SUSY, Casimir scaling and probabilistic properties of gluon and quark jets evolution

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Abstract

We study the new relation [1] between the anomalous dimensions, resummed through next-to-next-to-leading-logarithmic order, in the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi evolution equations for the first Mellin moments $D_{q,g}(\mu^2)$ of the fragmentation functions, which correspond to the average multiplicities of hadrons in jets initiated by quarks and gluons, respectively. This relation is shown to lead to probabilistic properties of the properly rescaled parton jet multiplicities obtained from standard ones by extracting the quark and gluon "color charges" $C_F$ and $C_A$, respectively.

The broad and elegant concept of supersymmetry (SUSY) is currently manifested in various branches of physics. For high energies it is pronounced in the properties of QCD supersymmetric extension rather than in the existence of supersymmetric partners. In particular, this corresponds to the SUSY-related properties of evolution kernels [2] discovered some time ago [3]. In the current paper we explore the recently found relation [1] for fragmentation kernels and suggest its probabilistic interpretation, bringing SUSY closer to observations.

The notion of fragmentation functions (FFs) $D_a(x, \mu^2)$ (hereafter $(a = q, g)$), $\mu$ is the factorization scale, was involved during the study of the inclusive production of single hadrons. Their $\mu^2$ dependence is governed by the timelike Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [2, 3].

The DGLAP equations are conveniently solved in Mellin space ($D_a(N, \mu^2) = \int dx x^{N-1} D_a(x, \mu^2)$ ($N = 1, 2, \ldots$) are FF Mellin moments)

$$\frac{\mu^2 d}{d\mu^2} \left( \begin{array}{c} D_s(N, \mu^2) \\ D_g(N, \mu^2) \end{array} \right) = \left( \begin{array}{cc} P_{qq}(N) & P_{gq}(N) \\ P_{gq}(N) & P_{gg}(N) \end{array} \right) \left( \begin{array}{c} D_s(N, \mu^2) \\ D_g(N, \mu^2) \end{array} \right),$$  \hspace{1cm} (1)

where $P_{ab}(N)$ (hereafter $(a, b = q, g)$ ) are anomalous dimensions and $D_s = (1/2n_f) \sum_{q=1}^{n_f}(D_q + D_{\bar{q}})$, with $n_f$ being the number of active quark flavors, is the quark singlet component. The quark non-singlet component is irrelevant for the present study.

The first Mellin moment $D_a(\mu^2) \equiv D_a(1, \mu^2)$ is under special interest. Up to corrections of orders beyond our consideration here, this corresponds to the average multiplicity $\langle n_h \rangle_a$ of hadrons in the jets initiated by parton $a$. Now there are a lot of experimental data on $\langle n_h \rangle_q$, $\langle n_h \rangle_g$, and their ratio $r = \langle n_h \rangle_g/\langle n_h \rangle_q$ for charged hadrons $h$ taken in $e^+e^-$ annihilation at different energies $\sqrt{s}$ of the center of mass, ranging from 10 to 209 GeV (see a list of
references in [4]). The study of $D_a$ contains a long story: the LO value of $r$, $C^{-1} = C_A/C_F$ with color factors $C_F = 4/3$ and $C_A = 3$, was found four decades ago [5,].

Usage of Eq. (1) with $N = 1$ for $D_a$ at fixed order in perturbation theory is problematic: $P_{ba} \equiv P_{ba}(N = 1)$ are ill defined and require resummation, which was performed for the leading logarithms (LL) [11], the next-to-leading logarithms (NLL) [12], and the next-to-next-to-leading logarithms (NNLL) [13].

In Ref. [14] (see also [14]), an unexpected relationship between the NNLL-resummed expressions for $P_{ba}$ has been found. Its existence in QCD is quite remarkable and interesting in its own right, because a similar relationship is familiar [3, 13, 15] from supersymmetric QCD (SQCD), where $C = 1$.

Consider now Eq. (1) for $N = 1$ with NNLL resummation, where [13]

$$P_{aa} = \gamma_0 (\delta_{ag} + K_0^{(1)} \gamma_0 + K_0^{(2)} \gamma_0^2), \quad P_{gq} = C (P_{gg} + A), \quad P_{qq} = C^{-1} (P_{qq} + A), \quad (2)$$

with $O(\gamma_0^3)$ accuracy, where $\gamma_0 = \sqrt{2 C_A a_s}$, $a_s = \alpha_s/(4\pi)$ is the couplant, $\delta_{ab}$ is the Kronecker symbol, and

$$K_0^{(1)} = \frac{2}{3} C, \quad K_0^{(2)} = - \frac{1}{12} \{ 11 + 2 \varphi (1 + 6 C) \}, \quad K_0^{(2)} = - \frac{1}{6} C [ 17 - 2 \varphi (1 - 2 C) ],$$

$$K_0^{(2)} = \frac{1193}{288} - 2 \zeta (2) - \frac{5 \varphi}{72} (7 - 38 C) + \frac{\varphi^2}{72} (1 - 2 C) (1 - 18 C), \quad A = K_0^{(1)} \gamma_0^2, \quad \varphi = \frac{n_f}{C_A}. \quad (3)$$

Eq. (2) is written in a form that allows us to glean a novel relationship (see [11]):

$$P_{qq} + C^{-1} P_{gg} = P_{gg} + CP_{gq}, \quad (4)$$

which is independent of $n_f$.

In SQCD the corresponding relation (i.e. [11] with $C = 1$) exists [3, 13, 15] for the anomalous dimensions $P_{ab}^{SUSY}(N)$ with arbitrary $N$ values [4]:

$$P_{qq}^{SUSY} (x) + P_{gq}^{SUSY} (x) = P_{gg}^{SUSY} (x) + P_{qq}^{SUSY} (x). \quad (5)$$

Beyond LO the property (5) is violated in the standard "dimension regularization" but it survives in the form of the "dimensional reduction" [16] and was used also to check real calculations (see Ref. [17] and discussion therein). It seems that the relation (5) is violated [19] at the NNLO level of accuracy but it needs some additional investigations.

It will be interesting to see if Eq. (4) also holds beyond $O(\gamma_0^3)$ in the case of the "dimensional reduction" [17]. The choice of a scheme in the above consideration is not so important because a difference in the results of various schemes is exactly canceled in Eq. (4).

Following to [20], Eq. (5) can be spelled out as an equality of the total probabilities of "quark" and "gluon" decays. We note that such probabilistic interpretation becomes to be very important directly in QCD [21, 22] for decoupling of orbital and total angular momenta in nucleon.

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1. One should stress that the multiplicities $D_a(\mu^2)$ obey to so-called "Casimir scaling", since their results are given by universal function times the quadratic Casimir operators, i.e. to $C_F$ and $C_A$ for the fundamental and adjoint representations of the color SU(3) group, respectively (see Refs. [6, 9] and discussions therein about the Casimir scaling, which appeared in the 1980s [10] in lattice calculations).

2. In fact it was observed for the splitting functions $P_{ab}^{SUSY}(x)$, which correspond to the $P_{ab}^{SUSY}(N)$ in the Bjorken $x$ space: $P_{ab}^{SUSY}(N) = \int_0^1 dx^{N-1} P_{ab}^{SUSY}(x)$. 

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Following to [20, 21], we can explore the probabilistic properties hidden in Eq. (4). To do it, we introduce new form of the quark $D_s$ and gluon $D_g$ multiplicities

$$D_s(\mu^2) = C_F D_s(\mu^2), \quad D_g(\mu^2) = C_A D_g(\mu^2),$$

where we extract the corresponding ”color charges” $C_F$ and $C_A$, respectively.

The new multiplicities obey to the following DGLAP equations

$$\mu^2 \frac{d}{d\mu^2} \left( \frac{D_s(\mu^2)}{D_g(\mu^2)} \right) = \left( \frac{P_{qq} P_{gg}}{P_{qg} P_{gg}} \right) \left( \frac{D_s(\mu^2)}{D_g(\mu^2)} \right),$$

where

$$P_{aa} = P_{aa}, \quad P_{qg} = C P_{qg}, \quad P_{gq} = C^{-1} P_{qq}$$

and the relation (4) becomes to be as follows

$$P_{qq} + P_{gq} = P_{gg} + P_{qg},$$

i.e. it exactly equals (for $N = 1$) to one in (5) obtained in the SQCD framework.

So, the new parton multiplicities $\overline{D}_a$ have same probabilistic properties as the original ones $D_a$ in the supersymmetric case bringing SUSY closer to observable quantities.

Since the parton multiplicities $\overline{D}_a$ are proportional to the standard ones $D_a$, the solution of the DGLAP equation (7) is the same as one done in Ref. [1] for the equation (1) at $N = 1$: after diagonalization of (7) there are two solutions in the form of so-called ”+” and ”−” components.

The ”−” component $D_−$ can be obtained as the general solution of a homogeneous differential equation. It has the following form [1]

$$\frac{D_−(\mu^2)}{D_−(\mu_0^2)} = \exp \left[ \int_{\mu_0^2}^{\mu^2} \frac{d\mu^2}{\mu^2} \left( \frac{C_F}{3} \beta_0^3 - A \right) \right] = \frac{T_−(\gamma_0(\mu^2))}{T_−(\gamma_0(\mu_0^2))},$$

where

$$T_−(\gamma_0) = \gamma_0^d \exp \left( -\frac{4}{3} C_\gamma \gamma_0 \right), \quad d_− = \frac{8C_A}{3\beta_0} C_\gamma$$

and $\beta_0$ and $\beta_1$ are the first two terms of QCD $\beta$-function:

$$\beta_0 = \frac{C_A}{3}[11 - 2\varphi], \quad \beta_1 = \frac{2C_A^2}{3}[17 - \varphi(5 + 3C)].$$

The ”+” component $D_+$ obeys [1] to the inhomogeneous differential equation. The general solution $\overline{D}_+$ of its homogeneous part is

$$\frac{\overline{D}_+(\mu^2)}{\overline{D}_+(\mu_0^2)} = \exp \left[ \int_{\mu_0^2}^{\mu^2} \frac{d\mu^2}{\mu^2} \gamma_0 \left( 1 + (2K_q^{(1)} + K_g^{(1)})\gamma_0 + (K_q^{(2)} + K_g^{(2)})\gamma_0^2 \right) \right] = \frac{T_+(\gamma_0(\mu^2))}{T_+(\gamma_0(\mu_0^2))},$$

where

$$T_+(\gamma_0) = \gamma_0^d \exp \left[ \frac{4C_A}{\beta_0\gamma_0} - \frac{4C_A}{\beta_0} (K_q^{(2)} + K_g^{(2)} - b_1) \gamma_0 \right], \quad d_+ = -\frac{4C_A}{\beta_0} (2K_q^{(1)} + K_g^{(1)})$$
and \( b_1 = \beta_1/(2C_A\beta_0) \).

Adding to \( D_+ \) a special solution of the inhomogeneous differential equation for \( D_+ \), we find its general solution \[1\]:

\[
D_+(\mu^2) = \left[ \frac{D_+(\mu_0^2)}{T_+(\gamma_0(\mu_0^2))} - \frac{4}{3} C_\varphi \frac{D_-(\mu_0^2)}{T_-(\gamma_0(\mu_0^2))} \int_{\gamma_0}^{\gamma_0(\mu^2)} \frac{d\gamma_0}{1 + b_1\gamma_0^2} \frac{T_-(\gamma_0)}{T_+(\gamma_0)} \right] T_+(\gamma_0(\mu^2)).
\] (15)

Returning to the parton basis, it is useful to decompose \( \overline{D}_a = \overline{D}_a^+ + \overline{D}_a^- \) into the large and small components \( \overline{D}_a^\pm \) proportional to \( D_\pm \), respectively. Defining \( \tau_\pm = \frac{\overline{D}_g^\pm}{\overline{D}_s^\pm} \) and using Eqs. (2), (3), and (13), we then have \( C_F \overline{D}_s^\pm = \mp D_\pm \) and

\[
\tau_+ = 1 + \mathcal{O}(\gamma_0^2), \quad \tau_- = -\frac{4}{3} n_f \gamma_0 + \frac{n_f}{18} [29 - 2\varphi(5 - 2C)] \gamma_0^2 + \mathcal{O}(\gamma_0^3).
\] (16)

Recalling that \( \tau = \overline{D}_g/\overline{D}_s \), we have

\[
\tau = \frac{\tau_+ + \tau_- \overline{D}_s^-/\overline{D}_s^+}{1 + \overline{D}_s^-/\overline{D}_s^+}
\] (17)

So, for the high energy asymptotics (i.e. \( \mu \to \infty \)), where the ”+-”-component strongly dominates, we have for the ratio \( \tau \):

\[
\tau \to \tau_+ = 1,
\] (18)

i.e. the new multiplicities of gluon and quark jets become to be equal in all know orders. This equality corresponds exactly to the Casimir scaling (i.e. to \( D_g^+/D_s^+ = C_A/C_F \)) mentioned above.

We think that the equality \( \tau_+ = 1 \) may be kept up to \( a_s^4 \sim \gamma_0^8 \) accuracy \[3\], where the corresponding splitting functions \( P_{ba} \) would contain Feynman diagrams coming with the quartic Casimir contributions (see similar investigations in Refs. \[6, 24\]).

The existence of the ”-”-component violates the equality (18) between the new multiplicities \( \overline{D}_s \) and \( \overline{D}_g \) already at the accuracy \( O(\gamma_0) \) that is essentially stronger then possible violation due the quartic Casimir contributions. We note that the ratio \( \tau_- \sim n_f \) and thus the equality (18) should be violated in pure gluodynamics essentially slowly, i.e. at \( a_s^4 \sim \gamma_0^8 \) accuracy by contributions of the quartic Casimirs.

We note also that the contribution of the ”-”-component is very important \[1, 4, 25\] for comparison of the theoretical predictions for the parton jet multiplicities with the experimental data, which belongs to the subasymptotic range. Indeed, as it was shown in \[4, 25\], the ”-”-component contribution gives the natural explanation of the longstanding discrepancy in the theoretical description of the data, which reviewed, for example, in Ref. \[26\]. It is also in full agreement with the study \[27\] of low \( x \) asymptotics of parton densities, where the existence of the corresponding ”-”-component leads to a good agreement between theoretical studies \[28\] and the experimental data \[29\] for the structure function \( F_2(x, Q^2) \) of the deep-inelastic scattering obtained by H1 and ZEUS Collaborations.

In summary, we studied the SUSY-like relation \[1\] between the NNLL-resummed first Mellin moments of the timelike DGLAP splitting functions in real QCD, in Eq. (4). In Eq.

\[3\]Note that the contributions from the quartic Casimir contributions may be negligible numerically \[23\] and the “Casimir scaling” may be fulfilled even above \( a_s^4 \sim \gamma_0^8 \) accuracy in approximated form.
we introduced the new quark and gluon jet multiplicities $\bar{D}_a$ which have probabilistic properties, same as for the standard multiplicities $D_a$ in the framework of the supersymmetric extension of QCD.

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