Pricing for Online Resource Allocation: Beyond Subadditive Values

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August 2, 2017

Abstract

We consider the problem of truthful online resource allocation to maximize social welfare in a stochastic setting. Sequential posted pricing has emerged as a desirable mechanism for this problem that is at once simple, easy to implement in practice, as well as approximately optimal in several cases. In this mechanism, the seller uses his knowledge of the demand distribution to determine and announce prices for individual resources or bundles of resources. Buyers then arrive in sequence and can purchase their favorite bundles while supplies last. Previous work shows that sequential posted pricing achieves good approximations when buyers’ values exhibit subadditivity. We consider settings where buyers desire bundles, that is, their values exhibit complementarities, and the seller faces a cost function for supplying the resource. We present both upper and lower bounds for the approximation factors achieved by sequential posted pricing in these settings.

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1 Introduction

Perhaps no economic system for allocating goods to consumers is as ubiquitous as posted prices. As a familiar example, consider a supermarket. The store determines prices for items, which may be sold individually or packaged into bundles. Customers arrive in arbitrary order and purchase the items they most desire at the advertised prices, unless they’re sold out. Many other domains have a similar sequential posted pricing format, from airfares to online retail to concert tickets. But do pricings result in allocations which give the goods to those who value them the most?

Prices are well regarded as market instruments for achieving allocative efficiency. For instance, when buyers’ values are publicly known and satisfy the so-called gross-substitutes condition, there always exists a Walrasian equilibrium—prices that clear the market and maximize welfare [24]. However, we are interested in settings where the prices are set by a benevolent monopolist who has incomplete knowledge of buyers’ values and no control over the order in which buyers arrive. A simple well-known example shows that in this case the allocation may not be perfectly efficient. Suppose that there is one item to sell and two buyers. The buyer that arrives first has a value of 1 for the item. The second buyer has a value of $1/\varepsilon$ with probability $\varepsilon$, and 0 otherwise. The optimal allocation achieves a social welfare of $2 - \varepsilon$ in expectation over the buyers’ values, however, any price placed on the item will result in an allocation with expected welfare at most 1, a gap of 2 from the optimum. For the case of a single item, this is the worst gap possible. What about more general settings with multiple items and buyers with complex preferences? What if the seller has a production cost for the item(s)?

These questions are at the heart of our work. Although sequential posted pricings have generated a robust literature within algorithmic mechanism design, the results primarily consider maximizing the seller’s revenue. Recently, elegant results of, e.g., Feldman et al. [17] have highlighted the power of pricings to maximize welfare. Yet a rich set of applications lies largely beyond the purview of that work. We investigate these questions, thereby aiming for a more complete understanding of the power and limitations of posted pricings for online resource allocation.

Application: job scheduling. The settings we consider are inspired by the classic job scheduling problem, in which jobs with various sizes must be assigned to a set of machines to maximize the total value (or weight) of assigned jobs. The problem of pricing cloud resources represents an immediate application. Following previous work [4, 6, 13], we consider a model in which a cloud provider has multiple copies of a single resource to allocate over time. Each buyer has a job that requires the use of the resource for some number of time steps. Buyers’ values for obtaining the resource may depend on when the resource is allocated to them. For example, some buyers may have a hard deadline by which their job must be completed, while others’ values may degrade steadily over time. Chawla et al. [13] proposed a simple pricing-based market mechanism for this problem. The seller announces in advance a price per unit of resource for each time unit, which may vary over time. Each buyer then considers available blocks of time that meet their requirements, and purchases the one that maximizes their value minus the price.

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1 The efficiency, a.k.a. social welfare, of an allocation is the sum over all buyers of the value each buyer derives from her allocation.

2 In economics literature, monopolies are usually assumed to be interested in maximizing their revenue. However, maximizing efficiency of allocation, a.k.a. social welfare, makes sense if the monopolist is an entity with social objectives such as the government. Alternately, it can be a proxy for maximizing market share and longer term revenues when there are multiple sellers in the market.
The recent push towards “time-of-use” (TOU) pricing in electricity markets provides another important and timely application. Traditionally, electricity has been sold at a flat rate based on usage. TOU pricing offers a means to better manage demand fluctuations and achieve more efficient utilization: prices are set higher or lower than normal during periods when very high or very low demand is expected, thereby incentivizing customers to move temporally flexible demand from peak to non-peak periods. To what extent can TOU pricing counter stochasticity in demand and achieve efficient allocation? As power grids increasingly utilize renewable sources of energy, the varying costs associated with different sources become increasingly significant. Do costs and supply fluctuations hurt the effectiveness of TOU pricing?

Simplicity versus optimality. These questions are closely related to a central theme in algorithmic mechanism design that studies the tradeoff between simplicity and optimality (see, e.g., [19, 15]). For welfare maximization, the optimal VCG mechanism has several desirable properties, such as deterministic outcomes and dominant strategy truthfulness, and yet is rarely observed in practice. The class of mechanisms we consider, sequential posted price mechanisms or SPMs, offers an attractive alternative. In an SPM, buyers observe prices before determining which bundle of items to buy. Therefore, these mechanisms are trivially incentive compatible in addition to being simple to understand and implement. Depending on the context, the prices can be static or adaptive (meaning they may change as supply declines), anonymous or non-anonymous (meaning they may vary across different buyers), and the seller may price individual items or bundles of items. We focus for the most part on static, anonymous pricings. These are particularly appealing in contexts where buyers arrive in the market in an online fashion, and indeed are the most prevalent form of posted pricings seen in practice.

Pricings and prophet inequalities. The single item allocation problem described earlier is closely related to so-called prophet inequalities [28, 21] from optimal stopping theory. There is much work on prophet inequalities under various feasibility constraints and objectives. (See, e.g., [11, 20, 25, 26].) Each such result directly implies an approximation to social welfare via SPMs for the corresponding resource allocation settings. However this connection is limited to settings where each buyer is single-valued, a.k.a. “single-parameter”, and in particular does not capture settings where buyers may have different values for different items or sets of items. Feldman et al. [17] were the first to extend these results to multi-parameter settings. They considered the setting of a seller with multiple items and buyers with XOS (or fractionally subadditive) valuations over sets of items. They showed that SPMs with static anonymous prices can obtain a 2-approximation to the expected optimal social welfare in this setting. Notably, Feldman et al.’s mechanisms are item pricings, i.e. they only price individual items, and are therefore particularly simple.

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3For example, the California Public Utilities Commision has proposed transitioning all residential electricity customers to time-of-use rates by 2019 [1, 2].

4Among other “problems”, the VCG mechanism’s outcome is unpredictable from the point of view of individual buyers without complete knowledge of their competitors’ values, and in this sense, the mechanism is not “obviously” strategyproof [23]. Furthermore, VCG is essentially an offline mechanism: it needs all of the buyers to participate simultaneously rather than in an online fashion in order to determine the allocation and payments. Finally, the incentive properties of the VCG mechanism do not always extend to settings where computing the optimal allocation is computationally intractable.

5The approximation guarantees of SPMs also depend on whether the buyers arrive in random order, or a seller-prescribed order, or adversarial order. In this paper, we focus on worst-case or adversarial arrival order.
Dealing with complementarities. In the applications we study, buyers' valuations exhibit complementarity. For example, suppose a buyer in the job scheduling setting has a job of length 10, that is it requires the resource for 10 units of time to complete. Then, allocating any fewer than 10 units to the buyer brings him 0 value. There are few results known in this case. Chawla et al. [13] studied the cloud pricing problem, focusing on the special case where each buyer has a time window and a length for his job, and gets a fixed positive value if and only if the job runs to completion within this time window. For this special case they showed that if the supply in each time period is large enough ($\tilde{O}(L^6/\varepsilon^3)$ for jobs up to length $L$), item pricings obtain a $1-\varepsilon$ approximation to social welfare. Feldman et al. [17] showed that in the absence of any supply assumptions, when buyers' valuations exhibit complementarities of up to size $L$, SPMs that price individual items can obtain a factor of $O(L)$ approximation to social welfare but no better. Is it possible to beat this factor by using more general schemes that price bundles?

1.1 Challenges, techniques, and our results

We consider settings where each buyer has arbitrary (correlated) valuations over sets of size up to $L$ for some parameter $L > 0$. For the most part we focus on the job scheduling setting where items are ordered and buyers value intervals. Our results for this setting are summarized in Table 1. We now discuss our techniques and findings.

We begin by reviewing the by now standard argument for proving prophet inequality style results, as typified in the work of Feldman et al. [17]. The social welfare of an allocation can be accounted for in two components—the seller’s share of the welfare, namely his revenue, and the buyers’ share of the welfare, namely their utility. Feldman et al. seek prices that are high enough so that if an item gets sold it generates enough revenue for the seller, and yet low enough so that if an item does not get sold then it generates enough utility for the buyer that is assigned the item in the optimal solution. When buyers have XOS (fractionally subadditive) valuations, it is possible to find prices such that the revenue and utility terms exactly balance out (each being half of the optimal social welfare) and we obtain a 2-approximation to the optimal social welfare. The beauty of this argument is that it depends only on whether or not an item has sold at the end of the SPM, and not on how the process plays out. It is therefore agnostic to the order of arrival of the agents.

For settings with complementarities, one might expect that the same approach works with bundle pricings, say by thinking of each bundle as a meta-item. The trouble is that meta-items share supply. A buyer arriving in the market may be unable to purchase a meta-item (i.e., bundle) $S$ because prior sales of other overlapping meta-items have depleted the supply of some item in $S$. The loss of the buyer’s utility in this event may not be adequately covered by the revenue generated by the other overlapping sales. This is not a hypothetical issue. Indeed we construct an example with buyers that desire bundles up to size $L$ in which bundle pricings cannot obtain any better than an $O(\log L / \log \log L)$ approximation. This holds also in the special case of the job scheduling setting, and even if buyers are single-minded, that is, each buyer desires a particular interval known to the seller in advance.

A partition into meta-items. To deal with the issue of shared supply, we ask whether there is a partition of the multiset of all items into meta-items that satisfies the following properties: (1)
Table 1: Old and new results at a glance. Here buyers desire intervals. “Bundle size” refers to the largest bundle desired by any buyer. (*) indicates that the pricing is adaptive, and otherwise it is static. $\alpha$ is the ratio of the maximum item supply to minimum item supply, and values are assumed to lie in the range $\{0\} \cup [1, \alpha_v]$. 

| Bundle Size | Supply       | Pricing         | Upper Bound                          | Lower Bound                          |
|-------------|--------------|-----------------|--------------------------------------|--------------------------------------|
| 1           | any item     | $O(L)$ [17]     | $\Omega(L)$ [17]                     | $\Omega(L)$ [17]                     |
| uniform     | item         | $O(\log L)$ (Cor 4.2) | $\Omega(\log L / \log \log L)$ (Thm 3.2) | $\Omega(\log L / \log \log L)$ (Thm 3.2) |
|             | partition    | $O(\log L)$ (Thm 3.7) | $\Omega(\log L / \log \log L)$ (Thm 3.2) | $\Omega(\log L / \log \log L)$ (Thm 3.2) |
|             | bundle       | $O(\log L)$ (Thm 3.7) | $\Omega(\log L / \log \log L)$ (Thm 3.2) | $\Omega(\log L / \log \log L)$ (Thm 3.2) |
| non-uniform | item         | $O(L)$ [17]     | $\Omega(L)$ [17]                     | $\Omega(L)$ [17]                     |
|             | partition    | $O(\log L \min\{\log \alpha_b, \log \alpha_v\})$ (Thm 4.5) | $\Omega(\log L / \log \log L)$ (Thm 3.2); $\Omega(\min\{\log \alpha_b, \log \alpha_v\})$ (Thm 4.1) | $\Omega(\log L / \log \log L)$ (Thm 3.2) |
|             | bundle       | $O(\log L)$ (Thm 3.7) | $\Omega(\log L / \log \log L)$ (Thm 3.2) | $\Omega(\log L / \log \log L)$ (Thm 3.2) |
| costs       | partition (*) | —               | $\Omega(L^{1/4})$ (Thm 4.6)          | —                                    |
|             | bundle (*)   | $O(\log L)$ (Thm 3.8) |                                       | —                                    |

no two meta-items share supply; (2) the partitioning is independent of the instantiation of buyers’ values; and (3) for each instantiation of buyers’ values, it is possible to generate an assignment of meta-items to buyers so that each buyer gets at most one meta-item, and not much social welfare is lost in expectation. If such a partitioning exists, it immediately implies a bundle pricing with a good approximation: we simply price each bundle corresponding to a meta-item just as Feldman et al.’s argument prices items; the only additional loss in welfare is due to point (3) above. The main component in our bundle pricing results is then to create meta-items satisfying the above properties. We obtain a logarithmic approximation nearly matching our lower bound.

**Pricing partitions.** Whereas item pricings are too weak in generating good social welfare, bundle pricings in their full generality can be fairly complex, offering exponentially (in the number of items) many bundles at different prices (or quadratically many when all the bundles are intervals). We therefore study the performance of **partition pricings** where the seller partitions the items into non-overlapping bundles and prices these bundles. In the case of job scheduling, this corresponds to pricing non-overlapping intervals. Partition pricings have simple menus, allow easy decision-making on part of the buyers, as well as easy inventory-keeping on the part of the seller. We show that partition pricings are weaker than general bundle pricings, but only by another logarithmic factor. These bounds require further ideas beyond the meta-items approach described above.

**The effect of costs.** When the seller faces costs on the items (with social welfare now equal to the total value of the allocation minus the cost of items allocated), partition pricings no longer perform

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7We note that for the Feldman et al. style argument to work, it does not matter how many meta-items a buyer desires or eventually buys in the mechanism. All that matters is that the near-optimal solution we are comparing against allocates at most one meta-item to each buyer—this allows the charging of lost utility against revenue.

8E.g., it is hard to imagine such a mechanism being implemented for the electricity pricing setting described previously.
well, even with adaptive prices, exhibiting a polynomial in $L$ gap from the optimum. However, we
can recover a logarithmic approximation using adaptive bundle pricings. We leave open the question
of whether such an approximation is possible using static bundle prices. We show, however, that
for unit-demand buyers static item pricings continue to obtain a 2 approximation. In fact, even
though different units of the same item can cost the seller different amounts, our pricing charges
the same price (up to an artificial supply constraint) for these units.

**Extensions to more general settings.** Is it possible to extend the above logarithmic approx-
imations to settings where buyers desire arbitrary bundles (not just intervals) of length up to $L$?
We believe that such a result, if possible, would require radically new techniques. As evidence, we
present an example (see Theorem 3.1) in which the gap between the welfare obtained by any SPM
and a fractional relaxation of the optimal welfare is a factor of $\Omega(L)$.\(^9\) The fractional relaxation
we use enforces supply constraints ex-ante instead of ex-post. We observe that the above approx-
imation arguments—and indeed, to the extent of our knowledge, all arguments known to date for
prophet inequality style results—apply equally well against this fractional relaxation. This lower
bound holds even if the seller is allowed to change bundle prices adaptively as supply decreases.

Finally, we conjecture that SPMs perform better and better as item supply increases. All of
our lower bound examples enforce a supply of 1. On the other hand, with $\Omega(L^2)$ units of each item
available in the job scheduling setting, it becomes possible to entirely separate supply constraints
across bundles, leading to a constant approximation via bundle pricings.\(^10\) Understanding how the
approximation factor changes as supply ranges between 2 and $L^2$ is an interesting open problem.

**1.2 Further related work**

SPMs were first studied for the problem of revenue maximization, where computing the optimal
mechanism turns out to be a computationally hard problem and no simple characterizations of
optimal mechanisms are known. A series of works [10, 7, 11, 12, 5, 27, 9, 8] showed that in settings
where buyers have subadditive values, SPMs achieve constant factor approximations to revenue.
In most interesting settings, good approximations to revenue necessarily require non-anonymous
pricings. As a result, techniques in this literature are quite different from those for welfare.

For welfare maximization, following the work of Feldman et al. [17], Duetting et al. [16] de-
veloped a general framework for obtaining approximations to social welfare through SPMs. They
show that if prices can be set in the full-information setting to meet certain balance conditions,
then this leads to an approximation in the Bayesian setting. However, this approach necessarily
leads to SPMs that are non-anonymous and adaptive, that is, the pricing offered to a buyer depends
both on the identity of the buyer and previous sales. As such their approach does not apply to the
problems we study.

There is a long line of work on mechanisms for the job scheduling setting [22, 18, 14, 3, 4, 13],
but with the exception of [13], this work considers the worst-case setting where the seller has no
knowledge about the buyers’ preferences.

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\(^9\)The example we construct has an “integrality gap” of $\Theta(L/\log L)$, so it only displays an $\Omega(\log L)$ gap between
SPMs and optimal integral assignments.

\(^10\)Also, Chawla et al. [13] show that when the supply is large enough, item pricings exhibit near-optimal perfor-

mance.
2 Model and definitions

We consider a seller with \( T \) items, indexed by \( t \in [T] \). In settings with costs, we use \( c(t, i) \) to denote the cost of the \( i \)th copy of item \( t \); we will typically abbreviate this as \( c_{ti} \). \( C_t(i) = \sum_{i'=1}^t c_{ti'} \) is the cumulative cost of the first \( i \) copies. In settings without costs, equivalently when the costs are 0 or \( \infty \), we denote the number of units of supply of item \( t \) by \( B_t = \text{arg\,min}\{i : c_{ti} < \infty\} \); let \( B = \{B_t\}_{t \in T} \) denote the vector of item supplies. Let \( B_{\text{max}} = \max_t B_t, B_{\text{min}} = \min_t B_t, \) and \( \alpha_b = B_{\text{max}}/B_{\text{min}} \).

There are \( n \) buyers, indexed by \( j \in [n] \). Buyer \( j \)'s valuation is denoted \( v_j \) and is drawn (independently of other buyers’ values) from distribution \( F_j \). The buyer’s value for a set \( S \subseteq T \) of items is then given by \( v_j(S) \). Observe that \( F_j \) is a joint distribution over \( 2^T \) dimensional vectors. We assume that each buyer’s value for each set lies in the range \( [0, \alpha_v] \). Let \( \bar{v} \) denote the vector of all values and \( F \) denote the joint distribution \( \times_j F_j \).

We say that a buyer \( j \) is single-minded if for all \( v_j \) in the support of \( F_j \), there exists a set \( S \subseteq T \) such that for all \( S' \subseteq T, v_j(S') = v_j(S) \) if \( S' \supseteq S \), and \( v_j(S') = 0 \) otherwise. We say that a buyer \( j \) is unit-demand if for all \( v_j \) in the support of \( F_j \) and for all \( S \subseteq [T], v_j(S) = \max_{S \subseteq S \subseteq S} v_j(S) \).

We next describe the job scheduling setting that we also call the interval demands setting. We use \([a, b]\) to denote the set (interval) \( \{a, a+1, \cdots, b\} \) of items for \( 1 \leq a \leq b \leq T \). We say that a buyer \( j \) desires intervals if the buyer’s value for a set of items depends on the value of the maximum-value interval in that set. Formally, we have for all \( v_j \) in the support of \( F_j \) and for all \( S \subseteq [T], v_j(S) = \max_{[a, b] \subseteq S} v_j([a, b]) \).

A feasible allocation \( X = \{X_j\} \) is an assignment of items, \( X_j \) to buyer \( j \) for \( j \in [n] \), such that for all \( t \in T, |\{j : X_j \ni t\}| \leq B_t \). The social welfare achieved by an allocation \( X \) is given by the sum of buyers’ values for their respective assignments minus the associated costs:

\[
\text{SW}(X, \bar{v}) = \sum_j v_j(X_j) - \sum_t C_t(|\{j : X_j \ni t\}|).
\]

We drop the argument \( \bar{v} \) when it is clear from the context. We use \( \text{OPT} \) to denote the optimal social welfare. This is the quantity relative to which we measure the performance of our mechanisms.

\[
\text{OPT} = E_{\bar{v} \sim F} \left[ \max_{\text{feasible } X} \text{SW}(X, \bar{v}) \right].
\]

We also consider a fractional (ex-ante) relaxation for the optimal social welfare. An allocation function \( X(\bar{v}) \) is fractionally (or ex-ante) feasible if for all \( t \in T, E_{\bar{v} \sim F} \left[ |\{j : X_j(\bar{v}) \ni t\}| \right] \leq B_t \). The fractional optimum is then defined to be

\[
\text{FracOpt} = \max_{\text{ex-ante feasible } X} E_{\bar{v} \sim F} \left[ \text{SW}(X(\bar{v}), \bar{v}) \right].
\]

Posted price mechanisms

We begin by describing SPMs with static anonymous prices. A static SPM is denoted by \((p, M)\) where \( p \) is a price vector and \( M \subseteq 2^T \) is a “menu” of bundles of items. We use \( p_S \) to denote the price of a bundle \( S \in M \). The mechanism proceeds as follows.

1. The seller announces the menu and pricing \((p, M)\). 

2. Buyers arrive in an arbitrary sequence.

3. When buyer \( j \) arrives she purchases a collection \( \kappa \subseteq \mathcal{M} \) of bundles maximizing \( v_j(\bigcup_{S \in \kappa} S) - \sum_{S \in \kappa} p_S(\mathcal{B}) \). The seller updates the remaining supply of items, \( \mathcal{B} = \mathcal{B} - 1_{\kappa} \), and depending on the supply, removes some subset of the bundles from the menu \( \mathcal{M} \).

Observe that the prices in the above mechanism are static and do not change as supply decreases, however, the collection of bundles offered can become smaller and smaller over time as items are sold. We are intentionally vague about how the seller restricts the supply of bundles. For our upper bounds, we design pricings where the seller offers a certain number of copies of each bundle and removes the bundle from the menu once those copies are bought. For our lower bounds, we allow the seller to impose a more general supply constraint, for example, for two overlapping bundles \( S_1 \) and \( S_2 \), the seller may offer up to two copies of each and up to three copies of both collectively.

We next define adaptive SPMs. Such a mechanism is again given by the pair \((p, \mathcal{M})\) where \( p \) is now a function that maps supply to bundle prices. We use \( p_S(\mathcal{B}') \) to denote the price of a bundle \( S \in \mathcal{M} \) when the supply vector is \( \mathcal{B}' \). The mechanism proceeds as follows.

1. The seller announces the menu and pricing \((p, \mathcal{M})\).

2. Buyers arrive in an arbitrary sequence.

3. When buyer \( j \) arrives she purchases a collection \( \kappa \subseteq \mathcal{M} \) of bundles maximizing \( v_j(\bigcup_{S \in \kappa} S) - \sum_{S \in \kappa} p_S(\mathcal{B}) \). The seller updates the remaining supply of items: \( \mathcal{B} = \mathcal{B} - 1_{\kappa} \), and for all \( t \in T \) with \( B_t = 0 \) and \( S \ni t \), removes \( S \) from \( \mathcal{M} \).

Observe that the allocation given by the above mechanisms is always feasible. Both mechanisms are deterministic, and trivially dominant strategy incentive compatible as long as the buyer has no control over when he arrives.\(^{11}\) We overload notation and use \( \text{SW}(p, \mathcal{M}) \) to denote the expected social welfare achieved by the above mechanisms, in expectation over \( \vec{v} \) drawn from \( F \).

We do not consider non-anonymous pricings in this paper.

**Item and partition pricings**

In general SPMs price arbitrary overlapping bundles of items and can have exponentially large menus. We consider special cases where the menus are small and simple. A **partition pricing** is an SPM \((p, \mathcal{M})\) where for all \( S, S' \in \mathcal{M}, S \cap S' = \emptyset \). In other words, a partition pricing prices non-overlapping bundles of items, and as a result \( |\mathcal{M}| \leq |T| \). We emphasize that the partition of items into non-overlapping bundles is done once prior to the arrival of the buyers, and is maintained throughout the mechanism. An **item pricing** is a further simplification where all sets \( S \in \mathcal{M} \) are singletons, i.e. \( |S| = 1 \). In other words, the seller places prices on individual items and the price of a bundle is simply the sum of prices of the individual items in the bundle.

**Partition pricings with variable supply.** In selling bundles, when different items are available in different quantities, we may run out of some items within a bundle earlier than the others. In this case, for static partition pricings, instead of removing the entire bundle from the menu, we allow the seller to sell whatever items remain at the same price as before. In this manner, the pricing remains static but allows us to fully utilize item supply without undue complexity. We continue to call such a pricing a static partition pricing.

\(^{11}\)Arriving earlier can be better.
3 The power of bundle pricings

In this section, we give upper and lower bounds on the approximation ratio achieved by bundle pricing for the objective of maximizing social welfare. A key parameter that determines these bounds is the maximum size of the bundles that buyers desire, call it $L$. We first observe that when buyers have non-interval valuations, bundle pricings can perform very poorly: they exhibit a linear in $L$ gap against the fractional (ex-ante) relaxation and logarithmic in $L$ against the optimal welfare.

**Theorem 3.1.** There exists an instance with single-minded buyers with general (non-interval) valuations over bundles up to size $L$, such that for any SPM $(p, \mathcal{M})$ we have

$$\text{FracOpt} = \Omega(L) \text{SW}(p, \mathcal{M}) \quad \text{and} \quad \text{Opt} = \Omega(\log L) \text{SW}(p, \mathcal{M}).$$

These gaps arise from a family of instances in which buyers are single-minded and each item has a single copy available. The bundles desired by buyers overlap in such a manner as to impose the following feasibility constraint: there is a partitioning of buyers into groups such that it is feasible to allocate to any subset within a particular group, but no two buyers from different groups can be simultaneously allocated. This is a familiar kind of gap example for sequential mechanisms. We present the details in Section 5.

We then focus for the remainder of the section on the job scheduling setting where buyers have interval valuations. Section 3.1 presents an $\Omega(\log L / \log \log L)$ lower bound on the performance of static bundle pricings in this setting. Given these lower bounds, we aim for a logarithmic upper bound via bundle pricings. At a high level, our approach is to transform the given problem into an item pricing problem by partitioning the multiset of items into appropriate bundles and selling each such bundle as a single “meta-item”. There are several details that go into this approach. First, in order to obtain a good approximation, we leverage known results for item pricing for “unit demand” buyers. Restricting each buyer to having unit-demand preferences over meta-items appears to greatly limit our selection of meta-items. Instead, we show in Section 3.2 that the unit-demand item pricing results and arguments extend to arbitrary valuation profiles as long as the performance of the item pricing is measured against an optimum that allocates at most one item to each buyer; we call such a solution the unit allocation optimum or $\text{UAOpt}$. In Section 3.3 we then present a partitioning into meta-items as well as a unit allocation algorithm over them that loses only a logarithmic factor in social welfare. Putting these together implies a logarithmic upper bound on the approximation factor achieved by bundle pricings (Section 3.4).

For the setting with costs, we need to be a little careful in our “reduction” from pricing bundles to pricing individual meta-items. In the absence of costs, we can leverage free disposal of items and allocate meta-items to buyers that are strict supersets of their desired bundles. This gives us more freedom in partitioning the multiset of items into meta-items. With costs, however, we need to be careful not to allocate items for which the buyer does not have sufficient value to cover the cost to the seller. In Section 3.5 we show how to recover a logarithmic approximation by pricing bundles adaptively.

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12While item pricing also provides good approximations in many other subadditive settings, these don’t seem directly relevant to our approach.
Figure 1: The bundles demanded by the buyers in the proof of Theorem 3.2, which shows a logarithmic gap between the optimal allocation and that achievable by a bundle pricing. A buyer demanding a bundle of size \( \ell \) has value drawn from \( \{0, \ell\} \). An optimal solution allocates every item for which some buyer has nonzero value; this is achieved by, e.g., allocating bundles greedily from the bottom up. In the pricing setting, bundles are allocated greedily from the top down (subject to willingness to pay).

### 3.1 Lower bound for static bundle pricings with interval valuations

Next we show that even for buyers with interval valuations, bundle pricings cannot obtain a better than logarithmic approximation if prices are required to be static.

**Theorem 3.2.** There exists an instance with single-minded buyers with interval valuations over bundles up to size \( L \), such that for any static bundle pricing \((p, M)\) we have

\[
\text{OPT} = \Omega(\log L / \log \log L) \text{SW}(p, M).
\]

**Proof.** We construct a family of instances parameterized by an integer \( b > 0 \). Each instance in the family has one unit each of \( L = 2^b \) items. There are \( 2L - 1 \) single-minded buyers with interval values. We may therefore identify buyers with the intervals they desire.

The buyers are divided into \( b + 1 \) groups, each corresponding to a particular interval length, as shown in Figure 1. Group \( b \) consists of a single buyer interested in buying the entire interval \([1, L]\) of items. Group \( b - 1 \) contains two buyers, one interested in the interval \([1, L/2]\), and the other in the interval \([L/2 + 1, L]\). More generally, group \( j \) contains \( 2^{b-j} \) buyers. The \( i \)th buyer in the group, denoted \((j, i)\), is interested in the interval \([(i - 1)2^j + 1, i2^j]\). Each buyer in group \( j \) has a value of \( 2^j \) for his corresponding interval with probability \( 1/b \), and value 0 otherwise. Observe that the buyers in each group partition the set of \( L \) items equally among themselves. Furthermore, the intervals desired by the buyers form a laminar family, that is, for any two intervals that overlap, one fully contains the other.

**Optimal welfare.** Consider the following allocation. Once the buyers’ values are instantiated, we consider the set of buyers with non-zero values for their respective intervals; call this set \( Y \).

We consider buyers in non-increasing order of interval length and allocate their interval to them if it is still available. It is easy to see that if an item is desired by at least one buyer in the set \( Y \), then it will be allocated. Furthermore, since the value of each buyer in \( Y \) is equal to the number of items allocated to them, each allocated item generates a value of 1. Therefore, the expected
optimal social welfare is

$$\text{OPT} = \sum_{t \in [T]} \Pr[t \text{ is allocated}] = \sum_{t \in [T]} \left\{ 1 - \left(1 - \frac{1}{b}\right)^{b+1} \right\} \geq \left(1 - \frac{1}{e}\right) \cdot L = \Theta(L).$$

Here the second equality follows by recalling that $t$ is contained in the interval of one buyer in each group, for a total of $b + 1$ buyers, and each has a $1/b$ chance of being included in the set $S$.

The welfare obtained by a bundle pricing. We assume that buyers arrive in the market in the order of increasing interval length, i.e. group 0 arrives first and group $b$ arrives last. Within each group the order can be arbitrary. We may assume without loss of generality that no bundles are sold at a price of 0. Given any static bundle pricing, it suffices to focus on the set of buyers which could afford to buy a subset of bundles covering their interval, conditioned on having a non-zero value and on the bundles being available. Let $Y$ be this set of buyers. Observe that $Y$ is determined by the pricing, i.e. before buyers’ values are realized, and that buyers within the set $Y$ may have overlapping bundles.

Each buyer $(j, i)$ in $Y$ is served if two conditions are satisfied: (1) the buyer’s value is non-zero, and (2) the buyer is not “blocked” by buyers who have been served previously. To simplify our analysis, we will allow the SPM to account for the value of a buyer $(j, i)$ as long as this buyer is not blocked by another buyer whose interval is fully contained in $(j, i)$’s interval.

Formally, let $A_{j,i}$ be an indicator variable denoting whether buyer $(j, i)$ belongs to the set $Y$. Let $Z_{j,i}$ be the number of buyers belonging to $Y$ whose intervals are fully contained in buyer $(j, i)$’s interval. If the realized value of any of the buyers in this set is non-zero, that buyer (or one of its predecessors) blocks $(j, i)$. (By assumption, a buyer in $Y$ can afford to purchase its desired interval if its value is non-zero.) Therefore, a necessary condition for $(j, i)$ to contribute to the social welfare is for each of the buyers included in the count $Z_{j,i}$ to have a value of 0.

Thus, the contribution of $(j, i)$ to the social welfare of the SPM is at most $q \cdot A_{j,i} \cdot (1-q)^{Z(j,i)} \cdot 2^j$ where $q := 1/b$. Summing over all buyers we get:

$$\text{SW} \leq \sum_{j=0}^b L/2^j \sum_{i=1}^L q \cdot A_{j,i} \cdot (1-q)^{Z(j,i)} \cdot 2^j$$

In order to simplify this sum, we first partition the buyers according to the scale of their $Z(j,i)$. Specifically, let $S_\ell = \{(j, i) : Z(j,i) \in [2^{\ell-1}, 2^\ell)\}$ for $1 \leq \ell \leq b + 1$. Then we have,

$$\text{SW} \leq \sum_{\ell} \sum_{(j,i) \in S_\ell} q \cdot A_{j,i} \cdot (1-q)^{Z(j,i)} \cdot 2^j \leq q \sum_{\ell} (1-q)^{2^{\ell-1}} \left( \sum_{(j,i) \in S_\ell} A_{j,i} \cdot 2^j \right)$$

We then employ the following lemma to bound the inner sum. We postpone its proof to Section 5.

**Lemma 3.3.** For any $\ell \geq 0$ it holds that, $\sum_{(j,i) \in S_\ell} A_{j,i} \cdot 2^j < 2L$.

Applying the lemma we continue with upper-bounding the welfare:

$$\text{SW} \leq q \sum_{\ell=0}^{b+1} (1-q)^{2^{\ell-1}} \cdot 2L$$
Recall that $q = 1/b$. In the sum above, the first $\log b$ terms are significant and contribute no more than 1 each. The remaining terms decrease geometrically and are easily bounded by a constant. Therefore, the entire sum can be bounded:

$$\text{SW} = O\left(\frac{1}{b} L \log b\right) = O\left(\frac{L \log \log L}{\log L}\right).$$

This concludes the proof.

### 3.2 Unit allocations and item pricings

In this section, we consider a setting where buyers’ valuations are completely arbitrary functions of bundles of items, and items have costs. Our goal is to argue that item pricings perform well relative to optimal allocations that allocate only singleton items to buyers. We begin by defining the unit allocations optimum or $\text{UAOpt}$.

An allocation $X$ is said to be $\text{UA}$-feasible if in addition to being feasible with respect to item supply, it satisfies $|X_j| \leq 1$ for all buyers $j$. We then have

$$\text{UAOpt} = \mathbb{E}_{\vec{v} \sim F}\left[\max_{\text{UA-feasible} \ X} \text{SW}(X, \vec{v})\right].$$

The main result of our section is the following lemma which shows that item pricings are 2-approximate for social welfare with respect to $\text{UAOpt}$. Our result has the additional useful property that the pricings we construct are static: for items with supply greater than 1, each copy of the item is sold at an identical price, even if different copies have different costs. For an item $t$ priced at $p_t$, we then sell up to $i$ copies where $i = \arg\max\{i' : c_{it'} \leq p_t\}$.

Our proof follows the approach of [17], although coming up with static prices as described above requires additional ideas. We cannot, in particular, treat each copy of an item as a different item altogether. We state the result but defer its proof to Section 5.

**Lemma 3.4.** For any value distribution $F$ and nondecreasing costs $\vec{c}$, there exist static item prices $p$ such that $\text{SW}(p) \geq \frac{1}{2} \text{UAOpt}$.

We obtain the following immediate corollary, since for unit-demand buyers, $\text{UAOpt} = \text{Opt}$. This extends Feldman et al. [17]’s result for unit-demand buyers to settings with costs.

**Corollary 3.5.** For any unit-demand value distribution $F$ and nondecreasing costs $\vec{c}$, there exist static item prices $p$ such that $\text{SW}(p) \geq \frac{1}{2} \text{Opt}$.

### 3.3 A “reduction” to unit allocations over meta-items

In this section we will start with an instance of the job scheduling setting where buyers have interval valuations. We will then argue that there exists a partition of the multiset of all items into meta-items such that a unit allocation over these meta-items achieve social welfare within an $O(\log L)$ factor of the optimal social welfare. We begin by partitioning items into “segments” and then partitioning the multi-set in each segment into “layers”. Each layer then forms a meta-item.

**Segments.** Let $S = \{S_1, S_2, \ldots, S_M\}$ denote a partition of the items into disjoint intervals: $[T] = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_M$. Each interval $S_m$ is called a segment and $S$ is called a segmentation.
Layers. Consider a single segment $S_m$. We will now partition this segment into layers. Each layer $i$ corresponds to the $i$th copies of the items in $S_m$. In particular, $L_{mi} = \{ t \in S_m : c_t < \infty \}$. We associate the tuple of costs $C_{mi} = (c_t : t \in S_m)$ with this layer. It may be the case that several layers are identical in terms of costs, i.e. $C_{mi} = C_{m' i}$. If this occurs, we say layer $i$ has a multiplicity (equal to the number of identical layers).

We will treat each layer of each segment as a separate meta-item. Let $UA\text{Opt}(S)$ denote the maximum social welfare achievable by allocating at most one meta-item defined by this segmentation to each buyer. In particular, we take the maximum over allocations $X$ such that (1) for every buyer $j$ and values $\vec{v}$, either $X_j(\vec{v}) = \emptyset$ or there exists a pair $(m, i)$ with $X_j(\vec{v}) \subseteq L_{mi}$, and (2) for every pair $(m, i)$ and values $\vec{v}$, there is at most one buyer $j$ such that $X_j(\vec{v}) \subseteq L_{mi}$.

We now present the main result of this section.

**Lemma 3.6.** Given any instance of the job scheduling problem, there exists a segmentation $S$ for the items $[T]$ such that

$$\text{Opt} \leq O(\log L) \cdot UA\text{Opt}(S).$$

**Proof.** Given a segmentation $S = \{S_1, S_2, \ldots, S_M\}$ we say that an allocation $X$ is $S$-respecting if no buyer is allocated items contained in more than one segment of $S$. We first show that there is a segmentation $S$ such that restricting to $S$-respecting allocations loses a factor of at most $O(\log L)$ in social welfare.

We begin by explicitly constructing $2(\log L + 1)$ different segmentations. Let $\ell$ be a power of 2 in the range $[1, L]$. Consider a partition of $[T]$ into intervals of length $\ell$ given by $A^{(\ell)}_k = [(k-1)\ell + 1, k\ell]$. We define the segmentation $S_1^{(\ell)} = \{A^{(\ell)}_k \cup A^{(\ell)}_{k+1} : k \text{ odd}\}$ and $S_2^{(\ell)} = \{A^{(\ell)}_k \cup A^{(\ell)}_{k+1} : k \text{ even}\}$.

Next we will construct $2(\log L + 1)$ different allocations, one for each segmentation above; call these allocations $X_1^{(\ell)}$ and $X_2^{(\ell)}$ respectively. Fix a realization $\vec{v}$ of the buyers’ valuation functions. Consider a particular buyer $j$, and suppose that the optimal allocation $\text{Opt}$ assigns an interval $I$ of length $\ell'$ to $j$. Let $\ell' \in (\ell/2, \ell]$ where $\ell$ is a power of 2. Furthermore, let $k$ be defined such that the interval $I$ begins in $A^{(\ell)}_k$. Then, $I \subset A^{(\ell)}_k \cup A^{(\ell)}_{k+1}$. If $k$ is odd, we assign the same interval $I$ to $j$ in $X_1^{(\ell)}$, that is, $X_1^{(\ell)}(j) = I$. In all the other allocations, $j$ is assigned nothing. Likewise, if $k$ is even, we assign the interval $I$ to $j$ in $X_2^{(\ell)}$ and nothing in the other allocations. By construction, each allocation is segmentation-respecting for its respective segmentation.

Observe that each buyer is assigned in exactly one of the $O(\log L)$ allocations defined above to the same interval as its assignment under $\text{Opt}$. Therefore, the sum over the allocations of the values obtained by buyers in each segmentation is exactly equal to the value of all the buyers in $\text{Opt}$. On the other hand, owing to the convexity of cumulative costs, the collective cost of the $O(\log L)$ allocations is smaller than the total cost incurred by the optimal allocation. As a consequence, the social welfare of one of the allocations is at least an $O(\log L)$ fraction of $\text{Opt}$:

$$\text{Opt} \leq O(\log L) \max_{z \in \{1, 2\}, \ell} \mathbb{E}_{\vec{v} \sim F}[\text{SW}(X_z^{(\ell)})].$$

For the remainder of this proof we focus on the segmentation and allocation that achieve the above maximum. Let these be $S$ and $X$ respectively. (We drop the subscript and superscript for notational convenience.)

Let us now consider a single segment $S_m$ within $S$. Fix a realization $\vec{v}$ of buyers’ values and let $J_m$ denote the set of buyers assigned to an interval in $S_m$ by the allocation $X$. We will now
construct a unit allocation of the layers of $S_m$ to buyers in $J_m$ that loses at most a constant factor in social welfare.

For $t \in S_m$, let $D_t = |\{j : X_j \ni t\}|$ be the number of copies of item $t$ allocated to buyers in $J_m$ by the allocation $X$. Let $D = \max_{t \in S_m} D_t$. We complete the proof by induction on $D$.

Consider the base case of $D = 1$. Recall that $X$ allocates at least $\ell/2$ items to every buyer in $J_m$, whereas $|S_m| = 2\ell$. Therefore, there are at most 4 buyers in $J_m$. We consider the one that contributes the most to the social welfare of $X$, that is, arg max$_{j \in J_m} \{v_j(X_j) - \sum_{t \in X_j} c_t\}$, and allocate the first layer $L_{mi}$ to this buyer.

Suppose $D = k > 1$. Let $J$ be a minimal set of buyers whose allocations together cover $L_{m1}$. Since $|S_m| \leq 4\ell$ and we select $J$ to be minimal, $|J| \leq 6$. To see this, let $J = \{j_1, \cdots, j_N\}$, where buyers are numbered by increasing start time in $X$. Note that there must be at least one item between $X_{j_n}$ and $X_{j_{n+2}}$ not covered by either allocation, so $|\{j_n : n \text{ even}\}| \leq |\{j_n : n \text{ odd}\}| \leq 3$. Choosing the buyer in $J$ which maximizes welfare obtains at least a sixth of the welfare of $X$ contributed by $J$. The remaining buyers, $J_m \setminus J$, together with the costs for all copies beyond the first, form an instance of the problem with $D - 1$ layers; by induction, there exists a way to allocate one buyer per remaining layer to obtain at least $\frac{1}{6}$ of the welfare that $X$ obtains from $J_m \setminus J$. This finishes the proof.

\[\Box\]

### 3.4 Bundle pricing for settings with no supply costs

With the results from Sections 3.2 and 3.3, it is straightforward to construct a bundle pricing for the interval values setting. We first observe that in the absence of costs on items, we can obtain a stronger notion of unit allocations of meta-items. In the previous section, the unit allocations over segmentations we constructed allocated a (potentially proper) subset of layers to buyers. In the absence of costs, we can allocate entire layers to buyers without hurting social welfare. In particular, \(\text{UAOpt}^S\) is now the maximum over allocations $X$ such that for every $j$ and $\bar{v}$, either $X_j(\bar{v}) = \emptyset$ or there exists a pair $(m, i)$ with $X_j(\bar{v}) = L_{mi}$.

Our bundle pricing then proceeds as follows. We apply Lemma 3.6 to obtain a segmentation $S$ such that $\text{UAOpt}(S)$ is at least an $O(\log L)$ fraction of $\text{Opt}$. We then set up an item pricing problem as follows.

The items in the new instance correspond to layers in the segments in $S$. Let $e_{mi}$ denote the item correspond to layer $L_{mi}$. For every buyer $j$ in the original instance, we construct a new buyer $\hat{j}$ in the new instance. The valuation of $\hat{j}$ is instantiated as follows. We draw a valuation $v_j$ for $j$. Then, for every subset $S$ of the new items, we set $v_j(S) = v_j(\cup_{(m,i) \in S} L_{mi})$.

It follows immediately from the construction that the unit allocation optimum for the new instance is at least as large as the social welfare obtained by a unit allocation of layers in the original instance:

\[\widehat{\text{UAOpt}} \geq \text{UAOpt}(S)\]

We then apply Lemma 3.4 to obtain an item pricing $p$ for items in the new instance. Finally, we construct a bundle pricing as follows. Each bundle in our menu $M$ corresponds to a layer $L_{mi}$. We offer one copy of this bundle at a price equal to the price of the corresponding item $e_{mi}$ in the new instance. Since buyer values over items in the new instance and those over bundles in the original instance are exactly coupled, offering the layers as bundles at the same prices $p$ gives the same allocation in both instances. Consequently, the bundle pricing $(p, M)$ gives us a social welfare of at least half of $\widehat{\text{UAOpt}}$. Furthermore, since buyers have the same values for different layers that
contain the same set of items, p assigns the same price to each such layer. Therefore, the pricing is static. Putting everything together, we obtain the following theorem.

**Theorem 3.7.** For any setting without item costs in which buyers have interval values over intervals of size at most $L$, there exists a static bundle pricing $(p, M)$ such that $\text{OPT} \leq O(\log L)\text{SW}(p, M)$.

### 3.5 Adaptive bundle pricing for settings with costs

In the presence of costs, the approach taken in Section 3.4 no longer works. There, we constructed a partition over items into segments and layers, and argued that there is a good unit allocation that allocates to each buyer at most one entire layer. In the presence of costs, Lemma 3.6 only gives us an allocation that allocates to each buyer a subset of a layer. Since different subsets have different costs, we cannot simply allocate the entire layer at once.

In order to solve this issue, we construct an SPM with a menu that contains every subinterval of every layer, instead of just the layers themselves. The prices of these subintervals reflect their different costs. Initially, we set prices to recover welfare from the first layer in each segment. Each time a bundle sells, the price on each bundle belonging to the same segment increases to recover welfare from the next layer.

**Theorem 3.8.** For any setting with item costs in which buyers have interval values over intervals of size at most $L$, there exists an adaptive bundle pricing $(p, M)$ such that $\text{OPT} \leq O(\log L)\text{SW}(p, M)$.

**Proof.** As in our proof of Theorem 3.7, we begin with the segmentation $S$ given by Lemma 3.6 such that $\text{UAOPT}(S)$ is at least an $O(\log L)$ fraction of $\text{OPT}$.

We first construct an item pricing instance as follows. Once again, we have a meta-item $e_{mi}$ for every layer $L_{mi}$. For every buyer $j$ in the original instance, we construct a buyer $\hat{j}$ in the new instance. We draw a value for $\hat{j}$ as follows. We draw a value $v_j$ for $j$ and then for every subset $Y$ of the items $\{e_{mi}\}$, we set

$$v_{\hat{j}}(Y) = \max_X \left\{ v_j(X) - \sum_{(t,i) \in X} c_{ti} \right\},$$

where the maximum is taken over subsets $X$ of the union of layers corresponding to items $e_{mi} \in Y$. Here we interpret items $t$ in $X$ as being associated with a specific copy $i$. In other words, for every such subset $X$, we consider the contribution of buyer $j$ to the social welfare (his value for the set minus the cost of the set) from allocating that subset to the buyer; the maximum such contribution is set to be the value of $\hat{j}$ for the set $Y$.

As before, it is straightforward to see that $\text{UAOPT}$ over the new instance is exactly equal to $\text{UAOPT}(S)$. We now apply Lemma 3.4 to obtain an item pricing $p$ for the items $\{e_{mi}\}$. We use this item pricing to associate the following prices for layers in the original instance. For each layer $L_{mi}$ and for each subset $Y$ of this layer, we assign a price of $p_Y^{(i)} = p_{e_{mi}} + \sum_{t \in Y} c_{ti}$.

Our adaptive bundle pricing then runs as follows. We keep track of how many subsets of segment $S_m$ have already been sold. After $i - 1$ subsets of the segment have been sold, we offer every subset $Y$ of the $i$th layer $L_{mi}$ at a price of $p_Y^{(i)}$.

We now claim that the social welfare generated by this pricing is precisely equal to the social welfare generated by the item pricing $p$ for the new instance when the buyer values and arrival order in the two settings are coupled. This can be proved by induction over the number of agents that have already arrived. Our inductive hypothesis is that at any point of time, the set of new items
Figure 2: The optimal allocation for the example in the proof of Theorem 4.1, which shows a gap logarithmic in buyers’ values and item supply between the optimal allocation and that achievable by any partition pricing. Buyers of type $i$ have value $2^i - 1$, and there are $2^L$ such buyers in total. Item $t$ has $2^t - 1$ copies. The optimal solution allocates $2^i - 1$ copies of item $i$ to buyers of type $i$, as shown. Notice that if we allow buyers of type $i$ to buy their bundles until item $i$ is exhausted, they will block all buyers with larger values.

available in the new instance is exactly the same as the set of unsold layers in the original instance. By the manner in which we construct buyers’ values in the new instance, buyers’ preferences in the two instances are exactly coupled and the claim follows.

This concludes the proof. □

4 The power of partition pricings

We now turn to partition pricings for the job scheduling setting, focusing on static pricings. Note that the $\Omega(\log L / \log \log L)$ lower bound of Theorem 3.2 in Section 3.1 applies also to partition pricings. The next subsection shows that partition pricings face an additional lower bound when different items have very different supply and when different buyers have very different values. Altogether, no static partition pricing can obtain an approximation factor better than $O(\max(\log L, \log \min\{\alpha_b, \alpha_v\}))$, where $\alpha_b$ is the ratio of the maximum item supply to the minimum item supply, and $\alpha_v$ is the ratio of the maximum to the minimum buyer values. In Section 4.2 we supply an upper bound which is only a quadratic factor worse: $O(\log L \log \min\{\alpha_b, \alpha_v\})$. Finally, in Section 4.3 we show that when items have costs, partition pricings can obtain no better than an $O(L^{1/4})$ approximation even with adaptive prices.

4.1 Lower bound for partition pricings with interval valuations

The lower bound in the previous subsection hinges on buyers desiring bundles of very different lengths. We now construct a lower bound for partition pricings which applies even when every buyer desires a constant fraction of all the items.

**Theorem 4.1.** There exists an instance with single-minded buyers with interval valuations such that for any static partition pricing $(p, M)$ we have

$$\text{OPT} = \Omega(\log \min\{\alpha_b, \alpha_v\}) \text{ SW}(p, M).$$
Here $\alpha_b$ is the ratio of the maximum to minimum item supply, and $\alpha_v$ is the ratio of the maximum to minimum non-zero buyer value.

Proof. Consider the following family of instances parameterized by an integer $L > 0$. We have $2L - 1$ items, where item $t$ comes in $B_t = 2^t - 1$ copies if $t \leq L$ and $B_t = 2^L - 1$ copies otherwise. There are $L$ types of buyers and $2^L$ buyers of each type. Each type $j$ buyer is single minded: she wants precisely the $2L - j$ items in the interval $(j, 2L - 1)$ and her value for them is $v_j = \frac{1}{2^j}$. See Figure 2. Buyers arrive in increasing order of interval length, in other words, the $2^L$ buyers of type $L$ arrive first and the $2^L$ buyers of type 1 arrive last.

Optimal welfare. Consider the following allocation. For each type $j$ of buyers, we allocate their desired bundle $(j, 2L - 1)$ to exactly $2^{j-1}$ of the buyers. This exactly satisfies the supply constraint. See Figure 2. It follows that the optimal social welfare is at least $\text{Opt} \geq \sum_j 2^{j-1} \times 2^{-j} = L/2$.

The welfare obtained by a static partition pricing. Recall that buyers arrive in increasing order of length, and consider the first time that a buyer can afford to buy a collection of the offered bundles to cover her interval. Let this be a type $j$ buyer. Recall that items in $(j, 2L - 1)$ have at most $2^j - 1$ copies available. However, the number of different buyers of type $j$ is $2^L$, each of which can now afford to buy the same collection of bundles as the aforementioned first buyer. Therefore, by the time all of the buyers of type $j$ have visited the market, all items in the interval $(j, 2L - 1)$ are sold out. Since all the remaining buyers desire these items also, no more buyers can be allocated their intervals. The social welfare of the pricing is therefore at most the total value of the $2^{-j} \times (2^j - 1) < 1$, implying a gap of $\Omega(L)$ with respect to the optimum. The lemma follows by observing that $\alpha_b = \alpha_v = 2^L$ for this instance.

4.2 Upper bound for static partition priceings with interval valuations

We now turn to upper bounds. We first observe that if all items have the same supply, that is $B_t = B$ for all $t \in [T]$, then the bundle pricing of Theorem 3.7 is really a static partition pricing. Indeed, every layer of any segment in this case is identical (and equal to the entire segment itself), and so the pricing constructed in the proof of the theorem assigns the same price to each such layer. We therefore immediately have the following corollary.

Corollary 4.2. For any setting with no item costs in which buyers have interval values over intervals of size at most $L$, and $B_t = B$ for all $t \in [T]$, there exists a static partition pricing $(p, M)$ such that $\text{Opt} \leq O(\log L)\text{SW}(p, M)$.

In the remainder of this section, we focus on settings where different items have different amounts of supply. We will prove the following two lemmas.

Lemma 4.3. For any setting without item costs in which buyers have interval values over intervals of size at most $L$, and values lie in the set $\{0\} \cup [1, \alpha_v]$, there exists a static partition pricing $(p, M)$ such that $\text{Opt} \leq O(\log L \log \alpha_v)\text{SW}(p, M)$.

Lemma 4.4. For any setting without item costs in which buyers have interval values over intervals of size at most $L$, and each item has between $B_{\min}$ and $B_{\max}$ copies, there exists a static partition pricing $(p, M)$ such that $\text{Opt} \leq O(\log L \log \alpha_b)\text{SW}(p, M)$ where $\alpha_b = B_{\max}/B_{\min}$.
The two lemmas together imply the following theorem.

**Theorem 4.5.** For any setting without item costs in which buyers have interval values over intervals of size at most \( L \), values lie in the set \( \{0\} \cup [1, \alpha_v] \), and each item has between \( B_{\min} \) and \( b_{\max} \) copies with \( \alpha_b = B_{\max}/B_{\min} \), there exists a static partition pricing \((p, M)\) such that \( \text{OPT} \leq O(\log L \log \min\{\alpha_v, \alpha_b\}) \text{SW}(p, M) \).

Proof of Lemma 4.3. We begin once again with the partition into bundles or meta-items given by the segmentation \( S \) of Lemma 3.6. Observe that for every segment \( S_m \) in \( S \), every layer of \( S_m \) is a subset of the previous layer. If we can obtain good social welfare via a bundle pricing of the layers that assigns the same price to every layer, this would form a static partition pricing.

Recall from Lemma 3.6 that \( UA_{\text{OPT}}(S) \) is at least an \( O(\log L) \) fraction of \( \text{OPT} \). Let \( X \) be a unit allocation over the layers that achieves social welfare equal to \( UA_{\text{OPT}}(S) \). Consider some power of 2 in the range \([v_{\min}, v_{\max}]\), say \( \ell \), and define \( W_\ell = E_{\mathcal{F} \sim \mathcal{F}} \left[ \sum_{j: v_j(X_j(\mathcal{F})) \in [\ell, 2\ell]} v_j(X_j(\mathcal{F})) \right] \). That is, \( W_\ell \) is the contribution to the social welfare of buyers that obtain a value between \( \ell \) and \( 2\ell \) from their allocation.

We have \( \sum_\ell W_\ell = UA_{\text{OPT}}(S) \), so for at least one value of \( \ell \), we have \( \text{OPT} \leq O(\log L \log \alpha_v)W_\ell \). We now claim that placing a price of \( \ell \) on each layer in each segment in \( S \) obtains a social welfare of \( \Theta(W_\ell) \). To see this, observe that \( W_\ell \leq 2\ell \cdot |\{j: v_j(X_j(\mathcal{F})) \in [\ell, 2\ell]\}| \). On the other hand, the social welfare of a partition pricing that sells every bundle on its menu at a flat rate of \( \ell \) is at least \( \ell \) times the number of bundles sold. Thus it remains to show that the partition pricing sells at least a constant fraction of the bundles allocated under \( W_\ell \). But this is just a maximum matching problem, for which the partition pricing produces an online greedy solution. The lemma then follows from recalling that the greedy algorithm achieves a 2-approximation for the maximum matching problem. \( \Box \)

Proof of Lemma 4.4. The proof of this lemma is very similar to that of Lemma 4.3. We focus on a segmentation \( S \) and an unit allocation \( X \). For a particular segment \( S_m \), let \( g_m(p) \) denote the expected number of buyers that are allocated by \( X \) to a subinterval of \( S_m \) and that receive a value of at least \( p \) from that allocation. Formally:

\[
g_m(p) = E_{\mathcal{F} \sim \mathcal{F}}[|\{j: X_j(\mathcal{F}) \subseteq S_m \text{ and } v_j(X_j(\mathcal{F})) \geq p\}|].
\]

Then, it holds that \( \text{SW}(X) = \sum_m \int_0^\infty g_m(p) \, dp \). Let \( h_m \) be the inverse function of \( g_m \). In other words, for any value \( p \) and \( b := g_m(p) \), we define \( h_m(b) = p \).\(^{13}\) Observe that \( h_m \) is defined over the domain \([0, B_{\max}]\). Now we can write:

\[
\text{SW}(X) = \sum_m \int_0^{B_{\max}} h_m(b) \, db = \sum_m \int_0^{B_{\min}} h_m(b) \, db + \sum_m \int_{B_{\min}}^{B_{\max}} h_m(b) \, db
\]

Consider the first term in the sum. This is the contribution to the social welfare of \( X \) from the first \( B_{\min} \) jobs allocated to each segment. We can recover a constant fraction of this quantity using a static partition pricing as in Corollary 4.2.

\(^{13}\) We are assuming for notational simplicity that the inverse is uniquely defined, although this is not necessary for our argument.
Let us now consider the second term. It is easy to see that for each segment $S_m$, there exists a $b_m \in [B_{\text{min}}, B_{\text{max}}]$, such that $b_m h_m(b_m) \geq \frac{1}{\ln \alpha} \int_{B_{\text{min}}}^{B_{\text{max}}} h_m(b) \, db$. Here, recall, $\alpha_b = B_{\text{max}} / B_{\text{min}}$. Let $p_m = h_m(b_m)$. Therefore, $\sum_m p_m g_m(p_m)$ is at least a $\ln \alpha$ fraction of the second term in $\text{SW}(X)$. Let $p_m = h_m(b_m)$. Therefore, $\sum_m p_m g_m(p_m)$ is at least a $\ln \alpha$ fraction of the second term in $\text{SW}(X)$.

We now argue how to recover a constant fraction of $\sum_m p_m g_m(p_m)$ from a static partition pricing. We simply set the price of each layer of each segment $S_m$ to $p_m/2$. Fix an instantiation of values and consider a particular segment $S_m$. Let $G$ denote the number of buyers assigned by $X$ to a subinterval of $S_m$ at this instantiation that obtain a value of at least $p_m$. (So, $g_m(p_m)$ is just the expectation of $G$ over the valuations.) Observe that if our pricing allocates $G'$ layers of $S_m$ for this instantiation of values, then we get a revenue of $G' p_m/2$. On the other hand, for each layer not allocated but that contributes towards $G$, the buyer allocated to this layer in $X$ gets a utility of at least $p_m/2$ and therefore, this set of buyers contributes at least $(G - G') p_m/2$ to the buyer utility. Therefore, the social welfare from our allocation is at least $G p_m/2$.

This concludes the proof.

### 4.3 Lower bound for items with costs

We now show that partition pricings perform poorly when buyers desire bundles and items have costs. This lower bounds holds even when buyers are single-minded and have interval values.

**Theorem 4.6.** There exists an instance with buyers with interval valuations over bundles up to size $L$, such that for any adaptive partition pricing $(p, M)$ we have

$$\text{Opt} = \Omega \left( \frac{L^{1/4}}{\epsilon} \right) \text{SW}(p, M).$$

**Proof.** We construct a family of instances parameterized by integer $L > 0$. We have $L$ items, each with one copy costing 1. That is, $c_t = 1$ and $c_t = \infty$ for $i > 2$ for all $t$. Denote $\epsilon = \frac{1}{L}$.

We have three types of buyers.

- **Type 1:** there are $O(L^2)$ type 1 buyers, one for each subinterval of $[L]$. A buyer corresponding to interval $I$ has deterministic value of $\epsilon$ for this interval. Note that such a buyer cannot contribute to welfare because his value always equals the cost of his interval.

- **Type 2:** there are $L^{1/4}$ type 2 buyers. For any $1 \leq a \leq L^{1/4}$, there exists one type 2 buyer that has a deterministic value of $a(1 + L^{1/4} \epsilon)$ for any interval with length $a$.

- **Type 3:** there are $\frac{L}{2}$ type 3 buyers. For any $\frac{L}{2} < b \leq L$, there exists one type 3 buyer who desires interval $[1, b]$: the buyer’s value is $b(1 + \epsilon)$ with probability $2/L$, or 0 otherwise.

**The optimal welfare.** With probability at least $1 - (1 - 2/L) L^{1/2}$, there exists a type 3 buyer with positive value. Thus

$$\text{Opt} \geq \left( 1 - (1 - 2/L) L^{1/2} \right) \frac{L \epsilon}{2} = \Omega(L \epsilon).$$

**Welfare of an adaptive partition pricing.** Recall that an adaptive partition pricing must fix the partition of items into intervals before buyers show up, although prices can vary over time. For any partition and setting of prices, we will adversarially pick an arrival order for buyers. We assume that the adversary generating the sequence of buyers can observe the prices set by the algorithm (in other words, the adversary is adaptive). The algorithm terminates the first time a type 3 buyer buys his interval, or if no buyers remain.
Let $\mathcal{M}$ denote the set of intervals in the partition pricing. First we observe that if at any time any interval is priced at or below its cost, the adversary can send in a type 1 buyer to buy that interval. This buyer generates no social welfare and precludes some other type 3 buyers from purchasing their interval. So, henceforth, we may assume that the pricing always prices all intervals above their cost.

Second, we note that a type 3 buyer cannot afford to buy a superset of her interval. If her interval is of size $b < L$, her value is $b(1 + \varepsilon) < b + 1$, which is in turn less than the price of any interval of larger length. Therefore, a type 3 buyer can buy its interval only if the interval has the same right endpoint as an interval in $\mathcal{M}$.

Let $X$ be the set of intervals that appear in the menu with length at most $L^{1/4}$; $X'$ be the first $L^{3/4}$ intervals in $X$; $Y$ be the set of intervals that appear in the menu with length greater than $L^{1/4}$. Observe that $|X'| + |Y| \leq 2L^{3/4}$.

Third, we note that if at any time no interval in $X'$ can be afforded by a type 2 buyer, then any type 3 buyer with a right endpoint the same as some interval in $X \setminus X'$ cannot afford to buy her interval. This is because the total surcharge of intervals in $X'$ in this case is at least $L^{3/4} \cdot L^{1/4} \varepsilon = L \varepsilon$. At this time, the adversary sends in any remaining type 3 buyer with a right endpoint not matching that of intervals in $X' \cup Y$.

Fourth, we note that if at any time an interval in $X'$ can be afforded by a type 2 buyer, we let that buyer arrive at this time and buy this interval. Then no type 3 buyer with a right endpoint the same as some interval in $X \setminus X'$ can be served since it is blocked. At this point, the adversary sends any remaining type 3 buyers in arbitrary sequence.

Putting all of these eventualities together, we note that the only type 3 buyers that can at some point of time potentially afford to buy their intervals are those whose right endpoints match one of the intervals in $X' \cup Y$. There are at most $2L^{3/4}$ such buyers. Therefore, the probability that the mechanism serves a type 3 buyer is at most $2L^{3/4} \times 2/L = 4L^{-1/4}$. Each such buyer contributes at most $L \varepsilon$ to the social welfare for a total contribution of $O(L^{3/4} \varepsilon)$. On the other hand, the total contribution of type 2 buyers to social welfare is at most $\sum_{a \leq L^{1/4}} a \cdot L^{1/4} \varepsilon = O(L^{3/4} \varepsilon)$.

We therefore get a gap of $\Omega(L^{1/4})$ between the social welfare obtained by the partition pricing and the optimal welfare. To conclude, we observe that in the example we constructed, buyers’ interval lengths range in $[1, L]$, and their values lie in $\{0\} \cup [1, L + 1]$, and their contributions to social welfare lie in $\{0\} \cup [L^{-3/4}, 1]$. In other words, the gap we achieve is polynomial in all of the natural parameters of the problem.

## 5 Deferred Proofs

**Theorem 3.1.** There exists an instance with single-minded buyers with general (non-interval) valuations over bundles up to size $L$, such that for any SPM $(p, \mathcal{M})$ we have

$$\text{FracOpt} = \Omega(L) \text{SW}(p, \mathcal{M}) \quad \text{and} \quad \text{Opt} = \Omega(\log L) \text{SW}(p, \mathcal{M}).$$

**Proof.** Let $L$ be a prime number. There are $L$ groups of $L$ buyers, with $L^2$ buyers in all. We have one unit of each of $L^2$ items available. Each buyer is single-minded and desires a bundle of size $L$. The bundles that the buyers desire overlap in such a manner as to impose the following feasibility constraint: we can serve any subset of the buyers within any single group simultaneously, but we cannot serve any two buyers from different groups. Specifically, each item belongs to the bundle of exactly one buyer in each group. Now suppose that each buyer has a value of 1 with probability
1/L and 0 otherwise. Then, note that a solution that serves all agents with value 1 satisfies the feasibility constraint ex-ante. It therefore obtains an expected social welfare of $L^2 \times 1/L \times 1 = L$. On the other hand, any sequential pricing either allocates to no buyer, or once it commits to allocating to a buyer can only allocate to other buyers within the same group. Therefore, its social welfare is bounded by $1 + (L - 1) \times 1/L \times 1 < 2$. Therefore, we obtain a gap of $\Omega(L)$ between the welfare obtained by any SPM and the ex-ante relaxation. The gap between the welfare of an SPM and the optimal integral social welfare is only $\Omega(\log L)$: with constant probability there exists a group in which log $L$ buyers have a value of 1; this is the group served by the optimal solution.

It remains to describe the bundles that impose the aforementioned feasibility constraint over buyers. We have $L^2$ items in all indexed by pairs of numbers in $\{0, \cdots, L - 1\}$. Each buyer is indexed by the pair $(a, b)$ with $a \in \{0, \cdots, L - 1\}$ and $b \in [L]$. Buyer $(a, b)$ desires the bundle of items $\{(i, a + ib \mod L) : i \in \{0, \cdots, L - 1\}\}$. It is now easy to see that buyers $(a, b)$ and $(a', b)$ with $a \neq a'$ have non-overlapping bundles. On the other hand, because $L$ is prime, buyers $(a, b)$ and $(a', b')$ with $b \neq b'$ share at least one item. In particular, there exists an index $i$ for which $a - a' = i(b' - b) \mod L$, and so, $a + ib = a' + ib'$ (mod $L$).

This completes our construction. \qed

Lemma 3.3. With the notation defined in Section 3.1, for any $\ell \geq 0$ it holds that, $\sum_{(j,i) \in S_\ell} A(j, i) \cdot 2^j < 2L$.

Proof. We prove the lemma by induction on $L$. The base case is when $L = 1$, in which case there is only one job which belongs to group $j = 0$, so it trivially holds that $\sum_{(j,i) \in S_\ell} A(j, i) \cdot 2^j \leq A(0, 1) \cdot 2^0 \leq 1$.

Assume now that the lemma holds when $L = 2^x$ for all $0 \leq x \leq r$. We will show that it also holds for $L = 2^{r+1}$. We partition the items into two intervals of length $2^r$: $I_1 = \{1, 2, \cdots, 2^r\}$ and $I_2 = \{2^r, 2^r+1, 2^r+2, \cdots, 2^{r+1}\}$. Notice that, with the exception of the job of length $L$, that is, job $(r+1, 1)$, every other job is completely contained either in $I_1$ or in $I_2$. Let $J_1$ and $J_2$ be set of jobs strictly contained in $I_1$ and $I_2$ respectively.

We consider two cases. If $(r+1, 1)$ does not belong to $S_\ell$, then we have

$$\sum_{(j,i) \in S_\ell} A(j, i) \cdot 2^j = \sum_{(i,j) \in S_\ell \cap I_1} A(i, j) \cdot 2^j + \sum_{(i,j) \in S_\ell \cap I_2} A(i, j) \cdot 2^j < L + L = 2L.$$  

Here the final inequality follows from the induction hypothesis applied to $I_1$ and $I_2$.

Alternately, suppose $(r+1, 1)$ belongs to $S_\ell$. Then $Z_{r+1,1} < 2^r$ by definition. But the interval of each buyer contributing to the count $Z_{r+1,1}$ either fully belongs to $I_1$ or to $I_2$. At least one of these sets, buyers in $Y$ whose interval is a subset of $I_1$, or those whose interval is a subset of $I_2$, is strictly smaller than $2^r/2$ in size. Without loss of generality, let this be the former set. Then none of the buyers $(j,i)$ in $I_1$ belong to $S_\ell$. Therefore,

$$\sum_{(j,i) \in S_\ell} A(j, i) \cdot 2^j = A(r+1, 1) \cdot 2^{r+1} + \sum_{(i,j) \in S_\ell \cap I_1} A(i, j) \cdot 2^j < L + L = 2L,$$

where once again we applied the inductive hypothesis to $I_2$. \qed

Lemma 3.4. For any value distribution $F$ and nondecreasing costs $\bar{c}$, there exist static item prices $p$ such that $SW(p) \geq \frac{1}{2} UA^\text{OPT}$.
Proof. Let \(v_{jt}\) denote the value of buyer \(j\) for the singleton item \(t\). Let \(x_{jti}(v_j)\) be the probability (over other jobs’ values) that UAOpt assigns the \(i\)th copy of item \(t\) to buyer \(j\), given that job \(j\) has value \(v_j\). Then we can write

\[
\text{OPT} = \sum_{t,i} \sum_{j} \mathbb{E}_{v_j \sim F_j} [(v_{jt} - c_{ti}) x_{jti}(v_j)].
\]

We now define the following two functions:

\[F_t(p) = \sum_{i} \sum_{j} \mathbb{E}_{v_j \sim F_j} [(v_{jt} - p)^+ x_{jti}(v_j)] \quad \text{and} \quad G_t(p) = \sum_{i} (p - c_{ti})^+.
\]

Let \(J_t(p) = |\{i : p \geq c_{ti}\}|\) be the number of copies of item \(t\) with cost at most \(p\). Intuitively, \(F_t(p)\) is the contribution to welfare\(^{14}\) from values above \(p\), and \(G_t(p)\) is the revenue from selling the first \(J_t(p)\) items at price \(p\).

We assume \(F_t(0) > 0\) and \(G_t(v_{\text{max}}) > 0\); if not, item \(t\) does not contribute to UAOpt and can be ignored. On the other hand, \(G_t(0) = 0\), and \(F_t(v_{\text{max}}) = 0\). Furthermore, these functions are sums of continuous functions, and therefore are themselves continuous. So we can choose \(p_t\) such that \(F_t(p_t) = G_t(p_t)\). Given this choice of \(p_t\), we show that \(\sum_t F_t(p_t) \geq \frac{1}{2}\text{OPT}:

\[
\text{OPT} = \sum_{t,i} \sum_{j} \mathbb{E}_{v_j \sim F_j} [(v_{jt} - c_{ti}) x_{jti}(v_j)]
\leq \sum_{t,i} \sum_{j} \mathbb{E}_{v_j \sim F_j} [((v_{jt} - p_t)^+ + (p_t - c_{ti})^+) x_{jti}(v_j)]
= \sum_{t} F_t(p_t) + \sum_{t} \sum_{i} (p_t - c_{ti})^+ \sum_{j} \mathbb{E}_{v_j \sim F_j} [x_{jti}(v_j)]
\leq \sum_{t} (F_t(p_t) + G_t(p_t)) = 2 \sum_{t} F_t(p_t).
\]

Finally, we show that these prices generate at least \(\sum_t F_t(p_t)\) in expected welfare. Let \(B_t = J_t(p_t)\), and let \(X_t\) be the (random) number of copies of item \(t\) sold. First, the total revenue generated by these prices is at least

\[
\text{REV}(p) \geq \sum_{t} \Pr[X_t = B_t] G_t(p_t)
= \sum_{t} \Pr[X_t = B_t] F_t(p_t) \quad (1)
\]

Fix a buyer \(j\). Let \(x_{jti}(v_j) = \sum_i x_{jti}(v_j)\) be the probability that \(j\) with value \(v_j\) is assigned to

\(^{14}\)Note, however, that \(F_t(p)\) actually overcounts this welfare, as \(p\) may be less than \(c_{ti}\) for some \(i\).
any copy of item $t$. Then its expected utility is

$$u_j(p) \geq \mathbb{E}_{X,v_j} \left[ \max_{t: X_t < B_t} (v_{jt} - p_t)^+ \right]$$

$$\geq \mathbb{E}_{X,v_j} \left[ \sum_{t: X_t < B_t} (v_{jt} - p_t)^+ x_{jt}(v_j) \right]$$

$$= \sum_t \Pr[X_t < B_t] \mathbb{E}_{v_j} \left[ (v_{jt} - p_t)^+ x_{jt}(v_j) \right]$$

$$= \sum_t \Pr[X_t < B_t] \sum_i \mathbb{E}_{v_j} \left[ (v_{jt} - p_t)^+ x_{jti}(v_j) \right].$$

Summing over all jobs, we have

$$\text{Util}(p) = \sum_j \mathbb{E}[u_j(p)]$$

$$\geq \sum_t \Pr[X_t < B_t] \sum_i \sum_j \mathbb{E}_{v_j} \left[ (v_{jt} - p_t)^+ x_{jti}(v_j) \right]$$

$$= \sum_t \Pr[X_t < B_t] F_t(p_t) \tag{2}$$

Finally, by summing (1) and (2), we have

$$\text{SW}(p) = \text{Util}(p) + \text{Rev}(p)$$

$$\geq \sum_t F_t(p_t).$$

Together with the above observation that $U_{\text{AOPT}} \leq 2 \sum_t F_t(p_t)$, this completes the proof. \qed

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