Confinement of matter-wave solitons on top of a pedestal-shaped potential

K. K. Ismailov¹, B. B. Baizakov¹ and F. Kh. Abdullaev¹,²

¹ Physical-Technical Institute, Uzbek Academy of Sciences, 100084, Tashkent, Uzbekistan
² Department of Theoretical Physics, National University of Uzbekistan, Tashkent 100174, Uzbekistan

Reflection of wave packets from downward potential steps and attractive potentials, known as a quantum reflection, has been explored for bright matter-wave solitons with the main emphasis on the possibility to trap them on top of a pedestal-shaped potential. In numerical simulations we observed that moving solitons return from the borders of the potential and remain trapped for sufficiently long time. The shuttle motion of the soliton is accompanied by shedding some amount of matter at each reflection from the borders of the trap, thus reducing its norm. The one- and two-soliton configurations are considered. A discontinuous jump of trajectories of colliding solitons has been discussed. The time-shift observed in a step-like decay of the moving soliton’s norm in the two-soliton configuration is linked to the trajectory jump phenomenon. The obtained results can be of interest for the design of new soliton experiments with Bose-Einstein condensates.

Introduction. Matter-wave solitons are macroscopic objects with particle-like properties, which can exhibit non-classical behavior when they interact at low velocity with downward potential steps and attractive potential wells. A classical particle would certainly accelerate in these potentials, while quantum particle has a finite probability to be turned back. Reflection from such potentials occurs without reaching the classical turning point. In this sense, quantum reflection is understood as a classically forbidden reflection.

A counter-intuitive phenomenon of quantum reflection of bright matter-wave solitons from attractive potential wells was theoretically considered in Ref. [1] and later experimentally observed in ⁸⁵Rb Bose-Einstein condensate [2]. The subject was further elaborated in subsequent studies for single solitons [3, 4] and two-soliton bound states [5]. Related phenomenon for dark solitons was considered in [6].

Significant quantum reflection occurs when the potential abruptly changes over a spatial domain, much smaller than the width of the wave packet. In ultra-cold quantum gases the de Broglie wavelength of atoms can be considerably greater than the spatial region over which the potential notably changes, thus suitable conditions for quantum reflection of slowly moving atoms can arise [3, 4]. The confining property of quantum-reflection traps in the form of a potential plateau is an interesting subject [6]. The survival probability of linear wave packets on top of a two-dimensional quantum reaction trap was studied in [7]. In experiments, a pedestal-shaped trap for solitons can be created using matter-wave-guides combined with red-detuned laser beams.

Nonlinear localized waves in pedestal-traps can exhibit novel dynamical features, not observed with linear wave packets, in particular with regard to their interactions and collisions. In the early days of soliton research, the term “soliton” was reserved only for solutions of integrable nonlinear equations. Later the term started to be used also for localized solutions of nonintegrable equations possessing certain solitonic properties. True solitons collide elastically and pass through one another preserving their shape, amplitude, and velocity. One of the fundamental properties of true solitons is that the trajectories of colliding solitons undergo a discontinuous jump, which has been theoretically predicted [11] and experimentally observed in optics [12] and Bose-Einstein condensates [13]. Below we show that soliton collisions on top of a pedestal-shaped potential can be a good platform for investigation of the aforementioned “trajectory jump” phenomenon, stemming from the nonlinear interaction of colliding wave packets. A relevant study for a soliton performing shuttle motion in a double-well potential structure with an elevated floor and emitting linear waves was reported in [14].

Our main objective in this work is to find out how long matter-wave bright solitons can be held on top of a pedestal - shaped potential while moving and experiencing repeated quantum reflections from its borders. We also show that by comparing the step-like decay of the amount of matter remaining on top of a pedestal-trap for one- and two-soliton configurations one can estimate the magnitude of the trajectory jump arising from soliton collisions. In the following sections we describe the model and present the results of numerical simulations of the governing Gross-Pitaevskii equation (GPE).

Model. In the experiments, the condensate is always held in some external magnetic or optical trapping potential. If the confinement in two of the spatial directions is much stronger than in the third direction, the condensate acquires a highly elongated cigar shaped form. The axial dynamics of the effectively one dimensional condensate is described by the reduced GPE, represented in normalized units as follows

\[ i\dot{\psi} + \frac{1}{2} \psi_{xx} - V(x)\psi + |\psi|^2\psi = 0, \] (1)

where \( \psi(x,t) \) is the mean field wave function of the condensate and \( V(x) \) is a plateau - shaped external potential

\[ V(x) = \frac{V_0}{2} \left[ \text{th} \left( \frac{x + h}{w} \right) - \text{th} \left( \frac{x - h}{w} \right) - 2 \right], \] (2)

where \( V_0 \) is the drop in the potential, which takes place over distance \( w \).
The dimensionless variables in Eq. (1) are related to corresponding variables in physical units as follows $t \to \sqrt{\omega_0 t}$, $x \to x/a_0$, $V(x) \to V(x)/\hbar \omega_0$, $\psi \to a_0^{1/2} \psi$, with $\omega_0$ being the frequency of radial confinement, $a_0 = \sqrt{\hbar/m \omega_0}$ is the harmonic oscillator length, $m$ is the atomic mass.

The strength (drop) of the potential $V_0$, the length of the plateau $2\hbar$ and width of the border region $W$ can be varied in numerical simulations to find out suitable conditions for quantum reflection. In absence of the external potential Eq. (1) has a fundamental one soliton solution, which will be used as initial condition for Eq. (1)

$$\psi(x,0) = \text{Sech}(x - x_0) e^{i\alpha(x-x_0)}, \quad (3)$$

where $x_0$ is the initial position of the soliton and $v$ is the initial velocity.

**Numerical results for one-soliton configuration.** To verify if a matter-wave soliton can be trapped for sufficiently long time on top of a pedestal-shaped potential, we perform numerical simulation of the GPE (1). As initial condition we use the fundamental soliton (3), placed at $x = x_0 = -h/2$, and set in motion with some velocity. The result is shown in Fig. 1. When the soliton is set in motion with sufficiently small velocity $v = 0.1$, it remains trapped on top of the potential, experiencing repeated quantum reflections form its borders at $x = \pm h$.

Each time the soliton is reflected from the trap’s border some amount of matter is transmitted to the adjacent region outside of the plateau in the form of linear waves, where they can freely propagate and eventually be absorbed in the domain boundaries. As a result of this emission a stepwise reduction of the number of atoms in the plateau-region $N(t) = \int_{-h}^{h} |\psi(x,t)|^2 dx$ takes place, as illustrated in Fig. 2a. The absolute velocity of the soliton performing shuttle motion on top of the pedestal-trap is not affected by the reflections, while its norm diminishes.

From numerical simulations with different parameters of the potential $V_0$, $w$ and initial the velocity $v$ we observe that the trapping time strongly depends on the values of these parameters. Namely, when the drop in the potential is large ($V_0 \gg 1$) and sharp ($w \ll 1$), the quantum reflection is significant. This means longer time for the soliton to remain trapped on top of the potential.

The stepwise time-dependence of the soliton’s norm, shown in Fig. 2a, can be used to estimate the coefficient of quantum reflection

$$R = N_{i+1}/N_i, \quad (4)$$

where $N_i, N_{i+1}$ are the norms (the amount of matter in the plateau region) of the soliton before and after the reflection correspondingly, $i$ is the step number in Fig. 2a starting from the top. It turns out that the ratio $N_{i+1}/N_i$ is almost the same for all $i = 1, 2, 3, ...$, meaning that only the small-amplitude tail of the soliton interacts with the trap borders, thus nonlinear effects are insignificant.

For qualitative interpretation of obtained results we recall that in the plane wave approximation, the reflection coefficient from a negative potential step $V(x) = -(V_0/2)(1 + \text{th}(x/w))$ is given by [13]

$$R_p = \left( \frac{\text{Sinh} \left[ \frac{2}{k_2 - k_1} w \right]}{\text{Sinh} \left[ \frac{2}{k_2 + k_1} w \right]} \right)^2, \quad (5)$$

where $k_1 = v$, $k_2 = \sqrt{v^2 + 2V_0}$. From Eq. (5) we conclude that at fixed parameters of the potential $V_0$ and $w$, the maximum reflection $R_p \to 1$ is observed for slowly moving soliton $k_1 \to 0$, $k_2 \to \sqrt{2V_0}$, because in this case the numerator and denominator become equal. At fixed velocity ($v$) and sharp potential drop ($w \ll 1$), the coefficient of reflection is given by $R_p \to (k_2 - k_1)^2/(k_2 + k_1)^2$ [12].

It is relevant to mention, that although the Eq. (4) is derived for plane waves using time-independent Schrödinger equation, our numerical simulations confirm its validity for moving nonlinear localized wave packets,
described by GPE (1). Since only the low-intensity tail of the soliton interacts with the trap’s borders, the plane wave formula (5) for the reflection coefficient appears to be a good approximation. In Fig. 2, we compare the prediction of Eq. (5) and the corresponding results from GPE according to Eq. (4). From the obtained results we conclude that the matter-wave soliton can be trapped towards the central quiescent one at \( \Delta x = 0 \) with a small velocity. The initial condition for this setup is

\[
\psi(x, 0) = \text{Sech}(x - x_0) e^{i\lambda(x-x_0)} + \text{Sech}(x).
\]

The two-soliton configuration is interesting because of the intriguing phenomenon of trajectory jump, associated with soliton collisions. The moving soliton experiences a discontinuous forward shift of its trajectory after the collision with the quiescent central soliton [17]

\[
\Delta x = \frac{1}{2A} \ln \left| \frac{\lambda_1 - \bar{\lambda}_2}{\lambda_1 - \bar{\lambda}_2} \right|,
\]

where \( A \) is the soliton amplitude. The complex parameter \( \lambda_1 \) is linked to the moving soliton velocity and amplitude as follows: \( \text{Re} \lambda_1 = -v/4, \text{Im} \lambda_1 = A/2 \). For the quiescent soliton \( \text{Re} \lambda_2 = 0, \text{Im} \lambda_2 = A/2 \). The estimate (7) for the parameter values \( A = 1, v = 0.1 \) yields \( \Delta x \approx 5.2 \), in good agreement with the GPE simulations shown in Fig. 3b. The central quiescent soliton jumps in the backward direction for the same distance and then remains immobile. Due to this phenomenon, the moving soliton reaches the border of the trap for less time as compared to the previously considered single soliton case. A similar “time advance effect” was noticed also in the transmission of solitons and two-soliton bound states through reflectionless potentials [3].

Figure 4 illustrates how the trajectory jump of solitons shows up in numerical simulations. The effect accumulates as the moving soliton repeatedly passes through the quiescent central soliton. Meantime, running into the borders of the pedestal-trap is accompanied by simultaneous emission of small amplitude coherent matter-waves by the moving soliton. Since the trajectory shift is linked to the nonlinear interactions of solitons, its magnitude decreases as the moving soliton’s norm diminishes due to the emission of linear waves at each reflection from the trap borders (see Fig. 4a). By comparing the step-like decay of the soliton’s norm for the one- and two-solitons configurations, one can estimate the magnitude of the trajectory jump phenomenon. In particular, by compar-

FIG. 3: a) Schematic showing two solitons (red solid line) positioned on top of a pedestal-shaped potential \( V(x) \), given by Eq. (2) for \( V_0 = 10, h = 20, w = 0.1 \) (blue dashed line). For visual convenience we plot \( V(x)/V_0 \). b) The density plot \( |\psi(x, t)|^2 \) corresponds to numerical solution of the GPE (1) with initial condition Eq. (6) and parameter values \( x_0 = -h/2, v = 0.1 \). The moving soliton experiences a discontinuous forward shift of the center-of-mass position, while the quiescent central soliton shifts in backward direction. The dashed lines show the soliton trajectories in the absence of interaction.

FIG. 4: a) The density plot showing the repeated passage of the moving soliton through the quiescent central soliton. The trajectory shifts become smaller as the moving soliton’s norm decays. The dashed line shows the initial position of the central soliton. The logarithmic scale is used to make the emission of linear waves by the moving soliton at the trap borders more visible. b) The magnitude of the trajectory shifts can be inferred from the stepwise decay of the moving soliton’s norm in a two-soliton configuration relative to the one-soliton configuration. The parameter values are the same as in previous figure.

An important question to be asked is whether the moving soliton preserves its true soliton nature after repeated collisions with the quiescent soliton. To clarify this issue we compare the shapes of the moving soliton at an ini-
tial time and after six collisions with the central soliton. As can be seen from Fig. 5 the moving soliton indeed preserves its original Sech form. This gives the evidence that collisions were elastic inherent to integrable systems. In the numerical simulations, the size of the integration domain was sufficiently large ($L = 24\pi$) and the velocity of the soliton was relatively small ($v \sim 0.1$). These were necessary conditions for the soliton to regain its Sech form after the reflection from the trap boundaries and before the collision with the central soliton.

![Fig. 5: a) The shapes of the solitons at initial time ($t = 0$) and after six collisions ($t = 1772$). The central quiescent soliton preserves its initial shape and position, while the moving soliton loses its norm due to emission of linear waves at the trap boundaries. b) Inspection of the moving soliton’s shape at $t = 1772$ (blue dashed line) shows that it retains the form $A\text{Sech}[A(x - x_0)]$ (red solid line) with $A = 0.52$, $x_0 = -10$.](image)

**Conclusions.** We have demonstrated that moving matter-wave solitons can be trapped on top of a pedestal-shaped potential for a sufficiently long time. The mechanism behind this effect is the quantum reflection, which occurs when a wave packet encounters with downward potential steps or attractive potentials, rapidly varying on a length scale much smaller than the width of the wave packet. The intensity of linear waves, emitted to outside regions of the trap potential, depends on the value of potential drop and sharpness of the border. In the two-soliton configuration, we observed the well-known phenomenon of trajectory jump in soliton collisions. As a consequence of the last phenomenon, a time shift in the stepwise decay of the soliton norm due to its quantum reflections on the trap boundaries has been revealed. The phenomena considered in this work can be observed in experiments with Bose-Einstein condensates, where a high level of control over matter-wave solitons has been achieved $[2, 10]$. Apart from basic scientific interest, the results may have applications in the fields involving coherent matter-waves.

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