DIMENSION OF IMAGES OF LARGE LEVEL SETS

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Abstract. Let $k$ be a natural number. We consider $k$-times continuously-differentiable real-valued functions $f : E \to \mathbb{R}$, where $E$ is some interval on the line having positive length. For $0 < \alpha < 1$ let $I_{\alpha}(f)$ denote the set of values $y \in \mathbb{R}$ whose preimage $f^{-1}(y)$ has Hausdorff dimension $\dim f^{-1}(y) \geq \alpha$. We consider how large can be the Hausdorff dimension of $I_{\alpha}(f)$, as $f$ ranges over the set $C^k(E, \mathbb{R})$ of all $k$-times continuously-differentiable functions from $E$ into $\mathbb{R}$. We show that the sharp upper bound on $\dim I_{\alpha}(f)$ is $1 - \frac{\alpha}{k}$.

1. Introduction

1.1. Big level sets. Let $f$ be a continuously-differentiable real-value function of a real variable defined on some interval $I \subset \mathbb{R}$, and consider its level sets $f^{-1}(y)$, for $y \in \mathbb{R}$.

In everyday experience of nice smooth functions mapping $\mathbb{R} \to \mathbb{R}$, one encounters only level sets that are discrete. But pathology is not far below the surface. For instance, for an arbitrary closed set $F \subset \mathbb{R}$, the function $f : x \mapsto \text{dist}(x, F)^{k+1}$ is $k$-times continuously-differentiable on $\mathbb{R}$ and has $f^{-1}(0) = F$. It is not difficult to construct examples for which infinitely many level sets are as large and wild as one might wish.

For how many $y$ could $f^{-1}(y)$ be ‘large’? This depends on the meaning of ‘large’. Clearly, at most a countable number of level sets may have positive length (i.e. positive one-dimensional Lebesgue measure). Simple examples such as the sine function show that all the nonempty level sets may be infinite. What about interpretations of ‘large’ intermediate between ‘infinite’ and of ‘positive length’?

1.2. Uncountable level sets. Consider the set

$$I_{\alpha}(f) := \{ y \in \mathbb{R} : f^{-1}(y) \text{ is uncountable} \}.$$
It may certainly happen that $I_u(f)$ is itself uncountable. For instance, the first coordinate of the map constructed in [31] is a $C^\infty$ function and has an uncountable number of uncountable level sets. For $y \in I_u(f)$, the closed set $f^{-1}(y)$ must contain accumulation points, and these points are critical points of $f$, so by a well-known result of Morse and Sard [30], $I_u(f)$ has length zero. In fact, for small $\delta > 0$, the intersection of $I_u(f)$ with any bounded interval may be covered by $O(1/\delta)$ intervals of length $o(\delta)$. If $f$ is smoother, say $k$ times continuously-differentiable, then $f$ is ‘flat to order $k$’ (i.e. all derivatives up to order $k$ are zero) at each accumulation point of $f^{-1}(y)$ for $y \in I_u(f)$, and it follows easily that $I_u(f)$ has Hausdorff dimension at most $1/k$. For background on Hausdorff measures and dimension, and fractals, see [34, 27, 19, 18, 9, 28].

1.3. Level sets of positive dimension. To explore beyond the merely uncountable, let us consider critical image sets of positive Hausdorff dimensions. We denote by $\dim S$ the Hausdorff dimension of a set $S \subset \mathbb{R}$. For $0 < \alpha \leq 1$, consider the set

$$I_\alpha(f) := \{ y \in \mathbb{R} : \dim f^{-1}(y) \geq \alpha \}.$$ 

How large could $I_\alpha(f)$ be?

The most natural measure of its size is its Hausdorff dimension:

$$L(\alpha, f) := \dim I_\alpha(f).$$

This is some number in the interval $[0, 1]$. It is nonincreasing as $\alpha$ increases:

$$\alpha_1 < \alpha_2 \implies L(\alpha_1, f) \geq L(\alpha_2, f).$$

We use $C^k$ to stand for ‘$k$ times continuously differentiable’, and $C^k(E, \mathbb{R})$ to denote the set of all $C^k$ functions from an interval $E$ into $\mathbb{R}$.

Our main result is the following:

**Theorem 1.** Let $k \in \mathbb{N}$ and $0 < \alpha < 1$, and let $E$ be a nonempty open interval on $\mathbb{R}$. Then

$$\sup \{ L(\alpha, f) : f \in C^k(E, \mathbb{R}) \} = \frac{1 - \alpha}{k}.$$ 

Observe that it suffices to prove this result for any one particular nonempty open interval $E$. We shall prove it for $E = (-2, 2)$. 
1.4. **Structure of the paper.** The rest of the paper is organised as follows. In Section 2 we show that

\[
\sup \{ L(\alpha, f) : f \in C^k(E, \mathbb{R}) \} \leq \frac{1 - \alpha}{k},
\]

and in Section 3 we show that

\[
\sup \{ L(\alpha, f) : f \in C^k(E, \mathbb{R}) \} \geq \frac{1 - \alpha}{k}.
\]

1.5. **Wider Context.** There has been substantial interest in the possible pathology of level sets of continuous functions. These have been studied for functions mapping \( \mathbb{R}^m \to \mathbb{R}^n \) and more generally for functions on and to more general topological spaces.

Notorious examples of *worst-case behaviour* have been known for a long time. For example, the components \( f_j \) \((j = 1, 2)\) of a Peano space-filling curve \( f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2 \) obviously have an interval of values \( y \) each having a nonempty perfect set contained in \( f^{-1}(y) \). The existence of such curves in suitable Hölder classes has been explored [13]. Components of Hölder-continuous arcs passing through product Cantor sets (similar to the example in [16]) can achieve level sets of any dimension less than 1 whose combined image has any dimension less than 1. Further the space \( C([0, 1], \mathbb{R}) \) contains a dense (residual) set of functions for each of which all but a countable number of the nonempty level sets are perfect [15, Theorem 2].

Generically, functions \( f \in C([0, 1], \mathbb{R}) \) have level sets of Hausdorff dimension zero [24]. The survey [5] summarises and extends work on *generic behaviour*, in various senses (Baire category, prevalence) with respect to the usual topology on various spaces of continuous functions \( C(X, Y) \), and focussing on the Hausdorff dimension (and other types of dimension) of level sets. This line of investigation goes back to Bruckner and Garg [15, 10, 11, 12, 15]. See also [2, 3, 4, 6, 22, 23, 24, 17, 14]. This activity forms part of a wider theme concerning the pathology of image sets, graphs of functions, and slices. See in addition [29, 33].

Special kinds of functions, such as the Hardy-Weierstrass nowhere-differentiable trigonometric series have been studied intensively [18, 21, 35, 13]. Beyer [8] refined results of Kaczmarz and Steinhaus and established facts about the Hausdorff dimension of level sets of certain Rademacher series. Bertoin [7] computed the Hausdorff dimension of level sets of a class of self-affine functions, building on work of Kono on occupation densities [25, 26].

Continuously-differentiable and highly-differentiable functions have attracted less attention from the point of view of pathological level sets,
although the existence of pathological non-generic behavior is notorious from other aspects, such as $C^\infty$ dimension [31], the structure of critical sets [1], dynamics (iteration) and conjugation and centralisers in the case of invertible maps, even in dimension one [32]. This behaviour places formidable obstacles in the way of those who attempt to solve open problems about general smooth functions. The many large level sets that appear in our main result are not found for generic functions in $C^k(E,\mathbb{R})$, which of course have zero-dimensional level sets. The example functions we construct in this paper become completely tame, with discrete level sets, if we add a small linear perturbation, replacing $f(x)$ by $f(x) + \epsilon x$.

The phenomenon exposed in the present paper may prompt others to investigate questions about the pathological behaviour of level sets of smooth functions in other dimensions, and using other senses of ‘large’, (both for the level sets themselves and for the image of the union). The work mentioned above about continuous functions involved interpretations of ‘large’ that are topological (such as Baire category), metric (such as various kinds of Hausdorff dimension) and measure-theoretic (related to Haar measure). This could be a large investigation. We content ourselves with the present result, which just uses classical concepts in the simplest context. Sufficient unto the day is the generality thereof.

2. Upper Bound for $L(\alpha,f)$

2.1. To prove (2), we have to prove the following:

**Lemma 1.** For $k \in \mathbb{N}$, $0 < \alpha < 1$ and $f \in C^k(E,\mathbb{R})$, we have

$$L(\alpha, f) \leq \frac{1 - \alpha}{k}.$$ 

**Proof.** Suppose this lemma is false. Then we can choose some $k \in \mathbb{N}$, $0 < \alpha < 1$, and $f \in C^k(E,\mathbb{R})$ such that

$$L(\alpha, f) > \frac{1 - \alpha}{k}.$$ 

Fix a number $\beta$ strictly between $L(\alpha, f)$ and $\frac{1 - \alpha}{k}$. Then $\mathcal{H}^\beta(I_\alpha(f))$ is positive, where $\mathcal{H}^\beta$ denotes Hausdorff measure of dimension $\beta$. By a theorem of Davies [19, 2.10.47] (see also [20]), we may choose a compact subset $Y \subset I_\alpha(f)$ having finite positive $\mathcal{H}^\beta(Y)$.

We have $\alpha + k\beta > 1$, so we can choose a positive number $\gamma < \alpha$ such that $\gamma + k\beta > 1$. Then for each $y \in Y$, the preimage $f^{-1}(y)$ has positive $\mathcal{H}^\gamma$ measure (infinite, in fact).
Let $T$ denote the set of condensation points of the critical set $(f')^{-1}(0)$ of $f$. Then $T$ is a closed subset of the interval $E$, and since $f$ is flat of order $k$ at each point of $T$, we have that
\[ |f(x_1) - f(x_2)| = o\left(|x_1 - x_2|^k\right), (x_1, x_2 \in T). \]
For each $y \in Y$, the set $f^{-1}(y) \setminus T$ of non-condensation points of its preimage is countable, so
\[ \mathcal{H}^\gamma(f^{-1}(y) \cap T) = \mathcal{H}^\gamma(f^{-1}(y)) > 0. \]
Let $X = T \cap f^{-1}(Y)$.

We now start from Federer [19], section 2.10.25. The theorem there states:

**Theorem 2.** If $f : X \to Y$ is a Lipschitzian map of metric spaces, $A \subset X$, $0 \leq k < \infty$ and $0 \leq m < \infty$, then
\[ \int_Y \mathcal{H}^k(A \cap f^{-1}(y)) \, d\mathcal{H}^m y \leq (\text{Lip } f)^m \frac{\alpha(k)\alpha(m)}{\alpha(k + m)} \mathcal{H}^{k+m}(A), \]
provided either $\{y : \mathcal{H}^k(A \cap f^{-1}(y)) > 0\}$ is the union of a countable family of sets with finite $\mathcal{H}^m$ measure, or $Y$ is boundedly compact. □

Here $\int^*$ denotes the upper integral, $\text{Lip } f$ is the Lipschitz constant of $f$, and $\alpha(k)$ is the constant
\[ \alpha(k) := \frac{\Gamma(\frac{1}{2})^k}{\Gamma(\frac{k}{2} + 1)}, \]
where $\Gamma$ denotes the Euler Gamma function.

2.2. In the proof of Theorem 2 (on page 189), Federer used the hypothesis that $f$ is Lipschitzian in order to bound the diameter of the image $f(S)$ of a small set $S$ by a constant multiple of $\text{diam}(S)$. If instead we can bound $\text{diam } f(S)$ (for, say, $\text{diam}(S) \leq 1$) by a constant times $(\text{diam } S)^s$, for some $s > 1$, then we can improve the inequality in Theorem 2 to
\[ \int_Y \mathcal{H}^k(A \cap f^{-1}(y)) \, d\mathcal{H}^m y \leq \text{const} \cdot \mathcal{H}^{k+sm}(A). \]
(Just replace $m$ by $sm$ at each occurrence in Federer’s argument.)

We can apply this, with the $X$ and $Y$ chosen above, and the replacements
\[ k \to \beta, m \to \gamma, s \to k, A \to X \]
so that it reads
\[ \int_Y \mathcal{H}^\gamma(X \cap f^{-1}(y)) \, d\mathcal{H}^\beta y \leq C \mathcal{H}^{\gamma+k\beta}(X). \]
Thus
\[
\int_Y \mathcal{H}^\gamma \left( f^{-1}(y) \right) \, d\mathcal{H}^\beta y \leq C \mathcal{H}^{\gamma+k\beta}(X),
\]
But \( \gamma + k\beta > 1 \), and \( X \subset \mathbb{R} \), so
\[
\int_Y \mathcal{H}^\gamma \left( f^{-1}(y) \right) \, d\mathcal{H}^\beta y = 0.
\]
This is impossible, because the integrand is positive (in fact +\( \infty \)) for all \( y \in Y \), and \( \mathcal{H}^\beta(Y) > 0 \).
Thus the lemma is proven. \( \square \)

3. Examples of large \( L(\alpha, f) \)

Fix \( k \in \mathbb{N} \).

3.1. In order to prove that the lower bound \((3)\) holds for each \( \alpha \) strictly between 0 and 1, it suffices to show by construction that for each given \( r, s \), and \( \epsilon \), with \( 2 \leq r \in \mathbb{N} \), \( 2 \leq s \in \mathbb{N} \), and
\[
0 < \epsilon < \frac{1}{rs-1},
\]
there is a function \( f \in C^k(E, \mathbb{R}) \) having
\[
L \left( \frac{\log_s r}{1 + \log_s \left( \frac{r}{1-\epsilon} \right)}, f \right) \geq \frac{1}{(k + \epsilon) \left\{ 1 + \log_s \left( \frac{r}{1-\epsilon} \right) \right\}}.
\]
To see this, observe first that the set of all numbers
\[
\alpha(r, s) := \frac{\log_s r}{1 + \log_s r}, \quad (2 \leq r \in \mathbb{N}, 2 \leq s \in \mathbb{N})
\]
is dense in the interval \((0, 1)\). So given any positive \( \alpha < 1 \), and any \( \eta > 0 \), we can choose \( r \) and \( s \) so that
\[
\alpha + \eta > \alpha(r, s) > \alpha,
\]
and then choose \( \epsilon > 0 \) so that
\[
\frac{1}{(k + \epsilon) \left\{ 1 + \log_s \left( \frac{r}{1-\epsilon} \right) \right\}} > \frac{1}{k(1 + \log_s r)} - \eta = \frac{1 - \alpha(r, s)}{k} - \eta,
\]
and
\[
\frac{\log_s r}{1 + \log_s \left( \frac{r}{1-\epsilon} \right)} > \alpha.
\]
Then for the function constructed we have
\[
L(\alpha, f) \geq L \left( \frac{\log_s r}{1 + \log_s \left( \frac{r}{1-\epsilon} \right)}, f \right) > \frac{1 - \alpha - \eta}{k} - \eta.
\]
Since this can be done for each positive $\eta$, we conclude that
\[
\sup \{ L(\alpha, f) : f \in C^k(E, \mathbb{R}) \} \geq \frac{1 - \alpha}{k},
\]
as stated.

### 3.2. Fix $r, s$ and $\epsilon$ with $2 \leq r \in \mathbb{N}$, $2 \leq s \in \mathbb{N}$, $0 < \epsilon < \frac{1}{2}$.

The $C^k$ function $f$ we are going to construct will map the interval $E = [-2, 2]$ to $\mathbb{R}$, will be $k$-flat on a certain Cantor-type set $A \subset [0, 1]$, and will be strictly increasing when restricted to $[-2, 0]$ and when restricted to $[1, 2]$. All the action will occur in $[0, 1]$. The only critical points of $f$ will be the points of $A$. The function will map $[0, 1]$ onto itself, with $f(0) = 0$ and $f(1) = 1$. The image $f(A) \subset [0, 1]$ will be another Cantor-type set $D$. For each $y \in D$, the preimage $f^{-1}(y)$ will consist of a Cantor-type subset $A_y$ of $A$, together with a set of isolated points. Each set $A_y$ will have the same Hausdorff dimension
\[
\gamma(r, s, \epsilon) := \log_s r \frac{1}{1 + \log_s \left(\frac{r}{1-\epsilon}\right)}.
\]
Thus
\[
I_{\gamma(r, s, \epsilon)}(f) = D,
\]
and hence $L(\gamma(r, s, \epsilon), f) = \text{dim } D$. The Hausdorff dimension of $D$ will be
\[
\frac{1}{(k + \epsilon) \left(1 + \log_s \left(\frac{r}{1-\epsilon}\right)\right)}.
\]

The construction involves the repetition at reducing scales of a single pattern. It involves generations of nested rectangles. The graph $f$ traverses each rectangle from bottom left to top right. The first generation has the single rectangle $[0, 1] \times [0, 1]$. Each rectangle of the $n$-th generation contains $rs$ rectangles of the next generation, arranged in $s$ rows of $r$ congruent rectangles. The rectangles of different generations are not similar, i.e. they do not have the same proportions of height to width. In other words, the rescaling from generation to generation is non-isotropic, non-conformal. In fact the ratio of height to width tends rapidly to zero as we progress through the generations. The rectangles of a given generation project vertically to pairwise-disjoint closed intervals, but their horizontal projections overlap. More precisely, the horizontal projections of each given row are identical, and the horizontal projections of distinct rows are pairwise-disjoint.
Figure 1 shows the pattern of the 12 rectangles of generation 1 for the case when \( r = 4 \) and \( s = 3 \).

We denote the vertical projection of a rectangle \( T \) by \( T_1 \), and its horizontal projection by \( T_2 \).

We use the following notation for features of the \( n \)-th generation:

- \( R_n \): the family of all its rectangles,
- \( c_n \): width of each rectangle, which is to depend only on \( n \),
- \( a_n \): height of each rectangle, also depending only on \( n \),
- \( P_n \): the family of all vertical projections \( T_1 \) of \( T \in R_n \) (on \( \mathbb{R} \)),
- \( Q_n \): the family of horizontal projections \( T_2 \) of \( T \in R_n \),

If \( n > 1 \) and \( T \in R_{n-1} \), then we use the following notation for features of the \( n \)-generation rectangles contained in \( T \):

- \( R_T \): the family of all these rectangles,
- \( R_{Tm} \): the \( m \)-th row of \( R_T \),
- \( R_{Tmp} \): the \( p \)-th rectangle of \( R_{Tm} \),
- \( P_T \): the family of vertical projections of \( R_T \),
$Q_T$: the family of horizontal projections of $R_T$,

$d_n$: horizontal space between adjacent rectangles $P_T$, also depending only on $n$,

$b_n$: vertical spacing between adjacent intervals of $Q_n$, also depending only on $n$.

In Figure 1, the second-generation rectangle $R_{Tmp}$, where $T = [0, 1] \times [0, 1]$, is labelled $Tmp$.

The first rows $R_{T1}$ of rectangles of the $n$-th generation are placed at the bottom left of their containing rectangle $T \in R_{n-1}$, and the last rows $R_{Ts}$ are placed at the top right. The remaining rows are evenly spaced vertically in $T$ (the space is $b_n$), and the horizontal gap between rows has the same width $d_n$ as the horizontal gap between the rectangles of each row $R_{Tm}$.

With this notation, we can define the Cantor-type sets $A$ and $D$ as

$$A := \bigcap_{n=1}^{\infty} \bigcup P_n, \quad D := \bigcap_{n=1}^{\infty} \bigcup Q_n.$$ 

For $n > 1$, and $T \in R_{n-1}$, the function $f$ is defined on the union of gaps $T_1 \setminus P_T$ in such a way that its graph links the top right vertex of each $n$-th generation rectangle $R_{Tmp}$ (apart from the last one, $P_{Tsr}$) to the bottom left vertex of the next to its right. (If $p < r$, the next rectangle is $R_{Tm(p+1)}$, whereas if $p = r$, the next rectangle is $R_{T(m+1)}$.)

This linking is illustrated in Figures 2 and 3. Figure 2 shows the link between the two rectangles of the same row (specifically, the first two rectangles of the first row of the next generation inside rectangle $T$). Figure 3 shows the link between the last rectangle of one row and the first rectangle of the next row.

![Figure 2](image1)

![Figure 3](image2)
Each of these linking graphs is defined by a formula of the form

\[ f(x) := \delta \cdot \phi \left( \frac{x - \mu}{\nu} \right) + \rho, \]

where the parameters \( \delta, \mu, \nu \) and \( \rho \) depend on \( T, m \) and \( p \), but the function \( \phi \) is always the same. To be more specific, \( \phi : [0, 1] \to [0, 1] \) is a \( C^k \) function that is flat of order \( k \) at 0 and 1, and decreases strictly monotonically from 1 to 0. A specific example of such a function is given by

\[ \phi(x) := 1 - \frac{\int_0^x t^k(1 - t)^k \, dt}{\int_0^1 t^k(1 - t)^k \, dt}. \]

The parameters are chosen so that the top right of \( R_{T_{mp}} \) has coordinates \((\rho, \mu)\), and the bottom left of the next rectangle has coordinates \((\rho + \nu, \mu + \delta)\). Thus \( \delta = -a_n \) when \( R_{T_{mp}} \) is not the last rectangle of its row \( R_{T_m} \), and \( \delta = b_n \) when \( R_{T_{mp}} \) is last in its row, i.e when \( m < s \) and \( p = r \). The parameter \( \nu \), the width of the horizontal gap between successive rectangles of the same generation, is \( d_n \), depending only on \( n \).

If we set

\[ K := \sup_{[0,1]} \phi^{(k)}, \]

then in the gap after \( R_{T_{mp}} \) we have

\[ \sup |f^{(k)}| \leq \frac{K\delta}{\nu^k} \leq \frac{K \max(a_n, b_n)}{d_n^k} < \frac{K a_{n-1}}{d_n^k}. \]

Thus as long as we ensure that \( a_{n-1}/d_n^k \to 0 \) as \( n \uparrow \infty \), we will end up with a \( C^k \) function \( f \), flat of order \( k \) on \( A \).

(If this seems puzzling, note that at later generations the horizontal gaps between rectangles will be far larger than the height of the rectangles and even much larger than the vertical gaps between rows of rectangles than they are in the figures. The rectangles become very very low, compared with their width.)

We now specify the specific values that determine the function \( f \) completely. We take \( a_1 = c_1 = 1 \), and then for \( n > 1 \) we define

\[ c_n := \frac{(1 - \epsilon)c_{n-1}}{rs} = \left( \frac{1 - \epsilon}{rs} \right)^{n-1}, \]

so that the intervals of each \( P_T \) take up all but a proportion \( \epsilon \) of the projection \( T_1 \). The horizontal gap-length \( d_n \) is then forced to be

\[ d_n = \frac{\epsilon c_{n-1}}{rs - 1} > \frac{\epsilon(1 - \epsilon)^{n-2}}{(rs)^{n-1}}. \]
Next, we take
\[ a_n := d_n^{k+\epsilon}, \]
and this ensures that \( a_{n-1}/d_n^k \to 0 \), as required. The value of the vertical gap-height \( b_n \) is then forced to be
\[ b_n = \frac{a_{n-1} - sa_n}{s - 1}. \]

The Cantor set \( D \) is covered by the \( s^n \) disjoint horizontal projections of the family \( R_{n+1} \), each of length \( a_{n+1} \), and for each \( n \) we have
\[ s^n a_{n+1}^\beta = s^n d_{n+1}^{(k+\epsilon)\beta} = \left( \frac{\epsilon rs}{(1-\epsilon)(rs - 1)} \right)^{(k+\epsilon)\beta}, \]
(independently of \( n \)) when
\[ \beta = \frac{1}{(k + \epsilon) \left\{ 1 + \log_s \left( \frac{r}{1-\epsilon} \right) \right\}}, \]
and we conclude that this value of \( \beta \) is the Hausdorff dimension of \( D \) (Compare Example 4.4, page 69, in Falconer’s text [18], of which our set \( D \) is a special case).

Now for \( y \in D \), we consider \( f^{-1}(y) \). The preimage of \( y \) consists of a countable subset of \([0, 1] \setminus A\), together with the Cantor set \( A_y \), and this Cantor set may be described as follows. The rectangle \([0, 1] \times [0, 1]\) contributes the interval \([0, 1]\) as the first approximation to \( A_y \). A rectangle \( T \in R_n \) contributes an interval \( T_1 \) to the \( n \)-th approximation to \( A_y \) if and only if \( y \in T_2 \). The set \( A_y \) is the intersection of these approximations, and each approximation covers \( A_y \) by \( r^n \) intervals of length \( c_n \). These intervals may be shrunk by a certain amount, and still cover, because for each \( T \),
\[ \text{diam}(A_y \cap T_1) < \text{diam}(T_1). \]
In fact, \( \inf A_y \cap T_1 \) is the left-hand end of the projection of the first rectangle of some row of the next generation inside \( T \), whereas \( \sup A_y \cap T_1 \) lies somewhere inside the projection of the last rectangle of the same row. However the ratio
\[ \lambda(r, s, \epsilon) := \frac{\text{diam}(A_y \cap T_1)}{\text{diam}(T_1)} \]
depends only on \( r, s \) and \( \epsilon \), (and lies between \( (r - 1)/rs \) and \( 1/s \)), so \( A_y \) is covered by \( r^{n-1} \) intervals of length \( \lambda c_n \). (In fact, a calculation shows that, perhaps surprisingly, the ratio
\[ \lambda = \frac{r - 1}{r s - 1}, \]
For each \( n \in \mathbb{N} \) we have that
\[
 r^{n-1} (\lambda c_n)^\alpha = \lambda^\alpha \left( \frac{(1-\epsilon)^{n-1-\alpha}}{s^\alpha} \right) = \lambda^\alpha,
\]
when
\[
 \alpha = \frac{\log s r}{1 + \log \left( \frac{s}{1-r} \right)},
\]
and we conclude that this value of \( \alpha \) is the Hausdorff dimension of \( A_y \) (— the set \( A_y \) is another case of Example 4.4 in [18]).

This completes the contruction, and the proof.

3.3. Remarks. 1. For \( f \in C^k(E, \mathbb{R}) \), we have
\[
 0 \leq L(1, f) \leq \inf_{0<\alpha<1} L(\alpha, f) = 0
\]
so the equality in Theorem 1 also holds when \( \alpha = 1 \).

2. The construction we gave shows that the Hausdorff dimension of the image of the critical set of a \( C^k \) function on \( \mathbb{R} \) may be arbitrarily close to \( 1/k \), i.e. that the bound mentioned in Subsection 1.2 is sharp.

3. At the other extreme, the construction also shows that the dimension of each one of an uncountable family of level sets may be arbitrarily close to 1.

4. By tweaking the construction, keeping \( r = 2 \) but increasing \( s \) at each step, one obtains a \( C^k \) function having a critical set \( A \) of dimension zero, with \( \dim f(A) = 1/k \).

5. By, instead, keeping \( s = 2 \) but increasing \( r \) at each step, one obtains a \( C^k \) function having an uncountable family of level sets of Hausdorff dimension 1.

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