More Membrane Matrix Model Solutions,
– and Minimal Surfaces in $S^7$

Joakim Arnlind and Jens Hoppe

Department of Mathematics
Royal Institute of Technology
Stockholm

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Abstract

New solutions to the classical equations of motion of a bosonic matrix-membrane are given. Their continuum limit defines 3-manifolds (in Minkowski space) whose mean curvature vanishes. Part of the construction are minimal surfaces in $S^7$, and their discrete analogues.
Some time ago [1], solutions of the bosonic matrix-model equations,

\[ \dot{X}_i = -\sum_{j=1}^{d} \left[ [X_i, X_j], X_j \right] \]

\[ \sum_{i=1}^{d} [X_i, \dot{X}_i] = 0 \]

were found where

\[ X_i(t) = x(t)R_{ij}(t)M_j, \]

with \( R(t) = e^{A\varphi(t)} \) a real, orthogonal \( d \times d \) matrix, \( x(t) \) and \( \varphi(t) \) being given via

\[ \frac{1}{2} x^2 + \frac{\lambda}{4} x^4 + \frac{L^2}{2x^2} = \text{const.} \]

\[ \varphi^2(t)x(t) = L(=\text{const}), \]

and the \( d \) hermitean \( N \times N \) matrices \( M_i \) satisfying

\[ \sum_{j=1}^{d'} \left[ [M_i, M_j], M_j \right] = \lambda M_i \]

\[ i = 1, \ldots, d'. \]

The reason for \( d' \) (rather than \( d \)) appearing in [2] was that in order to satisfy the two remaining conditions,

\[ A^2 \vec{M} = -\vec{M} \]

\[ \sum_{j=1}^{d} [M_j, (A\vec{M})_j] = 0 \]

– which have to be fulfilled in order for [2] to satisfy [1] – in an “irreducible” way (the matrix valued \( d\)-component vector \( \vec{M} \) can, of course, always be broken up to contain pairs of identical pieces) half – or more – of the matrices \( M_j \) were chosen to be zero, and (permuting the \( M_i \)’s such that the first \( d' \leq \frac{d}{2} \) are the non-zero ones) the non-zero elements of \( \mathcal{A} \) as \( A_{i+d', j} = 1 = -A_{i,i+d'} \), \( i, j = 1, \ldots, d' \); in particular, [3] was satisfied by having, for each \( j \), either \( M_j \) or \( (A\vec{M})_j \) be identically zero.

As, in the membrane context. \( d (\leq) 9 \), \( d' = 4 \) received particular attention, while the continuum limit of [1],

\[ \sum_j \{m_i, m_j\}, m_j = -\lambda m_i, \]

\[ \left\{ m_i, m_j \right\} := \frac{1}{\rho} (\partial_1 m_i \partial_2 m_j - \partial_2 m_i \partial_1 m_j) ; \quad g_{rs} := \partial_r \vec{m} \cdot \partial_s \vec{m} ; \quad \vec{m} = \vec{m}(\varphi^1, \varphi^2) \]

alias

\[ \frac{1}{\rho} \frac{\partial}{\partial s} \frac{g_{rs}}{\rho} \partial_s \vec{m} = -\lambda \vec{m} \]
is related to
\[
\frac{1}{\sqrt{g}} \partial^s \sqrt{g} g^{rs} \partial_s m^r = -2 \vec{m},
\]
\[
\vec{m}^2 = 1,
\]
(9)
i.e the problem of finding minimal surfaces in higher dimensional spheres (which for \( d' = 4 \) was proven \(^2\) to admit solutions of any genus).

In this letter, we would like to enlarge the realm of explicit solutions (of \( \mathbf{1} \), resp. its \( N \to \infty \) limit, resp \(^9\) while shifting emphasis from \( d' = 4 \) to \( d' = 8 \) (the case \( d' = 6 \), which can be used to obtain nontrivial solutions in the BMN matrix-model, will be discussed elsewhere).

Our first observation is that \(^9\) rather naturally admits solutions which avoid the “doubling mechanism”. While \( \mathcal{A} \) is kept to be an “antisymmetric permutation”-matrix in a maximal even-dimensional space, \(^9\) can be realized if \( \mathbf{M} := \{ M_j \}_{j=1}^d \) (with \( M_d = 0 \) if \( d \) is odd) can be written as a union of even-dimensional subsets of mutually commuting members. In order to give a first example, let us, for later convenience, define (for arbitrary odd \( N > 1 \)) \( N^2 \) independent \( N \times N \) matrices
\[
\begin{align*}
U^{(N)}_m &=: \frac{N}{4\pi M(N)} \omega^{m_1 m_2} g^{m_3} h^{m_2} \\
\end{align*}
\]
where \( \omega := e^{\frac{i \pi M(N)}{N}}, \vec{m} = (m_1, m_2), \)
\[
g = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \omega & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \omega^{N-1}
\end{pmatrix}, \quad h = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]
\(^10\) provides a basis of the Lie-algebra \( \text{gl}(N, \mathbb{C}) \), with
\[
\left[ U^{(N)}_m, U^{(N)}_n \right] = \frac{-iN}{2\pi M(N)} \sin \left( \frac{2\pi M(N)}{N} (m \times \vec{n}) \right) U^{(N)}_{\vec{m}+\vec{n}}
\]
(12)
(for the moment, we will put \( M(N) = 1 \), as only when \( N \to \infty \), \( \frac{M(N)}{N} \to \Lambda \in \mathbb{R} \), this “degree of freedom” is relevant).

Let now \( N = 3 \),
\[
\begin{align*}
\vec{M} &= \frac{1}{2} \left( U_{1,0} + U_{-1,0}, U_{1,0} - U_{-1,0}, U_{0,1} + U_{0,-1}, U_{0,1} - U_{0,-1}, U_{1,1} + U_{-1,-1}, U_{1,1} - U_{-1,-1} \right) \\
&=: (M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8).
\end{align*}
\]
\(^13\) satisfies \(^1\), \([M_1, M_2] = 0, [M_3, M_4] = 0, [M_5, M_6] = 0 \) and \([M_7, M_8] = 0 \) (note that we have implicitly reordered the elements of \( \mathcal{A} \)), and \( \vec{M}^2 = \mathbb{1} \).
One can rewrite the 8 $M_j$'s, being a basis of $\text{su}(3)$, in terms of the Cartan-Weyl basis \{h_1, h_2, e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}, e_{\alpha+\beta}, e_{-\alpha-\beta}\},

\begin{align*}
[h_1, h_2] &= 0 \\
h_{i}, e_{\alpha} &= \alpha_i e_{\alpha} \\
h_{i}, e_{-\alpha} &= -\alpha_i e_{-\alpha} \\
h_{i}, e_{\beta} &= \beta_i e_{\beta} \\
h_{i}, e_{-\beta} &= -\beta_i e_{-\beta} \\
h_{i}, e_{\alpha+\beta} &= (\alpha + \beta_i) e_{\alpha+\beta} \\
h_{i}, e_{-\alpha-\beta} &= -(\alpha + \beta_i) e_{-\alpha-\beta} \\
[e_\alpha, e_{-\alpha}] &= 4h_1 \\
[e_\beta, e_{-\beta}] &= -2h_1 + 2\sqrt{3}h_2 \\
[e_\alpha, e_{\alpha+\beta}] &= 2e_{\alpha+\beta} \\
[e_\alpha, e_{-\alpha-\beta}] &= -2e_{\alpha-\beta} \\
[e_{-\alpha}, e_{\alpha+\beta}] &= 2e_{\beta} \\
[e_{-\alpha}, e_{-\alpha-\beta}] &= -2e_{-\alpha-\beta} \\
[e_{\beta}, e_{\alpha+\beta}] &= 2e_{\alpha} \\
[e_{-\beta}, e_{\alpha+\beta}] &= -2e_{\alpha}.
\end{align*}

obtaining

\begin{align*}
M_1 &= \frac{3}{32\pi}(3h_1 + \sqrt{3}h_2) \\
M_2 &= \frac{3}{32\pi}(\sqrt{3}h_1 - 3h_2) \\
M_3 &= \frac{3}{32\pi}(e_\alpha + e_{-\alpha} + e_\beta + e_{-\beta} + e_{\alpha+\beta} + e_{-\alpha-\beta}) \\
M_4 &= \frac{3}{32\pi i}(e_\alpha - e_{-\alpha} + e_\beta - e_{-\beta} - e_{\alpha+\beta} + e_{-\alpha-\beta}) \\
M_5 &= \frac{3}{32\pi}(\sqrt{\omega}e_\alpha + \frac{1}{\sqrt{\omega}} e_{-\alpha} + e_\beta + e_{-\beta} + \sqrt{\omega}e_{\alpha+\beta} + \omega e_{-\alpha-\beta}) \\
M_6 &= \frac{3}{32\pi i}(\sqrt{\omega}e_\alpha - \frac{1}{\sqrt{\omega}} e_{-\alpha} + e_\beta - e_{-\beta} - \sqrt{\omega}e_{\alpha+\beta} + \omega e_{-\alpha-\beta}) \\
M_7 &= \frac{3}{32\pi}(\frac{1}{\sqrt{\omega}} e_\alpha + \sqrt{\omega}e_{-\alpha} + e_\beta + e_{-\beta} + \frac{1}{\sqrt{\omega}} e_{\alpha+\beta} + \frac{1}{\omega} e_{-\alpha-\beta}) \\
M_8 &= \frac{3}{32\pi i}(\frac{1}{\sqrt{\omega}} e_\alpha - \sqrt{\omega}e_{-\alpha} + e_\beta - e_{-\beta} - \frac{1}{\sqrt{\omega}} e_{\alpha+\beta} + \frac{1}{\omega} e_{-\alpha-\beta})
\end{align*}

where $\sqrt{\omega} = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (in this equation, (15)).

By considering arbitrary representations of $\text{su}(3)$ one can, also for higher $N (N \to \infty)$, obtain a set of matrices, given by (15), satisfying (4). When checking that (13) solves (4), one uses that, ($N$ arbitrary)

\begin{align*}
&\left[\left(\epsilon_{(N)}^{(\vec{m})}, \Gamma_{(N)}^{(\vec{n})}, U_{(N)}^{(\vec{n})} \right), \frac{N^2}{4\pi^2} \sin^2 \frac{2\pi}{N} \frac{2\pi}{\overline{\vec{m}} \times \overline{\vec{n}}} U_{(N)}^{(\vec{m})}\right],
\end{align*}

and $\sin^2 \frac{2\pi}{N} = \sin^2 \frac{2\pi}{N}$.

Similarly, one may take

\begin{align*}
\vec{M} = \frac{1}{2} \left( \frac{U_{\vec{m}} + U_{-\vec{m}}}{2}, \frac{U_{\vec{m}} - U_{-\vec{m}}}{2i}, \frac{U_{\vec{m}}' + U_{-\vec{m}}'}{2}, \frac{U_{\vec{m}}' - U_{-\vec{m}}'}{2i}, \frac{U_{\vec{n}} + U_{-\vec{n}}}{2}, \frac{U_{\vec{n}} - U_{-\vec{n}}}{2i}, \frac{U_{\vec{n}}' + U_{-\vec{n}}'}{2}, \frac{U_{\vec{n}}' - U_{-\vec{n}}'}{2i} \right),
\end{align*}

(17)
with
\[ \vec{m}' = \begin{pmatrix} -m_2 \\ m_1 \end{pmatrix} \quad \vec{n}' = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix}, \]

which is a solution of (14) for \( N = \hat{N} := \hat{m}^2 + \hat{n}^2 \) (which we assume to be odd), write the \( M_j \)'s (8 \( \hat{N} \times \hat{N} \) matrices) as \((\hat{N}^2\text{-dependent})\ linear combinations of a (\( \hat{N} \text{ “independent”}) basis of \( \text{gl}(\hat{N}, \mathbb{C}) \))

\[ M_j^{(\hat{N})} = \sum_{a=1}^{\hat{N}^2-1} \mu_j^a(\hat{N}) T_a^{(\hat{N})}, \]

and then define

\[ M_j^{(N)} := \sum_{a=1}^{\hat{N}^2-1} \mu_j^a(\hat{N}) T_a^{(N)} \]

(20)
to obtain corresponding solutions for \( N > \hat{N} \) (by letting \( T_a^{(N)} \) be \( N \)-dimensional representations of (19)).

In the case of \( \hat{m}^2 \) being equal to \( \hat{n}^2 \), this detour is not necessary, and (17) directly gives solutions of (4) for any (odd) \( N \). The reason is that, by using (16) the “discrete Laplace operator”

\[ \Delta_{\vec{M}}^{(N)} := \sum_{j=1}^{d} \left[ \cdot, M_j \right], M_j \]

when acting on any of the components of \( \vec{M} \), in each case yields the same scalar factor (“eigenvalue”)

\[ \frac{N^2}{4\pi^2} \left( \sin^2 \frac{2\pi}{N} (\vec{m} \times \vec{n}) + \sin^2 \frac{2\pi}{N} \vec{m}^2 + \sin^2 \frac{2\pi}{N} (\vec{m} \cdot \vec{n}) \right). \]

(22)
The \( N \to \infty \) limit of this construction gives (a solution of (7))/8, resp. (9)

\[ \vec{m}(\varphi^1, \varphi^2) = \frac{1}{2} \left( \cos \vec{m} \varphi, \sin \vec{m} \varphi, \cos \vec{m}' \varphi, \sin \vec{m}' \varphi, \cos \vec{n} \varphi, \sin \vec{n} \varphi, \cos \vec{n}' \varphi, \sin \vec{n}' \varphi, \right), \]

(23)
which for each choice
\[ \vec{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \quad \vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad \vec{m}' = \begin{pmatrix} -m_2 \\ m_1 \end{pmatrix} \quad \vec{n}' = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix} \]

describes a minimal torus in \( S^7 \).

Interestingly, the \( N \to \infty \) limit, (28), allows for non-trivial deformations (apart from the arbitrary constant that can be added to each of the 4 different arguments), namely

\[ \vec{m}_\gamma = \frac{1}{2} \left( \cos \gamma \cos \vec{m} \varphi, \cos \gamma \sin \vec{m} \varphi, \cos \gamma \cos \vec{m}' \varphi, \cos \gamma \sin \vec{m}' \varphi, \sin \gamma \cos \vec{n} \varphi, \sin \gamma \sin \vec{n} \varphi, \sin \gamma \cos \vec{n}' \varphi, \sin \gamma \sin \vec{n}' \varphi \right). \]

(24)
It is easy to check that (24) solves (9) (and (8), with an appropriate choice of \( \rho \), constant), but when “checking” (7) (which is identical to (8)) via the \( N \to \infty \) limit of (12), the \( \gamma \)-dependence of the \( m_j \) at first looks as if leading to a “contradiction” (it would, in the finite \( N \)-case), but the rationality of the structure-constants (\( \vec{m} \times \vec{n} \) instead of \( \frac{N}{2\pi} \sin \frac{2\pi}{N} (\vec{m} \times \vec{n}) \)) comes at rescue.

To come to the final observation of this note, rewrite (24) as

\[
\vec{m}_\gamma = \frac{1}{\sqrt{2}} \vec{x}^\gamma_+ + \frac{1}{\sqrt{2}} \vec{x}^\gamma_-
\]

with

\[
\begin{align*}
\vec{x}^\gamma_+ &= \frac{1}{2} \left( \cos(\vec{m} \cdot \vec{\varphi} + \gamma), \sin(\vec{m} \cdot \vec{\varphi} + \gamma), \cos(\vec{m}' \cdot \vec{\varphi} + \gamma), \sin(\vec{m}' \cdot \vec{\varphi} + \gamma), \\
& \quad \sin(\vec{n} \cdot \vec{\varphi} + \gamma), -\cos(\vec{n} \cdot \vec{\varphi} + \gamma), \sin(\vec{n}' \cdot \vec{\varphi} + \gamma), -\cos(\vec{n}' \cdot \vec{\varphi} + \gamma) \right) \\
\vec{x}^\gamma_- &= \frac{1}{2} \left( \cos(\vec{m} \cdot \vec{\varphi} - \gamma), \sin(\vec{m} \cdot \vec{\varphi} - \gamma), \cos(\vec{m}' \cdot \vec{\varphi} - \gamma), \sin(\vec{m}' \cdot \vec{\varphi} - \gamma), \\
& \quad -\sin(\vec{n} \cdot \vec{\varphi} - \gamma), \cos(\vec{n} \cdot \vec{\varphi} - \gamma), -\sin(\vec{n}' \cdot \vec{\varphi} - \gamma), \cos(\vec{n}' \cdot \vec{\varphi} - \gamma) \right)
\end{align*}
\]

While \( \gamma \), in this form, becomes irrelevant (insofar each of the 4 arguments in \( \vec{x}_+ := \vec{x}^{[0]}_+ \), as well as those in \( \vec{x}_- := \vec{x}^{[0]}_- \) can have an arbitrary phase-constant), not only their sum, (25), but (due to the mutual orthogonality of \( \vec{x}_+, \partial_1 \vec{x}_+, \partial_2 \vec{x}_+, \vec{x}_-, \partial_1 \vec{x}_- \) and \( \partial_2 \vec{x}_- \)) both \( \vec{x}_+ \) and \( \vec{x}_- \) separately, in fact any linear combination

\[
\vec{x}_\theta = \cos \theta \vec{x}_+ + \sin \theta \vec{x}_-
\]

gives a minimal torus in \( S^7 \).

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**References**

[1] J. Hoppe. “Some Classical Solutions of Membrane Matrix Model Equations”, [hep-th/9702169](https://arxiv.org/abs/hep-th/9702169), Proceedings of the Cargèse Nato Advanced Study Institute, May 1997, Kluwer 1999.

[2] H. B. Lawson, Jr. “Complete minimal surfaces in \( S^3 \)”, Ann. of Math. (2) 92 (1970), p 335–374.