A FRACTAL OPERATOR ASSOCIATED TO BIVARIATE FRACTAL INTERPOLATION FUNCTIONS

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Abstract. A general framework to construct fractal interpolation surfaces (FISs) on rectangular grids was presented and bilinear FIS was deduced by Ruan and Xu [Bull. Aust. Math. Soc. 91(3), 2015, pp. 435-446]. From the viewpoint of operator theory and the standpoint of developing some approximation aspects, we revisit the aforementioned construction to obtain a fractal analogue of a prescribed continuous function defined on a rectangular region in $\mathbb{R}^2$. This approach leads to a bounded linear operator analogous to the so-called $\alpha$-fractal operator associated with the univariate fractal interpolation function. Several elementary properties of this bivariate fractal operator are reported. We extend the fractal operator to the $L^p$-spaces for $1 \leq p < \infty$. Some approximation aspects of the bivariate continuous fractal functions are also discussed.

1. Introduction and Preliminaries

To fulfill a preparatory role, we shall take a cursory look at the requisite basic concepts in the theory of fractal interpolation and approximation. Our discussion will be interspersed with an, albeit incomplete, list of references.

Assume that $N > 2$ and $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \ldots, N\}$ is such that $x_0 < x_2 < \cdots < x_N$. Set $I = [x_0, x_N]$ and for $i = 1, 2, \ldots, N$, let $L_i : I \to [x_{i-1}, x_i]$ be a contractive homeomorphism such that

\[ L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i. \]

Let $F_i : I \times \mathbb{R} \to \mathbb{R}$ be a continuous map which satisfies the following

1. $F_i(x_0, y_0) = y_{i-1}$ and $F_i(x_N, y_N) = y_i$,
2. $|F_i(x, y) - F_i(x, y')| \leq r_i |y - y'|$ for all $x \in I$ and $y, y' \in \mathbb{R}$, where $r_i \in [0, 1)$.

Define $W_i(x, y) = (L_i(x), F_i(x, y))$ for $(x, y) \in I \times \mathbb{R}$. Then, $\{I \times \mathbb{R}; W_1, W_2, \ldots, W_N\}$ is an iterated function system, IFS for short [13]. Barnsley [2] proved that its unique invariant set $G$ is the graph of a continuous function $g : I \to \mathbb{R}$, referred to as a fractal interpolation function (FIF) such that for every $i = 0, 1, \ldots, N$ and $x \in I$, we have $g(x_i) = y_i$ and $g$ satisfies a self-referential equation

\[ g(L_i(x)) = F_i(x, g(x)), \quad i = 1, 2, \ldots, N. \]

As a new class of interpolants, FIFs demonstrated more advantages than the classical interpolants in fitting and approximation of naturally occurring functions that

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display some kind of self-similarity. Consequently, numerous papers on this subject have been published so far and we refer the reader to [3, 6, 16, 23] for some discussions, recent results and further references.

Most widely studied FIFs are obtained by considering each $L_i$ and $F_i$ in the following form.

$$L_i(x) = a_i x + b_i, \quad F_i(x, y) = \alpha_i y + q_i(x),$$

where $q_i : I \to \mathbb{R}$ is suitable continuous map and $\alpha_i \in \mathbb{R}$ is such that $|\alpha_i| < 1$. Let $f : I \to \mathbb{R}$ be a prescribed continuous function. By taking $q_i(x) = f(L_i(x)) - \alpha_i b(x)$, where $b$ is an appropriate continuous map, Navascués [18] observed that fractal analogue of a given continuous function $f$ can be constructed. The FIF associated with such an IFS is called the $\alpha$-fractal function for $f$ and is denoted by $f^{\alpha}_{\Delta, b}$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in (-1, 1)^N$ is referred to as the scale vector. Such an approach leads to an operator

$$F^{\alpha}_{\Delta, b} : f \mapsto f^{\alpha}_{\Delta, b},$$

which depends on a chosen partition $\Delta$, function $b$ and constants $\alpha_i$, and which sends each $f$ to its fractal perturbation $f^{\alpha}_{\Delta, b}$. If $b = Lf$, where $L$ is a bounded linear operator, then the operator $F^{\alpha}_{\Delta, L}$ is a bounded linear operator termed the fractal operator. This operator formulation of fractal functions somewhat hidden in the construction of FIFs enables them to interact with other traditional branches of mathematics including operator theory, complex analysis, harmonic analysis and approximation theory [18, 19, 20, 21, 22, 26]. More recently, the second author and collaborators identified suitable values of the parameters so that the $\alpha$-fractal function $f^{\alpha}_{\Delta, b}$ preserves the shape properties inherent in the source function $f$ [27]. This lead to the intersection of the two different traditions - theory of FIF and univariate constrained approximation - that were otherwise developing independently.

In the attempt to different extensions of FIF, it is natural to seek FIFs in higher-dimensional cases, in particular, the two-dimensional case aiming at a realistic modeling of rough surfaces. While it is straightforward to define a similar IFS as that in the one-dimensional case, it is hard to ensure that the invariant set of such IFS is the graph of a continuous function. There are various approaches for the construction of fractal interpolation surface (FIS) (an, albeit incomplete list of references [5, 7, 11, 17, 24]), each with their particular strengths and weaknesses. More recently, we have developed a few constructive approaches for solving constrained interpolation by fractal surfaces [8, 9]. However, a unified approach for developing fractal versions of various traditional methods of constrained bivariate interpolation is strongly felt.

To summarize, the approach to the univariate FIF wherein a source function is perturbed to obtain its fractal analogue and the fractal operator emerged thereby (i) enabled FIF to interact with other branches of mathematics (ii) played a key role in developing fractal versions of fundamental theorems in constrained approximation (iii) provided a unified approach to various shape preserving fractal interpolation schemes (iv) reinforced the ubiquity of fractal functions as claimed by the fractal researchers. On the other hand, to the best of our knowledge, there is no research reported in the direction of bivariate analogue of the $\alpha$-fractal operator arising from FISs. The reason perhaps is that a general framework for the construction of FISs was missing until the researches reported in [24]. Motivated by these facts, in
A FRAC TAL OPERATOR ASSOCIATED TO BIVARIATE FRAC TAL INTERPOLATION FUNCTIONS

broad terms, the object of the current paper is to provide foundational aspects of a bivariate version of the \( \alpha \)-fractal function and fractal operator. To be precise, the current paper is a realization that the general framework to generate FIS given in [24] can be used to obtain a fractal analogue of a continuous real-valued function defined on a rectangular region in \( \mathbb{R}^2 \).

Let \( -\infty < x_0 < x_N < \infty, -\infty < y_0 < y_M < \infty, I = [x_0, x_N] \) and \( J = [y_0, y_M] \). As is customary, we denote by \( C(I \times J, \mathbb{F}) \), the Banach space of all continuous functions \( f : I \times J \rightarrow \mathbb{F} \) supplied with the uniform norm, where \( \mathbb{F} \) is the real field \( \mathbb{R} \) or the complex field \( \mathbb{C} \). In Section 3 we shall define and establish some elementary properties of the fractal operator \( F_{\alpha,L} : C(I \times J, \mathbb{R}) \rightarrow C(I \times J, \mathbb{R}) \) that maps a bivariate continuous function \( f(x, y) \) to its fractal analogue \( f_{\alpha,L}(x, y) \).

Using the usual density arguments, we extend our fractal operator to \( L^p(I \times J, \mathbb{C}) \) in Section 4. Some approximation aspects of the continuous bivariate \( \alpha \)-fractal functions and bivariate “fractal polynomials” are investigated in Section 5. Our approach reveals a natural kinship with the research works of Navascues in the field of univariate \( \alpha \)-fractal functions scattered in the literature; see, for instance, [19] [18] [21]. However, we feel that the new approach to bivariate fractal functions introduced herein with only the surface being scratched, finds potential applications in approximation problems. Let us emphasize that the bivariate version is proposed not merely as an extension of the univariate case; but with an eye towards extending fractal surfaces to the territory of constrained approximation. We envisage that the constrained approximation with fractal surfaces is a problem involving larger resources than their traditional counterparts.

2. Auxiliary Apparatus

In this section, we revisit a general framework to construct FISs on rectangular grids; for details the reader is referred to [24].

Let \( I = [x_0, x_N] \) and \( J = [y_0, y_M] \). Suppose that an interpolation data set \( \{(x_i, y_j, z_{ij}) \in \mathbb{R}^3 : i = 0, 1, \ldots, N; j = 0, 1, \ldots, M \} \) such that \( x_0 < x_1 < \cdots < x_N \) and \( y_0 < y_1 < \cdots < y_M \) is provided. Following the notation in [24], we write \( \Sigma_N = \{1, 2, \ldots, N\}, \Sigma_{N,0} = \{0, 1, \ldots, N\}, \partial \Sigma_{N,0} = \{0, N\} \) and \( \text{int} \Sigma_{N,0} = \{1, 2, \ldots, N-1\} \). Similarly, we can define \( \Sigma_M, \Sigma_{M,0}, \partial \Sigma_{M,0} \) and \( \text{int} \Sigma_{M,0} \). Let \( I_i = [x_{i-1}, x_i] \) and \( J_j = [y_{j-1}, y_j] \) for \( i \in \Sigma_N \) and \( j \in \Sigma_M \). For any \( i \in \Sigma_N \), let \( u_i : I \rightarrow I_i \) be a contractive homeomorphism satisfying

\[
\begin{align*}
    u_i(x_0) &= x_{i-1}, \quad u_i(x_N) = x_i, \text{ if } i \text{ is odd}, \\
    u_i(x_0) &= x_i, \quad u_i(x_N) = x_{i-1}, \text{ if } i \text{ is even, and} \\
    |u_i(x_1) - u_i(x_2)| &\leq \alpha_i|x_1 - x_2|, \quad \forall x_1, x_2 \in I,
\end{align*}
\]

where \( 0 < \alpha_i < 1 \) is a given constant. Similarly, for any \( j \in \Sigma_M \), let \( v_j : J \rightarrow J_j \) be a contractive homeomorphism satisfying

\[
\begin{align*}
    v_j(y_0) &= y_{j-1}, \quad v_j(y_M) = y_j, \text{ if } j \text{ is odd}, \\
    v_j(y_0) &= y_j, \quad v_j(y_N) = y_{j-1}, \text{ if } j \text{ is even, and} \\
    |v_j(y_1) - v_j(y_2)| &\leq \beta_j|y_1 - y_2|, \quad \forall y_1, y_2 \in J,
\end{align*}
\]

where \( 0 < \beta_j < 1 \) is a given constant. By the definitions of \( u_i \) and \( v_j \), it is easy to check that

\[
u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i), \quad \forall i \in \text{int} \Sigma_{N,0},
\]
Let $K$ be a continuous function satisfying
\[ v_j^{-1}(y_j) = v_{j+1}^{-1}(y_j), \quad \forall j \in \text{int}\Sigma_M.0. \]
Let $\tau : \mathbb{Z} \times \{0, N, M\} \to \mathbb{Z}$ be defined by
\[ \tau(i, 0) = \begin{cases} i - 1, & \text{if } i \text{ is odd} \\ i, & \text{if } i \text{ is even.} \end{cases} \]
Let $K = I \times J \times \mathbb{R}$. For each $(i, j) \in \Sigma_N \times \Sigma_M$, let $F_{ij} : K \to \mathbb{R}$ be a continuous function satisfying
\[ F_{ij}(x, y, z) = z_{\tau(i,k),\tau(j,l)}, \quad \forall (k, l) \in \partial \Sigma_N \times \partial \Sigma_M, \text{ and} \]
\[ |F_{ij}(x, y, z') - F_{ij}(x, y, z'')| \leq \gamma_{ij}|z' - z''|, \quad \forall (x, y) \in I \times J, \text{ and } z', z'' \in \mathbb{R}, \]
where $0 < \gamma_{ij} < 1$ is a given constant.
Finally, for each $(i, j) \in \Sigma_N \times \Sigma_M$, we define $W_{ij} : K \to I_i \times J_j \times \mathbb{R}$ by
\[ W_{ij}(x, y, z) = (u_i(x), v_j(y), F_{ij}(x, y, z)). \]
Then $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ is an IFS.

**Theorem 2.1.** Let $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ be the IFS defined as above. Assume that the map $F_{ij}, (i, j) \in \Sigma_N \times \Sigma_M$ satisfies the following matching conditions
\[ \begin{align*} 
(1) \quad & \text{for all } i \in \text{int}\Sigma_N, j \in \Sigma_M \text{ and } x^* = u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i), \\
& F_{ij}(x^*, y, z) = F_{i+1,j}(x^*, y, z), \quad \forall y, z \in \mathbb{R}, \text{ and} \\
(2) \quad & \text{for all } i \in \Sigma_N, j \in \text{int}\Sigma_M, \text{ and } y^* = v_j^{-1}(y_j) = v_{j+1}^{-1}(y_j), \\
& F_{ij}(x, y^*, z) = F_{i,j+1}(x, y^*, z), \quad \forall x \in I_i, z \in \mathbb{R}. 
\end{align*} \]
Then there exists a unique continuous function $f : I \times J \to \mathbb{R}$ such that $f(x_i, y_j) = z_{ij}$ for all $(i, j) \in \Sigma_N \times \Sigma_M$, and $G = \bigcup_{(i, j)\in \Sigma_N \times \Sigma_M} W_{ij}(G)$, where $G = \{(x, y, f(x, y)) : (x, y) \in I \times J\}$ is the graph of $f$.

**Definition 2.2.** We call $G$ in the aforementioned theorem as the FIS and $f$ as the bivariate FIF with respect to the IFS $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ corresponding to the data set $\{(x_i, y_j, z_{ij}) \in \mathbb{R}^3 : i = 0, 1, \ldots, N; j = 0, 1, \ldots, M\}$.

**Remark 2.3.** Consider the set
\[ \mathcal{C}^*(I \times J, \mathbb{R}) := \left\{ g \in \mathcal{C}(I \times J, \mathbb{R}) : g(x_i, y_j) = z_{ij} \quad \forall (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0} \right\} \]
endowed with the supremum metric. Let us define an operator $T$, referred to as Read-Bajraktarević operator
\[ Tg(x, y) = F_{ij}\left( u_i^{-1}(x), v_j^{-1}(y), g(u_i^{-1}(x), v_j^{-1}(y)) \right), \quad (x, y) \in I_i \times J_j, \quad (i, j) \in \Sigma_N \times \Sigma_M. \]
The bivariate FIF $f$ in the previous definition is the unique fixed point of $T$. Consequently, $f$ satisfies the self-referential equation:
\[ f(x, y) = F_{ij}\left( u_i^{-1}(x), v_j^{-1}(y), f(u_i^{-1}(x), v_j^{-1}(y)) \right), \quad (x, y) \in I_i \times J_j, \quad (i, j) \in \Sigma_N \times \Sigma_M. \]
Under some assumptions on the elements in the IFS involved, the box counting dimension of the graph of the resulting bilinear FIS is studied in [14], which we shall recall here. Let $M = N$. Let $g$ be the bilinear function on $I \times J$ satisfying $g(x_i, y_j) = z_{ij}$ for all $(i, j) \in \partial \Sigma_{N,0} \times \partial \Sigma_{N,0}$. That is

$$
g(x, y) = \frac{1}{(x_N - x_0)(y_N - y_0)} \left[ (x_N - x)(y_N - y)z_0 + (x - x_0)(y_N - y)z_N 
+ (x_N - x)(y - y_0)z_0N + (x - x_0)(y - y_0)z_N \right].$$

Let $h : I \times J \to \mathbb{R}$ be a function satisfying $h(x_i, y_j) = z_{ij}$ for all $(i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}$ such that $h$ when restricted to $D_{ij}$ is bilinear for all $(i, j) \in \Sigma_N \times \Sigma_N$. Assume that $\{s_{ij} : (i, j) \in \Sigma_{N,0} \times \Sigma_{N,0}\}$ is a given data set with $|s_{ij}| < 1$ for all $i, j$. We define $S : I \times J \to \mathbb{R}$ such that $S(x_i, y_j) = s_{ij}$ for all $(i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}$ and that $S$ restricted to $D_{ij}$ is bilinear for all $(i, j) \in \Sigma_N \times \Sigma_M$. Define

$$F_{ij}(x, y, z) = S(u_i(x), v_j(y))(z - g(x, y)) + h(u_i(x), v_j(y)).$$

Assume that the scaling factors are steady, that is, for each $(i, j) \in \Sigma_N \times \Sigma_N$ all of $s_{i-1,j-1}$, $s_{i,j-1}$, $s_{i-1,j}$ and $s_{ij}$ are nonnegative or all of them are nonpositive. Let

$$\sum_{i,j \in \Sigma_N} |S(u_i(x_0), v_j(y_0))| = \sum_{i,j \in \Sigma_N} |S(u_i(x_0), v_j(y_0))| = \sum_{i,j \in \Sigma_N} |S(u_i(x_N), v_j(y_0))| =: \gamma$$

**Theorem 2.4.** [14] Let $f$ be the bilinear FIF determined with aforementioned assumptions on the IFS. If $\gamma > N$ and the interpolation points $\{(x_i, y_j, z_{ij})\}$ are not co-bilinear, then $\dim_B(\text{Graph}(f)) = 1 + \frac{\log \gamma}{\log N}$. Otherwise, $\dim_B(\text{Graph}(f)) = 2$.

### 3. Associated Fractal Linear Operator on $C(I \times J, \mathbb{R})$

Let $I = [x_0, x_N]$ and $J = [y_0, y_M]$. Let $f : I \times J \to \mathbb{R}$ be a given continuous function. Define a net $\Delta$ by

$$x_0 < x_1 < \cdots < x_N; \quad y_0 < y_1 < \cdots < y_M.$$ 

Let $L : C(I \times J, \mathbb{R}) \to C(I \times J, \mathbb{R})$ be a bounded linear operator satisfying $Lf \neq f$,

$$(Lf)(x_i, y_j) = f(x_i, y_j), \quad \forall (i, j) \in \partial \Sigma_{N,0} \times \partial \Sigma_{M,0}.$$ 

Let $\alpha : I \times J \to \mathbb{R}$ be a continuous function such that

$$||\alpha||_\infty := \sup \{|\alpha(x, y)| : (x, y) \in I \times J\} < 1.$$ 

Set $K = I \times J \times \mathbb{R}$ and define $F_{ij} : K \to \mathbb{R}$ by

$$F_{ij}(x, y, z) = \alpha(u_i(x), v_j(y))z + f(u_i(x), v_j(y)) - \alpha(u_i(x), v_j(y))(Lf)(x, y),$$

where $u_i \in C(I, \mathbb{R})$ and $v_j \in C(J, \mathbb{R})$ satisfy conditions prescribed in the previous section. In the sequel, we define $u_i$ and $v_j$ to be linear functions satisfying the required conditions, say

$$u_i(x) = a_i x + b_i, \quad v_j(y) = c_j y + d_j,$$
where constants involved are suitably determined. For each \((i, j) \in \Sigma_N \times \Sigma_M\), we define
\[
W_{ij} : K \to I_i \times J_j \times \mathbb{R}
\]
\[
W_{ij}(x, y, z) = (u_i(x), v_j(y), F_{ij}(x, y, z)).
\]  
(3.3)

Let us mention two examples for such an operator \(L : \mathcal{C}(I \times J, \mathbb{R}) \to \mathcal{C}(I \times J, \mathbb{R})\).

1. \((Lf)\)\((x, y) = f(x, y)\)\((t(x, y))\), where \(t \in \mathcal{C}(I \times J, \mathbb{R})\) is a fixed non-constant function such that \(t(x_i, y_j) = 1\), \(\forall (i, j) \in \partial \Sigma_{N,0} \times \partial \Sigma_{M,0}\). On calculating the operator norm of \(L\), we obtain \(\|L\| = \|t\|_\infty\).

2. \((Lf)\)\((x, y) = (f \circ t)(x, y)\), where \(t \in \mathcal{C}(I \times J, I \times J)\) is a fixed map \(t \neq Id\), the identity map and \(t(x_i, y_j) = (x_i, y_j)\), \(\forall (i, j) \in \partial \Sigma_{N,0} \times \partial \Sigma_{M,0}\). In this case, we get \(\|L\| = 1\).

It is straightforward to see that \(F_{ij}\) satisfies the matching conditions required in Theorem 2.4 and therefore we have the following.

**Theorem 3.1.** Let \(\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}\) be the IFS defined through (3.1)-(3.3) above. Then there exists a unique continuous function \(f^\alpha_{\Delta, L} : I \times J \to \mathbb{R}\) such that \(f^\alpha_{\Delta, L}(x, y) = f(x, y)\) for all \((i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\) and \(G = \bigcup_{(i,j) \in \Sigma_N \times \Sigma_M} W_{ij}(G)\), where \(G = \{(x, y, f^\alpha_{\Delta, L}(x, y)) : (x, y) \in I \times J\}\) is the graph of \(f^\alpha_{\Delta, L}\).

**Remark 3.2.** Being the fixed point of the RB-operator (Cf. Remark 2.3), \(f^\alpha_{\Delta, L}\) satisfies the self-referential equation:

\[
f^\alpha_{\Delta, L}(x, y) = F_{ij}\left(u_i^{-1}(x), v_j^{-1}(y), f^\alpha_{\Delta, L}(u_i^{-1}(x), v_j^{-1}(y))\right), \ \forall (x, y) \in I_i \times J_j.
\]

That is, for all \((x, y) \in I_i \times J_j\), where \((i, j) \in \Sigma_N \times \Sigma_M\) we have

\[
f^\alpha_{\Delta, L}(x, y) = f(x, y) + \alpha(x, y)f^\alpha_{\Delta, L}(u_i^{-1}(x), v_j^{-1}(y)) - \alpha(x, y)(Lf)(u_i^{-1}(x), v_j^{-1}(y)).
\]

The function \(f^\alpha_{\Delta, L}\) appeared in the previous remark is important enough to be dignified with a name of its own.

**Definition 3.3.** We call the aforementioned self-referential function \(f^\alpha_{\Delta, L}\) as (bi-variate) \(\alpha\)-fractal function, fractal perturbation or associate fractal function corresponding to \(f\) with respect to the operator \(L\) and the net \(\Delta\).

**Remark 3.4.** Theorem 2.4 establishes the box dimension of the graph of \(f^\alpha_{\Delta, L}\) for some special class of functions \(f\) and choice of parameters \(\alpha, \Delta\) and \(L\). Recently, the box counting dimension of the graph of a univariate \(\alpha\)-fractal function established in a more general setting in [1]. We believe that by modifying and adapting these results, the box dimension of the graph of \(f^\alpha_{\Delta, L}\) can be computed for a more general class and details will appear elsewhere.

**Definition 3.5.** For a fixed net \(\Delta\), a scale function \(\alpha\) and an operator \(L\), let us define the \(\alpha\)-fractal operator or simply fractal operator

\[
\mathcal{F}^\alpha_{\Delta, L} : \mathcal{C}(I \times J, \mathbb{R}) \to \mathcal{C}(I \times J, \mathbb{R}), \ \mathcal{F}^\alpha_{\Delta, L}(f) = f^\alpha_{\Delta, L}.
\]

Next let us recall a pair of lemmas and definitions that are fundamental in functional analysis; see, for instance, [4].

**Lemma 3.6.** If \(T\) is a bounded linear operator from Banach space into itself such that \(\|T\| < 1\), then \((Id - T)^{-1}\) exists and it is bounded.
Lemma 3.7. If $T$ is a bounded linear operator and $S$ is a compact operator on a normed linear space $X$, then $TS$ and $ST$ are compact operators.

Following [1], we shall use the product notation for the value of a linear functional on an element: $\langle x, f \rangle = \langle f, x \rangle = f(x)$ for $x$ in a normed linear space $X$ and $f$ in $X^*$, the dual of $X$. Let $X, Y$ be normed spaces and $T : X \to Y$ be a bounded linear operator. The adjoint or dual $T^*$ of $T$ is the unique map $T^* : Y^* \to X^*$ such that
$$\langle x, T^* g \rangle = \langle Tx, g \rangle, \quad \forall \ x \in X, \ g \in Y^*.$$ 

Definition 3.8. An operator $T$ is Fredholm if:

1. $\text{Range}(T)$ is closed.
2. $\ker(T)$ and $\ker(T^*)$ are finite-dimensional.

Further, the index of a Fredholm operator is defined as
$$\text{index}(T) = \dim(\ker(T)) - \dim(\ker(T^*))$$

Definition 3.9. Given a Banach space $X$, the annihilator of a subspace $L$ of $X^*$ in $X$ (or the pre annihilator of $L$) is
$$^aL = \{ x \in X : \langle x, f \rangle = 0 \ \forall \ f \in L \}.$$ 

The following theorem exhibits some elementary properties of the bivariate $\alpha$-fractal function and the corresponding fractal operator. This result is reminiscent of the univariate case scattered in the fractal literature; see, for instance, [19]. However, for the sake of completeness and record, we provide a fairly self-contained arguments.

Theorem 3.10. Let $\|\alpha\|_{\infty} = \sup \{ |\alpha(x, y)| : (x, y) \in I \times J \}$, and let $I d$ be the identity operator on $C(I \times J, \mathbb{R})$.

1. For any $f \in C(I \times J, \mathbb{R})$, the perturbation error satisfies
$$\| f_{\Delta, L}^\alpha - f \|_{\infty} \leq \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}} \| f - Lf \|_{\infty}.$$ 

In particular, if $\alpha = 0$, then $F_{\Delta, L}^\alpha = I d$.

2. The fractal operator $F_{\Delta, L}^\alpha$ is a bounded linear operator with respect to the uniform norm on $C(I \times J, \mathbb{R})$. Furthermore, the operator norm satisfies
$$\| F_{\Delta, L}^\alpha \| \leq 1 + \frac{\|\alpha\|_{\infty} \| I d - L \|}{1 - \|\alpha\|_{\infty}}.$$ 

3. For $\|\alpha\|_{\infty} < \| L \|^{-1}$, $F_{\Delta, L}^\alpha$ is bounded below. In particular, $F_{\Delta, L}^\alpha$ is one to one.

4. If $\|\alpha\|_{\infty} < (1 + \| I d - L \|)^{-1}$, then $F_{\Delta, L}^\alpha$ has a bounded inverse and consequently a topological automorphism (i.e., a bijective bounded linear map with a bounded inverse from $C(I \times J, \mathbb{R})$ to itself). Moreover,
$$\| (F_{\Delta, L}^\alpha)^{-1} \| \leq \frac{1 + \|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty} \| L \|}.$$ 

5. If $\|\alpha\|_{\infty} \neq 0$, then the fixed points of $L$ are the fixed points of the fractal operator $F_{\Delta, L}^\alpha$ as well.

6. If $1$ belongs to the point spectrum of $L$, then $1 \leq \| F_{\Delta, L}^\alpha \|.$

7. For $\|\alpha\|_{\infty} < \| L \|^{-1}$, the fractal operator $F_{\Delta, L}^\alpha$ is not a compact operator.
(8) If \( \| \alpha \|_\infty < (1 + \| Id - L \|)^{-1} \), then \( F_{\Delta, L}^\alpha \) is Fredholm and its index is 0.

**Proof.**

1. For \((x, y) \in I_i \times J_j\), from (3.4) we have
   \[ f_{\Delta, L}^\alpha(x, y) - f(x, y) = \alpha(x, y) \left[ f^\alpha(u_i^{-1}(x), v_j^{-1}(y)) - (L f)(u_i^{-1}(x), v_j^{-1}(y)) \right]. \]
   Therefore
   \[ |f_{\Delta, L}^\alpha(x, y) - f(x, y)| \leq \| \alpha \|_\infty \| f_{\Delta, L}^\alpha - L f \|_\infty. \]
   Since the above inequality is true for all \((x, y) \in I_i \times J_j\) where \((i, j) \in \Sigma_N \times \Sigma_M\), we conclude that
   \[ \| f_{\Delta, L}^\alpha - f \|_\infty \leq \| f_{\Delta, L}^\alpha - L f \|_\infty. \]

   Using the triangle inequality, we get
   \[ \| f_{\Delta, L}^\alpha - f \|_\infty \leq \| \alpha \|_\infty \left[ \| f_{\Delta, L}^\alpha - f \|_\infty + \| f - L f \|_\infty \right]. \]
   This demonstrates
   \[ \| f_{\Delta, L}^\alpha - f \|_\infty \leq \frac{\| \alpha \|_\infty \| Id - L \|}{1 - \| \alpha \|_\infty} \| f \|_\infty. \]

   For \( \| \alpha \|_\infty = 0 \), the previous inequality produces \( \| f_{\Delta, L}^\alpha - f \|_\infty = 0 \), and hence \( f_{\Delta, L}^\alpha = f \) for all \( f \in C(I \times J, \mathbb{R}) \). That is, \( F_{\Delta, L}^\alpha = Id \).

2. Let \( f, g \in C(I \times J, \mathbb{R}) \) and \( \beta, \gamma \in \mathbb{R} \). For \((x, y) \in I_i \times J_j\), we have
   \[ \beta f_{\Delta, L}^\alpha + \gamma g_{\Delta, L}^\alpha(x, y) = \beta f(x, y) + \beta \alpha(x, y) \left[ f_{\Delta, L}^\alpha(u_i^{-1}(x), v_j^{-1}(y)) - (L f)(u_i^{-1}(x), v_j^{-1}(y)) \right] + \gamma g(x, y) + \gamma \alpha(x, y) \left[ g_{\Delta, L}^\alpha(u_i^{-1}(x), v_j^{-1}(y)) - (L g)(u_i^{-1}(x), v_j^{-1}(y)) \right]. \]
   Adding the above two equations, one gets
   \[ (\beta f_{\Delta, L}^\alpha + \gamma g_{\Delta, L}^\alpha)(x, y) = (\beta f + \gamma g)(x, y) + \alpha(x, y). \]

   The previous equation reveals that \( \beta f_{\Delta, L}^\alpha + \gamma g_{\Delta, L}^\alpha \) is the fixed point of RB-operator
   \[ (Th)(x, y) = \alpha(x, y)h(u_i^{-1}(x), v_j^{-1}(y)) + (\beta f + \gamma g)(x, y) - \alpha(x, y)L(\beta f + \gamma g)(u_i^{-1}(x), v_j^{-1}(y)). \]

   Since the fixed point of the RB operator is unique, we have \((\beta f + \gamma g)^{\alpha}_{\Delta, L} = \beta f_{\Delta, L}^\alpha + \gamma g_{\Delta, L}^\alpha\), which reveals the linearity of the operator \( F_{\Delta, L}^\alpha \). From the previous item we write
   \[ \| f_{\Delta, L}^\alpha \|_\infty - \| f \|_\infty \leq \frac{\| \alpha \|_\infty \| Id - L \|}{1 - \| \alpha \|_\infty} \| f \|_\infty. \]
   It implies that
   \[ \| F_{\Delta, L}^\alpha(f) \|_\infty \leq \| f \|_\infty + \frac{\| \alpha \|_\infty \| Id - L \|}{1 - \| \alpha \|_\infty} \| f \|_\infty. \]
   Therefore \( F_{\Delta, L}^\alpha \) is a bounded linear operator.
Here $\alpha$ point of Remark 3.11

\[ \| f \|_{\infty} - \| f_{\Delta, L}^{\alpha} \|_{\infty} \leq \| f_{\Delta, L}^{\alpha} - f \|_{\infty} \leq \| \alpha \|_{\infty} \| f_{\Delta, L}^{\alpha} - Lf \|_{\infty} \]

\[ \leq \| \alpha \|_{\infty} (\| f_{\Delta, L}^{\alpha} \|_{\infty} + \| L \| \| f \|_{\infty}). \]

Hence we get 

\[ (1 - \| \alpha \|_{\infty} \| L \|) \| f \|_{\infty} \leq (1 + \| \alpha \|_{\infty}) \| f_{\Delta, L}^{\alpha} \|_{\infty}. \]

If $\| \alpha \|_{\infty} < \| L \|^{-1}$, then

\[ (3.6) \]

\[ \| f \|_{\infty} \leq \frac{1 + \| \alpha \|_{\infty}}{1 - \| \alpha \|_{\infty} \| L \|} \| f_{\Delta, L}^{\alpha} \|_{\infty}. \]

Thus $F_{\Delta, L}^{\alpha}$ is bounded below.

(4) From the hypothesis $\| Id - F_{\Delta, L}^{\alpha} \|_{\infty} \leq \frac{\| \alpha \|_{\infty} \| Id - L \|}{1 - \| \alpha \|_{\infty}} < 1$. Consequently, Lemma 3.3 dictates that $F_{\Delta, L}^{\alpha}$ has a bounded inverse. From (3.5), we infer that

\[ \| (F_{\Delta, L}^{\alpha})^{-1}(f) \|_{\infty} \leq \frac{1 + \| \alpha \|_{\infty}}{1 - \| \alpha \|_{\infty} \| L \|} \| f \|_{\infty}, \]

which in turn yields the required bound for the operator norm of $(F_{\Delta, L}^{\alpha})^{-1}$.

(5) Let $\| \alpha \|_{\infty} \neq 0$, and $f$ be a fixed point of $L$. From (3.5)

\[ \| f_{\Delta, L}^{\alpha} - f \|_{\infty} \leq \| \alpha \|_{\infty} \| f_{\Delta, L}^{\alpha} - f \|_{\infty}. \]

Since $\| \alpha \|_{\infty} < 1$, this implies that $f_{\Delta, L}^{\alpha} = f$.

(6) Choose $g \in C(I \times J, \mathbb{R})$ such that $Lg = g$ and $\| g \| = 1$. By the previous part of the theorem $F_{\Delta, L}^{\alpha}(g) = g$, and hence $\| F_{\Delta, L}^{\alpha}(g) \|_{\infty} = \| g \|_{\infty}$. The definition of the operator norm now yields $1 \leq \| F_{\Delta, L}^{\alpha} \|_{\infty}$.

(7) For $\| \alpha \|_{\infty} < \| L \|^{-1}$, we know that $F_{\Delta, L}^{\alpha} : C(I \times J, \mathbb{R}) \to C(I \times J, \mathbb{R})$ is one-one. Note that the range space of $F_{\Delta, L}^{\alpha}$ is infinite dimensional. We define the inverse map $(F_{\Delta, L}^{\alpha})^{-1} : F_{\Delta, L}^{\alpha}(C(I \times J, \mathbb{R})) \to C(I \times J, \mathbb{R})$. With this choice of $\alpha$, $F_{\Delta, L}^{\alpha}$ is bounded below, and hence it follows that $(F_{\Delta, L}^{\alpha})^{-1}$ is a bounded linear operator. Assume that $F_{\Delta, L}^{\alpha}$ is a compact operator. Then by Lemma 3.7, we deduce that the operator $T = F_{\Delta, L}^{\alpha}(F_{\Delta, L}^{\alpha})^{-1} : F_{\Delta, L}^{\alpha}(C(I \times J, \mathbb{R})) \to C(I \times J, \mathbb{R})$ is a compact operator, which is a contradiction to the infinite dimensionality of the space $F_{\Delta, L}^{\alpha}(C(I \times J, \mathbb{R}))$. Therefore, $F_{\Delta, L}^{\alpha}$ is not a compact operator.

(8) Under the hypothesis, range space of $F_{\Delta, L}^{\alpha}$ is closed. Furthermore, $F_{\Delta, L}^{\alpha}$ is invertible. Recall that if $T : X \to Y$ is invertible, then $T^*$ is also invertible [4]. Therefore $(F_{\Delta, L}^{\alpha})^*$ is invertible. As a consequence, $F_{\Delta, L}^{\alpha}$ is Fredholm. The index of a Fredholm operator is defined as

\[ \text{index}(F_{\Delta, L}^{\alpha}) = \dim \left( \ker(F_{\Delta, L}^{\alpha}) \right) - \dim \left( \ker(F_{\Delta, L}^{\alpha})^* \right). \]

Hence, the index is zero.

\[ \square \]

Remark 3.11. If $\alpha(x, y)$ is a non-zero constant function in $I \times J$ and $f$ is a fixed point of $F_{\Delta, L}^{\alpha}$, then $f$ is a fixed point of $L$ as well. This can be easily seen as follows. Here $\alpha(x, y) = \alpha \neq 0$, a constant function on $I \times J$. Now let $f$ be a fixed point of $F_{\Delta, L}^{\alpha}$, that is, $f_{\Delta, L}^{\alpha} = f$. For $(x, y) \in I \times J$, by the functional equation we have

\[ f(u_i(x), v_j(y)) = f(u_i(x), v_j(y)) + \alpha \left[ f(x, y) - Lf(x, y) \right], \]
from which it follows that $Lf = f$.

**Theorem 3.12.** Let $f \in C(I \times J, \mathbb{R})$.

1. If $\alpha_n \in C(I \times J, \mathbb{R})$ be such that $\|\alpha_n\|_\infty < 1$ and $\alpha_n \to 0$ as $n \to \infty$. Then the corresponding sequence of $\alpha$-fractal functions $f_{\alpha_n}^\Delta \to f$ as $n \to \infty$.  
2. If $L_n : C(I \times J, \mathbb{R}) \to C(I \times J, \mathbb{R})$ be a sequence of bounded linear operators such that $L_n f \neq f$, $(L_n f)(x_i, y_j) = f(x_i, y_j)$ for $(x_i, y_j) \in \partial \Sigma_{N,0} \times \partial \Sigma_{M,0}$ with respect to a net $\Delta$, and $L_n f \to f$. Then the corresponding sequence of $\alpha$-fractal functions $f_{\alpha_n}^\Delta \to f$ as $n \to \infty$ for any fixed admissible choice of the scale function $\alpha$.

**Proof.** From item (1) in the previous theorem, we note that the uniform error bounds for the process of approximation of $f$ with $f_{\alpha_n}^\Delta$ is given by

$$
\|f_{\alpha_n}^\Delta - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|f - Lf\|_\infty.
$$

From this the required results can be deduced. □

**Remark 3.13.** For an example of a sequence $(L_n)$ satisfying conditions required in the previous theorem, one can work with the two dimensional Bernstein operators. Let us recall that if $f(x, y)$ is continuous in the square $S = [0, 1] \times [0, 1]$, then

$$
\lim_{m,n \to \infty} B_{m,n}(x, y) = f(x, y),
$$

uniformly in $x$ and $y$, as $n, m$ approach infinity in any manner whatsoever. Here

$$
B_{n,m}(x, y) = \sum_{i=0}^{n} \sum_{k=0}^{m} f\left(\frac{i}{n}, \frac{k}{m}\right) p_{i,n}(x) p_{k,m}(y),
$$

where

$$
p_{j,s}(z) = \binom{s}{j} z^j (1 - z)^{s - j}.
$$

Similarly, one can work with many extensions of Bernstein operator known in literature.

A simple reformulation of the above theorem is given below.

**Theorem 3.14.** Let $f \in C(I \times J, \mathbb{R})$ and $\epsilon > 0$. Then there exists a bivariate fractal function $f_{\alpha_n}^\Delta$ obtained via the fractal perturbation process given above such that

$$
\|f - f_{\alpha_n}^\Delta\|_\infty < \epsilon.
$$

**Proof.** Choose $f_{\alpha_n}^\Delta$ or $f_{\alpha_n}^\Delta$ for a suitably large $n$. □

**Definition 3.15.** Given a Banach space $X$ and a bounded linear operator $T : X \to X$, a subspace $Y \subseteq X$ is called invariant under $T$ or $T$-invariant if $T(Y) \subseteq Y$. We call, $Y$ is an invariant subspace of $T$.

**Remark 3.16.** Clearly, $Y = \{0\}$ and $Y = X$ are $T$-invariant subspaces for every bounded linear operator $T : X \to X$. Consequently, one is interested only in the other invariant subspaces, so-called the non-trivial invariant subspaces.

Next pair of lemmas is fundamental in functional analysis; for instance, these are exercises in [4]. We give the proofs here for the sake of completeness.
Lemma 3.17. Let $X$ be a non-separable Banach space, then every bounded linear operator $T : X \to X$ has a non-trivial closed invariant subspace.

Proof. Suppose $T$ has no non-trivial closed invariant subspace. Choose a non-zero element $x$ of $X$ and define $Y_x = \text{span}\{x, T(x), T^2(x), \ldots\}$. Clearly, $T(Y_x) \subseteq Y_x$. For closedness, we define $M = \overline{\bigcup Y_x}$. It is obvious that $T(M) \subseteq M$, that is $M$ is a closed invariant subspace of $T$. Clearly, $M \neq \{0\}$. If $M = X$, then we obtain a subset (having rational coefficients in the linear combinations) of $X$ which is dense in $X$, contradicting the hypothesis that $X$ is non-separable. \hfill \Box

Lemma 3.18. If $Y$ is a closed invariant subspace of $T^*$, then $^a Y$ is a closed invariant subspace of $T$.

Proof. Let $y \in T(\text{"Y})$, where $y = T(x)$ for an element $x$ in $\text{"Y}$. Let $x^*$ be arbitrary in $Y$.

$$\langle y, x^* \rangle = \langle T(x), x^* \rangle = \langle x, T^*(x^*) \rangle = 0.$$ 

In the above, the first two equalities are obvious whereas the last follows from the hypothesis that $Y$ is an invariant subspace under $T^*$. Finally we have, $T(\text{"Y}) \subseteq ^a Y$, and being the intersection of null spaces of functional from $X^*$, $^a Y$ is closed. \hfill \Box

Theorem 3.19. There exists a non-trivial closed invariant subspace for the fractal operator $F_{\Lambda,L}^a : C(I \times J, \mathbb{R}) \to C(I \times J, \mathbb{R})$.

Proof. We know that the adjoint of $F_{\Lambda,L}^a$ denoted by $\left( F_{\Lambda,L}^a \right)^* \circ (C(I \times J, \mathbb{R}))^* \to (C(I \times J, \mathbb{R}))^*$ is a bounded linear operator. Since $(C(I \times J, \mathbb{R}))^*$ is a non-separable Banach space, from Lemma 3.17 it follows that $\left( F_{\Lambda,L}^a \right)^*$ has a non-trivial closed invariant subspace. Lemma 3.18 now yields that the fractal operator $F_{\Lambda,L}^a$ has a non-trivial closed invariant subspace. \hfill \Box

Remark 3.20. Having established the existence, it is natural to ask for a description of a closed invariant subspace of $F_{\Lambda,L}^a$. This still remains an open question.

The following remarks are straightforward, however worth recording.

Remark 3.21. Assume that $\alpha : I \times J \to \mathbb{R}$ is a nonzero constant function. Furthermore, assume that the mappings $u_i$ and $v_j$ are affine functions, that is, $u_i(x) = a_i x + b_i$ and $v_j(y) = c_j y + d_j$. Let an invariant subspace $A$ of $F_{\Lambda,L}^a$ satisfies the following condition, for any $g(x) = \beta x + \gamma$ and $h(y) = \delta y + \kappa$, $f(g(\cdot), h(\cdot)) \in A$ for all $f \in A$. Then $A$ is invariant subspace for $L$ as well. To see this, let $f \in A$. The functional equation can be written in the following form

$$\alpha L f(x, y) = f(u_i(x), v_j(y)) + \alpha f_{\Lambda,L}^a(x, y) - f_{\Lambda,L}^a(u_i(x), v_j(y)).$$

Since $F_{\Lambda,L}^a(A) \subseteq A$ and $f \in A$, we have $f_{\Lambda,L}^a \in A$. The condition on $A$ yields $f(u_i(x), v_j(y)) \in A$ and $f_{\Lambda,L}^a(u_i(x), v_j(y)) \in A$, since $u_i(x) = a_i x + b_i$ and $v_j(y) = c_j y + d_j$. Therefore, the function on the right side of the above equation is in $A$, that is, $L f$ is in $A$.

Remark 3.22. Consider $I = J = [0, 1]$. Let $F_{\Lambda,L}^a : C([0, 1]^2, \mathbb{R}) \to C([0, 1]^2, \mathbb{R})$ be the fractal operator corresponding to an admissible scale function $\alpha$. Let $L$ be the Bernstein operator, that is, $L f = B_{k,l}(f)$ and $A$ be the set of polynomials of degree at most $(m, n)$ where $m \geq k$ and $n \geq l$. Clearly, $A$ is invariant under the Bernstein operator $L$. Moreover, $A$, being finite dimensional, is a closed subspace of $C([0, 1]^2)$. 


One can easily check that functions in $\mathcal{A}$ also satisfy the condition stated in the above remark. We anticipate that in most cases the self-referential function $f_{\Delta,L}^\alpha$ has noninteger box counting dimension and consequently $f \mapsto f_{\Delta,L}^\alpha$ is a roughing operation; see also Remark 3.4. Therefore the class $F_{\Delta,L}^\alpha(\mathcal{A})$ contains non-smooth functions and $F_{\Delta,L}^\alpha(\mathcal{A}) \subseteq A$ does not hold, in general.

4. Extension to $L^p(I \times J, \mathbb{C})$ and some properties

Here we extend the notion of fractal function to $L^p(I \times J, \mathbb{C})$ spaces, $1 \leq p < \infty$. We demonstrate that corresponding to a given $f \in L^p(I \times J, \mathbb{C})$, there exists a self-referential function $	ilde{f}_{\Delta,L} \in L^p(I \times J, \mathbb{C})$. As an interlude, in Lemma 4.2 below, we define a fractal operator on the space of continuous complex valued functions on $I \times J$, denoted by $\mathcal{C}(I \times J, \mathbb{C})$, endowed with the $L^p$-norm.

**Theorem 4.1.** Let $\mathcal{C}(I \times J, \mathbb{R})$ be endowed with the $L^p$-norm, $1 \leq p < \infty$, and $f \in \mathcal{C}(I \times J, \mathbb{R})$. Further, let $L$ be a linear map bounded with respect to the $L^p$-norm on $\mathcal{C}(I \times J, \mathbb{R})$. Then the following inequality holds:

$$\|f - f_{\Delta,L}^\alpha\|_{L^p} \leq \|\alpha\|_\infty \frac{1}{1 - \|\alpha\|_\infty} \|f - Lf\|_{L^p}.$$

Consequently, the fractal operator $F_{\Delta,L}^\alpha$ is bounded.

**Proof.** Note that

$$\|f_{\Delta,L}^\alpha - f\|_{L^p}^p = \int_{I \times J} |f_{\Delta,L}^\alpha(x,y) - f(x,y)|^p dx dy \leq \sum_{i,j} \int_{I \times J} |f_{\Delta,L}^\alpha(x,y) - f(x,y)|^p dx dy.$$

Using the functional equation for $f_{\Delta,L}^\alpha$ given in (3.4) we obtain

$$\|f_{\Delta,L}^\alpha - f\|_{L^p}^p = \sum_{i,j} \int_{I \times J} |\alpha(x,y)\delta_{\alpha} f_{\Delta,L}^\alpha(x,y) - Lf(x,y)|^p dx dy \leq \sum_{i,j} \int_{I \times J} \|\alpha\|_\infty |f_{\Delta,L}^\alpha(x,y) - Lf(x,y)|^p dx dy.$$

Changing the variable $(x,y)$ to $(\tilde{x}, \tilde{y})$ through the transformation $\tilde{x} = u_i^{-1}(x)$, $\tilde{y} = v_j^{-1}(y)$ and using the change of variable formula for double integrals, we have

$$\|f_{\Delta,L}^\alpha - f\|_{L^p}^p \leq \sum_{i,j} \int_{I \times J} \|\alpha\|_\infty |f_{\Delta,L}^\alpha(\tilde{x}, \tilde{y}) - Lf(\tilde{x}, \tilde{y})|^p \frac{\partial(x,y)}{\partial(\tilde{x}, \tilde{y})} d\tilde{x} d\tilde{y}.$$

Therefore

$$\|f_{\Delta,L}^\alpha - f\|_{L^p}^p \leq \|\alpha\|_\infty \|f_{\Delta,L}^\alpha - Lf\|_{L^p} \sum_{i,j} |a_i| |c_j|.$$

and hence

$$\|f_{\Delta,L}^\alpha - f\|_{L^p} \leq \|\alpha\|_\infty \|f_{\Delta,L}^\alpha - Lf\|_{L^p}.$$

Using this and the triangle inequality

$$\|f_{\Delta,L}^\alpha - f\|_{L^p} \leq \|\alpha\|_\infty [\|f_{\Delta,L}^\alpha - f\|_{L^p} + \|f - Lf\|_{L^p}],$$
whence
\[ \|f - f_{\Delta, L}^\alpha\|_{L^p} \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|f - Lf\|_{L^p}. \]

From the above bound for the perturbation error we have
\[ \|f_{\Delta, L}^\alpha\|_{L^p} - \|f\|_{L^p} \leq \|f - f_{\Delta, L}^\alpha\|_{L^p} \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|f - Lf\|_{L^p}, \]
using which we infer that
\[ \|F_{\Delta, L}^\alpha(f)\|_{L^p} \leq \left[1 + \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|Id - L\| \right] \|f\|_{L^p}. \]

That is, \( F_{\Delta, L}^\alpha \) is a bounded operator. \( \square \)

In what follows, for the notational convenience, we may suppress the dependence on \( \Delta, L \) to denote the bivariate \( \alpha \)-fractal function corresponding to \( f \) by \( f^\alpha \) and the fractal operator by \( F^\alpha \).

**Lemma 4.2.** Let \( F^\alpha \) be a fractal operator on \( C(I \times J, \mathbb{R}) \), endowed with the \( L^p \)-norm. The operator \( F^\alpha_C : C(I \times J, \mathbb{C}) \to C(I \times J, \mathbb{C}) \) defined by
\[ F^\alpha_C(f) = F^\alpha_C(f_1 + if_2) = F^\alpha(f_1) + i F^\alpha(f_2) \]
is a bounded linear operator.

**Proof.** Since \( F^\alpha \) is linear, \( F^\alpha_C \) is linear. It remains to show that \( F^\alpha_C \) is a bounded operator. For this,
\[
\|F^\alpha_C(f)\|_{L^p}^p = \int_{I \times J} |F^\alpha_C(f)|^p \, dx \, dy \\
= \int_{I \times J} \left[ |F^\alpha(f_1)|^p + |F^\alpha(f_2)|^p \right] \, dx \, dy \\
\leq 2^\frac{p}{2} \int_{I \times J} \left[ |F^\alpha(f_1)|^p + |F^\alpha(f_2)|^p \right] \, dx \, dy \\
= 2^\frac{p}{2} \left[ \|F^\alpha(f_1)\|_{L^p}^p + \|F^\alpha(f_2)\|_{L^p}^p \right] \\
\leq 2^\frac{p}{2} \|F^\alpha\|^p \left[ \|f_1\|_{L^p}^p + \|f_2\|_{L^p}^p \right] \\
= 2^\frac{p}{2} \|F^\alpha\|^p \left[ \int_{I \times J} f_1^p \, dx \, dy + \int_{I \times J} f_2^p \, dx \, dy \right] \\
\leq 2^\frac{p}{2} + 1 \|F^\alpha\|^p \int_{I \times J} |f|^p \, dx \, dy \\
= 2^\frac{p}{2} + 1 \|F^\alpha\|^p \|f\|_{L^p}^p,
\]
thus we have
\[ \|F^\alpha_C(f)\|_{L^p} \leq 2^\frac{p}{2} + 1 \|F^\alpha\|^p \|f\|_{L^p}, \]
proving that \( F^\alpha_C \) is a bounded operator and \( \|F^\alpha_C\| \leq 2^\frac{p}{2} + 1 \|F^\alpha\|. \) \( \square \)
Remark 4.3. The case \( p = 2 \) is better behaved in the following sense. If \( p = 2 \), then in place of (11) we have

\[
\|F^\alpha(f)\|_{2}^{2} = \int_{I \times J} |F^\alpha(f)|^2 \, dx \, dy
\]

\[
= \int_{I \times J} \left[ |F^\alpha(f_1)|^2 + |F^\alpha(f_2)|^2 \right] \, dx \, dy
\]

\[
= \|F^\alpha(f_1)\|_{2}^{2} + \|F^\alpha(f_2)\|_{2}^{2}
\]

\[
\leq \|F^\alpha\|^2 \|f_1 + If_2\|_{2}^{2}
\]

\[
= \|F^\alpha\|^2 \|f\|_{2}^{2}.
\]

Consequently, \( \|F^\alpha\| \leq \|F^\alpha\| \), improving the bound obtained in the previous lemma.

Let us remind the following fundamental theorem.

Theorem 4.4. \([15]\) If an operator \( T : Z \to Y \) is linear and bounded, \( Y \) is a Banach space and \( Z \) is dense in \( X \), then \( T \) can be extended to \( X \) preserving the norm of \( T \).

In the sequel, we fix the notation \( M := 2^\frac{1}{2} + \frac{1}{2} \).

Theorem 4.5. \( \) Let \( 1 \leq p < \infty \), \( C(I \times J, \mathbb{R}) \) be equipped with \( \mathcal{L}_p \)-norm and \( F^\alpha : C(I \times J, \mathbb{C}) \to C(I \times J, \mathbb{C}) \) be the fractal operator defined in the previous lemma. Then there exists a bounded linear operator \( \mathcal{F}_C : L^p(I \times J, \mathbb{C}) \to L^p(I \times J, \mathbb{C}) \) such that its restriction to \( C(I \times J, \mathbb{C}) \) is \( F^\alpha \) and \( \|F^\alpha\| \leq \|\mathcal{F}_C\| = \|F^\alpha\| \leq M\|F^\alpha\| \).

Proof. It is well-known that \( C(I \times J, \mathbb{C}) \) is dense in \( L^p(I \times J, \mathbb{C}) \), for \( 1 \leq p < \infty \). From the previous lemma, we have a bounded linear operator \( \mathcal{F}_C : C(I \times J, \mathbb{C}) \to C(I \times J, \mathbb{C}) \). Now using the previous theorem we conclude that there exists a bounded linear operator \( \mathcal{F}_C : L^p(I \times J, \mathbb{C}) \to L^p(I \times J, \mathbb{C}) \) such that

\[
\|\mathcal{F}_C\| = \|\mathcal{F}_C\| \leq M\|F^\alpha\|.
\]

Furthermore, the previous theorem gives \( \mathcal{F}_C(f) = F^\alpha(f) = F^\alpha(f), \forall f \in C(I \times J, \mathbb{R}) \). Hence

\[
\|F^\alpha\| := \sup \{ \|F^\alpha(f)\|_{L^p} : f \in C(I \times J, \mathbb{R}), \|f\|_{L^p} = 1 \}
\]

\[
= \sup \{ \|\mathcal{F}_C(f)\|_{L^p} : f \in C(I \times J, \mathbb{R}), \|f\|_{L^p} = 1 \}.
\]

This implies that \( \|F^\alpha\| \leq \|\mathcal{F}_C\| \) and hence that \( \|F^\alpha\| \leq \|\mathcal{F}_C\| \leq M\|F^\alpha\| \). \( \square \)

Remark 4.6. In view of Remark 4.3, for \( p = 2 \), we have \( \|\mathcal{F}_C\| = \|F^\alpha\| \); for univariate counterpart see [19].

Lemma 4.7. Let \( L : C(I \times J, \mathbb{R}) \to C(I \times J, \mathbb{R}) \) be a bounded linear operator with respect to the \( \mathcal{L}_p \)-norm on \( C(I \times J, \mathbb{R}) \), \( 1 \leq p < \infty \). Then there exists a bounded linear operator \( \mathcal{L}_C : L^p(I \times J, \mathbb{C}) \to L^p(I \times J, \mathbb{C}) \) such that the restriction of \( \mathcal{L}_C \) to \( C(I \times J, \mathbb{R}) \) is \( L \) and \( \|L\| \leq \|\mathcal{L}_C\| \leq M\|L\| \). Moreover, the extension of \( \text{Id} - L \) is \( \text{Id} - \mathcal{L}_C \).

Proof. The first part of the lemma can be proved similar to Lemma 4.2 via \( \mathcal{L}_C \). For the rest, let \( f \in L^p(I \times J, \mathbb{C}) \) and a sequence of complex-valued continuous functions such that \( f_n \to f \) with respect to the \( \mathcal{L}_p \)-norm. Then

\[
(I - L)\mathcal{L}_C f_n = f_n - \mathcal{L}_C f_n \to f - \mathcal{L}_C f
\]

and hence \( (I - L)\mathcal{L}_C = \text{Id} - \mathcal{L}_C \). \( \square \)
Lemma 4.8. For each \( f \in C(I \times J, \mathbb{C}) \),
\[
\|F_C^\alpha(f) - f\|_{L^p} \leq M\|\alpha\|_\infty \|F_C^\alpha(f) - L_C(f)\|_{L^p}.
\]

Proof. Let \( f \in C(I \times J, \mathbb{C}) \), where \( f = f_1 + i f_2 \) for some \( f_1, f_2 \in C(I \times J, \mathbb{R}) \). Since \( F_C^\alpha(f) := F^\alpha(f_1) + i F^\alpha(f_2) \), we have
\[
\text{Re}(F_C^\alpha(f)) = F^\alpha(f_1), \quad \text{Im}(F_C^\alpha(f)) = F^\alpha(f_2).
\]
Furthermore, Using Theorem 4.1 and some basic inequalities, we have
\[
\|F_C^\alpha(f) - f\|_{L^p}^p = \int_{I \times J} |F_C^\alpha(f) - f|^p \, dx \, dy
\]
\[
= \int_{I \times J} \left[ |F^\alpha(f_1) - f_1|^2 + |F^\alpha(f_2) - f_2|^2 \right] \frac{1}{2} \, dx \, dy
\]
\[
\leq 2^p \int_{I \times J} \left[ |F^\alpha(f_1) - f_1|^p + |F^\alpha(f_2) - f_2|^p \right] \, dx \, dy
\]
\[
= 2^p \left[ \|F^\alpha(f_1) - f_1\|_{L^p}^p + \|F^\alpha(f_2) - f_2\|_{L^p}^p \right]
\]
\[
\leq 2^p \|\alpha\|_\infty^p \left[ \|F^\alpha(f_1) - L(f_1)\|_{L^p}^p + \|F^\alpha(f_2) - L(f_2)\|_{L^p}^p \right]
\]
\[
= 2^p \|\alpha\|_\infty^p \left[ \int_{I \times J} |F^\alpha(f_1) - L(f_1)|^p \, dx \, dy + \int_{I \times J} |F^\alpha(f_2) - L(f_2)|^p \, dx \, dy \right]
\]
\[
\leq 2^p + 1 \|\alpha\|_\infty^p \int_{I \times J} |F_C^\alpha(f) - L_C(f)|^p \, dx \, dy
\]
\[
= 2^p + 1 \|\alpha\|_\infty^p \|F_C^\alpha(f) - L_C(f)\|_{L^p}^p,
\]
hence the proof. \( \square \)

The proof of the next corollary follows from the way in which \( F_C^\alpha \) is defined using denseness of \( C(I \times J, \mathbb{C}) \) in \( L^p(I \times J, \mathbb{C}) \).

Lemma 4.9. For each \( f \in L^p(I \times J, \mathbb{C}) \),
\[
\|F_C^\alpha(f) - f\|_{L^p} \leq M\|\alpha\|_\infty \|F_C^\alpha(f) - L_Cf\|_{L^p}.
\]

Using the previous lemma, we deduce the next theorem in a similar way as that in Theorem 3.10. We avoid the proof, however recall that for a bounded linear operator \( T : X \to X \) in a Hilbert space \( X \), the following orthogonal decomposition holds:
\[
X = \text{Range}(T) \oplus \text{Ker}(T^*)
\]

Theorem 4.10. Let \( \|\alpha\|_\infty = \sup \{ |\alpha(x, y)| : (x, y) \in I \times J \} \), \( M\|\alpha\|_\infty < 1 \) and let \( \text{Id} \) be the identity operator on \( L^p(I \times J, \mathbb{C}) \).

1. For any \( f \in L^p(I \times J, \mathbb{C}) \), the perturbation error satisfies
\[
\|F_C^\alpha(f) - f\|_{L^p} \leq \frac{M^2\|\alpha\|_\infty \|\text{Id} - L\|_{L^p}}{1 - M\|\alpha\|_\infty} \|f\|_{L^p}.
\]

In particular, If \( \|\alpha\|_\infty = 0 \) then \( F_C^\alpha = \text{Id} \).

2. The norm of the fractal operator \( F_C^\alpha : L^p(I \times J, \mathbb{C}) \to L^p(I \times J, \mathbb{C}) \) satisfies
\[
\|F_C^\alpha\|_{L^p} \leq 1 + \frac{M^2\|\alpha\|_\infty \|\text{Id} - L\|}{1 - M\|\alpha\|_\infty}.
\]
Proof. Let \((e_n)\) be a Schauder basis of \(C([0, 1]^{d}, \mathbb{R})\), whose existence is hinted at the last paragraph. Choose \(\alpha\) such that \(||\alpha||_{\infty} < (1 + \|Id - L\|)^{-1}\), so that by Theorem 3.10 the fractal operator \(\mathcal{F}_{\alpha}^{*}L\) is a topological automorphism. If \(g \in C([0, 1]^{d}, \mathbb{R})\)
Lemma 5.6. Let \( F \) be the perturbed approximation class of fractal polynomials, see also [18]. Let \( P \) be a bivariate polynomials, then we denote by \( J \) a new class of functions by considering the fractal perturbation of functions in \( C \) and obtain the Schauder basis for \( \text{Schauder basis for } X \). Consider the fractal operator \( \Delta \).

Definition 5.3. Let \( F \) be a linear operator \( F : C(I \times J, \mathbb{R}) \rightarrow C(I \times J, \mathbb{R}) \) defined by \( f \mapsto f\Delta \); see Section [3]. Let \( p \in C(I \times J, \mathbb{R}) \) be a bivariate polynomial. Then \( F \Delta (p) = p\Delta \), denoted for simplicity by \( p\Delta \), is referred to as a bivariate fractal polynomial; see also [18]. Let \( \mathcal{P}(I \times J) \subset C(I \times J, \mathbb{R}) \) be the space of all bivariate polynomials, then we denote by \( \mathcal{P}\alpha(I \times J) \), the image space \( F\Delta \mathcal{P}(I \times J) \).

Notation 5.4. Let \( \mathcal{P}_{m,n}(I \times J) \) be the set of all bivariate polynomials of total degree at most \( m + n \) defined on \( I \times J \). That is,

\[
\mathcal{P}_{m,n}(I \times J) = \left\{ p(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j : a_{ij} \in \mathbb{R}, 0 \leq i \leq m \text{ and } 0 \leq j \leq n \right\}.
\]

We let \( \mathcal{P}\alpha_{m,n}(I \times J) = F\Delta \mathcal{P}_{m,n}(I \times J) \).

Remark 5.5. In fact, given an approximation class \( X \subset C(I \times J, \mathbb{R}) \), one can obtain a new class of functions by considering the fractal perturbation of functions in \( X \), that is, by considering \( F\Delta(X) \). For some reasons, perhaps the physical situation which the approximant is intended to model, finding an irregular approximant with a specified roughness (quantified in terms of the box counting dimension) from a subset of \( C(I \times J, \mathbb{R}) \) to a given \( f \in C(I \times J, \mathbb{R}) \) is of interest, and one may tackle it with the perturbed approximation class \( F\Delta(X) \).

Lemma 5.6. For any admissible choice of \( \alpha \in C(I \times J, \mathbb{R}) \), the space \( \mathcal{P}\alpha_{m,n}(I \times J) \) is finite dimensional. For \( \|\alpha\|_\infty < \|L\|^{-1} \), the dimension of \( \mathcal{P}\alpha_{m,n}(I \times J) \) is \( (m+n+2)^{2(m+n+1)} \).

Proof. It is well-known that the dimension of \( \mathcal{P}_{m,n}(I \times J) \) is \( (m+n+2)^2 \). However, for the sake of exposition let us provide an abbreviated argument here. By a straightforward counting argument, we see that there are \( \binom{k+2-1}{k-1} \) ways in which \( k \) indistinguishable exponents can be distributed to \( 2 \) distinguishable variables. Therefore it follows that the dimension of \( \mathcal{P}_{m,n}(I \times J) \) is \( \sum_{k=0}^{m+n} \binom{k+2-1}{k-1} = \binom{m+n+2}{2} = r \). Let \( \{p_1, p_2, \ldots, p_r\} \) be a basis for \( \mathcal{P}_{m,n}(I \times J) \). Since \( F\Delta \) is a linear map, it follows...
that $\mathcal{P}_{m,n}^\alpha$ is spanned by $\{p_1^\alpha, p_2^\alpha, \ldots, p_n^\alpha\}$. If $\|\alpha\|_\infty < \|L\|^{-1}$, then $\mathcal{F}_{\Delta,L}^\alpha$ is injective and hence $\{p_1^\alpha, p_2^\alpha, \ldots, p_n^\alpha\}$ is basis for $\mathcal{P}_{m,n}^\alpha(I \times J)$, completing the proof. \hfill $\Box$

Let us recall some basic concepts and a result from approximation theory; see, for instance, [10].

**Definition 5.7.** Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$, the field of real or complex numbers. Given a nonempty set $V \subseteq X$ and an element $x \in X$, distance from $x$ to $V$ is defined as $d(x, V) = \inf \{\|x - v\| : v \in V\}$. If there exists an element $v^* \in V$ such that $\|x - v^*\| = d(x, V)$, we call $v^*$ a best approximant to $x$ from $V$.

A subset $V$ of $X$ is called proximinal (proximal or existence set) if for each $x \in X$ a best approximant $v^*(x) \in V$ of $x$ exists.

**Theorem 5.8.** If $V$ is a finite dimensional subspace of the normed linear space $X$, then for each $x \in X$, there is a best approximant from $V$.

The following theorem is a direct consequence of the previous theorem and Lemma 5.6.

**Theorem 5.9.** Let $\mathcal{C}(I \times J, \mathbb{R})$ be endowed with the uniform norm. For each $f \in \mathcal{C}(I \times J, \mathbb{R})$, a best approximant $p^\alpha_f$ in $\mathcal{P}_{m,n}^\alpha(I \times J)$ exists.

**Theorem 5.10.** Let $\mathcal{C}(I \times J, \mathbb{R})$ be endowed with the uniform norm, $f \in \mathcal{C}(I \times J, \mathbb{R})$, and $L : \mathcal{C}(I \times J, \mathbb{R}) \to \mathcal{C}(I \times J, \mathbb{R})$, $L \neq Id$ be a bounded linear operator satisfying $(Lf)(x_i, y_j) = f(x_i, y_j)$, $\forall (i, j) \in \partial \Sigma_{N,0} \times \partial \Sigma_{M,0}$. For any $\epsilon > 0$, net $\Delta$ of the rectangle $I \times J$, there exists a bivariate fractal polynomial $p^\alpha$ such that

$$\|f - p^\alpha\|_\infty < \epsilon.$$

**Proof.** Let $\epsilon > 0$ be given. By the Stone-Weierstrass theorem, there exists a polynomial function $p$ in two variables such that

$$\|f - p\|_\infty < \frac{\epsilon}{2}.$$

Fix a net $\Delta$ of the rectangle $I \times J$, a bounded linear operator $L : \mathcal{C}(I \times J, \mathbb{R}) \to \mathcal{C}(I \times J, \mathbb{R})$, $L \neq Id$ satisfying $(Lf)(x_i, y_j) = f(x_i, y_j)$, $\forall (i, j) \in \partial \Sigma_{N,0} \times \partial \Sigma_{M,0}$. Choose $\alpha : I \times J \to \mathbb{R}$ as continuous function on $I \times J$ with $\|\alpha\|_\infty = \sup \{|\alpha(x, y)| : (x, y) \in I \times J\} < 1$ such that

$$\|\alpha\|_\infty < \frac{\epsilon}{2} + \|Id - L\| \|p\|_\infty.$$

Then we have

$$\|f - p^\alpha\|_\infty \leq \|f - p\|_\infty + \|p - p^\alpha\|_\infty.$$

$$\leq \|f - p\|_\infty + \|\alpha\|_\infty \|Id - L\| \|p\|_\infty.$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

$$= \epsilon.$$

In the above, the first inequality is just the triangle inequality, second follows from Theorem 3.10 and third is obvious. \hfill $\Box$
Remark 5.11. In the above proof, we selected \( \alpha \in C(I \times J, \mathbb{R}) \), for instance, constants, such that \( \| \alpha \|_{\infty} < \frac{\epsilon}{2} + \| Id - L \|_{\infty} \). In this case, \( \alpha \) may be “close” to 0 and hence \( p^\alpha \) may lose self-referentiality and behave as a traditional bivariate polynomial. Alternatively, one can fix \( \alpha \in C(I \times J, \mathbb{R}) \) such that \( \| \alpha \|_{\infty} < 1 \), but otherwise arbitrary and choose a bounded linear operator \( L : C(I \times J, \mathbb{R}) \to C(I \times J, \mathbb{R}) \), \( L \neq Id \) satisfying \( (Lf)(x, y) = f(x, y) \), \( \forall (i, j) \in \partial \Sigma_{N,0} \times \partial \Sigma_{M,0} \) such that
\[
\| Id - L \| < \frac{1 - \| \alpha \|_{\infty}}{\| \alpha \|_{\infty} \| p \|_{\infty}}.
\]
In this case, we expect that the graph of the corresponding fractal polynomial \( p^\alpha \) has the box dimension greater than 2, thus possesses a “fractality” in it and differs from the traditional bivariate polynomial.

In view of the previous theorem, we have

**Theorem 5.12.** The set of bivariate fractal polynomials with non-null scale vector is dense in \( C(I \times J, \mathbb{R}) \).

In the next theorem, we provide the denseness of a class of bivariate fractal polynomials which is a proper subset of the dense set considered above. This theorem reveals that one single scale vector is sufficient to obtain a bivariate fractal polynomial approximation of any bivariate continuous function.

**Theorem 5.13.** If \( \| \alpha \|_{\infty} < (1 + \| Id - L \|)^{-1} \), then \( P^\alpha(I \times J) \) is dense in \( C(I \times J, \mathbb{R}) \).

**Proof.** Note that under the given condition on \( \alpha \), \( F_{\Delta, L}^\alpha \) is a topological automorphism. Let \( f \in C(I \times J, \mathbb{R}) \). By the Stone-Weierstrass theorem, there exists a sequence of bivariate polynomials \( (p_n) \) such that \( p_n \to (F_{\Delta, L}^\alpha)^{-1}(f) \) in the uniform norm. Now since \( F_{\Delta, L}^\alpha \) is bounded, we obtain \( p_n^\alpha := F_{\Delta, L}^\alpha(p_n) \to f \) as \( n \to \infty \), and with it the proof. \( \square \)

**Definition 5.14.** Let \( C([-1, 1]^2, \mathbb{R}) \) be supplied with the uniform norm and \( f \in C([-1, 1]^2, \mathbb{R}) \). The Ditzian-Totik modulus of smoothness is defined as
\[
\omega_{r}^\phi(f; \delta_1, \delta_2) = \sup_{0 < h_i \leq \delta_i, i = 1, 2} \left| \sum_{h_1 \phi(x), h_2 \phi(y)} f(x, y) \right|,
\]
where \( \phi(x) = \sqrt{1 - x^2} \) and \( r \)-th symmetric difference of the function \( f \) is given by
\[
\sum_{h_1 \phi(x), h_2 \phi(y)} f(x, y) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f \left( x + h_1 \phi(x) \left( \frac{r}{2} - k \right), y + h_2 \phi(y) \left( \frac{r}{2} - k \right) \right)
\]
if \( (x \pm rh_1 \phi(x)/2, y \pm rh_2 \phi(y)/2) \in [-1, 1]^2 \), \( \sum_{h_1 \phi(x), h_2 \phi(y)} f(x, y) = 0 \) elsewhere.

**Theorem 5.15.** If \( f \) is real-valued continuous on \([-1, 1]^2 \), a sequence of bivariate polynomials \( (P_{m, n}(f))_{m, n \in \mathbb{N}} \) exists, with degree \( \leq m + n \), such that
\[
\| f - P_{m, n}(f) \| \leq C \omega_{2}^\phi(f; \frac{1}{m}, \frac{1}{n}),
\]
where \( C > 0 \) is independent of \( f, m \) and \( n \), and \( \omega_{2}^\phi(f; \frac{1}{m}, \frac{1}{n}) \) is the Ditzian-Totik modulus of smoothness with \( \phi(x) = \sqrt{1 - x^2} \).

**Notation 5.16.** We define \( E_{m, n}(f) := \inf \{ \| f - p \|_{\infty} : p \in P_{m, n}(I \times J) \} \) and \( E_{m, n}^\alpha(f) := \inf \{ \| f - p^\alpha \|_{\infty} : p^\alpha \in P_{m, n}^\alpha(I \times J) \} \).
Theorem 5.17. If \( f \) is real-valued continuous on \([-1,1]^2\), then the following estimate holds:

\[
E_{m,n}^\alpha(f) \leq C \left( \frac{1 + \|\alpha\|_\infty \|Id - L\| \|f\|_\infty}{1 - \|\alpha\|_\infty} \right) \omega_2^0 \left( f; \frac{1}{m}, \frac{1}{n} \right) + \frac{\|\alpha\|_\infty \|Id - L\| \|f\|_\infty}{1 - \|\alpha\|_\infty},
\]

where \( C > 0 \) is an absolute constant.

Proof. Let \( f \in C([-1,1]^2, \mathbb{R}) \). Let \( p_f \in \mathcal{P}_{m,n}([-1,1]^2) \) be a best approximant to \( f \). That is, \( E_{m,n}(f) = \|f - p_f\|_\infty \). Using the previous theorem, we estimate a bound for \( E_{m,n}^\alpha(f) \) in the following manner:

\[
E_{m,n}^\alpha(f) \leq \|f - (p_f)^\alpha\|_\infty
\leq \|f - p_f\|_\infty + \|p_f - (p_f)^\alpha\|_\infty
\leq E_{m,n}(f) + \frac{\|\alpha\|_\infty \|Id - L\|}{1 - \|\alpha\|_\infty} \|p_f\|_\infty
\leq E_{m,n}(f) \left( 1 + \frac{\|\alpha\|_\infty \|Id - L\|}{1 - \|\alpha\|_\infty} \right) + \frac{\|\alpha\|_\infty \|Id - L\|}{1 - \|\alpha\|_\infty} \|f\|_\infty
\leq \frac{1 + \|\alpha\|_\infty \|Id - L\| \|f\|_\infty}{1 - \|\alpha\|_\infty} E_{m,n}(f) + \frac{\|\alpha\|_\infty \|Id - L\| \|f\|_\infty}{1 - \|\alpha\|_\infty}
\]

hence by the previous theorem, we obtain the result. \( \square \)

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