Morita equivalence classes of blocks with elementary abelian defect groups of order 16

Charles W. Eaton

17th January 2018

Abstract

We classify the Morita equivalence classes of blocks with elementary abelian defect groups of order 16 with respect to a complete discrete valuation ring with algebraically closed residue field of characteristic two. As a consequence, blocks with this defect group are derived equivalent to their Brauer correspondent in the normalizer of a defect group and so satisfy Broué’s Conjecture.

Keywords: Donovan’s conjecture; Morita equivalence; finite groups; block theory.

1 Introduction

Throughout let \( k \) be an algebraically closed field of prime characteristic \( \ell \) and let \( \mathcal{O} \) be a discrete valuation ring with residue field \( k \) and field of fractions \( K \) of characteristic zero. We assume that \( K \) is large enough for the groups under consideration. We consider blocks \( B \) of \( \mathcal{O}G \) with defect group \( D \), for finite groups \( G \).

Our purpose is the description of the Morita and derived equivalence classes of (module categories for) blocks of finite groups with a given defect group. It is already known by [13] that Donovan’s conjecture holds for elementary abelian 2-groups, that is, for each \( n \in \mathbb{N} \) there are only finitely many Morita equivalence classes of blocks with defect group \( (C_2)^n \), and so in theory Morita equivalence classes of such blocks could be classified for any given \( n \). Here we consider the case \( n = 4 \) and achieve a complete classification. The main tool is the description given in [13] of the 2-blocks with abelian defect groups of the quasisimple groups. The number of irreducible ordinary and Brauer characters of blocks with defect group \( (C_2)^4 \) has already been determined in [27] and [13]. Our work continues [12] in which a classification is given for blocks with elementary abelian defect groups of order 8. The Morita equivalence classes of block with Klein four defect groups are known by [16] and [30]. Other \( \ell \)-groups where there are classifications are: cyclic \( \ell \)-groups, where the Morita equivalence classes can be characterised in terms of Brauer trees (in work by many, for which see [31]).

*This research was supported by the EPSRC (grant no. EP/M015548/1).
†School of Mathematics, University of Manchester, Manchester, M13 9PL, United Kingdom. Email: charles.eaton@manchester.ac.uk
abelian 2-groups of 2-rank at most three (see [13, 15] and [14]); dihedral, semidihedral and generalized quaternion 2-groups except where the block has two simple modules in the case that the defect group is generalized quaternion (see [17], and note that this classification is only known with respect to $k$); nonmetacyclic minimal nonabelian 2-groups $\langle x, y : x^{2r} = y^{2s} = [x, y] = [y, [x, y]] = 1 \rangle$, where $r \geq s \geq 1$, by [39] and [14]; 2-groups which are a direct product of cyclic factors all of different orders, whose automorphism groups are themselves 2-groups so the block must be nilpotent and so Morita equivalent to $OD$ by [6] and [35]; the remaining metacyclic 2-groups not listed above, which force the block to be nilpotent by [8]. Finally, principal blocks with defect group $C_3 \times C_3$ are classified (even up to Puig equivalence) in [21].

A significant challenge arises in our situation which does not arise for defect groups of order 8 in that we must address Morita equivalences in the case of a normal subgroup of index 3 where there are infinitely many possibilities for $N$ but the Morita equivalence class of the block of $N$ is fixed. We do so by following Külshammer’s analysis in [25] of possible crossed products in detail.

Before stating the main theorem, we recall the definition of a subpair and of the inertial quotient of $B$.

A $B$-subpair is a pair $(Q, b_Q)$ where $Q$ is a $p$-subgroup of $G$ and $b_Q$ is a block of $\mathcal{O}QC_G(Q)$ with Brauer correspondent $B$. When $D$ is a defect group for $B$, the $B$-subpairs $(D, b_D)$ are $G$-conjugate. Write $N_G(D, b_D)$ for the stabilizer in $N_G(D)$ of $b_D$. Then the **inertial quotient** of $B$ is $E = N_G(D, b_D)/DC_G(D)$, an $\ell'$-group unique up to isomorphism. To simplify further definitions, suppose that $D$ is abelian. We say $(Q, b_Q) \leq (R, b_R)$ for subpairs $(Q, b_Q)$ and $(R, b_R)$ if $Q \leq R$ and $(b_R)^{C_G(Q)} = b_Q$. If $x \in D$ and $b_x$ is a block of $C_G(x)$ we say that the Brauer element $(x, b_x) \in (D, b_D)$ if $(b_D)^{C_G(x)} = b_x$.

Note that if $D$ is abelian, then $B$ is nilpotent precisely when the inertial quotient is trivial.

The possible inertial quotients for a block with defect group $D$ are given in [27], and these are: 1 (corresponding to nilpotent blocks); $C_3$ with action as in $A_4 \times (C_2)^2$; $C_3$ consisting of 5th powers of a Singer cycle for $F_{16}$ (with only one fixed point in its action on $D$); $C_5$ consisting of 3rd powers of a Singer cycle; $C_7$ coming from a Singer cycle for $F_8$; $C_3 \times C_3$; $C_7 \times C_3$ coming from a Singer cycle and a field automorphism of $F_8$; $C_{15}$ coming from a Singer cycle of $F_{16}$. Each inertial quotient apart from $C_3 \times C_3$ has trivial Schur multiplier, and in this case it is $C_3$.

We say that a block with defect group $(C_2)^4$ is of type $E$ if it has inertial quotient $E$ where there is only one possible faithful action on $(C_2)^4$, and in the case that the inertial quotient is $C_3$, we say it has type $(C_3)^1$ when the action is as in $A_4 \times (C_2)^2$ and type $(C_3)^2$ when there is only one fixed point.

**Theorem 1.1** Let $B$ be a block of $\mathcal{O}G$ with elementary abelian defect group $D$ of order 16, where $G$ is a finite group. Then $B$ is Morita equivalent to precisely one of the following:

(a) a non-principal block of $(C_2)^4 \times 3_1^{1+2}$, where the centre of $3_1^{1+2}$ acts trivially, and we note that the two non-principal blocks are Morita equivalent;

(b) the principal block of precisely one of the following:

(i) $D$;
(ii) $(C_2)^2 \times A_4$;
(iii) $(C_2)^2 \times A_5$;
(iv) $D \rtimes C_3$ of type $(C_3)_2$;
(v) $D \rtimes C_5$;
(vi) $C_2 \times ((C_2)^3 \rtimes C_7)$;
(vii) $C_2 \times SL_2(8)$ (type $C_7$);
(viii) $A_4 \rtimes A_4$;
(ix) $A_4 \times A_5$ (type $C_3 \times C_3$);
(x) $A_5 \times A_5$ (type $C_3 \times C_3$);
(xi) $D \rtimes C_{15}$;
(xii) $SL_2(16)$ (type $C_{15}$);
(xiii) $C_2 \times ((C_2)^3 \rtimes (C_7 \rtimes C_3))$;
(xiv) $C_2 \rtimes J_1$ (type $C_7 \rtimes C_3$);
(xv) $C_2 \rtimes \text{Aut}(SL_2(8))$ (type $C_7 \rtimes C_3$).

If $B$ is a principal block, then it is Morita equivalent to one of the examples in case (b), i.e., the blocks in case (a) cannot be Morita equivalent to a principal block of any finite group.

Blocks are derived equivalent if and only if they have the same inertial quotient and number of simple modules.

Remarks 1.2 (i) It will be clear from the proof of Theorem 1.1 that blocks with defect group $(C_2)^4$ cannot be Morita equivalent to a block with non-isomorphic defect group. The same can be said of the inertial quotient, as the number of irreducible characters and number of simple modules together determine the inertial quotient.

(ii) Non-nilpotent blocks with defect group $(C_2)^4$ and just one simple module are studied in [29]. The structure of the centre of such a block is described there, and hopefully in the future there will be a classification-free proof that all such blocks are Morita equivalent to the blocks in part (a) of Theorem 1.1.

Corollary 1.3 Broue’s abelian defect group conjecture holds for blocks with defect group $(C_2)^4$, that is, if $B$ is a block of $\mathcal{O}G$ with such a defect group $D$, then $B$ is derived equivalent to its Brauer correspondent in $\mathcal{O}N_G(D)$.

Remark 1.4 We cannot say at present whether the blocks are splendid equivalent as we cannot say anything about the sources of the bimodules giving the Morita equivalences in Theorem 1.1.

The paper is structured as follows. In Section 2 we give many of the miscellaneous preliminary results necessary for the proof of Theorem 1.1. In Section 3 we cover some background on Picard groups and calculate the subgroup of trivial source bimodules in the special cases required for the analysis of possible crossed products, which we perform in Section 4. In Section 5 we prove Theorem 1.1.
2 Preliminary results

Let $G$ be a finite group, $N < G$ and let $b$ be a $G$-stable block of $ON$. The normal subgroup $G[b]$ of $G$ is defined to be the group of elements of $G$ acting as inner automorphisms on $b \otimes_O k$. We first collect some results concerning $G[b]$ that will be used when considering automorphism groups of simple groups.

**Proposition 2.1** Let $G$ be a finite group and $B$ a block of $OG$ with defect group $D$. Let $N < G$ with $D \leq N$ and suppose that $B$ covers a $G$-stable block $b$ of $ON$. Let $B'$ be a block of $OG[b]$ covered by $B$. Then

(i) $b$ is source algebra equivalent to $B'$, and in particular has isomorphic inertial quotient;

(ii) $B$ is the unique block of $OG$ covering $B'$.

**Proof.** Part (i) is [20, 2.2], noting that a source algebra equivalence over $k$ implies one over $O$ by [35, 7.8]. Part (ii) follows from [11, 3.5]. \hfill \Box

The following is a distillation of those results in [26] which are relevant here.

**Proposition 2.2** ([26]) Let $G$ be a finite group and $N < G$. Let $B$ be a block of $OG$ with defect group $D$ covering a $G$-stable nilpotent block $b$ of $ON$ with defect group $D \cap N$. Then there is a finite group $L$ and $M < L$ such that (i) $M \cong D \cap N$, (ii) $L/M \cong G/N$, (iii) there is a subgroup $D_L$ of $L$ with $D_L \cong D$ and $D_L \cap M \cong D \cap N$, and (iv) there is a central extension $\hat{L}$ of $L$ by an $\ell'$-group, and a block $\hat{B}$ of $\hat{O} \hat{L}$ which is Morita equivalent to $B$ and has defect group $\hat{D} \cong D_L \cong D$.

If $B$ is the principal block, then $\hat{B}$ is the principal block.

**Proof.** Guidance on the extraction of these results form [26] is given in [12, 2.2]. It remains to prove the claim regarding the principal block. Note that if $B$ is the principal block, then $b$ is also principal and so $N$ has a normal $\ell'$-complement. Then $O_{\ell'}(N)$ lies in the kernel of $B$ and the corresponding $\hat{B}$ is the principal block. \hfill \Box

Recall that a block of a finite group $G$ is quasiprimitive if every block of every normal subgroup that it covers is $G$-stable under conjugation.

**Corollary 2.3** Let $G$ be a finite group and $N < G$ with $N \not\leq Z(G)O_{\ell'}(G)$. Let $B$ be a quasiprimitive block of $OG$ with defect group $D$ covering a nilpotent block $b$ of $ON$. Then there is a finite group $H$ with $[H : O_{\ell'}(Z(H))] \leq [G : O_{\ell'}(Z(G))]$ and a block $B_H$ with defect group $D_H \cong D$ such that $B_H$ is Morita equivalent to $B$.

**Proof.** Let $b'$ be the block of $OZ(G)N$ covered by $B$ and covering $b$. Then $b'$ must also be nilpotent, and we may assume that $Z(G) \leq N$. Applying Proposition 2.2, we may take $H = \hat{L}$ and $B_H = \hat{B}$. Note that $[\hat{L} : O_{\ell'}(Z(\hat{L}))] \leq |\hat{L}| = [G : N]|D \cap N| < [G : O_{\ell'}(Z(G))].$ \hfill \Box

**Proposition 2.4** ([41]) Let $B$ be an $\ell$-block of $OG$ for a finite group $G$ and let $Z \leq O_{\ell}(Z(G))$. Let $\hat{B}$ be the unique block of $O(G/Z)$ corresponding to $B$. Then $\hat{B}$ is nilpotent if and only if $B$ is nilpotent.
Proof. The result in [11] is stated over \( k \), but it follows over \( O \) immediately. \( \square \)

Recall that a block \( B \) of \( OG \) is nilpotent covered if there is a finite group \( H \) with \( G < H \) and a nilpotent block of \( OH \) covering \( B \). Let \( D \) be a defect group for \( B \) and let \( b \) be the Brauer correspondent of \( B \) in \( ON_G(D) \). Following [37] \( B \) is inertial if it is basic Morita equivalent to \( b \), that is, if there is a Morita equivalence induced by a bimodule with endopermutation source.

**Proposition 2.5** ([37], [45]) Let \( G \) and \( N \) be finite groups and \( N < G \). Let \( b \) be a block of \( ON \) covered by a block \( B \) of \( OG \).

(i) If \( B \) is inertial, then \( b \) is inertial.
(ii) If \( b \) is nilpotent covered, then \( b \) is inertial.
(iii) If \( \ell \nmid [G : N] \) and \( b \) is inertial, then \( B \) is inertial.

Proof. (i) is [37, 3.13], (ii) is [37, 4.3] and (iii) is the main theorem of [45]. \( \square \)

We will make frequent use of the classification of Morita equivalence classes of blocks with Klein four defect groups throughout this paper without further reference:

**Proposition 2.6** ([16], [30], [7]) Let \( B \) be a block of \( OG \) for a finite group \( G \). If \( B \) has Klein four defect group \( D \), then it is source algebra equivalent to the principal block of one of \( OD \), \( OA_4 \) and \( OA_5 \).

We extract the results of [13] necessary for this paper:

**Proposition 2.7** ([13]) Let \( B \) be a block of \( OG \) for a quasisimple group \( G \) with elementary abelian defect group \( D \) of order dividing 16. Then one or more of the following occurs:

(i) \( G \cong SL_2(16) \), \( J_1 \) or \( 2G_2(q) \), where \( q = 2^{2m+1} \) for some \( m \in \mathbb{N} \), and \( B \) is the principal block;
(ii) \( G \cong Co_3 \) and \( B \) is the unique non-principal 2-block of defect 3;
(iii) \( G \) is of type \( D_n(q) \) or \( E_7(q) \) for some \( q \) of odd prime power order, \( O_2(G) = 1 \) and \( B \) is Morita equivalent to a block \( C \) of a \( OL \) where \( L = L_0 \times L_1 \leq G \) such that \( L_0 \) is abelian and the block of \( OL_1 \) covered by \( C \) has Klein four defect groups;
(iv) \( |O_2(G)| = 4 \) and \( D/O_2(G) \) is a Klein four group;
(v) \( B \) is nilpotent covered.

Proof. This follows from Proposition 5.3 and Theorem 6.1 of [13]. \( \square \)

One obstacle in classifying Morita equivalence classes over \( O \) rather than \( k \) is that the results of [22] only apply over \( k \). However in our situation we are lucky to be able to apply some work of Watanabe on perfect isometries to obtain the same result over \( O \) in certain crucial cases. For the benefit of the reader we state the relevant result of [42] here. First we need some more notation.

Let \( B \) be a block of \( OG \), where \( G \) is a finite group. Write \( L_K(G, B) \) for the group of generalized characters of \( B \) with respect to \( K \). Let \( \chi \) be a generalized character of \( B \). Fix a maximal \( B \)-subpair \( (D, b_D) \). Let \( \lambda \) be a generalized character of a defect group \( D \) of \( B \) such that whenever \( (x, b_x) \in (D, b_D) \) and \( z \in G \) such that \( (x, b_x)^z \in (D, b_D) \),
we have $\lambda(x) = \lambda(x^z)$. Define $\lambda \ast \chi$ as in [5], another generalized character of $B$. In the following, if $\lambda$ is a generalized character of a factor group of $D$, then we are implicitly considering its inflation to $D$.

**Proposition 2.8 (Lemma 3 of [42])** Let $B$ be a block of a finite group $G$ covering a $G$-stable block $b$ of $N \triangleleft G$. Suppose that $B$ has an abelian defect group $D$ and there is $Q \leq D$ such that $D = Q \times (D \cap N)$ and $G = N \times Q$. Let $b_D$ be a block of $C_G(D)$ with Brauer correspondent $B$, and write $B' = (b_D)^{C_G(Q)}$. If there is a perfect isometry $I : \mathcal{L}_K(C_G(Q), B') \to \mathcal{L}_K(G, B)$ satisfying $I(\lambda \ast \zeta) = \lambda \ast I(\zeta)$ for all $\lambda \in \text{Irr}(Q)$ and $\zeta \in \text{Irr}(B')$, then $B \cong \mathcal{O}Q \otimes \mathcal{O} b$ as $\mathcal{O}$-algebras.

**Proposition 2.9** Let $G$ be a finite group and let $B$ be a block of $\mathcal{O}G$ with elementary abelian defect group $D$ of order 16 and cyclic inertial quotient. Suppose $N \trianglelefteq G$ with $G = ND$. If $B$ covers a non-nilpotent $G$-stable block $b$ of $\mathcal{O}N$, then there is an elementary abelian 2-group $Q \leq D$ with $G = N \times Q$ such that $B$ is Morita equivalent to a block $C$ of $\mathcal{O}(N \times Q)$ with defect group $(D \cap N) \times Q \cong D$.

**Proof.** Let $b_D$ be a block of $C_G(D)$ with Brauer correspondent $B$, and write $E = N_G(D, b_D)/C_G(D)$ as described in the introduction. We may suppose $G \neq N$, so $|E| \leq 7$. By [40, Theorem 15] we have $l(B) = |E|$.

Following [43], we may write $D = D_1 \times D_2$ where $D_1 = C_D(N_G(D, b_D))$ and $D_2 = [N_G(D, b_D), D]$. We have $D \times E = D_1 \times (D_2 \times E)$. Since $D_2 \leq N$, $E$ is cyclic and $D$ is elementary abelian, we may choose $Q$ to be a direct factor of $D_1$.

By the main theorem of [43] there is a perfect isometry

$$I : \mathcal{L}_K(N_G(D, b_D), b^{N_G(D, b_D)}) \to \mathcal{L}_K(G, B)$$

such that $I(\lambda \ast \zeta) = \lambda \ast I(\zeta)$ for all $\lambda \in \text{Irr}(D_1)$ and $\zeta \in \mathcal{L}_K(N_G(D, b_D), b^{N_G(D, b_D)})$. We have $N_G(D, b_D) \leq C_G(D_1) \leq C_G(Q)$. Let $B' = (b_D)^{C_G(Q)}$. Now $B'$ also has inertial quotient $E$ and we may apply the same argument to obtain a perfect isometry

$$J : \mathcal{L}_K(N_G(D, b_D), b^{N_G(D, b_D)}) \to \mathcal{L}_K(C_G(Q), B')$$

such that $J(\lambda \ast \zeta) = \lambda \ast J(\zeta)$ for all $\lambda \in \text{Irr}(D_1)$ and $\zeta \in \mathcal{L}_K(N_G(D, b_D), b^{N_G(D, b_D)})$. We may then apply Proposition 2.8 to $I \circ J^{-1}$ and the result follows.

In the above note that if $b$ is Morita equivalent to a block $c$ of $\mathcal{O}M$ for some finite group $M$, then $C$ is Morita equivalent to the block $c \otimes \mathcal{O}Q$ of $\mathcal{O}(M \times Q)$.

**Lemma 2.10** Let $G$ be a finite group and $N \triangleleft G$ with $G/N$ of odd order (and solvable). Let $B$ be a block of $\mathcal{O}G$ covering a $G$-stable block $b$ of $\mathcal{O}N$ with defect group $D \cong (C_2)^4$. Suppose that $B$ covers no nilpotent block of any normal subgroup $M \triangleleft G$ with $N \leq M$. If $b$ is of type $C_3 \times C_3$ or $(C_3)_1$, then $B$ is also of one of these two types.

**Proof.** It suffices to consider the case that $[G : N]$ is an odd prime, say $w$. Note that $B$ and $b$ share the defect group $D$.

Suppose $C_G(D) = C_N(D)$. Then the inertial quotient of $B$ contains that of $b$ with index dividing $w$. Since $C_3 \times C_3$ is maximal amongst subgroups of odd order of $GL_4(2)$
and is the only subgroup containing $C_3$ as a normal subgroup the result follows in this case.

Suppose $C_G(D) \neq C_N(D)$. Let $(D,b_D)$ be a $b$-subpair and let $(D,B_D)$ be a $B$-subpair with $B_D$ covering $b_D$. If $C_G(D) \nsubseteq N_G(D,b_D)$, then $B_D$ covers $w$ conjugates of $b_D$ and $B_D$ is the unique block of $C_G(D)$ covering $b_D$, so $N_G(D,B_D) = C_G(D)N_G(D,b_D)$. Hence $N_G(D,B_D)/C_G(D) \cong N_N(D,b_D)/C_N(D)$ and we are done in this case. If $C_G(D) \leq N_G(D,b_D)$, then $N_G(D,B_D) \leq N_G(D,b_D)$ as $b_D$ is the unique block of $C_N(D)$ covered by $B_D$. Now $[N_G(D,B_D):C_G(D)]$ divides $[N_N(D,b_D):C_N(D)]$ and we are done. \hfill $\Box$

## 3 Automorphisms of some blocks and basic algebras

In this section we compute some subgroups of the Picard groups for basic algebras of the principal 2-blocks of $A_4 \times C_2 \times C_2$, $A_5 \times C_2 \times C_2$, $A_4 \times A_4$, $A_5 \times A_5$ and $A_4 \times A_5$. To do so we may make use of [4], where the Picard groups of the blocks with Klein four defect groups are calculated. We are grateful to the authors of [4] for allowing us to see an early draft.

We briefly introduce the notation that we need.

Let $N$ be a finite group and $b$ be a block of $\mathcal{O}N$ with abelian defect group $D$. Let $\mathcal{F}$ be the fusion system for $b$ on $D$, defined using a maximal $B$-subpair $(D,b_D)$. Write $E = N_N(D,b_D)/DC_N(D)$, the inertial quotient. $\text{Aut}(D,\mathcal{F})$ denotes the subgroup of $\text{Aut}(D)$ of automorphisms stabilizing $\mathcal{F}$. Write $\text{Out}(D,\mathcal{F}) = \text{Aut}(D,\mathcal{F})/\text{Aut}_\mathcal{F}(D)$.

Recall that a source idempotent for $b$ is a primitive idempotent $i$ of $b^D$ (for fixed points of $b$ under conjugation by elements of $D$) such that $\text{Br}_D(i) \neq 0$. Now let $A = i b i$ be a source algebra for $b$, so $A$ is a $D$-algebra and we may consider the fixed points $A^D$ under the action of $D$. Write $\text{Aut}_D(A)$ for the group of algebra automorphisms of $A$ fixing each element of the image of $D$ in $A$, and $\text{Out}_D(A)$ for the quotient of $\text{Aut}_D(A)$ by the subgroup of automorphisms given by conjugation by elements of $(A^D)^\times$. As observed in [4], by [36 14.9] $\text{Out}_D(A)$ is isomorphic to a subgroup of $\text{Hom}(E,k^\times)$.

The Picard group $\text{Pic}(b)$ of $b$ consists of isomorphism classes $b$-$b$-bimodules which induce Morita self-equivalences of $b$. For $b$-$b$-bimodules $M$ and $N$, the group multiplication is given by $M \otimes_b N$. Write $\mathcal{T}(b)$ for the subset of $\text{Pic}(b)$ consisting of bimodules with trivial source. It is shown in [4] that $\mathcal{T}(b)$ forms a subgroup of $\text{Pic}(b)$. By [4 1.1] there is an exact sequence

$$1 \rightarrow \text{Out}_D(A) \xrightarrow{\Psi} \mathcal{T}(b) \rightarrow \text{Out}(D,\mathcal{F})$$

(where the final map is not always a surjection). Writing $kb$ for $b \otimes_O k$, there is a group homomorphism $\zeta : \text{Pic}(b) \rightarrow \text{Pic}(kb)$ given by tensoring with $k$. Since trivial source modules over $k$ lift uniquely to $O$, $\text{Res}^{\text{Pic}(b)}_{\mathcal{T}(b)}(\zeta)$ is injective. As in [4] elements of $\text{Out}_D(b)$ correspond to trivial source direct summands of $\mathcal{O}N_i \otimes_{\mathcal{O}D} i\mathcal{O}N$ as $b$-$b$-bimodules inducing Morita equivalences. Since $\text{Res}^{\text{Pic}(b)}_{\mathcal{T}(b)}(\zeta)$ is injective and trivial source $kb$-$kb$-bimodules giving Morita self-equivalences of $kb$ lift uniquely to $b$-$b$-bimodules giving
Morita self-equivalences of \( b \), elements of \( \Out_D(A) \) correspond to trivial source direct summands of \( kNi \otimes_{kD} ikN \) inducing Morita equivalences, where we are identifying \( i \) with its image in \( kN \).

The reason we use \( \mathcal{T}(b) \) and \( \Pic(b) \) is that in considering crossed products of blocks with finite groups we must study the related groups \( \Out(b) \) and \( \Out(fbf) \) where \( fbf \) is a basic algebra for \( b \). There is a well-known embedding \( \Out(b) \to \Pic(b) \) which we describe below. | \( | \Pic(b) | \) (or \( | \Out(b) | \)) is not known to be finite in general, and the paper [4] tackles this interesting problem. However, as pointed out to us by Linckelmann, in constructing crossed products arising from blocks of normal subgroups of finite groups we need only consider automorphisms corresponding to elements of \( \mathcal{T}(b) \).

We give this argument now.

Let \( \varphi \in \Aut(b) \). Define the \( b-b \)-bimodule \( \varphi b \) by letting \( \varphi b = b \) as sets and defining \( a_1 \cdot m \cdot a_2 = \varphi(a_1)ma_2 \) for \( a_1, a_2, m \in b \). By [10] 55.11 inner automorphisms give isomorphic bimodules and \( \varphi \mapsto \varphi b \) gives rise to an injection \( \Out(b) \to \Pic(b) \). Now suppose that \( N \triangleleft G \) for \( G \) a finite group, and suppose that \( b \) is \( G \)-stable. Let \( \tau : G \to \Aut(\mathcal{O}N) \) be given by conjugation, and also write \( \tau \) for the map \( G \to \Aut(b) \) it induces. Let \( g \in G \). Now \( \tau(g) b \) is a direct summand of the permutation \( \mathcal{O}N-\mathcal{O}N \)-bimodule \( \tau(g) \mathcal{O}N \), so has trivial source, and \( \tau(g) b \in \mathcal{T}(b) \).

**Lemma 3.1** Let \( G_1 \) and \( G_2 \) be finite groups, \( B_j \) a block of \( kG_j \) and \( B = B_1 \otimes B_2 \) a block of \( k(G_1 \times G_2) \). Let \( D \) be a defect group of \( B \) with \( D = D_1 \times D_2 \), where \( D_j \) is a defect group of \( B_j \). Let \( i_j \) be a source idempotent of \( B_j \). Then, with a slight abuse of notation, \( i := i_1i_2 \) is a source idempotent of \( B \) and \( iBi \cong i_1B_1i_1 \otimes i_2B_2i_2 \).

Further \( \Out_D(B) \cong \Out_{D_1}(B_1) \times \Out_{D_2}(B_2) \).

**Proof.** We have \( B^D = B_1^{D_1} \otimes B_2^{D_2} \) and \( C_G(D) = C_G(D_1) \times C_G(D_2) \), so the first part follows since by [10] 10.37 \( i \) is primitive in \( B^D \) if and only if \( i_j \) is primitive in \( B_j^{D_j} \) for each \( j \).

We saw above that elements of \( \Out_D(B) \) correspond to trivial source direct summands of \( kG_1 \otimes_{kD_1} iG \) as \( B-B \)-bimodules inducing Morita equivalences. Now

\[
kG_1 \otimes_{kD_1} ikG \cong (kG_1i_1 \otimes_{kD_1} i_1kG_1) \otimes_k (kG_2i_2 \otimes_{kD_2} i_2kG_2).
\]

Suppose that \( M | kG_1 \otimes_{kD_1} ikG \) has trivial source and induces a Morita equivalence. Since \( M \) is indecomposable, by [10] 10.37 \( M \cong M_1 \otimes_k M_2 \), where \( M_j | kG_ji_j \otimes_{kD_j} i_jkG_j \) is an indecomposable \( B_j-B_j \)-bimodule. Then each \( M_j \) has trivial source. Further, \( M \otimes_B M^* \cong B \) iff \( (M_1 \otimes_{B_1} M_1^*) \otimes_k (M_2 \otimes_{B_2} M_2^*) \cong B_1 \otimes_k B_2 \). Note that \( M_1 \otimes_{B_1} M_1^* \) and \( M_2 \otimes_{B_2} M_2^* \) are indecomposable since \( B \) is. Hence \( M \otimes_B M^* \cong B \) precisely when \( M_j \otimes_{B_j} M_j^* \cong B_j \) for each \( j \), i.e., when \( M_1 \) and \( M_2 \) induce Morita equivalences, and we are done.

\[ \square \]

**Lemma 3.2** Let \( Q \) be a finite 2-group. Let \( G_1, G_2 \in \{ Q, A_4, A_5 \} \). Then the principal block \( B \) of \( \mathcal{O}(G_1 \times G_2) \) is equal to its own source algebra.

**Proof.** \( QQ \) and \( QA \) are basic algebras, so automatically equal to their own source algebras. \( B_0(\mathcal{O}A_5) \) is its own source algebra by [32] 12.2.10. The result follows by **Lemma 3.1** \( \square \)
Proposition 3.3 Let $Q$ be a finite abelian 2-group. Let $G_1, G_2 \in \{Q, A_4, A_5\}$, $b_i = B_0(OG_i)$ and $b = B_0(G_1 \times G_2)$, so $b = b_1 \otimes b_2$. Then for all $M \in \mathcal{T}(b)$ there is $\varphi \in \text{Aut}(b)$ such that $M = \varphi b$ in $\mathcal{T}(b)$ and there is an idempotent $f \in b$ with $bf$ basic and $\varphi(f) = f$.

(i) $\mathcal{T}(\mathcal{O}(A_4 \times Q)) \cong S_3 \times \text{Aut}(Q)$.
(ii) $\mathcal{T}(B_0(\mathcal{O}(A_5 \times Q))) \cong C_2 \times \text{Aut}(Q)$.
(iii) $\mathcal{T}(\mathcal{O}(A_4 \times A_4)) \cong S_3 \cap C_2$.
(iv) $\mathcal{T}(B_0(\mathcal{O}(A_5 \times A_4))) \cong S_3 \times C_2$.
(v) $\mathcal{T}(B_0(\mathcal{O}(A_5 \times A_3))) \cong C_2 \cap C_2$.

Remark 3.4 (i) In particular $\mathcal{T}(\mathcal{O}(A_1 \times C_2 \times C_2)) \cong S_3 \times S_3$ and $\mathcal{T}(B_0(\mathcal{O}(A_5 \times C_2 \times C_2))) \cong C_2 \times C_3$.
(ii) It follows that $\varphi$ defines an automorphism of $bf$.

Proof. Let $D$ be a defect group for $b$ with $D = D_1 \times D_2$ where $D_i$ is a defect group of $b_i$. By [14], $\mathcal{T}(\mathcal{O}(Q)) \cong \text{Aut}(Q)$, $\mathcal{T}(\mathcal{O}(A_4)) \cong S_3$ is given by permutations of the three isomorphism classes of simple modules and $\mathcal{T}(B_0(\mathcal{O}(A_3))) \cong C_2$ is given by permutation of the two isomorphism classes of nontrivial modules. Note that $\mathcal{T}(b_1) \times \mathcal{T}(b_2) \leq \mathcal{T}(b)$. If further $G_1 = G_2$, then $\mathcal{T}(b_1) \cap C_2 \leq \mathcal{T}(b)$.

In cases (i) and (ii) $\text{Aut}(D, \mathcal{F}) \cong S_3 \times \text{Aut}(Q)$ and $\text{Aut}_F(D) \cong C_3$, so $\text{Out}(D, \mathcal{F}) \cong C_2 \times \text{Aut}(Q)$. In cases (iii)-(v) $\text{Out}(D, \mathcal{F}) \cong S_3 \cap C_2$ and $\text{Out}_F(D) \cong C_3 \times C_3$, so $\text{Out}(D, \mathcal{F}) \cong C_2 \cap C_2$.

By Lemma 3.2, all algebras under consideration are equal to their source algebras and so by Lemma 3.1 $\text{Out}_D(A) = \text{Out}_{D_1}(b_1) \times \text{Out}_{D_2}(b_2)$. By [14, 1.5] $\text{Out}_{D_1}(b_1) = 1$ if $b_1 \cong kQ$ or $B_0(kA_3)$ and $C_3$ otherwise.

We have shown that $\mathcal{T}(b)$ is as stated in all cases except (iv). However, in this case $S_3 \times C_2 \leq \mathcal{T}(b)$ and there is an injection of $\mathcal{T}(b)$ into $C_3 \times (C_2 \cap C_2)$. This latter map cannot be surjective, which may be seen by considering centralizers of elements of order 2. Hence (iv) also holds.

Let $e$ be the block idempotent for $b$. Write $e = \sum_{t=1}^m \sum_{i=1}^{n_t} f_{t,i}$ where the $f_{t,i}$ are primitive idempotents and $bf_{t,i} \cong bf_{t,j}$ for each $i, j$. Let $M \in \mathcal{T}(B)$. By the above we may write $M = \varphi b$ where $\varphi \in \text{Aut}(b)$ is a composition of an automorphism induced by a group automorphism of $Q$ (where appropriate) and a permutation of the $\sum_{i=1}^{n_t} f_{t,i}$ and this may be taken so that for all $t \in \{1, \ldots, m\}$ we have $\varphi(f_{t,i}) = \varphi(f_{u,i})$ for some $u$. Defining $f = \sum_{t=1}^m f_{t,1}$ we have $\varphi(f) = f$ and we are done. □

4 Crossed products

An essential part of a reduction of Donovan’s conjecture to quasisimple groups is Külshammer’s analysis in [25] of the situation of a normal subgroup containing the defect groups of a block, which involves the study of crossed products of a basic algebra with an $\ell'$-group. In the general setting he finds finiteness results for the possible crossed products, but in our situation we are able to precisely describe the possibilities.

Background on crossed products may be found in [25], but we summarize what we need here. Let $X$ be a finite group and $R$ an $\mathcal{O}$-algebra. A crossed product of $R$ with
We claim each $kYH$ constitutes an $X$-algebra. Suppose that in [25, Section 3], $x$ with generator $fbf$ and $H$ is an $X$-algebra. Write $f$ as a crossed product of $R$ with $X$ and let $\alpha = \langle x \rangle$ (see [3, 2.8.4]), which vanishes since $H^3(X, U(Z(R)))$ vanishes for $X$ is trivial, and we may identify $X$ with a subgroup of $\text{Out}(R)$.

Now suppose that $X = \langle x \rangle$ is a cyclic $\ell'$-group. Let $i \in \mathbb{N}$. Following the strategy in [25, Section 3], $U(Z(R)) \cong U(Z(R)/J(Z(R))) \times (1 + J(Z(R)))$ and

$$H^i(X, U(Z(R))) \cong H^i(X, U(Z(R)/J(Z(R)))) \times H^i(X, 1 + J(Z(R)))$$

for each $i$. We have $H^i(X, 1 + J(Z(R))) = 0$ since $X$ is an $\ell'$-group. Now $Z(R)/J(Z(R))$ is a commutative semisimple $k$-algebra, which we denote $A$, and note as above that it is an $X$-algebra. Write $A = A_1 \times \cdots \times A_\ell$, where each $A_i$ is a product of simple algebras constituting an $X$-orbit. We have $H^i(X, U(A_i)) \cong H^i(X, U(A_1)) \times \cdots \times H^i(X, U(A_\ell))$. We claim each $H^i(X, U(A_j))$ vanishes, for as a $kX$-module $U(A_j)$ is induced from the trivial module of $kY$ for some $Y \leq X$, and so by Shapiro’s Lemma $H^i(X, U(A_j)) \cong H^i(Y, k^X)$ (see [3, 2.8.4]), which vanishes since $X$ is cyclic. We have shown that $H^i(X, U(Z(R))) = 0$ for each $i$.

Now suppose that we have a finite group $G$ and $N \triangleleft G$ with $G/N$ an $\ell'$-group. Suppose that $B$ is a block of $\mathcal{O}G$ covering a $G$-stable block $b$ of $\mathcal{O}N$. Define $X = G/N$ with generator $x$ and let $g_x \in x$.

Suppose that $\varphi \in \text{Aut}(b)$ and $f$ is an idempotent of $b$ such that $fbf$ is a basic algebra for $b$ and $\varphi(f) = f$. Then $\varphi$ may also be regarded as an element of $\text{Aut}(fbf)$ and we may consider $fBf$ as a crossed product of $fbf$ with $X$ via $\varphi$.

Suppose further that $\text{Aut}(R)$ splits over $\text{Inn}(R)$, say $\text{Aut}(R) = \text{Inn}(R) \times H$. Then in order to describe all crossed products of $R$ with $X$ it suffices to consider algebras $\langle a, R \rangle$ where $a$ is a unit and there is $h \in H$ such that for all $r \in R$ we have $ara^{-1} = h(r)$. Since $H^2(X, U(Z(R)))$ vanishes, we may assume that the map $\mu$ is trivial, and we may identify $X$ with a subgroup of $H$.

**Proposition 4.1** Let $G$ be a finite group and $N \triangleleft G$ with $G/N$ cyclic of odd prime order. Let $b$ be a $G$-stable block of $\mathcal{O}N$ with defect group $D \cong (C_2)^4$.

(i) If $b$ is Morita equivalent to $\mathcal{O}(A_4 \times C_2 \times C_2)$, then $B$ is Morita equivalent to $b$, $\mathcal{O}D$, $\mathcal{O}(A_4 \times A_4)$ or a non-principal block of $\mathcal{O}(C_2)^4 \times 3^{1+2}$, where the centre of $3^{1+2}$ acts trivially.

(ii) If $b$ is Morita equivalent to the principal block of $\mathcal{O}(A_5 \times C_2 \times C_2)$, then $B$ is Morita equivalent to $b$ or the principal block of $\mathcal{O}(A_5 \times A_4)$.

(iii) If $b$ is Morita equivalent to $\mathcal{O}(A_4 \times A_4)$, then $B$ is Morita equivalent to $b$ or $\mathcal{O}(A_4 \times C_2 \times C_2)$. 

10
(iv) If $b$ is Morita equivalent to the principal block of $\mathcal{O}(A_4 \times A_5)$, then $B$ is Morita equivalent to $b$ or the principal block of $\mathcal{O}(C_2 \times C_2 \times A_5)$.

(v) If $b$ is Morita equivalent to the principal block of $\mathcal{O}(A_5 \times A_5)$, then $B$ is Morita equivalent to $b$.

**Proof.** Let $B$ be a block of $G$ covering $b$. By Proposition[2.1] either $B$ is Morita equivalent to $b$, in which case we are done, or $B$ is the unique block of $G$ covering $b$. Hence we may assume the latter. Note that $B$ and $b$ share a defect group.

Define $X = G/N$ with generator $x$ and let $g \in x$. As described in the previous section conjugation $\tau(g)$ in $b$ by $g$ gives an element $\tau(g)b$ of $\mathcal{T}(b)$. By Proposition[3.3] in all cases (i)-(v) we may choose $\varphi \in \text{Aut}(b)$ with $\varphi \text{Inn}(b) = \tau(g)\text{Inn}(b)$ (and $\varphi b \cong \tau(g)b$) and an idempotent $f$ of $b$ such that $R := fbf$ is a basic algebra for $b$ and $\varphi(f) = f$. Then $\varphi$ may also be regarded as an element of $\text{Aut}(fbf)$ and we may consider $\Lambda := fBf$ as a crossed product of $fbf$ with $X$ via $\varphi$. Note that $\Lambda$ is Morita equivalent to $B$.

The weak equivalence classes of crossed products of $R$ with $X$ are in 1-1 correspondence with equivalence classes of homomorphisms $\alpha : X \to \text{Out}(R)$, hence to determine the possible Morita equivalence classes for $B$ we must determine the equivalence classes of homomorphisms $\alpha$ with image in $\langle \varphi \text{Inn}(R) \rangle$ when $\varphi$ is obtained as above from an element of $\mathcal{T}(b)$.

Now if $\ker(\alpha) = X$, then $\Lambda \cong \mathcal{O}X \otimes fbf$, which contradicts the fact that $B$ is Morita equivalent to $fbf$. By Proposition[3.3] $\mathcal{T}(b)$ has order divisible only by the primes 2 and 3 and so we may suppose that $|X| = 3$ and that $\ker(\alpha) = 1$.

We treat the five cases in turn. In many of the cases we will be making use of the example $PSL_3(7)$ where there is a block which is Morita equivalent to $\mathcal{O}A_4$ and covered by a nilpotent block of $PGL_3(7)$.

(i) Suppose $b$ is Morita equivalent to $\mathcal{O}(A_4 \times C_2 \times C_2)$. By Proposition[3.3] we have $\mathcal{T}(b) \cong S_3 \times S_3$ and so there are three possibilities for $\alpha$ up to equivalence (recall that $\alpha$ is assumed to be regarded). The three possible Morita equivalence types for $B$ are given by: $\mathcal{O}(A_4 \times A_4)$, realised when $N$ is $A_4 \times C_2 \times C_2$; $\mathcal{O}D$, realised when $G = PGL_3(7) \times C_2 \times C_2$; $N = PSL_3(7) \times C_2 \times C_2$; a non-principal block of $\mathcal{O}(C_2^4 \times 3^{1+2}_1)$, where the centre of $3^{1+2}_1$ acts trivially, achieved when $N$ is a maximal subgroup of $G$.

(ii) Suppose $b$ is Morita equivalent to the principal block of $\mathcal{O}(A_5 \times C_2 \times C_2)$. We have $\mathcal{T}(b) \cong C_2 \times S_3$ and so there is just one possibility for $\alpha$ up to equivalence, and this is achieved with $G = A_5 \times A_4$.

(iii) Suppose $b$ is Morita equivalent to $\mathcal{O}(A_4 \times A_4)$. We have $\mathcal{T}(b) \cong S_3 \times C_2$. There are two non-trivial possibilities for $\alpha$ up to equivalence. They give rise to an algebra Morita equivalent to $\mathcal{O}(A_4 \times C_2 \times C_2)$, realised with $G = A_4 \times PGL_3(7)$, and a non-principal block of $\mathcal{O}(C_2^4 \times 3^{1+2}_1)$.

(iv) Suppose $b$ is Morita equivalent to the principal block of $\mathcal{O}(A_4 \times A_5)$. We have $\mathcal{T}(b) \cong S_3 \times C_2$ and so there is just one possibility for $\alpha$ up to equivalence, and this is realised with $G = PGL_3(7) \times A_5$.

(v) Suppose $b$ is Morita equivalent to the principal block of $\mathcal{O}(A_5 \times A_5)$. We have $\mathcal{T}(b) \cong C_2 \times C_2$ and so there are no non-trivial possibilities for $\alpha$ (i.e., this case cannot occur without contradiction of the assumption that $B$ is the unique block covering $b$).
Corollary 4.2 Consider $G = (C_2)^4 \times 3_1^{1+2}$, where the centre of $3_1^{1+2}$ acts trivially. The 2-blocks of $O\mathcal{O}_G$ correspond to the simple modules of $Z(3_1^{1+2})$, and the two non-principal blocks are Morita equivalent. Further, these blocks are Morita equivalent to the two non-principal blocks of $\mathcal{O}((C_2)^4 \times 3_1^{1+2})$.

**Proof.** Let $B$ be any faithful 2-block of $G = (C_2)^4 \times 3_1^{1+2}$ or $(C_2)^4 \times 3_1^{1+2}$ and $l(B) = 1$. Take a maximal subgroup $N$ of $G$ and a block $b$ of $N$ covered by $B$. Then $N \cong ((C_2)^4 \times C_3) \times C_3$ or $(C_2)^4 \times C_9$ and $b$ is Morita equivalent to $\mathcal{O}(C_2 \times C_2 \times A_4)$. By Proposition [4.1] there is only one possibility for the Morita equivalence class of $B$ under the restriction that there is just one simple module.

\[\square\]

5 Proof of the main theorem

We first address the case where the defect group is normal.

**Lemma 5.1** Let $B$ be a block of $O\mathcal{O}_G$ for a finite group $G$ with normal defect group $D \cong (C_2)^4$. Then $B$ is Morita equivalent to a block as in (a) or (b)(i), (ii), (iv), (v), (vi), (viii), (xi) or (xiii) in Theorem 1.1.

**Proof.** This follows from the main result of [24], applying Lemma 4.2 when the inertial quotient is $C_3 \times C_3$. \[\square\]

For a block $B$, write $\text{IBr}(B)$ for the set of irreducible Brauer characters of $B$ and $l(B) = |\text{IBr}(B)|$.

The following lemma deals for example with the situation $SL_n(q) \cong N_\triangleleft G$ where $G$ is an extension by field automorphisms and the block of $SL_n(q)$ is nilpotent covered.

**Lemma 5.2** Let $G$ be a finite group and $N_\triangleleft G$ such that $G/N$ is solvable. Let $B$ be a quasiprimitive block of $O\mathcal{O}_G$ with abelian defect group $D$ covering a block $b$ of $O\mathcal{O}_N$ also with defect group $D$. If $b$ is nilpotent covered, then $B$ is Morita equivalent to a block of a finite group with normal defect group. In particular, if $D \cong (C_2)^4$, then $B$ is Morita equivalent to one of the blocks in (a) or (b)(i), (ii), (iv), (v), (vi), (viii), (xi) or (xiii) of Theorem 1.1.

**Proof.** By Proposition [2.5](ii) $b$ is inertial, i.e., basic Morita equivalent to its Brauer correspondent $c$ in $N_N(D)$. Let $M$ be the preimage in $G$ of $O_{\ell}(G/N)$ and $B_M$ the unique block of $M$ covered by $B$. By Proposition [2.5](iii) $B_M$ is inertial. Write $M_1$ for the preimage in $G$ of $O_{\ell}(G/M)$ and let $B_{M_1}$ be the unique block of $M_1$ covered by $B$. Note that $B_M$ and $B_{M_1}$ both have defect group $D$. Since $M_1/M$ is an $\ell$-group $B_{M_1}$ is the unique block of $M_1$ covering $B_M$. But then by [11.15.1] $M_1 = MD$, and so $M = M_1$. Since $G/N$ is solvable this implies that $M = G$, and $B$ is inertial. The last part follows by Lemma 5.1. \[\square\]

We prove Theorem 1.1.
**Proof.** Let $B$ be a block of $OG$ for a finite group $G$ with defect group $D \cong (C_2)^4$ with $([G : O_2(Z(G))], |G|)$ minimised in the lexicographic ordering such that $B$ is not Morita equivalent to any of the sixteen blocks listed in the theorem.

Suppose $N < G$ and $b$ is a block of $ON$ covered by $B$. Write $I = I_G(b)$ for the stabiliser of $b$ under conjugation. Then there is a unique block $B_I$ of $I$ covering $b$ with Brauer correspondent $B$ (the Fong-Reynolds correspondent) and $B_I$ is Morita equivalent to $B$. Further $B$ and $B_I$ share a defect group, hence by minimality $I = G$.

Applying this to all normal subgroups of $G$, we have that $B$ is quasiprimitive, that is, for every $N < G$ each block of $ON$ covered by $B$ is $G$-stable.

By Corollary 2.3 and minimality, if $N < G$ and $B$ covers a nilpotent block of $ON$, then $N \leq Z(G)O_2(G)$. In particular $O_2(G) \leq Z(G)$.

Note that $O^2(G)D = G$. This holds by [11, 15.1] since any block of $O^2(G)$ covered by $B$ is $G$-stable and $B$ is the unique block of $G$ covering it.

Following [2] write $E(G)$ for the layer of $G$, that is, the central product of the subnormal quasisimple subgroups of $G$ (the components). Write $F(G)$ for the fitting subgroup, which in our case is $F(G) = Z(G)O_2(G)$. Write $F^*(G) = F(G)E(G) < G$, the generalised fitting subgroup, and note that $C_G(F^*(G)) \leq F^*(G)$. Let $b^*$ be the unique block of $OF^*(G)$ covered by $B$.

We have $E(G) \neq 1$, since otherwise $F^*(G) = F(G) = Z(G)O_2(G)$ and $D \leq C_G(F^*(G)) \leq F^*(G)$, so that $D < G$, a contradiction by Lemma 5.1. Write $E(G) = L_1 \ast \cdots \ast L_t$, where each $L_i$ is a component of $G$ (we have shown that $t \geq 1$). Now $B$ covers a block $b_E$ of $OE(G)$ with defect group contained in $D$, and $b_E$ covers a block $b_i$ of $OL_i$. Since $b_E$ is $G$-stable, for each $i$ either $L_i < G$ or $L_i$ is in a $G$-orbit in which each corresponding $b_i$ is isomorphic (with equal defect). Since $B$ has defect four, it follows that if $t \geq 3$, then $B$ covers a nilpotent block of a normal subgroup generated by components of $G$, a contradiction. Hence $t \leq 2$, and in particular $G/F^*(G)$ is solvable by the Schreier conjecture.

We have $|F^*(G) \cap D| \geq 4$, since otherwise $B$ covers a nilpotent block of $F^*(G)$, a contradiction since $F^*(G)$ is not central in $G$.

In the next part of the proof we will show that $G$ (as a minimal counterexample) has a proper normal subgroup $N$ containing $D$ such that the unique block $b$ of $N$ covered by $B$ is of type $(C_3)_1$ or $(C_3) \times C_3$.

Suppose $|F^*(G) \cap D| = 4$. Then $F^*(G) \cap D$ is normal in $NG(D)$ and so any non-nilpotent block of $O^2(F^*(G)\langle D^g : g \in G \rangle)$ has type $(C_3)_1$, $(C_3)_2$ or $C_3 \times C_3$. We claim that $O^2(F^*(G)\langle D^g : g \in G \rangle)$ is a proper subgroup of $G$. For suppose $O^2(F^*(G)\langle D^g : g \in G \rangle) = G$. Since $O^2(G) = G$, then $F^*(G)\langle D^g : g \in G \rangle = G$. Since $G/F^*(G)$ is solvable it follows that $O^2(G) \neq G$. Since $G = O^2(G)D$, it follows that $G$ has a normal subgroup $H$ of index 2 containing $F^*(G)$ such that $G = H D$. Hence $B$ must have type $(C_3)_1$ and we may apply Proposition 2.3 to show that by minimality $B$ is Morita equivalent to a block on the list. Hence $O^2(F^*(G)\langle D^g : g \in G \rangle)$ is a proper subgroup of $G$ as claimed, and we take $N = O^2(F^*(G)\langle D^g : g \in G \rangle)$. As above $O^2(N) \neq N$ and $N$ has a normal subgroup of index 2, so that we may rule out the possibilities that $b$ has type $(C_3)_2$ or $C_3 \times C_3$.

Suppose that $|F^*(G) \cap D| = 8$. It follows from Proposition 2.7 that either $b^*$ has inertial quotient $C_3$ or $E(G)$ is isomorphic to one of $SL_2(8)$, $G_2(3^{2m+1})$, $J_1$ or $Co_3$. In the former case we may take $N = F^*(G)\langle D^g : g \in G \rangle$ and it is clear that $b$ must be of
type $(C_3)_1$. Since $O^2(G) = G$ we have $N \neq G$. On the other hand each of the groups $SL_2(8)$, $2G_2(3^{2m+1})$, $J_1$ and $C_03$ has odd order outer automorphism group, so $G$ has a direct factor of order 2, contradicting $O^2(G) = G$.

Hence we may suppose that $D \leq F^*(G)$. We examine the possibilities for $O_2(G)$.

If $|O_2(G)| = 16$, then $O_2(G) = D$, a contradiction by Lemma 5.1. If $|O_2(G)| = 8$, then as $E(G) \neq 1$, $B$ covers a nilpotent block of $E(G)$, a contradiction. If $|O_2(G)| = 4$, then $b^*$ must be of type $(C_3)_1$, $(C_3)_2$ or $C_3 \times C_3$. However $F^*(G)$ would have a normal subgroup of index 2 and so we may rule out the cases of type $(C_3)_2$ and $C_3 \times C_3$. If $F^*(G) = G$, then $B$ is Morita equivalent to a block in the list, a contradiction. Hence we may take $N = F^*(G)$.

Hence $|O_2(G)| = 1$ or 2, and $O_2(G) \leq Z(G)$.

Suppose that $t = 1$. By Proposition 2.7 one or more of:

1. $b^*$ has type $(C_3)_1$;
2. $F^*(G)$ is isomorphic to one of $C_2 \times SL_2(8)$, $C_2 \times 2G_2(3^{2m+1})$, $C_2 \times J_1$, $C_2 \times C_03$ or $SL_2(16)$, as in each of these cases the component must be simple (since the Schur multiplier is trivial); or
3. $b^*$ is nilpotent covered.

In case (1) we may take $N = F^*(G)$.

Suppose case (2) occurs. If $F^*(G) \cong C_2 \times J_1$, $C_2 \times C_03$ or $SL_2(16)$, then $Out(F^*(G)) = 1$ and so $G = F^*(G)$. By [23] the non-principal block of $C_03$ with elementary abelian defect group of order 8 is Morita equivalent to the principal block of $Aut(SL_2(8))$ and so in each of these three cases $B$ is Morita equivalent to a block in the list, a contradiction. If $F^*(G) \cong SL_2(8)$, then $G \cong S^2L_2(8)$ or $Aut(SL_2(8))$, again a contradiction. If $F^*(G) \cong C_2 \times 2G_2(3^{2m+1})$, then $G$ has two as a direct factor and by [12] 3.1 $B$ is Morita equivalent to $b^*$. Hence by minimality $G \cong C_2 \times 2G_2(3^{2m+1})$.

By [34] Example 3.3, which in turn uses [28], $b^*$ is Morita equivalent to the principal block of $C_2 \times 2G_2(3) \cong C_2 \times Aut(SL_2(8))$, again a contradiction to minimality.

If (3) occurs, then we may apply Lemma 5.2 to obtain a contradiction.

Now suppose that $t = 2$. Then $b_1$ and $b_2$ both have Klein four defect group and are non-nilpotent, and $b^*$ has type $C_3 \times C_3$. Hence we may take $N = F^*(G)$.

We have shown that there is a normal subgroup $N \triangleleft G$ containing $D$ and a block $b$ of $N$ covered by $B$ with type $(C_3)_1$ or $C_3 \times C_3$.

Write $J = G[b] \triangleleft G$, and let $B_J$ be the unique block of $OJ$ covering $b$ and covered by $B$. By Proposition 2.1, $B_J$ is source algebra equivalent to $b$, and so in particular is also of type $(C_3)_1$ or $C_3 \times C_3$. Hence we may assume (repeatedly applying the argument if necessary) that $G[b] = N$. Then by Proposition 2.1, $B$ is the unique block of $G$ covering $b$. Hence by [1] 15.1, $[G : N]$ is odd since $B$ and $b$ share a defect group, and so $G/N$ is solvable (note that it is not strictly necessary to directly use the odd order theorem here, as in all the cases above $N$ contains $F^*(G)$ and $G/F^*(G)$ is solvable).

Let $M \triangleleft G$ with $[G : M]$ is an odd prime. Let $B_M$ be the unique block of $M$ covered by $B$. By Lemma 2.10, $B_M$ has type $(C_3)_1$ or $C_3 \times C_3$.

Now by minimality $B_M$ is Morita equivalent to a block as in (a), (b)(ii), (b)(iii), (b)(viii), (b)(ix) or (b)(x) in the statement of the theorem. Suppose that $B_M$ is as in (a). Then $B_M$ is inertial as by minimality there is only one possibility for the Morita equivalence class of $B_M$ and of its Brauer correspondent in $N_M(D)$. So by Proposition 2.5 $B$ is also inertial and by Lemma 5.1 is Morita equivalent to one of the listed blocks,
a contradiction. We now have that $B_M$ is Morita equivalent to one of the blocks considered in Proposition 4.1, which we apply to see that $B$ is one of the blocks listed in the statement of the theorem.

To see that the blocks in cases (a),(b) (i)-(xv) represent distinct Morita equivalence classes it suffices to note that the blocks in case (b) have distinct Cartan matrices and the basic algebras for the blocks in (a) and (b)(i) are not isomorphic.

That the blocks in case (a) cannot be Morita equivalent to a principal block follows from 33, 6.13 that if the principal block has only one simple module, then it is nilpotent.

Finally, we reference the literature that tells us that representatives of the Morita equivalence classes with the same inertial quotient and number of simple modules are derived equivalent. In the cases below splendid derived equivalences are established between the relevant blocks defined with respect to $k$. Then by 38, 5.2 there is a splendid derived equivalence over $\mathcal{O}$.

By 38, §3 the principal blocks of $kA_4$ and $kA_5$ are splendid derived equivalent. It follows that the blocks in cases (ii) and (iii) are derived equivalent, and that the blocks in cases (viii), (ix) and (x) are derived equivalent. The principal blocks of $kSL_2(16)$ and $k((C_2)^4 \rtimes C_{15})$ (the normalizer of a Sylow 2-subgroup) are derived equivalent by 19, and so the blocks in cases (xi) and (xii) are splendid derived equivalent. That the principal blocks of $kJ_1$ and $k((C_2)^3 \rtimes (C_7 \rtimes C_3))$ are derived equivalent follows from 18, and a published proof may be found in 9 §6.2.3, with the observation that the blocks are splendid derived equivalent. Hence the blocks in cases (xiii) and (xiv) are derived equivalent. Finally, the splendid derived equivalence between the blocks in cases (xiii) and (xv) follows from 34 Remark 3.4 and 9 4.33.

\[ \square \]

ACKNOWLEDGEMENTS

I am deeply indebted to Jon Carlson, who wrote and ran MAGMA routines for calculating outer automorphism groups of the basic algebras defined over $k$, which helped me see their final structure and complete an earlier version of this paper. I also thank Markus Linckelmann for encouraging me to extend my results over $k$ to $\mathcal{O}$ and for helpful discussions. The papers of Watanabe are essential in completing the classification over $\mathcal{O}$, and I am indebted to Hu Xueqin for directing me to 43 and to Shigeo Koshitani for showing me 42 and 45. Finally I thank Cesare Ardito for his careful reading of the manuscript and for his helpful comments, and Michael Livesey for some useful discussions.

References

[1] J. L. Alperin, Local Representation Theory, Cambridge Studies in Advanced Mathematics 11, Cambridge university Press (1986).

[2] M. Aschbacher, Finite group theory, Cambridge Studies in Advanced Mathematics 10, Cambridge university Press (1986).
[3] D. Benson, *Representations and cohomology. I*, Cambridge Studies in Advanced Mathematics 30, Cambridge University Press (1991).

[4] R. Boltje, R. Kessar and M. Linckelmann, *On Picard groups of blocks of finite groups*, preprint

[5] M. Broué and L. Puig, *Characters and local structure in G-algebras*, J. Algebra 63 (1980), 306–317.

[6] M. Broué and L. Puig, *A Frobenius theorem for blocks*, Invent. Math. 56 (1980), 117–128.

[7] D. A. Craven, C. W. Eaton, R. Kessar and M. Linckelmann, *The structure of blocks with a Klein four defect group*, Math. Z. 268 (2011), 441–476.

[8] D. A. Craven and A. Glesser, *Fusion systems on small p-groups*, Trans. AMS 364 (2012), 5945–5967.

[9] D. A. Craven and R. Rouquier, *Perverse equivalences and Broué’s conjecture*, Adv. Math. 248 (2013), 1–58.

[10] C. W. Curtis and I. Reiner, *Methods of representation theory with applications to finite groups and orders, Volumes I and II*, John Wiley and Sons (1987).

[11] E. C. Dade, *Block extensions*, Ill. J. Math. 17 (1973), 198-272.

[12] C. W. Eaton, *Morita equivalence classes of 2-blocks of defect three*, Proc. AMS 144 (2016), 1961–1970.

[13] C. W. Eaton, R. Kessar, B. Külshammer and B. Sambale, *2-blocks with abelian defect groups*, Adv. Math. 254 (2014), 706-735.

[14] C. W. Eaton, B. Külshammer and B. Sambale, *2-blocks with minimal nonabelian defect groups, II*, J. Group Theory 15 (2012), 311–321.

[15] C. W. Eaton and M. Livesey, *Classifying blocks with abelian defect groups of rank 3 for the prime 2*, preprint (2017).

[16] K. Erdmann, *Blocks whose defect groups are Klein four groups: a correction*, J. Algebra 76 (1982), 505–518.

[17] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Mathematics 1428, Springer-Verlag (1990).

[18] H. Gollan and T. Okuyama, * Derived equivalences for the smallest Janko group*, preprint (1997).

[19] M. L. Holloway, *Derived equivalences for group algebras*, Ph.D. thesis, University of Bristol (2001).
[20] R. Kessar, S. Koshitani and M. Linckelmann, *Conjectures of Alperin and Broué for 2-blocks with elementary abelian defect groups of order 8*, J. Reine Angew. Math. **671** (2012), 85–130.

[21] S. Koshitani, *Conjectures of Donovan and Puig for principal 3-blocks with abelian defect groups*, Comm. Alg. **31** (2003), 2229-2243; *Corrigendum*, **32** (2004), 391–393.

[22] S. Koshitani and B. Külshammer, *A splitting theorem for blocks*, Osaka J. Math. **33** (1996), 343–346.

[23] S. Koshitani, J. Müller and F. Noeske, *Broué’s abelian defect group conjecture holds for the sporadic simple Conway group Co3*, J. Algebra **358** (2011), 354–380.

[24] B. Külshammer, *Crossed products and blocks with normal defect groups*, Comm. Alg. **13** (1985), 147–168.

[25] B. Külshammer, *Donovan’s conjecture, crossed products and algebraic group actions*, Israel J. Math. **92** (1995), 295–306.

[26] B. Külshammer and L. Puig, *Extensions of nilpotent blocks*, Invent. Math. **102** (1990), 17–71.

[27] B. Külshammer and B. Sambale, *The 2-blocks of defect 4*, Representation Theory **17** (2013), 226–236.

[28] P. Landrock and G. Michler, *Principal 2-blocks of the simple groups of Ree type*, Trans. Amer. Math. Soc. **260** (1980), 83–111.

[29] P. Landrock and B. Sambale, *On centers of blocks with one simple module*, J. Algebra **472** (2017), 339–368.

[30] M. Linckelmann, *The source algebras of blocks with a Klein four defect group*, J. Algebra **167** (1994), 821–854.

[31] M. Linckelmann, *The isomorphism problem for cyclic blocks and their source algebras*, Invent. Math. **125** (1996), 265–283.

[32] M. Linckelmann, *The block theory of finite groups I, II*, to appear, CUP.

[33] G. Navarro, *Characters and blocks of finite groups*, London Mathematical Society Lecture Note Series **250**, Cambridge University Press (1998).

[34] T. Okuyama, *Some examples of derived equivalent blocks of finite group*, preprint (1997).

[35] L. Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), 77–116.

[36] L. Puig, *Pointed groups and construction of modules*, J. Algebra **116** (1988), 7–129.
[37] L. Puig, *Nilpotent extensions of blocks*, Math. Z. 269 (2011), 115-136.

[38] J. Rickard, *Splendid equivalences: derived categories and permutation modules*, Proc. London Math. Soc. 72 (1996), 331–358.

[39] B. Sambale, 2-blocks with minimal nonabelian defect groups, J. Algebra 337 (2011), 261–284.

[40] B. Sambale, *Cartan matrices and Brauer’s k(B)-conjecture IV*, J. Math. Soc. Japan 69 (2017), 735–754.

[41] A. Watanabe, *On nilpotent blocks of finite groups*, J. Algebra 163 (1994), 128–134.

[42] A. Watanabe, *A remark on a splitting theorem for blocks with abelian defect groups*, RIMS Kokyuroku Vol.1140, Edited by H.Sasaki, Research Institute for Mathematical Sciences, Kyoto University (2000) 76–79.

[43] A. Watanabe, *On perfect isometries for blocks with abelian defect groups and cyclic hyperfocal subgroups*, Kumamoto J. Math. 18 (2005), 85-92.

[44] C. Wu, K. Zhang and Y. Zhou, *Blocks with defect group $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$*, preprint (2017).

[45] Y. Zhou, *On the $p'$-extensions of inertial blocks*, Proc. AMS 144 (2016), 41-54.

Charles Eaton
School of Mathematics
University of Manchester
Oxford Road
Manchester
M13 9PL
United Kingdom
charles.eaton@manchester.ac.uk