Local witness for bipartite quantum discord

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Recently, we have proposed a method for the local detection of quantum correlations on the basis of local measurements and state tomography at different instances in time [Phys. Rev. Lett. \textbf{107}, 180402 (2011)]. The method allows for the detection of quantum discord in bipartite systems when access is restricted to only one of the subsystems. Here, we elaborate the details of this method and provide applications to specific physical models. In particular, we discuss the performance of the scheme for generic complex systems by investigating thermal equilibrium states corresponding to randomly generated Hamiltonians. Moreover, we formulate an ergodicity-like hypothesis which links the time average to the analytically obtained average over the group of unitary operators equipped with the Haar measure.

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I. INTRODUCTION

The field of quantum information theory is dedicated to developing computational techniques with an advantage over classical methods using the laws of quantum mechanics [1]. A variety of tools for communication and computation science have been developed in the past years, ranging from quantum teleportation [2] and quantum dense coding [3] to efficient algorithms for quantum computers [4–6]. The fundamental resource for these applications is usually summarized under the term quantum correlations, even though it has proven difficult to identify a common resource to all of these applications. More precisely, ideas like quantum teleportation and the violation of Bell’s inequalities [7] are profoundly related to quantum entanglement [8]. Other applications could be linked directly to a resource named quantum discord [9–11], which is identical to entanglement for pure states but differs for statistical mixtures [12–14]. While for entanglement the term quantum correlation is suitable, not least in view of its connection to nonlocality, quantum discord indicates the presence of non-commuting local observables in the decomposition of the state which does not necessarily imply strong correlations [15–20]. However, regardless of its interpretation in terms of correlations, quantum discord has proven to be an important resource for certain tasks in quantum communication and computation [9–11, 14]. It is considered especially promising in the context of operations involving highly mixed states, which emerge naturally due to the inevitable influence of noise [21].

Several methods have been developed which allow for the detection of quantum discord with relatively small effort if all subsystems are under sufficient degree of control [15, 22–24]. Recently, we have shown that the quantum discord of a bipartite system can be witnessed by accessing only one of the two subsystems [25]. The method extends a general theoretical scheme for the detection of initial correlations in the dynamics of open quantum systems developed in Ref. [26], which has been recently realized experimentally [27, 28]. Typically, an open quantum system represents a well-controlled quantum system which is coupled to a complex, largely inaccessible environment and therefore constitutes a natural setting in which we could benefit from the method described in this paper.

Our strategy for the construction of a local witness for the quantum discord in a bipartite system is based on a local dephasing operation, describing measurements carried out on one of the subsystems, which leaves the marginal states invariant while erasing all quantum discord between the two subsystems. When the subsequent time evolution of the composite, bipartite system is changed by this dephasing operation, one can conclude that the original state has a non-vanishing quantum discord. A suitable local witness for quantum discord is thus given by any appropriate measure for the distance between the time-evolved reduced subsystem states obtained from the total system states corresponding to the evolution with and without local dephasing operation [25].

In the present paper we develop the details of this method and provide a study of its applications to thermal equilibrium states of generic complex quantum systems. In order to assess the performance of our witness for quantum discord we compare the actual dynamics under randomly generated Hamiltonians with the mean values and fluctuations obtained from the average over the unitary group equipped with the Hamiltonians with the mean values and fluctuations obtained from the average over the unitary group equipped with the Haar measure, employing results of Ref. [29]. We conclude with the formulation of a general ergodicity-type hypothesis which relates the average of the local witness over the unitary group to the time average of the witness obtained for a generic system dynamics.
II. QUANTUM DISCORD AND LOCAL DEPHASING OPERATION

Throughout this paper we deal with a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, composed of local Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with dimensions $d_A$ and $d_B$, respectively. A state $\rho$ of the composite system has zero discord with respect to subsystem $A$ if and only if it can be written as

$$\rho = \sum_i p_i |i\rangle\langle i| \otimes \rho_B^i,$$  \hspace{1cm} (1)

with a basis $\{|i\rangle\}$ of $\mathcal{H}_A$, a probability distribution $\{p_i\}$, and a set of arbitrary quantum states $\{\rho_B^i\}$. States of zero discord are considered as classical. In the following we use this asymmetric definition, expressing classicality with respect to subsystem $A$. The reduced density operator $\rho_A = \sum_i p_i |i\rangle\langle i| \otimes \rho_B^i$ is obtained from $\rho$ via the partial trace over subsystem $B$. We introduce the quantum operation

$$\Phi(X_A) = \sum_i |i\rangle\langle i| X_A |i\rangle\langle i|$$ \hspace{1cm} (2)

which represents a completely positive and trace preserving linear map acting on operators $X_A$ of subsystem $A$. The definition in Eq. (1) is then equivalent to the following statement: A state $\rho$ has zero discord if and only if the operation

$$(\Phi \otimes I_B)\rho = \sum_i \Pi_i \rho \Pi_i$$ \hspace{1cm} (3)

leaves the state invariant [30], where we have introduced the local projectors $\Pi_i = |i\rangle\langle i| \otimes I_B$ onto the eigenbasis of $\rho_A$, and $I_B$ denotes the identity operation on subsystem $B$. Equation (3) defines the local dephasing operation in the eigenbasis of $\rho_A$. This operation constitutes the central element for the local detection scheme and has a series of important properties [25]:

(i) The operation (3) can be interpreted as a nonselective measurement in the eigenbasis of $\rho_A$, which is fully accessible from $\rho$ by measurements in the local subsystem $A$.

(ii) None of the two reduced density operators $\rho_A$ and $\rho_B$ is affected by application of the local dephasing.

(iii) The state produced by the local dephasing operation is always classical.

Property (i) is easily confirmed: Assume that the reduced state $\rho_A$ has been obtained by state tomography. After diagonalization this yields the local eigenbasis $\{|i\rangle\}$. The nonselective measurement in this basis is described by the operation (2), which by extension to the total Hilbert space results in the local dephasing operation (3) associated with the state $\rho$. Furthermore, this operation describes complete decoherence in the basis $\{|i\rangle\}$: The diagonal elements of any operator represented in this basis are left unchanged while all off-diagonal terms are set to zero.

To prove property (ii) we write the total state as $\rho = \sum_{\alpha} R_A^\alpha \otimes R_B^\alpha$, where $R_A^\alpha$ and $R_B^\alpha$ are operators on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The state after application of the local dephasing operation will be denoted by $\rho' = (\Phi \otimes I_B)\rho$. Its corresponding reduced density operator $\rho_B'$ of subsystem $B$ will be unchanged, since only the identity operation is applied to this part of the Hilbert space:

$$\rho_B' = \text{Tr}_A \rho' = \text{Tr}_A \sum_{\alpha} \Phi(R_A^\alpha) \otimes R_B^\alpha$$

$$= \sum_{\alpha} \text{Tr} \{\Phi(R_A^\alpha)\} R_B^\alpha$$

$$= \sum_{\alpha} \text{Tr} \{R_A^{\alpha'}\} R_B^\alpha = \text{Tr}_A \rho = \rho_B.$$ \hspace{1cm} (4)

The reduced state of subsystem $A$ is not altered since the measurement is performed in its own eigenbasis:

$$\rho_A' = \text{Tr}_B \rho' = \text{Tr}_B \sum_{\alpha} |i\rangle\langle i| R_A^{\alpha'} |i\rangle\langle i| \otimes R_B^\alpha$$

$$= \sum_{\alpha} \sum_i |i\rangle\langle i| \text{Tr} \{R_B^\alpha\} |i\rangle R_A^{\alpha'} |i\rangle$$

$$= \sum_i p_i |i\rangle\langle i| = \rho_A.$$ \hspace{1cm} (5)

where $p_i = \sum_{\alpha} \text{Tr} \{R_B^\alpha\} |i\rangle R_A^{\alpha'} |i\rangle |i\rangle = \langle i | \rho_A | i \rangle$.

Finally, property (iii) is obvious since $\rho'$ can be readily cast into the form of Eq. (1) with $\rho_B' = \sum_{\alpha} |i\rangle R_A^{\alpha'} |i\rangle R_B^\alpha$.

The combination of all three properties leads to an additional interpretation: Performing a nonselective measurement in the local eigenbasis, i.e., applying the corresponding local dephasing operation to a state $\rho$ erases the quantum discord in $\rho$ while leaving its marginals unchanged.

We end this section with remarks on two special situations. First, in the case of degeneracies in the spectrum of $\rho_A$, the local basis $\{|i\rangle\}$ in Eq. (1) is not uniquely determined by the local state $\rho_A$. Performing a local dephasing operation in an arbitrary eigenbasis of $\rho_A$ can then change the given total state $\rho$ even if it has zero discord. However, in the following we may ignore the possibility of degenerate local states since they form a set of zero measure. Second, if the local state tomography yields a pure state $\rho_A = |\varphi\rangle\langle \varphi|$, no further action is required. It is already safe to conclude that no total correlations exist between the two subsystems and the total state is a product state, $\rho = |\varphi\rangle\langle \varphi| \otimes \rho_B$. Specifically, this situation is encountered in experiments if one of the subsystems is prepared in a pure state. Total correlations in terms of the distance to the corresponding product state can be witnessed on the basis of an arbitrary local operation using the contraction property of the trace distance [20].
III. LOCAL WITNESS FOR QUANTUM DISCORD

For simplicity, we assume that the composition of the systems $A$ and $B$ forms a closed system. We will see below that this assumption can be dropped. The dynamics of a closed system is described by a unitary time evolution operator $U_t$, propagating states from time 0 to time $t$. Tracing over subsystem $B$ yields the reduced density matrix at time $t$, $\rho_A(t) = \text{Tr}_B \{U_t \rho U_t^\dagger \}$. The local detection method is based on the following idea: First, the accessible part of the unknown initial state $\rho$ is measured, yielding the state $\rho_A$ and its eigenbasis. After the reference state $\rho'$ is produced by local nonselective measurement of the total state in this basis, we compare the dynamics of the two reduced states $\rho_A(t)$ and $\rho'_A(t) = \text{Tr}_B \{U_t \rho' U_t^\dagger \}$. The difference of these states can be quantified by an arbitrary operator distance

$$\text{dist}(t) = \|\rho_A(t) - \rho'_A(t)\|^2 = \|\text{Tr}_B \{U_t (\rho - \rho') U_t^\dagger \}\|^2.$$ \hspace{1cm} (6)

First, note that $\text{dist}(0) = 0$ due to property (ii) of the local dephasing map. On the other hand, if we find an instant of time $t > 0$ for which $\text{dist}(t) > 0$, we can conclude that $\rho$ and $\rho'$ must be different states. This in turn implies that $\rho$ has nonzero discord which enables us to locally witness bipartite quantum discord [25].

Since the states $\rho$ and $\rho'$ differ only in their quantum discord, a possible measure for the amount of discord is given by the distance [30]

$$D(\rho) = \|\rho - \rho'\|^2.$$ \hspace{1cm} (7)

Until this point all results are independent of the specific choice of distance. For later applications we choose the squared Hilbert-Schmidt norm $\|A\|^2 = \text{Tr} A^\dagger A$, which has also been used in a similar context under the term geometric measure for quantum discord [15, 31]. With this choice, Eq. (7) can be written as a difference of purities [25]. More generally, for any map of the form $\Phi(X_A) = \sum_\pi \pi_i X_A \pi_i$ with a complete set of mutually orthogonal projection operators $\pi_i$ we have:

$$\|\rho - (\Phi \otimes I_B) \rho\|^2 = \text{Tr} \{\rho^2\} - \text{Tr} \{[(\Phi \otimes I_B) \rho]^2 \} = P(\rho) - P((\Phi \otimes I_B) \rho),$$ \hspace{1cm} (8)

with the purity $P(\rho) = \text{Tr} \{\rho^2\}$. To prove this relation we write the left-hand side of this equation as

$$\|\rho - (\Phi \otimes I_B) \rho\|^2 = P(\rho) - 2 \text{Tr} \{\rho (\Phi \otimes I_B) \rho\} + P((\Phi \otimes I_B) \rho).$$ \hspace{1cm} (9)

Making use of the Kraus representation of $\Phi$, we obtain

$$P((\Phi \otimes I_B) \rho) = \text{Tr} \left\{ \sum_{a,\beta,i,j} \delta_{ij} \pi_i R_A^a \pi_j R_B^\beta \otimes R_B^\beta R_B^\beta \right\}$$
$$= \sum_{a,\beta,i} \text{Tr} \{ R_A^a \pi_i R_A^\beta \} \text{Tr} \{ R_B^\beta R_B^\beta \}$$
$$= \text{Tr} \left\{ \sum_{a,\beta,i} R_A^a \pi_i R_A^\beta \otimes R_B^\beta R_B^\beta \right\}$$
$$= \text{Tr} \left\{ \rho (\Phi \otimes I_B) \rho \right\}.$$ \hspace{1cm} (10)

which proves Eq. (8). Moreover, this adds a nice operational interpretation to the measure $D(\rho)$ in terms of the purity-decreasing effect of the local dephasing operation. Note that if the state $\rho$ is pure, the expression $D(\rho)$ yields the generalized concurrence [25], a well-known entanglement measure [32, 33], illustrating the equivalence of discord and entanglement in the case of pure states.

For distance measures which are contractive under the action of trace-preserving quantum operations, $\text{dist}(t)$ provides a lower bound for the quantum discord expressed by $D(\rho)$. Using for example the trace norm defined by $\|A\|_1 = \text{Tr} A^\dagger A$ we obtain:

$$D(\rho) \geq \text{dist}(t).$$ \hspace{1cm} (11)

Even though the Hilbert-Schmidt distance is not contractive under trace-preserving operations, one can derive a lower bound for the quantum discord in terms of $\text{dist}(t)$, employing the contractivity of the trace distance and well-known upper and lower bounds for the Hilbert-Schmidt distance in terms of the trace distance:

$$D(\rho) \geq \frac{1}{d_A d_B} \text{dist}(t).$$ \hspace{1cm} (12)

We note that this method can even be extended to general linear time-evolutions given by a family of quantum dynamical maps $\Lambda_t$, such that $\rho(t) = \Lambda_t(\rho)$ and $\rho'(t) = \Lambda_t(\rho')$, which yields

$$\text{dist}(t) = \|\text{Tr}_B \{\Lambda_t (\rho - \rho')\}\|^2.$$ \hspace{1cm} (13)

Thereby the scheme can be used to detect correlations also in bipartite systems under additional dissipation caused by the coupling to an external environment. For the rest of this paper, we will restrict to the case of unitary evolution.

IV. PERFORMANCE OF THE WITNESS AND EXAMPLES

The above method may fail to detect correlations depending on the time evolution $U_t$. Consider for instance the trivial case of two uncoupled subsystems. The time evolution factorizes, $U = U_A \otimes U_B$, where we omit the
time argument. In this case, no signature of the total state will be visible in the reduced system dynamics, which can be seen easily by decomposing \( \rho - \rho' = \sum_\alpha D_A^\alpha \otimes D_B^\alpha \).

\[
\text{Tr}_B \{ U(\rho - \rho')U^\dagger \} = \text{Tr}_B \left\{ \sum_\alpha U_A D_A^\alpha U_A^\dagger \otimes U_B D_B^\alpha U_B^\dagger \right\} = \sum_\alpha U_A D_A^\alpha U_A^\dagger \text{Tr} \{ D_B^\alpha \}
\]

\[
= U_A \text{Tr}_B \{ \rho - \rho' \} U_A^\dagger = 0. \tag{14}
\]

The question is thus, what is the performance of the method for generic systems? In order to answer this question we make use of a recently developed approach based on unitary average values [25, 29]. In order to obtain an estimate for the quantity \( \text{dist}(t) \), we replace \( U_t \) with a random unitary matrix \( U \) and determine the average integrating over the uniform Haar measure \( d\mu \). According to ensemble theory, the average value is expected to reflect the behavior of generic complex quantum systems. We denote unitary average values by angular brackets,

\[
\langle F(U) \rangle = \int d\mu(U) F(U). \tag{15}
\]

The Hilbert-Schmidt distance for an arbitrary pair of states \( \rho \) and \( \rho' \) yields the average value [25]

\[
\mu = \langle \| \text{Tr}_B \{ U(\rho - \rho')U^\dagger \} \|^2 \rangle = \frac{d_A^2 d_B^2 - d_B}{d_A^2 d_B - 1} \| \rho - \rho' \|^2 ,
\]

and the variance [29]

\[
s^2 = \text{Var} \langle \| \text{Tr}_B \{ U(\rho - \rho')U^\dagger \} \|^2 \rangle = c_1 \text{Tr}((\rho - \rho')^2)^2 + c_2 \text{Tr}((\rho - \rho')^4), \tag{16}
\]

with the coefficients \( c_1 \) and \( c_2 \) given by

\[
c_1 = \frac{2(15 - 4d_A^2 d_B + d_A^4 d_B^2)(d_A^2 - 1)(d_B^2 - 1)}{(36 - 13d_A^2 d_B^2 + d_A^2 d_B^4)(d_A^2 d_B^2 - 1)^2},
\]

\[
c_2 = \frac{-10d_A d_B (d_B^2 - 1)(d_A^2 - 1)}{d_A^2 d_B^2 (d_A^2 d_B^2 - 7)^2 - 36}. \tag{18}
\]

Inserting \( \rho' = (\Phi \otimes I_B)\rho \) into Eq. (16), we find that the average increase of the local distance is directly proportional to the squared Hilbert-Schmidt distance of the original state \( \rho \) to its locally dephased reference state \( \rho' \), which we had previously defined as \( D(\rho) \), a measure for quantum discord. This result also holds for a more general average, which is performed only over the eigenvectors of the Hamiltonian while the time dependence and the eigenvalue distribution are retained, see Refs. [25, 29].

From Eqs. (16)-(18) we find that for large \( d_B \) the relative fluctuations are given by

\[
\frac{s}{\mu} \approx \frac{2}{d_A^2 - 1}. \tag{19}
\]

This ratio is always smaller than one and decreases as \( s/\mu \sim 1/d_A \) for large \( d_A \). Thus we see that the standard deviation \( s \) is at most of the same order of magnitude as the mean value \( \mu \) [29]. This statement is confirmed by the numerical studies discussed below. Since the median (50%-quantile) of a random number always lies in the range \( \mu \pm s \), we find that the squared reduced system Hilbert-Schmidt distance is larger than \( \mu - s \) with a probability of at least 50%. We conclude that for generic systems the quantum discord in the initial state will be successfully detected by the present method with high probability.

The main purpose of the unitary average value is to demonstrate the general reliability of the presented method. However, we note that from Eq. (16) we see that if the unitary average of the local distance could be measured, it could be used not only to witness the discord in the initial states, but also to quantify it. Even though the number of gates needed for the realization of Haar-random unitary operators scales exponentially with the number of qubits involved, there have been efforts aiming at the realization of unitary averages with methods which are experimentally feasible [34–36].

### A. Simple example of pure states

As a first simple illustration of this method, we consider pure states \( \rho_z = |\Psi_z\rangle \langle \Psi_z| \), with

\[
|\Psi_z\rangle = \sqrt{z}|00\rangle + \sqrt{1 - z}|11\rangle, \tag{20}
\]

and \( 0 \leq z \leq 1 \). The reduced system state is given by \( \rho_A = z|0\rangle \langle 0| + (1 - z)|1\rangle \langle 1| \), and to produce the reference state by local dephasing we project onto the operators \( |0\rangle \langle 0| \) and \( |1\rangle \langle 1| \),

\[
\rho'_z = (\Phi \otimes I_B)\rho_z = \sum_{i=0,1} (|i\rangle \langle i| \otimes I_B)\rho_z(|i\rangle \langle i| \otimes I_B). \tag{21}
\]

Thus, we obtain the reference state

\[
(\Phi \otimes I_B)\rho_z = z|00\rangle \langle 00| + (1 - z)|11\rangle \langle 11|. \tag{22}
\]
The mixedness of the reduced state of $\rho_z$ stems from the entanglement in $|\Psi_z\rangle$. On the other hand, $\rho_z'$ is only classically correlated but passes its own mixedness on to the reduced state. Since both states yield the same reduced density matrix, the nature of the total state cannot be revealed on the basis of the reduced system at the initial time. However, if the subsequent time evolution in the subsystem is taken into account, it is possible to distinguish between the total states with and without quantum correlations.

The generic increase of the distance in the reduced system is given by Eq. (16), which leads to

$$\mu = \left\langle \| \text{Tr}_B \{ U(\rho_z - \rho_z')U^\dagger \} \| \right\rangle^2 = \frac{2}{5} \mathcal{D}(\rho_z),$$

(23)

where $\mathcal{D}(\rho_z) = 2(1 - z)z$ is proportional to the square of the concurrence.

The variance is given by Eq. (17), which for this state yields

$$s^2 = \text{Var} \left( \| \text{Tr}_B \{ U(\rho_z - \rho_z')U^\dagger \} \| \right) = \frac{38}{175} (z - 1)^2 z^2.$$

(24)

The relative error is constant for all values of $z$ and amounts to $s/\mu = \sqrt{19/56} \approx 0.58$. The relatively large value of the variance is explained by the low dimensions of system and environment. A plot showing the dependence of expectation value and variance on the parameter $z$ is given in Fig. 1.

### B. Random Gibbs states of $2 \times d_B$ systems

In this section we demonstrate the local detection scheme for Gibbs states of randomly generated $d$-dimensional Hamiltonians. Once such a random $H$ has been generated [37], the Gibbs state can easily be obtained as $\rho_G = e^{-\beta H}/Z$, with the partition function $Z = \text{Tr} e^{-\beta H}$, $\beta = 1/kT$, temperature $T$, and the Boltzmann constant $k$. We consider the total Hilbert space to be $2d_B$-dimensional, i.e., the system Hilbert space $\mathcal{H}_A$ is two-dimensional. Employing the product basis $\{|0\rangle, |1\rangle\} \otimes \{|\chi_i\rangle\}_{i=1}^{d_B}$, where $\{|\chi_i\rangle\}$ denotes an arbitrary fixed basis of $\mathcal{H}_B$, the Gibbs state $\rho_G$ can be written as

$$\rho_G = \sum_{i,j} a_{ij}^{00} |0\rangle \otimes |\chi_i\rangle \langle \chi_j| + \sum_{i,j} a_{ij}^{01} |0\rangle \otimes |\chi_i\rangle \langle \chi_j|$$

$$+ \sum_{i,j} a_{ij}^{10} |1\rangle \otimes |\chi_i\rangle \langle \chi_j| + \sum_{i,j} a_{ij}^{11} |1\rangle \otimes |\chi_i\rangle \langle \chi_j|.$$

(25)

Hence, the reduced density operator of subsystem $A$ can be represented by the matrix

$$\rho_A = \text{Tr}_B \rho_G = \left( \sum_i a_{ii}^{00} \sum_j a_{ij}^{01} \right).$$

(26)

On the basis of the eigenvectors $\{|0\rangle, |1\rangle\}$ of this $(2 \times 2)$-matrix, the local dephasing map is expressed as

$$\Phi \otimes \mathbb{I}_B \rho = \Pi_{\tilde{0}} \rho \Pi_{\tilde{0}} + \Pi_{\tilde{1}} \rho \Pi_{\tilde{1}},$$

(27)

with $\Pi_{\tilde{0}} = \tilde{|0\rangle\langle 0|} \otimes \mathbb{I}_B$. Application of this map to the original Gibbs state $\rho_G$ creates the reference state $(\Phi \otimes \mathbb{I}_B) \rho_G$. Next, we examine the dynamics of the distance of the two reduced system states by creating the corresponding time evolution operator $U_t = \exp\{-iHt\}$ from the same randomly generated Hamiltonian $H$. The distance is given as a function of $t$ by:

$$\text{dist}(t) = \| \text{Tr}_B \{ U_t (\rho_G - (\Phi \otimes \mathbb{I}_B) \rho_G) U_t^\dagger \} \|. $$

(28)

On the other hand we can obtain the unitary expectation value and its variance for the same quantity by Eqs. (16) and (17), which in this case yield

$$\mu = \left\langle \| \text{Tr}_B \{ U(\rho_G - (\Phi \otimes \mathbb{I}_B) \rho_G)U^\dagger \} \| \right\rangle^2$$

$$= \frac{3d_B}{4d_B^2 - 1} \| \rho_G - (\Phi \otimes \mathbb{I}_B) \rho_G \|^2$$

(29)
FIG. 3. (Color online) Comparison of the unitary average with the actual time evolution for the Gibbs states of four randomly picked Hamiltonians for a qubit coupled to environments with different dimensions at fixed temperature $\beta = 1$.

and

$$
\begin{align*}
&s^2 = \text{Var} \left( \left\| \text{Tr}_B \{ U (\rho_G - (\Phi \otimes I_B) \rho_G) U^\dagger \} \right\|^2 \right) \\
&= \frac{3(15 - 16d_B^2 + 16d_B^4)}{2(1 - 4d_B^2)^2(4d_B^2 - 9)} \left\| \rho_G - (\Phi \otimes I_B) \rho_G \right\|^4 \\
&- \frac{15d_B}{9 - 40d_B^2 + 16d_B^4} \text{Tr} \left\{ (\rho_G - (\Phi \otimes I_B) \rho_G)^4 \right\}.
\end{align*}
$$

(30)

We have carried out an extensive numerical study of various cases with many different parameter sets and initial states. In the following we present a selection of our results to illustrate the main features. Figure 2 shows a series of time evolutions including the corresponding unitary average value $\mu$ and the first standard deviation $s$ for six randomly generated $2 \times 2$ Hamiltonians at fixed temperature $\beta = 1$. The dependence on the environmental dimension is plotted in Fig. 3, while Fig. 4 displays the role of the inverse temperature $\beta$. From these simulations we can make a number of observations. First, the numerical analysis suggests that generic Gibbs states contain quantum discord since the function $\text{dist}(t)$ assumes nonzero values for all realizations, confirming measure-theoretic studies on the abundance of quantum discord [20, 38]. Second, for most of the examples the time evolution fits nicely into the margin given by the unitary average within one standard deviation, indicated by the highlighted areas. It is of course no surprise to find some deviating realizations as in the top right picture of Fig. 2. Third, as becomes obvious by comparison of Figs. 2 and 3, the unitary average value depends stronger on the dimensions of system and environment than on the actual Hamiltonian. The values in Fig. 2 differ only very little between the considered random examples, while in Fig. 3 we see that the average value $\mu$ and the standard deviation $s$ decrease significantly with increasing environmental dimension. This is mainly caused by the dimension-dependent factors in Eqs. (29) and (30).

Figure 4 shows how the witness $\text{dist}(t)$ changes for different temperatures. We see however that the overall functional shape remains similar which is due to the fact that the Hamiltonian is the same in all plots. The bottom right picture shows the asymptotic convergence of mean value and variance for decreasing temperature. In the high-temperature limit ($\beta \to 0$) the unitary average value, and with it the generic effect of the initial correlations on the reduced system vanishes as expected since the state becomes closer to a complete mixture, which is a state of zero discord. Note that correlations in the low-temperature limit of the Gibbs state can be used to reveal the structure of the ground state [39], which in turn can be associated with a quantum phase transition [40–43].

To conclude this section, we recall that a state of nonzero discord cannot be a factorized product state [44]. On the other hand, factorizing initial conditions are commonly assumed in the derivation of master equations for...
the dynamical description of open systems in terms of completely positive maps, see, e.g., Refs. [21, 26, 45] and references therein. Hence, the present method can also be used to detect deviations from this assumption [25]. Obviously, if the witness is nonzero, a dynamical map which is independent of the correlations does not exist. A study of the role of the total initial correlations in thermal equilibrium states is presented in Ref. [39].

C. An ergodicity-like relation

The foregoing study shows that unitary averages provide important and useful information about the time evolution, which may be experimentally observable. It was pointed out in Ref. [29] that the dimension \( d_B \) appearing in expressions for the averages must be chosen carefully. Formally, it is always possible to artificially increase the dimension of the Hilbert space by including an additional Hilbert space which is not coupled to the original system. Correspondingly, the dimension appearing in the expectation value must be regarded as an effective dimension, indicating the dimension of the subspace of the Hilbert space which actually affects the local dynamics. In general, a suitable, effective dimension \( d_B^{\text{eff}} \) may be defined via the equality

\[
\langle \| \text{Tr}_B \{ U (\rho - \rho') U^\dagger \} \|^2 \rangle_{\text{eff}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \| \text{Tr}_B \{ U_t (\rho - \rho') U_t^\dagger \} \|^2. \tag{31}
\]

Thus, we are led to an ergodicity-like hypothesis for complex generic systems expressing the equivalence of the unitary average value and the time average according to the given, actual Hamiltonian: For complex generic systems, the effective dimension coincides with the dimension of the Hilbert space. The effective dimensions of non-generic systems depend not only on the system parameters but also on the observable in question. For example, in a partly chaotic system with regular areas, some initial states may explore large parts of the state space in the course of their time evolution while for different initial conditions only a very limited fraction may be visited. The estimation of the dimension of quantum systems is a topic of growing interest [46].

V. CONCLUSION

The method discussed in this paper allows for the detection of quantum discord in bipartite systems when access to only one of the subsystems is possible. This situation emerges naturally in the context of open quantum systems and quantum communication protocols. The procedure was illustrated by application to thermal equilibrium states of random Hamiltonians. In order to estimate the performance of the method for generic systems we compared the time evolution with the value obtained by averaging over all unitary evolutions employing the Haar measure. The mean values as well as the fluctuations predicted by the Haar measure were found to be in good agreement with the actual time evolution. This fact led to the proposition of an ergodicity-like hypothesis, linking unitary average and time average, and to the introduction of an effective dimension of the underlying Hilbert space. Further studies are required, on the one hand to confirm this hypothesis with additional examples of generic systems and, on the other hand, to obtain the effective dimensions of non-generic systems which typically exploit only an effective subspace whose dimension is much lower than that of the total Hilbert space.

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