Efficient classical simulation of Clifford circuits with nonstabilizer input states

Kai Feng Bu\textsuperscript{1,2,*} and Dax Enshan Koh\textsuperscript{3,†}

\textsuperscript{1}School of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China
\textsuperscript{2}Department of Physics, Zhejiang University, Hangzhou, Zhejiang 310027, China
\textsuperscript{3}Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

We investigate the problem of evaluating the output probabilities of Clifford circuits with nonstabilizer product input states. First, we consider the case when the input state is mixed, and give an efficient classical algorithm to approximate the output probabilities, with respect to the $l_1$ norm, of a large fraction of Clifford circuits. The running time of our algorithm decreases as the inputs become more mixed. Second, we consider the case when the input state is a pure nonstabilizer product state, and show that a similar efficient algorithm exists to approximate the output probabilities, when a suitable restriction is placed on the number of qubits measured. This restriction depends on a magic monotone that we call the Pauli rank. We apply our results to give an efficient output probability approximation algorithm for some restricted quantum computation models, such as Clifford circuits with solely magic state inputs (CM), Pauli-based computation (PBC) and instantaneous quantum polynomial time (IQP) circuits.

I. INTRODUCTION

One of the main motivations behind the field of quantum computation is the expectation that quantum computers can solve certain problems much faster than classical computers. This expectation has been driven by the discovery of quantum algorithms which can solve certain problems believed to be intractable on a classical computer. A famous example of such a quantum algorithm is due to Shor, whose eponymous algorithm can solve the factoring problem exponentially faster than the best classical algorithms we know today [1, 2].

With the advent of noisy intermediate-scale quantum (NISQ) devices [3], an important near-term milestone in the field is to demonstrate that quantum computers are capable of performing computational tasks that classical computers cannot, a goal known as quantum supremacy [4, 5]. Several restricted models of quantum computation have been proposed as candidates for demonstrating quantum supremacy. These include boson sampling [6], the one clean qubit model (DQC1) [7, 8], instantaneous quantum polynomial-time (IQP) circuits [9], Hadamard-classical circuits with one qubit (HC1Q) [10], Clifford circuits with magic initial states and nonadaptive measurements [11–13], the random circuit sampling model [14, 15], and conjugated Clifford circuits (CCC) [16]. These models are potentially good candidates for quantum supremacy because they can solve sampling problems that are conjectured to be intractable for classical computers, and are conceivably easier to implement in experimental settings.

In contrast to the above models, quantum circuits with Clifford gates and stabilizer input states are not a candidate for quantum supremacy, because they can be efficiently simulated on a classical computer using the Gottesman-Knill simulation algorithm [17]. The Gottesman-Knill algorithm, however, breaks down and efficient classical simulability can be proved to be impossible (under plausible assumptions) when Clifford circuits are modified in various ways, under various notions of simulation [11–13, 16]. For example, it can be proved under plausible complexity assumptions that no efficient classical sampling algorithm exists that can sample from the output distributions of Clifford circuits with general product state inputs when the number of measurements made is of order $O(n)$ [11].

In this paper, we present two new efficient classical algorithms for approximately evaluating the output probabilities of Clifford circuits with nonstabilizer inputs. Our first algorithm shows that the output distribution of Clifford circuits with mixed product states can be efficiently approximated, with respect to the $l_1$ norm, for a large fraction of Clifford circuits. This algorithm explicitly reveals the role of mixedness of the input states in affecting the running time of the simulation, which decreases as the inputs become more mixed.

Our second algorithm shows that such an efficient approximation algorithm still exists in the case where the inputs are pure nonstabilizer states, as long as we impose a suitable restriction on the number of measured qubits. This restriction depends on a magic monotone called the Pauli rank that we introduce in this paper. This algorithm also explicitly links the simulation time to the amount of magic in the input states, and implies that for Clifford circuits with magic input states, it is possible in certain cases to achieve an efficient classical approximation of the output probability even when $O(n)$ qubits are measured. This is in contrast to the hardness result in [11], which shows that sampling from those output probabilities is hard. Finally, we apply our results to give an efficient approximation algorithm for some restricted quantum computation models, like Clifford circuits with solely magic state inputs (CM), Pauli-based computation (PBC) and instantaneous quantum polynomial time (IQP) circuits.

II. MAIN RESULTS

Let $P^n$ be the set of all Hermitian Pauli operators on $n$ qubits, i.e., operators that can be written as the $n$-fold tensor product of the single-qubit Pauli operators $\{I, X, Y, Z\}$ with sign $\pm 1$. The Clifford unitaries on $n$ qubits are the uni-
taries that maps Pauli operators to Pauli operators, that is, \( C_{l_n} = \{ U \in U(2^n) : UPU^\dagger \in P^m, \forall P \in P^n \} \). Stabilizer states are pure states of the form \( |0\rangle^{\otimes n} \otimes |\rho_i\rangle^{\otimes m} \) [18], where \( U \) is some Clifford unitary.

Here, we consider Clifford circuits with product input states \( |0\rangle^{\otimes n} \otimes |\rho_i\rangle^{\otimes m} \), and measurements on \( k \) qubits. If either \( m \) or \( k \) is \( O(\log n) \), the output probabilities can be efficiently simulated classically by the Gottesman-Knill theorem [11, 17]. However, if both \( m \) and \( k \) are greater than \( O(\log n) \), we show that the output probability of such circuits can still be approximated efficiently with respect to the \( l_1 \) norm for a large fraction of Clifford circuits.

![FIG. 1. A circuit diagram of Clifford circuits with product state inputs, which could be either pure or mixed.](image)

### A. Mixed input states

We first consider the case where all \( \rho_i \) are mixed states and give an efficient classical algorithm to approximate the output probabilities.

**Theorem 1.** Given a Clifford circuit \( C \) on \( n + m \) qubits with input state \( |0\rangle^{\otimes n} \otimes |\rho_i\rangle^{\otimes m} \) and measurement on each qubit in the computational basis, there exists a classical algorithm to approximate the output probabilities of the circuit up to \( l_1 \) norm \( \delta \) in time \((n+m)O(1)m^{\Theta(\sqrt{\delta}/\epsilon)}\lambda\) for at least \( 1 - \frac{\delta}{2} \) fraction of circuits \( C \), where \( \lambda = \min \{ \lambda_i \} \), with \( \lambda_i = 1 - \sqrt{2\text{Tr}[\rho_i^2]} - 1 \), is a measure of the mixedness of the input state \( \rho_i \).

The proof of the Theorem is presented in Appendix A. The theorem states that the efficiency of the classical simulation increases with the mixedness of the input states.

Next, we show that the result in Theorem 1 can be easily generalized to quantum circuits \( C \) which are slightly beyond Clifford circuits. To this end, we consider the Clifford hierarchy, a class of operations introduced by Gottesman and Chuang [19] that has important applications in fault-tolerant quantum computation and teleportation-based state injection. Let \( C_{l_n}^{(3)} \) be the third level of the Clifford Hierarchy, i.e., \( C_{l_n}^{(3)} = \{ U \in U(2^n) : UPU^\dagger \in C_{l_n}, \forall P \in P^n \} \). There are several important gates in the third level of Clifford Hierarchy, such as the \( \pi/8 \) gate (which we denote \( T \)) and the CCZ gate [20]. (Note that the set \( C_{l_n}^{(3)} \) is not closed under multiplication. For example, \( TH, T \in C_{l_n}^{(3)} \), but \( THT \notin C_{l_n}^{(3)} \).)

**Corollary 2.** Let \( C = C_1 \circ V \) be a quantum circuit with input states \( |0\rangle^{\otimes n} \otimes |\rho_i\rangle^{\otimes m} \), where the gates in the circuit \( C_1 \) are taken from the set of Clifford gates on \( n + m \) qubits \( C_{l_n+m} \), and \( V \) is taken from the third level of Clifford hierarchy \( C_{l_n}^{(3)} \) acting on \( n + 1, ..., n + m \)-th qubits. Assume that each qubit is measured in the computational basis. Then, Theorem 1 still holds if we replace \( C \) in Theorem 1 with \( C \) defined above.

The key property we use here is that the gates in the third level of the Clifford Hierarchy map Pauli operators to Clifford unitaries, which makes the proof of Theorem 1 still hold. (See a discussion of this in Appendix A.) Although \( C_{l_n}^{(3)} \) is not a group, the diagonal gates in \( C_{l_n}^{(3)} \), denoted as \( C_{l_n}^{(3)} \), forms a group [20, 21]. Since the \( T \) gate and CCZ gate both belong to \( C_{l_n}^{(3)} \), the result in Theorem 1 still holds for the quantum circuits \( C = C_1 \circ C_2 \) where gates in \( C_1 \) and \( C_2 \) are chosen from \( C_{l_n+m} \) and \( C_{l_n}^{(3)} \), respectively.

Since noise is inevitable in real physical experiments, it is important to consider the effects of noise in quantum computation. Recently, it has been demonstrated that if there is some noise on the random quantum gates [22] or measurements of IQP circuits [23], then there exists an efficient classical simulation of the output distribution of quantum circuits. In the rest of this subsection, we apply our results to two important subuniversal quantum circuits with noisy input states and give an efficient classical approximation algorithm for the output probabilities of the corresponding quantum circuits.

**Example 1**—First, we consider Clifford circuits with magic input states. It is well known that the Clifford + \( T \) gate set is universal for quantum computation. By magic state injection, circuits with this gate set can be efficiently simulated by Clifford circuits with magic state \( |T\rangle \) inputs, where \( |T\rangle = \frac{1}{\sqrt{2}}( |0\rangle + e^{i\pi/4} |1\rangle) \). It has been shown that postCM = postBQP [13], and thus output probabilities are \#P-hard approximate up to some constant relative error [24–26]. However, if there is some independent depolarizing error acting on each input magic state, e.g., the input state on each register is \( (1 - \varepsilon) T |0\rangle + \varepsilon |1\rangle \), then Theorem 1 implies directly that there exists a classical algorithm to approximate the output probability up to \( l_1 \) norm \( \delta \) in time \( n^O(\log(1/\delta)/\epsilon) \) for a large fraction of the CM circuits with noisy inputs.

**Example 2**—IQP circuits have a simple structure with input states \( |0\rangle^{\otimes n} \) and gates of the form \( H^{\otimes n}DH^{\otimes n} \), where the diagonal gates in \( D \) are chosen from the gate set \( \{ Z, S, T, CZ \} \). It has been shown that postIQP = postBQP [9] and thus, the output probabilities are \#P-hard to approximate up to some constant relative error [24–26]. Also, if there is some depolarizing noise acting on each input state \( |0\rangle \), i.e., each input state...
is a mixed state \((1 - \epsilon)|0\rangle|0\rangle + \epsilon \frac{1}{2}\), then Theorem 1 implies that there exists a classical algorithm to approximate the output probability up to \(l_1\) norm \(\delta\) in time \(n^{O(\log(1/\delta)/\epsilon)}\) for a large fraction of such IQP circuits. (The proof is presented in Appendix B in detail, which depends on the output distribution of IQP circuits in Appendix C.)

**B. Pure nonstabilizer input states**

As we can see, the running time in Theorem 1 blows up if the input state \(\rho_i\) is pure. Here, we consider the case where all \(\rho_i\) are pure nonstabilizer states, that is Clifford gates with the input state \(|0\rangle^\otimes n\otimes |i\rangle\).

For pure states \(|\psi\rangle\), the stabilizer fidelity [27] is defined as follows

\[
F(\psi) = \max_{\langle \phi \rangle} |\langle \phi | \psi \rangle|^2,
\]

where the maximization is taken over all stabilizer states. Here, we define

\[
\mu(\psi) := 2(1 - F(\psi)).
\]

It is easy to see that \(\mu(\psi) = 0\) if \(|\psi\rangle\) is a stabilizer state. Thus, \(\mu\) quantifies the distance between a given state to the set of stabilizer states. Since each \(|\psi_i\rangle\) is not a stabilizer state, it follows that \(\mu(\psi_i) > 0\).

Next, let us introduce the Pauli rank for pure single qubit states \(|\psi\rangle\). First, we write a pure state \(|\psi\rangle\) in terms of its Bloch sphere representation \(|\psi\rangle = \frac{1}{2} \sum_{x \in \{0, 1\}} \psi_x X^x Z^x\), where \(\psi_0 = 1\) and \(|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 = 1\). We define the Pauli rank \(\chi(\psi)\) to be the number of nonzero coefficients \(\psi_x\). By the definition of Pauli rank, it is easy to see that \(2 \leq \chi(\psi) \leq 4\), and that \(|\psi\rangle\) is a stabilizer state iff \(\chi(\psi) = 2\). Since each input state \(|\psi_i\rangle\) is a nonstabilizer state, it follows that \(\chi(\psi_i) = 3\) or \(4\).

For example, for the magic state \(|T\rangle\), the corresponding Pauli rank \(\chi = 3\). For \(n\)-qubit systems, the Pauli rank serves as a good candidate for a magic monotone as it is easier to compute than other magic monotones which require a minimization over all stabilizer states [28–30]. (See a discussion of Pauli rank for \(n\)-qubit systems in Appendix D.)

**Theorem 3.** Given a Clifford circuit \(C\) on \(n + m\) qubits with input state \(|0\rangle^\otimes n \otimes |i\rangle\) and measurements on \(k\) qubits in the computational basis with \(k \leq n + m - \sum_{j=1}^{m} \log_2 (\chi(\psi_j)/2)\) and \(\chi(\psi_i)\) being the Pauli rank of \(\psi_i\), there exists a classical algorithm to approximate the output probability up to \(l_1\) norm \(\delta\) in time \((n + m)^{O(1)} m^{O(\log(1/\delta)/\mu)}\) for at least a \(1 - \frac{2}{3}\) fraction of Clifford circuits \(C\), where \(\mu := \min_i \mu(\psi_i)\) and \(\mu(\psi_i)\) is defined as (2).

The proof is presented in Appendix D. The maximal number of allowed measured qubits in this algorithm decreases with the amount of the magic in the input states, which is quantified by the Pauli rank. Curiously, the running time of this algorithm scales with the decrease in the amount of magic of the input states quantified by fidelity. This is contrary to the intuition that quantum circuits with more magic are harder to simulate. Similarly, if the quantum circuits are slightly beyond the Clifford circuits, for example, \(C = C_1 \circ V\) where the gates in \(C_1\) are Clifford gates in \(\text{Cliff}_m\) and \(V\) is some unitary gate in the third level of the Clifford Hierarchy \(\text{Cliff}_m^{(3)}\), then the result in Theorem 3 still holds.

Combining Theorem 1 and 3, we have the following corollary for any product input state:

**Corollary 4.** Let \(C\) be a Clifford circuit on \(n + m_1 + m_2\) qubits with input states \(|0\rangle^\otimes n \otimes |\rho_1\rangle \otimes |\rho_2\rangle\), where each \(\rho_i\) is a mixed state, and each \(|\psi_j\rangle\) is a pure nonstabilizer state. Assume that measurements are performed on \(k\) qubits in the computational basis, where \(k \leq n + m_1 + m_2 - \sum_{j=1}^{m_2} \log_2 (\chi(\psi_j)/2)\) and \(\chi(\psi_j)\) is the Pauli rank of \(\psi_j\). Then, there exists a classical algorithm to approximate the output probability with respect to the \(l_1\) norm \(\delta\) in time \((n + m_1 + m_2)^{O(1)} (m_1 + m_2)^{O(\log(\sqrt{m_1}/\delta))}\) for at least \(1 - \frac{2}{3}\) fraction of Clifford circuits \(C\), where \(\varepsilon = \min \{\lambda, \mu\}\) and \(\lambda := \min_i \lambda_i, \mu := \min_j \mu(\psi_j)\).

Now, let us apply our results to some restricted quantum computation models, such as Clifford circuits with solely magic state inputs (CM) and Pauli-based measurement (PBC), which gives an efficient simulation of \(O(n)\) measurement with high probability.

**Example 3**—Theorem 3 implies the following result: for Clifford circuit \(C\) with input states \(|T\rangle^\otimes n\) and measurement on \(k\) qubits in computational basis with \(k \leq 1 - \log_2(3/2)\) \(n \approx 0.415n\), there exists a classical algorithm to approximate the output probability up to \(l_1\) norm \(\delta\) in time \(n^{O((3+\sqrt{2})/\log(\sqrt{2}))}\) for at least \(1 - \frac{2}{3}\) fraction of Clifford circuits \(C\), where \(\mu(T) = 1 - 1/\sqrt{2}\) and \(\chi(T) = 3\). This may be contrasted with the hardness result ruling out efficient classical sampling from this class of circuits [13].

**Example 4**—A Pauli-Based Computation (PBC) is defined as a sequence of measurement of some Pauli operators \(P_i \in P^i\), where the measurement outcome is \((-1)^{\sigma_i}\) with \(\sigma_i \in \{0, 1\}\) and the Pauli operators \(\{P_i\}\) are commuting with each other. Here, the initial state is \(|T\rangle\) (or \(|H\rangle = \cos \frac{\pi}{8} |0\rangle + \sin \frac{\pi}{8} |1\rangle\), which is equivalent to \(|T\rangle\) up to Clifford unitary [31]). After \(k\) steps, the probability of outcome \(P(\sigma_1, \ldots, \sigma_k) = \langle T^\otimes n | \Pi P | T^\otimes n \rangle\), where \(\Pi = 2^{-k} \prod_{i=1}^{k} \{I + (-1)^{\sigma_i} P_i\}\). Note that PBC was considered in the fault-tolerant implementation of quantum computation based on stabilizer codes, where the stabilizer codes provide a simple realization of nondestructive Pauli measurements [32, 33]. Besides, it has been proved that the quantum computation based on Clifford+\(T\) circuits can be simulated by PBC [31]. Thus, this implies that the output probability \(P(\sigma_1, \ldots, \sigma_k)\) is #P-hard to simulate. It has been shown that any PBC on \(n\) qubits can be classically simulated in \(2^{m \cdot \text{poly}(n)}\) time with \(c \approx 0.94\) [31]. Here,
Theorem 3 implies that if the measurement steps \( k \leq (1 - \log_2(3/2))n \approx 0.415n \), then there exists a classical algorithm to approximate the output probability up to \( L_1 \) norm \( \delta \) in time \( n^{O(2+\sqrt{2})\log(1/\delta)} \) for a large fraction of PBC.

### III. CONCLUSION

In this work, we investigated the problem of evaluating the output probabilities of Clifford circuits with nonstabilizer input states. First, we provided an efficient classical algorithm to approximate the output probability of the Clifford circuits with mixed input states and showed that the running time scales with the increase in the purity of input states. Second, we showed that a modification of this algorithm gives an efficient classical simulation for pure nonstabilizer states, under some restriction on the number of measured qubits that is determined by the Pauli rank of the input states. The Pauli rank we introduced in this work can be regarded as a good candidate for a magic monotone. We showed that these two results have several implications in other restricted quantum computation models such as Clifford circuits with magic input states, Pauli-based computation and IQP circuits.

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Appendix A: Proof of Theorem 1

1. Efficient evaluation of Fourier coefficients

First, let us define the Fourier transformation on a single qubit state, inspired by \cite{22}. Given a single qubit state \( \rho \in D(\mathbb{C}^2) \), we can write it in terms of its Bloch sphere representation

\[
\rho = \frac{1}{2} (\rho_{00} I + \rho_{10} X + \rho_{01} Z + \rho_{11} XZ),
\]

where \( \rho_{00} = 1 \) and \( |\rho_{10}|^2 + |\rho_{01}|^2 + |\rho_{11}|^2 \leq 1 \).

Given \( a, b \in \mathbb{F}_2 \), it is easy to verify that

\[
X^b Z^a \rho Z^a X^b = \frac{1}{2} \sum_{s, t \in \mathbb{F}_2} (-1)^{sa + tb} \rho_{st} X^s Z^t.
\]

Thus, we can define the Fourier transformation on the state \( \rho \) as follows

\[
E_{a \in \mathbb{F}_2, b \in \mathbb{F}_2} X^b Z^a \rho Z^a X^b (-1)^{sa + tb} = \frac{1}{2} \rho_{st} X^s Z^t.
\]

Note that for \( t = s = 0 \), the above Fourier transformation is equal to the completely depolarizing channel. And the equation (A2) is the inverse Fourier transformation of (A3).

Given the input states \( |0\rangle|0\rangle \otimes_{i=1}^{m} \rho_i \) with Clifford unitary \( U \), the output probability \( q(\bar{y}) \) is

\[
q(\bar{y}) = \langle \bar{y}| U|0\rangle|0\rangle \otimes_{i=1}^{m} \rho_i U^\dagger \langle \bar{y}',
\]

for any \( \bar{y} \in \mathbb{F}_2^{m+1} \). Let us denote the Pauli operators \( Z^b := \otimes_{i=1}^{m} Z_i^0, X^b := \otimes_{i=1}^{m} X_i^b \) for any \( \bar{a}, \bar{b} \in \mathbb{F}_2^m \) to be operators acting on the latter \( m \) qubits. Now, let us insert \( X^b Z^a \) into the \( m \) mixed states as follows

\[
q_{\bar{a}, \bar{b}}(\bar{y}) = \langle \bar{y}| U|0\rangle|0\rangle \otimes_{i=1}^{m} (X^b Z^a \otimes_{i=1}^{m} \rho_i Z_i^a X_i^b) U^\dagger \langle \bar{y}' \rangle = \langle \bar{y}| U|0\rangle|0\rangle \otimes_{i=1}^{m} (\otimes_{i=1}^{m} X_i^b Z_i^a \rho_i Z_i^a X_i^b) U^\dagger \langle \bar{y}' \rangle.
\]

Hence, the output probability \( q(\bar{y}) = q_{\bar{a}, \bar{b}}(\bar{y}) \). Then, let us take the Fourier transformation with respect to \( \bar{a}, \bar{b} \) and the corresponding Fourier coefficient is

\[
\hat{q}_{\bar{a}, \bar{b}} := E_{\bar{a} \in \mathbb{F}_2^m, \bar{b} \in \mathbb{F}_2^m} \langle \bar{a}, \bar{b}| \hat{q}_{\bar{a}, \bar{b}}(\bar{y}) = (-1)^{\bar{a}, \bar{b}} \langle \bar{y}| U|0\rangle|0\rangle \otimes_{i=1}^{m} (\otimes_{i=1}^{m} \rho_i Z_i^a X_i^b) U^\dagger \langle \bar{y}' \rangle
\]

\[
= \langle \bar{y}| U|0\rangle|0\rangle \otimes_{i=1}^{m} (\otimes_{i=1}^{m} E_{\bar{a} \in \mathbb{F}_2^m, \bar{b} \in \mathbb{F}_2^m} X_i^b Z_i^a \rho_i Z_i^a X_i^b) U^\dagger \langle \bar{y}' \rangle.
\]

By equation (A3), we have

\[
\hat{q}_{\bar{a}, \bar{b}} = \langle \bar{y}| U|0\rangle|0\rangle \otimes_{i=1}^{m} X_i^b Z_i^a U^\dagger \langle \bar{y}' \rangle \cdot \prod_{i=1}^{m} \left( \frac{\rho_{i0}}{2} \right),
\]

where \( \rho_{i0} \) is the coefficient of \( \rho_i \) in the corresponding Bloch sphere representation. Since \( U \) is a Clifford unitary, then

\[
U|0\rangle|0\rangle \otimes_{i=1}^{m} X_i^b Z_i^a U^\dagger = \prod_{i=1}^{n} \left( \frac{I + P_i}{2} \right) \prod_{j=1}^{m} Q_{ij},
\]

where the Pauli operators \( P_i := UZ_i U^\dagger \) for \( 1 \leq i \leq n \) and \( P_j := UX_j U^\dagger \) for \( 1 \leq j \leq m \) and they are commuting with each other. Thus, by Gottesman-Knill Theorem, the Fourier coefficients \( \hat{q}_{\bar{a}, \bar{b}} \) can be evaluated in classical \( O((n + m)^3) \) time.

2. Exponential decay of Fourier coefficients

Since \( \rho \) is a mixed state in \( D(\mathbb{C}^2) \), it can always be written as \( \rho = (1 - \lambda) \sigma + \frac{\lambda}{2} I \), where \( \sigma \) is a pure state and \( \lambda = 1 - \sqrt{2 \text{Tr}[\rho^2]} - 1 \). The pure state \( \sigma \) also has the Bloch sphere representation

\[
\sigma = \frac{1}{2} (\sigma_{00} I + \sigma_{10} X + \sigma_{01} Z + \sigma_{11} XZ),
\]

where \( \sigma_{00} = 1 \) and \( |\sigma_{10}|^2 + |\sigma_{01}|^2 + |\sigma_{11}|^2 = 1 \). We have the following relationship between the coefficients \( \rho_{st} \) and \( \sigma_{st} \) for any \( s, t \in \mathbb{F}_2 \),
Lemma 5. Given a mixed state \( \rho = (1 - \lambda)\sigma + \frac{1}{2}I \), where \( \rho, \sigma \) has Bloch sphere representation given by (A1) and (A7) respectively, then we have

\[
\rho_{st} = (1 - \lambda)^{w(s,t)} \sigma_{st},
\]

for any \( s,t \in \mathbb{F}_2 \), where \( w(s,t) \) is defined as

\[
w(s,t) = \begin{cases} 
0, & s = 0, t = 0 \\
1, & \text{otherwise}
\end{cases}
\]

Proof. This is because

\[
\rho_{st} = \text{Tr} [X^s \rho Z^t] = (1 - \lambda) \text{Tr} [X^s \sigma Z^t] + \lambda / 2 \text{Tr} [X^s Z^t] = (1 - \lambda) \sigma_{st} + \lambda \delta_{s0} \delta_{t0} = (1 - \lambda)^{w(s,t)} \sigma_{st},
\]

where \( w(s,t) \) is defined as (A9).

Each mixed input state \( \rho_i \) can be written as \( \rho_i = (1 - \lambda_i)\sigma_i + \frac{1}{2}I \) where \( \sigma_i \) is a pure state. Consider the quantum circuit with input state \( |0\rangle \otimes_{i=1}^m \sigma_i \) and Clifford unitary \( U \), the output probability \( p(\bar{y}) \) is equal to

\[
p(\bar{y}) = \langle \bar{y}| U|0\rangle \otimes_{i=1}^m \sigma_i U^\dagger \langle \bar{y} |.
\]

Similar to \( q(\bar{y}) \), we insert \( X^s Z^t \) into the circuit and define \( p_{st}^{ab} \) as follows

\[
p_{st}^{ab}(\bar{y}) = \langle \bar{y}| U|0\rangle \otimes_{i=1}^m X^{s_i} Z^{t_i} \sigma_i X^{b_i} \otimes \sigma_i X^{b_i} U^\dagger \langle \bar{y} |.
\]

Then the corresponding Fourier coefficient can also be expressed as follows,

\[
\hat{p}_{s,t}^{a,b} = \langle \bar{y}| U|0\rangle \otimes_{i=1}^m X^{s_i} Z^{t_i} \sigma_i U^\dagger \langle \bar{y} | \cdot \prod_{i=1}^m \left( \frac{\sigma_{s,t}^{(i)}}{2} \right),
\]

where \( \sigma_{s,t}^{(i)} \) is the coefficient of \( \sigma_i \) in the corresponding Bloch sphere representation. By Lemma 5, it is easy to see that

\[
|\hat{q}_{s,t}^{a,b}| \leq (1 - \lambda)^{w(s,t)} |\hat{p}_{s,t}^{a,b}|,
\]

where \( \lambda = \min \lambda_i \) and \( w(s,t) \) is defined as

\[
w(s,t) := \sum_t w(s,t).
\]

3. Good approximation with respect to \( l_1 \) norm

The following lemma regarding Clifford unitaries on \( n \) qubits is necessary the proof.

Lemma 6 ([34]). The uniform distribution of Clifford unitaries on \( n \) qubits is an exact 2-design, that is, for any \( A, B, W \), we have

\[
\mathbb{E}_{U \sim \mathbb{C}_n} U^\dagger A U W U^\dagger B U = \int_{U(2^n)} dU U^\dagger A U W U^\dagger B U,
\]

where \( \mathbb{E}_{U \sim \mathbb{C}_n} := \frac{1}{|\mathbb{C}_n|} \sum_{U \sim \mathbb{C}_n} \) and

\[
\int_{U(2^n)} dU U^\dagger A U W U^\dagger B U = \frac{\text{Tr} [AB] \text{Tr} [W] I}{2^n} + \frac{2^n \text{Tr} [A] \text{Tr} [B] - \text{Tr} [AB]}{2^n (2^n - 1)} \left( W - \text{Tr} [W] I \right).
\]

Now, let us prove Theorem 1. Let us define

\[
\hat{q}_{s,t}^{a,b}(\bar{y}) = \begin{cases} 
\hat{q}_{s,t}^{a,b}(\bar{y}), & \text{if } w(s,t) \leq l \\
0, & \text{otherwise}
\end{cases}
\]

(A17)
which gives an family of unnormalized probability distribution \( \{ q'_{\vec{a}, \vec{b}} \} \) as \( q'_{\vec{a}, \vec{b}}(\vec{y}) = \sum_{\vec{x}} \hat{q}_{\vec{a}, \vec{y}}(\vec{x}) \) for each \( \vec{y} \in \mathbb{F}_2^{n+m} \). Then we show that \( q'_{0,0}(\vec{y}) \) gives a good approximation of \( q_{0,0}(\vec{y}) \) with respect to \( l_1 \) norm

\[
\| q'_{0,0} - q_{0,0} \|_1 = \sum_{\vec{y} \in \mathbb{F}_2^{n+m}} |q'_{0,0}(\vec{y}) - q_{0,0}(\vec{y})|
\]

for a large fraction of Clifford circuits. First, since \( \hat{q}_{\vec{a}, \vec{y}}(\vec{y}) \) depends on the Clifford unitaries \( U \), denote it as \( \hat{q}_{\vec{a}, \vec{y}}(\vec{y})[U] \), then it is easy to show that

\[
\hat{q}_{\vec{a}, \vec{y}}(\vec{y})[U](-1)\vec{a}^{\dagger} \vec{b}^{\dagger} = \hat{q}_{\vec{a}, \vec{y}}(\vec{y})[U'],
\]

where \( U' = U \circ Z^{\vec{a}}X^{\vec{b}} \) is also a Clifford unitary for any \( \vec{a}, \vec{b} \in \mathbb{F}_2^n \) and \( Z^{\vec{a}}X^{\vec{b}} \) act on the \( n+1, \ldots, n+m \) qubits. Thus

\[
\mathbb{E}_{U \sim \text{Cl}_{n+m}} \left[ \left\| q'_{\vec{a}, \vec{y}} - q_{\vec{a}, \vec{y}} \right\|_1^2 \right] = \mathbb{E}_{U \sim \text{Cl}_{n+m}} \left[ \left\| q'_{\vec{a}, \vec{y}} - q_{\vec{a}, \vec{y}} \right\|_1 \right] = \mathbb{E}_{U \sim \text{Cl}_{n+m}} \mathbb{E}_{\vec{a} \in \mathbb{F}_2^n, \vec{b} \in \mathbb{F}_2^n} \left[ \left\| q'_{\vec{a}, \vec{y}} - q_{\vec{a}, \vec{y}} \right\|_1 \right].
\]

Moreover,

\[
\mathbb{E}_{U \sim \text{Cl}_{n+m}} \left[ \left( q'_{\vec{a}, \vec{y}} - q_{\vec{a}, \vec{y}} \right)^2 \right] \leq \mathbb{E}_{U \sim \text{Cl}_{n+m}} \left[ \sum_{\vec{a} \in \mathbb{F}_2^n, \vec{b} \in \mathbb{F}_2^n} \left( q'_{\vec{a}, \vec{y}} - q_{\vec{a}, \vec{y}} \right)^2 \right] \]

\[
= 2^{n+m} \sum_{\vec{a} \in \mathbb{F}_2^n} \sum_{\vec{b} \in \mathbb{F}_2^n} \mathbb{E}_{U \sim \text{Cl}_{n+m}} \left[ \left( q'_{\vec{a}, \vec{y}}(\vec{y}) - q_{\vec{a}, \vec{y}}(\vec{y}) \right)^2 \right] \]

\[
= 2^{n+m} \sum_{\vec{a} \in \mathbb{F}_2^n} \sum_{\vec{b} \in \mathbb{F}_2^n} \left( q'_{\vec{a}, \vec{y}}(\vec{y}) - q_{\vec{a}, \vec{y}}(\vec{y}) \right)^2 \leq 2^{n+m}(1 - \lambda)^{2l} \sum_{\vec{y} \in \mathbb{F}_2^{n+m}} \hat{p}_{\vec{a}, \vec{y}}^2(\vec{y}) \]

\[
\leq 2^{n+m}(1 - \lambda)^{2l} \sum_{\vec{y} \in \mathbb{F}_2^{n+m}} \hat{p}_{\vec{a}, \vec{y}}^2(\vec{y}) = 2^{n+m}(1 - \lambda)^{2l} \sum_{\vec{y} \in \mathbb{F}_2^{n+m}} \mathbb{E}_{U \sim \text{Cl}_{n+m}} \left[ \left( q'_{\vec{a}, \vec{y}}(\vec{y}) - q_{\vec{a}, \vec{y}}(\vec{y}) \right)^2 \right].
\]

where the first line comes from the Cauchy-Schwarz inequality, the third line comes from the Parseval identity, and the fourth line comes from the fact that \( |\hat{q}_{\vec{a}, \vec{y}}(\vec{y})| \leq (1 - \lambda)^{w(\vec{a}, \vec{y})} |\hat{p}_{\vec{a}, \vec{y}}(\vec{y})| \). According to Lemma 6, we have

\[
\mathbb{E}_{U \sim \text{Cl}_{n+m}} p_{\vec{a}, \vec{y}}^2(\vec{y}) \leq 2 \cdot 2^{-2(n+m)}.
\]

Thus

\[
\mathbb{E}_{U \sim \text{Cl}_{n+m}} \left[ \left\| q'_{0,0} - q_{0,0} \right\|_1^2 \right] \leq 2 \alpha e^{-\lambda l}.
\]

By Markov’s inequality, we have

\[
\mathbb{P}(U \sim \text{Cl}_{n+m} \left[ \left\| q'_{0,0} - q_{0,0} \right\|_1 \leq \sqrt{\alpha} e^{-\lambda l} \right] \geq 1 - \frac{2}{\alpha}.
\]

Therefore, to obtain the \( l_1 \) norm up to \( \delta \), we need take \( l = O(\log(\sqrt{\alpha}/\delta)/\lambda) \) and evaluate the Fourier coefficients \( \hat{q}_{\vec{a}, \vec{y}}(\vec{y}) \) with \( w(\vec{a}, \vec{y}) \leq l \), where total amount of such Fourier coefficients is \( \sum_{l \leq 3l} C_{l,m} \leq 3l' \). Thus, there exists a classical algorithm to approximate each output probability \( q(\vec{y}) \) in time \( O((n+m)^3)l' \leq (n+m)O(1)mO(\log (\sqrt{\alpha}/\delta)/\lambda) \) with \( l_1 \) norm less than \( \delta \) for at least \( 1 - \frac{2}{\alpha} \) fraction of Clifford circuits. Thus, we finish the proof of Theorem 1.

4. Slightly beyond Clifford circuits

Now, let us consider the quantum circuit \( C = C_1 \circ V \) with input state \( |0\rangle^m \otimes_{i=1}^m p_i \) and the gates in circuits \( C_1 \) taken from the set of Clifford gates on \( n \) qubits \( Cl_{n+m} \) and \( V \) is taken from the third level of Clifford hierarchy \( Cl^{(3)}_{m} \) acting on \( n+1, \ldots, (n+m) \)th
qubits. The proof of Corollary 2 is almost the same as that of Theorem 1. We only need to show the corresponding Fourier coefficients of $q_{\tilde{a}, \tilde{b}}$ also can be evaluated in $O((n + m)^3)$ time, where

$$q_{\tilde{a}, \tilde{b}}(\vec{y}) = \langle \vec{y} \mid UV \ket{0} \ket{0}^{\otimes n} \otimes (X^{\tilde{b}}Z^\alpha \otimes \rho_i Z^\alpha \chi_i) V^\dagger \mid \vec{y} \rangle = \langle \vec{y} \mid UV \ket{0} \ket{0}^{\otimes n} \otimes (\otimes_{i=1}^{m} X^{\tilde{b}} Z^\alpha \rho_i Z^\alpha X^{\tilde{b}}) U^\dagger V^\dagger \mid \vec{y} \rangle. \quad (A21)$$

and $V \in Cl_m^{(3)}$, $U \in Cl_{n+m}$. Then the Fourier coefficient $\hat{q}_{EF}(\vec{y})$ is equal to

$$\hat{q}_{EF} = \langle \vec{y} \mid UV \ket{0} \ket{0}^{\otimes n} \otimes \rho_i Z^\alpha U^\dagger \mid \vec{y} \rangle \cdot \prod_{i=1}^{m} \left( \frac{\rho_{sh}(i)}{2} \right). \quad (A22)$$

Since $V \in Cl_m^{(3)}$, then $V \otimes^{m}_{i=1} X^\alpha Z^\alpha U^\dagger \in Cl_m$. Thus,

$$\hat{q}_{EF} = \langle \vec{y} \mid U \ket{0} \ket{0}^{\otimes n} \otimes \rho_i Z^\alpha U^\dagger \mid \vec{y} \rangle \cdot \prod_{i=1}^{m} \left( \frac{\rho_{sh}(i)}{2} \right).$$

where $U, U' = V \otimes^{m}_{i=1} X^\alpha Z^\alpha U^\dagger$ are both Clifford unitaries. Thus, the Fourier coefficient $\hat{q}_{EF}$ can also be evaluated in $O((n + m)^3)$ time by Gottesman-Knill Theorem. Therefore, it is easy to prove Corollary 2 by following the proof of Theorem 1.

Appendix B: Efficient classical simulation of IQP circuits with noisy input states

In this section, we will prove the following proposition in Example 2:

**Proposition 7.** Given an IQP circuit $H^{\otimes n}DH^{\otimes n}$ with the diagonal unitaries chosen from the gate set $\{CZ, Z, S, T\}$, if there is depolarizing noise acting on each input state, i.e., input state is $(1 - \varepsilon)\ket{0} + \frac{\varepsilon}{4}I^{\otimes n}$, then there exists an efficient classical algorithm to approximate the output probabilities up to $l_1$ norm $\delta$ in time $n^{O(\log(1/\delta)/\varepsilon)}$ for at least $1 - \frac{2}{n}$ fraction of IQP circuits.

**Proof.** The proof is similar to that of Theorem 1. If the state $\rho$ has some specific form as $\rho = \frac{1}{2}(\rho_0 I + \rho_1 Z)$, then we can simplify the Fourier transformation (A3) as

$$E_{a \in \mathbb{F}_2} X^a \rho X^a (-1)^{aa} = \frac{1}{2}\rho Z^a. \quad (B1)$$

Given an IQP circuit $H^{\otimes n}DH^{\otimes n}$ with noisy input states $\rho^{\otimes n}$, $\rho = (1 - \varepsilon)\ket{0} + \frac{\varepsilon}{4}I$, and gates in $D$ chosen from the gate set $\{CZ, Z, S, T\}$, then the output probability $q(\vec{y})$ is equal to

$$q(\vec{y}) = \langle \vec{y} \mid H^{\otimes n}DH^{\otimes n} X^\alpha \rho^{\otimes n} X^\alpha H^{\otimes n} DH^{\otimes n} \mid \vec{y} \rangle. \quad (B2)$$

Similar to the proof of Theorem 1, we insert $X^\alpha$ into the circuits for any $\vec{a} \in \mathbb{F}_2^n$ and define $q_{\vec{a}}(\vec{y})$ as follows

$$q_{\vec{a}}(\vec{y}) = \langle \vec{y} \mid H^{\otimes n}DH^{\otimes n} X^\alpha \rho^{\otimes n} X^\alpha H^{\otimes n} DH^{\otimes n} \mid \vec{y} \rangle = \langle \vec{y} \mid H^{\otimes n}DH^{\otimes n} \otimes X^\alpha \rho X^\alpha H^{\otimes n} DH^{\otimes n} \mid \vec{y} \rangle \quad (B3)$$

Then let us take the Fourier transformation with respect to $\vec{a}$ and the corresponding Fourier coefficient is

$$\hat{q}_{\vec{a}}(\vec{y}) := E_{\vec{a} \in \mathbb{F}_2^n} q_{\vec{a}}(\vec{y}) (-1)^{\vec{a} \cdot \vec{y}} = E_{\vec{a} \in \mathbb{F}_2^n} \langle \vec{y} \mid H^{\otimes n}DH^{\otimes n} \otimes X^\alpha \rho X^\alpha H^{\otimes n} DH^{\otimes n} \mid \vec{y} \rangle \langle \vec{y} \mid (-1)^{\vec{a} \cdot \vec{y}} = \langle \vec{y} \mid H^{\otimes n}DH^{\otimes n} \otimes X^\alpha (E_{\vec{a} \in \mathbb{F}_2^n} \rho X^{\alpha a} (-1)^{aa}) H^{\otimes n} DH^{\otimes n} \mid \vec{y} \rangle = \langle \vec{y} \mid H^{\otimes n}DH^{\otimes n} \otimes Z^\alpha H^{\otimes n} DH^{\otimes n} \mid \vec{y} \rangle \prod_{i=1}^{n} \left( \frac{P_i}{2} \right) = \langle \vec{y} \mid H^{\otimes n}DH^{\otimes n} \otimes Z^\alpha H^{\otimes n} DH^{\otimes n} \mid \vec{y} \rangle \prod_{i=1}^{n} \left( \frac{1 - \varepsilon^{\alpha_i}}{2} \right), \quad (B4)$$

where the second last equality comes from (B1).

Besides,

$$DH^{\otimes n} \otimes Z^\alpha H^{\otimes n} D^* = D \otimes Z^\alpha D^* = D' \otimes Z^\alpha X^{\alpha} D'^* \quad (B5)$$
where the diagonal part $D$ can be written as $D' \circ \otimes_{i=1}^{n} T_i^{x_i}$ with $y_i \in \mathbb{F}_2$ and the gates in $D'$ chosen from the gate set $\{ CZ, Z, S \}$. It is easy to verify that

$$T_i^{x_i} T_i^{-y_i} = e^{-i\frac{\pi}{2} y_i S_i H_i X_i^{x_i}},$$

(B6)

for any $y_i, s_i \in \{0, 1\}$. That is, $DH_i \otimes_{i=1}^{n} T_i^{x_i} H_i D_i^{x_i}$ is a Clifford circuit. Thus, each Fourier coefficient can be evaluated in $O(n^3)$ by Gottesman-Knill Theorem.

We also consider the same IQP circuits with input states $|0\rangle^{\otimes n}$, then output probability $p(y) = \langle y| H_i^{\otimes n} D_i^{x_i} H_i^{\otimes n} |0\rangle^{\otimes n}$. Similarly, we insert the operator $X_i^y$ as follows

$$p_d(y) = \langle y| H_i^{\otimes n} D_i^{x_i} H_i^{\otimes n} X_i^y H_i^{\otimes n} |0\rangle^{\otimes n}.$$

(B7)

And the corresponding Fourier coefficient is

$$\hat{p}_d(y) := E_{\bar{a} \in \mathbb{F}_2^n} p_d(y) = \langle y| H_i^{\otimes n} D_i^{x_i} H_i^{\otimes n} Z_i^{x_i} H_i^{\otimes n} |0\rangle^{\otimes n} \cdot 2^{-n}.$$

(B8)

Comparing (B4) with (B8), we have the following relation

$$\hat{q}_d(y) = (1 - \epsilon)^{|y|} \hat{p}_d(y),$$

(B9)

where $|\bar{a}| = \sum_i s_i$ is the Hamming weight of $\bar{a} \in \mathbb{F}_2^n$.

Let us define

$$\hat{q}_d(y) = \begin{cases} \hat{q}_d(y), & |\bar{a}| \leq l, \\ 0, & \text{otherwise}, \end{cases}$$

(B10)

which gives an family of unnormalized probability distribution $\{ q_d(y) \}$ as $q_d(y) = \sum \hat{q}_d(y) (-1)^{\bar{a} \cdot \bar{y}}$ for each output $y \in \mathbb{F}_2^n$. Then we will show that $q_d(y) \approx q_0(y)$ gives a good approximation of $q_0(y)$ with respect to $L_1$ norm

$$\left\| q_d(y) - q_0(y) \right\|_1 = \sum_{y \in \mathbb{F}_2^n} |q_d(y) - q_0(y)|$$

for a large fraction of IQP circuits. We denote $D_n$ to be the set of of diagonal part of IQP circuits where the diagonal gates are chosen from $\{ CZ, Z, S, T \}$. Since $\hat{q}_d(y)$ depends on the IQP circuits, denote it as $\hat{q}_d(y) [D]$, then it is easy to verify that

$$\hat{q}_d(y) [D] (-1)^{\bar{a} \cdot \bar{y}} = \hat{q}_d(y) [D'],$$

where $D' = D \circ Z_i^x$ also belongs to $D_n$. Thus

$$E_{\bar{a} \in \mathbb{F}_2^n} \left\| q_d(y) - q_0(y) \right\|_1^2 = E_{\bar{a} \in \mathbb{F}_2^n} \left\| q_d(y) - q_0(y) \right\|_1^2 = E_{\bar{a} \in \mathbb{F}_2^n} E_{\bar{a} \in \mathbb{F}_2^n} \left\| q_d(y) - q_0(y) \right\|_1^2 .$$

And

$$E_{\bar{a} \in \mathbb{F}_2^n} \left\| q_d(y) - q_0(y) \right\|_1^2 \leq E_{\bar{a} \in \mathbb{F}_2^n} \left\| q_d(y) - q_0(y) \right\|_1^2 = 2^n \sum_{\bar{a} \in \mathbb{F}_2^n} (q_d(y) - q_0(y))^2$$

$$ \leq 2^n (1 - \epsilon)^2 \sum_{\bar{a} \in \mathbb{F}_2^n} \sum_{|\bar{a}| \geq l} p_d^2(y)$$

where the first line comes from the Cauchy-Schwarz inequality, the third line comes from Parvesal identity, and the fourth line comes from the fact that $\hat{q}_d(y) = (1 - \epsilon)^{|y|} \hat{p}_d(y)$. According to Lemma 8 in Appendix C, we have

$$E_{\bar{a} \in \mathbb{F}_2^n} \sum_{\bar{a} \in \mathbb{F}_2^n} p_d^2(y) \leq 2^{-(n-1)} .$$
Thus, we have

\[ \mathbb{E}_{D \sim D_n} \left\| q_0^e - q_0 \right\|_1^2 \leq 2e^{-2\ell}. \]

Therefore, by Markov’s inequality, we have

\[ \Pr_{D \sim D_n} \left[ \left\| q_0^e - q_0 \right\|_1 \leq \sqrt{\alpha}e^{-\ell} \right] \geq 1 - \frac{\alpha}{2}. \]

Therefore, to obtain the \( l_1 \) norm up to \( \delta \), we need take \( l = O(\log(\sqrt{\alpha}/\delta)/\epsilon) \) and the total computational complexity is \( O(n^3n^l) = n^{O(\log(\sqrt{\alpha}/\delta)/\epsilon)} \). □

Appendix C: Distribution of IQP circuits based on Gowers uniformity norm

Here we consider IQP circuits, which can be represented by \( H^\otimes n \hat{DH}^\otimes n \ket{0}^\otimes n \), where the gates in the diagonal part \( D \) are chosen from the gate set \( \{ CZ, Z, S \} \). Then the output distribution is \( p(y) = \langle y | H^\otimes n \hat{DH}^\otimes n \ket{0}^\otimes n \rangle^2 = | \tilde{f}(y) |^2 \) for any \( y \in \mathbb{F}_2^n \), where \( \tilde{f}(y) = \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} f(x)(-1)^{x \cdot y} \) and the function \( f \) can be expressed as

\[ f(x) = (-1)^{\sum_{i<j} \alpha_{ij} x_i x_j + \sum_i \beta_i x_i + \sum_{i,j} \gamma_{ij} x_i x_j e^{\pi i/4} \sum_i t_i}, \quad (C1) \]

where \( \alpha_{ij}, \beta_i, \gamma_i, t_i \in \mathbb{F}_2 \), denote the number of CZ between \( i \)th and \( j \)th qubits, \( Z \) gate on \( i \)th qubit, \( S \) gate on \( i \)th gate and \( T \) gate on \( i \)th gate. Since \( T^2 = S, S^2 = Z \) and \( Z^2 = I \), then there are at most one \( T, S, Z \) gate on each qubit respectively. Thus, \( \beta, \gamma, t \in \mathbb{F}_2^n \) and the Hamming weight \( |\beta|, |\gamma|, |t| \) is the number of \( Z, S \) and \( T \) gates in the IQP circuit.

In fact, the function \( f \) can be rewritten as follows

\[ f(x) = (-1)^{\sum_i A_i x_i} e^{\pi i/4 \sum_i t_i}, \quad (C2) \]

where \( A_{ii} = \gamma_i \) and \( A_{ij} = A_{ji} = \alpha_{ij} \) for \( i \neq j \). That is, the matrix \( A \) is a symmetric \( 0-1 \) matrix.

Now, let us introduce the Gowers uniformity norm here. Let \( G \) be a finite additive group and \( f : G \to \mathbb{C} \) and an integer \( d \geq 1 \). Then the Gowers uniformity norm \( \| f \|_{U^d(G)} \) [35] is defined as

\[ \| f \|_{U^d(G)}^d = \mathbb{E}_{h_0, \ldots, h_d \in G} \Delta_{h_0} \ldots \Delta_{h_d} f(x), \quad (C3) \]

where \( \Delta_h f(x) := f(x + h) \overline{f(x)} \). Here we take \( G = \mathbb{F}_2^n \) and the Fourier transformation for \( f : \mathbb{F}_2^n \to \mathbb{C} \) is defined as \( \hat{f}(y) = \mathbb{E}_{x \in \mathbb{F}_2^n} f(x)(-1)^{x \cdot y} \), where \( \mathbb{E}_{x \in \mathbb{F}_2^n} := \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} \). One important property of Gowers uniformity norm, which we will use in the following section to demonstrate the distribution of IQP circuits, is the following equality [35]

\[ \| f \|_{U^d([2^n])}^4 = \sum_{y \in \mathbb{F}_2^n} | \hat{f}(y) |^4. \quad (C4) \]

For IQP circuits with diagonal gates chosen from \( \{ CZ, Z, CCZ \} \) randomly, it has been proved that the average value of the second moment of output probability satisfies that \( \sum_y p_D^2(y) \leq \alpha 2^{-n} \), where \( \alpha \) is some constant [36]. Here, we consider the case where the gates in the diagonal part \( D \) are chosen uniformly, i.e., \( P(\alpha_{ij} = 1) = P(\beta_i = 1) = P(\gamma_i = 1) = P(t_i = 1) = 1/2 \), then we can give the exact value of average value of the second moment of the output probability of random IQP circuits.

**Lemma 8.** Given an IQP circuit, if the gates in the diagonal part \( D \) can be chosen uniformly, then

\[ \mathbb{E}_D \sum_{y \in \mathbb{F}_2^n} p_D^2(y) = 2^{-(n-1)} - 2^{-2n}. \quad (C5) \]

**Proof.** Due to the equation (C4), we have

\[ \sum_{y \in \mathbb{F}_2^n} | p(y) |^2 = \sum_{y \in \mathbb{F}_2^n} | \hat{f}(y) |^4 = \| f \|_{U^2([2^n])}^4. \quad (C6) \]

For the function \( f(x) = (-1)^{\beta \cdot x} e^{\pi i/4 \beta \cdot x} \), the Gowers uniformity norm \( \| f \|_{U^2([2^n])} \) can be expressed as follows
\[ \|f\|_{L^2(\mathbb{F}_2^n)}^{2} = \mathbb{E}_{\tilde{a}, \tilde{B}, \tilde{B}' \in \mathbb{F}_2^n} f(\tilde{a} \oplus \tilde{B}) f(\tilde{a} \oplus \tilde{B}') = \mathbb{E}_{\tilde{a}, \tilde{B}, \tilde{B}' \in \mathbb{F}_2^n} 2^{\tilde{a} \cdot \tilde{B}} e^{i\pi/4 \sum_{t} [(x_t \oplus a_t) + x_t - (x_t \oplus a_t)]} = \mathbb{E}_{\tilde{a}, \tilde{B}, \tilde{B}' \in \mathbb{F}_2^n} (-1)^{\tilde{a} \cdot \tilde{B}} e^{i\pi/4 \sum_{t} [(x_t \oplus a_t) + x_t - (x_t \oplus a_t)]} = \mathbb{E}_{\tilde{a}, \tilde{B}, \tilde{B}' \in \mathbb{F}_2^n} (-1)^{\tilde{a} \cdot \tilde{B}} \prod_{t=1}^{n} \mathbb{E}_{x_t \in \mathbb{F}_2} e^{i\pi/4 [x_t \oplus a_t + x_t - (x_t \oplus a_t)]} \] 

It is easy to verify that
\[ \mathbb{E}_{x \in \mathbb{F}_2} e^{i\pi/4 [(x \oplus a) + x - (x \oplus a)]} = \frac{1 + (-1)^{a \cdot b}}{2}, \] 
for any \( t, a, b \in \mathbb{F}_2 \). Thus, we have
\[ \|f\|_{L^2(\mathbb{F}_2^n)}^{2} = \mathbb{E}_{\tilde{a}, \tilde{B}, \tilde{B}' \in \mathbb{F}_2^n} (-1)^{\tilde{a} \cdot \tilde{B}} \prod_{t=1}^{n} \left[ \frac{1 + (-1)^{a \cdot b_t}}{2} \right]. \] 

The expected value of \( \sum_{p \in \mathbb{F}_2} P_{\mathcal{D}}^p(\mathcal{Y}) \) over the random IQP circuits is
\[ \mathbb{E}_{\mathcal{D}} \sum_{p \in \mathbb{F}_2} P_{\mathcal{D}}^p(\mathcal{Y}) = \mathbb{E}_{\mathcal{D}} \|f_{\mathcal{D}}\|_{L^2(\mathbb{F}_2^n)}^{2} = \mathbb{E}_{\{a_i, b_i, \gamma_i, \tau_i\}} \mathbb{E}_{\tilde{a}, \tilde{B}, \tilde{B}' \in \mathbb{F}_2^n} (-1)^{\tilde{a} \cdot \tilde{B}} \prod_{t=1}^{n} \left[ \frac{1 + (-1)^{a \cdot b_t}}{2} \right] = \mathbb{E}_{\tilde{a}, \tilde{B}, \tilde{B}' \in \mathbb{F}_2^n} \prod_{t \neq j} \left[ \frac{1 + (-1)^{a \cdot b_t + b \cdot a_j}}{2} \right] \prod_{t=1}^{n} \left[ \frac{1 + (-1)^{a \cdot b_t}}{2} \right] \left[ \frac{3 + (-1)^{a \cdot b_j}}{4} \right]. \] 

Since
\[ \frac{1 + (-1)^{a \cdot b_t}}{2} = \begin{cases} 
0, & (a_t, b_t) = (1, 1), \\
1, & \text{otherwise}, 
\end{cases} \] 
then the above equation is equal to
\[ \frac{1}{4^n} \sum_{(a_t, b_t) \in \{(0, 0), (0, 1), (1, 0)\}} \prod_{t \neq j} \left[ \frac{1 + (-1)^{a_t \cdot b_t + b_t \cdot a_j}}{2} \right] = \frac{1}{4^n} \sum_{(a_t, b_t) \in \{(0, 0), (0, 1), (1, 0)\}} \prod_{t \neq j} \left[ \frac{1 + (-1)^{a_t \cdot b_t + b_t \cdot a_j}}{2} \right], \] 
where the equality comes from the fact that
\[ a_t b_j + b_t a_j = (a_i + a_j)(b_i + b_j) - (a_t b_i + a_i b_j) = (a_i + a_j)(b_t + b_j), \] 
when \((a_i, b_i), (a_j, b_j)\) are chosen from \(\{(0, 0), (0, 1), (1, 0)\}\). Moreover, for \((a_t, b_t), (a_j, b_j) \in \{(0, 0), (0, 1), (1, 0)\}\), we have
\[ \frac{1 + (-1)^{(a_t + a_j)(b_t + b_j)}}{2} = \begin{cases} 
0, & (a_t, b_t, a_j, b_j) = (1, 0, 0, 1), (0, 1, 1, 0) \\
1, & \text{otherwise} \end{cases} \] 
Thus,
\[ \sum_{(a_t, b_t) \in \{(0, 0), (0, 1), (1, 0)\}} \prod_{t \neq j} \left[ \frac{1 + (-1)^{(a_t + a_j)(b_t + b_j)}}{2} \right] = \left( \sum_{(a_t, b_t) \in \{(0, 0), (0, 1)\}} 1 \right) + \left( \sum_{(a_t, b_t) \in \{(0, 0), (1, 0)\}} 1 \right) - 1 = 2^{n+1} - 1. \]
Therefore, we obtain the result that

\[ \mathbb{E}_D \sum_s p^2_D(s) = \frac{1}{4^n} \left( 2^n + 1 \right). \]

Besides, based on the Gowers uniformity norm, we can also give an approximation of the second moment for any IQP circuit.

**Proposition 9.** Given an IQP circuit with the diagonal gates chosen from \( \{ CZ, Z, S, T \} \), then the output probability of this circuit satisfies the following property,

\[ \sum_{y \in \mathbb{F}_2^n} p^2(y) \leq 2^{-c|\vec{t}| - \text{Rank}(A(\vec{t}))}, \tag{C8} \]

where the constant \( c = \log 4 > 0 \), \( A(\vec{t}) \) is the matrix obtained from \( A \) by removing the rows and columns \( i \) such that \( t_i = 1 \) and \( \text{Rank}(A(\vec{t})) \) denotes the rank of the matrix \( A(\vec{t}) \) in \( \mathbb{F}_2 \). Moreover, if \( \vec{t} = 0 \), i.e., there is no \( T \) gate, then

\[ \sum_{y \in \mathbb{F}_2^n} p^2(y) \leq 2^{-\text{Rank}(A)}. \tag{C9} \]

**Proof.** Due to the equation (C4) and Lemma 8, we have

\[ \sum_{y \in \mathbb{F}_2^n} p^2(y) = \sum_{y \in \mathbb{F}_2^n} |\hat{\beta}(\vec{x})|^4 = ||f||^4_{U^2(\mathbb{F}_2^n)} = \mathbb{E}_{\vec{a},\vec{b} \in \mathbb{F}_2^n} (-1)^{\vec{a}A\vec{b}} \prod_{i=1}^n \left[ 1 + (-1)^{a_i b_i} \right]. \]

Thus, we need estimate the Gower uniform norm \( ||f||_{U^2(\mathbb{F}_2^n)} \) for the phase polynomial \( f(\vec{x}) = (-1)^{\vec{b} \cdot \vec{a} \cdot \vec{x}} e^{i \pi / 4 |\vec{t}|} \) by the Hamming weight \( |\vec{t}| \) and the rank of the symmetric matrix \( A \).

Without loss of generality, we assume the first \( k = |\vec{t}| \) qubits have \( T \) gates, i.e., \( t_1 = \ldots = t_k = 1 \), and the remaining qubits do not have \( T \) gate, then we can decompose the symmetric matrix \( A \) as follows

\[ A = \begin{bmatrix} A_{k,k} & A_{k,n-k} \\ A_{n-k,k} & A_{n-k,n-k} \end{bmatrix}, \]

where \( A_{k,k} \) is a \( k \times k \) symmetric matrix, \( A_{n-k,n-k} \) is an \( (n-k) \times (n-k) \) symmetric matrix and \( A_{n-k,k} = A_{k,n-k}^T \). Similarly, we also decompose the vectors \( \vec{a}, \vec{b} \) as

\[ \vec{a} = \begin{bmatrix} \vec{a}_k \\ \vec{a}_{n-k} \end{bmatrix}, \vec{b} = \begin{bmatrix} \vec{b}_k \\ \vec{b}_{n-k} \end{bmatrix}, \]

where \( \vec{a}_k, \vec{b}_k \in \mathbb{F}_2^k \) and \( \vec{a}_{n-k}, \vec{b}_{n-k} \in \mathbb{F}_2^{n-k} \). Thus,

\[ ||f||^4_{U^2(\mathbb{F}_2^n)} = \mathbb{E}_{\vec{a},\vec{b} \in \mathbb{F}_2^n} (-1)^{\vec{a}A\vec{b}} \prod_{i=1}^n \left[ 1 + (-1)^{a_i b_i} \right] \]

\[ = \mathbb{E}_{\vec{a}_k,\vec{b}_k \in \mathbb{F}_2^k} (-1)^{\vec{a}_kA_k\vec{b}_k} \prod_{i=1}^k \left[ 1 + (-1)^{a_{ki} b_{ki}} \right] \mathbb{E}_{\vec{a}_{n-k},\vec{b}_{n-k} \in \mathbb{F}_2^{n-k}} (-1)^{\vec{a}_{n-k}A_{n-k,k} \vec{b}_{n-k} + \vec{a}_{n-k}A_{n-k,n-k} \vec{b}_{n-k} + \vec{a}_{n-k}A_{n-k,n-k} \vec{b}_{n-k} + \vec{a}_{n-k}A_{n-k,n-k} \vec{b}_{n-k}}. \]

Since

\[ \frac{1 + (-1)^{a_i b_i}}{2} = \begin{cases} 0, & (a_i, b_i) = (1, 1) \\ 1, & \text{otherwise} \end{cases}, \tag{C10} \]
then

$$\left| \mathbb{E}_{\vec{a}_k \vec{b}_k \in \mathbb{F}_2^n} (-1)^{\vec{a}_k \vec{A}_k \vec{B}_k} \left[ \prod_{i=1}^k \frac{1 + (-1)^{\vec{a}_i \vec{b}_i}}{2} \right] \mathbb{E}_{\vec{a}_n-k \vec{b}_n-k \in \mathbb{F}_2^n} (-1)^{\vec{a}_n-k \vec{A}_n-k \vec{B}_n-k + \vec{a}_n-k \vec{A}_n-k \vec{B}_n-k} \right|$$

$$\leq \frac{1}{4^k} \sum_{(a_i,b_i) \in \{(0,0),(0,1),(1,0),(1,1)\}^k} (-1)^{\vec{a}_k \vec{A}_k \vec{B}_k} \left[ \prod_{i=1}^k \frac{1 + (-1)^{\vec{a}_i \vec{b}_i}}{2} \right] \mathbb{E}_{\vec{a}_n-k \vec{b}_n-k \in \mathbb{F}_2^n} (-1)^{\vec{a}_n-k \vec{A}_n-k \vec{B}_n-k + \vec{a}_n-k \vec{A}_n-k \vec{B}_n-k} \right|$$

$$\leq \frac{1}{4^k} \sum_{(a_i,b_i) \in \{(0,0),(0,1),(1,0),(1,1)\}^k} \left| \mathbb{E}_{\vec{a}_n-k \vec{b}_n-k \in \mathbb{F}_2^n} (-1)^{\vec{a}_n-k \vec{A}_n-k \vec{B}_n-k + \vec{a}_n-k \vec{A}_n-k \vec{B}_n-k} \right|$$

$$\leq \left( \frac{3}{4} \right)^k \max_{\vec{x},\vec{y} \in \mathbb{F}_2^{n-k}} \left| \mathbb{E}_{\vec{a}_n-k \vec{b}_n-k \in \mathbb{F}_2^n} (-1)^{\vec{a}_n-k \vec{A}_n-k \vec{B}_n-k + \vec{a}_n-k \vec{A}_n-k \vec{B}_n-k + \vec{x} \vec{b}_n-k} \right|$$

Besides, for any $\vec{x},\vec{y} \in \mathbb{F}_2^{n-k}$,

$$\left| \mathbb{E}_{\vec{a}_n-k \vec{b}_n-k \in \mathbb{F}_2^n} (-1)^{\vec{a}_n-k \vec{A}_n-k \vec{B}_n-k + \vec{x} \vec{b}_n-k} \right|$$

$$= \left| \mathbb{E}_{\vec{a}_n-k \vec{b}_n-k \in \mathbb{F}_2^n} (-1)^{\vec{y} \vec{a}_n-k \vec{B}_n-k} \right|$$

$$= \left| \mathbb{E}_{\vec{a}_n-k \vec{b}_n-k \in \mathbb{F}_2^n} \delta_{\vec{a}_n-k \vec{b}_n-k} \right|$$

$$\leq \left\{ \vec{a}_n-k : A_{n-k,n-k} \vec{a}_n-k = \vec{x} \right\}$$

$$\leq \frac{1}{2^{n-k}} \left| \ker(A_{n-k,n-k}) \right|$$

$$= \frac{1}{2^{\text{rank}(A_{n-k,n-k})}}$$

where $\text{rank}(A_{n-k,n-k})$ denotes the rank of the matrix $A_{n-k,n-k}$ in $\mathbb{F}_2$. Therefore,

$$\left\| f \right\|_{U^2(\mathbb{F}_2)}^4 \leq \left( \frac{3}{4} \right)^k \frac{1}{2^{\text{rank}(A_{n-k,n-k})}} = 2^{-ck-\text{rank}(A_{n-k,n-k})},$$

where $c = \log \frac{4}{3}$.

Moreover, if $\vec{f} = 0$, then

$$\left\| f \right\|_{U^2(\mathbb{F}_2)}^4 = \mathbb{E}_{\vec{a} \in \mathbb{F}_2^n} \mathbb{E}_{\vec{b} \in \mathbb{F}_2^n} (-1)^{\vec{a} \vec{b}} \mathbb{E}_{\vec{a} \in \mathbb{F}_2^n} \delta_{\vec{a} \vec{b}} = \mathbb{E}_{\vec{a} \in \mathbb{F}_2^n} \delta_{\vec{a} \vec{b}} = \frac{\text{ker}(A)}{2^n} = 2^{-\text{rank}(A)}.$$

\[ \square \]

**Appendix D: Efficient classical simulation with pure nonstabilizer input states**

1. **Proof of Theorem 3**

**Lemma 10.** For any pure state $|\psi\rangle$ in $D(\mathbb{C}^2)$, the stabilizer fidelity can be expressed as

$$F(\psi) = \frac{1}{2} \left( 1 + \max_{P \in \{X,Y,Z\}} \left| \langle \psi | P | \psi \rangle \right| \right).$$

**Proof.** This follows directly from the fact the single-qubit stabilizer states are the eigenstates of $X,Y,Z$, that is, the stabilizer states have the form $|\phi\rangle\langle\phi| = \frac{1}{\sqrt{2}} |P\rangle\langle P|$, where $P \in \{X,Y,Z\}$. \[ \square \]

Thus $\mu(\psi)$ can also be expressed as

$$\mu(\psi) = 1 - \max_{P \in \{X,Y,Z\}} \left| \langle \psi | P | \psi \rangle \right|. \quad \text{(D2)}$$
Now, let us begin the proof of Theorem 3. Since $|\psi\rangle$ has the Bloch sphere representation as $|\psi\rangle\langle\psi| = \frac{1}{2} \sum_{t \in F_2} \psi_{\theta,\xi} X^t Z^t$, it is easy to see that

$$|\psi_{st}| \leq 1 - \mu(\psi).$$  

(D3)

for any $(s,t) \neq (0,0)$.

Without loss of generality, we assume the first $k$ qubits are measured as the swap gate belongs to $Cl_{n+m}$. Then the output probability is

$$q(\vec{y}) = \text{Tr} \left[ U |0\rangle\langle0| \otimes \otimes_{i=1}^{m} |\psi_i\rangle\langle\psi_i| U^\dagger |\vec{y}\rangle\langle\vec{y}| \otimes I_{n+m-k} \right],$$

(D4)

for any $\vec{y} \in F_2^k$, where $I_{n+m-k}$ denotes the identity on the $k+1,\ldots,(n+m)$th qubits. Let us insert the Pauli operator $X^\vec{b} Z^\vec{q}$ into the circuit and the corresponding output probability

$$q_{\vec{a},\vec{b}}(\vec{y}) = \text{Tr} \left[ U |0\rangle\langle0| \otimes \otimes_{i=1}^{m} X^\vec{a} Z^\vec{q} U^\dagger |\vec{y}\rangle\langle\vec{y}| \otimes I_{n+m-k} \right].$$

(D5)

The corresponding Fourier coefficient is

$$\hat{q}_{\vec{a},\vec{b}}(\vec{y}) = E_{\vec{d} \in F_2^m, \vec{e} \in F_2^m} q_{\vec{a},\vec{b}}(\vec{y}) (-1)^{\vec{d} \cdot \vec{a} + \vec{e} \cdot \vec{b}}.$$

(D6)

Now let us define the reference Hermitian operator with respect to $\psi$ as follows

$$O(\psi) := \frac{1}{2} (I + \text{sgn}(|\psi_{10}\rangle)X + \text{sgn}(|\psi_{01}\rangle)Z + \text{sgn}(|\psi_{11}\rangle)I),$$

(D7)

where the function $\text{sgn}$ is defined as $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = 0$ if $x = 0$. It is easy to verify that $\text{Tr} [O(\psi)] = 1, \text{Tr} [O(\psi)^2] = \frac{\chi(\psi)}{2}$, where $\chi(\psi)$ is the Pauli rank of $\psi$. Besides, we have

$$|O_{st}| = |\text{Tr} [X^t O Z^s]| = \text{sgn}(|\psi_{st}|).$$

(D8)

Combined with (D3), we have the following relation

$$|\psi_{st}| \leq (1 - \mu(\psi))^{w(st)} |O_{st}|,$$

(D9)

for any $s,t \in F_2$. We also define $O_{\vec{a},\vec{b}}$ as follows

$$O_{\vec{a},\vec{b}}(\vec{y}) = \text{Tr} \left[ U |0\rangle\langle0| \otimes \otimes_{i=1}^{m} X^\vec{a} Z^\vec{q} U^\dagger |\vec{y}\rangle\langle\vec{y}| \otimes I_{n+m-k} \right],$$

(D10)

where each $O_i$ is the reference Hermitian operator with respect to $\psi_i$ defined as (D7) and the corresponding Fourier coefficient is

$$\hat{O}_{\vec{a},\vec{b}}(\vec{y}) = E_{\vec{d} \in F_2^m, \vec{e} \in F_2^m} O_{\vec{a},\vec{b}}(\vec{y}) (-1)^{\vec{d} \cdot \vec{a} + \vec{e} \cdot \vec{b}} = \text{Tr} \left[ U |0\rangle\langle0| \otimes \otimes_{i=1}^{m} X^\vec{a} Z^\vec{q} U^\dagger |\vec{y}\rangle\langle\vec{y}| \otimes I_{n+m-k} \right] \prod_{i=1}^{m} \left( \frac{\psi_{i,\theta,\xi}^{(i)}}{2} \right).$$

(D11)

Thus, in terms of the relation (D9), we have

$$|\hat{q}_{\vec{a},\vec{b}}(\vec{y})| \leq (1 - \mu)^{w(\vec{a},\vec{b})} |\hat{O}_{\vec{a},\vec{b}}(\vec{y})|,$$

(D12)

where $\mu = \min_i \mu(\psi_i)$ and $w(\vec{a},\vec{b})$ is defined as (A14).

Let us define

$$d_{\vec{a},\vec{b}}(\vec{y}) = \begin{cases} \hat{q}_{\vec{a},\vec{b}}(\vec{y}), & w(\vec{a},\vec{b}) \leq l \times l \times l, \\ 0, & \text{otherwise} \end{cases}$$

(D13)

which gives a family of unnormalized probability distribution $\{ d_{\vec{a},\vec{b}}(\vec{y}) \}$ as $d_{\vec{a},\vec{b}}(\vec{y}) = \Sigma_{\vec{x},\vec{y}}^{l} d_{\vec{a},\vec{b}}(\vec{y}) (-1)^{\vec{x} \cdot \vec{a} + \vec{y} \cdot \vec{b}}$ for each output $\vec{y} \in F_2^k$.

Similar to the proof of Theorem 1, we show that $d_{0,0}(\vec{y})$ gives a good approximation of $d_{0,0}(\vec{y})$ with respect to $l_1$ norm for a large fraction of Clifford circuits.

It is easy to verify that the equations (A18) and (A19) still hold, and we can repeat the process of inequality (A20) and obtain the following inequality

$$E_{\vec{a} \in F_2^m, \vec{b} \in F_2^m} \left| d_{\vec{a},\vec{b}} - d_{\vec{a},\vec{b}} \right|_1 \leq 2^l (1 - \mu)^{2l} \sum_{\vec{y} \in F_2^k} E_{\vec{a} \in F_2^m, \vec{b} \in F_2^m} d_{\vec{a},\vec{b}}(\vec{y}).$$
By the Lemma 6, we have
\[
\mathbb{E}_{U \sim \text{Cl}_{n+m}} \, \alpha_{\delta}^2 (\gamma) \leq 2^{-n-m-k} \sum_{i=1}^{m} \frac{X(\psi)}{2} + 2^{-2k}.
\]
Since \( k \leq n + m - \sum_{i=1}^{m} \log_2 \left( \frac{X(\psi)}{2} \right) \), then we have
\[
\mathbb{E}_{U \sim \text{Cl}_{n+m}} \left\| q'_{0,0} - q_{0,0} \right\|_1^2 \leq 2e^{-2\mu l}.
\]
By Markov’s inequality, we have
\[
\Pr_{U \sim \text{Cl}_{n+m}} \left( \left\| q'_{0,0} - q_{0,0} \right\|_1 \leq \sqrt{\alpha e^{-\mu l}} \right) \geq 1 - \frac{2}{\alpha}
\]
Therefore, to obtain the \( l_1 \) norm up to \( \delta \), we need take \( l = O(\log(\sqrt{\alpha}/\delta)/\mu) \) and evaluate the Fourier coefficients \( q'_{\gamma}(\gamma) \) with \( w(\gamma) \leq l \), where the total amount of such Fourier coefficients is \( \sum_{l \leq 3} (3C_m^l \leq 3m^{l}) \). Thus, there exists a classical algorithm to approximate each output probability \( q(\gamma) \) in time \( O((n+m)^3 m^{O(1)} m^{O(1/\log(\sqrt{\alpha}/\delta)/\mu)}) \) with \( l_1 \) norm less than \( \delta \) for at least \( 1 - \frac{2}{\alpha} \) fraction of Clifford circuits. Thus, we finish the proof of Theorem 3.

Moreover, if the quantum circuit \( C \) is slightly beyond the Clifford circuits, e.g., \( C = C_1 \circ V \) where the gates in \( C_1 \) are Clifford gates and \( V \) is some unitary gate in third level of Clifford hierarchy, then the result in Theorem 3 still works, as the unitary in third level of Clifford hierarchy maps Pauli operators to Clifford unitaries and thus the discussion in Appendix A4 still works.

2. Property of Pauli rank

At the end of this section, let us introduce several basic properties of Pauli rank. For any pure state \( |\psi\rangle \) on \( n \) qubits, we have the Bloch sphere representation
\[
|\psi\rangle\langle \psi | = \frac{1}{2^n} \sum_{\vec{s}, \vec{t} \in \mathbb{Z}_2^n} \psi_{\vec{s}, \vec{t}} \hat{X}^{\vec{s}} \hat{Z}^{\vec{t}},
\]
where \( \psi_{0,0} = 1 \) and \( \sum_{(\vec{s}, \vec{t}) \neq (0,0)} |\psi_{\vec{s}, \vec{t}}|^2 = 2^n - 1 \). The Pauli rank is defined as the number of nonvanishing coefficients \( \psi_{\vec{s}, \vec{t}} \), that is,
\[
\chi(\psi) := | \{ (\vec{s}, \vec{t}) \in \mathbb{Z}_2^n \mid \psi_{\vec{s}, \vec{t}} \neq 0 \} |	ag{D14}\]
Then we have the following property for the Pauli rank.

**Proposition 11.** Given an \( n \)-qubit pure state \( |\psi\rangle \), we have
(i) \( 2^n \leq \chi(\psi) \leq 4^n \), \( \chi(\psi) = 2^n \) iff \( \psi \) is a stabilizer state.
(ii) \( \chi(\psi_1 \otimes \psi_2) = \chi(\psi_1) \chi(\psi_2) \).

**Proof.** (i) \( 2^n \leq \chi(\psi) \leq 4^n \) follows directly from the definition. We only need prove \( \chi(\psi) = 2^n \) iff \( \psi \) is a stabilizer state. In the backward direction, if \( \psi \) is a stabilizer state, then it can be written as \( |\psi\rangle\langle \psi | = \prod_{i=1}^{n} P_i^{1/2} P_i^{1/2} \), where \( P_i \in P^n \) and \( P_i \) are commuting with each other. Thus, the Pauli rank of \( |\psi\rangle \) is \( 2^n \). In the forward direction, if \( \chi(\psi) = 2^n \), then it can be represented as \( |\psi\rangle\langle \psi | = \prod_{i=1}^{n} P_i^{1/2} P_i^{1/2} P_i^{1/2} P_i^{1/2} \) where \( P_i \) is a Pauli operator, and each \( P_i \) is not equivalent in the sense that \( \text{Tr}(P_i P_j) = 0 \) for any \( i \neq j \). First, we show that \( P_i P_j = P_j P_i \) for any \( i, j \). Otherwise, there exists \( i_0, j_0 \) such that \( P_{i_0} P_{j_0} = -P_{j_0} P_{i_0} \). Since \( \psi \) is a pure state, then
\[
|\psi\rangle\langle \psi | = |\psi\rangle\langle \psi |^2 = \frac{1}{4^n} \sum_{i,j=1}^{2^n} P_i P_j \frac{1}{4^n} \sum_{i,j=1}^{2^n} \{ i,j \neq \{i_0, j_0\} \} P_j P_i = \frac{1}{2^n} \sum_{k=1}^{2^n} n_k P_k,
\]
where the third inequality comes from the fact that \( P_{i_0} P_{j_0} = -P_{j_0} P_{i_0} \). Since each \( P_i P_j \) is equal to \( \sum_{k=1}^{2^n} n_k P_k \) for some \( k \) and \( n_k \) is the summation the these phases \( \tilde{\epsilon}_{i,j} \), thus
\[
\sum_{k=1}^{2^n} |n_k| \leq | \{ (i, j) \mid 1 \leq i, j \leq 2^n, \{ i, j \} \neq \{i_0, j_0\} \} | = 4n - 2.
\]
Then there is some \( k_0 \) such that \( |n_{k_0}| \leq 2^n - 1 \), i.e., \( \frac{|n_{k_0}|}{2^n} < 1 \), which contradicts with the representation of \( \psi \). Thus, \( P_i \) are commuting with each other. Next, we prove that this set of \( \{ P_i \}_{i=1}^{2^n} \) can be generated by some subset \( S \) up to \( \pm \) sign. For
any $P_i$ not equal to identity, e.g., $P_2$, then there exists $U_1 \in Cl_n$ such that $U_1P_2U_1^\dagger = Z \otimes I_{n-1}$, and for any $i$, $U_iP_iU_i^\dagger$ must have the form $Z^{a_i} \otimes P_{i,n-1}$, where $P_{i,n-1} \in P^{n-1}$ and they are commuting with each other. The generating set $S = \{Z \otimes I_{n-1}\}$.

For some $P_{i,n-1}$ not equal to identity, e.g., $Z^{a_3} \otimes P_{3,n-1}$, there exists $U_2 \in Cl_{n-1}$ such that $U_2U_1P_3U_1^\dagger U_2^\dagger = Z^{a_3} \otimes Z \otimes I_{n-2}$, and $U_2U_1P_1U_1^\dagger U_2^\dagger = Z^{a_3} \otimes Z \otimes P_{i,n-2}$. Then the generating set is updated to $S = \{Z \otimes I_{n-1}, Z^{a_3} \otimes Z \otimes I_{n-2}\}$. Let us repeat the above process for another $n-2$ times, finally we will get some generating set $S = \{g_i\}_{i=1}^n$, where $g_i = Z^{a_i} \otimes \ldots \otimes Z^{a_{i-1}} \otimes Z \otimes I_{n-i}$. Moreover, the remaining Pauli operators must have the form $\pm \otimes_{i=1}^n Z^{a_i}$, which can be generated by the generating set $S$ up to sign. That is, there is a Clifford unitary map $U$ that maps $|\psi\rangle\langle\psi|$ to another pure state $|\psi'\rangle\langle\psi'| = \frac{1}{\sqrt{2^n}} \sum_{a \in \mathbb{F}_2^n} c_a Z^a$ where $c_{|a|} = \pm 1$, $|a| := \sum_i a_i 2^{i-1}$ and $c_0 = 1$. Repeating the argument (D15) and (D16) for the pure state $|\psi'\rangle\langle\psi'|$, we have $c_{|a|} = \prod_{i=1}^n c_{a_i}^{d_i}$. Thus $|\psi'\rangle\langle\psi'| = \prod_{i=1}^n \frac{1+c_{a_i}}{2} Z_i^a$ where $Z_i$ denotes the Pauli $Z$ operator acting on the $i$th qubit. Therefore $\psi$ is a stabilizer state. (ii) This property follows directly from the definition.

Based on the above proposition and the fact that the Pauli rank is invariant under conjugation by Clifford unitaries, it is easy to see that the Pauli rank is a good candidate to quantify the magic in a state. Here, using the Pauli rank as a magic monotone is advantageous because it is easier to compute than previous magic monotones [28–30], which typically involve a minimization over all stabilizer states.