Iterative method for the numerical solution of a system of integral equations for the heat conduction initial boundary value problem

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Abstract. The paper deals with a numerical algorithm to reduce the overall system of integral equations describing the heat transfer process at any geometrically complex area (both two-dimensional and three-dimensional), to the iterative solution of a system of independent one-dimensional integral equations. This approach has been called "string method" and has been used to solve a number of applications, including the problem of the detonation wave front for the calculation of heat loads in pulse detonation engines. In this approach "the strings" are a set of limited segments parallel to the coordinate axes, into which the whole solving area is divided (similar to the way the strings are arranged in a tennis racket). Unlike other grid methods where often for finding solutions, the values of the desired function in the region located around a specific central point here in each iteration step is determined by the solution throughout the length of the one-dimensional "string", which connects the two end points and set them values and determine the temperature distribution along all the strings in the first step of an iterative procedure.

1. Introduction

Integral equations, on contrary to the differential, have a number of advantages, which primarily include the ability to solve them with the help of iterative algorithms and evaluate the accuracy of the approximate solution with the vector of errors or discrepancy. With regard to integral equations of heat conduction additional advantage is no need to coordinate continuity of temperature and heat flux at the interface between different media. Thus, the integral equation simplifies considerably the numerical problem definition temperature fields in geometrically complicated articles having non-uniform switching, which, among others, can be attributed to composite materials (prepregs). Modes of heat treatment of products from such materials greatly affect their distortion and ultimately structural strength, particularly for products spatially extended. In this regard, more relevant than the direct problem of determining the temperature fields in the blanks for a given external thermal effects, and the inverse problem - finding such thermal effects, which would ensure a more uniform curing and thus a more regular product structure thickness.

2. A general view of integral equations for the of heat conduction initial boundary value problem
The author has developed an approach that allowed to bring integral equations describing the process of unsteady heat conduction [1-5], as in the two-dimensional and three-dimensional cases for bodies of arbitrary geometry. Consider the conventional two-dimensional formulation of the non-stationary heat conduction problems with heat sources:

\[ cp\frac{\partial u}{\partial t} = \partial \partial x (k, \partial u / \partial x) + \partial \partial y (k, \partial u / \partial y) + q, \]

with initial and boundary conditions in the case when set boundary heat flows:

\[
\begin{align*}
 u(x,y,0) &= u_0, \\
 W_f &= w_f,
\end{align*}
\]

where: \( u= u(x,y,t) \) – the temperature function, \( u_0= u_0(x,y) \) – the initial temperature function, \( k_x= k_x(x,y,u) \) – the thermal conductivity along the axis \( x \), \( k_y= k_y(x,y,u) \) – the thermal conductivity along the axis \( y \), \( \varepsilon = \varepsilon(x,y,u) \) – the specific heat, \( \rho = \rho(x,y,u) \) – the density, depending on the spatial variable and temperature and \( q= q(x,y,t) \) – the distribution function of the heat sources (power density), \( w_f \) - external heat flux density, defined in every point of the boundary of solving area.

The figure below shows the solving area, the initial and boundary conditions:

**Figure 1.** An illustration of two-dimensional heat conduction initial-boundary value problem

Integral equations are based on the fact that instead of the unknown function of temperature are introduced new functions (two in case of two-dimensional and three when three-dimensional), which are physically meaningful spatial derivative of the heat flux for each of the coordinate axes:

\[
\begin{align*}
 S_x &= [cp]^{-1} \partial / \partial x (k, \partial u / \partial x) - f_x, \\
 S_y &= [cp]^{-1} \partial / \partial y (k, \partial u / \partial y) - f_y.
\end{align*}
\]

For the two-dimensional case we have the following integral equation containing the unknown function as mentioned above:

\[
\begin{align*}
 u_0 + \int_0^t S_x dt + \int_0^t S_y dt + & \int_0^t \left[ \int \left[ \int [cpdx]^{-1} \int [cpdy]^{-1} \right] dx \right] dy + \int \left[ \int [cpdx]^{-1} \right] dx \left( w_f - w_{i_f} \right) \right) dt + \int \left[ \int [cp]^{-1} q \right] dt = \\
 = 0.5 \int \left[ \int [k_x]^{-1} k_x^{-1} k_y^{-1} \right] \left[ \int [cpdx] \right] dx \left( w_f - w_{i_f} \right) + 0.5 \int \left[ \int [k_x]^{-1} \right] dx \left( w_f - w_{i_f} \right) + 0.5 \int \left[ \int [k_x]^{-1} \right] dx \left( w_f - w_{i_f} \right) dt + \int \left[ \int [cp]^{-1} q \right] dt =
\end{align*}
\]
\[ \begin{align*}
  & = 0.5 \left\{ \int_{l} \left[ k \right]^{-1} r_{y} \int_{l} \left[ k \right]^{-1} r_{x} \right\} \int_{l} cpS_{x} dy dy - 0.5 \left\{ \int_{l} \left[ k \right]^{-1} r_{y} \int_{l} \left[ k \right]^{-1} r_{x} \right\} \int_{l} cpS_{y} dy dy + \\
  & + 0.5 w_{y} \left\{ \int_{l} \left[ k \right]^{-1} r_{y} \int_{l} \left[ k \right]^{-1} r_{y} \right\} \int_{l} cpS_{x} dy dy + 0.5 w_{x} \left\{ \int_{l} \left[ k \right]^{-1} r_{x} \int_{l} \left[ k \right]^{-1} r_{x} \right\} \int_{l} cpS_{y} dy dy; \\
  \end{align*}\]

where:

\[
\begin{align*}
  r_{x}^{\downarrow} &= \left\{ \int_{l} \left[ cpdx \right]^{-1} \int_{l} cpdx, \\
  r_{x}^{\uparrow} &= \left\{ \int_{l} \left[ cpdx \right]^{-1} \int_{l} cpdx, \\
  r_{y}^{\downarrow} &= \left\{ \int_{l} \left[ cpdy \right]^{-1} \int_{l} cpdy, \\
  r_{y}^{\uparrow} &= \left\{ \int_{l} \left[ cpdy \right]^{-1} \int_{l} cpdy. \\
  \end{align*}\]

There designations were introduced for a more transparent recording of integral equations:

\[
\begin{align*}
  \int_{l}^{0} g(t) dt &= \int_{l}^{0} g dt, \\
  \int_{l}^{x} f(x) dx &= \int_{l}^{x} f dx, \\
  \int_{l}^{x} f(x) dx &= \int_{l}^{x} f dx, \\
  \int_{l}^{x} f(x) dx &= \int_{l}^{x} f dx. \\
  \end{align*}\]

After solving the resulting system of integral equations and determining the unknown functions \( S_{x} \) and \( S_{y} \), the temperature distribution function can be calculated by simple integration:

\[ u = u_{0} + \int_{0}^{l} S_{x} dt + \int_{0}^{l} S_{y} dt + \int_{0}^{l} f_{x} dt + \int_{0}^{l} f_{y} dt + \int_{0}^{l} q/cpd, \]

where:

\[
\begin{align*}
  f_{x} &= \left\{ \int_{l} \left[ cpdx \right]^{-1} \int_{l} cpS_{x} dx + (w_{x}^{\downarrow} - w_{x}^{\uparrow}), \\
  f_{y} &= \left\{ \int_{l} \left[ cpdy \right]^{-1} \int_{l} cpS_{y} dy + (w_{y}^{\downarrow} - w_{y}^{\uparrow}). \\
  \end{align*}\]

3. **Principles of the string method in two dimension area**

The string method was set up as an iterative procedure in order to find the solution of integral equations (4) through its opening. This approach allows solving (on each iteration step) integral set of heat conduction equations for one-dimensional problems.

Common views of the resulting set of «strings», describing the original problem in a two-dimensional formulation, are shown below in figure 2:
In order to have clear principles of the method of strings, we rewrite the integral equation (4) in the form in which terms relating to "crosstalk" between these equations, moved to the right side:

\[
\int_0 S dt - 0.5 \left\{ \int [k_1]^{-1} r_{ij} \int [k_1]^{-1} r_{ij} \right\} cpS dx + 0.5 \left\{ \int [k_1]^{-1} r_{ij} \int [k_1]^{-1} r_{ij} \right\} \int cpS dx = \\
\quad = - \int_0 S dt - 0 \left\{ \int [cpdy]^{-1} \int [cpdy] \int cpS dy dt \right\} \\
\quad + 0.5 w_{1,0} \left\{ \int [k_1]^{-1} r_{ij} \int [k_1]^{-1} r_{ij} \right\} dx + 0.5 w_{1,0} \left\{ \int [k_1]^{-1} r_{ij} \int [k_1]^{-1} r_{ij} \right\} dx - \\
\quad \quad \quad - u_0 - \int [cp]^{-1} q dt;
\]

If bold members in (9) are set up to zero, we obtain exactly the one-dimensional integral equation of heat conduction. Therefore, we can consider the iterative procedure, on each step assumed to be known (taken from the previous step) of S function (for the first equation of the system) and S, (for the second equation), then the solution of (4) is reduced to the decision of an independent set of integral equations for dimensional problems with a new right-hand side.

4. Iterative method of solving one-dimensional heat conduction integral equations

Solution of one-dimensional integral equations also preferably performed using iterative procedures for the same reasons - the limited properties of the integral operator, which allows to evaluate the accuracy of the approximate solution. The best known of the methods used are the method of successive approximations and the method of steepest descent. The main drawback of these methods is that often the first few iterations of these methods give a good approximation, and their further use or lead to divergence or convergence to substantially zero iterative procedure. This is due to the fact that these methods work well only in the case of systems with diagonally dominant. The proposed method for solving the one-dimensional integral equation is based on the summation of a geometric progression, and can be used for solving systems for which there is no diagonal predominance.

Let us consider a one-dimensional integral equation of heat conduction in the form of an operator equation in a Hilbert space:

\[
H = H([x_1, x_2] | [0, T])
\]

\[
(f, g) = \int u f g dv
\]

\[
\|f\|_H^2 = (f, f);
\]

and:

\[
Ax = b
\]

where A is a bounded operator on the entire Hilbert space H:

\[
A: H \rightarrow H, \|A\|_H = C < \infty.
\]

**Definition.** The linear system (11) with the vector of the right side of b is called the fixed-sign, if all the elements of vectors b, Ab, ... A^n b, ... have the same sign:

\[
sign A^n b = sign b, \ n = 0, 1, 2, ...;
\]
**Theorem.** If the linear system is a fixed sign, its solution can be found as the sum of a convergent series of geometric progression:

\[
x = \lambda b + (E - \lambda A)\lambda b + (E - \lambda A)^2\lambda b + \ldots + (E - \lambda A)^n\lambda b + \ldots
\]

Note that if \( b \) is an eigenvector of \( A \) with eigenvalue \( m \), then we get a common geometric progression, which can ensure the convergence selection parameter smaller than \( \lambda < 1/m \). Therefore, the proof is based on the decomposition of the vector on the right part of the eigenvectors \( A \). In order to ensure the convergence of exponentially as convergence parameter selected value of \( \lambda = 1/m_{\text{max}} \), where \( m_{\text{max}} \) maximum eigenvalue of integral operator. Due to the limited conditions of the operator and fixed sign, the maximum eigenvalue of a positive and of course: \( 0 < m_{\text{max}} < ||A||_H \)

5. **Conclusions**

Description of initial boundary value problem of heat conduction in the form of integral equations to solve them allows you to use iterative algorithms, which on one hand can effectively parallelize the calculation process, but on the other - to control the accuracy of the calculations for the error function (residual). The proposed approach can be used to solve a variety of problems in which the accuracy of the calculation is a critical condition for the calculations - such as the modeling of heat treatment processes, products from metals and alloys, forecast heat load on the combustion chamber wall of aircraft and rocket engines, and others.

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