Shrinking targets for non-autonomous systems

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Abstract
In the present work we establish a Bowen-type formula for the Hausdorff dimension of shrinking target sets for non-autonomous conformal iterated function systems in arbitrary dimensions and satisfying certain conditions. In the case of dimension 1 we also investigate non-linear perturbations of linear systems and obtain sufficient conditions under which the perturbed systems satisfy the conditions in our hypotheses.

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1. Introduction

In [1], Rufus Bowen proved a dimension result for certain dynamically-defined sets in terms of a topological pressure function. Such formulae relating Hausdorff dimension to topological pressure came to be known as Bowen’s formula. Since Bowen’s original work many others have extended Bowen’s formula to several different contexts [2–5]. For a first introduction to Bowen’s equation see chapter 9 of [6]. The first results of such type for shrinking target sets appeared in a series of papers by Hill and Velani. In [7] they prove that if \( T \) is an expanding rational map of the Riemann sphere with Julia set \( J \), then for every \( z_0 \in J \) and \( \tau > 0 \), the Hausdorff dimension of the set

\[
W(\tau) = \bigcap_{m \geq 1} \bigcup_{n \geq m} \{ z \in J \mid T^n(z) \in B(z_0, |(T^n)'(z)|^{-\tau}) \}
\]

is the unique solution to the equation \( P(s) = 0 \), where \( P \) is a pressure function associated to the map \( T \) and the constant \( \tau \) [7].

In classical Diophantine approximation the set of \( \alpha \)-well approximable numbers is
It is well known that if $0 \leq \alpha \leq 2$ this set is $[0, 1] \setminus \mathbb{Q}$. By the Borel–Cantelli lemma, $\mathcal{D}_\alpha$ is a set of Lebesgue measure zero for all $\alpha > 2$. Thus, a natural question is the Hausdorff dimension of $\mathcal{D}_\alpha$. Jarník [8] and Besicovitch [9] both proved that the Hausdorff dimension of $\mathcal{D}_\alpha$ is $2\alpha$.

The sets $W(\tau)$ and $\mathcal{D}_\alpha$ are examples of shrinking target sets. In dynamical systems and metric Diophantine approximation shrinking target sets have been studied in various contexts. Two questions that often arise from shrinking target problems are dichotomy laws or Borel–Cantelli lemmas (see [10] or [11] for example), and Hausdorff dimension of such sets [12, 13]. In this paper we will focus on the latter.

1.1. Non-autonomous IFS

Recently, Rempe-Gillen and Urbański [14] expanded Bowen’s formula into the realm of non-autonomous iterated function systems (IFS).

An autonomous IFS consists of a countable indexing set $I$ called the alphabet, and a collection $(\varphi_a)_{a \in I}$ of contracting maps on some set $X \subseteq \mathbb{R}^d$. The Cartesian product $I^n$ is referred to as the set of words of length $n$, and for every $\omega \in I^n$ we define $\varphi^n\omega : X \to X$ by the composition

$$\varphi^n\omega = \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n},$$

where $\omega_j$ denotes the $j$-th coordinate of $\omega$.

As an example, consider the celebrated middle-third Cantor set $C$. Let $I = \{0, 2\}$ and for each $a \in I$ define $\varphi_a : [0, 1] \to [0, 1]$ as

$$\varphi_a(x) = \frac{x + a}{3}.$$

The middle-third Cantor set can now be written as

$$C = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^n} \varphi^n\omega [0, 1].$$

Note that if we consider the alphabet $I = \{0, 1, 2\}$ and define $\varphi_a$ as above, then each map $\varphi_a$ corresponds to one of the three inverse branches of the expanding map $T(x) = 3x \mod 1$ on $[0, 1]$. More precisely,

$$T(\varphi_a(x)) = x, x \neq 1$$

for all $a \in I$ and

$$T^n(\varphi^n\omega(x)) = x, x \neq 1$$

for all $\omega \in I^n$.

Instead of only considering one alphabet $I$, a non-autonomous IFS considers a countable collection $(I^n)_{n \in \mathbb{N}}$ of such alphabets. For each $n$ there is again a collection $(\varphi^{(n)}_{\omega})_{\omega \in I^n}$ of contractions on $X$. Letting $I^n$ denote the Cartesian product $I^{(1)} \times I^{(2)} \times \cdots \times I^{(n)}$ we define

$$\varphi^n\omega = \varphi^{(1)}_{\omega_1} \circ \varphi^{(2)}_{\omega_2} \circ \cdots \circ \varphi^{(n)}_{\omega_n},$$

where $\omega_j \in I^{(j)}$. 
In [14] the authors consider non-autonomous conformal iterated function systems $\Phi$ on $X \subset \mathbb{R}^n$ and their associated limit set

$$J = \bigcap_{n \geq 1} \bigcup_{\omega \in \mathcal{F}} \varphi_\omega(X).$$

Under suitable assumptions on $\Phi$, Rempe-Gillen and Urbański show a Bowen-type formula for the limit set, that is,

$$\text{HD} (J) = \sup \{ t \geq 0 \mid P(t) \geq 0 \},$$

where

$$P(t) = \liminf_{n \to \infty} \frac{1}{n} \log \left( \sum_{\omega \in \mathcal{F}} \|D\varphi_\omega^n\|^t \right) \quad (1.1)$$

In [15], the authors explore the shrinking target problem for a certain class of non-autonomous systems. Specifically, for a sequence $Q = (q_n)$ of integers no smaller than 2, define

$$T_n = x \mapsto q_n x \pmod{1} : [0, 1] \to [0, 1].$$

This sequence of maps gives rise to a non-autonomous dynamical system on $[0, 1]$ whose orbits are defined by

$$T^n = T_n \circ \cdots \circ T_1.$$

Given a sequence $\alpha = (\alpha_n) \in (0, \infty)^\mathbb{N}$ and letting $\alpha(n) = \alpha_1 + \cdots + \alpha_n$, the shrinking target associated to $Q$ and $\alpha$ is defined as

$$\mathcal{D}_Q(\alpha) = \bigcap_{m \geq 0} \bigcup_{n \geq m} \left\{ x \in [0, 1] \mid |T^n(x)| \leq e^{-\alpha(n)} \right\},$$

where $|x|$ denotes distance to the nearest integer. The pressure associated to $Q$ and $\alpha$ is given by

$$P_Q(t) = \limsup_{n \to \infty} \frac{1}{n} \log \left( (q_1 q_2 \cdots q_n)^{1-t} e^{-\alpha(n)} \right).$$

Note that $\mathcal{D}_Q(\alpha)$ can be rewritten in terms of a non-autonomous IFS. Indeed, if we define $I^{(n)} = \{0, 1, \ldots, q_n - 1\}$ and for, $a \in I^{(n)}$, $\varphi_a^{(n)} : [0, 1] \to [0, 1]$ as

$$\varphi_a^{(n)}(x) = q_n^{-1} (x + a),$$

then

$$\mathcal{D}_Q(\alpha) = \bigcap_{m \geq 0} \bigcup_{n \geq m} \bigcup_{\omega \in I^{(n)}} \varphi_\omega^n([0, 1]).$$

The main result in [15] is an extension of Bowen’s formula, namely that

$$\text{HD} \left( \mathcal{D}_Q(\alpha) \right) = \sup \{ t \geq 0 \mid P(t) \geq 0 \}.$$

Our main results, theorems 2 and 3, establish Bowen’s formula for a certain class of IFS coming from those considered in [14] satisfying certain natural conditions. This class generalizes those IFS in [15] in two important ways:
We consider a certain class of IFS in higher dimensions; that is, on subsets of $\mathbb{R}^d$, $d \geq 1$.

While we delay some necessary definitions until the next section, we state the precise results now.

**Theorem 2.** Let $\Phi$ be a non-autonomous conformal IFS on a compact, convex set $X \subseteq \mathbb{R}^d$ with nonempty interior satisfying OSC, ESC, UCC, LVC, and NEQ conditions. Suppose that the sequence $(\kappa_n)$ is bounded and that there exists an $h$-Ahlfors measure, $\mu_h$, where $h = \text{HD}(J)$, and $\text{supp}(\mu_h) = J$. Then $\text{HD}(\mathcal{D}) = b$.

**Theorem 3.** Let $\Phi$ be a conformal non-autonomous IFS satisfying the OSC, ESC, UCC, and NEQ conditions. If there exists an $h$-Ahlfors measure supported on $J$, $\mathcal{P}(t) = \mathcal{P}(t)$ on a neighbourhood of $b$, and

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\kappa_n}{\kappa_n} = 0,$$

then $\text{HD}(\mathcal{D}) = b$.

1.2. Organization.

In section 2 we establish our notation and basic definitions. In section 3 we prove an upper bound for the Hausdorff dimension of our sets of interest. The proof is fairly elementary and general. Our main results are in section 4. It begins by defining and describing all the conditions necessary in the hypothesis of our theorems. Then we state and prove the main theorems. In section 5 we pay special attention to one of the conditions in our hypotheses: the existence of Ahlfors measures. We prove sufficient conditions for their existence. Finally, in section 6 we focus on the case in Euclidean dimension $d = 1$ and investigate the ‘rigidity’ of IFSs satisfying our conditions. We prove that IFS preserve all the required conditions under sufficiently small perturbations.

2. Definitions and preliminaries

Let $X \subset \mathbb{R}^d$ be a compact, convex subset with nonempty interior and let $V$ be a bounded, open, connected set containing $X$. Consider a countable collection $\left( I^n \right)_{n \in \mathbb{N}}$ of finite alphabets which will be used to encode a non-autonomous iterated function system (IFS) in the following way. For every $n \in \mathbb{N}$ and every $j \in I^n$ we fix conformal contractions $\varphi_j^n: V \to V$ such that $\varphi_j^n(X) \subseteq X$; that is, there exists $\theta > 0$ such that for all $n \in \mathbb{N}$ and all $j \in I^n$ we have that

$$\varphi_j^n := \max_{j \in I^n} \left\| D\varphi_j^n \right\| \leq e^{-\theta},$$

and $D\varphi_j^n$ is a similarity.

Letting

$$I^n = \prod_{k=1}^n I^k,$$

we define for every $\omega = (\omega_1, \ldots, \omega_n) \in I^n$ the map

$$\varphi_\omega^n: V \to V, \quad \omega \mapsto \varphi_\omega^n,$$
\[ \varphi^n_\omega = \varphi^{(1)}_{\omega_1} \circ \varphi^{(2)}_{\omega_2} \circ \ldots \circ \varphi^{(n)}_{\omega_n}. \]

Furthermore, products of the form \( I^k \times I^{(k+1)} \times \ldots \times I^n \) will be denoted by \( I^{(k,n)} \), where \( n \) may be infinity. The set \( I^{(1,\infty)} \) will simply be denoted by \( I^\infty \). We will also make use of the shift map \( \sigma \), which takes a word \( \omega \in I^n \) to a word \( \sigma^k \omega \in I^{(k+1,n)} \) where

\[ \sigma^k \omega = (\omega_{k+1}, \ldots, \omega_n) \]

for all \( 0 \leq k < n \). The empty word \( \sigma^n \omega \) is used to encode the identity map, i.e., \( \varphi_{\sigma^n \omega} = id \) for all \( \omega \in I^n \).

On the other hand, for \( \omega \in I^{(m,n)} \) we will let \( \omega|_k \) denote the word \( (\omega_1, \ldots, \omega_k) \in I^{(m,n+k-1)} \) for all \( 1 \leq k \leq n - m + 1 \).

To define a shrinking target set we fix a sequence \( (\beta_n) \) of functions \( \beta_n : I^{(n,\infty)} \to (0, \infty) \). Let \( S_n \beta : I^{\infty} \to (0, \infty) \) be defined by

\[ S_n \beta \left( \xi \right) = \beta_1 \left( \xi \right) + \beta_2 \left( \sigma \xi \right) + \ldots + \beta_n \left( \sigma^{n-1} \xi \right). \]

The quantity above will determine the rate at which the shrinking targets shrink to zero radius in the following way: fix a sequence \( (\xi(n)) \) where \( \xi(n) \in I^{(n+1,\infty)} \), and a sequence \( (\chi(n)) \in X^\mathbb{N} \). For every \( \omega \in I^n \) we define the shrinking targets as

\[ B_\omega \left( \varphi^n_\omega \left( \chi(n) \right), \ e^{-S_n \beta \left( \omega \xi(n) \right)} \right). \]

The shrinking target set is then defined as

\[ \mathcal{D} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{\omega \in I^n} B_\omega. \]

As a special case one may consider the one where \( \beta_n \) is a constant function into \( (0, \infty) \), as it is done in [15].

We denote the Hausdorff dimension of a set \( A \) by \( \text{HD}(A) \). Let us also denote the diameter of a set \( A \) by \( |A| \).

Now for \( t \geq 0 \) we define the upper pressure

\[ \mathcal{P}_\beta(t) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} e^{-S_n \beta \left( \omega \xi(n) \right)}. \]

(2.1)

The lower pressure \( \mathcal{L}_\beta(t) \) is defined similarly by taking a limit inferior instead of a limit superior. If \( \mathcal{P}_\beta(t) = \mathcal{L}_\beta(t) \) holds, we denote this common value by \( P_\beta(t) \).

Now we briefly explore certain properties of the pressure functions. Note that for \( \epsilon > 0 \) we have that

\[ \mathcal{P}_\beta(t + \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \left( e^{-S_n \beta \left( \omega \xi(n) \right)} e^{-\epsilon S_n \beta \left( \omega \xi(n) \right)} \right) \]

\[ \leq \limsup_{n \to \infty} \left[ \frac{1}{n} \log \left( \sum_{\omega \in I^n} e^{-\epsilon S_n \beta} \right) \right] \]

\[ = \mathcal{P}_\beta(t), \]

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so the upper (as well as lower) pressure function is non-increasing. We say that the sequence \((\beta_n)\) is tame if the the upper pressure is strictly decreasing. Furthermore, assuming \(\#F^{(n)} \geq 2\) for all \(k\) it is immediate that \(P_{\beta}(0) \geq \log(2)\).

Now, if we assume that \(B > 0\) such \(\#I^{(n)} \leq B\) for all \(n \in \mathbb{N}\), and that (4.2) holds then

\[
\mathcal{P}_{\beta}(d) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in F} e^{-dS_{\beta}(\omega^{(n)})} \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( B^n \max \left\{ e^{-dS_{\beta}(\omega^{(n)})} : \omega \in I^n \right\} \right) \\
\leq \limsup_{n \to \infty} \frac{1}{n} \log \left( B^n e^{-nd\theta} \right) \\
\leq \log B - d\alpha + \limsup_{n \to \infty} \left( \frac{d}{n} \log \tau_n \right) \\
\leq \log B - d\alpha - d\theta
\]

It follows that \(\mathcal{P}_{\beta}(d) \leq 0\) if \(B \leq e^{d(\alpha + \theta)}\).

We observe that if \(\mathcal{P}_{\beta}\) is strictly decreasing, and \(\mathcal{P}_{\beta}(0) \cdot \mathcal{P}_{\beta}(d) < 0\), then there exists a unique number \(0 < b < d\) such that

\[
b = \inf \left\{ t \geq 0 \left| \mathcal{P}_{\beta}(t) < 0 \right. \right\} = \sup \left\{ t \geq 0 \left| \mathcal{P}_{\beta}(t) > 0 \right. \right\}.
\]

Note that such a unique number \(b\) still exists in \([0, \infty]\) when only assuming condition (4.2). We refer to such number as the Bowen parameter. The main objective of our analysis is to establish conditions under which \(HD(\mathcal{D}) = b\).

3. Upper bound

We say that a countable collection \((U_k)\) of subsets of \(\mathbb{R}^d\) is a \(\delta\)-cover of \(\mathcal{D}\) if \(\mathcal{D} \subset \bigcup U_k\) and \(\text{diam}(U_k) \leq \delta\) for all \(k\). We recall here the definition of \(t\)-dimensional Hausdorff measure.

\[
H^t(\mathcal{D}) = \liminf_{\alpha \to 0} \left\{ \sum_{k=1}^{\infty} \frac{[\text{diam}(U_k)]^t}{(U_k)_{k \geq 1}} \left| \text{is an } \alpha - \text{cover of } \mathcal{D} \right. \right\} \\
= \sup_{\alpha > 0} \inf \left\{ \sum_{k=1}^{\infty} \frac{[\text{diam}(U_k)]^t}{(U_k)_{k \geq 1}} \left| \text{is an } \alpha - \text{cover of } \mathcal{D} \right. \right\}.
\]

Hausdorff dimension is then defined as

\[
\text{HD}(\mathcal{D}) = \inf \left\{ t \geq 0 \left| H^t(\mathcal{D}) = 0 \right. \right\} = \sup \left\{ t \geq 0 \left| H^t(\mathcal{D}) = \infty \right. \right\}.
\]

**Theorem 1.** For any shrinking target set \(\mathcal{D}\) originating from a non-autonomous IFS and a tame sequence \(\beta\), we have that \(HD(\mathcal{D}) \leq b\).

**Proof.** Let \(t > b\). We will show that \(H^t(\mathcal{D}) = 0\). Note that for any \(N \geq 1\) the collection \(\left( \bigcup_{\omega \in F} B_{\omega} \right)_{n \geq N}\) covers \(\mathcal{D}\), so
Since \( t > b \) and \( \beta \) is tame we have that \( \overline{P}_\beta (t) < 0 \). Thus, for large enough \( M \),

\[
 n \geq M \implies \frac{1}{n} \log \sum_{\omega \in I^n} e^{-tS_n(\omega)} < \frac{1}{2} \overline{P}_\beta (t) < 0.
\]

Hence,

\[
 \sum_{\omega \in I^n} e^{-tS_n(\omega)} < e^{\overline{P}_\beta (t)} < 1.
\]

Thus,

\[
 \sum_{n \geq N} \sum_{\omega \in I^n} [\text{diam} (B_\omega)]^t \leq 2 \sum_{n \geq N} e^{\overline{P}_\beta (t)}.
\]

The right hand side of the inequality above is the tail of a converging geometric series. After fixing \( \epsilon > 0 \) we can choose \( N \) large enough so that

\[
 \sum_{n \geq N} \sum_{\omega \in I^n} [\text{diam} (B_\omega)]^t < \epsilon.
\]

This shows that \( H^t (\mathcal{D}) < \epsilon \). Since \( \epsilon > 0 \) and \( t > b \) were chosen arbitrarily, we have that

\[
 \text{HD} (\mathcal{D}) \leq b.
\]

\section{4. Lower bound}

For the proof of the lower bound we will need to impose some restrictions on our IFS. First we establish some preliminary definitions and results.

We define

\[
 \kappa_j^{(n)} = \min_{j \in I^n} \inf_{x \in X} \left| D_{\varphi_j^{(n)}} (x) \right|,
\]

\[
 \tau_j^{(n)} = \max_{j \in I^n} \sup_{x \in X} \left| D_{\varphi_j^{(n)}} (x) \right|,
\]

\[
 \kappa_n = \min_{\omega \in I^n} \inf_{x \in X} \left| D_{\varphi_\omega^{(n)}} (x) \right|,
\]

\[
 \tau_n = \max_{\omega \in I^n} \sup_{x \in X} \left| D_{\varphi_\omega^{(n)}} (x) \right|.
\]

It is easy to check that

\[
 \prod_{k=1}^n \kappa_k (\xi) \leq \kappa_n \leq \tau_n \leq \prod_{k=1}^n \tau_k (\xi). \tag{4.1}
\]

Let \( J \) be the limit set (attractor) of the IFS, i.e.,

\[
 J = \bigcap_{n \geq 1} \bigcup_{\omega \in I^n} \varphi_\omega^{(n)} (X).
\]

Consider the projection map \( \pi_\omega : I^{n+1, \infty} \to X \) where \( \pi_\omega (\xi) \) is defined as the element in the singleton set
For every $n \in \mathbb{N}$ and every $\omega \in I^\prime$, $\varphi^n_\omega (J_n) \subseteq J$; indeed,

$$\varphi^n_\omega (J_n) = \varphi^n_\omega \left( \bigcup_{\xi \in \{ 0, 1, \ldots, n+1 \}} \bigcap_{k \geq 1} \varphi^{(n+1,n+k)}_{\xi k} (X) \right) \subseteq \bigcup_{\xi \in \{ 0, 1, \ldots, n+1 \}} \bigcap_{k \geq 1} \varphi^{(n+1,n+k)}_{\xi k} (X) \subseteq J.$$

For every $n \in \mathbb{N} \cup \{ 0 \}$ we fix $\xi^{(n)} \in I^{(n+1,\infty)}$ and from this we define a sequence $\chi^{(n)} \in J_n$ as $\chi^{(n)} = \pi_n (\xi^{(n)})$. This implies that the balls $B_\omega = B \left( \varphi^n_\omega \left( \chi^{(n)} \right), e^{-S_{k+1}} \right)$ are centred at a point in $J$.

Furthermore, we make the following assumptions:

- For all $n \in \mathbb{N}$ and all $j \in I^{(n)}$, $\varphi^n_j$ is injective.
- Open set condition (OSC): for all $n \in \mathbb{N}$, and for all $i, j \in I^{(n)}$, $i \neq j$, $\varphi^n_j \left( \text{int} (X) \right) \cap \varphi^n_i \left( \text{int} (X) \right) = \emptyset$.
- Uniformly contracting condition (UCC): assume that for some $\theta > 0$ we have that $\tau^{(k)} \leq e^{-\theta}$, for all $k$.
- Exponentially shrinking condition (ESC): we assume that there exist numbers $\overline{\alpha}$ and $\underline{\alpha}$ such that

$$0 < \underline{\alpha} \leq \beta_k (\xi) + \log \sigma_k (\xi) \leq \beta_k (\xi) + \log \tau^{(k)} \leq \overline{\alpha},$$

for all $k$ and all $\xi \in I^{(k,\infty)}$. It is easy to check that

$$0 < n \underline{\alpha} \leq S_n \beta (\xi) + \log \sigma_n (\xi) \leq S_n \beta (\xi) + \log \tau_n \leq n \overline{\alpha},$$

for all $n$ and all $\xi \in I^{(n)}$. 

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• Non-empty quasi middle (NEQ): recall that for a set \( A \) in a metric space and \( \varepsilon > 0 \), the \( \varepsilon \)-thickening of \( A \) is
\[
B(A, \varepsilon) = \bigcup_{x \in A} B(x, \varepsilon).
\]

Now let
\[
X_\varepsilon := X \setminus B\left(\mathbb{R}^d \setminus X, \varepsilon\right).
\]

We assume that there exists \( \varepsilon > 0 \) for which \( J_n \cap X_\varepsilon \neq \emptyset \), for all \( n \).

Hence, assuming the NEQ condition we can choose the point \( x^{(n)} \) appearing in the definition of the balls \( B_\omega \) to be in \( J_n \cap X_\varepsilon \).

• Linear variation condition (LVC): the sequence \( (\beta_n) \) is said have the linear variation condition if
\[
\lim_{n \to \infty} \frac{1}{n} \left( \sup_{\xi \in I^\infty} S_n \beta(\xi) - \inf_{\xi \in I^\infty} S_n \beta(\xi) \right) = 0.
\]

We note that this condition implies that for all \( \varepsilon > 0 \) there exists \( N_\varepsilon \geq 1 \) such that for all \( n \geq N_\varepsilon \) and all \( \xi, \xi' \in I^\infty \) we have that
\[
\exp\left\{ -S_n \beta(\xi) - \varepsilon n \right\} \leq \exp\left\{ -S_n \beta(\xi') \right\} \leq \exp\left\{ -S_n \beta(\xi) + \varepsilon n \right\}.
\]

• Bounded distortion property (BDP): we assume that there exists \( K \geq 1 \) such that for every \( n \in \mathbb{N} \), every \( \omega \in I^n \), and every \( x, y \in X \),
\[
|D\varphi^{(n)}_\omega(x)| \leq K |D\varphi^{(n)}_\omega(y)|.
\]

It should be noted that a sufficient condition for BDP, one in terms of the maps \( \varphi^{(n)}_a \) and not in terms of the composition \( \varphi^{(n)}_\omega \), is if there exists \( \alpha > 0 \) such that
\[
\frac{|D\varphi^{(n)}_\omega(x)|}{|D\varphi^{(n)}_\omega(y)|} - 1 \leq K |x - y|^{\alpha},
\]
for all \( x, y \in X \), all \( n \in \mathbb{N} \), and all \( a \in I^n \).

Let us now examine some consequences of a conformal non-autonomous IFS having these properties. First we note that ESC and UCC imply that the radii of \( B_\omega \) decay exponentially fast; indeed \( e^{-S_n(\omega(\cdot))} \leq \mathcal{L}_n e^{-n\theta} \).

One geometric consequence of BDP is that for every ball \( B(x, r) \subseteq X \), for all \( n \in \mathbb{N} \), and for all \( \omega \in I^n \), we have that
\[
B\left(\varphi^{(n)}_\omega(x), K^{-1} \left\| D\varphi^{(n)}_\omega \right\| r \right) \subseteq \varphi^{(n)}_\omega \left( B(x, r) \right) \subseteq B\left(\varphi^{(n)}_\omega(x), K \left\| D\varphi^{(n)}_\omega \right\| r \right).
\]

For a proof of this fact see, for instance, [3].
We remark that conformality implies the bounded distortion property whenever \( d \geq 2 \). For \( d = 2 \) this follows from Koebe’s distortion theorem [16], and for \( d \geq 3 \) it is a consequence of Liouville’s theorem for conformal maps [17].

Another consequence of ESC, NEQ, and BDP is the following

**Claim 4.1.** For all \( \omega \in I^n \), and all \( n \) large enough, we have that \( B_{\omega} \subset \varphi_n^\omega (X) \).

**Proof.** Notice that the centre of the ball \( B_{\omega} \) is contained in \( \varphi_n^\omega (X) \), by condition NEQ. Now,

\[
\varphi_n^\omega (X) \supset \varphi_n^\omega (B (x^{(n)}, \varepsilon)) \\
\supset B (\varphi_n^\omega (x^{(n)}), K^{-1} \| D \varphi_n^\omega \| \varepsilon) \\
\supset B (\varphi_n^\omega (x^{(n)}), K^{-1} \lambda_n \varepsilon).
\]

Thus, it suffices to show that

\[
K^{-1} \lambda_n \varepsilon \geq e^{-S_n(\beta(\omega x^{(n)}))} \text{ for all } \omega \in I^n.
\]

Given condition ESC notice that the desired inequality holds for all \( n \geq K(h \alpha)^{-1} \).

Recall that a measure \( \mu_h \) is \( h \)-Ahlfors regular if there exists a constant \( C \geq 1 \) such that

\[
C^{-1} \leq \frac{\mu_h (B (x, r))}{r^h} \leq C, \tag{4.5}
\]

for all \( x \in \text{supp} (\mu_h) \) and all \( 0 < r \leq 1 \).

We establish the lower bound of the Hausdorff dimension under different sets of assumptions. For this purpose we appeal to the celebrated Frostmann lemma [18].

**Lemma 4.2 (Frostmann).** Let \( m \) be a Borel probability measure on \( X \). If there exist constants \( C > 0 \) and \( t \geq 0 \) such that for all \( x \in X \) and all \( r > 0 \)

\[
m (B (x, r)) \leq Cr^t,
\]

then \( \text{HD} (\text{supp} (m)) \geq t \).

**Theorem 2.** Let \( \Phi \) be a non-autonomous conformal IFS on a compact, convex set \( X \subseteq \mathbb{R}^d \) with nonempty interior satisfying OSC, ESC, UCC, LVC, and NEQ conditions. Suppose that the sequence \( \left( \frac{\lambda_n}{\lambda_{n+1}} \right) \) is bounded and that there exists an \( h \)-Ahlfors measure, \( \mu_h \), where \( h = \text{HD} (J) \), and \( \text{supp} (\mu_h) = J \). Then \( \text{HD} (\mathcal{D}) = b \).

**Proof.** Recall that \( \text{HD} (\mathcal{D}) \leq b \) has been proven in theorem 1. Let \( 0 < t < b \). Our strategy consists of constructing a measure \( m \) supported on a set \( K \subseteq \mathcal{D} \) satisfying the hypothesis of the Frostmann lemma with exponent \( t \). Choose an increasing sequence \( (n_l) \in \mathbb{N}^n \) such that

\[
\overline{P}_\beta (t) = \lim_{l \to \infty} \frac{1}{h_l} \log \sum_{\omega \in \mathcal{P}^l} e^{-S_{n_l} (\beta (\omega x^{(n_l)}))}.
\]

If necessary, we refine our subsequence so that it satisfies the following inequality for all \( l \):

\[
n_{l+1} \geq \frac{4h}{P (l)} \left( \text{const} + \sum_{k=1}^{l} m_k \right).
\]

Now define \( R_1 = I^{n_1} \). Assuming \( R_l \subseteq I^{n_l} \) has been defined, for every \( \omega \in R_l \) let
Indeed, proves the claim. □

Now we will focus on obtaining a lower bound on the cardinality of the sets \( R_{t+1}(\omega) \). We denote \( B \left( \varphi^n \left( \alpha \right), \frac{1}{2} e^{-S_n(\omega(\eta))} \right) \) by \( \frac{1}{2} B_\omega \).

**Claim 4.3.** Let \( \tau \in P^{n+1} \) and \( \omega \in P^n \). If \( n_{t+1} \geq \theta^{-1} [\log (2) + n_t (\tau) + \theta] \) then either

\[
\varphi^{n_{t+1}}(X) \cap \frac{1}{2} B_\omega = \emptyset
\]

or

\[
\varphi^{n_{t+1}}(X) \subseteq B_\omega.
\]

**Proof of claim.** Assume \( \varphi^{n_{t+1}}(X) \cap \frac{1}{2} B_\omega \neq \emptyset \). It suffices to show that \( 4 \left| \varphi^{n_{t+1}}(X) \right| \leq |B_\omega| \). Indeed,

\[
4 \left| \varphi^{n_{t+1}}(X) \right| \leq |B_\omega| \iff 2 \pi_{n_{t+1}} \leq e^{-S_n(\omega(\eta))}
\]

\[
\iff S_{n_{t+1}}(\omega(\eta)) + \log \pi_{n_{t+1}} \leq -\log 2
\]

\[
\iff S_{n_{t+1}}(\omega(\eta)) + \log \pi_{n_{t+1}} + \sum_{j=n_t+1}^{n_{t+1}} \log \pi(j) \leq -\log 2
\]

\[
\iff n \pi \pi + \sum_{j=n_t+1}^{n_{t+1}} \log \pi(j) \leq -\log 2
\]

\[
\iff n \pi \pi - \sum_{j=n_t+1}^{n_{t+1}} \theta \leq -\log 2
\]

\[
\iff n \pi \pi - (n_{t+1} - n_t) \theta \leq -\log 2
\]

\[
\iff n_{t+1} \geq \theta^{-1} [\log (2) + n_t (\tau) + \theta],
\]

where the 3rd, 4th, and 5th implications follow from (4.1), ESC, and UCC, respectively. This proves the claim. □

From the Ahlfors property of \( \mu_h \) we get that for all \( \omega \in R_t \)

\[
C^{-1} \left( \frac{1}{2} e^{-S_n(\omega(\eta))} \right)^h \leq \mu_h \left( \frac{1}{2} B_\omega \right)
\]

\[
\leq \# \left\{ \tau \in P^{n+1} \mid \varphi^{n_{t+1}}(X) \cap \frac{1}{2} B_\omega \neq \emptyset \right\} \max_{\tau \in P^{n+1}} \mu_h \left( \varphi^{n_{t+1}}(X) \right)
\]

\[
= \# \left\{ \tau \in P^{n+1} \mid \varphi^{n_{t+1}}(X) \subseteq B_\omega \right\} \max_{\tau \in P^{n+1}} \mu_h \left( \varphi^{n_{t+1}}(X) \right)
\]

\[
\leq \# R_{t+1}(\omega) \max_{\tau \in P^{n+1}} \mu_h \left( \varphi^{n_{t+1}}(X) \right)
\]

\[
\leq \# R_{t+1}(\omega) C \kappa_{n_{t+1}}^h,
\]
where the equation above follows from claim 4.3. Therefore, we obtain that

$$
\#R_{l+1}(\omega) \geq C^{-2} \left( e^{-S_{n_l}^\beta(\omega(n_l))} \right)^{\frac{1}{2}}.
$$

By redefining the constant $C$ we will write

$$
\#R_{l+1}(\omega) \geq C^{-1} \left( e^{-S_{n_l}^\beta(\omega(n_l))} \right)^{\frac{1}{2}}. \quad (4.8)
$$

Notice that $R_{l+1}(\omega) \neq \emptyset$ if we choose our subsequence $(n_l)$ to increase rapidly; indeed,

$$
\#R_{l+1}(\omega) \geq 1 \iff C^{-1} \left( e^{-S_{n_l}^\beta(\omega(n_l))} \right)^{\frac{1}{2}} \geq 1
$$

$$
\iff \prod_{k=n_l+1}^{n_l+1} \left( e^{-S_{n_l}^\beta(\omega(n_l))} \right)^{\frac{1}{2}} \geq C^{-1} \left( e^{-S_{n_l}^\beta(\omega(n_l))} \right)^{\frac{1}{2}}
$$

$$
\iff e^{-(n_l+1-n_l)\theta} \leq C^{-1} \left( e^{-S_{n_l}^\beta(\omega(n_l))} \right)^{\frac{1}{2}}
$$

$$
\iff (n_l+1-n_l)\theta \geq \frac{1}{\beta} \log(C) + n_l\alpha
$$

Now for every $\omega \in R_l$ define

$$
m_l(B_\omega) = (\#R_l)^{-1}.
$$

Assuming that $m_l(B_\omega)$ has been defined for every $\omega \in R_l$ we now define for every $\tau \in R_{l+1}(\omega)$

$$
m_{l+1}(B_\tau) = \frac{m_l(B_\tau)}{\#R_{l+1}(\omega)} \left( \#R_{l+1} \right)^{-1}. \quad (4.9)
$$

$$
m_{l+1}(B_\tau) = \left[ \prod_{k=1}^{l} \left( \#R_{k+1}(\omega) \right)^{-1} \right] \left( \#R_l \right)^{-1}. \quad (4.10)
$$

We can extend the functions $m_l$ to a measure on $X$ and let us take a weak limit $m$ of the sequence $(m_l)$. The function $m$ is then a Borel probability measure. Furthermore, notice that $\text{supp}(m) \subset \bigcup_{\omega \in R_l} B_\omega(X)$ for all $l$. This implies that
Now consider $\tau \in R_{t+1}$ and we have that $m(B_{\tau}) = m_{t+1}(B_{\tau})$. Furthermore, from $R_l \neq \emptyset$ it follows that $K \neq \emptyset$.

For $\tau \in R_{t+1}(\omega)$, the inequality (4.8) yields the following estimate for $m(B_{\tau})$

$$m(B_{\tau}) \leq \prod_{k=1}^{l} \left( \frac{\# R_{k+1}(\omega)_{\ell_k}}{\# R_1} \right)^{-1}$$

$$\leq \prod_{k=1}^{l} C \left( e^{-S_{R_k}^{(\beta)}} (\omega)_{\ell_k} \right)^{-h}$$

$$= C \prod_{k=1}^{l} \tau_{R_k}^{h} e^{h \beta} (\omega)_{\ell_k}.$$ 

Now consider $x \in K$ and a number $r$ such that $0 < r < \tau_{R_k}^{h} e^{h \beta} (\omega)_{\ell_k}$ for all $l \in \mathbb{N}$. Let

$$\ell(r) := \min_{l \in \mathbb{N}} \left\{ l \mid \max_{\tau \in R_{l+1}} e^{-S_{R_k}^{(\beta)} (\tau \ell_{r+1})} \leq r \right\},$$

and

$$\# \ell_{(r)+1} := \# \left\{ \tau \in R_{(r)+1} \mid B_{\tau} \cap B(x, r) \neq \emptyset \right\}.$$ 

Since $x \in K \subset \bigcup_{\tau \in R_{(r)+1}} B_{\tau}$ it follows that $|x - \varphi_{\tau}^{(r)+1}(x_{\ell_k})| \leq e^{-S_{R_k}^{(\beta)} (\tau \ell_{r+1})}$ for some $\tau \in R_{(r)+1}$. This implies that $\varphi_{\tau}^{(r)+1}(x_{\ell_k}) \in B(x, r)$ and it follows that $\# \ell_{(r)+1} \geq 1$.

Recall that $m$ is supported on $K \subset \bigcup_{\tau \in R_{(r)+1}} B_{\tau}$ and that $m(B_{\omega}) = m(B_{\omega})$ for all words $\omega \in R_l$ of the same length, so for all $\tau \in R_{(r)+1}$ we have that

$$m(B(x, r)) \leq \# \ell_{(r)+1} \max_{\tau \in R_{(r)+1}} m(B_{\tau})$$

$$\leq \# \ell_{(r)+1} C^{(r)} \prod_{k=1}^{l} \tau_{R_k}^{h} e^{h \beta} (\omega)_{\ell_k}$$

$$= \# \ell_{(r)+1} C^{(r)} \exp \left\{ h \sum_{k=1}^{(r)} \tau_{R_k}^{h} (\omega)_{\ell_k} \right\} \prod_{k=1}^{l} \tau_{R_k}^{h}.$$
We will use the following upper bound for \( \#I(r)_{+1} \).

**Claim 4.4.** \( \#I(r)_{+1} \leq C \left( \frac{r}{\Xi I(r)_{+1}} \right)^h. \)

**Proof of claim.** Notice that if \( B_r \cap B(x, r) \neq \emptyset \) we have that \( B_r \subset B(x, 2r) \) since

\[
e^{-S_{I(r)_{+1}}(\tau_{\xi(n)_{+1}})} \leq r.
\]

From the Ahlfors condition (4.5) and from claim 4.1 we get that

\[
C^h \geq \mu_h(B(x, r)) \geq \# \left\{ \tau \in \mathbb{P}_{I(r)_{+1}} : \varphi_{\tau}^{D_{I(r)_{+1}}}(X) \cap B(x, r) \neq \emptyset \right\} \min_{\tau \in \mathbb{P}_{I(r)_{+1}}} \mu_h \left( \varphi_{\tau}^{D_{I(r)_{+1}}}(X) \right) \geq \# \left\{ \tau \in \mathbb{P}_{I(r)_{+1}} \cap B(x, r) \neq \emptyset \right\} \min_{\tau \in \mathbb{P}_{I(r)_{+1}}} \mu_h \left( \varphi_{\tau}^{D_{I(r)_{+1}}}(X) \right) \geq \#I(r)_{+1} \min_{\tau \in \mathbb{P}_{I(r)_{+1}}} \mu_h \left( \varphi_{\tau}^{D_{I(r)_{+1}}}(X) \right) \geq C^{-1} \#I(r)_{+1} \Xi I(r)_{+1}^h.
\]

The result follows by solving for \( \#I(r)_{+1} \).

From the previous claim we obtain that

\[
m(B(x, r)) \leq C^{\ell(r)} \left( \frac{r}{\Xi I(r)_{+1}} \right)^h \exp \left\{ h \sum_{k=1}^{\ell(r)} S_{n_k} \beta \left( \tau_{n_k} \xi(n_k) \right) \right\} \prod_{k=1}^{\ell(r)-1} \Xi I(r)_{+1}^{h} \leq \text{const} \cdot r^l.
\]

By Frostman’s lemma it is enough to show that there exists \( \tau \in \mathbb{P}_{I(r)_{+1}} \) for which

\[
C^{\ell(r)} \left( \frac{r}{\Xi I(r)_{+1}} \right)^h \exp \left\{ h \sum_{k=1}^{\ell(r)} S_{n_k} \beta \left( \tau_{n_k} \xi(n_k) \right) \right\} \prod_{k=1}^{\ell(r)-1} \Xi I(r)_{+1}^{h} \leq \text{const} \cdot r^l h^{-1}
\]

holds, which is equivalent to showing that

\[
C^{\ell(r)} \left( \frac{r}{\Xi I(r)_{+1}} \right)^h \exp \left\{ \sum_{k=1}^{\ell(r)} S_{n_k} \beta \left( \tau_{n_k} \xi(n_k) \right) \right\} \prod_{k=1}^{\ell(r)-1} \Xi I(r)_{+1}^{h} \leq \text{const} \cdot r^l h^{-1}
\]

holds for some \( \tau \in \mathbb{P}_{I(r)_{+1}} \).

From the definition of \( \ell(r) \) it follows that \( \exp \left\{ -S_{n_0} \beta \left( \tau_{n_0} \xi(n_0) \right) \right\} > r \) for some \( \tau \in \mathbb{P}_{I(r)_{+1}} \). By comparing (1.1) and (2.1) we see that \( t < b \leq \text{HD}(J) = h \), so that \( t^{-1} h < 1 \). Hence, we have that

\[
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\]
\[ \exp \left\{ \left( 1 - \frac{1}{h} \right) S_{n(\ell)\beta} \left( \tau_{|n(\ell)\xi(n(\ell))} \right) \right\} < \ell^{h-1}, \]

for some \( \tau \in R_{n(\ell)+1} \). So it suffices to show that

\[ C^{(\ell)} \left( \frac{\mathcal{R}_{n(\ell)+1}}{\mathcal{E}_{n(\ell)+1}} \right) \exp \left\{ \sum_{k=1}^{\ell(\ell)} S_{\beta_k} \left( \tau_{|n(\ell)\xi(n(\ell))} \right) \right\} \prod_{k=1}^{\ell(\ell)-1} \mathcal{F}_{n_k+1} \]

\[ \leq \text{const} \cdot \exp \left\{ \left( 1 - \frac{1}{h} \right) S_{n(\ell)\beta} \left( \tau_{|n(\ell)\xi(n(\ell))} \right) \right\} \]

for some \( \tau \in R_{n(\ell)+1} \), which is equivalent to showing that

\[ C^{(\ell)} \left( \frac{\mathcal{R}_{n(\ell)+1}}{\mathcal{E}_{n(\ell)+1}} \right) \exp \left\{ \sum_{k=1}^{\ell(\ell)-1} S_{\beta_k} \left( \tau_{|n(\ell)\xi(n(\ell))} \right) \right\} \prod_{k=2}^{\ell(\ell)} \mathcal{F}_{n_k} \]

\[ \leq \text{const} \cdot \exp \left\{ -\frac{1}{h} S_{n(\ell)\beta} \left( \tau_{|n(\ell)\xi(n(\ell))} \right) \right\} \]

holds for some \( \tau \in R_{n(\ell)+1} \).

Since \( \mathcal{F}_{\beta} (t) > 0 \) we have (by choosing \( n_1 \) large enough if necessary) that

\[ \frac{1}{n(\ell)} \log \sum_{\omega \in \mathcal{F}(\ell)} \exp \left\{ -t S_{n(\ell)\beta} \left( \omega \xi(n(\ell)) \right) \right\} \geq \frac{3}{4} \mathcal{F}_{\beta} (t), \]

which implies that

\[ \sum_{\omega \in \mathcal{F}(\ell)} \exp \left\{ -t S_{n(\ell)\beta} \left( \omega \xi(n(\ell)) \right) \right\} \geq \exp \left\{ \frac{n(\ell)}{2} \mathcal{F}_{\beta} (t) \right\}. \]

By defining \( n_1 \) to be large enough if necessary it follows from inequality (4.4) that for any \( \tau \in R_{n(\ell)+1} \)

\[ \# \mathcal{P}(\ell) \exp \left\{ -t S_{n(\ell)\beta} \left( \tau_{|n(\ell)\xi(n(\ell))} + t \in \mathcal{P}(\ell) \right) \right\} \geq \sum_{\omega \in \mathcal{F}(\ell)} \exp \left\{ -t S_{n(\ell)\beta} \left( \omega \xi(n(\ell)) \right) \right\}. \]

Combining the last two inequalities we get that it suffices to show that

\[ \exp \left\{ \frac{1}{h} S_{\beta} \left( \tau_{|n(\ell)\xi(n(\ell))} \right) \right\} \geq (\# \mathcal{P}(\ell))^{-1/h} \exp \left\{ -\frac{1}{h} n(\ell) \right\} \exp \left\{ \frac{3n(\ell) \mathcal{F}_{\beta} (t)}{4h} \right\}. \]

This estimate yields the further sufficient condition

\[ C^{(\ell)} \left( \frac{\mathcal{R}_{n(\ell)+1}}{\mathcal{E}_{n(\ell)+1}} \right) \exp \left\{ \sum_{k=1}^{\ell(\ell)-1} S_{\beta_k} \right\} \prod_{k=2}^{\ell(\ell)} \mathcal{F}_{n_k} \leq \text{const} \cdot \exp \left\{ \frac{3n(\ell) \mathcal{F}_{\beta} (t)}{4h} \right\} \exp \left\{ -\frac{1}{h} n(\ell) \right\} \left( \# \mathcal{P}(\ell) \right)^{-1/h} \]

for some \( \tau \in R_{n(\ell)+1} \).
If we choose \( \varepsilon \) such that \( 0 < \varepsilon < \frac{\pi(1)}{4\eta} \) then it suffices to show that

\[
C^{(r)} \left( \frac{\overline{n}_{n_{(r)+1}}}{E_{n_{(r)+1}}} \right) \exp \left\{ \sum_{k=1}^{(r)-1} S_{n_k} \beta \left( \tau_{n_k} \xi(n_k) \right) \right\} \prod_{k=2}^{(r)-1} \overline{\tau}_{n_k} \leq \text{const} \cdot \exp \left\{ \frac{n^{(r)} \overline{\tau}_{\beta} (t)}{2h} \right\} \left( \# F^{(r)} \right)^{-1/h}
\]

for some \( \tau \in R_{(r)+1} \).

Now, since \( \text{supp} (\mu_h) = J \subseteq \bigcup_{\omega \in F^{(r)}} \phi_{\omega} (X) \) we have that

\[
1 = \sum_{\omega \in F^{(r)}} \mu_h (\phi_{\omega} (X)) \geq C^{-1} (\# F^{(r)}) \sum_{n_{(r)+1}}^{h} \overline{\tau}_{n_{(r)+1}}
\]

which yields the inequality

\[
(\# F^{(r)})^{-1/h} \leq C^{-1/h} \sum_{n_{(r)+1}}^{h} \overline{\tau}_{n_{(r)+1}}.
\]

Hence, it is enough to show that for some \( \tau \in R_{(r)+1} \)

\[
C^{(r)} \left( \frac{\overline{n}_{n_{(r)+1}}}{E_{n_{(r)+1}}} \right) \left( \frac{\overline{n}_{n_{(r)}}}{E_{n_{(r)}}} \right) \exp \left\{ \sum_{k=1}^{(r)-1} S_{n_k} \beta \left( \tau_{n_k} \xi(n_k) \right) \right\} \prod_{k=2}^{(r)-1} \overline{\tau}_{n_k} \leq \text{const} \cdot \exp \left\{ \frac{n^{(r)} \overline{\tau}_{\beta} (t)}{2h} \right\} \left( \# F^{(r)} \right)^{-1/h}.
\]

(4.11)

Since the sequence \( \overline{n}_{n_{(r)}} \) is bounded, this inequality follows by showing

\[
C^{(r)} \exp \left\{ \sum_{k=1}^{(r)-1} S_{n_k} \beta \left( \tau_{n_k} \xi(n_k) \right) \right\} \prod_{k=2}^{(r)-1} \overline{\tau}_{n_k} \leq \text{const} \cdot \exp \left\{ \frac{n^{(r)} \overline{\tau}_{\beta} (t)}{2h} \right\} \left( \# F^{(r)} \right)^{-1/h}.
\]

Furthermore, it is enough to show that

\[
\exp \left\{ \sum_{k=1}^{(r)-1} S_{n_k} \beta \left( \tau_{n_k} \xi(n_k) \right) \right\} \prod_{k=2}^{(r)-1} \overline{\tau}_{n_k} \leq \text{const} \cdot \exp \left\{ \frac{n^{(r)} \overline{\tau}_{\beta} (t)}{2h} \right\} \left( \# F^{(r)} \right)^{-1/h}.
\]

for some \( \tau \in R_{(r)+1} \) and that

\[
C^{(r)} \leq \exp \left\{ \frac{n^{(r)} \overline{\tau}_{\beta} (t)}{2h} \right\}.
\]

The first inequality is satisfied given condition (4.7). The second inequality is satisfied by choosing our rapidly increasing sequence \( (n_l) \) to satisfy \( n_l \gg l \). This completes the proof. \( \Box \)

**Theorem 3.** Let \( \Phi \) be a conformal non-autonomous IFS satisfying the OSC, ESC, UCC, and NEQ conditions. If there exists an \( h \)-Ahlfors measure supported on \( J \), \( \overline{\tau}_{\beta} (t) = \overline{P}_{\beta} (t) \) on a neighborhood of \( b \), and
\[ \lim_{n \to \infty} \frac{1}{n} \log \frac{\pi_n^{\beta}}{\pi_n} = 0, \]  
(4.12)

then \( \text{HD}(\mathcal{D}) = b \).

**Proof.** As before, we choose \( 0 \leq t < b \) in the neighborhood of \( b \) where \( P_\beta \) exists. It suffices to show that inequality (4.11) holds. This will follow from showing that the following three inequalities hold for some \( \tau \in R_{(\beta)+1}^\beta \):

\[ C^{(\beta)} \exp \left\{ \sum_{k=1}^{n_{(\beta)}-1} S_{n_k} \beta \left( \tau|n_k \xi(n_k) \right) \right\} \prod_{k=2}^{n_{(\beta)}-1} \pi_{n_k} \leq \text{const} \cdot \exp \left\{ \frac{n_{(\beta)} \beta}{6h} \right\}, \]  
(4.13)

\[ \frac{\pi_{n_{(\beta)}+1}}{\pi_{n_{(\beta)}}} \leq \exp \left\{ \frac{n_{(\beta)}}{6h} P_\beta (t) \right\}, \]

and

\[ \frac{\pi_{n_{(\beta)}+1}}{\pi_{n_{(\beta)}}} \leq \exp \left\{ \frac{n_{(\beta)}}{6h} P_\beta (t) \right\}. \]

The second inequality is equivalent to the inequality

\[ \frac{1}{n_{(\beta)}} \log \frac{\pi_{n_{(\beta)}}}{\pi_{n_{(\beta)+1}}} \leq \frac{P_\beta (t)}{6h}, \]

which is satisfied simply by choosing \( n_1 \) large enough. This can be achieved without loss of generality since \( P_\beta (t) > 0 \) and by assumption (4.12).

To prove the third inequality first we note that it is equivalent to

\[ \frac{n_{(\beta)+1}}{n_{(\beta)}} \frac{1}{n_{(\beta)+1}} \log \frac{\pi_{n_{(\beta)+1}}}{\pi_{n_{(\beta)}}} \leq \frac{P_\beta (t)}{6h}. \]

Let

\[ 0 < A \leq \frac{P_\beta (t)}{6h} \alpha (\log C + \alpha)^{-1}, \]

where \( C \) is the same constant as in (4.5). Since \( \overline{P}_\beta (t) = P_\beta (t) \) we have that (4.6) holds for every increasing sequence \( (n_l) \). Consider in particular an increasing sequence with the property

\[ n_{l+1} = \min \{ n \in \mathbb{N} | nA \geq \alpha (n_1 + \cdots + n_l) \}, \]  
(4.14)

for all \( l \in \mathbb{N} \).

Such a sequence satisfies the following claim.

**Claim 4.5.** The following inequality holds:

\[ A^{-1} \alpha \leq \frac{n_{l+1}}{n_l} \leq A^{-1} \alpha + 2. \]

**Proof.** Condition (4.14) implies that

\[ A \left( n_{l+1} - 1 \right) \leq \alpha (n_1 + \cdots + n_l) \leq An_{l+1}. \]
Therefore,
\[ A \left( n_{l+1} - n_l - 1 \right) \leq \alpha n_l \leq A \left( n_{l+1} - n_l + 1 \right) \]

Re-arranging terms algebraically we get
\[
\alpha n_l A^{-1} - 1 \leq n_{l+1} - n_l \leq \alpha n_l A^{-1} + 1,
\]
\[
n_l + \frac{\alpha n_l A^{-1}}{n_l} - 1 \leq n_{l+1} \leq n_l + \frac{\alpha n_l A^{-1}}{n_l} + 1,
\]
\[
\alpha A^{-1} + 1 - \frac{1}{n_l} \leq \frac{n_{l+1}}{n_l} \leq \alpha A^{-1} + 1 + \frac{1}{n_l}.
\]
\[ \alpha A^{-1} \leq \frac{n_{l+1}}{n_l} \leq \alpha A^{-1} + 2. \]

This proves the claim. \(\square\)

Since our sequence is chosen so that \(\frac{n_{l+1}}{n_l}\) is uniformly bounded, the desired inequality
\[
\frac{n_{(l+1)(r)}}{n_{(l)(r)} + 1} \log \frac{P_{\beta}(t)}{6h} \leq \frac{n_{(l)(r)}}{6h} P_{\beta}(t)
\]
follows again by choosing \(n_1\) large enough.

The remaining inequality
\[
C^{(r)} \exp \left\{ \sum_{k=1}^{(r)-1} S_{a_k} \beta \left( \tau | a_k \xi^{(a_k)} \right) \right\} \prod_{k=2}^{(r)-1} \pi_{a_k} \leq \exp \left\{ \frac{n_{(l)(r)}}{6h} P_{\beta}(t) \right\}
\]
is equivalent to showing
\[
\ell (r) \log (C) + \sum_{k=1}^{(r)-1} \left( S_{a_k} \beta \left( \tau | a_k \xi^{(a_k)} \right) + \log \pi_{a_k} \right) \leq \const + \frac{n_{(l)(r)}}{6h} P_{\beta}(t).
\]

Given ESC, it suffices to show
\[
\ell (r) \log (C) + \sum_{k=1}^{(r)-1} n_k \bar{\alpha} \leq \const + \frac{n_{(l)(r)}}{6h} P_{\beta}(t).
\]

Since \(\const > 0\), this inequality follows from showing
\[
n_{(l)(r)} \geq \frac{6h}{P_{\beta}(t)} \left[ \ell (r) \log (C) + \sum_{k=1}^{(r)-1} n_k \bar{\alpha} \right]. \tag{4.15}
\]

In view of claim 4.5, we have that \(n_l \geq A^{-1} \alpha (n_1 + \cdots + n_{l-1})\) for all \(l\). Now, condition (4.15) holds if
\[
A^{-1} \alpha \sum_{k=1}^{l-1} n_k \geq \frac{6h}{P_{\beta}(t)} \left[ l \cdot \log (C) + \bar{\alpha} \sum_{k=1}^{l-1} n_k \right]
\]
or, re-arranging terms, if
\[ A^{-1} \geq \frac{6h}{P_\gamma(t)} \left[ \log \left( \frac{C + \alpha}{\alpha} \right) \right]. \]

Since the sequence \((n_l)\) is increasing and assuming without loss of generality that \(n_1 \geq 2\), we have that \(l \leq n_1 + \cdots + n_{l-1}\) for all \(l\). Hence, it suffices to show that

\[ A^{-1} \geq \frac{6h}{P_\gamma(t)} \left[ \log \left( \frac{C + \alpha}{\alpha} \right) \right]. \]

This follows from our choice of \(A\) above.

Since all three inequalities in (4.13) hold, this completes the proof. \(\square\)

5. Ahlfors measures

Now we focus our attention on establishing sufficient conditions for the existence of an \(h\)-Ahlfors measure. Let us define for every \(n \in \mathbb{N}\),

\[ \rho_n = \max_{a,b \in I_n} \frac{\|D_{\phi_a}^{(n)}\|}{\|D_{\phi_b}^{(n)}\|}, \]

and

\[ Z_n(t) = \sum_{\omega \in I_n} \|D_{\phi_\omega}^{(n)}\|^t. \]

Following the analysis in [14] we obtain the following result.

**Theorem 5.1.** If the sequences \((\#I_n)_{n \geq 1}, (\rho_n)_{n \geq 1}, (Z_n(h))_{n \geq 1}\), and \((Z_n^{-1}(h))_{n \geq 1}\) are bounded, then there exists an \(h\)-Ahlfors measure supported on \(J\).

**Proof.** In the proof of theorem 3.2 in [14] the authors construct a measure \(\mu\) on \(J\) for which

\[ \mu(B(x,r)) \leq Cr^t \]

holds for every \(x \in X\) and \(r > 0\) and for every \(t \geq 0\) satisfying

\[ \liminf_{n \to \infty} Z_n^{-1}(t) \cdot (\#I_n)^{t/d} \cdot \frac{\min_{a \in I_n} \|D_{\phi_a}^{(n)}\|^t}{1 + \log \left( \max_{j \leq n} \rho_j \right) \min_{a \in I_n} \|D_{\phi_a}^{(n)}\|^t} > 0. \]

We claim that the measure \(\mu\) is \(h\)-Ahlfors. In order to prove the upper bound in the Ahlfors condition it suffices to show that the limit inferior above is positive for \(t = h\).

Let \(B\) be a bound for all the sequences in the hypothesis of the theorem. Since \(\#I_n \geq 2\) and \(\rho_n \leq B\), it suffices to show that

\[ \liminf_{n \to \infty} Z_n^{-1}(h) \min_{a \in I_n} \|D_{\phi_a}^{(n)}\|^h > 0. \]

Note that since the sequence \((Z_n^{-1}(h))_{n \geq 1}\) is bounded above by \(B\) we have that the sequence \((Z_n(h))_{n \geq 1}\) is bounded below by \(B^{-1} > 0\).

So it suffices to show the following...
Claim 5.2. The sequence
\[
\left( \min_{a \in \mathcal{F}(n)} \| D\varphi^{(n)}(a) \|^h \right)_{n \geq 1}
\]
is bounded below by a positive number.

Proof. Note that
\[
B^{-1} \leq Z_{n+1} (h)
\]
\[
= \sum_{\tau \in \mathcal{F}_{n+1}} \| D\varphi^{(n+1)}(\tau) \|^h
\]
\[
= \sum_{\omega \in \mathcal{I}^n} \sum_{a \in \mathcal{F}(a+1)} \| D\varphi^{(n+1)}\omega_a \|^h
\]
\[
\leq \sum_{\omega \in \mathcal{I}^n} \sum_{a \in \mathcal{F}(a+1)} \| D\varphi^{(n)}\omega \|^h \| D\varphi^{(a+1)}(a) \|^h
\]
\[
= Z_n (h) \sum_{a \in \mathcal{F}(n+1)} \| D\varphi^{(n+1)}(a) \|^h
\]
\[
\leq Z_n (h) \left( \# I_{n+1} \right) \max_{a \in \mathcal{F}(a+1)} \| D\varphi^{(n+1)}(a) \|^h
\]
\[
= Z_n (h) \left( \# I_{n+1} \right) \rho_{n+1} \min_{a \in \mathcal{F}(a+1)} \| D\varphi^{(n+1)}(a) \|^h
\]
Since the product \(Z_n (h) \left( \# I_{n+1} \right) \rho_{n+1}\) is uniformly bounded above, the claim follows. □

To prove that \(\mu(\mathcal{B}(x, r)) \geq C^{-1} r^t\) we shall now consider an arbitrary \(0 \leq r < \text{diam } (J)\) and \(x \in J\). Note that
\[
x \in \bigcap_{n \geq 1} \varphi^n_{\xi_n}(X),
\]
for some \(\xi \in I^{\infty}\). Define
\[
n = \max \left\{ k \in \mathbb{N} \mid \text{diam } \varphi^k_{\xi_k}(X) \geq r \right\}.
\]
It follows that
\[
\mu(\mathcal{B}(x, r)) \geq \mu \left( \varphi^{n+1}_{\xi_{n+1}}(X) \right)
\]
since \(\varphi^{n+1}_{\xi_{n+1}}(X) \subseteq \mathcal{B}(x, r)\).

We make the following

Claim 5.3. For all \(n \in \mathbb{N}\) and every \(\omega \in I^n\) the measure \(\mu\) satisfies
\[
\mu \left( \varphi^n_{\omega}(X) \right) \geq K^{-1} Z_n^{-1} (h) \| D\varphi^{n}_{\omega} \|^h,
\]
where \(K \geq 1\) is the distortion constant.
Proof. In [14] the measure $\mu$ is constructed as a weak limit of a sequence $(\mu^{(n)})_{n \geq 1}$ of measures where

$$
\mu^{(n)}(\varphi_\omega^n(X)) = \frac{\|D\varphi_\omega^n\|^h}{Z_n(h)}
$$

for all $\omega \in \mathcal{I}$. Now, for every $q \in \mathbb{N}$ and every $\omega \in \mathcal{I}$ we have that

$$
\mu^{(n+q)}(\varphi_\omega^{n+q}(X)) = \mu^{(n+q)}(\varphi_\omega^{n+q}(X) \cap \text{supp } (\mu^{(n+q)}))
$$

$$
= \mu^{(n+q)}\left(\bigcup_{\gamma \in \mathcal{I}(n+1, n+q)} \varphi_\omega^{n+q}(X)\right)
$$

$$
= \sum_{\gamma \in \mathcal{I}(n+1, n+q)} \mu^{(n+q)}(\varphi_\omega^{n+q}(X))
$$

$$
= Z_{n+q}^{-1}(h) \sum_{\gamma \in \mathcal{I}(n+1, n+q)} \|D\varphi_\omega^{n+q}\|^h
$$

$$
\geq Z_{n+q}^{-1}(h) \sum_{\gamma \in \mathcal{I}(n+1, n+q)} K^{-h} \|D\varphi_\omega^n\|^h \|D\varphi_\omega^{n+1, n+q}\|^h,
$$

where the last inequality follows from the BDP.

Furthermore, the inequality

$$
Z_{n+q}^{-1}(h) \sum_{\gamma \in \mathcal{I}(n+1, n+q)} \|D\varphi_\omega^{n+1, n+q}\|^h \geq Z_n^{-1}(h)
$$

follows from noting that

$$
Z_{n+q}(h) = \sum_{\tau \in \mathcal{I}(n+q)} \|D\varphi_\omega^{n+q}\|^h
$$

$$
= \sum_{\omega \in \mathcal{I}(n+q)} \sum_{\gamma \in \mathcal{I}(n+1, n+q)} \|D\varphi_\omega^{n+q}\|^h
$$

$$
\leq \sum_{\omega \in \mathcal{I}(n+q)} \sum_{\gamma \in \mathcal{I}(n+1, n+q)} \|D\varphi_\omega^n\|^h \|D\varphi_\omega^{n+1, n+q}\|^h
$$

$$
= Z_n(h) \sum_{\gamma \in \mathcal{I}(n+1, n+q)} \|D\varphi_\omega^{n+1, n+q}\|^h.
$$

This proves that

$$
\mu^{(n+q)}(\varphi_\omega^n(X)) \geq Z_{n+q}^{-1}(h) K^{-h} \|D\varphi_\omega^n\|^h
$$

for all $q \in \mathbb{N}$. Taking the limit as $q \to \infty$ proves the claim. \qed
From the claim above it follows now that
\[
\mu(B(x,r)) \geq \mu \left( \tilde{\varphi}^{n+1}_{\xi \mid_{n+1}}(X) \right) \\
\geq C^{-1} Z_{n+1}^{-1}(h) \|D\tilde{\varphi}^{n+1}_{\xi \mid_{n+1}}\|^h \\
\geq C^{-1} Z_{n+1}^{-1}(h) K^{-\beta} \|D\varphi^n_{\xi \mid_n}\|^\beta \|D\varphi^{(n+1)}_{\xi \mid_{n+1}}\|^h,
\]
where the last inequality follows from BDP.

By the mean value inequality we have that
\[
\|D\varphi^n_{\xi \mid_n}\| \text{diam } (X) \geq \text{diam } \left( \tilde{\varphi}^{n}_{\xi \mid_n}(X) \right) \geq r.
\]
Redefining \( C \) we obtain that
\[
\mu(B(x,r)) \geq C^{-1} Z_{n+1}^{-1}(h) r^\beta \|D\tilde{\varphi}^{(n+1)}_{\xi \mid_{n+1}}\|^h.
\]

From the hypothesis and claim 5.2 the product \( Z_{n+1}^{-1}(h) \|D\tilde{\varphi}^{(n+1)}_{\xi \mid_{n+1}}\|^h \) is uniformly bounded below by a positive number. This allows us to redefine \( C \), independent of \( x \) and \( r \), to obtain
\[
\mu(B(x,r)) \geq C^{-1} r^h,
\]
as desired. \( \square \)

6. Perturbations of linear systems in one dimension

Let \( X = [0, 1] \) and consider a piecewise linear non-autonomous IFS \( \Phi = \{ \varphi^n_{\epsilon} \}_{\epsilon \in \mathbb{N}, e \in \mathcal{F}(\epsilon)} \). Now consider a nonlinear perturbative system \( \Phi^\epsilon = \{ \varphi^n_{\epsilon} \}_{\epsilon \in \mathbb{N}, e \in \mathcal{F}(\epsilon)} \) satisfying
\[
\varphi^n_{\epsilon}(x) = \varphi^0_{\epsilon}(u^n_{\epsilon}) + \left( \varphi^0_{\epsilon} \right)' \int_{u^n_{\epsilon}}^x (1 + \gamma^n_{\epsilon}(t)) \, dt,
\]
where \( u^n_{\epsilon} \in [0, 1] \) and \( \gamma^n_{\epsilon} : [0, 1] \to (-\epsilon_\epsilon, \epsilon_\epsilon) \) is Hölder continuous and \( \epsilon_\epsilon > 0 \) is independent of \( \epsilon \in \mathcal{F}(\epsilon) \). Our goal is to establish sufficient conditions on \( \gamma^n_{\epsilon} \) for which the system \( \{ \varphi \} \) satisfies the hypothesis of theorems 2 or 3.

Observe that
\[
\left| \left( \varphi^n_{\epsilon} \right)'(x) \right| = \left| \left( \varphi^0_{\epsilon} \right)' \right| \left[ 1 + \gamma^n_{\epsilon}(x) \right],
\]
and
\[
\left| \left( \varphi^n_{\epsilon} \right)'(x) \right| = \prod_{k=1}^n \left| \left( \varphi^0_{\epsilon} \right)' \right| \left( \varphi^{(k+1)\epsilon}_{\omega}(x) \right) \\
= \left| \varphi^n_{\epsilon} \right| \prod_{k=1}^n \left[ 1 + \gamma^n_{\epsilon}(x) \right] \left( \varphi^{(k+1)\epsilon}_{\omega}(x) \right) \\
\leq \kappa \prod_{k=1}^n \left[ 1 + \gamma^n_{\epsilon}(x) \right] \left( \varphi^{(k+1)\epsilon}_{\omega}(x) \right).
\]
Now define
\[ \bar{\gamma}^{(n)} = \max_{x \in I^{(n)} \cap [0, 1]} \sup \{ |\gamma^{(n)}_n(x)| \} . \]

Then we have that for all \( x \in [0, 1] \)
\[ \mathcal{L}^{n} \prod_{k=1}^{n} \left[ 1 - \bar{\gamma}^{(k)} \right] \leq \left| \left( \tilde{\varphi}^{(n)}_j \right)' (x) \right| \leq \mathcal{R}^{n} \prod_{k=1}^{n} \left[ 1 + \bar{\gamma}^{(k)} \right] . \]

Now we impose some conditions on \( \epsilon \) that will guarantee \( \{ \tilde{\varphi} \} \) to satisfy the OSC. Let \( g^{(n)} \) be the size of the smallest ‘gap’ between images under the unperturbed system \( \Phi \) at level \( n \), i.e.,
\[ g^{(n)} := \min \left\{ |\varphi^{(n)}_j(x) - \varphi^{(n)}_i(y)| : x, y \in [0, 1] ; j, i \in I^{(n)}, j \neq i \right\} . \]

We will assume \( \Phi \) has the strong separation condition, i.e., that \( g^{(n)} > 0 \) for all \( n \).

**Lemma 6.1.** If \( 0 < \epsilon_n < \frac{g^{(n)}}{\mathcal{R}^{(n)}} \) for all \( n \), then \( \hat{\Phi} \) has the strong separation condition.

**Proof.** Observe that
\[ \left| \tilde{\varphi}^{(n)}_j(x) - \varphi^{(n)}_j(x) \right| = \left| \varphi^{(n)}_j(u^{(n)}_j) + \left( \varphi^{(n)}_j \right)' \int_{u^{(n)}_j}^{x} \left( 1 + \gamma^{(n)}_j(t) \right) \, dt \right| \]
\[ - \varphi^{(n)}_j(u^{(n)}_j) - \int_{u^{(n)}_j}^{x} \left( \varphi^{(n)}_j \right)' (t) \, dt \]
\[ \leq \left| \left( \varphi^{(n)}_j \right)' (1 + \epsilon_n) \right| \left| \left( \varphi^{(n)}_j \right)' \left( x - u^{(n)}_j \right) \right| \]
\[ \leq \epsilon_n \left| \varphi^{(n)}_j \right| \]
\[ \leq \epsilon_n g^{(n)} \]
\[ < \frac{g^{(n)}}{2} . \]

Note that the right hand side is independent of \( j \in I^{(n)} \). Now, it is an elementary fact in analysis that \( |a + b + c| \geq |a| - |b| - |c| \) for all \( a, b, c \in \mathbb{R} \). Using this inequality we show that
\[ \left| \tilde{\varphi}^{(n)}_j(x) - \tilde{\varphi}^{(n)}_i(y) \right| \geq \left| \varphi^{(n)}_j(x) - \varphi^{(n)}_i(y) \right| \]
\[ > \frac{g^{(n)}}{2} . \]

for all \( j \neq i \) and all \( x, y \in [0, 1] \). \( \square \)
Furthermore, define
\[
\overline{\kappa}_n = \max_{e \in I(n)} \sup_{x \in X} \left| \left( \varphi_n^{(e)} \right)'(x) \right| \\
\underline{\kappa}_n = \min_{e \in I(n)} \inf_{x \in X} \left| \left( \varphi_n^{(e)} \right)'(x) \right| \\
\overline{\kappa}_n = \max_{\omega \in I(n)} \sup_{x \in X} \left| \left( \varphi_n^{(\omega)} \right)'(x) \right| \\
\underline{\kappa}_n = \min_{\omega \in I(n)} \inf_{x \in X} \left| \left( \varphi_n^{(\omega)} \right)'(x) \right|.
\]

Observe that
\[
\prod_{j=1}^{n} \overline{\kappa}_j \leq \overline{\kappa}_n \leq \underline{\kappa}_n \leq \prod_{j=1}^{n} \underline{\kappa}_j.
\]

Now,
\[
\frac{\overline{\kappa}_n}{\underline{\kappa}_n} \leq \prod_{k=1}^{n} \frac{1 + \gamma^{(k)}}{1 - \gamma^{(k)}}.
\]

Since \( |\gamma| \leq 1 - \epsilon \) it follows that \( \overline{\kappa}_n > 0 \) for all \( n \). From this estimate we see that the sequence \( \left( \frac{\overline{\kappa}_n}{\underline{\kappa}_n} \right) \) is bounded if \( \left( \frac{\underline{\kappa}_n}{\overline{\kappa}_n} \right) \) is bounded and if \( \sup_{n \geq 1} \prod_{k=1}^{n} \frac{1 + \gamma^{(k)}}{1 - \gamma^{(k)}} < \infty \). Note that
\[
\sup_{n \geq 1} \prod_{k=1}^{n} \frac{1 + \gamma^{(k)}}{1 - \gamma^{(k)}} < \infty \iff \sum_{k \geq 1} \log \left( \frac{1 + \gamma^{(k)}}{1 - \gamma^{(k)}} \right) < \infty \iff \sum_{k \geq 1} \gamma^{(k)} < \infty,
\]
where the last step follows from the limit comparison test in calculus. Hence, we have the following

**Theorem 6.2.** Suppose that \( \Phi \) is a non-autonomous IFS of linear functions satisfying the hypothesis of theorem 2 and that \( (\gamma^{(n)}(x))_{e \in I(n), x \in X} \) is a sequence of Hölder-continuous functions from \([0, 1]\) into \((-\epsilon_n, \epsilon_n)\) for some \( \epsilon_n \in (0, 1) \). Furthermore, let \( \tilde{\Phi} \) be a nonlinear perturbation of \( \Phi \) defined as
\[
\tilde{\varphi}_n^{(e)}(x) = \varphi_n^{(e)}(u_n^{(e)}(x)) + \left( \varphi_n^{(e)} \right)' \int_{u_n^{(e)}}^{x} \left( 1 + \gamma^{(n)}(t) \right) dt.
\]

If
\[
\sum_{k \geq 1} \gamma^{(k)} < \infty,
\]
and either
\[(11a) \tilde{\varphi}_n^{(e)}([0, 1]) \subset \varphi_n^{(e)}([0, 1]), \quad \text{or} \]
\[(11b) 0 < \epsilon_n < \frac{\min_{x \in X} u_n^{(e)}(0)}{2 \epsilon_n},\]
then \( \tilde{\Phi} \) satisfies the hypothesis of theorem 2.
We wish to formulate a similar theorem for perturbed systems corresponding to theorem 3. If we now assume that the ESC and (4.12) hold, then we see that

\[ 0 \leq \frac{1}{n} \log \frac{\eta_n}{\kappa_n} \leq \frac{1}{n} \log \frac{\eta_n}{\kappa_n} + \frac{1}{n} \log \prod_{k=1}^{n} \left( 1 + \gamma_k^{(k)} \right) - \frac{1}{n} \log \prod_{k=1}^{n} \left( 1 - \gamma_k^{(k)} \right) \]

\[ \sim \frac{1}{n} \log \frac{\eta_n}{\kappa_n} + \sum_{k=1}^{n} \gamma_k^{(k)}. \]

From (4.12) it suffices to have \( \frac{1}{n} \sum_{k=1}^{n} \gamma_k^{(k)} \to 0 \), which holds whenever

\[ \lim_{k \to \infty} \gamma_k^{(k)} = 0. \]

**Theorem 6.3.** Suppose that \( \Phi \) is a non-autonomous IFS of linear functions satisfying the hypothesis of theorem 3 and that \( \left( \gamma_{(n)}^{(n)}(x) \right)_{n \in \mathbb{N}, x \in f(n)} \) is a sequence of Hölder-continuous functions from \([0, 1]\) into \((-\epsilon_n, \epsilon_n)\) for some \( \epsilon_n \in (0, 1) \). Furthermore, let \( \Phi \) be a nonlinear perturbation of \( \Phi \) defined as

\[ \tilde{\varphi}_{e}^{(n)}(x) = \varphi_{e}^{(n)}(u_{e}^{(n)}) + \left( \left( \varphi_{e}^{(n)} \right)' \right) \int_{u_{e}^{(n)}}^{x} \left( 1 + \gamma_{e}^{(n)}(t) \right) dt. \]

If

\[ \frac{1}{n} \sum_{k=1}^{n} \gamma_k^{(k)} \to 0, \]

in particular, if \( \lim_{k \to \infty} \gamma_k^{(k)} = 0 \), and either

(13a) \( \tilde{\varphi}_{e}^{(n)}(\Delta) \subset \varphi_{e}^{(n)}(\Delta) \), or
(13b) for all \( n \), \( \gamma_{e}^{(n)}(\Delta) \subset (-\epsilon_n, \epsilon_n) \) and \( 0 < \epsilon_n < \tfrac{\epsilon_{\alpha}}{2 \gamma_{e}^{(n)}} \),

then \( \Phi \) satisfies the hypothesis of theorem 3.

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