SCREENING OPERATORS AND PARABOLIC INDUCTIONS FOR AFFINE $\mathcal{W}$-ALGEBRAS

NAOKI GENRA

ABSTRACT. (Affine) $\mathcal{W}$-algebras are a family of vertex algebras defined by the generalized Drinfeld-Sokolov reductions associated with a finite-dimensional reductive Lie algebra $g$ over $\mathbb{C}$, a nilpotent element $f$ in $[g, g]$, a good grading $\Gamma$ and a symmetric invariant bilinear form $\kappa$ on $g$. We introduce free field realizations of $\mathcal{W}$-algebras by using Wakimoto representations of affine Lie algebras, where $\mathcal{W}$-algebras are described as the intersections of kernels of screening operators. We call these Wakimoto free fields realizations of $\mathcal{W}$-algebras. As applications, under certain conditions that are valid in all cases of type $A$, we construct parabolic inductions for $\mathcal{W}$-algebras, which we expect to induce the parabolic inductions of finite $\mathcal{W}$-algebras defined by Premet and Losev. In type $A$, we show that our parabolic inductions are a chiralization of the coproducts for finite $\mathcal{W}$-algebras defined by Brundan-Kleshchev. In type $BCD$, we are able to obtain some generalizations of the coproducts in some special cases.

1. Introduction

Let $g$ be a reductive Lie algebra, $f$ a nilpotent element in $[g, g]$, $\kappa$ a symmetric invariant bilinear form on $g$ and

$$\Gamma : g = \bigoplus_{j \in \mathbb{Z}} g_j$$

a good grading on $g$ for $f$. We associate with the (affine) $\mathcal{W}$-algebra $\mathcal{W}^\kappa(g, f; \Gamma)$ that is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded conformal vertex algebra defined by means of the (generalized) Drinfeld-Sokolov reduction [FF4, KRW]. The vertex algebra structure of $\mathcal{W}$-algebras doesn’t depend on the choice of the good grading $\Gamma$ for fixed $g, f, \kappa$, although the conformal grading does [BG] [AKM].

In this paper, we construct inclusions

$$\text{Ind}_I^g : \mathcal{W}^\kappa(g, f; \Gamma) \rightarrow \mathcal{W}^\kappa(l, f_l; \Gamma_l)$$

for Levi subalgebras $I$ of $g$, nilpotent elements $f_l$ in $[l, l]$ and good gradings $\Gamma_l$ on $l$ for $f_l$ that satisfy some conditions. We call the maps $\text{Ind}_I^g$ parabolic inductions of $\mathcal{W}$-algebras. We expect that our construction gives a chiralization of the parabolic induction for finite $\mathcal{W}$-algebras defined by Premet [P4] and Losev [Lo3]. In the case of $g = gl_N$, we show that these inclusions induce exactly the coproducts of the finite $\mathcal{W}$-algebras of Brundan-Kleshchev [BK2]. In the case of $g = so_N, sp_N$ with rectangular nilpotent elements, we obtain a generalization of the coproducts of the corresponding finite $\mathcal{W}$-algebras.

To state our results more precisely, let $\Pi$ be a set of simple roots of $g$ compatible with $\Gamma$, $\Pi_j$ the subset of $\Pi$ consisting of simple roots whose root vectors belong

---

This work was supported by Grant-in-Aid for JSPS Fellows (No.040070235Z) and the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.
to $g_j$ for $j \in \frac{1}{2} \mathbb{Z}$. Then $\Pi = \Pi_0 \sqcup \Pi_2 \sqcup \Pi_1$ by [EK]. According to Lusztig and Spaltenstein [LS], for any Levi subalgebra $l$ of $g$, each nilpotent orbit $O_l$ in $l$ defines a nilpotent orbit

$$O_g = \text{Ind}_l^g O_l \quad \text{in} \quad g,$$

which is called the induced nilpotent orbit of $O_l$. Let $\Pi_l \subset \Pi$ be the set of simple roots of $l$, $G$ a connected Lie group corresponding to $g$ and $L$ the Lie subgroup of $G$ such that $\text{Lie}(L) = l$.

**Lemma 1.1** (Lemma 6.4). Suppose that a good grading $\Gamma$ satisfies $\Pi \setminus \Pi_l \subset \Pi_l$. Then $O_g = G \cdot f$ is induced from $O_l = L \cdot f_l$ for a nilpotent element $f_l$ in $[l, l]$. Moreover, the restriction $\Gamma_l$ of $\Gamma$ to $l$ is a good grading on $l$ for $f_l$.

We note that the existence of $\Gamma$ in Lemma 1.1 is valid in all cases of type $A$ ([KOSW]), in the cases of rectangular nilpotent elements in type $BCD$ ([KOS], [Sp]), and in all cases of type $G$ ([EK], [GE]). However, there exist some induced nilpotent orbits $O_g = \text{Ind}_l^g O_l$ in type $E$ and $F$ such that no good grading on $g$ satisfies that $\Pi \setminus \Pi_l \subset \Pi_l$, see [EK], [GE].

**Theorem A** (Theorem 6.11, Proposition 6.11). Suppose that a good grading $\Gamma$ satisfies the condition that $\Pi \setminus \Pi_l \subset \Pi_l$.

1. For any symmetric invariant bilinear form $\kappa$ on $g$, there exists an injective vertex algebra homomorphism

$$\text{Ind}_l^g : \mathcal{W}^{\kappa}(g, f; \Gamma) \to \mathcal{W}^{\kappa}(l, f_l; \Gamma_l),$$

where $f_l$, $\Gamma_l$ are given in Lemma 6.14, $\kappa_1 = \kappa + \frac{1}{2}\kappa_0^\circ - \frac{1}{2}\kappa_0^2$, and $\kappa_0^\circ$, $\kappa_0^2$ are the Killing forms on $g$, $l$ respectively.

2. $\text{Ind}_l^g$ is unique vertex algebra homomorphism that satisfies $\mu = \mu_l \circ \text{Ind}_l^g$, where $\mu$, $\mu_l$ are the Miura maps [KW1] for $\mathcal{W}^{\kappa}(g, f; \Gamma)$, $\mathcal{W}^{\kappa}(l, f_l; \Gamma_l)$ respectively.

3. Let $l'$ be any Levi subalgebra of $g$ such that $\Pi \setminus \Pi_{l'} \subset \Pi_l$ and $l \subset l' \subset g$. Then the maps $\text{Ind}_{l'}^g$, $\text{Ind}_{l'}^f$ exist and $\text{Ind}_l^g = \text{Ind}_{l'}^g \circ \text{Ind}_{l'}^f$.

See Section 5.2 for the definition of the Miura map. In the case that $f$ is a principal nilpotent element, the map $\text{Ind}_l^g$ has been constructed in Theorem B 7.1 of [BFN].

For any $\frac{1}{2} \mathbb{Z}_{>0}$-graded conformal vertex algebra $V$, we can associate with an associative algebra Zhu$(V)$, called the (twisted) Zhu algebra Zhu$([ZD]$, [DK]). It is proved in [AHL], [DK] that Zhu$(\mathcal{W}^{\kappa}(g, f; \Gamma))$ is the finite $\mathcal{W}$-algebra associated with $g, f, \Gamma$ ([DL], [CG]), which we denote by $U(g, f; \Gamma)$. It is easy to see that any vertex algebra homomorphism $\alpha : V \to W$ induces an algebra homomorphism between the Zhu algebras, which we denote by $\text{Zhu}(\alpha) : \text{Zhu}(V) \to \text{Zhu}(W)$. For an algebra homomorphism $A$, we call a map $\alpha$ a chiralization of the map $A$ if $A = \text{Zhu}(\alpha)$. In the case of $\alpha = \text{Ind}_l^g$, we obtain an algebra homomorphism

$$\text{Zhu}(\text{Ind}_l^g) : U(g, f; \Gamma) \to U(l, f_l; \Gamma_l),$$

which is a unique injective algebra homomorphism that satisfies $\bar{\mu} = \bar{\mu}_l \circ \text{Zhu}(\text{Ind}_l^g)$, where $\bar{\mu}$, $\bar{\mu}_l$ are the Miura maps for $U(g, f; \Gamma)$, $U(l, f_l; \Gamma_l)$ respectively (Lemma 6.14). See [Ly] or Section 6.2 for the definition of the Miura map for $U(g, f; \Gamma)$. 
Given an induced nilpotent orbit $G \cdot f = \text{Ind}_P^G(L \cdot f_t)$ in $g$ with a good grading $\Gamma$ on $g$ for $f$ and a good grading $\Gamma_1$ on $l$ for $f_t$, Losev proved the existence of an injective algebra homomorphism

\[
U(g, f; \Gamma) \to \tilde{U}(l, f_t; \Gamma_1)
\]

in \cite{Lo3}, where $\tilde{U}(l, f_t; \Gamma_1)$ is a certain completion of $U(l, f_t; \Gamma_1)$. The map \cite{11} induces a functor from the category of $U(l, f_t; \Gamma_1)$-modules to the category of $U(g, f; \Gamma)$-modules, called the parabolic induction that was first introduced by Premet \cite{23}.

We conjecture that Zhu(\cite{Zhu}) coincides with \cite{11}, and this is the reason why we call the map $\text{Ind}_P^g$ the parabolic induction of $W$-algebras.

In the case of $gl_N$, any nilpotent element in $sl_N = [gl_N, gl_N]$ admits a good $\mathbb{Z}$-grading. These good $\mathbb{Z}$-gradings on $gl_N$ are classified by combinatoric objects called (even) pyramids $\pi$ introduced in \cite{EK}, which are sequences of the columns of $1 \times 1$ boxes such that each of rows in $\pi$ is a single connected strip (see Section 7 for details). For a pyramid $\pi$ consisting of $N$ boxes, we associate with a nilpotent element $f_\pi$ in $gl_N$, a good $\mathbb{Z}$ grading $\Gamma_\pi$ on $gl_N$ for $f_\pi$, and the finite $W$-algebra $U(gl_N, \pi) = U(gl_N, f_\pi; \Gamma_\pi)$. It was shown by Brundan and Kleshchev in \cite{BK2} that $U(gl_N, \pi)$ is isomorphic to a truncation of the Yangian $Y(gl_n)$ for some $n \geq 1$ and the coproduct of $Y(gl_n)$ induces an injective algebra homomorphism between finite $W$-algebras

\[
\Delta = \Delta_{\pi_1, \pi_2}^\pi : U(gl_N, \pi) \to U(gl_{N_1}, \pi_1) \otimes U(gl_{N_2}, \pi_2)
\]

for a pyramid $\pi$ that splits into sum of $\pi_1$ and $\pi_2$ along a column of $\pi$ (see e.g. Section 7.2), which we denote by $\pi = \pi_1 \oplus \pi_2$. This map $\Delta$ is called a coproduct of finite $W$-algebras and satisfies the coassociativity, i.e.

\[
(\text{Id} \otimes \Delta_{\pi_2, \pi_3}^\pi) \circ \Delta_{\pi_1, \pi_2 \oplus \pi_3}^\pi = (\Delta_{\pi_1, \pi_2}^\pi \otimes \text{Id}) \circ \Delta_{\pi_1 \oplus \pi_2, \pi_3}^\pi
\]

for a pyramid $\pi = \pi_1 \oplus \pi_2 \oplus \pi_3$. The coproduct $\Delta$ plays a fundamental role to produce representations of finite $W$-algebras of type $A$, see \cite{BK3}.

Consider a maximal Levi subalgebra $l$ in $gl_N$, that is, $l = gl_{N_1} \oplus gl_{N_2}$ for some $N_1, N_2 \in \mathbb{Z}_{\geq 1}$ such that $N = N_1 + N_2$. According to \cite{Kil OW}, it follows that any induced nilpotent orbit in $gl_N$ takes the form

\[
GL_N \cdot f_\pi = \text{Ind}_P^g(GL_{N_1}, f_{\pi_1} + GL_{N_2}, f_{\pi_2})
\]

for some pyramid $\pi = \pi_1 \oplus \pi_2$, where $f_{\pi_1} \in gl_{N_1}$ and $f_{\pi_2} \in gl_{N_2}$. Therefore, it is expected that $\Delta$ coincides with the special case of \cite{11} for $g = gl_N$ and $l = gl_{N_1} \oplus gl_{N_2}$.

For $k \in \mathbb{C}$, let us denote by $W_k(gl_N, \pi) = W_k(gl_N, f_\pi; \Gamma_\pi)$, where $\kappa$ is a symmetric invariant bilinear form on $gl_N$ such that $\kappa(u|v) = k \text{ tr}(uv)$ for all $u, v \in sl_N$.

The following assertion is obtained from Theorem A.

**Theorem B** (Theorem 7.1 Proposition 7.2). Let $\pi$ be a pyramid consisting of $N$ boxes such that $\pi = \pi_1 \oplus \pi_2$.

1. For any $k \in \mathbb{C}$, there exists an injective vertex algebra homomorphism

\[
\Delta = \Delta_{\pi_1, \pi_2}^\pi : W_k(gl_N, \pi) \to W_k^{\pi_1}(gl_{N_1}, \pi_1) \otimes W_k^{\pi_2}(gl_{N_2}, \pi_2),
\]

where $k + N = k_1 + N_1 = k_2 + N_2$ and $N_i$ is a number of boxes in $\pi_i$ for $i = 1, 2$. 
(2) \( \Delta \) is a unique vertex algebra homomorphism that satisfies \( \mu = (\mu_1 \otimes \mu_2) \circ \Delta \), where \( \mu_1, \mu_2 \) are the Miura maps for \( \mathcal{W}^k(\mathfrak{g}_N, \pi), \mathcal{W}^k(\mathfrak{g}_N, \pi_1), \mathcal{W}^k(\mathfrak{g}_N, \pi_2) \) respectively.

(3) \( \Delta \) is coassociative, i.e. \( (\text{Id} \otimes \Delta^{\pi_2 \oplus \pi_3}) \circ \Delta^{\pi} = (\Delta^{\pi_1 \oplus \pi_2} \otimes \text{Id}) \circ \Delta^{\pi} \) for \( \pi = \pi_1 \oplus \pi_2 \oplus \pi_3 \).

(4) \( \Delta \) is a chiralization of \( \hat{\Delta} \), that is, \( \text{Zhu}(\Delta) = \hat{\Delta} \).

See Section 3 for some examples of \( \Delta \). In the case that \( f_\pi \) is a principal nilpotent element, the coproduct \( \Delta \) is an injective map

\[
\mathcal{W}^k_N \to \mathcal{W}^k_{N_1} \otimes \mathcal{W}^k_{N_2}
\]

for \( N = N_1 + N_2 \) and \( k + N = k_1 + N_1 = k_2 + N_2 \), where \( \mathcal{W}^k_N \) is the \( \mathcal{W} \)-algebra of \( \mathfrak{g}_N \) with a principal nilpotent element and level \( k \). It seems that the existence of the map has been suggested in [FigSta].

In the case of \( \mathfrak{g}_N = \mathfrak{so}_N \) or \( \mathfrak{sp}_N \), any maximal Levi subalgebra of \( \mathfrak{g}_N \) takes the form \( \mathfrak{g} = \mathfrak{gl}_{N_1} \oplus \mathfrak{g}_{N_2} \) for some \( N_1, N_2 \in \mathbb{Z}_{\geq 1} \) such that \( N = 2N_1 + N_2 \). Applying Theorem A to this setting with rectangular nilpotent elements, we obtain some generalizations of the coproducts for \( \mathcal{W} \)-algebras of \( \mathfrak{g}_N \). See Theorem 3.3 for precise statements. We note that our results suggest the existence of certain coproducts for truncated twisted Yangians, which is obtained as Corollary 7.4 in some special cases. We refer to [R, Bro] for connections between twisted Yangians and finite \( \mathcal{W} \)-algebras of type BCD.

The basic tool for the proof of Theorem A is Wakimoto representations of \( \mathcal{W} \)-algebras, which we introduce in Section 4. For simplicity, we assume that \( \mathfrak{g} \) is a simple Lie algebra. Denote by \( \mathcal{W}^k(\mathfrak{g}, f; \Gamma) = \mathcal{W}_N(\mathfrak{g}, f; \Gamma) \) if \( k = \kappa(\theta)\theta/2 \) for the highest root \( \theta \) of \( \mathfrak{g} \). Wakimoto representations of the affine Lie algebra \( \widehat{\mathfrak{g}} \) are introduced by Wakimoto [Wak] in the case of \( \mathfrak{sl}_2 \) and Feigin-Frenkel [FF1] in general case, see also Section 3. The actions of \( \widehat{\mathfrak{g}} \) on Wakimoto representations are induced from an embedding of the affine vertex algebra \( \mathcal{V}^k(\mathfrak{g}) \) into the tensor product of the Heisenberg vertex algebra \( \mathcal{H} \) associated with a Cartan subalgebra \( \mathfrak{h} \) in \( \mathfrak{g} \) and \( \dim \mathfrak{n}_+ \), copies of the \( \beta\gamma \)-system, where \( \mathfrak{n}_+ = \text{Lie}(N_+ \cap \mathfrak{h}) \) and \( N_+ \) is the big cell of the flag manifold \( G/B \). The image of this embedding is the intersection of kernels of screening operators \( S_\alpha \) for all \( \alpha \in \Pi \) if \( k \) is a formal parameter ([FF3]). As explained in detail in Section 4 applying Drinfeld-Sokolov reductions to Wakimoto representations of \( \widehat{\mathfrak{g}} \), we obtain free fields realizations of \( \mathcal{W} \)-algebras \( \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \), which we call Wakimoto free fields realizations of \( \mathcal{W} \)-algebras \( \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \), \( \mathcal{H}^T \) instead of \( \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \), \( \mathcal{H} \) respectively. Let \( N \) be the nilpotent cone of \( \mathfrak{g} \), \( S_f \) the Slodowy slice of \( \mathfrak{g} \) through \( f \).

**Theorem C** (Theorem 6.3). The \( \mathcal{W} \)-algebras \( \mathcal{W}^T(\mathfrak{g}, f; \Gamma) \) over \( T \) may be embedded into the tensor products of \( \mathcal{H}^T \) and \( \frac{1}{2}\dim(N \cap S_f) \) copies of the \( \beta\gamma \)-system. These image can be identified with the intersections of kernels of screening operators \( Q_\alpha \) induced by \( S_\alpha \).

See Theorem 4.12 for the precise formulae of \( Q_\alpha \). In the case that \( f \) is a principal nilpotent element, screening operators \( Q_\alpha \) coincide with the ones constructed in [FF3]. In the case that \( \mathfrak{g}_0 \) is a Cartan subalgebra \( \mathfrak{h} \), screening operators \( Q_\alpha \) coincide with the ones constructed in [Ge].
Our strategy to prove Theorem A is simple. Under the assumption in Theorem A, we consider the specialization of inclusion maps
\[(1.3) \quad \bigcap_{\alpha \in \Pi} \text{Ker} \, Q_\alpha \hookrightarrow \bigcap_{\alpha \in \Pi_1} \text{Ker} \, Q_\alpha \hookrightarrow \bigcap_{\alpha \in \Pi_0} \text{Ker} \, Q_\alpha.\]
We show by using Theorem C that the first map is nothing but \(\text{Ind}_I \), i.e.
\[W^{\kappa_i}(l, f; \Gamma) \simeq \bigcap_{\alpha \in \Pi_1} \text{Ker} \, Q_\alpha.\]
Our assumption \((\Pi \setminus \Pi_1 \subset \Pi_1)\) is used here. Since the Miura map is injective by Fre [A3], Ge [C3], Theorem A therefore follows if we show that the map \(\text{Ind}_I \) satisfies the formula \(\mu = \mu_i \circ \text{Ind}_I \), which in fact follows from (1.3) and Theorem D.

**Theorem D** (Theorem 5.6). The specialization
\[\mu: W^k(\mathfrak{g}, f; \Gamma) \to V^\tau_n(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_1)\]
of an inclusion map
\[\bigcap_{\alpha \in \Pi} \text{Ker} \, Q_\alpha \hookrightarrow \bigcap_{\alpha \in \Pi_0} \text{Ker} \, Q_\alpha.\]
coincides with the Miura map.

Let us make some comment on the relationship between \(W^k(\mathfrak{g}(\mathfrak{n}), \pi)\) and the affine Yangian \(Y(\hat{\mathfrak{gl}}_n)\). In the case that \(f\) is a principal nilpotent element, an action of \(Y(\hat{\mathfrak{gl}}_1)\) on \(W^k\) was first suggested by Aldehy-Gaiotto-Tachikawa [AGT] and was studied by Maulik-Okounkov [MO] and Schiffmann-Vasserot [SV], see also [BFN] for the generalizations. The coproduct (1.2) is expected to be induced by the coproduct of \(Y(\hat{\mathfrak{gl}}_1)\) as an analogue of the finite cases. We hope to study the relationship between the coproduct \(\Delta\) in Theorem B and that of affine Yangian \(Y(\hat{\mathfrak{gl}}_n)\) in our future works.

The paper is organized as follows. In Section 2, we review the definitions of \(W\)-algebras. In Section 3, we recall Wakimoto representations of \(V^k(\mathfrak{g})\) and screening operators \(S_\alpha\). In Section 4, we introduce Wakimoto representations of \(W^k(\mathfrak{g}, f; \Gamma)\) and screening operators \(Q_\alpha\), and state the precise formulae of \(Q_\alpha\) in Theorem 4.12. In Section 5.1, we recall results in [Ge]. In Section 5.2, we recall the Miura map and prove Theorem C and Theorem D by using Lemma 5.3. In Section 6.1, we define \(W\)-algebras associated with reductive Lie algebras and conclude some results from Theorem C and Theorem D. In Section 6.2, we recall the definitions and properties of induced nilpotent orbits. In Section 6.3, we prepare some preliminary results in order to prove Theorem A. In Section 6.4, we prove Theorem A. In Section 6.5, we derive some results for finite \(W\)-algebras from Theorem A. In Section 7.1, we recall the definitions of pyramids. In Section 7.2, we prove Theorem B. In Section 7.3, we derive some generalizations of the coproducts for the \(W\)-algebras of type BCD from Theorem A. In Section 8, we give examples of Theorem B in the case that \(f\) is a principal, rectangular and subregular nilpotent element. In Appendix A, we prove Theorem 4.12. In Appendix B, we prove Lemma 5.3.

Acknowledgments The author is grateful to his supervisor Tomoyuki Arakawa for valuable discussions and lots of advice to improve this paper. He thanks to Shigernori Nakatsuka for useful comments and discussions. He wishes to express his
gratitude to Boris Feigin for sharing his ideas and useful discussions for Theorem B. He is grateful to Hiraku Nakajima for valuable comments, which triggered Theorem A. Some part of this work was done while he was visiting M.I.T. and National Research University Higher School of Economics in 2016. He is grateful to those institutes for their hospitality.

2. Affine \( \mathcal{W} \)-algebras

We recall the definitions of the (affine) \( \mathcal{W} \)-algebras, following [KRW]. Let \( g \) be a finite-dimensional simple Lie algebra over \( \mathbb{C} \), \( f \) a nilpotent element of \( g \) and \( \Gamma \) a good grading of \( g \) for \( f \) denoted by

\[
\Gamma: g = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} g_j,
\]

where the \( \frac{1}{2} \mathbb{Z} \)-grading \( \Gamma \) is called good for \( f \) if \( [g_i, g_j] \subset g_{i+j} \) for all \( i, j \in \frac{1}{2} \mathbb{Z} \), \( f \in g_{-1} \) and \( \text{ad} f: g_j \to g_{j-1} \) is injective for \( j \geq \frac{1}{2} \), surjective for \( j \leq \frac{1}{2} \). Then there exists a semisimple element \( h \in g \) such that the grading \( \Gamma \) of \( g \) is the eigenspace decomposition of \( \text{ad}(\frac{1}{2} h) \). By Jacobson-Morozov Theorem, there exists an \( sl_2 \)-triple \((e, h, f)\) in \( g \), and \( \text{ad}(\frac{1}{2} h) \) defines a \( \frac{1}{2} \mathbb{Z} \)-grading on \( g \), which is good for \( f \) called the Dynkin grading. Choose the Cartan subalgebra \( h \) containing \( h \) so that \( h \subset g_0 \). Let \( \Delta \) be the set of roots, \( \Delta_+ \) the set of positive roots such that \( \bigoplus_{\alpha \in \Delta_+} g_\alpha \subset g_{\geq 0} \), where \( g_\alpha \) is the root space of \( \alpha \in \Delta \). Let \( \Pi \) be the set of simple roots, \( \Delta_j = \{ \alpha \in \Delta \mid g_\alpha \subset g_j \} \) and \( \Pi_j = \Pi \cap \Delta_j \) for all \( j \in \frac{1}{2} \mathbb{Z} \). Set \( \Delta_0 = \Delta_0 \cap \Delta_+ \). Then

\[
\Delta = \bigcup_{j \in \frac{1}{2} \mathbb{Z}} \Delta_j, \quad \Delta_+ = \Delta_0^+ \sqcup \bigsqcup_{j > 0} \Delta_j, \quad \Pi = \Pi_0 \sqcup \Pi_1 \sqcup \Pi_2,
\]

see [EK]. Denote by \( \text{deg}_f \alpha = j \) if \( \alpha \in \Delta_j \). Fix a root vector \( e_\alpha \in g \) for each \( \alpha \in \Delta \) and a non-degenerate symmetric invariant bilinear form \((\cdot, \cdot)\) on \( g \) such that \((\theta \theta) = 2\) for the highest root \( \theta \) of \( g \). Then \( \kappa_\theta(u|v) = 2h^\vee (u|v) \) for all \( u, v \in g \), where \( \kappa_\theta \) is the Killing form on \( g \) and \( h^\vee \) is the dual Coxeter number of \( g \). Let \( \chi: g \to \mathbb{C} \) be a linear map defined by \( \chi(u) = (f|u) \) for \( u \in g \). Denote by \( n_\pm = \bigoplus_{\alpha \in \Delta_\pm} g_\alpha \) and \( b_\pm = h \oplus n_\pm \).

We follow [FBZ] [Ka] for the definitions of vertex algebras. We use the following notations:

\[
A(z) = \sum_{n \in \mathbb{Z}} A(n) z^{-n-1}, \quad \int A(z) \, dz = A(0)
\]

for any field \( A(z) \), and \( \delta(z - w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n \). Denote by \( A(z)B(z) \) : the normally ordered products and by \( A(z)B(w) \sim \sum_{n \geq 0} \frac{C_n(w)}{(z-w)^{n+1}} \) the operator product expansion for local fields \( A(z), B(z) \), where \( [A(z), B(w)] = \sum_{n \geq 0} \frac{1}{n!} C_n(w) \partial_z^n \delta(z-w) \). If \( A(z), B(z) \) are fields on a vertex (super)algebra, \( C_n(z) = (A(n)B)(z) \) for \( n \geq 0 \). For \( k \in \mathbb{C} \), let \( V^k(g) \) be the affine vertex algebra associated with \( g \) of level \( k \), whose generating fields \( u(z) \) for \( u \in g \) satisfy

\[
u(z)v(w) \sim \frac{[u, v](w)}{z-w} + \frac{k(u|v)}{(z-w)^2}.
\]
for all $u, v \in \mathfrak{g}$. Let $F_{\text{ch}}(\mathfrak{g}_{>0})$ be the charged fermion vertex superalgebra associated with $\mathfrak{g}_{>0}$, whose generating odd fields $\varphi_\alpha(z), \varphi^\alpha(z)$ for $\alpha \in \Delta_{>0}$ satisfy

$$\varphi_\alpha(z)\varphi^\beta(w) \sim \frac{\delta_{\alpha,\beta}}{z-w}, \quad \varphi_\alpha(z)\varphi_\beta(w) \sim 0 \sim \varphi^\alpha(z)\varphi^\beta(w)$$

for all $\alpha, \beta \in \Delta_{>0}$. The charged decomposition $F_{\text{ch}}(\mathfrak{g}_{>0}) = \bigoplus_{n \in \mathbb{Z}} F_{\text{ch}}^n$ is defined by the charged degree $\deg_{\text{ch}}(\varphi_\alpha(z)) = -1$ and $\deg_{\text{ch}}(\varphi^\alpha(z)) = 1$ for all $\alpha \in \Delta_{>0}$, where $F_{\text{ch}}^n = \{ A \in F_{\text{ch}}(\mathfrak{g}_{>0}) | \deg_{\text{ch}}(A) = n \}$. Let $\Phi(\mathfrak{g}_+)\check{\otimes} \mathfrak{g}_{>0}$ be the neutral vertex algebra associated with $\mathfrak{g}_+$, whose generating (even) fields $\Phi_\alpha(z)$ for $\alpha \in \Delta_+$ satisfy

$$\Phi_\alpha(z)\Phi_\beta(w) \sim \frac{\chi(e_{\alpha,\beta})}{z-w}$$

for all $\alpha, \beta \in \Delta_+$. Set

$$C_k = V^k(\mathfrak{g}) \otimes F_{\text{ch}}(\mathfrak{g}_{>0}) \otimes \Phi(\mathfrak{g}_+)$$

and $d = \int d(z) dz$, where $d(z) = d_{\text{st}}(z) + d_{\text{ne}}(z) + d_\chi(z)$ is an odd field on $C_k$ defined by

$$d_{\text{st}}(z) = \sum_{\alpha \in \Delta_{>0}} :e_\alpha(z)\varphi_\alpha(z) :, -\frac{1}{2}\sum_{\alpha, \beta, \gamma \in \Delta_{>0}} c_{\alpha,\beta}^\gamma :\varphi_\gamma(z)\varphi^\alpha(z)\varphi^\beta(z) :,$$

$$d_{\text{ne}}(z) = \sum_{\alpha \in \Delta_+} :\varphi^\alpha(z)\Phi_\alpha(z) :, \quad d_\chi(z) = \sum_{\alpha \in \Delta_+} \chi(e_\alpha)\varphi^\alpha(z),$$

where $c_{\alpha,\beta}^\gamma \in \mathbb{C}$ is the structure constant for $\alpha, \beta, \gamma \in \Delta_{>0}$. The charged decomposition $C_k = V^k(\mathfrak{g}) \otimes F_{\text{ch}}(\mathfrak{g}_{>0})^* \otimes \Phi(\mathfrak{g}_+)\check{\otimes} \mathfrak{g}_{>0}$ is induced from that of $F_{\text{ch}}(\mathfrak{g}_{>0})$. Since $d^2 = 0$ and $d \cdot C_k^p \subset C_k^{p+1}$, an odd vertex operator $d$ defines a differential of a cochain complex on $C_k$. The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ associated with $\mathfrak{g}, f, k, \Gamma$ is defined as the BRST cohomology of the complex $(C_k, d)$:

$$\mathcal{W}^k(\mathfrak{g}, f; \Gamma) = H(C_k, d),$$

called the (generalized) Drinfeld-Sokolov reduction. There exists a decomposition of the complex $C_k = C_- \otimes C_+$ such that $H(C_-, \mathbb{C}) = \mathbb{C}$ and $C_+$ has only non-negative charged degree. Moreover $\mathcal{W}^k(\mathfrak{g}, f; \Gamma) = H^0(C_+, d)$ (Kac-Wakimoto). A vertex algebra structure on $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ is induced from that of $C_k$ and does not depend on the choice of $\Gamma$. A conformal $\frac{1}{2}\mathbb{Z}$-grading on $C_k$ is defined by $\Delta(u) = 1 - j$ ($u \in \mathfrak{g}_j$), $\Delta(\varphi^\alpha) = 1 - \deg_T \alpha$, $\Delta(\varphi_\alpha) = \deg_T \alpha$ and $\Delta(\Phi_\alpha) = \frac{1}{2}$, where $\Delta(A)$ is the conformal weight of $A \in C_k$. This conformal grading is preserved by the differential $d$ and induces a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-grading on $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$, which depends on the choice of $\Gamma$.

Let $T = \mathbb{C}[k]$ and $V^T(\mathfrak{g})$ the affine vertex algebra over $T$, where we replace $k$ by a formal parameter $k$. Set $F^T_{\text{ch}}(\mathfrak{g}_{>0}) = F_{\text{ch}}(\mathfrak{g}_{>0}) \otimes T$ and $\Phi^T(\mathfrak{g}_+)\check{\otimes} \mathfrak{g}_{>0}$ is induced by the BRST cohomology of the complex $(C_T, d)$.

We have

$$\mathcal{W}^T(\mathfrak{g}, f; \Gamma) \otimes \mathbb{C}_k = \mathcal{W}^k(\mathfrak{g}, f; \Gamma),$$
where $\mathbb{C}_k$ is a 1-dimensional $T$-module defined by $k = k \in \mathbb{C}$. See e.g. [ACL]. For $k \in \mathbb{C}$, we call the functor $? \otimes \mathbb{C}_k$ the specialization.

3. Wakimoto representations for Affine Vertex Algebras

We introduce Wakimoto representations of $V^T(\mathfrak{g})$ and the screening operators $S_\alpha$ of $V^T(\mathfrak{g})$. We follow the construction given in [Fre].

3.1. Differential representations of $\mathfrak{g}$. Let $G$ be a connected simply-connected Lie group corresponding to $\mathfrak{g}$, $B_+$ the Borel subgroup corresponding to $\mathfrak{b}_+$, $B_-$ the opposite Borel subgroup and $N_+$ the unipotent subgroup corresponding to $\mathfrak{n}_+$. The left $G$-action on a flag variety $G/B_-$ induces a Lie algebra homomorphism $\rho_{G/B_-}: \mathfrak{g} \to \mathcal{D}_{G/B_-}$, where $\mathcal{D}_{G/B_-}$ is the ring of differential operators of regular functions on $G/B_-$. Let $p: G \to G/B_-$ be the canonical projection and $U = N_+ \cdot p(1)$ an $N_+$-orbit in $G/B_-$, where $1$ denotes the unit in $G$. An orbit $U$ is a unique open dense orbit in $G/B_-$ called the big cell. Since $N_+$ is unipotent, the exponential map $c(\mathfrak{n}_+): \mathfrak{n}_+ \to N_+$ is an isomorphism. The big cell $U \simeq N_+$ is then the affine space of the dimension $|\Delta_+|$ and the ring $\mathbb{C}[N_+]$ of regular functions on $N_+$ is a polynomial ring. A Lie algebra homomorphism $\rho: \mathfrak{g} \to \mathcal{D}_{N_+}$ is defined by the restriction of $\rho_{G/B_-}$ on $U$. Fix a coordinate system $\{x_\alpha\}_{\alpha \in \Delta_+}$ on $N_+$ by using $c(\mathfrak{n}_+)$ such that $h \cdot x_\alpha = -\alpha(h)x_\alpha$ for all $h \in \mathfrak{h}$ and $\alpha \in \Delta_+$. This coordinate is called homogeneous.

To describe the image of $\rho$, we introduce the frameworks in [Fre]. Fix a root vector $e_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta$. Denote by $f_\alpha = e_{-\alpha}$ and $h_\alpha = [e_\alpha, f_\alpha]$ for $\alpha \in \Delta_+$. Let $G^0 = p^{-1}(U) = N_+ \cdot B_-$ be a dense open submanifold in $G$. For $a \in \mathfrak{g}$, set a smooth curve $\gamma(t) = \exp(-ta)$ on $G$. Given $X \in G^0$,

$$\gamma(t)X = Z_+(t)Z_-(t)$$

for $|t| \ll 1$, where $Z_+(t) \in N_+$ and $Z_-(t) \in B_-$. A vector field $\zeta_\alpha$ is then given by the following formula:

$$(\zeta_\alpha f)(p(X)) = \frac{d}{dt}f(Z_+(t))\big|_{t=0}$$

for any smooth function $f$ defined in an open subset in $U$ around $p(X)$. Choose a faithful representation $V_0$ of $\mathfrak{g}$ and consider $X \in N_+$ as a matrix in $\text{GL}(V_0)$ whose entries are polynomials in $\mathbb{C}[N_+] = \mathbb{C}[x_\alpha]_{\alpha \in \Delta_+}$. We have

$$(1 - ta)X = Z_+(t)Z_-(t) \mod (t^2).$$

Hence $Z_+(t) = X + tZ$, $Z_- = 1 + tZ'$ $\mod (t^2)$, where $Z \in \mathfrak{n}_+$ and $Z' \in \mathfrak{b}_-$. We have

$$\zeta_\alpha \cdot X = -X(X^{-1}aX)_+,$$
where \((\cdot)_+ : g = n_+ \oplus b_+ \rightarrow n_+\) is the first projection. For \(a \in g\), \(\rho(a)\) is a derivation in \(\mathbb{C}[N_+]\) such that

\[
\rho(e_\alpha) = \sum_{\beta \in \Delta_+} P^\beta_\alpha(x) \partial_\beta = \partial_\alpha + \sum_{\beta \in \Delta_+ \setminus \{\alpha\}} P^\beta_\alpha(x) \partial_\beta,
\]

\[
\rho(h_\alpha) = - \sum_{\beta \in \Delta_+} \beta(h_\alpha) x_\beta \partial_\beta,
\]

\[
\rho(f_\alpha) = \sum_{\beta \in \Delta_+} Q^\beta_\alpha(x) \partial_\beta
\]

for all \(\alpha \in \Delta_+\), where \(\partial_\alpha = \partial/\partial x_\alpha\), \(x = (x_\alpha)_{\alpha \in \Delta_+}\) and \(P^\beta_\alpha(x), Q^\beta_\alpha(x) \in \mathbb{C}[N_+]\). For \(\lambda \in h^*\), we have a twisted Lie algebra homomorphism \(\rho_\lambda : g \rightarrow D_{N_+}\) by

\[
\rho_\lambda(e_\alpha) = \sum_{\beta \in \Delta_+} P^\beta_\alpha(x) \partial_\beta,
\]

\[
\rho_\lambda(h_\alpha) = - \sum_{\beta \in \Delta_+} \beta(h_\alpha) x_\beta \partial_\beta + \lambda(h_\alpha),
\]

\[
\rho_\lambda(f_\alpha) = \sum_{\beta \in \Delta_+} Q^\beta_\alpha(x) \partial_\beta + \lambda(h_\alpha) x_\alpha
\]

for all \(\alpha \in \Pi\).

### 3.2. Wakimoto representations of \(V^T(g)\)

For any finite set \(S\), let \(A_S\) be the (infinite-dimensional) Weyl vertex algebra associated with \(S\), whose generating fields \(a_\alpha(z), a^*_\alpha(z)\) for \(\alpha \in S\) satisfy

\[
a_\alpha(z)a^*_\beta(w) \sim \delta_{\alpha,\beta} \frac{z-w}{z-w}, \quad a_\alpha(z)a_\beta(w) \sim 0 \sim a^*_\alpha(z)a^*_\beta(w)
\]

for all \(\alpha, \beta \in S\). For a polynomial \(P(x) \in \mathbb{C}[N_+]\), we define a field \(P(a^*)(z)\) on \(A_{\Delta_+}\) by

\[
P(a^*)(z) := P(x)|_{x_\alpha = a^*_\alpha(z)} \quad (a \in \Delta_+).
\]

Since \(a^*_\alpha(z)\) and \(a^*_\beta(z)\) commute for all \(\alpha, \beta \in \Delta_+\), \(P(a^*)(z)\) is well-defined. Denote by \(P(a^*)\) the vector in \(A_{\Delta_+}\) corresponding to a field \(P(a^*)(z)\). We have

\[
a_\alpha(z) P(a^*)(w) \sim \frac{\partial_\alpha P(a^*)(w)}{z-w}.
\]

Let \(\mathcal{H} = V^{k+h^*}(h)\) be the Heisenberg vertex algebra associated with the Cartan subalgebra \(h\) of \(g\), whose generating fields \(b_\alpha(z)\) for \(\alpha \in \Pi\) satisfy

\[
b_\alpha(z)b_\beta(w) \sim \frac{(k + h^*)(\alpha|\beta)}{(z-w)^2}
\]

for all \(\alpha, \beta \in \Pi\). For \(\lambda \in h^*\), denote by \(\mathcal{H}_\lambda\) the highest weight \(\mathcal{H}\)-module with highest weight \(\lambda\). Let \(\mathcal{H}^T\) be the Heisenberg vertex algebra over \(T\) and \(\mathcal{H}^T_\lambda\) the highest weight \(\mathcal{H}^T\)-module with highest weight \(\lambda \in h^*\), where we replace \(k\) by a formal parameter \(k\) in \(T\). Set \(A^T_S = A_S \otimes_{\mathbb{C}} T\).
Lemma 3.1 (Fre). There exists an injective vertex algebra homomorphism \( \hat{\rho}: V^T(\mathfrak{g}) \to \mathcal{A}_{\Delta_+}^T \otimes \mathcal{H}^T \) over \( T \) with some \( c_\alpha \in \mathbb{C} \) for each \( \alpha \in \Pi \) such that

\[
\hat{\rho}(e_\alpha(z)) = \sum_{\beta \in \Delta_+} \: P_{\alpha}^\beta(a^*(z))a_\beta(z) := a_\alpha(z) + \sum_{\beta \in \Delta_+ \setminus \{\alpha\}} \: P_{\alpha}^\beta(a^*(z))a_\beta(z),
\]

\[
\hat{\rho}(h_\alpha(z)) = -\sum_{\beta \in \Delta_+} \beta(h_\alpha) : a_\beta^*(z)a_\beta(z) : + b_\alpha(z),
\]

\[
\hat{\rho}(f_\alpha(z)) = \sum_{\beta \in \Delta_+} \: Q_{\alpha}^\beta(a^*(z))a_\beta(z) + : b_\alpha(z)a_\alpha^*(z) : + ((e_\alpha | f_\alpha) k + c_\alpha) \partial a_\alpha^*(z)
\]

for all \( \alpha \in \Pi \). For any \( \alpha \in \Delta_+ \), \( \hat{\rho}(e_\alpha(z)) \), \( \hat{\rho}(h_\alpha(z)) \) also take the same forms.

The injective vertex algebra homomorphism \( \hat{\rho} \) provides a \( V^T(\mathfrak{g}) \)-module structure on any \( \mathcal{A}_{\Delta_+}^T \otimes \mathcal{H}^T \)-module, called a Wakimoto representation of \( V^T(\mathfrak{g}) \). The specialization of \( \hat{\rho} \) induces a vertex algebra homomorphism

\[
\hat{\rho}_k = \hat{\rho} \otimes \mathbb{C}_k: V^k(\mathfrak{g}) \to \mathcal{A}_{\Delta_+}^T \otimes \mathcal{H},
\]

which is also injective by [Fre].

3.3. Screening Operators for \( V^T(\mathfrak{g}) \). Let \( \rho^R: \mathfrak{n}_+ \to \mathcal{D}_{\mathfrak{n}_+} \) be the Lie algebra anti-homomorphism induced by the right action of \( \mathfrak{n}_+ \) on itself. Denote by

\[
(3.3) \quad \rho^R(e_\alpha) = \sum_{\beta \in \Delta_+} P_{\alpha}^\beta(\partial_\beta)\end{equation}

for \( \alpha \in \Delta_+ \), where \( P_{\alpha}^\beta(\partial_\beta) \) is a polynomial in \( \mathbb{C}[\mathfrak{n}_+] \). Since the left and right actions of \( \mathfrak{n}_+ \) on itself commute, we have

\[
[\rho(e_\alpha), \rho^R(e_\beta)] = 0
\]

for all \( \alpha, \beta \in \Delta_+ \). Let \( W^T(\lambda) = \mathcal{A}_{\Delta_+}^T \otimes \mathcal{H}_\lambda^T \) be a Wakimoto representation of \( V^T(\mathfrak{g}) \) for \( \lambda \in \mathfrak{h}^* \) and \( \tilde{T} = T_{k+h^\vee} \) the localization of \( T \) at a multiplicative set \( \{(k + h^\vee)^i \mid i \in \mathbb{Z}_{\geq 0}\} \), where \( k \) is a formal parameter in \( T \). Denote by \( W(\lambda) = W^T(\lambda) \otimes \mathbb{C}_k \), by \( V^\tilde{T}(\mathfrak{g}) = V^T(\mathfrak{g}) \otimes \tilde{T} \) and by \( W^\tilde{T}(\lambda) = W^T(\lambda) \otimes \tilde{T} \). Set \( W^\tilde{T}_0 = W^T(0), \mathcal{W}_0 = W(0) \) and \( W^\tilde{T}_0 = W^\tilde{T}(0) \). By [FF5], there exists an exact sequence

\[
(3.4) \quad 0 \to V^T(\mathfrak{g}) \xrightarrow{\phi} W^\tilde{T}_0 \bigoplus_{\alpha \in \Pi} W^\tilde{T}(\tilde{\alpha}),
\]

where \( \tilde{\alpha} = -(k + h^\vee)^{-1}_\alpha \in \mathfrak{h}^* \otimes \tilde{T} \) and \( S_\alpha: W^\tilde{T}_0 \to W^\tilde{T}(\tilde{\alpha}) \) is an intertwining operator defined by

\[
(3.5) \quad S_\alpha = \int : \rho^R(e_\alpha(z)) e^{-\frac{1}{k+h^\vee} f_\alpha(z)} : dz
\]

for \( \alpha \in \Pi \), where

\[
(3.6) \quad \rho^R(e_\alpha(z)) = \sum_{\beta \in \Delta_+} P_{\alpha}^\beta(\partial_\beta)\end{equation}

In other words,

\[
V^T(\mathfrak{g}) \simeq \bigcap_{\alpha \in \Pi} \ker S_\alpha.
\]
The intertwining operators $S_\alpha$ are called the screening operators for $V^T(\mathfrak{g})$. We note that these screening operators are only considered for generic $k = k$ in [FF5], but the same proof also holds when the base ring is $T$.

4. WAKIMOTO REPRESENTATIONS FOR AFFINE W-ALGEBRAS

4.1. Coordinates on $N_+$. Let $G_{>0}, G_{0}^+$ be the unipotent Lie subgroups in $N_+$ corresponding to $\mathfrak{g}_{>0}, \mathfrak{g}_{0}^+ = \mathfrak{g}_0 \cap n_+$ respectively. Since $\mathfrak{g}_{>0}$ is an ideal in $n_+$, a subgroup $G_{>0}$ is normal in $N_+$. Hence a set $G_{>0} \times G_{0}^+$ has a group structure and is isomorphic to $N_+$ by $G_{>0} \times G_{0}^+ \ni (a, b) \mapsto a \cdot b \in G_{>0} \cdot G_{0}^+ = N_+$. Let $c(\mathfrak{g}_{>0}), c(\mathfrak{g}_{0}^+)$ be homogeneous coordinates on $G_{>0}, G_{0}^+$ respectively. A coordinate $c(n_+)$ on $N_+$ is then defined by $c(n_+) = c(\mathfrak{g}_{>0}) \cdot c(\mathfrak{g}_{0}^+)$, which induces a ring isomorphism $\mathbb{C}[N_+] \cong \mathbb{C}[G_{>0}] \otimes \mathbb{C}[G_{0}^+]$. We call $c(n_+) = c(\mathfrak{g}_{>0}) \cdot c(\mathfrak{g}_{0}^+)$ a coordinate on $N_+$ compatible with the decomposition $N_+ = G_{>0} \times G_{0}^+$. By construction, we have

$$
\rho|_{\mathfrak{g}_{>0}} = \rho_{\mathfrak{g}_{>0}}, \quad \rho^R|_{\mathfrak{g}_{0}^+} = \rho^R_{\mathfrak{g}_{0}^+},
$$

where $\rho_{\mathfrak{g}_{>0}}$ is the Lie algebra homomorphism derived from the left action of $G_{>0}$ on $G_{>0}$ and $\rho^R_{\mathfrak{g}_{0}^+}$ is the Lie algebra anti-homomorphism derived from the right action of $G_{0}^+$ on $G_{0}^+$. Thus, we obtain:

**Lemma 4.1.** Suppose that $c(n_+)$ is compatible with the decomposition $N_+ = G_{>0} \times G_{0}^+$. Then

1. $\rho(u)$ belongs to $\mathcal{D}_{G_{>0}}$ for all $u \in \mathfrak{g}_{>0}$.
2. $\rho^R(u)$ belongs to $\mathcal{D}_{G_{0}^+}$ for all $u \in \mathfrak{g}_{0}^+$. In particular,

$$
\rho^R(e_\alpha) = \sum_{\beta \in \Delta^+_0} P^{x, R}_\alpha(x) \partial_\beta
$$

for all $\alpha \in \Delta^+_0$.

Let $Q$ be the root lattice of $\mathfrak{g}$ and $Q_+ \subset \mathfrak{Q}$ the set of linear combination with coefficients in $\mathbb{Z}_{\geq 0}$ of elements of $\Pi$. Define a $Q$-valued grading on $\mathcal{D}_{N_+}$ by

$$
\text{deg}_Q(\partial_\alpha) = -\alpha, \quad \text{deg}_Q x_\alpha = \alpha
$$

for $\alpha \in \Delta_+$, which induces a $Q_+$-grading on $\mathbb{C}[N_+]$. We define a $Q$-valued grading on $\mathfrak{g}$ by $\text{deg}_Q(\mathfrak{g}_a) = \alpha$ and $\text{deg}_Q(\mathfrak{g}_b) = 0$. Then $\rho$ and $\rho^R$ reverse the $Q$-gradings, i.e. $\text{deg}_Q \rho(u) = -\text{deg}_Q(u)$ for $u \in \mathfrak{g}$, and $\text{deg}_Q \rho^R(u) = -\text{deg}_Q(u)$ for $u \in n_+$. Therefore

$$
(4.1) \quad \text{deg}_Q P^\beta_\alpha(x) = \text{deg}_Q P^{x, R}_\alpha(x) = \beta - \alpha \in Q_+,
$$

$$
(4.2) \quad \text{deg}_Q Q^\beta_\alpha(x) = \beta + \alpha \in Q_+
$$

unless $P^\beta_\alpha(x) = P^{x, R}_\alpha(x) = Q^\beta_\alpha(x) = 0$. A $\frac{1}{2}\mathbb{Z}$-grading $\text{deg}_\Gamma$ on $\Delta$ may be extended to $Q$ linearly. Then the composition map $\text{deg}_\Gamma \circ \text{deg}_Q$ defines a $\frac{1}{2}\mathbb{Z}$-grading on $\mathcal{D}_{N_+}$, which we denote by $\text{deg}_\Gamma$ by abuse of notations. We have

$$
(4.3) \quad \text{deg}_\Gamma(\partial_\alpha) = -\text{deg}_\Gamma \alpha, \quad \text{deg}_\Gamma x_\alpha = \text{deg}_\Gamma \alpha
$$

for $\alpha \in \Delta_+$, which induces a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-grading on $\mathbb{C}[N_+]$. Then $\rho$ and $\rho^R$ reverse the gradings, i.e. $\text{deg}_\Gamma \rho(u) = -\text{deg}_\Gamma(u)$ for $u \in \mathfrak{g}$, and $\text{deg}_\Gamma \rho^R(u) = -\text{deg}_\Gamma(u)$ for
u ∈ n_+. We have

\[\text{deg}_\Gamma P_\alpha^\beta(x) = \text{deg}_\Gamma P_\alpha^\beta.R(x) = \text{deg}_\Gamma \beta - \text{deg}_\Gamma \alpha \geq 0,\]

\[\text{deg}_\Gamma Q_\alpha^\beta(x) = \text{deg}_\Gamma \beta + \text{deg}_\Gamma \alpha \geq 0\]

unless \(P_\alpha^\beta(x) = P_\alpha^\beta.R(x) = Q_\alpha^\beta(x) = 0.\)

**Lemma 4.2.** Suppose that \(c(n_+)\) is compatible with the decomposition \(N_+ = G_{>0} \times G_{0}^+.\) If \(\text{deg}_\Gamma \alpha = \text{deg}_\Gamma \beta,\) polynomials \(P_\alpha^\beta(x)\) and \(P_\alpha^\beta.R(x)\) belong to \(\mathbb{C}[G_{0}^+].\)

**Proof.** Since \(\text{deg}_\Gamma P_\alpha^\beta(x) = \text{deg}_\Gamma P_\alpha^\beta.R(x) = 0,\) they are concentrated in the homogeneous component of \(\mathbb{C}[N_+]\) with degree 0, which coincides with \(\mathbb{C}[G_{0}^+].\) This completes the proof. \(\Box\)

**Lemma 4.3.** Suppose that \(c(n_+)\) is compatible with the decomposition \(N_+ = G_{>0} \times G_{0}^+.\) For \(\alpha \in \Delta_{>0},\)

\[\rho(e_\alpha) = \partial_\alpha + \sum_{\beta \in \Delta_{>0}, \text{deg}_\beta > \text{deg}_\alpha} P_\alpha^\beta(x) \partial_\beta.\]

**Proof.** Let \(\alpha \in \Delta_{>0}.\) Then \(\rho(e_\alpha)\) belongs to \(D_{G_{>0}}\) by Lemma 4.1 so polynomials \(P_\alpha^\beta(x)\) are in \(\mathbb{C}[G_{>0}]\) for all \(\beta \in \Delta_{>0}.\) First, we assume that \(\text{deg}_\Gamma \beta < \text{deg}_\Gamma \alpha.\) If \(P_\alpha^\beta(x) \neq 0,\) we have \(\text{deg}_\Gamma P_\alpha^\beta(x) = \text{deg}_\Gamma \beta - \text{deg}_\Gamma \alpha \geq 0,\) which is contrary to our assumption. Therefore \(P_\alpha^\beta(x) = 0.\)

Next, we assume that \(\text{deg}_\Gamma \beta = \text{deg}_\Gamma \alpha.\) Then \(P_\alpha^\beta(x) \in \mathbb{C}[G_{>0}] \cap \mathbb{C}[G_{0}^+] = \mathbb{C}\) by Lemma 4.2. Therefore, \(P_\alpha^\beta(x)\) is a scalar. Hence, \(\text{deg}_\beta P_\alpha^\beta(x) = \beta - \alpha\) should be zero unless \(P_\alpha^\beta(x) = 0.\) Now, we have \(P_\alpha^\beta(x) = 0\) if \(\text{deg}_\Gamma \beta \leq \text{deg}_\Gamma \alpha\) except for \(\beta = \alpha.\) Since \(P_\alpha^\beta(x) = 1\) by construction, the lemma follows. \(\Box\)

### 4.2. The cohomology \(H(C_{g_{>0}}^T, d).\)

Let \(C_{g_{>0}}^T\) be a vertex superalgebra over \(T\) defined by

\[C_{g_{>0}}^T = \mathcal{A}_{g_{>0}}^T \otimes F_{ch}(g_{>0}) \otimes \Phi_T(g_{<0}).\]

Since \(\rho(u)\) belongs to \(D_{G_{>0}}\) for all \(u \in \mathfrak{g}_{>0}\) by Lemma 4.1, \(\hat{\rho}(u(z))\) is a field on \(\mathcal{A}_{g_{>0}}^T\) for all \(u \in \mathfrak{g}_{>0}.\) Since \(C_{g_{>0}}^T\) has a \(V^T(g_{>0})\)-module structure given by \(\hat{\rho},\)

\((C_{g_{>0}}^T, d)\) defines a cochain complex with respect to the charge degree on \(F_{ch}(g_{>0}).\)

To compute the cohomology \(H(C_{g_{>0}}^T, d),\) we introduce a \(1/2\)-grading on \(C_{g_{>0}}^T\) by

\[\text{deg}_{C_{g_{>0}}^T} (a_\alpha) = \text{deg}_{C_{g_{>0}}^T} (\varphi_\alpha) = -\text{deg}_\alpha,\]

\[\text{deg}_{C_{g_{>0}}^T} (a^*_\alpha) = \text{deg}_{C_{g_{>0}}^T} (\varphi^*_\alpha) = \text{deg}_\alpha,\]

and \(\text{deg}_{C_{g_{>0}}^T} (\partial A) = \text{deg}_{C_{g_{>0}}^T} (A),\) \(\text{deg}_{C_{g_{>0}}^T} (A:B) = \text{deg}_{C_{g_{>0}}^T} (A) + \text{deg}_{C_{g_{>0}}^T} (B)\) for all \(A, B \in C_{g_{>0}}^T.\)

We associate this grading with the subspaces

\[F_p C_{g_{>0}}^T = \{ A \in C_{g_{>0}}^T | 2 \text{deg}_{C_{g_{>0}}^T} (A) \geq p \} \]

of \(C_{g_{>0}}^T\) for \(p \in \mathbb{Z},\) which satisfy that

\[\bigcup_{p \in \mathbb{Z}} F_p C_{g_{>0}}^T = C_{g_{>0}}^T, \quad \bigcap_{p \in \mathbb{Z}} F_p C_{g_{>0}}^T = 0, \quad F_{p+1} C_{g_{>0}}^T \subset F_p C_{g_{>0}}^T, \quad d : F_p C_{g_{>0}}^T \subset F_{p+1} C_{g_{>0}}^T.\]
Hence, \( \{ F_pC^T_{\mathfrak{g} > 0} \}_{p \in \mathbb{Z}} \) is a filtration of the cochain complex \( (C^T_{\mathfrak{g} > 0}, d) \). Let \( \{ E_q \}_{q=0}^\infty \) be the spectral sequence induced by \( F_pC^T_{\mathfrak{g} > 0} \). Let \( \Delta(A) \) be the conformal weight of \( A \in C^T_{\mathfrak{g} > 0} \) defined by

\[
\Delta(a_\alpha) = \Delta(\varphi_\alpha) = 1 - \deg_\Gamma \alpha, \quad \Delta(a^*_\alpha) = \Delta(\varphi^\alpha) = \deg_\Gamma \alpha, \quad \Delta(\Phi_\alpha) = \frac{1}{2}.
\]

Set

\[
C^T_{\mathfrak{g} > 0}(n) = \text{Span}_C \{ A \in C^T_{\mathfrak{g} > 0} \mid \Delta(A) = n \}
\]

for \( n \in \frac{1}{2} \mathbb{Z} \). Since \( d \cdot C^T_{\mathfrak{g} > 0}(n) \subset C^T_{\mathfrak{g} > 0}(n) \),

\[
C^T_{\mathfrak{g} > 0} = \bigoplus_{n \in \frac{1}{2} \mathbb{Z}} C^T_{\mathfrak{g} > 0}(n)
\]

is the decomposition as a complex. Denote by \( F_pC^T_{\mathfrak{g} > 0}(n) \) the induced filtration on \( C^T_{\mathfrak{g} > 0}(n) \).

**Lemma 4.4.** \( F_pC^T_{\mathfrak{g} > 0}(n) = 0 \) for \( p > 2n \).

**Proof.** The vertex algebra \( C^T_{\mathfrak{g} > 0} \) is spanned by all vectors of the form

\[
A = : (Da^*)(Da)(D\varphi^*)(D\varphi)(D\Phi) :,
\]

where

\[
Da^* = (\partial^{m_i}_{a^{(i)}_1} a^{*}_{a^{(i)}_1}) \cdots (\partial^{m_r}_{a^{(r)}_1} a^{*}_{a^{(r)}_1}), \quad Da = (\partial^{m_i}_{a^{(i)}_1} a_{a^{(i)}_1}) \cdots (\partial^{m_r}_{a^{(r)}_1} a_{a^{(r)}_1}),
\]

\[
D\varphi^* = (\partial^{m_i}_{\varphi^{(i)}_1} \varphi^{a^{(i)}_1}) \cdots (\partial^{m_r}_{\varphi^{(r)}_1} \varphi^{a^{(r)}_1}), \quad D\varphi = (\partial^{m_i}_{\varphi^{(i)}_1} \varphi_{a^{(i)}_1}) \cdots (\partial^{m_r}_{\varphi^{(r)}_1} \varphi_{a^{(r)}_1}),
\]

\[
D\Phi = (\partial^{m_i}_{\Phi^{(i)}_1} \Phi^{a^{(i)}_1}) \cdots (\partial^{m_r}_{\Phi^{(r)}_1} \Phi^{a^{(r)}_1})
\]

for some \( m_i^{(i)}, r, s, t, u, v \in \mathbb{Z}_{\geq 0}, a^{(i)}_j, \varphi^{(i)}_j, \Phi^{(i)}_j \in \Delta_{> 0} \) (\( i = 1, 2, 3, 4 \)) and \( a^{(5)}_j, \varphi^{(5)}_j, \Phi^{(5)}_j \in \Delta_{\frac{1}{2}} \). Therefore, it suffices to see that \( 2\Delta(A) \geq p \) for all vectors \( A \in F_pC^T_{\mathfrak{g} > 0} \) of the above form.

By definition,

\[
\Delta(A) - \deg_{C^T_{\mathfrak{g} > 0}}(A) = \sum_{i,j} m_j^{(i)} + s + u + \frac{1}{2} v.
\]

Since \( A \in F_pC^T_{\mathfrak{g} > 0} \),

\[
2\Delta(A) \geq 2\Delta(A) - 2(\sum_{i,j} m_j^{(i)} + s + u + \frac{1}{2} v) = 2 \deg_{C^T_{\mathfrak{g} > 0}}(A) \geq p.
\]

This completes the proof. \( \square \)

**Proposition 4.5.** \( H(C^T_{\mathfrak{g} > 0}, d) \simeq \Phi^T(\mathfrak{g}_{\frac{1}{2}}) \).

**Proof.** The spectral sequence induced by \( F_pC^T_{\mathfrak{g} > 0}(n) \) converges for each \( n \in \frac{1}{2} \mathbb{Z} \) by Lemma 4.3 and so the total spectral sequence \( E_n \) does. By definition, it is easy to see that

\[
E_1 = H(C^T_{\mathfrak{g} > 0}, d_{st}) = H(A^T_{\Delta_{> 0}} \otimes F^T_{\mathfrak{ch}}(\mathfrak{g}_{> 0}), d_{st}) \otimes \Phi^T(\mathfrak{g}_{\frac{1}{2}}),
\]

where \( d_{st} = \int d_{st}(z) \, dz \). The cohomology \( H(A^T_{\Delta_{> 0}} \otimes F^T_{\mathfrak{ch}}(\mathfrak{g}_{> 0}), d_{st}) \) coincides with the semi-infinite cohomology of \( \mathfrak{g}_{> 0} \)-module with the coefficient of the Wakimoto
module $A^T_{\Delta > 0}$ over $T$, see [Fei]. Hence, the vanishing theorem given in [FF3, V2] can be applied to our cases:

$$H(A^T_{\Delta > 0} \otimes F^T_{\text{ch}}(\mathfrak{g}_{> 0}), d_{st}) = H^0(A^T_{\Delta > 0} \otimes F^T_{\text{ch}}(\mathfrak{g}_{> 0}), d_{st}) = T.$$ Therefore,

$$(4.6) \quad H(C^T_{\mathfrak{g}_{> 0}}, d) \simeq E_{\infty} = E_1 = \Phi^T(\mathfrak{g}_{\frac{1}{2}})$$

as required. \hfill \Box

**Corollary 4.6.** The isomorphism in Proposition 4.5 is a vertex algebra isomorphism over $T$.

**Proof.** First, notice that $H(C^T_{\mathfrak{g}_{> 0}}, d)$ has a vertex algebra structure inherited from that of $C^T_{\mathfrak{g}_{> 0}}$. We will show that the isomorphism (4.6) is a vertex algebra isomorphism. Set

$$(4.7) \quad \hat{\Phi}_\alpha(z) = \Phi_\alpha(z) + \sum_{\beta \in \Delta_{\frac{1}{2}}} \chi([e_\beta, e_\alpha])a_\beta(z)$$

for $\alpha \in \Delta_{\frac{1}{2}}$. Recall that $\deg_{C^T_{\mathfrak{g}_{> 0}}} (\Phi_\alpha) = 0$ and $\Delta(\Phi_\alpha) = \frac{1}{2}$. Moreover, $\deg_{C^T_{\mathfrak{g}_{> 0}}} (\hat{\Phi}_\alpha - \Phi_\alpha) = \frac{1}{2}$ and $\Delta(\hat{\Phi}_\alpha) = \frac{1}{2}$. By Lemma 3.1 and Lemma 4.3 we have

$$\hat{\rho}(e_\alpha(z)) = a_\alpha(z) + \sum_{\beta \in \Delta_{\geq 1}} :P^\beta_{\alpha}(a^*)(z) a_\beta(z) :$$

for $\alpha \in \Delta_{\frac{1}{2}}$. Hence,

$$d \cdot a^*_\alpha = d_{st} \cdot a^*_\alpha = \sum_{\beta \in \Delta_{\frac{1}{2}}} \int :\hat{\rho}(e_\beta(z))\varphi^\beta(z) : a^*_\alpha \ dz = \varphi^\alpha.$$ 

for $\alpha \in \Delta_{\frac{1}{2}}$. Therefore,

$$d \cdot \hat{\Phi}_\alpha = \sum_{\beta \in \Delta_{\frac{1}{2}}} \chi([e_\beta, e_\alpha])\varphi^\alpha + \sum_{\beta \in \Delta_{\frac{1}{2}}} \chi([e_\alpha, e_\beta])(d \cdot a^*_\beta) = 0.$$ 

This implies that the isomorphism (4.6) is given by the correspondence $\hat{\Phi}_\alpha \mapsto \Phi_\alpha$. We have

$$\hat{\Phi}_\alpha(z)\Phi_\beta(w) \sim \frac{\chi([e_\alpha, e_\beta])}{z - w} \sim \Phi(z)\Phi(w)$$

for all $\alpha, \beta \in \Delta_{\frac{1}{2}}$. Hence, the isomorphism (4.6) is a vertex algebra isomorphism. \hfill \Box

4.3. **Wakimoto free fields realizations of $W^T(\mathfrak{g}, f; \Gamma)$**. Given a $V^T(\mathfrak{g})$-module $M$, define a $C_T$-module $C_T(M) = M \otimes F^T_{\text{ch}}(\mathfrak{g}_{> 0}) \otimes \Phi^T(\mathfrak{g}_{\frac{1}{2}})$. Then $(C_T(M), d)$ is a cochain complex whose cohomology

$$H_\chi(M) = H(C_T(M), d)$$

has a structure of a $W^T(\mathfrak{g}, f; \Gamma)$-module by construction. Consider the case that $M$ is a Wakimoto representation $W^T(\lambda)$ of $V^T(\mathfrak{g})$. We have a $W^T(\mathfrak{g}, f; \Gamma)$-module

$$H_\chi(W^T(\lambda)) = H(W^T(\lambda) \otimes F^T_{\text{ch}}(\mathfrak{g}_{> 0}) \otimes \Phi^T(\mathfrak{g}_{\frac{1}{2}}), d)$$

for each $\lambda \in \mathfrak{h}^*$. 
Lemma 4.7. For all $\lambda \in \mathfrak{h}^* \otimes T$,

$$H_\chi(W^T(\lambda)) = H^0_\chi(W^T(\lambda)) \simeq A^T_{\Delta_0^+} \otimes \Phi^T(\mathfrak{g}_+^*) \otimes \mathcal{H}^T_{\lambda}.$$ 

Proof. By definition, $d \cdot (A^T_{\Delta_0^+} \otimes \mathcal{H}^T_{\lambda}) = 0$. Hence,

$$H_\chi(W^T(\lambda)) = A^T_{\Delta_0^+} \otimes H(C^T_{\mathfrak{g}^*}, d) \otimes \mathcal{H}^T_{\lambda}.$$ 

By Proposition 4.3, $H(C^T_{\mathfrak{g}^*}, d) \simeq \Phi(\mathfrak{g}_+^*)$. Therefore the assertion follows. \qed

Let

$$\sigma_\lambda : H_\chi(W^T(\lambda)) \xrightarrow{\sim} A^T_{\Delta_0^+} \otimes \Phi^T(\mathfrak{g}_+^*) \otimes \mathcal{H}^T_{\lambda}$$

be the isomorphism defined in Lemma 4.7 for $\lambda \in \mathfrak{h}^* \otimes T$. Denote by $\sigma_\lambda(A) = \sigma_\lambda([A])$ for all $A \in C_T(W^T(\lambda))$, where $[A]$ denotes the cohomology class of $A$ in $H_\chi(W^T(\lambda))$. Set $\sigma = \sigma_0$. Notice that $H_\chi(W^T_{\mathfrak{g}})$ has a vertex algebra structure inherited from that of $C_T(W^T_{\mathfrak{g}})$. Then $H_\chi(W^T(\lambda))$ is an $H_\chi(W^T_{\mathfrak{g}})$-module for all $\lambda \in \mathfrak{h}^* \otimes T$.

Corollary 4.8. The map $\sigma$ is an isomorphism of vertex algebras over $T$ defined by $\sigma(A) = A$ for $A = a_\alpha, a_\alpha^* (\alpha \in \Delta^+_0)$, $b_\alpha (\alpha \in \Pi)$, and $\sigma(\hat{\Phi}_\alpha) = \hat{\Phi}_\alpha (\alpha \in \Delta^+_0)$, and $\sigma_\lambda$ is an isomorphism of $A^T_{\Delta_0^+} \otimes \Phi^T(\mathfrak{g}_+^*) \otimes \mathcal{H}^T$-modules for all $\lambda \in \mathfrak{h}^*$, where $\hat{\Phi}_\alpha$ is defined by (4.7).

Proof. The corollary is immediate from by Corollary 4.6 and the construction of the isomorphism $\sigma_\lambda$ in Lemma 4.7. \qed

The same argument applies to the case that $M$ is a Wakimoto representation $W^\tilde{T}(\lambda)$ of $V^\tilde{T}(\mathfrak{g})$. Thus, we obtain:

Lemma 4.9. For all $\lambda \in \mathfrak{h}^* \otimes \tilde{T}$, we have a vertex algebra isomorphism

$$H_\chi(W^\tilde{T}(\lambda)) = H^0_\chi(W^\tilde{T}(\lambda)) \simeq A^T_{\Delta_0^+} \otimes \Phi^\tilde{T}(\mathfrak{g}_+^*) \otimes \mathcal{H}^\tilde{T}_{\lambda}$$

over $\tilde{T}$, where $X^\tilde{T} = X^T \otimes \tilde{T}$ for $X = A_{\Delta_+}, \Phi(\mathfrak{g}_+^*), \mathcal{H}_\lambda$.

Recall the exact sequence (4.4):

$$0 \rightarrow V^T(\mathfrak{g}) \xrightarrow{\tilde{\rho}} W^T_{\mathfrak{g}} \xrightarrow{\oplus S_{\alpha}} \bigoplus_{\alpha \in \Pi} W^\tilde{T}(\tilde{\alpha}).$$

Let $C_T(M) = M \otimes F^\tilde{T}_{\mathfrak{g}^*}(\mathfrak{g}^*_{\geq 0}) \otimes \Phi^\tilde{T}(\mathfrak{g}_+^*)$. Then we have

$$0 \rightarrow C_T \rightarrow C_T(W^T_{\mathfrak{g}}) \rightarrow \bigoplus_{\alpha \in \Pi} C_{\tilde{T}}(W^\tilde{T}(\tilde{\alpha})).$$

Recall that $H(C_T, d) = H^0(C_T, d) = W^T(\mathfrak{g}, f; \Gamma)$. According to Lemma 4.7 and Lemma 4.9 we have an exact sequence

$$(4.8) \quad W^T(\mathfrak{g}, f; \Gamma) \xrightarrow{\omega} A^T_{\Delta_0^+} \otimes \Phi^T(\mathfrak{g}_+^*) \otimes \mathcal{H}^T \bigoplus_{\alpha \in \Pi} A^T_{\Delta_0^+} \otimes \Phi^\tilde{T}(\mathfrak{g}_+^*) \otimes \mathcal{H}^\tilde{T}_{\alpha},$$

where

$$\omega : W^T(\mathfrak{g}, f; \Gamma) \rightarrow A^T_{\Delta_0^+} \otimes \Phi^T(\mathfrak{g}_+^*) \otimes \mathcal{H}^T$$
Hence, the right \( V \) induces that on Lemma 4.10. For all \( W \), the Wakimoto representation for a \( \alpha \)

\[
\text{Proof.} \quad \text{Since} \quad \rho(e_{\alpha}) \text{ and } \rho^R(e_{\beta}) \text{ commutes for all } \alpha, \beta \in \Delta_+, \text{ we have } d \cdot \rho^R(e_{\alpha}) = 0.
\]

Hence, the right \( V^T(\alpha) \)-action on a Wakimoto representation \( W^T_\alpha \) given by \( \hat{\rho}^R \) induces that on \( H_\chi(W^T_\alpha) \simeq A_{\Delta_0^+}^T \otimes \Phi^T(\mathfrak{g}_\Phi^+) \otimes H^T_\alpha \) by \( \sigma \circ \hat{\rho}^R \).

**Lemma 4.10.** For all \( \alpha \in \Pi \),

\[
Q_\alpha = \int : \sigma(\hat{\rho}^R(e_{\alpha}))(z) e^{-\Delta_+} f_{b_\alpha}(z) : dz.
\]

**Proof.** Since \( Q_\alpha \) is the intertwining operator induced by \( S_\alpha \) through the functor \( H_\chi(?) \), the assertion of the lemma follows. \( \square \)

**Lemma 4.11.** For all \( \alpha \in \Pi_0 \),

\[
Q_\alpha = \int : \hat{\rho}^R(e_{\alpha}))(z) e^{-\Delta_+} f_{b_\alpha}(z) : dz.
\]

In particular, \( Q_\alpha = S_\alpha \) on \( A_{\Delta_0^+}^T \otimes H^T \) and acts as \( 0 \) on \( \Phi^T(\mathfrak{g}_\Phi^+) \) for all \( \alpha \in \Pi_0 \).

**Proof.** By Lemma 4.11 (2),

\[
\hat{\rho}^R(e_{\alpha}) = \sum_{\beta \in \Delta_0^+} P_{\alpha}^{\beta, R}(a^*) a_{\beta}
\]

belongs to \( A_{\Delta_0^+} \). Since \( \sigma(A) = A \) for all \( A \in A_{\Delta_0^+} \), the assertion of the lemma follows from Lemma 4.10. \( \square \)

We recall that \( P_{\alpha}^{\beta, R}(a^*)(z) \) is the field on \( A_{\Delta_0^+} \) defined by (3.3) for the polynomial \( P_{\alpha}^{\beta, R}(x) \) that is defined by (3.1) and depends on the choice of coordinates \( c(n_+) \) on \( N_+ \). According to Lemma 4.2, it follows that \( P_{\alpha}^{\beta, R}(a^*)(z) \) is a field on \( A_{\Delta_0^+} \) if \( \deg_T \alpha = \deg_T \beta \).

**Theorem 4.12.** Suppose that the coordinate \( c(n_+) \) on \( N_+ \) is compatible with the decomposition \( N_+ = G_{>0} \times G_{>0}^\times \). Then, we have

\[
Q_\alpha = \sum_{\beta \in \Delta_0^+} \int : P_{\alpha}^{\beta, R}(a^*)(z) a_{\beta}(z) e^{-\Delta_+} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_0),
\]

\[
Q_\alpha = \sum_{\beta \in \Delta_\frac{1}{2}} \int : P_{\alpha}^{\beta, R}(a^*)(z) \Phi_{\beta}(z) e^{-\Delta_+} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_\frac{1}{2}),
\]

\[
Q_\alpha = \sum_{\beta \in \Delta_1} \chi(e_{\beta}) \int : P_{\alpha}^{\beta, R}(a^*)(z) e^{-\Delta_+} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_1).
\]

We will prove Theorem 4.12 in Appendix A.

By Theorem 5.5, it turns out that

\[
\mathcal{W}^T(\mathfrak{g}, f; \Gamma) \simeq \bigcap_{\alpha \in \Pi} \ker Q_\alpha.
\]
In particular, the map \( \omega \) is injective (Corollary 5.7). Thus, we have an exact sequence

\[
0 \to W^T(g, f; \Gamma) \to A_{\Delta_0^+}^\tau \otimes \Phi^T(g_\Delta) \otimes \mathcal{H}^T \oplus \bigoplus_{\alpha \in \Pi} A_{\Delta_0^+}^\tau \otimes \Phi^T(g_\Delta) \otimes \mathcal{H}^T.
\]

We note that \( A_{\Delta_0^+}^\tau \otimes \Phi^T(g_\Delta) \) coincides with \( \frac{1}{2} \dim(N \cap S_f) \) copies of the \( \beta \gamma \)-system, where \( N \) is the nilpotent cone of \( g \), \( S_f \) is the Slodowy slice of \( g \) through \( f \), since

\[
\frac{1}{2} \dim(N \cap S_f) = \frac{1}{2}(\dim g_0 + \dim g_\Delta - \dim h) = \dim g_0^+ + \frac{1}{2} \dim g_\Delta.
\]

Let \( f = f_{\text{prin}} = \sum_{\alpha \in \Pi} \epsilon_\alpha \) be a principal nilpotent element in \( g \) and \( \Gamma \) the Dynkin grading on \( g \) for \( f \). Then \( \Pi = \Pi_\lambda \subseteq \Delta_1 \) and \( \chi(\epsilon_\alpha) = 1 \) for all \( \alpha \in \Pi \). By Lemma 4.3 \( P^\lambda_{\alpha, \beta}(z) = \delta_{\alpha, \beta} \) for all \( \alpha, \beta \in \Pi \). By Theorem 5.5 and Theorem 5.7 \( W^T(g, f; \Gamma) \) is isomorphic to

\[
\bigcap_{\alpha \in \Pi_\lambda} \ker \int : e^{-\frac{1}{2\kappa_{\alpha}}} f_{\alpha}(z) : dz, \quad \bigcap_{\alpha \in \Pi_\lambda} \ker \int : e^{-\frac{1}{2\kappa_{\alpha}}} f_{\alpha}(z) : dz,
\]

which is a well-known result given in [FF], see also [FBZ].

In the case that \( g_0 = h \), we have \( \Pi = \Pi_\lambda \subseteq \Pi_1 \) and, by Lemma 4.3 \( P^\lambda_{\alpha, \beta}(z) = \delta_{\alpha, \beta} \) for all \( \alpha \in \Pi_1, \beta \in \Delta_1 \) and \( i = \frac{1}{2}, 1 \). By Theorem 5.5 and Theorem 5.7 \( W^T(g, f; \Gamma) \) is an exact sequence.

Let \( V^{\tau_k}(g_0) \) be the affine vertex algebra associated with \( g_0 \) and its invariant bilinear form \( \tau_k \) defined by

\[
\tau_k(u|v) = \kappa_0^k(u|v) + \frac{1}{2} \kappa_0^k(u|v) - \frac{1}{2} \kappa_{g_0}^k(u|v)
\]

for all \( u, v \in g_0 \), where \( \kappa_0^k, \kappa_{g_0}^k \) are the Killing forms on \( g, g_0 \) respectively. Denote by \( V^T(g_0) \) instead of \( V^{\tau_k}(g_0) \) when the base ring is \( T = \mathbb{C}[k] \). By [Fre], there exists an exact sequence

\[
0 \to V^T(g_0) \xrightarrow{\hat{\rho}_{g_0}} A_{\Delta_0^+}^\tau \otimes \mathcal{H}^T \oplus \bigoplus_{\alpha \in \Pi_0} A_{\Delta_0^+}^\tau \otimes \mathcal{H}^T,
\]

where

\[
(4.9) \quad \hat{\rho}_{g_0} : V^T(g_0) \to A_{\Delta_0^+}^\tau \otimes \mathcal{H}^T
\]

is an injective vertex homomorphism over \( T \), called a Wakimoto representation of \( V^T(g_0) \), and is defined by

\[
\hat{\rho}_{g_0}(e_\alpha(z)) = \sum_{\beta \in \Delta_0^+} : P^\beta_{\alpha}(a^\tau)(z) a_\beta(z) :,
\]

\[
\hat{\rho}_{g_0}(h_\alpha(z)) = - \sum_{\beta \in \Delta_0^+} \beta(h_\alpha) : a^\beta_\alpha(z) a_\beta(z) + b_\alpha(z),
\]

\[
\hat{\rho}_{g_0}(f_\alpha(z)) = \sum_{\beta \in \Delta_0^+} : Q^\beta_{\alpha}(a^\tau)(z) a_\beta(z) + b_\alpha(z) a^\alpha_\beta(z) + ((e_\alpha(f_\alpha)k + c_\alpha) a^\alpha_\beta(z)
\]

and \( \hat{\rho}_{g_0} \) is a homomorphism.
for all $\alpha \in \Pi_0$ and $\alpha' \in \Pi$. Thus, we have

\begin{equation}
V^T(g_0) \simeq \bigcap_{\alpha \in \Pi_0} \ker S_\alpha \mid \Delta_0^+ \otimes H^T.
\end{equation}

**Lemma 4.13.** Suppose that the coordinate $c(n_+)$ on $N_+$ is compatible with the decomposition $N_+ = G_{>0} \times G_0^+$. Then

\[ \bigcap_{\alpha \in \Pi_0} \ker Q_\alpha \simeq V^T(g_0) \otimes \Phi(g_+). \]

**Proof.** Since $Q_\alpha = S_\alpha$ on $\Delta_0^+ \otimes H^T$ and acts as 0 on $\Phi(T)$ for all $\alpha \in \Pi_0$ by Lemma 4.11, the assertion of the lemma follows from (4.10). \qed

5. Screening operators and Miura map

In this section we recall the screening operators $\tilde{Q}_\alpha$ ($\alpha \in \Pi_{>0}$) introduced in \cite{Ge} and clarify the relationship between $Q_\alpha$ and $\tilde{Q}_\alpha$. As a result, we show that $Q_\alpha = \tilde{Q}_\alpha$ for all $\alpha \in \Pi_{>0}$ (Theorem 5.5) and the screening operator $Q_\alpha$’s are compatible with the Miura map $\mu$ (Theorem 5.6).

5.1. Screening Operators $\tilde{Q}_\alpha$. We follow the construction given in \cite{Ge}. Though the construction of $Q$ was considered in \cite{Ge} for generic $k = k$, the same argument applies when the base ring is $T$. Let $Q_0 = \bigoplus_{\gamma \in \Pi_0} \mathbb{Z}\gamma$ be the root lattice of $g_0$, $\Pi^T = \{ \alpha \in \Delta_{>0} \mid \beta, \gamma \in \Delta_{>0} \text{ s.t. } \alpha = \beta + \gamma \}$ a set of indecomposable roots in $\Delta_{>0}$. Define an equivalence relation on $\Delta_{>0}$ by $\alpha \sim \beta \iff \alpha - \beta \in Q_0$, which may restrict to $\Pi^T$. Let $[\Pi^T] = \Pi^T/\sim$ be the quotient set and

\begin{equation}
[\alpha] = \{ \beta \in \Delta_+ \mid \beta - \alpha \in Q_0 \}
\end{equation}

the equivalence class of $\alpha \in \Pi^T$ in $[\Pi^T]$. Consider a map $b: \Pi_{>0} \ni \alpha \mapsto [\alpha] \in [\Pi^T]$.

**Lemma 5.1.** The map $b$ is bijective.

**Proof.** Since it is clear that $b$ is injective, we will show that $b$ is surjective. Let $\beta \in \Pi^T$ and $n = \text{ht} \beta$ the height of $\beta$. In the case that $n = 1$, we have $\beta \in \Pi_{>0}$ and $b(\beta) = [\beta]$. Next, we assume that $n > 1$. Then there exist $\beta_1, \beta_2 \in \Delta_+$ such that $\beta = \beta_1 + \beta_2$. Since $\beta \in \Pi^T$, we may assume that $\beta_2 \in \Delta_{>0}$. Then $\beta - \beta_1 = \beta_2 \in Q_0$ and $[\beta] = [\beta_1]$. We claim that $\beta_1 \in \Pi^T$. The reason is below. If there exist $\gamma_1, \gamma_2 \in \Delta_{>0}$ such that $\beta_1 = \gamma_1 + \gamma_2$, we have

$$g_\beta = [g_{\beta_1}, g_{\beta_2}] = [[g_{\gamma_1}, g_{\gamma_2}], g_{\beta_2}] = [[g_{\gamma_1}, [g_{\gamma_2}, g_{\beta_3}]], g_{\gamma_2}] + [g_{\gamma_1}, g_{\gamma_2}, g_{\beta_3}].$$

Then $\gamma_1 + \beta_2 \in \Delta_{>0}$ or $\gamma_2 + \beta_2 \in \Delta_{>0}$. Hence, it turns out that $\beta = \gamma_1 + \gamma_2 + \beta_2$ can be decomposed to the sum of two roots in $\Delta_{>0}$, which is contrary to our assumption that $\beta \in \Pi^T$. Therefore there exists $\beta_1 \in \Pi^T$ such that $[\beta] = [\beta_1]$ and $\text{ht}(\beta_1) < \text{ht}(\beta)$. By induction on $n$, it follows that there exists $\alpha \in \Pi_{>0}$ such that $b(\alpha) = [\beta]$, that is, $b$ is surjective. The proof of the lemma is now complete. \qed

Thanks to Lemma 5.1 we may identify $[\Pi^T]$ with $\Pi_{>0}$ through $b$. Set a vector space $\mathbb{C}^{[\alpha]} = \bigoplus_{\beta \in [\alpha]} \mathbb{C} \tilde{\nu}_\beta$ for each $\alpha \in \Pi_{>0}$ and define a $g_0$-action on $\mathbb{C}^{[\alpha]}$ by

$$u \cdot \tilde{\nu}_\beta = \sum_{\gamma \in [\alpha]} c_{\beta, u}^{\gamma} \tilde{\nu}_\gamma.$$

By \([Ge]\), we have a vertex algebra isomorphism
\[ (5.3) \]
\[ (5.4) \]
\[ (5.2) \]

Let \(\tilde{\tau}\) be the \(v\) for all \(V\) where \(V\) is well-defined, see Proposition 3.7 in \([Ge]\). We define the screening operator \(\tilde{\tau}\) an intertwining operator corresponding to \(\tilde{\tau}\) and \(\tilde{\alpha}\) is an irreducible \(\tilde{V}\) vertex algebra \((\text{see } [FHL]\) for the details). For a \(\tilde{M}\) is called an intertwining operator of type \(\tilde{M},V\) \((\text{see } [Ge]\) for the details). For a \(\tilde{M}\) and a vector \(A \in M\), we shall call \(Y^M_{M,V}(A,z)\) an intertwining operator corresponding to \(A\). In the present paper, we only consider intertwining operators of type \((M\), \(V\)). Let \(\tilde{V}_\beta(z) = \sum_{n \in \mathbb{Z}} \tilde{V}_\beta^\gamma z^{-n}\) be an intertwining operator corresponding to \(\tilde{\beta}\) defined by \(\tilde{V}_\beta^\gamma \cdot 1 = \delta_{\alpha,0} \tilde{\beta}\) \((n \geq 0)\) and \([u(z), \tilde{V}_\beta(w)] = \sum_{\gamma \in [\alpha]} c^\beta_{\gamma,u} \tilde{V}_\gamma(w) \delta(z - w)\) for all \(u \in \tilde{g}_0\), where \(1\) denotes the vacuum vector in \(V^T(\tilde{g}_0)\). Then \(\tilde{V}_\beta(z)\) is well-defined, see Proposition 3.7 in \([Ge]\). We define the screening operator
\[ \tilde{Q}_\alpha: V^T(\tilde{g}_0) \otimes \Phi^T(\tilde{g}_{\frac{1}{2}}) \to \tilde{M}_\alpha \otimes \Phi^T(\tilde{g}_{\frac{1}{2}}) \]
for \(\alpha \in \Pi_{>0}\) by
\[ (5.2) \]
\[ (5.3) \]

By \([Ge]\), we have a vertex algebra isomorphism
\[ (5.4) \]
5.2. Miura map. As mentioned in Section [2] there exists a subcomplex $C_+$ of $C_k$ such that $C_+$ has only non-negative charged degree and $\mathcal{W}^k(g, f; \Gamma) = H^0(C_+, d)$. Hence, we have

$$\mathcal{W}^k(g, f; \Gamma) = \text{Ker} d|_{C_+} \subset C_+ = V^\infty(g_{\leq 0}) \otimes \Phi(g_{\frac{1}{2}}).$$

Consider the projection map $g_{\leq 0} \rightarrow g_0$. It induces a surjective vertex algebra homomorphism $V^\infty(g_{\leq 0}) \rightarrow V^\infty(g_0)$, giving rise to a map

$$\mu : \mathcal{W}^k(g, f; \Gamma) \rightarrow V^\infty(g_0) \otimes \Phi(g_{\frac{1}{2}}),$$

which is called the Miura map for $\mathcal{W}^k(g, f; \Gamma)$ and injective for all $k \in \mathbb{C}$ by [Fr2, A3], see also [Ge]. Following [Fr2], the Miura map coincides with the specialization $\mu^T \otimes C_k$ of an injective vertex algebra homomorphism

$$\mu^T : \mathcal{W}^T(g, f; \Gamma) \rightarrow V^T(g_0) \otimes \Phi^T(g_{\frac{1}{2}}).$$

over $T$ induced by the formula (5.3).

**Proposition 5.2.** Suppose that $c(n_+)$ is compatible with the decomposition $N_+ = G_{>0} \times G_0^+$.

1. Let $\alpha \in \Pi_{>0}$ and $\beta \in \Delta_+$ such that $\deg_T \alpha = \deg_T \beta$. Then $P^\beta_R(\alpha)(x) = 0$ unless $\beta \in [\alpha]$.

2. Let $Q_\alpha$ be a screening operator defined in Theorem 4.12. Then

$$Q_\alpha = \sum_{\beta \in [\alpha]} \int : P^\beta_R(a^*)(z) \Phi_\beta(z) e^{-\frac{1}{\kappa + h^\rho}} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_\frac{1}{2}),$$

$$Q_\alpha = \sum_{\beta \in [\alpha]} \chi(\epsilon) \int : P^\beta_R(a^*)(z) e^{-\frac{1}{\kappa + h^\rho}} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_1).$$

**Proof.** To prove the assertion (1), we assume that $P^\beta_R(\alpha)(x) \neq 0$ for $\alpha \in \Pi_{>0}$ and $\beta \in \Delta_+$ such that $\deg_T \alpha = \deg_T \beta$. We will show that $\beta \in [\alpha]$. Since $P^\beta_R(\alpha)(x)$ is a polynomial in $\mathbb{C}[G_0^+]$ by Lemma 4.12 and $\deg_Q P^\beta_R(\alpha)(x) = \beta - \alpha$ by (4.1), we have $\beta - \alpha \in Q_0$, that is, $\beta \in [\alpha]$. Thus, $P^\beta_R(\alpha)(x) = 0$ unless $\beta \in [\alpha]$.

Next, we derive (2) from (1). By Theorem 4.12 we have

$$Q_\alpha = \sum_{\beta \in \Delta_+} \int : P^\beta_R(a^*)(z) \Phi_\beta(z) e^{-\frac{1}{\kappa + h^\rho}} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_\frac{1}{2}),$$

$$Q_\alpha = \sum_{\beta \in \Delta_1} \chi(\epsilon) \int : P^\beta_R(a^*)(z) e^{-\frac{1}{\kappa + h^\rho}} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_1).$$

Since $P^\beta_R(a^*)(z) = 0$ for all $\alpha \in \Pi_i$ and $\beta \in \Delta_i \setminus [\alpha]$ with $i = \frac{1}{2}, 1$, we may restrict the summation range in $Q_\alpha$ to $\{ \beta \in [\alpha] \}$. This completes the proof. \qed

Set

$$v_\beta = P^\beta_R(a^*) \otimes e^\alpha \in A^T_{\Delta_0} \otimes \mathcal{H}^T_\alpha$$

for all $\beta \in [\alpha]$, where $e^\alpha$ is a highest weight vector in $\mathcal{H}^T_\alpha$. Let

$$V^\beta(z) = \sum_{n \in \mathbb{Z}} V^\beta_n z^{-n} = : P^\beta_R(a^*) e^{-\frac{1}{\kappa + h^\rho}} f_{b_\alpha}(z) :$$
be an intertwining operator corresponding to \( v_\beta \). By definition, \( V_\beta^n \cdot 1 = \delta_{n,0} v_\beta \) for \( n \geq 0 \), where 1 denotes the vacuum vector in \( \mathcal{A}_{\Delta_0^+}^T \otimes \mathcal{H}^T \). Recall that \( \tilde{\rho}_{g_0} : V^T(g_0) \rightarrow \mathcal{A}_{\Delta_0^+}^T \otimes \mathcal{H}^T \) is a Wakimoto representation of \( V^T(g_0) \) defined in 4.13 and provides a \( V^T(g_0) \)-module structure on \( \mathcal{A}_{\Delta_0^+}^T \otimes \mathcal{H}^T_{\Delta_0} \). We shall often drop the vertex algebra homomorphism \( \tilde{\rho}_{g_0} \) for the \( V^T(g_0) \)-action on \( \mathcal{A}_{\Delta_0^+}^T \otimes \mathcal{H}^T_{\Delta_0} \) (e.g. \( u(z) \) in place of \( \tilde{\rho}_{g_0}(u(z)) \)) if no confusion may arise.

**Lemma 5.3.** For all \( u \in g_0 \) and \( \beta \in [\alpha] \),

\[
(u(z), V^\beta(w)) = \sum_{\gamma \in [\alpha]} c^\beta_{\gamma,u} V^\gamma(w) \delta(z - w),
\]

where 1 denotes the vacuum vector in \( \mathcal{A}_{\Delta_0^+}^T \otimes \mathcal{H}^T \). Computing their formal residues at \( z = 0 \), we have

\[ u(0)v_\beta = \sum_{\gamma \in [\alpha]} c^\beta_{\gamma,u} v_\gamma. \]

Hence, the free \( T \)-module \( \bigoplus_{\beta \in [\alpha]} \tilde{T} v_\beta \) has a structure of a \( g_0 \)-module, and is isomorphic to \( \tilde{T}^{[\alpha]} \) by \( \tilde{v}_\beta \mapsto v_\beta \) due to the irreducibility of \( \tilde{T}^{[\alpha]} \). Let \( M_\alpha \) be a \( V^T(g_0) \)-submodule generated by \( \bigoplus_{\beta \in [\alpha]} \tilde{T} v_\beta \) in \( \mathcal{A}_{\Delta_0^+}^T \otimes \mathcal{H}^T_{\Delta_0} \) for each \( \alpha \in \Pi_{>0} \), which is isomorphic to \( \tilde{M}_\alpha \) by \( \tilde{v}_\beta \mapsto v_\beta \) due to the irreducibility of \( \tilde{M}_\alpha \). Then \( V^\beta(z)u \) belongs to \( M_\alpha((z)) \) for all \( u \in V^T(g_0) \subseteq \mathcal{A}_{\Delta_0^+}^T \otimes \mathcal{H}^T \). Hence, we may identify \( V^\beta(z) : V^T(g_0) \rightarrow M_\alpha((z)) \) with an intertwining operator corresponding to \( v_\beta \) for all \( \beta \in [\alpha] \), satisfying that \( V^\beta_n \cdot 1 = \delta_{n,0} v_\beta \) (\( n \geq 0 \)) and \( (5.5) \) by Lemma 5.3. Therefore the isomorphism \( \tilde{M}_\alpha \simeq M_\alpha \) of \( V^T(g_0) \)-modules implies that \( \tilde{V}^\beta(z) = V^\beta(z) \). Thus, we obtain:

**Lemma 5.4.** For all \( \alpha \in \Pi_{>0} \), \( \tilde{M}_\alpha \simeq M_\alpha \) as \( V^T(g_0) \)-modules, which induces that \( \tilde{V}^\beta(z) = V^\beta(z) \) on \( V^T(g_0) \) for all \( \beta \in [\alpha] \).

Since \( \bigcap_{\alpha \in \Pi_0} \ker Q_\alpha = V^T(g_0) \otimes \Phi^T(\mathfrak{g}_0^+) \) by Lemma 4.13, we have

\[
\bigcap_{\alpha \in \Pi} \ker Q_\alpha = \bigcap_{\alpha \in \Pi_{>0}} \ker Q_\alpha |_{V^T(g_0) \otimes \Phi^T(\mathfrak{g}_0^+)}. 
\]

**Theorem 5.5.** Suppose that \( \mathfrak{c}(\mathfrak{n}_+) \) is compatible with the decomposition \( N_+ = G_{>0} \times G_0^+ \). Then \( Q_\alpha = \tilde{Q}_\alpha \) on \( V^T(g_0) \otimes \Phi^T(\mathfrak{g}_0^+) \) for all \( \alpha \in \Pi_{>0} \). In particular,

\[
\mathcal{W}^T(g, f; \Gamma) \simeq \bigcap_{\alpha \in \Pi} \ker Q_\alpha.
\]
Proof. By Proposition 5.2

\[ Q_\alpha = \sum_{\beta \in [\alpha]} \int V^\beta(z) \Phi_\beta(z) \, dz \quad (\alpha \in \Pi_+), \]

\[ Q_\alpha = \sum_{\beta \in [\alpha]} \chi(e_\beta) \int V^\beta(z) \, dz \quad (\alpha \in \Pi_1). \]

Hence, \( Q_\alpha \) is a map from \( V^T(g_0) \otimes \Phi^T(g_\frac{1}{2}) \) to \( M_\alpha \otimes \Phi^T(g_\frac{1}{2}) \) for all \( \alpha \in \Pi_{>0} \).

According to Lemma 5.4, we have \( \widetilde{M}_\alpha \simeq M_\alpha \) and \( \widetilde{V}^\beta(z) = V^\beta(z) \), which implies that \( Q_\alpha = \widetilde{Q}_\alpha \) on \( V^T(g_0) \otimes \Phi^T(g_\frac{1}{2}) \) for all \( \alpha \in \Pi_{>0} \) by (5.2) and (5.3). Therefore,

\[ \mathcal{W}^T(g, f; \Gamma) \simeq \bigcap_{\alpha \in \Pi_{>0}} \text{Ker} \, Q_\alpha = \bigcap_{\alpha \in \Pi_{>0}} \text{Ker} \, Q_\alpha \big|_{V^T(g_0) \otimes \Phi^T(g_\frac{1}{2})} = \bigcap_{\alpha \in \Pi} \text{Ker} \, Q_\alpha \]

by (5.4) and (5.6). This completes the proof.

\( \square \)

Theorem 5.6. Suppose that \( c(n_+) \) is compatible with the decomposition \( N_+ = G_{>0} \times G_0^+ \). Then the specialization of an inclusion map

\[ \bigcap_{\alpha \in \Pi} \text{Ker} \, Q_\alpha \hookrightarrow \bigcap_{\alpha \in \Pi_0} \text{Ker} \, Q_\alpha \]

coincides with the Miura map \( \mu \) for \( \mathcal{W}^k(g, f; \Gamma) \).

Proof. Since \( Q_\alpha = \widetilde{Q}_\alpha \) on \( V^T(g_0) \otimes \Phi^T(g_\frac{1}{2}) \) for all \( \alpha \in \Pi_{>0} \) by Theorem 5.5, the assertion of the corollary follows from the fact that the specialization of an inclusion map

\[ \bigcap_{\alpha \in \Pi_{>0}} \text{Ker} \, Q_\alpha \hookrightarrow V^T(g_0) \otimes \Phi^T(g_\frac{1}{2}) \]

coincides with the Miura map \( \mu \) for \( \mathcal{W}^k(g, f; \Gamma) \) by [Ge].

\( \square \)

Corollary 5.7. The map \( \omega \) defined in (4.8) and the specialization \( \omega_k = \omega \otimes \mathbb{C}_k \) of the map \( \omega \) are injective for all \( k \in \mathbb{C} \).

Proof. By (4.8) and Theorem 5.5, the image of \( \omega \) coincides with a \( \mathcal{W} \)-algebra \( \mathcal{W}^T(g, f; \Gamma) \). Since \( \omega \) holds conformal gradings, it induces a surjective endomorphism

\[ \omega(n) : \mathcal{W}^T(g, f; \Gamma)(n) \rightarrow \mathcal{W}^T(g, f; \Gamma)(n), \]

where \( \mathcal{W}^T(g, f; \Gamma)(n) \) is the homogeneous subspace of \( \mathcal{W}^T(g, f; \Gamma) \) with conformal weight \( n \) for all \( n \in \frac{1}{2} \mathbb{Z}_{>0} \). Since \( \mathcal{W}^T(g, f; \Gamma)(n) \) is finite-dimensional, \( \omega(n) \) is isomorphism for all \( n \in \frac{1}{2} \mathbb{Z}_{>0} \). Hence, \( \omega \) is injective.

By construction and Theorem 5.6, the specialization of injective maps

\[ \bigcap_{\alpha \in \Pi} \text{Ker} \, Q_\alpha \hookrightarrow \bigcap_{\alpha \in \Pi_0} \text{Ker} \, Q_\alpha \hookrightarrow A^T_{\Delta_0^+} \otimes \mathcal{H}^T \otimes \Phi^T(g_\frac{1}{2}) \]

induces vertex algebra homomorphisms

\[ \mathcal{W}^k(g, f; \Gamma) \xrightarrow{\mu} V^{\tau_k}(g_0) \otimes \Phi(g_\frac{1}{2}) \xrightarrow{\Delta} A^T_{\Delta_0^+} \otimes \mathcal{H} \otimes \Phi(g_\frac{1}{2}), \]
whose composition map coincides with $\omega_k$. According to the proof of Lemma 4.13 we have
\[
\iota = (\hat{\rho}_{g_0})_k \otimes \text{Id}_{\Phi(g_2)},
\]
where $(\hat{\rho}_{g_0})_k = \hat{\rho}_{g_0} \otimes C_k$ is a Wakimoto representation of $V^\kappa(g_0)$ and injective due to [Fre]. Since $\mu$ is injective, $\omega_k = \iota \circ \mu$ is also injective for all $k \in \mathbb{C}$.

The following diagram summarizes the correspondence between the screening operators $Q_\alpha$ and $Q_\alpha$ that we have discussed above.

\[
\begin{array}{cccccc}
\mathcal{W}_T(g, f; \Gamma) & \overset{\omega}{\longrightarrow} & \mathcal{A}^T_{\Delta_0} \otimes H^T \otimes \Phi^T(g_2) \\
\downarrow \mu^T & & & & \downarrow \text{Id} \\
V^T(g_0) \otimes \Phi^T(g_2) & \overset{\hat{\rho}_{g_0} \otimes \text{Id}}{\longrightarrow} & \mathcal{A}^T_{\Delta_0} \otimes H^T \otimes \Phi^T(g_2) \\
\bigoplus_{\alpha \in \Pi_{>0}} \tilde{M}_\alpha \otimes \Phi^T(g_2) & \sim & \bigoplus_{\alpha \in \Pi_{>0}} M_\alpha \otimes \Phi^T(g_2) & \subset & \bigoplus_{\alpha \in \Pi_{>0}} \mathcal{A}^T_{\Delta_0} \otimes H^T \otimes \Phi^T(g_2)
\end{array}
\]

6. Parabolic inductions

In this section, we state and prove our main theorem (Theorem 6.10). From now on, we assume that $g$ is a reductive Lie algebra.

6.1. $\mathcal{W}$-algebras for reductive Lie algebras. Let $g$ be a finite-dimensional reductive Lie algebra, $f$ a nilpotent element in $[g, g]$, $\kappa$ a symmetric invariant bilinear form on $g$ and $\Gamma$ a good grading for $f$ on $g$ satisfying that the center $z_g$ of $g$ lies in $g_0$. The definition of $\mathcal{W}$-algebras $\mathcal{W}^\kappa(g, f; \Gamma)$ naturally extends for $g, f, \Gamma$ and $\kappa$. We use the same notations: $g_0, \Delta, \Pi, \Delta^+, \Pi^+, \mathfrak{h}, n_+$ as in Section 2 and 4.1. Set

\[
g = z_g \oplus \bigoplus_{i=1}^m g^i,
\]

where $g^i$ is a simple Lie algebra. Let $g^i_j = g^i \cap g_j$ and $f_i \in g^i$ such that $f = \sum_{i=1}^m f_i$ corresponding to \[6.1\]. Then

\[
\Gamma_i : g^i = \bigoplus_{j \in \mathbb{Z}} g^i_j
\]
is good for $f_i$. We have an isomorphism of vertex algebras

\[
\mathcal{W}^\kappa(g, f; \Gamma) \simeq V^\kappa(z_g) \otimes \bigotimes_{i=1}^m \mathcal{W}^{k_i}(g^i, f_i; \Gamma_i),
\]

where $k_i = \kappa(\theta_i, \theta_i)/2 \in \mathbb{C}$ for the highest root $\theta_i$ in $g^i$. Let $\mathfrak{h}^i$ be a Cartan subalgebra of $g^i$ contained in $\mathfrak{g}_0^i$ and $h^i_\gamma$ the dual Coxeter number of $g^i$. We have $\mathfrak{h} = z_g \oplus \bigoplus_{i=1}^m \mathfrak{h}^i$. Denote by $\Delta, \Delta^+, \Pi$ the sets of roots, positive roots and simple roots in $g$ respectively. Let $n_+ = \bigoplus_{\alpha \in \Delta^+} g_\alpha$ and $g_0^+, n_+ = g_0 \cap n_+$. We also
have \( n_+ = \bigoplus_{i=1}^{m_+} n_+^i \) and \( g_0^+ = \bigoplus_{i=1}^{m_+} g_0^i \). Set \( \Delta_0^i = \Delta_i \cap \Delta^i \), \( \Pi_0^i = \Pi_j \cap \Delta^i \) and \((\Delta_0^i)^+ = \Delta_0^i \cap \Delta^i_+\). Set \( A_{\Delta_0^i} = \bigoplus_{i=1}^{m_+} A_{\Delta_i^j} \). Define

\[
\mathcal{H} = V^\times(\mathfrak{g}) \otimes \bigotimes_{i=1}^{m} \mathcal{H}_{\kappa_i+h_i'}(h^i), \quad V^{\tau_+}(\mathfrak{g}_0) = V^\times(\mathfrak{g}) \otimes \bigotimes_{i=1}^{m} V^{\tau_+}(g_0^i).
\]

Let \( T_i = \mathbb{C}[k_i] \) and \( T = \bigotimes_{i=1}^{m} T_i \), where \( k_i \) is a formal parameter. By Theorem 5.5

\[
W^T(\mathfrak{g}, f; \Gamma) \simeq \bigcap_{\alpha \in \Pi} \ker Q_\alpha \subset A^T_{\Delta_0^i} \otimes \Phi^T(g_0^i) \otimes \mathcal{H}^T,
\]

where \( Q_\alpha \) is a screening operator. Since \( V^T(\mathfrak{g}) \) commutes with all \( Q_\alpha \), we have an isomorphism of vertex algebras

\[
(6.3) \quad W^T(\mathfrak{g}, f; \Gamma) \simeq \bigcap_{\alpha \in \Pi} \ker Q_\alpha \subset A^T_{\Delta_0^i} \otimes \Phi^T(g_0^i) \otimes \mathcal{H}^T
\]

over \( T \). By Theorem 6.6 the Miura map

\[
\mu: W^T(\mathfrak{g}, f; \Gamma) \to V^{\tau_+}(g_0) \otimes \Phi(g_+^i)
\]

coincides with the map induced by the specialization of an inclusion map

\[
\bigcap_{\alpha \in \Pi} \ker Q_\alpha \hookrightarrow \bigcap_{\alpha \in \Pi_0} \ker Q_\alpha.
\]

6.2. Induced nilpotent orbits. Let \( \mathcal{N} \) be the set of all nilpotent elements in \([\mathfrak{g}, \mathfrak{g}]\). A Lie group \( G \) acts on \( \mathcal{N} \) by the adjoint action, which decompose \( \mathcal{N} \) into finitely many orbits, called nilpotent orbits in \( \mathfrak{g} \). See e.g. \[ CM \]. Let \( \mathfrak{p} \) be a parabolic subalgebra, i.e. \( \mathfrak{b} \subset \mathfrak{p} \). There exists the Levi decomposition \( \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \) such that \( \mathfrak{l} \) is a reductive Lie subalgebra and \( \mathfrak{u} \) is a nilpotent subalgebra. We have a root subsystem \( \Delta_l \subset \Delta \) such that

\[
(6.4) \quad \mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_l} \mathfrak{g}_\alpha.
\]

The reductive Lie subalgebra \( \mathfrak{l} \) is called a Levi subalgebra of \( \mathfrak{g} \) and uniquely determined by simple roots \( \Pi_l \) of \( \Delta_l \) up to conjugation. Denote by \( P, L \) the Lie subgroups of \( G \) corresponding to \( \mathfrak{p}, \mathfrak{l} \) respectively. The following results are due to Lusztig and Spaltenstein \[ LS \].

**Proposition and Definition 6.1** \([ LS \]). Let \( \mathcal{O}_l \) be a nilpotent orbit in \( \mathfrak{l} \). Then there exists a unique nilpotent orbit \( \mathcal{O}_g \) in \( \mathfrak{g} \) such that \( (\mathcal{O}_l + \mathfrak{u}) \cap \mathcal{O}_g \) is Zarsiki dense in \( \mathcal{O}_l + \mathfrak{u} \), and \( \mathcal{O}_g \) doesn’t depend on the choice of \( \mathfrak{p} \). The orbit \( \mathcal{O}_g \) is called the induced nilpotent orbit from \( \mathcal{O}_l \) and denoted by \( \text{Ind}_l^g \mathcal{O}_l \).

**Proposition 6.2** \([ LS \]). Let \( \mathcal{O}_l \) be a nilpotent orbit in \( \mathfrak{l} \) and \( \mathcal{O}_g = \text{Ind}_l^g \mathcal{O}_l \) the induced nilpotent orbit from \( \mathcal{O}_l \).

1. \( \mathcal{O}_g \) is a unique nilpotent orbit that has the dimension \( \dim \mathcal{O}_g = \dim \mathcal{O}_l + 2 \dim \mathfrak{u} \) and \((\mathcal{O}_l + \mathfrak{u}) \cap \mathcal{O}_g \neq \phi\).
2. Induced nilpotent orbits are transitive, i.e.

\[
\text{Ind}_l^g \mathcal{O}_l = \text{Ind}_l^g \text{Ind}_l^{l'} \mathcal{O}_l
\]

for any Levi subalgebra \( \mathfrak{l}' \) such that \( \mathfrak{l} \subset \mathfrak{l}' \subset \mathfrak{g} \).

To prove Lemma 6.4 we recall the properties of (good) gradings in \[ EK \].
Lemma 6.3 ([EK]). Let $\mathfrak{g}$ be a reductive Lie algebra, $f$ a nilpotent element of $[\mathfrak{g}, \mathfrak{g}]$. Let $\Gamma$ be a $\frac{1}{2}\mathbb{Z}$-grading on $\mathfrak{g}$ such that $f \in \mathfrak{g}_{-1}$, the center of $\mathfrak{g}$ lies in $\mathfrak{g}_0$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all $i, j \in \frac{1}{2}\mathbb{Z}$.

1. The following are equivalent.
   (a) $\text{ad}(f) : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j-1}$ is injective for $j \geq \frac{1}{2}$.
   (b) $\text{ad}(f) : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j-1}$ is surjective for $j \leq \frac{1}{2}$.
   (c) $\Gamma$ is good for $f$.

2. Suppose that $\Gamma$ is good for $f$. Then $\dim \mathfrak{g}^f = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{\frac{1}{2}}$.

Lemma 6.4. Let $\Gamma$ be a good grading on $f$ on $\mathfrak{g}$, $G \cdot f$ the nilpotent orbit in $\mathfrak{g}$ through $f$, $\Gamma$ a Levi subalgebra of $\mathfrak{g}$ with simple roots $\Pi$. Suppose that $\deg \alpha = 1$ for all $\alpha \in \Pi \setminus \Pi_\Gamma$. Then there exists a nilpotent element $f_1$ in $[l, l]$ such that $\Gamma_1$ is a good grading for $f_1$ and $G \cdot f = \text{Ind}^\mathfrak{g}_\mathfrak{l} L \cdot f_1$, where $\Gamma_1$ is the restriction of $\Gamma$ to $l$ and $L \cdot f_1$ is the nilpotent orbit in $l$ through $f_1$.

Proof. As in Section 6.1, there exists a root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ compatible with $\Gamma$. We may choose $\Delta_+^\Gamma$ such that $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+^\Gamma} \mathfrak{g}_\alpha \subset \mathfrak{g}_{\geq 0}$. We have a root subsystem $\Delta_1$ of $\Delta_+^\Gamma$ satisfying (6.3). Let $u = \bigoplus_{\alpha \in \Delta \setminus (\Delta_1 \cap \Delta_\Gamma)} \mathfrak{g}_\alpha$. Then $\mathfrak{p} = l \oplus u$ is a parabolic subalgebra including the opposite Borel subalgebra $\mathfrak{b}_-\mathfrak{l}$ and gives the Levi decomposition of $\mathfrak{p}$ whose Levi subalgebra is $l$. Denote by $l_j = l \cap \mathfrak{g}_j$ and by $u_j = u \cap \mathfrak{g}_j$. Since $\Pi \setminus \Pi_0 \subset \Pi_1$, we have $\mathfrak{g}_j = l_j \oplus u_j$ for all $j \leq \frac{1}{2}$. Choose $f_1 \in l_{-1}$ and $f_u \in u_{-1}$ such that $f = f_1 + f_u$ corresponding to $\mathfrak{g}_{-1} = l_{-1} \oplus u_{-1}$. Since $[f, \mathfrak{g}_j] = \mathfrak{g}_{j-1}$ for all $j \leq \frac{1}{2}$ and $[l, u] \subset u$, we have $[f, l_j] = l_{j-1}$ for all $j \leq \frac{1}{2}$. Note that the center of $l$ lies in $\mathfrak{h} \subset \mathfrak{g}_0$ and the formula $[l, l_j] \subset l_{j+1}$ is deduced from $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. These imply that $\Gamma_1$ is good for $f_1$ by Lemma 6.3 (1). Therefore $\dim l^{l_1} = \dim l_0 + \dim l_{\frac{1}{2}}$ by Lemma 6.3 (2). Since $\mathfrak{g}_j = l_j$ for $j = 0, \frac{1}{2}$, we have $\dim l^{l_1} = \dim \mathfrak{g}^f$. Hence,

$$\dim G \cdot f = \dim \mathfrak{g} - \dim \mathfrak{g}^f = \dim l + 2 \dim u - \dim l^{l_1} = \dim L \cdot f_1 + 2 \dim u.$$

By construction, $f \in G \cdot f \cap (L \cdot f_1 + u) \neq \emptyset$. Therefore $G \cdot f = \text{Ind}^\mathfrak{g}_\mathfrak{l} L \cdot f_1$ by Proposition 6.2.

Corollary 6.5. Under the conditions in Lemma 6.4, $\chi(u) = (f_1|u)$ for all $u \in l$.

Proof. We use the notations in the proof of Lemma 6.4. Since $(l \mid u) = 0$, we have $\chi(u) = (f_1|u) + (f_u|u) = (f_1|u)$ for all $u \in l$.

Remark 6.6. The condition $\Pi \setminus \Pi_0 \subset \Pi_1$ in Lemma 6.4 is valid for all cases of type $A$ by [Ku OW], rectangular nilpotent cases of type $BCD$ by [Kr OW], all cases of type $G$ and many cases of other exceptional types by [Ge].

6.3. Preliminary results. Continue to use the notations in Section 6.1 and 6.2. Under the condition $\Pi_0 \subset \Pi_1$, we have a nilpotent element $f_1$ in $[l, l]$ such that $\Gamma_1$ is good for $f_1$ by Lemma 6.3. Set $n_1^+ = l \cap n_+$ and $l_0^+ = l_0 \cap n_+$. Denote by $(\Delta_1)_+ = \Delta_1 \cap \Delta_+$ and by $(\Delta_1)_j = \Delta_1 \cap \Delta_j$. Let $N_1^+, L_0^+, L_{>0}$ be the Lie subgroup of $L$ corresponding to $n_1^+, l_0^+$, $l_{>0}$ respectively. Since $\mathfrak{g}_0 = l_0$, we have $L_0^+ = G_0^+$. Let $c(n_+^\alpha) = c(g_0^\alpha) \cdot c(g_0^\alpha)$ be a coordinate on $N_+^1$ compatible with the decomposition $N_+^1 = G_0^+ \times G_0^+$. Then $c(n_1^+) = c(n_+^\alpha)|_{N_1^+} = c(g_0^\alpha)|_{L_0^+} \cdot c(g_0^\alpha)$ is a coordinate on $N_1^+$ compatible with the decomposition $N_1^+ = L_{>0} \times L_0^+$. Let $\rho_1^G : n_1^+ \rightarrow D_{N_1^+}$...
Lemma 6.9. Suppose that $\Pi$ with simple roots $\alpha$. Proof. All roots in $\Pi$ by restriction of $\Gamma$ where $\mu$. Parabolic inductions. 6.4. $\square$

The assertion of the lemma immediately follows from Corollary 6.5. Proof. Since $\Pi \| \Pi_1 \subset \Pi_1$, we have $\Delta_+ \setminus (\Delta_1)_+ \subset A_1$. The assertion follows by (6.5) and Lemma 4.2.

Let $\Phi(I_\alpha)$ be the neutral vertex algebra associated with $I_\alpha$, which is defined by

$$\Phi^I_\alpha(z)\Phi^I_\beta(w) \sim \frac{(f_1|e_\alpha, e_\beta)|}{z-w}$$

for generating fields $\Phi^I_\alpha(z)$, $\Phi^I_\beta(z)$ with $\alpha, \beta \in (\Delta_1)_\alpha$. Lemma 6.8. Suppose that $\Pi \| \Pi_1 \subset \Pi_1$. Then $\Phi(I_\alpha) = \Phi(g_\alpha)$. Proof. The assertion of the lemma immediately follows from Corollary 6.5. $\square$

Let $(\Pi_1)_j = \Pi_1 \cap \Delta_j$. Recall that $[\alpha]$ is the subset of $\Delta_+$ defined by $[\alpha]_\alpha$ for $\alpha \in \Pi_1$. Lemma 6.9. Suppose that $\Pi \| \Pi_1 \subset \Pi_1$. Then all roots in $[\alpha]$ lie in $\Delta_+ I_\alpha$ for all $\alpha \in (\Pi_1)_\alpha$. Proof. All roots in $[\alpha]$ are spanned by $\alpha$ and simple roots in $\Pi_0 = (\Pi_1)_0$. Hence, the assertion of the lemma follows. $\square$

6.4. Parabolic inductions. Set the Killing forms $\kappa_\alpha^0$, $\kappa_\alpha^0$ on $g$, $l$ respectively.

Theorem 6.10. Let $\Gamma$ be a good grading for $f$ on $g$ and $l$ a Levi subalgebra of $g$ with simple roots $\Pi$. Suppose that $\Pi \| \Pi_1 \subset \Pi_1$. Let $\Gamma_1$ be a $\frac{1}{2}Z$-grading on $l$ defined by restriction of $\Gamma$ and $f_1$ the nilpotent element of $[l, l]$ chosen by Lemma 6.4. Then there exists an injective vertex algebra homomorphism

$$\text{Ind}_l^g : \mathcal{W}_f(g, f; \Gamma) \to \mathcal{W}_f^l(l, f_1; \Gamma_1),$$

where

$$\kappa_1 = \kappa + \frac{1}{2} \kappa_0^0 - \frac{1}{2} \kappa_0^0.$$ Moreover, the map $\text{Ind}_l^g$ is a unique vertex algebra homomorphism that satisfies

$$\mu = \mu_1 \circ \text{Ind}_l^g,$$

where $\mu$, $\mu_1$ are the Miura maps for $\mathcal{W}_f^l(g, f; \Gamma)$, $\mathcal{W}_f^l(l, f_1; \Gamma_1)$ respectively.

Proof. First, we consider the case that $g$ is a simple Lie algebra. By Theorem 6.5

$$\mathcal{W}_f^l(g, f; \Gamma) \simeq \bigcap_{\alpha \in \Pi} \ker Q_\alpha,$$
where $Q_\alpha$ is a screening operator, which acts on $\mathcal{A}^+_\Delta \otimes \Phi^T(\mathfrak{g}_\frac{1}{2}) \otimes \mathcal{H}^T$. Set

\[
I = \mathfrak{z}_l \oplus \bigoplus_{i=1}^{m_l} i^l,
\]

where $\mathfrak{z}_l$ is the center of $I$ and $i^l$ is a simple Lie algebra. Note that $\mathfrak{h}$ is also a Cartan subalgebra of $l$. Let $\theta_i$ be the highest root of $i^l$, $h_i^\vee$ the dual Coxeter number of $i^l$ and $\kappa_i = \kappa + \frac{1}{T} \kappa^2 - \frac{1}{T} \kappa_i^2$ a $T$-valued invariant bilinear form on $g$, where $T = \mathbb{C}[k]$ and $\kappa(u|v) = k(u|v)$. Then

\[
\kappa_i(u|v) = \begin{cases} \langle k + h_i^\vee \rangle(u|v) & (u, v \in \mathfrak{z}_l) \\ \kappa_i(u|v) & (u, v \in i^l), \end{cases}
\]

where $k_i = \frac{2}{\theta_i(h_i^\vee)}(k + h_i^\vee) - h_i^\vee$. We shall denote by $\mathcal{W}^T(l, f_i; \Gamma_l)$ instead of $\mathcal{W}^{\kappa_i}(l, f_i; \Gamma_l)$. Set $(\Pi_i)^i = \{ \alpha \in \Pi_i \mid \mathfrak{i}_\alpha \subset i^l \}$ and $(\Pi_i)^j = (\Pi_i)^i \cap (\Pi_i)^j$. As in Section 6.1 we have

\[
\mathcal{W}^T(l, f_i; \Gamma_l) \simeq \bigcap_{\alpha \in \Pi_i} \text{Ker} Q^i_\alpha,
\]

where $Q^i_\alpha$ is a screening operator, which acts on $\mathcal{A}^T_{\Delta^+} \otimes \Phi^T(I_{\frac{1}{2}}) \otimes \mathcal{H}^T$. By Theorem 4.12 and Proposition 5.2, we have

\[
Q^i_\alpha = \sum_{\beta \in (\Delta^+)^i} \int : P^\beta, R_{\alpha, i}(a^*)(z) a_\beta(z) e^{-\frac{1}{h_i + k_i}} f b^l_\alpha(z) : dz \quad (\alpha \in (\Pi_i)^i),
\]

\[
Q^i_\alpha = \sum_{\beta \in (\Delta^+)^i \cap [\alpha]} \int : P^\beta, R_{\alpha, i}(a^*)(z) \Phi_\beta(z) e^{-\frac{1}{h_i + k_i}} f b^l_\alpha(z) : dz \quad (\alpha \in (\Pi_i)^i),
\]

\[
Q^i_\alpha = \sum_{\beta \in (\Delta^+)^i \cap [\alpha]} (f_\beta) \int : P^\beta, R_{\alpha, i}(a^*)(z) e^{-\frac{1}{h_i + k_i}} f b^l_\alpha(z) : dz \quad (\alpha \in (\Pi_i)^i),
\]

and $b^l_\alpha(z) = \frac{2}{\theta_i(h_i^\vee)} b_\alpha(z)$ for all $\alpha \in (\Pi_i)^i$. Then

\[
e^{-\frac{1}{h_i + k_i}} f b^l_\alpha(z) : = : e^{-\frac{1}{h_i + k_i}} f b_\alpha(z) :
\]

for all $\alpha \in (\Pi_i)^i$ by definition of $k_i$. Hence, we have

\[
Q^i_\alpha = \sum_{\beta \in \Delta^+} \int : P^\beta, R_{\alpha}(a^*)(z) a_\beta(z) e^{-\frac{1}{h_i + k_i}} f b_\alpha(z) : dz \quad (\alpha \in (\Pi_i)^i),
\]

\[
Q^i_\alpha = \sum_{\beta \in [\alpha]} \int : P^\beta, R_{\alpha}(a^*)(z) \Phi_\beta(z) e^{-\frac{1}{k_i + h_i}} f b_\alpha(z) : dz \quad (\alpha \in (\Pi_i)^i),
\]

\[
Q^i_\alpha = \sum_{\beta \in [\alpha]} \chi(\mathfrak{e}_\beta) \int : P^\beta, R_{\alpha}(a^*)(z) e^{-\frac{1}{k_i + h_i}} f b_\alpha(z) : dz \quad (\alpha \in (\Pi_i)^i),
\]

thanks to Lemma 6.7, Lemma 6.8 and Lemma 6.9. Therefore $Q_\alpha = Q^l_\alpha$ for all $\alpha \in \Pi$ by Theorem 4.12 and Proposition 5.2. The specialization of inclusion maps

\[
\bigcap_{\alpha \in \Pi} \text{Ker} Q_\alpha \hookrightarrow \bigcap_{\alpha \in \Pi} \text{Ker} Q^i_\alpha \hookrightarrow \bigcap_{\alpha \in \Pi} \text{Ker} Q^l_\alpha
\]
induces vertex algebra homomorphisms
\[
W^k(g, f; \Gamma) \to W^{\kappa_i}(l, f_1; \Gamma_1) \to V^{\tau_{\kappa_i}}(g_0) \otimes \Phi(g_1).
\]

Let us denote by
\[
\text{Ind}_l^g : W^k(g, f; \Gamma) \to W^{\kappa_i}(l, f_1; \Gamma_1)
\]
the first map and by \(\mu_1\) the second map in (6.7). Then
\[
\mu = \mu_1 \circ \text{Ind}_l^g,
\]
where \(\mu, \mu_1\) are the Miura maps for \(W^k(g, f; \Gamma), W^{\kappa_i}(l, f_1; \Gamma_1)\) respectively by Theorem 5.6. Since \(\mu\) is injective, so is \(\text{Ind}_l^g\). Therefore the assertion of the theorem follows for any simple Lie algebra \(g\).

Next, consider arbitrary reductive Lie algebra \(g\). We use the decomposition (6.1). Let \(f_1^i\) be the image of \(f_1\) by the projection \(l \to l \cap g^i\) and \(\Gamma_1^i\) the good grading for \(f_1^i\) inherited from \(\Gamma_1\) provided that \(l \cap g^i \neq 0\). Following the argument in the above, since \(g^i\) is a simple Lie algebra, we have an injective homomorphism
\[
\text{Ind}_{W_{\kappa_i}}^{g^i} : W^k(g^i, f^i; \Gamma^i) \to W^{\kappa_i}(l \cap g^i, f_1^i; \Gamma_1^i)
\]
for all \(i \in I_1 := \{j \in \{1, \ldots, m\} | l \cap g^j \neq 0\}\), where \(k_i = \kappa(\theta^i)/2\) for the highest root \(\theta^i\) in \(g^i\) and
\[
\kappa_i(u|v) = k_i(u|v) + \frac{1}{2} \kappa_{g^i} - \frac{1}{2} \kappa_{l \cap g^i}(u|v)
\]
for all \(u, v \in l \cap g^i\). Since
\[
W^{\kappa_i}(l, f_1; \Gamma_1) = V^{\kappa_i}(g_0) \otimes \bigotimes_{i \in I_1} W^{\kappa_i}(l \cap g^i, f_1^i; \Gamma_1^i),
\]
we have an injective homomorphism
\[
\text{Ind}_l^g = \text{Id}_{V^{\kappa_i}(g_0)} \otimes \bigotimes_{i \in I_1} \text{Ind}_{W_{\kappa_i}}^{g^i} : W^k(g, f; \Gamma) \to W^{\kappa_i}(l, f_1; \Gamma_1),
\]
where \(\kappa_i = \kappa + \frac{1}{2} \kappa_{g^i} - \frac{1}{2} \kappa_{l \cap g^i}\). Then \(\text{Ind}_l^g\) satisfies \(\mu = \mu_1 \circ \text{Ind}_l^g\) by the property of \(\text{Ind}_{W_{\kappa_i}}^{g^i}\). The proof of the theorem is now complete except for the uniqueness of \(\text{Ind}_l^g\). Let \(\psi : W^k(g, f; \Gamma) \to W^{\kappa_i}(l, f_1; \Gamma_1)\) be a vertex algebra homomorphism such that \(\mu = \mu_1 \circ \psi\). Since \(\mu_1 \circ \text{Ind}_l^g = \mu = \mu_1 \circ \psi\) and \(\mu_1\) is injective, we have \(\text{Ind}_l^g = \psi\). This completes the proof. \(\square\)

**Proposition 6.11.** Let \(g\) be a reductive Lie algebra, \(f\) a nilpotent element in \(g\), \(\Gamma\) a good grading for \(f\) on \(g\). Let \(l, l'\) be Levi subalgebras of \(g\) such that \(l \subset l'\) and \(\Pi \setminus \Pi_l, \Pi \setminus \Pi_{l'} \subset \Pi_1\). Then
\[
\text{Ind}_l^g = \text{Ind}_l^{l'} \circ \text{Ind}_{l'}^g.
\]

**Proof.** Since \(\Pi_l \setminus \Pi_l \subset \Pi_1 \setminus \Pi_1 \subset \Pi_1\), a map \(\text{Ind}_{l'}^g\) exists by Theorem 6.11. By the characterization of \(\text{Ind}_l^g\), \(\text{Ind}_l^{l'}\) and \(\text{Ind}_{l'}^g\) given in Theorem 6.10 \(\mu_1 \circ \text{Ind}_l^g = \mu = \mu_{l'} \circ \text{Ind}_{l'}^g = (\mu_1 \circ \text{Ind}^{l'}_l) \circ \text{Ind}_{l'}^g\). Since \(\mu_1\) is injective, \(\text{Ind}_l^g = \text{Ind}_l^{l'} \circ \text{Ind}_{l'}^g\). Therefore the assertion follows. \(\square\)
If \( f \) is a principal nilpotent element in \([\mathfrak{g}, \mathfrak{g}]\), there exists a (unique) good grading \( \Gamma \). Then \( \Pi = \Pi_1 = \Delta_1 \). Let \( I \) be any Levi subalgebra of \( \mathfrak{g} \). Then \( \Pi_1 = (\Pi_1)_I \) and the principal nilpotent element \( f_I \) in \([I, I]\) is chosen by Lemma 6.4. By Theorem 6.10 we have an injective homomorphism, which is constructed in [BFN].

6.5. Chiralizations. Let \( V \) be any \( \frac{1}{2}\mathbb{Z}_{\geq 0} \)-graded vertex algebra. Denote by

\[
A \circ B = \sum_{j=0}^{\infty} \binom{\Delta(A)}{j} A_{(j-2)} B,
\]

\[
A \ast B = \sum_{j=0}^{\infty} \binom{\Delta(A)}{j} A_{(j-1)} B
\]

for \( A, B \in V \). Then a vector space \( \text{Zhu}(V) = V/(V \circ V) \) has a structure of an associative algebra by the multiplication induced by \( \ast \), called the (twisted) Zhu algebra of \( V \) [Zhu] [PZ] [DK]. We call \( V \) a chiralization of an associative algebra \( \text{Zhu}(V) \). Recall that \( \mathcal{W}^*(\mathfrak{g}, f; \Gamma) = H^0(C_+, d) \), see Section 2. By [A3] [DK], we have

\[
\text{Zhu}(H^0(C_+, d)) = H^0(\text{Zhu}(C_+), \tilde{d}),
\]

where \( \tilde{d} \) is the differential induced by \( d \) such that a complex \( (\text{Zhu}(C_+^\ast), \tilde{d}) \) defines the finite \( \mathcal{W} \)-algebra associated with \( \mathfrak{g}, f, \Gamma \), which we denote by \( U(\mathfrak{g}, f; \Gamma) \), i.e.

\[
\text{Zhu}(\mathcal{W}^*(\mathfrak{g}, f; \Gamma)) = U(\mathfrak{g}, f; \Gamma).
\]

See e.g. [Lo2] [Wan] for the definitions and properties of finite \( \mathcal{W} \)-algebras. Let \( \text{Zhu}(\Phi(\mathfrak{g}_1^*)) = \Phi(\mathfrak{g}_2^*) \). Note that \( \text{Zhu}(V^*(\mathfrak{g})) = U(\mathfrak{g}) \) and \( \text{Zhu}(C_+^\ast) = U(\mathfrak{g}_{\leq 0}) \otimes \Phi(\mathfrak{g}_2) \). The projection \( \mathfrak{g}_{\leq 0} \oplus \Phi(\mathfrak{g}_2) \rightarrow \mathfrak{g}_0 \oplus \Phi(\mathfrak{g}_2) \) induces an algebra homomorphism

\[
\bar{\mu} : U(\mathfrak{g}, f; \Gamma) \rightarrow U(\mathfrak{g}_0 \oplus \Phi(\mathfrak{g}_2)),
\]

called the Miura map for \( U(\mathfrak{g}, f; \Gamma) \). The following result is proved by [LA] in the case that \( \Gamma \) is \( \mathbb{Z} \)-graded but may be also applied in general case. We give the sketch of the proof, following the proof of Proposition 4 in Section 2.6 in [A3] (with slight generalization).

Lemma 6.12. \( \bar{\mu} \) is injective.

Proof. Recall that \( S_f \) is the Slodowy slice through \( f \) and is isomorphic to the Marsden-Weinstein quotient of a transversal slice \( f + \mathfrak{g}_{\geq -\frac{1}{2}} \) in \( \mathfrak{g} \simeq \mathfrak{g}^* \) by \( G_{\geq -\frac{1}{2}} \) ([KR]). There exists a filtration on \( U(\mathfrak{g}, f; \Gamma) \), called the Kazhdan filtration, such that the induced map

\[
\nu : \text{gr} U(\mathfrak{g}, f; \Gamma) \rightarrow \text{gr} U(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_2).
\]

can be identified with the restriction map

\[

\nu : \mathbb{C}[S_f] = \mathbb{C}[f + \mathfrak{g}_{\geq -\frac{1}{2}}][G_{\geq -\frac{1}{2}}] \rightarrow \mathbb{C}[f + \mathfrak{g}_0 \oplus \mathfrak{g}_{-\frac{1}{2}}].
\]

We will show that the restriction map \( \nu \) is injective. If \( P \in \mathbb{C}[S_f] \) lies in the kernel of \( \nu \), \( P(g \cdot u) = 0 \) for all \( g \in G_{\geq \frac{1}{2}} \) and \( u \in f + \mathfrak{g}_{\geq -\frac{1}{2}} \). Hence, it suffices to show that a map

\[
\xi : G_{\geq \frac{1}{2}} \times (f + \mathfrak{g}_0 \oplus \mathfrak{g}_{-\frac{1}{2}}) \rightarrow f + \mathfrak{g}_{\geq -\frac{1}{2}}
\]

defined by \( \xi(g, u) = g \cdot u \) is dominant (i.e. the image of \( \xi \) is Zariski dense). Let

\[
d\xi : \mathfrak{g}_{\geq \frac{1}{2}} \times \mathfrak{g}_0 \oplus \mathfrak{g}_{-\frac{1}{2}} \rightarrow \mathfrak{g}_{-\frac{1}{2}}.
\]
be the differential of $\xi$. Then the differential of $\xi$ at $(1, u) \in G_{\geq \frac{1}{2}} \times (f + g_0 \oplus g_{-\frac{1}{2}})$ is given by
\[
d\xi_{(1,u)}(a,b) = [a,u] + b
\]
and is an isomorphism if $u \in f + (g_0 \oplus g_{-\frac{1}{2}})_{\text{reg}}$, where
\[
(g_0 \oplus g_{-\frac{1}{2}})_{\text{reg}} = \{ v \in g_0 \oplus g_{-\frac{1}{2}} \mid g^v = 0 \}, \quad g^v = \{ w \in g_{\geq \frac{1}{2}} \mid [v,w] = 0 \}.
\]
Hence, $\xi$ is dominant. See e.g. [TY]. This completes the proof. \(\square\)

Let $V, W$ be any $\mathbb{Z}_{\geq 0}$-graded vertex algebras and $\psi: V \to W$ any vertex algebra homomorphism. Since $\psi(V \circ V) = \psi(V) \circ \psi(V) \subset W \circ W$, the map $\psi$ induces an algebra homomorphism
\[
\text{Zhu}(\psi): \text{Zhu}(V) \to \text{Zhu}(W).
\]
We shall say that $\psi$ is a chiralization of $\text{Zhu}(\psi)$. For $\psi = \mu$, we obtain a map $\text{Zhu}(\mu): U(g,f;\Gamma) \to U(g_0 \otimes \Phi(g_{\frac{1}{2}})).$

**Lemma 6.13.** $\text{Zhu}(\mu) = \bar{\mu}$. In particular, $\text{Zhu}(\mu)$ is injective.

**Proof.** The formula $\text{Zhu}(\mu) = \bar{\mu}$ is induced by (6.8). The injectivity of $\text{Zhu}(\mu)$ follows from Lemma 6.12. \(\square\)

For any map $\text{Ind}_1^p$ given in Theorem 6.10 it induces an algebra homomorphism
\[
\text{Zhu}(\text{Ind}_1^p): U(g,f;\Gamma) \to U(l,f_l;\Gamma_l)
\]
such that
\[
(6.9) \quad \text{Zhu}(\mu) = \text{Zhu}(\mu_1) \circ \text{Zhu}(\text{Ind}_1^p)
\]
by the characterization of $\text{Ind}_1^p$.

**Lemma 6.14.** $\text{Zhu}(\text{Ind}_1^p)$ is a unique injective algebra homomorphism that satisfies $\bar{\mu} = \bar{\mu}_1 \circ \text{Zhu}(\text{Ind}_1^p)$.

**Proof.** The assertion of the lemma immediately follows from (6.8) and Lemma 6.13. \(\square\)

Losev constructs an injective algebra homomorphism in [Lo3]
\[
(6.10) \quad U(g,f;\Gamma) \to \bar{U}(l,f_l;\Gamma_l)
\]
if $G \cdot f = \text{Ind}_1^p L \cdot f_l$, where $\bar{U}(l,f_l;\Gamma_l)$ is a certain completion of $U(l,f_l;\Gamma_l)$.

**Conjecture 6.15.** Under the condition that $\Pi \setminus \Pi_1 \subset \Pi_1$, $\text{Zhu}(\text{Ind}_1^p)$ coincides with the map (6.10).

The pull-back of Losev’s map (6.10) gives a functor from $U(l,f_1;\Gamma_1)$-mod to $U(g,f;\Gamma)$-mod, called a parabolic induction functor and first introduced by Premet in [P4]. Motivated by these results and Conjecture 6.15, we call a map $\text{Ind}_1^p$ given in Theorem 6.10 a parabolic induction for $W$-algebras, which induces a functor from $W^{\kappa_1}(l,f_1;\Gamma_1)$-mod to $W^{\kappa}(g,f;\Gamma)$-mod.
7. Coproducts

In this section, we consider parabolic inductions \( \text{ind}^f_{\pi} \) in the case that \( \mathfrak{g} = \mathfrak{gl}_N \) and the case that \( \mathfrak{g} \) is of type \( BCD \) and \( f \) is a rectangular nilpotent element. In the former case, we show that \( \text{ind}^f_{\pi} \) is a chiralization of a coproduct \( \Delta \) of finite \( W \)-algebras of type \( A \) constructed by [BK2], which we call coproducts for \( W \)-algebras of type \( A \). In the latter case, we give a structure of coproducts on \( W \)-algebras of type \( BCD \) with rectangular nilpotent elements, giving rise to coproducts of twisted Yangians of level \( l \).

7.1. Pyramids. To describe good gradings of \( \mathfrak{gl}_N \) combinatorially, we introduce pyramids, which are first introduced by [EK] to classify all good gradings of simple (classical) Lie algebras. Following [BK2], we only consider pyramids corresponding to good \( Z \)-gradings of \( \mathfrak{gl}_N \), which should be called even pyramids but shall call just pyramids.

Let \( q = (q_1, \ldots, q_l) \) be a sequence of positive integers and \( \pi \) a diagram defined by stacking \( q_1 \) boxes in the first column, \( q_2 \) boxes in the second column, \( \ldots \), \( q_l \) boxes in the right-most column. The diagram \( \pi \) is called a pyramid if each row of \( \pi \) consists of a single connected strip, i.e. \( 0 \leq 3t \leq l \) such that \( q_t \leq \cdots \leq q_l \) and \( q_{t+1} \geq \cdots \geq q_l \). For example,

- \( \pi = \begin{array}{c} 1 \end{array} \)
- \( \pi = \begin{array}{ccc} 2 \\ 3 \end{array} \)
- \( \pi = \begin{array}{cccc} 4 \\ 5 \\ 6 \\ 7 \end{array} \)

Fix a pyramid \( \pi \). Set the height \( n = \max(q_1, \ldots, q_l) \) of \( \pi \), the sequence \( p = (p_1, \ldots, p_n) \) of length of rows of \( \pi \) from top to bottom, and the number \( N = \sum_{i=1}^{l} q_i = \sum_{i=1}^{n} p_i \) of boxes in \( \pi \) (e.g. \( p = (1, 2, 3) \) and \( N = 6 \) in the above examples). We fix a numbering of boxes in \( \pi \) by \( 1, \ldots, N \) from top to bottom and from left to right, and denote by \( \text{row}(i) \) the row number of the box in which \( i \) appears and by \( \text{col}(i) \) the column number similarly. For example,

\[
\begin{align*}
\pi & = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \\
\text{row}(i) & = \begin{cases} 1 & \text{for } 1 \\ 2 & \text{for } 2 \\ 3 & \text{for } 3 \\ 4 & \text{for } 4 \end{cases} \\
\text{col}(i) & = \begin{cases} 2 & \text{for } 1 \\ 3 & \text{for } 2 \\ 4 & \text{for } 3 \\ 5 & \text{for } 4 \end{cases}
\end{align*}
\]

We have \( \text{row}(4) = 3 \) and \( \text{col}(4) = 2 \) etc. Let \( \{v_i\}_{i=1}^N \) be the standard basis of \( \mathbb{C}^N \) and \( \{e_{i,j}\}_{i,j=1}^N \) the standard basis of \( \mathfrak{gl}_N = \text{End}(\mathbb{C}^N) \) by \( e_{i,j}(v_k) = \delta_{j,k}v_i \). We attach to \( \pi \) a nilpotent element \( f_\pi \) by

\[
f_\pi(v_j) = \begin{cases} v_i & \text{if } \text{row}(i) = \text{row}(j) \text{ and } \text{col}(i) = \text{col}(j) + 1 \\ 0 & \text{otherwise} \end{cases}
\]

and a \( Z \)-grading \( \Gamma_\pi \) on \( \mathfrak{gl}_N \) by \( \deg_{\Gamma_\pi}(e_{i,j}) = \text{col}(j) - \text{col}(i) \). Then \( f_\pi \) has the standard Jordan form consisting of \( p_1 \) Jordan block, \( p_2 \) Jordan block, \( \ldots \), \( p_N \) Jordan block with all the diagonal 0, and \( \Gamma_\pi \) is good for \( f_\pi \). Set a Cartan subalgebra \( \mathfrak{h} = \bigoplus_{i=1}^N \mathbb{C}e_i \), the dual basis \( \{e_i\}_{i=1}^N \) of \( \mathfrak{h}^* \) by \( e_i(e_{j,j}) = \delta_{i,j} \), and the root system

\[
\Delta = \{\epsilon_i - \epsilon_j \in \mathfrak{h}^* \mid 1 \leq i \neq j \leq N\}, \quad \Pi = \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid i = 1, \ldots, N - 1\}
\]
as usual. Since \( \deg_{\Gamma_{\alpha}}(\epsilon_i - \epsilon_j) = \deg_{\Gamma_{\alpha}}(\epsilon_{i,j}) \), we have

\[
(7.2) \quad \Pi_0 = \{ \alpha_i \in \Pi \mid \text{row}(i) < \text{row}(N) \}, \quad \Pi_1 = \{ \alpha_i \in \Pi \mid \text{row}(i) = \text{row}(N) \}.
\]

In the case of \((7.1)\),

\[
f_{\pi} = e_{5,3} + e_{7,6} + e_{6,4} + e_{4,1},
\]

We split \(\pi\) into two pyramids \(\pi_1, \pi_2\) along a column, which we denote by \(\pi = \pi_1 \oplus \pi_2\). For example,

\[
\begin{bmatrix}
2 \\
3 \\
5 \\
1 \\
4 \\
6 \\
7
\end{bmatrix} =
\begin{bmatrix}
2 \\
3 \\
5 \\
1 \\
4 \\
6 \\
7
\end{bmatrix} \oplus
\begin{bmatrix}
2 \\
3 \\
5 \\
1 \\
4 \\
6 \\
7
\end{bmatrix}.
\]

For \(i = 1, 2\), let \(N_i\) be the number of boxes in \(\pi_i\) and \(l_i = gl_{N_i}\) the Lie subalgebra of \(gl_N\) spanned by all \(e_{j,j'}\), where \(j, j'\) run over numbers labeling \(\pi_i\). Then \(\Gamma_{\pi_i}\) is a good grading on \(l_i\) for \(f_{\pi_i}\), and is the restriction of \(\Gamma_{\pi}\) by construction. Denote by \(O_{\pi}\) a nilpotent orbit in \(gl_N\) through \(f_{\pi}\), by \(O_{\pi_i}\) a nilpotent orbit in \(l_i\) through \(f_{\pi_i}\), and by \(l = l_1 \oplus l_2\) a maximal Levi subalgebra of \(gl_N\). A combinatorial description of induced nilpotent orbits in \(gl_N\) given in [K] is compatible with our cases:

\[
O_{\pi} = \text{Ind}^{gl_N}_{l_1}(O_{\pi_1} + O_{\pi_2}).
\]

### 7.2. Coproducts for type \(A\)

Let \(\pi\) be a pyramid consisting of \(N\) boxes. Set

\[
\mathcal{W}^k(gl_N, \pi) = V^{k+N}(3_{gl_N}) \otimes \mathcal{W}(\mathfrak{s}l_N, f_{\pi}; \Gamma_{\pi}).
\]

Set the subset \(\Pi^i\) of \(\Pi\) consisting of simple roots in \(gl_{N_i}\). Then

\[
\Pi^1 = \{ \alpha_1, \ldots, \alpha_{N_1-1} \}, \quad \Pi^2 = \{ \alpha_{N_1+1}, \ldots, \alpha_{N-1} \}.
\]

Therefore \(\Pi \setminus (\Pi^1 \cup \Pi^2) = \{ \alpha_{N_1} \}\) and \(\deg_{\Gamma_{\alpha}} \alpha_{N_1} = 1\) by \(\text{row}(N_1) = \text{row}(N)\).

**Theorem 7.1.** Let \(\pi\) be a pyramid split into \(\pi_1 \oplus \pi_2\). Set the numbers \(N, N_1, N_2\) of boxes in \(\pi, \pi_1, \pi_2\) respectively \((N = N_1 + N_2)\). Then there exists an injective vertex algebra homomorphism

\[
\Delta = \Delta_{\pi_1, \pi_2}^{\pi} : \mathcal{W}^k(gl_N, \pi) \rightarrow \mathcal{W}^{k_1}(gl_{N_1}, \pi_1) \otimes \mathcal{W}^{k_2}(gl_{N_2}, \pi_2)
\]

for all \(k \in \mathbb{C}\), where \(k + N = k_1 + N_1 = k_2 + N_2\), such that

1. \(\Delta\) is a unique vertex algebra homomorphism that satisfies \(\mu = (\mu_1 \otimes \mu_2) \circ \Delta\), where \(\mu, \mu_1, \mu_2\) are the Miura maps for \(\mathcal{W}^k(gl_N, \pi), \mathcal{W}^{k_1}(gl_{N_1}, \pi_1), \mathcal{W}^{k_2}(gl_{N_2}, \pi_2)\) respectively.
2. \(\Delta\) is coassociative, i.e.

\[
(\text{Id} \otimes \Delta_{\pi_2, \pi_3}^{\pi}) \circ \Delta_{\pi_1, \pi_2, \pi_3}^{\pi} = (\Delta_{\pi_1, \pi_2}^{\pi_1} \otimes \text{Id}) \circ \Delta_{\pi_1, \pi_2, \pi_3}^{\pi}.
\]

for \(\pi = \pi_1 \oplus \pi_2 \oplus \pi_3\).
Proof. We use the notations in Section 7.1. Let $\Pi_1$ be the set of simple roots in $I = I_1 \oplus I_2$. Since $\{\alpha_N\} = \Pi \setminus \Pi_1 \subset \Pi_1$, we may apply Theorem 6.10 for a nilpotent element $f_\pi$ of $gl_N$. Hence, we have an injective vertex algebra homomorphism

$$\Delta = \Delta_{\pi_1, \pi_2} = \text{Ind}^{gl_N}_{gl_I} W^{k_1}(gl_N, \pi) \to W^{k_2}(gl_{N_1}, \pi_1) \otimes W^{k_2}(gl_{N_2}, \pi_2)$$

for all $k \in \mathbb{C}$, where $k + N = k_1 + N_1 = k_2 + N_2$, which satisfies the desired properties by the characterization of $\text{Ind}^{gl_N}_{gl_I}$ and Proposition 6.11. This completes the proof. \qed

We will call $\Delta$ a coproduct for $W$-algebras of type $A$. Let $U(gl_N, \pi) = U(gl_N, f_\pi; \Gamma)$ be the finite $W$-algebra associated with $gl_N, f_\pi, \Gamma$ for a pyramid $\pi$ consisting of $N$ boxes. It is known that $U(gl_N, \pi)$ is isomorphic to a truncation of a shifted Yangian by [BK2]. Following [BK2], for any pyramid $\pi$ split into $\pi_1 \oplus \pi_2$, we have an injective algebra homomorphism

$$\tilde{\Delta} = \tilde{\Delta}_{\pi_1, \pi_2} : U(gl_N, \pi) \to U(gl_{N_1}, \pi_1) \otimes U(gl_{N_2}, \pi_2),$$

called a coproduct for finite $W$-algebras of type $A$, where $N_i$ is the number of boxes in $\pi_i$.

**Proposition 7.2.** Zhu($\Delta$) = $\tilde{\Delta}$. Therefore $\Delta$ is a chiralization of $\tilde{\Delta}$.

Proof. Let $\pi$ be a pyramid split into $\pi_1 \oplus \pi_2$, $\Delta$ the corresponding coproduct for $W$-algebras, $\tilde{\Delta}$ the corresponding coproduct for finite $W$-algebras, $N$ the number of boxes in $\pi$, $N_i$ the number of boxes in $\pi_i$, $l$ the column length of $\pi$ and $l_i$ the column length of $\pi_i$ ($l = l_1 + l_2$). We split $\pi_i$ into individual columns, i.e.

$$\pi_i = \pi_{i_1}^1 \oplus \cdots \oplus \pi_{i_1}^{l_i},$$

$$\pi = \pi_1^1 \oplus \cdots \oplus \pi_1^{l_1} \oplus \pi_2^1 \oplus \cdots \oplus \pi_2^{l_2}$$

such that $\pi_{i_j}^j$ has only one column for all $i = 1, 2$ and $j = 1, \ldots, l_i$. By [BK2], the coproducts of finite $W$-algebras corresponding to (7.3), (7.4) are the Miura maps $\tilde{\mu}_i, \tilde{\mu}_j$ for $U(gl_{N_i}, \pi_i), U(gl_N, \pi)$ respectively. By coassociativity of $\tilde{\Delta}$, it satisfies

$$\tilde{\mu} = (\tilde{\mu}_1 \otimes \tilde{\mu}_2) \circ \tilde{\Delta},$$

which implies that $\Delta = \text{Zhu}(\Delta)$ by Lemma 6.14. This completes the proof. \qed

**7.3 Coproducts for type BCD.** Let $N$ be a positive integer and $g_N = so_N$ or $sp_N$. If $g_N = sp_N$, we assume that $N$ is even. Recall that all nilpotent orbits in $g_N$ are classified by orthogonal partitions of $N$ if $g_N = so_N$ and by symplectic partitions of $N$ if $g_N = sp_N$. See e.g. [CM]. In case of $so_{2M}$, we mean nilpotent orbits under the group $O_{2M}$ not $SO_{2M}$ here. Let $f$ be a rectangular nilpotent element in $g_N$, corresponding to a partition $p = (l^n)$ of $N$. A rectangular pyramid with the height $n$ and the width $l$ represents a good grading for $f$ on $g_N$ in the classification of good gradings of $g_N$ in [BK1], and we denote by $\pi^+$ if $g_N = so_N$ and by $\pi^-$ if $g_N = sp_N$. We fix a numbering of boxes in $\pi^-$ ($\epsilon = -1$) by $1, \ldots, M$ from top to bottom and from left to right, by $-1, \ldots, -M$ in central symmetry and by 0 in the central box if the central box exists, where $M = \lfloor \frac{N}{2} \rfloor$. For example,
Denote a basis of $\mathbb{C}^N$ by $\{v_i, v_{-i}\}_{i=1}^M$ if $N = 2M$, and by $\{v_i\}_{i=1}^{M+1}$ if $N = 2M+1$. Then $\text{End}(\mathbb{C}^N)$ has a basis consisting of all $e_{i,j}v_k = \delta_{j,k}v_i$. To describe $f$ from $\pi^e$ explicitly, we fix a basis of $\mathfrak{g}_N$ in $\text{End}(\mathbb{C}^N)$ as follows:

- $\mathfrak{so}_{2M+1} : e_{i,j} - e_{-j,-i}, e_{s,t} - e_{t,-s}, e_{s,-t} - e_{-t,s}, e_{i,0} - e_{0,-i}, e_{0,-i} - e_{-i,0}$
- $\mathfrak{so}_{2M} : e_{i,j} - e_{-j,-i}, e_{s,t} - e_{t,-s}, e_{s,-t} - e_{-t,s}$
- $\mathfrak{sp}_{2M} : e_{i,j} - e_{-j,-i}, e_{s,t} + e_{t,-s}, e_{s,-t} + e_{-t,s}, e_{i,1}, e_{-i,1}$

with $1 \leq i, j \leq M$ and $1 \leq s < t \leq M$. We attach to $\pi^e$ a nilpotent element $f_{\pi^e}$ by

$$f_{\pi^e}(v_j) = \begin{cases} \pm v_i & (\text{row}(i) = \text{row}(j) \text{ and } \text{col}(i) = \text{col}(j) + 1) \\ 0 & (\text{otherwise}) \end{cases}$$

where the sign $\pm$ is chosen such that $f_{\pi^e} \in \mathfrak{g}_N$. We split $\pi^e$ into three rectangular pyramids along two columns in line symmetry, which we denote by $\pi^e = \pi^e_1 \oplus \pi^e_2 \oplus \pi^e_3$, such that $\pi^e_2$ represents a symmetric partition of $N_2$ if $\mathfrak{g}_{N_2} = \mathfrak{so}_{N_2}$, or an orthogonal partition of $N_2$ if $\mathfrak{g}_{N_2} = \mathfrak{sp}_{N_2}$. For example,

$$\begin{bmatrix} 1 & 4 & 7 & 10 & -9 & -6 & -3 \\ 2 & 5 & 8 & 0 & 0 & 5 & 2 \\ 3 & 6 & 9 & 10 & -7 & -4 & -1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 10 & -9 & -6 & -3 \\ 8 & 0 & 0 & 5 & 2 \\ 9 & 10 & -7 & -4 & -1 \end{bmatrix}$$

Let $N_a$ be the number of boxes in $\pi^e_2$ ($N = 2N_1 + N_2$), and $\mathfrak{h} = \bigoplus_{i=1}^M \mathfrak{ch}_i$ a Cartan subalgebra of $\mathfrak{g}_N$, where $h_i = e_{i,-i} - e_{-i,i}$. Set the dual basis $\{e_i\}_{i=1}^M$ of $\mathfrak{h}^*$ by $\epsilon_i(h_j) = \delta_{i,j}$, and a set $\Pi = \{\alpha_i\}_{i=1}^M$ of simple roots by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, M - 1$ and

$$\alpha_M = \begin{cases} \epsilon_M & (\mathfrak{g}_N = \mathfrak{so}_{2M+1}), \\ 2\epsilon_M & (\mathfrak{g}_N = \mathfrak{sp}_{2M}), \\ \epsilon_{M-1} + \epsilon_M & (\mathfrak{g}_N = \mathfrak{so}_{2M}). \end{cases}$$

Let $I_1 \oplus I_2$ be a maximal Levi subalgebra of $\mathfrak{g}_N$ such that $\{\alpha_i\}_{i=1}^{N_1-1}$ is a set of simple roots in $I_1 = \mathfrak{g}_{N_1}$, and $\{\alpha_i\}_{i=N_1+1}^M$ is a set of simple roots in $I_2 = \mathfrak{g}_{N_2}$. We attach to $\pi^e_2$ a nilpotent element $f_{\pi^e_2}$ in $I_2$ by the same formula in (7.5), where $i, j$ run over the set of numbers labeling $\pi^e_2$. By [Ke], we have

$$\mathcal{O}_{\pi^e} = \text{Ind}_{\mathfrak{g}_N}^{\mathfrak{g}_N} (\mathcal{O}_{\pi^e_1} + \mathcal{O}_{\pi^e_2}),$$

where $\mathcal{O}_{\pi^e}$ denotes a nilpotent orbit in $\mathfrak{g}_N$ through $f_{\pi^e}$, and $\mathcal{O}_{\pi^e_2}$ denotes a nilpotent orbit in $I_2$ through $f_{\pi^e_2}$. Define a good Z-grading $\Gamma_{\pi^e}$ on $\mathfrak{g}_N$ for $f_{\pi^e}$ by $\deg_{\Gamma_{\pi^e}}(e_{i,j}) = \text{col}(j) - \text{col}(i)$ and a good Z-grading $\Gamma_{\pi^e_2}$ on $I_2$ for $f_{\pi^e_2}$ similarly. Then $\Gamma_{\pi^e_2}$ is the restriction of $\Gamma_{\pi^e}$ on $I_2$, and satisfies that $\{\alpha_{N_1}\} = \Pi \setminus \Pi_1 \subset \Pi_1$. Let

$$W^k(\mathfrak{g}_N, \pi^e_2) = W^k(\mathfrak{g}_N, f_{\pi^e}; \Gamma_{\pi^e}),$$

$$W^k(\mathfrak{g}_{N_1}, \pi^e_1) = V^{k+N_1}(3\mathfrak{gl}_{N_1}) \otimes W^k(\mathfrak{sl}_{N_1}, f_{\pi^e_2}; \Gamma_{\pi^e_2}).$$
and
\[ \gamma(\epsilon) = \begin{cases} 1 & (\epsilon = +), \\ 2 & (\epsilon = -). \end{cases} \]

Recall that the dual Coxeter number of \( g_N \) is \( N - 2, M + 1 \) if \( g_N = \mathfrak{so}_N, \mathfrak{sp}_{2M} \) respectively. The same proof as we use to prove Theorem 7.1 is applicable to the following.

**Theorem 7.3.** In the above settings, there exists an injective vertex algebra homomorphism
\[ \Delta^r: \mathcal{W}^{k}(g_N, \pi^r) \rightarrow \mathcal{W}^{k_1}(g_{N_1}, \pi_1^r) \otimes \mathcal{W}^{k_2}(g_{N_2}, \pi_2^r), \]
where \( k + h^r_Y = \gamma(\epsilon)(k_1 + N_1) = k_2 + h^r_{Y_2} \) and \( h^r_Y, h^r_{Y_2} \) are the dual Coxeter numbers of \( g_N, g_{N_2} \) respectively. Moreover, \( \Delta^r \) is a unique vertex algebra homomorphism that satisfies that \( \mu = (\mu_1 \otimes \mu_2) \circ \Delta^r \), where \( \mu, \mu_1, \mu_2 \) are the Miura maps for \( \mathcal{W}^{k}(g_N, \pi^r), \mathcal{W}^{k_1}(g_{N_1}, \pi_1^r), \mathcal{W}^{k_2}(g_{N_2}, \pi_2^r) \) respectively.

Suppose that the height \( n \) of \( \pi^r \) is even if \( g_N = \mathfrak{sp}_N \). Let \( l_a \) be the width of \( \pi_1^r \) (\( 2l_1 + l_2 = l = N/n \)). According to [BK2] and [Bro], it follows that \( U(g_{N_1}, f_{\Gamma_1}; \Gamma_1^r) \) is isomorphic to the Yangian \( Y_{l_1}(g_{n_1}) \) of level \( l_1 \), and \( U(g_N, f_{\pi^r}; \Gamma_{\pi^r}) \) is isomorphic to the twisted Yangian \( Y_{l}^r(g_{n}) \) of level \( l \).

**Corollary 7.4.** Suppose that the height \( n \) of \( \pi^r \) is even if \( g_N = \mathfrak{sp}_N \). Then there exists an injective algebra homomorphism
\[ \widetilde{\Delta}^r: Y_{l}^r(g_{n}) \rightarrow Y_{l_1}(g_{n_1}) \otimes Y_{l_2}^r(g_{n_2}). \]

**Proof.** The assertion of the corollary immediately follows from Theorem 7.3 and Lemma 6.13 \( \square \)

8. **Examples**

We describe \( \Delta \) in Theorem 7.1 explicitly in some examples.

8.1. **Principal nilpotent.** Let \( \pi_{\text{prin}} \) be a pyramid that represents a principal nilpotent element in \( g_N \), i.e. a pyramid consisting of one row of \( N \) boxes. Set a basis \( \{h_i = e_{i,i}\}_{i=1}^N \) of a Cartan subalgebra \( h \) of \( g_N \) and the associated Heisenberg vertex algebra \( \mathcal{H} = \mathcal{H}^{k+N}(h) \), in which
\[ h_i(z)h_j(w) \sim \frac{(k+N)\delta_{i,j}}{(z-w)^2} \]
holds for all \( i, j = 1, \ldots, N \). Consider fields \( W_i(z) \) on \( \mathcal{H} \) defined by the following formal products
\[ : (\hat{\partial} + h_1(z)) \cdot (\hat{\partial} + h_2(z)) \cdots (\hat{\partial} + h_N(z)) : = \sum_{i=0}^{N} W_i(z) \hat{\partial}^{N-i}, \]
where \( \hat{\partial} \) is defined by \( [\hat{\partial}, h_i(z)] = (k + N - 1)\partial_z h_i(z) \) for all \( i \). Then, a vertex subalgebra of \( \mathcal{H} \) generated by \( W_i(z) \) for all \( i = 1, \ldots, N \) is isomorphic to the \( \mathcal{W} \)-algebra \( \mathcal{W}_N^k = \mathcal{W}^{k}(g_N, \pi_{\text{prin}}) \) by [FL] (and [FF4]), which coincides with the image...
of the Miura map for $W^k_N$. Let $N_1$, $N_2$ be positive integers such that $N_1 + N_2 = N$ and $W^1_j(z)$, $W^2_j(z)$ fields on $H$ defined by
\[
\sum_{i=0}^{N_1} W^1_i(z) \hat{\delta}^{N_1-i} =: (\hat{\delta} + h_1(z)) \cdots (\hat{\delta} + h_{N_1}(z)) ;, \\
\sum_{i=0}^{N_2} W^2_i(z) \hat{\delta}^{N_2-i} =: (\hat{\delta} + h_{N_1+1}(z)) \cdots (\hat{\delta} + h_{N_1+N_2}(z)) ;.
\]
For $j = 1, 2$, a vertex subalgebra of $H$ generated by $W^j_i(z)$ for all $i = 1, \ldots, N_j$ is isomorphic to $W^k_{N_j}$, where $k + N = k_1 + N_1 = k_2 + N_2$. By construction,
\[
\sum_{i=0}^{N} W_i(z) \hat{\delta}^{N-i} =: \left( \sum_{i=0}^{N_1} W^1_i(z) \hat{\delta}^{N_1-i} \right) \left( \sum_{i=0}^{N_2} W^2_i(z) \hat{\delta}^{N_2-i} \right) ;,
\]
which induces an injective vertex algebra homomorphism
\[
\Delta: W^k_N \rightarrow W^k_{N_1} \otimes W^k_{N_2}
\]
for all $k \in \mathbb{C}$. This map $\Delta$ is a coproduct for $W^k_N$ corresponding to a splitting of a pyramid:
\[
1 \quad 2 \quad \cdots \quad N-1 \quad N = 1 \quad \cdots \quad N_1 \quad \oplus \quad \delta_1 + 1 \quad \cdots \quad N_2.
\]

8.2. Rectangular cases. We generalize the above construction to the case that $f$ is a rectangular nilpotent element in $\mathfrak{gl}_N$. We follow the framework in [AM]. Let $\pi$ be a rectangular pyramid of the width $l$ and the height $n$ and $N = nl$. The target space of the Miura map for $W^k(\mathfrak{gl}_N, \pi)$ is a tensor vertex algebra $V^\kappa(\mathfrak{gl}_n)^{\otimes l}$, where $\kappa$ is defined by
\[
\kappa(u|v) = \begin{cases} (k + nl) \text{tr}(uv) & (u, v \in \mathfrak{gl}_n) \\ (k + n(l - 1)) \text{tr}(uv) & (u, v \in \mathfrak{sl}_n). \end{cases}
\]
Denote by $u^{(t)}(z)$ a field $u(z)$ on the $t$-th component in $V^\kappa(\mathfrak{gl}_n)^{\otimes l}$ for all $u \in \mathfrak{gl}_n$ and $t = 1, \ldots, l$. Set a fields-valued matrix
\[
A_t(z) = \left( e^{(t)}_{j,i}(z) \right)_{i,j=1}^n
\]
for each $t = 1, \ldots, l$. Let $W_{i,j,t}(z)$ be a field on $V^\kappa(\mathfrak{gl}_n)^{\otimes l}$ defined by the formal product
\[
: (\hat{\delta} + A_1(z)) \cdots (\hat{\delta} + A_l(z)) : = \sum_{t=0}^l W_t(z) \hat{\delta}^{(l-t)}
\]
and $W_t(z) = (W_{i,j,t}(z))_{i,j=1}^n$, where a product (e.g. $A_t(z) \cdot A_j(z)$ etc) is computed by the usual product of matrices and $\hat{\delta}$ is defined by
\[
[\hat{\delta}, A_t(z)] = (k + n(l - 1)) \partial_t A_t(z), \quad \partial_t A_t(z) = \left( \partial_t e^{(t)}_{j,i}(z) \right)_{i,j=1}^n.
\]
Then a vertex subalgebra of $V^\kappa(\mathfrak{gl}_n)^{\otimes l}$ generated by $W_{i,j,t}(z)$ for all $i, j = 1, \ldots, n$ and $t = 1, \ldots, l$ is isomorphic to a $W$-algebra $W^k(\mathfrak{gl}_N, \pi)$ by [AM], and coincides with the image of the Miura map for $W^k(\mathfrak{gl}_N, \pi)$. For a splitting $\pi = \pi_1 \oplus \pi_2$ of
πi, let \( l_i \) be the width of \( π_i \) and \( N_i = nl_i \). Then \( π_i \) is a \( n \times l_i \) rectangular pyramid. Define fields \( W_{i,j,l}^1(z) \), \( W_{i,j,l}^2(z) \) on \( V^{n}(\mathfrak{g}_l)_i \) by

\[
\begin{align*}
\sum_{i=0}^{l_1} W_{i,j,l}^1(z) \hat{\partial}^{(l_1-t)} &= : (\hat{\partial} + A_1(z)) \cdot (\hat{\partial} + A_2(z)) \cdots (\hat{\partial} + A_{l_1}(z)) : \\
\sum_{i=0}^{l_2} W_{i,j,l}^2(z) \hat{\partial}^{(l_2-t)} &= : (\hat{\partial} + A_{i+1}(z)) \cdot (\hat{\partial} + A_{i+2}(z)) \cdots (\hat{\partial} + A_{l_1}(z)) :,
\end{align*}
\]

where \( W_{i,j,l}^1(z) = (W_{i,j,l}^1(z))_{i,j=1}^{n} \) and \( W_{i,j,l}^2(z) = (W_{i,j,l}^2(z))_{i,j=1}^{n} \). By construction,

\[
\sum_{i=0}^{l} W_{i,j,l}(z) \hat{\partial}^{(l-t)} = : \left( \sum_{i=0}^{l_1} W_{i,j,l}^1(z) \hat{\partial}^{(l_1-t)} \right) \left( \sum_{i=0}^{l_2} W_{i,j,l}^2(z) \hat{\partial}^{(l_2-t)} \right) :,
\]

which induces an injective vertex algebra homomorphism

\[
\Delta: \mathcal{W}^k(\mathfrak{g}_N, π) \rightarrow \mathcal{W}^{k_1}(\mathfrak{g}_{N_1}, π_1) \otimes \mathcal{W}^{k_2}(\mathfrak{g}_{N_2}, π_2)
\]

for all \( k \in \mathbb{C} \). This map \( Δ \) is a coproduct corresponding to \( π = π_1 \oplus π_2 \).

8.3. Subregular nilpotent. Let \( π \) be the pyramid with the sequence of column lengths \((2,1^{N-2})\). Then the nilpotent element \( f_π = \sum_{i=2}^{N-1} e_i \) is a subregular nilpotent element in \( \mathfrak{g}_N \). We have \( (\mathfrak{g}_N)_0 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \) and \( \mathfrak{g}_N = \bigoplus_{i=1}^{N} \mathbb{C} h_i \), where \( h_i = e_i \). The corresponding V-algebra \( \mathcal{W}^k(\mathfrak{g}_N, π) \) is then isomorphic to the tensor of the Feigin-Semikhatov algebra \( \mathcal{W}^{k_1}_N \) \((\text{FeiSem})\) and the Heisenberg vertex algebra \( V^{k+N}(\mathfrak{sl}_2) \) if \( k + N \neq 0 \) \((\text{Ge})\). From now on, we assume that \( k + N \neq 0 \).

Let \( H(z), Z(z), E(z), F(z) \) be fields on \( V^{\tau_k}((\mathfrak{g}_N)_0) = V^{k+N}(\mathfrak{sl}_2) \otimes V^{k+2}(\mathfrak{sl}_2) \) defined by

\[
\begin{align*}
H(z) &= h_1(z) - \frac{1}{N} \sum_{i=1}^{N} h_i(z), \quad Z(z) = \sum_{i=1}^{N} h_i(z), \quad E(z) = e_{1,2}(z), \\
F(z) &= : (\hat{\partial} + (h_1 - h_N)(z)) \cdot (\hat{\partial} + (h_1 - h_N-1)(z)) \cdots (\hat{\partial} + (h_1 - h_3)(z)) e_{2,1}(z) :,
\end{align*}
\]

where \( \hat{\partial} = (k + N - 1) \partial_2 \), which generate a vertex subalgebra of \( V^{\tau_k}((\mathfrak{g}_N)_0) \) isomorphic to \( \mathcal{W}^k(\mathfrak{g}_N, π) \) by \( \text{FeiSem} \). We split \( π \) as

\[
\begin{array}{cccccccc}
1 & 2 & 3 & \cdots & N-1 & N \\
\end{array}
\oplus
\begin{array}{cccc}
1 & 2 & \cdots & N-1 \\
N_1 + 1 & \cdots & N \\
\end{array}
\]

which we denote by \( π = π_1 \oplus π_2 \). Let \( Z = V^{k+N}(\mathfrak{sl}_2) \) and \( Z_1 \) (resp. \( Z_2 \)) a vertex subalgebra of \( Z \) generated by \( h_i(z) \) with \( i = 3, \ldots, N_1 \) (resp. \( i = N_1 + 1, \ldots, N \)).

Set \( N_2 = N - N_1 \). We have \( Z = Z_1 \otimes Z_2 \). Let \( H_1(z), Z_1(z), E_1(z), F_1(z) \) be fields on \( Z_1 \otimes V^{k+2}(\mathfrak{sl}_2) \) defined by

\[
\begin{align*}
H_1(z) &= h_1(z) - \frac{1}{N_1} \sum_{i=1}^{N_1} h_i(z), \quad Z_1(z) = \sum_{i=1}^{N_1} h_i(z), \quad E_1(z) = e_{1,2}(z), \\
F_1(z) &= : (\hat{\partial} + (h_1 - h_{N_1})(z)) \cdot (\hat{\partial} + (h_1 - h_{N_1-1})(z)) \cdots (\hat{\partial} + (h_1 - h_3)(z)) e_{2,1}(z) :.
\end{align*}
\]
which generate a vertex subalgebra of \( Z_1 \otimes V^{k+2}(\mathfrak{sl}_2) \) isomorphic to \( W^{k_1}(\mathfrak{gl}_{N_1}, \pi_1) \) by construction, where \( k + N = k_1 + N_1 \). For \( i = 0, \ldots, N_2 \), let \( W_i(z) \) be fields on \( Z_2 \) defined by

\[
: (\hat{\partial} - h_N(z)) \cdots (\hat{\partial} - h_{N_i+1}(z)) : = \sum_{i=0}^{N_2} W_i(z) \hat{\partial}^{N_2-i}.
\]

Since an automorphism \( \tau \) on \( Z_2 \) defined by \( h_i(z) \mapsto -h_{N_i+1+N-i}(z) \) (\( i = N_1 + 1, \ldots, N \)) implies the formula

\[
\sum_{i=0}^{N_2} \tau(W_i(z)) = : (\hat{\partial} + h_{N_1+1}(z)) \cdots (\hat{\partial} + h_{N}(z)) :,
\]

which is a formal product defining generating fields of the \( W^{k_2}_{N_2} \) introduced in Section 8.1, the fields \( W_i(z) \) with \( i = 1, \ldots, N_2 \) generate a vertex subalgebra of \( Z_2 \) isomorphic to \( W^{k_2}(\mathfrak{gl}_{N_2}, \pi_2) \), where \( k + N = k_2 + N_2 \). We have

\begin{align}
H(z) &= H_1(z) + \frac{N_2}{NN_1} Z_1(z) - \frac{1}{N} W_1(z), \\
Z(z) &= Z_1(z) + W_1(z), \\
E(z) &= E_1(z), \\
F(z) &= \sum_{i=0}^{N_2} \sum_{j=0}^{N_2-i} \binom{N_2-j}{i} : (W_j(z) \hat{\partial}^{N_2-j-i}) P_i(z) F_1(z) :,
\end{align}

where

\[
P_0(z) = 1, \quad P_i(z) =: (\hat{\partial} - h_1(z))^{i-1} h_1(z) :
\]

for \( i = 1, \ldots, N_2 \). Here, we use the following lemma:

**Lemma 8.1.**

\[
\sum_{j=0}^{N_2-i} \binom{N_2-j}{i} W_j(z) \hat{\partial}^{N_2-j-i} = \sum_{j_1 < \cdots < j_i} : (\hat{\partial} - h_N(z))^{j_1} \cdots : (\hat{\partial} - h_{N_i+1}(z)) :.
\]

for all \( i = 0, \ldots, N_2 \).

**Proof.** For \( 1 \leq n \leq N_2 \), \( 1 \leq j \leq n \) and \( 1 \leq t_1 < \cdots < t_n \leq N_2 \), we define fields \( W^n_j(u_1, \cdots, u_n) \) on \( Z_2 \) by the following formula:

\[
: (\hat{\partial} - u_t(z)) \cdots (\hat{\partial} - u_{t_n}(z)) : = \sum_{j=0}^{n} W^n_j(u_1, \cdots, u_n) \hat{\partial}^{n-j},
\]

where \( u_i(z) = h_{N-i+1}(z) \). Set \( W^0_0(\phi) = 0 \). The assertion of the lemma is equivalent to the formula

\[
\binom{n-j}{i} W^n_j(u_1, \cdots, u_n) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} W^{n-i}_j(u_1, \cdots, j_i, \cdots, j_n).
\]

for \( n = N_2 \), where \( (i, j) \) run over \( \{ (i, j) \in \mathbb{Z}^2 \mid 0 \leq i, j, i+j \leq n \} \). We will show the formula \ref{8.8} by induction on \( n \) and \( i+j \). If \( n = 1 \) or \( i+j = n \), it is easy to check
that the formula (8.8) follows. If \( n > 1 \) and \( i + j < n \), we have

\[
\binom{n-j}{i} W^n_j(u_1, \ldots, u_n) = \binom{n-j}{i} \binom{n-j}{i+1} W^n_j(u_1, \ldots, u_n)
\]

\[= \frac{i + 1}{n - i - j} \sum_{1 \leq j_1 < \cdots < j_{i+1} \leq n} W^{n-i-1}_{j_1} \cdots W^{n-i-1}_{j_{i+1}}(u_1, \ldots, u_n)
\]

\[= \frac{1}{n - i - j} \sum_{1 \leq j_1 < \cdots < j_i \leq n} W^{n-i}_j(u_1, \ldots, \cdots, u_{j_i}, \ldots, u_n)
\]

by using our inductive assumptions. This completes the proof.

Since \( h_1(z) = H_1(z) + \frac{1}{\text{deg} z} Z_1(z) \) is a field on \( W^k_i(\text{gl}_{N_1}, \pi_1) : P_i(z) F_1(z) : \) are fields on \( W^k_i(\text{gl}_{N_1}, \pi_1) \) for all \( i = 0, \ldots, N_2 \). Hence, the formula (8.4)–(8.7) induces an injective vertex algebra homomorphism

\[
\Delta: W^k_i(\text{gl}_{N}, \pi) \rightarrow W^{k_1}(\text{gl}_{N_1}, \pi_1) \otimes W^{k_2}(\text{gl}_{N_2}, \pi_2),
\]

which is the coproduct corresponding to \( \pi = \pi_1 \oplus \pi_2 \).

### Appendix A. Proof of Theorem 4.12

We assume that the coordinate on \( N_+ \) is compatible with the decomposition \( N_+ = G_{>0} \times G_{>0}^0 \). Recall that \( \sigma_\lambda: H_\lambda(A_{\Delta_0}^T + H_\lambda') \rightarrow A_{\Delta_0}^T + \Phi^T(\mathfrak{g}_{\Delta_0}) \otimes H_\lambda' \) is the isomorphism of \( A_{\Delta_0}^T + \Phi^T(\mathfrak{g}_{\Delta_0}) \otimes H_\lambda' \)-modules, on which the \( A_{\Delta_0}^T + \Phi^T(\mathfrak{g}_{\Delta_0}) \otimes H_\lambda' \)-action is defined by \( \sigma = \sigma_{a_0} \) in Corollary 4.8. Recall that \( \sigma(d \cdot A) = 0 \) for all \( A \in C_k(A_{\Delta_0}^T + H_\lambda') \).

**Lemma A.1.** \( \sigma(\varphi^\alpha) = 0 \) for all \( \alpha \in \Delta_+ \).

**Proof.** Since

\[
\varphi(\beta(z)) = a_\beta(z) + \sum_{\gamma \in \Delta_+ \cap \text{deg} \, \beta} P^\gamma_\beta(a^*(\gamma))a_\gamma(z):
\]

for all \( \beta \in \Delta_+ \) by Lemma 4.3.

\[
d \cdot a_\alpha = d_{\text{st}} \cdot a_\alpha = \varphi^\alpha + \sum_{\beta \in \Delta_+ \cap \text{deg} \, \beta < \text{deg} \, \alpha} P_\beta^\alpha(a^*) \varphi^\beta:
\]

for all \( \alpha \in \Delta_+ \). Since \( \sigma(d \cdot a_\alpha) = 0 \), we have

\[
\sigma(\varphi^\alpha) + \sum_{\beta \in \Delta_+ \cap \text{deg} \, \beta < \text{deg} \, \alpha} \sigma(P^\beta_\alpha(a^*))\sigma(\varphi^\beta) = 0
\]

for all \( \alpha \in \Delta_+ \). We will show that \( \sigma(\varphi^\alpha) = 0 \) for \( \alpha \in \Delta_+ \) by the induction on \( \text{deg} \, \alpha \). If \( \text{deg} \, \alpha = \frac{1}{2} \), it follows that \( \sigma(\varphi^\alpha) = 0 \) by (A.1). If \( \sigma(\varphi^\beta) = 0 \) for all \( \beta < \text{deg} \, \alpha \), we also have \( \sigma(\varphi^\alpha) = 0 \) by (A.1). This completes the proof.

**Lemma A.2.** \( \sigma(a_\alpha) = 0 \) for all \( \alpha \in \Delta_+ \).
Proof. Let $\theta$ be the highest root in $\Delta$. If $\deg_\Gamma \theta \leq 1$, there is nothing to prove. Assume that $\deg_\Gamma \theta > 1$. For $\alpha \in \Delta_{>1}$,

$$
\sigma(d_+: \varphi_\alpha) = \sigma(d_{\text{st}}_+: \varphi_\alpha) = \sigma(\bar{\rho}(e_\alpha)) + \sum_{\beta, \gamma \in \Delta_{>0}} c_{\alpha, \beta}^{\gamma} : \sigma(\varphi_\beta)\sigma(\varphi_\gamma) = = \sigma(\bar{\rho}(e_\alpha))
$$

by Lemma A.1. Since $\sigma(d_+: \varphi_\alpha) = 0$, we have

(A.2) \quad $\sigma(a_\alpha) + \sum_{\beta \in \Delta_{>\deg_\alpha}} : \sigma(P_\alpha^\beta(a^*)\sigma(a_\beta)) = 0$

for all $\alpha \in \Delta_{>1}$ by Lemma A.3. If $\deg_\Gamma \alpha = \deg_\Gamma \theta, \sigma(a_\alpha) = 0$ by A.2. If $\sigma(a_\beta) = 0$ for all $\deg_\Gamma \beta > \deg_\Gamma \alpha$, we also have $\sigma(a_\alpha) = 0$ by A.2. Therefore the assertion of the lemma follows inductively. \qed

Lemma A.3. $\sigma(a_\alpha) = -\chi(e_\alpha)$ for all $\alpha \in \Delta_1$.

Proof. For $\alpha \in \Delta_1$,

$$
d_+: \varphi_\alpha = d_{\text{st}}_+: \varphi_\alpha + d_\chi_+: \varphi_\alpha = \bar{\rho}(e_\alpha) + \sum_{\beta, \gamma \in \Delta_{>0}} c_{\alpha, \beta}^{\gamma} : \varphi_\alpha \varphi_\beta = + \chi(e_\alpha).
$$

Since $\sigma(d_+: \varphi_\alpha) = 0$, we have $\sigma(a_\alpha) = -\chi(e_\alpha)$ for all $\alpha \in \Delta_1$ by Lemma A.1 and Lemma A.2. This completes the proof. \qed

Lemma A.4. For $\alpha \in \Delta_1$,

$$
\sigma(\bar{\rho}^R(e_\alpha)) = - \sum_{\beta \in \Delta_{1}} (-\chi(e_\beta))P_\alpha^{\beta, R}(a^*).
$$

Proof. By Lemma A.2 and Lemma A.3

$$
\sigma(\bar{\rho}^R(e_\alpha)) = \sum_{\beta \in \Delta_{1}} : \sigma(P_\alpha^{\beta, R}(a^*)\sigma(a_\beta)) = : = - \sum_{\beta \in \Delta_{1}} \chi(e_\beta)\sigma(P_\alpha^{\beta, R}(a^*))
$$

for all $\alpha \in \Delta_1$. Since $P_\alpha^{\beta, R}(a^*) \in \mathcal{A}_{\Delta_1}$ for all $\alpha, \beta \in \Delta_1$ by Lemma 12, $\sigma(P_\alpha^{\beta, R}(a^*)) = P_\alpha^{\beta, R}(a^*)$. Therefore the assertion follows. \qed

Lemma A.5. For $\alpha \in \Delta_{\frac{1}{2}}$,

$$
\sigma(\bar{\rho}^R(e_\alpha)) = - \sum_{\beta \in \Delta_{\frac{1}{2}}} : P_\alpha^{\beta, R}(a^*)\Phi_\beta.
$$

Proof. Since $\deg_\Gamma P_\alpha^\gamma(x) = \deg_\Gamma P_\alpha^{\gamma, R}(x) = \frac{1}{2}$ for all $\alpha \in \Delta_{\frac{1}{2}}$ and $\gamma \in \Delta_{1}$, there exist $P_\alpha^{\gamma, \beta}(x), P_\alpha^{\gamma, R}(x) \in \mathbb{C}[G_+^1]$ for all $\beta \in \Delta_{\frac{1}{2}}$ such that

$$
P_\alpha^\gamma(x) = \sum_{\beta \in \Delta_{\frac{1}{2}}} P_\alpha^{\gamma, \beta}(x)x_\beta, \quad P_\alpha^{\gamma, R}(x) = \sum_{\beta \in \Delta_{\frac{1}{2}}} P_\alpha^{\gamma, R}(x)x_\beta
$$

for all $\alpha \in \Delta_{\frac{1}{2}}$. Therefore the assertion follows. \qed
for all $\alpha \in \Delta_\frac{1}{2}$ and $\gamma \in \Delta_1$. We also have $P_{\alpha,\beta}^\gamma(x) = a(\alpha, \beta, \gamma) \in \mathbb{C}$ by Lemma \[4.2\]. Denote by $\gamma^\alpha_{\alpha,\beta} = P_{\alpha,\beta}^\gamma(x) \in \mathbb{C}$. Then

$$
\rho(e_\alpha) = \partial_\alpha + \sum_{\beta \in \Delta_\frac{1}{2}, \gamma \in \Delta_1} \lambda^\gamma_{\alpha,\beta} x_\beta \partial_\gamma + \sum_{\gamma \in \Delta_{>1}} P_{\alpha,\beta}^\gamma(x) \partial_\gamma,
$$

$$
\rho^R(e_\alpha) = \sum_{\beta \in \Delta_\frac{1}{2}} P^R_{\alpha,\beta}(x) \partial_\beta + \sum_{\beta \in \Delta_\frac{1}{2}} \sum_{\gamma \in \Delta_1} P^\gamma_{\alpha,\beta}(x) x_\beta \partial_\gamma + \sum_{\gamma \in \Delta_{>1}} P^\gamma_{\alpha,\beta}(x) \partial_\gamma
$$

for $\alpha \in \Delta_\frac{1}{2}$. Since $\rho(e_\alpha)$ and $\rho^R(e_\alpha)$ commute,

$$
0 = [\rho(e_\alpha), \rho^R(e_\alpha)] = \sum_{\gamma \in \Delta_1} (P^\gamma_{\alpha,\beta}(x) - \sum_{\beta \in \Delta_\frac{1}{2}} \lambda^\gamma_{\alpha,\beta} P^\beta_{\alpha,\beta}(x)) \partial_\gamma + \sum_{\gamma \in \Delta_{>1}} (\cdots) \partial_\gamma
$$

for all $\alpha, \alpha' \in \Delta_\frac{1}{2}$, where $(\cdots)$ denotes some polynomials in $\mathbb{C}[N_+]$. Hence,

$$
P^\gamma_{\alpha,\beta}(x) = \sum_{\beta \in \Delta_\frac{1}{2}} \lambda^\gamma_{\beta,\beta} P^\beta_{\alpha,\beta}(x)
$$

for all $\beta \in \Delta_\frac{1}{2}$ and $\gamma \in \Delta_1$. Therefore,

$$
(A.3) \quad \sigma(\rho^R(e_\alpha)) = \sum_{\beta \in \Delta_\frac{1}{2}} : \sigma(P^\beta_{\alpha,\beta}(x)) \left( \sigma(a_\beta) - \sum_{\gamma \in \Delta_1} \chi(\varepsilon_\gamma) \lambda^\gamma_{\alpha,\beta} \tau(a_\beta) \right) :
$$

for all $\alpha \in \Delta_\frac{1}{2}$ by Lemma \[4.2\] and Lemma \[A.3\]. Next,

$$
\sigma(d \cdot \varphi^0) = \sigma(d_{st} \cdot \varphi^0) + \sigma(d_{ae} \cdot \varphi^0) = \sigma(a_\alpha) - \sum_{\beta \in \Delta_{>1}, \gamma \in \Delta_1} \chi(\varepsilon_\gamma) \lambda^\gamma_{\alpha,\beta} \tau(a_\beta) + \sigma(\Phi_\alpha)
$$

for $\alpha \in \Delta_\frac{1}{2}$ by Lemma \[4.1\] and Lemma \[A.3\]. Since $\sigma(d \cdot \varphi^0) = 0$, we have

$$
(A.4) \quad \sigma(\Phi_\alpha) = -\sigma(a_\alpha) + \sum_{\beta \in \Delta_{>1}, \gamma \in \Delta_1} \chi(\varepsilon_\gamma) \lambda^\gamma_{\alpha,\beta} \tau(a_\beta).
$$

By using the formula $[\rho(e_\alpha), \rho(e_\alpha')] = \rho([e_\alpha, e_\alpha']) = 0$, we obtain that

$$
\sum_{\gamma \in \Delta_1} (\lambda^\gamma_{\alpha,\alpha} - \lambda^\gamma_{\alpha,\alpha'} - c^\gamma_{\alpha,\alpha'}) \partial_\gamma + \sum_{\gamma \in \Delta_{>1}} (\cdots) \partial_\gamma = 0
$$

for all $\alpha, \alpha' \in \Delta_\frac{1}{2}$, where $c^\gamma_{\alpha,\alpha'} \in \mathbb{C}$ is the structure constant. Hence,

$$
(A.5) \quad \lambda^\gamma_{\beta,\alpha} = \lambda^\gamma_{\alpha,\beta} + c^\gamma_{\alpha,\beta}
$$

for all $\beta \in \Delta_\frac{1}{2}$ and $\gamma \in \Delta_1$. Therefore,

$$
(A.6) \quad \sigma(\Phi_\alpha) = \sigma(\Phi_\alpha) + \sum_{\beta \in \Delta_{>1}, \gamma \in \Delta_1} \chi([e_\alpha, e_\beta]) \tau(a_\beta) = -\sigma(a_\alpha) + \sum_{\beta \in \Delta_{>1}, \gamma \in \Delta_1} \chi(\varepsilon_\gamma) \lambda^\gamma_{\beta,\alpha} \tau(a_\beta)$$
Proof of Theorem 4.12

Recall from Lemma 4.10 that for all \( \alpha \in \Delta_\beta \) by Lemma 4.3. Since \( \sigma(\hat{\Phi}_\beta) = \Phi_\beta \) and \( \sigma(P_\alpha^{\beta,R}(a^*)) = P_\alpha^{\beta,R}(a^*) \) for all \( \alpha, \beta \in \Delta_\beta \) by Lemma 4.12, the assertion of the lemma follows. This completes the proof. \( \square \)

Proof of Theorem 4.12

Recall from Lemma 4.10 that \( Q_\alpha \) is the intertwining operator induced by \( S_\alpha \) through the functor \( H_\chi(?) \) and satisfies that

\[
Q_\alpha = \int : \sigma(\hat{\rho}(e_\alpha))(z) e^{-\frac{1}{K_{x,K}}} f_{b_\alpha}(z) : dz
\]

for all \( \alpha \in \Pi \). Hence,

\[
Q_\alpha = \sum_{\beta \in \Delta_\alpha} \int : P_\alpha^{\beta,R}(a^*)(z) a_\beta(z) e^{-\frac{1}{K_{x,K}}} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_0),
\]

\[
Q_\alpha = -\sum_{\beta \in \Delta_\alpha} \int : P_\alpha^{\beta,R}(a^*)(z) \Phi_\beta(z) e^{-\frac{1}{K_{x,K}}} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_1),
\]

\[
Q_\alpha = \sum_{\beta \in \Delta_\alpha} \chi(\epsilon_{\alpha}) \int : P_\alpha^{\beta,R}(a^*)(z) e^{-\frac{1}{K_{x,K}}} f_{b_\alpha}(z) : dz \quad (\alpha \in \Pi_1).
\]

by Lemma 4.11 Lemma 4.4 and Lemma 4.5. Since \( \text{Ker} Q_\alpha = \text{Ker}(\bar{Q}_\alpha) \), we may replace \( Q_\alpha \) by \( -Q_\alpha \) for all \( \alpha \in \Pi_{>0} \). This completes the proof.

APPENDIX B. PROOF OF LEMMA 5.3

We assume that the coordinate on \( N_+ \) is compatible with the decomposition \( N_+ = G_{>0} \times G_0^* \).

Lemma B.1. Let \( \alpha \in \Pi_0 \) and \( \beta, \gamma \in \Delta_{>0} \) such that \([\beta] = [\gamma]\). Then

\[
\partial_\gamma P_\alpha^\beta(x) = c_\gamma,\alpha^\beta, \quad \partial_\gamma Q_\alpha^\beta(x) = c_\gamma,\alpha^{-1}.
\]

Proof. Let \( \alpha \in \Pi_0, \gamma \in \Delta_{>0} \) and \( n = \text{deg}_\gamma \gamma > 0 \). Then

\[
\rho(e_\alpha) = \sum_{\sigma \in \Delta_+} P_\alpha^\sigma(x) \partial_\sigma
\]

by Lemma 4.3 Since \([\rho(e_\gamma), \rho(e_\alpha)] = \sum_{\beta \in [\gamma]} c_\gamma,\alpha^\beta \rho(e_\beta)\), we have

\[
\sum_{\tau \in \Delta_+} \partial_\gamma P_\alpha^\tau(x) \partial_\tau + \sum_{\sigma \in \Delta_{>0}} \sigma \tau \Delta_+ \sum_{\tau \in \Delta_+} \left( P_\alpha^\tau(x) \cdot \partial_\sigma P_\alpha^\gamma(x) \partial_\tau - \partial_\tau P_\alpha^\tau(x) \cdot P_\alpha^\gamma(x) \partial_\sigma \right)
\]

\[
= \sum_{\beta \in [\gamma]} c_\gamma,\alpha^\beta \left( \partial_\beta + \sum_{\sigma \in \Delta_{>0}} P_\alpha^\beta(x) \partial_\sigma \right).
\]

For \( \beta \in [\gamma] \), comparing with the coefficients of \( \partial_\beta \) in the above, we have

\[
\partial_\gamma P_\alpha^\beta(x) + \sum_{\sigma \in \Delta_{>0}} P_\alpha^\beta(x) \cdot \partial_\sigma P_\alpha^\beta(x) = c_\gamma,\alpha^\beta.
\]
Here, we use deg_{\gamma} \beta = deg_{\gamma} \gamma = n. Since deg_{\sigma} P_{\alpha}^\beta(x) = deg_{\sigma} \beta - deg_{\sigma} \alpha = n by \[4.4\], we have \( \partial_\sigma P_{\alpha}^\beta(x) = 0 \) for all \( \sigma \in \Delta_{>n} \). Therefore \( \partial_\sigma P_{\alpha}^\beta(x) = c_{\gamma,\alpha}^\beta \).

Next, we apply the formula \( \rho(f_\alpha) = \sum_{\tau \in \Delta_+} Q_{\alpha}^\tau(x) \partial_\tau \) to the formula \( [\rho(e_\gamma), \rho(f_\alpha)] = \sum_{\beta \in [\gamma]} c_{\gamma,-\alpha}^\beta \rho(e_\beta) \). We have

\[
\sum_{\tau \in \Delta_+} \partial_\tau Q_{\alpha}^\tau(x) \partial_\tau + \sum_{\sigma \in \Delta_{>n}} \sum_{\tau \in \Delta_+} (P_{\gamma}^\tau(x) \cdot \partial_\sigma Q_{\alpha}^\tau(x) \partial_\tau - \partial_\tau P_{\gamma}^\tau(x) \cdot Q_{\alpha}^\tau(x) \partial_\sigma) = \sum_{\beta \in [\gamma]} c_{\gamma,-\alpha}^\beta \left( \partial_\beta + \sum_{\tau \in \Delta_{>n}} P_{\beta}^\tau(x) \partial_\tau \right).
\]

Similarly to the proof of the first assertion, we have

\[
\partial_\gamma Q_{\alpha}^\beta(x) + \sum_{\sigma \in \Delta_{>n}} P_{\gamma}^\beta(x) \partial_\sigma Q_{\alpha}^\beta(x) = c_{\gamma,-\alpha}^\beta
\]

for all \( \beta \in [\gamma] \). Since \( deg_{\gamma} Q_{\alpha}^\beta(x) = deg_{\gamma} \beta + deg_{\gamma} \alpha = n \), it follows that \( \partial_\sigma Q_{\alpha}^\beta(x) = 0 \) for all \( \sigma \in \Delta_{>n} \). Hence, \( \partial_\tau Q_{\alpha}^\beta(x) = c_{\gamma,-\alpha}^\beta \).

**Lemma B.2.** For \( \alpha, \beta \in \Pi \),

\[ |\rho(f_\alpha), \rho^R(e_\beta)| = (\alpha|\beta)x_\alpha \cdot \rho^R(e_\beta). \]

**Proof.** Recall that \( G^o = p^{-1}(U) = N_+ \cdot B_- \) as in Section 3. Set smooth curves \( \gamma_1(t) = exp(-tf_\alpha) \) and \( \gamma_2(t) = exp(-te_\beta) \) on \( G \). Given \( X \in G^o \),

\[
\gamma_1(t)X = Z_+(t)Z_-(t), \quad X \gamma_2(t) = Z_+^R(t)
\]

for \( |t| \ll 1 \), where \( Z_+(t), Z_+^R(t) \in N_+ \) and \( Z_-(t) \in B_- \). The vector fields \( \zeta_{f_\alpha}, \zeta_{e_\beta}^R \) are then given by the formulae

\[
(\zeta_{f_\alpha})(p(X)) = \frac{d}{dt} f(Z_+(t))|_{t=0}, \quad (\zeta_{e_\beta}^R)(p(X)) = \frac{d}{dt} f(Z_+^R(t))|_{t=0}
\]

for any smooth function \( f \) defined in an open subset in \( U \) around \( p(X) \). Choose a faithful representation \( V_0 \) of \( g \) and consider \( X \in N_+ \) as a matrix in \( GL(V_0) \) whose entries are polynomials in \( C[N_+] \). We have

\[
(1 - tf_\alpha)X = Z_+(t)Z_-(t), \quad X(1 - te_\beta) = Z_+^R(t) \mod (t^2).
\]

Hence \( Z_+(t) = X + tZ, \quad Z_+^R(t) = X + tZ^R \) and \( Z_- = 1 + tZ' \) modulo \( (t^2) \), where \( Z, Z^R \in n_+ \) and \( Z' \in b_- \) modulo. Therefore, \( (B.1) \)

\[
\zeta_{f_\alpha} \cdot X = -X(X^{-1}f_\alpha)X_+, \quad \zeta_{e_\beta}^R \cdot X = -Xe_\beta,
\]

where \( (\cdot)_+ : g = n_+ \oplus b_- \rightarrow n_+ \) is the first projection. We have \( -(X^{-1}f_\alpha X) \leq 0 = Z' \), \( (\cdot) : g = n_+ \oplus b_- \rightarrow b_- \) is the second projection. Since

\[
\left( \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}(-xe_\beta))^n(f_\alpha) \right)_{\leq 0} = f_\alpha - xe_\beta([e_\beta, f_\alpha])_{\leq 0} = f_\alpha - x_\alpha h_\alpha \delta_{\alpha, \beta}
\]

for all \( \beta \in \Delta_+ \), we have \( (X^{-1}f_\alpha X)_{\leq 0} = f_\alpha - x_\alpha h_\alpha \). Therefore \( (B.2) \)

\[
Z' = -f_\alpha + x_\alpha h_\alpha.
\]
To compute the commutation relation between \( \zeta_{f_{\alpha}} \) and \( \zeta_{e_{\beta}} \), we compute \( \zeta_{f_{\alpha}} \circ \zeta_{e_{\beta}}^{R}(X) \) and \( \zeta_{e_{\beta}} \circ \zeta_{f_{\alpha}}(X) \). First,

\[
(\text{B.3}) \quad \zeta_{e_{\beta}} \circ \zeta_{f_{\alpha}}(X) = -(\zeta_{f_{\alpha}} \cdot X)e_{\beta}
\]
by \[\text{B.1}\]. Next,

\[
\zeta_{f_{\alpha}} \circ \zeta_{e_{\beta}}^{R}(X) = \left. \frac{\partial^2}{\partial t_1 \partial t_2} (p(\gamma_1(t_1)X_\gamma(t_2))) \right|_{t_1=t_2=0}.
\]

By using the Baker-Campbell-Hausdorff formula, we obtain that

\[
\gamma_1(t_1)X_\gamma(t_2) = Z_+(t_1) \exp(-t_2(1 + t_1(\alpha|\beta)x_{\alpha})e_{\beta})Z_{\leq 0}(t_1, t_2) \mod (t^2)
\]
for \(|t_1|, |t_2| \ll 1\) with some \(Z_{\leq 0}(t_1, t_2) \in B_-\). Here, we use \([Z', e_{\beta}] = \delta_{\alpha, \beta}h_{\alpha} + (\alpha|\beta)x_{\alpha}e_{\beta}\) by \[\text{B.2}\]. Therefore,

\[
\zeta_{f_{\alpha}} \circ \zeta_{e_{\beta}}^{R}(X) = \left. \frac{\partial^2}{\partial t_1 \partial t_2} \left( Z_+(t_1)(1 - t_2(1 + t_1(\alpha|\beta)x_{\alpha})e_{\beta}) \right) \right|_{t_1=t_2=0} = \epsilon_{e_{\beta}} \circ \zeta_{f_{\alpha}}(X) + (\alpha|\beta)x_{\alpha}(\zeta_{e_{\beta}}^{R} \cdot X).
\]

by \[\text{B.1}\] and \[\text{B.3}\]. Hence, \([\zeta_{f_{\alpha}}, \zeta_{e_{\beta}}^{R}] = (\alpha|\beta)x_{\alpha} \cdot \zeta_{e_{\beta}}^{R}\). Thus, \([\rho(f_{\alpha}), \rho^{R}(e_{\beta})] = (\alpha|\beta)x_{\alpha} \cdot \rho^{R}(e_{\beta})\). This completes the proof. \(\square\)

**Lemma B.3.** Let \( \alpha \in \Pi_0, \epsilon \in \Pi_{>0} \) and \( \beta \in [\epsilon] \). Then

1. \[
\sum_{\gamma \in \Delta^+_0} P_\alpha^\gamma(x) \partial_\gamma P_\epsilon^{\beta,R}(x) = \sum_{\gamma \in [\epsilon]} c_{\gamma,\alpha}^\beta P_\epsilon^{\gamma,R}(x),
\]
2. \[
\sum_{\gamma \in \Delta^+_0} Q_\alpha^\gamma(x) \partial_\gamma P_\epsilon^{\beta,R}(x) = \sum_{\gamma \in [\epsilon]} c_{\gamma,\alpha}^\beta P_\epsilon^{\gamma,R}(x) + (\alpha|\beta)x_{\alpha}P_\epsilon^{\beta,R}(x).
\]

**Proof.** Let \( \alpha \in \Pi_0, \epsilon \in \Pi_{>0} \). Since \([\rho(e_{\alpha}), \rho^{R}(e_{\epsilon})] = 0\), we have

\[
\sum_{\beta, \gamma \in \Delta^+_0} \left( P_\alpha^\gamma(x) \cdot \partial_\gamma P_\epsilon^{\beta,R}(x) - P_\epsilon^{\gamma,R}(x) \cdot \partial_\gamma P_\alpha^\beta(x) \right) \partial_\beta = 0.
\]

Hence

\[
(\text{B.4}) \quad \sum_{\gamma \in \Delta^+_0} P_\alpha^\gamma(x) \partial_\gamma P_\epsilon^{\beta,R}(x) = \sum_{\gamma \in \Delta^+_0} P_\epsilon^{\gamma,R}(x) \cdot \partial_\gamma P_\alpha^\beta(x)
\]

for all \( \beta \in \Delta^+_0 \). Let \( \beta \in [\epsilon] \). We assume that \( \deg_{\gamma} P_\epsilon^{\gamma,R}(x) \cdot \partial_\gamma P_\alpha^\beta(x) \neq 0 \). Then

\[
\deg_{\gamma} P_\epsilon^{\gamma,R}(x) = \deg_{\gamma} \gamma - \deg_{\epsilon} \epsilon \geq 0,
\]

\[
\deg_{\gamma} \partial_\gamma P_\alpha^\beta(x) = \deg_{\gamma} \beta - \deg_{\alpha} \alpha - \deg_{\gamma} \gamma = \deg_{\gamma} \epsilon - \deg_{\gamma} \gamma \geq 0.
\]

Hence, \( \deg_{\gamma} \gamma = \deg_{\gamma} \epsilon \), which implies that \( \gamma \in [\epsilon] \) by Proposition \[\text{B.2}\]. Since \( P_\epsilon^{\gamma,R}(x) \in \mathbb{C}[G_0^+] \) by Lemma \[\text{B.2}\], \( \partial_\gamma P_\epsilon^{\gamma,R}(x) = 0 \) for all \( \gamma \in \Delta^+_0 \). Hence,

\[
\sum_{\gamma \in \Delta^+_0} P_\alpha^\gamma(x) \cdot \partial_\gamma P_\epsilon^{\beta,R}(x) = \sum_{\gamma \in [\epsilon]} P_\epsilon^{\gamma,R}(x) \cdot \partial_\gamma P_\alpha^\beta(x) = \sum_{\gamma \in [\epsilon]} c_{\gamma,\alpha}^\beta P_\epsilon^{\gamma,R}(x)
\]

for all \( \beta \in [\epsilon] \) by \[\text{B.3}\] and Lemma \[\text{B.1}\]. Therefore the assertion of (1) follows.

Next, since \([\rho(f_{\alpha}), \rho^{R}(e_{\epsilon})] = (\alpha|\epsilon)x_{\alpha} \cdot \rho^{R}(e_{\epsilon})\) by Lemma \[\text{B.2}\], we have

\[
\sum_{\beta, \gamma \in \Delta^+_0} \left( Q_\alpha^\gamma(x) \cdot \partial_\gamma P_\epsilon^{\beta,R}(x) - P_\epsilon^{\gamma,R}(x) \cdot \partial_\gamma Q_\alpha^\beta(x) \right) \partial_\beta = (\alpha|\epsilon)x_{\alpha} \sum_{\beta \in \Delta^+_0} P_\epsilon^{\beta,R}(x) \partial_\beta.
\]
Hence
\[ \sum_{\gamma \in \Delta_+} Q^\gamma_\alpha(x) \cdot \partial_{\gamma} P^{\beta,R}_e(x) = \sum_{\gamma \in \Delta_+} P^{\gamma,R}_e(x) \cdot \partial_{\gamma} Q^\beta_\alpha(x) + (\alpha|\epsilon)x_\alpha P^{\beta,R}_e(x) \]
for all \( \beta \in \Delta_+ \). Similarly to the proof of the assertion of (1),
\[ \sum_{\gamma \in \Delta_+} Q^\gamma_\alpha(x) \cdot \partial_{\gamma} P^{\beta,R}_e(x) = \sum_{\gamma \in [\epsilon]} c^\beta_{\gamma,-\alpha} P^{\gamma,R}_e(x) + (\alpha|\epsilon)x_\alpha P^{\beta,R}_e(x) \]
for all \( \beta \in [\epsilon] \) by Lemma B.1. Therefore the assertion of (2) follows. This completes the proof. \( \square \)

Proof of Lemma 5.3 Let \( \epsilon \in \Pi_{>0} \) and \( \beta \in [\epsilon] \). For \( u, v \in \mathfrak{g}_0 \),
\[ [u(z), v(w)] = [u, v](w)\delta(z - w) + \tau_k(u|v)\partial_u \delta(z - w). \]
on \( V^T(\mathfrak{g}_0) \). Hence,
\[ ([u(y), v(z)], V^\beta(w)] = ([u, v](z), V^\beta(w)]\delta(y - z). \]
If \( u(z) \) and \( v(z) \) satisfy (B.5),
\[ [u(y), v(z)], V^\beta(w)] = \sum_{\gamma, \sigma \in [\epsilon]} (c^\beta_{\gamma,u}c^\beta_{\sigma,v} - c^\beta_{\gamma,v}c^\beta_{\sigma,u})V^\gamma(w)\delta(y - w)\delta(z - w). \]
First, we have \( \sum_{\sigma \in [\epsilon]} (c^\beta_{\gamma,u}c^\beta_{\sigma,v} - c^\beta_{\gamma,v}c^\beta_{\sigma,u}) = c^\beta_{\gamma,[u,v]} \) for all \( \gamma \in [\epsilon] \) by the Jacobi identity:
\[ [e_{\gamma}, [u, v]] = [e_{\gamma}, u], v] - [e_{\gamma}, v], u]. \]
Next, by using the formula \( \delta(y - w)\delta(z - w) = \delta(y - z)\delta(z - w) \), we have
\[ ([u(y), v(z)], V^\beta(w)] = \sum_{\gamma \in [\epsilon]} c^\beta_{\gamma,[u,v]}W^\gamma(w)\delta(y - z)\delta(z - w). \]
Hence, combining (B.5) with (B.6) and computing the residue at \( y = 0 \), we obtain that
\[ ([u, v](z), V^\beta(w)] = \sum_{\gamma \in [\epsilon]} c^\beta_{\gamma,[u,v]}W^\gamma(w)\delta(z - w). \]
Therefore \( [u, v](z) \) satisfies (B.5). Thus, it suffices to show that (B.5) follows for \( u = e_\alpha, h_{\alpha'}, f_\alpha \) for \( \alpha \in \Pi_0, \alpha' \in \Pi \). Recall that \( u(z) = \hat{\rho}_{\mathfrak{g}_0}(u(z)) \) is defined by (4.9).
First, we consider the case that \( u = h_\alpha \). Then
\[ [h_\alpha(z), V^\beta(w)] = -\sum_{\gamma \in \Delta_0^+} \gamma(h_\alpha) : a^*_\gamma(z)[a_\gamma(z), P^{\beta,R}_e(a^*)(w)]e^{-\frac{1}{\kappa x_{\alpha'}}} f_{h_\alpha}(w) : \]
\[ + : P^{\beta,R}_e(a^*)(w)[h_\alpha(z), e^{-\frac{1}{\kappa x_{\alpha'}}} f_{h_\alpha}(w)] :. \]
Recall that \( A(z)\delta(z - w) = A(w)\delta(z - w) \) for any field \( A(z) \). By (3.2),
\[ \sum_{\gamma \in \Delta_0^+} \gamma(h_\alpha) : a^*_\gamma(z)[a_\gamma(z), P^{\beta,R}_e(a^*)(w)] : = \sum_{\gamma \in \Delta_0^+} \gamma(h_\alpha)a^*_\gamma(w)\partial_\gamma P^{\beta,R}_e(a^*)(w)\delta(z - w). \]
Notice that \( \sum_{\gamma \in \Delta^+_0} \gamma \cdot x, \partial_\gamma \) defines the \( \mathbb{Q}_0 \)-valued Euler operator on \( \mathbb{C}[G^+_0] \) with respect to \( \text{deg}_\mathbb{Q} \). Since \( \text{deg}_\mathbb{Q} P_{\epsilon}^{\beta,R}(x) = \beta - \epsilon \) by (4.1), we have
\[
\sum_{\gamma \in \Delta^+_0} \gamma \cdot (h_\alpha) a^*_\gamma(w) \partial_\gamma P_{\epsilon}^{\beta,R}(a^*) (w) = (\beta(h_\alpha) - \epsilon(h_\alpha)) P_{\epsilon}^{\beta,R}(a^*) (w).
\]
Since (B.7)
\[
[b_\alpha(z), e^{-\frac{1}{k+h}} \int b_\epsilon(w)] = -\epsilon(h_\alpha) e^{-\frac{1}{k+h}} \int b_\epsilon(w) \delta(z - w),
\]
we have
\[
[h_\alpha(z), V^\beta(w)] = \sum_{\gamma \in [\epsilon]} c^\beta_{\gamma,h_\alpha} V^\gamma(w) \delta(z - w).
\]
Hence, (5.5) follows for \( u = h_\alpha \). Next, we consider the case that \( u = e_\alpha \). Then
\[
[e_\alpha(z), V^\beta(w)] = \sum_{\gamma \in \Delta^+_0} P_{\gamma}^\alpha(a^*) (w) \partial_\gamma P_{\epsilon}^{\beta,R}(a^*) (w) e^{-\frac{1}{k+h}} \int b_\epsilon(w) \delta(z - w)
\]
by (3.2). By applying Lemma B.3 (1) to the above formula, we obtain that
\[
[e_\alpha(z), V^\beta(w)] = \sum_{\gamma \in [\epsilon]} c^\beta_{\gamma,e_\alpha} V^\gamma(w) \delta(z - w).
\]
Therefore (5.5) follows for \( u = e_\alpha \). Finally, we consider the case that \( u = f_\alpha \). Then
\[
[f_\alpha(z), V^\beta(w)]
\]
\[
= : \left( \sum_{\gamma \in \Delta^+_0} Q^\alpha_\gamma(a^*) (w) \partial_\gamma P_{\epsilon}^{\beta,R}(a^*) (w) - (\alpha|\epsilon) a^*_\alpha(w) P_{\epsilon}^{\beta,R}(a^*) (w) \right) e^{-\frac{1}{k+h}} \int b_\epsilon(w) \delta(z - w)
\]
by (3.2) and (B.7). By applying Lemma B.3 (2) to the above formula, we obtain that
\[
[f_\alpha(z), V^\beta(w)] = \sum_{\gamma \in [\epsilon]} c^\beta_{\gamma,-\alpha} V^\gamma(w) \delta(z - w).
\]
Therefore (5.5) follows for \( u = f_\alpha \). This completes the proof.

References

[AGT] L. F. Alday, D. Gaiotto, Y. Tachikawa. Liouville correlation functions from four-dimensional gauge theories. Lett. Math. Phys., 91:167–197, 2010.

[A1] T. Arakawa. Representation theory of \( \mathcal{W} \)-algebras. Invent. Math., 169(2):219–320, 2007.

[A2] T. Arakawa. Two-sided BGG resolutions of admissible representations. Represent. Theory, 18:183–222, 2014.

[A3] T. Arakawa. Introduction to \( \mathcal{W} \)-algebras and their representation theory. In: Callegaro F., Carnovale G., Caselli F., De Concini C., De Sole A. (eds) Perspectives in Lie Theory. Springer INdAM Series, vol 19. Springer.

[ACL] T. Arakawa, T. Creutzig, A. Linshaw. \( \mathcal{W} \)-algebras as coset vertex algebras. arXiv:1801.03822

[AKM] T. Arakawa, T. Kuwabara, F. Malikov. Localization of affine \( \mathcal{W} \)-algebras. Comm. Math. Phys., 335(1):143–182, 2015.

[AM] T. Arakawa, A. Molev. Explicit generators in rectangular affine \( \mathcal{W} \)-algebras of type \( \mathcal{A} \). Lett. Math. Phys., 107(1):47–59, 2017.

[BFN] A. Braverman, M. Finkelberg, H. Nakajima. Instanton moduli spaces and \( \mathcal{W} \)-algebras. Astérisque, (385):vii+128pp, 2016.
[BT] J. de Boer, T. Tjin. Quantization and representation theory of finite W-algebras. Comm. Math. Phys., 158:485–516, 1993.

[Bro] J. Brown. Twisted Yangians and finite W-algebras. Transform. Groups, 14:87–114, 2009.

[BG] J. Brundan, S. M. Goodwin. Good grading polytopes. Proc. Lond. Math. Soc., (3), 94(1):155–180, 2007.

[BGK] J. Brundan, S. M. Goodwin, A. Kleshchev. Highest weight theory for finite W-algebras. Int. Math. Res. Not. IMRN, (15):53pp, 2008.

[BK1] J. Brundan, A. Kleshchev. Parabolic presentations of the Yangian Y(gl_n). Comm. Math. Phys., 254(1):191–220, 2005.

[BK2] J. Brundan, A. Kleshchev. Shifted Yangians and finite W-algebras. Adv. Math., 200(1):136–195, 2006.

[BK3] J. Brundan, A. Kleshchev. Representations of shifted Yangians and finite W-algebras. Mem. Amer. Math. Soc., 196(918):viii+107pp, 2008.

[CM] D. H. Collingwood, W. M. McGovern. Nilpotent orbits in semisimple Lie algebras. Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.

[DK] A. De Sole, V. G. Kac. Finite vs affine W-algebras. Transform. Groups, 14:87–114, 2009.

[DKV1] A. De Sole, V. G. Kac, D. Valeri. Classical Affine W-Algebras for gl_N and Associated Integrable Hamiltonian Hierarchies. Comm. Math. Phys., 348(1):265–319, 2016.

[DKV2] A. De Sole, V. G. Kac, D. Valeri. Structure of classical (finite and affine) W-algebras. J. Eur. Math. Soc., 18(9):1873–1908, 2016.

[EK] A. G. Elashvili, V. G. Kac. Classification of good gradings of simple Lie algebras. Lie groups and invariant theory, 85–104, Amer. Math. Soc. Transl. Ser. 2, 213, Amer. Math. Soc., Providence, RI, 2005.

[FL] V. A. Fateev, S. L. Lukyanov. Additional symmetries and exactly solvable models of two-dimensional conformal field theory. Sov. Sci. Rev. A. Phys., 15:1–117, 1990.

[Fei] B. L. Feigin. Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras. Uspekhi Mat. Nauk, 39(2(236)):195–196, 1984.

[FF1] B. L. Feigin, E. Frenkel. A family of representations of affine Lie algebras. Russ. Math. Surv., 43(5):221-222, 1988.

[FF2] B. L. Feigin, E. Frenkel. affine Kac-Moody algebras and semi-infinite flag manifolds. Comm. Math. Phys., 128(1):161–189, 1990.

[FF3] B. L. Feigin, E. Frenkel. Representations of affine Kac-Moody algebras, bosonization and resolutions. Lett. Math. Phys., 19(4):307–317, 1990.

[FF4] B. L. Feigin, E. Frenkel. Quantization of Drinfel’d-Sokolov reduction. Phys. Lett., B 246(1–2):75–81, 1990.

[FF5] B. L. Feigin, E. Frenkel. Integrable hierarchies and Wakimoto modules. Differential topology, infinite-dimensional Lie algebras, and applications, 27–60, Amer. Math. Soc. Transl. Ser. 2, 194. Adv. Math. Sci., 44. Amer. Math. Soc., Providence, RI, 1999.

[FeiSem] B. L. Feigin, A. M. Semikhatov. W^{(2)}-algebras. Nuclear Phys., B 698(3):409–449, 2004.

[FigSta] J. M. Figueroa-O‘Farrill, S. Stanciu. Poisson Lie groups and the Miura transformation. Modern Phys. Lett. A, 10(36):2767–2773, 1995.

[FKPRW] M. Finkelberg, J. Kamnitzer, K. Pham, L. Rybnikov, A. Weekes. Comultiplication for shifted Yangians and quantum open Toda lattice. Adv. Math., 327:349–389, 2018.

[Fre] E. Frenkel. Wakimoto modules, opers and the center at the critical level. Advances in Math., 195:297–404, 2005.

[FBZ] E. Frenkel, D. Ben-Zvi. Vertex algebras and algebraic curves. Second edition. Mathematical Surveys and Monographs, 88. American Mathematical Society, Providence, RI, 2004.

[FKRW] E. Frenkel, V. Kac, A. Radul, W. Wang. W_{1+∞} and W(gl_N) with central charge N. Commun. Math. Phys., 170:337–358, 1995.

[FHL] I. B. Frenkel, Y.-Z. Huang, J. Lepowsky. On axiomatic approaches to vertex operator algebras and modules. Mem. Amer. Math. Soc., 104(494), 1993.

[FZ] I. B. Frenkel, Y. Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. Duke Math. J., 66(1):123–168, 1992.

[GG] W. L. Gan, V. Ginzburg. Quantization of Slodowy slices. Int. Math. Res. Not., 5:243–255, 2002.
[Ge] N. Genra. Screening operators for \( W \)-algebras. *Selecta Math. (N.S.),* 23(3):2157–2202, 2017.

[GRU] S. M. Goodwin, G. Röhrle, G. Ubly. On 1-dimensional representations of finite \( W \)-algebras associated to simple Lie algebras of exceptional type. *LMS J. Comput. Math.*, 13:357–369, 2010.

[GE] W. A. de Graaf, A. Elashvili. Induced nilpotent orbits of the simple Lie algebras of exceptional type. *Georgian Math. J.*, 16(2):257–278, 2009.

[GNW] N. Guay, H. Nakajima, C. Wendlandt. Coproducts for the Yangian of an affine Kac-Moody algebra. *arXiv:1701.05288.*

[Ka] V. G. Kac. Vertex algebras for beginners. Second edition. University Lecture Series, 10. *American Mathematical Society, Providence, RI,* 1998.

[KRW] V. G. Kac, S.-S. Roan, M. Wakimoto. Quantum reduction for affine superalgebras. *Comm. Math. Phys.*, 241(2-3):307–342, 2003.

[KW1] V. G. Kac, M. Wakimoto. Quantum reduction and representation theory of superconformal algebras. *Adv. Math.*, 185(2):400–458, 2004.

[KW2] V. G. Kac, M. Wakimoto. Corrigendum to: “Quantum reduction and representation theory of superconformal algebras” [Adv. Math. 185(2004)400–458]. *Adv. Math.*, 193(2):453–455, 2005.

[Ke] G. Kemperken. Induced conjugacy classes in classical Lie algebras. *Abh. Math. Sem. Univ. Hamburg,* 53:53–83, 1983.

[Ko] B. Kostant. The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. *Amer. J. Math.*, 81:975–1032, 1959.

[Kr] H. Kraft. Parametrisierung von Konjugationsklassen in \( sl_n \). *Math. Ann.,* 234(3):209–220, 1978.

[Lo1] I. Losev. Quantized symplectic actions and \( W \)-algebras. *J. Amer. Math. Soc.*, 23(1):35–59, 2010.

[Lo2] I. Losev. Finite \( W \)-algebras. *Proceedings of the International Congress of Mathematicians.* Volume III, 1281–1307, Hindustan Book Agency, New Delhi, 2010.

[Lo3] I. Losev. 1-Dimensional representations and parabolic induction for \( W \)-algebras. *Adv. Math.*, 226(6):4841–4883, 2011.

[Lo4] I. Losev. Finite-dimensional representations of \( W \)-algebras. *Duke Math. J.*, 159(1):99–143, 2011.

[LS] G. Lusztig, N. Spaltenstein. Induced unipotent classes. *J. Lond. Math. Soc.*, 19(2):41–52, 1979.

[Ly] T. Lynch. Generalized Whittaker vectors and representation theory. PhD thesis, M.I.T., 1979.

[MO] D. Maulik, A. Okounkov. Quantum Groups and Quantum Cohomology. *arXiv:1211.1287.*

[OW] H. Ozeki, M. Wakimoto. On polarizations of certain homogenous spaces. *Hiroshima Math. J.*, 2:445–482, 1972.

[P1] A. Premet. Special transverse slices and their enveloping algebras. *Adv. Math.*, 170:1–55, 2002.

[P2] A. Premet. Enveloping algebras of Slodowy slices and the Joseph ideal. *J. Eur. Math. Soc.*, 9(3):487–543, 2007.

[P3] A. Premet. Primitive ideals, non-restricted representations and finite \( W \)-algebras. *Moscow Math. J.*, 7:743–762, 2007.

[P4] A. Premet. Commutative quotients of finite \( W \)-algebras. *Adv. Math.*, 225(1):269–306, 2010.

[R] E. Ragoucy. Twisted Yangians and folded \( W \)-algebras. *Internat. J. Modern. Phys. A,* 16:2411–2433, 2001.

[RS] E. Ragoucy, P. Sorba. Yangian realizations from finite \( W \)-algebras. *Comm. Math. Phys.*, 203:551–572, 1999.

[SV] O. Schiffmann, E. Vasserot. Cherednik algebras, \( W \)-algebras and the equivariant cohomology of the moduli space of instantons on \( \mathbb{A}^2 \). *Publ. Math. Inst. Hautes Études Sci.,* 118(1):213–342, 2013.

[Sp] N. Spaltenstein. Classes unipotentes et sous-groupes de Borel. Lecture Notes in Mathematics, 946. *Springer-Verlag, Berlin-New York,* ix+259 pp, 1982.
[TY] P. Tauvel, R. W. T. Yu. Lie algebras and algebraic groups. Springer Monographs in Mathematics. Springer-Verlag, Berlin, xvi+653 pp, 2005.

[T1] A. Tsymbaliuk. The affine Yangian of $\mathfrak{gl}_1$ revisited. Adv. Math., 304:583–645, 2017.

[T2] A. Tsymbaliuk. Classical limits of quantum toroidal and affine Yangian algebras. J. Pure Appl. Algebra, 221:2633–2646, 2017.

[V1] A. Voronov. Semi-infinite homological algebra. Invent. Math., 113(1):103–146, 1993.

[V2] A. Voronov. Semi-infinite induction and Wakimoto modules. Amer. J. Math., 121(5):1079–1094, 1999.

[Wak] M. Wakimoto. Fock representations of affine Lie algebra $A_1^{(1)}$. Comm. Math. Phys., 104:605–609, 1986.

[Wan] W. Wang. Nilpotent orbits and $\mathcal{W}$-algebras. Geometric representation theory and extended affine Lie algebras, 71–105, Fields Inst. Commun., 59, Amer. Math. Soc., Providence, RI, 2011.

[Za] A. B. Zamolodchikov. Infinite extra symmetries in two-dimensional conformal quantum field theory. Theor. Math. Phys., 65(3):347–359, 1985.

[Zhu] Y. Zhu. Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc., 9(1):237–302, 1996.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 JAPAN
E-mail address: gnr@kurims.kyoto-u.ac.jp