Numerical invariant associated with manifold

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Abstract: Let $M$ be a Riemannian manifold of $n$ dimension with the coordinate $(x^1, \ldots, x^n)$. The distance on $M$ are given by first fundamental metrical tensor $I = g_{ij}dx^i dx^j$, where $g_{ij}$ will be assume to be analytic function of $x^1, \ldots, x^n$ and let the distance element in this space be given by second fundamental quadratic form $II = \Omega_{ij} dx^i dx^j$, where $\Omega_{ij}$ will be assume to be analytic function of $x^1, \ldots, x^n$. In 1929, W.V.D.Hodge introduced the theory of harmonic integral. By using the theory of harmonic integral, he gave the topological definition of geometric genus $P_g$ of a surface. But we have observed that in the theory of harmonic integral, there is no place for second fundamental form of a surface. This motivates us to introduce the new type of differential form by using second fundamental metrical tensor. In this paper, we have introduced the RP-harmonic integral, Modified RP-harmonic integral and Generalized harmonic integral. By using the period matrix corresponding to the RP-harmonic integral, Modified RP-harmonic integral and Generalized harmonic integral, we have studied the numerical invariant of a manifold $M$. As analogous to geometric genus of a surface, we have defined invariant of a surface, we called as RP-geometric genus $P_{rp}$ and Generalized geometric genus $P_{gh}$.

Keywords: RP-harmonic integral, Modified RP-harmonic integral and Generalized harmonic integral.

1. Introduction
Let $M$ be a Riemannian manifold of $n$ dimension with the coordinate $(x^1, \ldots, x^n)$. The distance on $M$ are given by first fundamental metrical tensor

$$I = g_{ij}dx^i dx^j$$

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, where $g_{ij}$ will be assume to be analytic function of $x^1, \ldots, x^n$ and let the distance element in this space be given by second fundamental quadratic form

$$II = \Omega_{ij} dx^i dx^j$$

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, where $\Omega_{ij}$ will be assume to be analytic function of $x^1, \ldots, x^n$.

Definition 1.1. The properties of manifold $M$ which is invariant under homeomorphism is called as topological invariant of a manifold $M$.

Definition 1.2. A numerical invariant of a manifold $M$ is a number associated with $M$ which has the same value for any manifold homeomorphic to $M$.

From the definition 1.2, numerical invariant of a manifold $M$ is also a topological property of $M$. A complete set of invariant determines a manifold upto homeomorphism. Many mathematician studied the
topological properties of a manifold. But till date no decent complete set of invariants for a manifold is known.

In [9], Oswald Veblen studied the topological properties of a manifold by using simplicial complex. In [6] De Rham studied the topological properties of a manifold by using the theory of integration. In the investigation of the properties of a manifold by using the theory of integral, certain topological invariants such as the torsion group of a manifold can’t be explain by using the theory of integration on a manifold. To study the certain topological invariant of a manifold, W.V.D.Hodge introduced the theory of harmonic integral by using first fundamental metrical tensor [2]. The role of second fundamental metrical tensor is absent in the theory of harmonic integral [7,10,11]. This statement motivate us to introduced the generalised theory of harmonic integral which consider first fundamental metrical tensor and second fundamental metrical tensor.

The paper is organised as follows: In section 2, we have studied the properties of cycle on a manifold. In section 3, we have studied the properties of differential form on a manifold M. In section 4, we have studied the period of integral on a manifold M. In section 5, we have studied harmonic integral on a manifold. In section 6, we have studied RP-harmonic integral on a manifold. In section 7, we have studied modified RP-harmonic integral on a manifold. In section 8, we have studied generalised harmonic integral on a manifold. In section 9, we have studied the period matrix of harmonic form, RP-harmonic form, Generalized harmonic form. In section 10, we have studied the numerical invariant of a manifold.

Remark 1.3. We have assumed that M be a manifold of dimension n with metric given by equation 1 and equation 2.

Remark 1.4. R denote the set of real numbers, C denote the set of complex numbers, Q denote the set of rational numbers, Z denote the set of integers, N denotes set of natural numbers.

2. Cycle on a manifold
Let \( \xi_r = \{(x_1, \ldots, x_r) : x_i \in R \} \) be the Euclidean space of r dimension. Let \( P_0, \ldots, P_p \) are \( p + 1 \) independent points in \( \xi_r \) such that \( P_k = (x_1^k, \ldots, x_r^k) \). The rectilinear p-simplex or p-simplex or a rectilinear simplex of p dimensions is denoted by \( (P_0 \cdots P_p) \) and it is defined as follows:

\[
(P_0 \cdots P_p) = \{(x_1, \ldots, x_r) : x_i = \sum_{k=0}^{p} \lambda_k x_i^k, \text{where} \lambda_k > 0 \text{and} \sum_{k=0}^{p} \lambda_k = 1 \}
\]

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The closure of rectilinear simplex of p-dimensions \( (P_0 \cdots P_p) \) is denoted by \( [P_0 \cdots P_p] \) and it is defined as follows:

\[
[P_0 \cdots P_p] = \{(x_1, \ldots, x_r) : x_i = \sum_{k=0}^{p} \lambda_k x_i^k, \text{where} \lambda_k \geq 0 \text{and} \sum_{k=0}^{p} \lambda_k = 1 \}
\]

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The boundary of the p-simplex \( (P_0 \cdots P_p) \) is the difference between \( (P_0 \cdots P_p) \) and \( [P_0 \cdots P_p] \). Therefore, the boundary of p-simplex consists of \( \binom{p+1}{k+1} \) simplexes of k dimension, where \( k = 0, \ldots, p - 1 \). The polyhedral complex of n dimension is denoted by \( K_n \) and it is a finite collection of rectilinear simplexes in \( \xi_r \) of dimension 0, \ldots, n such that (i) no two simplexes of the set have a point in common; (ii) every simplex lying on the boundary of a simplex of the set belongs to the set.

Remark 2.1. (i) We can denote the any p-simplex \( (P_0 \cdots P_p) \) by \( (P_{i_0} \cdots P_{i_p}) \), where \( (i_0, \cdots, i_p) \) is any derangement of the number \( (0, \cdots, p) \).

(ii) There are two classes of derangements which are obtain by even permutation of \( (0, \cdots, p) \) and odd permutation of \( (0, \cdots, p) \).
(iii) An oriented simplex of \((P_0 \cdots P_p)\) is denoted by \(P_{i_0} \cdots P_{i_p}\) and it is obtained by associating one of the class of derangements of the suffixes to \((P_{i_0} \cdots P_{i_p})\) and for the other class, we denote it by \(-P_{i_0} \cdots P_{i_p}\).

(iv) If we orient all the simplexes of a complex in an arbitrary manner, we get an oriented simplex.

Let for each \(p = 0, \cdots, n\), \(E^i_p\) are an oriented simplex of polyhedral complex \(K_n\), where \(i = 1, \cdots, \alpha_p\).

Let \(G^n_p(Z) = \{C_p = \sum_{i=1}^{\alpha_p} a_i E^i_p; a_i \in Z\}\), where \(p = 0, \cdots, n\). Then \(G^n_p(Z)\) form an additive group with a free generator \(E^i_p\), where \(i = 1, \cdots, \alpha_p\). An element \(C_p = \sum_{i=1}^{\alpha_p} a_i E^i_p\) is called as integral \(p\)-chain of complex \(K_n\). Let \(G^n_p(R) = \{C_p = \sum_{i=1}^{\alpha_p} a_i E^i_p; a_i \in R\}\), where \(p = 0, \cdots, n\). Then \(G^n_p(R)\) form an additive group with a free generator \(E^i_p\), where \(i = 1, \cdots, \alpha_p\). An element \(C_p = \sum_{i=1}^{\alpha_p} a_i E^i_p\) is called as real \(p\)-chain of complex \(K_n\).

Let \(G^n_p(F) = \{C_p = \sum_{i=1}^{\alpha_p} a_i E^i_p; a_i \in F\}\), where \(p = 0, \cdots, n\). Then \(G^n_p(F)\) form an additive group with a free generator \(E^i_p\), where \(i = 1, \cdots, \alpha_p\). An element \(C_p = \sum_{i=1}^{\alpha_p} a_i E^i_p\) is called as \(p\)-chain of complex \(K_n\) over a field \(F\).

The boundary of integral \(p\)-chain \(E^i_p = P_{i_0} \cdots P_{i_p}\) is the mapping \(F: G^n_p(Z) \to G^{n-1}_p(Z)\) which is defined as follows: \(F(E^i_p) = C_{p-1} = \sum_{j=0}^{\beta_p} (-1)^j P_{i_0} \cdots P_{i_{j-1}} P_{i_{j+1}} \cdots P_{i_p}\). The boundary of real \(p\)-chain \(E^i_p = P_{i_0} \cdots P_{i_p}\) is the mapping \(F: G^n_p(R) \to G^{n-1}_p(R)\) which is defined as follows: \(F(E^i_p) = C_{p-1} = \sum_{j=0}^{\beta_p} (-1)^j P_{i_0} \cdots P_{i_{j-1}} P_{i_{j+1}} \cdots P_{i_p}\). The boundary of \(p\)-chain \(E^i_p = P_{i_0} \cdots P_{i_p}\) over a field \(F\) is the mapping \(F: G^n_p(F) \to G^{n-1}_p(F)\) which is defined as follows: \(F(E^i_p) = C_{p-1} = \sum_{j=0}^{\beta_p} (-1)^j P_{i_0} \cdots P_{i_{j-1}} P_{i_{j+1}} \cdots P_{i_p}\).

**Remark 2.2.** The boundary of \(-P_{i_0} \cdots P_{i_p}\) is denoted by \(-C_{p-1}\).

**Definition 2.3.** An integral \(P\)-chain whose boundary is zero is called as integral \(p\)-cycle.

**Definition 2.4.** A real \(P\)-chain whose boundary is zero is called as real \(p\)-cycle.

**Definition 2.5.** A \(P\)-chain on a field \(F\) whose boundary is zero is called as \(p\)-cycle on \(F\).

We know that every \(p\)-cycle is the boundary of \((p + 1)\)-chain not generally true. One can classify the cycle into two parts. If \(C_p\) be a cycle which is a boundary of \(C_{p+1}\) chain then \(C_p\) is called as bounding cycles and we say that \(C_p\) is homologous to zero and we denote it as \(C_p \sim 0\).

If \(C_p\) and \(C_p'\) are two \(p\)-chains such that \(C_p - C_p'\) is bounding cycle then \(C_p \sim C_p' \sim 0\). We denote it by \(C_p \sim C_p'\) and we call it as \(C_p\) is homologous to \(C_p'\).

### 2.1. Betti Number and torsion coefficients

Let \(G^n_p(Z)\) is an additive group of integral \(p\)-chain of \(K_n\). Let \(G^n_p(Z)\) is a subgroup of \(G^n_p(Z)\) of bounding \(p\)-cycle [3]. Then \(H_p = \frac{G^n_p(Z)}{G^n_p(Z)}\) form a group which is generated by \(R_p\) free generators. The group \(H_p\) is direct sum of \(H^p_{\theta}\) and \(H^p_{\theta'}\), where \(H^p_{\theta}\) is infinite abelian group with \(R_p\) free generator and \(H^p_{\theta'}\) is a finite abelian group generated by \(\theta_p\) generator of the orders \(e_{\theta_1}, \cdots, e_{\theta_p}\). The group \(H^p_{\theta'}\) is called as betti group of \(K_n\) and the group \(H^p_{\theta'}\) is called as the \(p^{th}\) torsion group of \(K_n\) and \(e_{\theta_1}, \cdots, e_{\theta_p}\) are called as the \(p^{th}\) torsion coefficient of \(K_n\).

**Theorem 2.6.(Covering theorem)[3]** If \(M\) be any manifold of class \(u\), there exist a complex \(K_n\) of class \(v\), for any given \(v (0 \leq v \leq u)\), with the property that every point of \(M\) lies on one and only one simplex of \(K\), and every simplex of \(K\) lies on \(M\).
Hence to study the topological invariants of a manifold $M$, complexes play an important role. Topological property of a manifold play an important role in the theory of multiple integral.

**Definition 2.7.** Let $M$ be a manifold. Let $B = \{ \Gamma_p^i : i = 1 \ldots R_p \}$ be a collection of $p$-cycle on a manifold $M$. If $\Gamma_p \not\in B$ be any $p$-cycle on $M$ such that $\Gamma_p$ is related to element of $B$ by a homology of division, of the form

$$\Gamma_p \approx \sum_{i=1}^{R_p} a_i \Gamma_p^i$$

where the coefficients $a_i$ are integers. If $\Gamma_p^j \in B$ be any $p$-cycle on $M$ such that $\Gamma_p^j$ is not related to element of $B - \{ \Gamma_p^j \}$ by a homology of division, of the form $\Gamma_p^j \not\approx \sum_{i=1, i \neq j}^{R_p} a_i \Gamma_p^i$, where the coefficients $a_i$ are integers. Then the set $B$ is called as fundamental base for the $p$-cycle of $M$ over a set of integers $Z$.

**Lemma 2.8** Let $M$ be a manifold. Let $B = \{ \Delta_p^i : i = 1, \ldots, R_p \}$ be a basis of $M$. Let $B_1 = \{ \Delta_p^i : i = 1, \ldots, R_p \}$ be a collection of $p$-cycle on a manifold $M$. If $\Delta_p \approx \sum_{i=1}^{R_p} a_{ij} \Gamma_p^i$ where $[a_{ij}]$ is a unimodular matrix of integer, then $B_1$ form a basis.

**Definition 2.7.** Let $M$ be a manifold. Let $B = \{ \Gamma_p^i : i = 1 \ldots R_p \}$ be a collection of $p$-cycle on a manifold $M$. If $\Gamma_p \not\in B$ be any $p$-cycle on $M$ such that $\Gamma_p$ is related to element of $B$ by a homology of division, of the form

$$\Gamma_p \approx \sum_{i=1}^{R_p} a_i \Gamma_p^i$$

where the coefficients $a_i \in F$. If $\Gamma_p^j \in B$ be any $p$-cycle on $M$ such that $\Gamma_p^j$ is not related to element of $B - \{ \Gamma_p^j \}$ by a homology of division, of the form $\Gamma_p^j \not\approx \sum_{i=1, i \neq j}^{R_p} a_i \Gamma_p^i$, where the coefficients $a_i$ are integers. Then the set $B$ is called as fundamental base for the $p$-cycle of $M$ over a field $F$.

3. **Differential form on a manifold.**

In this section, we have considered the basic theory of differential form in order to make this paper complete. For more detail (see [3]). If $A = \frac{1}{p!} A_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p}$ be any $p$-form on a manifold of $n$ dimensions then $p + 1$ form of $A$ is denoted by $A_x$ and it is called as the derived form or the exterior derivative of $A$ and it is obtain by using Stokes theorem which is expressed by using following equation:

$$\int_{C_p = \partial(C_{p+1})} A = \int_{C_{p+1}} A_x,$$

Where $C_p = F(C_{p+1})$ is the boundary of $C_{p+1}$ on $M$.

A $P$-form $A$ is called as closed form on $M$ if and only if $A_x = 0$.

**Remark 3.1.** We denote $A_x = 0$ by $A \rightarrow 0$.

Form which are the exterior derivatives of other forms are called null forms.

**Definition 3.2.** Let $M$ be a manifold. Let $B = \{ \psi_p^i : i = 1 \ldots R_p \}$ be a collection of closed $p$-form on a manifold $M$. If $\psi_p^i \not\in B$ be any closed $p$-form on $M$ such that $\psi_p^i \approx \sum_{i=1}^{R_p} a_i \psi_p^i$ where the coefficients $a_i$ are integers. If $\psi_p^j \in B$ be any closed $p$-form on $M$ such that $\psi_p^j$ is not related to element of $B - \{ \psi_p^j \}$ by the form $\psi_p^j \not\approx \sum_{i=1, i \neq j}^{R_p} a_i \psi_p^i$, where the coefficients $a_i$ are integers. Then the set $B$ is called as fundamental base for closed $p$-form on $M$ over a set of integer $Z$. 

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4. Period of integral

From Stokes theorem, if $A_x = 0$ then the value of $\int A$ over any bounding cycle is zero. Let $B = \{ \Gamma^i_p : i = 1 \cdots R_p \}$ be a basis for $p$-cycle on $M$. Let $\Gamma_p$ be any $p$-cycle of $M$ such that $\Gamma_p \approx \sum_{i=1}^{R_p} \lambda_i \Gamma^i_p$, $\lambda_i \in \mathbb{Z}$. Then for a suitable $\lambda \neq 0$, there exists a $(p + 1)$-chain $C_{p+1}$ such that $C_{p+1} \to \lambda \Gamma_p - \lambda \sum_{i=1}^{R_p} \lambda_i \Gamma^i_p$. Then by Stokes theorem, we have $\int_{\Gamma_p} A = \sum_{i=1}^{R_p} \lambda_i \int_{\Gamma^i_p} A$.

**Definition 4.1.** Let $B = \{ \Gamma^i_p : i = 1 \cdots R_p \}$ be a basis of $p$-cycle on $M$ and let $A$ be a $P$-form such that $\int_{\Gamma^i_p} A = \omega^i$, where $\omega^i$ is a certain constant. Let $\Gamma_p$ be any $p$-cycle of $M$ such that $\Gamma_p \approx \sum_{i=1}^{R_p} \lambda_i \Gamma^i_p$, where $\lambda_i \in \mathbb{Z}$. Then the period of integral on the cycle $\Gamma_p$ is defined as

$$\int_{\Gamma_p} A = \lambda_i \omega^i$$

If the $p$-cycle $\Gamma_p$ is a bounding cycle then period of the integral of $A$ over $\Gamma_p$ is zero.

**Definition 4.2.** If $A^p_1, \cdots, A^p_2$ are the number of closed $p$-form and let $B = \{ \Gamma_p^i : i = 1 \cdots R_p \}$ be a basis of $p$-cycle on $M$ then period matrix is defined by $\omega = \begin{bmatrix} \omega_{ij} \end{bmatrix}$, where $\omega_{ij} = \int_{\Gamma^i_p} A^p_j$.

**Theorem 4.3.** If $B = \{ \Gamma_p^i : i = 1 \cdots R_p \}$ be a basis of $p$-cycle on $M$ and $\omega^i$ where $i = 1, \cdots, R_p$ are arbitrary real numbers, there exists a $P$-form $\phi$ with the properties: (i) $\phi$ is regular and closed on $M$; and (ii) $\int_{\Gamma_p^i} A = \omega^i$.

**Theorem 4.4.** If $\phi$ is a closed form on a manifold $M$ whose integral has all its periods equal to zero then $\phi$ is a null form.

**Theorem 4.5.** A set of closed $p$-form $\psi^i_p$, where $i = 1, \cdots, R_p$ is a base for the $p$-form on $M$ if and only if there exists a set of closed $(n-p)$-forms $\psi_{n-p}^i$ where $i = 1, \cdots, R_{n-p}$ such that the matrix $\left\| \int_M \psi^i_p \psi_{n-p}^j \right\|$ is nonsingular.

5. Harmonic integral on a manifold

From the theorem 4.4, one can observe that $B^{th}$ betti number is equal to maximum number of closed form which are linearly independent. Hence by using theorem 4.3 and 4.4, one can express the topology of a manifold $M$ in terms of properties of integral on manifold $M$.

In 1929, W.V.D hodge, observed that certain topological invariants such as torsion group of a manifold have no place in the theory of integral [3]. To study the topological properties of a manifold, in 1929, W.V.D. Hodge introduce a new theory of integral by using the first fundamental metrical tensor.

**Definition 5.1.** Let $M$ be a Riemannian manifold of $n$ dimension with the coordinate $(x^1, \cdots, x^n)$. The distance on $M$ are given by first fundamental metrical tensor

$$I = g_{ij}dx^i dx^j$$

Where $g_{ij}$ will be assume to be analytic function of $x^1, \cdots, x^n$ and let the distance element in this space be given by second fundamental quadratic form

$$II = \Omega_{ij}dx^i dx^j$$

Where $\Omega_{ij}$ will be assume to be analytic function of $x^1, \cdots, x^n$. Let $P = \frac{1}{p!} p_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_p}$ be a $p$-form in $M$. Let $P^*_{i_1 \cdots i_{n-p}} = \frac{1}{p!} \sqrt{g} g^{i_1 k_1} \cdots g^{i_p k_p} p_{k_1 \cdots k_p} e_{j_1} \cdots e_{j_{n-p}}$. The form
\[ P^* = \frac{1}{(n-p)!} P_{i_1 \cdots i_{n-p}}^* dx^{i_1} \cdots dx^{i_{n-p}} \]

is determined uniquely by \( P \) and the metrical tensor \( g_{ij} \). Then form \( P^* \) is called the dual of the form \( P \).

**Definition 5.2.** Let \( M \) be a Riemannian manifold. A \( p \)-form \( P \) is called as harmonic form on a manifold \( M \) if it satisfy the following properties: (i) \( P \) is regular on \( M \); (ii) \( P \to 0 \) and \( P^* \to 0 \).

**Definition 5.3.** The integral of harmonic form is called as harmonic integral.

**Definition 5.4.** The tensor defined by the coefficient of a harmonic form is called as harmonic tensor.

### 6. RP-Harmonic integral on a manifold

From definition 5.2, we can observed that harmonic form on a manifold \( M \) considered only first fundamental metrical tensor. There is no place given to second fundamental metrical tensor. To study the numerical invariant of a manifold, we have introduced new type of differential form, we called it as RP-harmonic form.

**Definition 6.1.** Let \( M \) be a Riemannian manifold of \( n \) dimension with the coordinate \((x^1, \cdots, x^n)\). The distance on \( M \) are given by first fundamental metrical tensor

\[ I = g_{ij} dx^i dx^j \]

, where \( g_{ij} \) will be assume to be analytic function of \( x^1, \cdots, x^n \) and let the distance element in this space be given by second fundamental quadratic form

\[ II = \Omega_{ij} dx^i dx^j \]

, where \( \Omega_{ij} \) will be assume to be analytic function of \( x^1, \cdots, x^n \). Let \( P = \frac{1}{p!} P_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_p} \) be a \( p \)-form in \( M \). Let \( P_{i_1 \cdots i_{n-p}}^* = \frac{1}{p!} \sqrt{\Omega} \Omega^{j_1 \cdots j_p} P_{k_1 \cdots k_p} e^{i_1 \cdots i_p j_1 \cdots j_{n-p}} \). The form

\[ P^* = \frac{1}{(n-p)!} P_{i_1 \cdots i_{n-p}}^* dx^{i_1} \cdots dx^{i_{n-p}} \]

is determined uniquely by \( P \) and the metrical tensor \( \Omega_{ij} \). Then form \( P^* \) is called the dual of the form \( P \).

**Definition 6.2.** Let \( M \) be a Riemannian manifold. A \( p \)-form \( P \) is called as RP-harmonic form on a manifold \( M \) if it satisfy the following properties: (i) \( P \) is regular on \( M \); (ii) \( P \to 0 \) and \( P^* \to 0 \).

**Definition 6.3.** The integral of RP-harmonic form is called as RP-harmonic integral.

**Definition 6.4.** The tensor defined by the coefficient of a RP-harmonic form is called as RP-harmonic tensor.

### 7. Modified RP-Harmonic integral on a manifold

**Definition 7.1.** Let \( M \) be a Riemannian manifold of \( n \) dimension with the coordinate \((x^1, \cdots, x^n)\). The distance on \( M \) are given by first fundamental metrical tensor

\[ I = g_{ij} dx^i dx^j \]

, where \( g_{ij} \) will be assume to be analytic function of \( x^1, \cdots, x^n \) and let the distance element in this space be given by second fundamental quadratic form
\[ II = \Omega_{ij} dx^i dx^j \]

where \( \Omega_{ij} \) will be assume to be analytic function of \( x^1, \cdots, x^n \). Let \( P = \frac{1}{p!} P_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_p} \) be a \( p \)-form in \( M \).

**Definition 7.2.** Let \( M \) be a Riemannian manifold. A \( p \)-form \( P \) is called as Modified RP-harmonic form on a manifold \( M \) if it satisfy the following properties: (i) \( P \) is harmonic form on \( M \); (ii) \( P \) is RP-harmonic form on \( M \).

**Definition 7.3.** The integral of Modified RP-harmonic form is called as Modified RP-harmonic integral.

**Definition 7.4.** The tensor defined by the coefficient of a Modified RP-harmonic form is called as Modified RP-harmonic tensor.

### 8. Generalized Harmonic integral on a manifold

From section 6 and section 7, we can observed that harmonic form on a manifold \( M \) considered only first fundamental metrical tensor and RP-harmonic form on a manifold \( M \) considered only second fundamental metrical tensor. There is no place given to any tensor \( R_{ij} \) associated with a manifold \( M \).

To study the numerical invariant of a manifold, we have introduced new type of differential form, we called it as Generalized Harmonic integral on a manifold.

**Definition 8.1.** Let \( M \) be a Riemannian manifold of \( n \) dimension with the coordinate \((x^1, \cdots, x^n)\). The distance on \( M \) are given by first fundamental metrical tensor

\[ l = g_{ij} dx^i dx^j \]

where \( g_{ij} \) will be assume to be analytic function of \( x^1, \cdots, x^n \) and let the distance element in this space be given by second fundamental quadratic form

\[ II = \Omega_{ij} dx^i dx^j \]

where \( \Omega_{ij} \) will be assume to be analytic function of \( x^1, \cdots, x^n \). Let \( R_{ij} \) be a tensor associated with \( M \). Let \( P = \frac{1}{p!} P_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_p} \) be a \( p \)-form in \( M \). Let \( P^*_{i_1 \cdots i_p} = \frac{1}{p!} \sqrt{\text{det}R} \epsilon^{i_1 k_1 \cdots i_p k_p} R^{j k_1 \cdots k_p} P_{j i_1 \cdots i_p} \)

The form \( (12) \) \( P^* = \frac{1}{(n-p)!} P^*_{i_1 \cdots i_{n-p}} dx^{i_1} \cdots dx^{i_{n-p}} \)

is determined uniquely by \( P \) and the metrical tensor \( R_{ij} \). Then form \( P^* \) is called the dual of the form \( P \).

**Definition 8.2.** Let \( M \) be a Riemannian manifold. A \( p \)-form \( P \) is called as generalized harmonic form on a manifold \( M \) if it satisfy the following properties: (i) \( P \) is regular on \( M \); (ii) \( P \rightarrow 0 \) and \( P^* \rightarrow 0 \).

**Definition 8.3.** The integral of generalized harmonic form is called as generalized harmonic integral.

**Definition 8.4.** The tensor defined by the coefficient of a generalized harmonic form is called as generalized harmonic tensor.

### 9. Period matrix of harmonic form, RP-harmonic form and Generalized harmonic form

**Definition 9.1.** If \( A^i_p, \cdots, A^n_p \) are the harmonic \( p \)-form and \( B = \{T^i_p; i = 1, \cdots, R_p\} \) be a basis of \( p \)-cycle on \( M \) then the period matrix is denoted by \( W_H = [\omega_{ij}] \), where \( \omega_{ij} = \int_{T^i_p} A^j_p \).
Definition 9.2. If $A^i_p, \cdots, A^s_p$ are the RP-harmonic $p$-form and $B = \{ \Gamma^i_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M$ then the period matrix is denoted by $W_{RP} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$.

Definition 9.3. If $A^i_p, \cdots, A^s_p$ are the Generalized-harmonic $p$-form and $B = \{ \Gamma^i_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M$ then the period matrix is denoted by $W_{GH} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$.

Lemma 9.1. If $A^i_p, \cdots, A^s_p$ are the harmonic $p$-form and $B = \{ \Gamma^i_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M$ then the rank of period matrix $W_H = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$ is a numerical invariant of a manifold.

Proof: If $M_1$ and $M_2$ are two manifold such that $M_1$ is homeomorphic to $M_2$. Let $A^i_p, \cdots, A^s_p$ are the harmonic $p$-form and $B = \{ \Gamma^i_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M_1$ with a period matrix $W_H = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$. Let $C^i_p, \cdots, C^s_p$ are the harmonic $p$-form and $B = \{ \Gamma^{1i}_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M_2$ with a period matrix $W_H^{2} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^{1i}_p} C^j_p$. By using the definition of homeomorphism, we must have rank of period matrix $W_H = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$ is a numerical invariant of a manifold.

Lemma 9.2. If $A^i_p, \cdots, A^s_p$ are the RP-harmonic $p$-form and $B = \{ \Gamma^i_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M$ then the rank of period matrix $W_{RP} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$ is a numerical invariant of a manifold.

Proof: If $M_1$ and $M_2$ are two manifold such that $M_1$ is homeomorphic to $M_2$. Let $A^i_p, \cdots, A^s_p$ are the RP-harmonic $P$-form and $B = \{ \Gamma^i_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M_1$ with a period matrix $W_{RP} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$. Let $C^i_p, \cdots, C^s_p$ are the RP-harmonic $p$-form and $B = \{ \Gamma^{1i}_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M_2$ with a period matrix $W_{RP}^{2} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^{1i}_p} C^j_p$. By using the definition of homeomorphism, we must have rank of period matrix $W_{RP} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$ is a numerical invariant of a manifold.

Lemma 9.3. If $A^i_p, \cdots, A^s_p$ are the Generalized-harmonic $p$-form and $B = \{ \Gamma^i_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M$ then the rank of period matrix $W_{GH} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$ is a numerical invariant of a manifold.

Proof: If $M_1$ and $M_2$ are two manifold such that $M_1$ is homeomorphic to $M_2$. Let $A^i_p, \cdots, A^s_p$ are the Generalized-harmonic $P$-form and $B = \{ \Gamma^i_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M_1$ with a period matrix $W_{GH} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$. Let $C^i_p, \cdots, C^s_p$ are the Generalized-harmonic $p$-form and $B = \{ \Gamma^{1i}_p: i = 1, \cdots, R_p \}$ be a basis of $p$-cycle on $M_2$ with a period matrix $W_{GH}^{2} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^{1i}_p} C^j_p$. By using the definition of homeomorphism, we must have rank of period matrix $W_{GH} = [\omega_{ij}]$, where $\omega_{ij} = \int_{\Gamma^i_p} A^j_p$ is a numerical invariant of a manifold.

10. Geometric genus of a surface as a topological invariant by using harmonic integral

Let $\Gamma_1^1 \cdots \Gamma_p$ be a fundamental base for the $p$-cycle of the Manifold $M$. Let $B = [b_{ij}]$, where $b_{ij} = (\Gamma_i \cdot \Gamma_j)$. Let $A$ be the inverse of the matrix $B$. Let $\phi_1, \cdots, \phi_R$ are harmonic form on $M$. Let $\phi^1_i$ is a dual form of $\phi_i$. Assume that the harmonic form satisfies the relation $\oint_{\Gamma_j} \phi_i = \delta^1_i$. Since $\phi^1_i$ is the dual form
of the harmonic form \( \phi_i \), by using the definition of harmonic form we have \( \phi_i' = \sum_{j=1}^{R_p} c_{ij} \phi_j \), where \( c_{ij} \) depends on first fundamental metrical tensor.

If \( \phi \) and \( \psi \) are the harmonic form on \( M \) such that \( \int_{\Gamma_i} \phi = \omega_i \) and \( \int_{\Gamma_i} \psi = v_i \), where \( w_i, v_i \) are the period of the integral \( \phi \) and \( \psi \) on \( p \)-cycle \( \Gamma_1 \cdots \Gamma_{R_p} \) of the manifold then \( \int_M \phi \psi = \sum_{i,j=1}^{R_p} A_{ij} w_i v_i \).

Hence, \( \sum_{k=1}^{R_p} c_{ik} A_{kj} = \int_M \phi_j \phi_i = \int_M \phi_i \phi_j' = \sum_{k=1}^{R_p} c_{jk} A_{ki} \). Therefore, we get \( cA = A'c \), where \( c' \) is a transpose of \( c \).

We have assumed that \( \int_{\Gamma_i} \phi_i = \delta_i^j \), therefore we get \( \int_M \left( \sum_{i=1}^{R_p} \lambda_i \phi_i \sum_{i=1}^{R_p} \lambda_i \phi_i' \right) \) is a positive for all values of \( \lambda_i \).

From above we can conclude that \( cA \) is a symmetric matrix which must have positive definite quadratic form.

Therefore \( cA \) is diagonalizable. Let \( c = \lambda^{-1} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda \), where \( \lambda \) is a real number and \( \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \) is a diagonal matrix with \( \alpha = R_p - 2P_g - 1 \) positive elements and \( \beta = 2P_g + 1 \) negative elements. Then

\[
\lambda^{-1} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda A = A\lambda' \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda^{-1}
\]

Hence \( A\lambda' \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda \) is of the form \( \begin{bmatrix} P & \cdots & P \\ \vdots & \ddots & \vdots \\ P & \cdots & P \end{bmatrix} \) where \( P \) and \( Q \) are symmetric matrices of orders \( \alpha \) and \( \beta \) respectively. Here \( \begin{bmatrix} P & \cdots & P \\ \vdots & \ddots & \vdots \\ P & \cdots & P \end{bmatrix} \) is a positive definite form. Therefore \( A \) is transformed to a diagonal matrix \( \mu A\mu' \), it has \( \beta \) negative terms. The number \( \beta \) is an invariant of the matrix \( A \). Since \( \beta = 2P_g + 1 \).

Hence \( P_g \) is a topological invariant of a manifold.

**11. Topological invariant by using RP-harmonic integral**

Let \( \Gamma_1 \cdots \Gamma_{R_p} \) be a fundamental base for the \( p \)-cycle of the Manifold \( M \). Let \( B = [b_{ij}] \), where \( b_{ij} = (\Gamma_i \cdot \Gamma_j) \). Let \( A \) be the inverse of the matrix \( B \). Let \( \phi_1, \cdots, \phi_{R_p} \) are RP-harmonic form on \( M \). Let \( \phi_i' \) is a dual form of \( \phi_i \). Assume that the RP-harmonic form satisfies the relation \( \int_{\Gamma_i} \phi_i = \delta_i^j \). Since \( \phi_i' \) is the dual form of the harmonic form \( \phi_i \), by using the definition of RP-harmonic form we have \( \phi_i' = \sum_{j=1}^{R_p} c_{ij} \phi_j \), where \( c_{ij} \) depends on first fundamental metrical tensor.

If \( \phi \) and \( \psi \) are the RP-harmonic form on \( M \) such that \( \int_{\Gamma_i} \phi = \omega_i \) and \( \int_{\Gamma_i} \psi = v_i \), where \( w_i, v_i \) are the period of the integral \( \phi \) and \( \psi \) on \( p \)-cycle \( \Gamma_1 \cdots \Gamma_{R_p} \) of the manifold then \( \int_M \phi \psi = \sum_{i,j=1}^{R_p} A_{ij} w_i v_i \).

Hence, \( \sum_{k=1}^{R_p} c_{ik} A_{kj} = \int_M \phi_j \phi_i = \int_M \phi_i \phi_j' = \sum_{k=1}^{R_p} c_{jk} A_{ki} \). Therefore, we get \( cA = A'c \), where \( c' \) is a transpose of \( c \).

We have assumed that \( \int_{\Gamma_i} \phi_i = \delta_i^j \), therefore we get \( \int_M \left( \sum_{i=1}^{R_p} \lambda_i \phi_i \sum_{i=1}^{R_p} \lambda_i \phi_i' \right) \) is a positive for all values of \( \lambda_i \).

From above we can conclude that \( cA \) is a symmetric matrix which must have positive definite quadratic form.
Therefore $cA$ is diagonalizable. Let $c = \lambda^{-1} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda$, wher $\lambda$ is a real number and $\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ is a diagonal matrix with $\alpha = R_p - 2P_{rp} - 1$ positive elements and $\beta = 2P_{rp} + 1$ negative elements. Then

$$\lambda^{-1} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda A = A \lambda' \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda^{-1}$$

Hence $A \lambda \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda'$ is of the form $\begin{bmatrix} P & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q \end{bmatrix}$ where $p$ and $q$ are symmetric matrices of orders $\alpha$ and $\beta$ respectively. Here $\begin{bmatrix} P & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -q \end{bmatrix}$ is a positive definite form. Therefore $A$ is transformed to a diagonal matrix $\mu A \mu'$, it has $\beta$ negative terms. The number $\beta$ is an invariant of the matrix $A$. Since $\beta = 2P_{rp} + 1$.

Hence $P_{rp}$ is a topological invariant of a manifold.

12. Topological invariant by using Generalized harmonic integral

Let $\Gamma_1, \ldots, \Gamma_{R_p}$ be a fundamental base for the $p$-cycle of the Manifold $M$. Let $B = [b_{ij}]$, where $b_{ij} = (\Gamma_i \cdot \Gamma_j)$. Let $A$ be the inverse of the matrix $B$. Let $\phi_1, \ldots, \phi_{R_p}$ are Generalized harmonic form on $M$. Let $\phi'_i$ is the dual form of $\phi_i$. Assume that the generalized harmonic form satisfies the relation $\int_{\Gamma_i} \phi_i = \delta_i^j$.

Since $\phi'_i$ is the dual form of the generalised harmonic fom $\phi_i$, by using the definition of generalized harmonic form we have $\phi'_i = \sum_{j=1}^{R_p} c_{ij} \phi_j$, where $c_{ij}$ depends on first fundamental metrical tensor $c$.

If $\phi$ and $\psi$ are the generalized harmonic form on $M$ such that $\int_{\Gamma_i} \phi = \omega_i$ and $\int_{\Gamma_i} \psi = v_i$, where $\omega_i, v_i$ are the period of the integral $\phi$ and $\psi$ on $p$-cycle $\Gamma_1, \ldots, \Gamma_{R_p}$ of the manifold then $\int_M \phi \psi = \sum_{i,j=1}^{R_p} A_{ij} w_i v_i$.

Hence, $\sum_{k=1}^{R_p} c_{ik} A_{kj} = \int_M \phi_j \phi'_i = \int_M \phi_i \phi'_j = \sum_{k=1}^{R_p} c_{jk} A_{ki}$. Therefore, we get $cA = A c'$, where $c'$ is a transpose of $c$.

We have assumed that $\int_{\Gamma_i} \phi_i = \delta_i^j$, therefore we get $\int_M \left( \sum_{i=1}^{R_p} \lambda_i \phi_i, \sum_{i=1}^{R_p} \lambda_i \phi'_i \right)$ is a positive for all values of $\lambda_i$.

From above we can conclude that $cA$ is a symmetric matrix which must have positive definite quadratic form.

Therefore $cA$ is diagonalizable. Let $c = \lambda^{-1} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda$, wher $\lambda$ is a real number and $\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ is a diagonal matrix with $\alpha = R_p - 2P_{gh} - 1$ positive elements and $\beta = 2P_{gh} + 1$ negative elements. Then

$$\lambda^{-1} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda A = A \lambda' \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda^{-1}$$

Hence $A \lambda \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \lambda'$ is of the form $\begin{bmatrix} P & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q \end{bmatrix}$ where $p$ and $q$ are symmetric matrices of orders $\alpha$ and $\beta$ respectively. Here $\begin{bmatrix} P & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -q \end{bmatrix}$ is a positive definite form. Therefore $A$ is transformed to a diagonal matrix $\mu A \mu'$, it has $\beta$ negative terms. The number $\beta$ is an invariant of the matrix $A$. Since $\beta = 2P_{gh} + 1$.

Hence $P_{gh}$ is a topological invariant of a manifold.

13. Conclusion

We know that there are no decent set of invariant to classify the manifold. In this paper, we have introduced the RP-harmonic integral, Modified RP-harmonic integral and Generalized harmonic integral. By using the period matrix corresponding to the RP-harmonic integral, Modified RP-harmonic
integral and Generalized harmonic integral, we have studied the numerical invariant of a manifold \(M\). As analogous to geometric genus of a surface, we have defined invariant of a surface, we called as RP geometric genus \(P_{rp}\) and Generalized geometric genus \(P_{gh}\). The question is still open.

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