Consistency of isotropic modified Maxwell theory: 
Microcausality and unitarity

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Abstract
The Lorentz-violating isotropic modified Maxwell theory minimally coupled to standard Dirac theory is characterized by a single real dimensionless parameter which is taken to vanish for the case of the standard (Lorentz-invariant) theory. A finite domain of positive and negative values of this Lorentz-violating parameter is determined, in which microcausality and unitarity hold. The main focus of this article is on isotropic modified Maxwell theory, but similar results for an anisotropic nonbirefringent case are presented in the appendix.

Key words: Lorentz violation, quantum electrodynamics, microcausality, unitarity
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1. Introduction
There are two Lorentz-violating extensions of the standard theory of photons [1, 2, 3], which are both gauge invariant and power-counting renormalizable [4, 5]. The standard Lorentz-invariant Maxwell theory has a quadratic field strength term ($F^2$) in the Lagrange density and the first Lorentz-violating extension adds a CPT-odd Chern–Simons-type term ($m_{CS} \hat{k} \cdot A F$, with a fixed normalized “four-vector” $\hat{k}^\mu$ and mass scale $m_{CS}$). The second Lorentz-violating extension adds another $F^2$ term, which has different contractions than those of the standard Maxwell term.

The consistency of the CPT-violating Maxwell–Chern–Simons (MCS) theory [6] has been studied in Ref. [7] and the result is that certain choices of the parameters (specifically, time-like $\hat{k}^\mu$) lead to violation of microcausality and/or unitarity. The concern now is the consistency of the CPT-invariant modified Maxwell theory, in particular, the theory restricted to the isotropic sector.

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The isotropic modified Maxwell theory is described by a single real dimensionless parameter $\tilde{\kappa}_{tr}$ and is, therefore, one of the simplest possible examples of a CPT–even Lorentz-violating theory of photons. The standard Lorentz-invariant Maxwell theory has $\tilde{\kappa}_{tr} = 0$. Positive values of $\tilde{\kappa}_{tr}$ have been derived from an underlying small-scale structure of spacetime in the long-wavelength limit of (standard) photons \cite{8, 9}, so that isotropic modified Maxwell theory with small enough positive $\tilde{\kappa}_{tr}$ can be expected to be consistent. But the consistency of isotropic modified Maxwell theory for negative values of $\tilde{\kappa}_{tr}$ is an entirely open question. In fact, there are only partial results for $\tilde{\kappa}_{tr} \geq 0$ in the literature \cite{10, 11}, which makes it worthwhile to give a more or less comprehensive analysis of the isotropic case. In addition, there have been many experiments (selected references will be given in Section 6) which provide upper and lower bounds on the parameter $\tilde{\kappa}_{tr}$, simply assuming the theory to be consistent and open to experimental verification.

The outline of this article is as follows. A brief discussion of isotropic modified Maxwell theory is given in Sections 2.1 and 2.2. The pure-photon theory is extended by the introduction of a minimal coupling of this photon to a charged Dirac particle in Section 2.3. In short, the theory considered is a particular modification of standard quantum electrodynamics, with a modified kinetic term of the photon in the action. The corresponding gauge-field propagator in Feynman gauge is presented in Section 3. Microcausality (i.e., commutation of electric and magnetic field operators with certain spacelike separations) is established in Section 4, together with the global causality of the theory (e.g., absence of closed timelike loops). Reflection positivity of the Euclidean gauge-field propagator is demonstrated in Sections 5.1 and 5.2. The unitarity of the interacting theory is checked by the direct evaluation of the optical theorem for one particular process in Section 5.3. Concluding remarks are presented in Section 6. The results for an anisotropic nonbirefringent case are given in Appendix A.

2. Isotropic modified Maxwell theory

2.1. Action and nonbirefringent Ansatz

In this article, we consider modified Maxwell theory \cite{4, 5, 12} which has an action given by

$$S_{\text{modMax}} = \int_{\mathbb{R}^4} d^4x \, \mathcal{L}_{\text{modMax}}(x),$$

where

$$\mathcal{L}_{\text{modMax}}(x) = -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x) - \frac{1}{4} \tilde{\kappa}_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x),$$

and $F_{\mu\nu}(x) \equiv \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x)$ is the field strength tensor of the $U(1)$ gauge field $A_{\mu}(x)$. The photons propagate over a flat Minkowski spacetime with global Cartesian coordinates...
\((x^\mu) = (x^0, \mathbf{x}) = (c t, x^1, x^2, x^3)\) and metric \(g_{\mu\nu}(x) = \eta_{\mu\nu} \equiv \text{diag} (1, -1, -1, -1)\). The fixed spacetime-independent background field \(\kappa^{\mu\nu\rho\sigma}\) in the second term of (2.1b) manifestly breaks Lorentz invariance.

If \(\kappa^{\mu\nu\rho\sigma}\) is taken to have a vanishing double trace, \(\kappa^{\mu\nu}_{\mu\nu} = 0\), and to obey the same symmetries as the Riemann curvature tensor, the number of independent Lorentz-violating parameters is 19. To leading order, birefringence is controlled by 10 of these 19 parameters. We restrict our considerations to the nonbirefringent sector with 9 parameters, which is parameterized by the following Ansatz [13]:

\[
\kappa^{\mu\nu\rho\sigma} = \frac{1}{2} \left( \eta^{\mu\rho} \tilde{\kappa}^{\nu\sigma} - \eta^{\mu\sigma} \tilde{\kappa}^{\nu\rho} - \eta^{\nu\rho} \tilde{\kappa}^{\mu\sigma} + \eta^{\nu\sigma} \tilde{\kappa}^{\mu\rho} \right),
\]

(2.2)
The constant \(4 \times 4\) matrix \(\tilde{\kappa}^{\mu\nu}\) is symmetric and traceless. Here and in the following, natural units are used with \(\hbar = c = 1\), where \(c\) corresponds to the maximal attainable velocity of a standard Dirac particle (see Section 2.3).

### 2.2. Restriction to the isotropic case

Next, restrict the nonbirefringent modified Maxwell theory to the isotropic sector which is characterized by a purely timelike normalized four-vector \(\xi^\mu\) in a preferred reference frame and a single real dimensionless parameter \(\tilde{\kappa}_{\text{tr}}\):

\[
\tilde{\kappa}^{\mu\nu} = 2 \tilde{\kappa}_{\text{tr}} \left( \xi^\mu \xi^\nu - \frac{1}{4} \xi^\lambda \xi^\lambda \eta^{\mu\nu} \right),
\]

(2.3a)

\[
(\xi^\mu) = (1, 0, 0, 0),
\]

(2.3b)

\[
(\tilde{\kappa}^{\mu\nu}) = \frac{3}{2} \tilde{\kappa}_{\text{tr}} \text{diag} \left( 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right).
\]

(2.3c)

From (2.1)–(2.3), the Lagrange density becomes in terms of the standard electric field \(E^i \equiv F^{i0}\) and magnetic field \(B^i \equiv (1/2) \epsilon_{ijk} F^{jk}\):

\[
\mathcal{L}_{\text{modMax}}^\text{isotropic} [c, \tilde{\kappa}_{\text{tr}}] (x) = \frac{1}{2} \left( (1 + \tilde{\kappa}_{\text{tr}}) |\mathbf{E}(x)|^2 - (1 - \tilde{\kappa}_{\text{tr}}) |\mathbf{B}(x)|^2 \right),
\]

(2.4)

where the dependence on the fundamental constants \(c\) and \(\tilde{\kappa}_{\text{tr}}\) has been made explicit on the left-hand side, which will be useful for the discussion of unitarity later.

The field equations of modified Maxwell theory in momentum space,

\[
M^{\mu\nu} A_\nu = 0, \quad M^{\mu\nu} \equiv k^\lambda k_\lambda \eta^{\mu\nu} - k^\mu k^\nu - 2 \kappa^{\mu\rho\sigma\nu} k_\rho k_\sigma,
\]

(2.5)
give the following dispersion relation for the isotropic case:

\[
\omega(k) = B |k|, \quad B \equiv \sqrt{\frac{1 - \tilde{\kappa}_{\text{tr}}}{1 + \tilde{\kappa}_{\text{tr}}}},
\]

(2.6)
in terms of the norm of the momentum three-vector $k$, explicitly defined by $|k| \equiv (k_1^2 + (k_2)^2 + (k_3)^2)^{1/2}$. The additional constant $A \equiv B^{-1}$ has been used in Ref. [14], but, in the present article, we prefer to employ only the constant $B$.

The dispersion relation (2.6) yields the following phase velocity of electromagnetic waves:

$$v_{ph} \equiv \frac{\omega(k)}{|k|} = B.$$ (2.7)

This phase velocity equals the group velocity,

$$v_{gr} \equiv \left| \left| \frac{\partial \omega(k)}{\partial k} \right| \right| = B,$$ (2.8)

which implies that the shape of a wave package does not change with time. From the modified dispersion law (2.6), it is clear that the vacuum behaves like an effective medium with a refraction index

$$n \equiv \frac{|k|}{\omega(k)} = B^{-1},$$ (2.9)

which is frequency independent because $B^{-1}$ is a constant. This particular Lorentz-violating theory does not display dispersion effects of photons propagating in vacuum.

Unless stated otherwise, we henceforth restrict $\tilde{\kappa}_{tr}$ to the following half-open interval:

$$\tilde{\kappa}_{tr} \in I, \quad I \equiv (-1, 1],$$ (2.10)

since for $\tilde{\kappa}_{tr} \notin I$ the frequency (2.6) becomes complex (signaling unstable behavior). The front velocity, which corresponds to the velocity of the high-frequency forerunners of electromagnetic waves [15], is given by

$$v_f \equiv \lim_{k \to \infty} v_{ph} = B,$$ (2.11)

and is seen to be equal to both the phase and group velocity. For $\tilde{\kappa}_{tr} < 0$, the front velocity of light exceeds the maximum attainable velocity of the standard matter particles, $v_f > c \equiv 1$. This alerts us to the issue of causality, which will be discussed in Section 4.

2.3. Coupling to matter: Modified QED

For the coupling of the photon to matter, take the minimal coupling to a standard (Lorentz-invariant) spin-\(\frac{1}{2}\) Dirac particle with electric charge $e$ and mass $M$. That is, the theory considered is a particular deformation of quantum electrodynamics (QED) [1, 2, 3] given by the following action:

$$S_{\text{modQED}}^{\text{isotropic}}[c, \tilde{\kappa}_{tr}, e, M] = S_{\text{modMax}}^{\text{isotropic}}[c, \tilde{\kappa}_{tr}] + S_{\text{Dirac}}[c, e, M],$$ (2.12)
with the modified-Maxwell term \((2.1)-(2.4)\) for the gauge field \(A_\mu(x)\) and the standard Dirac term for the spinor field \(\psi(x)\),

\[
S_{\text{Dirac}}[c, e, M] = \int_{\mathbb{R}^4} d^4x \overline{\psi}(x) \left( \gamma^\mu (i \partial_\mu - e A_\mu(x)) - M \right) \psi(x),
\]

(2.13)

with standard Dirac matrices \(\gamma^\mu\) corresponding to the Minkowski metric \(\eta^{\mu\nu}\). As mentioned before, the fundamental constant \(c\) may be operationally defined as the maximum attainable velocity of the Dirac particle. For further discussion on Lorentz violation and the role of different particle species, see, e.g., Refs. \([16, 17, 18]\) and references therein.

3. Polarization sum and propagator

The polarization sum can be computed by solving the field equations \((2.5)\) for appropriate coupling constants \(\kappa^{\mu\rho\sigma\nu}\), while respecting the normalization condition

\[
\langle k, \lambda | : P^0 : | k, \lambda \rangle = \langle k, \lambda | \int d^3x : T^{00} : | k, \lambda \rangle \equiv \omega(k),
\]

(3.1)

where, as usual, the pair of colons stands for the normal ordering of the enclosed operators and \(| k, \lambda \rangle\) denotes a photon state with momentum three-vector \(k\) and polarization label \(\lambda\). The \(T^{00}\) component of the energy-momentum tensor can be cast in the following form \([5]\):

\[
T^{00} = \frac{1}{2} \left( |E|^2 + |B|^2 \right) - \kappa^{0jk0k} E^j E^k + \frac{1}{4} \kappa^{ijklm} \varepsilon^{jlpk} \varepsilon^{lmq} B^p B^q \\
= \frac{1}{2} \left( (1 + \kappa_{\text{tr}}) |E|^2 + (1 - \kappa_{\text{tr}}) |B|^2 \right),
\]

(3.2)

with the three-dimensional totally antisymmetric Levi-Civita symbol \(\varepsilon^{ijk}\) and the electric and magnetic field components \(E^i\) and \(B^j\) defined in the lines above \((2.4)\). The final expression in \((3.2)\) makes clear that, for \(|\kappa_{\text{tr}}| > 1\), the theory suffers from unavoidable instabilities if the coupling to matter \((2.12)\) is taken into account (see, e.g., Ref. \([19]\) for a general discussion of the energy-positivity condition).

Returning to the parameter domain \((2.10)\), the solution of the field equations and the resulting energy-momentum tensor component \((3.2)\) give the following expression for the polarization sum:

\[
\Pi^{\mu\nu} = \sum_{\lambda=1,2} \frac{1}{1 + \kappa_{\text{tr}}} \left( -\eta^{\mu\nu} - \frac{1}{|k|^2} k^\mu k^\nu + \frac{B}{|k|} \left( k^\mu \xi^\nu + \xi^\mu k^\nu \right) + \frac{2 \kappa_{\text{tr}}}{1 + \kappa_{\text{tr}}} \xi^\mu \xi^\nu \right),
\]

(3.3)
where the sum runs over the two physical polarizations $\lambda \in \{1, 2\}$, with the polarization vectors $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ being orthogonal to the momentum three-vector $k$. Expression (3.3) for $\tilde{\kappa}_{\text{tr}} = 0$ reproduces the standard result [3]. For later use, also the expression is given if terms with $k^\mu$ or $k^\nu$ are removed:

$$
\Pi^{\mu\nu} \Big|_{\text{truncated}} = \frac{1}{1 + \tilde{\kappa}_{\text{tr}}} \left(-\eta^{\mu\nu} + \frac{2\tilde{\kappa}_{\text{tr}}}{1 + \tilde{\kappa}_{\text{tr}}} \xi^\mu \xi^\nu\right). \quad (3.4)
$$

For the study of causality and unitarity in the quantum theory, the gauge-field propagator $G_{\mu\nu}$ is needed. Recall that $G_{\mu\nu}$ is the Green’s function of the free field equations of the gauge potential in momentum space, which requires a particular gauge fixing.

The Feynman gauge [3, 20, 21], for example, is implemented by the following gauge-fixing condition:

$$
\mathcal{L}_{\text{gf}}(x) = -\frac{1}{2} \left(\partial_\mu A_\mu(x)\right)^2. \quad (3.5)
$$

It has been proven in Ref. [22] that the Feynman gauge in standard QED is equivalent to the Coulomb gauge, if one computes transition probabilities, i.e., matrix-element squares. The Coulomb gauge is well-defined in the sense that any gauge field configuration can be brought into that form. Since isotropic modified Maxwell theory is a deformation of standard QED, we expect this to be valid also in our case, at least, for a small enough deformation parameter $|\tilde{\kappa}_{\text{tr}}|$.

The differential operator $(G^{-1})^{\mu\nu}$ in Feynman gauge reads

$$
(G^{-1})^{\mu\nu} = \eta^{\mu\nu} \partial^2 - 2 \kappa^{\mu\rho\sigma\nu} \partial_\rho \partial_\sigma, \quad (3.6)
$$

and the gauge-field propagator is

$$
G_{\nu\lambda} \big|_{\text{Feynman}} = -i \left\{ a \eta_{\nu\lambda} + b k_\nu k_\lambda + c \xi_\nu \xi_\lambda + d \left(k_\nu \xi_\lambda + \xi_\nu k_\lambda\right) \right\} K, \quad (3.7)
$$

with scalar propagator $K$ and coefficient functions $a, b, c, d$ given by

$$
K = \frac{1}{(1 - \tilde{\kappa}_{\text{tr}} \xi^2) k^2 + 2\tilde{\kappa}_{\text{tr}}(k \cdot \xi)^2}, \quad (3.8)
$$

$$
a = 1, \quad (3.9a)
$$

$$
b = \frac{\tilde{\kappa}_{\text{tr}} \xi^2}{1 + \tilde{\kappa}_{\text{tr}} \xi^2} \frac{1}{k^4} \left(- \left(1 + \tilde{\kappa}_{\text{tr}} \xi^2\right) k^2 + 2\tilde{\kappa}_{\text{tr}}(\xi \cdot k)^2\right), \quad (3.9b)
$$
\[
c = -\frac{2 \kappa_{tr}}{1 + \kappa_{tr} \xi^2}, \tag{3.9c}
\]
\[
d = \frac{2 \kappa_{tr} \xi \cdot k}{1 + \kappa_{tr} \xi^2} \frac{\xi \cdot k}{k^2}. \tag{3.9d}
\]
Casana et al. [11] have also obtained the gauge-field propagator in Feynman gauge from a more general Ansatz given by their Eq. (28) with two four-vectors \(U^\mu\) and \(V^\mu\). Their result for \((U^\mu) = (V^\mu) = (\sqrt{2 \kappa_{tr}}, 0, 0, 0)\) agrees with ours.

4. Microcausality

4.1. Commutators of gauge potentials and physical fields

The notion of microcausality can be condensed to the statement that the commutator of two field operators \(\Phi(x')\) and \(\Phi(x'')\) must vanish for spacelike separations, specifically, \([\Phi(x'), \Phi(x'')] = 0\) for \((x' - x'')^2 < 0\). This assures that information can only propagate along or inside null-cones. Translation invariance of the modified Maxwell theory implies the following structure of the gauge-field commutator:

\[
K^{\mu\nu}(x', x'') \equiv [A^\mu(x'), A^\nu(x'')] = [A^\mu(x' - x''), A^\nu(0)] = [A^\mu(x), A^\nu(0)], \tag{4.1}
\]
for \(x^\mu \equiv x'^\mu - x''^\mu\). The corresponding result in momentum space must be of the form

\[
K^{\mu\nu}(k) = \Xi^{\mu\nu}(k^0, k) \left(iD(k)\right), \tag{4.2}
\]
where \(\Xi^{\mu\nu}(k)\) respects the tensor structure of the commutator and \(D(k)\) is a scalar commutator function.

The commutator \(K^{\mu\nu}(k)\) can be computed either directly by Fourier decomposition of the gauge potential in positive and negative frequency parts or by extraction from the Feynman propagator. Both methods yield the same result:

\[
\Xi^{\mu\nu} = (1 + \kappa_{tr}) \Pi^{\mu\nu}, \tag{4.3a}
\]
\[
D(k)^{-1} = (1 + \kappa_{tr}) k^2_0 - (1 - \kappa_{tr}) |k|^2, \tag{4.3b}
\]
where \(\Pi^{\mu\nu}\) is the polarization sum (3.3). In fact, (3.3) gives \(\Xi_{00} = \Xi_{0m} = \Xi_{m0} = 0\) for \(m \in \{1, 2, 3\}\), so that only certain purely spatial components of \(K^{\mu\nu}\) may be nonzero.

By using (3.3), the momentum-space commutators of the electric and magnetic fields can be computed, which are then transformed to configuration space. The resulting commutators are

\[
[E_i(x), E_j(0)] = \left(\partial_0^2 \delta_{ij} - B^2 \partial_i \partial_j\right) (iD(x)), \tag{4.4a}
\]
\[ [E_i(x), B_j(0)] = \varepsilon_{ijk} \partial_0 \partial_k (iD(x)) , \tag{4.4b} \]

\[ [B_i(x), B_j(0)] = (\nabla^2 \delta_{ij} - \partial_i \partial_j) (iD(x)) , \tag{4.4c} \]

where \( B \) has been defined in (2.6). The scalar commutator function in configuration space can be written as follows

\[
D(x) = \oint_C \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(1 + \tilde{\kappa}_{tr} k_0^2 - (1 - \tilde{\kappa}_{tr}) |k|^2} \exp \left( ik_0 x_0 + i k \cdot x \right) , \tag{4.5} \]

where the poles are circled in the counterclockwise direction along a contour \( C \). The evaluation of the four-dimensional integral (4.5) leads to the following expression:

\[
D(x) = -\frac{1}{2\pi \sqrt{1 - \tilde{\kappa}_{tr}^2}} \text{sgn}(\tilde{x}_0) \delta((\tilde{x}_0)^2 - |x|^2) , \quad \tilde{x}_0 \equiv B x_0 , \tag{4.6} \]

with the sign function

\[
\text{sgn}(x) = \begin{cases} 
1 & \text{for } x > 0 \\
0 & \text{for } x = 0 \\
-1 & \text{for } x < 0 
\end{cases} . \tag{4.7} \]

The overall minus sign in (4.6) has its origin in the definition of the commutator function (4.5). In this definition, the first term in the argument of the exponential function enters with a plus sign. This convention has been chosen to conform with Ref. [7] and is different from the one used in, for example, Appendix A1 of Ref. [2]. The commutators of the physical electric and magnetic fields in (4.4) are, of course, independent of this convention. For \( \tilde{\kappa}_{tr} \to 0 \), these commutators are equal to those of standard QED, first obtained by Jordan and Pauli [23, 1].

According to (4.6), the commutators (4.4) vanish if the relative distance in Minkowski spacetime lies outside the modified null-cone,

\[(\tilde{x}_0)^2 - |x|^2 = 0 . \tag{4.8} \]

The same result holds in standard QED with the standard null-cone, \( x_0^2 - |x|^2 = 0 \). Modified Maxwell theory only has a different opening angle of the null-cone for the case of a nonzero value of the Lorentz-violating parameter \( \tilde{\kappa}_{tr} \).

Different from the commutators of Maxwell-Chern-Simons (MCS) theory with a spacelike parameter [7], the commutators (4.4)–(4.6) vanish everywhere except on the null-cone, since
the pure-photon sector of modified Maxwell theory is scale-invariant. The spacelike MCS theory, on the other hand, is characterized by a mass scale, called $m_{CS}$ in Section 1, which leads to nonvanishing commutators both on and inside the null-cone.

Returning to the isotropic modified Maxwell theory, consider, now, the interaction of photons and charged matter particles as given by the modified QED action (2.12) and take, for simplicity, a vanishing mass for the Dirac particle, $M = 0$. Then, the photon has a null-cone (1.8) and the Dirac particle a different one given by $x_0^2 - |x|^2 = 0$. Intuitively, there are no causality problems to be expected from having these two different null-cones. There may, of course, be nonstandard interaction processes, for example, vacuum Cherenkov radiation for $\kappa_{tr} > 0$ and photon decay for $\kappa_{tr} < 0$ (see Ref. [14] for detailed calculations).

4.2. Wick rotation

For the analytic properties of the gauge-field propagator (3.7), the behavior of the propagator pole structure under Wick rotation is an important issue. For $\bar{\kappa}_{tr}$ in the domain (2.10), the scalar part (3.8) of the propagator shows that by performing a Wick rotation the poles of the full propagator do not lie within the integration contour and that the Wick-rotated axes do not cross any poles.

But these properties hold only for $\bar{\kappa}_{tr} \in (-1, 1]$. A Wick rotation $k_4 = -ik^0$ from Minkowski spacetime to Euclidean space, for example, maps poles with positive real part in Minkowski spacetime to poles with negative imaginary part in Euclidean space or poles with positive imaginary part to poles with negative real part. Hence, an analytic continuation of the propagator from Minkowski spacetime to Euclidean space, or vice versa, is possible with the Wick rotation. The gauge-field propagator (3.7) is thus well-behaved for the above mentioned parameter domain. For $\bar{\kappa}_{tr} \notin (-1, 1]$, however, the $k_0$ poles in Minkowski spacetime lie on the imaginary axis, which implies that the corresponding energy becomes imaginary for this parameter domain.

4.3. Effective metric

Following up on earlier work about the coupling of Lorentz-violating theories to gravity [24], it has been shown in Refs. [25, 26] that the action of isotropic modified Maxwell theory from (2.1) can be cast in the following form:

$$S_{\text{modMax}}^\text{isotropic} = -(1 - \kappa_{tr}) \int_{\mathbb{R}^4} d^4x \frac{1}{4} \bar{\eta}^{\mu\nu} \bar{\eta}^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma},$$  \hspace{1cm} (4.9)

with an effective metric

$$\tilde{\eta}^{\mu\nu} = \eta^{\mu\nu} + \frac{2\kappa_{tr}}{1 - \kappa_{tr}} \xi^\mu \xi^\nu,$$  \hspace{1cm} (4.10)
for $\xi^\mu$ from $[23]$. The existence of such an effective metric has interesting implications.

First, recall that a spacetime manifold $M_4$ is said to be “stably causal” if and only if there exists a Lorentzian metric $g_{\mu\nu}(x)$ and a scalar function $\theta(x)$, defined everywhere on $M_4$, so that $\nabla_\mu \theta \neq 0$ and $g^{\mu\nu}(\nabla_\mu \theta)(\nabla_\nu \theta) > 0$. If a spacetime manifold is stably causal, it does not contain closed timelike or lightlike curves (cf. Section 6.4 of Ref. [27]).

For the isotropic case of modified Maxwell theory defined over Minkowski spacetime with standard global spacetime coordinates as given below $[21]$, we can choose the globally defined scalar function $\theta(x)$ to be given by the time coordinate $t$. Then, $(\nabla_\mu t) = (1, 0, 0, 0) \neq 0$ and the effective metric $(4.10)$ gives:

$$\tilde{\eta}^{\mu\nu} \nabla_\mu t \nabla_\nu t = \frac{1 + \tilde{\kappa}_{tr}}{1 - \tilde{\kappa}_{tr}},$$

which is positive for parameter $\tilde{\kappa}_{tr} \in (-1, 1]$, where the value $\tilde{\kappa}_{tr} = 1$ arises as the limit from below. As a result, there are no closed timelike or lightlike curves of the effective metric, along which the modified photons could propagate. This reflects the global causality of the theory considered, in particular, for the $\tilde{\kappa}_{tr} < 0$ case mentioned in the last paragraph of Section 2.2. See, e.g., Refs. [28, 29] for further discussion on Lorentz violation and causality.

5. Unitarity

5.1. Reflection positivity: Simple test

Reflection positivity $[30, 31]$ is an important property of the Euclidean theory. It assures, for example, the existence of an analytic continuation of the Euclidean propagators to Minkowski propagators, such that the theory in Minkowski spacetime has a positive semi-definite Hermitian Hamiltonian $\tilde{H}$ and, therefore, a unitary time evolution operator $\exp(-i \tilde{H} t)$.

Following the previous analysis of MCS theory $[7]$, we restrict the general discussion of reflection positivity to the special case of reflection positivity of a Euclidean two-point function. Concretely, reflection positivity of the Euclidean two-point function corresponds to the following inequality:

$$\langle 0 | \Theta(\phi(x_4, x)) \phi(x_4, x) | 0 \rangle \geq 0,$$

for a complex scalar field $\phi(x_4, x)$ in four-dimensional Euclidean space and the reflection operation $\Theta$: $\phi(x_4, x) \mapsto \phi^\dagger(-x_4, x)$.

With the Fourier decomposition of the scalar field operator, the following “weak” reflection-positivity condition for the scalar Euclidean propagator $S_E(k_4, k)$ can be derived $[7, 30, 31]$:

$$\tilde{\eta}^{\mu\nu} \nabla_\mu t \nabla_\nu t = \frac{1 + \tilde{\kappa}_{tr}}{1 - \tilde{\kappa}_{tr}},$$
\[ S_E(x_4) \equiv \int_{\mathbb{R}^3} d^3k \int_{-\infty}^{+\infty} dk_4 \exp(-ik_4 x_4) S_E(k_4, k) \equiv \int d^3k \ S_E(x_4, k) \geq 0. \quad (5.2) \]

With appropriate smearing functions, also a “strong” condition can be derived

\[ S_E(x_4, k) \geq 0. \quad (5.3) \]

The validity of both conditions will be investigated for the theory considered, isotropic modified Maxwell theory.

### 5.2. Reflection positivity and unitarity

If the gauge-field propagator is coupled to physical sources, i.e., a conserved current \( j^\mu(k) \), then it follows from current conservation (at the classical level) or the Ward identities (at the quantum level) that all terms of the propagator which contain a propagator four-momentum \( k_\mu \) vanish by contraction with \( j^\mu(k) \). Hence, what remains from the gauge-field propagator (3.7) after projecting on the physical subspace is the first term involving the metric tensor and the third term proportional to a bilinear combination of the fundamental “four-vector” \( \xi^\mu \). Only these two terms describe the physical degrees of freedom. The pole structure of the propagator with respect to its momentum is of crucial importance for unitarity. The relevant pole structure is in the scalar part (3.8) of the propagator. [The term proportional to \( \xi \xi \) has an additional pole at \( \tilde{\kappa}_{tr} = -1 \), which plays no role for our analysis and is excluded anyway by condition (2.10).] Hence, it is sufficient to restrict the unitarity analysis to the scalar part \( K \) of the propagator, given by (3.8).

Since we have shown in Section 1.2 that Wick rotation is possible, the scalar propagator part (3.8) is Wick-rotated to Euclidean space. The resulting Euclidean expression will be denoted by \( S_E \). Recall, that a Wick rotation gives

\[ x_4 = -ix^0, \quad k_4 = -ik^0. \quad (5.4) \]

With our conventions, \( S_E(k_4, k) \) is then given by the negative of the Wick-rotated scalar propagator function:

\[
S_E(k_4, k) = S_E(k_4, |k|) = \frac{1}{(1 + \tilde{\kappa}_{tr}) (k_4^2 + |k|^2) - 2 \tilde{\kappa}_{tr} k_4^2}
\]

\[
= \frac{1}{(1 - \tilde{\kappa}_{tr}) k_4^2 + (1 + \tilde{\kappa}_{tr}) |k|^2}. \quad (5.5)
\]

In order to show reflection positivity for the scalar part of the Euclidean propagator, the following expression needs to be examined:

\[
S_E(x_4, |k|) = \int_{-\infty}^{+\infty} dk_4 \exp(-ik_4 x_4) S_E(k_4, |k|). \quad (5.6)
\]
Performing the relevant integrals for $\tilde{\kappa}_{\text{tr}}$ in the open interval corresponding to (2.10) yields

$$S_E(x_4,|k|) = \frac{\pi}{\sqrt{1-\tilde{\kappa}_{\text{tr}}^2}} \frac{1}{|k|} \exp \left( - |x_4| B^{-1} |k| \right),$$

(5.7a)

$$S_E(x_4) = \frac{4\pi^2}{\sqrt{1-\tilde{\kappa}_{\text{tr}}^2}} \frac{B^2}{x_4^2}.$$  

(5.7b)

Both of these expressions for $S_E(x_4,|k|)$ and $S_E(x_4)$ are manifestly larger than zero for $\tilde{\kappa}_{\text{tr}} \in (-1,1]$, where the value $\tilde{\kappa}_{\text{tr}} = 1$ arises as the limit from below. Hence, reflection positivity (5.2)–(5.3) is guaranteed for this parameter domain. In turn, this implies unitarity of the pure-photon sector, provided $\tilde{\kappa}_{\text{tr}} \in (-1,1]$.

For $\tilde{\kappa}_{\text{tr}} \notin [-1,1]$, the corresponding results are

$$S_E(x_4,|k|) = -\frac{\pi}{\sqrt{\tilde{\kappa}_{\text{tr}}^2 - 1}} \frac{1}{|k|} \sin \left( |x_4| C |k| \right),$$

(5.8a)

$$S_E(x_4) = \frac{4\pi^3}{\sqrt{\tilde{\kappa}_{\text{tr}}^2 - 1}} \delta'(|x_4| C),$$

(5.8b)

with

$$C \equiv \sqrt{\frac{\tilde{\kappa}_{\text{tr}} + 1}{\tilde{\kappa}_{\text{tr}} - 1}}.$$  

(5.8c)

Both of these last expressions for $S_E(x_4,|k|)$ and $S_E(x_4)$ are not manifestly positive because of the presence of the sine function and the delta function derivative.\(^1\) As a result, unitarity is violated for this parameter choice, which is also obvious from the fact that the corresponding dispersion relation is imaginary.

More generally, it is clear that the pure-photon isotropic modified Maxwell theory (2.4) is unitary by the following simple argument (prefigured in the discussion of Section 4.3). As the electric field involves one derivative with respect to the spacetime coordinate $x^0 \equiv ct$, the Lagrange density (2.4) can be made to be proportional to the standard form (having $\tilde{\kappa}_{\text{tr}} = 0$) by the introduction of a rescaled velocity $c' \equiv c B$, with constant $B \equiv \sqrt{(1-\tilde{\kappa}_{\text{tr}})/(1+\tilde{\kappa}_{\text{tr}})}$ as defined in (2.6). Moreover, as the action always appears divided by the Planck constant $\hbar$, it is even possible to remove the remaining overall factor by the introduction of a rescaled constant $\hbar' \equiv \hbar/(1-\tilde{\kappa}_{\text{tr}})$.\(^2\) Now, standard Maxwell–Jordan–Pauli photons (even with phase velocity $c'$ and rescaled Planck constant $\hbar'$) have been proven to be unitary \[^3\].

---

\(^1\)Equation (5.8b) is to be understood as acting on a test function. The sign of the resulting expression depends on the test function and need not be positive.

\(^2\)Similar redefinitions bring the commutators (4.4) back to the standard Jordan–Pauli form [23].
The outstanding issue is whether or not the unitarity of the pure-photon isotropic modified Maxwell theory is affected by the standard minimal coupling (2.13) to matter (recall that this minimal coupling is governed by the gauge principle). As there is only a single local interaction term in the action, reflection positivity can be expected to hold [31], but this needs, of course, to be verified.

In that respect, it is highly relevant that two physical decay processes have already been calculated in Ref. [14]. The exact tree-level results for the corresponding decay rates were found to be well-behaved for parameter values in the domain (2.10). Clearly, this agrees with the conjecture that isotropic modified QED (2.12) is unitary for the proper values of the parameter $\tilde{\kappa}_{tr}$. Further evidence will be given in the next subsection.

5.3. Optical theorem

In order to explicitly check the unitarity of isotropic modified QED (2.12) for the parameter domain $\tilde{\kappa}_{tr} \in (-1, 1]$, we will consider one particular process (electron-positron annihilation). The general idea is that the total cross section or decay width of a physical process is related to the imaginary part of the respective forward scattering amplitude via the optical theorem [20, 21]. The optical theorem follows directly from the unitarity of the S–matrix. Hence, if unitarity does not hold, this can be expected to show up as a violation of the optical theorem.

Consider, then, the following (nonstandard) process involving definite polarization states of the charged particles: annihilation of a left-handed electron and a right-handed positron to a single photon. The optical theorem will be verified by comparing the imaginary part of the relevant forward electron-positron scattering amplitude to the total cross section for the production of a modified photon ($\tilde{\gamma}$) from a left-handed electron ($e_L^-$) and a right-handed positron ($e_R^+$):

\[
2 \text{Im} \left( \begin{array}{c}
  e_L^-, k_1 \\
  e_R^+, k_2 \\
  \tilde{\gamma}, k
  \end{array} \right) = \int d\Pi_1, \quad \text{(5.9)}
\]

where $d\Pi_1$ denotes the one-particle phase-space element of the modified photon $\tilde{\gamma}$ in the final state.

Let us take, for simplicity, massless fermions, so that the helicity of a particle is a physically well-defined property, that is, independent of the reference frame. (Recall that,
for the case of massive fermions, chirality is not equal to helicity. In that case, left- and right-handed particles carry both parallel and antiparallel spins with respect to the momenta of the particles.) The assumption of massless particles leads to a conserved axial vector current: \( \partial_\mu j^\mu_5(x) = 0 \) with \( j^\mu_5(x) = \overline{\psi}(x)\gamma^\mu\gamma_5\psi(x) \). As a result, we have \( k_\mu j^\mu_5(k) = 0 \) for a photon with momentum \( k_\mu \) coupling to the current \( j^\mu_5(k) \). Incidentally, we can neglect the anomalous nonconservation of the axial vector current \([20, 21]\), because the possible additional terms in our calculation would be of higher order in the gauge coupling constant \( e \).

In the following calculation, it is assumed that the LV parameter \( \tilde{\kappa}_{tr} \) is negative and that the massless particles which annihilate have a nonzero total energy and are not collinear. In addition, the chirality conventions of Ref. \([21]\) will be used. The forward scattering amplitude \( \mathcal{M}_1 \equiv \mathcal{M}(e^+_L e^+_R \rightarrow e^-_L e^-_R) \) is then given by

\[
\mathcal{M}_1 = e^2 \frac{\gamma_\nu}{2} \frac{1 - \gamma^5}{\gamma} \frac{1 - \gamma^5}{\gamma} \frac{\eta_{\mu\nu} + b k_\mu k_\nu + c \xi_\mu \xi_\nu + d (k_\mu \xi_\nu + \xi_\mu k_\nu)}{1 + i\epsilon},
\]

(5.10)

for the photon propagator of the isotropic modified Maxwell theory, that is, \( K, b, c, \) and \( d \) taking values from \([3.8]\) and \([3.9]\).

Introducing an integration over the momentum \( k^\mu \) of the virtual photon gives

\[
\int \frac{d^4k}{(2\pi)^4} \delta^{(4)}(k_1 + k_2 - k) \mathcal{M}_1 =
\]

\[
\int_{-\infty}^{+\infty} \frac{dk^0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \delta^{(4)}(k_1 + k_2 - k) e^2 \overline{u}(k_1)\gamma^\nu \frac{1 - \gamma^5}{\gamma} \frac{1 - \gamma^5}{\gamma} \frac{\eta_{\mu\nu} + b k_\mu k_\nu + c \xi_\mu \xi_\nu + d (k_\mu \xi_\nu + \xi_\mu k_\nu)}{1 + i\epsilon},
\]

(5.11)

with

\[
\frac{1}{N} \equiv \frac{1}{1 + \tilde{\kappa}_{tr}},
\]

(5.12)

and \( B \) from \([2.6]\).

For the imaginary part of the amplitude, only the propagator poles contribute, since Feynman’s \( i\epsilon \) prescription only becomes important at the poles. These poles are given by \( k^0 = +B|k| - i\epsilon \) and \( k^0 = -B|k| + i\epsilon \), with a positive infinitesimal \( \epsilon \). The following relation holds for the propagator pole with a positive real part:

\[
\frac{1}{k^0 - B|k| + i\epsilon} = \mathcal{P} \frac{1}{k^0 - B|k|} - i\pi \delta(k^0 - B|k|) \equiv \mathcal{P} \frac{1}{k^0 - \omega} - i\pi \delta(k^0 - \omega),
\]

(5.13)
where $P$ denotes the principal value. The first term on the far right-hand-side of (5.13) is real, whereas the second one is imaginary and puts the virtual photon on-shell. By energy conservation, only the positive-frequency pole is relevant and, in order to obtain the imaginary part, all $k^0$ have to be replaced by the frequency $\omega$ from the dispersion relation (2.6).

With $\omega$ defined in (5.13) and the further notation $\hat{M}_1 \equiv M(\epsilon_\ell^+ e_R^+ \rightarrow \tilde{\gamma})$, we finally get:

\[
2 \text{Im}(M_1) = -\int_{-\infty}^{+\infty} dk^0 \int \frac{d^3k}{(2\pi)^3} \delta^{(4)}(k_1 + k_2 - k) \delta(k^0 - \omega) \\
\times e^2 P(k_1) \gamma^\nu \frac{1 - \gamma^5}{2} v(k_2) P(k_2) \gamma^\mu \frac{1 - \gamma^5}{2} u(k_1) \\
\times \frac{1}{N} \eta_{\mu\nu} + b k_\mu k_\nu + c \xi_\mu \xi_\nu + d (k_\mu \xi_\nu + \xi_\mu k_\nu) \\
= -\int \frac{d^3k}{(2\pi)^3} \frac{d^3k}{2\omega} \delta^{(4)}(k_1 + k_2 - k) \\
\times e^2 P(k_1) \gamma^\nu \frac{1 - \gamma^5}{2} v(k_2) P(k_2) \gamma^\mu \frac{1 - \gamma^5}{2} u(k_1) \\
\times \frac{1}{N} \left( \eta_{\mu\nu} + b k_\mu k_\nu + c \xi_\mu \xi_\nu + d (k_\mu \xi_\nu + \xi_\mu k_\nu) \right) \\
= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k}{2\omega} \delta^{(4)}(k_1 + k_2 - k) (\hat{M}_1)^\nu (\hat{M}_1)^\mu \frac{1}{N} \left( - \eta_{\mu\nu} - c \xi_\mu \xi_\nu \right) \\
= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k}{2\omega} \delta^{(4)}(k_1 + k_2 - k) (\hat{M}_1)^\nu (\hat{M}_1)^\mu \left( \sum_{\lambda=1,2} (\varepsilon^{(\lambda)})^\nu_{\nu} (\varepsilon^{(\lambda)})^\mu_{\mu} \right) \\
= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k}{2\omega} \delta^{(4)}(k_1 + k_2 - k) \sum_{\lambda=1,2} |\hat{M}_1|^2 \, (5.14)
\]

where the last line employs the definition $\hat{M}_1(k) \equiv \varepsilon_\mu(k) (\hat{M}_1)^\mu(k)$. In the third step of the above derivation, the Ward identity has been used, so that all terms vanish for which the momentum $k_\mu$ is contracted with $(\hat{M}_1)^\mu$ or its Hermitian conjugate. Recall that the Ward identity [3, 20, 21] reads

\[
k_\mu M^\mu = 0 \, , \quad (5.15)
\]

for a general matrix element $M^\mu(k)$ to which an external photon [with polarization vector $\varepsilon_\mu(k)$ and momentum $k_\mu$] couples. What remains in the fourth step of (5.14) is the polarization sum (3.3) [or, more precisely, the truncated polarization sum (3.4)], since $N$ corresponds to the normalization factor $1/(1 + \tilde{\kappa}_{\text{tr}})$ and $-c$ to $2 \tilde{\kappa}_{\text{tr}}/(1 + \tilde{\kappa}_{\text{tr}})$ according to (3.9c).
Two technical remarks on the derivation (5.14) are in order. First, the $k_2 = 0$ poles in the coefficients $b$ and $d$ from (3.12) do not affect the result. Energy-momentum conservation, encoded by the four-dimensional $\delta$-function with argument $k_1 + k_2 - k$, prevents the pole to be reached in the integration over $k^0$. Second, the Ward identity holds not only for an external photon but also for a photon propagator coupled to a conserved current. Hence, the Ward identity could already be used in the first step of (5.14) to remove the terms with coefficients $b$ and $d$ altogether. As a result, there would appear no $k_2 = 0$ poles in the rest of the calculation.

The conclusion from (5.14) is that the imaginary part of the forward scattering amplitude of the process $e^-_L e^+_R \rightarrow e^-_L e^+_R$ is related to the total cross section for the annihilation process $e^-_L e^+_R \rightarrow \tilde{\gamma}$. This result verifies the validity of the optical theorem, at least, for the process considered.

More generally, it is clear that only the position of the physical propagator poles and the existence of the Ward identity are of importance for the validity of the optical theorem (see, in particular, the succinct discussion of standard QED unitarity in Chapter 9 of Ref. [3]). The form of the matrix element itself (whether it is, for example, polarized or unpolarized) plays no role.

Since both reflection positivity and the optical theorem have been verified in this section, we conclude that, most likely, unitarity of the modified QED theory (2.12) holds for $\tilde{\kappa}_{tr} \in (-1, 1]$. The only caveat we have is the assumed applicability of Feynman–Dyson perturbation theory. But perturbation theory appears to hold for modified QED [32], just as it does for the standard Lorentz-invariant theory [3, 20, 21].

6. Discussion and outlook

In this article, the microcausality and unitarity of the isotropic modified Maxwell theory (2.4) have been established for numerical values of the “deformation parameter” $\tilde{\kappa}_{tr}$ lying in the domain (2.10). In addition, evidence has been presented that these properties of the pure-photon sector carry over to the modified QED theory (2.12) of photons minimally coupled to standard Lorentz-invariant Dirac particles. These results rely on Feynman–Dyson perturbation theory.

The next question is precisely which numerical value of the $\tilde{\kappa}_{tr}$ domain holds experimentally, where $\tilde{\kappa}_{tr} = 0$ corresponds to exact Lorentz invariance. Moreover, having a nonzero $\tilde{\kappa}_{tr}$ singles out a preferred frame of reference, in which the Ansatz “four-vector” $\xi^\mu$ from (2.3b).

In Appendix A a similar domain is established for a particular parity-even anisotropic case of nonbirefringent modified Maxwell theory.
is purely timelike. We have no idea what the proper reference frame is. Here, the reference frame is taken to correspond to the sun-centered celestial equatorial frame (SCCEF). Another possible choice would be the frame in which the cosmic microwave background is isotropic. The strategy is, first, to establish whether or not \( \tilde{\kappa}_{tr} \) differs from zero and, then, to determine the relevant reference frame if the parameter is indeed nonzero.

Direct laboratory bounds on \( |\tilde{\kappa}_{tr}| \) in the SCCEF range from the \( 10^{-2} \) level of the first experiment [33] to the \( 10^{-7} \) and \( 10^{-8} \) levels of the two most recent experiments [34, 35]. Indirect laboratory bounds are much stronger, ranging from the \( 10^{-11} \) level [36] to the \( 5 \times 10^{-15} \) level [37]. Still better indirect earth-based bounds follow from the observation of ultra-high-energy-cosmic-ray (UHECR) primaries and TeV gamma-rays at the top of the Earth’s atmosphere: \(-0.9 \times 10^{-15} < \tilde{\kappa}_{tr} < 0.6 \times 10^{-19} \) at the two-\( \sigma \) level [14]. Future results on UHECRs and TeV gamma-rays may even improve this last two-sided bound by two orders of magnitude [38].

The tight experimental bounds on \( \tilde{\kappa}_{tr} \) can perhaps be understood as implying the extreme smoothness of space, if \( \tilde{\kappa}_{tr} \) arises as the excluded-spacetime-volume fraction of “defects” randomly embedded in flat Minkowski spacetime [9]. Specifically, calculations in simple models give a positive value for \( \tilde{\kappa}_{tr} \) proportional to \( (b/l)^4 \), where \( b \) corresponds to the typical size of the defect (this size being obtained from measurements in the ambient flat spacetime) and \( l \) to the typical minimal length between the individual defects (again, from measurements in the ambient flat spacetime). Remark that, \emph{a priori}, the excluded-spacetime-volume fraction \( (b/l)^4 \) can be of order unity, implying the same order of magnitude for the deformation parameter \( \tilde{\kappa}_{tr} \) [9].

This brings us, finally, to the structure of spacetime (and possibly the cosmological constant problem [39, 40]). In that respect, it is of direct relevance that the modified QED theory (2.12) can also be coupled to external gravitational fields. But, remarkably, modified QED cannot be coupled to dynamical gravitational fields [24, 25]. The main hurdle appears to be that the energy-momentum tensor \( T_{\mu\nu} \) of isotropic modified QED has an antisymmetric part; see Eq. (2.11b) of Ref. [25] with \( \xi^a \) from (2.3b) in this paper. The conclusion may be that either standard gravity rules out the particular theory (2.12) with explicit Lorentz violation or that the theory of gravity itself needs to be modified fundamentally.

\footnote{An alternative calculation of \( \tilde{\kappa}_{tr} \) relies on anomalous effects and gives a positive value proportional to the product of the fine-structure constant \( \alpha \) and the affected-spacetime-volume fraction from “punctures” \( (b = 0) \) randomly embedded in flat Minkowski spacetime [8]. Since, \emph{a priori}, the affected-spacetime-volume fraction can be of order unity, the deformation parameter \( \tilde{\kappa}_{tr} \) from anomalous effects can be of order \( \alpha \sim 10^{-2} \).}
A. Parity-even anisotropic case

A.1. Definition and dispersion relation

One particular anisotropic case of nonbirefringent modified Maxwell theory (2.1)–(2.2) is characterized by a purely spacelike normalized four-vector $\xi^\mu$ in a preferred reference frame and a single Lorentz-violating parameter $\tilde{\kappa}_{33}$:

$$\tilde{\kappa}^{\mu\nu} = \frac{4}{3} \tilde{\kappa}_{33} \left( \xi^\mu \xi^\nu - \frac{1}{4} \xi^\lambda \xi^\eta \eta^{\mu\nu} \right), \quad (A.1a)$$

$$\langle \xi^\mu \rangle = (0, 0, 0, 1), \quad (A.1b)$$

$$\langle \tilde{\kappa}^{\mu\nu} \rangle = \tilde{\kappa}_{33} \operatorname{diag} \left( \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1 \right), \quad (A.1c)$$

By choosing the momentum four-vector as

$$(k^\mu) = (\omega(k), k_\perp, 0, k_\parallel), \quad k_\parallel = k \cdot \xi, \quad k_\perp = |k - k_\parallel \xi|, \quad \xi = (0, 0, 1), \quad (A.2)$$

we obtain the following dispersion relation from the field equations (2.5):

$$\omega(k) = \sqrt{k_\perp^2 + D^2 k_\parallel^2}, \quad D \equiv \sqrt{\frac{1 - 2 \tilde{\kappa}_{33}/3}{1 + 2 \tilde{\kappa}_{33}/3}}. \quad (A.3)$$

The case considered can be expressed in terms of the standard-model-extension (SME) parameters [12] with the help of the “translation dictionary” from [41]:

$$\tilde{\kappa}_{1\text{tr}} = \frac{2}{9} \tilde{\kappa}_{33}, \quad (\tilde{\kappa}_{e-})^{(11)} = \frac{4}{9} \tilde{\kappa}_{33}, \quad (\tilde{\kappa}_{e-})^{(22)} = \frac{4}{9} \tilde{\kappa}_{33}. \quad (A.4)$$

Hence, the anisotropic case considered in this appendix involves a mixture of the three parity-even parameters $(\tilde{\kappa}_{e-})^{(11)}$, $(\tilde{\kappa}_{e-})^{(22)}$, and $\tilde{\kappa}_{1\text{tr}}$.

A.2. Microcausality

The commutators of electric and magnetic fields can be computed just as for the isotropic case in Section 4.1. The results involve a particular tensor structure and a scalar commutator function (here, distinguished by a bar)

$$\mathcal{D}(x) = \oint_C \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(1 + 2 \tilde{\kappa}_{33}/3)(k_0^2 - k_\perp^2) - (1 - 2 \tilde{\kappa}_{33}/3) k_\parallel^2} \exp \left( ik_0 x_0 + i k \cdot x \right). \quad (A.5)$$
For the issue of microcausality, the properties of $D(x)$ are important and the computation of the four-dimensional integral in (A.5) yields:

$$D(x) = -\frac{1}{2\pi\sqrt{1 - 4\tilde{\kappa}^2_{33}/9}} \text{sgn}(x_0) \delta \left(x_0^2 - x_\perp^2 - x_\parallel^2 / D^2\right).$$  \hfill (A.6)

Hence, analogously to the isotropic case of Section 4.1, the commutator function vanishes everywhere except on the modified null-cone,

$$x_0^2 - x_\perp^2 - x_\parallel^2 / D^2 = 0.$$  \hfill (A.7)

As a result, microcausality is a property also for this particular anisotropic case of nonbirefringent modified Maxwell theory, provided

$$\frac{2}{3} \tilde{\kappa}_{33} \in I, \quad I \equiv (-1, 1],$$  \hfill (A.8)

which matches the domain of the isotropic parameter (2.10). In fact, the formal structure of these two cases is similar — recall the definitions of the matrices $\tilde{\kappa}^{\mu\nu}$ in (2.3c) and (A.1c). Note also that these two cases of nonbirefringent modified Maxwell theory, for $\tilde{\kappa}_{tt} > 0$ and $\tilde{\kappa}_{33} > 0$, can be induced from a single Lorentz-violating term in the fermionic action [42].

A.3. Reflection positivity and unitarity

The simple test of reflection positivity from Section 5.1 works just as for the isotropic case, since the scalar part of the Euclidean propagator (here, distinguished by a bar) is

$$\overline{S}_E(k_4, k) = \frac{1}{(1 + 2\tilde{\kappa}_{33}/3)(k_4^2 + |k|^2) - (4\tilde{\kappa}_{33}/3)k_3^2} = \frac{1}{1 + 2\tilde{\kappa}_{33}/3} \frac{1}{k_4^2 + k_\perp^2 + D^2 k_\parallel^2}. \hfill (A.9)$$

Turning immediately to the strong reflection-positivity condition (5.3) for a deformation parameter $\tilde{\kappa}_{33}$ from (A.8), the calculation of the integral gives:

$$\overline{S}_E(x_4, k) = \int_{-\infty}^{+\infty} dk_4 \exp(i k_4 x_4) \overline{S}_E(k_4, k) = \frac{1}{1 + 2\tilde{\kappa}_{33}/3} \int_{-\infty}^{+\infty} dk_4 \frac{\exp(i k_4 x_4)}{k_4^2 + \omega(k)^2} = \frac{2}{1 + 2\tilde{\kappa}_{33}/3} \int_0^{\infty} dk_4 \cos(k_4 x_4) \frac{1}{k_4^2 + \omega(k)^2} = \frac{1}{1 + 2\tilde{\kappa}_{33}/3} \frac{\pi}{\omega(k)} \exp \left(-|x_4| \omega(k)\right), \hfill (A.10)$$

with $\omega(k)$ given by (A.3). Result (A.10), for $\tilde{\kappa}_{33}$ given by (A.8), proves the strong reflection-positivity condition for the scalar propagator. Based on the result for the isotropic case, unitarity can be expected to hold also for the particular case of anisotropic modified Maxwell theory (A.1) interacting with a standard matter sector (2.13).
A.4. Discussion

As shown in this appendix, the pure-photon sector of the parity-even anisotropic non-birefringent modified Maxwell theory characterized by parameters (A.1) has microcausality and unitarity for the $\tilde{\kappa}_{33}$ parameter domain (A.8). The same can be expected to hold for the modified QED theory of photons minimally coupled to standard Lorentz-invariant Dirac particles (2.13). The question, now, is which numerical value of $\tilde{\kappa}_{33}$ holds experimentally, where $\tilde{\kappa}_{33} = 0$ corresponds to having exact Lorentz invariance.

The isolated Lorentz-violating parameters ($\tilde{\kappa}_{e-}^{(11)}$) and ($\tilde{\kappa}_{e-}^{(22)}$) are already tightly bounded at the $10^{-17}$ level by direct laboratory experiments [35, 43, 44]. This implies that, at the present moment, the experimental limit on the $\tilde{\kappa}_{33}$ parameter of the particular case considered in this appendix is controlled by the less tight limits on $\tilde{\kappa}_{tr}$, which have been discussed in the third paragraph of Section 6.

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