On the stability of Dark Energy with Mass-Varying Neutrinos

Niayesh Afshordi,‡ Matias Zaldarriaga, and Kazunori Kohri

1Institute for Theory and Computation, Harvard-Smithsonian Center for Astrophysics, MS-51, 60 Garden Street, Cambridge, MA 02138, USA
2Jefferson Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138, USA
3Department of Earth and Space Science, Graduate School of Science, Osaka University, Osaka 560-0043, Japan

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An interesting dynamical model for dark energy which does not require extremely light scalar fields such as quintessence, and at the same time explains the (near-) coincidence between the neutrino and dark energy densities is the model of dark energy coupled to mass varying neutrinos (MaVaNs). Despite the attractions of this model, we show that, generically, this model contains a catastrophic instability which occurs when neutrinos become non-relativistic. As a result of this instability, as neutrinos become non-relativistic, they condense into neutrino nuggets which redshift away similar to cold dark matter, and thus cease to act as dark energy. Any stable MaVaNs dark energy model is extremely contrived, and is virtually indistinguishable from a cosmological constant.

I. INTRODUCTION

During the past decade, it has become clear that different cosmological observations, such as the dimming of distant supernovae Ia [1,2], anisotropies in the cosmic microwave background [3], and the large scale structure of the universe (e.g., [4]) cannot be explained with a cosmological model that contains only ordinary (baryonic+dark) matter evolving according to Einstein’s theory of general relativity. The most popular solution is to introduce an extra component with negative pressure, the so-called dark energy (e.g., [5]).

While the simplest candidate for dark energy is a cosmological constant (or vacuum energy) the extremely small size of the cosmological constant is difficult to explain unless one is willing to invoke a selection effect (e.g., [6]). Furthermore there is an additional strange coincidence, that the density of photons, neutrinos, baryons and dark matter and dark energy density all seem to become comparable at roughly the same time in the history of the universe even though they scale very differently with redshift.

Dynamical models of dark energy (e.g., quintessence [7], k-essence [8,9]) may be built to try to explain the coincidence. These models require an extremely light scalar field (with mass \(m \ll \sim 10^{−33} \text{eV}\)). An interesting alternative, which simultaneously alleviates the need for an extremely light scalar field, and explains the coincidence between the neutrino and dark energy densities has been recently proposed in the context of a theory of mass varying neutrinos (MaVaNs) [10,11] (also see [12]). In this model, dark energy is modulated by the density of neutrinos, which can simultaneously cause the evolution of both dark energy density and neutrino masses as a function of cosmic time. The time evolution however is not caused by a slowly rolling light scalar, but by a scalar (the so-called acceleron, \(A\)), which is heavy (relative to the Hubble expansion rate) but has a minimum that varies slowly as a function of the density of neutrinos. The model may also have interesting implications for neutrino oscillation experiments [10,13] and solar neutrino observations [14,15].

In this work, we investigate the evolution of perturbations in the MaVaNs model, which has not yet been consistently studied in the literature. The key difference between perturbations in MaVaNs and most other dark energy candidates is that (unlike quintessence or k-essence) all the dynamical properties of (non-relativistic) MaVaNs are set by the local neutrino density. In particular, pressure is a local function of neutrino density, implying that the hydrodynamic perturbations are adiabatic. Subsequently, the speed of sound for adiabatic perturbations at scales smaller than the Hubble radius is set by the instantaneous time derivatives of pressure and density [10]. We will show that this implies that hydrodynamic perturbation are unstable on all macroscopic scales once the neutrinos become non-relativistic.

The paper is organized as follows: in Sec. II we summarize properties of the homogeneous MaVaNs model for dark energy. Sec. III presents hydrodynamic and kinetic theory arguments for why perturbations of non-relativistic MaVaNs have an imaginary sound speed, and are thus unstable. In particular we will show that, when neutrinos are non-relativistic, free streaming is unable to stabilize perturbations on all scales with wavelengths larger than \(m_{A}^{-1}\) where \(m_{A}\) is the mass of the acceleron field. We will then move on to speculate about the end result of the instability. In Sec. IV we argue that the outcome of instability is inevitably a multi-phase medium with most of neutrinos in a dense phase, which we call neutrino nuggets, and dilute as pressureless matter. Sec. V presents a thermodynamic description of the phase transition and draws the analogy with the liquid-gas phase equilibrium. Finally, Sec. VI considers potential stable MaVaNs (or adiabatic dark energy) models, and Sec. VII concludes the paper. One thing will become
clear: After neutrinos become non-relativistic, the model has a serious microscopic instability. As a result of this instability, the fluid will cease to act as dark energy and will not be able to drive the acceleration of the universe.

II. PRELIMINARIES OF THE DARK ENERGY WITH MASS-VARYING NEUTRINOS

Within the MaVaNs theory of dark energy, the local energy density associated with dark energy is the sum of the energy densities of neutrinos and a light scalar field, $\mathcal{A}$, the so-called acceleron field, which also modulates the neutrino mass:

$$V = n_{\nu}m_{\nu}(\mathcal{A}) + V_0(\mathcal{A}),$$

where the neutrinos are assumed to be non-relativistic.

In the limit $n_{\nu} \gg m_\mathcal{A}^3$, where $m_\mathcal{A}$ is the mass of the acceleron field, the acceleron field responds to the average density of neutrinos, and relaxes at the minimum of the potential $V$ (the actual relaxation of $\mathcal{A}$ at the minimum, of course, depends on its evolution through the cosmic history). In this limit, both $m_\nu$ and $V_0$ become functions of the neutrino density, $n_\nu$, through

$$\partial V/\partial \mathcal{A} = \left(n_\nu + \frac{\partial V_0}{\partial m_\nu}\right)\frac{\partial m_\nu}{\partial \mathcal{A}} = 0 \Rightarrow n_\nu = -\frac{\partial V_0}{\partial m_\nu},$$

if $\partial m_\nu/\partial \mathcal{A} \neq 0$.

Assuming that the kinetic energy in the acceleron field is negligible, the equation of state for the neutrino/acceleron fluid is:

$$w \equiv \frac{\text{Pressure}}{\text{Density}} \simeq -\frac{V_0(\mathcal{A})}{V} = -1 + \frac{n_\nu m_\nu(A)}{V},$$

where we used the fact that non-relativistic neutrinos have negligible pressure. We note that $w \simeq -1$ (i.e. a cosmological constant), as long as the energy density in neutrinos is small enough compared to the energy of the acceleron field.

Fardon et al. [11] go on to propose a specific model for the mass varying neutrinos:

$$m_\nu(\mathcal{A}) = \frac{m_\nu^2}{\mathcal{M}(\mathcal{A})},$$

$$V_0(\mathcal{A}) = \Lambda^4 \ln \left[1 + \mathcal{M}(\mathcal{A})/\mu\right] + C,$$

where it is assumed that $\mathcal{M}(\mathcal{A})/\mu \gg 1$, and $\Lambda \sim 10^{-5}$eV characterizes the energy scale of dark energy and $C$ is just a constant needed to set the minimum of the potential to zero (i.e. no cosmological constant). Now, using Eq. 2, we find:

$$V_0 \simeq \Lambda^4 \ln \left(\frac{m_0}{m_\nu}\right) + C \Rightarrow n_\nu = \frac{\Lambda^4}{m_\nu},$$

where $m_0 \equiv m_\nu^2/\mu$, and Eq. 3 gives the equation of state:

$$w = -1 + \left[1 + \ln \left(\frac{m_0}{m_\nu}\right)\right]^{-1}.\quad(7)$$

Based on current observational constraints $|1 + w| \lesssim 0.2$ (e.g., [4]), and thus $m_0 \gtrsim 10^2 m_\nu$. Therefore, we restrict the rest of our analysis (with the exception of Sec. V) to the $m_0 \gg m_\nu$ regime.

For the model to be sensitive to the mean density of neutrinos, there needs to be many neutrinos within the Compton wavelength of $\mathcal{A}$, i.e. $N_\nu \sim n_\nu/m_\mathcal{A}^3 \gg 1$. The mass of $\mathcal{A}$ is given by:

$$m_\mathcal{A}^2 = \left(\frac{\partial^2 V}{\partial \mathcal{A}^2}\right)_{\text{min}} \equiv \frac{\Lambda^4}{m_\nu^2} \left(\frac{\partial m_\nu}{\partial \mathcal{A}}\right)^2.\quad(8)$$

We thus have

$$N_\nu \sim \frac{m_0^2}{\Lambda^2} \left(\frac{\partial m_\nu}{\partial \mathcal{A}}\right)^{-3}.$$\quad(9)

Observational bounds restrict the coupling $(\partial m_\nu/\partial \mathcal{A}) < 10^{-6}$ [11], which puts a lower bound on $N_\nu$ around $10^{24}$, justifying the assumption that $\mathcal{A}$ field is only sensitive to the mean neutrino number density.

For non-vanishing $\partial V/\partial m_\nu$, Eq. 2 can also be satisfied if $\partial m_\nu/\partial \mathcal{A} = 0$ for a value of $\mathcal{A}$ or $m_\nu$. This can happen if $V_0(\mathcal{A})$ reaches a minimum, which subsequently puts a maximum on $m_\nu < m_{\text{max}}$. $m_{\text{max}}$ corresponds to the mass of neutrinos in vacuum or when $n_\nu < \Lambda^4/m_{\text{max}}$.

In particular, the macroscopic dynamics of the original MaVaNs model [11] is quantified in terms of $\Lambda$, $m_0$ and $m_{\text{max}}$, where

$$V_0(m_\nu) = \Lambda^4 \ln \left(1 + \frac{m_0}{m_\nu}\right) + C, \quad m_\nu < m_{\text{max}}.\quad(10)$$

Again, $C = -\Lambda^4 \ln(1+m_0/m_{\text{max}})$ is added such that the minimum of the potential is at zero.

III. EMERGENCE OF AN IMAGINARY SPEED OF SOUND

A. Hydrodynamic picture

In the conventional quintessence models [6], similar to inflationary models, the scalar field is slowly rolling at the present epoch, and therefore its effective mass is smaller than the Hubble expansion rate. In contrast, the acceleron field sits at the instantaneous minimum of its potential, and the cosmic expansion only modulates this minimum through changes in the local neutrino density. The mass of the acceleron field can be much larger than the Hubble expansion rate. However, by the same token, the coherence length of acceleron, $m_\mathcal{A}^{-1}$, is much smaller than the present Hubble length, and thus, unlike quintessence, the perturbations on sub-Hubble scales $> m_\mathcal{A}^{-1}$ are adiabatic, i.e. obey the same equation of state as the homogeneous universe. Therefore, the speed of sound, $c_s^2$, for these perturbations is simply given by [10]:

$$c_s^2 = \frac{\dot{\rho}}{\rho} = \frac{\dot{w} + w \dot{\rho}}{\rho} = w - \frac{\dot{w}}{3H(1+w)} = \frac{\partial \ln m_\nu}{\partial \ln n_\nu},$$\quad(11)
where we used the continuity equation \( \dot{\rho} = -3H(1+w)\rho \), and in the last step, we used Eq. (2).

We can already see that for a vanishing or small enough \( \dot{\psi}, c_s^2 \) has the same sign as \( w \), which is negative. Clearly, this is a catastrophic instability in this theory, as the sub-Hubble perturbations on scales larger than \( m_{\Lambda}^{-1} (H < k < m_{\Lambda}) \) would grow as \( \exp(k|c_s|t) \) if \( c_s^2 < 0 \).

In particular, for the model introduced in [11], Eq. (6) gives:

\[
\rho + P = m_\nu n_\nu = \Lambda^4 = \text{const.} \Rightarrow c_s^2 = \frac{\dot{\rho}}{\rho} = -1. \quad (12)
\]

**B. Kinetic theory picture**

One may wonder if, similar to the gravitational instability of collisionless cold dark matter (or massive neutrinos with constant mass), our instability is stabilized at small scales due to free streaming of neutrinos. In this section we study the nature of this instability in the kinetic theory picture, and show that the imaginary speed of sound is indeed scale-independent in any non-relativistic MaVaNs models, making the instability most severe at the microscopic scales (~\( m_{\Lambda}^{-1} \)).

The action of a test particle with a dynamical mass \( m_\nu(x, \eta) \), in a flat FRW universe, is given by:

\[
S = -\int m_\nu(x, \eta) \sigma(\eta) \sqrt{1 - u^2} d\eta,
\]

where

\[
u^2 = u \cdot u = \frac{dx}{d\eta} \cdot \frac{dx}{d\eta}, \quad (14)
\]

and \( a \) and \( \eta \) are the scale factor and conformal time respectively. Thus, the single particle Lagrangian, Momentum, and Energy take the standard form

\[
\mathcal{L}_\nu = -m_\nu a \sqrt{1 - u^2}, \quad (15)
\]

\[
p = \frac{\partial \mathcal{L}_\nu}{\partial \dot{u}} = \frac{m_\nu a u}{\sqrt{1 - u^2}} = m_\nu \gamma a u, \quad (16)
\]

\[
\mathcal{E}_\nu = u \cdot p - \mathcal{L}_\nu = m_\nu \gamma a, \quad (17)
\]

where \( \gamma \equiv (1 - u^2)^{-1/2} \),

while we have dropped the \( x, \eta \) dependence for brevity.

The evolution of \( p \) is given by the Euler-Lagrange equation:

\[
p' = \frac{dp}{d\eta} = \frac{\partial \mathcal{L}_\nu}{\partial \dot{x}} = -a\gamma^{-1}\nabla m_\nu, \quad (19)
\]

where \( \nabla \) denotes the gradient with respect to the comoving coordinates \( x \), and we have neglected gravity.

Now the Boltzmann equation for the evolution of the phase space density of neutrinos, \( f(x, p) \), takes the form:

\[
\frac{\partial f}{\partial \eta} + u \cdot \nabla f - a\gamma^{-1}\nabla m_\nu \cdot \frac{\partial f}{\partial p} = 0. \quad (20)
\]

Let us also generalize Eq. (2) for relativistic neutrinos. Using the fact that neutrino momentum remains constant in a homogeneous background, the relaxation of the \( A \) field at the minimum of its potential leads to

\[
\frac{\partial V}{\partial A} = \left[ \int \frac{m_\nu}{\sqrt{p^2 + m_\nu^2}} f(x, p) d^3p + \frac{\partial V_0}{\partial m_\nu} \right] \frac{\partial m_\nu}{\partial A} = 0,
\]

which for \( \partial m_\nu/\partial A \neq 0 \) (\( m_\nu < m_{\text{max}} \)) yields

\[
n_\nu(\gamma^{-1}) = -\frac{\partial V_0}{\partial m_\nu} \approx \frac{\Lambda^4}{m_\nu}, \quad (22)
\]

for the Logarithmic potential.

In the absence of perturbations, neutrinos become non-relativistic as they are free-streaming. Therefore, as a result of momentum conservation, they are frozen into the relativistic Fermi-Dirac phase space density:

\[
\tilde{f}(p) = \frac{2(2\pi)^{-3}}{\exp\left|\frac{p}{T_{\nu}}\right| + 1} = \frac{2(2\pi)^{-3}}{\exp\left|\frac{p_{ph}}{T_{\nu}}\right| + 1}, \quad (23)
\]

where \( p_{ph} = |p|/a \) is the physical momentum, while \( T_\nu = T_{0,\nu}/a \) is the neutrino kinematic temperature, which matches the thermodynamic temperature in the relativistic regime. The factor of 2 in Eq. (23) accounts for both neutrinos and anti-neutrinos.

Let us study the linear perturbations of the Boltzmann equation (Eq. 20), in the sub-Hubble regime, i.e. for time/length scales much shorter than the Hubble time/length. We will show that the instability occurs on microscopic scales so this is perfectly adequate. In this regime, without loss of generality, we can assume \( a = 1 \). For plane-wave linear perturbations:

\[
\delta f = \Delta(p) \exp\left[i(k \cdot x - \omega \eta)\right], \quad (24)
\]

\[
\delta m_\nu = \Sigma \exp\left[i(k \cdot x - \omega \eta)\right], \quad (25)
\]

Eq. (20) gives:

\[
\omega \Delta(p) = (k \cdot u) \Delta(p) - \gamma^{-1} \left( k \cdot \frac{\partial f}{\partial p} \right) \Sigma, \quad (26)
\]

while \( \partial V/\partial A = 0 \) constraint (Eq. 21) yields

\[
\int \gamma^{-1} \Delta(p) d^3p + \lambda \Sigma = 0, \quad (27)
\]

where

\[
\lambda = \frac{\partial^2 V_0}{\partial m_\nu^2} + \int \frac{p^2}{(p^2 + m_\nu^2)^{3/2}} \tilde{f}(p) d^3p
\]

\[
= \frac{\partial^2 V_0}{\partial m_\nu^2} + \left( \frac{n_\nu}{m_\nu} \right) \left( \frac{p_{ph}^2}{m_\nu^2} \right) + O \left( \frac{p}{m_\nu} \right)^4. \quad (28)
\]

We first notice that that there is no preferred scale in Eq. (28), as only the combination \( \omega/k = c_s \) (speed of sound), appears in the equation. This is contrary to what happens for gravity where there is a scale, the Jean's
scale, below which random motions stabilize perturbations. The reason for the difference is that for scales $k < m_\nu$ the acceleron field adjusts to the local density rather than satisfy a Poisson like equation.

Taking the first term on the right hand side to the left, dividing by $\omega - k \cdot u$, and substituting from Eq. (21) yields:

$$\Delta(p) = \frac{\langle \mathbf{k} \cdot \frac{\partial \hat{f}}{\partial \mathbf{p}} \rangle}{\gamma(p)(c_s - k \cdot u)\lambda} \int \frac{d^3 p}{\gamma^2(p)(c_s - k \cdot u)}.$$

(29)

We can eliminate the integral of the unknown amplitude $\Delta(p)$ through multiplying both sides by $\gamma^{-1}(p)$, and integrating over $d^3 p$, which gives the characteristic equation for $c_s$

$$\lambda = \int \frac{d^3 p}{\gamma^2(p)(c_s - k \cdot u)} \left( \mathbf{k} \cdot \frac{\partial \hat{f}}{\partial \mathbf{p}} \right).$$

(30)

In the limit of $u^2 \ll |c_s|^2$, we can expand the argument of the integral in Eq. (30) in powers of $u$, and ignore $O(u^4)$ terms. The angular parts of the integrals can be taken using the spherical symmetry of $\hat{f}(p)$, which yields:

$$\lambda = \int (4\pi p^2 dp) \left( \frac{p}{3m_\nu c_s^2} \right) \left( \frac{\partial \hat{f}(p)}{\partial p} \right) \times \left[ 1 + \left( \frac{3}{5c_s^2} - \frac{3}{2} \right) \left( \frac{p^2}{m_\nu} \right)^2 + O \left( \frac{p^4}{m_\nu} \right) \right]$$

$$= -\frac{n_\nu}{m_\nu} \left[ 1 + (c_s^{-2} - 5/2) \frac{p^2}{m_\nu}^2 + O \left( \frac{p^4}{m_\nu} \right) \right],$$

(31)

where, in the last step, we used integration by parts. Combining Eqs. (28) and (31) yields:

$$c_s^2 = -\frac{n_\nu}{m_\nu} \left[ 1 + (c_s^2 + c_s^{-2} - 5/2) \frac{p^2}{m_\nu}^2 + O \left( \frac{p^4}{m_\nu} \right) \right].$$

(32)

Finally, following Eq. (22), the relation between $n_\nu$ and $m_\nu$ is also corrected at finite temperature:

$$n_\nu = -\left[ 1 + \frac{(p^2)}{2m_\nu^2} + O \left( \frac{p^4}{m_\nu} \right) \right] \frac{\partial V_0}{\partial m_\nu},$$

(33)

which can be substituted in Eq. (32) to give:

$$c_s^2 = m_\nu^{-1} \left( \frac{\partial^2 V_0}{\partial m_\nu^2} \right)^{-1} \times \left[ 1 + (c_s^2 + c_s^{-2} - 2) \frac{p^2}{m_\nu^2} + O \left( \frac{p^4}{m_\nu} \right) \right].$$

(34)

Note that in the limit of $p \ll m_\nu$, this result is equivalent to Eq. (34) for non-relativistic neutrinos, as $n_\nu \simeq -\partial V_0/\partial m_\nu$ in this limit, so that

$$m_\nu^{-1} \left( \frac{\partial^2 V_0}{\partial m_\nu^2} \right)^{-1} = \frac{\partial \ln m_\nu}{\partial \ln n_\nu} = -1,$$

(35)

where the last equality is for the logarithmic potential.

Moreover, we should point out that the sign of the sound speed is set by the sign of the second derivative of $V(A)$ at the extremum, $\partial^2 V / \partial A^2 \equiv m_A^2$. We have

$$\frac{\partial^2 V_0}{\partial m_\nu^2} \sim \frac{\partial^2 V}{\partial A^2} = \frac{\partial^2 V}{\partial A^2} \left( \frac{\partial m_\nu}{\partial A} \right)^{-2}$$

for $p^2 \ll m_\nu^2$. (36)

so that

$$c_s^2 = -\frac{n_\nu}{m_\nu m_A^2} \left( \frac{\partial m_\nu}{\partial A} \right)^2.$$

(37)

Therefore, any realistic MaVaNs scenario with $m_A^2 > 0$ becomes unstable to hydrodynamic perturbations in the non-relativistic regime.

In particular, for the Fermi-Dirac neutrino phase space density (Eq. 24), we have $(p^2)/T_\nu^2 = 15\zeta(5)/\zeta(3) \simeq 12.9$, and assuming the Logarithmic model of Sec. 11 $V_0 \simeq -\ln m_\nu + \text{const.}$, we end up with

$$c_s^2 = -1 + 51.8 \frac{(T_\nu/m_\nu)^2}{(T_\nu/m_\nu)^4} + O \left( \frac{T_\nu^2}{m_\nu^4} \right).$$

(38)

Thus, we see that for the logarithmic potential, the model is unstable if $T_\nu \lesssim m_\nu/7.2$. Using Eq. (34), it is easy to see that the unstable regime corresponds to $T_\nu \lesssim 1.2\Lambda$.

IV. THE OUTCOME OF THE INSTABILITY: NEUTRINO NUGGETS

As we argued above, there is no preferred macroscopic ($> m_A^{-1}$) scale in the linear problem. In particular, the free-streaming of neutrinos acts uniformly on all macroscopic scales, and may only stabilize the perturbations of relativistic neutrinos. Therefore, the inevitable outcome of the instability is the formation of non-linear structures in neutrino density or neutrino nuggets. As the instability is the fastest at smallest scales we expect nuggets to form instantly at scales larger than $m_A^{-1}$, after the onset of instability. The non-linear nuggets may then slowly merge to form larger nuggets. However, one may also envision a scenario in which non-relativistic neutrinos are continuously heated back up to relativistic temperatures, as a result of instability, and at the expense of reducing the vacuum energy of the $A$ field. We investigate this possibility below:

A. Is it possible to have a homogeneous and marginally stable phase?

Assuming that this scenario works, neutrinos would be constantly heated up at microscopic scales through scattering off non-linear neutrino nuggets. As the instability, and hence scattering occurs at microscopic scales, neutrinos will have a microscopic mean free path, which implies
that evolution should be adiabatic on macroscopic scales. The assumption of marginal stability then implies:
\[ c_s^2 = \frac{\dot{\rho}}{\rho} = 0 \Rightarrow P = \text{const}, \]

which is necessary to maintain homogeneity and marginal instability at the same time. However, using energy conservation, \( \dot{\rho} + 3H(\rho + P) = 0 \), we can see that \( \rho \) and \( P \) can only have a ΛCDM-like behavior:
\[ \rho = A + \frac{B}{a^3}, \]
\[ P = -A, \]

where \( A \) and \( B \) are integration constants. In particular, we require that
\[ \rho + P = n_\nu m_\nu (\gamma (1 + \frac{u^2}{3})) = \frac{B}{a^3}, \]

where we used the fact that \( P = n_\nu m_\nu (\gamma u^2)/3 \) for the neutrino gas. However, using the fact that \( n_\nu \propto a^{-3} \), we find
\[ m_\nu (\gamma (1 + \frac{u^2}{3})) = \frac{B}{n_{0,\nu}} = \text{const.} \Rightarrow m_\nu \leq \frac{B}{n_{0,\nu}}. \]

The fact that \( m_\nu \) reaches a maximum, implies that either \( V_0(m_\nu) \) (or \( V_0(A) \)) reaches a minimum, or \( \langle \gamma^{-1} \rangle \propto n_\nu^{-1} \propto a^{-3} \) (eq. 22). The latter, however, can only last for less than a Hubble time, as \( \gamma \sim 1 \) at the onset of instability, while \( \langle \gamma^{-1} \rangle < 1 \).

Therefore, we see that the only way to maintain homogeneity of marginally stable neutrino gas is for the acceleron to settle at its true minimum, following which the neutrinos are decoupled and become non-relativistic, while the evolution follows the standard ΛCDM scenario.

B. Production of Neutrino Nuggets: Qualitative description of a multi-phase outcome

In Sec. IV.A we argued that the instability cannot be followed by a homogeneous (single phase) state of the neutrino-acceleron fluid, unless the acceleron settles at its true minimum (i.e. vacuum). If this does not happen, the outcome can only be a multi-phase medium.

As result of the instability, the neutrino fluid should fragment into condensed neutrino nuggets (or the liquid /interactive phase), within which, neutrinos maintain relativistic energies to have a stable density distribution. The remaining neutrinos remain in a tenuous gas phase, and should remain relativistic to maintain their stability. Moreover during the instability there is a possibility of converting part of the energy into \( A \) particles. The \( A \) particles are not trapped inside the nuggets and their contribution to the energy density diminishes with the Hubble expansion.

As neutrinos in the “gas phase” continuously lose energy to Hubble expansion, and nuggets, due to their small volume fraction, cannot significantly affect the gas phase, most of the neutrinos should accrete into the nuggets. In fact, as we show in Sec. VII within the assumption of thermal equilibrium, this process instantly exhausts the neutrinos outside the nuggets, unless they can decouple (i.e., \( A \) settles at its true vacuum) within the gas phase.

The fraction of neutrinos in the gas phase is set by the equilibrium of evaporation rate from and accretion rate onto the surface of the nuggets. However, the large density contrast between inside and outside of the nuggets implies that only neutrinos with a large Lorentz factor (\( \gamma > m_{\text{out}}/m_{\text{in}} \)) or \( \gamma \) escape the nuggets. As \( \langle \gamma \rangle \) inside nuggets is limited by a few times its initial value at the onset of the instability (where \( \langle \gamma \rangle \approx 1 \)), the equilibrium can only be maintained when a small fraction of neutrinos remain outside the nuggets.

Thus we conclude that the most probable outcome of the instability is the formation of dense non-linear structures of neutrinos, surrounded by practically empty space where the acceleron has settled to its minimum. All this is happening at microscopic scales. As far as the expansion of the universe is concerned, the fluid energy density will redshift as matter and thus cannot drive the acceleration of the Universe.

C. Production of \( A \) particles

Another possible outcome of the instability is production of \( A \) particles. In fact, one may speculate that, since the characteristic time and length scale for the instability of non-relativistic MaVaNs is \( m_A^{-1} \), a significant fraction of neutrino/acceleron energy may end up in \( A \) particles (or accelerons). However, we should note that, at the onset of the instability, the neutrinos are still marginally relativistic, implying that the characteristic time for the instability may be significantly longer.

Since the speed of sound (and thus the instability) is modulated by the Hubble expansion, it is reasonable to approximate \( c_s^2 \) as:
\[ c_s^2 = -\alpha(H\Delta t) + O(H\Delta t)^2, \]

close to the onset of the instability, where \( \alpha \approx 1 \), is constant of order unity. The maximum growth occurs at the scale of \( m_A^{-1} \), and is given by
\[ \Delta \propto \exp(|c_s| m_A \Delta t) = \exp(\alpha H^{1/2} \Delta t^{3/2} m_A). \]

Therefore, the characteristic growth time for the instability is given by
\[ \Delta t_{\text{ins}} \sim \left(\frac{m_A}{H}\right)^{1/3} m_A^{-1} \gg m_A^{-1}, \]

since, by construction, acceleron is much heavier than Hubble scale in MaVaNs theories (see Sec. IV). This
implies that the production of $A$ particles cannot be a significant energy sink, at least during the initial stage of instability (or nugget formation).

Subsequent mergers of relativistic $m_A^{-1}$-sized nuggets may potentially lead to $A$ particle production. However, the above argument can also be used to show that the velocities of produced nuggets, $v_{\text{nug}}$, can never be relativistic:

$$v_{\text{nug}} \lesssim \frac{m_A^{-1}}{\Delta t_{\text{ins}}} \ll 1.$$  \hspace{1cm} (47)

V. NUGGET-GAS PHASE TRANSITION: THE THERMODYNAMIC PICTURE

In Sec. IV B we presented a qualitative argument for why most of neutrinos must condense into nuggets as they become non-relativistic. In this section, we present a more rigorous picture of the associated phase transition within the assumption of thermodynamic equilibrium.

However, we should first note that, in the homogeneous phase, the thermodynamic approximation is clearly invalid, as the cross-section for neutrino interaction is small, and thus, by construction, thermal relaxation takes much longer than the Hubble time [11]. However, in the relativistic regime, the distribution remains thermal as the expansion does not change the original shape of the thermal Fermi-Dirac distribution. After neutrinos become non-relativistic, the instability develops within microscopic time and length scales, and thus, it seems plausible to assume that the system reaches some sort of thermal equilibrium within microscopic times.

Assuming thermodynamic equilibrium, the phase space density of neutrinos is given by the Fermi-Dirac distribution:

$$dn_{\nu} = f(E_{\nu}) \frac{d^3p}{(2\pi)^3} = 2 \left[ \exp \left( \frac{E_{\nu}(p) - \mu}{T} \right) + 1 \right]^{-1} \frac{d^3p}{(2\pi)^3},$$  \hspace{1cm} (48)

where the additional factor of 2 accounts for both neutrinos and anti-neutrinos. $\mu$ is the chemical potential, which can be non-zero if the neutrino number is conserved, while $T$ stands for the thermodynamic temperature.

Similar to previous sections, we assume that the acceleron field, $A$, is relaxed at its minimum. Therefore, the system is subject to the constraint condition of Eq. (21), which in conjunction with the condition:

$$n_{\nu} = \int \frac{d^3p}{(2\pi)^3} f(p),$$  \hspace{1cm} (49)

set the equation of state of the system, as they fix all thermodynamic variables (e.g., $\mu$ and $m_{\nu}$) for given neutrino number density, $n_{\nu}$, and temperature, $T$. In particular, the pressure of the system is given by:

$$P = \frac{1}{3} n_{\nu} m_{\nu} \langle \gamma u^2 \rangle - V_0(m_{\nu}).$$  \hspace{1cm} (50)
Fig. 1 shows the pressure as a function of specific volume (≡ $n^{-1}$) for the system described by these equations for the logarithmic potential of Sec. IV with $n_0 = 10^3 \Lambda$ (Eq. 10). The solid curves show different temperatures of $T_\nu = 2^\ell \Lambda$ for integer $\ell$'s, where the highest plotted temperature is 1024 $\Lambda$. The curves become degenerate below $T_\nu \sim 0.5 \Lambda$, as the degeneracy pressure dominates the thermal pressure. The dotted curve corresponds to $\langle \gamma \rangle = 2$, which separates the relativistic and non-relativistic regimes.

A homogeneous medium becomes unstable if its pressure decreases with increasing density. In Sec. IV we saw that this happened when neutrinos become non-relativistic, which is also clearly seen in Fig. 1, where $\partial P/\partial n$ changes sign (i.e. isothermal curves reach a minimum) at the boundary of relativistic and non-relativistic regimes.

The dashed curve in Fig. 1 is the adiabat with $\mu = 0$ as $T \rightarrow \infty$, which approximates the cosmic history of MaVaNs for the logarithmic potential. We again see that the pressure reaches a minimum (rendering the system unstable), when the neutrinos become non-relativistic, which also happens close to where the adiabat crosses $T = \Lambda$ isothermal curve, as demonstrated in Sec. IV. Therefore, as the density of neutrinos decreases monotonically due to cosmic expansion, they become unstable and fragment into nuggets at $n_\nu \sim \Lambda^3$, as no stable phase is available at lower densities. This can also be seen in the asymptotic (large $n_\nu$) limit, as the kinetic pressure is simply given by $n_\nu T$, which is sub-dominant to the acceleron pressure for $m_\nu \gg m_0$, as $-V_0 = -\Lambda^3 \ln(1 + m_0/m_\nu) \propto -n_\nu^{1/2}$, using the mass-density relation in the relevant range. Therefore, $\partial P/\partial n_\nu$ remains negative at smaller densities, and so the system does not have any other stable phase, justifying the qualitative picture of Sec. IV.

A more interesting case of a finite maximum neutrino mass, with $m_{\text{max}} = 10^3 \Lambda$ is shown in Fig. 2. The curves are the same as in Fig. 1. Here, we see that for densities lower than a critical density (which depends on temperature), the neutrino mass is fixed at its maximum, implying that the acceleron field relaxes at its true vacuum. Therefore, we end up with a gas of free-streaming neutrinos with the acceleron settled at the minimum. As a result, the system shows two stable phases of interactive (or liquid) mass-varying neutrinos and non-interactive (or gas) neutrinos with $m_\nu = m_{\text{max}}$. The two phases can co-exist only if they have the same pressure, temperature and chemical potential.

Fig. 3 is a blow-up of the $T = 16\Lambda$ isothermal contour from Fig. 2. Points A and C on the isothermal contour are liquid and gas phases which can coexist at this temperature. A pictorial way of finding the coexistence states is via Gibbs construction. Introducing $\epsilon$ and $s$, as the energy and entropy per unit particle, the thermodynamic and extensivity relations subsequently take the form

$$\frac{d\epsilon}{d\mu} = T d s - P d n^{-1}, \quad (51)$$

$$\epsilon = T s - P n^{-1} + \mu, \quad (52)$$

which can be combined to give

$$d\mu = n^{-1} dP - s dT. \quad (53)$$

Integrating over the isothermal contour ABC, the condition of equal chemical potentials for the coexisting phases A and C takes the form:

$$\mu_C - \mu_A = \int_{A B C} n^{-1} dP = S_{AB} - S_{BC} = 0. \quad (54)$$

Therefore, the area enclosed in regions $S_{AB}$ and $S_{BC}$ must be the same (even though it may not look this way due to the logarithmic scale in Fig. 3).

A. Onset of phase transition: bubble nucleation vs. linear instability

Even though the neutrino-acceleron fluid does not become unstable to linear perturbations until it becomes non-relativistic, it can become unstable to perturbations of finite amplitude, otherwise known as bubble nucleation,
much earlier. We will argue however that this is not relevant in the cosmological context being studied.

Consider a bubble of true vacuum with size larger than $m_A^{-1}$ in the relativistic interactive (liquid) phase. If the pressure of the ambient medium is smaller than the vacuum pressure inside the bubble (which is set to zero in our treatment), the bubble rapidly expands until it is in pressure equilibrium with the ambient medium. This can in principle be achieved through contraction of the medium as a result of bubble nucleation.

However, the onset of bubble nucleation depends on the statistics of seed bubbles present in the system. In particular, the smaller the value of $m_A$, the larger will be the minimum size of the bubble that can trigger a phase transition. The bounds on $m_A$, which come from the requirement of no thermal acceleron production in early universe [11], imply more that $10^{24}$ neutrinos in a volume of $m_A^{-3}$, making it extremely unlikely for seed (vacuum) bubbles to exist in a Hubble volume. Cosmological fluctuations of neutrino fluid are also damped as a result of free-streaming, and thus cannot seed bubble nucleation. Therefore, the phase transition is unlikely to start until the pressure reaches its minimum, where small linear perturbations can seed the instability.

The outcome of the instability, which develops on microscopic time scales, can be uniquely obtained from conservation of total energy and particle densities. For example, for the case of $m_{\text{max}} \to \infty$ where pressure and density vanish in the vacuum, the final density of fragmented nuggets is roughly at the intersection of $\mu = 0$ adiabat (dashed curve in Fig. [1]) with the $P = 0$ axis, which is approximately where $T_\nu \sim \Lambda[\ln(m_0/\Lambda)]^{1/4}$.

**B. The fate of nugget-gas multi-phase state**

Finally, let us consider the fate of the system of coexisting neutrino nuggets and gas. Assuming that $m_\nu \gtrsim m_{\text{max}} \gtrsim \Lambda[21]$, the area of the region $S_{AB}$ (see Fig. [3]) is dominated by the non-relativistic regime and is given by:

$$S_{AB} \sim \int_{AB} n^{-1} dP \simeq m_{\text{max}} + O\left(\Lambda, \frac{m_{\text{max}}^2}{m_0}\right).$$

Neglecting the degeneracy pressure (which can be easily justified for $m_{\text{max}} \gtrsim \Lambda$), the area of region $S_{BC}$ is given by:

$$S_{BC} = \int_{\text{nug}} \Lambda^4 m_{\text{max}} n^{-1} T da = -T \ln(n_{\text{gas}} m_{\text{max}}/\Lambda^4).$$

Therefore, the Gibbs construction $S_{AB} = S_{BC}$ yields

$$n_{\text{gas}} m_{\text{max}} = \Lambda^4 \exp \left(\frac{-m_{\text{max}}}{T}\right) \ll \Lambda^4$$

for $T \lesssim \Lambda \lesssim m_{\text{max}}$.

As we anticipated in Sec. [III] as long as the phase transition occurs far enough from the true minimum of $V_0(m_\nu)$, the density of neutrinos in the gas phase (where $\mathcal{A}$ relaxes at its true vacuum) is exponentially suppressed. The subsequent evolution follows from requiring energy conservation:

$$d(\bar{n}_\nu a^3 T) = -\bar{n}_\nu a^3 m_{\text{max}} \left[\frac{1}{\ln(n_{\text{gas}} m_{\text{max}}/\Lambda^4)}\right]$$

$$= -n_{\text{gas}} T da^3 = \frac{n_{\text{gas}} m_{\text{max}} da^3}{\ln(n_{\text{gas}} m_{\text{max}}/\Lambda^4)},$$

which can be approximately solved to give:

$$\frac{n_{\text{gas}}}{\bar{n}_\nu} \propto \frac{T}{m_{\text{max}}} = -\frac{1}{\ln(n_{\text{gas}} m_{\text{max}}/\Lambda^4)}$$

$$\sim \frac{1}{m_{\text{max}}/\Lambda + 3 \ln(a/a_0)},$$

where $\bar{n}_\nu$ denotes neutrino cosmic mean density, and $a_0$ is the cosmological scale factor at phase transition (where $T \sim \Lambda$). Therefore, both the fraction of neutrinos in the gas phase, as well as the equilibrium temperature, remain essentially unchanged their cosmic history.

**VI. DISCUSSION**

**A. How about a light neutrino mass eigenstate?**

We have shown that unless the acceleron field sets the mass of a relativistic particle, hydrodynamic perturbations are unstable. The Big Bang model predicts a kinematic temperature of $T_\nu \simeq 0.7 T_{\text{CMB}} \simeq 1.7 \times 10^{-4} \text{ eV}$ for relic cosmological neutrinos (e.g., [18]). Current experimental limits on neutrino oscillations require the most massive neutrino eigenstate, and thus the acceleron/dark energy dynamics is modulated by the density of a relativistic neutrino eigenstate. Even this seems difficult. For the logarithmic potential, the stability of this scenario requires $T_\nu \gtrsim 1.2 \Lambda$ (Sec. [III]B), which is already constrained, up to a logarithmic factor, by the current dark energy density ($\Omega_{DE} \simeq 0.7$):

$$\left(2.3 \times 10^{-3} \text{ eV}\right)^4 = (\rho_{DE} \simeq \Lambda^4 \ln(m_0/m_\nu))$$

$$\lesssim 0.5 \ T_\nu^4 \ln(m_0/m_\nu) \simeq (1.4 \times 10^{-4} \text{ eV})^4 \ln(m_0/m_\nu),$$

requiring

$$\ln(m_0/m_\nu) \gtrsim 7 \times 10^4,$$

which does not seem realistic.

Even if we do not restrict $V_0(m_\nu)$ to a logarithmic form, assuming a Fermi-Dirac distribution (Eq. [23]), we have

$$1 + w \simeq \frac{0.6}{\rho_{DE}} T_\nu^4 \sim 10^{-5},$$

$$\left(\frac{m_\nu}{E_\nu}\right)^{1/4} \sim 3 \times 10^{-4}$$

for $m_\nu \gtrsim 10^{-2} \text{ eV}$.
for any relativistic MaVaNs theory, implying a very slow evolution that is indistinguishable from the ΛCDM scenario. One may hope to be able to test this model by measuring neutrino oscillations (or structure formation) at high redshifts, as neutrino mass scales as the inverse of the temperature of neutrino background in such models \((m_\nu \propto T_\nu^{-1} \propto (1 + z)^{-1}).\)

### B. Adiabatic dark energy and cosmological constant

In Sec. IV A we showed that an adiabatic dark energy model with \(c_s = 0\) behaves exactly like a ΛCDM cosmology. The argument can be easily generalized for any stable dark energy model (\(c_s^2 > 0\)) which satisfies the null energy condition (\(\rho_{DE} + P_{DE} > 0\)) in the following way. Combining the stability condition and conservation of energy we have

\[
c_s^2 = \frac{\dot{P}_{DE}}{\rho_{DE}} = -\frac{\dot{P}_{DE}}{3H(\rho_{DE} + P_{DE})} > 0,
\]

which can be integrated to give

\[
P_{DE}(t \to \infty) = P_{DE}(t_0) - 3\int_{t_0}^{\infty} H(t) c_s^2(t) [\rho_{DE}(t) + P_{DE}(t)] dt < P_{DE}(t_0) < 0.
\]

Since dark energy has negative pressure at the present time (\(P_{DE}(t_0) < 0\), Eq. (64) guarantees that the pressure remains negative and finite, even when all particle densities approach zero at future infinity. Therefore a stable adiabatic dark energy model will necessarily asymptote to a cosmological constant/vacuum energy at future infinity (or it goes unstable).

### VII. CONCLUSIONS

We have demonstrated that the model of dark energy with mass-varying neutrinos (MaVaNs) is subject to a linear instability with an imaginary speed of sound. The onset of instability is around the time that neutrinos become non-relativistic, and is due to the fact that, unlike quintessence perturbations, the fluctuations in the non-relativistic neutrino density evolve adiabatically. The acclerons field mediates a strong attractive force (stronger than gravity) between neutrinos. There are more than \(10^{24}\) neutrinos inside the Compton wavelength of the \(\mathcal{A}\) field. When neutrinos become non-relativistic, their random motions are not sufficient to stop collapse and the medium becomes unstable.

The question of the outcome of the instability, an inherently non-linear process, is a more difficult problem to address. We present qualitative arguments for why the outcome of the instability should be a multi-phase medium, where most of neutrinos end up in dense nuggets (interactive/liquid phase) with nearly constant density, while a small fraction may remain outside in a decoupled gas phase. We then confirm these arguments quantitatively within the assumption of thermodynamic equilibrium, which is plausible following the phase transition. Both nuggets, and free neutrinos (which are exponentially suppressed) dilute similar to cold dark matter particles, with negligible pressure, leaving a vacuum energy/cosmological constant as the only possible source of the observed cosmic acceleration after the onset of instability in the MaVaNs models.

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[14] M. Cirelli, M. C. Gonzalez-Garcia, and C. Pena-Garay (2005), hep-ph/0503028.
[15] V. Barger, P. Huber, and D. Marfatia (2005), hep-ph/0502196.
[16] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rept. 215, 203 (1992).
[17] R. Kubo, Statistical Mechanics (North-Holland, 1965).
[18] T. Padmanabhan, Structure formation in the universe (Cambridge, UK: Cambridge University Press, —c1993, 1993).
[19] S. Eidelman et al. (Particle Data Group), Phys. Lett. B592, 1 (2004).
[20] One can see that the ratio of acceleron kinetic to potential energy is \( \sim (1 + w)^2(H/m_A)^2 \ll 1 \).
[21] Note that \( m_0 \gtrsim m_{\text{max}} \) assumption is not fundamental, and is only made to simplify the expressions. The \( m_0 \lesssim m_{\text{max}} \) regime can be treated in exactly the same way which results in a similar behavior to the one obtained in the text.