Reflection and refraction in active dielectric materials

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Abstract.
In this work we study and analyze in detail the characteristics of the modulus and phase of the reflection and transmission coefficients in interfaces between isotropic media, when the incident electromagnetic wave is propagating from a transparent medium towards an active one. We also demonstrate analytically that Amplified Reflection is impossible if semi-infinite media are involved. Due to these coefficients, the oscillatory or monotonic character of the phase difference between p and s modes is shown as a function of the angle of incidence for different active media. A qualitative and quantitative comparison between our own results and those obtained by many authors on absorbing media is made. We consider that this work can clarify some aspects that can contribute in the use of ellipsometric techniques for the determination of optical properties of active media.

1. Introduction
Light reflection and refraction when a plane wave impinges from a linear, isotropic, transparent and homogeneous medium towards another similar medium (either lossless or lossy) have been studied profoundly and extensively. Media with low and high loss have been considered including the situations in which the refraction index of the transparent medium is lower or higher that the real part of the index of the second medium [1-3]. In the last situation, termed Internal Reflection (IR) it is possible to obtain Total Internal Reflection (TIR) if the second medium is lossless or Attenuated Internal Reflection (AIR) if it is lossy. As it is well known, the value of the reflectivity is one under TIR conditions. On the contrary, in AIR conditions reflectivity is lesser than one as it happens in transparent interfaces when TIR does not take place. On the other hand, if the refraction index of the first medium is lower than the real part of the index of the second medium, we will be dealing with External Reflection (ER) or Attenuated External Reflection (AER) if the second medium is lossy. In both cases, reflectivity is lower than one. Moreover, the phase shifts of the reflected waves depend not only on the optical characteristics of the media but also on the incidence angle and the polarization mode (perpendicular to the plane of incidence p or parallel to the plane of incidence p).

Lasers opened a new perspective for the study of optical properties of novel (or little-used) materials. As an example, active media were modeled as linear homogenous media where traveling...
waves increase their amplitudes as they propagate. Unfortunately, as recently pointed out by A. Siegman [4], “at least a dozen refereed publications have claimed that amplified total reflection from a single gainy interface is real”. We suppose that some authors [5-7] support their statement considering that, regardless of the number of interfaces, if lossless- lossy interfaces yield reflectivities lower than one, lossless-gainy interfaces must yield reflectivities higher than one. In his clarifying paper, Siegman makes a detailed study of the components of the wave numbers vectors in lossless, lightly lossy and active media. He concludes that Amplified Total Reflection cannot exist when a single interface is considered. Of course, this is what it is to be expected from a physical point of view.

In this work, after an introduction about the nature of active media and the propagation of plane waves through them, we show that the absolute values of the reflection coefficients in isotropic lossless-active interfaces are always lower than one (regardless of the gain of the active medium). Furthermore, we demonstrate analytically that their values depend on the square of the imaginary part of the refraction index. In consequence, the same absolute value of reflectivity is obtained, both for lossy or active media. In addition, there is only a change in the differential phase shift between p and s reflected modes. We also study the differential phase shift as a function of the incidence angle for different relationships among refractive indices. We show that the differential phase shift may have a monotonic or oscillating dependence with the incidence angle. Then we make the equivalent study for the transmission coefficients.

We consider that our work can clarify some aspects that could contribute to the comprehension and use of reflection and transmission ellipsometry as an optical technique for the characterization of interfaces or films with at least one active medium.

2. Plane waves in active media

Active materials are substances where an external agent, usually within a limited frequency band, can control polarization. In absorbent media, the energy from the electric field of the incident wave is dissipated in the relaxation of polarization towards thermal equilibrium. In contrast, in active materials the polarization is driven by external sources (“pumping”), thus increasing the amplitude of the electric field in the material (“gain”). Typical applications of active media are lasers and optical amplifiers. It is clear that the electric field in an active material will not increase its amplitude indefinitely, eventually reaching maximum amplitude (“saturation”). However, at low electric field amplitudes, the active material may be considered linear, i.e., the increase in the amplitude of the electric field is proportional to the amplitude of the incident field.

When the wavelength of the electromagnetic wave propagating in the medium is much longer than the characteristic dimensions of the molecular structure, the material can be treated as a continuum. Within this hypothesis, the constitutive equation of an absorbent material is described by a complex permittivity where its imaginary part is negative. If the time dependence of the incident field is harmonic, this means that there is a phase lag of the polarization with respect to the electric field. In the same way, in active materials the polarization leads the phase of the electric field, and is described by a complex permittivity where the imaginary part is positive. Therefore, the refracting index (and, in consequence, the wave number) of the media is complex, and the sign of the imaginary part indicates if there is absorption or gain.

This leads to the possibility of studying, with an analogous treatment, the propagation through Maxwell’s equations of conducting, absorbing and active media, by defining a complex dielectric constant

\[ \tilde{\varepsilon} = \varepsilon \pm i\varepsilon'' \] (1)

The sign of the imaginary part must be chosen according to the physical situation. The definition of complex dielectric constants leads, as it has been explained before, to complex wave number vectors. In this way, the dispersion relationship is given by

\[ k^2 = \mu_0 \omega^2 \tilde{\varepsilon} \] (2)

where \( k^2 \equiv \tilde{k} \cdot \tilde{k} \)
3. Reflection and refraction of plane waves on a single interface

We consider now that a plane wave impinges on a lossless-active planar interface. The lossless medium has a refractive index \( n \) and the active medium a complex refractive index \( \hat{n} \) (figure 1).

Proposing forward-propagating solutions, i.e.

\[
\vec{E} = \vec{E}_0 \exp \left[ i (\vec{k} \cdot \vec{r} - \omega t) \right]
\]

and considering that \( \hat{n}^2 = \hat{\varepsilon} / \varepsilon_v \), where \( \varepsilon_v \) is the dielectric constant in vacuum, from equation (1) and from the condition of active medium, it follows that the imaginary part of \( \hat{\varepsilon} \) and \( \hat{n} \) must be negative (i.e. \( \hat{n} = n' - i n'' \) with \( n'' > 0 \)). As in absorbing media, the waves that propagate in active media are inhomogeneous [1, 8] i.e. constant phase surfaces do not coincide with constant amplitude surfaces. If we consider the co-ordinate system shown in figure 1, the components of the wave number vectors normal to the interface are

\[
k_x = -k_r' = \sqrt{\mu \omega^2 \varepsilon - k_z^2}
\]

\[
k_z' = \sqrt{\mu \omega^2 \varepsilon^2 - k_z^2}
\]

where \( k_z = \sqrt{\mu \omega^2 \varepsilon \cos \alpha} \). As expected, from equation (5) it follows that the component of the transmitted wave number vector perpendicular to the interface is complex, i.e.

\[
k_x = k_x + ik_z.
\]

We must take into account that the electric field increases its amplitude in the direction of positive \( x \) values, as the wave travels through the medium. In this way, the explicit equations for the real and imaginary parts of \( k_x' \) as a function of the refractive indices and the incidence angle are

\[
k_x' = \sqrt{\frac{\mu \omega^2 \varepsilon}{2} \left((n'' - n'^2 - n^2 \sin^2 \alpha) + \left((n'' - n'^2 - n^2 \sin^2 \alpha)^2 + 4n'^2 n'' \right)^{1/2}\right)^{1/2}}
\]

\[
k_z' = \sqrt{\frac{\mu \omega^2 \varepsilon}{2} \left(-(n'' - n'^2 - n^2 \sin^2 \alpha) + \left((n'' - n'^2 - n^2 \sin^2 \alpha)^2 + 4n'^2 n'' \right)^{1/2}\right)^{1/2}}
\]

From equations (6), (7) and (8) it is easy to analyze their dependence with the real and imaginary parts of the refractive index. Indeed, the real and imaginary parts of \( k_x' \) depend on the square of the real and imaginary parts of \( \hat{n} \). Consequently, the only formal difference between propagation in active and in absorbing or conducting media is that in the former case the amplitude of the transmitted wave increases with \( x \). That is, the imaginary part of \( k_x' \) (if defined as in equation (6)) is negative (for active media) and positive (for absorbing or conducting media). In figure 2 we plot the real and imaginary parts of \( k_x' \) (relative to vacuum) as a function of the incidence angle when the second medium is lossless, absorbing and active.

As expected, both for ER and IR the imaginary parts of \( k_x' \) are symmetric with respect to horizontal axis, whereas the real parts are independent of the lossy or active character of the medium. For lossless media, \( k_x' \) is always positive if TIR is not present. Besides, \( k_x' \) is zero and \( k_z' \) positive in TIR regime. This corresponds to the existence of evanescent waves.
In the next section, we study in detail the reflection and transmission characteristics in lossless-active interfaces. We also show that Amplified Reflection can never exist when dealing with gainy semi-infinite media. We consider both ER ($n < n'$) and IR ($n > n'$) for different values of gain.

(a) 
(b) 

Figure 2. Real (solid line) and imaginary (dashed line) parts of the transmitted wave number vectors for transparent (black), active (red) or lossy (blue) media when light impinges from a transparent medium. Violet lines correspond either to active or lossy media and $k_v$ is the wave number vector in vacuum (a) External Reflection (b) Internal Reflection.

4. Reflection coefficients

Fresnel’s equations can be obtained for any type of medium from boundary conditions. If we define, as usual, the isotropic proper modes $s$ and $p$ (considering in this case the $x$ component), the reflection coefficients for both modes can be written in terms of the wave number vector and the dielectric constants [1]

$$ R_s = \frac{k_s - k_s'}{k_s + k_s'} $$

(9)

$$ R_p = \frac{\varepsilon k_s - \varepsilon k_s'}{\varepsilon k_s + \varepsilon k_s'} $$

(10)

In order to analyze the reflected field it is useful to write these coefficients in polar form. Thus, the electric field associated to the incident wave is

$$ E = (E_s \bar{\varepsilon}_s + E_p \bar{\varepsilon}_p) e^{i(k_s + k_s' - \omega t)} $$

(11)

and the reflected field is

$$ E^* = E_s [R_s] \left( \bar{\varepsilon}_s + \frac{R_p}{R_s} \frac{E_p}{E_s} e^{i(\delta_s - \delta')} \right) e^{i(k_s + k_s' - \omega t + \delta')} $$

(12)

where, taking into account equation (6),

$$ |R_s|^2 = \frac{(k_s - k_s')^2 + k_s'^2}{(k_s + k_s')^2 + k_s'^2} $$

(13)

$$ \delta_s = \arctan \left( \frac{-2k_s k_s'}{k_s^2 - k_s'^2} \right) $$

(14)
\[ |R_p|^2 = \left( \frac{\epsilon' k_x - \epsilon n' k_{x'}^r}{\epsilon k_{x'}^r + \epsilon' k_x} \right)^2 \left( \frac{\epsilon n' k_{x'}^r - \epsilon k_x}{\epsilon k_{x'}^r + \epsilon' k_x} \right)^2 \]

\( \delta_p = \arctan \left( \frac{2 \epsilon' k_x k_{x'}^r}{k_x^2 (\epsilon n'^2 + \epsilon'^2) + \epsilon'^2 (k_{x'}^r)^2 - k_x^2} \right) \)

**Figure 3.** Quadratic moduli reflection coefficients as a function of the incidence angle in external reflection \((n=1.3, \tilde{n}=1.7-in'')\) for different interfaces. (a) \(p\) polarization; (b) \(s\) polarization

\[ |R_s|^2 = \left( \frac{\epsilon k_x - \epsilon' k_{x'}^r}{\epsilon k_{x'}^r + \epsilon' k_x} \right)^2 \left( \frac{\epsilon' k_x - \epsilon n' k_{x'}^r}{\epsilon k_{x'}^r + \epsilon' k_x} \right)^2 \]

\[ |R_s|^2 = \left( \frac{\epsilon n' k_{x'}^r - \epsilon k_x}{\epsilon k_{x'}^r + \epsilon' k_x} \right)^2 \left( \frac{\epsilon k_x - \epsilon' k_{x'}^r}{\epsilon k_{x'}^r + \epsilon' k_x} \right)^2 \]

**Figure 4.** Quadratic moduli reflection coefficients as a function of the incidence angle in internal reflection \((n=1.7, \tilde{n}=1.3-in'')\) for different interfaces. (a) \(p\) polarization; (b) \(s\) polarization

It is clear, by inspection of equations (7), (8), (13) and (15), that the absolute values of the reflection coefficients are not only lower than one for both polarization modes (no matter whether the medium is absorbent or active) but also that they have the same numeric value when the moduli of the imaginary and real parts of the refraction index are equal. This is a direct consequence of the parity of the function: they are even functions of the imaginary part of \(\tilde{\epsilon}\) (the change in the sign of \(\epsilon''\) is related to the change of sign in \(k_{x'}^r\)). On the contrary, from equations (14) and (16) it follows that the phase shifts in reflection are odd functions of \(\epsilon''\). Consequently, reflected waves are mostly elliptically polarized but the sense is the opposite to that of absorbing media.
In figures 3 and 4 we show the absolute values of the reflection coefficients for both modes. Figure 3 corresponds to ER and figure 4 corresponds to IR. In ER regime, we observe that reflectivity always increases with \( n'' \) for both modes whereas in IR reflectivity can increase or decrease with \( n'' \) depending on the incidence angle and the polarization mode. Nevertheless, in both cases there is an incidence angle for which the reflectivity for the p mode is minimum. The incidence angle for which this minimum exists depends on the imaginary part of the complex refractive index (and corresponds to the Brewster’s angle \( \alpha_B \) when \( n'' = 0 \)).

![Figure 3](image3.png) $\rho_r$ vs. $\alpha$

![Figure 4](image4.png) $\rho_r$ vs. $\alpha$

**Figure 5.** External Reflection: (a) Modulus of the relative amplitude of the reflection coefficients (b) Differential reflection phase shift as a function of the incidence angle for different values of \( n'' \).

![Figure 6](image6.png) $\rho_r$ vs. $\alpha$

**Figure 6.** Internal Reflection: (a) Modulus of the relative amplitude of the reflection coefficients (b) Differential reflection phase shift as a function of the incidence angle for different values of \( n'' \).

As reflection ellipsometry is a technique based on measurements of polarization states of incident and reflected light, we calculate the quotient between the complex reflection coefficients in order to ascertain if it can be used in the determination of optical properties of active media. We define, as usual for transparent and absorbing media \( \rho_r \equiv \frac{R_p}{|R_p|} \) and \( \Delta_r \equiv \delta_p - \delta_s \), and we will consider their behavior when some variables are modified. In figures 5 and 6 we plot \( \rho_r \) and \( \Delta_r \) as a function of the incidence angle using \( n'' \) (related to the gain of the active medium) as a parameter for both ER and IR. As it is clearly shown in figures 5(a) and 6(a), \( \rho_r \) is extremely sensitive to the incidence angle near Brewster’s angle (\( \tan \alpha_B = n'/n \)) for low values of gain. With regard to the differential phase shift \( \Delta_r \), we can see that there is a decreasing monotonic behavior under ER conditions. On the contrary, under IR conditions, \( \Delta_r \) presents this behavior only for high values of \( n'' \). For low gain, it presents an
oscillatory behavior for $\alpha > \alpha_s$: the minimum is between $\alpha_s$ and the angle of total reflection $\alpha_r (\sin \alpha_r = n'/n)$, and the maximum between $\alpha_r$ and grazing incidence. Consequently, $\Delta$ is very sensitive to the incidence angle near Brewster’s condition for low gain.

For higher gain values, $\Delta_r$ and $\rho_r$ lose sensitivity to the incidence angle (for both ER and IR conditions). This suggests that this kind of interface could be used to convert linear polarization to circular polarization, i.e. as a retarder using only one reflection.

5. Transmission coefficients

Although transmissivity studies are experimentally feasible when a film or layer is considered, we analyze the transmission coefficients in order to make a complete treatment of a transparent-active interface in semi-infinite media. Working in a similar way as in the previous section, the expression for the transmitted field is

$$\hat{E}_t = E_s |T_s| e^{-ik_{y_s}z} \left( \hat{e}_s + \left| \frac{T_p}{E_s} \right| E_p e^{i\phi_p - \delta_e} \hat{e}_p \right) e^{i(k_{x_s}x + k_{z_s}z - \omega t + \phi_d)}$$

(17)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7}
\caption{External Reflection: (a) Modulus of the relative amplitude of the transmission coefficients (b) Differential transmission phase shift as a function of the incidence angle for different values of $n''$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8}
\caption{Internal Reflection: (a) Modulus of the relative amplitude of the transmission coefficients (b) Differential transmission phase shift as a function of the incidence angle for different values of $n''$.}
\end{figure}

where, taking into account equation (6)

$$|T_s|^2 = \frac{4k_s^2}{\left( k_x + k_{x_s} \right)^2 + k_{s_s}^2}$$

(18)
\[ \delta'_i = \arctan \left( \frac{-k'_i k'_s}{k'_x + k'_i k'_x} \right) \]  
\[ |T'_p|^2 = \frac{4k'^2_i \varepsilon \left( \varepsilon^o + \varepsilon^{n2} \right)^{1/2}}{\left( \varepsilon k'_s + \varepsilon' k'_s \right)^2 + \left( \varepsilon k'_i + \varepsilon' k'_s \right)^2} \]  
\[ \delta'_p = \arctan \left( \frac{-\varepsilon k'_i + \varepsilon' k'_x}{\varepsilon k'_s + \varepsilon' k'_s} \right) \]

As in the case of reflection, transmission coefficients for both modes are only modified by a change of sign in the phase shift when an active medium is considered instead of an absorbing one. Figures 7 and 8 show \( \rho_i = \left| T'_p / |T'_s| \right| \) and \( \Delta_i = \delta'_p - \delta'_s \) for the same interfaces that were considered in the previous section. In an analogous way to what it is observed in reflection, the phase shift between the p and s transmitted modes shows oscillations for certain values of \( n'' \). However, the sensitivity to the incident angle is very low.

6. Conclusions

Modeling active media with a complex dielectric constant, we have determined analytically the reflection and transmission coefficients for transparent-active interfaces when light impinges from the transparent medium. Starting from the reflection coefficients for both polarization modes, we demonstrated analytically that Amplified Reflection is not possible when considering a single interface.

The dependence of the absolute values of the reflection and transmission coefficients with the real and imaginary parts of \( k'_i \), confirm that they are numerically equal to those corresponding to absorbing media (with identical real part and the same absolute value of the imaginary part). This is due to the fact that the absolute values of the reflection and transmission coefficients are even functions of the imaginary part of \( \varepsilon \): if the sign of the imaginary part changes, the sign of the imaginary part of the transmitted wave number vector also changes. We also demonstrated that there is only a change in the sense of rotation of the polarization state for either the reflected or the transmitted light due to the odd parity in the imaginary part of \( \varepsilon \).

As in transparent-absorbing interfaces, the determination of the phase shift through internal reflection ellipsometry in the range between Brewster’s and total reflection angles seems to be a stable and sensitive method in order to determine the imaginary part of the refraction index when the medium gain is low. On the contrary, for high values of gain the phase shift and the quotient between the modulus of the reflection coefficients lose sensitivity to the incidence angle. This suggests that this kind of interface can be used as quarter-wave retarder.

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8. References

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