Quantification and Aggregation over Concepts of the Ontology

Pierre Carbonnelle  Matthias Van der Hallen  Marc Denecker
KU Leuven
Leuven, Belgium*
pierre.carbonnelle@kuleuven.be  matthias.vanderhallen@gmail.com  marc.denecker@kuleuven.be

We argue that in some KR applications, we want to quantify over sets of concepts formally represented by symbols in the vocabulary. We show that this quantification should be distinguished from second-order quantification and meta-programming quantification. We also investigate the relationship with concepts in intensional logic.

We present an extension of first-order logic to support such abstractions, and show that it allows writing expressions of knowledge that are elaboration tolerant. To avoid nonsensical sentences in this formalism, we refine the concept of well-formed sentences, and propose a method to verify well-formedness with a complexity that is linear with the number of tokens in the formula.

We have extended FO(·), a Knowledge Representation language, and IDP-Z3, a reasoning engine for FO(·), accordingly. We show that this extension was essential in accurately modelling various problem domains in an elaboration-tolerant way, i.e., without reification.

1 Introduction

The power of a KR language for compact expression of knowledge lies for an important extent in the way it allows us to abstract over certain types of objects, e.g., to quantify, count, or sum over them. Classical first-order logic (FO) is a much stronger KR language than Propositional Calculus (PC) because it allows to quantify over domain objects. Still, the abstraction power of FO is limited, e.g., it does not allow counting or sum over domain objects satisfying some condition. Many first-order modeling languages are therefore extended to support aggregations. Likewise, one can only quantify over individuals in the domain, not over relations and functions. This is resolved in second-order logic (SO).

In this paper, we argue that in some KR applications, we want to abstract (i.e., quantify, count, sum, . . . ) not over relations and functions, but over certain concepts in the problem domain, as the following example will show.

Example 1 Consider a corona testing protocol. A person is to be tested if she shows at least two of the following symptoms: fever, coughing and sneezing. The set of persons showing a symptom is represented by a unary predicate ranging over persons, i.e., hasFever/1, coughs/1, and sneezes/1. Leaving first-order logic behind us, assume that the set of symptoms is represented by the predicate symptom/1.

Testing is expressed as a unary predicate test1/1, ranging over persons. Informally, this predicate is to be defined along the following lines (while freely extending the syntax of FO):

\[ \forall p(\text{test1}(p) \iff 2 \leq \# \{ x | \text{symptom}(x) \land x(p) \} ) \]

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The question is: what is the nature of the variable \( x \) and the predicate \( \text{symptom} \)? Over what sorts of values does \( x \) range?

At first sight, one might think that \( x \) is a second-order variable ranging over the set of sets of persons, and that we want to count the sets of persons having one of the symptoms and containing person \( p \). A ‘possible world’ analysis refutes this idea. Consider a state of affairs where everybody has all 3 symptoms. Formally, in a structure \( \mathcal{S} \) abstracting such a state, the interpretation of each symptom predicate is identical: it is the set of all persons, \( \text{hasFever}^{\mathcal{S}} = \text{coughs}^{\mathcal{S}} = \text{sneezes}^{\mathcal{S}} = \{ y \mid \text{person}(y) \} \). Thus, there is only one set that contains person \( p \): the cardinality in Eq. (1) would be one, and the condition for the first test would not be satisfied for any \( p \), even though everyone should be tested. A different idea is that \( x \) is ranging over \( \text{symbols} \), and that we want to count the number of symptom symbols whose interpretation contains person \( p \). This sort of variable is generally found in meta programming, of the kind investigated in, e.g., Logic Programming [4] or Hilo [10]. While this view has merits, it is not totally satisfying. Assume, for insight, that the team working on this application is international, and the French group has introduced the predicate \( \text{estFievreux}/1 \), French for “has fever”. Thus, \( \text{hasFever}/1 \) and \( \text{estFievreux}/1 \) are \( \text{synonymous} \). Consider \( \text{John} \) whose only symptom is fever. Both \( \text{isFievreux}(\text{John}) \) and \( \text{hasFever}(\text{John}) \) are true. The number of symptom symbols whose interpretation contains \( \text{John} \) would be two, leading to the erroneous decision that \( \text{John} \) needs to be tested.

The view that we elaborate in this paper, is that the abstraction in this example is over \( \text{concepts} \). For context, consider that the first phase of a rigorous approach to building a KR specification is the selection of a formal ontology \( \Sigma \) of symbols representing relevant \( \text{concepts} \) of the application field. These concepts can be identified with the user’s \( \text{informal interpretations} \) of the symbols in the ontology of the domain. The informal interpretation is in practice a crucial concept in all knowledge representation applications: it is the basis of all actions of formal knowledge representation, and of all acts of interpreting formal results of computation in the application domain. Connecting concepts to informal interpretations provides a good intuitive explanation of what concepts are and how they are relevant in a KR context.

In classical first-order logic, a structure only has the extension of symbols. The extension of symbol \( \sigma \) in a structure is its \( \text{formal} \) interpretation, a.k.a. a value, in that structure. It is a relation or function, of a precise arity, over the universe of discourse; in formulae, it is denoted simply by \( \sigma \).

For our purpose, we extend a structure to also have the intension of symbols, i.e., an object in the domain of discourse representing its \( \text{informal} \) interpretation. This intension is \( \text{rigid} \) in the sense that it is the same in every extended structure. Since the informal meaning of symbols is known, the synonym relation \( \sim_s \) is known too, and it is an equivalence relation on \( \Sigma \). Synonymous symbols have the same arity and type. Synonymous symbols \( \sigma_1, \sigma_2 \) have the same intension, and, in each structure \( \mathcal{S} \), the same extension. In formulae, the intension of \( \sigma \) is denoted by ‘\( \sigma \)’. Finally, we introduce the \( \text{value functor} \) \( \$(\cdot)\) when applied to an intension, it returns its formal interpretation in the current structure.

In the example, the intension of the symbols \( \text{hasFever}/1 \) and \( \text{estFievreux}/1 \) are the same object denoted by ‘\( \text{hasFever} \)’ (or ‘\( \text{estFievreux} \)’) which is the abstraction of the concept of people having fever. The symbol \( \text{symptom} \) denotes the set containing the intensions ‘\( \text{hasFever} \)’, ‘\( \text{coughs} \)’ and ‘\( \text{sneezes} \)’. In any particular structure, the \( \text{hasFever}/1 \) symbol has an interpretation which is a set of elements abstracting the set of people having fever in the state of affairs. This set is also the value of the ‘\( \text{hasFever} \)’ intension, i.e., of \( \$(\text{hasFever}) \). Suppose we want to count the concepts in \( \text{symptom} \) whose value (a set) contains person \( p \). The following formula expresses the testing protocol:

\[
\forall p (\text{test}1(p) \iff 2 \leq \# \{ x \mid \text{symptom}(x) \land \$(x)(p) \})
\]

(2)

In the current state of the art of Knowledge Representation, one might consider, as we once did, applying the common KR technique of \( \text{reification} \), rather than laboriously extending KR languages as
we do. In this particular case, one could introduce object symbols Fever, Sneezing, Coughing (under the Unique Name Assumption), and a predicate, Has/2. We can now represent that Bob has fever by the atom Has(Bob, Fever). The testing protocols can then be formalized (in a proper extension of FO) as:

\[
\forall x (\text{symptom}(x) \iff x = \text{Fever} \lor x = \text{Sneezing} \lor x = \text{Coughing})
\]

(3)

\[
\forall p (\text{test1}(x) \iff 2 \leq \#\{x | \text{symptom}(x) \land \text{Has}(p, x)\})
\]

(4)

Still, we believe that our study is valuable for two reasons. First, we improve the scientific understanding of the problem, thus helping avoid using KR encoding tricks to circumvent it. Second, our logic is more elaboration tolerant. Per [26], “a formalism is elaboration tolerant to the extent that it is convenient to modify a set of facts expressed in the formalism to take into account new phenomena or changed circumstances”. Because it helps reduce development costs, elaboration tolerance is a desirable feature of formalisms used in AI applications, and in particular in knowledge representation. Consider a knowledge base that, initially, does not contain rule (1). Per standard KR practice, it would use predicates such as hasFever/1 to encode symptoms. Then, circumstances change, and rule (1) must be introduced. A formalism that requires rewriting the knowledge base using reified symptoms Has/2 is not elaboration tolerant. Our formalism, by contrast, would not require any change to the initial knowledge base.

To evaluate our approach, we have extended FO(·) to allow quantification and aggregation over concepts, and extended IDP-Z3, a reasoning engine for FO(·), accordingly. It proved essential to accurately model the knowledge of various problem domains in an elaboration tolerant way, i.e., without reification.

To summarize, the main contributions of the paper are as follows. We argue that in some KR applications, it is useful to abstract over concepts in quantification and aggregations. We analyze the connection to well-known intensional logic (Section 2), and explain how to distinguish this from second-order quantification and meta-programming (Section 6). We explain how to extend a simple predicate logic so that every ontology symbol has an intensional and an extensional component (Section 3). We show that this may lead to nonsensical sentences, and we restrict our syntax to avoid them. In Section 4 we extend FO(·), a model-based knowledge representation language, and IDP-Z3, a reasoning engine for FO(·), as we propose, and illustrate the benefits of this extension in different applications (Section 5).

2 “Concepts” in Logic

The notions that we study here, i.e., the notions of a “concept” and of its “value” in a particular state of affairs, are closely related, if not identical, to the broadly studied distinction between intension and extension, meaning and designation, or sense and reference as studied in philosophical and intensional logic. The close relation between the two urges us to detail the role of concepts or “intensions” in intensional logic, and to make the comparison with our approach.

In philosophical logic, the concepts of intension and extension have been investigated at least since Frege (for a historical overview, see [22]). A prototypical example concerns the “morning star” and “evening star”, i.e., the star visible in the east around sunrise, and in the west around sunset. These words have two different intensions (i.e., they refer to two different concepts), but as it happens, in the current state of affairs, their extensions (i.e., their interpretation) are the same: the planet also known as Venus.

Early key contributors in this study are Frege, Church, Carnap and Montague. The intensional logics of Montague, of Tichý, and of Gallin, are typed modal logics based on Kripke semantics in which expressions at any level of the type hierarchy have associated intensions. They provide notations to access the
intension and the extension of expressions. They also provide modal operators to talk about the different values that expressions may have in the current and in accessible worlds.

The principles of intensions of higher-order objects are similar to those of the base level (i.e., domain elements), leading Fitting in several papers [21, 22] to develop a simplified logic, called FOIL, where only expressions at the base level of the type hierarchy have intensions.

In intensional logic, the intension of an expression is modelled as a mapping from possible worlds to the interpretation of the expression. For example, the intension of EveningStar is the mapping from possible worlds to the extension of EveningStar in that world. We can express that the extension of MorningStar in the current world, equals the extension of EveningStar in a (possibly different) accessible world:

$$\lambda x(\diamond (\text{EveningStar} = x)))(\text{MorningStar})$$

Here the lambda expression binds variable x to the extension of MorningStar in the current world, and the “\(\diamond\)” modal operator is used to indicate an accessible world. This statement would be true, e.g., in the view of a scientist that may not have evidence that both are the same but accepts it as a possibility.

A comparison between the work presented in this paper and the intensional logics in philosophical logic, is not easy. Even if the notion of intension versus extension is underlying the problems that we study here, the focus in our study is on very different aspects of intensions than in philosophical logic. As a result, the logic developed here differs strongly from intensional logics.

An important difference is the lack of modal logic machinery in our logic to analyze the difference between intensions and extensions. In our logic, a symbol has an extension (a.k.a interpretation) in different possible worlds, but there are no modal operators to “talk” about the extension in other worlds than the current one.

Another important difference is that we provide an abstraction mechanism through which we can quantify over, and count, intensional objects (a.k.a concepts). In KR applications, problems such as the symptom example easily occur (see Section 5). They can be analyzed and demonstrated and solved in a logic much closer to standard logic.

Compared to FOIL, we associate intensions to predicate and function symbols of any arity even though, as hinted earlier, it poses additional challenges to ensure the syntactical correctness of formulae. We will address these by introducing guards. By contrast, FOIL associates intensions only to intensional objects of arity 0, like Morningstar, avoiding the issue. Finally, our research also includes the extension of an existing reasoning engine for the proposed logic, capable of reasoning with aggregates over concepts such as in Eq.(3).

### 3 FO(Concept)

Below, we describe how first-order logic, in its simplest form, can be extended to support quantification over concepts. The language, FO(Concept), is purposely simple, to explain the essence of our ideas. We present the syntax [3.1] of the extension, its semantics [3.2], and discuss its complexity and some alternative formulations [3.3].

#### 3.1 Syntax

A vocabulary \( \Sigma \) is a set of symbols with associated arity. We want to extend the FO syntax of terms over vocabulary \( \Sigma \) with these four new construction rules:

- \( n \) is a term if \( n \) is a numeral, i.e., a symbol denoting an integer;
• \#\{x_1, \ldots, x_n : \phi\} is a term if x_1, \ldots, x_n are variables and \phi a formula;
• \sigma is a term if \sigma \in \Sigma,
• S(x)(t_1, \ldots, t_n) is a term if x is a variable, and t_1, \ldots, t_n are terms over \Sigma.

The last rule is problematic, however. In a structure \mathcal{J} extended as we propose, \sigma ranges not only on the domain of \mathcal{J}, but also over the set of intension of the symbols in \Sigma: when the value of x is such an intension, S(x) is the extension of the associated symbols; but when x is not such an intension, S(x) is undefined. Furthermore, the value assigned to x may be the intension of a predicate symbol: in that case, S(x)(t_1, \ldots, t_n) is not a term. Finally, the value of x may be the intension of a function of arity m \neq n: in that case, S(x)(t_1, \ldots, t_n) is not a well-formed term.

For example, S(x)() is not a well-formed term in the following cases (among others):

- [x = 1] where 1 denotes a numeric element of the domain of discourse;
- [x = 'p] where p is a predicate symbol;
- [x = 'f] where f is a function symbol of arity 1.

Essentially, whereas in FO, arities of function and predicate symbols are known from the vocabulary, and taken into account in the definition of well-formed composite terms and atoms by requesting that the number of arguments matches the arity, this now has become impossible due to the lack of information about S(x). Thus, to define well-formed terms, we need additional information about the variables occurring in them. We formalize that information in a typing function.

We call \gamma a typing function if it maps certain variables x to pairs (k, n) where k is either PRED or FUNC, and n is a natural number. Informally, when a variable x is mapped to (k, n) by \gamma, we know that x is a concept, of kind \kappa and arity \eta. For a given typing function \gamma, we define \gamma[x : (k, n)] to be the partial function \gamma’ identical to \gamma except that \gamma’(x) = (k, n).

This allows us to define the notion of well-formed term and formula:

**Definition 1 (Well-formed term)** We define that a string e is a well-formed term over \Sigma given a typing function \gamma (denoted \gamma \vdash_e e) by induction:

- \gamma \vdash_e x if x is a variable;
- \gamma \vdash_e f(t_1, \ldots, t_n) if f is an n-ary function symbol of \Sigma and for each i, \gamma \vdash_e t_i;
- + \gamma \vdash_e n if n is a numeral, a symbol denoting the corresponding natural number n;
- \gamma \vdash_e \#\{x_1, \ldots, x_n : \phi\} if x_1, \ldots, x_n are variables and \gamma \vdash_e \phi;
- \gamma \vdash_e \sigma if \sigma \in \Sigma;
- \gamma \vdash_e S(x)(t_1, \ldots, t_n) if x is a variable, \gamma(x) = (\text{FUNC}, n) and for each i \in [1, n], \gamma \vdash_e t_i.

(The rules with a “+” bullet are those we add to FO’s definition of terms.)

**Definition 2 (Well-formed formula)** We define that a string \phi is a well-formed formula over \Sigma given \gamma (denoted \gamma \vdash \phi) by induction:

- \gamma \vdash \top, \gamma \vdash \bot;
- \gamma \vdash p(t_1, \ldots, t_n) if p is an n-ary predicate of \Sigma and for each i \in [1, n], \gamma \vdash t_i;
- \gamma \vdash (\neg \phi) if \gamma \vdash \phi;
- \gamma \vdash (\phi \lor \psi) if \gamma \vdash \phi and \gamma \vdash \psi;
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. $\gamma \vdash_f \exists x : \phi$ if $x$ is a variable and $\gamma \vdash_f \phi$;

. $\gamma \vdash_f t_1 = t_2$ if $\gamma \vdash_f t_1$ and $\gamma \vdash_f t_2$;

+ $\gamma \vdash_f t_1 \leq t_2$ if $\gamma \vdash_f t_1$ and $\gamma \vdash_f t_2$ and $t_1$ and $t_2$ are numeric terms or cardinality aggregates;

+ $\gamma \vdash_f (\text{if } x :: k/n \text{ then } \phi \text{ else } \psi)$ if $x$ is a variable, $k$ is either \textsc{pred} or \textsc{func}, $n$ is a natural number, $\gamma \vdash_x (k,n)$ $\vdash_f \phi$ and $\gamma \vdash_f \psi$;

+ $\gamma \vdash_f \#(x)_{(t_1,\ldots,t_n)}$ if $x$ is a variable, $\gamma(x) = (\text{pred},n)$ and for each $i \in [1,n]$, $\gamma \vdash (t_i)$.

Notice that, because of the aggregate term rule, these definitions are mutually recursive. Also, due to their constructive nature, well-formed terms and formulae can be arbitrarily large, but always finite.

Formulae of the form $\phi \land \psi, \phi \Rightarrow \psi, \phi \Leftrightarrow \psi, \forall x : \phi$ are shorthand for these formulae, and are not further discussed:

$\neg(-\phi \lor -\psi), -\phi \lor \psi, (\phi \land \psi) \lor (-\phi \land -\psi), -\exists x : -\phi$

The other comparison operators, $<, >, \geq$, can be defined similarly.

Let $\emptyset$ be the typing function with empty domain. We say that $\phi$ is a \textit{well-formed formula} over $\Sigma$ if $\emptyset \vdash_f \phi$.

Example 2 Here is a well-formed version of the cardinality sub-formula of Equation 2:

\[
\#\{x|\text{if } x :: \text{pred}/1 \text{ then Symptom}(x) \land \#(x)(p) \text{ else false}\}
\]

We say that $\text{Symptom}(x) \land \#(x)(p)$ is \textit{guarded} by the $x :: \text{pred}/1$ condition, and that the formula is \textit{well-guarded}.

Note that, in this logic, a value $\#(x)$ cannot occur in a formula without immediately being applied to a tuple of arguments.

### 3.2 Semantics

\textbf{Definition 3 (Intensional ontology and equivalence class $|\sigma|_\emptyset$)} We define an intensional ontology $\emptyset$ as a pair of a vocabulary $\Sigma$ and a synonym relation $\sim_s$ between symbols of same arity in the vocabulary. We denote the equivalence class of symbol $\sigma$ in $\emptyset$ by $|\sigma|_\emptyset$, and the set of such equivalence classes (or witnesses) by $C_\emptyset$.

In the philosophical papers about intensional logic, the set of concepts is open. Here, however, we want to quantify only over the concepts that are interpretations of symbols in $\Sigma$, not over any concepts. In essence, we want to quantify over $C_\emptyset$.

Hence, to define the semantics of FO(Concept), we extend the notion of a structure to include $C_\emptyset$ and an additional mapping from concepts to their value.

\textbf{Definition 4 (Total structure)} A (total) structure $\mathcal{I}$ over ontology $\emptyset$ consists of:

. an object domain $D$ containing the set of natural numbers $\mathbb{N}$,

. a (total) mapping from predicate symbols $p/n$ in $\Sigma$ to $n$-ary relations $p^\mathcal{I}$ over $D \cup \mathcal{C}_\emptyset$,

. a (total) mapping from function symbols $f/n$ in $\Sigma$ to $n$-ary functions $f^\mathcal{I}$ over $D \cup \mathcal{C}_\emptyset$,

+ a (total) mapping $\#^\mathcal{I}$ from concepts in $\mathcal{C}_\emptyset$ to relations and functions over $D \cup \mathcal{C}_\emptyset$. 

We partially define the truth value \(\nu\) of well-formed terms.

**Definition 6 (Value of a term)** We partially define the value \(\nu\) of well-formed \(t\) in \((\mathcal{I}, \nu)\) (denoted \(\nu t = \nu\)) by induction:

- \(\nu x = \nu(x)\) if \(x\) is a variable in the domain of \(\nu\);
- \(\nu f(t_1, \ldots, t_n) = \nu f([t_1]_\nu, \ldots, [t_n]_\nu)\) if \(f\) is an \(n\)-ary function symbol and \([t_1]_\nu, \ldots, [t_n]_\nu\) are defined;
- \(\nu n = \nu n\) if \(n\) is the integer denoted by \(n\);
- \(\nu \#\{x_1, \ldots, x_n : \phi\} = \nu m\) if \([\phi]_{\nu[x_1/a_1, \ldots, x_n/a_n]} \) is defined for every \(a_1, \ldots, a_n \in D \cup \mathcal{C}_\mathcal{O}\), \(m\) is an integer, and \(\#\{x_1, \ldots, x_n : \phi\} = \nu t\) if \([x]_\nu\) is a concept in \(\mathcal{C}_\mathcal{O}\) mapped by \(\nu\) to an \(n\)-ary function, and \([t_1]_\nu, \ldots, [t_n]_\nu\) are defined;
- \(\nu [t] = \nu [t]\) if \(\nu [t] = \nu [t]\) is undefined in all the other cases.

**Definition 7 (Truth value of a formula)** We partially define the truth value \(\nu\) of well-formed \(\phi\) in \((\mathcal{I}, \nu)\) (denoted \(\nu \phi = \nu\)) by induction:

- \(\nu t = \nu t\); \(\nu f = \nu f\);
- \(\nu p(t_1, \ldots, t_n) = \nu p([t_1]_\nu, \ldots, [t_n]_\nu)\) if \(p\) is an \(n\)-ary predicate symbol and \([t_1]_\nu, \ldots, [t_n]_\nu\) are defined;
- \(\nu \neg \phi = \neg \nu \nu [\phi] = \nu \phi\) if \([\phi]_\nu\) is defined;
- \(\nu \phi \lor \psi = \nu \nu \nu [\phi] = \nu \phi\lor \nu [\psi] = \nu \psi\) if \([\phi]_\nu\) and \([\psi]_\nu\) are defined;
- \(\nu \exists x : \phi = \nu \nu \exists d \in D \cup \mathcal{C}_\mathcal{O} : \nu [\phi]_{[x/a]} = \nu [\phi]_{[x/a]}\) if \([\phi]_{[x/a]} = \nu [\phi]_{[x/a]}\) is defined for every \(d \in D \cup \mathcal{C}_\mathcal{O}\);
- \(\nu [t_1 = t_2] = \nu [t_1]_\nu = \nu [t_2]_\nu = \nu [t_2]_\nu\) if \([t_1]_\nu\) and \([t_2]_\nu\) are defined;
- \(\nu [t_1 \leq t_2] = \nu [t_1]_\nu = \nu [t_2]_\nu = \nu [t_2]_\nu\) if \([t_1]_\nu\) and \([t_2]_\nu\) are defined and integers;
- \(\nu [\text{if } x : k/n \text{ then } \phi \text{ else } \psi] = \nu [\phi] = \nu [\psi]\) if \([x]_\nu\) is a concept in \(\mathcal{C}_\mathcal{O}\) mapped by \(\nu\) to an \(n\)-ary predicate or function according to \(k\), and \([\phi]_\nu\) is defined;
- \(\nu [\text{if } x : k/n \text{ then } \phi \text{ else } \psi] = \nu [\psi] = \nu [\psi]\) if \([x]_\nu\) is not a concept in \(\mathcal{C}_\mathcal{O}\) mapped by \(\nu\) to an \(n\)-ary predicate or function according to \(k\), and \([\psi]_\nu\) is defined;
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The truth value \( [\phi]_\nu \) is defined for every well-formed formula \( \phi \) over \( \Sigma \), every total structure \( \mathcal{I} \) over \( (\Sigma, \sim) \), and every variable assignment \( \nu \) that assigns a value to all free variables of \( \phi \).

This can be proven by parallel induction over the definitions of well-formed formulae (and terms) and of their value, applying the properties of \( \mathcal{I} \) over \( (\Sigma, \sim) \), and, for formulae having sub-formulae, of determining the well-formedness of each sub-formula; formula \( \$x(t_1, \ldots, t_n) \) requires an additional lookup of \( x \) in \( \gamma \). Notice that, at each step, the complexity of the reasoning does not depend on the size of the domain of discourse. Bounding the cost of both the syntactical analysis and of the variable lookup by \( \alpha \), the cost \( C \) of the computation for any formula \( \phi \) having sub-formulae \( \phi_i \) is bounded as follows: \( C(\phi) \leq \alpha + \sum_i C(\phi_i) \leq \alpha \times N(\phi) \), where \( N(\phi) \) is the number of tokens in the formula \( N(\phi) = 1 + \sum_i N(\phi_i) \). Thus, the complexity is linear with the number of tokens in the formula.

The complexity of decision and search problems in FO(Concept) is the same as in FO. This is because the domain \( D \) of structures is extended only with a fixed, constant-sized set \( \mathcal{C}_\phi \) of elements (and not over higher-order objects). For example, deciding the existence of a model of an FO (resp. FO(Concept)) theory with an input domain \( D \) (resp. \( D \cup \mathcal{C}_\phi \)) is an NP problem measured in the size of \( D \) (resp. \( D \cup \mathcal{C}_\phi \)).

FO could be extended to support quantification over concepts in other ways than the one presented above. Instead of the \texttt{if} .. \texttt{then} .. \texttt{else} .. construct, we could use \texttt{∨} (and \texttt{∧}) with non-strict evaluation. Well-guarded quantifications would then be written as follows:

\[
\exists x((x :: \text{PRED}/1) \land \$x(p)) \quad (6)
\]
\[
\forall x((x :: \text{PRED}/1) \implies \$x(p)) \quad (7)
\]

Also, instead of giving partial definitions of values (Def. 6 and 7), we could give total definitions by assigning an arbitrary value when the term (resp. formula) is undefined. We would then show that this arbitrary value is not relevant in well-formed formula.

4 Extending FO(\( \cdot \)) with intensions

We now discuss how we extended the Knowledge Representation language called FO(\( \cdot \)) \cite{17, 16} to support quantification over intensions.

FO(\( \cdot \)) is first-order logic extended with language constructs to make it more expressive:
+ **types**: the vocabulary may include custom types, in addition to the built-in types (e.g., `Int`, `Bool`). Each symbol has a type signature of the form `T1 × ... × Tn → T` specifying their domain and range. The range of a predicate is the set of Booleans `Bool`. Formula must be well-typed, i.e., predicates and functions must be applied to arguments of the correct type.

+ **equality**: `t1 = t2` is a formula, where `t1` and `t2` are terms of the same type.

+ **arithmetic over integers and rationals**: arithmetic operators (`+`, `-`, `*`, `/`) and comparisons (e.g., `≤`) are interpreted functions.

+ **binary quantification**: `∃ x ∈ P: p(x)`. (where `P` is a type or predicate) is equivalent to `∃ x: P(x) ∧ p(x)`. 

+ **aggregations**, such as `#{x ∈ P: p(x)}` (count of `x` in `P` satisfying `p`) and `sum{f(x) | x ∈ P}` (sum of `f(x)` over `x` in `P`).

+ **(inductive) definitions** [17]: FO(·) theory consists of a set of logic sentences and a set of (potentially inductive) definitions. Such a definition is represented as a set of rules of the form:

  \[ ∀ x_1 \in T_1, \ldots, x_n \in T_n: p(x_1, \ldots, x_n) \leftarrow F. \]

where `F` is an FO(·) formula.

Thanks to the support of inductive definitions, FO(·) is unique in combining the expressivity of classical logic and logic programming [16]. The reference manual of FO(·) is available online [1].

A Knowledge Base (KB) written in FO(·) cannot be run: like human knowledge, it is just a “bag of information”, formally describing models in a problem domain. Knowledge bases do not distinguish inputs from outputs, and allow reasoning in any direction. They are used to perform a variety of reasoning tasks (each with a particular set of inputs and outputs), using generic methods provided by reasoning engines, such as IDP-Z3 [2] and FOLASP [20]: they can find relevant questions to solve a particular problem, derive the consequences of new information, explain how they derived these consequences, find possible models, and find a model that minimizes a cost function [15].

These generic capabilities are used to easily build knowledge-based interactive systems that assist users in finding solutions to problems in a problem domain [13, 19].

To allow reasoning about concepts, we have extended FO(·) with the “’ ” operator (to refer to the intension of a symbol) and the “$( . )$” operator (to refer to the interpretation of a concept), as described in Section 3 for FO.

An issue arises in expressions of the form

\[ $(x)(t_1, \ldots, t_n) \]

`x` must be a `Concept`, the arguments `t_1, \ldots, t_n` must be of appropriate types and number for the predicate or function `$(x)`, and `$(x)(t_1, \ldots, t_n)` must be of the type expected by its parent expression.

To address this issue and to support the writing of well-guarded formulae, we have introduced types for the concepts having a particular type signature:

\[ Concept [T_1 \times \ldots \times T_n → T] \]

The interpretation of this type is the set of concepts with signature `T_1 \times \ldots \times T_n → T`. Note that `T_1, \ldots, T_n, T` themselves can be conceptual types.

The well-formedness and semantics of quantifications over a conceptual type:

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1. [https://fo-dot.readthedocs.io/](https://fo-dot.readthedocs.io/)
2. [http://idp-z3.be/](http://idp-z3.be/)
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\[ \exists x \in \text{Concept} [T_1 \times \ldots \times T_n \to T]: \; \$ (x)(t_1, \ldots, t_n). \]

is defined by extending the concept of guards (Section 3.1), and considering the following equivalent statement with guards:

\[ \exists x: \text{if } x :: [T_1 \times \ldots \times T_n \to T] \text{ then } \$ (x)(t_1, \ldots, t_n) \text{ else false.} \]

where \( x :: [T_1 \times \ldots \times T_n \to T] \) is a guard specifying the types of the arguments of \( x \), and the type of \( \$ (x)(t_1, \ldots, t_n) \). The definitions of the typing function \( \gamma \) (Section 3.1), and of well-formed formulae and their semantics are updated accordingly.

The syntax of FO(\( \cdot \)) allows applying the “\( \$(\cdot) \)” operator to expressions (not just to variables), e.g., in atom \( \$ (\text{expr}) \). The type of \( \text{expr} \) must be a conceptual type with appropriate signature.

We have updated the IDP-Z3 reasoning engine \(^3\) to support such quantification over concepts. IDP-Z3 transforms FO(\( \cdot \)) theories into the input language of the Z3 SMT solver \(^29\) to perform various reasoning tasks. A quantification

\[ \exists \; x \; \text{in Concept} [T_1 \times \ldots \times T_n \to T]: \; \text{expr}(x). \]

is transformed into a disjunction of \( \text{expr}(c) \) expressions, where \( c \) is a \( \text{Concept} [T_1 \times \ldots \times T_n \to T] \). Occurrences of \( \$ (c)(t_1, \ldots, t_n) \) within \( \text{expr}(c) \) are transformed into \( c(t_1, \ldots, t_n) \) where \( c \) is the symbol denoting concept \( c \) (Our implementation currently does not support synonymous concepts). The resulting formula is an FO sentence that can be submitted to Z3.

5 Examples in FO(\( \cdot \))

In our practice, we have identified a few examples where quantifications over concepts proved essential to accurately model the knowledge available within a domain in an elaboration tolerant way, i.e., without reification. These examples come from a broad range of applications. They are:

- the May 2021 DM Community challenge \(^4\) about deciding to perform additional testing of patients, based on a set of symptoms (similar to the example we used in the introduction);
- the “International Law to fight money laundering” example, where we want to represent the general rule that the national laws must be stricter than the EU directive.
- the “Word disambiguation” example, in which the word “child” in a statutory law could represent either the biological or legal child.
- the “template” example, in which intensional objects are used to define templates.

The examples are available online \(^5\) and can be run using the IDP-Z3 reasoning engine. Below, we discuss the International law example for illustration purposes.

Since 1990, the European Union has adopted legislation to fight against money laundering and terrorist financing. It creates various obligations for the parties in a business relationship, such as verifying the identity of the counter-party. The member states have to transpose the directive into national laws. The national laws must meet the minimum obligations set forth in the EU directive.

\(^29\) http://idp-z3.be/
\(^4\) https://dmcommunity.org/challenge/challenge-may-2021/
\(^5\) https://tinyurl.com/Intensions
In our simplified example, the EU directive requires the verification of identity in any transaction with a value above 1M€; a national law might set the threshold at 500K€ instead. Similarly, the EU directive might require a bank to send a report to their authority at least quarterly, but a country might require a monthly report.

Our goal is then to express the requirement that the national obligations are stricter than the EU ones. We choose an ontology in which “has a lower value” means “stricter”. We also use a mapping from the parameters of the national laws to their equivalent parameter in the EU law.

\[
\text{vocabulary} \{ \\
\quad \text{type} \text{ Country} \\
\quad \text{threshold}, \text{ period: Country } \rightarrow \text{ Int} \\
\quad \text{obligation: Concept[Country } \rightarrow \text{ Int} \rightarrow \text{ Bool} \\
\quad \text{thresholdEU, periodEU: } () \rightarrow \text{ Int} \\
\quad \text{mapping: Concept[Country } \rightarrow \text{ Int} \rightarrow \text{ Concept[() } \rightarrow \text{ Int]} \\
\}\n\]

\[
\text{theory} \{ \\
\quad \text{obligation := {‘threshold, ‘period}} \\
\quad \text{mapping := {‘threshold } \rightarrow \text{‘thresholdEU, ‘period } \rightarrow \text{‘periodEU}} \\
\]

// national law must be stricter than European law.
\[
\forall o \in \text{obligation}: \forall c \in \text{Country}: \$(o)(c) \leq \$(\text{mapping}(o))().
\]

Notice that an expression (not a variable) is applied to the value operator: \(\$(\text{mapping}(o))().\) Because \(o\) is a Concept[Country \(\rightarrow\text{Int}\)] by quantification, it is in the domain of mapping, and mapping \((o)\) has type Concept[()] \(\rightarrow\) Int. Its value is thus a nullary function of range Int.

6 Related Work

We discussed the relation between intensional logic and our work in Section 2. We now discuss the relation with other work.

6.1 Second-order quantification and relations

A contribution of this work is to clarify the utility of being able to quantify over concepts in the vocabulary and to show how it differs from quantifying over sets or functions as in second-order quantification. Certainly in the case of predicate or function intensions, it is our own experience that it is easy to confuse the two. We showed in the introduction the need for quantification over intensions as opposed to over second or higher-order objects (relations or functions). However, the quantification over intensions cannot replace quantification over second-order. A clear-cut example of second-order quantification and relation occurs in the graph mining problem [23].

Example 3 A graph homomorphism \(h\) is a function from the nodes of a graph, called the pattern \(p\), to nodes of a graph \(g\), that preserves the edges. The following expression intends to define \(\text{hom}/1\) as the set of homomorphisms of \(p\) in \(g\), where \(p(x,y)\) (resp. \(g(x,y)\)) denotes an edge between nodes \(x\) and \(y\).

\[
\forall h(\text{hom}(h) \iff \forall x\forall y(p(x,y) \leftrightarrow g(h(x), h(y))))
\]
The question is: over what sorts of values does $h$ range (in the context of a structure $J$)? Over the set of intensions of unary function symbols in $\Sigma$? There may be none! Or over the set of all binary functions from nodes of the pattern to nodes of the graph in $J$? Clearly, over the latter. Hence, $h$ is a second-order variable and $\text{hom}/I$ a second-order predicate symbol.

### 6.2 Metaprogramming

Variables ranging over concepts, as we propose, are similar to variables ranging over symbols, as found in meta programming. Indeed, meta programming also brings the benefit that we seek: formulating knowledge about concepts in an elaboration tolerant way (i.e., without reification).

Meta programming has been proposed in logic-based languages such as OWL [28 30], as well as in rule-based languages such as Prolog [4], HiLog [10], or webdamlog [1], but not in languages integrating the two types of formulation, such as FO(·). This integration is needed to support all the use cases that we describe. For example, the International Law example encodes a logic constraint that cannot be encoded in a rule-based system; the Template example encodes an inductive definition that cannot be encoded in logic-based systems. By contrast, our reasoning engine does support the two types of formulation, and all the use cases.

Furthermore, we formalize the guarding mechanism that ensures that sentences are well-formed. To the best of our knowledge, this mechanism is missing in all meta-programming implementations (and in particular in HiLog and webdamlog). Without this mechanism, the developer of a knowledge base cannot benefit from the automatic detection of syntactical errors.

### 7 Summary

As our examples illustrate, it is often useful in knowledge-intensive applications to quantify over concepts, i.e., over the intensions of the symbols in an ontology. An intension is an atomic object representing the informal interpretation of the symbol in the application domain.

First-order logic and FO(·) can be extended to allow such quantifications in an elaboration tolerant way. Appropriate guards should be used to ensure that formulae in such extensions are well-formed: we propose a method to verify such well-formedness with a complexity that is linear with the number of tokens in the formula.

While related to modal logic, the logic introduced here differs strongly from it. It has no modal operators to talk about the extension of a symbol in other worlds than the contextual one, but it offers mechanisms to quantify and count intensional objects.

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