Black Hole Solution of Quantum Gravity

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Abstract

We present a spherically symmetric and static exact solution of Quantum Einstein Equations. This solution is asymptotically (for large $r$) identical with the black hole solution on the anti–De Sitter background and, for some range of values of the mass possesses two horizons. We investigate thermodynamical properties of this solution.

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1 Introduction

In the recent papers [1] we presented a class of exact solutions of the regularised Wheeler-De Witt equation. In these papers we proposed also an interpretation of the resulting ‘wave functionals’ in terms of the modified gravitational field dynamics, namely, in the framework the quantum potential approach to the quantum mechanics originally proposed by David Bohm (see e.g., [2] and [3]) and extended to the case of quantum gravity in [4] and [5] (in minisuperspace) and in [6], [7] (for full theory.) The resulting, effective Hamiltonian constraint of quantum gravity with additional quantum potential terms was presented in [9]. This Hamiltonian was assumed to be a generator of dynamics of quantum gravity. In the present paper I present a spherically symmetric and static solution of such defined theory which can be regarded as a black hole solution of quantum gravity.

Let us recall the basic steps leading to the quantum Hamiltonian constraints. In the papers [1] two exact solutions of the Wheeler–De Witt equation were found:

$$\Psi_I = \exp\left(-\frac{3\rho(5)}{\Lambda}\mathcal{V}\right) \quad \text{Case I;} \quad (1)$$

$$\Psi_{II} = \exp\left(\frac{4\Lambda}{3\kappa^4\rho(5)}\mathcal{R}\right) \quad \text{Case II;} \quad (2)$$

where $\mathcal{V} = \int \sqrt{h}$ is the volume of the universe, $\mathcal{R} = \int \sqrt{h}R(3)$ its average curvature, and $\rho(5)$ and $\Lambda$ are the renormalisation and bare cosmological constants. Since the Wheeler–De Witt equation is linear, any complex combination of solutions (1, 2) is a solution of the form $\Psi = e^{\Gamma}e^{i\Sigma}$. Taking such a combination, substituting to the WDW equation, extracting real part, and identifying the derivative $\frac{\delta S}{\delta h_{ab}}$ with the momenta $p_{ab}$, we get the equation (in the gauge where the shift vectors $N^a = 0$)

$$0 = \mathcal{H}_\perp = \kappa^2 G_{abcd}p^{ab}p^{cd} + \mathcal{F}\left(\frac{27}{16}\rho(5)^2\kappa^2\sqrt{h} + \frac{1}{\kappa^2}\sqrt{h}R - \frac{8}{9}\kappa^6\rho(5)^2\sqrt{h}\left(-\frac{3}{8}R^2 + R_{ab}R^{ab}\right)\right), \quad (3)$$

$$\mathcal{F} = \frac{1}{2} \sin^2(\phi) \left\{ \cosh\left(\frac{3\rho(5)}{\Lambda}\mathcal{V} + \frac{4\Lambda}{3\kappa^4\rho(5)}\mathcal{R}\right) + \cos(\phi) \right\}^2, \quad (4)$$
where $\kappa$ is the gravitational constant, $G_{abcd}$ is the Wheeler–De Witt metric, and $\phi$ is a parameter measuring the rate of mixing of two solutions above. Generalizing the observation of Gerlach [8] (who considered the case $\Gamma = 0$), it is assumed that the quantum Hamiltonian $H_\perp$ is a generator of time evolution. It should be stressed that the above quantum Hamiltonian constraint is by no means arbitrary, and can be derived from exact solutions of quantum gravity given above.

2 The solution

In the static case, where momenta are equal to zero, one of the dynamical equation of the theory (corresponding to the 00 component of Einstein field equations) is the requirement that the Hamiltonian constraint vanishes, to wit

$$
\frac{27}{16} \rho^{(5)} \kappa^2 \sqrt{h} + \frac{1}{\kappa^2} \sqrt{h} R - \frac{8}{9} \kappa^6 \rho^{(5)} \sqrt{h} \left( -\frac{3}{8} R^2 + R_{ab} R^{ab} \right) = 0
$$

(5)

It is worth observing that the cosmological and renormalization constants appear only in combination $v_0 = \frac{\Lambda}{\rho^{(5)}}$ of dimension $m^3$. Thus the theory possesses two length scales, the Planck scale $l$ defined by $\kappa$, and $v_0^{1/3}$.

In the spherically symmetric, static case, we can take the three-metric on the hypersurfaces of constant time of the form

$$
h_{ab} = A(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
$$

(6)

Substituting this into equation (5) we find a differential equation for the function $A(r)$

$$
0 = A' \left[ \frac{8 v_0^2}{9 l^6} A \frac{A - 1}{A^3 r^3} + \frac{2}{l^2} \frac{1}{A^2 r} \right] + \left[ - \frac{4 v_0^2}{9 l^6} \frac{(A - 1)^2}{A^2 r^4} + \frac{2 A - 1}{l^2 A^2 r^2} \right] + \frac{27 l^2}{16 v_0^2}.
$$

(7)

It is a remarkable fact that the coefficients multiplying $(A')^2$ vanishes identically. To solve this equation it is convenient to substitute $A = (1 - f(r) r^2)^{-1}$. It turns out that the function $f$ satisfies the quadratic equation

$$
f^2 + \frac{9}{2} \frac{l^4}{v_0^2} f + \frac{81}{64} \frac{l^8}{v_0^4} = \frac{9}{4} \frac{l^6}{v_0^2} \alpha/r^3.
$$

(8)
where \( \alpha \) is an integration constant with the solution

\[
f(r)_\pm = -\frac{9 \ell^4}{4 v_0^2} \pm \frac{1}{2} \sqrt{\frac{243 \ell^8}{16 v_0^4} + \frac{9 \ell^6 \alpha}{v_0^2} \frac{1}{r^3}},
\]

and the coefficient \( A \) of the three-metric on constant time surface equals (we consider only the \( f_+ \) solution)

\[
A = \frac{1}{1 + \frac{9 \ell^4}{4 v_0^2} r^2 - \frac{r^2}{2} \sqrt{\frac{243 \ell^8}{16 v_0^4} + \frac{9 \ell^6 \alpha}{v_0^2} \frac{1}{r^3}}}.
\]

In the next step we must construct the full four-metric. This metric is of the form

\[
g_{\mu\nu} = -N(r)^2 dt^2 + A(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
\]

To find \( N \), one can make use of the dynamical equations presented in [9] (modified by terms proportional to spacial covariant derivatives of \( N \)), but there is a more straightforward way. Namely, we can consider the action

\[
I = -\frac{1}{4} \int N \mathcal{H}_\perp,
\]

which is the Hamiltonian action for general relativity in the static case and in the gauge \( N^a = 0 \). One can easily check that such a procedure produces the correct expression for Schwarzschild metric. It is convenient to make an anzatz

\[
N = N_0 A^{-1/2}.
\]

Then the action \( I \) takes the form

\[
I = -\int NF' dr,
\]

with

\[
F = \frac{v_0^2}{9\ell^6} \frac{(A - 1)^2}{A^2 r} + \frac{1}{2\ell^2} \frac{r}{A} (A - 1) + \frac{9\ell^2}{64 v_0^2} r^3.
\]

The expression from \( A \), \( I \), as well as the condition \( N_0 = const \) follow from \( I \).
Thus our solution is of the form (we take \( N_0 = 1 \) for a while)

\[
g_{\mu\nu} = - \left( 1 + \frac{9}{4} \frac{l^4}{v_0^2} r^2 - \frac{r^2}{2} \sqrt{\frac{243}{16} \frac{l^8}{v_0^4} + \frac{9}{16} \frac{l^6}{v_0^4} \frac{\alpha}{r^4}} \right) dt^2 + \\
+ \frac{1}{1 + \frac{9}{4} \frac{l^4}{v_0^2} r^2 - \frac{r^2}{2} \sqrt{\frac{243}{16} \frac{l^8}{v_0^4} + \frac{9}{16} \frac{l^6}{v_0^4} \frac{\alpha}{r^4}}} dr^2 + r^2 (d\theta + \sin^2 \theta d\phi^2)
\]

(14)

Let us now analyse the solution. First consider the asymptotics for large \( r \):

\[
A(r) \xrightarrow{r \to \infty} \left( 1 + \frac{r^2}{8} \frac{l^4}{v_0^2} (18 - \sqrt{243}) - \frac{l^2}{\sqrt{3}} \frac{\alpha}{r} \right)^{-1}.
\]

(15)

We see therefore that our solution is asymptotically equivalent to the black hole solution on the anti–De Sitter background with the effective cosmological constant \( \lambda = -\frac{l^4}{8v_0^2} (18 - \sqrt{243}) \). Thus we interpret \( \alpha \) as being proportional to the black hole mass \( M \), so that the last term becomes the familiar \( \frac{2l^2 M}{r} \). This means that

\[
\alpha = 2\sqrt{3} M
\]

The asymptotics at \( r \) tending to zero is regular

\[
A(r) \xrightarrow{r \to 0} \left( 1 - \frac{2l^3}{3v_0} \sqrt{2\sqrt{3}Mr} \right)^{-1}.
\]

(16)

In spite of that the invariants constructed from the Riemann tensor are divergent at \( r = 0 \). The singularity is of order \( (\text{Riemann})^2 \sim r^{-3} \) and is significantly milder than the singularity of Schwarzschild solution, where it is of order \( \sim r^{-6} \). This singularity is hard to avoid because the solution describes a point mass source.

Thus we have the four metric \( g_{\mu\nu} \) (14) that describes an exact static, spherically symmetric black hole solution of quantum gravity (with negative cosmological constant).

For large \( M \) the space–time described by (14) has two horizons at \( r_+ \) and \( r_- \), whose exact values can be found by solving quartic equation. When \( M \) decreases, the horizons come closer to each other, and finally they degenerate
to a single one at \( r_c \). For still smaller values of \( M \) there are no horizons. This behaviour is similar to the way the standard charged black hole solution behaves, when its mass decreases, approaching the charge from above.

It can be checked that the function \( A(r)^{-1} \) has only one minimum for positive \( r \). This means that at the point where horizons merge (for some specific \( M \)), the derivative \( \left. \frac{d}{dr} A(r)^{-1} \right|_{r_c} = 0 \). This is important, because, as we will see, it follows that the temperature of the extreme black hole vanishes.

One can check that the causal structure of our solution is similar to that of the Reissner–Nordström solution on the anti–De Sitter background; the corresponding Penrose diagram can be found, for example, in \([10]\).

### 3 Thermodynamics

One can analyse the thermodynamics of quantum black hole basically repeating the standard steps \([11]\) (see also \([10]\)). To find the temperature we impose regularity on the horizon of the Euclidean continuation of the metric \([14]\), \( t \to -i\tau \) defined for \( r > r_+ \)

\[
    ds^2_E = N_0 A^{-1}(r) dr^2 + A(r) dr^2 + r^2 d\Omega^2, \tag{17}
\]

with \( A(r) \) given by \([10]\). To avoid conical singularity on the horizon \( r = r_+ \), we take \( \tau \) periodic with the range \([0, 1] \). Then we can identify \( N_0 \) with the inverse temperature \( N_0 = T^{-1} \), which can be found from standard formula

\[
    T = \frac{1}{4\pi} \left. \left( \frac{\partial A^{-1}(r)}{\partial r} \right) \right|_{r=r_+}. \tag{18}
\]

As mentioned already, for critical mass, where two horizons merge and form the single critical one, the derivative of \( A^{-1} \) vanishes. Thus the extreme black hole has zero temperature.

Let us try to rewrite the above expression for temperature in a more explicit form. The equation for horizon reads

\[
    1 - r_+ f(r_+) = 0.
\]

Differentiating equation \([3]\) with respect to \( r \) and putting \( r = r_+ \) we find

\[
    f'(r_+) = -\frac{3\alpha}{2r_+^2} \frac{1}{r_+^2 + \frac{4v_0^2}{g^2}}.
\]

\[
    \frac{d}{dr} A(r)^{-1} \bigg|_{r_c} = 0.
\]
Moreover, it follows from (8) that

\[ \alpha = \frac{4v_0^2}{9l^6} \frac{1}{r_+} + \frac{2}{l^2} r_+ + \frac{9l^2}{16v_0} r_+^3. \]  

(19)

The final expression for the temperature reads

\[ T = \frac{1}{4\pi} \left\{ -\frac{2}{r_+} + \frac{3}{2} \frac{1}{r_+^2} + \frac{4v_0^2}{9l^6} \left( \frac{1}{r_+} + \frac{2}{l^2} r_+ + \frac{9l^2}{16v_0} r_+^3 \right) \right\}. \]  

(20)

For large masses i.e., large \( r_+ \) we have

\[ T \sim \frac{1}{4\pi} \left[ \frac{27l^2}{32v_0^2} r_+ + \frac{5}{8} \frac{1}{r_+} \right]. \]  

(21)

This last expression is up to numerical coefficients identical with the classical result for black hole on anti – De Sitter background [10].

In the next step we can compute the entropy of the solution. To this end we use the procedure proposed by Gibbons and Hawking [11], and we identify the action \( I_E \) (12) with free energy divided by temperature:

\[ I_E = \frac{M}{T} - S. \]  

(22)

However, as it is well known we must add boundary terms \( B \) at infinity and at \( r = r_+ \) which are fixed by boundary conditions of the variational problem for the action \( I_E \). Thus we consider

\[ I_E = - \int_{r_+}^{\infty} N_0 F' + B \]

The integral is a linear combination of constraints \( \mathcal{H}_\perp(r) \), and thus

\[ B = \frac{M}{T} - S. \]  

(23)

The variation of the action up to the terms that vanish on-shell equals

\[ \delta I_E = - \left[ N \delta F \right]_{r_+}^{\infty} + \delta B(\infty) + \delta B(r_+). \]  

(24)
Consider the variation at $r_+$. Using the condition

$$4\pi = N(r_+) \left( \frac{\partial A^{-1}(r)}{\partial r} \right)_{r=r_+}$$

we get the condition

$$\delta B(r_+) = 4\pi \left( \frac{\partial F}{\partial A^{-1}(r)} \right)_{r=r_+}.$$  

(26)

Using the condition $N(\infty) = 1/T$ and solving the boundary condition at infinity we find

$$B = \frac{M}{T} + 4\pi \int dr_+ \left( \frac{\partial F}{\partial A^{-1}(r)} \right)_{r=r_+} - S_0,$$

(27)

where $S_0$ is a constant to be fixed in a moment. The entropy of the black hole of outer radius $r_+$ is therefore equal

$$S = -4\pi \int dr_+ \left( \frac{\partial F}{\partial A^{-1}(r)} \right)_{r=r_+} + S_0 =$$

$$= \frac{8\pi v_0^2}{l^6} \log(r_+) + \frac{\pi}{l^2} r_+^2 + S_0$$

Assuming that the entropy vanishes for extreme black hole with $r_+ = r_c$, we finally get

$$S = \frac{8\pi v_0^2}{l^6} \log \left( \frac{r_+}{r_c} \right) + \frac{\pi}{l^2} (r_+^2 - r_c^2)$$

(28)

For large $r_+$ the entropy is thus given by the standard Bekenstein - Hawking formula, and is equal to $1/4$ of the area of the black hole. There is another logarithmic term, whose interpretation is not clear.

### 4 Conclusions

1. The reader may wonder why we have chosen solution $f_+$ in (8). The reason is that if solution $f_-$ was to represent a black hole with positive mass, then the coefficient $\alpha$ would have to be negative, and the space-time would develop additional circular singularity (for $r$ such that the expression in square root in (10) vanishes. If $\alpha$ is positive we have to do with a negative mass black hole, whose interpretation is not clear.
2. Of course the most important problem to be analyzed is the issue of Hawking radiation. To approach this problem one should in principle have in disposal a quantum theory of gravitational field coupled to some matter field. Solutions of quantum gravity coupled to the massless scalar field exist \[12\] and it is feasible to construct a quantum Hamiltonian in this case. It is clear however that if the quantum black hole does radiate and its mass decreases, the temperature tends to zero. Contrary to the standard black hole, where the evaporation process becomes more and more rapid as the mass decreases \( T \sim M^{-1} \), here the process gradually stops and the black hole leaves eventually a cold remnant of the size of order of Planck length (assuming that the ratio \( l^3/v_0 \) is of order 1.) These questions are currently under investigation.

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**References**

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