Exploring $HVV$ amplitudes with $CP$ violation by decomposition and on-shell scattering amplitude methods

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Abstract

$CP$ violation may play an important role in Baryogenesis in early universe, and should be examined at colliders comprehensively. We study $CP$ properties of $HVV$ vertexes between Higgs and gauge boson pairs with defining a $CP$ violation phase angle $\xi$, which indicates the mixture of $CP$-even and $CP$-odd Higgs states in $HVV$ in new physics. A series of $HVV$ amplitudes $H \to \gamma \gamma, H \to \gamma V \to \gamma \ell \ell$, and $H \to V V \to 4\ell$ with $CP$ phase angle are studied systematically, which explains explicitly why $CP$ violation could only be probed in $4\ell$ process independently. We get a novel amplitude decomposition relation which illustrates if two preconditions (multilinear momentum dependent vertexes and current $J_\mu$ of $V \to \ell^+ \ell^-$ is formally proportional to a photon’s polarization vector) are satisfied, an high-point amplitude can be decomposed into a summation of a series of low-point amplitudes. As a practical example, the amplitude of $H \to \gamma V \to \gamma \ell \ell$, and $H \to V V \to 4\ell$ processes can be decomposed into summation of many $H \to \gamma \gamma$ amplitudes. Meanwhile, we calculate these amplitudes in the framework of on-shell scattering amplitude method, with considering both massless and massive vector gauge bosons with $CP$ violation phase angle. The above two approaches provides consistent results and exhibit clearly the $CP$ violation $\xi$ dependence in the amplitudes.

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I. INTRODUCTION

There are two kinds of $CP$ violation sources in Standard Model (SM), one is weak $CP$ violation in Cabibbo-Kobayashi-Maskawa matrix [1, 2], the other is strong $CP$ violation related to topological charge in QCD vacuum [3–5]. Both of them have relations with Higgs Yukawa couplings. The CKM matrix originates from general Yukawa coupling matrices for three generations [6, 7], the $\theta$ angle in QCD vacuum could rotate to the complex phase of mass matrix by chiral transformation [8]. Even though, the SM Higgs boson is a $CP$-even scalar with $CP$-conserving interactions. By contrast, in new physics beyond SM (BSM), $CP$ violation usually relates to Higgs bosons. One reason is there often exist scalars and pseudoscalars instead of one single scalar in SM. A mixture of scalar and pseudoscalar is natural. Such as in Two Higgs Doublet Model (THDM) [9], Minimal Supersymmetric Standard model (MSSM) [10], and Composite Higgs Model [11], pseudoscalar always appears and there are no simple rule to forbid a mixture between the scalar and pseudoscalar. Except for theoretical naturalness and generality, one practical motivation for new $CP$ violation source comes from the matter-antimatter asymmetry observed in our Universe [12–14]. In electroweak Baryogenesis mechanism [15], $CP$ violation plus sphaleron transition [16] could produce baryon and lepton number violation during electroweak phase transition, but the $CP$ violation ratio in SM is too small to fulfil the quantity of the observed matter-antimatter asymmetry [17–19]. Therefore new $CP$ violation source must be added to obtain electroweak Baryogenesis.

We choose two model-independent framework to study $CP$ violation: one is a traditional way through additional effective Lagrangian terms, the other is using on-shell scattering amplitude method to analyze amplitudes.

Adding new $CP$-violating terms in the Lagrangian is an convenient, effective description of the new couplings beyond the SM. The new terms can be $CP$ conserved or $CP$ violated, but should obey the Lorentz and gauge invariance. For a specific new physics model, its new Higgs couplings can be simplified into this effective Lagrangian terms when other couplings are small enough to be omitted. Therefore, constraints on these new Higgs couplings in effective Lagrangian provide concrete limitations for model buildings with certain gauge symmetries.

On-shell method is a novel tool to deal with amplitudes directly, even with no Lagrangian
and Feynman diagram needed [20]. It starts from on-shell particle states instead of field, sets up constraints, exploits analytical properties such as poles and branch cuts, then gets an available amplitude. Specifically, a 3-point massless (or 1 massive 2 massless) amplitude could be fixed by locality and little group scaling [20–22], then a \( n + 1 \)-point tree amplitude could be constructed from \( n \)-point amplitudes through recursion relations. In this way all tree amplitudes could be obtained and they have clear mathematical structures.

We focus on \( H \to \gamma\gamma \), \( H \to \gamma\ell\ell \) and \( H \to 4\ell \) processes to analyze their BSM amplitudes. At Large Hadron collider (LHC), the \( H \to \gamma\gamma \) and \( H \to ZZ \to 4\ell \) processes are Higgs discovery channel [23, 24], which have the advantage of clean background and relative large signal. They are also golden channels for precise measurement of Higgs properties [25–28]. In our previous research, we notice that \( CP \) violation phase could not be probed solely in \( H \to \gamma\gamma \) or \( H \to \gamma\ell\ell \) processes without interference from background [29, 30]. By contrast, in \( H \to ZZ \to 4\ell \) processes \( CP \) violation could be probed lonely through its kinematic angles [28, 31–33]. These could be explained clearly at amplitude level after we get a compact formula. We explore the relations between these BSM amplitudes through the above two independent ways in this paper. A decomposition relation between these amplitudes is illustrated in an interesting diagrammatic way. Then we calculate the same amplitude from the on-shell method (BCFW recursion relation), which can be regarded as a parallel proof of the decomposition relation. Meanwhile, the massive spinor formalism are applied to prove it is also suitable for massive cases.

The outline of this paper is as follows. In section II, we show the amplitudes of SM \( HVV \) processes both at proton-proton collider and \( e^+e^- \) collider. In section III, we calculate BSM amplitudes in effective Lagrangian description. The \( H \to \gamma\gamma \), \( H \to \gamma\ell\ell \) and \( H \to 4\ell \) processes correspond separately to 3, 4, 5-point amplitudes. In section IV, we deduce decomposition relations for these amplitudes. In section V, we reproduce these BSM amplitudes in on-shell scattering amplitude approach. In section VI, the BSM amplitudes in on-shell scattering amplitude are generalized to massive spinor cases. Section VII is summary and discussion.
II. SM HVV HELICITY AMPLITUDES

Experimentally, SM/BSM $HVV$ couplings can be measured/exploited at the existed proton-proton collider such as the LHC or the future $e^+e^-$ collider such as the Circular Electron Positron Collider (CEPC), the International Linear Collider (ILC) and the Compact Linear Collider (CLIC), etc. To study the BSM $HVV$ couplings with $CP$ violation, its interference with the corresponding SM process (include or not include Higgs) may become the dominant contribution as generally the BSM couplings are assumed to be suppressed compared to the corresponding SM couplings. Before studying the BSM amplitudes, the amplitudes of SM $HVV$ process and the main background process are introduced to show a global view how amplitudes works for physics process at colliders. Analyzing amplitudes could unveil some mysteries that are not clear at the observable level. As followings we take specific $HZZ$ related processes as an example, at proton-proton collider and $e^+e^-$ collider separately. For the example at proton-proton collider, we focus on process-dependent amplitudes, show amplitudes of signal process and background process and discuss how these amplitudes are used for experimental predictions. For the example at $e^+e^-$ collider, we focus on process-independent amplitudes, that is, the amplitudes with all external particles outgoing, which relate to process-dependent amplitudes by crossing symmetry.

A. At proton-proton collider

The $gg \rightarrow H \rightarrow ZZ \rightarrow 4\ell$ process at LHC is sensitive to BSM $HVV$ couplings. We draw the Feynman diagrams of this signal channel and its main background as in Fig. 1. In Fig. 1(a) the Higgs production process $gg \rightarrow H$ are mediated by the top quark loop. Fig. 1(b) represents $gg \rightarrow ZZ \rightarrow 4\ell$ box process without Higgs, which is important in the off-shell Higgs region. Studying the interference between the signal and this continuum background in the off-shell Higgs region could give a stringent bound on Higgs width [34, 35], and also BSM $HZZ$ couplings [31, 35]. In Fig. 1 the $\ell$ and $\ell'$ have different flavors. If we study the process of $4\ell$ with same flavors, two more diagrams which describe another pairing of $4\ell$ should be added. Nevertheless, their amplitudes are similar [31].

The triangle top quark loop in $gg \rightarrow H$ could be integrated and described by an effective $ggH$ coupling such that the helicity amplitudes of $ggH$ are shown as [30]
\[ M^{gg \rightarrow H(1^+, 2^+)} = \frac{2c_g}{v}[12]^2, \]
\[ M^{gg \rightarrow H(1^-, 2^-)} = \frac{2c_g}{v}(12)^2. \] (1)

with
\[
\frac{c_g}{v} = \frac{1}{2} \sum_f \frac{\delta^{ab}}{16\pi^2} g_s^2 \frac{g^2}{4} \frac{m_f^2}{2 M W s W} \frac{1}{M_H^2} (2 + s_{12}(1 - \tau_H) C_0^{\gamma}(m_f^2)), \] (2)

where \( v = 246 \text{ GeV} \) is the vacuum expectation value of the Higgs, \( a, b = 1, ..., 8 \) are \( SU(3)_c \) adjoint representation indices for the gluons, \( \tau_H = 4m_f^2/M_H^2 \), and the \( C_0^{\gamma}(m^2) \) function is Passarino-Veltman three-point scalar functions [36]. The \( \langle ij \rangle \) and \( [ij] \) are followed as the conventions in Ref. [37, 38]:

\[
\langle ij \rangle \equiv \langle i^-|j^+ \rangle = u_-(p_i) u_+(p_j), \quad [ij] \equiv \langle i^+|j^- \rangle = u_+(p_i) u_-(p_j),
\]

\[
\langle ij \rangle [ji] = 2p_i \cdot p_j, \quad s_{ij} = (p_i + p_j)^2, \quad \epsilon_{\pm}^\mu(p_i, q) = \pm \frac{\langle q^\mu \gamma_\mu p_i^\pm \rangle}{\sqrt{2(q^\pm p_i^\pm)}}, \] (3)

where \( p_i \) are momentum of external legs, \( q \) is the reference momentum that reflect the freedom of gauge transformation, \( \epsilon_{\pm}^\mu(p_i, q) \) is for outgoing photons with \( \pm \) helicities. Notice that the gluons are incoming in Eq. (1), if let them outgoing, the amplitudes just need an exchange between \( \langle \rangle \) and \( [\rangle \rangle \) because of the crossing symmetry.
The helicity amplitudes of $H \rightarrow ZZ \rightarrow 4\ell$ are [31]

$$
M^{H \rightarrow ZZ \rightarrow 4\ell}(3^{-}, 4^{+}, 5^{-}, 6^{+}) = f \times l_{e}^{2} M_{W}^{2} \cos^{2} \theta_{W} \langle 35 \rangle [46],
$$

$$
M^{H \rightarrow ZZ \rightarrow 4\ell}(3^{-}, 4^{+}, 5^{-}, 6^{+}) = f \times l_{e} r_{e} M_{W}^{2} \cos^{2} \theta_{W} \langle 36 \rangle [45],
$$

$$
M^{H \rightarrow ZZ \rightarrow 4\ell}(3^{+}, 4^{-}, 5^{-}, 6^{+}) = f \times l_{e}^{2} r_{e} M_{W}^{2} \cos^{2} \theta_{W} \langle 45 \rangle [36],
$$

$$
M^{H \rightarrow ZZ \rightarrow 4\ell}(3^{+}, 4^{-}, 5^{-}, 6^{+}) = f \times r_{e}^{2} M_{W}^{2} \cos^{2} \theta_{W} \langle 46 \rangle [35],
$$

where the common factor $f$ is defined as

$$
f = -2i e^{3} \frac{1}{M_{W} \sin \theta_{W}} P_{Z}(s_{34}) P_{Z}(s_{56}),
$$

with

$$
P_{X}(s) = \frac{1}{s - M_{X}^{2} + i M_{X} \Gamma_{X}}
$$

are the propagator of particle $X$. So the total amplitude of $gg \rightarrow H \rightarrow ZZ \rightarrow 4\ell$ is

$$
M^{gg \rightarrow H \rightarrow ZZ \rightarrow 4\ell} = M^{gg \rightarrow H} \times P_{H}(s_{12}) \times M^{H \rightarrow ZZ \rightarrow 4\ell}.
$$

The amplitude of the box process $gg \rightarrow ZZ \rightarrow 4\ell$ in Fig.1(b) are complicated due to the box loop integration. Its full analytical form could be found in Ref. [25], which is coded in MCFM package [39]. Meanwhile we could do phase space integral and get numerical cross sections in MCFM package.

When studying the observable effects of the BSM $HHV$ couplings, we could add BSM amplitudes to MCFM package and get total and partial cross section results. As the analytical form of each amplitude is clear, each contribution for the cross section can be singly shown. For example, the interference contribution from SM Higgs process and BSM Higgs process can be calculated by selecting Re($M^{SM}_{H} M^{BSM}_{H}$) part in the code. In Higgs off-shell region, the interference contribution between continuum background and BSM process is also important, so the Re($M^{SM}_{box} M^{BSM}_{H}$) needs to be singly focused on. More details could be found in Ref. [31].

B. At $e^{+}e^{-}$ Collider

At $e^{+}e^{-}$ Collider, two main processes with $HZZ$ coupling are $ZH$ process and vector boson fusion process. Their Feynman diagrams are shown as in Fig. 2. In these two processes
the $e^+e^-$ are incoming particles and $b\bar{b}\ell^+\ell^-$ (or $b\bar{b}e^+e^-$) are outgoing particles. As crossing symmetry illustrates that an incoming particle could be replaced by an outgoing antiparticle and leave the S-matrix unchanged, we could calculate the amplitude with all particles outgoing firstly and then deduce the amplitude for physics process just by relabeling momenta, helicities and particle properties. We won’t go to details about using crossing symmetry since it is actually trivial once rules are set, but focus on how to write the process-independent amplitudes with all external particles outgoing. Later we also keep this convention for all BSM amplitudes. So the amplitudes we need are same for Fig. 2(a) and Fig. 2(b) if $\ell$ is assumed to be $e$ or both them are assumed to be massless. They could be written as

$$\mathcal{M}(1_b, 2_b, 3_{\ell^-}, 4_{\ell^+}, 5_{\ell^-}, 6_{\ell^+}) = \mathcal{M}(1_b, 2_b, I_H) \times P_H(s_{12}) \times \mathcal{M}(I_{H}', 3_{\ell^-}, 4_{\ell^+}, 5_{\ell^-}, 6_{\ell^+}), \quad (8)$$

where $I_H, I_H'$ represents the mediate $H$ which are broken into two parts and appear in each smaller amplitudes. The amplitude $\mathcal{M}(I_{H}', 3_{\ell^-}, 4_{\ell^+}, 5_{\ell^-}, 6_{\ell^+})$ has been calculated in Eq. (4) except flipping the momentum of the Higgs boson to be outgoing. Actually, because Higgs is a scalar, the amplitude will remain unchanged in this case. One new amplitude we should pay attention is $\mathcal{M}(1_b, 2_b, I_H)$, which is

$$\mathcal{M}(2^-_b, 3^-_b) = \frac{-i m_b}{v} \langle 12 \rangle,$$

$$\mathcal{M}(2^+_b, 3^+_b) = \frac{-i m_b}{v} [12]. \quad (9)$$

The external $b, \bar{b}$ quarks are assumed to be massless in high energy limit. The $Hb\bar{b}$ coupling is still fixed to be proportional to $m_b$. The amplitudes for massive particles are realized in massive spinor formalism and need to use little group indices [22, 40, 41], which relates $\langle 12 \rangle$ to bolded $\langle 12 \rangle$ for example. We study them later in Section VI.
III. BSM $HV V$ HELICITY AMPLITUDES

In this section firstly we introduce the BSM $HV V$ effective couplings and define the $CP$ violation phase. Then we calculate their amplitudes. At last we discuss the interference contribution from the BSM amplitudes.

A. $HV V$ effective couplings

In SMEFT [42–44], the complete form of higher-dimensional operators can be written as

$$\mathcal{L} = \mathcal{L}_{SM} + \frac{1}{\Lambda} \sum_k C_k^5 \mathcal{O}_k^5 + \frac{1}{\Lambda^2} \sum_k C_k^6 \mathcal{O}_k^6 + \mathcal{O}(\frac{1}{\Lambda^3}),$$

where $\Lambda$ is energy scale of new physics, and $C_k^i$ with $i = 5, 6$ are Wilson loop coefficients.

BSM $HV V$ ($V$ represents $\gamma, Z/W$ boson) vertices start from dimension-six operators $\mathcal{O}_k^6$. In Warsaw basis [43], they are

$$\mathcal{O}_{\Phi D}^6 = (\Phi^\dagger D^\mu \Phi)^* (\Phi^\dagger D^\mu \Phi),$$
$$\mathcal{O}_{\Phi W}^6 = \Phi^\dagger \Phi W_{\mu \nu}^I W^{I \mu \nu}, \quad \mathcal{O}_{\Phi B}^6 = \Phi^\dagger \Phi B_{\mu \nu}^{\mu \nu}, \quad \mathcal{O}_{\Phi W B}^6 = \Phi^\dagger \tau^I \Phi W_{\mu \nu}^I B^{\mu \nu},$$
$$\mathcal{O}_{\Phi ˜W}^6 = \Phi^\dagger \Phi ˜W_{\mu \nu}^I W^{I \mu \nu}, \quad \mathcal{O}_{\Phi ˜B}^6 = \Phi^\dagger \Phi ˜B_{\mu \nu}^{\mu \nu}, \quad \mathcal{O}_{\Phi ˜W B}^6 = \Phi^\dagger \tau^I \Phi ˜W_{\mu \nu}^I B^{\mu \nu},$$

where $\Phi$ is a doublet representation under the $SU(2)_L$ group and the aforementioned Higgs field $H$ is one of its four components; $D_\mu = \partial_\mu - igW_\mu^I T^I - ig'^Y B_\mu$, where $g$ and $g'$ are coupling constants, $T^I = \tau^I/2$, where $\tau^I$ are Pauli matrices, $Y$ is the $U(1)_Y$ generator; $W_{\mu \nu}^I = \partial_\mu W_\nu^I - \partial_\nu W_\mu^I - g\epsilon^{IJK} W_\mu^J W^K_\nu$, $B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, $\bar{X}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \rho \sigma \nu} X^{\rho \sigma}$.

After spontaneous symmetry breaking, we get $HV V$ effective interactions,

$$\mathcal{L}^{int} = -\frac{c_{VV}}{v} HV_{\mu \nu} V_{\mu \nu} - \frac{\tilde{c}_{VV}}{v} HV_{\mu \nu} \tilde{V}_{\mu \nu},$$

$c_{VV}, \tilde{c}_{VV}$ are real numbers that originate from Wilson loop coefficients, $V$ represents vector boson. A detailed formula about $c_{VV}, \tilde{c}_{VV}$ and Wilson loop coefficients $C_k^6$ could be found in Ref. [31]. A standard analysis based on SMEFT should be a global study involving all dimension-6 operators. Here we concentrate only on the new $HV V$ terms.

For the origin of the dimension-6 operators, they can come from the loop momentum integration in the loop diagrams with multi-outlegs. The virtual particles in the loop can be both SM particles and BSM new particles. The difference between the two cases is the SM
processes have definite dimension-6 coupling coefficients while the dimension-6 coefficients in New Physics (NP) are still to be determined.

The $CP$ violation phase could be defined as

$$\xi \equiv \tan^{-1}(\tilde{c}_{VV}/c_{VV}), \text{ when } \text{Arg}(\tilde{c}_{VV}/c_{VV}) = 0 \text{ or } \pi,$$

(13)

where $\xi = 0 \ (\pi/2)$ represents a pure $CP$-even (-odd) $HVV$ vertex. $\xi \neq 0$ means $CP$ violation and $\xi = \pi/2$ corresponds to maximal $CP$ violation if other Higgs vertices are supposed to be $CP$-even. In amplitudes we will see that $\xi$ appears as a phase, which changes sign under $CP$ transformation. That is why we name it as $CP$ violation phase. Meanwhile,

$$c_{VV}^S \equiv \sqrt{c_{VV}^2 + \tilde{c}_{VV}^2},$$

(14)

could be defined as the amplitude moduli, which is proportional to signal strength in collider experiment.

B. Helicity amplitudes

In following sections, for simplification, we only take the amplitude of Higgs decay with BSM $HVV$ vertex as an example to illustrate the decomposition relation. It is also the process-independent amplitude since Higgs boson is a scalar and the amplitude is free of its incoming or outgoing. Full amplitudes with Higgs production and decay, can be easily obtained by multiplying the BSM Higgs decay amplitudes with the partial amplitude of Higgs production $M_{gg}\rightarrow H$ in Eq. (1) and a Higgs propagator in Eq. (6) at the proton-proton collider, or by multiplying with the partial amplitude of $H \rightarrow b\bar{b}$ Eq. (9) and a Higgs propagator in Eq. (6) at the $e^+e^-$ collider, similar as Eq. (7) or Eq. (8) as discussed in section II.

FIG. 3: Feynman diagrams of $H \rightarrow \gamma\gamma$, $H \rightarrow V\gamma \rightarrow \ell\ell\gamma$ and $H \rightarrow VV \rightarrow 2\ell2\ell'$ from left to right. Each $HVV$ vertex is dotted as an effective coupling.
Feynman diagrams with effective $HVV$ couplings are shown in Fig. 3. After some calculations, the helicity amplitudes are as follows.

- For process $H \rightarrow \gamma \gamma$, 
  
  \[
  \mathcal{M}(2^+, 3^+) = \frac{2c_s}{v} e^{i\xi} [23]^2, \\
  \mathcal{M}(2^-, 3^-) = \frac{2c_s}{v} e^{-i\xi} [23]^2, \\
  \mathcal{M}(2^+, 3^-) = 0, \\
  \mathcal{M}(2^-, 3^+) = 0, 
  \]
  
  where we use $\mathcal{M}(2_{h_1}^+, 3_{h_1}^-)$ to represent $\mathcal{M}(1_{H}^{h_1}, 2_{\gamma}^{h_2}, 3_{\gamma}^{h_3})$ since $h_1$ is trivially zero for all cases, $h_i$s are helicities of external legs with momentum outgoing. The results show that helicities of the two photons should keep same sign because the spin of Higgs is zero and total angular momenta conserves. Under $CP$ transformation $\mathcal{M}(2^+, 3^+)$ changes to $\mathcal{M}(2^-, 3^-)$. Analytically it corresponds that $\langle ij \rangle$ changes to $[ij]$. Thus in Eq. (15), a general nonzero $\xi$ represents $CP$ violation.

- For process $H \rightarrow V \gamma \rightarrow \ell\ell\gamma$, 
  
  \[
  \mathcal{M}(2_{e^-}^-, 3_{e^+}^+, 4^-) = f_{V}^{-}(s_{23}) \times \frac{2c_s}{v} e^{-i\xi} [23][24]^2, \\
  \mathcal{M}(2_{e^-}^-, 3_{e^+}^+, 4^+) = f_{V}^{-}(s_{23}) \times \frac{2c_s}{v} e^{i\xi} [23][34]^2, \\
  \mathcal{M}(2_{e^-}^+, 3_{e^+}^-, 4^+) = f_{V}^{+}(s_{23}) \times \frac{2c_s}{v} e^{i\xi} [23][24]^2, \\
  \mathcal{M}(2_{e^-}^+, 3_{e^+}^-, 4^-) = f_{V}^{+}(s_{23}) \times \frac{2c_s}{v} e^{-i\xi} [23][34]^2, 
  \]
  
  where $s_{23} = (p_2 + p_3)^2$, $f_{V}^{-}(s) = \sqrt{2} e l_V P_V(s)$ and $f_{V}^{+}(s) = -\sqrt{2} e r_V P_V(s)$, $P_V(s) = \frac{1}{s_M^2}$ is the propagator of the gauge boson, $l_V$ and $r_V$ are the left-handed and right-handed couplings between vector boson and leptons, and leptons are supposed to be massless. The remaining helicity amplitudes are equal to zero and thus not listed.
For process $H \rightarrow V V \rightarrow 2\ell 2\ell'$,

$$M(2_{\ell^-}, 3_{\ell^+}, 4_{\ell^-}, 5_{\ell^+}) = f_V^-(s_{23}) f_V^+(s_{45}) \frac{2c_{VV}}{v} \left( e^{i\xi} \langle 23 \rangle \langle 45 \rangle [35]^2 + e^{-i\xi} \langle 23 \rangle \langle 45 \rangle (24)^2 \right),$$

$$M(2_{\ell^-}, 3_{\ell^+}, 4_{\ell^-}, 5_{\ell^+}) = f_V^+(s_{23}) f_V^-(s_{45}) \frac{2c_{VV}}{v} \left( e^{i\xi} \langle 23 \rangle \langle 45 \rangle [34]^2 + e^{-i\xi} \langle 23 \rangle \langle 45 \rangle (25)^2 \right),$$

$$M(2_{\ell^-}, 3_{\ell^+}, 4_{\ell^-}, 5_{\ell^+}) = f_V^+(s_{23}) f_V^+(s_{45}) \frac{2c_{VV}}{v} \left( e^{i\xi} \langle 23 \rangle \langle 45 \rangle [25]^2 + e^{-i\xi} \langle 23 \rangle \langle 45 \rangle (34)^2 \right),$$

$$M(2_{\ell^-}, 3_{\ell^+}, 4_{\ell^-}, 5_{\ell^+}) = f_V^+(s_{23}) f_V^+(s_{45}) \frac{2c_{VV}}{v} \left( e^{i\xi} \langle 23 \rangle \langle 45 \rangle [24]^2 + e^{-i\xi} \langle 23 \rangle \langle 45 \rangle (35)^2 \right),$$

where $VV$ could be $\gamma\gamma$, or $ZZ$, or $\gamma Z$, or $W^+W^-$. While when it represents $\gamma Z$ or $W^+W^-$, the original Lagrangian in Eq. (12) should be scaled by a factor of 2 on the whole to make the formula consistent. The remaining helicity amplitudes are equal to zero.

### C. Interference contribution

By using the compact form of the amplitudes and the definition of $CP$ violation phases angle, it is interesting to compare the SM $HVV$ amplitudes with BSM $HVV$ amplitudes, and then show how to extract these BSM contributions in collider experiments.

Firstly we compare the SM $H\gamma\gamma$ amplitudes with BSM $H\gamma\gamma$ amplitudes. The SM $H\gamma\gamma$ amplitudes can be obtained by replacing the coefficient $C_g$ with $C_\gamma$ in Eq. (1), where $C_\gamma$ represents the triangle loop integral from both top quark loop and $W$ boson loop [45]. The BSM $H\gamma\gamma$ amplitude are shown in Eq. (15). Comparing Eq. (1) with Eq. (15), their spinor structures are the same while the $CP$ violation phase and coefficients are different. Therefore, for interference between these two amplitudes, the kinematic observables (such as the shape of the angular distribution of the external particles) will remain unchanged except for an overall scale factor.

Next we compare the SM and BSM amplitudes in . (4) and Eq.(17) of $H \rightarrow ZZ \rightarrow 4\ell$ processes. Their spinor structures are completely different, as each SM amplitude has two brackets (including both $\langle \rangle$ and $\langle \rangle$) and one term, while the BSM amplitude have four brackets and two summed terms. The two more brackets in BSM amplitudes origin from the partial derivatives in dim-6 operators as shown in Eq. (12). Therefore, the extra momentum dependence of the BSM scattering amplitudes can be regarded as an indication of the momentum dependence of the BSM couplings. It is obvious that this interference contribution between BSM and SM amplitudes, which is proportional to the momentum of the external particles, will be enhanced in the high energy region. In other words, the
interference effects are expected to be searched sensitively in the off-shell Higgs High energy region [31].

IV. DECOMPOSITION OF HELICITY AMPLITUDES

Amplitudes of $CP$ violation $HVV$ processes in Eq. (15) (16) and (17) have similar structures. In $H \to \gamma\gamma$ and $H \to \ell\ell\gamma$ processes, there is only one term for each helicity amplitude. $CP$ violation phase shows as a global phase. However, in $H \to 4\ell$ process, two terms appear and $CP$ violation phases have reverse signs. To explore how amplitudes change when external legs increase, we find a decomposition relation for a particular type of $n$-particle effective interactions. Then we apply it to the $HVV$ effective interactions and derive the corresponding amplitudes.

A. Proof

![Diagram](image)

**FIG. 4:** Diagram (a) and (b) are two examples of $M_{\text{lower}}$ and $M_{\text{higher}}$, where the blob represents the same effective interaction. In $M_{\text{lower}}$, $k_i$ and $l_i$ characterize vector bosons and other particles. In $M_{\text{higher}}$, some external vector bosons are replaced with the current $J^{(2)}$ while the others are noted by $J^{(1)}$.

Consider the amplitudes $M_{\text{lower}}$ and $M_{\text{higher}}$ with $m$ and more than $m$ external lines, which both include an $m$-particle effective interactions, e.g. Fig.4. In $M_{\text{higher}}$, all propagators are vector bosons and they are attached to the $m$-particle effective vertex. Therefore, vector bosons will be crucial in the construction of $M_{\text{higher}}$. In this subsection, we assume that the vector bosons in $M_{\text{higher}}$ are massless to derive the decomposition relation. Since the
contraction of the massless fermion current with massless vector propagator are the same as with massive vector bosons, the decomposition relations are valid for the massive vector bosons. In the massive case, the vector bosons in $M_{\text{lower}}$ are still massless, but the vector bosons in $M_{\text{higher}}$ could be massive. For the convenience, we relabel the momenta of $M_{\text{lower}}$ from $\{p_1, \ldots, p_m\}$ to $\{l_1, \ldots, l_{m-n}; k_1, \ldots, k_n\}$, where $k$ corresponds to the momenta of gauge bosons and $l$ corresponds to the others. Now we write $M_{\text{lower}}$ in terms of polarization $\epsilon$ and vertex $\Gamma$,

$$M_{\text{lower}}(l_1, \ldots, l_{m-n}; k_1^{h_1}, \ldots, k_n^{h_n}) = \Gamma^{\mu_1, \ldots, \mu_n}(k_1, \ldots, k_n) \prod_i \epsilon_{\mu_i}(k_i, r_i),$$

(18)

where $h_i$ and $r_i$ are the helicity and reference momentum of gauge boson $i$. Here $\Gamma$ is not the conventional vertex in Feynman diagrams in SM.

The decomposition relation is based on two key points. One point is the BSM vertex $\Gamma$ is multilinear to the momenta of vector bosons,

$$\Gamma^{\mu_1, \ldots, \mu_n}(k_1, \ldots, k_n) = \sum_{j_1, \ldots, j_n} \Gamma^{\mu_1, \ldots, \mu_n}(q_{1j_1}, \ldots, q_{nj_n}),$$

(19)

where

$$k_i = \sum_{j_i=1}^{n_i} q_{ij_i}, \quad n_i = 1, 2.$$  

(20)

The other point is that the current $J_{\mu}$ is proportional to the polarization vector of a photon $\epsilon_{\mu}$. They can be written in a uniform notation as

$$J_{\mu}^{(n_i)}(q_1^{h_1}, \ldots, q_{m_i}^{h_{m_i}}) = F_{q_{ij_i}}(q_1^{h_1}, \ldots, q_{m_i}^{h_{m_i}})\epsilon_{\mu}^{H_{ij_i}}(q_{ij_i}, r_{ij_i}),$$

(21)

where $n_i$ is the number of external particles and $F$ is a factor. The helicity $H_{ij_i}$ is a function of $h_{ij_i}$. When $n_i = 1$, it reduces to polarization $\epsilon_{\mu}$ and $H_{i1} = h_{i1}$. In this case, the reference momentum $r_{ij_i}$ is arbitrary. When $n_i = 2$, it reduces to the current, whose expression will be given in the next subsection.

Now we express $M_{\text{higher}}$ in terms of $\Gamma$ and $J_{\mu}^{(n_i)}$. Combining Eqs. (19) and (21), it reduces
to

\[ M_{\text{higher}} = \Gamma^{\mu_1 \cdots \mu_n}(k_1, \ldots, k_n) \prod_i f^{(n_i)}_{\mu_i}(q_{h_1}^{i_1}, \ldots, q_{h_{m_i}}^{i_m}) \]

\[ = \sum_{j_1, \ldots, j_n} \Gamma^{\mu_1 \cdots \mu_n}(q_{l_1 j_1}, \ldots, q_{l_n j_n}) \prod_i F^{(n_i)}_{q_{i j_i}}(q_{h_1}^{i_1}, \ldots, q_{h_{m_i}}^{i_m}) e_{\mu}(q_{l_j}, r_{i j_i}) \]

\[ = \sum_{j_1, \ldots, j_n} F^{(n_i)}_{q_{i j_i}}(q_{h_1}^{i_1}, \ldots, q_{h_{m_i}}^{i_m}) \Gamma^{\mu_1 \cdots \mu_n}(q_{l_1 j_1}, \ldots, q_{l_n j_n}) \prod_i e_{\mu}(q_{l_j}, r_{i j_i}) \]

\[ (22) \]

where we ignored the momenta \( l \) in \( M_{\text{lower}} \). This is the decomposition relations for helicity amplitudes.

Here the multilinear property of the momenta dependence in the vertex \( \Gamma \) is crucial for the decomposition of helicity amplitudes. One may worry about the universality of this multilinear vertex which may limit the usage of the decomposition relation. Our argument is that this multilinear momentum dependent vertex is widely appeared in high dimensional couplings, both in SM framework with loops and effective high dimensional operators. As we will see in the next subsection, the momentum of the vector bosons in these vertices comes from the partial derivative of the vector boson in the Lagrangian, which is commonly exists especially in higher dimensional operators.

**B. Applications**

Now we want to know what kind of effective interactions will give multilinear vertices. Since \( \Gamma \) is multilinear, each momentum of vector boson should be linear in these vertices. It implies that there is no \( D_\mu \) in the effective interactions. All momenta come from the field strength tensor \( X_{\mu \nu} \).

In dimension-6 operators, there are only three kinds of effective operators \( \psi^2 X \phi, X^2 \phi^2 \) and \( X^3 \) fulfilling these conditions. The corresponding multilinear vertices are

\[ (\bar{\psi} \gamma^{\mu \nu} \psi) X_{\mu \nu} : \quad \Gamma^{\mu}(k_1) = [l_1^\mu k_1^\mu] + [l_1] \gamma^\mu k_1^\mu, \]

\[ \phi^2 X^{\mu \nu} X_{\mu \nu} : \quad \Gamma^{\mu \nu}(k_1, k_2) = k_1^{\mu} k_2^{\nu} - g^{\mu \nu} k_1 \cdot k_2, \]

\[ \text{tr}(X^{\mu}_\nu X^{\nu}_\rho X^{\rho}_\mu) : \quad \Gamma^{\mu \nu \rho}(k_1, k_2, k_3) = k_1^{\mu} k_2^{\nu} k_3^{\rho} + \cdots. \]

(23)

The decomposition relation Eq. (22) can be applied to these effective interactions.
From now on, we will return to the $HVV$ vertex from Eq. (12) and use the label \{p_1, \ldots, p_m\}. This vertex is bilinear to the momenta of vector bosons, which is
\[
\Gamma^{\mu\nu}(k, k') = -\frac{4}{\nu} [cVV (k^\nu k'^\mu - k \cdot k' g^{\mu\nu}) + \bar{c}VV \epsilon^{\mu\rho\sigma} k_\rho k'_\sigma],
\] (24)
where $k, k'$ are the momenta of the two vector bosons. So when $k = p_2 + p_3$ or $k' = p_4 + p_5$, or both, where $p_i$s are momentum of external legs, we have
\[
\Gamma^{\mu\nu}(k, k') = \Gamma^{\mu\nu}(p_2 + p_3, k') + \Gamma^{\mu\nu}(p_3, k')
\] (25)
\[
= \Gamma^{\mu\nu}(p_2 + p_3, p_4 + p_5) = \Gamma^{\mu\nu}(p_2, p_4) + \Gamma^{\mu\nu}(p_2, p_5) + \Gamma^{\mu\nu}(p_3, p_4) + \Gamma^{\mu\nu}(p_3, p_5).
\] (26)

On the other hand, we write down the current $J_\mu$ of $V \rightarrow \ell^+\ell^-$ in Fig. 5 explicitly,
\[
J^{(2)}_\mu(p_2^\frac{1}{2}, p_3^\frac{1}{2}) = \frac{f_\nu(s_{23})(2^\pm |\gamma_\mu| 3^\mp)}{\sqrt{2}}
\] (27)
\[
= \pm f_\nu(s_{23})(2^\pm |\gamma_\mu| 1^\pm(3, 2))
\] (28)
\[
= \pm f_\nu(s_{23})(2^\pm |\gamma_\mu| 1^\pm(2, 3),
\] (29)
where $\epsilon_\mu^\pm(3, 2) \equiv \epsilon_\mu^\pm(p_3, p_2)$ could be considered as a polarization vector of photon with external momentum $p_3$ and $p_2$ is the chosen reference momentum. Similarly, $\epsilon_\mu^\pm(2, 3)$ represents a photon with external momentum $p_2$ with reference momentum $p_3$. In principle, $J_\mu$ is a gauge-dependent quantity. As we ignore the mass of leptons, it could be considered as a gauge-independent quantity in our proof.

**FIG. 5:** The current $J_\mu$ of $V \rightarrow \ell^+\ell^-$.  

According to Eqs. (29) and (28), we know $H_{ij} = 2h_{ij}$, and the expression of $F^{(2)}$ are
\[
F^{(2)}_{q_1, q_2}(q_{11}^\frac{1}{2}, q_{22}^\frac{1}{2}) = f_\nu(k_1^2)|q_{11}q_{22}|, \quad F^{(2)}_{q_1, q_2}(q_{11}^\frac{1}{2}, q_{22}^\frac{1}{2}) = f_\nu(k_1^2)|q_{11}q_{22}|,
\] (30)
\[
F^{(2)}_{q_1, q_2}(q_{11}^\frac{1}{2}, q_{22}^\frac{1}{2}) = -f_\nu(k_1^2)|q_{11}q_{22}|, \quad F^{(2)}_{q_1, q_2}(q_{11}^\frac{1}{2}, q_{22}^\frac{1}{2}) = -f_\nu(k_1^2)|q_{11}q_{22}|,
\]
Based on these equations, we could decompose amplitudes of $H \rightarrow V\gamma \rightarrow \ell\ell\gamma$ as
\[
\mathcal{M}(2_-, 3^+, 4^-) = F^{(2)}_{p_2}(p_2^\frac{1}{2}, p_3^\frac{1}{2}) \mathcal{M}(2_-, 4^-) + F^{(2)}_{p_3}(p_2^\frac{1}{2}, p_3^\frac{1}{2}) \mathcal{M}(3^+, 4^-)
\] (31)
\[
= f_\nu(s_{23}) \times (23) \mathcal{M}(2_-, 4^-) + (23) \mathcal{M}(3^+, 4^-))
\]
In the last step, the reference momenta of photons are different, which do not affect the form of $\mathcal{M}(\gamma, \gamma)$ because the vertex $\Gamma^{\mu\nu}$ satisfy Ward identity. The other helicity amplitudes of $H \to V \gamma \to \ell \ell \gamma$ have similar decomposition. An illustrating diagram for Eq. (31) is shown in Fig. 6. Each amplitude of $H \to V \gamma \to \ell \ell \gamma$ is composed by two amplitudes of $H \to \gamma \gamma$. It degenerate to one term because the amplitude of $H \to \gamma \gamma$ with reverse helicities is equal to zero. So the $CP$ violation phase keeps as a global phase in $H \to V \gamma \to \ell \ell \gamma$ process.

Next we prove that decomposition relation is also suitable for process $H \to V V \to 4\ell$. That is

$$
\mathcal{M}(2^-_1, 3^+_2, 4^-_3, 5^+_4) \\
= F_{p_2}^{(2)}(p_2^{-\frac{1}{2}}, p_3^{+\frac{1}{2}})F_{p_4}^{(2)}(p_4^{-\frac{1}{2}}, p_5^{+\frac{1}{2}})\mathcal{M}(2^-_1, 4^-_3) \\
+ F_{p_2}^{(2)}(p_2^{-\frac{1}{2}}, p_3^{+\frac{1}{2}})F_{p_5}^{(2)}(p_4^{-\frac{1}{2}}, p_5^{+\frac{1}{2}})\mathcal{M}(2^-_1, 5^+_4) \\
+ F_{p_3}^{(2)}(p_2^{-\frac{1}{2}}, p_3^{+\frac{1}{2}})F_{p_4}^{(2)}(p_4^{-\frac{1}{2}}, p_5^{+\frac{1}{2}})\mathcal{M}(3^+_2, 4^-_3) \\
+ F_{p_3}^{(2)}(p_2^{-\frac{1}{2}}, p_3^{+\frac{1}{2}})F_{p_5}^{(2)}(p_4^{-\frac{1}{2}}, p_5^{+\frac{1}{2}})\mathcal{M}(3^+_2, 5^+_4) \\
= f_{V}^{\ell}(s_{23})f_{V}^{\ell}(s_{45}) \times ( \\
[23][45]\mathcal{M}(2^-_1, 4^-_3) + [23][45]\mathcal{M}(2^-_1, 5^+_4) \\
+ [23][45]\mathcal{M}(3^+_2, 4^-_3) + [23][45]\mathcal{M}(3^+_2, 5^+_4) ),
$$

where for the specific helicity states in Eq. (32), one term of $H \to 4\ell$ is decomposed into four terms. Furthermore, the four terms in Eq. (33) would degenerate to two terms since reverse-sign $H \to \gamma \gamma$ amplitudes are zero, as shown in Eq. (17). The illustrating diagrams are shown in Fig. 7.

One may think it’s ridiculous with the first glance at the decomposition of the scattering amplitude of $H \to 4\ell$ into the combination of four $H \to \gamma \gamma$ amplitudes, as the final decay products of leptons are changed strangely to be photons. We argue that this result is instructive and has profound physical meanings. The amplitude of the Higgs decay can be
considered as a function of the momentum of its decay products. Here we just find the form of the momentum dependence between the $H \to 4\ell$ and the $H \to \gamma\gamma$ amplitudes. One can easily generate this results into similar processes obeying the two aforementioned key preconditions. Our results provide a new viewpoint of the amplitude of the multiple decay of the Higgs.

C. $CP$ violation phase in helicity amplitudes

From the decomposition relations, we see that the amplitudes of $H \to \gamma\gamma$ are basis for other amplitudes. Since $\mathcal{M}(+,−) = \mathcal{M}(−,+)$ = 0, the left basis are $\mathcal{M}(+,+)$ and $\mathcal{M}(−,−)$. $CP$ violation phases are reverse in the two bases. In $H \to \gamma\gamma$ and $H \to V\gamma \to \ell\ell\gamma$ processes, $CP$ violation phase is a global phase in each amplitude. So generally speaking it is an unobservable phase if one doesn’t consider interference between this amplitude and the background amplitudes [29, 30]. In $H \to 4\ell$ process, two bases coexist in each amplitude, thus the $CP$ violation phase appear as a physical observable. Meanwhile, it means that the interference between $CP$-even term and $CP$-odd term exists at differential cross section level after squaring the amplitude. So the interference could be probed through kinematic angles [28, 31–33]. An obvious effect is a shift of azimuthal angle caused by the interference between $CP$-even and $CP$-odd term. So we see the $CP$ phase angle dependence clearly in $HV\ell\ell\gamma$ processes as the benefit of our amplitude decomposition relations.
V. BSM AMPLITUDES FROM ON-SHELL APPROACH (MASSLESS)

In on-shell approach, the amplitude is not derived from Lagrangian and Feynman rules. Instead, it is constructed directly from on-shell particle states. In this section, firstly we introduce spinor variables for particles, secondly we show how amplitudes of $H\gamma\gamma$ are represented and fixed, thirdly we get amplitudes of $H\gamma\ell\ell$ and $H4\ell$ through recursion relations.

A. Spinor variables

The right-handed and left-handed spinors in Eq. (3) have their two-component versions [37, 38]:

$$|i_\alpha\rangle \equiv \lambda_\alpha \equiv u_+(p_i) \equiv |i^+\rangle, \quad |i^{\dot{\alpha}}\rangle \equiv \bar{\lambda}_i^{\dot{\alpha}} \equiv u_-(p_i) \equiv |i^-\rangle,$$

$$\langle i^\alpha| \equiv \lambda^\alpha_i \equiv \overline{u_-(p_i)} \equiv (i^-), \quad \langle i^{\dot{\alpha}}| \equiv \bar{\lambda}^\dot{\alpha}_{i\dot{\alpha}} \equiv \overline{u_+(p_i)} \equiv (i^+),$$

where the spinor indices can be raised or lowered by antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$,

$$\lambda^\alpha = \epsilon^{\alpha\beta} \lambda_\beta, \quad \lambda_\alpha = \epsilon_{\alpha\beta} \lambda^\beta.$$  \hspace{1cm} (35)

In this notation,

$$\langle ij| \equiv \lambda^\alpha_i \lambda_{j\alpha}, \quad [ij] \equiv \bar{\lambda}_{i\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}_j.$$  \hspace{1cm} (36)

An on-shell momentum of a massless particle is represented as

$$p_{a\dot{a}} \equiv p_\mu \sigma_\mu^{a\dot{a}} = \lambda_a \bar{\lambda}_{\dot{a}},$$  \hspace{1cm} (37)

where $\sigma^\mu = (1, \vec{\sigma})$ with $\vec{\sigma}$ being the Pauli matrices.

B. Amplitude of $H\gamma\gamma$

A general three point amplitude with one massive and two massless particle interaction is shown in Fig. 8 [22], where $\alpha_i, i = 1, 2, ..., 2S$ are indices of spinors, $S$ represents spin of the massive particle, $h_2$, $h_3$ are helicities of the two massless particles.

For amplitude of $H\gamma\gamma$, as the massive particle $H$ is a scalar with zero spin, we don’t need to take care about its spinors, which makes the formula much simpler. A general ansatz is [22, 46, 47]

$$M_3(H, 2^{h_2}_\gamma, 2^{h_3}_\gamma) = e^{i \xi_{h_2:h_3}} \frac{g}{m^{h_2+h_3-1}} [23]^{h_2+h_3},$$  \hspace{1cm} (38)
\[(\alpha_1 \alpha_2 \cdots \alpha_{2S}) \rightarrow \quad \mathcal{M}^{h_2 h_3} \{\alpha_1 \alpha_2 \cdots \alpha_{2S}\}\]

FIG. 8: A general one massive and two massless particle interaction. The subscript \(S\) represents spin of the massive particle, \(h_2, h_3\) are helicities of two massless particles.

where \(\xi^{h_2, h_3}\) represents a helicity-related phase, \(g\) represents an overall coupling constant, \(m\) is the mass of the Higgs boson. As \(\langle 23 \rangle[32] = (p_2 + p_3)^2 = p_1^2 = m^2, \langle 23 \rangle = \frac{m^2}{[32]}\). The little group scaling [20, 21] requires \(h_2 + h_3 = 2h_2 = 2h_3\), so \(\mathcal{M}(2^+, 3^-) = \mathcal{M}(2^-, 3^+) = 0\). The non-zero amplitudes are only \(\mathcal{M}(2^+, 3^+)\) and \(\mathcal{M}(2^-, 3^-)\). It doesn’t lose generality to require \(\xi^{+, +} = -\xi^{-, -} = \xi'\) since their equal part could be absorbed into the redefinition of \(g\).

The inequality of \(|\mathcal{M}(2^+, 3^+)| \neq |\mathcal{M}(2^-, 3^-)|\) could also cause \(CP\) violation, however, this is not favored by physics assumption. Specifically, from Lagrangian in Eq. (12) we need \(C_{VV}\) and \(\bar{C}_{VV}\) to be real to keep the Lagrangian Hermitian conjugate, so it results in \(|\mathcal{M}(2^+, 3^+)| = |\mathcal{M}(2^-, 3^-)|\) as in Eq. (15). From the above discussion, the nonzero amplitudes are

\[
\mathcal{M}_3(1_H, 2^+, 3^+) = e^{i\xi'} \frac{g}{m} [23]^2 , \tag{39}
\]

\[
\mathcal{M}_3(1_H, 2^-, 3^-) = e^{-i\xi'} \frac{g}{m} [23]^2 , \tag{40}
\]

which is equal to Eq. (15) as long as we require \(\frac{g}{m} = \frac{2e^S}{v}\) and \(\xi' = \xi\).

C. Amplitudes of \(H \rightarrow \gamma \ell \ell\)

\[
\mathcal{M}_4(1_H, 2^{h_2}_{\ell^-}, 3_{\ell^+}^{h_3}, 4_{\gamma}^{h_4}) =
\]

FIG. 9: Factorization of \(H \rightarrow \gamma \ell \ell\). We take the mediate particle as \(\gamma\) for simplicity.

The amplitudes of \(H \rightarrow \gamma \ell \ell\) could be built from three point amplitudes by recursion
relations. For the amplitude of $H \to \gamma \ell \ell$, a factorization way is $H \to \gamma V, V \to \ell \ell$. Figure 9 shows this factorization. The mediate particle is taken as $\gamma$ to avoid amplitude of massive particles, its momentum is marked as “$I$”. We shift momenta of the 2, 4 external particles according to BCFW recursion relation approach [48–50]. That is

$$|\hat{2}| = |2|, \quad |\hat{4}| = |4| + z|2|, \quad |\hat{4}| = |4|, \quad |\hat{2}| = |2| - z|4|, \quad (41)$$

where $z$ is a complex number, and shifted momenta are hatted.

The corresponding analytical formula are

$$\mathcal{M}(1_H, 2_{\ell^-}, 3_{\ell^+}, 4_{\ell^+}) = P_\gamma(s_{23})\mathcal{M}(1_H, \hat{4}_{\ell^+}, -\hat{P}^-_{I_\gamma}, \hat{2}_{\ell^-}, 3_{\ell^+})$$

$$+ P_\gamma(s_{23})\mathcal{M}(1_H, \hat{4}_{\ell^+}, -\hat{P}^+_{I_\gamma}, \hat{2}_{\ell^-}, 3_{\ell^+}), \quad (42)$$

where $\hat{p}_I = p_1 + \hat{p}_4 = -(\hat{p}_2 + p_3)$ is momentum of the mediate photon, $P_\gamma(s_{23}) = 1/s_{23} = 1/(p_2 + p_3)^2$ is the propagator with unshifted momenta.

The helicity amplitudes of $\gamma \ell^- \ell^+$ are three point amplitudes with massless particles, which are fully fixed by little group scaling and dimension analysis,

$$\mathcal{M}(1_{\gamma}, 2_{\ell^-}, 3_{\ell^+}) = \tilde{e} \langle 12 \rangle^2 \langle 23 \rangle, \quad (43)$$

$$\mathcal{M}(1_{\gamma}, 2_{\ell^+}, 3_{\ell^-}) = \tilde{e} \langle 13 \rangle^2 \langle 23 \rangle, \quad (44)$$

$$\mathcal{M}(1_{\gamma}, 2_{\ell^-}, 3_{\ell^+}) = \tilde{e} \langle 13 \rangle^2 \langle 23 \rangle, \quad (45)$$

$$\mathcal{M}(1_{\gamma}, 2_{\ell^+}, 3_{\ell^-}) = \tilde{e} \langle 12 \rangle^2 \langle 23 \rangle, \quad (46)$$

where $\tilde{e} = -\sqrt{2}e$, Eqs. (43)(44) correspond to $\langle 23 \rangle = 0$ solution and Eqs. (45)(46) correspond to $\langle 23 \rangle = 0$ solution.

After inserting Eq. (40) and Eq. (43) into Eq. (42), we get

$$\mathcal{M}(1_H, 2_{\ell^-}, 3_{\ell^+}, 4_{\ell^-}) = \tilde{e}P_\gamma(s_{23}) \times \frac{2e_{\gamma V}}{v} e^{-i\xi} \langle \hat{4} \rangle^2 \langle \hat{I}_3 \rangle^2 \langle 23 \rangle, \quad (47)$$

where the last equation is because

$$\langle \hat{4} \hat{I}_3 \rangle = \langle \hat{4} \hat{p}_I \rangle = \langle \hat{4} \hat{p}_2 + p_3 \rangle = \langle \hat{4} \hat{p}_2 \rangle = \langle \hat{4} \hat{2} \rangle \langle 23 \rangle = \langle 42 \rangle \langle 23 \rangle, \quad (48)$$
and meanwhile an analytical continuum of $\langle -p \rangle = -|p\rangle$, $\langle -p \rangle = |p\rangle$ is adopted. So Eq. (47) is the same formula as the one derived in effective Lagrangian calculation (see Eq. (16)). It is worthy to notice that because $P_\gamma(s_{23}) = \frac{1}{(23)(32)}$, Eq. (47) is proportional to $(24)^2$ and thus has a singularity when $\langle 23 \rangle = 0$.

If we take the propagator $V$ as a $Z$ boson, we should consider a $H\gamma Z$ amplitude together with a $Z\gamma\gamma$ amplitude. The $H\gamma Z$ amplitude is an amplitude with two massive one massless particles, and the $Z\gamma\gamma$ amplitude is an amplitude with one massive two massless particles. They are more complex than the $H\gamma\gamma$ amplitude since the spin of $Z$ is 1. These two amplitudes should use bolded spinor variables [22, 46].

**D. Amplitudes of $H \rightarrow 4\ell$**

The amplitude of $H \rightarrow 4\ell$ is a five point amplitude, we could factorize it into two parts: a four point amplitude plus a three point amplitude. Each amplitude split into four parts as shown in Fig. 10.

$$M_5(1_H, 2^\ell, 3^\ell, 4^\ell, 5^\ell) = \hat{I}_\gamma 1_H 5^\ell + \hat{I}_\gamma 1_H 3^\ell + \hat{I}_\gamma 1_H 4^\ell + \hat{I}_\gamma 1_H 2^\ell$$

**FIG. 10:** Factorization of $H \rightarrow 4\ell$. The external legs are arranged in clockwise order.
In formula, it is
\[
\mathcal{M}_5(1_H, 2_e^-, 3_{\ell^+}, 4_{\ell^+}, 5_{\ell^+}) = P_\gamma(s_{23})\mathcal{M}(1_H, 4_{\ell^+}, 5_{\ell^+}, -\hat{P}_{I\gamma})\mathcal{M}(\hat{P}_{I\gamma}, \hat{2}_{e^-}, 3_{\ell^+})
\]
\[
+ P_\gamma(s_{23})\mathcal{M}(1_H, \hat{4}_{e^-}, 5_{\ell^+}, -\hat{P}_{I\gamma})\mathcal{M}(\hat{P}_{I\gamma}, \hat{2}_{e^-}, 3_{\ell^+})
\]
\[
+ P_\gamma(s_{45})\mathcal{M}(1_H, \hat{2}_{e^-}, 3_{\ell^+}, -\hat{P}_{I\gamma})\mathcal{M}(\hat{P}_{I\gamma}, \hat{4}_{\ell^+}, 5_{\ell^+})
\]
\[
+ P_\gamma(s_{45})\mathcal{M}(1_H, \hat{2}_{e^-}, 3_{\ell^+}, -\hat{P}_{I\gamma})\mathcal{M}(\hat{P}_{I\gamma}, \hat{4}_{\ell^+}, 5_{\ell^+}),
\]
(49)

which corresponds to diagram A, B, C, D respectively. Diagram A and B correspond to \((1, 4, 5) + (2, 3)\) factorization, Diagram C and D correspond to \((1, 2, 3) + (4, 5)\) factorization. We assumed \(\ell \neq \ell'\) for generality, so the factorizations of \((1, 2, 5) + (3, 4)\) and \((1, 3, 4) + (2, 5)\) are absent because of flavor symmetry. Next we calculate these four diagrams separately.

The formula for diagram A is
\[
P_\gamma(s_{23})\mathcal{M}(1_H, \hat{4}_{e^-}, 5_{\ell^+}, -\hat{P}_{I\gamma})\mathcal{M}(\hat{P}_{I\gamma}, \hat{2}_{e^-}, 3_{\ell^+})
\]
\[
= \frac{2e^S}{v} e^{-i\kappa} P_\gamma(s_{23}) P_\gamma(s_{45}) [\hat{45}] [\hat{4}\hat{5}] [\hat{23}] [\hat{24}]^2,
\]
(50)

where in the last step we have used
\[
\langle \hat{4}\hat{5} | \hat{2} | \hat{3} \rangle = \langle \hat{4} | \hat{p}_2 + p_3 | \hat{3} \rangle = \langle \hat{4} | \hat{p}_2 | \hat{3} \rangle = \langle \hat{4} \hat{2} | \hat{23} \rangle = \langle \hat{42} | \hat{23} \rangle = (51)
\]
as in Eq. (48) and
\[
P_\gamma(s_{45}) [\hat{45}] = \frac{-1}{\langle \hat{45} | [\hat{45}] } = \frac{-1}{\langle \hat{45} | [\hat{45}] } = P_\gamma(s_{45}) [\hat{45}],
\]
(52)

\(\langle \hat{23} \rangle = 0\) is chosen for three-point amplitude, which is also required in diagram B.

The formula for diagram B is
\[
P_\gamma(s_{23})\mathcal{M}(1_H, \hat{4}_{e^-}, 5_{\ell^+}, -\hat{P}_{I\gamma})\mathcal{M}(\hat{P}_{I\gamma}, \hat{2}_{e^-}, 3_{\ell^+})
\]
\[
= \frac{2e^S}{v} e^{i\kappa} P_\gamma(s_{23}) P_\gamma(s_{45}) [\hat{45}] [\hat{5}\hat{2}] [\hat{23}] \times \langle \hat{23} \rangle^2
\]
\[
= \frac{2e^S}{v} e^{i\kappa} P_\gamma(s_{23}) P_\gamma(s_{45}) [\hat{45}] [\hat{5}\hat{2}]^2 \times 0
\]
\[
= 0,
\]
(53)

where the 3-point amplitude is equal to zero because \(\langle \hat{23} \rangle = \langle \hat{2} \rangle = 0\).
The formula for diagram C is

\[ P_\gamma(s_{45})\mathcal{M}(1_H, \hat{2}_e^-, 3^+_e, -\hat{P}_{I\gamma}^+)\mathcal{M}(\hat{P}_{I\gamma}^-, \hat{4}_e^-, 5^+_e) \]

\[ = \frac{2e^S}{v} e^{i\xi} P_\gamma(s_{45})P_\gamma(s_{23})\langle 23 \rangle [3\hat{I}]^2 \times \frac{\langle \hat{I}4 \rangle^2}{\langle 45 \rangle} \]

\[ = \frac{2e^S}{v} e^{i\xi} P_\gamma(s_{45})P_\gamma(s_{23})\langle 23 \rangle \langle 45 \rangle |35|^2 , \quad (54) \]

where

\[ [3\hat{I}]\langle \hat{I}4 \rangle = [35]\langle 54 \rangle \]

and

\[ P_\gamma(s_{23})\langle 23 \rangle = \frac{-1}{\langle 23 \rangle[23]} \langle 23 \rangle = \frac{-1}{\langle 23 \rangle[23]} \langle 23 \rangle = P_\gamma(s_{23})\langle 23 \rangle \quad (56) \]

are used. \([45] = 0\) is required.

The formula for diagram D is

\[ P_\gamma(s_{45})\mathcal{M}(1_H, \hat{2}_e^-, 3^+_e, -\hat{P}_{I\gamma}^+)\mathcal{M}(\hat{P}_{I\gamma}^+, \hat{4}_e^-, 5^+_e) \]

\[ = \frac{2e^S}{v} e^{i\xi} P_\gamma(s_{45})P_\gamma(s_{23})\langle 23 \rangle [\hat{2}\hat{I}]^2 \frac{|\hat{I}5|^2}{[45]} \]

\[ = 0 , \quad (57) \]

where \([\hat{4}5] = [\hat{I}5] = 0\) makes the three point amplitude zero.

After summing up the results of four parts, that is adding Eq.s (50)(53)(54)(57) together, we get

\[ \mathcal{M}(1_H, 2_e^-, 3^+_e, 4^+e^-, 5^+_e) = \frac{2e^S}{v} e^{i\xi} P_\gamma(s_{23})P_\gamma(s_{45})[45][23] \langle 24 \rangle^2 \]

\[ + \frac{2e^S}{v} e^{i\xi} P_\gamma(s_{45})P_\gamma(s_{23})\langle 23 \rangle \langle 45 \rangle |35|^2 , \quad (58) \]

which has the same form as the one derived in effective Lagrangian calculation (see Eq. (17)). So we get a consistent result from the on shell approach. Boundary contributions here are supposed to be zero.

**VI. BSM AMPLITUDES FROM ON-SHELL APPROACH (MASSIVE)**

When the propagators are \(Z\) bosons or \(W\) bosons, the massless on-shell method won’t work and we use the little-group covariant massive spinor formalism [22, 40, 41]. In this section, we deduce \(HVV\) amplitudes according to the presumed \(HVV\) vertex. A more general method starting from an ansatz is left in Appendix A.
A. Massive spinor formalism

In massive spinor formalism [22], a massive momentum is decomposed into two light-like vectors and thus two pairs of massless spinors,

\[ p_{\alpha\dot{\alpha}} = \lambda_\alpha^I \tilde{\lambda}_{I\dot{\alpha}} = |p^I| |p_I|, \quad \text{and} \quad p^{I\dot{\alpha}} = -\tilde{\lambda}^I \lambda_\dot{\alpha} = -|p^I| |p_I|. \] (60)

Where \( I = 1, 2 \) is little-group index, \( p \) is bolded to denote massive momentum. The equation of motion reads,

\[ p|p^I| = m|p^I|, \quad \bar{p}|p^I| = m|p^I|, \quad [p^I|p = -m|p^I|, \quad \langle p^I|p = -m|p^I|. \] (61)

The polarized vector of a massive vector boson of momentum \( p \) and mass \( m \) is [51]

\[ \epsilon_{\mu}^{I J}(p) = \frac{1}{\sqrt{2m}} \langle p^I|\gamma_{\mu}|p^J| \rangle, \] (62)

which corresponds to two transverse and one longitudinal modes,

\[ \epsilon_{\mu}^+ \equiv \epsilon_{11}^+, \quad \epsilon_{\mu}^0 \equiv \frac{1}{2}(\epsilon_{12}^0 + \epsilon_{21}^0), \quad \epsilon_{\mu}^- \equiv \epsilon_{22}^- . \] (63)

B. Amplitudes of HVV

According to the HVV vertex in Eq. (24), the amplitude of HVV is

\[ \mathcal{M}(1_H, I_V, J_V) = \Gamma^{\mu \nu}(p_I, p_J)\epsilon_{\mu}(p_I)\epsilon_{\nu}(p_J) \] (64)

Since \( p_I^\mu p_J^\nu \epsilon_{\mu}(p_I)\epsilon_{\nu}(p_J) = 0 \), we could add this term in amplitude formula freely and make amplitude more symmetric.

\[ \mathcal{M}(1_H, I_V, J_V) = -\frac{4}{v} [c_{VV}(p_I^\mu p_J^\nu + p_I^\nu p_J^\mu - p_I \cdot p_J g^{\mu \nu}) + \tilde{c}_{VV} \epsilon^{\mu \nu \alpha \beta} p_{I\alpha} p_{J\beta} |\epsilon_{\mu}(p_I)\epsilon_{\nu}(p_J)| \]
\[ = -\frac{1}{v} [c_{VV} \text{tr}(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta) + i\tilde{c}_{VV} \text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \gamma^5)] p_{I\alpha} p_{J\beta} \epsilon_{\mu}(p_I)\epsilon_{\nu}(p_J) \]
\[ = -\frac{1}{v} c_{VV}^{S}[e^{-i\xi}\text{tr}(\sigma^\mu \sigma^\alpha \sigma^\nu \sigma^\beta) + e^{i\xi}\text{tr}(\tilde{\sigma}^\mu \tilde{\sigma}^\alpha \tilde{\sigma}^\nu \tilde{\sigma}^\beta)] p_{I\alpha} p_{J\beta} \epsilon_{\mu}(p_I)\epsilon_{\nu}(p_J). \] (65)

where

\[ 4(g^{\mu \alpha} g^{\nu \beta} - g^{\mu \nu} g^{\alpha \beta} + g^{\nu \beta} g^{\mu \alpha}) = \text{tr}(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta) = \text{tr}(\sigma^\mu \sigma^\alpha \sigma^\nu \sigma^\beta) + \text{tr}(\tilde{\sigma}^\mu \tilde{\sigma}^\alpha \tilde{\sigma}^\nu \tilde{\sigma}^\beta), \]
\[ -4i\epsilon^{\mu \alpha \beta} = \text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^5) = -\text{tr}(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^5) = -\text{tr}(\sigma^\mu \sigma^\alpha \sigma^\nu \sigma^5) + \text{tr}(\tilde{\sigma}^\mu \tilde{\sigma}^\alpha \tilde{\sigma}^\nu \tilde{\sigma}^5). \] (66)

(67)
are used. Eq. (65) shows a general formula for $HVV$ amplitudes, which constitutes two parts with opposite $CP$ violation phases. If one part is zero, the $CP$ violation phase degenerates to a trivial phase. After inserting Eq. (61)(63) into Eq. (65), we get

$$M(1_H, I_V, J_V) = \frac{2c^S_{VV}}{v} [e^{-i\xi} \langle IJ \rangle^2 + e^{i\xi} [IJ]^2].$$  \hspace{1cm} (68)$$

When one vector boson is a massless photon, the amplitudes become

$$M(1_H, I^+_\gamma, J_V) = \frac{2c^S_{\gamma V}}{v} e^{i\xi} [IJ]^2,$$

$$M(1_H, I^-_\gamma, J_V) = \frac{2c^S_{\gamma V}}{v} e^{-i\xi} \langle IJ \rangle^2,$$

where the compact form for a general amplitude decompose into two parts, each has a trivial $CP$ violation phase. When both of the two vector bosons are massless photons, the amplitudes become

$$M(1_H, I^+_\gamma, J^+_\gamma) = \frac{2c^S_{\gamma \gamma}}{v} e^{i\xi} [IJ]^2,$$

$$M(1_H, I^-_\gamma, J^-_\gamma) = 0,$$

$$M(1_H, I^-_\gamma, J^+_\gamma) = 0,$$

$$M(1_H, I^-_\gamma, J^-_\gamma) = \frac{2c^S_{\gamma \gamma}}{v} e^{i\xi} \langle IJ \rangle^2.$$  \hspace{1cm} (70)$$

So the general amplitude decompose into four parts, each with a trivial global $CP$ violation phase except for the zero ones.

The amplitude of $VI^-l^+$ with massless leptons is

$$M(I_V, 2^-_l, 3^+_l) = e_{l'} \langle 2|\gamma^\mu|3\rangle \epsilon_\mu(p_I) = \sqrt{2} e_{l'} \frac{\langle 2I|3I \rangle}{m_V},$$  \hspace{1cm} (71)$$

$$M(I_V, 2^+_l, 3^-_l) = e_{l'} \langle 2|\gamma^\mu|3\rangle \epsilon_\mu(p_I) = \sqrt{2} e_{l'} \frac{\langle 2I|3I \rangle}{m_V},$$  \hspace{1cm} (72)$$

$$M(I_V, 2^-_l, 3^-_l) = M(I_V, 2^+_l, 3^+_l) = 0.$$  \hspace{1cm} (73)$$

C. Amplitudes of $H \rightarrow 4\ell$ with massive propagators

We get the amplitude of $H \rightarrow VV \rightarrow 4\ell$ by simply gluing the amplitudes of $H \rightarrow VV$, $V \rightarrow \ell^+\ell^-$ and $V \rightarrow \ell'^+\ell'^-$ as shown in Fig. 11.
FIG. 11: Glue amplitudes by contracting little-group indices of the massive propagators.

When a propagator goes on-shell, the amplitude factorize into the tensor product of two subamplitudes.

\[
\lim_{p^2 \to m^2} \mathcal{M} = \frac{\mathcal{M}^{\{I_1...I_{2s}\}}_L \otimes \mathcal{M}^{\{J_1...J_{2s}\}}_R}{p^2 - m^2},
\]

where \(L, R\) represent left and right amplitudes for each gluing. For each propagator particle, the sign of its momentum is opposite in the left and the right amplitudes, as shown explicitly in Eq. (42). An analytical continuum \(|-p\rangle = -|p\rangle, \mid -p\rangle = |p\rangle\) is adopted. The gluing procedure is done by choosing the singlet of the little-group for the on-shell propagator

\[
\mathcal{M}^{\{I_1...I_{2s}\}}_L \otimes \mathcal{M}^{\{J_1...J_{2s}\}}_R = \mathcal{M}_{L,\{I_1...I_{2s}\}} \epsilon_{I_1J_1}... \epsilon_{I_{2s}J_{2s}} \mathcal{M}_{R,\{J_1...J_{2s}\}}.
\]

Since amplitude of \(H \to VV \to 4\ell\) has two propagators \(p_I\) and \(p_J\), we take the limit \(p_I^2 \to m_V^2\) and \(p_J^2 \to m_V^2\) simultaneously,

\[
\lim_{p_I^2 \to m_V^2, p_J^2 \to m_V^2} \mathcal{M}(1_H, 2_{\ell^-}, 3_{\ell^+}, 4_{\ell^-}, 5_{\ell^+})
= f^{-}_V(s_{23}) f^{-}_V(s_{45}) \mathcal{M}(I_V, 2_{\ell^-}, 3_{\ell^+}) \otimes \mathcal{M}(1_H, I_V, J_V) \otimes \mathcal{M}(I_V, 4_{\ell^-}, 5_{\ell^+})
= \frac{2c^\ell_V}{m_V} f^{-}_V(s_{23}) f^{-}_V(s_{45}) \left[ e^{-i\xi \lim_{p_I^2 \to m_V^2} \mathcal{M}^a + e^{i\xi} \lim_{p_J^2 \to m_V^2} \mathcal{M}^b} \right],
\]

where \(\mathcal{M}^a\) and \(\mathcal{M}^b\) in this limit are

\[
\lim_{p_I^2 \to m_V^2, p_J^2 \to m_V^2} \mathcal{M}^a = (2I^4)\langle I_{I_1} J_{J_1} \mid J_{J_1} 4\rangle [3I^2] \langle I_{I_2} J_{J_2} \mid J_{J_2} 5\rangle + (2I^4)\langle I_{I_1} J_{J_1} \mid J_{J_1} 5\rangle [3I^2] \langle I_{I_2} J_{J_2} \mid J_{J_2} 4\rangle
= - \sqrt{p_I^2} \sqrt{p_J^2} \langle 24 \rangle [3|p_I p_J|5] + (2|p_I|5)(3|p_I|4),
\]

\(26\)
\[
\lim_{p_I^2, p_J^2 \to m_V^2} 2\mathcal{M}^b = \langle 2I^1 \rangle [I_I, J_{J_1}] (J_{J_2} 4 \rangle [3I^2] [I_I, J_{J_2}] [J_{J_2} 5] + \langle 2I^1 \rangle [I_I, J_{J_1}] [J_{J_2} 5] [3I^2] [I_I, J_{J_2}] (J_{J_2} 4) \\
= -\sqrt{p_I^2} \sqrt{p_J^2} (\langle 2|p_I p_J|4 \rangle [35] + \langle 2|p_J|5 \rangle \langle 3|p_J|4 \rangle),
\]

where we used
\[
|p^I\rangle_\alpha \langle p_I|_\beta = -m_\delta_{\alpha \beta}, \quad |p^I|^\dagger_\alpha \langle p_I|_\beta = m_\delta_{\alpha \beta}.
\]

When the propagators go off-shell, we should use
\[
p_I = p_2 + p_3, \quad p_J = p_4 + p_5, \quad \sqrt{p_I^2} = \sqrt{p_J^2} = m_V,
\]
instead of \(p_I\) and \(p_J\). Now we get
\[
\mathcal{M}^a = m_V^2 [23] [45]^2, \quad \mathcal{M}^b = m_V^2 [23] [45] [35]^2.
\]

So the amplitude of \(H \to VV \to 4\ell\) is
\[
\mathcal{M}(1_H, 2_{-\ell}, 3_{+\ell}, 4_{-\ell}, 5_{+\ell}) = \frac{2\kappa^S_{VV}}{v} f_V^-(s_{23}) f_V^-(s_{45}) [e^{-i\xi} [23] [45]^2 + e^{i\xi} [23] [45] [35]^2],
\]
which is same as the former results in Eq. (17).

In Eq. (75), the particles are all on-shell especially for the propagator particles. In our specific process, before and after gluing, \(s_{23} = s_{45} = m_V^2\) should be required in the amplitude except for the propagator factor. So Eq. (82) is the amplitude of \(H \to VV \to 4\ell\) in the on-shell limit. By contrast, Eq. (17) is obtained from off-shell method and \(s_{23} \neq s_{45} \neq m_V^2\). Why do these two amplitudes are same? The reason is because the amplitude of \(H \to VV \to 4\ell\) is independent of the residue \(z\). Eq. (48) illustrate this point explicitly, the combined amplitude has no \(z\) dependence.

D. \(CP\) violation phase

In the on-shell way we get a compact form to show that the \(CP\) violation phase in \(HVV\) amplitude is not a trivial phase because of the BSM \(HVV\) vertex. It degenerates to a trivial phase once the helicity of the vector boson is fixed as in \(H\gamma\gamma\) and \(H \to \gamma V\) cases. By contrast, in the off-shell way, we only see the nontriviality of the \(CP\) violation phase in \(H \to 4\ell\) amplitude. It is because in the off-shell way we don’t deal with massive vector boson independently. Its full properties are exhibited indirectly in the four final states.
VII. SUMMARY AND DISCUSSION

The $HVV$ amplitudes with $CP$ violation in beyond Standard Model are analyzed in two ways. One way is the off-shell way under field theory framework, we decompose helicity amplitudes of $H \rightarrow \gamma V \rightarrow \gamma \ell \ell$ and $H \rightarrow VV \rightarrow 4\ell$ into helicity amplitudes of $H \rightarrow \gamma \gamma$. There are two preconditions for the decomposition relation. 1. The multilinear momentum dependence of the $HVV$ vertexes, which allow us to decompose the vertexes of the overall momentum into a summation of momentum of sub-processes. 2. The current of $J_\mu$ in $V \rightarrow \ell^+\ell^-$ is formally proportional to a photon’s polarization vector, which allow us to replace such a sub-process by an equivalent photon. The other way is through the on-shell scattering amplitude approach. For the massless propagator case, the 3-point amplitude of $H\gamma\gamma$ is the start point, then the 4-point amplitude of $H\ell\ell\gamma$ and the 5-point amplitude $H4\ell$ are obtained through recursion relations. For the massive propagator case, we adopt the little-group covariant massive-spinor formalism. It expresses $HVV$ amplitude firstly, then glue $V\ell\ell$ amplitudes to get final $H \rightarrow VV \rightarrow 4\ell$ amplitudes. We get consistent results through off-shell and on-shell ways.

The $CP$ violation phase in $H \rightarrow VV \rightarrow 4\ell$ amplitude is a nontrivial phase while in $H \rightarrow \gamma V \rightarrow \gamma \ell \ell$ and $H \rightarrow \gamma \gamma$ amplitudes it is a global trivial phase. In off-shell way, the decomposition relations shows that in $H \rightarrow VV \rightarrow 4\ell$ amplitude, it mix the $H \rightarrow \gamma \gamma$ amplitudes with different helicities, so it mix $H \rightarrow \gamma \gamma$ amplitudes with different dependences on $CP$ violation phases. In the on-shell way, the nontriviality of the $CP$ violation phase appears directly in $HVV$ amplitudes, which is not a helicity amplitude but a compact massive-spinor amplitude. It degenerates to a trivial phase when the helicity of at least one vector boson is fixed, such as that happens in $H\gamma V$ and $H\gamma \gamma$ amplitudes. The decays of $V \rightarrow \ell \ell$ maintains the nontriviality of the $CP$ violation phase. The on-shell way supplies a simpler and clearer viewpoint about the $CP$ violation phase in amplitudes. Our systematic analysis on the series of the amplitudes of $HVV$ processes exhibits the dependence of $CP$ violation phase, therefore, can be convenient for the future $CP$ violation searches in $HVV$ couplings.
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Appendix A: massive HVV amplitudes

In section VI B, we derived the massive HVV amplitudes by using the HVV vertex $\Gamma^{\mu\nu}$. Now we construct them directly. Consider the three massive amplitude $\mathcal{M}(1_H, I_V, J_V)$. In this case, Ref. [22] showed that the spinor space is spanned by two tensors, the symmetric tensor $\mathcal{O}_{\beta\gamma}$ and the antisymmetric tensor $\varepsilon_{\beta\gamma}$. We choose the first tensor

$$\mathcal{O}_{\beta\gamma} = p_{2\{\beta|\gamma\}} = |2J\rangle \langle \beta|2^{K3}|\gamma\rangle + (\beta \leftrightarrow \gamma).$$

(A.1)

Therefore, the three massive amplitude have a general form

$$\mathcal{M}(1_H, I_V, J_V) = \lambda_2^{\beta_1 \gamma_1} \lambda_2^{\beta_2 \gamma_2} \lambda_3^{\gamma_3 K_1} \lambda_3^{\gamma_4 K_2} \sum_{i=0}^{1} g_{\sigma_i} (\mathcal{O}^{2-i}\varepsilon^i)_{\{\beta_1\beta_2\}, \{\gamma_1\gamma_2\}}$$

(A.2)

$$= \lambda_2^{\beta_1 \gamma_1} \lambda_2^{\beta_2 \gamma_2} \lambda_3^{\gamma_3 K_1} \lambda_3^{\gamma_4 K_2} (g_{\sigma_0} (\mathcal{O}\varepsilon)_{\{\beta_1\beta_2\}, \{\gamma_1\gamma_2\}} + g_{\sigma_1} (\mathcal{O}\varepsilon)_{\{\beta_1\beta_2\}, \{\gamma_1\gamma_2\}}).$$

The second term ($\mathcal{O}\varepsilon$) is

$$\mathcal{O}_{\beta_1\gamma_1}\varepsilon_{\beta_2\gamma_2} + \mathcal{O}_{\beta_1\gamma_2}\varepsilon_{\beta_2\gamma_1} \rightarrow \langle 23 \rangle [23].$$

(A.3)

Since this term is symmetric between angle and square brackets, it doesn’t contribute to the $CP$ violation. The first term ($\mathcal{O}\mathcal{O}$) can be parametrized as

$$g_1 \mathcal{O}_{\beta_1\beta_2} \mathcal{O}_{\beta_1\gamma_2} + g_2 (\mathcal{O}_{\beta_1\gamma_1} \mathcal{O}_{\beta_2\gamma_2} + \mathcal{O}_{\beta_1\gamma_2} \mathcal{O}_{\beta_2\gamma_1})$$

$$= \frac{g_1}{2} (2\mathcal{O}_{\beta_1\beta_2} \mathcal{O}_{\gamma_1\gamma_2} - \mathcal{O}_{\beta_1\gamma_1} \mathcal{O}_{\beta_2\gamma_2} - \mathcal{O}_{\beta_1\gamma_2} \mathcal{O}_{\beta_2\gamma_1}) + \frac{2g_2 + g_1}{2} (\mathcal{O}_{\beta_1\gamma_1} \mathcal{O}_{\beta_2\gamma_2} + \mathcal{O}_{\beta_1\gamma_2} \mathcal{O}_{\beta_2\gamma_1})$$

(A.4)

$$= \frac{g_1}{2} m_V^4 (\varepsilon_{\beta_1\gamma_1}\varepsilon_{\beta_2\gamma_2} + \varepsilon_{\beta_1\gamma_2}\varepsilon_{\beta_2\gamma_1}) + \frac{2g_2 + g_1}{2} (\mathcal{O}_{\beta_1\gamma_1} \mathcal{O}_{\beta_2\gamma_2} + \mathcal{O}_{\beta_1\gamma_2} \mathcal{O}_{\beta_2\gamma_1}),$$
where we use Schouten identity

\[
2O_{\beta_1\beta_2} O_{\gamma_1\gamma_2} - O_{\beta_1 \gamma_1} O_{\beta_2 \gamma_2} - O_{\beta_1 \gamma_2} O_{\beta_2 \gamma_1}
= \lambda J_1^{2(\beta_1 \lambda K_1)} J_2^{3(\beta_2 \lambda K_2)} (2_{J_1} 3_{K_1} - 2_{J_2} 3_{K_2})
= \lambda J_1^{2(\beta_1 \lambda K_1)} J_2^{3(\beta_2 \lambda K_2)} (2_{J_1} 3_{K_1} + 2_{J_2} 3_{K_2})
= m_1^2 \lambda J_1^{2(\beta_1 \lambda K_1)} J_2^{3(\beta_2 \lambda K_2)} (J_1 3_{K_1} 2_{J_2} K_2)
= m_1^4 (\varepsilon_{\beta_1 \gamma_1} \varepsilon_{\beta_2 \gamma_2} + \varepsilon_{\beta_1 \gamma_2} \varepsilon_{\beta_2 \gamma_1}).
\]

(A.5)

Therefore, the \((OO)\) term gives two independent structure that contribute to the \(CP\) violation,

\[
O_{\beta_1 \gamma_1} O_{\beta_2 \gamma_2} + O_{\beta_1 \gamma_2} O_{\beta_2 \gamma_1} \rightarrow [23]^2,
\varepsilon_{\beta_1 \gamma_1} \varepsilon_{\beta_2 \gamma_2} + \varepsilon_{\beta_1 \gamma_2} \varepsilon_{\beta_2 \gamma_1} \rightarrow \langle 23 \rangle^2.
\]

(A.6)

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