Four-photon interference: a realizable experiment
to demonstrate violation of EPR postulates
for perfect correlations

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Short title: Violation of EPR for perfect correlations

Abstract
Bell’s theorem reveals contradictions between the predictions of quantum mechanics and the EPR postulates for a pair of particles only in situations involving imperfect statistical correlations. However, with three or more particles, contradictions emerge even for perfect correlations. We describe an experiment which can be realized in the laboratory, using four-photon entangled states generated by parametric down-conversion, to demonstrate this contradiction at the level of perfect correlations.

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1 Introduction

Einstein, Podolsky and Rosen [1] (EPR) presented their famous *gedanken-experiment* in 1935 with the aim of showing that quantum mechanics (QM) was not a complete description of physical reality. A complete description, in their view, would require the introduction of additional variables, usually referred to as hidden variables. They outlined a program to reproduce the predictions of QM using local hidden variable (LHV) theories.

This program was challenged by Bell [2] in 1964 when he proved that any hidden variable theory that incorporated the concepts of locality and reality would be inconsistent with certain predictions of QM. In particular, he showed [3] that it was possible to derive from the postulates of EPR an inequality which was violated by statistical predictions of QM for a pair of particles. This violation has been observed in a number of experiments involving interference of pairs of photons produced in an entangled state [4-7].

A shortcoming of all these experiments is that, with a pair of particles, Bell’s theorem reveals contradictions between the predictions of QM and EPR’s postulates only in situations involving imperfect statistical correlations: no contradictions appear with perfect correlations. A way to overcome this limitation, which was proposed by Greenberger, Horne and Zeilinger [8] (GHZ) is to use three or more particles in an entangled state. Greenberger *et al.* [9] (GHSZ) have shown that, in this case, contradictions emerge even at the level of perfect correlations. They also described a *gedankenexperiment* using three entangled photons to illustrate this point.

Recent work suggests that it is possible to generate entangled four-photon states by parametric down-conversion of two pump photons [10]. Based on this work, we describe a design for a four-photon interference experiment
which can be realized in the laboratory. We show how, with just two measurements, one can demonstrate that quantum mechanics contradicts LHV theories, even at the level of perfect correlations.

2 Generation of four photon entangled states

Parametric down conversion in a crystal exhibiting a $\chi^{(2)}$ nonlinearity makes it possible to convert a pump photon (frequency $\omega_p$) into a pair of highly correlated photons with frequencies $\omega_d^{(1)}$ and $\omega_d^{(2)}$, where $\omega_d^{(1)} + \omega_d^{(2)} = \omega_p$ [11-13]. These photons are generated almost simultaneously (within the correlation time $\tau_d$ of the down-converted photons). While the frequency of each down-converted photon may vary over an appreciable range, the sum of their frequencies is fixed to within the pump bandwidth. The down-converted photons are therefore described by an energy entangled state.

Recent work has shown that it should be possible to extend this process to generate entangled four-photon states from two pump photons by achieving the required phase-matching conditions in a non-linear crystal with two non-collinear pump beams [10]. While the susceptibility for this two-photon down-conversion process is low, the gain depends on the second power of the pump amplitude, so that it should be possible to obtain an appreciable yield by using pulsed pump beams with high peak power. Further improvements in yield may be possible by the use of a resonant cavity (See Appendix D for further details of the experiment).

In such an arrangement,

$$\omega_d^{(1)} + \omega_d^{(2)} + \omega_d^{(3)} + \omega_d^{(4)} = 2\omega_p,$$

(1)

where $\omega_d^{(1)},...,\omega_d^{(4)}$ are the frequencies of the down-converted photons and
\( \omega_p \) is the frequency of the pump photons, and

\[
\mathbf{k}_d^{(1)} + \mathbf{k}_d^{(2)} + \mathbf{k}_d^{(3)} + \mathbf{k}_d^{(4)} = \mathbf{k}_p^{(1)} + \mathbf{k}_p^{(2)},
\]

(2)

where \( \mathbf{k}_d^{(1)}, \ldots, \mathbf{k}_d^{(4)} \) are the wave vectors of the down-converted photons, and \( \mathbf{k}_p^{(1)} \) and \( \mathbf{k}_p^{(2)} \) are the wave vectors of the pump photons.

## 3 Four photon interferometer

The four-photon interferometer shown in figure 1 is an extension of a two-photon interferometer described by Franson [5] in which each of the four down-converted photons enters one of four interferometers, each with a short path \( (s_j) \) of length \( S_j \) and a long path \( (l_j) \) of length \( L_j \). The optical path difference \( \Delta L_j = L_j - S_j \) in each of the interferometers, which can be varied by translating the right-angle prisms, is greater than the coherence length \( c \tau_d \) of the down-converted photons, so that no second-order interference effects due to single photons are observed in the individual interferometers.

If, in a pair of interferometers, we consider the four processes leading to photon counts \( (s_i - s_j, s_i - l_j, l_i - s_j, l_i - l_j) \), and the difference of the optical path differences \( (\Delta L_{ij} = \Delta L_i - \Delta L_j) \) is less than the coherence length of the pump beam, the \( l_i - l_j \) and \( s_i - s_j \) processes are indistinguishable from each other. However, the other two processes \( (s_i - l_j \text{ and } l_i - s_j) \) can be distinguished from the \( l_i - l_j \) and \( s_i - s_j \) processes by the relative time lag of the photons [15]. It is then possible, with fast coincidence counters, to reject counts arising from the \( s_i - l_j \) and \( l_i - s_j \) processes, so that we are only concerned with coincidences due to the \( l_1 - l_2 - l_3 - l_4 \) and \( s_1 - s_2 - s_3 - s_4 \) processes in the four interferometers (See Appendix D for further comments on coincidence counts).
4 Four photon interference

A four-photon event is recorded when photons are detected in coincidence (within the detector response time) in all the four interferometers. Since the four photons are generated (almost) simultaneously, such a coincidence could either be due to four photons which all took the short path ($|s >_1 |s >_2 |s >_3 |s >_4$) or the long path ($|l >_1 |l >_2 |l >_3 |l >_4$). Following Feynman [14], we can compute the amplitude (see refs. [15] and [16]) for the arrival of four coincident photons by summing the amplitudes for these indistinguishable alternatives. We have

\[ |\Psi > = |s >_1 |s >_2 |s >_3 |s >_4 + \exp(i\Phi)|l >_1 |l >_2 |l >_3 |l >_4, \]  

(3)

where the relative phase $\Phi$ of the interfering $s_1 - s_2 - s_3 - s_4$ and $l_1 - l_2 - l_3 - l_4$ processes is the sum of the relative phases acquired by the individual photons in the four interferometers, so that

\[ \Phi = \phi_1 + \phi_2 + \phi_3 + \phi_4, \]  

(4)

where $\phi_i = (\omega_p/2c)\Delta L_i, (i = 1, ..., 4)$ is the phase difference between the beams traversing the two arms of the $i$th interferometer. The predicted coincidence count is obtained by squaring the amplitude and is therefore proportional to

\[ |(1/2)(1 + \exp(i\Phi))|^2 = (1/2)(1 + \cos(\Phi)), \]  

(5)

where the constant of proportionality includes the intensity of the source, the detector efficiency (See Appendix C) and the losses in the system. A formal field-theoretic analysis which leads to the same result is presented in Appendix A. If the detectors are as nearly alike as possible, we can assume
fair sampling, so that the number of coincidences actually measured in any situation is proportional to those expected for a perfect system.

As can be seen, QM predicts that the coincidence rate $R_c$ depends only on $\Phi$, the sum of the phase delays $\phi_i$ in the four interferometers. The coincidences will be perfectly correlated ($R_c = 1$) when $\Phi = 0$ and perfectly anticorrelated ($R_c = 0$) when $\Phi = \pi$. When $\Phi = 0$, detection of a photon in three interferometers would imply the coincident detection of a photon in the fourth interferometer. When $\Phi = \pi$, detection of a photon in three of the interferometers would preclude the coincident detection of a photon in the fourth interferometer. Let us define a parameter (analogous to the visibility for sinusoidal fringes)

$$Q = \frac{R_c(0) - R_c(\pi)}{R_c(0) + R_c(\pi)}$$

whose value quantum mechanics predicts to be unity. As we will see in the next section, LHV theories cannot explain this value of $Q$.

5 LHV predictions

It is convenient in discussing the four-photon interferometer to use the language of spins traditionally used in the EPR literature. Traversals of the long and short arms of an interferometer are thought of as basis states $|s>$ and $|l>$ corresponding to “spin up” and “spin down” along the $z$ axis. The superposition of these states with a phase difference $\phi$

$$|\psi> = 1/\sqrt{2}(|s> + \exp(i\phi)|l>),$$

in spin language, is a state on the equator of the Poincaré sphere of states of a spin-half particle where $|l>$ and $|s>$ are the North and South poles. The choice of a phase delay $\phi_i$ in the $i$th interferometer corresponds to the choice
of a direction in the $x - y$ plane along which one measures spin in Bohm’s version [17] of the EPR gedankenexperiment.

An LHV description [8] of the four photon interferometer requires the use of a space $\Lambda$, the space of complete states whose elements are written $\lambda$, with a probability measure $\rho$. The expectation value of coincidence counts is then

$$\frac{1 + E^\#(\phi_1, \phi_2, \phi_3, \phi_4)}{2},$$

where

$$E^\#(\phi_1, \phi_2, \phi_3, \phi_4) = \langle A(\phi_1)B(\phi_2)C(\phi_3)D(\phi_4) \rangle$$

$$= \int_\Lambda A_\lambda(\phi_1)B_\lambda(\phi_2)C_\lambda(\phi_3)D_\lambda(\phi_4) d\rho,$$

and $A_\lambda(\phi_1), B_\lambda(\phi_2), C_\lambda(\phi_3), D_\lambda(\phi_4)$ are four functions of $\lambda$ which take values $\pm 1$. Locality is built into the theory by the fact that $A_\lambda(\phi_1)$ is independent of $\phi_2, \phi_3, \phi_4$, $B_\lambda(\phi_2)$ is independent of $\phi_1, \phi_3, \phi_4$, and so on.

5.1 Perfect correlations: ideal experiment

Following GHZ [8, 9] we can show that an LHV theory cannot reproduce the predictions of QM even at the level of perfect correlations.

Proof: Let us suppose functions $A_\lambda(\phi_1), B_\lambda(\phi_2), C_\lambda(\phi_3), D_\lambda(\phi_4)$ exist, satisfying the relations

$$\langle A(\phi_1)B(\phi_2)C(\phi_3)D(\phi_4) \rangle = 1, \text{ for } \Phi = 0,$$

and

$$\langle A(\phi_1)B(\phi_2)C(\phi_3)D(\phi_4) \rangle = -1, \text{ for } \Phi = \pi.$$
Since the quantity in brackets can only take values ±1, it follows that everywhere in Λ (except possibly for a set of measure zero),

$$A_\lambda(\phi_1)B_\lambda(\phi_2)C_\lambda(\phi_3)D_\lambda(\phi_4) = 1, \text{ for } \Phi = 0,$$  \hspace{1cm} (12)

and

$$A_\lambda(\phi_1)B_\lambda(\phi_2)C_\lambda(\phi_3)D_\lambda(\phi_4) = -1, \text{ for } \Phi = \pi.$$

(13)

It then follows that

$$A_\lambda(-\phi)C_\lambda(\phi)D_\lambda(0)B_\lambda(0) = 1, \hspace{1cm} (14)$$

and

$$B_\lambda(0)A_\lambda(-\phi)D_\lambda(\phi)C_\lambda(0) = 1. \hspace{1cm} (15)$$

Multiplying equations (14) and (15), we get, since $A_\lambda(-\phi)^2 = B_\lambda(0)^2 = 1$,

$$C_\lambda(0)D_\lambda(0)C_\lambda(\phi)D_\lambda(\phi) = 1. \hspace{1cm} (16)$$

But

$$A_\lambda(0)B_\lambda(0)C_\lambda(0)D_\lambda(0) = 1. \hspace{1cm} (17)$$

Therefore

$$A_\lambda(0)B_\lambda(0)C_\lambda(\phi)D_\lambda(\phi) = 1. \hspace{1cm} (18)$$

However, if we set $\phi = \pi/2$ in equation (18), it contradicts equation (13). It follows that functions $A_\lambda(\phi_1), B_\lambda(\phi_2), C_\lambda(\phi_3), D_\lambda(\phi_4)$ satisfying equations (12) and (13) do not exist. Accordingly, LHV theories cannot reproduce the predictions of QM even at the level of perfect correlations.

5.2 Perfect correlations: real experiment

From section 4 we see that QM predicts a $Q$ of unity in an ideal experiment. However, in any real experiment, one would obtain a value for $Q$ less than
unity because of imperfections in the system. However, as noted by Ryff [18], “if a theorem is valid whenever we have perfect correlations, it cannot be totally wrong in the case of almost perfect correlations”. We show below that with four-photon interference, a value of \( Q \) greater than 0.5 is enough to rule out LHV theories. We do this by going beyond the original argument of GHZ [8, 9] to allow for experimental imperfections(\( Q < 1 \)). Mermin [19] has given an elegant and general analysis of the contradiction between quantum mechanics and LHV theories for \( n \) spin-1/2 particles in an entangled state and our bound on \( Q \) agrees with the restriction of Mermin’s analysis to the case of four particles.

Given two functions \( f \) and \( g \) on \( \Lambda \), let us define an inner product (or cross correlation)

\[
<f g> = \int_{\Lambda} f_{\lambda} g_{\lambda} d\rho.
\]

(19)

We will only need to deal with functions which satisfy the condition

\[
<f f> = 1.
\]

(20)

We then have the following lemma.

Lemma: Let \( f, g, h \) be three functions on \( \Lambda \) with values \( \pm 1 \). Then,

\[
<f h> \geq <f g> + <g h> - 1.
\]

(21)

We present a proof and a geometrical interpretation of this lemma in Appendix B.

Let us then suppose that there exist functions \( A_\lambda(\phi_1), B_\lambda(\phi_2), C_\lambda(\phi_3), D_\lambda(\phi_4) \) satisfying the relations

\[
<A(\phi_1)B(\phi_2)C(\phi_3)D(\phi_4)> = Q, \text{ for } \Phi = 0,
\]

(22)
and

\[ < A(\phi_1)B(\phi_2)C(\phi_3)D(\phi_4) >= -Q, \text{ for } \Phi = \pi, \] (23)

where \( 0 \leq Q \leq 1 \). (Note that in the limit, when \( Q \to 1 \), we recover equations (10) and (11).) We can no longer argue, as we did before, that the angular brackets in equations (10) and (11) can be removed. However, the lemma can be used to determine the maximum allowed value for \( Q \).

From equation (22), it follows that

\[ < (A(-\phi)C(\phi)D(0))(B(0)) >= Q, \] (24)

and

\[ < (B(0))(A(-\phi)D(\phi)C(0)) >= Q. \] (25)

If we use the lemma, with

\[
\begin{align*}
    f &= A(-\phi)C(\phi)D(0), \\
    g &= B(0), \\
    h &= A(-\phi)D(\phi)C(0),
\end{align*}
\] (26-28)

and remember that \((A_\lambda(-\phi))^2 = (B_\lambda(0))^2 = 1\), we get

\[ < C(0)D(0)C(\phi)D(\phi) >= 2Q - 1. \] (29)

However,

\[ < A(0)B(0)C(0)D(0) >= Q, \] (30)

so that, if we apply the lemma to these two relations, we find that

\[ < A(0)B(0)C(\phi)D(\phi) >= 3Q - 2. \] (31)

If then, we set \( \phi = \pi/2 \) and use equation (23), we find that

\[ (-Q) \geq 3Q - 2, \] (32)
from which it follows that
\[ Q \leq \frac{1}{2}. \quad (33) \]

This result proves that LHV theories cannot yield a value of \( Q \) greater than 0.5.

Note that in our adaptation of the original GHZ argument [8, 9], it is not necessary to set the individual phase differences \( \phi_1, ..., \phi_4 \) to 0 (or, more correctly, \( 2m\pi \)): it is only necessary to set \( \Phi \), the sum of these phase differences, to 0 (or \( 2m\pi \)). In practice, it is difficult (nearly impossible) to set the individual phase differences to any preassigned value, since the optical path differences in the individual interferometers are greater than the coherence lengths of the down-converted photons; our adaptation eliminates this problem and makes the experiment feasible.

In the actual experiment, one of the four interferometers is adjusted initially so that the coincidence rate is a maximum. The first measurement therefore corresponds to the condition \( \Phi = 2m\pi \). A phase shift of \( \pi \) is then introduced in any one of the interferometers and the event rate is measured at the resulting minimum. (Note that the introduction of a further phase shift of \( \pi \) in any of the interferometers would bring the event rate back to a maximum). The results of these two measurements are inserted in equation (6) to obtain the value of the quantity \( Q \). Any value greater than 0.5 represents a breakdown of LHV theories under perfect correlations.

The only data used correspond effectively to values of \( \Phi \) of 0 (eq. 22) and \( \pi \) (eq. 23). This is very much in the spirit of the original GHZ argument [8, 9], which relies only on perfect correlations.
5.3 Statistical correlations

This experiment also makes it possible to demonstrate violations of the original Bell inequality (which uses statistical correlations for two particles) in systems of four particles. In this case, it is not necessary to adjust the value of $\Phi$ for the first measurement so that the coincidence rate is a maximum; $\Phi$ can have any arbitrary value (say) $\Phi_0$. Three more measurements of the event rate are then made after introducing phase shifts of $\pi/2$, successively, in three of the interferometers. We then have four values of the event rate corresponding to values of $\Phi$ of $\Phi_0, \Phi_0 + \pi/2, \Phi_0 + \pi$ and $\Phi_0 + 3\pi/2$.

If one assumes that the fringe profile is sinusoidal, one can easily determine the fringe visibility from these four measurements. Whereas quantum mechanics predicts a visibility of unity, it has been shown [20, 21, 22] that LHV theories cannot explain a fringe visibility greater than $1/(2\sqrt{2})$. In this respect, a four-photon experiment offers a more probing test than three-photon experiments for which the critical visibility is 1/2. However, as explained in [21] this lower value for the critical visibility relies on the use of statistical correlations rather than perfect correlations.

6 Conclusion

While most theoretical studies related to EPR have involved spin-(1/2) particles, actual experiments have used optical analogs of such systems. In particular, all interferometric tests of Bell’s inequality carried out so far have used entangled two-photon states [4-7]. In this case, LHV theories do not contradict QM at the level of perfect correlations. Therefore, tests of LHV theories with two-photon states require measurements of statistical correlations. EPR experiments involving more than two particles (as in section 5.3)
utilize extensions of Bell’s inequality and, therefore, also involve statistical correlations. On the other hand, tests such as those described in section 5.2 are based on the GHZ analysis [8, 9] and, therefore, only involve perfect correlations. Since perfect correlations formed the basis of the original EPR criterion for “elements of reality”, the contradiction emerging from the GHZ analysis [8, 9] strikes at the heart of the EPR program. We have described a realizable experiment involving four-photon interference which demonstrates the conflict between EPR and QM even at the level of perfect correlations.

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Appendix A

The field theoretic analysis presented by Franson [5] can be easily extended to the case of four-photon interference. A field theoretic description has the advantage that it is manifestly local. We sketch the main ideas below using his notation.

We need only deal with scalar fields since the polarization is fixed throughout. The scalar field operator \( \psi(\vec{r}, t) \) is expanded in free space modes as

\[
\psi(\vec{r}, t) = \sum_{\vec{k}} \frac{a_{\vec{k}}}{\sqrt{V}} \exp(i(\vec{k} \cdot \vec{r} - \omega t)).
\] (34)

The time evolution of this operator is governed by the free Hamiltonian of the electromagnetic field and since

\[
\psi(x + c \Delta t, t) = \psi(x, t - \Delta t),
\] (35)

the particle it describes moves at the speed of light.

The field operator at the detector of the \( i \)th interferometer with the beam splitter removed is given by \( \psi_0(\vec{r}_i, t) \). For each pair \((i, j)\) of the interferometers, these operators satisfy the condition

\[
\psi_0(\vec{r}_i, t)\psi_0(\vec{r}_j, t \pm \Delta t)|0>= 0,
\] (36)

which is analogous to Franson's equation (5). With the beam splitter inserted in the interferometer, the field operator at the \( i \)th detector becomes

\[
\psi(\vec{r}_i, t) = (1/2)(\psi_0(\vec{r}_i, t) + \exp(i\phi_i)\psi_0(\vec{r}_i, t - \Delta t)),
\] (37)

where \( i = 1, ..., 4 \). The coincidence rate \( R_c \) for the four detectors \( D_1, D_2, D_3, D_4 \), with the beam splitters inserted, is then
\[ R_c = \eta_1 \eta_2 \eta_3 \eta_4 \times < 0|\psi_t^{\dagger}(r_1, t)\psi_t^{\dagger}(r_2, t)\psi_t^{\dagger}(r_3, t)\psi_t^{\dagger}(r_4, t)\psi_t(r_1, t)\psi_t(r_2, t)\psi_t(r_3, t)\psi_t(r_4, t)|0 >, \]  
(38)

where \( \eta_i \) is the efficiency of the \( i \)th detector. Substituting (37) in (38) and using (36), we obtain the result
\[ R_c = \left( \frac{R_{c0}}{2\eta} \right) \left( 1 + \cos(\Phi) \right), \]  
(39)

where
\[ R_{c0} = \eta_1 \eta_2 \eta_3 \eta_4 \times < 0|\psi_0^{\dagger}(\vec{r}_1, t)\psi_0^{\dagger}(\vec{r}_2, t)\psi_0^{\dagger}(\vec{r}_3, t)\psi_0^{\dagger}(\vec{r}_4, t)\psi_0(\vec{r}_1, t)\psi_0(\vec{r}_2, t)\psi_0(\vec{r}_3, t)\psi_0(\vec{r}_4, t)|0 > \]  
(40)

and \( \Phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 \).

**Appendix B**

Let us consider three real-valued functions \( f, g, h \) on \( \Lambda \). One can think of these functions as elements of a vector space. We only need to work in a three dimensional subspace containing \( f, g, h \). If \( f, g, h \) are unit vectors, \( \langle fg \rangle = \cos(\theta_{fg}), \langle gh \rangle = \cos(\theta_{gh}) \) and \( \langle fh \rangle = \cos(\theta_{fh}) \), where the angles \( \theta_{fg}, \theta_{gh}, \theta_{fh} \) are defined to be less than \( \pi \) and represent the angles between the unit vectors \( f, g, h \). These angles can also be interpreted as the lengths of the shortest geodesics between the tips of the vectors \( f, g, h \) on the unit sphere. From the triangle inequality, it then follows that
\[ \theta_{fh} \leq \theta_{fg} + \theta_{gh}. \]  
(41)
Since \( \cos(\theta) \) is a decreasing function of its argument for \( 0 \leq \theta \leq \pi \), we arrive at the result

\[
\arccos(< fh >) \geq \arccos(< fg >) + \arccos(< gh >). \tag{42}
\]

This inequality has a clear interpretation in terms of the triangle inequality on the unit sphere.

The inequality we use in the text is similar in spirit. We only need to deal with functions \( f, g, h \) which take values \( \pm 1 \) and, in this case, can make a stronger statement. (Such functions and their correlations are of interest in digital signal processing, in communication theory [23] and radio astronomy [24]). We then have

\[
< fg > = 2\Omega(fg) - 1, \tag{43}
\]

where \( \Omega(fg) \) is the volume of the domain \( \mathcal{D}(fg) \) in \( \Lambda \) where \( f \) and \( g \) agree. Similarly, \( < gh > = 2\Omega(gh) - 1 \) and \( < fh > = 2\Omega(fh) - 1 \). Since the domain of agreement \( \mathcal{D}(fh) \) between \( f \) and \( h \) includes at least \( \mathcal{D}(fg) \cap \mathcal{D}(gh) \), the intersection of the domains of agreement between \( f \) and \( g \) (\( \mathcal{D}(fg) \)) and between \( g \) and \( h \) (\( \mathcal{D}(gh) \)), we conclude that

\[
\Omega(fh) \geq \Omega(fg) + \Omega(gh) - 1. \tag{44}
\]

It follows immediately that

\[
< fh > \geq < fg > + < gh > - 1 \tag{45}
\]

It is worth noting that if we write \( F(x) = (1-x)/2 \), inequality (45) reads

\[
F(< fh >) \leq F(< fg >) + F(< gh >), \tag{46}
\]
which is similar to inequality (42), with $F(x)$ replacing the function $\arccos(x)$. Both these inequalities express the idea that if $f$ and $g$ are highly correlated and $g$ and $h$ are highly correlated, then $f$ and $h$ must be correlated to some extent.

**Appendix C: Detector efficiency**

In avalanche photodiodes, the only significant loss mechanism is reflection of the incident photons. The detection efficiency is therefore given by the relation

$$\eta = 1 - R,$$

where $R$ is the fraction of photons reflected at the surface of the photocathode. It is, therefore, possible to reduce the loss due to this cause to negligible levels by using a number of photodiodes in a light-trapping arrangement [29].

A simple trap-detector which can be used for photon counting uses only two photodiodes [30]. In this arrangement, as shown in Fig.2, the incident beam undergoes three reflections at the photodiodes before exiting. The fraction of the photons lost by reflection is then

$$R_2 = R^3,$$

and summation of the outputs of the two photodiodes should yield a detection efficiency

$$\eta_2 = 1 - R^3.$$  

With commercial avalanche photodiodes, for which $R$ is typically around 0.3, it should be possible to obtain an increase in detection efficiency from 70% to 97%.
Appendix D: Generation of four-photon states

In the usual parametric process, yielding two down-converted photons, a 25 mm long ADP crystal pumped by a 9 mW He-Cd laser ($\lambda = 325 nm$), yields, at a 2 mm aperture placed at a distance of 1 m from the crystal, a down-converted flux of $4 \times 10^5$ photons / second, for each beam [25], corresponding to a down-conversion efficiency of $3 \times 10^{-11}$. However, crystals such as beta-barium borate (BBO) are now available with a nonlinear coefficient 5 times higher than ADP. In addition, it should be possible to obtain an increase in down-conversion efficiency by placing the crystal in a short resonant cavity [26]. If we use a 1.5 cm long BBO crystal, placed in a short cavity with mirrors whose reflectivity is chosen so that the effective length of the crystal is increased to around 7.5 cm, it should be possible to obtain a down-conversion efficiency of $4.5 \times 10^{-10}$.

The nonlinear susceptibility involved in the production of the four-photon field is, to a first approximation, the square of the nonlinear susceptibility involved in the production of two down-converted photons [26], so that the down-conversion efficiency, in this case, would work out to $2 \times 10^{-19}$. However, with two pump beams, the gain depends on the second power of the pump amplitude [26]. As a result, the output with pulsed pump beams with high peak power can be several orders of magnitude greater than that obtained with continuous-wave excitation at the same average power.

With a laser generating pulses with a duration of 1 $\mu$s, at a repetition rate of 10 pulses/second, it should be possible to obtain a peak power that is $10^5$ times greater than the average power, and an improvement in down-conversion efficiency by a factor of this order. Accordingly, with an average power of 100 mW (corresponding to a peak power of 10 kW), it should be
possible to obtain a total down-converted flux in the four output beams of $4 \times 10^3$ photons/second, or $10^3$ photons/second in each beam. After allowing for losses, it should be possible to obtain a flux of $100$ photons/second at the output from each of the four interferometers, which should permit useful measurements.

The use of a pulsed pump beam might be expected, at first sight, to create problems connected with the spectral coherence of the pump beams and the time resolution of the detectors. However, with a pulse duration of $1 \ \mu s$, the coherence length of the pump beams, with a properly designed laser cavity, would be greater than $100$ m. On the other hand, the coherence length of the down-converted beams, which would be determined by the decay time of the cavity modes (in this case, about $0.5$ ns), would be less than $0.15$ m. Accordingly, it would be possible to avoid second-order interference fringes by working with an optical path difference greater than this value, without a significant loss in the visibility of fourth-order interference effects.

As mentioned earlier, with a crystal placed in a resonant cavity, the light beams have an intrinsic bandwidth determined by the bandwidth of the cavity. This is consistent with the picture that the four down-converted photons are produced simultaneously, but then escape independently within a time interval equal to the decay time of the cavity [27, 28], which, as mentioned earlier, is around $0.5$ ns. Since this time interval is much less than the time resolution of a fast photodetector (say, $1.5$ ns), the effects of such a deviation from simultaneity would not be noticeable.

Finally, we need to consider the probability of accidental coincidences. Since the output from each interferometer consists of a series of pulses with a duration of $1 \ \mu s$, each containing about $10$ photons, the probability of detecting a single photon in a time window of $1.5$ ns would be $0.015$. The ratio
of the probability of accidental coincidences at the four outputs, due to un-
correlated photons, to that for actual coincidences would be only marginally
higher, at around 0.02. This proportion of accidental coincidences should not
have a significant effect on the visibility of fourth-order interference effects
produced by the four down-converted beams.
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Figure Captions

Fig. 1. Schematic of the four-photon interferometer.

Fig. 2. Optical configuration for a single-photon trap detector using two avalanche photodiodes.
