SLOWLY VANISHING MEAN OSCILLATIONS: NON-UNIQUENESS OF BLOW-UPS IN A TWO-PHASE FREE BOUNDARY PROBLEM

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Dedicado a Carlos Kenig, un gran maestro y amigo en conmemoración de sus 70 años.

Abstract. In Kenig and Toro’s two-phase free boundary problem, one studies how the regularity of the Radon-Nikodym derivative $h = \frac{d\omega^-}{d\omega^+}$ of harmonic measures on complementary NTA domains controls the geometry of their common boundary. It is now known that $\log h \in C^{0,\alpha}(\partial\Omega)$ implies that pointwise the boundary has a unique blow-up, which is the zero set of a homogeneous harmonic polynomial. In this note, we give examples of domains with $\log h \in C(\partial\Omega)$ whose boundaries have points with non-unique blow-ups. Philosophically the examples arise from oscillating or rotating a blow-up limit by an infinite amount, but very slowly.

1. Introduction

In this note, we answer a question about uniqueness of blow-ups in non-variational two-phase free boundary problems for harmonic measure in the negative. Throughout, we let $\Omega^+ = \Omega \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ denote complementary unbounded domains with a common boundary $\partial\Omega = \partial\Omega^+ = \partial\Omega^-$. Furthermore, we require that $\Omega^\pm$ belong to the class of NTA domains in the sense of Jerison and Kenig [JK82]. Let $\omega^\pm$ denote harmonic measures on $\Omega^\pm$ with finite poles $X^\pm$ or with poles at infinity (see Kenig and Toro [KT99]). Finally, we assume $\omega^+ \ll \omega^- \ll \omega^+$ and let

\[
(1.1) \quad h = \frac{d\omega^-}{d\omega^+}
\]

denote the Radon-Nikodym derivative of harmonic measure on one side of the boundary with respect to harmonic measure on the other side. We are interested in understanding how different regularity assumptions on $h$ controls the geometry of $\partial\Omega$.

Following Kenig and Toro [KT06] and Badger [Bad11], we know if $\log h \in \text{VMO}(d\omega^+)$ (vanishing mean oscillation) or $\log h \in C(\partial\Omega)$ (continuous), then the boundary admits a finite decomposition into pairwise disjoint sets,

\[
(1.2) \quad \partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_d,
\]
where geometric blow-ups (tangent sets) of \( \partial \Omega \) centered at any \( Q \in \Gamma_d \) (1 \( \leq d \leq d_0 \)) are zero sets \( \Sigma_p \) of homogeneous harmonic polynomials (hhp) \( p : \mathbb{R}^n \to \mathbb{R} \) of degree \( d \). That is to say, given any boundary point \( Q \in \partial \Omega \) and any sequence of scales \( r_i > 0 \) with \( \lim_{i \to \infty} r_i = 0 \), there exists a subsequence \( r_{ij} \) and a hhp \( p \) of degree \( d \) such that

\[
\lim_{j \to \infty} \max_j \left\{ \text{excess} \left( \frac{\partial \Omega - Q}{r_{ij}} \cap B, \Sigma_p \right), \text{excess} \left( \Sigma_p \cap B, \frac{\partial \Omega - Q}{r_{ij}} \right) \right\} = 0
\]

for every ball \( B \) in \( \mathbb{R}^n \). Here \( \text{excess}(S,T) = \sup_{s \in S} \inf_{t \in T} |s - t| \) when \( S, T \subset \mathbb{R}^n \) are nonempty and \( \text{excess}(\emptyset, T) = 0 \); see \[BL15\] for more information about this mode of convergence of closed sets (the Attouch-Wets topology). Following \[Bad13\] and \[BET17\], we further know that the regular set \( \Gamma_1 \) is closed and has Hausdorff and Minkowski dimension at most \( n - 3 \).

We remark that the maximum degree \( d_0 \) witnessed in the decomposition \( (1.2) \) can be bounded in terms of the ambient dimension and the NTA constants of \( \Omega^\pm \). When \( n = 2 \), it is always the case that \( \partial \Omega = \Gamma_1 \). When \( n = 3 \), we have \( \partial \Omega = \Gamma_1 \cup \Gamma_3 \cup \cdots \cup \Gamma_{2d_1 + 1} \) (odd degrees only) and for every odd \( d \geq 1 \), there exist two-sided domains with \( \Gamma_d \neq \emptyset \). In dimensions \( n \geq 4 \), for every integer \( d \geq 1 \), even or odd, there exist two-sided domains with \( \Gamma_d \neq \emptyset \). See \[BET17\] for details and \[AMT20\], \[PT20\], \[TT22\] for additional results on the regularity of \( \Gamma_1 \).

One may ask: Are the blow-ups at each point in \( \partial \Omega \) unique? In other words, is the zero set \( \Sigma_p \) in \( (1.3) \) independent of choice of the sequence of scales \( r_i \)? Under a stronger free boundary regularity hypothesis, the answer is affirmative. Following Engelstein \[Eng16\] and \[BET20\], we know that if \( \log h \in C^{0,\alpha}(\partial \Omega) \) for some \( \alpha > 0 \) (Hölder continuous), then blow-ups are unique. Moreover, when \( \log h \in C^{0,\alpha}(\partial \Omega) \), the regular set \( \Gamma_1 \) is actually a \( C^{1,\alpha} \) embedded submanifold and the singular set \( \partial \Omega \setminus \Gamma_1 \) is \( (n - 3) \)-rectifiable in the sense of geometric measure theory (see e.g. \[Mat95\]). Below, we supply examples demonstrating that under the weaker regularity hypothesis \( \log h \in C(\partial \Omega) \), there may exist points in the boundary that have non-unique blow-ups.

**Theorem 1.1.** For each \( d \in \{1, 3\} \), there exist complementary NTA domains \( \Omega^\pm \subset \mathbb{R}^3 \) such that \( \log h \in C(\partial \Omega) \), but there exists a point \( p \in \Gamma_d \) at which geometric blow-ups of \( \partial \Omega \) are not unique.

**Remark 1.2.** In fact, the domains that we construct below have locally finite perimeter and Ahlfors regular boundaries: that is, there exists \( C > 0 \) (depending on \( \Omega \)) such that

\[
C^{-1} r^{n-1} \leq \mathcal{H}^{n-1}(\partial \Omega \cap B(Q, r)) \leq C r^{n-1}
\]

for all \( Q \in \partial \Omega \) and \( r > 0 \), where \( \Omega \subset \mathbb{R}^n \) and \( \mathcal{H}^{n-1} \) denotes the \( (n - 1) \)-dimensional Hausdorff measure. Even more, the boundaries of the domains are smooth surfaces outside of a single point.

The basic strategy is to start with a blow-up domain \( \Omega^+_p = \{ X \in \mathbb{R}^n : \pm p(X) > 0 \} \) associated to a hhp \( p \) of degree \( d \), which has log \( h \equiv 0 \) and \( 0 \in \Gamma_d \). We then deform the domain near the origin by introducing rotations/oscillations at each scale \( 0 < r \leq 1/100 \)
so that the magnitude of the oscillation at scale \( r \) vanishes as \( r \to 0 \). The tension in the proof becomes choosing the correct speed of vanishing. On the one hand, by choosing the speed to be sufficiently \textit{quick}, we can guarantee by making estimates on elliptic measure that the deformed domain has \( \log h \in C(\partial \Omega) \). On the other hand, by choosing the speed to be sufficiently \textit{slow}, we can guarantee that the deformed domain has uncountably many blow-ups at the origin, each of which are rotations of the original domain.

\textit{Remark 1.3.} By a suitable modification, the technique introduced in the case \( d = 3 \) can be used to show existence of domains with \( \log h \in C(\partial \Omega) \) and non-unique blow-ups at an isolated point \( Q \in \Gamma_d \) for any value of \( d \geq 2 \). When \( d \geq 3 \) is odd, the examples can be produced in \( \mathbb{R}^3 \). When \( d \geq 2 \) is even, the examples can be produced in \( \mathbb{R}^4 \).

In a related context, Allen and Kriventsov \cite{AK20} use conformal maps to construct domains \( \Omega^\pm = \{ u^\pm > 0 \} \subset \mathbb{R}^n \) associated to non-negative subharmonic functions \( u^\pm \) for which the Alt-Caffarelli-Friedman functional

\begin{equation}
\Phi(r, u^+, u^-) = \frac{1}{r^4} \int_{B_r(0)} \frac{|\nabla u^+|^2}{|X|^{n-2}} \int_{B_r(0)} \frac{|\nabla u^-|^2}{|X|^{n-2}}
\end{equation}

has a positive limit as \( r \to 0 \), but whose interface \( \partial \Omega = \partial \Omega^+ = \partial \Omega^- \) does not have a unique tangent plane at the origin. It would be interesting to know whether a suitable modification of their examples satisfy \( \log h \in C(\partial \Omega) \). For more on the connection between the ACF functional and two-phase free boundary problems for harmonic measure (originally observed by Kenig, Preiss, and Toro \cite{KPT09}), see \cite[§2.2]{AKN22} and the references within.

We handle the case \( d = 3 \) of Theorem 1.1 in §2 and the case \( d = 1 \) in §3.

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2. \textbf{The First Example: Non-Unique Singular Tangents}

2.1. \textbf{Description and Geometric Properties.} We begin with Szulkin’s example \cite{Szu79} of a degree 3 hlp,

\begin{equation}
s(x, y, z) = x^3 - 3xy^2 + z^3 - 1.5(x^2 + y^2)z,
\end{equation}

with the interesting feature that its zero set \( \Sigma_s \) is homeomorphic to \( \mathbb{R}^2 \). See Figure 2.1. Because \( \Sigma_s \) is a cone (\( s \) is homogeneous) and \( \Sigma_s \cap S^2 \) is a smooth curve\footnote{One can check that \( \nabla s(x, y, z) = 0 \Leftrightarrow (x, y, z) = (0, 0, 0) \).}, it follows that \( \Omega_s^\pm = \{ (x, y, z) \in \mathbb{R}^3 : \pm s(x, y, z) > 0 \} \) are complementary NTA domains. Note that the positive \( z \)-axis belongs to \( \Omega_s^+ \) and the negative \( z \)-axis belongs to \( \Omega_s^- \), since \( s(0, 0, \pm 1) = \pm 1 \).
Figure 2.1. Left: Szulkin $\Sigma_s$, viewed from the $z$-axis. Center: the curve formed by intersection of Szulkin $\Sigma_s$ and $S^2$, viewed from a different angle. Right: Szulkin $\Sigma_s$ inside of the annulus $1/2 < r < 1$, viewed from the $z$-axis.

Figure 2.2. Examples of twisted Szulkin domains $\Omega^\pm$ defined using various rotation functions $\theta(r)$.
Left: $\theta(r) = \log(-\log(r))$; the domains $\Omega^\pm$ are NTA and $\log h \in C(\partial \Omega)$.
Center: $\theta(r) = -\log(r)$; the domains $\Omega^\pm$ are NTA, but $\log h \not\in VMO(d\omega^+)$.
Right: $\theta(r) = (-\log(r))^2$; the domains $\Omega^\pm$ are not NTA.

To build $\Omega^\pm$, we deform $\Omega^\pm_s$ by rotating spherical shells $\Sigma_s \cap \partial B_r(0)$ in the $xy$-plane. More precisely, we put $\Omega^\pm = \{ \pm s_{\text{twist}} > 0 \}$, where $s_{\text{twist}} \equiv s \circ \Phi_{-\theta}$ and $\Phi_{\pm\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ are homeomorphisms given by

$$
\Phi_{\pm\theta}(x, y, z) = (x \cos(\pm\theta) - y \sin(\pm\theta), x \sin(\pm\theta) + y \cos(\pm\theta), z),
$$

$$
\theta \equiv \theta(r) := \log(-\log(r)) \quad \text{for all } 0 < r := \sqrt{x^2 + y^2 + z^2} \leq 1/100
$$

and we smoothly interpolate to $\theta(r) := 0$ for all $r \geq 1$. See Figure 2.2.

If $s_{\text{twist}}(x, y, z) = 0$, then $\Phi_{-\theta}(x, y, z) \in \Sigma_s$. Hence the interface $\Sigma = \partial \Omega^\pm = \Phi_{\theta}(\Sigma_s)$.

Remark 2.1. Let us collect some simple, but useful observations about $\theta$ and $\Phi_{\theta}$. 
Since $0 \in E$, let’s examine the $(1,1)$ entry of $E$. The second property is true by compactness of the torus $\mathbb{R}/2\pi$. Proof. The first property holds since $\theta(r) = \theta_0$ (mod $2\pi$), i.e. such that $\min_{k \in \mathbb{Z}} |\theta(r) - \theta_0 - 2\pi k| = 0$ for all $i \geq 1$.

(ii) For any sequence $r_i \downarrow 0$, there exists $\theta_0 \in [0, 2\pi)$ and a $r_{ij} \downarrow 0$ such that $\theta(r_{ij}) \to \theta_0$ (mod $2\pi$), i.e. $\lim_{j \to \infty} \min_{k \in \mathbb{Z}} |\theta(r_{ij}) - \theta_0 - 2\pi k| = 0$.

(iii) For all $0 < r \leq 1/100$, we have $|\nabla \theta| = 1/(-r^2 \log(r))$ and $|\partial_{ij} \theta| \leq C/(-r^2 \log(r))$ for all $1 \leq i, j \leq 3$.

(iv) For all $(x, y, z)$ with $0 < r \leq 1/100$, we can write $D \Phi(r) = R_\theta + E_\theta$, where

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a rotation matrix and the “error matrix” $E_\theta$ is such that $\|E_\theta\|_\infty \leq C/(-\log(r))$, where the norm is the sup norm on the entries of $E_\theta$.

(v) The map $\Phi_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ is a quasiconformal homeomorphism, with $\Phi_\theta^{-1} = \Phi_{-\theta}$. Moreover, $\Phi_\theta$ is asymptotically conformal at the origin.

Proof. The first property holds since $\theta(r)$ is continuous in $r$ and $\theta(r) \to \infty$ as $r \downarrow 0$. The second property is true by compactness of the torus $\mathbb{R}/2\pi$. The third property is a straightforward computation. By another straightforward (if tedious) computation, $D \Phi_\theta = R_\theta + E_\theta$, where $R_\theta$ is as above and $E_\theta$ is the rank $1$ matrix given by

$$E_\theta = \begin{pmatrix} -x \sin(\theta) - y \cos(\theta) \\ x \cos(\theta) - y \sin(\theta) \\ 0 \end{pmatrix} \begin{pmatrix} \theta_x & \theta_y & \theta_z \end{pmatrix}.$$ 

Let’s examine the $(1,1)$ entry of $E_\theta$. Since $\theta_x = \theta'(r)x/r = \theta'(r)x/r$ and $|x| \leq r$, we have

$$|x \theta_x \sin(-\theta) + y \theta_x \cos(-\theta)| \leq 2r|\theta'(r)| \leq 2/(-\log(r)).$$

The other non-zero entries of $E_\theta$ obey the same estimate. This gives the fourth property. To prove that $\Phi_\theta$ is quasiconformal (see e.g. [Hei06]), it suffices to check that $\Phi_\theta \in W^{1,n}_{\text{loc}}$ and there exists $1 \leq L < \infty$ such that the a.e. defined singular values $\lambda_1 \leq \lambda_2 \leq \lambda_3$ of $D \Phi_\theta$ satisfy $\lambda_3 \leq L\lambda_1$ a.e. These facts follow from property (iv) and the variational characterization of the minimum and maximum singular values. Furthermore, as $r \downarrow 0$, the maximum ratio of $\lambda_3/\lambda_1$ in $B_r$ goes to $1$. Therefore, $\Phi_\theta$ is asymptotically conformal at the origin.

The Hausdorff distance $\text{HD}(A, B) = \max\{\text{excess}(A, B), \text{excess}(B, A)\}$ for all nonempty sets $A, B \subset \mathbb{R}^n$. Note that $\text{HD}(\lambda A, \lambda B) = \lambda \text{HD}(A, B)$ for any dilation factor $\lambda > 0$.

**Lemma 2.2** (twisted Szulkin vs. rotations of Szulkin). If $r, \epsilon, R > 0$ and $0 < Rr \leq 1/100$, then $\text{HD}(\Sigma \cap B_{Rr}, R_{\theta(r)} (\Sigma \cap B_{Rr})) \leq C \max\{\epsilon r, \sup\{q\theta(q) - \theta(r) : \epsilon r \leq q \leq Rr\}\}$.

Proof. For any $p \in B_{\epsilon r}$, we have $\text{dist}(p, R_{\theta(r)} (\Sigma \cap B_{Rr})) \leq 2\epsilon r$ and $\text{dist}(p, \Sigma \cap B_{Rr}) \leq 2\epsilon r$, since $0 \in R_{\theta(r)} (\Sigma \cap B_{Rr})$ and $0 \in \Sigma$. Thus, the main issue is to estimate distances inside $B_{Rr} \setminus B_{\epsilon r}$.

Let $p \in \Sigma \cap B_{Rr} \setminus B_{\epsilon r}$, say $p \in \Sigma \cap \partial B_q$ with $\epsilon r \leq q \leq Rr$. Then we may write $p = R_{\theta(q)}x$ for some $x \in \Sigma$. Let’s estimate $\text{dist}(p, R_{\theta(r)} (\Sigma \cap B_{Rr}))$ from above by the distance of $p$ to
the point \( y = R_{\theta(r)} x \in R_{\theta(r)} \Sigma_s \cap \partial B_q \). Note that \( y = R_{\theta(r)} x = R_{\theta(r)} R_{-\theta(q)} p = R_{\theta(r)-\theta(q)} p \) and \( |y| = |p| = q \). Hence

\[
|p - y| \leq q(1, 0, 0) - (\cos(\theta(q) - \theta(r)), \sin(\theta(q) - \theta(r)), 0)]
\]

\[
= q(2 - 2\cos(\theta(q) - \theta(r)))^{1/2}
\]

\[
\leq Cq|\theta(q) - \theta(r)|,
\]

where the first inequality holds by geometric considerations and the last inequality used the Taylor series expansion for cosine.

A similar inequality holds starting from any \( p \in R_{\theta(r)} \Sigma_s \cap B_{Rr} \setminus B_{cr} \).

**Lemma 2.3.** With \( \theta(r) = \log(-\log(r)) \), the twisted Szulkin domains \( \Omega^\pm \) as defined above are chord-arc domains (i.e. NTA domains with Ahlfors regular boundaries). The interface \( \Sigma = \partial \Omega^\pm \) has a continuum of blow-ups at the origin, each of which is a rotation of \( \Sigma_s \) in the xy-plane.

**Proof.** The domains \( \Omega^\pm = \Phi_\theta(\Omega^\pm_s) \) are NTA, because global quasiconformal maps send NTA domains to NTA domains. Every boundary of an NTA domain is lower Ahlfors regular (see e.g. [Bad12, Lemma 2.3]). Thus, \( \Sigma \) is lower Ahlfors regular. To check upper Ahlfors regularity, first note that \( \Sigma_s \) is upper Ahlfors regular, since \( \Sigma_s \) can be covered by a finite number of Lipschitz graphs. Since \( \|\det(D\Phi_\theta)\|_{\infty} < \infty \), it follows that \( \Sigma = \Phi_\theta(\Sigma_s) \) is upper Ahlfors regular, as well.

Let’s address the blow-ups of \( \partial \Omega \) at the origin. Let \( r_i \downarrow 0 \) and suppose initially that \( \theta(r_i) = \theta_0 \mod 2\pi \) for all \( i \). Let \( \epsilon(r) \) be a function of \( r \) to be specified below. Let \( R \gg 1 \) be a large radius. By Lemma 2.2, the homogeneity of the Hausdorff distance, and the mean value theorem, we have

\[
\text{HD}(r_i^{-1} \Sigma \cap B_R, R_{\theta_0} \Sigma_s \cap B_R)
\]

\[
\leq Cr_i^{-1} \max \{\epsilon(r_i)r_i, \sup \{q|\theta(q) - \theta(r_i)| : \epsilon(r_i)r_i \leq q \leq Rr_i\}\}
\]

\[
\leq C \max \{\epsilon(r_i), \sup \{|t|\theta'(r_i) - \theta(r_i)| : \epsilon(r_i) \leq t \leq R\}\}
\]

\[
\leq C \max \{\epsilon(r_i), R(R - 1)r_i \sup \{|\theta'(r_i)| : \epsilon(r_i) \leq t \leq R\}\}.
\]

Our task is to choose \( \epsilon(r_i) \) so that

\[
\lim_{i \to \infty} \epsilon(r_i) = 0 \quad \text{and} \quad \lim_{i \to \infty} \sup \{r_i|\theta'(r_i)| : \epsilon(r_i) \leq t \leq R\} = 0.
\]

Since \( |\theta'(r)| = 1/(-r \log r) \), we have \( \sup \{r_i|\theta'(r_i)| : \epsilon(r_i) \leq t \leq R\} \leq 1/(-\epsilon(r_i) \log(Rr_i)) \) for all sufficiently large \( i \) (i.e. for all sufficiently small \( r_i \)). Thus, (2.4) is satisfied (e.g.) by choosing \( \epsilon(r) = |\log(r)|^{-1/2} \). It follows that \( \lim_{i \to \infty} \text{HD}(r_i^{-1} \Sigma \cap B_R, R_{\theta_0} \Sigma_s \cap B_R) = 0 \) for all \( R > 0 \). This implies that \( \Sigma/r_i \) converges to \( R_{\theta_0} \Sigma_s \) in the sense of (1.3).

In the general case, starting from any sequence \( r_i \downarrow 0 \), pass to a subsequence such that \( \theta(r_i) \to \theta_0 \mod 2\pi \). One can readily check that \( R_{\theta(r_i)} \Sigma_s \) converges to \( R_{\theta_0} \Sigma_s \) in the Attouch-Wets topology. Therefore, \( \Sigma/r_i \) converges to \( R_{\theta_0} \Sigma_s \) in the sense of (1.3) by the special case and the triangle inequality for excess. 

\[\square\]
Remark 2.4. For all exponents $0 < p < 1$, the twisted Szulkin domains defined using the rotation function $\theta(r) = (-\log(r))^p$ also satisfy the conclusions of Lemma 2.3. However, there is phase transition at $p = 1$. When $\theta(r) = -\log(r)$, one can show that the blow-ups of $\Sigma$ are no longer zero sets of $\nabla u$. The essential difference is that the “speed of rotation” vanishes as one zooms-in at the origin when $p < 1$, but the “speed of rotation” is constant when $p = 1$. When $p > 1$, the “speed of rotation” goes to infinity as one zooms-in at the origin and the associated twisted Szulkin domains $\Omega^\pm$ are not even NTA. See Figure 2.2.

2.2. Potential Theory for the First Example. Let $r_i \downarrow 0$ be an arbitrary sequence of radii going to zero and let $K \gg 1$. Recall that $\Sigma \cap (B_{Kr_1} \backslash B_{r_1/K}) = \Phi_\theta(\Sigma_s \cap (B_{Kr_1} \backslash B_{r_1/K}))$. Set

$$\tilde{u}_i^\pm(x) = \frac{u^\pm \circ \Phi^{-1}_\theta(r_i)x}{\omega^\pm(B_{r_1})},$$

where $u^\pm$ are the Green’s functions with poles at infinity for $\Omega^\pm$. Then in $\Omega^\pm_s \cap B_K \backslash B_{1/K}$, we have that $\tilde{u}_i^\pm$ satisfies

$$-\text{div}(B(r_i)x\nabla -) = 0, \quad B = (\det D\Phi_\theta)^{-1}(D\Phi_\theta)(D\Phi_\theta)^T$$

and $\Phi_\theta$ is as in (2.2).

To see that $B(r_i)x$ is Lipschitz regular, we note that Remark 2.1(iii) implies that $\|DB\| \leq \frac{C}{r \log(r_i)}$. Therefore, using the fundamental theorem of calculus along curves which stay in the annulus $B_K \backslash B_{1/K}$

$$\|B(r_i)x - B(r_iy)\| \leq Cr_i|x - y| \sup_{B_{Kr_1} \backslash B_{r_1/K}} \|DB\| \leq \frac{CK}{\log(r_i)}|x - y|, \forall x, y \in B_K \backslash B_{1/K},$$

where $C > 0$ is independent of $i, K$. This uniform Lipschitz continuity immediately implies the next result:

Lemma 2.5. Let $\alpha \in (0, 1), K > 1$. The sequence $\tilde{u}_i^\pm$ is pre-compact in $C^{1,\alpha}(\Omega^\pm_s \cap B_K \backslash B_{1/K})$. Furthermore, there exists a subsequence along which $\tilde{u}_i^\pm \to \kappa s$, uniformly on compacta, where $s$ is the Szulkin polynomial, for some $\kappa > 0$.

Proof. We see that $\tilde{u}_i^\pm$ solves an elliptic PDE with coefficients that are Lipschitz continuous and elliptic with coefficients independent of $i$. Furthermore,

$$\sup_{B_{4Kr_1}} \|\tilde{u}_i^\pm\| \leq C \Leftrightarrow \sup_{B_{4Kr_1}} \|u^\pm\| \leq C\frac{\omega^\pm(B_{r_1})}{r_i}.$$ 

The latter inequality holds (with a $C > 0$ that depends on $K$) by the Caffarelli-Fabes-Mortola-Salsa and doubling estimates on harmonic measure in NTA domains, see e.g. [JKS82]. Then Schauder theory tells us that $\tilde{u}_i^\pm$ are uniformly in $C^{1,\alpha}(\Omega^\pm_s \cap B_K \backslash B_{1/K})$ for any $\alpha \in (0, 1)$; see [GT01] Theorem 8.3. The precompactness follows.

Passing to a subsequence, we get that the sequences converges to functions $\tilde{u}_\infty^\pm$, which solves $-\text{div}(B_\infty \nabla \tilde{u}_\infty^\pm) = 0$ in $\Omega^\pm_s \cap B_K \backslash B_{1/K}$. From (2.6) we see that $B_\infty = \text{Id}$ and so,
invoking a diagonal argument, \( \tilde{u}_i^\pm \to \tilde{u}_\infty^\pm \), uniformly on compacta in \( \mathbb{R}^3 \). Furthermore, \( \tilde{u}_\infty^\pm \) are positive harmonic functions in \( \Omega^\pm_\infty \) that vanish on \( (\Omega^\pm)^c \).

Since \( (\Omega^\pm_s)^c \) are (global) NTA domains, the boundary Harnack inequality implies that there are scalars \( \kappa_\pm > 0 \) such that \( \tilde{u}_\infty^\pm = \kappa_\pm \) (see [KT99] Lemma 3.7 and Corollary 3.2).

To wrap up, let us again note that the points \( (0,0,\pm 1) \in \Omega^\pm_\infty \) are invariant under \( \Phi \).

Furthermore by symmetry \( u^+(0,0,1) = u^-(0,0,-1) \) and \( \omega^+(B_r) = \omega^-(B_r) \) for all \( r > 0 \). Thus, \( u^\pm_\infty(0,0,1) = u^-_\infty(0,0,-1) \) and this number determines the constant of proportionality with \( s \).

Finally, the proof of the continuity of \( \log h \) follows immediately:

**Proof of \( \log h \in C(\partial \Omega) \).** We note that away from the origin, \( \partial \Omega \) is smooth so continuity of the Radon-Nikodym derivative follows from classical potential theory. Furthermore, arguing by symmetry (that is, \(-\Omega^+ = \Omega^-\)) we have that \( \omega^+(B(0,r)) = \omega^-(B(0,r)) \) for all \( r > 0 \). Thus, recalling that \( u^\pm \) are the Green’s function for \( \Omega^\pm \) respectively, we are done if we can show that

\[
\lim_{\partial Q \ni Q \to 0} \frac{|\nabla u^+|(Q)}{|\nabla u^-|(Q)} = 1.
\]

(Recall that where \( \partial \Omega \) is smooth, \( C^{1,\alpha} \) is sufficient, the Radon-Nikodym derivative is given by the ratio of the derivatives of the Green functions [Kel12]).

Let \( Q_i \in \partial \Omega \) with \( Q_i \to 0 \) and let \( |Q_i| = r_i \downarrow 0 \). Let \( \tilde{u}_i^\pm \) be given by (2.5). Then

\[
\frac{\omega^+(B_{r_i})}{r_i^2} D\Phi_\theta(r_i x) \nabla \tilde{u}_i^\pm(x) = \nabla u^\pm(\Phi^{-1}_\theta(r_i x)).
\]

Let \( \tilde{Q}_i = \Phi_\theta(Q_i)/r_i \in \Sigma_s \cap \partial B_1 \). We have shown that

\[
\frac{|\nabla u^+|(Q_i)}{|\nabla u^-|(Q_i)} = \frac{|D\Phi_\theta(r_i \tilde{Q}_i) \nabla \tilde{u}_i^\pm(\tilde{Q}_i)|}{|D\Phi_\theta(r_i \tilde{Q}_i) \nabla \tilde{u}_i^-(\tilde{Q}_i)|}.
\]

Continuity of \( \log h \) follows from Lemma 2.5 (the lemma implies that \( \tilde{u}_i^\pm \to \kappa s \) in \( C^{1,\alpha}(\overline{\Omega}\cap B_2(\Omega)) \)) and the fact that along some subsequence \( D\Phi_\theta(r_i x) \to R_{\theta_0} \) for some \( \theta_0 \) (depending on the subsequence).

\[\square\]

### 3. The Second Example: Non-Unique Flat Tangents

#### 3.1. Description and Geometric Properties.

To show non-uniqueness at “flat points” we adapt an example from [Tor94]. We set \( \Omega^\pm = \{(x, y, z) \in \mathbb{R}^3 : \pm(z - v(x, y)) > 0\} \), where \( v : \mathbb{R}^2 \to \mathbb{R} \) is defined by setting \( v(0,0) = 0 \),

\[v(x, y) = x \log |\log(r)| \sin(\log(|\log(r)|)) \quad \text{when } 0 < r = (x^2 + y^2)^{1/2} \leq 1/100,
\]

and smoothly (e.g. \( C^{1,\alpha} \)) interpolating to \( v(x, y) = 1 \) when \( r \geq 1 \).

**Lemma 3.1** (see [Tor94] Example 2). *The graph domains \( \Omega^\pm \) are chord-arc domains.*

The interface \( \Sigma = \partial \Omega^\pm \) has a continuum of blow-ups at the origin, each of which is a plane \( z = mx \) with “slope” \(-\infty \leq m \leq \infty\).*
Remark 3.2. Moreover, $\Omega^\pm$ are vanishing chord-arc domains in the sense of $[KT03]$. This can be seen as follows. First, every pseudo blow-up (an Attouch-Wets limit $\Gamma$ of $(\Sigma - Q_i)/r_i$ with $Q_i \to Q$ and $r_i \downarrow 0$) is a plane. Indeed, on the one hand, if $\limsup_{i \to \infty} |Q_i - Q|/r_i = \infty$, then $\Gamma$ is a plane, because $\Sigma \setminus \{0\}$ is smooth. On the other hand, if $|Q_i|/r_i \leq C$ for all $i$, then $\Gamma$ is a translate of a blow-up at $Q$ (see $[BL15, \text{Lemma 3.7}]$), and thus, $\Gamma$ is a plane by Lemma 3.1. Because every pseudo blow-up is a plane, $\Sigma$ is locally Reifenberg vanishing. Now, $v \in W^{2,2}(\mathbb{R}^2)$ (see $[Tor94]$). Hence, by Sobolev embedding, the normal vector of the interface $\hat{n} \in \text{BMO}(\partial\Omega)$ with small BMO norm. Therefore, $\Omega^\pm$ are vanishing chord-arc domains; see e.g. $[KT97, \text{BEG}+22]$.

3.2. Potential Theory for the Second Example. Following the approach of §2.2, we now prove that $\log h \in C(\partial\Omega)$.

As before, because $\partial\Omega$ is smooth outside of any neighborhood of the origin, $\log h \in C^\infty$ on $\partial\Omega \setminus B_r(0)$ for any $r > 0$. Thus, the key point is to show that $\log h$ is continuous at the origin.

Let $H^\pm = \{ \pm z > 0 \}$ denote the open upper and lower half-spaces. Let $r_i \downarrow 0$ be arbitrary, $K \gg 1$ and write

$$\{z = v(x, y)\} \cap (B_{Kr_i} \setminus B_{r_i/K}) = \Phi^{-1}(\{z = 0\} \cap (B_{Kr_i} \setminus B_{r_i/K})), $$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is the homeomorphism given by

$$\Phi(x, y, z) \equiv (x, y, z - v(x, y)).$$

(3.1) Set $\tilde{u}_i^\pm(p) = \frac{u^\pm \circ \Phi^{-1}(r_i p)}{\omega^\pm(B_{r_i}(0))}$, where $u^\pm$ are the Green’s functions with poles at infinity for $\Omega^\pm$, and the $\omega^\pm$ are the corresponding harmonic measures. In $H^\pm \cap B_K \setminus B_{1/K}$, $\tilde{u}_i^\pm$ satisfies

$$-\text{div}(B(r_i p) \nabla \tilde{u}_i^\pm(p)) = 0, \quad B = (\det D\Phi)^{-1}(D\Phi)(D\Phi)^T.$$ 

Lemma 3.3. Let $\alpha \in (0, 1), K > 1$. The sequence $\tilde{u}_i^\pm$ is pre-compact in $C^{1,\alpha}(H^\pm \cap B_K \setminus B_{1/K})$. Furthermore, there exists a subsequence along which $\tilde{u}_i^\pm \to \kappa z_\pm$ for some $\kappa > 0$ uniformly on compact subsets of $\mathbb{R}^3$. $^2$

---

$^2$One could prove the weaker result that $\log h \in \text{VMO}(d\omega^\pm)$ using Remark 3.2 and standard properties of $A_\infty$ weights.
Proof. We claim that \( \tilde{u}_i^{\pm} \) solves an elliptic PDE with Lipschitz continuous coefficients in \( B_K \setminus B_{1/K} \cap H^{\pm} \). Indeed,
\[
|B(r;p) - B(r;q)| \leq Cr_i |p-q| \| DB \|_{L^\infty(B_{K^i r_i} \setminus B_{K^i r_i})} \leq CK r_i \frac{\log |\log(r_i)|}{\log(r_i)} |p-q| \leq CK |p-q|,
\]
by the fundamental theorem of calculus.

Arguing as in Lemma 2.5 above, \( \tilde{u}_i^{\pm} \) are uniformly in \( C^{1,\alpha}(H^{\pm} \cap B_K \setminus B_{1/K}) \) for any \( \alpha \in (0,1) \) and thus have the desired pre-compactness. Passing to a subsequence and invoking a diagonal argument \( \tilde{u}_i^{\pm} \to \tilde{u}_i^{\pm} \) uniformly on compacta. Furthermore \( \tilde{u}_i^{\pm} > 0 \) and solves \( -\text{div}(B_\infty \nabla \tilde{u}_i^{\pm}) = 0 \) in \( H^{\pm} \) and has \( \tilde{u}_i^{\pm}(x,y,0) = 0 \). We see in (3.2) that \( B_\infty \) is constant (as \( \log |\log(r_i)|/\log(r_i) \downarrow 0 \)) and so \( -\text{div}(B_\infty \nabla z) = 0 \). Again, up to scalar multiplication there is a unique signed solution of \( -\text{div}(B_\infty \nabla z) = 0 \) in \( H^{\pm} \) which vanishes on \( \{ z = 0 \} \) and that has subexponential growth at infinity. Continuing to follow the argument for Lemma 2.5 we conclude that \( \tilde{u}_i^{\pm} = \kappa_{\pm} z^{\pm} \), with \( \kappa_{+} = \kappa_{-} \). (Remember that \(-\{ z > v(x,y) \} = \{ z < v(x,y) \} \), because \( v \) is odd.) \( \square \)

Finally, the proof of the continuity of \( \log h \) in this context follows exactly as in (2.2) except that we must be more careful estimating \( |D\Phi(r_i \tilde{Q}_i) \nabla \tilde{u}^{\pm}(\tilde{Q}_i)| \). (We do not know that \( D\Phi(r,p) \) converges to a rotation as \( r_i \downarrow 0 \).) However, observe that \( \tilde{u}^{\pm} \equiv 0 \) on \( \{ z = 0 \} \), so we know that \( \nabla \tilde{u}^{\pm}(\tilde{Q}_i) \) is parallel to \( e_3 \). Thus, an elementary computation shows that
\[
\frac{|D\Phi(r_i \tilde{Q}_i) \nabla \tilde{u}^{\pm}(\tilde{Q}_i)|}{|D\Phi(r_i \tilde{Q}_i) \nabla \tilde{u}^{\pm}(\tilde{Q}_i)|} = \frac{\nabla \tilde{u}^{\pm}(\tilde{Q}_i)}{|\nabla \tilde{u}^{\pm}(\tilde{Q}_i)|} = \frac{|\nabla \tilde{u}^{\pm}(\tilde{Q}_i)|}{|\nabla \tilde{u}^{\pm}(\tilde{Q}_i)|} = 1.
\]
The quantity on the right hand side converges to 1 by Lemma 3.3. As in (2.2) it follows that \( \log h \in C(\partial \Omega) \).

4. Open Questions and Further Directions

We end by presenting some natural open questions. Our first question concerns the size of the set of non-uniqueness:

**Question 4.1.** Let \( \Omega^{\pm} \subset \mathbb{R}^n \) be complementary NTA domains with \( \log h \in C(\partial \Omega) \). Is it possible for
\[
NU(\Omega) := \{ Q \in \partial \Omega : \text{there is no unique (geometric) blow-up at } Q \}
\]
to have Hausdorff dimension \( n - 1 \)?

We note that a local version of [TT22, Theorem 1.1] implies that the set \( \Gamma_1 \) of flat points in \( \partial \Omega \) is uniformly rectifiable. Thus \( \omega(NU) = 0 = \mathcal{H}^{n-1}(NU \cap \Gamma_1) \). Further, by the main result of [BET17], \( \dim \partial \Omega \setminus \Gamma_1 \leq n - 3 \). Thus, \( \mathcal{H}^{n-1}(NU) = 0 \). On the other hand, the example of [AK20] suggests that \( \mathcal{H}^{n-2}(NU \cap \Gamma_1) > 0 \) may be possible.

The example in [2] (twisted Szulkin) shows that it is possible for all singular points to have non-unique blowups and for the set of singular points with non-unique blowups to
have positive $\mathcal{H}^{n-3}$-measure. (When $n \geq 4$, simply take $\Omega^\pm \times \mathbb{R}^{n-3}$.) This is sharp by [BET17]. Thus, the natural analogue of Question 4.1 is answered in the affirmative.

Our second question asks what are the possible tangent cones at points of non-unique blow-up:

**Question 4.2.** Let $C \subset G(n, n-1)$ be a compact, connected subset of the Grassmannian. Does there exist a pair of complementary NTA domains $\Omega^\pm$ with $\log h \in C(\partial \Omega)$ and a point $Q \in \partial \Omega$ at which $\text{Tan}(\partial \Omega, Q) = C$?

In §3, we showed that the set $\text{Tan}(\partial \Omega, 0)$ of blow-ups of the interface of the graph domains at the origin consists of all planes $z = mx$ with “slope” $-\infty \leq m \leq +\infty$. For any closed interval $I \subset \mathbb{R}$, it is not hard to adapt the example so that the blow-ups at the origin are exactly the planes $z = mx$ with $m \in I$. It is known that for any closed set $\Sigma \subset \mathbb{R}^n$ and $Q \in \Sigma$, the set $\text{Tan}(\Sigma, Q)$ of all tangent sets of $\Sigma$ at $Q$ is closed and connected in the Attouch-Wets topology [BL15]; the statement and proof of this fact was originally motivated by similar statement for tangent measures [Pre87, KPT09].

We may also ask a version of Question 4.2 at points where the blow-ups are homogeneous of higher degree:

**Question 4.3.** Let $\mathcal{H}_{n,d}$ be the set of degree $d$ homogeneous harmonic polynomials $p$ in $\mathbb{R}^n$ such that $\Omega_p^\pm = \{\pm p > 0\}$ are NTA domains. For each $n \geq 3$ and $d \geq 2$ and $C \subset \mathcal{H}_{n,d}$, which is compact and connected, does there exist complementary NTA domains $\Omega^\pm$ with $\log h \in C(\partial \Omega)$ and a point $Q \in \partial \Omega$ at which $\text{Tan}(\partial \Omega, Q) = \{\Sigma_p : p \in C\}$?

The condition that $\mathbb{R}^n \setminus \Sigma_p$ is a union of two NTA domains is necessary for $\Sigma_p$ to arise as a blow-up of the interface of complementary NTA domains. The first step to answering Question 4.3 may be to study the “moduli space” of $\mathcal{H}_{n,d}$ when $d \geq 2$. For example:

**Question 4.4.** If $p$ and $q$ lie in the same connected component of $\mathcal{H}_{n,d}$, is it true that $\Sigma_q$ is bi-Lipschitz equivalent to $\Sigma_p$?

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