The consequences of dependence between the formal area efficiency and the macroscopic electric field on linearity behavior in Fowler–Nordheim plots

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Abstract
This work presents a theoretical explanation for a crossover in the linear behavior in Fowler–Nordheim (FN) plots based on orthodox cold field electron emission (CFE) experimental data. It is characterized by a clear change in the decay rate of usually single-slope FN plots, and has been reported when non-uniform nano-emitters are subject to high macroscopic electric field $F_M$. We assume that the number of emitting spots, which defines an apparent formal area efficiency of CFE surfaces, depends on the macroscopic electric field. Non-uniformity is described by local enhancement factors $\{\gamma_j\}$, which are randomly assigned to each distinct emitter of a conducting CFE surface, from a discrete probability distribution $\rho(\gamma_j)$, with $j = 1, 2$. It is assumed that $\rho(\gamma_1) < \rho(\gamma_2)$, and that $\gamma_1 > \gamma_2$. The local current density is evaluated by considering a usual Schottky–Nordheim barrier. The results reproduce the two distinct slope regimes in FN plots when $F_M \in [2, 20] \, V \, \mu m^{-1}$ and are analyzed by taking into account the apparent formal area efficiency, the distribution $\rho$, and the slopes in the corresponding FN plot. Finally, we remark that our results from numerical solution of Laplace’s equation, for an array of conducting nano-emitters with uniform apex radii 50 nm but different local height, supports our theoretical assumptions and could used in orthodox CFE experiments to test our predictions.

Keywords: cold field electron emission, Schottky barrier, vacuum nanoelectronics, Fowler–Nordheim plots

(Some figures may appear in colour only in the online journal)
emission characteristics. To sidestep some of these difficulties, the cold field emission (CFE) community redirected efforts to study and produce different purpose LAFEs as nano-electronic devices, including carbon nano-structures which have near-ideal whisker-like shapes with hemispherical tips [4]. This choice is justified by a set of favorable properties like nanometer size tip, high chemical inerts, high electrical and thermal conductivity, and low manufacturing costs [3].

A relevant issue relating experimental and theoretical aspects of CFE studies is how to assess, with sufficient technologic reliability, several quantities related to the LAFE efficiency from measurable current–voltage characteristics. This is usually done using Fowler–Nordheim (FN) plots, which relates the macroscopic current density \( J_M \) to the applied (or macroscopic) electric field \( F_M \). The theory leading to Fowler–Nordheim-type (FN-type) equations suggests to draw FN-plots consisting of curves for \( \ln(J_M/F_M^2) \) versus \( 1/F_M \), but other variable combinations can be used as well (see for instance [1]). Actually, FN-plots may present a non-linear behavior and is necessary to set up a convenient theory that takes into account more realistic conditions under which a specific CFE experiment is performed in order to obtain a correct interpretation of the field enhancement factor (FEF) and other experimental outputs [5]. In this context, it’s important to discuss some general definitions as follow: the slope characterization parameter (alternatively called apparent FEF) is defined by

\[
\beta_{app} = -\frac{b\phi^{3/2}}{S^{lin}} \tag{1}
\]

where \( S^{lin} \) is the slope of a sufficient linear FN-plot for a given range of \( F_M \), \( \phi \) is the local work-function of the emitter, and \( b \) is the second Fowler–Nordheim (FN) constant (\( \approx 6.830 \, 890 \, eV \, m^{-3/2} \, V \, m^{-1} \)), the actual characteristic FEF, \( \gamma_C \), is defined as

\[
\gamma_C = \frac{F_C}{F_M} \tag{2}
\]

where \( F_C \) is the characteristic local barrier field. Then, the general relationship between \( \gamma_C \) and \( \beta_{app} \) has the form

\[
\gamma_C = \sigma_f \beta_{app} \tag{3}
\]

where \( \sigma_f \) is the relevant generalized slope correction factor.

Some situations can display nonlinear behavior in the corresponding FN-plots. This can be observed already in the pioneer work by Lauritsen who, in this PhD Thesis [6] obtained plots of the form log\( (i) \) versus 1/Voltage, where \( i \) is the macroscopic current emitted (see appendix C). He found experimentally that plots of the form log\( (i) \) versus 1/Voltage may be consisted of two straight lines, with a slight kink in the middle, using a single tip cylindrical wire geometry (see, for instance, figures 6 and 12 of that work). Lauritsen’s hypothesis was that the emission probably is coming from two different emission locations on the cylindrical wire, and that a crossover occurs in which location is dominant [6, 7]. Another example is related to the particular condition in which a large series resistance is found in the circuit between the high-voltage generator and the emitter’s regions. The interpretation of corresponding FN-plots was provided by Forbes and collaborators [8]. For both LAFE and single tip field emitters (STFEs), they showed that if the so-called CFE orthodox emission hypotheses [9] are not satisfied, the analysis of the results based on the elementary FN equation, as usually performed by experimentalists, can generate a spurious estimates for the true electrostatic FEF [9, 10]. On the other hand, recent theoretical works by one of authors [11, 12] explained how a slight positive curvature on FN-plots arises when a dependency between the apparent formal area efficiency (\( \alpha_f \)) and \( F_M \) is taken into account. For some assumptions of non-uniform conditions in the LAFES morphology, which amounts to consider a local FEF (\( \gamma \)) probability distribution \( \rho(\gamma) \) with exponential or Gaussian behavior, the orthodoxy test showed does not fail for practical circumstances. Despite this, it was possible to suggest experimental tests that can verify the proposed correction to the \( \beta_{app} \) values with statistical significance.

In this work, the authors investigate the conditions under which a clear crossover on the FN plots of CFE may appear, by assuming that it is only a consequence of the dependency between \( \sigma_f \) and \( F_M \). The electron emission from a conduction band on a particular LAFE location is described by FN-type equations with a Schottky–Nordheim (SN) barrier. Different from [11, 12], which considered continuous \( \gamma \) distributions, the present model assumes CFE through a non-uniform distribution of the local FEF \( \gamma \) on LAFE surface, which is described by a discrete asymmetric bimodal distribution for two distinct values \( \gamma_1 \) and \( \gamma_2 \), with \( \gamma_1 > \gamma_2 \) and \( \rho(\gamma_1) < \rho(\gamma_2) \). So, let us define

\[
q = \frac{\gamma_2}{\gamma_1} \tag{4}
\]

and

\[
r = \frac{\rho(\gamma_2)}{\rho(\gamma_1)}. \tag{5}
\]

The characteristic FEF of the LAFE is \( \gamma_1 \). From now on, whenever we mention this specific model we will indicate the characteristic FEF as \( \gamma_1 \), while \( \gamma_C \) will be used to refer to FEF in general conditions. Depending on the bimodal asymmetry parameter \( r \equiv \rho(\gamma_2)/\rho(\gamma_1) \), this contribution may lead to a clear crossover effect in the corresponding FN plots. Our results suggest that this simple mechanism, mimicking fluctuations of the individual emitter morphology on a LAFE surface, can justify a pronounced change in FN plots only as the emission is orthodox.

This paper is organized as follows. In section 2, the model and the equations for computing the local current density \( J_L \) are presented. We put this in perspective of previous studies discussing nonlinear behavior in the corresponding current–voltage measurements. Results are presented in section 3, focusing on the conditions where nonlinear FN plots can be found. We also discuss the results from numerical solution of Laplace’s equation, using an array of conducting nano-emitters with large apex radii (50 nm) but different heights. In section 4, the main conclusions are presented.
2. Current density calculations, model and previous works

The interpretation of experimental CFE outputs have often been done using the elementary FN-type equation, hereafter referred to as ‘elementary’ equations and theory, which considers the quantum-mechanical electron tunneling across a triangular barrier. However, it known since the 1950’s that this equation under-predicts current density by a factor of $10^2$–$10^3$ [13], specially in the case of bulk metals. A physically complete FN-type equation [14] for the local current density $J_L$ can be written as

$$J_L = \lambda_L a \phi^{-1} F_L^2 \exp(-\nu b \phi^{3/2}/F_L).$$

Here, $\nu$ is the barrier form correction factor associated with barrier shape, and $\lambda_L$ takes into account all other effects, including electronic structure, temperature, and corrections associated with integration over electronic states. In this work, we are restricted to the tunneling of electrons close to the Fermi level, so that we implicitly assume that $\nu$ takes into account this fact, and we refrain from explicitly adding a subscript ‘F’ to $\nu$. $a \approx 1.541 \times 10^{-6}$ A eV $V^{-2}$ and $b$ (the latter defined in introduction) are the first and second Fowler–Nordheim (FN) constants, respectively, while $\phi$ is the local work function and $F_L$ is the local electric field.

The correction associated with a SN barrier (used in Murphy–Good theory [15]), which accounts for the potential energy contribution resulting from the interaction of the electron with its image charge, is written as [13, 16]

$$\nu^{SN} \approx 1 - f + (1/6)f^2 \ln(f),$$

where $f=F_L/F_R$. Since $F_R \equiv (4\pi \epsilon_0 \phi_0^2)e^2$, where ‘$e’ is the positive elementary charge and $\epsilon_0$ is the electric constant, is the value of the external field for which height of the tunneling barrier vanishes, $f$ represents the scaled value of $F_L$. It plays a relevant role in CFE theory as a reliable criterion to test if the emission is orthodox or not [17]. Indeed, from a FN plot based on data points, it’s possible to derive values for $f^{extr}$ [9, 17] from the equation

$$f^{extr} = -\frac{s_f \eta(\phi)}{S^m(1/F_{M}^{ex})}.$$ (8)

If orthodox emission hypothesis is respected, all independent variables are linearly related to each other, and ‘$f’ can be used as a scaled value of the variable ‘$F_L’.” [9]. Then, in data analysis based on the orthodox emission hypothesis, equation (8) applies for all appropriate choices of independent and dependent variables and guarantees that the test for lack of orthodoxy works for any physically relevant form of FN plot. Let us remark that all quantities in equation (8) are directly accessible from CFE experiments or have been previously obtained for typical conditions in conductor materials [18]. The parameter $s_f(\phi) \equiv b \phi^{3/2}/F_R$ depends only on the work-function $\phi$, while $S^m$ is the slope of a sufficient linear FN-plot for a given range of the macroscopic electric field. The symbol $s_f$ represents the ‘fitting value’ of the slope correction function for the SN barrier, and can be approximated by $\approx 0.95$. It plays a similar role to the symbol $s$ in equation (3) and, since we restrict our work to SN barriers, it will replace $s$ from now on. Equation (8) provides estimates of the values of $f^{extr}$ that correspond to macroscopic-field values apparently inferred from experiment.

In this work, we constructed FN plots of the form $\ln(J_M/F_{M}^{ex})$ versus $1/F_M$. If the emission is orthodox, it’s possible to measure directly the values of $\gamma$, once the characteristic point ‘C’ over a LAFE device is defined as apex of the structure, representing the tip with the highest apex field.

Over an experimental LAFE surface, it is possible to find an almost continuous distribution of local $\gamma$ values. However, considering two most prominent emitting locations on LAFE, it is convenient to approximate such a distribution by a discrete one, with at most two distinct values of $\gamma$ ($j = 1, 2$), namely $\{\gamma_1 = \gamma_C, \gamma_2\}$, so that $\rho(\gamma_1) + \rho(\gamma_2) = 1$ with $\gamma_1 > \gamma_2$. Therefore, as already mentioned, our analysis is restricted to a bimodal distribution for the local FEFs of LAFE emitters. Indeed, any other location in the LAFE will be considered as having a FEF $\gamma_3 < \gamma_2$. Under this assumption, the corresponding local current density $J_{Lj} \approx 0$ so that we can restrict all following expressions to the values $j = 1$ and $2$.

Using equations (6) and (7), it is possible to write an expression for the site $j$ dependent local current density $J_{Lj}$ in a LAFE surface (see [12] and [19]) under the assumption of a SN barrier as

$$J_{Lj}(\phi, F_M, \gamma_j) = \lambda_L a \phi^{-1} \exp(\eta(\phi)/F_R^{ex}(\phi)/(\gamma F_M)^{\kappa}) \times \exp(-b \phi^{3/2}/(\gamma F_M)),$$

where $\kappa = 2 - \eta(\phi)/6$, the local field $F_L$ is replaced by $\gamma F_M$, and $F_M$ lies in the range $2 V \mu m^{-1} \leq F_M \leq 20 V \mu m^{-1}$, which are the typical conditions for CFE technologies that use nano-sized diameters. We remark that, depending on the barrier shape, $\lambda_L$ can assume values over a wide interval $0.005 < \lambda_L < 11$ [8]. In this work, we always consider $\lambda_L = 1$.

Summing up over the possible values of $\gamma_j$ the total $J_M$ current density is written as

$$J_M = iL/M = n_L \sum_{j=1,2} \rho(\gamma_j) J_{Lj}(\phi, F_M, \gamma_j) \Omega \Delta A_j^L/M.$$ (10)

where $i$ is the total emission current, and $\Omega \Delta A_j^L$ ($\Omega$ represents a typical notional area efficiency of a field emitter) is the notional emission area associated with the $j$th FEF-value which, in a first approximation, is considered to be independent of $F_M$ and the sum over $\rho(\gamma_j)$ is normalized. This approximation is very good since, for usual values of $F_M$ of the order of few $V \mu m^{-1}$, $\Omega \Delta A_j^L$ is only weakly dependent of $F_M$ (see section 3.3). Figure 1 shows a representation of the emitters used in LAFE and the corresponding ‘footprint’ of areas $L^2$.

We remember that equation (10) considers negligible the total emission contribution where the FEFs is effectively unity, i.e. at planar regions of footprint. For a plausible estimation of $\Omega$, which is expected to be much less than unit, we consider the following arguments: experimental values of macroscopic current density are often around 10 mA cm$^{-2}$. However,
consider only orthodox field emission), despite similar forms of FN plots have been obtained. For instance, in [25] the field emission properties of ‘flexible SnO2 nanoshuttle’ led to FN plots with a clear crossover presenting two quasi-linear sections. As pointed by Forbes [9], for both sections, as a consequence of the unorthodoxy emission (possible explanations include field-dependent changes in emitter geometry and/or changes in collective electrostatic screening effects), spurious FEF values have been found.

Lu [26] analyzed the field emission properties of well-aligned graphitic nano-cones synthesized on polished silicon wafers. The authors have investigated how the difference between the values of $\gamma$ corresponding to two types of emission sites on the LAFE surface affects the effective emission area for a given range of $F_M$ values. Unfortunately, some of their experimental outputs have shown also inconsistencies with the orthodox assumptions [9, 17]. For instance, consider the data shown in figure 2 of [26] together with the work function $\phi = 5\, \text{eV}$ of graphitic nano-cones. For anodes with diameter 1.5 mm, 2.0 mm, 2.5 mm and 3.0 mm and low $F_M$ regime (where a sufficient linear FN plot is obtained), we find, respectively, the following corresponding values for the scaled barrier field (see equation (8)): $f_{\text{exp}} = 0.46, 0.62, 0.79$ and 1.54. The first value has been found for $1/F_{\text{M exp}} = 0.06\, \text{um}\, \text{V}^{-1}$, while the three further values were found for $1/F_{\text{M exp}} = 0.0325\, \text{um}\, \text{V}^{-1}$. This suggests that, for all cases where non-linear behavior is observed in the corresponding FN plots, a closer investigation is required to provide a reliable interpretation of the results. In this specific study, this corresponds to the two smaller anodes. Moreover, for the larger anodes with nonuniform substrates, the orthodox test clearly fails, despite the linear behavior of the FN plots. Therefore, the corresponding FEFs indicated in these two cases and the corresponding emission areas extracted are questionable. Finally, it is important to emphasize that, very recently, Forbes provided a simple confirmation that the SN barrier is a better model for actual conducting emitters than the usual triangular barrier [28] to extract the emission areas. This can be noticed for a tungsten emitter (X89) data from Dyke and Trolan [20] and independent assessment of emitter area made by electron microscopy.

3. Results and discussions

3.1. Formal area efficiency: role of $\rho(\gamma)$ and $q$

Remembering that the formal area efficiency $\alpha_f$ is an experimentally accessible measure of the fraction of the LAFE surface that is actually emitting electrons, let us explicitly indicate its dependency on $F_M$ in equation (11) by writing

$$J_M = \alpha_f(F_M)J_{\text{SC}}.$$  

(13)

After some manipulations using equations (9)–(11) and equations (4)–(10), the following expression can be written (see appendix A):

$$\alpha_f(F_M) = \Omega \rho(\gamma)\{1 + \Gamma(q, r, \phi, F_M)\},$$  

(14)

where

\[ \text{Figure 1. Illustration of the single tips used in a LAFE with } j = 1 \text{ (left) and } j = 2 \text{ (right) (locations of an array of nanostructures) and the corresponding footprint of areas } (\Delta A_j). \text{ The related notional emission area } (\Omega \Delta A_j) \text{ is also indicated.} \]
\[ \Gamma(q, r, \phi, F_M) \equiv q^r \exp \left[ -b(q^{-1} - 1)^{1/2}/\Omega \Gamma_{M} \right]. \]  

Based on the actual experimental FEF values \cite{29}, we fix \( \gamma_1 = 690 \), while \( \gamma_2 \) is free to take different values. This is in accordance with the previous assumptions that the active LAFE emission sites fall into two classes, one of which is ‘more pointy’ than the other, and hence has a higher FEF. Changes in \( \gamma_2 \), with the corresponding changes in \( q \), are restricted to the condition that the electric field over the LAFE device does not exceed a few V/nm, while other complicated effects (as destruction of the LAFE device due to thermal effects) have been neglected.

Equation (14) makes it clear that \( \alpha_f \) depends on \( \rho(\gamma) \). This is illustrated in figure 2(a) that shows, for several values of \( q \) and for a typical value \( F_M = 10 \) V \( \mu \)m\(^{-1} \), the behavior of \( \alpha_f \) as \( \rho(\gamma) \) changes from \( 10^{-6} \) to \( 10^{-1} \). The values of \( \alpha_f \) were computed by using equations (14) and (15). For small values of \( q \) (e.g. \( q \lesssim 0.25 \)), figure 2(a) shows that \( \alpha_f \) assumes, approximately, the same values of \( \Omega \rho(\gamma) \). In this limit, \( \Gamma(q, r, \phi, F_M) \ll 1 \) for \( F_M = 10 \) V \( \mu \)m\(^{-1} \), and the only emitting spots on the LAFE surface are those with \( \gamma = \gamma_1 \) for all \( 10^{-6} \leq \rho(\gamma) \leq 10^{-1} \). This behavior is not observed for other values of \( q \geq 0.25 \) and smaller values of \( \rho(\gamma) \), when the contribution of the \( \gamma_1 = \gamma_2 \) regions for the electron emission become relevant as compared with \( \gamma = \gamma_1 \) regions. However, for larger values of \( \rho(\gamma) \), again the main emitting spots that contribute to \( \alpha_f \) are those with \( \gamma = \gamma_1 \). In this case, the curve bends upwards and \( \alpha_f \approx \Omega \rho(\gamma) \), which is observed as long as \( q \) is not so close to 1. Finally, when the limit \( q \rightarrow 1 \) is approached, the regions with \( \gamma = \gamma_2 \) contribute to \( \alpha_f \) for almost all range of values of \( \rho(\gamma) \). It is important to stress that, as \( q \) increases, a more uniform LAFE surface is built, with the presence of second-scale structures presenting close values of \( \gamma \). The results shown in figure 2(b) indicate the behavior of \( \alpha_f \) at a larger value \( F_M = 20 \) V \( \mu \)m\(^{-1} \). In this case, the results suggest that, for values of \( q \) close to unity, the regions of the LAFE surface \( \gamma = \gamma_2 \) also contribute to \( \alpha_f \) for low values of \( \rho(\gamma) \). As will be discussed in the next subsection, when \( \alpha_f = \Omega \rho(\gamma) \) and \( q \) is not so close to 1, \( \alpha_f \) depends on \( F_M \) leading to nonlinear behavior in the corresponding FN plots. Before discussing the behavior of the FN plots, we investigate how \( \alpha_f \) is related with \( q \) when both \( \rho(\gamma) \) and \( F_M \) are kept fixed.

Figure 3(a) shows the behavior of \( \alpha_f \) as a function of \( q \) for several values of \( \rho(\gamma) \) and \( F_M = 10 \) V \( \mu \)m\(^{-1} \). It’s possible to observe that, for higher values of \( \rho(\gamma) \), the wider is the interval where \( \alpha_f \) has a weak dependency on \( q \). In this regime, \( \alpha_f \approx \Omega \rho(\gamma) \) and, again, the regions which contributes to \( \alpha_f \) are only those with \( \gamma = \gamma_1 \). After the plateau, which increases as \( \rho(\gamma) \) increases, \( \alpha_f \) is expected to depends more strongly on \( q \). Figure 3(b) illustrate the behavior for \( F_M = 20 \) V \( \mu \)m\(^{-1} \). Now the plateau disappears for small values of \( \rho(\gamma) \) and, in this regime, \( \alpha_f \) depends on \( q \) in the entire displayed range. For larger values of \( \rho(\gamma) \), e.g. \( \rho(\gamma) \gtrsim 10^{-2} \), the plateau region is restored. However, even in this range of \( \rho(\gamma) \), it’s possible...
and the macroscopic electric field in the range of 2 V µm⁻¹ presents a constant behavior, suggesting that \( M \) does not depend exponentially on \( \gamma \), respectively.

In table 1, we list the characteristic FEF using the slopes (1) and (2). The data for an uniform LAFE with all local slopes are highlighted in (1) and (2)). The data for an uniform LAFE with all local slopes are highlighted in (1) and (2)). The data for an uniform LAFE with all local slopes are highlighted in (1) and (2)). The data for an uniform LAFE with all local slopes are highlighted in (1) and (2)).

In the inset, is shown the dependence of \( q \) for larger \( \gamma \) and \( F \). The correction, which was introduced very recently by one of authors \([11, 12]\), accounts for a nonlinear relationship between the macroscopic and the characteristic local current density, both of which are accessible experimentally, in principle.

Our results for the relation between the \( J_M \) and \( J_{AC} \) (see equations (10) and (12)) add valuable insights to the discussion about the physical reasons that are responsible for the crossover phenomenon in FN plots. Previous works suggest that the weak nonlinear dependency in FN plots could be traced back to a power law relation \( J_M \) to \( J_{AC} \), namely \( J_M \sim (J_{AC})^n \), where \( n \) presents a weak dependency on \( F \) but is strongly influenced by the LAFE geometry \([11, 12]\). This effect provides a more general method for a reliable assessment of the characteristic FEF \( \gamma_c \) from FN plots. A good approximation \( \gamma_{C}^{aprx} \) for the true FEF \( \gamma_c \) was derived in \([11, 12]\), which leads to

\[
\gamma_{C}^{aprx} = -\omega_n b_0^{-3/2} S_M = \omega_s J_F^{1/3},
\]

where \( s_I \) was introduced in equation (8). Under orthodox emission conditions the situation is that, if \( \alpha_f \) does not depend on \( F \), \( J_M \) generally over-predicts \( \gamma_c \) by approximately 5%. As anticipated in the section 2, \( s_I \approx 0.95 \) is verified for practical circumstances \([30]\). The correction \( F \), which was introduced very recently by one of authors \([11, 12]\), accounts for a nonlinear relationship between the macroscopic and the characteristic local current density, both of which are accessible experimentally, in principle.

In figure 4(b), we illustrate the behavior of \( J_M \) as a function of \( J_{AC} \) for the same parameters used in figure 4(a). We clearly identify that the same two slope patterns in the FN plots is observed for the dependency between \( J_M \) and \( J_{AC} \). Thus, it’s convenient to define \( \omega_1 \) and \( \omega_2 \) so that

\[
\gamma_{C}^{aprx} = -\omega_n b_0^{-3/2} S_M \quad (n = 1, 2),
\]

where \( \gamma_{C}^{aprx} \) and \( \gamma_{C}^{aprx} \) correspond to the approximations for the characteristic FEF using the slopes \( S_M \) and \( S_M^2 \), respectively. The results in figure 4(b), together with equations (13)–(15), suggest that:

\[
J_M \sim (J_{AC})^{(\omega_1} \quad (F < F^*),
\]

and

\[
J_M \sim (J_{AC})^{(\omega_2)} \quad (F > F^*).
\]

Here \( F^* \) denotes the value of the electric field at the crossover that separates the regions with two different slopes in FN plots as indicated in figure 4(b). In appendix B, we provide detailed derivation of the expressions that allow to extract the parameter ‘\( r \)’ from similar nonlinear FN plots in orthodox CFE experiments. ‘\( r \)’ is a function of \( F \), \( S_M, S_M^2 \) as well as of the local work function that through the exponent \( \kappa \).

The results in table 1 indicate that \( \omega = \omega_2 \approx 1.0 \) in the low \( F_M \) regime. The slope \( S_M \) provides information on the characteristic FEF, \( \gamma_c \sim \gamma \). In this regime, the results reinforce the interpretation that CFE is orthodox, as confirmed by the extracted value \( F_M^{1/\gamma} \) (see equation (8) of this work, and table 2 in [9], for \( \phi = 3.5 \text{ eV} \)). On the other hand, for
high values of $F_M$, table 1 indicates $\omega_2 > 1$, which means that, besides the regions with $\gamma = \gamma_1$, the regions with $\gamma = \gamma_2$ also contributes in a significant way to $\alpha_f$. This suggests an important result that might be suitable for experimental observation: when $\omega_2 > 1$ in the corresponding range of $F_M$, the slope $S_{M}$ provides information regarding the macroscopic FEF, $\gamma_2 < \gamma_1$. A good estimate of the real characteristic FEF would be $\gamma_{\text{approx}} = -\omega_2 \phi_{\text{SN}}$, for $F_M > F^*$. For this ansatz, the errors do not exceed 15%, as indicated in table 1 for $q \approx 0.43$. More interestingly, the values of $f_{2}^{\text{extr}}$ shown on table 1 (extracted from the range $F_M > F^*$), confirm that the emission is also orthodox.

At this point, we emphasize the importance of measuring $\omega$. To see this, let us consider two different LAFE devices: (i) the first one is characterized by uniform local FEFs with $\gamma_1 = \gamma_2 = 552$ (and $q = 1$); (ii) the second one is composed by regions with two distinct FEFs values, namely $\gamma_1 = 690$ and $\gamma_2 = 552$ ($q = 0.8$) and $\rho(\gamma_1) = 10^{-6}$. The device (i) represents an ideal homogeneous array composed by the same second-scale structures. Device (ii) represents an array where most of the second-scale structures are characterized by $\gamma = \gamma_2$, but there is a small probability to find regions with $\gamma = \gamma_1$, as already discussed in the characterization of a non-uniform LAFE surface. Both corresponding FN plots are shown in figure 4(a), but the two curves for $q = 1$ and $q = 0.8$ are actually indistinguishable. However, the results in the inset show that, while $\alpha_f$ is independent of $F_M$ in case (i), $\alpha_f$ does depend on $F_M$ for the device (ii). These observations culminate with the following conclusions: although FN plots present the same behavior for two distinct LAFE surfaces, in case (i) the corresponding slope provides the correct value of the characteristic FEF. On the other hand, the device (ii) has characteristic FEF $\gamma_C = \gamma > \gamma_2$. Thus, the linear aspect of the FN plot does not mean, necessarily, that the area of emission does not depend on the macroscopic field. Indeed, the results in the inset of figure 4(a) for device (ii) hints at change in the value of $\alpha_f$ by, at least, two orders of magnitude. Moreover, despite the linear aspect, and the orthodox CFE, the FN slope can not measure, necessarily, the characteristic FEF; $\gamma_C$. This reflects the importance of measure $\omega_m$, so that $\omega_m > 1$ suggests this behavior. Finally, we remark that if $\omega_m \approx 1$ for a given $F_M$ range in CFE experiments, it just indicates that $\alpha_f$ does not depend (or weakly depends) on the $F_M$ in that range.
of FEF for single emitters on a conducting plane, as a function of aspect ratio, can be found in [34]. The electric potential distribution on the integration domain was calculated using a finite element method scheme (software COMSOL v4.3b). This allows to calculate the electric field distribution over the LAFE device, as well as the local emitting current density using equation (9). We consider the same work function, $\phi = 3.5 \text{ eV}$ used in the previous section. Figure 5 shows the radial integration domain (emitting location) and the used boundary conditions for an idealized situation in which a single tip is placed in the center of a $L \times L$ location. The line at the right side boundary generates an enclosing cylindrical surface (ECS) when it is rotated by $2\pi$ around the position where the left boundary lies. In this way, the electric field component normal to this plane is locally zero everywhere. Since a similar geometry may be found in the neighboring locations, with the exception that the tips do not necessarily lie in the corresponding location centers, the resulting field may be distorted as a consequence of the superposition of individual field at each location. Thus, there is an overall screening effect inside each ECS. In this work we use $L = 5 \, h_1$ and $d = \sqrt{2} L$, so that the screening is negligible (the emitters can be considered as isolated) and the field lines can be considered parallel and vertically aligned [35].

Figure 6. Normalized local current density map ($J_L/J_{KC}$) for emitter with $\gamma_m = 678$ at macroscopic electric fields $2 \text{ V } \mu \text{m}^{-1}$ and $20 \text{ V } \mu \text{m}^{-1}$.

The line at $4V \text{ m}^{-1}$ for emitter, while hollow symbols indicate corresponds to probabilities to found decreases).

Figure 7. Comparison between $\alpha_f$ as a function of $F_M$ for two different conditions. Solid lines indicate the solutions obtained from equations (13)–(15), for $\Omega \approx 10^{-7}$, while hollow symbols indicate the results from numerical solution of Laplace’s equation.

Figure 8. Comparison of the FN plots for the same conditions shown in figure 7. Solid lines indicate results obtained from equations (13)–(15), for $\Omega \approx 10^{-7}$, while hollow symbols correspond to the numerical solution of Laplace’s equation.

The macroscopic current density was calculated as follow:

$$J_M = \frac{1}{L^2} \left\{ \rho(\gamma_1) \sum_{\text{cap}} j_{\text{cap}} L \Delta A_1^\text{cap} + \rho(\gamma_2) \sum_{\text{cap}} j_{\text{cap}} L \Delta A_2^\text{cap} \right\},$$

(20)

where the sum is computed over all spherical cap surface area and $\rho(\gamma_1)$ and $\rho(\gamma_2)$ correspond to probabilities to found a location of LAFE that contains a nanostructure with characteristic FEF $\gamma_1$ and $\gamma_2$, respectively. In this case, $\alpha_f$ may changes essentially for two reasons: (i) the emitters with FEFs $\gamma_2$ contribute to the overall current; (ii) the notional area on each emitter increases slowly as $F_M$ increases, as shown in figure 6. To illustrate this dependency, we have computed the normalized local current density map ($J_L/J_{KC}$) at macroscopic electric fields $2 \text{ V } \mu \text{m}^{-1}$ and $20 \text{ V } \mu \text{m}^{-1}$. In fact, it is possible to observe a clear increase of the notional area of a single nano-emitter, as first suggested by Abbott and Henderson [36] in 1939. In figure 7, we show a comparison for the dependence of $\alpha_f$ as a function of $F_M$ for two methodologies: the one based on equations (14) and (15), and that obtained by solving Laplace’s equation. In the latter, using the dimensions previously discussed, $\Delta \Omega R^2 L^2 \sim 10^{-7}$. Moreover, the results suggest that $\Delta \Delta A_1^\text{cap}$ is weakly dependent on $F_M$ in the low $F_M$ regime (for instance, considering $\rho(\gamma_1) = 10^{-6}$, this limit corresponds to $F_M \lesssim 4 \text{ V } \mu \text{m}^{-1}$ and increases as $\rho(\gamma_1)$ decreases). Then, in equation (10) we have used the reasonable proportionality $\Omega \Delta A_1^\text{cap} \sim \pi R^2$, which means to use $\Omega \approx 10^{-7}$ in equation (14). It’s possible to observe the good agreement between two results. A small deviation occurs in low $F_M$ regime, which can be justified because the emitting area of a single tip structure grows very slowly as the macroscopic electric field increases (see figure 6). However, an important result is that this very subtle effect does not affect the form of FN plots. Figure 8 shows the nonlinear behavior of FN plots for
actual emitters, considering $10^{-6} \leq \rho(\gamma) \leq 10^{-1}$ and $q = 0.51$, showing the excellent agreement with the results from equations (14) and (15).

4. Conclusions

In this work, we present a theoretical explanation for the crossover in the behavior of the FN plots, commonly found for large area field emitters with irregular morphology. The latter is assumed to lead to a more prominent emitting locations with FEFs distributed approximately as a bimodal distribution. Our results suggest an orthodox field electron emission for two quasi-linear sections of FN plots as the formal area efficiency is the sole cause of the crossover, in a typical range $F_M \in [2, 20]$ V $\mu$m$^{-1}$. For such situations, we propose a physically relevant ansatz leading to the interpretation of the slopes in FN plots as a function of the $\gamma$ and $r$ asymmetry parameters characterizing $\rho(\gamma)$. Finally, the results from solution of Laplace’s equation for an array of conducting nano-emitters supports our theoretical assumptions regarding the information provided by FN plots, which can be tested if CFE experiments are orthodox.

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Appendix A. Derivation of $\alpha_I$

According to equations (10) and (5), the macroscopic current density for a LAFE with two prominent emitter locations can be written as

$$J = \frac{m_e}{A_M} \left\{ \rho(\gamma)J_1^L + \rho(\gamma)J_2^L \right\}. \quad (A.1)$$

We emphasize that, in our theory, $\Delta A_L^j$ represents the footprint area of $j$th post-like emitter. $\Omega \Delta A_L$ represents the corresponding notional emission area. Then, using equation (9) (for $\lambda_L = 1$), assuming that $\Omega$ is weakly field dependent, and $\Delta A_L^j = \Delta A_L^j = \Delta A_L$, equation (A.1) becomes

$$J_M = \frac{m_e \Omega \Delta A_L}{A_M} \left[ \rho(\gamma)J_1^L + \rho(\gamma)J_2^L \right] \exp \left\{ -b\phi^{3/2}/\gamma F_M \right\} \exp \left\{ -b\phi^{3/2}r^q \gamma F_M \right\}. \quad (A.2)$$

Once the term $\exp \left\{ -b\phi^{3/2}/\gamma F_M \right\}$ appears in both terms, we take into account that $m_I \Delta A_L = A_M$ to simplify equation (A.2) to

$$J_M = \Omega \rho(\gamma) \left( 1 + q^r \exp \left\{ -b(\gamma - 1)\phi^{3/2}/(\gamma F_M) \right\} \right) J_{LC}. \quad (A.3)$$
where $I_{AC}$ is given by equation (12). Then, making use of the notation introduced in equation (13), the formal area efficiency can be given by:

$$
\alpha_f(F_M) = \Omega \rho(\gamma) \left[ 1 + q^r \exp\left(-b(q^r - 1)\phi^{3/2}/(\gamma F_M)\right) \right] 
= \Omega \rho(\gamma) \left[ 1 + \Gamma(q, r, \phi, F_M) \right].
$$  \hfill (A.4)

A generalization of equation (A.4) that consider a LAFE with a larger number of tips types, i.e. with $\{\gamma_j\}$, can be easily derived, leading to

$$
\alpha_f(F_M) = \Omega \rho(\gamma) \sum_{j=1}^{n} q_j^r \exp\left[-b(q_j^r - 1)\phi^{3/2}/(\gamma_j F_M)\right],
$$  \hfill (A.5)

where $q_j = \gamma_j/\gamma$ and $r_j = \rho(\gamma_j)/\rho(\gamma)$.

### Appendix B. Extraction of parameter ‘r’ from nonlinear FN plots in orthodox CFE experiments

If CFE experiments are orthodox and the FN plots present two clear-cut quasi-linear sections, it’s possible to provide an estimation of the parameter ‘r’ defined in equation (5). Let the macroscopic electric field at the crossover point that separates the regions with two different slopes be noted by $F^*$, as illustrated in figure 4(b). At this point, it is expected that the contribution for macroscopic current density from the locations with FEF $\gamma_i$ is the same as those from the locations with $\gamma_i$.

This lead to

$$
\rho(\gamma)(\gamma F^*)^r \exp[-b(\gamma F^*)^{3/2}/(\gamma F_M)] = \rho(\gamma_2)(\gamma_2 F^*)^r \exp[-b(\gamma_2 F^*)^{3/2}/(\gamma_2 F_M)].
$$  \hfill (B.1)

From equation (B.1), it’s possible to write the product $rq^r$ as

$$
q^r = g(F^*) \left[ \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right],
$$  \hfill (B.2)

where $g(F^*) \equiv \exp[-b(\gamma F^*)^{3/2}/F^*]$. From the expressions for the two distinct slopes in the same corresponding FN plot, $\gamma_1 = -s_b\phi^{3/2}/S_M$ and $\gamma_2 = -s_b\phi^{3/2}/S_M$, it’s possible to write

$$
\frac{1}{\gamma_1} - \frac{1}{\gamma_2} = -\frac{1}{s_b\phi^{3/2}}(S_M - S_M^*). \hfill (B.3)
$$

Finally, using equations (B.2) and (B.3), the parameter $r$ is given by:

$$
r = \exp\left[\left(S_M - S_M^*\right)/s_b F^*\right] \left[\frac{S_M}{S_M^*}\right]^{-3/2}.
$$  \hfill (B.4)

### Appendix C. Analysis of experimental Lauritsen plots (LPs)

In this section, we analyze the orthodoxy test in the data presented in the pioneering Lauritsen PhD Thesis [6] (see figure C1). For a tungsten emitter examined by Lauritsen a long time ago [6], a series of current–voltage measurements is available.

If $i$ is the total emission current, and $V$ the applied voltage, it’s possible to write:

$$
\log_{10}\left\{\frac{i}{V^2}\right\} = \log_{10}\{i\} + 2\log_{10}\left\{\frac{1}{V}\right\}. \hfill (C.1)
$$

Let us define:

$$
Y_{LP} \equiv \log_{10}\{i\},
$$  \hfill (C.2)

and

$$
X_{FNP} = X_{LP} \equiv \frac{1}{V}. \hfill (C.4)
$$

In equations (C.2)–(C.4), the subscript ‘FNP’ indicates the variables used in Fowler–Nordheim plots and ‘LP’ the corresponding ones used in Lauritsen plot. The analysis of $i – V$ data shown in figure C1 yields the results shown in table C1. In the later, we indicate the coordinates used in regions (I) and (II) of figure C1 and the related results. By assuming the local work function of tungsten $\phi = 4.5 eV$ ($\eta(\phi) = 4.6368$), it is possible to see that the Lauritsen measurements confirm an orthodox field electron emission [9]. The parameter ‘r’ can be estimated from equation (B.4) using $s_b \approx 0.95$. From figure C1, we have estimated the crossover abscise coordinate $1/V^* \approx 1.06 \times 10^{-4} V^{-1}$. Then, using equation (B.4) and table C1, $r \approx 14318$. Also from $S_M^*$ and $S_M^*$ values, $q \approx 0.53$.

The results of simulation of the LP plot using our model (see equation (A.3)), for $r \approx 14318$ and $q \approx 0.53$, are shown in figure C2 for $\gamma = 677$, confirming the similar form of LPs using $i – V$ variables and $J_M/F_M$.

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