Numerical studies of instability of generalized polytropic models of stellar disks

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Abstract. The distribution function for generalized polytropic models has been used to construct a series of numerical models of anisotropic disks. It has been shown with the help of simulation that such systems are unstable with respect to bar formation at any degree of radial elongation of star orbits. The result is completely at variance with the conclusions of earlier works, where similar models were studied. The pattern speeds and amplitudes of the forming bar were found, and the initial distributions of orbit precession rates were calculated. Such systems have been shown to fulfill all conditions for the onset of radial orbit instability, responsible for the formation of a slow bar.

1. Introduction

About 50% of spiral galaxies are known by the observation data to have a typical elongated large-scale structure, the bar. The bar properties are dependent on the Hubble type of the host galaxy [3]. As a rule, the surface brightness profile of the bar is planar in galaxies of earlier types and exponential in those of later ones. The bars are typically highly prolate. For instance, the bar in the Galaxy has the ratio of the major to minor semi-axes of about 3:1.

Some interrelated issues are still at the focus of astronomers’ attention. What are the mechanisms of bar formation? Why do they have the properties (pattern speed, size, density profile), which are observed?

As to the pattern speed, bars are divided into two types, fast and slow. The pattern speed of a slow bar is on the order of the average precession rate of star orbits in the disk. The pattern speed of a fast bar exceeds the maximal precession rate of orbits. Fast bars are intensively studied in a lot of theoretical and numerical experimental works. As a rule, the bars observed in galaxies are fast ones [11]. As to slow bars, which may form in the central regions of stellar disks, there are practically no works on the topic, either theoretical or experimental.

The mechanism of slow bar formation was suggested by Lynden-Bell [7] to be the radial orbit instability (ROI). In Lynden-Bell’s approach, orbits with small precession rates are considered, i.e. those having precession rate much less than the angular frequency of star revolution, \( \Omega_{pr} \ll \Omega \). In this case, an orbit can be thought as a “spoke” with a small precession rate. By virtue of Poincaré’s theorem on invariants, the integral to generalized moments and coordinates

\[
2J_f = \frac{1}{2\pi} \int \vec{p} \cdot d\vec{q}, \tag{1}
\]

will conserve; here \( J_f \) is the value of the adiabatic invariant over a single radial oscillation of a star. Since a star completes two radial oscillations during a revolution around the galaxy center, the integral in Eq. (1) is twice the value of the adiabatic invariant. Lynden-Bell [7] demonstrated...
that such orbits would coalesce and form a bar in the case when adiabatic invariant $J_f$ has a specific dependence on angular momentum $L$ of a star:

$$\left(\frac{\partial Q_{pr}}{\partial L}\right)_{J_f} > 0,$$

(2)

An efficient method of studying bar-like instabilities is simulations, allowing one to investigate various nonlinear effects, which are hard to take into account in a theory. Nonetheless, numerical simulations face several problems. The first one is the limited computing power. The number of stars in actual galaxies is $10^9 - 10^{11}$. Calculation of the evolution of such number of particles on a modern computer would take years. For this reason, the system is approximated with a smaller number of particles, $10^5 - 10^7$. The second problem growing out of the first one is the decrease of the system relaxation time with falling number of particles. After Chandrasekhar [14], the relaxation time is the time it takes a star to essentially change its characteristics (energy or trajectory). In the interval equal to the relaxation time, the system “forgets” the initial conditions. Such an effect arises due to discreteness of the matter distribution in a galaxy. Relaxation time $T_{rel}$ for three-dimensional systems grows with the number of particles, $N$, in the system [3]:

$$T_{rel} \propto N \log(N).$$

(3)

The estimate Eq. (3) proves to be quite incorrect for planar systems. In two-dimensional models, the relaxation time is of the order of the crossing time and is independent of the number of particles because relaxation in such systems is dominated by close encounters [10], whereas in three-dimensional systems, distant encounters contribute to particle trajectory deviations. Rybicki [10] demonstrated that the dependence of the relaxation time on the number of particles in two-dimensional systems arises if one invokes the potential smoothing procedure:

$$\Phi(r) \propto \frac{1}{r} \rightarrow \Phi(r) \propto \frac{1}{\sqrt{r^2 + \epsilon^2}}$$

(4)

where $\epsilon$ is the smoothing parameter of the system. The smoothing of the two-dimensional model potential allows to consider a two-dimensional flat disk as a three-dimensional disk of a finite small thickness $\epsilon$. In this case the relaxation time does depend on the number of particles [10] [4]:

$$T_{rel} \propto N.$$ (5)

The relaxation time in galaxies is greater than the Hubble time. In numerical simulations using the number of particles smaller by orders of magnitude than those in real galaxies, one has to find an estimate of the relaxation time and use it to set the integration time of the model.

The present work studied the equilibrium two-dimensional generalized polytropic models of stellar disks with the help of simulation. The models are described in detail in Section 2. Here we will focus on their most important property. Such models have constant degree of anisotropy, $\beta$, in the entire disk

$$\beta = \frac{\sqrt{\sigma_z^2(r)}}{\sqrt{\sigma_r^2(r)}} - 1$$

(6)

where $\sigma_z^2(r)$ and $\sigma_r^2(r)$ are the average squared velocities of stars at given radius $r$ in the azimuthal and radial directions, respectively; $\beta = 0$ corresponds to the isotropic velocity distribution, and $\beta = -1$ to strictly radial orbits. Parameter $\beta$ is chosen within minus one to zero at the initial stage of the model construction. Earlier, such models were considered by [9]. The authors found the instability boundary for $\beta \approx -0.5$. The present work revises the result completely. With models of high spatial resolution as an example, such systems are shown to be unstable at any degree of anisotropy of orbits, $\beta$. To test the ROI theory, we have evaluated the initial distributions of particle precession rates and shown that all conditions for the onset of the ROI are fulfilled in such systems.
2. Construction of equilibrium generalized polytropic model of a stellar disk

Generalized polytropic models are defined by the following phase density profile:

\[ f(E, L) = A \Theta(L_z) L^\beta (E_0 - E)^\alpha. \quad (7) \]

Here \( L \) is the angular momentum of a star \( (L = |\vec{L}| = |\vec{r} \times \vec{V}|) \), \( \Theta(L_z) \) is the Heaviside function, \( E \) is the total energy of a star \( (E = \Phi(r) + \frac{V^2}{2}) \), \( E_0 = \Phi(R) \) is the potential energy at the system boundary, \( R \) is the system size, \( V \) is the star velocity, \( r = |\vec{r}| \) is the star radius vector magnitude, \( \alpha, \beta \) are dimensionless parameters, \( A \) is the proportionality factor, dependent on the total system mass.

The model construction proceeded in several stages. It started with finding gravitational potential \( \Phi(r) \) to match the phase density profile Eq. (7) and involved one-parameter family of models where \( \beta = -\alpha/2 \) (for details see [9]). The model sizes were derived from the condition of the potential function being monotone. Then, particles were distributed in the coordinate and velocity spaces by methods of acceptance-rejection and the inverse transform. The initial coordinates and velocities of particles were chosen so that the model would remain planar in the course of simulations. Here, all particles have the same mass \( m = M_{\text{tot}}/N \) (where \( M_{\text{tot}} \) is the total system mass, \( N \) is the number of particles) and rotate counterclockwise. To simplify the calculations, the system of units was used, where \( G = 1 \), \( A \cdot C \cdot G = 1 \), \( \Phi(R) - \Phi(0) = 1 \).

Table 1 gives the resulting model sizes for different \( \beta \). Fig. 1 shows the view of generalized polytropic models in the \( xy \) plane for different degrees of anisotropy, \( \beta \), and \( N = 10^5 \). Fig. 2 shows the mass density profiles for the corresponding models, which were evaluated by two methods: velocity space integration of phase density Eq. (8) for the corresponding \( \beta \) [9]

\[ \Sigma(r) = A C r^\beta (E_0 - \Phi(r))^{\beta/2 + \alpha + 1}, \quad (8) \]

and summation of particle masses over concentric layers in numerical implementations of models.

| \( \beta \)  | 0.0  | -0.1 | -0.2 | -0.3 | -0.4 | -0.5 |
|----------|------|------|------|------|------|------|
| \( R \)  | \( 4.38 \cdot 10^{-1} \) | \( 3.51 \cdot 10^{-1} \) | \( 2.63 \cdot 10^{-1} \) | \( 1.78 \cdot 10^{-1} \) | \( 1.03 \cdot 10^{-1} \) | \( 4.63 \cdot 10^{-2} \) |

3. Solution of the \( N \)-body problem

For simulations we used the \( NEMO \) package [13] and its extension gyrfalcON [6]. The \( NEMO \) package allows numerical solution of particle motion equations and fast evaluation of gravitational forces, acting between particles. “Fast” in this case means that the calculation of forces in \( NEMO \) employs an algorithm with computing effort less than that of direct summation. Namely, the treecode algorithm [2] has the complexity \( O(N\log N) \) and its combination with fast multipole expansion [5] has the complexity \( O(N) \). Smoothing parameter \( \varepsilon \) was chosen in the optimal manner [8]. The integration step was adjusted to the chosen smoothing parameter [12].

\[ dt = \frac{\varepsilon_{\text{opt}}}{5v_r^{1/2}}, \quad (9) \]

where \( v_r^{1/2} \) is the escape velocity from the radius of the circle, containing a half of the model mass.
Figure 1. View of generalized polytropic models in the xy plane for different degrees of anisotropy, $\beta$, and $N = 10^5$.

Figure 2. Disk surface density Eq. (8) for $\beta = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ (dash lines) as a function of distance, normalized to the system size, $\tilde{r} = r/R$. Different symbols plot the density values, found by summation of particle masses over radial system layers at $N = 10^5$.

4. Bar registration
A bar was registered with the help of the Fourier transform of the density profile:

$$\bar{\Sigma}(p, m, t) = \frac{1}{N} \sum_{j=1}^{N} \exp \{i [p \ln(r_j) + m \varphi_j] \},$$

where $m$ is the azimuthal wavenumber, and $p$ is the radial wavenumber. The bar is identified as the harmonic with azimuthal number $m = 2$ radial number $p = 0$. Position angle $\alpha(t)$ and bar pattern speed $\Omega_p$ were found from the following relationships:

$$\tan \alpha(t) = \frac{\text{Im}(\bar{\Sigma}(2, 0, t))}{\text{Re}(\bar{\Sigma}(2, 0, t))}, \quad \sin \alpha(t) = \frac{\text{Im}(\bar{\Sigma}(2, 0, t))}{|\bar{\Sigma}(2, 0, t)|}, \quad \Omega_p = \frac{1}{m} \frac{d\alpha(t)}{dt}.$$
5. Instability of generalized polytropic models of stellar disks
Work [9] dealt with similar models for $\beta = 0.4$ and $\beta = 0.6$. Basing on the Fourier analysis of the density, the authors concluded the stability of models at $\beta > 0.5$, whereas a bar formed at $\beta = 0.5$. However, the authors used a small number of particles $N \approx 10^3$ and disregarded the possible numerical relaxation of the models. The present work evaluated the relaxation times of generalized polytropic models at different $\beta$ (see Fig. 3), basing on the analysis of deviations of test particles in a fixed potential of the numerical implementation of the model [1]. It follows from our estimates that at the number of particles, considered in [9], various effects arising due to the discrete initial distribution manifest themselves even before the bar formation. Our calculations of the evolution of generalized polytropic models with great number of particles $N = 10^5$ and $N = 5 \cdot 10^5$ for different $\beta$ have shown that the models prove to be unstable at any degree of orbit elongation, $\beta$, and the bar forms (see Fig. 4).

![Figure 3](image.png)

Figure 3. Relaxation time $T_{\text{rel}}$ vs. the number of particles in the system, $N$, at different values of $\beta$, found by deviations of particles in the fixed initial potential of the numerical implementation of the model [1].

6. The initial set of orbits
The present work tested the ROI conditions for generalized polytropic models of stellar disks. The initial distributions of precession rate $\Omega_{\text{pr}}$, adiabatic invariants $J_f$, orbit eccentricities $e$, and dependence $\Omega_{\text{pr}}(L)$ were found from the analysis of particle motion in a fixed initial potential. Fig. 5 shows the ratio of the average particle precession and revolution rates for given eccentricities and the number of particles with given eccentricity. Even in the most isotropic case at $\beta = 0$, most orbits are highly elongated. In addition, the greater the orbit eccentricity, the better the fulfillment of condition of a small precession rate, which is necessary for the onset of the ROI. Fig. 6 plots precession rate isolines in “adiabatic invariant – angular momentum” coordinates. The direction of the precession rate growth corresponds exactly to the second ROI condition. Similar dependences are observed for all other considered models.

7. Conclusions
Numerical simulations of the evolution of generalized polytropic models of stellar disks with great number of particles reveal that the models are unstable with respect to the bar-like perturbation. At that, the initial distributions of particle precession rates fulfill all necessary conditions for the onset of the ROI, which were formulated by Lynden-Bell [7]. Namely, almost all particles prove to
Figure 4. Evolution of the logarithm of amplitude magnitude of the Fourier transform of the disk surface density, $\ln|\tilde{\Sigma}(p, m = 2, t)|$, for models with different $\beta$. Heavy lines are the profiles of the density Fourier transform, which correspond to the final time instances ($T = 250$). The maximum of the density Fourier transform shifts with time to $p = 0$, which corresponds to the bar formation.

Figure 5. Average ratio of particle precession and revolution rates (heavy line) and the number of particles with given eccentricity (fine line) in each eccentricity layer with step $\Delta e = 0.01$ for the model with degree of anisotropy $\beta = 0$ and the number of particles $N = 10^5$.

Figure 6. Precession rate isolines for particles at degree of anisotropy $\beta = 0$. The adiabatic invariant is laid off as ordinate, and angular momentum is laid off as abscissa. The dash line corresponds to circular orbits ($J_f = L/2$). Arrows mark the direction of precession rate growth at a fixed adiabatic invariant.

have small precession rates and the rates to be a monotone function of the angular momentum at a fixed adiabatic invariant. Deriving the sufficient conditions, i.e. finding the resonant particles, precessing with the bar pattern speed in the self-consistent problem, is planned to be worked on later.
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