Feedback Interference Alignment: Exact Alignment for Three Users in Two Time Slots

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Abstract—We study the three-user interference channel where each transmitter has local feedback of the signal from its targeted receiver. We show that in the important case where the channel coefficients are static, exact alignment can be achieved over two time slots using linear schemes. This is in contrast with the interference channel where no feedback is utilized, where it seems that either an infinite number of channel extensions or infinite precision is required for exact alignment. We also demonstrate, via simulations, that our scheme significantly outperforms time-sharing even at finite SNR. — SUBMITTED TO ICC 2013.

I. INTRODUCTION

It is widely accepted that interference is now the dominant bottleneck in wireless networks. Conventional techniques, such as time-division, perform poorly as the number of users increases. Recently, the discovery of a new technique, known as interference alignment [1], [2], [3], has shown that, in certain scenarios, it is possible for each user to achieve half its interference-free rate. Since the discovery of interference alignment, an ongoing research theme in communications systems is to study the extent to which the gains predicted by theory can be realized in practice. In this paper, we contribute to this research theme through a novel utilization of feedback. In particular, two of the primary limitations of existing schemes for interference alignment are the necessity of high signal-to-noise ratios (SNRs) [2], [4], and the necessity of an arbitrarily large amount of time-variations and symbol extensions [2], [5] to induce alignment. In this paper, we use feedback to lift these restrictions.

We will focus on the single-antenna, three-user Gaussian interference channel with flat fading (i.e., the channel remains constant over the duration of the codeword). It is unknown how to induce alignment in this scenario except in the very high SNR regime via real interference alignment [4]. As we will show, if each transmitter has local feedback from its targeted receiver, together they can create an effective aligned channel over two adjacent time slots. This implies that each user can simultaneously achieve 1/2 degrees-of-freedom (DoF). Furthermore, we propose an optimization framework to maximize the sum rate and show that the average achievable rate nearly reaches half the interference-free rate for the entire SNR regime.

A. Related Work

It is well-known that feedback does not increase the capacity of point-to-point channels [6]. However, for multi-user networks, feedback can increase capacity. For instance, in the Gaussian multiple-access channel and broadcast channels, feedback has been shown to enlarge the capacity region [7], [8]. More recently, several groups have studied the impact of feedback on Gaussian interference channels. Suh and Tse considered the two-user Gaussian interference channel with local feedback and determined the capacity region to within two bits [9]. For interference channels with more than two users, since the capacity (or even constant gap approximations) continues to elude researchers, recent literature has focused on the DoF, both with and without feedback. Perhaps surprisingly, Cadambe and Jafar demonstrated that in the K-user interference channel, even global and perfect feedback cannot increase the degrees of freedom [10]. In other words, the K user interference channel has a maximum of K/2 DoF irrespective of the presence of feedback. In light of this negative result, a natural research direction is to explore the utility of feedback to simplify signaling schemes (in particular alignment-based schemes), and to improve finite SNR performance. This research direction is especially important because existing DoF-optimal schemes need arbitrarily large bandwidth [2], or channel precision [4]. Further, for static channels, DoF is a brittle metric since it is discontinuous at all rational channel gains [11]. In addition, a scheme that attains the maximum DoF may still perform poorly at finite SNR. In this paper, we will use feedback to address these issues by providing a simple alignment scheme for static interference channels with significant benefits at finite SNR.

The use of feedback for interference channels with time-varying channel gains was studied in [12], where it was shown that similar to the broadcast channels [13], feedback of stale channel information provides DoF gains. In contrast, the object of focus of our paper is the static interference channel, where the coding is done over a single coherence block. Very recently, Papailiopoulos, Suh, and Dimakis studied the static K-user Gaussian interference channels under two feedback models [14]. In the first, each transmitter gets channel output feedback from every receiver and K/2 DoF are easily achievable over two time-slots. In the second, the feedback is observed through a “backwards interference channel.” They
proved that $3/2$ DoF is achievable for the 3-user case over 4 time-slots and presented numerical evidence for the 4-user case. Geng and Viswanath obtained similar results for the second model and developed a generalization to full duplex communication \cite{15}. The key distinction between this work and our own is that we focus on local feedback, i.e., each transmitter only has channel output feedback from its intended receiver. For this setting, we will show that $3/2$ degrees of freedom are achievable with 2 time-slots with only local feedback information. Thus, we present a stronger version of \cite{14}, where global feedback was used to achieve the same degrees-of-freedom result. In addition, we optimize our scheme to significantly outperform standard techniques such as time-sharing even at finite SNRs.

II. PROBLEM STATEMENT

We will focus on the 3-user interference channel with (perfect) feedback. In this setting, there are three users indexed by $\ell \in \{1, 2, 3\}$. Given a blocklength of length $n$, each transmitter has a message $w_{\ell}$ that is drawn independently and uniformly over $\{1, 2, \ldots, 2^{nK}\}$. The channel input $x_{\ell}$ from the $\ell$th transmitter is the tuple $(x_{\ell}[1], x_{\ell}[2], \ldots, x_{\ell}[n]) \in \mathbb{R}^n$ whose entry to the channel input for the $i$th symbol, where $1 \leq i \leq n$. All channel inputs must satisfy an average power constraint, $\|x_{\ell}\|_2 \leq nP_{\ell}$. The channel output at the $k$th receiver is

$$y_k[t] = h_{k1}x_1[t] + h_{k2}x_2[t] + h_{k3}x_3[t] + z_k[t]$$

and the noise $z_k[t]$ is i.i.d. Gaussian with mean zero and unit variance. Let $H = \{h_{\ell i}\}$ denote the matrix of channel coefficients. We assume that $H$ is globally available at all transmitters and receivers. (In fact, as we will argue, our scheme can tolerate a significant delay in acquiring the channel state.) The channel output can be written concisely in vector notation

$$y[t] = Hx[t] + z[t]$$

where $y[t] = [y_1[t], y_2[t], y_3[t]]^T$, $x[t] = [x_1[t], x_2[t], x_3[t]]^T$, and $z[t] = [z_1[t], z_2[t], z_3[t]]^T$. In this short-hand notation, we also sometimes vectorize the power constraints as $P = [P_1, P_2, P_3]^T$. The key assumption is that each transmitter is given access to local feedback from its intended receiver. That is, the encoding function at each transmitter can be written in terms of the message and the channel outputs up to time $t-1$, $x_{\ell}[t] = E_{\ell t}(w_{\ell}, y_{\ell}[1], \ldots, y_{\ell}[t-1])$. The decoding function at the $k$th receiver uses its $n$ channel observations to estimate the message from the $k$th transmitter, $\hat{w}_k = D_k(y_k[1], \ldots, y_k[n])$.

We say a rate tuple $(R_1, R_2, R_3)$ is achievable for a given channel matrix $H$ if, for any $\epsilon > 0$ and $n$ large enough, there exist encoders and decoders such that $P((\hat{w}_1, \hat{w}_2, \hat{w}_3) \neq (w_1, w_2, w_3)) < \epsilon$. We will be primarily concerned with determining the maximum sum rate $R_1 + R_2 + R_3$. The sum-capacity denoted by $C(P)$ is the supremum of $R_1 + R_2 + R_3$ over the set of all achievable rate tuples $(R_1, R_2, R_3)$. When $P_1 = P_2 = P_3 = P$, we simply denote the capacity as $C(P)$. As usual, the sum degrees-of-freedom (DoF) are defined as

$$d \triangleq \lim_{P \to \infty} \frac{C(P)}{\frac{P}{2} \log P}.$$  

**Remark 1.** Strictly speaking, the DoF $d$ is a function of the channel matrix $H$. However, as we show in this paper, the DoF of the channel under consideration takes the same value\footnote{The DoF for the interference channel with feedback is not dependent on whether the channel gains are rational or irrational, unlike the case of the interference channel without feedback \cite{14}. This makes DoF a more “robust” performance metric for the channel in consideration.} for almost all values of $H$; we therefore suppress its dependence on the channel gain matrix in this paper.

III. FEEDBACK INTERFERENCE ALIGNMENT

We describe here a transmission strategy that takes advantage of the locally available feedback in order to improve the overall system performance. In our scheme, transmitters send linear combinations of new symbols and available feedback. Quite remarkably, we show that the proposed scheme can align all interference over two time slots and achieve the maximum DoF of the above channel almost surely. Then we focus on the finite SNR regime and investigate efficient algorithms to optimize performance.

A. The Achievable Scheme

Our scheme employs two channel uses in order to successfully transmit one symbol to each receiver. For notational convenience, we assume these time slots are consecutive. Note that the performance of our scheme will not change if these two time slots are separated by a longer delay (up to $n/2$), owing to delay in the feedback path. Let $x_{\ell}[1]$ denote the first symbol from transmitter $\ell$; receiver $\ell$ intends to estimate this symbol over two time slots. In the first slot ($t = 1$), transmitter $\ell$ simply sends $x_{\ell}[1]$. The $k$th receiver observes $y_k[1] = \sum_{\ell=1}^3 h_{k\ell}x_{\ell}[1] + z_k[1]$. Rewriting everything in the compact matrix notation introduced above, we get

$$y[1] = Hx[1] + z[1]$$

These noisy observations are then fed back to the receiver by the corresponding transmitters. In the second time slot ($t = 2$), transmitter $\ell$ observes $y_{\ell}[1]$ and transmits the linear combination $x_{\ell}[2] = t_{\ell}x_{\ell}[1] + f_{\ell}y_{\ell}[1]$ for some $t_{\ell}, f_{\ell} \in \mathbb{R}$ whose choice will soon be described. Let $t \triangleq [t_1, t_2, t_3]^T$, $f \triangleq [f_1, f_2, f_3]^T$ and $T = \text{diag}(t)$, $F = \text{diag}(f F)$. In matrix notation, the receivers observe

$$y[2] = H(Tx[1] + Fy[1]) + z[2] = H(Tx[1] + F(Hx[1] + z[1])) + z[2] = H(T + FH)x[1] + HFz[1] + z[2]$$

For ease of analysis, we will consider real-valued channels. All of the analysis extends naturally to complex-valued channels.
Remark 2. Note that our scheme only makes use of the channel state information $H$ during the second phase. Therefore, we can tolerate a delay of up to $n/2$ in learning the channel matrix.

\[
\begin{align*}
\text{Let } P &= \left\{ (t, f) \in \mathbb{R}^3 \times \mathbb{R}^3 : (t_\ell, f_\ell) \in P_\ell, \forall \ell \right\}. \\
\text{The achievable sum rate of the proposed scheme is}
\end{align*}
\]

\[
R_{\text{sum}} = \max_{(t, f) \in P} \sum_k R_k(t, f) \tag{11}
\]

### B. Interference Alignment and DoF

The natural strategy for the achievable scheme described above would be to solve (11). However, the associated optimization problem is non-convex and hard to analyze. In this section, to simplify analysis, we use the coarser metric of DoF in order to study the performance of our scheme.

In particular, we will show that it is possible, through an appropriate choice of $t$ and $f$, for receiver $k$ to null the interference and recover $\lambda_k x_k[1] + \tilde{z}_k$ where $\lambda_k \neq 0$ and $\tilde{z}_k$ depends only on $w_k$. Since we use two time-slots to achieve the above, this would automatically imply that receiver $k$ can decode $x_k[1]$ at 1/2 DoF as required. The interference alignment is implicit in this scheme because receiver $k$ receives (noisy) linear combinations of 3 scalars, $x_1[1], x_2[1], x_3[1]$ in two time-slots; resolving $x_k[1]$ is possible if the two interferers $x_1[1], j \neq \ell$ align. To obtain $x_k[1]$, receiver $k$ linearly combines $y_k[1], y_k[2]$ as $y_k[2] + d_k y_k[1]$. The goal is to choose $d \triangleq [d_1, d_2, \ldots, d_3]^T$, $t, f$ so that

\[
y_k[2] + d_k y_k[1] = \lambda_k x_k[1] + \tilde{z}_k, \quad \lambda_k \neq 0.
\]

The above condition for decidability can be expressed in matrix form as

\[
DH + HT + HFH = \Lambda, \quad \det(\Lambda) \neq 0 \tag{12}
\]

where $D = \text{diag}(d)$.

Remark 3. One can think of the vector $[1, d_k]$ as a (scaled) linear projection of the corresponding 2-dimensional observation at receiver $k$. It is worth noting that for fixed $t, f$, the optimal linear receiver that combines $y_k[1], y_k[2]$ to recover $x_k[1]$, assuming that the interference is treated as noise, is given by the MMSE receiver. The corresponding rate achieved is indicated in (7). However, for the purposes of analyzing the coarser metric of degrees of freedom, it suffices to choose the linear combination coefficients $d_k$ to zero-force the interference, i.e., to satisfy (12).

We next show that we can indeed design $D, T, F$ and $\Lambda$ so that the above condition is satisfied. For space considerations, we provide here only a sketch of the proof.

**Lemma 1.** There exist diagonal matrices $D, T, F$ and $\Lambda \in \mathbb{R}^{3 \times 3}$ such that $DH + HT + HFH = \Lambda, \det(\Lambda) \neq 0$ with probability 1.

**Proof:** Let $d_i, t_i, f_i, \lambda_i \in \mathbb{R}$ be the diagonal elements of $D, T, F$ and $\Lambda$ respectively and let

\[
\begin{align*}
\alpha &\triangleq [d_1, d_2, d_3, t_1, t_2, t_3, f_1, f_2, f_3]^T, \\
\lambda &\triangleq [\lambda_1, 0, 0, 0, \lambda_2, 0, 0, 0, \lambda_3]^T. 
\end{align*}
\]

Note that, for any fixed value of $\lambda$, the interference alignment condition $DH + HT + HFH = \Lambda$ is essentially a set
of linear equations in \(a\) with one equation corresponding to each entry of \(A\). Also, note that the condition \(\text{det}(A) \neq 0\) is equivalent to stating that \(\lambda_1 \lambda_2 \lambda_3 \neq 0\). Therefore, the conditions of the lemma can be equivalently written in terms of \(a\) and \(\lambda\) as

\[
\begin{cases}
\text{Ma} = \lambda \\
\lambda_1 \lambda_2 \lambda_3 \neq 0
\end{cases}
\]  

(14)

where

\[
M = 
\begin{bmatrix}
    h_{11} & 0 & 0 & h_{11} & 0 & 0 & h_{11}^2 & h_{12}h_{21} & h_{12}h_{23} \\
    h_{21} & 0 & h_{22} & 0 & h_{22} & 0 & h_{22}^2 & h_{23}h_{21} & h_{23}h_{23} \\
    0 & h_{31} & h_{31} & 0 & h_{31} & h_{31} & h_{31}^2 & h_{32}h_{21} & h_{32}h_{23} \\
    h_{12} & 0 & 0 & h_{12} & h_{12} & h_{12} & h_{12}^2 & h_{13}h_{21} & h_{13}h_{23} \\
    h_{22} & 0 & 0 & h_{22} & h_{22} & h_{22} & h_{22}^2 & h_{23}h_{21} & h_{23}h_{23} \\
    0 & h_{32} & h_{32} & 0 & h_{32} & h_{32} & h_{32}^2 & h_{33}h_{21} & h_{33}h_{23} \\
    h_{13} & 0 & 0 & h_{13} & h_{13} & h_{13} & h_{13}^2 & h_{13}h_{21} & h_{13}h_{23} \\
    0 & h_{33} & 0 & 0 & h_{33} & h_{33} & h_{33} & h_{33}^2 & h_{33}h_{21} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{33} & h_{33}
\end{bmatrix}
\]

For convenience, we can divide the 9 linear equations above into two parts: the first part contains 6 equations whose right-hand side should evaluate to 0, and the second part contains the remaining 3 equations whose right-hand side is equal to a non-zero value \(\lambda\). Denote as \(B \in \mathbb{R}^{6 \times 9}\) the rows of \(M\) that correspond to the first part, and let \(Q \in \mathbb{R}^{3 \times 9}\) be the matrix containing the remaining rows that correspond to \(\lambda_1, \lambda_2, \lambda_3\). Condition (14) becomes

\[
\begin{cases}
Ba = 0 \\
Qa = [\lambda_1, \lambda_2, \lambda_3]^T \\
\lambda_1 \lambda_2 \lambda_3 \neq 0
\end{cases}
\]

(15)

Let \(N(B)\) denote the nullspace of \(B\). By performing elementary row operations, it can be verified that the following matrix is a basis for the nullspace of \(B\).

\[
N = 
\begin{bmatrix}
    -1 & h_{22}(h_{21}h_{22} - h_{12}h_{23}) & h_{22}(h_{21}h_{22} - h_{12}h_{23}) & h_{12}h_{21}h_{23} & - h_{33} \\
    -1 & h_{32} & h_{32} & h_{12}h_{21} & - h_{33} \\
    -1 & 0 & h_{33} & h_{13} & - h_{33} \\
    1 & - h_{33} & h_{13} & h_{13} & h_{13} \\
    1 & 0 & h_{12} & h_{12} & h_{12} \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Any \(v \in N(B)\) can be written as \(Nw, w \in \mathbb{R}^3\). Condition (15) becomes

\[
\begin{cases}
w \in \mathbb{R}^3 \\
QNW = [\lambda_1, \lambda_2, \lambda_3]^T \\
\lambda_1 \lambda_2 \lambda_3 \neq 0
\end{cases}
\]

(16)

We have

\[
QN = 
\begin{bmatrix}
    0 & s_{12} & s_{13} \\
    0 & s_{22} & 0 \\
    0 & 0 & s_{33}
\end{bmatrix}
\]

(17)

where

\[
s_{12} = \frac{(h_{11}h_{23} - h_{13}h_{21})(h_{11}h_{32} - h_{12}h_{31})}{h_{13}h_{31}}
\]

(18)

\[
s_{13} = \frac{(h_{11}h_{23} - h_{13}h_{21})(h_{11}h_{32} - h_{12}h_{31})}{h_{12}h_{31}}
\]

(19)

\[
s_{22} = \frac{(h_{12}h_{23} - h_{13}h_{22})(h_{21}h_{23} - h_{22}h_{31})}{h_{13}h_{31}}
\]

(20)

\[
s_{33} = \frac{(h_{13}h_{32} - h_{12}h_{33})(h_{21}h_{33} - h_{23}h_{31})}{h_{12}h_{21}}
\]

(21)

If \(s_{22}, s_{33} \neq 0\), \(s_{12}s_{33} + s_{13}s_{22} \neq 0\), one can choose \(w^* = [1, 1/s_{22}, 1/s_{33}]^T\) in order to obtain

\[
QNw^* = \begin{bmatrix}
    s_{12} + s_{13} \\
    s_{22} \\
    s_{33}
\end{bmatrix}
\]

and

\[
\lambda_1 \lambda_2 \lambda_3 = \frac{s_{12}}{s_{22}} + \frac{s_{13}}{s_{33}} = \frac{s_{12}s_{33} + s_{13}s_{22}}{s_{22}s_{33}} \neq 0.
\]

Putting everything together, \(a^* = NW^*\) satisfies (14) if

\[
\begin{cases}
h_{ij} \neq 0, \forall i, j \\
h_{12}(h_{21}+h_{23}) \neq h_{23}(h_{11}+h_{12}) \\
s_{22}, s_{33} \neq 0, \\
s_{12}s_{33} + s_{13}s_{22} \neq 0
\end{cases}
\]

(22)

We have a solution when the conditions (22) are satisfied. Since these conditions are satisfied almost surely when \(h_{ij}\) are drawn from a continuous non-degenerate distribution, the lemma is proved.

\[\Box\]

C. Optimization at finite SNR

Interference alignment requires in (15) that \(Ba = 0\) and Lemma 1 guarantees that \(\lambda_1, \lambda_2, \lambda_3 \neq 0\) for some \(a_0 \in N(B)\) with probability 1. However, at finite SNR, satisfying condition (15) is not enough: It is more important to be able to control the magnitude of the effective channel gains \(|\lambda_i|^2\) and optimize performance under the given transmit power constraint.

1) Exact Interference Alignment: Maximizing the sum-rate under the exact interference alignment condition (15) corresponds to solving the following optimization problem

\[
Q1: \quad \text{maximize}_{a} \quad \sum_k \log(1 + P|q_k^Ta|^2/\sigma_k^2(a))
\]

subject to:

\[
Ba = 0
\]

\[
a \in P
\]

where \(q_k\) denotes the \(k\)th row of \(Q\), \(P = \{a \in \mathbb{R}^9 : (a_{\ell+3}, a_{\ell+6}) \in P_\ell, \ell = 1, 2, 3\}\) is the corresponding power constraint defined in (10) and \(\sigma_k^2(a)\) is the effective noise variance observed at the \(k\)th user that depends on \(a\) via (6).
As in most rate optimization problems, the main difficulty in solving $Q_1$ efficiently comes directly from the non-convexity of the objective. We relax this problem to maximizing $\sum |\lambda_i|^2 = ||Qa||^2$ subject to the same constraints.

Let $N \in \mathbb{R}^{3\times 3}$ denote a basis for the nullspace of $B$. A simple change of variables $a = Nw$, $w \in \mathbb{R}^3$ results in the following optimization problem

$$Q2: \maximize_{w \in P'} ||QNw||^2$$

where $P' = \{w \in \mathbb{R}^3 : Nw \in P\}$. Tracing back the definitions of $P$ and $P_r$, one can see that $P'$ can be written as the intersection of three ellipsoids – one for each transmit power constraint in (10). Even though maximizing a quadratic over the intersection of $k$ ellipsoids is still an intractable problem, it admits a natural semidefinite programming (SDP) relaxation with provable approximation guarantees [16].

A much simpler alternative comes directly from the singular value decomposition (SVD) of $S \triangleq QN$. One can first choose the direction of $w_0$ along the principal component of $S$, maximizing $||Sw||^2$ subject to $||w|| = 1$, and then scale by $c \in \mathbb{R}$ such that $c \cdot w_0 \in P'$.

2) A “max-SINR” approach: Even though interference alignment attains the optimal DoF, it can be too restrictive in the practical SNR range. Here, we remove the exact alignment requirement and propose a “departing direction” from the nullspace of $B$ (perfect IA) to its orthogonal complement in order to search for better solutions based on a max-SINR heuristic.

Let $R \in \mathbb{R}^{9\times 6}$ be a basis for the rowspace of $B$, denoted $\mathcal{R}(B^T)$. $a_N \in \mathcal{N}(B)$ be an interference alignment solution and $v \in \mathcal{R}(B^T)$ be a unit vector in an orthogonal direction. We would like to explore solutions of the form $a = a_N + \delta \cdot v$, for some $\delta \in \mathbb{R}$. Any direction $v$ we choose will definitely increase the aggregate interference power since we are moving away from alignment. In this context, we would like to choose $v$ in a “max-SINR” direction, that can favor useful signal more than interference.

A natural choice for a heuristic would be to seek for $v \in \mathcal{R}(B^T)$ that maximizes the ratio $||Qv||/||Bv||$. Writing $v = Rw$, $w \in \mathbb{R}^6$ we want to find

$$w^* = \arg \max_w \frac{||QRw||}{||BRw||}$$

$$= \arg \max_w \frac{w^T R^T Q^T QRw}{w^T R^T B^T BRw}. \quad (23)$$

This can be seen as a generalized eigenvector problem in which the optimal $w^*$ is given by the principal eigenvector of the matrix $(R^T B^T BR)^{-1} R^T Q^T QR$ and $v^* = Rw^*$.

Now, for any solution $a_N \in \mathcal{N}(B)$ we can use $v^*$ to search for rate maximizing solutions in their two dimensional span. In fact, one only needs to search for the “right” angle between $a_N$ and $v^*$. Let $u^* \triangleq a_N/||a_N||$. The candidate solutions to the rate optimization problem can be parametrized by $\theta \in [0, \pi]$ as

$$a(\theta) = \beta(\theta) [u^* \cos(\theta) + v^* \sin(\theta)] \quad (24)$$

where $\beta(\theta) \triangleq \sup \{c \in \mathbb{R}^+ : c[u^* \cos(\theta) + v^* \sin(\theta)] \in P\}$ is the appropriate scaling parameter to satisfy the power constraints and can be computed in closed form given the tractable structure of $P$. In view of (24), we can then write the achievable rate in (11) as a function of a single variable $\theta \in [0, \pi)$, denoted as $R_{sum}(a(\theta))$, and find

$$\theta^* = \arg \max_\theta R_{sum}(a(\theta)). \quad (25)$$

Notice that the above parametrization does not exclude perfect IA solutions ($\theta = 0$) and one expects that $\theta^*$ will go to zero as the SNR increases.

We propose next an efficient heuristic algorithm that can be used to optimize the sum rate of our feedback scheme across the entire SNR range. The algorithm takes as input the matrices $B \in \mathbb{R}^{6\times 9}$ and $Q \in \mathbb{R}^{3\times 9}$ (defined in the previous section) and computes the linear combination coefficients $t, f \in \mathbb{R}^3$ that will be used by the transmitters in the second time-slot.

**Algorithm: Max-SINR Feedback**

**Input:** matrices $B \in \mathbb{R}^{6\times 9}$, $Q \in \mathbb{R}^{3\times 9}$

**Output:** linear combination coefficients $t, f \in \mathbb{R}^3$

1) $N$ = basis for $\mathcal{N}(B)$, $R$ = basis for $\mathcal{R}(B^T)$
2) $w_N = \max$-eigenvector of $(QN)^T QN$;
3) $w_R = \max$-eigenvector of $(R^T B^T BR)^{-1} (QR)^T QR$;
4) $u^* = Nw_N/||Nw_N||$, $v^* = Rw_R/||Rw_R||$;
5) $\theta^* = \arg \max_{\theta} R_{sum}(\beta(\theta) [u^* \cos(\theta) + v^* \sin(\theta)])$;
6) $a = \beta(\theta^*) [u^* \cos(\theta^*) + v^* \sin(\theta^*)]$;
7) $t = \{a_4, a_5, a_6\}^T$, $f = \{a_7, a_8, a_9\}^T$;

**IV. NUMERICAL RESULTS**

In this section, we evaluate the performance of the proposed feedback scheme in terms of its average achievable sum rate. We consider real channel coefficients $H$ (that remain fixed for at least two time-slots) and compute the average sum rate over the ensemble of channel realizations with $h_{kk} \sim \mathcal{N}(0, 1)$. We compare with the following four schemes.

**Time Sharing:** Transmitters use the channel in a round robin fashion. The average achievable sum rate in this case is given by $R_{sum} = \sum_k \frac{1}{\mathcal{E}} \log(1 + 3P|h_{kk}|^2)$.

**Treat as Noise:** All the transmitters use the channel simultaneously and interference is treated as noise. The average sum rate is given by $R_{sum} = \sum_k \frac{1}{\mathcal{E}} \log \left( 1 + \frac{P|h_{kk}|^2}{\sum_{\ell \neq k} P|h_{\ell k}|^2} \right)$.

**Ergodic Alignment:** For time-varying channels, this scheme can achieve half the interference-free rate at any SNR by carefully pairing up channel matrices to cancel out interference [5]. We use the performance of this scheme as a benchmark, even though it is not achievable in our problem setting as the channel is static. The average sum rate is given by $R_{sum} = \sum_k \frac{1}{\mathcal{E}} \log(1 + 2P|h_{kk}|^2)$.
2-user Feedback: We consider a time sharing version of the two bit gap scheme proposed in \([9]\) for the two-user interference channel with feedback. Transmitters can use the channel in pairs and transmit with power \(3P/2\).

In our simulations, we plot the average achievable sum rates for all of the above schemes versus SNR measured in dB. Figure 1 shows the performance comparison in the standard setting where \(h_{k}\ell \sim \mathcal{N}(0,1)\). We observe that the proposed max-SINR feedback scheme gets – in two time slots – rates that are not too far from what ergodic IA could ultimately achieve across many independent channel realizations (if the channel were time-varying). Notice that the max-SINR heuristic is able to provide considerable gains across many independent channel realizations (if the channel were time-varying). Notice that the max-SINR heuristic is able to provide considerable gains in the low to medium SNR range while it maintains the the right slope as SNR increases.

Figure 2 shows the same comparison in a weaker interference power regime, where the cross channel gains are scaled such that \(10\log_{10}\mathbb{E}[|h_{k\ell}|^{2}] = -3\text{dB}\). In this setting, the performance gains obtained by feedback become more significant, especially for the max-SINR design that seems to achieve the right balance between useful signal and interference power.

![Fig. 1. Performance of the proposed feedback scheme. We assume real channel coefficients \(h_{k\ell} \sim \mathcal{N}(0,1)\) and we plot average sum rates.](image1)

![Fig. 2. Performance of the proposed feedback scheme with cross-channel gains at -3dB.](image2)

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