Transient Chaos in Strongly Monotone Dynamical Systems*

Jinxiang Yao
School of Mathematical Sciences
University of Science and Technology of China
Hefei, Anhui, 230026, P. R. China
jxyao@mail.ustc.edu.cn

Abstract
For strongly monotone dynamical systems, it is shown that the principal Lyapunov exponent $\lambda_1(x) \leq 0$ holds on a prevalent set in the measure-theoretic sense. The nonexistence of observable chaos is thus obtained as a by-product. Together with the existing works, we point out that transient chaos is ubiquitous in strongly monotone dynamical systems, which is a form of unobservable chaos introduced by Young [Commun. Pure Appl. Math. 66(2013)].

Keywords: Observable chaos; Transient chaos; Lyapunov exponents; Sharpened dynamics alternative; Improved prevalent dynamics; $C^1$-robustness; Monotone dynamical systems.

1 Introduction

Observable chaos (see e.g., Young [35, Section 1, p.1441], [36, Section 4], [37, Section 3]) indicates that the principal Lyapunov exponent $\lambda_1(x) > 0$ holds on a positive Lebesgue measure set, which implies that the instability persists for all future times and occur on a set large enough to be observable. As Young [35] pointed out that, the presence of a horseshoe does not imply the system has observable chaos. Such unobservable complicated dynamics are characterized as transient chaos in Young [35, Section 1, p.1442], [36, Section 4]. Transient chaos is ubiquitous in fluid, chemical, biological, and engineering systems. See also [1, 17, 27] (and references therein) etc. for more details of the theory and applications of transient chaos.

The ground-breaking work by Hirsch [9] and the extended work by Poláčik [18] and Smith and Thieme [25] showed that, generic precompact orbit of a strongly monotone semiflow converges to equilibria (see also [12]). Here, generic describes the properties hold residually i.e., on a countable intersection of open dense subsets in a Baire space. For strongly monotone

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discrete-time dynamical systems, Poláčik and Tereščák \[19\] first proved the **generic convergence to cycles** occurs provided that the mapping \( F \) is of class \( C^{1,\alpha} \) (i.e., \( F \) is a \( C^1 \)-map with a locally \( \alpha \)-Hölder derivative \( DF, \alpha \in (0,1) \)). Motivated by Tereščák \[28\], Wang and Yao \[31, 32\] extended the result to \( C^1 \)-smooth strongly monotone discrete dynamical systems. Recently, generic Poincaré-Bendixson Theorem was obtained by Feng et al. \[4, 5\] for smooth flows strongly monotone with respect to 2-cone. In the measure-theoretic sense, Enciso, Hirsch and Smith \[3\] investigated the prevalent behavior of strongly monotone semiflows and proved that the set of points that converge to equilibria is prevalent. Here, **Prevalent** describes properties of interest occurs for “almost surely” in an infinite-dimensional space from a probabilistic or measure-theoretic perspective. It is a natural generalization to separable Banach spaces of the notion of the full Lebesgue measure for Euclidean spaces (see definition in Section 2). In particular, on \( \mathbb{R}^d \), it is equivalent to the notion of “Lebesgue almost everywhere”. Wang et al. \[33\] proved the set of points that converge to periodic orbits is prevalent for discrete-time systems. Very recently, Wang et al. \[34\] further studied the prevalent behavior of flows with invariant k-cones and obtained an almost sure Poincaré-Bendixson theorem.

Although a lot of generic and prevalent asymptotic behavior theory have been built in monotone systems, one can not expect that monotone systems can only have simply dynamics. Smale \[21\] first smartly constructed a profound example to show that essentially arbitrary dynamics, including positive entropy, horseshoes, chaos in the sense of Li-Yorke and/or Devaney can be found in the codimension-1 invariant manifolds in monotone systems, even if in low-dimensional systems (see also this example in Smith \[23, Chapter 4, p.71-72\]). For the investigation of complicated dynamics in monotone dynamical systems, one may also refer to \[2, 22–24, 29, 30\] (and references therein) etc. for more details.

In the present paper, we shall focus on and prove the nonexistence of observable chaos for \( C^1 \)-smooth strongly monotone dynamical systems, as well as its \( C^1 \)-robustness for \( C^1 \)-perturbed systems. Together with the existing works on the presence of complexity in monotone systems, this implies the presence of transient chaos in monotone systems. That is to say, if the chaotic behavior exists, it is so unstable as to be unobservable. To be more precise, we formulate some standing hypotheses:

**(H1)** \((X, C)\) is a strongly ordered separable Banach Space.

**(H2)** \( F_0 : X \to X \) is a compact \( C^1 \)-map, such that for any \( x \in X \), the Fréchet derivative \( DF_0(x) \) is a strongly positive operator, i.e., \( DF_0(x)v \gg 0 \) whenever \( v > 0 \).

**Theorem A** (Nonexistence of observable chaos for mappings). **Assume that (H1)-(H2) hold. Assume also \( F_0 \) is pointwise dissipative with an attractor \( A \). Then the set**

\[
S_0 := \{ x \in X : \lambda_1(x, F_0) \leq 0 \},
\]

**is prevalent in** \( X \). **In particular, if** \( X = \mathbb{R}^d \), **then** \( F_0 \) **cannot have observable chaos.**

Motivated by Theorem A, we further consider \( C^1 \)-perturbations of the \( C^1 \)-smooth mapping
$F_0$, and obtain the nonexistence of observable chaos for the $C^1$-perturbed systems. More precisely, we present an additional standing hypothesis:

(H3) Let $J = [-\epsilon_0, \epsilon_0] \subset \mathbb{R}$, and $F : J \times X \to X; (\epsilon, x) \mapsto F_\epsilon(x)$ is a compact $C^1$-map, i.e., $DF(\epsilon, x)$ continuously depends on $\epsilon, x \in J \times X$.

The following theorem reveals that the nonexistence of observable chaos of $F_0$ is robust under the $C^1$-perturbation.

**Theorem B** ($C^1$-robustness for nonexistence of observable chaos). Assume that (H1)-(H3) hold. Assume also $F_0$ is pointwise dissipative with an attractor $A$. Let $B_1 \supset A$ be an open ball such that

$$\sup\{|F_\epsilon x - F_0 x| + |DF_\epsilon(x) - DF_0(x)| : \epsilon \in J, x \in B_1\}$$

sufficiently small. Then there exists a closed bounded set $M$ ($\text{Int} M \supset A$) and an integer $q > 0$ such that, for any $|\epsilon|$ sufficiently small,

(i). $F^q_\epsilon(M) \subset M$; and

(ii). The set $S_\epsilon := \{x \in M : \lambda_1(x, F_\epsilon) \leq 0\}$, is prevalent in $M$. In particular, if $X = \mathbb{R}^d$, then the set $S_\epsilon$ is of full Lebesgue measure in the compact domain $M$, and $F^q_\epsilon|_M$ cannot have observable chaos.

For a semiflow $\phi$, we present the following standing hypothesis:

(H2)$'$ $\phi : \mathbb{R} \times X \to X$ is a $C^1$-smooth strongly monotone semiflow with compact orbit closures. For some fixed $t_0 > 0$, the Fréchet derivative $D\phi_{t_0}(x)$ is compact for any $x \in X$ and $D\phi_{t_0}(e)$ is strongly positive for any equilibrium $e \in X$.

**Theorem C** (Nonexistence of observable chaos for semiflows). Assume that (H1) and (H2)$'$ hold. Then the set $S_\phi := \{x \in X : \lambda_1(x, \phi) \leq 0\},$ is prevalent in $X$. In particular, if $X = \mathbb{R}^d$, then $\phi$ cannot have observable chaos.

Theorem A-C concludes that observable chaos cannot exist in $C^1$-smooth strongly monotone systems, as well as for $C^1$-perturbed systems. This implies that the aforementioned unobservable complicated dynamics in strongly monotone systems are indeed transient chaos.

It deserves to point out that the existing works exhibited that any attractor for monotone systems cannot be chaotic in the sense of topological transitivity (see e.g., [8, 10, 11]). Here, Theorem A-C provide us a new point of view in measure-theoretic sense to understand such highly unstable complicated dynamics in monotone dynamical systems, which is a widespread phenomenon in such systems. Meanwhile, it also reveals an interaction relationship between the generic (prevalent) asymptotic behavior (in the whole space) and arbitrary dynamics (in the codimension-1 invariant manifolds) in monotone systems.
Our approach is motivated by the Sharpened $C^1$-dynamics alternative (Lemma 3.1) and its $C^1$-robustness (Lemma 3.6) in our recent work [32], which is a critical insight for the inherent structure of discrete-time strongly monotone systems. By appealing to the Sharpened $C^1$-dynamics alternative and its $C^1$-robustness and a useful tool of upper (lower) $\omega$-unstable sets introduced by Takáč [26], we obtain the Improved prevalent dynamics theorem (Theorem 3.3) and its $C^1$-robustness (Theorem 3.8). Together with a formula for the spectral radius of an operator (Lemma 4.1), we utilize the improved prevalent dynamics theorem to accomplish our approach.

This paper is organized as follows. In Section 2, we agree on some notations, give relevant definitions and preliminary results. In Section 3, we establish the improved prevalent dynamics theorem (Theorem 3.3) and its $C^1$-robustness (Theorem 3.8), which turn out to be crucial in the proof our main results (Theorem A-C). Section 4 is devoted to the proof of the main results.

2 Notations and Preliminary results

Let $(X, \norm{\cdot})$ be a Banach space. A cone $C$ is a closed convex subset of $X$ such that $\lambda C \subset C$ for all $\lambda > 0$ and $C \cap (-C) = \{0\}$. $(X, C)$ is said to be a strongly ordered Banach space if $C$ has nonempty interior $\text{Int} C$. For $x, y \in X$, we write $x \leq y$ if $y - x \in C$, $x < y$ if $y - x \in C \setminus \{0\}$, $x \ll y$ if $y - x \in \text{Int} C$. The reversed signs are used in the usual way. A subset $J' \subset X$ is called a simply ordered, open arc if there is an increasing homeomorphism $h$ from an open interval $I \subset \mathbb{R}$ onto $J'$ ($h$ is increasing if $\xi_1 < \xi_2$ implies $h\xi_1 \ll h\xi_2$). A mapping $h : X \to X$ is monotone (strongly monotone), if $x \leq y$ ($x < y$) implies $hx \leq hy$ ($hx \ll hy$). Similarly, a semiflow $\phi : \mathbb{R} \times X \to X$ is monotone (strongly monotone), if $x \leq y$ ($x < y$) implies $\phi_t(x) \leq \phi_t(y)$ ($\phi_t(x) \ll \phi_t(y)$) for all $t \geq 0$ ($t > 0$).

In this paper, we sometimes also need to deal with arguments for another solid cone $C_1(\subset C)$. Hence, for the sake of no confusion, we write $\leq_1, <_1, \ll_1$ as the corresponding order relation induced by the cone $C_1$ throughout the paper.

For a continuous map $h : X \to X$, the orbit of $x \in X$ is denoted by $O(x, h) = \{h^n x : n \geq 0\}$. The $\omega$-limit set of $x \in X$ is $\omega(x, h) \equiv \bigcap_{k \geq 0} \{h^n x : n \geq k\}$. We say that $h$ is $\omega$-compact in a subset $Y$ of $X$, if $O(x, h)$ is relatively compact for each $x \in Y$ and $\bigcup_{x \in Y} \omega(x, h)$ is relative compact. Given any $x \in X$, we define the upper and lower $\omega$-limit sets of $x$ by

$$\omega_+(x, h) \equiv \bigcap_{u \in X} \bigcup_{y \in X} \omega(y, h) \quad \text{and} \quad \omega_-(x, h) \equiv \bigcap_{u \in X} \bigcup_{y \in X} \omega(y, h),$$

respectively. If $h$ is $\omega$-compact in some neighbourhood of $x$ in $X$, $\omega_+(x, h)$ (resp. $\omega_-(x, h)$) is non-empty and compact (see Takáč [26, Proposition 3.1]). Write

$$U_+(h) = \{x \in X : \omega_+(x, h) \neq \omega(x, h)\} \quad \text{and} \quad U_-(h) = \{x \in X : \omega_-(x, h) \neq \omega(x, h)\}.$$
as the upper $\omega$-unstable set and lower $\omega$-unstable set in $X$, respectively. We denote the set $\mathcal{U}_q(h) = \mathcal{U}_+(h) \cap \mathcal{U}_-(h)$.

A point $x \in X$ is called a periodic point of $h$, if $h^p x = x$ for some integer $p \geq 1$. $p$ is then a period of $x$. Moreover, if $h^l x \neq x$ for $l = 1, 2, \cdots, p-1$, $x$ is said to be $p$-periodic. $p$ is the minimal period of $x$. Particularly, if $p = 1$, $x$ is a fixed point of $h$. $K$ is a cycle if $K = O(x, h)$ for some periodic point $x$. For a $C^1$-smooth map $h$ and $x \in X$, we define

$$\lambda_1(x, h) = \limsup_{n \to +\infty} \frac{\log \|Dh^n(x)\|}{n}$$

as the principal Lyapunov exponent of $x$ (with respect to $h$). We say a cycle $K = O(x, h)$ (of minimal period $p$) is linearly stable if $r_{\sigma}(Dh^p(x)) \leq 1$, where $r_{\sigma}(Dh^p(x))$ is the spectral radius of $Dh^p(x)$ ($x$ is also said to be a linearly stable $p$-periodic point of $h$). It is equivalent to $\lambda_1(x, h) \leq 0$ (see e.g., Hess and Poláčik [7, p.1316-1317]). Particularly, if $p = 1$, $x$ is called a linearly stable fixed point of $h$. Let $B \subset X$. $k$ is said to be a stable period for the restriction $h|_B$ if there is a linearly stable $k$-periodic point $x$ of $h$ such that the orbit $O(x, h) \subset B$. If $B = X$, we simply say that $k$ is a stable period of $h$. For brevity, we hereafter say $\omega(x, h)$ is a linearly stable cycle (of minimal period $p$), if $\omega(x, h)$ is a linearly stable cycle (of minimal period $p$) of $h$.

A continuous map $h : X \to X$ is pointwise dissipative if there is a bounded subset $B \subset X$ such that $B$ attracts each point of $X$. An invariant set $A$ is called an attractor of $h$ if $A$ is the maximal compact invariant set which attracts each bounded subset of $X$. If $h : X \to X$ is compact and pointwise dissipative, then there is a connected attractor $A$ of $h$ (see e.g., [6, Theorem 2.4.7]).

For a semiflow $\phi : \mathbb{R} \times X \to X$, the orbit of $x \in X$ is $O(x, \phi) = \{ \phi_t x : t \geq 0 \}$. The $\omega$-limit set of $x \in X$ is $\omega(x, \phi) = \bigcap_{s \geq 0} \{ \phi_{t+s} x : t \geq 0 \}$. A point $e \in X$ is called an equilibrium if $\phi_t(e) = e$ for any $t \geq 0$. We call a semiflow $\phi$ is $C^1$-smooth if the Fréchet derivative $D\phi_t(x)$ with respect to the state variable $x$ exists for each $x \in X$ and $t > 0$, and $x \mapsto D\phi_t(x)$ is continuous. For a $C^1$-smooth semiflow $\phi$ and $x \in X$, we define the principal Lyapunov exponent of $x$ (with respect to $\phi$) as

$$\lambda_1(x, \phi) = \limsup_{n \to +\infty} \frac{\log \|D\phi_t(x)\|}{t}.$$ 

An equilibrium $e$ is called linearly stable, if $\lambda_1(e, \phi) \leq 0$.

Let $M$ be a compact domain of $\mathbb{R}^d$ or a finite-dimensional Riemannian manifold and $h : M \to M$ be a $C^1$-map. We call that $h$ has observable chaos if $\lambda_1(\cdot, h) > 0$ holds on a positive Lebesgue measure set (see e.g., Young [35, Section 1, p.1441], [36, Section 4], [37, Section 3]). Similarly, a $C^1$-smooth semiflow $\phi : \mathbb{R} \times M \to M$ is said to have observable chaos if $\lambda_1(\cdot, \phi) > 0$ holds on a positive Lebesgue measure set. We say that a $C^1$-smooth semiflow $\phi_t$ (resp. mapping $h$) has transient chaos, if $\phi_t$ (resp. $h$) has horseshoes but no observable chaos.

In what follows, we introduce the definition and some significant properties of prevalence and shyness. A Borel subset $W \subset X$ is called shy if there exists a nonzero compactly supported Borel measure $\mu$ on $X$ such that $\mu(W + x) = 0$ for every $x \in X$. A Borel subset $W \subset X$ is
prevalent (in $X$) if its complement $X \setminus W$ is shy. Given a Borel subset $V \subset X$, we say that a Borel subset $W \subset X$ is prevalent in $V$ if $V \setminus W$ is shy.

Shyness has the following fundamental properties ( [13, 14]):

(i) Every Borel subset of a shy set is shy;
(ii) Every translation of a shy set is shy;
(iii) No nonempty open set is shy;
(iv) Every countable union of shy sets is shy;
(v) In finite-dimensional spaces, a Borel set $W$ is shy if and only if it has Lebesgue measure zero.

Now, we present a sufficient condition which guarantees a Borel subset $W \subset X$ to be shy.

**Proposition 2.1.** Let $W \subset X$ be a Borel subset and assume that there exists $v \gg 0$ such that $L \cap W$ is countable for every straight line $L$ parallel to $v$. Then $W$ must be shy in $X$.

**Proof.** See Enciso, Hirsch and Smith [3, Lemma 1].

The finite-dimensional space spanned by $v$ is also called a probe, which is a very useful tool to show prevalence (see e.g., [14]). In the following, we give a structure proposition of $U_-$ ($U_+$, resp.), which is crucial in our proof of prevalence in Section 3.

**Proposition 2.2.** Let $D \subset X$ be an open subset and $h : D \to D$ be strongly monotone. Assume that the set

$$D_0 = \{ x \in D : O(x, h) \text{ is relatively compact} \}$$

is dense in $D$. Let $J' \subset D_0$ be a simply ordered, open arc. Then the set $J'_- = J' \cap U_-(h)$ is at most countable. A corresponding result holds for $U_+(h)$.

**Proof.** See [26, Corollary 3.4].

**Lemma 2.3.** Let $h : X \to X$ be a $C^1$-map. If $z$ is a linearly stable $k$-periodic point of $h^q$, then $z$ is a linearly stable periodic point of $h$ of minimal period at most $kq$.

**Proof.** See the claim in the proof of [32, Corollary 2.5].

3 Improved prevalent dynamics/and its $C^1$-robustness for $C^1$-perturbed Systems

In this section, we will focus on the proof of the prevalence of “convergence to cycles whose minimal periods are uniformly bounded” (Theorem 3.3), as well as its $C^1$-robustness for $C^1$-perturbed systems (Theorem 3.8). Theorem 3.3 and Theorem 3.8 will play a crucial role in our approach for the nonexistence of observable chaos and its $C^1$-robustness in Section 4.

For $C^1$-smooth strongly monotone mapping $F_0$, we have the following sharpened dynamics alternative:
Lemma 3.1. (Sharpened $C^1$-dynamics alternative). Assume that (H1)-(H2) hold. Assume also $F_0$ is pointwise dissipative. Then there is an integer $m > 0$ such that, for any $x \in X$, either
\begin{itemize}
  \item[(a)] $\omega(x, F_0)$ is a linearly stable cycle of minimal period at most $m$; or,
  \item[(b)] there is a constant $\delta > 0$ such that, for any $y \in X$ satisfying $y < x$ or $y > x$,
$$\limsup_{n \to +\infty} \| F_0^n x - F_0^n y \| \geq \delta.$$ \end{itemize}

Proof. See Wang and Yao [32, Theorem A]. \qed

Remark 3.2. In fact, all the stable periods of $F_0$ are bounded above by $m$ (see [32, Corollary 2.5]).

Theorem 3.3. (Improved prevalent dynamics). Assume that (H1)-(H2) hold. Assume also $F_0$ is pointwise dissipative with an attractor $A$. Then there is an integer $m > 0$ such that the set
$$Q_0 := \{ x \in X : \omega(x, F_0) \text{ is a linearly stable cycle of minimal period at most } m \}$$
is prevalent in $X$. In particular, if $X = \mathbb{R}^d$, then the set $Q_0$ is of full Lebesgue measure in $X$.

Remark 3.4. For $C^1$-smooth strongly monotone discrete-time dynamical systems, Wang et al. [33, Theorem A] obtained the “convergence to linearly stable cycles” is a prevalent asymptotic behavior in the measure theoretic sense. Here, Theorem 3.3 improves [33, Theorem A] by showing that the set of minimal periods of linearly stable cycles is bounded by $m$. In addition, we succeed in proving the measurability of the sets involved by virtue of the boundedness of stable periods. Moreover, Theorem 3.3 is also shown to be robust under the $C^1$-perturbation (see Theorem 3.8).

In order to prove the improved prevalent dynamics in Theorem 3.3, it is important to prove first the set $Q_0$ is Borel. For this purpose, we need the following lemma.

Lemma 3.5. Let $B \subset X$ be a Borel subset and $h : B \to B$ be a $C^1$-map. Then the set
$$C_s := \{ x \in B : \omega(x, h) = z \text{ for some linearly stable fixed point } z \text{ of } h \}$$
is Borel.

Proof. Enciso, Hirsch and Smith [3, Appendix, Lemma 10] proved this lemma for $C^1$-semiflows. One can follow the exactly same arguments and obtain an analogous proof for $C^1$-maps. We omit it here. \qed

Proof of Theorem 3.3. Let the integer $m > 0$ be in Lemma 3.1. Define
$$Q_0 := \{ x \in X : \omega(x, F_0) \text{ is a linearly stable cycle of minimal period at most } m \},$$
and
$$Q'_0 := \{ x \in X : \omega(x, F_0^{ml}) \text{ is a linearly stable fixed point} \}.$$ We first show that $Q_0 = Q'_0$. It is clear that if $\omega(x, F_0)$ is a linearly stable cycle, then $\omega(x, F_0^{ml})$ is a linearly stable cycle. This entails that $Q_0 \subset Q'_0$. Notice from Remark 3.2 that all the stable periods of $F_0$ are bounded above by $m$. Then, $Q'_0 \subset Q_0$. Thus, $Q_0 = Q'_0$. 

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It follows from Lemma 3.5 that $Q_0'$ is Borel. So, $Q_0$ is also Borel. By virtue of Lemma 3.1, we can repeat the exactly same arguments in Poláčik and Tereščák [19, Section 5] to obtain that $X \setminus Q_0 \subset U_2(F_0)$. Proposition 2.2 implies that, for every simply ordered, open arc $J'$, $J' \cap U_2(F_0)$ is at most countable. So, $J' \setminus Q_0$ is also at most countable. Thus, Proposition 2.1 entails that $X \setminus Q_0$ is shy. Therefore, $Q_0$ is prevalent.

In particular, if $X = \mathbb{R}^d$, then $Q_0$ is of full Lebesgue measure in $X$, since the property (v) of shyness in Section 2. We have completed the proof.

Hereafter in this section, we consider the $C^1$-robustness for the improved prevalent dynamics.

**Lemma 3.6.** ($C^1$-robustness for sharpened $C^1$-dynamics alternative). Assume that (H1)-(H3) hold and $F_0$ is pointwise dissipative with an attractor $A$. Let $B_1$ be an open ball containing $A$. If

$$\sup \{ \| F_\epsilon x - F_0 x \| + \| D F_\epsilon (x) - D F_0 (x) \| : \epsilon \in J, x \in B_1 \} < \epsilon'$$

for some sufficiently small $\epsilon' > 0$, then there is a solid cone $C_1 \subset \text{Int} C$, an open bounded set $D_1$ ($B_1 \supset D_1 \supset A$) and integers $q, m_1 > 0$ such that, for each $|\epsilon|$ sufficiently small

(i). For each $n \geq q$, $F_\epsilon^n(D_1) \subset D_1$ and $F_\epsilon^n x \ll_1 F_\epsilon^n y$ whenever $x \ll_1 y$ (with $x, y \in D_1$).

(ii). For each $x \in D_1$, either

(a) $\omega(x, F_\epsilon^n)$ is a linearly stable cycle of minimal period at most $m_1$; or,

(b) there is a constant $\delta > 0$ such that, for any $y \in D_1$ satisfying $y \ll_1 x$ or $y \gg_1 x$,

$$\limsup_{n \to +\infty} \| F_\epsilon^{nq} x - F_\epsilon^{nq} y \| \geq \delta.$$

**Proof.** For the proof of item (i), we refer to Tereščák [28, Theorem 5.1]. For the proof of item (ii), see [32, Theorem 3.1].

**Remark 3.7.** In fact, for any $|\epsilon|$ sufficiently small, all the stable periods of $F_\epsilon^n|_{\overline{D_1}}$ are bounded above by $m_1$ (see [32, Proposition 2.4]). In addition, for any open bounded subset $D_2$ (satisfying $D_1 \supset \overline{D_2} \supset D_2 \supset A$), one can choose $\epsilon'$ smaller and let the integer $q > 0$ larger (if necessary) in Lemma 3.6 such that, both $D_1$ and $D_2$ satisfies items (i)-(ii) in Lemma 3.6 (see [32, Remark 2.3] or [28, Eq.(5.11) on p.19]). Moreover, one can also choose $D_2$ to be connected, since $A$ is connected. If $X = \mathbb{R}^d$, this leads $\overline{D_2}$ to be a compact domain.

Hereafter in this paper, we always reserve the open bounded subsets $D_1, D_2$ (with $D_1 \supset \overline{D_2} \supset D_2 \supset A$) and the integers $q, m_1 > 0$ as in Lemma 3.6 and Remark 3.7.

The following theorem reveals that the improved prevalent dynamics of $F_0$ (Theorem 3.3) is robust under the $C^1$-perturbation.

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Theorem 3.8. ($C^1$-robustness for improved prevalent dynamics). Let all hypotheses in Lemma 3.6 hold. Then there exists an open bounded set $D_1 \supset A$ and an integer $m > 0$ such that, for any $|\varepsilon|$ sufficiently small, the set

$$Q_\varepsilon := \{ x \in D_1 : \omega(x, F_\varepsilon) \text{ is a linearly stable cycle of minimal period at most } m \}$$

is prevalent in $D_1$. In particular, if $X = \mathbb{R}^d$, then the set $Q_\varepsilon$ is of full Lebesgue measure in $D_1$.

Proof. Let the open bounded subset $D_1$ and the integers $q, m_1 > 0$ be in Lemma 3.6. Clearly, it follows from (H3) that $F^q_\varepsilon : D_1 \rightarrow D_1$ is compact. Moreover, for each $|\varepsilon|$ sufficiently small, Lemma 3.6(i) directly implies that $F^q_\varepsilon : D_1 \rightarrow D_1$ is strongly monotone with respect to $C_1$. By Lemma 3.6(ii), we can repeat the exactly same arguments in the proof of Theorem 3.3 (with $F_0$ replaced by $F^q_\varepsilon$ there) to obtain that, the set

$$\tilde{Q}_\varepsilon := \{ x \in D_1 : \omega(x, F^q_\varepsilon) \text{ is a linearly stable cycle of minimal period at most } m_1 \}$$

(3.1)

is prevalent in $D_1$, for any $|\varepsilon|$ sufficiently small.

Now, we define

$$Q_\varepsilon := \{ x \in D_1 : \omega(x, F_\varepsilon) \text{ is a linearly stable cycle of minimal period at most } m \},$$

where $m = m_1q$. On one hand, it is clear that if $\omega(x, F_\varepsilon)$ is a linearly stable cycle, then $\omega(x, F^q_\varepsilon)$ is a linearly stable cycle. Then, the bound $m_1$ of stable periods of $F^q_\varepsilon|_{\mathcal{P}}$ in Remark 3.7 entails that, $Q_\varepsilon \subset \tilde{Q}_\varepsilon$. On the other hand, it follows from Lemma 2.3 that $\tilde{Q}_\varepsilon \subset Q_\varepsilon$. Then, we have proved $Q_\varepsilon = \tilde{Q}_\varepsilon$. Thus, $\tilde{Q}_\varepsilon$ is also prevalent in $D_1$, for any $|\varepsilon|$ sufficiently small.

In particular, if $X = \mathbb{R}^d$, then $Q_\varepsilon$ is of full Lebesgue measure in $X$, since the property (v) of shyness in Section 2. This completes the proof. □

Remark 3.9. We specially point out that, one can apply our theoretical result in this section (Theorem 3.3 and Theorem 3.8) to obtain that, the improved prevalence of convergence to periodic solutions with a uniform bound of minimal periods, for time-periodic parabolic equations and their perturbed systems (see e.g. the equations in [7,20,32,33]).

4 Proof of the Main Results

We focus on in this section and prove Theorem A-C. In order to prove these results, we need the following formula for the spectral radius of an operator.

Lemma 4.1. Let $T \in \mathcal{L}(X)$. Then $r_\sigma(T) = \inf \|T\|$, where the infimum is taken over all norms $\| \cdot \|$ on $X$ equivalent to $\| \cdot \|$.

Proof. See Holmes [15, Theorem on p.164]. For finite-dimensional case, see also Horn and Johnson [16, Lemma 5.6.10]. □
Now, we are in position to prove Theorem A-C.

Proof of Theorem A. Let the integer $m > 0$ be obtained in Theorem 3.3. Take $T := F_0^m$. We first assert that: If $x \in X$ with $\omega(x, T) = z$ for some linearly stable fixed point $z$ of $T$, then $\lambda_1(x, F_0) \leq 0$.

In fact, one has $r_\sigma(DT(z)) \leq 1$, since $z$ is a linearly stable fixed point of $T$. Then by Lemma 4.1, for any $\tilde{c} > 0$, there exists a norm $| \cdot |$ equivalent to $\| \cdot \|$ on $X$ such that

$$|DT(z)| < r_\sigma(DT(z)) + \tilde{c}.$$ 

Hence, $|DT(z)| < 1 + \tilde{c}$. Recall that $T$ is $C^1$. Then there exists a $\delta > 0$ such that,

$$|DT(y)| < 1 + 2\tilde{c},$$

for any $y \in X$ with $|y - z| < \delta$.

It follows from $\omega(x, T) = z$ that, there exists an integer $N > 0$ such that, $|T^n x - z| < \delta$ for any $n \geq N$. Thus, the chain rule shows that

$$\lambda_1(x, T) = \limsup_{n \to +\infty} \frac{\log |DT^n(x)|}{n} \leq \limsup_{n \to +\infty} \frac{\log |DT^N(x)| + \log |DT^{n-N}(T^N x)|}{n} \leq \limsup_{n \to +\infty} \frac{\log |DT^N(x)| + (n - N) \log(1 + 2\tilde{c})}{n} \leq \log(1 + 2\tilde{c}).$$

Since the arbitrary of $\tilde{c}$, we have $\lambda_1(x, T) \leq 0$.

For any integer $n \geq 1$, write $n = k_n m! + l_n$, where $k_n \geq 0$ and $l_n \in \{0, 1, 2, \cdots, m!-1\}$. Then

$$\lambda_1(x, F_0) = \limsup_{n \to +\infty} \frac{\log |DF_0^n(x)|}{n} = \limsup_{n \to +\infty} \frac{\log |DF_0^{k_n m! + l_n}(x)|}{k_n m! + l_n} \leq \limsup_{n \to +\infty} \frac{\log(M |DF_0^{k_n m!}|) - \log(k_n m! + l_n)}{k_n m! + l_n} = \frac{1}{m!} \limsup_{n \to +\infty} \frac{\log |DT^k(x)|}{k} = \frac{1}{m!} \lambda_1(x, T) \leq 0,$$

where $M := \max\{|DF_0^l(u)| : 0 \leq l \leq m!-1, u \in \overline{O(x, F_0)}\}$. Thus, we have proved the assertion.

Define

$$S_0 := \{x \in X : \lambda_1(x, F_0) \leq 0\}.$$

Recall that $F_0$ is $C^1$. Then the function $x \mapsto \frac{\log |DF_0^n(x)|}{n}$ is Borel-measurable, for any $n \geq 1$. Hence, $x \mapsto \lambda_1(x, F_0)$ is Borel-measurable. Thus, $S_0$ is Borel. Now, we define

$$Q_0 := \{x \in X : \omega(x, T) \text{ is a linearly stable fixed point}\}.$$ 

The assertion entails that $Q_0 \subset S_0$. It follows from the proof of Theorem 3.3 that $Q_0$ is prevalent in $X$. Then $S_0$ is also prevalent in $X$, since the property (i) of shyness in Section 2.

In particular, if $X = \mathbb{R}^d$, $S_0$ is of full Lebesgue measure in $X$, since the property (v) of shyness in Section 2. It follows that $F_0$ cannot have observable chaos. Thus, we have proved Theorem A. \hfill $\square$
Proof of Theorem B. Let the open bounded subsets \( D_1, D_2 \) (with \( D_1 \supset \overline{D}_2 \supset D_2 \supset A \)) and the integer \( q > 0 \) be in Lemma 3.6 and Remark 3.7. Define the closed bounded set \( M := \overline{D}_2 \). We note that \( F_t^q(M) \subset M \), since \( F_t^q(D_2) \subset D_2 \). Define \( S_{\epsilon, D_1} := \{ x \in D_1 : \lambda_1(x, F_t^q) \leq 0 \} \).

By virtue of (3.1), one can follow the exactly same proof of Theorem A to obtain that, \( S_{\epsilon, D_1} \) is prevalent in \( D_1 \). That is to say, \( D_1 \setminus S_{\epsilon, D_1} \) is shy. Now, we define \( S_{\epsilon} := \{ x \in M : \lambda_1(x, F_t) \leq 0 \} \).

Clearly, \( M \setminus S_{\epsilon} \subset D_1 \setminus S_{\epsilon, D_1} \). Moreover, the same reason of \( S_0 \) is Borel in the proof of Theorem A also entails that, \( S_{\epsilon} \) is Borel. So \( M \setminus S_{\epsilon} \) is shy, since the property (i) of shyness in Section 2. Thus, \( S_{\epsilon} \) is prevalent in \( M \).

In particular, if \( X = \mathbb{R}^d \), then \( M \) is a compact domain in \( \mathbb{R}^d \). Thus \( S_0 \) is of full Lebesgue measure in \( M \), since the property (v) of shyness in Section 2. It follows that \( F_t^q|_M \) cannot have observable chaos. The proof is completed.

Proof of Theorem C. Define the sets \( C_s := \{ x \in X : \omega(x, \phi) \text{ is a linearly stable equilibrium} \} \), and \( S_{\phi} := \{ x \in X : \lambda_1(x, \phi) \leq 0 \} \).

Enciso, Hirsch and Smith [3, Theorem 1] entails that, \( C_s \) is prevalent in \( X \). We are going to prove \( C_s \subset S_{\phi} \). Suppose that \( x \in C_s \) and \( \omega(x, \phi) = e \). Let \( \tilde{t} > 0 \). Then \( r_{\sigma}(D\phi_t(e)) \leq 1 \). Thus we can repeat the exactly same arguments in the proof of Theorem A (with \( F_0 \) and \( T \) replaced by \( \phi_t \) there) to obtain that, \( \lambda_1(x, \phi_{\tilde{t}}) \leq 0 \). Thus, \( \lambda_1(x, \phi) \leq 0 \) since the chain rule and the compactness of \( \overline{O}(x, \phi) \). Hence, \( x \in S_{\phi} \). That is to say, \( C_s \subset S_{\phi} \).

Recall that \( \phi \) is a \( C^1 \)-semiflow. Then \( x \mapsto \lambda_1(x, \phi) \) is Borel-measurable. Thus, \( S_{\phi} \) is Borel. Therefore, \( S_{\phi} \) is also prevalent in \( X \), since the property (i) of shyness in Section 2.

In particular, if \( X = \mathbb{R}^d \), \( S_{\phi} \) is of full Lebesgue measure in \( X \), since the property (v) of shyness in Section 2. It follows that \( \phi \) cannot have observable chaos. Thus, we have proved Theorem C.

\[ \square \]

Remark 4.2. A large number of existing works have shown that monotone dynamical systems may have complicated dynamics, including horseshoes (see e.g., [2, 21–24, 29, 30] and references therein). As a consequence of Theorem A-C, those chaotic behavior in monotone dynamical systems are all transient chaos.
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