AFFINE REFLECTION SUBGROUPS OF COXETER GROUPS

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Abstract. In this paper we study affine reflection subgroups in arbitrary Coxeter groups of finite rank. In particular, we study the distribution of roots in the root subsystems associated with affine reflection subgroups. We give a characterization of limit roots arising from affine reflection subgroups. We also give a characterization of when a Coxeter group may possess affine reflection subgroups.

1. Introduction

A Coxeter system \((W, R)\) consists of an abstract group \(W\) and a generating set \(R\) consisting of involutions. The group \(W\), called a Coxeter group, is generated by elements of \(R\) subject only to braid-relations on pairs of generators. Such a group can be realized, via the classical Tits representation, as a reflection group acting on a real vector space \(V\) with the term reflection taken only to be an involutory orthogonal transformation with respect to a certain bilinear form on \(V\) such that the \(-1\)-eigenspace of this orthogonal transformation is 1-dimensional and not wholly contained in the radical of that bilinear form. We represent such a reflection by an element in the set \(T := \bigcup_{w \in W} wRw^{-1}\), and call it the set of reflections in \(W\). A reflection subgroup of \(W\) is a subgroup of \(W\) generated by a subset of \(T\). It is known that a reflection subgroup of a Coxeter group is itself a Coxeter group. Among all reflection subgroups, those generated by affine reflections (an affine
reflection is a composite of a euclidean reflection with a translation in the direction of a vector in the radical of the bilinear form) are known as affine reflection subgroups. In this paper we study affine reflection subgroups in Coxeter groups with finitely many generators.

One of the most important tools devised in the study of Coxeter groups is the so called root system. A root system of a Coxeter group $W$ is the collection of all roots which are representative non-zero vectors in the $-1$-eigenspaces of the reflections in $W$. It is known that the group $W$ is finite if and only its root system is finite; and finite or not, the root system is a discrete set in $V$. However, through an approach pioneered in [19], it has been observed that the projections of roots onto suitably chosen hyperplanes are always contained in certain compact sets. These projections of roots are called normalized roots, and they may exhibit intricate asymptotic behaviours as observed in [19] and [10]. The set of accumulation points of normalized roots is denoted by $E(W)$, and its elements are called the limit roots of the Coxeter group $W$. It was observed in [19] that $E(W)$ is contained in the isotropic cone of $V$ (where the isotropic cone consists of all vectors which are orthogonal to themselves with respect to the given bilinear form), and there is a natural $W$-action on $E(W)$ which was later shown in [10] that this action was minimally faithful. It was also established in [19] that the limit roots arising from all infinite dihedral reflection subgroups of $W$ is dense in $E(W)$. In [10] it has been proven that the convex hull of $E(W)$ is the topological closure of the projection of the so-called imaginary cone onto the chosen hyperplane. The notion of an imaginary cone was first introduced in the context of Kac-Moody algebra as the cone generated by positive imaginary roots, and has been later adapted to the setting of Coxeter groups as a (potentially not strict) subset of the so-called dual of the Tits cone. The seminal results presented in [19] and [10] suggest that normalized roots and limit roots may play a non-trivial role in studying general infinite Coxeter groups and their associated root systems.

In this paper, we study the limit roots arising from affine reflection subgroups of a Coxeter group $W$. We prove that when $W$ is infinite but finitely generated, then a limit root $\eta$ arises from an affine reflection subgroup if and only if $\eta$ bilinear formed with finitely many roots to be strictly positive. We also give a definitive characterization of when a Coxeter group possesses affine reflection subgroups, and we prove that affine reflection subgroups are precisely infinite subgroups of affine parabolic subgroups. We then study the set of roots which when bilinear formed with an affine limit root to be strictly positive, and we show that such sets determine their corresponding limit roots. Furthermore, we prove that the containment of the imaginary cone in the dual of the Tits cone is strict in the case of a non-affine Coxeter group, with
the imaginary cone containing only the limit roots arising from affine reflection subgroups.

We stress that the same abstract Coxeter group \( W \) may be realized in a number of different ways, that is, different Coxeter systems may give rise to the same group \( W \) with distinct generating sets \( R \)'s, and possibly more importantly, different sets of reflections \( T \)'s. An example may be the dihedral group of order 20 and the direct product of \( \mathbb{Z}_2 \) with the dihedral group of order 10, in which the same abstract group can be realized differently as Coxeter group with incompatible sets of reflections. Having this in mind, in this paper, in most cases, we take the preference of referring to a Coxeter system \( (W, R) \) instead of just referring to a Coxeter group \( W \); and when we simply use the expression “let \( W \) be a Coxeter group”, then the reader should interpret it as a statement concerning not only the group \( W \), but more importantly the particular presentation afforded by a Coxeter system \( (W, R) \).

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2. Background Materials

In this section we collect a number of background results concerning Coxeter groups and their associated root systems. We stress that the definition of root system in this section differs with the definition contained in classical literature such as [24] (in particular, we remove the requirement that a root basis is a basis for the space bearing the classical Tits representation). However the results in this section are straightforward adaptation of classical results in the setting with the modified root systems. Before we formally begin, we set the following notations which are used throughout this paper.

For a set \( A \), define \( \text{PLC}(A) \), the positive linear combination of \( A \), to be the set given by
\[
\{ \sum_{a \in A} \lambda_a a \mid \lambda_a \geq 0 \text{ for all } a \in A \text{ and } \lambda_{a'} > 0 \text{ for some } a' \in A \};
\]
and we define \( \text{cone}(A) \), the positive cone spanned by \( A \), to be the set
\[
\text{cone}(A) = \text{PLC}(A) \cup \{0\};
\]
and we use the notation of \( \text{conv}(A) \) to denote the convex hull of \( A \).

**Definition 2.1.** ([26]) Suppose that \( V \) is a vector space over \( \mathbb{R} \) and let \( (\cdot, \cdot) \) be a bilinear form on \( V \) and let \( \Pi \) be a subset of \( V \). Then \( \Pi \) is called a root basis if the following conditions are satisfied:

\((C1)\) \( (a, a) = 1 \) for all \( a \in \Pi \), and if \( a, b \) are distinct elements of \( \Pi \) then either \( (a, b) = -\cos(\pi/m_{ab}) \) for some integer \( m_{ab} = m_{ba} \geq 2 \), or else \( (a, b) \leq -1 \) (in which case we define \( m_{ab} = m_{ba} = \infty \));

\((C2)\) \( 0 \notin \text{PLC}(\Pi) \).
If $\Pi$ is a root basis, then we call the triple $\mathcal{C} = (V, \Pi, (, ))$ a Coxeter datum. Throughout this paper we fix a particular Coxeter datum $\mathcal{C}$. (C1) implies that for each $a \in \Pi$, $a \notin \text{PLC}(\Pi \setminus \{a\})$. Furthermore, (C1) together with (C2) yield that $\{a, b\}$ is linearly independent for all distinct $a, b \in \Pi$. For each non-isotropic $a \in V$ (that is $(a, a) \neq 0$), define $\rho_a \in \text{GL}(V)$ by the rule: $\rho_a x = x - 2\frac{(x, a)}{(a, a)} a$, for all $x \in V$. Observe that $\rho_a$ is an involution, and $\rho_a a = -a$.

Let $G_{\mathcal{C}}$ be the subgroup of $\text{GL}(V)$ generated by $\{\rho_a | a \in \Pi\}$. Suppose that $(W, R)$ is a Coxeter system in the sense of [18] or [24] with $R = \{r_a | a \in \Pi\}$ being a set of involutions generating $W$ subject only to the condition that $(r_a r_b)^{m_{ab}} = 1$ for all $a, b \in \Pi$ with $m_{ab} \neq \infty$. A standard argument yields that there exists a group homomorphism $\phi_{\mathcal{C}} : W \to G_{\mathcal{C}}$ satisfying $\phi_{\mathcal{C}}(r_a) = \rho_a$ for all $a \in \Pi$. This homomorphism together with the $G_{\mathcal{C}}$-action on $V$ give rise to a $W$-action on $V$: for each $w \in W$ and $x \in V$, define $wx \in V$ by $wx = \phi_{\mathcal{C}}(w)x$. It can be easily checked that this $W$-action preserves $(, )$. Denote the length function of $W$ with respect to $R$ by $\ell$, and we have:

**Proposition 2.2.** ([22 Lecture 1]) Let $G_{\mathcal{C}}, W, R$ and $\ell$ be as the above, and let $w \in W$ and $a \in \Pi$. If $\ell(wr_a) \geq \ell(w)$ then $wa \in \text{PLC}(\Pi)$. □

An immediate consequence of the above proposition is the following important fact:

**Corollary 2.3.** ([22 Lecture 1]) Let $G_{\mathcal{C}}, W, R$ and $\phi_{\mathcal{C}}$ be as the above, and let $S := \{\rho_a | a \in \Pi\}$. Then the bijection $R \to S$ given by $r_a \mapsto \rho_a$ for $a \in \Pi$ extends to a group isomorphism $\phi_{\mathcal{C}} : W \to G_{\mathcal{C}}$. □

In particular, the above corollary yields that $(G_{\mathcal{C}}, S)$ is a Coxeter system isomorphic to $(W, R)$. We call $(W, R)$ the abstract Coxeter system associated to the Coxeter datum $\mathcal{C}$, and we call $W$ a Coxeter group of rank $\#R$ (where $\#$ denotes cardinality).

**Definition 2.4.** The root system of $W$ in $V$ is the set

$$\Phi = \{wa | w \in W \text{ and } a \in \Pi\}.$$  

The set $\Phi^+ = \Phi \cap \text{PLC}(\Pi)$ is called the set of positive roots, and the set $\Phi^- = -\Phi^+$ is called the set of negative roots.

From Proposition 2.2 we may readily deduce that:

**Proposition 2.5.** ([22 Lecture 3]) (i) Let $w \in W$ and $a \in \Pi$. Then

$$\ell(wr_a) = \begin{cases} 
\ell(w) - 1, & \text{if } wa \in \Phi^-; \\
\ell(w) + 1, & \text{if } wa \in \Phi^+.
\end{cases}$$

(ii) $\Phi = \Phi^+ \uplus \Phi^-$, where $\uplus$ denotes disjoint union.

(iii) $W$ is finite if and only if $\Phi$ is finite. □
Define $T = \bigcup_{w \in W} wRw^{-1}$. We call $T$ the set of reflections in $W$. If $x \in \Phi$ then $x = wa$ for some $w \in W$ and $a \in \Pi$. Direct calculations yield that $\rho_x = (\phi_r(w))\rho_a(\phi_r(w))^{-1} \in G_r$. Now let $r_x \in W$ be such that $\phi_r(r_x) = \rho_x$. Then $r_x = wr_nw^{-1} \in T$ and we call it the reflection corresponding to $x$. It is readily checked that $r_x = r_{-x}$ for all $x \in \Phi$ and $T = \{ r_x \mid x \in \Phi \}$. For each $t \in T$ we let $\alpha_t$ be the unique positive root with the property that $r_{\alpha_t} = t$. It is also easily checked that there is a bijection $\psi : T \to \Phi^+$ given by $\psi(t) = \alpha_t$.

Define functions $N : W \to P(\Phi^+)$ and $\overline{N} : W \to P(T)$ (where $P$ denotes power set) by setting $N(w) = \{ x \in \Phi^+ \mid wx \in \Phi^- \}$ and $\overline{N}(w) = \{ t \in T \mid \ell(wt) < \ell(w) \}$ for all $w \in W$. We call $\overline{N}$ the reflection cocycle of $W$. Standard arguments as those in [23] yield that for each $w \in W$,

$$\ell(w) = \# N(w)$$

and

$$\overline{N}(w) = \{ r_x \mid x \in N(w) \}.$$  

(2.1)

(2.2)

In particular, $N(r_a) = \{ a \}$ for $a \in \Pi$. Moreover, $\ell(wv^{-1}) + \ell(v) = \ell(w)$, for some $w, v \in W$ if and only if $N(v) \subseteq N(w)$.

A subgroup $W'$ of $W$ is a reflection subgroup of $W$ if $W' = \langle W' \cap T \rangle$ ($W'$ is generated by the reflections contained in it). For any reflection subgroup $W'$ of $W$, let

$$R(W') = \{ t \in T \mid \overline{N}(t) \cap W' = \{ t \} \}$$

and

$$\Pi(W') = \{ x \in \Phi^+ \mid r_x \in R(W') \}.$$  

(2.2)

(2.3)

It was shown by Dyer ([7]) and Deodhar ([5]) that $(W', R(W'))$ forms a Coxeter system:

**Theorem 2.6.**

(i) Suppose that $W'$ is an arbitrary reflection subgroup of $W$. Then $(W', R(W'))$ forms a Coxeter system. Moreover, $W' \cap T = \bigcup_{w \in W'} wR(W')w^{-1}$.

(ii) Suppose that $W'$ is a reflection subgroup of $W$, and suppose that $a, b \in \Pi(W')$ are distinct. Then

$$(a, b) \in \{ -\cos(\pi/n) \mid n \in \mathbb{N} \text{ and } n \geq 2 \} \cup (-\infty, -1].$$

And conversely if $\Delta$ is a subset of $\Phi^+$ satisfying the condition that

$$(a, b) \in \{ -\cos(\pi/n) \mid n \in \mathbb{N} \text{ and } n \geq 2 \} \cup (-\infty, -1]$$

for all $a, b \in \Delta$ with $a \neq b$, then $\Delta = \Pi(W')$ for some reflection subgroup $W'$ of $W$. In fact, $W' = \langle \{ r_a \mid a \in \Delta \} \rangle$.

**Proof.** (i) [7], Theorem 3.3].

(ii) [7], Theorem 4.4].
Let $(,')$ be the restriction of $(,)$ on the subspace $\text{span}(\Pi(W'))$ of $V$. Then $G' = (\text{span}(\Pi(W'))$, $\Pi(W')$, $(,')$) is a Coxeter datum with $(W', R(W'))$ being the associated abstract Coxeter system. Thus the notion of a root system applies to $G'$. We let $\Phi(W')$, $\Phi^+(W')$ and $\Phi^-(W')$ be, respectively, the set of roots, positive roots and negative roots for the datum $G'$. Then $\Phi(W') = W'\Pi(W')$ and Theorem 2.6 (i) yields that $\Phi(W') = \{ x \in \Phi \mid r_x \in W' \}$. Furthermore, we have $\Phi^+(W') = \Phi(W') \cap \text{PLC}(\Pi(W'))$ and $\Phi^-(W') = -\Phi^+(W')$. We call $R(W')$ the set of canonical generators of $W'$, and we call $\Pi(W')$ the set of canonical roots of $\Phi(W')$. In this paper a reflection subgroup $W'$ is called a dihedral reflection subgroup if $\#R(W') = 2$.

A subset $\Phi'$ of $\Phi$ is called a root subsystem if $r_x y \in \Phi'$ whenever $x, y$ are both in $\Phi'$. It is easily seen that there is a bijective correspondence between the set of reflection subgroups $W'$ of $W$ and the set of root subsystems $\Phi'$ of $\Phi$: $W'$ uniquely determines the root subsystem $\Phi(W')$, and $\Phi'$ uniquely determines the reflection subgroup $\{ \{ r_x \mid x \in \Phi' \} \}$.

A standard parabolic subgroup $W_M$ of $W$ is defined as follows, for $M \subseteq \Pi$, let $R_M := \{ r_\alpha \mid \alpha \in M \}$, and $W_M := \langle R_M \rangle$. It can be easily deduced that $(W_M, R_M)$ is a Coxeter system with an associated Coxeter datum $G_M := (\text{span}(M), \Pi, (,)_M)$, where $(,)_M$ is the restriction of $(,)$ to the subspace $\text{span}(M)$. Observed that $\Phi(W_M) = \Phi \cap \text{span}(M)$. We may also use the notation $W_{R_M}$ in place of $W_M$. A parabolic subgroup of $W$ is any conjugate of a standard parabolic subgroup of $W$.

The notion of a length function also applies to the Coxeter system $(W', R(W'))$, and we let $\ell_{(W', R(W'))} : W' \rightarrow \mathbb{N}$ be the length function for $(W', R(W'))$. If $w \in W'$ and $a \in \Pi(W')$ then applying Proposition 2.6 to the Coxeter datum $G' = (\text{span}(\Pi(W'))$, $\Pi(W')$, $(,')$) yields

$$\ell_{(W', R(W'))}(wa) = \begin{cases} \ell_{(W', R(W'))}(w) - 1, & \text{if } wa \in \Phi^-(W') \\ \ell_{(W', R(W'))}(w) + 1, & \text{if } wa \in \Phi^+(W') \end{cases}$$

where $(,')$ is the restriction of $(,)$ to $\text{span}(\Pi(W'))$.

Similarly the notion of a reflection cocycle also applies to the Coxeter system $(W', R(W'))$. Let $N_{(W', R(W'))} : W' \rightarrow \mathcal{P}(W' \cap T)$ denote the reflection cocycle for $(W', R(W'))$. Then for each $w \in W'$,

$$N_{(W', R(W'))}(w) = \{ t \in W' \cap T \mid \ell_{(W', R(W'))}(wt) < \ell_{(W', R(W'))}(w) \}.$$  

And we define $N_{(W', R(W'))}(w) = \{ x \in \Phi^+(W') \mid wx \in \Phi^-(W') \}$, for each $w \in W'$. It is shown in [6] that $N_{(W', R(W'))}(w) = N(w) \cap W'$ for arbitrary reflection subgroup $W'$ of $W$. Furthermore, it is readily seen that the bijection $\psi$ restricts to a bijection $\psi' : T \cap W' \rightarrow \Phi^+(W')$ given by $\psi'(t) = \alpha_t$. For $w \in W'$, applying (2.1) to the Coxeter datum $G' = (\text{span}(\Pi(W'))$, $\Pi(W')$, $(,')$) yields that

$$\ell_{(W', R(W'))}(w) = \#N_{(W', R(W'))}(w).$$  (2.3)
Furthermore, $\ell_{(W', R(W'))}(wv^{-1}) + \ell_{(W', R(W'))}(v) = \ell_{(W', R(W'))}(w)$, for some $w, v \in W'$, precisely when $N_{(W', R(W'))}(v) \subseteq N_{(W', R(W'))}(w)$.

3. Some basic results

**Proposition 3.1.** ([22], Lecture 3) In an arbitrary Coxeter group, each finite subgroup is contained in some finite parabolic subgroup. □

**Lemma 3.2.** Let $(W, R)$ be a Coxeter system in which $W$ is an irreducible affine Coxeter group (in the sense of [24]), with associated Coxeter datum $C = (V, \Pi, (, )$) and corresponding root system $\Phi$, and let $\text{Rad}$ denote the radical of the bilinear form $(, )$. Then each element of $\Phi$ is congruent modulo $\text{Rad}$ to an element of the root subsystem corresponding to some (fixed) finite standard parabolic subgroup of $W$.

**Proof.** First, recall that $\text{Rad} = \{ v \in V \mid (v, u) = 0, \text{ for all } u \in V \}$.

Since $W$ is irreducible affine, [24, Theorem 2.7] yields that the bilinear form $(, )$ is positive semidefinite on $V$, and [24, Proposition 2.6] yields that $\text{Rad} = Q$, where $Q := \{ v \in V \mid (v, v) = 0 \}$ is the isotropic cone of $(, )$ in $V$, and moreover $\dim \text{Rad} = 1$. It then follows readily that there exists a well-defined bilinear form $(, )_{V/\text{Rad}}$ on $V/\text{Rad}$ such that $(x + \text{Rad}, y + \text{Rad})_{V/\text{Rad}} = (x, y)$ for all $x, y \in V$. The fact that $\text{Rad} = Q$ implies that $V/\text{Rad}$ is positive definite with respect to $(, )_{V/\text{Rad}}$.

If $x \in V \setminus \text{Rad}$ then $r_x$ is a reflection on $V$ preserving $\text{Rad}$, thus $r_x$ induces a reflection $r'_x \in \text{GL}(V/\text{Rad})$ defined by

$$r'_x(y + \text{Rad}) = (y + \text{Rad}) - 2 \frac{(y + \text{Rad}, x + \text{Rad})_{V/\text{Rad}}}{(x + \text{Rad}, x + \text{Rad})_{V/\text{Rad}}} (x + \text{Rad}),$$

for all $y + \text{Rad} \in V/\text{Rad}$.

Set $W' = \{ r'_x \mid x \in \Phi \}$. Then $W'$ is a reflection group on $V/\text{Rad}$.

Since $(, )_{V/\text{Rad}}$ is positive definite, [24, Corollary 6.2] and [24, Theorem 6.4] yield that $W'$ is a finite Coxeter group. Let $\Phi'$ be the root system of $W'$ in $V/\text{Rad}$. It is readily checked that $\Phi' = \{ x + \text{Rad} \mid x \in \Phi \}$.

Let $\pi: \Phi \to \Phi'$ be the natural map given by $\pi(x) = x + \text{Rad}$. Choose a root basis $\Pi'$ for $\Phi'$, and for each $a' \in \Pi'$, choose a representative $x_{a'} \in \Phi$ such that $\pi(x_{a'}) = a'$.

Set $A = \{ x_{a'} \mid a' \in \Pi' \}$. Then $\text{span}(A)$ is a positive definite subspace of $V$ (of codimension 1). Let $W'' \subset W$ be the reflection subgroup generated by reflections from the set $\{ r_x \mid x \in A \}$. In particular, $W''$ is a reflection group acting on the positive definite space $\text{span}(A)$. Thus [24, Corollary 6.2] and [24, Theorem 6.4] yield that $W''$ is finite. Let $\Phi''$ be the root subsystem of $\Phi$ corresponding to $W''$. It is readily checked that each element of $\Phi$ is congruent modulo $\text{Rad}$ to an element of $\Phi''$.

Furthermore, for any fixed $w \in W$, since $w\Phi = \Phi$ and $w$ preserves $\text{Rad}$, it follows that each element of $\Phi$ is congruent modulo $\text{Rad}$ to an element of $w\Phi''$. Now since $W''$ is finite, Proposition 3.1 implies that $W''$ is
contained in a conjugate of a finite standard parabolic subgroup. Thus there exist some \( w \in W \) and \( M \subseteq \Pi \) such that \( wW''w^{-1} \subseteq W_M \) with \( W_M \) being finite. Consequently \( w\Phi'' \subseteq \Phi(W_M) \). Thus each element of \( \Phi \) is congruent modulo \( \text{Rad} \) to some element in \( w\Phi'' \), and, in particular, each element of \( \Phi \) is congruent modulo \( \text{Rad} \) to some element in \( \Phi(W_M) \), with \( W_M \) being finite.

The above lemma immediately enables us to give the following characterization of reflections in an affine Coxeter group.

**Corollary 3.3.** Let \((W,R)\) be a Coxeter system in which \( W \) is an affine Coxeter group, and let \( \mathcal{C} = (V,\Pi,\langle , \rangle) \) be the associated Coxeter datum. Then a reflection in \( W \) is either a Euclidean reflection or the composite of a Euclidean reflection with a translation in the direction of a non-zero vector in the radical of \( \langle , \rangle \).

From this point on, we let \( W \) be the abstract Coxeter group associated to the Coxeter datum \( \mathcal{C} = (V,\Pi,\langle , \rangle) \), and let \( \Phi \) and \( T \) be the corresponding root system and the set of reflections respectively.

**Lemma 3.4.** ([26, Lemma 6.1.1]) Let \( W \) be an irreducible Coxeter group of finite rank, and let \( \text{Rad} \) denote the radical associated with the bilinear form \( \langle , \rangle \). Then \( W \) is non-affine implies that \( \text{Rad} \cap \text{cone}(\Pi) = \{0\} \).

The following is a well-known result:

**Proposition 3.5.** ([19, Lemma 4.9]) Suppose that \( W \) is a Coxeter group of finite rank. Then \( W \) is infinite if and only if \( W \) contains an infinite dihedral reflection subgroup.

**Definition 3.6.** (i) Let \( W' \) be a reflection subgroup of \( W \), and let \( x,y \in \Phi(W') \). Then we say that \( x \) dominates \( y \) with respect to \( W' \) if

\[
\{ w \in W' \mid wx \in \Phi^-(W') \} \subseteq \{ w \in W' \mid wy \in \Phi^-(W') \}.
\]

If \( x \) dominates \( y \) with respect to \( W' \) then we write \( x \) dom\( y \) in case of \( x \) dom\( y \).

(ii) Let \( W' \) be a reflection subgroup of \( W \) and let \( x \in \Phi^+(W') \). Define \( D_{W'}(x) = \{ y \in \Phi^+(W') \mid y \neq x \text{ and } x \text{ dom } y \} \). If \( D_{W'}(x) = \emptyset \) then we call \( x \) elementary with respect to \( W' \). For each non-negative integer \( n \), define \( D_{W',n} = \{ x \in \Phi^+(W') \mid \#D_{W'}(x) = n \} \). In the case that \( W' = W \), we write \( D(x) \) and \( D_n \) in place of \( D_{W'}(x) \) and \( D_{W',n} \) respectively. If \( D(x) = \emptyset \) then we call \( x \) elementary.

It is clear from the above definition that

\[
\Phi^+ = \bigcup_{n \in \mathbb{N}} D_n.
\]
It was shown in [3] by Brink and Howlett that when a Coxeter group \( W \) is finitely generated then the set of elementary roots is finite. This finiteness property then enabled Brink and Howlett to establish that all finitely generated Coxeter groups are automatic. Brink later gave a complete construction of \( D_1 \) for all such Coxeter groups in [4]. Subsequently in [13], it was shown that in a finitely generated Coxeter group \( W \) the sets \( D_n \) are finite for all \( n \in \mathbb{N} \), and furthermore, each \( D_n \) is non-empty for all infinite Coxeter groups.

**Remark 3.7.** In a finitely generated Coxeter group \( W \), the decomposition of \( \Phi^+ \) in [3] implies that if \( x \in \Phi^+ \), then \( x \) can only dominate finitely many positive roots.

It is readily checked that dominance with respect to any reflection subgroup \( W' \) of a Coxeter group \( W \) is a partial ordering on \( \Phi(W') \). The following lemma summarizes some basic properties of dominance:

**Lemma 3.8.** ([13, Lemma 2.2]) (i) Let \( x, y \in \Phi \) be arbitrary. Then there is dominance between \( x \) and \( y \) if and only if \( (x, y) \geq 1 \).

(ii) Dominance is \( W \)-invariant, that is, if \( x \ dom \ y \) then \( wx \ dom \ wy \) for all \( w \in W \).

(iii) Let \( x, y \in \Phi \) be such that \( x \ dom \ y \). Then \( -y \ dom \ -x \).

**Proposition 3.9.** Suppose that \( x, y \in \Phi \) are distinct with \( x \ dom_W y \). Then there exists some \( w \in W \) such that \( wx \in \Phi^+ \), \( wy \in \Phi^- \) and \( (w(x - y), z) \leq 0 \) for all \( z \in \Phi^+ \).

From the above we may deduce a number of inequalities.

**Lemma 3.10.** Suppose that \( a, b, c \in \Phi \) such that \( a \ dom b \ dom c \). Then \( (a, c) \geq (b, c) \).

**Proof.** There is nothing to prove if either \( a = b \) or \( b = c \). Thus we may assume that the dominance relations are strict. Since \( a \) strictly dominates \( b \), Proposition 3.9 then yields that there exists some \( w \in W \) with \( wa \in \Phi^+ \), \( wb \in \Phi^- \), and \( (w(a - b), z) \leq 0 \) for all \( z \in \Phi^+ \). Since \( b \ dom c \) and \( wb \in \Phi^- \), it follows that \( wc \in \Phi^- \), and consequently \( (w(a - b), wc) \geq 0 \); that is \( (a - b, c) \geq 0 \).

**Lemma 3.11.** Suppose that \( a, b, c \in \Phi \) such that \( a \ dom b \ dom c \). Then \( (a, b) \leq (a, c) \).

**Proof.** Note that Lemma 3.8 yields that \( -c \ dom -b \ dom -a \), and then Lemma 3.10 yields that \( (-c, -a) \geq (-b, -a) \), which is equivalent to \( (a, b) \leq (a, c) \).

We close this section with the following easy but useful result.
Lemma 3.12. Suppose that $a, b \in \Phi^+$ with $(a, b) \leq -1$, and suppose that $x \in \Phi$ with $x \dom a$. Then $(x, b) \leq -1$.

Proof. Since $(a, b) \leq -1$, it follows that $a \dom -b$. Thus we have $x \dom a \dom -b$, and hence $(x, b) \leq -1$. □

4. LIMIT ROOTS

Throughout this section we fix a Coxeter datum $\mathcal{G} = (V, \Pi, (, ))$ and its associated Coxeter system $(W, R)$, and furthermore, we assume that $\# \Pi = \# R < \infty$. Let $\Phi$ be the corresponding root system. It is well known that $\Phi$ is an infinite set if and only if $W$ is an infinite group. It is also well known that $\Phi$ is a discrete set (with respect to the standard topology on $\mathbb{R}^n$), since $W$ acts discretely on $V$. Nevertheless, following the approach of studying the so-called normalized roots introduced in [19], we may study the distribution of elements in $\Phi$ if we consider such elements as representatives of directions whose corresponding reflections generate $W$. If we choose a suitable hyperplane $V' \subseteq V$ not passing through the origin and not parallel to any element in $\Phi$, and let $\hat{\Pi}$ be the set of intersection points of the lines $Rx$, $x \in \Pi$ and this hyperplane. Then the property $\Phi = \Phi^+ \uplus \Phi^-$ ensures that the set $\hat{\Phi}$ (normalized roots) consisting of the intersection points of the lines $Rx$, $(x \in \Phi)$ with this hyperplane, are contained in the simplex spanned by the elements of $\hat{\Pi}$. With $\Pi$ being a finite set, this simplex is compact, and we may study the asymptotic behaviours exhibited by sequences of elements of $\hat{\Phi}$.

We collect a few basic definitions and results from [19]:

Definition 4.1. ([19], [10]) An affine hyperplane $V_1$ of codimension 1 in $V$ is called transverse to $\Phi^+$ if for each $a \in \Pi$ the ray $\mathbb{R}_{>0}a$ intersects $V_1$ in exactly one point, and this unique intersection is denoted by $a_{V_1}$. Given a hyperplane $V_1$ transverse to $\Phi^+$, let $V_0$ be the hyperplane that is parallel to $V_1$ and contains the origin.

Remark 4.2. It follows from Proposition 2.5 (ii) and the requirement that $0 \notin \text{PLC}(\Pi)$ that it is always possible to find a hyperplane containing the origin that separates $\Phi^+$ and $\Phi^-$ (see [19] 5.2) for more details). By suitably translating this hyperplane it is always possible to find a hyperplane transverse to $\Phi^+$.

Let $V_1$ be a transverse hyperplane and let $V_0$ be as in the preceding definition. Let $V_0^+$ be the open half space induced by $V_0$ that contains $V_1$. Observe that $V_0^+ \text{ contains } \text{PLC}(\Pi)$. Since $\Phi^+ \subset \text{PLC}(\Pi) \subset V_0^+$, and $V_1$ is parallel to the boundary of $V_0^+$, it follows that $\#(V_1 \cap \mathbb{R}_{>0}\beta) = 1$ for each $\beta \in \Phi^+$. This leads to an alternative definition of transverse hyperplanes:

Lemma 4.3. ([19]) An affine hyperplane $V_1$ is transverse if and only if $\#(V_1 \cap \mathbb{R}_{>0}\beta) = 1$ for each $\beta \in \Phi^+$. □
Definition 4.4. Let $V_1$ be a transverse hyperplane in $V$, and let $V_0$ be obtained from $V_1$ as in Definition 4.1.

1. For each $v \in V \setminus V_0$, the unique intersection point of $R_v$ and the transverse hyperplane $V_1$ is denoted $\hat{v}$. The normalization map is $\pi_{V_1} : V \setminus V_0 \to V_1$, $\pi_{V_1}(v) = \hat{v}$. Set $\hat{\Phi} = \pi_{V_1}(\Phi)$, and the elements $\hat{x} \in \hat{\Phi}$ are called normalized roots.

2. Let $|\cdot|_1 : V \to \mathbb{R}$ be the unique linear map satisfying the requirement that $|v|_1 = 0$ for all $v \in V_0$, and $|v|_1 = 1$ for all $v \in V_1$.

Observe that $\pi_{V_1}(-x) = \pi_{V_1}(x)$ for all $x \in V$, and $\hat{y} = \frac{y}{|y|_1}$ for all $y \in V \setminus V_0$.

Also observe that $\hat{\Phi} \subseteq \text{conv}(\Pi)$, (recall that conv($X$) denotes the convex hull of a set $X$), and $\Pi = \{ \hat{x} \mid x \in \Phi \}$. Since $\Pi$ is a finite set (in which case the associated Coxeter group $W$ is finitely generated), then we see that $\hat{\Phi}$ is contained in the compact set $\text{conv}(\hat{\Pi})$. Consequently, if $\Pi$ is finite, then the accumulation points of $\hat{\Phi}$ are contained in $\text{conv}(\hat{\Pi})$.

Definition 4.5. Keep all the notations of the previous definition.

1. The set of limit roots $E(W)$ (with respect to $V_1$) is the set of accumulation points of $\hat{\Phi}$.

2. The isotropic cone $Q$ (associated to the bilinear form $(\cdot, \cdot)$) is the set

$$Q = \{ v \in V \mid (v, v) = 0 \}.$$

Furthermore, define the normalized isotropic cone $\hat{Q}$ (with respect to $V_1$) by $\hat{Q} = Q \cap V_1$.

Example 4.6. Suppose that $\Pi$ forms a basis for the space $V$. For each $v \in V$, there is a unique expression of the form $v = \sum_{a \in \Pi} v_a a$, $v_a \in \mathbb{R}$.

Then the hyperplane $V_1 := \{ v \in V \mid \sum_{a \in \Pi} v_a = 1 \}$ is a transverse hyperplane, and $V_0 := \{ v \in V \mid \sum_{a \in \Pi} v_a = 0 \}$ is the corresponding hyperplane obtained from translating $V_1$ to contain the origin. Observe that under these conditions the corresponding $|\cdot|_1$ has the property that $|v|_1 = \sum_{a \in \Pi} v_a$ for all $v \in V$.

Remark 4.7. For the rest of this section and the whole of the next section, we shall adopt the set up as in the preceding example, that is, we shall assume that $\Pi$ forms a basis for $V$, and we shall take

$$V_1 = \{ v \in V \mid \sum_{a \in \Pi} v_a = 1 \},$$

where for each $v \in V$, there is a unique expression of the form

$$v = \sum_{a \in \Pi} v_a a, \text{ where } v_a \in \mathbb{R}.$$
Theorem 4.8. ([19 Theorem 2.7]) Let $V_1$ be a transverse hyperplane, and let $\hat{Q}$ be the corresponding normalized isotropic cone. Then

1. $E(W) \neq \emptyset$ if and only if the Coxeter group $W$ is infinite.
2. $E(W) \subseteq \hat{Q}$.

□

Proposition 4.9. Let $\mathcal{C} = (V, \Pi, B)$ be a Coxeter datum in which $\# \Pi$ is finite, and suppose that the corresponding Coxeter group $W$ is infinite. Then the set of limit roots $E(W)$ is compact.

Proof. It follows from Theorem 4.8 that $E(W) \subseteq \hat{Q} \cap \text{conv}(\hat{\Pi})$.

Since $E(W)$, by definition, is topologically closed, and since $\text{conv}(\hat{\Pi})$ is bounded whenever $\# \Pi$ is finite, it follows that $E(W)$ is compact. □

Following the convention set in [19], we define

$$D = \bigcap_{w \in W} w(V \setminus V_0) \cap V_1 = V_1 \setminus \bigcup_{w \in W} wV_0,$$

and we define the $\cdot$ action of $W$ on $D$ as follows: for any $w \in W$ and $x \in D$,

$$w \cdot x = \hat{w}x.$$

Observe that the property that $WD = \bigcup_{w \in W} wD \subseteq V \setminus V_0$ guarantees that this $\cdot$ action of $W$ on $D$ is well-defined. Furthermore, we observe that each $w \in W$ acts continuously on $D$. The next result is taken from [19] which summarizes a number of key facts:

Proposition 4.10. ([19 Proposition 3.1]) Let $V_1$ be a transverse hyperplane, and let $\hat{\Phi}$ and $E(W)$ be the corresponding normalized roots and limit roots.

(i) $\hat{\Phi}$ and $E(W)$ are contained in $D$.

(ii) $\hat{\Phi}$ and $E(W)$ are stable under the $\cdot$ action of $W$; moreover

$$\hat{\Phi} = W \cdot \Pi.$$

(iii) The topological closure $\hat{\Phi} \cup E(W)$ is stable under the $\cdot$ action of $W$.

□

Furthermore, it has been observed in [19] that the $\cdot$ action has the following nice geometric description:

Proposition 4.11. ([19 Proposition 3.5]) Keep previous notations.

(i) Let $\alpha \in \Phi$, and $x \in D \cap Q$. Denote by $L(\hat{\alpha}, x)$ the line containing $\hat{\alpha}$ and $x$. Then

(a) if $(\alpha, x) = 0$, then $L(\hat{\alpha}, x)$ intersects $Q$ only at $x$, and $r_{\alpha} \cdot x = x$;
(b) if \((\alpha, x) \neq 0\), then \(L(\hat{\alpha}, x)\) intersects \(Q\) in two distinct points, namely, \(x\) and \(r_\alpha \cdot x\).

(ii) Let \(\alpha_1\) and \(\alpha_2\) be two distinct roots in \(\Phi\), \(x \in L(\alpha_1, \alpha_2) \cap Q\), and \(w \in W\). Then \(w \cdot x \in L(w \cdot \alpha_1, w \cdot \alpha_2) \cap Q\).

\[\Box\]

Given a transverse hyperplane \(V_1\), the limit roots coming from a given rank 2 reflection subgroup can be observed inside \(E(\Phi)\). Take two distinct positive roots \(a\) and \(b\), denote by \(W' = \langle r_a, r_b \rangle\) the dihedral reflection subgroup of \(W\) generated by the reflections \(r_a\) and \(r_b\) corresponding to the two positive roots. Let \(\Pi(W') = \{a', b'\}\) be the set of canonical roots for the dihedral reflection subgroup \(W'\). Then Theorem 2.6 yields that \(\hat{\Phi}'\) is also transverse with respect to \(\Phi'\). Let we denote \(E(W')\) be the set of limit roots of \(W'\) in the root system \(\Phi' = W'\Pi'\). Observe that the hyperplane \(V_1\) is also transverse with respect to \(\Phi'\). Let \(\Phi' = (V', \Pi', (, )')\) is also a Coxeter datum with associated root system \(\Phi' = W'\Pi'\).

Given a transverse hyperplane \(V_1\), the limit roots coming from a given rank 2 reflection subgroup can be observed inside \(E(\Phi)\). Take two distinct positive roots \(a\) and \(b\), denote by \(W' = \langle r_a, r_b \rangle\) the dihedral reflection subgroup of \(W\) generated by the reflections \(r_a\) and \(r_b\) corresponding to the two positive roots. Let \(\Pi(W') = \{a', b'\}\) be the set of canonical roots for the dihedral reflection subgroup \(W'\). Then Theorem 2.6 yields that \(\hat{\Phi}'\) is also transverse with respect to \(\Phi'\). Let we denote \(E(W')\) be the set of limit roots of \(W'\) in the root system \(\Phi'\) with respect to \(V_1\). Then the following was observed in 2.3 of [19]:

**Proposition 4.12.** Given the set up in the preceding paragraph,

(i) \(E(W') = Q \cap L(\hat{a}', \hat{b}') = E(W) \cap L(\hat{a}', \hat{b}')\);

(ii) the cardinality of \(E(W')\) is 0, 1, or 2, respectively, precisely when \(|(a', b')| < 1\), \(|(a', b')| = 1\), or \(|(a', b')| > 1\);

\[\Box\]

Proof. See the discussions in Section 2.3 of [19].

It turned out by considering the cardinality of \(E(W)\) we may easily ascertain whether \(W\) is affine or not.

**Proposition 4.13.** ([19 Corollary 2.16]) Suppose that \(W\) is an irreducible Coxeter group. Then \(E(W)\) is a singleton set if and only if \(W\) is an affine Coxeter group.

\[\Box\]

Utilizing the above observation, we may deduce the following characterization of affine Coxeter groups.

**Proposition 4.14.** Let \(W\) be an irreducible Coxeter group of finite rank. Then \(W\) is affine if and only if every infinite dihedral reflection subgroup of \(W\) is affine.

\[\Box\]

**Proof.** Suppose that \(W\) is irreducibly affine and is of rank \(n\). Then it follows readily that the signature of the bilinear form \((, )\) is \((n-1, 0)\), it follows that \(\hat{Q}\) consists of a single point (see also [19 Corollary 2.16]). If \(W'\) is any infinite dihedral reflection subgroup of \(W\), then \(E(W') \subseteq \hat{Q}\), and consequently, \(W'\) has a unique accumulation point, and thus \(W'\) is affine.
For the converse, we prove that the contra-positive is true. Suppose that \( W \) is an infinite irreducible Coxeter group of finite rank, and suppose that \( W \) is non-affine. It is enough to show that there exists an infinite dihedral reflection subgroup of \( W \) that is not affine.

Let \( W' \) be an arbitrary infinite dihedral reflection subgroup of \( W \). If \( W' \) is non-affine then we are done, and so we may assume that \( W' \) is affine.

Let \( \Pi(W') = \{ \alpha, \beta \} \) be the set of canonical roots of the root subsystem of \( W' \). Then Theorem 2 yields that \( \alpha, \beta \in \Phi^+ \), and \( (\alpha, \beta) = -1 \).

For the converse, we prove that the contra-positive is true. Suppose that \( W \) is infinite and non-affine by [19, Corollary 2.16]. If \( W \) is affine, then we are done, and so we may assume that \( W \) is non-affine. It is enough to show that there exists an infinite dihedral reflection subgroup of \( W \) such that \( \Pi(W) \) is infinite and non-affine. It is enough to show that there exists an infinite dihedral reflection subgroup of \( W \) such that \( \Pi(W) \) is infinite and non-affine. Let \( \Pi(W') = \{ \alpha, \beta \} \) be the set of canonical roots of the root subsystem of \( W' \). Then \( (\alpha, \beta) < 0 \), and there exists some \( \alpha \in \Pi \) such that \( \langle \alpha, \beta \rangle \) is strictly negative (since there is an \( \eta \in E(W) \) with \( r_a \cdot \eta \neq \eta \)).

We shall construct two roots in \( \Phi(W) \) with their bilinear form value being strictly less than \(-1\), and consequently, the dihedral reflection subgroup generated by the corresponding reflections will be infinite and non-affine.

Consider the Coxeter diagram of \( W' \):

Without loss of generality, we may assume that \( m_{ab} \leq m_{ac} \).

First consider the case that \( m_{ab} = 2 \). If \( m_{ac} = 3 \) then the requirement that \( W' \) is infinite and non-affine implies that \( m_{bc} \geq 7 \) (to exclude the \( A_3, B_3, H_3 \) and \( G_2 \) cases). Set \( \lambda = -2(b, c) > \sqrt{3} \). Set \( x = r_c r_b c = \lambda b + (\lambda^2 - 1)c \).

Consequently, \( (a, x) = \lambda(a, c) = -\frac{1}{2}(\lambda^2 - 1) \leq -1 \), and whence the dihedral reflection subgroup \( \langle r_a, r_x \rangle \) is infinite and non-affine. If \( m_{ac} = 4 \) then the requirement that \( W' \) is infinite and non-affine implies that \( m_{bc} \geq 5 \) (to exclude the \( C_3 \) case). Set \( \lambda = -2(b, c) \geq 2 \cos \frac{\pi}{5} = \frac{\sqrt{5} + 1}{2} \). Set \( x = r_c r_b c = \lambda b + \lambda c \).

Consequently, \( (a, x) = \lambda(a, c) \leq -\frac{\sqrt{2}(\sqrt{5} + 1)}{2} \leq -1 \), and whence the dihedral reflection subgroup \( \langle r_a, r_x \rangle \) is infinite and non-affine. If \( m_{ac} \in \{5, 6\} \), then requirement that \( W' \) is infinite and non-affine implies that \( m_{bc} \geq 4 \) (to exclude the \( H_3 \) and \( G_2 \) cases). Set \( \lambda = -2(a, c) \geq \frac{\sqrt{3} + 1}{2} \). Set \( x = r_c a = a + \lambda c \).

Consequently, \( (b, x) = \lambda(b, c) \leq -\frac{\sqrt{2}(\sqrt{5} + 1)}{2} \leq -1 \), and whence the dihedral reflection subgroup \( \langle r_b, r_x \rangle \) is infinite and non-affine. Set \( \lambda = -2(a, c) < \sqrt{3} \). Set \( x = r_c a = a + \lambda c \).

Consequently, \( (b, x) = (\lambda^2 - 1)(b, c) \leq -\frac{(\lambda^2 - 1)}{2} < -1 \),
and whence the dihedral reflection subgroup $\langle r_b, r_x \rangle$ is infinite and non-affine.

Next, consider the case $m_{ab} \geq 3$. If $m_{bc} = 2$, then by relabeling the vertices, we get into the situations previously considered. Thus we may assume that $m_{bc} \geq 3$. Furthermore, interchanging the labels of $a$ and $c$ if necessary, we may also assume that $3 \leq m_{ab} \leq m_{bc}$. First, assume that $m_{ab} = 3$. If $m_{bc} = 3$, then the requirement that $W''$ is infinite and non-affine implies that $m_{ac} \geq 4$ (to exclude the $A_3$ case). Then we have $(b, c) = -\frac{1}{2}, (a, c) \leq -\sqrt{\frac{3}{2}}$. Set $x := r_b c = c + b$. Note that then $(a, x) = (a, c) + (a, b) \leq -\sqrt{\frac{3}{2}} - \frac{1}{2} < -1$, and consequently, the dihedral reflection subgroup $\langle r_a, r_x \rangle$ is infinite and non-affine. If $m_{bc} \in \{4, 5, 6\}$, then the requirement that $W''$ is infinite and non-affine implies that $m_{ac} \geq 3$. Consequently, $(b, c) \leq -\sqrt{\frac{3}{2}}, (a, c) \leq -\frac{1}{2}$. Set $x := r_b c = c + \lambda b$, where $\lambda = -2(b, c) \geq \sqrt{3}$. Then $(a, x) = (a, c) + \lambda (a, b) \leq -\frac{1}{2} - \sqrt{\frac{3}{2}} \cdot \frac{1}{2} < -1$, and consequently the dihedral reflection subgroup $\langle r_a, r_x \rangle$ is infinite and non-affine. If $m_{bc} \geq 7$, set $\lambda = -2(b, c) > \sqrt{3}$. Set $x := r_b r_a b = r_b(b + \lambda c) = -b + \lambda (c + \lambda b) = (\lambda^2 - 1)b + \lambda c$. Then $(a, x) = (\lambda^2 - 1)(a, b) + \lambda (a, c) \leq (\lambda^2 - 1)(a, b) = -\frac{\lambda^2 - 1}{2} < -1$, and consequently the dihedral reflection subgroup $\langle r_a, r_x \rangle$ is infinite and non-affine.

Next, we consider the case that $m_{ab} = m_{bc} = 4$. Then the requirement that $W''$ is infinite non-affine forces that $m_{ac} \geq 3$. Now we have $(a, b) = (b, c) = -\sqrt{\frac{3}{2}}, (a, c) \leq -\frac{1}{2}$. Setting $x := r_b c = c + \sqrt{2} b$, we have $(a, x) = (a, c + \sqrt{2} b) = (a, c) + \sqrt{2}(a, b) \leq -\frac{1}{2} - \sqrt{2} \cdot \frac{\sqrt{2}}{2} = -\frac{3}{2} < -1$.

Finally, consider the case $m_{ab} \geq 4$ and $m_{bc} \geq 5$. Then $(a, b) \leq -\sqrt{\frac{3}{2}}$. Set $\lambda = -2(b, c)$, then $\lambda \geq 2 \cos \frac{\pi}{6} = \frac{\sqrt{3} + 1}{2}$. Furthermore, set $x := r_b c = c + \lambda b$, and then $(a, x) = (a, c) + \lambda (a, b) \leq \lambda (a, b) \leq -\frac{3}{2} \cdot \frac{\sqrt{3} + 1}{2} = -\sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}} < -1$.

$\square$

5. Connection with limit roots

Throughout this section each Coxeter group $W$ is understood to have an associated Coxeter datum $\mathcal{C} = (V, \Pi, (\ , \ ))$ with $\Pi$ being a finite set, and let $\Phi$ be the corresponding root system. As stated in Remark 4.7, we shall further assume that $\Pi$ forms a basis for $V$, and we let $V_1$ be the transverse hyperplane as in Remark 4.7.

Suppose that $W''$ is an irreducible affine reflection subgroup of $W$, and let $\Pi(W'') = \{ a_1, a_2, \ldots, a_s \}$. Let $\eta$ be the unique limit root in $E(W'')$. Since by Lemma 3.2 $\eta$ is in the radical of the bilinear form $(\ , \ )$ restricted to the subspace $\mathbb{R}a_1 \oplus \cdots \oplus \mathbb{R}a_s$, it follows that $(\eta, x) = 0$ for all $x \in \Phi(W') \subset \mathbb{R}a_1 \oplus \cdots \oplus \mathbb{R}a_s$. However, it is possible that $(\eta, x) > 0$ for a root $x \in \Phi^+ \setminus \Phi(W'')$, in this section, amongst other things, we shall prove that $(\eta, x) > 0$ for only finitely many positive
roots $x$. Note that Propositions 3.5, 4.13 and 4.14 together imply that if $W'$ is an irreducible affine reflection subgroup of $W$ then $W'$ has an affine dihedral reflection subgroup $W'' \subseteq W'$ with $E(W'') = E(W')$. Hence many discussions on general affine reflection subgroups can be simplified to discussions on affine dihedral reflection subgroups.

Before we give a characterization of those limit roots in $E$ arising from affine reflection subgroups, let us first look at some limit roots that possibly do not arise from affine reflection subgroups. The first candidate for such possibly non-affine limit roots might come from infinite non-affine dihedral reflection subgroups.

Let $a, b \in \Phi^+$ be such that $(a, b) = -\cosh \theta < -1$, and let $W' = \langle r_a, r_b \rangle$ be the dihedral reflection subgroup generated by the reflections corresponding to $a$ and $b$. Note first that $W'$ is infinite and non-affine. Proposition 4.12 then yields that $E(W')$ consists of two distinct limit roots. Direct calculations then show that the isotropic cone $Q$ intersects the subspace $R_a \oplus R_b$ consists of two lines $R((\cosh \theta + \sinh \theta) a + b)$ and $R((\cosh \theta - \sinh \theta) a + b) = R(a + (\cosh \theta + \sinh \theta) b)$, and if we let $\eta_1$ and $\eta_2$ be as following, then $E(W') = \{\eta_1, \eta_2\}$:

$$\eta_1 := \frac{(\cosh \theta + \sinh \theta) a_1}{(\cosh \theta + \sinh \theta) a_1 + |b_1|} \hat{a} + \frac{|b_1|}{(\cosh \theta + \sinh \theta) a_1 + |b_1|} \hat{b},$$

and

$$\eta_2 := \frac{(\cosh \theta - \sinh \theta) a_1}{(\cosh \theta - \sinh \theta) a_1 + |b_1|} \hat{a} + \frac{|b_1|}{(\cosh \theta - \sinh \theta) a_1 + |b_1|} \hat{b}.$$ 

For each $i \in \mathbb{N}$, we adopt the following notation

$$c_i := \frac{\sinh(i\theta)}{\sinh \theta}. \quad (5.1)$$

Then

$$(r_a r_b)^i a = c_{2i+1} a + c_{2i} b,$$

and

$$(r_b r_a)^i b = c_{2i} a + c_{2i+1} b.$$ 

Observe that

$$\lim_{i \to \infty} \frac{c_{2i+1}}{c_{2i}} = \lim_{i \to \infty} \frac{\sinh(2i\theta) \cosh \theta + \cosh(2i\theta) \sinh \theta}{\sinh(2i\theta)} \quad (5.2)$$

$$= \lim_{i \to \infty} (\cosh \theta + \coth(2i\theta) \sinh \theta)$$

$$= \cosh \theta + \sinh \theta.$$ 

Consequently, we see that

$$\eta_1 = \lim_{i \to \infty} (r_ar_b)^i a \in \mathbb{R}((\cosh \theta + \sinh \theta) a + b)$$

and
\[ \eta_2 = \lim_{i \to \infty} (r_br_a)^ib \in \mathbb{R}((\cosh \theta - \sinh \theta)a + b). \]

And it follows readily that

\[(\eta_1, a) = \frac{\sinh \theta}{(\cosh \theta + \sinh \theta)|a|_1 + |b|_1} > 0;\]

and

\[(\eta_2, b) = \frac{\sinh \theta(\cosh \theta - \sinh \theta)}{(\cosh \theta - \sinh \theta)|a|_1 + |b|_1} > 0.\]

Note that \((r_ar_b)^i \cdot \eta_1 = \eta_1\) and \((r_br_a)^i \cdot \eta_2 = \eta_2\) for all \(i \in \mathbb{N}\). Consequently, for all \(i \in \mathbb{N}\)

\[(\eta_1, (r_ar_b)^i a) > 0\text{ and } (\eta_2, (r_br_a)^i b) > 0. \tag{5.3} \]

However, the above discussion does not rule out the possibility that an affine dihedral reflection subgroup and an infinite non-affine dihedral reflection subgroup sharing the same limit root, a situation as illustrated in the following diagram.

In this diagram the dotted circle represents the normalized isotropic cone, and the normalized root subsystems of two infinite dihedral reflection subgroups, one affine and the other non-affine, are contained in the two straight lines, with the black dots schematically representing normalized roots, and the two red dots representing possible limit roots.

It turned out that this situation will not arise. We shall show that if \(\eta \in E(W)\), then \(\eta \in E(W')\) for some affine reflection subgroup \(W'\) if and only if \((\eta, x) > 0\) for only finitely many \(x \in \Phi^+\).

The following fundamental and well-known result for infinite Coxeter groups of finite rank states that for pairs of roots, the bilinear form takes only finitely many values in the unit radius interval about zero.

**Lemma 5.1.** [[1]] Proposition 4.5.5] *Suppose that* \(W\) *is a Coxeter group of finite rank. Then the set* \(\{(\alpha, \beta) \mid \alpha, \beta \in \Phi, \, |(\alpha, \beta)| < 1\} \) *is finite.*
In particular, there exists a fixed $\epsilon > 0$ such that $|\langle \alpha, \beta \rangle| > \epsilon$ whenever $\alpha, \beta \in \Phi$ satisfy $(\alpha, \beta) \neq 0$.

**Definition 5.2.** Suppose that $W$ is an infinite Coxeter group. For each $\eta \in E(W)$, define $\text{Pos}(\eta) \subseteq \Phi^+$ by

$$\text{Pos}(\eta) = \{ x \in \Phi^+ \mid (\eta, x) > 0 \}$$

**Definition 5.3.** A limit root $\eta \in E(W)$ is called an affine limit root if there exists an irreducible affine reflection subgroup $W' \leq W$ with $E(W') = \{ \eta \}$, and define

$$E_{aff} := \{ \eta \in E(W') \mid W' \text{ is an affine reflection subgroup of } W \},$$

the set of all affine limit roots in $W$.

**Remark 5.4.** Note that by Propositions 4.13 and 4.14, $\eta \in E(W)$ is affine if and only if there exists an irreducible affine dihedral reflection subgroup $W' \leq W$ with $E(W') = \{ \eta \}$.

**Theorem 5.5.** Let $W$ be an infinite Coxeter group with a finite generating set $R$. If $\eta \in E(W)$ is an affine limit root, then $\text{Pos}(\eta)$ is finite.

**Proof.** Let $D \leq W$ be an affine dihedral subgroup with $E(D) = \{ \eta \}$ and let $a, b \in \Phi^+$ be such that $\Pi(D) = \{ a, b \}$. Then $(a, b) = -1$ and $(\eta, a) = (\eta, b) = 0$. Letting $a_k = k(a + b) + a$ and $b_k = k(a + b) + b$ (for $k \geq 0$) we have that $\hat{a}_k \to \eta$ and $\hat{b}_k \to \eta$ and the root subsystem of $D$ satisfies $\Phi^+(D) = \{ a_k, b_k \mid k \geq 0 \}$. Note also that $a_k$ dom $a$ and $b_k$ dom $b$ for all $k \geq 0$.

Let $x \in \text{Pos}(\eta)$. Since $\hat{a}_k \to \eta$ and $\hat{b}_k \to \eta$, for sufficiently large $k$ we have $(x, a_k) > 0$ and $(x, b_k) > 0$. Fixing such a $k$, we have

$$0 < (x, a_k) + (x, b_k) = (x, (2k + 1)(a + b)) = (2k + 1)(x, a + b)$$

Therefore $(x, a + b) > 0$. Observe that $(x, a) > -1$, since otherwise by Lemma 3.8, $x$ dom $-a$ dom $-a_k$ for all $k \in \mathbb{N}$, which implies that $(x, a_k) \leq -1$ for all $k \in \mathbb{N}$, but this means that $(x, \eta) \leq 0$, contradicting $x \in \text{Pos}(\eta)$. Similarly, $(x, b) \geq -1$.

We now observe that there exists $\epsilon > 0$ such that $(x, a + b) > \epsilon$ for all $x \in \text{Pos}(\eta)$. To see this let $F$ denote the finite set

$$F = \{ (\alpha, \beta) \mid \alpha, \beta \in \Phi, \ |(\alpha, \beta)| < 1 \}$$

and define $\epsilon' = \min \{ r + s \mid r, s \in F, r + s > 0 \}$ and $\epsilon'' = 1 + \min F$. Setting $\epsilon = \min(\epsilon', \epsilon'')$ we have

$$(x, a + b) = (x, a) + (x, b) \geq \epsilon > 0$$

Therefore, since $(x, a_k) = k(x, a + b) + (x, a)$ and $(x, b_k) = k(x, a + b) + (x, b)$, there is a fixed $M \in \mathbb{N}$ such that

$$\forall x \in \text{Pos}(\eta) \text{ and } \forall k \geq M \text{ we have } (x, a_k) \geq 1 \text{ and } (x, b_k) \geq 1 \quad (5.4)$$
Now, for a contradiction, suppose that $\text{Pos}(\eta)$ is infinite. With $M$ as above, for fix some $k \geq M$ then (5.3) implies that there are dominance between the infinitely many $x \in \text{Pos}(\eta)$ and $a_k, b_k$. Since a given root can dominate only finitely many positive roots (see Remark 3.7), it follows that there are infinitely many $x \in \text{Pos}(\eta)$ satisfying $x \text{ dom } a_k$ and $x \text{ dom } b_k$, but then we would have $x \text{ dom } a$ and $x \text{ dom } b$, contradicting Lemma 3.12.

\[ \square \]

It turns out that the converse of Theorem 5.5 is also true, but before we can prove this we shall need a number of preparatory results first.

Recall that

\[ \mathcal{X} := \{ v \in \text{PLC}(\Pi) \mid (v, a) \leq 0, \text{ for all } a \in \Phi^+ \} \]

\[ = \{ v \in \text{PLC}(\Pi) \mid (v, a) \leq 0, \text{ for all } a \in \Pi \}, \]

and

\[ \mathcal{Z} := \bigcup_{w \in W} w \mathcal{X}, \]

where $\mathcal{Z}$ is the imaginary cone. The concept of the imaginary cone was first introduced in [25] in the context of Kac-Moody Lie algebras as the pointed cone spanned by the positive imaginary roots and later generalized to Coxeter groups. A definitive reference on the imaginary cones of Coxeter group can be found in [9]. We will show, amongst other things, that in a finitely generated Coxeter group $W$, the only limit roots in $\mathcal{Z}$ are precisely those limit roots arising from an affine reflection subgroup of $W$.

It has been shown in [10, Lemma 2.4] that if $(W, R)$ is an irreducible Coxeter system in which $W$ is a non-affine infinite Coxeter group then $\text{Int}(\mathcal{X}) \neq \emptyset$

If $x \in \text{Int}(\mathcal{X})$ then it is clear from the definition of $\mathcal{X}$ that $(x, a) < 0$ for all $a \in \Pi$, and in particular, $(x, x) < 0$.

The following seminal results were taken from [10] and [19]:

**Theorem 5.6.** Let $(W, R)$ be an irreducible Coxeter system in which $\#R < \infty$, and let $E(W)$ be the set of limit roots of $W$. Then

(i) $|wx|_1 > 0$ for all $w \in W$, and $x \in E$;

(ii) If $x \in E$, then $E = W \cdot x$, that is, the dot-action of $W$ on $E$ is minimal.

**Proof.**

(i) [19, Proposition 3.2];

(ii) [10, Theorem 3.1].

\[ \square \]

It can be readily observed that the irreducibility requirement in part (i) of above theorem can be removed, and in fact, we have:
Theorem 5.7. Let \((W, R)\) be Coxeter system in which \#\(R < \infty\), and let \(E(W)\) be the set of limit roots of \(W\). Then \(|wx| > 0\) for all \(w \in W\), and \(x \in E\).

Lemma 5.8. Suppose that \((W, R)\) is a Coxeter system in which \(W\) is an infinite Coxeter group of finite rank, and let \(E(W)\) be the set of limit roots for \(W\). Suppose that \(\eta \in E(W)\) such that \#\(\text{Pos}(\eta) < \infty\). Then there exists some \(w \in W\) with \(w\eta \in \mathcal{K}\).

Proof. We proceed with an induction on \#\(\text{Pos}(\eta)\).

If \#\(\text{Pos}(\eta) = 0\) then \(\eta \in \mathcal{K}\) and we are done by choosing \(w = 1\). If \#\(\text{Pos}(\eta) \neq 0\) then there exists some \(a \in \Pi\) such that \(a \in \text{Pos}(\eta)\). Note that \(\lambda r_a \cdot \eta \in E(W)\), and

\[
\text{Pos}(\lambda r_a \cdot \eta) = r_a(\text{Pos}(\eta) \setminus \{a\}),
\]

since \(r_a(\Phi^+ \setminus \{a\}) = \Phi^+ \setminus \{a\}\). Then the inductive hypothesis implies that there exists some \(w' \in W\) such that \(w'(r_a \cdot \eta) \in \mathcal{K}\), which by Theorem 5.7 means that \(w' \lambda r_a \eta = \lambda w'r_a \eta \in \mathcal{K}\) for some \(\lambda > 0\).

Setting \(w := w'r_a\) yields that \(\lambda w \eta \in \mathcal{K}\). Finally, since \(\mathcal{K}\) is a cone, it follows readily that \(w \eta \in \mathcal{K}\).

The following is taken from [15, Lemma 5.9], and for completeness we include a proof here.

Lemma 5.9. Suppose the \(x \in \mathcal{K} \cap Q\), and \(x \neq 0\). Then there exists a subset \(M\) of \(\Pi\) such that the restriction of the bilinear form \((\cdot, \cdot)\) on the subspace spanned by \(M\) has a nonzero radical \(\text{Rad}(M)\), and \(x \in \text{Rad}(M)\).

Proof. Let \(M := \{a \in \Pi \mid (x, a) = 0\}\), and for each \(a \in \Pi\), let \(\lambda_a\) be the canonical coefficient of \(a\) in \(x\). Thus \(x = \sum_{a \in \Pi} \lambda_a a\). Note that each \(\lambda_a\) is non-negative, as \(x \in \mathcal{K}\). Now

\[
0 = (x, x) = \sum_{a \in \Pi} \lambda_a (x, a). \tag{5.5}
\]

Note that those summands corresponding to \(a \in M\) in (5.5) are all zero. For \(a \in \Pi \setminus M\), \((x, a)\) is strictly negative. Since \((x, x) = 0\), it follows that \(\lambda_a = 0\) for all \(a \in \Pi \setminus M\), and consequently \(x \in \text{span}(M)\).

Finally, since \((x, a) = 0\) for all \(a \in M\), it follows that \(x \in \text{Rad}(M)\).

It turns out that the converse of Lemma 5.9 is also true.

Lemma 5.10. Suppose that \(M \subseteq \Pi\) satisfies the condition that the restriction of the bilinear form \((\cdot, \cdot)\) on \(\text{span}(M)\) has a non-zero radical \(\text{Rad}(M)\), and suppose that moreover, \(\text{Rad}(M) \cap \text{PLC}(M) \neq \emptyset\). Then \(\text{Rad}(M) \cap \text{PLC}(M) \subseteq \mathcal{K} \cap Q\).

Proof. Let \(x \in \text{Rad}(M) \cap \text{PLC}(M)\) be arbitrary. Then \((x, a) = 0\) for all \(a \in M\), and moreover, \((x, a') \leq 0\) for all \(a' \in \Pi \setminus M\) by the definition of \(\Pi\) and the fact that \(x \in \text{PLC}(M)\). Therefore \(x \in \mathcal{K}\). Next, to
establish that \( x \in \mathbb{Q} \), it is enough to observe that
\[
 x = \sum_{a \in M} \lambda_a a
\]
where \( \lambda_a \in \mathbb{R} \), and then it follows readily from the fact
\( x \in \text{Rad}(M) \) that
\[
 (x, x) = \sum_{a \in M} \lambda_a (x, a) = 0.
\]

**Proposition 5.11.** Suppose that \( \eta \in E(W) \) such that \( \text{Pos}(\eta) \) is a finite set. Then there exists \( M \subseteq \Pi \) such that the restriction of the bilinear form \((\cdot, \cdot)\) on \( \text{span}(M) \) has a non-zero radical \( \text{Rad}(M) \), and there exists some \( w \in W \) such that
\[
 w \cdot \eta \in \text{Rad}(M).
\]

**Proof.** By Lemma 5.8 there exists some \( w \in W \) such that \( w \cdot \eta \in \mathcal{K} \). Note that then \( w \cdot \eta \in \mathcal{K} \cap \mathbb{Q} \) and \( w \cdot \eta \neq 0 \). Consequently, it follows from Lemma 5.9 that there exists some \( M \subseteq \Pi \) such that the restriction of the bilinear form \((\cdot, \cdot)\) to the subspace spanned by \( M \) has a non-zero radical \( \text{Rad}(M) \), and \( w \cdot \eta \in \text{Rad}(M) \).

**Theorem 5.12.** Suppose that \( \eta \in E(W) \) such that \( \# \text{Pos}(\eta) < \infty \). Then there exists an affine reflection subgroup \( W' \) of \( W \) satisfying \( E(W') = \{\eta\} \).

**Proof.** Without loss of generality, we may assume that \( W \) is an irreducible Coxeter group. By Proposition 5.11 there exists \( M \subseteq \Pi \) such that the restriction of the bilinear form \((\cdot, \cdot)\) on the subspace spanned by \( M \) has a non-zero radical \( \text{Rad}(M) \), and there exists some \( w \in W \) satisfying the condition that \( w \cdot \eta \in \text{Rad}(M) \). Let \( W_M := \langle r_a \mid a \in M \rangle \) be the standard parabolic subgroup corresponding to the set \( M \). Observe that \( W_M \) is an infinite reflection subgroup of \( W \) and \( w \cdot \eta \in E(W_M) \). Furthermore, note that \( r_a \cdot (w \cdot \eta) = w \cdot \eta \) for all \( a \in M \), as \( (a, w \cdot \eta) = 0 \) (since \( w \cdot \eta \in \text{Rad}(M) \)). Consequently, \( W_M \cdot (w \cdot \eta) = \{w \cdot \eta\} \). Thus part (ii) of Theorem 5.6 yields that \( E(W_M) = \{w \cdot \eta\} \), and hence \( W_M \) is affine. Finally, observe that \( \eta \in E(w^{-1}W_M w) \), and \( w^{-1}W_M w \) being a conjugate of an affine reflection subgroup is itself affine, and we are done.

Combining Theorem 5.5 and Theorem 5.12 we immediately have the following characterization of affine limit roots in an infinite Coxeter group of finite rank.

**Theorem 5.13.** Let \( W \) be an infinite Coxeter group with the finite generating set \( R \), and let \( \eta \in E(W) \). Then \( \eta \in E(W') \) for some affine reflection subgroup \( W' \) of \( W \) if and only if \( \text{Pos}(\eta) \) is a finite set.

The next result is likely to be known to the experts of this field, but to the best of our knowledge, it has not appeared in the literature.

**Theorem 5.14.** Suppose that \((W, R)\) is a Coxeter system in which \( W \) is an infinite Coxeter group and \( R \) is a finite generating set. Then \( W \) contains an affine reflection subgroup if and only if \( W \) contains an affine standard parabolic subgroup. In particular, affine reflection
subgroups in $W$ are precisely the infinite subgroup of affine parabolic subgroups of $W$.

**Proof.** Suppose that $W$ has an affine standard parabolic subgroup. Then there is nothing to prove, for a standard parabolic subgroup of $W$ is, *a priori*, a reflection subgroup of $W$.

Conversely, suppose that $W$ has an affine reflection subgroup $W'$. Without loss of generality, we may assume that $W'$ is irreducible, and we let $E(W') = \{\eta\}$. Then Theorem 5.5 yields that $\# \text{Pos}(\eta) < \infty$, and, in turn, it follows from Proposition 5.11 that there exists some $M \subseteq \Pi$ such that the restriction of the bilinear form $(\ ,\ )$ on $\mathbb{R}M$, the subspace spanned by $M$ has a non-zero radical $\text{Rad}M \subseteq \mathbb{R}M$, and there exists some $w \in W$ such that $w \cdot \eta \in \text{Rad}M \cap \text{PLC}(M)$. In particular, $w \cdot \eta \neq 0$. That is, $\text{Rad}(M) \cap \text{cone}(M) \neq \{0\}$, which by Lemma 3.4 meant that $W_M := \langle r_a \mid a \in M \rangle$, the standard parabolic subgroup generated by the reflections corresponding to elements of $M$ is affine. In this case, we can readily see that $W' \subseteq w^{-1}W_M w$.

**Corollary 5.15.** Let $W$ be an infinite Coxeter group. Then $W$ contains an affine reflection subgroup if and only if there exists a connected subset $M_0 \subseteq \Pi$ such that the restriction of the bilinear form $(\ ,\ )$ on $\text{span}(M_0)$ satisfies $\text{cone}(M_0) \cap \text{Rad}(M_0) \neq \{0\}$, where $\text{Rad}(M_0) \subseteq \text{span}(M_0)$ is the radical of the restricted bilinear form.

**Proof.** Suppose that $W$ has an affine reflection subgroup $W'$. By Theorem 5.14 it remains to prove that there exists a connected subset $M_0$ of $M$ that satisfies the condition. Suppose $M = M_1 \cup M_2$ where $M_1$ and $M_2$ are both collections of connected components of $M$. Denote $V_1 = \text{span}(M_1)$ and $V_2 = \text{span}(M_2)$. Then by the definition of a Coxeter datum we have:

$$(x_1, x_2) = 0,$$

whenever $x_1 \in M_1$ and $x_2 \in M_2$. Let $0 \neq v \in \text{cone}(M) \cap \text{Rad}V_M$, then $v = \sum_{\alpha \in M} c_\alpha \alpha$, where $c_\alpha \geq 0$ for all $\alpha \in M$.

If $c_\alpha = 0$ for all $\alpha \in M_2$ (resp. $M_1$), then $v \in \text{cone}(M_1) \cap \text{Rad}V_1$ (resp. $\text{cone}(M_2) \cap \text{Rad}V_2$). Else, let us consider $v_1 = \sum_{\alpha \in M_1} c_\alpha \alpha$, and $v_2 = \sum_{\alpha \in M_2} c_\alpha \alpha$. Then $v = v_1 + v_2$. By (5.6) for any $x_1 \in V_1$, $(x_1, v_2) = 0$, thus $(x_1, v_1) = 0$. Hence $v_1 \in \text{cone}(M_1) \cap \text{Rad}V_1$. 
Apply the process above for finite times, we can find the desired connected subset $M_0$.

Conversely, suppose that $M \subseteq \Pi$ is a connected subset such that $\text{Rad}(M) \cap \text{cone}(M) \neq \{0\}$. Then Lemma 5.16 yields that $W_M$ is an affine reflection subgroup of $W$. \hfill $\square$

In the special case that $W'$ is a standard parabolic subgroup and also an affine subgroup of an infinite Coxeter group $W$, we have the following:

**Lemma 5.16.** Suppose that $W$ is an infinite Coxeter group, and suppose that $M \subseteq \Pi$ is a connected subset such that $W' = \langle r_a | a \in M \rangle$ is an affine subgroup. Let $E(W') = \{\eta\}$. Then $\text{Pos}(\eta) = \emptyset$.

**Proof.** Let $V'$ be the subspace spanned by $M$, and let $(\cdot, \cdot)'$ be the restriction of $(\cdot, \cdot)$ on $V'$. Then as in the proof of Lemma 3.2 the isotropic cone and the radical of $(\cdot, \cdot)'$ coincide, and is one dimensional. Furthermore, $\eta$ is in this radical. Now let $x \in \Phi^+$ be arbitrary. Then $x = \sum_{a \in M} \lambda_a a + \sum_{b \in \Pi \setminus M} \lambda_b b$, where all the $\lambda_a \geq 0$ and $\lambda_b \geq 0$. Thus

$$(\eta, x) = \sum_{a \in M} \lambda_a (\eta, a) + \sum_{b \in \Pi \setminus M} \lambda_b (\eta, b).$$

Since $\eta \in \text{PLC}(M)$ and $(b, a) \leq 0$ for all $b \in \Pi \setminus M$ and all $a \in M$, it follows from the above that $(\eta, x) \leq 0$. \hfill $\square$

The converse of this lemma is also true:

**Lemma 5.17.** For $\eta \in E(W)$, if $\text{Pos}(\eta) = \emptyset$, then $\{\eta\} = E(W')$, where $W'$ is an irreducible standard parabolic affine subgroup.

**Proof.** It follows from Theorem 5.12 that $\{\eta\} = E(W')$, where $W'$ is an affine reflection subgroup. If $W'$ is not a standard parabolic subgroup, then Theorem 5.14 yields that $W' \subseteq wW''w^{-1}$, where $W''$ is a standard parabolic affine subgroup and $w \neq 1$. Let $E(W'') = \{\eta''\}$, then $\eta = w \cdot \eta'' \neq \eta''$.

Write $w = w'r_a$ with $l(w) = l(w') + 1$. Without loss of generality, we can assume that $r_a \cdot \eta'' \neq \eta''$, which is equivalent to $(a, \eta'') \neq 0$. Since $\eta'' \in E(W'')$, and $W''$ is an affine standard parabolic subgroup, it follows from Lemma 5.16 that $(a, \eta'') < 0$. Moreover, $l(w'r_a) = l(w') + 1$ yields that $w'a \in \Phi^+$. Hence we have:

$$(w'a, \eta) = (w'a, w\eta'') = (a, r_a \eta'') = -(a, \eta'') > 0,$$

whence $w'a \in \text{Pos}(\eta)$, contradicting $\text{Pos}(\eta) = \emptyset$. \hfill $\square$

Combining the two lemmas we have:

**Theorem 5.18.** For $\eta \in E(W)$, $\text{Pos}(\eta) = \emptyset$ if and only if $\eta$ is the limit root of an irreducible standard parabolic affine subgroup. \hfill $\square$

In the case that $\eta$ is the limit root arising from an affine reflection subgroup of $W$, we can describe $\text{Pos}(\eta)$ in more details:
Theorem 5.19. Suppose that \( \{ \eta \} = E(W') \) where \( W' \) is an affine reflection subgroup of \( W \). Let \( w \in W \) be of minimal length such that \( W' \subseteq wW''w^{-1} \), where \( W'' \) is an irreducible standard parabolic affine subgroup of \( W \). Then \( \text{Pos}(\eta) = N(w^{-1}) \).

Proof. It follows from Theorem 5.14 that the element \( w \) in the theorem exists.

If \( l(w) = 1 \), there exists a simple root \( \alpha \) such that \( \eta = r_\alpha \cdot \eta'' \), where \( \{ \eta'' \} = E(W'') \). By the minimality of \( l(w) \) we have \( \eta \neq \eta'' \), hence \( (a, \eta'') \neq 0 \). It then follows from Lemma 5.16 that \( (a, \eta) = -(a, \eta'') > 0 \), and hence \( a \in \text{Pos}(\eta) \).

On the other hand, in the proof of Lemma 5.8 we have observed the following fact: if \( \# \text{Pos}(\eta) \neq 0 \), then there exists some \( a \in \Pi \) such that \( a \in \text{Pos}(\eta) \), and moreover:

\[
\text{Pos}(r_\alpha \cdot \eta) = r_\alpha(\text{Pos}(\eta) \setminus \{ a \}).
\]

Since \( \text{Pos}(\eta'') = \emptyset \), we can easily deduce that \( \text{Pos}(\eta) = \{ a \} = N(r_\alpha) \).

Now we proceed an induction on \( l(w) \). Suppose \( l(w) = n > 1 \), and \( \eta'' := w\eta \). Set \( w = r_{a_1}w' = r_{a_1}r_{a_2} \cdots r_{a_n} \), such that \( l(w) = l(w') + 1 \), and \( a_1, \ldots, a_n \in \Pi \). Then \( (w')^{-1}a_1 \in \Phi^+ \). Set \( \eta' = r_{a_1} \cdot \eta \). By induction hypothesis,

\[
\text{Pos}(\eta') = N((w')^{-1}) = \{ a_2, r_{a_2}a_3, \ldots, r_{a_2} \cdots r_{a_{n-1}}a_n \}.
\]

Now we have:

\[
(a_1, \eta) = -(a_1, \eta') = -(a_1, w' \eta'') = -((w')^{-1}a_1, \eta'') > 0
\]

So \( a_1 \in \text{Pos}(\eta) \). Hence \( N((w')^{-1}) = \text{Pos}(\eta') = r_{a_1}(\text{Pos}(\eta) \setminus \{ a_1 \}) \). And finally,

\[
\text{Pos}(\eta) = r_{a_1}N((w')^{-1}) \cup \{ a_1 \}
\]

\[
= \{ a_1, r_{a_1}a_2, r_{a_1}r_{a_2}a_3, \ldots, r_{a_1}r_{a_2} \cdots r_{a_{n-1}}a_n \}
\]

\[
= N(w^{-1}),
\]

which is the desired result. \( \square \)

Definition 5.20. Define

\[
E_{\text{nonaff}} := \{ \eta \in E(W') \mid W' \text{ is a non-affine dihedral reflection subgroup of } W \}
\]

the set of all non-affine dihedral limit roots of \( W \).

The following characterization of the imaginary cone of a finitely generated Coxeter group was given in [14]:

Proposition 5.21. Suppose that \((W, R)\) is a Coxeter system in which \( R \) is a finite set, and let \( \mathcal{L} \) be the imaginary cone of \( W \). Then

\[
\mathcal{L} = \{ v \in U^* \mid (v, a) \leq 0 \text{ for all but finitely many } a \in \Phi^+ \},
\]
where $U^*$ is the dual of the Tits cone, and
\[
U^* = \bigcap_{w \in W} w(\text{PLC}(\Pi) \cup \{0\}).
\]

Proof. \[14\] Lemma 4.4 and Proposition 4.22. \[\square\]

Combining Theorem \[5.13\] and Proposition \[5.21\] we immediately arrive at the following conclusion:

**Corollary 5.22.** Suppose that $(W, R)$ is a Coxeter system in which $R$ is a finite set, and let $\mathcal{Z}$ be the imaginary cone of $W$. Then
\[
E(W) \cap \mathcal{Z} = E_{\text{aff}}
\]
furthermore, if $W$ is not an affine Coxeter group, then
\[
E_{\text{nonaff}} \subseteq U^* \setminus \mathcal{Z} \neq \emptyset.
\]

\[\square\]

**References**

1. A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics, GTM 231, Springer, 2005
2. N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4, 5 et 6*, Hermann, Paris, 1968
3. B. Brink and R. B. Howlett, *A finiteness property and an automatic structure of Coxeter groups*, Math. Ann. 296 (1993), 179–190.
4. B. Brink, *The set of dominance-minimal roots*, J. Algebra 206 (1998), 371–412.
5. V. Deodhar, *On the root system of a Coxeter group*, Comm. Algebra, 10(6): 611–630, 1982.
6. M. Dyer, *Hecke algebras and reflections in Coxeter groups*, PhD thesis, University of Sydney, 1987.
7. M. Dyer, *Reflection Subgroups of Coxeter Systems*, J. Algebra 135 (1990), 57–73.
8. M. Dyer, *On the “Bruhat graph” of a Coxeter system*, Compositio Math. 78 (1991), no. 2, 185–191.
9. M. Dyer. “Imaginary cone and reflection subgroups of Coxeter groups”. arXiv: 1210.5206 [math.RT], preprint, 2012.
10. Matthew Dyer, Christophe Hohlweg and Vivien Ripoll. “Imaginary cones and limit roots of infinite Coxeter groups”. In: *Math. Z.* 284 (2016), no.3–4, pp. 715–780.
11. T. Edgar, *Dominance and regularity in Coxeter groups*, PhD thesis, University of Notre Dame, 2009 (downloadable from [http://etd.nd.edu/ETD-db/](http://etd.nd.edu/ETD-db/)).
12. X. Fu, *Root systems and reflection representations of Coxeter groups*, PhD thesis, University of Sydney, 2010.
13. X. Fu, “The Dominance Hierarchy of Root Systems of Coxeter Groups”, *J. Algebra* 366 (2012), 187–204.
14. X. Fu, “Coxeter groups, imaginary cone and dominance”, *Pacific J. Math.* 262 (2013), no.2, 339–363.
15. X. Fu and L. Reeves, “Mapping the Davis complex into the imaginary cone”, arXiv: 1405.04837 [math.GR], preprint, 2017.
16. J.-Y. Hée, *Le cône imaginaire d’une base de racines sur $\mathbb{R}$*, unpublished.
17. J.-Y. Hée, *Sur la torsion de Steinberg-Rec des groupes de Chevalley et des groupes de Kac-Moody*, PhD thesis, Université de Paris-Sud, Orsay, 1993.
18. H. Hiller, *Geometry of Coxeter Groups*. Research Notes in Mathematics 54, Pitman (Advanced Publishing Program), Boston-London, 1981.

19. C. Hohlweg, J. P. Labbé and V. Ripoll, “Asymptotical behaviour of roots of infinite Coxeter groups”. In: *Canad. J. Math.* 66 Vol. 2, (2014), pp. 323–353.

20. C. Hohlweg, J. P. Préaux and V. Ripoll, “On the Limit Set of Root Systems of Coxeter Groups and Kleinian Groups”. arXiv: 1305.0052 [math.GR], preprint, 2013.

21. R. B. Howlett, *Normalizers of parabolic subgroups of reflection groups*, J. London Math. Soc. (2) 21 (1980), no. 1, 62–80.

22. R. B. Howlett, *Introduction to Coxeter groups*, Lectures given at ANU, 1996 (available at [http://www.maths.usyd.edu.au/res/Algebra/How/1997-6.html](http://www.maths.usyd.edu.au/res/Algebra/How/1997-6.html)).

23. R. B. Howlett, P. J. Rowley and D. E. Taylor, *On outer automorphisms of Coxeter groups*, Manuscripta Math. 94 (1997), 499–513.

24. J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge studies in advanced mathematics 29, 1990.

25. V. G. Kac, *Infinite-dimensional Lie algebras*, third edition, Cambridge University Press, Cambridge, 1990.

26. D. Krammer, *The conjugacy problem for Coxeter groups*, PhD thesis, Universiteit Utrecht, 1994.