INTEGRABLE FIELD THEORY WITH BOUNDARY CONDITIONS

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1. Introduction

The purpose of these talks is to review some of the ideas surrounding the topic of integrability, either classical or quantum, when two-dimensional field theory (i.e. one space—one time) is restricted to a half-line, or to a segment of the line.

There are some simple questions which may be asked but they do not turn out to have simple answers. Moreover, there are some surprises and unexpected constraints once particular models are examined. The affine Toda field theories, or $\sigma$-models provide excellent illustrations.

For example, suppose an integrable field theory describes a collection of distinguishable particles (real affine Toda field theory has this property—the scalar particles are distinguished by conserved charges of non-zero spin), what is the spectrum of particle energies if the system is enclosed on the interval $[-L,L]$?

Alternatively, if the theory is confined to the space region $x < 0$, and a particle approaches $x = 0$ from the left one might expect the minimal effect of the boundary to be a reversal of all the momentum-like conserved quantities and the preservation of all energy-like conserved quantities. If that is the case, the particle preserves its identity but reverses its direction. It might be expected that the ‘out’ state consisting of a single particle is proportional to the ‘in’ state with the momentum reversed. In other words:

$$|k >_{\text{out}} = K_a(k)|k >_{\text{in}}$$  \hspace{1cm} (1.1)

where $a$ is a particle label. It might happen (for example in the $\sigma$-models) that the particles belong to multiplets whose members are distinguishable only via spin-zero charges. In those cases, the reflection factor $K_a$ appearing in Eq.(1.1) will be a matrix to allow for a mixing of states at the boundary. Labels corresponding to multiplicity will generally be suppressed unless there is a specific reason to display them.
If the particles of an integrable theory are distinguishable scalars then the possible momenta for a particle of type $a$ confined to the interval $[-L, L]$ might be expected to be given by the solutions to

$$e^{4ikL} K_a^{(L)}(k) K_a^{(-L)}(-k) = 1$$  \hspace{1cm} (1.2)

where the superscripts on the reflection factors take into account possibly different boundary conditions at the two ends of the interval. Eq.(1.2) may be a false assumption but it is based on the idea that in an integrable theory factorization is a fundamental property, and the effects of the two boundaries ought to be independent of each other. It is clearly true for a particle described by a free field with linear boundary conditions

$$\partial_x \phi|_{x=\pm L} = a_{\pm} \phi|_{\pm L}.$$  \hspace{1cm} (1.3)

But otherwise, of course, such a claim requires proof. If it is correct then Eq.(1.2) provides a motivation for the study of the reflection factors themselves, even in the simplest of models.

Much is now known about classical and quantum integrability for field theories defined over the whole line in a two-dimensional space-time. Several classes of models (e.g. affine Toda theory, or $\sigma$-models) have been particularly well studied, as has the sine-Gordon model. The latter is essentially the $a_1^{(1)}$ affine Toda theory with a purely imaginary field. On the other hand, the situation in which the line is truncated to a half-line, or to an interval, is hardly explored except for the choice of Neumann, or periodic boundary conditions. In particular, until recently, it was not known what boundary conditions were compatible with integrability, even classically.

There has been renewed interest in this topic following investigations in condensed matter physics in which boundaries play a significant rôle. In particular, the sine-Gordon model has been studied afresh, with a number of new results obtained, notably by Ghoshal and Zamolodchikov,\textsuperscript{1,2} and others\textsuperscript{3} (see Fendley, Saleur and Warner\textsuperscript{4}). The sine-Gordon model is the simplest of the affine Toda field theories, based on the root data of the Lie algebra $a_1$, and it is therefore natural to explore the question of boundary conditions within the context of other models in the same class\textsuperscript{5,6}.

However, before doing so, it is worthwhile to review and explore some of the properties which are expected to hold for the reflection factors in quantum field theory and which were conjectured some time ago by Cherednik\textsuperscript{7}.

2. Quantum integrability: conjectures

The question of quantum integrability in the presence of a boundary is difficult to tackle. It appears the best one can hope to do at present is to set out a set of hypotheses which at least allow consistent conjectures to be made which may themselves be checked subsequently in various ways. For example, in certain circumstances perturbation theory might be relevant. The principal ideas were set out by Cherednik\textsuperscript{7} many years ago and have been supplemented recently in work of Ghoshal and Zamolodchikov\textsuperscript{1} concerning
the sine-Gordon theory, and by Fring and Köberle\textsuperscript{5} and by Sasaki\textsuperscript{6} concerning the affine Toda theories. Consider first the situation with a single boundary.

The principal idea is that particle states in the presence of the boundary (taken to be at $x = 0$) continue to be eigenstates of energy and other energy-like conserved charges. However, an initial state containing a single particle moving towards the boundary will evolve into a final state with a single particle moving away from the boundary. Thus

$$|a, v\rangle_{\text{out}} = K_{ab}(\theta)|b, -v\rangle_{\text{in}}, \quad (2.1)$$

where the states $a, b$ correspond to multiplets of particles distinguishable merely via spin-zero charges, and $K_{ab}$ is a matrix which may mix the particles as a result of the reflection from the boundary. For real affine Toda theory the particles are all distinguishable and therefore there should be a set of reflection factors, one for each particle, corresponding to each integrable boundary condition. The velocity of a particle is reversed on reflection, and this is equivalent to reversing its rapidity (defined by $v = \tanh \theta$). The first major task is to determine a set of $K$ factors for a specified boundary condition.

This situation is represented pictorially by:

\begin{center}
\begin{tikzpicture}
    \draw[thick] (0,0) -- (0,4);
    \node at (0,4) [circle,fill,inner sep=1.5pt] {};
    \node at (0,4) [below] {$a, v$};
    \node at (0,0) [below] {$a, -v$};
\end{tikzpicture}
\end{center}

where the vertical line represents the world-line of the boundary at $x = 0$.

It is also supposed that particles scatter factorizably, but independently of the boundary, when they are far from it. As mentioned before, such an assumption requires proof within any particular model. It is a plausible assumption since the boundary would not be expected to affect particles moving towards it until they were ‘close’ by. However, if it is true then the usual arguments\textsuperscript{8} require the S-matrix describing the scattering of two-particle states:

$$|\theta_a, \theta_b\rangle_{\text{out}} = S_{ab}(\theta_a - \theta_b)|\theta_a, \theta_b\rangle_{\text{in}} \quad (2.2)$$

to satisfy a Yang-Baxter equation

$$S_{ab}(\theta_a - \theta_b)S_{ac}(\theta_a - \theta_c)S_{bc}(\theta_b - \theta_c) = S_{bc}(\theta_b - \theta_c)S_{ac}(\theta_a - \theta_c)S_{ab}(\theta_a - \theta_b). \quad (2.3)$$

As in Eq.(2.1), the subscripts denote particle types, the matrix labels have been omitted deliberately.

Eq.(2.3) may be represented pictorially by:
This is something of a constraint unless the S-matrix factors are merely sets of phases (as they are in a situation where the particles are distinguishable), but it fails to determine the S-matrix completely. If the particles are distinguishable, Eq.(2.3) is an identity. The general solution to the Yang-Baxter equation is not known although general classes of solution have been found, including those related to the theory of quantum groups (see for instance the reprint volume by Jimbo\(^9\)).

On the other hand, a two-particle state consisting of two incoming particles will eventually evolve into a state containing the two outgoing particles. However, each of the particles will scatter from the boundary and, inevitably from each other. But, the order of the individual scatterings and reflections should not matter because they depend (supposedly inessentially) on the initial condition setting up the two-particle state. If it is supposed that these events also take place factorisably then one obtains the reflection Yang-Baxter equation. Algebraically, the relationship is

$$K_a(\theta_a)S_{ab}(\theta_b + \theta_a)K_b(\theta_b)S_{ab}(\theta_b - \theta_a) = S_{ab}(\theta_b - \theta_a)K_b(\theta_b)S_{ab}(\theta_b + \theta_a)K_a(\theta_a). \quad (2.4)$$

For affine Toda field theory where the particles are distinguishable, $K$ and $S$ are diagonal, and the boundary Yang-Baxter equation is satisfied identically.

Pictorially, Eq.(2.4) would be represented by:
For the whole line theory there is a consistent bootstrap principle, in the sense that there is a consistent set of couplings between the particles, signalled by the presence of poles in the S-matrix at certain (imaginary) relative rapidities, and these may be used to relate the S-matrix elements to each other. The set of bootstrap relations take the form

$$S_{dc}(\theta_d - \theta_c) = S_{da}(\theta_d - \theta_a)S_{db}(\theta_d - \theta_b) \quad (2.5)$$

where

$$\theta_a = \theta_c - i\bar{\theta}_{ac}^b \quad \theta_b = \theta_c + i\bar{\theta}_{bc}^a \quad (2.6)$$

$\bar{\theta} = \pi - \theta$, and the coupling angles are the angles of the triangle with side-lengths equal to the masses of particles $a$, $b$ and $c$ participating in the coupling $ab \to \bar{c}$. Not every pole in the S-matrix factors signal the existence of a bound state, however. In the real affine Toda theories corresponding to data for the $a^{(1)}$, $d^{(1)}$ or $e^{(1)}$ series of affine root systems, the relevant poles are those of odd order with a coefficient of the correct sign$^{10}$. Moreover, the couplings themselves follow Dorey’s rule$^{11}$, which is intimately related to the properties of root systems and the Coxeter element of the Weyl group. For the other theories, based on affine root data corresponding to the non simply-laced diagrams, the situation is rather different. These cases, demonstrate some remarkable phenomena in the sense that the quantum bound states do not appear to have mass ratios equal to the classical values (in the ade cases they do). Rather, the masses ‘float’ between the classical values corresponding to dual pairs (see later in section(3)). The details may be found in Delius et al. and elsewhere$^{12}$. For these, the geometrical setting for the couplings (that is, a suitable generalization of Dorey’s rule) is not known.

Assuming the family of whole line couplings remains relevant in the presence of a boundary, the bootstrap implies relations between the various reflection factors. At the special imaginary relative rapidities corresponding to a bound state, the two particle $a$, $b$ states have quantum numbers identically equal to the quantum numbers of particle $c$. One might think picturesquely of either $a$, $b$ separately reflecting from the boundary in advance (or after) the bound state forms, or the particle $c$ reflects from the boundary. Algebraically, the reflection bootstrap equation is:

$$K_c(\theta_c) = K_a(\theta_a)S_{ab}(\theta_b + \theta_a)K_b(\theta_b), \quad (2.7)$$

where $\theta_a, \theta_b$ are given in Eq.(2.6).
This too has a pictorial representation:

There is also the possibility of bound states involving a particle and the boundary—with their own coupling angles and bootstrap property (see\textsuperscript{1,13,14}). The idea is that the boundary can be excited, and that this possibility should be indicated by the existence of poles in $K_a(\theta_a)$. For example, particle $a$ and boundary-type $\alpha$ might have a reflection factor $K^\alpha_a$ with a suitable pole at $\theta_a = i\psi^{\beta\alpha}_{aa}$, which is to be interpreted as a signal for a boundary bound state labelled $\beta$, say. One would expect the ‘physical’ strip for $\psi^{\beta\alpha}_{aa}$ to lie in the range $[0, \pi/2]$. A two-particle state consisting of $a$ and another particle $b$ would also encode the pole in $\theta_a$, in which case the the particle $b$ should be regarded as either reflecting from the boundary state $\beta$ directly, or, alternatively, reflecting from the boundary state $\alpha$. In the latter case it scatters twice with particle $a$, once before, and once after its reflection from the boundary state $\alpha$. Of course, such an interpretation also relies heavily on the factorisation assumption. Algebraically, one would write,

$$K^\beta_b(\theta_b) = S_{ab}(\theta_b + i\psi^{\beta\alpha}_{aa})K^\alpha_b(\theta_b)S_{ab}(\theta_b - i\psi^{\beta\alpha}_{aa}).$$  \hspace{1cm} (2.8)$$

Eq.(2.8) gives a set of consistency conditions which would need to be satisfied by the boundary bound states and their associated reflection factors. Almost no work has been done to determine the rules corresponding to the boundary bound state poles. Nor has there been a systematic determination of the full set of reflection factors for any specific model. It might be imagined that one could begin with a full set of ground state factors and simply calculate the rest self-consistently. However, that turns out to be easier said than done because it has simply proved impossible, so far, properly to analyse the pole structure of the reflection factors. It has even been suggested that the reflection factors might have singularities of square-root type\textsuperscript{15}. If that is genuinely so, then the whole question of the analytic structure needs to be re-examined. Even a free field with a linear boundary condition of the type given in Eq.(1.3), has a reflection factor of the form

$$K(k) = \frac{ik - a}{ik + a},$$ \hspace{1cm} (2.9)$$

containing a pole at $k = ia$. This pole corresponds to an exponentially decreasing wave function in the region $x < 0$, provided $a$ is positive, and therefore a normalizable solution.
to the field equation representing a bound state. Some ideas concerning boundary bound states within the sine-Gordon model are to be found in Ghoshal and Zamolodchikov’s paper, while some conjectures concerning the affine Toda theories are to be found elsewhere.

Pictorially, the boundary bound state bootstrap would be represented by something like:

Finally, there are the Crossing relations

\[ S_{ab}(i\pi - \Theta) = S_{\bar{a}\bar{b}}(\Theta) = S_{\bar{a}b}(\Theta) \]  

where \( \Theta = \theta_a - \theta_b \),

\[ K_a(\theta - i\pi/2)K_{\bar{a}}(\theta + i\pi/2)S_{aa}(2\theta) = 1; \]

and the Unitarity relations

\[ S_{ab}(\Theta) = S_{ab}^{-1}(-\Theta) \quad K_{a}(\theta) = K_{a}^{-1}(-\theta). \]

These are written in a manner appropriate to a model in which all particles are distinguishable (as were the bootstrap relations). If the particles belong to multiplets these relations need to be suitably modified; in the cases of Eqs.(2.11), (2.10) a crossing matrix is needed.

Even for a collection of relatively simple models such as affine Toda theory, there are many known solutions to the reflection bootstrap equations. However, it is not clear how to relate them to the possible choices of boundary condition. Presumably, it will be necessary to apply a semi-classical approximation, or to use perturbation theory, although the latter may be difficult in situations where there are no small parameters associated with the boundary potential. There has been some work in this direction by Kim, but only concerning the Neumann boundary condition. Quantum versions of the conserved quantities have also been investigated recently by Penati and Zanon.
whose calculations suggest that the boundary parameters will require renormalisation. It is notable, however, that at least in the $a^{(1)}_r$ series of cases the boundary parameters renormalise together\(^{17}\). However, it will be seen later on that the permissible boundary conditions, compatible with integrability are not usually continuously connected to the Neumann condition. Rather, the integrable boundary conditions are ‘isolated’. For this reason, perturbation using a boundary parameter ought probably to be treated with great caution.

**A classical bootstrap**

One interesting fact is the following\(^{19}\). If the classical limit is taken in Eq.\((2.7)\) then it is certainly expected that the S-matrix becomes unity. However, in the presence of a boundary, the corresponding limit of the reflection factors need not necessarily be unity; remember, even a free particle must rebound from the boundary, and a linear boundary condition of the type given in Eq.\((1.3)\) leads to a reflection factor of the form given in Eq.\((2.9)\). Rather, the classical limit $K_0$ might be expected to satisfy the classical limit of the reflection bootstrap equation \((2.7)\). Moreover, these classical limits are themselves computable via a standard linearised scattering problem involving an ‘effective potential’ determined by the (presumably static) lowest energy solution to the classical field equations. The quantum theory would have to be constructed in terms of perturbations around the basic solution but perhaps surprisingly the classical problem already has a rich structure (including, in some cases bound states)\(^{13,19}\). Some other comments on this idea will be made later, in section (5), and further details and developments may be found in a recent paper by Bowcock\(^{20}\).

**A strategy**

Thus, the following strategy suggests itself. First, determine the possible boundary conditions compatible with integrability, at least in the sense of maintaining as many of the desirable features usually associated with integrability on the whole line, such as an infinity of conserved quantities in involution. Second, determine the classical reflection factors associated with each of the allowed possibilities. Third, use these classical limits as a guide to guess suitable solutions to all the above consistency relations. Finally, check the hypothetical reflection factors against perturbation theory.

The first part of this programme can be carried through fully for the affine Toda theories and that work forms the principal content of the next sections. The second part has been carried out in many examples and it is fascinating to see the classical limit of the bootstrap relation actually working in practice for the ‘classical’ reflection factors. Often these rely on wonderful identities. Ultimately, though, the beautiful structure can be traced to the properties of what might be called ‘solitons’ in the real affine Toda field theories. These will be discussed briefly in section (5).

On the other hand, it must be said that although this is a clear enough strategy, there are a number of serious obstacles in the way of carrying it out fully. In particular, the final part of the programme needs to be tackled and the necessary perturbation theory has been neglected up to now.
3. The affine Toda theories

The affine Toda field theories (for a recent review, see\textsuperscript{21}) are scalar theories with Lagrangian

\[ \mathcal{L}_0 = \frac{1}{2} \partial_{\mu} \phi \cdot \partial^{\mu} \phi - \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \phi}. \]  

(3.1)

In this Lagrangian \( m \) and \( \beta \) are two constants (which may be removed from the classical field equations by a rescaling of the fields and the space-time coordinates); the important information is carried by the set of vectors \( \alpha_i \) and the set of integers \( n_i \).

The vectors \( \alpha_i, \ i = 1, 2, \ldots, r \) are a set of simple roots for a Lie algebra \( g \), meaning that they are linearly independent and any other root may be expressed as an integer linear combination of these with either all coefficients positive, or all coefficients negative. In particular, the special root \( \alpha_0 \) is a linear combination of the simple roots,

\[ \alpha_0 = - \sum_{i=1}^{r} n_i \alpha_i \]

where the choice of coefficients depends on \( g \). For the ade series of cases, \( \alpha_0 \) is always the ‘lowest’ root. In terms of the extended Dynkin diagrams classifying Kac-Moody algebras, \( \alpha_0 \) is the Euclidean part of the extra, ‘affine’ root. For the full list of extended Dynkin diagrams, see the book by Kac\textsuperscript{22}. The diagrams fall into two classes: there are the \( a, d, e \)-type (including \( a_{2n}^{(2)} \)) which are symmetric under the root transformation

\[ \alpha \rightarrow 2 \frac{\alpha}{|\alpha|^2}; \]

(3.2)

and the others which come in pairs related to each other by (3.2). The first type are summarised in the first diagram below, in which the circle representing \( \alpha_0 \) is indicated by a 0, and the simple roots are represented by circles labelled by the integers \( n_i \). The second type are listed in the subsequent set of diagrams, in pairs.

The classical field theories are integrable in all cases. Each of the theories may be described by a Lax pair (the details will appear in the next section) and, as a consequence each theory has a characteristic set of conserved quantities: infinitely many, in fact, in involution with one another. The classical theory was developed over a number of years and by many authors\textsuperscript{23}. More recent developments building on the older work, including a description of the field theory data (masses and couplings) are described in the author’s Banff lecture notes\textsuperscript{21} where a fuller set of references may be found.

As far as the whole line is concerned, the quantum field theory of the real affine Toda field theory was studied first by Arinshtein, Fateev and Zamolodchikov\textsuperscript{24}, for the case corresponding to the data for the affine algebra \( a_{11}^{(1)} \). They discovered the remarkable fact that the quantum spectrum of states is essentially described by the classical parameters. Their result was generalised eventually to include all the sets of affine root data, but the generalisation turns out to be not entirely straightforward.

The two classes, self-dual and dual pairs, behave differently after quantisation. As far as the dual pairs are concerned the behaviour is a generalisation of the results obtained
by Arinshtein, Fateev and Zamolodchikov, although much more intricate, and has been developed in a number of papers \cite{25,10,26}. For these cases, there is also a beautiful formula for the S-matrix elements, due to Dorey\cite{11,27}. On the other hand, unexpectedly, the dual pairs behave very differently\cite{12}. The two sets of theories exhibit a remarkable type of strong-weak coupling duality. However, they achieve the duality differently. As far as the first set is concerned, the duality is an invariance in the sense that the set of conjectured S-matrices has a symmetry under the mapping

$$\beta \rightarrow \frac{4\pi}{\beta}.$$  \hfill (3.3)

Moreover, for these models the quantum spectrum of states is essentially identical with the classical spectrum in the sense that the masses come in the same ratio before and after quantisation, and the classical three-point couplings and the pairwise formation of bound states are in one to one correspondence. However, for the second set the duality is much more subtle. For these, for each pair it seems there is a single quantum field theory, in the sense that the S-matrix (all these hypotheses are checked only for S-matrices) would be the same computed from the classical data corresponding to either of the classical lagrangians of the pair. The mapping Eq.(3.3) remains a symmetry of the S-matrix but, for small $\beta$ perturbation theory based on one of the classical theories corresponds to the large $\beta$ limit of the other. The mass spectrum of these models depends on the coupling and the $\beta$ dependent masses interpolate the classical mass parameters of the members of the pair. The classical couplings of the two models in each pair are not the same (even in number!) and the rules governing the formation of bound states are extremely subtle. A more detailed account of how this works in the simplest of cases ($g_2^{(1)} - d_4^{(3)}$) may be found elsewhere\cite{21}.

This kind of duality is not to be confused with the well-known particle-soliton duality which is exhibited by the sine-Gordon model, according to which the model may be alternatively formulated completely differently—as the massive Thirring model\cite{28}. It is not known if the other affine Toda theories enjoy a similar feature. They certainly have ‘solitons’ in the sense of complex solutions with localised energy, and these solitons have interesting properties\cite{29}. There are conjectured S-matrices for them\cite{30} and, in some cases, complete solutions to the bootstrap constraints\cite{31}, including ‘breathers’. However, the question of the existence of an alternative formulation in which the solitons are fundamental remains open.

It will be seen later that the two sets of theories also behave differently when a boundary is involved. The self-dual set (with the exception of the sine-Gordon model and the models based on data from $a_{2n}^{(1)}$) does not appear to allow extra parameters to be introduced via boundary conditions. On the other hand, most of the theories corresponding to the dual-pairs allow limited freedom at the boundary in the sense of a small number of free parameters. In some of the latter cases there is the possibility of a continuous deformation away from the Neumann boundary condition. For that reason, these may be good examples within which to develop the perturbation theory needed for checking conjectured quantum reflection factors.
The self dual affine root systems
4. Affine Toda theory on a half-line

If the Toda field theory is restricted to a half-line (say, $x \leq 0$) then there must be a boundary condition at $x = 0$. In other words, the Lagrangian must be modified and
might for example take the form:

$$\mathcal{L}_B = \theta(-x)\mathcal{L}_0 - \delta(x)\mathcal{B}(\phi),$$

where $\mathcal{L}_0$ is the Lagrangian for the whole line theory, Eq.(3.1), and it has been assumed that the boundary term depends only on the fields, not on their derivatives. The latter assumption is not strictly necessary but making it simplifies the discussion.

As a consequence of (4.1), the field equations are restricted to the region $x \leq 0$ and are supplemented by a boundary condition at $x = 0$:

$$x \leq 0 : \quad \partial^2 \phi_a = - \sum_{i=0}^{r} (\alpha_i)_a n_i e^{\alpha_i \cdot \phi}$$

$$x = 0 : \quad \partial_x \phi_a = - \frac{\partial \mathcal{B}}{\partial \phi_a}.$$

The first question to ask is the following: What choices of $\mathcal{B}$ are compatible with integrability?

The question may be tackled via several routes. One way, probably the simplest, is to consider the densities of the conserved charges which integrate to yield the conserved charges for the full-line theory, and discover how to modify them to preserve ‘enough’ charges to maintain integrability. Another is to develop a generalisation of the standard Lax pair approach, including modifications arising from the boundary condition, and to use it to investigate the charges and their mutual Poisson brackets. The first approach is somewhat limited because it is clearly not feasible to study more than the first couple of low spin conserved charges. Nevertheless, even a partial investigation already leads to some surprising conclusions and there is a strong suspicion that exploring the first few conserved quantities beyond energy is probably enough for all practical purposes. The second approach is also needed in order to be certain that conditions found by studying low spin charges are in fact all that are necessary. It is clearly preferable. However, a full inverse scattering procedure in the presence of boundary conditions will be needed eventually.

The addition of a boundary effectively removes translational invariance and it is no longer expected, therefore, that momentum should be conserved. Lorentz invariance is also inevitably lost. Nevertheless, the total energy is given by

$$\tilde{E} = \int_{-\infty}^{0} dx \mathcal{E}_0 + \mathcal{B}(\phi_0),$$

where $\mathcal{E}_0$ is the usual energy density corresponding to $\mathcal{L}_0$, and is easily seen to be conserved whatever the choice of $\mathcal{B}$ might be. (The subscript on the argument of $\mathcal{B}$ emphasises the fact that for this extra term the field is evaluated at $x = 0$.) These two elementary remarks already demonstrate that the best one might hope for is that parity even charges (like energy) might continue to be conserved whilst parity odd charges (like momentum) cannot be.
It has already been remarked that integrable theories have infinitely many conserved charges labelled by their spins. For the affine Toda theories based on the root data for the algebra $g$, the possible spins are the exponents of the algebra modulo its Coxeter number. For example, the Lie algebras $a_r$ has Coxeter number $h = r + 1$ and the associated affine Toda field theory has classically conserved charges whose spins are $s = \pm 1, \pm 2, \ldots, \pm r \mod (r + 1)$. Spin $\pm 1$ corresponds to the energy-momentum vector, and the conserved quantities of other spins correspond to other conserved tensors of higher rank. It is expected half of these (energy-like), at most, could be conserved on the half-line.

For the whole-line theory, it is convenient to think in terms of light-cone coordinates and densities for spin $s$ satisfying,

$$\partial_\pm T_{\pm(s+1)} = \partial_\pm \Theta_{\pm(s-1)}.$$  

However, on the half-line the energy-like combinations are the relevant candidates and the quantities $\hat{P}_s$ (the spin label will continue to be used even though Lorentz invariance has been lost) defined by

$$\hat{P}_s = \int_{-\infty}^{0} dx \,(T_{s+1} + T_{-s-1} - \Theta_{s-1} - \Theta_{-s+1}) - \Sigma_s(\phi_0), \quad (4.3)$$

where the additional term must be chosen to satisfy

$$T_{s+1} - T_{-s-1} + \Theta_{s-1} - \Theta_{-s+1} = \frac{\partial \phi_a}{\partial t} \frac{\partial \Sigma_s}{\partial \phi_a}. \quad (4.4)$$

Eq((4.4)) is remarkably strong. Ghoshal and Zamolodchikov appear to be the first to use such an argument for the case of the spin-three charge in the sine-Gordon model.

For low spin charges, such as occur in the $a_n^{(1)}$ theories ($s = 2$ for $n > 1$), and the $a_1^{(1)}$ and $d_n^{(1)}$ cases ($s = 3$), it is straightforward to examine (4.4) directly, and many of the details are available elsewhere. The conclusion is the following. For all of these models, the boundary potential must take the form

$$\mathcal{B} = \sum_{0}^{r} A_t e^{\alpha_i \cdot \phi/2} \quad (4.5)$$

where, either every coefficient vanishes (the Neumann condition) or, every coefficient is non-zero with magnitude $2\sqrt{n_i}$, except for the case $a_1^{(1)}$, where the two coefficients are arbitrary.

It is tempting to conjecture that the form of the boundary potential provided by (4.5) is universal. This is indeed so, but the restrictions on the coefficients are not quite applicable in every case. The Lax pair approach reveals that the strong restrictions on the coefficients $A_t$ apply to every ade-type model but are not quite universal. The second class of models, based on the non simply-laced algebras, most occurring as dual pairs, allows a small amount of freedom in the choice of boundary data (see the table
reproduced at the end of the section). However, it is only in the sine-Gordon case that the maximum freedom is permitted. Note also, in most cases, setting the field to a specific value at the boundary will not be compatible with integrability in the sense described. In other words, Dirichlet boundary conditions are not generally permitted.

Actually, even in the sinh-Gordon case there is a question of stability. Recall the Bogomolny bound argument and consider the total energy for a time independent solution to the theory restricted to a half-line:

\[
\hat{E} = \int_{-\infty}^{0} dx \left( \frac{1}{2} (\phi')^2 + e^{\sqrt{2} \phi} + e^{-\sqrt{2} \phi} - 2 \right) + A_1 e^{\phi_0/\sqrt{2}} + A_0 e^{-\phi_0/\sqrt{2}}
\]

\[
= \int_{-\infty}^{0} dx \frac{1}{2} \left( \phi' - \sqrt{2} e^{\phi/\sqrt{2}} + \sqrt{2} e^{-\phi/\sqrt{2}} \right)^2 + \int_{-\infty}^{0} dx \sqrt{2} \phi' \left( e^{\phi/\sqrt{2}} - e^{-\phi/\sqrt{2}} \right) + \ldots
\]

\[
\geq -4 + (A_0 + 2) e^{-\phi_0/\sqrt{2}} + (A_1 + 2) e^{\phi_0/\sqrt{2}}.
\]

It is clear that the energy is bounded below provided \( A_0 \geq -2 \) and \( A_1 \geq -2 \). Further details on the question of stability in this and other cases are to be found in Fujii and Sasaki. There is a similar Bogomolny style argument for the \( a_2^{(2)} \) (or Bullough-Dodd) model. However, if there exists such an argument for the other cases, it does not appear to have been written down.

The form of (4.5) has been discovered by examining low spin charges but there is always the possibility that some higher spin charge will violate integrability unless further, more stringent, conditions are imposed. To ensure compatibility with infinitely many charges it will be necessary to adopt a different approach and to develop the Lax pair idea beyond its formulation for the whole line.

First, the basic idea of a Lax pair requires the discovery of a ‘gauge field’ whose curvature vanishes if and only if the field equations for the fields \( \phi_a \) are satisfied. Explicitly, for affine Toda theory, the Lax pair may be chosen to be:

\[
a_0 = H \cdot \partial_1 \phi/2 + \sum_0^r \sqrt{m_i}(\lambda E_{\alpha_i} - 1/\lambda E_{-\alpha_i}) e^{\alpha_i \cdot \phi/2}
\]

\[
a_1 = H \cdot \partial_0 \phi/2 + \sum_0^r \sqrt{m_i}(\lambda E_{\alpha_i} + 1/\lambda E_{-\alpha_i}) e^{\alpha_i \cdot \phi/2},
\]

where \( H_a, E_{\alpha_i} \) and \( E_{-\alpha_i} \) are the Cartan subalgebra and the generators corresponding to the simple roots, respectively, of the simple Lie algebra providing the data for the Toda theory. The coefficients \( m_i \) are related to the \( n_i \) by \( m_i = n_i \alpha_i^2 / 8 \). The conjugation properties of the generators are chosen so that

\[
a_1^\dagger(x, \lambda) = a_1(x, 1/\lambda) \quad a_0^\dagger(x, \lambda) = a_0(x, -1/\lambda).
\]

Using the Lie algebra relations

\[
[H, E_{\pm \alpha_i}] = \pm \alpha_i E_{\pm \alpha_i}, \quad [E_{\alpha_i}, E_{-\alpha_i}] = 2 \alpha_i \cdot H/(\alpha_i^2),
\]
the zero curvature condition for (4.6)

\[ f_{01} = \partial_0 a_1 - \partial_1 a_0 + [a_0, a_1] = 0 \]

leads to the affine Toda field equations:

\[ \partial^2 \phi = - \sum_{r} n_i \alpha_i e^{a_i \cdot \phi}. \quad (4.8) \]

**Modified Lax pair**

For the purposes of the following discussion (which follows very closely the article by Bowcock et al.\textsuperscript{33}) the boundary of the half-line will be placed at \( x = a \).

To construct a modified Lax pair including the boundary condition derived from (4.1), it was found in to be convenient to consider an additional special point \( x = b \) (\( b > a \)) and two overlapping regions \( R_- : x \leq (a + b + \epsilon)/2 \); and \( R_+ : x \geq (a + b - \epsilon)/2 \). The second region will be regarded as a reflection of the first, in the sense that if \( x \in R_+ \), then

\[ \phi(x) \equiv \phi(a + b - x). \quad (4.9) \]

The regions overlap in a small interval surrounding the midpoint of \([a, b]\). In the two regions define:

\[
\begin{align*}
R_- : & \quad \tilde{a}_0 = a_0 - \frac{1}{2} \theta(x - a) \left( \partial_1 \phi + \frac{\partial B}{\partial \phi} \right) \cdot H \quad \tilde{a}_1 = \theta(a - x)a_1 \\
R_+ : & \quad \tilde{a}_0 = a_0 - \frac{1}{2} \theta(b - x) \left( \partial_1 \phi - \frac{\partial B}{\partial \phi} \right) \cdot H \quad \tilde{a}_1 = \theta(x - b)a_1.
\end{align*}
\]

(4.10)

Then, it is clear that in the region \( x < a \) the Lax pair (4.10) is the same as the old but, at \( x = a \) the derivative of the \( \theta \) function in the zero curvature condition enforces the boundary condition

\[ \frac{\partial \phi}{\partial x} = -\frac{\partial B}{\partial \phi}, \quad x = a. \quad (4.11) \]

Similar statements hold for \( x \geq b \) except that the boundary condition at \( x = b \) differs by a sign in order to accommodate the reflection condition (4.9).

On the other hand, for \( x \in R_- \) and \( x > a \), \( \tilde{a}_1 \) vanishes and therefore the zero curvature condition merely implies \( \tilde{a}_0 \) is independent of \( x \). In turn, this fact implies \( \phi \) is independent of \( x \) in this region. Similar remarks apply to the region \( x \in R_+ \) and \( x < b \). Hence, taking into account the reflection principle (4.9), \( \phi \) is independent of \( x \) throughout the interval \([a, b]\), and equal to its value at \( a \) or \( b \). For general boundary conditions, a glance at (4.10) reveals that the gauge potential \( \tilde{a}_0 \) is different in the two regions \( R_\pm \). However, to maintain the zero curvature condition over the whole line the values of \( \tilde{a}_0 \) must be related by a gauge transformation on the overlap. Since \( \tilde{a}_0 \) is in fact independent of \( x \in [a, b] \) on both patches, albeit with a different value on each patch,
the zero curvature condition effectively requires the existence of a gauge transformation \( K \) with the property:
\[
\partial_0 K = K \hat{a}_0(t, b) - \hat{a}_0(t, a) K.
\] (4.12)

The group element \( K \) lies in the group \( G \) with Lie algebra \( g \), the Lie algebra whose roots define the affine Toda theory.

The conserved quantities on the half-line \((x \leq a)\) are determined via a generating function \( \hat{Q}(\lambda) \) given by the expression
\[
\hat{Q}(\lambda) = \text{tr} \left( U(-\infty, a; \lambda) K U^\dagger(-\infty, a; 1/\lambda) \right),
\] (4.13)

where \( U(x_1, x_2; \lambda) \) is defined by the path-ordered exponential:
\[
U(x_1, x_2; \lambda) = P \exp \int_{x_1}^{x_2} dx_1 a_1.
\] (4.14)

To further understand the nature of \( K \), it is convenient to make a couple of assumptions which turn out to be no more restrictive as far as the boundary potential is concerned than the investigation of the low spin charges. Suppose \( K \) is time independent, and also independent of \( \phi \) in a functional sense. Then, (4.12) simplifies to
\[
K \hat{a}_0(t, b) - \hat{a}_0(t, a) K = 0
\]
or, in terms of the explicit expression for \( \hat{a}_0 \),
\[
\frac{1}{2} \left[ K(\lambda), \frac{\partial B}{\partial \phi} \cdot H \right]_+ = - \left[ K(\lambda), \sum_0^r \sqrt{m_i} (\lambda E_{\alpha_i} - 1/\lambda E_{-\alpha_i}) e^{\alpha_i \cdot \phi/2} \right]_-, \quad (4.15)
\]

where the field-dependent quantities are evaluated at the boundary \( x = a \). Eq.(4.15) is very strong, not only determining \( B \) but also \( K \) almost uniquely. The details of many solutions, including a catalogue of the restrictions on \( B \) have been found\(^{33}\). Here, just two examples will be given for \( K \), and a list of the parameter restrictions for \( B \). For \( K \) the overall scale is a matter of convenience only.

\[ a_1^{(1)} : \quad K(\lambda) = \left( \lambda^2 - \frac{1}{\lambda^2} \right) I + \left( \begin{array}{cc} 0 & \lambda A_1 - \frac{4a}{\lambda} \\ \lambda A_0 - \frac{4a}{\lambda} & 0 \end{array} \right) \]

\[ a_n^{(1)} : \quad K(\lambda) = I + 2 \sum_{\alpha > 0} \prod C_i^{l_i(\alpha)} \left[ \frac{(\lambda E_{\alpha_i} - 1/\lambda E_{-\alpha_i})}{1 + C\lambda^h} + \frac{(-1/\lambda)^{l_i(\alpha)} E_{-\alpha_i}}{1 + C/\lambda^{-h}} \right], \]

where, in the latter expression, \( C_i = A_i/2, C = \prod_i C_i \), each positive root in the sum may be decomposed as a sum of simple roots and \( l_i(\alpha) \) denotes the number of times \( \alpha_i \) appears in the decomposition, \( l(\alpha) = \sum_i l_i(\alpha) \).

As far as the boundary potential is concerned, the conjecture mentioned above appears to be correct for the \( ade \) series of models, implying the strongly restricted boundary
parameters. For all the others, the form of the boundary potential is the same but the restrictions on the parameters are less severe. The diagrams below represent the possibilities. The symbols next to the circles representing simple roots indicate the type of coefficient the corresponding term may have in the boundary potential. Where there is only a discrete choice, it is labelled by $\epsilon$, denoting $\pm$, and any choice of signs is permitted. These would be the only labels for the ade diagrams. Where there is a continuous parameter associated with a term, it is labelled by $x$ or $y$. The labels above the circles represent one set of possibilities, those below represent an alternative set; the Neumann condition is possible in all cases (but not Dirichlet). In very few cases is there the possibility of continuously deforming away from the Neumann condition while maintaining integrability.

Finally, once $\mathcal{K}(\lambda)$ is determined, it is necessary to demonstrate its compatibility with the classical $r$ matrix which itself determines the Poisson brackets between the generating functions for the conserved charges defined for the whole line theory (see for example the articles by Olive and Turok$^{37}$).

Explicitly,
\[
\{U(\lambda) \otimes U(\mu)\} = [r(\lambda/\mu), U(\lambda) \otimes U(\mu)],
\]
where $r$ has the form
\[
r(s) = \sum_i r_i(s) g_i \otimes g_i^\dagger,
\]
and $U(\lambda)$ is defined in (4.14). The quantities $g_i$ represent the Lie algebra generators for an algebra whose root system defines a particular Toda model. Calculating the Poisson brackets between two charges of the form given by (4.13), will clearly require a consistency condition to be satisfied involving $r$ and $\mathcal{K}$. In earlier work,$^{38}$ the compatibility relation appears as the main equation to be satisfied by $\mathcal{K}(\lambda)$ whereas here, $\mathcal{K}$ has been determined independently via Eq.(4.15).

The necessary checking has been carried out$^{33}$ and $\mathcal{K}$ is indeed compatible with $r$. In other words, it satisfies the following:
\[
\left[ r(\lambda/\mu), \mathcal{K}^{(1)}(\lambda) \mathcal{K}^{(2)}(\mu) \right] = \mathcal{K}^{(1)}(\lambda) \tilde{r}(\lambda\mu) \mathcal{K}^{(2)}(\mu) - \mathcal{K}^{(2)}(\mu) \tilde{r}(\lambda\mu) \mathcal{K}^{(1)}(\lambda),
\]
where
\[
\mathcal{K}^{(1)}(\lambda) = \mathcal{K}(\lambda) \otimes 1, \quad \mathcal{K}^{(2)}(\mu) = 1 \otimes \mathcal{K}(\mu),
\]
and
\[
\tilde{r}(s) = \sum_i r_i(s) g_i \otimes g_i.
\]
The relationship between $r$ and $\mathcal{K}$ is one which would probably repay further study; in a sense $\mathcal{K}$ is a fundamental object and presumably one could argue that the classical $r$ matrix must be chosen to be compatible with it. Indeed it appears remarkable that $\mathcal{K}$ and $r$ are compatible given they have been determined independently, and given the seemingly strong assumptions made to derive the expressions for $\mathcal{K}$ in the various examples. Even in the quantum case, as discussed earlier, there is a set of reflection bootstrap equations which would actually allow a computation of the complete set of S-matrix factors if the full set of reflection factors had been already independently determined.
5. Classical reflection factors

Following the programme described earlier, the next step is to discover the classical reflection factors for each of the permissible boundary conditions. This is an unfinished
part of the programme but at least a method for carrying it out can be relatively
easily, and briefly, described. First, one needs to know the lowest energy static solution
for a given theory with a specific boundary condition. In general, this solution will
not be $\phi_a = 0$. These are, in effect stationary ‘solitons’ in the following sense. The
real affine Toda theories cannot have finite energy localised solutions because all static
solutions diverge somewhere. Nevertheless, where there is a boundary these solutions
are of relevance because the divergence can be placed harmlessly on the other side of it.
I.e. for $x < 0$ there are non-singular solutions each of which is a portion of a ‘soliton’.
Similarly, on a finite interval, periodic solutions may be used whose singularities lie
outside the interval.

The simplest example of this is the sinh-Gordon model.\textsuperscript{19} There, the equation for the
static background is

\[
\begin{align*}
\phi'' &= -\sqrt{2} \left( e^{\sqrt{2} \phi} - e^{-\sqrt{2} \phi} \right) \quad x < 0 \\
\phi' &= -\sqrt{2} \left( \epsilon_1 e^{\phi/\sqrt{2}} - \epsilon_0 e^{-\phi/\sqrt{2}} \right) \quad x = 0, \quad A_i = 2\epsilon_i
\end{align*}
\]

from which, on integrating the first equation once, and comparing with the boundary
condition, one obtains

\[
\begin{align*}
\phi' &= \sqrt{2} \left( e^{\phi/\sqrt{2}} - e^{-\phi/\sqrt{2}} \right) \quad x < 0 \\
e^{\sqrt{2} \phi} &= \frac{1 + \epsilon_0}{1 + \epsilon_1} \quad x = 0.
\end{align*}
\]

Hence, the static solution takes the form

\[
e^{\phi/\sqrt{2}} = \frac{1 + e^{2(x-x_0)}}{1 - e^{2(x-x_0)}},
\]

with

\[
\coth x_0 = \sqrt{\frac{1 + \epsilon_0}{1 + \epsilon_1}}.
\]

The expression given in (5.4) assumes $\epsilon_0 > \epsilon_1$; if that is not the case, it is necessary
to adjust the solution by shifting $x_0$ by $i\pi/2$. Provided $x_0$ is positive the singularity in
Eq.(5.3) is irrelevant.

Other examples for the series $a_n^{(1)}$ have been calculated by Bowcock.\textsuperscript{20}

Once the static background is known the classical reflection factors are sought by
linearising the field equation and the boundary condition, and calculating the reflection
of a plane wave in the effective potential due to the static background.

Thus, within the sinh-Gordon model the classical background is given by Eq.(5.3)
and the linearised wave equation and boundary condition in this background have the
form

\[
\begin{align*}
\partial^2 \phi^{(0)} &= -4 \left( 1 + \frac{2}{\sinh^2 2(x-x_0)} \right) \phi^{(0)} \quad x < 0 \\
\partial_1 \phi^{(0)} &= - (\epsilon_0 \tanh x_0 + \epsilon_1 \coth x_0) \phi^{(0)} \quad x = 0.
\end{align*}
\]
The solitonic nature of the static solution is now fairly evident since it leads to a ‘sech’ effective potential which is well-known to be exactly solvable; this potential is also known to be related to solitons in various ways.

The classical scattering data for this potential is computable in terms of the parameters in the boundary term. It is convenient to set \( \phi^{(0)} = e^{-i\omega t}\Phi(z) \), in which case the solution to (5.5) takes the form

\[
\Phi(z) = a(i\lambda - \coth(z - z_0))e^{i\lambda z} + cc, \quad \lambda = \sinh \theta,
\]

where the ratio of coefficients \( a^*/a \) can be computed from the boundary condition. The reflection coefficient may be read off and turns out to be

\[
K = \frac{1 - i\lambda (i\lambda)^2 + i\lambda \sqrt{1 + \epsilon_0 \sqrt{1 + \epsilon_1}} + (\epsilon_0 + \epsilon_1)/2}{1 + i\lambda (i\lambda)^2 - i\lambda \sqrt{1 + \epsilon_0 \sqrt{1 + \epsilon_1}} + (\epsilon_0 + \epsilon_1)/2} = - (1)^2 \left[(1 + a_0 + a_1)(1 - a_0 + a_1)(1 + a_0 - a_1)(1 - a_0 - a_1)\right]^{-1},
\]

where in the last step it has been convenient to set

\[
\epsilon_i = \cos a_i \pi, \quad |a_i| \leq 1, \quad i = 0, 1,
\]

and to use the notation

\[
(z) = \frac{\sinh \left(\frac{\theta}{2} + \frac{i\pi z}{2h}\right)}{\sinh \left(\frac{\theta}{2} - \frac{i\pi z}{2h}\right)}, \quad h = 2.
\]

To extend beyond the restriction on the \( a_i \), it is necessary to continue the formula (5.6) by making the substitution \( a_i \to a_i + 2 \). The result Eq.(5.6) is remarkably similar to the conjectured quantum reflection coefficient for the lightest sine-Gordon breather which has been provided by Ghoshal. However, it remains unclear how the parameters of the classical boundary potential are related to the parameters in his conjecture.

A similar calculation may be made in other cases. However, a more general procedure is the following: Once it is realised that the background is a ‘soliton’, the corresponding ‘multisolitons’ must provide solutions in which solitons scatter from the boundary. In particular, a triple, or possibly other multi-soliton, solution containing the static background as one of its components, should, in a suitable limit, directly provide the classical reflection factors. At least in principle, all the multi-soliton solutions are known because they ought to be simply related to the complex solitons mentioned before. From these, the classical reflection data can be determined.

Once the classical reflection data has been determined it should be possible to check the classical version of the bootstrap condition.

For example, in the \( d_5^{(1)} \) case, with the following configuration of boundary coefficients \( a = -2 \) and \( b = -2\sqrt{2} \):
the classical static soliton is given by

\[ \phi = (\alpha_2 + \alpha_3)\psi, \quad e^{\psi/2} = \frac{1 - e^{2x}}{1 + e^{2x}}, \]  

(5.8)

where \( \alpha_2 \) and \( \alpha_3 \) are the simple roots corresponding to the two central circles of the above Dynkin-Kač diagram. Notice the solution actually diverges just at the boundary; nevertheless, its energy is finite because of a cancellation between the integral of the energy density over the half-line and the boundary potential.

The calculation of the classical reflection factors is quite complicated and will not be reproduced here. However, the result is the following:

\[ K_s = K_\bar{s} = -(2)(6) \quad K_1 = \frac{(3)(5)}{(1)(7)} \quad K_2 = -(4)^2 \quad K_3 = (1)(3)(5)(7), \]  

(5.9)

where the notation of Eq.(5.7) has been used with \( h = 8 \) (the Coxeter number of \( d_5 \)), and where the particle labels are the standard ones for affine Toda field theory.\(^{10}\) (The labels \( s \) and \( \bar{s} \) are reserved for the particles corresponding to one of the forks on the diagram above; 2 and 3 label the particles associated with the centre circles; particle 1 is associated with one of the circles of the other fork, and the final circle corresponds to the extended affine root and has no particle associated with it.) This particular model has three-point couplings:

\[ ss1 \quad ss3 \quad ss2 \quad 112 \quad 123 \quad 332 \]

and it is not hard to check that the bootstrap associated with these couplings is satisfied by the classical reflection factors given in Eq.(5.9).

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