THE FULL RENORMALIZATION HORSESHOE FOR MULTIMODAL MAPS: THE CONTINUITY OF THE ANTI-RENORMALIZATION OPERATOR FOR MULTIMODAL MAPS

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Abstract. In this paper, we consider the renormalization operator $R$ for multimodal maps. We prove the renormalization operator $R$ is a self-homeomorphism on any totally $R$-invariant set. As a corollary, we prove the existence of the full renormalization horseshoe for multimodal maps.

1. Introduction

Renormalization has been an important idea and tool in dynamical systems. Feigenbaum’s renormalization conjecture says that a certain renormalization operator has a hyperbolic fixed point. In fact, the original case considered by Feigenbaum is the periodic doubling case [5]. And such a conjecture was also formulated by Coullet and Tresser independently from Feigenbaum. For the periodic doubling case, Lanford [9] proved the existence of the hyperbolic fixed point with computer assistant, Sullivan [21] and McMullen [13] proved the uniqueness of the fixed point and the exponential contraction of $R$. Finally, Lyubich [10] considered the renormalization operator $R$ on the space $QG$ of quadratic-like germs, he defined a complex structure on $QG$ and then proved the hyperbolicity of the renormalization horseshoe. In [11], Lyubich proved the set of infinitely renormalizable real polynomials has Lebesgue measure zero. It implies his famous result: a typical real polynomial is either regular or stochastic. Avila and Lyubich [1] generalize the result to analytic unimodal case by introducing a method of path holomorphic structure and cocycles. There are also parallel results about the renormalization conjecture for critical circle maps, see [23, 24, 25].

In [18], Smania introduced multimodal maps of type $N$ and proved that deep renormalizations of infinitely renormalizable multimodal maps are multimodal maps of type $N$ for some positive integer $N$. Let $I = [-1, 1]$. A multimodal map $f : I \to I$ is called a multimodal map of type $N$, if there exists unimodal maps $f_0, \cdots, f_{N-1}$ with following properties:

1. $f_j : I \to I$ is a unimodal map fixing $-1$;
2. $f = f_{N-1} \circ \cdots \circ f_0$;
3. $0$ is a quadratic critical point of $f_j$ such that $f_j(0) \geq 0$ and $f_j''(0) < 0$.

We will call $(f_0, f_1, \cdots, f_{N-1})$ a unimodal decomposition of $f$. For convenience, we will also assume that $f$ is even, i.e. $f(x) = f(-x)$ for all $x \in I$. Since we
A multimodal map $f$ of type $\mathbf{N}$ is called renormalizable if there exists a periodic interval $J$ of period $p$ of $f$ such that $f^p|_J$ is affinely conjugate to a multimodal map of type $\mathbf{N}$. There is a canonical way to normalize $f^p|_J$ to be a multimodal map of type $\mathbf{N}$, and we call the normalized map $Rf$ the renormalization of $f$ and the smallest integer $p$ is called the renormalization period of $f$. We say $f$ is infinitely renormalizable with bounded combinatorics if $f$ is infinitely renormalizable and the renormalization period $p_k$ of $R^k f$ is bounded.

Let $\mathcal{I}$ be the set of all the infinitely renormalizable real-analytic multimodal maps of type $\mathbf{N}$ equipped with the $C^3$-topology. Then the renormalization operator $R$ for multimodal maps of type $\mathbf{N}$ induces a dynamical system $R : \mathcal{I} \to \mathcal{I}$. In [19, 20], Smania considered the sub-dynamical system : $R|_{\mathcal{I}_p} : \mathcal{I}_p \to \mathcal{I}_p$, where $\mathcal{I}_p$ is the set of infinitely renormalizable multimodal maps of type $\mathbf{N}$ with combinatorics bounded by $p$. He proved that the $\omega$-limit set $\Omega_p$ of the renormalization operator $R|_{\mathcal{I}_p}$ is compact and $R|_{\Omega_p} : \Omega_p \to \Omega_p$ is topologically conjugate to a full shift of finite elements.

In this paper, we prove the renormalization operator of multimodal maps of type $\mathbf{N}$ has a full horseshoe:

**Theorem A.** Let $\mathcal{I}$ be the set of all the infinitely renormalizable multimodal maps of type $\mathbf{N}$ and $\Sigma$ be the set of all the renormalization combinatorics for multimodal maps of type $\mathbf{N}$. Then there exists a precompact subset $A \subset \mathcal{I}$ such that the restriction $R|_A$ of $R$ is topologically conjugate to a two-sided full shift on $\Sigma^\mathbb{Z}$.

See section 3 for a definition of the renormalization combinatorics.

To prove the full renormalization horseshoe for multimodal maps of type $\mathbf{N}$, there are two main difficulties. One is to prove infinitely renormalizable multimodal maps has complex bounds, which has been done by Shen[16]. Since such complex bounds has been built, we can modify the argument of Avila-Lyubich[1] to get a semi-conjugacy desired in Theorem A:

**Theorem B.** Let $\mathcal{I}$ and $\Sigma$ be as in the assumptions of Theorem A. Then there exists a precompact subset $A \subset \mathcal{I}$ such that $R(A) = A$ and a continuous bijection $h$ which gives a topological semi-conjugacy between $R|_A$ and a two-sided full shift on $\Sigma^\mathbb{Z}$.

Another difficulty is to prove the inverse $h^{-1}$ of the semi-conjugacy in Theorem B is continuous. For the uniformly bounded combinatorics case, the proof is easy. However, it is not trivial to deal with the unbounded combinatorics case. To this end, we prove the following dichotomy:

**Key Lemma.** Let $\{f_k\}$ be a sequence of bi-infinitely renormalizable multimodal maps of type $\mathbf{N}$ which is precompact under $C^3$-topology. If the renormalization periods $p_k$ of $f_k$ tend to infinity, then each limit of $R f_k$ is either a polynomial of degree $2^n$ or with bounded real trace.

It is worth mentioning that Avila-Lyubich proved this theorem in the unimodal case[1], which inspired us so much. Even in that case, the proof is nontrivial and complicated.

As a corollary of the Key Lemma, we prove
Theorem C. For any totally $R$-invariant precompact subset $A' \subset I$, the restriction $R^{-1}|_{A'}$ of the anti-renormalization operator $R^{-1}$ to $A'$ is continuous.

Let us now describe the organization of the paper.

The proof of Theorem A will be postponed to section 5. In section 2, we recall some background of real box maps and use the distortion results for real box maps to prove some compactness lemmas which are the indispensable tools in the proof of the Key Lemma. We recall the definition of renormalization combinatorics in section 3 and then prove Theorem C. From section 4 to the end of this paper, we will use the idea of path holomorphic space together with Theorem C, following Avila-Lyubich, to show the existence of the full renormalization horseshoe for multimodal maps of type $N$. We study the complexification of the multimodal maps of type $N$ and its renormalization operator in section 4, the external and inner structure will be discussed there. The path holomorphic structure on each hybrid leaf will be defined in section 5 and we modify the argument of Avila and Lyubich [1] to show that the renormalization operator contracts exponentially fast along the real-symmetric hybrid leaves by virtue of the complex bounds.

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2. Renormalization Operator and infinitely renormalizable maps

In this section, we will first introduce the definition of renormalization of multimodal maps of type $N$ and recall some results about the real bounds for real box maps. Then we will prove the Main Theorem:

Key Lemma. Let $\{f_k\}$ be a sequence of bi-infinitely renormalizable multimodal maps of type $N$ which is precompact under $C^3$-topology. If the renormalization periods $p_k$ of $f_k$ tend to infinity, then each limit of $R f_k$ is either a polynomial of degree $2^n$ or with bounded real trace.

2.1. The extended maps and renormalization. Fix a multimodal map $f$ of type $N$ and a unimodal decomposition $(f_0, f_1, \cdots, f_{N-1})$ of $f$. Let $I_N = \{(x,j) \mid x \in I, 0 \leq j < N\}$, following Smania [19], we define the extended map $F$ of $f$:

$$F : I_N \rightarrow I_N$$

$$(x,j) \mapsto (f_j(x), j + 1 \mod N).$$

Clearly, the extended map of $f$ is not unique since $f$ can have several unimodal decompositions. The extended map is a real box map.

Definition 2.1. A closed interval $J \ni 0$ is called a $k$-periodic interval of an extended map $F$ if it satisfies:

1. $F^k(J \times \{0\}) \subset J \times \{0\}$,
2. $J \times \{0\}, F(J \times \{0\}), \cdots, F^{k-1}(J \times \{0\})$ are closed intervals with disjoint interiors,
3. for every $1 \leq j \leq N - 1$, there exists exactly one $1 \leq m < k$ such that $0 \times \{j\} \in F^m(J \times \{0\})$,
4. $k > N$.

Let $p = k/N$, we also say $J$ is a $p$-periodic interval of $f$. 

Let $F$ be an extended map of a multimodal map $f$ of type $N$. Consider a maximal $k$-periodic interval $J$ of $F$, i.e., $F^k(\partial J \times \{0\}) \subset \partial J \times \{0\}$ (if $F$ has a $k$-periodic interval, then it must have a maximal $k$-periodic interval). Then there exists a canonical affine transformation $A_0 : J \to \mathbb{I}$ such that $A_0(0) = 0$ and $A_0 \circ f^{k/N} \circ A_0^{-1} : \mathbb{I} \to \mathbb{I}$ is a multimodal map of type $N$. To see this, for every $0 \leq j < N$, let $m_j$ be the integer such that $0 \times \{j\} \in F^{m_j}(J \times \{0\})$ and $J_j$ be the interval such that $J_j \times \{j\}$ is the symmetrization of $F^{m_j}(J \times \{0\})$ with respect to $(0, j)$. There is a periodic point $z_j$ of $f$ on the boundary of $J_j$, let $A_j : J_j \to \mathbb{I}$ be the affine transformation such that $A_j(0) = 0$, $A_j(z_j) = -1$ and $A_j : J_j \times \{j\} \to \mathbb{I}$, $A_j(x, j) = (A_j(x), j)$ for all $0 \leq j < N$. For convenience, we make a convention that $m_N = m_0 = 0$, $J_N = J_0$, $A_N = A_0$, $\tilde{A}_N = \tilde{A}_0$. Then for every $0 \leq j \leq N - 1$, $F_j := \tilde{A}_{j+1} \circ f^{m_j + 1 - m_j} \circ \tilde{A}_j^{-1} : \mathbb{I} \times \{j\} \to \mathbb{I} \times \{j + 1 \mod N\}$ is a unimodal map with critical point $0$ and $\tilde{A}_0 \circ f^k \circ \tilde{A}_0^{-1} = F_{N-1} \circ \cdots \circ F_1 \circ F_0$, it implies $A_0 \circ f^{k/N} \circ A_0^{-1} : \mathbb{I} \to \mathbb{I}$ is also a multimodal map of type $N$. If $f$ does not have a periodic interval with period strictly smaller that $k/N$, then we call $A_0 \circ f^{k/N} \circ A_0^{-1}$ is the \textit{real renormalization} of $f$ and denote it by $Rf$. Clearly, the renormalization of $f$ does not depend on the unimodal decomposition of $f$. If $Rf$ is again renormalizable, then we will say $f$ is \textit{twice renormalizable}. If this procedure can be done infinitely many times, then $f$ will be called \textit{infinitely renormalizable}. In this paper, we mainly concern about the infinitely renormalizable maps.

\textbf{Definition 2.2.} A multimodal map $f$ of type $N$ is called \textit{anti-renormalizable}, if there exists a renormalizable multimodal map $g$ of type $N$ such that $Rg = f$.

Since Smania had proved the renormalization operator $R$ is an injection \cite{20} Proposition 2.2, the anti-renormalization operator is also well-defined. Similar to infinitely renormalizable maps, we can define infinitely anti-renormalizable maps, and a multimodal map is called \textit{bi-infinitely renormalizable} if it is both infinitely renormalizable and anti-infinitely renormalizable.

\section{Background in real box maps.} Throughout Section 2, we will assume $f$ is a bi-infinitely renormalizable multimodal map of type $N$ with renormalization period $p > 2$ and fix an extended map $F$ of $f$. Set $c_j = (0, j) \in \mathbb{I} \times \{j\}$ for all $0 \leq j \leq N - 1$. Assume $(\alpha, 0)$ is the $F^n$- fixed point closest to $c_0$ and $(-\alpha, 0)$ is the reflection of $(\alpha, 0)$ with respect to $c_0$. Set

$$I_0 := (\alpha, -\alpha) \times \{0\}.$$

Let $(\beta, 0)$ be the preimage of $F^{-n}(\alpha, 0)$ closest to the point $(-1, 0)$ and define a set

$$E_0 := I \times \{0\} \setminus \{(\beta, 0), (\beta, 0), (\alpha, 0), (-\alpha, 0)\}.$$

\textbf{Definition 2.3.} An open subset $B$ of $\mathbb{I}_N$ is called nice if $\bigcup_{k \geq 1} F^k(\partial B) \cap B = \emptyset$.

The concept of nice interval was first introduced by Martens \cite{12}. For a nice symmetric interval $B$, we denote $D_B$ the first entry domain of $B$ under the iterates of $F$, that is,

$$D_B = \{x \in \mathbb{I}_N \mid F^k(x) \in B \text{ for some integer } k \geq 1\}.$$
For any \( x \in D_B \), the minimal positive integer \( k = k(x) \) such that \( F^k(x) \in B \) is called the first entry time of \( x \). The first entry map to \( B \) is defined as:

\[
R_B : D_B \rightarrow B
x \rightarrow F^{k(x)}(x).
\]

The restriction of \( R_B \) to \( D_B \cap B \) is called the first return map. For \( x \in D_B \), we shall use \( \mathcal{L}_x(B) \) to denote the connected component of \( D_B \) containing \( x \). Let \( \mathcal{L}_x^1(B) = \mathcal{L}_x(B) \), and for any positive integer \( j \geq 2 \), let \( \mathcal{L}_x^j(B) = \mathcal{L}_x(\mathcal{L}_x^{j-1}(B)) \), whenever it makes sense.

For a symmetric nice interval \( c \in I \subset \mathbb{I}_N \), the scaling factor of \( I \) is defined as:

\[
\lambda_I := \frac{|I|}{|\mathcal{L}_c(I)|}.
\]

For a nice open set \( K \cap \omega(c) \neq \emptyset \) (where \( \omega(c) \) is the \( \omega \)-limit set of \( c \)), let \( \mathcal{M}(K) \) be the collection of intervals which are pullbacks of components of \( K \). Shen defined the limit scaling factor of \( K \) as:

\[
\Lambda_K := \sup_I \lambda_I,
\]

where the supremum is taken over all symmetric nice intervals in \( \mathcal{M}(K) \).

We say \( B_0 \ni B_1 \ni B_2 \ni \cdots \) is a nest if there exists \( x \in B_0 \) such that \( B_{n+1} = \mathcal{L}_x(B_n) \) for all \( n \in \mathbb{N} \).

A sequence of nice symmetric intervals \( B_1 \ni B_2 \ni \cdots \ni B_L \ni c \) is called a central cascade with respect to \( c \), if \( B_{j+1} = \mathcal{L}_c(B_j) \) for all \( 1 \leq j \leq L - 1 \) and \( R_{B_{j+1}}(c) \in B_{j+1} \) for each \( 1 \leq j \leq L - 2 \). Such a central cascade is called maximal if \( R_{B_{L-1}} \) displays a non-central return, i.e., \( R_{B_{L-1}}(c) \notin B_L \). We say that the central cascade is of saddle node type if \( R_{B_1 \mid B_2} \) has all the critical points in \( B_L \), and does not have a fixed point.

**Remark 2.1.** If \( B_1 \ni B_2 \ni \cdots \ni B_L \) is a maximal central cascade, then \( B_j \ni B_{j+1} \ni \cdots \ni B_L \) is also a maximal central cascade for all \( 2 \leq j \leq L - 1 \).

By a chain we mean a sequence of open intervals \( \{G_s\}_{s=0}^k \) such that \( G_{s+1} \) is a component of \( F^{-1}(G_s) \) for every \( 0 \leq s \leq k - 1 \). The order of the chain is the number of the integers \( s \) with \( 0 \leq s < n \) such that \( G_s \) intersects \( \text{Crit}(F) \) and the intersection multiplicity is the maximal number of the intervals \( G_s \) \((0 \leq s \leq n)\) which have a non-empty intersection.

We shall need the following known results:

**Theorem 2.1** ([22] Theorem A). There exists \( 1 < \lambda = \lambda(||F||_{C^3}) \) with the following property. Let us consider a nest \( B_0 \ni B_1 \ni B_2 \ni \cdots \). If \( R_{B_k} \) does not display a central return, then

\[
\lambda_{B_{k+1}} = \frac{|B_{k+1}|}{|B_{k+2}|} > \lambda.
\]

Real bounds for \( S \)-unimodal maps were proved earlier by Martens [12].

We say an open interval \( I \) is a \( \delta \)-neighborhood of an interval \( J \), which is denoted by \( (1 + 2\delta)J \), if \( J \subset I \) and each component \( I \setminus J \) has length equal to \( \delta |J| \).

**Lemma 2.1** ([15] Proposition 2.2). For any \( p, q \in \mathbb{N} \) and any \( \delta > 0 \), there exists a constant \( \delta_1 = \delta_1(\delta, p, ||F||_{C^3}) > 0 \) such that the following holds. Let \( G = \{G_j\}_{j=0}^s \)
and $G' = \{G'_j\}_{j=0}^s$ be chains such that $G_j \Subset G'_j$ for all $0 \leq j \leq s$. Assume the order of $G'$ is at most $p$ and

$$\#\{j \mid G'_j \supseteq G_s\} \leq q.$$ 

If $(1 + 2\delta)G_s \subset G'_s$, then $(1 + 2\delta_1)G_0 \subset G'_0$. Moreover, $\delta_1 \to \infty$ as $\delta \to \infty$.

For two intervals $I$ and $J$, we briefly say $J$ is geometrically deep inside $I$ if there is a large $\delta$ such that $(1 + 2\delta)J \subset I$.

**Remark 2.2.** If $F^q|_I : I \to I'$ is a first return map and $J'$ is geometrically deep inside $I'$, then it follows immediately from Lemma 2.1 that $J := (F^q|_I)^{-1}(J')$ is geometrically deep inside $I$.

All the central cascades have been proved to be essentially saddle-node in the following sense by Shen[14]:

**Lemma 2.2** ([19 Proposition 5.1, Theorem 5.4]). For any $\delta > 0$ and $\rho > 0$, there exists $b = b(\delta, \rho, \|F\|_{C^3})$ and $\eta = \eta(\delta, \rho, \|F\|_{C^3}) > 1$ with the following property. Consider a central cascade $B_1 \supset B_2 \supset \cdots \supset B_L$, assume $B_1 \supset (1+2\delta)B_2, \Lambda B_1, \rho$, and $L > 3b$. Then

1. the central cascade $B_b \supset B_{b+1} \supset \cdots \supset B_{L'}$ is of saddle-node type for some $L - b < L' \leq L$;
2. for any $x \in B_{L'}$, we have
   $$|R_{B_1}(x) - x| \geq \frac{|B_1|}{b};$$
3. for each $1 \leq j \leq L - 1$, we have the Yoccoz equality:
   $$\frac{1}{\eta k^2} < \frac{|B_j \setminus B_{j+1}|}{|B_1|} < \frac{1}{\eta k^2}$$
   where $k = \min(j, L - j)$.

### 2.3. Admissible intervals and transition maps.

We say an interval $T$ is an admissible interval if $T \in \mathcal{M}(\mathbb{E}_0)$. An admissible interval $T'$ is called a pullback of $T$ if $T' \in \mathcal{M}(T)$. More precisely, we say $T'$ is a $k$-pullback of $T$ if $F^k(T') \subset T$ and $F^k(\partial T') \subset \partial T$.

For an admissible interval $T$, let $\mathcal{A}_T : T \to \text{int}(\mathbb{I})$ be an orientation-preserving affine homeomorphism. Let $T'$ be a $k$-pullback of $T$, then the transition map of $T$ and $T'$ is defined as

$$G_{T,T'} := \mathcal{A}_T \circ F^k \circ \mathcal{A}_T^{-1} : \text{int}(\mathbb{I}) \to \text{int}(\mathbb{I}).$$

Let $\{G_0\}_{s=0}^k$ be the chain from $T'$ to $T$, that is, a sequence of open intervals such that $G_k = T$, $G_0 = T'$ and $G_{s+1}$ is a component of $F^{-1}(G_s)$ for every $0 \leq s \leq k - 1$. For a critical point $c \in \text{Crit}(F)$, if $c \notin G_s$ for every $1 \leq s \leq k - 1$, then we say the transition map $G_{T,T'}$ is short (with respect to $c$), otherwise $G_{T,T'}$ is called long (with respect to $c$). For a long transition map, let $1 \leq m_1 < \cdots < m_l \leq k - 1$ be all the integers such that $G_{m_j} \ni c$, set $T_{\ell+1-j} = G_{m_j}$, then we have a canonical decomposition (with respect to $c$):

$$G_{T,T'} = G_{T,T_1} \circ G_{T_1,T_2} \circ G_{T_{l-1},T_{l}} \circ G_{T_{l},T'}. $$

We consider the principal nest:

$$I_0 := (\alpha, -\alpha) \times \{0\} \ni c_0, I_1 = \mathcal{L}_{c_0}(I_0), \cdots, I_n = \mathcal{L}_{c_0}(I_n), \cdots,$$
and $I_\infty = \bigcap I_n$ is a periodic interval since $f$ is renormalizable with respect to 0.

To describe the geometric properties of $F$, we need the following definition.

**Definition 2.4.** For each admissible interval $I$, let $U(I)$ be the union of all the components of $D_I$ which intersect $I \cap \omega(c_0)$. We say $I$ has $C$-bounded geometry if

1. $((1 + 2C^{-1})I - (1 - 2C^{-1})I) \cap \omega(c_0) = \emptyset$;
2. for each component $J$ of $I \setminus \partial U(I)$, $|J| > C^{-1}|I|$.

An admissible interval which satisfies condition (1) is called $C$-nice.

**Remark 2.3.** As $f$ is infinitely renormalizable, for any $c \in \text{Crit}(F)$, $\omega(c) = \omega(c_0) = P(F)$, where $P(F)$ is the postcritical set of $F$.

Let $m(0) = 0$ and let $m(1) < m(2) < \cdots < m(\kappa)$ be all the non-central return moments, i.e., $R_{m(k) - 1}$ displays a non-central return. The integer $\kappa$ is called the height of $F$.

**Lemma 2.3.** For any $q > 0$ and $\rho > 0$, there exists $C_1 = C_1(\rho, \|F\|_{C^1}) > 0$ and $C_2 = C_2(\rho, q, \|F\|_{C^1}) > 0$ with the following properties. If $I_{m_0} \subset I_k$, then

1. $1 + C_1^{-1} < I_{m(k)} < \rho$ for all $0 \leq k \leq \kappa$;
2. $I_0$ is $C_1$-nice;
3. for any $t \in \mathbb{N}$ with inf $|t - m(k)| \leq q$, $I_t$ has $C_2$-bounded geometry.

**Proof.** Theorem 2.4 implies (1). Statements (2) and (3) follows from [16, Proposition 5.10] and [16, Theorem 5.5] respectively.\[\square\]

Given an interval $J \subset \mathbb{R}$, let $\mathbb{C}_J := \mathbb{C} \setminus (\mathbb{R} \setminus J)$ denote the plane slit along two rays. Following Shen[16], we define the Epstein class as following. For any $C > 0$ and $\eta > 0$, the class $\mathcal{K}(C, 1 + \eta)$ consists of diffeomorphisms $\phi : I \to I$ with following properties:

1. the $C^{1+1/2}$-norm of $\phi$ is at most $C$;
2. $\phi^{-1}|_{\text{int}(I)}$ extends to a real symmetric $(1+\eta)$-qc map from $\mathbb{C}_{\text{int}(I)}$ into itself.

For any $u \in [-1/2, 1/2]$, let $Q_u(z) = u(z^2 - 1) + z$. For any $v \in (0, 2]$, let $P_v(z) = v(z^2 - 1) + 1$. Let $\mathcal{SE}(C, 1 + \eta, M)$ denote the set of all functions $\Phi : I \to I$ which can be written as

$\Phi = \psi_m \circ \phi_m \circ \cdots \circ \psi_1 \circ \phi_1$

for some $m \leq M$, where for each $1 \leq j \leq m$, $\phi_j \in \mathcal{K}(C, 1 + \eta)$; and $\psi_j = Q_{u_j}$ for some $u_j \in [-1/2, 1/2]$ or $\psi_j = P_{v_j}$ for some $v_j \in [1/C, 2]$. We say $\Phi : I \to I$ is in the Epstein class if $\Phi \in \mathcal{SE}(C, 1, M)$ for some $C > 0$ and $M > 0$.

**Remark 2.4.** For any $C > 0, \eta > 0$ and $N > 0$, $\mathcal{SE}(C, 1 + \eta, M)$ is compact in $C^1$-topology. If $\Phi_k \in \mathcal{SE}(C, 1 + 1/k, M)$ converges to $\Phi$ in $C^1$-topology, then $\Phi \in \mathcal{SE}(C, 1, M)$.

**Lemma 2.4.** If $F^k : J \to F^k(J)$ is a diffeomorphism, then $F^{-k} : F^k(J) \to J$ extends to a conformal map from $\mathbb{C}_{F^k(J)}$ into $\mathbb{C}_J$.

**Proof.** Since $F$ is infinitely anti-renormalizable, there exists a sequence $\{H_j\}_{j=1}^\infty$ of infinitely renormalizable extended maps with following properties:

- there exist positive integers $m_1, m_2, \cdots$ and multi-intervals $J_1, J_2, \cdots$ such that $H_j^{m_i}|_{J_i}$ is affinely conjugate to $F$;
- each fiber of $J_i$ has length less than $\lambda^{-j}$ with $\lambda > 1$ for all $j = 1, 2, \cdots$, where $\lambda$ is given by Theorem 2.1.
Let \( \Psi_j \) be the affine conjugacy between \( F \) and \( H_j^{m_j}|_{J_j} \). Then \( H_j^{m_j} : \Psi_j(J) \rightarrow \Psi_j(F^k(J)) \) is a diffeomorphism, by \[16\] Proposition 5.7, \( H_j^{-k m_j} : \Psi_j(F^k(J)) \rightarrow \Psi_j(J) \) can extend to a \( \exp(O(\lambda^{-j})) \)-qc map from \( \mathbb{C}_{\Psi_j(F^k(J))} \) into \( \mathbb{C}_{\Psi_j(J)} \). Use the affine conjugacy, we conclude \( F^{-k} : F^k(J) \rightarrow J \) can extend to a \( \exp(O(\lambda^{-j})) \)-qc map \( \Phi_j \) from \( \mathbb{C}_{F^k(J)} \) into \( \mathbb{C}_J \) for all \( j \in \mathbb{N} \). By the compactness of normalized \( K \)-qc maps, \( \Phi_j \) converges uniformly to a conformal map \( \Phi \) from \( \mathbb{C}_{F^k(J)} \) into \( \mathbb{C}_J \). Clearly, \( \Phi|_{F^k(J)} = F^{-k} \).

**Lemma 2.5.** There exists \( C = C(\delta, M) > 0 \) with the following property. Let \( \{G^s_s\}_{s=0}^k \) and \( \{G^s_J\}_{s=0}^k \) be chains satisfying:
- \( G^s_s \supset G_s \) for all \( 0 \leq s \leq k \) and \( G_0 \cap \omega(c_0) \neq \emptyset \);
- the multiplicity of \( \{G^s_s\}_{s=0}^k \) is at most \( M \);
- \( G^s_k \supset (1 + 2\delta)G_k \) and \( |f_k(G_0)| \geq \delta|G_k| \).

For any \( 0 \leq s \leq k \), let \( \gamma_s : G_s \rightarrow I \) be the orientation-preserving affine homeomorphism. Then the map
\[
\gamma_k \circ f^k \circ \gamma_0^{-1} : I \rightarrow I
\]
belongs to the Epstein class \( SE(C, 1, 2MN) \).

**Proof.** We will use a similar argument in the proof of Lemma\[2.4\] Since \( F \) is infinitely anti-renormalizable, there exists a sequence \( \{H_j\}_{j=1}^\infty \) of infinitely renormalizable extended maps with following properties:
- there exist positive integers \( m_1, m_2, \ldots \) and multi-intervals \( J_1, J_2, \ldots \) such that \( H_j^{m_j}|_{J_j} \) is affinely conjugate to \( F \);
- each fiber of \( J_i \) has length less than \( \lambda^{-j} \) with \( \lambda > 1 \) for all \( j = 1, 2, \ldots \), where \( \lambda \) is given by Theorem \[2.1\].

Let \( \Psi_j \) be the affine conjugacy between \( F \) and \( H_j^{m_j}|_{J_j} \). Put \( \tilde{G}_{j,s} = \Psi_j(G^s_s) \) and \( \tilde{G}_{j,s} = \Psi_j(G^s_s) \) for all \( 0 \leq s \leq k \). Then for any \( j \in \mathbb{N} \), two chains \( \{G^s_{j,s}\}_{s=0}^k \) and \( \{\tilde{G}_{j,s}\}_{s=0}^k \) satisfy the following conditions:
- \( G^s_{j,s} \supset \tilde{G}_{j,s} \) for all \( 0 \leq s \leq k \) and \( G^s_{j,0} \cap \Psi_j(\omega(c_0)) \neq \emptyset \);
- the multiplicity of \( \{G^s_{j,s}\}_{s=0}^k \) is at most \( M \);
- \( \tilde{G}_{j,k} \supset (1 + 2\delta)\tilde{G}_{j,k} \) and \( |f_k(\tilde{G}_{j,0})| \geq \delta|\tilde{G}_{j,k}| \).
- \( \tilde{G}_{j,k} \) is \( \lambda^{-j} \)-nice and the transition map
\[
\tilde{\gamma}_{j,k}(H_j^{m_j}|_{J_j})^k \circ \tilde{\gamma}_{j,0}^{-1} : I \rightarrow I
\]
belongs to \( SE(C, 1 + \eta, 2MN) \), where \( \tilde{\gamma}_{j,0,k} = \gamma_k \circ \Psi_j^{-1} \) and \( \tilde{\gamma}_{j,0,0} = \gamma_0 \circ \Psi_j^{-1} \).

Thus \( \gamma_k \circ f^k \circ \gamma_0^{-1} : I \rightarrow I \) belongs to \( SE(C, 1 + \eta, 2MN) \). As \( \eta \) is arbitrary, \( \gamma_k \circ f^k \circ \gamma_0^{-1} : I \rightarrow I \) belongs to the Epstein class \( SE(C, 1, 2MN) \).

**Lemma 2.6.** There exists a constant \( C' = C'(C) > 0 \) with the following property. Let \( c \in \text{Crit}(F) \) and \( c \in T \) be a \( C \)-nice admissible interval. If \( T' = L_c(T) \) and \( |R_T(T')| \geq C^{-1}|T| \), then \( T' \) is \( C' \)-nice and the transition map \( G_{T,T'} \in SE(C', 1, 8N) \).
Proof. Let \( \{G_x\}_{x=0}^k \) be the chain from \( T' \) to \( T \) and \( \{G'_x\}_{x=0}^k \) be the chain such that \( G'_x = (1 + 2C^{-1})T \) and \( G'_0 \supset T' \).

By \[16\] Lemma 3.8, the chain \( \{G'_x\}_{x=0}^k \) has order at most \( N \) and multiplicity at most 4. As \( |F^k(G_0)| = |R_T(T')| > C^{-1}|T| = C^{-1}|G_k| \), it follows Lemma 2.5 that there exists \( C' > 0 \) such that \( G_{T,T'} \in SE(C', 1, 8N) \).

Now we prove \( T' \) is \( C'' \)-nice for some \( C'' > 0 \). To this end, let \( \{\tilde{G}_x\}_{x=0}^k \) be the chain such that \( \tilde{G}_k = (1 + C^{-1})T \) and \( \tilde{G}_0 \supset T' \). A straightforward way to prove \( T' \) has \( C'' \)-nice property is using Lemma 2.4 which is left to the readers.

We will use another way, by using the compactness of the Epstein class, which also works for subsequent Lemmas. By a similar argument in the previous paragraph, we can conclude \( \gamma_k \circ F^k \circ \gamma_0^{-1} : I \to I \) is in the Epstein class \( SE(C', 1, 8N) \), where \( \gamma_k : \tilde{G}_k \to I \) and \( \gamma_0 : \tilde{G}_0 \to I \) are orientation-preserving affine homeomorphisms.

We shall prove \( T' \) is well inside \( \tilde{G}_0 \), that is, \( \tilde{G}_0 \) contains a definite neighborhood of \( T' \). For otherwise, there exists a sequence \( \{F_j\}_{j=0}^\infty \) of extended maps with following properties:

- for any \( j \in \mathbb{N} \), \( T(F_j) \) is a \( C \)-nice admissible interval of \( F_j \);
- for any \( j \in \mathbb{N} \), there exists \( x_j \in T(F_j) \) and \( z_j \in \partial G_0(F_j) \) such that \( |x_j - z_j||T'(F_j)|^{-1} \to 0 \) as \( j \to \infty \);
- for any \( j \in \mathbb{N} \), \( |F_j^{k_j}(x_j) - F_j^{k_j}(z_j)| > \frac{1}{2}C'1|T(F_j)| \).

Let \( \gamma_{j,0} : G_0(F_j) \to I \) and \( \gamma_{j,k_j} : G_{k_j}(F_j) \to I \) be the orientation-preserving homeomorphisms and \( \tilde{x}_j = \gamma_{j,0}(x_j), \tilde{z}_j = \gamma_{j,0}(z_j) \). Without loss of generality, we can assume \( \tilde{x}_j \to \tilde{x} \in I, \tilde{z}_j \to \tilde{z} \in I \) and \( \Phi_j := \gamma_{j,k_j} \circ F_j^{k_j} \circ \gamma_{j,0}^{-1} \) converges to some \( \tilde{\Phi} \) in \( C^1 \)-topology. Then by the properties of \( \{F_j\} \) we have

- \( |\tilde{x}_j - \tilde{z}_j| \to 0 \) as \( j \to \infty \), and then \( \tilde{x} = \tilde{z} \);
- \( |\Phi_j(\tilde{x}_j) - \Phi_j(\tilde{z}_j)| > \frac{1}{2}C^{-1}C \frac{1}{1 + C} = \frac{1}{2(1 + C)} \).

Then by the uniform convergence, it follows \( 0 = \tilde{\Phi}(\tilde{x}) - \tilde{\Phi}(\tilde{z}) > \frac{1}{2(1 + C)} \), which is ridiculous. Thus \( T' \) is well inside \( \tilde{G}_0 \), which implies there exists \( C'' > 0 \) such that \( (1 + 2/C'')T' \cap I = \emptyset \) since \( \tilde{G}_0 \setminus T' \cap \omega(c_0) = \emptyset \).

A similar argument shows that \( T' \setminus (1 - 2/C'')T' \cap \omega(c_0) = \emptyset \). Hence, \( T' \) is \( C'' \)-nice. Enlarge \( C' \) so that \( C'' > C' \), and we are done. \( \square \)

Corollary 2.1. There exists a constant \( C' = C'(C,d) > 0 \) with the following property. Let \( c \in \text{Crit}(F) \) and \( c \in T \) be a \( C \)-nice admissible interval. If \( T' = L_c^+(T) \) and \( R_T \circ \cdots \circ R_{L^{d-1}_c(T)}(T') \) has length at least \( C^{-1}|T| \), then \( T' \) is \( C' \)-nice and the transition map \( G_{T,T'} \in SE(C', 1, 8dN) \).

Proof. It follows immediately by Lemma 2.6 and induction. \( \square \)

2.4. Compactness for transition maps.

Lemma 2.7. For any \( \theta > 0 \), there exists a constant \( \xi = \xi(\theta, \|F\|_{C^1}) > 0 \) such that the following holds. Let \( B, B' \in M(I_0) \), assume \( B'' = L_x(B) \) for some \( x \in B \) and \( B \supset (1 + 2\xi)B' \). Then \( L_{c_0}(B) \supset (1 + 2\theta) L_{c_0}(B') \).

Proof. See \[16\] Proposition 4.1. \( \square \)
Lemma 2.8. Let $T_1, T_2, \cdots, T_L$ be a maximal central cascade with respect to some $c \in \text{Crit}(F)$. Assume there exists a positive integer $C > 1$ such that

- $T_1$ is $C$-nice with limit scaling factor $\Lambda_{T_1} < C$;
- $T_1 \supset (1 + 2C^{-1})T_2$ and $\min(\ell, L - \ell) \leq C$;
- $|T_1| \leq C|R_T(T_\ell)|$.

Then there exists a positive integer $C' = C'(C, \|F\|_{C^3})$ such that the transition map $G_{T_1, T_\ell}$ belongs to the Epstein class $\mathcal{E}(C', 1, 8C'N)$.

Proof. Let $b = b(C^{-1}, C, \|F\|_{C^3})$ and $L - b < L' \leq L$ be as in Lemma 2.2. Since $T_1$ is $C$-nice and $|T_1| \leq C|R_T(T_\ell)|$, by Corollary 2.4 if $\ell \leq \max(3b, C)$ is not large, then there exists a positive integer $C' > \max(3b, C)$ such that $G_{T_1, T_\ell} \in \mathcal{E}(C', 1, 8C'N) \subset \mathcal{E}(C', 1, 8C'N)$.

Now we suppose that $L \geq \ell > \max(3b, C)$. By Lemma 2.2, $T_b, T_{b+1}, \cdots, T_{L'}$ is of saddle-node type and for any $x \in T_{L'}$,

$$|R_{T_b}(x) - x| \geq \frac{1}{b}|T_1|.$$

Clearly, $x, R_{T_b}(x), \cdots, R_{T_L}(x)$ lie in order. Thus, we have $R_{T_b}(T_b) \subset T_{L'} \setminus T_{L'}$. Let $J$ be the component of $T_{L'} \setminus T_{L'}$ containing $R_{T_b}(T_b)$. Then $R_{T_b}^{-2b}|J$ maps $J$ diffeomorphically onto a component $\tilde{J}$ of $T_b \setminus T_{L' - \ell + 2b}$ since all the critical points of $R_{T_b}$ are contained in $T_{L'}$. By Lemma 2.2, $(R_{T_b}^{-2b}|J)^{-1} : \tilde{J} \to J$ can extend to a conformal mapping from $\mathbb{C}_J$ onto $\mathbb{C}_J$. We shall prove that the diffeomorphism $R_{T_b}^{-2b}|J$ has uniformly bounded distortion, which implies $\gamma_1 \circ R_{T_b}^{-2b}|J \circ \gamma_0^{-1} \in K(C''', 1)$ for some $C'' > 0$ where $\gamma_0 : J \to \mathbb{I}$ and $\gamma_1 : J \to \mathbb{I}$ are orientation preserving affine homeomorphisms.

Since $L' - \ell + 2b \leq L - \ell + 2b \leq C + 2b$, by Lemma 2.2 or Yoccoz’s Lemma, there exists $C_1 = C_1(C, b) > 1$ such that

$$\frac{1}{C_1}|T_1| \leq |T_{L' - \ell + 2b - 1} - T_{L' - \ell + 2b}|.$$

By Corollary 2.4, there exists $C' > 1$ such that $G_{T_1, T_b}$ lies in a compact set $\mathcal{E}(C', 1, 8bN)$ and $T_b$ is $C'$-nice. Thus we can extend $R_{T_b}^{-2b}|J$ to be a diffeomorphism onto a $C_2$-neighborhood of $\tilde{J}$, where $C_2 = \min(1/C', 1/C_1)$. By real Koebe principle, $R_{T_b}^{-2b}|J$ has uniformly bounded distortion.

Finally, since $G_{T_1, T_b}$ and $G_{T_{b-1}, T_\ell}$ both lie in $\mathcal{E}(C', 1, 8bN)$, we conclude

$$G_{T_1, T_\ell} = G_{T_1, T_b} \circ \gamma_1 \circ R_{T_b}^{-N'-N} \circ \gamma_0^{-1} \circ G_{T_{b-1}, T_\ell}$$

belongs to $\mathcal{E}(C', 1, 8C'N)$ by enlarging $C'$.

The following lemmas will play an important role in the proof of the Key Lemma.

Lemma 2.9. Let $c \in \text{Crit}(F)$. Assume $c \in T$ is an admissible interval with

$$|T| < \Lambda|L_\infty(I_\infty)|$$

for some $\Lambda > 0$ and $c_0 \in T'$ is a $k$-pulback of $T$ with $k \leq Np$. Let $\{G_s\}_{s=0}^k$ be the chain from $T'$ to $T$. If $G_s = T_1, T_2, \cdots, T_\ell = G_{s_1}$ is a central cascade in the decomposition of the chain $\{G_s\}_{s=0}^k$ with following properties:

1. if $T_1, T_2, \cdots, T_\ell$ is a maximal central cascade, then either $L = \ell$ or $T_{\ell+1} \neq G_s$ for all $0 \leq s < s_1$;
(2) there exists positive integer \( C > 1 \) such that \( T_1 \) is \( C \)-nice with scaling factor 
\[ \lambda_{T_1} > 1 + 2C^{-1} \]
and 
\[ |T_1| < C|F^s(I_\infty)|, \]

(3) either \( c_0 \notin G_s \) for \( s_1 \leq s \leq s_\ell \) or \( T_1, T_2, \ldots, T_\ell \) is a cascade with respect to \( c_0 \).

then there exists a positive integer \( C' = C'(\Lambda, C, \|F\|_{C^3}) \) such that

- \( T_\ell \) is \( C' \)-nice with \( \lambda_{T_\ell} > 1 + 2C'^{-1} \);
- \( |T_\ell| < C'|F^s(I_\infty)|; \)
- \( G_{T_1, T_\ell} \in SE(C', 1, 8C'N). \)

Proof. First, we prove there exists \( \rho = \rho(\Lambda, \|F\|_{C^3}) \) such that the limit scaling factor \( \Lambda_{T_1} \) of \( T_1 \) is less than \( \rho \). Fix some constant \( \theta > \Lambda \) and let \( \xi = \xi(\theta, \|F\|_{C^3}) \) be given by Lemma 2.7. We claim that \( \Lambda_{T_1} \leq \rho := 2\xi + 1 \). For otherwise, there exists a critical point \( c' \) and an admissible interval \( I \ni c' \) such that \( I \supset c' \) such that \( I \supset (1 + 2\xi)\mathcal{L}(I) \).

By Lemma 2.7,
\[ \Lambda \mathcal{L}(I_\infty) \subset (1 + 2\theta)\mathcal{L}(I_\infty) \subset (1 + 2\theta)\mathcal{L}(\mathcal{L}(I)) \subset \mathcal{L} \subset T, \]
which contradicts with the assumption.

Let us now consider the maximal central cascade \( T_1, T_2, \ldots, T_L \). We shall prove there exists \( q = q(\Lambda, C, \|F\|_{C^3}) \) such that \( Q := \min(\ell, L - \ell) \leq q \).

Set \( J_i := F^i(I_\infty) \) for all \( i \geq 1 \). Let \( b = b(C^{-1}, \rho, \|F\|_{C^3}) \), \( L - b < L' \leq L \) and \( \eta = \eta(C^{-1}, \rho, \|F\|_{C^3}) \) be given by Lemma 2.2. Without loss of generality, we may assume \( L > \ell > 10b \).

Case 1. \( J_{s_L} \subset T_{L'} \). It follows from Lemma 2.2 that \( T_{b_1}, T_{b_2}, \ldots, T_{L'} \) is a central cascade of saddle-node type and for any \( x \in T_{L'} \),
\[ |R_{T_{b_1}}(x) - x| > \frac{1}{b}|T|, \]
Note that \( R_{T_{b_1}}(x) \notin T_{L'} \) for any \( x \in J_{s_L} \). Indeed, for such \( x \in J_{s_1} \subset T_{L'}, x, R_{T_{b_1}}(x), \ldots, R_{T_{b_1}}(x) \) lie in order. So if \( R_{T_{b_1}}(x) \in T_{L'}, \) then \( |R_{T_{b_1}}(x) - x| > N' \times \frac{1}{b}|T| = |T| \), which is a contradiction. Thus we obtain
\[ J_{s_L} = R_{T_{b_1}}^{-1}(J_{s_1}) = R_{T_{b_1}}^{-1-b} \circ R_{T_{b_1}}(J_{s_1}) \subset R_{T_{b_1}}^{-1-b}(T_{L'} - b \setminus T_{L'}) = T_{L' - \ell + 1} \setminus T_{L' - \ell + 1 + b}. \]

Hence,
\[ C^{-1} < \frac{|J_{s_L}|}{|T|} \leq \frac{|T_{L' - \ell + 1} \setminus T_{L' - \ell + 1 + b}|}{|T|}. \]

By Lemma 2.2, we have
\[ \frac{|T_{L' - \ell + 1} \setminus T_{L' - \ell + 1 + b}|}{|T|} \leq \frac{b\eta}{Q^2}, \]

So \( Q \) is bounded in terms of \( C \) and \( \|F\|_{C^3} \).

Case 2. Set \( M = \max\{2b, \sqrt{C\eta} + 1\} \). If \( J_{s_L} \subset T_{\ell + m} \setminus T_{\ell + m + 1} \) for some \( m > M \) with \( \ell + m < L' \), then \( J_{s_L} = R_{T_{b_1}}^{-1}(J_{s_1}) \subset T_{m + 1} \setminus T_{m + 2} \). It follows from Lemma 2.2 that
\[ C^{-1} < \frac{|J_{s_1}|}{|T_1|} \leq \frac{|T_{m + 1} \setminus T_{m + 2}|}{|T_1|} \leq \max\{\frac{\eta}{(m + 1)^2}, \frac{\eta}{(L - m - 1)^2}\}. \]
As $\frac{\eta}{(m+1)^2} < \frac{\eta}{M^2} < C^{-1}$, we have $(L-m-1)^2 < C\eta$, and then

$$\ell \leq L - m < \sqrt{C\eta} + 1.$$  

**Case 3.** Now suppose $J_{s_1} \subset T_{t+m} \setminus T_{t+m+1}$ for some $m \leq M$. Take a maximal integer $s_1 \leq r < s_1 + Np$ such that $J_r \subset T_1$, where $p$ is the renormalization period of $f$. Such an $r$ exists since $J_{s_1} \subset T_1$. Clearly, $R_{T_t} (J_r) = J_{s_1+Np} = J_{s_2}$.

**Claim.** $J_r \subset T_1 \setminus T_2$.

Let $u = s_t - s_{t-1}$, then $R_{T_t} |T_2 = F^u |T_2$. If the claim fails, i.e., $J_r \subset T_2 = G_{s_{t-1}}$, then $T_{t+1}$ is the component of $F^{-u}(T_t)$ containing $J_r$ and $u + r = Np + s_1$. If $u \leq s_1$, then $G_{s_{t-1}} - u$ is the component of $F^{-u}(T_t)$ containing $J_r$. This implies $T_t = G_{s_{t-1}} - u$, which contradicts with condition (1). Thus, $u > s_1$. However, it is also impossible. Indeed, if $u > s_1$, then $r = Np + s_1 - u < Np$. Then $G_{s_{t-1} + Np - r} \supset J_{Np} \supset \mathfrak{c} \mathfrak{a}$ since $J_r \subset G_{s_{t-1}}$. A direct computation shows that

$$s_{t-1} + Np - r = s_{t-1} + s_1 - u = s_{t-1} - 1 + (s_t - s_{t-1}) = s_1 - s_t \in \{0, 1\} \cap \mathbb{N}.$$  

By condition (3), $T_1, T_2, \ldots, T_t$ should be a central cascade with respect to $\mathfrak{c} \mathfrak{a}$. Then by the definition of $r$, $r \geq Np$ since $J_{Np} \subset T_1$. This is a contradiction, as $u + r = Np + s_1$ and $u > s_1$. Hence, the claim follows.

Let $D \supset J_r$ be the component of the first return domain to $T_1$ and let $F^u |D : D \to T_1$ be the corresponding return map. By the definition of $r$, we have $r = Np + s_1 - r$.

Set $D' := (F^u |D)^{-1}(T_{t+m} \setminus T_{t+m+1})$, by the Markov property, $F^u (\mathcal{L}_{J_r}(D))$ must be contained in $T_{t+m} \setminus T_{t+m+1}$. It follows $D' \supset \mathcal{L}_{J_r}(D)$. We claim $(1 + 2\xi) \mathcal{L}_{J_r}(D) \not\subset D$. For otherwise, by Lemma 2.7, $\mathcal{L}_{\mathfrak{c} \mathfrak{a}}(D) \supset (1 + 2\xi) \mathcal{L}_{\mathfrak{c} \mathfrak{a}}(\mathcal{L}_{J_r}(D))$. So

$$\frac{1}{\Lambda \leq |I_{\infty}|}{|T|} \leq \frac{|\mathcal{L}_{\mathfrak{c} \mathfrak{a}}(\mathcal{L}_{J_r}(D))|}{|\mathcal{L}_{\mathfrak{c} \mathfrak{a}}(D)|} < \frac{1}{1 + 2\xi} < \frac{1}{2\Lambda}.$$  

This is absurd. By Lemma 2.2, we have

$$\frac{|T_{t+m} \setminus T_{t+m+1}|}{|T_t|} \geq \frac{1}{\min \{\ell, L - \ell\}^2} = \frac{1}{Q^2}.$$  

It follows from Lemma 2.1 that there exists $\delta_1 = \delta(Q)$ such that $(1 + 2\delta_1) \mathcal{L}_{J_r}(D) \subset D$ and $\delta_1 (Q) \to \infty$ as $Q \to \infty$. As $\delta_1 \leq \xi$, there exists $q = q(\xi, C, |F|_{C^3}) = q(\Lambda, C, |F|_{C^3})$ such that $\min \{\ell, L - \ell\} = Q \leq q$. Thus we are done.

Hence, $T_1$ satisfies the following properties:

1. $T_1$ is $C$-nice with limit scaling factor $\Lambda_1 \leq \rho$;
2. $T_1 \supset (1 + 2C^{-1}) T_2$;
3. $C|R_{T_1}(T_2)| \geq C|F^{s_1}(I_{\infty})| > |T_1|$;
4. $Q = \min \{\ell, L - \ell\} \leq q$.

By Lemma 2.8, there exists a positive integer $C' > 1$ such that $G_{T_1, T_2} \in \mathcal{S}(C', 1, 8C'N)$. As $|T_1| < C|F^{s_1}(I_{\infty})|$ and $T_1$ is $C$-nice, it follows easily there exists $C'' = C''(\Lambda, C, |F|_{C^3})$ such that $|T_1| < C''|F^{s_1}(I_{\infty})|$ and $T_1$ is $C''$-nice from the fact that $G_{T_1, T_2}$ lies in a compact set. The argument is similar to that we used in the proof of Lemma 2.6.

We only prove $|T_1| < C''|F^{s_1}(I_{\infty})|$. Indeed, if there does not exist such $C''$, then there exists a sequence $(F_i)$ of extended maps with following properties:

- $\{G_i := G_{T_{F_i}, T_{F_i}(F_i)}\}$ converges uniformly to some map $G$;
- there exist $x_1(i), x_2(i) \in F^{s_1}(I_{\infty}(F_i))$ such that $\hat{x}_1(i) - \hat{x}_2(i) \to 0$ as $i \to \infty$, where $\hat{x}_1(i) = A_{T_{F_i}(F_i)}(x_1(i))$ and $\hat{x}_2(i) = A_{T_{F_i}(F_i)}(x_2(i))$;
Without loss of generality, we can assume $\hat{x}_1(i) \to \hat{x} \in [-1,1]$. Since $|G_i(\hat{x}_1(i)) - G_i(\hat{x}_2(i))| > C^{-1}$, take a limit, we obtain $0 = |G(\hat{x}) - G(\hat{x})| > C^{-1}$. This is absurd.

As $Q = \min(t,L - \ell)$ is uniformly bounded, by (3) of Lemma 2.2 $\lambda_T$ is uniformly bounded both below and above. Hence, enlarge $C'$ to be large enough, then the conclusions of this lemma follow.

\[
\begin{align*}
\text{Lemma 2.10 (Long transition maps). Assume } c & \in T \text{ is an admissible interval with } \\
\quad |T| & < \Lambda|L_c(I_\infty)| \\
\text{for some } \Lambda > 0 \text{ and } T' \ni c_0 \text{ is a } k\text{-pullback of } T \text{ with } k \leq Np. \text{ If there exists a positive integer } C > 1 \text{ such that } \\
\quad (1) & \text{ } T \text{ is } C\text{-nice with } \lambda_T > 1 + 2C^{-1}; \\
\quad (2) & \text{ } |T| < C|F^k(I_\infty)|. \\
\text{Then there exists a positive integer } C' & = C'(\Lambda, C, \|F\|_{C^3}) > 1 \text{ such that } G_{T,T'} \in SE(C', 1, 8C'N).
\end{align*}
\]

Proof. First, a similar argument in the proof of Lemma 2.9 shows that there exists $\rho = \rho(\Lambda, \|F\|_{C^3})$ such that the limit scaling factor $\Lambda_T$ of $T$ is less than $\rho$.

Let $\{G_s\}_{s=0}^k$ be the chain from $G_0 = T'$ to $G_k = T$.

**Statement (M).** Let $Y = G_y, Y' = G_{y'}$ ($0 \leq y' < y \leq k$) be two symmetric admissible intervals in the chain such that

\[
\begin{align*}
\quad (1) & \text{ } Y \text{ is } C\text{-nice with } \lambda_Y > 1 + 2C^{-1}; \\
\quad (2) & \text{ } |Y| < C|F_y(I_\infty)|.
\end{align*}
\]

If $\#(\text{Crit}(F)) \cap \bigcup_{s=y+y+1}^{y-1} G_s \leq M$, then there exists a positive integer $C' = C'(\Lambda, C, \|F\|_{C^3}) > 1$ such that

\[
\begin{align*}
\quad & \text{ } Y' \text{ is } C'\text{-nice with } \lambda_{Y'} > 1 + 2C'^{-1}; \\
\quad & \text{ } |Y'| < C'|F_{y'}(I_\infty)|; \\
\quad & G_{Y,Y'} \in SE(C', 1, 8C'N).
\end{align*}
\]

Proof of Statement (0). By Lemma 2.3 the diffeomorphism

\[
F^{-(y-y'-1)} : F^{-y'}(Y') \to F(Y')
\]

can extend to a conformal map from $C_{F^{-y'}(Y')} \to C_{F(Y')}$. Since $Y$ is $C$-nice, $F^{-y'-1}|_{F(Y')}$ can be extended to a diffeomorphism onto $(1 + 2C^{-1})Y$. By Koebe’s distortion theorem, $F^{-y'-1}|_{F(Y')}$ has uniformly bounded distortion, which implies that $G_{Y,Y'} \in SE(C', 1, 2)$ for some $C' > 0$. Then it follows easily $|Y'| < C'|F_{y'}(I_\infty)|$ by the compactness of $SE(C', 1, 2)$.

Now we prove there exists $C'' > 0$ such that $\lambda_{Y'} > 1 + 2C''^{-1}$. Assume the critical point contained in $Y'$ is $c'$. Let $B' = L_{c'}(Y')$ and $r_{B'}$ be the return time from $B'$ to $Y'$, then $r_{B'} > y - y'$. For otherwise, $G_{y'+r_{B'} \cup Y'}$ contains a critical point, a contradiction. Thus, $F^{-y'}(B')$ lies in a first return component $B$ to $Y$.

Since $\lambda_Y > 1 + 2C^{-1}$ and $\lambda_Y \leq \Lambda_T < \rho$, we have

\[
\rho^{-1}|Y| < |L_c(Y)| < \frac{C}{C+2}|Y|,
\]
where $\hat{c}$ is the critical point in $Y$. Thus, there exists $\tilde{C} > 0$ such that $\tilde{C}^{-1}|Y| < |B| < (1 - \tilde{C}^{-1})|Y|$. As $F^u \cdot \gamma \cdot F^{v'} \cdot \gamma'$ has uniformly bounded distortion, there exists $C'' > 0$ such that $(1 + 2C''^{-1})|B'| < |Y'|$. Enlarge $C'$ so that $C' > C''$, the conclusion of Statement (0) follows.

Statement $(M - 1) \Rightarrow$ Statement $(M)$. 

**Case 1.** $c_0 \in \bigcup_{y=1}^{y-1} G_s$. Let $y' < v < y$ and $y' < v' < y$ be the largest and smallest integer such that $c_0 \in G_{\nu} \subset G_v$ respectively. Consider the canonical decomposition of $G_{G_v,G_{\nu}}$ with respect to $c_0$, that is,

$$G_v = T_1^1 \supseteq \cdots \supseteq T_1^1 \supseteq T_1^2 \supseteq \cdots \supseteq T_1^\ell = G_{\nu},$$

where $T_1^1, \ldots, T_1^{\ell}$ is a central cascade satisfying condition (1) in Lemma 2.9. $j = 1, \ldots, \ell$. As $|T| < LL_c(I_0)$, by Theorem 2.1 and Lemma 2.7, we know $\ell$ is bounded in terms of $\Lambda$ and $\|F\|_{C^1}$. Let $s'_i$ be the integer such that $T_i^j \supseteq G_{s'}$ for all $1 \leq j \leq \ell$, $1 \leq i \leq L_j$. By Statement (0), there exists a positive integer $\tilde{C}_1 > 1$ such that

P1(1). $T_1^1$ is $\tilde{C}_1$-nice with $\lambda_{T_1^1} > 1 + 2\tilde{C}_1$;

P2(1). $|T_1^1| < \tilde{C}_1 |F^{s'_1}(I_\infty)|$;

P3(1). $G_{Y,G_{\nu}} \in SE(\tilde{C}_1, 1, 8\tilde{C}_1 \mathbb{N})$.

It follows from Lemma 2.9 that

H1(1). $T_{L_1}^1$ is $\tilde{C}_1$-nice with $\lambda_{T_{L_1}^1} > 1 + 2\tilde{C}_1$;

H2(1). $|T_{L_1}^1| < \tilde{C}_1 |F^{s'_1}(I_\infty)|$;

H3(1). $G_{T_{L_1}^1,T_{L_1}^1} \in SE(\tilde{C}_1, 1, 8\tilde{C}_1 \mathbb{N})$.

By Statement $(M - 1)$, there exists a positive integer $\tilde{C}_2 > 1$ such that

P1(2). $T_1^2$ is $\tilde{C}_2$-nice with $\lambda_{T_1^2} > 1 + 2\tilde{C}_2$;

P2(2). $|T_1^2| < \tilde{C}_2 |F^{s'_2}(I_\infty)|$;

P3(2). $G_{T_{L_1}^1,T_{L_1}^1} \in SE(\tilde{C}_2, 1, 8\tilde{C}_2 \mathbb{N})$.

By induction, we can obtain two finite sequences of positive integers $\{\tilde{C}_j\}_{j=2}^\ell$ and $\{\tilde{C}_j\}_{j=2}^\ell$ such that for all $2 \leq j \leq \ell$, the following holds.

P1(j). $T_1^j$ is $\tilde{C}_j$-nice with $\lambda_{T_1^j} > 1 + 2\tilde{C}_j$;

P2(j). $|T_1^j| < \tilde{C}_j |F^{s'_j}(I_\infty)|$;

P3(j). $G_{T_{L_1}^j,T_{L_1}^j} \in SE(\tilde{C}_j, 1, 8\tilde{C}_j \mathbb{N})$;

H1(j). $T_{L_1}^j$ is $\tilde{C}_j$-nice with $\lambda_{T_{L_1}^j} > 1 + 2\tilde{C}_j$;

H2(j). $|T_{L_1}^j| < \tilde{C}_j |F^{s'_j}(I_\infty)|$;

H3(j). $G_{T_{L_1}^j,T_{L_1}^j} \in SE(\tilde{C}_j, 1, 8\tilde{C}_j \mathbb{N})$.

Consider the transition map $G_{G_{\nu},Y'}$, by Statement $(M - 1)$, there exists a positive integer $\tilde{C}_{\ell+1}$ such that

- $Y'$ is $\tilde{C}_{\ell+1}$-nicewith $\lambda_{Y'} > 1 + 2\tilde{C}_{\ell+1}$;

- $|Y'| < \tilde{C}_{\ell+1} |F^{y'}(I_\infty)|$;

- $G_{G_{\nu'},Y'} \in SE(\tilde{C}_{\ell+1}, 1, 8\tilde{C}_{\ell+1} \mathbb{N})$. 


Let $C' = \tilde{C}_{\ell+1} + \Sigma_{j=1}^{\ell+1} (\tilde{C}_j + \tilde{C}_j)$, then the conclusions of Statement (M) follow.

**Case 2.** $c_0 \notin \bigcup_{s=y'+1}^{y-1} G_s$. Let $y' < v < y$ be the largest integer such that $\text{Crit}(F) \cap G_v \neq \emptyset$. Assume the critical point lying in $G_v$ is $c$ and let $y' \leq \nu' \leq v$ be the smallest integer such that $G_{\nu'} \ni c$. Consider the canonical decomposition of $G_{G_{\nu}G_{\nu'}}$, with respect to $c$, that is,

$$G_v = T_1^1 \ni \cdots \ni T_1^1 \ni T_2^2 \ni \cdots \ni T_1^2 \ni \cdots \ni T_1^\ell \ni G_{\nu'}.$$

where $T_1^1, \cdots, T_1^\ell$ is a central cascade satisfying condition (1) in Lemma 2.9, $\nu = 1, \cdots, \ell$. Then the proof is essentially the same as that of Case 1.

**Lemma 2.11.** Let $T$ be a component of $E_0$ and $T' \ni c_0$ be an $(N_p - b)$-pullback of $T$ for some $0 \leq b \leq N_p$. If there exists $\Lambda > 0$ such that $|I_0| < \Lambda |I_\infty|$, then there exists a positive integer $C' = C'(\Lambda, b, \|F\|_{C_3})$ such that $G_{T', T} \in \mathcal{SE}(C', 1, 8C'N)$.

**Proof.** First, we show there exists a constant $\Lambda' > 0$ such that $\Lambda_0 \leq \Lambda'$. Some constant $\hat{\Lambda} \geq \Lambda_0$ and let $M = \xi(\hat{\Lambda}, \|F\|_{C_3})$ be given by Lemma 2.7. Then $\Lambda_0 = \Lambda'$ and there exists a critical point $c$ and an admissible interval $I \ni c$ such that $I \ni (1 + 2\xi)\mathcal{L}_c(I)$. Let $\Lambda_\infty \subset (1 + \theta)\mathcal{L}_c(I_\infty) \subset (1 + 2\theta)\mathcal{L}_c(I) \subset \mathcal{L}_c(I)$.

By Markov property, $\mathcal{L}_c(I) \subset I_0$, thus $|I_0| \geq \Lambda |I_\infty|$, which contradicts with the assumption.

Recall that $F$ is bi-infinitely renormalizable with renormalization period large than $2N$. As $\Lambda_0 \leq \Lambda'$, by [16] Theorem 5.6, $|T| > C^{-1}$ and $T$ has $C$-bounded geometry for some $C > 0$. Let $\{G_s\}_{s=0}^{N_p-b}$ be the chain from $T'$ to $T$ and $s_0$ be the largest integer such that $G_{s_0} \cap \text{Crit}(F) \neq \emptyset$. Put $T_0 = G_{s_0}$. We shall prove there exists a positive integer $C_0 > 1$ such that

1. $G_{T,T_0} \in \mathcal{SE}(C_0, 1, 8C_0N)$;
2. $T_0$ is $C_0$-nice with $\lambda(T) > 1 + 2C_0^{-1}$;
3. $|T_0| < C_0|\mathcal{L}_c(I_\infty)|$, where $c$ is the critical point in $T_0$;
4. $|T_0| < C_0|\mathcal{L}_c(I_\infty)|$.

Then the conclusion will follow easily. Indeed, by Lemma 2.10 (2), (3) and (4) implies there exists a positive integer $C'' > 1$ such that $G_{T_0, T} \in \mathcal{SE}(C'', 1, 8C''N)$. Let $C' = C_0 + C''$, then $G_{T, T'} \in \mathcal{SE}(C', 1, 8C'N)$.

Now we prove (1). To this end, we first show there exists $\tilde{C} > 0$ such that $|F^{N_p-b}(I_\infty)| > \tilde{C}^{-1}$. We claim $|I_0| > \tilde{C}^{-1}$ for some $\tilde{C} = \tilde{C}(\|F\|_{C_3}) > 0$. For otherwise, there exists a sequence $\{F_j\}$ of extended maps such that

- $F_j^N$ converges to a map $H$ in $C^3$-topology;
- $(F_j)^N(\alpha(F_j), 0) < -1$ and $(\alpha(F_j), 0) \to (0, 0)$ as $j \to \infty$.

This implies $-1 \leq H'(0, 0) = 0$, which is absurd. Let $x_0 \in 0I_\infty$ be a periodic point of $F$, let $\tilde{x}_0, \tilde{z}_0 \in F^{N_p-b}(I_\infty)$ such that $F^b(\tilde{x}_0) = x_0$ and $F^b(\tilde{z}_0) = c_0$. By the Mean Value Theorem, there exists $\tilde{t} \in T$ such that $|\tilde{x}_0 - \tilde{z}_0| = |x_0 - c_0| > \tilde{C}^{-1}$. As $(|F^b(\tilde{t})|)$ is uniformly bounded above in terms of $\|F\|_{C_3}$, $|\tilde{x}_0 - \tilde{z}_0|$ has uniform lower bound. So there exists $\tilde{C} > 0$ such that $|F^{N_p-b}(I_\infty)| > \tilde{C}^{-1}$.

Put $u = N_p - b - s_0$. By Lemma 2.7, the diffeomorphism $F^{-u-1} : T \to F(T_0)$ can extend to a conformal map form $C_T$ into $C_{F(T_0)}$. Since $T$ is $C$-nice, by real Koebe principle, $F^{u-1}$ has uniformly bounded distortion. This implies $\gamma_1 \circ F^{u-1} \circ
\[ \gamma_0 : \mathbb{I} \to \mathbb{I} \text{ belongs to } \mathcal{K}(C_1, 1) \text{ for some } C_1 > 1, \text{ where } \gamma_1 : T \to \mathbb{I} \text{ and } \gamma_0 : F(T_0) \to \mathbb{I} \text{ are orientation-preserving homeomorphism.} \]

Thus, \( G_{T,T_0} \in \mathcal{SE}(C_1,1,2) \). Then (2) and (4) follows easily by a compactness argument which we used frequently in the proof of Lemma 2.6 and 2.9.

To prove (3), we consider a nest \( I_0 = T_0, I_1 = \mathcal{L}_c(I_0'), \cdots, I_{j+1} = \mathcal{L}_c(I_j'), \cdots \). Let \( m'(1) < m'(2) < m'(<k) \) be all the non-central return moments, i.e., \( R_{m'(k)} \) displays a non-central return. Note that \( k' \) is bounded in terms of \( \Lambda \) and \( \|F\|_{C^3} \). Indeed, for any \( 1 \leq j \leq k' \), let \( V_j = \mathcal{L}_c(I_{m'(j)+1}) \) and \( U_j = \mathcal{L}_c(I_{m'(j)+2}) \), then \( V_{j+1} \subset U_j \). By Theorem 2.4 and Lemma 2.7, there exists \( \lambda' > 1 \) such that \( \lambda' U_j \subset V_j \). Hence, if \( k' \) is not bounded, then \( |I_\infty|/|I_0| \) should be small, a contradiction. Since \( \Lambda_T \leq \Lambda_{e_0} < \Lambda' \), by Lemma 2.5, \( |I_{m'(k')+1}| \asymp |I_0'| \). We claim \( |\mathcal{L}_c(I_\infty)| \asymp |I_{m'(k')+1}| \).

For otherwise, by Lemma 2.7, \( I_\infty = \mathcal{L}_c(I_\infty) \) will be geometrically deep inside \( \mathcal{L}_c(I_{m'(k')+1}) \subset I_0 \). This is a contradiction. So \( |\mathcal{L}_c(I_\infty)| \asymp |I_{m'(k')+1}| \asymp |I_0'| = |T_0| \). \( \square \)

Let us now recall the definition of Kozlovski-Shen-vanStrien’s enhanced nest (see [7] Section 8).

**Lemma 2.12.** Let \( T \ni c \) be an admissible interval. Then there is a positive integer \( \nu \) with \( f^\nu(c) \in T \) such that the following holds. Let \( T' \) be the component of \( f^{-\nu}(T) \) containing \( c \) and let \( \{G_j\}_{j=0}^\nu \) be the chain from \( T' \) to \( T \). Then

1. \( \#\{0 \leq j \leq \nu - 1 : \text{Crit}(F) \cap G_j \neq \emptyset\} \leq N^2; \)
2. \( T' \cap \omega(c) \) is contained in the component of \( f^{-\nu}(\mathcal{L}_c(T)) \).

**Proof.** See [7] Lemma 8.2]. \( \square \)

For an open set \( B \) and a point \( x \in B \), we use \( \text{Comp}_x(B) \) to denote the component of \( B \) containing \( x \). For each admissible interval \( T \ni c \), let \( \nu = \nu(T) \) be the smallest integer with the properties specified by Lemma 2.11. Following Kozlovski-Shen-vanStrien, we define

\[ \mathcal{G}(T) = \text{Comp}_c(f^{-\nu}(\mathcal{L}_{f^{\nu}(c)}(T))), \]

\[ \mathcal{H}(T) = \text{Comp}_c(f^{-\nu}(T)). \]

Let \( T \) be an admissible interval and \( T' \) be a \( k \)-pullback of \( T \) for some \( k > 0 \). Consider the chain \( \{G_s\}_{s=0}^{k} \) from \( T' \) to \( T \), we call \( T' \) a kid of \( T \) if \( G_s \cap \text{Crit}(F) = \emptyset \) for \( 1 \leq s \leq k \).

For a nice symmetric interval \( B \), let \( \hat{L}_x(B) \) denote the component of \( D_B \cup B \) containing \( x \).

**Definition 2.5.** Given an admissible interval \( T \ni c \), by a successor of \( T \), we mean an admissible interval of the form \( \mathcal{L}_c(T) \), where \( T \) is a kid of \( \hat{L}_x(T) \) for some \( c' \in \text{Crit}(F) \).

Since \( F \) is (infinitely) renormalizable, each admissible interval \( T \) has a smallest successor and we denote it by \( \Gamma(T) \).

Then we can define Kozlovski-Shen-vanStrien’s enhanced nest (briefly, KSS nest) as following: let \( E_0 = I_0 \) and for each \( k \geq 0 \),

\[ L_k = \mathcal{G}(E_k), \]
\[ M_{k,0} = \mathcal{H}(E_k), \]
\[ M_{k,j+1} = \Gamma(M_{k,j}) \text{ for } 0 \leq j \leq 5N - 1, \]
For each \( j \geq 0 \), let \( r_j \) be the first return time from \( \mathcal{L}_{c_0}(E_j) \) to \( E_j \). Since \( F \) is renormalizable, there is a smallest nonnegative integer \( \chi \) such that \( r_\chi = Np \). Let \( m_j \) be the integer such that \( F^{m_j}(E_{j+1}) \subset E_j \) and \( F^{m_j}(\partial E_{j+1}) \subset \partial E_j \) for \( 0 \leq j \leq \chi - 1 \).

**Proposition 2.1.** There exists \( C > 0 \) such that \( E_j \) is \( C \)-nice for \( 0 \leq j \leq \chi \), \( m_{j+1} \geq 2m_j \) and \( 3r_{j+1} \geq m_j \) for \( 0 \leq j \leq \chi - 2 \).

**Proof.** See Proposition 8.1 and Lemma 8.3 in [7]. \( \square \)

**Fact 1.** \( E_\chi \subset I_{m(\kappa)-1} \).

**Proof.** It follows easily from the fact that the return time \( r_\chi = Np \) to \( E_\chi \) is strictly larger than the return time to \( I_{m(\kappa)-1} \). \( \square \)

**Corollary 2.2.** \( Np \geq \max(0, \sum_{j=0}^{\chi-5} m_j) \).

**Proof.** By the definition, \( Np = r_\chi > r_{\chi-1} \). It follows easily Proposition 2.1 that

\[
r_{\chi-1} \geq 1/3 m_{\chi-2} \geq 4/3 m_{\chi-4} > 2m_{\chi-5} \geq m_{\chi-5} + 2m_{\chi-6} \geq \cdots \geq \sum_{j=0}^{\chi-5} m_j.
\]

\( \square \)

Now we are ready to prove the Key Lemma.

**Proof of the Key Lemma.** Without loss of generality, we may assume that the renormalization sequences \( \mathcal{R} f_k \) converges to \( g_\infty \). For all \( k \in \mathbb{N} \), let \( F_k \) be the extended map of \( f_k \) and \( \Phi_k \) be the orientation-preserving linear map such that \( \Phi_k(I_\infty(F_k)) = [-1,1] \). To simplify the notation, we denote \( \kappa_k = \kappa(F_k) \) the height of \( F_k \), \( I_\infty^k = I_\infty(F_k) \) for all \( k \in \mathbb{N} \).

**Case 1.** If \( \sup_k |I_{m(\kappa_k)}(F_k)|/|I_\infty^k| \to \infty \), \( F_k^{Np_k}|I_{m(\kappa_k)+1}: I_{m(\kappa_k)+1} \to I_{m(\kappa_k)} \) can be extended to a polynomial-like map \( g_k : U_k \to V_k \) such that \( \text{mod}(V_k \setminus J(g_k)) \to \infty \) as \( k \to \infty \). Thus \( g_\infty \) is a polynomial of degree \( 2^\mathbb{N} \).

**Case 2.** Assume \( \sup_k \kappa(F_k) = \infty \) and \( \sup_k |I_{m(\kappa_k)}(F_k)|/|I_\infty^k| < \Lambda \) for some \( \Lambda > 0 \). In this case, we may assume \( \kappa_k \to \infty \). Since \( \Phi_k(I_\infty^k) = [-1,1] \) for all \( k \), we obtain

\[
2 \leq |\Phi_k(I_{m(\kappa_k)}(F_k))| \leq 2\Lambda.
\]

Then for each \( j \in \mathbb{N} \), \( \Phi_k(I_{m(\kappa_k-j)}(F_k)) \) are bounded intervals (the bound depends on \( j \)). In particular, \( \Phi_k(E_{\chi}(F_k)) \) are bounded intervals since \( E_{\chi}(F_k) \subset I_{m(\kappa_k)-1} \), and so are \( \Phi_k(E_{\chi}(F_k)^-5) \).

For every \( j, k \in \mathbb{N} \), let \( T_{k,j} \) be the component of \( F_k^{-Np_k}(I_{m(\kappa_k-j)}(F_k)) \) containing \( c_0 \), by Corollary 2.2 \( T_{k,j} \subset E_{\chi}(F_k)^-5 \). Thus \( |\Phi_k(T_{k,j})| < \infty \). Passing to a subsequence (use diagonal argument) we may assume \( \Phi_k(I_{m(\kappa_k-j)}(F_k)) \) and \( \Phi_k(T_{j,k,j}) \) converge respectively to closed intervals \( D_j \) and \( D_j' \) for all \( j \in \mathbb{N} \). Moreover, we have \( \sup_j |D_j'| < \infty \) since \( \sup_j |\Phi_k(T_{j,k,j})| < \infty \). By Theorem 2.1 \( |D_j| \to \infty \), and then \( \bigcup_j D_j = \mathbb{R} \). It follows from Lemma 2.10 that \( g_\infty \) has an analytic proper extension
$g_\infty : D_j' \to D_j$ for every $j \in \mathbb{N}$. Hence $g_\infty$ has a maximal analytical extension to $\bigcup_j D_j'$ which is a bounded interval.

**Case 3.** Now we assume $\sup_k \kappa(F_k) < \infty$, $\sup_k |I_m(\kappa_k)| < \Lambda |I^k_\infty|$ for some $\Lambda > 0$ and $p_k \to \infty$ as $k \to \infty$. In this case, we have $\sup_k |I_0(F_k)|/|I^k_\infty| < \tilde{\Lambda}$ for some $\tilde{\Lambda} > 0$.

For any $i \in \mathbb{N}$, let $T_{k,i}$ be the component of $F^{-i}_k(\mathfrak{C}_0(F_k))$ containing $c_0$. Since for any $0 \leq j \leq i$, $F_k^{\mathfrak{N}p_k}(\beta T_{k,Np_k-j}) \subset \{\pm 1, \pm \alpha(F_k), \pm \beta(F_k)\} \times \{0\} =: \mathfrak{A}_k$, there exists $a_k \in \mathfrak{A}_k$ such that $(F_k^{-\mathfrak{N}p_k}(a_k) \cap T_{k,Np_k-j}) \geq \frac{1}{N}$. Thus there are at least $\max(\frac{k^2}{N}, 0)$ critical points of $F_k^{\mathfrak{N}p_k}$ in $T_{k,Np_k-j}$.

Similar to Case 2, we can assume $F_k(T_{k,Np_k-j})$ converges to $D_i$ for every $i \in \mathbb{N}$. Also we can assume $F_k(I_0(F_k))$ converges to a bounded interval $D^\infty$, then $\bigcup_i D_i \subset D^\infty$. It follows from Lemma 2.11 that $g_\infty$ has an analytic extension to $D_i$ for all $i \in \mathbb{N}$. If $g_\infty$ has an analytic extension to some $\Omega \ni \bigcup_i D_i$, then $g_\infty$ must be a constant function since $g_\infty'$ has infinitely many zeros in $\bigcup_i D_i$. Hence, the real trace of $g_\infty$ is contained in $D^\infty$.

\[ \square \]

**3. The Continuity of the Anti-renormalization Operator $R^{-1}$**

In this section, we will recall the definition of renormalization combinatorics introduced by Smania [19] section 2] and prove Theorem C.

Let $J$ and $J'$ be two disjoint intervals which are contained in $\bigcup_{j=0}^{N-1} \mathbb{I} \times \{j\}$, we say $J \prec J'$ if there exists $j_0$ such that $J, J' \subset \mathbb{I} \times \{j_0\}$ and $J$ lies to the left of $J'$.

**Definition 3.1.** Denote by $(A, A^{\text{Crit}}, \pi, P, m)$ the **combinatorial data** which contains

1. $A = \bigcup_{j=0}^{N-1} B_j$ where $B_j$ is a collection of disjoint intervals contained in $\mathbb{I} \times \{j\}$ with $\#B_j = m$ for $0 \leq j \leq N - 1$;
2. $A^{\text{Crit}} = \{J \in A \mid J \ni (0, j) \text{ for some } 0 \leq j \leq N - 1\}$ and $\#(A^{\text{Crit}} \cap B_j) = 1$ for all $0 \leq j \leq N - 1$;
3. $\pi : A \to A$ is a bijection with the following property: if $c \in A^{\text{Crit}}$, then $a \prec b \prec c$ implies $\pi(a) \prec \pi(b) \prec \pi(c)$ and $c \prec b \prec a$ implies $\pi(a) \prec \pi(b) \prec \pi(c)$;
4. For any $a \in A$ there exists $c \in A^{\text{Crit}}$ so that $\pi^j(c) = a$, for some $j \geq 0$;
5. $(0, 0) \in P \in A^{\text{Crit}}$.

**Definition 3.2.** Two combinatorial data $\sigma = (A, A^{\text{Crit}}, \pi, P, m)$ and $\tilde{\sigma} = (\tilde{A}, \tilde{A}^{\text{Crit}}, \tilde{\pi}, \tilde{P}, \tilde{m})$ are equivalent if there exists a bijection $\phi : A \to \tilde{A}$ such that

1. $\phi(A^{\text{Crit}}) = \tilde{A}^{\text{Crit}}$;
2. for any $x, y \in A$, $x \prec y$ if and only if $\phi(x) \prec \phi(y)$;
3. $\phi \circ \pi = \tilde{\pi} \circ \phi$;
4. $\phi(P) = \tilde{P}$.

We use $\mathcal{M} = \mathcal{M}(\sigma)$ to denote the equivalence classes of $\sigma$ and let $\Sigma'$ be the set of all the combinatorics.

Now we are going to define the product of two combinatorics.
Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two combinatorics, their product $\mathcal{M}_1 \ast \mathcal{M}_2$ is defined as following:

Assuming $\sigma_s = (A_s, A^{\text{Crit}}_s, \pi_s, P_s, m_s)$ is a representative of $\mathcal{M}_s (s = 1, 2)$, we define intervals $J^s_i := \pi^s_s(P_s)$ for $s = 1, 2$ and $0 \leq i < m_s \mathcal{N}$. Choose a family $\{\psi_i\}_{i=0}^{m_s \mathcal{N} - 1}$ of orientation preserving homeomorphisms such that

1. $\psi_i : J^1_i \to \mathcal{N} \times \{i \mod \mathcal{N}\}$ for $0 \leq i < m_1 \mathcal{N}$;
2. if $J^1_i \in A^{\text{Crit}}_1$, then $\psi_i(0, i \mod \mathcal{N}) = (0, i \mod \mathcal{N})$.

If $i = j \mod \mathcal{N}$, then we define $I_{i,j} := \psi_i^{-1}(J^2_j)$. Then we can define a new combinatorial data $\sigma = (A, A^{\text{Crit}}, P, \pi, m_1 m_2)$ such that

- $A = \{I_{i,j} \mid i = j \mod \mathcal{N}, 0 \leq i < m_1 \mathcal{N}, 0 \leq j < m_2 \mathcal{N}\}$;
- $A^{\text{Crit}} = \{J \in A \mid J \ni (0,j) \text{ for some } 0 \leq j < \mathcal{N} - 1\}$;
- $\pi : A \to A$ such that
  \[
  \pi(I_{i,j}) = \begin{cases} 
  I_{i+1,j+1} \mod (m_1 \mathcal{N}, m_2 \mathcal{N}), & \text{if } I_{i,j} \in A^{\text{Crit}} \\
  I_{i+1,j} \mod (m_1 \mathcal{N}, m_2 \mathcal{N}), & \text{otherwise}
  \end{cases}
  \]
- $P = I_{0,0} \ni (0,0)$.

Note that $\sigma$ is a combinatorial data and its equivalence class $\mathcal{M} := [\sigma]$ does not depend on the choices in the above construction. Finally, $\mathcal{M}$ is defined to be the product $\mathcal{M}_1 \ast \mathcal{M}_2$.

A combinatorial $\mathcal{M}$ is said to be primitive if it does not have decomposition $\mathcal{M} = \mathcal{M}_1 \ast \mathcal{M}_2$. Let $\Sigma \subset \Sigma'$ be the set of all the primitive combinatorics.

**Definition 3.3.** Let $f$ be a multimodal map of type $\mathcal{N}$ and consider an extended map $F$ induced by a decomposition $(f_0, \cdots, f_{\mathcal{N}-1})$ of $f$. If $P$ is a maximal periodic interval for $F$ of period $k$, then we can associate the following combinatorial data $\sigma = (A, A^{\text{Crit}}, P, \pi, m_1 m_2)$ such that

- $A = \{F^i(P) : 0 \leq i < k\}$;
- $A^{\text{Crit}} = \{F^i(P) : c \in F^i(P) \text{ for some critical point } c \text{ of } F\}$;
- $\pi : A \to A$ is defined by $\pi(F^i(P)) = F^{i+1 \mod \mathcal{N}}(P)$.

If $\mathcal{M} = [\sigma]$ is primitive, then we say $f$ is renormalizable with renormalization combinatoric $\mathcal{M}$.

We say $f$ is a multimodal maps of type $\mathcal{N}$ with combinatorics $(\mathcal{M}_k)_{k \geq 0} \in \Sigma^\mathcal{N}$ if $f$ is infinitely renormalizable and $\mathcal{R}^k f$ is renormalizable with renormalization combinatoric $\mathcal{M}_k$ for all $k \geq 0$.

Note that for any combinatorics $\mathcal{M} \in \Sigma^\mathcal{N}$ there exists a real polynomial in $\mathcal{I}$ with combinatorics $\mathcal{M}$ (see [19] section 2.1 and section 5.1). By Kozlovski-Shen-vanStrien’s combinatorial rigidity theorem [3], such a real polynomial is unique.

**Lemma 3.1.** Assume $\mathcal{M}^m \to \mathcal{M}$ in $\Sigma^\mathcal{N}$. If for every $m \in \mathbb{N}$, $f_m$ is an infinitely renormalizable multimodal map of type $\mathcal{N}$ with combinatorics $\mathcal{M}^m$, then any limit point $g$ of $\{f_m\}$ is infinitely renormalizable with combinatorics $\mathcal{M}$.

**Proof.** Without loss of generality, we may assume $f_m \to g$. Let $J^k_m$ be the restrictive interval of the $k$-th pre-renormalization. As $\mathcal{M}^m \to \mathcal{M}$, $J^m_l = J_l$ for all sufficiently large $m$ and $l \leq k$. Thus the periods of $J^m_m$ are the same, and so we can assume $J^m_m$ converges to a periodic interval $0 \in J^k_l$ for $g$. For otherwise, 0 will be a superattracting periodic point of $g$. It follows that the post-critical set of $g$ is contained in a solenoidal attractor. Thus $g$ only has repelling periodic point.
Hence the restrictive intervals and the post-critical set move continuously, which implies \( g \) is infinitely renormalizable with combinatorics \( \mathcal{M} \).

**Lemma 3.2.** If a sequence \( \{ f_k \} \) of infinitely renormalizable multimodal maps of type \( N \) converges to an infinitely renormalizable multimodal map \( f \) of type \( N \), then \( \mathcal{R} f_k \) converges to \( \mathcal{R} f \).

**Proof.** Let \( J = [-a, a] \) be the renormalization interval of \( f \), then one of \( \{-a, a\} \) is a repelling periodic point of \( f \). Without loss of generality, we assume \( a \) is a repelling periodic point of \( f \) with period \( p \). Since \( f_k \) converges to \( f \), there exists \( a_k \) such that \( a_k \) is a repelling periodic point of \( f_k \) with period \( p \) and \( a_k \) converges to \( a \). It follows easily \( f_k^p([−a_k, a_k]) \) is affine conjugate to a multimodal maps of type \( N \) by the uniform convergency. Thus the renormalization period \( \hat{p}_k \) of \( f_k \) is at most \( p \) for all \( k \) large.

Without loss of generality, we can assume the renormalization period of \( f_k \) is \( q \leq p \) for all \( k \in \mathbb{N} \) and the corresponding restrictive interval \( [-\hat{a}_k, \hat{a}_k] \) converges to \( [-\hat{a}, \hat{a}] \). Since \( f_k \) converges to \( f \) in \( C^3 \)-topology, \( f^q([-\hat{a}, \hat{a}]) \subset [-\hat{a}, \hat{a}] \). Clearly \( -\hat{a} \neq \hat{a} \), for otherwise \( \hat{a} = 0 \) will be a superattracting periodic point of \( f \), which is impossible as \( f \) is infinitely renormalizable. Thus, \( [-\hat{a}, \hat{a}] \) is a periodic interval of \( f \). Note that \( f^q([-\hat{a}, \hat{a}]) \) is a pre-renormalization of \( f \). Hence, \( \mathcal{R} f_k \) converges to \( \mathcal{R} f \). \( \square \)

**Theorem C.** For any totally \( R \)-invariant precompact subset \( \mathcal{A}' \subset \mathcal{I} \), the restriction \( \mathcal{R}^{-1}|_{\mathcal{A}'} \) of the anti-renormalization operator \( \mathcal{R}^{-1} \) to \( \mathcal{A}' \) is continuous.

**Proof.** For any sequence \( \{ f_k \} \subset \mathcal{A}' \) converging to \( f \in \mathcal{A}' \), we prove \( \mathcal{R}^{-1} f_k \to \mathcal{R}^{-1} f \) as \( k \to \infty \). Let \( N p_k \) be the renormalization period of \( f_k \) for all \( k \in \mathbb{N} \).

**Claim** \( \sup_k p_k < \infty \).

**Proof of the Claim:** Arguing by contradiction, we may assume \( p_k \to \infty \). Then by the Key Lemma, we know that either

\[
 f = \lim_{k \to \infty} \mathcal{R}(\mathcal{R}^{-1} f_k)
\]

is a polynomial of degree \( 2^n \) or it has bounded real trace. But both of these two cases are impossible. Indeed, every polynomial of degree \( 2^n \) cannot be anti-renormalizable due to the degree. On the other hand, \( f \) is infinitely anti-renormalizable, let \( g_{-k} = \mathcal{R}^{-k} f \). By Theorem 2.1, we know \( g_{-k} \) can be extended to \( [-\lambda^k, \lambda^k] \) for all \( k \in \mathbb{N} \). Since \( f \) is affinely conjugate to an iteration of \( g_{-k} \) for all \( k \in \mathbb{N} \), \( f \) cannot have bounded real trace.

Since the renormalization periods \( N p_k \) of \( \mathcal{R}^{-1} f_k \) are bounded, then by a similar argument in the proof of Lemma 3.1 we can conclude all the limit point \( g \) of \( \{ \mathcal{R}^{-1} f_k \} \) is renormalizable and \( \mathcal{R} g = \lim_{k_n} \mathcal{R}(\mathcal{R}^{-1} f_{k_n}) = f \). Thus \( \mathcal{R}^{-1} f_k \) converges to \( \mathcal{R}^{-1} f \).

\( \square \)

The following corollary follows immediately from Lemma 3.2 and Theorem C.

**Corollary 3.1.** For any totally \( R \)-invariant precompact subset \( \mathcal{A}' \subset \mathcal{I} \), the restriction \( \mathcal{R}|_{\mathcal{A}'} \) of the renormalization operator \( \mathcal{R} \) to \( \mathcal{A}' \) is a self-homeomorphism.
4. Polynomial-like extension for multimodal maps of type $N$

In order to prove Theorem B, we will use the complex method, that is, we extend $f$ to the complex plane and use the tools in complex dynamics. In this section, we will first recall some definitions and results in polynomial dynamics. Then we prove each hybrid leaf is homeomorphic to a standard model space $E_N$, which is simply connected.

We say $P$ is a polynomial of type $N$ if there exists quadratic polynomials $P_j = a_j z^2 - a_j - 1$ ($j = 0, \ldots, N-1$) such that $P = P_{N-1} \circ \cdots \circ P_0$. And $(P_0, \ldots, P_{N-1})$ is called a quadratic decomposition of $P$.

A polynomial-like map $f : U \to V$ of degree $d$ is a holomorphic proper map of degree $d$ where $U \subseteq V$ are quasidisks. The filled Julia set of $f$ is

$$K(f) := \{ z \mid f^n(z) \in U, \forall n \geq 0 \}$$

and the boundary of $K(f)$ is called the Julia set of $f$. The idea of polynomial-like map was first introduced by Douady and Hubbard [4].

In this paper, we will consider a special kind of polynomial-like maps which is called polynomial-like map of type $N$.

**Definition 4.1.** A polynomial-like map $f$ is called a polynomial-like map of type $N$ if there exists quasidisks $U = U_0 \subset U_1 \subset \cdots \subset U_{N-1} \subset U_n = V$ and holomorphic branched double covering $f_j : U_j \to U_{j+1}$ with a unique critical point $z = 0$ ($j = 0, \ldots, N-1$) such that $f = f_{N-1} \circ \cdots \circ f_0 : U \to V$ is a polynomial-like representative of $f$.

We will normalize the polynomial-like map of type $N$ so that $0, -1 \in U$ and $f(-1) = -1$.

Given a polynomial-like map $f$, the corresponding polynomial-like germ $\lbrack f \rbrack$ is an equivalence class of polynomial-like maps $\tilde{f}$ such that $K(\tilde{f}) = K(f)$ and $\tilde{f} = f$ near the filled Julia set $K(f)$. The modulus of a polynomial-like germ is defined as:

$$\text{mod } f = \sup \text{mod}(V \setminus \overline{U}),$$

where the supremum is taken over all polynomial-like representatives $\tilde{f} : U \to V$ of $f$.

In this paper, we will not distinguish the notion of a polynomial-like map and its corresponding polynomial-like germ, and let $C_N$ be the family of all the normalized polynomial-like germs of type $N$ with connected Julia set. For any $\delta > 0$, $C_N(\delta)$ is used to denote the set of $f \in C_N$ with $\text{mod } f \geq \delta$. Note that $\text{mod } f = \infty$ if and only if $f$ is exactly a polynomial.

4.1. Topology and Complex analytic structure. Given a Jordan disk, let $B_U$ be the Banach space of functions which are holomorphic on $U$ and continuous up to the boundary $\partial U$ and denote $\| \cdot \|_U$ the $L_\infty$-norm of $B_U$.

Now we define the topology of $C_N$ as following. We say $f_k$ converges to $f$ in $C_N$ if and only if there exists quasidisk $W \supseteq K(f_k)$ such that $f_k, k = 1, 2, \ldots$ and $f$ are well defined on $\overline{W}$ for all sufficiently large $k$ and $\| f_k - f \|_{W} \to 0$ as $k \to \infty$. We say $K \subset C_N$ is closed if for every sequence $\{f_k\} \subset K$, the limit points of $\{f_k\}$ also belong to $K$.

Two polynomial-like germ $f$ and $g$ are called hybrid equivalent, if there exists quasiconformal map $h : \mathbb{C} \to \mathbb{C}$ such that $h \circ f = g \circ h$ near $K(f)$ and $\partial h = 0$ almost everywhere on $K(f)$.
For \( b = (b_0, \cdots, b_{N-1}) \in \mathbb{C}^N \), let \( P_b := (b_{N-1} z^2 - b_{N-1} - 1) \cdots \circ (b_0 z^2 - b_0 - 1) \), \( \mathcal{H}(b) := \{ f \mid f \text{ is hybrid equivalent to } P_b \} \) and let \( \mathcal{H}(b) \) be the component of \( \mathcal{H}(b) \) containing \( P_b \). Such an \( \mathcal{H}(b) \) is called a hybrid leaf and it has a natural topology induced from the topology of \( \mathcal{C}_N \). A hybrid leaf is called real symmetric if it contains a real map. Let \( \mathcal{H}(b, \epsilon) \) denote the set of all the \( f \in \mathcal{H}(b) \) with \( \text{mod } f \geq \epsilon \), then \( \mathcal{H}(b, \epsilon) \) is a precompact set and any precompact subset \( K \) of \( \mathcal{H}(b) \) is contained in some \( \mathcal{H}(b, \epsilon) \) (see [14 section 5]).

**Theorem 4.1.** For every polynomial-like map \( f \) of type \( N \), there exists a polynomial of type \( N \) hybrid equivalent to \( f \).

**Proof.** The proof is based on quasiconformal surgery. See [19 Proposition 4.1], [6, Theorem A] and also [17].

On the contrary, we have

**Theorem 4.2.** If a polynomial-like map \( f : U \rightarrow V \) is hybrid equivalent to some polynomial \( P \) of type \( N \), then \( f \) is a polynomial-like map of type \( N \).

**Proof.** Let \( (P_0, \cdots, P_{N-1}) \) be a quadratic decomposition of \( P \) and \( h \) be a hybrid conjugacy between \( f \) and \( P \). Select quasidisks \( U_0 \subset U_1 \subset \cdots \subset U_N \) such that \( U_0 = U \) and \( U_N = V \). Let \( W_N = h(V) \), \( W_N = P_{N-1}^{-1}(W_N) \), \( W_{N-2} = P_{N-2}^{-1}(W_{N-1}) \), \( \cdots \), \( W_0 = P_0^{-1}(W_1) = h(U) \). Choose quasiconformal mappings \( \varphi_j : U_j \rightarrow W_j \) such that \( \varphi(U_{j-1}) = W_{j-1} \) and \( \varphi_j(0) = 0 \) for all \( 1 \leq j < N \). Then \( h^{-1} \circ P_{N-1} \circ \varphi_N : U_{N-1} \rightarrow U_N \) is a quasiregular map. Thus, we can choose a quasiconformal map \( \psi_{N-1} : U_{N-1} \rightarrow U_{N-1} \) such that \( \psi_{N-1}(0) = 0 \) and

\[
f_{N-1} := h^{-1} \circ P_{N-1} \circ \varphi_N \circ \psi_{N-1} : U_{N-1} \rightarrow U_N
\]

is a holomorphic proper map. Similarly, there exists quasiconformal map \( \psi_{N-2} : U_{N-2} \rightarrow U_{N-2} \) such that \( \psi_{N-2}(0) = 0 \) and

\[
f_{N-2} := \psi_{N-1}^{-1} \circ \varphi_N^{-1} \circ P_{N-2} \circ \varphi_{N-1} \circ \psi_{N-2} : U_{N-2} \rightarrow U_{N-1}
\]

is a holomorphic proper map. By induction, there exist quasiconformal maps \( \psi_j : U_j \rightarrow U_j \) such that \( \psi_j(0) = 0 \) and

\[
f_j := \psi_{j+1}^{-1} \circ \varphi_j^{-1} \circ P_j \circ \varphi_{j+1} \circ \psi_j : U_j \rightarrow U_{j+1}
\]

are holomorphic proper maps \( 1 \leq j \leq N - 2 \). Set \( f_0 := \psi_1^{-1} \circ \varphi_2^{-1} \circ P_0 \circ h \), then \( f_{N-1} \circ \cdots \circ f_1 \circ f_0 = h^{-1} \circ P_{N-1} \circ \cdots \circ P_1 \circ P_0 \circ h = f \). Now we only need to check whether \( f_0 \) is holomorphic. Denote \( f_{N-1} \circ \cdots \circ f_2 \circ f_1 \) by \( G \), by differentialing we obtain a.e. \( z \)

\[
0 = \frac{\partial f}{\partial z} = \frac{\partial G}{\partial w} \frac{\partial f_0}{\partial z} + \frac{\partial G}{\partial w} \frac{\partial f}{\partial w} = \frac{\partial^2 f_0}{\partial z^2}
\]

It follows from Weyl’s Lemma that \( f_0 \) is holomorphic and we are done.

\[
\square
\]

Modifying Lyubich’s argument of complex analytic variety in [10] and [11], we can define the complex analytic structure of \( \mathcal{H}(0) \) as following. Set \( \mathcal{B}_U^0 := \{ f \in \mathcal{B}_U \mid f^{(j)}(0) = 0, j = 1, 2, \cdots, 2N-1 \} \), then it is also a Banach space under the \( L_\infty \)-norm. If \( f : U \rightarrow V \) is a polynomial-like representative of \( f \in \mathcal{H}(0) \), it is easy to see \( g \in \mathcal{B}_U^0(f, \epsilon) \) has a polynomial-like restriction on a quasidisk slightly smaller than \( U \) for \( \epsilon \) sufficiently small (where \( \mathcal{B}_U^0(f, \epsilon) \) is an \( \epsilon \)-neighborhood of \( f \) in \( \mathcal{B}_U^0 \)).
Thus, we have a natural continuous inclusion $\mathcal{J}_{U,f,\epsilon} : \mathcal{B}_{U}^{\epsilon}(f,\epsilon) \to \mathcal{H}(0)$ and $\mathcal{B}_{U}^{\epsilon}(f,\epsilon)$ is called a Banach slice of $\mathcal{H}(0)$ centered at $f$. For convenience of notion, we use $\mathcal{S}_{U}$ to stand for a Banach slice without specifying $f$ and $\epsilon$. Roughly speaking, $\mathcal{J}_{U} : \mathcal{S}_{U} \to \mathcal{H}(0)$ can be understood as the local chart of $\mathcal{H}(0)$ and Lyubich [10, 11] proved:

**Lemma 4.1 (Lyubich).** The family of local charts $\mathcal{J}_{U}$ satisfies the following properties:

1. **Countable base and Compactness.** There is a countable family of Banach Slices $\mathcal{S}_{i} = \mathcal{S}_{U}$, such that for any $f \in \mathcal{H}(0)$, the Banach Slice $\mathcal{S}_{U}$ centered at $f$ is compactly contained in some $\mathcal{S}_{i}$.

2. **Lifting of analyticity.** If $W \subset U$, then the inclusion map $\mathcal{J}_{U,W} : \mathcal{S}_{U} \to \mathcal{B}_{W}$ is complex analytic. Moreover, let $U \supseteq V$. Let us consider a locally bounded map $\phi : V \to \mathcal{B}_{V}$ defined on a domain $V$ in some Banach space. Assume that the map $\mathcal{J}_{V,W} \circ \phi : V \to \mathcal{B}_{W}$ is analytic, then the map $\mathcal{J}_{V,U} \circ \phi : V \to \mathcal{B}_{U}$ is also analytic.

3. **Density.** If $W \subset U$, then $\mathcal{B}_{U}$ is dense in $\mathcal{B}_{W}$.

A space with properties $P_{1}$, $P_{2}$, $P_{3}$ is called a complex analytic variety, then $\mathcal{H}(0)$ is a complex analytic variety.

A map $\phi : \mathbb{D} \to \mathcal{H}(0)$ is called analytic if for any $z \in \mathbb{D}$, there exists a small disk $\mathbb{D}(z,\delta)$ and a Banach slice $\mathcal{S}_{U}$ such that $\phi(\mathbb{D}(z,\delta)) \subset \mathcal{S}_{U}$ and the restriction $\phi|_{\mathbb{D}(z,\delta)} : \phi(\mathbb{D}(z,\delta)) \to \mathcal{S}_{U}$ is analytic in the Banach sense. Clearly, $\phi$ is an analytic implies that $\phi$ is continuous.

### 4.2. External maps of polynomial-like germs

A real analytic circle map $g : \mathbb{T} \to \mathbb{T}$ is called expanding if there exists $k \geq 1$ such that $|Dg^{k}(z)| > 1$ for all $z \in \mathbb{T}$ where $D$ denotes the derivative with respect to $z$.

Let $\mathcal{E}_{\mathbb{N}}$ be the family of real analytic expanding circle covering maps $g : \mathbb{T} \to \mathbb{T}$ of degree $2^{\mathbb{N}}$ normalized so that $g(1) = 1$. $\mathcal{E}_{\mathbb{N}}$ is simply connected ([11, Lemma 2.1]). Since $g$ is real analytic and expanding, it can be extended to be a holomorphic covering $g : U \to V$ of degree $2^{\mathbb{N}}$ where $U \subset V$ are annular neighborhood of $\mathbb{T}$.

Similarly, we can define the modular of expanding circle maps as:

$$\text{mod } g = \sup_{z \in U} |g(z)|,$$

where the supremum is taken over all extensions $g : U \to V$ of $g$.

We will use the Inductive limit topology of $\mathcal{E}_{\mathbb{N}}$ (see [10, Appendix 2]). In this topology, a sequence $g_{k} \in \mathcal{E}_{\mathbb{N}}$ converges to $g \in \mathcal{E}_{\mathbb{N}}$ if there exists a neighborhood $W$ of $\mathbb{T}$ such that all the $g_{k}$ admit a holomorphic extension to $W$, and $g_{k} \to g$ uniformly on $W$, i.e. $\sup_{z \in W} |g_{k}(z) - g(z)| \to 0$.

Let $f \in \mathcal{C}_{\mathbb{N}}$, consider the Böttcher coordinate $\phi_{f} : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus \overline{\mathbb{T}}$, then $g = \phi_{f} \circ f \circ \phi_{f}^{-1}$ is well defined in a small outer neighborhood of $\mathbb{T}$, by Schwarz reflection principle, $g$ can be extended to a holomorphic expanding map of degree $2^{\mathbb{N}}$ in a neighborhood of $\mathbb{T}$. Such a map $g$ is unique up to a rotation conjugation, thus it can be normalized so that $g \in \mathcal{E}_{\mathbb{N}}$ and called an external map of $f$. Unfortunately, such an external map $g$ may not be unique. However, we can construct a canonical external map $g \in \mathcal{E}_{\mathbb{N}}$ from a polynomial-like germ $f \in \mathcal{C}_{\mathbb{N}}$. Indeed, we prove the following theorem.
Theorem 4.3. For every $b \in \mathbb{C}^N$, there exists a homeomorphism $I_b : \mathcal{E}_N \to \mathcal{H}(b)$. Moreover, $\mod I_b(g) = \mod g$ for all $g \in \mathcal{E}_N$.

Proof. Firstly, given $g \in \mathcal{E}_N$, we choose a continuous path $\{g_t\}$ connecting $g_0 = z^{2N}$ and $g_1 = g$. Then we can construct a continuous family of $K$-q.c maps $h_t : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus \mathbb{D}$ with Beltrami differential $\nu_t$ continuously depending on $t$ such that $h_0 = \text{id}$ and $h_t \circ g_0 = g_t \circ h_t$ near the unit circle $\mathbb{T}$ in the following way:

Since $\{g_t\}$ is compact, there exist representatives $g_t : W_t^1 \setminus \mathbb{D} \to W_t^2 \setminus \mathbb{D}$ where $W_t^1, W_t^2$ are quasidisks with $\mod W_t^2 \setminus W_t^1 \geq \epsilon_0$ for some $\epsilon_0 > 0$. By Grötzsch’s extremal problem, we know $W_t^1$ contains a Euclid disk $\mathbb{D}(0, r_1)$ with $r_1 > 1$. As $g_t$ is uniformly expanding near $\mathbb{T}$, we can choose $1 < r_0 < r_1$ such that $g_t^{-1}(\mathbb{D}(0, r_0)) \subset \mathbb{D}(0, r_0)$ for all $t$. Let $\gamma_t^2 \equiv \partial \mathbb{D}(0, r_0)$ and $\gamma_t^1 = g_t^{-1}(\gamma_t^2)$, define $h_t \equiv \text{id}$ outside $\mathbb{D}(0, r_0)$ and we can lift $h_t$ to $\partial \mathbb{D}(0, r_0^{1/d})$ such that $g_t \circ h_t = g_0$. Then by extension, we can construct a $K$-q.c maps $h_t : A_0 \to A_t$ where $A_t$ is the annulus bounded by $\gamma_t^1$ and $\gamma_t^2$. Moreover, $h_t$ depends continuously on $t$. Finally, we can lift $h_t$ to $\mathbb{C} \setminus \mathbb{D}$ by respecting the dynamics so that the Beltrami differentials $\nu_t$ of $h_t$ satisfying $\|\nu_t\| \leq k = \frac{K - 1}{K + 1}$ for all $t$.

Now we are going to construct a continuous path $\{f_t\} \subset \mathcal{H}(b)$ from the path $\{g_t\}$. Consider the Böttcher coordinate $\xi_b : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus K(P_b)$, we define Beltrami differentials $\mu_t$ on $\mathbb{C}$ such that $\mu_t = (\xi_b)_* \nu_t$ on $\mathbb{C} \setminus K(P_b)$ and $\mu_t = 0$ on $K(P_b)$. By the Measurable Riemann Mapping Theorem, we can obtain a continuous path $\{q_t\}$ of $K$-q.c maps such that $\partial q_t = \mu_t \partial q_t$ and $q_t$ fixes $0$. Then $Q_t := q_t \circ P_t \circ q_t^{-1}$ defines a polynomial-like map of degree $2N$ which is hybrid to $P_t$ and $Q_0 = P_b$. By Theorem 4.2, $Q_t$ is affinely conjugate to a map in $\mathcal{H}(b)$. Thus we can use a continuous family of affine transformations $\{A_t\}$ to normalize $Q_t$, so that $f_t := A_t \circ Q_t \circ A_t^{-1} \in \mathcal{H}(b)$. Let $\phi_t := A_t \circ q_t$, then $f_t = \phi_t \circ P_t \circ \phi_t^{-1}$.

Lemma 4.2. For every $t \in [0, 1]$, $f_t$ is an external map of the polynomial-like map $f_t$.

Proof. Let $\psi_t : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus K(f_t)$ be the continuous family of Riemann maps normalized so that $\psi_0 = \xi_b$ and $\hat{g}_t = \psi_t^{-1} \circ f_t \circ \psi_t : \mathbb{T} \to \mathbb{T}$ fixes $1$, hence $\hat{g}_t$ is an external map of $f_t$.

To see the existence of $\psi_t$, we consider another normalized family of Böttcher coordinates $\hat{\psi}_t : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus K(f_t)$ such that $\hat{\psi}_t(\infty) = \infty, \hat{\psi}_t'(\infty) > 0$. It is a continuous family due to the semi-upper continuity of $K(f_t)$ and the semi-lower continuity of $J(f_t)$ (see [10] Lemma 4.15 [for an example]). Then $G_t|_{\gamma} = \hat{\psi}_t^{-1} \circ f_t \circ \hat{\psi}_t$ is continuous in $t$ where $\gamma$ is a Jordan curve in $\mathbb{C} \setminus \bar{\mathbb{D}}$. By the Schwarz reflection and Maximum Principle, $G_t : \mathbb{T} \to \mathbb{T}$ is a continuous family. Let $z(t)$ be a fixed point of $G_t$, so that $z(t)$ is continuous and $z(0) = 1$. Then $\hat{\psi}_t := \hat{\psi}_t \circ e^{i\arg(z(t))} : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus K(f_t)$ is continuous in $t$ and and $\hat{g}_t = \psi_t^{-1} \circ f_t \circ \psi_t : \mathbb{T} \to \mathbb{T}$ fixes $1$.

It remains to show that $g_t = \hat{g}_t$ for all $t \in [0, 1]$. Clearly $\sigma_t := \psi_t^{-1} \circ \phi_t \circ \xi_b : \mathbb{C} \setminus \bar{\mathbb{D}} \to \mathbb{C} \setminus \bar{\mathbb{D}}$ is a quasiconformal map conjugating $g_0$ to $\hat{g}_t$. Since $1$ is the fixed point of $g_t$, $\lambda_t(1)$ must be one of the fixed point of $\hat{g}_t$ by the conjugacy. Let $t_0 = \sup\{t \mid \lambda_t(1) = 1, x \leq t\}$, we claim $t_0 = 1$. Indeed, $\lambda_0 = \text{id}$ implies $\lambda_0(1) = 1$, thus $0 \leq t_0 \leq 1$ exists. By continuity, we obtain $\lambda_t(1) = 1$. Hence $t_0 = 1$, for otherwise, there exists $\ell$ slightly large than $t_0$ such that $\lambda_\ell(1) = 1$ for all $x \leq \ell$, which contradicts with the definition of $t_0$. We conclude that $\lambda_t = \text{id}$ and hence $g_t = \hat{g}_t$ for all $t \in [0, 1]$. 


Let $T_b(g) := f_1$ and we should check that the polynomial-like map $f_1$ we constructed above does not depend on the choice of the path $\{g_t\}$.

To this end, we first show that once the connecting path $g_t$ is chosen, the path $f_t$ does not depend on the choice of $h_t$. Suppose $h_t$ is another $K$-qc conjugation from $g_0 = z^{2N}$ to $g_t$, let $\tilde{f_t}$ and $\tilde{\phi}_t$ be the corresponding polynomial-like map and hybrid conjugation. Then $\tilde{\eta}_t = \tilde{\phi}_t \circ \phi_t^{-1}$ is a hybrid conjugation from $f_t$ to $\tilde{f_t}$ near $K(f_t)$. Let $\psi_t : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \setminus K(f_t)$ and $\tilde{\psi}_t : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \setminus K(\tilde{f}_t)$ be the Riemann mappings in the above construction of the external maps respectively.

Let $\tilde{\eta}_t = \tilde{\psi}_t^{-1} \circ \tilde{\eta}_t \circ \psi_t$, then $\tilde{\eta}_t(1)$ is a fixed point of $g_t$ and $g_t \circ \tilde{\eta}_t = \tilde{\eta}_t \circ g_t$ on $\mathbb{T}$. Indeed, $\tilde{\eta}_t$ is a $K$-qc map defined on $U \setminus \overline{D}$, then it can be extended to $U \setminus \overline{D}$. It is continuous in $t$ and $\tilde{\eta}_t(1) = 1$, and hence $\tilde{\eta}_t(1) = 1$.

**Lemma 4.3.** Let $g : \mathbb{T} \to \mathbb{T}$ be an expanding circle map, $h$ is an automorphism of $\mathbb{T}$ such that $h \circ g = g \circ h$ and $h(1) = 1$, then $h = id$.

**Proof.** It is trivial when $g = m_d(x) = dx \mod 1$, where $d = 2^N$. It is well-known any expanding circle map is quasisymmetrically conjugate to $dx \mod 1$ (see [10] the proof of Lemma 3.8), so the conclusion follows easily. □

**Lemma 4.4 ([2] Lemma 2.1).** Let $S$ be a hyperbolic Riemann surface with boundary $\gamma$ and $H : S \to S$ be a $K$-qc map homotopic to the identity rel the boundary, then $d_S(x, H(x)) \leq C(K)$ where $d_S$ is the hyperbolic distance and $C(K)$ is a constant only depend on $K$.

By the above two Lemmas, we conclude that $\tilde{\eta}_t = id$ on $\mathbb{T}$ and

$$d_{U \setminus K(f_t)}(\tilde{\eta}_t(x), \tilde{\psi}_t \circ \psi_t^{-1}(x)) = d_{U \setminus K(f_t)}(\tilde{\psi}_t^{-1} \circ \tilde{\eta}_t(x), \psi_t^{-1}(x))$$

$$= \left\{ \begin{array}{ll}
\psi_t \circ \psi_t^{-1}(x), & x \in \mathbb{C} \setminus K(\tilde{f}_t) \\
\tilde{\eta}_t(x), & x \in K(\tilde{f}_t)
\end{array} \right.$$  \hspace{1cm} where $x = \psi_t(z)$ and $\tilde{\eta}_t(x) \to K(\tilde{f}_t)$, $\tilde{\psi}_t \circ \psi_t^{-1}(x) \to K(f_t)$ as $z \to K(f_t)$. Hence $d_{E_{uc}(\tilde{\eta}_t(x), \psi_t^{-1}(x))} \to 0$ as $z \to K(f_t)$. By [4] Lemma 2, we obtain a quasiconformal map:

$$Q_t(x) = \left\{ \begin{array}{ll}
\psi_t \circ \psi_t^{-1}(x), & x \in \mathbb{C} \setminus K(\tilde{f}_t) \\
\tilde{\eta}_t(x), & x \in K(\tilde{f}_t)
\end{array} \right.$$  \hspace{1cm} It follows from Weyl’s lemma that $Q_t$ is conformal, and thus $Q_t$ is affine. But $\tilde{f}_0 = f_0 = P_b$ and $Q_0 = id$, by the continuity, we conclude that $\tilde{f}_t = f_t$ for all $t \in [0, 1]$.

Let us now show that the endpoint $f = f_1$ does not change whether the path $\{g_t\}$ is alternated. Given two paths $\{g_t\}$ and $\{\tilde{g}_t\}$ connecting $z^d$ and $g$, by the simply connectedness of $\mathcal{E}_N$, we can choose a homotopy $g^s_t$ with $g^0_t = g_t$ and $g^1_t = \tilde{g}_t$. For every $s \in [0, 1]$, let $f^s_t$ be the polynomial-like map corresponding to the path $\{g^s_t\}$, then $f^s_t$ are hybrid equivalent and $g$ is the external map of $f^s_1$. By a similar argument as above, we can show that there exists a continuous family of affine transformations $A_s$ such that $A_0 = id$ and $A_s \circ f^s_1 \circ A_s^{-1} = f^s_1$. Hence, by the continuity, $f^s_1 = f^s_1$ for all $s \in [0, 1]$.

The construction of $T_b(g)$ implies the continuity of $T_b$. 

Let us now prove that $I_b$ is a bijection. For every $f \in \mathcal{H}(b)$, choose a path $(f_t) \subset \mathcal{H}(b)$ to connect $P_0$ and $f$. We will prove that $(f_t)$ has a unique lift in $\mathcal{E}_N$, and this implies that $I_b$ is a bijection. Consider a continuous family of conformal mappings $\varphi_t : C \setminus \overline{D} \to C \setminus K(f_t)$ such that $\varphi_t(\infty) = \infty$ and $\varphi_t'(\infty) > 0$. Then $\tilde{g}_t := \varphi_t^{-1} \circ f_t \circ \varphi_t : \mathbb{T} \to \mathbb{T}$ is an analytic expanding map of degree $2^N$ for all $t \in [0, 1]$ and $\tilde{g}_0 = z^{2^N}$. We can choose a continuous family of conformal maps $A_t : C \setminus \overline{D} \to C \setminus \overline{D}$ so that $g_t := A_t^{-1} \circ \tilde{g}_t \circ A_t : \mathbb{T} \to \mathbb{T}$ belongs to $\mathcal{E}_N$ and $A_0 = \text{id}$. It is easy to check that $I_b(g_t) = f_t$ by the definition, and so $(g_t)$ is a lift of $(f_t)$. If $(f_t)$ has another lift $(\tilde{g}_t)$, then $\tilde{g}_0 = z^{2^N}$ since $P_0$ has a unique preimage $z \mapsto z^{2^N}$. Let $\psi_t : C \setminus \overline{D} \to C \setminus K(f_t)$ and $\hat{\psi}_t : C \setminus \overline{D} \to C \setminus K(f_t)$ be the conformal maps so that $f_t \circ \psi_t = \psi_t \circ g_t$, $\hat{\psi}_0 = \xi_b$ and $f_t \circ \hat{\psi}_t = \hat{\psi}_t \circ \tilde{g}_t = \hat{\psi}_t \circ \hat{g}_t = \xi_b$ respectively. Set $\eta_t := \psi_t^{-1} \circ \hat{\psi}_t$, we get $\eta_t \circ \hat{g}_t = g_t \circ \eta_t$ and $\eta_0 = \text{id}$. Thus $\eta_t(1)$ is one of the fixed points of $g_t$. But $\eta_t(1) = 1$, so by continuity, $\eta_t(1) = 1$ for all $t \in [0, 1]$. It follows from the Theorem of Boundary Correspondence that $\eta_t$ can only be the identity. Hence $\psi_t = \hat{\psi}_t$, and this implies $g_t = \tilde{g}_t$ for all $t \in [0, 1]$. In other words, $(g_t)$ is the unique lift of $(f_t)$.

For every $\epsilon > 0$, consider the restriction $I_b|_{\mathcal{E}_N(\epsilon)} : \mathcal{E}_N(\epsilon) \to \mathcal{H}(b, \epsilon)$ of $I_b$ to $\mathcal{E}_N(\epsilon)$. It is a continuous bijection, so by the compactness of $\mathcal{E}_N(\epsilon)$, it is a homeomorphism. If $f_n \to f$ in $\mathcal{H}(b)$, then $\{f_n\}_{n=1}^{\infty} \cup \{f\}$ is a compact subset of $\mathcal{H}(b)$, thus it is contained in some $\mathcal{H}(b, \epsilon_0)$. Hence $g_n := I_b^{-1}(f_n) \in \mathcal{E}_N(\epsilon_0)$ for all $n \in \mathbb{N}$. Since $\mathcal{E}_N(\epsilon_0)$ is compact, every subsequence of $g_n$ has a limit point, and by the continuity and the bijectivity of $I_b$, the limit point must be $I_b^{-1}(f)$. This implies $I_b^{-1}(f_n)$ converges to $I_b^{-1}(f)$, thus $I_b^{-1}$ is continuous. \qed

By Theorem 4.3 for every $f \in \mathcal{C}_N$, there exists a unique $b$ such that $f \in \mathcal{H}(b)$ and we denote it by $\chi(f)$.

### 4.3. Complex renormalization for multimodal maps of type $N$

Let us now define the complex renormalization for multimodal maps of type $N$.

We say a multimodal map $f$ of type $N$ has a polynomial-like extension if there exists quasidisks $U_0, U_1, \ldots, U_N$ such that $f : U_0 \to U_N$ is a polynomial-like map, $\mathbb{I} \subset U_j$ and $f_j : U_j \to U_{j+1}$ is holomorphic proper for all $0 \leq j \leq N - 1$ where $(f_j : \mathbb{I} \to \mathbb{I})_{j=0}^{N-1}$ is a unimodal decomposition of $f$.

**Definition 4.2.** A multimodal map $f$ of type $N$ is called complex renormalizable if it is real renormalizable and both itself and its real renormalization $R_f$ have polynomial-like extensions. The germ of the polynomial-like extension of $R_f$ will be called the complex renormalization of $f$.

In [16], Shen proved the complex bounds for all the infinitely renormalizable real analytic box map without critical points of odd order:

**Theorem 4.4 ([16] Theorem 3').** There exists $\epsilon_0 > 0$ with the following property. If $F$ is a compact family of infinitely renormalizable multimodal maps of type $N$, then there exists $K > 0$ such that for any $k > K$ and $f \in F$, $R^k_f$ has a polynomial-like extension $R^k_f : U \to V$ with

$$\text{mod } V \setminus U \geq \epsilon_0.$$ 

Hence, all the maps in $I$ are actually infinitely complex renormalizable (see also [16] Theorem 3'), so from now on we don’t distinguish the terminology of real
renormalization and complex renormalization for multimodal maps. We mention here that Clark, Trejo and van Strien \cite{3} proved the complex bounds for all the infinitely renormalizable real analytic box map recently.

For polynomial-like germs of type $N$, one can still easily define the pre-renormalization just as a first return map. However, there is not a canonical way to normalize such a first return map to the normalized form. A usual way to do this, is to use the external marking. Nevertheless, we can define the renormalization for polynomial like germ $f$ which is in a hybrid leaf of a complex renormalizable multimodal map $f_\ast$ of type $N$ as following: Choose a path $\{f_t\}_{t \in [0,1]}$ in this hybrid leaf to connect $f_s$ and $f$, let $h_t$ be the hybrid conjugacy between $f_s = f_0$ and $f_t$, then $f_t = h_t^{-1} \circ f_s \circ h_t$. Suppose the renormalization period of $f_\ast$ is $p$, then $f_t^p = h_t^{-1} \circ f_s^p \circ h_t$ restricting to some small region is a pre-renormalization for $f_t$. Finally, there exists a continuous family $\{\Lambda_t\}$ of affine maps such that $\Lambda_t \circ f_s^p \circ \Lambda_t^{-1} \in \mathcal{H}(Rf_s)$ for all $t \in [0,1]$ such that $\Lambda_0 \circ f_s^p \circ \Lambda_0^{-1} = Rf_s$ and we define $Rf := \Lambda_1 \circ f^p \circ \Lambda_1^{-1}$. As $\mathcal{H}(f_s)$ is homeomorphic to $\mathcal{E}_N$, it is simply connected, so the definition of $Rf$ does not depend on the choice of the path $\{f_t\}$.

5. Path holomorphic structure on hybrid leaves

In this section, we will use the method of path holomorphic space developed in \cite{1} by Avila and Lyubich. Following Avila-Lyubich \cite{1}, we define the path holomorphic structure on all the real-symmetric hybrid leaves. Under the path holomorphic structure, the renormalization operator between two hybrid leaves is contracting with respect to the corresponding Caratheódy metric. Use Avila and Lyubich’s idea of cocycles, one can transfer the beau bounds (uniform a priori complex bounds) for real maps to the beau bounds for entire hybrid leaves of real maps. Altogether the contracting property for the renormalization operator and the beau bounds, we show the exponential contraction of the renormalization operator along the real-symmetric leaves. Some proofs in this section are similar to the unimodal case, so we will skip these proofs. For details, we refer the readers to section 3 – 8 in \cite{1}.

**Definition 5.1.** Let $X$ be a topological space, a path holomorphic structure $\text{Hol}(X)$ on $X$ is a family of continuous paths $\Gamma = \{\gamma : \gamma : D \to X \text{ continuous}\}$ such that

1. $\Gamma$ contains all constant maps;
2. for any $\gamma \in \Gamma$ and holomorphic map $\phi : D \to D$, the composition $\gamma \circ \phi$ belongs to $\Gamma$.

A topological space $X$ equipped with a path holomorphic structure is called a path holomorphic space. Every element in $\text{Hol}(X)$ will be called a holomorphic path.

For two path holomorphic space $X$ and $Y$, we say $\Phi : X \to Y$ is a path holomorphic map if $\Phi$ maps each holomorphic path in $X$ to a holomorphic path in $Y$. We denote by $\text{Hol}(X, Y)$ the set consisting of all the path holomorphic map from $X$ to $Y$.

**Definition 5.2** (Holomorphic path in $\mathcal{H}(0)$). Let $\{f_\lambda\}_{\lambda \in \mathbb{D}}$ be a continuous path in $\mathcal{H}(0)$, we say $\{f_\lambda\}_{\lambda \in \mathbb{D}}$ is a holomorphic path if there exists a holomorphic motion $h_\lambda(z) : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ such that

1. $h_0 = \text{id}$;
2. $f_\lambda \circ h_\lambda = h_0 \circ f_0$ on $K(f_0)$;
3. $\partial h_\lambda = 0$ a.e on $K(f_0)$.
Lemma 5.1. A map \( \lambda : \mathbb{D} \to \mathcal{H}(0) \) is a holomorphic path if and only if it is analytic.

Proof. If \( \lambda : \mathbb{D} \to \mathcal{H}(0) \) is analytic, then for any \( \lambda_0 \in \mathbb{D} \), there exist a sufficiently small round disk \( \mathbb{D}(\lambda_0, r) \) and \( r \) such that \( \phi(\mathbb{D}(\lambda_0, r)) \subseteq \mathcal{S}_U \). Then we can easily construct an analytic family of quasiconformal maps \( \{ h_\lambda : \mathbb{D}(\lambda_0, r) \to \mathbb{D}(\lambda_0, r') \} \) such that \( h_\lambda = \lambda \) on \( \mathbb{D}(\lambda_0, r) \) and \( \partial h_\lambda = 0 \) a.e on \( \mathbb{D}(\lambda_0, r) \).

Note that \( \{ f_\lambda \} \) is a holomorphic path if and only if it is a Beltrami path. Let \( \mathcal{H}(0) \) be the set of all the holomorphic paths in \( \mathcal{H}(0) \), then \( \mathcal{H}(0) \) is a path holomorphic structure on \( \mathcal{H}(0) \).

The following lemma explains the relation between path holomorphic structure and analytic structure in \( \mathcal{H}(0) \).

Definition 5.3 (Locally holomorphic path). Let \( \{ f_\lambda \} \) be a continuous path in \( \mathcal{H}(0) \), we say \( \{ f_\lambda \} \) is a locally holomorphic path if for any \( \lambda_0 \in \mathbb{D} \), there exists a disk \( \mathbb{D}(\lambda_0, r) \) and a holomorphic motion \( h_\lambda(z) : \mathbb{D}(\lambda_0, r) \times \mathbb{C} \to \mathbb{C} \) such that

1. \( h_{\lambda_0} = \text{id}; \)
2. \( f_\lambda \circ h_\lambda = h_{\lambda_0} \circ f_{\lambda_0} \) on \( K(f_{\lambda_0}) \);
3. \( \partial h_\lambda = 0 \) a.e on \( K(f_{\lambda_0}) \).

Definition 5.4 (Beltrami path). For every \( b \in \mathcal{C} \), a path \( \{ f_\lambda \} \subseteq \mathcal{H}(b) \) is called a Beltrami path if there exists a holomorphic motion \( h_\lambda : \mathbb{C} \to \mathbb{C} \) over \( \mathbb{D}(\lambda_0, r) \), based on \( \lambda_0 \), that provides a hybrid conjugacy between \( f_{\lambda_0} \) and \( f_\lambda \).

The proof of Lemma 5.1 has implied the following corollary:

Corollary 5.1. A continuous path \( \{ f_\lambda \} \subseteq \mathcal{H}(0) \) is a holomorphic path if and only if it is a Beltrami path.

Now we are going to use the homeomorphism \( I_b \circ I_0^{-1} \) to define the path holomorphic structure on the hybrid leaf \( \mathcal{H}(b) \) for each \( b \).
**Definition 5.5** (Path holomorphic structure on \(\mathcal{H}(b)\)). For every \(b \in \mathcal{C}\), a continuous path \(\{f_\lambda\}_{\lambda \in \mathbb{D}} \subset \mathcal{H}(b)\) is a holomorphic path if \(\{\mathcal{I}_b \circ \mathcal{I}_0^{-1}(f_\lambda)\}_{\lambda \in \mathbb{D}}\) is a holomorphic path in \(\mathcal{H}(0)\).

Let \(h_\mathbb{D}(\cdot, \cdot)\) be the hyperbolic metric on \(\mathbb{D}\) and let \(d_\mathbb{D}(\cdot, \cdot) := \frac{e^{h_\mathbb{D}} - 1}{e^{h_\mathbb{D}} + 1}\). By convexity, \(d_\mathbb{D}\) is a metric on \(\mathbb{D}\).

For each hybrid leaf \(\mathcal{H}(b)\), following Avila-Lyubich [1], we define

\[
d_{\mathcal{H}(b)}(f_1, f_2) = \sup_{\phi \in \text{Hol}(\mathcal{H}(b), \mathbb{D})} d_\mathbb{D}(\phi(f_1), \phi(f_2)),
\]

for any \(f_1, f_2 \in d_{\mathcal{H}(b)}\). It is a well-defined Caratheôdory metric on the path holomorphic space \(\mathcal{H}(b)\). (See [1] Theorem 4.2 and Lemma 4.1.)

Since the homeomorphism \(\mathcal{I}_b \circ \mathcal{I}_0^{-1}\) and the renormalization operator \(\mathcal{R}\) map Beltrami paths to Beltrami paths, we obtain:

**Corollary 5.2.** For every \(b \in \mathcal{C}\), a continuous path \(\{f_\lambda\}_{\lambda \in \mathbb{D}} \subset \mathcal{H}(b)\) is a holomorphic path if and only if it is a Beltrami path.

**Lemma 5.2.** For every \(b_1, b_2 \in \mathcal{C}\), the renormalization operator \(\mathcal{R} : \mathcal{H}(b_1) \to \mathcal{H}(b_2)\) is path holomorphic.

Recall that \(\mathcal{I}\) is the set of all the infinitely renormalizable multimodal maps of type \(N\). A hybrid leaf \(\mathcal{H}(b)\) is called real-symmetric if it contains a polynomial-like extension for some multimodal maps of type \(N\). Let \(\hat{\mathcal{I}} = \bigcup_{f \in \mathcal{I}} \mathcal{H}(\chi(f))\), where \(\mathcal{H}(\chi(f))\) is the real-symmetric hybrid leaf containing \(f\). A family \(\mathcal{F} \subset \hat{\mathcal{I}}\) is said to have beau bounds if there exists \(\epsilon_0 > 0\) such that for any \(\delta > 0\) there is a moment \(n_\delta\) so that \(\text{mod}(\mathcal{R}^n f) \geq \epsilon\) for all \(n \geq n_\delta\) and any \(f \in \mathcal{F}\) with \(\text{mod}(f) \geq \delta\). By Theorem 4.1 [1], \(\mathcal{I}\) has beau bounds. Let us restate the following two theorems in [1] to our situation. For more details, we refer the readers to section 6–8 in [1].

**Theorem 5.1.** [1] Theorem 6.2. There exists \(\epsilon_0 > 0\) with the following property. For any \(\gamma > 0\) and \(\delta > 0\) there exists \(N = N(\gamma, \delta)\) such that for any two maps \(f, \bar{f} \in \mathcal{C}_N(\delta) \cap \hat{\mathcal{I}}\) in the same real-symmetric hybrid leaf we have

\[
\mathcal{R}^k f, \mathcal{R}^k \bar{f} \in \mathcal{H}_{b_k}(\epsilon_0), \text{ and } d_{\mathcal{H}_{b_k}}(\mathcal{R}^k f, \mathcal{R}^k \bar{f}) < \gamma, \ k \geq N,
\]

where \(b_k = \chi(\mathcal{R}^k f)\).

**Proof.** For each real-symmetric hybrid leaf \(\mathcal{H}(b)\), we can associate a cocycle \(G = G_b\) with values in \(\text{Hol}(\mathcal{H}(0), \mathcal{H}(0))\) as following: for each \(b \in \mathcal{C}\), let \(\Psi_b := \mathcal{I}_b \circ \mathcal{I}_0^{-1} : \mathcal{H}(b) \to \mathcal{H}(0)\) and we define

\[
G^{m,n}(\Psi_b(f)) := \Psi_{b_{n-m}}(\mathcal{R}^{n-m}(f)),
\]

where \(b_{n-m} = \chi(\mathcal{R}^{n-m}(f))\). Let \(\mathcal{G}\) be the set of all such cocycles which correspond to real-symmetric hybrid leaves. Similar to the unicritical case, one can show that \(\mathcal{G}\) satisfies the hypotheses of [1] Theorem 6.3] (see [1] Lemma 6.4, Lemma 6.5) for an example). Then the Theorem follows from [1] Theorem 6.3.

**Theorem 5.2.** [1] Theorem 5.1. Let \(\mathcal{F} \subset \mathcal{C}_N\) be a family of infinitely renormalizable maps with beau bounds which is forward invariant under renormalization. If \(\mathcal{F}\) is
a union of entire hybrid leaves then there exists \( \lambda < 1 \) such that whenever \( f, \tilde{f} \in \mathcal{F} \) are in the same hybrid leaf, we have

\[
d_{\mathcal{H}_{b_k}}(R^k f, R^k \tilde{f}) \leq C\lambda^k, \quad k \in \mathbb{N},
\]

where \( b_k = \chi(R^k f) \) and \( C > 0 \) only depends on \( \text{mod } f \) and \( \text{mod } \tilde{f} \).

Combine these two theorem, we conclude that

**Corollary 5.3.** For any two \( f, \tilde{f} \in \mathcal{I} \) in the same real-symmetric hybrid leaf, then there exists \( C > 0 \) and \( 0 < \lambda < 1 \) such that for sufficiently large \( k \in \mathbb{N} \),

\[
d_{\mathcal{H}_{b_k}}(R^k f, R^k \tilde{f}) \leq C\lambda^k.
\]

where \( b_k = \chi(R^k f) \) and \( C > 0 \) only depends on \( \text{mod } f \) and \( \text{mod } \tilde{f} \).

Now we are going to prove Theorem B.

**Theorem B.** Let \( \mathcal{I} \) and \( \Sigma \) be as in the assumptions of Theorem A. Then there exists a precompact subset \( A \subset \mathcal{I} \) and a topological semi-conjugacy between \( R|_A \) and a two-sided full shift on \( \Sigma^Z \).

**Proof.** The proof is similar to Avila and Lyubich’s, but for completeness we give a proof here. There exists \( \epsilon_0 > 0 \) with the following property due to the beau bounds for \( \mathcal{I} \). For every real-symmetric polynomial \( P_b \in \mathcal{I} \), we have \( \text{mod } R^k P_b \geq \epsilon_0 \). Given \( \mathcal{M} = (M_k)_{k \in \mathbb{Z}} \), for any \( k_0 < 0 \), there is a unique real-symmetric polynomial \( P_{k_0} \) with combinatorics \( (M_k)_{k \geq k_0} \). Then for any \( l \geq k_0 \), set \( f_{l,k_0} = R^{l-k_0} P_{k_0} \). Clearly, \( f_{l,k_0} \) is infinitely renormalizable with combinatorics \( (M_k)_{k \geq l} \) and \( \text{mod } f_{l,k_0} \geq \epsilon_0 \). By the precompactness of \( C_N(\epsilon_0) \), we may select a subsequence \( k(j) \to -\infty \) such that \( f_{l,k(j)} \) converges to some \( f_l \in C_N(\epsilon_0) \). It follows from Lemma 3.1 that \( f_l \) is infinitely renormalizable with combinatorics \( (M_k)_{k \geq l} \). Using the diagonal procedure (going backwards in \( l \)), we ensure that \( R f_{l-1} = f_l \). Then \( f_0 \) is a bi-infinitely renormalizable map with combinatorics \( \mathcal{M} \) and \( \text{mod } R^k f_0 \geq \epsilon_0 \) for all \( k \in \mathbb{Z} \).

Let us now prove that the \( f_0 \) we constructed in the above paragraph is unique. If \( f_0 \) is another bi-infinitely renormalizable map with combinatorics \( \mathcal{M} \) and \( \text{mod } R^k f_0 \geq \epsilon_0 \) for all \( k \in \mathbb{Z} \). By the combinatorial rigidity, \( R^k f_0 \) and \( R^k \tilde{f}_0 \) are in the same real-symmetric hybrid leaf for all \( k \in \mathbb{Z} \). It follows from Theorem 5.2 that

\[
d_{\mathcal{H}_{b_k}}(R^k f_0, R^k \tilde{f}_0) = d_{\mathcal{H}_{b_k}}(R^l R^{k-1} f_0, R^l R^{k-1} \tilde{f}_0) \leq C\lambda^l, \quad l \in \mathbb{N},
\]

and let \( l \to +\infty \) we get \( R^k f_0 = R^k \tilde{f}_0 \) for all \( k \in \mathbb{Z} \). In particular, \( f_0 = \tilde{f}_0 \).

Now we define \( h : \Sigma^Z \to \mathcal{F} \) such that \( h(\mathcal{M}) = f_0 \) where \( f_0 \) is a bi-infinitely renormalizable map with combinatorics \( \mathcal{M} \). Since the existence and uniqueness of \( f_0 \) has been proven, the map \( h \) is well-defined. Let \( A := h(\Sigma^Z) \), then \( h : \Sigma^Z \to A \) is surjective.

To see the injectivity of \( h \), we assume that if there exist \( \mathcal{M}^1 \neq \mathcal{M}^2 \) such that \( h(\mathcal{M}^1) = h(\mathcal{M}^2) = f \). Then by the definition of \( h \), \( f \) is bi-infinitely renormalizable with combinatorics \( \mathcal{M}^1 \) and \( \mathcal{M}^2 \). Clearly, \( \mathcal{M}^1_k = \mathcal{M}^2_k \) for all \( k \geq 0 \). However, by the injectivity of the renormalization operator \( R \) (see [20], Proposition 2.2)), \( R^k f \) is a singleton for \( k \in \mathbb{Z}_- \) and then \( \mathcal{M}^1_k = \mathcal{M}^2_k \) for all \( k \in \mathbb{Z}_- \). Thus \( \mathcal{M}^1 = \mathcal{M}^2 \).

Finally, the continuity of \( h \) follows easily from Lemma 3.1 and we are done. \( \square \)

To prove Theorem A, it remains to prove the following lemma:

**Lemma 5.3.** If \( f_k \to f \) in \( A \), then \( h^{-1}(f_k) \to h^{-1}(f) \) in \( \Sigma^Z \).
Proof. Let $\mathcal{M} = (\mathcal{M}_l)_{l \in \mathbb{Z}}$ and $\mathcal{M}^k = (\mathcal{M}^k_l)_{l \in \mathbb{Z}}$ be the combinatorics of $f$ and $f^k$ ($k \in \mathbb{N}$) respectively. It suffices to show that for any $l_0 \in \mathbb{N}$, $\mathcal{M}^k_l = \mathcal{M}_l$ for all $-l_0 \leq l \leq l_0$ and $k$ sufficiently large.

Since the periodic points of an infinitely renormalizable map are all repelling, the post-critical set moves continuously in a neighborhood of $f$. Thus it is clear that $\mathcal{M}^k_l = \mathcal{M}_l$ for $0 \leq l \leq l_0$ and $k$ large.

To prove $\mathcal{M}^k_l = \mathcal{M}_l$ for all $-l_0 \leq l < 0$ and $k$ sufficiently large, we just need to prove $R^{-l}f^k \to R^{-l}f$ for $-l_0 \leq l < 0$. By Theorem C, $R^{-l}$ is continuous and we are done. □

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