INTEGRAL EQUIENERGETIC NON-ISOSPECTRAL CAYLEY GRAPHS

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Abstract. We prove that the Cayley graphs $X(G, S)$ and $X^+(G, S)$ are equienergetic for any abelian group $G$ and any symmetric subset $S$. We focus on two subfamilies: unitary Cayley graphs $G_R = X(R, R^*)$, where $R$ is a commutative ring, and semiprimitive generalized Paley graphs $\Gamma(k, q) = X(F_q, \{x^k : x \in F_q^*\})$. We prove that under mild conditions, $\{G_R, G_R^+\}$ and $\{\Gamma(k, q), \Gamma^+(k, q)\}$ are pairs of equienergetic non-isospectral graphs (generically integral, connected and non-bipartite). Then, we obtain conditions such that $\{G_R, \bar{G}_R\}$ and $\{\Gamma(k, q), \bar{\Gamma}(k, q)\}$ are equienergetic non-isospectral graphs. Finally, we characterize all (integral) equienergetic non-isospectral triples $\{G_R, G_R^+, \bar{G}_R\}$ and $\{\Gamma(k, q), \Gamma^+(k, q), \bar{\Gamma}(k, q)\}$ such that all the graphs are also Ramanujan.

1. Introduction

This paper deals with the spectrum and the energy of Cayley graphs and Cayley sum graphs. We mainly focus on two subfamilies: unitary Cayley (sum) graphs over rings and generalized Paley (sum) graphs. Our main goal is to give a general construction of infinite pairs of integral equienergetic non-isospectral graphs with some nice extra properties like being connected, non-bipartite or Ramanujan (or all of them). One of the graphs of the pairs can be taken either with or without loops.

If $\Gamma$ is a graph of $n$ vertices, the eigenvalues of $\Gamma$ are the eigenvalues $\{\lambda_i\}_{i=1}^n$ of its adjacency matrix. The spectrum of $\Gamma$, denoted $\text{Spec}(\Gamma)$, is the set of all the different eigenvalues $\{\lambda_i\}$ of $\Gamma$ counted with their multiplicities $\{e_i\}$. The spectrum is symmetric if for every eigenvalue $\lambda$, its opposite $-\lambda$ is also an eigenvalue with the same multiplicity as $\lambda$. The graph is called integral if $\text{Spec}(\Gamma) \subset \mathbb{Z}$, i.e. if all of its eigenvalues are integers. The energy of $\Gamma$ is defined by $E(\Gamma) = \sum_{i=1}^n |\lambda_i|$. We refer to the books [5] or [6] for a complete viewpoint of spectral theory of graphs, and to [8] for a survey on energy of graphs.

Equienergetic non-isospectral graphs. Let $\Gamma_1$ and $\Gamma_2$ be two graphs with the same number of vertices. The graphs are called isospectral (or cospectral) if $\text{Spec}(\Gamma_1) = \text{Spec}(\Gamma_2)$ and equienergetic if $E(\Gamma_1) = E(\Gamma_2)$. It is clear by the definitions that isospectrality implies equienergeticity, but the converse is false. Thus, we are interested in the construction of

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equienergetic pairs of graphs which are non-isospectral. The smallest such pair is given by the 4-cycle $C_4$ and two copies of $K_2, K_2 \otimes K_2$ of 4-vertices or, if one wants connected graphs, by the 5-cycle $C_5$ and the 5-wheel $W_5 = C_4 + K_1$ (the join of $C_4$ with an edge) of 5-vertices. In fact, we have $\text{Spec}(C_4) = \{0^2, 2^2\}$ and $\text{Spec}(K_2 \otimes K_2) = \{-1^2, 1^2\}$ and also $\text{Spec}(C_5) = \{2 \sqrt{5-1}, \sqrt{5+1}^2, -\sqrt{5+1}^2\}$ and $\text{Spec}(W_5) = \{\sqrt{5+1}, 0^2, -\sqrt{5+1}, -2\}$. Hence, we get $E(C_4) = E(K_2 \otimes K_2) = 4$ and $E(C_5) = E(W_5) = 2(\sqrt{5} + 1)$. Notice that the first pair is integral while the second not.

Although there are many examples of pairs of equienergetic non-isospectral graphs in the literature, there are few systematic constructions. The authors are only aware of the following four:

(a) **Kronecker products.** In 2004, Balakrishnan ([3]) showed that the graphs $\Gamma \otimes K_2 \otimes K_2$ and $\Gamma \otimes C_4$ are equienergetic and non-isospectral, where $\Gamma$ is a non-trivial graph and $\otimes$ denotes the Kronecker product. Here, the first graph is not connected.

(b) **Iterated line graphs.** In the same year, Ramane et al ([19], 2004) proved that for two $k$-regular graphs $\Gamma_1$ and $\Gamma_2$ with the same number of vertices and $k \geq 3$, the iterated line graphs $L^r(\Gamma_1)$ and $L^r(\Gamma_2)$ are equienergetic for every $r \geq 2$. Thus, if $\Gamma_1$ and $\Gamma_2$ are connected and non-isospectral, then $L^r(\Gamma_1)$ and $L^r(\Gamma_2)$ are connected equienergetic non-isospectral graphs.

(c) **Gcd-graphs.** Five years later Ilic ([11], 2009) obtained families of $k$ hyperenergetic equienergetic non-isospectral gcd-graphs for any $k \in \mathbb{N}$. Namely, given $n = p_1 \cdots p_k$ with $p_1, \ldots, p_k$ primes, take the graphs $X_n(1, p_1), X_n(p_1, p_2), \ldots, X_n(p_{k-1}, p_k)$ where $X_n(p_{j-1}, p_j)$ have vertex set $\mathbb{Z}_n$ and edge set $E_j = \{(a, b) : (a - b, n) \in \{p_{j-1}, p_j\}\}$, where $p_0 = 1$.

(d) **Doubles.** More recently, Ganie, Pirzada and Iványi ([7], 2014) constructed several pairs of equienergetic non-isospectral graphs using the bipartite double $G^*$ and the double $D[G]$ of $G$, and the iterated doubles $G^{k*}, D^k[G]$ of them. In particular, they proved that if $G$ is bipartite, $\{G^*, D[G], G \otimes K_2\}$ is a triple of equienergetic non-isospectral graphs. This suggests that we may restrict to the search of equienergetic non-isospectral pairs to non-bipartite graphs.

Some of these methods can be combined to form new pairs of equienergetic non-isospectral graphs. For instance, in [10] it is proved that if $G_1$ and $G_2$ are $k$-regular graphs of the same order then $\{L^2(G_1)^*, L^2(G_1)^*\}, \{L^2(G_1)^*, (L^2(G_1))^*\}$ and $\{(L^2(G_1))^*, (L^2(G_1))^*\}$ are all pairs of bipartite equienergetic non-isospectral graphs.

Integral graphs were first considered by Harary and Schwenk in 1973 ([9]) when they posed the question “Which graphs have integral spectra?” The question in this generality seems very hard. A survey of integral graphs from 2002 is [4] focusing on trees, cubic graphs, 4-regular graphs and graphs of small size. In [1] the authors prove that only a small fraction of graphs of $n$-vertices are integral. More precisely, if $I(n)$ denotes the number of integral graphs of $n$-vertices then

$$I(n) \leq 2 \frac{n(n-1)}{\phi(n)},$$

where $2 \frac{n(n-1)}{\phi(n)}$ is the number of graphs of $n$-vertices. Also, integral graphs may be of interest in the design of the network topology of perfect state transfer networks (see [1] and references therein).
In this paper we present a new general construction to produce infinite pairs or triples of integral equienergetic non-isospectral regular graphs (typically connected and non-bipartite), using Cayley graphs, their complements and Cayley sum graphs, that we introduce next. In this line of thought, the question of whether it is possible to give a general construction of equienergetic non-isospectral trees is still pending (see the ‘open questions’ at the end of Chapter 8 in [14]).

Cayley graphs. Let $G$ be a finite abelian group and $S$ a subset of $G$ with $0 \notin S$. The Cayley graph $X(G, S)$ is the directed graph whose vertex set is $G$ and $v, w \in G$ form a directed edge of $\Gamma$ from $v$ to $w$ if $w - v \in S$. Since $0 \notin S$ then $\Gamma$ has no loops. Analogously, the Cayley sum graph $X^+(G, S)$ has the same vertex set $G$ but now $v, w \in G$ are connected in $\Gamma$ by an arrow from $v$ to $w$ if $v + w \in S$. Notice that if $S$ is symmetric, that is $-S = S$, then $X(G, S)$ and $X^+(G, S)$ are $|S|$-regular simple graphs. Notice, however, that $X^+(G, S)$ may contain loops. In this case, there is a loop on vertex $x$ provided that $x + x \in S$. For an excellent survey of spectral properties of general Cayley graphs the reader can see [16].

Two special cases of these graphs are obtained when: (a) $G$ is the finite field of $q$ elements $\mathbb{F}_q$ and $S$ is the set of nonzero $k$-th powers of $\mathbb{F}_q$ and (b) $G$ is a finite ring $R$ and $S$ is its group of units. In the first case we get the generalized Paley graph
\begin{equation}
\Gamma(k, q) = X(\mathbb{F}_q, R_k) \quad \text{with} \quad R_k = \{x^k : x \in \mathbb{F}_q^*\}
\end{equation}
and the generalized Paley sum graph $\Gamma^+(k, q) = X^+(\mathbb{F}_q, R_k)$. We will refer to them simply as $GP$-graphs and $GP^+$-graphs respectively. Note that $\Gamma(1, q) = K_n$ is the complete graph of $q$ vertices and that $\Gamma(2, q) = P(q)$ is the classical Paley graph (for $q \equiv 1 \pmod{4}$). In the second case we get
\begin{equation}
G_R = X(R, R^*) \quad \text{and} \quad G_R^+ = X^+(R, R^*),
\end{equation}
the unitary Cayley graphs and the unitary Cayley sum graphs, respectively. In this paper we will work with these two kind of Cayley graphs as well as with their complements.

Clearly, if the characteristic of the ring $R$ is 2, the graphs $X(R, S)$ and $X^+(R, S)$ are the same, so in the sequel we will assume that $\text{char}(R) \neq 2$. Also, notice that in odd characteristic, since 2 is always a unit, multiplication by 2 is a bijection of $R$. Thus, for any $y \in S$ we have that $y = 2x \in S$ for some unique $x \in R$ and, hence, there are loops in exactly $|S|$ vertices of $X^+(R, S)$.

Outline and results. We now summarize the structure and main results of the paper. In Section 2, we study the spectra of the graphs $X(G, S)$ and $X^+(G, S)$ for any abelian group $G$. In Proposition 2.5 we compute the multiplicities of the eigenvalues of $X^+(G, S)$. In Theorem 2.6, we show that the graphs $X(G, S)$ and $X^+(G, S)$ are equienergetic provided that $S$ is symmetric. These graphs are generically non-isospectral. Under certain conditions on the characters of $G$, the spectrum of $X(G, S)$ determines that of $X^+(G, S)$ and the graphs $X(G, S)$ and $X^+(G, S)$ are in fact non-isospectral (see Proposition 2.11).

In the next section we consider unitary Cayley graph over rings; i.e. $G_R = X(R, R^*)$ and $G_R^+ = X^+(R, R^*)$, with $R$ a commutative finite ring and $R^*$ the group of units of $R$. The spectrum of $X(R, R^*)$ is known (see [12]). By using this, we compute the spectrum of $X^+(R, R^*)$ and show that $X(R, R^*)$ and $X^+(R, R^*)$ are equienergetic and non-isospectral integral graphs. In the case that $R$ is a local ring we have to require that $|R| \neq 2|\mathfrak{m}|$,
where \( m \) is the maximal ideal (Proposition 3.1), while in the general case we need \( |R| \) to be odd (Theorem 3.4).

In Section 4 we compute the spectrum of GP-sum graphs \( \Gamma^+(k, q) \). We first consider arbitrary GP+-graphs. In Theorem 4.1 we give the spectrum \( \text{Spec}(\Gamma^+(k, q)) \) in terms of Gaussian periods using the known spectrum of \( \Gamma(k, q) \) obtained in [17]. In Corollary 4.2 we get that if \( k \mid \frac{q^2 - 1}{2} \) then \( \Gamma(k, q) \) and \( \Gamma^+(k, q) \) are equienergetic non-isospectral graphs, which are integral if we further assume that \( k \mid \frac{q^4 - 1}{p - 1} \). For semiprimitive GP+-graphs (see (4.5)) we give the spectrum explicitly in Theorem 4.3. Finally, we consider the graphs \( \Gamma^+(3, q) \) and \( \Gamma^+(4, q) \) and we compute their spectra in Theorems 4.5 and 4.6, respectively.

In Section 5 we take into account the complementary graphs

\[
\bar{\Gamma} = X(G, S^c \setminus \{0\})
\]

of \( \Gamma = X(G, S) \), either for \( \Gamma \) a unitary Cayley graph \( G_R \) or a GP-graph \( \Gamma(k, q) \). Both \( \Gamma \) and \( \bar{\Gamma} \) are loopless. Using the known expressions for the energies of \( G_R \) and \( \bar{G}_R \) we obtain a general arithmetic condition on \( R \) for \( G_R \) to be equienergetic with its complement \( \bar{G}_R \) (see (5.3)). In Proposition 5.1 we obtain explicit conditions when \( R \) is local or a product of two local rings, such that \( E(G_R) = E(\bar{G}_R) \). As a consequence, in Theorem 5.5 we get that if \( m \) is a prime power, then the complete \( m \times m \) bipartite graph \( K_{m,m} \) of \( m^2 \) vertices is equienergetic and non-isospectral with its complement and similarly, the crown graph \( C_{m,1} \) of \( 2m \) vertices is equienergetic and non-isospectral with its complement. Relative to generalized Paley graphs, in Proposition 5.7 we compute the energy for semiprimitive GP-graphs \( \Gamma = \Gamma(k, q) \). Using this, in Remark 5.9 we exhibit the triple \( \{\Gamma, \bar{\Gamma}, \Gamma^+\} \) of equienergetic non-isospectral graphs, where \( \Gamma, \bar{\Gamma} \) are strongly regular graphs while \( \Gamma^+ \) not.

In the last section, we consider the construction of equienergetic non-isospectral pairs of graphs such that at least one of them is Ramanujan. We will use known characterizations of Ramanujan unitary Cayley graphs \( G_R \) due to Liu and Zhou ([15], 2012) and of Ramanujan semiprimitive GP-graphs \( \Gamma(k, q) \) due to the authors ([17], 2018). First, we obtain necessary and sufficient conditions for the pairs \( \{G_R, \bar{G}_R\} \) and \( \{G_R, \bar{G}_R\} \) to be equienergetic non-isospectral where at least one of the graphs is Ramanujan, distinguishing the cases when \( R \) is a local ring or not (see Theorems 6.1 and 6.4). In Corollaries 6.2 and 6.6 we characterize all possible triples \( \{G_R, \bar{G}_R, \bar{G}_R\} \) of equienergetic non-isospectral Ramanujan graphs, with \( R \) any commutative ring with identity. Finally, in Theorem 6.7 we get the characterization of all the triples of equienergetic non-isospectral graphs \( \{\Gamma(k, q), \bar{\Gamma}(k, q), \bar{\Gamma}(k, q)\} \), with \( (k, q) \) a semiprimitive pair, which are Ramanujan.

2. EQUIENERGY OF \( X(G, S) \) AND \( X^+(G, S) \)

Here we study the equienergeticity of the graphs \( X(G, S) \) and \( X^+(G, S) \). Let \( G \) a finite abelian group and \( S \) a subset of \( G \) not containing 0. It is well-known that the spectra of \( X(G, S) \) and \( X^+(G, S) \) can be computed by using the irreducible characters \( \hat{G} \) of \( G \). Given \( \chi \in \hat{G} \), i.e. a group homomorphism \( \chi : G \to S^1 \subset \mathbb{C}^\ast \), one can define

\[
e_{\chi} = \chi(S) = \sum_{s \in S} \chi(s) \quad \text{and} \quad v_{\chi} = (\chi(g))_{g \in G}.
\]

Note that \( e_{\chi} \in \mathbb{C} \) and \( v_{\chi} \in (S^1)^n \) if \( |G| = n \).
We have the following well-known result.

**Lemma 2.1.** In the previous notations:

(a) The eigenvalues of \(X(G, S)\) are \(\{e_\chi\}_{\chi \in \hat{G}}\) and \(e_\chi\) has associated eigenvector \(v_\chi\).

(b) The eigenvalues of \(X^+(G, S)\) are either \(e_\chi = 0\) or \(\{\pm|e_\chi|\}_{\chi \in \hat{G}} \subset \mathbb{R}\), such that:

(i) If \(e_\chi = 0\), their corresponding eigenvectors are \(v_\chi\) and \(v_{\chi^{-1}}\).

(ii) If \(e_\chi \neq 0\), the eigenvector associated to \(\pm|e_\chi|\) is \(|e_\chi|v_\chi \pm e_\chi v_{\chi^{-1}}\).

Notice that \(|S|\) is always an eigenvalue of both \(X(G, S)\) and \(X^+(G, S)\). In fact, denoting by \(\chi_0\) the principal character of \(G\) (i.e. \(\chi_0(g) = 1\) for every \(g \in G\)) then we have \(e_{\chi_0} = |S|\). Moreover, \(|S|\) is the principal eigenvalue of both graphs, i.e. \(\lambda_0 = \lambda_0^+ = |S|\) since \(X(G, S)\) and \(X^+(G, S)\) are \(|S|\)-regular. In general, if \(\Gamma\) is \(k\)-regular, then \(k\) is an eigenvalue of \(\Gamma\) and \(k \geq |\lambda|\) for an \(\lambda \in \text{Spec}(\Gamma)\), hence \(k\) is called the trivial or principal eigenvalue of \(\Gamma\).

The lemma shows that \(X(G, S)\) and \(X^+(G, S)\) are generically non-isospectral graphs. However, in some special cases \(X(G, S)\) and \(X^+(G, S)\) could be isospectral or even the same graph (for instance, if \(G = \mathbb{F}_{2m}\) or \(\mathbb{Z}_{2m}\) and \(S\) is any subset of \(G \smallsetminus \{0\}\)).

### 2.1. Equienergeticity

Here we give simple conditions on \(G\) and \(S\) for \(X(G, S)\) and \(X^+(G, S)\) to be equienergetic. We will need the following definition in the sequel.

**Definition 2.2.** If \(G\) is an abelian group and \(S\) is a symmetric subset of \(G\) with \(0 \notin S\), then we say that \((G, S)\) is an abelian symmetric pair.

We begin by showing that there is a simple relation between the eigenvalue \(e_\chi\) associated to a character \(\chi\) of \(G\) as in (2.1) and the corresponding one associated to its inverse \(\chi^{-1}\).

**Lemma 2.3.** Let \(S\) be a subset of a finite abelian group \(G\) such that \(0 \notin S\). Then \(e_{\chi^{-1}} = \overline{e_\chi}\) and hence \(e_\chi = 0\) if and only if \(e_{\chi^{-1}} = 0\), for all \(\chi \in \hat{G}\). Moreover, if \(S\) is symmetric, then \(e_\chi \in \mathbb{R}\) and \(e_{\chi^{-1}} = e_\chi\) for all \(\chi \in \hat{G}\).

**Proof.** If \(\chi \in \hat{G}\), we have that

\[
e_{\chi^{-1}} = \chi^{-1}(S) = \sum_{g \in S} \chi^{-1}(g) = \sum_{g \in S} \overline{\chi(g)} = \sum_{g \in S} \chi(g) = \chi(S) = \overline{e_\chi}.
\]

and thus \(e_\chi = 0\) if and only if \(e_{\chi^{-1}} = 0\). Now, if \(S\) is a symmetric set, then the adjacency matrix of \(X(G, S)\) is symmetric. Thus, \(e_\chi \in \mathbb{R}\) and hence \(e_{\chi^{-1}} = \overline{e_\chi} = e_\chi\). \(\square\)

**Remark 2.4.** The condition \(\chi = \chi^{-1}\) is equivalent to \(\chi\) being a real character of \(G\), i.e. \(\chi(g) \in \mathbb{R}\) for all \(g \in G\). For finite abelian groups, a real character takes only values in \(\mathbb{S}^1 \cap \mathbb{R} = \{\pm 1\}\). Also, if \(|G|\) is odd then \(\chi_0\) is the only real irreducible character of \(G\). In fact, suppose \(\chi\) is a nontrivial real character of \(G\) and \(\chi(g_0) = -1\) for some \(g_0 \in G\), then \(1 = \chi(e) = \chi(g_0^{(G)}) = (-1)^{|G|} = -1\).

An equivalence relation \(\sim\) between irreducible characters of \(G\) is given by

\[
\chi \sim \chi' \iff e_\chi = e_{\chi'}
\]

where \(\chi, \chi' \in \hat{G}\). We denote by \(\hat{G}/\sim\) the set of equivalence classes

\[
[\chi] = \{\chi' \in \hat{G} : \chi' \sim \chi\} = \{\chi' \in \hat{G} : e_{\chi'} = e_\chi\}.
\]
We will also need to consider the following set of characters
\[ (2.3) \quad \overline{\chi} = \{ \chi' \in \hat{G} : e_{\chi'} = -e_{\chi} \} \]
and the associated subsets
\[ (2.4) \quad \{ \rho \in [\chi] : \rho^{-1} = \rho \}, \quad \{ \rho \in [\chi] : \rho^{-1} \neq \rho \}, \quad \{ \rho \in [\chi] : e_{\rho} = -e_{\chi} \}, \quad \{ \rho \in [\chi] : e_{\rho} = -e_{\chi} \}. \]
Hence,
\[ [\chi] = [\chi]_{R} \cup [\chi]_{R^c} \quad \text{and} \quad \overline{\chi} = \overline{\chi}_{R} \cup \overline{\chi}_{R^c}. \]

We denote by \( m(\lambda) \) (resp. \( m^+(\lambda) \)) the multiplicity of the eigenvalue \( \lambda \) in \( X(G, S) \) (resp. \( X^+(G, S) \)), with the convention that \( m(\lambda) = 0 \) (resp. \( m^+(\lambda) = 0 \)) if \( \lambda \) is not an eigenvalue of \( X(G, S) \) (resp. \( X^+(G, S) \)).

It is clear that if \( e_{\chi} \neq 0 \) then
\[ (2.5) \quad m(e_{\chi}) = \# [\chi] \quad \text{and} \quad m(-e_{\chi}) = \# \overline{\chi}, \]
by the independence of the characters. We now compute the multiplicities of the eigenvalues for general Cayley sum graphs.

**Proposition 2.5.** Let \((G, S)\) be a finite abelian symmetric pair. If \( \chi \in \hat{G} \), then the eigenvalues of \( X^+(G, S) \) are \( \pm e_{\chi} \in \mathbb{R} \) with multiplicities given by
\[ (2.6) \quad m^+(e_{\chi}) = \# [\chi] R + \frac{1}{2} \# [\chi] R^c + \frac{1}{2} \# \overline{\chi} R^c, \quad m^+(e_{\chi}) = \# \overline{\chi} R + \frac{1}{2} \# [\chi] R^c + \frac{1}{2} \# \overline{\chi} R^c. \]
Moreover, the relation with the multiplicities of \( X(G, S) \) is given by \( m^+(0) = m(0) \) and
\[ (2.7) \quad m(e_{\chi}) + m(-e_{\chi}) = m^+(e_{\chi}) + m^+(e_{\chi}) \]
for \( e_{\chi} \neq 0 \).

**Proof.** By Lemma 2.1, the eigenvalues of \( X^+(G, S) \) are either \( e_{\chi} = 0 \) (with eigenvectors \( v_{\chi} \) and \( v_{\chi}^{-1} \)) or \( \pm |e_{\chi}| \), with eigenvector \( V_{\chi}^\pm = |e_{\chi}| v_{\chi} \pm e_{\chi} v_{\chi}^{-1} \), for \( \chi \in \hat{G} \). Moreover, since \( S \) is symmetric the eigenvalues are real (the adjacency matrix of \( X^+(G, S) \) is symmetric) and hence given by \( \pm e_{\chi} \). We will see that the only characters that can contribute to the multiplicity of \( \pm |e_{\chi}| \) are those \( \rho \in \hat{G} \) such that either \( \rho \in [\chi] \) or \( \rho \in [\overline{\chi}] \). For clarity, we split the proof into cases.

(a) Suppose first that \( \rho \in [\chi] R \), that is \( \rho \sim \chi \) and \( \rho^{-1} = \rho \). If \( e_{\rho} > 0 \), then \( |e_{\rho}| = e_{\rho} \) is an eigenvalue with eigenvector \( V_{\rho}^+ = 2e_{\rho} v_{\rho} \), by Lemma 2.1. Moreover, \( -|e_{\rho}| \) is not an eigenvalue because we would have \( V_{\rho}^- = e_{\rho} v_{\rho} - e_{\rho} v_{\rho}^{-1} = 0 \). Similarly, if \( e_{\rho} < 0 \), then \( |e_{\rho}| = -e_{\rho} \) is not an eigenvalue, although \( -|e_{\rho}| = e_{\rho} \) is an eigenvalue with eigenvector \( V_{\rho}^- = -2e_{\rho} v_{\rho} \). Therefore, each real character \( \rho \sim \chi \) contributes 1 to the multiplicity of \( e_{\chi} \) and 0 to the one of \( -e_{\chi} \).

(b) Now, assume that \( \rho \in [\overline{\chi}] R^c \), that is \( \rho \sim \chi \) and \( \rho^{-1} \neq \rho \). If \( e_{\rho} > 0 \), then \( |e_{\rho}| = e_{\rho} \) is an eigenvalue with eigenvector \( V_{\rho}^+ = e_{\rho} (v_{\rho} + v_{\rho}^{-1}) \). Notice that, by Lemma 2.3, \( e_{\rho}^{-1} = e_{\rho} \), since \( S \) is symmetric. Thus, \( e_{\rho}^{-1} \) is an eigenvalue with eigenvector
\[ V_{\rho}^- = |e_{\rho}^{-1}| v_{\rho}^{-1} + e_{\rho}^{-1} v_{\rho} = e_{\rho} (v_{\rho} + v_{\rho}^{-1}) = V_{\rho}^+. \]
Hence, $\rho^{-1}$ does not contribute to the multiplicity of $e_\chi$. Therefore, the characters $\rho$ and $\rho^{-1}$ both contribute only one to the multiplicity of the eigenvalue $e_\chi$. On the other hand, $-|e_\rho| = -e_\rho = |e_{\rho^{-1}}|$ is an eigenvalue with eigenvectors

$$V^-_\rho = e_\rho(v_{\rho^{-1}} - v_\rho) \quad \text{and} \quad V^\rho_{\rho^{-1}} = e_\rho(v_\rho - v_{\rho^{-1}}).$$

Then $V^\rho_{\rho^{-1}} = -V^-_\rho$ and, thus, the characters $\rho$ and $\rho^{-1}$ contribute only one to $m^+(e_\rho)$.

If $e_\rho < 0$, then $|e_\rho| = -e_\rho = |e_{\rho^{-1}}|$ is an eigenvalue of $X^+(G, S)$ with eigenvectors $V^-_\rho = e_\rho(v_{\rho^{-1}} - v_\rho)$ and $V^\rho_{\rho^{-1}} = e_\rho(v_\rho - v_{\rho^{-1}})$, and thus $V^\rho_{\rho^{-1}} = -V^-_\rho$. On the other hand, $-|e_\rho| = e_{\rho'} = |e_{\rho^{-1}}|$ is an eigenvalue of $X^+(G, S)$ with eigenvector $-e_\rho(v_\rho + v_{\rho^{-1}})$, and therefore the characters $\rho$ and $\rho^{-1}$ contribute only one to the multiplicity of the eigenvalue $\pm e_\rho$.

(c) Suppose now that $\rho \in \overline{\chi}$, that is $\rho \in \hat{G}$ and $e_\rho = -e_\chi$. By proceeding similarly as before we have that if $\rho$ is a real character this contributes in 1 to the multiplicity of $-e_\chi$ and does not contribute to the multiplicity of $e_\chi$ and on the other hand, if $\rho$ is a non real character then $\rho$ and $\rho^{-1}$ contribute in 1 to the multiplicity of the eigenvalues $\pm e_\chi$.

By putting together the information in (a), (b) and (c) we get (2.6).

Now, if $e_\chi \neq 0$ then, by (2.4) and (2.5) we have

$$m(e_\chi) + m(-e_\chi) = \#[\chi] + \#[\overline{\chi}] = (\#[\chi]_R + \#[\overline{\chi}]_R) + (\#[\overline{\chi}]_R + \#[\chi]_R).$$

Therefore, by (2.6), we have $m(e_\chi) + m(-e_\chi) = m^+(e_\chi) + m^+(-e_\chi)$, as it was to be shown.

Finally suppose that $\chi \in \hat{G}$ with $e_\chi = 0$. If $\chi^{-1} = \chi$, the contribution of $\chi$ to the multiplicity of 0 is one, since $v_\chi = v_{\chi^{-1}}$. On the other hand, if $\chi^{-1} \neq \chi$, then $e_{\chi^{-1}} = 0$ by Lemma 2.3. In this case both $e_\chi$ and $e_{\chi^{-1}}$ are eigenvalues with eigenvectors $v_\chi$ and $v_{\chi^{-1}}$ and thus $m^+(0) = m(0)$.

We are now in a position to show that Cayley graphs and Cayley sum graphs defined over the same abelian symmetric pair are always equienergetic.

**Theorem 2.6.** Let $(G, S)$ be a finite abelian symmetric pair. Then, the graphs $X(G, S)$ and $X^+(G, S)$ are equienergetic with the same number of edges.

**Proof.** Denote by $\hat{G}/\approx$ the set of equivalence classes of the relation $\approx$ given by $\chi \approx \chi'$ if and only if $e_\chi = \pm e_{\chi'}$. That is, $\hat{G}/\approx$ equals $(\hat{G}/\sim)/\{\pm 1\}$. By (2.7) in Proposition 2.5 we have

$$E(X(G, S)) = \sum_{\chi \in \hat{G}} |e_\chi| = \sum_{\chi \in \hat{G}/\approx} \{m(e_\chi) + m(-e_\chi)\} |e_\chi| = \sum_{\chi \in \hat{G}/\approx} \{m^+(e_\chi) + m^+(-e_\chi)\} |e_\chi| = E(X^+(G, S)).$$

The remaining assertion is clear since both graphs has vertex set $G$ and are $|S|$-regular. \hfill $\square$

**Example 2.7.** Let $S$ be a symmetric subset of $\mathbb{Z}_n$ not containing 0. Then, the graphs $X(\mathbb{Z}_n, S)$ and $X^+(\mathbb{Z}_n, S)$ are equienergetic. This includes the cases of $n$-cycles $C_n$ and...
2. Non-isospectrality. Recall that one of our goals is to construct pairs of equienergetic non-isospectral graphs of the form \( \Gamma = X(G, S) \) and \( \Gamma^+ = X^+(G, S) \).

We first give a simple condition for a pair \( \Gamma \) and \( \Gamma^+ \) to be isospectral which follows directly from (2.7) in Proposition 2.5.

**Corollary 2.8.** Let \( \Gamma = X(G, S) \) and \( \Gamma^+ = X^+(G, S) \). If \( \text{Spec}(\Gamma) \) and \( \text{Spec}(\Gamma^+) \) are both symmetric then \( \Gamma \) and \( \Gamma^+ \) are isospectral.

**Remark 2.9.** In general, we have \( e_-\chi = -e_+\chi \) for every character \( \chi \) of a group \( G \). Let \((G, S)\) be an abelian symmetric pair such that \( -\chi \in \hat{G} \) for all \( \chi \in \hat{G} \). Then, the sets \([\chi]_R\) and \([\hat{\chi}]_R\) have equal cardinality, and the same happens with \([\chi]_{R^e}\) and \([\hat{\chi}]_{R^e}\). This implies, by (2.5) and (2.6), that
\[
m(\varepsilon_\chi) = m(-\varepsilon_\chi) \quad \text{and} \quad m^+(\varepsilon_\chi) = m^+(-\varepsilon_\chi).
\]

Thus, \( X(G, S) \) and \( X^+(G, S) \) have symmetric spectra and hence, by Corollary 2.8, they are isospectral.

**Example 2.10.** Consider the group \( G = \mathbb{Z}_{2n} \) with \( n \) odd and let \( S \) be any symmetric subset of \( \mathbb{Z}_{2n} \setminus \{0\} \). In this case, \( -\chi \in \hat{\mathbb{Z}}_{2n} \) for any irreducible character \( \chi \in \hat{\mathbb{Z}}_{2n} \). Thus, by the previous remark, the graphs \( X(\mathbb{Z}_{2n}, S) \) and \( X^+(\mathbb{Z}_{2n}, S) \) with \( n \) odd are isospectral.

Now, as a consequence of Proposition 2.5, we obtain the following condition for non-isospectrality.

**Proposition 2.11.** Let \((G, S)\) be a finite abelian symmetric pair such that the principal character \( \chi_0 \) is the only real character of \( G \). For each \( \chi \in \hat{G} \setminus \{\chi_0\} \), with \( e_\chi \neq 0, \pm |S| \), we have
\[
m^+(e_\chi) = m^+(-e_\chi) = \frac{1}{2}\{m(e_\chi) + m(-e_\chi)\}. \tag{2.8}
\]

In particular, \( \text{Spec}(X(G, S)) \) determines \( \text{Spec}(X^+(G, S)) \). If, in addition, \( G \) has a non-trivial character \( \chi \) such that \( -e_\chi \) is not an eigenvalue of \( X(G, S) \), then \( X(G, S) \) and \( X^+(G, S) \) are non-isospectral.

**Proof.** By Remark 2.4 and (2.4) we have that \([\chi]_R = [\hat{\chi}]_R = \emptyset\). Hence, by Proposition 2.5, we get that \( m^+(e_\chi) = m^+(-e_\chi) \) and thus \( 2m^+(e_\chi) = m(e_\chi) + m(-e_\chi) \), from which (2.8) follows.

Now, if \( \chi_0 \neq \chi \in \hat{G} \) such that \( -e_\chi \) is not an eigenvalue of \( X(G, S) \), then
\[
0 = m(-e_\chi) < \frac{1}{2}m(e_\chi) = m^+(-e_\chi).
\]

This implies the last assertion in the statement. \( \square \)
Example 2.12. (i) The odd cycles and odd paths with loops at the ends are equienergetic non-isospectral graphs. Indeed, \(C_{2n+1} = X(\mathbb{Z}_{2n+1}, \{\pm 1\})\) and \(P_{2n+1} = X^+(\mathbb{Z}_{2n+1}, \{\pm 1\})\) are equienergetic by Example 2.7, and non-isospectral since one has loops and the other not. However, we can also use Proposition 2.11 to see this. In fact, the spectra of \(C_n\) is well-known, for \(n \geq 1\) we have
\[
\text{Spec}(C_{2n+1}) = \{2 \cos \left( \frac{2\pi i}{2n+1} \right) \} \quad 0 \leq j \leq 2n
\]
If \(\omega = e^{\frac{2\pi i}{2n+1}}\) denotes the \((2n+1)\)-th primitive root of unity and \(\chi\) is the associated character, then \(-e^{\chi} = -2Re(\omega) \notin \text{Spec}(C_{2n+1})\), as we wanted to show.

Note that \(\text{Spec}(C_{2n+1})\) is integral if and only if \(n = 1\). In this case we have
\[
\text{Spec}(C_3) = \{2, -1, -1\} \quad \text{and} \quad \text{Spec}(\hat{P}_3) = \{2, 1, -1\},
\]
since \(\hat{P}_3\) has adjacency matrix \(A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\), and hence \(E(C_3) = E(\hat{P}_3) = 4\). In this way, the graphs \(C_3\) and \(\hat{P}_3\)

\[
\begin{array}{c}
\triangle \\
C_3 \\
\end{array}
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array}
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array}
\begin{array}{c}
\hat{P}_3 \\
\end{array}
\]

are connected 2-regular integral equienergetic non-isospectral, one having a cycle and no loops while the other is acyclic with loops. This is the smallest possible example of this kind, unless one disregard the cycles. In this case one can take even smaller graphs, namely \(K_2 = P_2\) (1-regular) and \(\hat{P}_2\) (2-regular). We have \(\text{Spec}(P_2) = \{1, -1\}\) and \(\text{Spec}(\hat{P}_2) = \{2, 0\}\), hence \(E(P_2) = E(\hat{P}_2) = 2\). Thus, they are integral equienergetic non-isospectral regular graphs with different regularity degree, one with loops while the other not.

(ii) More generally, consider the subset \(S_r = \{\pm r \pmod{n}\}\) of \(\mathbb{Z}_n\). If \((r, n) = 1\) then \(X(\mathbb{Z}_n, S_r)\) is isomorphic to \(X(\mathbb{Z}_n, S_1)\) and \(X^+(\mathbb{Z}_n, S_r)\) is isomorphic to \(X^+(\mathbb{Z}_n, S_1)\), since the \(\mathbb{Z}_n\)-automorphism which sends 1 to \(r\) maps \(S_1\) in \(S_r\). Then all the graphs \(\{X(\mathbb{Z}_n, S_1), X^+(\mathbb{Z}_n, S_1)\}_{r=1}^{(r, n)=1}\) are equienergetic and non-isospectral.

Since the principal character \(\chi_0 \in \hat{G}\) gives rise to the principal eigenvalue in \(\Gamma = X(G, S)\) from now on we will denote the eigenvalues of \(\Gamma\) by \(\lambda_0, \ldots, \lambda_{n-1}\) instead of \(\lambda_1, \ldots, \lambda_n\), in order to have \(\lambda_0 = e^{\chi_0}\).

In the sequel, we will use the following concept.

Definition 2.13. If \(\Gamma\) is a graph, we will say that \(\text{Spec}(\Gamma)\) is almost symmetric if the multiplicities of an eigenvalue \(\lambda\) and of its opposite \(-\lambda\) are the same, except for \(\lambda_0\), i.e. \(m(\lambda) = m(-\lambda)\) for every \(\lambda \neq \lambda_0\) (note that \(\lambda = 0\) automatically satisfies this and it is allowed that \(m(-\lambda_0) = 0\)). If in addition \(|m(\lambda_0) - m(-\lambda_0)| = 1\) holds, then we say that \(\text{Spec}(\Gamma)\) is strongly almost symmetric.

It is known that the sum of the eigenvalues of non-directed graphs equals 0. This is not true for the sum graphs \(\Gamma = X^+(G, S)\). However, if \(\Gamma\) is almost symmetric, we have \(\sum \lambda_i = |m(\lambda_0) - m(-\lambda_0)|\lambda_0\). If, further, \(\Gamma\) is strongly almost symmetric then \(\sum \lambda_i = \lambda_0\).

For strongly almost symmetric graphs, connectivity is equivalent to non-bipartiteness.
Proposition 2.14. If $\Gamma$ is a strongly almost symmetric graph, then $\Gamma$ is connected if and only if $\Gamma$ is non-bipartite.

Proof. Since $\Gamma$ is strongly almost symmetric, we have $m(\lambda_0) = t + 1$ and $m(-\lambda_0) = t$ for some $t \in \mathbb{N}_0$, where $\lambda_0$ is the principal eigenvalue of $\Gamma$. If $\Gamma$ is connected, then $t = 0$ and hence $-\lambda_0$ is not an eigenvalue of $\Gamma$ and hence $\Gamma$ is non-bipartite. Conversely, if $\Gamma$ is non-bipartite, $t = 0$ and hence $m(\lambda_0) = 1$, thus $\Gamma$ is connected. \qed

Corollary 2.15. Let $p$ be an odd prime and $s \in \mathbb{N}$. The graphs $G_{p^s} := X(\mathbb{Z}_{p^s}, \mathbb{Z}_{p^s}^*)$ and $G_{p^s}^+ := X^+(\mathbb{Z}_{p^s}, \mathbb{Z}_{p^s}^*)$ are integral equienergetic and non-isospectral. Moreover, $G_{p^s}^+$ is strongly almost symmetric with loops.

Proof. The graphs $G_{p^s}$ and $G_{p^s}^+$ are equienergetic by Theorem 2.6, since $(\mathbb{Z}_{p^s}, \mathbb{Z}_{p^s}^*)$ is an abelian symmetric pair, and non-isospectral by Proposition 2.11. In fact, it is known that $X(\mathbb{Z}_{p^s}, \mathbb{Z}_{p^s}^*)$ is the complete $p$-partite graph with integral spectrum

$$\text{Spec}(G_{p^s}) = \{[p^s - p^{s-1}], [0]p^s-p, [-p^{s-1}]p^{-1}\}. \quad (2.9)$$

In particular, if $s = 1$, $X(\mathbb{Z}_p, \mathbb{Z}_p^*) = K_p$ with

$$\text{Spec}(K_p) = \{[p-1], [-1]p^{-1}\}. \quad (2.10)$$

Thus, (2.9) implies that $e_\chi = -p$ or $e_\chi = 0$ for every non-trivial irreducible character $\chi$ of $\mathbb{Z}_{p^s}$. In this way, for those characters with $e_\chi = -p$, we have that $-e_\chi = p \notin \text{Spec}(G_{p^s})$ (since $p \neq 2$). Hence, by the previous corollary, $X(\mathbb{Z}_{p^s}, \mathbb{Z}_{p^s}^*)$ and $X^+(\mathbb{Z}_{p^s}, \mathbb{Z}_{p^s}^*)$ are non-isospectral. Moreover, by (2.8), the spectrum of the sum graph is given by

$$\text{Spec}(G_{p^s}^+) = \{[p^s - p^{s-1}], [p^{s-1}]\frac{p+1}{2}, [0]p^s-p, [-p^{s-1}]\frac{p-1}{2}\} \quad (2.11)$$

where we have that $|S| = e_{\chi_0}$ is always an eigenvalue of both $\Gamma$ and $\Gamma^+$. From the above expression on can see that $\text{Spec}(G_{p^s}^+)$ is strongly almost symmetric. \qed

Remark 2.16. Note that under the hypothesis of Proposition 2.11, $\text{Spec}(X^+(G, S))$ does not determine $\text{Spec}(X(G, S))$. However, if in addition one has that $m(-e_\chi) = 0$ or $m(-e_\chi) = m(e_\chi)$ for every $\chi \in \hat{G}$, then $\text{Spec}(X^+(G, S))$ determines $\text{Spec}(X(G, S))$.

3. Unitary Cayley graphs over finite rings

Here we produce pairs of equienergetic non-isospectral pairs of unitary Cayley (sum) graphs over rings, namely $G_R = X(R, R^*)$ and $G_R^+ = X^+(R, R^*)$.

Let $R$ be a finite commutative ring, $R^*$ its group of units, and consider $G_R = X(R, R^*)$. By the well-known Artin’s structure theorem we have that

$$R = R_1 \times \cdots \times R_s \quad (3.1)$$

where each $R_i$ is a local ring, that is having a unique maximal ideal $m_i$, with $|m_i| = m_i$ say. Moreover, one also has the decomposition $R^* = R_1^* \times \cdots \times R_s^*$. This implies that

$$G_R = G_{R_1} \otimes \cdots \otimes G_{R_s} \quad \text{and} \quad G_R^+ = G_{R_1}^+ \otimes \cdots \otimes G_{R_s}^+ \quad (3.2)$$

with $G_{R_i} = X(R_i, R_i^*)$, $G_{R_i}^+ = X^+(R_i, R_i^*)$, and where $\otimes$ denotes the Kronecker product of graphs.
The spectrum of $G_R = X(R, R^*)$ is known. If $R$ is as in (3.1), put

$$\lambda_C = (-1)^{|C|} \frac{|R^*|}{\prod_{j \in C} (|R_j^*|/m_j)}$$

for each subset $C \subseteq \{1, \ldots, s\}$. From [12] (see also [15]), the eigenvalues of $G_R$ are

$$\lambda = \begin{cases} 
\lambda_C, & \text{repeated } \prod_{j \in C} (|R_j^*|/m_j) \text{ times}, \\
0, & \text{with multiplicity } |R| - \prod_{i=1}^s (1 + \frac{|R_i|}{m_i}),
\end{cases}$$

where $C$ above runs over all the subsets of $\{1, 2, \ldots, s\}$. Note that, a priori, different subsets $C$ can give the same eigenvalue.

For a finite commutative local ring $R$ we can give the spectrum of the unitary Cayley sum graph $G_R^+ = X^+(R, R^*)$ from the known one of $G_R = X(R, R^*)$.

**Proposition 3.1.** Let $(R, m)$ be a finite local ring with $|R| = r$ and $|m| = m$. If $r \neq 2m$ then the spectrum of $G_R^+ = X^+(R, R^*)$ is strongly almost symmetric given by

$$\text{Spec}(G_R^+) = \{[r - m]^1, [m] \frac{-1}{2m}, [0] \frac{-1}{m}, [-m] \frac{1}{2m}\}$$

if $R$ is not a field ($m > 1$) and by

$$\text{Spec}(G_R^+) = \{[r - 1]^1, [1] \frac{-1}{r - 1}, [-1] \frac{1}{r - 1}\}$$

if $R$ is a field ($m = 1$). Moreover, $G_R = X(R, R^*)$ and $G_R^+$ are integral equienergetic non-isospectral graphs.

**Proof.** Since $R$ is local, we have $R^* = R \setminus m$ and $s = 1$ in the decompositions (3.1) and (3.2). Thus, by (3.3) and (3.4), the eigenvalues of $X(R, R^*)$ are given by $\lambda_{\emptyset} = |R^*|$ with multiplicity 1, $\lambda_{\{1\}} = -\frac{|R|}{|R^*|} m$ with multiplicity $\frac{|R^*|}{m}$ and 0 with multiplicity $r - (1 + \frac{m}{r})$, where $\frac{m}{r} \neq 1$ since $m$ is a proper ideal of $R$. Hence, we have

$$\text{Spec}(G_R) = \begin{cases} 
\{[r - m]^1, [0] \frac{-1}{m}, [-m] \frac{1}{m}\} & \text{if } R \text{ is not a field (} m > 1), \\
\{[r - 1]^1, [1] \frac{-1}{r - 1}, [-1] \frac{1}{r - 1}\} & \text{if } R \text{ is a field (} m = 1).
\end{cases}$$

This implies that there is some $\chi \in \hat{R}$ such that $e_{\chi} = -m$ and hence $-e_{\chi} = m \notin \text{Spec}(G_R)$ $r \neq 2m$. Thus, by (2.8) in Proposition 2.11 we obtain (3.5) and (3.6) as desired, from which it follows that the spectrum is strongly almost symmetric.

Finally, the graphs $G_R$ and $G_R^+$ are equienergetic by Theorem 2.6, since $(R, R^*)$ is an abelian symmetric pair, and integral non-isospectral by (3.5) – (3.7).

Two particular cases of finite local rings are finite fields and Galois rings. By the previous result we have the following.

**Example 3.2.** Let $p$ be an odd prime and $s, t \in \mathbb{N}$. Let $R = GR(p^s, t)$ be the finite Galois ring of $p^{st}$ elements. Its maximal ideal is $m = (p)$ and hence $m = p^{(s-1)t}$ and
\[ |R^*| = p^{(s-1)t}(p^t - 1). \] In particular, if \( t = 1 \) then \( R \) is the local ring \( \mathbb{Z}_{p^t} \) while if \( s = 1 \) we have that \( R \) is the finite field \( \mathbb{F}_{p^t} \). Thus,

\[
\begin{align*}
Spec(G_R) &= \{[p^{(s-1)t}(p^t - 1)]^1, [0]^{p^{(s-1)t}-1}, [-p^{(s-1)t}]^{p^t-1}\}, \\
Spec(G^+_R) &= \{[p^{(s-1)t}(p^t - 1)]^1, [p^{(s-1)t}]^{\frac{p^t}{2}}, [0]^{p^{(s-1)t}-1}, [-p^{(s-1)t}]^{\frac{p^t}{2}}\}.
\end{align*}
\]

Note that in the case that \( R \) is a field, since \( s = 1 \), \( 0 \) is not an eigenvalue. We have that \( X(R, R^*) \) and \( X^+(R, R^*) \) are integral equienergetic non-isospectral graphs and \( X^+(R, R^*) \) is strongly almost symmetric.

**Remark 3.3.** Notice that Corollary 2.15, obtained by a different method, is a particular case of Proposition 3.1 or Example 3.2.

We now give our first general family of pairs of equienergetic and non-isospectral graphs. This result generalizes the previous proposition.

**Theorem 3.4.** Let \( R \) be a finite commutative ring with \(|R|\) odd. Then, \( G_R = X(R, R^*) \) and \( G^+_R = X^+(R, R^*) \) are integral equienergetic non-isospectral graphs. Moreover, \( G^+_R \) is strongly almost symmetric.

**Proof.** By Theorem 2.6, the graphs \( X(R, R^*) \) and \( X^+(R, R^*) \) are equienergetic. Since \(|R|\) is odd, the only real character of \( R \) is the trivial one. Thus, by Proposition 2.11, to show that \( X(R, R^*) \) and \( X^+(R, R^*) \) are non-isospectral it is enough to show that there is some non-trivial irreducible character \( \chi \) of \( R \) such that \(-e_\chi\) is not an eigenvalue of \( X(R, R^*) \).

Let \( R = R_1 \times \cdots \times R_s \) be the Artin’s decomposition of \( R \) in local rings as in (3.1), with \(|R_i| = r_i \) and \(|m_i| = m_i \) for \( i = 1, \ldots, s \).

Suppose first that \( R \) is reduced, that is \( R \) has no non-trivial nilpotent elements. In this case, the rings \( R_i \) are all fields (hence \( m_i = 1 \)) for all \( i \). In this way, by (3.3), the eigenvalues of \( X(R, R^*) \) have the form

\[ \lambda_C = (-1)^{|C|} \prod_{j \notin C} (r_j - 1), \]

where \( C \subseteq \{1, \ldots, s\} \). Notice that if \( C = \{1, \ldots, s\} \), then

\[ \lambda_C = (-1)^s \]

is an eigenvalue of \( X(R, R^*) \) while \(-\lambda_C = (-1)^{s+1} \) is not. In fact, \( r_i > 2 \) for every \( i = 1, \ldots, s \), since \(|R|\) is odd. This implies that there is some non-trivial character \( \chi \in \hat{R} \) such that \(-e_\chi \notin Spec(G_R) \). By Proposition 2.11, we have that \( X(R, R^*) \) and \( X^+(R, R^*) \) are non-isospectral.

Now, suppose that none of the rings \( R_i \) are fields, hence \( m_i > 1 \) for every \( i = 1, \ldots, s \). Then, the eigenvalues of \( R \) are 0 and \( \lambda_C \) as given in (3.3) and (3.4), that in this case we denote by \( \mu_D \) with \( D \subseteq \{1, \ldots, s\} \). Taking \( D = \{1, \ldots, s\} \) we have that

\[ \mu_D = (-1)^s m_1 \cdots m_s \]

is an eigenvalue of \( G_R \) while \(-\mu_D \) is not. Proceeding as before, this implies that \( G_R \) and \( G^+_R \) are non-isospectral.

In the general case, we can always write

\[ R = F \times L \]
where $F = F_1 \times \cdots \times F_s$ is a reduced ring with $F_i$ a field for all $i = 1, \ldots, s$ and $L = L_1 \times \cdots \times L_t$ where each $L_i$ is a local ring which is not a field, for all $j = 1, \ldots, t$. Hence, $R^* = F^* \times L^*$ and

$$G_R = G_F \otimes G_L.$$

If $C = \{1, \ldots, s\}$ and $D = \{1, \ldots, t\}$, we can take the eigenvalues $\lambda_C$ of $G_F$ and $\mu_D$ of $G_L$ as before. Since it is known that the eigenvalues of the Kronecker product is the product of the eigenvalues of its factors, we have that

$$\lambda_C \mu_D = (-1)^{s+t} m_1 \cdots m_t$$

is an eigenvalue of $G_R$ while $-\lambda_C \mu_D$ is not. Therefore, $G_R$ and $G_R^+$ are non-isospectral.

Finally, the integrality of $Spec(G_R)$ is known and this clearly implies the integrality of $Spec(G_R^+)$. The last assertion follows directly from Proposition 2.11.

\[ \Box \]

4. Generalized Paley sum graphs

Now we consider the group $G$ to be a finite field of $q = p^m$ elements. Note that if $p = 2$ then $X(\mathbb{F}_{2^m}, S) = X^+(\mathbb{F}_{2^m}, S)$ for any subset $S$ of $\mathbb{F}_{2^m}$. However, in odd characteristic we have that $X(\mathbb{F}_q, S) \neq X^+(\mathbb{F}_q, S)$, generically. So, in this section we will assume that $p$ is an odd prime.

In the case of the graphs $\Gamma = X(\mathbb{F}_q, S)$ and $\Gamma^+ = X^+(\mathbb{F}_q, S)$, with $S$ symmetric and $\mathbb{F}_q$ a non-prime field, the relation (2.8) in Proposition 2.11 between the multiplicities of $\Gamma$ and $\Gamma^+$ hold. The reason for this is that in general if $|G|$ is odd then the only real character is the trivial one. Thus, these graphs $\Gamma$ and $\Gamma^+$ are non-isospectral to each other.

If one considers $S$ to be the set of $k$-th powers we get the generalized Paley (sum) graphs $\Gamma(k, q) = X(\mathbb{F}_q, R_k)$ and $\Gamma^+(k, q) = X^+(\mathbb{F}_q, R_k)$ where $R_k = \{x^k : x \in \mathbb{F}_q^*\}$ and $q = p^m$, that we call GP-graphs and GP$^+$-graphs, respectively. The spectrum of $\Gamma(k, q)$ is described in [18]. From this, we will compute the spectrum of $\Gamma^+(k, q)$ and produce another general family of pairs of integral equienergetic non-isospectral graphs.

**Arbitrary GP$^+$-graphs.** It is usual to assume $k \mid q - 1$ when studying GP-graphs, since $\Gamma(k, q) = \Gamma((k, q - 1), q)$. Also, $\Gamma(k, q)$ is simple if $q$ is even or $k \mid \frac{q - 1}{2}$ if $p$ is odd and $\Gamma(k, q)$ is connected if $n = \frac{q - 1}{k}$ is a primitive divisor of $q - 1$. If in addition $k \mid \frac{q - 1}{2}$ the spectrum of $\Gamma(k, q)$ is integral.

The spectrum of $\Gamma(k, q)$ is known and it is given in terms of Gaussian periods

\begin{equation}
\eta_i^{(N, q)} = \sum_{x \in C_i^{(N, q)}} e^{\frac{2\pi i}{p} Tr_{q/p}(x)}, \quad 0 \leq i \leq N, \tag{4.1}
\end{equation}

where $N = (\frac{q - 1}{p - 1}, k)$ and $C_i^{(N, q)} = \omega^i \langle \omega^N \rangle$ is the coset in $\mathbb{F}_q$ of the subgroup $\langle \omega^N \rangle$ of $\mathbb{F}_q^*$ with $\omega$ is a primitive element of $\mathbb{F}_q$ over $\mathbb{F}_p$. Using this and the previous proposition we will get the spectrum of the GP sum graphs $\Gamma^+(k, q)$.

We set first some notations. Let $n = \frac{q - 1}{k}$ and $\eta_0 = \eta_k^{(N, q)}$, $\ldots$, $\eta_{k-1} = \eta_{k-1}^{(N, q)}$ be the Gaussian periods. Let $\eta_i, \ldots, \eta_k$ be the different Gauss periods not equal to $n$ and, for $0 \leq i \leq k - 1$ define the numbers

\begin{equation}
\mu = \# \{0 \leq i \leq k - 1 : \eta_i = n\} \quad \text{and} \quad \mu_i = \# \{0 \leq j \leq k - 1 : \eta_i = \eta_j\} \geq 1. \tag{4.2}
\end{equation}
By Theorem 2.1 in [18] the spectrum of the GP-graph $\Gamma(k, q)$ is given by
\begin{equation}
Spec(\Gamma(k, q)) = \{[n]^{1+\mu_n}, [\eta_1]^{\mu_1 n}, \ldots, [\eta_k]^{\mu_k n}\}.
\end{equation}

It may well happen that $\eta_{ij} = -\eta_{ik}$ for some values. Thus, we rearrange the order of the Gauss periods such that
\[\eta_1 = -\eta_{i_2}, \ldots, \eta_{i_{2t-1}} = -\eta_{i_{2t}}, \eta_{i_{2t+1}}, \ldots, \eta_{i_s}, \quad 0 \leq t \leq \left\lfloor \frac{s}{2} \right\rfloor,
\]
and $\eta_{ij} \neq -\eta_{ik}$ for any $j, k \geq 2t$.

We are now in a position to give the promised result, which follows directly from (4.3) and Propositions 2.5 and 2.11.

**Theorem 4.1.** Let $q = p^m$ with $p$ an odd prime and $m \geq 2$. In the previous notations, the spectrum of $\Gamma^+(k, q)$ is strongly almost symmetric and given by
\begin{equation}
Spec(\Gamma^+(k, q)) = \{[n]^{1+\mu_n}, [-n]^{\frac{m}{2}}, [\eta_1]^{\mu_1 n}, \ldots, [\eta_{2s}]^{\mu_{2s} n}\}
\end{equation}
\[\cup \{[\pm i_{2s+1}]^{\sqrt{\mu_{2s+1} n}}, \ldots, [\pm \eta_s]^{\sqrt{\mu_s n}}\}
\]
where $n = \frac{q-1}{2}$ and $\mu'_{2s-1} = \mu'_{2s} = \frac{1}{2}(\mu_{2s-1} + \mu_{2s})$ for $1 \leq \ell \leq t$. If in addition $k \mid \frac{q-1}{2}$, then $Spec(\Gamma^+(k, q))$ is integral.

The following facts are automatic from previous results.

**Corollary 4.2.** Let $q = p^m$ with $p$ an odd prime and $m \geq 2$. Assume that $k \mid \frac{q-1}{2}$. Then the graphs $\Gamma(k, q)$ and $\Gamma^+(k, q)$ are equienergetic and non-isospectral. If in addition $k \mid \frac{q-1}{2}$ then both graphs are integral.

**Semiprimitive GP-graphs.** Semiprimitive GP-graphs were defined in [18], since they are related with semiprimitive cyclic codes associated to them. They are the graphs $\Gamma(k, q)$ where $q = p^m$ with $m$ even such that
\begin{equation}
k \mid p^t + 1 \quad \text{for some } t \mid \frac{m}{2} \quad \text{and} \quad k \neq p^\frac{m}{2} + 1.
\end{equation}
We say that $(k, q)$ is a semiprimitive pair (of integers).

By Corollary 4.2 (or by Theorems 2.6 and 4.3) we know that $\Gamma(k, q)$ and $\Gamma^+(k, q)$ are integral equienergetic and non-isospectral graphs for every semiprimitive pair of integers $(k, q)$. We will next obtain the spectra of all the semiprimitive GP sum graphs $\Gamma^+(k, q)$. Proposition 2.11 allows us to find the spectrum of $\Gamma^+ = \Gamma^+(k, q)$ from the spectrum of $\Gamma = \Gamma(k, q)$ for $(k, q)$ a semiprimitive pair, and this last spectrum is actually known ([18]).

We need the following notation; define the sign
\begin{equation}
\sigma = (-1)^{s+1}
\end{equation}
where $s = \frac{m}{2t}$ and $t$ is the least integer $j$ such that $k \mid p^j + 1$ (hence $s \geq 1$).

**Theorem 4.3.** Let $q = p^m$ with $p$ an odd prime, $m \geq 2$ even and put $n = \frac{q-1}{2}$. If $(k, q)$ is a semiprimitive pair then the spectra of $\Gamma^+(k, q)$ is integral almost symmetric given by
\begin{equation}
Spec(\Gamma^+(k, q)) = \{[n]^{1}, [\lambda_1]^{\frac{1}{2}}, [\lambda_2]^{\frac{1}{2}(k-1)n}, [-\lambda_1]^{\frac{1}{2}}, [-\lambda_2]^{\frac{1}{2}(k-1)n}\},
\end{equation}
where
\begin{equation}
\lambda_1 = \frac{\sigma(k-1)\sqrt{q-1}}{k}, \quad \lambda_2 = -\frac{\sigma\sqrt{q+1}}{k},
\end{equation}
with $\sigma$ as in (4.6).
Proof. By Theorem 3.3 from [18], the nontrivial eigenvalues of \( \Gamma(k, q) \) are as in (4.8) with multiplicities \( m_1 = n \) and \( m_2 = (k-1)n \), respectively. By Proposition 2.11, we only need to show that \( \lambda_1 \neq -\lambda_2 \). Suppose that \( \lambda_1 = -\lambda_2 \) then we have that \( \sigma \sqrt{q}(k-1) - 1 = \sigma \sqrt{q} + 1 \), and thus \( \sigma \sqrt{q}(k-2) = 2 \) which is impossible, since \( p \) is an odd prime and \( m \geq 2 \).

Example 4.4. Let \( p \) be an odd prime and let \( \ell, m \) be non-negative integers such that \( \ell \mid m \) with \( m \ell = \frac{m}{4} \) even and \( \ell \neq \frac{m}{4} \). The pair \((p^\ell + 1, p^m)\) is clearly semiprimitive. The graphs \( \Gamma(p^\ell + 1, p^m) \) were studied in [17] (denoted \( \Gamma_{p,m}(\ell) \) there). By Theorem 4.3, the spectrum of \( \Gamma^+ = \Gamma^+(p^\ell + 1, p^m) \) is integral and is given by

\[
\text{Spec}(\Gamma^+) = \{[n_\ell]^1, [\nu_\ell]^{\frac{p^\ell - 1}{2}}, [-\nu_\ell]^{\frac{p^\ell - 1}{2}}, [\mu_\ell]^{\frac{p^m - 1}{2}}, [-\mu_\ell]^{\frac{p^m - 1}{2}}\} \subset \mathbb{Z},
\]

where \( \varepsilon_\ell = (-1)^{\frac{p^m - 1}{2}} = -\sigma \) and

\[
n_\ell = \frac{p^m - 1}{p^\ell + 1}, \quad \nu_\ell = \frac{\varepsilon_\ell p^\ell - 1}{p^\ell + 1}, \quad \mu_\ell = -\frac{\varepsilon_\ell p^m + \ell + 1}{p^\ell + 1}.
\]

The graphs \( \Gamma^+(k, q) \) with \( 1 \leq k \leq 4 \). Generalized Paley graphs with \( k = 1, 2 \) are trivial, since \( \Gamma(1, q) = K_q \) and \( \Gamma(2, q) = P(q) \), \( q \equiv 1 \pmod{4} \), are the complete and the classical Paley graphs. The next GP-graphs to consider, aside from the semiprimitive ones, are those with \( k = 3, 4 \).

We now give the spectrum of the graphs \( \Gamma^+(k, q) \) for the lowest values of \( k = 1, 2, 3, 4 \), using that the spectra of \( \Gamma(k, q) \) are already known. The spectra of \( \Gamma(1, q) \) and \( \Gamma(2, q) \) are folklore and the spectra of \( \Gamma(3, q) \) and \( \Gamma(4, q) \) was recently computed in [18]. Notice that \( \Gamma^+(1, q) = \Gamma(1, q) \) and the spectra of \( \Gamma^+(2, q) \) can be calculated from (4.7)–(4.8) since the pair \( (2, q) \) is semiprimitive. This only left us with the cases \( k = 3, 4 \).

Theorem 4.5. Let \( q = p^m \) with \( p \) prime and \( m \in \mathbb{N} \) such that \( 3 \mid \frac{q - 1}{p - 1} \) and \( q \geq 5 \) and put \( n = \frac{q - 1}{3} \). Thus, the graph \( \Gamma^+(3, q) \) has integral almost symmetric spectrum given by:

(a) If \( p \equiv 1 \pmod{3} \) then \( m = 3t \) for some \( t \in \mathbb{N} \) and

\[
\text{Spec}(\Gamma^+(3, q)) = \{[n_\ell]^1, [\pm \frac{\sqrt{3} - 1}{3}]^\frac{n}{2}, [\pm \frac{-2}{3} \sqrt{3} - 1]^\frac{n}{2}, [\pm \frac{-2}{3} \sqrt{3} - 1]^\frac{n}{2}\}
\]

where \( a, b \) are integers uniquely determined by

\[
4 \sqrt{q} = a^2 + 27b^2, \quad a \equiv 1 \pmod{3} \quad \text{and} \quad (a, p) = 1.
\]

(4.9)

(b) If \( p \equiv 2 \pmod{3} \) then \( m = 2t \) for some \( t \in \mathbb{N} \) and

\[
\text{Spec}(\Gamma^+(3, q)) = \begin{cases} 
\{[n_\ell]^1, [\pm \frac{\sqrt{3} - 1}{3}]^\frac{n}{2}, [\pm \frac{-2}{3} \sqrt{3} - 1]^\frac{n}{2}\} & \text{for } m \equiv 0 \pmod{4}, \\
\{[n_\ell]^1, [\pm \frac{2}{3} \sqrt{3} - 1]^\frac{n}{2}, [\pm \frac{-2}{3} \sqrt{3} - 1]^\frac{n}{2}\} & \text{for } m \equiv 2 \pmod{4}.
\end{cases}
\]

This and the next result are straight consequences of Theorem 4.1 and Corollary 4.2 of this paper and Theorems 6.1 and 6.3 in [18], respectively.

Theorem 4.6. Let \( q = p^m \) with \( p \) prime and \( m \in \mathbb{N} \) such that \( 4 \mid \frac{q - 1}{p - 1} \) and \( q \geq 5 \) with \( q \neq 9 \) and put \( n = \frac{q - 1}{4} \). Thus, the graph \( \Gamma^+(4, q) \) has integral almost symmetric spectrum given as follows:
(a) If $p \equiv 1 \pmod{4}$ then $m = 4t$ for some $t \in \mathbb{N}$ and

\[ \text{Spec}(\Gamma^+(4, q)) = \{[n]^1, \left[ \pm \frac{\sqrt{7+4d} \cdot \sqrt{7-1}}{4} \right]^\frac{3m}{t}, \left[ \pm \frac{\sqrt{7-4d} \cdot \sqrt{7-1}}{4} \right]^\frac{3m}{t}, \left[ \pm \frac{\sqrt{7+2c} \cdot \sqrt{7-1}}{4} \right]^\frac{3m}{t}, \left[ \pm \frac{\sqrt{7-2c} \cdot \sqrt{7-1}}{4} \right]^\frac{3m}{t} \} \]

where $c, d$ are integers uniquely determined by

\[ (4.10) \quad \sqrt{q} = c^2 + 4d^2, \quad c \equiv 1 \pmod{4} \quad \text{and} \quad (c, p) = 1. \]

(b) If $p \equiv 3 \pmod{4}$ then $m = 2t$ for some $t \in \mathbb{N}$ and

\[ \text{Spec}(\Gamma^+(4, q)) = \begin{cases} \{[n]^1, \left[ \pm \frac{\sqrt{7-1}}{4} \right]^\frac{3m}{t}, \left[ \pm \frac{3\sqrt{7+1}}{4} \right]^\frac{3m}{t} \} & \text{for } m \equiv 0 \pmod{4}, \\ \{[n]^1, \left[ \pm \frac{\sqrt{7+1}}{4} \right]^\frac{3m}{t}, \left[ \pm \frac{3\sqrt{7-1}}{4} \right]^\frac{3m}{t} \} & \text{for } m \equiv 2 \pmod{4}. \end{cases} \]

Note that the situations in items (b) of the two previous theorems correspond to semiprimitive cases. It is reassuring to check that in these cases the spectra coincide with the ones given by Theorem 4.3.

We now illustrate the previous theorems in the non-primitive cases.

**Example 4.7.** (i) Let $p = 7$ and $m = 3$, hence $q = 7^3 = 343$ and $p \equiv 1 \pmod{3}$. The integers $a = b = 1$ satisfy the conditions in (a) of Theorem 4.5. By Theorem 6.1 in [18] and Theorem 4.5 we have

\[ \text{Spec}(\Gamma(3, 7^3)) = \{[114]^1, [9]^{114}, [2]^{114}, [-12]^{114} \}, \]

\[ \text{Spec}(\Gamma^+(3, 7^3)) = \{[114]^1, [\pm 9]^{57}, [\pm 2]^{57}, [\mp 12]^{57} \}. \]

(ii) Let $p = 5$ and $m = 4$, hence $q = 5^4 = 625$ and $p \equiv 1 \pmod{4}$. The integers $a = -3, b = 2$ satisfy the conditions in (a) of Theorem 4.6. By Theorem 6.4 in [18] and Theorem 4.6 we have

\[ \text{Spec}(\Gamma(4, 5^4)) = \{[156]^1, [16]^{156}, [1]^{156}, [-4]^{156}, [-14]^{156} \}, \]

\[ \text{Spec}(\Gamma^+(5, 7^4)) = \{[156]^1, [\pm 16]^{78}, [\pm 1]^{78}, [\mp 4]^{78}, [\mp 14]^{78} \}. \]

**Remark 4.8.** By the results in this section, we have obtained that \( \{\Gamma(k, q), \Gamma^+(k, q)\} \) for \((k, q)\) semiprimitive, \(\{\Gamma(3, q), \Gamma^+(3, q)\}\) and \(\{\Gamma(4, q), \Gamma^+(4, q)\}\) for any \(q\), are integral equienergetic non-isospectral pairs of graphs with the same number of vertices and edges.

5. ENERGY AND COMPLEMENTARY GRAPHS

Here we address the computation of the energies of the graphs studied in the previous sections as well as of their complements. We recall that the energy of a graph \(\Gamma\) with eigenvalues \(\{\lambda_1, \ldots, \lambda_n\}\) is given by

\[ E(\Gamma) = \sum_{i=1}^{n} |\lambda_i|. \]

Furthermore, we are interested in determining which of these Cayley graphs are equienergetic with their own complements (in the non self-complementary case). We will find some of them and, as a consequence, we will exhibit triples \(\{\Gamma, \Gamma^+, \overline{\Gamma}\}\) of equienergetic non-isospectral Cayley graphs with the same number of vertices and edges both for unitary Cayley graphs and generalized Paley graphs.
We recall that if \( \Gamma \) is a \( k \)-regular graph of \( n \) vertices with non-principal eigenvalues \( \{ \lambda \} \) the complementary graph \( \bar{\Gamma} \) is an \((n - k - 1)\)-regular graph with non-principal eigenvalues \( \{-1 - \lambda\} \). Also, the complement of a Cayley graph \( \Gamma = X(G, S) \) is the Cayley graph \( \bar{\Gamma} = X(G, S^c \setminus \{0\}) \).

**Unitary Cayley graphs.** The energy of a unitary Cayley graph \( G_R = X(R, R^*) \) with \( R \) a finite commutative ring with unity and that of its complement \( \bar{G}_R \) are already known. They were computed by Kiani et al in 2009 ([12], see also [2]). If \( R \) is as in (3.1) then
\[
E(G_R) = 2^s|R^*|,
\]
(5.2)
\[
E(\bar{G}_R) = 2(|R| - 1) + (2^s - 2)|R^*| - \prod_{i=1}^{s} |R_i|/m_i + \prod_{i=1}^{s} (2 - |R_i|/m_i).
\]
In this way, the condition \( E(G_R) = E(\bar{G}_R) \) for equienergy between the graph and its complement is, by (5.2), given by
\[
2(|R| - |R^*| - 1) = \prod_{i=1}^{s} |R_i|/m_i - \prod_{i=1}^{s} (2 - |R_i|/m_i).
\]
(5.3)
We now classify all unitary Cayley graphs \( G_R \) which are equienergetic with their complements, for \( R \) a finite commutative ring with only one or two factors in the artinian decomposition, i.e. for \( s = 1, 2 \) in (3.1).

**Proposition 5.1.** Let \( R \) be a finite commutative ring with unity.

(a) If \( R \) is local with maximal ideal \( m \) then \( E(G_R) = E(\bar{G}_R) \) if and only if \( |R| = m^2 \) where \( m = |m| \) is a prime power (i.e. \( R \) is not a field).

(b) If \( R = R_1 \times R_2 \), then \( E(G_R) = E(\bar{G}_R) \) if and only if \( R_1 = \mathbb{F}_q \) and \( R_2 = \mathbb{F}_{q'} \) are both finite fields (not necessarily distinct).

**Proof.** (a) Let \( r = |R| \) and \( m = |m| \). Notice that \( |R^*| = r - m \) in this case. By (5.3), \( G_R \) and \( \bar{G}_R \) are equienergetic if and only if
\[
2(m - 1) = 2 - \frac{r}{m} = (2 - \frac{r}{m}) = 2(\frac{r}{m} - 1).
\]
This equality holds if and only if \( r = m^2 \). Since \( R/m \) is a field, \( r \) and \( m \) are prime powers.

(b) Now, assume that \( R = R_1 \times R_2 \). Let \( m_1, m_2 \) be the maximal ideals of \( R_1 \) and \( R_2 \), respectively, and put \( r_i = |R_i|, m_i = |m_i| \) and \( q_i = \frac{m_i}{m} \) for \( i = 1, 2 \). Thus, (5.3) takes the form
\[
2(r_1 r_2 - (r_1 - m_1)(r_2 - m_2) - 1) = \frac{r_1 r_2}{m_1 m_2} - (2 - \frac{r_1}{m_1})(2 - \frac{r_2}{m_2}).
\]
Thus, taking into account that \( r_i = q_i m_i \) we have
\[
2(m_1 m_2 q_1 q_2 - m_1 m_2 (q_1 - 1)(q_2 - 1)) - 2 = q_1 q_2 - (2 - q_1)(2 - q_2).
\]
Notice that if we write \( 2 - q_i = 1 - (q_i - 1) \) for \( i = 1, 2 \), then we have that \( (2 - q_1)(2 - q_2) = 1 - (q_1 - 1) - (q_2 - 1) + (q_1 - 1)(q_2 - 1) \) and hence
\[
(2m_1 m_2 - 1)(q_1 q_2 - (q_1 - 1)(q_2 - 1)) = 1 + (q_1 - 1) + (q_2 - 1).
\]
In this way, we get
\[
(2m_1 m_2 - 1)(q_1 + q_2 - 1) = q_1 + q_2 - 1.
\]
Since \( q_i \geq 1 \) for \( i = 1, 2 \), by cancellation we obtain \( 2m_1m_2 - 1 = 1 \) which clearly holds if and only if \( m_1 = m_2 = 1 \). Therefore \( R_1 \) and \( R_2 \) are finite fields, as asserted. \( \square \)

**Example 5.2.** (i) If \( R \) is a quadratic extension of \( \mathbb{Z}_p \), with \( p \) prime, then \( E(G_R) = E(\overline{G}_R) \).

In fact, we have \( r = p^{2t} \), \( m = p^t \). In particular, since the maximal ideal of \( \mathbb{Z}_{p^{2t}} \) is \( \mathbb{Z}_{p^{2t-1}} \), the condition \( r = m^2 \) in (a) of the proposition holds if and only if \( 2t - 1 = t \), i.e. \( t = 1 \). Thus, we can take

\[
R = \mathbb{Z}_{p^2} \quad \text{or} \quad R = \mathbb{Z}_p[x]/(x^2).
\]

The Cayley graphs \( G_R \) for these rings are isomorphic since there is an isomorphism from one to the other taking units onto units. By (a) of Proposition 5.1 we have that \( G_R \) and \( \overline{G}_R \) are equienergetic non-isospectral graphs.

(ii) More generally,

\[
R = \mathbb{F}_{p^t}[x]/(x^2)
\]

is a local ring for any prime \( p \) and \( t \in \mathbb{N} \), whose maximal ideal is the principal ideal \( (x) = x\mathbb{F}_{p^t} \). Thus, we have \( r = m^2 \) and hence by (a) in the proposition, \( G_R \) and \( \overline{G}_R \) are equienergetic non-isospectral graphs.

**Remark 5.3.** In (a) of the proposition, if \( (R, \mathfrak{m}) \) is a local ring then \( G_R \) is the complete \( m \)-multipartite graph

\[
K_{m \times m} = K_{m,m,\ldots,m}
\]

of \( m^2 \) vertices, each part having \( m \) vertices, where \( m = |\mathfrak{m}| \). In (b) of the proposition, if \( R = \mathbb{F}_q \times \mathbb{F}_{q'} \) then \( G_R \) is both a \( q \)-partite and \( q' \)-partite graph. In the particular case when \( R = \mathbb{F}_2 \times \mathbb{F}_q \), the graph \( G_R \) is isomorphic to the complete bipartite graph \( K_{2 \times q} = K_{q,2} \) with a perfect matching removed, i.e. \( G_R \) is the crown graph \( C_{q,1} \) with \( 2q \) vertices (in particular, it is distance regular).

The following is a direct consequence of Theorem 3.4, Proposition 5.1 and Example 5.2.

**Corollary 5.4.** If \( R = \mathbb{F}_{p^t}[x]/(x^2) \) or \( R = \mathbb{F}_q \times \mathbb{F}_{q'} \) with \( p, q \) and \( q' \) odd then \( \{G_R, \overline{G}_R, \overline{G}_R\} \) are integral equienergetic non-isospectral graphs.

Putting together the results of Proposition 5.1 and the comments in Example 5.2 and Remark 5.3 we have the following.

**Theorem 5.5.** Let \( m = p^t \) be a prime power.

(a) The complete \( m \)-multipartite graph \( K_{m \times m} \) of \( m^2 \) vertices is equienergetic and non-isospectral with its complement.

(b) The crown graph \( C_{m,1} \) of \( 2m \) vertices is equienergetic and non-isospectral with its complement.

**Generalized Paley graphs.** Here we study the energy of GP-graphs (and hence of the GP\(^+\)-graphs too) and of their complements. We first deal with the semiprimitive case and then with the graphs \( \Gamma(3, q) \) and \( \Gamma(4, q) \).

For a general graph \( \Gamma(k, q) \), it is hard to give an explicit expression for the energy. However, we can give a divisibility property.

**Proposition 5.6.** Let \( \Gamma = \Gamma(k, q) \) and put \( n = \frac{q+1}{k} \). If \( k \mid \frac{q+1}{p-1} \) then \( n \mid E(\Gamma) \) and \( n \mid E(\overline{\Gamma}) \).
Proof. First note that if \( k \mid \frac{q^2 - 1}{p^2 - 1} \) then both \( \Gamma \) and \( \bar{\Gamma} \) are integral. Since \( \bar{\Gamma} \) is regular of degree \( q - n - 1 = (n - 1)k \), by (4.3), we have that
\[
E(\Gamma) = (\mu n + 1)n + \sum_{i=1}^{s} n\mu_i|\eta_i| = n\alpha,
\]
(5.4)
\[
E(\bar{\Gamma}) = (\mu n + 1)(k - 1)n + \sum_{i=1}^{s} n\mu_i|\eta_i + 1| = n\beta,
\]
with \( \alpha, \beta \in \mathbb{N} \), as it was to be shown. \( \square \)

Semiprimitive pairs \((k, q)\). We now consider semiprimitive GP-graphs and their complements. The complement of \( \Gamma(k, q) \) is also a Cayley graph (although not a GP one); in fact,
\[
\bar{\Gamma}(k, q) = X(\mathbb{F}_q, (R_k)^c \setminus \{0\}).
\]
Thus, we fix the following two families of graphs
\[
(5.5) \quad \mathcal{G} = \{ \Gamma(k, q) : (k, q) \text{ is semiprimitive}\} \quad \text{and} \quad \mathcal{G} = \{ \bar{\Gamma}(k, q) : \Gamma \in \mathcal{G}\}.
\]
Notice that \( \Gamma(2, q) = \bar{\Gamma}(2, q) \), since it is known that Paley graphs are self-complementary. However, for \( k > 2 \), these families of graphs are mutually disjoint. That is, we have \( \mathcal{G} \cap \mathcal{G} = \mathcal{P} = \{ \Gamma(2, q) \} \). These families were previously considered in [18] (see also [17] for an interesting subfamily) where some of their spectral and structural properties were studied.

We now give the energy of the graphs \( \Gamma(k, q) \) and \( \bar{\Gamma}(k, q) \) with \((k, q)\) a semiprimitive pair, improving the result in Proposition 5.6 in this case.

**Proposition 5.7.** The energy of \( \Gamma = \Gamma(k, q) \in \mathcal{G} \cup \mathcal{G} \) is given by
\[
E(\Gamma(k, q)) = \begin{cases} 
2n(k - 1)|\lambda_2| & \text{if } \sigma = 1 \text{ and } \Gamma \in \mathcal{G} \cup \mathcal{G}, \\
2n|\lambda_1| & \text{if } \sigma = -1 \text{ and } \Gamma \in \mathcal{G}, \\
2n(k - 1)(1 + \lambda_2) & \text{if } \sigma = -1 \text{ and } \Gamma \in \mathcal{G},
\end{cases}
\]
(5.6)
where \( n = \frac{q - 1}{k} \), \( \lambda_1 \), \( \lambda_2 \) are as in (4.8) with \( \sigma \) defined in (4.6). In particular, \( 2n \mid E(\Gamma) \).

Proof. The spectra of \( \Gamma = \Gamma(k, q) \) and \( \bar{\Gamma} = \bar{\Gamma}(k, q) \) for \((k, q)\) a semiprimitive pair are known (see Theorem 3.3 in [18]). They are given by
\[
\text{Spec}(\Gamma) = \{ [n]_1^1, [\lambda_1]^n, [\lambda_2]^{(k - 1)n} \},
\]
\[
\text{Spec}(\bar{\Gamma}) = \{ [(k - 1)n]^1, [(k - 1)\lambda_2]^n, [-1 - \lambda_2]^{(k - 1)n} \}.
\]
From these expressions, (5.6) in the statement follows directly. \( \square \)

Let \((k, q)\) a semiprimitive pair. Hence \( q = p^m \) with \( m \) even. Recall that \( \sigma = (-1)^{s+1} \) with \( s = \frac{m}{2} \) where \( t \) is the least integer \( j \) such that \( k \mid p^j + 1 \). We next show that for half the cases, \( \{ \Gamma, \bar{\Gamma} \} \) form an equienergetic non-isospectral pair of graphs.

**Corollary 5.8.** For any semiprimitive pair \((k, q)\) with \( k > 2 \), the graphs \( \Gamma(k, q) \) and \( \bar{\Gamma}(k, q) \) are equienergetic non-isospectral graphs if and only if \( \sigma = 1 \).
Proof. Suppose that $\sigma = 1$. That the graphs $\Gamma$ and $\bar{\Gamma}$ are equienergetic is obvious from (5.6). Also, since $k > 2$, $\Gamma$ and $\bar{\Gamma}$ are non-isospectral since they have different trivial eigenvalues. Now, if we assume $\sigma = -1$, one can check that $E(\Gamma) = E(\bar{\Gamma})$ holds if and only if $k = 2$, which is not possible, thus completing the proof. \qed

Remark 5.9. By Corollaries 4.2 and 5.8, for any semiprimitive pair $(k, q)$ with $k > 2$ such that $k \mid \frac{q^2 - 1}{q - 1}$ and $\sigma = 1$ we have that

$$\{\Gamma(k; q), \Gamma^+(k; q), \bar{\Gamma}(k; q)\}$$

form a triple of equienergetic non-isospectral graphs (all integral if further $k \mid \frac{q^2 - 1}{q - 1}$). Moreover, $\Gamma(k; q)$ and $\bar{\Gamma}(k; q)$ are strongly regular graphs. In fact, they are Latin square graphs (see Proposition 3.5 in [18]), while $\Gamma^+(k; q)$ is not strongly regular. To our best knowledge, this is the first general triple of this kind in the literature (compare with the triple in [7] mentioned in the introduction).

Example 5.10. Let $p$ be a prime and $t \geq 1$. One can check that the pairs $(3, p^{2t})$ with $p \equiv 2 \pmod{3}$ ($t \geq 2$ if $p = 2$), $(3, p^{2t})$ with $p \equiv 3 \pmod{4}$ ($t \geq 2$ if $p = 3$), and $(p^t + 1, p^{2t})$ with $t \mid t$, are semiprimitive pairs. Thus, for each prime $p$, the following

- $\{\Gamma(3, p^{2t}), \Gamma^+(3, p^{2t}), \bar{\Gamma}(3, p^{2t})\}_{t \geq 2}$ with $p \equiv 2 \pmod{3}$,
- $\{\Gamma(4, p^{2t}), \Gamma^+(4, p^{2t}), \bar{\Gamma}(4, p^{2t})\}_{t \geq 2}$ with $p \equiv 3 \pmod{4}$,
- $\{\Gamma(p^t + 1, p^{2t}), \Gamma^+(p^t + 1, p^{2t}), \bar{\Gamma}(p^t + 1, p^{2t})\}_{t \geq 2}$ with $t \mid t$,

are infinite families of triples of integral equienergetic non-isospectral graphs.

6. EQUIENERGETIC NON-ISOSPECTRAL RAMANUJAN GRAPHS

In this final section we consider Ramanujan graphs. We will show that there exist infinite families of equienergetic non-isospectral pairs of graphs, both Ramanujan or one being Ramanujan and the other not. We will use the results of the previous sections and known characterizations of Ramanujan unitary Cayley graphs $G_R$ due to Liu-Zhou ([15], 2012) and of Ramanujan semiprimitive GP-graphs $\Gamma(k; q)$ due to the authors ([17], 2018).

Recall that a connected $n$-regular graph $\Gamma$ is Ramanujan if

$$\lambda(\Gamma) \leq 2\sqrt{n - 1}, \quad (6.1)$$

where $\lambda(\Gamma)$ is the greatest absolute value of the non-principal eigenvalues. There is a more general definition which applies to regular digraphs with loops. A $k$-regular digraph $\Gamma$ is Ramanujan if it satisfies (6.1) and also its adjacency matrix can be diagonalized by a unitary matrix. However, the adjacency matrix of a sum graph $X^+(G, S)$ with $G$ abelian is diagonalizable by a unitary matrix (see Proposition 2 in [13]). So, in the cases we are interested in, namely $G^+_R$ and $\Gamma^+(k; q)$, this condition is automatic and we just have to check (6.1).

Since $G^+_R$ and $\Gamma^+(k; q)$ are strongly almost symmetric graphs, by Theorems 3.4 and 4.1 the graphs $G_R$ and $G^+_R$ are both Ramanujan or both not Ramanujan, respectively. Similarly for $\Gamma(k; q)$ and $\Gamma^+(k; q)$.

Unitary Cayley graphs. Here we will use the characterizations of Ramanujan graphs of the form $G_R$ and $G^+_R$, for $R$ a finite commutative ring, given by Liu and Zhou in 2012 ([15]). For convenience, we distinguish the cases when $R$ is a local ring or not.
Let \( R \) be a local ring. If \( R \) is a local ring with maximal ideal \( m \), by Theorems 11 and 15 in [15], we respectively have that \( G_R \) is Ramanujan if and only if
\[
(6.2) \quad r = 2m \quad \text{or else} \quad r \geq \left( \frac{m}{2} + 1 \right)^2 \quad \text{and} \quad m \neq 2,
\]
where \( r = |R|, m = |m| \), and that \( \bar{G}_R \) is always Ramanujan.

**Theorem 6.1.** Let \((R, m)\) be a finite commutative local ring and put \( r = |R| \) and \( m = |m| \). Then, we have

(a) \( \{G_R, G^+_R\} \) are always equienergetic, non-isospectral if \( r \) is odd, and both Ramanujan if and only if \( r \) and \( m \) satisfy (6.2).

(b) \( \{G_R, \bar{G}_R\} \) are equienergetic non-isospectral graphs if and only if \( r = m^2 \), where \( \bar{G}_R \) is Ramanujan. The graph \( G_R \) is Ramanujan provided \( r \) and \( m \) satisfy conditions (6.2).

**Proof.** (a) The graphs \( G_R \) and \( G^+_R \) are equienergetic by Theorem 2.6 and non-isospectral if \( r \neq 2m \), by Proposition 3.4. The comments at the beginning of the section imply that \( G_R \) and \( G^+_R \) are both Ramanujan or both not Ramanujan.

(b) We know that the graphs \( G_R, \bar{G}_R \) are non-isospectral and by (a) in Proposition 5.1 they are equienergetic if and only if \( r = m^2 \). The remaining assertions follow by the comments before the proposition. \( \square \)

The following is a direct consequence of the previous proposition.

**Corollary 6.2.** Let \( R \) be a local ring with \( r = m^2 \) and \( m \neq 2 \). Then \( \{G_R, G^+_R, \bar{G}_R\} \) are equienergetic non-isospectral Ramanujan graphs.

**Proof.** Note that \( r = m^2 \) and \( m \neq 2 \) imply \( r \neq 2m \). Thus, the graphs \( G_R \) and \( G^+_R \) are Ramanujan if and only if they satisfy the second condition in (6.2), which is clearly implied by \( r = m^2 \). \( \square \)

**Example 6.3.** Let \( R = F_q[x]/(x^2) \) with \( q \neq 2 \). Then, by (ii) in Example 5.2 and the previous proposition, \( \{G_R, G^+_R, \bar{G}_R\} \) is a triple of equienergetic non-isospectral Ramanujan graphs.

Let \( R \) be a non-local ring. Consider \( R \) a finite commutative ring with 1 which is not local. Hence, \( R = R_1 \times \cdots \times R_s \) with \( R_i \) local and \( s \geq 2 \). In this case we can give characterizations for the pairs \( \{G_R, G^+_R\} \) and \( \{G_R, \bar{G}_R\} \) to be equienergetic non-isospectral Ramanujan graphs.

We will need the following condition
\[
(6.3) \quad q_1 \leq q_2 \leq 2(q_1 + \sqrt{(q_1 - 2)q_1}) - 1
\]
for \( q_1, q_2 \) prime powers.

**Theorem 6.4.** Let \( R = R_1 \times R_2 \times \cdots \times R_s \) be a commutative non-local finite ring \( (s \geq 2) \).

(a) If \( |R| \) is odd then \( \{G_R, G^+_R\} \) are equienergetic non-isospectral Ramanujan graphs if and only if \( R = F_3 \times F_3 \times F_3, R = F_3 \times Z_9, R = F_3 \times Z_5[\alpha]/(x^3) \) or \( R = F_{q_1} \times F_{q_2} \) with \( q_1, q_2 \) odd satisfying (6.3).

(b) If \( s = 2 \) then \( \{G_R, \bar{G}_R\} \) are equienergetic non-isospectral graphs if and only if \( R = F_{q_1} \times F_{q_2} \). In this case, \( G_R \) is Ramanujan if and only if \( R = F_3 \times F_3, R = F_3 \times F_4, \)
\[
R = \mathbb{F}_4 \times \mathbb{F}_4 \text{ or } R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \text{ with } q_1, q_2 \text{ prime powers satisfying (6.3) while } \bar{G}_R \text{ is Ramanujan if and only if } q_1 \neq 2 \text{ and }
\]
\[
(6.4) \quad \frac{(q_1 - 1)(q_2 - 1) - 1}{q_1 - 1} \leq \sqrt{(2q_1 - 3)^2 + (4q_1q_2 - 9) - (2q_1 - 3)}.
\]

Therefore, \(\{G_R, G^+_R, \bar{G}_R\}\) are equienergetic non-isospectral Ramanujan graphs if and only if \(R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}\) with \(q_1, q_2\) odd satisfying (6.3) and (6.4).

**Proof.** (a) Since \(|R|\) is odd, the graphs \(G_R\) and \(G^+_R\) are equienergetic and non-isospectral by Theorem 3.4. Also, the only possibilities for \(G_R\) (and hence for \(G^+_R\)) to be Ramanujan are (b) and (e) with \(s = 3\) and (g) with \(s = 2\) in Theorem 12 in [15]. Thus, \(R\) must be one of the four cases in the statement.

(b) The graphs \(G_R, \bar{G}_R\) are non-isospectral, and by (b) in Proposition 5.1 they are equienergetic if and only if \(R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}\) is the product of two finite fields, possibly the same. In this situation, \(G_R\) is Ramanujan if we are in the conditions (b), (c), (d), (f) or (g) in Theorem 12 of [15] (with \(s = 2\)). Hence
\[
R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \text{ with } (q_1, q_2) = (3, 3), (3, 4), (4, 4)
\]
or any prime powers satisfying (6.3). By Theorem 16 in [15] \(\bar{G}_R\), only condition (d) is allowed in this case and hence \(\bar{G}_R\) is Ramanujan for \(R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}\) if \(q_1, q_2\) satisfy (6.4).

The remaining assertion clearly follows by items (a) and (b).

**Example 6.5.** Consider \(R = \mathbb{F}_q \times \mathbb{F}_q\) with \(q\) odd. By the previous theorem, \(\{G_R, G^+_R, \bar{G}_R\}\) is a triple of equienergetic non-isospectral Ramanujan graphs if and only if \(q\) satisfies (6.3) and (6.4). For \(q_1 = q_2 = q\), (6.3) is automatic and (6.4) reads
\[
\frac{(q - 1)^2 - 1}{q - 1} \leq \sqrt{2q(4q - 3) - (2q - 3)}
\]
which only holds only for \(q = 3, 5, 7, 9, 11\) and 13.

**R the ring \(\mathbb{Z}_n\).** We now consider the particular case of \(R\) the ring of integers modulo \(n\).

**Corollary 6.6.** Let \(R = \mathbb{Z}_n\) with \(n\) odd. The triples \(\{G_R, G^+_R, \bar{G}_R\}\) are equienergetic non-isospectral Ramanujan graphs if and only if

(a) \(R = \mathbb{Z}_{2\ell}\) with \(\ell \neq 2\) or \(R = \mathbb{Z}_p, \mathbb{Z}_{p^2}\) with \(p\) an odd prime (in the local case) or,

(b) \(R = \mathbb{Z}_3 \times \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_7\) or \(\mathbb{Z}_5 \times \mathbb{Z}_7\) (in the non-local case).

**Proof.** It follows by the previous results in Sections 3 and 5 and from Corollaries 14 and 17 in [15].

**Generalized Paley graphs.** Here we exhibit infinite pairs or triples of equienergetic non-isospectral semiprimitive GP-graphs which are Ramanujan.

In a previous work, we have characterized all semiprimitive Ramanujan GP-graphs. In fact, by Theorem 4.1 in [18], we have that a semiprimitive GP-graph \(\Gamma(k, q)\) is Ramanujan if and only if \((k, q)\) is one of the given in Table 1 below.

Moreover, \(\bar{\Gamma}(k, p^m)\) is Ramanujan for all semiprimitive pair \((k, p^m)\). Note that in cases (ii) – (viii) the spectra of \(\Gamma(k, q)\) and \(\bar{\Gamma}(k, q)\) are integral.
Table 1. Ramanujan semiprimitive GP-graphs $\Gamma(k, q)$ with $q = p^m$.

| case | pair $(k, q = p^m)$ | conditions |
|------|---------------------|------------|
| (i)  | $(2, q)$            | $q$ odd and $q \equiv 1 \pmod{4}$ (Paley graphs) |
| (ii) | $(3, 2^m)$          | $t \geq 2$ |
| (iii)| $(3, p^m)$          | $p \equiv 2 \pmod{3}$, $p \neq 2$, and $t \geq 1$ |
| (iv) | $(4, 3^m)$          | $t \geq 2$ |
| (v)  | $(4, p^m)$          | $p \equiv 3 \pmod{4}$, $p \neq 3$, and $t \geq 1$ |
| (vi) | $(5, 4^m)$          | $t \geq 2$ |
| (vii)| $(5, p^m)$          | $p \equiv 2, 3 \pmod{5}$, $p \neq 2$, and $t \geq 1$ |
| (viii)| $(5, p^m)$         | $p \equiv 4 \pmod{5}$ and $t \geq 1$ |

As a direct consequence of this and the results in the previous sections, we have the following characterization of all the equienergetic non-isospectral pairs and triples all of them Ramanujan.

Theorem 6.7. Let $\Gamma(k, q)$ be a semiprimitive generalized Paley graph, with $q = p^m$. Then, we have:

(a) $\{\Gamma(k, q), \Gamma^+(k, q)\}$ are equienergetic non-isospectral Ramanujan graphs if and only if $(k, q)$ are as given in Table 1 with $p$ odd.

(b) $\{\Gamma(k, q), \bar{\Gamma}(k, q)\}$ are equienergetic non-isospectral Ramanujan graphs if and only if $(k, q)$ are as given in Table 1 with $t$ odd and $k \neq 2$.

Therefore, $\{\Gamma(k, q), \Gamma^+(k, q), \bar{\Gamma}(k, q)\}$ are integral equienergetic non-isospectral Ramanujan graphs if and only if $(k, q)$ are as given in Table 1 with $p, t$ odd and $k \neq 2$.

Proof. Item (a) follows directly from Corollary 4.2 and Table 1. To probe item (b), notice that the condition $t$ odd implies that $\sigma = 1$ for all of the semiprimitive pairs in (b). Thus, the statement follows directly from Corollary 5.8 and Table 1. The case $(2, q)$, $q \equiv 1 \pmod{4}$ is excluded here since the classic Paley graph $\Gamma(2, q)$ is self-complementary, so they $\Gamma(2, q)$ and $\bar{\Gamma}(2, q)$ are trivially isospectral. \(\square\)

7. Final remarks

We have shown that the Cayley graphs $\Gamma = X(G, S)$ and $\Gamma^+ = X^+(G, S)$ are equienergetic for any finite abelian group $G$ and any symmetric subset $S$, and that they are generically non-isospectral. By considering the subfamilies of unitary Cayley graphs $G_R = X(R, R^*)$ and generalized Paley graphs $\Gamma(k, q)$ we get infinite families of pairs $\{\Gamma_i, \Gamma_i^+\}$ of integral equienergetic non-isospectral graphs (non-bipartite and, since regular, with the same number of edges), which can be taken to be connected and non-bipartite. Furthermore, it is also possible to obtain infinite pairs as before, taking both graphs Ramanujan, or one of them Ramanujan and the other not.

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