EMBEDDING OF HIGSON COMPACTIFICATION INTO THE PRODUCT OF ADELIC SOLENOIDS.

ALEXANDER DRANISHNIKOV AND JAMES KEESLING

ABSTRACT. The Higson compactification of any simply connected proper geodesic metric space admits an embedding into a product of adelic solenoids that induces an isomorphism of 1-dimensional cohomology.

1. INTRODUCTION

The Higson corona of a metric space was introduced by Higson and Roe [Ro1, Ro2] to bring an approximation of the $K$-theory of its Roe algebra $C^*_{roe}(X)$ by the topological $K$-theory of the corona. The $K$-theory of the Roe algebra is the recipient in the coarse assembly map and therefore, it plays an important role in a coarse approach to the Novikov and the Baum-Connes conjectures. It turns out that for injectivity of the coarse assembly map it is sufficient to have the Higson compactification acyclic [Ro1]. James Keesling noticed that the Higson compactification of $\mathbb{R}^n$ is not acyclic in dimension one [K]. Later Dranishnikov and Ferry have constructed nontrivial second cohomology classes of the Higson compactification of $\mathbb{R}^2$ [DF].

It turns out that all the above cohomology classes of the Higson compactifications are divisible. This leaves a hope that all cohomology groups of the Higson compactification are vector spaces over $\mathbb{Q}$. In that case the Higson compactification would be acyclic with respect to any finite coefficient group. In [DFW] Dranishnikov, Ferry, and Weinberger proved the Novikov Higher Signature conjecture for manifolds whose fundamental group $\Gamma$ admits a finite classifying space $B\Gamma$ with trivial Čech cohomology of the Higson compactification $\overline{ET}$ of the universal cover $\overline{E}\Gamma$ with $\mathbb{Z}_2$ coefficients. In the same paper it was proven that $\hat{H}^*(\overline{ET};\mathbb{Z}_p) = 0$ for all prime $p$ whenever the asymptotic dimension $\text{asdim}\,\Gamma$ of the group $\Gamma$ is finite.

Date: May 10, 2022.

2000 Mathematics Subject Classification. Primary 53C23, 54F15; Secondary 22C05, 54F15, 55N05.

The first author was supported by Simons Foundation.
In this paper we construct an embedding of $E\Gamma$ into the product of adelic solenoids $\prod \Sigma_p$ which induces isomorphism of rational cohomology. In particular, our embedding implies that $\check{H}^1(E\Gamma) = \oplus \mathbb{Q}$ and, hence, $\check{H}^1(E\Gamma; \mathbb{Z}_p) = 0$ for all $p$.

We think that it is a reasonable to believe that our embedding induces an epimorphism of cohomology groups for higher dimensions. This conjecture seems plausible in the case when $\dim E\Gamma < \infty$. We note that from [DFW] one can derive our conjecture for groups with $\text{asdim} \Gamma < \infty$. We recall that always $\dim \nu(E\Gamma) \leq \text{asdim} \Gamma$ [DKU] and $\dim \nu(E\Gamma) = \text{asdim} \Gamma$ [Dr] when the latter is finite. Here $\nu(E\Gamma)$ is the Higson corona: $\nu(E\Gamma) = E\Gamma \setminus E\Gamma$.

There is a long open question whether there exists a metric spaces $X$ with $\dim \nu X < \infty$ and $\text{asdim} \Gamma = \infty$. Perhaps a more challenging related question here is whether the Novikov conjecture holds true for groups with $\dim \nu \Gamma < \infty$. We recall that the Novikov conjecture for groups with $\text{asdim} \Gamma < \infty$ was proven more than two decades ago in [Yu].

2. Preliminaries

2.1. Higson compactification. Let $(X, d)$ be a metric space with a base point $x_0$. By $B(r, x)$ we denote the open ball of radius $r$ with the center in $x$. We call a map $f : X \to Y$ to a metric space $Y$ slowly oscillating if for any $r > 0$ the diameter of $f(B(x, r))$ tends to $0$ as $d(x, x_0)$ tends to infinity. The Higson compactification $\bar{X}$ of $X$ is characterized by bounded slowly oscillating continuous functions $f : X \to \mathbb{R}$. If $C_h$ is the ring of all such functions, then $\bar{X}$ is homeomorphic to the closure of $X$ under the embedding

$$(f)_{f \in C_h} : X \to \prod_{f \in C_h} [\inf f, \sup f].$$

We use notation $\nu X = \bar{X} \setminus X$ for the Higson corona. The following is straightforward.

2.1. Proposition. For a metric space $X$ a continuous map $f : X \to C$ to a compact metric space $C$ extends continuously to the Higson corona if and only if $f$ is slowly oscillating.

A metric space $X$ is called proper if the distance to each point in $X$ is a proper function. A metric space $X$ is called geodesic if for any pair of points $x, y \in X$ there is an isometric embedding $\xi : [0, d(x, y)] \to X$ with $\xi(0) = x$ and $\xi(d(x, y)) = y$.

The following can be found in [DKU].
2.2. Proposition. In a proper metric space $X$ closed subsets $Y$ and $Y'$ have common points in the Higson corona, $\text{Cl}_X Y \cap \text{Cl}_X (Y') \cap \nu X \neq \emptyset$, if and only if $d(Y \setminus B(x_0, r), Y') \to \infty$ as $r \to \infty$.

2.2. Solenoids. We start from the definition of the $p$-adic solenoid $\Sigma_p$ as the inverse limit:

$$\Sigma_p = \lim_{\leftarrow} \{ S^1 \xrightarrow{\ell} S^1 \xrightarrow{\ell} S^1 \leftarrow \cdots \}$$

of unit circles $S^1 \subset \mathbb{C}$ where the bonding maps are group homomorphisms defined by the map $z \mapsto z^p$. Thus, $\Sigma_p$ is an abelian topological group. We use $+$ for the group operation in $\Sigma_p$. Let $p_0 : \Sigma_p \to S^1$ be the projection onto the first circle in the inverse sequence. Note that the subgroup $p_0^{-1}(1) = A_p \subset \Sigma_p$ is the group of $p$-adic integers. We use the notation $A_p$ for $p$-adic integers instead of more common notation from algebra $\mathbb{Z}_p$, since the later is used in topology (and already was used in this paper) for the $\mathbb{Z} \mod p$ group.

Let $u : \mathbb{R} \to S^1 \subset \mathbb{C}$ be the universal cover given by the exponential map $u(t) = e^{2\pi it}$. Thus, $u$ is a group homomorphism with $u^{-1}(1) = \mathbb{Z}$. Then the lift $\tilde{u} : \mathbb{R} \to \Sigma_p$ that takes $0$ to the unit of $\Sigma_p$ is an injective group homomorphism. Thus, the group of reals is a subgroup of the solenoid, $\mathbb{R} \subset \Sigma_p$. We note that $\mathbb{R} \cap A_p = \mathbb{Z}$. Since $\mathbb{Z}$ is dense in $A_p$, it follows that $\mathbb{R}$ is dense in $\Sigma_p$. Also we note that all path components of $\Sigma_p$ are the $\mathbb{R}$-cosets $a + \mathbb{R}$, $a \in \Sigma_p$. They are the bijective images of the reals $g : \mathbb{R} \to \Sigma_p$ that are obtained as different lifts of the universal cover $u$ with respect to $p_0$. Another way to define the $p$-adic solenoid is taking the quotient group $\Sigma_p = (\mathbb{R} \times A_p)/\mathbb{Z}$ for the natural diagonal embedding $\mathbb{Z} \to \mathbb{R} \times A_p$.

The adelic solenoid is the quotient $\Sigma_Q = A_Q/\mathbb{Q}$ of the adels of the rationals [BV]. It can be viewed as the quotient group $\Sigma_Q = (\mathbb{R} \times \hat{\mathbb{Z}})/\mathbb{Z}$ where $\hat{\mathbb{Z}}$ is the profinite completion of the integers and $\mathbb{Z} \to \mathbb{R} \times \hat{\mathbb{Z}}$ is the diagonal map for natural inclusions $\mathbb{Z} \subset \mathbb{R}$ and $\mathbb{Z} \subset \hat{\mathbb{Z}}$. We note that the image of the inclusion $\mathbb{Z} \subset \hat{\mathbb{Z}}$ is dense as well as the image of the inclusion $\mathbb{R} \subset \Sigma_Q$. The adelic solenoid admits the following description as the inverse limit

$$\Sigma_Q = \lim_{\leftarrow} \{ S^1 \xrightarrow{\ell^2} S^1 \xrightarrow{\ell^3} S^1 \leftarrow \cdots \}$$

where the bonding homomorphism from the $n$th circle to the previous is taking to the power $n$, $z \mapsto z^n$. Again we denote by $p_0 : \Sigma_Q \to S^1$ the projection to the first circle. All the above facts about $\Sigma_p$ hold true for $\Sigma_Q$ with replacement of $p$-adic integers by the profinite completion.
of \( \mathbb{Z} \). In this paper we will be using the notations \( \Sigma_\infty \) for \( \Sigma_\mathbb{Q} \) and \( A_\infty \) for \( \hat{\mathbb{Z}} \).

2.3. Proposition. For the Higson compactification \( \bar{X} \) of a simply connected geodesic metric space \( X \) there is an isomorphism

\[
\check{H}^1(\bar{X}) = \bigoplus_A \mathbb{Q}.
\]

Proof. The idea of the proof is taken from [K]. It suffices to show that each element \( \alpha \in \check{H}^1(\bar{X}) \) is \( p \)-divisible for all \( p \). Let \( \phi : \bar{X} \to S^1 \) be a map representing \( \alpha \), \( \alpha = \phi^*(1) \), where \( 1 \in \mathbb{Z} = H^1(S^1) \). Since \( X \) is simply connected, there is a lift \( f : X \to \Sigma_p \) with respect to \( p_0 \). We fix a metric on \( \Sigma_p \) and define \( \varepsilon(d) \) to be the maximal diameter of a component of \( p_0^{-1}(B(s, d)) \) for \( s \in S^1 \) and the standard metric on \( S^1 \). Clearly, \( \lim_{d \to 0} \varepsilon(d) = 0 \). Since the balls \( B(x, r) \) in a geodesic metric space are connected, we obtain

\[
diam f(B(x, r)) \leq \varepsilon(diam(\phi(B(x, r))).
\]

Then the slow oscillation of \( \phi \) implies that \( f \) is slow oscillating. Hence there is a continuous extension \( \bar{f} : \bar{X} \to \Sigma_p \) of \( f \). Then \( \phi = p_0 \bar{f} \) and, hence, \( \alpha = \bar{f}^*(1) \) where \( 1 \in \mathbb{Z}[\frac{1}{p}] = \check{H}^1(\Sigma_p) \). Thus, \( \alpha \) is \( p \)-divisible. \( \square \)

2.3. Lipschitz extension. The following fact is well-known. It can be proven by use of Helly’s theorem and the transfinite induction [CGM], [Dr].

2.4. Proposition. For any metric space \( X \) and any \( \lambda \)-Lipschitz map \( f : A \to Y \), \( A \subset X \), where \( Y = \mathbb{R} \) or \( Y = [a, b] \) there is a \( \lambda \)-Lipschitz extension \( \bar{f} : X \to Y \).

3. Main Theorem

3.1. Theorem. Every simply connected proper geodesic metric space \( X \) admits an embedding of its Higson compactification into the product of adelic solenoids

\[
F : \bar{X} \to \prod_A \Sigma_\infty
\]

that induces an isomorphism of 1-dimensional Čech cohomology.

We denote as \( [X, Y]_0 \) pointed homotopy classes of maps \( f : (X, x_0) \to (Y, y_0) \).

3.2. Proposition. Let \( x_0 \in X \) be a base point of a simply connected metric space \( X \) such that its inclusion \( x_0 \to \bar{X} \) is a cofibration and let
0 ∈ Σ_p, p ∈ ℕ ∪ {∞}, and 1 ∈ S^1 be the corresponding base points. Then

\[ [\bar{X}, \Sigma_p]_0 = [\bar{X}, S^1]_0 = \check{H}^1(\bar{X}). \]

**Proof.** The map \( p_0 : Z \to S^1 \) defines the map \( \Psi : [\bar{X}, \Sigma_p]_0 \to [\bar{X}, S^1]_0 \) as \( \Psi([f]) = [f p_0] \) for the pointed homotopy class \([f]\) of a map \( f : (\bar{X}, x_0) \to (\Sigma_p, 0) \). We define the inverse map \( \Phi \) as follows. Given a map \( \phi : (\bar{X}, x_0) \to (S^1, 1) \) we consider its lift \( \tilde{f} : (\bar{X}, x_0) \to (\Sigma_p, 0) \) with respect to \( p_0 \) constructed in the proof of Proposition 2.3. If \( \phi \) has a pointed homotopy to a map \( \phi' \), then the lifting of that homotopy will be a pointed homotopy, since the fiber \( p^{-1}_0(1) \) is disconnected. This proves \( \Phi \) is well-defined. Clearly, \( \Psi \Phi = 1 \). The uniqueness of lift of connected space \( X \) with the prescribed value 0 for the base point implies that \( f \) is the only lift of \( f p_0 \). Thus, \( \Phi \Psi = 1 \).

Since \( x_0 \to X \) is a cofibration, we obtain \( [\bar{X}, S^1]_0 = [\bar{X}, S^1] = \check{H}^1(\bar{X}). \)

We can choose a metrics on the solenoids \( \Sigma_p, p \in \mathbb{N} \cup \{\infty\} \), in such a way that \( p_0 : \Sigma_p \to S^1 \) is 1-Lipschitz and the inclusion map \( \mathbb{R} \to \Sigma_p \) is Lipschitz.

For a slowly oscillating map \( f : X \to \Sigma_p \) we denote by \( \tilde{f} : \bar{X} \to \Sigma_p \) its extension to the Higson corona. Thus, \( g|_X = g \) for any continuous map \( g : \bar{X} \to \Sigma_p \). We note that for connected space \( X \) with \( x_0 \in X \) any continuous map \( f : (X, x_0) \to (\Sigma_p, 0) \) lands in the subgroup \( \mathbb{R} \subset \Sigma_p \).

**3.3. Proposition.** Let \( X \) be a simply connected proper metric space with a base point. For any elements \([f], [g] \in [\bar{X}, \Sigma_p]_0\), where \( \bar{X} \) is the Higson compactification, and for any \( n \in \mathbb{N} \) we have

\[
\begin{align*}
n \Psi([f]) &= \Psi([nf|_X]) \quad \text{and} \\
\Psi([f]) + \Psi([g]) &= \Psi([f|_X + g|_X]).
\end{align*}
\]

**Proof.** We recall that the restriction of \( p_0 : \Sigma_p \to S^1 \) to \( \mathbb{R} \) equals the exponential map \( \exp(t) = e^{2\pi it} \). Then

\[
\begin{align*}
n \Psi([f]) &= n[p_0 f] = [(p_0 f)^n] = [(p_0 f|_X)^n] = [(\exp(f|_X))^n] = \\
&= [\exp(nf|_X)] = [p_0(nf|_X)] = [p_0nf|_X] = \Psi([nf|_X])
\end{align*}
\]

and

\[
\begin{align*}
\Psi([f]) + \Psi([g]) &= [p_0 f] + [p_0 g] = [p_0 f|_X + p_0 g|_X] = [p_0 f|_X \cdot p_0 g|_X] = \\
&= [\exp(f|_X) \cdot \exp(g|_X)] = [\exp(f|_X + g|_X)] = [p_0(f|_X + g|_X)] = \Psi([f|_X + g|_X]).
\end{align*}
\]

Here the multiplication \( \cdot \) is taken in the unit circle \( S^1 \). \( \square \)
For subsets $A$ and $B$ of a metric space $(X, d)$ we denote by
\[
dist(A, B) = \inf\{d(a, b) \mid a \in A, \ b \in B\}
\]
the distance from $A$ to $B$.

3.4. Proposition. Let $X$ be a proper metric space with a base point $x_0 \in X$, let $c \in \nu X$ be a fixed point, and let $A, B \subset \nu X$ be two closed disjoint subsets. For any $z \in A_p \setminus \{0\}$, $p \in \mathbb{N} \cup \{\infty\}$, there is a slow oscillating function $f_z : X \to \mathbb{R} \subset \Sigma_p$ with the continuous extension $\tilde{f}_z : X \to \Sigma_p$ such that $f_z(x_0) = 0$, $\tilde{f}_z(c) = z$, and $f_z(A) \cap \tilde{f}_z(B) = \emptyset$.

Proof. Suppose that $c \notin B$. Then we enlarge $A$ by adding $c$ to it. We will define slowly oscillating function $f_z : X \to \mathbb{R} \subset \Sigma_p$ such that $\tilde{f}_z$ takes $B$ to 0 and takes $A$ to $z$.

Let $A', B' \subset X$ be closed disjoint subsets with $A = \text{Cl}_X(A') \cap \nu X$ and $B = \text{Cl}_X(B') \cap \nu X$. Let
\[
r_n = \dist(A' \setminus B(x_0, 2^n), B' \setminus B(x_0, 2^n)).
\]
In view of Proposition 2.2 we may assume that $r_n$ is increasing with $r_n \to \infty$.

We consider the partition of $X \setminus B(x_0, 1)$ into closed annuli $A_n$ centered at $x_0$ with outer radius $2^{n+1}$ and with the inner radius $2^n$. Consider
\[
C = A' \cap \bigcup_{n=1}^{\infty} A_{2n-1} \quad \text{and} \quad C' = A' \cap \bigcup_{n=1}^{\infty} A_{2n}.
\]
Since $A' = C \cup C'$ and $c \in \text{Cl}_X(A')$, either $c \in \text{Cl}_X(C)$ or $c \in \text{Cl}_X(C')$. Without loss of generality we may assume the first.

Let $z_n \in \mathbb{R}$ be a sequence converging to $z$ in $\Sigma_p$. We may assume that $z_n \to \infty$ in $\mathbb{R}$ and $z_n \leq \min\{\sqrt{r_n}, 2^{n-1}\}$. We define $f_z(C_n) = z_n$ and $f_z(B') = 0$. Let
\[
\lambda_n = \max\left\{\frac{1}{\sqrt{r_n}}, \frac{1}{2n}\right\}.
\]
Note that the restriction of $f_z$ to $(C \cup B') \cap A_{2n-1}$ is $\lambda_n$-Lipschitz. Indeed, for $x \in B'$ and $y \in C_n$ we have $d(x, y) \geq r_n$ and
\[
|f_z(y) - f_z(x)| = z_n \leq \sqrt{r_n} \leq \frac{1}{\sqrt{r_n}}d(x, y).
\]
By Proposition 2.4 the above restriction of $f_z$ can be extended to a $\lambda_n$-Lipschitz map
\[
f_z^{2n-1} : A_{2n-1} \to [0, z_n].
\]
We show that the union map
\[
f_z^{2n-1} \cup f_z^{2n+1} : A_{2n-1} \cup A_{2n+1} \to [0, z_{n+1}]
\]
is $\lambda_n$-Lipschitz. For if, $x \in A_{2n+1}$ and $y \in A_{2n-1}$ we have $d(x, y) \geq 2^{2n}$ and, hence,

$$|f^{2n+1}_z(x) - f^{2n-1}_z(y)| \leq 2^{n+1} \leq 2^n = \frac{1}{2^n} 2^{2n} \leq \frac{1}{2^n} d(x, y) \leq \lambda_n d(x, y).$$

By Proposition 2.24 the map $f^{2n-1}_z \cup f^{2n+1}_z : A_{2n-1} \cup A_{2n+1} \to \mathbb{R}$ can be extended to $\lambda_n$-Lipschitz map $f_{2n} : A_{2n} \to \mathbb{R}$. We extend $f^0_z$ continuously to $B(x_0, 1)$ with $f_z(x_0) = 0$. The union $\bigcup_k f^k_z$ defines a slow oscillating function $f_z : X \to \mathbb{R}$ with the required properties. Indeed,

since the inclusion of the subgroup $\mathbb{R} \to \Sigma_p$ is a Lipschitz function, the map $f_z : X \to \Sigma_p$ is slowly oscillating. Then by Proposition 2.21 there is a continuous extension $\bar{f}_z : \tilde{X} \to \Sigma_p$. Note that $\bar{f}_z(B) = 0$. Since the sequence $z_k \subset Z = \mathbb{R} \cap A_p \subset \Sigma_p$ converges to $z$, we obtain $\bar{f}_z(c) = \bar{f}_z(A) = z \neq 0$.

A family of elements $\{x_\alpha\} \subset A$ of an abelian group is called linearly independent if for any finite subfamily $\{x_\alpha\}$ satisfying the equality $\sum_i n_i x_\alpha = 0$ with $n_i \in \mathbb{Z}$ it follows that $n_i = 0$ for all $i$.

3.5. Proposition. The group $A_p/\mathbb{Z}$, $p \in \mathbb{N} \cup \{\infty\}$, contains a family of linearly independent elements $\{u_\alpha\}$ of cardinality continuum.

Proof. First we show it for finite $p$. The group of $p$-adic rational numbers $\mathbb{Q}_p$ contains $A_p$ and $\mathbb{Q}$ as subgroups with $A_p \cap \mathbb{Q} = \mathbb{Z}$. The group $\mathbb{Q}_p$ is divisible and, hence, is a vector space over $\mathbb{Q}$. Thus, $\mathbb{Q}_p = (\mathbb{Q}_p/\mathbb{Q}) \oplus \mathbb{Q}$. Let $\{v_\alpha\} \subset \mathbb{Q}_p$ be a basis for the summand $\mathbb{Q}_p/\mathbb{Q}$. Then for each $v_\alpha$ there is $k = k(\alpha)$ such that $p^k v_\alpha \in A_p$. We define $u_\alpha = p^{k(\alpha)} v_\alpha$. Suppose that $\sum_i n_i u_i \in \mathbb{Z}$. Then $\sum_i n_i p^{k(i)} v_i \in \mathbb{Z} \subset \mathbb{Q}$. Therefore, $n_i p^{k(i)} = 0$ for all $i$. Hence $n_i = 0$ for all $i$. Thus the family $\{u_\alpha\}$ is linearly independent over $\mathbb{Z}$.

We note that $A_\infty \cong \prod_p A_p$ where the product is taken over all prime numbers $p$. We take a family $\{z_\alpha\} \subset A_\infty$ in such way that $\pi_p(z_\alpha) = u_\alpha$ where $\pi_p : \prod_p A_p \to A_p$ is the projection onto the factor. Then $\{z_\alpha\}$ forms a linearly independent family in $A_\infty/\mathbb{Z}$.

Proof of Theorem 3.1. We fix points $x_0 \in X$ and $c \in \nu X$. Since $X$ is a reasonably nice space, we may assume that the inclusion $x_0 \to X$ is a cofibration. Otherwise we attach to $X$ an interval $[x_0, x_1]$ along some $x_1 \in X$.

Let $\{z_\alpha \in A_\infty \mid \alpha \in A'\}$ be the family form Proposition 3.5 of the cardinality of continuum $c$. 

EMBEDDING 7
It is not difficult to show that the weight of the Higson corona $\nu X$ is $c$. Let $U$ be a basis of topology of $\nu X$ of cardinality $c$. Define

$$P = \{(U, U') \in U \times U \mid U \cap U' = \emptyset\}$$

and let $\gamma : P \to A'$ be a bijection. For every pair $(U, U') \in P$ we apply Proposition 3.4 with $A = U$, $B = U'$ and $z = z_{U(U',U')}$ to obtain a function $f_z : X \to \mathbb{R}$ that extends to a map $\tilde{f}_z : (\tilde{X}, x_0) \to (\Sigma_p, 0)$ with $f_z(x_0) = 0$, $\tilde{f}_z(c) = z$, and $\tilde{f}_z(A) \cap \tilde{f}_z(B) = \emptyset$.

We denote by $f_\alpha = f_{z_\alpha}$, $\alpha \in A'$. We claim that the cohomology classes $\{[p_0 f_\alpha]\}_{\alpha \in A'}$ in $H^1(\tilde{X})$ are linearly independent. Suppose that $\sum n_i [p_0 f_\alpha_i] = 0$. Then by Proposition 3.3

$$\sum n_i [p_0 f_\alpha_i] = \sum n_i \Psi[f_\alpha_i] = \sum \Psi[n_i f_\alpha_i] = \Psi(\sum n_i f_\alpha_i) = 0.$$  

Hence there is a rel $x_0$ homotopy of $p_0 \sum n_i f_\alpha_i$ to the constant map to 1 in $S^1$. The lift of this homotopy is a rel $x_0$ homotopy in $\Sigma_\infty$ of $\sum n_i f_\alpha_i$ to the constant map to 0. Therefore, $\sum n_i \tilde{f}_\alpha_i(c) = \sum n_i z_\alpha_i$ lies in the path component of 0. Thus, $\sum n_i z_\alpha_i \in \mathbb{Z}$. The linear independence of $\{z_\alpha + \mathbb{Z}\} \in A_\infty/\mathbb{Z}$ implies that all $n_i = 0$.

We complete the cohomology classes $\{[p_0 f_\alpha]\}_{\alpha \in A'}$ in $\tilde{H}^1(\tilde{X}) = \oplus \mathbb{Q}$ to a basis $\{[\phi_\alpha]\}_{\alpha \in A''}$ where $\phi_\alpha : (\tilde{X}, x_0) \to (S^1, 1)$. Let $f_\alpha : (\tilde{X}, x_0) \to (\Sigma_\infty, 0)$ be the lift of $\phi_\alpha$ defined by the map $\Phi$ of Proposition 3.2. Let $A = A' \cup A''$. Since the map

$$F = (f_\alpha)_{\alpha \in A} : \tilde{X} \to \prod_A \Sigma_\infty$$

restricted to the Higson corona $\nu X$ separates the disjoint closures of basis sets, it separates distinct points and, hence, it is an embedding on $\nu X$. We note that the restriction $F' = F|X : X \to \prod_A \mathbb{R} \subset \prod_A \Sigma_\infty$ to $X$ lands in $\prod_A \mathbb{R}$.

We approximate $F$ by embedding as follows. Fix $\omega$ coordinates $\alpha_1, \ldots, \alpha_n, \cdots \in A$ and consider $\|\|_\infty$ metric on $\mathbb{R}^\omega$. We denote by $f_i = f_{\alpha_i}$. Let

$$d_i(t) = \sup_{x \in X \setminus B(x_0,t)} \text{diam}(f_i(B(x,1))).$$

Since $f_i$ is extendible to the Higson corona, it is slowly oscillating. Hence $\lim_{t \to \infty} d_i(t) = 0$. We recall that the limitation topology on a functional space $C(X, Y)$ with metrizable $Y$ is defined by the collection of open balls $B_{\rho}(f, 1)$ with respect to all compatible metrics $\rho$ on $Y$ where the ball in $C(X, Y)$ is defined by means of the sup metric $[Bo]$. We note that the functional space $C(X, \mathbb{R}^\omega)$ taken with the limitation topology is a Baire space $[T],[Bo]$. This allows to prove that the map
Embedding 9

\[ f = (f_i)_{i=1}^N : X \rightarrow \mathbb{R}^\omega \] can be approximated by embedding \( g : X \rightarrow \mathbb{R}^\omega \) in such a way that

\[ \rho(t) = \sup_{x \in X \setminus B(x_0, t)} \| f(x) - g(x) \|_{\infty} \rightarrow 0 \]

as \( t \rightarrow \infty \). Then the \( i \)th coordinate of \( g \), the map \( g_i : X \rightarrow \Sigma_p \), is slowly oscillating for each \( i \). Indeed, by the triangle inequality

\[ \lim_{t \rightarrow \infty} \sup_{x \in X \setminus B(x_0, t)} \text{diam}(g_i(B(x, 1))) \leq \lim_{t \rightarrow \infty} (d_i(t) + 2\rho(t)) = 0. \]

Note that the extension \( \bar{g}_i : \bar{X} \rightarrow \Sigma_\infty \) of \( g_i \) coincides with \( \bar{f}_i \) on \( \nu X \).

We replace the map \( F \) with a new map

\[ F' = (\bar{g}_i)_{i=1}^\infty \times (\bar{f}_\alpha)_{\alpha \in A \setminus \{\alpha_1, \ldots, \alpha_n\} : \bar{X} \rightarrow \prod_{\mathcal{A}} \Sigma_\infty \]

which is an embedding. \( \square \)

3.6. Remark. All coordinate maps \( \bar{f}_\alpha \) in the embedding \( F : \bar{X} \rightarrow \prod_{\mathcal{A}} \Sigma_\infty \) have the property that the restriction \( \bar{f}_\alpha \) to \( X \) lands in the subgroup \( \mathbb{R} \subset \Sigma_\infty \). Moreover, by the construction \( \bar{f}_\alpha(X) \subset \mathbb{R}_+ \).

4. Embedding into the product of Knaster continua

The proof of Theorem 3.11 in the case of finite \( p \) brings the following.

4.1. Theorem. For any \( p \) every simply connected proper geodesic metric space \( X \) admits an embedding of its Higson compactification into the product of \( p \)-adic solenoids

\[ F : \bar{X} \rightarrow \prod_{\mathcal{A}} \Sigma_p \]

that induces a rational isomorphism of 1-cohomology,

\[ \hat{H}^1(\prod_{\mathcal{A}} \Sigma_p) \otimes \mathbb{Q} \rightarrow \hat{H}^1(\bar{X}) \otimes \mathbb{Q} = \hat{H}^1(\bar{X}) = \bigoplus_{\mathcal{A}} \mathbb{Q}. \]

We call a map \( f : X \rightarrow Z \) essential if every map \( g : X \rightarrow Z \) homotopic to \( f \) is surjective. We call a subset \( X \subset \prod Z_\alpha \) essential if its projection on each factor \( p_\alpha : X \rightarrow Z_\alpha \) is essential. Since the embedding \( F : \bar{X} \rightarrow \prod \Sigma_p \) in Theorem 4.1 is essential, we obtain

4.2. Corollary. For any \( p \) every simply connected proper geodesic metric space \( X \) admits an essential embedding of its Higson compactification into a product of \( p \)-adic solenoids.

4.3. Proposition. For any \( p \in \{\infty\} \cup \mathbb{N} \setminus \{1\} \) any surjective map \( f : Y \rightarrow \Sigma_p \), of a connected compact Hausdorff space is essential.
Proof. Assume that $f$ is homotopic to $g$ which is not onto. Then for some projection $\pi : \Sigma_p \to S^1$ onto the circle in the inverse limit presentation of $\Sigma_p$ the composition $\pi \circ g$ is not onto. Hence it is homotopic to a constant map. The lifting of this homotopy defines a homotopy of $g$ to the disconnected fiber $A_p$ and hence to a point. Thus, we have a homotopy of $f$ to a constant map. Since $\Sigma_p$ has more than one path component, this is impossible for a surjective map. \hfill \Box

We define a continuum $K_p = \Sigma_p / \sim$ to be the quotient space under the identification $x \sim -x$. A theorem of Bellamy [Be] says that $K_2$ is homeomorphic to the Knaster continuum, also known as the Bucket handle continuum. We note that the subgroup $\mathbb{R} \subset \Sigma_p$ is taken under this identification to a path component of $K_p$ which is an injective image of the ray $\mathbb{R}_+$ with the initial point $q(0)$ where $q : \Sigma_p \to K_p$ is the quotient map.

Since for complex numbers the conjugation commutes with taking the power, $z^n = \overline{z^n}$, we have a commutative diagram

$$
\begin{array}{ccc}
S^1 & \xleftarrow{z^n} & S^1 \\
q \downarrow & & q \downarrow \\
I & \xleftarrow{\tilde{n}} & I
\end{array}
$$

where $q$ is the projection onto the interval, the orbit space of the conjugation, and $\tilde{n}$ is the induced map. Note that $\tilde{n} : I \to I$ is the $n$ times folding the interval. Thus, there is the inverse limit presentation of $K_p$ that fits into the diagram:

$$
\begin{array}{ccccccc}
S^1 & \xleftarrow{z_{n_1}} & S^1 & \xleftarrow{z_{n_2}} & S^1 & \xleftarrow{z_{n_3}} & \cdots & \xleftarrow{} & \Sigma_p \\
q \downarrow & & q \downarrow & & q \downarrow & & q \downarrow \\
I & \xleftarrow{\tilde{n}_1} & I & \xleftarrow{\tilde{n}_2} & I & \xleftarrow{\tilde{n}_3} & \cdots & \xleftarrow{} & K_p.
\end{array}
$$

We identify the first interval in the bottom inverse sequence with $[0, 1]$, the second with $[0, n_1]$, the third with $[0, n_1n_2]$ and so on to view the bonding maps as 'isometric' folding. Moreover, we view each folding map $\tilde{n}_k : [0, n_1n_2 \cdots n_k]$ as a retraction onto $[0, n_1n_2 \cdots n_{k-1}]$. Thus

$$
\mathbb{R}_+ = \bigcup_k [0, n_1n_2 \cdots n_k]
$$

is naturally embedded into $K_p$ as the path component of $q(0)$.

4.4. Proposition. For any $p \in \{\infty\} \cup \mathbb{N} \setminus \{1\}$ any surjective map $f : Y \to K_p$ of a compact Hausdorff space is essential.
Proof. Let $q : \Sigma_p \to K_p$ be the quotient map. Note that $q$ is open and all preimages $q^{-1}(y)$ are 2-point sets except $q^{-1}(q(0)) = 0$. We show that the compact space $Z$ in the pull-back in the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{f'} & \Sigma_p \\
q' & \downarrow & \downarrow q \\
Y & \xrightarrow{f} & K_p
\end{array}
\]

is connected. Suppose that $Z = U_1 \cup U_2$ is a disjoint union of nonempty open sets. For $i = 1, 2$ we define subsets $A_i = \{y \in Y \mid (q')^{-1}(y) \subset U_i\}$. Since $U_i$ are closed and open and $q'$ is an open map the sets $A_1 = Y \setminus q'(U_2)$ and $A_2 = Y \setminus q'(U_1)$ are closed and open. Since $q'$ has single point fibers at least one of $A_i$ is not empty, let it be $A_1$. Since $Y$ is connected, we obtain $Y = A_1$. Then $U_2 = \emptyset$ which contradicts to the assumption.

Assume that $f$ is homotopic by means of homotopy $H : Y \times I \to K_p$ to a non-surjective map $g$. Note that for $y \in Y$ with $f(y) \in \mathbb{R}_+$ we have $H(y \times I) \subset \mathbb{R}_+$. We may assume that the inner metric on $\mathbb{R}_+$ induced by a metric on $K_p$ is the standard metric on $\mathbb{R}_+$. Thus $\mathbb{R}_+$ makes infinitely many waves in $K_p$ of amplitude 1. Then the compactness of $Y$ and the continuity of $H$ imply that there is an upper bound $b$ on the diameter of a path $H(y, -) : I \to \mathbb{R}_+$ in $\mathbb{R}_+$. Let $Y_0 = f^{-1}([0, 2b])$ and let $a \in K_p \setminus g(Y)$. We consider a new homotopy $H_0 : Y \times I \to K_p$ defined by the region under the graph of a function $\phi : Y \to [0, 1]$ with $\phi(Y_0) = 0$ and $\phi(f^{-1}(a)) = 1$. Thus, the homotopy $H_0$ is stationary on $Y_0$ and does not move other points through $0 \in \mathbb{R}_+ \subset K_p$. The homotopy $H_0$ defines a homotopy $H'_0 : Z \times I \to K_p$. Let $Z_0 = (q')^{-1}(Y_0)$. By the homotopy lifting property of a fibration $q\mid : \Sigma_p \setminus \{0\} \to K_p \setminus \{0\}$ there is a lift $\bar{H}_0 : (Z \setminus Z_0) \times I \to \Sigma_p$ of $H'_0|_{(Y \setminus Y_0) \times I}$. Since $q$ is open, the extension of $\bar{H}_0$ to $\bar{H}_1 : Z \times I \to \Sigma_p$ by the stationary homotopy of $f'|_{Z_0}$ is a homotopy of $f' : Z \to \Sigma_p$ to a map with the image in a proper subset. This contradicts to Proposition 1.3.

4.5. Theorem. For any $p$ and any simply connected finite dimensional proper geodesic metric space $X$ its Higson compactification can be essentially embedded into the product of continua $K_p$.

Proof. The embedding follows from the fact that the family of functions $\mathcal{C} = \{q\tilde{f}_\alpha \mid \alpha \in A\}$, where $q : \Sigma_p \to K_p$ is the quotient map, separates disjoint closures of basis sets by the argument of Proposition 3.4.

In view of Proposition 4.4 all maps $q\tilde{f}_\alpha$ are essential. □
References

[Be] D. Bellamy, A tree-like continuum without the fixed point property, Houston Math. J. 6 (1979), 1-13.

[Bo] Ph. Bowers, Limitation Topologies on Function Spaces, Trans. Amer Math. Soc. V. 314, No 1 (1989), 421-431.

[BV] J. Burgos, A. Verjovsky, Adelic solenoid I: Structure and topology, arXiv:1603.05676v4 [math.CV] 2 Aug 2019.

[CGM] A. Connes, M. Gromov, H. Moscovici, Group cohomology with Lipschitz control and higher signatures, GAFA v. 3 (1993), 1-78.

[Dr] A. Dranishnikov, Asymptotic topology, Russian Math. Surveys 55 (2000), no. 6, 1085-1129.

[DF] A. Dranishnikov, S. Ferry, On the Higson-Roe corona, Russian Math. Surveys 52 (1997), no. 5, 1017-1028.

[DFW] A. Dranishnikov, S. Ferry, and S. Weinberger, An Etale Approach to the Novikov Conjecture Comm.Pure Appl. Math, 61 no.2, 2008, 139-155.

[DKU] A. Dranishnikov, J. Keesling, V. Uspenski, On the Higson corona of uniformly contractible spaces, Topology 37 (1998), no. 4, 791–803.

[HR] Higson, Nigel; Roe, John Analytic K-homology. Oxford Mathematical Monographs. Oxford Science Publications. Oxford University Press, Oxford, 2000.

[K] J. Keesling, The one-dimensional Čech cohomology of the Higson compactification and its corona. Topology Proc. 19 (1994), 129–148.

[Ro1] Roe, John, Coarse cohomology and index theory on complete Riemannian manifolds. Mem. Amer. Math. Soc. 104 (1993), no. 497.

[Ro2] Roe, John, Lectures on coarse geometry. University Lecture Series, 31. American Mathematical Society, Providence, RI, 2003.

[T] H. Torunczyk, Characterizing Hilbert space topology. Fund. Math. 111 (1981), 247-262.

[Yu] Guoliang Yu, The Novikov conjecture for groups with finite asymptotic dimension, Ann. of Math. (2) 147 (1998), no. 2, 325–355.

Alexander N. Dranishnikov, Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, FL 32611-8105, USA
Email address: dranish@ufl.edu

James E. Keesling, Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, FL 32611-8105, USA
Email address: kees@ufl.edu