Optimal Weights in a Two-Tier Voting System with Mean-Field Voters

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Abstract

We analyse two-tier voting systems with voters described by a multi-group mean-field model that allows for correlated voters both within groups as well as across group boundaries. The objective is to determine the optimal weights each group receives in the council to minimise the expected quadratic deviation of the council vote from a hypothetical referendum of the overall population. The mean-field model exhibits different behaviour depending on the intensity of interactions between voters. When interaction is weak, we obtain optimal weights given by the sum of a constant term equal for all groups and a term proportional to the square root of the group’s population. When interaction is strong, the optimal weights are in general not uniquely determined. Indeed, when all groups are positively coupled, any assignation of weights is optimal. For two competing clusters of groups, the difference in total weights must be a specific number, but the assignation of weights within each cluster is arbitrary. We also obtain conditions for both interaction regimes under which it is impossible to reach the minimal democracy deficit due to the negativity of weights.

Keywords: two-tier voting systems, probabilistic voting, mean-field models, democracy deficit, optimal weights

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1 Introduction

This article studies yes-no-voting in two-tier voting systems. In two-tier voting systems, the overall population is subdivided into $M$ groups (such as the member states of the European Union). Each group sends a representative to a council which makes decisions for the union. The representatives cast their vote (‘aye’ or ‘nay’) according to the majority in their respective group. For groups of different sizes, it is natural to assign different voting weights to the representatives. To determine these weights is the problem of ‘optimal’ weights. The assignation of the weights should follow some fixed objective. One criterion studied in the literature is to minimise the ‘democracy deficit’, i.e. the deviation of the council vote from a hypothetical referendum across the entire population (see e.g. [4, 9, 18, 10, 19, 12]).

Suppose the overall population is of size $N$, whereas each group has $N_\lambda$ voters, where the subindex $\lambda$ stands for the group $\lambda \in \{1, \ldots, M\}$. Let the two voting alternatives be recorded as $\pm 1$ for ‘aye’ and $-1$ for ‘nay’. The vote of voter $i \in \{1, \ldots, N_\lambda\}$ in group $\lambda$ will be denoted by the variable $X_{\lambda i}$. We will refer to all $(x_{11}, \ldots, x_{1N_1}, \ldots, x_{M1}, \ldots, x_{MN_M}) \in \{-1, 1\}^N$ as voting configurations.

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**Definition 1.** For each group \( \lambda \), we define the voting margin \( S_\lambda := \sum_{i=1}^{N_\lambda} X_{\lambda i} \). The overall voting margin is \( S := \sum_{\lambda=1}^{M} S_\lambda \).

Each group casts a vote in the council by applying the majority rule to the group vote:

**Definition 2.** The council vote of group \( \lambda \) is given by

\[
\chi_\lambda := \begin{cases} 
+1, & \text{if } S_\lambda > 0, \\
-1, & \text{otherwise.}
\end{cases}
\]

The representative of group \( \lambda \) votes ‘aye’ if there is a majority in group \( \lambda \) on the stated proposal. Each group \( \lambda \) is assigned a voting weight \( w_\lambda \). The weighted sum \( \sum_{\lambda=1}^{M} w_\lambda \chi_\lambda \) is the council vote. The council vote is in favour of a proposal if \( \sum_{\lambda=1}^{M} w_\lambda \chi_\lambda > 0 \). Weights \( w_1, \ldots, w_M \) together with a relative quota \( q \) constitute a weighted voting system for the council, in which a coalition \( C \subset \{1, 2, \ldots, M\} \) is winning if

\[
\sum_{i \in C} w_i \geq q \sum_{i=1}^{M} w_i.
\]

We will assume throughout this article that \( q = 1/2 \), i.e. a simple majority suffices in the council.

It is reasonable to choose the voting weights \( w_\lambda \) in the council in such a way, that the difference between the council vote and a hypothetical referendum

\[
\Delta = \Delta(w_1, \ldots, w_M) := \left| S - \sum_{\lambda=1}^{M} w_\lambda \chi_\lambda \right|
\]

is as small as possible. However, there is clearly no choice of the weights which makes \( \Delta \) small uniformly over all possible voting configurations. All we can hope for is to make \( \Delta \) small ‘on average’. More precisely, we try to minimise the expected quadratic deviation of \( \sum_{\lambda=1}^{M} w_\lambda \chi_\lambda \) from \( S \).

To follow this approach, we have to clarify what we mean by ‘expected’ deviation, i.e. there has to be some notion of randomness underlying the voting procedure.

While the votes cast are assumed to be deterministic and rational, obeying the voters’ preferences which we do not model explicitly, the proposal put before them is assumed to be unpredictable, i.e. randomly selected. Since each yes/no question can be posed in two opposite ways, one to which a given voter would respond ‘aye’ and one to which they would respond ‘nay’, it is reasonable to assume that each voter votes ‘aye’ with the same probability they vote ‘nay’.

This leads us to the following definition:

**Definition 3.** A voting measure is a probability measure \( \mathbb{P} \) on the space of voting configurations \( \{-1, 1\}^N = \prod_{\lambda=1}^{M} \{-1, 1\}^{N_\lambda} \) with the symmetry property

\[
\mathbb{P}(X_{11} = x_{11}, \ldots, X_{MN_m} = x_{MN_m}) = \mathbb{P}(X_{11} = -x_{11}, \ldots, X_{MN_m} = -x_{MN_m})
\]

for all voting configurations \( (x_{11}, \ldots, x_{MN_m}) \in \{-1, 1\}^N \). By \( \mathbb{E} \) we will denote the expectation with respect to \( \mathbb{P} \).
The simplest voting measure is the $N$-fold product of the measures

$$P_b(1) = P_b(-1) = \frac{1}{2},$$

which models independence between all the voting results $X_{\lambda i}$. In this much analysed case, known as the Impartial Culture (see e.g. [7], [8] or [13]), we have

$$P(X_{11} = x_{11}, \ldots, X_{MN_M} = x_{MN_M}) = \prod_{\lambda=1}^{M} \prod_{i=1}^{N} P_0(X_{\lambda i} = x_{\lambda i}) = \frac{1}{2^N}.$$

Once a voting measure is given, the quantities $X_{\lambda i}$, $S_\lambda$, $\chi_\lambda$, etc. are random variables defined on the same probability space $\{-1, 1\}^N$. This article treats the class of voting measures called the mean-field model (MFM) which extends the Impartial Culture by allowing correlations both between voters in the same group as well as correlations across group borders. The MFM has been extensively studied and applied to the social sciences. Models from statistical mechanics were first used by Föllmer [5] to study social interactions. The MFM specifically was first employed in [2]. See e.g. [3, 6, 16, 15] for other applications.

We next define and discuss the democracy deficit and the concept of optimal weights in the council. Afterwards, we introduce and discuss the MFM in Section 1.3 and its regimes in Section 1.4. In Sections 2 and 3, we discuss the optimal weights under the MFM for weak and strong interactions between voters, respectively. Section 4 treats several clusters of independent groups, and Section 5 contains the proofs of the results presented in this paper.

### 1.1 Democracy Deficit

With the concept of a voting measure at our disposal, we can formally define the democracy deficit. The presentation in this section and the next one follows [12].

**Definition 4.** The democracy deficit given a voting measure $P$ and a set of weights $w_1, \ldots, w_M$ is defined by

$$\Delta_1 = \Delta_1(w_1, \ldots, w_M) := \mathbb{E} \left[ \left( S - \sum_{\lambda=1}^{M} w_\lambda \chi_\lambda \right)^2 \right].$$

We call $(w_1, \ldots, w_M)$ optimal weights if they minimise the democracy deficit, i.e.

$$\Delta_1(w_1, \ldots, w_M) = \min_{(v_1, \ldots, v_M) \in \mathbb{R}^M} \Delta_1(v_1, \ldots, v_M).$$

Note that the democracy deficit depends on the voting measure in addition to the weights. We observe that minimising the democracy deficit implies that the magnitude and sign of the council vote approximates well the magnitude and sign of the popular vote. Thus, we do not merely wish to achieve agreement between the two outcomes in the binary sense but a rather stronger property: the voters should observe that the council follows the opinions prevalent in the population as closely as possible.

If we multiply each weight by the same positive constant and keep the relative quota $q$ fixed, we obtain an equivalent voting system. If the weights $w_\lambda$ minimise the democracy deficit $\Delta_1$, then the (equivalent) weights
for any $\sigma > 0$ minimise the ‘renormalised’ democracy deficit $\Delta_\sigma$ defined by

$$\Delta_\sigma = \Delta_\sigma (v_1, \ldots, v_M) := \mathbb{E}\left[\left(\frac{S}{\sigma} - \sum_{\lambda=1}^{M} v_\lambda \chi_\lambda\right)^2\right].$$

Whenever we speak of the uniqueness of the vector of optimal weights, it shall be understood to mean ‘uniqueness up to multiplication by a positive constant’.

It is, therefore, irrelevant whether we minimise $\Delta_1$ or $\Delta_\sigma$ as long as $\sigma > 0$. In this article, we will compute optimal weights as $N$ tends to infinity. As a rule, in this limit the minimising weights for $\Delta_1$ will also tend to infinity, it is therefore useful to minimise $\Delta_\sigma$ with an $N$-dependent $\sigma$ to keep the weights bounded. For the MFM, the two possible choices for $\sigma$ will turn out to be $\sqrt{N}$ and $N$. Which one of these is appropriate will depend on the parameters of the model (see Section 1.4).

### 1.2 Optimal Weights

Our objective is to choose the weights to minimise the democracy deficit. By taking partial derivatives of $\Delta_\sigma$ with respect to each $w_\lambda$, we obtain a system of linear equations that characterizes the optimal weights. Indeed, for $\lambda = 1, \ldots, M$,

$$\sum_{\nu=1}^{M} \mathbb{E}(\chi_\lambda \chi_\nu) w_\nu = \frac{1}{\sigma} \mathbb{E}(\chi_\lambda S).$$

(2)

Defining the matrix $A$, the weight vector $w$ and the vector $b$ on the right hand side of (2) by

$$A := (A_{\lambda,\nu})_{\lambda,\nu=1,\ldots,M} := \mathbb{E}(\chi_\lambda \chi_\nu)$$

(3)

$$w := (w_\nu)_{\nu=1,\ldots,M}$$

$$b := (b_\lambda)_{\lambda=1,\ldots,M} := \frac{1}{\sigma} \mathbb{E}(\chi_\lambda S).$$

we may write (2) in matrix form as

$$A w = b.$$ (4)

A solution $w$ of (4) is a minimum if the matrix $A$, the Hessian of $\Delta_\sigma$, is positive definite. In this case, the matrix $A$ is invertible and consequently there is a unique tuple of optimal weights, namely the unique solution of (4).

If the groups vote independently of each other, the matrix $A$ is diagonal. As we will see, this happens for some mean-field models. However, we will mainly focus on the case where there is correlated voting across group boundaries.

In the general case the matrix $A$ is indeed invertible under rather mild conditions.

**Definition 5.** We say that a voting measure $\mathbb{P}$ on $\prod_{\lambda=1}^{M} \{-1,1\}^{N_\lambda}$ is **sufficiently random** if

$$\mathbb{P}(\chi_1 = s_1, \ldots, \chi_M = s_M) > 0 \quad \text{for all } s_1, \ldots, s_M \in \{-1,1\}.$$ (5)

Note that (5) is not very restrictive. For example, if the support $\text{supp} \mathbb{P}$ of the measure $\mathbb{P}$ is the whole space $\{-1,1\}^{N}$, then $\mathbb{P}$ satisfies (5). As a matter of fact, all versions of the MFM studied in this article are sufficiently random for finite $N$. However, asymptotically, this property is lost in some cases.
Proposition 6. Let $\mathbb{P}$ be a voting measure and let $A$ be defined by (3).

1. The matrix $A$ is positive semi-definite.
2. $A$ is positive definite if $\mathbb{P}$ is sufficiently random.

Proof. This is Proposition 12 in [12]. It follows immediately from the previous proposition that the following holds:

Theorem 7. If the voting measure $\mathbb{P}$ is sufficiently random, the optimal weights minimising the democracy deficit $\Delta_\sigma$ are unique and given by

$$w = A^{-1} b.$$ (6)

Although for finite $N$, the MFM is sufficiently random, this result is of rather limited utility as it is practically impossible to compute the ingredients like $E(\chi_\lambda \chi_\nu)$ and $E(S \chi_\lambda)$ for finite $N$. An alternative is to compute these quantities approximately for $N \to \infty$ and use these in Sections 2 and 3. Next, we define the class of voting measures we analyse in this article.

1.3 Mean-Field Model

In statistical mechanics, the mean-field model\(^1\) is usually defined for a single set of spins or binary random variables. There is an energy function, also called Hamiltonian, that assigns each spin configuration $x = (x_1, \ldots, x_N) \in \{-1, 1\}^N$ a real number

$$H(x) := -\frac{1}{2N} \left( \sum_{i=1}^{N} x_i \right)^2.$$ (7)

This energy function determines the ‘cost’ of the configuration. Less costly configurations are thought of as more common. The only parameter of the model is an inverse temperature parameter $\beta \geq 0$. The probability measure on the space of configurations $\{-1, 1\}^N$ is a Gibbs measure that assigns each configuration the probability

$$\mathbb{P}(x) := Z^{-1} \exp(-\beta H(x)).$$

The minus sign makes the configurations with lower energy levels $H(x)$ more probable under the measure $\mathbb{P}$. The normalisation constant $Z$ depends on both $\beta$ and the number of spins.

In [9], one of the authors of this article used this single-group model to study two-tier voting systems. The limitation of such an approach is that each group is described by a separate single-group model, thus precluding the possibility of studying correlated voting across group boundaries.

In order to study the interaction between voters belonging to different groups of constituencies, we need to define a model with several different sets of spins that potentially interact with each other in different ways. Instead of a single inverse temperature parameter, there is a coupling matrix that describes the interactions. We will call this matrix

$$J := (J_{\lambda\mu})_{\lambda,\mu=1,\ldots,M}.$$\(^2\)

\(^1\)This model is also called the ‘Curie-Weiss model’, named after the physicists Pierre Curie and Pierre Weiss.
Just as in the single-group model, there is a Hamiltonian function that assigns each voting configuration a certain energy level. This energy level can be interpreted as the cost of a given voting configuration in terms of the conflict between different voters. Voters tend to vote in such a way that the conflict is minimised. For each voting configuration \((x_{11}, \ldots, x_{MN}) \in \{-1, 1\}^N\), we define

\[
H(x_{11}, \ldots, x_{MN}) := -\frac{1}{2} \sum_{\lambda, \mu=1}^{M} \frac{J_{\lambda\mu}}{\sqrt{N_\lambda N_\mu}} \sum_{i=1}^{N_\lambda} \sum_{j=1}^{N_\mu} x_{\lambda i} x_{\mu j}.
\] (8)

Instead of each voter interacting with each other voter in the exact same way, voters in different groups \(\lambda, \mu\) are coupled by a coupling constant \(J_{\lambda\mu}\). These coupling constants subsume the inverse temperature parameter \(\beta\) found in the single-group model. We note that depending on the signs of the coupling parameters \(J_{\lambda\mu}\) different voting configurations have different energy levels assigned to them by \(H\). If all coupling parameters are positive, there are two voting configurations that have the lowest energy levels possible: \((-1, \ldots, -1)\) and \((1, \ldots, 1)\). All other voting configurations receive higher energy levels. The highest levels are those where voters are evenly split (or closest to it in case of odd group sizes). This represents the assumed tendency of voters to cooperate with each other if they are positively coupled.

**Definition 8.** Let \(J\) be a positive semi-definite \(M \times M\) matrix. The mean-field probability measure \(P\) which gives the probability of each of the \(2^N\) voting configurations, is defined by

\[
P(X_{11} = x_{11}, \ldots, X_{MN} = x_{MN}) := Z^{-1} e^{-H(x_{11}, \ldots, x_{MN})}
\] (9)

for each \(x_{\lambda i} \in \{-1, 1\}\) and \(Z\) is a normalisation constant which depends on \(N\) and \(J\).

The mean-field measure is indeed a voting measure, as can be seen in the definition of the Hamiltonian \(H\). Throughout this article, we will assume that as the overall population grows, so do the group populations, and that their relative sizes compared to the overall population converge to fixed limits:

**Definition 9.** We define the group size parameters for each group \(\nu\):

\[
\alpha_\nu := \lim_{N \to \infty} \frac{N_\nu}{N}.
\]

We will consider two variations of this model. The first one, called the homogeneous model, is similar to the single-group model in that the interaction between voters is the same independently of the identities of the voters in question. The coupling matrix is given by \(J = (\beta)_{\lambda,\nu=1,\ldots,M}\) with \(\beta \geq 0\). With this \(J\), the main difference between the Hamiltonian of of the multi-group model in (8) and the Hamiltonian the single-group model in (7) is the normalisation by the group sizes \(N_\lambda\) instead of the overall population \(N\). However, we note that the multi-group model reduces to the single-group model if we set \(M = 1\) and \(N_1 = N\). We also mention that the Impartial Culture model of independent voters is a special of the homogeneous coupling model corresponding to \(\beta = 0\).

The second variation of the model features different coupling constants for voters belonging to different groups. We will assume that the coupling matrix \(J\) is positive definite. This implies in particular that the voters within each group are positively coupled, but voters belonging to different groups may be negatively coupled – a scenario we will consider in Sections 2 and 3. We also mention that the homogeneous coupling
model is not a special case of the heterogeneous model as the coupling matrix \((\beta)_{\lambda,\nu=1,\ldots,M}\) is not positive definite for any \(\beta \geq 0\).

Since the representatives vote in the council according to the majority rule, we have to understand the behaviour of the model in terms of the voting margins

\[
S = (S_1, \ldots, S_M) = \left( \sum_{i_1=1}^{N_1} X_{1i_1}, \ldots, \sum_{i_M=1}^{N_M} X_{1i_M} \right)
\]

(10)

in order to perform our analysis of the optimal weights. We will see that the voting margins show distinct patterns of behaviour depending on the parameters of the model. We can distinguish three different ‘regimes’ of the model.

### 1.4 Regimes of the Mean-Field Model

In the field of statistical physics, the regimes of the mean-field model are called ‘temperature regimes’ because in the single-group model the model only has a single parameter which can be interpreted as the inverse temperature of the spin system. In the present context, different temperatures correspond to different intensities of interaction between voters. A high temperature means there is a lot of disorder or confusion, and the voters mostly make up their own minds. However, there may still be some rather weak tendency to vote alike. We will call this the ‘weak interaction regime’. At low temperatures, voters want to align with others. As a result, votes will be strongly correlated. We will call this the ‘strong interaction regime’.

In the single-group mean-field model, \(\beta < 1\) is the ‘high temperature regime’, \(\beta = 1\) the ‘critical regime’, and \(\beta > 1\) the ‘low temperature regime’. In the homogeneous multi-group model, the number of groups also plays a role:

**Definition 10.** For homogeneous coupling matrices, we define the **weak interaction regime** to be \(\beta < 1/M\), the **critical regime** to be \(\beta = 1/M\), and the **strong interaction regime** to be \(\beta > 1/M\).

The parameter space of the mean-field model with heterogeneous coupling is

\[
\Phi := \{ J \mid J \text{ is an } M \times M \text{ positive definite matrix} \}.
\]

We will write \(I\) for the identity matrix. Its dimensions should be clear from the context.

**Definition 11.** For heterogeneous coupling matrices, the **weak interaction regime** is the set of parameters

\[
\Phi_h := \{ \phi \in \Phi \mid I - J \text{ is positive definite} \}.
\]

The **critical regime** is the set of parameters

\[
\Phi_c := \{ \phi \in \Phi \mid I - J \text{ is positive semi-definite but not positive definite} \}.
\]

The **strong interaction regime** is the set of parameters

\[
\Phi_l := \Phi \setminus (\Phi_h \cup \Phi_c).
\]
Of the three regimes, the critical regime is by far the smallest. We will exclusively deal with the other two regimes which are of more practical importance. Of these two regimes, the weak interaction regime is sufficiently random not only for finite $N$ but also asymptotically, and thus guarantees a unique set of optimal weights by Theorem 7.

**Proposition 12.** Suppose that $\mathbb{P}$ is an MFM measure in the weak interaction regime. Then $\mathbb{P}$ is sufficiently random in the limit $N \to \infty$.

### 2 Optimal Weights for the Weak Interaction Regime

In the weak interaction regime, the voters do not exert much influence over each other. Thus, the typical voting margins $S$ are of order $\sqrt{N}$ in each component. See Section 5.1 for the limit theorems. As a direct consequence, we normalise the democracy deficit by $\sigma = \sqrt{N}$. The problem of optimal weights always has a unique solution in the weak interaction regime by Proposition 32.

First, we calculate the optimal weights for homogeneous coupling matrices. For heterogeneous coupling matrices, the weights are also uniquely determined by Proposition 12. The optimal weights are given by the matrix equation $w = A^{-1}b$. Each group's optimal weight is a linear combination of the square roots of the relative group sizes $\alpha_\nu$. As we have to invert $A$ in order to solve the linear equation system (2), it is difficult to analyse the properties of these optimal weights for arbitrary heterogeneous coupling matrices. We will therefore study the class of heterogeneous coupling matrices in which the absolute value of the coupling constants can only take two different values. We will be using the asymptotic results for the MFM found in the Appendix.

#### 2.1 Homogeneous Coupling

Under homogeneous coupling, in the weak interaction regime, the optimal weights are uniquely determined, and we can calculate the weights explicitly using the results in Section 5.

One of the key ingredients for the optimal weights is the correlation between the council votes of two different groups $\lambda, \nu$: set $a := \mathbb{E}(\chi_\lambda \chi_\nu)$. Note that in the homogeneous model, this correlation does not depend on the specific groups $\lambda, \nu$ selected. As we will show in Section 5.4, $a$ is given by

$$a = \frac{2}{\pi} \arcsin \frac{\beta}{1 - (M - 1) \beta}.$$

In particular, for $\beta = 0$, when all voters are independent, $a = 0$. The correlation coefficient $a$ is strictly increasing in $\beta$ and $a \nearrow 1$ as $\beta \nearrow 1/M$. This presages the breakdown of the Central Limit Theorem 24 as $\beta$ approaches the critical value $1/M$, as the optimal values are no longer unique if we simply substitute the values corresponding to the critical regime in the linear equation system (2).

Let $\eta := \sum_{\lambda=1}^{M} \sqrt{\alpha_\lambda}$.

**Theorem 13.** Let $J$ be homogeneous and $\beta < 1/M$. Then the optimal weight for each group $\lambda = 1, \ldots, M$ is given by

$$w_\lambda = C_1 \sqrt{\alpha_\lambda} + C_2 \eta, \quad (11)$$

---

\(^2\)In fact, $\Phi_c$ has zero Lebesgue measure in the respective parameter space for each of the two classes of coupling matrices.
where
\[
C_1 = (1 + (M - 1) a) (1 - M \beta),
\]
\[
C_2 = (1 + (M - 1) a) \beta - a.
\]
The coefficient $C_1$ is positive, and $C_2 \geq 0$ with equality if and only if $\beta = 0$.

**Proof.** The proof can be found in Section 5.4.

This result concerning the optimal weights under homogeneous coupling in the weak interaction regime is typical of a variety of positive coupling scenarios. The optimal weights are given by the sum of a term which is proportional to the square root of the group's size and a summand which is equal for all groups. The coefficient of the proportional term is strictly positive, hence larger groups always receive higher weights. The constant term is also positive, except for the case that all voters are independent. Thus, we recover Penrose’s square root law first presented in [17] with $\beta = 0$. For dependent voters, there is no pure square root law.

In the formula (11) for the optimal weights, we have the constants $C_1$ and $C_2$, which only depend on the coupling matrix $J$ (including $J$’s dimensions), and the terms $\sqrt{\alpha_{\lambda}}$ and $\eta$, which only depend on the distribution of group sizes. Let us fix $J$ and hence $C_1, C_2$. The two extremes are:

1. There is a single large group that concentrates almost the entire population, and all other groups have very small fractions of the overall population. Then $\eta$ is close to 1, and each of the small groups receives a weight of approximately $C_2$. The single large group is assigned a weight of $C_1 + C_2$.

2. All groups are the same size. Unsurprisingly, the optimal weights are equal.

Thus, for each $1/M > \beta > 0$, there is a bounded range of possible quotients $\max_{\lambda} w_{\lambda}/\min_{\nu} w_{\nu}$ of optimal weights. If $\beta = 0$, then small groups receive weights close to 0, and $\max_{\lambda} w_{\lambda}/\min_{\nu} w_{\nu}$ is not bounded above. Nevertheless, a weight proportional to the square root of the population is still more beneficial to small groups when compared to weights proportional to the population.

### 2.2 Positive Coupling between Groups

In the first of the scenarios for positive definite coupling matrices, interactions within each group are of the same strength for each group. Similarly, interactions between voters of different groups are also uniform. This gives rise to a coupling matrix of the form

\[
J = (J_{\lambda\nu})_{1 \leq \lambda, \nu \leq M} = \begin{cases} j_0, & \lambda = \nu, \\ \tilde{j}, & \lambda \neq \nu. \end{cases}
\]  

(12)

We will assume that voters in different groups interact positively with each other or are independent, i.e. $\tilde{j} \geq 0$. According to (25), the positive definiteness of $J$ is equivalent to $j_0 > \tilde{j}$, so voters within each group are more dependent on each other than on voters in other groups.

By Definition 11 the weak interaction regime is the condition that $I - J$ is positive definite. By (24), this is itself equivalent to the two conditions

\[
j_0 < 1 \quad \text{and} \quad \tilde{j} < j_0 \wedge \frac{1 - j_0}{M - 1}.
\]  

(13)

Let $a$ stand for the expectation $E(\chi_1\chi_2)$ and $\eta := \sum_{\lambda=1}^{M} \sqrt{\alpha_{\lambda}}$. 


Theorem 14. For a coupling matrix as in (12) with $j_0 > \bar{j}$ and parameter ranges corresponding to the weak interaction regime stated in (13), the optimal weights are given by

$$w_\lambda = D_1 \sqrt{\alpha_\lambda} + D_2 \eta$$

for each group $\lambda = 1, \ldots, M$, where

$$D_1 = (1 + (M - 1) a) (1 - j_0 - (M - 1) \bar{j})$$
$$D_2 = (1 + (M - 2) a) \bar{j} - a (1 - j_0)$$

The theorem is proved in Section 5.5.

Proof. The theorem implies that the optimal weights are composed of a summand proportional to the square root of the group’s population and a constant summand equal for all groups. The constant summand is 0 if and only if the groups are independent. Thus, we recover the square root law from [9] for $\bar{j} = 0$. For dependent voters across group boundaries, there is no pure square root law. Instead, the optimal weight is given by a term equal for each group and a term proportional to the square root of the group’s population. Note that for $\bar{j} = 0$ voters within each group are not independent, contrary to homogeneous coupling with $\beta = 0$. The remarks after Theorem 13 apply almost verbatim to this scenario with $\bar{j}$ instead of $\beta$ and independent groups instead of independent voters.

2.3 Two Antagonistic Clusters of Groups

Consider two clusters of groups. Let each contain $M_i, i = 1, 2$, groups so that $M_1 + M_2 = M$. Without loss of generality, assume that the cluster $C_1$ contains the first $M_1$ groups and $C_2$ the last $M_2$. Let the coupling matrix have the block matrix form

$$J = \begin{pmatrix} J_1 & B^T \\ B & J_2 \end{pmatrix},$$

where $J_i \in \mathbb{R}^{M_i \times M_i}, i = 1, 2$, has diagonal entries equal to $j_0$ and off-diagonal entries equal to $\bar{j}$, and $B = -\bar{j} 1_{M_1 \times M_2}$. We use the notation $1_{m \times n}$ to denote an $m \times n$ matrix with all entries equal to 1.

Hence, voters belonging to the same group have a coupling of $j_0$, voters in different groups of the same cluster are coupled positively with strength $\bar{j}$, and voters belonging to groups in different clusters are negatively coupled with strength $-\bar{j}$. The requirement that $J$ be positive definite is equivalent to $j_0 > \bar{j}$. The condition characterising the weak interaction regime is once again given by (13).

Let $a$ stand for the intra-cluster correlation $\mathbb{E}(\chi_1 \chi_2)$ and $\bar{\eta} := \sum_{\lambda \in C_1} \sqrt{\alpha_\lambda} - \sum_{\lambda \in C_2} \sqrt{\alpha_\lambda}$. Let $\mathbb{1}_A$ be the indicator function of some set or condition $A$.

The optimal weights are as follows:

Theorem 15. For a coupling matrix as in (14) with $j_0 > \bar{j}$ and parameter ranges corresponding to the weak interaction regime stated in (13), the optimal weights are given by

$$w_\lambda = D_1 \sqrt{\alpha_\lambda} + (-1)\mathbb{1}_{\lambda \in C_2} D_2 \bar{\eta}$$

(15)
for each group $\lambda = 1, \ldots, M$, where

$$D_1 = (1 + (M - 1) a) (1 - J_0 - (M - 1) \bar{J}) ,$$
$$D_2 = (1 + (M - 2) a) \bar{J} - a (1 - J_0) .$$

The coefficient $D_1$ is positive, and $D_2 \geq 0$ with equality if and only if $\bar{J} = 0$.

Proof. The theorem is proved in Section 5.6.

The optimal weights have identical coefficients $D_1$ and $D_2$ to the scenario with positive entries in the coupling matrix (or single-cluster model). However, instead of $\eta$, the sum of all $\sqrt{\alpha_\lambda}$, we have $\bar{\eta}$, the difference between the sum of all $\sqrt{\alpha_\lambda}$ belonging to each cluster. In most cases, either $\bar{\eta}$ or $-\bar{\eta}$ will be negative. Therefore, there are cases where one or more groups are assigned a negative voting weight. This happens when the term $(-1)^{1(\lambda \in C_2)} D_2 \bar{\eta}$ is negative and larger in absolute value than $D_1 \sqrt{\alpha_\lambda}$. Even if in a real-life situation a group were to accept being assigned a negative weight, their voting behaviour would not minimise the democracy deficit due to incentives to misrepresent their true preferences. We have:

**Corollary 16.** Under the same assumptions as in the Theorem 15, if there is a group $\lambda$ of sufficiently small size $\alpha_\lambda$, it is impossible to reach the minimal democracy deficit given by the solution of (2).

If there are no groups small enough for a negative weight, then the democracy deficit can be minimised as in the previous scenario. We will say that $C_1$ is ‘larger’ and ‘more uniformly sized’ than $C_2$ if $\bar{\eta} > 0$, even though strictly speaking $\bar{\eta} > 0$ can hold even if cluster 1 represents less than half the overall population. By the formula (15), groups belonging to the larger of the two clusters receive a weight composed of the sum of a term proportional to their population’s square root, $D_1 \sqrt{\alpha_\lambda}$, and a constant term, $D_2 |\bar{\eta}|$, equal for each group in that cluster. The groups belonging to the smaller cluster also receive a weight given by such a sum, however, the constant term is $-D_2 |\bar{\eta}|$. As in the last section, if the groups are independent, then $D_2 = 0$, and we recover the square root law. In addition to that, if $\bar{\eta} = 0$, then the weights are proportional to the square roots, even if the groups are not independent. $\bar{\eta} = 0$ can occur even if there are different numbers of groups in each cluster and they represent different proportions of the overall population.

### 2.4 Hostile World

Now we consider a scenario where all groups are antagonistic towards each other. The voters within each group interact positively with a coupling constant $J_0 > 0$, but voters belonging to different groups interact negatively with coupling constant $-\bar{J} \leq 0$. According to (25), the requirement that $J$ be positive definite is equivalent to

$$\bar{J} < \frac{J_0}{M - 1} .$$

The weak interaction condition that $I - J$ be positive definite, on the other hand, is

$$J_0 < 1 \quad \text{and} \quad \bar{J} < J_0 \wedge (1 - J_0) . \quad (17)$$

Let $a$ stand for $-E(\chi_1 \chi_2) \geq 0$ and $\bar{\eta} := \sum_{\lambda=1}^{M} \sqrt{\alpha_\lambda}$.

In all previous classes of coupling matrices, $a$ could range all the way from 0 to 1, as we let $\beta \nearrow 1/M$ or $\bar{J} \nearrow J_0$. In the hostile world scenario, $a$ is bounded above by $\frac{1}{M - 1}$ as a consequence of the positive definiteness constraint (10) on $J$. 

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Theorem 17. For a coupling matrix with diagonal entries $j_\lambda$ and off-diagonal entries $-\bar{j}$ such that the positive definiteness condition (16) holds and parameter ranges corresponding to the weak interaction regime stated in (17), the optimal weights are given by

$$w_\lambda = D'_1 \sqrt{\alpha_\lambda} - D'_2 \eta$$

for each group $\lambda = 1, \ldots, M$, where

$$D'_1 = (1 - (M - 1) a) (1 - j_0 + (M - 1) \bar{j}),$$

$$D'_2 = (1 - (M - 2) a) \bar{j} - a (1 - j_0).$$

The coefficient $D'_1$ is positive, and $D'_2 \geq 0$ with equality if and only if $\bar{j} = 0$.

Proof. The proof can be found in Section 5.7.

The optimal weights given by formula (18) are the sum of a term proportional to the square root of the group’s population and a negative offset equal for all groups. This offset is the product of a factor $D'_2$ which depends on the coupling matrix $J$ and a factor $\eta$ which depends on the distribution of the groups’ sizes. Very small groups may receive a negative weight, whereas the largest groups always receive a positive weight. An impossibility to minimise the democracy deficit results similarly to the two-cluster scenario above:

Corollary 18. Under the same assumptions as in the Theorem 17, if there is a group $\lambda$ of sufficiently small size $\alpha_\lambda$, it is impossible to reach the minimal democracy deficit given by the solution of (2).

As in the other scenarios, letting the groups be independent by setting $\bar{j} = 0$ reduces (18) to the square root law.

3 Optimal Weights for the Strong Interaction Regime

In the strong interaction regime, the coupling between voters induces a pronounced tendency to vote alike. Thus, the typical voting margins $S$ are of order $N_\lambda$ in each component. As a direct consequence, we normalise the democracy deficit by $\sigma = N$. Contrary to the weak interaction regime, the optimal weights are not necessarily uniquely determined. In fact, as stated in Theorem 19, whenever voters belonging to any pair of groups are positively coupled, any $M$-tuple of positive weights is optimal. This is due to results like the limit Theorem 27 which states that, asymptotically, the per capita voting margins

$$(S_1/N_1, \ldots, S_M/N_M)$$

assume only two possible values, with probability 1/2 each. One of the values lies in the positive orthant, the other, in the negative orthant.

In particular, this solves the question of optimal weights in the strong interaction regime for homogeneous coupling matrices, as well as for heterogeneous coupling matrices with positive entries. We next consider the three scenarios from Section 2.

\footnote{In fact, the following weaker assumption suffices: let there be for any two groups $\lambda$ and $\nu$ a sequence $\kappa_0 = \lambda, \ldots, \kappa_m = \nu$ such that $J_{\kappa_i, \kappa_{i+1}} > 0$ holds for all $i = 0, \ldots, m - 1$. This implies that all groups are indirectly coupled.}
3.1 Positive Coupling between Groups

Recall from Section 2.2 the coupling matrix $J$ given in (12) and the condition of positive definiteness $j_0 > \bar{j}$. The strong interaction regime is by Definition 11 equivalent to the matrix $I - J$ not being positive semi-definite. By (25), this is itself equivalent to

$$j_0 > 1 \quad \text{or} \quad \bar{j} > \frac{1 - j_0}{M - 1}.$$  \hspace{1cm} (20)

Under this setup, the optimal weights are not uniquely determined:

**Theorem 19.** For a coupling matrix as in (12) with $j_0 > \bar{j} > 0$ and parameter ranges corresponding to the strong interaction regime stated in (20), any $M$-tuple of positive weights is optimal.

**Proof.** This follows directly from the fact that $A = 1_{M \times M}$ and all entries of $b$ are equal. \qed

When voters of all groups are positively coupled, asymptotically, the council votes will be almost surely unanimous by Theorem 27. Any distribution of weights among the groups gives rise to the same council votes.

3.2 Two Antagonistic Clusters of Groups

Consider the coupling matrix with two clusters of groups first introduced in Section 2.3 with $j_0 > \bar{j}$. The strong interaction regime is equivalent to the condition (20).

The per capita voting margins (19) concentrate asymptotically in two orthants. However, contrary to the cases of homogeneous and positive coupling between groups, these values do not lie in the positive and negative orthant. Rather, they are located in the two orthants where the coordinates belonging to each cluster have the same sign and the two clusters are of opposite signs.

The optimal weights are not unique; however, contrary to the homogeneous model and the heterogeneous model with positive coupling between groups, there is a binding condition on the total weight of the groups belonging to each cluster:

**Theorem 20.** For a coupling matrix as in (14) with $j_0 > \bar{j} > 0$ and parameter ranges corresponding to the strong interaction regime stated in (20), any $M$-tuple of positive weights satisfying

$$\sum_{\lambda \in C_1} w_\lambda - \sum_{\lambda \in C_2} w_\lambda = \Theta$$  \hspace{1cm} (21)

is optimal. The difference between the cluster weights $\Theta$ depends on the parameters of the model.

**Remark 21.** If the function $F$ defined in (24) has exactly two global minima, $m$ located in the orthant with positive coordinates $1, \ldots, M_1$ and negative coordinates $M_1 + 1, \ldots, M$ and $-m$, then $\Theta = \sum_{\lambda \in C_1} \alpha_\lambda |m_\lambda| - \sum_{\lambda \in C_2} \alpha_\lambda |m_\lambda|$.

**Proof.** The matrix $A$ has block form

$$A = \begin{pmatrix} 1_{M_1 \times M_1} & -1_{M_1 \times M_2} \\ -1_{M_2 \times M_1} & 1_{M_2 \times M_2} \end{pmatrix},$$

and $b$ has identical entries for $\nu \in C_1$ and the negative of this value for $\nu \in C_2$. \qed

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The voters belonging to different clusters have a strong tendency to vote opposite to each other. Asymptotically, the groups in cluster 1 will votes ‘yes’ if and only if the groups in cluster 2 vote ‘no’, almost surely. Under the uniqueness assumption in Remark 21, the absolute per capita voting margin \( E(|S_\lambda|/N_\lambda) \) converges to the constant \( m_\lambda \). Hence, we can interpret \( m_\lambda \in (0,1) \) as a measure of how large the typical majority is, i.e. a measure of the cohesion within the group. As such, we can interpret the terms \( \sum_{\lambda \in C_\alpha} \alpha_\lambda m_\lambda \) in the optimality condition (21) as follows: a group contributes to the overall weight of its cluster by being large and cohesive in its vote. Also, Theorem 20 makes no prescription as to how the joint weight of a cluster is to be distributed among the groups. Similarly to the model with positive couplings, it is irrelevant how the weights are assigned among groups that vote the same almost surely.

3.3 Hostile World

The strong interaction regime in a hostile world leads to some complicated yet interesting behaviour of the model. Instead of the two values that hold the entire probability mass for the per capita voting margins (19), there are potentially more than two different points. For example, if \( M \) is even and all groups are of the same size \( \alpha_\lambda = 1/M \), then there are \( \binom{M}{M/2} \) points the voting margin \( S \) normalised by the group populations assumes with positive (and equal) probability. Specifically, these points are located in the orthants with precisely half the coordinates positive and the other half negative. We can interpret this to mean that in a hostile world we have ever-changing coalitions that maintain the balance between the two alternatives being voted on. As all groups are hostile toward each other, there are no permanent alliances as in the other scenarios we considered previously. As a consequence of the shifting coalitions, the linear equation system (2) has a unique solution as the matrix \( A \) is not singular. The optimal weights are given by \( w = 0 \). As null weights lead to a council incapable of reaching consensus on any proposal, this is yet another case where in practice it is impossible to reach the minimal democracy deficit.

If \( M \) is odd and the groups are of the same size, it is impossible to achieve a perfect balance between the alternatives. Instead, the closest possible approximation is realised, in which \((M+1)/2\) groups vote for one alternative and the rest vote for the other. Contrary to \( M \) even, here the optimal weights are unique and positive: \( w_\lambda = m_\lambda \frac{M+1}{M} \). If we consider the limit of \( M \to \infty \), the asymmetry disappears, as a difference of one group in the council vote becomes insignificant.

The hostile world scenario illustrates that the optimal weights in the strong interaction regime are uniquely determined in some cases.

4 Independent Clusters of Groups

We have analysed both the weak and strong interaction regimes. As we have seen, the regime of the model determines the asymptotic behaviour of the voting margins. In particular, it determines whether the group voting margins are of order \( \sqrt{N} \) or order \( N \). It is not possible to have some voting margin \( S_\lambda \) that grows like \( \sqrt{N_\lambda} \) and some \( S_\nu \) that behaves like \( N_\nu \), unless the groups are independent. If we posit \( K \geq 2 \) clusters of groups which are independent of each other, then these clusters can be in different regimes. This assumption corresponds to a coupling matrix of block form. Let \( M_1, \ldots, M_K \) be the number of groups in each cluster.

---

4To obtain the fraction of voters voting with the majority, use the transformation \((1 + m_\lambda)/2\).
Then the coupling matrix has the form

\[ J = \begin{pmatrix}
J_1 & 0 & \ldots & 0 \\
0 & J_2 & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & J_K
\end{pmatrix}, \tag{22}\]

where \( J_i \) is the coupling matrix of cluster \( C_i \). Let \( A_i \) be \((\mathbb{E}(\chi_{\lambda \nu}))_{\lambda, \nu \in C_i}\) and \( b^{(i)} = (b_{\lambda})_{\lambda \in C_i} \). We will call the vector of optimal weights for cluster \( i \) \( w^{(i)} \).

**Theorem 22.** Let there be \( K \) independent clusters with a coupling matrix as in (22). Set \( \sigma := \sqrt{N} \) if all clusters are in the weak interaction regime and \( \sigma := N \) otherwise. Then, for all clusters \( i = 1, \ldots, K \), the optimal weights for all groups \( \lambda \in C_i \) are given by the linear equation system

\[ A_i w^{(i)} = b^{(i)}. \]

An immediate consequence of this theorem is that if there are two independent clusters, the first in the weak interaction regime, the second, in the strong interaction regime, then the second cluster will receive all the weight as the overall population goes to infinity.

**Corollary 23.** Let \( K = 2 \) and let the first cluster be in the weak and the second in the strong interaction regime. Also assume that all entries of \( J_2 \) are non-negative. Then the total weight of cluster 1 is \( O\left(\frac{1}{\sqrt{N}}\right) \), and the total weight of cluster 2 is a positive number.

This corollary illustrates that clusters in the weak interaction regime, whose voters interact loosely with each other, receive little weight compared to clusters in the strong interaction regime. Why does this happen? We have to think about whose opinion the representative of a group represents in the council. The answer is they only represent the difference in votes between the alternative that won, ‘yes’ or ‘no’, and the alternative that lost, the absolute voting margin. Hence, on average, the representatives cast their vote in the council in the name of a number of people in their group that corresponds to the expected absolute voting margin in their group. The expected per capita absolute voting margin in the weak interaction regime behaves like \( 1/\sqrt{N} \), whereas in the strong interaction regime it is constant with respect to \( N \). That is the reason the strong interaction regime representatives should receive more weight in the council: they stand for more people in favour of or against the proposal.

We also see that if there is a cluster of a single group, then that group will either have a weight proportional to the square root of its population if it is in the weak interaction regime, or a weight proportional to its population if it is in the strong interaction regime. Hence we recover the previous results found in [9][10][14].

### 5 Appendix

#### 5.1 Limit Theorems

In this section, we discuss two limit theorems that describe the asymptotic (i.e. large population) behaviour of the mean-field model. The symbol \( \xrightarrow{\text{w}} \) stands for convergence in distribution as the size of the overall
population (and hence by Definition [3] each group’s population) goes to infinity. We need these theorems to be able to determine the optimal weights that minimise the democracy deficit. These are known results (see e.g. [11]), and hence we will not prove them in this article.

Let \( \mathcal{N}((0, \ldots, 0), C) \) stand for a centred multivariate normal distribution with covariance matrix \( C \). Let for each \( x \in \mathbb{R}^M \) the symbol \( \delta_x \) stand for the Dirac measure in \( x \).

In the weak interaction regime, we have a central limit theorem:

**Theorem 24.** For homogeneous and heterogeneous coupling matrices, in their respective weak interaction regimes, we have

\[
\left( \frac{S_1}{\sqrt{N_1}}, \ldots, \frac{S_M}{\sqrt{N_M}} \right) \xrightarrow{w} \mathcal{N}((0, \ldots, 0), C),
\]

where the covariance matrix \( C \) is given by

\[
C = I + \Sigma,
\]

and the matrix \( \Sigma \) is \( \left( \frac{\beta}{1-M\beta} \right)_{\lambda,\mu=1,\ldots,M} \) if \( J \) is homogeneous, and \( \Sigma = (J^{-1} - I)^{-1} \), which is positive definite, if \( J \) is heterogeneous. For heterogeneous coupling, \( C \) also has the simpler representation \( (I - J)^{-1} \).

In the weak interaction regime, the correct normalising powers of the group populations are the square roots. This is due to the weak interaction between voters in this regime. Since \( S_\nu \) is the voting margin in group \( \nu \), normalising by \( \sqrt{N_\nu} \) suffices to approach a limiting distribution as \( N_\nu \to \infty \). This means that typically \( S_\nu \) is fairly close to 0 (a split vote).

A limit theorem holds for the strong interaction regime as well. However, contrary to the weak interaction regime, here the limiting distribution is not concentrated in the origin. The voters show a strong tendency to align, and thus the typical voting configurations show large majorities for or against a proposal. The precise size of a typical majority depends on the parameters. For homogeneous couplings, we need:

**Definition 25.** We call the equation

\[
\beta \sum_\lambda \tanh \left( \frac{x}{\sqrt{\alpha_\lambda}} \right) = x \tag{23}
\]

the Curie-Weiss equation. Let for all \( \beta \geq 0 \) \( m(\beta) \) be the largest solution to (23).

We characterise the value of \( m(\beta) \):

**Proposition 26.** For \( \beta \leq 1 \), there is a unique solution to (23), which is 0. For \( \beta > 1/M \), there are three solutions \( -x_1, 0, x_1 \), such that \( x_1 \in (0, 1) \). Therefore,

\[
m(\beta) = 0 \iff \beta \leq 1/M,
\]

and \( m(\beta) \in (0, 1) \) if and only if \( \beta > 1/M \).

With this definition of \( m(\beta) \) we can state the limit theorem for homogeneous coupling matrices:
**Theorem 27.** For homogeneous coupling matrices, in the strong interaction regime, we have

\[
\left( \frac{S_1}{N_1}, \ldots, \frac{S_M}{N_M} \right) \xrightarrow{w} \frac{1}{2} \left( \delta_{(-m(\beta), \ldots, -m(\beta))} + \delta_{(m(\beta), \ldots, m(\beta))} \right).
\]

For heterogeneous coupling matrices, the situation is more complicated. There are currently no general results for the strong interaction regime. Some special cases, such as \( M = 2 \) and \( M \in \mathbb{N} \) with groups of equal size have been solved. The problem lies in proving rigorously where the global minima of the function

\[
F(y) = \frac{1}{2} y^T \sqrt{\alpha} J^{-1} \sqrt{\alpha} y - \sum_{\lambda=1}^{M} \alpha \ln \cosh y_\lambda, \quad y \in \mathbb{R}^M,
\]  

are located. In the formula above, the \( M \times M \) matrix \( \sqrt{\alpha} \) is diagonal with its entries equal to \( \sqrt{\alpha_\lambda} \). \( F \) is bounded below and continuous. Hence it has global minima. The main issue is their uniqueness. However, we do not need the full strength of a limit theorem for the strong interaction regime for most of our results. The following observation suffices:

**Lemma 28.** Let the model be in the strong interaction regime. Then the minima of the function \( F \) defined in (24) are located in specific orthants of \( \mathbb{R}^M \):

1. In the positive coupling scenario defined in Section 3.1, the global minima are found in the positive and the negative orthant.

2. In the two cluster scenario defined in Section 3.2, the global minima are found in the two orthants with positive coordinates for \( \lambda \in C_1 \) and negative entries for \( \lambda \in C_2 \) and vice versa.

3. In the hostile world scenario defined in Section 3.3 with equal group sizes, global minima are located in \( \left( \frac{M}{M/2} \right) \) orthants if \( M \) is even and \( \left( \frac{M}{(M+1)/2} \right) \) if \( M \) is odd. The orthants in question are those where half the coordinates (or \( (M \pm 1)/2 \) if \( M \) is odd) are positive and the other half (or \( (M \mp 1)/2 \) if \( M \) is odd) are negative.

So in the strong interaction regime, with positive coupling, the voters tend to vote alike for the most part. Thus, we have to normalise the voting margins by \( N_\nu \) instead of \( \sqrt{N_\nu} \) to approach a limiting distribution as \( N_\nu \to \infty \). Due to the symmetry condition (1), it is of course equally likely that the votes will be mostly negative or mostly positive.

### 5.2 Positive Definiteness of the Coupling and Covariance Matrices

In Sections 2 and 3 we deal with two different types of matrices for which we need to determine under what conditions they are positive definite, or alternatively, not positive semi-definite. Let in the following discussion \( e_1, \ldots, e_M \) be the canonical basis vectors of \( \mathbb{R}^M \).

Let \( A \) be an \( M \times M \) matrix with identical diagonal entries, say \( a \), and identical off-diagonal entries which we will call \( b \). These matrices have eigenvalues \( a - b \) and \( a + (M-1)b \). The eigenvalue \( a - b \) has the multiplicity \( M - 1 \) with eigenvectors \( e_1 - e_2, \ldots, e_1 - e_M \). The eigenvalue \( a + (M-1)b \) has the eigenvector \( e_1 + \cdots + e_M \). Hence, the matrix \( A \) is positive definite if and only if

\[
a - b > 0 \quad \text{and} \quad a + (M-1)b > 0,
\]  

(25)
and not positive semi-definite if and only if at least one of the values is strictly negative.

The second type of matrix is an \( M \times M \) matrix with block matrix form

\[
C = \begin{pmatrix} A_1 & B \\ B^T & A_2 \end{pmatrix},
\]

where \( A_i \) is an \( M_i \times M_i \) matrix with the same structure as \( A \) above with diagonal entries \( a \) and off-diagonal entries \( b \), and \( M_1 + M_2 = M \). Let \( B \) equal \( -b \mathbf{1}_{M_1 \times M_2} \). The matrix \( C \) also has the eigenvalues \( a - b \) and \( a + (M - 1)b \). The eigenvectors are now given by \( \mathbf{e}_1 - \mathbf{e}_2, \ldots, \mathbf{e}_1 - \mathbf{e}_{M_1}, \mathbf{e}_1 + \mathbf{e}_{M_1 + 1}, \ldots, \mathbf{e}_1 + \mathbf{e}_M \), and \( \mathbf{e}_1 + \cdots + \mathbf{e}_{M_1} - (\mathbf{e}_{M_1 + 1} + \cdots + \mathbf{e}_M) \). Hence the condition for positive definiteness (25) is the same.

Depending on the sign of \( b \) in the considerations above, one or the other inequality in (25) will be binding.

5.3 Entries of the Linear Equation System (2)

In order to calculate the optimal weights, we first need the general form of the entries in the matrix \( A \) in (2).

Lemma 29. The entries of \( A \) are

\[
A_{\lambda \mu} = 4P(S_\lambda, S_\nu > 0) - 1 \quad \text{for all } \lambda, \mu = 1, \ldots, M.
\]

Proof. This is a straightforward calculation.

We show that

Proposition 30. Let \( C = (c_{\kappa \nu})_{\kappa, \nu=1,\ldots,M} \) be the covariance matrix defined in Theorem 24. In the weak interaction regime, we have

1. \( E(\chi_\kappa \chi_\nu) \approx \frac{2}{\pi} \arcsin \frac{c_{\kappa \nu}}{\sqrt{c_{\kappa \kappa} c_{\nu \nu}}} \) for all \( \kappa \neq \nu \),
2. \( E(\chi_\kappa S_\kappa) \approx \sqrt{\frac{2}{\pi c_{\kappa \kappa}}} N_\kappa \) for all \( \kappa \),
3. \( E(\chi_\kappa S_\nu) \approx \sqrt{\frac{2}{\pi c_{\kappa \kappa}}} N_\nu c_{\kappa \nu} \) for all \( \kappa \neq \nu \).

Proof. By Lemma 29, we have

\[
E(\chi_\kappa \chi_\nu) \approx 4P \left( \frac{S_\kappa}{\sqrt{N_\kappa}}, \frac{S_\nu}{\sqrt{N_\nu}} > 0 \right) - 1.
\]

We need to calculate the two-dimensional marginal distribution of \( \left( \frac{S_\kappa}{\sqrt{N_\kappa}}, \frac{S_\nu}{\sqrt{N_\nu}} \right) \). This distribution is bivariate normal with mean 0 and covariance matrix

\[
C_{\kappa \nu} = \begin{pmatrix} c_{\kappa \kappa} & c_{\kappa \nu} \\ c_{\kappa \nu} & c_{\nu \nu} \end{pmatrix}.
\]

For convenience sake, we set \( X' := \frac{S_\kappa}{\sqrt{N_\kappa}} \) and \( Y' := \frac{S_\nu}{\sqrt{N_\nu}} \). We standardise by dividing by the standard deviations:

\[
X := \frac{X'}{\sqrt{c_{\kappa \kappa}}}, \quad Y := \frac{Y'}{\sqrt{c_{\nu \nu}}}.
\]
so that both $X$ and $Y$ have marginal standard normal distributions. The correlation between them is given by

$$\rho := \frac{E(XY)}{\sqrt{\text{c}_{\kappa \kappa} \text{c}_{\nu \nu}}} = \frac{\text{c}_{\kappa \nu}}{\sqrt{\text{c}_{\kappa \kappa} \text{c}_{\nu \nu}}}$$

We set

$$Z := \frac{Y - \rho X}{\sqrt{1 - \rho^2}}$$

and note that $X$ and $Z$ are independent: $X$ and $Z$ are both normal and their covariance is

$$E(XZ) = \frac{E(XY) - \rho E(X^2)}{\sqrt{1 - \rho^2}} = \frac{-\rho \rho}{\sqrt{1 - \rho^2}} = 0.$$ 

It is easily verified that the distribution of $Z$ is standard normal. We let $\phi$ represent the density function of the standard normal distribution and calculate

$$P(X', Y' > 0) = P(X, Y > 0) = P\left(X > 0, \frac{Y - \rho X}{\sqrt{1 - \rho^2}} > 0\right)$$

$$= \int_0^\infty \int_{-\rho x}^{\rho x} \phi(x) \phi(z) \, dz \, dx = \frac{1}{2\pi} \int_0^\infty \int_{-\rho x}^{\rho x} e^{-\frac{x^2 + z^2}{2}} \, dz \, dx.$$ 

We switch to polar coordinates and the last integral above equals

$$\frac{1}{2\pi} \int_0^{\pi/2} \int_{\rho \sin \varphi}^{\rho \sin \varphi} e^{-\frac{r^2}{2}} r \, d\varphi \, dr = \frac{1}{4} + \frac{1}{2\pi} \arcsin \rho.$$ 

We next show the third result and note that the second one is a special case of the third. We set $X := \frac{S}{\sqrt{N_{\kappa \kappa}}}$ and $Y := \frac{S}{\sqrt{N_{\nu \nu}}}$ and use the conditional expectation

$$E(Y|X) = \frac{c_{\kappa \nu}}{c_{\kappa \kappa}} X,$$

which can be easily verified (for a proof see Chapter 4 of [1]). We are interested in $E\left(\chi_\kappa \frac{S}{\sqrt{N_{\nu \nu}}}\right)$, which is equal to $E(\text{sgn}(X)Y)$, therefore, we need to calculate $E(Y|X > 0)$ and $E(Y|X < 0)$. Their difference is the expectation we are looking for.

$$E(Y|X > 0) = \int \int 1\{X > 0\} Y P^{X,Y}(dx, dy) = \int 1\{X > 0\} \int Y P^{Y|X}(dy) P^X(dx)$$

$$= \int 1\{X > 0\} E(Y|X = x) P^X(dx) = \int_0^\infty \frac{c_{\kappa \nu}}{c_{\kappa \kappa}} x \frac{1}{\sqrt{2\pi c_{\kappa \kappa}}} e^{-\frac{x^2}{2c_{\kappa \kappa}}} \, dx$$

$$= \frac{c_{\kappa \nu}}{\sqrt{2\pi c_{\kappa \kappa}}}.$$
A very similar calculation yields

\[ \mathbb{E}(Y \mathbb{1}_{\{X < 0\}}) = -\frac{c_{K\nu}}{\sqrt{2\pi} c_{KK}}. \]

Therefore, we have

\[ \mathbb{E}\left(\chi_K \frac{S_{\nu}}{\sqrt{N_{\nu}}}\right) = \frac{\sqrt{2} c_{K\nu}}{\sqrt{\pi} c_{KK}}. \]

**Corollary 31.** Let \( C = (c_{K\nu})_{K,\nu=1,...,M} \) be the covariance matrix defined in Theorem 24. In the weak interaction regime, the linear equation system \( \mathbf{2} \) reads

\[ \left( \frac{2}{\pi} \arcsin \frac{c_{K\nu}}{\sqrt{c_{KK}}} \right)_{K,\nu=1,...,M} w = \sqrt{\frac{2}{\pi}} \left( \sqrt{c_{KK}} \sqrt{\alpha_{\nu}} + \sum_{\nu \neq K} c_{K\nu} \sqrt{\alpha_{\nu}} \right)_{\nu=1,...,M}. \] 

(26)

We next show that the linear equation system \( \mathbf{2} \) has a unique solution in the weak interaction regime.

**Proposition 32.** The matrix \( \left( \mathbb{E}(\chi_K \chi_{\nu}) \right)_{K,\nu=1,...,M} \) is non-singular in the weak interaction regime.

*Proof.* The covariance matrix \( C = I - J \) is positive definite. Thus, the limiting distribution is sufficiently random and by Proposition 6 the claim follows. \( \square \)

### 5.4 Proof of Theorem 13

The procedure to calculate the optimal weights is similar for both classes of coupling matrices. The details differ and we will comment on them in the subsequent proofs.

We take the covariance matrix \( C \) from Theorem 24 and calculate the entries of the matrix \( A \) using Proposition 30:

\[ (A)_{\lambda\nu} = \begin{cases} 1, & \lambda = \nu, \\ \frac{2}{\pi} \arcsin \frac{c_{K\nu}}{\sqrt{c_{KK} c_{\nu\nu}}}, & \lambda \neq \nu. \end{cases} \]

We note that the argument in the arcsin function above is strictly increasing in \( \beta \) and arcsin is strictly increasing on \((0, 1)\). Thus the off-diagonal entries \( a \) range from 0 for \( \beta = 0 \) to 1 as \( \beta \rightarrow 1/M \).

The entries of the vector \( b \) are given by

\[ b_\lambda = \mathbb{E}\left(\chi_\lambda \frac{S_{\lambda}}{\sqrt{N}}\right) \approx \mathbb{E}\left(\chi_\lambda \frac{S_{\lambda}}{\sqrt{N_{\lambda}}}\right) \sqrt{\alpha_\lambda} + \sum_{\nu \neq \lambda} \mathbb{E}\left(\chi_\lambda \frac{S_{\nu}}{\sqrt{N_{\nu}}}\right) \sqrt{\alpha_\nu} = \sqrt{\frac{2}{\pi c_{\lambda\lambda}}} \left[ c_{\lambda\lambda} \sqrt{\alpha_\lambda} + \sum_{\nu \neq \lambda} c_{\nu\lambda} \sqrt{\alpha_\nu} \right] \propto (1 - M \beta) \sqrt{\alpha_\lambda} + \beta \eta, \]

where \( \eta \) is as defined in Section 2.1. Above, we dropped the multiplicative constant which is identical for all \( \lambda \).

We invert the matrix \( A \),

\[ (A^{-1})_{\lambda\nu} = \frac{1}{D_A} \begin{cases} 1 + (M - 2) a, & \lambda = \nu, \\ -a, & \lambda \neq \nu. \end{cases} \]

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where $D_A$ is the positive determinant of $A$ which will not play a role in the calculation of the optimal weights. Dropping common multiplicative constants and simplifying,

\[
    w_\lambda = (A^{-1} b)_\lambda \propto (1 + (M - 2) a) b_\lambda - a \sum_{\nu \neq \lambda} b_\nu \\
    = (1 + (M - 1) a) (1 - M\beta) \sqrt{\alpha_\lambda} + [(1 + (M - 1) a) \beta - a] \eta.
\]

The claim regarding the sign of $C_1$ is clear, since $C_1 > 0$ is equivalent to $\beta < 1/M$. For $C_2 \geq 0$, we need the following auxiliary lemma:

**Lemma 33.** For all $x \in [0, 1]$, the inequality $x \leq \sin \left( \frac{\pi}{2} x \right)$ is satisfied. It holds with equality if and only if $x \in \{0, 1\}$.

**Proof.** Set $f(x) := \sin \left( \frac{\pi}{2} x \right) - x$.

The function $f$ has the values $f(0) = f(1) = 0$, and the derivative $f'(x) = \frac{\pi}{2} \cos \left( \frac{\pi}{2} x \right)$ has the values $f'(0) = \frac{\pi}{2} - 1 > 0$, $f'(1) = -1 < 0$. The second derivative $f''(x) = -\left( \frac{\pi}{2} \right)^2 \sin \left( \frac{\pi}{2} x \right)$ is non-positive on the interval $[0, 1]$ and strictly negative on $(0, 1]$. Thus, we conclude that $f'$ is strictly decreasing on $[0, 1]$, and there is some $x_0 \in (0, 1)$ such that $f$ is strictly increasing on $(0, x_0)$ and strictly decreasing on $(x_0, 1)$. In particular, $f > 0$ holds for all $x \in (0, 1)$.

The inequality $D_2 \geq 0$ is equivalent to

\[
    a \leq \frac{\beta}{1 - (M - 1) \beta}.
\]

We multiply both sides by $\pi/2$ and take the sine function of both sides. The claim $D_2 \geq 0$ follows from Lemma 33.

### 5.5 Proof of Theorem 14

By Theorem 24, the covariance matrix $C$ of the normalised voting margins is $(I - J)^{-1}$. We invert the matrix and obtain

\[
    C = (c_{\lambda\nu}) = \frac{1}{D_{I - J}} \begin{cases} 
    1 - j_0 - (M - 2) \bar{j}, & \lambda = \nu, \\
    \bar{j}, & \lambda \neq \nu,
    \end{cases}
\]

where $D_{I - J}$, the determinant of $I - J$, will not play an important role in the calculation of the optimal weights. We also note that all diagonal entries are equal and so are all off-diagonal entries.

Next we calculate the entries of the linear equation system (2) using the results from Theorem 30. The entries of matrix $A$ have the form

\[
    (A)_{\lambda\nu} = \begin{cases} 
    1, & \lambda = \nu, \\
    \frac{\pi}{2} \arcsin \frac{\bar{j}}{1 - j_0 - (M - 2) \bar{j}}, & \lambda \neq \nu.
    \end{cases}
\]
We set \( a \) equal to the second expression above, i.e. the off-diagonal entries of \( A \). The entries of the vector \( b \) are given by

\[
b_\lambda = E \left( \chi_\lambda \frac{S}{\sqrt{N}} \right) \approx E \left( \chi_\lambda \frac{S_\lambda}{\sqrt{N_\lambda}} \right) \sqrt{\alpha_\lambda} + \sum_{\nu \neq \lambda} E \left( \chi_\lambda \frac{S_\nu}{\sqrt{N_\nu}} \right) \sqrt{\alpha_\nu} \\
= \sqrt{\frac{2}{\pi c_\lambda}} \left[ c_{\lambda \lambda} \sqrt{\alpha_\lambda} + \sum_{\nu \neq \lambda} c_{\nu \lambda} \sqrt{\alpha_\nu} \right] \approx (1 - j_0 - (M - 1) \bar{j}) \sqrt{\alpha_\lambda} + \bar{\eta},
\]

where \( \eta \) is as defined in Section 2.2. We dropped the multiplicative constant which is identical for all \( \lambda \).

We invert the matrix \( A \),

\[
(A^{-1})_{\lambda \nu} = \frac{1}{D_A} \cdot \begin{cases} 
1 + (M - 2) a, & \lambda = \nu, \\
-a, & \lambda \neq \nu,
\end{cases}
\]

and proceed to calculate the optimal weights. Dropping common multiplicative constants and simplifying,

\[
w_\lambda = (A^{-1} b)_\lambda \approx (1 + (M - 2) a) b_\lambda - a \sum_{\nu \neq \lambda} b_\nu \\
\approx (1 + (M - 1) a) (1 - j_0 - (M - 1) \bar{j}) \sqrt{\alpha_\lambda} + [(1 + (M - 2) a) \bar{j} - a (1 - j_0)] \eta.
\]

The positivity of \( D_1 \) follows immediately, since the second factor \( 1 - j_0 - (M - 1) \bar{j} \) is positive in the weak interaction regime as shown previously. As for \( D_2 \), the inequality \( D_2 \geq 0 \) is equivalent to

\[
a \leq \frac{\bar{j}}{1 - j_0 - (M - 2) \bar{j}},
\]

and thus the claim follows from Lemma 33.

### 5.6 Proof of Theorem 15

The proof of this theorem proceeds along the same lines as that of Theorem 14. The main difference is the inversion of the coupling matrix \( I - J \). The block matrix form allows us to calculate its inverse using the Schur complement formula

\[
(I - J)^{-1} = \begin{pmatrix}
I - J_1 & -B \\
-B^T & I - J_2
\end{pmatrix}^{-1}
= \begin{pmatrix}
(I - J_1 - B (I - J_2)^{-1} B^T)^{-1} & 0 \\
0 & (I - J_2 - B^T (I - J_1)^{-1} B)^{-1}
\end{pmatrix}
\begin{pmatrix}
I & BJ_2^{-1} \\
B^T J_1^{-1} & I
\end{pmatrix},
\]

since the matrices \( I - J_i \) are both invertible in the weak interaction regime. After lengthy but straightforward calculations, we obtain

\[
C = (c_{\lambda \nu}) = \frac{1}{D_{I - J}} \cdot \begin{pmatrix}
1 - j_0 - (M - 2) \bar{j}, & \lambda = \nu, \\
\bar{j}, & \lambda, \nu \in C_i, i = 1, 2, \\
-j, & \lambda \in C_i, \nu \notin C_i, i = 1, 2.
\end{pmatrix}
\]

Afterwards, the calculation of the optimal weights proceeds along the same lines as in the last section, carefully keeping track of the signs of the terms.
5.7 Proof of Theorem 17

We show the claim regarding the upper bound

\[ a \leq \frac{1}{M - 1}. \]  

(27)

The upper bounds given in (16) and (17) can be consolidated into

\[ j_0 < 1 \quad \text{and} \quad j < \frac{j_0}{M - 1} \wedge (1 - j_0). \]

We set

\[ z := \frac{j}{1 - j_0 + (M - 2)j_0}, \]

and show that \( z \leq \frac{1}{M - 1} \) holds. Note that for any \( s, t > 0 \), the function \( f_{s,t}(x) := \frac{x}{s+tx} \) is strictly increasing.

We distinguish the cases where the binding upper bound is

1. \( \frac{j_0}{M - 1} < 1 - j_0 \). This inequality is equivalent to \( j_0 < \frac{M - 1}{M} \), which itself is equivalent to the first inequality below:

\[
z \leq \frac{j_0}{1 - j_0 + \frac{(M - 2)j_0}{M - 1}} = \frac{j_0}{M - 1 - j_0} \leq \frac{M - 1}{M - 1 + \frac{M - 1}{M}} = \frac{1}{M - 1}.
\]

2. \( \frac{j_0}{M - 1} \geq 1 - j_0 \). This inequality is equivalent to \( j_0 \geq \frac{M - 1}{M} \), which itself is equivalent to the first inequality below:

\[
z \leq \frac{1 - j_0}{1 - j_0 + (M - 2)(1 - j_0)} = \frac{1}{M - 1}.
\]

Now the claim (27) follows from the inequalities

\[ 0 \leq \sin \left( \frac{\pi}{2} x \right) - x \leq \sin \left( \frac{\pi}{2} \frac{1}{M - 1} \right) - x \]

for all \( x \leq \frac{1}{M - 1} \), which follow from Lemma 33. The first inequality is strict if \( x > 0 \).

As a direct consequence, we see that the coefficient \( D_1' \) is positive and \( D_2' \geq 0 \) holds.

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