MINIMAL TRIANGULATIONS OF SIMPLOTOPE

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Abstract. We derive lower bounds for the size of simplicial covers of simplotope, which are products of simplices. These also serve as lower bounds for triangulations of such polytopes, including triangulations with interior vertices. We establish that a minimal triangulation of a product of two simplices is given by a vertex triangulation, i.e., one without interior vertices. For products of more than two simplices, we produce bounds for products of segments and triangles. Our analysis yields linear programs that arise from considerations of covering exterior faces and exploiting the product structure of these polytopes. Aside from cubes, these are the first known lower bounds for triangulations of simplotope with three or more factors. We also construct a minimal triangulation for the product of a triangle and a square, and compare it to our lower bound.

1. Introduction

A classical problem in discrete geometry is to determine the size of a minimal triangulation of a given polytope. For instance, a polytope that has received considerable attention is the $d$-dimensional cube; see e.g., [3, 4, 9, 11, 13] for upper and lower bounds on the size of many kinds of minimal decompositions of the cube. Minimal triangulations also serve a practical purpose as well; they can be used in simplicial algorithms for finding fixed points (e.g., see [17, 19]) as well as for economic applications [15], since smaller triangulations lead to more efficient algorithms. By a triangulation, we mean a decomposition of a polytope $P$ into simplices that meet face-to-face, and we allow the vertex set of the triangulation to include more points than just the vertices of $P$. (Triangulations that only use vertices of $P$ will be called vertex triangulations.)

In this paper, we study minimal triangulations of simplotope, which are products of simplices [7]. The $d$-cube is thus a special kind of simplotope, the product of $d$ 1-dimensional simplices (segments). Simplotope are of special interest in economics, since in a non-cooperative $n$-person game, the space of strategies is the product of simplices (one for each player)— and finding a Nash-equilibrium is equivalent to finding a fixed point of a function on this space, e.g., [18]. Simploptopes and their triangulations also appear in algebraic geometry [1, 2, 14] and optimization [6]. Orden and Santos [10] used an “efficient” 38-simplex triangulation of a simplotope— the product of a 3-cube and a triangle— to construct triangulations of arbitrarily high-dimensional cubes with few simplices; however, it should be noted the concept of an efficient triangulation is different from (though related to) the minimal triangulation. The idea of triangulating products...
of polytopes by using triangulations of simplotopes as building blocks is present also in \cite{8, 12}.

Aside from cubes, very little is known about minimal triangulations of simplotopes, especially for triangulations that are not vertex triangulations. It is well-known (e.g., \cite{10}) that the product of two simplices of dimensions $a$ and $b$ must be triangulated with exactly $(a+b)$ simplices if it is a vertex triangulation; one of our results in this paper is that a vertex triangulation is indeed minimal. Work has been done to enumerate the many different vertex triangulations of such a product \cite{16}. DeLoera, Rambau and Santos \cite{5} give a recent survey of the enumerative and structural properties of triangulations of simplotopes.

2. Results

Let $\Delta^d$ denote the standard $d$-dimensional simplex, the convex hull of the $d+1$ unit vectors in $\mathbb{R}^{d+1}$. Thus $x = (x_1, ..., x_{d+1})$ in $\Delta^d$ satisfies $x_i \geq 0$, $\sum_i x_i = 1$.

A simplotope is the product of simplices. Let $\Pi(c_1, \ldots, c_n)$ denote the simplotope that is the product $\Delta^{c_1} \times \cdots \times \Delta^{c_n}$. Thus, each point in $\Pi(c_1, \ldots, c_n)$ corresponds to a selection of one point in each factor $\Delta^{c_i}$. For example, a point in $\Pi(2, 2)$ corresponds to a choice of a pair of points, one from each triangle. See Figure I.

Two kinds of simplotopes will receive special attention: (i) products of two simplices of any dimension and (ii) arbitrary products of segments and triangles. We use the notation $\Pi_{s,t}$ as an adjective to refer to any product of $s$ segments and $t$ triangles. For instance $\Pi(1,1,2,1,2,2)$ and $\Pi(2,1,2,2,1,1,2)$ are both $\Pi_{3,4}$ simplotopes, and $\Pi(1,1)$ is a $\Pi_{2,0}$ simplotope, i.e., a square. We let $\Pi^*_{s,t}$ as a noun to denote the specific product $\Pi(1, \ldots, 1, 2, \ldots, 2)$ with $s$ segments and $t$ triangles.

We shall obtain lower bounds for the size of a minimal triangulation by studying the associated concept of a cover. Given a $d$-dimensional polytope $P$, a collection of $d$-simplices is a (simplicial) cover of $P$ if the union of the simplices is $P$ and the vertices of each simplex are vertices of $P$. Thus a vertex triangulation is a special kind of cover of $P$. The following result provides the key to connect the study of covers to the study of triangulations, and it may be surprising, in light of the fact that a general triangulation of $P$ may not be a cover of $P$.

**Theorem 2.1** (Bliss-Su). Let $P$ be a polytope. Let $C(P)$ be the size of the minimal cover of $P$, using only simplices spanned by the vertices of $P$. Let $T(P)$ be the size of a minimal triangulation of $P$, possibly using vertices that are not vertices of $P$. Then

$$C(P) \leq T(P).$$

Bliss and Su proved this result in \cite{3} by consider a piecewise linear map taking vertices of a triangulation to the vertices of a cover, using a Sperner labelling of the vertices, and showing that this map has degree 1. They used it to obtain bounds for minimal triangulations of cubes; we develop new techniques to extend their ideas to find bounds for minimal covers and triangulations of simplotopes.

Our first result shows in Theorem 4.1 that the minimal triangulation of the product of 2 simplices is any vertex triangulation (indeed, the standard triangulation). Previously it
was known that any two vertex triangulations must have the same number of simplices, but had not been known if adding extra vertices could reduce the size of a triangulation.

We then develop a technique for studying arbitrary products of segments and triangles; it will be apparent that these ideas can be extended to other kinds of simplotopes if needed. Our results are summarized in Table 1 in which we provide lower bounds for triangulations of $\Pi_{s,t}^*$ for several small dimensions. Aside from cubes, these are the first known lower bounds for triangulations of simplotopes with three or more factors.

As an example, the efficient 38-simplex triangulation of $\Pi_{3,1}^*$ by Orden and Santos [10] (for vertex triangulations) compares favorably to our lower bound of 32 in Table 1 (for triangulations that allow extra vertices).

This paper is organized as follows. Section 3 establishes terminology and background on triangulating simplotopes, including coordinate representations, volume considerations, and the standard triangulation, that serve as a foundation for the rest of the paper. In Section 4 we prove that for a product of two simplices, any vertex triangulation is minimal. Then in Section 5 we then set up some of the tools we will use to study arbitrary products of segments and triangles. These tools allow us in Section 6 to develop a recurrence relation to be used in a linear program to give the lower bounds found in the table in Figure 1. Finally, in Section 7 we exhibit a minimal triangulation of $\Pi_{2,1}^*$ and compare it to our lower bound.
3. Background

Coordinate Representations. There are two different coordinate representations that we will find convenient for representing simplotope.

The standard coordinate representation expresses each point \( v \) of \( \Pi(c_1, \ldots, c_n) \) as a \((c_1 + \ldots + c_n + n)\)-tuple in \( \mathbb{R}^{c_1 + \ldots + c_n + n} \):

\[
v = (x^1; x^2; \ldots; x^n)
\]

where each \( x^i \) is a \((c_i + 1)\)-vector in \( \Delta^{c_i} \). We write \( x^i = (x^i_1, \ldots, x^i_{c_i + 1}) \) and say the coordinates within each \( x^i \) are in the same factor. To distinguish different factors, we separate the coordinates in different factors by semi-colons. Note that for each point \( v \), we have that \( x^i_j \geq 0 \) for all \( i \) and \( j \) and \( x^i_1 + \cdots + x^i_{c_i + 1} = 1 \) for all \( i \). The latter relations imply that although we represent \( \Pi(c_1, \ldots, c_n) \) as an object in \( \mathbb{R}^{c_1 + \ldots + c_n + n} \), the simplotope has dimension \( c_1 + \ldots + c_n \). Additionally, if all \( x^i_j \) are integers \((0\ or\ 1)\), then \( v \) is a vertex of \( \Pi(c_1, \ldots, c_n) \).

For example, the vector \((0.2, 0.3, 0.5; 0.1, 0.3, 0.6)\) represents a point in the interior of \( \Pi(2, 2) = \Pi_{0,2} \) in the standard coordinate representation. Notice the two triplet factors in this vector sum to one. Another point in \( \Pi(2, 2) \) is represented by the vector \((0, 1, 0; 0, 0, 1)\), and it is a vertex of \( \Pi(2, 2) \) because each coordinate is an integer.

![Figure 1](image_url)  

**Figure 1.** A Schlegel diagram of \( \Delta^2 \times \Delta^2 \), a 4-dimensional polytope, labeled by standard coordinates in \( \mathbb{R}^6 \). The two grey points lie on the same 2-face.

For some purposes we will want to use a reduced coordinate representation that expresses points of \( \Pi(c_1, \ldots, c_n) \) by vectors in \( \mathbb{R}^{c_1 + \ldots + c_n} \). This is similar to the standard representation, except for each \( i \) exactly one of \( x^i_1, \ldots, x^i_{c_i + 1} \) is removed, i.e., in each factor, one of its coordinates is removed. Which coordinates are removed can be specified by picking a vertex of the simplotope and “forgetting” all the coordinates that are non-zero for that vertex. We call this the reduced coordinate representation of \( \Pi(c_1, \ldots, c_n) \) with respect to a vertex \( v \) and call the process reduction with respect to \( v \).
As an example, consider reducing the standard representation of $\Pi(2, 2) = \Pi_{0, 2}$ with respect to the vertex $v = (0, 0, 1; 0, 1, 0)$. Then the point $(0.2, 0.3, 0.5; 0.1, 0.3, 0.6)$ is expressed in reduced coordinates (with respect to $v$) as $(0.2, 0.3, 0.1, 0.6)$. In this reduction, the last coordinate in the first factor and the second coordinate in the last factor are removed, because those correspond to the non-zero coordinates of $v$.

Note that no information is lost because the removed coordinates can always be recovered by remembering that the standard coordinates in each factor must sum to 1.

Any collection of points $v_1, \ldots, v_k$ in standard coordinates can be represented in matrix form as

$$M(v_1, \ldots, v_k) = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$$

in which the rows of the matrix are the given points in standard coordinates. Similarly, we let $M_v(v_1, \ldots, v_k)$ denote the reduced matrix whose rows are the given points in reduced coordinates with respect to a vertex $v$. There is a linear transformation (a projection) taking $M$ to $M_v$.

The simplotope $\Pi(c_1, \ldots, c_n)$ is defined by the half-spaces $x_{i,j}^j \geq 0$, one half-space for each $i, j$ combination, intersected with the hyperplanes $\sum_{j=1}^{c_i} x_{i,j}^j = 1$, one hyperplane for each $i$. Note that a $k$-face is the intersection of the simplotope and $c_1 + \ldots + c_n - k$ of the hyperplanes $x_{i,j}^j = 0$. See Figure 2 for an example for $\Pi(2)$, which is just an equilateral triangle.

**Theorem 3.1.** The set of points, $v_1, \ldots, v_m$ in $\Pi(c_1, \ldots, c_n)$ lie on the same $k$-face if and only if their standard matrix representation $M(v_1, \ldots, v_m)$ has at least $c_1 + \ldots + c_n - k$ columns consisting of only zeros.

For example, with $c_1 = 2$ and $c_2 = 2$ (see Figure 1), the two points $(1, 0, 0; 0.5, 0.5)$ and $(0, 0, 1; 1, 0, 0)$ have two coordinates, $x_2^1$ and $x_2^2$, that are zero in both points. Therefore they lie in the same $2 + 2 - 2 = 2$-face.
Proof. By definition, the points $v_1, \ldots, v_m$ lie on the same $k$-face of the simplotope $\Pi(c_1, \ldots, c_n)$ if and only if each point lies on the intersection of $c_1 + \cdots + c_n - k$ hyperplanes of the form $x_j^i = 0$. For each such hyperplane $x_j^i = 0$, the matrix representation $M(v_1, \ldots, v_m)$ for these points will have corresponding $ij$-th column equal to zero. □

Exterior Faces of Simplices. For simplotope, we say a $(c_1 + \cdots + c_n)$-simplex of a triangulation of $\Pi(c_1, \ldots, c_n)$ has an exterior $j$-face if the simplex has $j + 1$ vertices in the same $j$-face of $\Pi(c_1, \ldots, c_n)$.

The Standard Triangulation. There is a standard triangulation of any simplotope $\Pi(c_1, \ldots, c_n)$ of size

$$\frac{(c_1 + c_2 + \cdots + c_n)!}{c_1!c_2!\cdots c_n!}.$$  

To demonstrate, we introduce a new coordinate system that will allow for a very simple permutation description of the simplices in the standard triangulation. Note that this coordinate representation will be different than either the standard or reduced coordinate systems defined earlier. We can represent a point $w$ in $\Pi(c_1, \ldots, c_n)$ by a vector $$w = (y^1; y^2; \ldots; y^n)$$ where $y^i = (y^i_1, \ldots, y^i_{c_i})$, each $y^i_j \in [0,1]$, and

$$y^i_{j-1} \geq y^i_j$$

for all $i \in \{1, \ldots, n\}$ and $j \in \{2, \ldots, c_i\}$. Hence the coordinates within each factor, from left to right, are non-increasing. For example, in $\Pi(2, 2)$ we have that $y^1_1 \geq y^1_2 \geq y^2_1$, and that $y^2_1 \geq y^2_2 \geq y^2_3$. However, the relative sizes of the coordinates of $y^1$ and $y^2$ are unrelated. Note that the restrictions on each factor $y^i$ define a point in the simplex that is the convex hull of the $c_i + 1$ points

$$u_1 = (0, 0, 0, \ldots, 0)$$
$$u_2 = (1, 0, 0, \ldots, 0)$$
$$\vdots$$
$$u_{c_i+1} = (1, 1, 1, \ldots, 1).$$

Hence, $w$ defines a point of a simplotope. Note that if all $y^i_j$ are integers, then $w$ is a vertex of $\Pi(c_1, \ldots, c_n)$.

By placing further restrictions on the coordinates of the standard representation of the simplotope, we may obtain a subdivision that is a triangulation. Consider any ordering of all the coordinates $y^i_j$ that is consistent with the inequalities in (2). An example of such an ordering if $c_1 = 2$ and $c_2 = 2$ is

$$y^2_1 \geq y^2_2 \geq y^1_1 \geq y^1_2 \geq y^2_3 \geq y^1_3.$$  

One may check that: (i) each such ordering defines a simplex, (ii) every point in the simplotope satisfies at least one such ordering, and is therefore in (at least) one of the simplices, (iii) these simplices meet face-to-face.

Therefore these simplices form a triangulation of the simplotope, called the standard triangulation. The number of simplices in the triangulation is equal to the number of
ways of arranging the $y_i$'s, subject to the prior arrangement of the coordinates within each factor being in non-increasing order. This amounts to choosing positions in the ordering for the coordinates within each factor, and this is given by the multi-choose expression $\binom{n}{r}$.

The standard triangulation is the largest possible vertex triangulation of a simplotope; this follows from noting the simplices of this triangulation are all of class 1 (to be defined in the next section) and hence have the smallest possible volume.

**Class.** One way to obtain bounds for the number of simplices required to triangulate $\Pi(c_1, \ldots, c_n)$ is to use volume estimates. For example, the volume of the smallest $k$-simplex with vertices at lattice points is $\frac{1}{k!}$, and therefore the largest triangulation of any polytope with vertices at lattice points is at most the volume of the polytope divided by $\frac{1}{k!}$, or $k!$ times the volume of the polytope.

For convenience, we use a kind of normalized volume so that such volumes are integers. We define the class of a $d$-dimensional set in $\mathbb{R}^d$ to be the volume of that set multiplied by $d!$. Then if $\alpha$ is a simplex whose vertices are specified (in a reduced coordinate system) by the rows of a matrix $M_v$, the class of a $\alpha$ will be $|\det[1|M_v]|$, which denotes the absolute value of determinant of the matrix formed by augmenting $M_v$ with a column of ones in front. So in a vertex triangulation or a cover, the class of any simplex must be an integer, because it is the determinant of an integer matrix $[1|M_v]$ filled with 1’s and 0’s. Although our definition of class appears to depend on the choice of a reduced coordinate system, the choice of $v$ does not matter:

**Theorem 3.2.** Let $\alpha$ be a simplex of a vertex triangulation of a simplotope. Then the class of $\alpha$ is independent of the choice of reduction $v$ used to compute it.

**Proof.** This follows easily from noting that $[1|M_v]$ and $[1|M_v]$ are related by elementary column operations that do not involve scaling by numbers other than 1 or $-1$, since entries in certain columns are replaced by the first column minus the sum of the columns of one factor. \qed

**4. Products of Two Simplices**

These ideas can be used to demonstrate the well-known result that for the product of two simplices, $\Pi(a, b)$, every vertex triangulation has the same number of simplices (e.g., see [3, 10]). From the standard matrix representation and Theorem 3.1, one can verify that any simplex $\sigma$ using vertices of $\Pi(a, b)$ must have an exterior facet—the standard matrix $M$ of $\sigma$ has $a+b+1$ rows each with two 1’s in them, and $a+b+2$ columns, so some column must have no more than one 1 in it. Removing the row of that 1, if needed, produces an exterior facet with a single 0 coordinate (there is at most one 0 coordinate because the $\sigma$ is non-degenerate). In Proposition 5.3 we will show that if a simplex has an exterior facet, then the class of the simplex is equal to the class of that facet. But that facet is a simplex in a facet of the simplotope, hence in a product of two simplices of lower total dimension. Then induction can be used to show that every simplex in $\Pi(a, b)$ has class one. Hence, to cover the entire volume of $\Pi(a, b)$, one must use a number of simplices equal to the number of simplices in the standard triangulation. Hence every vertex triangulation is a minimal triangulation.
Since our argument above is a covering argument, it actually shows a stronger result via Theorem 2.1 that has not been proved before:

**Theorem 4.1.** A minimal triangulation of a product of two simplices cannot be smaller than a vertex triangulation (e.g., the standard triangulation).

Could the introduction of extra vertices inside the 7-dimensional polytope $\Delta^3 \times \Delta^4$ allow for a smaller triangulation than the standard triangulation? The theorem above says it will not.

Thus the product of two simplices is uninteresting in the sense that the standard triangulation is a minimal triangulation. This is not true for the product of three or more simplices, because in it there exist simplices of class larger than one. For the product of three segments (the 3-cube) the minimal triangulation is 5 while the standard triangulation is 6, because the former uses a class 2 simplex. In the product of three triangles, it is possible to have simplices of class 4.

## 5. The Product of Segments and Triangles

We now focus on $\Pi^{s,t}_{s',t'}$, the product of $s$ segments and $t$ triangles. Our goal is to determine a lower bound on the number of simplices required in a cover of $\Pi^{s,t}_{s',t'}$. As discussed earlier, such a lower bound is also a lower bound for triangulations of $\Pi^{s,t}_{s',t'}$, even when interior vertices are allowed.

Notice that every face of $\Pi^{s,t}_{s',t'}$ is also a simplotope that is the product of segments and triangles. Consider a face $E_{s',t'}$ that is a $\Pi^{s,t}_{s',t'}$ face of $\Pi^{s,t}_{s',t'}$. Note that while $t \geq t'$, it is not necessarily true that $s \geq s'$. This is because when one coordinate of a triangle factor is fixed at 0, then possible values for the remaining two coordinates span one edge of that triangle factor, i.e., a segment. Thus the $s'$ segments in the face can come from either the $s$ segment or the $t$ triangle factors of $\Pi^{s,t}_{s',t'}$. For example, a $\Pi_{0,2}$ simplotope (see Figure 1) can have a $\Pi_{2,0}$ face (i.e., a square). This happens when one column in both triangle factors is fixed at zero, and they both effectively become segment factors, whose product is a $\Pi_{2,0}$ simplotope.

Let $q$ denote the number of segment factors in $E_{s',t'}$ that come from segment factors in $\Pi^{s,t}_{s',t'}$. This leaves $s' - q$ of the segment factors of $E_{s',t'}$ to come from triangle factors of $\Pi^{s,t}_{s',t'}$.

Let $Q(s,t,s',t')$ denote the number of $\Pi^{s,t}_{s',t'}$ faces in $\Pi^{s,t}_{s',t'}$.

**Theorem 5.1.**

$$Q(s,t,s',t') = \binom{t}{t'} \sum_{q=0}^{s'} 2^{s-q}3^{t-t'} \binom{s}{q} \binom{t-t'}{s' - q}.$$  

**Proof.** A $\Pi^{s,t}_{s',t'}$ face of $\Pi^{s,t}_{s',t'}$ is a product of $t'$ triangle factors, $s'$ segment factors, and then one vertex from each of the factors from $\Pi^{s,t}_{s',t'}$ that are not already being used. We count how many ways we can make these choices.

There are $\binom{t}{t'}$ ways to choose $t'$ triangle factors from the $t$ triangle factors. The $s'$ segments can be chosen from the $s$ segments or from the $t - t'$ triangles not already chosen as triangle factors above. Again, let $q$ represent the number of the $s'$ segments chosen from the $s$ segments. The other $s' - q$ are picked from the $t - t'$ triangles.
Thus there are \((s_q)\) ways of picking from the segments and \(3^{s-q}(t'-t)\) ways from picking from the triangles, since there are three ways to pick a single segment from a triangle factor.

Finally, we can fix all the remaining factors by picking a vertex of each. There are two ways to pick a vertex of the remaining \(s-q\) segments, and three ways of picking a vertex from the remaining \(t-t'-s'-q\) triangles. So there are \(2^{s-q}\) ways for picking vertices from the segments not already used, and \(3^{t-t'-s'-q}\) ways of choosing vertices of the triangles not already used.

Multiplying these quantities together and summing up over all possible values of \(q\) yields the desired result. \(\Box\)

We remark that another way to derive this formula is to consider the generating function

\[(x + 2)^s(y + 3x + 3)^t\]

and note that the coefficient of the \(x^{s'}y^{t'}\) term counts the number of ways that a \(\Pi_{s',t'}\) face can be produced as a product of segments and triangles that come from faces of the segment and triangle factors of \(\Pi_{s,t}\). On the other hand, this coefficient is just \(Q(s, t, s', t')\).

For instance, in the case \(t = t' = 0\), the simplotope has no triangle factors. Because it consists only of segment factors, it is an \(s\)-cube. Using Theorem 5.1, we see that for this case

\[Q(s, 0, s', 0) = 2^{s-s'}\binom{s}{s'},\]

which is the formula for the surface \(\mathbb{R}^{s'}\)-volume of the unit \(s\)-cube.

**Counting Exterior Faces.** Consider a cover of \(\Pi_{s,t}^*\), which consists of \((s+2t)\)-dimensional simplices. Any such simplex \(\alpha\) of class \(c\) may or may not have, in a given \(\Pi_{s',t'}\) face, an exterior \((s'+2t')\)-dimensional face of class \(c'\). Over all \(\Pi(s',t')\) faces, \(\alpha\) may have several exterior \((s'+2t')\)-dimensional class \(c'\) faces. Let

\[F(s, t, c, s', t', c')\]

denote the maximum number of such faces over all possible \(\alpha\). Although we may not know \(F\) explicitly we will derive a bounds for \(F\) later.

Let \(V(s, t)\) denote the largest possible class of a simplex in a cover of \(\Pi_{s,t}^*\). We can now formulate an inequality that a cover of \(\Pi_{s,t}^*\) must satisfy. This inequality will form the basis of a linear program that we will solve.

**Theorem 5.2.** Given a cover of a \(\Pi_{s,t}^*\), let \(x_c\) be the number of simplices of class \(c\) in that cover. Then for any \(s', t'\) pair

\[
\sum_{c=1}^{V(s,t)} \frac{c \cdot x_c}{(s'+2t')!} F(s, t, c, s', t', c) \geq \frac{Q(s, t, s', t')}{2^{t'}}
\]

for \(t'\) between \(0\) and \(t\), and \(s'\) between \(0\) and \(s + t - t'\).
Proof. This is a volume bound. On the right side, we have the number of \(\Pi_{s',t'}\) faces \(Q(s,t,s',t')\) multiplied by the volume of these faces. This volume is easy to calculate because it is a product; each segment multiplies the volume by 1, and each triangle multiplies the volume by \(1/2\).

The sum on the left side is an upper bound for all the exterior faces of simplices that could cover the volume of \(\Pi_{s',t'}\) faces represented by the right side. Notice that if a \(d\)-face of the simplotope is covered by a collection of simplices, then every \((d-1)\)-facet of that face must also be covered by the same collection. Therefore, on the left side, we only need to count faces of simplices that are exterior facets, or that are exterior facets of exterior facets, or exterior facets of exterior facets of exterior facets, etc. As we will show in Proposition 5.3, an exterior facet will always have the same class as the simplex itself. Therefore we need only to consider exterior faces that are the same class as the simplex itself, i.e., for which \(c' = c\). Each of these faces has a volume of \(\frac{c}{(s'+2t')!}\). \(\square\)

The Uniqueness of Shadow-Footprint Pairs. Let \(\alpha\) be an \((s+2t)\)-simplex of class \(c \neq 0\) (non-degenerate) in \(\Pi_{s,t}^*\). Suppose that \(\sigma\) is an exterior face of \(\alpha\) with dimension \(k\). By Theorem 3.1 the standard matrix representation of \(\sigma\) must have \(s+2t-k\) columns consisting entirely of zeros. Choose one vertex \(v\) of \(\sigma\), and consider the reduced coordinate system with respect to \(v\) along with the reduced matrix representation, \(M_v\). In the rows of \(M_v\), put \(v\) in the first row, followed by the vertices of \(\sigma\), followed by the rest of the vertices. Rearrange the columns so that the coordinates that must be zero in \(\sigma\) are in the final columns. The result is

\[
M_v = \begin{bmatrix}
0 & \ldots & 0 & 0 & \ldots & 0 \\
A & \text{zeros} \\
C & B
\end{bmatrix},
\]

which is a row of zeros followed by the blocks labeled \(A\), \(B\), and \(C\) as shown. Note that \(A\) is a \(k \times k\) block, and \(B\) is a \((s+2t-k) \times (s+2t-k)\) block. Also, observe that the rows that \(A\) inhabits must be linearly independent, or else \(\sigma\) would not be a simplex. Therefore we can row reduce \(M_v\) to get \(M_v(\sigma)\) by adding multiples of the rows of \(A\) to zero out all the rows of \(C\). We are left with

\[
M_v(\sigma) = \begin{bmatrix}
0 & \ldots & 0 & 0 & \ldots & 0 \\
A & \text{zeros} \\
\text{zeros} & B
\end{bmatrix}.
\]

Note that \(|\det[1|M_v]|\) is the class of \(\alpha\), so \(|\det[1|M_v(\sigma)]|\) is also the class of \(\alpha\). But \(|\det[1|M_v(\sigma)]| = |\det(A)\det(B)|\). From this we can make a few observations:
Proposition 5.3. The class of any exterior face of an simplex divides the class of the simplex. Furthermore, the class of any exterior facet of a simplex equals the class of the simplex.

Proof. Let $\sigma$ be an exterior face. The first statement follows noting the class of the simplex is the product $|\det(A)\det(B)|$, and the class of $\sigma$ is $|\det(A)|$. If $\sigma$ is a facet, then matrix $B$ must be a 1x1-matrix, with entry 1 or 0. Since $B$ is non-degenerate, it must have determinant 1. Thus the class of the simplex is just $|\det(A)|$, which is the class of the $\sigma$. $\square$

Let $\sigma_\perp$ be the simplex spanned by the origin and the last $s + 2t - k$ rows of $M_\nu(\sigma)$. Let $\pi_\sigma$ denote the linear projection of $\alpha$ onto $\sigma_\perp$ that takes the vertices of $\sigma$ to the origin and takes the last $s + 2t - k$ rows of $M_\nu$ to the corresponding rows of $M_\nu(\sigma)$.

Proposition 5.4. The projection $\pi_\sigma$ is one-to-one on vertices of $\alpha$ that are not in $\sigma$.

Proof. The vertices of $\alpha$ that are not in $\sigma$ are represented by the last $s + 2t - k$ rows of $M_\nu$, the submatrices $B$ and $C$. The projection $\pi_\sigma$ simply takes $C$ and zeros it out completely, so if $\pi_\sigma$ were not one-to-one, there would be two identical rows of $B$. But $\det(B) \neq 0$ because the class $c \neq 0$. $\square$

Figure 3. A diagram of a shadow-footprint pair. The simplex, shown in grey, is part of triangular prism $\Pi_{1,1}$. Both $\tau$ and $\sigma$ are faces of the simplex that are part of $\Pi_{2,0}$ faces of the simplotope. The footprint of $\tau$ with respect to $\sigma$ is shown with a white dotted line, and the shadow of $\tau$ with respect to $\sigma$ is shown with a black dotted line.

Given another exterior $k$-face $\tau$ of $\alpha$, the footprint of $\tau$ with respect to $\sigma$ is the intersection $\sigma \cap \tau$, denoted by $\phi_\sigma(\tau)$. The shadow of $\tau$ with respect to $\sigma$ is $\pi_\sigma(\tau)$. The shadow will always be a subset of $\sigma_\perp$. The footprint will always be a subset of $\sigma$.

Theorem 5.5. No two distinct exterior faces have both the same footprint and shadow with respect to some exterior face $\sigma$. 
Proof. Consider two exterior faces $\tau_1$ and $\tau_2$ that have the same footprint and shadow with respect to $\sigma$. Since their footprints are the same, then $\tau_1$ and $\tau_2$ have the same set of vertices in common with $\sigma$. Since their shadows are the same, by Proposition 5.4, $\tau_1$ and $\tau_2$ have the same set of vertices of $\alpha$ that are not in $\sigma$. So $\tau_1$ and $\tau_2$ are identical. \qed

This theorem is the key insight that will allow us to bound $F(s, t, c, s', t', c')$.

**Zero, Free, and Dependent Coordinates.** We introduce some terminology that will make subsequent proofs clearer. Over any subset $A$ of the simplotope, some of the coordinates of points in $A$ in the standard representation may vary, and some may be fixed at 0 or 1 (e.g., if $A$ is a subset of a face of the simplotope). Call the coordinates that are fixed at 0 (over all of $A$) the zero coordinates of $A$, the coordinates fixed at 1 the dependent coordinates of $A$, and remaining coordinates that vary over the points of $A$ the free coordinates of $A$.

Notice that choosing a subset of the coordinates to be zero coordinates corresponds to a choice of face of the simplotope.

As an example, let $A$ be a $\Pi_{2,0}$ face of a $\Pi_{0,2}$ simplotope (which is a square face of the product of two triangles). This must have one zero coordinate and two free coordinates in each triangle factor of $\Pi_{0,2}$. Also, because there are 3 ways of picking the zero coordinate in each factor, the 9 resulting ways of picking both zero coordinates correspond to the 9 square faces of the product of two triangles.

For a given $\Pi_{s', t'}$ face, every triplet of free coordinates in a triangle factor of the simplotope contributes a triangle factor to that face. Likewise, every pair of free coordinates from a segment factor of the simplotope contributes a segment factor to the face. Finally, a pair of free coordinates matched with a zero coordinate from a triangle factor of the simplotope contributes a segment factor to the face.

Now consider a non-degenerate simplex $\alpha$ in a $\Pi_{s, t}$, and let $\sigma$ be an exterior face of $\alpha$ that lives in a $\Pi_{s', t'}$ face of the simplotope. Notice that the zero and dependent coordinates of $\sigma$ are free coordinates of $\sigma_\perp$, and likewise zero and dependent coordinates of $\sigma_\perp$ are free coordinates of $\sigma$. Thus, a segment factor of the simplotope must correspond to either a segment factor of $\sigma$ or a segment factor of $\sigma_\perp$. A triangle factor of the simplotope can correspond either to a triangle factor of $\sigma$, a triangle factor of $\sigma_\perp$, or to a pair of segment factors: one in $\sigma$, and one in $\sigma_\perp$. See Figure 4.

When taking the footprint or shadow of an exterior face $\tau$ of $\alpha$, every free coordinate in $\tau$ corresponds to exactly one free coordinate in either the footprint or the shadow.

**Combinatorial Upper Bound on $F(s, t, c, s', t', c')$.** We can determine some values of $F$. First, note that $F(s, t, c, s', t', c') = 0$ if $s'+2t' > s+2t$, since clearly we can’t have a face with a higher dimension than that of the simplex. If $s = s'$ and $t = t'$, then we just have the simplex itself so $F(s, t, c, s, t, c) = 1$. We also define all values of $F$ where $s < 0, t < 0, s' < 0, t' < 0, c < 1$, or $c' < 1$ to be zero. If $c > V(s, t)$ then $F$ is zero. If $c'$ does not divide $c$, by proposition 5.3 $F$ is zero.

To obtain an upper bound for $F$ we need to establish some facts about parallel and tri-positioned faces. We say that two faces of a simplotope are parallel if they have exactly
the same free coordinates. For example, in $\Pi(2,1)$, the line connecting the points
\[(0,1;0,1;0,0,1), (0,1;0,1;0,0,0)\]
is parallel to the line connecting the points
\[(0,1;1,0;0,0,1), (0,1;1,0;1,0,0)\].
This is because in both cases each line has two free coordinates, the 5th and 7th coordinates. All other coordinates are either zero or dependent in both lines.

**Proposition 5.6.** Let $\alpha$ be a non-degenerate simplex in $\Pi^*_{s,t}$, and let $\sigma$ be an exterior face of $\alpha$. Then any face of $\Pi^*_{s,t}$ parallel and distinct to the one containing $\sigma$ can contain at most 1 vertex of $\alpha$.

**Proof.** Let $E$ be the set of points in $\alpha$ on a given face parallel to $\sigma$; then $E$ has exactly same free coordinates as $\sigma$. Under the map $\pi_\sigma$, all the free coordinates of $\sigma$ are sent to zero, so therefore all the free coordinates of $E$ are sent to zero. Since $E$ under the map $\pi_\sigma$ no longer has any free coordinates, it is completely fixed. So $\pi_\sigma$ maps $E$ onto a single point. By Proposition 5.4, we know $\pi_\sigma$ is one-to-one on the vertices of $\alpha$ that are not in $\sigma$, so $E$ can contain at most one vertex of $\alpha$. \qed

This yields an immediate corollary about parallel faces.

**Proposition 5.7.** Let $\alpha$ be a non-degenerate simplex in $\Pi^*_{s,t}$. No two distinct exterior faces of $\alpha$ of dimension greater than or equal to 1 can be contained in parallel faces of $\Pi^*_{s,t}$.

We say that three faces of a simplotope are **tri-positioned** if: (i) each of the three faces have exactly the same zero coordinates except in one triangle factor of the simplotope, and (ii) in that triangle factor, each of the faces have exactly one zero coordinate and it is a different coordinate for each face. For example, the three square facets of a triangular prism ($\Pi_{1,1}$) are tri-positioned, as are the three edges of any triangular facet.

**Proposition 5.8.** Let $\alpha$ be a non-degenerate simplex in $\Pi^*_{s,t}$. No three distinct exterior faces of $\alpha$ of dimension greater than or equal to 2 can be contained in tri-positioned faces of $\Pi^*_{s,t}$.

**Proof.** Let $\sigma$, $\tau$, and $\rho$ be three tri-positioned exterior faces of $\alpha$ in three different $\Pi_{s',t'}$ faces of $\Pi^*_{s,t}$. Each of $\sigma$, $\tau$, and $\rho$ has $s' + 2t' + 1$ vertices (and they may share some in common). A naive count of all the vertices with possible over-counting yields $3s' + 6t' + 3$ vertices. Each vertex in this naive count is in at most two of $\sigma$, $\tau$, and $\rho$ since no point can simultaneously satisfy condition (ii) for all three tri-positioned faces. Since each vertex is counted in the naive count at most twice, so there are at least $\lceil \frac{3}{2}s' + \frac{3}{2} \rceil + 3t'$ distinct vertices among the three faces.

As a collection of points, these vertices have $s + 2t - (s' + 2t' + 1)$ common zero coordinates. So we have $\lceil \frac{3}{2}s' + \frac{3}{2} \rceil + 3t'$ points on the same $(s' + 2t' + 1)$-face. Because a simplex cannot have more than $k + 1$ vertices in a $k$-face, this is a contradiction unless
s' = 0 and t' = 0 (a zero-dimensional face) or s' = 1 and t' = 0 (a one-dimensional face).
As long as the dimension is greater than or equal to 2, σ and τ and ρ cannot all be faces of α.

Now we can develop a combinatorial upper bound on F.

**Theorem 5.9.** If the dimension \( s' + 2t' \geq 2 \), then

\[
F(s, t, c, s', t', c') \leq \binom{t}{t'} \sum_{q=0}^{\min(s, s')} \binom{s}{q} (t - t')^{s' - q} 2^{s' - q}.
\]

Otherwise, \( F(s, t, c, 1, 0, 1) \leq s + 3t \) and \( F(s, t, c, 0, 0, 1) = 1 \).

**Proof.** Consider a simplex \( \alpha \) of the simplotope \( \Pi_{s', t'}^* \). We will bound the number of exterior faces of \( \alpha \) that are a part of a \( \Pi_{s', t'}^* \) face of the simplotope.

Recall that each of the \( s' \) segment factors of the \( \Pi_{s', t'}^* \) face can arise from a segment factor of the simplotope, or a triangle factor of the simplotope in which one coordinate is not free. Let \( q \) denote the number of segment factors of the \( \Pi_{s', t'}^* \) face that arise from segment factors of the simplotope. Thus \( s' - q \) segment factors arise from triangle factors of the simplotope.

First, it is clear that two faces of \( \alpha \) cannot lie on the same \( \Pi_{s', t'}^* \) face of \( \Pi_{s, t}^* \). Nor, from Proposition 5.7, can two faces of \( \alpha \) be in parallel faces of \( \Pi_{s, t}^* \). This means two different exterior faces of \( \alpha \) must have a distinct (but not necessarily disjoint) set of free coordinates.

Thus we just need to count the number of valid ways of picking free coordinates for the faces of \( \alpha \). Let us first choose the triangle factors of the face of \( \alpha \), which amounts to setting all three coordinates in a triangle factor to be free coordinates. There are \( \binom{t}{t'} \) ways of doing this.

Now let us pick the segment factors. Notice \( q \) can assume values from 0, where all the segment factors come from triangles, to \( \min(s, s') \). (Recall that \( s' \) may be bigger than \( s \).) Then there are \( \binom{s}{q} \) ways of choosing segment factors of the face from the segment factors of the simplotope. Finally, we need to pick \( s' - q \) segment factors of the face from the triangles factors of the simplotope. There are \( t - t' \) triangles factors left over that can be used to get segment factors of the face; we have \( \binom{t - t'}{s' - q} \) ways of doing this. Notice each way of choosing a segment from a triangle could have up to two possibilities corresponding to having a zero coordinate in two of three places in a triangle factor. We know we can not have a zero coordinate in the last place in this triangle factor, because then we would have three tri-positioned faces, and by Proposition 5.8 this is impossible if the dimension \( s' + 2t' \geq 2 \). Therefore, we get an extra factor of \( 2^{s' - q} \). That gives the desired upper bound.

In the special case \( (s', t') = (1, 0) \), Proposition 5.8 does not hold, so \( \alpha \) may have 3 exterior edges contained in tri-positioned edges (of one triangular face), when there are three vertices of \( \alpha \) whose factors have identical coordinates except for one triangle factor, and in that factor, the 3 possibilities for edges are specified by a choice of a pair of vertices. Other exterior edges of \( \alpha \) may arise from having two vertices whose factors have identical coordinates except for one segment factor. In view of Proposition 5.7 there are no other exterior edges of \( \alpha \) possible, since two vertices of \( \alpha \) whose coordinates
differ in more than one factor cannot be exterior because it will not have enough zero coordinates. Since each triangle factor gives rise to at most 3 exterior edges, and each segment factor gives rise to at most 1 exterior edge, and every exterior edge is class 1, we have \( F(s,t,c,1,0,1) \leq s + 3t \).

For the special case \((s',t') = (0,0)\), a \((0,0)\)-face is a vertex and can contain at most one vertex of \(\sigma\), hence \( F(s,t,c,0,0,1) = 1 \). \(\square\)

A corner simplex is defined by a vertex \(v\) of the simplotope and all the “neighbor” vertices connected to \(v\) by edges of the simplotope. It is the simplex with the maximum number of exterior faces in the sense that it achieves the bound given by Theorem 5.9.

**Theorem 5.10.** The bound given by Theorem 5.9 is achieved by a corner simplex.

**Proof.** Let \(v\) be a vertex in the simplotope \(\Pi_{s,t}^*\); its standard coordinate representation consists of a 1 in a single coordinate of every factor and 0’s every entry otherwise. Any neighbor vertex of \(v\) will have the same coordinates as \(v\) in every factor except one; in that factor, there will be two coordinates transposed, one containing 1 and the other containing 0. We call that factor the neighbor factor for that neighbor vertex. Let \(\alpha\) be the corner simplex consisting of \(v\) and all its neighbors; let \(M\) be its standard coordinate representation with \(v\) as the first row.

The reduced representation is obtained from the standard representation by removing one column from each factor— in particular, the columns of \(v\) that contain a 1. Note that these columns have at most one or two 0’s in them (one 0 if the column comes from a segment factor and two 0’s if the column comes from a triangle factor), because there can be at most one or two neighbor vertices that differ from \(v\) in a given factor. The upshot of this is the following remark: any 3 rows chosen from \(M\) will not have any zero coordinate removed when reduced relative to \(v\).

Note that the reduced coordinate representation matrix \(M_v\) of \(\alpha\) is very simple: the first row is now all zeroes and the other rows will contain exactly one 1 and can be arranged like so:

\[
\begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}
\]

We now determine the number of exterior faces of \(\alpha\) that could lie in a \(\Pi_{s',t'}^*\) face of \(\Pi_{s,t}^*\).

Note that any set of \(s' + 2t' + 1\) rows we choose will result in a \((s' + 2t')\)-dimensional simplex \(\tau\). If the first row is chosen, then \(\tau\) will be exterior, because the first row has \(s + 2t\) zero coordinates (standard or reduced), and each of the \(s' + 2t'\) other rows will block a different coordinate from being a zero coordinate. So there will be exactly \(s + 2t - (s' + 2t')\) total zero coordinates (and not fewer) in \(\alpha\); hence by Theorem 3.1 \(\tau\) is an exterior \((s' + 2t')\)-dimensional face of \(\alpha\).

This construction is the only way the selection of \(s' + 2t' + 1\) rows forming \(\tau\) can be exterior as long as \(s' + 2t' + 1 \geq 3\), because replacing the first row by any other row will decrease the number of zero coordinates by blocking a new coordinate in the neighbor
Consider first the case \( s' + 2t' \geq 2 \). Then \( \tau \) (constructed above) has the same dimension as an exterior \( \Pi_{s',t'} \) face, but it may not lie in a \( \Pi_{s',t'} \) face unless that face has \( t' \) triangle factors and \( s' \) segment factors. To get \( t' \) triangle factors in that face, we take them from triangle factors of the simplotope. This consists of picking the two rows that have a 1 in the triangle factor’s two columns. We do this \( t' \) times, and we have \( t \) triangles to choose from. Thus there are \( \binom{t'}{t} \) ways of doing this. Now we only need to pick \( s' \) more vertices, which will represent segment factors of the face. These can come from triangle or segment factors of the simplotope. Let \( q \) be the number of vertices that come from segment factors. Then there are \( \binom{s'}{q} \) ways to pick from segment factors, and \( \binom{t'-t'}{s'-q} \) ways of picking from triangle factors. But for each vertex picked from triangle factors there are two ways of choosing it, since each triangle factor consists of two rows. This produces an extra \( 2^{s'-q} \) term. Taking the product of these possibilities, we get the formula in Theorem 5.9.

This estimate is too small in the case that \( (s',t') = (1,0) \). But this case is easy to calculate. Clearly any vertex paired with \( v \) will result in \( s + 2t - 1 \) zero coordinates for the edge, so it will be exterior in a 1-face of the simplotope. This totals \( s + 3t \) exterior edges so far. We also have an exterior edge between every pair of vertices that have the same neighbor factor. So this adds \( t \) more exterior edges, for a total of \( s + 3t \), as desired. \( \Box \)

6. Recurrence Relation

In general, we expect \( F(s,t,c,s',t',c') \) to be smaller when \( c \) grows (big simplices don’t have as many exterior faces). The upper bound for \( F \) above does not depend on \( c \) at all, and therefore is only good when \( c \) is small. To get good bounds on \( F \) for higher values of \( c \) we use a recurrence relation based on footprint and shadow considerations.

Consider a simplex \( \alpha \) of class \( c \) in \( \Pi_{s,t,c} \), and suppose it has at least one exterior \( \Pi_{s',t'} \)-face \( \sigma \). Consider another exterior \( \Pi_{s',t'} \)-face \( \tau \). We know from Theorem 5.5 that every such face \( \tau \) has a unique footprint/shadow combination with respect to \( \sigma \). Hence, counting the number of footprint/shadow combinations will yield an upper bound on the number of such faces \( \tau \). The footprint must be exterior because of intersections of exterior faces are exterior (or empty) from Proposition 6.1 below. The shadow must be exterior from the Proposition 6.2 below. Therefore, we count possible footprints, which is bounded by the function of \( F \) using parameters for \( \sigma \), and count possible shadows bounded by \( F \) using the parameters for \( \sigma_\perp \).

Proposition 6.1. Let \( \alpha \) be a non-degenerate simplex in \( \Pi(c_1, \ldots, c_n) \), and let \( \sigma, \tau \) be two exterior faces of \( \alpha \). Then \( \sigma \cap \tau \) is also an exterior face of \( \alpha \).

Proof. Let the dimensions of \( \alpha, \sigma, \) and \( \tau \) be \( N, n, \) and \( m \) respectively. We see from Theorem 3.1 that \( \sigma \) has \( N - n \) zero coordinates and \( \tau \) has \( N - m \) zero coordinates.
Let $p$ be the dimension of the intersection of $\sigma$ and $\tau$, and let $q$ be the number of zero coordinates $\sigma$ and $\tau$ have in common. Then, the intersection $\sigma \cap \tau$ would have $2N - m - n - q$ zero coordinates. To show that $\sigma \cap \tau$ is exterior, we need to show that $\sigma \cap \tau$ has $N - p$ zero coordinates, or in other words, to show that $q = N - m - n + p$.

Suppose, for the sake of contradiction, $q > N - m - n + p$. Then the union of vertices of $\sigma$ and $\tau$ has more than $N - m - n + p$ zero coordinates, which implies by Theorem 3.1 that these vertices must lie in a face of dimension less than $m + n - p$. However, this union contains $m + 1 + n + 1 - (p + 1) = m + n - p + 1$ vertices, which is too many points for a non-degenerate $\alpha$ to lie in a face of dimension $m + n - p - 1$ or less.

On the other hand if $q < N - m - n + p$, then intersection $\sigma \cap \tau$, which has $2N - m - n - q$ zero coordinates, would have more than $N - p$ zero coordinates. This means the intersection is on at most a $(p - 1)$-face of the simplotope, which is impossible if the dimension of the intersection is $p$.

In either case we obtain a contradiction, hence $q = N - m - n + p$, as desired.

**Proposition 6.2.** In the simplex $\alpha$ of a simplotope $\Pi(c_1, \ldots, c_n)$, the shadow of some exterior face $\tau$ with respect to some exterior face $\sigma$ is an exterior face.

**Proof.** Again let the dimensions of $\alpha$, $\sigma$, and $\tau$ be $N$, $n$, and $m$, respectively. Let $p$ be the dimension of the intersection of $\sigma$ and $\tau$, so that the dimension of the shadow is $m - p$. Finally, let $q$ the number of zero coordinates that $\sigma$ and $\tau$ share in common. Then $q + n$ is the number of zero coordinates in the shadow of $\tau$, because it includes every coordinate not fixed at zero in $\sigma$, as well all the coordinates that are fixed at zero in both $\sigma$ and $\tau$. From the proof of Proposition 6.1 we know the intersection has $N - p$ zero coordinates. Also from above, $N - p = 2N - m - n - q$, which implies $q = N - m - n + p$, so $q + n = N - m + p$. Therefore the number of zero coordinates in the shadow is equal to $N - (m - p)$. This implies that the shadow is exterior by Theorem 3.1.

A good way to grasp counting footprint-shadow pairs is to use a footprint-shadow diagram, like the one in Figure 4. The lines, both vertical and slanted ones, represent segment factors, while the triangles represent triangle factors. In Figure 4 $\alpha$ is a simplex that is part of a cover of $\Pi_{3,3}$, so it is represented by three vertical lines and three triangles. For the purposes of this example, we have chosen a face $\sigma$ of $\alpha$ in a $\Pi_{3,1}$ face, so it is represented in the diagram by three lines and one triangle; moreover, we have chosen $\sigma$ so that one of its segment factors arises from a triangle factor of $\alpha$, and this.

![Figure 4](image_url)
is represented by the slanted line in \( \sigma \)'s row. Using the diagram, it is easily deduced that \( \sigma_\perp \) consists of two segment factors and one triangle factor. Note how \( \sigma_\perp \) and \( \sigma \) complement each other with respect to \( \alpha \); where there is a factor in one, there is not a factor in the other. The one exception is under the first triangle factor of \( \alpha \). Both \( \sigma \) and \( \sigma_\perp \) have segment factor there, but they are different segment factors, represented by different slants of the line. The triangle factor in this case is decomposed into two segment factors. Using a footprint-shadow diagram can help give intuition on how to derive the following recurrence relation.

**Theorem 6.3.** The quantity \( F(s, t, c, s', t', c') \) is less than or equal to

\[
\max_e \sum_{w=0}^{e'} \sum_{k=1}^{c'} \sum_{j=0}^{t'} \sum_{i=0}^{s'} F(s', t', c', i, j, k) \cdot F \left( s'' - i + 2w, t' - j - w, \frac{c'}{k} \right),
\]

where \( s'' = s' - s + 2e, \) \( t'' = s + t - s' - t' - e, \) \( w' = \min(s' - s + e, t'), \) \( i' = \min(s' + t' - j, s' + w), \) and the maximum ranges over \( e \) from \( \max(0, s - s') \) to \( \min(s + t - s' - t', s) \).

**Proof.** Fix a simplex \( \alpha \) in \( \Pi_{s,t}^* \), and suppose \( \alpha \) is of class \( c \). Recall that \( F(s, t, c, s', t', c') \) counts the number of faces of \( \alpha \) in a certain family, namely, the number of faces of \( \alpha \) that live in a \( \Pi_{i,j}^* \) face of \( \Pi_{s,t}^* \) and that are of class \( c' \).

We fix one such face of this family, call it \( \sigma \). (If there is no such face, the above inequality will trivially hold.) Any other \( \tau \) in this family will have a unique footprint-shadow pair with respect to \( \sigma \), so it suffices count the number of possible footprint-shadow pairs, where the footprint is an exterior face of \( \sigma \) and the shadow is an exterior face of \( \sigma_\perp \).

To do this, we consider footprints that lie in some \( \Pi_{i,j}^* \) face and are of class \( k \), then count the number of shadows that could be associated with this footprint, then sum up...
over all possible \(i, j, k\). Since not all \(e\) may be achieved by some \(\sigma\), we take the maximum over all \(e\) as a bound.

The number of possible footprints that lie in some \(\Pi_{i,j}\) face and are of class \(k\) is less than or equal to:

\[
F(s', t', c', i, j, k).
\]

This follows from the definition of \(F\), noting that a footprint is an exterior face of \(\sigma\) in a \(\Pi_{i,j}\) face, and \(\sigma\) is a simplex in a \(\Pi_{s',t'}\) face of class \(c'\).

For each such footprint, there are a number of shadows that could be associated with this footprint, and bounding this number is complicated by the fact that the number of segment and triangle factors of a shadow is not uniquely determined by a footprint. See, for instance, the \(\tau\) in Figures 4 and 5, which have the same footprint dimensions \(i\) and \(j\) but shadows with different numbers of segment and triangle factors. These numbers depend on a quantity \(w\), which we define to be the number of triangle factors of \(\tau\) that are chosen from triangle factors of \(\alpha\) that support segment factors of \(\sigma\).

Recall that \(e\) was defined above as the number of segment factors of \(\sigma_{\perp}\) that correspond to segment factors of \(\Pi_{s',t'}^{s'}.\) Then \(s - e\) is the number of segment factors of \(\alpha\) that support segment factors of \(\sigma\), so the number of triangle factors of \(\alpha\) that support segment factors of \(\sigma\) is \(s' - (s - e)\). Thus \(w\) varies between 0 and \(s' - s + e\). Then for a given footprint, the number of shadows is bounded by

\[
\sum_{w=0}^{s' - s + e} F(s'', t'', c, s' - i + 2w, t' - j - w, \frac{c'}{k}).
\]

This bound follows from the definition of \(F\), noting that a shadow of \(\tau\) is an exterior face of \(\sigma_{\perp}\), which lives in a \(\Pi_{s',t'}\) simplotope face \(H\) of \(\Pi_{s,t}^{s'}.\) What remains is to figure out what are the appropriate dimensions of \(\sigma_{\perp}\) and the shadow, and what kind of simplotope faces they live in.

From the argument above, we know the number of triangle factors of \(\alpha\) that support segment factors of \(\sigma\) (hence also \(\sigma_{\perp}\)) is \(s' - s + e\), the total number of segment factors of \(H\) is obtained by adding \(e\) to this, hence \(s'' = s' - s + 2e\). The number of triangle factors of \(H\) is \(t\) (the number of triangle factors of \(\alpha\)) minus \(t'\) (the number of triangle factors used by \(\sigma\)) minus \(s' - s + e\) (the number of triangle factors of \(\sigma\) supporting segment factors of \(\sigma_{\perp}\)). Therefore \(t'' = t - t' - (s' - s + e) = s + t - s' - t' - e\).

The class of \(\sigma_{\perp}\) is the class of \(\alpha\) divided by the class of \(\sigma\), i.e., \(c/c'\).

The total dimensions of the shadow and footprint must sum to the dimension of \(\tau\), which lives in a \(\Pi_{s',t'}\) face, and if this were true individually for the segment factors and triangle factors by themselves, then the number of segment and triangle factors of the shadow would be \(s' - i\) and \(t' - j\), respectively. However, as noted above, in some instances a segment supporting the footprint face (a subface of \(\sigma\)) may combine with a segment supporting the shadow face (a subface of \(\sigma_{\perp}\)) to form a triangle factor of \(\alpha\). This can only happen when \(\sigma\) and \(\sigma_{\perp}\) are on supporting faces that have segment factors from the same triangle factor of \(\alpha\); we saw there are at most \(s' - s + e\) such triangle factors. Note that \(w\), as defined above, is the number of factors of \(\alpha\) for which the footprint and shadow have segment factors from the same triangle factor of \(\alpha\). Then \(w\) can vary between 0 (when \(\tau = \sigma\)) and \(\min(s' - s + e, t')\), and for a given \(w\), there
are $2w$ more segments in the footprint and shadow dimensions, and $w$ fewer triangles. Thus, the number of segment factors of the shadow must be $s' - i + 2w$, so that when we combine this with the $i$ segments of the footprint and the remove of the segments caused by splitting factors of $\alpha$, we have $s'$ total segment factors. Likewise, the shadow must have $t' - j - w$ triangle factors.

Finally, the class of the shadow and footprint must multiply to the class of $\tau$, so the class of the shadow must be $c'/k$.

Now we sum over all possible $i$, $j$, and $k$. As long as we count every possible footprint-shadow combination, we will have an upper bound on the number of exterior faces $F$.

The index $j$ can run from 0 to at most $t'$, the number of triangle factors supporting $\sigma$. The index $i$ is at least $w$ because there are at least that many segment factors supporting $\sigma$. The index $i$ cannot be more than $s' + t' - j$, the number of segment and triangle factors supporting $\sigma$ minus the number of triangles used by $j$, and $i$ cannot also be more than $s' + w$, the number of segments supporting $\alpha$ together with the number of triangle factors of $\alpha$ that support segment factors of $\sigma$. Hence $i$ is at most $\min(s' + t' - j, s' + w)$. The index $k$ can run from 0 to $c'$ (in fact $k$ must be a divisor of $c'$ but that is already accounted for in the initial conditions for $F$).

Finally, what are the possible values of $e$? Recall that $e$ is the number of segment factors of $\Pi^*_{s,t}$ that are segment factors of $\sigma_\perp$. The smallest $e$ occurs in the case where all $s'$ segment factors of $\sigma$ are from segment factors of $\Pi^*_{s,t}$; this leaves either 0 or $s - s'$ segment factors for $\sigma_\perp$, whichever is larger. The largest $e$ occurs in the case where as many of the $s'$ segment factors of $\sigma$ come from triangle factors of $\Pi^*_{s,t}$ as possible. The number of triangle factors available is at most $t - t'$, so this leaves at least $\max(s' - t + t', 0)$ segments of $\sigma$ to come from segment factors of $\Pi^*_{s,t}$, or at most $s - \max(s' - t + t', 0) = \min(s - s' + t - t', s)$ segment factors from which $\sigma_\perp$ may be supported. 

Notice that when $t = t' = 0$, then $e$ is fixed at $s - s'$ and $j, w$ are fixed at zero, so we have

$$F(s, 0, c, s', 0, c') \leq \sum_{i,j,k} F(s', 0, c', i, 0, k) \cdot F(s - s', 0, c, s' - i, 0, c'/k),$$

the same recurrence relation derived by Bliss and Su for cubes $[3]$.

**Examples.** To understand Theorem 6.3, it is helpful to consider examples.

First, consider a prism $\Delta^1 \times \Delta^2 = \Pi^*_{1,1}$. Suppose we want an upper bound on the number of exterior faces a class 1 simplex $\alpha$ can have inside a square face of the prism. Then $s = 1, t = 1, s' = 2, t' = 0$ and $c = c' = 1$ and we want to compute $F(1, 1, 1, 2, 0, 1)$. The first three numbers indicate that we have a class 1 simplex $\alpha$ in $\Pi^*_{1,1}$, and the last three numbers indicate that we are looking for exterior faces of $\alpha$ in $\Pi^*_{2,0}$ faces of the prism (square faces, see Figure 3). From Theorem 5.9 we know that the maximum number of exterior faces is 2, shown in Figure 3 as $\sigma$ and $\tau$, square faces of a corner simplex $\alpha$. As we will see, our recurrence relation produces an upper bound of 3.

By making appropriate substitutions, we see in Theorem 6.3 that the following indices are fixed: $j = 0$, $k = 1$, $w = 0$, and $e = 0$ (because $\sigma$ lies on a square face and must use
the segment factor of $\alpha$, so $\sigma_\perp$ cannot). Then we have

\[
F(1, 1, 1, 2, 0, 1) \leq \sum_{i=0}^{2} F(2, 0, 1, i, 0, 1) \cdot F(1, 0, 1, 2-i, 0, 1)
\]

\[
\leq F(2, 0, 1, 0, 0, 1)F(1, 0, 1, 2, 0, 1)
+ F(2, 0, 1, 1, 0, 1)F(1, 0, 1, 1, 0, 1)
+ F(2, 0, 1, 2, 0, 1)F(1, 0, 1, 0, 0, 1).
\]

We see that $F(1, 0, 1, 2, 0, 1) = 0$ because there cannot be a square face of a line segment. In the second term, $F(2, 0, 1, 1, 0, 1) = 2$ since a simplex in a square can have two exterior faces that are segments. In the third term, $F(2, 0, 1, 2, 0, 1) = 1$ by definition. We know $F(1, 0, 1, 1, 0, 1) = 1$ and $F(1, 0, 1, 0, 0, 1) = 1$ by Theorem 5.9. Hence $F(1, 1, 1, 2, 0, 1) \leq 0 + 2 \cdot 1 + 1 \cdot 1 = 3$.

Let us look at how this is interpreted geometrically, using Figure 5. We count footprints and shadows with respect to $\sigma$, a $\Pi_{2,0}$ face we picked arbitrarily. The first term above corresponds to trying to get an entire square face from the shadow. As in Figure 3, the shadow is just a segment, so it cannot have a square face, therefore the first term is zero. The second term corresponds to using a segment exterior face of both the shadow and the footprint. There are two ways to do this, so this term is equal to two. The final term corresponds to ways in which the footprint is in a square face.

Consider another example: the triangle cross a square $\Pi_{2,1}^*$, shown in Figure 6 below. We know $F(2, 1, 1, 1, 1, 1)$ is, for a class 1 simplex $\alpha$ in $\Pi_{2,1}^*$, the maximum number of class 1 facets of $\alpha$ that live in prism faces. Since $s = 2, t = 1, s' = 1, t' = 1$, then in the recurrence we have $e = 1, w = 0, k = 1$ and $j$ runs from 0 to 1, and $i$ runs from 0 to 1. So there will be four terms in the sum:

\[
F(2, 1, 1, 1, 1) \leq \sum_{j=0}^{1} \sum_{i=0}^{2-j} F(1, 1, 1, i, j, 1) \cdot F(1, 0, 1, 1 - i, 1 - j, 1)
\]

\[
\leq F(1, 1, 1, 0, 0, 1)F(1, 0, 1, 1, 1, 1)
+ F(1, 1, 1, 1, 0, 1)F(1, 0, 1, 0, 1, 1)
+ F(1, 1, 1, 0, 1, 1)F(1, 0, 1, 1, 0, 1)
+ F(1, 1, 1, 1, 1, 1)F(1, 0, 1, 0, 0, 1)
= 1 \cdot 0 + 4 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 2.
\]

Finally, for a larger example, examine one term in the recursive formula for $F(3, 3, 2, 3, 1, 1)$. Here, $\alpha$ is a class 1 simplex in a cover of $\Pi_{3,3}^*$, and we are searching for exterior $\Pi_{3,1}$ faces of class 1. This example is illustrated in Figure 4. With the choice of $\sigma$ in the figure, we see that $e = 1$ since $\sigma_\perp$ has one segment factor that comes from a segment factor in $\alpha$. The face $\tau$ is counted in the product $F(3, 1, 1, 2, 0, 1)F(2, 1, 1, 1, 1, 1)$, when $s' = 3, t' = 1, i = 2, j = 0$, and $w = 0$. We see the triple $(3, 1, 1)$ represents $\sigma$, the triple $(2, 0, 1)$ represents the footprint, the $(2, 1, 1)$ represents $\sigma_\perp$, and the $(1, 1, 1)$ represents the shadow. Figure 5 represents the same situation but with a different $\tau$; this $\tau$ has $w = 1$ and is counted in the product $F(3, 1, 1, 2, 0, 1)F(2, 1, 1, 3, 0, 1)$. 
A Linear Program. Using the inequalities derived in Theorem 5.2 we can use a linear program to minimize the number of simplices that meet the requirements of the inequalities, i.e.,:

$$\min \sum_{c=1}^{c=V(s,t)} x_c \text{ subject to the inequalities in Theorem 5.2.}$$

To get the best results, we need good bounds on $V(s,t)$, which is the largest possible class of a simplex in a cover of $\Pi_{s,t}^*$. The is a hard problem, related to the Hadamard maximum determinant problem. Specific values are known for cubes of small dimension. We use these values of cubes as upper bounds on the values for simploptopes, because $\Pi_{s,t}^*$ can be seen as a subset of a $s + 2t$-cube, so the largest simplex in $\Pi_{s,t}^*$ must be less than or equal to the largest simplex in a $s + 2t$-cube.

However, for many values of $(s,t)$, this is not a very good bound. Therefore for specific low dimensions, we ran a brute force computer program that checked the classes of all possible simplices that could be part of a triangulation and returned the highest one. By this method we see that $V(1,1) = 1$, $V(0,2) = 1$, $V(2,1) = 2$, $V(1,2) = 3$, and $V(0,3) = 4$. Although this improvement is only for low dimensions of $V(s,t)$, it can greatly affect higher dimensions of our bounds because of the recursion that takes place in determining $F$.

In determining values of $F$, we used upper bounds that result from the recurrence relation, using the combinatorial upper bound on $F$ as a base case of the recurrence.

Using lp.solve, we solved the linear program to obtain the values shown in Table 1.

![Figure 6. A Schlegel diagram of a triangle cross a square, labeled by a reduced coordinate representation.](image)
least 9 simplices are required to cover $\Pi_{2,1}^*$. However, we shall show that no cover of 9 simplices exists, using the following two key propositions:

**Proposition 7.1.** Every simplex in a cover of $\Pi_{2,1}^*$ has an exterior facet, and therefore the largest class of a simplex in a cover of $\Pi_{2,1}^*$ is class 2.

*Proof.* Consider how 5 vertices of $\Pi_{2,1}^*$ can be chosen as corners of a simplex. For each, the triangle factor must be one of $(0,0,1)$, $(0,1,0)$, or $(1,0,0)$ in standard coordinates. Each of these three choices must be picked at least once to get 5 affinely independent points, so there must be one of these choices that is chosen at most once, hence the other two are chosen exactly 4 times total. The corresponding four points have a zero coordinate in common, which means, by Theorem 3.1, the simplex has to have an exterior facet. By Theorem 5.3, the class of the simplex must be equal to the class of this facet. The exterior faces of $\Pi_{2,1}^*$ are $\Pi_{1,1}^*$ and $\Pi_{3,0}^*$ simplotopes. The largest class that a simplex can have in either of these is 2, which occurs for a $\Pi_{3,0}^*$ face (and cannot occur for a $\Pi_{1,1}^*$ face). \[\square\]

The center of $\Pi_{2,1}^*$ is the point fixed by all isomorphisms of $\Pi_{2,1}^*$ with itself; in reduced coordinates is $(1/2; 1/2; 1/3, 1/3)$.

**Proposition 7.2.** Any class 2 simplex of a cover of $\Pi_{2,1}^*$ will contain the center of $\Pi_{2,1}^*$ in the interior of a facet of the simplex.

*Proof.* From the proof of Proposition 7.1 we know that a class 2 simplex must have an exterior facet of class 2 in one of the $\Pi_{3,0}^*$ facets of the simplotope. Therefore, every class 2 simplex must consist of one of these class 2 facets coned to a point not on that facet. There are 24 possible class 2 simplices, because there are two ways of choosing a class 2 simplex facet from a $\Pi_{3,0}^*$ facet, there are three possible $\Pi_{3,0}^*$ facets to choose from, and there are four ways of picking a point not on that facet once it is chosen. However, we now show that each of these 24 class 2 simplices is isomorphic up to rotations and reflections about the center.

Fix a class 2 simplex and consider its standard matrix representation $M$. Permuting the columns associated to each factor corresponds to rotations and reflections of the simplotope that leave the center fixed. Furthermore, every permutation corresponds to a different class 2 simplex because otherwise two columns of $M$ would be identical, which cannot occur if the simplex is non-degenerate. There are $24 = 2! \times 2! \times 3!$ such permutations that permute columns of $M$ within each factor. These account for all of the 24 class 2 simplices determined above.

Since the class 2 simplices are isomorphic, without loss of generality, we can check that the proposition holds for one class 2 simplex. Consider the following class 2 simplex spanned by these vertices in reduced coordinates: $(0; 0; 0, 0)$, $(0; 1; 0, 1)$, $(1; 0; 0, 1)$, $(1; 1; 0, 0)$, $(0; 0; 1, 0)$. Suppose some convex combination of these points produced the center point using coefficients $a_1$, $a_2$, $a_3$, $a_4$, and $a_5$ respectively. Such a combination
solves the system of equations
\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix}
= \begin{bmatrix}
1/2 \\
1/2 \\
1/3 \\
1/3
\end{bmatrix}.
\]

Furthermore, as a convex combination we know \(a_1 + a_2 + a_3 + a_4 + a_5 = 1\). These equations can be solved to find \(a_1 = 0, a_2 = 1/6, a_3 = 1/6, a_4 = 1/3, \) and \(a_5 = 1/3\). Since exactly one of the coefficients is zero, the center lies on a three dimensional face of the simplex but no two dimensional face, so the center must be interior to a facet of the simplex. \(\square\)

This yields an immediate corollary by noting that three half-spaces through one point must have an overlapping pair:

**Corollary 7.3.** Given any three class 2 simplices in \(\Pi^*_2,1\), at least two of the three simplices must overlap, i.e., there is a point interior to both.

With these, we obtain the main result of this section.

**Theorem 7.4.** The minimal cover of \(\Pi^*_2,1\) is 10 simplices, and this can be accomplished with a triangulation.

*Proof.* Proposition [7.1] shows that the largest class of a simplex in \(\Pi^*_2,1\) is two. Table [1] shows that a cover must have at least 9 simplices. Then there must be at least three class 2 simplices in the cover; otherwise, two class 2 simplices and seven class 1 simplices would not be enough to cover \(\Pi^*_2,1\) (which has total class 12).

On the other hand, there must be at least six class 1 simplices in any cover, because any pair of opposite prism facets of \(\Pi^*_2,1\) require six tetrahedra to cover them that are facets of simplices in \(\Pi^*_2,1\). These are distinct simplices, because no simplex can have exterior facets in parallel facets of \(\Pi^*_2,1\), by Proposition [5.7].

Thus a size 9 cover must consist of three class 2 simplices and six class 1 simplices. Since their total class is 12, these would have to cover \(\Pi^*_2,1\) without overlapping interiors; however, Corollary [7.3] shows that this is impossible. Therefore, a cover of \(\Pi^*_2,1\) must be size 10 or more.

We now exhibit a size 10 triangulation, then describe its construction. Order the vertices of \(\Pi^*_2,1\) numerically by their reduced coordinates, and label them by the twelve symbols 1, 2, 3, 4, ..., 8, 9, 0, #, *. Thus: 1 represents (0; 0; 0, 0), 2 represents (0; 0; 0, 1), 3 represents (0; 0; 1, 0), 4 represents (0; 1; 0, 0), and so on, until finally * represents (1; 1; 1, 0).

Using these symbols, we now specify 10 simplices of a minimal triangulation of \(\Pi^*_2,1\) by their 5 vertices:

\[
\begin{bmatrix}
[1850*], 1450*, 1456*, 1356*, [1358*], 1398*, 1798*, 1708*, \\
#850*, 13582
\end{bmatrix}.
\]

Two simplices are said to be adjacent if they meet face-to-face along a facet or, equivalently, if they differ in just one vertex. The 8 simplices in the first row above form
a cycle—each is adjacent to the simplices next to it in the displayed row, with row ends 1850* and 1708* also adjacent. The bracketed simplices 1850* and 1358* play a special role in this triangulation—they are the “fat” simplices of class 2, while all other simplices are class 1. In addition 1850* and 1358* are adjacent to each other along the common facet 158*, and they are each adjacent to one of the simplices in the bottom row: #850* is the “corner” simplex at vertex # and is adjacent to 1850*, and 13582 is the corner simplex at vertex 2 and is adjacent to 1358*. These are all the adjacencies in the triangulation. Every other facet of a simplex lies in an exterior facet of the simplotope.

Figure 7. Two replacements used in our construction of a 10 simplex cover of $\Pi_{2,1}^s$. Cones over these 3-dimensional simplices produce 4-dimensional simplices that are part of a triangulation of $\Pi_{2,1}^s$.

Here is how we constructed this triangulation. Start with the standard triangulation of $\Pi_{2,1}^s$ consisting of 12 simplices, where every simplex is spanned by the two vertices $(0;0;0,1)$ and $(1;1;1,0)$ and three other vertices. Consider the convex hull of the following subset of points of $\Pi_{2,1}^s$:

$$(0;0;0,0), (0;0;0,0), (0;0;0,0), (1;0;0,0), (1;0;0,0),$$

$$(0;0;0,1), (0;0;0,1), (1;0;0,1), (1;0;0,1), (1;0;0,1), (1;0;0,1).$$

This is a cone over a cube, with apex at $(1;1;1,0)$. The standard triangulation of $\Pi_{2,1}^s$ triangulates this cone by six simplices, which arise as a cone over a standard triangulation
of the 3-dimensional cube, depicted at top left of Figure \ref{fig7}. The triangulation $T$ of this cone can be replaced with a new triangulation $T'$ formed by the cone of $(1; 1; 1, 0)$ over a triangulation of the cube of size 5; see top right of Figure \ref{fig7}. This produces a size 11 cover of $\Pi^*_2, 1$. In fact, it is a triangulation, because both $T$ and $T'$ meet the rest of the triangulation in exactly the same way, i.e., the facets of $T$ and $T'$ that are on the boundary of the cone but not exterior to $\Pi^*_2, 1$ are exactly the same. Referring to the top left diagram of Figure \ref{fig7}, those facets are cones of $(1; 1; 1, 0)$ over any of the triangles on the three faces of the cube containing $(0; 0; 0)$. Note that these triangles are unchanged in the top right diagram of Figure \ref{fig7}.

Now, consider the convex hull of the following points:

\[(0; 0; 1, 0), (0; 1; 1, 0), (1; 0; 1, 0), (1; 1; 1, 0), (0; 0; 0, 1), (0; 1; 0, 1), (1; 0; 0, 1), (0; 0; 0, 0).\]

This is a cone of $(0; 0; 0, 0)$ over a cube with a corner cut off; see the bottom left diagram of Figure \ref{fig7}. In the size 11 cover of $\Pi^*_2, 1$ above, this cone has a triangulation $W$ by 5 simplices, and it is the cone over the triangulation depicted in the bottom left diagram of Figure \ref{fig7}. However, we can replace $W$ by $W'$, the cone over the triangulation in the bottom right diagram of Figure \ref{fig7} which has just 4 simplices. The replacement of $W$ by $W'$ produces a triangulation, because the facets of $W$ and $W'$ that are on the boundary of the cone but not exterior to $\Pi^*_2, 1$ are exactly the same. In the bottom left diagram of Figure \ref{fig7} such facets are formed by the dotted triangle, and the two vertical facets bordering the dotted triangle. Since these facets are unchanged in the bottom right diagram, we obtain the triangulation described above that has just 10 simplices. 

\[\Box\]

8. Discussion

Our results in Table \ref{tab1} may possibly be improved by considering additional information that is not contained in our linear program.

For instance, in Section \ref{sec7} we were able to improve the bounds for the triangle cross square by noting that the larger simplices in that polytope must overlap. Such considerations may close the gap between upper and lower bounds for minimal triangulations of simplotopes in several specific dimensions. For example, for $\Delta^2 \times \Delta^2 \times \Delta^1$, our lower bound is 20 and the standard triangulation gives a construction of size 30. For $\Delta^2 \times \Delta^2 \times \Delta^2$, our lower bound is 50, and by comparison the standard construction has size 90.

Also, the bounds that we used in our linear program for $V(s, t)$ only relied on known results for the volumes of simplices in cubes (i.e., the Hadamard determinant problem); however, bounds for $V(s, t)$ might perhaps be improved by restricting attention to volumes of simplices in simplotopes (embedded in cubes).

Another direction that may improve our bounds slightly is to consider corner simplices, as was done in \ref{sec8}. The idea is that corner simplices in a cube have a large number of exterior faces, and no other simplices have nearly as many. There ought to be an analogous result for simplotopes.
Finally, we remark that our methods may be generalized further to consider products of simplices of dimension greater than 2, by more extensive bookkeeping of the interactions between simplices of various dimensions as we did here for simplices and triangles.

Acknowledgements. The authors wish to thank Deborah Berg for helpful comments.

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