TWO RESULTS ON DOMINO AND RIBBON TABLEAUX

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Abstract. Inspired by the spin-inversion statistic of Schilling, Shimozono and White [8] and Haglund et al. [2] we relate the symmetry of ribbon functions to a result of van Leeuwen, and also describe the multiplication of a domino function by a Schur function.

1. Introduction

Lascoux, Leclerc and Thibon [5] defined spin-weight generating functions $G^{(n)}_{\lambda/\mu}(X; q)$ (from hereon called ribbon functions) for ribbon tableaux. They proved that these functions were symmetric functions using the action of the Heisenberg algebra on the Fock space of $U_q(\hat{sl}_n)$. For the $n = 2$ case of domino tableaux, a combinatorial proof of the symmetry and in fact a description of the expansion of $G^{(n)}_{\lambda/\mu}(X; q)$ in terms of Schur functions is given by the Yamanouchi domino tableaux of Carré and Leclerc [1]. More recently, Schilling, Shimozono and White [8] and separately Haglund et al. [2] have described the spin statistic of a ribbon tableau in terms of an inversion number on the $n$-quotient. This article gives two applications of this inversion number towards the ribbon functions.

Our first application is a proof of the symmetry of ribbon functions using a result of van Leeuwen [6] developed from his spin-preserving Knuth correspondence for ribbon tableaux. The result says roughly that the spin generating functions for adding horizontal ribbon strips above or below a lattice path vertical on both ends are equal. Another ‘elementary’ but more systematic proof of the symmetry of ribbon functions will appear in [3].

Our second application is an imitation of Stembridge’s concise proof of the Littlewood-Richardson rule [9] for the domino tableau case. We describe the expansion of $s_\nu(X)G^{(2)}_{\mu/\rho}(X; q)$ in the basis of Schur functions in terms of $\nu$-Yamanouchi domino tableaux. This description appears to be new and also gives a shorter proof of the result of Carré and Leclerc [1], corresponding to $\nu = (0)$, the empty partition.

In the last section we describe explicitly a bijection in terms of words required to prove the symmetry of ribbon functions.

We refer the reader to [5, 4] for the necessary definitions and notation concerning ribbon tableaux, spin and ribbon functions. We will always think of our partitions and tableaux as being drawn in the English notation.

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2. Spin-inversion statistic

We will use the spin-inversion statistic from [2] as its description is considerably shorter than the one in [8], and we will only be interested in how spin changes rather than its...
exact value. Let \( \text{quot}_n(T) = (T^{(0)}, \ldots, T^{(n-1)}) \) denote the \( n \)-quotient of a ribbon tableau \( T \) (which may have skew shape). With the \( n \)-core fixed, semistandard ribbon tableaux are in bijection with such \( n \)-tuples of usual tableaux. The diagonal \( \text{diag}(s) \) of a cell \( s \in \text{quot}_n(T) \) is equal to the diagonal of \( T \) on which the head of the corresponding ribbon \( \text{Rib}(s) \) lies.

For a cell \( s \in T^{(i)} \) it is given by \( \text{diag}(s) = nc(s) + c_i \) for some offsets \( c_i \) depending on the \( n \)-core of \( \text{sh}(T) \). Here \( c(s) = j - i \) is the usual content of a square \( s = (i, j) \). An inversion is a pair of entries \( T(x) = a, T(y) = b \) such that \( a < b \) and \( 0 < \text{diag}(x) - \text{diag}(y) < n \). We denote by \( \text{inv}(T) = \text{inv}(\text{quot}_n(T)) \) the number of inversions of \( \text{quot}_n(T) \). We have:

**Lemma 1.** Given a skew shape \( \lambda/\mu \), there is a constant \( e(\lambda/\mu) \) such that for every standard \( n \)-ribbon tableau \( T \) of shape \( \lambda/\mu \), we have \( \text{spin}(T) = e(\lambda/\mu) - \text{inv}(\text{quot}_n(T)) \).

We shall use a particular diagonal reading order on our tableaux. Let \( T \) be a ribbon tableau. The reading word \( \tau(T) \) is given by reading the diagonals of \( \text{quot}_n(T) \) in descending order, where in each diagonal the larger numbers are read first. We will regularly abuse notation by allowing ourselves to identify ribbons in \( T \) with squares of the \( n \)-quotient \( \text{quot}_n(T) \). We will also identify a skew shape \( \lambda/\mu \) which is a horizontal ribbon strip with the corresponding horizontal ribbon strip tableau \( T \) satisfying \( \text{sh}(T) = \lambda/\mu \).

### 3. Symmetry of ribbon functions

We fix the length \( n \geq 1 \) of our ribbons throughout.

Recall that the standard way to prove that a Schur function is symmetric is to give an involution \( \alpha_i \) on semistandard tableaux of shape \( \lambda \) which swaps the number of \( i \)'s and \( (i+1) \)'s, for each \( i \). This is known as the Bender-Knuth involution. Our first aim is to study the symmetry of the ribbon functions \( G_{\lambda/\mu}^{(n)}(X; q) \) from the perspective of the \( n \)-quotient.  

This symmetry is equivalent to the existence of a ribbon Bender-Knuth involution \( \sigma_i \) on ribbon tableaux \( T \) which changes the number of \( i \)'s and \( i+1 \)'s while preserving spin.

We call a skew shape \( \lambda/\mu \) a double horizontal ribbon strip if it can be tiled by two horizontal ribbon strips. Let \( R_{\lambda/\mu}^{a,b} \) be the set of ribbon tableaux of shape \( \lambda/\mu \) filled with \( a \) 1's and \( b \) 2's. To obtain a ribbon Bender-Knuth involution, it suffices to find a spin preserving bijection between \( R_{\lambda/\mu}^{a,b} \) and \( R_{\lambda/\mu}^{b,a} \) for every \( a \) and \( b \) and every double horizontal strip \( \lambda/\mu \). Let \( T \in R_{\lambda/\mu}^{a,b} \). Suppose some tableau \( T^{(i)} \) of the \( n \)-quotient contains a column with two squares, then those two squares must be 1 on top of a 2.

We first show that we may reduce to the case that the \( n \)-quotient contains no such columns. If \((x, y)\) is an inversion of \( T \) we say that the inversion involves \( x \) and \( y \). Let \( \text{inv}_x(T) \) denote the number of inversions of \( T \) which involve \( x \).

**Lemma 2.** Let \( T \) be a ribbon tableau and \( \text{quot}_n(T) \) contain two squares \( x \) and \( y \) in the same column such that \( T(x) = i \) and \( T(y) = i + 1 \). Let \( T' \) be a semistandard ribbon tableau obtained from \( T \) by changing a \( i \) to a \( i + 1 \). Then \( \text{inv}_x(T) + \text{inv}_y(T) = \text{inv}_x(T') + \text{inv}_y(T') \).

**Proof.** We first note that \( \text{diag}(x) = \text{diag}(y) + n \). Thus the only relevant inversions come from squares \( z \) satisfying \( \text{diag}(z) > \text{diag}(x) > \text{diag}(y) \) and \( T(z) \in \{i, i+1\} \). We check directly that regardless of the value of \( T(z) \), the cell \( z \) contributes exactly one inversion to \( \text{inv}_x(T) + \text{inv}_y(T) \) and thus to \( \text{inv}_x(T') + \text{inv}_y(T') \) as well. \( \square \)

Lemma 2 combined with Lemma 1 shows that to prove that all ribbon functions are symmetric functions we only need to check it for horizontal ribbon strips \( \lambda/\mu \). For a
horizontal ribbon strip \(\lambda/\mu\), let \(I_{\lambda/\mu} \subseteq \mathbb{Z}\) be the set of diagonals such that \(\text{quot}_n(\lambda/\mu)\) contains a cell. It follows from Lemma 1 that the symmetry of \(G^{(n)}_{\lambda/\mu}(X;q)\) implies the symmetry for all horizontal strips \(\nu/\rho\) with the same set of diagonals \(I_{\nu/\rho} = I_{\lambda/\mu}\) — only the constant \(e(\nu/\rho)\) has changed. It is easy to see that given a set of diagonals \(I \subseteq \mathbb{Z}\), we can find a horizontal ribbon strip \(\lambda/\mu\) such that \(I_{\lambda/\mu} = I\) and so that \(\lambda/\mu\) is tileable using vertical ribbons only. Thus the symmetry of all ribbon functions reduces to the symmetry of ribbon functions \(G^{(n)}_{\lambda/\mu}(X;q)\) corresponding to a horizontal ribbon strip \(\lambda/\mu\) tileable only using vertical ribbons. In fact it is clear that we need only check this symmetry for such shapes which are connected.

4. Connection with a result of van Leeuwen

Curiously, the symmetry of these special ribbon functions follows from a result of van Leeuwen concerning adding ribbons above and below a fixed lattice path. We identify the steps of an infinite lattice path \(P\) going up and right with a doubly infinite sequence \(p = \{p_i\}_{i=\infty}^{-\infty}\) of 0's and 1's, where a 0 corresponds to a step to the right and a 1 corresponds to a step up. We may think of such lattice paths as the boundary of a shape (or partition) in which case the bit string is known as the edge sequence [7]. For our purposes, the indexing of \(\{p_i\}\) is unimportant.

Van Leeuwen’s result is the following [6, Claim 1.1.1].

**Proposition 3.** Let \(p = \{p_i\}_{i=\infty}^{-\infty}\) be a lattice path which is vertical at both ends. Let \(R_p\) denote the generating function

\[
R_p(X,q) = \sum_S q^{\text{spin}(S)} X^{|S|}
\]

where the sum is over all horizontal ribbon strips \(S\) that can be attached below \(p\). Let \(\tilde{p}\) denote \(p\) reversed. Then

\[
R_p(X,q) = R_{\tilde{p}}(X,q).
\]

Note that the generating functions \(R_p(X,q)\) are finite, since only finitely many horizontal ribbon strips can be placed under a lattice path which is vertical at both ends. The lattice path \(\tilde{p}\) should be thought of as rotating \(p\) upside-down, so that \(R_{\tilde{p}}(X,q)\) enumerates the ways of adding a horizontal ribbon strip above \(p\) (see [6]).

We will also need the following technical lemma [6, Lemma 5.2.2] to make a calculation with spin. For a set \(I \subseteq \mathbb{Z}\) of diagonals, we denote \(\text{spin}_I(T)\) to be the sum of the spins of the ribbons of \(T\) whose heads lie on the diagonals of \(I\).

**Lemma 4** (6). Let \(\lambda, \mu, \nu\) be partitions so that \(\lambda/\mu, \lambda/\nu, \mu/\nu\) are all horizontal ribbon strips. Let \(I, J \subseteq \mathbb{Z}\) be the set of diagonals occurring in \(\lambda/\mu\) and \(\mu/\nu\) respectively. Then

\[
\text{spin}_I(\lambda/\nu) - \text{spin}(\lambda/\mu) = \text{spin}_J(\lambda/\nu) - \text{spin}(\mu/\nu).
\]

**Proposition 5.** Let \(\lambda/\nu\) be a connected skew shape which is tileable by vertical ribbons only. Then \(G^{(n)}_{\lambda/\nu}(x_1,x_2;q)\) is a symmetric function.

**Proof.** In the notation of Proposition 3 we pick \(p\) so that \(\lambda/\nu\) is the shape obtained by adding as many vertical ribbons as possible below \(p\) to give a horizontal lattice strip. Alternatively, we can think of \(\lambda/\nu\) as the bounded region obtained by shifting the lattice path upwards \(n\) steps. Let \(m = |\lambda/\nu|/n\). Let \(S_1\) be a horizontal ribbon strip with \(a \leq m\)
ribs added below $p$ which we assume has shape $\mu/\nu$. Filling $S_1$ with 1’s there is a unique way to add another horizontal ribbon strip $S_2$ filled with 2’s to give a tableau $T \in \mathcal{R}_{\lambda/\nu}^{a,b}$.

Since $\text{spin}_{1}(\lambda/\nu) = (n - 1)|I|$ for any valid set of diagonals $I \subseteq I_{\lambda/\nu}$, we have $\text{spin}(S_2) = (n - 1)(2a - m) + \text{spin}(S_1)$ by Lemma 4. Summing over all $S_1$, we get

$$G_{\lambda/\nu}(x_1, x_2; q) = x_2^m q^{-(n - 1)m} R_p \left( \frac{x_1}{x_2}; q^{2(n - 1)}, q^2 \right).$$

However, we can also obtain the tableau $T$ by counting the horizontal ribbon strip $S_2$ containing 2 first, so a similar argument gives $G_{\lambda/\nu}(x_1, x_2; q) = x_1^m q^{-(n - 1)m} R_p \left( \frac{x_2}{x_1}; q^{2(n - 1)}, q^2 \right)$. Since $R_p = R_p$ by Proposition 3 we obtain $G_{\lambda/\nu}(x_1, x_2; q) = G_{\lambda/\nu}(x_2, x_1; q)$. □

The following theorem follows immediately from Proposition 5 and earlier discussion.

**Theorem 6.** Let $\lambda/\mu$ be any skew shape tileable by $n$-ribons. Then $G_{\lambda/\mu}(X; q)$ is a symmetric function.

Theorem 6 was first shown by Lascoux, Leclerc and Thibon [3] using an action of the Heisenberg algebra on the Fock space of $U_q(\mathfrak{sl}_n)$.

5. **Generalised Yamanouchi domino tableaux**

In this section we imitate a proof of the Littlewood Richardson rule due to Stembridge [3], which we apply to domino tableaux. We fix $n = 2$ throughout this section. Define the generalised (domino) $q$-Littlewood Richardson coefficients $c_{\mu/\rho,\nu}^{\lambda}(q)$ by

$$s_{\nu}(X) G_{\mu/\rho}(X; q) = \sum_{\lambda} c_{\mu/\rho,\nu}^{\lambda}(q) s_{\lambda}(X).$$

Let $\{\sigma_r\}$ denote a set of fixed domino Bender-Knuth involutions which exist by Theorem 4. Let $w = w_1 w_2 \cdots w_k$ be a sequence of integers. Then the weight $\text{wt}(w) = (\text{wt}_1(w), \text{wt}_2(w), \ldots)$ is the composition of $k$ such that $\text{wt}_i(w) = |\{j \mid w_j = i\}|$. If $T$ is a ribbon tableau, let $T_{\geq j}$ and $T_{>j}$ denote the set of ribbons lying in diagonals which are $\geq j$ and $> j$ respectively (and similarly for $T_{<j}$ and $T_{\leq j}$). These are not tableaux, but the compositions $\text{wt}(T_{\geq j})$ and $\text{wt}(T_{>j})$ are well defined, in the usual manner.

**Definition 7.** Let $\lambda$ be a partition. A word $w = w_1 w_2 \cdots w_k$ is $\lambda$-Yamanouchi if for any initial string $w_1 w_2 \cdots w_l$, and any integer $l$, we have $\text{wt}_l(w_1 w_2 \cdots w_l) + \lambda_l \geq \text{wt}_{l+1}(w_1 \cdots w_l) + \lambda_{l+1}$. A domino tableau $D$ is $\lambda$-Yamanouchi if its reading word $r(D)$ is $\lambda$-Yamanouchi.

One can check that (0)-Yamanouchi is essentially the notion of Yamanouchi introduced by Carré and Leclerc [1].

**Theorem 8.** The generalised $q$-Littlewood Richardson coefficients are given by

$$c_{\mu/\rho,\nu}^{\lambda}(q) = \sum_Y q^{\text{spin}(Y)}$$

where the sum is over all $\nu$-Yamanouchi domino tableaux $Y$ of shape $\mu/\rho$ and weight $\lambda$.

**Proof.** Our proof will follow Stembridge’s proof [3] nearly step by step. We will prove the Theorem in the variables $x_1, \ldots, x_m$ and will always think of a tableau $D$ in terms of its 2-quotient. By definition,

$$G_{\mu/\rho}(X; q) = \sum_D q^{\text{spin}(D)} x^D$$
where the sum is over all semistandard domino tableaux of shape \( \mu/\rho \) filled with numbers in \([1, m]\). Let \( a_{\lambda+\delta} \) denote the alternating sum \( \sum_w (-1)^w x^{w(\lambda+\delta)} \) where the sum is over all permutations \( w \in S_m \). Then

\[
(1) \quad a_{\lambda+\delta}q^{\mu/\rho}(X; q) = \sum_w \sum_D q^{\text{spin}(D)}(-1)^w x^{D+w(\lambda+\delta)}
\]

\[
(2) \quad = \sum_D q^{\text{spin}(D)} \sum_w (-1)^w x^{w(D+\lambda+\delta)}
\]

(3) \quad = \sum_D q^{\text{spin}(D)} a_{D+\lambda+\delta}.

To obtain (2) we have used Theorem 5 to see that the weight generating function for domino tableaux with fixed spin is \( w \) invariant. We call \( D \) a Bad Guy if

\[
\lambda_k + \text{wt}_k(D_{>j}) < \lambda_{k+1} + \text{wt}_{k+1}(D_{\geq j})
\]

for some \( j \) and \( k \). Of all such pairs \( (j, k) \), we pick one that maximises \( j \) and amongst those we pick the smallest \( k \). Thus the reading word of \( r(D_{>j}) \) is \( \lambda \)-Yamanouchi and the \( j \)-th diagonal of \( D \) contains a \( k+1 \) (and possibly a \( k \)) while the \( (j+1) \)-th diagonal contains no \( k \).

Now let \( S \) be the set of dominoes obtained from \( D_{<j} \) by including the \( k \) on the \( j \)-th diagonal if any. Set \( S^* = \sigma_k(S) \). This makes sense since the squares of \( S \) containing a \( k \) or \( k+1 \) forms a double horizontal strip which is actually of skew shape, so we can apply the Bender-Knuth involution. Now since \( \text{sh}(S) = \text{sh}(S^*) \) we can attach \( S^* \) back onto \( D_{\geq j} \) to obtain a tableau \( D^* \). We check that \( D^* \) is a semistandard domino tableau. This is the case as only \( k \)'s and \( k+1 \)'s are changed into each other, and the boundary diagonals \( j \) and \( j+1 \) only contain \( k+1 \)'s (there are two conditions to check, one for each tableau of the 2-quotient). Also note that if there is a \( k \) in diagonal \( j \) of \( S \) then there must be a \( k+1 \) immediately below it, so it will always remain a \( k \) in \( S^* \).

It follows immediately from the construction that \( D \mapsto D^* \) is an involution on the set of Bad Guys. We check that it is spin-preserving by counting the number of inversions. Since we have assumed that \( \sigma_k \) preserves spin, the only inversions that we have to be concerned about are those where \( D(x) = k+1 \) and \( D(y) = k \) and \( \text{diag}(x) = j-1 \) and \( \text{diag}(y) = j \). But if the \( j \)-th diagonal contains a \( k \), then there is a \( k+1 \) immediately below it, so by Lemma 2 it can be ignored for calculations of spin in \( D, D^* \) and also \( S \) and \( S^* \). So \( \text{spin}(D) = \text{spin}(D^*) \).

Now,

\[
a_{D+\lambda+\delta} = -a_{D^*+\lambda+\delta},
\]

since \( s_k(D + \lambda + \delta) = D^* + \lambda + \delta \), so the contributions of the Bad Guys to the sum cancel out. The tableaux which are not Bad Guys are exactly the \( \lambda \)-Yamanouchi tableaux. Dividing both sides of (3) by \( a_\delta \) and using the bialternant formula \( s_\lambda(X) = a_{\lambda+\delta}/a_\delta \) now gives the Theorem.

Unfortunately, this proof seems to fail for ribbon tableaux with \( n > 2 \). The similarly defined involution \( T \mapsto T^* \) no longer preserves either semistandard-ness or spin.

We should remark also that Carré and Leclerc’s algorithm mapping a domino tableau \( D \) to a pair \((Y, T)\) of a Yamanouchi domino tableau and a usual Young tableau can also be interpreted in terms of the 2-quotient.
6. Word sequence formulation of ribbon function symmetry

We end the paper by describing explicitly the bijection needed to prove symmetry of ribbon functions in terms of certain sequences. Let \( n \geq 1 \) be an integer.

**Definition 9.** A \((1, 2, \emptyset)\)-word is a sequence \((a_1, a_2, \ldots, a_m)\) where each \( a_i \in \{1, 2, \emptyset\} \), such that whenever \( a_i = 2 \), then \( a_{i+n} \neq 1 \). The form \( F_a \) of a sequence \((a_1, a_2, \ldots, a_m)\) is the finite set \( F_a = \{ i \in [1, m] \mid a_i = \emptyset \} \). The weight \( \text{wt}(a) \) of such a word \( a = (a_1, \ldots, a_m) \) is \((\mu_1, \mu_2)\) where \( \mu_i = \# \{ j : a_j = i \} \).

**Definition 10.** A \( n \)-local inversion of a \((1, 2, \emptyset)\)-word \((a_1, a_2, \ldots, a_m)\) is a pair \((i, j)\) satisfying \( 1 \leq i < j \leq m \) and \( j - i < n \) such that \( a_i = 2 \) and \( a_j = 1 \). We let \( \text{linv}_n(w) \) denote the number of \( n \)-local inversions of \( w \).

The following proposition makes the connection between \((1, 2, \emptyset)\)-words and a ribbon Bender Knuth involution.

**Proposition 11.** The symmetry of ribbon functions is equivalent to the following identity on \((1, 2, \emptyset)\)-words for each positive integer \( m \), form \( F \subset [1, m] \) and weight \((\mu_1, \mu_2)\):

\[
\sum_{a : \text{wt}(a) = (\mu_1, \mu_2)} q^{\text{linv}_n(a)} = \sum_{a : \text{wt}(a) = (\mu_2, \mu_1)} q^{\text{linv}_n(a)}
\]

where the sum is over all \((1, 2, \emptyset)\)-words with length \( m \), form \( F \) and specified weight.

**Proof.** We have already established that we need only be concerned with tableaux which are horizontal ribbon strips filled with ribbons labelled 1 and 2. Our \((1, 2, \emptyset)\)-words are simply the (reversed) reading words of these ribbon tableaux where the form \( F \) keeps track of the empty diagonals. The Proposition follows immediately from Lemma 1. \( \square \)

We remark that when the form \( F \) is the emptyset, a bijection giving (11) is obtained by reversing the sequence and changing 2’s to 1’s and vice versa.

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