On a local-global principle for $H^3$ of function fields of surfaces over a finite field.

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Abstract

Let $K$ be the function field of a smooth projective surface $S$ over a finite field $F$. In this article, following the work of Parimala and Suresh, we establish a local-global principle for the divisibility of elements in $H^3(K, \mathbb{Z}/\ell)$ by elements in $H^2(K, \mathbb{Z}/\ell)$, $\ell \neq \text{car}(K)$.

1 Introduction

Let $K$ be a field of one of two following types:

(i) $K$ is the function field of a smooth projective surface $S$ over a finite field $\mathbb{F}$ of characteristic $p$;

(ii) $K$ is the function field of a regular (relative) curve, proper over a ring of integers of a $p$-adic field.

Recall that for a field $k$, the $u$-invariant $u(k)$ is defined as a maximal dimension of an anisotropic quadratic form over $k$ [7]. Another arithmetical invariant which one can associate to $k$ is the period-index exponent: an integer $d$ (if it exists), such that for any element $\alpha$ of the Brauer group $Br k$ we have $\text{ind}\alpha | (\text{per}\alpha)^d$. Recall that $\text{ind}\alpha$ and $\text{per}\alpha$ have the same prime factors, so that for a fixed $\alpha \in Br k$ one can find an integer $d = d(\alpha)$ as above.

For $k = K$ of one of the types above, these invariants are now understood. More precisely, if $K$ is of type (ii), Saltman [14, 15] showed that $\text{ind}\alpha | (\text{per}\alpha)^2$ for $(\text{per}\alpha, p) = 1$. For $K$ of type (i), using the techniques of twisted sheaves and also some Saltman’s results on the ramification of $\alpha$, Lieblich [11] established that $\text{ind}\alpha | (\text{per}\alpha)^2$ as well. For the $u$-invariant, Parimala and Suresh [12] established that $u(K) = 8$ for $K$ of type (ii) and $p \neq 2$ (in the case (i) one easily sees that $u(K) = 8$ as well). This result has been also obtained by different methods by Harbater, Hartmann and Krashen [11] and Heath-Brown and Leep [3, 10] (which also contains the case $p = 2$). One of crucial steps in the proof of Parimala and Suresh is to establish a local-global principle for the divisibility of elements in $H^3(K, \mathbb{Z}/\ell)$ by symbols $\alpha \in H^2(K, \mathbb{Z}/\ell)$, $\ell \neq p$, which also uses Saltman’s results [14, 16] on the
classification of the ramification points of \( \alpha \). In [13], they also apply such a local-global principle to establish the vanishing of the third unramified cohomology group of conic fibrations over \( S \).

In this note, we follow the arguments of Parimala and Suresh and we establish the local-global principle in the general case for \( K \) of type \((i)\). The main technical difficulty is that for \( \alpha \) a symbol there is no so-called «hot» points in the classification of Saltman, which was also the case considered in [16, 17]. Our main result is the following:

**Theorem 1.1.** Let \( K \) be the function field of a smooth projective surface \( S \) over a finite field and let \( \ell \) be a prime, \( \ell \neq \text{char}(K) \). Assume that \( K \) contains a primitive \( \ell \)th root of unity. Let \( \xi \in H^3(K, \mathbb{Z}/\ell) \) and \( \alpha \in H^2(K, \mathbb{Z}/\ell) \) be such that the union \( \text{ram}_S(\xi) \cup \text{ram}_S(\alpha) \) is a simple normal crossings divisor. Assume that for any point \( x \in S^{(1)} \) there exists \( f_x \in K_x^* \) in the field of fractions of the completion of the local ring of \( S \) at \( x \), such that \( \xi = \alpha \cup f_x \) in \( H^3(K_x, \mathbb{Z}/\ell) \). Then there exists a function \( f \in K^* \) such that \( \xi = \alpha \cup f \) in \( H^3(K, \mathbb{Z}/\ell) \).

In section 2, we first fix the notations and recall some elements of Saltman’s approach on the ramification of \( \alpha \in Br_k \), then we give some additional properties of so-called «hot» points. This allows us to deduce the local-global principle 1.1 in section 3.

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## 2 Classification of the ramification points, complements on 'hot' points

### 2.1 Notations and first properties

#### 2.1.1 Residues and unramified cohomology

Let \( A \) be a discrete valuation ring of rank one with fraction field \( K \) and the residue field \( \kappa \), let \( \pi \) be a uniformizing parameter of \( A \). For any \( i \geq 1 \), \( j \in \mathbb{Z} \) and \( n \) an integer invertible in \( \kappa \) we have the residue maps in Galois cohomology

\[
H^i(K, \mu_n^{\otimes j}) \xrightarrow{\partial_A} H^{i-1}(\kappa, \mu_n^{\otimes j-1}).
\]

If \( \partial_A(x) = 0 \) for \( x \in H^i(K, \mu_n^{\otimes j}) \) we say that \( x \) is unramified. In this case we define the specialisation \( \bar{x} \) of \( x \) as \( \bar{x} = \partial_A(x \cup \pi) \).

If \( A \) is a regular ring with fraction field \( K \), an element \( x \in H^i(K, \mu_n^{\otimes j}) \) is called unramified on \( A \) if it is unramified with respect to all discrete valuations corresponding to height one prime ideals in \( A \).

Let \( k \) be a field. For \( L \) a function field over \( k \), \( n \) an integer invertible in \( k \), \( i \geq 1 \) and \( j \in \mathbb{Z} \) we denote
\[ H^i_{nr}(L/k; \mu_n^{\otimes j}) = \bigcap_A \text{Ker}[H^i(L, \mu_n^{\otimes j}) \xrightarrow{\partial_A} H^{i-1}(k_A, \mu_n^{\otimes j-1})], \]

where \( A \) runs through all discrete valuation rings of rank one with \( k \subset A \) and fraction field \( L \). Here we denote by \( k_A \) the residue field of \( A \) and by \( \partial_A \) the residue map. If \( X \) is an integral variety over \( k \), we denote \( H^i_{nr}(X, \mu_n^{\otimes j}) \) by \( H^i_{nr}(k(X)/k, \mu_n^{\otimes j}) \), where \( k(X) \) is the function field of \( X \). If \( X \) is a smooth and projective variety, then we also have

\[ H^i_{nr}(X, \mu_n^{\otimes j}) = \bigcap_{x \in X^{(1)}} \text{Ker} \partial_x, \]

where we denote \( \partial_x \) by \( \partial_{O_{x,x}} \) (see \([1]\)).

We will use the following vanishing results for varieties over finite fields.

**Proposition 2.1.** Let \( \mathbb{F} \) be a finite field and let \( \ell \) be a prime different from the characteristic of \( \mathbb{F} \).

(i) If \( C/\mathbb{F} \) is a smooth projective curve, then \( H^2_{nr}(C, \mu_\ell) = 0 \).

(ii) If \( S/\mathbb{F} \) is a smooth projective surface, then \( H^3_{nr}(S, \mu_\ell^{\otimes 2}) = 0 \).

**Proof.** For (i) note that \( H^2_{nr}(C, \mu_\ell) \) is the \( \ell \)-torsion subgroup of \( Br C \) (see \([1]\)), the group which is zero as \( C \) is a smooth projective curve over a finite field \([3]\). The statement (ii) is established in \([2]\) p.790.

**Remark 2.2.** A part of the Kato conjecture states that \( H^{d+1}_{nr}(X, \mu_\ell^{\otimes d}) = 0 \) for \( X \) a smooth projective variety of dimension \( d \), defined over a finite field \( \mathbb{F} \). This conjecture has been recently established by Kerz and Saito \([9]\), using also the arguments of Jannsen \([6]\).

### 2.1.2 Function fields of surfaces over a finite field

In the rest of this section we use the following notations: \( \mathbb{F} \) is a finite field of characteristic \( p \), \( \ell \) is a prime different from \( p \), \( S \) is a smooth projective surface over \( \mathbb{F} \) and \( K \) is the function field of \( S \). If \( P \) is a point of \( S \), \( \hat{A}_P \) denotes the completion of the local ring \( A_P \) at \( P \) at its maximal ideal, \( K_P \) is the field of fractions of \( \hat{A}_P \).

**We assume that \( K \) contains \( \ell \)-th roots of unity.**

We denote by \( \alpha \) a (fixed for what follows) element of \( H^2(K, \mathbb{Z}/\ell) \). For any \( x \in S^{(1)} \), a codimension one point of \( S \) we have the residue maps in Galois cohomology, as in the previous section:

\[ \partial_x : H^i(K, \mathbb{Z}/\ell) \to H^{i-1}(\kappa(x), \mathbb{Z}/\ell), \quad i \geq 1 \]
where $\kappa(x)$ is the residue field of $x$; the set of points $x \in S^{(1)}$ such that $\partial_x(\alpha)$ is non zero is finite. We define the ramification divisor of $\alpha$ as $\text{ram}_S\alpha = \sum_{x \in S^{(1)}, \partial_x(\alpha) \neq 0} x$.

We write

$$\text{ram}_S\alpha = \sum_{i=1}^m C_i$$

where $C_i \subset S, i = 1 \ldots m$ are integral curves and we denote $u_i = \partial_{C_i}(\alpha), u_i \in H^1(\kappa(C_i), \mathbb{Z}/\ell) = \kappa(C_i)^*/\kappa(C_i)^{\ast\ell}$.

If $P \in S$ is a closed point, we will say that $\alpha$ is unramified at $P$ if $P \notin \text{ram}_S\alpha$.

**Lemma 2.3.** Let $P \in S$ be a closed point. If $\alpha$ is unramified at $P$, then $\alpha$ is trivial in $H^2(K_P, \mathbb{Z}/\ell)$.

**Proof.** As $\alpha$ is unramified at $P$, we have that $\alpha$ comes from $H^2_{et}(A_P, \mathbb{Z}/\ell)$ (cf. [H §3.6, §3.8]). It is then trivial in $H^2(K_P, \mathbb{Z}/\ell)$, as the group $H^2_{et}(A_P, \mathbb{Z}/\ell) = H^2(\kappa(P), \mathbb{Z}/\ell)$ is zero as $\kappa(P)$ is a finite field. \qed

In the rest of this section we assume that $\text{ram}_S\alpha$ is a simple normal crossings divisor.

### 2.2 Classification of points

Let $P \in S$ be a closed point. We recall the classification of [16] with respect to the divisor $\text{ram}_S\alpha$:

1. if $P \notin \text{ram}_S\alpha$ then it is called a neutral point;

2. if $P$ is on only one irreducible curve in the ramification divisor, then it is called a curve point.

3. As the divisor $\text{ram}_S\alpha$ is a simple normal crossings divisor, the remaining case is when $P$ is on two curves of the ramification divisor, in this case it is called a nodal point. Assume that $P$ is on the curves $C_i$ and $C_j$ for some $i \neq j$. Recall that we denote $u_i = \partial_{C_i}(\alpha)$. By the reciprocity law (see [S]), $\partial_P(u_i) = -\partial_P(u_j)$. Then the following cases may occur:

   (a) $u_i$ and $u_j$ are ramified at $P$. Then $P$ is called a cold point;

   (b) $u_i$ and $u_j$ are unramified at $P$. Denote $u_i(P), u_j(P) \in H^1(\kappa(P), \mathbb{Z}/\ell)$ the specialisations of $u_i$ (resp. $u_j$) at $P$.

      i. If $u_i(P)$ and $u_j(P)$ are trivial, $P$ is called a cool point;

      ii. if $u_i(P)$ and $u_j(P)$ are non trivial and generate the same subgroup of $H^1(\kappa(P), \mathbb{Z}/\ell)$, then $P$ is called a chilly point;

      iii. if $u_i(P)$ and $u_j(P)$ do not generate the same subgroup of $H^1(\kappa(P), \mathbb{Z}/\ell)$, then $P$ is called a hot point. Since $\kappa(P)$ is a finite field, we have $H^1(\kappa(P), \mathbb{Z}/\ell) \simeq \mathbb{Z}/\ell$. We then get that for a hot point one of the specialisations $u_i(P)$ and $u_j(P)$, say $u_j(P)$, should be trivial. Then
we will call \( P \) a **hot non neutral point** on \( C_i \) and **hot neutral point** on \( C_j \).

Consider a graph whose vertices are curves in the ramification locus and whose edges correspond to chilly or hot points. We will call a **hot chilly circuit** a loop in this graph or a connected component with more than one hot point on curves in this component.

### 2.3 Local description

**Lemma 2.4.** Let \( P \in S \) be a closed point.

(i) If \( P \in C_i \) is a curve point with \( \pi_i \) a local parameter of \( C_i \) in \( K \), then in \( H^2(K_P, \mathbb{Z}/\ell) \) we have \( \alpha = u \cup \pi_i \) for some unit \( u \in A_P \).

(ii) If \( P \in C_i \cap C_j \) is a nodal point with \( \pi_i, \pi_j \) local parameters of \( C_i \) and \( C_j \) in \( K \), then in \( H^2(K_P, \mathbb{Z}/\ell) \) we have

1. \( \alpha = 0 \) if \( P \) is a cool point;
2. \( \alpha = u\pi_j \cup v\pi_i^s \) for some units \( u, v \in A_P \), \( 1 \leq s \leq \ell - 1 \), if \( P \) is a cold point;
3. \( \alpha = u \cup \pi_i \pi_j^s \) for some unit \( u \in A_P \), \( 1 \leq s \leq \ell - 1 \), if \( P \) is a chilly point;
4. \( \alpha = u \cup \pi_i \), for some unit \( u \in A_P \), if \( P \) is a hot point, non neutral on \( C_i \).

**Proof.** For (i) we write (see [14, Proposition 2.1])

\[
\alpha = \alpha' + (u, \pi_i)
\]

where \( \alpha' \) is unramified at \( P \) and \( u \) is a unit in \( A_P \). By lemma 2.3, \( \alpha' \) is zero in \( H^2(K_P, \mathbb{Z}/\ell) \), so that we get (i). For (ii), all the statements but the last one are in [13, Lemma 1.3]. To see the last one, we write (see [14, Proposition 2.1])

\[
\alpha = \alpha' + (u, \pi_i) + (v, \pi_j)
\]

where \( \alpha' \) is unramified at \( P \), \( u, v \) are units in \( A_P \) and the image of \( v \) in \( \kappa(C_j) \) is \( \partial C_j(\alpha) \). As above, \( \alpha' \) is zero by lemma 2.3. The element \( v \) is also zero in \( A_P \) because \( v(P) \) is zero in \( H^1(\kappa(P), \mathbb{Z}/\ell) \) by the definition of a hot point, so that \( v \in H^1(\hat{A}_P, \mathbb{Z}/\ell) = H^1(\kappa(P), \mathbb{Z}/\ell) \) is zero as well.

Let \( C_i \) be a curve in the ramification divisor \( \text{ram}_S \alpha \) and let \( \pi_i \) be a prime defining \( C_i \) in \( K \). We define a residual class \( \beta_i \) of \( \alpha \) at \( C_i \) as \( \beta_i = \partial C_i(\alpha \cup \pi_i) \) (see [13, Remark 2.6], [16] p.820).

**Proposition 2.5.** If \( C_i \in \text{ram}_S \alpha \) contains no hot points, the element \( \beta_i \) is trivial over \( L_i = \kappa(C_i)(\sqrt[\ell]{u_i}) \).
Proof. Using proposition 2.1 it is sufficient to see that $\beta_i$ is unramified over $L_i$. Let $v$ be a discrete valuation on $L_i$, let $v'$ be the induced valuation on $\kappa(C_i)$ and let $e$ be the valuation of a uniformizing parameter of $\kappa(C_i)$ in $L_i$. We have the following commutative diagram (cf. [1, Proposition 3.3.1]):

$$
\begin{array}{c}
H^2(L_i, \mathbb{Z}/\ell) \xrightarrow{\partial_v} H^1(\kappa(v), \mathbb{Z}/\ell) \\
\text{res} \downarrow \quad \quad \downarrow e \cdot \text{res} \\
H^2(\kappa(C_i), \mathbb{Z}/\ell) \xrightarrow{\partial_{v'}} H^1(\kappa(v'), \mathbb{Z}/\ell).
\end{array}
$$

We may assume that the valuation $v'$ corresponds to a point $P \in C_i$. We have the following cases:

1. if $P$ is a curve point, in $\hat{A}_P$ we have $\alpha = u \cup \pi_i$ for some unit $u$ by the local description 2.1 and so $\partial_{v'}(\beta_i) = \partial_P \partial_C(u \cup \pi_i \cup \pi_i) = 0$.

2. if $P$ is a cool point, $\alpha$ is trivial in $\hat{A}_P$ by the local description, hence $\partial_{v'}(\beta_i) = 0$.

3. if $P$ is a cold point, the extension $L_i/\kappa(C_i)$ is ramified at $P$, hence $l|e$. As $\kappa(v')$ is a finite field, $\partial_{v'}(\beta_i)$ becomes trivial in $\kappa(v)$.

4. if $P$ is a chilly point, from the local description we deduce that $\partial_{v'}(\beta_i) = u_{ij}(P)$ which equals to $u_i(P)^s$ for some $1 \leq s \leq l - 1$ by the definition of a chilly point, hence trivial over $\kappa(v)$.

Thus $\beta_i$ is unramified over $L_i$ and hence trivial.

3 The local-global principle

In this section we prove the local-global principle 1.1.

3.1 Additional notations

Let $\xi \in H^3(K, \mathbb{Z}/\ell)$ be as in the theorem 1.1. As the divisor $\text{ram}_S(\xi) \cup \text{ram}_S(\alpha)$ is a simple normal crossings divisor, we can write

$$
\text{ram}_S(\xi) \cup \text{ram}_S(\alpha) = \sum_{i=1}^{n} C_i
$$

where $n \geq m$ and $C_i$ are integral curves.

We denote $\mathcal{C} = \{C_1, \ldots, C_m\}$ and $\mathcal{T} = \{C_1, \ldots, C_{m+1}, \ldots, C_n\}$.

Let $\mathcal{P}$ a finite set of closed points on $S$, consisting of all the points of intersections $C_i \cap C_j$ and at least one point from each component $C_i$, let $B$ be the semilocal ring of $S$ at $\mathcal{P}$ and let $\pi_i$ be a (fixed from the beginning) prime defining $C_i$ in $B$.
Recall that we denote \( u_i = \partial_{C_i}(\alpha) \in \kappa(C_i) \). We put as before

\[
\beta_i = \partial_{C_i}(\alpha \cup \pi_i) \in H^2(\kappa(C_i), \mathbb{Z}/\ell)
\]

and

\[
f_i = f_{C_i} = \pi_i^r f_i' \in K_{C_i},
\]  

where \( v_{C_i}(f_i') = 0 \).

### 3.2 Plan of the proof

Vanishing of the third unramified cohomology of \( S \) from proposition 2.1 implies that to establish theorem 1.1 it is sufficient to prove that there exists \( f \in K \) such that for any \( x \in S^{(1)} \), \( \partial_x(\xi) = \partial_x(\alpha \cup f) \) in the residue field \( \kappa(x) \). This is done essentially in two steps.

1. First we get the condition above for \( x = C_i \); to do this we first work locally around the double points \( P \).

2. Next we adjust the element \( f \) from the previous step to get the condition for other curves in the support of \( \text{div}(f) \).

### 3.3 Analyzing curves and points in the ramification divisor

#### 3.3.1 Local description at nodal points

**Proposition 3.1.** Let \( P \in C_i \cap C_j \) be a nodal point. Then

1. If \( P \) is a chilly point, then for any \( 0 \leq r_i \leq \ell - 1 \) there exists \( 0 \leq r_j \leq \ell - 1 \) such that \( \xi - \alpha \cup \pi_i^{r_i} \pi_j^{r_j} \) is unramified on \( \hat{A}_P \);

2. If \( P \) is a hot point, non neutral on \( C_i \), then \( \xi - \alpha \cup \pi_i^{r_i} \pi_j^{t_j} \) is unramified on \( \hat{A}_P \) for any \( 0 \leq r_i \leq \ell - 1 \), where \( t_j \) is defined in (2).

**Proof.** The first part is in [13, Lemma 3.1]. To see the second, let us first write \( t_j' = v_{\pi_j}(f_i) = v_{\pi_i}(f_i') \).

By the reciprocity law (cf. [5]), we have

\[
\partial_P(\partial_{C_i}(\xi)) = -\partial_P(\partial_{C_j}(\xi)),
\]

and so

\[
\partial_P(\partial_{C_i}(\alpha \cup f_i)) = -\partial_P(\partial_{C_j}(\alpha \cup f_j)).
\]

In \( \hat{A}_P \) we have \( \alpha = u \cup \pi_i \) by the local description. This gives

\[
\partial_P(\partial_{C_i}(u \cup \pi_i \cup \pi_i^{r_i} f_i')) = -\partial_P(\partial_{C_j}(u \cup \pi_i \cup \pi_j^{t_j} f_j')).
\]

Moreover, one can see that for \( r_j \neq t_j \mod \ell \), \( \xi - \alpha \cup \pi_i^{r_i} \pi_j^{r_j} \) is ramified on \( \hat{A}_P \).
The left side is
\[ \partial_P(u(C_i) \cup f_i'(C_i)^{-1}) = -(t_j' + \epsilon)u(P), \]
where we denote \( u(C_i), f_i'(C_i) \) the images of \( u, f_i' \) in \( \kappa(C_i) \) and \( \epsilon = v_P(f_i'(C_i)/\pi_j^t(C_i)) \).

The right side is
\[ -t_j \partial_P(u(C_j) \cup \pi_i(C_j)) = -t_ju(P), \]
where we denote \( u(C_j), \pi_i(C_j) \) the images of \( u, \pi_i \) in \( \kappa(C_j) \). We then get \( t_j = t_j' + \epsilon \) as \( u(P) \) is not an \( t^{th} \) power. We then can write \( f_i = \pi_i^t \pi_j^{-\epsilon} f_i'' \), where \( v_P(f_i''(C_i)) = \epsilon \).

Claim: \( \partial_C(\xi - \alpha \cup \pi_i^r \pi_j^s) = 0 \) in the completion \( \kappa(C) \).

In fact, in \( \kappa(C) \), we have
\[
\partial_C(\xi) = \partial_C(u \cup \pi_i \cup \pi_i^r \pi_j^{-\epsilon} f_i'') = \partial_C(u \cup \pi_i \cup \pi_i^r \pi_j^{-\epsilon} f_i'') + \partial_C(u \cup \pi_i \cup \pi_j^{-\epsilon} f_i'') = \\
= \partial_C(\alpha \cup \pi_i^r \pi_j^s) - u(C_i) \cup \pi_j(C_i)^{-\epsilon} f_i''(C_i)
\]
(for the last equality we observe that \( \partial_C(u \cup \pi_i \cup \pi_i) \) is zero in \( \kappa(C) \)) and \( u(C_i) \cup \pi_j(C_i)^{-\epsilon} f_i''(C_i) \) is unramified at \( P \) and, hence, zero in \( \kappa(C) \).

From the claim we deduce that \( \xi - \alpha \cup \pi_i^r \pi_j^s \) is ramified on \( \hat{A}_P \) at most at \( \pi_j \); using reciprocity (cf. [13, Lemma 1.2]), it is then unramified.

Using the proposition above, we would like to associate an integer \( r_i \) to each curve \( C_i \) globally, and not only locally in each double point. To do this, we will need to avoid the hot chilly circuits.

**Lemma 3.2.** Let \( P \in S \) be a chilly or a hot nodal point. Let \( S' \to S \) be the blowing-up of \( S \) at \( P \). Then for any point \( x \in S^{(1)} \) there exists \( f_x \in K_x^* \) such that \( \xi = \alpha \cup f_x \in H^3(K_x, \mathbb{Z}/\ell) \).

**Proof.** We need only to consider the case when \( x = E \) is an exceptional divisor of the blowing-up. If \( P \) is a chilly point, then the statement is established in [13, Theorem 3.4]. Assume that \( P \) is a hot point, which is non neutral on \( C_i \). Denote again \( C_i, C_j \) the strict transforms of \( C_i \) and \( C_j \), let \( P_i = C_i \cap E \) and \( P_j = C_j \cap E \). Using the local description, we easily check that \( P_j \) is a hot point, non neutral on \( E \) and that \( P_j \) is a chilly point. Then, taking \( f_E = \pi_i^t \pi_j^s \) we see that \( \xi - \alpha \cup f_E \) is unramified on \( K_E \) (for any \( r_i \)). In fact, we have an injection of local rings \( \hat{A}_P \to O_{S', E} \), so that \( \hat{A}_P \) is contained on the completion of \( O_{S', E} \) at its maximal ideal. Since \( \xi - \alpha \cup f_E \) is unramified on \( \hat{A}_P \) by proposition 3.11 it is unramified at \( E \).

**Lemma 3.3.** There exists a smooth and projective surface \( S' \to S \), obtained as a blowing-up of \( S \) in some hot and chilly points, such that on \( S' \) there is no hot chilly circuits for \( \alpha \) and such that we still have that for any point \( x \in S^{(1)} \) there exists \( f_x \in K_x^* \) with \( \xi = \alpha \cup f_x \in H^3(K_x, \mathbb{Z}/\ell) \).

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Proof. In fact, suppose that we have a hot chilly circuit and let \( P \in C_i \cap C_j \) be a hot or a chilly point on a curve in this circuit. If \( P \) is a hot point, which is non neutral on \( C_i \), let us consider the blowing-up of \( P \). Let \( E \) be the exceptional divisor of the blowing up at \( P \), and denote again \( C_i, C_j \) the strict transforms of \( C_i \) and \( C_j \), let \( P_i = C_i \cap E \) and \( P_j = C_j \cap E \). As in the previous lemma, \( P_j \) is a hot point, non neutral on \( E \) and \( P_i \) is a chilly point. We may then suppose that there is a chilly point on curves in the circuit. By a construction of [16, 2.9] we may break the circuit by successif blowing-ups of this point, and (some of) the double points above. Thus by successif blowing-ups of hot and chilly points we can obtain a surface \( S' \) with no hot chilly circuits. We may find a corresponding function \( f_x \) for each exceptional divisor by the previous lemma.

Up to replacing \( S \) by \( S' \) as in the previous lemma, from now on, we assume that there is no hot chilly circuits.

Now we can choose constants \( r_i \) globally.

**Lemma 3.4.** Assume that there is no hot chilly circuits. Then we may associate to each curve \( C_i \in \mathcal{C} \) an integer \( 0 \leq r_i \leq \ell - 1 \) such that \( \xi - \alpha \cup \pi_i^r \pi_j^r \) is unramified on \( \hat{A}_P \) for any nodal point \( P \). If there is a hot point on \( C_i \) then \( r_i = t_i \).

**Proof.** Since there is no hot chilly circuits, we may successively choose the integers \( r_i \) as in the statement using lemma [3.1]. \( \square \)

### 3.3.2 First choice of \( f \)

For \( 1 \leq i \leq m \), let \( r_i \) be as in lemma [3.4]. For \( i \geq m + 1 \), we put \( r_i = v_{C_i}(f_{C_i}) \).

**Proposition 3.5.** Assume that there is no hot chilly circuits. Then there exists a function \( f \in K^* \) such that

\[
\text{div}(f) = \sum_{i=1}^{n} r_i C_i + F,
\]

where \( F \) does not pass through any point in \( \mathcal{P} \), and such that \( \xi - \alpha \cup f \) is unramified on \( \hat{A}_P \) for all \( P \in \mathcal{P} \).

**Proof.** By [13, Lemma 3.2], for any non hot point \( P \in C_i \cap C_j \) for \( C_i, C_j \in \mathcal{T} \), one can associate \( w_P \in A_P \), such that \( \xi - \alpha \cup w_P \pi_i^r \pi_j^r \) is unramified at \( \hat{A}_P \), where the \( r_i \)’s are as above. We put \( w_P = 1 \) if \( P \) is a hot point. Then \( \xi - \alpha \cup w_P \pi_i^r \pi_j^r \) is unramified at \( \hat{A}_P \) by lemma [3.4]. For any other point \( P \in \mathcal{P} \) we put \( w_P = 1 \).

Next we choose the function \( f \) satisfying the conditions of the proposition by the same construction as in [13, Lemma 3.3]. We recall shortly this construction. First consider \( g = \prod_{i=1}^{r} \pi_i^r \). Then we can choose an element \( u_P \in A_P \) for any \( P \in \mathcal{P} \) as follows. If \( P \notin C_i \) for all \( i \), we put \( u_P = 1 \). If \( P \) is on only one \( C_i \) we put...
$u_P = g_P - r_i$ and if $P \in C_i \cap C_j$ we put $u_P = g_P - r_i - r_j$. Now let $w \in B$ be such that $w(P) = w_P / u_P$ for all $P \in \mathcal{P}$. Then one checks that $f = wg$ satisfies the conditions of the proposition.

3.3.3 A better choice of $f$

To choose an adjusting term we will need to make a more precise choice of the function $f$:

**Proposition 3.6.** After possibly blowing up the surface $S$, there exists an element $f \in K^*$ such that

(i) $\text{div}(f) = \sum_{i=1}^n r_i C_i + E$;

(ii) $\xi - \alpha \cup f$ is unramified at $C_i$ for any $C_i \in \mathcal{T}$;

(iii) $E$ does not pass through any point of $\mathcal{P}$;

(iv) $(E \cdot C_i)_P$ is a multiple of $\ell$ at any intersection point of $E$ and $C_i$, $i = 1, \ldots n$.

**Proof.** Let us consider $f$ as in proposition 3.5. First we claim that for $i \leq m$ the residue

$\gamma_i := \partial_{C_i}(\xi - \alpha \cup f)$

may be written as $u_i \cup b_i$ for some constant $b_i$. In fact, it is sufficient to prove that $\gamma_i$ is trivial over $\kappa(C_i)(\sqrt{u_i})$. We have $\partial_{C_i}(\xi) = \partial_{C_i}(\alpha \cup f) = r_i \beta_i + u_i \cup g_i$ for some $g_i$ and $\partial_{C_i}(\alpha \cup f) = u_i \cup h_i$ for some $h_i$. If $C_i$ has hot points, then $r_i = t_i$ by construction and the claim is clear. If $C_i$ has no hot points, then $\beta_i$ is trivial over $\kappa(C_i)(\sqrt{u_i})$ by proposition 2.5 hence so is $\gamma_i$.

Note that for $i > m$ the residue $\gamma_i := \partial_{C_i}(\xi - \alpha \cup f)$ is trivial by the choice of $f$.

We proceed next exactly as in [13, Theorem 3.4]. We have

1. $\xi - \alpha \cup f$ is unramified at $\hat{A}_P$ for all $P \in \mathcal{P}$ by the choice of $f$, so $\gamma_i$ is unramified at all points $P \in \mathcal{P} \cap C_i$. Hence $b_i$ is a norm from $\kappa(C_i)\sqrt{u_i}$.

2. By weak approximation, we then can find $a_i \in \kappa(C_i)$ which is a norm from $\kappa(C_i)(\sqrt{u_i})$ and such that $a_i b_i(P) = 1$ for all $P \in \mathcal{P} \cap C_i$. As $u_i \cup b_i = u_i \cup a_i b_i$ we may assume that $b_i(P) = 1$ for all $P \in \mathcal{P} \cap C_i$.

3. We take $b \in B$ a unit such that $b_i$ is the image of $b$ in $B/\pi_i$.

4. Changing $f$ by $bf$, the conditions (i) -- (iii) are now satisfied. By [13, Theorem 3.4. p.15] $(E \cdot C_i)_P$ is a multiple of $\ell$, after possibly some blowing ups of the points in the intersection of $E$ and $C_i$.
3.4 Adjusting term for other curves

Let \( f \) and \( E \) be as in proposition 3.6. Let \( P' \) be a finite set of closed points on \( S \), consisting of \( P \) and all intersection points of \( C_i \) and \( E \), and at least one point from each component of \( E \) and at least one non hot point from each \( C_i \).

Proposition 3.7. There exists \( u \in K^* \) and \( x \in K^* \) a norm from \( K(\sqrt{u}) \), such that

(i) \( \text{div}(u) = -E + E' + \ell U \), where \( E' \) does not pass through any point in \( P' \setminus (\bigcup_{i \neq j} C_i \cap C_j) \);

(ii) the image of \( u \) in \( \kappa(C_i)^*/\kappa(C_i)^{\ell} \) equals to \( u_i \);

(iii) \( \text{div}(x) = -E + E'' + \ell U' \) where \( E'' \) does not pass through any point in \( P' \) and any intersection point \( C_i \cap E' \);

(iv) if \( D \) is an irreducible curve in the \( \text{Supp}(E'') \), then the specialization of \( \alpha \) at \( D \) is unramified at every discrete valuation of \( \kappa(D) \) centered on a closed point of \( D \).

Proof. Let \( P'' = P' \setminus (\bigcup_{i \neq j} C_i \cap C_j) \) and let \( B' \) be the semilocal ring of \( S \) at \( P'' \). By [17, Proposition 0.3], one can find \( v \in B' \) such that \( v = u_i \mod \pi_i \). By [13, Lemma 2.1], under the assumption that \((E \cdot C_i)_P\) is a multiple of \( \ell \) at any intersection point of \( E \) and \( C_i \), \( i = 1, \ldots n \), one can find an element \( z \in K^* \) such that:

1. \( z \) is a unit at \( C_i \) and maps to an \( \ell^{\text{th}} \) power in \( \kappa(C_i)^* \) for all \( i \);
2. \( \text{div}(z) = -E + Z_1 + \ell Z_2 \) where the support of \( Z_1 \) does not contain any point in \( P \).

Then the element \( u = vz \) satisfies the conditions (i) – (ii) of the proposition. By [13, Lemma 2.3] we can construct \( x \) satisfying (iii). One then verifies (iv) as in [13, Lemma 2.4].

3.4.1 End of the proof of theorem 1.1

Let \( x \) be as in proposition 3.7. We will now show that \( \xi - \alpha \cup (fx) \) is unramified at any codimension one point \( D \in S \). We have

\[
\text{div}(fx) = \sum_{i=1}^{n} r_i C_i + E'' + \ell U'.
\]

1. If \( D \in \mathcal{T} \), then \( \xi - \alpha \cup f \) is unramified at \( D \) by the previous section. If \( D = C_i \) with \( i \leq m \), then \( \alpha = \alpha' + (u, \pi_i) \), where \( \alpha' \) is unramified on \( C_i \), by the choice of \( u \). Hence \( \partial_D(\alpha \cup x) = \partial_{C_i}(u \cup \pi_i \cup x) = 0 \) as \( x \) is a norm from \( K(\sqrt{u}) \). If \( D = C_i \) with \( i > m \), then \( \alpha \) and \( x \) are unramified at \( D \), then so is \( \alpha \cup x \).
2. If $D \notin \mathcal{T} \cup \text{Supp}(fx)$ or $D \in \text{Supp}(U')$ then $\xi - \alpha \cup (fx)$ is unramified at $D$.

3. Assume now that $D \in \text{Supp}(fx) \setminus (\mathcal{T} \cup U')$, i.e. $D \in \text{Supp}(E'')$. We have that $\xi$ and $\alpha \cup f$ are unramified at $D$. So it is sufficient to show that $\partial_D(\alpha \cup x) = 0$ which follows from proposition 3.7(iv).

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