Abstract. We prove that the extrinsic Hausdorff dimension is always greater than or equal to the intrinsic Hausdorff dimension in models of triangulated random surfaces with action which is quadratic in the separation of vertices. We furthermore derive a few naive scaling relations which relate the intrinsic Hausdorff dimension to other critical exponents. These relations suggest that the intrinsic Hausdorff dimension is infinite if the susceptibility does not diverge at the critical point.
1 Introduction

The geometrical properties of random surfaces can be studied from two points of view. One can look at the surfaces in an ambient imbedding space as an outside observer or one can study the surfaces as an observer living in the two-dimensional world defined by the surface. The first point of view, the extrinsic one, is appropriate in string theory and in the statistical theory of membranes while the second point of view is the only meaningful one in quantum gravity. There are distinct observables and critical exponents associated with these two viewpoints. Some of these exponents were evaluated for multicritical branched polymers in [1].

In this letter we consider the gaussian dynamical triangulation model [2] and prove that the intrinsic Hausdorff dimension is always smaller than or equal to the extrinsic Hausdorff dimension. This is a rather simple consequence of the fact that the geodesic distance between two points on a surface is almost always greater than or equal to the Euclidean distance in imbedding space between the same points. We expect the result to be model independent and it is certainly fulfilled in all known examples. The proof presented below uses however the quadratic nature of the action in an essential way.

Recently the intrinsic Hausdorff dimension of discretized pure 2-dimensional gravity has been studied numerically and found to be infinite [3] (see [4, 5] for related older simulations) and in perturbation theory [6] using the Liouville formulation. Unfortunately our inequality for the Hausdorff dimensions has no direct bearing on this case. The inequality is only meaningful when the extrinsic Hausdorff dimension is defined and this requires \( d \geq 1 \). Presumably the inequality holds for gravity coupled to matter with central charge \( c \geq 1 \). There does not seem to be any good reason to expect the inequality to be valid for the analytically continued external Hausdorff dimension to \( d < 1 \).

In the second half of the paper we discuss alternative definitions of the intrinsic Hausdorff dimension and derive naive scaling relations which relate the intrinsic Hausdorff dimension to other critical exponents in any imbedding dimension. Provided our scaling assumptions are valid, it is hard to escape the conclusion that the Hausdorff dimension is infinite if the critical exponent of the susceptibility is negative.

2 Intrinsic and extrinsic Hausdorff dimension

Let \( T_N \) denote the collection of all triangulations of the sphere with \( N \) vertices, one of which is singled out and called the marked vertex. We label the marked vertex...
by 0 and label the others arbitrarily by \( i = 1, 2, \ldots, N - 1 \). A random surface in \( \mathbb{R}^d \) based on a given triangulation \( T \in T_N \) is a mapping from the vertices of \( T \) into \( \mathbb{R}^d \), \( i \mapsto x_i \). To each such surface we assign the gaussian action

\[
S_T = \sum_{(ij) \in \mathcal{L}(T)} (x_i - x_j)^2,
\]

(1)

where \( \mathcal{L}(T) \) is the collection of all nearest neighbour pairs of vertices in \( T \). The canonical partition function is defined by

\[
Z_N = \sum_{T \in T_N} \rho(T) \int e^{-S_T} N^{-1} \prod_{i=1}^{N-1} dx_i,
\]

(2)

where \( \rho \) is a non-negative weight factor for triangulations of the form discussed in [7] and we have removed the translational degree of freedom by fixing the vertex \( i = 0 \) at the origin in \( \mathbb{R}^d \). The Gibbs state associated with \( Z_N \) is defined by

\[
\langle \cdots \rangle_N = \frac{1}{Z_N} \sum_{T \in T_N} \rho(T) \int \cdots e^{-S_T} N^{-1} \prod_{i=1}^{N-1} dx_i
\]

\[
= Z_N^{-1} (2\pi)^{d/2} \sum_{T \in T_N} \rho(T) \det C_T,
\]

(3)

where \( C_T \) is the modified adjacency matrix of \( T \), see [7]. We remind the reader of the graph theoretical result that we shall use later on:

\[
\det C_T = \# \{ B : B \text{ is a spanning tree subgraph of } T \}.
\]

(4)

For a proof of this see e.g. [8].

The mean square extent of a surface with \( N \) vertices is defined by

\[
R_N^2 = \langle N^{-1} \sum_{i=0}^{N-1} x_i^2 \rangle_N.
\]

(5)

If there is a number \( \delta_{ext} > 0 \), such that asymptotically

\[
R_N^2 \sim N^{2/\delta_{ext}},
\]

(6)

then we call \( \delta_{ext} \) the extrinsic Hausdorff dimension of the random surfaces in the given model. It is clear that \( \delta_{ext} \geq 2 \) if it exists. If \( R_N \) grows more slowly than any power of \( N \), then it is customary to say that the extrinsic Hausdorff dimension is infinite.

The mean square extent of surfaces based on regular square triangulations of the torus can be evaluated and the result is

\[
R_N^2 \sim \ln N,
\]

(7)
in all dimensions $d$, see e.g. [9]. Branched polymers are easily seen to have extrinsic Hausdorff dimension 4, see e.g. [1]. In dimension $d=-2$ the natural analytic continuation of $\delta_{\text{ext}}$ can be evaluated in the case when the weight factor $\rho(T)$ is given by the order of the symmetry group of $T$ to the power $-1$ [5]. The result is $\delta_{\text{ext}} = \infty$. A very suggestive calculation in one dimension gives the same result [10]. Several numerical studies have been made of $\delta_{\text{ext}}$ and the results are rather inconclusive, see [4, 5], but indicate that the extrinsic Hausdorff dimension is infinite for negative imbedding dimensions but becomes finite around $d = 2$ and may decrease to 4 at large $d$, where one expects a branched polymer phase. It is possible that the extrinsic Hausdorff dimension is nonuniversal, i.e. depends on $\rho$.

Next we define the intrinsic Hausdorff dimension. Let $i$ be a vertex in $T \in T_N$ and denote by $d_T(i)$ the geodesic distance from the marked vertex $0 \in T$ to $i$, i.e. $d_T(i)$ is the smallest number $n$ of links $l_1, \ldots l_n$ in $T$ such that 0 is an endpoint of $l_1$, $i$ is an endpoint of $l_n$ and $l_j$ and $l_{j+1}$ share exactly one vertex for $j = 1, 2, \ldots, n-1$. We define the intrinsic Hausdorff dimension $\delta_{\text{int}}$ (if it exists) by the asymptotic formula

$$< N^{-1} \sum_{i=0}^{N-1} d_T(i)^2 >_N \sim N^{2/\delta_{\text{int}}}.$$  \hfill (8)

The intrinsic Hausdorff dimension of multicritical branched polymers was evaluated in [1], see also [3]. In the case of regular triangulations $\delta_{\text{int}}$ is trivially 2. We shall now prove that for any choice of weight factor $\rho$ for which the intrinsic and extrinsic Hausdorff dimensions exist they satisfy the inequality

$$\delta_{\text{int}} \leq \delta_{\text{ext}}.$$  \hfill (9)

Let $i \in T$ and $n = d_T(i)$. Let $j(\alpha), \alpha = 0, 1, \ldots, n$, $j(0) = 0$, $j(n) = i$, be the vertices in a geodesic path in $T$ from 0 to $i$. Then

$$x_i = \sum_{\alpha=1}^{n} (x_{j(\alpha)} - x_{j(\alpha-1)}),$$  \hfill (10)

so

$$x_i^2 \leq n \sum_{\alpha=1}^{n} (x_{j(\alpha)} - x_{j(\alpha-1)})^2.$$  \hfill (11)

Hence,

$$< \sum_{i=0}^{N-1} x_i^2 >_N \leq Z_N^{-1} \sum_{T \in T_N} \rho(T) \sum_{i=0}^{N-1} d_T(i) \sum_{\alpha=1}^{d_T(i)} \int (x_{j(\alpha)} - x_{j(\alpha-1)})^2 e^{-S_T} \prod_{i=1}^{N-1} dx_i.$$  \hfill (12)

Now we use that

$$(x_{j(\alpha)} - x_{j(\alpha-1)})^2 e^{-S_T} \leq K e^{-S_T},$$  \hfill (13)
where $K$ is a constant and the modified action $S'_T$ is defined by leaving out the term $(x_{j(\alpha)} - x_{j(\alpha-1)})^2$ in the action (7), i.e.

$$S'_T = \sum_{(k,l) \in \mathcal{L}(T')} (x_k - x_l)^2,$$

and here $T'$ is the graph obtained by removing the link $(j(\alpha), j(\alpha-1))$ from $T$. Note that $T'$ is not a triangulation, but the formula (7) is still valid for $\det C_{T'}$. Any tree that spans $T'$ also spans $T$, so

$$\det C_{T'} \leq \det C_T.$$

We furthermore claim that

$$\det C_T \leq 3 \det C_{T'}.$$

Hence,

$$R^2_N \leq K'N^{-1} < \sum_{i=0}^{N-1} d_T(i)^2 >_N,$$

where $K'$ is another constant and the inequality (11) follows for any imbedding dimension $d$.

In order to prove the claim we note that the trees that span $T$ fall into two disjoint classes: those which contain the link $\lambda = (j(\alpha - 1), j(\alpha))$ and the ones that do not contain $\lambda$. The trees in the second class are precisely those that span $T'$. Consider a tree $B$ in the first class. Let $k$ be the third vertex in one of the two triangles in $T$ that share the link $\lambda$. If we remove $\lambda$ from $B$, we obtain two disjoint trees $B_1$ and $B_2$ with $j(\alpha - 1) \in B_1$ and $j(\alpha) \in B_2$. Let us assume that $k \in B_1$. Define a tree $B'$ by removing the link $\lambda$ from $B$ and replacing it with the link $(k, j(\alpha))$. If $k \in B_2$, we define $B'$ by replacing $\lambda$ with $(k, j(\alpha - 1))$. In both cases $B'$ is a spanning tree of $T'$ and the desired result follows, since the mapping $B \mapsto B'$ is at most two to one.

### 3 Scaling relations for intrinsic exponents

The above definition of the Hausdorff dimension is not the only possible one. It is not unreasonable to define a Hausdorff dimension in the grand canonical ensemble relating the average area of surfaces to a diverging correlation length as the critical point is approached. One can also, in the case of the intrinsic Hausdorff dimension, consider the volume $V$ of a ball of radius $n$ and use the relation between the average value of $V$ and $n$ to define a Hausdorff dimension. It is by no means clear that the two intrinsic Hausdorff dimensions defined in this way in the grand canonical ensemble are the same or identical to the dimension defined in the canonical ensemble in the
previous section. Here we shall study the two definitions of the intrinsic Hausdorff dimension in the grand canonical ensemble and give a naive derivation of their relation to other intrinsic critical exponents.

Let $S_{2,n}$ be the collection of all triangulations with two marked vertices a distance $n$ apart. The intrinsic two point function $G_{\mu}(n)$ is defined by

$$G_{\mu}(n) = \sum_{T \in S_{2,n}} e^{-\mu|T|} W(T),$$

where $W(T)$ is a weight factor coming from $\rho(T)$ and the "matter fields":

$$W(T) = \rho(T) \int e^{-S_T} \prod_{i=1}^{\vert T \vert-1} dx_i.$$  \hspace{1cm} (19)

Here $\vert T \vert$ denotes the number of vertices in $T$. With this definition of $W$ there is a $\mu_c > 0$, such that the two point function is finite for $\mu > \mu_c$ and the sum defining $G_{\mu}(n)$ diverges for $\mu < \mu_c$ \cite{7}. The arguments presented below do not make any use of the detailed form of $W$ and are valid for the analytic continuation of $W$ to $d < 1$.

The susceptibility is defined by

$$\chi(\mu) = \sum_{n=1}^{\infty} G_{\mu}(n)$$

and the partition function by

$$Z(\mu) = \sum_{T \in S_{1}} e^{-\mu|T|},$$

where $S_{1}$ is the collection of all triangulations with one marked vertex. If $\chi$ diverges at $\mu_c$, we assume that there is $\gamma > 0$ such that $\chi(\mu) \sim (\mu - \mu_c)^{-\gamma}$, but if $\chi(\mu_c) < \infty$, then we assume that $\chi(\mu_c) - \chi(\mu) \sim (\mu - \mu_c)^{-\gamma}$ for some $\gamma < 0$.

By subadditivity arguments, see \cite{11}, one can show that

$$G_{\mu}(n) \sim e^{-m(\mu)n},$$

where $m(\mu) \geq 0$. We assume that

$$m(\mu) \sim (\mu - \mu_c)^{\nu}$$

for some $\nu > 0$ as $\mu \to \mu_c$, and define the correlation length $\xi$ by

$$\xi(\mu) = m(\mu)^{-1}.$$  \hspace{1cm} (24)

Assuming that triangulations with linear size $\xi$ give a dominating contribution to the two point function and assuming that there is $\delta_1 > 0$ such that

$$G_{\mu}(\xi)^{-1} \sum_{T \in T_{2,\xi}} |T| e^{-\mu|T|} W(T) \sim \xi^{\delta_1},$$

$$6$$
it follows that
\[ \delta_1 \nu = 1. \]  
(26)

If \( T \in S_1 \), let \( D_T(n) \) be the number of points in \( T \) at a distance \( n \) from the marked point. Assume that there is a \( \delta_2 \geq 0 \) such that
\[ Z^{-1} \sum_{T \in T_1} D_T(\xi)e^{-\mu|T|}W(T) \sim \xi^{\delta_2 - 1}. \]  
(27)
The numbers \( \delta_1 \) and \( \delta_2 \) are the two candidates for an intrinsic Hausdorff dimension.

Let us next assume that
\[ G_\mu(\xi) \sim \xi^{-\eta} \]  
(28)
and the scaling limit of \( G_\mu \) exists. Then we obtain a Fisher scaling relation, which in the present context takes the form
\[ \nu(1 - \eta) = \max \{0, \gamma\} \]  
(29)
by the same proof as in [12].

Finally note that if \( Z \) is finite at the critical point, then
\[ \eta = 1 - \delta_2 \]  
(30)
by Eqs. (27) and (28), since
\[ \sum_{T \in T_1} D_T(\xi)e^{-\mu|T|}W(T) = G_\mu(\xi). \]  
(31)
Suppose now \( \gamma < 0 \). Then
\[ \nu(1 - \eta) = 0, \]  
(32)
so \( \eta = 1 \) or \( \nu = 0 \). If \( \eta = 1 \), then \( \delta_2 = 0 \) which presumably does not happen in any sensible model. We conclude that \( \nu = 0 \) which implies that the Hausdorff dimension \( \delta_1 \) is infinite. If we in addition assume that \( \delta_1 = \delta_2 \), then \( \eta = -\infty \).

If \( \gamma > 0 \) and we assume that \( \delta_1 = \delta_2 \), then we can solve equations (26), (29) and (30) for \( \gamma \) and the result is
\[ \gamma = 1. \]  
(33)
This, however, is impossible since the inequality \( \gamma \leq \frac{1}{2} \) holds in all models of the type we are considering [7, 11]. We conclude that the Hausdorff dimensions \( \delta_1 \) and \( \delta_2 \) must be different or at least one of the scaling assumptions is not valid.

For ordinary branched polymers one finds [1] \( \gamma = \frac{1}{2}, \nu = \frac{1}{2} \) and \( \eta = 0 \) by a direct calculation and hence \( \delta_1 = 2 \) and \( \delta_2 = 1 \) by the scaling relations. In this case \( \delta_{\text{int}} = 2 \) as remarked in [3], which throws some doubt on the validity of the scaling relation (31).
4 Discussion

The intrinsic Hausdorff dimension calculated in \[3\] is defined in almost the same way as \(\delta_2\) but in a different ensemble. It is therefore possible that the scaling relations in the last section and the fact that \(\gamma < 0\) for pure 2d-gravity explain the numerical results of \[3\]. In fact, all numerical evidence is consistent with \(\gamma < 0\) and \(\delta_{\text{int}} = \infty\) for \(d < 1\), since \(\delta_{\text{ext}}\) seems to be infinite for \(d \leq 1\).

One must bear in mind that the exponents that we have considered here give a very incomplete picture of what a typical surface looks like. This is best illustrated by the fact that \(\delta_{\text{int}} = 2\) both for ordinary branched polymers and for surfaces with a regular triangulation.

An observable of great interest is the number of connected components of the boundary of a ball of radius \(n\) as \(n\) gets large. This quantity was studied numerically in \[3\] and found to increase rapidly with \(n\). This means that in some sense the surfaces of 2d-gravity are similar to branched polymers. However, these surfaces are not ordinary branched polymers, since they have susceptibility exponent \(\gamma = -\frac{1}{2}\), whereas \(\gamma = \frac{1}{2}\) for branched polymers with positive weight. A clarification of this issue would be of utmost importance.

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