LOWER ESTIMATES FOR DISTANCES FROM A GIVEN QUANTUM CHANNEL TO CERTAIN CLASSES OF QUANTUM CHANNELS

M. E. Shirokov and A. V. Bulinski

UDC 519.248.3

Abstract. By using estimates for the variation of quantum mutual information and the relative entropy of entanglement, we obtain ε-exact lower estimates for distances from a given quantum channels to sets of degradable, antidegradable, and entanglement-breaking channels. As an auxiliary result, we obtain ε-exact lower estimates for the distance from a given two-particle state to the set of all separable states.

Keywords and phrases: quantum state, quantum channel, coherent information, separable states, relative entropy of entanglement.

AMS Subject Classification: 81P45, 94A40

1. Introduction and basic concepts. Estimates of the variation of entropy characteristics of quantum states and channels are usually used for the study of problems in which the uniform continuity of these characteristics is of a high importance. Note that the well-known Fannes estimates for the variation of the von Neumann entropy and the Aliska–Fanness estimates for the variation of the quantum conditional entropy play an important role in the proofs of many results of the quantum information theory (see [1, 4, 6, 7, 13]).

In this paper, we show that estimates of the variation also can be used for obtaining lower estimates of the distance from a given quantum state (channel) to a certain class of states (channels). In other words, estimates of the variation allow us to estimate the size of a neighborhood of a given state (channel) that does not contain states (channels) of a certain type. The accuracy of these estimates plays a key role in this “nonstandard” application of estimates of the variation.

Let \( H \) be a finite-dimensional Hilbert space and \( \mathcal{B}(H) \) and \( \mathcal{S}(H) \) be the spaces of all linear operators in \( H \) with the operator norm \( \| \cdot \| \) and the trace norm \( \| \cdot \|_1 = \text{Tr} | \cdot | \), respectively. We denote by \( \mathcal{S}_+(H) \) the cone of positive operators in \( \mathcal{S}(H) \) and by \( \mathcal{G}(H) \) the convex set of density operators, i.e., operators from \( \mathcal{S}_+(H) \) with a unit trace that describe quantum states (see [6, 13]).

Let \( I_H \) be the identity operator in the Hilbert space \( H \) and \( \text{Id}_H \) be the identity transformation of the Banach space \( \mathcal{S}(H) \).

The von Neumann entropy of a quantum state \( \rho \in \mathcal{G}(H) \) defined by the formula \( H(\rho) = \text{Tr} \eta(\rho) \), where \( \eta(x) = -x \log x \) for \( x > 0 \) and \( \eta(0) = 0 \), is a nonnegative, concave, continuous function on the set \( \mathcal{G}(H) \) (see [6, 9, 13]). In this paper, we also use the binary entropy \( h_2(x) = \eta(x) + \eta(1 - x) \).

The quantum relative entropy of states \( \rho \) and \( \sigma \) from \( \mathcal{S}_+(H) \) is defined by the expression (see [9])

\[
H(\rho||\sigma) = \sum_{i=1}^{+\infty} \langle i | \rho \log \rho - \rho \log \sigma | i \rangle,
\]

where \( \{ |i\rangle \}_{i=1}^{+\infty} \) is an orthonormal basis consisting of eigenvectors of the state \( \rho \) if \( \text{supp} \rho \subseteq \text{supp} \sigma \) and \( H(\rho||\sigma) = +\infty \) otherwise. (In this paper, we use the Dirac notation (see, e.g., [6]), in which an orthonormal set of vectors is traditionally denoted by \( \{ |i\rangle \}_{i \in I} \), where \( I = \{1, 2, \ldots, n\} \) or \( I = \mathbb{N} \).

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 138, Quantum Computing, 2017.
If quantum systems \( A \) and \( B \) are described by Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively, then the composite system \( AB \) is described by the tensor product of the spaces \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \). If \( \omega_{AB} \) is a state from \( \mathcal{S}(\mathcal{H}_{AB}) \), then its partial states are \( \omega_A = \text{Tr}_B \omega_{AB} \) and \( \omega_B = \text{Tr}_A \omega_{AB} \), where \( \text{Tr} \) is the partial trace of \( \mathcal{H}_B \) (and similarly for \( A \)).

The quantum mutual information of the composite quantum system in a state \( \omega_{AB} \) is defined by the expressions (see [8])

\[
I(A:B)_\omega = H(\omega_{AB} \| \omega_A \otimes \omega_B) = H(\omega_A) + H(\omega_B) - H(\omega_{AB}).
\]

Using the Alisa—Fannes method (optimized by Winter in [14]), Shirokov proved in [11] that

\[
|I(A:B)_\rho - I(A:B)_\sigma| \leq 2\varepsilon \log d + 2g(\varepsilon)
\]

for any states \( \rho \) and \( \sigma \) from \( \mathcal{S}(\mathcal{H}_{AB}) \), where

\[
\varepsilon = \frac{1}{2} \| \rho - \sigma \|_1, \quad d = \min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\}, \quad g(\varepsilon) = (1 + \varepsilon) h_2 \left( \frac{\varepsilon}{1 + \varepsilon} \right).
\]

The estimate (1) is \( \varepsilon \)-exact (for large \( d \)).

2. On the “inverse” use of estimates of the variation. The following lemma shows how estimates of the variation of various characteristics of quantum states (channels) can be used to obtain lower estimates of the distance between quantum states (channels) and distance from a given state (channel) to a certain class of states (channels).

**Lemma 1.**

(A) Let \( X \) be a set with the metric\(^1\) \( D \) and \( F \) be a function on \( X \) such that

\[
|F(x_1) - F(x_2)| \leq A\varepsilon + r(\varepsilon), \quad \varepsilon = D(x_1, x_2)
\]

for all \( x_1, x_2 \in X \), where \( A > 0 \) and \( r \) is a nondecreasing function on \( \mathbb{R}_+ \) such that \( r(0) = 0 \). Then

\[
D(x_1, x_2) \geq A^{-1} (\Delta - r(A^{-1} \Delta)), \quad \Delta = |F(x_1) - F(x_2)|.
\]

(B) If \( F(x) \leq 0 \) for all \( x \in X_0 \subset X \) and \( F(x_*) > 0 \) for some \( x_* \in X \), then

\[
\inf_{x \in X_0} D(x_*, x) \geq A^{-1} (G - r(A^{-1} G))
\]

for all positive \( G \leq F(x_*) \).

**Proof.** (A) Since \( h(\varepsilon) = \varepsilon + A^{-1} r(\varepsilon) \) is an increasing nonnegative function such that \( h(0) = 0 \), we have

\[
\varepsilon \geq h^{-1}(A^{-1} \Delta) \geq A^{-1} \Delta - A^{-1} r(A^{-1} \Delta),
\]

where \( h^{-1} \) is the inverse function to the function \( h \), and the last inequality follows from the monotonicity of \( r \):

\[
h^{-1}(t) \geq t - A^{-1} r(t) \iff t \geq h(t - A^{-1} r(t)) = t - A^{-1} r(t) + A^{-1} r(t - A^{-1} r(t)),
\]

where \( t = A^{-1} \Delta \) (we can assume that \( r(x) = 0 \) if \( x < 0 \)).

(B) This assertion follows from the above argument, since for any \( x \in X_0 \) we have

\[
G \leq F(x_*) \leq A\varepsilon + r(\varepsilon), \quad \varepsilon = D(x_*, x).
\]

\(^1\)The statements of the lemma are valid for any nonnegative function \( D \) on \( X \times X \).
3. Lower estimates of the distance from a given two-particle state to the set of all separable states. Let \( \rho_{AB} \) be an arbitrary state in \( \mathcal{S}(\mathcal{H}_{AB}) \). Due to the second part of Lemma 1, the lower estimate of the distance from the state \( \rho_{AB} \) to the set \( \mathcal{S}_s(\mathcal{H}_{AB}) \) of all separable states of \( \mathcal{S}(\mathcal{H}_{AB}) \), i.e., the magnitudes

\[
D_s(\rho_{AB}) = \inf_{\sigma \in \mathcal{S}_s(\mathcal{H}_{AB})} \| \rho - \sigma \|_1,
\]

can be obtained by using estimates of the variation of any indicator of entanglement on \( \mathcal{S}(\mathcal{H}_{AB}) \), i.e., a nonnegative function \( E \) on \( \mathcal{S}(\mathcal{H}_{AB}) \) such that \( E^{-1}(0) = \mathcal{S}_s(\mathcal{H}_{AB}) \) (under the condition that this estimate has the form considered in Lemma 1). In particular, we can use any asymptotically continuous measure of entanglement \( E \) on \( \mathcal{S}(\mathcal{H}_{AB}) \) (see [10]).

The choice of a specific function \( E \) for a given problem is determined by the following requirements:

(i) the existence of a sufficiently exact estimate of the variation of the function \( E \);

(ii) the possibility of computing of \( E(\rho_{AB}) \) for an arbitrary state \( \rho_{AB} \) or the presence of a computable lower estimate for \( E(\rho_{AB}) \).

The first requirement is due to the desire of obtaining a sufficiently exact lower estimate for \( D_s(\rho_{AB}) \) and the second to the possibility of computable estimates.

Among well-known measures of entanglement on \( \mathcal{S}(\mathcal{H}_{AB}) \), the optimal choice of the function \( E \) is the choice of the relative entropy of entanglement \( E_R \), which is defined for any state \( \rho \) from \( \mathcal{S}(\mathcal{H}_{AB}) \) by the following expression (see [10, 12]):

\[
E_R(\rho) = \inf_{\sigma \in \mathcal{S}_s(\mathcal{H}_{AB})} H(\rho || \sigma).
\]  

(2)

Recently, Winter in [14] obtained the following \( \varepsilon \)-exact estimate of the variation of \( E_R \):

\[
|E_R(\rho) - E_R(\sigma)| \leq \varepsilon \log d + g(\varepsilon)
\]  

(3)

for any states \( \rho \) and \( \sigma \) from \( \mathcal{S}(\mathcal{H}_{AB}) \), where

\[
d = \min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\}, \quad \varepsilon = \frac{1}{2} \| \rho - \sigma \|_1, \quad g(\varepsilon) = (1 + \varepsilon) h_2 \left( \frac{\varepsilon}{1 + \varepsilon} \right).
\]

This estimate essentially refines the estimate of the variation of \( E_R \) obtained in [3].

Similarly, with all measures of entanglement, the relative entropy of entanglement is hard to compute for an arbitrary state \( \rho \in \mathcal{S}(\mathcal{H}_{AB}) \); however, it has an easily calculated lower estimate (see [10]):

\[
E_R(\rho) \geq I_c(\rho) = \max \left\{ I(A|B)_\rho, I(B|A)_\rho \right\} = \max_{X=A,B} H(\rho_X) - H(\rho),
\]  

(4)

where \( I(X|Y)_\rho = -H(X|Y)_\rho \) is coherent information on the state \( \rho \).

Applying Lemma 1 to the function \( F = E_R \) and using (3) and (4), we obtain the following statement.

**Proposition 1 (A. V. Bulinsky).** Let \( \rho \) be any state from \( \mathcal{S}(\mathcal{H}_{AB}) \). Then

\[
D_s(\rho) \geq 2 \frac{E_R(\rho)}{\log d} - \frac{2}{\log d} g \left( \frac{E_R(\rho)}{\log d} \right),
\]  

(5)

where

\[
d = \min\{ \dim \mathcal{H}_A, \dim \mathcal{H}_B \}, \quad g(\varepsilon) = (1 + \varepsilon) h_2 \left( \frac{\varepsilon}{1 + \varepsilon} \right).
\]

If \( I_c(\rho) > 0 \), where \( I_c(\rho) \) is defined by the formula (4), then

\[
D_s(\rho) \geq 2 \frac{I_c(\rho)}{\log d} - \frac{2}{\log d} g \left( \frac{I_c(\rho)}{\log d} \right).
\]  

(6)
If $\mathcal{H}_A = \mathcal{H}_B$ and $\rho$ is the maximally entangled, pure state from $\mathcal{G}(\mathcal{H}_{AB})$, then $E_R(\rho) = I_c(\rho) = \log d$ and, therefore, (5) and (6) give the same estimate

$$D_s(\rho) \geq 2 - \frac{2g(1)}{\log d} = 2 - \frac{4\log 2}{\log d}$$

which gives an alternative proof of the well-known fact that $D_s(\rho)$ is close to 2 for large $d$ (see [15]). We also note that the inequality (7) shows that both estimates (5) and (6) are asymptotically exact.

Applying Lemma 1 to the estimate of the variation (1) and to its generalization to quantum conditional mutual information (see [11, Corollary 1]), one can obtain lower estimates for

(i) $\| \cdot \|_1$-distance from a given two-particle state $\rho_{AB}$ to the set of all product states (i.e., states of the form $\sigma_A \otimes \sigma_B$);

(ii) $\| \cdot \|_1$-distance from a given three-particle state $\rho_{ABC}$ to the set of all short Markov chains (i.e., states $\sigma_{ABC}$ such that $\sigma_{ABC} = \text{Id}_A \otimes \Phi(\sigma_{AB})$ for some channel $\Phi : B \to BC$; see [5]).

4. Lower estimates of the distances from a given channel to the sets of degradable, antidegradable, and entanglement-breaking channels. A quantum channel $\Phi$ from a system $A$ to a system $B$ is a completely positive, trace-preserving linear mapping $\mathcal{G}(\mathcal{H}_A) \to \mathcal{G}(\mathcal{H}_B)$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces associated with the systems $A$ and $B$ (see [6, 13]). For brevity, we write $\Phi : A \to B$.

The set $\mathcal{F}(A, B)$ of all quantum channels from a system $A$ into a system $B$ is usually equipped with the metric generated by the norm of complete boundedness

$$\| \Phi \|_\circ \doteq \sup_{\rho \in \mathcal{G}(\mathcal{H}_{AB}), \| \rho \|_1 = 1} \| \Phi \otimes \text{Id}_R(\rho) \|_1$$

on the set of all completely bounded mappings from $\mathcal{G}(\mathcal{H}_A)$ into $\mathcal{G}(\mathcal{H}_B)$.

For any quantum channel $\Phi : A \to B$, the Steinspring theorem ensures the existence of a Hilbert space $\mathcal{H}_E$ (neighborhood) and an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \text{Tr}_E V \rho V^*$$

The quantum channel $\hat{\Phi} : A \to E$ defined by the expression

$$\hat{\Phi}(\rho) = \text{Tr}_B V \rho V^*$$

is said to be complementary to the channel $\Phi$ (see [6, Chap. 6]).

A channel $\Phi : A \to B$ is called degradable if $\hat{\Phi} = \Theta \circ \Phi$ for some channel $\Theta : B \to E$. A channel $\Phi : A \to B$ is called antidegradable if $\hat{\Phi}$ is a degradable channel (see [2]). We denote by $\mathcal{F}_d(A, B)$ and $\mathcal{F}_a(A, B)$ the sets of all degradable and antidegradable channels between the systems $A$ and $B$, respectively. These sets have a nonempty intersection: for example, the erasing channel

$$\Phi_p(\rho) = \begin{bmatrix} (1 - p)\rho & 0 \\ 0 & p \text{Tr}\rho \end{bmatrix}, \quad p \in [0, 1],$$

from a $d$-dimensional system $A$ to a $(d + 1)$-dimensional system $B$ is simultaneously degradable and antidegradable at $p = 1/2$.

The set $\mathcal{F}_a(A, B)$ contains the important subset $\mathcal{F}_{eb}(A, B)$ of channels that break entanglement, i.e., channels $\Phi : A \to B$ such that $\Phi \otimes \text{Id}_R(\omega_{AR})$ is a separable state from $\mathcal{G}(\mathcal{H}_{BR})$ for any state $\omega_{AR}$, where $R$ is an arbitrary quantum system (see [2]).

\footnote{Strictly speaking, the norm (8) coincides with the norm of complete boundedness of the dual mapping $\Phi^* : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ to the mapping $\Phi$ (see [6, 7, 13]).}

240
An important property of any degradable (respectively, antidegradable) channel $\Phi$ is the nonnegativity (respectively, nonpositivity) of the coherent information

$$I_c(\Phi, \rho) = H(\Phi(\rho)) - H(\hat{\Phi}(\rho))$$

for any input state $\rho$ (see [6, 13]). Note that

$$I_c(\Phi, \rho) = I(B:R)_{\Phi\otimes\text{Id}_R(\hat{\rho})} - H(\rho),$$

where $\mathcal{H}_R \cong \mathcal{H}_A$ and $\hat{\rho}$ is a pure state in $\mathcal{S}(\mathcal{H}_{AR})$ such that $\hat{\rho}_A = \rho$. It is easy to obtain from (1) the following estimate of the variation of the coherent information as a function of the channel:

$$|I_c(\Phi, \rho) - I_c(\Psi, \rho)| \leq 2\varepsilon \log d + 2g(\varepsilon),$$

where $\Phi$ and $\Psi$ are quantum channels from $A$ to $B$,

$$\varepsilon = \frac{1}{2}\|\Phi - \Psi\|_o, \quad d = \min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\}, \quad g(\varepsilon) = (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right).$$

The following proposition contains the lower estimates for the quantities

$$D_d(\Phi) \doteq \inf_{\Psi \in \mathcal{D}_d(A,B)} \|\Phi - \Psi\|_o, \quad D_a(\Phi) \doteq \inf_{\Psi \in \mathcal{D}_a(A,B)} \|\Phi - \Psi\|_o, \quad D_{eb}(\Phi) \doteq \inf_{\Psi \in \mathcal{D}_{eb}(A,B)} \|\Phi - \Psi\|_o,$$

which determine the radii of the largest open channel neighborhoods $\Phi$ that do not contain degradable, antidegradable, and entanglement-breaking channels, respectively.

**Proposition 2** (M. E. Shirokov). Let $\Phi : A \to B$ be a quantum channel,

$$d = \min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\}, \quad g(\varepsilon) = (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right).$$

(A) If there exists an input state $\rho$ such that

$$I_c(\Phi, \rho) > 0,$$

then

$$D_a(\Phi) \geq \frac{I_c(\Phi, \rho)}{\log d} - \frac{2}{\log d \log d} g\left(\frac{I_c(\Phi, \rho)}{2 \log d}\right),$$

$$D_{eb}(\Phi) \geq 2 \frac{I_c(\Phi, \rho)}{\log d} - \frac{2}{\log d \log d} g\left(\frac{I_c(\Phi, \rho)}{2 \log d}\right).$$

(B) If there exists an input state $\rho$ such that

$$L(\Phi, \rho) \doteq H(\rho) - H(\hat{\Phi}(\rho)) > 0,$$

then

$$D_{eb}(\Phi) \geq 2 \frac{L(\Phi, \rho)}{\log d} - \frac{2}{\log d \log d} g\left(\frac{L(\Phi, \rho)}{2 \log d}\right).$$

(C) If $R \cong A$ and $\omega$ is an arbitrary state from $\mathcal{S}(\mathcal{H}_{AR})$, then

$$D_{eb}(\Phi) \geq \frac{2}{\log d \log d} g\left(\frac{E_R(\Phi \otimes \text{Id}_R(\omega))}{\log d}\right),$$

where $E_R$ is the relative entropy of entanglement in $\mathcal{S}(\mathcal{H}_{BR})$.

(D) If there exists an input state $\rho$ such that

$$I_c(\Phi, \rho) < 0,$$

then

$$D_d(\Phi) \geq \frac{-I_c(\Phi, \rho)}{\log d} - \frac{2}{\log d \log d} g\left(\frac{-I_c(\Phi, \rho)}{2 \log d}\right).$$
The inequalities (13)–(17) are \( \varepsilon \)-exact estimates in the sense that for each of these inequalities and any \( \varepsilon > 0 \), there exist a channel \( \Phi \) and a state \( \rho \) such that the difference between the right- and left-hand sides of the inequality is less than \( \varepsilon \).

**Remark 1.** The estimates (14) and (15) are computable: they can be considered as weakened versions of the estimate (16), difficult for computing in the general case (see Example 2 below, in which all these estimates are explicitly calculated).

**Proof.** The inequalities (13) and (17) are obtained by applying the second part of Lemma 1 to the estimate of the variation (12).

To prove the inequalities (14)–(16), we note that

\[
D_{eb}(\Phi) \geq \sup_{\omega_{AR}} D_{s}(\Phi \otimes \text{Id}_R(\omega_{AR})),
\]

where the supremum is taken over all states from \( \mathcal{S}(\mathcal{H}_{AR}) \), and \( D_{s}(\rho_{BR}) \) is the distance from the state \( \rho_{BR} \) to the set of all separable states in \( \mathcal{S}(\mathcal{H}_{BR}) \) (see Sec. 2). Due to the standard arguments based on the property of convexity, this supremum can be taken over pure states \( \omega_{AR} \). Thus, the estimates (14)–(16) follow from Proposition 1 since it is easy to see that

\[
I(R)B_{\Phi \otimes \text{Id}_R(\omega_{AR})} = I_c(\Phi, \omega_A), \quad I(B)R_{\Phi \otimes \text{Id}_R(\omega_{AR})} = L(\Phi, \omega_A)
\]

for any pure state \( \omega_{AR} \).

The \( \varepsilon \)-exactness of the estimates (13) and (17) can be proved by using the family of erasing channels (11). It is known that the channel \( \Phi_\rho \) is degradable if \( p \leq 1/2 \) and antidegradable if \( p \geq 1/2 \) and, moreover, \( I_c(\Phi, \rho) = (1 - 2p)H(\rho) \) for \( p \in [0, 1] \) (see [6, Chap. 10]). Therefore, if \( \Phi = \Phi_{1/2-x} \) and \( \rho \) is a chaotic state, then the right-hand side of (13) is equal to

\[
2x - \frac{2g(x)}{\log d} \quad \text{provided that} \quad D_{s}(\Phi_{1/2-x}) \leq \|\Phi_{1/2-x} - \Phi_{1/2}\|_\diamond = 2x.
\]

The inequalities (13) and (17) are obtained by applying Lemma 1 to the estimate of the variation (12). The \( \varepsilon \)-exactness of the estimates (13) and (17) can be proved by using the family of erasing channels (11). It is known that the channel \( \Phi_\rho \) is degradable if \( p \leq 1/2 \) and antidegradable if \( p \geq 1/2 \) and, moreover, \( I_c(\Phi, \rho) = (1 - 2p)H(\rho) \) for \( p \in [0, 1] \) (see [6, Chap. 10]). Therefore, if \( \Phi = \Phi_{1/2-x} \) and \( \rho \) is a chaotic state, then the right-hand side of (13) is equal to

\[
2x - \frac{2g(x)}{\log d} \quad \text{provided that} \quad D_{s}(\Phi_{1/2-x}) \leq \|\Phi_{1/2-x} - \Phi_{1/2}\|_\diamond = 2x.
\]

The \( \varepsilon \)-exactness of the estimates (14)–(16) follows from Proposition 1 since it is easy to see that

\[
I(R)B_{\Phi_{1/2-x}} = I_c(\Phi_{1/2-x}, \omega_A), \quad I(B)R_{\Phi_{1/2-x}} = L(\Phi_{1/2-x}, \omega_A)
\]

for any pure state \( \omega_{AR} \).

The following example shows the \( \varepsilon \)-exactness of the estimates (14)-(16).

**Example 1.** If \( \dim \mathcal{H}_A \leq \dim \mathcal{H}_B \), \( \Phi = \text{Id}_A \) is the identical embedding of the set \( \mathcal{S}(\mathcal{H}_A) \) into \( \mathcal{S}(\mathcal{H}_B) \), and \( \rho \) is a chaotic state in \( \mathcal{S}(\mathcal{H}_A) \), then the inequality (13) implies

\[
D_{s}(\text{Id}_A) \geq 1 - \frac{2g(1/2)}{\log d_A} \approx 1 - \frac{1.9}{\log d_A},
\]

where \( d_A = \dim \mathcal{H}_A \), while all the inequalities (14)–(16) yield the same estimate

\[
D_{eb}(\text{Id}_A) \geq 2 - \frac{2g(1)}{\log d_A} \approx 2 - \frac{2.8}{\log d_A}.
\]

Thus, the radius of an open ball centered at \( \text{Id}_A \), which does not contain antidegradable (respectively, entanglement-breaking) channels, is close to 1 (respectively, to 2) for large dimensions \( d_A \). On the other hand, for \( x = 1/2 \), from (18) and the definition of the norm of complete boundedness (8), it follows that

\[
D_{s}(\text{Id}_A) \leq 1, \quad D_{eb}(\text{Id}_A) \leq 2.
\]

for any dimension \( d_A \).

**Example 2.** If \( \Phi_p \) is an erasing channel (11), then it is easy to see that

\[
I_c(\Phi_p, \rho) = (1 - 2p)H(\rho), \quad L(\Phi_p, \rho) = (1 - p)H(\rho) - h_2(p),
\]

242
where \( h_2 \) is the binary entropy (see [6, Chap. 6]). Therefore, the inequalities (14) and (15) with a chaotic state \( \rho \) imply respectively the inequalities

\[
D_{eb}(\Phi_p) \geq 2(1 - 2p) - \frac{2}{\log d_A} g(1 - p),
\]

\[
D_{eb}(\Phi_p) \geq 2(1 - p) - \frac{2}{\log d_A} \left( h_2(p) + g \left( (1 - p) - \frac{h_2(p)}{\log d_A} \right) \right),
\]

where we assume that \( g(x) = 0 \) for \( x < 0 \).

Since

\[
\Phi_p \otimes \text{Id}_R(\omega) = (1 - p)\omega \oplus p|\varphi\rangle \langle \varphi| \otimes \text{Tr}_A \omega,
\]

where \( \varphi \) is a unit vector in \( \mathcal{H}_B \oplus \mathcal{H}_A \), using the basic properties of the relative entropy and the convexity of \( E_R \), we can show that

\[
E_R(\Phi_p \otimes \text{Id}_R(\omega)) = (1 - p)E_R(\omega).
\]

Therefore, the inequality (16) with maximally entangled pure state \( \omega \) implies

\[
D_{eb}(\Phi_p) \geq 2(1 - p) - \frac{2}{\log d_A} g(1 - p).
\]

Since

\[
D_{eb}(\Phi_p) \leq \| \Phi_p - \Phi_1 \|_\diamond = 2(1 - p),
\]

we see that the inequalities (15) and (16) yield \( \varepsilon \)-exact estimates of \( D_{eb}(\Phi_p) \) for large dimension \( d_A \) (in contrast to the inequality (14)). We also see that the inequality (16) gives the most accurate estimate of \( D_{eb}(\Phi_p) \) for all \( p \) (as expected). Unfortunately, the applicability of this estimate to an arbitrary channel \( \Phi \) is restricted by the difficulty of computing of \( E_R \).

**Acknowledgment.** The work of M. E. Shirokov was supported by the Russian Science Foundation (project No. 14-21-00162).

**REFERENCES**

1. R. Alicki and M. Fannes, “Continuity of quantum conditional information,” J. Phys. A: Math. Gen., 37, No. 5, L55–L57 (2004).
2. T. S. Cubitt, M. B. Ruskai, and G. Smith, “The structure of degradable quantum channels,” J. Math. Phys., 49, 102104 (2008).
3. M. J. Donald and M. Horodecki, “Continuity of the relative entropy of entanglement,” Phys. Lett. A, 264, 257–260 (1999).
4. M. Fannes, “A continuity property of the entropy density for spin lattice systems,” Commun. Math. Phys., 31, 291–294 (1973).
5. P. Hayden, R. Jozsa, D. Petz, and A. Winter, “Structure of states which satisfy strong subadditivity of quantum entropy with equality,” Commun. Math. Phys., 246, No. 2, 359–374 (2004); arXiv:quant-ph/0304007
6. A. S. Holevo, Quantum Systems, Channels, Information. A Mathematical Introduction, De Gruyter, Berlin (2012).
7. D. Leung and G. Smith, “Continuity of quantum channel capacities,” Commun. Math. Phys., 292, 201–215 (2009).
8. G. Lindblad, “Entropy, information and quantum measurements,” Commun. Math. Phys., 33, 305–322 (1973).
9. G. Lindblad, “Expectation and entropy inequalities for finite quantum systems,” Commun. Math. Phys., 39, No. 2, 111–119.
10. M. B. Plenio and S. Virmani, “An introduction to entanglement measures,” *Quantum Inf. Comput.*, **7**, 1–51 (2007).
11. M. E. Shirokov, *Tight continuity bounds for the quantum conditional mutual information, for the Holevo quantity and for capacities of a channel*, e-print arXiv:1512.09047.
12. V. Vedral and M. B. Plenio, “Entanglement measures and purification procedures,” *Phys. Rev. A.*, **57**, 1619–1633 (1998).
13. M. M. Wilde, *From classical to quantum Shannon theory*, e-print arXiv:1106.1445
14. A. Winter, “Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints,” *Commun. Math. Phys.*, **347**, No. 1, 291–313 (2016).
15. A. Winter, Private communication.

M. E. Shirokov
Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow, Russia
E-mail: msh@mi.ras.ru

A. V. Bulinski
Moscow Institute of Physics and Technology (State University), Moscow, Russia
E-mail: andrey.bulinski@yandex.ru