Spin(9) Average of SU(N) Matrix Models
I. Hamiltonian

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\textbf{Abstract}

We apply a method of group averaging to states and operators appearing in (truncations of) the Spin(9) × SU(N) invariant matrix models. We find that there is an exact correspondence between the standard supersymmetric Hamiltonian and the Spin(9) average of a relatively simple lower-dimensional model.

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1 Introduction

Due to its relevance to M-theory, reduced Yang-Mills theory, and membrane theory, considerable effort has been put into investigating the structure of Spin(9) × SU(N) invariant supersymmetric matrix models (see e.g. [1] for a review). Despite this, a concrete knowledge of the conjectured zero-energy eigenfunction of the Hamiltonian \( H \) is still lacking.

In [2] a certain truncation of the Spin(9) invariant model was introduced, based on a coordinate split of \( \mathbb{R}^9 \) into \( \mathbb{R}^7 \times \mathbb{R}^2 \). The corresponding Hamiltonian \( H_D = \{Q_D, Q_D^\dagger\} \), which is essentially just a set of supersymmetric harmonic oscillators, can be interpreted as a two-dimensional supersymmetric SU(N) matrix model with a seven-dimensional space of parameters. Recently, a deformation of the standard matrix model – based on the same coordinate split – was considered that produces a \( G_2 \times U(1) \) invariant supersymmetric Hamiltonian \( \tilde{H} \) in which \( H_D \) plays a central role [3]. The explicit knowledge of the structure of \( H_D \) and its eigenfunctions made it possible to prove in a straightforward manner that \( \tilde{H} \) and \( H \) have similar spectra.

In this paper we calculate the Spin(9) average of the truncated Hamiltonian \( H_D \) and find that it is essentially equal to the full supersymmetric Hamiltonian \( H \). The correspondence is made exact by a slight modification of \( H_D \).

Motivated by this result, we also expect that the wavefunctions obtained by averaging the eigenfunctions of \( H_D \) (or slight modifications of those) could be related to the Spin(9) invariant eigenfunctions of \( H \). Calculating the average of such eigenstates, however, is a technically more difficult problem, to be addressed in a forthcoming paper [4].

2 Group averaging

First, let us define what we mean by group averaging (the notion is well-known in the literature; see e.g. [5] and references therein for a general approach and various applications).

Assume that we are given a unitary representation \( U(g) \) of a compact Lie group \( G \) acting on a complex separable Hilbert space \( \mathcal{H} \). Then, given any state \( \Psi \in \mathcal{H} \) and linear operator \( A \) acting on \( \mathcal{H} \), we define the corresponding
$G$-averaged state $[\Psi]_G$ resp. operator $[A]_G$ by

$$[\Psi]_G := \int_{g \in G} U(g) \Psi \, d\mu(g)$$

resp.

$$[A]_G := \int_{g \in G} U(g) AU(g)^{-1} \, d\mu(g),$$

where $\mu$ denotes the unique normalized left- and right-invariant (Haar) measure on $G$. Due to the translation invariance of $\mu$, $[\Psi]_G$ will be invariant under the action of $U(g)$, and $[A]_G$ will commute with $U(g)$.

One can also extend the above definition to generalized (non-normalizable) states, e.g. Schwartz distributions $\psi \in \mathcal{D}'$, by taking

$$\langle \psi \rangle_{G}(\phi) := \int_{g \in G} \psi(U(g)\phi) \, d\mu(g)$$

for any test function $\phi \in \mathcal{D} = C_{0}^\infty(\Omega)$.

### 3 The model and its group actions

We are interested in the supersymmetric matrix model described by the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^{9n}) \otimes \mathcal{F}, \quad \mathcal{F} = \bigotimes_{A=1}^{n} \mathcal{F}^{(A)} = \mathbb{C}^{2^{8n}}$$

and the Hamiltonian

$$H = p_{sA}p_{sA} + \frac{1}{2}(f_{ABC}x_{sB}x_{sC})^2 + \frac{i}{2} x_{sC} f_{ABC} \gamma^s_{\alpha\beta} \theta_{\alpha A} \theta_{\beta B} = -\Delta + V + H_F,$$

where we sum over corresponding indices $s,t,\ldots = 1,\ldots,9$, $A,B,\ldots = 1,\ldots,n := N^2 - 1$, $\alpha,\beta,\ldots = 1,\ldots,16$. $\gamma^s$ generate (a matrix representation of) the Clifford algebra over $\mathbb{R}^9$ acting irreducibly on $\mathbb{R}^{16}$, while $\theta_{\alpha A}$ generate the Clifford algebra over $\mathbb{R}^{16} \otimes \mathbb{R}^n$, i.e. $\{\theta_{\alpha A}, \theta_{\beta B}\} = 2\delta_{\alpha\beta}\delta_{AB}$, acting irreducibly on $\mathcal{F}$. The coordinates $x_{sA}$, canonically conjugate to $p_{sA} = -i\partial_{sA}$, comprise a set of 9 traceless hermitian matrices $(X_1,\ldots,X_9) = X \in \mathbb{R}^9 \otimes \mathbb{R}^n$, and we use the isomorphism $i \cdot \mathfrak{su}(N) \cong \mathbb{R}^n$ to map seamlessly between such

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\[1\] Note that the action of $G$ on distributions is given by $(U(g)\psi) (\phi) = \psi(U(g^{-1})\phi)$. 

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a matrix $E$ and its coordinate representation $e_A$ in a basis where the SU($N$) structure constants $f_{ABC}$ are totally antisymmetric.

$H$ is invariant under the action of SU($N$), where the corresponding representation on $H$ is generated by the anti-hermitian operators 

$$\tilde{J}_A = i J_A = i f_{ABC} \left( x_{sB} p_{sC} - \frac{i}{4} \theta_{aB} \theta_{aC} \right)$$

$$= \sum_{B<C} f_{ABC} \left( x_{sB} \partial_{sC} - x_{sC} \partial_{sB} + \frac{1}{2} \theta_{aB} \theta_{aC} \right) = \tilde{L}_A + \tilde{M}_A,$$

with $\tilde{L}_A$ and $\tilde{M}_A$ generating the representation of $\mathfrak{su}(N) \hookrightarrow \mathfrak{so}(n)$ on $L^2(\mathbb{R}^{9n})$ and $\mathcal{F}$, respectively.

Furthermore, $H$ is also invariant under Spin(9), generated by 

$$\tilde{J}_{st} = i J_{st} = i \left( x_{sA} p_{tA} - x_{tA} p_{sA} - \frac{i}{8} \gamma^{st} \theta_{aA} \theta_{bA} \right) = \tilde{L}_{st} + \tilde{M}_{st},$$

with

$$\tilde{L}_{st} = \sum_A \tilde{L}_{st}^{(A)} = x_{sA} \partial_{tA} - x_{tA} \partial_{sA}$$

and

$$\tilde{M}_{st} = \sum_A \tilde{M}_{st}^{(A)} = \sum_{a<\beta} \frac{1}{2} [\gamma_{st}]_{a\beta} \theta_{aA} \theta_{bA}.$$

Note that the spinor representation of Spin(9) is generated by $\frac{1}{4} \gamma^{st} := \frac{1}{4} [\gamma^{s}, \gamma^{t}]$ acting by left multiplication on the Clifford algebra generated by the $\gamma$’s, i.e. left multiplication by the matrix $[\gamma^{st}] \in \mathfrak{so}(16)$ acting on the spinor space $\mathbb{R}^{16}$. This action is in turn represented on the Fock space $\mathcal{F}^{(A)} = \mathbb{C}^{2^n}$ by the spinor representation of $\mathfrak{spin}(16) = \frac{1}{2} \cdot \mathfrak{so}(16)$.

The full, exponentiated, action of $g = e^{\epsilon_{st}} \frac{1}{2} \gamma^{st} \in \text{Spin}(9)$ on a state $\Psi \in \mathcal{H}$, i.e. a wavefunction $\Psi : \mathbb{R}^9 \otimes \mathbb{R}^n \rightarrow \mathcal{F}$, is then given by

$$(U(g)\Psi)(X) = (e^{\epsilon_{st}} \tilde{J}_{st}\Psi)(X) = e^{\epsilon_{st} \tilde{M}_{st}} \Psi(e^{\epsilon_{st} \tilde{L}_{st}} X) = \tilde{R}_g \Psi(R_{g^{-1}}(X)),$$

where $\tilde{R}_g := e^{\epsilon_{st} \frac{1}{2} \gamma_{a\beta} \theta_{aA} \theta_{bA}}$ is the unitary representative of $g$ acting on $\mathcal{F}$, and $R_{g}(x) := gxg^{-1}$ is the corresponding rotation $R_g \in \text{SO}(9)$ acting on vectors $x = x_t \gamma_t \in \mathbb{R}^9$ considered as grade-1 elements of the Clifford algebra. This follows by considering the infinitesimal action on a function $f : \mathbb{R}^9 \rightarrow \mathbb{R}$,
i.e. \((\bar{L}_{st} f)(x) = \text{grad} f \cdot \bar{L}_{st}(x)\), and (using \(x \cdot y = \frac{1}{2} \{x, y\}\))

\[
\bar{L}_{st} x = (x_s \partial_t - x_t \partial_s)(x_u \gamma_u) = x_s \gamma_t - x_t \gamma_s
\]

\[
= (x \cdot \gamma_s)\gamma_t - (x \cdot \gamma_t)\gamma_s = -\frac{1}{2} [\gamma^{st}, x].
\]

Consider now an operator of the form

\[\mathcal{B} = [B(X)]_{\alpha\beta} \theta_{\alpha A} \wedge \theta_{\beta B}\]

where \(B(X)\) is a symmetric 16 \times 16 matrix and \(A \wedge B := \frac{1}{2} [A, B]\). The infinitesimal action is

\[
[\hat{J}_{st}, \mathcal{B}] = \left([\bar{L}_{st}, B(X)]\right)_{\alpha\beta} \theta_{\alpha A} \wedge \theta_{\beta B} + [B(X)]_{\alpha\beta} [\bar{M}_{st}, \theta_{\alpha A} \wedge \theta_{\beta B}],
\]

which, using

\[
[\bar{M}_{st}, \theta_{\alpha A} \wedge \theta_{\beta B}] = \frac{1}{8} \gamma^{st}_{\alpha'\beta'} \left[\theta_{\alpha'C} \wedge \theta_{\beta'C}, \theta_{\alpha A} \wedge \theta_{\beta B}\right]
\]

\[
= \frac{1}{2} \gamma^{st}_{\epsilon\alpha} \theta_{\epsilon A} \wedge \theta_{\beta B} - \gamma^{st}_{\beta\epsilon} \theta_{\alpha A} \wedge \theta_{\epsilon B},
\]

exponentiates to

\[
e^{\epsilon_{st} \hat{J}_{st}} B e^{-\epsilon_{st} \hat{J}_{st}} = \left[e^{\frac{\epsilon_{st} \gamma^{st}}{2}} B \left(e^{-\frac{\epsilon_{st} \gamma^{st}}{2}} X e^{\frac{\epsilon_{st} \gamma^{st}}{2}}\right) e^{-\frac{\epsilon_{st} \gamma^{st}}{2}}\right]_{\alpha\beta} \theta_{\alpha A} \wedge \theta_{\beta B}
\]

\[
= [g B(R_g^T(X)) g^{-1}]_{\alpha\beta} \theta_{\alpha A} \wedge \theta_{\beta B}.
\]

Regarding the supersymmetry of \(H\), it is for the following sufficient to know that there is a set of hermitian supercharge operators \(Q_\alpha\) such that \(H = Q^2_\alpha\) on the subspace of SU\((N)\) invariant states, \(\mathcal{H}_{\text{phys}}\), which is the physical Hilbert space of the theory.

In order to arrive at a conventional Fock space formulation of the model it is necessary to make certain choices which break the explicit Spin(9) symmetry. After introducing a split of the coordinates into \((x', z)\), with

\[x' = (x_{j=1,...,7}) \in \mathbb{R}^7, \quad z_A := x_{8A} + ix_{9A},\]

and a representation of \(\theta_{\alpha A}\) in terms of creation and annihilation operators \(\lambda, \lambda^\dagger\), together with a suitable representation of \(\gamma^a\) (see e.g. Appendix A of \([3]\)), it is rather natural to single out a certain part of the supercharges, resulting in a truncation of \(H\) to the Hamiltonian \([2, 3]\)

\[H_D = -4 \bar{\partial}_z \cdot \partial_z + \bar{z} \cdot S(x') z + 2W(x') \lambda \lambda^\dagger = -\Delta_{89} + V_D + W_D,\]
where each of these terms will be explained in the next section. This operator constitutes a set of $2n$ supersymmetric harmonic oscillators in $x_{8A}$ and $x_{9A}$ whose frequencies are the square root of the eigenvalues of the positive semidefinite matrix operator $S(x') = \sum_{j=1}^{7} \text{ad}_{X_j} \circ \text{ad}_{X_j}$. Thus, $H_D$ can be considered as acting on a smaller Hilbert space over the $z$-coordinates,

$$\hat{\mathcal{H}} = L^2(\mathbb{R}^{2n}) \otimes \mathcal{F},$$

with $x_j$ entering as parameters, and has with respect to $\hat{\mathcal{H}}$ the complete basis of eigenstates

$$\psi_{k,\sigma}(x', z) = \pi^{-n/2} (\det S(x'))^{1/4} H_k(x', z) e^{-\frac{1}{2} \bar{z} S(x') \frac{1}{2} z} \xi_{x'}^\sigma. \quad (2)$$

$H_k(x', z)$ denote products of normalized Hermite polynomials in $S(x')^{1/4} x_8$ and $S(x')^{1/4} x_9$, while $\xi_{x'}^\sigma \in \mathcal{F}$, $\sigma \in \{0, 1\}^{8n}$, form the basis of eigenvectors of $W_D$ (see [2, 3] for details).

As pointed out in [3], both $H_D$ and its nondegenerate eigenstates are SU($N$) invariant (covariant) in the sense that they are unchanged under the simultaneous action of SU($N$) on $\hat{\mathcal{H}}$ and the parameters $x_j$.

### 4 The averaged Hamiltonian

We would like to apply group averaging w.r.t. $G = \text{Spin}(9)$ to the truncated Hamiltonian $H_D$ and its $\hat{\mathcal{H}}$-eigenstates [2] (which are generalized states w.r.t. the full Hilbert space $\mathcal{H}$).

Note that averaging the supercharge $Q_D$ corresponding to $H_D$ gives zero in the same way that, for the supercharges $Q_\alpha$ corresponding to $H$ and transforming like spinors, $[Q_\alpha]_G = [g_{\beta\alpha} Q_\beta]_G = 0$, taking $g = -1$.

#### 4.1 Laplacian part

The principal part of $H_D$ is the Laplace operator on $\mathbb{R}^2 \otimes \mathbb{R}^n$,

$$\Delta_{89} = \Delta_8 + \Delta_9 = \partial_{8A} \partial_{8A} + \partial_{9A} \partial_{9A}.$$
In order to average this operator, consider first \( x_1^2 = (x \cdot \gamma_1)^2 \) in \( \mathbb{R}^d \), for which

\[
[x_1^2]_{\text{Spin}(d)} = \int_{g \in \text{Spin}(d)} U(g)(x \cdot \gamma_1)^2 U(g)^{-1} \, d\mu(g)
\]

\[
= \int (R^T_g(x) \cdot \gamma_1)^2 \, d\mu(g) = \frac{1}{d} \sum_{j=1}^d (x \cdot R_{gh_j}(\gamma_1))^2 \, d\mu(g)
\]

\[
= \frac{1}{d} \int \sum_{j=1}^d (x \cdot R_{gh}^j)^2 \, d\mu(g) = \frac{1}{d} [x^2]_{\text{Spin}(d)} = \frac{1}{d} |x|^2,
\]

where we used the invariance of \( \mu \) to insert \( h_j \in \text{Spin}(d) \) s.t. \( R_{h_j}(\gamma_1) = \gamma_j \).

Analogously, one finds \( \partial_1^2 \) Spin(\( d \)) = \( \frac{1}{d} \Delta_{\mathbb{R}^d} \). Hence,

\[
[\Delta_{89}]_{\text{Spin}(9)} = \frac{2}{9} \Delta_{\mathbb{R}^9}.
\]

### 4.2 Potential part

Denoting the norm in \( i \cdot \text{su}(N) \) by \( || \cdot || \), so that for such a matrix \( E \leftrightarrow e_A, ||E||^2 = e_A e_A \), we have

\[
V_D = \tilde{z}_A S(x')_{A'A'} z_{A'} = \tilde{z}_A f_{ABC} x_j B f_{A'B'C} x_j B' z_{A'} = \sum_{\substack{a=8,9 \\ j=1,\ldots,7}} \|[X_a, X_j]\|^2.
\]

Using that any pair \( (\gamma_a, \gamma_j) \) of orthonormal vectors can be rotated into any other orthonormal pair \( (\gamma_s, \gamma_t) = (R_h(\gamma_a), R_h(\gamma_j)) \) by some \( R_h, h \in \text{Spin}(9) \), we find

\[
\left[\|[X_a, X_j]\|^2\right]_G = \int_G U(g)\|[X \cdot \gamma_a, X \cdot \gamma_j]\|^2 U(g)^{-1} \, d\mu(g)
\]

\[
= \int \|[R^T_g(X) \cdot \gamma_a, R^T_g(X) \cdot \gamma_j]\|^2 \, d\mu(g)
\]

\[
= \int \|[X \cdot R_{gh}(\gamma_a), X \cdot R_{gh}(\gamma_j)]\|^2 \, d\mu(g)
\]

\[
= \int \|[X \cdot R_g(\gamma_s), X \cdot R_g(\gamma_t)]\|^2 \, d\mu(g) = \left[\|[X_s, X_t]\|^2\right]_G.
\]

Therefore,

\[
[V_D]_G = \sum_{\substack{a=8,9 \\ j=1,\ldots,7}} \left[\|[X_a, X_j]\|^2\right]_G = \frac{14}{36} \sum_{s<t} \left[\|[X_s, X_t]\|^2\right]_G = \frac{7}{18} [V]_G = \frac{7}{18} V.
\]
4.3 Fermionic part

The fermionic part of $H_D$, given in terms of Fock space operators $\lambda_{\alpha' A} \equiv \frac{1}{2}(\theta_{\alpha' A} + i \theta_{8+\alpha' A})$, $\alpha' = 1, \ldots, 8$, is $W_D = 2x_J f_{CAB} (\delta_{\alpha' 8} \delta_{\beta' j} - \delta_{\alpha' j} \delta_{\beta' 8}) \lambda_{\alpha' A} \lambda_{\beta' B}$.

With our choice of representation of the $\gamma$ matrices (see Appendix A of [3]), we find

$$W_D = i \sum_{\rho=8,16} x_J f_{CAB} \gamma^j_{\rho\beta} \theta_{\rho A} \theta_{\beta B} = ix_J f_{CAB} [P \gamma^j]_{\alpha\beta} \theta_{\alpha A} \theta_{\beta B},$$

where $P$ is a projection matrix s.t. $P_{8,8} = P_{16,16} = 1$ and zero otherwise. Furthermore, one can verify that $P$ can be written as a product of three commuting projectors of the form $\frac{1}{2}(1 \pm E_\mu)$, $E_\mu^2 = 1$, in the Clifford algebra:

$$P = \frac{1}{8} (1 - \gamma_1 \gamma_2 \gamma_3 I_7) (1 - \gamma_2 \gamma_5 \gamma_7 I_7) (1 - \gamma_3 \gamma_6 \gamma_7 I_7) = \frac{1}{8} (1 - CI_7), \quad (3)$$

where $I_7 := \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6 \gamma_7$, and

$$C := \gamma^{123} + \gamma^{165} + \gamma^{246} + \gamma^{435} + \gamma^{147} + \gamma^{367} + \gamma^{257}$$

defines an octonionic structure. By choosing different signs for $E_\mu$ in the three projectors one obtains all $8 = 2^3$ projection matrices of that form. Also note that $\gamma_1$, $\gamma_5$, and $\gamma_6$ share a particular property in the expression (3).

The action (1) yields

$$[W_D]_G = \int_G i(R_g^T(X) \cdot \gamma_j) f_{CAB} [gP \gamma^j g^{-1}]_{\alpha\beta} \theta_{\alpha A} \theta_{\beta B} \ d\mu(g)$$

$$= \frac{1}{8} \sum_{p=1}^8 \int_G i(X \cdot R_{h_p}(\gamma_j)) f_{CAB} [gh_p Ph_p^{-1} g^{-1} R_{h_p}(\gamma^j)]_{\alpha\beta} \theta_{\alpha A} \theta_{\beta B} \ d\mu(g),$$

where we insert 8 different $h_p \in \text{Spin}(7)$ such that $R_{h_p}(\gamma_j) = \sigma_{p,j} \gamma_j \forall j$, and $\sigma_{p,j} \in \{+, -\}$ are signs chosen so that $\sum_p h_p Ph_p^{-1} = 1$, e.g. according to the
This is possible with \( h_p \in \text{Spin}(7) \) (and not only \( \text{Pin}(7) \)) because \( \gamma_4 \) does not appear explicitly in (3) except in \( I_7 \), which with the choice of signs above is invariant, i.e. \( h_p I_7 h_p^{-1} = R_{h_p}(\gamma_1) R_{h_p}(\gamma_2) \ldots R_{h_p}(\gamma_7) = I_7 \). Hence,

\[
[W_D]_G = \frac{1}{8} \sum_{j=1}^{7} \int_G i(\mathbf{X} \cdot R_{g}(\gamma_j))_{C} f_{CAB} \left[ g \left( \sum_p \sigma_p \sigma_p h_p P h_p^{-1} \right) g^{-1} R_{g}(\gamma^j) \right]_{\alpha\beta} \theta_{\alpha A} \theta_{\beta B} \, d\mu(g)
\]

\[
= \frac{1}{8} \sum_{j=1}^{7} \int_G i(\mathbf{X} \cdot R_{g h_j}(\gamma_j))_{C} f_{CAB} \left[ R_{g h_j}(\gamma^j) \right]_{\alpha\beta} \theta_{\alpha A} \theta_{\beta B} \, d\mu(g)
\]

\[
= \frac{17}{4} [H_F]_G,
\]

again using some appropriately chosen \( h'_j \in \text{Spin}(9) \).

5 Result

In total, we have

\[
[H_D]_G = [-\Delta_{89}]_G + [V_D]_G + [W_D]_G = -\frac{2}{9} \Delta_{99(N^2-1)} + \frac{7}{2} V + \frac{7}{4} H_F.
\]

The relative coefficients of the terms of the resulting operator do not match those of \( H \). In fact, \([H_D]_G\) has a discrete spectrum on \( \mathcal{H}_{\text{phys}} \) (contrary to
$H$ whose spectrum covers the whole positive axis \[6\)]. This can be seen by rescaling the coordinates by \((\sqrt{7}/2)^{1/3}\), obtaining up to a constant

$$[H_D]_G \sim -\Delta + V + \kappa H_F = (1 - \kappa)(-\Delta + V) + \kappa H \geq (1 - \kappa)(-\Delta + V),$$

with \(\kappa = \sqrt{7}/4 < 1\). The observation follows since $H$ is a positive operator (by supersymmetry) and $-\Delta + V$ has a purely discrete spectrum \[7\].

On the other hand, we can of course define a rescaled operator

$$H_D' := -\frac{9}{2} \Delta_{s9} + \frac{2 \cdot 9}{7} V_D + \frac{4 \cdot 9}{7} W_D$$

for which the average then is $[H_D']_{\text{Spin}(9)} = H$. Unlike $H_D$ which is positive due to supersymmetry, $H_D'$ has energies tending to negative infinity in certain regions of the $x'$ parameter space (note that its \(\hat{\kappa}\)-eigenstates are still given by \(2\), but with a rescaled frequency $S$). However, considering the action on $\text{Spin}(9) \times \text{SU}(N)$ invariant states $\Psi = U(g)\Psi$, we have

$$\langle \Psi, H_D' \Psi \rangle = \int \langle U(g^{-1})\Psi, H_D' U(g^{-1})\Psi \rangle d\mu(g)$$

$$= \left\langle \Psi, \int U(g)H_D' U(g)^{-1} d\mu(g) \Psi \right\rangle = \langle \Psi, [H_D'] \Psi \rangle$$

$$= \langle \Psi, H \Psi \rangle = \|Q_\alpha \Psi\|^2 \geq 0. \quad (4)$$

Hence, we conclude that these quadratic forms coincide on the subspace $\mathcal{H}_{\text{inv}}$ of invariant states, so that $H_D'$ and $H$ are actually the same operator on that subspace. Furthermore, because a zero-energy state of $H$ must be $\text{Spin}(9)$ invariant \[8\] it is therefore sufficient to check that it is annihilated by $H_D'$, i.e. that

$$(-7\Delta_{s9} + 4\hat{z} \cdot S(x')z + 16W(x')\lambda\lambda^\dagger) \Psi(x', z) = 0 \quad \forall x'.$$

Also note that the same holds for any linear combination, $(\alpha H + \beta H_D') \Psi = 0$.

\footnote{The reader who is worried about the unboundedness of the operators $H$ and $H_D'$ may consider the dense subspace $\mathcal{H}_{\text{inv}} \cap C_0^\infty$, where \[4\] makes perfect sense, and then conclude that the Friedrichs extensions of $H_D'$ and $H$ on $\mathcal{H}_{\text{inv}}$ are equal.}
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