About the possibility of synchronization in dynamical systems

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Abstract. The principle possibility of two metal pendulums synchronization oscillating in the parallel planes, the distance between the suspension points of which is fixed, is proved. This behavior is possible only due to the taking into account two physical factors. The first one is an effect of electromagnetic interaction between them. The second one is accounting the electromagnetic radiation of each pendulum, which leads to nonlinear attenuation. The system of nonlinear dynamical equations of the movement is obtained and its decision are numerically analyzed.

1. Introduction

In this paper we will continue the discussion of a fundamental explanation of the phenomenon of synchronization in the system of two oscillating pendulums. However, contrary to [1], [2] papers, where the main scheme and the idea of calculations for pendulums oscillating in one plane were outlined, in this paper we should also discuss two pendulums, but now moving in the parallel planes.

For describing of the dynamic motion in this case, we should consider the Figure 1, which illustrates the entire geometry of the problem. If the lengths of suspension are equal to $l_1$ and $l_2$ then due to the geometry, illustrated by the Figure 1, we have the following equality:

$$ R = l_2 - l_1 + b. $$

![Figure 1. The geometry of the problem.](image-url)
In the case of identical pendulums, when \( l_1 = l_2 = l, m_1 = m_2 = m \), from the interrelation (1), it follows that:

\[
\varphi_1 = \varphi_2 + \delta, \quad \dot{\varphi}_1 = \dot{\varphi}_2,
\]

(2)

where \( \delta \) - is a phase of oscillation. The equalities (2) can be considered the conditions for perfect synchronization of two pendulums. For the case of various pendulums, when \( l_1 \neq l_2 \), the synchronization condition should be written only in the form of equality of angular frequencies \( \dot{\varphi}_1 = \dot{\varphi}_2 \). Although, we should notice that in some sources dedicated to the theory of synchronization, this condition is written as \( \dot{\varphi}_1 = \frac{p}{q} \dot{\varphi}_2 \), where \( p, q \) are the close whole numbers. In this case

\[
\varphi_1 = \omega_1 t + \alpha_1, \quad \varphi_2 = \omega_2 t + \alpha_2,
\]

(3)

where \( \alpha_{1,2} \) are the stationary phases. It means that in the case of synchronization, the distance \( R \) becomes fixed and due to (1), we have that:

\[
R = \sqrt{l_1^2 + l_2^2 + b^2 - 2l_1 l_2 \cos(\alpha_1 - \alpha_2)} = \text{const}.
\]

(4)

We should describe the dynamic motion in the language of generalized angular coordinates of \( \varphi_1 \) and \( \varphi_2 \).

2. Materials and methods

For deriving equation we should use the principle of conservation of the total power of the system i.e. the condition \( \sum \dot{E} + \sum \dot{Q} + \sum \dot{W} = 0 \), where «point» traditionally refers to time differentiation.

Due to the Figure 1, the total energy of the system can be presented as

\[
E = T + U = U_0 + \frac{m_1 l_1 \dot{\varphi}_1^2}{2} + \frac{m_2 l_2 \dot{\varphi}_2^2}{2} - g \left( m_1 l_1 \cos \varphi_1 + m_2 l_2 \cos \varphi_2 \right) + \frac{U_{12}(R) + U_{21}(R)}{2} = \text{const},
\]

(5)

where \( U_0 = \left( m_1 + m_2 \right) g H \), \( l_{1,2c} \) - is the distance from the point of suspension pendulums to its center of attraction and \( H \) is the height of the suspension. On the assumption with the potential energy of pendulum interaction has an electromagnetic character and due to the general principles of classical electrodynamics (see, for example, [3-4]), we can write that:

\[
U = U_{EM} = \frac{1}{c^2} \int \int_{V_1, V_2} \frac{\mathbf{j}_1 \cdot \mathbf{j}_2}{\mathbf{R}} dV_1 dV_2,
\]

(6)

where \( \mathbf{R} \) is a vector introduced as

\[
\mathbf{R} = \mathbf{R} + \mathbf{r}_1 - \mathbf{r}_2,
\]

(7)

\( V_1 \) and \( V_2 \) are the volumes of the balls, \( c \) - is the light of the velocity and the current densities are the following \( \mathbf{j}_1 = \rho_e \mathbf{v}_1 = \rho_e \dot{\varphi}_1 (l_1 \mathbf{v}_1 + \mathbf{r}_1 \mathbf{v}_1) \), \( \mathbf{j}_2 = \rho_e \dot{\varphi}_2 (l_2 \mathbf{v}_2 + \mathbf{r}_2 \mathbf{v}_2) \), which appears due to the acceleration of motion balls, \( \rho_e \) - is a nonequilibrium density of electrons in the ball, \( l_{1,2c} \) - is the distance from the point of suspension to the center of mass of corresponding system ball + rod.
Assuming that \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) are the unit vectors, directed along angular velocities \( \omega_1 \) and \( \omega_2 \) in the result, we obtain the following energy of interaction

\[
U_{EM} = -\frac{\phi_1 \phi_2 (\mathbf{k}_1 \cdot \mathbf{k}_2) \rho^2 V^2 l_1 l_2 \cos (\phi_1 - \phi_2)}{c^2 \xi},
\]

(8)

where \( \xi \) is a numerical dimensionless ratio the order of unity, not too important for the further decision. The scalar product of \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) should be written in the form of \( \mathbf{k}_1 \cdot \mathbf{k}_2 = \cos \psi = \pm 1 \). We obtain for the linear velocities the following \( \mathbf{v}_1 = \phi_1 l_1 \mathbf{\tau}_1 \), and \( \mathbf{v}_2 = \phi_2 l_2 \mathbf{\tau}_2 \), where \( \mathbf{\tau}_{1,2} \) are the unit tangent vectors to the motion trajectory, which could be written in the form of expansion in fixed basis of Cartesian coordinates as:

\[
\begin{align*}
\mathbf{\tau}_1 &= i \cos \phi_1 + j \sin \phi_1, \\
\mathbf{\tau}_2 &= i \cos \phi_2 + j \sin \phi_2.
\end{align*}
\]

We should go farther in this matter. The power of electromagnetic radiation of pendulums \( W \) due to the acceleration motion should be easily found thanks to the Liepar - Wiechert potentials (see [4-5]). In our case we can write their as

\[
\begin{align*}
\psi(r,t) &= \frac{enV}{R\left[1 - \left(\frac{v \cdot n}{c}\right)^2\right]} \approx \frac{enV}{R\left[1 + \left(\frac{v \cdot n}{c}\right)^2\right]}, \\
A(r,t) &= \frac{enVv}{Rc\left[1 - \left(\frac{v \cdot n}{c}\right)^2\right]} \approx \frac{enVv}{Rc\left[1 + \left(\frac{v \cdot n}{c}\right)^2\right]},
\end{align*}
\]

(9)

where \( n \) is the free electron nonequilibrium concentration, thanking to the oscillations of pendulum, and \( V \) is a volume of a sphere. Substituting potentials (9) into the definitions

\[
E = -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \psi, \quad B = \text{rot} A,
\]

(10)

we obtain the following:

\[
\begin{align*}
E &= \frac{enV}{R^3} \cdot \frac{enV}{c^2} \cdot \frac{\psi}{R} + \frac{enV}{R^3} \left(\frac{v \cdot n}{c}\right)^2 + \frac{enV}{R^3} \left(\frac{v R}{c}\right)^2, \\
B &= \frac{enV}{cR^3} \left(\nabla \times R\right),
\end{align*}
\]

(11)

where \( R = r - r_0(t) \), and \( r_0(t) \) is a motion trajectory. Hence \( \dot{R} = -\dot{r}_0 = -v \) and the acceleration \( \ddot{v} = \ddot{\psi} + \frac{v^2}{l} \mathbf{n} \), where \( \mathbf{n} \) is a normal unit vector to the motion trajectory. Then take into account (11) for the intensity of electromagnetic radiation \( I \) in the accordance with [3] we have that:
According to the determination of radiation power \( W = \int IR^2 dO = 4\pi IR^2 \) we can write that:

\[
W = \frac{(enVl)^2}{2c^3} \left( \phi^2 + \phi^4 \right) \approx \frac{(enVl)^2}{2c^3} \phi^4.
\]

The inequality \( |\phi_{1,2}| \ll |\phi_{1,2}^2| \) is confirmed by numerical calculation (see Figure 2).

\[ \frac{d\phi}{d\tau} \]

\[ 0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \]

\[ 1200 \quad 1400 \quad 1600 \quad 1800 \quad 2000 \]

\[ \tau \]

Figure 2. The dependence \( \phi_{1,2} \) practically merges with abscissa \( \tau \) and strong oscillations are illustrated in the figure.

Neglecting the dissipative medium properties of \( \dot{Q} \) in the ideal case (for example, in the vacuum), we obtain that:

\[
\sum \dot{E} + \sum W = 0.
\]

Taking into account (5), (6) and (13) and after simple differentiation, we are coming to the required system of differential equations:

\[
\begin{align*}
\phi_1^* + \sin \phi_1 - \lambda_1 \left( \phi_2^* \cos (\phi_1 - \phi_2) - \phi_2^2 \sin (\phi_1 - \phi_2) \right) + \kappa_1 \phi_1^3 &= 0, \\
\phi_2^* + \sin \phi_2 - \lambda_2 \left( \phi_1^* \cos (\phi_1 - \phi_2) + \phi_1^2 \sin (\phi_1 - \phi_2) \right) + \kappa_2 \phi_2^3 &= 0.
\end{align*}
\]

Where the dimensionless parameters are the following

\[
\lambda_1 = \frac{\xi \cos \psi \rho^2 V^2 l}{m_1 \omega_{10}^2 c^2 b l_1}, \quad \lambda_2 = \frac{\xi \cos \psi \rho^2 V^2 l}{m_2 \omega_{20}^2 c^2 b l_2}, \quad \kappa_1 = \frac{(enV)^2}{2m_1 \omega_{10}^2 c^3}, \quad \kappa_2 = \frac{(enV)^2}{2m_2 \omega_{20}^2 c^3}.
\]
and the frequencies \( \omega_{0,1,2} = \sqrt{\frac{g}{l_{1,2c}}} \). Here the dimensionless time is \( \tau = \omega_{0} t \). As we can see from the equations (15), the stationary point is \( \varphi_{s,1,2} = 0 \). This solution is asymptotically stable, despite the oscillating character of the moving pendulums. The dynamics of oscillating pendulums is such that oscillations in coordinates \( \varphi_{1} - \varphi_{2} \) are concentrated near the critical point \( \varphi_{s,1,2} = 0 \). This fact is reflected in the Figures 3 - 5 as a result of numerical solution of the system (15) at the initial conditions:

\[
\varphi_{1}(0) = -\frac{\pi}{6}, \quad \varphi_{2}(0) = \frac{\pi}{4}, \quad \dot{\varphi}_{1}(0) = \dot{\varphi}_{2}(0) = 0. \tag{17}
\]

\[\text{Figure 3. At the short times } \tau \in [500, 1000] \text{ dependence } \varphi_{1}'(\varphi_{1}') \text{ has a random nature.}\]
Figure 4. In this figure dependence \( \phi_2(\phi_1') \) is shown at the interval time \( \tau \in [739900,740000] \). The values of parameters are 
\[ \lambda_1 = 10^{-2}, \lambda_2 = 2 \cdot 10^{-2}, \]
\[ \kappa_1 = 10^{-3}, \kappa_2 = 2 \cdot 10^{-3}. \]

Figure 5. In this figure dependence \( \phi_2(\phi_1) \) is shown at the interval time \( \tau \in [739900,740000] \). The values of parameters are 
\[ \lambda_1 = 10^{-2}, \lambda_2 = 2 \cdot 10^{-2}, \]
\[ \kappa_1 = 10^{-3}, \kappa_2 = 2 \cdot 10^{-3}. \]
3. Results and discussion

The synchronization phenomenon described above we’d like discuss now more detail. Indeed, for two identical pendulums we can use the system of equations (15) for the case when \( \cos \psi = 1 \) (recall that \( \cos \psi = k_1 \cdot k_2 = \pm 1 \)) and in dimensionless units we have the following system

\[
\begin{align*}
\dot{\varphi}_1^* + \sin \varphi_1 + \lambda \left( -\varphi_1^* \cos \left( \varphi_1 - \varphi_2 \right) + \varphi_2^* \sin \left( \varphi_1 - \varphi_2 \right) \right) + \kappa \varphi_1^3 &= 0, \\
\dot{\varphi}_2^* + \sin \varphi_2 + \lambda \left( -\varphi_2^* \cos \left( \varphi_1 - \varphi_2 \right) - \varphi_1^* \sin \left( \varphi_1 - \varphi_2 \right) \right) + \kappa \varphi_2^3 &= 0.
\end{align*}
\]

(18)

As we know (see above) the synchronization condition means that \( \varphi_1 = \pm \varphi_2 + \delta, \ \varphi'_1 = \varphi'_2, \) where \( \delta \) is some phase. For the arbitrary of the moment time when we are bring to the stationary point nearer the solutions can be write in the form

\[
\varphi_1 (\tau) \approx \varphi_\delta + \delta \tau, \ \varphi_2 (\tau) \approx \varphi_\delta - \delta \tau.
\]

(19)

Summarizing equations (18), we are finding that

\[
\begin{align*}
\left( \varphi_1 + \varphi_2 \right)^* + 2 \sin \left( \frac{\varphi_1 + \varphi_2}{2} \right) \cos \left( \frac{\varphi_1 - \varphi_2}{2} \right) + \\
+ \lambda \left( \left( \varphi_1 + \varphi_2 \right)^* \cos \left( \varphi_1 - \varphi_2 \right) + \left( \varphi_1^* + \varphi_2^* \right) \sin \left( \varphi_1 - \varphi_2 \right) \right) + \kappa \left( \varphi_1^* + \varphi_2^* \right) \left( \varphi_1^2 - \varphi_2^2 + \varphi_1^2 \right) &= 0.
\end{align*}
\]

(20)

In the accordance of the decision (19) from (20) immediately follows the conclusion that the classical synchronization conditions are observed. Indeed, since we are taking arbitrary moment of the time the area of identity equation (20) have place at implementation of the condition that the functions \( \varphi_1, \varphi_2 \) are independent. Moreover, the condition which in the monographs [9-10] it’s noted we can use due to the strict mathematical terminology in the theory of oscillations of dynamical systems.

Really, for any two-dimensional dissipative dynamical systems the conditions of the solution curves there can be only two types: stable critical points and stable maximum cycles. In our case, it is only a stable critical point. When we are talking about the phase space with dimensionality \( n=2 \), dynamic behavior of the system becomes much more complicated. In addition to critical points and maximum cycles, there can exist much more complex attained sizes in the form two-dimensional invariant torus, corresponding quasiperiodic motion. Such phenomenon characterizing not one, but two rationally independent frequencies.

When considering the general case the phase space is more than three (\( n > 3 \)), multidimensional attracting torus will be unstable, and under the influence of any perturbations in dynamical systems, they should collapse within a certain visible period of the time. However, in the case of the appearance mode of motion with a large number of incommensurate frequencies, the destruction of torus will be prevented by the so-called hunting of the phases, which will lead not to destruction, but to the appearance of synchronization (see [10], §30). The same effect also occurs in our two-dimensional case described above.

4. Conclusion

In conclusion, we’d like note the following statements.

1. It was given the strong evidence of the possibility of synchronizing a pair of interacting pendulums, which is based on two purely physical factors. 1. Its energy electromagnetic interaction between metal balls. 2. The electromagnetic friction radiation caused by a differential motion of the balls along a path curve.
2. The numerically it is shown that synchronization occurs after a well defined period of the time \( t_{\text{synch}} \) as a function of specific physical and geometric parameters of the problem.

3. It is given numerical solution of the equations (15), illustrating full picture of synchronization (see Figures 3-10).

**References**

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