Lifshitz black holes with a time-dependent scalar field in Horndeski theory

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In arbitrary dimensions, there has been an intense activity to promote the ideas underlying the gauge-gravity duality to non-relativistic physics. The hope is to gain a better understanding of some strongly coupled condensed matter physics phenomena observed in laboratories, for a review see e. g. [1]. In this context, the so-called Schrödinger or Lifshitz spacetimes are the natural candidates to the be the gravity duals for non-relativistic scale invariant theories, [2–4]. In the present work, we are concerned with the Lifshitz spacetimes given by

\[ ds^2 = -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2_{D-2}, \tag{1} \]

where the dynamical exponent \( z \) reflects the anisotropy of the scaling symmetry

\[ t \rightarrow \lambda^2 t, \quad r \rightarrow \frac{r}{\lambda}, \quad \vec{x} \rightarrow \lambda \vec{x}. \]

Here, \( \vec{x} \) denotes a \((D-2)\)-dimensional vector. As it is now well-known, for \( z \neq 1 \), in order for the Einstein gravity to accommodate the Lifshitz spacetime, some extra matter source is required, like \( p \)-form gauge fields or some Proca model, see e. g. [2]. There also exists the option of considering higher-order gravity theories for which there exist examples of Lifshitz black holes without source, see e. g. [6, 7]. By Lifshitz black holes, we mean a black hole geometry whose asymptotic behavior reproduces the Lifshitz spacetime [1]. In this work, we deal with a source of the Einstein equations given by a real scalar field with its usual kinetic term together with a nonminimal kinetic coupling. More precisely, we consider the following \( D \)-dimensional action

\[ S = \int \sqrt{-g} d^D x \left( R - 2\Lambda - \frac{1}{2} \left( \alpha g_{\mu\nu} - \eta G_{\mu\nu} \right) \nabla^2 \phi \phi R \right). \tag{2} \]

where \( R \) and \( G_{\mu\nu} \) stand respectively for the Ricci scalar and the Einstein tensor. This model is part of the so-called Horndeski action which is the most general tensor-scalar action yielding at most to second-order field equations in four dimensions [5]. The action also enjoys the shifting symmetry \( \phi \rightarrow \phi + \text{const.} \). The field equations obtained by varying the action with respect to the two dynamical fields \( g_{\mu\nu} \) and \( \phi \) read

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2} \left[ \alpha T^{(1)}_{\mu\nu} + \eta T^{(2)}_{\mu\nu} \right], \tag{3a} \]

\[ \nabla_\mu \left[ (\alpha g^{\mu\nu} - \eta G^{\mu\nu}) \nabla_\nu \phi \right] = 0, \tag{3b} \]

where the stress tensors \( T^{(i)}_{\mu\nu} \) are defined by

\[ T^{(1)}_{\mu\nu} = \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \phi \nabla^\lambda \phi \right), \tag{4} \]

\[ T^{(2)}_{\mu\nu} = \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi R - 2 \nabla_\lambda \phi \nabla_\mu \phi R^{\lambda}_\nu - \nabla^\lambda \phi \nabla^\rho \phi R_{\mu\lambda\nu\rho} - \left( \nabla_\mu \nabla^\lambda \phi \right) \left( \nabla_\nu \nabla_\lambda \phi \right) \nabla^\phi \phi + \frac{1}{2} g_{\mu\nu} \nabla^2 \phi - g_{\mu\nu} \left( - \frac{1}{2} \left( \nabla^\lambda \phi \right) \left( \nabla_\nu \phi \right) \nabla_\lambda \phi + \frac{1}{2} \left( \nabla^2 \phi \right)^2 - \nabla_\lambda \phi \nabla^\lambda \phi R^{\lambda}_{\mu\nu} \right). \]

The first exact black hole solution of these equations without cosmological constant was found in [9]. However, in this case, the scalar field solution becomes imaginary outside the horizon. Recently, this problem has been circumvented by adding a cosmological constant term yielding to asymptotically locally \( (\Lambda) dS \) (and even flat for \( \alpha = 0 \)) black hole solutions with a real scalar field outside the horizon [10]. The field equations [3] admit other interesting solutions with a nontrivial and regular time-dependent scalar field on a static and spherically symmetric spacetime [11]. Interestingly enough, this solution in the particular case of \( \Lambda = \eta = 0 \) reduces to an unexpected stealth configuration on the Schwarzschild metric [11].

Because of the anisotropy symmetry, the \((D-2)\)-dimensional base manifold of Lifshitz spacetimes [1] is planar. For this reason, we will restrict ourselves in looking for black hole solutions whose horizon topology is planar. Moreover, in order to escape from the no-hair theorem established in [12], we will also impose by hand that the radial component of the conserved current vanishes identically without restricting the radial dependence of the
scalar field, that is
\[ \alpha g^{rr} - \eta G^{rr} = 0. \] (5)

Note that in all the references previously cited \[8,11\], the different authors also consider this restriction that simplifies the field equations. Under these two hypotheses (planar base manifold and the condition \[11\]), we will see that the only solution with a static radial scalar field is a planar AdS black hole already reported in Ref. \[10\]. Nevertheless, in order to extend the space of admissible planar black hole solutions, we turn on the time dependence of the scalar field. In this case, the condition \[11\] will impose that the time dependence of the scalar field is linear. In doing so, we will effectively obtain a Lifshitz black hole solution with a linear time-dependent scalar field for a specific value of the dynamical exponent \( z = \frac{1}{3} \). The plan of the paper is organized as follows. In the next section, we provide the general analysis for a static and radial scalar field, and see that the only solution with a planar base manifold is the AdS black hole solution obtained in \[10\]. In Sec. III, we turn on the time dependence of the scalar field and construct a Lifshitz black hole solution with a time-dependent scalar field characterized by a dynamical exponent \( z = \frac{1}{3} \). The last section is devoted to our conclusions.

II. GENERAL ANALYSIS WITH A STATIC SCALAR FIELD

Let us consider the following Ansatz
\[ ds^2_{D,\gamma} = -h(r)dt^2 + \frac{dt^2}{f(r)} + r^2d\Omega^2_{D-2,\gamma}, \quad \phi = \phi(r) \] (6)
where \( d\Omega^2_{D-2,\gamma} \) represents the line element of a \((D-2)\)-dimensional sphere, plane or hyperboloid which corresponds respectively to \( \gamma = 1.0 \) or \( \gamma = -1.0 \). Here, we are mainly interested on the planar case \( \gamma = 0 \) but we prefer to keep this general form in order to stress the particularity of considering the planar case \( \gamma = 0 \). In this case, the condition on the radial component of the current conservation \[11\] permits to relate the two metric functions as
\[ f = h \left( \frac{2\alpha r^2 + (D-3)(D-2)\gamma \eta}{(D-2)\eta [h' r + (D-3) h]} \right). \] (7)

We are now in position to solve the Einstein equations \[11\]: their \((r, r)\) component allows to express \( \psi_{\text{static}} := \phi' \) as
\[ \psi(r)_{\text{static}}^2 = -\frac{4 r^2 (\Lambda \eta + \alpha)(D-2) [r h' + h (D-3)]}{[2 \alpha r^2 + (D-2)(D-3) \eta \gamma]^2 h} \].

Substituting these two expressions in the remaining independent Einstein equations, that is the \((t, t)\) or \((i, i)\) component, one obtains a second order differential equation for the metric function \( h \). Under the following substitution
\[ h(r) = -\frac{\mu}{r^{D-3}} + \frac{2}{r^{D-3}} \int \frac{j(r) dr}{2\alpha r^2 + \gamma (D-2)(D-3)}. \] (8)

where \( \mu \) is an integration constant, the differential equation becomes a third-order algebraic equation for the function \( j(r) \),
\[ \varepsilon_{\text{static}} := j \left[ r^2 (\Lambda \eta - \alpha) - \gamma \eta (D-2)(D-3) \right] \]
\[ + \frac{C_0 j^{3/2}}{r^{2-\alpha}} = 0, \] (9)

where \( C_0 \) is a second integration constant. From this last expression, it is clear that in the case \( \gamma = 0 \), the only possibility for the function \( j \), apart from the trivial case \( j = 0 \) which is of little interest, is for \( j \propto r^D \) whose full integration yields the planar AdS black hole solution reported previously in \[10\]. It is also interesting to stress from the expression \[9\] that in the planar case \( \gamma = 0 \), the point \( \alpha = \eta A \) is degenerate in the sense that it will impose \( C_0 = 0 \), and in turn, the function \( j \) will be undetermined. Hence, for \( \gamma = 0 \) and for \( \alpha = \eta A \), any metric functions \( f \) and \( h \) satisfying the constraint \[2\] will be solution of the field equations.

In what follows, we will see that a way in order to extend the space of admissible planar black hole solutions is to turn on the time dependence of the scalar field.

III. LIFSHITZ BLACK HOLE WITH A TIME-DEPENDENT SCALAR FIELD

As seen previously, Lifshitz spacetimes \[11\] with a static radial scalar field can not source the particular Horndeski considered here \[2\]. Inspired by the work done in \[11\], we wonder whether there exist Lifshitz static black hole solutions with a nontrivial time-dependent scalar field \( \phi = \phi(t, r) \) satisfying the field equations \[3\]. In doing so, we consider again the same Ansatz for the metric \[6\]. We firstly note that the \((t, r)\) component of the Einstein equations gives
\[ \left\{ \begin{array}{l} 2 (D-2) \eta \phi' f h r - \phi \left[ (\eta (D-2)(D-3) (f - \gamma) \right. \\
-2 \alpha r^2) h + (D-2) \eta r h' f \right] \phi' = 0, \end{array} \right. \] (10)

where \( () \) denotes the derivative with respect to the time \( t \) and \( (') \) the derivative with respect to the radial coordinate. Apart from the trivial option \( \phi' = 0 \) that does not yield interesting result, this equation is easily integrated as
\[ \phi(t, r) = \zeta(r) + q(t) e^{\chi(r)}, \] (11)
where \( \zeta \) (resp. \( q \)) is a function of the radial coordinate (resp. of the time), and where we have defined
\[
\chi(r) = \frac{1}{2} \int \left[ \eta (D - 2)(D - 3)(f - \gamma) - 2 \alpha r^2 \right] \frac{dr}{(D - 2) \eta f r} + \frac{h'}{h} dr. \tag{12}
\]

One can see that under our hypothesis, the expression between the brackets vanishes, yielding a scalar field to be given by
\[
\phi(t, r) = \zeta(r) + q(t) .
\]

Injecting this expression into the conservation equation, this implies that the scalar field must be linear in time
\[
\phi(t, r) = \zeta(r) + \phi_1 t, \tag{13}
\]
where \( \phi_1 \) is an integration constant. The \((r, r)\) component of the Einstein equations allows to express \( \varphi := \zeta' \) as
\[
\varphi(r) = \frac{(D - 2) \eta \phi_1^2 h' r}{2 \alpha r^2 + (D - 2)(D - 3) \gamma \eta h^2} + \psi(r)^2, \tag{14}
\]
and the remaining independent Einstein equation, given by the \((t, t)\) or \((i, i)\) component, yields a second order differential equation for the metric function \( h \). As before, through the substitution, the metric function \( h \) will be given by where now \( j \) is a solution of the following third-order algebraic equation
\[
\frac{1}{8} (D - 3) \eta \phi_1^2 [(D - 2)(D - 3) \gamma \eta + 2 \alpha r^2]^2 r^{D-4} + \varepsilon_{\text{static}} = 0, \tag{15}
\]
where \( \varepsilon_{\text{static}} \) is defined by

Here, it is interesting to note that, in contrast with the purely static case, the point defined by \( \alpha = \eta \Lambda \) and \( \gamma = 0 \) is not degenerate if one considers time-dependent scalar field, \( \phi_1 \neq 0 \). Indeed, in this case, we obtain a Lifshitz black hole solution with a dynamical exponent \( z = \frac{3}{4} \) given by
\[
ds^2 = -r^2 g(r) dt^2 + \frac{dr^2}{r^2 g(r)} + r^2 d\sigma_{D-2}^2, \tag{16a}
\]
\[
g(r) = 1 - \frac{M}{r^2}, \tag{16b}
\]
\[
\phi(t, r) = \int \varphi(r) dr + \phi_1 t, \tag{16c}
\]
where
\[
\varphi(r)^2 = \frac{1}{r^2 f(r)} \left[ \frac{\phi_1^2}{f(r)} - 3 \phi_1^2 (D - 3) \frac{4 r^2}{(3 D - 7)} \frac{\Lambda}{\eta} \right], \tag{17}
\]
and, where the coupling constants are tied as
\[
\alpha = \frac{1}{6} (D - 2) \eta (3 D - 7), \quad \Lambda = \frac{\alpha}{\eta}. \tag{18}
\]

Two remarks can be made concerning this solution. Firstly, the limiting case \( M = 0 \) is well-defined and hence the field equations may accommodate pure Lifshitz spacetimes only with a time-dependent scalar field. It is also interesting to note that the same metric is also a particular solution of the Horndeski equations without the Einstein-Hilbert-\( \Lambda \) pieces, that it satisfies the equations
\[
\alpha T^{(1)}_{\mu \nu} + \eta T^{(2)}_{\mu \nu} = 0, \tag{19}
\]
provided that the scalar field is given by
\[
\phi(t, r) = \phi_1 \left( \int \varphi(r) dr + t \right), \tag{20}
\]
\[
\varphi(r) = \pm \frac{1}{f(r)} \frac{1}{r^2} \sqrt{\frac{(3 D - 7) - 3 (D - 3) f(r)}{(3 D - 7)}},
\]
and for the coupling constants given by

### IV. CONCLUSIONS

Here, we have considered a particular case of the Horndeski theory whose gravity theory is given by the Einstein piece and, whose matter source is described by a scalar field with its usual kinetic term as well as an additional nonminimal kinetic coupling. For this model and for a static scalar field, we have shown that besides a planar AdS black hole, this system can not accommodate other solutions with a planar base manifold. In order to circumvent this problem, we have seen that turning on the time dependence of the scalar field is primordial to obtain Lifshitz solutions. We have effectively derived a Lifshitz black hole solution for a particular value of the dynamical exponent \( z = \frac{3}{4} \). There is a priori no physical reasons to explain the occurrence of this particular value of the dynamical exponent. It will be interesting to see wether others sectors of the Horndeski action may accommodate more general Lifshitz solutions with others values of the dynamical exponents.

We also believe that, because of structure of the solutions obtained in this paper as well as those derived in, the model considered here or maybe more general sectors of the Horndeski theory can be a good laboratory in order to gain more insight concerning the black hole solutions with scalar field. The task of finding interesting solutions can also be explored for the recent theory formulated and involving more than one scalar field but still yielding to second-order field equations.

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