

As mentioned in the Introduction the nature of electric and magnetic fields is discussed in section 1.10 in [7]. There, it is concluded that these fields are the 4D fields. If one applies the LT to BT’s equation (1.109) (it is our equation (21)), e.g., to the electric field 4D vector then, as discussed above, both $F_{\alpha\beta}$ and $w_{\beta}$ (their $w$ is our $v$) have to be transformed. The equation (30) would be obtained and equation (37) would hold. This is not noticed by Blandford and Thorne, [7], and they believe as all others that their equation (1.113) with the 3D vectors (the same as equation (11.149) in [1]) is the mathematically correct “Relationship Between Fields Measured by Different Observers.” Thus, although they deal with 4D GQs they still consider that in the 4D spacetime, in the same way as in the 3D space, the 3D vectors are the physical quantities, whereas the 4D quantities
are considered to be only mathematical, auxiliary, quantities. This is visible in the treatment of the Lorentz force in [7]. In the usual formulations the physical meaning of 3D vectors $\mathbf{E}$ and $\mathbf{B}$ is determined by the Lorentz force as a 3D vector $\mathbf{F}_L = q\mathbf{E} + qu \times \mathbf{B}$ and by Newton’s second law $\mathbf{F} = dp/dt$, $p = m\gamma u$. BT start with the correct equation (1.106) $(dp^\mu/d\tau = (q/c)F^\mu u_\mu$, our notation), but then instead of to use the decomposition of $F^{\mu\nu}$, their equation (1.110), our equation (1.107), which, as discussed above, is synchronization dependent and even meaningless in the \{\gamma_\mu\} basis, see equations (7) and (8). Obviously BT do not know for the \{\gamma_\mu\} basis. Finally they get “the familiar Lorentz-force form” in terms of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$, their equation (1.108).

Thus, the same as in the usual approaches. However, in the 4D spacetime, as mentioned above, the Lorentz force $K_L$ is given by equation (1.106) \{\gamma_\mu\} in terms of $\mathbf{F}$ and $u$. Using the decomposition of $\mathbf{F}$ (15), the Lorentz force $K_L$ becomes

$$K_L = (q/c)(1/c)E \wedge v + (IB) \cdot v \cdot u,$$

where $u$ is the velocity vector of a charge $q$ (it is defined to be the tangent to its world line). Note that there are two velocity vectors in $K_L$ if it is expressed in terms of fields $E$ and $B$, because $E$ and $B$ are determined relative to the observer with velocity vector $v$. If $K_L$ is represented as a CBGQ in the standard basis it is

$$K_L = K_L^{\mu}\gamma_\mu = (q/c)F^{\mu\nu}u_\nu\gamma_\mu = (q/c)\{(1/c)(E^\mu v^\nu - E^\nu v^\mu) + \varepsilon^{\lambda\mu\nu\rho}v_\lambda B_\rho\}u_\nu\gamma_\mu,$$

where $F^{\mu\nu}$ is from equation (20). In contrast to the usual expression for the Lorentz force with the 3D fields $\mathbf{E}$ and $\mathbf{B}$, $\mathbf{F}_L = q\mathbf{E} + qu \times \mathbf{B}$, the Lorentz force with the 4D fields $E$ and $B$ (55) or (56) contains not only the 4D velocity $u$ of a charge $q$ but also the 4D velocity $v$ of the observer who measures 4D fields. It can be simply checked that for $K_L^{\mu}\gamma_\mu$ (56) the relation (57) holds

$$K_L = K_L^{\mu}\gamma_\mu = K_L^{\mu}\gamma'_\mu = K_L^{\mu}r_\mu = K_{L'}^{\mu}r'_\mu,$$

as for any other 4D CBGQ. In the 4D spacetime, the physical meaning of $E^\mu$ and $B^\mu$ is determined by the Lorentz force $K_L$ (56), i.e., $K_L^{\mu}\gamma_\mu$ (56) and by the 4D expression for Newton’s second law

$$K_L^{\mu}\gamma_\mu = (dp^\mu/d\tau)\gamma_\mu, \quad p^\mu = mu^\mu,$$

$p^\mu$ is the proper momentum (components) and $\tau$ is the proper time. All components $E^\mu$ and $B^\mu$, thus $E^0$ and $B^0$ as well, are equally well physical and measurable quantities by means of the mentioned $K_L^{\mu}$ (56) and the 4D expression for Newton’s second law (58) (with $K_L^{\mu}$ instead of some arbitrary $K^{\mu}$). Hence, in the 4D spacetime, contrary to the assertion from [7], the use of the mathematically correct 4D GQs as in (55) or (56) cannot lead to “the familiar Lorentz-force form.”
Furthermore, BT in [7], state: “Only after making such an observer-dependent “3+1 split” of spacetime into space plus time do the electric field and magnetic field come into existence as separate entities.” But, as shown above, in the 4D spacetime “3+1 split” is ill-defined. It does not hold in the \{r_\mu\} basis and even in the \{\gamma_\mu\} basis it is not a Lorentz covariant procedure, i.e., the 3-surface of simultaneity for one observer (with 4D velocity \(w\)) cannot be transformed by the LT into the 3-surface of simultaneity for a relatively moving inertial observer (with 4D velocity \(w'\)). If for one observer \(w^\mu = (1, 0, 0, 0)\) then for a relatively moving inertial observer it holds that \(w'^\mu = (\gamma, -\beta\gamma, 0, 0)\)). Hence, it cannot be mathematically correct that both \(E^0_w = 0\) and \(E^0_{w'} = 0\), but it is necessary \(E^0_{w'} \neq 0\), as in (30) or (54). This means that their equation (1.107) is not correct. It does not follow from equation (1.109), our equation (21) (without unit 4D vectors). Also, equation (1.113) cannot be obtained by a mathematically correct procedure from equation (1.110). Simply, in the 4D spacetime there is no room for the 3D quantities; an independent physical reality has to be consistently attributed to the 4D GQs and not to the usual 3D quantities. Obviously, an important statement from Chapter 1 in [7] that is already mentioned above: “We shall state physical laws, e.g. the Lorentz force law, as geometric, coordinate-free relationships between these geometric, coordinate free quantities,” has to be changed in this way:

In the 4D spacetime physical laws, e.g. the Lorentz force law, are geometric, coordinate-free relationships between the 4D geometric, coordinate free quantities.

The 3D fields \(E\) and \(B\) and the Lorentz force \(F_L\) (\(F_L = qE + qu \times B\)) are also geometric quantities but in the 3D space, which means that they do not have well-defined mathematical and physical meaning in the 4D spacetime.

In addition, BT in [7], consider, as almost the whole physics community, that the Lorentz contraction and the time dilation are the intrinsic relativistic effects. But, as already mentioned, in [3], [5] and in Appendix in [2], it is exactly proved that such an opinion is not correct since both the Lorentz contraction and the time dilation are ill-defined in the 4D spacetime. Instead of them the 4D GQs, the position 4D vector, the distance 4D vector between two events and the spacetime length have to be used, since they are properly defined quantities in the 4D spacetime.

6.1. Additional comments about the 4D Lorentz force

Here it is at place to give some additional comments about the Lorentz force \(K_L\) (55) or (56) as a 4D GQ. It is visible from (55) or (56) that the Lorentz force ascribed by an observer comoving with a charge, \(u = v\), i.e., if the charge and the observer world lines coincide, then \(K_L\) is purely electric, \(K_L = qE\). In the general case when \(u\) is different from \(v\), i.e. when the charge and the observer have distinct world lines, \(K_L\) (55) or (56) can be written in terms of \(E\) and \(B\) as a sum of the \(v\) - orthogonal part, \(K_{L\perp} (K_{L\perp} \wedge v = 0)\) and \(v\) - parallel part,
with the vectors $E$ and $B$ the $v$ - orthogonal part, $K_{L\perp}$, from (59) plays the role of the usual Lorentz force lying on the 3D hypersurface orthogonal to $v$, whereas $K_{L\parallel}$ from (59) is related to the work done by the field on the charge. This can be seen specifying (60) to the $\gamma_0$ - frame, $v = c\gamma_0$, in which $E^0 = B^0 = 0$. In the $\gamma_0$ - frame it is possible to compare the 4D vector $K_L$ with the usual 3D Lorentz force, $F_L = qE + qu \times B$, which yields

\begin{align*}
K^0_L\gamma_0 &= K^0_{L\parallel}\gamma_0 = -(q/c)E^iu_i\gamma_0, \quad K^0_{L\perp} = 0, \\
K^i_L\gamma_i &= K^i_{L\perp}\gamma_i = q(E^i + \varepsilon^{ijk}u_jB_k)\gamma_i, \quad K^i_{L\parallel}\gamma_i = 0
\end{align*}  

(60)

It is visible from (60) that $K^0_L$ is completely determined by $K^i_{L\parallel}$, whereas the spatial components $K^i_L$ are determined by $K^{0\perp}_L$. However, as already mentioned several times, in this 4D geometric approach only both parts taken together, i.e., the whole $K_L = K^0_L + K^i_L$ does have a definite physical meaning and it defines the 4D Lorentz force both in the theory and in experiments.

In section 2.5 in [14], under the title “The Lorentz force and the motion of charged particle in the electromagnetic field $F$” the definition of $K_L$ in terms of $F$ is exclusively used ($K_L = (q/c)F \cdot u$) without introducing the electric and magnetic fields. Observe that the 4D GQs $K^i_L$, $p$, $u$ transform in the same way, like any other 4D vector, i.e., according to the LT and not according to the awkward AT of the 3D force $F$, e.g., equations (12.66) and (12.67) in [4], and the 3D momentum $p$, i.e., the 3D velocity $u$. In [29], under the title “Four Dimensional Geometric Quantities versus the Usual Three-Dimensional Quantities: The Resolution of Jackson’s Paradox,” it is shown that only with the use of the 4D Lorentz force (55), (56) or (59), the torque bivector $N = (1/2)N^{\mu\nu}\gamma_{\mu} \wedge \gamma_{\nu}$, $N^{\mu\nu} = x^{\mu}\gamma_{\nu} - x^{\nu}\gamma_{\nu}$ and the angular momentum bivector $M = (1/2)M^{\mu\nu}\gamma_{\mu} \wedge \gamma_{\nu}$, $M^{\mu\nu} = m(x^{\mu}u^{\nu} - x^{\nu}u^{\mu})$ there is no apparent electrodynamic paradox with the torque and that the principle of relativity is naturally satisfied. The mentioned paradox is described in [30] and it consists in the fact that there is a 3D torque $N$ and thus $dL/dt$ ($N = dL/dt$) in one inertial frame, but no 3D angular momentum $L'$ and no 3D torque $N'$ in another relatively moving inertial frame. Similar electrodynamic paradoxes with the 3D torque appear in the Trouton-Noble paradox, see, e.g., [31], and the “charge-magnet paradox” [32]. Using the above mentioned 4D GQs, 4D Lorentz force, the torque and angular momentum bivectors it is explicitly shown in [33], [14], for the Trouton-Noble paradox and [34], [2] for Mansuripur’s paradox that there is no paradox and consequently there is no need for some “resolutions” of the paradoxes, e.g., by the introduction of the Einstein-Laub force, [32], or by the introduction of some “hidden” quantities, e.g., [35].
7. The shortcomings in the derivations of the AT of E and B
and in the treatment of the angular momentums in KS [8]

7.1. The shortcomings in the derivations of the AT of E and B

Similar mistakes as in BT [7] are made by Klajn and Smolić (KS) in section 3 in [8]. KS [8] use the tensor formalism with the abstract index notation but, nevertheless, they consider as in [7] that the 3D vectors are well-defined physical quantities in the 4D spacetime whereas the 4D quantities are only mathematical, auxiliary, quantities. In the first part of section 3 in [8] they derive the transformations of the 3D E and B, their equations (25) and (26), in the same way as in [1]. The shortcomings of such a derivation are discussed in detail in our section 1, the objections 1), 2) and in section 2.1, the objections 3), 4) and 5). As in [7], KS [8] also know only for the standard basis and not for the \{r_{\nu}\} basis in which, according to equations (1) and (2), the usual identification, equation (24) in [8], i.e., our equation (1), is meaningless even in their specific inertial reference frame \(R\), what is the \(\gamma_0\)-frame in our notation. For the same reasons, contrary to their assertion, it is not true that the identifications (2) hold for a relatively moving inertial frame \(R'\) too. As already discussed at the end of section 1, their \(F_{ab}\), our \(F\), is represented as in (1) and it contains not only components but a basis as well, which means that their relation \(F_{ab} \rightarrow F_{\mu\nu}\) is not mathematically correct. In the second part of section 3 in [8] they deal, as they say, with “an alternative approach” in which the observers which measure the electric and magnetic fields are explicitly introduced.

The mathematical incorrectness of their derivation can be best seen, e.g., from their discussion at the end of section 3 and equations (34) - (37) in [8]. They, KS, construct the electric 4-vectors, in the same way as it is made by BT in [7]. In [8] it is assumed that if the 4-velocity of the observer in \(R\) is in the \(\gamma_0\) direction, \(v = c\gamma_0\), and consequently \(E = F \cdot \gamma_0\) with the components \(E^{\mu} = (E^0 = 0, E^1, E^2, E^3)\), then the same relations must hold for a relatively moving inertial observer, \(o' = c\gamma_0'\) and \(E'^{\mu} = (E'^0 = 0, E'^1, E'^2, E'^3)\). In their notation, for the observer \(o\), with \(\alpha^{\mu} = (c, 0)\), \(E^{\alpha}(o) = F^{\alpha b}o_b\) so that \(E^{\alpha}(o) = (0, E)\) and it is supposed that the same holds for the observer \(o'\), \(E'^{\alpha}(o') = F^{\alpha b}o'_b\) so that \(E'^{\alpha}(o') = (0, E')\). In [8] it is stated: “The 4-vector \(E^{\alpha}(o)\) is related to the electric field 3-vector as measured by \(o\), and the same holds for \(E^{\alpha}(o')\) and the observer \(o'\).” We remark that the same relation has to hold for the observers \(S''\), \(S'''(a'', o'''\text{ etc.}, since it is the definition of the vector \(E\). However, it is not understood by KS that \(E'\) and \(v'\) from \(E' = F \cdot v'/c = F' \cdot \gamma_0'\) are not the Lorentz transforms and they have nothing to do with the LT of \(E\) and \(v\) from \(E = F \cdot v/c\). The reason is that \(\gamma_0\) is transformed by the LT as in equation (1):

\[\gamma_0 = \gamma(\gamma_0' - \beta \gamma_1').\]  

(61)

As it is discussed in section 5 the unit vector in the time direction \(\gamma_0\) (from \(v = c\gamma_0\), \(E = F \cdot \gamma_0\)) for the observer \(S\) is not transformed by the LT into the unit vector in the time direction \(\gamma_0'\) for the observer \(S'\) (from \(v' = c\gamma_0', E' = F' \cdot \gamma_0'\)), which means that if \(v\) is in the \(\gamma_0\) direction then, as said above, \(v'\)
cannot be in the $\gamma'_0$ direction. One can take any observer as the starting one for which $E$ is defined as in $E = F \cdot v/c$ and then to find the electric field vector $E'$ for a relatively moving observer $S'$ one has to perform the active LT of that $E$ in a mathematically correct way, i.e., for the active LT as in equations (27) - (32). In addition, it is worth mentioning that their notation $E^\mu(o) = (0, E)$, $E^\mu(o') = (0, E')$, etc. is not correct not only because the temporal component in $E^\mu(o')$ cannot be zero, but for other reasons too. Firstly, usually $E$ denotes the 3D electric field in which the components $E_{x,y,z}$ are multiplied by the unit 3D vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, whereas in $E^\mu(o)$ they have to be $E_1, E_2, E_3$, which in the geometric quantity $E^\mu(o)$ would need to be multiplied by the spatial unit 4D vectors. In the 4D spacetime there are no 3D vectors. Moreover, as already said, the standard basis is implicitly assumed in the whole paper [8]. But, an observer can use different bases. Particularly, if the $\{r_\mu\}$ basis is used then, as seen from (25), the temporal component of $E^\mu(o)$ in the $\{r_\mu\}$ basis, $E_0^{r} \neq 0$, even if it is = 0 in the standard basis. This is not taken into account in their formulation and with their notation.

Let us explain the shortcomings and misconceptions in their derivations in another way too. The whole their reasoning is clearly visible from their equation (35). In that equation, in their notation, they have on the r.h.s. $E^\mu = (E^0 = 0, E^1, E^2, E^3)$ and also on the l.h.s. $E'^\mu = (E'^0 = 0, E'^1, E'^2, E'^3)$, i.e., the temporal component of the electric 4D vector is taken to be zero for both relatively moving inertial observers $o$ and $o'$. Then it is stated in [8] that the only LT that satisfies the equation (35) is the 3-rotation transformation. However, they erroneously consider that in both relatively moving inertial frames the temporal components have to be zero. This is completely equivalent to the treatment from [7] in which it is supposed that the “3+1 split” of the spacetime into space + time holds in both relatively moving inertial frames, i.e., that it is a Lorentz covariant procedure. As already explained several times, in the 4D spacetime the physical quantities are the 4D geometric quantities and not the 3D vectors, which means that the LT will necessary transform the electric 4D-vector with $E^0 = 0$ into the electric 4D-vector with $E'^0 \neq 0$. In the 4D spacetime, as stated above, all components of the 4D vectors $E$ and $B$ including $E^0$ and $B^0$ are equally well physical and measurable quantities by means of the equations (56) and (58). Their, [8], equation (35) has to have on the l.h.s. $E'^0 \neq 0$. Only in that case it will be a mathematically correct LT (boost) of the 4D electric vector from the $\gamma_0$ - frame (the r.h.s. of (35)) and the components will be given by equation (30). Thus, the mathematically incorrect equations (34) and (35) in [8] has to be replaced with our mathematically correct LT (boost) (53) and (54), i.e., (37). In that case, as stated at the end of section 5.1, the relation (37) holds as for any other 4D vector. From the mathematical viewpoint under the passive LT both the components and the basis are Lorentz transformed but the 4D vector $E$ remained unchanged. The 4D rotation of the basis is performed, e.g., for the standard basis, $\gamma_\mu \rightarrow \gamma'_\mu$. The components of that unchanged $E$ are determined relative to that new basis, $E^\mu \rightarrow E'^\mu$. Hence, in (34) as in (34) $E'^\mu = A^\mu_\nu E^\nu$. In contrast to the statement from [8], the components $E^\mu$ and $E'^\mu$ refer to the measurements by the observers in two relatively moving inertial
frames of reference, two different bases $\gamma_\mu$ and $\gamma'_\mu$. The vector $E = E^\nu \gamma_\mu$ is a genuine 4D vector. KS [8] do not properly differ between the passive LT and the active LT. As explained above, section 4.1, under the active LT $E$ is transformed into a new electric field vector $E'$, (29); the components are changed, (30), but the basis remains unchanged as in (31). From the physical viewpoint the measurements are made by the observers in one frame, one basis, but they are made on two different 4D vectors $E$ and $E'$, which are connected by the active LT as in (31). It is visible from (31) and (54) that the components of the new vector $E'$ in the old basis are the same as the components of the old vector $E$ in the new basis $\gamma'_\mu$, as it has to be. This is not understood by KS [8]. In [8] it is also used an unusual and in some way an awkward notation with primed quantities and primed indices. Instead of such a strange notation they could simply use the primed components and bases.

The title of [8] is “Subtleties of invariance, covariance and observer independence.” From the above discussion it can be concluded that, contrary to the assertions from section 3 in [8], the correct LT of the electric field are our equations, (27) - (32), i.e., with matrices, (53) and (54). The “subtle” point that is not understood by KS [8] is that the LT are properly defined on the 4D spacetime and they cannot transform three spatial components for one observer again into three spatial components for relatively moving observer. In other words, in the 4D spacetime the 4D vectors are properly defined and not the 3D vectors. Hence, in the 4D spacetime the 3-rotation transformation is meaningless and it has nothing to do with the mathematically correct LT (the rotation in the 4D spacetime) since the 3D vectors are not well-defined quantities.

The fact that in [8] the 3D $E$ and $B$ are considered as well-defined physical quantities in the 4D spacetime causes an incorrect expression for the Lorentz force law, their equation (33), $d(mu^a)/d\tau = qF^a_b u^b \equiv qE^a(u)$. The correct expression for the Lorentz force is $qF^a_b u^b$, but it is completely incorrect to argue that it is $\equiv qE^a(u)$, where : “$E_a(u)$ stands for the combination of both electric and magnetic 3D vectors (the familiar 3D vector representation of Lorentz’s law).” If $E_a(u)$ is expressed with 3D vectors how then it can be identical to the 4D vector $qF^a_b u^b$. In the 4D spacetime the mathematically correct formulation of the Lorentz force law is given in our section 6 by equations (55) or (56), i.e., (59) and (60).

Observe also an important difference between our equations (18), (19) and equations (27), (30) in [8]. In our formulation the starting equation for the introduction of the 4D $E$ and $B$ is equation (18), i.e., a mathematical theorem that holds for any antisymmetric tensor of the second rank. In that theorem the 4D vector $v$ is a time-like 4D vector, which means that it is not necessary in the time direction. $E$ and $B$ are the space-like vectors given by equation (19). It is not so in [8]. They first define $E^a(o)$ and $B^a(o)$ by equation (27) in which $o_0$ is explicitly in the time-direction ($o^0 = (c, 0)$). Then, they construct the Faraday tensor $F_{ab}$ using such $o_0$. It corresponds to the case that it is chosen $v = c\gamma_0$ in our (18) and (19). However, (18) and (19) hold in the same measure if $v$ is not $c\gamma_0$, but it is obtained by the LT from $c\gamma_0$, i.e., $v = c(\gamma\gamma_0 - \beta\gamma_1)$. That $v$ is not in the time direction, but it is still a time-like 4D vector, $v^2$ is again $= c^2$. 25
Their, [8], definitions (27) and (30) with \( \sigma^\mu = (c, \mathbf{0}) \) are the real cause of all other mathematical incorrectnesses in [8], which are discussed in this section. Instead of (27) and (30) from [8] one has to use (18) and (19) and it has to be in that order. In addition, their, [8], reference [12] is not correct. The title of that paper, reference [21] here, is: The proof that the standard transformations of E and B are not the Lorentz transformations.

7.2. The shortcomings in the treatment of the angular momentums in [8]

There are even more mathematical incorrectnesses in the treatment of the angular momentums in section 4 in [8]. They start the consideration with equation (38) in which the components (implicitly taken in the standard basis) \( J_{\mu\nu} \) of the angular momentum tensor \( J_{ab} \) in \( \mathcal{R} \) (our \( \gamma_0 \)-frame) are identified with the components of two 3D vectors \( \mathbf{K} \) and \( \mathbf{J} \). As they say: “\( \mathbf{K} \) is the boost 3D vector describing the movement of the particle’s center of mass, while \( \mathbf{J} \) is the angular momentum 3D vector.” In the usual covariant approaches, e.g., [1], [4], [36] the 3D vectors are considered as primary physical quantities that determine the components \( F_{\mu\nu} \) of the electromagnetic field tensor. In the same way in [8] the components of the 3D vectors \( \mathbf{K} \) and \( \mathbf{J} \) are considered as primary physical quantities that determine the components \( J_{\mu\nu} \) of the angular momentum tensor. Firstly, as already discussed several times, such an identification of the components (in the standard basis) of \( J_{\mu\nu} \) with the components of the 3D vectors \( \mathbf{K} \) and \( \mathbf{J} \), their equation (38), is synchronization dependent and even meaningless in the \( \{r_\mu\} \) basis. The objections 1), 2) from our section 1 and 3), 4) from section 2.1 hold in the same measure for their treatment of the angular momentums. However, in this case, there are some additional objections. The first one refers to the physical interpretation of the components of \( J_{\mu\nu} \) in their equation (38). According to their interpretation the three components of \( \mathbf{K} \), i.e., the “time-space” components of \( J_{\mu\nu} \), are not the angular momentum components as are the “space-space” components \( J_{x,y,z} \). This is also visible from their equation (39), which, as they state, defines the angular momentum 4-vector. If that equation would be written with CBGQs in their specific reference frame \( \mathcal{R} \) then the components of that angular momentum 4-vector would be \((0,J_x,J_y,J_z) \) \( (J^0 = 0, J^i = (1/2)\varepsilon^{ijk}J_{jk}) \). Only the “space-space” components of \( J_{\mu\nu} \) define the spatial components of the angular momentum 4-vector \( J^a(o) \). The temporal component of that 4D-vector is = 0. However, if one uses the \( \{r_\mu\} \) basis instead of the standard basis then, in the same way as in relations (7) and (8), one would get that, e.g., the “time-space” component \( K_{x,r} \) of the component form of the angular momentum four-tensor in the \( \{r_\mu\} \) basis, \( J_{\mu\nu(r)} \), would be expressed as the combination of the “time-space” component \( K_x \) and the “space-space” components \( J_y \) and \( J_z \) of the same angular momentum four-tensor in the \( \{\gamma_\mu\} \) basis, i.e., that one whose components are given by equation (38) in [8],

\[
K_{x,r} = K_x + J_z - J_y.
\] (62)

This is the reason why we use the quotation marks in “time-space” and “space-space.” Furthermore, if the LT of the components \( J_{\mu\nu} \) from equation (38) in [8]
is performed and the same identification is used in the relatively moving inertial frame of reference $R'$ then the AT of the components of the 3D vectors $K$ and $J$ are obtained

$$
J'_x = J_x, \quad J'_y = \gamma(J_y + \beta K_z), \quad J'_z = \gamma(J_z - \beta K_y),
$$
$$
K'_x = K_x, \quad K'_y = \gamma(K_y - \beta J_z), \quad K'_z = \gamma(K_z + \beta J_y).
$$

(63)

As can be seen from (63) these transformations are the same as the AT for $B_i$ and $E_i$, respectively. Here, they are written for the motion along the $x^1$ axis. The essential point is that in (63) the transformed components $J'_i$ are expressed by the mixture of components, $J_k$, $K_k$ and vice versa. The above relation for $K_x$ (62) and the relations (63) clearly show that it is not correct to consider that only three “space-space” components of $J_{\mu\nu}$ implicitly taken in the standard basis are the components of the physical angular momentum. From the mathematical viewpoint all six independent components of $J_{\mu\nu}$ are completely equivalent and they necessarily have to have the same physical interpretation. Strictly speaking the components taken alone are not physical. In this case the physical quantity is the angular momentum four tensor $J_{ab}$ as an abstract 4D GQ or its representation in some basis the 4D CBGQ that contains not only the components as in equation (38) in [8] but the chosen basis as well. In equations (40) and (41) they, KS in [8], define the orbital angular momentum $L^a$ and the spin $S^a$, respectively. Then they get equation (42) in which the total angular momentum is written as the sum of the orbital angular momentum and the spin 4-vector. Observe that again only the “space-space” part of $J_{ab}$ is used to define $L^a$ and $S^a$. In order to get that $J^a$ can be written as a sum of $L^a$ and $S^a$, their equation (42), they define $L^a$ in such a way that it contains both, the 4-velocity of the observer $o_b$ and the 4-velocity of the particle $u_b$, their equation (40). Hence it is not correct to write $L^a(o)$ since it depends on the particle’s 4-velocity $u$ as well. Even in the $R$ frame in which $o^\mu = (c, 0, 0, 0)$ the temporal component of $L^\mu$ will be different from zero and one cannot get the usual expression for the spatial components of the orbital angular momentum. Hence, e.g., $J_x$ from their equation (38) is not equal to the sum of the usual $L_x$ and $S_x$. Similarly, in the treatment of the spin, equations (43) - (45), KS [8] consider that the spin 3D vector $s$ is a well-defined physical quantity in the 4D spacetime and that it transforms according to the transformations given by their equation (45) (equation (11.159) in [1]), which are typical AT of the 3D vectors.

The treatment of the angular momentums in [8] is very similar to the treatment of the angular momentum and torque in Jackson’s paper [30]. There, [30], Jackson deals with the usual covariant definition of the angular momentum four-tensor (orbital) $M^{\mu\nu} = \pi^{(\mu} p^{\nu)} - \pi^{\nu} p^{\mu}$. The components $L_i$ of the 3D orbital angular momentum $L = \mathbf{r} \times \mathbf{p}$ are identified with the “space-space” components of $M^{\mu\nu}$ and the components $K_i$ of another 3D vector $K$ are identified with the three “time-space” components of $M^{\mu\nu}$. In [30], in contrast to [8], it is not given any physical interpretation for $K_i$. It is assumed that $L_i$ and
$K_i$ transform as the “space-space” and “time-space” components respectively of the usual covariant angular momentum four-tensor $M^{\mu\nu}$, see [30] and section 3 in the first paper in [29]. These AT of the components of $L$ are the same as the AT of $J$ in (62) but with $L_i$ replacing $J_i$, which means that the transformed components $L'_i$ are expressed by the mixture of components, $L_k$, $K_k$ and vice versa. The same situation happens with the 3D torque $N$ and the torque bivector $N = (1/2) N^{\mu\nu\gamma\delta} \gamma_{\mu} \wedge \gamma_{\nu}$ in the above mentioned considerations of different electrodynamic paradoxes, [31, 32]. In [30], and also in, e.g., [31, 32, 35], only the “space-space” components of $M^{\alpha\beta}$ ($L_i$) and $N^{\alpha\beta}$ ($N_i$) are considered to be the physical angular momentum and torque respectively, because they are associated with actual rotation in the 3D space of the object. On the other hand, the “time-space” components of $M^{\alpha\beta}$ ($K_i$) and $N^{\alpha\beta}$ (let us denote them as $R_i$) are not considered to be of the same physical nature as $L_i$ and $N_i$. In all usual treatments it is considered that $K_i$ and $R_i$ are not the physical angular momentum and torque respectively, because they are not associated with any overt rotation in the 3D space of the object, see, particularly, the paper by Griffiths and Hnizdo in [35] and Jackson’s paper [30]. However, as already discussed above, the relations (62) and (63) reveal that such usual interpretation of the components of $M^{\alpha\beta}$ and $N^{\alpha\beta}$ is apparently incorrect; how it can be physically acceptable that in the relation, e.g., $L'_y = \gamma (L_y + \beta K_z)$, $L_y$ and $L'_y$ are the components of a physical angular momentum, whereas it is not so with $K_z$. The same objection refers to the treatment of the angular momentums in [8].

8. Briefly about the mathematically correct 4D angular momentums

In contrast to treatment of the angular momentums in [8], the mathematically correct definitions with the 4D GQs of the orbital angular momentum bivector are given, e.g., in section 2 in the first paper in [29] (section 4 in the second paper)

$$ M = x \wedge p, \quad M = (1/2) M^{\mu\nu} \gamma_{\mu} \wedge \gamma_{\nu}, \quad M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \quad (64) $$

in connection with the discussion of Jackson’s paradox and also in [33] and [34] in the mathematically correct treatment of the Trouton-Noble paradox and Mansuripur’s paradox respectively. The same definitions but in the tensor formalism with the abstract index notation are given in section 2 in [10]. In a complete analogy with the decomposition of $F$ into $E$, $B$ and $v$, equations (18) - (21), the mathematical theorem from section 3.3 can be used for the decomposition of the bivector $M$ into two 4D vectors $M_t$ and $M_s$ and $v$, the 4D velocity vector of a family of observers who measures $M$

$$ M = (v/c) \wedge M_t + (v/c) \cdot (M_s I), \quad M_t = (v/c) \cdot M, \quad M_s = I(M \wedge v/c), \quad (65) $$

28
With the condition
\[ M_s \cdot v = M_t \cdot v = 0. \tag{66} \]

Only three components of \( M_s \) and three components of \( M_t \) are independent since \( M \) is antisymmetric. If \( M, M_s \) and \( M_t \) are written as CBGQs in the \( \{\gamma_\mu\} \) basis then there components are

\[
M^{ij} = (1/c)[(v^k M^j_k - v^j M^k_k)] + \epsilon^{ijk\sigma} M_{s\sigma},
\]

\[
M^j = (1/c)\epsilon^{ijk\mu} M_{i\mu} v, \quad M_t^j = (1/c)M^{ij} v. \tag{67}
\]

Similarly as for \( E \) and \( B \) it can be concluded from \( (65) \) - \( (67) \) that both \( M_s \) and \( M_t \) depend not only on \( M \) but also on \( v \). Hence, it can be said that the bivector \( M \) is the primary quantity for the angular momentums. Both vectors \( M_s \) and \( M_t \) are physical angular momentums which contain the same physical information as the bivector \( M \) only when they are taken together. In the \( \gamma_0 \) - frame \( v^\mu = (c, 0, 0, 0) \), \( M_s^0 = 0 \) and only the spatial components \( M_s^i \) and \( M_t^i \) remain, \( M_s^i = M_{0i}^0 \). 

\[
M_s^i = (1/2)c \epsilon^{ijk} M_{jk}, \quad M^{23} = x^2 p^3 - x^3 p^2, \quad M^3 = M^{12},
\]

\[
M_s^3 = M_t^3.
\]

Therefore \( M_s \) can be called the "space-space" angular momentum and \( M_t \) the "time-space" angular momentum. \( M_s^i \) and \( M_t^i \) correspond to the components of \( L \) and \( K \) that are introduced, e.g., in \[30\] and discussed in the preceding section. However, as already mentioned, Jackson \[30\], as all others, considers that only the 3D \( L \) is a physical quantity whose components transform according to equation (11) in \[30\], i.e., equation (63) here but with \( L_i \) replacing \( J_i \). In contrast to it the 4D vectors \( M_s \) and \( M_t \) transform under the LT as any other 4D vectors transform, i.e., the components in the standard basis transform like in equation (32). Under the active LT, e.g., the 4D vector \( M_s \) transforms again into the "space-space" angular momentum \( M_s^i \) and there is no mixing with \( M_t^i \).

It is shown in \[37, 38, 10, 2\] that the same consideration as for the orbital angular momentum can be applied to the intrinsic angular momentum. The primary quantity with definite physical reality for the intrinsic angular momenta is the spin bivector \( S \) (four-tensor \( S^{ab} \) in \[37, 38, 10\]), which, as in \( (65) \) - \( (67) \), can be decomposed into the usual "space-space" intrinsic 4D angular momentum vector \( S \), the "time-space" 4D intrinsic angular momentum vector \( Z \) and the unit time-like 4D vector \( u/c \), where \( u \) is the 4D velocity vector of the particle

\[
S = (1/c)[Z \wedge u + (SI) \cdot u],
\]

\[
Z = S \cdot u/c, \quad S = I(S \wedge u), \tag{68}
\]

equation (58) in \[2\], or with \( S^{ab} \), equation (8) in \[10\]. It holds that \( Z \cdot u = S \cdot u = 0 \); only three components of \( Z \) and three components of \( S \) are independent since \( S \) is antisymmetric. \( S \) and \( Z \) depend not only on \( S \) but on \( u \) as well. Only in the particle’s rest frame, the \( K' \) frame, and the \( \{\gamma'_{\mu}\} \) basis, \( u = c\gamma_0 \) and \( S^{00} = Z^{00} = 0, \quad S^{ij} = (1/2c)\epsilon^{ijk} S_{jk}^', \quad Z'^i = S^{0i} \). According to equation (68), a new "time-space" 4D spin \( Z \) is introduced and it is a physical quantity in the same measure as it is the usual "space-space" 4D spin \( S \). Both 4D vectors \( S \) and \( Z \) transform under the LT as any other 4D vector transforms, i.e., the
components in the standard basis transform like in equation (32). The 4D vector $S$ transforms again to $S'$ and there is no mixing with $Z$. As already stated the transformations of the 3-spin from equation (45) in [8] (equation (11.159) in [1]) are a typical example of the AT and they have nothing to do with the mathematically correct LT of the 4D intrinsic angular momentum vector $S$.

Hence, the correct introduction of the total angular momentum has to be expressed in terms of the primary quantities as

$$J = M + S,$$

(69)
or, in the tensor notation as $J^{ab} = M^{ab} + S^{ab}$, and not in the form of equation (42) in [8]. Only in the case that $v = u$, i.e., the observer is comoving with the particle, one could have $J^a = M^a + S^a$, which stands instead of equation (42) in [8]. However, together with that equation we have another equally important and physical equation $J^a = M^a + Z^a$. This is a fundamental difference between our approach which exclusively deals with 4D GQs and the treatment from [8].

In [10] (earlier in [37]) a fundamental result is obtained by a consistent application of the 4D GQs and the relations like (20) and (21). First, the generalized Uhlenbeck-Goudsmit hypothesis is formulated as the relation which connects the dipole moment tensor $D^{ab}$ and the spin four-tensor $S^{ab}$, $D^{ab} = g_S S^{ab}$, equation (9) in [10], instead of the usual relation between the 3D vectors, the magnetic moment $m$ and the spin 3D vector $S$, $m = \gamma S$. Then, both $D^{ab}$ and $S^{ab}$ are decomposed like in (20) into the dipole moment 4-vectors $m^a$, $d^a$, equation (2) in [10], and the intrinsic angular momentum 4-vectors, the usual $S^a$ and the new one $Z^a$, equation (8) in [10], which is equation (68) here. It is obtained in a mathematically correct procedure that $d^a$, the electric dipole moment of a fundamental particle, is determined by $Z^a$ and not, as generally accepted, by the spin 3D vector $S$. The connections between the dipole moments $m^a$ and $d^a$ and the corresponding intrinsic angular momentums $S^a$ and $Z^a$, respectively, are given by equation (10) in [10]

$$m^a = cg_S S^a, \quad d^a = g_S Z^a.$$  

(70)

In the particle’s rest frame and the $\{e'_\mu\}$ basis, $u^a = c e_0'$ and $d'^a = m'^a = 0$, $d^0 = g_S Z^0$, $m^0 = cg_S S^0$.

Furthermore, an important result is obtained in [38] by using the mathematical theorem from section 3.3. In that paper, [38], we have reported the relativistic generalizations of the usual commutation relations for the components of the 3D orbital angular momentum $L$. From the Lie algebra of the Poincaré group we know that

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\hbar(\epsilon^{\mu\nu\rho\sigma}M^{\rho\sigma} + g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho}).$$

(71)

Taking into account the decomposition of the components $M^{\mu\nu}$, (67), into $M^\mu_s$ and $M^\mu_t$ (they are now operators), where, for a macroscopic observer, $v^\mu$ can be taken as the classical velocity of the observer (the components), i.e., not the
operator. This leads to the new commutation relations, equation (3) in \[38\],

\[
\begin{align*}
[M^s_\mu, M^s_\nu] &= (i\hbar/c)\varepsilon^{\mu\nu\alpha\beta} M^a_s v_\alpha v_\beta, \\
[M^a_\mu, M^a_\nu] &= (i\hbar/c)\varepsilon^{\mu\nu\alpha\beta} M^a_s v_\alpha v_\beta, \\
[M^a_\mu, M^s_\nu] &= (i\hbar/c)\varepsilon^{\mu\nu\alpha\beta} M^a_t v_\alpha v_\beta,
\end{align*}
\]

(72)

which, in the $\gamma_0$-frame, where $M^0_s = M^0_t = 0$, reduce to the usual commutators for the components of $L$ and $K$ (as operators), see, e.g., \[39\] equations (2.4.18) - (2.4.20). It is worth noting that the same commutation relations (72) can be obtained using $M^a_\mu$ and $M^a_\mu$ expressed in terms of $M^{\mu\nu}$, equation (67), and the relativistic generalization of the fundamental commutation relations, i.e., the worldspace fundamental commutation relations, see, e.g., \[40\],

\[
\begin{align*}
[x^\mu, p^\nu] &= i\hbar \delta_{\mu\nu}, \\
[x^\mu, x^\nu] &= [p^\mu, p^\nu] = 0.
\end{align*}
\]

(73)

The same commutators as in (72) have to hold for the intrinsic angular momentums (the components) $S^\mu$ and $Z^\mu$; $S^\mu$ replaces $M^a_\mu$, $Z^\mu$ replaces $M^a_\mu$ and the velocity of the particle (the components) $u^\mu$ replaces the velocity of the observer $v^\mu$, equation (4) in \[38\],

\[
\begin{align*}
[S^\mu, S^\nu] &= (i\hbar/c)\varepsilon^{\mu\nu\alpha\beta} S^a u_\alpha u_\beta, \\
[Z^\mu, Z^\nu] &= (i\hbar/c)\varepsilon^{\mu\nu\alpha\beta} Z^a u_\alpha u_\beta.
\end{align*}
\]

(74)

Usually, e.g., \[41\], only the commutators $[L_i, L_j]$ and $[S_i, S_j]$ appear.

Taking into account the relations (70), i.e., in components, $m^\mu = cg S^\mu$, $d^\mu = gS Z^\mu$ one can express the commutation relations for $m^\mu$ and $d^\mu$ in terms of those for $S^\mu$ and $Z^\mu$,

\[
\begin{align*}
[m^\mu, m^\nu] &= c^2 g^2 S^\mu S^\nu, \\
[d^\mu, d^\nu] &= g^2 Z^\mu Z^\nu.
\end{align*}
\]

what is equation (5) in \[38\].

9. The electromagnetic field of a point charge in uniform motion

It is worth mentioning that KS \[8\] and also the majority of physicists consider that if the electric field would be transformed by the LT again into the electric field as in (30), i.e., as if in their relation (35) $E_0'$ would be different from zero, then it would imply, \[8\]: “that moving electrons produce no magnetic field.” In section 5.6 in \[34\] the electromagnetic field of a point charge in uniform motion is treated in detail. There it is shown that the formulation of that problem with the 4D fields and their LT (29), (30) is mathematically completely correct but its physical interpretation is different than in the usual formulation with the 3D fields and their AT. The above assertion from \[8\] is caused by their incorrect assumption that for both relatively moving inertial observers $o$ and $o'$ the temporal component of the electric 4D vector is zero $E^0 = E_0' = 0$, i.e., that $E^\mu (o) = (0, E)$ and $E^\mu (o') = (0, E')$. The consideration presented in 5.6.2 - 5.6.2.2 in \[34\] explicitly shows that their assertion is not correct and that the
formulation with the 4D fields that transform according to the LT \cite{29,30} simply explains the existence of the electric and magnetic fields for a moving electron.

9.1. The bivector field $F$

Here we shall briefly quote the main results from \cite{34}. In the 4D formulation the primary quantity is the the bivector field $F$. The expression for $F$ for an arbitrary motion of a point charge is given in \cite{14} by equations (10) (coordinate-free quantities) and (11) (CBGQs). Particularly, for a charge $Q$ moving with constant 4D velocity vector $u$, $F$ is given by equation (12) in \cite{14} (coordinate-free quantities), i.e., equation (65) in \cite{34}

$$F(x) = G(x \wedge (u/c)), \quad G = kQ/|x \wedge (u/c)|^3,$$  \hspace{1cm} (76)

where $k = 1/4\pi\varepsilon_0$. $G$ is a number, a Lorentz scalar. The geometric character of $F$ is contained in $x \wedge (u/c)$. If that $F$ is written as a CBGQ in the standard basis it is

$$F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu; \quad F^{\mu\nu} = G(1/c)(x^\mu u^\nu - x^\nu u^\mu), G = kQ/[(x^\mu u_\mu)^2 - c^2 x^\mu x_\mu]^{3/2}. \hspace{1cm} (77)$$

In order to find the explicit expression for $F$ from (77) in the $S'$ frame in which the charge $Q$ is at rest one has simply to put into (77) that $u = c\gamma_0'$ with $\gamma_0' = (1,0,0,0)$. Then, $F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$ and

$$F = F^{i0}(\gamma_i' \wedge \gamma_0'), \quad G = kQ/(x^i x_0')^{3/2}. \hspace{1cm} (78)$$

In $S'$ and in the standard basis, the basis components $F^{\mu\nu}$ of the bivector $F$ are obtained from (77) and they are:

$$F^{i0} = -F^{0i} = kQx^i/(x^ix_0')^{3/2}, \quad F^{ij} = 0. \hspace{1cm} (79)$$

In the charge’s rest frame there are only components $F^{i0}$, which are the same as the usual components of the 3D electric field $E$ for a charge at rest.

In the same way we find the expression for $F$ in the $S$ frame in which the charge $Q$ is moving, i.e., $u = u^\mu\gamma_\mu$ with $u^\mu/c = (\gamma, \beta, 0, 0)$. Then

$$F = G\gamma[(x^1 - \beta x^0)(\gamma_1 \wedge \gamma_0) + x^2(\gamma_2 \wedge \gamma_0) + x^3(\gamma_3 \wedge \gamma_0) - \beta x^2(\gamma_1 \wedge \gamma_2) - \beta x^3(\gamma_1 \wedge \gamma_3)], \quad G = kQ/\gamma^2(x^1 - \beta x^0)^2 + (x^2)^2 + (x^3)^2)^{3/2}. \hspace{1cm} (80)$$

In $S$ and in the standard basis, the basis components $F^{\mu\nu}$ of the bivector $F$ are again obtained from (77) and they are

$$F^{10} = G\gamma[(x^1 - \beta x^0), \quad F^{20} = G\gamma x^2, \quad F^{30} = G\gamma x^3, \quad F^{21} = G\gamma^2 x^2, \quad F^{31} = G\gamma^2 x^3, \quad F^{32} = 0. \hspace{1cm} (81)$$
The expression for $F$ as a CBGQ in the $S$ frame can be found in another way as well, i.e., to make the LT of the quantities from (78). Observe that the CBGQs from (78) and (80), which are the representations of the bivector $F$ in $S'$ and $S$ respectively, are equal, $F$ from (78) = $F$ from (80); they are the same quantity $F$ from (78), for observers in $S'$ and $S$. It can be seen from (81) that $F^0$ and $F_0^1$ are different from zero for a moving charge and they are the same as the usual components of the 3D fields $E$ and $B$, respectively. But, as already discussed and as seen from (9) and (77) only the whole $F$, which contains components and the bivector basis, is properly defined physical quantity.

9.2. The expressions for the 4D $E$ and $B$

The general expressions

From the known $F$ (77) and the relations (21) we can construct in a mathematically correct way the 4D vectors $E$ and $B$ for a charge $Q$ moving with constant velocity $u$. If written as CBGQs in the standard basis they are given by equation (73) in [34]

$$E = E^\mu \gamma_\mu = (G/c^2)[(u^\nu v_\nu)x^\mu - (x^\nu v_\nu)u^\mu] \gamma_\mu,$$
$$B = B^\mu \gamma_\mu = (G/c^3)\varepsilon^{\mu\alpha\beta\gamma}x_\nu u_\alpha v_\beta \gamma_\mu, \tag{82}$$

where $G$ is from (77). The vectors $E$ and $B$ are explicitly observer dependent, i.e., dependent on $v$. For the same $F$ the vectors $E$ and $B$ will have different expressions depending on the velocity of observers who measure them. It is visible from (82) that $E$ and $B$ depend on two velocity 4D vectors $u$ and $v$, whereas the usual 3D vectors $E$ and $B$ depend only on the 3-velocity of the charge $Q$. Note also that although $E$ and $B$ as the CBGQs from (82) depend not only on $u$ but on $v$ as well the electromagnetic field $F$ from (77) does not contain the velocity of the observer $v$. This result directly proves that the electromagnetic field $F$ is the primary quantity from which the observer dependent $E$ and $B$ are derived. The expressions for $E$ and $B$ from (82) correctly describe fields in all cases simply specifying $u$ and $v$ and this assertion holds not only for the $\{\gamma_\mu\}$ basis but for the $\{r_\mu\}$ basis as well, i.e., the relation like (87) holds for the expressions from (82). However, observe that, as already mentioned several times, the 4D fields $E$ and $B$ and the usual 3D fields $E$ and $B$ have the same physical interpretation only in the $\gamma_0$ - frame with the $\{\gamma_\mu\}$ basis in which $E^0 = B^0 = 0$. In section 5.6.2.1 in [34] the general expression (82) for the 4D $E$ and $B$ is specified to the case when the $\gamma_0$ - frame is the rest frame of the charge $Q$, the $S'$ frame, $v = c\gamma'_0 = u$, whereas in section 5.6.2.2 the same is made in the case when the $\gamma_0$ - frame is the laboratory frame, the $S$ frame, $v = c\gamma_0$, in which the charge $Q$ is moving, $u^\mu = (\gamma c, \beta \gamma c, 0, 0)$.

The $\gamma_0$ - frame is the rest frame of the charge $Q$, the $S'$ frame.
If the $\gamma_0$ - frame is the $S'$ frame, $v = c\gamma_0' = u$, then \[ E'_{\gamma_i} = Gx'^i, \quad E'^0 = 0, \quad G = kQ/(x'^1x'_1)^{3/2}; \quad B = B'^\mu\gamma'_\mu = 0. \] (83)

The components in (83) agree, as it is expected, with the usual result with the 3D fields, e.g., with components in equation (11) in the first paper in [32]. Now comes the essential difference relative to all usual approaches. In order to find the representations of $E$ and $B$ in $S$, i.e., the CBGQs $E'^\mu\gamma'_\mu$ and $B'^\mu\gamma'_\mu$, we can either perform the LT of $E'^\mu\gamma'_\mu$ and $B'^\mu\gamma'_\mu$ that are given by (83), or simply to take in (82) that both the charge $Q$ and the “fiducial” observers are moving relative to the observers in $S$; $u^\mu = (\gamma c, \beta\gamma c, 0, 0)$. This yields equation (84) ((75) in [34]), i.e., the CBGQs $E'^\mu\gamma'_\mu$ and $B'^\mu\gamma'_\mu$ in $S$ with the condition that the “fiducial” observers are in $S'$, $v = c\gamma_0'$, which is the rest frame of the charge $Q$, $u = c\gamma_0'$.

$$E = E'^\mu\gamma'_\mu = G[\beta\gamma^2(x^1 - \beta x^0)\gamma_0 + \gamma^2(x^1 - \beta x^0)\gamma_1 + x^2\gamma_2 + x^3\gamma_3], \quad B = B'^\mu\gamma'_\mu = 0,$$  

(84)

where $G$ is that one from (80). The result (84) significantly differs from the result obtained by the AT, equations (12a), (12b) in [32]. Under the LT the electric field vector transforms again to the electric field vector and the same for the magnetic field vector. It is worth mentioning that, in contrast to the conventional results, it holds that $E'^\mu\gamma'_\mu$ from (83) is $= E'^\mu\gamma'_\mu$ from (75) in [34]; they are the same quantity $E$ for all relatively moving inertial observers. The same holds for $B$, $B'^\mu\gamma'_\mu$ from (83) is $= B'^\mu\gamma'_\mu$ from (84) and they are $= 0$ for all observers. Furthermore, observe that in $S'$ there are only the spatial components $E'^i$, whereas in $S$, as seen from (84), there is also the temporal component $E'^0$ as a consequence of the LT.

The $\gamma_0$ - frame is the laboratory frame, the $S$ frame

Now, let us take that the “fiducial” observers are in $S$, $v = c\gamma_0$, in which the charge $Q$ is moving, $u^\mu = (\gamma c, \beta\gamma c, 0, 0)$. In contrast to the previous case, both $E$ and $B$ are different from zero. The expressions for the CBGQs $E'^\mu\gamma'_\mu$ and $B'^\mu\gamma'_\mu$ in $S$ can be simply obtained from (82) taking in it that $v = c\gamma_0$ and $w^\mu = \gamma c\gamma_0 + \beta\gamma c\gamma_1$. This yields that $E'^0 = B'^0 = 0$ (from $v = c\gamma_0$) and the spatial parts are

$$E = E'^i\gamma_i = G\gamma[(x^1 - \beta x^0)\gamma_1 + x^2\gamma_2 + x^3\gamma_3],$$

$$B = B'^i\gamma_i = (G/c)[0\gamma_1 - \beta\gamma x^3\gamma_2 + \beta\gamma x^2\gamma_3],$$

(85)

where $G$ is again as in (80). The 4D vector fields $E$ and $B$ from (85) can be compared with the usual expressions for the 3D fields $E$ and $B$ of an uniformly moving charge, e.g., from equations (12a), (12b) in [32]. It is visible that they
are similar, but $E$ and $B$ in (85) are the 4D fields and all quantities in (85) are correctly defined in the 4D spacetime, which transform by the LT, whereas the fields in equations (12a), (12b) in [32] are the 3D fields that transform according to the AT.

In order to find the representations of $E$ and $B$ in $S'$, i.e., the CBGQs $E'_{\mu\gamma}^\mu$ and $B'_{\mu\gamma}^\mu$, we can either perform the LT of $E_{\mu\gamma}^\mu$ and $B_{\mu\gamma}^\mu$ that are given by (85), or simply to take in (82) that relative to $S'$ the “fiducial” observers are moving with $v = v_\gamma', v^\mu = (c\gamma', -\beta\gamma c, 0, 0)$, and the charge $Q$ is at rest relative to the observers in $S'$, $u^\mu = (c, 0, 0, 0)$. This yields the CBGQs $E'_{\mu\gamma}^\mu$ and $B'_{\mu\gamma}^\mu$ in $S'$ with the condition that the “fiducial” observers are in $S$, $v = c\gamma_0$,

$$E = E_{\mu\gamma}^\mu = G\gamma [-\beta x_1'\gamma_0' + x_1'\gamma_1' + x_2'\gamma_2' + x_3'\gamma_3'],$$
$$B = B_{\mu\gamma}^\mu = (G/c)[0\gamma_0' + 0\gamma_1' - \beta\gamma x_3'\gamma_2' + \beta\gamma x_2'\gamma_3'],$$

(86)

where $G$ is as in (83). Again, as in the case that $v = c\gamma_0$, it holds that $E_{\mu\gamma}^\mu$ from (85) is = $E'_{\mu\gamma}^\mu$ from (86); they are the same quantity $E$ for all relatively moving inertial observers. The same holds for $B_{\mu\gamma}^\mu$ from (85) which is = $B'_{\mu\gamma}^\mu$ from (86) and they are both different from zero. Note that in this case there are only the spatial components $E^i$ in $S$, whereas in $S'$ there is also the temporal component $E^0$ as a consequence of the LT.

It is visible from (86) that if the $\gamma_0$ - frame is the lab frame ($v = c\gamma_0$) in which the charge $Q$ is moving then $E_{\mu\gamma}^\mu$ and $B'_{\mu\gamma}^\mu$ in the rest frame of the charge $Q$, the $S'$ frame, are completely different than those from (83); in (86) $B_{\mu\gamma}^\mu$ is different from zero and the representation of $E$ contains also the term $E^0\gamma_0'$.

It has to be emphasized that all four expressions for $E$ and $B$, (89), (81), (85) and (86), are the special cases of $E$ and $B$ given by (82). They all give the same $F$ from (77), which is the representation (CBGQ) of $F$ given by the basis free, abstract, bivector (76).

10. Comparison with the experiments

The approach with 4D GQs and their mathematically correct LT is in a true agreement, independent of the chosen inertial reference frame and of the chosen basis in it, with experiments in electromagnetism. This is already explicitly shown in [22, 12] for the motional emf, in [23] for the Faraday disk and in [33, 14] for the Trouton-Noble experiment.

A nice example that illustrates the fundamental difference between the LT like (29), (30) and the AT (4), i.e., between the approach with 4D GQs and the usual approach with the 3D vectors is presented in the discussion of the motional electromotive force (emf) in sections 5 - 5.2 in [22].

In section 5.1 in [22] the motional emf $\varepsilon$ is calculated using the 3D Lorentz force, $F_L = qE + q U \times B$, and the AT for the 3D $E$ and $B$, equation (11.149) in

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The emf $\varepsilon$ of a complete circuit is defined by means of $F_L$ that acts on a charge $q$, which is at rest relative to the section $dl$ of the circuit

$$\varepsilon = \oint \frac{F_L}{q} \cdot dl,$$  \hspace{1cm} (87)

equation (26) in [22]. Observe that it is implicitly assumed in (87) that the integral is taken over the whole circuit at the same moment of time in $S$, say $t = 0$. Then it is assumed that in the laboratory frame $S$ a conducting bar is moving in a steady uniform magnetic field (3D vector) $B = -Bk$ with velocity 3D vector $U$ parallel to the $x$ axis. The length of the bar is $l$ and it moves parallel to the $y$ axis. There is no external applied electric field in $S$, $E = 0$ and the components of $B$ are $(0, 0, -B)$, which yields that the emf $\varepsilon$ is

$$\varepsilon = \int_0^l UB \, dy = UBl,$$  \hspace{1cm} (88)

equation (27) in [22]. Note that in $S$ the emf $\varepsilon$ is determined only by the contribution of the magnetic part of the 3D Lorentz force $F_L$, i.e., $qU \times B$. On the other hand, in $S'$ the conducting bar is at rest. The usual explanation of this kind. If in $S$ $E = 0$ and the components of $B$ are $(0, 0, -B)$ then, according to the AT of the 3D $E$ and $B$, equation (11.148) in [1], the observer in the $S'$ frame sees $E'_y = \gamma UB$ and $B'_z = -\gamma B$. Hence in $S'$ there is not only the magnetic field but an induced electric field as well. The calculation of $\varepsilon'$ in $S'$ yields that the contribution of $B'_z$ to the emf $\varepsilon'$ is zero and only the contribution of $E'_y$ remains, which is

$$\varepsilon' = \int_0^l \gamma UB \, dy = \gamma UBl,$$  \hspace{1cm} (89)

equation (29) in [22]. Observe that the integral in (89) is again taken at the same moment of time but now $t'$ in $S'$, which can be arbitrarily chosen, say $t' = 0$, or $t' = 10s$, ... . The moments of time $t$ in $S$ and $t'$ in $S'$ are not connected in any way. The LT cannot transform the moment of time $t$ in $S$ again, exclusively, to some $t'$ in $S'$. According to the LT, to one $t$ in $S$ will correspond many $t'$ in $S'$ depending on the spatial position in $S'$; $t = \gamma(t' + Ux'/c^2)$. This remark clearly shows that the usual definition of $\varepsilon$, [87], is not relativistically correct definition.

It is visible that the emf $\varepsilon'$ in $S'$ is not equal to the emf $\varepsilon$ determined in $S$.

$$\varepsilon = UBl, \quad \varepsilon' = \gamma UBl, \quad \varepsilon' \neq \varepsilon.$$  \hspace{1cm} (90)

$\varepsilon'$ is not much different from $\varepsilon$ only if $U \ll c$, i.e., $\gamma \simeq 1$. This means the principle of relativity is not satisfied; the emf obtained by the application of the AT for the 3D $E$ and $B$ is different for relatively moving 4D observers. That result explicitly shows that the AT of the 3D $E$ and $B$ are not the correct relativistic transformations, i.e., they are not the LT. Thus, it is not true that the conventional formalism correctly describes even such simple experiment.
This is a simple but completely correct calculation, which reveals a fundamental flaw in the usual formulations with the 3D \( E \) and \( B \) and their AT. The fact that \( \varepsilon \) and \( \varepsilon' \) do not significantly differ for low velocities is completely irrelevant; the principle of relativity is not satisfied in the usual approach.

On the other hand, in section 5.2 in [22] the emf \( \varepsilon \) is calculated using the 4D GQs. The Lorentz force \( K_L \) is defined by equations (55) or (56). These expressions reveal the fundamental difference between \( K_L \) and the 3D Lorentz force \( F_L \); \( K_L \) contains not only the 4-velocity \( u \) of a charge \( q \) but also the 4-velocity \( v \) of the observer who measures 4D fields. Then the emf \( \varepsilon \) is defined by equation (35) in [22] as an invariant 4D quantity, the Lorentz scalar,

\[
\varepsilon = \int \left( K_L/q \right) \cdot dl,
\]

where vector \( dl \) is the infinitesimal spacetime length and \( \Gamma \) is the spacetime curve. In the laboratory frame \( S \) as the \( \gamma_0 \) frame, the observers are at rest \( v = c \gamma_0 \), whereas the conducting bar is moving with velocity vector \( u \), \( u^\mu = (\gamma c, \gamma U, 0, 0) \).

Furthermore, \( E = 0 \) and \( B^\mu = (0, 0, 0, -B) \). Hence, \( K_L^0 = K_L^1 = K_L^3 = 0 \), but \( K_L^2 = \gamma q UB \), yielding that \( \varepsilon = \int_0^L \gamma UBdy = \gamma UBl \) (equation (36) in [22]); the emf \( \varepsilon \) is determined by the contribution of the magnetic part of \( K_L \), i.e., \( (q/c) [(IB) \cdot v] \cdot u \).

Now comes the main difference relative to the usual approaches with the 3D quantities. The expression for \( \varepsilon \) (91) is independent of the chosen reference frame and of the chosen basis in it. Hence, \( \varepsilon \) is the same in \( S \) and in the relatively moving \( S' \) frame:

\[
\varepsilon = \int \left( K_L^\mu/q \right) dl_\mu = \int \left( K_{L'}^\mu/q \right) dl'_\mu = \gamma UBl,
\]

(equation (37) in [22]) and the same holds if the \( \{r_\mu\} \) basis is used. This means that the observers in \( S \) and \( S' \) are ‘looking’ at the same physical quantity \( \varepsilon \) defined by (91).

Obviously, in contrast to the usual approaches, the principle of relativity is naturally satisfied in the approach with 4D GQs and their mathematically correct LT, like (50). This result (92) for \( \varepsilon \) can be checked directly performing the LT of all vectors from \( S \) to \( S' \) as in [22]. In \( S' \) the 3-velocity \( U \) of a charge \( q \) is zero, but the velocity vector \( u \) is not, \( u = c \gamma_0 \). From the viewpoint of the observers in \( S' \) the velocity vector \( v \) of the “fiducial” observers contains not only the temporal component as in \( S \) \( (v = c \gamma_0) \), but also the spatial component, \( v'^\mu = (\gamma c, -\gamma U, 0, 0) \). According to the LT, like (50), there is no mixing of components of vectors of the electric and magnetic fields. This means that in \( S' \), as in \( S \), there is no electric field!!

In this particular case the LT yield that the components \( B'^\mu \) in \( S' \) are the same as \( B^\mu \) in \( S \), \( B'^\mu = B^\mu = (0, 0, 0, -B) \), and the same holds for the components of the Lorentz force, \( K_L^0 = K_L^1 = K_L^3 = 0 \) and \( K_L^2 = K_L^2 = \gamma q UB \). In \( S' \), as in \( S \), there is only the magnetic part of the Lorentz force and again only that part determines the emf \( \varepsilon \), equation (37) in [22]. Note that in this calculation all quantities are invariant under the passive LT, e.g., \( B = B^\nu \gamma_\nu = B'^\nu \gamma'_\nu \).
\( v = v^\nu \gamma_\nu = v^\nu' \gamma'_\nu, \quad K = K^\nu \gamma_\nu = K^\nu' \gamma'_\nu, \) etc., and the same holds if the \( \{ r_\mu \} \) basis is used.

The same result as in section 5.2 in [22] is obtained in [12] but exclusively dealing with \( F \) and not with its decompositions (18) and (20).

The result that the conventional theory with the 3D \( E \) and \( B \) and their AT, equations (11.148) and (11.149) in [1], i.e., here (41), yields different values for the motional emf \( \varepsilon \) for relatively moving inertial observers, \( \varepsilon = UBL \) in \( S \) and \( \varepsilon' = \gamma UBL \) in \( S' \), equation (90), whereas the approach with 4D GQs and their LT, e.g., (30), yields always the same value for \( \varepsilon, \varepsilon = \gamma UBL \), equation (92), is very strong evidence that the usual approach is not relativistically correct. It is for the experimentalists to find the way to measure the emf \( \varepsilon \) with a great precision in order to see that in the laboratory frame \( \varepsilon = \gamma UBL \) and not simply \( \varepsilon = UBL \).

Such an experiment would be a crucial experiment that could verify from the experimental viewpoint the validity of the formulation of the electromagnetism with the 4D GQs and their mathematically correct LT, like (29), (30).

Completely the same conclusions about the fundamental difference between the conventional theory with the 3D \( E \) and \( B \) and the theory with 4D GQs are obtained in [23], where an important experiment, the Faraday disk, is considered in detail. Particularly important and instructive comparison with experiments is the comparison with the Trouton-Noble experiment that is presented [33]. That comparison is also given in the formulation with \( F \) in section 4 in [12]. In these papers, it is shown that in the treatment with 4D GQs the Trouton-Noble paradox does not appear. The presented explanations are in a complete agreement with the principle of relativity and with the Trouton-Noble experiment without the introduction of any additional torque, which must be necessarily introduced in all usual approaches with the 3D quantities.

Furthermore, in [13], the constitutive relations and the magnetoelectric effect in moving media are explained in a completely new way using 4D GQs. In equation (17) in [13] it is shown how the polarization vector \( P(x) \) depends on \( E, B, u \), the bulk velocity vector of the medium and \( v \), the velocity vector of the observer who measures fields

\[
P^\mu \gamma_\mu = (\varepsilon_0 \chi E/c)(1/c)(E^\nu v^\nu - E^\nu v^\nu) + \varepsilon^{\mu \nu \alpha \beta} v_\alpha B_\beta u_\nu \gamma_\mu, \quad (93)
\]

whereas in equation (18) in [13] the same is shown for the magnetization vector \( M(x) \),

\[
M^\mu \gamma_\mu = \varepsilon_0 \chi B[(B^\mu v^\nu - B^\nu v^\mu) + (1/c)\varepsilon^{\mu \nu \alpha \beta} E_\alpha v_\beta]u_\nu \gamma_\mu. \quad (94)
\]

Both equations are written with CBGQs, whereas the corresponding equations with AQs are equations (13) and (14) in [13]. In this geometric approach, the relations (93) and (94) replace the constitutive relations with the 3D vectors, equations (23) and (24) that are derived in [13] and which are equivalent to Minkowski’s constitutive relations given by equation (22) in [13]. The equations (13) and (14) in [13] are derived from the basic constitutive relations for moving media, equations (11) and (12) in [13], which are written in terms of the primary quantities for the electric and magnetic fields, i.e., the electromagnetic field bivector \( F \), the primary quantity for the the polarization and magnetization,
i.e., the generalized magnetization-polarization bivector $\mathcal{M}$ and the electric and magnetic susceptibility $\chi_E, \chi_B$:

$$\mathcal{M} \cdot u = \varepsilon_0 \chi_E F \cdot u,$$

(95)

$$(IM) \cdot u = (\chi_B/\mu_0 c^2) u \cdot (IF)$$

(96)

Then, the decompositions of $F$ and the similar one for $\mathcal{M}$ are used to derive equations (13) and (14) in [13]. If equations (13) and (14) in [13] with AQs are written in terms of CBGQs in the standard basis then equations (93) and (94) are obtained. The last term in (93) and that one in (94) describe the magnetoelectric effect in a moving dielectric. The last term in (93) shows that a moving dielectric becomes electrically polarized if it is placed in a magnetic field, the Wilsons’ experiment [39]. Let us take that the laboratory frame, the $S$ frame, is the $\gamma_0$-frame ($v = c\gamma_0$) in which the material medium, the $S'$ frame, is moving with velocity $u$. If in equation (93) it is chosen that $E^\mu = (0, 0, 0, 0), B^\mu = (0, 0, 0, -B^3), u^\mu = (\gamma_u c, \gamma_u U^1, 0, 0)$, then, in $S$, equation (93) becomes equation (20) in [13],

$$P^\mu = (0, 0, P^2 = \varepsilon_0 \chi_E \gamma_u U^1 B^3, 0).$$

(97)

The components in (97) correspond to the “translational” version of Wilsons’ experiment [42]. Similarly, the last term in (94) shows that a moving dielectric becomes magnetized if it is placed in an electric field, Röntgen’s experiment [43]. If in equation (94) it is chosen that $B^\mu = (0, 0, 0, 0), E^\mu = (0, 0, -E^2, 0), u^\mu = (\gamma_u c, \gamma_u U^1, 0, 0)$, then, in $S$, equation (94) becomes

$$M^\mu = (0, 0, 0, M^3 = (\chi_B/\mu_0 c^2) \gamma_u U^1 E^2).$$

(98)

The components in (98) correspond to the “translational” version of Röntgen’s experiment [43]. Observe that in this geometric approach all quantities are correctly defined 4D quantity that correctly transform under the LT. The term in (93) and (94) that describes the magnetoelectric effect is obtained without any transformations by the correct mathematical procedure from the fundamental constitutive relations (95) and (96). It is not so in all previous approaches, e.g., [44], in which the 3D $E, B, P, M, D, H$, etc. and their AT are used considering them as that they are the mathematically correct LT. In sections 5.1 and 5.2 the constitutive relations with 4D GQs, the relations (93) and (94) here, are compared with Minkowski’s constitutive relations with the 3D vectors, i.e., with the equivalent relations (23) - (25) in [13]. It is shown that there are important differences between them, which could be experimentally examined.

11. Discussion and Conclusions

The main point in the whole paper is explicitly expressed by the motto at the beginning of the text. In the 4D spacetime physical laws are geometric,
coordinate-free relationships between the 4D geometric, coordinate-free quantities. This point of view is also adopted in the nice textbook [7] but not in the consistent way. They still introduce the 3D vectors and their transformations, e.g., in section 1.10 in [7] and this is discussed in section 6 here. Similarly happens in [8], which is discussed in section 7 here. A fully consistent application of this viewpoint is adopted in Oziewicz’s papers, see, e.g., [11]. The same viewpoint is adopted in all my papers given in the references and including the present paper. Here, in this paper, the mathematically correct proofs are given that the electric and magnetic fields are properly defined vectors on the 4D spacetime, sections 3.1 and 3.3. According to Oziewicz’s proof from section 3.1, e.g., \( \mathbf{E}(\mathbf{r},t) \) (written in the usual notation) must have four components (some of them can be zero) since it is defined on the 4D spacetime and not, as usually considered, only three components. In section 3.3 it is taken into account that, as proved in [14], the primary quantity for the whole electromagnetism is the electromagnetic field bivector \( \mathbf{F} \). The decomposition of \( \mathbf{F} \) given by equation (15) expresses \( \mathbf{F} \) in terms of observer dependent electric and magnetic 4D vectors \( \mathbf{E} \) and \( \mathbf{B} \), which are given by equation (19). Both equations (15) and (19) are with the abstract, coordinate-free quantities. This is in a sharp contrast with the usual covariant approaches, e.g., [1, 4, 36] in which it is considered that \( F^{\alpha\beta} \) (the components implicitly taken in the standard basis) is physically well-defined quantity. Moreover, these components are considered to be six independent components of the 3D \( \mathbf{E} \) and \( \mathbf{B} \), see equations (1) and (2). Then, as described in section 1, in these approaches [1, 4, 36], the transformations of the components of \( \mathbf{E} \) and \( \mathbf{B} \) \( [4] \) are obtained supposing that they transform under the LT as the components of \( F^{\alpha\beta} \) transform, equation (3). The objections to such treatment are given in section 1, the objections 1), 2) and section 2.1, the objections 3), 4) and 5). From the mathematical viewpoint all these objections are well-founded since they are based on the following facts: 1) The bivector \( \mathbf{F}(\mathbf{x}) \), as described in detail in [14] and very briefly in section 3.2 here, is determined, for the given sources, by the solutions of the equation (11), i.e., (12) (with CBGQs in the \( \{\gamma_{\mu}\} \) basis) and not by the components of the 3D \( \mathbf{E} \) and \( \mathbf{B} \). It is a 4D GQ and not only components. It yields a complete description of the electromagnetic field without the need for the introduction either the field vectors or the potentials. 2) As seen from section 2 and particularly from equations (7) and (8) the identification of the components of the 3D \( \mathbf{E} \) and \( \mathbf{B} \) with the components of \( F^{\alpha\beta} \) is synchronization dependent. Moreover, it is completely meaningless in the “r” synchronization, i.e., in the \( \{r_{\mu}\} \) basis. Both bases, the commonly used standard basis with Einstein’s synchronization and the \( \{r_{\mu}\} \) basis with the “r” synchronization are equally well physical and relativistically correct bases.

Furthermore, it is proved in section 4.1 with the coordinate-free quantities and the active LT and in section 4.2 with CBGQs and the passive LT that the mathematically correct LT of, e.g., the electric field vector are given by (29) - (32) and not by the AT of the 3D vectors equations (11.148) and (11.149) in [1], i.e., equation (11) or equation (33) here.

In section 5.1 the same fundamental difference between the correct LT and the usual AT of the 3D vectors is explicitly exposed using matrices. The equa-
tions $(50)$ - $(54)$ refer to the correct LT of the components in the standard basis of the electric field 4D vector in which the transformed components $E'_{\mu}$ are obtained as $E'_{\mu} = c^{-1}F^\mu_{\nu\nu}'v'_{\nu}$, i.e., both $F^\mu_{\nu\nu}$ and the velocity of the observer $v = c\gamma_0$ are transformed by the matrix of the LT $A^\mu_{\nu}$ (the boost in the direction $x^1$). It is visible from equation $(54)$ that the same components are obtained as $E'_{\mu} = A^\mu_{\nu}E_{\nu}$ and they are the same as in $(50)$. This means that under the mathematically correct LT the electric field 4D vector transforms again only to the electric field 4D vector as any other 4D vector transforms. As stated at the end of section 5.1 if $E$ is written as a CBGQ then again holds the relation $(37)$ as for any other CBGQ. On the other hand equation $(48)$ refers to the AT in which the transformed components $E'_{\mu}$ are obtained as $E'_{\mu} = c^{-1}F^\mu_{\nu\nu}'v_{\nu}$, i.e., only $F^\mu_{\nu\nu}$ is transformed by the LT but not the velocity of the observer $v = c\gamma_0$. These transformed components $E'_{\mu}$ are the same as in equation $(33)$. The transformed spatial components $E'_{i}$ are the same as are the transformed components of the usual 3D vector $E$, i.e., as in equation (11.148) in [1]. However, according to these transformations the 4D vector with $E^0 = 0$ is transformed in such a way that the transformed temporal component is again zero, $E'^0 = 0$. Hence, as stated in section 5.1, such transformations cannot be the mathematically correct LT.

It can be concluded from the whole consideration in this paper that in the 4D spacetime an independent physical reality has to be attributed to the 4D geometric quantities, coordinate-free quantities or the CBGQs, e.g., the electromagnetic field bivector $F$, the 4D vectors of the electric $E$ and magnetic $B$ fields, etc., and not to the usual 3D quantities, e.g., the 3D $E$ and $B$. This is the answer to the question what is the nature of the electric and magnetic fields. Furthermore, the mathematically correct LT are properly defined on the 4D spacetime. They can correctly transform only the 4D quantities like $E$ and $B$, the transformations $(29)$ - $(32)$, according to which, e.g., the electric field 4D vector transforms again only to the electric field 4D vector as any other 4D vector transforms. The LT cannot act on the 3D quantities like the 3D $E$ and $B$, which means that the usual transformations of the 3D quantities, e.g., the 3D vectors $E$ and $B$, equations (11.148) and (11.149) in [1], i.e., equation $(4)$ or equation $(42)$ here, are not the LT, but the mathematically incorrect transformations in the 4D spacetime, i.e., the AT. This is the answer to the question how the fields transform.

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