Abstract.

We derive the non-improvable Grand Lebesgue Space norm estimations for multivariate and multidimensional operator of infimal convolution.

Key words and phrases. Infimal convolution, Gaussian density, upper and lower estimates and limits, dilation operator, Lebesgue-Riesz and Grand Lebesgue Space norm and spaces, fundamental function, examples.

1 Notations. Statement of problem. Previous results.

Let \( f_1(x), f_2(x), x \in \mathbb{R}^d \) be two numerical valued functions. The following function \( g = f_1 \boxplus f_2 : \mathbb{R}^d \to \mathbb{R} \)

\[
g(x) = [f_1 \boxplus f_2](x) \overset{\text{def}}{=} \inf_{y \in \mathbb{R}^d} \{ f_1(y) + f_2(x - y) \}
\]

(1)

is named infimal convolution of \( f_1, f_2 \) ones. More generally, \( g_m[f_1, f_2, \ldots, f_m](x) = \)
\[ g_m(x) := \Box^m_{j=1} f_j(x) := (((f_1 \Box f_2) \Box f_3) \ldots \Box f_m)(x), \ x \in \mathbb{R}^d \]  \hspace{1cm} (2)

or equally

\[ g_m(x) := \Box^m_{j=1} f_j(x) := \inf \left\{ \sum_{j=1}^m f_j(y_j), \ \sum_{j=1}^m y_j = x, \right\}, \ x, y_j \in \mathbb{R}^d. \]  \hspace{1cm} (3)

This operation has many applications in convex analysis, theory of optimization etc., see [19], [22] and so one.

**Our aim in this short report is to estimate the Lebesgue - Riesz norm for this infimal convolution through ones for its components.**

We improve the previous results obtained in particular in an article [15].

Recall that the mentioned Lebesgue - Riesz norm for the (measurable) function \( h : \mathbb{R}^d \rightarrow \mathbb{R} \) is defined as follows

\[ ||h||_p \overset{\text{def}}{=} \left[ \int_{\mathbb{R}^d} |h(x)|^p \, dx \right]^{1/p}, \ p \geq 1, \]

\[ ||h||_{\infty} \overset{\text{def}}{=} \sup_{x \in \mathbb{R}^d} |f(x)|. \]

As ordinary, \( L_p(\mathbb{R}^d) = L_p = \{ f : \mathbb{R}^d \rightarrow \mathbb{R}, ||f||_p < \infty \} \).

**Grand Lebesgue Spaces (GLS).**

Let \((a,b) = \text{const}, \ a \geq 1, b \in (a, \infty); \) the case \( b = +\infty \) is also not excluded.

Let also \( p \in [a, b) \) or \( p \in [a, b]; \) evidently, the last case take place iff the value \( b \) is finite and greatest than \( a \). Let \( \psi_{(a,b)}(p) = \psi = \psi(p) \) be a function defined in the domain \((a,b), \) not necessarily be finite in each point inside of the interval \((a,b), \) such that \( \inf \psi(p) > 0. \)

We can and will suppose without loss of generality \( a = \inf\{p, \psi(p) < \infty\}; \) \( b = \sup\{p, \psi(p) < \infty\}, \) so that \( \text{Dom}[\psi] = [a, b) \) or \( \text{Dom}[\psi] = [a, b], \) of course iff \( b < \infty. \)

When \( a > 1, b < \infty, \) we define formally \( \psi(p) = +\infty \) for the values \( p \notin (a,b). \)

Denote also

\[ U\Psi \overset{\text{def}}{=} \cup_{a \geq 1, \ b \in (a, \infty)} \Psi_{(a,b)}. \]  \hspace{1cm} (4)

**Definition 1.1.** The Grand Lebesgue Space (GLS) \( G\psi = G_{\psi_{(a,b)}} \) consists of all the numerical valued (complex, in general case) measurable functions \( \{ h \}, \) \( h : \mathbb{R}^d \rightarrow \mathbb{R} \) having a finite norm

\[ ||h||_{G\psi} \overset{\text{def}}{=} \sup_{p \in \text{Dom}[\psi]} \left\{ \frac{||h||_p}{\psi(p)} \right\}. \]  \hspace{1cm} (5)
By definition, \( C/\infty := 0 \).

The function \( \psi = \psi(p) \) is named ordinary generating function for this Grand Lebesgue Space \( G\psi \).

These GLS spaces are rearrangement-invariant Banach functional spaces in the classical sense and were investigated in particular in many works, see e.g. [1], [2], [3], [4], [7], [8], [9], [10], [11], [12], [13], [14] - [17], [20], [21] etc.

They were applied in particular in the theory of probability, especially in the theory of random processes and fields; in the functional analysis - operators theory, theory of partial differential equations (PDE) etc.

The belonging of the function \( f = f(x), x \in \mathbb{R}^d \) is closely related with its tail behavior, where a tail function \( T_f(u), u \geq 1 \) for the (measurable) function \( f \) is defined as ordinary

\[
T_f(u) = \text{mes} \{ x, x \in \mathbb{R}^d, |f(x)| \geq u \}, \quad u \geq 1,
\]

and with finiteness of its norm in appropriate Orlicz - Luxemburg space. A most popular class of these spaces:

\[
\psi_s(p) := p^{1/s}, \quad s = \text{const} > 0; \quad p \in [1, \infty).
\]

It is known for instance that

\[
f \in G\psi_s \iff \exists C = C(s) > 0, \quad T_f(u) \leq \exp \left( -C(s)u^s \right), \quad u \geq 1. \tag{6}
\]

The value \( s = 2 \) correspondent to the famous subgaussian case.

Another example (degenerate generating function). Define for some constant \( r \geq 1 \)

\[
\psi_r(p) = 1, \quad p = r; \quad \psi_r(p) = +\infty \tag{7}
\]

otherwise. The \( G\psi_r \) norm of the function \( f : \mathbb{R} \to \mathbb{R} \) coincides with the classical Lebesgue - Riesz one:

\[
||f||_{\psi_r} = ||f||_r.
\]

Recall also that the so-called fundamental function \( \phi[G\psi](\delta) \delta \geq 0 \) for these spaces is defined as follows

\[
\phi[G\psi](\delta) \overset{def}{=} \sup_{p \in \text{Dom}(f)} \left\{ \frac{\delta^{1/p}}{\psi(p)} \right\}. \tag{8}
\]

It is investigated in particular in [18].
2 Main result: Lebesgue - Riesz spaces.

Let $p$ be certain fixed number from the interval $[1, \infty)$. Introduce the following important variable

$$K(d, m, p) \overset{def}{=} \sup_{\sum_{j=1}^{m} \|f_j\| \in (0, \infty)} \left\lbrace \frac{\|\Box_{j=1}^{m} f_j\|_{p}}{\sum_{j=1}^{m} \|f_j\|_{p}} \right\rbrace. \quad (9)$$

We set ourselves a goal to calculate the exact value of these important for us variable.

**Theorem 2.1.**

$$K(d, m, p) = m^{d/p}, \ m = 1, 2, \ldots \quad (10)$$

**Proof.** Upper estimate.

Introduce the well known dilation operator

$$T\lambda[f](x) \overset{def}{=} f(\lambda x), \ \lambda \in (0, \infty), \ f \in L_p(R^d), \ x \in R^d.$$ 

One has

$$\|T\lambda[f]\|_{p} = \lambda^{-d/p} \|f\|_{p}, \ f \in L_p(R^d). \quad (11)$$

Further, one can assume without loss of generality that $f_j(x) \geq 0$. Denote as above

$$g_m(x) := \left[ \Box_{j=1}^{m} f_j \right](x).$$

Evidently,

$$g_m(x) \leq \sum_{j=1}^{m} f_j \left( \frac{x}{m} \right) = \sum_{j=1}^{m} T_{1/m}[f_j](x).$$

We deduce by virtue of triangle inequality

$$\|g_m\|_p \leq \sum_{j=1}^{m} \|T_{1/m}[f_j]\|_p.$$ 

It remains to use the relation (11):

$$\|g_m\|_p \leq m^{d/p} \sum_{j=1}^{m} \|f_j\|_p. \quad (12)$$

**For example:** a Hilbert norm estimate, i.e. when $p = 2$:
\[ \|g_m\|_2 \leq m^{d/2} \sum_{j=1}^{m} \|f_j\|_2. \]

**Lower estimate.**

It is easily to verify that the equality in (12) is attained if for example \( f_j(x) = G(x) \), where \( G(x), x \in \mathbb{R}^d \) is famous Gaussian density function

\[ G(x) = \exp \left(-\|x\|^2\right), \quad \|x\|^2 = \sum_{k=1}^{d} x_k^2. \]

In detail, we find solving the following extremal problem

\[ \sum_{k=1}^{m} G(y_k) \to \min \quad \sum_{k=1}^{m} y_k = x \]

that \( y_k = x/m \). Therefore

\[ \square_{k=1}^{m} G(x) = m G(x/m), \quad \| \square_{k=1}^{m} G(\cdot) \|_p = K(d, m, p) \sum_{k=1}^{m} \|G(\cdot)\|_p. \]

This completes the proof of theorem 2.1.

### 3 Main result: Grand Lebesgue Space approach.

Let now the function \( \psi = \psi_{a,b}(p) \) be certain function from the set \( \Psi_{(a,b)}, 1 \leq a < b \leq \infty \), see(4). We suppose that it may be represented as follows

\[ \psi(p) = \frac{\nu(p)}{\zeta(p)}, \quad p \in (a, b) \]  
\[ (13) \]

for appropriate such a functions \( \nu(\cdot), \zeta(\cdot) \) belonging at the same set \( \Psi_{(a,b)}. \) For instance, \( \nu(p) = \psi(p), \zeta(p) = 1. \)

Let once again the function \( g_m(x), x \in \mathbb{R}^d \) be defined in (2) or equally in (3).

**Theorem 4.1.** Suppose \( \forall j = 1,2,...,m \) \( f_j \in G\psi. \) Then the function \( g_m \) belongs to the Grand Lebesgue Space \( G\nu \) and herewith

\[ \|g_m\|_{G\nu} \leq \phi_{G\zeta}(m^d) \cdot \sum_{j=1}^{m} \|f_j\|_{G\psi}, \]  
\[ (14) \]

where (we recall) \( \phi_{G\zeta}(\delta) \) is the fundamental function of the Grand Lebesgue space \( G\zeta, \) see (8).

Furthermore, the last estimate (14) is essentially unimprovable, see an example further.
Proof. Let $p$ be an arbitrary number from the segment $(a,b)$. We get from the direct definition of the norm in the GLS:

$$||f_j||_p \leq ||f_j||_{G\psi} \cdot \psi(p) = ||f_j||_{G\psi} \cdot \frac{\nu(p)}{\xi(p)}.$$  

One can apply the estimation (12):

$$||g_m||_p \leq m^{d/p} \cdot \frac{\nu(p)}{\xi(p)} \cdot \sum_{j=1}^{m} ||f_j||_{G\psi}, \quad (15)$$

following

$$\frac{||g_m||_p}{\nu(p)} \leq \frac{m^{d/p}}{\xi(p)} \cdot \sum_{j=1}^{m} ||f_j||_{G\psi}, \quad p \in (a,b). \quad (16)$$

It remains to take the $\sup$ over $p \in (a,b)$:

$$||g_m||_{G\nu} \leq \phi_{G\xi}(m^d) \cdot \sum_{j=1}^{m} ||f_j||_{G\psi}, \quad (17)$$

Q.E.D.

An example: choose $a = 1, b \in (a, \infty), \nu(p) = \psi(p)$, hence $\xi(p) = 1$ and following $\phi_{G\xi}(m^d) = m^d$. We deduce

$$||g_m||_{G\psi} \leq m^d \cdot \sum_{j=1}^{m} ||f_j||_{G\psi}.$$

Note in conclusion that the last estimate is essentially non - improvable, for instance, when $\psi(p) = \psi_{(r)}(p), \ r \in (1,b)$.

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