Minimal surfaces in finite volume hyperbolic 3-manifolds $N$ and in $M \times S^1$, $M$ a finite area hyperbolic surface

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MINIMAL SURFACES IN FINITE VOLUME HYPERBOLIC
3-MANIFOLDS $N$ AND IN $M \times S^1$, $M$ A FINITE AREA
HYPERBOLIC SURFACE

By P. COLLIN, L. HAUSWIRTH, and H. ROSENBERG

Abstract. We consider properly immersed finite topology minimal surfaces $\Sigma$ in complete finite volume hyperbolic 3-manifolds $N$, and in $M \times S^1$, where $M$ is a complete hyperbolic surface of finite area. We prove $\Sigma$ has finite total curvature equal to $2\pi$ times the Euler characteristic $\chi(\Sigma)$ of $\Sigma$, and we describe the geometry of the ends of $\Sigma$.

1. Introduction. The global theory of properly embedded doubly-periodic minimal surfaces was developed by Meeks and Rosenberg [12]. In this paper they described the geometry of the annular ends of such a minimal surface, in the quotient of $\mathbb{R}^3$ by a $\mathbb{Z}^2$ group of independent translations, leaving the surface invariant, i.e., in $T^2 \times \mathbb{R}$, $T^2$ a flat 2-torus. They then proved the total curvature of the surface $\Sigma$ in $T^2 \times \mathbb{R}$ is $2\pi \chi(\Sigma)$. The finite total curvature theorem was then extended to properly embedded finite topology minimal surfaces in complete flat 3-manifolds [13]. The study was then pursued in $M \times \mathbb{R}$, $M$ a closed Riemannian surface [14]. The theory of properly embedded minimal surfaces in $H^2 \times \mathbb{R}$, $H^2$ the hyperbolic plane, was initiated in [10].

In this paper we will study properly immersed minimal surfaces in hyperbolic 3-manifolds of finite volume and in $M \times S^1$, $M$ a hyperbolic surface of finite area. We describe examples and we study the geometry of finite topology minimal surfaces. One of our main results says that a properly immersed finite topology minimal surface $\Sigma$ in a complete hyperbolic finite volume 3-manifold $N$ has each of its annular ends asymptotic to a totally geodesic annulus in a cusp end of $N$. We then prove the total curvature of $\Sigma$ equals $2\pi \chi(\Sigma)$.

The classical proof of Meeks and Rosenberg [12, 13], used techniques particular to flat euclidean 3-space; in particular, complex analysis (the Gauss map of a minimal surface in $\mathbb{R}^3$ is conformal), and the simple geometric structure of finite total curvature minimal annular ends in $\mathbb{R}^3$. We do not have these tools in a hyperbolic manifold or $M \times S^1$, $M$ a hyperbolic surface. In these spaces we construct explicit barriers which enable us to trap properly immersed minimal annular ends in regions of space where we can control their geometry. Previous results needed

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to assume the annular ends are embedded. We are able to relax this to immersed since we trap the ends in small regions of the ambient space.

The paper is organized as follows. In Section 2, we explain the geometry of the ends of the ambient spaces $N$ and $M \times S^1$ and we describe the geometry of the standard minimal annular ends in these spaces. We then state the Theorem 2.1, the main theorem of the paper, and we state several applications. In Section 3, we describe some examples of properly embedded minimal surfaces of finite topology in $M \times S^1$. We start with $M$ a 3-punctured sphere, then $M$ a sphere with $2n$ punctures, and $M$ a once punctured torus. We hope to convey to the reader the wealth of interesting examples in these spaces. In Section 4, we construct a one-parameter family of minimal surfaces by solving the equation of a mean curvature graph. In Section 5, we describe some properties of the standard examples $A(p, q)$ in the cusp ends $M$ of $M \times S^1$. We begin the study of a lift $E \subset \mathbb{H}^2 \times \mathbb{R}$ of an annular end $A$ of $M$. We prove that a subend of $A$ is trapped between two standard ends $A(p, q)$ that are close at infinity; “close” will be defined later. In Section 6, we prove the Dragging Lemma. In Section 7, we study compact annuli that we will use in the proof of the theorem. In Section 8, we study the limit of a family of Scherk type graphs in $\mathbb{H}^2$ which are converging to 0 and we use this sequence to prove that the third coordinate of a periodic end with a horizontal period has a limit at infinity. In Section 8, we prove that a trapped subend of $A$ is a killing graph, hence it is stable and has bounded curvature. Then in Sections 8, 9, and 10, we prove the main theorem. Some techniques here are inspired by the ideas of Colding and Minicozzi [1].

Since this paper was written, there have been two papers that apply our finite total curvature Theorem 2.1. In [2] the authors prove there is a compact embedded minimal surface in any complete hyperbolic 3-manifold of finite volume. Since such a non-compact manifold $N$ is “not convex at infinity”, minimization techniques do not produce such a minimal surface. To understand this the reader can verify that on a complete hyperbolic 3-punctured 2-sphere, there is no simple closed geodesic. In dimension 3, a min-max technique, together with several maximum principles in the cusp ends of $N$, will produce compact embedded minimal surfaces.

Another theorem obtained is that a properly embedded minimal surface in $N$ of bounded curvature has finite topology [2]. Thus the geometry of the ends is given by Theorem 2.1. Since stable surfaces have bounded curvature, the above theorem applies to them. Also they obtain:

**Theorem.** [2] A proper minimal embedding of a 3-punctured sphere in $N$ is totally geodesic.

In [11], the authors obtain a lower bound for the area of a closed orientable embedded minimal surface in $N$. They prove the area is at least $2\pi$ when the Heegaard genus of $N$ is at least 6. For non-orientable minimal surfaces the lower area bound is $\pi$. This theorem uses Theorem 2.1 of the present paper.
2. Minimal ends in ambient hyperbolic spaces and results. Let \( N \) denote a complete hyperbolic 3-manifold of finite volume. An end of \( N \) is modeled on the quotient of a horoball of the hyperbolic 3-space \( \mathbb{H}^3 \), by a \( \mathbb{Z}^2 \) parabolic subgroup of the isometry group of \( \mathbb{H}^3 \) leaving the horoball invariant. More precisely we consider the model of the half-space of \( \mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3; y > 0\} \) with the metric \( ds^2 = \frac{dx^2 + dy^2 + dt^2}{y^2} \). Then an end of \( N \) has a sub-end isometric to

\[
\mathcal{M}(-1) = \{(x,y,t) \in \mathbb{R}^3; y \geq y_0 > 0\}
\]

modulo a \( \mathbb{Z}^2 \)-parabolic subgroup of isometries of \( \mathbb{H}^3 \) leaving the planes \( \{y = c\} \) invariant. The horosphere \( \{y = \text{constant}\} \) quotient to tori \( T(y) \) in \( \mathcal{M}(-1) \); \( T(y) \) has constant mean curvature one. Let \( c \) be a closed geodesic of \( T(1) \). Then \( A(-1) = \{(x,y,t) \in \mathbb{R}^3; y \geq 1 \text{ and } (x,t) \in c\} \) is a minimal annulus immersed in \( \mathcal{M}(-1) \), which we will call a standard cusp-end in \( \mathcal{M}(-1) \).

A complete surface \( M \) of constant curvature \( K = -1 \) and finite area has finite total curvature hence \( M \) is conformally diffeomorphic to a compact surface punctured in a finite number of points. Each end of \( M \) (called a cusp end), denoted by \( C \), is an annular end isometric to the quotient of a horodisk \( H \) in the hyperbolic plane \( \mathbb{H}^2 \) by a parabolic isometry \( \psi \).

To describe the geometry of such ends we model \( \mathbb{H}^2 \) by the upper half-plane

\[
\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2; y > 0\}
\]

with the metric \( ds^2 = \frac{dx^2 + dy^2}{y^2} \). Then a cusp end \( C \) of \( M \) is isometric to \( H/[\psi] \), where \( H = \{(x,y) \in \mathbb{R}^2; y \geq 1\} \) is a horodisk and \( \psi(x,y) = (x + \tau, y) \), for some \( \tau \neq 0 \).

In \( M \times S^1 \), with the product metric, the ends become \( \mathcal{M} := C \times S^1 \), and are foliated by constant mean curvature tori \( T(y_1) = c(y_1) \times S^1 \), where \( c(y_1) = \{(x,y) \in \mathbb{H}^2; y = y_1\} / [\psi] \). We consider \( S^1 = \mathbb{R} / [T(h)], T(h) \) the translation of \( \mathbb{R} \) by some \( h > 0 \) and

\[
\mathcal{M} = \bigcup_{y \geq y_0} T(y) = (H/[\psi]) \times (\mathbb{R} / [T(h)])
\]

\[
= \{(x,y,t) \in \mathbb{R}^3; y \geq y_0 \geq 1\} / [\psi, T(h)].
\]

Thus the ends of \( N \) and those of \( M \times S^1 \) share many properties. Both are parametrized by the same half-space of \( \mathbb{R}^3 \), and foliated by constant mean curvature tori \( T(y) \) (curvature one half in \( \mathcal{M} \) and one in \( \mathcal{M}(-1) \)). \( \mathcal{M}(-1) \) has constant sectional curvature \( -1 \) and the tori \( T(y) \) shrink exponentially when one flows by the geodesics \( y \) increasing. In \( \mathcal{M} \), the horizontal cycles \( c(y) \) shrink exponentially along the \( y \) increasing flow and the \( t \) cycles are of constant length \( h \). Subsequently we will develop the geometry of surfaces in these ends.

Now let \( \Sigma \) be a properly embedded minimal surface in \( N \) or \( M \times S^1 \) of finite topology; and such that \( \Sigma \) has a finite number of annular ends \( \{A_j\} \) for \( 1 \leq j \leq k \).
Since $\Sigma$ is proper, each end $A_j$ of $\Sigma$ is in some end $M$ of $M \times S^1$ or in some end $M(-1)$ of $N$. We denote by $E$ a connected component of a lift of an end $A$ of $\Sigma$, $E$ in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{H}^3$.

We will now describe the model ends of minimal annuli in $M$ and $M(-1)$. In $M(-1)$ the model end is the standard cusp end $A(-1)$ we previously defined.

In $M$, there are essentially three model ends. In $M$, we define $A(p,q)$ to be the annular end that is the quotient of a (euclidean) half-plane $E(p,q)$ orthogonal to the plane $\{(x,y,t) \in \mathbb{R}^3; y = 1\}$ and of slope $qh/p\tau$. For $(p,q) = (1,0)$, the end

$$E_{(1,0)}(t_0) = \{(x,y,t) \in \mathbb{R}^3; y \geq 1, \quad t = t_0\}$$
and $A_{(1,0)} = E_{(1,0)}/[\psi]$ is a cusp end of $M$ (horizontal). For $(p,q) = (0,1)$, it is the product of a horizontal geodesic ray of $M$ and $S^1$. Then $E_{(0,1)}(x_0) = \{(x,y,t) \in \mathbb{R}^3; y \geq 1, \quad x = x_0\}$ and $A_{(0,1)} = E_{(0,1)}/[T(h)]$.

For $(p,q) \neq \{(0,1), (1,0)\}$, we think of $A(p,q)$ as a helicoid with axis at the cusp at infinity. It is the quotient of

$$E_{(p,q)}(c_0) = \{(x,y,t) \in \mathbb{R}^3; y \geq 1, \quad p\tau t - qh.x = c_0\}$$
and $A_{(p,q)} = E_{(p,q)}/[\psi, T(h)]$.

We will prove that a properly immersed annular end $A$ in $M$ or in $M(-1)$ has finite total curvature and is asymptotic to a standard end $A_{(p,q)}$ in $M$ or a standard cusp end $A(-1)$ in $M(-1)$. The main theorem of the paper is:

**Theorem 2.1.** Consider a complete surface $M$ with curvature $K = -1$ and finite area and $N$ a complete hyperbolic 3-manifold of finite volume. Let $\Sigma$ be a properly immersed minimal surface in $N$ or in $M \times S^1$ of finite topology. Then the surface $\Sigma$ has finite total curvature and each end $A$ of $\Sigma$ is asymptotic to a standard cusp-end $A(-1)$ in $M(-1)$ or to a standard end $A_{(p,q)}$ in $M$.

(i) $A_{(p,0)}$ a horizontal cusp $C \times \{t_0\}$
(ii) $A_{(0,q)}$ a vertical plane $\gamma \times S^1$
(iii) $A_{(p,q)}$ a helicoidal end with axis at infinity.

Moreover

$$\int_{\Sigma} KdA = 2\pi \chi(\Sigma).$$

Combining the formula for the total curvature of $\Sigma$ in theorem 2.1, with the Gauss equation, we obtain topological obstructions for the existence of proper minimal immersions of a finite topology surface $\Sigma$ into $N$ or $M \times S^1$, of a given topology.

**Corollary 2.2.** There is no proper minimal immersion of a plane $\mathbb{R}^2$ into $N$ or $M \times S^1$. 
COROLLARY 2.3. In $N$ there is no proper minimal immersion of the sphere $S^2$ with $n$ punctures; $n = 0, 1, \text{ or } 2$.

COROLLARY 2.4. A proper minimal immersion of $S^2$ with two punctures (an annulus) in $M \times S^1$, is necessarily $\gamma \times S^1$, $\gamma$ a complete geodesic of $M$.

Proof of Corollaries 1, 2, and 3. Suppose $\Sigma$ is an orientable surface of genus $g$ with $n$ punctures, $n \geq 0$. Then $\chi(\Sigma) = 2 - 2g - n$, so if $\Sigma$ can be properly minimally immersed in $N$ or $M \times S^1$, it follows from Theorem 2.1 and the Gauss equation

$$\int_{\Sigma} K_{\Sigma} = 2\pi(2 - 2g - n) = \int_{\Sigma} K_e + \int_{\Sigma} K_\sigma,$$

where $K_e$ and $K_\sigma$ are the extrinsic and sectional curvatures of $\Sigma$ respectively. Since $-1 \leq K_\sigma \leq 0$ in $M \times S^1$, $K_\sigma = -1$ in $N$, and $K_e \leq 0$, we have

(2.1) $2 - 2g - n \leq 0$

and equality if and only if $K_e = K_\sigma = 0$.

This equality cannot occur in $N$ (since $K_\sigma = -1$) and equality in $M \times S^1$ yields $\Sigma$ is vertical and $g$ is 0 or 1. When $g = 0$, then $n = 2$ and $\Sigma = \gamma \times S^1$, where $\gamma$ is a complete, non-compact geodesic of $M$. When $2 < 2g + n$ then if $g = 0$, one cannot have $n \leq 2$. This proves Corollaries 1 and 3.

So excluding the equality case we discussed above, there is no proper minimal immersion of $S^2$ with 0, 1, or 2 punctures, in $N$ or $M \times S^1$. This proves Corollary 3. \hfill \square

In $N$, one obtains an area estimate. If $\Sigma$ is properly minimally immersed in $N$ then

$$2\pi(2 - 2g - n) = \int_{\Sigma} K_e - |\Sigma|$$

where $|\Sigma|$ denote the area of $\Sigma$. Thus $|\Sigma| = \int_{\Sigma} K_e + 2\pi(2g + n - 2) \leq 2\pi(2g + n - 2)$ and equality holds precisely when $\Sigma$ is totally geodesic. This estimate motivates the question: do such totally geodesic immersions exists in $N$?

By equation (2.1) we have $0 < 2g + n - 2$, so if $2g + n - 2 \leq 0$, then the immersion $\Sigma$ does not exist in $N$. This discussion leads us to ask: If $2g > 2 - n$, can $\Sigma$ be properly minimally immersed in $N$?

3. **Examples in** $M \times S^1$. The first examples in $M \times S^1$ that come to mind are the horizontal slices $\Sigma = M \times \{c\}$ and the vertical annuli (or totally geodesic tori), $\Sigma = \gamma \times S^1$, $\gamma$ a complete geodesic (perhaps compact) of $M$. 

We describe five examples; $M$ will be a sphere with three or four punctures or a once punctured torus, and have a complete hyperbolic metric of finite area. Denote by $\mathbb{S}ph(k)$, $k = 3$ or $4$ such a hyperbolic sphere.

**Example 1.** $\Sigma$ an embedded minimal surface in $\mathbb{S}ph(3) \times S^1$ with three ends, two helicoidal and the other horizontal. The domains and notation we now introduce will be used in all the examples we describe.

Let $\Gamma$ be the ideal triangle in the disk model of $H^2$ with vertices $A = (0, 1)$, $B = (0, -1)$, $C = (-1, 0)$ and sides $a, b, c$ as indicated in Figure 1.

Let $\Sigma$ be the minimal graph over the domain $D$ bounded by $\Gamma$, taking the values $0$ on $b$ and $c$ and $h > 0$ on $a$. Extend $\Sigma$ to an entire minimal graph $\tilde{\Sigma}$ over $H^2$ by rotation by $\pi$ in all the sides of $\Gamma$, and the sides of the triangles thus obtained. More precisely, given a minimal surface $\Sigma$ bounded by horizontal and vertical geodesic arcs in $H^2 \times \mathbb{R}$, one can extend $\Sigma$ by “Schwarz reflection” about the boundary arcs. For each boundary arc $C$ of $\Sigma$, there is an isometry of $H^2 \times \mathbb{R}$ which is a rotation by $\pi$ about $C$. This extends $\Sigma$ smoothly through the interior of $C$. One then does all the symmetries through the geodesic boundary arcs of the new surface.

In Figure 2, we indicate some of the reflected triangles and the values of the graph $\tilde{\Sigma}$ on their sides. The triangle $A, B, C_1$ is the reflected triangle of $A, B, C$ in the geodesic $a$. We denote by $b_1$ the geodesic with end points $B$ and $C_1$ and by $c_1$ the geodesic with end points $C$ and $C_1$ (see Figure 2).
Let $D$ be the domain bounded by $\Gamma$. Let $\psi_A$ be the parabolic isometry with fixed point $A$ which takes the geodesic $c$ to $c_1$ and let $a_1 := \psi_A(a); \psi_A = R_{c_1} R_a$, where $R_{\gamma}$ denotes reflection in the geodesic $\gamma$. Let $\psi_B$ be the parabolic isometry of $\mathbb{H}^2$ leaving $B$ fixed, taking $b$ to $b_1$ and $a$ to $c_2; \psi_B = R_{b_1} R_a$ (see Figure 2).

Notice that the group of isometries of $\mathbb{H}^2 \times \mathbb{R}$, generated by $T(2h) \circ \psi_A$ and $T(2h) \circ \psi_B$, leaves $\tilde{\Sigma}$ invariant.

Let $M$ be the 3-punctured sphere obtained by identifying the sides of $D \cup R_a(D)$ by $\psi_A, \psi_B$ (c with $c_1$, $b$ with $b_1$). $M$ is hyperbolic and has finite area.

Let $\Sigma_2$ be the subgraph in $\tilde{\Sigma}$ over $D \cup R_a(D)$. Then the multi-graph $\bigcup_{k \in \mathbb{Z}} T(k2h)(\Sigma_2)$ passes to the quotient $M \times ([\mathbb{R}/[T(2h)])$ to give a complete embedded minimal surface $\Sigma$ with 3-ends; two helicoidal and the other horizontal. $\Sigma$ is a 3-punctured sphere, has total curvature $-2\pi$ and $\Sigma$ is stable ($\Sigma$ is transverse to the killing field $\partial/\partial t$).

**Example 2.** $\Sigma$ an embedded minimal surface in $\text{Sph}(4) \times S^1$, where $\text{Sph}(4)$ is the sphere with four ends. Let $Q$ be the ideal quadrilateral $D \cup R_a(D)$, and define $F = Q \cup R_{c_1}(Q)$. Let $M$ be the quotient of $F$ obtained by identifying the sides of $\partial F$ as follows:

1. Identify $c$ with $R_{c_1}(c)$ by the parabolic isometry at $A$ taking $c$ to $R_{c_1}(c)$,
2. Identify $b$ with $R_{c_1}(b)$ by the hyperbolic isometry taking $b$ to $R_{c_1}(b)$ and,
3. Identify $b_1$ with $R_{c_1}(b_1)$ by the parabolic isometry at $C_1$ taking $b_1$ to $R_{c_1}(b_1)$.
$M$ is a 4-punctured sphere. A more symmetric picture of $M$ is obtained by changing the picture by the isometry taking $A$ to $A$ and $C_1$ to $B$ as indicated in Figure 3.

Then the graph of $\tilde{\Sigma}$ over $F$ yields an embedded minimal surface $\Sigma$ in $M \times (\mathbb{R}/[T(4h)])$. $\Sigma$ has two horizontal ends and two helicoidal ends of type $E(1, 1)$. This surface $\Sigma$ is also stable.

**Example 3.** A compact singly periodic Scherk surface in $M \times S^1$, where $M$ is a once punctured torus. This surface is constructed in [10] and we briefly describe it here. Let $Q = D \cup R_\alpha(D)$ and $\gamma_1, \gamma_2$ be minimizing geodesics joining opposite sides of $D$; see Figure 4. In $\mathbb{H}^2 \times \mathbb{R}$, we desingularize the intersection of the planes $\gamma_1 \times \mathbb{R}$ and $\gamma_2 \times \mathbb{R}$ in the usual manner to create a Scherk surface invariant under a vertical translation. We next describe this surface. Let $\alpha$ and $\beta$ be the segments of $\gamma_1, \gamma_2$ in the first and fourth quadrants respectively. Form a polygon in $\mathbb{H}^2 \times \mathbb{R}$ by joining to $(\alpha \times \{h\}) \cup (\alpha \times \{0\})$ by the two vertical segments joining $\alpha(\Gamma) \times \{0\}$ to $\alpha(\Gamma) \times \{h\}$, and by joining $\beta(\Gamma) \times \{0\}$ to $\beta(\Gamma) \times \{h\}$; $\alpha(\Gamma)$ denotes the endpoint of $\alpha$ on $\Gamma$ (similarly for $\beta(\Gamma)$). This polygon bounds a least area minimal disk $D_1$; see Figure 5.

Successive symmetries in all the horizontal sides yields a “Scherk” type surface in $\mathbb{H}^2 \times \mathbb{R}$ that is bounded by 4 vertical geodesics and which is invariant by vertical translation by $2h$.

We now identify opposite sides of $Q$ by the hyperbolic translations $T(\gamma_1)$, $T(\gamma_2)$, along $\gamma_1$ and $\gamma_2$. This gives a once punctured torus $M$. The Scherk surface
Example 4. A singly periodic Scherk surface with 4 vertical annular ends in $M \times S^1$, where $M$ is a once punctured torus. This example has been constructed in $\mathbb{H}^2 \times \mathbb{R}$ quotiented by a vertical translation $T(2h)$ by Morabito and Rodriguez [15]. We briefly describe a construction in $M \times S^1$. 

passes to the quotient to give a compact minimal surface $\Sigma$ without boundary in $M \times (\mathbb{R}/[T(2h)])$. 

Figure 4. $M$ is a once punctured torus.

Figure 5. A least area disk $D_1$. 
First we “rotate” Example 3 by $\pi/4$. Let $\alpha_1, \alpha_2$ be the complete geodesics joining the opposite vertices of $\partial Q$; $\alpha_1, \alpha_2$ are the $x$ and the $y$-axis in the unit disc model. Again we construct least-area disks bounded by the polygon of Figure 6.

We know that when the vertical geodesic segments diverge along $\beta$, the least-area solutions converge to a complete embedded surface in $\mathbb{H}^2 \times \mathbb{R}$ with boundary $\{(x,0)/x \geq 0\} \cup \{(y,0)/y \geq 0\} \cup \{(x,h)/x \geq 0\} \cup \{(y,h)/y \geq 0\}$. The symmetries of this surface along all the edges yields a singly periodic Scherk surface in $Q \times \mathbb{R}$, invariant by $T(2h)$.

As in Example 3, we identify the opposite sides of $Q$ by hyperbolic translations to obtain a torus $M$ with one puncture. This gives the Scherk surface $\Sigma$ in $M \times (\mathbb{R}/[T(2h)])$, with four vertical annular ends.

We remark that one can quotient $\partial Q$ by parabolic isometries to obtain this Scherk surface in $M \times \mathbb{S}^1$, where $M$ is now a 4-punctured sphere.

Example 5. A helicoid with helicoidal ends in $M \times \mathbb{S}^1$, where $M$ is a once punctured torus. It is convenient to describe this example in $M \times \mathbb{S}^1$, where $M$ is the once punctured torus obtained from the ideal quadrilateral $Q_1$ in $\mathbb{H}^2$ with the 4 vertices $(\pm \frac{1}{\sqrt{2}}, \pm \frac{i}{\sqrt{2}})$, by identifying opposite sides.

Let $S$ be the third quadrant of $Q_1$: $S = \{(x,y) \in Q_1; x \leq 0, y \leq 0\}$. For $h > 0$, let $\Sigma_1$ be the minimal graph over $S$ with boundary values indicated in Figure 7.

Let $\Sigma_3$ be the reflection of $\Sigma_1$ through $\beta$ (see Figure 7), $\Sigma_3$ is between heights $h$ and $2h$ and is a graph over the second quadrant of $Q_1$. Then rotate $\Sigma_1 \cup \Sigma_3$ by $\pi$ through the vertical axis between $(0,0)$ and $(0,2h)$, to obtain $\Sigma_2 \cup \Sigma_4$, where $\Sigma_4$ is a graph over the fourth quadrant of $Q_1$. $\Sigma$ is the union of the four pieces $\Sigma_1$, through $\Sigma_4$, identified along the boundaries as follows.

First we consider identifying opposite sides of $Q_1$ by the hyperbolic translations sending the opposite side to the other. Then we can quotient by $T(2h)$ or by
$\Sigma_1$ be a minimal graph over $S$. The first quotient gives a non-orientable surface in $M \times \mathbb{S}^1$ with one helicoid type end. The second gives an orientable surface of total curvature $-8\pi$ with two helicoidal type ends (it is a double cover of the first example). Topologically the first example is the connected sum of a once punctured torus and a projective plane. The second surface is 2 punctured orientable surface of genus two.

The reader can see the helicoidal structure of $\Sigma$ by going along a horizontal geodesic on $\Sigma$ at $h = 0$, from one puncture to the other. Then spiral up $\Sigma$ along a helice going to the horizontal geodesic at height $h$. Continue along this geodesic to the other (it’s the same) puncture and spiral up the helices on $\Sigma$ to height $2h$. If we do this correctly, then we are back where we started.

4. **Barriers in** $M \times \mathbb{S}^1$. We construct barriers by solving the mean curvature equation of ruled surfaces. These barriers will be used to prove the Trapping Theorem in Section 5. In the model $\mathbb{H}^2 \times \mathbb{R} = \{(x, y, t) \in \mathbb{R}^3; y > 0\}$, we consider surfaces

$$X : (u, v) \rightarrow (u, \alpha(v), v + \lambda u)$$

for a $C^2$ real positive function of one variable $v \rightarrow \alpha(v)$ defined on some interval $I$.

**Lemma 4.1.** The mean curvature $H$ of the surface $X : (u, v) \rightarrow (u, \alpha(v), v + \lambda u)$ immersed in $\mathbb{H}^2 \times \mathbb{R} = \{(x, y, t) \in \mathbb{R}^3; y > 0\}$ with the metric $ds^2 = \ldots$
\[
\frac{dx^2 + dy^2}{y^2} + dt^2 \text{ is given by }
\]
\[
2H = \frac{-\alpha''}{Z^3} \left[ \alpha''(1 + \lambda^2 \alpha^2) + \alpha(1 + \lambda^2 (\alpha')^2) \right].
\]

**Proof.** In the model of \( \mathbb{H}^2 \times \mathbb{R} = \{(x, y, t) \in \mathbb{R}^3; y > 0\} \) with the metric \( ds^2 = \frac{dx^2 + dy^2}{y^2} + dt^2 \), the non-zero terms of the connection are given by
\[
\nabla_{\partial/\partial x} \partial/\partial x = \frac{1}{y} \partial/\partial y \\
\nabla_{\partial/\partial y} \partial/\partial y = -\frac{1}{y} \partial/\partial y \\
\nabla_{\partial/\partial y} \partial/\partial x = \nabla_{\partial/\partial y} \partial/\partial x = -\frac{1}{y} \partial/\partial x.
\]

The tangent space is generated by
\[
dX(\partial/\partial u) = E_1 = \partial/\partial x + \lambda \partial/\partial t = (1, 0, \lambda) \\
dX(\partial/\partial v) = E_2 = \alpha'(v) \partial/\partial y + \partial/\partial t = (0, \alpha'(v), 1)
\]
The direct unit normal vector is given by \( N = V/Z \) with \( V = E_1 \wedge E_2 \),
\[
V = -\lambda \alpha'(v) \alpha(v)^2 \partial/\partial x - \alpha(v)^2 \partial/\partial y + \alpha'(v) \partial/\partial t \\
= (\lambda \alpha'(v) \alpha(v)^2 - \alpha(v)^2, \alpha'(v)) \\
Z^2 = |V|^2 = (\alpha'(v))^2 (1 + \lambda^2 \alpha(v)^2) + \alpha(v)^2.
\]

We compute the mean curvature by the divergence formula
\[
-2H = \operatorname{div}(N) = \operatorname{div} \left( \frac{V}{Z} \right) = \frac{1}{Z^3} \left( Z^2 \operatorname{div}(V) - \frac{1}{2} V(Z^2) \right).
\]

We compute the first term
\[
\operatorname{div}(V) = -\frac{\partial}{\partial x} (\lambda \alpha^2(v) \alpha'(v)) - \lambda \alpha^2(v) \alpha'(v) \operatorname{div} \left( \frac{\partial}{\partial x} \right) \\
- \frac{\partial}{\partial y} (\alpha^2(v)) - \alpha^2(v) \operatorname{div} \left( \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial t} (\alpha'(v)) + \alpha'(v) \operatorname{div} \left( \frac{\partial}{\partial t} \right).
\]

Using \( \operatorname{div} \left( \frac{\partial}{\partial x} \right) = \operatorname{div} \left( \frac{\partial}{\partial t} \right) = 0 \) and \( \operatorname{div} \left( \frac{\partial}{\partial y} \right) = -\frac{2}{y} \) with \( \alpha(v) = y \) and \( v = t - \lambda x \),
a direct computation gives
\[
\operatorname{div}(V) = \lambda^2 \alpha^2 \alpha'' - 2 \alpha - \alpha^2 \left( -\frac{2}{\alpha} \right) + \alpha'' \\
= (1 + \lambda^2 \alpha^2) \alpha''.
\]
For the second term
\[
\frac{1}{2} \frac{\partial}{\partial x} (Z^2) = -\lambda \alpha' \alpha'' (1 + \lambda^2 \alpha^2) \\
\frac{1}{2} \frac{\partial}{\partial y} (Z^2) = \alpha (1 + \lambda^2 \alpha^2) \\
\frac{1}{2} \frac{\partial}{\partial t} (Z^2) = \alpha' \alpha'' (1 + \lambda^2 \alpha^2).
\]

Hence \( \frac{1}{2} V(Z^2) = (1 + \lambda^2 \alpha^2) \lambda^2 \alpha^2 \alpha'' - \alpha^3 (1 + \lambda^2 \alpha^2) + \alpha^2 \alpha'' (1 + \lambda^2 \alpha^2) \). Finally we obtain
\[
\text{div}(N) = \text{div}(V/Z) \\
= \frac{1}{Z^3} [(1 + \lambda^2 \alpha^2) \alpha'' (\alpha^2 + \alpha^2 + \lambda^2 \alpha^2 \alpha^2) \\
- (1 + \lambda^2 \alpha^2) \lambda^2 \alpha^2 \alpha'' + \alpha^3 (1 + \lambda^2 \alpha^2) + \alpha^2 \alpha'' (1 + \lambda^2 \alpha^2)] \\
= \frac{\alpha^2}{Z^3} [\alpha'' (1 + \lambda^2 \alpha^2) + \alpha (1 + \lambda^2 \alpha^2)] = -2H. \quad \square
\]

We study the geometry of surfaces \( X : (u, v) \to (u, \alpha(v), v + \lambda u) \) which are minimal. We notice they are ruled surfaces foliated by curves \( v \to (0, \alpha(v), v) \) where \( \alpha \in C^2(I), \alpha > 0, \lambda \geq 0 \).

First consider the case when \( \lambda = 0 \). The solution \( \alpha(v) = T \sin v \) gives the family of minimal surfaces which are defined up to a vertical translation by
\[
S^0_T = \{(u, T \sin v, v) \in \mathbb{R}^3; \ u \in \mathbb{R}, \ v \in [0, \pi]\}.
\]

This surface is foliated by the horizontal horocycles \( u \to (u, \alpha(v), v) \); it is described in Hauswirth [7], then by Toubiana and Sa Earp [17], Daniel [5] and Mazet, Rodriguez, Rosenberg [10] (see Figure 8). By the nature of the curve \( v \to (0, T \sin v, v) \), the surfaces \( S^0_T \) foliate the slab \( S = \{(x, y, t) \in \mathbb{R}^3; \ 0 < y, \ 0 \leq t \leq \pi\} \).

The general case where \( \lambda \neq 0 \) depends on a function \( \alpha \), that is a solution of the equation
\[
\alpha'' (1 + \lambda^2 \alpha^2) + \alpha (1 + \lambda^2 (\alpha')^2) = 0.
\]

This equation has a first integral \( (1 + \lambda^2 \alpha^2)(1 + \lambda^2 \alpha^2) = T \) for some fixed constant \( T > 1 \). Since \( \alpha'' < 0 \), the curves \( v \to (0, \alpha(v), v) \) are convex. For fixed \( T > 1 \), the function \( \alpha_T(v) \) has its maximum value at \( \sup \alpha_T(v) = \lambda^{-1} \sqrt{T - 1} \). The function \( \alpha_T \) is positive on a set \([0, v_0(T)]\). The solution \( \alpha_T(v) \) with initial data \( \alpha_T(0) = 0 \) and \( \alpha'(0) = \lambda^{-1} \sqrt{T - 1} \) defines a one-parameter family of minimal surfaces
\[
S^\lambda_T = \{(u, \alpha_T(v), v + \lambda u) \in \mathbb{R}^3; \ u \in \mathbb{R}, \ v \in [0, v_0(T)]\}.
\]
For large values of $T$ and a fixed constant $M > 0$, we look for the set of values where $0 \leq \alpha_T(v) \leq M$. On this set we remark that

$$\alpha'^2 = \lambda^{-2} \left( \frac{T}{1 + \lambda^2 \alpha^2} - 1 \right) \geq \lambda^{-2} \left( \frac{T}{1 + \lambda^2 M^2} - 1 \right).$$

This inequality implies that $\alpha'_T(v) \to \infty$ when $T \to \infty$. The part of the curve $(0, \alpha_T(v), v)$ contained in the domain $0 < y \leq M$ converges to the half-geodesic \{$(0, y, 0) \in \mathbb{R}^3; \ 0 < y \leq M$\}.

We summarize this discussion in Figure 9 and we will use the following lemma:

**Lemma 4.2.** (a) The family of surfaces $S^0_T$, foliates the slab $\mathbb{H}^2 \times [0, \pi]$ and when $T$ goes to infinity the surfaces $S^0_T$ converge on compact sets to the horizontal section $\mathbb{H}^2 \times \{0\}$.

(b) The one-parameter family of surfaces $S^\lambda_T$ converges on compact sets to \{$(x, y, t) \in \mathbb{R}^3; \ y > 0$ and $t = \lambda x$\}.

5. Trapping theorem for minimal ends. In this section $\Sigma$ will denote a properly immersed minimal surface in $M \times S^1$ of finite topology. Hence each end of $\Sigma$ is an annular end. Since $\Sigma$ is properly immersed, each end $A_0$ of $\Sigma$ is contained in some end $M$ of $M \times S^1$. 
Lemma 5.1. There is \( y_0 \geq 1 \) and a subend \( A \) of \( A_0 \) such that \( \partial A \subset \mathcal{T}(y_0) \), \( A \subset \bigcup_{y \geq y_0} \mathcal{T}(y) \).

Proof. Since \( \Sigma \) is properly immersed each end of \( \Sigma \) has a subend \( A_0 \) contained in some \( \mathcal{M} = \bigcup_{y \geq 1} \mathcal{T}(y) \). Since \( A_0 \) is transverse to almost every \( \mathcal{T}(y) \), let \( y_0 > 1 \) be such that \( \partial A_0 \subset \bigcup_{1 \leq y < y_0} \mathcal{T}(y) \), and \( A_0 \) is transverse to \( \mathcal{T}(y_0) \). Then \( A_0 \cap \mathcal{T}(y_0) = C_1 \cup \ldots \cup C_k \), where each \( C_j \) an immersed Jordan curve in \( \mathcal{T}(y_0) \).

\( A_0 \) is proper and so \( A_0 \cap \mathcal{T}(y) \neq \emptyset \) for large \( y \), \( A_0 \cap \{ y \leq y_0 \} \) is compact. If each \( C_j \) bounds an immersed disk in \( A_0 \) then \( A_0 \) is compact; this is a contradiction. Hence at least one of the \( C_j \) is not null homotopic in \( A_0 \). Observe that there is at most one such \( C_j \). For if \( C_i \) and \( C_j \) are not trivial then they bound a compact domain \( F \) in \( A_0 \) disjoint from \( \partial A_0 \). \( F \) cannot be contained in \( \bigcup_{1 \leq y < y_0} \mathcal{T}(y) \) since then \( F \) would touch some \( \mathcal{T}(y_1), y_1 < y_0 \), on the mean convex side of \( \mathcal{T}(y_1) \), a contradiction. So \( F \subset \bigcup_{y \geq y_0} \mathcal{T}(y) \). But then, \( \partial A_0 \) and \( C_i \) or \( C_j \) (\( C_i \) say) would bound a compact domain \( F_1 \) on \( A_0 \) and \( C_j \subset A_0 - F_1 \). \( A_0 - F_1 \) is an annular sub-end of \( A_0 \) with boundary \( C_i \) contained in \( \partial \mathcal{T}(y_0) \). Since \( A_0 - F_1 \) intersects \( \mathcal{T}(y_0) \) also at \( C_j \), there is a compact domain \( F_2 \) of \( A_0 - F_1 \) contained in \( \bigcup_{1 \leq y \leq y_0} \mathcal{T}(y) \) with \( \partial F_2 \subset \mathcal{T}(y_0) \); a contradiction; see Figure 10.

Now it is clear that if each \( C_\ell, \ell \neq i \) bounds a disk \( D \) on \( A_0 \) that is contained in \( \bigcup_{y \geq y_0} \mathcal{T}(y) \). It follows that the connected component of \( A \) in \( \bigcup_{y \geq y_0} \mathcal{T}(y) \) that has \( C_i \) in its boundary has no other \( C_\ell, \ell \neq i \), in its boundary. This proves the lemma. \( \square \)

By a change of coordinates on \( \mathcal{M} \), we can assume that the end \( A \) is in \( \bigcup_{y \geq 1} \mathcal{T}(y) \) and \( \partial A \subset \mathcal{T}(1) \). Let \( E \) be a connected component of the lift of \( A \) to
$\mathbb{H}^2 \times \mathbb{R}$. The boundary $\partial E \subset P := \{(x,y,t) \in \mathbb{R}^3; y = 1\}$ and $E$ is transverse to $P$. There is $(p,q)$ such that the curve $\partial E$ is invariant by the isometry of $\mathbb{H}^2 \times \mathbb{R}$

$$\psi^p \circ T(h)^q : (x,y,t) \rightarrow (x+p\tau,y,t+qh).$$

We say that $A$ and $E$ are of type $(p,q)$. The curve $\partial A$ is a curve of the torus $\mathbb{T}(1)$. We prove in the following lemma that $(p,q) \neq (0,0)$.

**Lemma 5.2.** The end $E$ is topologically a half-plane and $\partial E$ is a non-compact curve in $P$.

**Proof.** Assume the contrary, and let $E$ be a lifting of $A$ to $\mathbb{H}^2 \times \mathbb{R}$, $E$ an immersed annulus in $\{y \geq 1\}$, $\partial E \subset H = \{y = 1\}$. We know the coordinate $y$ is a proper function on $E$.

Denote by $\Pi : \mathbb{H}^2 \times \mathbb{R} \rightarrow \mathbb{H}^2 = \mathbb{H}^2 \times \{0\}$, the vertical projection. Let $\gamma_1$, $\gamma_2$ be disjoint geodesics of $\mathbb{H}^2$, disjoint from $\Pi(\partial E)$, and that separate $\Pi(\partial E)$ from the point at infinity of $\mathbb{H}^2$; see Figure 11 (the set $\Pi(\partial E)$ is compact). Let $\Omega \subset \mathbb{H}^2$ be the domain of $\mathbb{H}^2$ bounded by $\gamma_1 \cup \gamma_2$, so $\Pi(\partial E) \cap \Omega = \emptyset$.

For $a \in \mathbb{R}$, solve the Dirichlet problem on $\Omega$ to find a minimal graph over $\Omega$, with asymptotic values $+\infty$ on $\gamma_1 \cup \gamma_2$, and $a$ on $\partial_\infty(\Omega)$ (see Theorem 4.9 of Mazet, Rodriguez, and Rosenberg [9]). To apply Theorem 4.9, one must choose the geodesics far enough apart. More precisely, let $x_1$, $x_2$ be the end points of $\gamma_1$ on $\partial_\infty \mathbb{H}^2$ and $x_3$, $x_4$ those of $\gamma_2$. After a conformal isometry of the disk $\mathbb{H}^2$, one can assume $(x_1,x_2,x_3)$ are the points on $((0,-1),(1,0),(0,1))$ (see Figure 12). When $x_4$ is on the open arc at infinity from $x_3$ to $(-1,0)$ the condition $2\alpha < \gamma$ of Theorem 4.9 in [9] is satisfied.

An equivalent formulation of the condition on $\gamma_1$ and $\gamma_2$ is that the distance between $\gamma_1$ and $\gamma_2$ (the length of the geodesic arc orthogonal to $\gamma_1$ and $\gamma_2$) be strictly larger than the distance between $\gamma_1$ and $\gamma_2$; when $x_4 = (-1,0)$; the 4 points then form an ideal regular quadrilateral.

By varying $a$ we obtain a first point of contact of the graph with $E$; this is a contradiction. \hfill \Box
Figure 11. An annular end in $\mathcal{M}$.

Figure 12. The geodesics $\gamma_1$ and $\gamma_2$. 
Now we prove that an end $E$ of type $(p, q), (p, q) \neq (0, 0)$ is trapped between two ends of type $E_{(p, q)}$:

**Theorem 5.3.** (The Trapping Theorem) Let $A \subset \Sigma$ be a properly immersed end in $\mathcal{M}$ with $\partial A \subset \mathbb{T}(1)$ and $A$ transverse to $\mathbb{T}(1)$. If $\partial A$ is a curve of type $(p, q)$ in $\mathbb{T}(1)$, then $A$ is contained in a slab on $\mathcal{M}$ bounded by two standard ends $A_{(p, q)}$.

*Proof.* We use the model of $\mathcal{M} = (H \times \mathbb{R})/\{\psi, \mathbb{T}(h)\}$ and a connected component $E$ of a lifting of $A$ in $\mathbb{H}^2 \times \mathbb{R}$. We prove that $E$ is contained in a slab bounded by two half-planes $E_{(p, q)}$ in $H \times \mathbb{R}$ where $H = \{(x, y) \in \mathbb{R}^2; y \geq 1\}$.

*Case $(p, q) = (0, q)$.* First we begin with the case where the curve $\partial A$ is of type $(0, q)$. This means that the boundary $\partial E$ is a periodic curve invariant by vertical translation $T(h)^q = T(qh)$. From this invariance of $\partial E$, we know there exists $x_{\text{Min}}$ and $x_{\text{Max}}$ such that $\partial E \subset \{(x, 1, t); x_{\text{Min}} \leq x \leq x_{\text{Max}}\}$.

Let $Q = \{(x, y) \in \mathbb{R}^2; x \geq x_{\text{Max}} \text{ and } y > 0\}$. Foliate $Q$ by the geodesics $\gamma_T$ whose end-points at infinity are $(x_{\text{Max}}, 0)$ and $(T, 0)$; $T > x_{\text{Max}}$. For $|T - x_{\text{Max}}| < 2$, $\gamma_T \cap \{y \geq 1\} = \emptyset$. Define $S_T = \gamma_T \times \mathbb{R}$; so $S_T \cap E = \emptyset$ for $|T - x_{\text{Max}}| < 2$.

Now let $T$ increase to $\infty$, so $S_T$ converges to $\{(x_{\text{Max}}, y); y > 0\} \times \mathbb{R}$. Since $E$ is periodic, $S_T$ must be disjoint from $E$ for all $T > 1$; otherwise there would be a first point of contact (i.e., the two surfaces cannot have a first contact point at infinity), contradicting the maximum principle (see Figure 13).

The same argument using $x_{\text{Min}}$ shows $E$ is trapped between two standard ends of type $E_{(0, 1)}$.

*Case $(p, q) = (p, 0)$.* Now $\partial E$ is a curve invariant by $\psi^p(x, y, t) = (x + p\tau, y, t)$. Let $t_{\text{Min}}$ and $t_{\text{Max}}$ satisfy:

$$\partial E \subset \{(x, 1, t); zt_{\text{Min}} \leq t \leq t_{\text{Max}}\}.$$

![Figure 13. An annular end in $\mathcal{M}$.](image-url)
The same argument with By Lemma 4.2, the family $S_{T} >$ For $|T| \leq 1$, we have $S_{T}^{0}(t) \cap (H \times \mathbb{R}) = \emptyset$, and $\partial E$ is below height $t = t_{\text{Max}}$. By Lemma 4.2, the family $S_{T}^{0}(t_{\text{Max}})$ converges on compact sets to the horizontal section $t = t_{\text{Max}}$. For $t > t_{\text{Max}}$, $t$ large, $T_{0}$ given, we have $S_{T_{0}}^{0}(t) \cap E = \emptyset$.

If $S_{T_{0}}^{0}(t_{\text{Max}}) \cap E \neq \emptyset$, then since $E$ is periodic, there would be a first $t_{1}$ such that $S_{T_{0}}^{0}(t_{1}) \cap E \neq \emptyset$, contradicting the maximum principle. Thus $E$ is below $t = t_{\text{Max}}$. The same argument with $S_{T}^{0}(t) := \{(u, T \sin \nu, v + t) \in \mathbb{R}^{3}; \ u \in \mathbb{R}, \ v \in [0, \pi]\}$ with $t \leq t_{\text{Min}}$ shows $E$ is above $t = t_{\text{Min}}$. Thus $E$ is trapped between two standard ends of type $E_{(p,0)}$.

Case $(p, q) \neq (0, q), (p, 0)$. Now we use the family of barriers $S_{T}^{\lambda}$. $\partial E$ is invariant by the isometry $\psi^{p} \circ T(h)^{q} : (x, y, t) \rightarrow (x + p\tau, y, t + qh)$ on $y = 1$. Thus there exists $c_{\text{Min}}, c_{\text{Max}}$ such that
\[
\partial E \subset \{(x, 1, t) \in \mathbb{R}^{3}; \ c_{\text{Min}} \leq p\tau t - qhx \leq c_{\text{Max}}\}.
\]
We use $S_{T}^{\lambda}$ of Lemma 4.2 with $\lambda = \frac{qh}{p\tau}$; see Figure 9:

$S_{T}^{\lambda}(t) := \{(u, \alpha_{T}(v), v + \lambda u + t) \in \mathbb{R}^{3}; \ u \in \mathbb{R}, \ v \in [0, v_{0}(T)]\}$ with $t \geq c_{\text{Max}}/(p\tau)$.

For $T > 1$ fixed, there is $t_{0} > c_{\text{Max}}/(p\tau)$ large so that $S_{T}^{\lambda}(t_{0}) \cap E = \emptyset$. Decreasing $t$ from $t_{0}$ to $c_{\text{Max}}/(p\tau)$ we conclude (there is no first point of contact) that $E$ is below $S_{T}^{\lambda}(c_{\text{Max}}/(p\tau))$ for any $T$. Let $T \rightarrow \infty$; the $S_{T}^{\lambda}(c_{\text{Max}}/(p\tau))$ converge to $\{(x, y, t) \in \mathbb{R}^{3}; \ p\tau t - qhx = c_{\text{Max}}\}$, hence
\[
E \subset \{(x, y, t) \in \mathbb{R}^{3}; \ p\tau t - qhx \leq c_{\text{Max}}\}.
\]

The same argument with
\[
S_{T}^{\lambda}(t) := \{(u, \alpha_{T}(v), v + \lambda u - v_{0}(T) + t) \in \mathbb{R}^{3}; \ u \in \mathbb{R}, \ v \in [0, v_{0}(T)]\}
\]
with $t \leq c_{\text{Min}}/(p\tau)$ shows that
\[
E \subset \{(x, y, t) \in \mathbb{R}^{3}; \ p\tau t - qhx \geq c_{\text{Min}}\},
\]
which completes the proof of the theorem. \qed
6. The Dragging Lemma.

**Dragging Lemma 6.1.** Let \( g : \Sigma \to N \) be a properly immersed minimal surface in a complete 3-manifold \( N, \partial \Sigma = \emptyset \). Let \( A \) be a compact surface (perhaps with empty boundary) and \( f : A \times [0, 1] \to N \) a \( C^1 \)-map such that \( f(A \times \{t\}) = A(t) \) is a minimal immersion for \( 0 \leq t \leq 1 \). If \( \partial(A(t)) \cap g(\Sigma) = \emptyset \) for \( 0 \leq t \leq 1 \) and \( A(0) \cap g(\Sigma) \neq \emptyset \), then there is a \( C^1 \) path \( \gamma(t) \) in \( \Sigma \), such that \( g \circ \gamma(t) \in A(t) \cap g(\Sigma) \) for \( 0 \leq t \leq 1 \). Moreover we can prescribe any initial value \( g \circ \gamma(0) \in A(0) \cap g(\Sigma) \).

**Proof.** When there is no chance of confusion we will identify in the following \( \Sigma \) and its image \( g(\Sigma), \gamma \subset \Sigma \) and \( g \circ \gamma \) in \( g(\Sigma) \subset N \). In particular when we consider embeddings of \( \Sigma \) there is no confusion.

Let \( \Sigma(t) = g(\Sigma) \cap A(t) \) and \( \Gamma(t) = f^{-1}(\Sigma(t)) \), \( 0 \leq t \leq 1 \) the pre-image in \( A \times [0, 1] \).

We consider \( p_0 \in g(\Sigma) \cap A(0) \), and pre-images \( z_0 \in g^{-1}(p_0) \) and \( (q_0, 0) \in f^{-1}(p_0) \). We will obtain the arc \( \gamma(t) \in \Sigma \) in a neighborhood of \( z_0 \) by a lift of an arc \( \eta(t) \) in a neighborhood of \( (q_0, 0) \) in \( \Gamma([0, 1]) \), i.e., \( g \circ \gamma(t) = f \circ \eta(t) \). We will extend the arc continuously by iterating the construction.

Since \( \Sigma \) is properly immersed, each \( A(t) \) is compact and \( \partial A(t) \neq \emptyset \), it follows from the maximum principle that \( \Gamma(t) \) is a finite union of compact smooth immersions of topological circles \( \Gamma_1(t), \ldots, \Gamma_k(t) \), \( k \) may depend on \( t \). By hypothesis, \( \Gamma(t) \cap (\partial A \times [0, 1]) = \emptyset \) for all \( t \). The maximum principle and the fact that \( A(t) \neq g(\Sigma) \) for all \( t \), assures that \( \Gamma(t) \) cannot contain an isolated point for \( 0 \leq t \leq 1 \).

Suppose \( \Sigma \) is a Riemannian surface and \( K \subset \Sigma \) is a compact domain. Given \( d_1 > 0 \), there exists \( d_2 > 0 \) such that if there exists a loop \( \gamma \subset K \) with \( \text{diam}(\gamma) < d_2 \) then \( \gamma \) is the boundary of a disk \( D \) in \( K \) with \( \text{diam}(D) < d_1 \).

Next consider the compact minimal surface \( A(t) \). Minimal surfaces are locally stable so there exists \( d_3 > 0 \) and \( d_4 > 0 \), such that if \( \gamma \) is a loop on \( A(t) \), \( \gamma \) the image of a loop on \( A \), and \( \text{diam}(\gamma) < d_3 \), then \( \gamma \) bounds a disk \( D \) on \( A(t) \) and there is no other minimal disk in \( N \), with \( \partial D = \gamma \), \( \text{diam}(D) < d_4 \), other than \( D \).

Now consider loops in \( \Gamma(t) \) that lift to loops in \( \Sigma \). By the previous paragraph we know that such loops cannot be too small.

**Claim 1.** We will see that for each \( t \) with \( \Gamma(t) \neq \emptyset \), \( t < 1 \) there is a \( \delta(t) > 0 \) such that if \( (q, t) \in \Gamma(t) \), then there is a \( C^1 \) arc \( \eta(\tau) \) defined for \( t \leq \tau \leq t + \delta(t) \) such that \( \eta(t) = (q, t) \) and \( \eta(\tau) \in \Gamma(\tau) \) for all \( \tau \).

First suppose \( (q, t) \in \Gamma(t) \) is a point where \( A(t) = f(A \times \{t\}) \) and \( g(\Sigma) \) are transverse at \( f(q, t) \). Let us consider the \( C^1 \) immersions

\[
F : A \times [0, 1] \to N \times [0, 1] \quad \text{with} \quad F(q', \tau) = (f(q', \tau), \tau)
\]

\[
G : \Sigma \times [0, 1] \to N \times [0, 1] \quad \text{with} \quad G(z, \tau) = (g(z), \tau).
\]
Let $\hat{M} = F(A \times [0, 1]) \cap G(\Sigma \times [0, 1])$ and $M = F^{-1}(\hat{M})$. $F(A \times [0, 1])$ and $G(\Sigma \times [0, 1])$ are transverse at $p = F(q,t)$. Thus $\hat{M}$ is a 2-dimensional surface of $N \times [0, 1]$ near $p$, and $M$ is a 2-dimensional surface of $A \times [0, 1]$ in a neighborhood of $(q,t)$, transverse to $A \times \{t\}$ at $(q,t)$.

Suppose now, that $0 < t$. In a neighborhood $U$ of $(q,t)$ in $M$, $M$ meets $A \times \{\tau\}$, $\tau$ near $t$, transversally in smooth arcs in $\Gamma(\tau)$ and the angle the tangent plane of $M$ makes with the slices $A \times \{\tau\}$ is bounded away from zero in $U$. Let $Y$ be the unit vector field tangent to $U$ which is a conormal vector of the curves $\Gamma(\tau)$ in $U$.

There is a $\delta = \delta(q,t) > 0$, such that for each $(q',t) \in \Gamma(t)$, a distance along $\Gamma(t)$ at most $\delta$ from $(q,t)$, an arc of the integral curve of $Y$ through $(q',t)$ can be parametrized by a $C^1$ map $\tau \to \eta(\tau) \in M$ defined for $\tau \in [t-\delta(q), t+\delta(q)]$, with $\eta(t) = (q',t)$ and $\eta(\tau) \in \Gamma(\tau)$.

The curve $f \circ \eta(\tau)$ then lifts to a curve $\gamma(\tau)$ in $\Sigma$ such that $g \circ \gamma(\tau) = f \circ \eta(\tau)$.

When $t = 0$, the above argument gives $\eta(\tau)$ and $\gamma(\tau)$ defined for $0 \leq \tau \leq \delta(q,0)$, satisfying at each $(q',0)$, $\delta$-close to $(q,0)$ (assuming transversality of $A(0)$ and $g(\Sigma)$ at $f(q,0)$).

We will find a $\delta > 0$ that works in a neighborhood of a singular point $(q,t) \in \Gamma(t)$, where there is a $z \in \Sigma$ such that $f(q,t) = g(z)$ and $T_{f(q,t)}A(t) = T_{g(z)}g(\Sigma)$. We consider singularities of $\Gamma(t)$ where $A(t)$ and $g(\Sigma)$ are tangent. Near a singularity $(q,t) \in \Gamma(t)$, $\Gamma(t)$ contains $2k$ differentiable curves intersecting at $(q,t)$ with non-zero angles, $k \geq 1$.

Let $V$ be a neighborhood of $(q,t)$ in $A \times \{t\}$. The set $\Gamma(t) \cap V$ is $2k$ smooth curves on $A \times \{t\}$, meeting at $(q,t)$ with non-zero angles. Let $\alpha : [-\epsilon, \epsilon] \to V \cap \Gamma(t) \subset A \times \{t\}$, $\alpha(s) = (a(s), t)$ be a regular parametrization of one of these curves with $\alpha(0) = (q,t)$ and $\alpha(\pm \epsilon) \in \partial V$.

For $s \in (-\epsilon, \epsilon)$, $s \neq 0$, $A(t)$ and $g(\Sigma)$ are transverse at $f(\alpha(s))$. As we previously observed, there is a $\delta(s) = \delta(a(s), t) > 0$ and a $C^1$ map $\eta_s(\tau)$ defined for $t - \delta(s) \leq \tau \leq t + \delta(s)$, (here we assume $t > 0$) such that $\eta_s(t) = (a(s), t) = \alpha(s)$ and $\eta_s(\tau) \in \Gamma(\tau)$.

For $-\epsilon \leq s \leq \epsilon$, we choose $\delta(s)$ so that it is $C^1$, strictly increasing from $\delta(0) = 0$ to $\delta(\epsilon) = \delta(a(\epsilon), t)$ and $\delta(-s) = \delta(s)$ with $\delta'(0) = 0$. Then $\tilde{\eta}_+(s) = \eta_s(t + \delta(s))$ is a $C^1$ path in $M$ joining $(q,t)$ to $(a(\epsilon), t + \delta(\epsilon))$ with $\tilde{\eta}_+(s) \in \Gamma(t + \delta(s))$ for each $s$ between 0 and $\epsilon$.

In the same manner, one constructs a $C^1$ path $\tilde{\eta}_-(s)$, for $-\epsilon \leq s \leq 0$, joining $(a(-\epsilon), t - \delta(\epsilon))$ to $(q,t)$, with $\tilde{\eta}_-(0) = (q,t)$, and $\tilde{\eta}_-(s) \in \Gamma(t - \delta(s))$. Then reparametrizing $\tilde{\eta}_+(s)$ by $\tau = t + \delta(s)$ and $\tilde{\eta}_-(s)$ by $\tau = t - \delta(s)$ we obtain a piecewise $C^1$ path $\eta(\tau) \in \Gamma(\tau)$ joining monotonically $(a(-\epsilon), t - \delta(\epsilon))$ to $(a(\epsilon), t + \delta(\epsilon))$ as desired (see Figure 14).

We then do this construction for each of the $k$-arcs of $V \cap \Gamma(t)$ through $(q,t)$ and the minimum value of the $\delta$’s of each arc completes the proof of the claim.

Since $\Gamma(0) \neq \emptyset$, Claim 1 shows that the set of $t$ for which $\eta(t)$ is defined is a non-empty open set. This defines an arc $\gamma(\tau)$ as a lift of $f \circ \eta(\tau) \subset A(\tau)$ in a
neighborhood of $\gamma(t) \in \Sigma$. To complete the proof of the Dragging lemma, it suffices to show the set of $t$ for which $\gamma(t)$ is defined is closed. Assume that there is a point $t_0$ such that the arc $\gamma(t)$ is defined in a $C^1$ manner for $t < t_0$. By compactness of $A$, the arc $\gamma(t)$ accumulates at a point $(q, t_0) \in \Gamma(t_0)$. If $(q, t_0)$ is a transversal point, one can easily find (by the structure we previously described near a transversal intersection) a piecewise $C^1$ path $\gamma(t)$ from some $\gamma(t_1), t_1$ near $t_0$ to $(q, t_0)$. In the case where $(q, t_0)$ is a singular point, since this point is isolated, the curve accumulates in only one point and extends continuously to the limit $(q, t_0)$. To ensure a $C^1$ path through $t_0$, we need a more careful analysis at $(q, t_0)$.

Claim 2. Suppose the path $\gamma(t)$ satisfies the conditions of the Dragging lemma for $0 \leq t < t_0 < 1$. Then $\gamma(t)$ can be extended to $0 < t < t_0 + \delta$, to be $C^1$ and satisfy the conditions of the Dragging lemma, for some $\delta > 0$.

If $(q, t_0)$ is a transversal point, $M$ has a structure of a manifold and if $t_0 - \delta(t_0) < t_1 < t_0$ and $\eta(t_1) = (q_1, t_1)$ is in a neighborhood of $(q, t_0)$, we can find a $C^1$ arc that joins $\eta(t_1)$ to $(q, t_0) \in \Gamma(t_0)$. Next we extend the arc for $t_0 \leq t \leq t_0 + \delta(t_0)$.

If $(q, t_0)$ is a singular point, we consider a neighborhood $V \subset A$ of $q$ and $\Gamma(t_0)$ intersects $\partial V$ in $2k$ transversal points $q_1, \ldots, q_{2k}$. We consider $V \times [t_1, t_0]$ with $t_0 - \delta(t_0) < t_1 < t_0$. By transversality at $(q_1, t_0), \ldots, (q_{2k}, t_0)$, the analytic set $\Gamma(t_1)$ intersects $\partial V$ in $2k$ points and $V$ in $k$ analytical arcs $\alpha_1, \ldots, \alpha_k$. We suppose that $\eta(t_1) \in \alpha_1 \subset V \times \{t_1\}$. We construct below a monotonous $C^1$ arc from $\eta(t_1)$ to a point $(\hat{q}, t_2)$ on $\partial V \times \{t_2\}$ for some $t_1 < t_2 < t_0$ and by transversality an arc from $(\hat{q}, t_2)$ to a point $(q', t_0) \in \partial V \times \{t_0\}$, using the fact that $t_0 - \delta(t_0) < t_2$. Next we can extend the arc in a $C^1$ manner from $(q', t_0)$ to some point in $\Gamma(t_0 + \delta(t_0))$. 

Figure 14. Neighborhood of a singular point.
Figure 15. The curve $\eta(\tau)$ passing through several singularities.

We consider $(\tilde{q}_1, t_1), \ldots, (\tilde{q}_\ell, t_1)$ singular points of $\Gamma(t_1) \cap V \times \{t_1\}$ and we denote by $W_1, \ldots, W_\ell$ neighborhoods of $\tilde{q}_1, \ldots, \tilde{q}_\ell$ in $A \cap V$. The arc $\alpha_1$ cannot have double points in $V$ without creating small loops. Hence $\alpha_1$ passes through each $W_1, \ldots, W_\ell$ at most one time, before joining a point of $\partial V$ (We can restrict $V$ in such a way that there are no small loops in $V$).

First we assume that there is $t_2$ such that for any $t \in [t_1, t_2]$, the curve $\Gamma(t)$ has exactly one isolated singularity in each neighborhood $W_i \times \{t\}$ with the same type as $\tilde{q}_i \in \Gamma(t_1)$ ($i = 1, \ldots, \ell$) and $t_2 < t_1 + \delta(t_1)$. If we parametrize $\alpha_1 : [s_0, s_{2\ell+1}] \to \Gamma(t_1)$, we can find $s_1, \ldots, s_{2\ell}$ such that $\alpha_1(s_{2k-1}), \alpha_1(s_{2k}) \in \partial W_k$ and $I_k = [s_{2k-2}, s_{2k-1}]$ are intervals parametrizing transversal points in $\Gamma(t_1)$.

The manifold structure of $M$ gives an immersion $\psi_j : I_j \times [t_1, t_1 + \delta] \to M$, $t_1 + \delta < t_2$ and $j = 1, \ldots, \ell + 1$. In the construction of $\eta$ up to $t_1$, the singular points are isolated; then we can assume $\eta(t_1)$ is a regular point of $\Gamma(t_1)$, hence it is contained in an $\alpha_1(I_j)$. We construct the beginning of the arc $\eta(\tau)$ as the graph parametrized by $\phi_j(s, \tau(s))$ with $\tau$ an increasing function from $t_1$ to $t_1 + \delta/n$ as $s$ varies from $\tilde{s} \in I_j$, corresponding to the initial point $\eta(t_1) = \alpha_1(\tilde{s})$, to $s_{2j-1}$. Next we pass through the singularity $(\tilde{q}_j, t_1 + 2\delta/n)$ by constructing an arc which joins the point $\phi_j(s_{2j-1}, t_1 + \delta/n) \in \Gamma(t_1 + \delta/n) \cap \partial W_j$ to the point $\phi_{j+1}(s_{2j}, t_1 + 3\delta/n) \in \Gamma(t_1 + 3\delta/n) \cap \partial W_j$ (see Figure 14). For a suitable value of $n$ we can iterate this construction, passing through the singularities $\tilde{q}_j, \tilde{q}_{j+1}, \ldots$, until we join a point $(\tilde{q}, t_2)$ of $\partial V \times \{t_2\}$ and then we extend the arc up to $t_0$ by transversality outside $V$.

Now we look for this interval $[t_1, t_2]$. Let $t_1 < t'_1 < t_0$ and $\Gamma(t'_1)$ have several singularities in some neighborhood $W_k$, or a unique singularity of index less than one of the $\tilde{q}_k$. We consider in this $W_k$ a finite collection of neighborhoods of
isolated singularities $W'_{k,1}, \ldots, W'_{k,\ell'}$. We observe, by transversality that there are the same number of components of $\Gamma(t_1)$ and $\Gamma(t'_1)$ in $W_k$ (see Figure 15). Hence each $W'_{k,j}$ contains a number of components of $\Gamma(t'_1)$ strictly less than the number of components of $\Gamma(t_1)$ in $W_k$. The index of the singularity is strictly decreasing along this procedure. We can iterate this analysis up to a point where each singularity cannot be reduced to a simple one. This gives the interval $[t_1, t_2]$. \hfill \Box

7. Compact minimal annuli. We now introduce the compact stable horizontal minimal annulus $F_0$ bounded by circles in vertical planes $P(c)$ and $P(-c)$ where $P(c) = \{(x, y, t) \in \mathbb{R}^3; y > 0 \text{ and } x = c\}$. We also foliate a tubular neighborhood $\text{Tub}(F_0)$ of $F_0$ by compact minimal annuli $F_s, \ -1 \leq s \leq 1$ and certain small balls $B_\rho$ containing horizontal minimal annuli $C_\ell$.

We will now construct the stable compact annulus $F_0$. Let $\eta$ be a circle of radius one in the vertical plane $P(c)$, centered at $(c, y, 0)$. The metric induced on $P(c)$ is euclidean so circles make sense. As $y \to \infty$, dist$(\eta, P(0)) \to 0$. The disk of least area in $\mathbb{H}^2 \times \mathbb{R}$ bounded by $\eta$ is the disc in $P(c)$ bounded by $\eta$ (by the maximum principle). The area of this disk does not depend $y$. So for $y$ large, there is a compact annulus in $\mathbb{H}^2 \times \mathbb{R}$ with one boundary in $P(0)$ and the other boundary $\eta$, whose area is less than the area of the disk in $P(c)$ bounded by $\eta$. Assume $y$ is large. Then by the Douglas criterium (see Remark 7.1) there is a least area annulus $F_+$ having one boundary $\eta$ and the other in $P(0)$. Since $F_+$ has least area with respect to this boundary condition, $\partial F_+ \cap P(0)$ is orthogonal to $P(0)$. Hence $F_0$, the symmetry of $F_+$ through $P(0)$, union $F_+$, is a smooth compact minimal annulus orthogonal to $P(0)$ and $F_+ \cap P(0)$ is convex. The normal vector along this curve takes on all directions in the plane $P(0)$.
Remark 7.1. We explain the Douglas theorem for annuli. Consider two disjoint smooth Jordan curves $C_1$, $C_2$ in a complete Riemannian 3-manifold $M$, each null homotopic. Let $a_i$ be the infimum of the areas of immersed disks $D_i$ in $M$, $\partial D_i = C_i$, $i = 1, 2$. The Douglas criterium says that one can find an immersed minimal annulus $A$ in $M$, bounded by $C_1 \cup C_2$ if one can find an annulus $B$ in $M$, bounded by $C_1 \cup C_2$ whose area is strictly less than $a_1 + a_2$. Here is the idea of the proof. Consider a sequence $B_n$ of annuli bounded by $C_1 \cup C_2$, with $b = \lim_{n \to \infty} (|B_n|) = \inf\{|B|; \ B \text{ an annulus in } M \text{ with } \partial B = C_1 \cup C_2\}$.

The curves on $B_n$ that are not null homotopic on $B_n$, have lengths bounded away from zero as $n \to \infty$. Otherwise one could cut $B_n$ along very short curves and then one could attach small disks to the two short curves. This would produce...
two disks $D_i$, $\partial D_i = C_i$, $i = 1, 2$, and $|D_1| + |D_2| < a_1 + a_2$; a contradiction. Then one can show a subsequence of the $B_n$ converges to an immersed minimal annulus $A$ with boundary $C_1 \cup C_2$.

Let $\sigma$ be symmetry through $P(0)$, $\eta_- = \sigma(\eta)$, $F_- = \sigma(F_+)$. Observe that $F_0$ has least area with boundary $\eta \cup \eta_-$. For if $B$ is an annulus with $\partial B = \eta \cup \eta_-$, write $B = B_+ \cup B_-$ where $B_+ = \{(x,y,t) \in \mathbb{R}^3; y > 0$ and $0 \leq x \leq c\} \cap B$, and $B_- = \{(x,y,t) \in \mathbb{R}^3; y > 0$ and $-c \leq x \leq 0\} \cap B$. We know that the Area$(B_+) = |B_+| \geq \frac{|F_0|}{2}$ and $|B_-| \geq \frac{|F_0|}{2}$ so $|B| \geq |F_0|$. Thus $F_0$ is a stable annulus as desired. Let $\gamma_1$ be the geodesic joining $(c, y, 0)$ to $(-c, y, 0)$. We assume $y$ large so that $\gamma_1 \cap F_0 = \emptyset$.

Let $\eta(s)$ be equidistant circles of $\eta$ in $P(c)$, for $|s|$ small, $\eta(0) = \eta$. Let $\eta_-(s) = \sigma(\eta(s))$ be equidistant circles in $P(-c)$, $\eta_- (0) = \eta_-$. Since $F_0$ is strictly stable, there is a $\delta > 0$ so that for $|s| \leq \delta$, there is a foliation of a tubular neighborhood $\text{Tub}(F_0)$, of $F_0$ by compact minimal annuli $F(s)$, with $\partial F(s) = \eta(s) \cup \eta_-(s)$. Choose $\delta$ sufficiently small so that

$$\text{dist} \left( \text{Tub}(F_0), \gamma_1 \right) > 0.$$

Let $\text{Slab}(c) = \{(x, y, t) \in \mathbb{R}^3; y > 0$ and $|x| \leq c\}$. We denote by $F_0^-$ the bounded component of $\text{Slab}(c) - F_0$, and by $F_0^+$ the other component of $\text{Slab}(c) - F_0$. The annuli $F_s$ are inside $F_0^-$ for $s \in [-1, 0]$ and inside $F_0^+$ for $s \in [0, 1]$.

We consider $\text{Tub}^-(F_0) = \cup_{s \in [-1, 0]} F_s$ and $\text{Tub}^+(F_0) = \cup_{s \in [0, 1]} F_s$; domains of $\mathbb{H}^2 \times \mathbb{R}$.

We consider the curves $S_+ = P(0) \cap F_{1/2}$ and $S_- = P(0) \cap F_{-1/2}$. There exists a constant $\rho > 0$, such that for any $q$ of $S_+$ (or $S_-$) the geodesic ball $B_\rho(q)$ of geodesic radius $\rho$ centered at $q$ is contained in $\text{Tub}^+(F_0)$ (resp. $\text{Tub}^-(F_0)$).

We can find $\ell > 0$ such that any geodesic ball of radius $\rho$ centered at $q$ contains a small compact minimal annulus $C_\ell$ bounded by two geodesic circles contained in $P(\ell) \cap B_\rho$ and $P(-\ell) \cap B_\rho$. We say in the following that $C_\ell$ is centered at $q \in S_+ \cup S_-$. We denote by $q_1$ the point $\gamma_1 \cap P(0)$ (see Figure 18).

In summary, we fix $\rho > 0$ such that

1. $3\rho < \text{dist}(F_0, \gamma_1)$,
2. $B_{2\rho}(q_1) \cap \text{Tub}^-(F_0) = \emptyset$,
3. $B_\rho(q) \subset \text{Tub}^-(F_0)$ for any $q \in S_-$,
4. $B_\rho(q) \subset \text{Tub}^+(F_0)$ for any $q \in S_+$,

Now we clearly have the following:

Claim. Any continuous curve $\gamma$ in the interior of $\text{Tub}^+(F_0) \cap \text{Slab}(\ell)$ (or $\text{Tub}^-(F_0) \cap \text{Slab}(\ell)$) joining $F_0$ to $F_1$ (or $F_{-1}$) intersects a compact annulus of the family $C_\ell(q) \subset \text{Tub}^+(F_0)$ (resp. $\text{Tub}^-(F_0)$) for some point $q \in S_+$ (reps. $q \in S_-$).
The next proposition gives at least two components of $\Sigma$ in $F_0^-$ when $F_0$ is tangent to $\Sigma$ at some point.

**Proposition 7.2.** Let $\Sigma$ be a properly immersed minimal half-plane in $\text{Slab}(\ell)$. Suppose $\Sigma$ is tangent to $F_0$ at $p$ and $\partial \Sigma \cap F_0 = \emptyset$. Then there are at least two connected components of $\Sigma$ in $F_0^-$. More precisely if $\Sigma_1(-)$ and $\Sigma_2(-)$ are distinct local components of $\Sigma$ in $F_0^-$ then $\Sigma_1(-)$ and $\Sigma_2(-)$ are in distinct components of $\Sigma \cap F_0^-$. 

**Proof.** If the proposition fails, then we can find a path $\alpha_0$ in $\Sigma \cap F_0^-$, joining a point $x \in \Sigma_1(-)$ and $y \in \Sigma_2(-)$. Then join $x$ to $y$ by a local path $\beta_0$ in $\Sigma$ going...
through \( p \), but \( \beta_0 \subset F_0^\ominus \) except at \( p \) (see Figure 19). Let \( \Gamma = \alpha_0 \cup \beta_0 \subset F_0^\ominus \). Since \( \Sigma \) is a half-plane, \( \Gamma \) bounds a disk \( D \) in \( \Sigma \). By construction \( D \) contains points in the interior of \( F^0_0 \).

Hence there is a compact component of \( D \) in \( F^0_0 \) with boundary in \( F_0 \). By the maximum principle \( D \cap F_t \neq \emptyset, 0 \leq t \leq 1 \) and there is at least one point \( p_1 \) of \( D \cap F_1 \). Using compact annuli \( C_\ell \) inside \( \text{Tub}^+(F_0) \), we can find an annulus \( C_\ell(q) \) which intersects \( D \) (by the claim). Now translate this catenoid in the interior \( F^0_0 \) to a point outside the convex hull of \( F_0 \). Apply the Dragging lemma to obtain points of \( D \) outside the convex hull. This contradicts the maximum principle. \( \square \)

8. A family of graph barriers. In this section we study a one parameter family of surfaces \( \Sigma_n \) graphs on a sequence of domains \( \Omega_n \) of \( \mathbb{H}^2 \) bounded by two geodesics. In the unit disk model of \( \mathbb{H}^2 = \{ (x,y) \in \mathbb{R}^2; x^2 + y^2 < 1 \} \), we consider two geodesics \( \gamma_n \) and \( \gamma_{-n} \) passing through the points \((-1 + 1/n, 0) \) and \((1 - 1/n, 0) \) and both orthogonal to \( \{ y = 0 \} \). We consider the domain \( \Omega_n \) bounded by \( \gamma_n \) and \( \gamma_{-n} \) (see Figure 20, Left). We solve the minimal graph equation for a function \( u_n : \Omega_n \to \mathbb{R} \) with \( u_n = +\infty \) on \( \gamma_n \cup \gamma_{-n} \) and \( u_n = 0 \) on \( \partial_{\infty} \Omega_n \), the boundary at infinity of \( \Omega_n \). This example has been constructed in Theorem 4.9 of Mazet, Rodriguez, and Rosenberg [9] when \( n \) is large enough.

The graph \( u_n \) has a line of curvature \( \Gamma_n \) over the geodesic \( \gamma_0 = \{ (x,y) \in \mathbb{H}^2; x = 0 \} \). The following proposition describes the limit of the graphs \( \Sigma_n \) when \( n \to \infty \).

**Proposition 8.1.** The sequence of solutions of the minimal graph equation in the sequence of domains \( \Omega_n \) with boundary data \( u_n = +\infty \) on \( \gamma_n \cup \gamma_{-n} \) and \( u_n = 0 \) on \( \partial_{\infty} \Omega_n \), converges uniformly to the horizontal section \( \mathbb{H}^2 \times \{ 0 \} \).
Proof. There is a general maximum principle in Section 4 of [9] which can be used. However we give a complete geometrical argument. The sequence of domains $\Omega_n$ is an increasing sequence in $\mathbb{H}^2$; $\Omega_n \subset \Omega_{n+1}$. The maximum principle assures that the sequence is decreasing with $0 \leq u_{n+1}(q) \leq u_n(q)$ for any $q \in \Omega_n$. Hence the sequence of graphs $\Sigma_n$ converges to an entire graph of a function $u_0 : \mathbb{H}^2 \to \mathbb{R}$. We will prove that $u_0 \equiv 0$.

It suffices to prove that $u_0 = 0$ on the geodesic $\gamma_0$. If not we can assume that $\sup_{\gamma_0} u_n = a_n \geq b > 0$ and there is $p \in \gamma_0$ such that $u_0(p) = b$.

This point $p$ exists because $u_0$ takes value 0 at infinity of $\gamma_0$. This comes from the fact that $(u_n)$ is a decreasing sequence hence $u_n = 0$ at infinity of $\Omega_n$ for any $n \in \mathbb{N}$.

We consider the sequence of minimal surfaces $\Sigma_n$ graphs of a function $v_n$ on a domain $V_n$ bounded by $\gamma_n$, with boundary data $v_n = +\infty$ on $\gamma_n$ and $v_n = b$ on $\partial_{\infty} V_n$. This family of graphs is well known and Mazet, Rodriguez and Rosenberg proved in Appendix A [9] that the sequence $v_n$ converges uniformly to $v_0 \equiv b$, when $n \to \infty$.

We restrict the function $v_n : V_n \to \mathbb{R}$ to the domain $W_n$ bounded by the geodesic $\gamma_n$ and the geodesic $\gamma_0$. On $W_n$ we claim that the maximum principle applies to show that $v_n \geq u_0$. To see this its suffices to check for the inequality on the boundary of the domain $W_n$. On $\gamma_n$, the function $v_n = +\infty > u_0$ and on $\gamma_0$, we have $v_n \geq b \geq u_0$ at the boundary at infinity $\partial_{\infty} V_n$ we have $v_n = b > 0 = u_0$.

Now let $n \to \infty$ to show that $v_n \to v_0 = b$ and $v_0 \geq u_0$. This proves by symmetry that the function $u_0 \leq b$. The point $p$ of $\gamma_0$ where $u_0(p) = b$ is an interior maximum point of the function, hence $u_0 = b$. This contradicts the fact that $u_0$ take the value 0 at the boundary at infinity of $\mathbb{H}^2$. \qed
We consider an end $E_{(1,0)}(c)$ contained in a slab $S = \{(x,y,t) \in \mathbb{R}^3; \ y \geq 1 \text{ and } -c_1 \leq t \leq c_1\}$ and we use Proposition 8.1 to obtain $C^0$ convergence:

**Proposition 8.2.** An end $A$ of type $(p,0)$ of a properly immersed minimal surface $\Sigma$ in $M \times S^1$ has third coordinate which has a limit at infinity, i.e., that $A$ converges in the $C^0$ norm to $A_{(p,0)}$ at height $a \in ]-c_1,c_1[$.

**Remark 8.3.** In Section 9, we will prove that $A$ is a graph on $A_{(p,0)}$. By stability argument (see Finite total curvature, Section 9), $A$ is a graph converging uniformly in the $C^2$ norm to the cusp $A_{(p,0)}$ at height $\{t=a\}$.

**Proof.** A covering $E$ of $A$ is contained in a slab bounded by $E_{(1,0)}(-c_1)$ and $E_{(1,0)}(c_1)$. We study the intersection of $E$ with the level section $E_{(1,0)}(c) = \{(x,y,t) \in \mathbb{R}^3; \ y \geq 1 \text{ and } t = c\}$ with $c \in ]-c_1,c_1[$. If $\Gamma$ is a compact component of $A \cap E_{(1,0)}(c)$ then $\Gamma \cap \partial A \neq \emptyset$. Otherwise $\Gamma$ bounds a disc $D$ or a subend $A_0$. In both cases, the maximum principle of Proposition 8.1 applies and $A = A_{(p,0)}$, i.e., $A$ is a flat standard end of height $c$.

Varying the value $c \in ]-c_1,c_1[$, there is a value $a \in ]-c_1,c_1[$, such that the intersection $A \cap E_{(1,0)}(c)$ has a non-compact component denoted by $\Gamma$. By Proposition 8.1, $\Gamma \cap \partial E \neq \emptyset$. In the lift $E$ of $A$, we consider two lifts of $\Gamma$ denoted by $\bar{\Gamma}_1$ and $\bar{\Gamma}_2 := \psi \circ \bar{\Gamma}_1$. The curves $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and a compact arc $\Gamma_3 \subset \partial E$ bound a fundamental domain.

We consider a graph obtained by a translation $\sigma_n$ of $\Sigma_n$ of Section 8 such that the geodesic $\gamma_{-n}$ translates to a fixed geodesic $\bar{\gamma}$ which does not intersect $\Gamma_3 \subset \partial E$ and $\Sigma_n$ is above $E$ with boundary data $u_n = a$ at the boundary at infinity. The graph $\Sigma_n$ has a line of curvature which is a graph over the translation of the geodesic $\gamma_0$ denoted by $\sigma_n \circ \gamma_0$ and the boundary curve $\gamma_n$ is sent to $\sigma_n \circ \gamma_n$. We remark that $\sigma_n \circ \gamma_0$ is a distance $n$ from $\bar{\gamma}$ and $\sigma_n \circ \gamma_n$ at a distance $2n$ from $\bar{\gamma}$.

We let $n \to \infty$ and fix the geodesic $\bar{\gamma}$, using the horizontal isometry $\sigma_n$. These graphs are congruent to the graphs of the sequence $u_n$ of Proposition 8.1. We know that the graph over $\sigma_n \circ \gamma_0$ is converging to the height $a$, hence we see that the end $A$ cannot have a point above the height $a$ at infinity. We do the same with a symmetric graph with value $-\infty$ on the geodesic $\gamma_{-n}$ and $\gamma_{+n}$. This proves that the end $A$ is trapped between two graphs which have third coordinate going to the same value $a$. Hence the end $A$ converges in the $C^0$ norm to a cusp end $t = a$. □

9. **Proof of the theorem in $M \times S^1$**. The surface $\Sigma$ is properly immersed. By Lemmas 5.1, 5.2 and Theorem 5.3, each end $A$ lifts to $E$ a half-plane trapped between two standard ends $E_{(p,q)}$. First we prove the theorem for an end of type $(0,p)$ and then we adapt the arguments to the general case.

**Ends of type $(0,p)$: the vertical case.** Since $E$ is trapped between two vertical planes and the distance between two vertical planes tends to zero as $y \to \infty$, we can assume that there exists a constant $\ell > 0$ define in Section 7, such that $E \subset$
Slab($\ell/2$) where Slab($c$) = $\{(x,y,t) \in \mathbb{R}^3; \ y > 0 \text{ and } |x| \leq c\}$, and $\partial E \subset \{y = y_0\}$. We will prove that when $p \in E$, and $y(p) > 3 + y_0$, then the killing field $\frac{\partial}{\partial x}$ is transverse to $E$ at $p$. Then this subend of $E$ is stable, hence it has bounded curvature. We will prove later that this gives the theorem in this case.

Suppose on the contrary that some $p$, with $y(p) > 3 + y_0$, has $\frac{\partial}{\partial x}|_p \in T_pE$. The annulus $F_0$ meets $P(0)$ orthogonally and the normal vector to this curve of intersection is in the plane $P(0)$ and takes all directions in this plane as one goes once around the curve.

Since $\frac{\partial}{\partial x}|_p \in T_pE$, the normal vector to $E$ at $p$ is in the plane $P(x = x(p))$. Thus we can translate $F_0$ to $p$ (call $F_0$ this translated $F_0$) to be tangent to $E$ at $p$. By a translation of less than $\ell/2$ we can assume $x(p) = 0$, so now $E \subset \text{Slab}(\ell)$.

Recall that $\gamma_1$ is the geodesic joining the centers of the boundary circles of $F_0$ and $q_1 = \gamma_1 \cap P(0)$. Write $q_1 = (0,y_1,0)$ with $y_1 > 2 + y_0$.

The convex hull of the foliation of $\text{Tub}^-(F_0) \cup \text{Tub}^+(F_0) \cup F_0$ has $y$ coordinate at least the minimum of the $y$-coordinate of the boundary circles of $F(t)$, i.e., $y \geq y_0 + 1/2$ on the convex hull. Now we proved in the Section 7, that this foliation contains a family of geodesic balls $B_\rho(q)$ of radius $\rho > 0$ centered at points $q \in S_+ \cup S_-$. We choose this constant $\rho$ such that

$$3\rho < \text{dist}(F_0,\gamma_1) \text{ and } 4\rho < 1.$$ 

Each such geodesic ball $B(q)$ contains a compact annulus $C_\ell(q)$ bounded by geodesic circles of radius $\delta$ contained in $P(\ell)$ and $P(-\ell)$.

**Step 1: Construction of arcs on $E$.** We know by Proposition 7.2, that there are at least two connected components $\Sigma_1, \Sigma_2$ of $E - F_0$ that have $p$ in their closure, and $\Sigma_1, \Sigma_2 \subset F_0^-$. Clearly, by the maximum principle, each of $\Sigma_1, \Sigma_2$ intersects each of the catenoids in the local foliation $F_s$ about $F_0$ in $F_0^-$. In particular there is a $\tilde{q} \in S_-$ such that $C_\ell(\tilde{q}) \cap \Sigma_1 \neq \emptyset$. Now translate $C_\ell(\tilde{q})$ along the geodesic joining $\tilde{q}$ to $q_1$ and apply the Dragging lemma to obtain a point $p_1 \in \Sigma_1 \cap C_\ell(q_1) \subset B_\rho(q_1)$.

The same argument gives a point $p_2 \in \Sigma_2 \cap C_\ell(q_1)$. Recall that $p_1$ and $p_2$ cannot be joined by an arc in $E \cap F_0^-$ (we will use this later). Now we construct a loop $\mu$ in $E$.

For a value $k_0 \in \mathbb{N}$ which will be defined in Step 2, we consider $\Gamma_+$ to be the euclidean segment joining $q_1 = (0,y_1,0)$ to $(0,y_1,k_0h + 2\rho)$, together with the segment joining $(0,y_1,k_0h + 2\rho)$ to $z = (0,y_0,k_0h + 2\rho)$. We will connect the point $p_1$ and $p_2$ by an arc in $E$ which stays in a tubular neighborhood of $\Gamma_+ \cup \partial E$. We note by $\text{Tub}_\rho(\Gamma_+)$ the tubular neighborhood of geodesic radius $\rho$ along $\Gamma_+$. We parametrize the curve $\Gamma_+$ in a piecewise $C^1$-monotone manner by $q(\tilde{t})$, $0 \leq \tilde{t} \leq 1$ and we move $B_\rho(q_1)$ along $q(\tilde{t})$, from $q_1$ to $z = (0,y_0,k_0h + 2\rho)$, by $B_\rho(q(\tilde{t}))$. Each ball $B_\rho(q(\tilde{t}))$, $q \in \Gamma_+$ contains the catenoid $C_\ell(q(\tilde{t}))$ and the Dragging lemma then gives two continuous paths $\sigma_1^+(\tilde{t}), \sigma_2^+(\tilde{t})$ starting at $p_1,p_2$ respectively such that $\sigma_i^+(\tilde{t}) \in E$ for $0 \leq \tilde{t} \leq 1$. 

We apply the Dragging lemma up to the value $q(1) = z$ and $\sigma^+_i(1) \in B_\rho(z)$ for $i = 1, 2$. Since $\tilde{p}_1 = \sigma^+_1(1)$ and $\tilde{p}_2 = \sigma^+_2(1)$ are in $\partial E \cap B_\rho(z)$, we can find a path $\sigma^+_{12}$ in $\partial E$ from $\tilde{p}_1$ to $\tilde{p}_2$. We have $t(\tilde{p}_1), t(\tilde{p}_2) \in [k_0 h + \rho, k_0 h + 3 \rho]$. We will prove in Step 2, that we can find a path $\sigma^+_{12} \in \partial E$ from $\tilde{p}_1$ to $\tilde{p}_2$ such that for all $p \in \sigma^+_{12}$, $t(p) \in [\rho, k_0 h + 3 \rho]$. 

Under this hypothesis we have constructed a path $\mu^+$ in $E$ from $p_1$ to $p_2$ which is

$$
\mu^+ = \sigma^+_1 \cup \sigma^+_{12} \cup \sigma^+_2.
$$

The arcs $\sigma^+_1(\tilde{t}), \sigma^+_2(\tilde{t})$ are contained in $T_\rho(\Gamma_+)$. The arcs of $\sigma^+_1$ and $\sigma^+_2$ from $p_1$ to $F_0$ and $p_2$ to $F_0$ are disjoint (see Proposition 7.2) since $\sigma^+_1 \subset \Sigma_1$ and $\sigma^+_2 \subset \Sigma_2$ in $\text{Tub}^- (F_0)$. 

Thus the paths are quasi-monotone along the segment of $\Gamma_+$ in $\text{Tub}(\Gamma_+)$: once the catenoids $C_\ell(q)$, $q \in \Gamma_+$ have advanced along $\Gamma_+$ a distance $2 \rho$, the paths $\sigma^+_1$ and $\sigma^+_2$ do not return to the $\rho$-ball where they started. 

If the arcs $\sigma^+_1(\tilde{t})$ and $\sigma^+_2(\tilde{t})$ remain disjoint for $\tilde{t} \leq 1$, we do not change $\mu^+$. If the arcs intersect then at the first point of intersection $p_3$ we replace $\mu^+$ by the path on $\sigma^+_1$ from $p_1$ to $p_3$ union the path on $\sigma^+_2$ from $p_2$ to $p_3$. Such a point $p_3$ is necessarily outside $F_{-1/2}$. 

**Step 2: The boundary $\partial E$.** Now we study the boundary of the annulus and the function $t : \partial E \to \mathbb{R}$ the restriction of the third coordinate in the model of the half-plane. We parametrize the boundary curve $\partial E$ by the immersion $C : \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$, $C(s) = (x(s), y_0, t(s))$ with period

$$
C(s + 1) = (x(s + 1), y_0, t(s + 1)) \longrightarrow (x(s), y_0, t(s) + h).
$$

The diameter is defined by

$$
G := \sup_{s_1, s_2 \in [0, 1]} |t(s_1) - t(s_2)|
$$

and choose $k_0 \in \mathbb{N}$ such that $K = k_0 h \geq G$. We consider the intersection of $\partial E$ with a transverse plane to the curve $P(\alpha) := \{(x, y, t) \in \mathbb{R}^3; \ y = y_0, \ t = \alpha\}$. Since $C$ is a proper immersed curve, we have a finite number of intersection points

$$
C(s) \cap P(\alpha) = \{C(s_1), \ldots, C(s_\ell)\}.
$$

We claim that $(s_i)_{1 \leq i \leq \ell} \in [s_1 - k, s_1 + k]$. To see this we remark that if $s_1 + 1 + k' \geq s \geq s_1 + k' \geq s_1 + k$, we have

$$
t(s) - \alpha = t(s) - t(s_1 + k') + t(s_1 + k') - t(s_1) \geq k' \tau - G \geq k_0 \tau - G > 0.
$$

Hence independently of the choice of $\alpha$, two points of $\partial E$ with the same $t$ coordinate are connected by a sub-arc $\Gamma$ of $\partial E$ with $t(\Gamma) \subset [\alpha - K, \alpha + K]$. Two
points of $\partial E$ with coordinate $t_1 \leq t_2$ can be connected in $\partial E$ by a sub-arc $\Gamma$ with $t(\Gamma) \in [t_1 - K, t_2 + K]$.

**Step 3: A loop $\mu$ in $E$.** In Step 1, we constructed an arc $\mu^+ = \sigma_1^+ \cup \sigma_{12}^+ \cup \sigma_2^+$ which joins the points $p_1$ and $p_2$ and $\mu^+ \subset \text{Tub}_\rho(\Gamma_+) \cup \partial E$. Now do this construction in the half-space $\{t \leq 0\}$ to obtain a path $\mu^-$ joining $p_1$ to $p_2$ with similar properties. Let $\Gamma_-$ be the segment from $q_1 = (0, y_1, 0)$ to $(0, y_1, -k_0 h - 2\rho)$, together with the segment joining $(0, y_1, -k_0 h - 2\rho)$ to $z = (0, y_0, -k_0 h - 2\rho)$. Move $B_\rho(q_1)$ in a $C^1$-monotone manner along $\Gamma_-$ and we use the Dragging lemma as before to construct arcs $\sigma_1^-(\bar{t}), \sigma_2^-(\bar{t})$ in $E$. We note by $\text{Tub}_\rho(\Gamma_-)$ the tubular neighborhood of geodesic radius $\rho$ along $\Gamma_-$. We follow the arc $\sigma_1^-(\bar{t}), \sigma_2^-(\bar{t})$ up to points of $\partial E$. As in step 2, we construct the arc $\sigma_{12}^-$, so that points $p \in \sigma_{12}^-$ have coordinate $t(p) \subset [-k_0 h - \rho, -\rho]$.

Finally we consider in Figure 21 the arc

$$
\mu^- = \sigma_1^- \cup \sigma_{12}^- \cup \sigma_2^-
$$

and we let $\mu$ be the loop $\mu^+ \cup \mu^-$. $\mu$ is contained in $\text{Tub}_\rho(\Gamma_+ \cup \Gamma_-)$. If the arcs $\sigma_1^-(\bar{t})$ and $\sigma_2^-(\bar{t})$ remain disjoint for $\bar{t} \leq 1$, we do not change $\mu^-$. If the arcs intersect then at the first point of intersection $p_4$ we replace $\mu^-$ by the path on $\sigma_1^-$ from $p_1$ to $p_4$ union the path on $\sigma_2^-$ from $p_2$ to $p_4$. Such a point $p_4$ is necessarily outside $E_{-1/2}$.
The end $E$ is an immersed half-plane $X : \Omega = \{(u, v) \in \mathbb{R}^2; v \geq 0\} \to \mathbb{H}^2 \times \mathbb{R}$ with $X(\Omega) = E$. The loop $\mu \subset E$ is immersed and we denote by $\hat{\mu} = X^{-1}(\mu)$ the pre-image of $\mu$ in $\Omega$.

In the Dragging lemma, we constructed the arc $\mu$ locally and then extended it. The pre-image $\hat{\mu}$ is locally embedded in $\Omega$. The arc $\hat{\mu}$ can have self-intersections. If $\hat{\mu}$ is one of them, we consider the sub-arc $\gamma$ of $\hat{\mu}$ with end points $\hat{p}$. This sub-arc $\gamma$ bounds a disk in $\Omega$. We remove these sub arcs to obtain a piecewise $C^1$ connected curve in $\Omega$ without self-intersecting points. This defines a closed Jordan curve which bounds a disk $D$ in $\Omega$. The immersion $X(D)$ is a minimal disk in $\mathbb{H}^2 \times \mathbb{R}$ with boundary an immersed connected curve contained in $\text{Tub}_\rho(\Gamma_+ \cup \Gamma_-) \cup \partial E$. Now we analyze the geometry of the disk $X(D)$.

Consider the plane defined by $P(0) = \{(x, y, t) \in \mathbb{R}^3; t = 0\}$. This plane separates $\Gamma_+ \cup \Gamma_-$ in two connected components.

We denote by $\hat{\mu} = \partial X(D)$ the boundary of the minimal disk. Let $\hat{\mu}_1 = \hat{\mu} \cap (\sigma_+^1 \cup \sigma_1^-)$ and $\hat{\mu}_2 = \hat{\mu} \cap (\sigma_+^2 \cup \sigma_2^-)$ be the connected components of the loop in $E$ containing $p_1$ and $p_2$ respectively. The end points of $\hat{\mu}_1, \hat{\mu}_2$ are in different half-spaces determined by $\{t = 0\}$ (one end point has $t > \rho$ and the other $t < -\rho$). Thus the plane $P(0)$ intersects $\hat{\mu}_1$ and $\hat{\mu}_2$, each one in an odd number of points.

Now we will obtain a contradiction by proving that $P(0) \cap \hat{\mu}_1$ is an even number of points. One translates horizontal catenoids $C_t(q), q \in E(0) \cap P(0)$, starting far from $\mu$ to see that before $C_t$ touches a $\rho$-tubular neighborhood of $\mu$, one does not touch the disk $X(D)$. Hence $X(D) \cap P(0)$ is contained in $F_0^- \cap \text{Tub}_\rho(\Gamma_+ \cup \Gamma_-)$.

In $F_0^-$ the sub arc $\hat{\mu}_1 \subset \Sigma_1$ and $\hat{\mu}_2 \subset \Sigma_2$ cannot be connected. Hence a connected arc $\gamma \subset X(D) \cap P(0)$ must have end points either in $\Sigma_1$ or in $\Sigma_2$. This means that there are an even number of point of $\hat{\mu}_1 \cap P(0)$ on $\partial D = \mu$. This contradicts the odd intersection number of each arc with $P(0)$.

This proves that $E$ is a graph for $y \geq y_0 + 2R$.

**Ends of type $(p, q)$, tilted planes.** Next we prove the theorem when $E$ is trapped between two tilted (not horizontal) planes $E(p, q)$. We can suppose $E$ is contained a tilted slab $S$ of the form, for some $c_1 > 0$:

$$S = \{(x, y, t) \in \mathbb{R}^3; y > 0 \text{ and } -c_1 \leq p \tau t - qhx \leq c_1\}.$$ 

Since $S$ is converging to a vertical slab as $y \to \infty$, there is a $y_0 > 1$ so that if $p \in E$, $y(p) \geq y_0$, then the catenoid $C_t(p)$ in $B_\rho(p)$, has both its boundary circles outside of $S$. To see this, we use an isometry which leaves the slab $S$ invariant and takes $p \in E$ to a point $\bar{p} = (x, y, 0)$. Observe that $S \cap \{|t| \leq 1\}$ is in a vertical slab bounded by $P(-c_2)$ and $P(c_2)$, where $c_2$ depends on $p\tau$ and $qh$. Then for any point $\bar{p} = (x, y, 0)$ with $|x| \leq c_2$, $C_t(\bar{p}) \subset B_\rho(\bar{p})$ has boundary circles outside of the slab bounded by $P(d)$ and $P(-d)$ for $d \geq c_2$, and $y$ greater than some $y_0$ (using that $(x, y, t) \to (\lambda x, \lambda y, t)$ is an isometry). This property is invariant by changing $\bar{p} = (x, y, 0)$ to $p = (x + p\tau, y, t + qh)$.  


We will prove that a sub-end of $E$ is transverse to $\frac{\partial}{\partial x}$ for large $y$. Suppose this is not the case. We proceed exactly as in the case $E$ is trapped between vertical planes to find $p_1, p_2 \in E \cap B_\rho(q_1)$, $q_1$ the center of a horizontal catenoid $F_0$, and $p_1, p_2$ cannot be joined by a path in $E$ that is inside $F_0$. The proof is modified in our choice of $\Gamma = \Gamma_+ \cup \Gamma_-$, and a loop in $E$ passing through $p_1$ and $p_2$.

We denote by $\vec{u}$ the unit vector director of the straight line $\{(x, y, t) \in \mathbb{R}^3; \ p\tau t - qhx = 0$ and $y = y_0\}$. For a value $k_0 \in \mathbb{N}$ which depends on the diameter of the periodic boundary curve, we consider $\Gamma_+$ be the euclidean segment joining $q_1 = (0, y_1, 0)$ to $q_1 + (k_0 h + 2\rho)\vec{u}$, together with the segment joining $q_1 + (k_0 h + 2\rho)\vec{u}$ to $z = (0, y_0, 0) + (k_0 h + 2\rho)\vec{u}$. We connect the point $p_1$ and $p_2$ by an arc in $E$ which stays in a tubular neighborhood of $\Gamma_+ \cup \partial E$.

Let $\Gamma_-$ be the segment from $q_1 = (0, y_1, 0)$ to $q_1 - (k_0 h + 2\rho)\vec{u}$, together with the segment joining $(0, y_1, 0) - (k_0 h + 2\rho)\vec{u}$ to $z = (0, y_0, 0) - (k_0 h + 2\rho)\vec{u}$. We connect the point $p_1$ and $p_2$ by an arc in $E$ which stays in a tubular neighborhood of $\Gamma_- \cup \partial E$.

Then the argument is the same to obtain a contradiction with $\Gamma = \Gamma_+ \cup \Gamma_-.

**Ends of type $(p,0)$, horizontal planes.** Let $E$ be a half-plane end (lifting of $A \subset \mathcal{M}$) to $\mathbb{H}^2 \times \mathbb{R}$, between the planes $t = \pm d$, with $\partial E \subset \{y = 1\}$, where $\partial E$ is invariant by the isometry $(x, y, t) \rightarrow (x + \tau, y, t)$. By Proposition 8.2, we can assume $t \rightarrow 0$ on $E$ as $y \rightarrow \infty$. So for $y_0$ large, the sub-end of $E$ given by $y \geq y_0$ is between planes $t = \pm c$ for any small $c > 0$.

Let $\eta$ be a circle of radius one in $\{t = c\}$ and let $\eta_-$ be $\eta$ translated vertically to a circle in $\{t = -c\}$. For $c$ small enough, $\eta \cup \eta_-$ bounds a stable (rotational) annulus $F_0$. $F_0$ is a bigraph over $\{t = 0\}$. Now we assume $y_0$ chosen so that $E$ is between $t = \pm c$ for $y \geq y_0$ and then $\partial F_0 \subset \{t = \pm c\}$.

As in Section 6, where $E$ was trapped between two vertical planes and $F_0$ was a horizontal catenoid, we define $B_\rho(q), C_\ell$ in the same manner, with $C_\ell$ a vertical catenoid. We choose $y_0$ large enough so that $E$ is between $t = \pm \ell/2$ and $C_\ell$ has its boundary circles in $t = \pm \ell$ for $y \geq y_0 + 3$.

Suppose $p$ is in $E$, $y(p) \geq y(0) + 3$, and $E$ has a vertical tangent plane at $p$. Then one places a vertical catenoid $F_0$ to be tangent to $E$ at $p$ (after a small translation) and one obtains $p_1, p_2 \in E \cap B_\rho(q), q$ the center of $F_0$, such that $p_1, p_2$ cannot be joined by a path in $E$ that is inside $F_0$.

For a value $k_0 \in \mathbb{N}$ which depends on the diameter of the periodic boundary curve, we consider $\Gamma_+$ be the euclidean segment joining $q_1 = (0, y_1, 0)$ to $(k_0 h + 2\rho, y_1, 0)$, together with the segment joining $(k_0 h + 2\rho, y_1, 0)$ to $z = (k_0 h + 2\rho, y_0, 0)$. We connect the points $p_1$ and $p_2$ by an arc in $E$ which stays in a tubular neighborhood of $\Gamma_+ \cup \partial E$.

Let $\Gamma_-$ be the segment from $q_1 = (0, y_1, 0)$ to $(-k_0 h - 2\rho, y_1, 0)$, together with the segment joining $(-k_0 h - 2\rho, y_1, 0)$ to $z = (-k_0 h - 2\rho, y_0, 0)$. We connect the
point \( p_1 \) and \( p_2 \) by an arc in \( E \) which stays in a tubular neighborhood of \( \Gamma \cup \partial E \). We apply now the same argument to obtain a contradiction.

**Finite total curvature.** We proved that a minimal annulus is trapped in Slab and is a killing multigraph outside a compact set \( K_0 \subset M \times S^1 \). These graphs are stable, hence they have bounded Gaussian curvature. They are contained in a euclidean slab whose hyperbolic width tends to zero at infinity.

In the horizontal case with \( A \) asymptotic to \( A(p,0) \), the end \( A \) has a limit for its third coordinate. Since the curvature is bounded, \( A \) is a vertical graph of a function \( f : A(p,0) \to \mathbb{R} \), with \( f \) converging to 0 in a \( C^2 \) manner. The end \( A \) is converging to the cusp \( C \times \{0\} \) and the curve \( T(y) \cap A = \gamma(y) \) is a topological circle converging to a finite covering of a quotient \( c(y)/[\psi] \). The curve \( \gamma(y) \) has uniform bounded curvature and its length goes to zero. Thus \( \int_{\gamma(y)} k_g ds \to 0 \) as \( y \to \infty \).

In the case of ends of type \( (0,q) \) and \( (p,q) \), the ends are horizontal multigraphs on some \( A(0,p) \). Since \( A \) converges in a \( C^2 \) manner to \( A(p,0) \), the curves \( \gamma(y) = T(y) \cap A \) converge to a finite covering of a quotient of a vertical geodesic by the translation \( T(h) \). This implies that the curvature of \( \gamma(y) \) converges uniformly to zero as \( y \to \infty \).

We apply the Gauss-Bonnet formula on an exhaustion of \( M \times S^1 \) by a sequence of compact \( K_n \), with boundary of \( K_n \) the union of mean curvature one tori \( T_1(n), \ldots, T_k(n) \), in each end \( M \subset M \times S^1 \) and \( \gamma_{k,n} = T_k(n) \cap \Sigma \).

\[
\int_{K_n \cap S} K dA + \int_{\gamma(k,n)} k_g ds = 2\pi \chi(\Sigma).
\]

When \( n \to \infty \), the integral of the curvature on \( \gamma(k,n) \) tends to zero and we obtain the finite total curvature formula

\[
\int_S K dA = 2\pi \chi(\Sigma).
\]

10. **Proof of the theorem in \( N \).** Now we complete the proof of Theorem 1.1 when the ambient space is \( N \). The idea is the same as in \( M \times S^1 \). Let \( A \) be an annular end in \( M(-1) \), minimal and properly immersed. By Lemma 4.1 (the same proof) we can suppose \( A \subset \cup_{y \geq 1} T(y) \), \( \partial A \) is an immersed closed curve and \( A \) is transverse to \( T(1) \) along \( \partial A \). Let \( E \) be a connected lift of \( A \) to \( \mathbb{H}^3 \), so \( \partial E \subset \{(x,1,t) \in \mathbb{R}^3 \} \). Observe that \( E \) is a half-plane, not an annulus. Suppose, on the contrary that \( E \) is an immersed annulus, so \( \partial E \subset \{(x,1,t) \in \mathbb{R}^3 \} \) is compact. Let \( D \) be the convex hull of \( \{(x,0,t) \in \mathbb{R}^3 \}; (x,1,t) \in \partial E \} \) in the \( y = 0 \) plane. Let \( L \) be a line of the plane \( Q = \{y = 0\} \), disjoint from \( D \).

Let \( C \) be a small circle in \( Q \) in the half-space \( \mathcal{H} \) of \( Q - L \) disjoint from \( D \). \( C \) bounds a totally geodesic hyperbolic plane in \( \mathbb{H}^3 \) (it is a hemisphere orthogonal to \( Q \) along \( C \) in our model). For \( C \) small, this plane is disjoint from \( E \). Let the circle \( C \) grow in \( \mathcal{H} \) and converge to \( L \). By the maximum principle, there is no
first contact point of the planes bounded by these circles with \( E \) (the planes do not touch \( \partial E \)). Since the hyperbolic planes bounded by the circles converge to \( L \times \mathbb{R}^+ \), it follows that \( E \) is on one side of \( L \times \mathbb{R}^+ \). Hence \( E \) is contained in the cylinder \( \text{Cyl} = \{(x,y,t) \in \mathbb{R}^3; (x,0,t) \in \partial D, y > 0 \} \).

For \( y \) large, the diameter of \( \text{Cyl} \) tends to zero, i.e., the diameter of \( \text{Cyl} \cap \{y = \text{const}\} \) tends to zero. So we could touch \( E \) by a catenoid at an interior point of \( E \), which is a contradiction.

Now we know \( E \) is a half-plane. After an isometry of \( \mathbb{H}^3 \), we can assume \( \partial E \) is invariant under the parabolic isometry, \((x,y,t) \to (x + \tau, y, t)\), and \( \partial E \subset \{y = 1\} \). So the \( t \) coordinate is bounded on \( \partial E \). The same convex hull argument as in the previous annular case, then shows the \( t \) coordinate has the same bound on \( E \); \(|t| \leq c\), for some \( c > 0 \), (one takes \( L \) to be a horizontal line in \( \{y = 0\} \), above height \( c \), and considers circles \( C \) in \( \{y = 0\} \) above \( L \). When \( C \) converges to \( L \) in \( \{y = 0\} \), the hyperbolic planes in \( \mathbb{H}^3 \) bounded by \( C \), are disjoint from \( E \) and converge to \( L \times \mathbb{R}^+ \)). So \( E \) is trapped between two horizontal planes \( t = \pm c \).

The distance between these horizontal planes tends to zero as \( y \to \infty \). Now we will prove that for \( y \) large, the Killing field \( \frac{\partial}{\partial t} \) is transverse to \( E \). Hence a sub-end of \( E \) has bounded curvature. This will complete the proof as follows. The sub-end is a vertical graph over the plane \( t = 0 \), that converges to zero in the \( C^2 \)-topology. The graph function is the distance to the plane \( t = 0 \). Thus the geodesic curvature of the curve in \( E \), given by \( \text{Cyl} = E \cap \{y = \text{constant}\} \) is bounded. Also the length of this curve \( C_y \) tends to zero in \( C_y \) modulo \( (x,y,t) \to (x + \tau, y, t) \). This yields the formula for the finite total curvature of \( \Sigma \) in \( N \): apply Gauss-Bonnet to the compact part of \( \Sigma \) bounded by the curves \( C_y \) in the ends and let \( y \to \infty \).

Thus it suffices to prove \( E \) is transverse to \( \frac{\partial}{\partial t} \) for \( y \) large. The proof of this is the same as in Section 7, for an end trapped between two horizontal planes. More precisely, for an end \( E \) in \( \mathcal{M} \) between two horizontal planes that are close, the distance between the planes \(|t| = c\) tends to zero as \( y \to \infty \), so one can put a vertical catenoid \( F_0 \), whose boundary circles are of radius one and in the horizontal planes \(|t| = d > c\), when the center \( q \) of \( F_0 \) has \( y(q) \) larger than some \( y_0 \).

One chooses \( \rho, \ell \) as in Sections 6 and 8, and using the Dragging lemma, one shows that if \( E \) has a vertical tangent plane at \( p, y(p) \) large, then one finds \( p_1, p_2 \in B_\rho(q) \cap E \), that cannot be joined by a path in \( E \cap F_0^- \). One defines \( \Gamma = \Gamma_+ \cup \Gamma_- \) and the same proof now gives a contradiction.

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