ON THE DIOPHANTINE EQUATIONS

\[ X^3 + Y^3 + Z^3 + aU^k = \sum_{i=0}^{n} a_i U_i^{t_i}; \quad k = 3, 4 \]

FARZALI IZADI AND MEHDI BAGHALAGHDAM

Abstract. In this paper, elliptic curves theory is used for solving the Diophantine equations

\[ X^3 + Y^3 + Z^3 + aU^k = \sum_{i=0}^{n} a_i U_i^{t_i}; \quad k = 3, 4 \]

where \( n, t_i \in \mathbb{N} \cup \{0\} \), and \( a \neq 0, a_i \), are fixed arbitrary rational numbers. We try to transform each case of the above Diophantine equations to a cubic elliptic curve of positive rank, then get infinitely many integer solutions for each case. We also solve these Diophantine equations for some values of \( n, a, a_i, t_i \), and obtain infinitely many solutions for each case, and show among the other things that how sums of four, five, or more cubics can be written as sums of four, five, or more biquadrates as well as sums of 5th powers, 6th powers and so on.

1. Introduction

The authors in two different papers, used elliptic curves to solve two Diophantine equations

\[(1.1) \quad X^4 + Y^4 = 2U^4 + \sum_{i=1}^{n} T_i U_i^{\alpha_i}, \]

where \( n, \alpha_i \in \mathbb{N} \), and \( T_i \), are appropriate rational numbers, and

\[(1.2) \quad \sum_{i=1}^{n} a_i x_i^6 + \sum_{i=1}^{m} b_i y_i^3 = \sum_{i=1}^{n} a_i X_i^6 \pm \sum_{i=1}^{m} b_i Y_i^3, \]

where \( n, m \in \mathbb{N} \) and \( a_i, b_i \neq 0 \in \mathbb{Q} \). (see [1], [2])

In this paper, we are interested in the study of the Diophantine equations:

\[(1.3) \quad X^3 + Y^3 + Z^3 + aU^k = \sum_{i=0}^{n} a_i U_i^{t_i}; \quad k = 3, 4 \]

where \( n, t_i \in \mathbb{N} \cup \{0\} \), and \( a \neq 0, a_i \), are fixed arbitrary rational numbers. This shows that how sums of four, five, or more cubics can be written as sums of four, five, or more biquadrates as well as sums of 5th powers, 6th powers and so on.

2010 Mathematics Subject Classification. 11D45, 11D72, 11D25, 11G05 and 14H52.

Key words and phrases. Diophantine equations, High power Diophantine equations, Elliptic curves.
We conclude this introduction with a standard fact which is needed in section 3. (see [4])

**Lemma 1.1.** Let $K$ be a field of characteristic not equal to 2. Consider the equation

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2,$$

with $a, b, c, d \in K$.

Let $x = \frac{2q(v+c)+du}{u^2}$, $y = \frac{4q^2(v+c)+2q(du+cu^2)-(d^2u^2)}{u^6}$.

Define $a_1 = \frac{d}{q}$, $a_2 = c - (\frac{d^2}{4q^2})$, $a_3 = 2qb$, $a_4 = -4q^2a$, $a_6 = a_2a_4$.

Then $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.

The inverse transformation is $u = \frac{2q(v+c)-(\frac{d^2}{4q})}{y}$, $v = -q + \frac{u(ux-d)}{2q}$.

The point $(u, v) = (0, q)$ corresponds to the point $(x, y) = \infty$ and $(u, v) = (0, -q)$ corresponds to $(x, y) = (-a_2, a_1a_2 - a_3)$.

2. The Diophantine equation (DE) $X^3 + Y^3 + Z^3 + aU^3 = \sum_{i=0}^{n} a_i U_i^{t_i}$

**Main Theorem 2.1.** Consider the DE (1.3) for the case $k = 3$, where $n, t_i \in \mathbb{N} \cup \{0\}$, $a \neq 0$, $a_i$, are fixed arbitrary rational numbers.

We try to transform this DE to a cubic elliptic curve of positive rank.

Let $Y^2 = X^3 + FX^2 + GX + H$, be an elliptic curve in which the coefficients $F, G, H$, are all functions of $a, a_i, t_i$ and the other rational parameters $P_i, t, s$, yet to be found later. If the elliptic curve has positive rank, depending on the values of $P_i, s$, the DE (1.3) has infinitely many integer solutions.

**Proof 2.1.** Firstly, it is clear that if we find rational solutions for each case of the Diophantine equations (1.3), then by canceling the denominators of $X, Y, Z, U, U_i$, and by multiplying both sides of these Diophantine equations by appropriate number $M$, we may obtain integer solutions for each case.

Let $X = -Z + t$, $Y = -Z - t$, where $Z, t$ are rational variables. By substituting these variables in the DE (1.3), and some simplifications, we get:

$$t^2 = (\frac{a}{6Z})U^3 + (\frac{-Z^2}{6} - \sum_{i=0}^{n} a_i U_i^{t_i})$. 

(2.1)
ON THE DIOPHANTINE EQUATIONS $X^3 + Y^3 + Z^3 + aU^k = \sum_{i=0}^{n} a_i U_i^k$; $k = 3, 4$  

By multiplying both sides of the Eq. (2.1), by $(\frac{a}{6Z})^2$, and letting  

(2.2) \[ Y' = t.(\frac{-a}{6Z}), \]

and  

(2.3) \[ X' = U.(\frac{a}{6Z}), \]

we get:  

(2.4) \[ Y'^2 = X'^3 + (-\frac{a^2}{216} - (\frac{a^2}{216Z^3}). \sum_{i=0}^{n} a_i U_i^k). \]

Now by choosing appropriate values for $Z$ (equal to $s$), and $U_i$ (equal to $P_i$), such that the rank of the elliptic curve (2.4) to be positive, and by calculating $t, U, X, Y,$ from the relations (2.2), (2.3), $X = -Z + t$ and $Y = -Z - t$, some simplifications and canceling the denominators of $X, Y, Z, U, U_i$, we obtain infinitely many integer solutions for the DE (1.3). The proof is completed.

Now we work out some examples.

2.1. Application to examples.

**Example 1.** $X^3 + Y^3 + Z^3 + U^3 = U_0^3 + U_1^3 + U_2^3$

i.e., the sum of 4 cubics can be written as the sum of 3 cubics.

The cubic elliptic curve (2.4) is $Y'^2 = X'^3 - (\frac{a^2}{216} - (\frac{a^2}{216Z^3}).\sum_{i=0}^{n} a_i U_i^k).

Rank=1.

Generator: $P = (X', Y') = (\frac{643}{90}, \frac{2578}{135}).$

$t, U) = (\frac{-5156}{9}, \frac{643}{3}).$

Solution:

$9^3 + 18^3 + 27^3 + 5201^3 = 45^3 + 1929^3 + 5111^3.$

**Example 2.** $X^3 + Y^3 + Z^3 + U^3 = U_0^4$

i.e., fourth power of an integer is written as a sum of four cubics.

The cubic elliptic curve (2.4) is $Y'^2 = X'^3 - (\frac{a^2}{216} - (\frac{a^2}{216Z^3}).

Now letting $Z = -2$, and $U_0 = 7$, yields
\[ Y'^2 = X'^3 + \frac{2393}{1728}. \]
Rank=2.
Generators: \( P_1 = (X', Y') = \left( \frac{115}{48}, \frac{249}{64} \right), \ P_2 = (X'', Y'') = \left( \frac{1320481}{291600}, \frac{1528666129}{157464000} \right). \)
\( (t, U) = \left( \frac{747}{16}, \frac{-115}{4} \right), \) for the point \( P_1, \)
\( (t', U') = \left( \frac{1528666129}{24300}, \frac{-1320481}{242} \right), \) for the point \( P_2. \)

Solutions:

\[ 779^3 + (-715)^3 + (-32)^3 + (-460)^3 = 56^4. \]
\[ 7774550645^3 + (-7512110645)^3 + (-131220000)^3 + (-3565298700)^3 = 5103000^4. \]

**Example 3.** \( X^3 + Y^3 + Z^3 + U^3 = U^4_0 + U^4_1 + U^4_2 + U^4_3 \)
i.e., the sum of 4 cubics is written as the sum of 4 biquadrates.

The cubic elliptic curve \((2.4)\) is \[ Y'^2 = X'^3 - \frac{1}{216} - \frac{(U_0^4 + U_1^4 + U_2^4 + U_3^4)}{216Z^4}. \]
Now letting \( Z = -5, \ U_0 = 1, \ U_1 = 2, \ U_2 = 3, \ U_3 = 4, \) yields
\[ Y'^2 = X'^3 + \frac{229}{27000}. \]
Rank=1.
Generator: \( P = (X', Y') = \left( \frac{397}{7200}, \frac{14821}{3997200} \right). \)
\( (t, U) = \left( \frac{14821}{5324}, \frac{-367}{242} \right). \)

Solution:

\[ 1823404^3 + 519156^3 + (-1171280)^3 + (-355256)^3 = 10648^4 + 21296^4 + 31944^4 + 42592^4. \]

**Example 4.** \( X^3 + Y^3 + Z^3 + U^3 = U^5_0 + U^5_1 + U^5_2 + U^5_3 \)
i.e., the sum of 4 cubics is written as the sum of 4 fifth powers.

The cubic elliptic curve \((2.4)\) is \[ Y'^2 = X'^3 - \frac{1}{210} - \frac{(U_0^5 + U_1^5 + U_2^5 + U_3^5)}{216Z^5}. \]
Now letting \( Z = -6, \ U_0 = 1, \ U_1 = 2, \ U_2 = 3, \ U_3 = 4, \) yields
\[ Y'^2 = X'^3 + \frac{271}{11664}. \]
Rank=1.
Generator: \( P = (X', Y') = \left( \frac{47}{324}, \frac{917}{5832} \right). \)
\( (t, U) = \left( \frac{917}{162}, \frac{-37}{9} \right). \)
ON THE DIOPHANTINE EQUATIONS \( X^3 + Y^3 + Z^3 + aU^k = \sum_{i=0}^{n} a_i U_i^3 ; \ k = 3, 4 \)

Solution:

\[ 90672^3 + 2640^3 + (-46656)^3 + (-31968)^3 = 216^5 + 432^5 + 648^5 + 864^5. \]

Example 5. \( X^3 + Y^3 + Z^3 + U^3 = U_0^6 + U_1^6 + U_2^6 + U_3^6 \)
i.e., the sum of 4 cubics is written as the sum of 4 sixth powers.

The cubic elliptic curve \( [2.4] \) is \( Y^2 = X^3 - \frac{1}{216} - \left( \frac{U_0^6 + U_1^6 + U_2^6 + U_3^6}{216} \right) \).

Now letting \( Z = -7, U_0 = 1, U_1 = 2, U_2 = 3, U_3 = 4 \), yields
\[ Y^2 = X^3 + \frac{4547}{74088}. \]

Rank=2.

Generators:
\[ P_1 = (X', Y') = \left( \frac{284747}{318480}, \frac{111956735}{127042398} \right), P_2 = (X'', Y'') = \left( \frac{-162932615}{160744500}, \frac{1819882008883}{721489} \right). \]

\( (t, U) = \left( \frac{3524819}{111956735}, \frac{-1138900}{7581} \right), (t', U') = \left( \frac{-1333042310}{921489}, \frac{-1526057}{1701} \right). \)

Solutions:
\[ 2529478892^3 + (-1724877038)^3 + (-402300927)^3 + (-2158667007)^3 = 7581^6 + 15162^6 + 22743^6 + 30324^6. \]

\[ 489320985767551^3 + 5232371404673^3 + 484195324491480^3 + (-24726678586112)^3 = 5943504^6 + 11887008^6 + 17830512^6 + 23774016^6. \]

If we take \( Z = -7, U_0 = U_1 = 3, U_2 = U_3 = 4 \), the above elliptic curve becomes
\[ Y^2 = X^3 - \frac{1}{216} - \left( \frac{3^6 + 3^6 + 4^6 + 4^6}{216(-7)^3} \right) = X^3 + \frac{9307}{74088}. \]

Rank=2.

Generators:
\[ P_1 = (X', Y') = \left( \frac{1654081}{3572100}, \frac{3201764021}{6751269000} \right), P_2 = (X'', Y'') = \left( \frac{1526057}{71442}, \frac{-666521155}{6751269} \right). \]

\( (t, U) = \left( \frac{160744500}{3201764021}, \frac{-1654081}{85050} \right), (t', U') = \left( \frac{-1333042310}{921489}, \frac{-1526057}{1701} \right). \)

Solutions: (for the DE \( X^3 + Y^3 + Z^3 + U^3 = 2U_0^6 + 2U_2^6 \)):
\[ 21634877605^3 + (-10382762605)^3 + (-5626057500)^3 + (-15631065450)^3 = 2(85050)^6 + 2(113400)^6. \]
\[-1330791887^3 + 1335292733^3 + (-2250423)^3 + (-288424773)^3 = 2(1701)^6 + 2(2268)^6.\]

**Example 6.** \(X^3 + Y^3 + Z^3 + U^3 = U_0^7 + U_1^7 + U_2^7 + U_3^7\)
i.e., the sum of 4 cubics is written as the sum of 4 seventh powers.

The cubic elliptic curve (2.4) is \(Y'^2 = X'^3 - \frac{1}{216} - \left(\frac{U_0^7 + U_1^7 + U_2^7 + U_3^7}{216Z^3}\right)\).
Now letting \(Z = 2, U_0 = 1, U_1 = 2, U_2 = 3, U_3 = 4\), yields \(Y'^2 = X'^3 - \frac{1559}{144}\).
Rank=1.
Generator: \(P = (X', Y') = \left(\frac{55825}{1921}, -\frac{52754917}{237276}\right)\).
\((t, U) = \left(\frac{52754017}{19773}, \frac{223300}{507}\right)\).

Solution:
\[365855455514133^3 + (-366404379540849)^3 + 274462013358^3 + 60441190910100^3 = 59319^7 + 118638^7 + 177957^7 + 237276^7.\]

**Example 7.** \(X^3 + Y^3 + Z^3 + U_0^3 + U_3^3 = U_1^8 + U_2^8 + U_3^8 + U_4^8 + U_5^8\)
i.e., the sum of 5 cubics is written as the sum of 5 eighth powers.

The cubic elliptic curve (2.4) is \(Y'^2 = X'^3 - \frac{1}{216} - \left(\frac{-U_0^7 + U_1^7 + U_2^7 + U_3^7 + U_4^7 + U_5^7}{216Z^3}\right)\).
Now letting \(Z = 2, U_0 = 1, U_1 = 2^3, U_2 = 2, U_3 = 3^3, U_4 = 4^3, U_5 = 4\), yields \(Y'^2 = X'^3 - \frac{14946019}{3799136}\).
Rank=1.
Generator: \(P = (X', Y') = \left(\frac{4905677718694537513}{249\,711\,379\,389\,655\,3424}, -\frac{7515135371775048887228789899}{394\,600\,381\,245\,463\,819\,147\,517\,2032}\right)\).
\((t, U) = \left(\frac{7515135371775048887228789899}{328\,833\,651\,037\,886\,515\,966\,264\,336}, \frac{4905677718694537513}{208\,092\,816\,158\,029\,452}\right)\).

Solution:
\[X = \frac{6857468069699275855316261227}{328\,833\,651\,037\,886\,515\,966\,264\,336}, \quad Y = \frac{-81728026785082919141318571}{328\,833\,651\,037\,886\,515\,966\,264\,336}, \quad U = \frac{4905677718694537513}{208\,092\,816\,158\,029\,452}.\]
ON THE DIOPHANTINE EQUATIONS $X^3 + Y^3 + Z^3 + aU^k = \sum_{i=0}^n a_i U_i^{t_i}; k = 3, 4$

$Z = 2, U_0 = 1, U_1 = \frac{2}{3}, U_2 = 2, U_3 = \frac{1}{3}, U_4 = 3, U_5 = \frac{4}{3}$.

**Remark 2.2.** By choosing the other points on the above elliptic curves such as $nP$ ($n = 3, 4, \cdots$), $P$ is one of the elliptic curves generators, we obtain infinitely many solutions for each case of the above Diophantine equations.

3. The DE $X^3 + Y^3 + Z^3 + aU^4 = \sum_{i=0}^n a_i U_i^{t_i}$

**Main Theorem 3.1.** Consider the DE (1.3) for the case $k = 4$, where $n, t \in \mathbb{N} \cup \{0\}$, and $a \neq 0, a_i$, are fixed arbitrary rational numbers.

Let $Y^2 = X^3 + FX^2 + GX + H$, be an elliptic curve in which the coefficients $F, G, H$, are all functions of $a, a_i, t_i$ and the other rational parameters $P_i, t, s$, yet to be found later. If the elliptic curve has positive rank, depending on the values of $P_i, s$, the DE (1.3) has infinitely many integer solutions.

**Proof 3.1.** Let $X = -Z + t, Y = -Z - t$, where $Z, t$ are rational variables. By substituting these variables in the DE (1.3), and some simplifications, we get:

\begin{equation}
(3.1) \quad t^2 = \left(\frac{a}{6Z}\right)U^4 + \left(-\frac{Z^2}{6} - \frac{\sum_{i=0}^n a_i U_i^{t_i}}{6Z}\right).
\end{equation}

Now by choosing appropriate values for $U_i$ (equal to $P_i$), and $Z$ (equal to $s$), such that the rank of the quartic elliptic curve (3.1) to be positive (Then we get infinitely many rational solutions for $U, t$), by calculating $U, t, X, Y, U_i, Z$, from the relations (3.1), $X = -Z + t, Y = -Z - t, U_i = P_i, Z = s$, after some simplifications and canceling the denominators of the values obtained for variables, we obtain infinitely many integer solutions for the DE (1.3). The proof is completed.

**Remark 3.2.** (If in the quartic elliptic curve (3.1),

\begin{equation}
(3.2) \quad Q := \left(-\frac{Z^2}{6} - \frac{\sum_{i=0}^n a_i U_i^{t_i}}{6Z}\right),
\end{equation}

to be square (It is done by choosing appropriate values for $Z$, and $U_i$), say $q^2$, we may use the lemma 1.1 for transforming this quartic to a cubic elliptic curve of the form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, where $a_i \in \mathbb{Q}$.
Then we solve the cubic elliptic curve just obtained of the rank $\geq 1$, and get infinitely many solutions for the above DE.

Generally, it is not essential that $Q$ to be square, because we may transform the quartic (3.1) to a quartic that its constant number is square if the rank of the quartic (3.1) is positive. The only important thing is that the rank of the quartic elliptic curve (3.1), to be positive for getting infinitely many nontrivial solutions for the above DE. See the example 8.)

Now let us take
\[ q^2 = \frac{-Z^2}{6} - \sum_{i=0}^{n} a_i U_i t_i / 6Z \]  
(3.3)

Then for the quartic elliptic curve (3.1), we have
\[ t^2 = \left( \frac{a}{6Z} \right) U^4 + q^2. \]  
(3.4)

With the inverse transformation
\[ U = \frac{2qX'}{Y'}, \]  
(3.5)

and
\[ t = -q + \frac{U^2 X'}{2q}, \]  
(3.6)

the corresponding cubic elliptic curve is
\[ Y'^2 = X'^3 + \left( \frac{aZ}{9} + \frac{a.(\sum_{i=0}^{n} a_i U_i t_i)}{9Z^2} \right) X'. \]  
(3.7)

Then if the elliptic curve (3.7) has positive rank (This is done by choosing appropriate values for $U_i$, and $Z$), by calculating $U$, $t$, $X$, $Y$, from the relations (3.5), (3.6), $X = -Z + t$, and $Y = -Z - t$, after some simplifications and canceling the denominators of values obtained for variables, we obtain infinitely many integer solutions for the DE (1.3). Thus we conclude that we must choose appropriate values for $U_i$, and $Z$, such that the rank of the elliptic curve (3.7) to be positive, then obtain infinitely many solutions for the DE (1.3).

3.1. Application to examples.

Example 8. $X^3 + Y^3 + Z^3 = U^4 + a_0 U_0^{t_0} + a_1 U_1^{t_1}$
ON THE DIOPHANTINE EQUATIONS $X^3 + Y^3 + Z^3 + aU^k = \sum_{i=0}^{n} a_i U_i^k$; $k = 3, 4$

First of all, we solve the DE $X^3 + Y^3 + Z^3 = U^4 + a_0 U_0^t + a_1$.
Let $X = -Z + t$ and $Y = -Z - t$. By substituting these variables in the above DE, and some simplifications, we get

$$(3.8) \quad t^2 = \left(\frac{-1}{6Z}\right)U^4 + \left(\frac{-Z^2}{6} - \frac{a_0 U_0^t + a_1}{6Z}\right).$$

Let us transform the quadratic (3.8) to a cubic elliptic curve. Let

$$(3.9) \quad q^2 = \left(\frac{-Z^2}{6} - \frac{a_0 U_0^t + a_1}{6Z}\right).$$

Now with the inverse transformation

$$(3.10) \quad U = \frac{2qX'}{Y'},$$

and

$$(3.11) \quad t = -q + \frac{U^2 X'}{2q},$$

the quadratic (3.8), maps to the cubic elliptic curve

$$(3.12) \quad Y'^2 = X'^3 + \left(\frac{2q^2}{3Z}\right)X'.$$

Then if the above elliptic curve has positive rank (This is done by choosing appropriate arbitrary values for $q$, and $Z$, such as the rank of the above elliptic curve to be $\geq 1$), we obtain infinitely many rational solutions for the DE $X^3 + Y^3 + Z^3 = U^4 + a_0 U_0^t + a_1$.

After multiplying both sides of the above DE by appropriate number $M$, we obtain integer solutions for the main DE $X^3 + Y^3 + Z^3 = U^4 + a_0 U_0^t + a_1 U_1^t$.

As an example, if we take $Z = -2$, $q = 7$, $a_0 = 1$, $U_0 = 2$, $t_0 = t_1 = 4$, ($a_1 = 580$),
the elliptic curve (3.12) becomes

$Y'^2 = X'^3 - \frac{49}{3}X'.$

Rank=1.
Generator: $P = (X', Y') = (\frac{300}{361}, \frac{24730}{6859})$.
$(t, U) = (\frac{46590103}{6113729}, \frac{7980}{2373})$.

Solution:

$$(-210128161627205)^3 + 3597367626432969^3 + (-74804282402882)^3 = 48803517420^4 + 580.(15124197817)^4 + 30248395634^4.$$
By taking $Z = -1$, $q = 12$, $a_0 = 1$, $U_0 = 3$, $t_0 = t_1 = 6$, ($a_1 = 136$), the elliptic curve (3.12) becomes $Y'^2 = X'^3 - 96X'$. 

Rank=1. 
Generator: $P = (X', Y') = (-8, 16)$. 
$2P = (25, -115)$. 
$(t, U) = (\frac{8652}{529}, -\frac{120}{23})$, (for the point $2P$). 

Solution: 

$4856749^3 + (-4297067)^3 + (-279841)^3 = 63480^4 + 1587^6 + 136(529)^6$. 

Letting $Z = -1$, $q = 9$, $a_0 = 1$, $U_0 = 2$, $t_0 = t_1 = 8$, ($a_1 = 231$), in the elliptic curve (3.12), yields $Y'^2 = X'^3 - 54X'$. 

Rank=1. 
Generator: $P = (X', Y') = (-2, 10)$. 
$(t, U) = (\frac{-261}{25}, -\frac{18}{5})$. 

Solution: 

$(-3687500)^3 + 4468750^3 + (-390625)^3 = 56250^4 + 250^8 + 231.(125)^8$. 

**Example 9.** $Y_1^3 + Y_2^3 + Y_3^3 = X_1^5 + X_2^5 + X_3^5$ ($a = 0$) 

i.e., the sum of 3 cubics can be written as the sum of 3 fifth powers.

We solve this DE with another new method. 
Let: $Y_1 = t + v$, $Y_2 = t - v$, $Y_3 = \beta t$, $X_1 = t + x_1$, $X_2 = t - x_1$, $X_3 = \alpha t$. 

By substituting these variables in the above DE, we get 

$$2t^3 + 6tv^2 + \beta^3 t^3 = (2t^5 + 10x_1^4t + 20x_1^2t^3) + \alpha^5 t^5.$$ 

Then after some simplifications and clearing the case of $t = 0$, we obtain 

$$v^2 = \frac{2 + \alpha^5}{6}t^4 + \frac{20x_1^2 - 2 - \beta^3}{6}t^2 + \left(\frac{5}{3}\right)x_1^4.$$ 

Note that $(\frac{5}{3})x_1^4$, is not a square. Then we may not use from the lemma 1.1 for transforming this quartic to a cubic elliptic curve, but we do this work
ON THE DIOPHANTINE EQUATIONS $X^3 + Y^3 + Z^3 + aU^k = \sum_{i=0}^{\alpha} a_i U_i^k$; $k = 3, 4$

by another method. Let us take $x_1 = 1$, $\alpha = \beta = 2$. Then the quartic (3.14) becomes

(3.15) \[ v^2 = \frac{17}{3} t^4 + \frac{5}{3} t^2 + \frac{5}{3}. \]

By searching, we see that the above quartic has two rational points $P_1 = (1, 3)$, and $P_2 = (7, 117)$, among others. Let us put $T = t - 1$. Then we get

(3.16) \[ v^2 = \frac{17}{3} T^4 + \frac{68}{3} T^3 + \frac{107}{3} T^2 + 26T + 9. \]

Now with the inverse transformation

(3.17) \[ T = \frac{6(X + \frac{107}{3}) - 26^2}{Y}, \]

and

(3.18) \[ v = -3 + \frac{T(XT - 26)}{6}, \]

the quartic (3.16), maps to the cubic elliptic curve

(3.19) \[ Y^2 + \frac{26}{3} XY + 136Y = X^3 + \frac{152}{9} X^2 - 204X - \frac{10336}{3}. \]

Rank=2.

Generators: $P_1 = (X', Y') = \left( -\frac{152}{9}, \frac{280}{27} \right)$, $P_2 = (X'', Y'') = \left( -\frac{44}{3}, \frac{20}{9} \right)$.

To square the left hand of (3.19), let us put $M = Y + \frac{13}{3}X + 68$. Then the cubic elliptic curve (3.19) transforms to the Weierstrass form

(3.20) \[ M^2 = X^3 + \frac{107}{3} X^2 + \frac{1156}{3}X + \frac{3536}{3}. \]

Rank=2.

Generators: $G_1 = (X', M') = \left( -\frac{44}{3}, \frac{20}{3} \right)$, $G_2 = (X'', M'') = \left( -\frac{152}{9}, \frac{140}{27} \right)$.

Thus we conclude that we could transform the main quartic (3.15) to the cubic elliptic curve (3.20) of the rank equal to 2.

Because of this, the above cubic elliptic curve has infinitely many rational points and we may obtain infinitely many solutions for the above DE too. Since $G_1 = (X', M') = \left( -\frac{44}{3}, \frac{20}{3} \right)$, we get $(t, v) = (7, -117)$, that is on the (3.15), by calculating $X_i, Y_i$, from the above relations and after some simplifications and canceling the denominators of $X_i, Y_i$, we obtain a solution.
for the above DE:

\((-110)^3 + 124^3 + 14^3 = 8^5 + 6^5 + 14^5.\)

It is interesting to see that \((-110) + 124 + 14 = 8 + 6 + 14\), too.

Also we have \(2G_2 = (\frac{373}{36}, -\frac{21721}{216})\).

By using this new point \(2G_2 = (\frac{373}{36}, -\frac{21721}{216})\), we get \((t, v) = (\frac{11}{47}, \frac{2943}{2209})\).

Solution:

\(359227580^3 + (-251874598)^3 + 107352982^3 = 128122^5 + (-79524)^5 + 48598^5.\)

The Sage software has been used for calculating the rank of the elliptic curves. (see [3])

REFERENCES

[1] F. Izadi and M. Baghalaghdam, On the Diophantine equation \(X^4 + Y^4 = 2U^4 + \sum_{i=1}^{n} T_i U_{\alpha_i}^4\), submitted, (2016).
[2] F. Izadi and M. Baghalaghdam, On the six-three power Diophantine equations, submitted, (2016).
[3] SAGE software, available from http://sagemath.org.
[4] L. C. Washington, Elliptic Curves: Number Theory and Cryptography, Chapman-Hall, (2008).

Farzali Izadi, Department of Mathematics, Faculty of Science, Urmia University, Urmia 165-57153, Iran  
E-mail address: f.izadi@urmia.ac.ir

Mehdi Baghalaghdam, Department of Mathematics, Faculty of Science, Azarbaijan Shahid Madani University, Tabriz 53751-71379, Iran  
E-mail address: mehdi.baghalaghdam@azaruniv.edu