Multiple Positive Solutions of Second-Order Nonlinear Difference Systems with Repulsive Singularities

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1. Introduction

Difference systems are widely used in modeling real-life phenomena [1] and references therein. In this paper, we establish the existence positive solutions for the following nonlinear difference systems:

$$-\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = f(n, x(n)) + e(n),$$

with the boundary conditions:

$$x(0) = x(T), \quad p(0)\Delta x(0) = p(T)\Delta x(T),$$

where $q(n) = \text{diag}(q_1(n), q_2(n), \ldots, q_N(n))$, $p(n) = \text{diag}(p_1(n), p_2(n), \ldots, p_N(n))$, $e = (e_1, e_2, \ldots, e_N)T$, and $f = (f_1, f_2, \ldots, f_N)T$, $N \geq 1$. By a periodic solution, we mean a function $x = (x_1, x_2, \ldots, x_N)T$, solving (1) and (2) and such that $x(n) \neq 0$ for all $n$. We call boundary condition (2) the periodic boundary conditions which are important representatives of nonseparated boundary conditions. For convenience, we denote by $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$ the sets of all integer numbers, natural numbers, and real numbers, respectively. For $a, b \in \mathbb{Z}$, let $\mathbb{Z}(a) = \{a, a+1, \ldots\}$, $\mathbb{Z}[a, b] = \{a, a+1, \ldots, b\}$ when $a \leq b$. As usual, $\Delta$ denotes the forward difference operator defined by

$$\Delta x(n) = x(n+1) - x(n).$$

In particular, the nonlinearity $f(x, x(n)) : \mathbb{N} \times \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^N$ may have a repulsive singularity at $x = 0$, from the physical explanation, which means that $\lim_{x \to 0} f(x, x) = +\infty$, uniformly in $n \in \mathbb{Z}[1, T]$, $i = 1, 2, \ldots, N$.

Such repulsive singularity appears in many problems of applications such as the Brillouin focusing systems and nonlinear elasticity [2].

System (1) can be viewed as a discretization of the following more general class of the Sturm singular second-order differential system:

$$-(p(t)x')' + q(t)y = f(t, y) + e(t).$$

Such systems, even in case $p \equiv 1$, where they are referred to as being of Klein-Gordon or Schrödinger type, appear in many scientific areas including fluid mechanics, gas dynamics, and quantum field theory. During the last few decades, the study of the existence of periodic solutions for singular differential equations has deserved the attention of many researchers [3–11]. Tracing back to 1987, Lazer and Solimini
[5] investigated the singular model:

\[ x'' + \frac{h(t)}{x^\alpha} = g(t), \tag{5} \]

where \( \lambda > 0, h, g \) are \( T \)-periodic functions and the mean value of \( g \) is negative, \( g < 0 \). One of the common conditions to guarantee the existence of positive periodic solution is a so-called strong force condition (corresponds to the case \( \lambda \geq 1 \) in (5)) [11, 12]. For example, if we consider the system:

\[ \ddot{x} + V'(t,x) = f(t), \tag{6} \]

with \( V(t,x) = 1/|x|^\alpha \); the strong force condition holds for \( \alpha \geq 2 \). On the other hand, the existence of positive periodic solutions of the singular differential equations has been established with a weak force condition (corresponds to the case \( 0 < \lambda < 1 \) in (5)) [13–15].

From then on, some classical tools have been used to study singular differential equations in the literature, including the degree theory [6, 11, 16], the method of the upper and lower solutions [8, 17], Schauder’s fixed point theorem [14], some fixed point theorems in cones for completely continuous operators [13, 18], and a nonlinear Leray-Schauder alternative principle [19].

For the existence of periodic solutions of difference equations, some results have been obtained using the variational methods or the topological methods [1, 20–25]. For example, by minimax principle, Guo and Yu [23] discussed the existence of periodic solutions for difference equation:

\[ -\Delta^2 x(n-1) + f(n,x(n)) = 0, \tag{7} \]

where the nonlinearity \( f \) is of superlinear or sublinear growth at infinity. Based on the method of the upper and lower solutions, Atici and Cabada [21] studied the existence of periodic solutions for difference equation:

\[ -\Delta^2 x(n-1) + q(n)x(n) = f(n,x(n)). \tag{8} \]

In [26], Zhou and Liu investigated the following autonomous difference equations:

\[ \Delta^2 x(n-1) + f(x(n)) = 0. \tag{9} \]

By Conley index theory, the author showed that the suitable assumptions of asymptotically linear nonlinear are enough to guarantee the existence of periodic solutions.

In this paper, we establish two different existence results of positive periodic solutions for (1) and (2) and proof of the existence of positive solutions; the first one is based on an application of a nonlinear alternative of Leray-Schauder, which has been used by many authors [19, 27, 28] and references therein; the second one is based on a fixed point theorem in cones. Our main motivation is to obtain new existence results for positive periodic solutions of the system:

\[
\begin{align*}
-\Delta[p_1(n-1)\Delta x(n-1)] + q_1(n)x(n) &= (x^2 + y^2)^{-\alpha/2} + \mu(x^2 + y^2)^{\beta/2} + e_1(n), \\
-\Delta[p_2(n-1)\Delta y(n-1)] + q_2(n)y(n) &= (x^2 + y^2)^{-\alpha/2} + \mu(x^2 + y^2)^{\beta/2} + e_2(n).
\end{align*}
\tag{10}
\]

satisfying the initial conditions:

\[ \varphi_i(0) = \varphi_i(1) = 0; \psi_i(0) = 0, p_i(0)\psi_i(1) = 1. \tag{12} \]

Let

\[ D_i = \varphi_i(T) + p_i(T)\Delta\psi_i(T) - 2. \tag{13} \]

Throughout this paper, we always assume that

(H) For each \( i = 1, 2, \cdots, N \), \( p_i(n) > 0, q_i(n) \geq 0, q_i(\cdot) \notin 0 \), \( n \in \mathbb{Z}[1, T] \)

**Lemma 1** (see [29]). If (H) holds, then \( D_i > 0 \).
Lemma 2 (see [29]). Assume (H) holds. For the solution of the problem:

\[
\begin{aligned}
-\Delta [p_i(n-1)\Delta x(n-1)] + q_i(n)x(n) = e_i(n), \quad n \in \mathbb{Z}[1, T], \\
x(0) = x(T), \quad p_i(0)\Delta x(0) = p_i(T)\Delta x(T),
\end{aligned}
\]  

(14)

holds, where

\[
G_i(n, s) = \frac{\psi_i(T)\varphi_i(n)\varphi_i(s) - p_i(T)\Delta \varphi_i(T)}{D_i} \psi_i(n)\psi_i(s) + \left\{ \begin{array}{l}
p_i(T)\Delta \varphi_i(T) - \varphi_i(n)\varphi_i(s) - \frac{\varphi_i(T)}{D_i} - \frac{\varphi_i(n)}{D_i}\varphi_i(s), \quad 0 \leq s \leq n \leq T + 1, \\
p_i(T)\Delta \varphi_i(T) - \varphi_i(s)\varphi_i(n) - \frac{\varphi_i(T)}{D_i} - \frac{\varphi_i(s)}{D_i}\varphi_i(n), \quad 0 \leq n \leq s \leq T + 1,
\end{array} \right.
\]  

(16)

is the Green’s function; the number \(D_i\) is defined by (13).

Lemma 3 (see [29]). Under condition (H), the Green’s function \(G_i(n, s)\) of the boundary value problem (14) is positive, i.e., \(G_i(n, s) > 0\) for \(n, s \in \mathbb{Z}[0, T]\).

We denote

\[
A_i = \min_{n, s \in \mathbb{Z}[0, T]} G_i(n, s), \quad B_i = \max_{n, s \in \mathbb{Z}[0, T]} G_i(n, s), \quad \sigma_i = A_i/B_i.
\]  

(17)

Obviously, \(B_i > A_i > 0\) and \(0 < \sigma_i < 1\).

Remark 4. If \(p_i(t) = 1, q_i(t) = \alpha > 0\), then Green’s function \(G_i(n, s)\) of the boundary value problem (14) has the form:

\[
G_i(n, s) = \begin{cases}
\frac{\beta^{n-s} + \beta^{-nN}}{(\beta - \beta^{-1})(\beta^n - 1)}, & 0 \leq s \leq n \leq T + 1, \\
\frac{\beta^{n-s} + \beta^{n+N}}{(\beta - \beta^{-1})(\beta^n - 1)}, & 0 \leq n \leq s \leq T + 1,
\end{cases}
\]  

(18)

where \(\beta = (\alpha + 2 + \sqrt{\alpha(\alpha + 2)})/2\). If \(n\) is even, a direct calculation shows that

\[
A_i = \frac{2\beta^{T/2}}{(\beta - \beta^{-1})(\beta^T - 1)}, \\
B_i = \frac{1 + \beta^T}{(\beta - \beta^{-1})(\beta^T - 1)},
\]  

(19)

\[
\sigma_i = \frac{2\beta^{T/2}}{1 + \beta^T} < 1.
\]

3. Main Results

In this section, we state and prove the new existence results for (1). In order to prove our main results, the following nonlinear alternative of Leray-Schauder is needed, which can be found in [30].

Lemma 5. Assume \(\Omega\) is a relatively compact subset of a convex set \(E\) in a normed space \(X\). Let \(a \mathbb{I} : \Omega \rightarrow E\) be a compact map with \(0 \in \Omega\). Then, one of the following two conclusions holds:

(i) \(T\) has at least one fixed point in \(\Omega\)
(ii) There exist \(u \in \partial \Omega\) and \(0 < \lambda < 1\) such that \(u = \lambda a\).

Let

\[
X_1 = \{ x : \mathbb{Z}[0, T + 1] \rightarrow \mathbb{R} | x(0) = x(T), p(0)\Delta x(0) = p(T)\Delta x(T) \}
\]  

(20)

Then, \(X_1\) is a Banach space with the norm

\[
\| x \| = \max_{n \in \mathbb{Z}[1, T]} x(n).
\]  

(21)

We take

\[
X = X_1 \times X_1 \times \cdots \times X_1 (N \text{ copies}),
\]  

(22)

with the norm

\[
| x | = \max \{ \| x_1 \|, \| x_2 \|, \cdots, \| x_N \| \}.
\]  

(23)

Define

\[
\gamma_i(n) = \sum_{s=1}^{T} G_i(n, s)e_i(s),
\]  

(24)

which corresponds to the unique solution of (14), and the
operator \( \mathcal{A} : X \to X \) by \( \mathcal{A}_x = (\mathcal{A}_1 x, \mathcal{A}_2 x, \ldots, \mathcal{A}_N x)^T \), where

\[
(\mathcal{A}_x)(n) = \sum_{i=1}^{T} G_i(n, s) f_i(s, x(s) + \gamma(s)), i = 1, 2, \ldots, N. \tag{25}
\]

Now, we present the first existence result of the positive solution to problem (1).

**Theorem 6.** Suppose that condition (H) holds and \( \gamma_* \geq 0 \). Furthermore, we assume that

(H.1) For each constant \( L > 0 \), there exists a function \( \phi \in L(\mathbb{Z}[1, T]) \) such that each component \( f_i \) of \( f \) satisfies

\[
\phi(n) = \mathcal{A}_x(n) \geq \mathcal{A}_x(0) = \phi(0)/C_{138}
\]

for all \( i = 1, 2, \ldots, N \), Here,

\[
K_i(n) = \sum_{s=1}^{T} G_i(n, s) k_i(s),
\]

\[
\sigma = \min_{i=1,2, \ldots, N} \{ \sigma_i \},
\]

\[
\gamma_* = \min_{i=1,2, \ldots, N} \gamma(n),
\]

\[
\gamma^* = \max_{i=1,2, \ldots, N} \gamma(n).
\]

Then, (1) and (2) has at least one positive periodic solution \( x \) with \( x(n) > \gamma(n) \) for all \( n \in \mathbb{Z}[0, T] \) and \( 0 < |x - \gamma| < r \).

**Proof.** We first show that

\[
-\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = f(n, x(n) + \gamma(n)),
\]

together with (2) has a positive solution \( x \) satisfying \( x(n) + \gamma(n) \) for all \( n \in \mathbb{Z}[0, T] \) and \( 0 < |x - \gamma| < r \). If this is true, it is easy to see that \( u(n) = x(n) + \gamma(n) \) will be a positive solution of (1) and (2) with \( 0 < |u - \gamma| < r \) since

\[
-\Delta[p(n-1)\Delta u(n-1)] + q(n)u(n) = -\Delta[p(n-1)\Delta(x(n-1) + \gamma(n-1))] + q(n)(x(n) + \gamma(n)) = f(n, x(n) + \gamma(n)) + e(n) = f(n, u(n)) + e(n).
\]

Since (H.3) holds, let \( J_0 = \{ j_0, j_0 + 1, \ldots \} \), we can choose

\[
(H.2) \text{For each component } f_i \text{ of } f, \text{ there exist nonnegative functions } g_i(x), h_i(x), \text{ and } k_i(n) \text{ such that}
\]

\[
0 \leq f_i(n, x) \leq \{ g_i(x) + h_i(x) \} k_i(n) \text{ for all } (n, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N_{>0},
\]

and \( g_i(x) > 0 \) is nonincreasing and \( h_i(x)/g_i(x) \) is nondecreasing in \( x \)

(H.3) There exists a positive number \( r \) such that \( \sigma r + \gamma_* > 0 \) and

\[
g_i(y_1, \ldots, y_n, \sigma r + \gamma_* \ldots, \gamma^*) \left( 1 + \left( \frac{h_i(r + \gamma^*, \ldots, r + \gamma^*)}{g_i(r + \gamma^*, \ldots, r + \gamma^*)} \right) \right)^n > K_i^2,
\]

\[
\begin{cases}
\text{if } j \in J_0, \\
\text{if } \text{Fix } j \notin J_0.
\end{cases}
\]

\[
\begin{cases}
-\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = \lambda f_i(n, x(n) + \gamma(n)) + \frac{q(n)}{j},
\end{cases}
\]

where \( \lambda \in [0, 1] \) and for each \( i = 1, 2, \ldots, N \),

\[
\begin{cases}
f_i(n, x), & \text{if } x \geq \frac{1}{j}, \\
f_i \left( n, x_1, \ldots, x_j, \frac{1}{j}, x_{j+1}, \ldots, x_N \right), & \text{if } x \leq \frac{1}{j}.
\end{cases}
\]

Problem (29) and (2) are equivalent to the following fixed point problem:

\[
\begin{cases}
x_i(n) = \lambda \sum_{s=1}^{T} G_i(n, s) f_i(s, x(s) + \gamma(s)) + \frac{1}{j} = \lambda \left( \mathcal{A}_x(n) + \frac{1}{j} \right),
\end{cases}
\]

for each \( i = 1, 2, \ldots, N \), here, we used the fact

\[
\sum_{s=1}^{T} G_i(n, s) q_i(s) = 1, i = 1, 2, \ldots, N.
\]
We claim that any fixed point \( x \) of (34) for any \( \lambda \in [0, 1] \) must satisfy \( |x| \neq r \). Otherwise, assume that \( x \) is a fixed point of (34) for some \( \lambda \in [0, 1] \) such that \( |x| = r \). Without loss of generality, we assume that \( |x| = r \) for some \( l = 1, 2, \cdots, N \).

Thus, we have

\[
x_i(n) - x = \lambda \sum_{j=1}^{N} G_{i}(n, s)f_j(s, x(s) + \gamma(s))ds \geq \lambda \sum_{j=1}^{N} f_j(x(s) + \gamma(s))ds \geq \lambda \sum_{j=1}^{N} f_j(x(s))ds \geq \lambda \sum_{j=1}^{N} f_j(x)ds = \lambda j.
\]

From this claim, the nonlinear alternative of Leray-Schauder guarantees that

\[
x(n) = (\sigma f)(x(n)) + \frac{1}{j},
\]

has a fixed point, denoted by \( x^j(n) \), in \( B_r = \{x \in X : |x| < r\} \), i.e.,

\[
-\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = f^j(n, x(n) + \gamma(n)) + \frac{q(n)}{j},
\]

has a periodic solution \( x^j \) with \( |x^j| < r \).

Next, we claim that these solutions \( x^j(n) + \gamma(n) \) have a uniform positive bound, that is, there exists a constant \( \delta > 0 \), independent of \( j \in I_0 \), such that

\[
\min_{n \in B_{\delta}} \{x^j(n) + \gamma(n)\} \geq \delta,
\]

for all \( j \in I_0 \). To see this, we know from (H1) that there exists a continuous function \( \phi_r + y^r \) such that each component \( f_j \) of \( f \) satisfies \( f_j(n, x) \geq \phi_r \) for all \( (n, |x|) \in \mathbb{Z}[1, T] \times (0, r + y^r) \). Now, let \( x^{i_1} \gamma(n) \) be the unique solution to

\[
-\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = \Phi(n),
\]

with (2), here \( \Phi(n) = (\phi_{i_1} \gamma(n), \cdots, \phi_{i_N} \gamma(n))^T \). Then, we have

\[
x^{i_1} \gamma(n) + \gamma(n) = \sum_{i=1}^{T} G_{i}(n, s)\phi_{i_1} \gamma(s) + \gamma(n) \geq \Phi_\gamma + \gamma(n) > 0
\]

for each \( i = 1, \cdots, N \), here

\[
\Phi_\gamma = \inf_{n} \Phi(n), \quad \Phi_\gamma(n) = \sum_{i=1}^{T} G_{i}(n, s)\phi_{i_1} \gamma(s).
\]

Next, we show that (43) holds for \( \delta = \Phi_\gamma + \gamma(n) > 0 \). To see this, for each \( i = 1, \cdots, N \), since \( x^{i_1} \gamma(n) + \gamma(n) \leq r + y^r \) and \( x^{i_1} \gamma(n) + \gamma(n) \geq 1/j \), we have

\[
x^{i_1} \gamma(n) + \gamma(n) = \sum_{i=1}^{T} G_{i}(n, s)\phi_{i_1} \gamma(s) + \gamma(n) + 1/j \geq \Phi \gamma + \gamma(n) \geq \Phi + \gamma(n) = \delta.
\]

The fact \( |x(n)| < r \) and (43) show that for each \( i = 1, 2, \cdots, N \), \( x_i(T) \) is a bounded family on \( \mathbb{Z}[1, T] \). Moreover, we have

\[
x^{i_1}(T) = x_{i_1}(T), \quad p_i(T)\Delta x^{i_1}(T) = p_i(T)\Delta x^{i_1}(T).
\]
which implies that
\[
x_i^j(T+1) = \frac{p_i(0)}{p_i(T)} \Delta x_i^j(0) + x_i^j(T), \quad j \in J_0.
\] (49)

Thus, the Arzela–Ascoli theorem guarantees that \(\{x_i^j\}_j \in J_0\) has a subsequence, \(\{x_i^{jk}\}_{k \in N}\), converging uniformly on \(\mathbb{Z}[0, T+1]\) to a function \(x_i\). Let \(x = (x_1, \ldots, x_N), x(n)\) satisfies \(\delta \leq x_i(n) + y_i(n) < \sigma + \gamma^*\) for all \(n \in \mathbb{Z}[1, T]\) and \(i = 1, \ldots, N\). Moreover, \(x_i^j\) satisfies the integral equation:
\[
x_i^j(n) = \sum_{s=1}^{T} G_i(n, s) f_i(s, x^j(s) + y(s)) + \frac{1}{j^k}, \quad i = 1, \ldots, N.
\] (50)

Letting \(k \to \infty\), we arrive at
\[
x_i(n) = \sum_{s=1}^{T} G_i(n, s) f_i(s, x(s) + y(s)), \quad i = 1, 2, \ldots, N,
\] (51)

here, we have used the fact that \(f(n, x)\) is with respect to \((n, x)\) with \(n \in \mathbb{Z}[1, T]\) and \(x > 0\) satisfying \(\delta \leq |x| \leq \sigma + \gamma^*\). Therefore, \(x_i\) is a positive periodic solution of (1) and satisfies \(0 < |x| \leq r\).

**Corollary 7.** Assume that (H) holds, \(\alpha > 0, \beta \geq 0\). Then, for each \(e_1, e_2\) with \(\gamma_\ast \geq 0\), we have

(i) if \(\beta < 1\), then (10) has at least one positive periodic solution for each \(\mu > 0\);  
(ii) if \(\beta \geq 1\), then (10) has at least one positive periodic solution for each \(0 < \mu < \mu_1\), where \(\mu_1\) is some positive constant.

**Proof.** We will apply Theorem 6. To this end, assumption (H1) is fulfilled by \(\partial_0 = (\sqrt{2L})^{-n}\). If we take
\[
g_1(x, y) = g_2(x, y) = (x^2 + y^2)^{-\alpha/2},
\]
\[
h_1(x, y) = h_2(x, y) = \mu (x^2 + y^2)^{\beta/2},
\] (52)

and \(k_1(n) = k_2(n) = 1\), then (H2) is satisfied. Let
\[
\omega_1(n) = \sum_{s=1}^{T} G_1(n, s), \omega_2(n) = \sum_{s=1}^{T} G_2(n, s).
\] (53)

Then, the existence condition (H3) becomes
\[
\mu < \frac{r(\sigma r + \gamma^*)^2 + \gamma^*^{2\alpha}}{2^{2\alpha + 2}(r + \gamma^*)^{\alpha + \beta}}, \quad i = 1, 2,
\] (54)

for some \(r > 0\). So, (10) has at least one positive periodic solution for
\[
0 < \mu < \mu_1 = \min_{i=1,2} \sup_{r>0} \frac{r[(\sigma r + \gamma^*)^2 + \gamma^*^{2\alpha}]}{2^{2\alpha + 2}(r + \gamma^*)^{\alpha + \beta}}, \quad i = 1, 2.
\] (55)

Note that \(\mu_1 = \infty\) if \(\beta < 1\) and \(\mu_1 < \infty\) if \(\beta \geq 1\). We have (i) and (ii).

In more general, we can obtain the following result.

**Corollary 8.** Assume that (H) holds and there exist functions \(a, \tilde{a}, b, \tilde{b}\) and \(\alpha, \beta > 0\) such that, for \(i = 1, 2, \ldots, N, \)
\[
a(n) = b(n)|x|^\beta \leq f_i(n, x) = \tilde{a}(n) = \tilde{b}(n)|x|^\beta.
\] (56)

Then, for each \(e\) with \(\gamma_\ast \geq 0\), we have

(i) if \(\beta < 1\), then (10) has at least one positive periodic solution for each \(\mu > 0\);  
(ii) if \(\beta \geq 1\), then (10) has at least one positive periodic solution for each \(0 < \mu < \mu_2\), where \(\mu_2\) is some positive constant.

By using a fixed point theorem for compact maps on conical shells [31], we established the second positive periodic solution for (1). Recall that a compact operator means an operator which transforms every bounded set into a relatively compact set and introducing the definition of a cone.

**Definition 9.** Let \(X\) be a Banach space and let \(K\) be a closed, nonempty subset of \(X\). \(K\) is a cone if

(i) \(au + bv \in K\) for all \(u, v \in K\) and all \(\alpha, \beta > 0\)
(ii) \(u, -u \in K\) implies \(u = 0\)

**Lemma 10** (see [31]). Let \(X\) be a Banach space and \(K\) a cone in \(X\). Assume \(\Omega_1, \Omega_2\) are open subsets of \(X\) with \(\emptyset = \Omega_1, \Omega_2 \in \Omega_2^\ast\). Let
\[
\Phi : K \cap (\overline{\Omega_2} \setminus \overline{\Omega_1}) \longrightarrow K
\] (57)

be a continuous and completely continuous operator such that

(i) \(\|\Phi x\| \leq \|x\|\) for \(x \in K \cap \partial \Omega_1\)

(ii) There exist \(\Psi \in K \setminus \{0\}\) such that \(x \neq \Phi x + \lambda \Psi\) for \(x \in K \cap \partial \Omega_2\) and \(\lambda > 0\)
Then, $F$ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. The same conclusion remains valid if (i) holds on $K \cap \partial \Omega_2$, and (ii) holds on $K \cap \partial \Omega_1$.

Define

$$K = \left\{ x = (x_1, \ldots, x_N) \in X : \min_{t \in [0, T]} x_i(t) \geq \sigma \|x\| \text{ for all } n \in \mathbb{Z}[0, T], i = 1, \ldots, N \right\}. \quad (59)$$

Then, one can readily verify that $K$ is a cone in $X$.

$$\sigma R \leq g_i^1(R + y^*, \ldots, R + y^*) \left\{ 1 + \left( \frac{h_i^1(R + y^* + \cdots, \sigma R + y^* + \cdots, R + y^*)}{g_i^1(R + y^*, \ldots, R + y^*)} \right) \right\} \leq K_i. \quad (61)$$

Then, problems (1) and (2) have another one positive periodic solution $\bar{x}$ with $r < |x - y| \leq R$.

**Proof.** Let $\mathcal{A}x = (\mathcal{A}_1x, \ldots, \mathcal{A}_N x)^T$, $\mathcal{A}x$ is given by (25), then, it is easy to verify that $\mathcal{A}x$ is well defined and maps $X$ into $K$. Moreover, $\mathcal{A}x$ is continuous and completely continuous, and let $K$ be a cone in $X$ defined by (59). Define the

$$\Omega_1 = \{ x \in X : |x| < r \}, \quad \Omega_2 = \{ x \in X : |x| < R \}. \quad (62)$$

As in the proof of Theorem 3.1, we only need to show that (29) has a positive periodic solution $u \in X$ with $u(n) + y(n) > 0$ and $r < |u| \leq R$. We claim that

(i) $|\mathcal{A}x| \leq |x|$ for $x \in K \cap \partial \Omega_1$ \hspace{1cm} (63)

(ii) There exist $\psi \in K \setminus \{0\}$ such that $x \neq \mathcal{A}x + \lambda \psi$ for $x \in K \cap \partial \Omega_2$ and $\lambda > 0$

We start with (i). In fact, if $x \in K \cap \partial \Omega_1$, then $|x| = r$ and $\sigma_r + y_1 \leq x_1(n) + y(n) \leq r + y^*$ for all $t \in [0, T]$. Fix $i \in \{1, 2, \ldots, N\}$, thus, we have

$$\mathcal{A}(x)(t) = \frac{\sum_{j=1}^{T} G_i(n,s)f_i(s,x(s) + y(s))}{\int_{\mathbb{R}^+} G_i(n,s)k_i(s)g_i(x(s) + y(s))ds + \lambda} \leq \frac{\sum_{j=1}^{T} G_i(n,s)k_i(s)g_i(x(s)) + y(s)}{\int_{\mathbb{R}^+} G_i(n,s)k_i(s)g_i(x(s) + y(s))ds + \lambda} \leq g_i(x(s) + y(s))$$

$$\leq g_i(x(s) + y(s)) \left\{ 1 + \frac{h_i(x(s) + y(s))}{g_i(x(s) + y(s))} \right\} \leq g_i(x(s) + y(s)) \left\{ 1 + \frac{h_i(x(s) + y(s))}{g_i(x(s) + y(s))} \right\} \sum_{j=1}^{T} G_i(n,s)k_i(s)$$

$$\leq g_i(x(s) + y(s)) \left\{ 1 + \frac{h_i(x(s) + y(s))}{g_i(x(s) + y(s))} \right\} \sum_{j=1}^{T} G_i(n,s)k_i(s) + \frac{\sum_{j=1}^{T} G_i(n,s)k_i(s)}{g_i(x(s) + y(s))} K_i$$

Hence, $\min_{0 \leq t \leq T} x_i(t) > \sigma R$; this is a contradiction and we prove the claim.

Now, Lemma 3.7 guarantees that $\mathcal{A}$ has at least one fixed point $x \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ with $r \leq |x| \leq R$.

Let us consider again the example (10) in Corollary 7 for the superlinear case.

**Corollary 12.** Assume in (10) that $p_i, q_i (i = 1, 2)$ satisfy (H), for each $e_1, e_2$ with $y_i \geq 0, \beta > 1$, then, for each $\mu$ with $0 < \mu < \mu_i$, where $\mu_i$ is given as in Corollary 7, problem (10) has at least two different positive solutions. To verify

**Theorem 11.** Suppose conditions (H), (H$_1$)–(H$_3$) hold. Furthermore, assume that the following two conditions are satisfied:

(H$_4$) There exist continuous, nonnegative functions $g_i^1(x)$, $h_i^1(x)$ and $k_i^1(n)$ such that

$$f_i(n,x) \geq \{ g_i^1(x) + h_i^1(x) \} k_i^1(n) \text{ for all } (n,x) \in [0, T] \times \mathbb{R}^+_\{0\}, \quad (60)$$

where $g_i^1(x) > 0$ is nonincreasing and $h_i^1(x)/g_i^1(x)$ is nondecreasing in $x$.

(H$_5$) There exists $R > r$ such that

$$x_i(n) = (\mathcal{A}x)(n) + \lambda \leq \sum_{j=1}^{T} G_i(n,s) f_i(s,x(s) + y(s))ds + \lambda$$

$$\geq \sum_{j=1}^{T} G_i(n,s) k_i(s) g_i(x(s) + y(s)) \left\{ 1 + \frac{h_i(x(s) + y(s))}{g_i(x(s) + y(s))} \right\} \sum_{j=1}^{T} G_i(n,s) k_i(s)$$

$$+ \lambda \geq \frac{h_i(x(s) + y(s))}{g_i(x(s) + y(s))} \sum_{j=1}^{T} G_i(n,s) k_i(s)$$

$$+ \lambda \geq \frac{h_i(R + y^*, \ldots, R + y^*)}{g_i(R + y^*, \ldots, R + y^*)} K_i$$

$$R > r \quad (65)$$
(H₂), one may take
\begin{align}
g^1_i(x, y) = g^2_i(x, y) &= \frac{1}{2} (x^2 + y^2)^{-\alpha/2}, \\
h^1_i(x, y) &= \frac{1}{2} (x^2 + y^2)^{\beta/2},
\end{align}
and \( k^1_i(n) = k^2_i(n) = 1 \). If \( \beta > 1 \), then the existence condition (H₂) becomes
\begin{align}
\mu \geq \frac{2^{\beta+2}(R + \gamma \ast)^\alpha \sigma R - 2 \omega_\ast}{[(\sigma R + \gamma \ast)^2 + (\sigma_2 R + \gamma \ast)^2]^{(\alpha + \beta)/2} \omega_\ast},
\end{align}
and such (67) is satisfied. The proofs of main results are based on a nonlinear alternative principle of Leray-Schauder and a fixed point theorem in cones. It is interesting that the singularity \( f \) is applicable to the case of a weak singularity as well as the case of a strong singularity. In the next research, we will continue to study the periodic problem to the difference systems like (10) where \( f \) may have attractive singularity at \( x = 0 \), and whether the condition \( \gamma \ast \geq 0 \) can be removed.

\section{Data Availability}

The data used to support the findings of this study are included within the article.

\section{Conflicts of Interest}

The authors declare that they have no conflicts of interest.

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