Spin-orbit coupling and the Landau level spectrum of ABA-stacked trilayer graphene

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Abstract. We review the tight-binding model of ABA-stacked trilayer graphene. The presence of mirror reflection symmetry of the crystal structure leads to a decomposition of the band structure into separate monolayerlike and bilayerlike parts. Owing to a lack of spatial inversion symmetry, next-nearest layer coupling induces gaps in the monolayerlike and the bilayerlike parts of the spectrum, and also shifts the monolayerlike and bilayerlike bands with respect to each other. We calculate the Landau level spectrum of ABA-stacked trilayer graphene taking into account terms within the single-particle picture that break level degeneracy including next-nearest layer coupling and spin-orbit coupling.

1. Introduction
Soon after the isolation of individual graphene flakes \cite{1}, observation of the integer quantum Hall effect in monolayers \cite{2, 3} and bilayers \cite{4} led to an explosion of interest in the nature of chiral charge carriers in them. The electronic spectrum of monolayer graphene \cite{5} consists of two Dirac cones with linear dispersion supporting chiral quasiparticles with Berry’s phase $\pi$. The Dirac cones are located at two inequivalent corners of the Brillouin zone, denoted $K_+$ and $K_-$ points, and also referred to as valleys \cite{6}. In bilayer graphene \cite{4, 7, 8}, the band structure is also gapless at the $K$ points with a parabolic dispersion supporting chiral quasiparticles corresponding to Berry’s phase $2\pi$.

In graphene monolayers and bilayers, both space and time inversion operations connect electronic states at $K_+$ to those at $K_-$ \cite{9, 10, 11}. Thus, the energy spectrum is degenerate between the two valleys, even when time reversal symmetry is broken by the presence of a magnetic field \cite{12}, as long as spatial inversion symmetry is not broken by an external field produced, for example, by an external gate or doping \cite{7, 8, 13, 14}. This holds for general ABA-stacked (Bernal) multilayer graphene composed of $N$ layers as long as $N$ is even (bilayer graphene corresponds to $N = 2$). The situation is different in ABA-stacked multilayer graphene with $N$ odd, beginning with trilayer graphene $N = 3$, because their crystal structures do not obey spatial inversion symmetry \cite{15, 11, 12, 16}.

Instead of spatial inversion symmetry, odd-$N$ multilayers possess mirror reflection symmetry of their crystal structure in the plane of their central layer. It can not guarantee valley degeneracy when time reversal symmetry is broken by a magnetic field, because it does not connect states at the two valleys, but it does allow the electronic Hamiltonian to be decomposed into separate
Figure 1. (a) Schematic of the crystal structure of ABA-stacked trilayer graphene with six atomic sites in the unit cell, $A$ (white circles) and $B$ (black circles) on each layer, showing the Slonczewski-Weiss-McClure parameterization [25] of couplings $\gamma_0-\gamma_5$. (b) Brillouin zone of trilayer graphene (shaded hexagon) showing momentum $\mathbf{p} = (p_x, p_y)$ measured with respect to the corner of the Brillouin zone $K_\pm$. Crosses indicate reciprocal lattice points.

monolayer- and bilayerlike parts [17, 18, 19, 20, 21, 22, 23, 24]. Each part approximately satisfies an effective inversion symmetry corresponding to that of the real lattice of monolayer or bilayer graphene, respectively, and this ensures approximate valley degeneracy of the electronic spectrum of odd-$N$ multilayers even for finite magnetic field [12]. Mirror symmetry is broken by interlayer asymmetry induced by an external electric field, which breaks valley degeneracy of the Landau level spectrum [12]. In this paper, we focus instead on intrinsic trilayer graphene ($N = 3$), when symmetry is not broken by an external electric field, and describe terms in the single-particle picture that break the effective inversion symmetry and the approximate valley degeneracy of the Landau level spectrum. They include next-nearest layer coupling and spin-orbit coupling. We begin with a review of the tight-binding model of ABA-stacked trilayer graphene in Section 2, followed by a discussion of the Landau level spectrum in the presence of next-nearest layer coupling in Section 3 and in the presence of spin-orbit coupling in Section 4.

2. Tight-binding model of ABA-stacked trilayer graphene

We consider ABA-stacked trilayer graphene to consist of three layers of carbon atoms, each arranged with a honeycomb crystal structure (as in monolayer graphene), as shown in Fig. 1. The unit cell consists of six atomic sites: pairs $\{A_1, B_1\}$, $\{A_2, B_2\}$, and $\{A_3, B_3\}$ in the bottom, center, and top layers, respectively. The relative arrangement of the layers is Bernal ($A-B$) stacking [25], in that half of the atomic sites have a counterpart directly above or below them in the adjacent layer: $B_1$ is directly below $A_2$, and $A_2$ is directly below $B_3$.

Our tight-binding model [7, 17, 24] takes into account a single $p_z$ orbital per atomic site. There are six orbitals per unit cell and six bands in the electronic band structure. We adapt the Slonczewski-Weiss-McClure parameterization [25, 26] of couplings between atomic sites including $\gamma_0$ which describes nearest-neighbor ($Ai-Bi$ for $i = 1, 2, 3$) coupling within each layer, $\gamma_1$ which describes strong nearest-layer coupling between sites ($B_1-A_2$ and $A_2-B_3$) that lie directly above or below each other, and $\gamma_3$ ($\gamma_4$) which describes weaker nearest-layer coupling between sites $A_1-B_2$ and $B_2-A_3$ ($A_1-A_2$, $B_1-B_2$, $A_2-A_3$, and $B_2-B_3$). If the model would include only these couplings, it would satisfy an effective inversion symmetry (as explained in detail later) guaranteeing the presence of a degeneracy point at each of two inequivalent corners, $K_{\pm}$, of the hexagonal Brillouin zone [6]. However, the effective inversion symmetry and the degeneracy tends to be broken by next-nearest-layer coupling $\gamma_2$ (between $A_1$ and $A_3$) and $\gamma_5$ (between $B_1$
and $B3$, as well as parameter $\delta$ describing a difference of on-site energy between sites $A1$, $B2$, $A3$, and $B1$, $A2$, $B3$. Parameter $\Delta$ that is often used in models of bulk graphite [25] is related to $\delta$ by $\Delta = \delta + \gamma_2 - \gamma_3$.

In a basis with components $\psi_{A1}$, $\psi_{B1}$, $\psi_{A2}$, $\psi_{B2}$, $\psi_{A3}$, $\psi_{B3}$, the ABA-stacked trilayer Hamiltonian [17, 24, 27, 28] is

\[
\tilde{H} = \begin{pmatrix}
U_1 & v\pi \dagger & -v_4 \pi \dagger & v_3 \pi & \gamma_2/2 & 0 \\
v\pi & U_1 + \delta & \gamma_1 & -v_4 \pi \dagger & 0 & \gamma_5/2 \\
v_4 \pi \dagger & -v_4 \pi & U_2 + \delta & v\pi \dagger & -v_4 \pi & \gamma_1 \\
v_3 \pi \dagger & -v_4 \pi & v\pi & U_2 & v_3 \pi \dagger & -v_4 \pi \\
\gamma_2/2 & 0 & -v_4 \pi \dagger & v_3 \pi & U_3 & v\pi \dagger \\
0 & \gamma_5/2 & \gamma_1 & -v_4 \pi \dagger & v\pi & U_3 + \delta
\end{pmatrix},
\]

(1)

where $\pi = \xi p_x + i p_y$ and $\pi^\dagger = \xi p_x - i p_y$ are operators related to the in-plane momentum $p = (p_x, p_y)$ measured with respect to the $K$ point [6], and $\xi = \pm 1$ is the valley index. Effective velocities are defined as $v = (\sqrt{3}/2)a\gamma_3/\hbar$, $v_3 = (\sqrt{3}/2)a\gamma_3/\hbar$, and $v_4 = (\sqrt{3}/2)a\gamma_4/\hbar$, and $U_i = -eV_i$ describe different on-site energies in each layer $i = \{1, 2, 3\}$ corresponding to different layer potentials $V_i$ created by external fields.

Mirror reflection symmetry of the lattice in the plane of its central layer suggests that a basis involving linear combinations of orbitals from the top and bottom layers would be convenient [20, 21, 22, 24]. There are a pair that are odd with respect to mirror reflection symmetry,

\[
\phi_{1,m} = (\psi_{A1} - \psi_{A3})/\sqrt{2},
\]

(2)

\[
\phi_{2,m} = (\psi_{B1} - \psi_{B3})/\sqrt{2},
\]

(3)

and four that are even with respect to mirror reflection symmetry,

\[
\phi_{1,b} = (\psi_{A1} + \psi_{A3})/\sqrt{2},
\]

(4)

\[
\phi_{2,b} = \psi_{B2},
\]

(5)

\[
\phi_{3,b} = \psi_{A2},
\]

(6)

\[
\phi_{4,b} = (\psi_{B1} + \psi_{B3})/\sqrt{2}.
\]

(7)

Then, in a basis with components $\phi_{1,m}$, $\phi_{2,m}$, $\phi_{1,b}$, $\phi_{2,b}$, $\phi_{3,b}$, $\phi_{4,b}$, the ABA-stacked trilayer Hamiltonian [24] is

\[
H = \begin{pmatrix}
H_m & D \\
D^\dagger & H_b
\end{pmatrix},
\]

(8)

\[
H_m = \begin{pmatrix}
\Delta_2 - \gamma_2/2 & v\pi^\dagger \\
v\pi & \Delta_2 - \gamma_5/2 + \delta
\end{pmatrix},
\]

\[
H_b = \begin{pmatrix}
\Delta_2 + \gamma_2/2 & \sqrt{2}v_3 \pi & -\sqrt{2}v_3 \pi \dagger \\
\sqrt{2}v_3 \pi \dagger & -2\Delta_2 & v\pi \dagger \\
v\pi & -\sqrt{2}v_4 \pi & -2\Delta_2 + \delta & \sqrt{2}\gamma_1
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
\Delta_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta_1
\end{pmatrix}.
\]

(9)

In this Hamiltonian we use two parameters to take into account differences in the on-site energies $U_1$, $U_2$, $U_3$ of the three layers [24]:

\[
\Delta_1 = (U_1 - U_3)/2,
\]

(10)

\[
\Delta_2 = (U_1 - 2U_2 + U_3)/6.
\]
and we set the average on-site energy \([U_1 + U_2 + U_3]/3\) to zero. The first parameter, \(\Delta_1\), describes the average energy difference between the outer layers [29, 17, 30, 24, 12], while the second, \(\Delta_2\), describes the difference between the energy of the central layer as compared to the average energy of the outer layers. Parameter \(\Delta_1 (\Delta_2)\) is odd (even) with respect to mirror reflection.

The Hamiltonian Eq. (8) has a two by two block \(H_m\) written in a basis \(\phi_{1,m}, \phi_{2,m}\) of orbitals that are odd with respect to mirror reflection. It is similar to the Dirac-like Hamiltonian of monolayer graphene, and it contributes two monolayerlike bands near zero energy. There is a four by four block \(H_b\) with components \(\phi_{1,b}, \phi_{2,b}, \phi_{3,b}, \phi_{4,b}\) that are even with respect to mirror reflection. It is similar to the Hamiltonian of bilayer graphene [7] except that terms proportional to \(\gamma_1, \gamma_3, \gamma_4\) describing interlayer coupling appear with a factor \(\sqrt{2}\). Block \(H_b\) gives four bilayerlike bands: two bands near zero energy and two bands split away from zero by energy \(\epsilon \approx \pm \sqrt{2}\gamma_1\) at the K point. Parameters \(\gamma_3\) and \(\gamma_4\) have a similar effect as in bilayer graphene [7], with \(\gamma_3\) giving trigonal warping of the bands and \(\gamma_4\) contributing to electron-hole asymmetry.

The only parameter in the Hamiltonian Eq. (8) that is odd with respect to mirror reflection symmetry is the asymmetry \(\Delta_1\) between the outer layers [29, 17, 30, 24, 12]. Therefore, it is the only parameter that appears in the off-diagonal blocks \(D\). It connects the odd and even parts of the Hamiltonian, \(H_m\) and \(H_b\), and hybridizes the monolayerlike and bilayerlike bands. The electronic spectrum for finite \(\Delta_1\) has been discussed previously for zero [29, 17, 30, 24, 27, 28] and finite magnetic field [12], and we refer the reader there for further details.

For \(\Delta_1 = 0\), the odd and even parts of the Hamiltonian Eq. (8) are separate giving superimposed monolayerlike and bilayerlike bands. Furthermore, for \(\gamma_2 = \gamma_5 = \delta = \Delta_2 = 0\), the monolayer part and the bilayerlike part assume an effective inversion symmetry corresponding to that of the real lattice of monolayer or bilayer graphene, respectively. This symmetry, combined with time-reversal symmetry, guarantees degeneracy of the spectrum at the K point. Finite \(\gamma_2, \gamma_5, \delta, \) and \(\Delta_2\) break the effective inversion symmetry, and their effect is to induce gaps in the monolayerlike and the bilayerlike parts of the spectrum, but also to shift the monolayerlike and bilayerlike bands with respect to each other. At the K point (\(p = 0\)), and for \(\Delta_1 = 0\), the band energies are \(\epsilon = \Delta_2 - \gamma_2/2\) and \(\epsilon = \Delta_2 - \gamma_5/2 + \delta\) for the monolayer part, and \(\epsilon = \Delta_2 + \gamma_2/2\) and \(\epsilon = -2\Delta_2\) for the bilayer part (as well as the two split bands at \(|\epsilon| \approx \sqrt{2}\gamma_1\)).

### 3. Landau level spectrum in the presence of next-nearest layer coupling

In order to focus on the low-energy electronic spectrum, we perform a Schrieffer-Wolff transformation [31, 7] to write an effective four-component Hamiltonian describing the four bands that approach zero energy. This eliminates components \(\phi_{3,b}, \phi_{4,b}\) corresponding to atomic sites that are strongly coupled by \(\gamma_1\) and produce the split bands at \(\epsilon \approx \pm \sqrt{2}\gamma_1\) in the bilayerlike spectrum. Then, in basis with components \(\phi_{1,m}, \phi_{2,m}, \phi_{1,b}, \phi_{2,b}\), the approximate four-component ABA-stacked trilayer Hamiltonian [12, 32] is

\[
\begin{align*}
  h &= \begin{pmatrix} h_m & d \\ d^\dagger & h_b \end{pmatrix}, \\
  h_m &= \begin{pmatrix} a_m + b_m & v^\pi \\ v^\pi & a_m - b_m \end{pmatrix}, \\
  h_b &= \begin{pmatrix} a_b + b_b & 0 \\ 0 & a_b - b_b \end{pmatrix} + \frac{v^2}{2\gamma_1^2} \begin{pmatrix} (r_b + s_b) \pi^\dagger \pi & 0 \\ 0 & (r_b - s_b) \pi \pi^\dagger \end{pmatrix} \\
  & - \frac{(v^2 + 2v_4^2)}{\sqrt{2}\gamma_1} \begin{pmatrix} 0 & (\pi)^2 \\ \pi^2 & 0 \end{pmatrix} + \sqrt{2}v_3 \begin{pmatrix} 0 & \pi^\dagger \\ \pi & 0 \end{pmatrix}, \\
  d &= \Delta_1 \begin{pmatrix} 1 \\ v_4 \pi/\gamma_1 \\ -v_4 \pi^\dagger/\sqrt{2}\gamma_1 \\ 0 \end{pmatrix},
\end{align*}
\]
where

\[ a_m = \Delta_2 - (\gamma_2 + \gamma_5)/4 + \delta/2, \]
\[ b_m = (\gamma_5 - \gamma_2)/4 - \delta/2, \]
\[ a_b = -\Delta_2/2 + \gamma_2/4, \]
\[ b_b = 3\Delta_2/2 + \gamma_2/4, \]
\[ r_b = (\gamma_5 - \gamma_2)/4 + \delta + 4\gamma_1 v_4/v, \]
\[ s_b = -3\Delta_2 - (\gamma_2 + \gamma_5)/4. \]

We assume \( |\varepsilon|, |vp|, |\Delta_1|, |a_m|, |b_m|, |a_b|, |b_b| \) \( \ll |\gamma_1| \) and generally keep terms in the four-component Hamiltonian \( h \) up to order \( 1/\gamma_1 \) [26]. Terms in the two-component bilayerlike part \( h_b \) are analogous to those in the two-component Hamiltonian of bilayer graphene [7, 33]. The first term in Eq. (13) indicates the role of next-layer coupling in shifting the bilayer bands with respect to the monolayer bands (\( a_b \) term) and in opening a gap in the bilayerlike spectrum (\( b_b \) term). Although it is of order \( 1/\gamma_1^2 \), the second term in Eq. (13) is kept because it describes electron-hole asymmetry (\( r_b \) term) and weak degeneracy breaking of the bilayerlike Landau levels (\( s_b \) term). The quadratic term in Eq. (14) dominates over a broad range of energy \( |\varepsilon| \ll |\gamma_1| \) and the linear term Eq. (14) describes trigonal warping of the bands. For simplicity, we neglect \( \gamma_3 \) and \( \gamma_4 \) in the following.

As mentioned above, the spectrum for finite \( \Delta_1 \) has been discussed previously for zero [29, 17, 30, 24, 27, 28] and finite magnetic field [12]. Here, we focus on \( \Delta_1 = 0 \). The odd and even (with respect to mirror reflection) parts of the Hamiltonian produce separate monolayerlike \( \epsilon_m \) and bilayerlike \( \epsilon_b \) bands. At zero magnetic field, valley degeneracy is preserved [32],

\[ \epsilon_m = a_m \pm \sqrt{b_m^2 + v^2 p^2}, \]
\[ \epsilon_b \approx a_b \pm \sqrt{b_b^2 + v^4 p^4/(2\gamma_1^2)}, \]

where we neglected small contributions from the terms containing \( r_b \) and \( s_b \). The parameters \( a_m/b \) produce shifts of the monolayerlike and bilayerlike spectra with respect to each other,
that are spin degenerate. The monolayerlike levels $\epsilon_{n}^{(n)}$ given by

$$\epsilon_{m}^{(n)} = a_m \pm \sqrt{b_m^2 + n\Gamma^2},$$

$$\epsilon_{b}^{(n)} \approx a_b + \xi b_{1} \frac{\Gamma^2}{4\gamma_1^2} \pm \sqrt{b_b^2 + n(n+1)\Gamma^2 \frac{2\gamma_1^2}{n^2}},$$

are spin degenerate. The monolayerlike levels $\epsilon_{m}^{(n)}$ are valley degenerate, but parameter $s_b$ introduces weak splitting of the bilayerlike levels between valleys $K_+$ ($\xi = +1$) and $K_-$ ($\xi = -1$). Here, $\Gamma = \sqrt{2\hbar v^2 e B} = \sqrt{2\hbar v/\lambda_B}$ is an energy scale related to the inverse of the magnetic length, $\lambda_B = \sqrt{\hbar/(eB)}$, and we consider $\{|e|, \sqrt{n} \Gamma, |a_m/b|, |b_m/b|, |b_b|, |s_b| \ll |\gamma_1|$. There are three “zero” energy levels per valley and spin component, hence twelve in total [17].

In the presence of next-layer coupling, we find their level splitting to be given by

$$\epsilon_{m}^{(0)} = a_m + \xi b_m,$$

$$\epsilon_{b}^{(0)} = a_b + \xi b_b + (r_b + \xi s_b) \frac{\Gamma^2}{2\gamma_1^2},$$

$$\epsilon_{b}^{(-1)} = a_b + \xi b_b.$$

Valley degeneracy is broken more strongly for the “zero” energy levels, Eqs. (26-28), by parameters $b_{m/b}$, than it is for the higher levels. As in bilayer graphene [7], there is also a
values are \( v \) indicate level degeneracy, including spin and valley degeneracy. In both plots, other parameter values are \( v = 1.0 \times 10^6 \text{ms}^{-1} \), \( \gamma_1 = 0.39\text{eV} \), \( \gamma_2 = \gamma_4 = \Delta_1 = \Delta_2 = \delta = 0 \). Plots were made using approximations Eqs. (22), (23) and Eqs. (24-28) including, for clarity, only levels up to \( n = 8 \).

weak splitting of the bilayerlike levels \( \epsilon_b^{(0)} \) and \( \epsilon_b^{(-1)} \). The spectrum Eqs. (24-28) is plotted in Fig. 2(b) for \( \gamma_2 = \gamma_5 = 0 \) (and \( a_m = b_m = a_b = b_b = 0 \) where numbers in parenthesis indicate level degeneracy. Fig. 3(b) shows the spectrum for \( \gamma_2 = -0.020\text{eV} \), \( \gamma_5 = +0.038\text{eV} \) (the zero-field band structure has a gap, Fig. 3(a)) and fig. 4(b) shows the spectrum for \( \gamma_2 = -0.020\text{eV} \), \( \gamma_5 = -0.038\text{eV} \) (the zero-field band structure is gapless, Fig. 4(a)). In Figs. 3 and 4, valley degeneracy splitting of the “zero” energy levels can be seen for finite \( \gamma_2 \) and \( \gamma_5 \), but level splitting due to \( s_b \) and \( r_b \) is not resolved. The energies of the “zero” energy levels correspond to the band energies at the \( K \) point \((p = 0)\) for zero field. The monolayerlike levels are only two-fold (spin) degenerate, reading \( \epsilon^{(0)} = a_m + b_m = \Delta_2 - \gamma_2/2 \) at valley \( K_+ \), and \( \epsilon^{(0)} = a_m - b_m = \Delta_2 - \gamma_5/2 + \delta \) at \( K_- \). The “zero” energy bilayerlike levels are almost four-fold degenerate, \( \epsilon_b^{(0)} \approx \epsilon_b^{(-1)} = a_b + b_b = \Delta_2 + \gamma_2/2 \) at \( K_+ \), and \( \epsilon_b^{(0)} \approx \epsilon_b^{(-1)} = a_b - b_b = -2\Delta_2 \) at \( K_- \).

4. Landau level spectrum in the presence of spin-orbit coupling

In this section, we consider the influence of intrinsic spin-orbit terms that exist in samples in the absence of external electric field, e.g. we neglect the influence of a transverse electric field that breaks reflection symmetry and produces an additional, Rashba-like term [34, 35, 36, 37, 38]. In monolayer graphene, the only spin-orbit term that exists at the center of each valley in the absence of external fields was introduced by Kane and Mele [34],

\[
\mathcal{H}_{KM} = \alpha \sigma_z S_z ,
\]

(29)

where \( \alpha \) is a parameter and Pauli matrices \( \Pi_z \), \( \sigma_z \), \( S_z \) act in \( K_+ / K_- \) valley, \( A / B \) lattice, and \( 1 / 1 \) spin space, respectively. This terms couples the out-of-plane component of electronic spin with the out-of-plane component of pseudospin. It doesn’t break spin and valley degeneracy, but it breaks the \( A / B \) sublattice symmetry, opening a band gap and realizing a new, topological state of matter, a quantum spin Hall insulator [34].

In trilayer graphene, in addition to the Kane-Mele term Eq. (29), there is a second type of intrinsic spin-orbit coupling present at low-energy at the center of the valley [32],

\[
\mathcal{H}_{ABA} = \beta \Pi_z S_z .
\]

(30)
It is allowed because of the lack of spatial inversion symmetry of the lattice. It couples the out-of-plane component of electronic spin with the out-of-plane component of the valley ‘spin’ degree of freedom. This “valley-Zeeman” effect produces Zeeman-like spin splitting of the energy bands at each valley, of magnitude proportional to the parameter $\beta$, with an opposite sign of the effective Zeeman field in the two valleys.

In the presence of these spin-orbit terms, the approximate four-component ABA-stacked trilayer Hamiltonian Eq. (11) is modified [32]. Per spin and valley, the monolayerlike block, in basis $\phi_{1,m}, \phi_{2,m}$, becomes

$$
\begin{aligned}
\hat{h}_m &= \left( a_m + b_m + s\xi_\alpha m + s\xi_\beta m \quad \frac{v_\pi}{v_\pi} \right),
\end{aligned}
$$

(31)

where the bilayerlike block of Eq. (11), in basis $\phi_{1,b}, \phi_{2,b}$, is

$$
\begin{aligned}
\hat{h}_b &= \left( a_b + b_b + s\xi_\alpha b + s\xi_\beta b \quad -v^2 \left( \frac{\pi}{2} \right)^2 / (2\gamma_1) \right),
\end{aligned}
$$

(32)

while the bilayerlike block of Eq. (11), in basis $\phi_{1,b}, \phi_{2,b}$, is

$$
\begin{aligned}
\hat{h}_b &= \left( a_b + b_b + s\xi_\alpha b + s\xi_\beta b \quad -v^2 \pi^2 / (\sqrt{2}\gamma_1) \right),
\end{aligned}
$$

(32)

where $s = \pm 1$ labels spin components, $\xi = \pm 1$ valleys. Parameters $\alpha_{m/b}$ indicate the strength of the Kane-Mele term Eq. (29) in the monolayerlike and bilayerlike parts of the Hamiltonian, respectively, and $\beta_{m/b}$ indicate the strength of the “valley-Zeeman” term Eq. (30). For simplicity, we neglect parameters $r_b$ and $s_b$ in Eq. (32). At zero magnetic field [32],

$$
\begin{aligned}
\epsilon_m &= a_m + s\xi_\beta m + \sqrt{(b_m + s\xi_\alpha m)^2 + v^2 p^2}, \\
\epsilon_b &= a_b + s\xi_\beta b + \sqrt{(b_b + s\xi_\alpha b)^2 + v^4 p^4 / (2\gamma_1^2)},
\end{aligned}
$$

(33, 34)

where $\{\epsilon_{m/b}, v_p, |a_{m/b}|, |b_{m/b}|, |\alpha_{m/b}|, |\beta_{m/b}|\} \ll \gamma_1$. The Kane-Mele spin-orbit term, responsible for factors $\alpha_{m/b}$, has been discussed in detail elsewhere [34, 39, 40, 36, 37, 41, 42, 43, 44, 45, 46, 47, 48]. Here, we note that, in trilayers, it can conspire with the $b_{m/p}$ parameters to break spin degeneracy even at zero magnetic field.

In the presence of a finite perpendicular magnetic field $B$, the Landau levels $n \geq 1$ are given by

$$
\begin{aligned}
\epsilon^{(n)}_m &= a_m + s\xi_\beta m + \sqrt{(b_m + s\xi_\alpha m)^2 + v^2 p^2 + n\Gamma^2}, \\
\epsilon^{(n)}_b &= a_b + s\xi_\beta b + \sqrt{(b_b + s\xi_\alpha b)^2 + v^4 p^4 / (2\gamma_1^2) + n(n+1)\Gamma^2},
\end{aligned}
$$

(35, 36)

where $\{\epsilon_{m/b}, \sqrt{n\Gamma}, |a_{m/b}|, |b_{m/b}|, |\alpha_{m/b}|, |\beta_{m/b}|\} \ll \gamma_1$. There are three “zero” energy levels per valley and spin component (hence twelve in total) given by

$$
\begin{aligned}
\epsilon^{(0)}_m &= a_m + s\xi_\beta m, \\
\epsilon^{(-1)}_b &= \epsilon^{(0)}_b = a_b + s\xi_\beta b + s\xi_\alpha b + s\xi_\beta b.
\end{aligned}
$$

(37, 38)

Parameters $\beta_{m/b}$ arise from the “valley-Zeeman” spin-orbit term unique to odd-$N$ multilayers and they describe a Zeeman-like spin splitting of the energy bands at each valley, with an opposite sign of the effective Zeeman field in the two valleys. A crude estimate, using the tight-binding model, found $|\alpha_{m/b}| = |\beta_{m/b}| \sim 0.1\text{meV}$ [32]. Using $\alpha_m = \alpha_b = \beta_m = -\beta_b \equiv \alpha$ (with a large value $\alpha = 2\text{meV}$ chosen for illustrative purposes), the electronic spectrum in the presence of spin-orbit in ABA-stacked trilayer graphene is plotted in Fig. 5, neglecting the effect of next-layer coupling that was discussed in Section 3.
Figure 5. The low-energy spectrum in the presence of spin-orbit coupling showing (a) energy bands in the vicinity of the $K_+$ point at zero magnetic field and (b) Landau level spectrum, taking into account both valleys. Numbers in parenthesis indicate level degeneracy, including spin and valley degeneracy. In both plots, $a_m = a_b = b_m = b_b = 0$, $\alpha_m = \alpha_b = \beta_m = -\beta_b \equiv \alpha$ with the value $\alpha = 2\text{meV}$ chosen for illustrative purposes. Other parameter values are $v = 1.0 \times 10^6 \text{ms}^{-1}$, $\gamma_1 = 0.39 \text{eV}$, $\gamma_3 = \gamma_4 = \Delta_1 = \Delta_2 = \delta = 0$. Plots were made using the analytic approximations Eqs. (33), (34) and Eqs. (35-38) including, for clarity, only levels up to $n = 8$.

5. Conclusions
In this paper, we calculated the Landau level spectrum of ABA-stacked trilayer graphene, showing how degeneracy of the levels is affected by next-nearest layer coupling and spin-orbit coupling. We considered effects within the single-particle picture, neglecting spontaneous symmetry breaking driven by electron-electron interactions that is likely to occur in quantum Hall states of trilayer graphene in a similar fashion as monolayer \cite{10, 49, 50, 51, 52, 53, 54} and bilayer \cite{55, 56} graphene.

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We consider ABA-stacked trilayer graphene in this paper. Note that ABC-stacked trilayer graphene has inversion symmetry, guaranteeing valley degeneracy of its Landau levels in the single-particle picture.