REMARKS ON SUPERENERGY TENSORS

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We define (super)\(^n\)-energy tensors for non-gravitational fields. The possibility of interchange of superenergy between gravitational and other fields is considered.

1 The Bel-Robinson tensor and its properties.

The Bel-Robinson (BR) tensor \( T^{\alpha\beta\lambda\mu} \) is defined by

\[
T^{\alpha\beta\lambda\mu} \equiv C^{\alpha\rho\lambda\sigma} C^{\beta\mu}_{\rho\sigma} + \ast C^{\alpha\rho\lambda\sigma} \ast C^{\beta\mu}_{\rho\sigma}
\]

where \( C^{\alpha\rho\lambda\sigma} \) is the Weyl tensor and \( \ast \) indicates the usual dual operation:

\[
\ast C^{\alpha\beta\lambda\mu} \equiv \frac{1}{2} \eta^{\alpha\beta\rho\sigma} C_{\rho\sigma}^{\lambda\mu} = \frac{1}{2} \eta^{\lambda\mu\rho\sigma} C_{\rho\sigma}^{\alpha\beta}.
\]

Its main properties are:

A It is completely symmetric and traceless

\[
T^{\alpha\beta\lambda\mu} = T^{(\alpha\beta\lambda\mu)}, \quad T^{\alpha\lambda\mu\nu} = 0.
\]

B For any timelike unit vector \( \vec{u} \), we can define the BR super-energy (s-e) density relative to \( \vec{u} \) as

\[
W_T (\vec{u}) \equiv T^{\alpha\beta\lambda\mu} u^\alpha u^\beta u^\lambda u^\mu.
\]

For any \( \vec{u} \) this is a positive quantity as follows from the expression

\[
W_T (\vec{u}) = E_\alpha E^\alpha + H_\alpha H^\alpha \geq 0
\]

where \( E_\alpha (\vec{u}) \equiv C^{\alpha\beta\lambda\mu} u^\beta u^\mu \) and \( H_\alpha (\vec{u}) \equiv \ast C^{\alpha\beta\lambda\mu} u^\beta u^\mu \) are the well-known electric and magnetic parts of the Weyl tensor, respectively.

Furthermore, \( W_T (\vec{u}) \) is zero for some \( \vec{u} \) if and only if (iff) the Weyl and BR tensors vanish

\[
\exists \vec{u} \text{ such that } W_T (\vec{u}) = 0 \iff T^{\alpha\beta\lambda\mu} = 0 \iff C^{\alpha\beta\lambda\mu} = 0.
\]

C The scalar \( W_T (\vec{u}) \) still satisfies a stronger inequality, which has been called the dominant super-energy property (DSEP):

for all future-pointing vectors \( \vec{v}, \vec{w}, \vec{x}, \vec{y} \), we have

\[
T_{\alpha\beta\lambda\mu} v^\alpha w^\beta x^\lambda y^\mu \geq 0.
\]

This is equivalent to the “dominance” of the completely timelike component of \( T \) in any orthonormal basis \( \{ e_\mu \} \), that is, \( W_T (\vec{e}_\alpha) = T_{\alpha\beta\lambda\mu} e^\alpha e^\beta e^\lambda e^\mu \geq 0 \). This property is very useful for it can provide s-e estimates and bounds on the “growth” of the Weyl tensor.

D Another fundamental property is that the BR tensor is “conserved” in vacuum, in the sense that \( T \) is divergence-free

\[
R_{\alpha\beta} = \Lambda g_{\alpha\beta} \implies \nabla_\alpha T^{\alpha\beta\lambda\mu} = 0
\]

where \( R_{\alpha\beta} \) is the Ricci tensor and \( g_{\alpha\beta} \) the metric tensor.

\(^a\)Dedicated to Professor Lluís Bel on the occasion of the Spanish Relativity Meeting 1998.
The Bel tensor: properties and decomposition

The Bel tensor is defined by

\[ 2 B^{\alpha \beta \lambda \mu} \equiv R^{\rho \lambda \sigma} R^\beta _{\rho \sigma} + R^\lambda _{\rho \sigma} R^{\beta \rho \sigma} + R^\mu _{\rho \sigma} R^{\beta \rho \sigma} + R^\rho _{\rho \sigma} R^{\beta \rho \sigma} \]  

where \( R^{\rho \lambda \sigma} \) is the Riemann tensor and the duals act on the left or the right pair of indices according to their position. The properties of \( B \) are:

\[ A_{\alpha \beta \lambda \mu} = B_{\beta \alpha \lambda \mu} = B_{\alpha \beta \mu \lambda} = B_{\lambda \mu \alpha \beta} = 0, \quad B^\alpha _{\beta \lambda \mu} = 0, \quad \text{but} \quad B^\alpha _{\beta \lambda \mu} \neq 0 \]  

The Bel tensor density is defined by

\[ W_B(\vec{u}) \equiv B^{\alpha \beta \lambda \mu} u^\alpha u^\beta u^\lambda u^\mu, \]  

so that \( W_B(\vec{u}) = (EE)^2 + (HH)^2 + (HE)^2 + (EH)^2 \geq 0 \) where \( EE, EH, HE \) and \( HH \) are the electric-electric, electric-magnetic, magnetic-electric and magnetic-magnetic parts of the Riemann tensor, which are 2-index symmetric tensors spatial relative to \( \vec{u} \). The notation \( (A)^2 \equiv A_{\mu \nu \rho \sigma} A^{\mu \nu \rho \sigma} > 0 \) for any such spatial tensor. Again, \( W_B \) vanishes iff the whole Riemann and Bel tensors vanish too.

The decomposition of the Riemann tensor into irreducible parts induces a canonical decomposition

\[ B^{\alpha \beta \lambda \mu} = T^{\alpha \beta \lambda \mu} + Q^{\alpha \beta \lambda \mu} + M^{\alpha \beta \lambda \mu} \]  

where \( Q \) and \( M \) are called the matter-gravity and the pure matter s-e tensors, respectively. See [8] for their definitions and properties. Here I only remark that \( Q \) is not typical (it does not satisfy any of the properties A-D above) and that \( M \) is a good s-e tensor for the matter content, it satisfies all the properties of the Bel tensor including DSEP, and the pure-matter s-e density \( W_M(\vec{u}) \) (defined as usual) is positive and vanishes iff \( R_{\mu \nu} = 0 \). Furthermore, the Bel s-e density decomposes as the simple sum:

\[ W_B = W_T + W_M \]  

It is noticeable that \( B \) has non-zero divergence in general but is divergence-free in the absence of matter (then \( B = T \)). Hence, the question arises of how to define s-e tensors for matter fields to see if the conservation of s-e can be restored (\( M \) is not satisfactory because it has no sense in Special Relativity).
3 Definition of (super)$n$-energy tensors for arbitrary physical fields

Consider any $m$-covariant tensor $t_{\mu_1...\mu_m}$ as an $r$-fold $(n_1, \ldots, n_r)$-form (with $n_1 + \ldots + n_r = m$) by separating the $m$ indices into $r$ blocks, each containing $n_A$ ($A = 1, \ldots, r$) completely antisymmetric indices. This can always be done because, even if $t_{\mu_1...\mu_m}$ has no antisymmetries, it can be seen as an $m$-fold $(1, \ldots, 1)$-form.

Several examples are: $F_{\mu\nu} = F_{[\mu\nu]}$ is a simple $(2,2)$-form, while $\nabla_{\rho} F_{\mu\nu}$ is a double $(1,2)$-form; the Riemann tensor is a double symmetrical $(2,2)$-form (the pairs can be interchanged) and the Ricci tensor is a double symmetrical $(1,1)$-form; a tensor such as $t_{\mu\nu\rho\sigma} = t_{(\mu\nu\rho\sigma)}$ is a triple symmetrical $(1,1,1)$-form, etcetera. I shall denote $t_{\mu_1...\mu_m}$ schematically by $t_{[n_1]...[n_r]}$ where $[n_A]$ indicates the $A$-th block with $n_A$ antisymmetrical indices. Then, one can define the duals by using the * acting on each of these blocks, obtaining the tensors (obvious notation and 4 dimensions):

\[ t_{[4-n_1]...[n_r]} \ldots t_{[n_1]...[4-n_r]} \]

There are $2^r$ tensors in this set (including $t_{[n_1]...[n_r]}$). Contracting the first index of each block with $\bar{u}$ for all these tensors we get the “electric-magnetic” decomposition of $t_{\mu_1...\mu_m}$. The electric-magnetic parts will be denoted by

\[ (E \ldots E)_{[n_1]...[n_r-1]}; \ldots; (E \ldots E H)_{[n_1]...[n_{r-1}] [3-n_r]} \]

These $2^r$ tensors are spatial relative to $\bar{u}$ and all of them determine $t_{\mu_1...\mu_m}$ uniquely and completely. Besides, $t_{\mu_1...\mu_m}$ vanishes if all its E-H parts do. Let us define the “semi-square” ($t_{[n_1]...[n_r]} \times t_{[n_1]...[n_r]}$) by contracting all indices but one of each block in the product of $t$ with itself

\[ (t \times t)_{\lambda \mu_1...\lambda \mu_r} \equiv \prod_{A=1}^{r} \frac{1}{(n_A - 1)!} t_{\lambda_1 \rho_2...\rho_{n_1} \ldots \lambda_r \sigma_2...\sigma_{n_r}} t_{\mu_1 \rho_2...\rho_{n_1} \ldots \mu_r \sigma_2...\sigma_{n_r}}. \]

The basic s-e tensor of $t$ is defined as the sum of the $2^r$ semi-squares constructed with $t_{[n_1]...[n_r]}$ and all its duals. Explicitly:

\[ 2T_{\lambda_1 \mu_1...\lambda_r \mu_r} \{ t \} \equiv (t_{[n_1]...[n_r]} \times t_{[n_1]...[n_r]})_{\lambda_1 \mu_1...\lambda_r \mu_r} + \ldots + \]

\[ \left( t_{[4-n_1]...[4-n_r]} \times t_{[4-n_1]...[4-n_r]} \right)_{\lambda_1 \mu_1...\lambda_r \mu_r} \]

A Expression $B$ is a $2r$-covariant tensor, symmetric on each $(\lambda_A \mu_A)$-pair, and if $t_{[n_1]...[n_r]}$ is symmetric in the interchange of $[n_A]$-blocks, then $T\{ t \}$ is symmetric in the interchange of the corresponding $(\lambda_A \mu_A)$-pairs. Moreover, $T\{ t \}$ is traceless in any $(\lambda_A \mu_A)$-pair coming from $[2]$-blocks (i.e. if $n_A = 2$).
The super-energy density \( W(\vec{u}) \equiv T_{\lambda_1 \mu_1 \ldots \lambda_r \mu_r} \{ t \} u^{\lambda_1} u^{\mu_1} \ldots u^{\lambda_r} u^{\mu_r} \) is positive for \( 2W(\vec{u}) = (EE \ldots E)^2 + (E \ldots EH)^2 + \ldots + (HH \ldots H)^2 \geq 0 \), and

\[
\exists \vec{u} \text{ such that } W(\vec{u}) = 0 \iff T_{\lambda_1 \mu_1 \ldots \lambda_r \mu_r} \{ t \} = 0 \iff t_{\mu_1} \ldots t_{\mu_m} = 0. \tag{4}
\]

Actually, \( W(\vec{u}) \) is the sum of the squares \( |t^{\mu_1} \ldots t^{\mu_m}|^2 \) of all the components of \( t \) in any orthonormal basis which includes \( \vec{u} = \vec{e}_0 \).

C The DSEP holds for the general \( T \{ t \} \), which is one of the main justifications and most important properties of definition \( \{ 3 \} \).

Example 1: The massless scalar field. Let \( \phi \) be a scalar field satisfying \( \nabla^\mu \nabla_\mu \phi = 0 \). First, by considering \( \nabla_\mu \phi \) as the basic field, one can construct the tensor \( \{ 3 \} \) and after expanding the duals we get

\[
T_{\lambda \mu} \{ \nabla \phi \} = \nabla_\lambda \phi \nabla_\mu \phi - \frac{1}{2} g_{\lambda \mu} \nabla_\rho \phi \nabla^\rho \phi
\]

which is the standard energy-momentum tensor. This tensor is identically divergence-free. Second, one can use the double symmetric \((1,1)\)-form \( \nabla_\alpha \nabla_\beta \phi \) as the basic object, and construct the corresponding tensor \( \{ 3 \} \) which becomes (this tensor was previously found by Bel in Special Relativity, see also \( \{ 11 \} \))

\[
T_{\alpha \beta \lambda \mu} \{ \nabla \nabla \phi \} = \nabla_\alpha \nabla_\lambda \phi \nabla_\beta \phi + \nabla_\alpha \nabla_\mu \phi \nabla_\lambda \nabla_\beta \phi -
- g_{\alpha \beta} \nabla_\rho \phi \nabla_\mu \phi - g_{\lambda \mu} \nabla_\alpha \nabla_\rho \phi \nabla_\beta \phi + \frac{1}{2} g_{\alpha \beta} g_{\lambda \mu} \nabla_\sigma \phi \nabla_\rho \phi \nabla^\sigma \nabla^\rho \phi. \tag{5}\]

This is the basic s-e tensor of the scalar field. Its divergence can be easily computed and there appear several terms proportional to the Riemann and Ricci tensors. Therefore, it is a conserved tensor in flat spacetime. Now, one can go on and build the \((super)\)\(^2\)-energy tensor \( T_{\alpha \beta \lambda \mu \tau \nu} \{ \nabla \nabla \nabla \phi \} \) associated with the triple \((1,1,1)\)-form \( \nabla_\alpha \nabla_\beta \nabla_\mu \phi \), and so on. This produces an infinite set of basic \((super)\)\(^n\)-energy tensors, one for each natural number \( n \). The following fundamental result holds: “the basic \((super)\)\(^n\)-energy (tensor) of the scalar field vanishes iff the \((n+1)\)th covariant derivative of \( \phi \) is zero”.

Example 2: The source-free electromagnetic field. Let \( F_{\mu \nu} \) be a 2-form satisfying Maxwell’s equations \( \nabla_\rho F_{\rho \nu} = 0, \nabla_\rho \tilde{F}_{\rho \nu} = 0 \). The tensor \( \{ 3 \} \) for \( F \) is

\[
T_{\lambda \mu} \{ F \} = F_{\lambda \rho} F_{\mu \rho} - \frac{1}{4} g_{\lambda \mu} F_{\rho \sigma} F^{\rho \sigma}
\]

which is the standard divergence-free energy-momentum tensor. To define the s-e tensor of the electromagnetic field one takes the double \((1,2)\)-form \( \nabla_\alpha F_{\mu \nu} \) as basic field and construct the corresponding expression \( \{ 3 \} \)

\[
T_{\alpha \beta \lambda \mu} \{ \nabla F \} = \nabla_\alpha F_{\lambda \rho} \nabla_\beta F_{\mu \rho} + \nabla_\alpha F_{\mu \rho} \nabla_\beta F_{\lambda \rho} - g_{\alpha \beta} \nabla_\sigma F_{\lambda \rho} \nabla_\tau F_{\mu \rho} -
- \frac{1}{2} g_{\lambda \mu} \nabla_\alpha F_{\sigma \rho} \nabla_\beta F^{\sigma \rho} + \frac{1}{4} g_{\alpha \beta} g_{\lambda \mu} \nabla_\tau F_{\sigma \rho} \nabla^\tau F^{\sigma \rho}. \tag{6}\]

\[
\]
This tensor is not symmetric in the interchange of $\alpha\beta$ with $\lambda\mu$ and is traceless in $\lambda\mu$. It does not coincide with previous s-e tensors for $\nabla F$, such as that of (which is simply $T_{\alpha\beta\lambda\mu}(\nabla F) + T_{\lambda\mu\alpha\beta}(\nabla F)$). Its divergence (with respect to the third index) is not zero in general, but it vanishes in flat spacetime. Again one can construct (super)$^n$-energy tensors associated to the higher derivatives of $F$, and the important result is: “the basic (super)$^n$-energy (tensor) of the electromagnetic field vanishes iff the $n^{th}$ covariant derivative of $F$ is zero”.

Example 3: The gravitational field. If the gravitational field is described by the Riemann tensor, then the s-e tensor (3) is exactly the Bel tensor (2). Similarly, the corresponding expression (3) for the Weyl tensor coincides with the BR tensor (1). One can also construct the s-e tensor for the Ricci tensor as basic field. This tensor has similar properties to those of the pure-matter s-e part $M$ of the Bel tensor, but it is not the same. In fact

$$2M_{\alpha\beta\lambda\mu} = T_{\alpha\lambda\beta\mu}(\tilde{R}) + T_{\alpha\mu\beta\lambda}(\tilde{R}) - T_{\alpha\beta\lambda\mu}(\tilde{R}) + \frac{R^2}{72} (4g_{\alpha(\lambda}g_{\mu)\beta} - g_{\alpha\beta}g_{\lambda\mu})$$

where $\tilde{R}$ indicates the trace-free Ricci tensor and $R$ is the scalar curvature. However, it is remarkable that $M_{\alpha\beta\lambda\mu}$ can be certainly obtained as basic s-e tensor for a field involving only the Ricci part of the curvature, namely

$$M_{\alpha\beta\lambda\mu} = T_{\alpha\beta\lambda\mu} \{ R,... - C,... \}$$

where $R,... - C,...$ denotes the double symmetric $(2,2)$-form $R_{\alpha\beta\lambda\mu} - C_{\alpha\beta\lambda\mu}$ (see e.g. for its explicit expression in terms of $R_{\mu\nu}$). This together with the Example 2 above makes it clear that the basic s-e tensor (3) for a given field is not the unique one with the same good properties.

What is the arbitrariness in the definition of s-e tensors? To answer this, let us consider only the super-energy tensors (4-index s-e tensors). From the definition (3), the basic $T_{\alpha\beta\mu\lambda}$ satisfies $T_{\alpha\beta\mu\lambda} = T_{\beta\alpha\mu\lambda} = T_{\alpha\beta\mu\lambda}$. The general s-e tensor formed with index permutations exclusively (to keep DSEP) is then

$$T_{\alpha\beta\lambda\mu} \equiv c_1 T_{\alpha\beta\lambda\mu} + c_2 T_{\alpha\lambda\beta\mu} + c_3 T_{\alpha\mu\lambda\beta} + c_4 T_{\lambda\beta\alpha\mu} + c_5 T_{\mu\beta\lambda\alpha} + c_6 T_{\lambda\mu\alpha\beta}$$

which does not have any symmetry in general. However, it is automatically symmetric in the interchange of $(\alpha\beta) \leftrightarrow (\lambda\mu)$ whenever the original $T_{\alpha\beta\mu\lambda}$ has this property, in which case we can redefine $\hat{c}_3 = c_1 + c_6$, $\hat{c}_2 = c_2 + c_5$, $\hat{c}_3 = c_3 + c_4$, $\hat{c}_4 = \hat{c}_5 = c_6 = 0$. In general, if the original $T_{\alpha\beta\mu\lambda}$ satisfies the DSEP, then the general $T_{\alpha\beta\lambda\mu}$ will also satisfy the DSEP (and a fortiori the positivity of the super-energy and the essential property (3)) at least when the constants satisfy $c_1, c_2, c_3, c_4, c_5, c_6 \geq 0$. Thus, subject to this condition there is a six-parameter family (reduced to a three-parameter one when the interchange
between pairs holds for $T_{\alpha\beta\lambda\mu}$ of s-e tensors satisfying the fundamental DSEP. Furthermore, $T_{\alpha\beta\lambda\mu}$ is symmetric in $\alpha\beta$ iff $c_2 = c_4$ and $c_3 = c_5$ (or $\hat{c}_2 = \hat{c}_3$ in the special case), and symmetric in $\lambda\mu$ iff $c_2 = c_3$ and $c_4 = c_5$ (respectively $\hat{c}_2 = \hat{c}_3$). This provides a three-parameter (resp. two-parameter) family of s-e tensors with the same symmetries as the basic one; $T_{\alpha\beta\lambda\mu}$ is symmetric in the interchange of $(\alpha\beta) \leftrightarrow (\lambda\mu)$ iff $c_1 = c_6$ and $c_3 = c_4$ (resp., in general). In particular, this proves that there exists a two-parameter family of s-e tensors for the gravitational field satisfying the same properties of the Bel tensor, and another two-parameter family for the scalar field $T^{(\alpha\beta)}$. A possible way to avoid this arbitrariness is to take the completely symmetric part $T_{\alpha\beta\lambda\mu} = \frac{1}{2} (T_{\alpha\beta\lambda\mu} + T_{\beta\alpha\lambda\mu})$. In the gravitational case, this provides the s-e tensor $T_{\alpha\beta\lambda\mu} = M_{\alpha\beta\lambda\mu}$ (no matter-gravity part), which has been also put forward recently by Robinson.

4 Interchange of super-energy between different physical fields

The important question of whether or not super-energy has any physical reality or interest may find an answer by studying its interchange between fields (as energy does) and its possible conservation. One of the most striking features of the Bel tensor is that it is conserved in the absence of matter (when it coincides with the BR tensor). Similarly, the s-e tensors (5) and (6) are conserved in the absence of gravity (in flat spacetime). The natural question arises: can these tensors be combined to produce a conserved quantity?

To answer it, let us consider the propagation of discontinuities of the Riemann and other fields. The notation and conventions we use are those in Ref. $\Sigma$ denotes a null hypersurface whose first fundamental form is $\bar{g}_{ab}$, $(a, b, \ldots = 1, 2, 3$ are indices in $\Sigma$). The normal one-form to $\Sigma$ is $n_\mu$ $(n_\mu n^\mu = 0)$, and a basis of tangent vectors to $\Sigma$ is denoted by $\vec{e}_a$ $(n_\mu e^\mu_a = 0)$. Obviously $\vec{n} = n^a \vec{e}_a$ and $\bar{g}_{ab} n^a = 0$. As the null vector $\vec{n}$ is in fact tangent to $\Sigma$, one needs to choose a vector field transversal to $\Sigma$ which is called the rigging and denoted by $\vec{\ell}$ $(n_\mu \ell^\mu = 1)$. There are many different choices for $\vec{\ell}$, but given any of them we can define $\omega^a_\mu$ as the three one-form fields completing with $n_\mu$ the basis dual to $\{\vec{\ell}, \vec{e}_a\}$, that is, $\ell^\mu \omega^a_\mu = 0$, $\vec{e}_b \omega^a_\mu = \delta^a_b$. We also put $\bar{g}^{ab} \equiv g^{\mu\nu} \omega^a_\mu \omega^b_\nu$. Notice that this is not the inverse of $\bar{g}_{ab}$, which does not exist because $\Sigma$ is null and the first fundamental form is degenerate. This last fact also implies that there is no canonical metric connection associated to $\bar{g}_{ab}$ in $\Sigma$. However, given any rigging one can define the so-called rigged connection $\bar{\Gamma}^a_{bc}$ in $\Sigma$ by means of $\bar{\Gamma}^a_{bc} \equiv \omega^a_\mu e^b_\mu \nabla_\mu e^c_\nu$, which is obviously torsion-free ($\bar{\Gamma}^a_{bc} = \bar{\Gamma}^a_{cb}$). The covariant derivative associated to the rigged connection will be written as $\vec{\nabla}$. The discontinuity of any object $v$ across $\Sigma$ will be denoted by $[v]$, as is customary. A very important point is that all the formulae appearing in what
follows are independent of the choice of the rigging vector $\ell$.

**The vacuum case.** As is well-known, the discontinuity across $\Sigma$ of the Riemann tensor is determined by a symmetric tensor $B_{ab}$ on $\Sigma$ (in principle, six independent components), as follows

$$[R_{\alpha\beta\lambda\mu} = B_{ab} (n_\alpha \omega^a_\beta - n_\beta \omega^a_\alpha) (n_\lambda \omega^b_\mu - n_\mu \omega^b_\lambda). \quad (7)$$

However, in vacuum only two independent discontinuities survive because $\bar{g}^{ab}B_{ab} = 0$ and $n^a B_{ab} = 0$. Then, one can show that

$$\nabla_a \left( B^2 n^b n^c n^d \right) = 0,$$

where $B^2 \equiv B^{ab}B_{ab} = \bar{g}^{ac}\bar{g}^{bd}B_{ab}B_{cd} > 0$. Moreover, the object inside brackets is directly related to the BR tensor by

$$2B^2 n^a n^b n^c n^d = \omega^a_\alpha \omega^b_\beta \omega^c_\lambda \omega^d_\mu \left[ T^{\alpha\beta\lambda\mu} \right].$$

Therefore, the discontinuity of the BR tensor is conserved along $\Sigma$ in vacuum.

**The Einstein-Maxwell case.** If there is only electromagnetic field $F_{\mu\nu}$ in the spacetime, and this is continuous, the discontinuities of the Riemann tensor take the same form (7) and besides

$$[\nabla_\lambda F_{\mu\nu}] = n_\lambda f_{\mu\nu}, \quad f_{\mu\nu} = f_a \left( n_\mu \omega^a_\nu - n_\nu \omega^a_\mu \right), \quad n^a f_a = 0$$

so that there only appear two independent discontinuities in the first derivatives of $F$. It can be proved that

$$\nabla_a \left\{ (B^2 + f^2) n^a n^b n^c n^d \right\} = 0, \quad (8)$$

where $f^2 \equiv f_a f_a = \bar{g}^{ab}f_a f_b > 0$. The interesting thing is that this quantity can be related to the s-e tensor by

$$2f^2 n^a n^b n^c n^d = \omega^a_\alpha \omega^b_\beta \omega^c_\lambda \omega^d_\mu \left[ T^{\alpha\beta\lambda\mu} \{ \nabla F \} \right].$$

Notice that in this case we have $[B_{\alpha\beta\lambda\mu}] = [T_{\alpha\beta\lambda\mu}]$, because the pure-matter part is continuous. Thus, if a Maxwell field propagates in vacuum, the sum of the discontinuities of the Bel tensor and of the s-e tensor for the electromagnetic field defined in (7) is conserved along $\Sigma$. A version of this fundamental result and of relation (8) were obtained by Lichnerowicz back in 1960.

**The case with a massless scalar field.** Assume finally that there only exists a massless scalar field $\phi$ in the spacetime whose gradient is continuous so that the energy-momentum tensor and the pure-matter part of the Bel tensor are also continuous, $[M_{\alpha\beta\lambda\mu}] = 0$. Then, expression (7) still holds and furthermore

$$[\nabla_\mu \nabla_\nu \phi] = V n_\mu n_\nu.$$
which provides a unique discontinuity freedom. Then, one can arrive at
\[ \nabla_a \left\{ \left( k_1 B^2 + k_2 V^2 \right) n^a n^b n^c n^d \right\} = 0, \quad \forall k_1, k_2 \]
and where, analogously as before, if the scalar field vanishes at one side of \( \Sigma \)
\[ 2V^2 n^a n^b n^c n^d = \omega^a_{\alpha} \omega^b_{\beta} \omega^c_{\lambda} \omega^d_{\mu} \left[ T^{\alpha \beta \lambda \mu} \{ \nabla \phi \} \right] \]
for the s-e tensor defined in (3). However, in this case, given that \( k_1 \) and \( k_2 \)
are arbitrary constants, the discontinuities are divergence-free separately. In
other words, under the stated assumptions, the discontinuities of the Bel tensor
and of the s-e tensor for the scalar field defined in (3) are conserved along \( \Sigma \)
independently.

Similar studies can be carried out for the higher-order (super)\( n \)-energy
tensors. The relevance and interpretation of all these results are under current
investigation.

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