Fuzzy Fréchet Manifold

Ahmad Ghanawi Jasim 1, Al-Nafie Z. D. 2

1 Department of Mathematics, College of Education for Pure Science, University of Babylon, Iraq.
2 Department of Mathematics, College of Education for Pure Science, University of Babylon, Iraq.

Email: agjhalnajar@gmail.com

Abstract. In this paper, we introduced the concepts of fuzzy Fréchet space, fuzzy Fréchet continuous mapping between fuzzy Fréchet spaces, and the differentiability of fuzzy Fréchet continuous mapping between fuzzy Fréchet spaces. Moreover, we constructed a fuzzy Fréchet manifold and proved some properties on it.

1. Introduction

Zadeh in 1965 first introduced the concept of a fuzzy set [1]. C. L. Chang [2], C. K. Wong [3], R. Lowen [4], and others developed a theory of fuzzy topological spaces. A. K. Katsaras and D. B. Liu introduced the concept of a fuzzy topological vector space in [5]. A. K. Katsaras [6] first introduced the idea of a fuzzy seminorm. I. Sadeqi and F. Solaty Kia [7] studied the notation of fuzzy seminormed space and obtained new results. M. Ferraro and D. H. Foster [8] introduced the concept of $C^1$ fuzzy manifold. The concept of fuzzy Banach manifold was introduced by Halakatti and Archana Halijol [9].

In this paper, we defined the concepts of the open ball and closed ball in fuzzy seminormed space. The fuzzy topology generated by a family of open balls in fuzzy seminormed space. The converge sequence and Cauchy sequence in fuzzy topology induced by a family of fuzzy seminorms. Complete fuzzy topology vector space whose fuzzy topology induced by a family of fuzzy seminorms and, fuzzy Fréchet space. Moreover, we introduced fuzzy Fréchet continuous mapping (in short, $FF$—continuous) between fuzzy Fréchet spaces, The fuzzy derivative of $FF$—continuous mapping between fuzzy Fréchet spaces, Smooth mapping between fuzzy Fréchet spaces, and diffeomorphism mapping between fuzzy Fréchet spaces. Also, We introduced the concepts of fuzzy Fréchet chart, Transition map between two fuzzy Fréchet charts, $FF$—smoothly compatible fuzzy Fréchet charts, fuzzy Fréchet atlas, and $FF$—Smooth Fuzzy Fréchet manifold.

2. Preliminaries

Definition (2.1)[14, 15]: Let $Z$ be a set. A fuzzy subset $A$ of $Z$ is defined as, $A = \{(z, \mu_A(z)), \forall z \in Z\} = \mu_A$ where $\mu_A : Z \rightarrow [0,1]$. 

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Remark (2.2): A fuzzy set with constant membership function \( \mu_{k_c}(z) = c, \forall z \in Z \) is denoted by \( k_c \). The fuzzy set \( k_1 \) corresponds to the set \( Z \) and the fuzzy set \( k_0 \) corresponds to the empty set \( \emptyset \).

Definition (2.3)[10]: A fuzzy point in a non-empty set \( Z \) is a fuzzy set with membership function \( \mu_{w_\alpha}(z), z \in Z \) defined by
\[
\begin{align*}
\mu_{w_\alpha}(z) &= \alpha \text{ for } w = z \\
\mu_{w_\alpha}(z) &= 0, \text{ otherwise}
\end{align*}
\]

Where \( 0 < \alpha \leq 1 \). The point \( w \) is said to be support of \( w_\alpha \) and \( \alpha \) its value. \( w_\alpha \) is said to belong to a fuzzy set \( D \) written \( w_\alpha \in D \) if and only if \( \alpha \leq \mu_D(w) \). It is clear that \( w_\alpha \in D \) for some \( 0 < \alpha \leq 1 \) if and only if \( w \in \text{supp}(D) = \{ w \in Z : \mu_D(w) > 0 \} \).

Definition (2.4)[8]: Let \( S \) be a mapping from a set \( Z \) to a set \( W \) and let \( M \) be a fuzzy set in \( W \) then \( S^{-1}(M) \) is fuzzy set in \( Z \) with membership function defined by
\[
\mu_{S^{-1}(M)}(z) = \mu_M(S(z)), z \in Z
\]

Conversely, let \( N \) be a fuzzy set in \( Z \) then \( S(N) \) is fuzzy set in \( W \) with membership function defined by
\[
\begin{align*}
\mu_{S(N)}(w) &= \sup_{z \in S^{-1}(w)} \mu_N(z), \\
\text{if } S^{-1}(w) \text{ is a non empty} & \quad \mu_{S(N)}(w) = 0, \text{ otherwise}
\end{align*}
\]

Where \( S^{-1}(w) = \{ z : S(z) = w \} \).

Definition (2.5)[1]: A fuzzy topology on a set \( Z \) is a family \( \tau \) of fuzzy subsets in \( Z \) which satisfies the following conditions:
1. \( k_0, k_1 \in \tau \)
2. If \( A, B \in \tau \) then \( A \cap B \in \tau \)
3. If \( A_j \in \tau, \forall j \in J \) (if some index set) then \( \bigcup_{j \in J} A_j \in \tau \)

The pair \( (Z, \tau) \) is called a fuzzy topological space and the members of \( \tau \) are called open fuzzy subsets.

If a fuzzy topology defined above satisfies Lowen’s definition [6], then it said to be a proper fuzzy topology.

Definition (2.6)[1]: A fuzzy set in a fuzzy topological space \( Z \) is said to be closed if it is complement is an open fuzzy set.

Definition (2.7)[2]: A fuzzy set \( A \) in a fuzzy topological space \( (Z, \tau) \) is called neighborhood of a point \( x \in Z \) if there exists an open fuzzy set \( B \) with \( B \subseteq A \) and \( \mu_A(x) = \mu_B(x) > 0 \).

Theorem (2.8)[12]: A fuzzy set \( A \) in a fuzzy topological space \( Z \) is open if and only if \( A \) is a neighborhood of \( x \) for each \( x \in Z \) with \( \mu_A(x) > 0 \).

Definition (2.9)[12]: Let \( \tau \) be a fuzzy topology on a set \( Z \). A subfamily \( \mathcal{B} \) of \( \tau \) is called a base for \( \tau \) if each member of \( \tau \) can be expressed as the union of members of \( \mathcal{B} \).

Theorem (2.10)[6]: A collection \( \mathcal{H} \) of fuzzy sets in a set \( Z \) is a base for a proper fuzzy topology on \( Z \) if it holds the following statements:

1. \( \sup_{D \in \mathcal{H}} \{ \mu_D(z) \} = 1, \forall z \in Z \).
2. If \( D_1, D_2 \in \mathcal{H} \) then \( D_1 \cap D_2 \in \mathcal{H} \).
3. For each \( 0 \leq r < 1 \) and each \( D \in \mathcal{H} \), \( k_r \cap D \in \mathcal{H} \).

If a base \( \mathcal{H} \) for an improper fuzzy topology on \( Z \) the statement (3) is unnecessary.
Definition (2.11): A fuzzy topological space \((Z, \tau)\) is said to be fuzzy Hausdorff if for \(x, y \in Z\) and \(x \neq y\) there exists two fuzzy neighborhoods \(A\) and \(B\) in \(\tau\) of \(x\) and \(y\), respectively such that \(A \cap B = \emptyset\).

Definition (2.12)[2]: Let \((Z, \tau_1)\) and \((W, \tau_2)\) be two fuzzy topological spaces. A mapping \(S: (Z, \tau_1) \rightarrow (W, \tau_2)\) is called fuzzy continuous at some point \(x \in Z\) if \(S^{-1}(A)\) is a fuzzy neighborhood of \(x\) for each fuzzy neighborhood \(A\) of \(S(x)\). \(S\) is called fuzzy continuous if \(S\) is fuzzy continuous at every \(x \in Z\). This equivalent to inverse of every fuzzy open subset of \(W\) is fuzzy open in \(Z\).

Definition (2.13)[1]: Let \(Z, W\) be fuzzy topological spaces. A map \(S\) of \(Z\) onto \(W\) is said to be a fuzzy homeomorphism if and only if \(S\) is bijection and both \(S\) and \(S^{-1}\) is fuzzy continuous.

Definition (2.14)[5]: A fuzzy topological vector space (FTVS) is a vector space \(Z\) over the field \(K\) of real or complex numbers, \(Z\) equipped with fuzzy topology \(\tau\) and \(K\) equipped with the usual topology \(\kappa\), such that the mappings

1) \((x, y) \rightarrow (x+y)\) of \((Z, \tau) \times (Z, \tau)\) onto \((Z, \tau)\)
2) \((ax) \rightarrow (ax)\) of \((K, \kappa) \times (Z, \tau)\) onto \((Z, \tau)\) are fuzzy continuous.

Definition (2.15)[11]: Let \(Z\) be a vector space over a field \(K\). A fuzzy set \(P\) in \(Z \times \mathbb{R}\) (i.e. \(P: Z \times \mathbb{R} \rightarrow [0,1]\)) is said to be fuzzy seminorm (FSN) on \(Z\) if the following conditions are satisfied: \(\forall z, w \in Z\) and \(\forall r, s \in \mathbb{R}\)

1) \(P(z, r) = 0, \forall r \leq 0\)
2) \(P(\beta z, r) = P\left(z, \frac{r}{|\beta|}\right), \forall \beta \in K \setminus \{0\}, \forall r > 0\)
3) \(P(z + w, r + s) \leq \min\{P(z, r), P(w, s)\}\)
4) \(\lim_{r \to 0} P(z, r) = 1, \lim_{r \to 0} P(z, r) = 0\) and \(P(z, r)\) is non-decreasing w.r.t. \(r\) for each \(z \in Z\).

And \((Z, P)\) is said to be fuzzy seminormed space (FSNS).

Remark (2.16)[11]: A fuzzy seminorm \(P\) is a fuzzy norm if \(P(z, r) = 1, \forall r > 0\) then \(z = 0\).

Definition (2.17)[11]: A family \(Q\) of fuzzy seminorms on a vector space \(Z\) is said to be separating if to each \(z \neq 0\) corresponds at least one \(P \in Q\) and \(r > 0\) such that \(P(z, r) \neq 1\).

Remark (2.18)[11]: Let \(Z\) be a vector space over a field \(K\) and \(P\) a fuzzy seminorm on a vector space \(Z\). For \(z \neq 0\), \(P(z, .): (0, \infty) \rightarrow [0,1]\) is continuous.

3. Fuzzy Fréchet Space

Definition (3.1): Let \((Z, P)\) be a fuzzy seminormed space.

1) The open ball \(B_\delta(z, r)\) with center \(z \in Z\) and radius \(0 < \delta < 1\) is defined as follows:
\[B_\delta(z, r) = \{w \in Z: P(z - w, r) > 1 - \delta\}\]
2) The closed ball \(B_\delta[z, r]\) with center \(z \in Z\) and radius \(0 < \delta < 1\) is defined as follows:
\[B_\delta[z, r] = \{w \in Z: P(z - w, r) \geq 1 - \delta\}\]

Definition (3.2): Let \(A = \{P_\lambda\}_{\lambda \in J}\) be a family of fuzzy seminorms on a vector space \(Z\). Then the \(\lambda\) th open ball of radius \(0 < \delta < 1\) centered at \(z \in Z\) is
\[B_\delta^\lambda(z, r) = \{w \in Z: P_\lambda(z - w, r) > 1 - \delta\}, r > 0\]

Let \(B\) be the family of open balls in \(Z\)
\[B = \{B_\delta^\lambda(z, r): \lambda \in J, 0 < \delta < 1, z \in Z\}\]

The fuzzy topology \(\tau_\Lambda\) generated by \(B\) is called The fuzzy topology induced by \(A = \{P_\lambda\}_{\lambda \in J}\). The following concepts are based on [3].

Definition (3.3): Let \(Z\) be a fuzzy topological vector space whose fuzzy topology \(\tau_\Lambda\) induced by a family of fuzzy seminorms \(A = \{P_\lambda\}_{\lambda \in J}\). A sequence \(\{z_n\}_{n \in \mathbb{N}}\) in \(Z\) converge to a point
\[ z \in Z, \text{i.e. } z_n \to z \text{ as } n \to \infty \text{ if and only if } P_A(z_n - z, r) \to 1 \text{ as } n \to \infty, \forall r > 0 \text{ (or, } 0 < \delta < 1, r > 0, \exists n_0 \in \mathbb{Z}^+ \text{ such that } P_A(z_n - z, r) > 1 - \delta, \forall n \geq n_0) \forall P_A \in \Lambda. \]

**Definition (3.4):** Let \( Z \) be a fuzzy topological vector space whose fuzzy topology \( \tau_A \) induced by a family of fuzzy seminorms \( A = \{ P_A \}_{\alpha \in J} \). A sequence \( \{ z_n \}_{n \in \mathbb{N}} \) in \( Z \) is Cauchy if and only if \( P_A(z_n - z_m, r) \to 1 \text{ as } n, m \to \infty, \forall r > 0 \) (or, \( P_A(z_n - z_{n+p}, r) \to 1 \text{ as } n \to \infty, \forall r > 0 \) \( p = 1, 2, 3, \ldots \)) \( \forall P_A \in \Lambda. \)

**Definition (3.5):** Let \( Z \) be a fuzzy topological vector space whose fuzzy topology \( \tau_A \) induced by a family of fuzzy seminorms \( A = \{ P_A \}_{\alpha \in J} \) is (sequentially) complete if every Cauchy sequence is converge.

**Definition (3.6):** A fuzzy Fréchet space (in short, \( FFS \)) is a complete fuzzy topological vector space whose fuzzy topology induced by a countable separating family of fuzzy seminorms.

We will denote for \( Z, H \) and \( G \) are fuzzy Fréchet spaces whose fuzzy topologies \( \tau_A, \tau_F \) and \( \tau_K \), induced by a countable separating families of fuzzy seminorms \( A = \{ P_A \}_{\alpha \in J} \), \( F = \{ P_F \}_{\alpha \in I} \) and \( K = \{ P_K \}_{\gamma \in K} \), respectively.

**Definition (3.7):** Let \( S: Z \to H \) be a mapping, \( S \) is said to be \( FF \) continuous at \( z_0 \in Z \) if for any sequence \( \{ z_n \}_{n \in \mathbb{N}} \) in \( Z \) such that \( z_n \to z_0 \) as \( n \to \infty \), implies \( S(z_n) \to S(z_0) \) as \( n \to \infty \). If \( S \) be a \( FF \) continuous at each point in \( Z \), then \( S \) is said to be \( FF \) continuous on \( Z \).

**Theorem (3.8):** Let \( S: Z \to H \) be a \( FF \) continuous on \( Z \) and let \( T: H \to G \) be a \( FF \) continuous on \( H \) then \( T \circ S \) be a \( FF \) continuous on \( Z \).

**Proof:** Let \( S: Z \to H \) be a \( FF \) continuous on \( Z \) and let \( T: H \to G \) be a \( FF \) continuous on \( H \). Let \( z_0 \in Z \) and \( \{ z_n \}_{n \in \mathbb{N}} \) be a sequence in \( Z \) such that \( z_n \to z_0 \) as \( n \to \infty \).

Since \( S \) is \( FF \) continuous on \( Z \) then \( S(z_n) \to S(z_0) \) as \( n \to \infty \).

Since \( T: H \to G \) be a \( FF \) continuous on \( H \) then \( T(S(z_n)) \to T(S(z_0)) \) as \( n \to \infty \).

So \( (T \circ S)(z_n) \to (T \circ S)(z_0) \) as \( n \to \infty \).

Hence \( T \circ S \) be a \( FF \) continuous on \( Z \).

**Definition (3.9):** Let \( U \) be an open fuzzy subset of \( Z \) and let \( S: U \subseteq Z \to H \) be \( FF \) continuous mapping defined on the support of \( U = \{ x \in Z: \mu_U(x) > 0 \} \), the fuzzy derivative of \( S \) at \( x \) lie in the support of \( U \) in the direction \( h \in Z \) is defined by

\[
DS(x)h = \lim_{t \to 0, t \notin B} \frac{S(x+th)-S(x)}{t}, \quad B = \{ t \in \mathbb{R}: t \neq 0 \}
\]

We say that \( S \) is \( FF \) differentiable of class \( C^1 \) on \( U \) if the limit exists \( \forall x \) in the support of \( U, \forall h \in Z \) and if \( DS: (U \subseteq Z) \times Z \to H \) is \( FF \) continuous (jointly as a function on a subset of the product).

**Definition (3.10):** Let \( U \) be an open fuzzy subset of \( Z \) and let \( S: U \subseteq Z \to H \) be \( FF \) continuous mapping defined on the support of \( U \), the second fuzzy derivative of \( S \) is the fuzzy derivative of the first fuzzy derivative, we differentiate \( DS(x)h \) with respect to \( x \) only, in the direction \( k \in Z \).
\[ D^2S(x)(h,k) = \lim_{t \to 0} \frac{DS(x + tk)h - DS(x)h}{t} \]

We say that \( S \) is a FF differentiable of class \( C^2 \) on \( U \) if the limit exists \( \forall x \) in the support of \( U, \forall h, k \in Z \) and if \( D^2S; (U \subseteq Z) \times Z \times Z \to H \) is FF continuous.

**Definition (3.11):** Let \( U \) be an open fuzzy subset of \( Z \) and let \( S: U \subseteq Z \to H \) be FF continuous mapping defined on the support of \( U \), the third fuzzy derivative of \( S \) is the fuzzy derivative of the second fuzzy derivative, we differentiate \( D^2S(x)(h,k) \) with respect to \( x \) only, in the direction \( m \in Z \),

\[ D^3S(x)(h,k,m) = \lim_{t \to 0} \frac{D^2S(x + tm)(h,k) - D^2S(x)(h,k)}{t} \]

We say that \( S \) is a FF differentiable of class \( C^3 \) on \( U \) if the limit exists \( \forall x \) in the support of \( U, \forall h, k, m \in Z \) and if \( D^3S; (U \subseteq Z) \times Z \times Z \times Z \to H \) is FF continuous.

**Definition (3.12):** Let \( U \) be an open fuzzy subset of \( Z \) and let \( S: U \subseteq Z \to H \) be FF continuous mapping defined on the support of \( U \), the \( k \)th fuzzy derivative of \( S \) is the fuzzy derivative of \( D^{k-1}S(x)(h_1,h_2,...,h_{k-1}) \) with respect to \( x \) only, in the direction \( h_k \in Z \),

\[ D^kS(x)(h_1,h_2,...,h_k) = \lim_{t \to 0} \frac{D^{k-1}S(x+t)(h_1,h_2,...,h_{k-1},h_k) - D^{k-1}S(x)(h_1,h_2,...,h_{k-1})}{t} \]

We say that \( S \) is a FF \( k \)-differentiable of class \( C^k \) on \( U \) if the limit exists \( \forall x \) in the support of \( U, \forall h_1, h_2, ..., h_k \in Z \) and if \( D^kS; (U \subseteq Z) \times Z \times \times (k\text{-times}) \times Z \to H \) is FF continuous.

**Definition (3.13):** Let \( S: Z \to H \) be FF continuous mapping we say that \( S \) is a FF \( k \)-differentiable of class \( C^k \) (FF-smooth, infinite FF differentiable) if \( S \) is a FF \( k \)-differentiable of class \( C^k \), \( \forall k \geq 1 \).

**Definition (3.14):** A bijection map \( S: Z \to H \) is said to be FF diffeomorphism if \( S \) and \( S^{-1} \) are FF differentiable of class \( C^\infty \).

**4. FF Smooth Fuzzy Fréchet Manifolds**

**Definition (4.1):** Let \( \mathcal{M} \) be a fuzzy topological space. A pair \((V, \psi)\) where \( V \) is a fuzzy open subset of \( \mathcal{M} \) such that, \( \supp(\psi_V(z)) = 1, \forall z \in \mathcal{M} \) and \( \psi \) is a fuzzy homeomorphism mapping defined on the support of \( V \), \( \supp(V) = \{z \in \mathcal{M}: \mu_V(z) > 0\} \), which maps \( V \) onto an open fuzzy subset \( \psi(V) \) in some fuzzy Fréchet space \( Z \), called a fuzzy Fréchet chart on \( \mathcal{M} \). \((V, \psi)\) called a fuzzy Fréchet chart at \( z \in \mathcal{M} \) if and only if \( z \in \supp(V) \).

**Definition (4.2):** Let \( \mathcal{M} \) be a fuzzy topological space. If \((V_1, \psi_1)\) and \((V_j, \psi_j)\) two fuzzy Fréchet charts on \( \mathcal{M} \) then \( \psi_j \circ \psi_1^{-1}: \psi_1(V_1 \cap V_j) \to \psi_j(V_1 \cap V_j) \) called the transition map from \( \psi_1 \) to \( \psi_j \). In addition, \( \psi_1(V_1 \cap V_j) \) and \( \psi_j(V_1 \cap V_j) \) are open fuzzy subsets in some fuzzy Fréchet space \( Z \).

**Definition (4.3):** Let \( \mathcal{M} \) be a fuzzy topological space and let \((V_i, \psi_i)\), \((V_j, \psi_j)\) are two fuzzy Fréchet charts on \( \mathcal{M} \), then \((V_1, \psi_1)\) and \((V_j, \psi_j)\) are said to be FF smoothly compatible if the transition map \( \psi_j \circ \psi_1^{-1} \) is FF diffeomorphism.

**Definition (4.5):** Let \( \mathcal{M} \) be a fuzzy topological space. A collection \( A = \{ (V_i, \psi_i) \}_{i \in I} \) of fuzzy Fréchet charts on \( \mathcal{M} \) such that \( V_i, i \in I \) cover \( \mathcal{M} \) i. e. \( \bigcup_{i \in I} V_i = \mathcal{M} \) called a fuzzy Fréchet atlas on \( \mathcal{M} \).

**Definition (4.6):** A fuzzy Fréchet manifold \( \mathcal{M} \) is a fuzzy topological space modeled on some fuzzy Fréchet space \( Z \).
Definition (4.7): A fuzzy Fréchet atlas $A = \{(V_i, \psi_i)\}_{i \in I}$ on a fuzzy topological space $\mathcal{M}$ is called $FF$-smooth fuzzy Fréchet atlas if any two fuzzy Fréchet charts in $A$ are $FF$-smoothly compatible with each other.

Definition (4.8): Two $FF$-smooth fuzzy Fréchet atlas $A_1 = \{(V_i, \psi_i)\}_{i \in I}$ and $A_2 = \{(V_j, \psi_j)\}_{j \in J}$ on a fuzzy topological space $\mathcal{M}$ are said to be $FF$-smoothly compatible if each fuzzy Fréchet chart in $A_1$ is $FF$-smoothly compatible with each fuzzy Fréchet chart in $A_2$.

Remark (4.9): It is easy to see that the relation of $FF$-smoothly compatible between fuzzy Fréchet atlases is an equivalence relation.

Definition (4.10): A $FF$-smooth fuzzy Fréchet atlas $A = \{(V_i, \psi_i)\}_{i \in I}$ on a fuzzy topological space $\mathcal{M}$ is said to be maximal if each fuzzy Fréchet chart is $FF$-smoothly compatible with each fuzzy Fréchet chart in $A$ is already in $A$. A maximal $FF$-smooth fuzzy Fréchet atlas also called $FF$-smooth structure or $FF$-differentiable structure.

Definition (4.11): A $FF$-smooth fuzzy Fréchet manifold is a pair $(\mathcal{M}, A)$ where $\mathcal{M}$ is a fuzzy topological space and a $FF$-smooth structure $A$. Sometimes we say just $\mathcal{M}$ is $FF$-smooth fuzzy Fréchet manifold.

Remark (4.12): If $\mathcal{M}$ is a $FF$-smooth fuzzy Fréchet manifold, any fuzzy Fréchet chart in the given maximal $FF$-smooth fuzzy Fréchet atlas will be called $FF$-smooth fuzzy Fréchet chart and the corresponding map will be called $FF$-smooth fuzzy coordinate map.

Lemma (4.13): Let $\mathcal{M}$ be fuzzy Fréchet manifold and let $A = \{(V_i, \psi_i)\}_{i \in I}$ be a fuzzy Fréchet atlas on $\mathcal{M}$. If two fuzzy Fréchet charts $(U, \psi)$ and $(W, \theta)$ on $\mathcal{M}$ are both $FF$-smoothly compatible with the fuzzy Fréchet atlas $A = \{(V_i, \psi_i)\}_{i \in I}$, then they are $FF$-smoothly compatible with each other.

Proof: Let $(U, \psi)$ and $(W, \theta)$ are fuzzy Fréchet charts on fuzzy Fréchet manifold $\mathcal{M}$ both $FF$-smoothly compatible with the fuzzy Fréchet atlas $A = \{(V_i, \psi_i)\}_{i \in I}$ on $\mathcal{M}$. We need to show that $\theta \cdot \psi^{-1}: \psi(U \cap W) \to \theta(U \cap W)$ is $FF$-diffeomorphism.

Let $z = \psi(q)$ be arbitrary point $\psi(U \cap W)$ such that $q$ lies in the support of $U \cap W$. Since $A = \{(V'_i, \psi'_i)\}_{i \in I}$ is the fuzzy Fréchet atlas on $\mathcal{M}$ then $q$ lies in the support of $V'_i$ for some $i$. Thus $q$ lies in the support of $V'_i \cap U \cap W$ for some $i$. Since $(U, \psi)$ and $(W, \theta)$ are both $FF$-smoothly compatible with the fuzzy Fréchet atlas $A = \{(V_i, \psi_i)\}_{i \in I}$ then both maps $\psi_i \circ \psi^{-1}$ and $\theta \circ \psi_i^{-1}$ are $FF$-smooth for some $i$. Then $\theta \circ \psi^{-1} = (\theta \circ \psi_i^{-1}) \circ (\psi_i \circ \psi^{-1})$ is $FF$-smooth map at $z$. Since $z$ arbitrary then $\theta \circ \psi^{-1}$ is $FF$-smooth map. Since $(\theta \circ \psi^{-1})^{-1} = \psi \circ \theta^{-1}$ then as above $(\theta \circ \psi^{-1})^{-1}$ is $FF$-smooth map. So that $\theta \circ \psi^{-1}$ is $FF$-diffeomorphism.

Theorem (4.14): Let $\mathcal{M}$ be fuzzy Fréchet manifold and let $A = \{(V_i, \psi_i)\}_{i \in I}$ be a fuzzy Fréchet atlas on $\mathcal{M}$. The family of open fuzzy sets $W \subset \mathcal{M}$ such that $W \subset V_i$, $(V_i, \psi_i) \in A$ and $\psi_i(W)$ is an open fuzzy subset in some fuzzy Fréchet space $Z$ is a base for a proper fuzzy topology on $\mathcal{M}$.

Proof: Let $\mathcal{H} = \{W \subset \mathcal{M} \text{ such that } W \subset V_i, (V_i, \psi_i) \in A \text{ and } \psi_i(W) \text{ is an open fuzzy subset in some fuzzy Fréchet space } Z\}$

Since for every $(V_i, \psi_i) \in A$, $V_i \subset \mathcal{M}$, $V_i \subset V_i$ and $\psi_i(V_i)$ is an open fuzzy set in some fuzzy Fréchet space $Z$ then each $V_i$ is an element of $\mathcal{H}$. Thus, we get
\[
\sup_{z \in \mathcal{M}} \{ \mu_w(z) \} = 1, \forall z \in \mathcal{M}
\]

Now, let \( U, W \) be open fuzzy sets in \( \mathcal{H} \) then there are two fuzzy Fréchet charts \((V_1, \psi_1)\) and \((V_2, \psi_2)\) in \( A \) such that \( U \subset V_1, W \subset V_2 \) and \( \psi_1(U), \psi_2(W) \) are open fuzzy subsets of some fuzzy Fréchet space \( Z \).

Since \( U \subset V_1 \) and \( W \subset V_2 \) then \( U \cap W \subset V_1 \) and \( U \cap W \subset V_2 \) then
\[
\psi_1(U \cap W) = \psi_1(U \cap W \cap V_1 \cap V_2)
\]
\[
= \psi_1(U) \cap \psi_1(W \cap V_1 \cap V_2)
\]
\[
= \psi_1(U) \cap \psi_1 \cdot \psi_2^{-1}(\psi_2(W) \cap \psi_2(V_1 \cap V_2))
\]

Therefore \( \psi_1(U \cap W) \) is an open fuzzy subset in some fuzzy Fréchet space \( Z \). Hence \( U \cap W \in \mathcal{H} \).

Finally, for each 0 \( \leq c < 1 \). Let \( W \in \mathcal{H} \) then there is a fuzzy Fréchet chart \((V_i, \psi_i)\) in \( A \) such that \( W \subset V_i, \psi_i(W) \) is an open fuzzy subset of some fuzzy Fréchet space \( Z \).

Let \( D = K_c \circ W \), since \( D \subset W \) and \( W \subset V_i \) then \( D \subset V_i \) such that \( (V_i, \psi_i) \in A \) and \( \psi_i(W) \) is an open fuzzy subset in some fuzzy Fréchet space \( Z \). And
\[
\psi_i(D) = \psi_i(W \cap K_c)
\]
\[
= \psi_i(W \cap K_c \cap V_i)
\]
\[
= \psi_i(W) \cap \psi_i(K_c \cap V_i)
\]
\[
= \psi_i(W) \cap \psi_i \cdot \psi_i^{-1}(\psi_i(K_c) \cap \psi_i(V_i))
\]

Therefore \( \psi_i(W) \) is an open fuzzy subset in some fuzzy Fréchet space \( Z \). Hence \( D \in \mathcal{H} \) and by Theorem (2.10), \( \mathcal{H} \) is a base for a proper fuzzy topology on \( \mathcal{M} \). So we can say that a fuzzy Fréchet atlas on \( \mathcal{M} \) induce a fuzzy topology on \( \mathcal{M} \) denoted by \( \tau_A \).

**Theorem (4.15):** Let \( \mathcal{M} \) be fuzzy Fréchet manifold and let \( A = \{ (V_i, \psi_i) \}_{i \in I} \) be a fuzzy Fréchet atlas on \( \mathcal{M} \). A fuzzy subset \( V \) of \( \mathcal{M} \) is an open fuzzy subset in the induced fuzzy topology \( \tau_A \) by fuzzy Fréchet atlas \( A \) if and only if \( V \) intersect some fuzzy Fréchet chart \((V_i, \psi_i) \in A \) and \( \psi_i(V \cap V_i) \) is an open fuzzy subset in a fuzzy Fréchet space \( Z \).

**Proof:** Let \( V \) be a fuzzy subset of \( \mathcal{M} \) such that if \( V \) intersect some fuzzy Fréchet chart \((V_i, \psi_i) \in A \) and \( \psi_i(V \cap V_i) \) is an open fuzzy subset in a fuzzy Fréchet space \( Z \). We get
\[
V = \bigcup_{i \in I} (V \cap V_i) \quad (4.15.1)
\]

For each fuzzy Fréchet chart \((V_i, \psi_i) \in A \) intersect \( V, V \cap V_i \subset V_i \). By assumption, \( \psi_i(V \cap V_i) \) is an open fuzzy subset in a fuzzy Fréchet space \( Z \). Then, by Theorem (4.14) we get \( V \cap V_i \) is open in the induced fuzzy topology on \( \mathcal{M} \). By (4.15.1) we get \( V \) is an open fuzzy subset in the induced fuzzy topology \( \tau_A \) by fuzzy Fréchet atlas \( A \).

Conversely, let \( V \) is an open fuzzy subset in the induced fuzzy topology \( \tau_A \) by fuzzy Fréchet atlas \( A \) and let \((V_i, \psi_i) \) be a fuzzy Fréchet chart in \( A \), then we get \( V \cap V_i \) is an open fuzzy subset in the induced fuzzy topology \( \tau_A \) since \( \psi_i \) which maps \( V_i \) onto an open fuzzy subset \( \psi_i(V_i) \) in some fuzzy Fréchet space \( Z \) then we get, \( \psi_i(V \cap V_i) \) is an open fuzzy subset in a fuzzy Fréchet space.

**Lemma (4.16):** Let \( \mathcal{M} \) be fuzzy Fréchet manifold, \( A_1 = \{ (V_i, \psi_i) \}_{i \in I} \) be a fuzzy Fréchet atlas on \( \mathcal{M} \) and Let \( \mathcal{R} \) be fuzzy Fréchet manifold, \( A_2 = \{ (V_i, \psi_i) \}_{i \in L} \) be a fuzzy Fréchet atlas on \( \mathcal{R} \).

Then the family of pairs \((V_i \times V_j, \psi_i \times \psi_j)\) is a fuzzy Fréchet chart on \( \mathcal{M} \times \mathcal{R} \).

**Proof:** By definition of fuzzy Fréchet chart we get
sup_i{\mu_{V_i}(y)} = 1, \forall y \in \mathcal{M} and sup_i{\mu_{V_i}(w)} = 1, \forall w \in \mathcal{R}

\text{Since } \mu_{V \times V_i}(y) = \min \{ \mu_{V_i}(z), \mu_{V_i}(w) \}, y = (z, w)

\text{Then we get } sup_i{\mu_{V \times V_i}(y)} = sup_i{\min \{ \mu_{V_i}(z), \mu_{V_i}(w) \}}

= \min \{ sup_i{\mu_{V_i}(z)}, sup_i{\mu_{V_i}(w)} \} = 1

That is the domains of the fuzzy Fréchet charts \((V_i \times V, \psi_i \times \psi_i)\) cover \(\mathcal{M} \times \mathcal{R}\). Hence, the family of pairs \((V_i \times V, \psi_i \times \psi_i)\) is a fuzzy Fréchet atlas on \(\mathcal{M} \times \mathcal{R}\).

**Theorem (4.17):** Let \(\mathcal{M}\) be a FF-smooth fuzzy Fréchet manifold then every FF-smooth fuzzy Fréchet atlas on \(\mathcal{M}\) is contained in a unique maximal FF-smooth fuzzy Fréchet atlas. 

**Proof:** Let \(\mathcal{A} = \{(V_i, \psi_i)\}_{i \in I}\) be FF-smooth fuzzy Fréchet atlas on \(\mathcal{M}\) and let \(\mathcal{B}\) be the set of all fuzzy Fréchet charts that are FF-smoothly compatible with each fuzzy Fréchet charts in \(\mathcal{A}\). To show that \(\mathcal{B}\) is FF-smooth fuzzy Fréchet atlas on \(\mathcal{M}\), we need to show that any two fuzzy Fréchet charts of \(\mathcal{B}\) are FF-smoothly compatible with each other.

Let \((U, \varphi)\) and \((W, \chi)\) be fuzzy Fréchet charts in \(\mathcal{B}\) on fuzzy Fréchet manifold \(\mathcal{M}\) both FF-smoothly compatible with \(\mathcal{A}\) = \(\{(V_i, \psi_i)\}_{i \in I}\). We need to show that

\[\chi \circ \varphi^{-1}: \varphi(U \cap W) \to \chi(U \cap W)\]

is FF-diffeomorphism. Let \(z = \varphi(q)\) be arbitrary point \(q \in \varphi(U \cap W)\) such that \(q\) lies in the support of \(U \cap \mathcal{W}\). Since \(A = \{(V_i, \psi_i)\}_{i \in I}\) the fuzzy Fréchet atlas on \(\mathcal{M}\) then \(q\) lies in the support of \(V_i\) for some \(i\). Thus \(q\) lies in the support of \(V_i \cap U \cap \mathcal{W}\) for some \(i\). Since \((U, \varphi)\) and \((W, \chi)\) are both FF-smoothly compatible with the fuzzy Fréchet atlas \(A = \{(V_i, \psi_i)\}_{i \in I}\) then both maps \(\psi_i \circ \varphi^{-1}\) and \(\chi \circ \psi_i^{-1}\) are FF-smooth for some \(i\). Then \(\chi \circ \varphi^{-1} = (\chi \circ \psi_i^{-1}) \circ (\psi_i \circ \varphi^{-1})\) is FF-smooth map at \(z\). Since \(z\) arbitrary then \(\chi \circ \varphi^{-1}\) is fuzzy FF-smooth map. Since \((\chi \circ \varphi^{-1})^{-1} = \varphi \circ \chi^{-1}\) then as above \((\chi \circ \varphi^{-1})^{-1}\) is FF-smooth map. So that \(\chi \circ \varphi^{-1}\) is FF-diffeomorphism. Next, to show that \(\mathcal{B}\) is maximal FF-smooth fuzzy Fréchet atlas, note that every fuzzy Fréchet chart that is FF-smoothly compatible with every fuzzy Fréchet chart in \(\mathcal{B}\) must, in particular, be FF-smoothly compatible with every fuzzy Fréchet chart in \(\mathcal{A}\), so it already in \(\mathcal{B}\). This shows the existence of maximal FF-smooth fuzzy Fréchet atlas containing \(\mathcal{A}\).

Finally, let \(\mathcal{D}\) be any other maximal FF-smooth fuzzy Fréchet atlas containing \(\mathcal{A}\), then each fuzzy Fréchet chart in \(\mathcal{D}\) is FF-smoothly compatible with every fuzzy Fréchet chart in \(\mathcal{A}\), so \(\mathcal{D} \subseteq \mathcal{B}\). By maximality of \(\mathcal{D}\) we get \(\mathcal{D} = \mathcal{B}\).

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