Multiplication Operators Between Iterated Logarithmic Lipschitz Spaces of a Tree

Robert F. Allen, Flavia Colonna and Andrew Prudhom

In memory of Maurice Heins

Abstract. In this article, we characterize the bounded and the compact multiplication operators between distinct iterated logarithmic Lipschitz spaces, and between the Lipschitz space and an iterated logarithmic Lipschitz space of an infinite tree. In addition, we provide operator norm estimates and show that there are no isometries among such operators.

Mathematics Subject Classification. Primary 47B38, 05C05.

Keywords. Multiplication operators, trees, iterated logarithmic Lipschitz spaces.

1. Introduction

Let $X$ and $Y$ be Banach spaces of complex-valued functions defined on a set $\Omega$. For a complex-valued function $\psi$ on $\Omega$ we define the multiplication operator with symbol $\psi$ to be the linear operator $M_\psi f = \psi f$ for all $f \in X$.

A primary objective in the study of operators with symbol is to relate the function-theoretic properties of the symbol to the properties of the operator.

In recent years, the study of such operators on spaces of functions defined on discrete structures has been conducted. In particular, the first two authors, among others, have studied operators on spaces of functions defined on infinite rooted trees. These spaces, under certain restrictions, are often considered as discrete analogs to classical spaces of analytic functions defined on the open unit disk $\mathbb{D}$. A connection between functions on a homogeneous tree and analytic functions on $\mathbb{D}$ was made explicit through a particular embedding of the tree into $\mathbb{D}$ in [7].

Motivated by the wide interest in the study of linear operators on the Bloch space of analytic functions on $\mathbb{D}$, whose elements are characterized by a Lipschitz condition under an appropriate choice of metrics (see [8]), the Lipschitz space $\mathcal{L}$ on a tree $T$ was introduced in [9]. The study of multiplication and composition operators on $\mathcal{L}$ conducted in [4,9] led, in [3], to the
study of the weighted Lipschitz space $L_w$ and of the multiplication operators acting on $L_w$. For related work on such operators see [1]. Due to the analogy between the Bloch space and the Lipschitz space in their respective environments, the weighted Lipschitz space can be viewed as the discrete analog of the weighted Bloch space studied in [15].

The process that gave rise to $L_w$ was used to define the iterated logarithmic Lipschitz spaces, $L^{(k)}$ for $k \in \mathbb{N}$, of which $L_w = L^{(1)}$. The multiplication operators on $L^{(k)}$ were studied in [2]. These spaces can be viewed as discrete versions of the logarithmic Bloch spaces, on which the weighted composition operators were studied in [11].

Research on the multiplication operators acting between the space $L^{\infty}$ of bounded functions on a tree equipped with the supremum norm $\| \cdot \|_{\infty}$, and $L$ was carried out in [10]. In [5,6], the first author with Craig and Pons, respectively, studied multiplication and composition operators on the weighted Banach space $L^{\infty}_\mu$ of functions on a tree for a given positive weight $\mu$. In [12,13], the authors developed a discrete version of the Hardy spaces on homogeneous trees and studied the multiplication and composition operators on such spaces.

In this paper, we expand the research carried out in [2] by focusing on the multiplication operators acting either between distinct iterated logarithmic Lipschitz spaces, or between the Lipschitz space and an iterated logarithmic Lipschitz space. The techniques used also provide improvements on known results in some cases.

1.1. Preliminary Definitions and Notation

By a tree $T$ we mean a locally finite, connected, and simply connected graph, which, as a set, we identify with the collection of its vertices. By a function on a tree we mean a complex-valued function on the set of its vertices. Two vertices $v$ and $w$ are called neighbors if there is an edge $[v, w]$ connecting them, and we use the notation $v \sim w$. A vertex is called terminal if it has a unique neighbor. A path is a sequence of vertices $[v_0, v_1, \ldots]$ such that $v_k \sim v_{k+1}$ and $v_{k-1} \neq v_{k+1}$, for all $k$. Define the length of a finite path $[v = v_0, v_1, \ldots, w = v_n]$ to be the number of edges $n$ connecting $v$ to $w$. The distance between vertices $v$ and $w$ is the length $d(v, w)$ of the unique path connecting $v$ to $w$.

Given a tree $T$ rooted at $o$, the length of a vertex $v$ is defined as $|v| = d(o, v)$. For a vertex $v \in T$, a vertex $w$ is called a descendant of $v$ if $v$ lies in the path from $o$ to $w$. The vertex $v$ is then called an ancestor of $w$. For $v \in T$ with $v \neq o$, we denote by $v^-$ the unique neighbor which is an ancestor of $v$. The vertex $v$ is called a child of $v^-$. For $v \in T$, the set $S_v$ consisting of $v$ and all its descendants is called the sector determined by $v$. The set $T\setminus\{o\}$ will be denoted by $T^*$.

In this paper, we shall assume the tree $T$ to be without terminal vertices (and hence infinite), and rooted at a vertex $o$. In addition, $k, m,$ and $n$ are natural numbers, and $\psi$ a fixed function on $T$. We define the discrete derivative of a function $f$ on $T$ as
\[ f'(v) = \begin{cases} f(v) - f(v^-) & \text{if } v \neq o, \\ 0 & \text{if } v = o. \end{cases} \]

2. Lipschitz-Type Spaces

In [9], the Lipschitz space was defined as the set of functions on \( T \) which are Lipschitz as maps between the metric space \((T,d)\) and the Euclidean space \( C \). It was shown that a function \( f \in \mathcal{L} \) if and only if \( f' \in L^\infty \), and \( \|f'\|_\infty \) is precisely the Lipschitz number of \( f \). It was further shown that \( \mathcal{L} \) is a Banach space under the norm
\[
\|f\|_\mathcal{L} = |f(o)| + \|f'\|_\infty.
\]

The following result provides a bound on point evaluation in the Lipschitz space.

**Lemma 2.1.** [9, Lemma 3.4(a)] If \( f \in \mathcal{L} \) and \( v \in T \), then
\[
|f(v)| \leq |f(o)| + |v|\|f'\|_\infty.
\]
In particular, if \( \|f\|_\mathcal{L} \leq 1 \), then \( |f(v)| \leq |v| \) for each \( v \in T^* \).

The bounded multiplication operators on \( \mathcal{L} \) were characterized as follows.

**Theorem 2.2.** [9, Theorem 3.6] The operator \( M_\psi \) is bounded on \( \mathcal{L} \) if and only if \( \psi \in L^\infty \) and \( \psi' \in L^\infty_w \), where \( w \) is the weight defined by the length, that is, \( w(v) = |v| \).

This result motivated the study in [3] of the weighted Lipschitz space \( \mathcal{L}_w \) defined as the set of functions \( f \) on \( T \) such that \( f' \in L^\infty_w \). Thus, the bounded functions in \( \mathcal{L}_w \) are precisely those that induce bounded multiplication operators on \( \mathcal{L} \). The bounded multiplication operators on \( \mathcal{L}_w \) were characterized in the following result.

**Theorem 2.3.** [3, Theorem 4.1] The operator \( M_\psi \) is bounded on \( \mathcal{L}_w \) if and only if \( \psi \in L^\infty \) and \( \psi' \in L^\infty_\mu \), where \( \mu(v) = |v| \log |v| \) for \( v \in T^* \).

In fact, this process can be continued, thus creating the iterated logarithmic Lipschitz spaces, defined in [2]. For \( x \geq 1 \) define the recursive sequence \( \ell_j(x) \) by
\[
\ell_j(x) = \begin{cases} x & \text{if } j = 0, \\ 1 + \log x & \text{if } j = 1, \\ 1 + \log \ell_{j-1}(x) & \text{if } j \geq 2. \end{cases}
\]
In addition, the sequence \( \Lambda_k(x) \) is defined as
\[
\Lambda_k(x) = \begin{cases} 1 & \text{if } k = 0, \\ \prod_{j=0}^{k-1} \ell_j(x) & \text{if } k \geq 1. \end{cases}
\]
Definition 2.4. Let $T$ be a tree rooted at $o$. For non-negative integer $k$, the iterated logarithmic Lipschitz space $\mathcal{L}^{(k)}$ is the set of functions $f$ on $T$ satisfying the condition

$$\sup_{v \in T^*} |f'(v)|\Lambda_k(|v|) < \infty.$$ 

For $f \in \mathcal{L}^{(k)}$, define

$$\|f\|_k = |f(o)| + \sup_{v \in T^*} |f'(v)|\Lambda_k(|v|).$$

Notice that $\mathcal{L} = \mathcal{L}^{(0)}$ and $\mathcal{L}_w = \mathcal{L}^{(1)}$. In [9, Proposition 3.3] and [2, Proposition 2.2] it was shown that $\mathcal{L}^{(k)}$ is a functional Banach space under the norm $\|f\|_k$. Moreover, the point-evaluation functionals were proven to be bounded. With Lemma 2.1, we obtain the following result.

Lemma 2.5. If $f \in \mathcal{L}^{(k)}$ and $v \in T^*$, then

$$|f(v)| \leq \begin{cases} (1 + |v|)\|f\|_k & \text{if } k = 0, \\ \ell_k(|v|)\|f\|_k & \text{if } k \geq 1. \end{cases}$$

Moreover, if $\|f\|_k \leq 1$, then $|f(v)| \leq \ell_k(|v|)\|f\|_k$ for all $k \geq 0$.

The iterated logarithmic Lipschitz spaces are continuously embedded in one another, as shown in the next result.

Theorem 2.6. If $m \leq n$, then $\mathcal{L}^{(m)} \subseteq \mathcal{L}^{(n)}$ and $\|f\|_m \leq \|f\|_n$ for all $f \in \mathcal{L}^{(n)}$.

Proof. It suffices to show that each space $\mathcal{L}^{(k)}$ is contained in the immediate predecessor space $\mathcal{L}^{(k-1)}$. Let $f$ be a function on $T$. Since

$$|f'(v)| \leq |v||f'(v)| \leq |v|(1 + \log |v|)|f'(v)|$$

for all $v \in T^*$, we have

$$\sup_{v \in T^*} |f'(v)|\Lambda_k(|v|) \leq \sup_{v \in T^*} |f'(v)|\Lambda_1(|v|) \leq \sup_{v \in T^*} |f'(v)|\Lambda_2(|v|).$$

Thus $\|f\|_0 \leq \|f\|_1 \leq \|f\|_2$, and so $\mathcal{L}^{(2)} \subseteq \mathcal{L}^{(1)} \subseteq \mathcal{L}^{(0)}$.

Next, assume $k > 2$. Then

$$\sup_{v \in T^*} |f'(v)|\Lambda_k(|v|) = \sup_{v \in T^*} |f'(v)|\ell_{k-1}(|v|)\Lambda_{k-1}(|v|)$$

$$= \sup_{v \in T^*} |f'(v)|(1 + \log \ell_{k-2}(|v|))\Lambda_{k-1}(|v|)$$

$$\geq \sup_{v \in T^*} |f'(v)|\Lambda_{k-1}(|v|).$$

Thus, we have $\|f\|_{k-1} \leq \|f\|_k$, which implies $\mathcal{L}^{(k)} \subseteq \mathcal{L}^{(k-1)}$. \hfill \Box

The following result is inspired by [14, Lemma 2.10], where it was proved for Banach spaces of analytic functions on $D$.

Lemma 2.7. [5, Lemma 2.5] Let $X$ and $Y$ be Banach spaces of functions on $T$. Suppose that

1. the point-evaluation functionals of $X$ are bounded;
2. the closed unit ball of $X$ is a compact subset of $X$ in the topology of pointwise convergence;
3. **T : X → Y is bounded when X and Y are given the topology of pointwise convergence.**

Then **T** is a compact operator if and only if given a bounded sequence (**f**<sub>n</sub>) in **X** converging to 0 pointwise, the sequence (**Tf**<sub>n</sub>) converges to zero in the norm of **Y**.

### 3. Boundedness and Operator Norm

In this section, we characterize the bounded multiplication operators between distinct iterated logarithmic Lipschitz spaces, and, in particular, between the Lipschitz and weighted Lipschitz spaces. In addition, we provide estimates on the operator norm.

The boundedness of **M**<sub>ψ</sub> of **L**<sup>(k)</sup> was proven for **k** = 0 in [9, Theorem 3.6] and for **k** ≥ 1 in [2, Theorem 3.1], showing that the bounded functions in **L**<sup>(k)</sup> induce bounded multiplication operators on **L**<sup>(k+1)</sup>.

**Theorem 3.1.** Let **k** be a non-negative integer. Then the operator **M**<sub>ψ</sub> is bounded on **L**<sup>(k)</sup> if and only if **ψ** ∈ **L**<sup>∞</sup> ∩ **L**<sup>(k+1)</sup> for every non-negative **k**. Furthermore, the following estimates hold:

\[
\max\{\|\psi\|_k, \|\psi\|_\infty\} \leq \|M\psi\| \leq \|\psi\|_\infty + \sup_{v \in T^*} |\psi'(v)|\Lambda_{k+1}(|v|).
\]

To characterize the boundedness of **M**<sub>ψ</sub> acting between distinct iterated logarithmic spaces, we define the following two quantities. For **ψ** a function on **T**, and **m** and **n** distinct non-negative integers, define

\[
\mu_{\psi, m, n} = \sup_{v \in T^*} |\psi'(v)|\ell_m(|v|)\Lambda_n(|v|),
\]

\[
\nu_{\psi, m, n} = \sup_{v \in T^*} \frac{\left|\psi(v^-)\right|\Lambda_n(|v|)}{\Lambda_m(|v|)}.
\]

**Theorem 3.2.** Let **m** and **n** be distinct non-negative integers. Then the operator **M**<sub>ψ</sub> : **L**<sup>(m)</sup> → **L**<sup>(n)</sup> is bounded if and only if **µ**<sub>ψ, m, n</sub> and **ν**<sub>ψ, m, n</sub> are both finite. Furthermore, under these conditions,

\[
\|\psi\|_n \leq \|M\psi\| \leq |\psi(o)| + \mu_{\psi, m, n} + \nu_{\psi, m, n}.
\]

**Proof.** Suppose **µ**<sub>ψ, m, n</sub> and **ν**<sub>ψ, m, n</sub> are both finite. Let **f** ∈ **L**<sup>(m)</sup> such that \(\|f\|_m \leq 1\). By Lemma 2.5, for **v** ∈ **T**

\[
|(\psi f)'(v)|\Lambda_n(|v|) \leq |\psi'(v)||f(v)|\Lambda_n(|v|) + |\psi(v^-)||f'(v)|\Lambda_n(|v|)
\]

\[
\leq |\psi'(v)|\ell_m(|v|)\Lambda_n(|v|)|f||f| + \frac{|\psi(v^-)||f'(v)|\Lambda_n(|v|)\Lambda_m(|v|)}{\Lambda_m(|v|)}
\]

\[
\leq |\psi'(v)|\ell_m(|v|)\Lambda_n(|v|)|f||ff| + \frac{|\psi(v^-)|\Lambda_n(|v|)}{\Lambda_m(|v|)}|f||f|
\]

\[
\leq \mu_{\psi, m, n} + \nu_{\psi, m, n}.
\]

So **ψf** ∈ **L**<sup>(n)</sup> with

\[
\|M\psi f\|_n = |\psi(o)f(o)| + \sup_{v \in T^*} |(\psi f)'(v)|\Lambda_n(|v|) \leq |\psi(o)| + \mu_{\psi, m, n} + \nu_{\psi, m, n}.
\]
Thus $M \psi : L^m \to L^n$ is bounded, and the upper bound of the operator norm is established.

Next, suppose $M \psi : L^m \to L^n$ is bounded. Then $\psi = M \psi 1$ is an element of $L^n$. So $\| \psi \|_n \leq \| M \psi \|$, establishing the lower bound of the operator norm.

To prove the finiteness of $\nu_{\psi, m, n}$, fix a vertex $v \in T$ and define for all $w \in T$

$$f_v(w) = \frac{\chi_v(w)}{\Lambda_m(|v| + 1)},$$

where $\chi_v$ denotes the characteristic function of $\{ v \}$. Thus

$$|f'_v(w)| = \begin{cases} 
 1 / \Lambda_m(|v| + 1) & \text{if } w = v \text{ or } w^- = v, \\
 0 & \text{otherwise.}
\end{cases}$$

From this, we obtain $f_v \in L^m$ since $\sup_{w \in T^*} |f'_v(w)| \Lambda_m(|w|) = 1$. Since $M \psi$ is bounded,

$$\frac{|\psi(v)| \Lambda_n(|v| + 1)}{\Lambda_m(|v| + 1)} = \sup_{w \in T^*} |(\psi f_v)'(w)| \Lambda_n(|w|) \leq \| M \psi \|.$$

Thus, since $v$ is arbitrary, we have

$$\nu_{\psi, m, n} = \sup_{v \in T^*} \frac{|\psi(v^-)| \Lambda_n(|v|)}{\Lambda_m(|v|)} \leq \| M \psi \| < \infty. \quad (1)$$

Finally, to prove the finiteness of $\mu_{\psi, m, n}$, define

$$g(v) = \begin{cases} 
 0 & \text{if } v = o, \\
 \ell_m(|v|) & \text{if } v \neq o.
\end{cases}$$

The function $g$ is an element of $L^m$ for all $m \geq 0$ since $\| g \|_0 = 1$ and for $m \geq 1$, $\| g \|_m \leq 2 \prod_{j=1}^{m} \ell_j(2)$, as shown in [2, proof of Theorem 3.1]. From the boundedness of $M \psi$, we have

$$\sup_{v \in T^*} |(\psi g)'(v)| \Lambda_n(|v|) \leq \| M \psi \| \| g \|_m.$$

On the other hand, for $v \in T^*$ we obtain

$$|\psi'(v)| \ell_m(|v|) \Lambda_n(|v|) = |\psi'(v)| g(v) \Lambda_n(|v|)$$

$$\leq |(\psi g)'(v)| \Lambda_n(|v|) + |\psi(v^-)| g'(v) \Lambda_n(|v|)$$

$$\leq \| M \psi \| \| g \|_m + \frac{|\psi(v^-)| \Lambda_n(|v|)}{\Lambda_m(|v|)} \| g \|_m.$$

Taking the supremum over all $v \in T^*$, we obtain

$$\mu_{\psi, m, n} \leq (\| M \psi \| + \nu_{\psi, m, n}) \| g \|_m.$$

Thus $\mu_{\psi, m, n}$ is finite. \qed
4. Compactness

In this section, we characterize the compact multiplication operators among those studied in Sect. 3. As was the case for boundedness of \( M_\psi \) acting from \( \mathcal{L}^{(m)} \) to \( \mathcal{L}^{(n)} \), the characterizing quantities for compactness are precisely \( |\psi'(v)|\ell_m(|v|)\Lambda_n(|v|) \) and \( \frac{|\psi(v^-)|\Lambda_n(|v|)}{\Lambda_m(|v|)} \). The characterization for the bounded multiplication operators used the growth behavior, or “Big-Oh”, of these quantities. As is seen in other situations, the compactness is characterized by the asymptotic behavior, or “little-oh”, of these quantities. To this end, define for \( v \in T^* \)

\[
\mu_{\psi,m,n}(v) = |\psi'(v)|\ell_m(|v|)\Lambda_n(|v|),
\]
\[
\nu_{\psi,m,n}(v) = \frac{|\psi(v^-)|\Lambda_n(|v|)}{\Lambda_m(|v|)}.
\]

Observe the reuse of notation from Sect. 3, since

\[
\mu_{\psi,m,n} = \sup_{v \in T^*} \mu_{\psi,m,n}(v)
\]

and likewise for \( \nu_{\psi,m,n} \).

**Theorem 4.1.** Let \( m \) and \( n \) be distinct non-negative integers. Then the bounded operator \( M_\psi : \mathcal{L}^{(m)} \to \mathcal{L}^{(n)} \) is compact if and only if

\[
\lim_{|v| \to \infty} \mu_{\psi,m,n}(v) = 0 \quad \text{and} \quad \lim_{|v| \to \infty} \nu_{\psi,m,n}(v) = 0.
\]

**Proof.** First, suppose \( \mu_{\psi,m,n}(v) \to 0 \) and \( \nu_{\psi,m,n}(v) \to 0 \) as \( |v| \to \infty \). Let \( (f_k) \) be a bounded sequence in \( \mathcal{L}^{(m)} \) which converges to 0 pointwise, and define \( s = \sup_{k \in \mathbb{N}} \|f_k\|_m \). By Lemma 2.7, it suffices to show that \( \|M_\psi f_k\|_n \to 0 \) as \( k \to \infty \).

Fix \( \varepsilon > 0 \). Since \( f_k \to 0 \) pointwise, \( |f_k(o)| < \frac{s}{3\varepsilon} \) for all \( k \) sufficiently large. By assumption, there exists \( M > 0 \) such that

\[
|\psi'(v)|\ell_m(|v|)\Lambda_n(|v|) < \frac{\varepsilon}{3s}
\]

and

\[
\frac{|\psi(v^-)|\Lambda_n(|v|)}{\Lambda_m(|v|)} < \frac{\varepsilon}{3s}
\]

for all \( |v| \geq M \).

Applying Lemma 2.5, and normalizing \( f_k \) if \( \|f_k\|_m > 1 \), for all \( |v| \geq M \)

\[
|(|\psi f_k)'(v)|\Lambda_n(|v|)| \leq |\psi'(v)||f_k(v)|\Lambda_n(|v|) + |\psi(v^-)||f_k'(v)|\Lambda_n(|v|) \\
\leq |\psi'(v)|\ell_m(|v|)\Lambda_n(|v|) + \frac{|\psi(v^-)|\Lambda_n(|v|)}{\Lambda_m(|v|)}\|f_k\|_m \\
< \frac{2\varepsilon}{3}.
\]

Since \( (f_k) \) converges to 0 on \( \{v \in T : |v| \leq M\} \), so does the sequence

\[
|(|\psi f_k)'(v)|\Lambda_n(|v|))
\]
So for sufficiently large \( k \), and all \( v \in T^* \), we have \( |(ψf_k)'(v)|Λ_n(|v|) < \frac{2ε}{3} \). Thus

\[
||M_ψf_k||_n = |ψ(o)||f_k(o)| + \sup_{v \in T^*} |(ψf_k)'(v)|Λ_n(|v|) < ε
\]

for all \( k \) sufficiently large. Thus \( ||M_ψf_k||_n \to 0 \) as \( k \to ∞ \), proving the compactness of \( M_ψ \).

Conversely, suppose \( M_ψ : \mathcal{L}^{(m)} \to \mathcal{L}^{(n)} \) is compact. Let \( (v_k) \) be a sequence in \( T \) such that \( 1 \leq |v_k| \to ∞ \). For \( k \in \mathbb{N} \cup \{0\} \) and \( w \in T \), define

\[
g_k(w) = \frac{X_{v_k}(w)}{Λ_m(|v_k|)}.
\]

Observe that \( g_0(w) = X_{v_0}(w) \), thus \( g_0 \in \mathcal{L}^{(0)} \) with \( ||g_0||_0 \leq 1 \). For \( k \geq 1 \), \( g_k \in \mathcal{L}^{(m)} \) with \( ||g_k||_m = 1 \), as shown in the proof of Theorem 3.2 (with a minor modification). Moreover, \( (g_k) \) converges to 0 pointwise. Since \( M_ψ \) is compact, by Lemma 2.7, it follows that \( ||M_ψg_k||_n \to 0 \) as \( k \to ∞ \). Observing that

\[
|(ψg_k)'(v)| = \frac{|ψ(v_k^-)|}{Λ_m(|v_k|)}
\]

we obtain

\[
\frac{|ψ(v_k^-)|Λ_n(|v_k|)}{Λ_m(|v_k|)} = |(ψg_k)'(v_k)|Λ_n(|v_k|) \leq ||M_ψg_k||_n \to 0.
\]

Thus \( ν_{ψ,m,n}(v) \to 0 \) as \(|v| \to ∞ \).

Lastly, corresponding to the above sequence \((v_k)\), where without loss of generality we now assume \(|v_k| > 3\), define

\[
h_k(w) = \begin{cases} 0 & \text{if } |w| = 0 \text{ or } 1, \\ \ell_m(|w|) & \text{if } 2 \leq |w| < |v_k| - 1, \\ \ell_m(|v_k|) & \text{if } |w| \geq |v_k| - 1. \end{cases}
\]

From [2, Theorem 6.2] we have \( h_k \in \mathcal{L}^{(m)} \) and \( ||h_k||_m \) is bounded for all \( m \). Moreover, \( (h_k) \) converges to 0 pointwise. Therefore, by the compactness of \( M_ψ \), we have

\[
|ψ'(v_k)|\ell_m(|v_k|)Λ_n(|v_k|) = |(ψh_k)'(v_k)|Λ_n(|v_k|) \leq ||M_ψh_k||_n \to 0,
\]

as \( k \to ∞ \). Thus \( μ_{ψ,m,n}(v) \to 0 \) as \(|v| \to ∞ \), completing the proof. \( \square \)

5. Isometries

From [9, Theorem 9.1] and [2, Theorem 5.1], the only isometric multiplication operators acting on \( \mathcal{L}^{(k)} \) are induced by unimodular constant functions, for all \( k \geq 0 \). In this section, we show there are no isometries among the multiplication operators acting between distinct iterated logarithmic spaces.

**Theorem 5.1.** Let \( m \) and \( n \) be distinct non-negative integers. Then \( M_ψ : \mathcal{L}^{(m)} \to \mathcal{L}^{(n)} \) is not an isometry.
Proof. Assume $M_\psi$ is an isometry. Taking as test functions the constant function 1 and $g(v) = \frac{1}{2} \chi_o(v)$, we obtain
\[ \|\psi\|_n = \|M_\psi 1\|_n = \|1\|_m = 1 = \|g\|_m = \|M_\psi g\|_n = \frac{1}{2} \|\psi \chi_o\|_n = |\psi(o)|. \]
So
\[ \sup_{v \in T^*} |\psi'(v)| \Lambda_n(|v|) = \|\psi\|_n - |\psi(o)| = 0. \]
Since $\Lambda_n(|v|) \geq 1$ for all $v \in T^*$, it follows that $\psi'$ is identically zero. Thus $\psi$ is a unimodular constant function.

Fix a vertex $w$ in $T$ such that $|w| \geq 2$. Then
\[ \Lambda_m(|w|) = \|\chi_w\|_m = \|M_\psi \chi_w\|_n = |\psi(w)| \Lambda_n(|w|) = \Lambda_n(|w|). \]
Observe that the sequence $\Lambda_k(|w|)$ is strictly increasing in $k$ since $|w| \geq 2$. So it can not be the case that $\Lambda_m(|w|) = \Lambda_n(|w|)$. Therefore, $M_\psi$ is not an isometry. \[ \square \]

Acknowledgements
We wish to thank the referee for his/her insightful comments on improving the manuscript. The third author was supported by the McNair Scholars program at the University of Wisconsin-La Crosse.

References
[1] Allen, R.F., Colonna, F., Easley, G.: Multiplication operators between Lipschitz type spaces of a tree. Internat. J. Math. Math. Sci., 2011, 1–36 (2011)
[2] Allen, R.F., Colonna, F., Easley, G.: Multiplication operators on the iterated logarithmic Lipschitz spaces of a tree. Mediterr. J. Math. 9, 575–600 (2012)
[3] Allen, R.F., Colonna, F., Easley, G.: Multiplication operators on the weighted Lipschitz space of a tree. J. Oper. Theory 69, 209–231 (2013)
[4] Allen, R.F., Colonna, F., Easley, G.: Composition operators on the Lipschitz space of a tree. Mediterr. J. Math. 11, 97–108 (2014)
[5] Allen, R.F., Craig, I.M.: Multiplication operators on weighted Banach spaces of a tree. Bull. Korean Math. Soc. 54, 747–761 (2017)
[6] Allen, R.F., Pons, M. A.: Composition operators on weighted Banach spaces of a tree. Bull. Malays. Math. Sci. Soc. (2016). doi: 10.1007/s40840-016-0428-x
[7] Cohen, J.M., Colonna, F.: Embeddings of trees in the hyperbolic disk. Complex Variables Theory Appl. 24, 311–335 (1994)
[8] Colonna, F.: Bloch and normal functions and their relation. Rend. Circ. Mat. Palermo, II XXXVIII, 161–180 (1989)
[9] Colonna, F., Easley, G.R.: Multiplication operators on the Lipschitz space of a tree. Integr. Equ. Oper. Theory 68, 391–411 (2010)
[10] Colonna, F., Easley, G.R.: Multiplication operators between the Lipschitz space and the space of bounded functions on a tree. Mediterr. J. Math. 9, 423–438 (2012)
[11] Hosokawa, T., Dieu, N.Q.: Weighted composition operators on the logarithmic Bloch spaces with iterated weights. Nihonkai Math. J. 20, 57–72 (2009)

[12] Muthukumar, P., Ponnusamy, S.: Discrete analogue of generalized Hardy spaces and multiplication operators on homogenous trees. Anal. Math. Phys. (2016). doi:10.1007/s13324-016-0141-9

[13] Muthukumar, P., Ponnusamy, S.: Composition operators on the discrete analogue of generalized Hardy space on homogenous trees. Bull. Malays. Math. Sci. Soc. (2016). doi:10.1007/s40840-016-0419-y

[14] Tjani, M.: Compact composition operators on Möbius invariant Banach spaces, Ph.D. thesis, Michigan State University (1996)

[15] Yoneda, R.: The composition operators on weighted Bloch space. Arch. Math. (Basel) 78, 310–317 (2002)

Robert F. Allen
Department of Mathematics and Statistics
University of Wisconsin-La Crosse
La Crosse WI 54601
USA
e-mail: rallen@uwltax.edu

Flavia Colonna
Department of Mathematical Sciences
George Mason University
Fairfax VA 22030
USA
e-mail: fcolonna@gmu.edu

Andrew Prudhom
Department of Mathematics
University of North Carolina at Chapel Hill
Chapel Hill NC 27599
USA
e-mail: prudhom@live.unc.edu

Received: March 4, 2017.
Revised: August 12, 2017.
Accepted: September 15, 2017.