THE INEXACT LOG-EXPONENTIAL REGULARIZATION METHOD FOR MATHEMATICAL PROGRAMS WITH VERTICAL COMPLEMENTARITY CONSTRAINTS

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Abstract. We study the convergence of the log-exponential regularization method for mathematical programs with vertical complementarity constraints (MPVCC). The previous paper assume that the sequence of Lagrange multipliers are bounded and it can be found by software. However, the KKT points cannot be computed via Matlab subroutines exactly in many cases. We note that it is realistic to compute inexact KKT points from a numerical point of view. We prove that, under the MPVCC-MFCQ assumption, the accumulation point of the inexact KKT points is Clarke (C-) stationary point. The idea of inexact KKT conditions can be used to define stopping criteria for many practical algorithms. Furthermore, we introduce a feasible strategy that guarantees inexact KKT conditions and provide some numerical examples to certify the reliability of the approach. It means that we can apply the inexact regularization method to solve MPVCC and show the advantages of the improvement.

1. Introduction. In this paper, we consider the following mathematical program with vertical complementarity constraints (MPVCC)

$$\begin{align*}
\min \quad & f(z) \\
\text{s.t.} \quad & \min \{F_{i1}(z), F_{i2}(z), \ldots, F_{il}(z)\} = 0 \quad \forall \ i = 1, 2, \ldots, m, \\
& g_i(z) \leq 0 \quad \forall \ i = 1, 2, \ldots, p, \\
& h_i(z) = 0 \quad \forall \ i = 1, 2, \ldots, q,
\end{align*}$$

(1)

where $f: \mathbb{R}^n \to \mathbb{R}$, $g_i: \mathbb{R}^n \to \mathbb{R}$, $h_i: \mathbb{R}^n \to \mathbb{R}$, and $F_{ij}: \mathbb{R}^n \to \mathbb{R}$ are twice continuously differentiable functions. We have first studied the MPVCC by Scheel and Scholtes [21], which appears in economic equilibrium systems, engineering design,

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optimal control, etc. The constrained min-max-min problem (CM³P) [25] and the Stackelberg game with an equilibrium constraint [27] can be written as MPVCC. We allow different \( l_i \) for each vertical complementarity constraint, such as some min-max-min problems in [25]. For simplicity, we may take the single \( l \) and it has no effect in our theoretical analysis. Note that the vertical complementarity constraints in (1) can be equivalently rewritten as

\[
F_{i1}(z) \geq 0, \quad F_{i2}(z) \geq 0, \ldots, \quad F_{il}(z) \geq 0, \quad \prod_{j=1}^{l} F_{ij}(z) = 0, \quad i = 1, 2, \ldots, m,
\]

which implies that there is no feasible solution satisfying all inequalities strictly. Due to the special constraints, the MPVCC is viewed as a difficult nonlinear optimization problem (NLP), since some theoretical properties and numerical methods of NLP are invalid to deal with MPVCC.

In fact, several ideas have been discussed to overcome the difficulties of MPVCC. Among various methods, regularization methods have been proven to be one prominent method. The efficiency of the regularization methods depend on replacing the vertical complementarity conditions by appropriate regularized conditions with a parameter \( t > 0 \). Notice, if \( t \downarrow 0 \), then the regularized problems approach the original MPVCC. Therefore, we can get stationary points of MPVCC by constructing effective numerical methods to solve regularized problems.

It is obvious that mathematical programs with complementarity constraints (MPCC) [22] is a particular MPVCC. There exist several regularization methods to solve MPCC, but not all regularization methods can directly apply to MPVCC. Apparently, we cannot use the NCP function introduced in [17] and functions in [16] to formulate the special structure (2). Thus, the regularization schemes by Kanzow & Schwartz [10] and Lin & Fukushima can not be applied to solve (1). Furthermore, by comparing the complementarity constraints of MPCC with (2), there are more difficulties in the proofs of convergence properties of the regularization schemes for MPVCC. Consequently, we may not get the convergence results of the regularized problems for MPVCC, as we have previously known in [8]. Based on the above argument, we emphasize that some regularization methods for MPCC fail in MPVCC.

During the last decade, two regularization methods which still obtain the C-stationarity in the limit have been extended to solve MPVCC. In [14], they propose a regularization method which can be seen an extension of the global regularization method by Scholtes [22]. The authors assume that the KKT-points of the regularized problems exist and get C-stationary points under the MPVCC Linear independence constraint qualification (MPVCC-LICQ) assumption. Based on the log-exponential function, another regularization method [27] was given for MPVCC. It can be seen an extension of Yin & Zhang [26] and Li, Tan & Li [13] to MPVCC. The accumulation point of the KKT-points of regularized problem is also C-stationarity when the KKT-points and Lagrange multipliers are convergent. However, practical applications do not always follow these theories. In spite of quite a number of contributions in theoretical results, there are unrealistic to compute the KKT-points directly in a practical algorithm. Since the KKT-points are not naturally available as a numerical point. Moreover, the defect lead to the termination criteria does not coincide with the usual used in practice.

An effective way to deal with the defect is to replace KKT-points by the inexact KKT points of the regularized problems. However, there were some difficulties in the
proofs of the convergence results, since the sequence of inexact KKT points lose the sign structure of some multipliers. The main purpose of this paper is to obtain the anticipant stationary point by the regularization method for MPVCC and present a feasible strategy to solve MPVCC. Hence, we expect to find a regularization method that may maintain the stationarity of MPVCC in the convergence results.

We focus on the log-exponential regularization method proposed by [27] which still converge to C-stationarity. We show that the inexact KKT points of the log-exponential regularization problem converge to C-stationary point under the MPVCC Mangasarian-Fromovitz constraint qualification (MPVCC-MFCQ). From Example 1, the accumulation point can not be M-stationary point while maintaining the assumptions. We emphasize that our theoretical result is inspired by the feasibility of generating the inexact KKT points. Consequently, from the practical and theoretical points of view, we introduce a feasible strategy based on Newtonian penalty-barrier Lagrangian method. The sequence of inexact KKT points can be generated by the approach. Finally, the numerical comparisons with other methods are discussed with some examples. We certify the reliability and applicability of the approach.

The paper is organized as follows. In Section 2, we define $\varepsilon$-KKT concept as our notion of the inexact KKT point and give some stationarity concepts of MPVCC and constraint qualifications that we shall need. The main theoretical contribution is summarized in Section 3, which improve the convergence results by considering $\varepsilon$-KKT points. The feasible strategy that guarantees $\varepsilon$-KKT points is introduced in Section 4, showing that the log-exponential regularization method is effective to solve the MPVCC in a practical point of view. The applicability of our approach is illustrated through the numerical results presented in Section 5. We conclude the paper in Section 6.

We give some notations needed in this paper. $N$ denotes the set of natural number. For a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we denote $\nabla f(z)$ as the transposed Jacobian of $f$ and $\nabla^2 f(z)$ as the Hessian. We denote that $\alpha_i$ or $[\alpha]_i$ is the $i$-th element of $\alpha^T = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. $e_i$ denotes the $i$-th column of identity matrix and $e = \sum e_i$. The support of $z \in \mathbb{R}^n$ is defined as $\text{supp}(z) = \{i \mid z_i \neq 0\}$. $\|\cdot\|_\infty$ and $\|\cdot\|$ denote respectively the infinity norm and the Euclidean norm of a vector.

2. Preliminaries. Firstly, we recall several definitions and results from nonlinear program, which are used in our regularized problem. Consider the following nonlinear program (NLP)

$$\begin{align*}
\min \quad & f(z) \\
\text{s.t.} \quad & g_i(z) \leq 0 \quad \forall i = 1, \ldots, p, \\
& h_{i}(z) = 0 \quad \forall i = 1, \ldots, q,
\end{align*}$$

where $f$, $g_i$, $h_{i} : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions. We denote the active index of $g$ at $z$ as

$$\mathcal{I}_g(z) = \{i \mid g_i(z) = 0\}.$$ 

We say that a feasible point $z^*$ is a KKT point of NLP, i.e., $z^*$ satisfies the Karush-Kuhn-Tucker (KKT) conditions. It means that there exist multipliers $\lambda \in \mathbb{R}^p$, $\mu \in$
\[ \nabla f(z^*) + \sum_{i=1}^{p} \lambda_i \nabla g_i(z^*) + \sum_{i=1}^{q} \mu_i \nabla h_i(z^*) = 0, \]

where \( \lambda \geq 0 \) and \( \text{supp}(\lambda) \subseteq \mathcal{I}_p(z^*) \).

However, the numerical algorithms rarely terminate in the KKT point. Note that the stopping criteria of nonlinear program algorithms are usually approximate conditions in software, like [3, 4, 5, 6, 24]. In a recent work [9], Kanzow only need to compute inexact KKT points of variational inequalities to investigate some classes of quasi-variational inequalities. From [2], we can fulfill the inexact KKT conditions depending on the following conditions.

**Definition 2.1.** [11, 12] Let \( \varepsilon \) be a positive constant and \( z^* \in \mathbb{R}^n \). If there exist multipliers \( \lambda \in \mathbb{R}^p, \mu \in \mathbb{R}^q \) such that

\[
\| \nabla f(z^*) + \sum_{i=1}^{p} \lambda_i \nabla g_i(z^*) + \sum_{i=1}^{q} \mu_i \nabla h_i(z^*) \|_{\infty} \leq \varepsilon,
\]

\[
g_i(z^*) \leq \varepsilon, \quad \lambda_i \geq -\varepsilon, \quad | \lambda_i g_i(z^*) | \leq \varepsilon, \quad i = 1, \ldots, p,
\]

\[
h_i(z^*) \leq \varepsilon, \quad i = 1, \ldots, q,
\]

we call that \( z^* \) is an \( \varepsilon \)-KKT of NLP.

Note that \( \varepsilon \)-KKT is also denoted as AKKT(\( \varepsilon \)) point (Approximate KKT point).

In practice, different constant \( \varepsilon > 0 \) can be used in the inexact KKT conditions. For simplicity, we may take the single \( \varepsilon \) in the definition. Clearly, the \( \varepsilon \)-KKT conditions can be formulated in particular class of methods. According to the analysis of interior-point methods, SQP-type methods and semismooth Newton methods, it follows that the definition of \( \varepsilon \)-KKT point is enough to cover the whole circumstances.

The most common stationary point of NLP is KKT point. Note that there exist several meaningful stationary points of MPVCC. In the following, we give some definitions of stationarity concepts of MPVCC.

**Definition 2.2.** Let \( z^* \) be a feasible point of (1).

(i) We say that \( z^* \) is weakly stationarity [21], if there exist multipliers \( \lambda^* \in \mathbb{R}^p, \mu^* \in \mathbb{R}^q, \Gamma^* \in \mathbb{R}^{m \times l} \) such that

\[
\begin{align*}
\nabla L(z^*, \lambda^*, \mu^*, \Gamma^*) &= 0, \\
\Gamma^*_{ij} F_{ij}(z^*) &= 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, l, \\
\lambda^* &\geq 0, \quad \sum_{i=1}^{p} \lambda_i^* g_i(z^*) = 0,
\end{align*}
\]

where \( L(z, \lambda, \mu, \Gamma) = f(z) + \sum_{i=1}^{p} \lambda_i g_i(z) + \sum_{i=1}^{q} \mu_i h_i(z) - \sum_{i=1}^{m} \sum_{j=1}^{l} \Gamma_{ij} F_{ij}(z). \)

(ii) \( z^* \) is C-stationarity [21], if there exist multipliers satisfying (3) and

\[
\Gamma^*_{ij} \Gamma^*_{ij} \geq 0, \quad (j, \bar{j}) : F_{ij}(z^*) = F_{\bar{j} \bar{j}}(z^*) = 0.
\]

(iii) \( z^* \) is M-stationarity [15], if there exist multipliers satisfying (3) and

either \( \Gamma^*_{ij} \Gamma^*_{ij} = 0 \) or \( \Gamma^*_{ij} > 0, \quad \Gamma^*_{\bar{j} \bar{j}} > 0 \), if \( \exists \ j \neq \bar{j} : F_{ij}(z^*) = F_{\bar{j} \bar{j}}(z^*) = 0. \)

(iv) \( z^* \) is S-stationarity [21], if there exist multipliers satisfying (3) and

\[
\Gamma^*_{ij} \geq 0, \quad \exists \ j \neq \bar{j} : F_{ij}(z^*) = F_{\bar{j} \bar{j}}(z^*) = 0.
\]
The relations among these above stationary concepts can be stated as follows:

S-stationarity \( \Rightarrow \) M-stationarity \( \Rightarrow \) C-stationarity \( \Rightarrow \) weakly stationarity.

Next, we introduce the conceptions of the definition of positive-linearly dependent vectors which can be used in some MPVCC CQs.

**Definition 2.3.** [20, 23] The vectors \( \{ \alpha_i \}_{i=1}^p \cup \{ \beta_i \}_{i=1}^q \) are said to be positive-linearly dependent if there exist scalars \( a \in \mathbb{R}^p \) and \( b \in \mathbb{R}^q \) such that \( a_i \geq 0 \) for all \( i = 1, \ldots, p \), \( (a, b) \neq 0 \) and

\[
\sum_{i=1}^p a_i \alpha_i + \sum_{i=1}^q b_i \beta_i = 0.
\]

Otherwise, we say that the vectors are positive-linearly independent.

In order to facilitate the notation, we denote several useful index sets as follows:

\[
\mathcal{I}_{F_{ij}}(z) = \{ i \mid F_{ij}(z) = 0 \},
\]

\[
\mathcal{I}_F(z) = \{ (i, j) \mid F_{ij}(z) = 0 \}.
\]

Now, we shall introduce MPVCC-MFCQ which are well-known.

**Definition 2.4.** [21] Let \( z^* \) be a given feasible point of (1). MPVCC-MFCQ holds at \( z^* \) if the gradient vectors

\[
\{ \nabla h_i(z^*) \mid i = 1, 2, \ldots, q \} \cup \{ \nabla F_{ij}(z^*) \mid (i, j) \in \mathcal{I}_F(z^*) \}
\]

are linearly independent and there exists a vector \( d \) such that

\[
\nabla g_i(z^*)^T d < 0, \quad i \in \mathcal{I}_g(z^*),
\]

\[
\nabla h_i(z^*)^T d = 0, \quad i = 1, \ldots, q,
\]

\[
\nabla F_{ij}(z^*)^T d = 0, \quad (i, j) \in \mathcal{I}_F(z^*).
\]

Note that it is not hard to present an equivalent definition for MPVCC-MFCQ (cf. [8]). A feasible point \( z^* \) of (1) satisfies MPVCC-MFCQ if the gradients

\[
\{ \nabla g_i(z^*) \mid i \in \mathcal{I}_g(z^*) \} \cup \{ \nabla h_i(z^*) \mid i = 1, 2, \ldots, q \} \cup \{ \nabla F_{ij}(z^*) \mid (i, j) \in \mathcal{I}_F(z^*) \}
\]

are positive-linearly independent.

Notice that the extra pair of curly brackets around those vectors means no sign constraints occur in the definition of positive-linear dependence.

### 3. An inexact log-exponential regularization method

In this section, we focus on the following regularized problem \( NLP(t) \) [27]:

\[
\min_z f(z)
\]

s.t. \( g_i(z) \leq 0 \ \forall \ i = 1, 2, \ldots, p, \)

\( h_i(z) = 0 \ \forall \ i = 1, 2, \ldots, q, \)

\( \phi_i(z; t) = 0 \ \forall \ i = 1, 2, \ldots, m, \)

where

\[
\phi_i(z; t) = \begin{cases} 
-t \ln(\sum_{j=1}^t \exp(-F_{ij}(z)/t)), & t > 0, \\
\min\{F_{i1}(z), F_{i2}(z), \ldots, F_{it}(z)\}, & t = 0
\end{cases}
\]
for \( i = 1, 2, \ldots, m \). The above approximation function is used recently in [18, 19, 27]. It follows from [19] that
\[
\lim_{t \to 0} \phi_i(z; t) = \min\{F_{i1}(z), F_{i2}(z), \ldots, F_{il}(z)\} \quad \forall i = 1, \ldots, m
\]
and \( \phi_i(z; t) \) is continuously differentiable for all \( t > 0 \). The gradients with respect to \( z \) are given by
\[
\nabla \phi_i(z; t) = \sum_{j=1}^{l} \nu_{ij}(z; t) \nabla F_{ij}(z)
\]
and
\[
\nu_{ij}(z; t) = \frac{\exp(-F_{ij}(z)/t)}{\sum_{\tau=1}^{l} \exp(-F_{i\tau}(z)/t)} \in (0, 1), \quad \sum_{j=1}^{l} \nu_{ij}(z; t) = 1 \forall i = 1, \ldots, m. \tag{5}
\]

Now, we establish the relation between the solutions of problem (1) and (4) under some classical MPVCC conditions. To illustrate this, recall that the main convergence results in [27] are as follows: given a sequence \{\( t_k \)\} \( \searrow 0 \), a corresponding sequence of KKT-points \((z^k, \lambda^k, \mu^k, \delta^k)\) of \( NLP(t_k) \) with \( z^k \to z^* \) and the sequence \((z^k, \lambda^k, \mu^k, \delta^k)\) converges to \((z^*, \lambda^*, \mu^*, \delta^*)\), then \( z^* \) is a \( C \)-stationarity of (1). It can be observed that KKT-points of \( NLP(t_k) \) are computed via Matlab subroutines, which, in many cases, are not so reliable. In the subsequent analysis, we still get the convergence results by replacing KKT-points by \( \varepsilon_k \)-KKT points of \( NLP(t_k) \). In Section 4, we will recommend a feasible strategy that guarantees \( \varepsilon_k \)-KKT points of \( NLP(t_k) \). In Section 5, the numerical results of some examples certify the applicability of the regularization method for MPVCC.

In order to establish the convergence result of the regularized problem \( NLP(t) \), we need the following result.

**Lemma 3.1.** Let \{\( t_k \)\} \( \searrow 0 \), \{\( \varepsilon_k \)\} \( \searrow 0 \), \{\( z^k \)\} be a sequence of \( \varepsilon_k \)-KKT points of \( NLP(t_k) \) and assume that \( z^k \to z^* \). Then, it holds that
\[
\lim_{k \to +\infty} \nu_{ij}(z^k; t_k) = 0 \quad \forall (i,j) \text{ satisfying } F_{ij}(z^*) > 0.
\]

**Proof.** Note that
\[
\nu_{ij}(z^k; t_k) = \frac{\exp(-F_{ij}(z^k)/t_k)}{\sum_{\tau=1}^{l} \exp(-F_{i\tau}(z^k)/t_k)} = \frac{\exp((F_i(z^k) - F_{ij}(z^k))/t_k)}{\sum_{\tau=1}^{l} \exp((F_i(z^k) - F_{i\tau}(z^k))/t_k),}
\]
where \( F_i(z) = \min\{F_{i1}(z), \ldots, F_{il}(z)\} \) for all \( i = 1, \ldots, m \).

Denote the new index set \( B_i(z) = \{j \mid F_{ij}(z) = F_i(z), \ j = 1, \ldots, l\} \). By definition, we have \(|B_i(z^k)| \geq 1\) for all \( k \) and thus
\[
\sum_{\tau=1}^{l} \exp((F_i(z^k) - F_{i\tau}(z^k))/t_k) \geq \sum_{\tau \in B_i(z^k)} \exp((0)/t_k) \geq 1 \forall k,
\]
where \(|B_i(z^k)| \) is the number of elements of the index set \( B_i(z^k) \). For all \((i,j)\) with \( F_{ij}(z^*) > 0 = F_i(z^*) \), we have \( F_i(z^k) - F_{ij}(z^k) \to F_i(z^*) - F_{ij}(z^*) < 0 \) and thus
\[
\exp((F_i(z^k) - F_{ij}(z^k))/t_k) \to 0.
\]
Hence, for \((i,j)\) satisfying \( F_{ij}(z^*) > 0 \), it holds that \( \lim_{k \to +\infty} \nu_{ij}(z^k; t_k) = 0. \)
Notice that Lemma 3.1 is different to the result in [27]. We do not need \{z^k\} is the feasible point of regularized problem. The following result shows that, under the MPVCC-MFCQ assumption, the \(\varepsilon_k\)-KKT points of the NLP\((t_k)\) converge to C-stationarity of MPVCC as \(\{t_k\} \searrow 0\) and \(\{\varepsilon_k\} \searrow 0\).

**Theorem 3.2.** Let \(\{t_k\} \searrow 0\), \(\{\varepsilon_k\} \searrow 0\), \(\{z^k\}\) be a sequence of \(\varepsilon_k\)-KKT points of NLP\((t_k)\) and assume that \(z^k \to z^*\). If MPVCC-MFCQ holds at \(z^*\), then \(z^*\) is a C-stationary point of (1).

**Proof.** Since \(\{z^k\}\) are \(\varepsilon_k\)-KKT points of NLP\((t_k)\), the multipliers \((\lambda^k, \mu^k, \delta^k)\) satisfies

\[
\|\nabla f(z^k) + \sum_{i=1}^p \lambda_i^k \nabla g_i(z^k) + \sum_{i=1}^q \mu_i^k \nabla h_i(z^k) - \sum_{i=1}^m \delta_i^k \nabla \phi_i(z^k ; t_k)\|_\infty \leq \varepsilon_k,
\]

\[
\|g_i(z^k)\| \leq \varepsilon_k, \quad \lambda_i^k \geq -\varepsilon_k, \quad |\lambda_i^k g_i(z^k)| \leq \varepsilon_k, \quad i = 1, \ldots, p,
\]

\[
|h_i(z^k)| \leq \varepsilon_k, \quad i = 1, \ldots, q, \quad |\phi_i(z^k ; t_k)| \leq \varepsilon_k, \quad i = 1, \ldots, m.
\]

Obviously, \(z^*\) is a feasible point of (1). In view of the definition \(\nu_{ij}(z^k ; t_k)\) and Lemma 3.1, it holds that \(\lim_{k \to +\infty} \nu_{ij}(z^k ; t_k) = 0\) for every \((i, j)\) satisfying \(F_{ij}(z^*) > 0\).

According to (5), we may assume that

\[
\lim_{k \to +\infty} \nu_{ij}(z^k ; t_k) = \nu_{ij}^*(z^*) \text{ for } (i, j) \in \mathcal{I}_F(z^*)
\]

and obtain that, for all \(i = 1, \ldots, m\),

\[
\sum_{j=1}^l \nu_{ij}^*(z^*) = 1,
\]

where

\[
\nu_{ij}^*(z^*) = \begin{cases} 
\nu_{ij}^*(z^*), & (i, j) \in \mathcal{I}_F(z^*), \\
0, & \text{else}.
\end{cases}
\]

Then we obtain that

\[
\|\nabla f(z^k) + \sum_{i=1}^p \lambda_i^k \nabla g_i(z^k) + \sum_{i=1}^q \mu_i^k \nabla h_i(z^k) - \sum_{j=1}^l \sum_{i=1}^m \lambda_{ij}^k \nabla F_{ij}(z^k) - \sum_{j=1}^l \sum_{i=1}^m \delta_{ij}^k \nu_{ij}(z^k ; t_k) \nabla F_{ij}(z^k)\|_\infty \leq \varepsilon_k,
\]

where, for all \(j = 1, \ldots, l\),

\[
\gamma_{ij}^k = \begin{cases} 
\delta_{ij}^k \nu_{ij}(z^k ; t_k), & i \in \mathcal{I}_{F_{ij}}(z^*), \\
0, & \text{else},
\end{cases} \quad \delta_{ij}^k = \begin{cases} 
\delta_{ij}^k, & i \in \mathcal{I}_{F_{ij}}(z^*), \\
0, & \text{else}.
\end{cases}
\]

and we denote the complement set of \(\mathcal{I}_{F_{ij}}(z^*)\) by \(\bar{\mathcal{I}}_{F_{ij}}(z^*)\).

Next, we prove that the multipliers \((\lambda^k, \mu^k, \gamma^k, \bar{\delta}^k)\) from (8) are bounded. If the sequence is unbounded, then there exist a subsequence \(K\) such that, for \(k \in K\),

\[
\frac{(\lambda^k, \mu^k, \gamma^k, \bar{\delta}^k)}{\|(\lambda^k, \mu^k, \gamma^k, \bar{\delta}^k)\|} \to (\bar{\lambda}, \bar{\mu}, \bar{\gamma}, \bar{\delta}) \neq 0.
\]
Then, (8) is divided by \( \| (\lambda^k, \mu^k, \gamma^k, \delta^k) \| \) on the both sides and passing to the limit with \( k \to +\infty \). According to Lemma 3.1, we have

\[
\lim_{k \to +\infty} \frac{\delta^k_i}{\| (\lambda^k, \mu^k, \gamma^k, \delta^k) \|} \nu_{ij}(z^k; t_k) = 0 \quad \forall i = 1, \ldots, m, \ j = 1, \ldots, l
\]

and

\[
\sum_{i=1}^{p} \tilde{\lambda}_i \nabla g_i(z^*) + \sum_{i=1}^{q} \tilde{\mu}_i \nabla h_i(z^*) - \sum_{j=1}^{l} \sum_{i=1}^{m} \tilde{\gamma}_{ij} \nabla F_{ij}(z^*) = 0,
\]

where \( \tilde{\lambda}_i \geq 0 \) for all \( i = 1, \ldots, p \). If \( \tilde{\lambda}_i > 0 \), it means that \( \lambda^k_i > c \) for some constant \( c > 0 \) and \( k \) sufficiently large. This yields

\[
0 \leq |g_i(z^k)| \leq \frac{\varepsilon_k}{\| \lambda^k \|} \leq \frac{\varepsilon_k}{c} \to 0.
\]

because \( \varepsilon_k \downarrow 0 \) and thus \( i \in I_g(z^*) \). Clearly, we have \( supp(\tilde{\gamma}) \subseteq I_F(z^*) \). If \( (\bar{\lambda}, \bar{\mu}, \bar{\gamma}) \neq 0 \), (9) contradicts the MPVCC-MFCQ assumption. If \( (\bar{\lambda}, \bar{\mu}, \bar{\gamma}) = 0 \), then we may assume that there exist a \( \delta_{i0} \neq 0 \). In the view of the definition of \( \gamma^k_{ij} \) and \( \delta^k_i \), we note that there exists \( j_0 \) such that \( F_{i0,j_0}(z^*) = 0 \), \( \nu_{i0,j_0}^*(z^*) = \alpha \) and \( \alpha \in (0, 1] \). It implies

\[
\tilde{\gamma}_{i0,j_0} = \lim_{k \to +\infty} \frac{\delta^k_i \nu_{i0,j_0}^*(z^k; t_k)}{\| (\lambda^k, \mu^k, \gamma^k, \delta^k) \|} = \delta_{i0} \nu_{i0,j_0}^*(z^*) = \alpha \delta_{i0} \neq 0,
\]

which contradicts to assumption \( \tilde{\gamma} = 0 \). Thus we prove \( (\lambda^k, \mu^k, \gamma^k, \delta^k) \) from (8) are bounded. Considering a converging subsequence, it converges to \( (\lambda^*, \mu^*, \gamma^*, \delta^*) \). We have \( \lambda^* \geq 0 \), \( supp(\lambda^*) \subseteq I_g(z^*) \) and \( supp(\gamma^*) \subseteq I_F(z^*) \). It is obvious that \( z^* \) is at least a weakly stationary point of (1). Recall that C-stationarity requires in addition that \( \gamma^*_{ij} \gamma^*_{ij} \geq 0 \) if \( F_{ij}(z^*) = F_{ij'}(z^*) = 0 \). Note that

\[
\gamma^*_{ij} \gamma^*_{ij} = \delta^2_{i} \nu^*_{ij}(z^*) \nu^*_{ij'}(z^*) \geq 0 \quad \text{for} \ (j, j') \ \text{satisfy} \ F_{ij}(z^*) = F_{ij'}(z^*) = 0.
\]

This yields that \( z^* \) is a C-stationary point of (1). \( \Box \)

The following example illustrates that we cannot improve the conclusion of Theorem 3.2 concerning the stationarity of MPVCC while maintaining the assumptions.

**Example 1.** Consider the MPVCC problem

\[
\min_z \quad -z_1 - z_3 - z_4 \\
\text{s.t.} \quad -z_2 \leq 0, \quad z_1^2 - z_2 \leq 0, \\
\quad \min\{z_1, z_3, z_4\} = 0.
\]

Clearly, MPVCC-MFCQ holds at \( z^* = (0, 0, 0, 0) \). The regularized problem \( NLP(t) \) is given by

\[
\min_z \quad -z_1 - z_3 - z_4 \\
\text{s.t.} \quad -z_2 \leq 0, \quad z_1^2 - z_2 \leq 0, \\
\quad -t \ln(\exp(-z_1/t) + \exp(-z_3/t) + \exp(-z_4/t)) = 0
\]
with parameter \( t > 0 \). Let \( t_k = k^{-\frac{1}{3}} \), \( \varepsilon_k = 2k^{-\frac{1}{3}} \) and \( z_k = (k^{-1}, k^{-2}, k^{-1}, k^{-1}) \). The \( \varepsilon_k \)-KKT conditions of NLP(\( t_k \)) are as follow:

\[
\| \nabla L(z^k, \lambda^k, \delta^k) \|_\infty = \left\| \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\| + \lambda^k_1 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \lambda^k_2 \begin{pmatrix} 2z^k_1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \delta^k \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq \varepsilon_k,
\]

\[
z^k_2 \geq -\varepsilon_k, \quad \lambda^k_1 \geq -\varepsilon_k, \quad | \lambda^k_1 z^k_2 | \leq \varepsilon_k,
\]

\[
(z^k_1)^2 - z^k_2 \leq \varepsilon_k, \quad \lambda^k_2 \geq -\varepsilon_k, \quad | \lambda^k_2 (z^k_1)^2 - z^k_2 | \leq \varepsilon_k,
\]

\[
| \phi(z^k; t_k) | = | -t_k \ln(\exp(-z^k_1/t_k) + \exp(-z^k_2/t_k) + \exp(-z^k_3/t_k)) | \leq \varepsilon_k.
\]

Notice that \( \ln(3 \exp(-k^{-\frac{2}{3}})) \) is an increasing function with respect to \( k \) (\( k = 1, 2, \ldots \)) and

\[
0 \leq \ln(3 \exp(-1^{-\frac{2}{3}})) \leq 2, \quad 0 \leq \lim_{k \to +\infty} \ln(3 \exp(-k^{-\frac{2}{3}})) \leq 2.
\]

Hence, \( | \phi(z^k; t_k) | = | k^{-\frac{1}{3}} \ln(3 \exp(-k^{-\frac{2}{3}})) | \leq 2k^{-\frac{1}{3}} = \varepsilon_k \). Thus, \( z^k \) together with the multipliers \( (\lambda^k_1, \lambda^k_2, \delta^k) = (0, 0, -3) \) are \( \varepsilon_k \)-KKT points of NLP(\( t_k \)). However, for \( t_k \downarrow 0 \), the sequence \( z^k \) converges to \( z^* \), which is C-stationarity but not M-stationarity. Since the MPVCC multipliers \( (\lambda^*, \Gamma^*) \) satisfies the equation

\[
\nabla L(z^*, \lambda^*, \Gamma^*) = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \lambda^*_1 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \lambda^*_2 \begin{pmatrix} 2z^*_1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \Gamma^* \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \Gamma^*_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0, \quad (10)
\]

it implies that \( \Gamma^*_1 = -1, \quad \Gamma^*_2 = -1, \quad \Gamma^*_3 = -1 \) and \( \lambda^*_1 = -\lambda^*_2 \). There are multipliers \( \Gamma^*_1 = -1, \quad \Gamma^*_2 = -1, \quad \Gamma^*_3 = -1 \), \( \lambda^*_1 = -\lambda^*_2 = 0 \) satisfying the equation \( (10) \) and the conditions

\[
-\lambda^*_1 z^*_2 = 0, \quad \lambda^*_1 \geq 0, \quad \lambda^*_2 ((z^*_1)^2 - z^*_2) = 0, \quad \lambda^*_2 \geq 0,
\]

\[
\Gamma^*_1 z^*_1 = 0, \quad \Gamma^*_2 z^*_2 = 0, \quad \Gamma^*_3 z^*_3 = 0,
\]

\[
\Gamma^*_1 \Gamma^*_2 \geq 0, \quad \Gamma^*_1 \Gamma^*_3 \geq 0, \quad \Gamma^*_2 \Gamma^*_3 \geq 0.
\]

Note that we cannot construct multipliers \( \Gamma^*_1 = -1, \quad \Gamma^*_2 = -1, \quad \Gamma^*_3 = -1 \) satisfying the M-stationary condition. Then \( z^* \) is only C-stationarity but not M-stationarity.

This example means that we cannot obtain the M-stationarity of Theorem 3.2. According to the definition of \( \varepsilon_k \)-KKT point, we can find more points satisfy inexact KKT conditions. The existence of \( \varepsilon_k \)-KKT point of NLP(\( t_k \)) is discussed in the following section. We will design a feasible strategy to obtain these inexact KKT points.

4. A feasible strategy that generates \( \varepsilon \)-KKT points. As mentioned previously, it is unrealistic to compute the exact KKT points from a practical point of view, since the usual stopping criteria may fail to satisfy the classical KKT conditions. The using of inexact KKT conditions is able to define stopping criteria for many practical algorithms. In Section 3, we have shown that the limit point is still C-stationary, if we replace KKT points by \( \varepsilon \)-KKT points of the regularized problem (4). It is in principle the best result one can hope for. Motivated by this result, our aim is to obtain a C-stationary point by computing a sequence of \( \varepsilon \)-KKT points.
of the regularized problem. Therefore, the key difficulty is how to compute such a sequence of \( \varepsilon \)-KKT points of \( NLP(t) \) as \( t \searrow 0 \) and \( \varepsilon \searrow 0 \). In order to overcome the difficulty, we propose a feasible strategy to generate sequences which satisfy \( \varepsilon \)-KKT conditions. Now, we focus on the basic strategy for MPVCC as follows.

**Algorithm 1: A basic strategy for MPVCC**

Step 1 Choose a starting vector \( z^0 \), positive sequences \( \{t_k\} \searrow 0 \), a stopping tolerance \( \varepsilon' > 0 \), a parameter \( t_{\min} > 0 \) and set \( k = 0 \).

Step 2 While \( t_k \geq t_{\min} \) do

Choose \( \varepsilon_k \). Solve the regularization \( NLP(t_k) \) and use \( z^k \) as starting vector.

Find an \( \varepsilon_k \)-inexact solution \( z^{k+1} \) of \( NLP(t_k) \). Set \( k \leftarrow k + 1 \).

end while

Step 3 Return the final iterate \( z_{\text{opt}} = z^k \) and function value at \( z_{\text{opt}} \).

The basic strategy enables us to compute a solution of MPVCC by solving a sequence of nonlinear programs. We solve the \( \varepsilon_k \)-KKT point \( z^{k+1} \) of the regularized problems \( NLP(t_k) \) and repeat until \( t_k \) tends to zero. However, the rules for updating parameters \( t_k \) and \( \varepsilon_k \) are not discussed. Furthermore, the authors [27] guarantee such a sequence of \( z^{k+1} \) by the solver \textit{fmincon} in Matlab which are not fully reliable. Hence, the purpose of this section is to present a feasible strategy that gives an effective way to update \( t_k \), \( \varepsilon_k \) and generates sequences that satisfy the \( \varepsilon_k \)-inexact conditions.

We recall the regularized problem \( NLP(t) \)

\[
\min_z f(z) \quad \text{s.t.} \quad g_i(z) \leq 0 \quad \forall \, i = 1, 2, \ldots, p, \\
g_i(z) = 0 \quad \forall \, i = 1, 2, \ldots, q, \\
\phi_i(z; t) = 0 \quad \forall \, i = 1, 2, \ldots, m. 
\] (11)

By adding the slack variables, it can be equivalently written in the form

\[
\min_{\bar{x}} \bar{f}(x) \quad \text{s.t.} \quad H(x; t) = 0, \, x \geq 0, 
\] (12)

where \( \bar{f}(x) = f(z) \),

\[
H(x; t) = \begin{pmatrix} g(z) + s \\ h(z) \\ \Phi(z; t) \end{pmatrix}, \Phi(z; t) = \begin{pmatrix} \phi_1(z; t) \\ \vdots \\ \phi_m(z; t) \end{pmatrix},
\]

\( z = \bar{y} - y, \, \bar{y} \geq 0, \, y \geq 0, \, s \geq 0, \, x = (\bar{y}, y, s)^T \in \mathbb{R}^n \) and \( \bar{n} = 2n + p \). Furthermore, if there exists a sequence \((x, \lambda, \omega)\) such that \( x \geq -\varepsilon e, \, \omega \geq -\varepsilon e \),

\[
\max \{ \| \nabla \bar{f}(x) + \nabla H(x; t) \lambda - \omega \|_\infty, \| H(x; t) \|_\infty, \| XWe \|_\infty \} \leq \varepsilon, \quad (13)
\]

with

\[
e = (1, \cdots, 1)^T \quad \text{and} \quad X = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}, \quad W = \begin{pmatrix} \omega_1 & \cdots & \omega_n \end{pmatrix}
\]
are diagonal matrices, then we obtain that $x$ is $\varepsilon$-KKT points of (12). In the subsequent analysis, we turn to (13) which plays an important role in the following feasible strategy.

Instead of solving problem (12) as $t \searrow 0$, for a given penalty parameter $\rho > 0$, a barrier parameter $\mu > 0$ and $t > 0$, we solve the following problem

$$
\min_x \psi_{\rho,\mu}(x; t),
$$

where

$$
\psi_{\rho,\mu}(x; t) = \tilde{f}(x) + \frac{\rho}{2} \|H(x; t)\|^2 - \mu \sum_{i=1}^{\bar{n}} \ln(x_i),
$$

with $x > 0$. The objective function of unconstrained subproblem is a penalty-barrier function that involves both primal and dual variables. The process is then implemented by decreasing $\mu$ to zero, increasing $\rho$ to infinity and enforcing the parameter $t$ decreasing to zero. The optimality condition $\nabla \psi_{\rho,\mu}(x; t) = 0$ is equivalent to

$$
\nabla \tilde{f}(x) + \nabla H(x; t)\lambda - w = 0, \quad H(x; t) - \frac{\lambda}{\rho} = 0, \quad XWe = \mu e.
$$

The equivalent optimality conditions (15) motivate the approach for solving problem (14) to check (13). Therefore, we design a feasible strategy that generates a sequence such that (13) to get a solution of MPVCC. We are inspired by the Newtonian penalty-barrier Lagrangian method [1]. However, it differs from [1] by allowing the updating of parameter $t$. From the analysis of basic strategy, it is necessary to give an effective way to calculate parameters $t$ and $\varepsilon$ at each iteration. We use the information of the barrier parameter $\mu$ and penalty parameter $\rho$ to update the variable $\varepsilon$. The feasible strategy can be stated as Algorithm 2.

**Remark 1.** By direct calculus, the equation system in Step 4 can be changed to

$$
[P^i + c_i I + (X^i)^{-1}W^i + \rho \nabla H(x^i; t_k) \nabla H(x^i; t_k)^T]d^i = -\nabla \psi_{\rho_k,\mu_k}(x^i; t_k).
$$

Due to add the suitable diagonal matrix $c_i I$, we can guarantee that the matrix on the left hand side of (16) is positive definiteness. It means that the direction $d^i$ in Step 5 is a descent direction for $\psi_{\rho_k,\mu_k}$ in each iteration.

Note that the Step 5 and Step 6 are similar to that of the method in [1]. However, the above approach is different from the method in [1], since the Step 5 and Step 6 are related to the iteration of variable $t_k$. In the feasible strategy, as $k$ and $\rho_k$ increase to infinity, we define parameters $t_k$ and $\varepsilon_k$ which decrease to zero. Furthermore, comparing with the method in [1], these reducing steps of $t_k$ and $\varepsilon_k$ in our approach simplify the form and requirement for the stopping criterion.

Next, we focus on the feasibility of the strategy. From the similar analysis in [1], we know that the strategy is well defined and generates sequences that satisfy $x^i > 0$, $\omega^i > 0$. If we assume $\{x^i\}$ is bounded, then there exists an iteration index $i_{(k)}$ such that

$$
\|\nabla \psi_{\rho_k,\mu_k}(x^{(i_{(k)})}; t_k)\|_\infty \leq \varepsilon_k, \quad \text{for all } k.
$$

Since the reducing steps of parameter $t_k$ is in the inner iteration $k$. We consider $t_k$ as inner parameters which the same as $\mu_k$ and $\rho_k$. Therefore, for all $k$, the discussion about the existence of $i_{(k)}$ such that (17) on our feasible strategy follows the similar argument as in [1]. The bounded assumption is general when the constraints are modified by $a \leq x \leq \bar{a}$. 

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**THE INEXACT LOG-EXPONENTIAL REGULARIZATION METHOD FOR MPVCC**

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[1] Reference or citation here.
Algorithm 2: A feasible strategy for MPVCC

Step 0. (Initialization) Given an initial point \((x^0, \lambda^0, \omega^0)\) with \(x^0 > 0, \omega^0 > 0, \lambda^0 > 0\), a parameter \(t_0 > 0\), constant numbers \(\beta_{k+1} \geq 1 \geq \beta_{k+1} > 0, \theta_0 > 0, 1 > \zeta > 0\), penalty parameter \(\rho_0 > 0\), and the stopping tolerance \(\epsilon' > 0\). Choose an initial barrier parameter \(\rho_0 > 0\), \(\sigma_0 = \max\{0.99, 1 - \mu_0\}\), and \(\epsilon_0 = 10\max\{\mu_0, \rho_0^{-1}\}\). Let \(i = 0, k = 0\).

Step 1. (Check Optimal of the penalty-barrier problem) If \(\|\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)\|_\infty \leq \epsilon_k\), then we set \(x^i = \rho_k H(x^i; t_k), \omega^i = \mu_k(X^i)^{-1}e\) and go to Step 2. Otherwise, go to Step 4.

Step 2. (Termination criterion) If \(\|H(x^i; t_k)\|_\infty \leq \epsilon_k\) and \(\epsilon_k \leq \epsilon',\) stop.

Step 3. (Compute the new parameters) Set \(\mu_{k+1} = \min\{\mu_k, \mu_k^{1.5}\}\), \(\rho_{k+1} = \max\{\rho_k, \rho_k^{1.5}\}, \epsilon_{k+1} = 10\max\{\mu_{k+1}, \rho_{k+1}^{-1}\}, \theta_{k+1} \in (0, \theta_k], \beta_{k+1} \in (0, \beta_k], \beta_{k+1} \geq \beta_{k+1} \geq \beta_{k+1}, \sigma_{k+1} = \max\{0.99, 1 - \mu_{k+1}\}, t_{k+1} = \zeta t_k\) and \(k \leftarrow k + 1\).

Step 4. (Compute the search direction) Compute \((d^i, \lambda)\) by

\[
\begin{bmatrix}
P^i + c_I I + (X^i)^{-1}W^i & \nabla H(x^i; t_k) \\
\nabla H(x^i; t_k)^T & -\rho_k^{-1} I
\end{bmatrix}
\begin{bmatrix}
d^i \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
-\nabla f(x^i) + \mu_k(X^i)^{-1}e \\
-H(x^i; t_k)
\end{bmatrix},
\]

where \(P^i = \nabla^2 f(x^i) + \sum_{i=1}^{\bar{m}} \lambda_i^j \nabla^2 H_i(x^i; t_k), \bar{m} = p + q + m\) and \(c_I I\) with \(c_i > 0\) is obtained in [24]. Set \(\omega = (X^i)^{-1}(\mu_k e - W^i d^i)\).

Step 5. (Check adequacy of the search direction)

If \(\|d^i\| = 0\), then we set \(x^i = \rho_k H(x^i; t_k), \omega^i = \mu_k(X^i)^{-1}e\) and go to Step 2.

If \(\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)^T d^i \geq -\theta_k \|\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)\| \|d^i\|\), set \(d^i = -\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)\) and go to Step 6.

If \(\|d^i\| > \beta_{k+1} \|\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)\|\), set \(d^i = \frac{\beta_{k+1} \|\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)\|}{\|d^i\|} \|\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)\|\).

If \(\|d^i\| < \beta_{k+1} \|\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)\|\), set \(d^i = \frac{\beta_{k+1} \|\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)\|}{\|d^i\|} \|\nabla \psi_{\rho_k, \mu_k}(x^i; t_k)\|\).

Step 6. (Line search strategy) Let \(\alpha = 1\) be the trial step and \(\alpha_i > 0\) satisfying \(\psi_{\rho_k, \mu_k}(x^i + \alpha d^i; t_k) \leq \psi_{\rho_k, \mu_k}(x^i; t_k) + 0.001\alpha \nabla \psi_{\rho_k, \mu_k}(x^i; t_k)^T d^i\).

Set \(x^{i+1} = x^i + \alpha d^i, \lambda^{i+1} = \lambda^i + \alpha_i (\lambda - \lambda^i), \omega^{i+1} = \omega^i + \alpha_i (\omega - \omega^i)\), where \(\alpha_i = \max\{\alpha \in (0, 1) \mid \omega^i + \alpha (\omega - \omega^i) \geq (1 - \sigma_k)\omega^i\}\).

Set \(i \leftarrow i + 1\) and go to Step 1.

Return the final iterate \(x_{\text{opt}} = x^i\) and function value at \(x_{\text{opt}}\).

We note that the limit point of inexact KKT points generated by the log-exponential regularization method is feasible for the problem (1), but it is not
automatically true for the penalty-barrier Lagrangian method. It seems unavoidable for exterior penalty methods. In the following theorem, we show that the approach terminates finitely or the accumulation point of \{x^{i(k)}\} is infeasible.

**Theorem 4.1.** Assume that \(x^*\) is a limit point of \(\{x^{i(k)}\}\). Then the following statements hold:

(i) \(x^*\) is a stationary point of \(\|H(x; 0)\|^2\) subject to \(x \geq 0\).

(ii) If \(\|H(x^*; 0)\|^2 < \varepsilon'\), then there exists \(x^{i(k)}\) such that the stopping criterion and (13) holds.

**Proof.** (i) According to the above analysis, we obtain that, for every \(k\), there exists \(i(k)\) such that

\[
\|\nabla \psi_{\rho_k, \mu_k}(x^{i(k)}; t_k)\|_\infty \leq \varepsilon_k.
\]

(18)

Recalling the definition of \(\mu_k, \rho_k, \varepsilon_k\), and \(t_k\), we have \(\lim_{k \to +\infty} \mu_k = 0, \lim_{k \to +\infty} \rho_k = \infty, \lim_{k \to +\infty} \varepsilon_k = 0, \text{ and } \lim_{k \to +\infty} t_k = 0\). Obviously,

\[
\lim_{k \to +\infty} \nabla \psi_{\rho_k, \mu_k}(x^{i(k)}; t_k) = 0.
\]

Hence,

\[
\lim_{k \to +\infty} \frac{1}{\rho_j} \nabla \tilde{f}(x^{i(k)}) + \nabla H(x^{i(k)}; t_k)H(x^{i(k)}; t_k) - \frac{\mu_k}{\rho_k}(X^{i(k)})^{-1}e = 0.
\]

(19)

A stationary point \(x^*\) of

\[
\min_x \|H(x; 0)\|^2 \text{ s.t. } x \geq 0
\]

is defined as \(0 \in \partial \|H(\cdot; 0)\|^2 (x^*) + N_{\mathbb{R}_+^n}(x^*), \) where we use the Clarke subdifferential and an arbitrary normal cone since \(\mathbb{R}_+^n\) is convex. This is equivalent to the existence of a subgradient

\[
\xi \in \partial \|H(\cdot; 0)\|^2 (x^*) = 2\partial H(x^*; 0)H(x^*; 0)
\]

such that

\[
\xi_j \begin{cases} 
= 0, & \text{if } x^*_j > 0, \\
\geq 0, & \text{if } x^*_j = 0.
\end{cases}
\]

To prove the existence of this subgradient we infer from (19) that

\[
\lim_{k \to +\infty} \left[ \nabla H(x^{i(k)}; t_k)H(x^{i(k)}; t_k) \right]_j - \frac{\mu_k}{\rho_k} \left( \frac{1}{x^{i(k)}_j} \right) = 0.
\]

Here, by taking the limit \(k \to +\infty\), we know

\[
\frac{\mu_k}{\rho_k} \frac{1}{x^{i(k)}_j} \begin{cases} 
\to 0, & \text{if } x^*_j > 0, \\
\geq 0, & \text{if } x^*_j = 0.
\end{cases}
\]

From [26, Theorem 2.1], the result can be extended to MPVCC. We have, for all \(j = 1, \ldots, \bar{n}\),

\[
\lim_{t^{i(k)}_k \to x^*} \text{dist} \{[\nabla H(x^{i(k)}; t_k)]_j, [\partial H(x^*; 0)]_j\} = 0.
\]

Thus, we have that \(\nabla H(x^{i(k)}; t_k)\) converges at least on a subsequence to some element \(J \in \partial H(x^*; 0)\), then \(\xi = 2JH(x^*; 0)\) has exactly the properties we are looking for.
(ii) If \( \|H(x^*;0)\|^2 < \varepsilon' \), we first show that \( \|H(x^{i(k)};t_k)\|_\infty \leq \varepsilon' \) for infinitely many indices \( k \). We prove this assertion by contradiction. Therefore we assume that there is only finite indices \( k \) such that \( \|H(x^{i(k)};t_k)\|_\infty > \varepsilon' \) for \( k \) large enough. However, passing to the limit with \( k \to +\infty \), we get \( \|H(x^*;0)\|_\infty \geq \varepsilon' \) which is a contradiction to \( \|H(x^*;0)\|_\infty \leq \|H(x^*;0)\|^2 < \varepsilon' \). Hence, we have

\[
\|H(x^{i(k)};t_k)\|_\infty \leq \varepsilon'
\]

for infinitely many indices \( k \). From the definition of \( \varepsilon_k \) and \( \mu_k \), it is obvious that

\[
\|X^{i(k)}W^{i(k)}e\|_\infty = \|\mu_k e\|_\infty \leq \varepsilon_k \leq \varepsilon',
\]

for \( k \) large enough. Together with (18)-(21), the proof is completed. \( \square \)

From Theorem 4.1, we note that the sequence \( x^{i(k)} \) that generated by feasible strategy satisfies conditions (13) when the accumulation point of this sequence is feasible. It implies that \( x^{i(k)} \) is an \( \varepsilon_k \)-KKT points of (12). To simplify the notation, we denote \( (x^{i(k)}, \lambda^{i(k)}, \omega^{i(k)}) \) by \( (x^k, \lambda^k, \omega^k) \).

Based on the above analysis of algorithm, we know that the feasible strategy can generate \( \varepsilon_k \)-KKT points \( (x^k, \lambda^k, \omega^k) \) of (12). It is necessary to show the relationship between the stationary points of (11) and (12). From the previous assumption of algorithm, we know that the sequence \( x^k \) is bounded which is trivial and used in many methods. Together with the continuity of \( g \), it is clear that \( g(z^k) \) are bounded. If the multiplier happens to be bounded as \( k \to +\infty \), we know that it is reasonable to get some inexact KKT points of (11) by computing \( \varepsilon_k \)-KKT points of (12). Therefore, the next result establishes that the multipliers generated by Algorithm 2 are bounded under the MPVCC-MFCQ.

**Theorem 4.2.** Let \( (x^k, \lambda^k, \omega^k) \) denote the infinite \( \varepsilon_k \)-KKT sequence generated by Algorithm 2. Assume that \( x^* \) is a limit point of \( \{x^k\} \). If MPVCC-MFCQ holds at \( z^* \), then the sequence of multipliers \( (\lambda^k, \omega^k) \) is bounded.

**Proof.** From the definition of Algorithm 2 and \( x^k \to x^* \), we have \( \omega^k \geq 0 \), \( x^k \geq 0 \) and

\[
\max \{ \|\nabla \tilde{f}(x^k) + \nabla H(x^k; t_k)x^k - \omega^k\|_\infty, \|H(x^k; t_k)\|_\infty, \|X^kW^ke\|_\infty \} \leq \varepsilon_k.
\]

The inequality \( \|\nabla \tilde{f}(x^k) + \nabla H(x^k; t_k)x^k - \omega^k\|_\infty \leq \varepsilon_k \) implies that

\[
\| \nabla f(z^k) + \sum_{i=1}^p \lambda_{g,i}^k \nabla g_i(z^k) + \sum_{i=1}^q \lambda_{h,i}^k \nabla h_i(z^k) + \sum_{i=1}^m \lambda_{\phi,i}^k \nabla \phi_i(z^k; t_k) - \omega^{1,k} \|_\infty \leq \varepsilon_k,
\]

\[
\| -\nabla f(z^k) - \sum_{i=1}^p \lambda_{g,i}^k \nabla g_i(z^k) - \sum_{i=1}^q \lambda_{h,i}^k \nabla h_i(z^k) - \sum_{i=1}^m \lambda_{\phi,i}^k \nabla \phi_i(z^k; t_k) - \omega^{2,k} \|_\infty \leq \varepsilon_k,
\]

\[
\| \lambda_{g,k} - \omega^{s,k} \|_\infty \leq \varepsilon_k,
\]

where \( \lambda^k = (\lambda_{g,k}, \lambda_{h,k}, \lambda_{\phi,k})^T \) and \( \omega^k = (\omega^{1,k}, \omega^{2,k}, \omega^{s,k})^T \).
Suppose that the sequence \((\lambda^k, \omega^k)\) is unbounded. There exist a subsequence \(\mathcal{K}\) such that

\[
\frac{(\lambda^k, \omega^k)}{|| (\lambda^k, \omega^k) ||} \rightarrow (\hat{\lambda}, \hat{\omega}) \neq 0 \text{ for } k \in \mathcal{K}.
\]

Then, (22) and (23) are divided by \(|| (\lambda^k, \omega^k) ||\) on the both sides and passing to the limit with \(k \rightarrow +\infty\). Combining with the definition of \(\nu_{ij}(z^k; t_k)\), we have that \(\hat{\lambda}^g = \hat{\omega}^3\)

\[
\sum_{i=1}^{p} \hat{\lambda}_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \hat{\lambda}_i^h \nabla h_i(z^*) + \sum_{j=1}^{m} \sum_{i=1}^{l} \hat{\gamma}_{ij} \nabla F_{ij}(z^*) = \hat{\omega}^1,
\]

\[
\sum_{i=1}^{p} \hat{\lambda}_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \hat{\lambda}_i^h \nabla h_i(z^*) + \sum_{j=1}^{m} \sum_{i=1}^{l} \hat{\gamma}_{ij} \nabla F_{ij}(z^*) = -\hat{\omega}^2,
\]

where \(\lambda_i^{g,k} \nu_{ij}(z^k; t_k) \sum_{j} \hat{\lambda}_i^g \nu_{ij}(z^*) = \hat{\gamma}_{ij}\) and \(\text{supp}(\hat{\gamma}) \subseteq \mathcal{I}_F(z^*)\). From (6) and (7), for any \(i = 1, \ldots, m\), it has at least one \(j\) such that \(1 \geq \nu_{ij}(z^*) > 0\). Thus, in case \(\hat{\lambda}^g \neq 0\) this implies \(\hat{\gamma} \neq 0\).

According to \(\hat{\omega} \geq 0\), it is clear that

\[
\sum_{i=1}^{p} \hat{\lambda}_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \hat{\lambda}_i^h \nabla h_i(z^*) + \sum_{j=1}^{m} \sum_{i=1}^{l} \hat{\gamma}_{ij} \nabla F_{ij}(z^*) = 0,
\]

\(\hat{\omega}^1 = \hat{\omega}^2 = 0\) and \(\hat{\lambda}^g = \hat{\omega}^s \geq 0\). Since \(\mathcal{I}(s^*) = \mathcal{I}_g(z^*)\) and \(s_i^n \hat{\omega}_i^s = 0\), we have \(\text{supp}(\hat{\lambda}^g) \subseteq \mathcal{I}_g(z^*)\). It contradicts the MPVCC-MFCQ assumption.

Theorem 4.2 is useful for illustrating that an approximate KKT point of the reformulated problem (12) also yields an approximate KKT point of the original problem (11). If \((x^k, \lambda^k, \omega^k)\) is an \(\varepsilon_k\)-KKT point of (12) generated by Algorithm 2, we have the multipliers \(\lambda^k = (\lambda_1^g, \lambda_1^h, \lambda_2^g, \lambda_2^h, \lambda_3^g, \lambda_3^h)^T \in \mathbb{R}^m\), \(\omega^k = (\omega_1^g, \omega_1^h, \omega_2^g, \omega_2^h, \omega_3^g, \omega_3^h)^T \in \mathbb{R}^n\) such that

\[
\| \nabla f(z^k) + \sum_{i=1}^{p} \lambda_i^{g,k} \nabla g_i(z^k) + \sum_{i=1}^{q} \lambda_i^{h,k} \nabla h_i(z^k) \| \leq \varepsilon_k,
\]

\[
\| -\nabla f(z^k) - \sum_{i=1}^{p} \lambda_i^{g,k} \nabla g_i(z^k) - \sum_{i=1}^{q} \lambda_i^{h,k} \nabla h_i(z^k) \| \leq \varepsilon_k,
\]

\[
\| \lambda_i^{g,k} - \omega_i^{s,k} \| \leq \varepsilon_k,
\]

\[
-\bar{y}_i^k \leq \varepsilon_k, -\bar{y}_i^k \leq \omega_i^{1,k} \geq -\varepsilon_k, \quad \omega_i^{2,k} \geq -\varepsilon_k.
\]

\[
| y_i^k \omega_i^{1,k} | \leq \varepsilon_k, \quad | g_i^k \omega_i^{2,k} | \leq \varepsilon_k \quad \forall i = 1, \ldots, n,
\]

\[
-\bar{s}_i^k \leq \varepsilon_k, \quad \omega_i^{s,k} \geq -\varepsilon_k, \quad | s_i^k \omega_i^{s,k} | \leq \varepsilon_k, \quad | g_i(z^k) + s_i^k | \leq \varepsilon_k \quad \forall i = 1, \ldots, p,
\]

\[
| h_i(z^k) | \leq \varepsilon_k \quad \forall i = 1, \ldots, q, \quad | \phi_i(z^k; t_k) | \leq \varepsilon_k, \quad \forall i = 1, \ldots, m.
\]
Due to (24) and the definition of $\| \cdot \|_{\infty}$, we have, for all $i = 1, \ldots, n$,
\[-\varepsilon_k + \omega_i^{1,k} \leq \| \nabla f(z^k) + \sum_{i=1}^{p} \lambda_i^{g,k} \nabla g_i(z^k) + \sum_{i=1}^{q} \lambda_i^{h,k} \nabla h_i(z^k) + \sum_{i=1}^{m} \lambda_i^{\phi,k} \nabla \phi_i(z^k; t_k) \|_{\infty} \leq \varepsilon_k - \omega_i^{2,k}.
\]
Hence, according to $\omega^k \geq -\varepsilon_k e$, we obtain
\[\| \nabla f(z^k) + \sum_{i=1}^{p} \lambda_i^{g,k} \nabla g_i(z^k) + \sum_{i=1}^{q} \lambda_i^{h,k} \nabla h_i(z^k) + \sum_{i=1}^{m} \lambda_i^{\phi,k} \nabla \phi_i(z^k; t_k) \|_{\infty} \leq 2\varepsilon_k.
\]
According to (25) and (26), it is clear that $\lambda_i^{g,k} \geq \omega_i^{s,k} - \varepsilon_k \geq -2\varepsilon_k$ and $g_i(z^k) \leq 2\varepsilon_k$ for all $i = 1, \ldots, p$. We notice that
\[| (g_i(z^k) + s^k_i)(\lambda_i^{g,k} - \omega_i^{s,k}) | = | g_i(z^k)\lambda_i^{g,k} + (\lambda_i^{g,k} - \omega_i^{s,k})s^k_i + \omega_i^{s,k}s^k_i - (g_i(z^k) + s^k_i)\omega_i^{s,k} | \leq \varepsilon_k^2.
\]
By calculation, $| g_i(z^k)\lambda_i^{g,k} | \leq (1 + \varepsilon_k + | s^k_i | + | \omega_i^{s,k} |)\varepsilon_k$. From the bounded assumption of $\{x^k\}$ and Theorem 4.2, we know that $| s^k_i |$ and $| \omega_i^{s,k} |$ are bounded. There exists a positive constant $C_1$ such that $| s^k_i | + | \omega_i^{s,k} | \leq C_1$ for $k$ sufficiently large. We know that the multipliers $(\lambda_i^{g,k}, \lambda_i^{h,k}, \lambda_i^{\phi,k})$ satisfies, for $\varepsilon_k = \max\{2\varepsilon_k, (1 + \varepsilon_k + C_1)\varepsilon_k\}$,
\[\| \nabla f(z^k) + \sum_{i=1}^{p} \lambda_i^{g,k} \nabla g_i(z^k) + \sum_{i=1}^{q} \lambda_i^{h,k} \nabla h_i(z^k) + \sum_{i=1}^{m} \lambda_i^{\phi,k} \nabla \phi_i(z^k; t_k) \|_{\infty} \leq \varepsilon_k,
\]
\[g_i(z^k) \leq \varepsilon_k, \lambda_i^{g,k} \geq -\varepsilon_k, | \lambda_i^{g,k}g_i(z^k) | \leq \varepsilon_k \forall i = 1, \ldots, p,
\]
\[h_i(z^k) \leq \varepsilon_k \forall i = 1, \ldots, q, \phi_i(z^k; t_k) \leq \varepsilon_k \forall i = 1, \ldots, m,
\]
which means that $z^k$ is a $\varepsilon_k$-KKT points of (11). Thus, we know that feasible strategy is effective for getting the inexact KKT points of (11). It is also shown the practical advantages of the weaker formulation of approximate stationary conditions as termination criteria. According to the above analysis, the following result establishes the convergence of MPVCC.

**Corollary 1.** Let $(x^k, \lambda^k, \omega^k)$ denote the infinite $\varepsilon_k$-KKT sequence generated by Algorithm 2.
(i) There exists a subsequence $K_1$ and a limit point $x^*$ such that $x^k \to x^*$ for $k \in K_1$.
(ii) If MPVCC-MFCQ holds at $z^*$, then $z^*$ is a C-stationary point of (1).

5. **Numerical examples.** The purpose of this section is to show that our feasible strategy works. We write an experimental code implementing Algorithm 2. It is not to test the code, but to validate applicability of the approach. We apply Algorithm 2 as a solver to compute an inexact KKT point of the subproblems. We focus on the Stackelberg game with a equilibrium constraint [27] and Min-max-min problems with max-min constraints [25], since these problems can be formulated as (1). The numerical results is mainly to verify the reliability of the method for MPVCC. Our experiments are implemented on MATLAB 8.4.
Example 2. [27] Consider the following Stackelberg game with an equilibrium constraint
\begin{align*}
\min & \quad y_1^2 + (z_1 - 1)^2 + (y_2 - z_2 + 1)^2 + e^{y_3} + y_4 
\text{s.t.} & \quad z_1^2 + 2z_2 = 2, \ z_1 + z_2 \leq 2, \ y \in \text{SOL}(z),
\end{align*}
where \( y \in \text{SOL}(z) \) is an equilibrium constraint \([7, 27]\).

In view of \([7], \text{Proposition 1}\), for given a point \( z, \ y \in \text{SOL}(z) \) is equivalent to the equations
\[
F(y, z) = \begin{pmatrix}
\min \{ y_1 + z_1, (Ag - e)_1, (Bx - e)_1 \} \\
\min \{ y_2 + z_2, (Ag - e)_2, (Bx - e)_2 \} \\
\min \{ y_3, (C^T(y + z) - e)_1, (D^T(y + z) - e)_1 \} \\
\min \{ y_4, (C^T(y + z) - e)_2, (D^T(y + z) - e)_2 \}
\end{pmatrix}
= 0,
\]
where \( y = (y_3, y_4)^T, \ x = (y_1, y_2)^T, \ A = (\frac{2}{3} \ 1), \ B = (\frac{1}{3} \ \frac{2}{3}), \ C = (\frac{1}{3} \ \frac{2}{3}), \ D = (\frac{2}{3} \ \frac{1}{3}) \) and \( e = (1, 1)^T \). Clearly, we can reformulate (27) as MPVCC problem
\begin{align*}
\min & \quad f(y, z) = y_1^2 + (z_1 - 1)^2 + (y_2 - z_2 + 1)^2 + e^{y_3} + y_4 \\
\text{s.t.} & \quad h(z) = z_1^2 + 2z_2 - 2 = 0, \ g(z) = z_1 + z_2 - 2 \leq 0, \\
& \quad F(y, z) = 0.
\end{align*}

In our experiment, we choose the different \( t \) to obtain the optimal solution \((y_1^*, y_2^*, y_3^*, y_4^*, z_1^*, z_2^*)\) of NLP\((t)\). Table 1 summarizes the computational results, where \( f^t \) is the value of objective function at \((y_1^t, y_2^t, y_3^t, y_4^t, z_1^t, z_2^t)\).

**Table 1. The numerical results for Example 2**

| \( t \) | \((y_1^*, y_2^*, y_3^*, y_4^*, z_1^*, z_2^*)\) | \( f^t \) |
|---|---|---|
| 0.2 | ( 0.0592, -0.4868, 0.3863, 0.2741, 1.0058, 0.4937) | 1.7496 |
| 0.01 | (0.0000, -0.4999, 0.4011, 0.1994, 1.0001, 0.4999) | 1.6929 |
| 0.005 | ( 0.0000, -0.5000, 0.3997, 0.1998, 1.0000, 0.5000) | 1.6901 |

The results for different values of the parameter \( t \), shown in Table 1, are in good agreement with the analysis in Section 3. We know that \( h(z^t) = 0, \ g(z^t) \leq 0, \ |F(y^t, z^t)|_i = 0.0000 \ \forall i = 1, \ldots, 4 \) at \( t = 0.005 \). It implies that \((y_1^t, y_2^t, y_3^t, y_4^t, z_1^t, z_2^t)\) is an approximate solution of (28) with \( t = 0.005 \). The discussion of this example is presented in Zhang et al. [27] to obtain the approximate optimal solution \((0, -0.5, 1, 0, 1, 0.5)\) by the solver \textit{fmincon}. By verification, we know that \((0, -0.5, 1, 0, 1, 0.5)\) is an M-stationary point of (28). By our algorithm, we find that \((y_1^t, y_2^t, y_3^t, y_4^t, z_1^t, z_2^t) \rightarrow (0, -0.5, 0.4, 0.2, 1, 0.5)\) with \( t \) decreasing to zero. Note that \((0, -0.5, 0.4, 0.2, 1, 0.5)\) is not only an M-stationary point of (28), but also S-stationarity. Furthermore, the approximate optimal function value \( f^t = 1.6901 \) (Table 1) is less than the optimal values \( f = 2.7183 \) in [27]. Hence, the numerical results imply that the Stackelberg equilibrium problem can be computed more successfully by our algorithm.

Next, we consider the constrained min-max-min problems \([25]\):
\begin{align*}
\min_{z} \quad & \left\{ f(z) = \max_{1 \leq i \leq m} \min_{1 \leq j \leq l_i} \{ f_{ij}(z) \} \right\} \\
\text{s.t.} & \quad g(z) = \max_{1 \leq k \leq k} \min_{1 \leq j \leq p_i} \{ g_{ij}(z) \} \leq 0,
\end{align*}

\( y \in \text{SOL}(z) \) is an equilibrium constraint \([7, 27]\).
where \( f_{ij} : \mathbb{R}^n \to \mathbb{R} \), \( g_{ij} : \mathbb{R}^n \to \mathbb{R} \) are twice continuously differentiable. Note that the problem (29) can be equivalently rewritten as the following problem

\[
\begin{align*}
\min & \quad y_0 \\
\text{s.t.} & \quad \min_{1 \leq j \leq l} \{ f_{ij}(z) + s_i - y_0 \} = 0, \quad s_i \geq 0, \quad i = 1, \ldots, m, \\
& \quad \min_{1 \leq j \leq p} \{ g_{ij}(z) + y_i \} = 0, \quad y_i \geq 0, \quad i = 1 + m, \ldots, k + m,
\end{align*}
\]

(30)

which means that it can be reformulated as MPVCC. We solve the log-exponential regularization of problem (30) by our feasible strategy (Algorithm 2). In what follows, we give three examples to show that it is effective to solve (29).

**Example 3.** [25] Consider the min-max-min problem, where

\[
\begin{align*}
& f(z) = 100(z_2 - z_1^2)^2 + (1 - z_1)^2, \\
& g_{11}(z) = 0.75 + (z_1 - 1)^2 - (z_2 - 1), \quad g_{12}(z) = -0.25 - (z_1 - 1)^2 - (z_2 - 1), \\
& g_{13}(z) = 0.75 - (z_2 - 1.5)^2 - (z_1 - 1), \quad g_{14}(z) = 0.75 - (z_2 - 1.5)^2 + (z_1 - 1), \\
& g_{21}(z) = 0.5 + z_2 - 1 + 2(z_1 - 1)^2, \\
& g_{22}(z) = 1 - (z_2 - 1) - 2(z_1 - 1) - 4(z_1 - 1)^2, \\
& g_{23}(z) = 1 - (z_2 - 1) + 2(z_1 - 1) - 4(z_1 - 1)^2.
\end{align*}
\]

We rewrite the problem as the following MPVCC

\[
\begin{align*}
\min & \quad 100(z_2 - z_1^2)^2 + (1 - z_1)^2 \\
\text{s.t.} & \quad y_1 \geq 0, \quad \min \{ g_{11}(z) + y_1, \ g_{12}(z) + y_1, \ g_{13}(z) + y_1, \ g_{14}(z) + y_1 \} = 0, \\
& \quad y_2 \geq 0, \quad \min \{ g_{21}(z) + y_2, \ g_{22}(z) + y_2, \ g_{23}(z) + y_2 \} = 0.
\end{align*}
\]

Our algorithm finds the solution (1.2001, 1.4401) and obtains the final objective function value 0.0400. For this example, we observe that the solution (1.1999, 1.4402) founded in [25] is not a feasible point of (29), since \( g(1.1999, 1.4402) = \max_{1 \leq i \leq 13, 1 \leq j \leq p} \{ g_{ij}(1.1999, 1.4402) \} = 1.5996 - 4 > 0 \). Although the same objective function value is obtained, our method performs more well.

**Example 4.** [25] Consider the min-max-min problem, where

\[
\begin{align*}
& f(z) = z_1^2 + (z_2 - 2)^2, \\
& g_{11}(z) = z_1^2 + z_2^2 - 4, \quad g_{21}(z) = -z_1 - z_2 - 2, \quad g_{22}(z) = z_1.
\end{align*}
\]

Note that \( z^* = (0, 2) \) is the global minimizer of the problem and the optimal value is 0. The problem can be written as the following MPCC

\[
\begin{align*}
\min & \quad z_1^2 + (z_2 - 2)^2 \\
\text{s.t.} & \quad z_1^2 + z_2^2 - 4 \leq 0, \\
& \quad y_1 \geq 0, \quad \min \{ -z_1 - z_2 - 2 + y_1, \ z_1 + y_1 \} = 0.
\end{align*}
\]

**Example 5.** [25] Consider the min-max-min problem, where

\[
\begin{align*}
& f(z) = \max \{ (z_1 - 1)^2 + z_2^2, \ (z_1 - 0.5)^2 + z_2^2 \}, \\
& g_{11}(z) = 1.5 + 0.5z_1^2 - z_2, \quad g_{12}(z) = 0.5 + 0.5z_1^2 - z_2, \\
& g_{13}(z) = -1.5 + (z_2 - 1)^2 - z_1, \quad g_{14}(z) = -1.5 + (z_2 - 1)^2 + z_1, \\
& g_{21}(z) = -5 + z_2 + z_1^2, \quad g_{22}(z) = 2 - z_2 + 2z_1 + 2z_1^2, \\
& g_{23}(z) = -4 + z_2 - 2z_1 + 2z_1^2.
\end{align*}
\]
Notice that \( z^* = (0.75, 0) \) is the optimal solution of the problem and the optimal value is 0.065. We rewrite the problem as the following MPVCC:

\[
\begin{align*}
\min \quad & y_0 \\
\text{s.t.} \quad & (z_1 - 1)^2 + z_2^2 \leq y_0, \quad (z_1 - 0.5)^2 + z_2^2 \leq y_0, \\
& y_1 \geq 0, \quad \min \{g_{11}(z) + y_1, \ g_{12}(z) + y_1, \ g_{13}(z) + y_1, \ g_{14}(z) + y_1\} = 0, \\
& y_2 \geq 0, \quad \min \{g_{21}(z) + y_2, \ g_{22}(z) + y_2, \ g_{23}(z) + y_2\} = 0.
\end{align*}
\]

In numerical experiment, we choose the same starting points in our approach and the solver \textit{fmincon}. As a measure between \( z \) and the exact solution \( z^* \), we introduce the indicator \( \text{Gap} = (1 - \frac{\|z - z^*\|}{\|z^*\|})\% \). Table 2 summarizes the computational results of Example 4 and Example 5. From the results in Table 2, we find that our algorithm is comparable to the other methods.

**Table 2. The numerical results for Example 4, 5**

| Example | Algorithm | \( z \) | \( f \) | Gap |
|---------|-----------|--------|--------|-----|
| 4       | Algorithm 2 | (0.0000, 2.0000) | 0.0000 | 100% |
|         | \textit{fmincon} | (0.0004, 2.0000) | 0.0000 | 99.98% |
|         | ADH | (0.0000, 1.9988) | 0.0000 | 99.94% |
|         | AH | (-0.0000, 1.9999) | 0.0000 | 100% |
|         | Polak \(^1\) | (0.0000, 1.8708) | 0.0167 | 93.54% |
| 5       | Algorithm 2 | (0.7500, 0.0000) | 0.0625 | 100% |
|         | \textit{fmincon} | (0.7500, 0.0003) | 0.0625 | 99.96% |
|         | ADH | (0.7500, 0.0000) | 0.0625 | 100% |
|         | AH | (0.7500, 0.0000) | 0.0625 | 100% |
|         | Polak \(^1\) | (0.7500, 0.0000) | 0.0625 | 100% |

By the Algorithm 2, we get more exact optimal solution for Example 4. The method \textit{Polak} \(^1\) may be leads to a non-optimal solution as Example 4. The computational results of Example 5 indicate that the optimal values are the same. We find that the Algorithm 2 can obtain the better optimal solutions than using the solver \textit{fmincon} to solve the log-exponential regularized subproblems for MPVCC directly. Therefore, Algorithm 2 is effective for solving MPVCC and produces the results that are intended.

6. **Conclusion.** In this paper, we discuss the convergence properties of a log-exponential regularization scheme for MPVCC. Due to the fact that the exact KKT points is not useful from a practical standpoint. We consider the \( \varepsilon \)-KKT points of the regularized problems and show that the accumulation point is \( C \)-stationarity for MPVCC which is an expectable convergence result. Note that we only need a weaker assumption regarding the choice of \( \varepsilon \). Furthermore, we present an feasible strategy that obtains \( \varepsilon \)-KKT sequences of the regularized problems. The numerical experiments illustrate that the approach is effective. The inexact log-exponential regularization method for solving MPVCC is useful from a practical point of view.

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