A Universal Interacting Crossover Regime in Two-Dimensional Quantum Dots

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Interacting electrons in quantum dots with large Thouless number \( g \) in the three classical random matrix symmetry classes are well-understood. When a specific type of spin-orbit coupling known to be dominant in two dimensional semiconductor quantum dots is introduced, we show that a new interacting quantum critical crossover energy scale emerges and low-energy quasiparticles generically have a decay width proportional to their energy. The low-energy physics of this system is an example of a universal interacting crossover regime.

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The idea of a crossover between two symmetry classes will play a central role in this paper. Consider a system in which the Hamiltonian is crossing over from the GOE (T intact, no spin-orbit coupling), the unitary or GUE (T broken), and the symplectic or GSE (T intact, with spin-orbit coupling). More recently, other classes have been identified for disordered superconductors and quantum dots constructed from two-dimensional semiconductor heterostructures with spin-orbit coupling. We will focus on a symmetry class in the latter case (which we call the Aleiner-Falko (AF) class) in which, after a canonical transformation, the single-particle \( S_z \) is conserved, while \( S^2 \) is not.

The idea of a crossover between two symmetry classes will play a central role in this paper. Consider a system in which the Hamiltonian is crossing over from the GOE to the GUE, achieved, e.g., by turning on the orbital effects of a magnetic field. For Thouless number \( g \gg 1 \) of the original GOE, the \( g \times g \) crossover Hamiltonian is

\[
H_X(\alpha) = H_{\text{GOE}} + \frac{\alpha}{\sqrt{g}} H_{\text{GUE}}
\]

where \( \alpha \) is the crossover parameter. Properties of eigenvector correlations have been computed in the crossover. For \( g \gg 2\alpha^2 = g_X \gg 1 \), the following ensemble-averaged correlations hold for the eigenstates \( \psi_i(i) \), where \( \mu \neq \nu \) label the states and \( i, j, k, l \) the original orthogonal labels:

\[
\langle \psi_{\mu}^*(i) \psi_{\nu}(j) \rangle = \langle \psi_{\mu}^*(i) \psi_{\nu}^*(j) \psi_{\mu}(k) \psi_{\nu}(l) \rangle = \frac{\delta_{\mu\nu}\delta_{ij}}{g^2} + \frac{1}{g^2} \delta_{\mu\nu}\delta_{kl} E_X \delta/\pi + \frac{E_X \delta/\pi}{g^2} \delta_{\mu\nu}(\epsilon_{g} - \epsilon_{S}) \delta_{ij}\delta_{kl}
\]

The last term on the second line shows the extra correlations induced in the crossover. The crossover scale \( E_X = g_X \delta/\pi \) represents a window within which GUE correlations have spread, while GOE correlations remain at high energies.

The crossover from the spin-rotation invariant GOE to the AF class is a GOE \( \rightarrow \) GUE crossover, where the “magnetic flux” has opposite signs for opposite eigenvalues of \( S_z \). If the linear size of the system is \( L \) and the spin-orbit scattering length is \( \xi \gg L \), this new AF symmetry class manifests itself below \( E_X \simeq (\frac{\delta}{\pi}) \frac{1}{\lambda} \Delta T \). The crossover to the fully symplectic GSE occurs at the parametrically smaller energy scale of \( (\frac{\delta}{\pi}) \frac{1}{\lambda} \Delta T \), set here to 0.

Turning from single-particle physics to interactions, for small to moderate \( r_s \), the “Universal Hamiltonian” is known to contain all the relevant couplings at low energies in the renormalization group sense.

\[
H_U = \sum_{\alpha, s} \epsilon_{\alpha,s} \hat{c}_{\alpha,s}^\dagger \hat{c}_{\alpha,s} + \frac{U_0}{2} \hat{N}^2 - JS^2 + \lambda T \hat{T}
\]

Here \( \hat{N} \) is the total particle number, \( S \) is the total spin, and \( T = \sum_{\alpha, \beta, \gamma} T_{\alpha\beta\gamma} \). \( H_U \) has a charging energy \( U \), an exchange energy \( J \) and a superconducting coupling \( \lambda \). This last term is absent in the GUE, while the exchange term disappears in the GSE. In the large-\( g \) limit only interaction terms which are invariant under the symmetries of the one-body Hamiltonian appear in \( H_U \). At larger \( r_s \), the system enters a quantum critical regime connected with the impending Pomeranchuk transition.

In the GOE, ignoring the Cooper coupling (two-dimensional semiconductor quantum dots do not superconduct) and tuning \( J \) one sees the mesoscopic Stoner effect. Take for illustration an evenly spaced set of levels with level spacing \( \delta \). Since both \( S^2 \) and \( S_z \) are conserved, we can focus on the ground state with \( S_z = S \), and find its energy for an even number of particles to be \( E_{gs}(S) = S^2 \delta - JS(S + 1) \). Defining \( J = J/\delta \) and minimizing with respect to \( S \) leads to steps from \( S = 0 \) to \( S = 1 \) at \( J = \frac{1}{2} \), from \( S = 1 \) to \( S = 2 \) at \( J = \frac{3}{4} \) etc. Including the mesoscopic (sample-to-sample) fluctuations of the energies and the matrix elements leads to probability distributions for spin \( S \).

Note that the \( S_z = S \) ground state is electronically uncorrelated (a single Slater determinant, not a superposition).
At strong spin-orbit coupling\cite{8, 9, 16}, noting that previously\cite{17} in the restricted case we end up with the Hamiltonian of Eq. (4) with $
abla$. The ground and excited states of this Hamiltonian have a nonzero density at low energies and fermionic quasiparticles become very broad at low energies (as long as ground state $S_z \neq 0$). Finally, the mesoscopic Stoner effect is smoothly pushed to higher $J$ as spin-orbit coupling increases.

We will set $J_z = 0$ in Eq. (1) henceforth, since that is the correct starting point for a Hamiltonian deep in the Thouless band\cite{10, 13, 14}. We decompose both the interaction terms by introducing the Hubbard-Stratanovich fields $h(t)$, $q(t) = X(t) + iY(t)$, and $q^*(t)$ to get the $T = 0$ imaginary time action

$$A = \int_{-\infty}^{\infty} dt \sum_{\alpha, \beta, \beta'} \tilde{\psi}_\alpha \left[ (\partial_t + \epsilon_\alpha) \delta_{\alpha \beta'} - \frac{\nabla}{2} (\sigma_2)_{\alpha \beta'} \right] \psi_{\alpha'} + \frac{h^2 + g^2}{4J} - \frac{q(t)}{2} \sum_{\alpha, \beta} M_{\alpha \beta} \tilde{\psi}_{\alpha \uparrow} \psi_{\beta \downarrow} - \frac{q^*(t)}{2} \sum_{\alpha, \beta} M^*_{\alpha \beta} \tilde{\psi}_{\beta \downarrow} \psi_{\alpha \uparrow}(T)$$

$X(t), Y(t)$ are fluctuating fields and are integrated out, but $h(t)$ acquires an expectation value (also called $h$). For our picket fence spectrum $\epsilon_n = \delta(n - 1/2)$ with chemical potential at 0, at $h = 0$ all states $n \leq 0$ are occupied while all states $n \geq 1$ are empty. When $h$ lies between $h_n = \delta(2n - 1)$ and $h_{n+1} = \delta(2n + 1)$, at $T = 0$ the single-particle states between $-n + 1$ and $n$ singly occupied, and $S_z = n \equiv S$. We will find saddle points for $h$ in each of the intervals $h_n < h < h_{n+1}$, thereby obtaining the ground state energy as a function of $S = n$, and look for the lowest one.

First integrate out the fermion fields and obtain a quadratic effective action for $h$, $X$, and $Y$.

$$A_{eff} = \int_{-\infty}^{\infty} d\omega \left[ \frac{|X(\omega)|^2 + |Y(\omega)|^2}{2J} (1 - J\chi_R(i\omega)) + \frac{X(\omega)Y(-\omega) - Y(\omega)X(-\omega)}{4J} \chi_I(i\omega) \right]$$

The cross terms are a consequence of $|S_x, S_y| = iS_z$, and $\chi_R$ and $\chi_I$ are the real and imaginary parts of $\chi$, the fermionic transverse spin-susceptibility.

$$\chi(i\omega) = \sum_{m,n} [M_{mn}]^2 \frac{N_F(\epsilon_m) - N_F(\epsilon_n)}{i\omega + \epsilon_n - \epsilon_m}$$

where $N_F$ is the Fermi occupation. To make further progress, we assume that $E_X \gg \delta$, which allows us to convert the sums over states into integrals, and also replace the sample-specific value of $|M_{mn}|^2$ by its ensemble average. Such self-averaging occurs naturally in the large-$N$ limit\cite{13, 14}. The dominant contribution to $\chi$ is

$$\chi(i\omega) = \frac{1}{\delta} \frac{E_X - iE_S \text{sgn}(\omega)}{|\omega| + E_X - ih \text{sgn}(\omega)}$$

where $E_S = 2S\delta$ and $\text{sgn}(\omega)$ is the sign of $\omega$. Next, the integration over $X$ and $Y$ results in a fluctuation contribution to the effective action

$$A_{fluc} = \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \log \left( \frac{|\omega| + E_{QCX})^2 + (h - J\epsilon_S)^2}{(|\omega| + E_X)^2 + h^2} \right)$$
where the argument of the logarithm is the determinant of the matrix of the quadratic form of $X$ and $Y$. $A_{\text{fluc}}$ is logarithmically divergent due to a limitation of the ensemble averages Eqs. (12) for large energy separations

$$A_{ef}(S, h) = S^2\delta + \frac{h^2}{4J} - hS + \frac{1}{\pi}\left(\frac{\Delta h}{E_{QCX}} - \hbar + \frac{\Delta h}{E_{QCX}}h\tan^{-1}\frac{h}{E_{X}} + \frac{E_{X}}{2}\log\left(1 + \frac{h^2}{E_{X}}\right) - \frac{E_{QCX}}{2}\log\left(1 + \frac{(\Delta h)^2}{E_{QCX}^2}\right)\right)$$

Eq. (12) is one of the central results of this paper. Note that the term in the brackets is of order $1/S$ compared to the first three terms ($h$ will turn out to be order $S$). From it we find that the saddle point value of $h_0$ satisfies

$$h_0 = \tilde{J}E_S + \frac{2J}{\pi}\left(\tan^{-1}\frac{\Delta h_0}{E_{QCX}} - \tan^{-1}\frac{h_0}{E_X}\right)$$

We must also ensure that $h_0 < h_0 < h_{n+1}$ for $S = n$. Using $h = h_0$ in Eq. (12) gives the ground state energy for that $S$. Even though this result has been derived for $E_X \gg \delta$ and $S \gg 1$, note that one recovers the correct spin-rotation-invariant result for all $S$ on taking the $E_X \to 0$ limit first in Eq. (12) and solving it to obtain the ground state energy. Fig. 1 shows regions in the $E_X, J$ plane with different ground state $S_x = S$. Note the smooth approach to the spin-rotation invariant results as $E_X \to 0$.

Now consider the low-lying excitations, which are bosonic spin excitations and fermionic quasiparticles. The transverse spin correlator

$$D(t) = -(T_tS_+(t)S_-(0)) = -\frac{1}{4J^2}(T_tq^+(t)q(0)) + \frac{1}{J}$$

as a function of Matsubara frequency (at $T = 0$) is

$$D(i\omega) = \frac{1}{\delta|\omega| + E_{QCX} + i\Delta h\text{sgn}(|\omega|)}$$

The last term in the denominator is never more than a fraction of $\delta$ and will be ignored below. Going over to the retarded commutator with the standard replacement $i\omega \to \omega + i0^+$, we find the spectral function for transverse spin excitations

$$B(\omega) = -2\text{Im}(D_\text{ret}(\omega)) = \frac{2\omega E_X + E_S E_{QCX}}{\delta(\omega^2 + E_{QCX}^2)}$$

For $S = 0$ this reproduces the scaling function computed earlier [17]. More importantly, for $S \neq 0$ there is a nonzero density of spin excitations even as $E_{QCX} \ll 1$ (but $\omega \gg \delta$). These excitations are related to the $2S + 1$-degenerate ground states of the spin-rotation-invariant system (since $S^2$ is not conserved here, the ground state contains a superposition of many values of $S^2$). These low-energy excitations make the system strongly correlated in the electronic sense, that is, the ground and low-lying states are no longer described even approximately by single Slater determinants.

The decay of an electron of energy $\varepsilon$ to leading order occurs via the emission of a single spin excitation. Using the Fermi Golden Rule, we get the decay rate

$$\Gamma(\varepsilon) \simeq 2\pi J^2 \int_0^\varepsilon d\varepsilon' |M(\varepsilon - \varepsilon')|^2 \rho(\varepsilon')$$

where $\rho(\varepsilon')$ is the electronic density of states at energy $\varepsilon'$ and $M$ is the matrix of Eq. (10). Using $\rho(\varepsilon') = 1/\delta$, and Eqs. (10) we obtain

$$\Gamma(\varepsilon) \simeq \frac{J^2 E_X}{\delta(\varepsilon^2 - E_{QCX}^2)} \left[\frac{E_X}{2}\log\left(1 + \frac{\varepsilon^2}{E_{QCX}^2}\right) - \log\left(1 + \frac{\varepsilon^2}{E_X^2}\right)\right] + 4E_S E_{QCX} \left[\tan^{-1}\frac{\varepsilon}{E_{QCX}} - \tan^{-1}\frac{\varepsilon}{E_X}\right]$$

This is the central result of this paper. At low energies $\varepsilon \ll E_{QCX}$, $\rho, |M|^2$, and $B$ are constant, leading to a decay rate which goes as $\Gamma(\varepsilon) \simeq \frac{16J^2 S^2}{E_X E_{QCX}} \varepsilon$, which can exceed $\varepsilon$ for $J \to 1$, leading to ill-defined quasiparticles. At high energies $\varepsilon \gg E_X$ (but $\varepsilon \ll E_T$), the decay rate due to spin excitations goes to a constant. There will be an additional Fermi liquid broadening [10], $\Gamma \simeq \varepsilon^2/E_T$ due to neglected interactions. Thus we have an unusual situation in which the quasiparticles are better defined at high energies than at low energies, because the high-energy physics is controlled by the weakly inter-
acting spin-rotation-invariant $H_0$.\cite{17}

In summary, we have established that a crossover from the spin-rotation-invariant GOE to the AF class in two-dimensional semiconductor quantum dots makes the ground and low-lying states of the system strongly correlated in the electronic sense, with a nonzero density of states for transverse spin excitations at very low energies, and a decay rate $\Gamma(\epsilon) \simeq \epsilon$ for low-energy fermionic quasiparticles (for ground state $S_z \neq 0$). The results are universal since all energies are $\ll E_T$, and apply to disordered as well as ballistic/chaotic dots.

Random matrix crossovers thus offer us access to universal interacting crossover regimes\cite{17}. Such regimes are dominated by many-body correlations and are distinct from single-particle RMT ensembles\cite{3,4,5}, and offer the possibility of tuning the many-body quantum crossover scale $E_{Q\text{CX}}$ by varying the single-particle $E_X$.

Another example of such a universal interacting crossover regime is a superconducting dot vertically tunnel-coupled to a normal dot with an orbital flux\cite{20}. The crossover scale is varied by tuning the tunneling strength. In our problem one can imagine a pair of vertically coupled quantum dots with one of them fabricated of a different material having a much larger spin-orbit coupling than the other. $E_X$ (and thus $E_{Q\text{CX}}$) would be tuned by varying the tunneling strength.

Let us turn to some caveats. We have focused on an even number of electrons in the dot but expect similar results for an odd number. Our large-$N$ approach is valid for $E_X \gg \delta$ and $S \gg 1$. However, even in this regime, in a tiny region of width $\Delta J \approx \delta^2/(E_X S)$ around the transition between $S_z$ steps, the value of $h_0$ is such that the fermionic susceptibility exceeds $1/J$, and a small saddle-point value for $|q|$ will be generated, leading to a more complicated effective action. Apart from potentially missing effects in this region, the large-$N$ approach seems to capture most of the physics. We have assumed a spectrum with equal level spacing, and the ensemble averaged form for the matrix elements $M_{\alpha\beta}$. In real dots in the crossover, both fluctuate\cite{3,4,5}. One thus expects mesoscopic (sample-to-sample) fluctuations in the ground state value of $S_z = S_3$\cite{9,13,16} leading to a probability distribution for $S$ given $J$ and $E_X$. Our methods can be used to complement exact diagonalizations\cite{21} in this case.

Many open questions remain, such as how to characterize the state at and near the transition between $S_z$ steps, how the state responds to an in-plane $B$ field, whether the strong electronic correlation has any signatures in the zero-bias conductance, and whether one can classify universal interacting crossover regimes into universality classes. The author hopes to explore these and other questions in future work.

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