A remark on finding the coefficient of the dissipative boundary condition via the enclosure method in the time domain

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Abstract

An inverse problem for the wave equation outside an obstacle with a dissipative boundary condition is considered. The observed data are given by a single solution of the wave equation generated by an initial data supported on an open ball. An explicit analytical formula for the computation of the coefficient at a point on the surface of the obstacle which is nearest to the center of the support of the initial data is given.

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1 Introduction

We consider an inverse obstacle scattering problem which is described by the classical wave equation outside an obstacle with a dissipative boundary condition.

First we formulate the problem. Let $D$ be a nonempty bounded open subset of $\mathbb{R}^3$ with $C^2$-boundary such that $\mathbb{R}^3 \setminus \overline{D}$ is connected. Let $\gamma$ be a function belonging to $L^\infty(\partial D)$ and satisfy $\gamma \geq 0$. Let $0 < T < \infty$. Let $B$ be an open ball satisfying $\overline{B} \cap \overline{D} = \emptyset$. We denote by $\chi_B$ the characteristic function of $B$; $p$ and $\eta$ the center and (very small) radius of $B$, respectively.

Let $u = u_B(x,t)$ denote the weak solution of the following initial boundary value problem for the classical wave equation:

$$
\begin{cases}
\partial_t^2 u - \Delta u = 0 & \text{in } (\mathbb{R}^3 \setminus D) \times ]0, T[,

u(x, 0) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D},

\partial_t u(x, 0) = \chi_B(x) & \text{in } \mathbb{R}^3 \setminus \overline{D},

\frac{\partial u}{\partial \nu} - \gamma(x) \partial_t u = 0 & \text{on } \partial D \times ]0, T[.
\end{cases}
$$

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Here $\nu$ denotes the outward normal to $D$ on $\partial D$. The solution class is taken from [3] and see also [6] for its detailed description.

The presence of $\gamma$ affects on the energy of the solution of (1.1). A formal computation yields

$$
\mathcal{E}'(t) = - \int_{\partial D} \gamma(x) |\partial_t u|^2 dS \leq 0,
$$

where

$$
\mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^3 \setminus \overline{D}} (|\partial_t u|^2 + |\nabla u|^2) dx, \ t \in [0, T].
$$

Thus the solution of (1.1) is a model of the wave that loses the energy on the surface of the obstacle. The distribution of $\gamma$ represents the state of the surface of the obstacle.

In this paper we consider the following problem.

**Problem.** Fix a large $T$ (to be determined later). Assume that both $D$ and $\gamma$ are unknown. Extract information about the location and shape of $D$ together with $\gamma$ from the wave field $u_B(x, t)$ given at all $x \in B$ and $t \in ]0, T[$.

For this problem we have already some solution obtained in [6]. Note that therein the initial data $\partial_t u(x, 0)$ is given by $\chi_B(x)$ multiplied by a function $f(x)$. However, we are considering a very small $B$ and so we simplify the problem setting as above.

Set

$$
w(x) = w_B(x, \tau) = \int_0^\tau e^{-\tau t} u_B(x, t) dt, \ x \in \mathbb{R}^3 \setminus \overline{D}, \ \tau > 0.
$$

Let $v = v_B(\cdot, \tau) \in H^1(\mathbb{R}^3)$ denote the weak solution of the modified Helmholtz equation

$$
(\triangle - \tau^2)v + \chi_B = 0 \text{ in } \mathbb{R}^3.
$$

We introduce two conditions on $\gamma$:

(A1) $\exists C' > 0$ $\gamma(x) \leq 1 - C'$ a.e. $x \in \partial D$;

(A2) $\exists C' > 0$ $\gamma(x) \geq 1 + C'$ a.e. $x \in \partial D$.

Define

$$
I_B(\tau) = \int_B (w - v) dx, \ \tau > 0.
$$

The function $\tau \mapsto I_B(\tau)$ is called the indicator function in the enclosure method.

The enclosure method goes back to [5] in which an inverse boundary value problem for the Laplace equation using a single set of the Cauchy data has been considered. Since then the basic idea of the enclosure method has been developed and applied to several inverse obstacle problems governed by partial differential equations. It is a method that finds a domain enclosing unknown object from the asymptotic behaviour of the indicator function like above. In particular, we have already established the following result.

**Theorem 1.1([6]).** If $T$ satisfies

$$
T > 2 \text{dist}(D, B),
$$

then we have:

if (A1) is satisfied, then there exists a $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$

$$
I_B(\tau) > 0;
$$

(1.5)
if (A2) is satisfied, then there exists a \( \tau_0 > 0 \) such that, for all \( \tau \geq \tau_0 \)
\[
I_B(\tau) < 0. \tag{1.6}
\]

In both cases the formula
\[
\lim_{\tau \to \infty} \frac{1}{2\tau} \log |I_B(\tau)| = -\text{dist} (D, B), \tag{1.7}
\]
is valid.

Formula (1.7) gives us an information about the geometry (location) of unknown obstacle. More precisely, recall the first reflector
\[
\Lambda_{\partial D}(p) = \{ q \in \partial D \mid |q - p| = d_{\partial D}(p) \},
\]
where \( d_{\partial D}(p) = \inf_{y \in \partial D} |y - p| \). Since \( \text{dist} (D, B) = d_{\partial D}(p) - \eta \), (1.7) gives us the largest sphere whose exterior encloses \( D \). On the sphere there exists a point on \( \Lambda_{\partial D}(p) \). In addition, given \( \omega \in S^2 \) taking a small open ball \( B' \) centered at \( p + s\omega \) with a small \( s \in [0, d_{\partial D}(p)] \), we can make a decision whether the point \( p + d_{\partial D}(p)\omega \) belongs to \( \partial D \) or not by sounding \( u_{B'} = u_{B|_{B=B'}} \) on \( B' \) over a large but finite time interval via formula (1.7) with \( I_{B'} \equiv I_B|_{B=B'} \).

The conclusions (1.5) and (1.6) are byproducts and describe qualitative property of \( \gamma \) which affects the state of the surface of unknown obstacle, roughly speaking, whether \( \gamma \gg 1 \) or \( \gamma << 1 \) in terms of the signature of the value of the indicator function for large \( \tau \).

The proof of Theorem 1.1 is based on the following lower and upper estimates of the indicator function as \( \tau \to \infty \) (see [6, 8]):

- if \( \gamma(x) \geq 0 \) for a.e. \( x \in \partial D \), then
  \[
  I_B(\tau) \geq \int_{\partial D} \left( \frac{\partial v}{\partial v} - \tau \gamma v \right) v\, dS + O(\tau^{-1}e^{-\tau T}); \tag{1.8}
  \]
- if \( \gamma(x) \geq C' \) a.e. \( x \in \partial D \) for a positive constant \( C' \), then
  \[
  I_B(\tau) \leq \int_{\partial D} \left( \frac{\partial v}{\partial v} - \tau \gamma v \right) v\, dS + \frac{1}{\tau} \int_{\partial D} \frac{1}{\tau} \left| \frac{\partial v}{\partial v} - \tau \gamma v \right|^2 \, dS + O(\tau^{-1}e^{-\tau T}). \tag{1.9}
  \]

However, these inequalities do not yield us the quantitative information about the distribution of \( \gamma \). The aim of this paper is to fill this gap and obtain an explicit formula which explains the reason for the validity of (1.5) and (1.6) quantitatively.

In what follows we denote by \( B_r(x) \) the open ball centered at \( x \) with radius \( r \). To describe the formula it has better to introduce some notion in differential geometry. Let \( q \in \Lambda_{\partial D}(p) \). Let \( S_q(\partial D) \) and \( S_q(\partial B_{d_{\partial D}(p)}(p)) \) denote the shape operators (or Weingarten maps) at \( q \) of \( \partial D \) and \( \partial B_{d_{\partial D}(p)}(p) \) with respect to \( v_q \) and \( -v_q \), respectively (see [13] for the notion of the shape operator). Since \( q \) attains the minimum of the function: \( \partial D \ni y \mapsto |y - p| \), we have always \( S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D) \geq 0 \) as the quadratic form on the common tangent space at \( q \).

We introduce the following condition which is stronger than (A1):
given by the Robin boundary condition

Now we are ready to state the main result in this paper.

**Theorem 1.2.** Assume that $\partial D$ is $C^3$ and $\gamma \in C^2(\partial D)$. Assume that $\gamma$ satisfies (A1) or (A2). Let $T$ satisfy (1.4). Assume that the set $\Lambda_{\partial D}(p)$ consists of finite points and

$$
det (S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) > 0 \forall q \in \Lambda_{\partial D}(p).$$

Then, we have, as $\tau \to \infty$

$$
\lim_{\tau \to \infty} \tau^4 e^{2\tau \text{dist}(D, B)} I_B(\tau) = \frac{\pi}{2} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \sum_{q \in \Lambda_{\partial D}(p)} \frac{1}{\sqrt{\det (S_q(\partial B_{d_{\partial D}(p)})(p)) - S_q(\partial D)}} \frac{1 - \gamma(q)}{1 + \gamma(q)}.
$$

Note that, in [7] we have considered the case when the boundary condition in (1.1) is given by the Robin boundary condition

$$
\frac{\partial u}{\partial \nu} - \beta(x)u = 0 \quad \text{on } \partial D \times [0, T[,
$$

where $\beta$ is a real-valued function on $\partial D$. We gave an explicit asymptotic formula of the indicator function in this case. However, the formula contains: complicated information about the shape of $\partial D$, more precisely, third-and fourth-order derivatives of a local representation of $\partial D$ at all the points on $\Lambda_{\partial D}(p)$; the term that contains $\beta$ does not appear as the leading profile of the indicator function. These together with (1.11) suggest us that information about the values of $\gamma$ is visible rather than those of $\beta$.

We mention here how to make use of Theorem 1.2 in remote sensing. Let $p \in \mathbb{R}^3 \setminus \overline{D}$. Assume that we have a known point $q \in \Lambda_{\partial D}(p)$. How can one find the value of $\gamma$ at $q$ by using the wave phenomena governed by the wave equation. From point $p$ let us go a little bit forward to $q$. We denote by $p'$ the point. Choose a small open ball $B'$ centered at $p'$ and generate the wave $u_{B'}$. Compute the indicator function $I_{B'}(\tau)$ by using $v_{B'}$ and $w_{B'}$ via (1.2) by observing $u_{B'}$ on $B'$ over time interval $[0, T[\tau$ for a $T'$ satisfying (1.4) with $B$ replaced with $B'$. Since the set $\Lambda_{\partial D}(p')$ consists of only the single point $q$ and (1.10) for $p$ replaced with $p'$ is satisfied, applying (1.11) to the present situation, one gets the quantity

$$
\mathcal{F}_{B'}(q) = \frac{\pi}{2} \left( \frac{\eta'}{d_{\partial D}(p')} \right)^2 \sum_{q \in \Lambda_{\partial D}(p')} \frac{1}{\sqrt{\det (S_q(\partial B_{d_{\partial D}(p')}(p')) - S_q(\partial D)}} \frac{1 - \gamma(q)}{1 + \gamma(q)},
$$

where $\eta'$ denotes the radius of $B'$ and we have $d_{\partial D}(p') = d_{\partial D}(p) - |p - p'|$. Note also that we have

$$
det (S_q(\partial B_{d_{\partial D}(p')}(p')) - S_q(\partial D)) = \lambda^2 - 2H_{\partial D}(q)\lambda + K_{\partial D}(q),
$$

where $\lambda = 1/d_{\partial D}(p')$, $H_{\partial D}(q)$ and $K_{\partial D}(q)$ are the mean and Gauss curvatures at $q$ with respect to $\nu_q$.

Thus if the curvatures $H_{\partial D}(q)$ and $K_{\partial D}(q)$ are known, then from $\mathcal{F}_{B'}(q)$ one can find $(1 - \gamma(q))/(1 + \gamma(q))$ and thus $\gamma(q)$ itself provided $\gamma(q) \neq 1$. 


By the way, if we do not know the two curvatures at \( q \), how can one find \( \gamma(q) \) in the observed wave. One simply way is to increase the number of observed waves.

Let \( p_1, p_2 \) and \( p_3 \) are three points on the segment \( p \to q \) different from its end points. Choose three small balls \( B_1, B_2 \) and \( B_3 \) with a common radius \( \eta' \) centered at \( p_1, p_2 \) and \( p_3 \), respectively. Applying the same procedure to \( I_B'(\tau) \) with \( B' = B_j, j = 1, 2, 3 \) mentioned above, one gets three quantities:

\[
\mathcal{F}_j \equiv \frac{2}{\pi} \left( \frac{d_{\partial D}(p_j)}{\eta'} \right)^2 \mathcal{F}'(q)|_{B' = B_j} = \frac{A}{\sqrt{\lambda_j^2 - 2H\lambda_j + K}}, \quad j = 1, 2, 3, \tag{1.12}
\]

where \( A = (1 - \gamma(q))/(1 + \gamma(q)) \), \( H = H_{\partial D}(q) \) and \( K = K_{\partial D}(q); \lambda_j = 1/d_{\partial D}(p_j) \), \( j = 1, 2, 3 \). Then, we have

\[
\begin{pmatrix}
-(\lambda_1\mathcal{F}_1^2 - \lambda_2\mathcal{F}_2^2)

-(\lambda_2\mathcal{F}_2^2 - \lambda_3\mathcal{F}_3^2)
\end{pmatrix}
\begin{pmatrix}
\mathcal{F}_1^2 - \mathcal{F}_2^2

\mathcal{F}_2^2 - \mathcal{F}_3^2
\end{pmatrix}
\begin{pmatrix}
2H

K
\end{pmatrix}
= \begin{pmatrix}
\mathcal{F}_2^2\mathcal{F}_1^2 - \mathcal{F}_1^2\mathcal{F}_2^2

\mathcal{F}_3^2\mathcal{F}_2^2 - \mathcal{F}_2^2\mathcal{F}_3^2
\end{pmatrix}. \tag{1.13}
\]

Solving this system, we will obtain \( H \) and \( K \). Then \( A^2 \) is uniquely determined by one of three equations (1.12) or

\[
A^2 = \frac{1}{3} \sum_{j=1}^3 \mathcal{F}_j^2(\lambda_j^2 - 2H\lambda_j + K).
\]

Since we can know \( \gamma(q) > 1 \) or \( \gamma(q) < 1 \) from one of three equations (1.12), taking the square root of the both side we obtain \( (1 - \gamma(q))/(1 + \gamma(q)) \) and hence \( \gamma(q) \) itself. Note that the discriminant \( M \) for (1.13) has the form

\[
M = (\lambda_3 - \lambda_2)\mathcal{F}_3^2\mathcal{F}_2^2 + (\lambda_2 - \lambda_1)\mathcal{F}_2^2\mathcal{F}_1^2 + (\lambda_1 - \lambda_3)\mathcal{F}_1^2\mathcal{F}_3^2.
\]

However, in general, one can not ensure the non vanishing of \( M \) exactly since one can not make the signature of three numbers \( \lambda_3 - \lambda_2, \lambda_2 - \lambda_1, \lambda_1 - \lambda_3 \) the same.

It should be pointed out that there is a well-known result due to Majda [12] in the context of the Lax-Phillips scattering theory for the wave equation. The boundary condition is the same one as that of (1.1). It is assumed that the obstacle is strictly convex. He considered the high frequency asymptotics for the scattering amplitude which can be measured at infinity. Therefore it is the case when \( T = \infty \). He clarified its leading term as the frequency goes to infinity. The geometrical information about the obstacle contained in the leading term is only the Gauss curvature. Our result contains also the mean curvature as mentioned above. The information about the coefficient \( \gamma \) contained in the formula in the back-scattering case is essentially same as (1.11). Note also that if \( D \) is convex, then \( \Lambda_{\partial D}(p) \) consists of a single point and (1.10) is satisfied.

Finally we present one simple corollary of Theorem 1.2. To make the dependence on \( \gamma \) clear we denote the indicator function \( I_B(\tau) \) by \( I_B(\tau; \gamma) \). Assume that we have \( \gamma_1 \) and \( \gamma_0 \) belonging to \( C^2(\partial D) \) and satisfying (A1)’ or (A2). Under the same assumption on \( \partial D \) and \( \Lambda_{\partial D}(p) \) in Theorem 1.2, (1.11) for \( \gamma = \gamma_1, \gamma_2 \) yields

\[
\lim_{\tau \to \infty} I_B(\tau; \gamma_1) = \frac{\sum_{q \in \Lambda_{\partial D}(p)} k_q \frac{1 - \gamma_1(q)}{1 + \gamma_1(q)}}{\sum_{q \in \Lambda_{\partial D}(p)} k_q \frac{1 - \gamma_0(q)}{1 + \gamma_0(q)}}, \tag{1.14}
\]
where
\[ k_q = \frac{1}{\sqrt{\det (S_q(\partial B_{\partial D}(p)) - S_q(\partial D))}}. \]

Then, from the right-hand side on (1.14) together with assumption (A1)' or (A2) for \( \gamma_0 \) we can easily obtain the following estimates and formula.

**Corollary 1.1.** Assume that \( \partial D \) is \( C^3 \). Let \( \gamma_0 \) and \( \gamma_1 \) belong to \( C^2(\partial D) \) and satisfy (A1)' or (A2). Let \( T \) satisfy (1.4). Assume that the set \( \Lambda_{\partial D}(p) \) consists of finite points and satisfies (1.10). Then, the limit \( \lim_{\tau \to -\infty} I_B(\tau; \gamma_1)/I_B(\tau; \gamma_0) \) exists and we have

\[
\min_{q \in \Lambda_{\partial D}(p)} \frac{1 - \gamma_1(q)}{1 + \gamma_1(q)} \leq \lim_{\tau \to -\infty} \frac{I_B(\tau; \gamma_1)}{I_B(\tau; \gamma_0)} \leq \max_{q \in \Lambda_{\partial D}(p)} \frac{1 + \gamma_1(q)}{1 - \gamma_0(q)}. \tag{1.15}
\]

In particular, if \( \Lambda_{\partial D}(p) \) consists of a single point \( q \in \partial D \), we have

\[
\lim_{\tau \to -\infty} \frac{I_B(\tau; \gamma_1)}{I_B(\tau; \gamma_0)} = \frac{1 - \gamma_1(q)}{1 + \gamma_1(q)} \quad \frac{1 - \gamma_0(q)}{1 + \gamma_0(q)}. \tag{1.16}
\]

Estimates on (1.15) give us some global information about the values of \( \gamma_1 \) relative to \( \gamma_0 \) at all the points on \( \Lambda_{\partial D}(p) \) without knowing the curvatures. We need just two observed waves generated by the same initial data on one day which is the case \( \gamma = \gamma_0 \) and another day the case \( \gamma = \gamma_1 \). Formula (1.16) gives a deviation of the value of \( \gamma_1 \) from \( \gamma_0 \) at a monitoring point on the surface of the obstacle. Note that given an arbitrary \( q \in \partial D \) if \( p = q + s\nu_q \) and \( s \) is a sufficiently small positive number, then \( \Lambda_{\partial D}(p) = \{ q \} \) and (1.10) is satisfied.

This paper is organized as follows. In Section 2 we give a proof of Theorem 1.2. It is based on a rough asymptotic formula of the indicator function as \( \tau \to -\infty \) which has been derived in [6]. The point of the proof of Theorem 1.2 is to clarify the asymptotic profile of the second term in the formula. It is stated as Theorem 2.1 and proved in Section 3. In final section we mention a conclusion together with further problems. In Appendix we give a proof of Lemma 3.1 which is essential for that of Theorem 2.1. The proof employs a reflection argument developed in [11] for a characterization of the right-end point of the scattering kernel. We have already used the argument or its modification in the framework of the enclosure method in the time domain, see [7, 9].

## 2 Proof of Theorem 1.2

Let \( R = w - v \). The function \( R \) satisfies

\[
\begin{cases}
(\Delta - \tau^2)R = e^{-\tau T}F & \text{in } R^3 \setminus \overline{D}, \\
\frac{\partial R}{\partial \nu} - \tau \gamma R = - \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) + e^{-\tau T}G & \text{on } \partial D,
\end{cases}
\tag{2.1}
\]
where
\[
\begin{align*}
F &= F(x, \tau) = \partial_t u(x, T) + \tau u(x, T), \\
G &= G(x) = \gamma(x) u(x, T).
\end{align*}
\]  
(2.2)

From [6] we have already known that, as \( \tau \to \infty \)
\[
I_B(\tau) = J(\tau) + E(\tau) + O(\tau^{-1} e^{-\tau T}),
\]  
(2.3)

where
\[
J(\tau) = \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) v \, dS
\]  
(2.4)

and
\[
E(\tau) = \int_{\mathbb{R}^3 \setminus \overline{D}} (|\nabla R|^2 + \tau^2 |R|^2) \, dx + \tau \int_{\partial D} \gamma |R|^2 \, dS.
\]  
(2.5)

See also [8] for a brief explanation about the derivation of formula (2.3).

Thus the essential part of the proof of Theorem 1.2 should be the study of the asymptotic behaviour of \( J(\tau) \) and \( E(\tau) \) as \( \tau \to \infty \). The asymptotic behaviour of \( J(\tau) \) can be reduced to study a Laplace-type integral [1]. For that of \( E(\tau) \) we have the following result which enables us to make a reduction of the study to a Laplace-type integral.

**Theorem 2.1.** Assume that \( \partial D \) is \( C^3 \) and \( \gamma \in C^2(\partial D) \). Assume that:
- \( \gamma \) has a positive lower bound;
- the set \( \Lambda_{\partial D}(p) \) consists of finite points and (1.10) is satisfied;
- there exists a point \( q \in \Lambda_{\partial D}(p) \) such that \( \gamma(q) \neq 1 \).

Let \( T \) satisfy
\[
T > \text{dist}(D, B).
\]  
(2.6)

Then, we have, as \( \tau \to \infty \)
\[
\lim_{\tau \to \infty} \frac{E(\tau)}{\int_{\partial D} \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) \frac{1 - \gamma}{1 + \gamma} v \, dS} = 1.
\]  
(2.7)

The proof of Theorem 2.1 is given in Section 3. We continue to proceed the proof of Theorem 1.2.

It is well known that the Laplace method under the assumption that \( \Lambda_{\partial D}(p) \) is finite and satisfies (1.10), we have
\[
\lim_{\tau \to \infty} \tau e^{2\pi d_{\partial D}(p)} \int_{\partial D} A(x) \frac{e^{-2\tau |x - p|}}{|x - p|^2} \, dS
\]  
(2.8)

\[
= \frac{\pi}{d_{\partial D}(p)^2} \sum_{q \in \Lambda_{\partial D}(p)} \frac{A(q)}{\sqrt{\det (S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D))}}.
\]  
where \( A \in C^1(\partial D) \). See [1], for example. The point is that the Hessian of the function \( \partial D \ni x \mapsto |x - p| \) at \( q \in \Lambda_{\partial D}(p) \) is given by the operator \( S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D) \). See, for example, [7] for this point.

The weak solution \( v \) of (1.3) is explicitly given by the formula
\[
v(x) = \frac{1}{4\pi} \int_B \frac{e^{-\tau |x - y|}}{|x - y|} \, dy.
\]
By the mean value theorem for the modified Helmholtz equation [2], we have, for all \( x \in \mathbb{R}^3 \),
\[
\frac{1}{4\pi} \int_B \frac{e^{-\tau|x-y|}}{|x-y|} \, dy = \frac{\varphi(\tau \eta) \, e^{-\tau|x-p|}}{\tau^3 |x-p|},
\]
where \( \varphi(\xi) = \xi \cosh \xi - \sinh \xi \). Therefore \( v \) outside \( B \) has the explicit form
\[
v(x) = \frac{\varphi(\tau \eta) \, e^{-\tau|x-p|}}{\tau^3 |x-p|}
\] (2.9)
and note that
\[
\varphi(\tau \eta) = \frac{\tau \eta e^{\tau \eta}}{2} (1 + O(\tau^{-1})).
\] (2.10)

Set
\[
\tilde{v}(x) = \frac{e^{-\tau|x-p|}}{|x-p|}.
\]
Let \( x \in \partial D \). We have
\[
\frac{\partial \tilde{v}}{\partial \nu} = \left( \tau + \frac{1}{|x-p|} \right) \frac{p-x}{|x-p|} \cdot \nu \tilde{v}
\] (2.11)
and thus
\[
\frac{\partial \tilde{v}}{\partial \nu} - \tau \gamma \tilde{v} = \tilde{v} \left\{ \tau \left( \frac{p-x}{|x-p|} \cdot \nu - \gamma \right) + \frac{1}{|x-p|} \frac{p-x}{|x-p|} \cdot \nu \right\}.
\] (2.12)
Then, it follows from (2.8) that
\[
\lim_{\tau \to \infty} e^{2\tau d_{\partial D}(p)} \int_{\partial D} \left( \frac{\partial \tilde{v}}{\partial \nu} - \tau \gamma \tilde{v} \right) \tilde{v} \, dS = \frac{\pi}{d_{\partial D}(p)^2} \sum_{q \in \Lambda_{\partial D}(p)} \frac{1 - \gamma(q)}{\sqrt{\det (S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D))}}.
\] (2.13)
Note that we have made use of the fact that, for all \( x \in \Lambda_{\partial D}(p) \) the unit vector \( (p-x)/|x-p| \) coincides with \( \nu \) at \( x \).

Since \( v \) has the form (2.9), (2.4) gives
\[
\int_{\partial D} \left( \frac{\partial \tilde{v}}{\partial \nu} - \tau \gamma \tilde{v} \right) \tilde{v} \, dS = \left( \frac{\tau^3}{\varphi(\tau \eta)} \right)^{2} J(\tau).
\]
Substituting this into (2.13) and using (2.10), we obtain
\[
\lim_{\tau \to \infty} \tau^4 e^{2\tau \text{dist}(D,B)} J(\tau)
= \frac{\pi}{4} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \sum_{q \in \Lambda_{\partial D}(p)} \frac{1 - \gamma(q)}{\sqrt{\det (S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D))}}.
\] (2.14)
Similarly, we obtain
\[
\lim_{\tau \to \infty} \tau^4 e^{2\tau \text{dist}(p,B)} \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) \frac{1 - \gamma}{1 + \gamma} v \, dS
= \frac{\pi}{4} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \sum_{q \in \Lambda_{\partial D}(p)} \frac{1}{\sqrt{\det (S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D))}} \frac{(1 - \gamma(q))^2}{1 + \gamma(q)}
\] (2.15)
provided $T$ satisfies (2.6). Note that this yields also that, for sufficiently large $\tau$ the denominator of (2.7) is positive under the condition $\gamma(q) \neq 1$ for a $q \in \Lambda_{\partial D}(p)$.

Since we have

$$\tau^4 e^{2\tau \text{dist}(D,B)} O(\tau^{-1} e^{-\tau T}) = O(\tau^3 e^{-\tau (T - \text{dist}(D,B)})$$

and

$$1 - \gamma(q) + \frac{(1 - \gamma(q))^2}{1 + \gamma(q)} = \frac{2}{1 + \gamma(q)},$$

from (2.3), (2.14), (2.7) and (2.15), we obtain (1.11) provided $T$ satisfies (1.4).

**Remark 2.1.** It should be emphasized that formula (2.7) is valid for $T$ satisfying (2.6) which includes the case when (1.4) is not satisfied. However, this is reasonable. Formula (2.7) is concerned with the energy of the reflected solution arrived at the surface of the obstacle, thus something about the incident field should be encoded therein if $T$ is greater than the first arriving time $\text{dist}(D,B)$ at the surface of the obstacle. Formula (2.7) clarifies its principle term explicitly.

**Remark 2.2.** Theorem 2.1 suggests us inequalities (1.8) and (1.9) are best possible in some sense. The reason for this is the following. From (2.12) for $\gamma \equiv 1$, roughly speaking, we have

$$\frac{\partial v}{\partial \nu} \sim \tau v \quad \text{on} \quad \Lambda_{\partial D}(p)$$

and thus we can expect

$$J(\tau) \sim \tau \int_{\partial D} (1 - \gamma)v^2 dS$$

and also

$$\frac{1}{\tau} \int_{\partial D} \frac{1}{\gamma} \left| \frac{\partial v}{\partial \nu} - \tau \gamma v \right|^2 dS \sim \tau \int_{\partial D} \frac{1 - \gamma}{\gamma} v^2 dS.$$

Hence we can expect

$$J(\tau) + \frac{1}{\tau} \int_{\partial D} \frac{1}{\gamma} \left| \frac{\partial v}{\partial \nu} - \tau \gamma v \right|^2 dS \sim \tau \int_{\partial D} \left\{ (1 - \gamma) + \frac{(1 - \gamma)^2}{\gamma} \right\} v^2 dS.$$

On the other hand, we can expect also

$$\int_{\partial D} \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) \frac{1 - \gamma}{1 + \gamma} v dS \sim \tau \int_{\partial D} \frac{(1 - \gamma)^2}{1 + \gamma} v^2 dS.$$

Thus from (2.3), (2.4) and (2.7) we can expect

$$I_B(\tau) \sim \tau \int_{\partial D} \left\{ (1 - \gamma) + \frac{(1 - \gamma)^2}{1 + \gamma} \right\} v^2 dS.$$

Then, we see that (1.8) and (1.9) correspond to the following trivial inequalities

$$1 - \gamma \leq (1 - \gamma) + \frac{(1 - \gamma)^2}{1 + \gamma} \leq (1 - \gamma) + \frac{(1 - \gamma)^2}{\gamma}.$$
3 Proof of Theorem 2.1

We denote by \( x^r \) the reflection across \( \partial D \) of the point \( x \in \mathbb{R}^3 \setminus D \) with \( d_{\partial D}(x) < 2\delta_0 \) for a sufficiently small \( \delta_0 > 0 \). It is given by \( x^r = 2q(x) - x \), where \( q(x) \) denotes the unique point on \( \partial D \) such that \( d_{\partial D}(x) = |x - q(x)| \). Note that \( q(x) \) is \( C^2 \) for \( x \in \mathbb{R}^3 \setminus D \) with \( d_{\partial D}(x) < 2\delta_0 \). The function \( \tilde{\gamma} \) is \( C^2 \) therein and coincides with \( \gamma(x) \) for \( x \in \partial D \). It is easy to see that \( \partial \tilde{\gamma} / \partial \nu = 0 \) on \( \partial D \).

Choose a cutoff function \( \phi_\delta \in C^2(\mathbb{R}^3) \) with \( 0 < \delta < \delta_0 \) which satisfies \( 0 \leq \phi_\delta(x) \leq 1; \phi_\delta(x) = 1 \) if \( d_{\partial D}(x) < \delta; \phi_\delta(x) = 0 \) if \( d_{\partial D}(x) > 2\delta; |\nabla \phi_\delta(x)| \leq C\delta^{-1}; |\nabla^2 \phi_\delta(x)| \leq C\delta^{-2} \).

Define
\[
R_0(x) = \frac{1 - \tilde{\gamma}(x)}{1 + \tilde{\gamma}(x)} \phi_\delta(x)v(x^r).
\]

Since we have
\[
v(x^r) = v(x), \quad \frac{\partial}{\partial \nu} v(x^r) = -\frac{\partial v}{\partial \nu}(x) \quad \text{on} \ \partial D,
\]
we obtain
\[
\frac{\partial R_0}{\partial \nu} = \frac{\partial}{\partial \nu} \left( \frac{1 - \tilde{\gamma}}{1 + \tilde{\gamma}} \right) v - \frac{1 - \gamma \partial v}{1 + \gamma \partial v} = \frac{-1 - \gamma \partial v}{1 + \gamma \partial v}
\]
and thus
\[
\frac{\partial R_0}{\partial \nu} - \tau \gamma R_0 = -\frac{1 - \gamma \partial v}{1 + \gamma \partial v} - \tau \gamma \frac{1 - \gamma}{1 + \gamma} v.
\]  \tag{3.1}

Now define
\[
R_1 = R - R_0.
\]

A combination of (3.1) and the boundary condition in (2.1) gives
\[
\frac{\partial R_1}{\partial \nu} - \tau \gamma R_1 = - \left( \frac{\partial v}{\partial \nu} - \tau v \right) + \frac{1 - \gamma \partial v}{1 + \gamma \partial v} + \tau \gamma \frac{1 - \gamma}{1 + \gamma} v + e^{-\tau T} G
\]
\[
= -\frac{2\gamma}{1 + \gamma} \left( \frac{\partial v}{\partial \nu} - \tau v \right) + e^{-\tau T} G,
\]
that is
\[
\frac{\partial R_1}{\partial \nu} - \tau \gamma R_1 = -\frac{2\gamma}{1 + \gamma} \left( \frac{\partial v}{\partial \nu} - \tau v \right) + e^{-\tau T} G. \tag{3.2}
\]

Define
\[
v_1 = \frac{2\gamma}{1 + \gamma} \left( \frac{\partial v}{\partial \nu} - \tau v \right). \tag{3.3}
\]
It follows from (2.1) and (3.2) that \( R_1 \) satisfies
\[
\begin{align*}
(\Delta - \tau^2)R_1 &= -(\Delta - \tau^2)R_0 + e^{-\tau T}F & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\
\frac{\partial R_1}{\partial \nu} - \tau \gamma R_1 &= -v_1 + e^{-\tau T}G & \text{on } \partial D.
\end{align*}
\] (3.4)

It follows from (2.1) and (2.5) that
\[
E(\tau) = \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) RdS - e^{-\tau T} \left( \int_{\partial D} GRdS + \int_{\mathbb{R}^3 \setminus \overline{D}} FRd\mathbf{x} \right)
\]

and thus
\[
E(\tau) = \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) \frac{1 - \gamma}{1 + \gamma} v dS
\]
\[
+ \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) R_1 dS - e^{-\tau T} \left( \int_{\partial D} GRdS + \int_{\mathbb{R}^3 \setminus \overline{D}} FRd\mathbf{x} \right).
\] (3.5)

In Lemma 2.1 and (2.28) in [6] we have already known that, as \( \tau \to \infty \)
\[
\begin{align*}
\| R \|_{L^2(\mathbb{R}^3 \setminus \overline{D})} &= O(e^{-\tau \text{dist}(D,B)} + e^{-\tau T}), \\
\| \nabla R \|_{L^2(\mathbb{R}^3 \setminus \overline{D})} &= O(\tau(e^{-\tau \text{dist}(D,B)} + e^{-\tau T})), \\
\| R \|_{L^2(\partial D)} &= O(\tau^{1/2}(e^{-\tau \text{dist}(D,B)} + e^{-\tau T})).
\end{align*}
\]

Noting also the form (2.2) of \( F \) and \( G \) and the solution class of (1.1) (see [6]), we see that
(3.5) becomes
\[
E(\tau) = \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) \frac{1 - \gamma}{1 + \gamma} v dS
\]
\[
+ \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - \tau \gamma v \right) R_1 dS + \tau e^{-\tau T}(e^{-\tau \text{dist}(D,B)} + e^{-\tau T}).
\] (3.6)

Now we study the asymptotic behaviour of \( \| R_1 \|_{L^2(\partial D)} \) as \( \tau \to \infty \). Multiplying both sides of the first equation on (3.4) with \( R_1 \) and applying integration by parts, we obtain
\[
\int_{\mathbb{R}^3 \setminus \overline{D}} (|\nabla R_1|^2 + \tau^2 |R_1|^2) dx + \tau \int_{\partial D} \gamma |R_1|^2 dS
\]
\[
+ e^{-\tau T} \left( \int_{\partial D} GR_1 dS + \int_{\mathbb{R}^3 \setminus \overline{D}} FR_1 dS \right)
\]
\[
= \int_{\partial D} v_1 R_1 dS + \int_{\mathbb{R}^3 \setminus \overline{D}} (\Delta - \tau^2)R_0 \cdot R_1 d\mathbf{x}.
\]
Rewrite this as

\[
\int_{\mathbb{R}^3 \setminus D} \left( 2|\nabla R_1|^2 + 2\tau^2 \left| R_1 + \frac{e^{-\tau T} F}{2\tau^2} \right|^2 \right) \, dx \\
+ 2\tau \int_{\partial D} \gamma \left| R_1 - \frac{1}{2\tau \gamma} (v_1 - e^{-\tau T} G) \right|^2 \, dS \\
= \frac{e^{-2\tau T}}{2\tau^2} \int_{\mathbb{R}^3 \setminus D} |F|^2 \, dx + \frac{1}{2\tau} \int_{\partial D} \frac{1}{\gamma} |v_1 - e^{-\tau T} G|^2 \, dS \\
+ 2 \int_{\mathbb{R}^3 \setminus D} (\Delta - \tau^2) R_0 \cdot R_1 \, dx.
\]  

(3.7)

Since we have

\[
\left\{ \begin{array}{l}
\tau^2 |R_1|^2 \leq 2\tau^2 \left( R_1 + \frac{e^{-\tau T}}{2\tau^2} \right)^2 + \frac{e^{-2\tau T}|F|^2}{2\tau^2} , \\
\tau \gamma |R_1|^2 \leq 2\tau \gamma \left| R_1 - \frac{1}{2\tau \gamma} (v_1 - e^{-\tau T} G) \right|^2 + \frac{1}{2\tau \gamma} |v_1 - e^{-\tau T} G|^2 ,
\end{array} \right.
\]

it follows from (3.7) that

\[
E_1(\tau) \leq \frac{e^{-2\tau T}}{\tau^2} \int_{\mathbb{R}^3 \setminus D} |F|^2 \, dx + \frac{1}{\tau} \int_{\partial D} \left| v_1 - e^{-\tau T} G \right|^2 \, dx \\
+ 2 \int_{\mathbb{R}^3 \setminus D} (\Delta - \tau^2) R_0 \cdot R_1 \, dx,
\]

(3.8)

where

\[
E_1(\tau) = \int_{\mathbb{R}^3 \setminus D} (|\nabla R_1|^2 + \tau^2 |R_1|^2) \, dx + \tau \int_{\partial D} \gamma |R_1|^2 \, dS.
\]

Here we make use of a rough estimate:

\[
|v_1 - e^{-\tau T} G|^2 \leq 2(|v_1|^2 + e^{-2\tau T}|G|^2).
\]

Then (3.8) gives

\[
E_1(\tau) \leq \frac{1}{\tau} \int_{\partial D} \frac{2}{\gamma} |v_1|^2 \, dx + 2 \int_{\mathbb{R}^3 \setminus D} (\Delta - \tau^2) R_0 \cdot R_1 \, dx + O(e^{-2\tau T}).
\]

(3.9)

Here we claim

**Lemma 3.1.** If \( \delta = \tau^{-1/2} \), then we have

\[
\left| \int_{\mathbb{R}^3 \setminus D} (\Delta - \tau^2) R_0 \cdot R_1 \, dx \right| \leq C\tau^{-1/2} J_0(\tau)^{1/2} E_1(\tau)^{1/2},
\]

(3.10)
where $C$ is a positive constant and

$$J_0(\tau) = \int_{\partial D} \frac{\partial v}{\partial \nu} \, dS = \int_{D} (|\nabla v|^2 + \tau^2 |v|^2) \, dx.$$  

For the proof of (3.10) see Appendix. Let us go ahead. A combination of (3.9) and (3.10) gives

$$|E_1(\tau)|^{1/2} - C^{1/2} J_0(\tau)^{1/2} \leq \frac{1}{\tau} \int_{\partial D} \frac{2}{\gamma} |v_1|^2 \, dx + C^2 \tau^{-1} J_0(\tau) + O(e^{-2\tau T}).$$

Since

$$|E_1(\tau)|^{1/2} - C^{1/2} J_0(\tau)^{1/2} + C^2 \tau^{-1} J_0(\tau) \geq \frac{1}{2} E_1(\tau),$$

we finally obtain

$$E_1(\tau) \leq \frac{1}{\tau} \int_{\partial D} \frac{4}{\gamma} |v_1|^2 \, dx + 4C^2 \tau^{-1} J_0(\tau) + O(e^{-2\tau T})$$

and hence

$$\tau^2 \int_{\partial D} |R_1|^2 \, dS \leq C \int_{\partial D} \left( |v_1|^2 + \frac{\partial v}{\partial \nu} \right) \, dS + O(\tau e^{-2\tau T}).$$

Note that we have made use of the fact that $\gamma$ has a positive lower bound.

From (2.8) and (2.11) we have

$$\lim_{\tau \to \infty} e^{2r_{\partial D}(p)} \int_{\partial D} \frac{\partial \tilde{v}}{\partial \nu} \, dS = \frac{\pi}{d_{\partial D}(p)^2} \sum_{q \in \Lambda_{\partial D}(p)} \frac{1}{\det (S_q (\partial B_{\partial D}(p)) - S_q (\partial D))}.$$  

Since $(p - x)/|x - p| = 1$ on $\Lambda_{\partial D}(p)$, it follows from (2.8) and (2.12) for $\gamma \equiv 1$ that

$$\lim_{\tau \to \infty} \tau^{-1} e^{2r_{\partial D}(p)} \int_{\partial \Omega} \left| \frac{\partial \tilde{v}}{\partial \nu} - \tau \tilde{v} \right|^2 \, dS = 0.$$  

Since we have (2.9) and (3.3), it follows from (3.11) that

$$\left( \frac{\tau^3}{\varphi(\tau \eta)} \right)^2 \tau \int_{\partial D} |R_1|^2 \, dS$$

$$\leq C \left( \sup_{x \in \partial D} \frac{2\gamma}{1 + \gamma} \int_{\partial D} \left| \frac{\partial \tilde{v}}{\partial \nu} - \tau \tilde{v} \right|^2 \, dS + \tau^{-1} \int_{\partial D} \frac{\partial \tilde{v}}{\partial \nu} \, dS \right) + \left( \frac{\tau^3}{\varphi(\tau \eta)} \right)^2 O(\tau e^{-2\tau T}).$$
Now applying (3.12) and (3.13) to this right-hand side and then, using (2.10), we obtain
\[
\lim_{\tau \to \infty} \tau^5 e^{2\tau \text{dist}(D,B)} \| R_1 \|_{L^2(\partial D)}^2 = 0. \tag{3.14}
\]

Now we study the asymptotic behaviour of the second term on (3.6).

We have
\[
\left| \int_{\partial D} \left( \frac{\partial v}{\partial v} - \tau \gamma v \right) R_1 dS \right| \left| \int_{\partial D} \left( \frac{\partial v}{\partial v} - \tau \gamma v \right) \frac{1 - \gamma}{1 + \gamma} v dS \right| \leq \tau^{3/2} e^{\gamma \text{dist}(D,B)} \left\| \frac{\partial v}{\partial v} - \tau \gamma v \right\|_{\partial D} \cdot \tau^{5/2} e^{\gamma \text{dist}(D,B)} \| R_1 \|_{L^2(\partial D)} \tag{3.15}
\]

Again from (2.8) and (2.12) we obtain
\[
\lim_{\tau \to \infty} \tau^{-1} e^{2\tau d_{\partial D}(p)} \int_{\partial D} \left| \frac{\partial \tilde{v}}{\partial v} - \tau \gamma \tilde{v} \right|^2 dS
\]
\[
= \frac{\pi}{d_{\partial D}(p)^2} \sum_{q \in \Lambda_{\partial D}(p)} (1 - \gamma(q))^2 \sqrt{\det S_q(\partial B_{d_{\partial D}(p)}(p) - S_q(\partial D))}
\]
and hence (2.10) gives
\[
\lim_{\tau \to \infty} \tau^3 e^{2\tau \text{dist}(D,B)} \left\| \frac{\partial v}{\partial v} - \tau \gamma v \right\|_{L^2(\partial D)}^2
\]
\[
= \frac{\pi}{4} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \sum_{q \in \Lambda_{\partial D}(p)} (1 - \gamma(q))^2 \sqrt{\det (S_q(\partial B_{d_{\partial D}(p)}(p) - S_q(\partial D))}
\]

Applying this, (2.15) and (3.14) to (3.15), we obtain
\[
\lim_{\tau \to \infty} \frac{\int_{\partial D} \left( \frac{\partial v}{\partial v} - \tau \gamma v \right) R_1 dS}{\int_{\partial D} \left( \frac{\partial v}{\partial v} - \tau \gamma v \right) \frac{1 - \gamma}{1 + \gamma} v dS} = 0. \tag{3.16}
\]

Note also that, if \( T \) satisfies (2.6), then, as \( \tau \to \infty \)
\[
\tau^4 e^{2\tau \text{dist}(D,B)} \cdot \tau e^{-\tau T} (e^{-\tau \text{dist}(D,B)} + e^{-\tau T})
\]
\[
= \tau^5 e^{-\tau T} (T - \text{dist}(D,B)) + \tau^5 e^{-2\tau T} (T - \text{dist}(D,B)) \to 0.
\]

Now applying this, (2.15), (3.16) to (3.6), we obtain (2.7).
4 Conclusions and further problems

In this paper we gave a remark on the previous application [6] of the enclosure method to an inverse obstacle problem using the wave governed by the wave equation. We believe that the argument developed in this paper is quite simple and shall be a prototype for applications to other inverse obstacle problems.

By the way, recently the enclosure method in the time domain has been applied also to the Maxwell system [9, 10]. In particular, in [10] we have already obtained a result corresponding to Theorem 1.1. It is assumed that the electromagnetic field as the solution of the Maxwell system satisfies the Leontovich boundary condition on the surface of an unknown obstacle. The Leontovich boundary condition is described by a single positive function defined on the surface. Thus, it would be interesting to find a formula for the function similar to (1.11) in Theorem 1.2. This belongs to our next project.

And also developing several applications of the enclosure method in the time domain to transmission problems for other wave equations and systems should be expected. See [8] for a survey on recent results obtained by using the enclosure method.

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5 Appendix. Proof of Lemma 3.1

This is an application of a reflection argument developed in [11]. First of all we compute $(\triangle - \tau^2)R_0$. Define

$$\tilde{\phi}_\delta(x) = \frac{1 - \tilde{\gamma}(x)}{1 + \tilde{\gamma}(x)} \phi_\delta(x).$$

Using (4.15) in [11] (see also [7]), we have

$$(\triangle - \tau^2)R_0 = \tilde{\phi}_\delta(x) \sum_{i,j} d_{\partial D}(x)a_{ij}(x)(\partial_i \partial_j v)(x^r)$$

$$+ \sum_j \left( \sum_k b_{jk}(x)(\partial_{x_k} \tilde{\phi}_\delta)(x) + d_j(x)\tilde{\phi}_\delta(x) \right) (\partial_j v)(x^r)$$

$$+ (\Delta \tilde{\phi}_\delta)(x)v(x^r),$$

where $a_{ij}(x)$, $b_{jk}(x)$ and $d_j(x)$ with $i, j, k = 1, 2, 3$ are independent of $\phi_\delta(x)$, $v$ and $\tau$; $a_{ij}(x)$ and $b_{jk}(x)$ are $C^1$ and $d_j(x)$ is $C^0$ for $x \in \mathbb{R}^3 \setminus D$ with $d_{\partial D}(x) < 2\delta_0$.

Thus we have to study three integrals:

$$I = \sum_{i,j} \int_{\mathbb{R}^3 \setminus D} \tilde{\phi}_\delta(x)d_{\partial D}(x)a_{ij}(x)(\partial_i \partial_j v)(x^r) \cdot R_1(x)dx,$$
\[ J = \sum_{j,k} \int_{\mathbb{R}^3 \setminus D} b_{jk}(x)(\partial_k \tilde{\phi}_\delta)(x)(\partial_j v)(x^r) \cdot R_1(x) dx \]
\[ + \sum_j \int_{\mathbb{R}^3 \setminus D} d_{jk}(x) \tilde{\phi}_\delta(x)(\partial_j v)(x^r) R_1(x) dx. \]

and
\[ K = \int_{\mathbb{R}^3 \setminus D} (\Delta \tilde{\phi}_\delta)(x) v(x^r) \cdot R_1(x) dx. \]

In what follows, the symbol \( C \) denotes several constants independent of \( \delta \) and \( \tau \).

### 5.1 Estimating \( I \)

The change of variables \( x^r = y \) yields
\[ I = \sum_{i,j} \int_D \tilde{\phi}_\delta(y^r) d_{\partial D}(y^r) a_{ij}(y^r)(\partial_i \partial_j v)(y) \cdot R_1(y^r) J(y) dy, \]
where \( J(y) \) denotes the Jacobian. Since \( d_{\partial D}(y^r) = 0 \) on \( \partial D \), integration by parts yields
\[ I = - \sum_{i,j} \int_D \partial_i(\tilde{\phi}_\delta(y^r) d_{\partial D}(y^r) a_{ij}(y^r) R_1(y^r) J(y))(\partial_j v)(y) dy. \]

Set
\[ R_1'(y) = R_1(y^r) \]

and
\[ D_\delta = \{ y \in D \mid d_{\partial D}(y) < 2\delta \}. \]

Noting \( d_{\partial D}(y^r) = d_{\partial D}(y) \), we can easily obtain
\[ |I| \leq C(\| R_1' \|_{L^2(D_\delta)} + \delta \| \nabla R_1' \|_{L^2(D_\delta)}) \| \nabla v \|_{L^2(D_\delta)}. \]
\[ (A.1) \]

### 5.2 Estimating \( J \)

Since
\[ J = \sum_{j,k} \int_D b_{jk}(y^r)(\partial_k \tilde{\phi}_\delta)(y^r) \partial_j v(y) \cdot R_1(y^r) J(y) dy \]
\[ + \sum_j \int_D d_{jk}(y^r) \tilde{\phi}_\delta(y^r) \partial_j v(y) R_1(y^r) J(y) dy, \]
we have
\[ |J| \leq C \delta^{-1} \| R_1' \|_{L^2(D_\delta)} \| \nabla v \|_{L^2(D_\delta)}. \]
\[ (A.2) \]

### 5.3 Estimating \( K \)

Since
\[ K = \int_D (\Delta \tilde{\phi}_\delta)(y^r) v(y) \cdot R_1(y^r) J(y) dy, \]
we have
\[ |K| \leq C \delta^{-2} \| R_1' \|_{L^2(D_\delta)} \| v \|_{L^2(D_\delta)}. \]
\[ (A.3) \]
Summing up (A.1)-(A.3), we obtain
\[
\left| \int_{\mathbb{R}^3 \setminus \overline{D}} (\Delta - \tau^2) R_0 \cdot R_1 \, dx \right| \leq C \left( \delta \| \nabla R_1^r \|_{L^2(D_0)} \| \nabla v \|_{L^2(D)} + \delta^{-1} \| R_1^r \|_{L^2(D_0)} \| \nabla v \|_{L^2(D)} + \delta^{-2} \| R_1^r \|_{L^2(D_0)} \| v \|_{L^2(D)} \right). \tag{A.4}
\]

Here note that
\[
\begin{cases}
\| \nabla v \|_{L^2(D)} \leq J_0(\tau)^{1/2}, \\
\| v \|_{L^2(D)} \leq \tau^{-1} J_0(\tau)^{1/2}
\end{cases} \tag{A.5}
\]
and
\[
\begin{cases}
\| \nabla R_1^r \|_{L^2(D_0)} \leq CE_1^0(\tau)^{1/2}, \\
\| R_1^r \|_{L^2(D_0)} \leq \tau^{-1} E_1^0(\tau)^{1/2},
\end{cases} \tag{A.6}
\]
where
\[
E_1^0(\tau) = \int_{\mathbb{R}^3 \setminus \overline{D}} (|\nabla R|^2 + \tau^2 |R|^2) \, dx.
\]

Applying (A.5) and (A.6) to the right-hand side of (A.4), we obtain
\[
\left| \int_{\mathbb{R}^3 \setminus \overline{D}} (\Delta - \tau^2) \tilde{v} \cdot \epsilon \, dx \right|
\leq C(\delta + \delta^{-1} \tau^{-1} + \delta^{-2} \tau^{-2})(J_0(\tau)E_1^0(\tau))^{1/2}.
\]

Therefore, choosing \( \delta = \tau^{-1/2} \), we obtain
\[
\left| \int_{\mathbb{R}^3 \setminus \overline{D}} (\Delta - \tau^2) R_0 \cdot R_1 \, dx \right| \leq C\tau^{-1/2}(J_0(\tau)E_1^0(\tau))^{1/2}.
\]

Since \( E_1(\tau) \geq E_1^0(\tau) \), this yields the desired estimate.

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