ON THE ENDOMORPHISM RINGS OF ABELIAN GROUPS AND THEIR JACOBSON RADICAL

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Dedicated to the memory of Professor J. Szendrei

Abstract. We give a characterization of abelian groups which are direct sums of cyclic groups with closed Jacobson radical of their endomorphism rings.

1. Introduction and Results

The question when the Jacobson radical of a topological ring is closed has drawn the attention of researchers since the beginning of the theory of topological rings (see for example [6], [7], [15] and [16]). In [6] Kaplansky has proved that the Jacobson radical of a compact ring is closed. Later in [7] he extended this result to locally compact rings.

An interesting construction which connects the theory of rings, the theory of abelian groups and the theory of topological rings is the ring of endomorphisms of an abelian group considered as a topological ring with the finite topology. It is tempting to use the Jacobson radical to study rings of endomorphisms. But Bourbaki’s example of a free module $M_R$ over a ring $R$ for which the Jacobson radical $J(End(M_R))$ of the ring of endomorphisms $End(M_R)$ is not closed (see [2], Chapter III, §6, p.110) is well known. The proof essentially uses properties of the ring $R$. The examples of abelian groups $A$ for which $J(End(A))$ is not closed were constructed independently in [3] and [13]. Moreover, in [13] an example of an abelian group $A$ is given which is a direct sum of cyclic groups for which the Jacobson radical $J(End(A))$ is not closed. All such examples raise the following natural question:

Question 1. When does the ring of endomorphisms of an abelian group which is a direct sum of cyclic groups have closed Jacobson radical?

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A right module $M_R$ over a ring $R$ is called a quasi-injective module (see [4] or [9], Chapter 4, p.104), if every partial endomorphism of $M_R$ can be extended to a full endomorphism.

The following results give a complete answer to Question 1.

**Theorem 1.** Let $M_R$ be a right quasi-injective module over a ring $R$. The Jacobson radical $\mathfrak{J}(\text{End}(M_R))$ of the ring $\text{End}(M_R)$ is closed with respect to the finite topology.

**Theorem 2.** Let $A$ be a direct sum of cyclic groups. Then $\mathfrak{J}(\text{End}(A))$ is closed if and only if every primary component of $A$ is bounded.

Now let $A$ be an abelian $p$-group. Denote $A_n = A[p^n]$ for each $n \in \mathbb{N}$. Thus the family $\{T(A_n)\}$ is a fundamental system of neighborhoods of zero of a Hausdorff ring topology $T_L$ on $\text{End}(A)$. This topology is called the Liebert topology. We note that each $T(A_n)$ is an ideal of $\text{End}(A)$.

The following question arises naturally.

**Question 2.** Classify abelian $p$-groups whose endomorphism rings are locally compact with respect to the Liebert topology.

**Theorem 3.** Let $A$ be an abelian $p$-group. The following statements are equivalent:

(i) $(\text{End}(A), T_L)$ is compact.

(ii) $(\text{End}(A), T_L)$ is locally compact.

(iii) $(\text{End}(A), T_{\text{fin}})$ is compact.

A ring topology $T$ on a ring $\text{End}(A)$ is called admissible (see [1]) if $T \geq T_{\text{fin}}$.

**Theorem 4.** Let $A$ be a countable $p$-elementary group. Then the ring $R = \text{End}(A)$ has a nonadmissible topology.

Let $\mathcal{A}$ be the class of all abelian groups. Assume that, for every group $A \in \mathcal{A}$, there is defined a topology $t(A)$ under which $A$ is a topological group (not necessarily Hausdorff). We call $t = \{t(A)|A \in \mathcal{A}\}$ a functorial topology (see [5], p.33) if every homomorphism in $\mathcal{A}$ is continuous. An important example of a functorial topology is the Bohr topology on abelian groups. Recall that if $A$ is an abelian group, then the Bohr topology $T_{\text{Bohr}}$ is the largest precompact topology on $A$. It is well known that $(A, T_{\text{Bohr}})$ is Hausdorff. For further information about Bohr topologies on abelian groups see in [14].

**Theorem 5.** Let $(A, T)$ be a functorial topology on a group $A$ and $\mathcal{B}$ the set of all neighborhoods of zero of $(A, T)$. For each $V \in \mathcal{B}$ the set $\{\alpha \in \text{End}(A)|\alpha(A) \subset V\}$ is denoted by $P(V)$. 
Then the family $\{ P(V) \}_{V \in \mathfrak{B}}$ defines a Hausdorff right bounded ring topology on $\text{End}(A)$.

**Theorem 6.** Let $A$ be an elementary countable $p$-group and let $\mathcal{T}_{\text{Bohr}}$ be the Bohr topology on $A$. If $U$ is the ring topology on $\text{End}(A)$ associated with $\mathcal{T}_{\text{Bohr}}$, then their supremum $\mathcal{T}_{\text{Bohr}} \lor \mathcal{T}_{\text{fin}}$ is a nonmetrizable admissible topology.

Now we can ask the following natural question.

**Question 3.** Under which condition on the abelian group $A$ is the ring topology $U$ on $\text{End}(A)$ discrete? metrizable?

A particular answer gives the next result.

**Theorem 7.** Let $A$ be one of the following groups:

(i) a torsion-free group of cardinality $\leq 2^{\aleph_0}$;
(ii) a free group of cardinality $> 2^{\aleph_0}$.

Then $(\text{End}(A), U)$ is discrete in the case (i) and non-discrete in the case (ii).

2. Preliminaries

The symbol $\omega$ stands for the set of all natural numbers including zero. The set of all prime natural numbers is denoted by $\mathbb{P}$. All rings are assumed to be associative with identity. We denote the Jacobson radical of a ring $R$ by $\mathfrak{J}(R)$. All modules are unitary right modules. By $\overline{A}$ we denote the closure of the subset $A$ of a topological space.

An abelian group $A$ is called bounded if there exists a natural number $n$ such that $na = 0$ for all $a \in A$. We denote the cyclic group of finite order $n$ with additive operation by $\mathbb{Z}(n)$.

We freely use facts about summable families in $(\text{End}(A), \mathcal{T}_{\text{fin}})$ (see [2], Chapter III, §5): If $A$ is an abelian group and $K$ is a finite subset of $A$, then put

$$T(K) = \{ \alpha \in \text{End}(A) \mid \alpha K = 0 \}.$$  

The finite topology on $\text{End}(A)$ is given by the family $\{ T(K) \}$, where $K$ runs over all finite subsets of $A$ as a fundamental system of neighborhoods of zero.

A family $\{ x_i \}_{i \in I}$ of elements in $\text{End}(A)$ is called summable (see [2], p. 80) if there exists $x \in \text{End}(A)$ such that for any finite subset $K$ of $A$ there exists a finite subset $I_0 \subset I$ such that $\Sigma_{i \in I_1} x_i \in x + T(K)$ for any finite subset $I_1$ containing $I_0$. 

It is well known (see [2], Theorem 1, p. 82) that a family \( \{ x_i \}_{i \in I} \) of \( \text{End}(A) \) is summable in \( (\text{End}(A), \mathcal{T}_{fin}) \) if and only if for each \( a \in A \) there exists a finite subset \( I_0 \) of \( I \) such that \( x_i(a) = 0 \) for all \( i \notin I_0 \).

All necessary notions from the theory of abelian groups, ring theory and the theory of topological groups can be found in [5], [9] and [2, 12], respectively.

3. Proofs

Proof of Theorem 1. Put \( \mathfrak{J}_M = \mathfrak{J}(\text{End}(M_R)) \). It is well known (see [9], §4.4) that \( \mathfrak{J}_M \) consists of endomorphisms with large kernels. Assume that there exists \( \alpha \in \mathfrak{J}_M \setminus \mathfrak{J}_M \). Since \( \ker(\alpha) \) is not large, there exists \( 0 \neq m \in M \) such that \( \ker(\alpha) \cap mR = 0 \). Let \( \gamma \in T(m) \) such that \( \alpha + \gamma = \beta \in \mathfrak{J}_M \). Then \( \ker(\beta) \cap mR \neq 0 \). If \( s \in R \) such that \( 0 \neq ms \in \ker(\beta) \), then \( \alpha(ms) = 0 \), hence \( 0 \neq ms \in \ker(\alpha) \cap mR \), a contradiction. \( \square \)

Recall that the set of prime natural numbers is \( \mathbb{P} \).

Lemma 1. Let \( A = \mathbb{Z}(p^m) \oplus \mathbb{Z}(p^n) \), where \( p \in \mathbb{P} \) and \( m < n \). Let \( \tau : \mathbb{Z}(p^m) \to p^{n-m} \mathbb{Z}(p^n) \subset \mathbb{Z}(p^n) \) be an embedding homomorphism. Then

\[
\alpha_{mn} : (x, y) \mapsto (0, \tau(x)) \in \mathfrak{J}(\text{End}(A)).
\]

Proof. Since \( \alpha_{mn}(A) \subset p^{n-m}A \) by the definition of \( \alpha_{mn} \), we obtain that \( \beta \alpha_{mn}(A) \subset p^{n-m}A \) for each \( \beta \in \text{End}(A) \). By induction on \( k \in \omega \), we get that

\[
(\beta \alpha_{mn}) \cdots (\beta \alpha_{mn}) \in p^{(n-m)k}A.
\]

This implies that \( (\text{End}(A))\alpha_{mn} \) is a nilpotent left ideal, hence \( \alpha_{mn} \in \mathfrak{J}(\text{End}(A)) \). \( \square \)

Lemma 2. Let \( A = \bigoplus_{i \in \omega} \mathbb{Z}(p^{k_i}) \), where \( p \in \mathbb{P} \) and \( k_i < k_{i+1} \) for \( i \in \omega \). Then \( \mathfrak{J}(\text{End}(A)) \) is not closed.

Proof. Let \( \beta_i = \alpha_{k_ik_{i+1}} \) be the endomorphism in Lemma 1 where \( i \in \omega \). We extend \( \beta_i \) to an endomorphism \( \gamma_i \) of \( A \), setting \( \gamma_i = \beta_i \) on \( B_i = \mathbb{Z}(p^{k_i}) \oplus \mathbb{Z}(p^{k_{i+1}}) \) and \( \beta_i(\mathbb{Z}(p^{k_l})) = 0 \) for \( l \neq i \) or \( l \neq i + 1 \).

The projection of \( A \) onto \( B_i \) is denoted by \( e_i \). Consider the embedding \( \psi_i : \text{End}(B_i) \to \text{End}(A) \), where

\[
\alpha \mapsto \psi_i(\alpha) = \begin{cases} 
\alpha & \text{if } \alpha \in B_i; \\
0 & \text{if } \alpha \notin B_i.
\end{cases}
\]
Thus $\psi_i(End(B_i)) = e_i(End(A))e_i$. Furthermore,

$$\gamma_i = \psi_i(\beta_i) \in \mathcal{J}(e_i(End(A))e_i) = \mathcal{J}(End(A)) \cap e_i(End(A))e_i \subset \mathcal{J}(End(A)).$$

We claim that the family $\{\gamma_i\}_{i \in \omega}$ is summable. For if $a \in A$, then there exists $n \in \omega$ such that $a \notin B_i$ for $i > n$. Then $\gamma_i(a) = 0$ for $i > n$, hence $\{\gamma_i\}_{i \in \omega}$ is summable.

Let $\gamma = \sum_{i \in \omega} \gamma_i$. We claim that $\beta = \sum_{i \in \omega} \beta_i$ is not right quasi-regular. Assume on the contrary that there exists $\beta' \in End(A)$ such that $\beta + \beta' + \beta\beta' = 0$. Let $x = (\theta 00\ldots) \in A$ and let $\beta'(\theta 00\ldots) = (x_0x_1\cdots)$, where $\theta$ is a generator of $\mathbb{Z}(p^\infty)$. Then

$$(0\theta 0\ldots) + (x_0x_1x_2\cdots) + (0x_0x_1\cdots) = 0,$$

so $x_0 = 0$, $x_1 = -\theta$, $\ldots$, $x_n = (-1)^n\theta$, $\ldots$ ($n \geq 2$), a contradiction.

The last property implies that the ideal $\mathcal{J}(End(A))$ is not closed. □

**Lemma 3.** If $A = \bigoplus_{\alpha \in \Omega} H_\alpha$, where $H_\alpha = \mathbb{Z}(p^n)$ for all $\alpha \in \Omega$, then $\mathcal{J}(End(A)) = pEnd(A)$ and $\mathcal{J}(End(A))^n = 0$.

**Proof.** Since $(pEnd(A))^n = 0$, we have $pEnd(A) \subset \mathcal{J}(End(A))$. Denote a generator of $\mathbb{Z}(p^n)_\alpha$ by $\theta_\alpha$, where $\alpha \in \Omega$.

Assume that there exists an element $j \in \mathcal{J}(End(A)) \setminus pEnd(A)$. Then there exists $\beta \in \Omega$ such that $j(\beta) \notin pA$. Indeed, otherwise for each $\alpha \in \Omega$ there exists $a_\alpha \in A$ such that $j(\alpha) = pa_\alpha$. The mapping $q$ from $\{\theta_\alpha\}_{\alpha \in \Omega}$ in $A$ which sends $\theta_\alpha$ in $a_\alpha$ has an extension to an endomorphism $q_1$ of $A$ and $j = pq_1$, a contradiction.

Let $j(\beta) = k_1\theta_\alpha + \cdots + k_m\theta_{\alpha_m}$. We can assume without loss of generality that $(k_1,p) = 1$. There exists a natural number $k$ such that $kk_1\theta_\alpha = \theta_{\alpha_1}$. Then $kj \in \mathcal{J}(End(A))$ and $kj(\alpha) = \theta_{\alpha_1} + \cdots + k_m\theta_{\alpha_m}$.

Clearly we can assume that $j(\beta) = \theta_{\alpha_1} + \cdots + k_m\theta_{\alpha_m}$.

Let $j_1 \in End(A)$, such that $j_1(\alpha_1) = \beta$ and $j_1(\alpha_\gamma) = 0$ for $\gamma \neq \alpha_1$. Define $j_2 \in End(A)$, such that $j_2(\beta) = \theta_\beta$ and $j_2(\alpha_\gamma) = 0$ for $\gamma \neq \beta$. Then

$$j_1j_2(\beta) = j_1j(\beta) = j_1(\theta_{\alpha_1} + k_2\theta_{\alpha_2} + \cdots + k_m\theta_{\alpha_m}) = \theta_\beta$$

and $j_1j_2(\gamma) = 0$ for $\gamma \neq \alpha$. Therefore $0 \neq j_2 = j_1j_2 \in \mathcal{J}(End(A))$ is an idempotent, a contradiction. □

**Lemma 4.** If $A$ is a bounded abelian group, then $\mathcal{J}(End(A))$ is a nilpotent ideal.
Proof. We can assume without loss of generality that $A$ is a $p$-group. It follows from Prüfer’s theorem (see [5], Theorem 17.3, p. 88) that there exists a decomposition $A = A_{n_1} \oplus \cdots \oplus A_{n_k}$ such that $n_1 > \cdots > n_k$ and $A_{n_i}$ is a direct sum of copies of $\mathbb{Z}(p^{n_i})$ for $1 \leq i \leq k$.

We prove our lemma by induction on $k$. For $k = 1$ the statement of the lemma follows from Lemma 3.

Assume by induction that our lemma has been proved for $k - 1$. Consider the abelian groups $B = A_{n_1} \oplus \cdots \oplus A_{n_{k-1}}$ and $C = A_{n_k}$ and also the following matrix rings

$$U = \begin{pmatrix} \text{Hom}(B,B) & \text{Hom}(C,B) \\ \text{Hom}(B,C) & \text{Hom}(C,C) \end{pmatrix}$$
and $$W_U = \begin{pmatrix} \mathfrak{J} (\text{End}(B)) & \text{Hom}(C,B) \\ \text{Hom}(B,C) & \mathfrak{J} (\text{End}(C)) \end{pmatrix}.$$

Clearly, $A = B \oplus C$ and $\text{End}(A) \cong U$.

We claim that the Jacobson radical $\mathfrak{J}(U)$ is $W_U$.

First we prove the following two inclusions:

$$\text{Hom}(C,B) \text{Hom}(B,C) \subset \mathfrak{J}(\text{End}(B));$$

$$\text{Hom}(B,C) \text{Hom}(C,B) \subset \mathfrak{J}(\text{End}(C)).$$

Indeed, let $\alpha \in \text{Hom}(C,B)$, $\beta \in \text{Hom}(B,C)$ and $b \in B$. This yields that $\beta(b) \in C$, hence $\alpha \beta(b) \in pB$. Therefore $\alpha \beta(B) \subset pB$.

If $\gamma \in \text{End}(B)$, then $\gamma \beta \alpha(B) \subset pB$. It follows that $(\gamma \beta \alpha)^n = 0$. This implies that $\alpha \beta \in \mathfrak{J}(\text{End}(B))$, hence the first equation of (1) is proved.

Now, let $\alpha \in \text{Hom}(B,C)$, $\beta \in \text{Hom}(C,B)$ and $c \in C$. Clearly, then $\beta(c) \in pB$, so $\alpha \beta(b) \in pC$. We prove in a similar way that

$$\alpha \beta \in \mathfrak{J}(\text{End}(C)).$$

It is easy to see that the map

$$\psi : U \to \left( \text{End}(B)/\mathfrak{J}(\text{End}(B)) \right) \times \left( \text{End}(C)/\mathfrak{J}(\text{End}(C)) \right),$$

defined by

$$\left( \begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix} \right) \mapsto \left( \alpha_{11} + \mathfrak{J}(\text{End}(B)), \alpha_{22} + \mathfrak{J}(\text{End}(C)) \right).$$

is a ring homomorphism, by (1). Moreover $\mathfrak{J}(U) \subset \text{Ker} (\psi) = W_U$.

It suffices to show that the ideal $W_U$ is nilpotent.

Obviously $T_1 = \begin{pmatrix} \mathfrak{J}(\text{End}(B)) & 0 \\ \text{Hom}(B,C) & 0 \end{pmatrix}$ is a left ideal of the ring $U$ and from the inductive assumption we get that it is nilpotent of class $2n$. Similarly the same is true for the left ideal $T_2 = \begin{pmatrix} 0 & \text{Hom}(C,B) \\ \text{Hom}(B,C) & \mathfrak{J}(\text{End}(C)) \end{pmatrix}$. This yields that $U_W = T_1 + T_2$ is a nilpotent ideal of the ring $U$. \qed
Corollary 1. The Jacobson radical of the ring of endomorphisms of a bounded abelian group is closed.  

Lemma 5. If \( A \) is a free group, then \( \mathfrak{J}(\text{End}(A)) = 0 \).

Proof. Put \( \mathfrak{J} = \mathfrak{J}(\text{End}(A)) \). Assume on the contrary that \( \mathfrak{J} \neq 0 \). Let \( \{a_\alpha\}_{\alpha \in \Omega} \) be a basis of \( A \) and let \( e_\alpha \in \text{End}(A) \) such that \( e_\alpha(a_\beta) = \delta_{\alpha\beta}a_\alpha \), \( (\beta \in \Omega) \) where \( \delta_{\alpha\beta} \) is the Kronecker \( \delta \) function.

Clearly \( e_\alpha(\text{End}(A))e_\alpha \cong \mathbb{Z} \) and \( e_\alpha(\mathfrak{J})e_\alpha = 0 \). Let \( b \in A \) and \( g \in \text{End}(A) \) such that \( g(a_\alpha) = b \) for a fix \( \alpha \in \Omega \). Thus

\[
e_\alpha \mathfrak{J} g e_\alpha(a) = e_\alpha \mathfrak{J}(b) = 0,
\]

hence \( e_\alpha \mathfrak{J} = 0 \). Let \( 0 \neq j \in \mathfrak{J}, \beta \in \text{End}(A) \) and \( h \in \text{End}(A) \), such that \( j(a_\beta) = k_1a_{\alpha_1} + \cdots \), where \( k_1 \neq 0 \), \( h(a_{\alpha_1}) = a_\alpha \) and \( h(a_\gamma) = 0 \) for all \( \gamma \neq \alpha_1 \). Then \( e_\alpha h j(a_\beta) = k_1a_\alpha \neq 0 \), a contradiction. \( \square \)

Lemma 6. Let \( M = R \otimes S \) be an \((R,S)\)-bimodule, where \( R \) and \( S \) are rings. Put \( U = (R \otimes S) \) and \( W_U = (\mathfrak{J}(R) \otimes S) \). Then \( \mathfrak{J}(U) = W_U \).

Proof. Clearly \( W_U \) is a quasiregular ideal of the matrix ring \( U \), hence is contained in \( \mathfrak{J}(U) \). The map

\[
\Gamma : (r \otimes s) \mapsto (r + \mathfrak{J}(R), s + \mathfrak{J}(S))
\]

is a ring homomorphism \( U \rightarrow (R/\mathfrak{J}(R)) \times (S/\mathfrak{J}(S)) \) with \( \text{Ker}(\Gamma) = W_U \).

This implies that \( W_U \subset \mathfrak{J}(U) \). The proof is complete. \( \square \)

Lemma 7. Let \( A = B \oplus C \), where \( B \) is a free group and \( C \) is a torsion group. Then

(i) \( \text{End}(A) \cong_{\text{top}} \begin{pmatrix} \text{End}(A) & 0 \\ \text{Hom}(B,C) & \text{End}(C) \end{pmatrix} \), where \( \text{End}(A), \text{Hom}(B,C), \) and \( \text{End}(C) \) are endowed with the finite topology;

(ii) \( \mathfrak{J} \left( \begin{pmatrix} \text{End}(B) & 0 \\ \text{Hom}(B,C) & \text{End}(C) \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & 0 \\ \text{Hom}(B,C) & \mathfrak{J}(\text{End}(C)) \end{pmatrix} \right) \);

(iii) \( \mathfrak{J}(\text{End}(A)) \) is closed if and only if \( \mathfrak{J}(\text{End}(C)) \) is closed.

Proof. (i) Follows from Lemma 1 in \([13]\). (ii) Follows from Lemma \([6]\). (iii) Since \( \text{Hom}(B,C) \) is always closed, the assertion follows from (ii). \( \square \)

Proof of Theorem \([2]\) Let \( A = B \oplus C \), where \( B \) is a free group and \( C \) is a torsion group. The proof follows from Lemmas \([11,5,7]\) \( \square \)

\(^1\)Closedness of \( \mathfrak{J}(\text{End}(A)) \) can be deduced from the results from \([11]\), but we are giving a different proof.
In the sequel $A$ denotes an abelian $p$-group. Denote $A_n = A[p^n]$ for each $n \in \mathbb{N}$. Thus the family $\{T(A_n)\}$ is a fundamental system of neighborhoods of zero of a Hausdorff ring topology $\mathcal{T}_L$ on $End(A)$. This topology is called the Liebert topology. We note that each $T(A_n)$ is an ideal of $End(A)$. Indeed, if $\alpha \in End(A)$ and $a \in A_n$, thus $\alpha(a) \in A[p^n]$, hence $T(A_n)\alpha(A_n) = 0$, i.e. $T(A_n)\alpha \subset T(A_n)$.

**Lemma 8.** The ring $(End(A), \mathcal{T}_L)$ is a complete 1st countable topological ring.

**Proof.** Indeed, $T(A_n) = \cap_{a \in A_n} T(a)$, hence $T(A_n)$ is closed in the finite topology. Since $End(A)$ endowed with the finite topology is complete, it is complete with respect to the Liebert topology. □

**Corollary 2.** For any $p$-group $A$ the ring $(End(A), \mathcal{T}_L)$ is the inverse limit of discrete rings $End(A)/T(A_n)$.

**Lemma 9.** (see [10] or [5], Vol. II, p. 224, Exercise 8) For any $p$-group the Jacobson radical of $(End(A), \mathcal{T}_L)$ is closed.

**Proof.** Follows from Lemma 8 and Corollary on the page 46 from [8]. □

**Lemma 10.** Let $A$ be an abelian $p$-group. Then $(End(A), \mathcal{T}_L)$ is discrete if and only if $A$ is a bounded group.

**Proof.** There exists $n \in \mathbb{N}$ such that $T(A_n) = 0$. Obviously, $A$ is a reduced group. If $A$ is unbounded, then $A$ contains a subgroup isomorphic to $\mathbb{Z}(p^k)$ as a direct summand, so $A = \mathbb{Z}(p^k) \oplus B$, where $k > n$. Set $\alpha \in End(A)$, $\alpha \in T(B)$, $\alpha(x) = p^{k-n}\mathbb{Z}(p^k)$. Thus $\alpha \neq 0$ and $\alpha \in T(A_n)$, a contradiction. □

Recall (see [11]), that a ring topology $\mathcal{T}$ on a ring $End(A)$ is called admissible if $\mathcal{T} \geq \mathcal{T}_{fin}$.

**Lemma 11.** The Liebert topology on $End(A)$ is admissible.

**Proof.** We have to show that the Liebert topology $\mathcal{T}_L$ is stronger than the finite topology $\mathcal{T}_{fin}$. Let $K$ be a finite subset of $A$. There exists $n \in \mathbb{N}$ such that $p^nK = 0$. Then $K \subset A_n$, hence $T(K) \supset T(A_n)$. □

**Proof of Theorem 3** (ii) $\Rightarrow$ (iii) There exists $n \in \mathbb{N}$ such that $T(A_n)$ is compact in $(End(A), \mathcal{T}_L)$. Since $\mathcal{T}_L$ is admissible, $T(A_n)$ is compact in $(End(A), \mathcal{T}_{fin})$.

We claim that the maximal divisible subgroup $D$ of $A$ is the direct sum of a finite number of copies of $\mathbb{Z}(p^\infty)$. Assume on the contrary that $A = B \oplus C$, where $B = \mathbb{Z}(p^\infty) \oplus \cdots \oplus \mathbb{Z}(p^\infty) \oplus \cdots$. Consider the shift homomorphism $\alpha : B \rightarrow B$, where $(x_1x_2x_3 \cdots) \mapsto (0x_1x_2 \cdots)$. Extend
\(\alpha\) to an endomorphism of \(A\) setting \(\alpha(C) = 0\) and \(\alpha_{1_B} = \alpha\). Then \(p^a\alpha^m \in T(A_n)\) for \(m \in \mathbb{N}\). It is easy to see that the elements \(p^a\alpha^ma\) are different for all \(m \in \mathbb{N}\), where \(a = \left(\frac{1}{p^a} + \mathbb{Z}\right)0\left(\ldots\right)\) (here the group \(\mathbb{Z}(p^\infty)\) is the subgroup of \(\mathbb{R}/\mathbb{Z}\) generated by the subset \(\{\frac{1}{p^n} + \mathbb{Z} \mid n \in \mathbb{N}\}\)). Thus \(T(A_n)\) will not be compact in the finite topology, a contradiction.

Let \(A = B \oplus C\), where \(C\) is a direct sum of a finite number of copies of \(\mathbb{Z}(p^\infty)\) and \(C\) is reduced. We prove that \(B\) is finite. Let \(B' = \oplus_{i \geq 1} (\bigoplus_m \mathbb{Z}(p^i))\) be a basic subgroup of \(B\). Obviously, \(B'\) is unbounded. Let \(i_0 \in \mathbb{N}\) such that \(m_{i_0} \geq 1\) and \(i_0 \geq n + 1\). There exists a subgroup \(K\) of \(A\) such that \(A = \mathbb{Z}(p^{i_0}) \oplus K\). Let \(a\) be a generator of \(\mathbb{Z}(p^{i_0})\). If \(i_1 > i_0\) and \(m_{i_1} \geq 1\), then there exists an embedding \(q_{i_1}\) of \(\mathbb{Z}(p^{i_0})\) in \(\mathbb{Z}(p^{i_1}) \subset \bigoplus m_{i_1} \mathbb{Z}(p^{i_1})\). Extend \(q_{i_1}\) to an endomorphism of \(A\) in the obvious way and keep the same notation. Then \(p^a q_{i_1} \in T(A_n)\) and the set \(\{p^a q_{i_1}(a) \mid i_1 \geq 1, m_{i_1} \geq 1\}\) is infinite, a contradiction.

**(iii) \(\Rightarrow\) (i)** If \((\text{End}(A), \mathcal{T}_{\text{fin}})\) is compact, it is easy to show that \((\text{End}(A), \mathcal{T}_L)\) is compact. \(\square\)

**Remark 1.** Let \(R\) be a ring. If \(e\) is an idempotent of \(R\), then

\[\text{Ann}_r(e) = R(1 - e) = \{x - ex \mid x \in R\}.\]

**Proof of Theorem 4.** Let \(\mathcal{T}\) be the ring topology having a fundamental system of neighborhoods of zero \(\text{Ann}_r(K)\), where \(K\) is a finite subset of \(R\). Let us show that \(\mathcal{T}\) is not admissible. Note that the family \(\{\text{Ann}_r(e)\}\) is a fundamental system of neighborhoods of zero of \((R, \mathcal{T})\), where \(e\) runs over all idempotents of \(I_\omega\). Indeed, \(Kx = 0\) if and only if \(RKx = 0\) for any finite subset \(K \subset R\). Since \(R\) is regular, \(RK = Re\) for an idempotent \(e \in R\). Obviously, \(e \in K\).

Now we will indicate a suitable fundamental system of neighborhoods of zero of \((R, \mathcal{T}_{\text{fin}})\). Consider a basis \(\{v_n \mid n \in \omega\}\) over \(\mathbb{F}_p\). If \(e \in I_\omega\), then \(\text{Ann}_r(e) \in \mathcal{T}_{\text{fin}}\).

Indeed, \(T(eA) = \text{Ann}_r(e)\). If \(K\) is a finite subset of \(A\), then \(T(K) = T(\{K\}) = \text{Ann}_r(e)\), where \(e\) is an idempotent of \(R\) such that \(e|_{\{K\}} = id_{\{K\}}\) and \(e(K') = 0\) for some complement \(K'\) of \(K\).

We claim that \(\mathcal{T}\) is not admissible. Assume the contrary. Let \(0 \neq e \in I_\omega\). Then there exists an idempotent \(f \in I_\omega\) such that

\[(1 - f)R = \text{Ann}_r(f) \subset \text{Ann}_r(e).\]

Thus \((1 - e)Rf = 0\). Since \(R\) is a prime ring, either \(1 - f = 0\) or \(e = 0\), a contradiction. \(\square\)

**Proof of Theorem 5.** Each set \(P(V)\) is a right ideal of the multiplicative semigroup of \(\text{End}(A)\). Indeed, \(\alpha\beta(A) \subset \alpha(A) \subset V\) for each \(\alpha \in P(V)\).
and $\beta \in \text{End}(A)$, hence $P(V)$ is a right ideal of the multiplicative semigroup of $\text{End}(A)$.

Furthermore, if $V_1, V_2 \in \mathcal{B}$ and $V_1 \subset V_2$, then $P(V_1) \subset P(V_2)$. Therefore, if $V_1, V_2, V_3 \in \mathcal{B}$ and $V_3 \subset V_1 \cap V_2$, then $P(V_3) \subset P(V_1) \cap P(V_2)$.

If $V_1, V_2 \in \mathcal{B}$ and $V_2 - V_2 \subset V_1$, then $P(V_2) - P(V_2) \subset P(V_1)$. If $\alpha \in \cap_{\mathcal{B}} P(V)$, then $\alpha(A) \cap \cap_{\mathcal{B}} V = 0$, hence $\alpha = 0$.

It follows that the family $\{P(V)\}_{V \in \mathcal{B}}$ defines a Hausdorff right bounded ring topology on $\text{End}(A)$.

Proof of Theorem 6. The family $T(K)$ when $K$ runs over all finite subgroups of $A$, forms a fundamental system of neighborhoods of zero of $(\text{End}(A), \mathcal{T}_{\text{fin}})$. Indeed, $T(K) = T([K])$ and $[K]$ is a finite subgroup of $A$ for every finite subset $K$ of $A$.

Note that $\mathcal{T}_{\text{Bohr}}$ is not admissible. Indeed, let $A = \langle a \rangle \oplus B$ and let $P(V) \subset T(a)$ for some cofinite subgroup $V$ of $A$, where $a \neq 0$. There exists $\alpha \in \text{End}(A)$ such that $\alpha(A) \subset V$ and $\alpha(a) \neq 0$, hence $\alpha \in P(V \setminus T(a))$, a contradiction.

Let us prove that $(A, \mathcal{T}_{\text{Bohr}} \vee \mathcal{T}_{\text{fin}})$ is nonmetrizable. Assume on the contrary that there exists a fundamental system $\{P(V_i) \cap T(K_i)\}_{i \in \omega}$ of neighborhoods of zero, where each $V_i$ is a cofinite subgroup of $A$ and each $K_i$ is a finite subgroup of $A$. Consider for each $i \in \omega$ the decomposition $A = V_i' \oplus (V_i \cap K_i) \oplus K_i' \oplus W_i$, where $V_i' \oplus (V_i \cap K_i) = V_i$ and $K_i = (V_i \cap K_i) \oplus K_i'$. We note that each $V_i'$ is infinite.

Construct by induction the sequence $\{a_i \in V_i' \mid i \in \omega\}$ of linearly independent system of vectors over $\mathbb{F}_p$. Let $\lambda : A \to \langle a_0 \rangle$, where $a_k \mapsto a_0$ for all $k \in \omega$. Then $W$ is an open subgroup of $(A, \mathcal{T})$. By assumption there exists $i \in \omega$ such that $P(V_i) \cap T(K_i) \subset P(W)$. Let $\gamma \in P(V_i) \cap T(K_i)$, $\gamma|_{V_i'} = \text{id}_{V_i'}$ and

$$\gamma[(V_i \cap K_i) \oplus K_i' \oplus W_i] = 0.$$  

Thus $\gamma \in T(K_i)$ and $\gamma(A) \subset V_i' \subset V_i$, hence $\gamma \in P(V_i) \cap T(K_i)$. It follows that $\gamma \in P(W)$, hence $\gamma(a_i) = a_i \in W$ and so $0 = \lambda(a_0) = a_0$, a contradiction.

Proof of Theorem 7. (i) Embed $A$ in $\mathbb{T}$. Then $A$ is a dense subgroup of $\mathbb{T}$, because $A$ is endowed with the topology inherited from the compact topology on $\mathbb{T}$. Consequently $A$ contains a neighborhood $V$ of zero which does not contain a nonzero subgroup. Then $P(V) = 0$, hence $(\text{End}(A), \mathcal{U})$ is discrete.

(ii) Every neighborhood of zero of $(A, \mathcal{T}_{\text{Bohr}})$ contains a non-zero subgroup. Assume the contrary. Let $V$ be a neighborhood of zero of $(A, \mathcal{T}_{\text{Bohr}})$ which does not contain any non-zero subgroup. Set $V_0 = V$ and construct by induction closed symmetric neighborhoods of zero of
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Let \((A, \mathcal{T}_{Bohr})\) such that \(V_n + V_n \subset V_{n-1}\) for \(n \in \omega\). The closure \(\overline{V_n}\) of each \(V_n\) in \(\hat{A}\), where \(\hat{A}\) is the completion of \((A, \mathcal{T}_{Bohr})\), is a neighborhood of zero of \(\hat{A}\). Then \(K = \cap \overline{V_n}\) is a compact subgroup of \(\hat{A}\) and \(\hat{A}/K\) is metrizable.

Let \(\phi : \hat{A} \to \hat{A}/K\) be the canonical homomorphism. We have that \(K \cap A \subset \overline{\cap A} = V\), hence \(K \cap A = 0\). It follows that \(\hat{A}/K\) contains an isomorphic copy of \(A\), hence \(|\hat{A}/K| > 2^{\aleph_0}\), a contradiction. \(\square\)

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