On recovering solutions for SPDEs from their averages

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Abstract

We study linear stochastic partial differential equations of parabolic type. We consider a
new boundary value problem where a Cauchy condition is replaced by a prescribed average
of the solution either over time and probabilistic space for forward SPDEs and over time for
backward SPDEs. Well-posedness, existence, uniqueness, and a regularity of the solution
for this new problem are obtained. In particular, this can be considered as a possibility to
recover a solution of a forward SPDE in a setting where its values at the initial time are
unknown, and where the average of the solution over time and probability space is observable,
as well as the input processes.

Keywords: Stochastic partial differential equations (SPDEs), non-local condition, backward
SPDEs, inverse problems, recovery of initial value.

1 Introduction

The paper studies boundary value problems for stochastic partial differential equations of the
second order. These equations have many applications and were widely studied; see e.g. [1, 2, 7, 8, 10, 12, 14, 15, 19, 22, 23, 24, 28, 29, 30, 31, 32, 35, 37, 38, 39]. Forward parabolic SPDEs are
usually considered with a Cauchy condition at initial time, and backward parabolic SPDEs are
usually considered with a Cauchy condition at terminal time. However, there are also results
for SPDEs with boundary conditions that mix the solution at different times that may include
initial time and terminal time. This category includes stationary type solutions for forward
SPDEs; see, e.g., [3, 4, 6, 20, 27, 30, 31, 36], and the references therein. Related results were
obtained for periodic solutions of SPDEs in [5, 21, 23]. Some results for parabolic equations
and stochastic PDEs with non-local conditions replacing the Cauchy condition were obtained in \cite{9,11,13,15,17}.

The present paper addresses these and related problems again. We consider forward and backward SPDEs with the Dirichlet condition at the boundary of the state domain; the equations are of a parabolic type. For forward SPDEs, a Cauchy condition at initial time is replaced by a condition requiring a prescribed average of the solution over time and probabilistic space. For backward SPDEs, a Cauchy condition at terminal time is replaced by a condition requiring a prescribed average of the solution over time. This is a novel setting for SPDEs; for deterministic parabolic equations, a related result was obtained in \cite{18}. We obtained sufficient conditions for existence and regularity of the solutions in $L^2$-setting (Theorems \ref{thm1} and \ref{thm2} below). This result can be interpreted as a possibility to recover a parabolic diffusion from its time-average when the values at the initial time are unknown.

\section{The problem setting and definitions}

Assume that we are given a standard complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a right-continuous filtration $\mathcal{F}_t$ of complete $\sigma$-algebras of events, $t \geq 0$, such that $\mathcal{F}_0$ is the $\mathbb{P}$-augmentation of the set $\{\emptyset, \Omega\}$. We are given also a $N$-dimensional Wiener process $w(t)$ with independent components; it is a Wiener process with respect to $\mathcal{F}_t$.

Let $D \subset \mathbb{R}^n$ be an open bounded domain with a $C^2$-smooth boundary $\partial D$. Let $T > 0$ be given, and let $Q = D \times [0,T]$.

We consider the following boundary value problem in $Q$ for forward SPDEs

\begin{align*}
d_t u &= (Au + \varphi) \, dt + \sum_{i=1}^{N} [B_i u + h_i] \, dw_i(t), \quad t \geq 0, \quad (2.1) \\
u(x,t,\omega)|_{x \in \partial D} &= 0, \quad (2.2) \\
\mathbb{E} \left( \kappa u(x,T) + \int_0^T \varrho(t) u(x,t) \, dt \right) &= \mu(x). \quad (2.3)
\end{align*}

Here $\mu$, $\varphi$, and $h_i$ are given inputs, $u$ is a sought out solution.

In addition, we will study the following boundary value problem in $Q$ for backward SPDEs

\begin{align*}
d_t u + (Au + \varphi) \, dt + \sum_{i=1}^{N} B_i \chi_i \, dt &= \sum_{i=1}^{N} \chi_i dw_i(t), \quad t \geq 0, \quad (2.4) \\
u(x,t,\omega)|_{x \in \partial D} &= 0 \quad (2.5) \\
\kappa u(x,0) + \int_0^T \varrho(t) u(x,t,\omega) \, dt &= \psi(x) \quad \text{a.s. for } x \in D. \quad (2.6)
\end{align*}

Here $\mu$ and $\varphi$ are given inputs, and the set of functions $(u, \chi_1, \ldots, \chi_N)$ is a sought out solution.
Here \( u = u(x,t,\omega), \psi = \psi(x,\omega), \varphi = \varphi(x,t,\omega), h_i = h_i(x,t,\omega), \chi_i = \chi_i(x,t,\omega), (x,t) \in Q, \omega \in \Omega. \)

In \((2.3)\), \( \kappa \in \mathbb{R}, \rho(t) \) is a measurable and bounded non-random function.

In these SPDEs, \( A \) and \( B \) are differential operators defined as

\[
A v = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{n} a_{ij}(x) \frac{\partial v}{\partial x_j}(x) \right) + a_0(x)v(x),
\]

\[
B_i v = \frac{dv}{dx}(x) \bar{\beta}_i(x,t,\omega) + \beta_i(x,t,\omega)v(x), \quad i = 1, \ldots, N,
\]

where functions \( \beta_j(x,t,\omega) : \mathbb{R}^n \times [0,T] \times \Omega \rightarrow \mathbb{R}^n, \bar{\beta}_i(x,t,\omega) : \mathbb{R}^n \times [0,T] \times \Omega \rightarrow \mathbb{R} \), and \( \varphi(x,t,\omega) : \mathbb{R}^n \times [0,T] \times \Omega \rightarrow \mathbb{R} \) are progressively measurable with respect to \( \mathcal{F}_t \) for all \( x \in \mathbb{R}^n \). The function \( \psi(x,\omega) : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R} \) is \( \mathcal{F}_T \)-measurable for all \( x \in \mathbb{R}^n \).

Conditions for the coefficients

To proceed further, we assume that Conditions \((2.1)-(2.4)\) remain in force throughout this paper.

**Condition 2.1** The functions \( a_{ij}(x) : D \rightarrow \mathbb{R} \) and \( a_0(x) : D \rightarrow \mathbb{R} \) are continuous and bounded, and there exist continuous bounded derivatives \( \partial a_0(x)/\partial x_i, \partial a_{ij}(x)/\partial x_i \), \( i,j = 1, \ldots, n \). In addition, we assume that the matrix \( a = \{ a_{ij} \} \) is symmetric. The functions \( \bar{\beta}_i(x,t,\omega) \) and \( \beta_i(x,t,\omega) \) are bounded and differentiable in \( x \) for a.e. \( t,\omega \), and the corresponding derivatives are bounded.

It follows from this condition that there exist modifications of \( \beta_i \) such that the functions \( \beta_i(x,t,\omega) \) are continuous in \( x \) for a.e. \( t,\omega \). We assume that \( \beta_i \) are such functions.

**Condition 2.2** \( \bar{\beta}_i(x,t,\omega) = 0 \) for \( x \in \partial D, i = 1, \ldots, N. \)

**Condition 2.3** [Superparabolicity condition \((35)\)] There exists a constant \( \delta > 0 \) such that

\[
y^T a(x) y - \frac{1}{2} \sum_{i=1}^{N} |y^T \bar{\beta}_i(x,t,\omega)|^2 \geq \delta |y|^2 \quad \forall y \in \mathbb{R}^n, \quad (x,t) \in D \times [0,T], \quad \omega \in \Omega. \quad (2.7)
\]

If \( \kappa \neq 0 \) and \( \rho \equiv 0 \), then the boundary value problems above are ill-posed, with ill-posed Cauchy conditions \( u(x,T) = \mu(x) \) or \( u(x,0) = \psi(x) \) respectively. This case is excluded from consideration by imposing the following restrictions.

**Condition 2.4** (i) \( \rho(t) \geq 0 \) a.e. and \( \kappa \geq 0. \)

(ii) For problem \((2.7)-(2.8)\), we assume that there exists \( T_1 \in (0,T] \) such that

\[
\text{ess inf}_{t \in [0,T_1]} \rho(t) > 0.
\]
(iii) For problem \((2.4)-(2.6)\), we assume that the function \(g(t)\) is continuous at \(t = T\) and that there exists \(T_1 \in [0,T)\) such that

\[
\text{ess inf}_{t \in [T_1, T]} g(t) > 0.
\]

We do not exclude an important special case where the function \(\varphi\) is deterministic, and \(h_i \equiv 0, B_i \equiv 0\) for all \(i\). In this case, the boundary value problem is deterministic, and \(\chi_i \equiv 0\) \((\forall i)\) for backward equations.

**Spaces and classes of functions**

We denote by \(|\cdot|\) the Euclidean norm in \(\mathbb{R}^k\), and \(\overline{G}\) denote the closure of a region \(G \subset \mathbb{R}^k\). We denote by \(\|\cdot\|_X\) the norm in a linear normed space \(X\), and \((\cdot, \cdot)_X\) denote the scalar product in a Hilbert space \(X\).

Let us introduce some spaces of real valued functions.

Let \(G \subset \mathbb{R}^d\) be an open domain. For \(q \geq 1\), we denote by \(L_q(G)\) the usual Banach spaces of classes of equivalency of measurable by Lebesgue functions \(v : G \to \mathbb{R}\), with the norms \(\|v\|_{L_q(G)} = \left(\int_G |v(x)|^q dx\right)^{1/q}\). For integers \(m \geq 0\), we denote by \(W^m_q(G)\) the Sobolev spaces of functions that belong to \(L_q(G)\) together with the distributional derivatives up to the \(m\)th order, \(q \geq 1\), with the norms \(\|v\|_{W^m_q(G)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_{L_q(G)}^q\right)^{1/q}\). Here \(D^k = D_{k_1} \cdots D_{k_d}\) is the partial derivative of the order \(|k| = \sum_{i=1}^d k_i\), where \(k = (k_1, \ldots, k_d) \in \mathbb{Z}^d\), \(k_i \geq 0\), \(D_i = \partial^{k_i}/dx^{k_i}\).

Let \(H^0 \triangleq L_2(D)\), and let \(H^1 \triangleq W^1_2(D)\) be the closure in the \(W^1_2(D)\)-norm of the set of all smooth functions \(u : D \to \mathbb{R}\) such that \(u|_{\partial D} \equiv 0\). Let \(H^2 = W^2_2(D) \cap H^1\) be the space equipped with the norm of \(W^2_2(D)\). The spaces \(H^k\) are Hilbert spaces, and \(H^k\) is a closed subspace of \(W^2_2(D)\), \(k = 1, 2\).

Let \(H^{-1}\) be the dual space to \(H^1\), with the norm \(\|\cdot\|_{H^{-1}}\) such that if \(u \in H^0\) then \(\|u\|_{H^{-1}}\) is the supremum of \((u,v)_{H^0}\) over all \(v \in H^1\) such that \(\|v\|_{H^1} \leq 1\). \(H^{-1}\) is a Hilbert space.

Let \(C_0(\overline{D})\) be the Banach space of all functions \(u \in C(\overline{D})\) such that \(u|_{\partial D} \equiv 0\) equipped with the norm from \(C(\overline{D})\).

We shall write \((u,v)_{H^0}\) for \(u \in H^{-1}\) and \(v \in H^1\), meaning the obvious extension of the bilinear form from \(u \in H^0\) and \(v \in H^1\).

We denote by \(\tilde{\ell}_k\) the Lebesgue measure in \(\mathbb{R}^k\), and we denote by \(\tilde{\mathcal{B}}_k\) the \(\sigma\)-algebra of Lebesgue sets in \(\mathbb{R}^k\).

We denote by \(\tilde{\mathcal{P}}\) the completion (with respect to the measure \(\tilde{\ell}_1 \times \mathcal{P}\)) of the \(\sigma\)-algebra of subsets of \([0,T] \times \Omega\), generated by functions that are progressively measurable with respect to \(\mathcal{F}_t\).
We introduce the spaces
\[ C_k \triangleq C \left( [s, T); H^k \right), \quad W^k \triangleq L^2([0, T], B_1; H^k), \quad k = -1, 0, 1, 2, \]
and the spaces
\[ Y^k(s, T) \triangleq W^1(s, T) \cap C_{k-1}, \quad k = 1, 2, \]
with the norm \( \| u \|_Y \triangleq \| u \|_{W^k} + \| u \|_{C_{k-1}}. \)

For \( \theta \in [0, T], \) we introduce a space \( U_{\theta} \) of functions \( \varphi \in W^0 \) such that \( \varphi(\cdot, t) = \varphi(\cdot, \theta) + \int_{\theta}^{t} \hat{\varphi}(\cdot, s) ds \) for \( t \in [\theta, T] \) for some \( \hat{u} \in L_1([\theta, T]; H^0), \) with the norm
\[ \| \varphi \|_{U_{\theta}} \triangleq \| \varphi \|_{W^0} + \| \varphi(\cdot, \theta) \|_{H^0} + \int_{\theta}^{T} \| \hat{\varphi}(\cdot, t) \|_{H^0} dt. \]
In particular, \( \varphi(\cdot, t) \) is continuous in \( H^0 \) in \( t \in [T-\theta, T]. \) If \( \theta = T \) then \( U_{\theta} = W^0 = L_2(D \times [0, T]). \)

In addition, we introduce the spaces
\[ X^k \triangleq L^2([0, T] \times \Omega, \bar{\mathcal{P}}; \ell_1 \times \mathbf{P}; H^k), \]
\[ Z^k_{\ell} \triangleq L^2(\Omega, \mathcal{F}_t, \mathbf{P}; H^k), \]
\[ C^k_{Z} \triangleq C \left( [0, T]; Z^k_{Z} \right), \quad k = -1, 0, 1, 2, \]
and the spaces
\[ Y^k \triangleq X^k \cap C_{k-1}^{Z}, \quad k = 1, 2, \]
with the norm \( \| u \|_{Y^k(s, T)} \triangleq \| u \|_{X^k} + \| u \|_{C_{k-1}^{Z}}. \)

We introduce Banach spaces \( Z^k_{P} = L_p(\Omega, \mathcal{F}, \mathbf{P}; H^k), \) \( p \in [1, +\infty]. \)

The spaces \( W_k, X^k, \) and \( Z^k_{\ell}, \) are Hilbert spaces.

**Proposition 2.1** Let \( \zeta \in X^0, \) and let a sequence \( \{ \zeta_k \}_{k=1}^{+\infty} \subset L^\infty([0, T] \times \Omega, \ell_1 \times \mathbf{P}; C(D)) \) be such that all \( \zeta_k(\cdot, t, \omega) \) are progressively measurable with respect to \( \mathcal{F}_t, \) and \( \| \zeta - \zeta_k \|_{X^0} \to 0 \)
as \( k \to +\infty. \) Let \( t \in [0, T] \) and \( j \in \{1, \ldots, N\} \) be given. Then the sequence of the integrals
\[ \int_{0}^{t} \zeta_k(x, s, \omega) dw_j(s) \]
converges in \( Z^0_{\ell} \) as \( k \to \infty, \) and its limit depends on \( \zeta, \) but does not depend on \( \{ \zeta_k \}. \)

**Proof** follows from completeness of \( X^0 \) and from the equality
\[ E \int_{0}^{t} \| \zeta_k(\cdot, s, \omega) - \zeta_m(\cdot, s, \omega) \|_{H^0}^2 ds = \int_{D} dx E \left[ \int_{0}^{t} (\zeta_k(x, s, \omega) - \zeta_m(x, s, \omega)) dw_j(s) \right]^2. \]

**Definition 2.1** For \( \zeta \in X^0, \) \( t \in [0, T], \) \( j \in \{1, \ldots, N\}, \) we define \( \int_{0}^{t} \zeta(x, s, \omega) dw_j(s) \) as the limit in \( Z^0_{\ell} \) as \( k \to \infty \) of a sequence \( \int_{0}^{t} \zeta_k(x, s, \omega) dw_j(s), \) where the sequence \( \{ \zeta_k \} \) is such as in Proposition 2.1.

Sometimes we shall omit \( \omega. \)
The definition of solution for forward SPDEs

**Definition 2.2** Let \( u \in Y^1, \varphi \in X^{-1}, \) and \( h_i \in X^0. \) We say that equations (2.1)-(2.2) are satisfied if

\[
\begin{align*}
    u(\cdot, t, \omega) &= u(\cdot, 0, \omega) + \int_0^t (Au(\cdot, s, \omega) + \varphi(\cdot, s, \omega)) \, ds \\
    &\quad + \sum_{i=1}^N \int_0^t [B_i u(\cdot, s, \omega) + h_i(\cdot, s, \omega)] \, dw_i(s)
\end{align*}
\]

for all \( t \in [0, T], \) and this equality is satisfied as an equality in \( Z_T^{-1}. \)

Note that the condition on \( \partial D \) is satisfied in the sense that \( u(\cdot, t, \omega) \in H^1 \) for a.e. \( t, \omega. \) Further, \( u \in Y^1, \) and the value of \( u(\cdot, t, \omega) \) is uniquely defined in \( Z_T^0 \) given \( t, \) by the definitions of the corresponding spaces. The integrals with \( dw_i \) in (2.8) are defined as elements of \( Z_T^0. \) The integral with \( ds \) in (2.8) is defined as an element of \( Z_T^{-1}. \) In fact, Definition 2.2 requires for (2.1) that this integral must be equal to an element of \( Z_T^0 \) in the sense of equality in \( Z_T^{-1}. \)

The definition of solution for backward SPDEs

**Definition 2.3** Let \( u \in Y^1, \chi_i \in X^0, i = 1, ..., N, \) and \( \varphi \in X^{-1}. \) We say that equations (2.4)-(2.5) are satisfied if

\[
\begin{align*}
    u(\cdot, t, \omega) &= u(\cdot, T, \omega) + \int_t^T (Au(\cdot, s, \omega) + \varphi(\cdot, s, \omega)) \, ds \\
    &\quad + \sum_{i=1}^N \int_t^T B_i \chi_i(\cdot, s, \omega) \, ds - \sum_{i=1}^N \int_t^T \chi_i(\cdot, s) \, dw_i(s)
\end{align*}
\]

for all \( r, t \) such that \( 0 \leq r < t \leq T, \) and this equality is satisfied as an equality in \( Z_T^{-1}. \)

Similarly to Definition 2.2 the integral with \( ds \) in (2.9) is defined as an element of \( Z_T^{-1}. \) Definition 2.3 requires for (2.4) that this integral must be equal to an element of \( Z_T^0 \) in the sense of equality in \( Z_T^{-1}. \)

3 The main result

For the case of forward SPDEs, the following result was obtained.

**Theorem 3.1** Assume that \( \theta = T \) if \( \kappa = 0 \) and \( \theta \in [0, T) \) if \( \kappa \neq 0. \) Problem (2.1)-(2.3) has a unique solution \( u \in Y^1 \) for any \( \mu \in H^2, h_i \in X^1, \) and any \( \varphi \in X^0 \) such that \( \bar{\varphi} \in U_\theta, \) where
ϕ(x,t) \triangleq E\varphi(x,t,\omega). Moreover,

$$\|u\|_{Y^1} \leq C \left( \|u\|_{H^2} + \|\varphi\|_{X^0} + \|\bar{\varphi}\|_{U_{\theta}} + \sum_{i=1}^{N} \|h_i\|_{X^1} \right),$$

Here c > 0 depends only on n, T, D, θ, κ, w, and on the coefficients of equation (2.1).

For the case of backward SPDEs, the following result was obtained.

**Theorem 3.2** Assume that

$$\bar{\beta}_i \equiv 0, \quad \beta_i(x,t,\omega) \equiv \bar{\beta}_i(t,\omega), \quad i = 1, \ldots, N.$$ Let ε > 0 be given. Assume that θ = 0 if κ = 0 and θ ∈ (0, T] if κ ≠ 0. Then problem (4.1)-(4.3) has a unique solution \((u, \chi_1, \ldots, \chi_N)\) in the class \(Y^1 \times (X^0)^N\), for any \(\psi \in Z^2_T\) and any \(\varphi \in X^0\) such that \(\|\varphi(\cdot, t)\|_{Z^2_T} \equiv 0\) for \(t \in [0, \theta]\) and \(\varphi(\cdot, t) \in Z^2_T\). In addition,

$$\|u\|_{Y^1} + \sum_{i=1}^{N} \|\chi_i\|_{X^0} \leq C \left( \|\varphi\|_{X^0} + \|\psi\|_{Z^2_T} \right),$$

for these \(\psi\) and \(\varphi\). Here \(C > 0\) depends only on δ, T, n, N, ε, θ, λ, κ, w, and on the supremums of the coefficients and derivatives coefficients of equation (2.4).

## 4 Proofs

### 4.1 Proof of Theorem 3.1

We consider the following boundary value problem in \(Q\)

\[
d_t u = (Au + \varphi) dt + \sum_{i=1}^{N} [B_i u + h_i] dw_i(t), \quad t \geq 0, \quad (4.1)
\]

\[
u(x,t,\omega)|_{x \in \partial D} = 0 \quad (4.2)
\]

\[
u(x,0) = \xi(x). \quad (4.3)
\]

**Lemma 4.1** Assume that Conditions (2.1), (2.2) are satisfied. Then problem (4.1)-(4.3) has a unique solution \(u\) in the class \(Y^k\) for any \(\varphi \in X^{k-2}, h_i \in X^{k-1}, \xi \in Z^{k-1}_s, \) and

$$\|u\|_{Y^k} \leq C \left( \|\varphi\|_{X^{k-2}} + \|\xi\|_{Z^{k-1}} + \sum_{i=1}^{N} \|h_i\|_{X^{k-1}} \right),$$

where \(C > 0\) does not depend on \(\xi\) and \(\varphi\).
Lemma 4.2 \cite{18} The linear operator $M : H^0 \to H^2$ is a continuous bijection; in particular, the inverse operator $M_0^{-1} : H^2 \to H^0$ is also continuous. In addition, the linear operator $M : U_0 \to H^2$ is continuous.

It can be noted that the classical results for parabolic equations imply that the operators $M_0 : H^k \to H^{k+1}$, $k = 0, 1$, and $M : W^0 \to H^2$, are continuous for $\kappa = 0$, and the operators $M_0 : H^k \to H^k$, $k = 0, 1$, and $M : W^0 \to H^1$, are continuous for $\kappa > 0$; see Theorems III.4.1 and IV.9.1 in \cite{26} or Theorem III.3.2 in \cite{25}. The proof of continuity of the operator $M_0 : H^0 \to H^2$
claimed in Lemma 4.2 can be found in [18]. In addition, it was shown in [18] that the operator $M : U_\theta \to H^2$ is continuous.

Since problem (4.1)-(4.3) is linear and the functions $a$ and $a_0$ are non-random, it follows that if $u = \mathcal{L}\xi + L\varphi + \sum_{i=1}^N \mathcal{H}_i h_i$ and $\bar{u}(c, t) \triangleq \mathbb{E}u(x, t)$, then $\bar{u} = \bar{\mathcal{L}}\xi + \bar{L}\varphi$, where $\bar{\varphi}(x, t) = \mathbb{E}\varphi(x, t)$. It follows from the definitions of $M_0$ and $M$ that

$$\mu = M_0 \xi + M\bar{\varphi}.$$ 

Hence

$$\xi = M_0^{-1}(\mu - M\bar{\varphi})$$

is uniquely defined, and

$$u = \mathcal{L}\xi + L\varphi + \sum_{i=1}^N \mathcal{H}_i h_i = \mathcal{L}M_0^{-1}(\mu - M\bar{\varphi}) + L\varphi + \sum_{i=1}^N \mathcal{H}_i h_i. \quad (4.4)$$

is an unique solution of problem (2.1)-(2.3) in $Y^1$. By the continuity of the operator $M : U_\theta \to H^2$ and other operators in (4.4), the desired estimate for $u$ follows. This completes the proof of Theorem 3.1. $\square$

### 4.2 Proof of Theorem 3.2

Let $\varphi \in X^{-1}$ and $\xi \in Z_T^0$. Consider the problem

$$dtu + (Au + \varphi) dt + \sum_{i=1}^N B_i \chi_i(t) dt = \sum_{i=1}^N \chi_i(t) dw_i(t), \quad t \leq T, \quad \text{(4.5)}$$

$$u(x, t, \omega)|_{x \in \partial D}, \quad \text{(4.6)}$$

$$u(x, T, \omega) = \xi(x, \omega). \quad \text{(4.7)}$$

**Lemma 4.3** For $k = 1, 2$, problem (4.5)-(4.7) has a unique solution $(u, \chi_1, ..., \chi_N)$ in the class $Y^k \times (X^0)^N$ for any $\varphi \in X^{k-2}$, $\xi \in Z_T^{k-1}$, and

$$\|u\|_{Y^k} + \sum_{i=1}^N \|\chi_i\|_{X^{k-1}} \leq C \left(\|\varphi\|_{X^{k-2}} + \|\xi\|_{Z_T^{k-1}}\right),$$

where $C > 0$ does not depend on $\varphi$ and $\psi$; it depends on $\delta, T, n, N, D,$ and on the supremums of the coefficients and derivatives listed in Condition 2.1.

Note that the solution $u = u(\cdot, t)$ is continuous in $t$ in $L_2(\Omega, \mathcal{F}, P, H^{k-1})$, since $Y^k = X^1 \cap C^0$. 

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For $k = 1$, the result of Lemma 4.3 can be found in [8] or in [14] (Theorem 4.2); this is an analog of the so-called "the first energy inequality", or "the first fundamental inequality" known for deterministic parabolic equations; see, e.g., inequality (3.14) in [25], Chapter III.

For $k = 2$, the lemma above represents a reformulation of Theorem 3.1 from [19] or Theorem 3.4 in [14], or Theorem 4.3 in [16]; this is an analog of the so-called "the second energy inequality", or "the second fundamental inequality" known for the deterministic parabolic equations (see, e.g., inequality (4.56) in [25], Chapter III. In the cited papers, this result was obtained under some strengthened version of Condition 2.3; this was restrictive. In [19], this result was obtained without this restriction, i.e. under Condition 2.3 only.

Remark 4.1 Thanks to Theorem 3.1 from [19], Condition 3.5 from [15] and Condition 4.1 from [16] can be replaced by less restrictive Condition 2.3; all results in [15, 16] are still valid.

Introduce operators $L_B : X^{-1} \to Y^1$, $\mathcal{L}_B : Z^0 \to Y^1$, $H_i : X^0 \to Y^1$, $\mathcal{H}_i : Z^0_T \to Y^1$ such that $u = L_B \varphi + \mathcal{L}_B \xi$ and $\chi_i = H_i \varphi + \mathcal{H}_i \xi$, where $(u, \chi_1, \ldots, \chi_N)$ is the solution of problem (4.5)-(4.7) in the class $Y^2 \times (X^1)^N$. By Lemma 4.3 these linear operators are continuous.

Let a linear operator $M_0 : H^0 \to Z^1_T$ be defined such that $(M_0 \xi)(x) = \kappa u(x, 0) + \int_0^T \theta(t) \bar{u}(x, t)dt$, where $\kappa = L_B \xi \in Y^1$.

Further, let a linear operator $M : X^0 \to Z^1_T$ be defined such that $(M \varphi)(x) = \kappa u(x, 0) + \int_0^T \theta(t) \bar{u}(x, t)dt$, where $u = L_B \varphi \in Y_1$, $\varphi \in X^0$.

In this notations, $\psi = M_0 \xi + M \varphi$ for a solution $u$ of problem (4.5)-(4.7), i.e. with $\xi = u(\cdot, T)$.

Lemma 4.4 The operator $M_0^{-1} : Z^2_{T-\varepsilon} \to Z^0_{T-\varepsilon}$ is continuous.

Proof of Lemma 4.4 It is known that there exists an orthogonal basis $\{v_k\}_{k=1}^\infty$ in $H^0$, i.e. such that $(v_k, v_m)_{H^0} = 0$, $k \neq m$, $\|v_k\|_{H^0} = 1$, and such that $v_k$ is the solution in $H^1$ of the boundary value problem

$$Av_k = -\lambda_k v_k, \quad v_k|_{\partial D} = 0, \quad (4.8)$$

for some $\lambda_k \in \mathbb{R}$, $\lambda_k \to +\infty$ as $k \to +\infty$; see e.g. Ladyzhenskaya (1985), Chapter 3.4. In other words, $\lambda_k$ and $v_k$ are the eigenvalues and the corresponding eigenfunctions of the eigenvalue problem (4.8).

Let $\xi$ and $\psi$ be expanded as

$$\xi(x, \omega) = \sum_{k=1}^\infty \alpha_k(\omega) v_k(x), \quad \psi(x, \omega) = \sum_{k=1}^\infty \gamma_k(\omega) v_k(x), \quad \chi_i(x, t, \omega) = \sum_{k=1}^\infty \chi_{ik}(t, \omega) v_k(x),$$

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where \( \{\alpha_k\}_{k=1}^\infty \) and \( \{\gamma_k\}_{k=1}^\infty \) and square-summable random sequences such that

\[
E \sum_{k=1}^N \alpha_k^2 = E\|\xi\|_{H^0}^2 = \|\xi\|_{Z_T^2}^2 < +\infty, \quad E \sum_{k=1}^N \gamma_k^2 = E\|\psi\|_{H^0}^2 = \|\psi\|_{Z_T^2}^2 < +\infty.
\]

By the choice of \( \xi \), we have that \( u = L\xi \). Applying the Fourier method, we obtain that

\[
u(x,t) = \sum_{k=1}^\infty y_k(t)v_k(x), \quad (4.9)
\]

where \( y_k(t) \) are the solutions of backward stochastic differential equations

\[
dy_k(t) + \left[-\lambda_k y_k(t) + \sum_{i=1}^N \beta_i(t)Y_{ik}(t)\right] \, dt = \sum_{k=1}^N Y_{ik}(t) \, dw_i(t), \quad y_k(T) = \alpha_k.
\]

Let

\[
F(t,s) = \exp \left(-\frac{1}{2} \sum_{i=1}^N \int_s^t \beta_i(r)^2 \, dr + \sum_{i=1}^N \int_s^t \beta_i(r) \, dw_i(r)\right), \quad \Phi_k(t,s) = e^{-\lambda_k(t-s)} F(t,s).
\]

We have that

\[
y_k(t) = \Phi_k(t,0)^{-1} E\{\Phi_k(T,0)\alpha_k|\mathcal{F}_t\} = E\{\Phi_k(T,t)\alpha_k|\mathcal{F}_t\};
\]

see e.g. Proposition 6.2.1 \[33\], p.142. Hence

\[
y_k(t) = e^{-\lambda_k(T-t)} Y_k(t),
\]

where

\[
Y_k(t) = E \{F(T,t)\alpha_k|\mathcal{F}_t\}.
\]

On the other hand,

\[
\psi(x) = \sum_{k=1}^\infty \gamma_k v_k(x) = \int_0^T \varrho(t)u(x,t) \, dt + \kappa u(x,0) = \int_0^T \varrho(t)y_k(t)v_k(x) \, dt + \kappa \sum_{k=1}^\infty y_k(0)v_k(x).
\]

Hence \( \alpha_k \) must be such that \( y_k \) satisfy the equation

\[
\gamma_k = \int_0^T \varrho(t)y_k(t) \, dt + \kappa y_k(0) \quad (4.10)
\]

which can be rewritten as

\[
\gamma_k = \int_0^T \varrho(t)y_k(t) \, dt + \kappa e^{-\lambda_k T} Y_k(0). \quad (4.11)
\]
or

\[ \gamma_k = \int_0^T \varrho(t)e^{-\lambda_k(T-t)}Y_k(t)dt + \kappa e^{-\lambda_k T}Y_k(0). \]  (4.12)

We consider this as an equation for unknown \( \alpha_k = Y_k(T) = y_k(T) \). Let us prove its solvability and uniqueness. We need to show existence of \( \bar{Y}_k \in \mathbb{R} \) and \( \mathcal{F}_T \)-adapted square integrable random processes \( \hat{Y}_{kd} \) such that

\[ \alpha_k = \bar{Y}_k + \sum_{d=1}^N \int_0^T \hat{Y}_{kd}(s)dw_d(s). \]

In this case,

\[ Y_k(t) = \bar{Y}_k + \sum_{d=1}^N \int_0^t \hat{Y}_{kd}(s)dw_d(s). \]

Let \( \hat{\gamma}_{kd} \) be \( \mathcal{F}_T \)-adapted square integrable random processes such that

\[ \gamma_k = E\gamma_k + \sum_{d=1}^N \int_0^{T-\varepsilon} \hat{\gamma}_{kd}(s)dw_d(s), \]

We assume, without a loss of generality, that \( \lambda_1 \geq \lambda \). This is possible because one can always replace the operator \( A \) by the operator \( A + rI \), for an arbitrarily large \( r > 0 \); in this case, the solution \( u \) of the original problem will be transformed as \( u(x, t, \omega)e^{-r(t-t)} \), with some adjustment of \( \varrho \) and \( \varphi \).

Clearly, it suffices to show existence of \( \hat{Y}_{kd} \) and \( \bar{Y}_k \in \mathbb{R} \). Let \( q_k(s) \triangleq \int_s^T \varrho(t)e^{-\lambda_k(T-t)}dt \). By Condition 2.1 there exist \( \kappa > 0 \) and \( T_1 \in [0, T) \) such that

\[ q_k(s) \geq \kappa \int_{T_1 \vee s}^T e^{-\lambda_k(T-t)}dt = \frac{1 - e^{-\lambda_k(T-T_1 \vee s)}}{\lambda_k} \geq \frac{1 - e^{-\lambda(T-s)}}{\lambda_k}. \]

By the definitions,

\[ E\gamma_k = \int_0^T \varrho(t)e^{-\lambda_k(T-t)}\bar{Y}_kdt + \kappa e^{-\lambda_k T}\bar{Y}_k, \]

and

\[ \bar{Y}_k = \left[ q_k(0) + \kappa e^{-\lambda_k T} \right]^{-1} E\gamma_k. \]

Hence

\[ \bar{Y}_k^2 \leq \text{const} \lambda_k^2 E\gamma_k^2. \]
In addition,
\[
\int_0^{T-\varepsilon} \tilde{\gamma}_{kd}(s) dw_d(s) = \int_0^T q(t) e^{-\lambda_k (T-t)} dt \int_0^t \tilde{Y}_{kd}(s) dw_d(s)
\]
\[
= \int_0^T \tilde{Y}_{kd}(s) dw_d(s) \int_s^T q(t) e^{-\lambda_k (T-t)} dt.
\]

Hence
\[
\tilde{Y}_{kd}(s) = \mathbb{I}_{\{s \leq T-\varepsilon\}} \frac{\tilde{\gamma}_{kd}(s)}{q_k(s)}
\]
and
\[
|\tilde{Y}_{kd}(s)| \leq \mathbb{I}_{\{s \leq T-\varepsilon\}} \frac{\lambda_k |\tilde{\gamma}_{kd}(s)|}{k(1 - e^{-\lambda_k (T-s)})}.
\]

Hence equation (4.12) has an unique solution \(\alpha_k \in L_2(\Omega, \mathcal{F}_{T-\varepsilon}, \Omega)\) and there exists \(C > 0\) such that
\[
\mathbb{E} \sum_{k=1}^\infty \alpha_k^2 \leq C \mathbb{E} \sum_{k=1}^\infty \gamma_k^2 \lambda_k^2.
\] (4.13)

Further, we have that
\[
A \psi = \sum_{k=1}^\infty \gamma_k A v_k = -\sum_{k=1}^\infty \gamma_k \lambda_k v_k \quad \text{a.s.}
\]
and
\[
\mathbb{E} \|A \psi\|_{H^0}^2 = \mathbb{E} \sum_{k=1}^\infty \gamma_k^2 \lambda_k^2.
\]

Hence (4.13) can be rewritten as
\[
\mathbb{E} \|\xi\|_{Z_{T-\varepsilon}^0}^2 \leq C \mathbb{E} \|A \psi\|_{H^0}^2 \leq C \|\psi\|_{Z_{T-\varepsilon}^0}^2
\] (4.14)
for some \(C_1 > 0\) and \(C > 0\) that are independent on \(\psi\). Thus, (4.14) implies that the operator \(\mathcal{M}^{-1}_{0} : Z_{T-\varepsilon}^2 \rightarrow Z_{T-\varepsilon}^0\) is continuous. This completes the proof of Lemma 4.4. \(\square\)

Lemma 4.5 Solution of problem (4.5)-(4.7) for \(\varphi = 0\) and \(\xi \in Z_{T-\varepsilon}^0\) is such that
\[
\|u(\cdot, 0)\|_{Z_{T-\varepsilon}^0} \leq C\|\xi\|_{Z_{T-\varepsilon}^0},
\]
where \(C > 0\) does not depend on \(\xi\); it depends on \(\delta, T, n, N, D\), and on the supremums of the coefficients and derivatives listed in Condition 2.7.
Proof of Lemma 4.6. We have that $u(\cdot, 0) = \sum_{k=1}^{\infty} v_k(x) y_k(0)$. Hence
\[
\|Au(\cdot, 0)\|_{Z_T^0} = \mathbf{E} \left( \sum_{k=1}^{\infty} \lambda_k^2 y_k(0) \right)^2 = \sum_{k=1}^{\infty} \lambda_k^2 e^{-\lambda_k T} \mathbf{E}(F(T, 0) \alpha_k)^2 
\leq \sum_{k=1}^{\infty} \lambda_k^2 e^{-\lambda_k T} \mathbf{E}(F(T, 0) \alpha_k)^2 
\]
We have that $\|F(T, 0) \alpha_k\|^2 \leq c_F \mathbf{E} \alpha_k^2$, where $c_F = \mathbf{E}(T, 0)^2$. Hence
\[
\|Au(\cdot, 0)\|_{Z_T^0} \leq c_F \sum_{k=1}^{\infty} \lambda_k^2 e^{-\lambda_k T} \mathbf{E} \alpha_k^2.
\]
Hence
\[
\|Au(\cdot, 0)\|_{Z_T^0} \leq C\|\xi\|_{Z_T^0},
\]
where $C > 0$ does not depend on $\xi$; it depends on $\delta, T, n, N, D$, and on the supremums of the coefficients and derivatives listed in Condition 2.1. Further, for any $\lambda \in \mathbb{R}$, we have that $h \triangleq Au(\cdot, 0) + \lambda u(\cdot, 0) \in H^0$. By the properties of the elliptic equations, it follows that there exists $\lambda \in \mathbb{R}$ and $c = c(\lambda) > 0$ such that
\[
\|u(\cdot, 0)\|_{H^2} \leq c\|h\|_{H^0} \leq c(\|Au(\cdot, 0)\|_{H^0} + \|\lambda u(\cdot, 0)\|\|H^0\|) \quad \text{a.s.},
\] (4.15)
see e.g. Theorem II.7.2 and Remark II.7.1 in Ladyzhenskaya (1975), or Theorem III.9.2 and Theorem III.10.1 in Ladyzhenskaya and Ural’ceva (1968). By (4.15), we have that
\[
\|u(\cdot, 0)\|_{Z_T^0} \leq c_1(\|Au(\cdot, 0)\|_{Z_T^0} + \|\xi\|_{Z_T^0}) \leq c_2\|\xi\|_{Z_T^0}
\] (4.16)
for some $c_i > 0$ that are independent on $\xi$ but depend on the parameters of the SPDE. This completes the proof of Lemma 4.6. □

Lemma 4.6 The operators $\mathcal{M}_0 : Z_T^0 \to Z_T^0$ and $\mathcal{M}_0 : Z_{T-\varepsilon}^2 \to Z_{T-\varepsilon}^0$ are continuous.

Proof of Lemma 4.6. By Lemmas 4.3 and 4.5, the operator $\mathcal{M}_0 : Z_T^0 \to Z_T^2$ is continuous. Further, let $\xi \in Z_{T-\varepsilon}^0$. Since $\chi_i(t) = 0$ for $t > T - \varepsilon$ and $\chi_i = \mathcal{H}_i \xi$, $\xi \in Z_{T-\varepsilon}^0$, it follows that the operator $\mathcal{M}_0 : Z_{T-\varepsilon}^2 \to Z_{T-\varepsilon}^0$ is continuous. This completes the proof of Lemma 4.6. □

Let $X_\theta$ be the space of all $\varphi \in X^0$ such that $\|\varphi(\cdot, t)\|_{Z_T^0} = 0$ for $t \in [0, \theta)$, with the norm $\|\varphi\|_{X_\theta} = \|\varphi\|_{X^0}$.

Let $Y_\varepsilon$ be the space of all $\varphi \in X^0$ such that $\varphi(\cdot, t) \in Z_{T-\varepsilon}^0$, with the norm $\|\varphi\|_{Y_\varepsilon} = \|\varphi\|_{X^0}$.

We consider $X_\theta \cap Y_\varepsilon$ as a linear normed space with the norm $\|\varphi\|_{Y_\varepsilon} = \|\varphi\|_{X^0}$. Clearly, this is a Banach space.
Lemma 4.7 The operators $\mathcal{M} : \mathcal{X}_\theta \to Z_T^2$, $\mathcal{M} : \mathcal{Y}_\epsilon \to Z_{T-\epsilon}^1$, and $\mathcal{M} : \mathcal{X}_\theta \cap \mathcal{Y}_\epsilon \to Z_T^2$ are continuous.

Proof of Lemma 4.7 Let us show that the operator $\mathcal{M} : \mathcal{X}_\theta \to Z_T^2$ is continuous. By Lemma 4.3, the operator $\mathcal{M} : X^0 \to Z_T^2$ is continuous if $\kappa = 0$; in this case, we can select $\theta = T$ and $X^0 = X^0$.

Let us show that the operator $\mathcal{M} : \mathcal{X}_\theta \to Z_T^2$ is continuous for the case where $\kappa \neq 0$ and $\theta \neq 0$. Without a loss of generality, let us assume that $\kappa = 1$, $\mu = M\varphi = u(x,0)$, and $\varphi(t) \equiv 0$; it suffices because the boundary value problem is linear.

Let $\varphi \in X^0_\theta$ and $u = \mathcal{L}_B\varphi$. For $\tilde{\theta} \in (0, \theta)$, consider a modification of problem (4.5)-(4.7)

$$d\tilde{u} + \left[Au + \sum_{i=1}^N B_i\chi_i(t)\right] dt = \sum_{i=1}^N \chi_i(t)dw_i(t), \quad t \in (0, \tilde{\theta}),$$

$$\tilde{u}(x,t,\omega)|_{x \in \partial D},$$

$$\tilde{u}(x,\tilde{\theta},\omega) = u(x,\tilde{\theta},\omega).$$

By the semi-group property of backward SPDEs from Theorem 6.1 from [14], we obtain that $\tilde{u}|_{t \in [0, \tilde{\theta}]} = u|_{t \in [0, \theta]}$. By Lemma 4.3, we have for $\tau \in [\tilde{\theta}, \theta]$ that

$$\inf_{t \in [\tilde{\theta}, \theta]} \|u(\cdot, t)\|_{Z_T^2}^2 \leq \frac{1}{\theta - \tilde{\theta}} \int_{\tilde{\theta}}^\theta \|u(\cdot, t)\|_{Z_T^2}^2 dt \leq \frac{C_1}{\theta - \tilde{\theta}} \|\varphi\|_{X^0}^2,$$

where $C_1 > 0$ is independent of $\varphi$. Hence the exists $\tau \in [\tilde{\theta}, \theta]$ such that

$$\|u(\cdot, \tau)\|_{Z_T^2}^2 \leq C\|\varphi\|_{X^0}^2,$$

where $C = 2C_1(\theta - \tilde{\theta})^{-1}$. Repeating the proof for the continuity of the operator $\mathcal{M}_0 : Z_T^2 \to Z_T^2$ to the new time interval $[0, \tau]$, we obtain that the operator $\mathcal{M} : \mathcal{X}_\theta \to Z_T^2$ is continuous.

Further, let $\varphi \in \mathcal{Y}_\epsilon$. In this case, $\chi_i(t) = 0$ for $t > T - \epsilon$ and $\chi_i = H_i\varphi$ and $\mathcal{M}\varphi \in Z_{T-\epsilon}^1$. It follows that the operator $\mathcal{M} : \mathcal{Y}_\epsilon \to Z_{T-\epsilon}^1$ is continuous. Then the continuity of the operator $\mathcal{M} : \mathcal{X}_\theta \cap \mathcal{Y}_\epsilon \to Z_{T-\epsilon}^2$ follows. This completes the proof of Lemma 4.7. 

We are now in the position to complete the proof of Theorem 3.2. By the assumptions, $\varphi \in \mathcal{X}_\theta \cap \mathcal{Y}_\epsilon$. It follows from the definitions of $\mathcal{M}_0$ and $\mathcal{M}$ that

$$\psi = \mathcal{M}_0\xi + \mathcal{M}\varphi.$$

Since the operator $\mathcal{M} : X^0_\theta \to Z_T^2$ and $\mathcal{M}_0^{-1} : Z_T^2 \to Z_T^0$ are continuous, it follows that $\mathcal{M}\varphi \in Z_T^2$ and

$$\xi = \mathcal{M}_0^{-1}(\psi - \mathcal{M}\varphi) \quad (4.17)$$
is uniquely defined. Hence

\[ u = L_B^L \xi + L_B^L \varphi = L_B^L M_0^{-1}(\psi - M \varphi) + L_B^L \varphi, \]

\[ \chi_i = H_i^L \varphi + H_i^L \xi = H_i^L \varphi_i + H_i^L M_0^{-1}(\psi - M \varphi), \quad i = 1, \ldots, N, \quad (4.18) \]

is a unique solution of problem (2.4)-(2.6) in \( Y_1 \). By the continuity of the operator \( M : X_{\theta} \cap Y_{\varepsilon} \rightarrow Z_{T - \varepsilon}^2 \) and other operators in (4.18), the desired estimate for \( u \) follows. This completes the proof of Theorem 3.2. □

Discussion and future development

(i) Consider the case where \( \kappa = 0 \) and \( \varrho \equiv 1 \). By Theorem 3.2, the corresponding boundary problem is well-posed as a backward SPDE (or BSPDEs), even without a Cauchy condition that is usually associated with BSPDEs. This gives a new information about the nature of BSPDEs.

(ii) The current proof is based on eigenfunctions expansion; we think that it is possible to obtain a similar result using a different proof for the case where the operator \( A \) is not necessarily symmetric and has coefficients depending on time. We leave this for the future research.

(iii)

(iv) It would be interesting to extend the result for the case of backward SPDEs on the case where \( \beta_i = \beta_i(x,t,\omega) \) or where the operators \( B_i \) are differential operators. So far, the approach based on eigenfunctions expansion used here does not allow this.

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