Global stability of a delayed and diffusive virus model with nonlinear infection function

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**ABSTRACT**

This paper studies a delayed viral infection model with diffusion and a general incidence rate. A discrete-time model was derived by applying nonstandard finite difference scheme. The positivity and boundedness of solutions are presented. We established the global stability of equilibria in terms of $R_0$ by applying Lyapunov method. The results showed that if $R_0$ is less than 1, then the infection-free equilibrium $E_0$ is globally asymptotically stable. If $R_0$ is greater than 1, then the infection equilibrium $E_*$ is globally asymptotically stable. Numerical experiments are carried out to illustrate the theoretical results.

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1. Introduction

Mathematical models that describe within-host dynamics have been proposed and studied by constructing corresponding differential equations to get a better understanding of viral processes, particularly, their global dynamics behaviour has been investigated [1, 2, 14, 17, 21, 28, 29, 32, 35–37]. For example, Wang and Zhou [32] studied the following model:

\begin{align}
\frac{dT}{dt} &= s - dT(t) - (1 - \epsilon) \beta TV,
\frac{dI}{dt} &= (1 - \alpha)(1 - \epsilon) \beta T(t - \tau_1) V(t - \tau_1) - \delta I,
\frac{dC}{dt} &= \alpha(1 - \epsilon) \beta T(t - \tau_2) V(t - \tau_2) - \mu C,
\frac{dV}{dt} &= k_1 \delta I(t - \tau_3) + k_2 \mu C(t - \tau_4) - cV,
\end{align}

where $T$, $I$, $C$ and $V$ represent the concentrations of the uninfected cell, shorted-lived infected cells, chronically infected cells and free virus particles, respectively. $s$ is the source term for uninfected cells. $\beta$ represents the infection rate. $\epsilon$ is the efficacy of the therapy. $d$, $\delta$, $\mu$ and $c$ are the mortality rates of uninfected cells, short infected cells, chronically infected cells and virus, respectively. The fractions $\alpha$ and $(1 - \alpha)$ are the probabilities that, upon infection, an uninfected cell will become either chronically infected or short-lived.
infected. $k_1 = N_1(1 - \gamma_1)$ and $k_2 = N_2(1 - \gamma_2)$ where $N_1$ and $N_2$ are the average numbers of virions produced in the lifetime of short-lived and chronically infected cells, respectively. $\gamma_1$ and $\gamma_2$ are the efficacy of the therapy. $\tau_1, \tau_2, \tau_3$ and $\tau_4$ are the intracellular delays. The global dynamics of the model have been studied by constructing Lyapunov functionals. For more details, one can refer to [32]. The key assumption in model (1) is that cells and viruses are well mixed, and the mobility of viruses was ignored. So, in order to study the influences of spatial structures of virus dynamics, Wang et al. [30] studied the following model:

$$\begin{align*}
\frac{\partial T(x,t)}{\partial t} &= s - dT(x,t) - \beta T(x,t)V(x,t), \\
\frac{\partial I(x,t)}{\partial t} &= \beta T(x,t) - \tau) V(x,t - \tau) - \delta I(x,t), \\
\frac{\partial V(x,t)}{\partial t} &= D\Delta V(x,t) + pI(x,t) - cV(x,t),
\end{align*}$$

(2)

where $T(x,t), I(x,t)$ and $V(x,t)$ represent the densities of uninfected cells, infected cells and free virus at position $x$ and at time $t$, respectively. $\tau$ is the intracellular delay. $D$ is the diffusion coefficient and $\Delta$ is the Laplacian operator.

The bilinear incidence rate $\beta TV$ used in models (1) and (2) is a simple description of the infection. Though the incidence rates $\beta TV^q, \frac{\beta TV}{1+aT+bV+abTV}$ and $\frac{\beta TV}{1+\alpha T+cV+abTV}$ are improved forms which are more commonly used [22, 26, 31], the general incidence rates $f(T, V) V, f(T, V)$ and $f(T, I, V) V$ can help us gain the unification theory by the omission of unessential details [7, 8, 11, 13]. So, motivated by [30, 32], we consider the following model with a general incidence rate which is similar to the one in [7]

$$\begin{align*}
\frac{\partial T(x,t)}{\partial t} &= s - dT(x,t) - (1 - \epsilon)f(T(x,t), V(x,t))V(x,t), \\
\frac{\partial I(x,t)}{\partial t} &= (1 - \alpha)(1 - \epsilon)f(T(x,t - \tau_1), V(x,t - \tau_1))V(x,t - \tau_1) - \delta I(x,t), \\
\frac{\partial C(x,t)}{\partial t} &= \alpha(1 - \epsilon)f(T(x,t - \tau_2), V(x,t - \tau_2))V(x,t - \tau_2) - \mu C(x,t), \\
\frac{\partial V(x,t)}{\partial t} &= D\Delta V(x,t) + k_1\delta I(x,t - \tau_3) + k_2\mu C(x,t - \tau_4) - cV(x,t).
\end{align*}$$

(3)

$f(T, V) V$ is the general incidence rate and satisfies the following hypotheses:

- $f(T, V) \geq 0$, and $f(T, V) = 0$ if and only if $T = 0$;
- there exists $\eta > 0$ such that $f(T, V) V \leq \eta T$ for all $T, V \geq 0$;
- $\frac{\partial f(T, V)}{\partial T} > 0$ for any fixed $V \geq 0$;
- $\frac{\partial f(T, V)}{\partial V} \leq 0$ for any fixed $T \geq 0$;
- $\frac{\partial (f(T, V) V)}{\partial V} > 0$ for any fixed $T \geq 0$.

(4)

It is easy to check that a class of functions $f(T, V) V$ satisfying (4) includes some common used nonlinear incidence functions such as $f(T, V) V = \frac{\beta TV}{1+bV+cTV}$, $f(T, V) V = \frac{\beta TV}{1+aT+bV}$ and $f(T, V) V = \frac{\beta TV}{1+aT+bV+abTV}$ for $\beta, a, b, c > 0$. 

The initial conditions of model (3) are given as
\[ T(x, s) = \phi_1(x, s) \geq 0, \quad I(x, s) = \phi_2(x, s) \geq 0, \quad C(x, s) = \phi_3(x, s) \geq 0, \]
\[ V(x, s) = \phi_4(x, s) \geq 0, \quad (x, s) \in \bar{\Omega} \times [-\tau, 0], \tag{5} \]
and we have considered model (3) with homogeneous Neumann boundary conditions
\[ \frac{\partial V}{\partial n} = 0, \quad t > 0, \ x \in \partial \Omega, \tag{6} \]
where \( \tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}, \ \Omega \) is a bounded interval in \( \mathbb{R} \) and \( \frac{\partial}{\partial n} \) denotes the outward normal derivative on \( \partial \Omega \).

As we know, it is difficult or even impossible to find the exact analytical solutions for most nonlinear models, such as (3). In order to perform numerical simulations, we need to seek an efficient discrete method to discretize such nonlinear continuous models. However, some classical numerical discrete methods are unsuccessful in preserving the quantitative properties of corresponding continuous model. For example, for classical forward Euler’s method, if the step size is selected small enough and the positivity conditions can be satisfied, then it can be shown that local asymptotic stability for an equilibrium is preserved while in some special cases numerical instability or Hopf bifurcation may appear. Thus how to design a feasible discrete scheme so that the same quantitative behaviours of solutions to the corresponding continuous models can be efficiently preserved is a challenging and interesting task. Recently, an interesting method which is called non-standard finite difference (NSFD) scheme has been proposed by Mickens [18, 19]. NSFD has been applied to obtain discrete-time epidemic models [4, 6, 9, 10, 20, 24, 25, 34] and references therein. Hence, motivated by the work of [18, 19], we apply the NSFD scheme on model (3), then we obtain
\[
\begin{align*}
\frac{T^{m+1}_n - T^n_m}{\Delta t} &= s - dT^{m+1}_n - (1 - \epsilon)f(T^n_m, V^n_m)V^n_m, \\
\frac{I^{m+1}_n - I^n_m}{\Delta t} &= (1 - \alpha)(1 - \epsilon)f(T^n_{m-1}, V^{m-1}_{n-1})V^{m-1}_{n-1} - \delta I^{m+1}_n, \\
\frac{C^{m+1}_n - C^n_m}{\Delta t} &= \alpha(1 - \epsilon)f(T^n_{m-2}, V^{m-2}_{n-2})V^{m-2}_{n-2} - \mu C^{m+1}_n, \\
\frac{V^{m+1}_{n-1} - V^n_m}{\Delta t} &= D\frac{V^{m+1}_{n+1} - 2V^n_m + V^{m-1}_{n-1}}{\Delta x^2} + k_1I^{m+1}_{n-1} + k_2\mu C^{m+1}_{n-1} + \epsilon V^n_m. \tag{7}
\end{align*}
\]
Set \( x \in \Omega = [a, b] \) be a bounded interval in \( \mathbb{R} \). Let \( \Delta t > 0 \) be the time step size and \( \Delta x = \frac{b-a}{N} \) be the space step size with \( N \) being a positive integer. Assume that there exist four integers \( m_i \in \mathbb{N} \) (\( i = 1, 2, 3, 4 \)) with \( \tau_i = m_i\Delta t \). Denote the mesh grid point as \( \{(x_m, t_n), m = 0, 1, 2, \ldots, N, n \in \mathbb{N} \} \) with \( x_m = a + m\Delta x \) and \( t_n = n\Delta t \). \( (T^m_n, I^m_n, C^m_n, V^m_n) \) are the approximations of solution \( (T(x_m, t_n), I(x_m, t_n), C(x_m, t_n), V(x_m, t_n)) \) at the discrete-time points. For simplicity, let \( U_n = (U^0_n, U^1_n, \ldots, U^N_n)^T \) be the approximation solutions at the time \( t_n \), where \( U \in \{T, I, C, V\} \) and the notation \( \cdot^T \) denotes the transposition of a vector. The initial conditions of model (7) are
\[
\begin{align*}
T^m_s &= \phi_1(x_m, t_s) \geq 0, \quad I^m_s = \phi_2(x_m, t_s) \geq 0, \quad C^m_s = \phi_3(x_m, t_s) \geq 0, \\
V^m_s &= \phi_4(x_m, t_s) \geq 0, \quad \text{for all } s = -l, -l + 1, \ldots, 0, \quad l = \max \{m_i\}, \tag{8}
\end{align*}
\]
and the discrete boundary conditions are
\[ V_n^{-1} = V_n^0, \quad V_n^{N+1} = V_n^N, \quad \text{for } n \in \mathbb{N}. \]

The main purpose of this paper is to demonstrate the discretized model (7) derived by applying NSFD scheme can efficiently preserves the global dynamical properties to the original model (3). The rest of this paper is organized as follows. In Section 2, we study the dynamical behaviour of the continuous model (3). In Section 3, we investigate the global dynamics of discrete model (7). An example, along with numerical simulations is presented in Section 4 to validate the theoretical results. A brief conclusion ends the paper.

2. Dynamical analysis of model (3)

2.1. Preliminaries

Let \( \mathbb{X} = C(\tilde{\Omega}, \mathbb{R}^4) \) be the space of continuous functions from the topological space \( \tilde{\Omega} \) into the space \( \mathbb{R}^4 \). Denote \( C = C([-\tau, 0], \mathbb{X}) \) be the Banach space of continuous functions from \([-\tau, 0]\) into \( \mathbb{X} \) with the usual supremum normal, and \( C_+ = C([-\tau, 0], \mathbb{X}_+) \). When convenient, we identify an element \( \phi \in C \) as a function from \( \tilde{\Omega} \times [-\tau, 0] \) into \( \mathbb{R}^4 \) defined by \( \phi(x, s) = \phi(s)(x) \). We adopt the notation that for \( \sigma > 0 \), a function \( u(\cdot) : [-\tau, \sigma) \to \mathbb{X} \) induces functions \( u_t \in C \) for \( t \in [0, \sigma) \), defined by \( u_t(\theta) = u(t + \theta) \) for \( \theta \in [-\tau, 0] \). Let \( D = (0, 0, 0, D)^T \). It follows from [3] that the \( \mathbb{X} \)-realization of \( D \Delta \) generates an analytic semi-group \( T(t) \) on \( \mathbb{X} \).

**Theorem 2.1:** For any given initial data \( \phi \in C \) satisfying the condition (5), there exists a unique solution of problem (3)–(6) defined on \([0, +\infty)\) and this solution remains non-negative and bounded for all \( t \geq 0 \).

**Proof:** For any \( \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T \in C_+ \) we define \( F = (F_1, F_2, F_3, F_4) : C_+ \to \mathbb{X} \) as follows. For any \( x \in \tilde{\Omega} \),

\[
\begin{align*}
F_1(\varphi)(x) &= s - d\varphi_1(x, 0) - (1 - \epsilon)f(\varphi_1(x, 0), \varphi_4(x, 0))\varphi_3(x, 0), \\
F_2(\varphi)(x) &= (1 - \alpha)(1 - \epsilon)f(\varphi_1(x, -\tau_1), \varphi_4(x, -\tau_1))\varphi_4(x, -\tau_1) - \delta\varphi_2(x, 0), \\
F_3(\varphi)(x) &= \alpha(1 - \epsilon)f(\varphi_1(x, -\tau_2), \varphi_4(x, -\tau_2))\varphi_4(x, -\tau_2) - c\varphi_3(x, 0), \\
F_4(\varphi)(x) &= k_1\varphi_2(x, -\tau_3) + k_2\varphi_3(x, -\tau_4) - c\varphi_4(x, 0).
\end{align*}
\]

We now reformulate (3)–(6) as the abstract functional differential equation

\[
\begin{cases}
\frac{d\Phi}{dt} = A\Phi + F(\Phi_t), & t > 0, \Phi_t \in C \\
\Phi_0 = \varphi \in C_+,
\end{cases}
\] (9)

where \( \Phi = (T, I, C, V)^T \) and \( A\Phi = (0, 0, 0, D\Delta V)^T \). It is clear that \( F \) is locally Lipschitz in \( \mathbb{X} \). It follows from [5, 15, 16, 27, 33] that system (9) admits a unique local solution on \( t \in [0, T_{\text{max}}) \), where \( T_{\text{max}} \) is the maximal existence time for the solution of system (9).

In order to demonstrate the boundedness of solutions. Define \( U(x, t) = T(x, t) + I(x, t + \tau_1) + C(x, t + \tau_2) \) and \( d_0 = \min\{d, \delta, \mu\} \), it then follows from model (3) that

\[
\frac{\partial U(x, t)}{\partial t} = s - dT(x, t) - \delta I(x, t + \tau_1) - \mu C(x, t + \tau_2)
\]
Then we have
\[ U(x, t) \leq \max \left\{ \frac{s}{d_0}, \max_{x \in \Omega} \{ \varphi_1(x, 0) + \varphi_2(x, -\tau_1) + \varphi_3(x, -\tau_2) \} \right\} : = \eta_1, \]

implying \( T, I \) and \( C \) are bounded.

From model (3)-(6), we deduce that \( V \) satisfies
\[
\begin{cases}
\frac{\partial V}{\partial t} - D \Delta V \leq k_1 \delta \eta_1 + k_2 \mu \eta_1 - c V, \\
\frac{\partial V}{\partial n} = 0, \\
V(x, 0) = \phi_4(x, 0) \geq 0.
\end{cases}
\]

Let \( \tilde{V}(t) \) be a solution to the ordinary differential equation
\[
\begin{cases}
\frac{d\tilde{V}}{dt} = k_1 \delta \eta_1 + k_2 \mu \eta_1 - c \tilde{V}, \\
\tilde{V}(0) = \max_{x \in \Omega} \phi_4(x, 0).
\end{cases}
\]

Then we get \( \tilde{V}(t) \leq \max \left\{ \frac{(k_1 \delta + k_2 \mu) \eta_1}{c}, \max_{x \in \Omega} \phi_4(x, 0) \right\}, \forall t \in [0, T_{\max}). \) Thus \( V(x, t) \leq \tilde{V}(t) \) follows from the comparison principle [23]. Therefore,
\[ V(x, t) \leq \max \left\{ \frac{(k_1 \delta + k_2 \mu) \eta_1}{c}, \max_{x \in \Omega} \phi_4(x, 0) \right\} : = \eta_2, \quad \forall (x, t) \in \Omega \times [0, T_{\max}). \]

Based on the above analysis, we have demonstrated that \( T(x, t), I(x, t), C(x, t) \) and \( V(x, t) \) are bounded in \( \Omega \times [0, T_{\max}) \). It then follows from the standard theory for semilinear parabolic systems [12] that \( T_{\max} = +\infty \).

Let \( [0, M]_C := \{ \varphi \in C : 0 \leq \varphi(s, x) \leq M, \forall \ s \in [-\tau, 0], x \in \Omega \} \) with \( 0 = (0, 0, 0, 0) \) and \( M = \left( \frac{s}{d_0}, \frac{s \eta_1}{d_0 \mu}, \frac{s \eta_1 \alpha (1-\epsilon)}{d_0 \mu}, \frac{s \eta_1 (1- \epsilon)(k_1(1-\alpha)+k_2 \alpha)}{d_0 \epsilon} \right) \). For any \( \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in [0, M]_C \) and any \( \rho \geq 0 \), we obtain
\[
\begin{align*}
\varphi(x, 0) + \rho F(\varphi)(x) &= \begin{pmatrix}
\varphi_1(x, 0) + \rho (s - d \varphi_1(x, 0) - (1-\epsilon)f(\varphi_1(x, 0), \varphi_4(x, 0)) \varphi_4(x, 0)) \\
\varphi_2(x, 0) + \rho (1-\alpha)(1-\epsilon)f(\varphi_1(x, -\tau_1), \varphi_4(x, -\tau_1)) \varphi_4(x, -\tau_1) - \rho \delta \varphi_2(x, 0) \\
\varphi_3(x, 0) + \rho \alpha (1-\epsilon)f(\varphi_1(x, -\tau_2), \varphi_4(x, -\tau_2)) \varphi_4(x, -\tau_2) - \rho c \varphi_3(x, 0) \\
\varphi_4(x, 0) + \rho k_1 \delta \varphi_2(x, -\tau_3) + \rho k_2 \mu \varphi_3(x, -\tau_4) - \rho c \varphi_4(x, 0)
\end{pmatrix}.
\end{align*}
\]
Recall from (4) that \( f(T, V)V \leq \eta T \) for all \( T, V \geq 0 \). Therefore, for any \( 0 \leq \rho \leq \min\{\frac{1}{(1-\epsilon)\eta + d}, \frac{1}{\delta}, \frac{1}{\mu}, \frac{1}{\epsilon}\} \), we have

\[
\varphi(x, 0) + \rho F(\varphi)(x) \geq \begin{pmatrix}
(1 - \rho(d + (1 - \epsilon)\eta))\varphi_1(x, 0) \\
(1 - \rho\delta)\varphi_2(x, 0) \\
(1 - \rho\mu)\varphi_3(x, 0) \\
(1 - \rho\epsilon)\varphi_4(x, 0)
\end{pmatrix} \geq \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix} = 0
\]

and

\[
\varphi(x, 0) + \rho F(\varphi)(x) \leq \begin{pmatrix}
\varphi_1(x, 0) + \rho(s - d\varphi_1(x, 0)) \\
\rho(1 - \alpha)(1 - \epsilon)\eta\varphi_1(x, -\tau_1) + (1 - \rho\delta)\varphi_2(x, 0) \\
\rho\alpha(1 - \epsilon)\eta\varphi_1(x, -\tau_2) + (1 - \rho\mu)\varphi_3(x, 0) \\
\rho k_1 \delta \varphi_2(x, -\tau_3) + \rho k_2 \mu \varphi_3(x, -\tau_4) + (1 - \rho\epsilon)\varphi_4(x, 0)
\end{pmatrix} \leq \begin{pmatrix}
s \\
\frac{d}{\delta} \\
\frac{d\delta}{\delta} \\
\frac{d\mu}{\delta} \\
\frac{s(1 - \epsilon)(k_1(1 - \alpha) + k_2\alpha)}{\delta}
\end{pmatrix} = M.
\]

Thus we can demonstrate that for small enough \( \rho, \varphi(0) + \rho F(\varphi) \in [0, M]_C \), which implies that

\[
\lim_{\rho \to 0^+} \frac{1}{\rho} \text{dist}(\varphi(0) + \rho F(\varphi), [0, M]_C) = 0, \quad \forall \varphi \in C.
\]

We are now prepared to refer to key results from the literature. Denote \( K = [0, M]_C \), \( S(t, s) = \mathcal{T}(t - s) \) and \( B(t, \varphi) = F(\varphi) \), it then follows from [15] that (3)–(6) has a unique mild solution \( \Phi(t, \varphi) \in [0, M]_C \) for \( t \in [0, \infty) \). Furthermore, since the semigroup \( \mathcal{T}(t) \) is analytic [3]. Thus it follows from [33] that the mild solution is classic for \( t \geq \tau \). This completes the proof.

The dynamical outcomes of model (3) will be determined by the basic reproduction number \( R_0 \), which is given by

\[
R_0 = \frac{(1 - \epsilon)(k_1(1 - \alpha) + k_2\alpha)f(T_0, 0)}{c}, \quad \text{with } T_0 = \frac{s}{d}.
\]

It is clear that model (3) always has an infection-free equilibrium \( E_0 = (T_0, 0, 0, 0) \) and any positive equilibrium, denoted by \( E_* = (T_*, I_*, C_*, V_*) \), must satisfy

\[
\begin{align*}
s &= dT_* + (1 - \epsilon)f(T_*, V_*), \\
(1 - \alpha)(1 - \epsilon)f(T_*, V_*) &= \delta I_*, \\
\alpha(1 - \epsilon)f(T_*, V_*) &= \mu C_*, \\
k_1 \delta I_* + k_2 \mu C_* &= cV_*. 
\end{align*}
\]
Simple calculations show that

\[ I_* = \frac{(1 - \alpha)cV_*}{\delta k_1(1 - \alpha) + k_2 \alpha}, \quad C_* = \frac{\alpha cV_*}{\mu k_1(1 - \alpha) + k_2 \alpha}, \]

\[ V_* = \frac{(s - dT_*)(k_1(1 - \alpha) + k_2 \alpha)}{c}, \]

and \( T_* \) satisfies

\[ H(T_*) := (1 - \epsilon)f\left( T_*, \frac{(s - dT_*)(k_1(1 - \alpha) + k_2 \alpha)}{c} \right) - \frac{c}{k_1(1 - \alpha) + k_2 \alpha} = 0. \]

It is easy to show that

\[ H(0) = -\frac{c}{k_1(1 - \alpha) + k_2 \alpha} < 0, \quad H(T_0) = \frac{c(\mathcal{R}_0 - 1)}{k_1(1 - \alpha) + k_2 \alpha}. \]

According to (4), calculating shows \( H'(T) > 0 \). Thus there exists a unique \( T_1 \in (0, T_0) \) such that \( H(T_1) = 0 \) if and only if \( \mathcal{R}_0 > 1 \). So, a unique infection equilibrium \( E_* \) exists when \( \mathcal{R}_0 > 1 \).

**Theorem 2.2:** If \( \mathcal{R}_0 \leq 1 \), then the only equilibrium is the infection-free equilibrium \( E_0 = (T_0, 0, 0, 0) \). If \( \mathcal{R}_0 > 1 \), then there exists a unique infection equilibrium \( E_* = (T_*, I_*, C_*, V_*) \).

**2.2. Stabilities of equilibria**

**Theorem 2.3:** If \( \mathcal{R}_0 \leq 1 \), then the infection-free equilibrium \( E_0 \) of model (3) is globally asymptotically stable.

**Proof:** Define a Lyapunov functional

\[ L_1 = \int_{\Omega} \left\{ (k_1(1 - \alpha) + k_2 \alpha) \left( T - T_0 - \int_0^T \frac{f(T_0, 0)}{f(s, 0)} \, ds \right) + k_1 I(x, t) + k_2 C(x, t) 
\]

\[ + V(x, t) + k_1(1 - \alpha)(1 - \epsilon) \int_{t - \tau_1}^t f(T(x, s), V(x, s)) V(x, s) \, ds 
\]

\[ + k_2 \alpha(1 - \epsilon) \int_{t - \tau_2}^t f(T(x, s), V(x, s)) V(x, s) \, ds 
\]

\[ + k_1 \delta \int_{t - \tau_3}^t I(x, s) \, ds + k_2 \mu \int_{t - \tau_4}^t C(x, s) \, ds \right\} \, dx. \]

For convenience, we let \( u = u(x, t) \) and \( u_{\tau_i} = u(x, t - \tau_i) \) ( \( i = 1, 2, 3, 4 \)) for any \( u \in \{ T, I, C, V \} \). Calculating the time derivative of \( L_1 \) along a solution of model (3), we obtain

\[ \frac{dL_1}{dt} = \int_{\Omega} \left\{ (k_1(1 - \alpha) + k_2 \alpha) \left( 1 - \frac{f(T_0, 0)}{f(T, 0)} \right) (dT_0 - dT - (1 - \epsilon)f(T, V) 
\]

\[ + k_1 \left( (1 - \alpha)(1 - \epsilon)f(T_{\tau_1}, V_{\tau_1}) V_{\tau_1} - \delta I \right) + k_2 \left( \alpha(1 - \epsilon)f(T_{\tau_2}, V_{\tau_2}) V_{\tau_2} - \mu C \right) \]
+ DΔV + k_1 \delta I_{I_3} + k_2 \mu C_{I_4} - cV \\
+ k_1 (1 - \alpha)(1 - \epsilon) \left( f(T, V)V - f(T, V_\ast)V_{V_\ast} \right) \\
+ k_2 \alpha(1 - \epsilon) \left( f(T, V)V - f(T, V_{I_3})V_{I_3} \right) + k_1 \delta(I - I_{I_3}) + k_2 \mu(C - C_{I_4}) \bigg\} \ dx \\
= \int_{\Omega} \left\{ \left( k_1 (1 - \alpha) + k_2 \alpha \right) \left( 1 - \frac{T}{T_0} \right) \left( 1 - \frac{f(T_0, 0)}{f(T, 0)} \right) + \left( \mathcal{R}_0 \frac{f(T_0, 0)}{f(T, 0)} - 1 \right) cV \right\} \ dx + D \int_{\Omega} \Delta V \ dx.

Recall that \( \int_{\Omega} \Delta V \ dx = 0 \) and (4), we obtain

\[
\frac{dL_1}{dt} \leq \int_{\Omega} \left\{ \left( k_1 (1 - \alpha) + k_2 \alpha \right) \left( 1 - \frac{T}{T_0} \right) \left( 1 - \frac{f(T_0, 0)}{f(T, 0)} \right) + (\mathcal{R}_0 - 1) cV \right\} \ dx.
\]

Recall the condition (4), it is easy to show that \( \frac{dL_1}{dt} \leq 0 \) whenever \( \mathcal{R}_0 \leq 1 \). Moreover, it can be shown that the largest invariant set \( \{ \frac{dL_1}{dt} = 0 \} \) is the singleton \( \{ E_0 \} \). By LaSalle’s Invariance Principle, the infection-free equilibrium \( E_0 \) of model (3) is globally asymptotically stable when \( \mathcal{R}_0 \leq 1 \). This completes the proof.

**Theorem 2.4:** If \( \mathcal{R}_0 > 1 \), then the infection equilibrium \( E_\ast \) is globally asymptotically stable.

**Proof:** Constructing a Lyapunov functional \( L_2 \) as follows:

\[
L_2 = \int_{\Omega} \left\{ \left( k_1 (1 - \alpha) + k_2 \alpha \right) \left( T - T_\ast - \int_{T_1}^T \frac{f(T_\ast, V_\ast)}{f(s, V_\ast)} \ ds \right) + k_1 \left( I - I_\ast - I_\ast \ln \frac{I}{I_\ast} \right) \\
+ k_2 \left( C - C_\ast - C_\ast \ln \frac{C}{C_\ast} \right) + \left( V - V_\ast - V_\ast \ln \frac{V}{V_\ast} \right) \\
+ k_1 (1 - \alpha)(1 - \epsilon)f(T_\ast, V_\ast)V_\ast \int_{T_1}^T \ h \left( \frac{f(T(x, s), V(x, s))}{f(T_\ast, V_\ast)} \right) \ ds \\
+ k_2 \alpha(1 - \epsilon)f(T_\ast, V_\ast)V_\ast \int_{T_1}^T \ h \left( \frac{f(T(x, s), V(x, s))}{f(T_\ast, V_\ast)} \right) \ ds \\
+ k_1 \delta I_{I_3} \int_{T_1}^T \ h \left( \frac{I(x, s)}{I_{I_3}} \right) \ ds + k_2 \mu \int_{T_1}^T \ h \left( \frac{C(x, s)}{C_\ast} \right) \ ds \right\} \ dx,
\]

where \( h(x) = x - 1 - \ln x \geq 0 \) for all \( x > 0 \) and with a global minimum \( h(1) = 0 \). In the calculation that follows we will use the equilibrium equations

\[
s = (1 - \epsilon)f(T_\ast, V_\ast)V_\ast + dT_\ast, \quad (1 - \alpha)(1 - \epsilon)f(T_\ast, V_\ast)V_\ast = \delta I_{I_3},
\]

\[
\alpha(1 - \epsilon)f(T_\ast, V_\ast)V_\ast = \mu C_\ast, \quad k_1 \delta I_\ast + k_2 \mu C_\ast = cV_\ast,
\]
\[
\frac{dL_2}{dt} = \int_{\Omega} \left\{ dT_{\ast}(k_{1}(1 - \alpha) + k_{2}\alpha) \left( 1 - \frac{T}{T_{\ast}} \right) \left( 1 - \frac{f(T_{\ast}, V_{\ast})}{f(T, V_{\ast})} \right) + (k_{1}(1 - \alpha) + k_{2}\alpha) \times (1 - \epsilon) \left( \frac{f(T_{\ast}, V_{\ast})}{f(T, V_{\ast})} \right) \left( f(T_{\ast}, V_{\ast})V_{\ast} - f(T, V) \right) + k_{1}(1 - \alpha)(1 - \epsilon)f(T_{\ast}, V_{\ast})V_{\ast} \times \left( 1 - \frac{I}{I_{\ast}} \right) \left( \frac{f(T_{\tau_{1}}, V_{\tau_{1}})V_{\tau_{1}}}{f(T_{\ast}, V_{\ast})V_{\ast}} - \frac{I}{I_{\ast}} \right) + k_{2}\alpha(1 - \epsilon)f(T_{\ast}, V_{\ast})V_{\ast} \left( 1 - \frac{C_{\ast}}{C} \right) \times \left( \frac{f(T, V)V}{f(T_{\ast}, V_{\ast})V_{\ast}} - \frac{C}{C_{\ast}} \right) + \left( 1 - \frac{V_{\ast}}{V} \right) \left[ k_{1}\delta I_{\ast} \left( \frac{I_{\tau_{3}}}{I_{\ast}} - \frac{V}{V_{\ast}} \right) + k_{2}\mu C_{\ast} \left( \frac{C_{\tau_{3}}}{C_{\ast}} - \frac{V}{V_{\ast}} \right) \right] + k_{1}\delta I_{\ast} \left( \frac{f(T, V)V}{f(T_{\ast}, V_{\ast})V_{\ast}} - \ln \frac{f(T, V)V}{f(T_{\ast}, V_{\ast})V_{\ast}} - \frac{f(T_{\tau_{1}}, V_{\tau_{1}})V_{\tau_{1}}}{f(T_{\ast}, V_{\ast})V_{\ast}} - \ln \frac{f(T_{\tau_{1}}, V_{\tau_{1}})V_{\tau_{1}}}{f(T_{\ast}, V_{\ast})V_{\ast}} \right) + k_{2}\mu C_{\ast} \left( \frac{f(T, V)V}{f(T_{\ast}, V_{\ast})V_{\ast}} - \ln \frac{f(T, V)V}{f(T_{\ast}, V_{\ast})V_{\ast}} - \frac{f(T_{\tau_{2}}, V_{\tau_{2}})V_{\tau_{2}}}{f(T_{\ast}, V_{\ast})V_{\ast}} + \ln \frac{f(T_{\tau_{2}}, V_{\tau_{2}})V_{\tau_{2}}}{f(T_{\ast}, V_{\ast})V_{\ast}} \right) + k_{1}\delta I_{\ast} \left( \frac{I}{I_{\ast}} - \ln \frac{I}{I_{\ast}} - \frac{I_{\tau_{3}}}{I_{\ast}} + \ln \frac{I_{\tau_{3}}}{I_{\ast}} \right) + k_{2}\mu C_{\ast} \left( \frac{C}{C_{\ast}} - \ln \frac{C}{C_{\ast}} - \frac{C_{\tau_{3}}}{C_{\ast}} + \ln \frac{C_{\tau_{3}}}{C_{\ast}} \right) + D \left( 1 - \frac{V_{\ast}}{V} \right) \Delta V \right) dx
\]

and note that \( \int_{\Omega} \Delta V(x, t) \, dx = 0 \), and \( \int_{\Omega} \frac{\Delta V(x, t)}{V(x, t)} \, dx = \int_{\Omega} \frac{\| \nabla V(x, t) \|^2}{V^2(x, t)} \, dx \). Then, calculating the time derivative of \( L_2 \) along a solution of model (3), we obtain

\[
\int_{\Omega} \frac{\Delta V(x, t)}{V(x, t)} \, dx = \int_{\Omega} \frac{\| \nabla V(x, t) \|^2}{V^2(x, t)} \, dx
\]
equilibrium see that the discrete model (7) has the same equilibria as model (3): the infection-free

3.1. Preliminary results

3. Dynamical analysis of model (7)

In this section, we dedicate to the investigation of the discrete model (7). It is easy to see that the discrete model (7) has the same equilibria as model (3): the infection-free equilibrium $E_0 = (T_0, 0, 0, 0)$ and the infection equilibrium $E_* = (T_*, I_*, C_*, V_*)$. In the following, we first show that the solution of model (7) is non-negative and bounded. To this end, rewriting the discrete model (7) yields

$$
\begin{align*}
T_{n+1}^m &= T_n^m + \Delta t \left[ s - dT_{n+1}^m - (1 - \epsilon)f(T_{n+1}^m, V_{n+1}^m) \right], \\
I_{n+1}^m &= I_n^m + \Delta t (1 - \omega)(1 - \epsilon)f(T_{n+1}^{m-1}, V_{n+1}^{m-1})V_{n+1}^{m-1}V_{n+1}^m, \\
C_{n+1}^m &= C_n^m + \Delta \alpha(1 - \epsilon)f(T_{n+1}^{m-1}, V_{n+1}^{m-1})V_{n+1}^{m-1}V_{n+1}^m, \\
AV_{n+1} &= V_n + \Delta t k_1 \delta I_{n+1}^m + \Delta t k_2 \mu C_{n+1}^m,
\end{align*}
$$

(12)

where matrix $A$ of dimension $(N + 1) \times (N + 1)$ is given by

$$
\begin{pmatrix}
c_1 & c_2 & 0 & \cdots & 0 & 0 & 0 \\
c_2 & c_3 & c_2 & \cdots & 0 & 0 & 0 \\
0 & c_2 & c_3 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_3 & c_2 & 0 \\
0 & 0 & 0 & \cdots & c_2 & c_3 & c_2 \\
0 & 0 & 0 & \cdots & 0 & c_2 & c_1
\end{pmatrix}
$$

with $c_1 = 1 + c\Delta t + D\Delta t/(\Delta x)^2$, $c_2 = -D\Delta t/(\Delta x)^2$ and $c_3 = 1 + c\Delta t + 2D\Delta t/(\Delta x)^2$. It is easy to show that $A$ is a strictly diagonally dominant matrix. Hence, $A$ is non-singular.

We then obtain that

$$V_{n+1} = A^{-1} \left( V_n + \Delta t k_1 \delta I_{n+1}^m + \Delta t k_2 \mu C_{n+1}^m \right).$$

**Theorem 3.1:** For any $\Delta t > 0$ and $\Delta x > 0$, the solutions of the discrete model (7) remain nonnegative and bounded for all $n \in \mathbb{N}$. 

Proof: We can claim that $T_n > 0$ for all $n \in \mathbb{N}$. In fact, assuming the contrary and letting $n_1 > 0$ be the first time such that $T_{n_1} \leq 0$ and $T_n > 0$, $I_n > 0$, $C_n > 0$, $V_n > 0$ for all $n < n_1$. Since,

$$T_{n_1-1}^m = T_{n_1}^m - \Delta t \left[ s - dT_{n_1}^m - (1 - \epsilon)f(T_{n_1}^m, V_{n_1-1}^m) \right].$$

Note that the conditions of (4), we then obtain $T_{n_1-1}^m \leq 0$, which contradicts our assumption and so $T_n > 0$ for all $n \in \mathbb{N}$. Moreover, it is easy to prove that the sequences $\{I_n\}$, $\{C_n\}$ and $\{V_n\}$ are non-negative by using mathematical induction.

Next, we establish the boundedness of solutions. To this end, we define a sequence $\{G_n\}$ as follows:

$$G_n^m = T_n^m + I_{n+m_1}^m + C_{n+m_2}^m.$$

Then, we have

$$G_{n+1}^m - G_n^m \leq \Delta t (s - \xi G_{n+1}^m),$$

where $\xi = \min\{d, \delta, \mu\}$. Thus we obtain

$$G_{n+1}^m \leq \frac{1}{1 + \xi \Delta t} G_n^m + \frac{s \Delta t}{1 + \xi \Delta t}.$$

By using induction, we easily obtain

$$G_n^m \leq \left( \frac{1}{1 + \Delta t \xi} \right)^n G_0^m + \frac{s}{\xi} \left[ 1 - \left( \frac{1}{1 + \Delta t \xi} \right)^n \right].$$

Thus $\lim_{n \to \infty} \sup G_n^m \leq \frac{s}{\xi}$, for all $m \in \{0, 1, \ldots, N\}$, which implies that $\{G_n\}$ is bounded. Therefore, $\{T_n\}$, $\{I_n\}$ and $\{C_n\}$ are bounded.

By the last equation of model (7) that

$$\sum_{m=0}^{N} V_n^m = \frac{1}{1 + c \Delta t} \left[ \sum_{m=0}^{N} V_n^m + \Delta t \left( \sum_{m=0}^{N} k_1 \delta I_{n+1-m_3}^m + \sum_{m=0}^{N} k_2 \mu C_{n+1-m_4}^m \right) \right].$$

Note that $\{I_n\}$ and $\{C_n\}$ are bounded, there exists two positive constant $\rho_1, \rho_2$ such that $I_n^m \leq \rho_1, C_n^m \leq \rho_2$ for $m \in \{0, 1, \ldots, N\}$. Thus we have

$$\sum_{m=0}^{N} V_{n+1}^m \leq \frac{1}{1 + c \Delta t} \left( \sum_{m=0}^{N} V_{n}^m + \Delta t (N+1) (k_1 \delta \rho_1 + k_2 \mu \rho_2) \right).$$

By induction, we have

$$\sum_{m=0}^{N} V_{n}^m \leq \frac{1}{(1 + c \Delta t)^n} \sum_{m=0}^{N} V_0^m + \frac{(k_1 \delta \rho_1 + k_2 \mu \rho_2)(N+1)}{c} \left( 1 - \frac{1}{(1 + c \Delta t)^n} \right) \leq \sum_{m=0}^{N} V_0^m + \frac{(k_1 \delta \rho_1 + k_2 \mu \rho_2)(N+1)}{c},$$

implying $\{V_n\}$ is bounded. This completes the proof. ■
3.2. Global stability

Theorem 3.2: For any $\Delta t > 0$, $\Delta x > 0$, if $R_0 \leq 1$, then the infection-free equilibrium $E_0$ of system (7) is globally asymptotically stable.

Proof: Consider the following discrete Lyapunov functional:

$$G_n = \sum_{m=0}^{N} \left\{ \frac{1}{\Delta t} \left[ (k_1(1 - \alpha) + k_2 \alpha) \left( T_n^m - T_{n-1}^m - \int_{T_{n-1}}^{T_n} \frac{f(T_0,0)}{f(s,0)} \, ds \right) + k_1 I_n^m + k_2 C_n^m ight. \right. $$

$$+ (1 + c \Delta t) V_n^m \right] + k_1(1 - \alpha)(1 - \epsilon) \sum_{j=n-m_1}^{n-1} f(T_{j+1}, V_j^m) V_j^m$$

$$+ k_2 \alpha(1 - \epsilon) \sum_{j=n-m_2}^{n-1} f(T_{j+1}, V_j^m) V_j^m + k_1 \delta \sum_{j=n-m_3}^{n-1} I_j^m + k_2 \mu \sum_{j=n-m_4}^{n-1} C_j^m \left. \right\}.$$ 

Then we have

$$G_{n+1} - G_n$$

$$= \sum_{m=0}^{N} \left\{ \frac{1}{\Delta t} \left[ (k_1(1 - \alpha) + k_2 \alpha) \left( T_{n+1}^m - T_n^m - \int_{T_n}^{T_{n+1}} \frac{f(T_0,0)}{f(s,0)} \, ds \right) \right. \right.$$ 

$$+ k_1 \delta (I_{n+1}^m - I_n^m) + k_2 \mu (C_{n+1}^m - C_n^m) + (1 + c \Delta t) (V_{n+1}^m - V_n^m) \right\]$$

$$+ k_1(1 - \alpha)(1 - \epsilon) \left( \sum_{j=n+1-m_1}^{n} f(T_{j+1}, V_j^m) V_j^m - \sum_{j=n-m_1}^{n-1} f(T_{j+1}, V_j^m) V_j^m \right)$$

$$+ k_2 \alpha(1 - \epsilon) \left( \sum_{j=n+1-m_2}^{n} f(T_{j+1}, V_j^m) V_j^m - \sum_{j=n-m_2}^{n-1} f(T_{j+1}, V_j^m) V_j^m \right)$$

$$+ k_1 \delta \left( \sum_{j=n+1-m_3}^{n} I_{j+1}^m - \sum_{j=n-m_3}^{n-1} I_j^m \right) + k_2 \mu \left( \sum_{j=n+1-m_4}^{n} C_{j+1}^m - \sum_{j=n-m_4}^{n-1} C_j^m \right) \left. \right\}$$

$$\leq \sum_{m=0}^{N} \left\{ (k_1(1 - \alpha) + k_2 \alpha) \left( 1 - \frac{f(T_0,0)}{f(T_{n+1},0)} \right) \left( s - d T_{n+1}^m - (1 - \epsilon) f(T_{n+1}^m, V_n^m) V_n^m \right) \right.$$ 

$$+ k_1 \left( (1 - \alpha)(1 - \epsilon) f(T_{n+1-m_1}^m, V_{n-m_1}^m) V_{n-m_1}^m - \delta I_{n+1}^m \right)$$

$$+ k_2 \left( \alpha(1 - \epsilon) f(T_{n+1-m_2}^m, V_{n-m_2}^m) V_{n-m_2}^m - \mu C_{n+1}^m \right) + D \frac{V_{n+1}^m - 2 V_n^m + V_{n-1}^m}{(\Delta x)^2}$$

$$+ k_1 \delta I_{n+1-m_3}^m + k_2 \mu C_{n+1-m_4}^m - C_{n+1}^m + c V_{n+1}^m + c (V_{n+1}^m - V_n^m)$$

$$+ k_1(1 - \alpha)(1 - \epsilon) \left( f(T_{n+1}^m, V_n^m) V_n^m - f(T_{n+1-m_1}^m, V_{n-m_1}^m) V_{n-m_1}^m \right)$$
\[ + k_2 \alpha (1 - \epsilon) \left( f(T_{n+1}^m, V_n^m) V_n^m - f(T_{n+1-m_2}^m, V_{n-m_2}^m) V_{n-m_2}^m \right) \]
\[ + k_1 \delta (I_{n+1}^m - I_{n+1-m_3}^m) + k_2 \mu \left( C_{n+1} - C_{n+1-m_4} \right) \]
\[ = \sum_{m=0}^{N} \left\{ dT_0(k(1 - \alpha) + k_2 \alpha) \left( 1 - \frac{T_{n+1}}{T_0} \right) \left( 1 - \frac{f(T_0, 0)}{f(T_{n+1}, 0)} \right) \right. \]
\[ + cV_n \left( \frac{(1 - \epsilon)(k(1 - \alpha) + k_2 \alpha)f(T_0, 0) f(T_{n+1}, V_n^m) f(T_{n+1}, 0)}{f(T_{n+1}, 0)} - 1 \right) \]
\[ \leq \sum_{m=0}^{N} \left\{ dT_0(k(1 - \alpha) + k_2 \alpha) \left( 1 - \frac{T_{n+1}}{T_0} \right) \left( 1 - \frac{f(T_0, 0)}{f(T_{n+1}, 0)} \right) + cV_n^m (\mathcal{R}_0 - 1) \right\}. \]

The last inequality is followed by the condition (4). Thus if \( \mathcal{R}_0 \leq 1 \), then we have \( G_{n+1} - G_n \leq 0 \), for all \( n \in \mathbb{N} \), which implies that \( G_n \) is a monotone decreasing sequence. Since \( G_n \geq 0 \), there is a limit \( \lim_{n \to \infty} G_n \geq 0 \) which implies that \( \lim_{n \to \infty} (G_{n+1} - G_n) = 0 \), from which we get \( \lim_{n \to \infty} T_n^m = T_0 \) and \( \lim_{n \to \infty} V_n^m (\mathcal{R}_0 - 1) = 0 \). We discuss two cases: (i) if \( \mathcal{R}_0 < 1 \), from model (7), we obtain \( \lim_{n \to \infty} I_n^m = 0 \), \( \lim_{n \to \infty} C_n^m = 0 \), for all \( m \in \{0, 1, \ldots, N\} \); (ii) if \( \mathcal{R}_0 = 1 \), by \( \lim_{n \to \infty} T_n^m = T_0 \) and from model (7), we have \( \lim_{n \to \infty} I_n^m = 0 \), \( \lim_{n \to \infty} C_n^m = 0 \), \( \lim_{n \to \infty} V_n^m = 0 \). Thus concluding the above discussion implies that \( E_0 \) is globally asymptotically stable. This completes the proof. □

**Theorem 3.3:** For any \( \Delta t > 0 \) and \( \Delta x > 0 \), if \( \mathcal{R}_0 > 1 \), then the infection equilibrium \( E_* \) of model (3) is globally asymptotically stable.

**Proof:** Define

\[ \tilde{G}_n = \sum_{m=0}^{N} \left\{ \frac{1}{\Delta t} \left[ (k(1 - \alpha) + k_2 \alpha) \left( T_n^m - T_* - \int_{T_*}^{T_n^m} \frac{f(T_*, V_*)}{f(s, V_*)} ds \right) \right. \right. \]
\[ + k_1 h \left( I_n \right) + k_2 C_n h \left( C_n \right) + (1 + c \Delta t) h \left( \frac{V_n}{V_*} \right) \]
\[ + \sum_{j=n-m_1}^{n-1} k_1 (1 - \alpha)(1 - \epsilon)f(T_*, V_*) V_* h \left( \frac{f(T_{j+1}, V_j^m) V_j^m}{f(T_*, V_*) V_*} \right) \]
\[ + \sum_{j=n-m_2}^{n-1} k_2 \alpha (1 - \epsilon)f(T_*, V_*) V_* h \left( \frac{f(T_{j+1}, V_j^m) V_j^m}{f(T_*, V_*) V_*} \right) \]
\[ + k_1 \delta I_* \sum_{j=n-m_3}^{n-1} h \left( \frac{I_{j+1}}{I_*} \right) + k_2 \mu C_* \sum_{j=n-m_4}^{n-1} h \left( \frac{C_{j+1}}{C_*} \right) \left\} \right. \]
\[ \right\}, \]

where \( h(x) = x - 1 - \ln x \geq 0 \) for all \( x > 0 \). Obviously, \( \varphi(x) \) has a global minimum at \( x = 1 \) and \( h(1) = 0 \).
Recall that model (12) and the infection equilibrium conditions (11), we then have the difference of $\tilde{G}_n$ satisfies

$$\tilde{G}_{n+1} - \tilde{G}_n$$

$$= \sum_{m=0}^{n} \left\{ \frac{1}{\Delta t} \left[ (k_1(1-\alpha) + k_2\alpha) \left( T_{m+1}^n - T_n^m - \int_{T_m^n}^{T_{m+1}^n} \frac{f(s, V_s)}{f(s, V_*)} \, ds \right) \right. \\
+ k_1 \left( I_{m+1}^n - I_m^n + I_* \ln \frac{I_m^n}{I_{m+1}^n} \right) + k_2 \left( C_m^n - C_n^m + C_* \ln \frac{C_m^n}{C_n^m} \right) \right. \\
+ (1 + c\Delta t) \left( V_{m+1}^n - V_m^n + V_* \ln \frac{V_m^n}{V_{m+1}^n} \right) \left. \right] + k_1(1-\alpha)(1-\epsilon)f(T_*, V_*) V_* \\
\times \left( \sum_{j=n+1-m_1}^{n} \varphi \left( \frac{f(T_{j+1}^m, V_{j+1}^m) V_{j+1}^m}{f(T_*, V_*) V_*} \right) \right) - \sum_{j=n-m_1}^{n-1} h \left( \frac{f(T_{j+1}^m, V_{j+1}^m) V_{j+1}^m}{f(T_*, V_*) V_*} \right) \\
+ k_2(1-\epsilon)f(T_*, V_*) V_* \sum_{j=n+1-m_2}^{n} h \left( \frac{f(T_{j+1}^m, V_{j+1}^m) V_{j+1}^m}{f(T_*, V_*) V_*} \right) \\
+ k_2(1-\epsilon)f(T_*, V_*) V_* \sum_{j=n-m_2}^{n-1} h \left( \frac{f(T_{j+1}^m, V_{j+1}^m) V_{j+1}^m}{f(T_*, V_*) V_*} \right) \\
+ k_1\delta I_* \left( \sum_{j=n+1-m_3}^{n} h \left( \frac{I_j+1^m}{I_*} \right) - \sum_{j=n-m_3}^{n-1} h \left( \frac{I_j+1^m}{I_*} \right) \right) \\
+ k_2\mu C_* \left( \sum_{j=n+1-m_4}^{n} h \left( \frac{C_{j+1}^m}{C_*} \right) - \sum_{j=n-m_4}^{n-1} h \left( \frac{C_{j+1}^m}{C_*} \right) \right) \right\}$$

$$\leq \sum_{m=0}^{N} \left\{ \frac{1}{\Delta t} \left[ (k_1(1-\alpha) + k_2\alpha) \left( 1 - \frac{f(T_*, V_*)}{f(T_{m+1}^n, V_*)} \right) (T_{m+1}^n - T_m^n) \right. \\
+ k_1 \left( 1 - \frac{I_*}{I_{m+1}^n} \right) (I_{m+1}^n - I_m^n) + k_2 \left( 1 - \frac{C_*}{C_{n+1}^m} \right) (C_m^n - C_{n+1}^m) \right. \\
+ \left. \left( 1 - \frac{V_*}{V_{m+1}^n} \right) (V_{m+1}^n - V_m^n) + c\Delta t \left( V_{m+1}^n - V_m^n + V_* \ln \frac{V_m^n}{V_{m+1}^n} \right) \right] \right\}$$

$$+ k_1(1-\alpha)(1-\epsilon) \left[ h \left( \frac{f(T_{m+1}^n, V_{m+1}^n) V_{m+1}^n}{f(T_*, V_*) V_*} \right) - h \left( \frac{f(T_{m+1}^n-m_1, V_{m+1}^n-m_1) V_{m+1}^n-m_1}{f(T_*, V_*) V_*} \right) \right] \\
+ k_2(1-\epsilon) \left[ h \left( \frac{f(T_{m+1}^n, V_{m+1}^n) V_{m+1}^n}{f(T_*, V_*) V_*} \right) - h \left( \frac{f(T_{m+1}^n-m_2, V_{m+1}^n-m_2) V_{m+1}^n-m_2}{f(T_*, V_*) V_*} \right) \right] \\
+ k_1\delta I_* \left[ h \left( \frac{I_{m+1}^n}{I_*} \right) - h \left( \frac{I_{m+1}^n-m_3}{I_*} \right) \right] + k_2\mu C_* \left[ h \left( \frac{C_{m+1}^n}{C_*} \right) - h \left( \frac{C_{m+1}^n-m_4}{C_*} \right) \right] \right\}
\[
\sum_{m=0}^{N} \left\{ k_1 (1 - \alpha) + k_2 \alpha \left( 1 - \frac{f(T_n, V_n)}{f(T_{n+1}^m, V_n)} \right) \right. \\
\times \left( s - d T_{n+1}^m (1 - \epsilon) f(T_{n+1}^m, V_{n+1}^m) V_n^m \right) \\
+ k_1 \left( 1 - \frac{I_n}{I_{n+1}} \right) \left( 1 - \alpha \right) (1 - \epsilon) (1 - \delta I_{n+1}) \left( \alpha (1 - \epsilon) f(T_{n+1-m_1}, V_{n-m_1}) - \mu C_{n+1} \right) \\
+ \left. k_2 \left( 1 - \frac{C_n}{C_{n+1}} \right) \alpha (1 - \epsilon) f(T_{n+1-m_2}, V_{n-m_2}) - \mu C_{n+1} \right) + \left( 1 - \frac{V_n}{V_{n+1}} \right) \\
\times \left( D \frac{V_{n+1}}{n+1} - \frac{2 V_{n+1}^{m+1} + V_{n+1}^{m-1}}{(\Delta x)^2} + k_1 \delta I_n + k_2 \mu C_{n+1-m_4} - c V_{n+1}^m \right) \\
+ k_1 (1 - \alpha) (1 - \epsilon) \left[ h \left( \frac{f(T_{n+1}, V_{n+1}^m)}{f(T_n, V_n)} \right) - h \left( \frac{f(T_{n+1-m_1}, V_{n-m_1})}{f(T_n, V_n)} \right) \right] \\
+ k_2 \alpha (1 - \epsilon) \left[ h \left( \frac{f(T_{n+1-m_2}, V_{n-m_2})}{f(T_n, V_n)} \right) - h \left( \frac{f(T_{n+1-m_1}, V_{n-m_1})}{f(T_n, V_n)} \right) \right] \\
+ k_1 \delta I_n \left[ h \left( \frac{I_{n+1}}{I_n} \right) \right] + k_2 \mu C_n \left[ h \left( \frac{C_{n+1}}{C_n} \right) \right] \\
\frac{c \left( V_{n+1} - V_n + V_n \ln \frac{V_n}{V_{n+1}} \right)}{n+1} \right. \\
\left. \sum_{m=0}^{N} \right\} d T_n \left( k_1 (1 - \alpha) + k_2 \alpha \right) \left( 1 - \frac{T_{n+1}}{T_n} \right) \left( 1 - \frac{f(T_n, V_n)}{f(T_{n+1}, V_n)} \right) \\
+ k_1 (1 - \alpha) (1 - \epsilon) f(T_n, V_n) V_n \left[ \left( 1 - \frac{f(T_n, V_n)}{f(T_{n+1}, V_n)} \right) \left( 1 - \frac{f(T_{n+1}, V_n)}{f(T_n, V_n)} \right) \right] \\
+ \left( 1 - \frac{I_n}{I_{n+1}} \right) \left( \frac{f(T_{n+1}, V_n)}{f(T_n, V_n)} V_n - \frac{T_{n+1}}{I_n} \right) \\
+ \left( 1 - \frac{I_n}{I_{n+1}} \right) \left( \frac{I_n}{I_{n+1}} V_n - \frac{V_n}{V_n} \right) + \left( \frac{V_n}{V_n} \right) - \frac{V_n}{V_n} + \ln \frac{V_n}{V_n} \right) \\
+ h \left( \frac{f(T_{n+1}, V_n)}{f(T_n, V_n)} V_n \right) - h \left( \frac{f(T_{n+1}, V_n)}{f(T_n, V_n)} V_n \right) \\
+ \left( \frac{I_{n+1}}{I_n} \right) - h \left( \frac{I_{n+1}}{I_n} \right) \right] + k_2 \alpha (1 - \epsilon) f(T_n, V_n) V_n \left[ \left( 1 - \frac{f(T_n, V_n)}{f(T_{n+1}, V_n)} \right) \left( 1 - \frac{f(T_{n+1}, V_n)}{f(T_n, V_n)} \right) \right] \\
\times \left( 1 - \frac{f(T_n, V_n)}{f(T_{n+1}, V_n)} V_n \right) + \left( 1 - \frac{C_n}{C_{n+1}} \right) \left( \frac{f(T_{n+1}, V_{n+1})}{f(T_n, V_n)} \right) V_n \right) \\
+ \left( 1 - \frac{V_n}{C_{n+1}} \right) \left( \frac{C_{n+1-m_4}}{V_n} \right) - \frac{V_n}{V_n} + \ln \frac{V_n}{V_n} \right) 
\]
\[
+ h \left( \frac{f(T_{n+1}^m V_{n+1}^m V_{n+1}^m)}{f(T_*, V_*) V_*} \right) - h \left( \frac{f(T_{n+2}^m V_{n+2}^m V_{n+2}^m)}{f(T_*, V_*) V_*} \right) \\
+ h \left( \frac{C_{n+1}^m}{C_*} \right) - h \left( \frac{C_{n+1-m_4}^m}{C_*} \right) \right] - \frac{DV_*}{(\Delta x)^2} \sum_{m=0}^{N-1} \left( \frac{V_{n+1}^m - V_{n+1}^m}{V_{n+1}^m} \right)^2 \\
= \sum_{m=0}^{N} \left\{ dT_* (k_1 (1 - \alpha) + k_2 \alpha) \left( 1 - \frac{T_{n+1}^m}{T_*} \right) \left( 1 - \frac{f(T_*, V_*)}{f(T_{n+1}^m, V_*)} \right) \right.
+ k_1 (1 - \alpha) (1 - \epsilon) f(T_*, V_*) V_* \left[ 3 - \frac{f(T_*, V_*)}{f(T_{n+1}^m, V_*)} - \frac{I f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} \right] \\
- \frac{V_{n+1}^m I_*}{V_{n+1}^m} + \frac{f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} - \frac{V_{n+1}^m}{V_*} + \frac{V_{n+1}^m}{V_*} + \ln \frac{V_{n+1}^m}{V_*} + \ln \frac{f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} \\
+ \ln \frac{V_{n+1}^m}{V_*} + \ln \frac{C_{n+1-m_4}^m}{C_{n+1}^m} \right\} - \frac{DV_*}{(\Delta x)^2} \sum_{m=0}^{N-1} \left( \frac{V_{n+1}^m - V_{n+1}^m}{V_{n+1}^m} \right)^2 \\
= \sum_{m=0}^{N} \left\{ dT_* (k_1 (1 - \alpha) + k_2 \alpha) \left( 1 - \frac{T_{n+1}^m}{T_*} \right) \left( 1 - \frac{f(T_*, V_*)}{f(T_{n+1}^m, V_*)} \right) \right.
+ k_1 (1 - \alpha) (1 - \epsilon) f(T_*, V_*) V_* \left[ 3 - \frac{f(T_*, V_*)}{f(T_{n+1}^m, V_*)} - \frac{I f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} \right] \\
- \frac{V_{n+1}^m I_*}{V_{n+1}^m} + \frac{f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} - \frac{V_{n+1}^m}{V_*} + \frac{V_{n+1}^m}{V_*} + \ln \frac{V_{n+1}^m}{V_*} + \ln \frac{f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} \\
+ \ln \frac{V_{n+1}^m}{V_*} + \ln \frac{C_{n+1-m_4}^m}{C_{n+1}^m} \right\} - \frac{DV_*}{(\Delta x)^2} \sum_{m=0}^{N-1} \left( \frac{V_{n+1}^m - V_{n+1}^m}{V_{n+1}^m} \right)^2 \\
= \sum_{m=0}^{N} \left\{ dT_* (k_1 (1 - \alpha) + k_2 \alpha) \left( 1 - \frac{T_{n+1}^m}{T_*} \right) \left( 1 - \frac{f(T_*, V_*)}{f(T_{n+1}^m, V_*)} \right) \right.
+ k_1 (1 - \alpha) (1 - \epsilon) f(T_*, V_*) V_* \left[ 3 - \frac{f(T_*, V_*)}{f(T_{n+1}^m, V_*)} - \frac{I f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} \right] \\
- \frac{V_{n+1}^m I_*}{V_{n+1}^m} + \frac{f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} - \frac{V_{n+1}^m}{V_*} + \frac{V_{n+1}^m}{V_*} + \ln \frac{V_{n+1}^m}{V_*} + \ln \frac{f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} \\
+ \ln \frac{V_{n+1}^m}{V_*} + \ln \frac{C_{n+1-m_4}^m}{C_{n+1}^m} \right\} - \frac{DV_*}{(\Delta x)^2} \sum_{m=0}^{N-1} \left( \frac{V_{n+1}^m - V_{n+1}^m}{V_{n+1}^m} \right)^2 \\
= \sum_{m=0}^{N} \left\{ dT_* (k_1 (1 - \alpha) + k_2 \alpha) \left( 1 - \frac{T_{n+1}^m}{T_*} \right) \left( 1 - \frac{f(T_*, V_*)}{f(T_{n+1}^m, V_*)} \right) \right.
+ k_1 (1 - \alpha) (1 - \epsilon) f(T_*, V_*) V_* \left[ 3 - \frac{f(T_*, V_*)}{f(T_{n+1}^m, V_*)} - \frac{I f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} \right] \\
- \frac{V_{n+1}^m I_*}{V_{n+1}^m} + \frac{f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} - \frac{V_{n+1}^m}{V_*} + \frac{V_{n+1}^m}{V_*} + \ln \frac{V_{n+1}^m}{V_*} + \ln \frac{f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_*) V_*} \\
+ \ln \frac{V_{n+1}^m}{V_*} + \ln \frac{C_{n+1-m_4}^m}{C_{n+1}^m} \right\} - \frac{DV_*}{(\Delta x)^2} \sum_{m=0}^{N-1} \left( \frac{V_{n+1}^m - V_{n+1}^m}{V_{n+1}^m} \right)^2 
\]
- \ h \left(\frac{I_n f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{f(T_{n+1}^m, V_{n+1}^m)}\right) + \ h \left(\frac{f(T_{n+1}^m, V_{n+1}^m)}{f(T_{n+1}^m, V_{n+1}^m)}\right) \\
+ \frac{V_n}{V_*} \left(1 - \frac{f(T_{n+1}^m, V_{n+1}^m)}{f(T_{n+1}^m, V_{n+1}^m)}\right) \left(\frac{f(T_{n+1}^m, V_{n+1}^m)}{f(T_{n+1}^m, V_{n+1}^m)} - 1\right) \\
+ k_2 \alpha (1 - \epsilon) f(T_{n+1}^m, V_{n+1}^m) \left[ - \ h \left(\frac{f(T_{n}^m, V_{n}^m)}{f(T_{n+1}^m, V_{n+1}^m)}\right) - \ h \left(\frac{V_* C_{n+1}^m}{V_{n+1}^m + C_{n+1}^m}\right)\right] \\
- \ h \left(\frac{C_n f(T_{n+1}^m, V_{n+1}^m) V_{n+1}^m}{C_{n+1} f(T_{n+1}^m, V_{n+1}^m)}\right) \left(\frac{f(T_{n+1}^m, V_{n+1}^m)}{f(T_{n+1}^m, V_{n+1}^m)} - 1\right) \\
+ \frac{V_n}{V_*} \left(1 - \frac{f(T_{n+1}^m, V_{n+1}^m)}{f(T_{n+1}^m, V_{n+1}^m)}\right) \left(\frac{f(T_{n+1}^m, V_{n+1}^m)}{f(T_{n+1}^m, V_{n+1}^m)} - 1\right) \right] \\
- \frac{D V_*}{(\Delta x)^2} \sum_{m=0}^{N-1} \left(\frac{V_{n+1}^m - V_{n+1}^m}{V_{n+1}^m + V_{n+1}^m}\right)^2.

It follows from the condition (4) that \(1 - \frac{f(T_{n+1}^m, V_{n+1}^m)}{f(T_{n+1}^m, V_{n+1}^m)}\left(\frac{f(T_{n+1}^m, V_{n+1}^m)}{f(T_{n+1}^m, V_{n+1}^m)} - 1\right) \leq 0. \) Recall that \(\varphi(x) \geq 0\) for all \(x > 0,\) we then obtain \(\tilde{G}_{n+1}^m - G_n^m \leq 0,\) for all \(n \in \mathbb{N}.\) This implies that \(\tilde{G}_n^m\) is monotone decreasing sequence. Since \(\tilde{G}_n^m > 0,\) there is a limit \(\lim_{n \to \infty} \tilde{G}_n^m > 0.\) Hence, \(\lim_{n \to \infty} T_n^m = T_* \), \(\lim_{n \to \infty} C_n^m = C_* \), \(\lim_{n \to \infty} V_n^m = V_* \), for all \(m \in \{0, 1, \ldots, N\},\) which implies that \(E_*\) of model (7) is globally asymptotically stable. This completes the proof.  

### 4. Numerical simulations

In this section, we perform numerical simulation to validate the main theoretical results obtained in previous sections. To this end, we reduce model (3) to the one with \(f(T, V) = \frac{\beta T V}{1 + a V},\) then the model reads as

\[
\begin{aligned}
\frac{\partial T(x, t)}{\partial t} &= s - dT(x, t) - (1 - \epsilon) \frac{\beta T V}{1 + a V}, \\
\frac{\partial I(x, t)}{\partial t} &= (1 - \alpha)(1 - \epsilon) \frac{\beta T(x, t - \tau_1) V(x, t - \tau_1)}{1 + a V(x, t - \tau_1)} - \delta I(x, t), \\
\frac{\partial C(x, t)}{\partial t} &= \alpha(1 - \epsilon) \frac{\beta T(x, t - \tau_2) V(x, t - \tau_2)}{1 + a V(x, t - \tau_2)} - \mu C(x, t), \\
\frac{\partial V(x, t)}{\partial t} &= D \Delta V(x, t) + k_1 \delta I(x, t - \tau_3) + k_2 \mu C(x, t - \tau_4) - c V(x, t),
\end{aligned}
\]

with the homogeneous Neumann boundary conditions

\[
\frac{\partial V}{\partial x} = 0, \quad t > 0, \quad x \in \partial \Omega
\]

and initial conditions

\[
T(x, s) = \phi_1(x, s) \geq 0, \quad I(x, s) = \phi_2(x, s) \geq 0, \quad C(x, s) = \phi_3(x, s) \geq 0,
\]
\[ V(x, s) = \phi_3(x, s) \geq 0, \quad (x, s) \in \tilde{\Omega} \times [-\tau, 0], \quad \tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}. \]

The corresponding discrete model to (13) is given by

\[
\begin{align*}
\frac{T_m^{n+1} - T_m^n}{\Delta t} &= s - d T_m^{n+1} - (1 - \epsilon) \frac{\beta T_m^{n+1} V_m^n}{1 + a V_m^n}, \\
\frac{I_m^{n+1} - I_m^n}{\Delta t} &= (1 - \alpha)(1 - \epsilon) \frac{\beta T_m^{n+1-m_1} V_m^{n-m_1}}{1 + a V_m^{n-m_1}} - \delta I_m^{n+1}, \\
\frac{C_m^{n+1} - C_m^n}{\Delta t} &= \alpha(1 - \epsilon) \frac{\beta T_m^{n+1-m_2} V_m^{n-m_2}}{1 + a V_m^{n-m_2}} - \mu C_m^{n+1}, \\
\frac{V_m^{n+1} - V_m^n}{\Delta t} &= D \frac{V_m^{n+1} - 2 V_m^n + V_m^{n-1}}{\Delta x^2} + k_1 \delta I_m^{n-m_3+1} + k_2 \mu C_m^{n-m_4+1} - c V_m^{n+1},
\end{align*}
\]

(14)

with the discrete boundary condition

\[ V_n^{-1} = V_0^n, \quad V_n^N = V_{N+1}^n, \quad \text{for } n \in \mathbb{N} \]

and the discrete initial conditions

\[ T_s^m = \phi_1(x_m, t_s) \geq 0, \quad I_s^m = \phi_2(x_m, t_s) \geq 0, \quad C_s^m = \phi_3(x_m, t_s) \geq 0, \]

\[ V_s^m = \phi_4(x_m, t_s) \geq 0, \text{for all } s = -l, -l + 1, \ldots, 0, \quad l = \max_{1 \leq i \leq 4}\{m_i\}. \]

Figure 1. \( R_0 < 1 \), the infection-free equilibrium \( E_0 \) is globally asymptotically stable.
Figure 2. $R_0 > 1$, the infection equilibrium $E_*$ is globally asymptotically stable.

Let $\Omega = [0, 20]$, $D = 0.05$, $\Delta t = 0.1$ and $\Delta x = 1$. For convenience, we set $a = 0.00001$, $\tau_1 = 3.5$, $\tau_2 = 2.5$, $\tau_3 = 1.5$ and $\tau_4 = 5$ in the following simulations. Some of the parameter values for these simulations are selected from [32]. The numerical implementation of (13) is carried out using the NSFD scheme described in (14). We first select the parameter values: $s = 1000$, $\beta = 8 \times 10^{-7}$, $d = 0.01$, $\alpha = 0.195$, $\epsilon = 0.5$, $a = 0.5$, $\delta = 0.7$, $\mu = 0.07$, $k_1 = 100$, $k_2 = 4.11$, $c = 13$. By a simple calculation, we have $R_0 = 0.2502 < 1$. Hence, in this case, the infection-free equilibrium $E_0 = (10^5, 0, 0, 0)$ is globally asymptotically stable, which means that the virus is cleared and the infection dies out. Figure 1 validates the above analysis.

Next, we select $s = 10^4$ and keep the other parameters are the same with Figure 1. In this case, we have $R_0 = 2.5016 > 1$ and the unique infection equilibrium $E_* = (5.1980 \times 10^5, 5.5223 \times 10^3, 1.3377 \times 10^4, 3.0032 \times 10^4)$ is globally asymptotically stable, which means that the virus persists in the host and the infection becomes chronic. Figure 2 confirms this observation.

5. Conclusion

In this paper, we have formulated a delayed and diffusive viral infection model incorporating shorted-lived and chronically infected cells and general nonlinear incidence function.
Then, by applying NSFD scheme, we presented an efficient numerical method for the corresponding continuous model. Theoretically, we have shown that the stability conditions for the equilibria are identical in case of both the continuous and discrete models. Specifically, if \( R_0 \leq 1 \), then the infection-free equilibrium \( E_0 \) is globally asymptotically stable; if \( R_0 > 1 \), then the infection equilibrium \( E_\ast \) is globally asymptotically stable. The results show that the NSFD scheme has the advantage that the positivity, boundedness and global properties of solutions for original continuous model are efficiently preserved.

As far as we know, there are few delayed and diffusive virus models considering both the shorted-lived and chronically infected cells, and no theoretical analysis has been made on this kind of models. Here, our main contribution is to construct suitable Lyapunov functional for both the continuous-time and discrete-time virus models, and present a general method to analyse this kind of models. Based on the above obtained results, one can extend this method to more complicated models.

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