LONG-TIME SOLVABILITY FOR THE 2D DISPERSE SQG EQUATION WITH IMPROVED REGULARITY

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Abstract. In this paper, we study the long-time existence and uniqueness (solvability) for the initial value problem of the 2D inviscid dispersive SQG equation. First we obtain the local solvability with existence-time independent of the amplitude parameter $A$. Then, assuming more regularity and using a blow-up criterion of BKM type and a space-time estimate of Strichartz type, we prove long-time solvability of solutions in Besov spaces for large $A$ and arbitrary initial data. In comparison with previous results, we are able to consider improved cases of the regularity and larger initial data classes.

1. Introduction. We are concerned with the 2-dimensional (2D) inviscid dispersive SQG equation

$$
\begin{cases}
\partial_t \theta + (u \cdot \nabla) \theta + Au^2 = 0, & \text{in } \mathbb{R}^2 \times (0, \infty) \\
u = (u_1, u_2) = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), \\
\theta(x, 0) = \theta_0(x), & \text{in } \mathbb{R}^2,
\end{cases}
$$

(1.1)

where $\theta = \theta(x, t)$ is a scalar function, which represents the potential temperature of the fluid (or a buoyancy field), $u$ stands for the velocity field and $A > 0$ is the amplitude parameter. The Riesz transforms $\mathcal{R}_l, l = 1, 2$, are defined by means of the Fourier transform as

$$\mathcal{R}_l f(\xi) := \frac{i\xi_l}{|\xi|} \hat{f}(\xi).$$

Equation (1.1) models the evolution of a surface temperature or surface buoyancy field in a rapidly rotating stratified fluid, which plays the same role as the conserved potential vorticity driving the interior dynamics. The presence of an environmental horizontal gradient $-\mathcal{R}_1 \theta = \partial_x A^{-1} \theta$, where $A := \sqrt{-\Delta}$, represents the meridional

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advection of a large-scale buoyancy, which comes from the meridional variation of the Coriolis parameter \[26\].

In recent years, the unforced SQG (i.e. \(A = 0\)) has spurred a lot of mathematical research since the nondissipative (inviscid) SQG equation have properties that are similar to those of the 3D Euler (E) system in vorticity form \[14\], although the former equation has been shown to possess global finite energy weak solutions \[29\]. Moreover, in the presence of a fractional dissipation given by \((\sqrt{-\Delta})^\alpha\), \(0 < \alpha \leq 2\), the issue of **global regularity** has been the object of numerous studies. In particular, the critical case \((\alpha = 1)\) is challenging since the balance between the nonlinearity and the dissipative term is the same no matter the scale at which one zooms in, so that, in this sense, this case is the 2D analogue of the 3D Navier-Stokes (NS) system. The global regularity of the critical dissipative SQG was an outstanding open problem until the independent breakthroughs in \[6\] and \[23\]. Unlike, this problem is still open for the supercritical dissipative SQG \((0 < \alpha < 1)\). For extensive details and further issues related to the SQG, the reader is referred to, e.g., \[8\], \[10\], \[11\], \[15\], \[16\], \[17\], \[21\], \[29\], \[35\] and their references.

Just as for the 3D Euler equations, and the supercritical dissipative SQG and Navier-Stokes equations, the long-time existence and uniqueness (i.e., solvability) of the inviscid SQG equation is an outstanding open problem in several settings. In fact, the problem is subtle since the 3D Euler equations can exhibit ill-posedness in Sobolev and Besov spaces, even locally-in-time, depending on the regularity index (see \[5\] for further details). However, the dispersive forcing in \(1.1\) has a crucial effect on the dynamics and makes this equation analogous to the 3D Euler equation with Coriolis forcing term \(\Omega (e_3 \times u)\), i.e. the Euler-Coriolis (EC) system

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p + \Omega (e_3 \times u) &= 0, \\
\nabla \cdot u &= 0, \quad u|_{t=0}(x) = u_0(x), \quad x \in \mathbb{R}^3,
\end{align*}
\]

where the parameter \(|\Omega|^{-1}\) is the Rossby number. The viscous case of (EC) then forms the 3D Navier-Stokes-Coriolis (NSC) system and both these systems are basic oceanographic and atmospheric models dealing with large-scale phenomena (cf. \[13\]). At the same time, the dispersive forcing in \(1.1\) does not contribute to energy decay (see Remark 2.6) similarly to the Coriolis forcing in (EC), so that the 2D analogues of (EC) and (NSC) are, respectively, equation \(1.1\) and the dissipative dispersive SQG (viscous case of \(1.1\)). The main advantage of these systems is that in the limit of vanishing Rossby number \((\Omega \to \infty)\) a stabilization effect arising from the Coriolis term yields the long-time well-posedness of strong solutions for arbitrary initial data \[2, 7, 19, 24\].

In particular, Koh, Lee and Takada \[24\] recently showed that (EC) has a unique local-in-time solution \(u\) in the space \(C([0,T]; H^s(\mathbb{R}^3)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^3))\) for \(s > s_0 = \frac{3}{2} + 1\) and \(u_0 \in H^s(\mathbb{R}^3)\). Also, assuming more regularity on \(u_0\), \(s > s_1 = \frac{5}{2} + 1\), they proved that the solution \(u\) can be uniquely continued up to an arbitrarily large time \(T_{\Omega}\) for speeds of rotation \(|\Omega|\) sufficiently large; the proof relies crucially on their Strichartz-type estimates. For the viscous case (NSC), similar estimates are used to show global well-posedness in Sobolev spaces (see \[2\], \[3\], \[13\], \[19\]).

Bearing in mind the previous results for (EC) (and also those for 3D Euler equations, see \[5\], \[9\], \[27\], \[36\]), the cases \(s_0\) and \(s_1\) emerge as important values of regularity for local and long-time solvability of (EC), respectively, in finite \(L^2\)-energy spaces of Sobolev and Besov types, i.e., \(H^s_p\) and \(B^s_{p,q}\)spaces with \(p = 2\). These cases were treated in \[1\], where the results of \[24\] were extended by employing
the framework of Besov spaces. There, the local solvability were proved for initial data in the borderline Besov space $B_{2,1}^s$ and small existence-time independent of $\Omega \in \mathbb{R}$. It is then also shown that, for large $|\Omega|$, it is possible to obtain long-time solvability of (EC) in $B_{2,1}^s$ with the improved regularity $s = s_1$.

In view of the results for (EC) in $H^s_0$ and $B_{p,q}^s$-spaces, and making a dimensional analysis, a suitable regularity index for the long-time solvability of (1.1) would be $s_1 = \frac{5}{2} + 2$ for $p = 2$. Thus, exploring the dispersive effect (i.e., for large values of $A$), our main aim is to show the long-time solvability of (1.1) in Besov spaces with the improved regularity $s = s_1$. Our starting point are the results developed for (1.1) which are related to those mentioned above for (EC), mainly the papers [18] and [33]. In [18], Elgindi and Widmayer proved sharp dispersive estimates and applied it to show existence of strong solutions with $A = 1$ and existence-time $T \sim \varepsilon^{-4/3}$ where $\|\theta_0\|_{H^{s+1}}, \|\theta_0\|_{W^{3+\beta,1}} \leq \varepsilon$, with $\delta, \beta > 0$. In fact, by means of a scaling argument, their result works well for arbitrary $T > 0$ and $\theta_0 \in H^{4+\delta} \cap W^{3+\beta,1}$ provided that $A \geq C(\varepsilon T^{1/2} + \varepsilon^2 T^{3/2})^2$ where $C$ is a universal constant.

The dispersive and corresponding Strichartz estimates were later generalized by Wan and Chen [33] to show long-time solvability of strong solutions for (1.1) with an improved regularity taking $\theta_0 \in H^s$ and $s > 3$, as well as for a Boussinesq-type system derived from it. Moreover, in a follow-up to [33], Wan [32] refined the condition on the size of the dispersive forcing in [33] to obtain long-time solvability and showed that it suffices that $A \geq C T^2 \|\theta_0\|_{H^s}^2$, for $s > 3$ and some universal constant $C > 0$. We remark that analogous results to those of (NSC) have been shown for dissipative dispersive SQG, particularly dealing with the issues of global regularity [22], as well as the global well-posedness and asymptotic behavior of solutions with large dispersive forcing [7].

However, the methods of [32, 33] only treat the cases for regularity $s > 3$ and do not reach the case $s = 3$. Notice that for $s > 3$ we have the continuous inclusion $H^s(\mathbb{R}^2) \hookrightarrow B_{2,1}^s(\mathbb{R}^2)$. Thus, our main goal is to improve these results to obtain long-time solvability for $s = 3$, although we borrow from their Strichartz estimates. We prove the following results.

**Theorem 1.1.** Let $s$ and $q$ be real numbers such that $s > 2$ with $1 \leq q \leq \infty$ or $s = 2$ with $q = 1$.

(i) (Local solvability) Let $\theta_0 \in B_{2,q}^s(\mathbb{R}^2)$. There exists $T = T(\|\theta_0\|_{B_{2,q}^s}) > 0$ such that (1.1) has a unique solution $\theta \in C([0, T]; B_{2,q}^s(\mathbb{R}^2)) \cap C^1([0, T]; B_{2,q}^{s-1}(\mathbb{R}^2))$, for all $A \in \mathbb{R}$.

(ii) (Long-time solvability) Let $T \in (0, \infty)$ and $\theta_0 \in B_{2,q}^{s+1}(\mathbb{R}^2)$. There exists $A_0 = A_0(T, \|\theta_0\|_{B_{2,q}^{s+1}}) > 0$ such that if $|A| \geq A_0$ then (1.1) has a unique solution $\theta \in C([0, T]; B_{2,q}^{s+1}(\mathbb{R}^2)) \cap C^1([0, T]; B_{2,q}^s(\mathbb{R}^2))$.

Our proofs also provide a relation between the strength of the dispersive forcing and the increased time of existence (see (4.1)), where in order to deal with the critical case $s = 3$ we start from the local well-posedness in item (i) of Theorem 1.1 and later use a blow-up criterion. In general lines then, as in [1, 24], we employ the following basic steps: construction of approximate solutions $\{\theta^\varepsilon\}_{\varepsilon \in (0,1)}$; a priori estimates uniformly w.r.t the amplitude parameter $A$; passing to the limit for obtaining local-in-time solvability; blow-up criterion; and finally long-time solvability.

Here we carry out estimates in Besov spaces with suitable regularity for (1.1). In order to obtain a solution as the limit of $\{\theta^\varepsilon\}_{\varepsilon \in (0,1)}$, we control the approximations $\theta^\varepsilon$ via estimates based on localizations and $B_{2,1}^s$-norms (see, e.g., Proposition 3.3
and the proof of Theorem 1.1) and using the embedding $B^{1} 2_{1}(\mathbb{R}^{2}) \hookrightarrow L^{\infty}(\mathbb{R}^{2})$ which is not verified for $H^{1}(\mathbb{R}^{2})$. Then, for large values of $A$ and $s = 3$, we obtain the long-time solvability by showing a blow-up criterion and controlling globally the integral $\int_{0}^{\infty} \|\nabla u(\tau)\|_{L^{\infty}} d\tau$ by using Besov-norms of $\theta$.

We remark that Chae-Lee [12] showed the global existence and uniqueness for the dissipative SQG in the Besov space $B^{2-\gamma}_{2,1} (0 < \gamma < 1)$ for small initial data. This result also applies to the case $\gamma = 0$ and is thus linked to our local result. However, the approximation used in [12] is different than our parabolic regularization approach, which we used specifically to obtain a local result with existence time independent of the dispersion parameter $A$. This independence allows us to show the extension to long-time solvability without restriction on the size of initial data.

In this regard our work is also motivated by Vishik’s result [31] where the long-time solvability for the 2D Euler equation was shown in Besov spaces of borderline regularity $B^{2/p+1}_{p,1}$ with $1 < p < \infty$. See also Chae [9] for local solvability of $n$-dimensional Euler equations in borderline Besov spaces $B^{n/p+1}_{p,1}$.

Given that the dimensionality of the inviscid SQG is analogous to that of the 3D Euler equation it is very hard to obtain long-time solvability for arbitrary initial data in the same borderline Besov spaces of [31] and in that of [12] with $\gamma = 0$. For this reason we consider it relevant to show that the stabilization effect from large dispersive forcing allows the long-time solvability for arbitrary initial data with the improved regularity $s = s_1 = 3$, as is the case in the 3D Euler-Coriolis equations. Nevertheless, it is open whether this regularity is sharp.

This paper is organized as follows. In Section 2 we give some preliminaries about Besov spaces and Strichartz estimates. Section 3 is devoted to the parabolic approximation scheme $\{\theta^\epsilon\}_{\epsilon > 0}$ where we show its local existence uniformly with respect to the parameters $\epsilon$ and $A$. The proof of Theorem 1.1 is performed in Section 4 where items (i) and (ii) are addressed in subsections 4.1 and 4.2, respectively.

## 2. Besov spaces and Strichartz estimates

In this section we review some facts about Besov spaces. For further details on these spaces, the reader is referred to [4]. Moreover, we give some suitable estimates of Strichartz type.

### 2.1. Besov spaces.

We denote the Schwartz class and its dual (space of tempered distributions) respectively by $S(\mathbb{R}^{2})$ and $S'(\mathbb{R}^{2})$. For $f \in S'(\mathbb{R}^{2})$, $\hat{f}$ stands for the Fourier transform of $f$. Let $\phi_0$ be a radial function in $S(\mathbb{R}^{2})$ such that $\text{supp} \hat{\phi}_0 \subset \{\xi \in \mathbb{R}^{2} : \frac{1}{2} \leq |\xi| \leq 2\}$, $0 \leq \hat{\phi}_0(\xi) \leq 1$ for all $\xi \in \mathbb{R}^{2}$, and

$$\sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^{2}\setminus\{0\}, \quad (2.1)$$

where $\phi_j(x) := 2^{2j}\phi_0(2^jx)$. For $k \in \mathbb{Z}$, let $S_k \in \mathcal{S}$ be defined via Fourier transform by

$$\hat{S_k}(\xi) = 1 - \sum_{j \geq k+1} \hat{\phi}_j(\xi),$$

and set $\psi = S_0$. Let $\mathcal{P}$ denote the set of polynomials with two variables. For each $j \in \mathbb{Z}$, consider the Fourier localization operator $\Delta_j : S'(\mathbb{R}^{2}) \to S'(\mathbb{R}^{2})$ defined by $\Delta_j \hat{f} = \hat{\phi}_j \hat{f}$. Using (2.1), we can decompose $f$ as

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f \quad \text{in } S'(\mathbb{R}^{2})/\mathcal{P},$$
which is called the Littlewood-Paley decomposition of $f$.

For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the homogeneous and inhomogeneous Besov spaces are defined respectively by

$$B_{p,q}^s(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2)/\mathcal{P} : \| f \|_{B_{p,q}^s} := \| \{ 2^{js} \| \Delta_j f \|_{L^p} \}_{j \in \mathbb{Z}} \|_{\ell^p(\mathbb{Z})} < \infty \right\}$$

and

$$B_{p,q}^s(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \| f \|_{B_{p,q}^s} := \| \{ 2^{js} \| \Delta_j f \|_{L^p} \}_{j \in \mathbb{N}} + \| \psi \ast f \|_{L^p} < \infty \right\}.$$  

The spaces $B_{p,q}^s$ and $B_{p,q}^s$ endowed with $\| \cdot \|_{B_{p,q}^s}$ and $\| \cdot \|_{B_{p,q}^s}$ are Banach spaces. For $s > 0$, the following equivalence of norms holds:

$$\| f \|_{B_{p,q}^s} \sim \| f \|_{B_{p,q}^s} + \| f \|_{L^p}. \quad (2.2)$$

In view of (2.2), without loss of generality, we can assume that

$$\| f \|_{B_{p,q}^s} := \| f \|_{B_{p,q}^s} + \| f \|_{L^p}. \quad (2.3)$$

In the next lemma, we recall the Bernstein inequality.

**Lemma 2.1.** Let $1 \leq p \leq \infty$ and let $f \in L^p$ be such that $\text{supp} \ \hat{f} \subset \{ \xi \in \mathbb{R}^2 : 2^{-s} \leq |\xi| < 2^s \}$. Then, we have the estimates

$$C^{-1}2^{|k|} \| f \|_{L^p} \leq \| D^k f \|_{L^p} \leq C2^{|k|} \| f \|_{L^p},$$

where $C = C(k)$ is a positive constant.

**Remark 2.2.** Using the above lemma, one can prove the equivalence

$$\| f \|_{B_{p,q}^{s+1}} \sim \| D^k f \|_{B_{p,q}^s}.$$  

Moreover, for $s > n/p$ with $1 \leq p, q \leq \infty$, or $s = n/p$ with $1 \leq p \leq \infty$ and $q = 1$, we have that (see, e.g., [4])

$$\| f \|_{L^\infty} \leq C \| f \|_{B_{p,q}^s}.$$  

Then,

$$\| \nabla f \|_{L^\infty} \leq C \| \nabla f \|_{B_{p,q}^{s+1}} \leq C \| f \|_{B_{p,q}^s},$$

where $s > n/p + 1$ with $1 \leq p, q \leq \infty$ or $s = n/p + 1$ with $1 \leq p \leq \infty$ and $q = 1$.

Some Leibniz-type rules in Besov spaces are the subject of the lemma below (see [9]).

**Lemma 2.3.** Let $s > 0$, $1 \leq p_1, p_2 \leq \infty$, $1 \leq r_1, r_2 \leq \infty$ and $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$. Then, there exists a universal constant $C > 0$ such that

$$\| fg \|_{B_{p,q}^s} \leq C(\| g \|_{L^{p_2}} \| f \|_{B_{p_1,q}^s} + \| f \|_{L^{r_2}} \| g \|_{B_{p_1,q}^s}),$$

$$\| fg \|_{B_{p,q}^s} \leq C(\| g \|_{L^{p_2}} \| f \|_{B_{p_1,q}^s} + \| f \|_{L^{r_2}} \| g \|_{B_{p_1,q}^s}).$$

We will obtain estimates in a setting involving Besov norms of the terms in the right-side hand of the approximate problem (3.2). In this direction, we will need some estimates for the heat semigroup.

**Lemma 2.4.** (see [25]) Let $1 \leq p, q \leq \infty$ and $s_0 \leq s_1$. We have the estimate

$$\| e^{t \Delta} f \|_{B_{p,q}^s} \leq C(1 + t^{-\frac{1}{2}(s_1 - s_0)}) \| f \|_{B_{p,q}^{s_0}},$$

for all $f \in B_{p,q}^{s_0}(\mathbb{R}^n)$, where $C$ is a positive universal constant.
Lemma 2.8. Commutator estimates in $\mathcal{B}_{p,q}^\ast$ and $B_{p,q}^\ast$ will also be useful to obtain convergence of our approximate solutions. Recall the commutator operator

$$[v \cdot \nabla, \Delta_j]u = v \cdot \nabla(\Delta_j u) - \Delta_j(v \cdot \nabla u).$$

We have the following estimates (see [9, 30, 34]):

**Lemma 2.5.** Let $1 < p < \infty$ and $1 \leq q \leq \infty$.

(i) Let $s > 0$, $v_1 \in \mathcal{B}_{p,q}^\ast(\mathbb{R}^n)$ and $v_2 \in \mathcal{B}_{p,q}^\ast(\mathbb{R}^n)$. Assume further that $\nabla v_1 \in L^\infty(\mathbb{R}^n)$, $\nabla \cdot v_1 = 0$ and $\nabla v_2 \in L^\infty(\mathbb{R}^n)$. Then, there exists a positive universal constant $C$ such that

$$\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \| [v_1 \cdot \nabla, \Delta_j] v_2 \|^q_{L^p} \right)^{1/q} \leq C \left( \| \nabla v_1 \|_{L^\infty} \| v_2 \|_{\mathcal{B}_{p,q}^\ast} + \| \nabla v_2 \|_{L^\infty} \| v_1 \|_{\mathcal{B}_{p,q}^\ast} \right).$$

(ii) Let $s > -1$, $v_1 \in \mathcal{B}_{p,q}^{s+1}(\mathbb{R}^n)$ and $v_2 \in \mathcal{B}_{p,q}^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Assume further that $\nabla v_1 \in L^\infty(\mathbb{R}^n)$ and $\nabla \cdot v_1 = 0$. Then, there exists a positive universal constant $C$ such that

$$\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \| [v_1 \cdot \nabla, \Delta_j] v_2 \|^q_{L^p} \right)^{1/q} \leq C \left( \| \nabla v_1 \|_{L^\infty} \| v_2 \|_{\mathcal{B}_{p,q}^s} + \| v_2 \|_{L^\infty} \| v_1 \|_{\mathcal{B}_{p,q}^{s+1}} \right).$$

2.2. Strichartz estimates. We will employ the Strichartz estimates of [33], linked to the dispersive term $Au_2$ of (1.1), which will allow us to obtain long-time solvability for (1.1).

Firstly, we note that the term $Au_2$ does not contribute to the energy decay, which is in direct analogy with the Coriolis term in the rotating Euler equations.

**Remark 2.6.** Since $|\hat{\theta}(\xi)|^2 = \hat{\theta}(\xi)\hat{\theta}(-\xi)$, by Plancherel’s formula we have

$$\int_{\mathbb{R}^2} \Lambda^s \mathcal{R}_1 \theta(x) \Lambda^s \theta(x) \, dx = \langle \Lambda^s \mathcal{R}_1 \theta, \Lambda^s \theta \rangle_{\mathbb{R}^2} = \int_{\mathbb{R}^2} i \xi_1 |\xi|^{2s+1} |\hat{\theta}(\xi)|^2 \, d\xi = 0.$$

From the remark above, one has that global weak solutions can be obtained as in the inviscid SQG, see e.g. [16] and [29].

The lemma below contains the Strichartz estimate of [33].

**Lemma 2.7.** Let $4 \leq \gamma \leq \infty$ and $2 \leq r \leq \infty$ be such that

$$\frac{1}{\gamma} + \frac{1}{2r} \leq \frac{1}{4}.$$

Then, there holds

$$\| e^{\mathcal{R}_1 t f} \|_{L^\gamma(\mathbb{R}^+, L^r(\mathbb{R}^2))} \leq C \| f \|_{\mathcal{B}_{2,1}^{1-\frac{2}{\gamma}}(\mathbb{R}^2)}, \quad \text{for all } f \in \mathcal{B}_{2,1}^{1-\frac{2}{\gamma}}(\mathbb{R}^2).$$

From Lemma 2.7 and the change of variable $t \to At$, we get

$$\| e^{\mathcal{R}_1 At f} \|_{L^\gamma(\mathbb{R}^+, L^r(\mathbb{R}^2))} \leq C |A|^{-\frac{1}{\gamma}} \| f \|_{\mathcal{B}_{2,1}^{1-\frac{2}{\gamma}}(\mathbb{R}^2)}, \quad \mbox{(2.4)}$$

**Lemma 2.8.** Let $A \in \mathbb{R}$, $4 \leq \gamma \leq \infty$ and $2 \leq r \leq \infty$ be such that

$$\frac{1}{\gamma} + \frac{1}{2r} \leq \frac{1}{4}. \quad \mbox{(2.5)}$$

Then, there holds

$$\| G_{\pm}(At)f \|_{L^\gamma(0, \infty; L^r)} \leq C |A|^{-\frac{1}{\gamma}} \| f \|_{L^2}.$$
for all \( f \in L^2(\mathbb{R}^2) \), where

\[
\mathcal{G}_\pm(t)f(x) := \int_{\mathbb{R}^2} e^{i\xi \cdot x + it\hat{\phi}(\xi)} \hat{f}(\xi) \, d\xi
\]

and \( \hat{\phi} \) is a compactly supported smooth function in \( \mathbb{R}^2 \).

**Proof.** Proceeding similarly as Lemma 3.1 in [33] and using the change of variable \( t \to At \), it follows the result. \( \square \)

**Lemma 2.9.** Let \( s, t, A \in \mathbb{R}, 1 \leq q \leq \infty, 4 \leq \gamma \leq q \) and \( 2 \leq r \leq \infty \). Assume also (2.5). Then,

\[
\|e^{\pm tA\mathcal{R}_1}f\|_{L^\gamma(0,\infty;\dot{B}^s_{r,q})} \leq C|A|^{-\frac{1}{2}}\|f\|_{\dot{B}^s_{r,q}+1-\frac{2}{r}} ,
\]

for all \( f \in \dot{B}^s_{r,q}+1-\frac{2}{r}(\mathbb{R}^2) \).

**Proof.** As in [33], we use the change of variable \( \xi = 2^j \eta \) to get

\[
\int_{\mathbb{R}^2} e^{i\xi \cdot x + it\hat{\phi}(\xi)} \hat{f}(\xi) \, d\xi = \sum_{|j'-j| \leq 1} \int_{\mathbb{R}^2} e^{i2^j\eta \cdot x + it\hat{\phi}(\eta)} \hat{f}(2^j\eta) \, d\eta
\]

\[
= \sum_{|j'-j| \leq 1} \int_{\mathbb{R}^2} e^{i2^j\eta \cdot x + it\hat{\phi}(\eta)} \hat{g}(2^j\eta) \, d\eta
\]

\[
= \sum_{|j'-j| \leq 1} \mathcal{G}_\pm(t)g(2^j x),
\]

where \( g = \Delta_{j'}f(2^{-j}x) \). Then

\[
e^{\pm tA\mathcal{R}_1}\Delta_j f(x) = \sum_{|j'-j| \leq 1} \mathcal{G}_\pm(t)g(2^j x)
\]

and

\[
\|e^{\pm tA\mathcal{R}_1}\Delta_j f\|_{L^r} \leq \sum_{|j'-j| \leq 1} \|\mathcal{G}_\pm(t)g(2^j x)\|_{L^r}
\]

\[
\leq \sum_{|j'-j| \leq 1} 2^{-\frac{2}{r}j} \|\mathcal{G}_\pm(t)g\|_{L^r}.
\]

Multiplying by \( 2^{sj} \) and applying the \( l^q(\mathbb{Z}) \)-norm, we have

\[
\|\{2^{sj} \|e^{\pm tA\mathcal{R}_1}\Delta_j f\|_{L^r}\}_{j \in \mathbb{Z}}\|_{l^q(\mathbb{Z})} \leq \sum_{|j'-j| \leq 1} \{2^{(s-\frac{2}{r})j} \|\mathcal{G}_\pm(t)g\|_{L^r}\}_{j \in \mathbb{Z}}\|_{l^q(\mathbb{Z})}.
\]
Since $\gamma \le q$, we apply $L^\gamma(0, \infty)$ and use Lemma 2.8 to get
\[
\|e^{tR_1} f\|_{L^\gamma(0, \infty; B^s_{p,q})} \le \sum_{|j'-j| \le 1} \left\| \left\{ 2^{(s-\frac{j}{2})j} \| \mathbf{G}_{\pm}(t) g \|_{L^r} \right\}_{j \in \mathbb{Z}} \right\|_{L^\gamma(0, \infty)} \\
\le \sum_{|j'-j| \le 1} \left\{ 2^{(s-\frac{j}{2})j} \| \mathbf{G}_{\pm}(t) g \|_{L^\gamma(0, \infty; L^r)} \right\}_{j \in \mathbb{Z}} \\
\le C \sum_{|j'-j| \le 1} \left\{ 2^{(s-\frac{j}{2})j} \| g \|_{L^2} \right\}_{j \in \mathbb{Z}} \\
= C \sum_{|j'-j| \le 1} \left\{ 2^{(s+1-\frac{j}{2})j} \| \Delta^j f \|_{L^2} \right\}_{j \in \mathbb{Z}} \\
\le C \| f \|_{B^{s+\frac{1}{2}}_{2,q}}.
\]

Finally, we use the change of variable $t \to At$ to obtain (2.6).

3. Regularized problem and uniform estimates. In order to prove the item (i) of Theorem 1.1, we will first show the local existence and uniqueness of mild-solutions of the regularized problem constructed from (1.1) with viscous approximations. For this, we consider for each $0 < \varepsilon < 1$ the following problem
\[
\begin{cases}
\partial_t \theta^\varepsilon + (u^\varepsilon \cdot \nabla) \theta^\varepsilon + A u^\varepsilon - \varepsilon \Delta \theta^\varepsilon = 0, \\
u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) = (-R_2 \theta^\varepsilon, R_1 \theta^\varepsilon), \\
\theta^\varepsilon(0, x) = \theta_0(x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, 
\end{cases}
\tag{3.1}
\]
where $\theta_0 \in B_{2,2}^s(\mathbb{R}^2)$ is the initial data. Considering the heat semigroup, we can represent the above equation in the following integral formulation
\[
\theta^\varepsilon(t) = e^{t \Delta} \theta_0 - \int_0^t e^{(t-\tau) \Delta} A u^\varepsilon(\tau) \; d\tau - \int_0^t e^{(t-\tau) \Delta} (u^\varepsilon(\tau) \cdot \nabla) \theta^\varepsilon(\tau) \; d\tau. 
\tag{3.2}
\]

We will prove that the above problem possesses a unique solution for each $\varepsilon > 0$ in a suitable class involving Besov spaces. For the second term in the right-side hand of (3.2), we have the following estimates.

**Lemma 3.1.** Let $0 < \varepsilon < 1$, $0 < T < \infty$, $1 \le p, q \le \infty$, $A \in \mathbb{R}$ and $s \in \mathbb{R}$.

(i) There exists $C > 0$ (independent of $T$) such that
\[
\sup_{0 \le t \le T} \left\| \int_0^t e^{(t-\tau) \Delta} A v(\tau) \; d\tau \right\|_{B_{p,q}^s} \le C|A|T \sup_{0 \le t \le T} \| v(t) \|_{B_{p,q}^s},
\tag{3.3}
\]
for all $v \in C([0, T]; B_{p,q}^s(\mathbb{R}^2))$.

(ii) There exists $C > 0$ (independent of $T$) such that
\[
\left\| \int_0^t e^{(t-\tau) \Delta} A v(\tau) \; d\tau \right\|_{L^1(0, T; B_{p,q}^{s+1})} \le C|A|T \| v(t) \|_{L^1(0, T; B_{p,q}^{s+1})},
\tag{3.4}
\]
for all $v \in L^1(0, T; B_{p,q}^{s+1}(\mathbb{R}^2))$.

**Proof.** By Lemma 2.4, we have that
\[
\left\| \int_0^t e^{(t-\tau) \Delta} A v(\tau) \; d\tau \right\|_{B_{p,q}^s} \le C|A| \int_0^t \| v(t) \|_{B_{p,q}^s} \; dt \le C|A|T \sup_{0 \le t \le T} \| v(t) \|_{B_{p,q}^s}.
\]
Proof. Considering Lemma 3.2, it follows that
\[
\left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} A \phi(\tau) \, d\tau \right\|_{L^1(0,T;B_{p,q}^{s+1})} \leq C |A| \int_0^T \int_0^t \| \phi(\tau) \|_{B_{p,q}^{s+1}} \, d\tau \, dt \\
\leq C |A| T \| \phi(t) \|_{L^1(0,T;B_{p,q}^{s+1})}.
\]
This proves (ii). \( \Box \)

For the third term in the right-hand side of (3.2), we get the following estimates.

**Lemma 3.2.** Let \( 0 < \varepsilon < 1 \), \( 0 < T < \infty \) and \( 1 \leq p \leq \infty \). Assume that \( s > \frac{2}{p} \) with \( 1 \leq q \leq \infty \) or \( s = \frac{2}{p} \) with \( q = 1 \). There exists \( C > 0 \) (independent of \( T \)) such that the following inequalities hold
\[
\sup_{0 \leq t \leq T} \left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} (u(\tau) \cdot \nabla) \theta(\tau) \, d\tau \right\|_{B_{p,q}^{s+1}} \leq C \sup_{0 \leq t \leq T} \left\| u(t) \right\|_{B_{p,q}^{s+1}} \left\| \theta \right\|_{L^1(0,T;B_{p,q}^{s+1})}.
\]
and
\[
\left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} (u(\tau) \cdot \nabla) \theta(\tau) \, d\tau \right\|_{L^1(0,T;B_{p,q}^{s+1})} \leq C (T + \varepsilon^{-\frac{1}{2}} T^2) \sup_{0 \leq t \leq T} \left\| u(t) \right\|_{B_{p,q}^{s+1}} \left\| \theta \right\|_{L^1(0,T;B_{p,q}^{s+1})},
\]
for all \( u \in C([0,T];B_{p,q}^{s+1}(\mathbb{R}^2)) \) and \( \theta \in L^1(0,T;B_{p,q}^{s+1}(\mathbb{R}^2)) \).

**Proof.** Considering \( s, p \) and \( q \) as in the hypotheses and using Lemma 2.4 and Lemma 2.3, it follows that
\[
\left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} (u(\tau) \cdot \nabla) \theta(\tau) \, d\tau \right\|_{B_{p,q}^{s+1}} \leq \int_0^t \left\| (u(\tau) \cdot \nabla) \theta(\tau) \right\|_{B_{p,q}^{s+1}} \, d\tau \\
\leq C \int_0^t \left\| u(\tau) \right\|_{B_{p,q}^{s+1}} \left\| \theta(\tau) \right\|_{B_{p,q}^{s+1}} \, d\tau \\
\leq C \sup_{0 \leq t \leq T} \left\| u(t) \right\|_{B_{p,q}^{s+1}} \left\| \theta \right\|_{L^1(0,T;B_{p,q}^{s+1})}.
\]
Applying the supremum over \([0,T]\), we arrive at (3.5). On the other hand, notice that
\[
\left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} (u(\tau) \cdot \nabla) \theta(\tau) \, d\tau \right\|_{L^1(0,T;B_{p,q}^{s+1})} \\
= \int_0^T \left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} (u(\tau) \cdot \nabla) \theta(\tau) \, d\tau \right\|_{B_{p,q}^{s+1}} \, dt \\
\leq C \int_0^T \left( 1 + (\varepsilon(t-\tau))^{-\frac{1}{2}} \right) \left\| u(\tau) \right\|_{B_{p,q}^{s+1}} \left\| \theta(\tau) \right\|_{B_{p,q}^{s+1}} \, d\tau \, dt \\
\leq C \sup_{0 \leq t \leq T} \left\| u(t) \right\|_{B_{p,q}^{s+1}} \left\{ T \left\| \theta \right\|_{L^1(0,T;B_{p,q}^{s+1})} \right\}.
\[
\varepsilon^{-\frac{1}{2}} \int_0^T \|\theta(\tau)\|_{B^{s+1}_{p,q}} \int_\tau^T (t-\tau)^{-\frac{1}{2}} \, dt \, d\tau \leq C(T + \varepsilon^{-\frac{1}{2}} T^{\frac{1}{2}}) \sup_{0 \leq t \leq T} \|u(t)\|_{B^{s+1}_{p,q}} \|\theta\|_{L^1(0,T;B^{s+1}_{p,q})},
\]
which gives (3.6).

Now, we prove that (3.1) has a local-in-time solution in Besov spaces with existence time independent of \(A \in \mathbb{R}\) and \(\varepsilon \in (0,1)\).

**Proposition 3.3.** Let \(\varepsilon \in (0,1)\) and \(A \in \mathbb{R}\). Assume that \(s > 2\) with \(1 \leq q \leq \infty\) or \(s = 2\) with \(q = 1\), and that \(\theta_0 \in B^{s}_{2,q}(\mathbb{R}^2)\). Then, there exists a time \(T = T(\|\theta_0\|_{B^{s}_{2,q}}) > 0\) such that (3.1) has a unique strong solution \(\theta^\varepsilon\) satisfying

\[
\theta^\varepsilon \in C([0,T];B^{s}_{2,q}(\mathbb{R}^2)) \cap AC([0,T];B^{-1}_{2,q}(\mathbb{R}^2)) \cap L^1(0,T;B^{s+1}_{2,q}(\mathbb{R}^2)).
\]

Furthermore, \(\{\theta^\varepsilon\}_{\varepsilon \in (0,1)}\) is bounded in \(C([0,T];B^{s}_{2,q}(\mathbb{R}^2))\).

**Proof.** We divide the proof of Proposition 3.3 in three parts.

**First part (Local existence and uniqueness of \(\theta^\varepsilon\)).** In the first part, we will prove that there exist \(T_{\varepsilon,A} = T_{\varepsilon,A}(\varepsilon,A,\|\theta_0\|_{B^{s}_{2,q}}) > 0\) and a unique solution \(\theta^\varepsilon \in C([0,T_{\varepsilon,A}];B^{s}_{2,q}(\mathbb{R}^2)) \cap L^1(0,T_{\varepsilon,A};B^{s+1}_{2,q}(\mathbb{R}^2))\) to (3.1). For this, we recall the mild formulation for (3.1)

\[
\theta^\varepsilon(t) = e^{\varepsilon t \Delta} \theta_0 - \int_0^t e^{\varepsilon (t-\tau) \Delta} Au_0^\varepsilon(\tau) \, d\tau - \int_0^t e^{\varepsilon (t-\tau) \Delta} (u^\varepsilon(\tau) \cdot \nabla) \theta^\varepsilon(\tau) \, d\tau. \quad (3.7)
\]

Using Lemma 2.4, we obtain the following estimate

\[
\|e^{\varepsilon t \Delta} \theta_0\|_{L^1(0,T;B^{s+1}_{2,q})} = \int_0^T \|e^{\varepsilon t \Delta} \theta_0\|_{B^{s+1}_{2,q}} \, dt \\
\leq C \int_0^T (1 + (\varepsilon t)^{-\frac{1}{2}}) \|\theta_0\|_{B^{s}_{2,q}} \, dt \\
= C(T + \varepsilon^{-\frac{1}{2}} T^{\frac{1}{2}}) \|\theta_0\|_{B^{s}_{2,q}},
\]

for all \(0 < T < \infty\). Thus, for all \(0 < T < \infty\), we have that there exists \(C_0 > 0\) such that

\[
\sup_{0 \leq t \leq T} \|e^{\varepsilon t \Delta} \theta_0\|_{B^{s}_{p,q}} + (T + \varepsilon^{-\frac{1}{2}} T^{\frac{1}{2}}) \|e^{\varepsilon t \Delta} \theta_0\|_{L^1(0,T;B^{s+1}_{2,q})} \leq C_0 \|\theta_0\|_{B^{s}_{p,q}}.
\]

Now, consider the map

\[
\Gamma(\theta^\varepsilon)(t) = e^{\varepsilon t \Delta} \theta_0 - \int_0^t e^{\varepsilon (t-\tau) \Delta} Au_0^\varepsilon(\tau) \, d\tau - \int_0^t e^{\varepsilon (t-\tau) \Delta} (u^\varepsilon(\tau) \cdot \nabla) \theta^\varepsilon(\tau) \, d\tau
\]
and the complete metric space

\[
W_T = \left\{ \theta \in C([0,T];B^{s}_{p,q}) \cap L^1(0,T;B^{s+1}_{p,q}) : \|\theta\|_{W_T} \leq 2C_0 \|\theta_0\|_{B^{s}_{p,q}} \right\},
\]

where

\[
\|\theta\|_{W_T} := \sup_{0 \leq t \leq T} \|\theta\|_{B^{s}_{p,q}} + (T + \varepsilon^{-\frac{1}{2}} T^{\frac{1}{2}}) \|\theta\|_{L^1(0,T;B^{s+1}_{p,q})}.
\]
We will show that the map \( \Gamma \) is a contraction on \( W_T \) for some \( T > 0 \). By Lemma 3.1, Lemma 3.2 and the continuity of the Riesz transforms \( R_i \)'s in Besov spaces, we get constants \( C_1 > 0 \) and \( C_2 > 0 \) such that
\[
\|\Gamma(\theta^\varepsilon) - \Gamma(\bar{\theta})\|_{W_T} \\
\leq \left( C T |A| \|u_2^\varepsilon - \bar{\theta}\|_{W_T} + C_1 T |A| + C_2 (T + \varepsilon^{-\frac{1}{2}} T_{\frac{1}{2}})\right) \|\theta^\varepsilon\|_{W_T} \\
\leq C_0 \|\theta_0\|_{B_{2,q}^s} (1 + 2 C_1 T |A| + 8 C_2 (T + \varepsilon^{-\frac{1}{2}} T_{\frac{1}{2}})\|\theta_0\|_{B_{2,q}^s})
\]
for all \( \theta^\varepsilon, \bar{\theta} \in W_T \). On the other hand, using Lemma 2.4 and (3.8), we can estimate
\[
\|\Gamma(\theta^\varepsilon)\|_{W_{T,\varepsilon,A}} \leq \|\Gamma(\theta^\varepsilon)\|_{W_T} \leq \left( C T |A| \|u_2^\varepsilon - \bar{\theta}\|_{W_T} + \frac{1}{2} |\theta^\varepsilon| - |\bar{\theta}|\right) \|\theta^\varepsilon\|_{W_T}
\]
for all \( \theta^\varepsilon, \bar{\theta} \in W_{T,\varepsilon,A} \). Thus, we can applied the Banach Fixed Point Theorem in order to obtain a unique solution \( \theta^\varepsilon \in W_{T,\varepsilon,A} \) for (3.1).

**Second part (Strong property of \( \theta^\varepsilon \)).** We now prove that \( \theta^\varepsilon \in W_{T,\varepsilon,A} \) is a strong solution for (3.1) in the class
\[
C([0, T_{\varepsilon,A}]; B_{2,q}^s) \cap AC([0, T_{\varepsilon,A}]; B_{2,q}^{s-1}) \cap L^1(0, T_{\varepsilon,A}; B_{2,q}^{s+1}).
\]

In fact, by the above estimates and using that
\[
\theta^\varepsilon \in C([0, T_{\varepsilon,A}]; B_{2,q}^s) \cap L^1(0, T_{\varepsilon,A}; B_{2,q}^{s+1}),
\]
it is not difficult to see that
\[
Au_2^\varepsilon + (u^\varepsilon \cdot \nabla)\theta^\varepsilon \in L^1(0, T_{\varepsilon,A}; B_{2,q}^s) \quad \text{and} \quad -\varepsilon \Delta \omega^\varepsilon \in L^1(0, T_{\varepsilon,A}; B_{2,q}^{s-1}),
\]
where
\[
\omega^\varepsilon(t) := - \int_0^t \varepsilon^{(t-\tau)} \{ Au_2^\varepsilon(\tau) + (u^\varepsilon(\tau) \cdot \nabla)\theta^\varepsilon(\tau) \} \, d\tau.
\]
Thus \( \partial_t \omega^\varepsilon \in L^1(0, T_{\varepsilon,A}; B_{2,q}^{s-1}) \), and therefore, \( \omega^\varepsilon \in AC([0, T_{\varepsilon,A}]; B_{2,q}^{s-1}) \). Also, \( e^{\varepsilon t} \theta_0 \in AC([0, T_{\varepsilon,A}]; B_{2,q}^{s-1}) \). By standard arguments (see, e.g., [20, 28]), we have
that \( \theta^\varepsilon \in AC([0, T_{\varepsilon}; A]; B^{s-1}_{2,q}) \). Moreover, the uniqueness statement can be obtained from the fact that \( \theta^\varepsilon \) is the unique solution for (3.1) in \( W_{T_{\varepsilon,A}} \).

**Third part (Boundedness of \( \theta^\varepsilon \in C([0, T]; B^{s}_{2,q}(\mathbb{R}^{2})) \)).** Finally, we are going to prove the result. For that, we apply the Littlewood-Paley operator \( \Delta_j \) to the equation in (3.1) and we take the \( L^2 \)-norm product with \( \Delta_j \theta^\varepsilon(t) \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_j \theta^\varepsilon(t) \|_{L^2}^2 + \varepsilon \langle -\Delta_j \theta^\varepsilon(t), \Delta_j \theta^\varepsilon(t) \rangle_{L^2} = -\langle \Delta_j (u^\varepsilon(t) \cdot \nabla) \theta^\varepsilon(t), \Delta_j \theta^\varepsilon(t) \rangle_{L^2},
\]

where we have used Plancherel’s Theorem (see Remark 2.6) applied to the Littlewood-Paley localizations

\[
\langle \Delta_j R_1 \theta^\varepsilon, \Delta_j \theta^\varepsilon \rangle_{L^2} = i \int_{\mathbb{R}^2} \frac{\xi_1}{|\xi|} |\hat{\phi}_j(\xi)|^2 |\hat{\theta}(\xi)|^2 \, d\xi = 0.
\]

Since the second term in the right-hand side of (3.10) is non-negative,

\[
\langle (u^\varepsilon(t) \cdot \nabla) \Delta_j \theta^\varepsilon(t), \Delta_j \theta^\varepsilon(t) \rangle_{L^2} = 0
\]

and recalling the definition of the commutator \([u^\varepsilon(t) \cdot \nabla, \Delta_j]\), we get

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_j \theta^\varepsilon(t) \|_{L^2}^2 \leq \langle [u^\varepsilon(t) \cdot \nabla, \Delta_j] \theta^\varepsilon(t), \Delta_j \theta^\varepsilon(t) \rangle_{L^2}.
\]

Thus,

\[
\frac{d}{dt} \| \Delta_j \theta^\varepsilon(t) \|_{L^2} \leq \| [u^\varepsilon(t) \cdot \nabla, \Delta_j] \theta^\varepsilon(t) \|_{L^2}.
\]

By Hölder inequality, (3.12), Lemma 2.5 and the continuity of \( R_1 \) in Besov spaces, we have that

\[
\frac{d}{dt} \| \theta^\varepsilon(t) \|_{B^s_{2,q}} = \| \theta^\varepsilon(t) \|^{1-q}_{B^s_{2,q}} \sum_{j \in \mathbb{Z}} 2^{sj(q-1)} \| \Delta_j \theta^\varepsilon(t) \|_{L^2}^{q-1} 2^{sj} \frac{d}{dt} \| \Delta_j \theta^\varepsilon(t) \|_{L^2} \\
\leq \| \theta^\varepsilon(t) \|^{1-q}_{B^s_{2,q}} \| \theta^\varepsilon(t) \|_{B^s_{2,q}}^{q-1} \left( \sum_{j \in \mathbb{Z}} 2^{sjq} \left( \frac{d}{dt} \| \Delta_j \theta^\varepsilon(t) \|_{L^2} \right) \right)^{\frac{q}{2}} \\
\leq \left( \sum_{j \in \mathbb{Z}} 2^{sjq} \left( \| [u^\varepsilon(t) \cdot \nabla, \Delta_j] \theta^\varepsilon(t) \|_{L^2} \right) \right)^{\frac{q}{2}} \\
\leq C \left( \| [u^\varepsilon(t) \cdot \nabla], \theta^\varepsilon(t) \|_{L^\infty} \| \theta^\varepsilon(t) \|_{B^s_{2,q}} + \| [\nabla \theta^\varepsilon(t), \theta^\varepsilon(t)] \|_{L^\infty} \| u^\varepsilon(t) \|_{B^s_{2,q}} \right) \\
\leq C \left( \| u^\varepsilon(t) \|_{B^s_{2,q}} \| \theta^\varepsilon(t) \|_{B^s_{2,q}} + \| \theta^\varepsilon(t) \|_{B^s_{2,q}} \| u^\varepsilon(t) \|_{B^s_{2,q}} \right) \\
\leq C \| \theta^\varepsilon(t) \|_{B^s_{2,q}}^2.
\]

Also, taking the \( L^2 \)-norm product with \( \theta^\varepsilon(t) \) in the first equation of (3.1) and using the fact \( \nabla \cdot u^\varepsilon = 0 \), it follows that

\[
\frac{1}{2} \frac{d}{dt} \| \theta^\varepsilon(t) \|_{L^2}^2 + \varepsilon \langle -\Delta \theta^\varepsilon(t), \theta^\varepsilon(t) \rangle_{L^2} = 0.
\]

Therefore,

\[
\frac{d}{dt} \| \theta^\varepsilon(t) \|_{L^2} \leq 0.
\]
Recalling (2.3) and combining (3.13) and (3.14), we obtain the estimate
\[
\frac{d}{dt} \| \theta^\varepsilon(t) \|_{B_{2,q}^s} \leq C \| \theta^\varepsilon(t) \|_{B_{2,q}^s}^2,
\]
which yields
\[
\| \theta^\varepsilon(t) \|_{B_{2,q}^s} \leq \frac{\| \theta_0 \|_{B_{2,q}^s}}{1 - C \| \theta_0 \|_{B_{2,q}^s}} t \quad \text{for all } 0 \leq t < \frac{1}{C \| \theta_0 \|_{B_{2,q}^s}}.
\]
Taking \( T = T \left( \| \theta_0 \|_{B_{2,q}^s} \right) = \frac{1}{2C \| \theta_0 \|_{B_{2,q}^s}} \), it holds that
\[
\| \theta^\varepsilon(t) \|_{B_{2,q}^s} \leq 2 \| \theta_0 \|_{B_{2,q}^s} \quad \text{for all } 0 \leq t \leq T.
\]
(3.15)

The existence time \( T > 0 \) can be taken independent of \( \varepsilon \in (0,1) \) and \( A \in \mathbb{R} \). In fact, if \( T_{\varepsilon,A} < T \), using (3.9) and (3.15) we can take \( T'_{\varepsilon,A} = T'_{\varepsilon,A}(\|u_0\|_{B_{2,q}^s}) > 0 \) sufficiently small and obtain a solution for (3.1) with initial data \( \theta^\varepsilon(T_{\varepsilon,A}) \in B_{2,q}^s(\mathbb{R}^2) \) on the interval \([T_{\varepsilon,A}, T_{\varepsilon,A} + T'_{\varepsilon,A}]\). Thus, the solution \( \theta^\varepsilon \) can be extended to \([0, T_{\varepsilon,A} + T'_{\varepsilon,A}]\). In case of being necessary, the same argument can be repeated in order to extend \( \theta^\varepsilon \) to \([0, T_{\varepsilon,A} + 2T'_{\varepsilon,A}], [0, T_{\varepsilon,A} + 3T'_{\varepsilon,A}] \) and so on. Therefore, we obtain a solution \( \theta^\varepsilon \) for (3.1) on \([0, T] \) verifying estimate (3.15).

4. Proof of Theorem 1.1.

4.1. Proof of item (i). Let \( s \) such that \( s > 2 \) if \( 1 \leq q \leq \infty \) or \( s = 2 \) if \( q = 1 \). We will show that there exists a limit \( \theta \in C([0, T]; B_{2,q}^s(\mathbb{R}^2)) \) such that
\[
\theta^\varepsilon(t) \to \theta(t) \quad \text{in } B_{2,q}^s \quad \text{uniformly for } t \in [0, T].
\]
(4.1)

For this, let \( 0 < \varepsilon_1 < \varepsilon_2 < 1 \) and consider the following system
\[
\begin{aligned}
&\partial_t (\theta^{\varepsilon_1} - \theta^{\varepsilon_2}) - \varepsilon_1 \Delta (\theta^{\varepsilon_1} - \theta^{\varepsilon_2}) + (\varepsilon_2 - \varepsilon_1) \Delta \theta^{\varepsilon_2} = -A(u^{\varepsilon_1}_2 - u^{\varepsilon_2}_2) \\
&- ((u^{\varepsilon_1}_2 - u^{\varepsilon_2}_2) \cdot \nabla) (\theta^{\varepsilon_1} - \theta^{\varepsilon_2}) + (\varepsilon_2 - \varepsilon_1) \Delta \theta^{\varepsilon_2} = -A(u^{\varepsilon_1}_2 - u^{\varepsilon_2}_2)
\end{aligned}
\]
\[
\begin{aligned}
&u^{\varepsilon_1} = (-R_2 \theta^{\varepsilon_1}, R_1 \theta^{\varepsilon_1}), \quad u^{\varepsilon_2} = (-R_2 \theta^{\varepsilon_2}, R_1 \theta^{\varepsilon_2}), \\
&(\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(0, x) = 0.
\end{aligned}
\]
(4.2)

Next, we obtain estimates in \( B_{2,q}^{s-1} \) for the difference \( \theta^{\varepsilon_1} - \theta^{\varepsilon_2} \) uniformly in \([0, T].\)

Applying the \( L^2 \)-norm product in the first equation in (4.2) with \( \theta^{\varepsilon_1} - \theta^{\varepsilon_2} \), we have
\[
\frac{1}{2} \frac{d}{dt} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2}^2 - \varepsilon_1 (\Delta (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t), (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t))_{L^2}
\]
\[
= - (\varepsilon_2 - \varepsilon_1) (\Delta (\theta^{\varepsilon_2}(t), (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t))_{L^2} - A((u^{\varepsilon_1}_2 - u^{\varepsilon_2}_2)(t), (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t))_{L^2}
\]
\[
- \langle ((u^{\varepsilon_1}_2 - u^{\varepsilon_2}_2)(t) \cdot \nabla) \theta^{\varepsilon_1}(t), (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \rangle_{L^2}
\]
\[
- \langle ((u^{\varepsilon_2}_2(t) \cdot \nabla) \theta^{\varepsilon_2}(t), (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \rangle_{L^2}.
\]

By (3.11) and since \( u^{\varepsilon_k}_2 = R_1 \theta^{\varepsilon_k}, k = 1, 2 \), it follows that \( \nabla \cdot u^{\varepsilon_1} = \nabla \cdot u^{\varepsilon_2} = 0 \), and then
\[
A((u^{\varepsilon_1}_2 - u^{\varepsilon_2}_2, \theta^{\varepsilon_1} - \theta^{\varepsilon_2})_{L^2} = \langle (u^{\varepsilon_2}_2 \cdot \nabla) (\theta^{\varepsilon_1} - \theta^{\varepsilon_2}), (\theta^{\varepsilon_1} - \theta^{\varepsilon_2}) \rangle_{L^2} = 0.
\]

Thus,
\[
\frac{1}{2} \frac{d}{dt} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2}^2 - \varepsilon_1 (\Delta (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t), (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t))_{L^2}
\]
\[
= (\varepsilon_2 - \varepsilon_1) (\Delta (\theta^{\varepsilon_2}(t), (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t))_{L^2}
\]
\[
- \langle ((u^{\varepsilon_1}_2(t) - u^{\varepsilon_2}_2(t)) \cdot \nabla) \theta^{\varepsilon_1}(t), (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \rangle_{L^2}.
\]
By Cauchy-Schwartz inequality, the continuity of $R_1$’s in $L^2$, Remark 2.2 and the condition
\[ -\varepsilon_1 \langle \Delta (\theta^{\varepsilon_1} - \theta^{\varepsilon_2}), \theta^{\varepsilon_1} - \theta^{\varepsilon_2} \rangle_{L^2} > 0, \]
we have that
\[ \frac{1}{2} \frac{d}{dt} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2}^2 \leq (\varepsilon_2 - \varepsilon_1) - \Delta \| \theta^{\varepsilon_2}(t) \|_{L^2} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2} + C \| \nabla \theta^{\varepsilon_1}(t) \|_{L^\infty} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2}^2. \]

Then,
\[ \frac{d}{dt} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2} \leq \varepsilon_2 \| - \Delta \theta^{\varepsilon_2}(t) \|_{L^2} + C \| \theta^{\varepsilon_1}(t) \|_{B^s_2,q} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2}. \]

We integrate over $(0, t)$ and employ (3.15) in order to obtain
\[ \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2} \leq \varepsilon_2 \int_0^t \| - \Delta \theta^{\varepsilon_2}(\tau) \|_{L^2} \ d\tau + C \int_0^t \| \theta^{\varepsilon_1}(\tau) \|_{B^s_2,q} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(\tau) \|_{L^2} \ d\tau \]
\[ \leq C \varepsilon_2 T \| \theta^{\varepsilon_2} \|_{L^\infty(0,T;B^s_2,q)} + C \| \theta^{\varepsilon_1} \|_{L^\infty(0,T;B^s_2,q)} \int_0^t \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(\tau) \|_{L^2} \ d\tau \]
\[ \leq C \varepsilon_2 T \| \theta_0 \|_{B^s_2,q} + C \| \theta_0 \|_{B^s_2,q} \int_0^t \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(\tau) \|_{L^2} \ d\tau. \]

Applying Gronwall’s inequality, we arrive at
\[ \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2} \leq C \varepsilon_2 T \| \theta_0 \|_{B^s_2,q} \exp \left( C \| \theta_0 \|_{B^s_2,q} T \right), \]
and therefore
\[ \sup_{0 \leq t \leq T} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{L^2} \leq C \varepsilon_2 T \| \theta_0 \|_{B^s_2,q} \exp \left( C \| \theta_0 \|_{B^s_2,q} T \right) \to 0, \]
as $\varepsilon_2 \to 0^+$. Let $0 < \gamma < 1$ and $s_1, s_2, s_3 \geq 0$ be such that $s_3 = (1 - \gamma)s_1 + \gamma s_2$. By interpolation, we estimate
\[ \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{B^{s_3}_{2,q}} \leq C \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{B^{s_2}_{2,q}}^{1-\gamma} \| (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t) \|_{B^{s_1}_{2,q}}^\gamma. \]

Taking $s_1 = 0$, $s_2 = s$ and $s_3 = \theta s_2$ in (4.4), and using (3.15), $B^0_{2,2} = L^2$ and (4.3), we get the convergence
\[ \| \theta^{\varepsilon_1} - \theta^{\varepsilon_2} \|_{L^\infty(0,T;B^{s_3}_{2,q})} \leq C \| \theta_0 \|_{B^{s_3}_{2,q}} \| \theta^{\varepsilon_1} - \theta^{\varepsilon_2} \|_{L^\infty(0,T;L^2)} \to 0, \]
as $\varepsilon_2 \to 0^+$.

From the uniqueness of the limit in the sense of distribution and completeness, it follows that $\theta^\varepsilon \to \theta$ in $L^\infty(0,T;B^{s}_{2,q})$ for all $0 < \tilde{s} < s$. We deduce the convergence (4.1) by considering $\tilde{s} = s - 1$ and using that $\theta^\varepsilon \in C([0,T];B^{s-1}_{2,q}(\mathbb{R}^2))$.

Moreover, by (3.15), $\{ \theta^\varepsilon \}_{\varepsilon \in (0,1)}$ is bounded in $L^\infty(0,T;B^s_{2,q}(\mathbb{R}^2))$. Thus, we can obtain a subsequence $\theta_{\varepsilon_j} \to \theta$ weakly-* in $L^\infty(0,T;B^s_{2,q}(\mathbb{R}^2))$, as $\varepsilon_j \to 0$. It follows that
\[ \theta \in L^\infty(0,T;B^s_{2,q}(\mathbb{R}^2)) \cap C([0,T];B^{s-1}_{2,q}(\mathbb{R}^2)) \]
and
\[ \| \theta \|_{L^\infty(0,T;B^{s}_{2,q})} \leq \liminf_{\varepsilon_j \to 0} \| \theta_{\varepsilon_j} \|_{L^\infty(0,T;B^{s}_{2,q})} \leq 2 \| \theta_0 \|_{B^{s}_{2,q}}. \]
In what follows, we show that \( \theta \) verifies (1.1). Firstly, we use Lemma 2.3, the continuity of \( R \) in Besov spaces, Remark 2.2, (3.15) and (4.6) to estimate

\[
\int_0^t \| (u_s' \cdot \nabla) \varphi(\tau) - (u(\tau) \cdot \nabla) \theta(\tau) \|_{B_{2,q}^{s-2}} \, d\tau \\
= \int_0^t \| ((u_s - u)(\tau) \cdot \nabla) \varphi(\tau) + (u(\tau) \cdot \nabla)(\varphi - \theta)(\tau) \|_{B_{2,q}^{s-1}} \, d\tau \\
\leq C \int_0^t \| (u_s - u)(\tau) \|_{B_{2,q}^{s-2}} \| \varphi(\tau) \|_{B_{2,q}^{s-1}} + \| u(\tau) \|_{B_{2,q}^{s-2}} \| (\varphi - \theta)(\tau) \|_{B_{2,q}^{s-1}} \, d\tau \\
\leq C \int_0^t \left( \| \varphi(\tau) \|_{B_{2,q}^{s-1}} + \| \theta(\tau) \|_{B_{2,q}^{s-1}} \right) \| (\varphi - \theta)(\tau) \|_{B_{2,q}^{s-1}} \, d\tau \\
\leq CT \| \theta_0 \|_{B_{2,q}^{s-1}} \sup_{0 \leq t \leq T} \| (\varphi - \theta)(t) \|_{B_{2,q}^{s-1}} \to 0, \text{ as } \varepsilon \to 0^+ ,
\]

and so

\[
\int_0^t (u_s' \cdot \nabla) \varphi(\tau) \, d\tau \to \int_0^t (u(\tau) \cdot \nabla) \theta(\tau) \, d\tau \text{ in } L^\infty(0,T;B_{2,q}^{s-2}), \text{ as } \varepsilon \to 0^+. \tag{4.7}
\]

Notice also that

\[
\varepsilon \int_0^t \| - \Delta \varphi(\tau) \|_{B_{2,q}^{s-2}} \, d\tau \leq \int_0^t \| \varphi(\tau) \|_{B_{2,q}^{s-1}} \, d\tau \\
\leq \varepsilon T \| \varphi \|_{L^\infty(0,T;B_{2,q}^{s-1})} \leq C \varepsilon T \| \theta_0 \|_{B_{2,q}^{s-1}} \to 0
\]

and

\[
\int_0^t \| A(u_s' - u_2)(\tau) \|_{B_{2,q}^{s-2}} \, d\tau \leq CT \| A \| \sup_{0 \leq t \leq T} \| (\varphi - \theta)(t) \|_{B_{2,q}^{s-1}} \to 0,
\]

as \( \varepsilon \to 0^+ \). Then

\[
\varepsilon \int_0^t - \Delta \varphi(\tau) \, d\tau \to 0 \text{ in } L^\infty(0,T;B_{2,q}^{s-2}), \\
\int_0^t A u_s(\tau) \, d\tau \to \int_0^t A u_2(\tau) \, d\tau \text{ in } L^\infty(0,T;B_{2,q}^{s-2}), \text{ as } \varepsilon \to 0^+. \tag{4.8}
\]

Using (4.7) and (4.8), we can pass the limit in (3.1) as \( \varepsilon \to 0^+ \) to get

\[
\theta(t) - \theta_0 = - \int_0^t \{ A u_2(\tau) + (u(\tau) \cdot \nabla) \theta(\tau) \} \, d\tau \text{ in } B_{2,q}^{s-2}(\mathbb{R}^2). \tag{4.9}
\]

In view of the above estimates and (4.5), we have that both sides of (4.9) belong to \( C([0,T];B_{2,q}^{s-1}(\mathbb{R}^2)) \). It follows that (4.9) holds in \( B_{2,q}^{s-1}(\mathbb{R}^2) \) and \( \theta \) is a solution of (1.1) belonging to \( AC([0,T];B_{2,q}^{s-1}(\mathbb{R}^2)) \cap L^\infty(0,T;B_{2,q}^{s-1}(\mathbb{R}^2)) \).

Now, we claim that \( \theta \in C([0,T];B_{2,q}^{s-1}(\mathbb{R}^2)) \) for initial data \( \theta_0 \in B_{2,q}^{s-1}(\mathbb{R}^2) \). We denote \( u_k := S_k \theta \) for each \( k \in \mathbb{N} \), and we will show that the sequence \( \{ u_k \}_{k \in \mathbb{N}} \) converges to the solution \( \theta \) of (1.1) in \( L^\infty(0,T;B_{2,q}^{s-1}(\mathbb{R}^2)) \). By Lemma 2.5, we have that \( \partial_t \theta \in L^\infty(0,T;B_{2,q}^{s-1}(\mathbb{R}^2)) \). Therefore,

\[
\theta \in W^{1,\infty}([0,T];B_{2,q}^{s-1}(\mathbb{R}^2)) \subset C([0,T];B_{2,q}^{s-1}(\mathbb{R}^2)).
\]

Now, we apply the operator \( \Delta_j \) to the equation in (1.1) and afterwards take the \( L^2 \)-norm product with \( \Delta_j \theta \) to get

\[
\langle \partial_t \Delta_j \theta, \Delta_j \theta \rangle_{L^2} + \langle \Delta_j (u \cdot \nabla) \theta, \Delta_j \theta \rangle_{L^2} = 0,
\]
where we have used the property (3.11). Then,
\[
\frac{1}{2} \frac{d}{dt} \| \Delta_j \theta(t) \|_{L^2}^2 = \langle [u(t) \cdot \nabla, \Delta_j \theta(t)], \Delta_j \theta(t) \rangle_{L^2}.
\]

By the Cauchy-Schwartz inequality, it follows that
\[
\frac{d}{dt} \| \Delta_j \theta(t) \|_{L^2} \leq \| [u(t) \cdot \nabla, \Delta_j \theta(t)] \|_{L^2}.
\]

Integrating over \((0, t)\), we have that
\[
\| \Delta_j \theta(t) \|_{L^2} \leq \| \Delta_j \theta_0 \|_{L^2} + \int_0^t \| [u(\tau) \cdot \nabla, \Delta_j \theta(\tau)] \|_{L^2} \, d\tau.
\]

Thus,
\[
\| \theta(t) - w_k(t) \|_{B_{2,q}^{s+1}} \leq C \left( \sum_{j \geq k} 2^{sjq} \| \Delta_j \theta(t) \|_{L^2}^q \right)^{\frac{1}{q}}
\leq C \left( \sum_{j \geq k} 2^{sjq} \| \Delta_j \theta_0 \|_{L^2}^q \right)^{\frac{1}{q}}
+ C \int_0^t \left( \sum_{j \geq k} 2^{sjq} \| [u(\tau) \cdot \nabla, \Delta_j \theta(\tau)] \|_{L^2}^q \right)^{\frac{1}{q}} \, d\tau =: I_1 + I_2.
\]

Since \( \theta_0 \in B_{2,q}^s(\mathbb{R}^2) \), it follows that \( I_1 \) converges to zero as \( k \to \infty \). And by Lemma 2.5, the embedding \( B_{2,q}^{s+1}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2) \) and the fact \( \theta(t) \in B_{2,q}^s(\mathbb{R}^2) \) for \( t \geq 0 \), we have that \( I_2 \) converges to zero as \( k \to \infty \). Therefore, \( w_k \to \theta \) in \( L^\infty(0, T; B_{2,q}^{s+1}(\mathbb{R}^2)) \).

Moreover, notice that
\[
\| w_k(\tilde{t}) - w_k(t) \|_{B_{2,q}^{s+1}} = \| S_k(\theta(\tilde{t}) - \theta(t)) \|_{B_{2,q}^{s+1}}
\leq C \left( \sum_{j = -1}^{k+1} 2^{sjq} \| \Delta_j (\theta(\tilde{t}) - \theta(t)) \|_{L^2}^q \right)^{\frac{1}{q}}
\leq C 2^{k+1} \| \theta(\tilde{t}) - \theta(t) \|_{B_{2,q}^{s+1}}.
\]

This proves that \( w_k \in C([0, T]; B_{2,q}^s(\mathbb{R}^2)) \), because \( \theta \in C([0, T]; B_{2,q}^{s+1}(\mathbb{R}^2)) \). Thus, the limit \( \theta \) also belongs to \( C([0, T]; B_{2,q}^s(\mathbb{R}^2)) \), showing the desired claim. Moreover, since \( \theta \) satisfies
\[
\partial_t \theta = -(u \cdot \nabla) \theta - Au_2 \in C([0, T]; B_{2,q}^{s-1}(\mathbb{R}^2)),
\]
it follows that \( \theta \) is a strong solution in the class
\[
C([0, T]; B_{2,q}^s(\mathbb{R}^2)) \cap C^1([0, T]; B_{2,q}^{s-1}(\mathbb{R}^2)).
\]

**Uniqueness.** Let \( \theta^{(1)} \) and \( \theta^{(2)} \) be two solutions of (1.1) with initial data \( \theta_0 \in B_{2,q}^s(\mathbb{R}^2) \). We denote \( u^{(i)} := -R_2 \theta^{(i)}, u_2^{(i)} := R_1 \theta^{(i)}, u^{(i)} := (u^{(i)}_1, u_2^{(i)}), i = 1, 2, \)
\[ \tilde{u} := u^{(1)} - u^{(2)} \text{ and } \tilde{\theta} := \theta^{(1)} - \theta^{(2)}. \] Thus, considering the difference of the equations for \( \theta^{(1)} \) and \( \theta^{(2)} \), it follows that

\[
\begin{align*}
\begin{cases}
\partial_t \tilde{\theta} + (u^{(1)} \cdot \nabla) \tilde{\theta} + (\tilde{u} \cdot \nabla) \theta^{(2)} + A\tilde{u}_2 = 0, \\
\tilde{u} = (\tilde{u}_1, \tilde{u}_2) = (-\mathcal{R}_2 \tilde{\theta}, \mathcal{R}_1 \tilde{\theta}), \\
\tilde{\theta}(0, x) = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2.
\end{cases}
\tag{4.10}
\end{align*}
\]

We compute the \( L^2 \)-norm product of the first equation in (4.10) with \( \tilde{\theta} \) to get

\[
\frac{1}{2} \frac{d}{dt} \| \tilde{\theta} \|_{L^2}^2 + \langle (u^{(1)} \cdot \nabla) \tilde{\theta}, \tilde{\theta} \rangle_{L^2} + \langle (\tilde{\theta} \cdot \nabla) \theta^{(2)} , \tilde{\theta} \rangle_{L^2} + A\|\tilde{u}_2\|_{L^2} = 0.
\]

Since \( \nabla \cdot u^{(1)} = 0 \) and \( \tilde{u}_2 = \mathcal{R}_1 \tilde{\theta} \), then

\[
\langle (u^{(1)} \cdot \nabla) \tilde{\theta}, \tilde{\theta} \rangle_{L^2} = \langle \tilde{u}_2, \tilde{\theta} \rangle_{L^2} = 0.
\]

Thus

\[
\frac{1}{2} \frac{d}{dt} \| \tilde{\theta} \|_{L^2}^2 = - \langle (\tilde{\theta} \cdot \nabla) \theta^{(2)} , \tilde{\theta} \rangle_{L^2}
\]

and

\[
\frac{d}{dt} \| \tilde{\theta} \|_{L^2} \leq \| (\tilde{\theta} \cdot \nabla) \theta^{(2)} \|_{L^2}
\]

\[
\leq \| \nabla \theta^{(2)} \|_{L^\infty} \| \tilde{\theta} \|_{L^2}.
\]

Using the embedding \( B_{2,q}^{s-1} \hookrightarrow L^\infty \) and integrating over \((0, t)\), it follows that

\[
\| \tilde{\theta} \|_{L^2} \leq C\| \theta^{(2)} \|_{L^\infty(0, T; B_{2,q}^s)} \int_0^t \| \tilde{\theta}(\tau) \|_{L^2} \, d\tau.
\]

By \( \| \theta^{(2)} \|_{L^\infty(0, T; B_{2,q}^s)} < \infty \) and Gronwall’s inequality, we have that \( \| \tilde{\theta} \|_{L^2} = 0 \) for a.e. \( t \in [0, T] \), and therefore, we conclude that \( \theta^{(1)} \equiv \theta^{(2)} \).

4.2. **Proof of item (ii).** In this section we prove the long-time solvability of (1.1) for large values of \( A \). We start with a proposition containing a blow-up criterion.

**Proposition 4.1.** Let \( s \) and \( q \) be such that \( s > 2 \) with \( 1 \leq q \leq \infty \) or \( s = 2 \) with \( q = 1 \). Assume that \( \theta_0 \in B_{2,q}^s(\mathbb{R}^2) \) and \( \theta \) is the corresponding solution of (1.1) in the class \( C([0, T); B_{2,q}^s(\mathbb{R}^2)) \cap C^1([0, T); B_{2,q}^{s-1}(\mathbb{R}^2)) \) satisfying \( \int_0^T \| \nabla u(\tau) \|_{L^2} \, d\tau < \infty \), where \( u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) \). Then, there exists \( T' > T \) such that \( \theta \) can be extended to \([0, T')\) with \( \theta \in C([0, T'); B_{2,q}^s(\mathbb{R}^2)) \cap C^1([0, T'); B_{2,q}^{s-1}(\mathbb{R}^2)) \).

**Proof.** By item (i) of Theorem 1.1, we know that the existence time \( T > 0 \) is independent of \( A \). Taking the \( L^2 \)-inner product and using the fact that the dispersive term does not contribute to the energy estimates (see Remark 2.6), we get the energy equality

\[
\| \theta(t) \|_{L^2}^2 = \| \theta_0 \|_{L^2}^2 \quad \text{for all } \ t \in [0, T].
\tag{4.11}
\]

Applying \( \Delta_j \) in (1.1), multiplying the result by \( \Delta_j \theta \) and using \( \langle (u(t) \cdot \nabla) \Delta_j \theta, \Delta_j \theta \rangle_{L^2} = 0 \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_j \theta(t) \|_{L^2}^2 = - \langle \Delta_j (u(t) \cdot \nabla) \theta(t), \Delta_j \theta(t) \rangle_{L^2} = \langle [(u(t) \cdot \nabla), \Delta_j] \theta(t), \Delta_j \theta(t) \rangle_{L^2}.
\]

Integrating over \((0, t)\) and after estimating the \( L^2 \)-inner product, we arrive at

\[
\| \Delta_j \theta(t) \|_{L^2} \leq \| \Delta_j \theta_0 \|_{L^2} + \int_0^t \| [(u(t) \cdot \nabla), \Delta_j] \theta(\tau) \|_{L^2} \, d\tau.
\tag{4.12}
\]
Now, multiplying (4.12) by $2^{sj}$ and taking the $l^q(\mathbb{Z})$-norm, it follows that
\[
\| \theta(t) \|_{B_{2,q}^s} \leq \| \theta_0 \|_{B_{2,q}^s} + \int_0^t \left( \sum_{j \in \mathbb{Z}} 2^{sjq} \| [(u(t) \cdot \nabla), \Delta_j] \theta(\tau) \|_{L^2}^q \right)^{\frac{1}{q}} d\tau.
\]

Estimating the commutator operator via Lemma 2.5 (i), we obtain a constant $C > 0$ such that
\[
\| \theta(t) \|_{B_{2,q}^s} \leq \| \theta_0 \|_{B_{2,q}^s} + C \int_0^t \| \nabla \theta(\tau) \|_{L^\infty} \| u(\tau) \|_{B_{2,q}^s} + \| \nabla u(\tau) \|_{L^\infty} \| \theta(\tau) \|_{B_{2,q}^s} d\tau.
\]

By Remark 2.2, (4.6) and the continuity of $\mathcal{R}_i$ in Besov spaces, there exist positive constants $C_3$ and $C_4$ such that
\[
\| \theta(t) \|_{B_{2,q}^s} \leq \| \theta_0 \|_{B_{2,q}^s} + C_3 t \| \theta_0 \|_{B_{2,q}^s}^2 + C_4 \int_0^t \| \nabla u(\tau) \|_{L^\infty} \| \theta(\tau) \|_{B_{2,q}^s} d\tau.
\]

By Gronwall’s inequality, we get
\[
\| \theta(t) \|_{B_{2,q}^s} \leq \left( \| \theta_0 \|_{B_{2,q}^s} + C_3 t \| \theta_0 \|_{B_{2,q}^s}^2 \right) \exp \left\{ C_4 \int_0^t \| \nabla u(\tau) \|_{L^\infty} d\tau \right\}
\leq \| \theta_0 \|_{B_{2,q}^s} \left( 1 + C_3 t \| \theta_0 \|_{B_{2,q}^s} \right) \exp \left\{ C_4 \int_0^T \| \nabla u(\tau) \|_{L^\infty} d\tau \right\},
\]
for all $t \in [0, T)$. By standard arguments, using $\int_0^T \| \nabla u(\tau) \|_{L^\infty} d\tau < \infty$ and (4.14), we are done. \qed

**Long-time solvability**

For $\theta_0 \in B_{2,q}^{s+1}(\mathbb{R}^2)$, let $\theta$ be the solution of (1.1) in the class $C([0, T^*); B_{2,q}^{s+1}(\mathbb{R}^2)) \cap C^1([0, T^*); B_{2,q}^{s+2}(\mathbb{R}^2))$ with maximal existence time $T^* > 0$. If $T^* = \infty$, we are done. Assume that $T^* < \infty$. By Duhamel’s principle, we have
\[
\theta(t) = e^{-AR(t)} \theta_0 - \int_0^t e^{-AR(t-\tau)} (u \cdot \nabla \theta)(\tau) d\tau.
\]

For $0 \leq t \leq T^*$, define
\[
\mathcal{M}(t) = \int_0^t \| \nabla u(\tau) \|_{L^\infty} d\tau.
\]

We first consider the case $s = 2$ with $q = 1$. We can estimate
\[
\mathcal{M}(t) \leq C \max_{l=1,2} \int_0^t \| \mathcal{R}_l e^{-AR(t-\tau)} \theta_0 \|_{B_{1,1}^1} d\tau + C \max_{l=1,2} \int_0^t \left\| \int_0^\tau \mathcal{R}_l e^{-AR_1(\tau'-\tau)} (u(\tau') \cdot \nabla) \theta(\tau') d\tau' \right\|_{B_{1,1}^1} d\tau
\]
\[
:= K_1 + K_2.
\]
For $K_1$, we use estimate (2.4) with $r = \infty$, Hölder’s inequality and the continuity of $\mathcal{R}_t$ in Besov spaces to get

$$K_1 = C \sum_{j \in \mathbb{Z}} 2^j \int_0^t \max_{l=1,2} \|\mathcal{R}_t e^{-AR_{l1}\tau} \Delta_j \theta_0\|_{L^\infty} \, d\tau$$

$$\leq C t^{1 - \frac{1}{q}} \sum_{j \in \mathbb{Z}} 2^j \max_{l=1,2} \|e^{-AR_{l1}\tau} \Delta_j \mathcal{R}_t \theta_0\|_{L^\gamma(\mathbb{R}^2;L^\infty)}$$

$$\leq C t^{1 - \frac{1}{q}} |A|^{-\frac{1}{q}} \sum_{j \in \mathbb{Z}} 2^j \max_{l=1,2} \|\Delta_j \mathcal{R}_t \theta_0\|_{B^{1}_{2,1}}$$

$$\leq C t^{1 - \frac{1}{q}} |A|^{-\frac{1}{q}} \|\mathcal{R}_t \theta_0\|_{B^{2}_{2,1}}$$

$$\leq C t^{1 - \frac{1}{q}} |A|^{-\frac{1}{q}} \|\theta_0\|_{B^{2}_{2,1}}.$$ 

For $K_2$, we have

$$K_2 \leq C \max_{l=1,2} \sum_{j \in \mathbb{Z}} 2^j \int_0^t \int_0^\tau \|\mathcal{R}_t e^{-AR_{l1}(\tau'-\tau)\Delta_j (u(\tau') \cdot \nabla) \theta(\tau')}\|_{L^\infty} \, d\tau' \, d\tau$$

$$= C \max_{l=1,2} \sum_{j \in \mathbb{Z}} 2^j \int_0^t \int_0^\tau \|e^{-AR_{l1}(\tau'-\tau)} \mathcal{R}_t \Delta_j (u(\tau') \cdot \nabla) \theta(\tau')\|_{L^\infty} \, d\tau' \, d\tau'$$

$$\leq C t^{1 - \frac{1}{q}} \max_{l=1,2} \sum_{j \in \mathbb{Z}} 2^j \int_0^t \|e^{-AR_{l1}(\tau'-\tau)} \mathcal{R}_t \Delta_j (u(\tau') \cdot \nabla) \theta(\tau')\|_{L^\gamma(\mathbb{R}^2;L^\infty)} \, d\tau'$$

$$\leq C t^{1 - \frac{1}{q}} |A|^{-\frac{1}{q}} \int_0^t \max_{l=1,2} \sum_{j \in \mathbb{Z}} 2^j \|\mathcal{R}_t \Delta_j (u(\tau') \cdot \nabla) \theta(\tau')\|_{B^{1}_{2,1}} \, d\tau'$$

$$\leq C t^{1 - \frac{1}{q}} |A|^{-\frac{1}{q}} \int_0^t \|\theta(\tau')\|_{B^{2}_{2,1}}^2 \, d\tau'.$$

Here we have used the inequality

$$\max_{l=1,2} \sum_{j \in \mathbb{Z}} 2^j \|\mathcal{R}_t \Delta_j (u(\tau') \cdot \nabla) \theta(\tau')\|_{B^{2}_{2,1}} \leq C \sum_{j \in \mathbb{Z}} 2^j \|\Delta_j (u(\tau') \cdot \nabla) \theta(\tau')\|_{B^{2}_{2,1}}$$

$$\leq C \|\theta(\tau')\|_{B^{2}_{2,1}}^2,$$

which can be shown using Bony’s paraproduct as in [33] and the continuity of $\mathcal{R}_t$ in Besov spaces. Thus, for each $0 < t < T^*$, we have

$$\mathcal{M}(t) \leq C t^{1 - \frac{1}{q}} |A|^{-\frac{1}{q}} \left( \|\theta_0\|_{B^{2}_{2,1}} + \int_0^t \|\theta(\tau')\|_{B^{2}_{2,1}}^2 \, d\tau' \right)$$

$$\leq C t^{1 - \frac{1}{q}} |A|^{-\frac{1}{q}} \left( \|\theta_0\|_{B^{2}_{2,1}} + \|\theta_0\|_{B^{2}_{2,1}}^2 \left( 1 + C_3 T \|\theta_0\|_{B^{2}_{2,1}} \right) \int_0^t e^{C_4 \mathcal{M}(\tau')} \, d\tau' \right)$$

$$\leq C t^{1 - \frac{1}{q}} |A|^{-\frac{1}{q}} \|\theta_0\|_{B^{2}_{2,1}} \left( 1 + \|\theta_0\|_{B^{2}_{2,1}} \left( 1 + C_3 T \|\theta_0\|_{B^{2}_{2,1}} \right) \right) \int_0^t e^{C_4 \mathcal{M}(t)} \, d\tau'.$$

Now, we deal the case $s > 2$ with $1 \leq q \leq \infty$. For each $1 \leq q \leq \infty$, we take $2 < r \leq \infty$ such that $q \leq r$. Note that since $s-1 > 1$ we have the non-homogeneous embedding $B^{s-1\infty}_{\infty,\infty} \hookrightarrow W^{1,\infty}$, so that we can estimate $\|\nabla \theta\|_{L^\infty}$ by the $B^{s-1\infty}_{\infty,\infty}$-norm.
of \( \theta \). Thus, we have that

\[
\mathcal{M}(t) \leq C \max_{i=1,2} \int_0^t \| R_i e^{-A R_i t} \theta_0 \|_{B^{s-1}_{\infty, \infty}} \, dt \\
+ C \max_{i=1,2} \int_0^t \left\| \int_0^\tau R_i e^{-A R_i (\tau' - \tau)} (u(\tau') \cdot \nabla) \theta(\tau') \, d\tau' \right\|_{B^{s-1}_{\infty, \infty}} \, d\tau \\
:= K_3 + K_4.
\]

We now estimate \( K_3 \). Using the embedding \( \dot{B}^{s-1}_{\infty,q} \rightarrow \dot{B}^{s-1}_{\infty,\infty} \), Hölder’s inequality and Lemma 2.9, we get

\[
\int_0^t \left\| R_i e^{-A R_i t} \theta_0 \right\|_{B^{s-1}_{\infty, \infty}} \, dt \leq \int_0^t \left\| e^{-A R_i t} R_i \theta_0 \right\|_{\dot{B}^{s-1}_{\infty, (\infty)}} \, dt \\
\leq t^{1-\frac{s}{2}} \left\| e^{-A R_i t} R_i \theta_0 \right\|_{L^\infty(\mathbb{R} ; \dot{B}^{s-1}_{\infty, \infty})} \\
\leq C t^{1-\frac{s}{2}} |A|^{-\frac{s}{2}} \left\| R_i \theta_0 \right\|_{\dot{B}^{\frac{s}{2}}_{2,q}} \\
\leq C t^{1-\frac{s}{2}} |A|^{-\frac{s}{2}} \left\| \theta_0 \right\|_{\dot{B}^{\frac{s}{2}}_{2,q}}.
\]

Also, by Lemma 2.8, we have

\[
\int_0^t \left\| R_i e^{-A R_i t} \theta_0 \right\|_{L^\infty} \, dt \leq t^{1-\frac{s}{2}} \left\| e^{-A R_i t} R_i \theta_0 \right\|_{L^\infty(\mathbb{R} ; L^\infty)} \\
\leq C t^{1-\frac{s}{2}} |A|^{-\frac{s}{2}} \left\| R_i \theta_0 \right\|_{L^2} \\
\leq C t^{1-\frac{s}{2}} |A|^{-\frac{s}{2}} \left\| \theta_0 \right\|_{L^2}.
\]

Therefore,

\[
K_3 \leq C t^{1-\frac{s}{2}} |A|^{-\frac{s}{2}} \left\| \theta_0 \right\|_{\dot{B}^{\frac{s}{2}}_{2,q}}.
\]

We proceed to estimate \( K_4 \). Using Hölder’s inequality, the embedding \( \dot{B}^{s-1}_{\infty,q} \hookrightarrow \dot{B}^{s-1}_{\infty,\infty} \) and Lemma 2.9, we have that

\[
\int_0^t \left\| \int_0^\tau R_i e^{-A R_i (\tau' - \tau)} (u(\tau') \cdot \nabla) \theta(\tau') \, d\tau' \right\|_{B^{s-1}_{\infty, \infty}} \, d\tau \\
\leq C \int_0^t \int_0^\tau \left\| e^{-A R_i (\tau' - \tau)} R_i (u(\tau') \cdot \nabla) \theta(\tau') \right\|_{B^{s-1}_{\infty, q}} \, d\tau' \, d\tau \\
= C \int_0^t \int_0^\tau \left\| e^{-A R_i (\tau' - \tau)} R_i (u(\tau') \cdot \nabla) \theta(\tau') \right\|_{B^{s-1}_{\infty, q}} \, d\tau' \, d\tau' \\
\leq C t^{1-\frac{s}{2}} \int_0^t \left\| e^{-A R_i (\tau' - \tau)} R_i (u(\tau') \cdot \nabla) \theta(\tau') \right\|_{L^\infty(\tau', \dot{B}^{s-1}_{\infty, q})} \, d\tau' \\
\leq C t^{1-\frac{s}{2}} |A|^{-\frac{s}{2}} \int_0^t \left\| R_i (u(\tau') \cdot \nabla) \theta(\tau') \right\|_{\dot{B}^{\frac{s}{2}}_{2,q}} \, d\tau' \\
\leq C t^{1-\frac{s}{2}} |A|^{-\frac{s}{2}} \int_0^t \left\| \theta(\tau') \right\|_{\dot{B}^{\frac{s}{2}}_{2,q}}^2 \, d\tau'.
\]

Here we have used the inequality

\[
\left\| R_i (u(\tau') \cdot \nabla) \theta(\tau') \right\|_{\dot{B}^{\frac{s}{2}}_{2,q}} \leq C \left\| (u(\tau') \cdot \nabla) \theta(\tau') \right\|_{\dot{B}^{\frac{s}{2}}_{2,q}} \leq C \left\| \theta(\tau') \right\|_{\dot{B}^{\frac{s}{2}+1}_{2,q}},
\]
which can be shown by using Bony’s paraproduct and the continuity of \( \mathcal{R}_t \) in Besov spaces. Also, by Lemma 2.8, the continuity of \( \mathcal{R}_t \) in \( L^2 \) and the embedding \( \dot{B}_{2,q}^{s+1} \hookrightarrow \dot{B}_{2,q}^{s} \), we have

\[
\int_0^t \left\| e^{-\Delta R_1(\tau'-\tau)} \mathcal{R}_t(u(\tau') \cdot \nabla) \theta(\tau') \right\|_{L^\infty_t(\tau',t;L^\infty_s)} d\tau' \\
\leq C |A|^{-\frac{1}{5}} \int_0^t \left\| \mathcal{R}_t(u(\tau') \cdot \nabla) \theta(\tau') \right\|_{L^2_s} d\tau' \\
\leq C |A|^{-\frac{1}{5}} \int_0^t \left\| (u(\tau') \cdot \nabla) \theta(\tau') \right\|_{L^2_s} d\tau' \\
\leq C |A|^{-\frac{1}{5}} \int_0^t \| \theta(\tau') \|^2_{B^s_{2,q}} d\tau'.
\]

It follows that

\[
K_4 \leq C t^{1-\gamma} |A|^{-\frac{1}{5}} \int_0^t \| \theta(\tau') \|^2_{B^s_{2,q}} d\tau'.
\]

Thus, for each \( 0 < t < T^* \), we have

\[
\mathcal{M}(t) \leq C t^{1-\frac{1}{5}} |A|^{-\frac{1}{5}} \left( \| \theta_0 \|_{B^s_{2,q}} + \int_0^t \| \theta(\tau') \|^2_{B^s_{2,q}} d\tau' \right)
\leq C t^{1-\frac{1}{5}} |A|^{-\frac{1}{5}} \left( \| \theta_0 \|_{B^s_{2,q}} + \| \theta_0 \|_{B^s_{2,q}}^2 \left( 1 + C_3 T \| \theta_0 \|_{B^s_{2,q}} \right) \right) \int_0^t e^{C_4 \mathcal{M}(\tau)} d\tau' \\
\leq C t^{1-\frac{1}{5}} |A|^{-\frac{1}{5}} \| \theta_0 \|^2_{B^s_{2,q}} \left( 1 + \| \theta_0 \|_{B^s_{2,q}} \left( 1 + C_3 T \| \theta_0 \|_{B^s_{2,q}} \right) \right)^2 \left( 1 + C_4 T \mathcal{M}(t) \right). 
\]

Therefore, for both cases of \( s \) and \( q \) such that \( s = 2 \) with \( q = 1 \) or \( s > 2 \) with \( 1 \leq q < \infty \), we have that there exists \( C_5 > 0 \) such that

\[
\mathcal{M}(t) \leq C_5 t^{1-\gamma} |A|^{-\frac{1}{5}} \left( 1 + \| \theta_0 \|_{B^s_{2,q}} \left( 1 + C_3 T \| \theta_0 \|_{B^s_{2,q}} \right) \right)^2 \left( 1 + C_4 T \mathcal{M}(t) \right). 
\]  

Next, for each \( 0 < T < \infty \) we define \( \bar{T} = \sup D_T \), where

\[
D_T = \{ t \in [0,T] \cap [0, T^*] \mid \mathcal{M}(t) \leq C_5 T^{1-\gamma} \| \theta_0 \|_{B^s_{2,q}} \}.
\]

We first show that \( \bar{T} = \min\{T, T^*\} \). We proceed by contradiction. So assume on the contrary that \( \bar{T} < \min\{T, T^*\} \). We have that there exists \( T_1 \) such that \( \bar{T} < T_1 < \min\{T, T^*\} \). It follows that \( \theta \in C([0,T_1]; B^{s+1+\gamma}_{2,q}(\mathbb{R}^2)) \), \( \mathcal{M}(t) \) is uniformly continuous on \([0,T_1]\) and

\[
\mathcal{M}(\bar{T}) \leq C_5 \bar{T}^{1-\gamma} \| \theta_0 \|_{B^s_{2,q}}. 
\]

We now take \( |A| \) large enough so that

\[
|A|^{\frac{1}{5}} \geq 2 \left( 1 + \| \theta_0 \|_{B^s_{2,q}} \left( 1 + C_3 T \| \theta_0 \|_{B^s_{2,q}} \right) \right)^2 \left( 1 + C_4 C_5 \bar{T}^{1-\gamma} \| \theta_0 \|_{B^s_{2,q}} \right). 
\]
Using (4.15), (4.16) and (4.17), we can estimate
\[
\mathcal{M}(\tilde{T}) \leq C_{5}(\tilde{T})^{1 - \frac{1}{\delta}} |A|^{-\frac{1}{\delta}} \|\theta_{0}\|_{B_{2,q}^{s+1}}
\times \left(1 + \|\theta_{0}\|_{B_{2,q}^{s+1}} \left(1 + C_{3}\tilde{T}\|\theta_{0}\|_{B_{2,q}^{s+1}}\right)^{2} \tilde{T} \exp \left(C_{4}\mathcal{M}(\tilde{T})\right)\right)
\leq C_{5}\tilde{T}^{1 - \frac{1}{\delta}} \|\theta_{0}\|_{B_{2,q}^{s+1}} |A|^{-\frac{1}{\delta}}
\times \left(1 + \|\theta_{0}\|_{B_{2,q}^{s+1}} \left(1 + C_{3}T\|\theta_{0}\|_{B_{2,q}^{s+1}}\right)^{2} T \exp \left(C_{4}C_{5}T^{1 - \frac{1}{\delta}} \|\theta_{0}\|_{B_{2,q}^{s+1}}\right)\right)
\leq \frac{1}{2} C_{5}\tilde{T}^{1 - \frac{1}{\delta}} \|\theta_{0}\|_{B_{2,q}^{s+1}}.
\]

Thus, we can choose \( T_{3} \) such that \( \tilde{T} < T_{3} < T_{1} \) with \( \mathcal{M}(T_{3}) \leq C_{5}\tilde{T}^{1 - \frac{1}{\delta}} \|\theta_{0}\|_{B_{2,q}^{s+1}} \). This contradicts the definition of \( \tilde{T} \). It follows that \( \tilde{T} = \min\{T, T^{*}\} \) when \( A \) verifies (4.17). If \( T^{*} < T \), we have that \( T^{*} = \tilde{T} = \sup D_{T} \) and
\[
\mathcal{M}(t) = \int_{0}^{t} \|\nabla u(\tau)\|_{L^{\infty}} \, d\tau \leq C_{5}T^{1 - \frac{1}{\delta}} \|\theta_{0}\|_{B_{2,q}^{s+1}} < \infty, \quad \text{for all } 0 \leq t < T^{*}.
\]

It follows that \( \mathcal{M}(T^{*}) < \infty \) which contradicts the maximality of \( T^{*} \) because the blow-up criterion. This concludes the proof.

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