Periodic waves of a discrete higher order nonlinear Schrödinger equation*

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Abstract

The Hirota equation is a higher order extension of the nonlinear Schrödinger equation by incorporating third order dispersion and one form of self steepening effect. New periodic waves for the discrete Hirota equation are given in terms of elliptic functions. The continuum limit converges to the analogous result for the continuous Hirota equation, while the long wave limit of these discrete periodic patterns reproduces the known result of the integrable Ablowitz-Ladik system.

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1 Introduction

The Hirota equation \[13\]

\[
iu_t + c_1 (pu_{xx} + q|u|^2u) + ic_2 (pu_{xxx} + 3q|u|^2u_x) = 0, \quad i^2 = -1, \quad c_2pq \neq 0,
\]

(1)
in which \(p, q, c_1, c_2\) are real constants and \(u\) is a complex function of \(x\) and \(t\), is quite important in the propagation of short pulses in optical fibers and lattice dynamics \[16, 25, 21\]. Indeed, this is one of the only two integrable cases \[24, 23\] of a class of equations \[14, 15\]

\[
iu_t + \alpha_1 u_{xx} + \alpha_2 |u|^2u + i \left( \beta_1 u_{xxx} + \beta_2 |u|^2u_x + \beta_3 \left(|u|^2\right)_x u \right) = 0, \quad (\alpha_j, \beta_j) \text{ real},
\]

(2)

which incorporates the higher-order correction terms needed when the nonlinear Schrödinger equation (NLS) becomes insufficient or inadequate to describe the relevant physical phenomena.

Just like the NLS, equation (1) is invariant under a Galilean transformation \[23, 10\] which could allow one to set \(c_1 = 0, c_2 = 1\) without loss of generality. It is however preferable to retain both parameters \(c_1, c_2\) in order to better display the two limits,

(a) the nonlinear Schrödinger equation \((c_1 = 1, c_2 = 0)\),

\[
iu_t + pu_{xx} + q|u|^2u = 0,
\]

(3)

(b) the modified Korteweg-de Vries equation (mKdV) \((c_1 = 0, c_2 = 1, \overline{\pi} = u)\),

\[
u_t + pu_{xxx} + 3qu^2u_x = 0.
\]

(4)

If the time \(t\) is kept continuous and only the space variable \(x\) is made discrete, one can build a discretization of (1) which admits a discrete Lax pair \[22\] and therefore remains integrable. With the notation \(U_n = U(X, t), \ X = nh\), this semi-discrete equation is

\[
iU_{n,t} + \frac{1}{2} \left[ \nu(U_{n+1} + U_{n-1}) + i\mu(U_{n+1} - U_{n-1}) \right] (1 + \delta U_n U_n^-) - \nu U_n = 0,
\]

(5)

and its continuum limit to (1) involves a convenient linear change of coordinates aimed at avoiding a \(U_{n,x}\) term in the definition of (5),

\[
\begin{cases}
U_n = U(X, t) = u(x, t), \ X = nh = x + c_0t, \\
\delta = \frac{q}{2p} h^2, \ \nu = 2c_1ph^{-2}, \ \mu = 6c_2ph^{-3}, \ c_0 = 6c_2ph^{-2}.
\end{cases}
\]

(6)

Bright and dark solitary (or localized) waves will occur when the signs of the dispersive and nonlinear terms in the NLS portion of (5) are of the same and opposite signs respectively. The explicit expressions have been given earlier \[19, 20\]. In particular, bright and rational solitons on a continuous wave background are also obtained for the case of dispersive and nonlinear terms of the same sign.

The main goal of the present paper is to deduce new, special periodic solutions of (5). First we shall start with solutions of the continuous Hirota equation (1), some of which, surprisingly, have not yet been documented. The discrete analogue of these solutions will yield exact results for (5).
2 Continuum and discrete one-soliton solutions

We begin by making a rather puzzling remark on localized pulses of these higher order envelope equations. In the defocusing regime \((pq < 0)\), where conventional wisdom will dictate the presence of dark solitons, a localized pulse which rises above the mean level of a continuous background can exist. This solution,

\[
\begin{aligned}
&u = e^{i(K x - \omega t)} a_0 \left[ \frac{k \sinh ka}{\cosh k(x - ct) + \cosh ka} + iY \right], \\
&K = \frac{c_1}{3c_2}, \\
&\omega = c_1 p \left( 2Y^2 - \frac{k^2}{3} + \frac{k^2 \coth^2 k a}{2} \right) - c_2 p K^3, \\
&c = c_2 p \left( k^2 - \frac{3}{2}(k \coth k a)^2 + 3K^2 - 6Y^2 \right),
\end{aligned}
\]

\((7)\)

can be obtained with the method in \([7, \text{App. A}])\). A discrete equivalent of \((7)\) is not yet known.

3 Elliptic travelling wave solution in the continuum case

Doubly periodic solutions of the continuous Hirota equation \((1)\) have been documented earlier \([8, 9]\), they are analytically identical to those \([5]\) of the Sasa-Satsuma case \([24]\) of the integrable higher NLS (i.e. \(3\beta_1 \alpha_2 = \beta_2 \alpha_1, \beta_2 = 2 \beta_3\) in \((2)\)). Most of these solutions can be expressed as either a purely real elliptic function, multiplied by the sinusoidal (or exponential) phase of wave oscillations. We propose a new form of solution expressed with nonzero real and imaginary parts. We shall present this solution first in the notations of Halphen (Appendix A). Although this notation appears to be more abstract and less common, it enjoys the advantage of displaying clearly the ternary symmetry that exists in the complex plane between e.g. the three Jacobi functions \((\text{cn}, \text{dn}, \text{sn})\). We then employ the more conventional Jacobi elliptic functions.

Assuming a solution of \((1)\) in the form of a traveling wave (see Appendix A),

\[
\begin{aligned}
&u = a_1 \left[ h_\alpha(x - ct) + ib_1 h_\beta(x - ct) \right] e^{i(K(x - ct) - \omega t)},
\end{aligned}
\]

\((8)\)

one finds the unique solution

\[
\begin{aligned}
&a_1^2 = -\frac{2p}{q}, \\
&\omega/p = c_1 (2b_1^2 - 2b_1 K - K^2 - 3e_\alpha) + c_2 (2K(3e_\alpha + K^2) + 6b_1 (e_\beta - e_\gamma) + 6K^2), \\
&c/p = 2c_1 (b_1 + K) + 3c_2 (e_\alpha - K^2 - 2b_1 K - 2b_1^2), \\
&b_1 (e_\gamma - e_\alpha - b_1^2)(e_\gamma - e_\beta) = 0,
\end{aligned}
\]

\((9)\)

which, as displayed in the last line, splits into three subcases

- \((b_1 = 0)\): the same elliptic solution \([8]\) as for the pure NLS case,
- \((e_\gamma - e_\alpha - b_1^2 = 0)\): another elliptic solution, different from the previous one,
- \((e_\gamma - e_\beta) = 0\): a trigonometric solution, identical to the well known dark one-soliton

\[
\begin{aligned}
&u = a_0 \left[ \frac{k}{2} \tanh \frac{k}{2}(x - ct) + ib_1 \right] e^{i(K x - \omega t)}.
\end{aligned}
\]

\((10)\)
In each case, the solution depends on three arbitrary constants, e.g. respectively, \((e_\gamma, e_\alpha, K)\), \((e_\alpha, b_1, K)\), \((e_\alpha, b_1, K)\), i.e. three less arbitrary constants than the general travelling wave solution whose expression is still unknown. This lack of arbitrary constants is responsible for the factorized form of the last line in (9).

To the best of our knowledge, the elliptic solution with \(b_1 \neq 0\) (second case) is new.

In the more familiar Jacobi notation [17], this elliptic solution for \(b_1 \neq 0\) is a periodic oscillation of amplitude \(A_1\),

\[
u = A_1 \left[ \text{cn}(r(x-ct),k) + i B_1 \frac{\text{sn}(r(x-ct),k)}{\text{dn}(r(x-ct),k)} \right] e^{i(K(x-ct)-\omega t)},
\]

where the wave vector \(r\), the Jacobi modulus \(k\), the velocity \(c\) and the angular frequency \(\omega\) are related by

\[
\begin{cases}
  k^2 = 1 - B_1^2, \\
  A_1^2 = \frac{2p}{q} r^2(1 - B_1^2), \\
  c/p = 2c_1(K-rB_1) + c_2(r^2 - 3K^2 + 6KrB_1 - 8r^2B_1^2), \\
  \omega/p = c_1(-K^2 + 2KrB_1 + 4r^2B_1^2 - r^2) \\
  + c_2(2Kr^2 + 6r^3B_1 + 2K^3 - 6Kr^2B_1 - 4Kr^2B_1^2),
\end{cases}
\]

and the three parameters \((a_\alpha, b_1, K)\) are arbitrary.

Remark. The nonintegrable higher NLS equation (2) also admits the solution (8), however with one less arbitrary constant since \(K\) is then fixed.

4 Particular travelling waves in the discrete case

If one assumes for (5) the same kind of solution as for (11), namely

\[
U = a_1 \left[ h_\alpha(X-ct) + i b_1 \frac{h_\beta(X-ct)}{h_\gamma(X-ct)} \right] e^{i(K(x-ct)-\omega t)},
\]

one finds solutions similar to those of the continuous case. In particular, the elliptic solution with \(b_1 \neq 0\) is

\[
\begin{cases}
  a_1^2 = -\frac{h_\beta^2(h)}{\delta N_\alpha}, \quad e_\gamma = e_\alpha + b_1^2, \\
  \omega = \nu \left( 1 - \frac{h_\beta(h) h_\gamma^3(h)}{N_\alpha} \right) + \mu b_1 (e_\beta - e_\gamma) \frac{h_\alpha(h)}{N_\alpha}, \\
  c = h_\gamma(h) \mu \frac{h_\alpha(h) h_\gamma(h) + \nu b_1 h_\beta(h)}{N_\alpha}, \text{ arbitrary } = (e_\alpha, b_1, K),
\end{cases}
\]

in which \(N_\alpha\) denotes the nonzero constant

\[
N_\alpha = h_\alpha^4(h) - (e_\alpha - e_\beta)(e_\alpha - e_\gamma) = 2 h_\alpha(h) h_\beta(h) h_\gamma(h) h_\alpha(2h).
\]

The corresponding expression in the Jacobi notation is

\[
U = A_1 \left[ \text{cn}(r(X-ct),k) + i B_1 \frac{\text{sn}(r(X-ct),k)}{\text{dn}(r(X-ct),k)} \right] e^{i(K(x-ct)-\omega t)},
\]

in which the relations linking \((r, k, c, \omega, K, B_1, A_1)\) are easily deduced from (14) using the correspondence (24).
5 Special limits

5.1 The continuum limit

The continuum limit $h \to 0$ is easier to perform on (14). The fixed constants $(\delta, \mu, \nu)$ behave as indicated in (6), the elliptic functions of $h$ expand as

$$h_\alpha(h) = h^{-1} - \frac{e_\alpha h^2}{2} + O(h^4), \quad N_\alpha = h^{-4} - 2e_\alpha h^{-2} + O(h^0),$$

$$1 - \frac{h_\beta(h) h_\gamma(h)}{N_\alpha} \sim 1 - \left(1 - \frac{e_\beta h^2}{2} - \frac{3e_\gamma h^2}{2} + \frac{e_\alpha(2h)^2}{2}\right) \sim \frac{5e_\beta + 7e_\gamma}{2} h^2,$$

therefore

$$c = 6c_2ph^{-3}h \left(1 - \frac{3e_\beta + 5e_\gamma}{2} h^2 + O(h^4)\right) + 2c_1ph^{-2}b_1 h^2,$$

$$X - ct = x - (c - c_0)t = x - p(2c_1b_1 - 3c_2(3e_\beta + 5e_\gamma))t + O(h^2),$$

and the solution (14) indeed goes to the solution (9).

5.2 The long wave limit

In the long wave limit $k \to 1$, the solution (16) goes to (19)

$$\begin{align*}
U &= a_2 \frac{\sin kh}{h} \operatorname{sech} k(X - ct) e^{i(KX - \omega t)}, \quad a_2^2 = \frac{2p}{q}, \\
\omega &= \nu + (\mu \sin Kh - \nu \cos Kh) \cosh kh, \\
c &= (\mu \cos Kh + \nu \sin Kh) \frac{\sinh kh}{k}, \quad \text{arbitrary} = (k, K).
\end{align*}\tag{19}$$

6 Conclusion

Periodic waves of the discrete Hirota equation have been generated from their counterparts for the continuous Hirota equation. Equation (5) combines the main features of the integrable discretization of NLS [2, 3, 18],

$$i \frac{\partial \psi_n}{\partial t} + \frac{\psi_{n+1} + \psi_{n-1} - 2\psi_n}{h^2} + (\psi_{n+1} + \psi_{n-1})\psi_n^* = 0, \tag{20}$$

and the discrete modified Korteweg-de Vries equation [12]

$$\frac{\partial \psi_n}{\partial t} = \frac{(\psi_{n+1} - \psi_{n-1})(1 + \psi_n^2)}{h^2}. \tag{21}$$

The most general travelling wave solution of the continuous Hirota equation is an hyperelliptic function of genus two, as detailed in Appendix B. The elliptic solution (9) is a very particular case of this general traveling wave. Future work should address the question of incorporating the freedom of the three missing arbitrary constants in the present elliptic solution.
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Appendix A. The basic elliptic functions

In order to fully display the ternary symmetry of the Jacobi notation, and consequently make the practical computations much easier, following Halphen [11], we will choose the basis functions as

\[ h_\alpha(u) = \sqrt{\wp(u) - e_\alpha}, \quad \alpha = 1, 2, 3, \]  

(22)

in which \( \wp \) is the Weierstrass elliptic function defined by

\[ \wp^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 4\wp^3 - g_2\wp - g_3. \]  

(23)

The link with more usual (but less symmetric) notation such as \((\text{cs}, \text{ds}, \text{ns})\) or \((\text{sn}, \text{cn}, \text{dn})\) is classical [4, §16.10, 16.20, 18.9.11], e.g. [11, Chap II, (16) p. 46],

\[ \frac{\text{cs}(z)}{h_1(u)} = \frac{\text{ds}(z)}{h_2(u)} = \frac{\text{ns}(z)}{h_3(u)} = \frac{u}{z} = \frac{1}{\sqrt{e_1 - e_3}}, \]  

(24)

and our main results will also be displayed in this more usual notation, familiar to the physicists.

Since the three factors \( \wp - e_j, j = 1, 2, 3 \) of \( \wp^2 \) are indistinguishable, any solution depending on \((h_1, h_2, h_3)\) remains a solution after any permutation \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\). Therefore, in the complex plane, a \( h_1 \) solution represents any of the twelve familiar functions pq of Jacobi (p,q among s,c,d,n). However, on the real axis of \( x \) or of \( x - ct \), depending on the values of \((e_1, e_2, e_3)\), less than twelve functions will be admissible.

For deriving the continuum limits, it is useful to consider the Laurent expansion at the origin [11, Chap VII p. 237]

\[ h_\alpha(u) = \frac{1}{u} - e_\alpha \frac{u}{2} + \left(g_2 - 5e_\alpha^2\right) \frac{u^3}{40} + \frac{5g_3 - 7e_\alpha g_2 u^5}{2^6.5.7} + \frac{28g_2^2 - 225e_\alpha g_3 - 105e_\alpha^2 g_2 u^7}{2^9.3.5^2.7} u^7 + O(u^9). \]  

(25)

Full details on this symmetric notation of Halphen can be found in Ref. [6].

In the main text, \((\alpha, \beta, \gamma)\) denotes any permutation of \((1, 2, 3)\).
Appendix B

The traveling waves of (1) are defined by the reduction

$$u = \sqrt{M(\xi)e^{i(\varphi(\xi) - \omega t)}}, \ \xi = x - ct,$$

in which \(M, \varphi\) are real functions and \(c, \omega\) real constants.

The two real functions \((M, \varphi)\) obey the sixth order differential system

$$c_2 \left[ -\frac{3\varphi' M''}{2M} + \frac{3\varphi' M'^2}{4M^2} - \frac{3\varphi'' M'}{2M} - \varphi'' + \frac{\varphi^3}{p} - \frac{3q\varphi' M}{p} \right]$$

$$+ ic_2 \left[ \frac{M''}{2M} - \frac{3M' M''}{4M^2} + \frac{3M'^3}{8M^3} - \frac{3\varphi'^2 M'}{2M} + \frac{3qM'}{2p} - 3\varphi' \varphi'' \right]$$

$$+ c_1 \left[ \frac{M''}{2M} - \frac{M'^2}{4M^2} - \varphi'^2 + \frac{qM}{p} \right] + ic_1 \left[ \varphi'' + \varphi' \frac{M'}{M} \right]$$

$$+ \frac{c\varphi' + \omega}{p} - i\frac{cM'}{2pM} = 0. \quad (27)$$

All the first integrals of the system (27) can be generated systematically from the Lax pair (1) with the result,

$$K_1 = c_2^2 \left( -\frac{M''}{3} + \frac{M'^2}{4M} + \varphi^2 M - \frac{qM^2}{2p} \right) - \frac{2}{3} c_1 c_2 \varphi' M + c_2 \frac{cM}{3p}, \quad (28)$$

$$K_2 = c_2^2 \left( -\varphi' M'' + \frac{\varphi'^2 M'^2}{2M} + \varphi'' M' + 2\varphi^3 M \right)$$

$$+ c_1 c_2 \left( M'' - \frac{M'^2}{M} + \frac{qM^2}{2p} - 4\varphi'^2 M \right) + 2c_1 \varphi' M - (c_1 c + c_2 \omega) \frac{M}{p}, \quad (29)$$

$$K_3 = c_2^2 \left( \frac{M''}{4M} - \frac{M'^2}{4M^2} - \varphi^2 M'' + \frac{qM}{p} MM'' + \frac{M'^4}{16M^3} \right)$$

$$+ \frac{3\varphi'^2 M'^2}{2M} - \frac{cM'^2}{2p} + 2\varphi' \varphi'' M' + \varphi^4 M + \frac{q^2 M^3}{p^2} \right)$$

$$+ c_1 c_2 \left( \varphi' M'' - \frac{3\varphi'M'^2}{2M} - \varphi'' M' - 2\varphi^3 M \right)$$

$$- \frac{c_2^2}{p} M'' + \left( \frac{c_2^2}{4} + \frac{c_2 c}{2p} \right) \left( \frac{M'^2}{M} + 4\varphi'^2 M \right) - 2\frac{c_1 c + c_2 \omega}{p} \varphi' M$$

$$+ \frac{c_2^2}{2p^2} qM^2 + \frac{c_1 p \omega + c^2}{p^2} M. \quad (30)$$

The associated spectral curve, derived from the monodromy matrix \(M\),

$$\det(M - \mu) = \mu^2 - P(\lambda) = 0,$$

$$P(\lambda) = - \left( 4c_2 p \lambda^3 + 2c_1 p \lambda^2 + c \lambda - \frac{\omega}{2} \right)^2 - pq \left( 6K_1 \lambda^2 - K_2 \lambda + \frac{K_3}{2} \right), \quad (31)$$

is hyperelliptic and it has genus two. Therefore the general solution is a singlevalued function of \(\xi\) expressed with genus two hyperelliptic functions. To our knowledge, neither the separating variables nor the explicit expression of this general solution are known.
When the polynomial $P(\lambda)$ has a double root, the solution is elliptic and our goal here is to derive explicitly this elliptic travelling wave solution. It depends on the following arbitrary constants: the origins $\xi_0, \varphi_0$ of $\xi$ and $\varphi$, plus four constants out of $c, \omega, K_1, K_2, K_3$. After this is achieved, we want to derive the corresponding solution of the discrete equation just by the natural discretization $x \to nh$ of this elliptic solution of the continuum equation.

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