TCFHs, hidden symmetries and M-theory backgrounds

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Abstract

We present the TCFH of 11-dimensional supergravity and so demonstrate that the form bilinears of supersymmetric solutions satisfy a generalisation of the conformal Killing-Yano equation with respect to the TCFH connection. We also compute the Killing-Stäckel, Killing-Yano and closed conformal Killing-Yano tensors of all spherically symmetric M-branes that include the M2-brane, M5-brane, KK-monopole and pp-wave and demonstrate that their geodesic flows are completely integrable by giving all independent conserved charges in involution. We then find that all form bilinears of pp-wave and KK-monopole solutions generate (hidden) symmetries for spinning particle probes propagating on these backgrounds. Moreover, there are Killing spinors such that some of the 1-, 2- and 3-form bilinears of the M2-brane solution also generate symmetries for spinning particle probes. We also explore the question on whether the form bilinears are sufficient to prove the integrability of particle probe dynamics on 11-dimensional supersymmetric backgrounds.
1 Introduction

Killing-Stäckel (KS) tensors and (conformal) Killing-Yano ((C)KY) forms have a long and distinguished history in general relativity as they have been used to investigate the integrability and separability properties of many classical equations, like the geodesic, Hamilton-Jacobi, Klein-Gordon, Dirac and Maxwell equations, on black hole spacetimes, see selected references [1]-[9] and reviews [10, 11]. In particular KS tensors generate (hidden) symmetries for relativistic particle probes propagating on gravitational backgrounds and so symmetries of the geodesic flow. While KY forms, which can be thought of as the “square root” of KS tensors, generate (hidden) symmetries for spinning particle probes [12] propagating on gravitational backgrounds [13].

More recently, it has been demonstrated in [14] that the conditions imposed by the Killing spinor equations (KSEs) on the (Killing spinor) form bilinears of any supergravity theory, which may include higher order curvature corrections, can be arranged as a twisted covariant form hierarchy (TCFH) [15]. This means that these conditions can be written as

\[ \mathcal{D}_X^F \Omega = i_X \mathcal{P} + X \wedge \mathcal{Q} , \]

for every spacetime vector field \( X \), where \( \Omega \) is a multiform with components the form bilinears, \( \mathcal{P} \) and \( \mathcal{Q} \) are appropriate multi-forms which depend on the bilinears and the fields of the theory. Note that \( X \) also denotes the associated 1-form constructed from the vector field \( X \) after using the spacetime metric \( g \), \( X(Y) = g(X,Y) \). Furthermore \( \mathcal{D}_X^F \) is a connection on the space of forms which depends on the fluxes \( F \) of the supergravity theory that it is not necessarily form degree preserving. A consequence of the TCFH is that the form bilinears \( \Omega \) satisfy a generalisation of the CKY equation with respect to \( \mathcal{D}_X^F \) connection as one can easily verify by skew-symmetrising and taking the contraction with respect to the spacetime metric of (1). This raises the question on whether the form bilinears generate symmetries for appropriate probes propagating on supersymmetric backgrounds. This question has been explored before in 4- and 5-dimensional minimal supergravities [16] as well in type II 10-dimensional supergravities [17].

One purpose of this paper is to present the full TCFH of 11-dimensional supergravity. We shall find that the reduced holonomy of the minimal\(^1\) TCFH connection is included in \( SO(10,1) \times GL(517) \times GL(495) \) while the reduced holonomy of the maximal TCFH connection is included in \( GL(528) \times GL(496) \). The latter holonomy is the same as that of the maximal connection of IIA and IIB TCFH [17]. Then we shall explore the question on whether the TCFH conditions can be identified with the invariance conditions of a probe action under transformations generated by the form bilinears. As the supersymmetric backgrounds of 11-dimensional supergravity have not been classified, we shall focus our investigation on the M-brane solutions\(^2\) which include the M2- and M5-branes as well as the pp-wave and KK-monopole.

Before we proceed with the investigation of the TCFH for M-branes, we shall give the KS tensors and KY forms associated with the complete integrability of the geodesic

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\(^1\)See [14] for the definition of these connections.

\(^2\)These have been instrumental in the understanding of string dualities [18, 19].
flow of spherically symmetric M-brane solutions, i.e. those that depend on a harmonic function with one centre. The geodesic equations of these backgrounds are separable in angular variables. Here we shall present all independent conserved charges which are in involution. Moreover we shall demonstrate that a relativistic particle probe propagating on spherically symmetric M-branes admits an infinite number of hidden symmetries generated by KS tensors. In addition, we shall find that the spinning particle probe action admits $2^8$, $2^7$ and $2^4$ symmetries generated by KY forms on the pp-wave, M2-brane and M5-brane backgrounds, respectively. Spinning particle probes exhibit enhanced worldline supersymmetry propagating on the KK-monopole.

After this, we shall return to investigate under which conditions the form bilinears of M-brane backgrounds, which may now depend on a general harmonic function and so they are not necessarily spherically symmetric, generate symmetries for spinning particle type of probes. For this we match the conditions required for a transformation generated by the form bilinears to leave a spinning particle probe action invariant with the TCFH conditions on the form bilinears. We shall find that all form bilinears of pp-wave and KK-monopole backgrounds generate symmetries for the spinning particle probes. This is because as a consequence of the TCFH and the vanishing of the 4-form field strength for these solutions, the form bilinears are covariantly constant with respect to the Levi-Civita connection. Furthermore we demonstrate that there are Killing spinors such that the 1-form, 2-form and 3-form bilinears of the M2-brane are KY forms and so generate symmetries for spinning particle probes propagating on this background. A similar analysis for the M5-brane reveals that only the 1-form bilinear generates symmetries for spinning particle probes. To demonstrate these results, we have computed all the form bilinears of M-brane backgrounds using spinorial geometry [20].

This paper is organised as follows. In section 2, we present the TCFH of 11-dimensional supergravity and give the reduced holonomy of TCFH connections. In section 3, we give the KS and KY tensors of spherically symmetric M-brane backgrounds and prove the complete integrability of their geodesic flows. In section 4, we identify the form bilinears of M-branes that generate symmetries for probe actions, and in section 5 we give our conclusions. In appendix A, we give the form bilinears of the M5-brane. In appendix B, we explore the symmetries of spinning particle probes with 4-form couplings.

2 The TCFH of D=11 supergravity

The supercovariant connection of 11-dimensional supergravity [21] is

$$D_\mu = \nabla_\mu + \frac{1}{288} (\Gamma_{\mu \nu_1 \nu_2 \nu_3} F_{\nu_1 \nu_2 \nu_3} - 8 F_{\mu \nu_1 \nu_2 \nu_3} \Gamma^{\nu_1 \nu_2 \nu_3}) ,$$

(2)

where $\nabla$ is the spin connection of the spacetime metric, $F$ is the 4-form field strength of the theory and $\epsilon$ is a spin$(10,1)$ Majorana spinor. The reduced holonomy of supercovariant connection on generic backgrounds is included in $SL(32,\mathbb{R})$ [22, 23, 24].

Supersymmetric backgrounds with $N$ Killing spinors, $\epsilon^r$, $r = 1, \ldots, N$, are those that admit $N$ linearly independent solutions to the KSE, $D_\mu \epsilon^r = 0$. Given $N$ Killing spinors, one can construct the form bilinears

$$f^{rs} = \langle \epsilon^r, \epsilon^s \rangle , \quad k^{rs}_\mu = \langle \epsilon^r, \Gamma_\mu \epsilon^s \rangle , \quad \omega^{rs}_{\mu \nu} = \langle \epsilon^r, \Gamma_{\mu \nu} \epsilon^s \rangle , \quad \varphi^{rs}_{\mu_1 \mu_2 \mu_3} = \langle \epsilon^r, \Gamma_{\mu_1 \mu_2 \mu_3} \epsilon^s \rangle ,$$

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\[ \theta_{\mu_1\mu_2\mu_3\mu_4} = \langle \epsilon^r, \Gamma_{\mu_1\mu_2\mu_3\mu_4} \epsilon^s \rangle, \quad \tau_{\mu_1\mu_2\mu_3\mu_4} = \langle \epsilon^r, \Gamma_{\mu_1\mu_2\mu_3\mu_4} \epsilon^s \rangle. \]

Note that the form bilinears \( k, \omega \) and \( \tau \) are symmetric in the exchange of \( \epsilon^r \) and \( \epsilon^s \) while the rest are skew-symmetric. There is no a classification of supersymmetric solutions of 11-dimensional supergravity. However there are many partial results. For example the maximally supersymmetric solutions have been classified in [25] and the KSE has been solved for one Killing spinor in [26, 27, 20], see review [28] for the current state of the art.

The TCFH of 11-dimensional supergravity for the form bilinears which are symmetric in the exchange of the two Killing spinors has been given in [14]. Here we shall present the TCFH for all form bilinears. The TCFH of 11-dimensional supergravity expressed in terms of the minimal connection \( D^F_\mu \) reads

\[
D^F_\mu k_{\nu} := \nabla_\mu k_{\nu} = \frac{1}{6} F_{\mu
u\alpha\beta\omega^{\alpha\beta}} - \frac{1}{6!} \epsilon_{\mu
u\alpha_1...\alpha_5 \rho_1...\rho_5} \tau^{\rho_1...\rho_5} \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5},
\]

\[
D^F_\mu \omega_{\nu_1\nu_2} := \nabla_\mu \omega_{\nu_1\nu_2} - \frac{1}{2 \cdot 3!} F_{\mu
u_1\nu_2\rho_3} \tau^{\rho_1\rho_2\rho_3} \nu_{\nu_1\nu_2} = -\frac{1}{3} F_{\mu\nu_1\nu_2\rho} k^\rho,
\]

\[
D^F_\mu \tau_{\nu_1\nu_2\nu_3\nu_4\nu_5} := \nabla_\mu \tau_{\nu_1\nu_2\nu_3\nu_4\nu_5} + \frac{5}{2} \epsilon_{\mu\nu_1\nu_2\nu_3\nu_4} \omega_{\nu_3\nu_4\nu_5} - \frac{5}{6} F_{\mu\nu_1\nu_2\nu_3} \tau_{\nu_4\nu_5} \tau^{\rho_1\rho_2\rho_3} = -\frac{5}{10} F_{\mu\nu_1\nu_2\nu_3\nu_4} k^\rho + \frac{5}{6} F_{\mu\nu_1\nu_2\nu_3} \omega_{\nu_4\nu_5} + \frac{5}{6} \tau_{\mu\nu_1\nu_2\nu_3} F_{\nu_4\nu_5} \tau^{\rho_1\rho_2\rho_3} = -\frac{1}{3} g_{\mu\nu_1\nu_2\nu_3} \theta^{\rho_1\rho_2\rho_3} F_{\nu_4\nu_5} \tau_{\rho_1\rho_2\rho_3} + \frac{5}{18} g_{\mu\nu_1\nu_2} \tau_{\rho_1\rho_2\rho_3} F_{\nu_3\nu_4\nu_5} \theta^{\rho_1\rho_2\rho_3},
\]

where for simplicity we have suppress the indices \( r \) and \( s \) on the form bilinears with label the independent Killing spinors. In our conventions \( \epsilon_{0123456789} = -1 \), \( \epsilon_{\mu_1...\mu_7} = 1 \), \( \epsilon_{\mu_1...\mu_9} F_{\mu_1...\mu_9} \) and \( \Gamma_\mu := \Gamma_{0...9} \), where \( \Gamma \) denotes the 11th direction. Clearly the equations above are of the form stated in (1), where \( \Omega \) is the multiform spanned by the form bilinears (3), \( \Omega \) can be read from the terms in the right hand side of (4) that explicitly contain the spacetime metric \( g \) and \( P \) is spanned by the remaining terms in the right hand side of (4). Clearly (4) provides a geometric interpretation of the conditions induced by the KSE on the form bilinears as it relates them to a generalisation of the CKY equations.

Viewing \( D^F_\mu \) as degree non-preserving connection on k-forms, \( k = 0, 1, 2, 3, 4, 5 \), the reduced holonomy of \( D^F_\mu \) factorises as the connection preserves the subspaces of \( k \)-degree forms for \( k = 1, 2, 5 \) and for \( k = 3, 4, 5 \), i.e. it preserves the subspaces of the form bilinears which are symmetric and skew-symmetric under the exchange of the two Killing spinors.
This is also the case for the maximal connection defined in [14] which we do not consider here in detail. In addition, $D^F_\mu$ preserves the subspace of 1-forms, and the subspace of 2- and 5-forms, and acts trivially on 0-forms. As a result the reduced holonomy of $D^F_\mu$ is included in $SO(10,1) \times GL(517) \times GL(495)$ group. Note that the reduced holonomy of the maximal connection is included in $GL(528) \times GL(496)$ as it does not preserve the subspace of 1-forms but instead it mixes them with the subspace of 2- and 5-forms and it acts non-trivially on 0-forms. The reduced holonomy of the maximal connection is the same as that of the maximal TCFH connections of type IIA and type IIB supergravities [17]. Of course for special backgrounds the holonomy of $D^F_\mu$ reduces further.

3 Symmetries of probes on M-brane backgrounds

3.1 Symmetries and integrability

We shall begin with a summary of the key properties of Killing-Stäckel (KS) tensors and Killing-Yano (KY) forms. As this has already appeared in the form required here elsewhere [17], we shall be brief. Consider the action of a relativistic particle probe propagating on a spacetime $M$ with metric $g$

$$A = \frac{1}{2} \int dt \, g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu ,$$

where $\dot{x}$ denotes the derivative of the coordinate $x$ with respect to $t$. The equations of motion are those of the geodesic flow on $M$ with affine parameter $t$. Given a rank $k$ Killing-Stäckel (KS) tensor on $M$, i.e. a symmetric $(0,k)$ tensor $d$ on $M$ which satisfies that equation

$$\nabla_{(\mu} d_{\nu_1 \nu_2 \cdots \nu_k)} = 0 ,$$

where $\nabla$ is the Levi-Civita connection of $g$, the action (5) is invariant under the infinitesimal transformations

$$\delta x^\mu = \epsilon d_{\nu_1 \cdots \nu_{k-1}}^\mu \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_{k-1}} ,$$

with parameter $\epsilon$. The associated conserved charge is

$$Q(d) = d_{\nu_1 \nu_2 \cdots \nu_k} \dot{x}^{\nu_1} \dot{x}^{\nu_2} \cdots \dot{x}^{\nu_k} .$$

For $k = 1$, $d$ is a Killing vector field. The symmetrised tensor product of two KS tensors is also a KS tensor. Hidden symmetries are those generated by rank $k \geq 2$ KS tensors $d$ with $d \neq g$.

A conformal Killing-Yano (CKY) tensor is a $k$-form on a spacetime $M$ with metric $g$ which satisfies the condition

$$\nabla_\mu \alpha_{\nu_1 \nu_2 \cdots \nu_k} = \frac{1}{k+1} d\alpha_{\nu_1 \nu_2 \cdots \nu_k} - \frac{k}{n-k+1} g_{\mu [\nu_1} \delta \alpha_{\nu_2 \cdots \nu_k]} .$$

If $\delta \alpha = 0$, then $\alpha$ is a KY form while if $d\alpha = 0$, $\alpha$ is a closed conformal Killing-Yano (CCKY) form. It turns out that if $\alpha$ is KY, then the Hodge dual $*\alpha$ is CCKY form. KY
forms are the “square roots” of KS tensors. In particular if \( \alpha \) and \( \beta \) are k-KY forms, then 
\[
\alpha(\mu_{\lambda_1 \cdots \lambda_{k-1}} \beta_{\nu})_{\lambda_1 \cdots \lambda_{k-1}} \text{ is a rank 2 KS tensor.}
\]

A spinning particle probe propagating on a spacetime \( M \) with metric \( g \) is described by the action
\[
A = -\frac{i}{2} \int dt d\theta \; g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu ,
\]
where \( t \) and \( \theta \) are the even and odd coordinates of the worldline superspace, respectively, \( x \) are worldline superfields \( x = x(t, \theta) \) and \( D^2 = i\partial_t \). Spinning particles are supersymmetric extensions of relativistic particles.

Given a KY form, \( \alpha \), on \( M \), the infinitesimal transformation
\[
\delta x^\mu = \epsilon \alpha^\mu_{\nu_1 \cdots \nu_{k-1}} D x^{\nu_1} \cdots D x^{\nu_{k-1}} ,
\]
with parameter \( \epsilon \) leaves the spinning particle action (10) invariant. The associated conserved charge is
\[
Q(\alpha) = (k+1) \alpha^\nu_{\mu_1 \cdots \mu_{k-1}} \partial_t x^{\mu_1} \cdots D x^{\mu_{k-1}} - \frac{i}{k+1} \left( d\alpha \right)_{\nu_1 \nu_2 \cdots \nu_{k+1}} D x^{\nu_1} D x^{\nu_2} \cdots D x^{\nu_{k+1}} .
\]
Observe that \( Q(\alpha) \) is preserved, \( DQ(\alpha) = 0 \), subject to the equations of motion of (10). Note that if \( d\alpha = 0 \) and so \( \alpha \) is covariantly constant (or equivalently parallel) with respect to the Levi-Civita connection, then
\[
\tilde{Q}(\alpha) = \alpha^\nu_{\mu_1 \cdots \mu_{k}} D x^{\mu_1} D x^{\mu_2} \cdots D x^{\mu_{k}} ,
\]
is also conserved subject to the field equations of (10), \( \partial_t \tilde{Q}(\alpha) = 0 \). There are several generalisations of the KS and CKY tensors, see e.g. [30]-[37].

The commutator algebra of transformations (11) generated by spacetime forms has been examined in detail in [29]. Given two symmetries (11) generated by the \( k \)-form \( \alpha \) and \( \ell \)-form \( \beta \), the commutator contains two types of terms. One terms depends on the Nijenhuis tensor of \( \alpha \) and \( \beta \) and the other term is the transformation
\[
\delta x^\mu = \epsilon (\alpha \cdot s \beta)^\mu_{\nu_1 \cdots \nu_{k+\ell-4}} \partial_t x^{\nu_1} D x^{\nu_2} \cdots D x^{\nu_{k-2}} D x^{\lambda_1} \cdots D x^{\lambda_{k+\ell-4}} ,
\]
generated by the tensor
\[
(\alpha \cdot s \beta)^\mu_{\nu_1 \cdots \nu_{k+\ell-4}} = \alpha^\mu_{\nu} [\lambda_1 \cdots \lambda_{k-2} \beta^\nu_{[\nu} | \lambda_{k-1} \cdots \lambda_{k+\ell-4}] + \alpha^\mu_{\nu} [\lambda_1 \cdots \lambda_{k-2} \beta^\nu_{\mu]} | \lambda_{k-1} \cdots \lambda_{k+\ell-4}] ,
\]
where \( \epsilon \) is an infinitesimal parameter. Clearly if \( \alpha \) and \( \beta \) are rank 2 KY tensors, then \( \alpha \cdot s \beta \) is a KS tensor. In the case that both \( \alpha \) and \( \beta \) are covariantly constant with respect to the Levi-Civita connection, the Nijenhuis tensor vanishes and so the transformation (14) is a symmetry of the spinning particle action (10). This will be the case for all symmetries generated by the form bilinears of pp-wave and KK-monopole solutions.

Consider a dynamical system with \( 2n \)-dimensional phase space \( P \). This is completely integrable, according to Liouville, provided that \( P \) admits \( n \) independent functions (observables) \( Q_r, r = 1, \ldots, n, \) including the Hamiltonian, in involution. \( Q_r \) are independent
provided that the map $Q : P \to \mathbb{R}^n$, where $Q = (Q_1, \ldots, Q_n)$, has rank $n$. Moreover $Q_r$ are in involution, iff $\{Q_r, Q_s\}_{PB} = 0$, i.e. Poisson bracket of any two $Q_r$s’ vanishes.

Returning to the relativistic particle, the conserved charges (8) can be written in phase space variables as

$$Q(d) = d^\nu_1 \cdots d^\nu_k p_{\nu_1} \cdots p_{\nu_k},$$  \hspace{1cm} (16)

where $p_\mu$ is the conjugate momentum of $x^\mu$ and we have raised the indices of $d$ with the spacetime metric $g$. These clearly commute with the Hamiltonian $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ as they are constants of motion. Furthermore the Poisson bracket algebra of two constants of motion $Q(d_1)$ and $Q(d_2)$ is $\{Q(d_1), Q(d_2)\}_{PB} = Q([d_1, d_2]_{NS})$, where

$$([d_1, d_2]_{NS})^{\nu_1 \cdots \nu_{k+\ell-1}} = k d_1^{\mu(\nu_1 \cdots \nu_{k-1}} \partial_\mu d_2^{\nu_{k+\ell-1})} - \ell d_2^{\mu(\nu_1 \cdots \nu_{k-1}} \partial_\mu d_1^{\nu_{k+\ell-1})},$$ \hspace{1cm} (17)

is the Nijenhuis-Schouten bracket of the KS tensors $d_1$ and $d_2$. Observe that if $d_1$ is a vector, then $[d_1, d_2]_{NS} = \mathcal{L}_{d_1}d_2$, i.e. the Nijenhuis-Schouten bracket is the Lie derivative of $d_2$ with respect to the vector field $d_1$. Therefore two charges are in involution provided that the Nijenhuis-Schouten bracket of the associated KS tensors vanishes.

In the examples that follow below, the complete integrability of the geodesic flow of the spacetimes considered is due to the large number of isometries that these spacetimes admit. As the Lie algebra of these isometries is not abelian, the associated conserved charges are not in involution. Nevertheless, it is possible to use these charges to construct new ones associated with KS tensors which are in involution, see the example below.

### 3.2 Complete integrability of black hole geodesic flow

Before we proceed to investigate the symmetries of probes on M-theory backgrounds, let us present some examples. The standard example is the integrability of the geodesic flow of the Kerr black hole. However more suitable for the results that follow are the examples of Schwarzschild and Reissner-Nordström black holes in four and higher dimensions. The metric of both these solutions in four dimensions can be written as

$$g = -A(r)t^2 + A^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$ \hspace{1cm} (18)

The associated geodesic equations of the metric above can be explicitly separated in the stated coordinates. However it is instructive to provide a symmetry argument for the complete integrability of the geodesic equations.

The isometry group of the above backgrounds is $\mathbb{R} \times SO(3)$. There are two commuting isometries given by $k_0 = \partial_t$ and $k_1 = \partial_\phi$ which give rise to the conserved charges $K_0 = p_t$ and $K_1 = p_\phi$. These together with the Hamiltonian $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ give three conserved charges in-involution. Note that $[K_r, H]_{NS} = \mathcal{L}_{k_r}g^{\mu\nu}p_\mu p_\nu = 0$, $r = 0, 1$, as $k_r$ are isometries.

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\(^3\)Complete integrability is related to the separability of the equations of motion of a dynamical system. In the phase space coordinates $(Q_1, \ldots, Q_n, \psi_1, \ldots, \psi_n)$ defined by the charges $Q_1, \ldots, Q_n$ and the action-angle coordinates $(\psi_1, \ldots, \psi_n)$ adapted to the Hamiltonian vector fields, $X_{Q_i} = \partial_{\psi_i}$, the time evolution of the system is at most linear.
It remains to find a fourth conserved charge in involution for the complete integrability of the geodesic system. For this consider the Killing vector fields

\[ k_1 = \partial_\phi , \quad k_2 = -\sin \phi \cot \theta \partial_\phi + \cos \phi \partial_\theta , \quad k_3 = \cos \phi \cot \theta \partial_\phi + \sin \phi \partial_\theta , \quad (19) \]

which generate the \( SO(3) \) isometry group and notice that \([k_a, k_b] = -\epsilon_{abc}k_c\). Then another conserved charge can be constructed utilising the (quadratic) Casimir operator of the Lie algebra of \( SO(3) \) which can be used to construct a symmetric tensor that commutes with all the isometries of the background. As the quadratic Casimir is proportional to the identity matrix in the basis chosen for the Lie algebra, the associated symmetric tensor is

\[ d = \delta^{ab}k_a \otimes k_b = \frac{1}{\sin^2 \theta} (\partial_\phi)^2 + (\partial_\theta)^2 . \quad (20) \]

This is a KS tensor because \( k_a \) are Killing vectors. Thus

\[ Q(d) = \frac{1}{\sin^2 \theta} (\partial_\phi)^2 + (\partial_\theta)^2 , \quad (21) \]

is a conserved charge of the geodesic flow of the metric (18). It turns out that \( K_r, H \) and \( Q(d) \) are independent and in involution implying that the geodesic equations are completely integrable for any function \( A = A(r) \) in (18).

The metric (18) also admits a CCKY 2-form [11]. This is given by

\[ \beta = r dt \wedge dr , \quad (22) \]

which can be verified after a computation. The dual

\[ \alpha = *\beta = r^3 \sin \theta d\theta \wedge d\phi , \quad (23) \]

is a KY 2-form. As a result it generates a symmetry for the spinning particle action (10) given by the infinitesimal variation in (11). There are four additional KY 1-forms constructed from the Killing vector fields \( k_0, k_1, k_2, k_3 \) using the metric. All of which generate symmetries for the action (10). One can also square the KY tensor (23) to construct a KS tensor. It turns out that this is not independent from (20).

The analysis we have done can be extended to black holes in higher than four dimensions. Indeed consider the metric

\[ g = -A(r)dt^2 + A^{-1}(r)dr^2 + r^2 g(S^n) , \quad (24) \]

where \( g(S^n) \) is the round metric on \( S^n \) with \( n \geq 2 \). Again the geodesic equation can be separated in angular coordinates and the geodesic flow is completely integrable. The above metric admits a \( \mathbb{R} \times SO(n + 1) \) group of isometries. Viewing \( S^n \) embedded as the hypersurface, \( \sum_i (x^i)^2 = 1 \), in \( \mathbb{R}^{n+1} \), the Killing vectors of the spacetime metric \( g \) can be written as

\[ k_0 = \partial_t , \quad k_{ij} = x_i \partial_j - x_j \partial_i , \quad i < j , \quad (25) \]

where \( i, j = 1, \ldots, n+1 \) and \( x_i = x^i \). Note that \( k_{ij} \) are tangent to \( S^n \) as \( (d(x^2 - 1))(k_{ij}) = 2x_k dx^k (k_{ij}) = 0 \). The associated conserved charges are \( Q_0 = p_t \) and \( Q_{ij} = x_i p_j - x_j p_i \),
where \( p_i \) is the momentum on \( S^n \) and so \( x^i p_i = 0 \). These conserved charges are not in involution. However

\[
Q_0 \, , \quad D_m = \frac{1}{4} \sum_{i,j \geq n+2-m} (Q_{ij})^2 \, , \quad m = 2, \ldots, n+1 ,
\]

(26)

are independent conserved charges of the geodesic flow and in involution which together with the Hamiltonian, \( H \), of the geodesic motion imply the complete integrability of the geodesic flow of the metric (24).

To explain the choice of \( D_m \) charges in (26), note that \( D_{n+1} \) is the Hamiltonian of the geodesic flow on \( S^n \) and it is constructed using the quadratic Casimir operator of \( \mathfrak{so}(n+1) \). The \( \mathfrak{so}(n+1) \) algebra admits a decomposition

\[
\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \cdots \subset \mathfrak{so}(n) \subset \mathfrak{so}(n+1) .
\]

(27)

The \( D_m \) conserved charge is constructed using the quadratic Casimir operator of the \( \mathfrak{so}(m) \) subalgebra of \( \mathfrak{so}(n+1) \). At each stage as the quadratic Casimir operator of \( \mathfrak{so}(m) \) is invariant under \( \mathfrak{so}(m) \), it is also invariant under the \( \mathfrak{so}(m-1) \) subalgebra of \( \mathfrak{so}(m) \). Therefore the quadratic Casimir operator of \( \mathfrak{so}(m-1) \) commutes with that of \( \mathfrak{so}(m) \). As a consequence \( D_{m-1} \) is in involution with \( D_m \). This method of constructing observables in involution has been generalised and used in [38] to investigate the integrability of geodesic flows on homogeneous manifolds.

Moreover a direct computation reveals that \( \beta \) in (22) is a CCKY form for the metric (24) and therefore its dual \( \alpha \) is a KY \( n \)-form. It turns out that \( \beta \) in (22) is a CCKY with respect to a metric as in (24) with \( g(S^n) \) now replaced with the metric, \( g(N) \), of any \( n \)-dimensional manifold \( N \) provided it is independent from the coordinates \( r \) and \( t \).

### 3.3 Hidden symmetries and spherically symmetric M-branes

Next let us turn to investigate the symmetries of relativistic and spinning particle probes described by the actions (5) and (10), respectively, propagating on M-branes. The focus will be on those KS and KY tensors which give rise to conserved charges related to the integrability of the geodesic flow on some of these backgrounds.

#### 3.3.1 M-theory pp-waves

The M-theory pp-wave solution is

\[
g = 2du(dv + \frac{1}{2}h(y,v)du) + \delta_{ij}dy^i dy^j ,
\]

(28)

with \( F = 0 \), where \((u, v, y)\) are the coordinates of 11-dimensional spacetime and \( h \) is a Harmonic function on \( \mathbb{R}^n \), \( \partial_y^2 h = 0 \). As the \( \partial_y^2 h = 0 \) condition appears in other M-brane backgrounds below, the solutions of this equation that we shall be considering on \( \mathbb{R}^n \), \( n > 2 \), are

\[
h = q_0 + \sum_{m=1}^{\ell} \frac{q_m}{|y - y_m|^{n-2}} , \quad q_0 = 0, 1 ,
\]

(29)
where $q_m$ are constants, $|·|$ is the Euclidean norm on $\mathbb{R}^n$ and $y_m$ are the centres or positions of the harmonic function $h$.

Here we shall investigate the symmetries of probes propagating on a spherically symmetric pp-wave, i.e. a pp-wave that depends on a harmonic function with one centre. After a coordinate transformation to put the centre at 0, $h = \frac{q}{|y|}$, where $q$ is constant denoting the momentum of the pp-wave. This solution has an $\mathbb{R}^2 \times SO(9)$ symmetry generated by the Killing vector fields $k_+ = \partial_u$, $k_- = \partial_v$ and

$$k_{ij} = y_i \partial_j - y_j \partial_i, \quad i < j,$$

where $y_i = y^i$. The latter vector fields are generated by the action of $SO(9)$ on the $y$ coordinates.

Clearly all the above vector fields generate symmetries for the probe action (5) with conserved charges

$$Q_\pm = Q(k_\pm) = p_\pm, \quad Q_{ij} = Q(k_{ij}) = y_ip_j - y_jp_i.$$  

In addition, one can demonstrate with a direct calculation that

$$d_{i_1\ldots i_k} = y^{j_1} \ldots y^{j_k} a_{j_1\ldots j_q, i_1\ldots i_k},$$

are KS tensors of the pp-wave spacetime provided that the constant tensor $a$ satisfies the condition

$$\frac{4}{q} a(j_1\ldots j_q, i_1) = a(j_1\ldots j_q, i_1) = 0.$$  

These in turn generate transformations as those in (7) which leave the action (5) invariant. The associated conserved charges are given in (8) or equivalently in (16). It is evident from the above analysis that a probe described by the action (5) and propagating on this pp-wave spacetime, and so the geodesic flow, admits infinite number of hidden symmetries. Note that KS and KY tensors on 4-dimensional pp-wave spacetimes have been investigated before, see e.g. [39, 40].

Although the probe (5) admits an infinite number of symmetries propagating on a pp-wave background, it does not immediately imply that the dynamics is completely integrable. Clearly the conserved charges $Q_{ij} = Q(k_{ij})$ and $Q_\pm = Q(k_\pm)$ generated by the vector fields $k_{ij}$ and $k_\pm$ are not in involution-the Poisson bracket algebra of $Q_{ij}$ is $so(9)$. However $Q_{ij}$ can be used to construct conserved charges which are in involution. In particular, one can show that the 10 conserved charges

$$Q_\pm, \quad D_m = \frac{1}{4} \sum_{i,j \geq 10-m} (Q_{ij})^2,$$

are in involution. These together with the Hamiltonian of the geodesic system $H = \frac{1}{4} g_{\mu\nu} p_\mu p_\nu$ give 11 independent conserved charges in involution leading to the complete

\footnote{For $q = k$, the $(0, 2q)$ tensors that lie in the irreducible representation of $GL(9)$ associated with the 2 rows and $q$ columns Young tableau solve the condition on $a$. A similar statement is true for the KS tensors of the M2- and M5-branes below.}
integrability of the geodesic flow. As in the black hole analysis, \( D_0 \) is the Hamiltonian of the geodesic flow on \( S^8 \) which is constructed from the quadratic Casimir operator of \( \mathfrak{so}(9) \).

Turning to the investigation of the symmetries of the probe \( (10) \) propagating on a pp-wave, one has to determine the KY tensors of the background. One can verify after some calculation that

\[
\beta(\varphi) = y_idx^i \wedge \varphi \wedge du \wedge dv ,
\]

are CCKY forms for any constant k-form \( \varphi \) on \( \mathbb{R}^9 \), where \( y_i = y^i \). As a result \( \alpha(\varphi) = \ast \beta(\varphi) \) are KY forms. These generate the transformations \( (11) \) which leave the spinning particle action \( (10) \) invariant with associated conserved charges given in \( (12) \). Therefore the probe \( (10) \) propagating on a pp-wave background admits \( 2^8 \) linearly independent conserved charges\(^5 \) generated by the KY forms \( \alpha(\varphi) \).

### 3.3.2 M2-branes

The M2-brane solution [42] can be expressed as

\[
g = h^{-\frac{2}{3}} \eta_{ab} d\sigma^a d\sigma^b + h^\frac{2}{3} \delta_{ij} dy^i dy^j , \quad F = \pm d\sigma^0 \wedge d\sigma^1 \wedge d\sigma^2 \wedge dh^{-1} ,
\]

where \( \sigma^a, a = 0, 1, 2 \), are the worldvolume coordinates of the brane, \( y^i, i = 1, \ldots, 8 \), are the transverse coordinates and \( h \) is a harmonic function \( \partial_y^2 h = 0 \) on the transverse space \( \mathbb{R}^8 \). An explicit expression for \( h \) is as in \( (29) \) with \( q_0 = 1 \) and \( n = 8 \).

For the spherically symmetric M2-brane solution that we shall consider in this section \( h = 1 + \frac{q}{|y|} \). This solution is invariant under the action of the \( SO(1, 2) \ltimes \mathbb{R}^3 \times SO(8) \) group, where the Poincaré group acts on the worldvolume coordinates of the M2-brane while \( SO(8) \) acts on the transverse coordinates with standard rotations. The Killing vector fields are \( k_a = \partial_a, k_{ab} = \sigma_a \partial_b - \sigma_b \partial_a \) and \( k_{ij} = y_i \partial_j - y_j \partial_i \), where \( \sigma_a = \eta_{ab} \sigma^b \) and \( y^i = y_i \). It is clear that the probe \( (5) \) propagating on this background admits symmetries generated by these vector fields and the associated conserved charges are

\[
Q_a = Q(k_a) = p_a , \quad Q_{ab} = \sigma_a p_b - \sigma_b p_a , \quad Q_{ij} = Q(k_{ij}) = y_i p_j - y_j p_i .
\]

As for the pp-wave, the probe \( (5) \) admits additional symmetries generated by KS tensors. To find these tensors we use an ansatz which preserves the worldvolume Poincaré symmetry of the solution. Then after some computation one can verify that

\[
d_{a_1 \ldots a_{2m} i_1 \ldots i_k} = h^{\frac{2}{3}(k-2m)} y^{j_1} \ldots y^{j_{2m}} a_{j_1 \ldots j_{2m} i_1 \ldots i_k} \eta(a_1 a_2 \ldots, \eta_{a_{2m-1} a_{2m}}) ,
\]

are KS tensors provided that the constant tensors \( a \) satisfy

\[
a_{(j_1 \ldots j_{2m}) \ldots i_k} = a_{j_1 \ldots (j_{2m}) \ldots i_k} = 0 .
\]

These in turn give additional conserved charges \( (8) \) for the relativistic particle probe \( (5) \). Therefore the probe \( (5) \), and so the geodesic flow on this M2-brane, admits an infinite number of hidden conserved charges.

\(^5\)The maximal number of independent KY k-forms [41] on a \( n \)-dimensional spacetime is \( (n + 1)!/((k + 1)!(n - k)!) \).
The dynamics of the relativistic particle (5) propagating on this M2-brane background, and so the geodesic flow, is completely integrable. Indeed one can verify after some calculation that the conserved charges

\[ Q_a, \quad D_m = \frac{1}{4} \sum_{i,j \geq 9-m} (Q_{ij})^2, \quad m = 2, \ldots, 8, \]  

are in involution. These together with the Hamiltonian of the relativistic particle (5) yield 11 independent conserved charges in involution.

Next let us turn to investigate the symmetries of the spinning particle probe (10) propagating on the spherically symmetric M2-brane. Clearly the Killing vector fields of the M2-brane generate symmetries for the probe (10). Additional symmetries are generated by the KY forms of this M2-brane. To find these, we adapt an ansatz which is invariant under the worldvolume Poincaré group of the M2-brane. Then after some computation, one finds that

\[ \beta(\varphi) = h^{-1/2} \eta^{ij} y_i dy_j \wedge \varphi \wedge dvol(\mathbb{R}^{2,1}), \]  

are CCKY tensors of the M2-brane for any constant k-form \( \varphi \) on \( \mathbb{R}^8 \), where \( dvol(\mathbb{R}^{2,1}) \) is the volume form of \( \mathbb{R}^{2,1} \). As a result \( \alpha(\varphi) = \ast \beta(\varphi) \) are KY tensor and so spinning particle action (10) is invariant the under transformation (11) generated by \( \alpha(\varphi) \). The associated constants of motion are given in (12). These KY tensors generate 27 linearly independent hidden symmetries for the action (10).

### 3.3.3 M5-branes

The M5-brane solution [43] is

\[ g = h^{-1/2} \eta^{ab} d\sigma^a d\sigma^b + h^{1/2} \delta_{ij} dy^i dy^j, \quad F = \pm \ast_5 dh, \]

where \( \sigma^a, a = 0, \ldots, 5 \), are the worldvolume coordinates, \( y^i, i = 1, \ldots, 5 \), are the transverse coordinates, the Hodge duality operation has been taken with respect to the flat metric on the transverse space \( \mathbb{R}^5 \) and \( h \) is a harmonic function, \( \partial^2 h = 0 \), on \( \mathbb{R}^5 \). \( h \) is given in (29) with \( n = 5 \) and \( q_0 = 1 \).

For the spherically symmetric M5-brane solution that we consider here, \( h \) has one centre and so it can be arranged such that \( h = 1 + \frac{1}{|y|^2} \). Such a solution admits a \( SO(1,5) \times \mathbb{R}^6 \times SO(5) \) isometry group. The Killing vector fields are \( k_a = \partial_a, \quad k_{ab} = \sigma_a \partial_b - \sigma_b \partial_a \) and \( k_{ij} = y_i \partial_j - y_j \partial_i \), where \( y_i = y^i \) and \( \sigma_a = \eta_{ab} \sigma^b \). The transformations generated by these vector fields leave invariant the relativistic particle action (5) and the associated conserved charges are

\[ Q_a = p_a, \quad Q_{ab} = \sigma_a p_b - \sigma_b p_a, \quad Q_{ij} = y_i p_j - y_j p_i. \]

As for M2-branes, relativistic particles propagating on the above M5-brane background admit additional symmetries associated with KS tensors. Adapting again an ansatz which is invariant under the worldvolume Poincaré symmetry and after some computation one finds that

\[ d_{a_1 \ldots a_{2m} i_1 \ldots i_k} = h^{1/2}(2k-m) y^{i_1} \ldots y^{i_k} a_{i_1 \ldots i_k} \eta(a_1 a_2 \ldots \eta(a_{2m-1} a_{2m}). \]
are KS tensors provided that the constant tensors $a$ satisfy
\begin{equation}
 a_{(j_1...j_q,i_1)...i_k} = a_{(j_2...j_q,i_1)...i_k} = 0 .
\end{equation}
Clearly, these generate infinite many hidden symmetries for the relativistic particle action (5). So the geodesic flow on the spherically symmetric M5-brane has infinite many conserved charges.

Furthermore, one can show that the dynamics of relativistic particles propagating on this M5-brane is completely integral. Indeed one can verify that the 10 conserved charges $Q_{ij}, D_m = \frac{1}{4} \sum_{i,j \geq 6-m} (Q_{ij})^2, m \geq 2,\ldots, 5$,
\begin{equation}
(46)
\end{equation}
are in involution. These together with the Hamiltonian of (5) yield 11 independent conserved charges in involution as required for complete integrability.

As for the M2-brane, the spinning particle action (10) admits, in addition to the symmetries generated by the Killing vectors field of the M5-brane, hidden symmetries generated by KY forms. To find these we adapt and ansatz which is invariant under the worldvolume Poincaré group of the M5-brane. Then after some computation, one can verify that
\begin{equation}
\beta(\psi) = h^{2(k-1)} y_i d y^i \wedge \psi \wedge d \text{vol}(\mathbb{R}^{5,1}) ,
\end{equation}
are CCKY forms for any constant k-form $\psi$ on $\mathbb{R}^5$. As a result $\alpha(\psi) = * \beta(\psi)$ are KY forms and so generate symmetries (11) for the spinning particle probe (10) with conserved charges (12). These KY forms generate $2^4$ linearly independent hidden symmetries.

### 3.3.4 KK-monopoles

The KK-monopole solution is
\begin{equation}
g = \eta_{ab} d \sigma^a d \sigma^b + g_{(4)} , \quad g_{(4)} = h^{-1}(d \rho + \omega)^2 + h \delta_{ij} d y^i d y^j ,
\end{equation}
with $F = 0$, where $\sigma^a, a = 0,\ldots, 6$, are the worldvolume coordinates and $g_{(4)}$ is in general the Gibbons-Hawking hyper-Kähler metric with $* \delta dh = d \omega$. $h$ is a harmonic action on $\mathbb{R}^3$, $\partial_\rho^2 h = 0$. An expression for $h$ can be found in (29) for $n = 3$.

Here we shall consider the KK monopole solution with $g_{(4)}$ the Taub-NUT metric. In such a case $h$ has one centre and so one can set without loss of generality $h = 1 + \frac{a}{y^1}$. The isometry group of the solution is $SO(1, 6) \ltimes \mathbb{R}^7 \times SO(2) \times SO(3)$. As for the solutions investigated already, the Killing vector fields generated by the Poincaré subgroup acting on the worldvolume coordinates are $k_a = \partial_a$ and $k_{ab} = \sigma_a \partial_b - \sigma_b \partial_a$. To give the vector fields generated by the $SO(2) \times SO(3)$ subgroup, write the Taub-NUT metric $g_{(4)}$ is angular coordinates as
\begin{equation}
g_{(4)} = h^{-1}(d \rho + q \cos \theta d \phi)^2 + h (d r^2 + r^2 (d \theta^2 + \sin^2 \theta d \phi^2)) ,
\end{equation}
with $|y| = r$. Then the Killing vector fields generated by $SO(2) \times SO(3)$ are given by
\begin{equation}
k_0 = \partial_\rho , \quad k_1 = \partial_\phi , \quad k_2 = -\sin \phi \cot \theta \partial_\phi + \cos \phi \partial_\theta + \frac{q}{\sin \theta} \partial_\rho ,
\end{equation}

13
\[ \tilde{k}_3 = \cos \phi \cot \theta \partial_\phi + \sin \phi \partial_\theta - q \cos \phi \sin \theta \partial_\rho . \]  

(50)

The \( SO(3) \) Killing vector fields are as in (19) with the addition of a component along \( \partial_\rho \) because \( \omega \) is not invariant under (19) but instead it is invariant up to a gauge transformation.

As the relativistic particle action (5) is invariant under all these isometries, the associated conserved charges are \( Q_a = p_a \), \( Q_{ab} = \sigma_a p_b - \sigma_b p_a \), \( \tilde{Q}_0 = p_\rho \) and \( \tilde{Q}_r = \tilde{k}_r^i p_i \), \( r = 1, 2, 3 \), where \( \sigma_a = \eta_{ab} \sigma^b \) and \( \tilde{k}_r \) are given in (50). The background admits several KS tensors. As the solution is a product \( \mathbb{R}^{6,1} \times N \), where \( N \) is the Taub-NUT manifold, one can consider the KS tensors of \( \mathbb{R}^{6,1} \) and \( N \) separately. One can easily verify that the symmetric tensors

\[ d_{a_1...a_k} = \sigma^{b_1} ... \sigma^{b_q} c_{b_1...b_q,a_1...a_k} , \]

(51)

are KS tensors\(^6\) provided that the constants \( c \) satisfy \( c_{b_1...(b_q,a_1...a_k)} = 0 \). \( N \) also admits three KS tensors given in [44] which we shall not explicitly state them here. They are constructed from the Kähler forms and the KY tensor of \( N \) given below. All these isometries and KS tensors generate infinite number of symmetries for the relativistic particle action (5).

The dynamics of the relativistic particle probe (5), or equivalently the geodesic flow, is completely integrable on this background. Indeed the commuting isometries of the KK-monopole are \( k_a = \partial_\sigma \), \( k_0 = \partial_\rho \) and \( k_1 = \partial_\phi \). These together with the Hamiltonian of the geodesic system give ten conserved charges in involution. There is an additional independent conserved charge in involution associated to the quadratic Casimir of \( SO(3) \) and constructed using the Killing vector fields (50) as

\[ D = \frac{1}{\sin^2 \theta} (p_\phi - q \cos \theta p_\sigma)^2 + p_\theta^2 + q^2 p_\sigma^2 , \]

(52)

which proves the statement. The integrability of the geodesic flow on the Taub-NUT space has been known for sometime, see [44].

The \( SO(1,6) \ltimes \mathbb{R}^7 \times SO(2) \times SO(3) \) isometries mentioned above also generate symmetries for the spinning particle probe (10) propagating on the KK-monopole background. Such probes have additional hidden symmetries. For example, it is well known that \( g_{GH} \) is a hyper-Kähler metric for any (multi-centred) harmonic function \( h \). The associated Kähler forms are

\[ \kappa_{(i)} = (d\rho + \omega) \wedge dy^i - \frac{1}{2} h \epsilon^{i,j,k} dy^j \wedge dy^k . \]

(53)

These 2-forms are anti-self-dual on the transverse directions of the KK-monopole, parallel with respect to the Levi-Civita connection and the associated complex structures satisfy the algebra of imaginary unit quaternions. As a result these Kähler forms can be thought as KY tensors and so generate symmetries (11) for the probe action (10) with conserved charges (12).

\(^6\)There is a systematic investigation of KS tensors on Minkowski spacetime as well as on some black hole spacetimes. For example there is a 20 dimensional space of rank 2 conformal KS tensors on 4-dimensional Minkowski spacetime, see for a summary [47]. But the approach adopted here suffices.
The KK monopole admits additional KY tensors. These are those of $\mathbb{R}^{6,1}$ and those of $N$. Observe that

$$\alpha = \frac{1}{k!} (\chi_{a_1 \ldots a_k} + \sigma^b \varphi_{b,a_1 \ldots a_k}) \, d\sigma^{a_1} \wedge \cdots \wedge d\sigma^{a_k},$$

are KY tensors of $\mathbb{R}^{6,1}$ for any constant tensors $\chi, \varphi$ with the latter to satisfy $\varphi_{b,a_1 \ldots a_k} = \varphi[b,a_1 \ldots a_k]$ [41]. If $N$ is the Taub-Nut space, it is known [44], see also [45], that

$$\tilde{\alpha} = (d\rho + q \cos \theta d\phi) \wedge dr + r(2r + q)(1 + \frac{r}{q}) \sin \theta d\theta \wedge d\phi,$$

is the KY form. All these KY forms generate symmetries for the spinning probe action (10). Incidentally the three KS tensors mentioned above are constructed from squaring $\tilde{\alpha}$ with $\kappa_{(i)}$.

4 Hidden symmetries from the TCFH

4.1 Hidden symmetries and M-theory pp-waves

Assuming that the pp-wave propagates in the 5th direction$^7$ and allowing the pp-wave metric (28) to depend on a (multi-centred) harmonic function as in (29) with $q_0 = 0$ and $n = 9$, the Killing spinors of the background are constant, $\epsilon = \epsilon_0$ and satisfy the condition$^8$ $\Gamma_{05} \epsilon_0 = \pm \epsilon_0$. To solve this condition, we shall use spinorial geometry and write $\epsilon_0 = \eta + e_5 \wedge \lambda$, where $\eta$ and $\lambda$ are Majorana$^9$ spinors, i.e. $\eta, \lambda \in \Lambda^s(\mathbb{R} \langle e_1, \ldots, e_4 \rangle)$ with the reality condition imposed by the anti-linear operation $\Gamma_{6789}^*$. Choosing the plus sign in the condition for $\epsilon_0$, this can be solved to yield $\epsilon = \epsilon_0 = \eta$, i.e. $\Gamma_{05} \epsilon_0 = \epsilon_0$ implies that $\lambda = 0$.

Given the solution of the condition on $\epsilon$ implied by the KSE, it is straightforward to compute all the bilinears of the background. In particular one finds that $f^{rs} = 0$ for all Killing spinors and the rest of the form bilinears (3) can be written as

$$\left(e^0 - e^5\right) \wedge \phi^{rs},$$

where

$$\phi^{rs} = \frac{1}{k!} \langle \eta^r, \Gamma_{i_1 \ldots i_k} \eta^s \rangle_H e^{i_1} \wedge \cdots \wedge e^{i_k}, \quad k = 0, 1, 2, 3, 4,$$

$\langle \cdot, \cdot \rangle_H$ is the Hermitian inner product restricted on the Majorana representation of spin(9) and $i_1, \ldots, i_k = 1, 2, 3, 4, 6, 7, 8, 9, 0$. Moreover $(e^0, e^5, e^i)$ is a pseudo-orthonormal frame.

$^7$This choice of worldvolume directions for the pp-wave, and those of the rest of M-branes below, may seem unconventional. But they are convenient as they are aligned with the basis used for the description of spinors in the context of spinorial geometry that we utilise to solve the conditions on the Killing spinors.

$^8$All gamma matrices considered from in section 4 and appendix A are in a frame basis.

$^9$Note that the reality condition on $\epsilon$ in the spinorial geometry basis is $\Gamma_{6789}^* \epsilon = \epsilon$ which in turn implies that $\eta$ and $\lambda$ are real as well.
such that \(-e^0 + e^5 = \sqrt{2} \, du\), \(e^0 + e^5 = \sqrt{2}(dv + \frac{1}{2} hdu)\) and \(e^i = dy^i\) after a relabelling of the transverse coordinates \(y\) of the spacetime. For example \(k^{rs} = \langle \eta^r, \eta^s \rangle_H \, (e^0 - e^5)\) and so on.

It remains to specify the \(k\)-form bilinears \(\phi^{rs}\). It turns out that these span all the constant forms on the transverse space of the pp-wave up and including those of degree 4. To see this decompose \(\phi^{rs} = e^5 \wedge \alpha^{rs} + \beta^{rs}\), where \(\alpha^{rs}\) and \(\beta^{rs}\) have components only along the directions transverse to \(e^5\). The tensor product of two Majorana spin\((8)\) representations, \(\Delta_{16}\), can be decomposed as

\[
\Delta_{16} \otimes \Delta_{16} = \bigoplus_{k=0}^{4} \Lambda^k(\mathbb{R}^8).
\]

Therefore the forms \(\beta^{rs}\) which are up to degree 4 span all forms of the same degree on \(\mathbb{R}^8\) subspace transverse to \(e^5\). On the other hand the Hodge duals of the forms \(\alpha^{rs}\) span all forms of degree 5 and higher in \(\mathbb{R}^8\). Thus the space of all bilinears of a pp-wave spans a \(2^8\)-dimensional vector space.

As for the pp-waves we have been considering the 4-form field strength \(F\) vanishes, all the form bilinears are covariantly constant with respect to the Levi-Civita connection. As a result all of them generate symmetries for the spinning particle probe action (10). The associated conserved charges are given in (12). They also generate symmetries for string probes as well similar to those investigated in [17]. The algebra of symmetries can be of W-type and has been described in [29].

### 4.2 Hidden symmetries and the KK-monopole

Choosing the worldvolume directions of the KK-monopole along 0125678 and allowing \(h\) in (48) to be any multi-centred harmonic function as in (29) with \(n = 3\), the Killing spinors \(\epsilon = \epsilon_0\) of the background satisfy \(\Gamma_{3489} \epsilon_0 = \pm \epsilon_0\), where \(\epsilon_0\) is a constant spinor. To solve this condition with the plus sign, we shall use spinorial geometry and write \(\epsilon_0 = \eta^1 + \epsilon_{34} \wedge \lambda^1 + \epsilon_3 \wedge \eta^2 + \epsilon_4 \wedge \lambda^2\), where \(\eta\) and \(\lambda\) are Dirac spinors of spin\((6,1)\), i.e. \(\eta, \lambda \in \Lambda^*(\mathbb{C}(e_1, e_2, e_5))\). To begin, let us assume that \(\epsilon_0\) is a complex spinor and impose the reality condition at the end. Then \(\Gamma_{3496} \epsilon_0 = \epsilon_0\) implies that \(\eta^2 = \lambda^2 = 0\) and so \(\epsilon_0 = \eta + \epsilon_{34} \wedge \lambda\), where \(\eta = \eta^1\) and \(\lambda = \lambda^3\). The reality condition on \(\epsilon_0\), \(\Gamma_{6789} * \epsilon_0 = \epsilon_0\), implies that \(\lambda = -\Gamma_{679}\). Therefore the spinors that solve the Killing spinor condition are

\[
\epsilon_0 = \eta - \epsilon_{34} \wedge \Gamma_{679}\eta^*,
\]

where \(\eta\) is any Dirac spin\((6,1)\) spinor.

The non-vanishing Killing spinors bilinears read

\[
\begin{align*}
    f^{rs} &= 2 \text{Re} \langle \eta^r, \eta^s \rangle, \\
    k^{rs} &= 2 \text{Re} \langle \eta^r, \Gamma_a \eta^s \rangle e^a, \\
    \omega^{rs} &= \frac{1}{2} \text{Re} \langle \eta^r, \Gamma_{ab} \eta^s \rangle e^a \wedge e^b - 2 \text{Re} \langle \eta^r, \lambda^s \rangle (e^3 \wedge e^4 - e^4 \wedge e^8) \\
    &\quad - 2 \text{Im} \langle \eta^r, \eta^s \rangle (e^3 \wedge e^8 + e^4 \wedge e^9) - 2 \text{Im} \langle \eta^r, \lambda^s \rangle (e^3 \wedge e^9 - e^4 \wedge e^8), \\
    \varphi^{rs} &= \frac{1}{3} \text{Re} \langle \eta^r, \Gamma_{abc} \eta^s \rangle e^a \wedge e^b \wedge e^c - 2 \text{Im} \langle \eta^r, \Gamma_a \eta^s \rangle (e^3 \wedge e^8 + e^4 \wedge e^9) \wedge e^a
\end{align*}
\]
\[ -2\text{Im}(\eta^*, \Gamma_a \lambda^s)(e^3 \wedge e^9 - e^4 \wedge e^8) \wedge e^a, \]
\[ \theta^{rs} = \frac{1}{12} \text{Re}(\eta^r, \Gamma_{abcd} \eta^s)e^a \wedge e^b \wedge e^c \wedge e^d - \text{Re}(\eta^r, \Gamma_{ab} \lambda^s)e^a \wedge e^b \wedge (e^3 \wedge e^4 - e^8 \wedge e^9) \]
\[ - \text{Im}(\eta^r, \Gamma_{ab} \eta^s)e^a \wedge e^b \wedge (e^3 \wedge e^8 + e^4 \wedge e^9) - \text{Im}(\eta^r, \Gamma_{ab} \lambda^s)e^a \wedge e^b \wedge (e^3 \wedge e^9 - e^4 \wedge e^8) \]
\[ + 2\text{Re}(\eta^r, \eta^s)e^3 \wedge e^4 \wedge e^8 \wedge e^9, \]
\[ \tau^{rs} = \frac{1}{60} \text{Re}(\eta^r, \Gamma_{a_1 \ldots a_5} \eta^s)e^{a_1} \wedge \cdots \wedge e^{a_5} + 2\text{Re}(\eta^r, \Gamma_a \eta^s)e^a \wedge e^3 \wedge e^4 \wedge e^8 \wedge e^9 \]
\[ - \frac{1}{3} \text{Re}(\eta^r, \Gamma_{abc} \lambda^s)e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^4 - e^8 \wedge e^9) \]
\[ - \frac{1}{3} \text{Im}(\eta^r, \Gamma_{abc} \eta^s)e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^8 + e^4 \wedge e^9) \]
\[ - \frac{1}{3} \text{Im}(\eta^r, \Gamma_{abc} \lambda^s)e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^9 - e^4 \wedge e^8), \]
(60)

where \( \langle \cdot, \cdot \rangle \) is the Dirac inner product, \( a, b, c = 0, 1, 2, 5, 6, 7, \bar{z}, \) \( e^a = dx^a, \) and \( e^i, i = 3, 4, 8, 9, \) is an orthonormal frame of \( g(4) \) in (48), e.g.

\[ e^3 = h^{-\frac{1}{2}}(dp + \rho), \quad e^4 = h^\frac{1}{2}dy^4, \quad e^7 = h^\frac{1}{2}dy^7, \quad e^8 = h^\frac{1}{2}dy^8, \]
(61)
after a relabelling of the coordinates of the spacetime. The bilinears of the spinors \( \eta \) span all real forms on the worldvolume \( \mathbb{R}^{6,1} \) of the KK-monopole solution. The argument is similar to that produced for the pp-wave.

As for the KK-monopole solution the 4-form field strength vanishes \( F = 0, \) a consequence of the TCFH is that all the form bilinears in (60) are covariantly constant with respect to the Levi-Civita connection. As a result they generate symmetries (11) for the spinning particle probe (10). The conserved charges are given in (12). The algebra of symmetries can be a W-type of algebra [29].

### 4.3 Hidden symmetries and the M2-brane

Choosing the M2-brane worldvolume directions along 05\( \bar{s} \), the Killing spinors of the solution are \( \epsilon = h^{-\frac{1}{2}}\epsilon_0 \), where \( \epsilon_0 \) is a constant spinor satisfying the condition \( \Gamma_{055} \epsilon_0 = \pm \epsilon_0 \) and \( h \) is a (multi-centred) harmonic function as in (29) with \( n = 8 \). To solve the condition with the plus sign use spinorial geometry to write \( \epsilon_0 = \eta + e_5 \wedge \lambda, \) where \( \eta, \lambda \in \Lambda^*(\mathbb{R}^4) \). Then the condition \( \Gamma_{055} \epsilon_0 = \epsilon_0 \) implies that \( \eta, \lambda \in \Lambda^6(\mathbb{R}^4) \), i.e. \( \eta, \lambda \) are Majorana-Weyl spin(8) spinors, where the reality condition is imposed with the anti-linear map \( \Gamma_{6789} \).

Using the solution of the condition on the Killing spinors and setting \( \phi^{rs} = h^{-\frac{1}{2}}\tilde{\phi}^{rs} \) for all bilinears \( \phi^{rs} \), one can easily find

\[ \tilde{f}^{rs} = -\langle \eta^r, \lambda^s \rangle_H + \langle \lambda^r, \eta^s \rangle_H, \]
\[ k^{rs} = \left( \langle \eta^r, \eta^s \rangle_H + \langle \lambda^r, \lambda^s \rangle_H \right)e^0 + \left( -\langle \eta^r, \eta^s \rangle_H + \langle \lambda^r, \lambda^s \rangle_H \right)e^5 + \left( \langle \eta^r, \lambda^s \rangle_H + \langle \lambda^r, \eta^s \rangle_H \right)e^5, \]
\[ \tilde{\omega}^{rs} = \left( \langle \eta^r, \lambda^s \rangle_H + \langle \lambda^r, \eta^s \rangle_H \right)e^0 \wedge e^5 + \left( \langle \eta^r, \eta^s \rangle_H - \langle \lambda^r, \lambda^s \rangle_H \right)e^0 \wedge e^5 \]
\[ + \left( -\langle \eta^r, \lambda^s \rangle_H - \langle \lambda^r, \eta^s \rangle_H \right)e^5 \wedge e^5 + \frac{1}{2} \left( -\langle \eta^r, \Gamma_{ij} \lambda^s \rangle_H + \langle \lambda^r, \Gamma_{ij} \eta^s \rangle_H \right)e^i \wedge e^j, \]
\[ \tilde{\varphi}^{rs} = \frac{1}{2} \left( \langle \eta^r, \Gamma_{ij} \eta^s \rangle_H + \langle \lambda^r, \Gamma_{ij} \lambda^s \rangle_H \right)e^0 \wedge e^i \wedge e^j + \frac{1}{2} \left( -\langle \eta^r, \Gamma_{ij} \eta^s \rangle_H + \langle \lambda^r, \Gamma_{ij} \lambda^s \rangle_H \right)e^5 \wedge e^i \wedge e^j \]
\[ + \frac{1}{2} \left( \langle \eta^r, \Gamma_{ij} \lambda^s \rangle_H + \langle \lambda^r, \Gamma_{ij} \eta^s \rangle_H \right)e^3 \wedge e^i \wedge e^j + \left( -\langle \eta^r, \lambda^s \rangle_H + \langle \lambda^r, \eta^s \rangle_H \right)e^0 \wedge e^5 \wedge e^5, \]

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A similar investigation reveals that to begin as the 1-form bilinears components, $\theta^a$, $\omega^a$, spinors such that contain explicitly the spacetime metric $g_e$ and only along the worldvolume directions of M2-brane and that there are Killing spinors $\Phi^a$ terms proportional to $F$ such that in the TCFH it is required that the terms in the TCFH connection that are proportional to $\eta^a$, $\Gamma_{i_1 \ldots i_4}^a$ is the pseudo-orthonormal frame with $e^a = h^{-1/3} d\sigma^a$, $a = 0, 5, 6$, and $e^i = h^{1/6} d\sigma^i i, j, k, \ell = 1, 2, 3, 4, 6, 7, 8, 9$, after an appropriate relabelling of the coordinates of the spacetime. As the product of two positive chirality Majorana-Weyl spin(8) representations, $\Delta^+_8$, is decomposed as

$$\otimes^2 \Delta^+_8 = \Lambda^0(\mathbb{R}^8) \oplus \Lambda^2(\mathbb{R}^8) \oplus \Lambda^{4+}(\mathbb{R}^8),$$

it is expected that the form bilinears above span all the 0-, 2- and self-dual 4-forms along the transverse directions of the M2-brane.

It remains to find which of the above form bilinears are KY tensors with respect to the Levi-Civita connection so that generate symmetries for the spinning particle probe (10). To begin as the 1-form bilinears $k$ are Killing they generate symmetries for the action (10) and the associated conserved charges are given in (12). For the bilinear $\omega$ to be a KY form, it is required that the terms in the TCFH connection that are proportional to $F$ as well as those that in the TCFH that contain explicitly the spacetime metric $g$ must vanish. After some investigation, these terms vanish provided that the components, $\tau_{abcij}$, of the form bilinear $\tau$ are zero, $\tau_{abcij} = 0$. This in turn implies that $\omega_{ij} = 0$. Setting $\omega_{ij} = 0$, $\omega = 1/2 \omega_{ab} e^a \wedge e^b$ is a KY tensor and generates a symmetry transformation (11) for the action (10) with associated conserved charge given in (12). Note that $\omega$ has components only along the worldvolume directions of the M2-brane. There are Killing spinors such that $\omega \neq 0$, even though $\omega_{ij} = 0$, as a consequence of the decomposition (63). A similar investigation reveals that $\tau$ cannot be a KY form as the conditions arising from the analysis of the TCFH imply that $\tau = 0$.

Next $\varphi$ is a KY form with respect to the Levi-Civita connection provided that the terms proportional to $F$ in the TCFH connection as well as those in the TCFH that contain explicitly the spacetime metric $g$ vanish. This is the case provided that the components, $\theta_{abij}$ of $\theta$ vanish, $\theta_{abij} = 0$. This in turn implies that $\varphi_{aij} = 0$. Therefore $\varphi = 1/8 \varphi_{abcde} e^a \wedge e^b \wedge e^c$ is a KY form and so generates a symmetry for the spinning particle probe action (10) with conserved charge (12). Note again that the KY form $\varphi$ has components only along the worldvolume directions of M2-brane and that there are Killing spinors such that $\varphi \neq 0$ even though $\varphi_{aij} = 0$ as a consequence of (63). A similar investigation concludes that $\theta$, as $\tau$, cannot be a KY form.

### 4.4 Hidden symmetries and the M5-brane

Choosing the worldvolume directions of the M5-brane along 012567, the Killing spinors of the background are $\epsilon = h^{-1/3} \epsilon_0$, where the constant spinor $\epsilon_0$ satisfies the condition
\[ \Gamma_{34890} e_0 = \pm e_0 \] and \( h \) is a multi-centred harmonic function as in (29) with \( n = 5 \). To continue it is convenient to solve the condition on \( e_0 \) with a plus sign by taking \( e_0 \) to be complex and impose the reality condition on \( e_0 \) at the end. Indeed for \( e_0 \) complex, one can use spinorial geometry to write

\[ e_0 = \eta^1 + e_{34} \wedge \lambda^1 + e_3 \wedge \eta^2 + e_4 \wedge \lambda^2, \]

where \( \eta^1, \eta^2, \lambda^1, \lambda^2 \in \Lambda^* (\mathbb{C} \langle e_1, e_2, e_5 \rangle) \). Then the condition \( \Gamma_{34890} e_0 = e_0 \) implies that

\[ \eta^1, \eta^2, \lambda^1, \lambda^2 \in \Lambda^* (\mathbb{C} \langle e_1, e_2, e_5 \rangle), \]

i.e. \( \eta^1, \eta^2, \lambda^1, \lambda^2 \) are positive chirality spinors on \( \text{spin}(5,1) \). Next imposing the reality condition on \( e_0 \), \( \Gamma_{6789} * e_0 = e_0 \), one finds that \( \lambda^1 = -\Gamma_{67}(\eta^1)^* \) and \( \lambda^2 = -\Gamma_{67}(\eta^2)^* \). Hence the spinors that solve the Killing spinor condition are

\[ e_0 = \eta^1 - e_{34} \wedge \Gamma_{67}(\eta^1)^* + e_3 \wedge \eta^2 - e_4 \wedge \Gamma_{67}(\eta^2)^*, \]

(64)

where \( \eta^1, \eta^2 \) are any positive chirality \( \text{spin}(5,1) \) spinors. The form bilinears of the M5-brane expressed in terms of the \( \eta^1 \) and \( \eta^2 \) spinors can be found in appendix A.

The 1-form bilinears \( k^{rs} \) are isometries and so generate symmetries for the spinning particle probe action (10). Next for the bilinear \( \omega \) to be a KY tensor, and so generate a symmetry for the spinning particle probe (10), the term that contains \( F \) in the minimal TCFH connection \( D^F \) and the term proportional to the spacetime metric \( g \) in the TCFH (4) must vanish. This is the case provided that the component, \( \tau_{ijklst} \), of \( \sigma \) vanishes. However this in turn implies that \( \omega = 0 \) and so \( \omega \) does not generate a symmetry. It turns out \( \theta \), like \( \omega \), does not generate a symmetry for the probe (10) because the conditions required by the TCFH for \( \theta \) to be a KY form are too restrictive and yield \( \theta = 0 \). Next \( \phi \) is a KY form as a consequence of TCFH provided that \( \theta_{abc} = \theta_{aijk} = 0 \). This implies \( \phi_{aij} = 0 \) and leaves the possibility that the remaining component of \( \phi \) \( \phi = \frac{1}{3} \phi_{abc} e^a \wedge e^b \wedge e^c \) is a KY form. However after some computation one can verify that there are no Killing spinors such that \( \phi \neq 0 \). A similar conclusion holds for the \( \tau \) form bilinear.

### 5 Concluding Remarks

We have presented the TCFH of 11-dimensional supergravity and we have demonstrated that the form bilinears of supersymmetric backgrounds of the theory satisfy a generalisation of the CKY equation with respect to a connection that depends on the 4-form field strength. We have also given the reduced holonomy of the minimal and maximal TCFH connections for generic backgrounds.

As KY forms with respect to the Levi-Civita connection generate symmetries for spinning particle actions, we investigated the question on whether the form bilinears of 11-dimensional supergravity generate symmetries for suitable particle probes propagating on supersymmetric backgrounds. For this we focused on M-branes which include the pp-wave, M2- and M5-brane, and KK-monopole solutions. As all the form bilinears of pp-wave and KK-monopole solutions are covariantly constant with respect to the Levi-Civita connection, they generate symmetries for the spinning particle action with only a metric coupling. For the M2-brane, there are Killing spinors such that the 1-form, 2-form and 3-form bilinears are KY tensors and therefore generate symmetries for the same spinning particle action. For the M5-brane only the 1-form bilinears generate symmetries for the spinning particle action.

We also took the opportunity to demonstrate the complete integrability of the geodesic flow of spherically symmetric pp-wave, M2- and M5-brane, and KK-monopole solutions.
For this we presented a large class of KS and KY tensors on all these backgrounds. Rela-
tivistic particles on these solutions admit an infinite number of symmetries generated
by KS tensors. We have also explicitly given all independent and in involution conserved
charges of the geodesic flow on these backgrounds.

Comparing the symmetries required for the integrability of the geodesic flow on M-
brane backgrounds with the symmetries generated by the form bilinears, one arrives at
the conclusion that these contribute at different sectors in the probe dynamics. If a form
bilinear generates a symmetry for a particle probe, it will generate a symmetry on all
M-brane backgrounds including those that depend on multi-centred harmonic functions.
One does not expect that the geodesic flow on generic such backgrounds to be completely
integrable. Therefore generically form bilinears of supersymmetric backgrounds are not
responsible for the integrability properties of a probe. Nevertheless generate additional
symmetries for probes, e.g. additional worldvolume supersymmetries, which characterise
the dynamics.

As the TCFH connection depends on the 4-form field strength of 11-dimensional super-
gravity, one should also consider spinning particle probes that exhibit a 4-form coupling.
The expectation would be that in this way one can better match the TCFH with the con-
ditions for invariance of the probe action under transformations generated by the form
bilinears. Such a probe action has been presented in appendix B. However under some
reasonable assumptions on the couplings and on the transformations constructed from
the form bilinears, one finds that the conditions for invariance of the probe action are
too strong for M-brane backgrounds and they do not match with those of TCFH. This
does not exhausts all possibilities of matching an 11-dimensional TCFH with the con-
ditions required for the form bilinear to generate a symmetry for a spinning particle probe
constructed using the results of [46]. One can choose different TCFHs associated with
the same supergravity theory as well as different probe actions. So it remains an open
question whether such a matching of conditions can be achieved in 11 dimensions.

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Appendices

A M5-brane bilinears

Using the solution (64) of the condition on the Killing spinors and setting $\phi^{rs} = h^{-1} \hat{\phi}^{rs}$
for all bilinears $\phi^{rs}$, one can easily find

$$\hat{k}^{rs} = 2(\text{Re}(\eta^{1r}, \Gamma_a \eta^{1s}) - \text{Re}(\eta^{2r}, \Gamma_a \eta^{2s})) e^a ,$$

$$\hat{\omega}^{rs} = 2(\text{Re}(\eta^{1r}, \Gamma_a \eta^{2s}) + \text{Re}(\eta^{2r}, \Gamma_a \eta^{1s})) e^a \wedge e^3$$

$$+ 2(\text{Im}(\eta^{1r}, \Gamma_a \lambda^{2s}) - \text{Im}(\eta^{2r}, \Gamma_a \lambda^{1s})) e^a \wedge e^4$$

$$+ 2(\text{Im}(\eta^{1r}, \Gamma_a \eta^{2s}) - \text{Im}(\eta^{2r}, \Gamma_a \eta^{1s})) e^a \wedge e^8.$$
\[ \dot{\varphi}^{rs} = \frac{1}{3} \left( \text{Re} \langle \eta^{1r}, \Gamma_{abc} \eta^{1s} \rangle + \text{Re} \langle \eta^{2r}, \Gamma_{abc} \eta^{2s} \rangle \right) e^a \wedge e^b \wedge e^c \]

\[ -2 \text{Re} \langle \eta^{1r}, \Gamma_{a} \lambda^{2s} \rangle (e^3 \wedge e^4 - e^8 \wedge e^9) \wedge e^a \]

\[ + 2 \text{Re} \langle \eta^{2r}, \Gamma_{a} \lambda^{2s} \rangle (e^3 \wedge e^4 + e^8 \wedge e^9) \wedge e^a \]

\[ - 2 \text{Im} \langle \eta^{1r}, \Gamma_{a} \eta^{1s} \rangle (e^3 \wedge e^8 + e^4 \wedge e^9) \wedge e^a \]

\[ + 2 \text{Im} \langle \eta^{2r}, \Gamma_{a} \eta^{2s} \rangle (e^3 \wedge e^8 - e^4 \wedge e^9) \wedge e^a \]

\[ - 2 \text{Im} \langle \eta^{1r}, \Gamma_{a} \lambda^{1s} \rangle (e^3 \wedge e^9 - e^4 \wedge e^8) \wedge e^a \]

\[ + 2 \text{Im} \langle \eta^{2r}, \Gamma_{a} \lambda^{2s} \rangle (e^3 \wedge e^9 + e^4 \wedge e^8) \wedge e^a \]

\[ - 2 \text{Re} \langle \eta^{1r}, \Gamma_{a} \eta^{2s} \rangle - \text{Re} \langle \eta^{2r}, \Gamma_{a} \eta^{1s} \rangle \right) e^3 \wedge e^c \wedge e^a \]

\[ - 2 \text{Re} \langle \eta^{1r}, \Gamma_{a} \lambda^{2s} \rangle + \text{Re} \langle \eta^{2r}, \Gamma_{a} \lambda^{1s} \rangle \right) e^4 \wedge e^c \wedge e^a \]

\[ - 2 \text{Im} \langle \eta^{1r}, \Gamma_{a} \eta^{1s} \rangle e^8 \wedge e^c \wedge e^a \]

\[ - 2 \text{Im} \langle \eta^{2r}, \Gamma_{a} \eta^{2s} \rangle + \text{Im} \langle \eta^{2r}, \Gamma_{a} \lambda^{1s} \rangle e^9 \wedge e^c \wedge e^a \]

\[ - 2 \text{Im} \langle \eta^{1r}, \Gamma_{a} \lambda^{2s} \rangle + \text{Im} \langle \eta^{2r}, \Gamma_{a} \lambda^{1s} \rangle \right) e^9 \wedge e^c \wedge e^a \]

\[ \dot{\theta}^{rs} = \frac{1}{3} \left( \text{Re} \langle \eta^{1r}, \Gamma_{abc} \eta^{2s} \rangle + \text{Re} \langle \eta^{2r}, \Gamma_{abc} \eta^{1s} \rangle \right) e^a \wedge e^b \wedge e^c \wedge e^3 \]

\[ + \frac{1}{3} \left( \text{Re} \langle \eta^{1r}, \Gamma_{abc} \lambda^{2s} \rangle - \text{Re} \langle \eta^{2r}, \Gamma_{abc} \lambda^{1s} \rangle \right) e^a \wedge e^b \wedge e^c \wedge e^4 \]

\[ + \frac{1}{3} \left( \text{Im} \langle \eta^{1r}, \Gamma_{abc} \eta^{2s} \rangle - \text{Im} \langle \eta^{2r}, \Gamma_{abc} \eta^{1s} \rangle \right) e^a \wedge e^b \wedge e^c \wedge e^8 \]

\[ + \frac{1}{3} \left( \text{Im} \langle \eta^{1r}, \Gamma_{abc} \lambda^{2s} \rangle - \text{Im} \langle \eta^{2r}, \Gamma_{abc} \lambda^{1s} \rangle \right) e^a \wedge e^b \wedge e^c \wedge e^9 \]

\[ + \frac{1}{3} \left( \text{Re} \langle \eta^{1r}, \Gamma_{abc} \lambda^{1s} \rangle \right) e^a \wedge (e^3 \wedge e^4 \wedge e^5 - e^8 \wedge e^9 \wedge e^c) \]

\[ + 2 \text{Re} \langle \eta^{1r}, \Gamma_{a} \lambda^{2s} \rangle \right) e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^4 \wedge e^5 - e^8 \wedge e^9 \wedge e^c) \]

\[ - 2 \text{Im} \langle \eta^{1r}, \Gamma_{a} \eta^{1s} \rangle e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^9 \wedge e^5 - e^8 \wedge e^9 \wedge e^c) \]

\[ - 2 \text{Im} \langle \eta^{2r}, \Gamma_{a} \eta^{2s} \rangle e^a \wedge (e^3 \wedge e^9 \wedge e^5 - e^8 \wedge e^9 \wedge e^c) \]

\[ + 2 \text{Im} \langle \eta^{2r}, \Gamma_{a} \lambda^{1s} \rangle \right) e^a \wedge (e^3 \wedge e^9 \wedge e^5 - e^8 \wedge e^9 \wedge e^c) \]

\[ - 2 \text{Re} \langle \eta^{1r}, \Gamma_{a} \lambda^{1s} \rangle \right) e^a \wedge (e^3 \wedge e^9 \wedge e^5 + e^4 \wedge e^8 \wedge e^c) \]

\[ + 2 \text{Re} \langle \eta^{2r}, \Gamma_{a} \lambda^{2s} \rangle \right) e^a \wedge (e^3 \wedge e^9 \wedge e^5 + e^4 \wedge e^8 \wedge e^c) \]

\[ + 2 \text{Im} \langle \eta^{1r}, \Gamma_{a} \lambda^{2s} \rangle \right) e^a \wedge (e^3 \wedge e^9 \wedge e^5 + e^4 \wedge e^8 \wedge e^c) \]

\[ - 2 \text{Im} \langle \eta^{2r}, \Gamma_{a} \lambda^{1s} \rangle \right) e^a \wedge (e^3 \wedge e^9 \wedge e^5 + e^4 \wedge e^8 \wedge e^c) \]

\[ \dot{\tau}^{rs} = - \frac{1}{3} \text{Re} \langle \eta^{1r}, \Gamma_{abc} \lambda^{1s} \rangle e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^4 - e^8 \wedge e^9) \]

\[ + \frac{1}{3} \text{Re} \langle \eta^{2r}, \Gamma_{abc} \lambda^{2s} \rangle e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^4 + e^8 \wedge e^9) \]

\[ - \frac{1}{3} \text{Im} \langle \eta^{1r}, \Gamma_{abc} \eta^{1s} \rangle e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^8 + e^4 \wedge e^9) \]

\[ + \frac{1}{3} \text{Im} \langle \eta^{2r}, \Gamma_{abc} \eta^{2s} \rangle e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^8 - e^4 \wedge e^9) \]

\[ - \frac{1}{3} \text{Im} \langle \eta^{1r}, \Gamma_{abc} \lambda^{1s} \rangle e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^9 - e^4 \wedge e^8) \]

\[ + \frac{1}{3} \text{Im} \langle \eta^{2r}, \Gamma_{abc} \lambda^{2s} \rangle e^a \wedge e^b \wedge e^c \wedge (e^3 \wedge e^9 + e^4 \wedge e^8) \]

\[ - \frac{1}{3} \text{Re} \langle \eta^{1r}, \Gamma_{abc} \eta^{2s} \rangle - \text{Re} \langle \eta^{2r}, \Gamma_{abc} \eta^{1s} \rangle \right) e^a \wedge e^b \wedge e^c \wedge e^3 \wedge e^2 \wedge \cdots \]
where after a relabelling of the spacetime coordinates $e^a = h^{-1/6} dx^a$, $a = 0, 1, 2, 5, 6, 7$, and $e^i = h^{1/3} dy^i$, $i = 3, 4, 8, 9, z$, is a pseudo-orthonormal frame of the M5-brane metric (42).

B Spinning particle probes with 4-form couplings

Following the use of spinning particle probes on 4- and 5-dimensional minimal supergravity backgrounds that exhibit 2-form couplings [16], one may be tempted to generalise these to spinning particle probes that exhibit 4-form couplings. Such a generalisation is desirable as the TCFH connection of 11-dimensional supergravity exhibits terms that depend of the 4-form field strength $F$. One way to generalise (10) is to adapt the general construction of [46] and introduce a fermionic superfield $\psi$ of mass dimension $[1/2]$. Insisting for the couplings of the action to be dimensionless, a minimal choice for an action with a 4-form coupling is

$$S = -\frac{1}{2} \int dt d\theta \left[ i g_{\mu\nu} D_\mu x^\nu \partial_1 x^{\sigma} - \frac{i}{12} F_{\mu\nu\rho\sigma} \psi^{\mu\nu\rho\sigma} \partial_1 x^{\sigma} + \beta \psi_{\mu\nu\rho} \nabla \psi^{\mu\nu\rho} \right], \quad (2)$$

with $\beta$ a constant which will be specified later,

$$\nabla \psi^{\mu\nu\rho} = D_\psi^{\mu\nu\rho} + 3 D x^\chi \Gamma^{[\mu}_{\chi \lambda} \psi^{\lambda] [\nu \rho]}, \quad (3)$$

and $\Gamma$ are the Christoffel symbols of the spacetime metric $g$. The numerical coefficient of the coupling $F \nabla \psi D x$ could be arbitrary but the above choice will suffice. Also one could add additional terms in the action like $F \nabla \psi D x$ which we shall explore later. Other terms include couplings of the type $\nabla F \psi D x$. After a superspace partial integration these can be re-expressed in terms of the $F \psi \partial_1 x$ and $F \nabla \psi D x$ couplings.

The variation of the action (2) can be expressed as

$$\delta S = - \int dt d\theta \left[ g_{\mu\nu} \delta x^{\mu} S^{\nu} + \Delta \psi_{\mu\nu\rho} S^{\mu\nu\rho} \right], \quad (4)$$

where

$$\Delta \psi^{\mu\nu\rho} = \delta \psi^{\mu\nu\rho} + 3 \delta x^\chi \Gamma^{[\mu}_{\chi \lambda} \psi^{\lambda] [\nu \rho]}, \quad (5)$$
\( \delta x \) and \( \delta \psi \) are arbitrary variations of the fields and

\[
S^\mu = -i \nabla_\nu D x^\nu - \frac{i}{24} \nabla^\mu F_{\nu \rho \sigma \lambda} \psi^{\nu \rho \sigma \lambda} \partial_\lambda x^\lambda - \frac{i}{24} \nabla_\lambda F^\mu_{\nu \rho \sigma} \psi^{\nu \rho \sigma} \partial_\lambda x^\lambda - \frac{i}{24} F^\mu_{\nu \rho \sigma} \nabla \psi^{\nu \rho \sigma} + \frac{3}{2} \beta \psi_{\nu \rho \sigma} D x^\lambda R^\mu_{\lambda \nu \tau} \psi^{\tau \rho \sigma},
\]

\[
S^{\mu \nu \rho} = \beta \nabla \psi^{\mu \nu \rho} - \frac{i}{24} F^{\mu \nu \rho} \lambda \partial_\lambda x^\lambda,
\]

are the equations of motion of \( x \) and \( \psi \), respectively.

The action (2) is manifestly invariant under one supersymmetry. For a probe described by the action (2) propagating on an M-brane background with spacetime metric \( g \) and 4-form field strength \( F \) to exhibit additional symmetries that are generated by the form bilinears \( \omega \) and \( \tau \) of 11-dimensional supergravity, one can consider the infinitesimal transformations

\[
\delta x^\mu = \alpha \omega^\mu \nu D x^\nu + \alpha c_1 \omega_{\nu \rho \sigma} \psi^{\nu \rho \sigma},
\]

\[
\delta \psi^{\mu \nu \rho} = \alpha \tau^{\mu \nu \rho} \sigma \lambda D x^\sigma D x^\lambda + \alpha c_2 \tau^{\mu \nu \rho} \sigma \lambda \psi^{\nu \rho \sigma} \kappa D x^\kappa,
\]

where \( \alpha \) is the supersymmetry parameter assigned mass dimension \([-1/2]\) and \( c_1, c_2 \) are constants. These transformations are the most general ones allowed such that the infinitesimal variations have the same mass dimension as those of the associated fields, and \( \omega \) and \( \tau \) are dimensionless.

For the TCFH on \( \omega \) to be interpreted as an invariance condition for the probe action (2), the conditions that arise for the invariance of this action under the infinitesimal variations (7) should match the TCFH. For this first notice that the equations of motion (6) contain the spacetime curvature \( R \). As such terms do not arise in the TCFH, these terms in the invariance conditions must vanish. This requires that \( \beta = 0 \). Moreover if the action had contained a \( F \nabla \psi D x \) coupling, this would have given rise to a \( FR \) term in the equations of motion. Because the TCFH does not contain such a term, the \( F \nabla \psi D x \) coupling was neglected from the beginning. The remaining conditions that arise from the invariance of the action (2) with \( \beta = 0 \) under the infinitesimal transformations (7) read

\[
\nabla_\mu \omega_{\nu \rho} - \frac{1}{12} F_{\mu \lambda \sigma \rho \kappa} \psi^{\lambda \sigma \rho \kappa} = 0,
\]

\[
\nabla_\mu \omega_{\nu \rho} - \frac{1}{12} F_{\mu \lambda \sigma \rho \kappa} \psi^{\lambda \sigma \rho \kappa} = 0,
\]

\[
(2c_1 + c_2) F_{\lambda \nu \rho \kappa} \sigma \lambda \rho \kappa \omega_{\mu \nu \rho} + \nabla_\mu F_{\psi \rho \sigma \kappa} \omega_{\nu \rho \sigma \kappa} = 0,
\]

\[
F_{\mu \nu \rho \kappa} \omega_{\nu \rho \sigma \kappa} + \nabla_\lambda F_{\nu \rho \sigma \kappa} \omega_{\mu \nu \rho \sigma \kappa} = 0,
\]

\[
(8)
\]

The first condition matches the expression of the TCFH connection on \( \omega \). However the second condition is rather strong on both M2- and M5-brane backgrounds to admit non-trivial solutions. Moreover this condition persists even if \( \beta \neq 0 \) and the curvature terms are included. This does not exclude the possibility that there may be backgrounds such that the TCFH matches with the conditions (8) but if this is the case, such examples will be restricted.

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