A GENERAL THEOREM ON TEMPORAL FOLIATION OF CAUSAL SETS

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Abstract. We show a general theorem of existence of temporal foliations in a general causal set, under mild constraints. Then we study automorphisms of infinite causal sets (which satisfy further requirements) and show that they fall under one of two types:

1) Automorphisms that induce automorphisms of spacelike hypersurfaces in some given foliation (i.e. spacelike automorphisms), or
2) Translation in time.

These results might be useful for quantization of the aforementioned causal sets.

Introduction

The fact that the absence of closed time like curves (CTC) in a Lorentzian manifolds implies (together with an extra assumption) the existence of a foliation (or slicing) of the manifold into spacelike hypersurfaces is not trivial. For a causal set (see e.g. [B], for the definition of a causal set), the analogous result is implicitly assumed to hold, although no proof is given.

One of the reasons of this fact is the (hidden) assumption in (most) causal set models of space-time, namely that they are finite, in which case a temporal foliation clearly exist. In this paper we show that every causal set admits a foliation by spacelike slices.

The paper is organized as follows: in section one we set our notations and lay down the formulation of the problem. In sections two and three we state the main results and their proofs, then we lay our conclusions.

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1. Causal spaces

In the following, we call (a generalized) causal space a nonempty set \( \mathcal{M} \) of events endowed with a causality relation, that is, a partial order on \( \mathcal{M} \), noted \( \prec \). This order is reflexive, transitive and antisymmetric binary relation on \( \mathcal{M} \). An important fact about this definition is that it excludes the possibility of CTC’s (closed timelike curves) which would violate the antisymmetry of \( \prec \).

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Here we do not put restrictions on the cardinality of $\mathcal{M}$. Later, we will treat separately the discrete and continuum cases, respectively. The discrete case is characterized by the following fact:

For all $x \in \mathcal{M}$, if there exists $y \in \mathcal{M}$, $y \neq x$ such that $x \prec y$ then there exists a $z \in \mathcal{M}$, $z \neq x$ such that $\forall t \in \mathcal{M}$, $x \prec t, t \prec z$, and $x \neq t \Rightarrow z = t$.

A causal space can be of any cardinality, including the cardinality of the continuum.

2. Temporal foliation

Let us define Temporal foliation. The symbols $\neg, \wedge, \vee$ denote respectively the logical operators: not, and, or.

2.1. Definition. Let $\mathcal{M}$ be a partially ordered set (POSET). A temporal foliation of $\mathcal{M}$ is a partition of $\mathcal{M}$ in sets $X_i, i \in I$ (called spacelike slices), ($I \neq \emptyset$ is a totally ordered set by $<\rangle$) such that:

i) $\mathcal{M} = \bigcup_{i \in I} X_i$.

ii) $(\forall i \in I)(\forall x \in X_i)(\forall y \in X_i)((x \prec y) \wedge (y \prec x))$.

iii) $(\forall i, j \in I, i < j, i \neq j)\{(\exists x \in X_i)(\exists y \in X_j)(x \prec y) \wedge (\forall z \in X_i)(\forall t \in X_j)(\neg (t \prec z))\}$.

Condition (ii) expresses that each $X_i$ is an antichain. The last condition says that each $X_i$ precedes all $X_j$ for all $i < j, i \neq j$.

In the case of Minkowski spacetime, the set $I$ is $\mathbb{R}$, and the $X_i$ are then isomorphic to $\mathbb{R}^3$. In the general case the $X_i$ correspond to Cauchy hypersurfaces, and in the causal set terminology we shall refer to the $X_i$ by spacelike slices.

By a chain in a causal space (or a poset) $\mathcal{M}$ we mean a subset $C$ which is linearly ordered by the order induce from $\mathcal{M}$. Similarly, by an antichain we mean a subset $A$ of $\mathcal{M}$ such that any two elements $x, y \in A$ we have $\neg (x \prec y)$ and $\neg (y \prec x)$.

A subset $Y$ of a poset $\mathcal{M}$ is bounded if there exists an element $a \in \mathcal{M}$ such that for all $x \in Y$ we have $x \prec a$. In this case $a$ will be called a majorant of $Y$.

A maximal element of $Y$ (as above) is an element $b \in Y$ such that for every $x \in Y$ if $b \prec x$ then $x = b$.

Given an element $x$ in a poset $\mathcal{M}$ and a subset $X$ of $\mathcal{M}$, we say that $x$ is said to be incomparable to $X$ if and only if for every element $y \in X$ we have $\neg (x \prec y)$ and $\neg (y \prec x)$, i.e. $x$ and $y$ are incomparable. $x$ and $X$ are said to be comparable otherwise, i.e. there exists at least one $y \in X$ such that $x$ and $y$ are comparable.

We recall the following well known result:

2.2. Theorem. (ZORN LEMMA) Let $X$ be any partially ordered set such that any bounded chain $C$ in $X$ has a maximal element.

This result is known to be equivalent to the axiom of choice, which we assume.

Let us state our first result:
2.3. Theorem. Consider an antichain \( X_c \) in \( \mathcal{M} \). Then there exists a temporal foliation of \( \mathcal{M} \) containing such that

\[ (*) \quad \text{\( X_c \) is contained in one of the spacelike slices of the foliation.} \]

Proof. By a partial foliation we mean a set of spacelike slices (not necessarily covering the whole universe \( \mathcal{M} \)) satisfying axioms (ii) and (iii) of definition 2.1. Let us denote by \( \mathcal{X} \) the set of all partial foliations satisfying the condition (*)\). \( \mathcal{X} \) is non-empty since the partial foliation \( F_0 := \{ X_c \} \) consisting of just one spacelike slice \( X_c \) belongs to \( \mathcal{X} \). Now consider the set of all partial foliations containing a spacelike hypersurface (hereafter a slice) \( X \supset X_c \). Given two partial foliations \( F_1 \) and \( F_2 \) we say that \( F_1 \sqsubseteq F_2 \) if every spacelike slice from \( F_1 \) is contained in one spacelike slice from \( F_2 \), more precisely, we have

For all \( X \in F_1 \), there exists one \( Y \in F_2 \) such that \( X \subset Y \).

Clearly \( \sqsubseteq \) is a partial order on \( \mathcal{X} \). Any totally ordered subset \( \mathcal{Y} \) of \( \mathcal{X} \) is bounded from above by \( F_{\text{sup}} \), where \( F_{\text{sup}} \) is such that:

\[ X \in F_{\text{sup}} \text{ iff there exists an chain (for inclusion) } (X_Y)_{Y \in \mathcal{Y}} \text{ such that } X = \bigcup_{Y \in \mathcal{Y}} X_Y. \]

Then it is easy to check that \( F_{\text{sup}} \) is indeed a majorant of \( \mathcal{Y} \).

It follows from Zorn’s lemma that \( \mathcal{X} \) admits a maximal element \( F_{\text{max}} \). We show that \( \mathcal{F} \) satisfies the requirements of the theorem. Actually axioms (ii) and (iii) hold by our hypothesis (since \( F_{\text{max}} \) is an element of \( \mathcal{X} \)). It remains to prove that \( F_{\text{max}} \) satisfies axiom (i).

Assume not, so there exists some \( x \in \mathcal{M} \) that is not contained in any spacelike slice of the foliation \( F_{\text{max}} \). Two cases to be considered:

Case1: \( x \) is incomparable to all spacelike slices of the foliation \( F_{\text{max}} \). Then adding \( x \) to any slice produces a partial foliation strictly larger than \( F_{\text{max}} \) (in the order relation \( \sqsubseteq \)) which contradicts our assumption that \( F_{\text{max}} \) is a maximal element of \( \mathcal{X} \).

Case2: \( x \) is comparable to at least one spacelike slice of \( F_{\text{max}} \). Let \( F_{\text{max}^-} \) be the set of spacelike slices from \( F_{\text{max}} \) which have predecessors to \( x \), i.e. for each \( X \in F_{\text{max}^-} \) there exists a \( y \in X \) such that \( y \prec x \). Also, let \( F_{\text{max}^+} \) be the set of spacelike slices from \( F_{\text{max}} \) which have successors to \( x \). If there exists any spacelike slice in \( F_{\text{max}} \) which lies strictly between slices in \( F_{\text{max}^-} \) and \( F_{\text{max}^+} \), then we add \( x \) to any of them, thus producing a partial foliation strictly larger than \( F_{\text{max}} \). If there is no such slice, we may add the slice \( \{ x \} \) to \( F_{\text{max}} \), (such that, \( \{ x \} \) lies between the slices in \( F_{\text{max}^-} \) and \( F_{\text{max}^+} \) respectively) also producing a partial foliation larger than \( F_{\text{max}} \). In all cases, we get a contradiction.

\[ \square \]

3. Automorphisms of infinite causal sets

3.1. Some assumptions. We want our causal sets to imitate well behaved space-times. As a first approximation we will consider that our causal sets models some spacetime which is very similar to Minkowski. Consideration of more general space-times will be considered in a future work.
Consider a non empty poset (or equivalently a causal space) \( M \), equipped with a partial order (or equivalently a causal relation) \( \prec \). Then for any \( x \in M \), we denote by Past(\( x \)) the set \( \text{Past}(x) := \{ y \in M | y \prec x \} \). Similarly by Future(\( x \)) we denote the following set \( \text{Future}(x) := \{ y \in M | x \prec y \} \).

Recall that a causal set \( C \) is a locally finite partially ordered set (Poset). By local finiteness we mean that for any \( x, y \in C \) we have \( \text{Past}(y) \cap \text{Future}(y) \) is finite. Let \( C \) be an infinite countable causal set (i.e. \( \text{Card}(C) = \aleph_0 \)). Then we say that \( C \) is a well behaved causal set if and only if:

(a) Let \( C \subset C \) be any maximal chain (where by a maximal chain we mean a chain which is maximal with respect to the inclusion relation among all chains of \( C \)) and \( A \subset C \) be any antichain. Then

\[
A \subset \bigcup_{x \in C} (\text{Past}(x) \cup \text{Future}(x)).
\]

(b) For any \( x \in C \) and any antichain \( A \), we have both:

\( \text{Card}(\text{Past}(x) \cap A) < \infty \) and \( \text{Card}(\text{Future}(x) \cap A) < \infty \).

(c) \( C \) contains an infinite antichain.

### 3.2. Automorphisms

In this section we will use the theorem above in order to get new results about automorphisms about well behaved causal sets. We will use the results obtained in this section towards a new quantization scheme of (infinite) causal sets.

Let \( C \) be an infinite well behaved causal set, with a causality relation denoted by \( \prec \). By an automorphism of a poset, or more specifically a causal set \( C \) (with the causal relation denoted by \( \prec \)), we mean a bijective map \( \Phi : C \rightarrow C \) such that for all \( x, y \in C \) we have: \( x \prec y \) iff \( f(x) \prec f(y) \).

Our goal is to show the following:

### 3.3. Theorem

Let \( C \) and \( \Phi \) be as above. There exists a foliation \( F \) of \( C \) such that either:

(a) For every spacelike slice \( X \) of \( F \), \( \Phi(X) = X \), (so \( \Phi \) is an automorphism acting inside each spacelike slice of \( F \)), or

(b) There exists a chain \( C \subset C \) and an integer \( k \neq 0 \) such that for every \( x \in C \), \( x \) and \( \Phi^k(x) \) are related (i.e. comparable), and for every spacelike slice \( X \subset C \) in \( F \), there exists a spacelike slice \( Y \) in \( F \) such that \( \Phi(X) = Y \), \( X \neq Y \).

**Proof.** There are two cases:

Case 1- \( \exists x \in C \) such that \( \Phi(x) \) and \( x \) are related, i.e. \( x \prec \Phi(x) \) or \( \Phi(x) \prec x \).

Case 2- \( \forall x \in C \), \( \Phi(x) \) and \( x \) are unrelated.

First, we consider case 1:

\( \exists x \in C \) such that \( \Phi(x) \prec x \) or \( x \prec \Phi(x) \). Let \( x_0 \) be such an element, and assume for definiteness that \( x_0 \prec \Phi(x_0) \), the other case being completely similar. It follows easily that the orbit of \( x_0 \) (i.e. the set \( \{ \Phi^k(x) | k \in \mathbb{Z} \} \)) is a chain. Let \( X_0 := \{ x_0 \} \), and let \( F_0 \) be the partial foliation of \( C \) containing \( X_0 \) as its only spacelike slice.

Now let us consider the set \( \mathcal{E} \) of partial foliations \( F \) containing \( X_0 \), meeting furthermore the following conditions:
For every spacelike slice \( X \in F \), \( \forall x \in X, \Phi(x) \notin X \).  \((*)\)

For every spacelike slice \( X \) in \( F \) there exists one space like slice \( Y \neq X \) such that \( \Phi(X) = Y \).  \((**)\)

Clearly, \( \mathcal{E} \) is non empty because \( F_0' \), the partial foliation of \( \mathcal{C} \) whose elements are \( \{\Phi^k(x)\}, k \in \mathbb{Z} \) is in \( \mathcal{E} \).

Now let us define an order relation "\( \sqsubseteq \)" on \( \mathcal{E} \) such that \( F \sqsubseteq F' \) if each spacelike slice in \( F \) is contained in some spacelike slice in \( F' \).

Now it is easy to see, similarly to the proof of the theorem \ref{theo:2.3} that \( \mathcal{E} \) equipped with the partial order \( \sqsubseteq \) satisfies the conditions of Zorn’s Lemma. In fact, every \( \sqsubseteq \)-chain \( \mathcal{Y} \) of partial foliations \( (F_i)_{i \in I} \) is bounded above by the foliation \( F_{\sup} \) which satisfies the condition

\[
X \in F_{\sup} \iff \text{there exists an chain (for inclusion) } (X_F)_{F \in \mathcal{Y}} \text{ such that } X = \bigcup_{F \in \mathcal{Y}} X_F.
\]

We have further the following:

Claim: \( F_{\sup} \) satisfies further the conditions \((*)\) and \((**)\).

Proof of claim:

Assume that there exists a spacelike slice \( X \) in \( F_{\sup} \) such that:

\[
\exists x \in X(\Phi(x) \in X).
\]

Then, since \( X \) is of the form \( X = \bigcup_{F \in \mathcal{Y}} X_F \) for some chain (with respect to inclusion) \( (X_F)_{F \in \mathcal{Y}} \), there is some \( F \in \mathcal{Y} \) such that \( x \in X_F \). Also, since we assumed that \( \Phi(x) \in X \) too, there exists some \( F' \in \mathcal{Y} \) such that \( \Phi(x) \in X_{F'} \). Since \( \mathcal{Y} \) is a chain, we have either \( X_F \subset X_{F'} \) or \( X_{F'} \subset X_F \). At any rate, there exists then some \( F'' \in \mathcal{Y} \) such that \( x, \Phi(x) \in X_{F''} \), contradicting our assumption on \( F'' \) since \( F'' \in \mathcal{Y} \subset \mathcal{E} \).

Now we show that \( F_{\sup} \) satisfies further the condition \((**)\). Let \( X \) be a spacelike slice in \( F_{\sup} \) as before so \( X = \bigcup_{F \in \mathcal{Y}} X_F \) where we keep the same notation and assumptions. Then \( \Phi(X) = \bigcup_{F \in \mathcal{Y}} \Phi(X_F) \) and \( (\Phi(X_F))_{F \in \mathcal{Y}} \) is a chain for inclusion (clear). In fact, for every \( F \), \( \Phi(X_F) = Y_F \) where \( Y_F \neq X_F \) and \( Y_F \) is a spacelike slice in \( F \), hence since \( \mathcal{Y} \) is a chain for \( \sqsubseteq \), we have in fact that \( (Y_F)_{F \in \mathcal{Y}} \) is a chain for inclusion. So \( \Phi(X) = Y \) where \( Y = \bigcup_{F \in \mathcal{Y}} Y_F \) and \( Y \) is a spacelike slice of \( F_{\sup} \) so condition \((**)\) holds.

It follows from the above claim that \( F_{\sup} \) is in \( \mathcal{E} \), so applying Zorn’s Lemma we get that there must exist a maximal partial foliation \( F_{\max} \) in \( \mathcal{E} \).

Now we claim that \( F_{\max} \) is a foliation of \( \mathcal{C} \), i.e. in addition to (ii) and (iii) of \ref{theo:2.1} it satisfies (i).

Assume that \( F_{\max} \) is not a foliation, i.e. that there exists some \( x \in \mathcal{C} \) such that for no spacelike slice \( X \) in \( F_{\max} \) we have \( x \in X \). Let \( O(x) \) be the orbit of \( x \). Then we have two possibilities:

(a) \( O(x) \) meets some spacelike slice \( X \) of \( F_{\max} \), or

(b) \( O(x) \) does not meet any spacelike slice in \( F_{\max} \).

In case (a), we let \( y \) be one point of intersection of \( O(x) \) with some spacelike slice \( Y \) of \( F_{\max} \). So \( x = \Phi^k(y) \) for some \( k \in \mathbb{Z} \) and hence necessarily \( x \in \Phi^k(Y) \in F_{\max} \).
which contradicts the assumption that $x$ is not contained in any spacelike slice of $F_{\text{max}}$, so case (a) does not occur.

Now consider case (b). In this case we have two subcases:

1. There exist $(Y_k)_{k \in \mathbb{Z}}$ such that for some $y \in O(x)$ $y$ is incomparable to $Y_{k_0}$ (for some $k_0 \in \mathbb{Z}$) and $Y_{k_0}$ lies strictly between slices in $F_{\text{max}}^-$ and $F_{\text{max}}^+$ respectively, where $F_{\text{max}}^-$ is the set of slices in $F_{\text{max}}$ that precede $y$ (i.e. contain a predecessor to $y$) and $F_{\text{max}}^+$ is the set of slices in $F_{\text{max}}$ that succeed $y$, and $\Phi^j(y)$ is incomparable to $Y_{k_0+\ell}$ and $Y_{k+\ell} = \Phi^j(Y_k)$ for all $k, \ell \in \mathbb{Z}$. In this case it suffices to add $\Phi^j(y)$ to $Y_{k_0+\ell}$ for every $\ell \in \mathbb{Z}$ to obtain a new partial foliation $F$ strictly greater (in the order $\sqsubseteq$) than $F_{\text{max}}$, thus getting a contradiction.

2. Case (i) does not occur. In this case we have also two subcases:
   1. For some $y \in O(x)$ we have $y$ is comparable to every spacelike slice $Y$ in $F_{\text{max}}$. In this case we add the slice $\{y\}$ to the partial foliation $F_{\text{max}}$ in the same way as in the proof of Theorem 2.3, namely we add the slice $\{y\}$ in the cut defined by $(Y < \{y\}, \{y\} < Y)$ where $<$ is the strict order relation induced on the spacelike slices introduced in the proof of Theorem 2.3. The other elements of $O(x)$ will be added in the cuts accordingly, namely if $Y < \{y\} < Y'$ then $\Phi^k(Y') < \{\Phi^k(y)\} < \Phi^k(Y')$. This is possible since $\Phi$ is an automorphism.
   2. Case (ii-i) does not occur. If there exists slices $Y, Y'$ in $F_{\text{max}}$ such that $y < z < y'$ for some $y \in Y$, $y', Y'$ and $z \in O(x)$ and such that $Y < Y'$ and there is no intermediate slice (in $F_{\text{max}}$) between $Y$ and $Y'$, then we can still add $\{z\}$ to the $F_{\text{max}}$ and then add the other elements of $O(x)$ in the same way in the corresponding places, and get a partial foliation strictly greater that $F_{\text{max}}$, contradiction. Otherwise, let $Y, Y'$ be slices in $F_{\text{max}}$ and $z \in O(x)$ as above but assume now that there are slices intervening between them. In this case, we can still add $\{z\}$ between $Y$ and $Y'$ in any way, and we add the other element of $O(x)$ correspondingly. This is possible again since $\Phi$ is an automorphism.

In all cases we obtain a partial foliation which is strictly greater than $F_{\text{max}}$ contradicting the hypotheses. So $F_{\text{max}}$ is a foliation as required.

2. Now let us consider the second case where for every $x$ in $\mathcal{C}$, $x$ and $\Phi(x)$ are unrelated.

Here we have two subcases:

a) For every $x \in \mathcal{C}$, and for every integer $k \neq 0$, $x$ and $\Phi^k(x)$ are unrelated.

b) There exists some $x \in \mathcal{C}$, and some integer $k \neq 0, 1, -1$ such that $x$ and $\Phi^k(x)$ are related.

We consider case (a) first: In this case, we consider an $x_0 \in \mathcal{C}$ (since $\mathcal{C}$ is not empty). Let $O(x_0)$ be the orbit of $x_0$. Then $O(x_0)$ is an antichain by assumption. Then $F_0 := \{O(x_0)\}$ is a partial foliation of $\mathcal{C}$. Let $\mathcal{E}$ be the set of partial foliations $F = \{X_i | i \in I\}$ where $I \subseteq \mathbb{Z}$ such that

For every $i \in I$, all $x \in X_i$ we have $\Phi^k(x) \in X_i$, \quad (\dagger)

Then $\mathcal{E}$ is not empty since $F_0 \in \mathcal{E}$. Now we repeat the same steps in the proof of Theorem 2.3 to this case. More precisely, we have that any chain (for the relation
defined similarly to above) which is bounded above has a majorant, and hence we conclude using Zorn’s Lemma that $E$ has a maximal element $F_{\text{max}}$. Now assume again that $F_{\text{max}}$ is not a foliation, so there exists some $x \in C$ which is not in a spacelike slice from $F_{\text{max}}$.

Let again $O(x)$ be the orbit of $x$ under the action of $\Phi$. Similarly to the proof of the previous Theorem, we need to consider two cases, namely $x$ is comparable to some spacelike slice in $F_{\text{max}}$ or $x$ is not comparable to any spacelike slice from $F_{\text{max}}$.

In either case, it is possible to add all elements of $O(x)$ to some convenient slice of $F_{\text{max}}$ or add a new slice containing $O(x)$ in a similar way to the end of the proof of Theorem 2.3. The fact that $\Phi$ is an order automorphism allows such a procedure, hence we obtain a foliation strictly larger than $F_{\text{max}}$, contradiction.

Next we handle case (b).

Let $k_0 \neq 0, 1, -1$ be the least (in absolute value) integer such that, for some $x \in C$, $x < \Phi^{k_0}(x)$ or $\Phi^{k_0}(x) < x$, and let $x_0 \in C$ such a realization (i.e. $x_0 < \Phi^{k_0}(x_0)$ or $\Phi^{k_0}(x_0) < x_0$).

Using our assumptions on the causal set that we presented at the beginning of this section, we can still show the existence of a foliation $F$ of $C$ satisfying the following:

1. $F$ satisfies the axioms of foliation, namely (i), (ii) and (iii) of Definition 2.1.
2. $F$ satisfies the clause (b) of the Theorem.

Let again $E$ be the set of all partial foliations of $C$ satisfying (*) and (**) above. Again, we show that $E$ is not empty.

We consider the element $x_0$. By our hypotheses, $x_0 < \Phi^{k_0}(x_0), k_0 > 1$ (say) and for $0 < \ell < k_0 x_0$ and $\Phi^{\ell}(x_0)$ are unrelated.

First let us show the following claim:

**Claim:** Under case (b) above we have: for every $y \in C$, there exists some $k \in \mathbb{N}, k \neq 0$ such that $y < \Phi^k(y)$.

**Proof of claim.**

Assume there is some $y \in C$ such that $O(y)$ is an antichain. If $O(y) = \{y\}$ (i.e. $y$ is a fixed point), then in this case, for every $k$, $y < \Phi^k(y)$ trivially, and we are done. Otherwise, $O(y)$ is an infinite antichain, and $x_0 < z$ (or $z < x_0$) for some $z \in O(y)$, (by our assumptions on $C$). It suffices then to consider $\Phi^k(x_0)$ for arbitrarily large $k$ to conclude that $\Phi^k(x_0) < z_k$, for $z_k \in O(y)$. Then by assumption (a) on $C$ we have that $\text{Future}(x) \cap O(y)$ is in fact infinite, where $x \in O(x_0)$ such that $x$ and $x_0$ are comparable, (since by letting $\ell \to -\infty$, $x = \Phi^\ell(x_0)$ we get $\text{Card}($Future$(x) \cap O(y))$ larger than any positive integer, so as to respect assumption (a) on $C$). The contradiction we got proves the claim.

Let us consider the set $\Sigma$ defined as the set of minimal elements $x$ such that $x_0 < x \& (\Phi^\ell(x_0) \neq x)$ for all $\ell > 0$. Then $\Sigma$ is an antichain, and in fact, $\Sigma$ is of the form $\text{Future}(x_0) \cap A$ where $A$ is some antichain, hence $\Sigma$ must be finite by our assumptions. Observe that $\Sigma$ is not empty, since if it were, then it would follow that for all $x \in \text{Future}(x_0)$, $x$ is of the form $\Phi^\ell(x_0)$. Then we consider a maximal chain $C$ containing $x_0$, and by assumption, $C \subset O(x_0)$, and also any element to the future or past of any element of $C$ must be in $O(x_0)$, but $O(x_0) \cap A_0$ is finite,
where $A_0$ is any infinite antichain in $C$ (such an antichain exists by assumption (c) before 3.2), which contradicts the property (b) before 3.2. Now we consider the set

$\Sigma_0 = \{y \in \Sigma | \neg (\Phi^\ell(x_0) \prec y), \ell > 0\}$. $\Sigma_0$ is not empty, otherwise $\Phi^\ell(\Sigma) = \Sigma$, which would contradict the above claim.

Now we let $F_0$ be the partial foliation defined as follows: We take any $z \in \Sigma_0$ as above, and consider the slice $S'_0 = \{\Phi(x_0), z\}$. Let $S_0 = \Phi^{-1}(S'_0)$ and $F_0 = \{\Phi^\ell(S_0) | \ell \in \mathbb{Z}\}$. Then $F_0$ is a partial foliation of $C$ satisfying the $(\ast)$ and $(\ast\ast)$ (immediate), and hence $F_0 \in \mathcal{E}$, so $\mathcal{E} \neq \emptyset$, as required. Finally we are able to complete the proof as usual, using Zorn Lemma to get a maximal element $F_{\text{max}}$ in $\mathcal{E}$. It is easily seen, by repeating the steps of the proof for the first case, that $F_{\text{max}}$ is in fact a foliation, and not only a partial foliation, which satisfies the clause (b) of the statement of the Theorem.

\[\square\]

4. Conclusion

In this paper we have presented a general theorem on temporal foliation of causal sets, and even general posets. Then we considered a special class of causal sets, namely infinite causal sets satisfying some regularity properties, and we were able to deduce some results concerning automorphisms of causal sets, namely that they fall in two classes:

1. Spatial automorphisms,
2. Time translations.

It looks plausible that such results might be useful for quantization of these types of causal sets, and that they might be generalized to continuous space-times (satisfying certain restricting hypotheses).

References

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