Maximal rank root subsystems of hyperbolic root systems.

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Abstract. A Kac-Moody algebra is called hyperbolic if it corresponds to a generalized Cartan matrix of hyperbolic type. We study root subsystems of root systems of hyperbolic algebras. In this paper, we classify maximal rank regular hyperbolic subalgebras of hyperbolic Kac-Moody algebras.

Introduction

A generalized Cartan matrix $A$ is called a matrix of hyperbolic type if it is indecomposable symmetrizable of indefinite type, and if any proper principal submatrix of the corresponding symmetric matrix $B$ is of finite or affine type. In this case $B$ is of the signature $(n, 1)$.

Consider a generalized Cartan matrix $A$ of hyperbolic type. Following Kac [5], we can construct a Kac-Moody algebra $\mathfrak{g}(A)$. According to Vinberg [8], the Weyl group of the root system $\Delta(A)$ is a Coxeter group. A fundamental chamber of the Weyl group is an $n$-dimensional hyperbolic Coxeter simplex of finite volume, whose dihedral angles are in the set $\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}\}$ (zero angle can also appear if $n = 2$).

In analogy with the finite-dimensional theory (see [1]), we say a subalgebra $\mathfrak{g}_1 \subset \mathfrak{g}(A)$ to be regular if $\mathfrak{g}_1$ is invariant with respect to some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$. In other words, $\mathfrak{g}_1 \subset \mathfrak{g}(A)$ is regular if it has a basis composed of some elements of $\mathfrak{h}$ and some root vectors of $\mathfrak{g}(A)$ (with respect to $\mathfrak{h}$). We are interested in maximal rank regular subalgebras that can be constructed as Kac-Moody algebras $\mathfrak{g}_1(A_1)$ for some generalized Cartan matrix $A_1$ of hyperbolic type.

Any subalgebra of this type of the Kac-Moody algebra $\mathfrak{g}(A)$ has a root system $\Delta_1(A_1) \subset \Delta(A)$ such that

$$(*) \quad \text{if } \alpha, \beta \in \Delta_1 \text{ and } \alpha + \beta \in \Delta, \text{ then } \alpha + \beta \in \Delta_1$$

Conversely, suppose we have a hyperbolic root system $\Delta_1$ in a hyperbolic root system $\Delta(A)$, and $(*)$ holds. Then we can construct a subalgebra of $\mathfrak{g}(A)$ we are interested in.

By hyperbolic root system we mean a root system of a Kac-Moody algebra constructed on a generalized Cartan matrix of hyperbolic type.

Let $\Delta$ be a hyperbolic root system. A root system $\Delta_1 \subset \Delta$ is called a root subsystem of $\Delta$ if the condition $(*)$ holds.

The classification of root subsystems of finite root systems is due to Dynkin [1].

In this paper we classify maximal rank hyperbolic root subsystems of hyperbolic root systems.
Consider a maximal rank hyperbolic root subsystem $\Delta_1$ of a hyperbolic root system $\Delta$. Let $W_1$ and $W$ be the Weyl groups of $\Delta_1$ and $\Delta$ respectively. Let $F_1$ and $F$ be fundamental chambers of $W_1$ and $W$. Then $F_1$ and $F$ are hyperbolic Coxeter simplices of finite volume. The groups $W_1$ and $W$ are generated by the reflections with respect to the facets of $F_1$ and $F$ respectively. Since $W_1$ is a subgroup of $W$, the simplex $F_1$ is composed of several copies of $F$. Moreover, any two copies of $F$ having a common facet are symmetric with respect to this facet.

By reflection group we mean a group generated by reflections. Introduce a partial ordering $\geq$ on the set of reflection subgroups of $W$ by setting $G \geq H$ if $H \subset G$. A decomposition $(F, F_1)$ of a simplex $F$ into several copies of $F$ is called minimal if $W_1$ is a maximal proper reflection subgroup of $W$. All the minimal decompositions of hyperbolic Coxeter simplices of finite volume are listed in [2], [3] and [6].

From now on by simplex we mean a hyperbolic Coxeter simplex of finite volume, whose dihedral angles are in the set $\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}\}$ (zero angle can also appear if $n = 2$).

In Section 1 (Th. 1) we prove that any minimal decomposition of a hyperbolic simplex corresponds to some root subsystem of a hyperbolic root system. In Section 2 (Th. 2) we prove that any decomposition of a hyperbolic simplex corresponds to some root subsystem. The complete classification of maximal rank hyperbolic root subsystems is contained in Fig. 1–19.

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1 Maximal subgroups

We use the following notation: $A$ is a generalized Cartan matrix of hyperbolic type; $\Delta$ is the corresponding root system; $\alpha_1, ..., \alpha_{n+1}$ are simple roots; $F$ is a fundamental chamber of $\Delta$; $L = \sum_{i=1}^{n+1} \mathbb{Z}\alpha_i$ is the corresponding root lattice. The Weyl group $W_F$ of $\Delta$ is generated by the reflections with respect to the facets of the simplex $F$. The simple roots vanish on the facets of $F$. Furthermore, $\Delta^\vee$ and $L^\vee$ are the root system and the root lattice for the generalized Cartan matrix $A^t$ (the fundamental simplex of the Weyl group of $\Delta^\vee$ is the same as of $\Delta$, but the lengths of simple roots are different in these systems). $\Delta_1 \subset \Delta$ is a hyperbolic root system whose root lattice $L_1$ is a maximal rank sublattice of $L$; $F_1$ is a fundamental simplex of the Weyl group $W_{F_1}$ of the root system $\Delta_1$.

We will use the following description of root system (see [5]). A hyperbolic root system $\Delta$ consists of two disjoint parts: the set of real roots $\Delta^r$ and the set of imaginary roots $\Delta^{im}$, where

$\Delta^r = W(\alpha_1) \cup \ldots \cup W(\alpha_{n+1})$, \quad $\Delta^{im} = \{ \alpha \in L \mid (\alpha | \alpha) \leq 0 \}$. 

2
**Lemma 1.** Let \((F, F_1)\) be a minimal decomposition. The following four conditions are equivalent:

(i) \(L_1\) is a proper sublattice of \(L\).

(ii) \(\Delta_1 = \Delta \cap L_1\).

(iii) \(\Delta_1\) is a root subsystem of \(\Delta\).

(iv) The condition \((\ast)\) holds for \(\Delta_{re}^1 \subset \Delta_{re}\).

**Proof.** (i) \(\rightarrow\) (ii) For \(\Delta_{im}^1\) the statement is evident.

Suppose that there exists \(\alpha \in \Delta_{re}^1\) such that \(\alpha \in L_1\) and \(\alpha \not\in \Delta_1\). Consider a subgroup \(G\) of \(W_F\) generated by the reflections with respect to all the roots contained in \(L_1\). Clearly, \(W_{F_1} \subset G\). Since \(W_{F_1}\) is maximal in \(W_F\) and \(G \neq W_F\), we have \(G = W_F\). Hence, any simple root of \(\Delta\) can be written as \(\sum_{i=1}^{n+1} c_i \beta_i\), where \(\beta_i \in L_1\) and \(c_i \in \mathbb{Z}\). Thus, any simple root of \(\Delta\) belongs to \(L_1\), and \(L = L_1\).

(ii) \(\rightarrow\) (iii) Suppose that \(\alpha, \beta \in \Delta_1\). Then \(\alpha, \beta, \alpha + \beta \in L_1\). If \(\alpha + \beta \in \Delta\) then \(\alpha + \beta \in \Delta \cap L_1\). Therefore, \(\alpha + \beta \in \Delta_1\).

(iii) \(\rightarrow\) (iv) The proof is evident.

(iv) \(\rightarrow\) (i) Assume that \(L_1 = L\).

Suppose that the simplex \(F_1\) has a decomposed dihedral angle, i.e. some mirror of a reflection contained in \(W_F\) decomposes the dihedral angle of \(F_1\). Let \(\alpha, \beta \in \Delta_{re}^1\) be the roots vanishing on the facets of this dihedral angle (the roots are the outward normals to the facets of the angle). Then \(\alpha - \beta \in L_1\) vanishes on one of the mirrors decomposing the dihedral angle. Hence, there exists \(c > 0\) such that \(c(\alpha - \beta) \in \Delta_{re}^1\). By the assumption \(L = L_1\), thus, \(c(\alpha - \beta) \in L_1\). Without loss of generality we can assume that \(\alpha\) and \(\beta\) are simple roots of \(\Delta_1\). Then \(\alpha - \beta \not\in \Delta_{im}^1\). The lattice \(L_1\) is generated by simple roots of \(\Delta_1\), thus, \(c \in \mathbb{Z}\). Since \((\ast)\) holds for \(\Delta_{re}^1 \subset \Delta_{re}\) and \(\alpha - \beta \not\in \Delta_{im}^1\), we have \(c \neq 1\). Since \((\alpha \mid \beta) \leq 0\), if \(c \geq 2\) then \(c(\alpha - \beta)\) is more than two times longer than \(\alpha\). This is impossible, since \(c(\alpha - \beta)\) and \(\alpha\) are not mutually orthogonal.

Suppose now that \(F_1\) has no decomposed dihedral angle. Then the pair \((F, F_1)\) is one of the six pairs listed in Table 1. Since no of these simplices has a dihedral angle different from \(\frac{\pi}{2}\) and \(\frac{\pi}{3}\), each simplex contained in Table 1 corresponds to a unique root system. A direct calculation shows that for each of these six pairs roots of the subsystem generate an index two sublattice of the root lattice.

\[\square\]

**Remark.** The proof of the second implication does not need the decomposition to be minimal. It will be convenient for the study of non-minimal decompositions.

Some simplices correspond to several root systems. Indeed, suppose that \(F\) has at least one dihedral angle different from \(\frac{\pi}{2}\) and \(\frac{\pi}{3}\). Then there exists
at least two ways to define the lengths of roots (see section 3). We will prove that for any minimal decomposition \((F, F_1)\) we can find a root system \(\Delta\) (with fundamental simplex \(F\)) such that the roots correspondent to \(F_1\) generate a proper sublattice of \(L\). By Lemma 1, the condition \((*)\) holds for the root system correspondent to \(F_1\).

First, suppose that \(F_1\) contains exactly two copies of \(F\).

Lemma 2. Suppose that \([W_F : W_{F_1}] = 2\). Let \(\Delta\) be any root system with fundamental simplex \(F\). Then the roots of \(\Delta\) vanishing on the facets of \(F_1\) generate a proper sublattice \(L_1\) of \(L\) (or \(L^\vee\) respectively). The index of the sublattice equals two, three or four.

Proof. The simplex \(F_1\) is a union of \(F\) and \(F'\), where \(F'\) is an image of \(F\) under the reflection with respect to some facet of \(F\). Let \(\alpha_1\) be a root vanishing on this facet. All but one facets of \(F_1\) are facets of \(F\). Thus, exactly one of \(\alpha_2, ..., \alpha_{n+1}\) is not orthogonal to \(\alpha_1\).

Suppose that \(\alpha_2\) is not orthogonal to \(\alpha_1\). We can assume that \(|a_{21}| \leq |a_{12}|\) (if \(|a_{21}| \geq |a_{12}|\), consider the matrix \(A^t\) instead of \(A\)). Then the facets of \(F_1\) correspond to the roots \(\alpha_2 - a_{12}\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n+1}\).

Since \(A\) is a matrix of hyperbolic type, \(a_{12} = -1, -2, -3\) or \(-4\) (\(-4\) occurs only if \(F_1\) and \(F\) are non-compact triangles). In case of \(a_{12} = -1\) the angle between the facets correspondent to \(\alpha_2\) and \(\alpha_2 - a_{12}\alpha_1\) equals \(\frac{2\pi}{3}\), and the group \(W_{F_1}\) coincides with \(W_F\).

Hence, we have either \(a_{12} = -2\), or \(a_{12} = -3\), or \(a_{12} = -4\). Therefore, \(L_1 = \mathbb{Z}(\alpha_2 - a_{12}\alpha_1) + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 + ... + \mathbb{Z}\alpha_{n+1}\) is a proper sublattice of index

| \(F\) | \(F_1\) | \([W_F : W_{F_1}]\) |
|------|------|----------------|
| \(\bullet\) | \(\square\) | 5 |
| \(\bullet\) | \(\times\) | 12 |
| \(\bullet\) | \(\times\) | 10 |
| \(\bullet\) | \(\times\) | 20 |
| \(\bullet\) | \(\times\) | 272 |
| \(\bullet\) | \(\times\) | 527 |

Table 1. Minimal decompositions without decomposed dihedral angles.
Consider the general case.

**Theorem 1.** Let $W_{F_1}$ be a maximal subgroup of $W_F$. Consider any root system $\Delta$ with fundamental simplex $F$. Then the roots of $\Delta$ (or $\Delta^\vee$) vanishing on the facets of $F_1$ generate a proper sublattice $L_1$ of $L$ (or $L^\vee$ respectively).

**Proof.** Since $W_{F_1}$ is a subgroup of $W_F$, $F_1$ is a union of several copies of $F$, where two copies having a common facet are symmetric with respect to this facet.

All the minimal decompositions of simplices are described in [2], [3] and [6]. Considering these decompositions case by case, one can find that for any minimal decomposition $(F, F_1)$ the simplex $F_1$ has at least one vertex $A$ whose stabilizer in $W_F$ coincides with its stabilizer in $W_{F_1}$. In other words, all but one facets of $F_1$ are the facets of $F$ (the intersection of these facets is the vertex $A$).

Let $\alpha_1, ..., \alpha_n$ be the roots vanishing on the common facets of $F$ and $F_1$. Let $\alpha_{n+1}$ be the roots correspondent to the rest facets of $F$ and $F_1$ respectively. The root $\alpha_{F_1}$ is a linear combination of roots $\alpha_1, ..., \alpha_{n+1}$. The index of the sublattice generated by $\alpha_1, ..., \alpha_n, \alpha_{F_1}$ is equal to the coefficient of $\alpha_{n+1}$.

Now take a minimal decomposition $(F, F_1)$ and a root system $\Delta$ with a fundamental simplex $F$. Compute the coefficient of $\alpha_{n+1}$ when $\alpha_{F_1}$ is represented as a linear combination of roots $\alpha_1, ..., \alpha_{n+1}$. If the coefficient is not equal to one then we obtain a proper sublattice (the coefficient can not be negative, since $\alpha_{F_1}$ is a positive root; moreover, it can not be equal to zero, otherwise simple roots of the root system would be linearly dependent).

Suppose that the coefficient equals one. Turn over all the arrows in the Dynkin diagram of $\Delta$. In other words, consider a root system $\Delta^\vee$. A direct calculation shows that in this root system the coefficient we are interested in is not equal to one, and we have a proper sublattice of $L^\vee$.

The sublattice is usually of index two. More precisely, sublattices of index different from two occur only in the dimensions two and three (see Fig. [4]).

2 Non-maximal subgroups

In this section, we prove that for any non-minimal decomposition $(F, F_1)$ there exist a root system $\Delta$ whose simple roots vanish on the facets of $F$ and the root system $\Delta_1$ whose simple roots vanish on the facets of $F_1$ such that $\Delta_1$ is a root subsystem of $\Delta$. In general, for some decompositions $(F, F_1)$ there exist more than one pair of root systems $\Delta_1 \subset \Delta$ satisfying the condition described above.

**Lemma 3.** Let $\Delta_1$ be a root subsystem of $\Delta$, and $\Delta_2$ be a root subsystem of $\Delta_1$. Then $\Delta_2$ is a root subsystem of $\Delta$. 

}\]
Proof. Suppose that $\alpha, \beta \in \Delta_2$ and $\alpha + \beta \in \Delta$. Since $\Delta_2 \subset \Delta_1$, we have $\alpha, \beta \in \Delta_1$. The condition (*) holds for $\Delta_1 \subset \Delta$. Thus, $\alpha + \beta \in \Delta_1$. Since (*) holds for $\Delta_2 \subset \Delta_1$, we have $\alpha + \beta \in \Delta_2$. Therefore, (*) holds for $\Delta_2 \subset \Delta$.

The assumption of Lemma 3 is not necessary (see section 3). However, we have the following

Lemma 4. Suppose that $\Delta_2 \subset \Delta_1 \subset \Delta$ and $\Delta_2 \subset \Delta_1$ is not a root subsystem. Then $\Delta_2 \subset \Delta$ is not a root subsystem either.

Proof. Since $\Delta_2 \subset \Delta_1$ is not a root subsystem, there exist $\alpha, \beta \in \Delta_2$ such that $\alpha + \beta \in \Delta_1$ and $\alpha + \beta \notin \Delta_2$. Since $\Delta_1 \subset \Delta$, we have $\alpha + \beta \in \Delta$. Therefore, (*) does not hold for $\Delta_2 \subset \Delta$.

Lemma 3 shows it is sufficient to find a sequence of root systems $\Delta_k \subset \Delta_{k-1} \subset \ldots \subset \Delta_1 \subset \Delta$, such that $\Delta_k$ corresponds to $F_1$, $\Delta$ corresponds to $F$, and for any $i \leq k$ the decomposition correspondent to $\Delta_i \subset \Delta_{i-1}$ is minimal. Such a sequence can be constructed for almost all non-minimal decompositions. The exclusions are two four-dimensional decompositions (see Fig. 10) and one five-dimensional decomposition (see Fig. 13). Root subsystems for these three decompositions are shown in Section 3.

We have proved the following

Theorem 2. Let $F$ and $F_1$ be finite volume hyperbolic Coxeter simplices having no dihedral angles different from $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$ and 0. Let $W_{F_1}$ and $W_F$ be the groups generated by the reflections with respect to the facets of $F_1$ and $F$ respectively. Suppose that $W_{F_1} \subset W_F$. Then there exist a root system $\Delta$ whose simple roots vanish on the facets of $F$ and the root system $\Delta_1$ whose simple roots vanish on the facets of $F_1$ such that $\Delta_1 \subset \Delta$ is a root subsystem.

## 3 Classification of maximal rank root subsystems

There exist finitely many Coxeter hyperbolic simplices, and no hyperbolic simplex exists in the dimension greater than 9 (see [4]).

Some Coxeter simplices correspond to several root systems. To list all the root systems correspondent to Coxeter simplex, consider the Coxeter diagram of the simplex and assign each multiple edge and some bold edges by an arrow (in other words, it is sufficient to define the lengths of roots). To obtain a Dynkin diagram of a root system, the arrows should satisfy the only necessary condition: if the Coxeter diagram contains a cycle without bold edges, then the number of arrows pointing clockwise must be equal to the number of arrows pointing counterclockwise. This condition should be hold by double edges as well as by triple edges (recall that an angle $\frac{\pi}{6}$ is shown in Dynkin diagram by a triple edge, but in Coxeter diagram this angle is shown by a 4-fold edge).
If the Coxeter diagram contains a cycle with a bold edge, the condition slightly changes (this occurs only if $n = 2$). If all the bold edges are indirected (i.e. the corresponding roots have the same length), then the condition coincides with one described above. If there is an oriented bold edge, then there are two possibilities: either there are two bold edges with different orientations and the third angle equals $\frac{\pi}{3}$ or 0, or there is exactly one oriented bold edge and other two are 2-fold edges directed to the other side.

In general case the way to list all root systems for a given Weyl group is described in [7].

To obtain a complete classification of root systems we do the following. Consider a minimal decomposition $(F, F_1)$. Assign the Coxeter diagram of $F$ by arrows in all possible ways and consider all the root systems correspondent to the simplex $F_1$. Do this for each minimal decomposition and consider the superpositions of minimal decompositions. This algorithm leads to the complete list of maximal rank hyperbolic root systems contained in hyperbolic root systems.

To classify regular subalgebras it is sufficient to check the condition (\*) for each pair $\Delta_1 \subset \Delta$. In case of minimal decomposition we can use Lemma 1; it is sufficient to show that $\Delta_1$ generates a proper sublattice of $L$. As it was mentioned above, in case of non-minimal decompositions the positive answer usually can be obtained applying Lemma 3. Note that in this case $\Delta_1 = \Delta \cap L_1$, where $L_1$ is a root lattice for $\Delta_1$. Lemma 4 helps to make calculations shorter: it shows immediately that (\*) does not hold for a long list of pairs $\Delta_1 \subset \Delta$. In the rest cases we check (\*) directly. There are 19 pairs of roots systems satisfying (\*) and corresponding to non-minimal decompositions. It turns out that $\Delta_1 = \Delta \cap L_1$ for all these 19 cases. Combining this result with Lemma 4 and the remark to this lemma, we have the following

**Theorem 3.** Let $\Delta_1 \subset \Delta$ be two hyperbolic root systems of the same rank. Let $L_1 \subset L$ be the corresponding root lattices. Then the following three conditions are equivalent:

(i) $\Delta_1 = \Delta \cap L_1$.

(ii) $\Delta_1 \subset \Delta$ is a root subsystem.

(iii) Condition (\*) holds for $\Delta_1^{\text{re}} \subset \Delta^{\text{re}}$.

Below we list all the maximal rank hyperbolic root subsystems of hyperbolic root systems.

In Fig. 1–19 we use the following notation:

Two diagrams are joined if the Weyl group correspondent to the lower diagram is a subgroup of the Weyl group correspondent to the upper diagram. Each edge correspondent to a minimal decomposition is assigned with an index of the subgroup. If the decomposition is minimal, the lower system is a subsystem of the upper one, and the index of the sublattice differs from 2, then the edge is attached with the index of the sublattice (the number in brackets).

Types of edges:

- decomposition is minimal, (\*) holds;
decomposition is minimal, (\ast) does not hold;

\[ \text{decomposition is non-minimal, (\ast) holds, but (\ast) does not hold for at least one intermediate minimal decomposition.} \]

A root system is not joined with a root subsystem if the decomposition is non-minimal and (\ast) holds for each intermediate minimal decomposition (see Lemma 3).

### 3.1 Triangles

There are exactly three minimal decompositions of compact Coxeter hyperbolic triangles having no angles different from \( \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \) and \( \frac{\pi}{6}. \) Considering all possible lengths of simple roots, we obtain

![Figure 1](image1.png)

There are six commensurability classes of non-compact Coxeter hyperbolic triangles having no angles different from \( \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \) and 0. Five of these classes contain a unique triangle each, thus, these classes produce no root subsystem. The rest commensurability class is described in Fig. 2.

![Figure 2](image2.png)
3.2 Tetrahedra

There exist four commensurability classes of Coxeter hyperbolic tetrahedra having no dihedral angles different from $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$ (see [4]). Two of these classes contain a unique tetrahedron each, thus, this classes produce no root subsystem. The rest two commensurability classes are described in Fig. 5–7 and Fig. 8–9.
To obtain diagrams in Fig. 6 one can turn over all the arrows on the diagrams shown in Fig. 5.

Figure 6.

The rest possibilities to assign the arrows (that correspond to the root systems containing real roots of three different lengths) are shown in Fig. 7.

Figure 7.

Root systems correspondent to the second commensurability class are shown in Fig. 8 and 9.

Figure 8.
3.3 Four-dimensional simplices

According to [4] (see also [2]), there are exactly two commensurability classes of four-dimensional Coxeter hyperbolic simplices having no dihedral angles different from $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$. One of these classes contains a unique simplex, thus, this class produces no root subsystem. The rest commensurability class is described in Fig. 10–12.
To obtain diagrams in Fig. 11 one can turn over all the arrows on the diagrams shown in Fig. 10.

The rest possibilities to assign the arrows (that correspond to the root systems containing real roots of three different lengths) are shown in Fig. 12.

3.4 Five-dimensional simplices

According to [4] (see also [2]), there are exactly three commensurability classes of five-dimensional Coxeter hyperbolic simplices. Two of these classes contain a unique simplex each, thus, these classes produce no root subsystem. The rest commensurability class is described in Fig. 13–15.
To obtain diagrams in Fig. 14 one can turn over all the arrows on the diagrams shown in Fig. 13.

The rest possibilities to assign the arrows (that correspond to the root systems containing real roots of three different lengths) are shown in Fig. 15.

Figure 13.

Figure 14.

The rest possibilities to assign the arrows (that correspond to the root systems containing real roots of three different lengths) are shown in Fig. 15.
3.5 Simplices of dimensions 6-9

There are no non-minimal decompositions in the dimensions 6-9 (see [4]). All root subsystems in these dimensions are shown in Fig. 16–19.

Figure 15.

Figure 16. Six-dimensional root subsystems.

Figure 17. Seven-dimensional root subsystems.
Figure 18. Eight-dimensional root subsystems.

Figure 19. Nine-dimensional root subsystems.
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