Activated zero-error classical communication over quantum channels assisted with quantum no-signalling correlations

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We study the activated quantum no-signalling-assisted zero-error classical capacity by first allowing the assistance from some noiseless forward communication channel and later paying back the cost of the helper. This activated communication model considers the additional forward noiseless channel as a catalyst for communication. First, we show that the one-shot activated capacity can be formulated as a semidefinite program and we derive a number of striking properties of this capacity. We further present a sufficient condition under which a noisy channel can be activated. Second, we find that one-bit noiseless classical communication is able to fully activate any classical-quantum channel to achieve its asymptotic capacity, or the semidefinite (or fractional) packing number. Third, we prove that the asymptotic activated capacity cannot exceed the original asymptotic capacity of any quantum channel. We also show that the asymptotic no-signalling-assisted zero-error capacity does not equal to the semidefinite packing number for quantum channels, which differs from the case of classical-quantum channels.

1 Introduction

A fundamental problem of information theory is to determine the capacity of a communication channel, which describes the capability of the channel for delivering information from the sender to the receiver. Shannon first discussed this problem in the zero-error setting and described the zero-error capacity of a channel as the maximum rate at which it can be used to transmit information perfectly [1]. It is well-known that the Shannon zero-error capacity is extremely difficult to compute even for very simple classical channels. Nevertheless, this capacity is upper bounded by the Lovász $\vartheta$ function [2] which is efficiently computable by semidefinite programming [3].

Recently the zero-error information theory has been studied in the quantum setting and many interesting phenomena were observed. For instance, it was shown that shared entanglement could sometimes improve the zero-error capacity of a classical channel [4, 5],

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and the entanglement-assisted zero-error channel coding was studied in [6–8]. Furthermore, both the zero-error classical and quantum capacities can be super-activated [9–12]. Another notable fact is that the entanglement-assisted zero-error capacity of a classical channel is also upper-bounded by the Lovász ϑ function [13, 14], and this result can be generalized to quantum setting by using a quantum version of Lovász ϑ function [14]. Recently the separation between the quantum Lovász ϑ function and entanglement-assisted zero-error capacity of quantum channels was shown in [15], while the case of classical channels remains unknown.

As more general resources, the no-signaling (NS) correlations have been considered to assist the zero-error communication in [20, 21], respectively. The no-signalling correlations arise in the research of the relativistic causality of quantum operations [16–19]. Cubitt et al. [20] first introduced classical no-signalling correlations into the zero-error classical communication. One of us and Winter [21] further introduced quantum no-signalling correlations into the zero-error communication problem. They formulated the one-shot capacity as a semidefinite program (SDP) which depends only on the non-commutative bipartite graph of the given channel (an operator space that is given by the linear span of the Choi-Kraus operators of the channel). Furthermore, Duan, Severini, and Winter studied the zero-error communication via quantum channel assisted by unlimited noiseless feedback and showed the induced capacity also depends only on the non-commutative bipartite graph [32].

In this paper, we further develop the theory of quantum NS-assisted communication by introducing the activated communication model. The model is introduced in Section 3 and it considers the additional forward noiseless channel as a catalyst for communication. For a quantum channel \( \mathcal{N} \), we can “borrow” a noiseless classical channel \( \mathcal{I} \), then we can use \( \mathcal{N} \otimes \mathcal{I} \) to transmit information. After the communication finishes we “pay back” the capacity of \( \mathcal{I} \). This kind of communication method was suggested in [31], and was highly relevant to the notion of potential capacity recently studied by Winter and Yang [33]. We further show that the one-shot activated zero-error capacity can also be formulated as an SDP. In Section 4, we show a striking result that one bit can even fully activate any cq channel to achieve its asymptotic NS-assisted zero-error capacity (or the fractional packing number). In Section 5, we further show that there is no activation in the asymptotic regime and the one-shot activated capacity is better than the super-dense coding bound in [21].

2 Zero-error communication over quantum channels

A quantum channel \( \mathcal{N}_{A'\rightarrow B} \) is a completely positive (CP) and trace-preserving (TP) linear map from operators on a finite-dimensional Hilbert space \( A' \) to operators on a finite-dimensional Hilbert space \( B \). The Choi-Jamiołkowski matrix of \( \mathcal{N} \) is \( J^\mathcal{N} = \sum_{ij} |i\rangle\langle j|_A \otimes \mathcal{N}(|i\rangle\langle j|_{A'}) = (\text{id}_A \otimes \mathcal{N})|\Phi\rangle\langle \Phi| \), where \( A \) and \( A' \) are isomorphic Hilbert spaces with respective orthonormal basis \( \{|i\rangle_A\} \) and \( \{|j\rangle_{A'}\} \), \( |\Phi\rangle = \sum_i |i\rangle_A |i\rangle_{A'} \), and \( \text{id}_A \) is the identity map.

In the communication task, Alice wants to send the classical messages to Bob using the composite channel \( \mathcal{M}_{A\rightarrow B'} = \Pi_{AB\rightarrow A'B'} \circ \mathcal{N}_{A'\rightarrow B} \), where \( \Pi \) is a quantum bipartite operation that generalizes the usual encoding scheme \( \mathcal{E} \) and decoding scheme \( \mathcal{D} \) (see Figure. 1). We say such \( \Pi \) is an NS-assisted code if it can be implemented by local operations with quantum no-signalling correlations. Here, the no-signalling constraint means that Alice and Bob cannot use the bipartite operation \( \Pi \) to communicate classical information. The NS-assisted codes have also been applied to study the ordinary classical and quantum communication over quantum channels (e.g., [21–30]).
We denote $M_{0,\text{NS}}(\mathcal{N})$ as the maximum number of bits can be transmitted perfectly over a single use of quantum channel $\mathcal{N}$ with NS-assisted codes, i.e.,

$$M_{0,\text{NS}}(\mathcal{N}) := \sup \{ \log \ell : \Pi \circ \mathcal{N} = \mathcal{I}_\ell, \Pi \text{ is an NS-assisted code} \},$$

where $\mathcal{I}_\ell(\rho) = \sum_{i=0}^{\ell-1} \text{Tr}(\rho|i\langle i|)\langle i|i|$ is the classical noiseless channel. Also, throughout this paper, $\log$ denotes the binary logarithm $\log_2$.

As showed in [21], for a channel $\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger$, the one-shot NS-assisted zero-error capacity $M_{0,\text{NS}}(\mathcal{N})$ only depends on the non-commutative bipartite graph $K$ of the channel, i.e.,

$$M_{0,\text{NS}}(\mathcal{N}) = M_{0,\text{NS}}(K),$$

where $K = \text{span}\{E_k\}$ is the linear span of the Choi-Kraus operators of the channel. For a quantum channel $\mathcal{N}$ with non-commutative bipartite graph $K$, the one-shot NS-assisted zero-error capacity is given by [21]

$$M_{0,\text{NS}}(\mathcal{N}) = M_{0,\text{NS}}(K) = \log \Upsilon(K),$$

where $\Upsilon(K)$ is given by the following SDP:

$$\Upsilon(K) = \max \text{ Tr } S_A$$

s.t. $0 \leq U_{AB} \leq S_A \otimes 1_B$,  
$\text{Tr} A U_{AB} = 1_B$,  
$\text{Tr} P_{AB}(S_A \otimes 1_B - U_{AB}) = 0.$

Here, $P_{AB}$ denotes the projection onto the support of the Choi-Jamiołkowski matrix of $\mathcal{N}$, which is uniquely determined by the non-commutative bipartite graph $K$. Due to this fact, we will not distinguish between the notations $\Upsilon(\mathcal{N})$ and $\Upsilon(K)$ in the rest of this paper.

Then by the regularization, the NS-assisted zero-error capacity is

$$C_{0,\text{NS}}(\mathcal{N}) = \sup_{n \geq 1} \frac{1}{n} \log \Upsilon(K^\otimes n).$$

The sup in Eq. (5) can be replaced by $\lim$ based on the lemma about existence of limits in [34].

3 One-shot activated zero-error communication

3.1 Activated one-shot zero-error capacity

The model of activated communication is described as follows. For a quantum channel $\mathcal{N}$ assisted by NS codes, we can first borrow a noiseless classical channel $\mathcal{I}_\ell$ whose capacity is
log \ell, then we can use \( N \otimes I \) coherently to transmit classical messages. After the communication finishes, we just pay back the capacity of \( I \) (see Figure 2.) The communication model follows the idea of potential capacities of quantum channels introduced by Winter and Yang [33].

\[
m \in \{1, \ldots, M\} \rightarrow \mathcal{E} \quad \begin{array}{c}
\vdots \\
I_\ell \\
\mathcal{N} \\
D \\
\end{array} \rightarrow \hat{m} \in \{1, \ldots, M\}
\]

Figure 2: Activated classical communication.

**Definition 1** For a quantum channel \( N \) with non-commutative bipartite graph \( K \), the one-shot activated no-signalling assisted zero-error classical capacity is defined as the following:

\[
\mathcal{M}_{0,NS}^a(N) = \mathcal{M}_{0,NS}^a(K) := \sup_{\ell \geq 1} [\mathcal{M}_{0,NS}(K \otimes \Delta_\ell) - \log \ell],
\]

where \( \Delta_\ell \) is the non-commutative graph of the noiseless channel \( I_\ell(\rho) = \sum_{i=0}^{\ell-1} \text{Tr}(\rho |i\rangle\langle i|) |i\rangle\langle i| \).

**Definition 2** For a quantum channel \( N \) with non-commutative bipartite graph \( K \), the asymptotic activated no-signalling zero-error classical capacity is given the following regularization:

\[
C_{0,NS}^a(N) = C_{0,NS}^a(K) := \sup_{n \geq 1} \frac{1}{n} \mathcal{M}_{0,NS}^a(K^{\otimes n}).
\]

To provide a feasible formulation of the activated capacity \( \mathcal{M}_{0,NS}^a(N) \), let us first introduce a slightly revised SDP of \( \Upsilon(K) \) as follows,

\[
\hat{\Upsilon}(K) = \max \text{Tr} S_A \\
\text{s.t.} \quad 0 \leq U_{AB} \leq S_A \otimes 1_B, \\
\text{Tr}_A U_{AB} \leq 1_B, \\
\text{Tr} P_{AB} (S_A \otimes 1_B - U_{AB}) = 0.
\]

The only difference between \( \hat{\Upsilon}(K) \) and \( \Upsilon(K) \) is that now \( \text{Tr}_A U_{AB} \) is only required to be less than or equal to \( 1_B \), and an equality is not necessary. However, we will see that such a small revision is of crucial importance. The dual SDP of \( \hat{\Upsilon}(K) \) is given by

\[
\hat{\Upsilon}(K) = \min \text{Tr} T_B \\
\text{s.t.} \quad V_{AB} \leq 1_A \otimes T_B, \\
\text{Tr}_B V_{AB} \geq 1_A, \quad T \geq 0, \\
(1 - P)_{AB} V_{AB} (1 - P)_{AB} \leq 0.
\]

Note that by strong duality, the values of both the primal and the dual SDPs coincide.

**Theorem 3** For any quantum channel \( N \) with non-commutative bipartite graph \( K \),

\[
\mathcal{M}_{0,NS}^a(N) = \log \hat{\Upsilon}(K).
\]
The intuition of this theorem is that the additional noiseless channel may play a role as a catalyst during the communication task.

To prove the achievable part, it’s important to observe that the additional noiseless channel indeed provides a larger solution space of \( \Upsilon(K \otimes \Delta_\ell) \). Let us first consider the case \( \ell = 2 \) and assume that the optimal feasible solution of \( \hat{\Upsilon}(K) \) is \( \{S_A, U_{AB}\} \). Let us choose

\[
S_{AA'} = S_A \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)_{A'}
\]

and

\[
U_{AA'BB'} = U_{AB} \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|)_{A'B'} + \bar{U}_{AB} \otimes (|01\rangle\langle 01| + |10\rangle\langle 10|)_{A'B'},
\]

where \( \bar{U}_{AB} = \frac{S_A}{\text{Tr}S_A} \otimes (1_B - \text{Tr}_A U_{AB}) \).

This construction ensures that

\[
\text{Tr}_{AA'} U_{AA'BB'} = \text{Tr}_A((U_{AB} + \bar{U}_{AB}) \otimes 1_{B'}) = 1_{BB'}.
\]

Moreover, we have

\[
S_{AA'} \otimes 1_{BB'} - U_{AA'BB'} = (S_A \otimes 1_B - U_{AB}) \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|)_{A'B'} + (S_A \otimes 1_B - \bar{U}_{AB}) \otimes (|01\rangle\langle 01| + |10\rangle\langle 10|)_{A'B'},
\]

which directly means that

\[
S_{AA'} \otimes 1_{BB'} - U_{AA'BB'} \geq 0.
\]

Also, noting that \( \text{Tr} P_{AB}(S_A \otimes 1_B - U_{AB}) = 0 \) and \( \text{Tr} 1_{A'B'}(|01\rangle\langle 01| + |10\rangle\langle 10|)_{A'B'} = 0 \), we have that

\[
\text{Tr}(P_{AB} \otimes 1_{A'B'})(S_{AA'} \otimes 1_{BB'} - U_{AA'BB'}) = 0.
\]

Therefore, \( \{S_{AA'}, U_{AA'BB'}\} \) is a feasible solution of \( \Upsilon(K \otimes \Delta_2) \), which means that

\[
\sup_{\ell \geq 2} \frac{\Upsilon(K \otimes \Delta_\ell)}{\ell} \geq \frac{\Upsilon(K \otimes \Delta_2)}{2} \geq \frac{\text{Tr} S_{AA'}}{2} = \hat{\Upsilon}(K).
\]

On the other hand, to prove the converse part, we will use the fact that \( \hat{\Upsilon}(K \otimes \Delta_\ell) = \ell \hat{\Upsilon}(K) \), which is provided in the following Lemma 4. This fact directly implies that

\[
\sup_{\ell \geq 2} \frac{\Upsilon(K \otimes \Delta_\ell)}{\ell} \leq \sup_{\ell \geq 2} \frac{\hat{\Upsilon}(K \otimes \Delta_\ell)}{\ell} = \hat{\Upsilon}(K).
\]

Finally, by Eq. (18) and Eq. (19), we can conclude that

\[
\mathcal{M}^{a}_{0,\text{NS}}(N) = \mathcal{M}^{a}_{0,\text{NS}}(K) = \log \hat{\Upsilon}(K).
\]

A simple but useful property of \( \hat{\Upsilon} \) is shown as follows.

**Lemma 4** For any non-commutative bipartite graph \( K \), we have

\[
\hat{\Upsilon}(K \otimes \Delta_\ell) = \ell \hat{\Upsilon}(K).
\]
Proof. On one hand, it is evident from the super-multiplicativity that \( \tilde{\Upsilon}(K \otimes \Delta_\ell) \geq \ell \tilde{\Upsilon}(K) \). On the other hand, note that an optimal solution for SDP (9) for \( \hat{K} \) always holds when \( \Upsilon(K) \). In other words, \( \Upsilon(K) \) is activatable. Indeed, in the previous proof we have shown the following stronger result: \( \Upsilon(K) \leq \text{Tr} T_B \otimes 1_{B'} = \ell \tilde{\Upsilon}(K) \). Therefore, \( \Upsilon(K) \leq \text{Tr} T_B \otimes 1_{B'} = \ell, \forall \ell \geq 2 \).

3.2 Activation via noisy quantum channels

We further discuss the activation via noisy quantum channels.

Proposition 5. Let us consider two quantum channels \( \mathcal{N}_1 \) with non-commutative bipartite graphs \( K_1 \) and \( K_2 \), respectively. If \( \Upsilon(K_2) = 1 \geq \frac{1}{\Upsilon(K_1)} \), then

\[
\mathcal{M}_{0,NS}(K_1 \otimes K_2) - \mathcal{M}_{0,NS}(K_2) \geq \mathcal{M}_{0,NS}^0(K_1).
\]

In other words, \( K_2 \) can activate \( K_1 \) if \( K_1 \) is activatable. In particular, this inequality always holds when \( \Upsilon(K_2) \geq 2 \).

Proof. Let us assume that the optimal solution to the SDP (8) of \( \tilde{\Upsilon}(K_1) \) is \( \{S_A, U_{AB}\} \), while the optimal solution to the SDP (4) of \( \Upsilon(K_2) \) is \( \{S_A', U_{A'B'}\} \).

Then we can choose

\[
S_{AA'} = S_A \otimes S_A',
\]
\[
U_{AA'B'B'} = U_{AB} \otimes U_{A'B'} + \hat{U}_{AB} \otimes V_{A'B'},
\]

where \( V_{A'B'} = (S_A \otimes 1_{B'} - U_{A'B'})/(\text{Tr} S_A - 1) \) and \( \hat{U}_{AB} = S_A/\text{Tr} S_A \otimes (1_B - \text{Tr}_A U_{AB}) \).

Furthermore, we have

\[
S_{AA'} \otimes 1_{BB'} - U_{AA'B'B'} = (S_A \otimes 1_B - U_{AB}) \otimes U_{A'B'} + [(\text{Tr} S_A - 1 - \frac{1}{\text{Tr} S_A}) S_A \otimes 1_B] \otimes V_{A'B'}.
\]

Then, one can check that the constructed solutions satisfy

\[
S_{AA'} \otimes 1_{BB'} - U_{AA'B'B'} \geq 0, \text{Tr}_{AA'} U_{AA'B'B'} = 1_{BB'},
\]
\[
(P_{AB} \otimes P_{A'B'})(S_{AA'} \otimes 1_{BB'} - U_{AA'B'B'}) = 0.
\]

Therefore, \( \{S_{AA'}, U_{AA'B'B}\} \) is a feasible solution to the SDP (4) of \( \Upsilon(K_1 \otimes K_2) \), which means that

\[
\Upsilon(K_1 \otimes K_2) \geq \tilde{\Upsilon}(K_1) \Upsilon(K_2),
\]

\( \square \)

If we only consider using the channel \( \mathcal{N} \) to activate itself, we have the following result from the above proposition.
For any quantum channel $\mathcal{N}$ with non-commutative bipartite graph $K$, if $\Upsilon(K) \geq \frac{1 + \sqrt{5}}{2}$, then
\[
\frac{\Upsilon(K \otimes K)}{\Upsilon(K)} \geq \hat{\Upsilon}(K).
\]
(29)

Note that $\Upsilon(K) \geq \frac{1 + \sqrt{5}}{2}$ means $\Upsilon(K) - 1 - \frac{1}{\Upsilon(K)} \geq 0$. Thus the result follows directly from Proposition 5.

### 4 Activated zero-error capacity of classical-quantum channel

A classical-quantum (cq) channel $\mathcal{N} : i \rightarrow \rho_i (1 \leq i \leq n)$ is a CPTP map with classical inputs $\{i\}_{i=1}^n$ and quantum outputs $\{\rho_i\}_{i=1}^n$. The non-commutative bipartite graph of a cq channel is called a cq graph. Given a cq channel $\mathcal{N} : i \rightarrow \rho_i (1 \leq i \leq n)$ with cq graph $K$, its one-shot NS-assisted zero-error capacity (quantified as messages) can be simplified to
\[
\Upsilon(K) = \max \sum_i s_i \\
\text{s.t. } 0 \leq s_i, 0 \leq R_i \leq s_i(1 - P_i), \\
\sum_i (s_i P_i + R_i) = 1.
\]
(30)

where $P_i$ is the projection onto the support of $\rho_i$ for $1 \leq i \leq n$.

Moreover, it was shown in [21] that the asymptotic no-signalling assisted zero-error classical capacity of a cq channel is equal to the semidefinite (fractional) packing number first suggested by Harrow [21].

**Lemma 6 (Theorem 4 in [21])** For any classical-quantum channel $\mathcal{N} : i \rightarrow \rho_i (1 \leq i \leq n)$ with cq graph $K$,
\[
C_{0,NS}(\mathcal{N}) = \log A(K),
\]
(31)

with
\[
A(K) = \max \sum_i s_i \\
\text{s.t. } 0 \leq s_i, \sum_i s_i P_i \leq 1.
\]
(32)

where $P_i$ the projection onto the support of $\rho_i$ for $1 \leq i \leq n$.

This result is a classical-quantum generalization of the fact that the fractional packing/covering number [1, 35] of the bipartite graph (induced by the classical channel) equals to its NS-assisted zero-error capacity [20]. Moreover, Shannon proved that the feedback-assisted zero-error capacity of a classical channel is also given by the fractional packing number [1].

For any channel $\mathcal{N}$ with cq graph $K$, the one-shot activated capacity $\mathcal{M}_{0,NS}^a(\mathcal{N}) = \log \hat{\Upsilon}(K)$ can be simplified to
\[
\hat{\Upsilon}(K) = \max \sum_i s_i \\
\text{s.t. } 0 \leq s_i, 0 \leq R_i \leq s_i(1 - P_i), \\
\sum_i (s_i P_i + R_i) \leq 1.
\]
(33)
**Theorem 7** For any classical-quantum channel \( N \) with cq graph \( K \),

\[
M^{0,\text{NS}}_{a}(N) = \log A(K). \quad (34)
\]

In other words, for any cq channel, the asymptotic NS-assisted zero-error capacity (or the semidefinite packing number) can be achieved via activated NS codes in the one-shot regime, i.e.,

\[
C^{a}_{0,\text{NS}}(N) = M^{a}_{0,\text{NS}}(N) = \log A(K). \quad (35)
\]

**Proof** First, we will show \( A(K) \geq \hat{\Upsilon}(K) \). Suppose that optimal solution of the SDP (33) of \( \hat{\Upsilon}(K) \) is \( \{ s_i, R_i \} \). Then,

\[
\sum_{i} s_{i} P_{i} \leq 1 - \sum_{i} R_{i} \leq 1, \quad (36)
\]

which means that \( \{ s_{i} \} \) is a feasible solution for \( A(K) \). So we have \( A(K) \geq \hat{\Upsilon}(K) \).

Second, let us assume the optimal solution of SDP (32) is \( \{ s_{i} \} \), let \( R_{i} = 0 \) for all \( i \). It is easy to check that \( \{ s_{i}, R_{i} \} \) is a feasible solution of SDP (33), which means that \( A(K) \leq \hat{\Upsilon}(K) \). Therefore, for any cq graph \( K \), it holds that

\[
\hat{\Upsilon}(K) = A(K). \quad (37)
\]

To see the existence of activation, let us consider an example here.

**Example** For the simplest possible cq channel \( N \), which has only two inputs and two pure output states \( P_{1} = |\psi_{1}\rangle \langle \psi_{1}| \). Without loss of generarity, we assume that \( |\psi_{0}\rangle = \alpha|0\rangle + \beta|1\rangle \) and \( |\psi_{1}\rangle = \alpha|0\rangle - \beta|1\rangle \) with \( \alpha \geq \beta = \sqrt{1 - \alpha^2} \). In [21], it has been solved that \( \Upsilon(K) = 1 \) and \( A(K) = \frac{1}{\alpha^2} \). Hence, by Theorem 7, we know

\[
\hat{\Upsilon}(K) = \frac{\Upsilon(N \otimes \Delta_{2})}{2} = \frac{1}{\alpha^2} > \Upsilon(K) = 1. \quad (38)
\]

Furthermore, we have

\[
C^{a}_{0,\text{NS}}(N) = M^{a}_{0,\text{NS}}(N) = -2 \log \alpha > M_{0,\text{NS}}(N) = 0. \quad (39)
\]

5 Activated zero-error communication in the asymptotic regime

5.1 Asymptotic zero-error capacity

As we find the activation phenomenon of zero-error communication in the one-shot regime, it’s natural to wonder whether there exists an activation in the asymptotic regime. In the following theorem, we prove that the answer is negative.

**Theorem 8** For any channel \( N \) with non-commutative bipartite graph \( K \) with positive zero-error capacity, let \( n_{0} \) be the smallest integer such that \( \Upsilon(K^{\otimes n_{0}}) \geq 2 \). Note that \( n_{0} \) always exists and depends only on \( K \). Then for any \( n \geq n_{0} \), we have

\[
2 \leq \hat{\Upsilon}(K^{\otimes(n-n_{0})}) \leq \Upsilon(K^{\otimes n}) \leq \hat{\Upsilon}(K^{\otimes n}). \quad (40)
\]

Moreover,

\[
C^{a}_{0,\text{NS}}(K) = \sup_{n \geq 1} \log \sqrt{n} \hat{\Upsilon}(K^{\otimes n}) = \lim_{n \to \infty} \log \sqrt{n} \hat{\Upsilon}(K^{\otimes n}) = C_{0,\text{NS}}(K). \quad (41)
\]
\textbf{Proof} Eq. (40) is immediately from Theorem 5. Then,
\begin{equation}
\lim_{n \to \infty} \log \sqrt[n]{\hat{\Upsilon}(K^{\otimes n})} = \lim_{n \to \infty} \log \sqrt[n]{\Upsilon(K^{\otimes n})}.
\end{equation}
(42)

To prove Eq. (41), the technique is based on a lemma about the existence of limits in [34]. On one hand, \( \log \hat{\Upsilon}(K^{\otimes n}) \leq 2n \log d \). On the other hand, since \( \hat{\Upsilon}(K) \) is super-multiplicative, then \( \log \hat{\Upsilon}(K^{\otimes m}) \geq \log \hat{\Upsilon}(K^{\otimes m}) + \log \hat{\Upsilon}(K^{\otimes n}) \). Therefore,
\begin{equation}
\sup_{n \geq 1} \frac{\log \hat{\Upsilon}(K^{\otimes n})}{n} = \lim_{n \to \infty} \frac{\log \hat{\Upsilon}(K^{\otimes n})}{n} = C_{0,NS}(K).
\end{equation}
(43)
\[\Box]\n
5.2 Lower bound of the activated capacity

We further explore the lower bound for the activated zero-error communication assisted with NS codes and study the equivalent conditions for a quantum channel to have a positive capacity. We find that the one-shot activated capacity is always larger than or equal to the super-dense coding bound in Theorem 25 in [21].

\textbf{Proposition 9} Let \( K \) be a non-commutative bipartite graph with Choi-Jamiołkowski projection \( P_{AB} \), and let \( Q_{AB} = I_{AB} - P_{AB} \) be the orthogonal complement of \( P_{AB} \). Let \( P_{B} = \text{Tr}_{A} P_{AB} \), then the following are equivalent:
\begin{enumerate}
  \item \( C_{0,NS}(K) > 0 \);
  \item \( A(K) > 1 \);
  \item \( P_{B} < d_{A}I_{B} \);
  \item \( \text{Tr}_{A} Q_{AB} \) is positive definite;
  \item \( M_{0,NS}^{a}(K) > 0 \) (or \( \hat{\Upsilon}(K) > 1 \)).
\end{enumerate}

As a matter of fact, we have
\begin{equation}
C_{0,NS}(\mathcal{N}) \geq M_{0,NS}^{a}(\mathcal{N}) \geq \log \frac{d_{A}}{\| \text{Tr}_{A} P_{AB} \|_{\infty}}.
\end{equation}
(44)

\textbf{Proof} In [21], \( i \), \( ii \), \( iii \) and \( iv \) are proved to be equivalent. We focus on \( v \) here.

On one hand, it is easy from \( v \) to \( i \) by Theorem 8. On the other hand, to prove \( i \) to \( v \), it is equivalent to prove \( iii \) to \( v \). We first apply the standard super-dense coding protocol to obtain a cq channel \( \mathcal{N}_{s} \) with \( d_{A}^{2} \) outputs \( \{(U_{m} \otimes I_{B})J_{AB}(U_{m} \otimes I_{B})^{\dagger}\} \), and the projections are given by \( \{(U_{m} \otimes I_{B})P_{AB}(U_{m} \otimes I_{B})^{\dagger}\} \), where \( U_{m} \) are generalized Pauli matrices acting on \( A \). Therefore,
\begin{equation}
A(\mathcal{N}_{s}) = \frac{d_{A}^{2}}{\sum_{m=1}^{d_{A}}(U_{m} \otimes I_{B})P_{AB}(U_{m} \otimes I_{B})^{\dagger}}\frac{d_{A}}{\| \text{Tr}_{A} P_{AB} \|_{\infty}}.
\end{equation}
(45)
Hence, \( \hat{\Upsilon}(K) \geq \hat{\Upsilon}(\mathcal{N}_{s}) = A(\mathcal{N}_{s}) \). And when \( \text{Tr}_{A} P_{AB} < d_{A}I_{B} \) strictly holds, the right-hand side of the above equation is strictly larger than 1. This also means that 1 bit is enough to activate any one-shot useless non-trival channel. \[\Box\]
Example Let us consider the amplitude damping channel $\mathcal{N}_r(\rho) = \sum_{i=0}^1 E_i \rho E_i \dagger$ with
\[
E_0 = |0\rangle\langle 0| + \sqrt{1 - r}|1\rangle\langle 1|, \quad E_1 = \sqrt{r}|0\rangle\langle 1|, \quad 0 \leq r \leq 1.
\] (46)

The classical communication capability of this class of channels has been studied in [36–38], but its classical capacity, zero-error capacity, NS-assisted zero-error capacity are all unknown. In particular, when $r = 3/4$, let $S_A = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ and $U_{AB} = \frac{1}{4}(|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|) + \frac{3}{4}|00\rangle\langle 10|$, it is not difficult to check that $\{S_A, U_{AB}\}$ is an feasible solution to the SDP (8) of $\hat{\Upsilon}(\mathcal{N}_r)$. Hence,
\[
\mathcal{M}_{0,NS}(\mathcal{N}_r) \geq \log \frac{\text{Tr} S_A}{2} = \log \frac{9}{8} > M_{0,NS}(\mathcal{N}_r) = 0.
\] (47)

By applying the super-dense coding bound [21], we know $C_{0,NS}(\mathcal{N}_r) \geq \log d = \log \frac{10}{9}$. And it’s clear that the one-shot activated capacity is better than the super-dense coding bound in [21], i.e.,
\[
\mathcal{M}_{0,NS}(\mathcal{N}_r) \geq \log \frac{9}{8} > \log \frac{10}{9} = \frac{d}{\|\text{Tr}_A P_{AB}\|_{\infty}}.
\] (48)

5.3 $C_{0,NS}$ and semidefinite packing number

As the asymptotic capacity of classical-quantum channel is given by the semidfinite (or fractional) packing number $A(K)$ in Eq.32, will it hold even for general quantum channels? In [21], the semidfinite packing number for a general quantum channel was also introduced, i.e.,
\[
A(\mathcal{N}) = A(K) = \max \text{Tr} S_A \quad \text{s.t.} \quad 0 \leq S_A, \text{Tr}_A P_{AB}(S_A \otimes 1_B) \leq 1_B.
\] (49)

The difficulty is that we currently do not know efficient methods to calculate the asymptotic no-signalling zero-error capacity.

We will exhibit an example to disprove it.

**Proposition 10** There exists a quantum channel $\mathcal{N}$ with non-commutative bipartite graph $K$ such that $\hat{\Upsilon}(K) > A(K)$. Consequently,
\[
C_{0,NS}(\mathcal{N}) \neq A(\mathcal{N}).
\] (50)

**Proof** Let $K$ correspond to the quantum channel $\mathcal{N}(\rho) = \sum_{i=0}^2 E_i \rho E_i \dagger$ with $E_0 = \frac{1}{\sqrt{2}}|0\rangle\langle 0| + \frac{1}{\sqrt{2}}|2\rangle\langle 0|$, $E_1 = \sqrt{\frac{50}{99}}|0\rangle\langle 2| + \sqrt{\frac{1}{99}}|1\rangle\langle 1| + \sqrt{\frac{1}{99}}|2\rangle\langle 2|$ and $E_2 = \sqrt{\frac{98}{99}}|0\rangle\langle 1|$. By solving SDPs on Matlab [39, 40], we find that
\[
\hat{\Upsilon}(\mathcal{N}) \approx 1.1767 > 1.1751 \geq A(\mathcal{N}).
\] (51)

Then, it leads to
\[
C_{0,NS}(\mathcal{N}) \geq \mathcal{M}_{0,NS}(\mathcal{N}) > \log A(\mathcal{N}).
\] (52)
6 Discussions

There are many related interesting open problems, of which we highlight a few here. For example, an interesting problem to characterize the activated entanglement-assisted classical communication over quantum channels. Also, it would be interesting to study the asymptotic NS-assisted zero-error capacity, which is still difficult to calculate since it can be larger than the semidefinite (fractional) packing number. Perhaps one could consider other quantum generalizations of the fractional packing number.

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