Concerning the Nature of the Cosmic Ray Power Law Exponents

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We have recently shown that the cosmic ray energy distributions as detected on earthbound, low flying balloon or high flying satellite detectors can be computed by employing the heats of evaporation of high energy particles from astrophysical sources. In this manner, the experimentally well known power law exponents of the cosmic ray energy distribution have been theoretically computed as $2.701178$ for the case of ideal Bose statistics, $3.000000$ for the case of ideal Boltzmann statistics and $3.151374$ for the case of ideal Fermi statistics. By “ideal” we mean virtually zero mass (i.e. ultra-relativistic) and noninteracting. These results are in excellent agreement with the experimental indices of $2.7$ with a shift to $3.1$ at the high energy $\sim \text{PeV}$ “knee” in the energy distribution. Our purpose here is to discuss the nature of cosmic ray power law exponents obtained by employing conventional thermal quantum field theoretical models such as quantum chromodynamics to the cosmic ray sources in a thermodynamic scheme wherein gamma and zeta function regulation is employed. The key reason for the surprising accuracy of the ideal boson and ideal fermion cases resides in the asymptotic freedom or equivalently the Feynman “parton” structure of the ultra-high energy tails of spectral functions.

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I. INTRODUCTION

As in our preliminary work[1], we seek to understand the nature of the power law exponents $\{\alpha\}$ which are employed to describe the energy distributions observed in the cosmic rays continually bombarding our planet and coming from astrophysical sources[2–4]. From the quantum field theory viewpoint we regard the cosmic rays as standard model hadrons evaporating from sources and moving away from such sources as a gaseous blowing wind. Such a solar wind exists issuing from the center of our own planetary system. These evaporating winds no doubt also blow away from other astrophysical objects such as neutron stars.

The starting point for defining cosmic ray power law exponents was purely experimental. It is known[5] that the energy distribution law of cosmic ray nuclei in the energy range $5 \text{ GeV} < E < 100 \text{ TeV}$ via the differential flux per unit time per unit area per steradian per unit energy obeys

$$\frac{d^4N}{dt dA d\Omega dE} \approx \frac{1.8 \text{ nucleons}}{\text{sec cm}^2 \text{ sr GeV}} \left(\frac{1 \text{ GeV}}{E}\right)^{\alpha}$$

wherein the experimental power law exponent $\alpha \approx 2.7$. At the “knee” of the distribution, i.e. at energy $E \sim 1 \text{ PeV}$, there is a shift in the power law exponent to the value $\alpha \approx 3.1$. In [1], we had computed theoretically the ideal Bose index. Here we also compute the ideal Fermi statistical index so that they read together as:

$$\alpha_{\text{Bose}} = 2.701178 \quad \text{and} \quad \alpha_{\text{Fermi}} = 3.151374 .$$

(2)

It would be well within experimental error to regard the knee as a crossover between statistics which in concrete physical evaporation terms merely means a crossover in the composition of cosmic ray emission winds blowing away from astrophysical sources. The critical values in Eq.(2) are ideal in the sense that the particles are ultra-relativistic $E \approx c|p|$ and noninteracting. One might ponder why a non-interacting theory is so close to experimental reality. The answer resides in the asymptotic freedom in the form of Feynman parton structure[6] of the ultra-high energy tails of spectral functions.

To describe cosmic ray sources in terms of thermal quantum field theoretical models, it is of some convenience to employ gamma and zeta function regulators whose definitions are reviewed in Sec.II wherein the ideal power law exponents are derived. That interactions apparently have little effect on the power law exponents would seem to imply that the quantum spectral functions are of the Feynman form[6] with Bose and Fermi operators being composites of quark operators. In the concluding Sec.III these points are qualitatively discussed.

II. GAMMA AND ZETA REGULATORS

A. Mathematical Details

The mathematics of gamma and zeta regulators resides in the properties of classical special functions[7]. Starting with the statistical index

$$\eta = 1 \quad \text{Bose}, \quad \eta = 0 \quad \text{Boltzmann} \quad \text{and} \quad \eta = -1 \quad \text{Fermi},$$

(3)
the general zeta function regulators are defined as
\[ Z(s, \eta) = \int_0^\infty \frac{x^s e^{-x \eta}}{e^x - \eta} \left[ \frac{dx}{x} \right] \quad \text{for } \Re s > 1 \quad (4) \]
and by analytic continuation in \( s \) elsewhere. The Boltzmann regulator is the Euler gamma function
\[ Z(s, 0) = \int_0^\infty x^s e^{-x} \left[ \frac{dx}{x} \right] = \Gamma(s). \quad (5) \]
The Bose regulator is determined by the Riemann zeta function defined by
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (6) \]
or its analytic continuation in \( s \) where the sum does not converge via
\[ Z(s, 1) = \int_0^\infty x^s \sum_{n=1}^{\infty} e^{-ns} \left[ \frac{dx}{x} \right] = \Gamma(s)\zeta(s). \quad (7) \]
The Fermi regulator
\[ Z(s, -1) = \int_0^\infty x^s \sum_{n=1}^{\infty} (-1)^{n-1} e^{-ns} \left[ \frac{dx}{x} \right], \]
\[ Z(s, -1) = \Gamma(s) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \]
\[ Z(s, -1) = \left[ 1 - \frac{2}{2^s} \right] \Gamma(s)\zeta(s). \quad (8) \]
Note that the Bose and Fermi regulators are rigorously yet simply related by
\[ Z(s, -1) = \left[ 1 - \frac{2}{2^s} \right] Z(s, 1). \quad (9) \]

### B. Ideal power law exponents

The density of states per unit energy per unit volume for ultra-relativistic particles is proportional to the square of the energy. The mean energy per particle in an ideal gas is thereby
\[ E_\eta = \frac{\int_0^\infty \frac{e^2 dx}{e^{\alpha_\eta k_B T} - \eta}}{\int_0^\infty \frac{e^2 dx}{e^{\alpha_\eta k_B T} - \eta}} = \alpha_\eta k_B T, \quad (10) \]
wherein the regulators determining the mean energies are
\[ \alpha_\eta = \left[ \frac{Z(4, \eta)}{Z(3, \eta)} \right]. \quad (11) \]

In detail,
\[ \alpha_0 = \frac{\Gamma(4)}{\Gamma(3)} = \frac{3!}{2!} = 3, \]
\[ \alpha_1 = \frac{\Gamma(4)\zeta(4)}{\Gamma(3)\zeta(3)} \approx 2.701178, \]
\[ \alpha_{-1} = \left[ \frac{1 - (1/8)}{1 - (1/4)} \right] \alpha_1 = \frac{7\alpha_1}{6} \approx 3.151374, \quad (12) \]
as in Eq.(2).

To establish \( \alpha \) as a power law exponent when the energy \( E = \alpha k_B T \), one must compute the entropy as
\[ E = \alpha k_B T = \alpha k_B \frac{dE}{dS} \Rightarrow S = \alpha k_B \ln \left( \frac{E}{E_\eta} \right), \quad (13) \]
and employ the heat of vaporization to compute the evaporation energy spectrum
\[ e^{-S/k_B} = \left( \frac{E_\eta}{E} \right)^\alpha \quad (14) \]
as in Eq.(1).

### III. Conclusions

A more detailed interacting quantum field theoretical calculation of \{\( \alpha \)\} power law exponents involves the construction of single particle spectral functions in the context of thermal quantum field theory. While we have here computed the \{\( \alpha \)\} indices for the free Fermi and free Bose field theories, the results are already in quite satisfactory agreement with experimental cosmic ray power law exponents. The reason for this remarkable agreement would appear to be due to a Feynman “parton” structure for the high energy asymptotic tails of the single particle spectral functions. In this case that structure would be described by free non-interacting particles. Following Feynman’s physical reasoning and employing the dispersion relations in a finite temperature many body quantum field theory context, we are presently computing the renormalized energy dependent power law exponent \( \alpha(E) \) for interacting theories.

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