Structural reliability analysis with temporal and spatial variations based on polynomial chaos expansion

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Abstract. Performance of engineering structures varies with time and space due to the uncertainties in time and space domain. This paper presents a polynomial chaos expansion (PCE) based method to evaluate the reliability problem with temporal and spatial variations. The sequential quadratic programming (SQP) is employed to obtain the samples of spatial response extreme value at discrete time instants, and then the surrogate model of spatial response extreme value can be constructed by PCE based on those samples. Therefore, the structural response hypersurface in time and space domain is transformed into a trajectory of spatial response extreme value in time domain and the reliability analysis can be achieved by Monte Carlo simulation (MCS). Three examples are used to demonstrate the performance of the presented method in accuracy and efficiency.

1. Introduction

Uncertainty is a potential deficiency caused by lack of knowledge at any state of design and operation of product, which may lead to large deviation or unpredictability of structural response. Reliability methods are useful tools to quantify and manage uncertainties on products. Reliability, as an essential quality metric in engineering, can be defined as the probability that a product will perform its intended function within a specified time and under specified conditions [1]. In the past decades, many typical reliability methods have been developed, e.g., first order reliability method (FORM) [2], second order reliability method (SORM) [3], response surface method (RSM) [4], subset simulation (SS) [5] and Monte Carlo simulation (MCS) [6], etc. It is noted that MCS is the most universal method but computationally expensive and its estimation is usually regarded as the true value. The above methods are considered as static methods since the effect of time factor on products is not taken into account. However, time-dependent uncertainties such as material properties deterioration and stochastic loading exist widely in real engineering. In this case, static reliability methods will lead to large errors in the estimation and prediction.

Time-dependent reliability analysis has attracted many attention in recent years due to its ability to tackle the reliability problem involving time-dependent uncertainties. The existing popular time-dependent reliability methods are mainly classified into two categories: the outcrossing rate based methods and the extreme value based methods. The outcrossing rate based methods focus on the crossing events when limit state function (LSF) falls from the safe domain to the failure domain and using the outcrossing rate to approximate the failure rate. Unlike the outcrossing rate based methods, the extreme value based methods study the extreme behavior of LSF in specific time and then perform time-dependent reliability analysis based on the information of response extreme value. Rice [7] first
proposed the formula for calculating the probability of first-passage based on crossing events. Since then, the research on the outcrossing rate based methods has made great progress. Hagen and Tvedt [8] developed a reliability formula based on parallel system theory to achieve the outcrossing rate. Andrieu-Renaud et al. [9] proposed a method called PHI2, which also views the time-dependent reliability problem as parallel system. PHI2 method has high efficiency since the FORM is employed to compute the outcrossing rate, but it may be inaccurate when LSF is high nonlinearity. Jiang et al. [10] derived the computational model of outcrossing rate of series, parallel and composite systems by referring to the concept of PHI2 method. These methods are all based on the assumption that the crossing events are independent of each other, which may be invalid in some application. Hu and Du [11] developed a joint upcrossing rates method to relax the assumption and improve the accuracy. Most of the outcrossing rate based methods require assumption which results in limitations and low accuracy. The extreme value based methods can achieve higher accuracy compared with the outcrossing rate based methods since the absence of assumption. Hu and Du [12] use the saddlepoint approximation method to estimate the distribution of response extreme value for time-dependent reliability problem in which stochastic process is general strength or stress variable. Wang and Wang [13] proposed an approach named the nested extreme response surface (NERS). The key of this method is to predict the time when the response will approach its extreme value by building the nested time prediction model with the efficient global optimization (EGO) and Kriging. Hu and Du [14] improved the NERS method by developing a mixed EGO method, which can obtain the extreme value of all training points simultaneously by constructing a surrogate for response about random variables and time. Considering that NERS and mixed EGO methods are all double-loop process, Hu and Mahadevan [15] proposed a single-loop Kriging surrogate model method to remove the optimization loop.

Although the time-dependent reliability methods have taken into account the effect of time factor on structures, the influence of spatial variables are ignored. However, structures may be affected by both temporal and spatial variables in practice. In this situation, the time-dependent reliability methods may be inapplicable or inaccurate. So far, only a few researches are investigated on the reliability analysis with temporal and spatial variations. Wei and Du [16] combined the FORM and SORM to obtain the samples of reliability and sensitivity vector at each discrete time instants, and accordingly, Kriging was used to transform the response of reliability problem with temporal and spatial variations into an equivalent Gaussian stochastic process. Shi et al. [17] transformed the reliability problem with multiple temporal and spatial parameters into a static reliability problem by finding the effective first-crossing points of each parameter that control whether structure fails. Hu and Mahadevan [18] adopted Kriging combined with singular value decomposition to evaluate the failure probability of multidisciplinary system with temporal and spatial response. Ahmad and Simonovicv [19] predicted the temporal and spatial variations of flood risk by using 3D fuzzy set theory based on the tool of system dynamics simulation and hydrodynamic modelling.

This study presents a new method for the reliability analysis with temporal and spatial variations. It takes advantages of PCE and SQP. This paper is organized as follows: the definition of reliability problem with temporal and spatial variations is given in Section 2. Then, a PCE based method for reliability problem with temporal and spatial variations is introduced in Section 3. Three examples are employed to demonstrate the performance of the proposed method in Section 4. The conclusions are drawn in Section 5.

2. Reliability problem with temporal and spatial variations
The performance of engineering structure is effected by the uncertainties in time and space domain. Hence, temporal and spatial variables should be simultaneously introduced into the LSF when performing the reliability analysis. The LSF of reliability problem with temporal and spatial variations is defined as

\[ G = g(X,S,t) \] (1)
where $G$ is the response, $X = [x_1, x_2, \ldots, x_n]$ is random vector with $n$ dimensions, $S = [s_1, s_2, \ldots, s_m]$ is an $m$-dimensional spatial variable vector, and $t$ is the time factor. Unlike the random variables $X$, the temporal variable $t$ and spatial variables $S$ do not have probability distributions, but vary within the intervals $[L, \bar{T}]$ and $[\underline{S}, \bar{S}]$, respectively.

It can be seen from equation (1) that $G$ is a stochastic process as well as a random field since the input involves both $t$ and $S$. Hence, the response of reliability problem with temporal and spatial variations is a general time-dependent random field. A failure of structure occurs when $G < 0$ in time interval $[t, \bar{T}]$ and space intervals $[\underline{S}, \bar{S}]$. The failure probability can be defined by

$$P_f = \Pr \{ g(X, S, t) < 0, \exists S \in [\underline{S}, \bar{S}], t \in [t, \bar{T}] \}$$

in which $\exists$ means “there exists at least one”.

Reliability problem with temporal and spatial variations is a general time-dependent random field problem, which is more complicated than time-dependent reliability problem whose response is a single stochastic process. In this paper, a PCE based method is developed to perform uncertainty analysis for it.

3. Proposed method

3.1. Overview

The degradation of structural reliability over time is interesting. For the reliability problem with temporal and spatial variations, its response is a hypersurface family in time and space domain when the input random variables $X$ take different realizations. Therefore, in order to evaluate the reliability over time, the response hypersurface in time and space domain can be transformed into a trajectory of spatial response extreme value in time domain.

$$Y(X, t) = \min_{S \in [\underline{S}, \bar{S}]} g(X, S, t)$$

Then, failure probability defined in equation (2) can be converted into

$$P_f = \Pr \{ Y(X, t) < 0, \exists t \in [t, \bar{T}] \}$$

However, it is very difficult to obtain $Y(X, t)$ directly. Therefore, PCE is employed to construct the surrogate model of the spatial response extreme value at each discrete time instant to approximate $Y(X, t)$. Then, MCS is used to achieve the reliability estimation.

3.2. Acquisition of spatial response extreme value

Time $t$ is discretized into a series of discrete time instants firstly for approximating $Y(X, t)$, the discrete time instants can be obtained by

$$t_i = i\Delta t, \quad i = 1, 2, \ldots, N_t$$

where $\Delta t$ is the step size and it is usually set as a constant; $N_t$ is the total number of discrete time instants and $N_t = (\bar{T} - t_0) / \Delta t + 1$.

The spatial response extreme value at each discrete time instant is a function of spatial variables when the time input random variables $X$ in equation (3) are fixed at a realization $x^*$

$$Y(x^*, t_i) = \min_{S \in [\underline{S}, \bar{S}]} g(x^*, S, t_i), \quad i = 1, 2, \ldots, N_t$$
Since the spatial response extreme value expressed by equation (6) is only related to spatial variables that vary within upper and lower bounds of the respective intervals. Therefore, the problem of calculation of spatial response extreme value at each discrete time instant is considered to be transformed into a constrained optimization problem with the upper and lower bounds of spatial variables as constraint conditions

\[
\min \quad g(x^*, S, t_i), i = 1, 2, \ldots, N_i
\]
\[
\text{s.t.} \quad S \leq x \leq \bar{S}
\]

(7)

where \( S \) and \( \bar{S} \) are the lower and upper bounds of spatial variables \( S \), respectively.

In order to solve the constrained optimization problem expressed in equation (7), SQP is employed in this paper. Figure 1 is an example that spatial response extreme value at each discrete time instant in time and space domain. According that, a trajectory of spatial response extreme value changing with time can be approximated, which is shown in Figure 2.

![Figure 1](image1.png) ![Figure 2](image2.png)

**Figure 1.** Spatial response extreme value at discrete time.

**Figure 2.** Trajectory of spatial response extreme value.

When the input random variables \( X \) take different realizations, various trajectories of spatial response extreme value can be obtained. Then, the surrogate model of relationship between the input random variables and spatial response extreme value at each discrete time instant can be built by PCE based on the sample trajectories of spatial response extreme value.

3.3. **Polynomial chaos expansion theory**

Response extreme value is a random variable according to the extreme value theory. The main concept of PCE is that a random variable can be represented by a sum of polynomial chaos basis. Therefore, PCE can be adopted to approximate the spatial response extreme value.

In PCE theory, a random model response \( Y(X) \) with mutually independent input random variables can be approximated as follows

\[
Y(X) = \sum_{i=0}^{\infty} \alpha_i \Phi_i(X)
\]

(8)

where \( X = [x_1, x_2, \ldots, x_m] \) is an m-dimensional random vector; \( \alpha_i \) are the coefficients to be determined; \( \Phi_i(X) \) are the multivariable polynomial chaos expansion basis.

Generally, the accurate of PCE may be satisfied when it is expanded to a certain order. Hence, the series in equation (8) is usually truncated and only a finite number of terms are retained for computational purpose. A \( p \) order truncated PCE with m-dimensional random variables can be expressed as
\[ Y(\mathbf{X}) = \sum_{i=0}^{P-1} \alpha_i \Phi_i(\mathbf{X}) \] (9)

where \( P \) is the number of multivariable polynomial chaos expansion basis, and \( P = C_m^n \).

Multivariable polynomial chaos expansion basis \( \Phi_i(\mathbf{X}) \) are usually expressed as the tensorization of one-dimensional polynomial chaos expansion basis

\[ \Phi_i(\mathbf{X}) = \prod_{j=1}^{m} \phi_{i_j}(X_j) \] (10)

where \( \phi_{i_j}(x_j) \) are one-dimensional polynomial chaos expansion basis with orthogonality; \( i_j \) represents the order of the corresponding one-dimensional polynomial chaos expansion basis, and \( i_j \in N^m, \sum_{j=1}^{m} i_j \leq p \).

The type of orthogonal polynomial chaos expansion basis can be determined according to the distribution form of the random variables. For example, when random variable follows a normal distribution, Hermite polynomial is used as the polynomial chaos expansion basis. Then, once the unknown coefficients \( \alpha_i \) are obtained, the PCE approximation of spatial response extreme value can be achieved. In this work, the ordinary least-squares is employed to calculate the coefficients \( \alpha_i \).

3.4. Implementation procedure

The implementation procedure of the proposed PCE based method for reliability problems with temporal and spatial variations is shown in Figure 3.

![Flowchart of the proposed method](image-url)
The detail steps are presented as follows:

1. Parameter setting. Set the sample size $N$ of input random variable, the discretization step $\Delta t$ of time and the highest order $p$ of PCE.

2. Discretization of time. Discrete time $t$ into $N_t$ instants $t_i$ with $\Delta t = i\Delta t$, $i = 1, 2, \ldots, N_t$.

3. Generation of sample point. A sample point $x_j$ ($j = 1$) is generated according to the distribution of input random variables by Latin Hypercube Sampling (LHS) method.

4. Calculation of the spatial response extreme value. Calculate the spatial response extreme value $Y(x_j, t_i)$ at each discrete time instant $t_i$ by SQP when the sample point is $x_j$.

5. Update of training sample sets. The input random variables sample set $T$ and spatial response extreme value sample set $Y$ are updated with $T = [T, x_j]$ and $Y = [Y, Y(x_j, t_i)]$. If $j \neq N$, return to (3); else if $j = N$, then go to (6).

6. PCE model construction. Build the PCE model based on $[T, Y]$ by taking advantage of MATLAB toolbox UQLAB [20].

7. Calculation of failure probability. MCS is employed to calculate the failure probability $P_f$ based on the PCE model.

4. Examples

Compared with MCS, the performance of the proposed method is demonstrated in three examples.

4.1. Example 1: mathematical example

The first example is a mathematical example [16] involves two random variables and two spatial variables. The performance function is given by

$$g(X, S, t) = 8 + 10x_1 + 12x_2 + x_1x_2 + 0.1s_1s_2x_1^2 - 0.2x_2^2 \cos(t + \pi / 2) + \sin(t)$$

where $x_i, i = 1, 2$ are independent random variables and $s_i \sim N(0, 0.2^2)$; $s_i, i = 1, 2$ are spatial variables and $s_i \in [1.5, 2.5]$; $t \in [0, 2\pi]$ rad is temporal variable.

In this example, the step size $\Delta t$ was set to 0.05 resulting in $N_t = 126$ discrete time instants. The highest order $p$ of PCE was set to 2. The Hermite polynomial is used as the polynomial chaos expansion basis since the input random variables follows normal distribution. $N = 5, 10, 15$ sample points of random variables were generated by LHS to investigate the performance of the proposed method with different sample sizes.

MCS is employed as the benchmark approach to compare with the proposed method. MCS was performed based on equation (3) to get accurate results as references for the results of the proposed method. $10^8$ samples of random variables were used by MCS in this paper.

Table 1 lists the results of the proposed method and MCS over five time intervals when $N=10$. The cumulative probability of failure and error of the proposed method compared with MCS are shown in Figure 4 and Figure 5, respectively.

| Time interval | PCE ($10^{-3}$) | MCS ($10^{-3}$) | Error (%) |
|---------------|-----------------|----------------|-----------|
| [0,3.0]       | 4.539           | 4.62           | 1.75      |
| [0,3.5]       | 6.411           | 6.64           | 3.45      |
| [0,4.0]       | 9.525           | 9.56           | 0.37      |
| [0,4.5]       | 11.589          | 11.77          | 1.54      |
| [0,2\pi]      | 11.813          | 12.00          | 1.56      |
Figure 4. Failure probabilities of example 1 with different sample sizes.

Figure 5. Errors of example 1 with different sample sizes.

Figure 4 shows that the failure probability curves of the proposed method over time with different sample sizes all have the same trend as that of MCS and very close to MCS. This indicates that the accuracy of the proposed method is good. It can be seen from Figure 5 that the error of the proposed method can be reduced with the increase of the sample size. However, the errors for $N=10$ are almost the same as that for $N=15$, and the maximum error is 3.48%. Therefore, the proposed method only needs 10 samples to achieve a similar accuracy to MCS using $10^5$ samples, which demonstrates the efficiency of the proposed.

4.2. Example 2: a slider mechanism

After illustrating the performance of the proposed method by using the mathematical example, a slider mechanism [16] shown in Figure 6 is adopted in this section.

The performance function of the slider mechanism is defined as follow

$$g = 1.1 - \left( x_{\text{actual}} - x_{\text{required}} \right)$$

(12)

where $x_{\text{actual}}$ and $x_{\text{required}}$ are the actual position and required position of the slider, respectively.

$x_{\text{actual}}$ and $x_{\text{required}}$ can be achieved by the following formula

$$x_{\text{actual}} = L_4 \cos(\theta_0 + \omega t) + \sqrt{L_2^2 - (h + L_4 \sin(\theta_0 + \omega t))^2}$$

(13)
\[ x_{\text{required}} = 15 \cos(\omega t) + \sqrt{35^2 - (15 + 15 \sin(\omega t))^2} \]  

respectively, in which \( \omega = \text{1 rad/s} \) is the angular velocity.

The detailed distribution information of the input variables of example 2 is listed in Table 2.

Table 2. Distribution information of variables in example 2.

| Variable | Type     | Mean/Lower bound | Std/Upper bound |
|----------|----------|------------------|-----------------|
| \( t \)  | Time     | 0                | 0.2\pi          |
| \( h(\text{m}) \) | Space   | 14.9             | 15.1            |
| \( \theta_0(\degree) \) | Space   | 0                | 5               |
| \( L_1(\text{m}) \) | Normal  | 15               | 0.15            |
| \( L_2(\text{m}) \) | Normal  | 35               | 0.35            |

In this example, time \( t \) was discretized into \( N_t = 41 \) discrete time instants with a step size \( \Delta t = 0.005 \pi \). The highest order \( p \) of PCE was set to 2. Three different sample sizes \( N = 5, 10, 15 \) were generated by LHS to build PCE model.

Table 3. Failure probability of example 2 (\( N=10 \)).

| Time interval | PCE (\( 10^{-3} \)) | MCS (\( 10^{-3} \)) | Error (%) |
|---------------|----------------------|----------------------|-----------|
| [0,0.05\pi]   | 6.812                | 6.97                 | 2.27      |
| [0,0.1\pi]    | 8.769                | 8.92                 | 1.70      |
| [0,0.15\pi]   | 11.877               | 11.81                | 0.57      |
| [0,0.2\pi]    | 16.933               | 16.64                | 1.76      |

Figure 7. Failure probabilities of example 2 with different sample sizes.

Figure 8. Errors of example 2 with different sample sizes.

Table 3 shows the comparison of the computational results of the failure probability between the proposed method and MCS in four time intervals when \( N=10 \). The cumulative probability of failure and error of the proposed method with three different sample sizes \( N = 5, 10, 15 \) are shown in Figure 7 and Figure 8, respectively. It can be seen from the above results that the cumulative probability of failure of the proposed method using three different sample sizes over the whole time interval is consistent with that of MCS. The accuracy of the proposed method can be improved by increasing the number of samples. The errors of the proposed method for \( N=10 \) and \( N=15 \) are smaller than those for \( N=5 \), and the errors for \( N=15 \) are similar to those for \( N=10 \). Therefore, 10 samples are sufficient for this example and the maximum error is 3.6\%. The proposed method can reduce the computational cost while meeting the accuracy.
4.3. Example 3: a four-bar function generator

As shown in Figure 9, a four-bar function generator is adapted from [21], which is used to generate a motion function $y_{desired}(x) = 60^\circ + 60^\circ \times \sin\left(0.75\left(x - 97^\circ\right)\right)$.

![Figure 9. A four-bar function generator.](image)

The angle $x$ is taken as the input and regarded as the time variable $t$, and the angle $y$ is the output.

The distribution information of this four-bar function generator is listed in Table 4.

Table 4. Distribution information of variables in example 3.

| Variable | Type    | Mean/Lower bound | Std/Upper bound |
|----------|---------|------------------|-----------------|
| $t$      | Time    | 97               | 217             |
| $L_1$ (mm) | Space   | 99.85            | 100.15          |
| $L_2$ (mm) | Space   | 54.85            | 55.15           |
| $L_3$ (mm) | Normal  | 144.1            | 0.05            |
| $L_4$ (mm) | Normal  | 72.5             | 0.05            |

The actual output angle of the four-bar function generator is defined by

$$y_{actual}(t) = 2 \arctan\left(\frac{-P_2 \pm \sqrt{P_2^2 + P_1^2 - P_3^2}}{P_3 - P_1}\right)$$  \hspace{1cm} (15)

in which $P_1$, $P_2$, and $P_3$ are given by

$$P_1 = 2L_4(L_1 - L_2 \cos t)$$
$$P_2 = -2L_2L_4 \sin t$$
$$P_3 = L_4^2 + L_2^2 + L_4^2 - L_5^2 - 2L_4L_2 \cos t$$  \hspace{1cm} (16)

The performance function of the four-bar function generator is defined by whether the motion error between the actual output angle and the desired output angle exceeds a given threshold

$$g = c - \text{abs}\left(y_{actual} - y_{desired}\right)$$  \hspace{1cm} (17)

where $c$ is the given threshold.

The failure probability of the four-bar function generator with eight different given threshold $c$ is calculated in this example, and $c=1, 1.025, 1.05, 1.075, 1.1, 1.125, 1.15, \text{ and } 1.175$. The number of sample $N$ was set to 5, 10 and 15, respectively. The highest order $p$ of PCE was set to 2. The results of failure probability of the four-bar function generator when $N=5$ is listed in Table 5.
Table 5. Failure probability of example 3 ($N=5$).

| $c$   | PCE ($10^{-3}$) | MCS ($10^{-3}$) | Error (%) |
|-------|-----------------|-----------------|-----------|
| 1     | 0.9788          | 0.9786          | 0.02      |
| 1.025 | 0.9196          | 0.9197          | 0.01      |
| 1.05  | 0.7811          | 0.7810          | 0.01      |
| 1.075 | 0.5596          | 0.5602          | 0.11      |
| 1.1   | 0.3170          | 0.3172          | 0.06      |
| 1.125 | 0.1351          | 0.1354          | 0.22      |
| 1.15  | 0.0420          | 0.0422          | 0.47      |
| 1.175 | 0.0093          | 0.0095          | 2.11      |

A large given threshold indicates that the accepted motion error between the actual angle and the desired angle is relaxed, so the failure probability of the four-bar function generator decreases with the increase of the given threshold. The results of the proposed method plotted in Figure 10 are also consistent with this fact. In addition, it can be seen from Figure 11 that the maximum error of the proposed method over the whole time interval is 2.45% when $N=5$. In terms of sample size, the proposed method only needs 5 samples to obtain an accurate result compared with MCS with $10^5$ samples. Therefore, the proposed method demonstrates its good performance in accuracy and efficiency.

5. Conclusions

In this work, a PCE based method for the reliability problems with temporal and spatial variations is developed. Firstly, time variable is discretized into a series of discrete time instants and the samples of spatial response extreme value at each discrete time instant can be obtained by SQP. According to these samples, the surrogate models of spatial response extreme value at discrete time instants are built by PCE. Therefore, the structural response hypersurface in time and space domain is transformed into a trajectory of spatial response extreme value in time. Then, the reliability or failure probability with temporal and spatial variations can be evaluated according to the trajectories of spatial response extreme value combined with MCS. The proposed method is compared with MCS in three examples, which proves that it has good performance in accuracy and efficiency.

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