Decomposition of fractional quantum Hall states: New symmetries and approximations

Ronny Thomale¹, Benoit Estienne², Nicolas Regnault³, and B. Andrei Bernevig¹

¹ Department of Physics, Princeton University, Princeton, NJ 08544, USA
² Institute for Theoretical Physics, Universiteit van Amsterdam Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands and
³ LPA, Département de Physique, ENS, CNRS, 24 rue Lhomond, 75005 Paris, France

(Dated: October 26, 2010)

We provide a detailed description of a new symmetry structure of the monomial (Slater) expansion coefficients of bosonic (fermionic) fractional quantum Hall states first obtained in Ref. 1, which we now extend to spin-singlet states. We show that the Haldane-Rezayi spin-singlet state can be obtained without exact diagonalization through a differential equation method that we conjecture to be generic to other FQH model states. The symmetry rules in Ref. 1 as well as the ones we obtain for the spin singlet states allow us to build approximations of FQH states that exhibit increasing overlap with the exact state (as a function of system size). We show that these overlaps reach unity in the thermodynamic limit even though our approximation omits more than half of the Hilbert space. We show that the product rule is valid for any FQH state which can be written as an expectation value of parafermionic operators.

PACS numbers: 73.43.f, 11.25.Hf

I. INTRODUCTION

Our understanding of the physics of the fractional quantum Hall effect (FQHE) has benefited greatly from the existence of model wave functions. Laughlin’s trial wave function for the ν = 1/3 filled FQH state provided the first paradigm to understand the emergent behavior of interacting electrons in a strong magnetic field. The current understanding of trial wave functions predominantly uses the conformal field theory (CFT) connection first proposed in Ref. 3. For every existing CFT, one can build a FQH trial wave function by taking the expectation value of branch-cut free primary fields in the CFT. The Read-Rezayi (RR) states are a product of this line of reasoning. Both spin-polarized as well as spin-singlet states can be obtained this way, most prominent examples of which are the Haldane-Rezayi (HR), the Non-Abelian spin singlet (NASS), and the Halperin states. FQH trial wave functions are essential to understanding the physically important concepts of fractional Abelian (in the Laughlin and composite fermion states) and non-Abelian statistics (in the Moore-Read (MR) and RR states).

The central drawback of the CFT-motivated trial wave function approach is the lack of both an explicit decomposition of a trial state in a second quantized many-body basis and of a first quantized closed form expression for the state. As a consequence, Monte Carlo methods, while useful for Laughlin states, cannot be applied for most non-Abelian states. Any quantitative analysis of these trial states has hence so far relied on exact diagonalization (ED) methods. In these methods, one starts with a trial Hamiltonian and generates the (lowest) Landau level (LLL) Hilbert space. The computational effort of diagonalization depends algebraically on the Hilbert space dimension, which grows factorially with system size. This sets the size limit that is reachable from ED. It is hence essential to use all available symmetries contained in the trial state and in the associated trial Hamiltonian to find the smallest subblock structure of the Hamiltonian matrix in terms of the non-interacting basis. One symmetry is the reflection of angular momentum \( L_z \rightarrow -L_z \) which, for a sphere geometry, is equivalent to the indistinguishability of the north and south pole. Other symmetries such as total \( L^2 \) multiplet structure exist in some cases. However, they are rather obvious in general and do not gain us deep insight in the structure of the FQH states. For the Laughlin 1/3 state, previous attempts to calculate the weights of the free many-body wave functions in the full interacting state failed. These works obtain only \( O(1/N!) \) of the \( O(N!) \) coefficients, and hence represent a set of measure zero of the Laughlin state expansion. In a recent paper, two of us found that a large series of FQH trial states obey a new type of symmetry for their free basis expansion coefficients. The symmetry relates a subset of the coefficients of the expansion in free many-body states of a given FQH state to products of state coefficients from smaller system size. This was developed for bosonic and spin-polarized fermionic states. In particular, it was observed in Ref. 7 that the overlap of the exact FQH state with the state approximated by the "product rule" symmetry increases with system size asymptotically towards unity.

In this paper, we give a detailed account of a general differential equation method used in Ref. 1 to access the monomial (Slater) decomposition of bosonic (fermionic) FQH states. We provide a detailed description of the very condensed derivation in Ref. 1 of the expansion coefficients for bosonic and polarized fermionic states. From there, we explain how the trial state can be numerically generated at a level intended for the novice reader. Next, we present an extended proof of the product rule symmetry for FQH trial states (and for all Jack polynomials).
previously summarized in Ref. [1]. We then extend the product rule symmetry, which allows to generate even more expansion coefficients than previously allowed. We also generalize the whole approach to spinful trial states, and illustrate it in detail for the HR state. We first derive an annihilation operator for the HR state from which we develop a recurrence relation for the expansion coefficients. We investigate the product rule symmetry analogue for spinful states and extract the entanglement spectrum of the HR state. For a spinful trial state, we find that the particle number $N$, angular momentum $L$, and spin multiplet $S$ are the quantum numbers of the reduced density matrix subblocks. The article is organized as follows. In Section II we discuss the recurrence relation of Jack polynomials that leads to the monomial expansion coefficients for bosonic FQH trial states. We elaborate on numerical subtleties for certain negative Jack parameters $\alpha$. In these cases, denominator divergences appear in the recurrence formula: they are accompanied by an (at least) similarly vanishing numerator. In Section III the Slater expansion coefficients of spin-polarized fermionic FQH states are derived from a fermionic version of the Laplace-Beltrami operator. This is the expanded version of previous calculations presented in Ref. [1]. The approach is used to develop a recurrence formula for fermionic FQH trial states. In Section IV we provide a largely expanded proof of the product rule symmetry. For non-Abelian bosonic states, we extend the product rule to treat general cases of cutting through a multiply occupied root partition orbital. In Section V we generalize the entire approach to the spinful HR state. We derive the recurrence formula, show the product rule property, and compute the entanglement spectrum of this spinful trial state. In Section VI we take a general viewpoint on the product rule symmetry from conformal field theory. We show that the product rule manifests itself as a generic property of all FQH states which can be written as an expectation value of parafermionic operators hence including a large set of both spin polarized and spin unpolarized FQH states. Finally, we conjecture in Section VII that the product rule symmetry is a structural property of the majority of FQH trial states including fermionic or bosonic states and spin-polarized or spin-unpolarized states, and potentially serves as an important ingredient to density matrix renormalization group approaches for FQH systems.

II. BOSONIC STATES

FQH states are analytic functions of the positions of electrons in a magnetic field. The single-particle orbitals in the Landau Level are given by $\phi_m(z) = (2\pi m!2^m)^{-1/2}z^m \exp(-|z|^2/4)$ with angular momentum $L_z = m \hbar$, although from now we will neglect the trivial Gaussian multiplication factors. A non-interacting $N$-particle basis state can be indexed by a partition $\lambda$ - an ordered list of the $L_z$ angular momentum of the occupied orbitals. The corresponding occupation number configuration is $n(\lambda) = (n_m(\lambda), m = 0, 1, 2, \ldots)_{15,16}$, where $m$ labels the individual single-particle orbitals and $n_m(\lambda)$ is the multiplicity of orbital $m$ in $\lambda$. We consider FQH states decomposed in this many-body basis, either of bosons (permanents) or fermions (slaters) with expansion coefficients $c_\lambda$. The central task of this paper is to develop methods to compute these expansion coefficients.

We now define a two-body operation on the many-body basis that is important for the purpose of the paper: for a pair of particles in the orbitals $m_1$ and $m_2$, with $m_1 < m_2 - 1$, the elementary squeezing operation consists of the two particles shifted to different momentum orbitals as $n_{m_1,2} \rightarrow n_{m_1,2} - 1$, $n_{m_1,2+1} \rightarrow n_{m_1,2+1} + 1$. This means that both particles in the $m_1, m_2$ orbitals are shifted ”inwards” the partition (as shown in Fig. 1). The squeezing defines a partial ordering relation between two partitions $\lambda > \mu$ when $\mu$ is generated by squeezing operations acting on $\lambda$. This ordering yields a tree hierarchy a complete example of which is shown in Appendix B. By contrast, when $\lambda$ and $\mu$ do not relate by squeezing, no ordering relation is set between these partitions.

The trial FQH states we consider are all squeezed polynomials. They possess a unique partition, called the root partition, dominating all other partitions. This means that all partitions with possible (but not guaranteed) non-zero weight are generated by subsequent squeezing operations acting on the root partition. In many cases, this already allows us to omit a significant (more than half) part of the Hilbert space (see Table I).

In this Section, we focus on the bosonic FQH states. The non-interacting basis is given by monomials $A(z_1, \ldots, z_N) = $Per$(z_i^{\lambda_j})/\prod n_m(\lambda)!$, where $\lambda_j$ is the momentum index of the $j$th particle in the partition $\lambda$ and Per is the permanent. It was shown that the $N$-particle bosonic RR $k$ series of states (which includes the Laughlin and MR state) are a special class of symmetric polynomials. Specifically, this class is called the Jack polynomials $J_{\lambda}^\alpha(z_1, \ldots, z_N)$ of parameter $\alpha = -\frac{m}{2}$ and root partition $\lambda = [k0^{r-1}k \ldots k0^{r-1}k]$. The Jack wave functions can be related to WA$k$-1 conformal field theories and can be classified in terms of symmetric polynomial categories. Moreover, the quasi-particle excitations of the trial state systems can also be written as coherent state superpositions of Jacks. This provides a complementary view to that of other approaches for FQH quasi-particle excitation states.
Algorithmic steps to generate the Jack state

(i) Generate squeezed monomial basis $M_{\lambda_i}: M_1 = M_\lambda$ is the root partition
   Order basis states by the integers $\sum_i 2^{\lambda_i + N - i}$
   Compute all $\rho_\kappa(\alpha)$ (Eq. 4)
(iii) Loop over all $\kappa$: Loop over all pairs of elements $l_i < l_j \in \kappa$
   For each pair unsqueeze to upper dominant partitions $\mu > \kappa$ and read off $c_{\lambda\mu}$
(iv) Compute contribution to $c_{\lambda\kappa}$ by Eq. 5
   if $\rho_\kappa(\alpha) = \rho_\lambda(\alpha)$ compute the limit prescription $\lim_{\alpha \to \alpha - \epsilon}$

TABLE II. Sketched how to use Eq. 3 to generate bosonic FQH states in monomial basis and Jacks of arbitrary parameter $\alpha$.

| nbr particles | full dim. | squeezed dim. |
|---------------|-----------|---------------|
| 4             | 18        | 16            |
| 5             | 73        | 59            |
| 6             | 338       | 247           |
| 7             | 1656      | 1111          |
| 8             | 8512      | 5302          |
| 9             | 45207     | 26376         |
| 10            | 246448    | 135670        |
| 11            | 1371535   | 716542        |
| 12            | 7764392   | 3868142       |
| 13            | 44585180  | 21265884      |
| 14            | 259140928 | 118741369     |
| 15            | 1521967986| 671906876     |
| 16            | 9020077206| 3846342253    |
| 17            | 53885028921| 22243294360  |

TABLE I. Size of the monomial basis for the bosonic Laughlin state $\nu = 1/2$ up to $N = 17$ particles. The second column is the complete size. The third column is the number of partitions allowed by the squeezing operation from the root partition 1010101...0101.

Jacks are eigenstates of the Laplace Beltrami (LB) operator:

$$\mathcal{H}_{LB} = \sum_i \left( z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left( z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right).$$

Until recently, the spectrum of the LB operator had been studied in detail for only positive $\alpha$ in the context of the Calogero-Sutherland model. Recent progress has shown that the LB operator has well-defined polynomial solutions for certain negative $\alpha$. In particular, as stated above, some of them are found to correspond to bosonic FQH trial states for the ground state, quasi-electron and quasi-hole excitations.

We expand the Jacks into the monomial basis:

$$J^\alpha_N = \sum_{\kappa \subseteq \lambda} c_{\lambda\kappa}(\alpha) M_{\kappa},$$

where $\kappa$ runs over all monomial partitions squeezed from the root partition $\lambda$. There is a known recurrence relation for the expansion coefficients $c_{\lambda\kappa}(\alpha)$:

$$c_{\lambda\kappa}(\alpha) = \frac{2/\alpha}{\rho_\lambda(\alpha) - \rho_\kappa(\alpha)} \sum_{\kappa < \mu \subseteq \lambda} ((l_i + t) - (l_j - t)) c_{\mu\kappa}(\alpha),$$

where $\kappa = [l_1, \ldots, l_t, \ldots, l_j, \ldots]$ and $\mu = [l_1, \ldots, l_t + t, \ldots, l_j - t, \ldots]$ denote partitions. We arrange the momentum orbitals denoted above in decreasing order from left to right, i.e. $l_1 \geq l_2 \geq l_t \geq \ldots$ in $\kappa$, and a possible rearrangement occurs in $\mu$ depending on $t$. All partitions $\mu$ are understood to be reordered in this way. The sum in (3) extends over all partitions $\mu$ strictly dominating $\kappa$ but being dominated (squeezed from) or equal to $\lambda$ that can be generated by unsqueezing (i.e. the inverse operation to squeezing). The $\rho$’s are defined as:

$$\rho_\lambda(\alpha) = \sum_i \lambda_i \left( \lambda_i - 1 - \frac{2}{\alpha} (i - 1) \right).$$

We now explain an easily implementable computer algorithm that allows one to obtain a large number of bosonic FQH states with high precision. From (4) we can uniquely index every partition by $\sum_i 2^{\lambda_i + N - i}$. For any numerical implementation, we order the basis according to this index, so that the look-up of a partition in the basis list becomes logarithmic in effort. By recurrence, we can always compute the coefficient of a partition from those coefficients of the partitions dominating it. The number of dominating partitions is small. Averaged over all partitions, it approximately scales with the number of fluxes ($\sim$ number of orbitals) times the square of the number of particles i.e. $N^2 N_\phi$. Thus, to compute all expansion coefficients, the procedure gives linear effort in the monomial basis dimension. The algorithm to generate the Jack state is sketched in Table II.
In contrast to positive $\alpha$ for which this formula was originally derived, there are minor caveats for certain negative $\alpha$. Situations occur in which the denominator in (3) vanishes for certain partitions. In these cases, one can find that two different partitions $\mu_1$ and $\mu_2$ obey $\rho_{\mu_1}(\alpha) = \rho_{\mu_2}(\alpha)$, a situation that can be proved to never arise for positive $\alpha$. An elementary example would be the 6-particle partitions $\mu_1(-4/3) = [5,5,4,1,1,0]$ and $\mu_2(-4/3) = [3,3,3,2,2]$. However, this denominator divergence is always regularized by a vanishing numerator. Under a limiting prescription $\lim_{\alpha \to 0} |\alpha - \alpha - \epsilon|$, the quotient either gives 0 or a rational value. Numerically, we let $\alpha$ slightly deviate from its exact value, vary it, and find a plateau value, which is then identified as the resulting expansion coefficient for the partition. This type of $\rho$ degeneracy does happen neither for the $r = 2$ RR series nor for the Gaffnian state. As before, the squeezing operation shifts two

$$\alpha^\mu_{\rho}$$

arise for positive $\alpha$, and $\rho_{\mu_1}(\alpha)$ originally derived, there are minor caveats for certain

III. POLARIZED FERMIONIC STATES

Similar to the bosonic case in Sec. II, we start with single particle orbitals of the Landau level defined before, i.e. $\phi_m(z) = (2\pi m!2^m)^{-1/2} z^m \exp(-|z|^2/4)$ with angular momentum $L_z = mh$ and $m$ the labeling index for all single particle orbitals. However, for the many-body state, we now assume that the particles described by the first quantized wave functions obey fermionic statistics. As a consequence, the non-interacting free fermion basis is given by Slater determinant states: $s_\lambda = A_\lambda(z_1^{\lambda_1} z_2^{\lambda_2} \ldots z_N^{\lambda_N}) = \text{Det}(z_i^{\lambda_j})$. $s_\lambda$ is the unnormalized orthogonal Slater determinant, where $A$ denotes the antisymmetrization over all $z$ coordinates. Different normalizations can be applied to put the polynomial wave function on different manifolds such as the plane or the sphere. As in the bosonic case, we describe the free many body states by partitions (or occupation numbers). We again assume the partition $\lambda = [\lambda_1, \ldots, \lambda_N]$ to be ordered by decreasing order in angular momentum $\lambda_i$ of the $i$th particle. As before, the squeezing operation shifts two particles inwards (towards each other) in the partition. For fermions, multiple occupancy is forbidden due to the Pauli principle.

In first quantized notation, bosonic and fermionic trial states can be transformed into each other by multiplication with a Vandermonde determinant. In terms of single particle coordinates, this polynomial is the Jastrow factor, which is the antisymmetric homogeneous polynomial of degree 1. Starting from a Jack polynomial $J_\lambda^\alpha$, the transformation reads $J_\lambda^\alpha \rightarrow S_\lambda^\alpha := J_\lambda^\alpha \prod_{i<j}(z_i - z_j)$. The $S_\lambda^\alpha$ polynomials are the exact fermionic analogue of the bosonic (Jack) trial state $J_\lambda^\alpha$. For example, the $\nu = 1/2$ bosonic Laughlin state (Jack of $(k, r) = (1, 2)$) becomes the $\nu = 1/3$ fermionic Laughlin state. The filling always changes from bosonic filling $\nu = p/q$ to fermionic filling $\nu = p/(p + q)$. However, in second quantized notation, multiplication by the Vandermonde determinant does not transform a single monomial to a single Slater. To obtain a one-to-one correspondence between a bosonic basis and fermionic Slaters, one would first have to transform the monomials to Schur functions. However, this involves knowledge of all the Kostka numbers, a longstanding unsolved mathematical problem with no known efficient algorithm. There are two ways to remedy this problem. First, we can use the knowledge that the transformation from monomials to Schur functions is exactly given by the $J^{\alpha=1}$ Jack polynomial coefficients, which we can compute from Eq. 3.

However, we try to tackle the fermionic trial states from a different angle. We define the decomposition of the $S_\lambda^\alpha$ polynomials into Slaters:

$$S_\lambda^\alpha(z_1, \ldots, z_N) = J_{\lambda\mu}^\alpha \prod_{i<j}(z_i - z_j) = \sum_{\mu \leq \lambda} b_{\lambda\mu} s_\mu,$$  \hspace{1cm} (5)$$

To avoid confusion, $\lambda\mu$ denotes the bosonic root partition and $\lambda$ the associated fermionic root partition. All partitions $\mu$ are squeezed from the fermionic partition $\lambda$ that is related to the bosonic partition by $\lambda_i = \lambda\mu_i + (N - \mu)$. We now use that the Jack part of $S_\lambda^\alpha$ is an eigenstate of the LB operator, i.e. $H_{LB} J_{\lambda\alpha}^\alpha = E_{\lambda\alpha}(\alpha) J_{\lambda\alpha}^\alpha$. We then relate the derivatives acting on $J_{\lambda\alpha}^\alpha$ to derivatives on $S_\lambda^\alpha$ (details are given in Appendix C):
the two-particle basis of Slater determinants, i.e. towards

\[ E_{\lambda_0}(\alpha)S_{\lambda_0}^\alpha = \prod_{k<l}(z_k - z_l) \left[ \sum_i \left( \frac{z_i}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \right] J_{\lambda_0}^\alpha \]

\[ = \left[ \sum_i \left( \frac{z_i}{\partial z_i} \right)^2 - 2 \sum_{i,m \neq i} \frac{z_i}{z_i - z_m} \frac{z_i}{\partial z_i} + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \right] S_{\lambda_0}^\alpha \]

\[ + \left[ \sum_{i,m \neq i} \frac{z_i(z_i + z_m)}{(z_i - z_m)^2} + \sum_{i,m,n \neq i} \frac{z_i^2}{(z_i - z_m)(z_i - z_n)} + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left( \sum_{m \neq i} \frac{z_i}{z_i - z_m} - \sum_{m \neq j} \frac{z_j}{z_j - z_m} \right) \right] S_{\lambda_0}^\alpha. \]

Simplifying several polynomial sums that yield constants as shown in Appendix A we can define a fermionic Laplace Beltrami operator that diagonalizes \( S_{\lambda_0}^\alpha \), i.e. \( H_{LB}(\alpha)S_{\lambda_0}^\alpha(z_1, \ldots, z_N) = E_{\lambda}(\alpha)S_{\lambda}^\alpha(z_1, \ldots, z_N) \), with

\[ E_{\lambda}(\alpha) = \sum_i \lambda_i \left( \lambda_i - 2 \left( \frac{1}{\alpha} - 1 \right) i \right) + \left( \frac{1}{\alpha} - 1 \right) \left( \frac{1}{\alpha} - 1 \right) \left( N + 1 \right) \sum_i \lambda_i - N (N - 1) \right), \]

\[ H_{LB}(\alpha) = H_K + \frac{1}{2} \left( \frac{1}{\alpha} - 1 \right) H_I = \sum_i \left( \frac{z_i}{\partial z_i} \right)^2 + \frac{1}{2} \left( \frac{1}{\alpha} - 1 \right) \left[ \sum_{i \neq j} \frac{z_i + z_j}{z_i - z_j} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) - 2 \frac{z_i^2 + z_j^2}{(z_i - z_j)^2} \right]. \]

We now diagonalize the above operator in the basis of Slater determinants. The action of the kinetic part yields \( \sum_i H_KS_{\mu} = (\sum_i \mu_i^2)S_{\mu} \), where the \( \mu_i \) denotes the polynomial power of the \( i \)th particle in the partition. The action of the interaction part \( H_I \) is non-diagonal in Slater determinant basis and demands further calculation. First we realize that, due to its two-body nature, the action of \( H_I \) on any Slater determinant decomposes into the sum of two-particle interaction terms. It is thus sufficient to look at the action on the two-particle Slater determinant \( S_{\mu}\alpha = \mu_1 \mu_2 \alpha_{\mu_1 \mu_2} \). Assume \( \mu_1 > \mu_2 \), and define \( p = \mu_1 - \mu_2 \):

\[ \frac{H_I}{z_1 \mu_1 \mu_2} = \left( \frac{z_1 + z_2}{z_1 - z_2} \right)^2 \left( s_1^p + s_2^p \right) - 2 \frac{z_1^2 + z_2^2}{(z_1 - z_2)^2} \left( s_1^p - s_2^p \right) \]

\[ = \frac{1}{z_1 - z_2} \left( p(z_1^{p+1} + z_2^{p+1} + z_1^p z_2 + z_2^p z_1) - 2 \sum_{s=1}^{p/2} \left( z_1^{p-s} z_2^s + z_2^{p-s} z_1^s + z_1^{p-s} z_2^s + z_2^{p-s} z_1^s \right) \right) \]

\[ = \frac{1}{z_1 - z_2} \sum_{s=1}^{p/2} \left( z_1^{p-s} z_2^s (z_2 - z_1) + z_2^{p-s} (z_1^{p-2} - z_1^{p-2}) + z_1^{p-s} z_2 (z_2 - z_1) + z_2^{p-s} z_1 (z_2 - z_1) \right) \]

\[ = 2 \sum_{s=1}^{p/2} \left( z_1^{p-s} - z_2^{p-s} \right) \sum_{t=1}^{(s-1)/2} \left( z_1^{s-t} z_2^{-t} - z_2^{s-t} z_1^{-t} \right) \]

The two terms are already grouped to yield two-particle Slater determinants. Collecting all prefactors, this gives:

\[ H_I S_{\mu_1 \mu_2} = (\mu_1 - \mu_2) S_{\mu_1 \mu_2} + 2 \sum_{s=1}^{(\mu_1 - \mu_2)/2} (\mu_1 - \mu_2 - 2s) S_{\mu_1 - s, \mu_2 + s}. \]

Eq. 10 has a particular form: it only scatters "inwards" the two-particle basis of Slater determinants, i.e. towards decreasing relative momentum of the particles, and thus to a squeezed partition. Let us now look at the total
unsqueezed partitions $\theta$:

$$b_{\lambda^0}(\alpha) = \frac{2(\frac{1}{\alpha} - 1)}{\rho^F_{\lambda^0}(\alpha) - \rho^F_{\lambda^0}(\lambda)} \sum_{\mu < \nu \leq \lambda} (\mu_1 - \mu_j) b_{\lambda^0}(-1)^{N_{SW}}$$

where $\rho^F_{\lambda^0}(\alpha) = \sum_{\lambda^0} \lambda_1^0 (\lambda^0 + 2(1 - 1/\alpha))$. Similar to the bosonic recurrence formula in [3], the sum in [11] extends over all partitions $\theta = [\mu_1, \ldots, \mu_1 + s, \ldots, \mu_j - s, \ldots, \mu_N]$ that dominate the partition $\mu = [\mu_1, \ldots, \mu_N]$ and are squeezed from the root partition $\lambda$. A new factor $(-1)^{N_{SW}}$ appears: a sign according to the even/oddness of the number of transpositions (swaps) of particles from a given dominating partition $\theta$ back to $\mu$. This term arises since the reordering of the partition in Slater determinants may cause a minus sign due to the fermionic anticommutation relations. $N_{SW}$ starts from zero for partition $\mu$ and advances by one unit every time the momentum of the unsqueezed electron passes through the value of the momentum for another electron. As a further difference from the bosonic Jack recurrence relation, the terms summed in Eq. 11 do not explicitly depend on the partition $\theta$. This is because the rescaling of the $s$ in 10 exactly cancels the term's dependence on $s$. For $\alpha = -(k + 1)$, 11 gives the coefficients of the fermionic Read-Rezayi states (an example computation of the partition coefficient for the $\nu = 1/3$ Laughlin state is shown in Fig. 2). The complete decomposition of the $N = 4$ particle $\nu = 1/3$ Laughlin state is given in Appendix B. As in the bosonic case, this is an advance in the numerical computation of the coefficients: the computational effort required to generate the state scales linearly with basis size. This approach has already been applied to increase the maximally reachable system in finite size studies.

### IV. PRODUCT RULES

The coefficients of the monomials (Slaters) in the bosonic (fermionic) FQH states exhibit a hidden symmetry found in Ref. 5 and named product rule. The product rule is valid in any quantum mechanical normalization, be it on the plane, sphere, cylinder, disk or any other genus-0 geometry. It is however easiest to see and explain in the basis of Eq. 3 and Eq. 11 for which we have already developed the formalism of the previous sections. Once a state is obtained in this basis, it is only a matter of specific change in the normalization of the free many-body wave functions to go between different genus-0 geometries. The product rule found in [3] and explained in detail in this section is valid not only for FQH Jacks but for all Jacks at any $\alpha$, of any partition $\lambda$. Furthermore, as shown in Section VI below, the product rule is a property of quantum Hall trial states even beyond Jack polynomials.

We consider a Jack state generated from (3) (or 11) that can serve as a suitable example to demonstrate the product rule. We discuss the fermionic MR state of $N = 10$ particles. This state can be written as a linear superposition of Slater determinants squeezed from the root partition $n(\lambda) = 110011011001110011100$. We pick a configuration squeezed from $\lambda$ that has the special property that two parts of the partition can be identified as squeezed from individual root partitions for smaller systems sizes (see Fig. 2). Let us consider 01101000101111010. We observe that the first 7 orbitals from the left, i.e. 011010, can be squeezed from the $N = 4$ particle 011010. The remainder right part, i.e. 00101111010, can be squeezed from the $N = 6$ particle 01100110011. We find that the product of the two coefficients obtained from the $N = 6$ part and the disconnected $N = 4$ part ($-70$ and $-2$ respectively) gives the coefficient $(-70) \cdot (-2) = 140$ of the $N = 10$ partition. The product rule (symmetry) allows the computation of a certain set of coefficients of an $N$-particle state from the knowledge of the state for $N - 1$ particles. This hints at similarities with Feynman disconnected diagram summation in interacting systems, where the total contribution is given by the product of the disconnected components (Fig. 3). As we show below, the product rule approximation succeeds in keeping the essential part of the correlation of the FQH state. We first prove the product rule for fermionic Jacks by induction principle; a similar
proof can be obtained for the bosonic Jacks.

Basic induction case. Assume we start with any fermionic polynomial \( S_N^\lambda(z_1, \ldots, z_N) \) of a configuration \( \mu \leq \lambda \) that can be divided in two disconnected sets: \( \mu = (\mu_A, \mu_B) \), with \( N_A \) particles in the first subpart \( A \) and \( N_B = N - N_A \) particles in the second subpart \( B \). This means that \( \mu_A \) is squeezable from an \( N_A \)-particle root partition \( \lambda_A = [\lambda_1, \ldots, \lambda_{N_A}] \), and \( \mu_B \) is squeezed from the \( N_B \)-particle root partition \( \lambda_B = [\lambda_{N_A+1}, \ldots, \lambda_N] \). The basic induction case is given by any partition of the monomials in \( S_N^\lambda(z_1, \ldots, z_N) \) for which the product rule holds. Trivially, for this purpose we can choose the root partition \( \lambda \). By definition, it has coefficient 1, and we can think of it as being separated into any product of two subpart root partitions. Again, all these have coefficient 1 by definition, so that the product rule holds for the root partition itself.

Induction hypothesis. We now assume the product rule is valid for all \( \lambda \) with \( \mu \leq \lambda \) and \( \mu < \lambda \). As shown in Eq. (10) the coefficients of \( S_N^\lambda \) are given as a recursion from partitions that dominate \( \mu \). By construction, any partition dominating \( \mu \) and entering (10) is also separable according to \( \theta = (\theta_A, \theta_B) \). As the unsqueezed operation is a two-body operation, the sum over all dominating partitions \( \theta \) can be decomposed into \( \sum \theta: \mu < \theta \leq \lambda = \sum_{\mu_A < \theta_A \leq \lambda_A} + \sum_{\mu_B < \theta_B \leq \lambda_B} \). In particular, the summation over the individual partition entries \( \mu_i \) only mixes \( \mu_j \) of the left hand side \( A \) and right hand side \( B \) separately, while the remainder right (left) part remains unchanged. Partition-wise, the first sum reads \((\theta_A, \mu_B)\), while the second reads \((\mu_A, \theta_B)\). Finally, we assume that all partitions dominating \( \mu \) satisfy the product rule:

\[
b_{\lambda_A, \mu_B}(\alpha) = b_{\lambda_A, \lambda_B}(\alpha)b_{\lambda(\mu_A, \mu_B)}(\alpha), \tag{12}
\]

where \((\theta_A, \lambda_B)\) denotes the partition formed by \( \theta_A \) and the remainder root state partition of part \( B \), \( \lambda_B \). This holds vice versa for \((\lambda_A, \theta_B)\).

Induction proof. Let us consider the coefficient \( b_{\lambda, \mu} \). By induction hypothesis, we can rewrite Eq. (10) as (we skip the argument \( \alpha \) in the notation for the coefficients \( b \)):

\[
b_{\lambda_A, \mu_B}(\alpha) = \frac{2(\frac{1}{n} - 1)}{\rho_A^F - \rho_B^F} \left( \sum_{\theta_A: \mu_A < \theta_A \leq \lambda_A} (\mu_i^A - \mu_j^A)b_{\lambda(\theta_A, \mu_B)}(-1)^{N_{SW}} + \sum_{\theta_B: \mu_B < \theta_B \leq \lambda_B} (\mu_i^B - \mu_j^B)b_{\lambda(\mu_A, \theta_B)}(-1)^{N_{SW}} \right)
\]

Writing out the coefficients \( b_{\lambda, \mu} \) according to Eq. (10) we then use \( \rho_A^F(\alpha) - \rho_B^F(\alpha) = \rho_A^F(\alpha) - \rho_B^F(\alpha) + \rho_A^F(\alpha) - \rho_A^F(\alpha) + \rho_A^F(\alpha) - \rho_A^F(\alpha) \) to get

\[
b_{\lambda_A, \mu_B}(\alpha) = \frac{2(\frac{1}{n} - 1)}{\rho_A^F(\alpha) - \rho_B^F(\alpha) + \rho_A^F(\alpha) - \rho_A^F(\alpha) + \rho_A^F(\alpha) - \rho_B^F(\alpha)} \left( \frac{2(\frac{1}{n} - 1)}{\rho_A^F(\alpha) - \rho_B^F(\alpha)} + \frac{2(\frac{1}{n} - 1)}{\rho_A^F(\alpha) - \rho_B^F(\alpha)} \right)
\]

A similar line of reasoning applies to the bosonic case. We have thus proved the product rule symmetry for this type of piece-separable configurations. This is valid for all bosonic and fermionic Jack polynomials, and hence for all Read-Rezayi states. However, the product rule even applies to a much larger range of polynomials (Section VII). Following similar steps as above, the product rule can also be explicitly derived for spin-unpolarized states such as the Haldane-Rezayi state discussed below in Section VII.

For bosonic states, there are certain classes of partitions where it is not immediately clear how to apply the product rule symmetry. Let us look again at the

\[
N = 10 \text{ particle MR state in bosonic notation; this state is squeezed from the root partition } \lambda = 2002020202.
\]

Partitions type I. An easy application of the product exists for configurations such as \( p_1 = 040000600 \). We identify the first 4 orbitals to be squeezed from \( \lambda_A = 2020 \), while the remainder orbitals are squeezed from \( \lambda_B = 020202 \); we find the product rule to hold (see also Fig. 8).

Partitions type II. Let us analyze the configuration \( p_2 = 023000401 \). We can split the configuration in a 5 particle separation to the left and a 5 particle separation to the right of the cut in Fig. 4. Both parts are disconnected in terms of squeezing operations on the particles.
However, what are the root partitions from which we generate the subparts? We have to split one doubly occupied orbital of the associated root partition 2020202 as in Fig. 3. We double copy this orbital and distribute the particles in both subparts. We consider 20201 and 10202 as the root partitions for subpart A and B: together they make up 20202020202 but the orbital where the bold particle 1 is placed is taken to belong to both parts A and B. At the same time, we double copy the 5th orbital of \( p_2 \), i.e. 02300 and 00401 (see Fig. 4). Following this recipe, we find that the product rule holds for these type II configurations. We can trace back separable configurations of type \( p_2 \) to product rule compositions of smaller system size (Fig. 4).

Let us investigate the accuracy of the approximated state for \( N \) particles built from the product rule using the exact states up to \( N - 1 \) particles. To begin with, an important quantity is the number of monomials (or Slater determinants) whose configurations conform to the product rule. Tables III and IV show the ratio between this number and the total size of the squeezed Hilbert space for the Laughlin state \( (\nu = 1/2 \text{ and } \nu = 1/3) \) and the MR state, respectively. As a rule of thumb, the product rule allows to construct more than a third and less than a half of the total Hilbert space. This ratio decreases with increasing system size, but remains a finite > 1/3 fraction of the Hilbert space.

The important question is how much of the exact state is kept in this part of the Hilbert space generated by the product rule. We compute the overlap between the exact state for \( N \) particles and the state constructed only from the product rule. The overlap is taken using the scalar product of the sphere geometry. We find that, involving only type I partitions for the Laughlin state (Table III), the overlap has > 99.9% overlap with the exact state. This tells us that the monomials that are generated by the product rule contain almost all of the exact state by overlap despite comprising only \( \sim 1/3 \) of its Hilbert space. In all cases we consider, the overlap has the peculiar feature that it increases with system size (by contrast, any comparison between a model state and the ground state of some realistic interaction would exhibit the opposite behavior).

As such, this provides indication that the product rule symmetry of quantum Hall trial states becomes exact in the thermodynamic limit. Fermionic states show a very similar behavior (see Table III for the Laughlin state at \( \nu = 1/3 \)). The overlaps are also very high but not as good as their bosonic counterpart. In our opinion, the product rule should be an essential ingredient of future DMRG studies.

For now, we have only considered partitions subject to the product rule construction which can be decomposed into a pair of ground state partitions of smaller systems size. In fact, one can further improve the procedure as we find for the specific case of Laughlin states. The product rule can be applied not only when considering disconnected squeezing sequences from the root partition, but also from a partition such as 10010001 with a cut between the two consecutive particles in the 8th and 9th orbital. In this case, to reconstruct the Slater determinant weight, one needs to glue together two Laughlin states with one quasi-hole excitation each (which are also Jack polynomials with the same \( \alpha \) parameter as the ground state). In this example, this corresponds to considering the Jacks of roots 10010001 and 10001. The only missing information we additionally need to know is the weight of a Slater determinant in a Laughlin state which is obtained from the root configuration by a single squeezing of two neighboring particles ...1001... into two consecutive occupied orbitals ...0110... It can be shown that this weight is always equal to 3 in the basis described in Eq. (10). The improvement of the overlap including this additional rule is shown in Table III.


\[
\begin{array}{c|cccccccccc}
N & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \\
\hline
\text{squeezed dim.} & 119 & 1070 & 10751 & 116287 & 1326581 & 15756587 & 193181910 & 242992112 \\
\text{prod. rules (type I)} & 27.73\% & 25.23\% & 23.58\% & 22.48\% & 21.63\% & 20.95\% & 20.40\% & 19.95\% \\
\text{overlap (type I)} & 0.8858 & 0.9070 & 0.9188 & 0.9262 & 0.9311 & 0.9344 & 0.9367 & 0.9383 \\
\text{prod. rules (type I + type II)} & 72.27\% & 70.09\% & 68.48\% & 66.98\% & 65.79\% & 64.82\% & 64.01\% & 63.34\% \\
\text{overlap (type I + type II)} & 0.9875 & 0.9895 & 0.9919 & 0.9931 & 0.9944 & 0.9947 & 0.9956 & 0.9963 \\
\end{array}
\]


**TABLE IV.** Shown: percentage of Hilbert space that is constructed from the product rule and overlap between the complete MR state and the state built from product rule on the sphere geometry. \(N\) is the number of particles and the overlap is defined as the absolute value of the scalar product. The first row of data is the total dimension of the squeezed Hilbert space for different system sizes. The second and third row are the overlap results obtained when only type I partitions are taken into account, while fourth and fifth row show the results obtained by involving both type I and type II partitions.

\[
\begin{array}{c|cccccccc}
\nu & 3 & 6 & 9 & 12 & 15 & 18 & 21 \\
\hline
\nu = 1/2 \text{ prod. rules} & 43.76\% & 42.01\% & 40.76\% & 39.93\% & 39.52\% & 39.32\% & 39.24\% & 39.19\% \\
\nu = 1/2 \text{ overlap} & 0.9933 & 0.9947 & 0.9947 & 0.9956 & 0.9963 & 0.9968 & 0.9972 & 0.9977 & 0.9979 \\
\nu = 1/3 \text{ prod. rules} & 47.68\% & 46.41\% & 45.33\% & 44.45\% & 43.73\% & 43.11\% & 42.56\% & 42.08\% & 41.65\% \\
\nu = 1/3 \text{ overlap} & 0.9502 & 0.9534 & 0.9557 & 0.9573 & 0.9585 & 0.9593 & 0.9599 & 0.9603 & 0.9605 \\
\nu = 1/3 \text{ prod. rules + qh} & 73.92\% & 72.52\% & 71.35\% & 70.40\% & 69.61\% & 68.90\% & 68.27\% & 67.70\% & 67.18\% \\
\nu = 1/3 \text{ overlap + qh} & 0.9938 & 0.9944 & 0.9949 & 0.9952 & 0.9955 & 0.9956 & 0.9957 & 0.9958 & 0.9958 \\
\end{array}
\]

**TABLE III.** Shown: Percentage of Hilbert space that can be constructed from the product rule and overlap between the exact full state and the state built from the product rule for sphere geometry. \(N\) is the number of particles and the overlap is defined as the absolute value of the scalar product. The first two rows of data are the results for the \(\nu = 1/2\) bosonic Laughlin state for different numbers of particles. The third and fourth row of data are the results for the \(\nu = 1/3\) fermionic Laughlin state relying only on the knowledge of the Slatters determined by ground state partitions, while the fifth and sixth row additionally take into account information stemming from the single quasi-hole excitations.

\section{V. \textsc{Haldane-Rezayi State}}

\subsection*{A. Basic properties}

We now turn to the generalization of the bosonic and fermionic states involving the spin degree of freedom of the constituent particles. In the following, we discuss the Haldane-Rezayi (HR) spin singlet state. The HR state was originally proposed as a trial state for the incompressible plateau state at \(\nu = 5/2\). As opposed to the spin-polarized Moore-Read (Pfaffian) state at identical filling, it is a spin singlet. The degree of spin polarization is still an experimental issue that is not yet settled in the \(\nu = 5/2\) state. The general belief, supported by numerical evidence from exact diagonalization studies, is that the Moore-Read state is the promising candidate to explain the \(\nu = 5/2\) state. The HR state attained considerable attention since it can be interpreted as describing the transition point between the strong and weak pairing phases of a spin-singlet \(d\)-wave BCS superconductor. This physical interpretation is supported by the realization that the bulk CFT is a non-unitary \(c = -2\) theory, which is expected to have gapless bulk excitations.\(^{55}\)

The HR state is given by:

\[
\Phi_{\text{HR}} = A_{z,w} \left( \frac{1}{(z_1 - w_1)^2} \frac{1}{(z_2 - w_2)^2} \cdots \right) \times \prod_{i<j}^N (z_i - z_j)^2 \prod_{i<j}^N (w_i - w_j)^2 \prod_{i,j}^N (z_i - w_j)^2,
\]

where \(N\) is the number of spin \(\uparrow\) and spin \(\downarrow\) particles with positions denoted by \(z\) and \(w\), \(A_{z,w}\) is the antisymmetrizer over the \(z\) particles and \(w\) particles separately. Due to the antisymmetric non-\(U(1)\) prefactor in the first line of (15) and the evenness of the Jastrow factors, the state in (15) is fermionic although a bosonic variant can also be written. Let us look at the clustering conditions at the level of first quantization. The wave function dies as the 2nd power of the difference between two equal spin coordinates. However, the prefactor of the Jastrow factors removes the 2nd order inter-spin zeroes induced by
the last Jastrow factor, allowing for configurations where one \( \uparrow \) and one \( \downarrow \) spin sit on the same site.

In order to span the Hilbert space of spinful states, we again start from the single particle Landau orbitals. However, in this case, the single particle orbitals possess an additional spin quantum number \( \sigma \) taking on the values \( \uparrow \) or \( \downarrow \). The free many-body basis is given by spinful Slater determinants \( s_{\lambda A; \lambda B} = \mathcal{A}(z_1^{\lambda_A; \lambda B_1}, \ldots, z_N^{\lambda_A; \lambda B_1}, w_1^{\lambda_A; \lambda B_1}, \ldots, w_N^{\lambda_A; \lambda B_1}) \). \( \lambda_A \) and \( \lambda_B \) label the momenta of the \( i \)th \( \uparrow \) spin and \( i \)th \( \downarrow \) spin particles, respectively.

The squeezing operation is defined as before, and applies equally to squeezing between \( \uparrow \) or \( \downarrow \) particles. The root partition of the HR state is given by \( X000X \ldots 0X \), where \( X = \uparrow \downarrow \), i.e. \( X \) denotes an orbital occupied by both spins. This is consistent with the first quantized cluttering condition and with the filling \( \nu = 1/2 \) in the highest partially populated Landau level. We consider the decomposition of the HR state as Slater determinants:

\[
\Phi_{HR} = \sum_{\lambda_A \uparrow, \lambda_B \downarrow \leq \text{X000X000X} \ldots} c_{\lambda A \uparrow, \lambda B \downarrow} s_{\lambda A \uparrow, \lambda B \downarrow}, \tag{16}
\]

where the sum extends over all spinful partitions obtained by squeezing operations on the root partition \( X000X \ldots 0X \). Let us consider the 4-particle HR root configuration \( X000X \). It can be expressed in partition language as \([4 \uparrow, 4 \downarrow, 0 \uparrow, 0 \downarrow]\) or \([4 \uparrow, 0 \uparrow] \times [4 \downarrow, 0 \downarrow]\). The latter is a factorization in \( \uparrow \)-spin and \( \downarrow \)-spin partitions. It differs from the former by a minus sign. Thus, to avoid ambiguities of global minus signs due to different orderings of fermionic operators, we order all partition entries (i.e. \( \uparrow \) and \( \downarrow \) spin) first by momenta in decreasing order. For a given momentum, we write the \( \uparrow \)-spin entry before the \( \downarrow \)-spin entry. Whenever a factorized partition notation of \( \uparrow \) spin and \( \downarrow \) spin momenta appears in the following text, it is only for reasons of presentation. The ordering of fermions should always be interpreted as explained above. Moreover, Slater determinants differing by an overall spin rotation have equal coefficients. This is so since the HR state is a spin singlet and thus spin rotationally invariant.

\section*{B. Differential action for the HR state}

From conformal field theory considerations, \( \Phi_0(z, w) = \frac{1}{(z_i - w_j)} \) satisfies the following differential equation:

\[
\left( \frac{1}{2} \frac{\partial^2}{\partial z_i^2} - \sum_{j \neq i} \frac{1}{(z_i - w_j)^2} + \frac{1}{z_i - w_j} \frac{\partial}{\partial w_j} \right) \Phi_0 = 0,
\]

with the same equation for \( z \leftrightarrow w \). Following section \ref{eq:diffaction}, we use this equation to obtain an operator for which \( \Phi_{HR} \) is an eigenstate. We rewrite the derivatives acting on \( \Phi_0 \) as derivatives acting on \( \Phi_{HR} \). This amounts to taking into account the additional derivative acting on Jastrow factors for both spin species. Using the intermediate steps explained in Appendix \ref{app:jastrow}, we derive the following differential equation:

\[
\begin{align*}
\left[ \frac{1}{2} \sum_{i} \left( z_i \frac{\partial}{\partial z_i} \right)^2 + \left( w_i \frac{\partial}{\partial w_i} \right)^2 - (3N - 2) \sum_{i} \left( z_i \frac{\partial}{\partial z_i} + w_i \frac{\partial}{\partial w_i} \right) - \frac{1}{2} \sum_{i,k} \frac{z_i + w_k}{z_i - w_k} \left( z_i \frac{\partial}{\partial z_i} - w_k \frac{\partial}{\partial w_k} \right) \\
- \frac{1}{4} \sum_{i \neq j} \left( \frac{z_i + z_j}{z_i - z_j} \left( z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right) + w_i + w_j \right) \left( w_i \frac{\partial}{\partial w_i} - w_j \frac{\partial}{\partial w_j} \right) + 4N(2N^2 - 3N + 1) \right] \Phi_{HR} = 0. \tag{18}
\end{align*}
\]

The above equation contains only 2-body interactions. The interaction terms are symmetric with respect to \( \uparrow \)-spin and \( \downarrow \)-spin variables. Both the inter and intraspin interaction are of Laplace-Beltrami type and are familiar from our previous calculations of the polarized fermionic states.

\subsection*{1. Equal spin action}

Let us first compute the action of the terms in Eq. (18) consisting of equal spin interactions. Once solved for one species, e.g. the \( \uparrow \) spin variables \( z_i \), this also applies for the \( \downarrow \) spin terms. This part of the HR operator is given by:

\[
\frac{z_i + z_j}{z_i - z_j} \left( z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right), \tag{19}
\]

which we previously encountered as part of the fermionic LB operator. However, the term \( \sim 1/(z_i - z_j)^2 \) of the LB operator is missing. This is an important difference. It implies that the operator in Eq. (18) does not map a single \textit{general} Slater determinant of arbitrary degree into another Slater determinant. The equivalent of (10) cannot be written in the current case by using only the operator in Eq. (18). Acting with the operator in Eq. (18) on a single Slater determinant usually leads to a fraction. However, the special sum of Slater determinants that comprise the HR state \textit{does} map back into a sum of Slater
Determinants. This is so since the HR state has two Jastrow terms in equal spins; they cancel any fractions that might appear upon the action of (19) on a Slater (see Appendix D). A single Jastrow factor would not. Hence a mapping of a linear superposition of Slaters constrained in this way maps, under action of (19), back to the space of Slaters. Details of this are given in Appendix D. We hence force the identity:

$$\sum_{i\neq j} \frac{z_i + z_j}{z_i - z_j} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \sum_{\mu} a_\mu s_\mu = \sum_{\mu} \mu s_\mu,$$  \tag{20}

where $s_\mu$ is defined as the Slater determinant of one spin species. The partition of the other spin species is omitted in typing but should be implicated. Since from the expression of the HR state we know that the linear combination of states that form the HR state have a double zero $\sum_\mu a_\mu s_\mu (z_1 \ldots z_M) \sim \prod_{i,j} (z_i - z_j)^2$, the forced identity (20) is clearly true. The coefficients $\mu$ are then given in terms of the coefficients $a_\mu$. We find

$$\sum_{i<j} \sum_{\mu<\lambda} (\mu_i - \mu_j) a_\mu s_\mu + 2a_\mu (\mu_i - \mu_j) \sum_{\theta<\mu} s_\theta (-1)^{N_{SW}},$$  \tag{21}

where $\theta_A, \theta_B$ are the two partitions of spin $\uparrow$ and $\downarrow$, respectively. They are obtained by squeezing only opposite spins in $\lambda_{A\uparrow}, \lambda_{B\downarrow}$; $s$ parametrizes the changed partition component obtained from squeezing spin $\uparrow$ with spin $\downarrow$: $\theta_A = [1, \lambda_{A\uparrow}, \ldots, \lambda_{i,A} + s, \ldots, \lambda_{N,A}]$ and $\theta_B = [1, \lambda_{B\downarrow}, \ldots, \lambda_{j,B} - s, \ldots, \lambda_{N,B}]$. Thus, the summation takes each possible pairwise combination of spin $\uparrow$ and spin $\downarrow$ particles, and squeezes them. $N_{SW}$ denotes the number of swaps needed to reorder the partition.

2. Different spin action

In order to compute the term in (15) involving action on both spin species, we consider one single spinful Slater determinant $s_{\lambda_A\uparrow, \lambda_B\downarrow}$ as defined before. Following the calculation detailed in Appendix D, the spin-rotated part $s_{\lambda_B\uparrow, \lambda_A\downarrow}$ always appears with the same coefficient. In this way, we find that the action of the $\uparrow \leftrightarrow \downarrow$ part of the operator also maps back to Slaters:

$$\sum_{i,k} \frac{z_i + w_k}{z_i - w_k} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial w_k} \right) s_{\lambda_A\uparrow, \lambda_B\downarrow} = \sum_{i,j,\sigma; \lambda_i,\sigma > \lambda_j,\sigma} (\lambda_i,\sigma - \lambda_i,\bar{\sigma}) s_{\lambda_A\uparrow, \lambda_B\downarrow} + \sum_{\theta_A, \theta_B; i,j,\sigma; \lambda_i,\sigma > \lambda_j,\sigma} (\lambda_i,\sigma - \lambda_i,\bar{\sigma}) s_{\theta_A\uparrow, \theta_B\downarrow} (2 - \delta_{\theta_A, \theta_B} \delta_{\lambda_i,\sigma - \lambda_j,\sigma}) \cdot (-1)^{N_{SW} + \lambda_{SW}},$$  \tag{22}

where $\theta = [\mu_1, \ldots, \mu_i - s, \mu_{i+1}, \ldots, \mu_j + s, \ldots, \mu_N]$, and $N_{SW}$ again denotes the number of swaps needed to reorder the partition.

C. Recurrence relation

We are now ready to compute the full action of the operator in Eq. (18) on the Haldane-Rezaii state. The remainder terms are of non-interacting kinetic type and straightforward to compute. By power counting, we find:

$$\sum_{i} \left( z_i \frac{\partial}{\partial z_i} + w_i \frac{\partial}{\partial w_i} \right) \Phi_{HR} = 4N(N - 1) \Phi_{HR}.$$  \tag{23}

The second order derivative terms give

$$\sum_{i} \left( z_i \frac{\partial}{\partial z_i} \right)^2 \Phi_{HR} = \sum_i \lambda_i^2 \Phi_{HR}.$$  \tag{24}

In the intra-spin term, the sums over differences of $\lambda$’s can be computed as

$$\sum_{i<j} (\lambda_i - \lambda_j) = \sum_{i} (N + 1 - 2i) \lambda_i,$$  \tag{25}

while there is no similar closed form expression for the inter-spin term. Summing up all terms of (18), we find the following equation:
\[
\sum_{\lambda_A \uparrow, \lambda_B \downarrow \leq \mathbf{X}000X000X \ldots} c_{\lambda_A \uparrow, \lambda_B \downarrow} s_{\lambda_A \uparrow, \lambda_B \downarrow} \left( 2N(3N^2 - 4N + 1) - \frac{1}{2} \sum_i (\lambda_{A,i}^2 + \lambda_{B,i}^2 + 2i\lambda_{A,i} + 2i\lambda_{B,i}) \right)
\]
\[+ \frac{1}{2} \sum_{i,j,\sigma; \lambda_i > \lambda_j, \sigma} (\lambda_i - \lambda_j, \sigma) \right) \]
\[= - \sum_{\lambda_A \uparrow, \lambda_B \downarrow \leq \mathbf{X}000X000X \ldots} c_{\lambda_A \uparrow, \lambda_B \downarrow} \left( \sum_{\theta_A < \lambda_A} (\lambda_{A,i} - \lambda_{A,j}) s_{\theta_A \uparrow, \lambda_B \downarrow} (-1)^{N_{SW}} + \sum_{\theta_B < \lambda_B} (\lambda_{B,i} - \lambda_{B,j}) s_{\lambda_A \uparrow, \theta_B \downarrow} (-1)^{N_{SW}} \right)
\]
\[+ \frac{1}{2} \sum_{i,j,\sigma; \lambda_i > \lambda_j, \sigma} (\lambda_i, \sigma - \lambda_j, \sigma) s_{\theta_A \uparrow, \theta_B \downarrow} (-1)^{N_{SW} + N_{SW,4}}. \tag{26} \]

In Eq. (26) we have grouped all diagonal terms to the left and all interactions to the right. Accordingly, the right hand side is made of three parts: parts of sums of spin-\(\uparrow\) partitions squeezed from the spin-\(\uparrow\) partition \(\lambda_A\), parts of sums of spin-\(\downarrow\) partitions squeezed from the spin-\(\downarrow\) partition \(\lambda_B\), and parts of sums of spin-\(\uparrow\) and spin-\(\downarrow\) partitions squeezed only from particles of different spin from the partition \([\lambda_A \uparrow, \lambda_B \downarrow]\). We now define the quantities

\[\rho_{\lambda_A \uparrow, \lambda_B \downarrow} = \frac{1}{2} \sum_i (\lambda_{A,i}^2 + \lambda_{B,i}^2 + 2i\lambda_{A,i} + 2i\lambda_{B,i}) \right) - \frac{1}{2} \sum_{i,j,\sigma; \lambda_i > \lambda_j, \sigma} (\lambda_i, \sigma - \lambda_j, \sigma). \tag{27} \]

We can then immediately show that the root partition obeys \(\rho_{\mathbf{X}000X \ldots X} = 2N(3N^2 - 4N + 1)\), and thus corresponds to the constant terms on the left side of (26). Now, equating the coefficient of every Slater determinant, we derive a recurrence relation for the coefficients of the decomposition:

\[c_{\mu_A \uparrow, \mu_B \downarrow} = \frac{1}{\rho_{\mathbf{X}000X000X \ldots} - \rho_{\lambda_A \uparrow, \lambda_B \downarrow}} \left( \sum_{\theta_A ; \mu_A < \theta_A \leq \mathbf{X}000X000X \ldots} (\mu_{A,i} - \mu_{A,j} + 2t)c_{\theta_A \uparrow, \mu_B \downarrow} (-1)^{N_{SW}} \right)
\]
\[+ \sum_{\theta_B ; \mu_B < \theta_B \leq \mathbf{X}000X000X \ldots} (\mu_{B,i} - \mu_{B,j} + 2t)c_{\mu_A \uparrow, \theta_B \downarrow} (-1)^{N_{SW}} \right.
\]
\[+ \frac{1}{2} \sum_{\theta_A, \theta_B : [\mu_A \uparrow, \mu_B \downarrow] < [\theta_A, \theta_B] \leq \mathbf{X}000X000X \ldots} (\mu_{\sigma} - \mu_{\sigma} + 2t)c_{\theta_A \uparrow, \theta_B \downarrow} (-1)^{N_{SW} + N_{SW,4}}. \tag{28} \]

where the last sum over the inter-spin terms extends over all partitions \(\theta_A, \theta_B\) with the property \([\theta_A, \theta_B] = [\mu_{A,1}, \mu_{A,2}, \ldots, \mu_{A,N}, \mu_{B,1}, \mu_{B,2}, \ldots, \mu_{B,N}]\) with \(\mu_{A,i} \geq \mu_{B,j}\), or \([\theta_A, \theta_B] = [\mu_{A,1}, \mu_{A,2}, \ldots, \mu_{A,N}, \mu_{B,1}, \mu_{B,2}, \ldots, \mu_{B,N}]\) \(\mu_{B,i} \geq \mu_{A,j}\). One explicit example of computation is presented in Fig. 5 and a complete 4-particle HR Slater decomposition is shown in Appendix E.

In analogy to (3) and (10), the Slater decomposition coefficients of the HR state can be read off from (28). The HR state can be generated with linear effort in Hilbert space dimension.

D. Zero-weight partitions

The recurrence relations we encountered for the spin polarized bosonic (fermionic) states in Eq. (3) (Eq. (10) only produced accidental (about \(1/N!\) of the total Hilbert space dimension) zero weights for squeezed monomial (Slater) configurations. By contrast, we observe an extensive number of zero weight configurations from the HR recurrence relation that hints at further structure in the HR coefficients beyond the spinful squeezed Slater basis. The key observation is that the HR state can be written as

\[\Phi_{HR} = \prod_{i<j} (z_i - z_j)^2 (w_i - w_j)^2 P_i(z_i, w_j), \tag{29} \]

where \(P_i\) is an angular momentum \(L = 0\) polynomial squeezed from the root partition \(X0X0X \ldots X0X\).

Interpreted in this way, the spinful \(2N\)–particle HR partitions are generated from the space of partitions of \(P_1\) squeezed from \(X0X0X \ldots X0X\) times partitions from the \(N\)-particle bosonic \(\nu = 1/2\) Laughlin factor squeezed from 1010101 \(\ldots 101\) for \(\uparrow\) spins and \(\downarrow\) spins, respectively.
Thus, the available range of possible momentum imbalance is equal to the total momentum added to both spin species is equal. Multiplying a partition of a given momentum imbalance by which partitions squeezed from the HR root partition can be efficiently denied or accepted.

### E. Spinful Entanglement spectra

We investigate the $2N$ particle HR partition $X000X000X\ldots$, where we count the momentum in terms of polynomial powers beginning from $m = 0$ at the north pole extending to $m = 4(N - 1)$ at the south pole. We define the momentum imbalance between total $\uparrow$ spin and $\downarrow$ spin particle momentum within a partition $\lambda$: $|\sum_\lambda \lambda_\uparrow - \sum_\lambda \lambda_\downarrow|$. The total momentum summed over $\uparrow$ and $\downarrow$ spins, i.e. $\sum_\lambda \lambda_\sigma$, is $4N(N - 1)$. For $N = 2$, the partitions of maximum momentum imbalance are given by $\downarrow 0 X 0 \uparrow$ and $\uparrow 0 X 0 \downarrow$ (both with momentum imbalance $4 + 2 - (2 + 0) = 4$). They are obtained by just squeezing $\uparrow$ spins to the left and $\downarrow$ spins to the right, and vice versa. If we arrange all $\uparrow$ and $\downarrow$ spins to generate the highest momentum imbalance between the spin species (which means to arrange all $\uparrow$ and $\downarrow$ spins around opposite poles), we find a partition structure $\downarrow 0 \downarrow 0 \ldots \downarrow 0 X 0 \uparrow 0 \uparrow \ldots 0 \uparrow$. Thus, we find a maximum imbalance of $2N(N - 1)$ for partitions squeezed from the HR root partition.

Let us now construct the HR state starting from $P_1$. Multiplying a $P_1$ partition of a given momentum imbalance with a $\nu = 1/2$ Laughlin partition for $\uparrow$ and $\downarrow$ spins does not change the momentum imbalance, as the total momentum added to both spin species is equal. Thus, the available range of possible momentum imbalance in HR partitions is given by the partitions squeezed from $X0X0X0 \ldots X0X$. There, arranging the different spin species to opposite poles gives a partition structure $\downarrow \downarrow \ldots \downarrow X \uparrow \uparrow \ldots \uparrow$. The maximum momentum imbalance is $N(N - 1)$, and thus only one half of the maximum imbalance for partitions squeezed from the HR root partition. As a consequence, we can remove all squeezed partitions in the HR state with momentum imbalance $> N(N - 1)$, as they must have zero weight. For $N = 2$, this applies for the partition $\downarrow 0 X 0 \uparrow$ and its spin-rotated counterpart $\uparrow 0 X 0 \downarrow$, as the momentum imbalance is 4, while the maximum allowed momentum imbalance is 2 (see Appendix E). While this rule is easily implemented numerically, there are even more 0 weight partitions squeezed from the HR root partition, which cannot be written in a similar closed form as the momentum imbalance constraint. Rather, we have to explicitly check whether a partition squeezed from the HR root partition can be generated from $P_1$ times Laughlin partitions. We have sketched the algorithm in Tab. V by which partitions squeezed from the HR root partition can be efficiently denied or accepted.

We now focus our attention on the entanglement spectrum, a quantity recently used to identify the topolog-
Algorithmic steps to accept / deny a 2N-particle HR-squeezed partition $\mu$

For a given partition $\mu$

(i) Check whether total momentum imbalance $|\sum_i \mu_i^\uparrow - \sum_i \mu_i^\downarrow|$ is $\leq N(N-1)$.

(ii) Loop over all $\uparrow$ and $\downarrow$ N-particle Laughlin partitions $\kappa$

(ii.1) Divide partition $\mu$ by $\kappa$: Check whether all momenta are still in allowed range and no equal spin momenta are mapped onto each other (fermionic state)

(ii.2) Check whether the resulting partition can be squeezed from $P_1$

| TABLE V. Sketched algorithm to remove zero-weight partitions squeezed from the HR root partition X000X000X ... We apply the structure of the HR state according to [29]. By this procedure, the total Hilbert space is reduced to about 83% of the full squeezed Hilbert space. |

| Algorithmic steps to accept / deny a 2N-particle HR-squeezed partition $\mu$ |
|---|
| For a given partition $\mu$ |
| (i) Check whether total momentum imbalance $|\sum_i \mu_i^\uparrow - \sum_i \mu_i^\downarrow|$ is $\leq N(N-1)$. |
| (ii) Loop over all $\uparrow$ and $\downarrow$ N-particle Laughlin partitions $\kappa$ |
| (ii.1) Divide partition $\mu$ by $\kappa$: Check whether all momenta are still in allowed range and no equal spin momenta are mapped onto each other (fermionic state) |
| (ii.2) Check whether the resulting partition can be squeezed from $P_1$ |

Algorithmic steps to accept / deny a 2N-particle HR-squeezed partition $\mu$

For a given partition $\mu$

(i) Check whether total momentum imbalance $|\sum_i \mu_i^\uparrow - \sum_i \mu_i^\downarrow|$ is $\leq N(N-1)$.

(ii) Loop over all $\uparrow$ and $\downarrow$ N-particle Laughlin partitions $\kappa$

(ii.1) Divide partition $\mu$ by $\kappa$: Check whether all momenta are still in allowed range and no equal spin momenta are mapped onto each other (fermionic state)

(ii.2) Check whether the resulting partition can be squeezed from $P_1$

TABLE V. Sketched algorithm to remove zero-weight partitions squeezed from the HR root partition X000X000X .... We apply the structure of the HR state according to [29]. By this procedure, the total Hilbert space is reduced to about 83% of the full squeezed Hilbert space.

![Graph showing entanglement spectrum for the N = 12 Haldane-Rezayi state, half sphere cut.](image)

FIG. 6. (Color online) Entanglement spectrum for the $N = 12$ Haldane-Rezayi state, half sphere cut, $N_A = 6$, $S_A^z = 0$ sector, entanglement levels plotted versus the angular momentum $L_A^z$. Levels relating to different spin multiplets, denoted by $S_A^z$, are given in red (singlets), crossed green (triplets), and square blue (quintuplets). A large part of the main entanglement weight resides in the singlet sector.

The conformal field theory (CFT) approach to quantum Hall states provides a powerful framework for understanding the quantum entanglement properties of these systems. In this context, the entanglement spectrum is a key tool for probing the underlying Hilbert space structure and identifying the dominant entanglement levels.

VI. CONFORMAL FIELD THEORY

We investigate the product rule property of FQH states from the perspective of conformal field theory (CFT). For the FQHE, we are concerned with chiral CFTs in 2 dimensions: all the fields are holomorphic, i.e. they depend on $z = x + iy$, and not on $\bar{z}$. This is an illustration of broken time-reversal symmetry in this context. For an in-depth introduction to CFT, we engage the reader to Ref. [30]. In this Section we will present the key notions that are needed to derive the product rule of quantum Hall wave functions from CFT.

The conformal dimensions are quantities that appear...
in the two-point correlation functions of the fields $\Phi_i$ present in the CFT. Conformal invariance fixes any two point function to be of the form

$$\langle \Phi_1(z_1)\Phi_2(z_2) \rangle = \frac{C_{1,2}}{(z_1 - z_2)^{\Delta_1 + \Delta_2}}$$

(30)

and its scaling properties are described by the so-called conformal dimension $\Delta_i$.

Primary fields are of special importance in a CFT. Among the infinite number of fields of a given CFT, primary fields can be thought of as highest weight fields. All non-primary fields ("descendants") are created by acting on primary fields with certain types of lowering operators - the (negative) Virasoro modes. The two point function of primary fields $\Phi_i$ and $\Phi_j$ reads:

$$\langle \Phi_i(z_1)\Phi_j(z_2) \rangle = \frac{\delta_{ij}}{(z_1 - z_2)^{2\Delta_i}}.$$  

(31)

It is non vanishing only if $\Phi_i$ and $\Phi_j$ are conjugate from one another, namely if the identity field 1 is produced in their fusion. The identity is a very special field, and for our purposes it will be the only primary field with a vanishing conformal dimension. As a consequence it is its own conjugate, $1 = 1$. In some cases, such as the logarithmic $c = -2$ ghost system corresponding to the Haldane-Rezayi state encountered in Section V, this is not the case, and one has to treat this cases more carefully as discussed below.

Conformal invariance also fixes the form of the correlation function between 3 primary fields:

$$\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3) \rangle = \frac{D_{1,2,3}}{(z_1 - z_2)^{\Delta_1 + \Delta_2}(z_2 - z_3)^{\Delta_2 + \Delta_3}(z_1 - z_3)^{\Delta_1 + \Delta_3}}$$

(32)

where $\Delta = \Delta_1 + \Delta_2 + \Delta_3$. When the fusion coefficient $D_{1,2,3}$ is non-zero, the field $\Phi_3$ is said to be produced in the fusion $\Phi_1 \times \Phi_2$. The fusion rules encode which primary fields are produced in any fusion. For instance a fusion rule of the form

$$\Phi_1 \times \Phi_2 = \Phi_a + \Phi_b + \Phi_c$$

(33)

tells us that the fusion of $\Phi_1$ and $\Phi_2$ only produces three primaries: $\Phi_a, \Phi_b$ and $\Phi_c$. Implicitly, all their descendants will also appear.

In order to compute correlation functions, fusion rules are not sufficient. An operator product expansion (OPE) is a refinement of the fusion rules. It is a formal (and exact) expansion of the product of two fields, and is an implicit way to define any correlation functions of a CFT by iteratively reducing the number of fields involved. It has the following generic form:

$$\Phi_1(z_1)\Phi_2(z_2) = \sum_k \frac{C_{1,2}^k}{(z_1 - z_2)^{\Delta_1 + \Delta_2 - \Delta_k}} \Phi_k(z_2),$$

(34)

where the sum is formally over all fields of the CFT. Thanks to conformal invariance, it is sufficient to know only the singular terms - those for which $\Delta_1 + \Delta_2 - \Delta_k > 0$ - and there will be a finite number of them for any rational CFT. $N$-point OPEs are a straightforward generalization:

$$\Phi_1(z_1) \cdots \Phi_N(z_N) = \sum_k F_k(z_1, \cdots, z_N) \Phi_k(0).$$

(35)

We are now in a position to derive the key property of CFT correlation functions that is responsible for the product rule. Consider a generic $N = N_A + N_B$-point correlation function of primary fields. For simplicity we denote the complex coordinates of the fields by $x_i$ and $y_j$:

$$C(x_i | y_j) = \langle \Phi_1(x_1) \cdots \Phi_{N_A}(x_{N_A}) \Phi_{N_A+1}(y_1) \cdots \Phi_N(y_{N_B}) \rangle.$$  

(36)

A particle cut between $N_A$ and $N_B$ particles is directly related to the asymptotic behavior of the correlation function as we spatially separate the two sets of variables $x_i$ and $y_j$. We consider the limit $\gamma \rightarrow \infty$ of $C(x_i | y_j)$. Using the $N_A$ points OPE (35) for $\Phi_1(x_1) \cdots \Phi_{N_A}(x_{N_A})$ around 0 we obtain

$$C(x_i | y_j) = \sum_k F_k(x_i) \langle \Phi_k(0) \Phi_{N_A+1}(y_1) \cdots \Phi_N(y_{N_B}) \rangle,$$

(37)

Thus conformal invariance tells us that the correlation function $\langle \Phi_k(0) \Phi_{N_A+1}(y_1) \cdots \Phi_N(y_{N_B}) \rangle$ - if it is non-vanishing - scales as $\gamma^{-\Delta_{N_A+1} - \cdots - \Delta_{N} - \Delta_k}$ as $\gamma \rightarrow \infty$. From this we can immediately infer that the behavior is dominated by the (primary) field $\Phi_\gamma$ (and its conjugate $\Phi_{\bar{\gamma}}$), which is the field with lowest conformal dimension $\Delta_\gamma$ appearing in both fusions:

$$\Phi_1 \times \cdots \times \Phi_{N_A} \rightarrow \Phi_\gamma$$

$$\Phi_{N_A+1} \times \cdots \times \Phi_{N} \rightarrow \Phi_{\bar{\gamma}}.$$  

(38, 39)

In addition, for a primary field $\Phi_\gamma$, the function $F_a$ appearing in the $N_A$ points OPE (35) is given by the correlation function

$$F_a(x_1, \cdots, x_{N_A}) = \langle \Phi_1(x_1) \cdots \Phi_{N_A}(x_{N_A}) \Phi_{\gamma}(\infty) \rangle$$

$$\lim_{x \rightarrow \infty} x^{2\Delta_{\gamma}} \langle \Phi_1(x_1) \cdots \Phi_{N_A}(x_{N_A}) \Phi_{\gamma}(x) \rangle.$$  

(40)

Finally we obtain the asymptotic behavior for $C(x_i | y_j)$ as $\gamma \rightarrow \infty$ given by

$$\gamma^{-\Delta_{N_A+1} - \cdots - \Delta_{N} - \Delta_\gamma} C(x_i | y_j) \sim \langle \Phi_1(x_1) \cdots \Phi_{N_A}(x_{N_A}) \Phi_{\gamma}(\infty) \rangle \times$$

$$\langle \Phi_\gamma(0) \Phi_{N_A+1}(y_1) \cdots \Phi_N(y_{N_B}) \rangle$$

(41)

for any correlation function of primary fields $C(x_i | y_j)$. Using first principles of CFT, we hence have derived a factorized form of the correlation function spatially separating two sets of variables $x_i$ and $y_j$. We will see in the following that this property implies both squeezing and the product rule for FQH wave functions.
A. Spinless FQH wave functions and parafermionic CFTs

1. Parafermionic CFTs

Many fully polarized QH wave functions can be written in terms of parafermionic CFTs. This includes Laughlin's, Moore-Read's and the Read-Rezayi states (as well as all Jack states), but also any generalized parafermionic states, such as \( N = 1 \) superconformal or \( S_3 \) wave functions.

Here we constrain ourselves to a condensed derivation required to obtain the product rule from CFT. More details about parafermions in the context of FQHE can be found in Ref. [24]. These parafermionic CFTs, denoted as \( \mathcal{Z}(r) \), contain a set of \( k \) parafermionic primary fields \( \{ \Psi_0 = 1, \Psi_1, \ldots, \Psi_{k-1} \} \) with the following fusion rules:

\[
\Psi_n \times \Psi_m \rightarrow \Psi_{n+m} \quad \text{mod} \quad k
\]  

(42)

The conformal dimension of the field \( \Psi_n \) is \( \Delta_n = \frac{2n(k-n)}{k} \), and its conjugate is \( \Psi_{k-n} \).

Parafermionic FQHE wave functions take the following form:

\[
P(z_1, \ldots, z_N) = \langle \Psi(z_1) \cdots \Psi(z_N) \rangle \prod_{i<j} (z_i - z_j)^{r/k}.
\]  

(43)

Using \([\ref{11}]\) for \( N = N_A + N_B \) in the case in which all the primary fields \( \Phi_i \) are taken to be parafermionic fields \( \Psi \) in a \( \mathcal{Z}(r) \) theory, we have:

\[
\gamma_{N_B \Delta_1 + \Delta_a} C(x_i | y_j) \sim \langle \Psi(x_1) \cdots \Psi(x_{N_A}) \Psi_{-a}(\infty) \rangle \langle \Psi(0) \Psi(y_1) \cdots \Psi(y_{N_B}) \rangle
\]  

(44)

where \( a = N_A \mod k \) corresponds to the sector in which the \( N_A \) particles live after the cut, and \( \Delta_a = \frac{2a(k-a)}{k} \). Equivalently, for the wave function this reads \( P(x_i | y_j) = C(x_i | y_j) \prod_{i<j} (y_i - y_j)^{r/k} \prod_{i<j} (x_i - x_j)^{r/k} \gamma_{(2N_{A}N_{B}+N_{A}(N_{B}-k)-a(k-a))} P_N(x_i | y_j) \sim \gamma_{(2N_{A}N_{B}+N_{A}(N_{B}-k)-a(k-a))} P_{N_{A}}(x_i) P_{N_{B}}(y_j) \) :

\[
P^{(a)}_{N_{A}}(x_i) P^{(a)}_{N_{B}}(-1/y_j) \left( \prod_{i} y_i \right)^{r/(N-k)}.
\]  

(45)

In particular, in the neutral sector \( a = 0 \) we have

\[
P_N(x_i | y_j) \sim \gamma_{N_B N_a(N_B - k)/2} \prod_{i} (y_i)^{r N_A / k} P_{N_A}(x_i) P_{N_B}(y_j),
\]  

(46)

where we introduced \( N_a(N) = r(N-k)/k \).

2. Squeezing and product rule

From \([\ref{15}]\) one can derive the following properties: Squeezing. Consider the power of \( \gamma \) in \([\ref{15}]\) for \( N_B = 1 \):

\[
P_N(z_1, z_2, \cdots, z_{N-1}, \gamma z_N) \sim \gamma_{r(N-k)/k}.
\]  

(47)

This shows that for any monomial \( m_a \) entering the decomposition of \( P_N \) we have \( m_1 \leq \lambda \frac{r(N-k)}{k} \). By iteration on \( N_B \), one finds that (i) the root partition is \( \lambda = (k^0 - 1)(r^1 - \cdots - r^k) \) and (ii) any other partition \( \mu \) obeys \( m_1 + \cdots + m_\mu \leq \lambda_1 + \cdots + \lambda_\mu \), i.e. is obtained by squeezing from \( \lambda \).

Product rule. In the limit we consider we send the \( N_B \) particles to infinity:

\[
\lim_{N_B \to \infty} \gamma_{-\frac{r}{k}(2N_{A}N_{B}+N_{A}(N_{B}-k)-a(k-a))} P_N(x_i | y_j) = \prod_{j=1}^{N_B} y_j^{r N_{A}/k}
\]  

(48)

only the monomials \( m_a \) such that

\[
\mu_1 + \cdots + \mu_{N_B} = \lambda_1 + \cdots + \lambda_{N_B}
\]  

(49)

survives. This kills all monomials obtained by squeezing through the cut between \( N_A \) and \( N_B \) particles, and leaves the others invariant. Now that squeezing has been established, the product rule is simply equivalent to the monomial decomposition of the factorization property:

\[
\lim_{N_B \to \infty} \gamma_{-\lambda_1 - \cdots - \lambda_{N_B}} P_N(x_i | y_j) = \prod_{j=1}^{N_{B}} y_j^{r N_{A}/k}
\]  

(50)

where \( \tilde{P}_{N_B} \) stands for the north (south) pole reflection of \( P_{N_B} \).

B. Spin singlet states

The same argument from above applies to spin-unpolarized states, such as the Haldane-Rezayi, Halperin, and NASS states. Their CFT description involves several types of electron operators, typically consisting of a parafermion field and a vertex operator of a set of chiral boson fields. These boson fields \( \Phi_x \) and \( \Phi_s \) usually describe charge and spin associated with the particles. The electron creation operators take the generic form:

\[
V_{\uparrow}(z) = \Psi_{\uparrow}(z) : e^{i \frac{r}{k} (\sqrt{\beta + \gamma} \Phi_+ + \sqrt{\beta - \gamma} \Phi_-)} : \quad (51)
\]

\[
V_{\downarrow}(w) = \Psi_{\downarrow}(w) : e^{i \frac{r}{k} (\sqrt{\beta - \gamma} \Phi_+ - \sqrt{\beta + \gamma} \Phi_-)} : \quad (52)
\]

where \( \Psi_{\uparrow} \) and \( \Psi_{\downarrow} \) can be trivial fields (Halperin), Gepner parafermions (NASS) or ghosts (Haldane-Rezayi). The values of the rational numbers \( \beta \) and \( \gamma \) entering the vertex operators depend on this CFT.

The spin polarized wave function assumes the form

\[
P(z_i, w_j) = \langle \Psi_{\uparrow}(z_1) \Psi_{\uparrow}(w_1) \cdots \Psi_{\uparrow}(z_N) \Psi_{\downarrow}(w_N) \rangle \prod_{i<j} (z_i - z_j)^{\beta}(w_i - w_j)^{\beta} \prod_{i<j} (z_i - w_j)^{\gamma}. \quad (53)
\]

Taking the asymptotic factorized behavior as \( n \) up spin and \( m \) down spin electrons are taken to infinity as in \([\ref{11}]\), we first obtain a weak form of squeezing: there exists an
integer \( N(n, m) \) dominating all the partitions \((\mu^+, \mu^-)\) in the sense that
\[
\mu_1^+ + \cdots + \mu_n^+ + \mu_1^- + \cdots + \mu_m^- \leq N(n, m),
\]
(54)
which is due to the asymptotic behavior \( P \sim \gamma^{N(n,m)} \).

This integer \( N(n, m) \) can be expressed in terms of the CFT data \( \alpha, \beta, \Delta_\uparrow, \Delta_\downarrow \) and \( \Delta_s \).

Second, we obtain a product rule if a partition \( \mu = (\mu_1^+, \mu_1^-) \) is separable, i.e. \( \mu = \mu_A + \mu_B \). With \( \mu_A \) (\( \mu_B \)) squeezed from \( \lambda_A \) (\( \lambda_B \)), the corresponding monomial (slater) coefficient is \( m_{\lambda \mu} = m_{\lambda_A \mu_1} \times m_{\lambda_B \mu_2} \). In order to obtain a stronger form of squeezing, a more detailed analysis is involved. One would have to specify the OPEs of the operators \( \Psi_\uparrow \) and \( \Psi_\downarrow \), and in particular work out the dimension of the field \( \Phi_\alpha \) appearing in the fusion of \( n \) fields \( \Psi_\uparrow \) and \( m \) fields \( \Psi_\downarrow \), as we did for the spin-polarized case. The detailed analysis of the NASS and Halperin states is beyond the scope of this paper.

We treat the HR state in Appendix F.

VII. CONCLUSION

In this paper we have given an extended derivation of a recurrence formula for the Slater decomposition of fermionic Jack polynomial states. We have given a rigorous and detailed account on the product rule symmetry for spin-polarized quantum Hall trial states first presented in Ref. [1]. We generalized the whole approach to spinful states and specifically derived a recurrence relation for the spinful slater determinant decomposition of the Haldane Rezayi state for which we have computed its spinful geometric entanglement spectrum. The product rule symmetry is found to be a deep general property of quantum Hall trial states, involving both fermionic and bosonic as well as spinful and spin-polarized states. While for states described by parafermionic conformal field theory (which include but transcend the Jack polynomials) we were able to show that the product rule comes out of the parafermionic fusion properties, for many other FQH states the product rule holds even though the states cannot be described by analogue conformal field theory. We showed that the product rule can be used as an increasingly good approximation of the FQH state.

While the product rule symmetry is not exactly valid for ground states built from realistic interactions (such as the Coulomb potential), we would like to investigate if an approximated version of this symmetry also manifests itself in these cases. Such property would greatly help to improve any density matrix renormalization group algorithm designed for fractional quantum Hall systems.

ACKNOWLEDGMENTS

We thank F.D.M. Haldane, R. Santachiara, K. Schoutens, and S. Simon for valuable discussions. RT, BE, and BAB thank the Ecole normale superieure (Paris) for hospitality. RT and BAB thank the center for international collaborations in Beijing where parts of this work have been done. RT is supported by a Feodor Lynen Fellowship of the Humboldt Foundation and Alfred P. Sloan Foundation funds. BE was supported by FOM of the Netherlands. NR was supported by Agence Nationale de la Recherche under Grant No. ANR-JCJC-0003-01. BAB acknowledges Princeton University startup funds, Alfred P. Sloan Foundation funds, and NSF CAREER DMR-0952428.

Appendix A: Helpful Formulas

We present several formulas used to simplify certain polynomial terms. They follow from elementary algebra and are easily derived by exploiting the symmetry with respect to the summation indices. In Sec. III we use
\[
\sum_{i \neq m} \frac{z_i}{z_i - z_m} = \frac{1}{2} N(N - 1),
\]
(A1)
\[
\sum_{i, m, n} \frac{z_i^2}{(z_i - z_m)(z_i - z_n)} = \frac{1}{3} N(N - 1)(N - 2) \quad (A2)
\]
In Sec. V and Appendix D we use
\[
\sum_{i, k \neq l} \frac{z_i^2}{(z_i - w_k)(z_i - w_l)} + \sum_{j \neq i; k} \frac{z_i z_j - w_k(z_i - z_j)}{(z_i - w_k)(z_j - w_k)} + \sum_{i, k \neq l} \frac{w_i^2}{(w_i - z_k)(w_i - z_l)} + \sum_{j \neq i; k} \frac{w_i w_j - z_k(w_i + w_j)}{(w_i - z_k)(w_j - z_k)}
\]
\[= 2N^2(N - 1) \quad (A3)
\]

Appendix B: Example for Monomial decomposition

In Fig. 1 we give an example of how to use the squeezing and product rule properties to obtain the \( N = 4 \)-particle \( \nu = 1/3 \) Laughlin state.

Appendix C: Fermionic Laplace-Beltrami operator

We want to rephrase the derivatives on \( J_{\lambda B}^\nu \) as a derivative action on \( S_\lambda^\nu \). This can be done in a compact form since they only differ by a Jastrow factor multiplication. The first derivative yields:
\[
z_i \frac{\partial}{\partial z_i} S_\lambda^\nu = \left[ z_i \frac{\partial}{\partial z_i} J_{\lambda B}^\nu \right] \prod_{k < l} (z_k - z_l) + \sum_{m \neq i} \frac{z_i}{z_i - z_m} S_\lambda^\nu.
\]
(C1)
Similarly, plugging in the previous result for the first derivative, the second derivative action can be rewritten as:

\[
\left( \frac{\partial}{\partial z_i} \right)^2 S_N^\alpha = \left[ \left( \frac{\partial}{\partial z_i} \right)^2 J_{1b}^\alpha \right] \prod_{k<l}(z_k - z_l) + \sum_{i \neq m} \frac{2z_i^2}{z_i - z_m} \frac{\partial}{\partial z_i} S_N^\alpha + \sum_{i \neq m} \frac{z_i}{z_i - z_m} - \frac{z_i^2}{(z_i - z_m)^2} - \sum_{m \neq n, n \neq i} \frac{z_i^2}{(z_i - z_m)(z_i - z_n)} \right] S_N^\alpha. \tag{C2}
\]

One observes that the \(m = 2, n = 2\) case is special: the three-body terms have an identical coefficient equal to 3. These 3-body terms reduce to a constant once the above equation for \(z\) is added to the similar equation for \(w\). By use of (A3), this leads to the expression (18).

**2. Recurrence formula**

*a. Equal spin action*

From the differential equation (18), we know that under the action of the equal and different-spin terms, the sum over Slatters contained in the HR state must yield another sum of Slatters. Thus, to deduce a decomposition formula we enforce this condition. Without loss of generality, we can consider the operator action on a two-particle Slater state, and explicitly demand

\[
\frac{z_1 + z_2}{z_1 - z_2} \left( \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right) \sum_{n_1, n_2} a_{n_1, n_2} s_{n_1, n_2} = \sum_{n_1, n_2} b_{n_1, n_2} s_{n_1, n_2}, \tag{D3}
\]

where we use \(n_1, n_2\) as momentum indices of the two-particle same-spin Slater determinants. Working out the derivative action and expanding the expression inside the equality, we find

\[
\sum_{n_1, n_2} (a_{n_1, n_2} (n_1 - n_2) + a_{n_1, n_2 - 1} (n_1 - n_2 + 2))
\]

\[
(z_1^{n_1+1} z_2^{n_2} + z_2^{n_1+1} z_1^{n_2}) = \sum_{m_1, m_2} (b_{m_1, m_2} - b_{m_1+1, m_2-1}) (z_1^m z_2^m + z_2^m z_1^m). \tag{D4}
\]

This equation can be worked out iteratively, starting from the maximum polynomial degree. In terms of the \(b\) coefficients in the final Slater superposition, the coefficients \(a\) are given by

\[
b_{\mu_1, \mu_2} = (\mu_1 - \mu_2) a_{\mu_1, \mu_2} + 2 \sum_i a_{\mu_1 + i, \mu_2 - i} (\mu_1 - \mu_2 + 2i). \tag{D5}
\]

Since the interaction is pairwise, this can be directly generalized to larger particle numbers of Slatters, and finally yields expression (21).

We now illustrate that the degree of the Jastrow factor in the polynomial determines whether the action of Eq. (D3) causes fractions or not. As an example, we consider the \(sl_{3,0} = z_1^3 - z_2^3\) two-particle Slater determinant.
The action of the operator in Eq. (D3) gives:

$$\frac{z_1 + z_2}{z_1 - z_2} \left( z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} \right) \text{sl}_{3,0} \equiv 3 \frac{z_1 + z_2}{z_1 - z_2} \left( z_1^3 + z_2^3 \right),$$

which cannot be decomposed into polynomials without fractions. As stated before, this case does not occur for the HR state, since there is an equal-spin Jastrow factor of 2nd power. Let us now consider the polynomial which is constructed from sl$_{3,0}$ times a Jastrow factor:

$$\frac{z_1 + z_2}{z_1 - z_2} \left( z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} \right) \text{sl}_{3,0}(z_1 - z_2)$$

$$= \frac{z_1 + z_2}{z_1 - z_2} (3(z_1^3 + z_2^3)(z_1 - z_2) + z_1 z_2(z_1^2 - z_2^2) + z_1 z_2^2(z_1 - z_2)).$$

Since all terms from above cancel the $z_1 - z_2$ fraction, the polynomial composed out of a Slater times a Jastrow factor gives no fractions upon the action of the operator in Eq. (D3). This holds for any higher power of Jastrow factors multiplied with the Slater determinant.

b. Inter-spin action

To compute the action on the inter-spin term, we again constrain ourselves to a two-particle mixed-spin Slater partition $z_1^n w_1^m + z_1^m w_1^n$, and assume $n \geq m$ (we remember that the $m \geq n$ Slater has the same coefficient):

$$\frac{z_1 + w_1}{z_1 - w_1} \left( z_1 \frac{\partial}{\partial z_1} - w_1 \frac{\partial}{\partial w_1} \right) \left( z_1^n w_1^m + z_1^m w_1^n \right)$$

$$= (n - m) \left( z_1^n w_1^m + z_1^m w_1^n \right)$$

$$+ 2(n - m) \sum_{i=1}^{(m-n)/2-1} \left( z_1^{n-i} w_1^{m+i} + z_1^{m+i} w_1^{n-i} \right)$$

$$+ 2(n - m) z_1^{(n+m)/2} w_1^{(n+m)/2}.$$  \hspace{1cm} (D8)

For $(n - m) \equiv 1 \text{mod} 2$, the upper limit of the sum is a half integer. In this case, the sum is evaluated as an analytical extension from an integer to a half integer upper limit. To each of these Slatters, we also add the counterpart $n \leftrightarrow m$. To prevent double counting, whenever $m = n$ the counterpart must not be added. In the language of single Slatters, this demands the division of the equal-momentum Slater prefactor by 2 in comparison to other Slatters. For the general particle case, this yields the following recipe: Sum over all combinations of pairs $(i, k)$ with one coordinate from the $\uparrow$ spin partition and one from the $\downarrow$ spin partition. If $\lambda_i > \lambda_k$ ($n > m$ as above) take the prefactor as given in (D8). If $\lambda_i < \lambda_k$, change $\uparrow \leftrightarrow \downarrow$ and add it (the spin-rotated Slater must have the same coefficient). Produce all allowed squeezings and add the Slater terms. If one pair $(i, k)$ and some squeezed partition produces an $m = n$ state (i.e. a doubly occupied single particle state), and the remainder $\uparrow$ spin and $\downarrow$ spin partition are equivalent as well, this factor must be divided by two (or rather, in our case, changed from 2 to 1). We then obtain (22).

### Appendix E: Example for spinful Slater decomposition

In Fig. 8 we give an example of the HR state decomposed into squeezed partition from the root partition. We show the full decomposition of the $N = 4$ HR state into spinful partitions squeezed from X000X. We observe an example of a partition squeezed from X000X that has zero coefficient, i.e. u0X0d: it cannot be constructed from $P_1 = X0X$ times 101 for $\uparrow$ spins and 101 for $\downarrow$ spins.

### Appendix F: Haldane-Rezayi, CFT analysis

As mentioned before in Section V, the CFT corresponding to the HR state is a logarithmic theory that contains two fields $1$ and $\tilde{1}$ with vanishing conformal dimension. This particularity makes the CFT treatment more complicated. Fortunately, the factorization can already be obtained from the explicit form of the correlation function:

$$\langle \partial \bar{\theta}(z_1) \bar{\partial} \bar{\theta}(w_1) \cdots \partial \bar{\theta}(z_N) \partial \bar{\theta}(w_N) \tilde{1} \rangle =$$

$$\sum_\sigma \epsilon(\sigma) \prod_i \frac{1}{(z_i - w_{\sigma(i)})^2}. \hspace{1cm} (F1)$$

The r.h.s. trivially obeys the following asymptotic behavior as $N_B$ up-spins and $N_B$ down-spins are taken to infinity:

$$\langle \partial \bar{\theta}(z_1) \bar{\partial} \bar{\theta}(w_N) \cdots \partial \bar{\theta}(w_{N_A}) \partial \bar{\theta}(\gamma z_{N_A+1}) \cdots \partial \bar{\theta}(\gamma w_{N_A+N_B}) \rangle \sim$$

$$\gamma^{-2N_B} \langle \partial \bar{\theta}(z_1) \cdots \partial \bar{\theta}(w_{N_A}) \rangle \langle \partial \bar{\theta}(z_{N_A+1}) \cdots \partial \bar{\theta}(w_{N_A+N_B}) \rangle. \hspace{1cm} (F2)$$

This yields the product rule (for a cut through an empty orbital) using the same steps as in the spin-polarized case. Cuts through an occupied orbital can be analysed in a similar way, by looking at the asymptotic behavior as $N_B$ up-spins and $N_B + 1$ down-spins are sent to infinity.

1. B. A. Bernevig and N. Regnault, Phys. Rev. Lett. 103, 206801 (2009).
2. R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
3. G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
4. N. Read and E. Rezayi, Phys. Rev. B 59, 8084 (1999).
5. F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. 60,
D. Poilblanc, arXiv:1005.2123.
A. M. Turner, F. Pollmann, and E. Berg, arXiv:1008.4346.
L. Fidkowski and A. Kitaev, arXiv:1008.4138.
H. Yao and X.-L. Qi, Phys. Rev. Lett. 105, 080501 (2010).
F. Pollmann and J. E. Moore, arXiv:0910.0051.

Z. Liu, H.-L. Guo, V. Vedral, and H. Fan, arXiv:1007.0840.
P. M. P. Di Francesco and D. Senechal, Conformal Field Theory (Springer New York, ADDRESS, 1997).
D. Gepner, Nucl. Phys. B. 290, 10 (1987).
FIG. 7. Monomial decomposition of $N = 4$ particle $\nu = 1/3$ Laughlin state. The particle positions are denoted by 1. The coefficients of the Slater partitions are computed according to (10), with $\alpha = -2$. The arrows denote a squeezing relation from the upper to the lower partition. In total, there are 4 squeezing levels till the maximally squeezed partition is reached.

FIG. 8. Spinful Slater decomposition of the $N = 4$ Haldane-Rezayi state. The notation corresponds to $X = \uparrow \downarrow$ on the same orbital, and $u \equiv \uparrow$, $d \equiv \downarrow$. The spinful partitions are written in spin-mixed fashion and ordered with respect to momentum. The coefficients are computed corresponding to Eq. 28. The two appearing zero weight partitions are a manifestation of the zero-weight rule according to (29). The arrows denote that a given partition can be squeezed from the partition above. There are three squeezing levels until the maximally squeezed partitions are reached.