5-dimensional Bourgeois contact structures are tight

Jonathan Bowden, Fabio Gironella, Agustin Moreno

Abstract

Given a contact structure on a manifold $V$ together with a supporting open book decomposition, Bourgeois gave an explicit construction for a contact structure on $V \times \mathbb{T}^2$. We prove that all such structures are universally tight in dimension 5, independent on whether the original contact manifold is tight or overtwisted.

1 Introduction

In [Bou02b], Bourgeois showed that, whenever $(V, \xi)$ is a contact manifold endowed with a supporting open book decomposition (which always exist by work of Giroux [Gir02]), then the manifold $V \times \mathbb{T}^2$ carries a natural contact structure. Besides being an appealing and elegant construction, it solved the question of whether odd dimensional tori admit contact structures, something that was open after Lutz showed in [Lut79] that $\mathbb{T}^5$ is contact, more than 20 years before. Therefore, this was, at the time, one of the first important steps in understanding the basic workings of higher-dimensional contact topology.

It was not until recently that Borman-Eliashberg-Murphy [BEM15] proved that contact structures in higher-dimensions exist in abundance (i.e. whenever the obvious topological obstructions disappear) by generalizing the 3-dimensional dichotomy between tight vs. overtwisted to higher dimensions. While overtwisted contact manifolds are flexible and satisfy an $h$-principle, tight contact structures behave more rigidly, and their classification is much more subtle. This then raises the natural question regarding which side of the classification the Bourgeois contact structures belong to.

It was observed in [LMN18] that, in any odd dimension, the Bourgeois contact manifold associated to $(V, \xi)$ can actually be tight, even if $(V, \xi)$ is overtwisted. Moreover, in [Gir17a], it was shown that if $V$ is a 3-manifold with non-zero first Betti number, then there exists a supporting open book such that the associated Bourgeois contact structure is (hyper)tight. In this paper, we prove that, at least
in dimension 5, these are particular instances of a more general fact. Namely, 5-dimensional Bourgeois contact structures are rigid, inherently geometric objects, independently of the rigid or flexible nature of \((V, \xi)\):

**Theorem 1.** Let \((V, \xi)\) be any 3-dimensional contact manifold together with any supporting open book. Then, the associated 5-dimensional Bourgeois contact manifold \((V \times \mathbb{T}^2, \xi_{BO})\) is universally tight.

While there are many well-known ways of making contact/symplectic manifolds more flexible (e.g. by adding a Lutz twist, or by taking the flexibilization of a Weinstein manifold), the above theorem shows that the Bourgeois construction can be interpreted as a “tightifying” procedure. The fact that the contact structures are universally tight is a simple consequence of the fact that finite covers on either factor of the product again yield Bourgeois contact structures. Thus tightness for every Bourgeois contact structure implies tightness for any finite cover. In the case that the initial contact manifold is of dimension three, so that its fundamental group is residually finite (and therefore the same applies to its product with \(\mathbb{T}^2\)), this is equivalent to universal tightness.

It is also worth mentioning that the above result is sharp with respect to taking branched covers. This is a consequence of the following argument by Patrick Massot and Klaus Niederkrüger, based on ideas from [Pre07] (cf. also with [Nie14, Theorem I.5.1], due to Presas). Let \(\Sigma_g\) be an orientable surface of genus \(g \geq 1\). Consider the contact branched cover on \(V \times \Sigma_g \to V \times \mathbb{T}^2\) that is given by some branched cover on the second factor. In the case that the branched cover on the surface has precisely two branch points and \(V\) is an overtwisted contact 3-manifold, one can consider an arc \(\gamma\) on \(\mathbb{T}^2\) joining the two branching points such that it lifts to \(\Sigma_g\) as a closed circle. One can then check that there is an embedded Plastikstufe inside \(V \times \gamma\) that is given parallel transport, with respect to a natural contact connection, along \(\gamma\) of any overtwisted disk on the fiber \(V \times \{q_0\}\). It follows that \(V \times \Sigma_g\) is therefore overtwisted by [CMP19, Hua17]. The interested reader can also consult [Gir17a, Observation 5.10] for further details.

**Outline of the argument.** The proof of Theorem 1 presented here involves some geometric group theory and hyperbolic geometry as well as some holomorphic curve techniques. For the sake of the reader, we give here a rapid sketch of the argument, in the case of a “generic” surface \(\Sigma\). “Non-generic” surfaces (namely the disk, the annulus and pairs of pants) need to be dealt with separately, in a case by case fashion but do not provide any serious difficulties.

Denote by \(BO(\Sigma, \phi)\) the Bourgeois contact manifold associated to an open book \(OBD(\Sigma, \phi)\) supporting a contact structure \(\xi\), where \(\Sigma\) is the page, and \(\phi\) is the monodromy. The first ingredient for the proof in the “generic” surface case is the
construction of a strong symplectic cobordism between Bourgeois contact structures; this is done in Section 3. More precisely, Theorem 3.1 is a “stabilized” version of the analogous result for open books, which has been proved (independently) in [Avd12, Klu18]. We point out that, while the symplectic form on the strong cobordism of Theorem 3.1 is exact, the Liouville vector field associated to the natural global primitive is not inwards pointing along the negative ends. However, the strong convexity property means that there exist local primitives of the symplectic form at the negative ends, such that the associated local Liouville vector fields are indeed inwards pointing. For simplicity, we shall refer to a strong symplectic cobordism with an exact symplectic form as pseudo-Liouville. In our specific setting, the pseudo-Liouville cobordism given by Theorem 3.1 satisfies a useful additional property, namely the compatibility of the local and global Liouville forms on a subset of the negative boundary (see Theorem 3.1 for a precise statement).

The second ingredient is a factorization result for the original monodromy $\phi$. More precisely, we show in Lemma 4.1 that $\phi$ can be factorized into a composition $\phi = \phi_1 \circ \phi_2$, such that the associated abstract open book for each factor $\phi_i$ satisfies that the binding has infinite order in the fundamental group. This passes through a factorization of any monodromy in two pseudo-Anosov diffeomorphisms (cf. with [CH08], where it is shown that one can achieve pseudo-Anosov monodromy via positive stabilizations of the open book), and through a hyperbolic Dehn filling theorem due to Thurston. When combined with an observation on the Reeb dynamics of the Bourgeois forms (see Observation 2.7 below), Lemma 4.1 implies that the Bourgeois contact manifolds $BO(\Sigma, \phi_i)$ associated to each factor is hypertight; see Corollary 4.2.

Then, we assume by contradiction that $BO(\Sigma, \phi)$ is overtwisted. Attaching an exact cobordism on top of the corresponding cobordism $(Q, \Omega)$ in Figure 1, one can suppose that the contact form at the positive end is adapted to a Plastikstufe. A standard application of the holomorphic curve machinery à-la [Hof93, Nie06, AH09] gives then a holomorphic plane in the symplectization of one of the negative ends. We point out that, while bubbles are ruled out by exactness, holomorphic caps at the negative ends are excluded via the explicit properties of the cobordism $(Q, \Omega)$, as well as the specific Reeb dynamics at the negative ends; this is a subtle point. Now, the existence of such holomorphic plane contradicts hypertightness of each connected component of the concave boundary $(Q, \Omega)$, thus concluding the proof.

**Fillability and Higher Dimensions.** While Theorem 1 is one of the first steps in understanding the Bourgeois construction, several relevant questions remain open, some of which have already been asked in the literature (e.g. [LMN18]). Besides tightness in any dimension, the question concerning symplectic fillability of these contact structures seems the most pressing. Indeed, one of the first applications of
the theory of holomorphic curves is that symplectically fillable contact manifolds are tight, as proven in [Eli90, Gro85] for the 3-dimensional case and in [Nie06, BEM15] for the higher dimensional one. Thus, it is natural to wonder about whether Bourgeois contact structures are, at least weakly, symplectically fillable. While there exist partial results in this direction in [LMN18] (see also [MNW13, Example 1.1]), a complete answer is yet to be found. See also Remark 6.1 for subtleties in finding weak fillings. We hope to address the question of fillability in further work.

**Question 1.** Are all Bourgeois contact structures tight in all odd dimensions?

**Question 2.** Are all Bourgeois contact structures weakly fillable?

While we do not expect Bourgeois contact structures to be always strongly fillable, the strong fillability of any Bourgeois contact structure in dimension 5 can actually be reduced to that of $BO(T^*S^1, \tau)$, where $\tau$ is the Dehn twist along the zero section. We give here a quick sketch of the argument, independently found by Francisco Presas.

Using Theorem 3.1, one can reduce the strong fillability question for $BO(\Sigma, \phi)$ to that of $BO(\Sigma, \tau)$, with $\tau$ a (positive or negative) Dehn twist along a simple closed curve. Now, the rest of the surface can be obtained from an annular collar neighbourhood of the curve by attaching 1-handles. In other words, the contact manifold $OBD(\Sigma, \tau)$ is obtained from $OBD(T^*S^1, \tau)$ by subcritical Weinstein handle attachment along the binding of $OBD(T^*S^1, \tau)$. Thus, by an adaptation of the proof of [LMN18, Theorem A.(b)], subcritical Weinstein cobordisms between the original open books translates into Weinstein cobordisms between the associated Bourgeois contact manifolds. We thus obtain a strong cobordism with convex end $BO(\Sigma, \phi)$ and concave end a disjoint union of copies of $BO(T^*S^1, \tau^{\pm1})$. Since $BO(T^*S^1, \tau)$ is contactomorphic to $BO(T^*S^1, \tau^{-1})$ by [LMN18, Theorem B], this proves the claim.
We point out that, while both $BO(T^*S^1, \tau^{\pm 1})$ are weakly fillable, it is far from clear whether they are strongly fillable. Notice also that the same arguments hold in higher dimensions, provided $\phi$ is a product of powers of Dehn-Seidel twists.

**Question 3.** Is $BO(T^*S^1, \tau)$ strongly fillable?

The argument sketched above shows that an affirmative answer to Question 3 implies that every 5-dimensional Bourgeois contact structure is strongly fillable. We should also point out that it was shown in [LMN18] that $BO(T^*S^1, \tau)$ is not subcritically Weinstein fillable.

The preceding discussion appears to suggest the following strategy for showing that Bourgeois contact structures in dimension 5 are weakly fillable, and hence tight. One first decomposes the monodromy into powers of positive and negative Dehn twists, and using the cobordism $(Q, \Omega)$ of Theorem 3.1, we have a cobordism from a disjoint union of contact manifolds $BO(\Sigma, \tau^{\pm 1})$ where $\tau$ is a single Dehn twist. These contact manifolds are indeed weakly fillable and it appears that we would be done. However, there are non-trivial cohomological obstructions to gluing weak fillings in general and these obstructions are indeed a problem for gluing these fillings to the negative ends of the cobordism $(Q, \Omega)$. For a more detailed account of why this gluing fails see Remark 6.1.

**Outline of Paper.** In Section 2 we review the Bourgeois construction, slightly generalising the standard presentation. Section 3 contains our main technical result generalising results of Avdek/Klukas. Section 4 gathers some results from three-dimensional topology and some geometric group theory related to the mapping class group. In Section 5 we prove Theorem 1 and Section 6 contains some further discussion and open problems.

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2 The Bourgeois construction

Consider a smooth manifold $V^{2n-1}$ and an open book decomposition $(B, \theta)$, together with a defining map $\Phi: V \to \mathbb{D}^2$ having each $z \in \text{int}(\mathbb{D}^2)$ as regular value. Here, $B \subset V$ is a closed codimension-2 submanifold, $\theta = \Phi/|\Phi|: V \setminus B \to S^1$ is a fiber bundle, and $\Phi$ is such that $\Phi^{-1}(0) = B$.

A 1-form $\alpha$ on $V$ is said to be adapted to $\Phi$ if it induces a contact structure on the regular fibers of $\Phi$ and if $d\alpha$ is symplectic on the fibers of $\theta = \Phi/|\Phi|$. In particular, if $\xi$ is a contact structure on $V$ supported by $(B, \theta)$, in the sense of [Gir02], then (by definition) there is such a pair $(\alpha, \Phi)$ with $\alpha$ defining $\xi$.

With this notation Bourgeois’ construction can be reformulated as follows:

**Theorem 2.1.** [Bou02b] Consider an open book decomposition $(B, \theta)$ of $V^{2n-1}$, represented by a map $\Phi = (\Phi_1, \Phi_2): V \to \mathbb{R}$ as above, and let $\alpha$ be a 1-form adapted to $\Phi$. Then, $\beta := \alpha + \Phi_1 dq_1 - \Phi_2 dq_2$ is a contact form on $M := V \times \mathbb{T}^2$, where $(q_1, q_2)$ are coordinates on $\mathbb{T}^2$.

The contact form $\beta$ on $M = V \times \mathbb{T}^2$ as above will be a called Bourgeois form associated to $(\alpha, \Phi)$ in the following. An immediate corollary of this formulation (and of Gray’s stability theorem) is the following uniqueness statement, which will be useful in the following sections:

**Corollary 2.2.** If $\alpha_0$ and $\alpha_1$ are homotopic, among 1-forms adapted to $\Phi$, then the associated $\beta_0$ and $\beta_1$ on $M = V \times \mathbb{T}^2$ induce the same contact structure up to isotopy.

**Remark 2.3.** The contact structure determined by a Bourgeois contact form is stable up to contactomorphism under finite covers of the torus factor. For up to precomposing by an automorphism of $\mathbb{T}^2$ any such cover is of the form

$$(q_1, q_2) \mapsto (kq_1, q_2).$$

Pulling back gives a contact form $\beta_k = \alpha + k\Phi_1 dq_1 - \Phi_2 dq_2$ and a straight forward calculation shows that linear interpolation gives a family of contact forms.

**Remark 2.4.** As done in [LMN18], one could reformulate Theorem 2.1 (and Corollary 2.2) using the language of ideal Liouville domains introduced in [Gir17b], without the need to fix the additional data of $\Phi$ and of talking about 1-forms adapted to $\Phi$. However, in our situations, one always has a natural choice for $\Phi$, and all the 1-forms we will deal with are adapted to it, so that the ad-hoc uniqueness statement in Corollary 2.2 is actually all we need.
Abstract open books and Bourgeois contact structures. For the proof of Theorem 1, it is also useful to see the Bourgeois construction above as a map that associates to an abstract contact open book $OBD(\Sigma, \psi)$ a contact form $\beta_{\Sigma, \psi}$ on the product of its underlying smooth manifold with $T^2$. We briefly recall here the construction in order to fix some notation; the reader can consult for instance [Gei08, Section 7.3] for further details.

Consider a Liouville domain $(\Sigma^n, \lambda)$, together with an exact symplectomorphism $\psi$ of $(\Sigma, d\lambda)$ (i.e. $\psi^* \lambda = \lambda - dh$, for some smooth $h : \Sigma \to \mathbb{R}^+$), fixing pointwise a neighborhood of the boundary $B := \partial \Sigma$. One can then consider the mapping torus $\Sigma_\psi$ of $(\Sigma, \psi)$, and the abstract open book $V_{\Sigma, \psi} := (B \times \mathbb{D}^2 \sqcup \Sigma_\psi) / \sim$ where $\sim$ identifies $(p, \varphi) \in \partial(B \times \mathbb{D}^2)$ with $[p, \varphi] \in \partial \Sigma_\psi$. One can also construct a fiberwise Liouville form $\lambda_\psi$ on the mapping torus $\pi_0 : \Sigma_\psi \to S^1$ of $(\Sigma, \psi)$. For large $K \gg 0$ the form $\alpha_K = K \pi_0^* d\varphi + \lambda_\psi$ is contact on $\Sigma_\psi$. The form $\alpha_K$ can then be extended to a contact form on all of $V_{\Sigma, \psi}$ by setting it to be

$$h_1(r) \lambda|_B + h_2(r) d\varphi \text{ on } B \times \mathbb{D}^2,$$

for a well chosen pair of functions $(h_1, h_2)$. We denote the resulting contact form on $V_{\Sigma, \psi}$ by $\alpha_{\Sigma, \psi}$.

In what follows, we will call the manifold $V_{\Sigma, \psi}$ together with the contact structure $\ker(\alpha_{\Sigma, \psi})$ an abstract contact open book, and denote it simply with $OBD(\Sigma, \psi)$; sometimes we will also use the explicit contact form $\alpha_{\Sigma, \psi}$.

We point out that there is a well defined map $\Phi_{\Sigma, \psi} : V_{\Sigma, \psi} \to \mathbb{D}^2$ given by extending the projection to the circle on $\Sigma_\psi$ by setting

$$\Phi_{\Sigma, \psi}|_{B \times \mathbb{D}^2}(p, r, \varphi) = \rho(r)e^{i2\pi \varphi} \in \mathbb{D}^2,$$

for some non-decreasing function $\rho$ satisfying $\rho(r) = r$ near $0$ and $\rho(r) = 1$ near $r = 1$. Notice also that $\alpha_{\Sigma, \psi}$ is naturally adapted to $\Phi_{\Sigma, \psi}$ (as defined above).

We then denote by $\beta_{\Sigma, \psi}$ the Bourgeois form on $M_{\Sigma, \psi} := V_{\Sigma, \psi} \times T^2$ associated to $(\alpha_{\Sigma, \psi}, \Phi)$ as in Theorem 2.1, by $\xi_{\Sigma, \psi}$ the contact structure it defines. Finally we let $BO(\Sigma, \psi) := (M_{\Sigma, \psi}, \xi_{\Sigma, \psi})$.

Hypertightness for Bourgeois Contact Forms. In the following sections, we will also need the hypertightness criterion for $\alpha_{\Sigma, \psi}$ from [Gir17a, Corollary 6.3]. In order to state it, we first give a definition which will be useful to encode the key behavior of the dynamics of some Reeb vector fields for $\xi_{\Sigma, \psi}$:
Definition 2.5. Let \((V \times T^2, \xi)\) be a contact manifold, and \(\mathcal{B}\) any subset of the set of closed Reeb orbits of a contact form \(\beta\). We say that \(\beta\) has \(T^2\)-trivial Reeb dynamics concentrated in \(\mathcal{B}\) if the image of every closed Reeb orbit not in \(\mathcal{B}\) under the projection \(V \times T^2 \to T^2\) is homotopically non-trivial.

Remark 2.6. It is a trivial consequence of the above definition that, if every Reeb orbit in \(\mathcal{B}\) has non-trivial image via the map induced by the projection \(V \times T^2 \to V\) at the \(\pi_1\)-level, then the Reeb flow associated to the contact form \(\beta\) has no contractible closed Reeb orbit, hence \((V \times T^2, \ker(\beta))\) is hypertight.

Let us now go back to the case of Bourgeois contact forms. An straightforward computation then gives the following:

Observation 2.7. \([\text{Gir17a}]\) The Bourgeois contact form \(\beta_{\Sigma, \psi}\) for \(\xi_{\Sigma, \psi}\) has \(T^2\)-trivial Reeb dynamics concentrated in the set \(\mathcal{B}\) consisting of the submanifolds \(\gamma_B \times \{q\} \subset V \times T^2\), for all \(q \in T^2\) and all \(\gamma_B\) closed Reeb orbit of \((B, \alpha_{\Sigma, \psi}|_B)\). In particular, if the binding \((B, \alpha_{\Sigma, \psi}|_B)\) of the natural open book of \(V_{\Sigma, \psi}\) admits no contractible Reeb orbits inside \(V_{\Sigma, \psi}\), then the Bourgeois contact structure \(\xi_{\Sigma, \psi}\) is hypertight.

Notice that, in the 3-dimensional case, Observation 2.7 implies that, if the binding consists of a collection of loops each having infinite order in \(\pi_1(V)\), then the associated Bourgeois contact structure is hypertight.

We point out, though, that we will not make use of Observation 2.7 in the proof of Theorem 1; rather, we will apply directly Remark 2.6 on a another contact form, which still defines the Bourgeois contact structure up to isotopy (see Lemma 5.1 below).

3 Cobordisms of Bourgeois contact structures

Let \((\Sigma^{2n-2}, \lambda)\) be a Liouville manifold, and let \(\phi\) be an exact symplectomorphism relative to the boundary. Notice that the boundary \((B, \alpha_B) := (\partial \Sigma, \lambda|_{\partial \Sigma})\) can naturally be seen as the “binding” submanifold of the associated open book. For each \(q \in T^2\), we also let then \(B_q\) be \(B \times \{q\} \subset V_{\Sigma, \phi} \times T^2 = M_{\Sigma, \phi}\).

The aim of this section is to give a proof of the following result:

Theorem 3.1. There is a smooth cobordism \(Q\) from \(M_{\Sigma, \psi} \sqcup M_{\Sigma, \phi}\) to \(M_{\Sigma, \psi\circ \phi}\) (recall Figure 1). This cobordism is smoothly a product \(Q_0 \times T^2\), where \(Q_0\) a smooth cobordism from \(V_{\Sigma, \psi} \sqcup V_{\Sigma, \phi}\) to \(V_{\Sigma, \psi\circ \phi}\). Moreover, there is a symplectic form \(\Omega\) on \(Q\) which satisfies the following properties:

1. \(\Omega\) admits local Liouville forms \(\lambda_+\) and \(\lambda_-\) near \(M_{\Sigma, \psi\circ \phi}\) and \(M_{\Sigma, \psi} \sqcup M_{\Sigma, \phi}\) respectively, satisfying:
(a) $\lambda_+$ induces a contact form on $M_{\Sigma, \psi \phi}$ which defines, up to isotopy, the contact structure $\ker(\beta_{\Sigma, \psi \phi})$.

(b) $\lambda_- \induces a contact form on M_{\Sigma, \psi} \sqcup M_{\Sigma, \phi}$, which has (on each connected component) $T^2$-trivial Reeb dynamics concentrated in $\{ B_q \}_{q \in T^2}$, and is homotopic to $\beta_{\Sigma, \phi}$, through contact forms whose restriction to each $B_q$ is $\alpha_B$ (up to a positive scalar multiple);

2. $\Omega$ admits a global primitive $\nu$ which coincides with $\lambda_+$ on at the convex boundary and such that $\nu|_{B_q} = \lambda_-|_{B_q}$ for each $B_q \subset M_{\Sigma, \psi} \sqcup M_{\Sigma, \phi}$.

Item 1 means in particular that, up to attaching cobordisms at its ends, $(Q, \Omega)$ is a strong symplectic cobordism with convex boundary $BO(\Sigma, \phi \circ \psi)$ and concave boundary $BO(\Sigma, \phi) \sqcup BO(\Sigma, \psi)$.

Notice that we do not claim that the global 1-form $\nu$ defines a contact structure at the concave boundary. In other words, the cobordism we give is not claimed to be Liouville (see Remark 3.2), but just pseudo-Liouville (as defined in the introduction).

Lastly, we point out that Theorem 3.1 can be thought of as a “stabilized” version of [Klu18, Theorem 1] and [Avd12, Proposition 8.3]. In fact, smoothly (but not symplectically), the cobordism $Q$ is just the product of the cobordism from [Klu18, Avd12] with $T^2$.

We now give a proof of Theorem 3.1, following very closely, with some adaptations, the proof given in [Klu18].

3.1 Some preliminaries

We start by completing the Liouville domain $(\Sigma, \omega = d\lambda)$ and by defining an auxiliary function $\tau$ on the completion.

Let $Y$ be the Liouville vector field on $(\Sigma, \omega = d\lambda)$ defined by $\iota_Y \omega = \lambda$. By the definition of Liouville manifold, $Y$ is positively transverse to $B = \partial \Sigma$. Consider a collar neighborhood $(-\delta, 0] \times B$ of $B$ inside $\Sigma$, with coordinates $(t, q) \in (-\delta, 0] \times B$, where $Y = \partial_t$. We then extend $(\Sigma, \lambda)$ to a complete Liouville manifold $(\hat{\Sigma}, \hat{\lambda})$ given by setting $\hat{\Sigma} = \Sigma \cup [0, +\infty) \times B$ and

$$\hat{\lambda} = \begin{cases} 
\lambda & \text{on } \Sigma \\
\ e^t \lambda|_B & \text{on } [0, +\infty) \times B.
\end{cases}$$

For simplicity, we will denote the contact form $\lambda|_B$ by $\alpha_B$ in the following. Let also $\hat{\omega} = d\hat{\lambda}$. We denote by $\hat{Y}$ the natural extension of this Liouville vector field on $\Sigma$ to $\hat{\Sigma}$.

Let now $\tau: \hat{\Sigma} \rightarrow \mathbb{R}_{>0}$ be a smooth function such that:

1. $\tau = -\delta$ on $\hat{\Sigma} \setminus (-\delta, +\infty) \times B$;
2. $\frac{\partial \tau}{\partial t} > 0$ and $\tau = \tau(t)$ on $(-\delta, +\infty) \times B$;

3. $\tau(t, q) = t$ on $[0, +\infty) \times B$.

Notice in particular that $Y$ is gradient-like for $\tau$ on $(-\delta, +\infty) \times B$.

A toroidal pair of pants cobordism in dimension 4. We now describe a strong 4-dimensional symplectic cobordism with concave end consisting of two disjoint copies of the standard tight contact structure $\xi_{std}$ on $T^3$ and convex end consisting of a single copy of it. Consider the unit disk cotangent bundle $D^*T^2$ of $T^2$, together with its standard symplectic structure $\omega_{std} = d\lambda_{std}$, where in coordinates $\lambda_{std} = p_1 dq_1 + p_2 dq_2$ is the standard Liouville form. To be precise, we need to work with scalar multiples $K\omega_{std}$ and $K\lambda_{std}$, where $K$ is a positive real constant that will be determined later on in the proof. This gives a Weinstein (hence strong) symplectic filling of $(T^3, \xi_{std})$. We denote by $X$ the Liouville vector field $p_1 \partial_{p_1} + p_2 \partial_{p_2}$.

Consider the submanifold $D^*_{\epsilon} T^2$ of $D^*T^2$ made of those covectors of norm less than a certain $\epsilon < 1/10$, and denote by $j_{\pm}: D^*_{\epsilon} T^2 \to D^*T^2$ the symplectomorphisms

$$(p_1, q_1, p_2, q_2) \mapsto (p_1 \pm 1/2, q_1, p_2, q_2).$$

For ease of notation, we consider the inclusion

$$j = j_- \sqcup j_+: (D^* T^2, K\omega_{std}) \sqcup (D^* T^2, K\lambda_{std}) \to (D^*T^2, K\omega_{std}).$$

Then, the desired cobordism is $(C, \omega_C) := (D^* T^2 \setminus j(D^* T^2), K\omega_{std})$. The Liouville field on a neighbourhood of the convex boundary is just given by $X$, whereas the one near the concave boundary is given by $j_+ X$, which is the vector field $(p_1 + 1/2) \partial_{p_1} + p_2 \partial_{p_2}$ on the image of $j_{\pm}$ respectively.

Remark 3.2. Observe that the Liouville vector field $X$ is not inward pointing along the boundary of the image of $j$. Here is where the Liouville condition fails, as explained after the statement of the theorem. In fact, one can prove that there is no Liouville cobordism having two disjoint copies of $(T^3, \xi_{std})$ as concave boundary and as convex boundary $(T^3, \xi_{std})$. This can be shown using the classification (up to symplectic deformation and blowups) of strong fillings of $(T^3, \xi_{std})$ in [Wen10], the nearby Lagrangian conjecture for $T^2$ proven in [DRGI16] and the fact that the Lagrangian Floer homology of the zero section of a cotangent bundle is non-trivial. As a consequence, it seems rather unlikely that the cobordism in Theorem 3.1 can be upgraded to a Liouville cobordism.

Lastly, we consider an auxiliary smooth function $f: T^*T^2 \to \mathbb{R}$, which satisfies:

1. $f = p_1^2 + p_2^2$ on $T^*T^2 \setminus D^*T^2$,
Figure 2: Picture of the submanifold (with corners) $P_{\text{top}} \subset \hat{\Sigma} \times T^{\ast}T^2$; here, we identify $T^{\ast}T^2 = \mathbb{R}^2 \times T^2$ and $\hat{\Sigma} \setminus \Sigma = B \times \mathbb{R}^2$, with coordinate $t \in [0, +\infty)$.

2. $f = (p_1 \mp 1/2)^2 + p^2_2$ on the image of $j_{\pm}$ respectively,

3. $\epsilon < f < 1$ on the interior of $C$.

Notice that Items 1 and 2 imply in particular that $X$ is gradient-like for $f$ in a neighborhood of $\partial(D^*T^2)$, and $j_*X$ is gradient like for $f$ on all the image of $j$ (and not only on a neighbourhood of the concave boundary of the cobordism $C$).

### 3.2 Description of the cobordism

We start by describing the strong cobordism $(Q, \Omega)$ as obtained by gluing two pieces, which are described separately. We then conclude by verifying that it also satisfies the desired properties as in Theorem 3.1.

**Piece 1.** Consider $(\hat{\Sigma} \times T^{\ast}T^2, \hat{\Omega})$, where $\hat{\Omega} = \hat{\omega} + K\omega_{\text{std}}$, where $\omega_{\text{std}}$ denotes the standard symplectic form on $T^{\ast}T^2$ and $K > 0$ will be determined later on. We then define

\[
P_{\text{top}} := \left\{ (x, q, p) \in \hat{\Sigma} \times T^{\ast}T^2 \mid \tau \geq 0, \tau^2 + f^2 \geq \epsilon^2, \tau^2 + |p|^2 \leq 1 \right\}, \tag{2}
\]

where $|p| = \sqrt{p_1^2 + p_2^2}$; see Figure 2. The boundary of $(P_{\text{top}}, \hat{\Omega})$ has the following two “distinguished portions”:

\[
\partial_-P_{\text{top}} := \left\{ (x, p, q) \in \hat{\Sigma} \times T^{\ast}T^2 \mid \tau \geq 0, \tau^2 + f^2 = \epsilon^2 \right\},
\]

\[
\partial_+P_{\text{top}} := \left\{ (x, p, q) \in \hat{\Sigma} \times T^{\ast}T^2 \mid \tau \geq 0, \tau^2 + |p|^2 = 1 \right\}
\]

Moreover, there are Liouville forms on $P_{\text{top}}$ as follows:
\[ P'_{\text{bot}} = \Sigma \times \mathbb{T}^2 \]

Figure 3: Picture of the submanifold (with corners) \( P'_{\text{bot}} \subset \hat{\Sigma} \times T^*\mathbb{T}^2 \).

1. \( \nu_{\text{top}} := \iota_{(Y + X)}\hat{\Omega} \) on all of \( P_{\text{top}} \);

2. \( \lambda_{\text{top}} := \iota_{(Y + j, X)}\hat{\Omega} \) near \( \partial_{-}P_{\text{top}} \).

We also denote by \( \lambda_{\text{top}}^\pm \) the restriction of \( \nu_{\text{top}} \) near \( \partial_{\pm}P_{\text{top}} \).

An explicit computation shows that \( (Y + X) \notin \partial_{+}P_{\text{top}} \) and \( (Y + j, X) \notin \partial_{-}P_{\text{top}} \), so that \( \lambda_{\text{top}}^\pm \) induces a contact structure on \( \partial_{\pm}P_{\text{top}} \). Notice however that \( \nu_{\text{top}} \) does not induce a contact structure on \( \partial_{-}P_{\text{top}} \).

**Piece 2.** Consider the subset

\[ P'_{\text{bot}} := \{(x, q, p) \in \hat{\Sigma} \times T^*\mathbb{T}^2 \mid \tau \leq 0, \epsilon \leq f \leq 1\} = \Sigma \times C \subset \hat{\Sigma} \times T^*\mathbb{T}^2, \]

as pictured in Figure 3. Here, there are again “distinguished portions” of the boundary, whose union is just \( \Sigma \times \partial C \):

\[ \partial_{-}P'_{\text{bot}} := \left\{ (x, p, q) \in \hat{\Sigma} \times T^*\mathbb{T}^2 \mid \tau \leq 0, f = \epsilon \right\}, \]

\[ \partial_{+}P'_{\text{bot}} := \left\{ (x, p, q) \in \hat{\Sigma} \times T^*\mathbb{T}^2 \mid \tau \leq 0, f = 1 \right\}; \]

Moreover, we have the obvious symplectic form \( \Omega'_{\text{bot}} \) on \( P'_{\text{bot}} \subset \Sigma \times T^*\mathbb{T}^2 \) given by restricting \( \omega + K\omega_{\text{std}} = d\lambda + Kd\lambda_{\text{std}} \). We set \( \nu'_{\text{bot}} := \lambda + K\lambda_{\text{std}} \) on all \( P'_{\text{bot}} \) to be the obvious primitive. We then have the following local Liouville forms:

1. \( \lambda_{\text{bot}}^+ = \nu'_{\text{bot}} = \lambda + K\lambda_{\text{std}} \) on a neighborhood of \( \partial_{+}P'_{\text{bot}} \); and

2. \( \lambda_{\text{bot}}^- = \lambda + K(j^{-1})^*\lambda_{\text{std}} \) on a neighborhood of \( \partial_{-}P'_{\text{bot}} \).

**Remark 3.3.** In the case of trivial monodromies \( \phi = \psi = \text{Id} \) in the hypothesis of Theorem 3.1, the desired \((Q, \Omega)\) is just given by gluing \( P_{\text{top}} \) and \( P'_{\text{bot}} \) along the subsets where \( \{\tau = 0\} \). Thus, in this case we have

\[ Q = \left\{ \tau \geq 0, \tau^2 + f^2 \geq \epsilon^2, \tau^2 + |p|^2 \leq 1 \right\} \cup \{ \tau \leq 0, \epsilon \leq f \leq 1 \} \subset \hat{\Sigma} \times T^*\mathbb{T}^2. \]

Notice that the symplectic structures and Liouville forms on the two pieces trivially glue in this case (to give the restriction of the ones of the ambient space \( \hat{\Sigma} \times T^*\mathbb{T}^2 \)).
However, the case of trivial monodromies is not very interesting for our purposes, because we already know that the resulting Bourgeois contact structures are Stein fillable according to [LMN18], and therefore tight.

We now modify $P'_{bot}$ by cutting and regluing to take into account the monodromies $\phi$ and $\psi$ as follows. Consider the two submanifolds

\[ T_1 := \{(p_1, q_1, p_2, q_2) \in \mathbb{R}^4 | p_1 \in [-1, -1/2 - \epsilon], p_2 = 0\}, \]
\[ T_2 := \{(p_1, q_1, p_2, q_2) \in \mathbb{R}^4 | p_1 \in [1/2 + \epsilon, 1], p_2 = 0\}, \]

and denote by $T$ their disjoint union $T_1 \sqcup T_2$. We can then cut $P'_{bot}$ along $\Sigma \times T$ and glue back via the diffeomorphism $\Psi$ of $\Sigma \times T$ defined as follows:

\[ \Psi(x, q) = \begin{cases} 
(\phi^{-1}(x), q) & \text{if } q \in T_1, \\
(\psi(x), q) & \text{if } q \in T_2.
\end{cases} \]

See Figure 4. We denote the result of this cut and paste procedure by $P_{bot}$. Notice that there is a natural fiber bundle structure $\pi: P_{bot} \to C$, with fiber $\Sigma$. Moreover, as $\phi$ and $\psi$ act trivially near $\partial \Sigma$, $P_{bot}$ is just a product $B \times (-\epsilon, 0] \times C$ near its boundary component $B \times C$. Lastly, we point out that there are again two “distinguished portions” of the boundary of $P_{bot}$, denoted by $\partial_- P_{bot}$ and $\partial_+ P_{bot}$, which come from $\partial_- P'_{bot}$ and $\partial_+ P'_{bot}$ respectively.

As both $\phi$ and $\psi$ are symplectomorphisms of $(\Sigma, \omega)$, the symplectic form $\omega$ on $\Sigma$ induces a 2-form $\omega_{fib}$ on $P_{bot}$ which is symplectic on each fiber of $\pi$. We then consider $\Omega_{bot} = \omega_{fib} + K\pi^* \omega_{std}$, where $\omega_{std}$ is the standard symplectic structure on $C \subset T^* \mathbb{T}^2$.

Since the monodromies are given by exact symplectomorphisms, relative to the boundary, one also has the following:

**Lemma 3.4.** There is a primitive $\lambda_{fib}$ of $\omega_{fib}$ on $P_{bot}$ that is Liouville on each fiber of $\pi$ and such that:

1. $\lambda_{fib} = e^t \lambda_B = e^t \alpha_B$ on the neighborhood $B \times (-\epsilon, 0] \times C$ of $B \times C$ in $P_{bot}$;
2. The restriction to \( \partial_+ P_{\text{bot}} = \Sigma_{\phi, \psi} \times T^2 \) and \( \partial_- P_{\text{bot}} = (\Sigma_\phi \sqcup \Sigma_\psi) \times T^2 \) coincides, respectively, with \( \lambda_{\phi, \psi} \) and \( \lambda_\phi \sqcup \lambda_\psi \) (defined as in Section 2). Consequently we have the following two Liouville forms for \( \Omega_{\text{bot}} \):

1. \( \nu_{\text{bot}} := \lambda_{\text{fib}} + K \pi^* \lambda_{\text{std}} \) on all \( P_{\text{bot}} \);

2. \( \lambda_{\text{bot}}^- := \lambda_{\text{fib}} + K \pi^* (j_\lambda \lambda_{\text{std}}) \) on a neighborhood of \( \partial_- P_{\text{bot}} \).

We also denote by \( \lambda_{\text{bot}}^+ \) the restriction of \( \nu_{\text{bot}} \) to a neighborhood of \( \partial_+ P_{\text{bot}} \).

An explicit computation shows that, for \( K > 0 \) sufficiently large, \( \lambda_{\text{bot}}^\pm \) induces a contact structure on \( \partial_{\pm} P_{\text{bot}} \); we hence fix \( K \) such that this contact condition at the boundary is satisfied. Note however that \( \nu_{\text{bot}} \) does not induce a contact structure on \( \partial_- P_{\text{bot}} \).

The cobordism \( Q \). Notice that, as the monodromies are trivial near \( \partial \Sigma \), the pieces \( P_{\text{top}} \) and \( P_{\text{bot}} \) fit together well along \( \{ \tau = 0 \} \subset P_{\text{top}} \) and \( \{ \tau = 0 \} \subset P_{\text{bot}} \) and we denote the glued manifold by \( Q \).

Obviously, \( Q \) is smoothly a product \( Q_0 \times T^2 \). Moreover, the boundary of \( Q \) decomposes naturally into positive and negative parts \( \partial_- Q := \partial_- P_{\text{bot}} \cup \partial_- P_{\text{top}} \) and \( \partial_+ Q := \partial_+ P_{\text{bot}} \cup \partial_+ P_{\text{top}} \) respectively.

From the symplectic point of view, \( Q \) inherits a symplectic structure \( \Omega \) from \( P_{\text{bot}} \) and \( P_{\text{top}} \). Moreover, the Liouville forms \( \nu_{\text{top}}, \lambda_{\text{top}}^+ \) and \( \lambda_{\text{top}}^- \) on \( P_{\text{top}} \) glue together with their \( \text{bot} \)-counterparts on \( P_{\text{bot}} \), giving a global primitive \( \nu \) for the symplectic form \( \Omega \) and local Liouville forms \( \lambda_+ \) and \( \lambda_- \) on the respective boundary components.

For each \( q \in T^2 \), we now denote by \( B_q^\pm \) the submanifold (actually, circles, when \( n = 2 \)) given by the subset \( \{ \tau = \max \partial_\pm Q(\tau) \} \cap (Q_0 \times \{ q \}) \) of \( Q = Q_0 \times T^2 \). The rest of the proof of Theorem 3.1 is now slightly technical. We hence summarize all that is left to prove in the following lemma:

**Lemma 3.5.** The 1-forms \( \nu, \lambda_+ \) and \( \lambda_- \) described above satisfy the following properties:

1. The restrictions \( \lambda_+|_{\partial_+ Q} \) and \( \lambda_-|_{\partial_- Q} \) have \( T^2 \)-trivial Reeb dynamics concentrated in \( B^\pm = \{ B_q^\pm \}_{q \in T^2} \);

2. For each \( q \in T^2 \), we have \( \nu|_{B_q^-} = \lambda_-|_{B_q^-} \);

3. The contact manifolds \( (\partial_+ Q, \ker(\lambda_+|_{\partial_+ Q})) \) and \( (\partial_- Q, \ker(\lambda_-|_{\partial_- Q})) \) are contactomorphic to

\[
(M_{\Sigma_{\phi, \psi}}, \ker(\beta_{\Sigma_{\phi, \psi}})) \quad \text{and} \quad (M_{\Sigma_{\phi}}, \ker(\beta_{\Sigma_{\phi}})) \cup (M_{\Sigma_{\psi}}, \ker(\beta_{\Sigma_{\psi}}))
\]

respectively. Moreover, under this contactomorphism, \( \partial_+ Q_0 \) and \( \partial_- Q_0 \) are identified with \( V_{\Sigma_{\phi, \psi}}, V_{\Sigma_{\phi}} \sqcup V_{\Sigma_{\psi}} \) respectively, and \( B^+_q \) is identified with \( B^\pm \times \{ q \} \), where \( B^+, B^- \) is the binding of \( V_{\Sigma_{\phi, \psi}}, V_{\Sigma_{\phi}} \sqcup V_{\Sigma_{\psi}} \) respectively.
Proof (Lemma 3.5). Item 2 follows from a straightforward computation; its proof is hence omitted.

We give an explicit proof of Item 1 in the case of $\partial_{+}Q$; the proof in the case of the other connected components is completely analogous.

We start by analyzing the Reeb dynamics on $\partial_{+}P_{\text{top}}$. Recall in Section 3.1 we defined a real valued function $\tau$ on $\hat{\Sigma}$, which just coincides with the coordinate $t \in [0, +\infty)$ on $\hat{\Sigma} \setminus \text{int}(\Sigma) = [0, +\infty) \times B$. Then, $\partial_{+}P_{\text{top}}$ admits a parametrization

$$B \times D^{*}T^{2} \rightarrow \partial_{+}P_{\text{top}} \subset [0, +\infty) \times B \times D^{*}T^{2}$$

$$(x, p_{1}, q_{1}, p_{2}, q_{2}) \mapsto (\sqrt{1 - |p|^{2}}, x, p_{1}, q_{1}, p_{2}, q_{2})$$

where $|p| = \sqrt{p_{1}^{2} + p_{2}^{2}}$. In this parametrization, $\lambda_{+}|_{\partial_{+}Q}$ becomes

$$e^{\sqrt{1 - |p|^{2}} \alpha_{B} + Kp_{1}dq_{1} + Kp_{2}dq_{2}},$$

and an explicit computation shows that its Reeb vector field is of the form

$$\alpha_{B} + Kp_{1}dq_{1} + Kp_{2}dq_{2},$$

and its Reeb vector field is of the form

$$F(|p|)R_{B} - G(|p|) (p_{1}\partial_{q_{1}} + p_{2}\partial_{q_{2}}),$$

where $F, G: [0, 1) \rightarrow \mathbb{R}_{>0}$ are smooth functions such that $F(0) = 1$, $G(0) = 0$ and $G(|p|) > 0$ if $|p| > 0$. In particular, it has $T^{2}$-trivial Reeb dynamics concentrated in $B^{2} = \{p_{1} = p_{2} = 0\} \cap \partial_{+}P_{\text{top}}$, which consists exactly of orbits of the form $B^{2}_{q}$ for $q \in T^{2}$, as desired. Notice moreover that the orbit of every point in $\partial_{+}P_{\text{top}}$ is entirely contained in $\partial_{+}P_{\text{top}}$ (i.e. does not enter $\partial_{+}P_{\text{bot}}$).

We now analyze the orbits in $\partial_{+}P_{\text{bot}}$. Recall that on the total space of $\pi: P_{\text{bot}} \rightarrow C$ we have

$$\lambda_{+} = \lambda_{\text{fib}} + K\tau \lambda_{\text{std}} = \lambda_{\text{fib}} + K\tau (\mathcal{L}_{\omega_{\text{std}}}) \cdot$$

Hence, the sub-bundle $\ker(d\lambda_{+}|_{\partial_{+}P_{\text{bot}}})$ of $T(\partial_{+}P_{\text{bot}})$ projects, via the differential of the projection $\partial_{+}P_{\text{bot}} \rightarrow S^{*}T^{2}$ induced by $\pi$, to the sub-bundle of $T(S^{*}T^{2})$ given by the kernel of $d(\sin \theta dq_{1} + \cos \theta dq_{2})$. Here, $\theta$ is the angular coordinate associated to the coordinates $(p_{1}, p_{2})$ on the cotangent fibers of $T^{*}T^{2}$. In particular, the Reeb orbit of every point in $(\partial_{+}P_{\text{bot}}, \lambda_{+}|_{\partial_{+}P_{\text{bot}}})$ stays entirely in $\partial_{+}P_{\text{bot}}$ and, in the case of closed orbits, has homotopically non-trivial projection on $T^{2}$. This concludes the proof of Item 1.

The only thing left to prove is Item 3. We describe how to obtain a contactomorphism of a connected component of $(\partial_{+}Q, \ker(\lambda_{+}|_{\partial_{+}Q}))$ with $(M_{\Sigma, \phi_{0}}, \xi_{\Sigma, \phi_{0}})$; the other components of $\partial Q$ can be dealt with in a similar fashion. As already remarked above, $\partial_{Q} = \partial_{+}Q_{0} \times T^{2}$; more precisely, the parametrization given above and Lemma 3.4 give a natural identification $\partial_{+}Q_{0} = V_{\Sigma, \phi_{0}}$ as smooth manifolds. Consider then the following linear interpolation of 1-forms on $V_{\Sigma, \phi_{0}}$:

$$\alpha_{s} := \begin{cases} (1 - s)e^{\sqrt{1 - |p|^{2}} \alpha_{B} + s(h_{1}\alpha_{B} + h_{2}d\theta)} & \text{on } B \times \mathbb{D}^{2} \subset \partial_{+}Q_{0} \\ \alpha_{\Sigma, \phi_{0}} & \text{on } \Sigma_{\phi_{0}} \subset \partial_{+}Q_{0} \end{cases},$$
here, $h_1$ and $h_2$ are as in Section 2. Notice that the two 1-forms that are being interpolated on $B \times \mathbb{D}^2$ both coincide with $\alpha_{\Sigma,\phi \psi}$ on a neighborhood of $B \times \partial \mathbb{D}^2$, so that $\alpha_s$ above is well defined. Moreover, for each $s \in [0, 1]$, the 1-form is adapted to the natural open book map $\Phi_{\Sigma,\phi \psi} : V_{\Sigma,\phi \psi} \to \mathbb{R}^2$.

Now, an explicit computation shows that $\lambda_+ |_{\partial_+ Q}$ on $\partial_+ Q = M_{\Sigma,\phi \psi}$ is the Bourgeois form associated to $(\alpha_0, \Phi_{\Sigma,\phi \psi})$ (as in Theorem 2.1), whereas $\beta_{\Sigma,\phi \psi}$ is the one associated to $(\alpha_1, \Phi)$. Corollary 2.2 then tells us that they define isotopic contact structures on $\partial_+ Q = M_{\Sigma,\phi \psi}$, as desired. \hfill \Box

\section{Factorizing the monodromy}

Let $\Sigma$ denote a connected orientable surface with boundary. We will denote the mapping class group as $\text{MCG}(\Sigma)$, which is defined to be the set of isotopy classes of orientation preserving homeomorphisms of $\Sigma$; note that these homeomorphisms are not required to fix the boundary components. This group is naturally isomorphic to the group of isotopy classes of hmeomorphisms of the corresponding punctured surface. One may also consider $\text{MCG}(\Sigma, \partial \Sigma)$ of mapping classes fixing the boundary.

There is a natural map $\text{MCG}(\Sigma, \partial \Sigma) \to \text{MCG}(\Sigma)$ whose kernel is generated by boundary parallel Dehn twists.

We will refer to a surface as sporadic if it is either a disc, an annulus or a pair of pants. These cases correspond to the mapping class group being virtually abelian. The aim of this section is to prove the following:

\begin{lemma}[Factorization Lemma] Let $\phi$ be a mapping class in $\text{MCG}(\Sigma, \partial \Sigma)$ for a non-sporadic surface $\Sigma$. Then $\phi$ can be factored as $\phi = \phi_1 \circ \phi_2$, where, for each $i = 1, 2$, $\phi_i$ is such that each connected component of the binding of $V_i := OBD(\Sigma, \phi_i)$ has infinite order in $\pi_1(V_i)$.
\end{lemma}

A direct consequence of the Factorization Lemma and Observation 2.7 is the following:

\begin{corollary} Let $\phi$ be a mapping class of a compact, orientable, non-sporadic surface $\Sigma$ with boundary. Then $\phi$ can be factored as $\phi = \phi_1 \circ \phi_2$, with $\phi_1, \phi_2$ such that the Bourgeois contact manifolds $BO(\Sigma, \phi_1)$ and $BO(\Sigma, \phi_2)$ are hypertight.
\end{corollary}

In order to prove Lemma 4.1, we start by recalling some results from geometric group theory and 3-dimensional hyperbolic geometry, respectively, in Sections 4.1 and 4.2; the actual proof of Lemma 4.1 is then given in Section 4.3.
4.1 Some geometric group theory

Recall that, by the Nielsen-Thurston classification theorem (see for instance [FM12, Theorem 13.2]), every element in $\text{MCG}(\Sigma)$ or $\text{MCG}(\Sigma, \partial \Sigma)$ is either pseudo-Anosov, reducible or of finite order.

We also recall that a quasi-homomorphism on a group $G$ is a function $H : G \to \mathbb{R}$ which satisfies

$$D(H) := \sup_{g, h \in G} |H(gh) - H(g) - H(h)| < \infty$$

The quantity $D(H)$ is called the defect of $H$. Observe that any bounded function is trivially a quasi-homomorphism. A quasi-homomorphism is called homogeneous if $H(g^k) = kH(g)$ for all $k \in \mathbb{Z}$ and $g \in G$. It is a standard fact that any quasi-homomorphism can be made homogenized by an averaging process analogous to the defintion of the Poincaré translation number.

The following lemma is direct consequence of (the proofs of) [BF02, Theorem 1] and [BF07, Proposition 5].

**Lemma 4.3.** Let $\Sigma$ be a connected orientable surface with boundary which is not sporadic. Then there exists a pseudo-Anosov map $f$ on $\Sigma$ fixing the boundary and a homogeneous quasi-homomorphism $H$ on $G = \text{MCG}(\Sigma, \partial \Sigma)$ such that $H$ is unbounded on the cyclic subgroup of $G$ generated by $f$, and $H$ vanishes on the cyclic subgroups generated by either finite order or reducible elements in $G$.

In other words, if $H$ is non-zero on $\langle \phi \rangle$, then $\phi$ must be pseudo-Anosov, and the set of such cyclic subgroups is non-empty.

**Corollary 4.4.** Let $\phi$ be an arbitrary mapping class on a non-sporadic compact, orientable, surface $\Sigma$ with boundary, and let $f$ be as in Lemma 4.3. Then, for sufficiently large $k$, the mapping class $f^k \phi \in \text{MCG}(\Sigma, \partial \Sigma)$ is pseudo-Anosov.

**Proof.** Let $H$ be as in Lemma 4.3. By the quasi-homomorphism property, we have

$$|H(f^k \phi)| \geq k|H(f)| + |H(\phi)| - 2D(H),$$

for each $k$.

Because $H(f) \neq 0$, the right hand side grows linearly with $k$, thus there is $K_0 > 0$ such that $H(f^k \phi) \neq 0$ for each $k > K_0$. Moreover, by the properties of $H$, each such $f^k \phi$ cannot be reducible or finite order, hence must be pseudo-Anosov.

4.2 Some hyperbolic geometry

We recall the following theorem on hyperbolic mapping tori due to Thurston [Thu98]:

**Theorem 4.5.** [Thu98] Let $\Sigma$ be a compact, orientable surface with boundary and negative Euler characteristic. If $\phi$ is a pseudo-Anosov map on $\Sigma$, then the interior of the associated mapping torus has a complete hyperbolic structure of finite volume.
We will also need another result due to Thurston on Dehn fillings of hyperbolic manifolds; an introductory account, as well as a detailed proof, can be found for instance in [Mar16, Chapter 15]. For the reader’s ease, we give here a statement of such a theorem which is adapted to the specific setting of Section 5 in which we will apply it.

Let $N$ be an orientable 3-manifold with boundary $\partial N$ a finite union $T_1 \sqcup \cdots \sqcup T_c$ of 2-dimensional tori. For each $i = 1, \ldots, c$, let also $m_i, l_i$ be generators of $\pi_1(T_i)$. For any $c$-tuple $s = (s_1, \ldots, s_c)$ of Dehn filling parameters, i.e. of pairs $s_i = (p_i, q_i)$ of coprime integers, one can consider the compact (boundary-less) 3-manifold $N_{\text{fill}}$ obtained by Dehn filling the boundary tori with parameters $s = (s_1, \ldots, s_c)$; more explicitly, for each $i = 1, \ldots, c$, a solid torus $P_i := D^2 \times S^1$ is glued to $N$ via the (unique up to isotopy) gluing map $\partial P_i \to T_i$ sending a meridian of $\partial P_i$ to a curve in the class $p_im_i + q_il_i \in \pi_1(T_i)$.

**Theorem 4.6.** [Thu97] In the setting described above, there is a compact set $K \subset \mathbb{R}^2$ such that, if every Dehn filling parameter $s_i$ is in $\mathbb{R}^2 \setminus K$, the closed 3-manifold $N_{\text{fill}}$ obtained by Dehn filling $N$ with parameters $s = (s_1, \ldots, s_c)$ admits a finite-volume complete hyperbolic structure $g$. Moreover, the cores of the filling solid tori are closed geodesics of $(N_{\text{fill}}, g_{\text{hyp}})$.

Notice that, each $s_i$ being a pair of coprime integers, the theorem just says that the set $X \subset \mathbb{Z}^{2c}$ of $c$-tuples of Dehn filling parameters $s$ for which $N_{\text{fill}}$ is not-hyperbolic is finite.

**Remark 4.7.** Since the fundamental group of a closed hyperbolic manifold is torsion-free and its closed geodesics are all non-contractible, the cores of the Dehn filling tori will have infinite order in $\pi_1(N_{\text{fill}})$.

### 4.3 Proof of the Factorization Lemma

**Proof.** Let $f$ be a pseudo Anosov map on $\Sigma$ as in Lemma 4.3. According to Corollary 4.4, $f^k\phi$ is pseudo Anosov on $\Sigma$ for sufficiently large $k$. We then write $\phi = F \circ G$, where

$$F = f^{-k}, \quad G = f^k\phi,$$

where both are pseudo Anosov for $k \gg 0$. By Theorem 4.5, the interiors of the mapping tori associated to $(\Sigma, F)$ and $(\Sigma, G)$ carry complete hyperbolic structures.

Let $\gamma_1, \ldots, \gamma_n$ be the components of the boundary $\partial \Sigma$. For each $i = 1, \ldots, n$, we then denote by $c_i$ a curve in $\Sigma$ which is parallel to $\gamma_i$ and contained in $\Sigma$; we can assume, up to isotopy, that they are pairwise disjoint. We also denote by $\tau_1, \ldots, \tau_n$ the corresponding right-handed Dehn twists, and

$$\tau := \tau_1 \cdots \tau_n.$$
Observe that $\tau^r = \tau_{i_1}^r \ldots \tau_{i_n}^r$ for every $r \in \mathbb{Z}$, since the $c_i$’s are disjoint.

Let $\phi_1 := F\tau^r$ and $\phi_2 := \tau^{-r}G$. It is easy to check that the 3-manifolds $OBD(\Sigma, \phi_1)$ and $OBD(\Sigma, \phi_2)$ correspond to Dehn fillings of, respectively, the mapping tori $\Sigma_F$ and $\Sigma_G$ with respect to Dehn filling parameters $s(r) = (s_1(r), \ldots, s_n(r))$ and $t(r) = (t_1(r), \ldots, t_n(r))$ such that $|s(r)|,|t(r)| \to +\infty$ for $r \to +\infty$. Thus, for sufficiently large $r$, the hyperbolic Dehn filling Theorem 4.6 implies that $OBD(\Sigma, \phi_1)$ and $OBD(\Sigma, \phi_2)$ carry hyperbolic structures and that the binding components (which coincide with the cores of the Dehn filling tori) are geodesics. In particular the latter have infinite order in the fundamental group (see Remark 4.7). In other words, we have found the desired decomposition $\phi = \phi_1 \circ \phi_2$ as posited in Lemma 4.1.

5 Proof of the main theorem

Recall that, in this section, the page $\Sigma$ is 2-dimensional. The proof of Theorem 1 relies on the following lemma, which is an analogue of the well-known fact that the convex end of a Liouville cobordism with hypertight concave end must be tight [Hof93, AH09]:

Lemma 5.1. Suppose the connected components of the bindings of both $OBD(\Sigma^2, \phi)$ and $OBD(\Sigma^2, \psi)$ have infinite order in the corresponding fundamental groups. Then, $BO(\Sigma, \phi \circ \psi)$ is tight.

The proof of the lemma follows from standard holomorphic curve arguments, except for ruling out holomorphic caps at the negative ends. Here are the details.

Proof. Let $(Q, \Omega)$ be a symplectic cobordism as in Theorem 3.1. According to Remark 2.6, Item 1b of Theorem 3.1 and our hypothesis on $OBD(\Sigma, \phi)$ and $OBD(\Sigma, \psi)$, the Reeb flow of $\lambda_{-}|_{\partial Q}$ has no contractible periodic orbits. We now show that this implies that $BO(\Sigma, \phi \circ \psi)$ is tight.

We assume by contradiction that its convex boundary $BO(\Sigma, \phi \circ \psi)$ is overtwisted. According to [BEM15], this implies the existence of an embedded Plastikstufe $\mathcal{P}S$, as defined in [Nie06]. Up to attaching a topologically trivial Liouville cobordism to $(Q, \Omega)$ along its positive end, we may then assume that the induced contact form at the positive end is (a positive multiple of) a contact form $\alpha_{PS}$ which is “adapted” to $\mathcal{P}S$, i.e. it has the normal form described in [Nie06, Proposition 4] near its core.

Take a sequence of smooth functions $f^{(k)}$ which $C^\infty$-converges to the constant function $f^{(\infty)} \equiv 1$, such that the contact form $\lambda_{-}^{(k)} := f^{(k)}\lambda_{-}$ is non-degenerate, and $\lambda_{-}^{(\infty)} = \lambda_{-}$. The non-degeneracy will mean that we can apply SFT compactness theorem directly for these perturbed forms. Taking limits we will then deduce the general case.
Attaching a cobordism at the negative ends using the local Liouville vector fields associated to $\lambda^{(\infty)}$, we obtain the Liouville completion $\hat{Q}^{(\infty)}$ of $\lambda^{(\infty)}$. Observe that the Liouville completion $\hat{Q}^{(k)}$ of $\lambda^{(k)}$ is smoothly the same as $\hat{Q}^{(\infty)}$, although with a different symplectic form at the negative ends, so that we work in a fixed smooth manifold. We denote by $\hat{\Omega}^{(k)}$ the completion of the symplectic form $\Omega$ in $\hat{Q}^{(k)}$, so that it coincides with $d(e^t\lambda^{(k)})$ at the negative ends. Since $\Omega = d\nu$ in $Q$, we deduce that, for each $k$, there is a 1-form $\hat{\nu}^{(k)}$ on $\hat{Q}^{(k)}$ such that $\hat{\Omega}^{(k)} = d(\hat{\nu}^{(k)})$, in such a way that $\hat{\nu}^{(k)} \to \hat{\nu}^{(\infty)}$ in the $C^\infty$-topology, where $\hat{\nu}^{(\infty)}$ is a primitive for $\hat{\Omega}^{(\infty)}$ on the homotopically equivalent manifolds $\hat{Q}^{(\infty)} \simeq Q \simeq \hat{Q}^{(k)}$. Moreover, since the primitive $\nu$ agrees with $\lambda_\infty = \lambda^{(\infty)}$ on the copies of the binding $B_q$ in the negative end (recall the statement of Theorem 3.1), by pulling back primitives and comparing to the symplectisation, one sees that we can assume that $\hat{\nu}^{(\infty)}|_{B_q} = c^t\lambda_\infty|_{B_q}$ on the negative end.

The negative ends of $\hat{Q}^{(k)}$ are nondegenerate contact forms. We can therefore take an $\hat{\Omega}^{(k)}$-compatible almost complex structure $J^{(k)}$, converging to a $\hat{\Omega}^{(\infty)}$-compatible $J^{(\infty)}$, all of them extending the local model of [Nie06], and cylindrical in the cylindrical ends. We have a Bishop family of Fredholm regular $J^{(k)}$-holomorphic disks in $\hat{Q}^{(k)}$ with Lagrangian boundary, stemming from the core of the Plastikstufe. Analogously to [Nie06, Proposition 10], one can check that the exactness of the symplectic form near the positive end, and hence near the Plastikstufe, provides uniform bounds on the Hofer energy, defined as in [Wen16, Page 115]. By SFT compactness (using the nondegeneracy condition at the negative ends), we thus obtain a non-trivial $J^{(\infty)}$-holomorphic building configuration, with potentially multiple levels. Since the symplectic form on $\hat{Q}^{(k)}$ is exact, there are no bubbles in the building. Also, there is no boundary bubbling, as shown in [Nie06]. We conclude, as in [Hof93, AH09], that it must contain non-trivial components in the negative ends.

For sufficiently large $k$, one can rule out holomorphic caps (which is not automatic from standard arguments, since $\hat{Q}^{(k)}$ is only pseudo-Liouville) as follows. After taking a subsequence of $k$’s, assume the existence of a sequence of $J^{(k)}$-holomorphic caps $C^{(k)}$ for $k \to +\infty$, considered as maps to the compactification $\overline{Q}^{(k)}$, and having boundary in the negative boundary of $\overline{Q}^{(k)}$. The Hofer energy bounds on the Bishop family provide universal bounds on the action of the boundary orbits $\gamma_k$ of each $C^{(k)}$. So, after passing to a further subsequence, by the Arzela–Ascoli theorem, we have that $\gamma_k \to \gamma_\infty$ converges to a periodic Reeb orbit of the Reeb flow of $\lambda_\infty$ in the $C^\infty$-topology.

Now as each $\gamma_k$ is nullhomotopic in the cobordism $\overline{Q}^{(k)} \cong Q$, the same is true of $\gamma_\infty$. Projecting to $T^2$, using the globally defined projection, we see that the image of $\gamma_\infty$ in $T^2$ is also nullhomotopic. We conclude by Theorem 3.1 that the Reeb orbit $\gamma_\infty$
must be a binding component \( B_q \). Since \( \hat{\nu}^{(\infty)} \) is positive along binding components, and hence on \( \gamma_{\infty} \), the same is true by continuity for \( \hat{\nu}^{(k)} \) restricted to \( \gamma_k \), for \( k \gg 0 \). Then integrating the exact symplectic form \( \hat{\Omega}^{(\infty)} \) along the holomorphic cap \( C^{(k)} \), and using Stokes’s theorem, we obtain that \( C^{(k)} \) has negative Hofer energy for all \( k \gg 0 \), which is absurd.

We conclude that the \( J^{(k)} \)-holomorphic building configuration contains a \( J^{(k)} \)-holomorphic plane \( P^{(k)} \) in the bottom level. Again, by passing to a subsequence and using Arzela-Ascoli, we obtain a contractible Reeb orbit in the negative symplectization of \( \lambda_{-} = \lambda_{-}^{(\infty)} \). But there are no such orbits at the negative ends, and this finishes the proof.

We can now proceed to the proof of the tightness of the Bourgeois contact structures in dimension 5.

Proof (Theorem 1). We start by proving the result in the “generic” case of non-sporadic page \( \Sigma \). We then deal with sporadic pages on a case by case basis.

Case 1: non-sporadic \( \Sigma \). By Lemma 4.1 we may factorise the monodromy \( \phi = \phi_1 \circ \phi_2 \), where the components of the bindings in \( OBD(\Sigma, \phi_1) \) and \( OBD(\Sigma, \phi_2) \) have infinite order. Then, according to Lemma 5.1, we conclude that \( BO(\Sigma, \phi) \) is tight.

Case 2: \( \Sigma \) is a disk. In this case, the monodromy \( \phi \) is necessarily isotopic to the identity. In other words, the resulting contact 3-manifold is \( (S^3, \xi_{std}) \) and the open book structure is the one induced by the subcritical Stein-filling \( \mathbb{D}^4 \). According to [LMN18, Theorem A.(b)], the associated Bourgeois contact structure is Stein fillable, and hence tight.

Case 3: \( \Sigma \) is an annulus. The mapping class group of the annulus is generated by a single positive Dehn twist around the core circle. If the monodromy is a non-negative power of such generator, then the resulting contact 3-manifold is Stein fillable; then, according to [MNW13, Example 1.1], the associated Bourgeois contact structure is weakly fillable, and hence tight. If the power is negative, according to [LMN18, Theorem B], the Bourgeois contact structure associated to \( OBD(\Sigma, \phi) \) is contactomorphic to that associated to \( OBD(\Sigma, \phi^{-1}) \), so we obtain tightness for this case.

Case 4: \( \Sigma \) is a pair of pants. For simplicity, enumerate from 1 to 3 the connected components of \( \partial \Sigma \). For \( i = 1, 2, 3 \), let \( \tau_i \) be a positive Dehn twist along the \( i \)-th connected component of \( \partial \Sigma \); Notice that the monodromy \( \phi \) is necessarily of the form \( \tau_1 \circ \tau_2 \circ \tau_3 \). We then define \( \tau := \tau_1 \circ \tau_2 \circ \tau_3 \) and, for any \( N \in \mathbb{N}_{>0} \), we can decompose
\( \phi \) as \( \phi = F \circ G \), with \( F := \phi \circ \tau_N = \prod_{i=1}^{3} \tau_i^{N+a_i} \) and \( G := \tau^{-N} = \prod_{i=1}^{3} \tau_i^{-N} \). We then use the following result, whose proof is postponed:

**Lemma 5.2.** If \( N > 0 \) is big enough, each binding component of \( \text{OBD}(\Sigma, F) \) and of \( \text{OBD}(\Sigma, G) \) is of infinite order in \( \pi_1(\text{OBD}(\Sigma, F)) \) and \( \pi_1(\text{OBD}(\Sigma, G)) \), respectively.

Combining Lemma 5.1 and Lemma 5.2 we conclude that \( BO(\Sigma, \phi) \) is tight, as desired.

Lastly, we prove the lemma used above in the case of a pair of pants:

**Proof (Lemma 5.2).** We deal only with the case of \( \text{OBD}(\Sigma, F) \); the proof for the manifold \( \text{OBD}(\Sigma, G) \) is completely analogous.

We first point out that, as explained in detail for instance in [Ozb07, Section 3], \( \text{OBD}(\Sigma, F) \) can be seen as obtained by Dehn surgery on the total space of \( S^2 \times S^1 \to S^2 \) along three \( S^1 \)-fibers, with coefficients \( r_i := -\frac{1}{N+a_i} \), for each \( i = 1, 2, 3 \). In other words, \( \text{OBD}(\Sigma, F) \) is the Seifert manifold

\[
\{0, (0_1,0); (N + a_1, -1), (N + a_2, -1), (N + a_3, -1)\}
\]

Moreover, the orbit space \( O \) of the Seifert fibration of \( \text{OBD}(\Sigma, F) \) is a 2-dimensional orbifold, with underlying topological surface \( S^2 \), and the binding \( B \) of \( \text{OBD}(\Sigma, F) \) consists of a union of fibers of the Seifert fibration.

We recall that there is a notion of *orbifold Euler characteristic* \( \chi_{\text{orb}} \) for orbifolds that behaves multiplicatively under finite covers of orbifolds. In our special case of the base orbifold \( B \) of the Seifert fibered space \( \text{OBD}(\Sigma, F) \), we have

\[
\chi_{\text{orb}}(O) = \chi(S^2) - \sum_{i=1}^{3} \left( 1 + \frac{1}{N+a_i} \right) = -1 - \sum_{i=1}^{3} \frac{1}{N+a_i}.
\]

From now on, let \( N > 0 \) be so big that \( \chi_{\text{orb}}(O) < 0 \). In particular, \( \text{OBD}(\Sigma, F) \) is finitely covered by a circle bundle \( X \) over a hyperbolic surface \( S \), in such a way that fibers of \( X \to S \) are mapped to fibers of \( \text{OBD}(\Sigma, F) \to O \) (see [Sco83] for instance). Now, \( S \) being hyperbolic, the fibers of \( X \) are of infinite order in \( \pi_1(X) \). As \( X \) covers \( \text{OBD}(\Sigma, F) \) in a compatible way with their Seifert bundle structures, it follows the fibers of \( \text{OBD}(\Sigma, F) \), hence its binding too, are of infinite order in its fundamental group, as desired.

We finally note that all these arguments remain valid when we pull-back under any finite cover of \( V \times \mathbb{T}^2 \) of the first factor. Since finite covers over the second factor do not change the contact structure up to contactomorphism (cf. Remark 2.3), such covers also preserve tightness. Now any finite cover is itself covered by a
composition of covers of the respective factors. Consequently, the contact structure remains tight under any finite cover on the first factor. Since the fundamental group of any closed 3-manifold is residually finite (cf. [Hem87]) so is $\pi_1(V \times \mathbb{T}^2)$ and hence tightness on finite covers is equivalent to tightness on the universal cover of $V \times \mathbb{T}^2$ and universal tightness follows. This concludes the proof.

6 Further Discussion and Open Questions

Whilst our main result is a first step in understanding the nature of the contact structures given by Bourgeois’ construction, it is an important problem to understand more precisely the dependence of the Bourgeois structure on the starting open book decomposition. It is a direct consequence of their definition that all Bourgeois contact structures are contact deformations of the almost contact structure $\xi_V \oplus T\mathbb{T}^2$ (i.e. the endpoint $\eta_1$ of a path $(\eta_t)_{t \in [0,1]}$ of hyperplane fields starting at $\eta_0 = \xi_V \oplus T\mathbb{T}^2$ and such that $\eta_t$ is contact for $t > 0$). As a consequence one can then construct weak cobordisms between $BO(\Sigma, \phi)$ and $BO(\Sigma', \phi')$, as in [MNW13, Example 1.1], for any $OBD(\Sigma', \phi')$ and $OBD(\Sigma, \phi)$ supporting the same contact structure. Besides sharing the formal homotopy class, the main result in this paper shows in particular that the tight vs overtwisted classification type of any 5-dimensional $BO(\Sigma, \phi)$ is independent of the open book.

On the other hand, in [Bou02a, Corollaries 10.6 and 10.8], Bourgeois used cylindrical contact homology with respect to noncontractible homotopy classes of Reeb orbits, in order to distinguish infinitely many Bourgeois contact manifolds arising from open books supporting the standard contact structure on $S^3$; and similarly for $\mathbb{T}^3$. Further instances of different open books supporting the same contact structure that induce non-contactomorphic Bourgeois contact manifolds can also be found in [LMN18, Example 1.5].

Given the results in this paper, a natural question is then the following:

**Question 4.** Can we find further contactomorphisms of Bourgeois contact manifolds, beyond the inversion of the monodromy from [LMN18]? More ambitiously, can we classify the contactomorphism type of all the Bourgeois contact manifolds arising from some fixed contact structure, especially via rigid holomorphic curves invariants?

A further interesting question relates to Giroux torsion. This is a standard notion in dimension 3, and was generalized to higher dimensions in [MNW13]. It was shown in [MNW13] to be an obstruction to strong fillability, and it was proven in [Mor17, Theorem 1.7] that it can be detected by an SFT-type contact invariant, algebraic torsion (defined in [LW11]). The result in this paper can be summarized
as: the Bourgeois construction “kills” overtwistedness. One can wonder if it also “kills” Giroux torsion.

**Question 5.** Does there exist a Bourgeois contact manifold with Giroux torsion?

An affirmative answer to Question 5 would provide an obstruction to strong fillability, but it would not immediately provide an obstruction to weak fillability, since Giroux torsion only provides obstructions to weak fillability under suitable cohomological conditions (see [MNW13, Corollary 8.2] or [Mor17, Corollary 1.8]).

**Remark 6.1.** (Obstructions to gluing weak cobordisms) Observe that in order to show that Bourgeois contact structures are tight – at least in dimension 5 –, one could be tempted to argue as follows. By decomposing the monodromy into powers of positive and negative Dehn twists, and using the cobordism \((Q, \omega)\) of Theorem 3.1, we end up with having to find fillings of \(BO(\Sigma, \tau^{\pm 1})\) where \(\tau\) is a single Dehn twist.

Notice that \(BO(\Sigma, \tau)\) is indeed weakly fillable, according to [LMN18, Theorem A.(a)]; more precisely, one can arrange the cohomology class of the symplectic form induced at the boundary of the filling to be \(\epsilon[\omega_{T^2}]\), where \(\omega_{T^2}\) is an area form on \(T^2\) and \(\epsilon > 0\). Since \((Q, \Omega)\) is exact at each end, one can then consider the perturbed version \((Q, \Omega + \epsilon\omega_{T^2})\), which has weakly dominated boundary. Then, [MNW13, Lemma 1.10] guarantees that the weak filling of \(BO(\Sigma, \tau)\) glues to this perturbed version. Now, in order to find an analogous filling for the case of \(BO(\Sigma, \tau^{-1})\), one would hope to appeal to [LMN18, Theorems A.(a) and B]. However, by carefully tracing the signs from the contactomorphism provided in [LMN18], one can see that the corresponding weak filling does not glue correctly to \((Q, \Omega + \epsilon\omega_{T^2})\). More precisely, the contactomorphism swaps the orientation of the \(T^2\)-factors. Of course, in the case where the monodromy is a product of powers of Dehn twists of the same sign, then indeed we do find weak fillings.

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School of Mathematical Sciences, Monash University, Melbourne, Australia.

Email: jonathan.bowden@monash.edu

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary.

Email: fabio.gironella@renyi.hu

Institut für Mathematik, Universität Augsburg, Augsburg, Germany.

Email: agustin.moreno2191@gmail.com