Generic Theory of Geometrodynamics  
from Noether’s theorem for the $\text{Diff}(M)$ symmetry group

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Abstract. We work out the most general theory for the interaction of spacetime geometry and given source fields, commonly referred to as geometrodynamics. The minimum set of postulates to be introduced is that (i) the action principle should apply and that (ii) the total action should be form-invariant under the diffeomorphism group. The second postulate thus implements the Principle of General Relativity. According to Noether’s theorem, this symmetry gives rise to a conserved Noether current, from which the complete set of theories compatible with both postulates can be deduced. Provided that the system has no other symmetries, this finally results in a new generic Einstein-type equation, which comprises the canonical energy-momentum tensor as the relevant source quantity. For the case of massive spin particles, we show that this entails an increased weighting of the kinetic energy over the mass in their roles as the source of gravity as compared to the metric energy momentum tensor, which constitutes the source of gravity in Einstein’s General Relativity. We furthermore show that a massive vector field necessarily acts as a source for torsion of spacetime.

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1. Introduction

The covariant Hamiltonian formalism in the realm of classical field theories was recently shown to allow generalized canonical transformations which also include arbitrary chart transition maps within the underlying spacetime manifold $M$ [1]. With this framework at hand, it is now possible to isolate the complete set of theories which are based on the action principle and implement the Principle of General Relativity, i.e., the condition for a system to be form-invariant under chart diffeomorphisms. Systems complying with the latter principle are thus required to have the $\text{Diff}(M)$ group as an intrinsic symmetry group. As known from Noether’s theorem, each symmetry is associated with a pertaining conserved Noether current $j^N_M$. The main favor of the Noether approach is that the condition for a conserved Noether
current directly leads to the respective field equations, hence one avoids all the ambiguities that emerge from constructing a Lagrangian or an equivalent covariant Hamiltonian.

With our Noether approach, it is now possible to work out the most general form of an Einstein-type equation for a given system satisfying the Principle of General Relativity—including a spacetime with torsion and without the restriction to a covariantly conserved metric. We thereby isolate the complete set of possible theories of geometrodynamics and derive a new form of a generic Einstein-type equation.

Similar to all gauge theories, the Noether approach to geometrodynamics provides the coupling of the fields of the given system to the spacetime geometry, but does not fix the Hamiltonian $\mathcal{H}_q$ resp. the Lagrangian $\mathcal{L}_R$ describing the dynamics of the “free” (uncoupled) gravitational field, hence the gravitational field dynamics in classical vacuum. The Hilbert Lagrangian $\mathcal{L}_{R,H}$—which entails the Einstein tensor of conventional General Relativity—is the simplest example. Based on analogy with other classical field theories, Einstein himself already proposed a Lagrangian $\mathcal{L}_R$ quadratic in the Riemann tensor \cite{2}, which will be discussed here as an amendment to the conventional Einstein approach.

The source term of gravity is shown to be given by the canonical energy-momentum tensor, provided that the given system has no additional symmetries—such as a SU($N$) symmetry—besides the Diff$(M)$ symmetry. This entails an increased weighting of the kinetic energy over the mass in their roles as sources of gravity. Also, a massive vector field is shown to necessarily induce a torsion of spacetime.

After reviewing in section 2 the formalism of canonical transformations in the covariant Hamiltonian description of classical field theories, we proceed in section 3 with the canonical transformation representation of finite Diff$(M)$ symmetry transformations. We then formulate in section 4 Noether’s theorem \cite{3} in the realm of classical Hamiltonian field theory. In order to work out the conserved Noether current for the Diff$(M)$ symmetry transformation, the finite transformation is reformulated in section 5 as the pertaining infinitesimal transformation. The detailed discussion of the conserved Noether current then follows in section 6. As the result, the most general Einstein-type equation of geometrodynamics is presented in section 7, where we draw upon previous publications \cite{1, 4, 5}. We discuss in section 8 possible Lagrangians $\mathcal{L}_R$ for the dynamics of the free gravitational field and propose a generalized field equation quadratic and linear in the Riemann tensor, of which the Einstein tensor is just a particular part.

2. Canonical transformations under a dynamic spacetime

2.1. Relative tensors and their transformation rules

The extended Hamiltonian formalism of field theory involves the description how dynamical quantities transform under a change of the reference frame $x \mapsto X$. This requires to generalize the transformation rules of absolute tensors to those of relative tensors. If a dynamical
quantity $t_{\alpha_1...\alpha_n}^{\beta_1...\beta_m}$ transforms to the space-time location $X$ according to

$$T_{\xi_1...\xi_n}^{\eta_1...\eta_m}(X) = t_{\alpha_1...\alpha_n}^{\beta_1...\beta_m}(x) \left| \frac{\partial X^{\xi_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial X^{\xi_n}}{\partial x^{\alpha_n}} \frac{\partial X^{\eta_1}}{\partial x^{\beta_1}} \cdots \frac{\partial X^{\eta_m}}{\partial x^{\beta_m}} \right|^w \frac{\partial x}{\partial X},$$

with $\left| \frac{\partial x}{\partial X} \right|$ the determinant of the Jacobi matrix of the transformation $x \mapsto X$.

then $t$ is referred to as a relative tensor of weight $w$. From Eq. (1), one concludes directly that the product of a relative $(i, n)$ tensor $t$ of weight $w_1$ with a relative $(j, m)$ tensor $s$ of weight $w_2$ yields a relative $(i + j, n + m)$ tensor $t \otimes s$ of weight $w_1 + w_2$. For $w = 0$, this definition includes the transformation rule for conventional tensors, which are also called absolute tensors if the distinction is to be stressed. The particular case of a relative tensor of weight $w = 1$ is also referred to briefly as a tensor density. With scalars denoting the particular class of tensors of rank zero, Eq. (1) also defines the transformation rule for scalars of weight $w$. Accordingly, a scalar of weight $w = 1$ is called a scalar density. For instance, the determinant of the contravariant representation of the metric tensor is a relative scalar of weight $w = -2$

$$G^{\mu\nu}(X) = g_{\alpha\beta}(x) \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta} \Rightarrow (\det G^{\mu\nu})(X) = (\det g^{\alpha\beta})(x) \left| \frac{\partial x}{\partial X} \right|^2.$$

Correspondingly, the covariant metric tensor transforms as

$$G_{\mu\nu}(X) = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu},$$

hence as a relative scalar of weight $w = +2$

$$(\det G_{\mu\nu})(X) = (\det g_{\alpha\beta})(x) \left| \frac{\partial x}{\partial X} \right|^2.$$

The square root of the determinant of a $(0, 2)$ tensor then transforms as a scalar density. With $g$ denoting the determinant of the covariant representation of the metric tensor $g_{\mu\nu}$,

$$g \equiv \det g_{\alpha\beta} < 0,$$

the transformation rule for $\sqrt{-g(x)} \mapsto \sqrt{-G(X)}$ follows as

$$\sqrt{-G} = \sqrt{-g} \left| \frac{\partial x}{\partial X} \right|.$$

$\sqrt{-g}$ thus represents a relative scalar of weight $w = 1$, i.e. a scalar density.

From the general rule for the derivative of the determinant of a (not necessarily symmetric) matrix $A$ with respect to a component $a_{jk}$ of $A$

$$\frac{\partial \det A}{\partial a_{jk}} = (A^{-1})_{kj} \det A \quad \Leftrightarrow \quad \frac{\partial \det A}{\partial A} = A^{-T} \det A,$$
the derivatives of $\sqrt{-g}$ with respect to a component of the covariant and of the contravariant metric are obtained as

$$\frac{\partial \sqrt{-g}}{\partial g_{\mu \nu}} = \frac{1}{2} g^{\nu \mu} \sqrt{-g}, \quad \frac{\partial \sqrt{-g}}{\partial g^{\mu \nu}} = -\frac{1}{2} g_{\nu \mu} \sqrt{-g}.$$ \hspace{1cm} (3)

The volume form $d^4x$ transforms as a relative scalar of weight $w = -1$

$$d^4X = \frac{\partial (x^0, \ldots, x^3)}{\partial (x^0, \ldots, x^3)} d^4x = |\frac{\partial X}{\partial x}| d^4x = \frac{1}{|\frac{\partial X}{\partial x}|} d^4x.$$ \hspace{1cm} (4)

In conjunction with Eq. (2), one concludes that the product $\sqrt{-g} d^4x$ transforms as a scalar of weight $w = 0$, hence as an absolute scalar

$$\sqrt{-g} d^4X = \sqrt{-g} d^4x.$$ \hspace{1cm} (5)

$\sqrt{-g} d^4x$ is thus referred to as the invariant volume form.

Another frequently used identity is

$$\frac{\partial}{\partial x^a} \left( \frac{\partial x^a}{\partial X^\beta} \left| \frac{\partial X}{\partial x} \right| \right) = \frac{\partial}{\partial x^a} \left( \frac{\partial x^a}{\partial X^\beta} \frac{1}{\left| \frac{\partial X}{\partial x} \right|} \right) = \frac{\partial^2 x^a}{\partial X^\beta \partial x^a} \frac{1}{\left| \frac{\partial X}{\partial x} \right|} - \frac{1}{\left| \frac{\partial X}{\partial x} \right|^2} \frac{\partial}{\partial x^a} \frac{\partial X^\beta}{\partial X^a}.$$ \hspace{1cm} (6)

By virtue of the general identity

$$\frac{\partial}{\partial \left( \frac{\partial x^a}{\partial X^b} \right)} = \frac{\partial X^b}{\partial x^a} \left| \frac{\partial X}{\partial x} \right|,$$

the $X^b$-derivative of $|\frac{\partial x}{\partial X}|$ in Eq. (5) is converted into

$$\frac{\partial}{\partial x^a} \frac{\partial x^a}{\partial X^b} = \frac{\partial}{\partial \left( \frac{\partial x^a}{\partial X^b} \right)} \frac{\partial x^a}{\partial X^b} = \frac{\partial^2 x^a}{\partial X^b \partial x^a} \left| \frac{\partial X}{\partial x} \right|.$$ \hspace{1cm} (7)

Inserting Eq. (7) into (6) then yields

$$\frac{\partial}{\partial x^a} \left( \frac{\partial x^a}{\partial X^b} \left| \frac{\partial X}{\partial x} \right| \right) \equiv 0.$$ \hspace{1cm} (8)

The conventional Lagrangians of field theories represent Lorentz scalars in order to be relativistically correct, hence to maintain their form under Lorentz transformations. Therefore they must transform as absolute scalars, $L' = L$. According to Eq. (2), this means for the extended Lagrangians $\tilde{L} = L \sqrt{-g}$

$$\tilde{L} = \sqrt{-g} \left( \frac{\partial x^a}{\partial X^b} \left| \frac{\partial X}{\partial x} \right| \right) = \tilde{L} \left( \frac{x_0, \ldots, x^3}{\partial X^0, \ldots, X^3} \right).$$
An extended Lagrangian $\tilde{L}$ thus represents a relative scalar of weight $w = 1$, hence a scalar density. As a consequence, the action integral maintains its form under a change of the reference frame $x \mapsto X$

$$\int_{\Omega} \tilde{L} \, d^4X = \int_{\Omega} \tilde{L}(\frac{\partial(x^0, \ldots, x^3)}{\partial(X^0, \ldots, X^3)}) \, d^4X = \int_{\Omega} \tilde{L} \, d^4x. \quad (9)$$

In the Hamiltonian formalism, the dual quantities $\pi^\mu(x)$ of the derivatives $\partial \phi/\partial x^\mu$ of a set of scalar fields $\phi$ are defined on the basis of a conventional Lagrangian $L$ as

$$\pi^\mu(x) = \frac{\partial L}{\partial (\partial \phi/\partial x^\mu)}.$$

The corresponding definition in terms of an extended Lagrangian $\tilde{L} = L \sqrt{-g}$ is then

$$\tilde{\pi}^\mu(x) = \pi^\mu(x) \sqrt{-g} = \frac{\partial \tilde{L}}{\partial (\partial \phi/\partial x^\mu)} \quad \text{and} \quad \tilde{\Pi}^\mu(X) = \Pi^\mu(X) \sqrt{-G} = \frac{\partial \tilde{L}'}{\partial (\partial \phi/\partial x^\mu)} \quad (10)$$

Therefore, $\tilde{\pi}^\mu$ can be regarded as the dual of the derivative $\partial \phi/\partial x^\mu$ with regard to the extended Lagrangian $\tilde{L}$. While $\pi^\mu$ transforms as an absolute tensor, the related $\tilde{\pi}^\mu = \pi^\mu \sqrt{-g}$ transforms as a relative vector of weight $w = 1$, hence as a vector density

$$\tilde{\Pi}^\mu(X) = \pi^\mu(x) \frac{\partial X^\mu}{\partial x^\nu} = \frac{\tilde{\Pi}^\mu(X)}{\sqrt{-G}} = \frac{\tilde{\pi}^\mu(x) \partial X^\mu}{\sqrt{-g} \partial x^\nu},$$

hence by virtue of Eq. (2)

$$\tilde{\Pi}^\mu(X) = \tilde{\pi}^\mu(x) \frac{\partial X^\mu}{\sqrt{-g}} = \tilde{\pi}^\mu(x) \frac{\partial X^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial X}. \quad (11)$$

Equation (11) is the general transformation rule for the relative vector $\tilde{\pi}^\mu(x)$ under a change of the reference frame. It coincides with the transformation rule for the corresponding absolute tensors, $\pi^\mu(x)$ and $\Pi^\mu(X)$, which must equally hold in order to be consistent. Note that according to Eq. (10), in the extended Hamiltonian formalism it is the vector density $\tilde{\pi}^\mu$ which represents the canonical conjugate of $\phi$ and the dual of $\partial \phi/\partial x^\mu$. With $\Phi(X) = \phi(x)$, Eq. (11) yields in conjunction with (5)

$$\tilde{\Pi}^\alpha(X) \frac{\partial \Phi}{\partial X^\alpha} \, d^4X = \tilde{\pi}^\beta(x) \frac{\partial \Phi}{\partial X^\beta} \frac{\partial X^\alpha}{\partial x^\nu} \left[ \frac{\partial x^\nu}{\partial X} \right] \, d^4X = \tilde{\pi}^\alpha(x) \frac{\partial \phi}{\partial x^\nu} \, d^4x. \quad (12)$$

Together with Eq. (9), one finds that the extended Hamiltonian $\tilde{H}$ also transforms as a scalar density

$$\left( \tilde{\pi}^\alpha(x) \frac{\partial \phi}{\partial x^\nu} - \tilde{H} \right) \, d^4x = \left( \tilde{\Pi}^\alpha(X) \frac{\partial \Phi}{\partial X^\alpha} - \tilde{H}' \right) \, d^4X$$

$$= \left( \tilde{\Pi}^\alpha(X) \frac{\partial \Phi}{\partial X^\alpha} - \tilde{H}' \right) \left[ \frac{\partial X}{\partial x} \right] \, d^4x.$$
If one inserts back Eq. (2)

\[
\left( \tilde{r}^{\alpha}(x) \frac{\partial \phi}{\partial x^\alpha} - \tilde{\mathcal{H}} \right) \frac{1}{\sqrt{-g}} = \left( \tilde{\Pi}^{\alpha}(X) \frac{\partial \Phi}{\partial X^\alpha} - \tilde{\mathcal{H}}' \right) \frac{1}{\sqrt{-G}},
\]

then

\[
\pi^a(x) \frac{\partial \phi}{\partial x^a} - \mathcal{H} = \Pi^a(x) \frac{\partial \Phi}{\partial x^a} - \mathcal{H}',
\]

hence the conventional action for a static spacetime is recovered for the vectors \( \pi, \Pi \) and scalars \( \mathcal{H}, \mathcal{H}' \).

2.2. Canonical transformation rules for generating functions of type \( \tilde{\mathcal{F}}^\mu_1, \tilde{\mathcal{F}}^\mu_2, \) and \( \tilde{\mathcal{F}}^\mu_3 \)

The requirement of form-invariance of the action functional under a transformation of the fields and the space-time geometry is

\[
\delta \int_{\Omega} \left( \tilde{\pi}^{\alpha} \frac{\partial \phi}{\partial x^\alpha} + \tilde{p}^a\phi \frac{\partial \phi}{\partial x^a} - \tilde{\mathcal{H}} - \frac{\partial \tilde{\mathcal{F}}^a}{\partial x^a} \right) d^4x = \delta \int_{\Omega} \left( \tilde{\Pi}^{\alpha} \frac{\partial \Phi}{\partial X^\alpha} + \tilde{P}^a \frac{\partial \Phi}{\partial X^a} - \tilde{\mathcal{H}}' \right) d^4X. \tag{14}
\]

As the action integral is to be varied, Eq. (14) implies that the integrands may differ by the divergence of a vector function \( \tilde{\mathcal{F}}^a_1 \) whose variation vanishes on the boundary \( \partial \Omega \) of the integration region \( \Omega \) within space-time.

\[
\delta \int_{\Omega} \frac{\partial \tilde{\mathcal{F}}^a_1}{\partial x^a} d^4x = \delta \int_{\partial \Omega} \tilde{\mathcal{F}}^a_1 dS_a = 0. \tag{15}
\]

The addition of a term \( \frac{\partial \tilde{\mathcal{F}}^a_1}{\partial x^a} \) to the integrand which can be converted into a surface integral—commonly referred to briefly as a surface term—thus does not modify the variation of the action integral. This means that the integrand is only determined up to the divergence of the functions \( \tilde{\mathcal{F}}^\mu_1(\Phi, \phi, A, a, x) \). With the transformation rule of the volume form from Eq. (4), and \( \tilde{\mathcal{F}}^a_1 \) to be taken at \( x \), the integrand condition for an extended canonical transformation thus writes

\[
\tilde{\pi}^{\alpha} \frac{\partial \phi}{\partial x^\alpha} + \tilde{p}^a \phi \frac{\partial \phi}{\partial x^a} - \tilde{\mathcal{H}} - \left( \tilde{\Pi}^{\alpha} \frac{\partial \Phi}{\partial X^\alpha} + \tilde{P}^a \frac{\partial \Phi}{\partial X^a} - \tilde{\mathcal{H}}' \right) \left| \frac{\partial X}{\partial x} \right| = \frac{\partial \tilde{\mathcal{F}}^\beta}{\partial \phi} \frac{\partial \phi}{\partial x^\beta} + \frac{\partial \tilde{\mathcal{F}}^\beta}{\partial \Phi} \frac{\partial \Phi}{\partial x^\beta} \frac{\partial ^\alpha X^\beta}{\partial x^\beta} + \frac{\partial \tilde{\mathcal{F}}^\beta}{\partial A^a} \frac{\partial A^a}{\partial x^\beta} + \frac{\partial \tilde{\mathcal{F}}^\beta}{\partial A^a} \frac{\partial A^a}{\partial x^\beta} \frac{\partial A^a}{\partial X^\beta} + \frac{\partial \tilde{\mathcal{F}}^\beta}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} \frac{\partial \phi}{\partial \phi} \Bigg|_{\text{expl}}. \tag{16}
\]

Comparing the coefficients yields the transformation rules

\[
\begin{align*}
\tilde{r}^{\mu}(x) &= \frac{\partial \tilde{\mathcal{F}}^\mu_1}{\partial \phi}, & \tilde{\Pi}^\mu(X) &= -\frac{\partial \tilde{\mathcal{F}}^\mu_1}{\partial \phi} \frac{\partial ^\mu X^\alpha}{\partial X^\alpha} \left| \frac{\partial X}{\partial x} \right|, \\
\tilde{p}^{\nu}(x) &= \frac{\partial \tilde{\mathcal{F}}^\mu_1}{\partial A^a}, & \tilde{P}^{\nu}(X) &= -\frac{\partial \tilde{\mathcal{F}}^\mu_1}{\partial A^a} \frac{\partial ^\nu X^\alpha}{\partial X^\alpha} \left| \frac{\partial X}{\partial x} \right|, \\
\tilde{\mathcal{H}}' &= \left( \tilde{\mathcal{H}} + \frac{\partial \tilde{\mathcal{F}}^\mu_1}{\partial x^\alpha} \left| \frac{\partial x}{\partial \phi} \right| \right) \left| \frac{\partial X}{\partial x} \right| \Rightarrow \tilde{\mathcal{H}}' d^4X = \left( \tilde{\mathcal{H}} + \frac{\partial \tilde{\mathcal{F}}^\mu_1}{\partial x^\alpha} \left| \frac{\partial x}{\partial \phi} \right| \right) d^4x. \tag{17}
\end{align*}
\]
The generating function $\tilde{F}_1^\mu(\Phi, \phi, A, a, x)$ may be Legendre-transformed into an equivalent generating function $\tilde{F}_2^\mu(\Pi, \phi, P, a, x)$ according to

$$\tilde{F}_2^\mu = \tilde{F}_1^\mu + \left(\Phi \Pi^\mu + A_\alpha P^{\mu\alpha}\right) \frac{\partial x^\nu}{\partial X^{\beta}} \frac{\partial X^{\beta}}{\partial x^\nu}.$$  

(18)

In order to derive the divergence of $\tilde{F}_2^\mu$, we make use of the identity (18) for the right-hand side factor. Thus

$$\frac{\partial \tilde{F}_2^\alpha}{\partial x^\alpha} = \frac{\partial \tilde{F}_2^\alpha}{\partial x^\alpha} - \left| \frac{\partial X}{\partial x^{\beta}} \right| \frac{\partial \tilde{F}_2^\alpha}{\partial X^{\beta}} \left(\Phi \Pi^\beta + A_\alpha \tilde{P}^{\alpha\beta}\right).$$  

(19)

Inserting Eq. (19) into the integrand condition (16), we encounter the modified integrand condition for a generating function of type $\tilde{F}_2^\mu$, to be taken at the spacetime event $x$

$$\tilde{F}_2^\beta \frac{\partial \phi}{\partial x^\beta} + \tilde{P}^{\alpha\beta} \frac{\partial a_\alpha}{\partial x^\beta} - \tilde{H} + \left(\Phi \Pi^\beta \frac{\partial \tilde{F}_2^\alpha}{\partial X^{\beta}} + A_\alpha \frac{\partial \tilde{P}^{\alpha\beta}}{\partial X^{\beta}} + \tilde{H}'\right) \left| \frac{\partial X}{\partial x^\beta} \right|, $$

(20)

and hence the transformation rules

$$\tilde{\Pi}^\mu(x) = \frac{\partial \tilde{F}_2^\mu}{\partial \phi}, \quad \delta^\nu \Phi(X) = \frac{\partial \tilde{F}_2^\alpha}{\partial \Pi^\nu} \frac{\partial X^\alpha}{\partial \phi} \frac{\partial x}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\nu},$$

$$\tilde{P}^{\mu\nu}(x) = \frac{\partial \tilde{F}_2^\mu}{\partial a_\nu}, \quad \delta^\mu A_\alpha(X) = \frac{\partial \tilde{F}_2^\alpha}{\partial P^{\mu\nu}} \frac{\partial X^\alpha}{\partial a_\nu} \frac{\partial x}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\mu},$$

$$\tilde{H}' = \left(\tilde{H} + \frac{\partial \tilde{F}_2^\alpha}{\partial x^\alpha} \left| \frac{\partial X}{\partial x^\beta} \right| \frac{\partial x}{\partial X^\beta} \right) \Rightarrow \tilde{H}' \, d^4 x = \left(\tilde{H} + \frac{\partial \tilde{F}_2^\alpha}{\partial x^\alpha} \left| \frac{\partial X}{\partial x^\beta} \right| \frac{\partial x}{\partial X^\beta} \right) d^4 x.$$  

(21)

While the extended Hamiltonian does not necessarily represent a scalar density, the total integrands in the action integrals (14) must be world scalars in order to keep their form under general space-time transformations. This ensures the canonical field equations to emerge as tensor equations. We remark that the extended generating function (18) reduces to a conventional one [4] — which does not define a mapping of the spacetime $x \mapsto X$ — if Eq. (2) is inserted and all fields are taken at $x$:  

$$F_2^\mu = F_1^\mu + \Phi \Pi^\mu + A_\alpha P^{\mu\alpha}.$$  

(22)

The generating function $\tilde{F}_1^\mu(\Phi, \phi, A, a, x)$ may also be Legendre-transformed into an equivalent generating function of type $\tilde{F}_3^\mu(\bar{\varphi}, \Phi, \bar{p}, A, x)$ according to

$$\tilde{F}_3^\mu = \tilde{F}_1^\mu - \phi \bar{\varphi}^\mu - a_\alpha \bar{p}^{\mu\alpha},$$

(23)

hence

$$\frac{\partial \tilde{F}_3^\alpha}{\partial x^\alpha} = \frac{\partial \tilde{F}_1^\alpha}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha} \left(\phi \bar{\varphi}^\alpha + a_\alpha \bar{p}^{\alpha\beta}\right).$$  

(24)
Inserting Eq. (24) into the integrand condition (16), we encounter the modified integrand condition for a generating function of type \( \tilde{F}^\mu_3 \), to be taken at the spacetime event \( x \)

\[
- \phi \delta^\mu_\xi \frac{\partial \tilde{\pi}^\xi}{\partial x^\beta} - a_\alpha \delta^\mu_\xi \frac{\partial \tilde{P}^\alpha_\xi}{\partial x^\beta} - \tilde{\mathcal{H}} - \left( \tilde{\Pi}^\beta \frac{\partial \Phi}{\partial X^\beta} + \tilde{P}^{\alpha\beta} \frac{\partial A^\alpha}{\partial x^\beta} - \tilde{\mathcal{H}}' \right) \frac{\partial X^\beta}{\partial x^\beta} \\
= \frac{\partial \tilde{F}^\eta_3}{\partial \Phi} \frac{\partial X^\beta}{\partial x^\eta} + \frac{\partial \tilde{F}^\eta_3}{\partial A_\alpha} \frac{\partial X^\beta}{\partial X^\alpha} + \frac{\partial \tilde{F}^\beta_3}{\partial \tilde{\pi}^\eta} \frac{\partial \tilde{P}^\alpha_\xi}{\partial x^\beta} + \frac{\partial \tilde{F}^\eta_3}{\partial \tilde{\pi}^\alpha} \frac{\partial \tilde{P}^\beta_\xi}{\partial x^\beta} + \frac{\partial \tilde{F}^\eta_3}{\partial x^\alpha} \tag{25}
\]

and hence the transformation rules by comparing the coefficients

\[
\tilde{\Pi}^\mu(X) = - \frac{\partial \tilde{F}^\eta_3}{\partial \Phi} \frac{\partial X^\mu}{\partial x^\eta} \quad \quad \tilde{\mathcal{H}}' = \left( \tilde{\mathcal{H}} + \frac{\partial \tilde{F}^\alpha_3}{\partial x^\alpha} \right) \frac{\partial X^\beta}{\partial x^\beta} \\
\tilde{P}^{\alpha\nu}(X) = - \frac{\partial \tilde{F}^\eta_3}{\partial A_\alpha} \frac{\partial X^\nu}{\partial x^\eta} \quad \quad \Rightarrow \tilde{\Pi}^\mu d^4X = \left( \tilde{\mathcal{H}} + \frac{\partial \tilde{F}^\alpha_3}{\partial x^\alpha} \right) d^4x. \tag{26}
\]

3. Finite Diff(M) symmetry transformation

For a system of a scalar field \( \phi \) and a vector field \( a_\mu \), the generating function of type \( \tilde{F}^\mu_3 \) for the finite canonical transformation of the Diff(M) symmetry group is

\[
\tilde{F}^\mu_3 \bigg|_{x} = - \tilde{\phi} \Phi - \tilde{\mathcal{H}} + \tilde{\pi}^{\alpha\mu} \frac{\partial X^\mu}{\partial x^\alpha} - \tilde{\mathcal{H}} + \tilde{\mathcal{H}}' \quad \quad \Rightarrow \tilde{F}^\mu_3 \bigg|_{x} = \tilde{\mathcal{H}} + \tilde{\mathcal{H}}' \quad \quad \tilde{\mathcal{H}}' = \left( \tilde{\mathcal{H}} + \tilde{\mathcal{H}}' \right) \frac{\partial X^\beta}{\partial x^\beta} \tag{27}
\]

Here and in the following, a tilde denotes that the respective quantity represents a tensor density, i.e., a relative tensor of weight \( w = 1 \). The transformed metric tensor \( G_{\xi\lambda}(X) \) is symmetric, which induces the tensor \( \tilde{K}^{\alpha\mu}_{\xi\lambda} \) to be symmetric in its first index pair, \( \alpha, \beta \). The general rules for extended generating functions of type \( \tilde{F}^\mu_3 \) from Eqs. (26) and similar ones for the real tensors \( \tilde{\mathcal{H}}, G, \tilde{q}, \) and for the connection \( \Gamma \):

\[
\begin{align*}
\delta^\mu_\alpha g_{\alpha\beta} &= \frac{\partial \tilde{F}^\mu_3}{\partial \tilde{q}^{\alpha\beta}}, \\
\delta^\mu_\alpha \gamma_{\alpha\beta} &= \frac{\partial \tilde{F}^\alpha_3}{\partial \tilde{q}^{\alpha\beta}}, \\
\tilde{K}^{\xi\lambda}_{\alpha\mu} &= \frac{\partial \tilde{F}^\xi_3}{\partial G_{\xi\lambda}} \frac{\partial X^\alpha}{\partial x^\lambda} \frac{\partial x^\beta}{\partial X^\mu}, \\
\tilde{Q}^{\xi\lambda}_{\alpha\mu} &= \frac{\partial \tilde{F}^\xi_3}{\partial \Gamma^\xi_{\alpha\lambda}} \frac{\partial X^\mu}{\partial x^\mu} \frac{\partial x^\beta}{\partial X^\lambda} \frac{\partial x^\lambda}{\partial x^\lambda} \tag{28}
\end{align*}
\]

yield the following particular rules for the specific generating function (27)

\[
\begin{align*}
\phi(x) &= \Phi(X), \\
\tilde{\Pi}^\mu(X) &= \tilde{\pi}^\mu(x) \frac{\partial X^\mu}{\partial x^\mu} \frac{\partial x^\beta}{\partial X^\beta}, \\
\tilde{P}^{\alpha\nu}(X) &= \tilde{p}^{\alpha\nu}(x) \frac{\partial X^\nu}{\partial x^\nu} \frac{\partial x^\beta}{\partial X^\beta} \tag{29}
\end{align*}
\]
and
\[ g_{\alpha \beta}(x) = G_{\varepsilon \lambda}(X) \frac{\partial X^\xi}{\partial x^\alpha} \frac{\partial X^\lambda}{\partial x^\beta}, \]
\[ \gamma'_{\alpha \beta}(x) = \Gamma'_{\varepsilon \lambda}(X) \frac{\partial X^\xi}{\partial x^\alpha} \frac{\partial X^\lambda}{\partial x^\beta} + \frac{\partial x^\eta}{\partial x^\xi} \frac{\partial^2 X^\nu}{\partial x^\alpha \partial x^\beta}, \]
\[ \tilde{K}^{\varepsilon \mu}(X) = \tilde{k}^{\varepsilon \mu}(x) \frac{\partial X^\xi}{\partial x^\alpha} \frac{\partial X^\lambda}{\partial x^\beta} \frac{\partial \chi^\nu}{\partial x^\alpha} \frac{\partial x^\mu}{\partial X^\lambda}, \quad \tilde{Q}^{\varepsilon \mu}(X) = \tilde{q}^{\varepsilon \mu}(x) \frac{\partial X^\xi}{\partial x^\alpha} \frac{\partial X^\lambda}{\partial x^\beta} \frac{\partial \chi^\nu}{\partial x^\alpha} \frac{\partial x^\mu}{\partial X^\lambda}. \] (30)

Finally, the transformation rule for the covariant Hamiltonian is given by
\[ \mathcal{H}'_\text{ex} = \mathcal{H}_x + \left[ \frac{\partial \mathcal{F}_\text{ex}}{\partial x^\alpha} \right] \left[ \frac{\partial x^\alpha}{\partial X^\beta} \right]. \] (31)

Thus, the transformation following from the generating function \( \mathcal{F}_\text{ex} \) defines a symmetry transformation, in the sense that any action integral in one chart of the manifold \( M \) is mapped to an action integral of the same form in another chart. In other words, the transformation implements the Principle of General Relativity.

4. Generalized Noether Theorem

For a sample system of a scalar field \( \phi \) and a vector field \( a_\mu \), the infinitesimal transformation rules are derived from the generating function
\[ \tilde{J}^{\mu}_N(x) = -\bar{\rho}^\mu(x) - \bar{p}^{\mu a}(x) - \tilde{k}^{\varepsilon \mu} G_{\varepsilon \lambda}(x) - \bar{q}^{\alpha \beta \mu} \bar{m}^{\gamma a}(x) - \bar{e} \tilde{j}^{\mu}_N(x). \] (32)

For \( \epsilon = 0 \), Eq. (32) thus generates the identity transformation for all dynamical quantities involved. All contributions of the general transformation rules (26), (28), and (31) that are associated with a non-identical mapping of fields and spacetime must be encoded in the particular expression for \( \tilde{j}^{\mu}_N(x) \). The general form of the transformation rules are
\[ \delta \phi = -\epsilon \frac{\partial \mathcal{F}^{\mu N}}{\partial \phi}, \quad \delta \bar{\rho}^\mu = \epsilon \frac{\partial \mathcal{F}^{\mu N}}{\partial \phi}, \quad \delta \bar{p}^{\mu a} = -\epsilon \frac{\partial \mathcal{F}^{\mu N}}{\partial a_\mu}, \quad \delta \bar{q}^{\alpha \beta \mu} = \epsilon \frac{\partial \mathcal{F}^{\mu N}}{\partial q^{\alpha \beta \mu}} \] (33)

and
\[ \delta \mathcal{H}_\text{CT}^{\mu} = -\epsilon \frac{\partial \mathcal{F}^{\mu N}}{\partial x^\alpha}. \]

On the other hand, for a closed system, where the Hamiltonian does not explicitly depend on \( x \), the variation of the Hamiltonian emerging from the variation of the fields follows as
\[ \delta \mathcal{H} = \frac{\partial \mathcal{H}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{H}}{\partial \bar{\rho}^\mu} \delta \bar{\rho}^\mu + \frac{\partial \mathcal{H}}{\partial \bar{p}^{\mu a}} \delta \bar{p}^{\mu a} + \frac{\partial \mathcal{H}}{\partial \bar{q}^{\alpha \beta \mu}} \delta \bar{q}^{\alpha \beta \mu} + \frac{\partial \mathcal{H}}{\partial \mathcal{H}_\text{CT}^{\mu}} \delta \mathcal{H}_\text{CT}^{\mu} + \frac{\partial \mathcal{H}}{\partial g^{\varepsilon \mu}} \delta g^{\varepsilon \mu} + \frac{\partial \mathcal{H}}{\partial \bar{q}^{\alpha \beta \mu}} \delta \bar{q}^{\alpha \beta \mu} + \frac{\partial \mathcal{H}}{\partial \tau^{\varepsilon \mu}} \delta \tau^{\varepsilon \mu} + \frac{\partial \mathcal{H}}{\partial \bar{m}^{\alpha \beta \mu}} \delta \bar{m}^{\alpha \beta \mu}. \]

Inserting the standard form of the canonical field equations, the variation of \( \mathcal{H} \) is expressed along the system’s spacetime evolution
\[ \delta \mathcal{H} = \frac{\partial \mathcal{H}}{\partial x^\alpha} \delta x^\alpha + \frac{\partial \mathcal{H}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{H}}{\partial \bar{\rho}^\mu} \delta \bar{\rho}^\mu + \frac{\partial \mathcal{H}}{\partial \bar{p}^{\mu a}} \delta \bar{p}^{\mu a} + \frac{\partial \mathcal{H}}{\partial \bar{q}^{\alpha \beta \mu}} \delta \bar{q}^{\alpha \beta \mu} + \frac{\partial \mathcal{H}}{\partial \mathcal{H}_\text{CT}^{\mu}} \delta \mathcal{H}_\text{CT}^{\mu} + \frac{\partial \mathcal{H}}{\partial g^{\varepsilon \mu}} \delta g^{\varepsilon \mu} + \frac{\partial \mathcal{H}}{\partial \bar{q}^{\alpha \beta \mu}} \delta \bar{q}^{\alpha \beta \mu} + \frac{\partial \mathcal{H}}{\partial \tau^{\varepsilon \mu}} \delta \tau^{\varepsilon \mu} + \frac{\partial \mathcal{H}}{\partial \bar{m}^{\alpha \beta \mu}} \delta \bar{m}^{\alpha \beta \mu}. \]
With the transformation rules (33), this writes in terms of the derivatives of the Noether current
\[
\delta \mathcal{H} = \epsilon \frac{\partial \tilde{\mathcal{H}}^\alpha}{\partial \dot{x}^\alpha} \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\rho}^\beta} + \epsilon \frac{\partial \phi}{\partial \dot{x}^\alpha} \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}} + \epsilon \frac{\partial \tilde{\mathcal{H}}^\beta}{\partial \dot{\phi}^\gamma} \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma} + \epsilon \frac{\partial d_x}{\partial x^\alpha} \frac{\partial \tilde{J}_N^\alpha}{\partial x^\alpha} + \epsilon \frac{\partial \tilde{\mathcal{H}}^{\alpha \beta}}{\partial \dot{\phi}^\gamma} \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma}
\]
\[
+ \epsilon \frac{\partial \tilde{\mathcal{H}}^{\gamma \beta}}{\partial \dot{\phi}^\gamma} \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma} + \epsilon \frac{\partial \tilde{\mathcal{H}}^{\gamma \beta}}{\partial \dot{\phi}^\gamma} \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma} + \epsilon \frac{\partial \tilde{\mathcal{H}}^{\gamma \beta}}{\partial \dot{\phi}^\gamma} \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma} + \epsilon \frac{\partial \tilde{\mathcal{H}}^{\beta \gamma}}{\partial \dot{\phi}^\gamma} \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma}
\]
\[
= \epsilon \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{x}^\alpha} - \epsilon \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma} \bigg|_{\text{expl}}.
\]
Both variations, \(\delta \tilde{\mathcal{H}}\) and \(\delta \tilde{\mathcal{H}}|_{\text{CT}}\), must coincide in order for the canonical transformation to define a symmetry transformation. Then, the vector function \(\tilde{J}_N^\alpha\) defines the associated conserved Noether current
\[
\delta \tilde{\mathcal{H}} = \delta \tilde{\mathcal{H}}|_{\text{CT}} \iff \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{x}^\alpha} - \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma} \bigg|_{\text{expl}} = - \frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma} \bigg|_{\text{expl}},
\]
hence
\[
\frac{\partial \tilde{J}_N^\alpha}{\partial \dot{\phi}^\gamma} = 0.
\]

5. Infinitesimal \emph{Diff}(M) symmetry transformation

In order to work out the particular form of the conserved Noether current \(\tilde{J}_N^\alpha\) which is associated with this symmetry transformation, the generating function of the corresponding \emph{infinitesimal} canonical transformation needs to be set up first. To this end, a function \(h^\mu(x)\) must be introduced, which defines an \emph{infinitesimal local transition map} in space-time \(x^\mu \mapsto X^\mu\)
\[
X^\mu = x^\mu + \epsilon h^\mu(x),
\]
with \(\epsilon \ll 1\). To first order in \(\epsilon\), the space-time dependent coefficients of (27) are then expressed according to
\[
\frac{\partial X^\mu}{\partial x^\nu} = \delta^\mu_\nu + \epsilon \frac{\partial h^\mu}{\partial x^\nu}, \quad \frac{\partial x^\mu}{\partial X^\nu} = \delta^\mu_\nu - \epsilon \frac{\partial h^\nu}{\partial x^\mu} - \epsilon^2 \frac{\partial^2 h^\nu}{\partial x^\mu \partial x^\nu}, \quad \frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\rho} = \epsilon \frac{\partial^2 h^\rho}{\partial x^\nu \partial x^\mu}.
\]
Accordingly, the finite transformation rules (29) and (30) of the fields can be expressed up to first order in \(\epsilon\) as
\[
\delta \phi(x) = \Phi(x) - \phi(x) = -\epsilon h^\rho \frac{\partial \phi}{\partial x^\rho}
\]
\[
\delta a_\alpha(x) = A_\alpha(x) - a_\alpha(x) = -\epsilon \left( h^\rho \frac{\partial a_\alpha}{\partial x^\rho} + a_\rho \frac{\partial h^\rho}{\partial x^\alpha} \right)
\]
\[
\delta g_{\xi \lambda}(x) = G_{\xi \lambda}(x) - g_{\xi \lambda}(x) = -\epsilon \left( h^\rho \frac{\partial g_{\xi \lambda}}{\partial x^\rho} + g_{\rho \gamma} \frac{\partial h^\gamma}{\partial x^\lambda} + g_{\rho \delta} \frac{\partial h^\delta}{\partial x^\lambda} \right)
\]
\[
\delta \gamma^\rho_{\lambda\tau}(x) = \Gamma^\rho_{\lambda\tau}(x) - \gamma^\rho_{\lambda\tau}(x) = -\epsilon \left( h^\rho \frac{\partial \gamma^\rho_{\lambda\tau}}{\partial x^\rho} - \gamma^\beta_{\lambda\rho} \frac{\partial h^\rho}{\partial x^\beta} + \gamma^\rho_{\beta\tau} \frac{\partial h^\rho}{\partial x^\beta} + \gamma^\rho_{\beta\gamma} \frac{\partial h^\rho}{\partial x^\beta} + \gamma^\rho_{\beta\tau} \frac{\partial h^\rho}{\partial x^\beta} \right).
\]
The corresponding rules for the conjugate momenta are

\[ \delta \tilde{\pi}^\mu (x) = \tilde{\Pi}^\mu (x) - \tilde{\pi}^\mu (x) = \epsilon \left[ -h^\mu_\rho \frac{\partial \tilde{\pi}^\rho}{\partial \chi^\nu} + \tilde{\pi}^\rho \left( \frac{\partial h^\mu_\rho}{\partial \chi^\nu} - \delta^\mu_\rho \frac{\partial h^\nu_\tau}{\partial \chi^\tau} \right) \right] \]

\[ \delta \tilde{p}^{\alpha \mu} (x) = \tilde{P}^{\alpha \mu} (x) - \tilde{p}^{\alpha \mu} (x) = \epsilon \left[ -h^\mu_\rho \frac{\partial \tilde{p}^{\alpha \mu}}{\partial \chi^\nu} + \tilde{p}^{\alpha \nu} \left( \frac{\partial h^\mu_\rho}{\partial \chi^\nu} - \frac{\partial h^\alpha_\rho}{\partial \chi^\mu} \right) \right] \]

\[ \delta \tilde{k}^{\lambda \tau \mu} (x) = \tilde{K}^{\lambda \tau \mu} (x) - \tilde{k}^{\lambda \tau \mu} (x) = \epsilon \left[ -h^\mu_\rho \frac{\partial \tilde{k}^{\lambda \tau \mu}}{\partial \chi^\nu} + \tilde{k}^{\lambda \rho \tau} \left( \frac{\partial h^\mu_\rho}{\partial \chi^\nu} - \frac{\partial h^\tau_\rho}{\partial \chi^\mu} \right) \right] \]

\[ \delta \tilde{q}^{\lambda \tau \mu} (x) = \tilde{Q}^{\lambda \tau \mu} (x) - \tilde{q}^{\lambda \tau \mu} (x) = \epsilon \left[ -h^\mu_\rho \frac{\partial \tilde{q}^{\lambda \tau \mu}}{\partial \chi^\nu} - \tilde{q}^{\lambda \rho \tau} \left( \frac{\partial h^\mu_\rho}{\partial \chi^\nu} - \delta^\mu_\rho \frac{\partial h^\tau_\rho}{\partial \chi^\mu} \right) \right] \]  \hfill (37)

As a particular feature of the covariant Hamiltonian formalism of field theories, merely the divergences of the momentum fields are determined by the system’s Hamiltonian and not the individual components of the canonical momentum tensor. The transformation rule for the divergence of the momenta is set up on the basis of Eqs. (37)

\[ \frac{\partial \tilde{\Pi}^\mu}{\partial \chi^\nu} - \frac{\partial \tilde{\pi}^\mu}{\partial \chi^\nu} = \epsilon \left[ -h^\mu_\rho \frac{\partial \tilde{\pi}^\rho}{\partial \chi^\nu} + \tilde{\pi}^\rho \left( \frac{\partial h^\mu_\rho}{\partial \chi^\nu} - \delta^\mu_\rho \frac{\partial h^\nu_\tau}{\partial \chi^\tau} \right) \right] \]

\[ = -\epsilon \left( h^\mu_\rho \frac{\partial \tilde{\pi}^\rho}{\partial \chi^\nu} + \tilde{\pi}^\rho \frac{\partial h^\mu_\rho}{\partial \chi^\nu} \right) \]

\[ = -\epsilon \frac{\partial}{\partial \chi^\mu} \left( h^\mu_\rho \frac{\partial \tilde{\pi}^\rho}{\partial \chi^\nu} \right) \]

The transformation rule for the momenta \( \tilde{\pi}^\mu \) from Eq. (37) can thus equivalently be expressed as

\[ \delta \tilde{\pi}^\mu = \tilde{\Pi}^\mu - \tilde{\pi}^\mu = -\epsilon h^\nu_\rho \frac{\partial \tilde{\pi}^\rho}{\partial \chi^\nu}, \]  \hfill (38)

which amounts to replacing the initial momentum tensor \( \tilde{\pi}^\mu \) by an equivalent tensor with the same divergence. Similarly, the divergences of the momenta \( \tilde{p}^{\alpha \mu} \) transform as

\[ \frac{\partial \tilde{P}^{\alpha \mu}}{\partial \chi^\nu} - \frac{\partial \tilde{p}^{\alpha \mu}}{\partial \chi^\nu} = \epsilon \left[ -h^\mu_\rho \frac{\partial \tilde{p}^{\alpha \mu}}{\partial \chi^\nu} + \tilde{p}^{\alpha \nu} \left( \frac{\partial h^\mu_\rho}{\partial \chi^\nu} - \frac{\partial h^\alpha_\rho}{\partial \chi^\mu} \right) \right] \]

\[ = -\epsilon \left( h^\mu_\rho \frac{\partial \tilde{p}^{\alpha \mu}}{\partial \chi^\nu} + \tilde{p}^{\alpha \nu} \frac{\partial h^\mu_\rho}{\partial \chi^\nu} \right) \]

\[ = -\epsilon \frac{\partial}{\partial \chi^\mu} \left( h^\mu_\rho \frac{\partial \tilde{p}^{\alpha \mu}}{\partial \chi^\nu} \right) \]

The transformation rule for the momenta \( \tilde{p}^{\alpha \mu} \) from Eq. (37) can thus equivalently be expressed as

\[ \delta \tilde{p}^{\alpha \mu} = \tilde{P}^{\alpha \mu} - \tilde{p}^{\alpha \mu} = -\epsilon \left( h^\nu_\rho \frac{\partial \tilde{p}^{\alpha \mu}}{\partial \chi^\nu} + \frac{\partial h^\alpha_\rho}{\partial \chi^\mu} \right). \]  \hfill (39)
The corresponding rules apply for the momenta ̃κ_λ^μ and ̃q_η^λμ. Summarizing, the canonical transformation rules (43) for an infinitesimal local translation (35) are

\[\begin{align*}
\delta^\mu \phi &= -\epsilon \frac{\partial j^\mu_\phi}{\partial x^\rho} \\
\delta \eta_\alpha &= \epsilon \frac{\partial j^\mu_\eta}{\partial x^\rho} \\
\delta^\mu \partial_\alpha &= -\epsilon \frac{\partial j^\mu_{\partial_\alpha}}{\partial x^\rho} \\
\delta \bar{p}^\mu &= -\epsilon \frac{\partial j^\mu_\bar{p}}{\partial x^\rho} \\
\delta \tilde{\kappa}^\mu_\lambda &= -\epsilon \frac{\partial j^\mu_{\tilde{\kappa}^\mu_\lambda}}{\partial x^\rho} \\
\delta \tilde{q}^\mu_\eta &= -\epsilon \frac{\partial j^\mu_{\tilde{q}^\mu_\eta}}{\partial x^\rho}
\end{align*}\]

The infinitesimal transformation rule for the covariant Hamiltonian follows from Eq. (41)

\[\delta \bar{H}_{|_x} = \left( \mathcal{H}_{|_x} + \frac{\partial \mathcal{F}^\mu_3}{\partial x^\rho} \right) \left( 1 - \epsilon \frac{\partial h^\rho}{\partial x^\tau} \right) - \epsilon h^\rho \frac{\partial \mathcal{F}^\rho_4}{\partial x^\tau},\]

hence

\[\delta \bar{H}_{|_x} = \frac{\partial \mathcal{F}^\mu_3}{\partial x^\rho} \left( 1 - \epsilon \frac{\partial h^\rho}{\partial x^\tau} \right) - \epsilon h^\rho \frac{\partial \mathcal{F}^\rho_4}{\partial x^\tau},\]

The conserved Noether current \(\bar{j}^\mu_\lambda\), which defines the particular infinitesimal symmetry transformation rules (40) and (41) according to the general rules from Eqs. (33), follows as

\[\bar{j}^\mu_\lambda = h^\beta \tilde{B}^\mu_\beta + \frac{\partial h^\beta}{\partial x^\rho} \tilde{C}^\mu_\beta + \frac{\partial^2 h^\beta}{\partial x^\rho \partial x^\tau} \tilde{q}^\mu_\beta \]

wherein \(\tilde{B}^\mu_\beta\) and \(\tilde{C}^\mu_\beta\) are given by

\[\begin{align*}
\tilde{B}^\mu_\beta &= \pi^\mu \frac{\partial \phi}{\partial x^\rho} + \bar{p}^\mu \frac{\partial a_\alpha}{\partial x^\rho} + \bar{k}^{\alpha \mu \lambda} \frac{\partial g_{\alpha \lambda}}{\partial x^\rho} + \bar{q}_\eta^{\alpha \mu \lambda} \frac{\partial \gamma^\eta}{\partial x^\rho} \\
&- \delta^\rho_\beta \left( \pi^\rho \frac{\partial \phi}{\partial x^\rho} + \bar{p}^\rho \frac{\partial a_\alpha}{\partial x^\rho} + \bar{k}^{\alpha \mu \lambda} \frac{\partial g_{\alpha \lambda}}{\partial x^\rho} + \bar{q}_\eta^{\alpha \mu \lambda} \frac{\partial \gamma^\eta}{\partial x^\rho} \right) \\
\tilde{C}^\mu_\beta &= \bar{p}^\mu a_\beta + 2 \bar{k}^{\lambda \beta \mu} g_{\lambda \beta} + \bar{q}_\eta^{\lambda \beta} \gamma^\eta_{\lambda \beta} + \bar{q}_\eta^{\alpha \lambda \beta} \gamma^\eta_{\lambda \beta} - \bar{q}_\eta^{\eta \mu \beta} \gamma^\eta_{\eta \beta}
\end{align*}\]
\( \tilde{B}_\beta^\mu \) is actually a local Hamiltonian representation of the canonical energy-momentum pseudo-tensor of the total system. Correspondingly, \( C_\beta^{\alpha\mu} \) also does not have tensor property. Yet, the final field equations emerging from the vanishing divergence of the Noether current will all turn out to be tensor equations. Under a static spacetime geometry, the canonical energy-momentum tensor \( B_\alpha^\beta \) is well-known to represent a conserved Noether current, hence \( \partial B_\alpha^\beta / \partial x^\beta = 0 \), if the given system is invariant under a translation along the coordinate \( x^\beta \). The Noether current (42) thus generalizes this correlation to a dynamic spacetime geometry, with \( h^\mu = h^\mu(x) \) now an arbitrary vector function of spacetime.

6. Discussion of the conserved Noether current

For the particular case \( h^\beta = \text{const.} \), the canonical energy-momentum tensor directly provides a conserved Noether current if the given system is invariant under global translations in spacetime. For the general case, the divergence of the Noether current (42) is obtained as

\[
\frac{\partial \tilde{j}_N^\mu}{\partial x^\nu} = h^\mu \frac{\partial \tilde{B}_\beta^\mu}{\partial x^\nu} + \frac{\partial h^\mu}{\partial x^\nu} \left( \tilde{B}_\beta^\alpha + \frac{\partial \tilde{C}_\beta^{\alpha\mu}}{\partial x^\nu} \right) + \frac{\partial^2 h^\mu}{\partial x^\alpha \partial x^\nu} \left( \tilde{C}_\beta^{\alpha\eta} + \frac{\partial \tilde{q}_\beta^{\alpha\eta \mu}}{\partial x^\nu} \right) + \frac{\partial^3 h^\mu}{\partial x^\alpha \partial x^\nu \partial x^\eta} \tilde{q}_\beta^{\alpha \eta \mu} = 0. \tag{45}
\]

With this equation involving a vanishing partial derivative of the Noether current \( \tilde{j}_N \), it establishes a proper (local) conservation law. Yet, the field equations emerging from Eq. (45) will turn out to be tensor equations and thus hold invariantly in any reference frame. As \( h^\beta(x) \) is supposed to be an arbitrary function of \( x \), Eq. (45) entails four separate conditions for each order of derivatives of \( h^\beta(x) \). These will be worked out in the following sections.

6.1. Condition 1: term proportional to the third partial derivatives of \( h^\beta \)

The term proportional to the third derivative of \( h^\beta \) is the canonical momentum \( \tilde{q}_\beta^{\alpha\eta \mu} \) alone, hence the dual of the partial \( x^\nu \)-derivative of the connection \( \gamma^\beta_{\alpha\eta} \). A necessary and sufficient condition for this term to vanish is that the (generally non-zero) momentum \( \tilde{q}_\beta^{\alpha\eta \mu} \) is skew-symmetric in one of the index pairs formed out of \( \alpha, \eta, \) and \( \mu \). We choose here the last index pair, namely \( \eta \) and \( \mu \), and define

\[
\tilde{q}_\beta^{\alpha\eta \mu} = -\tilde{q}_\beta^{\alpha\mu \eta} \tag{46}
\]

which implies that \( \tilde{q}_\beta^{\alpha\eta \mu} \) need not in addition be skew-symmetric in \( \alpha \) and \( \eta \). Equation (45) then simplifies to

\[
\frac{\partial \tilde{j}_N^\mu}{\partial x^\nu} = h^\mu \frac{\partial \tilde{B}_\beta^\mu}{\partial x^\nu} + \frac{\partial h^\mu}{\partial x^\nu} \left( \tilde{B}_\beta^\alpha + \frac{\partial \tilde{C}_\beta^{\alpha\mu}}{\partial x^\nu} \right) + \frac{\partial^2 h^\mu}{\partial x^\alpha \partial x^\nu} \left( \tilde{C}_\beta^{\alpha\eta} + \frac{\partial \tilde{q}_\beta^{\alpha\eta \mu}}{\partial x^\nu} \right) + \frac{\partial^3 h^\mu}{\partial x^\alpha \partial x^\nu \partial x^\eta} \tilde{q}_\beta^{\alpha \eta \mu} = 0. \tag{47}
\]

with \( \tilde{C}_\beta^{\alpha\eta} \) inserted from Eq. (44) in the last term. Due to the symmetry of the second partial derivatives of \( h^\beta \) in \( \alpha \) and \( \eta \), the term \( \tilde{q}_\beta^{\alpha\eta \mu} \gamma^\mu_{\beta\eta} \) contained in the definition of \( \tilde{C}_\beta^{\alpha\eta} \) drops out of the right-hand side of (47) by virtue of the skew-symmetry condition (46).
6.2. Condition 2: terms proportional to the second partial derivatives of $h^\beta$

A zero divergence of the Noether current for any symmetry transformation—hence for arbitrary functions $h^\beta(x)$—requires in particular that the sum of terms related to the second derivatives of $h^\beta(x)$ in Eq. (47) vanishes. As no symmetries in $\alpha$ and $\eta$ are implied in those terms, one encounters the condition

$$
\frac{\partial \tilde{q}_\beta^{\alpha\mu}}{\partial x^\mu} + \tilde{p}^\eta a_\beta + 2\tilde{k}^{\lambda\eta} g_{\lambda\beta} - \tilde{q}_\xi^{\alpha\lambda} \gamma^\xi_{\beta\lambda} - \tilde{q}_\beta^{\xi\eta} \gamma^\eta_{\xi\lambda} = 0 \quad (48)
$$

We can express Eq. (48) equivalently as the tensor equation

$$
\tilde{q}_\beta^{\alpha\mu} - \tilde{p}^\eta a_\beta + 2\tilde{k}^{\lambda\eta} g_{\lambda\beta} - \tilde{q}_\xi^{\alpha\lambda} \gamma^\xi_{\beta\lambda} - \tilde{q}_\beta^{\xi\eta} \gamma^\eta_{\xi\lambda} = 0, \quad (49)
$$

with $s^\eta_{\tau\mu} \equiv \gamma^\eta_{[\tau\mu]}$ the Cartan torsion tensor. It agrees with the corresponding field equation (56) of Ref. [1]. Its implications will be discussed in Sects. 7.3 and 7.2.

With Eq. (48), the condition (47) for the divergence of the Noether current now further simplifies to

$$
\frac{\partial \tilde{J}^\mu_N}{\partial x^\mu} = h_\beta \frac{\partial \tilde{B}^\mu_\beta}{\partial x^\mu} + \partial h^\beta \left( \tilde{B}^\alpha_\beta + \frac{\partial \tilde{C}^{\alpha\mu}}{\partial x^\mu} \right). \quad (50)
$$

6.3. Condition 3: terms proportional to the first partial derivatives of $h^\beta$

For a generally conserved Noether current, the coefficient proportional to the first derivative of $h^\beta$ in Eq. (50) must vanish as well, hence

$$
\tilde{B}^\alpha_\beta + \frac{\partial \tilde{C}^{\alpha\mu}}{\partial x^\mu} = 0. \quad (51)
$$

Equation (51) writes in expanded form with $\tilde{C}^{\alpha\mu}_\beta$ from Eq. (44)

$$
\tilde{B}^\alpha_\beta + \frac{\partial}{\partial x^\mu} \left( \tilde{p}^{\alpha\mu} a_\beta + 2\tilde{k}^{\lambda\mu\eta} g_{\lambda\eta} + \tilde{q}_\xi^{\alpha\lambda\mu} \gamma^\xi_{\lambda\beta\eta} + \tilde{q}_\eta^{\alpha\lambda\mu} \gamma^\eta_{\beta\lambda\nu} - \tilde{q}_\eta^{\lambda\mu\eta} \gamma^\eta_{\lambda\nu} \right) = 0,
$$

which is expressed equivalently inserting Eq. (48)

$$
\tilde{B}^\alpha_\beta + \frac{\partial}{\partial x^\mu} \left( \tilde{q}^{\alpha\eta\lambda\mu} \gamma^\eta_{\lambda\beta\nu} + \tilde{q}_\eta^{\alpha\lambda\mu} \gamma^\eta_{\lambda\beta\nu} \right) = 0.
$$

This equation reduces due to the skew-symmetry of $\tilde{q}_\beta^{\alpha\eta\mu}$ in its last index pair to

$$
\tilde{B}^\alpha_\beta + \frac{\partial}{\partial x^\mu} \left( \tilde{q}_\eta^{\lambda\mu\eta} \gamma^\eta_{\lambda\beta\nu} \right) = 0 \quad (52)
$$

As $\tilde{B}^\alpha_\beta$—defined by Eq. (43)—is the local representation of the canonical energy-momentum tensor of the total system of source fields and dynamic spacetime, Eq. (52) establishes a correlation of this (pseudo-)tensor with the dynamic spacetime. The explicit form of this equation will be discussed in Sect. 7.2.
6.4. Condition 4: term proportional to $h^\beta$

Finally, the term proportional to $h^\beta$ in Eq. (50) must separately vanish

$$\frac{\partial B^\mu_\beta}{\partial x^\mu} = 0$$  \hspace{1cm} (53)

Equation (53) thus establishes a \textit{local} energy and momentum conservation law of the \textit{total} system of scalar and vector fields and the dynamic spacetime. It is turns out to coincide with the divergence of Eq. (52) by virtue of the skew-symmetry of $\tilde{q}_{\eta}^{\lambda\rho\mu}$ in its last index pair

$$0 = \frac{\partial}{\partial x^\mu} \left( \tilde{B}^\mu_\beta + \tilde{q}_{\eta}^{\lambda\mu\alpha} \gamma^\eta_{\lambda\beta} + \tilde{q}_{\eta}^{\lambda\mu\alpha} \frac{\partial \gamma^\eta_{\lambda\beta}}{\partial x^\alpha} \right)$$

$$= \frac{\partial B^\mu_\beta}{\partial x^\mu} + \frac{\partial^2 \tilde{q}_{\eta}^{\lambda\mu\alpha}}{\partial x^\mu \partial x^\alpha} \gamma^\eta_{\lambda\beta} + \frac{\tilde{q}_{\eta}^{\lambda\mu\alpha} \partial \gamma^\eta_{\lambda\beta}}{\partial x^\alpha} + \frac{\tilde{q}_{\eta}^{\lambda\mu\alpha} \partial \gamma^\eta_{\lambda\beta}}{\partial x^\alpha} + \tilde{q}_{\eta}^{\lambda\mu\alpha} \frac{\partial^2 \gamma^\eta_{\lambda\beta}}{\partial x^\alpha \partial x^\alpha}$$

$$= \frac{\partial B^\mu_\beta}{\partial x^\mu}.$$  \hspace{1cm} (54)

Equation (53) is thus equivalent to a vanishing divergence of Eq. (52) as the divergence of its last term vanishes identically.

6.5. Discussion of Eq. (52)

In order to analyze Eq. (52), we write it in expanded form with the definition of $\tilde{B}^\alpha_\beta$ from Eq. (43) and the partial divergence of $\tilde{q}_{\eta}^{\lambda\rho\mu}$ from Eq. (48)

$$\tilde{R}^\alpha \frac{\partial \phi}{\partial x^\beta} + \tilde{p}^{\xi\alpha} \frac{\partial a_{\xi\beta}}{\partial x^\alpha} - \tilde{p}^{\xi\alpha} a_{\eta\xi\beta} + \tilde{k}^{\xi\lambda\rho}_{\eta\beta} \frac{\partial g_{\xi\lambda\beta}}{\partial x^\alpha} - 2 \tilde{k}^{\xi\alpha}_{\eta\lambda\beta} g_{\lambda\rho} \gamma^\eta_{\xi\beta}$$

$$+ \tilde{q}_{\eta}^{\lambda\rho\mu} \left( \frac{\partial \gamma^\eta_{\lambda\beta}}{\partial x^\rho} - \frac{\partial \gamma^\eta_{\lambda\beta}}{\partial x^\rho} + \gamma^\eta_{\rho\lambda} \gamma^\rho_{\xi\beta} \right)$$

$$- \tilde{H} = 0.$$  \hspace{1cm} (55)

The term proportional to $\tilde{q}_{\eta}^{\xi\lambda\alpha}$ is exactly the Riemann tensor, defined by

$$R^\eta_{\xi\beta\lambda} = \frac{\partial \gamma^\eta_{\xi\beta\lambda}}{\partial x^\gamma} - \frac{\partial \gamma^\eta_{\xi\beta\lambda}}{\partial x^\gamma} + \gamma^\eta_{\rho\lambda} \gamma^\rho_{\xi\beta},$$  \hspace{1cm} (56)

hence, after merging the partial derivatives with the $\gamma$-dependent terms into covariant derivatives:

$$\tilde{R}^\alpha \frac{\partial \phi}{\partial x^\beta} + \tilde{p}^{\xi\alpha} a_{\xi\beta} + \tilde{k}^{\xi\lambda\rho}_{\eta\beta} g_{\xi\lambda\beta} - \tilde{q}_{\eta}^{\xi\lambda\alpha} R^\eta_{\xi\beta\lambda}$$

$$- \tilde{H} = 0.$$  \hspace{1cm} (57)
The sum in parentheses stands for the Lagrangian \( \tilde{L} \) of the total dynamical system consisting of the scalar field, the vector field, and the spacetime geometry. This Lagrangian must be a world scalar density in order for Eq. (56) to be a tensor equation, hence to be form-invariant under the Diff\((M)\) symmetry group. The only way to achieve this in congruence with the terms in the first line is to split the Hamiltonian \( \tilde{\mathcal{H}} \) into the system Hamiltonian \( \tilde{\mathcal{H}}_0(\tilde{\pi}, \phi, \tilde{p}, a, \tilde{k}, g, \tilde{q}, \gamma) \) and the gauge Hamiltonian \( \tilde{\mathcal{H}}_G(\tilde{p}, a, \tilde{k}, g, \tilde{q}, \gamma) \) (see Ref. [1]):

\[
\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_G,
\]

with

\[
\tilde{\mathcal{H}}_G = \tilde{p}^\alpha \xi^\gamma a_{\xi^\gamma} + 2 \tilde{k}^\xi^\alpha g_{\xi^\alpha} + \tilde{q}^\xi^\alpha g_{\xi^\alpha}.
\]

The partial derivatives in Eq. (56) are thus converted into covariant derivatives, whereas the partial derivative of the connection reemerges as one-half the Riemann tensor:

\[
\begin{align*}
\tilde{\pi}^\mu &\frac{\partial \phi}{\partial x^\mu} + \tilde{\pi}^\alpha a_{\alpha;\mu} + \tilde{k}^\alpha g_{\alpha;\mu} - \tilde{q}^\alpha R^\alpha_{\alpha;\mu} - \tilde{H}_0 = 0. \\
\end{align*}
\]

Of course, the system Hamiltonian \( \tilde{\mathcal{H}}_0 \) describing the dynamics of the given scalar and vector fields must be generally covariant. The total Lagrangian \( \tilde{L} \) is thus expressed as

\[
\tilde{L} = \tilde{\pi}^\tau \frac{\partial \phi}{\partial x^\tau} + \tilde{p}^\alpha a_{\alpha;\tau} + \tilde{k}^\alpha g_{\alpha;\tau} - \frac{1}{2} \tilde{q}^\alpha R^\alpha_{\alpha;\tau} - \tilde{\mathcal{H}}_0.
\]

The field equations (67) and (59) will be rewritten in the following on the basis of this Lagrangian.

7. Discussion of the field equations in the Lagrangian description

7.1. Lagrangian representation of field equation (59)

In order to express the field equation (59) in terms of the Lagrangian (60), we define the canonical momenta as

\[
\begin{align*}
\tilde{\pi}^\mu &= \frac{\partial \tilde{L}}{\partial (\frac{\partial \phi}{\partial x^\mu})}, \\
\tilde{p}^\mu &= \frac{\partial \tilde{L}}{\partial a_{\alpha;\mu}}, \\
\tilde{k}^\alpha &= \frac{\partial \tilde{L}}{\partial g_{\alpha;\mu}}, \\
\tilde{q}^\alpha &= \frac{\partial \tilde{L}}{\partial R^\alpha_{\alpha;\mu}},
\end{align*}
\]

which yield the Noether condition (56) in the equivalent form

\[
\begin{align*}
\frac{\partial \tilde{L}}{\partial (\frac{\partial \phi}{\partial x^\mu})} &\frac{\partial \phi}{\partial x^\mu} + \frac{\partial \tilde{L}}{\partial a_{\alpha;\mu}} a_{\alpha;\mu} + \frac{\partial \tilde{L}}{\partial g_{\alpha;\mu}} g_{\alpha;\mu} + 2 \frac{\partial \tilde{L}}{\partial R^\alpha_{\alpha;\mu}} R^\alpha_{\alpha;\mu} = \delta^\mu
\end{align*}
\]

We may now split the Lagrangian \( \tilde{L} \) of the total system into a Lagrangian \( \tilde{L}_0 \) for the dynamics of the base fields \( \phi \) and \( a_\mu \), a Lagrangian \( \tilde{L}_g \) for the dynamics of the metric \( g_{\mu\nu} \),
and a Lagrangian $\hat{L}_R$ for the dynamics of the free gravitational field $R^\mu_{\alpha\beta\gamma}$ according to $L_1 = \hat{L}_0 + \hat{L}_R + \hat{L}_e$. As no derivative with respect to the metric appears in Eq. (62), we are allowed to divide all terms by $\sqrt{-g}$, whereby the field equation acquires the form

$$2 \frac{\partial L_R}{\partial R^\mu_{\alpha\beta\gamma}} R^\alpha_{\beta\mu} + \frac{\partial L_g}{\partial g_{\alpha\beta\gamma}} g_{\alpha\beta\gamma} - \delta^\mu_{\nu} (L_R + L_g) = - \frac{\partial L_0}{\partial \phi} \frac{\partial \phi}{\partial x^\nu} - \frac{\partial L_0}{\partial a_{\alpha\nu}} a_{\alpha\nu} + \delta^\mu_{\nu} L_0. \quad (63)$$

The left-hand side of Eq. (63) can be regarded as the canonical energy-momentum tensor pertaining to $L_R + L_g$. With the right-hand side the negative canonical energy-momentum tensor of the system $L_0$,

$$\theta^\nu_\mu = \frac{\partial L_0}{\partial \phi} \frac{\partial \phi}{\partial x^\nu} + \frac{\partial L_0}{\partial a_{\alpha\nu}} a_{\alpha\nu} - \delta^\mu_{\nu} L_0. \quad (64)$$

Eq. (63) thus establishes an energy-momentum balance relation. The energy-momentum tensor of the total system $L_0 + L_R + L_g$ then equals zero.

For the particular case $L_g = 0$, hence for a covariantly conserved metric, $g_{\alpha\beta\gamma} \equiv 0$, the respective terms drop out in Eq. (63)

$$2 \frac{\partial L_R}{\partial R^\mu_{\alpha\beta\gamma}} R^\alpha_{\beta\mu} - \delta^\mu_{\nu} L_R = -\theta^\mu_{\nu}. \quad (65)$$

It applies for all Lagrangians $L_R$ which (i) describe the dynamics of the “free” gravitational field for $\theta^\mu_{\nu} \equiv 0$ and (ii) entail a consistent field equation with regard to its trace and its covariant derivatives. We will discuss this point in Sect. 8 for some sample Lagrangians.

### 7.2. Lagrangian representation of the $x^i$-derivative of field equation (48)

Calculating the $x^i$-derivative of Eq. (48), the second partial derivative term of $\tilde{q}_{\alpha\beta}^{\alpha\mu}$ vanishes identically by virtue of Eq. (46):

$$\frac{\partial^2 \tilde{q}_{\alpha\beta}^{\alpha\mu}}{\partial x^i \partial x^j} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x^j} \left( \tilde{p}^{\alpha\eta}_\beta a_\beta + 2 \tilde{\kappa}^{\lambda\eta\gamma}_{\beta\alpha} g_{\lambda\beta\gamma} + \tilde{q}_{\beta}^{\alpha\lambda\eta\gamma_{\beta\alpha}} \gamma_{\beta\alpha} - \tilde{\gamma}_{\beta\alpha}^{\alpha\lambda\eta\gamma_{\beta\alpha}} \right) = 0. \quad (66)$$

This writes in expanded form, after inserting the divergence of $\tilde{q}_{\alpha\beta}^{\alpha\mu}$ from Eq. (48)

$$\frac{\partial \tilde{p}^{\alpha\eta}_\beta}{\partial x^j} a_\beta + \tilde{p}^{\alpha\eta}_\beta \frac{\partial a_\beta}{\partial x^j} + 2 \tilde{\kappa}^{\lambda\eta\gamma}_{\beta\alpha} g_{\lambda\beta\gamma} + 2 \tilde{\kappa}^{\lambda\eta\gamma}_{\beta\alpha} \frac{\partial g_{\lambda\beta\gamma}}{\partial x^j}$$

$$- \left( \tilde{p}^{\alpha\eta}_\beta a_\beta + 2 \tilde{\kappa}^{\lambda\eta\gamma}_{\beta\alpha} g_{\lambda\beta\gamma} - \tilde{q}_{\beta\alpha}^{\alpha\lambda\eta\gamma_{\beta\alpha}} \gamma_{\beta\alpha} - \tilde{q}_{\beta\alpha}^{\alpha\lambda\eta\gamma_{\beta\alpha}} \right) \frac{\partial \gamma_{\beta\alpha}}{\partial x^j} + \tilde{q}_{\beta\alpha}^{\alpha\lambda\eta\gamma_{\beta\alpha}} \frac{\partial \gamma_{\beta\alpha}}{\partial x^j} = 0. \quad (66)$$

The partial derivatives in Eq. (66) can now be converted into covariant derivatives:

$$\tilde{p}^{\alpha\eta}_\beta a_\beta + \tilde{p}^{\alpha\eta}_\beta a_\beta + 2 \tilde{\kappa}^{\lambda\eta\gamma}_{\beta\alpha} g_{\lambda\beta\gamma} + 2 \tilde{\kappa}^{\lambda\eta\gamma}_{\beta\alpha} g_{\lambda\beta\gamma} - 2 \tilde{s}_{\eta\gamma}^{\lambda\eta\gamma_{\beta\alpha}} \tilde{p}^{\alpha\eta}_\beta a_\beta + 2 \tilde{\kappa}^{\lambda\eta\gamma}_{\beta\alpha} g_{\lambda\beta\gamma}$$

$$+ \tilde{q}_{\beta\alpha}^{\alpha\lambda\eta\gamma_{\beta\alpha}} \left( \frac{\partial \gamma_{\beta\alpha}}{\partial x^j} + \gamma_{\beta\alpha} \gamma_{\beta\alpha} \right) - \tilde{q}_{\beta\alpha}^{\alpha\lambda\eta\gamma_{\beta\alpha}} \left( \frac{\partial \gamma_{\beta\alpha}}{\partial x^j} + \gamma_{\beta\alpha} \gamma_{\beta\alpha} \right) = 0.$$

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**Generic Theory of Geometrodynamics**

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Due to the skew-symmetry of $\tilde{q}_{\xi}^{\alpha\lambda\eta}$ in its last index pair, the last two terms are one-half the Riemann curvature tensor $\tilde{R}_{\xi}$, respectively. This finally yields the second rank tensor equation

$$
\left(\tilde{p}_{\alpha}^{\eta}a_{\beta} + 2\tilde{k}_{\lambda\eta}^{\alpha}g_{\eta\beta}\right)_{\eta} - 2s_{\eta}^{\xi}\left(\tilde{p}_{\alpha}^{\eta}a_{\beta} + 2\tilde{k}_{\lambda\eta}^{\alpha}g_{\eta\beta}\right) - \frac{1}{2}\tilde{q}_{\xi}^{\alpha\lambda\eta}R_{\beta\lambda\eta}^{\xi} + \frac{1}{2}\tilde{q}_{\beta}^{{\xi}\lambda\eta}R_{\xi\lambda\eta}^{\alpha} = 0.
$$

(67)

With

$$
\tilde{p}_{\alpha}^{\eta} = 2\tilde{k}_{\lambda}^{\alpha\eta}
$$

from Ref. [1] and Eqs. (61), the Noether condition (67) has the Lagrangian representation

$$
\frac{\partial\tilde{L}_{\alpha}}{\partial a_{\alpha}} + \frac{1}{2}\frac{\partial L_{\alpha}}{\partial g_{\alpha\beta}}g_{\beta\eta} + \frac{2}{2}\frac{\partial L_{\alpha}}{\partial g_{\alpha\gamma}}g_{\gamma\eta} + \frac{\partial L_{\alpha}}{\partial R_{\alpha\beta\gamma}^{\xi}}R_{\eta\beta\gamma}^{\xi} - \frac{\partial L_{\alpha}}{\partial R_{\alpha\beta\gamma}^{\xi}}R_{\eta\beta\gamma}^{\xi} = 0.
$$

(68)

On first sight, Eq. (68) appears to define a second Noether condition in addition to Eq. (62). Yet, as it turns out, both conditions are equivalent. In order to prove this, we sum up both equations:

$$
\frac{\partial\tilde{L}_{\alpha}}{\partial a_{\alpha}} + \frac{1}{2}\frac{\partial L_{\alpha}}{\partial g_{\alpha\beta}}g_{\beta\eta} + \frac{2}{2}\frac{\partial L_{\alpha}}{\partial g_{\alpha\gamma}}g_{\gamma\eta} + \frac{\partial L_{\alpha}}{\partial R_{\alpha\beta\gamma}^{\xi}}R_{\eta\beta\gamma}^{\xi} - \frac{\partial L_{\alpha}}{\partial R_{\alpha\beta\gamma}^{\xi}}R_{\eta\beta\gamma}^{\xi} = \delta_{\alpha}^{\gamma} \tilde{L}_{\alpha}.
$$

(69)

The resulting equation (69) represents an identity according to Eq. (A.3) of Corollary [2], hence, Eq. (68) holds exactly if Eq. (62) is satisfied.

For metric compatibility and dividing by $\sqrt{-g}$, Eq. (68) simplifies to

$$
2\frac{\partial L_{\alpha}}{\partial g_{\eta\beta}}g_{\eta\beta} - \delta_{\alpha}^{\nu}L_{\alpha} - \delta_{\alpha}^{\nu}L_{\alpha} + \frac{\partial L_{\alpha}}{\partial R_{\nu\eta\beta}^{\xi}}R_{\nu\eta\beta}^{\xi} - \frac{\partial L_{\alpha}}{\partial R_{\nu\eta\beta}^{\xi}}R_{\nu\eta\beta}^{\xi} = -2\frac{\partial L_{\nu}}{\partial g_{\eta\beta}}g_{\eta\beta} + \delta_{\nu}^{\alpha}L_{\nu} + \frac{\partial L_{\alpha}}{\partial a_{\nu}}a_{\nu} + \frac{\partial L_{\alpha}}{\partial a_{\nu}}a_{\nu}.
$$

where the derivatives of the Lagrangian densities $\tilde{L} = L \sqrt{-g}$ with respect to the metric are replaced by the corresponding derivatives of the Lagrangians $L$ as

$$
\frac{\partial L_{\alpha}}{\partial g_{\eta\beta}}g_{\eta\beta} = 2\frac{\partial L_{\alpha}}{\partial g_{\eta\beta}}g_{\eta\beta} = 2\frac{\partial L}{\partial g_{\eta\beta}}g_{\eta\beta} = 2\delta_{\nu}^{\alpha}L_{\nu}.
$$

(70)

7.3. Lagrangian representation of field equation

The field equation (49) has the Lagrangian representation

$$
\left(\frac{\partial\tilde{L}_{\alpha}}{\partial R_{\alpha\beta\gamma}^{\xi}}\right)_{\alpha\eta} = \frac{1}{2}\frac{\partial L_{\alpha}}{\partial a_{\alpha\beta}}g_{\alpha\beta} + \frac{\partial L_{\alpha}}{\partial g_{\alpha\beta}}g_{\eta\beta} + \frac{\partial L_{\alpha}}{\partial R_{\alpha\beta\gamma}^{\xi}}s_{\tau\mu}^{\eta} + \frac{\partial L_{\alpha}}{\partial R_{\alpha\beta}^{\xi}}s_{\mu}^{\xi\beta}.
$$

(71)
Splitting again the total Lagrangian \( \tilde{L}_t \) into the sum of the Lagrangian \( \tilde{L}_R \) for the free gravitational field, the Lagrangian \( \tilde{L}_g \) for the non-metricity, and the Lagrangian \( \tilde{L}_0 \) describing the dynamics of the scalar and vector field, this yields

\[
\left( \frac{\partial \tilde{L}_R}{\partial R^\mu_{a\beta\eta}} \right)_\mu = \frac{1}{2} \frac{\partial \tilde{L}_0}{\partial a_\eta} a_\eta + \frac{\partial \tilde{L}_g}{\partial g_{\tau\rho\beta}} g_{\tau\eta} + \frac{\partial \tilde{L}_R}{\partial R^\mu_{a\beta\eta}} s^\beta_{\eta\mu} + \frac{\partial \tilde{L}_R}{\partial R^\mu_{a\beta\eta}} s^\tau_{\eta\mu\tau} \tag{72}
\]

For the usual case of a covariantly conserved metric, hence for metric compatibility, we have \( \tilde{L}_g \equiv 0 \), hence, we can divide by \( \sqrt{-g} \).

\[
\left( \frac{\partial \tilde{L}_R}{\partial R_{a\beta\eta \mu}} \right)_\mu - \frac{\partial \tilde{L}_R}{\partial R_{a\tau\rho \mu}} s^\beta_{\eta \mu} - \frac{\partial \tilde{L}_R}{\partial R_{a\beta\eta \mu}} s^\tau_{\eta \mu \tau} = \frac{1}{4} \left( \frac{\partial \tilde{L}_0}{\partial a_\beta} a^\alpha - \frac{\partial \tilde{L}_0}{\partial a_\beta} a^\beta \right). \tag{73}
\]

We will show in the next section that leftmost term, hence the divergence associated with \( \tilde{L}_R \), then vanishes for the Hilbert Lagrangian \( \tilde{L}_{R,H} \). For that case, Eq. (73) yields an algebraic equation for the torsion emerging from the vector field \( a_\mu \). Due to the non-vanishing right-hand side of Eq. (73), the vector field necessarily acts as a source of torsion. For all other choices of \( \tilde{L}_R \), we encounter a non-algebraic equation for the torsion dynamics.

8. Sample Lagrangians

8.1. Sample \( \tilde{L}_R \)

The dynamics of the Riemann tensor \( R^\mu_{a\beta\eta} \) in classical vacuum is not determined by Noether’s theorem. Consequently, \( \tilde{L}_R \) must be provided on the basis of physical reasoning. As a physically reasonable example, we chose the Lagrangian equivalent of \( \tilde{\mathcal{H}}_{\eta}(\tilde{q}, g) \) from Ref. [1], which is obtained by means of the Legendre transformation

\[
\tilde{L}_R = -\frac{1}{2} g_{\eta} a^\mu a^\rho R_{a\mu\rho\tau} - \tilde{\mathcal{H}}_{\eta} \tag{74}
\]

hence

\[
\tilde{L}_R = \frac{g_1}{4} R^\mu_{a\beta\eta \mu} \left( R^\tau_{\eta\xi\lambda} - \tilde{R}^\tau_{\eta\xi\lambda} \right) s^\rho_{\xi\lambda} s^\tau_{\eta\lambda} - 6 g_1 g_2^2 \tag{75}
\]

with

\[
\tilde{R}^\tau_{\eta\xi\lambda} = 2 g_2 \left( \delta^\tau_{\eta} g_{\xi\lambda} - \delta^\tau_{\xi} g_{\eta\lambda} \right)
\]

the Riemann tensor of the maximally symmetric 4-dimensional manifold, which can be regarded as the “ground state” of spacetime [6]. Thus

\[
\frac{\partial \tilde{L}_g}{\partial R^\mu_{a\beta\eta \mu}} = \frac{g_1}{2} \left[ R^\mu_{\eta\xi\lambda} s^\rho_{\xi\lambda} s^\tau_{\eta\lambda} + g_2 \left( \delta^\rho_{\eta} s^\mu_{\xi\lambda} - \delta^\mu_{\eta} s^\rho_{\xi\lambda} \right) \right]. \tag{76}
\]

As a scalar, this Lagrangian satisfies the particular representation of the identity (A.1)

\[
2 \frac{\partial \tilde{L}_R}{\partial g_{\eta\xi\lambda} s^\rho_{\xi\lambda}} = \frac{\partial \tilde{L}_R}{\partial R^\tau_{\eta\xi\lambda}} R^\mu_{\eta\xi\lambda} - \frac{\partial \tilde{L}_R}{\partial R^\mu_{\eta\xi\lambda}} R^\tau_{\eta\xi\lambda} + 2 \frac{\partial \tilde{L}_R}{\partial R^\mu_{a\beta\eta \mu}} R^\rho_{a\beta\eta \mu}. \tag{77}
\]
Its left-hand side evaluates to
\[
2 \frac{\partial L_R}{\partial g^{\mu \nu}} g^{\alpha \beta} = -g_1 R^{\rho \sigma \mu \nu} R_{\rho \sigma \mu \nu} + g_1 g_2 \left( R^{\mu}_{\nu} + R^{\nu}_{\mu} \right),
\]
which agrees with the terms obtained from the right-hand side of Eq. (77):
\[
\frac{\partial L_R}{\partial R^{\nu}_{\alpha \beta \mu}} R^{\tau}_{\nu \alpha \beta \mu} + \frac{\partial L_R}{\partial R^{\nu}_{\alpha \beta \mu}} R^{\tau}_{\alpha \beta \mu} + \frac{\partial L_R}{\partial R^{\nu}_{\alpha \beta \mu}} R^{\tau}_{\alpha \beta \mu} - \frac{\partial L_R}{\partial R^{\tau}_{\alpha \beta \mu}} R^{\mu}_{\tau \alpha \beta} - \frac{\partial L_R}{\partial g^{\mu \nu}} g^{\alpha \beta} = -g_1 R^{\rho \sigma \mu \nu} R_{\rho \sigma \mu \nu} + g_1 g_2 \left( R^{\mu}_{\nu} + R^{\nu}_{\mu} \right),
\]
To set up the pertaining field equation for the spacetime dynamics, the Lagrangian
\[
L_{\tau \mu \nu} \text{ which agrees with the terms obtained from the right-hand side of Eq. (77)}
\]
and finally, replacing the coupling constant \( g_2 = \Lambda/3 \),
\[
g_1 \left( R^{\alpha \beta \mu \nu}_{\eta} R_{\alpha \beta \mu \nu}^{\eta} - \frac{1}{4} \delta^\mu_\nu R^{\alpha \beta \mu \nu}_{\eta} R_{\alpha \beta \mu \nu}^{\eta} \right) - \frac{1}{8 \pi G} \left( R^{\mu}_{\nu} - \frac{1}{2} \delta^\mu_\nu R + \Lambda \delta^\mu_\nu \right) = \theta_\nu^\mu. \tag{78}
\]
Neglecting the term quadratic in the Riemann tensor reduces \( L_R \) to the Hilbert Lagrangian \( L_{R,H} \)
\[
L_{R,H} = \frac{g_1}{2} R^{\mu \nu}_{\alpha \beta \tau} \hat{R}_{\alpha \beta \tau} - 6 g_1 g_2^2, \quad 6 g_1 g_2^2 = \frac{\Lambda}{8 \pi G}. \tag{79}
\]
The Lagrangian of Eq. (75) is thus the sum of the Hilbert Lagrangian plus a Lagrangian quadratic in the Riemann tensor. The latter was already proposed by A. Einstein in a personal letter to H. Weyl [2]. From (79) the Einstein tensor in Eq. (78) follows directly according to the generic field equation (68). The derivative of \( L_{R,H} \) with respect to the Riemann tensor follows as
\[
\frac{\partial L_{R,H}}{\partial R_{\rho \sigma \mu \nu}} = \frac{1}{32 \pi G} \left( g^{\rho \sigma} g^{\alpha \mu} - g^{\rho \mu} g^{\alpha \sigma} \right), \tag{80}
\]
whose covariant derivative vanishes for metric compatibility. The field equation (73) then yields the algebraic equation
\[
\left( g^{\rho \sigma} g^{\alpha \mu} - g^{\rho \mu} g^{\alpha \sigma} \right) s^\beta_{\tau \mu} = \left( g^{\rho \sigma} g^{\alpha \mu} - g^{\rho \mu} g^{\alpha \sigma} \right) s^\tau_{\beta \mu} = 8 \pi G \left( \frac{\partial L_0}{\partial \alpha_{\eta \beta}} a^\eta - \frac{\partial L_0}{\partial a_{\eta \beta}} a^\eta \right). \tag{81}
\]
Thus, a dynamical spin-1 particle field \( a_\mu \) always acts as a source of torsion of spacetime. The right-hand side will be specified for the Proca system in Sect. 8.3. Obviously, the spin-0 particle field \( \phi \), i.e., the Klein-Gordon system of the following section does not act as a source of torsion as Eq. (81) is identically satisfied for \( s^\beta_{\tau \mu} \equiv 0 \).

For the Lagrangian (75) quadratic in the Riemann tensor, one encounters the Poisson-type equation
\[
\frac{g_1}{2} R^{\rho \sigma \mu \nu}_{\eta} + \left( g^{\rho \sigma} g^{\alpha \mu} - g^{\rho \mu} g^{\alpha \sigma} \right) s^\beta_{\tau \mu} - \left( g^{\rho \sigma} g^{\alpha \mu} - g^{\rho \mu} g^{\alpha \sigma} \right) s^\tau_{\beta \mu} = 8 \pi G \left( \frac{\partial L_0}{\partial a_{\alpha \beta}} a^\eta - \frac{\partial L_0}{\partial a_{\eta \beta}} a^\eta \right). \tag{82}
\]
8.2. Klein-Gordon Lagrangian $\mathcal{L}_0$

The Klein-Gordon Lagrangian $\mathcal{L}_0(\phi, \partial\phi, g^{\mu\nu})$ for a system of a real scalar field $\phi$ in a dynamic spacetime is given by

$$\mathcal{L}_0 = \frac{1}{2} \left( \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} g^{\alpha\beta} - m^2 \phi^2 \right).$$

(83)

The mixed tensor representation of the canonical energy-momentum tensor $\theta_{\nu}^\mu$ of the Klein-Gordon system (83) is given by

$$\theta_{\nu}^\mu = \frac{\partial \mathcal{L}_0}{\partial \left( \frac{\partial\phi}{\partial x^\nu} \right)} \frac{\partial\phi}{\partial x^\alpha} - \delta_{\nu}^\mu \mathcal{L}_0.$$  

(84)

With

$$\frac{\partial \mathcal{L}_0}{\partial \left( \frac{\partial\phi}{\partial x^\nu} \right)} \frac{\partial\phi}{\partial x^\alpha} = \frac{1}{2} \delta_{\nu}^\alpha \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} g^{\alpha\beta} = \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} g^{\alpha\beta},$$

Eq. (84) sums up to

$$\theta_{\nu}^\mu = \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} g^{\alpha\beta} - \frac{1}{2} \delta_{\nu}^\mu \left( \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} g^{\alpha\beta} - m^2 \phi^2 \right).$$

This agrees with the metric energy-momentum tensor of this system for $\tilde{\mathcal{L}}_0 = \mathcal{L}_0 \sqrt{-g}$

$$T_{\nu}^\mu = \frac{2}{\sqrt{-g}} \frac{\partial \tilde{\mathcal{L}}_0}{\partial g^{\mu\nu}} g^{\mu\nu} = \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} \delta^\alpha_{\nu} \delta^\beta_{\mu} g^{\alpha\beta} - \frac{1}{2} \left( \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} g^{\alpha\beta} - m^2 \phi^2 \right) \delta^\mu_{\nu} g_{\bar{\nu}}$$

$$= \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} g^{\alpha\beta} - \frac{1}{2} \left( \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} g^{\alpha\beta} - m^2 \phi^2 \right) \delta^\mu_{\nu}$$

= $\theta_{\nu}^\mu$.

Hence, both energy-momentum tensors coincide for the Klein-Gordon system. On the basis of the Hilbert Lagrangian (79), we thus encounter the conventional Einstein equation

$$R^\mu_\nu - \frac{1}{2} g^\mu_\nu R + \Lambda g^\mu_\nu = -8\pi G T^\mu_\nu,$$

(85)

with a symmetric Ricci tensor $R^\mu_\nu = R^\nu_\mu$ and no source term for a spacetime torsion.

8.3. Proca Lagrangian

The Proca Lagrangian $\mathcal{L}_0$ writes

$$\mathcal{L}_0 = -\frac{1}{4} f_{\alpha\beta} f_{\gamma\delta} g^{\alpha\epsilon} g^{\beta\eta} + \frac{1}{2} m^2 a_\alpha a_\eta g^{\alpha\epsilon}, \quad f_{\alpha\beta} = a_{\beta\alpha} - a_{\alpha\beta} = -f_{\beta\alpha},$$

(86)

with $f_{\alpha\beta}$ denoting the skew-symmetric field tensor. With the Lagrangian (86), the general Eq. (A.1) takes on the particular form

$$\frac{\partial \mathcal{L}_0}{\partial g^{\mu\nu}} g^{\mu\nu} + \frac{\partial \mathcal{L}_0}{\partial f_{\gamma\delta}} f_{\epsilon\eta} g^{\epsilon\eta} - \frac{\partial \mathcal{L}_0}{\partial f_{\gamma\delta}} f^{\gamma\eta} g_{\nu\eta} - \frac{\partial \mathcal{L}_0}{\partial a_\nu} a_\nu \equiv 0.$$  

(87)
The derivatives with respect to the metric yield
\[
\frac{\partial L_0}{\partial g^{\mu\nu}} g^{\mu\nu} + \frac{\partial L_0}{\partial g^{\mu\nu}} g^{\mu\nu} = -\frac{1}{4} f_{\alpha\beta} f_{\gamma\delta} \left[ \left( \delta^{\alpha}_{\nu} \delta^{\beta}_{\lambda} g^{\gamma\delta} + g^{\alpha\beta} \delta^{\gamma}_{\delta} \right) g^{\mu\nu} + \left( \delta^{\alpha}_{\lambda} \delta^{\beta}_{\nu} g^{\gamma\delta} + g^{\alpha\beta} \delta^{\gamma}_{\nu} \right) g^{\mu\nu} \right] \\
+ \frac{1}{2} m^2 a_\epsilon a_\epsilon \left( \delta^\nu_\lambda \delta^\mu_\delta g^{\nu\delta} + \delta^\mu_\lambda \delta^\nu_\delta g^{\mu\nu} \right) \\
= -\frac{1}{4} \left( f_{\alpha\beta} f_{\gamma\delta} g^{\beta\delta} + f_{\alpha\gamma} f_{\nu\delta} g^{\alpha\delta} + f_{\alpha\beta} f_{\gamma\delta} g^{\beta\gamma} + f_{\alpha\beta} f_{\gamma\delta} g^{\beta\delta} + f_{\alpha\beta} f_{\gamma\delta} g^{\beta\gamma} \right) \\
+ \frac{1}{2} m^2 \left( a_\alpha a_\beta g^{\alpha\beta} + a_\alpha a_\beta g^{\alpha\beta} \right) \\
= -\frac{1}{4} \left( f_{\alpha\beta} f_{\gamma\delta} + f_{\alpha\beta} f_{\gamma\delta} \right) + m^2 a_\epsilon a_\epsilon.
\]

The derivatives with respect to the fields in \( L_0 \) yield
\[
\frac{\partial L_0}{\partial f_{\mu\nu}} f_{\lambda\alpha} + \frac{\partial L_0}{\partial f_{\lambda\alpha}} f_{\mu\nu} + \frac{\partial L_0}{\partial a_\alpha} a_\alpha = -\frac{1}{4} \left[ \left( \delta^\alpha_\nu \delta^\beta_\lambda f_{\epsilon\eta} + f_{\alpha\beta} \delta^\delta_\epsilon \delta^\delta_\nu \right) f_{\epsilon\lambda} + \left( \delta^\beta_\mu \delta^\mu_\nu f_{\epsilon\eta} + f_{\alpha\beta} \delta^\delta_\epsilon \delta^\delta_\eta \right) f_{\epsilon\nu} \right] g^{\alpha\beta} \\
+ \frac{1}{2} m^2 \left( a_\epsilon a_\xi + a_\alpha a_\beta g^{\alpha\beta} \right) \\
= -\frac{1}{4} \left( f_{\epsilon\eta} f_{\lambda\alpha} g^{\epsilon\eta} + f_{\epsilon\eta} f_{\lambda\alpha} g^{\epsilon\eta} + f_{\epsilon\eta} f_{\lambda\alpha} g^{\epsilon\eta} + f_{\epsilon\eta} f_{\lambda\alpha} g^{\epsilon\eta} \right) \\
+ \frac{1}{2} m^2 \left( a_\epsilon g^{\epsilon\eta} + a_\alpha g^{\alpha\beta} \right) a_\alpha \\
= -\frac{1}{4} \left( f^{\mu\nu} f_{\lambda\alpha} + f^{\mu\nu} f_{\lambda\alpha} \right) + m^2 a_\epsilon a_\epsilon.
\]

Clearly, both groups cancel, which verifies Eq. (87).

For the Lagrangian density \( \tilde{L}_0 = L_0 \sqrt{-g} \), the derivatives with respect to the metric yield an additional term, namely
\[
\frac{\partial \sqrt{-g}}{\partial g^{\nu\lambda}} g^{\mu\nu} + \frac{\partial \sqrt{-g}}{\partial g^{\nu\lambda}} g^{\mu\nu} = -\delta^\mu_\nu \sqrt{-g}.
\]

Hence, making use of the symmetry of the metric \( g^{\nu\lambda} \) and of the skew-symmetry of the field tensor \( f_{\nu\lambda} \), one encounters from the derivatives of \( \tilde{L}_0 \) with respect to the metric the metric energy-momentum tensor
\[
T_\nu^\mu = \frac{2}{\sqrt{-g}} \frac{\partial \tilde{L}_0}{\partial g^{\nu\lambda}} g^{\mu\lambda} = -f_{\nu\lambda} f^{\mu\lambda} + m^2 a_\epsilon a_\epsilon - \delta^\mu_\nu \tilde{L}_0.
\]

By virtue of the identity (87), the metric energy-momentum tensor can be equivalently obtained from the derivatives with respect to the fields:
\[
T_\nu^\mu = \frac{2}{\sqrt{-g}} \frac{\partial \tilde{L}_0}{\partial g^{\nu\lambda}} g^{\mu\lambda} = 2 \frac{\partial L_0}{\partial f_{\mu\nu}} f_{\nu\lambda} + \frac{\partial L_0}{\partial a_\mu} a_\nu - \delta^\mu_\nu L_0 \\
= \theta_\nu^\mu + \frac{\partial L_0}{\partial f_{\mu\nu}} f_{\nu\lambda} + \frac{\partial L_0}{\partial a_\mu} a_\nu,
\]

wherein \( \theta_\nu^\mu \) denotes the canonical energy-momentum tensor
\[
\theta_\nu^\mu = \frac{\partial L_0}{\partial f_{\mu\nu}} f_{\nu\lambda} - \delta_\nu^\mu L_0 = -\frac{1}{2} f_{\nu\lambda} f^{\mu\lambda} + \frac{1}{4} \delta_\nu^\mu \left( f_{\alpha\beta} f_{\epsilon\eta} g^{\epsilon\eta} g^{\alpha\beta} - 2m^2 a_\alpha a_\beta g^{\alpha\beta} \right).
\]
For the Proca system, both energy-momentum tensors do not agree but differ by two additional terms

\[ T_{\mu}^{\nu} = \theta_{\mu}^{\nu} + \frac{\partial L_0}{\partial f_{\mu}^{\lambda}} f_{\nu}^{\lambda} + \frac{\partial L_0}{\partial a_{\mu}} a_{\nu} = \theta_{\mu}^{\nu} - \frac{1}{2} f_{\nu}^{\lambda} f^{\mu\lambda} + m^2 a_{\nu} a^{\mu}. \]  

Our conclusion is that $\theta^{\mu\nu}$ represents the correct source term for a Proca system. The canonical energy-momentum tensor $\theta^{\mu\nu}$ thus entails an increased weighting of the kinetic energy over the mass as compared to the metric energy momentum tensor $T^{\mu\nu}$ in their roles as the source of gravity. This holds independently of the particular model for the “free” (uncoupled) gravitational field, whose dynamics is encoded in the Lagrangian $L_R$ of the generic Einstein-type equation (65). With

\[ \frac{\partial L_0}{\partial a_{\mu;\nu}} = f^{\mu\nu}, \]

the field equation (82) for the Lagrangian $L_R$ from Eq. (75) follows as

\[ \frac{g_{\nu;\mu}}{2} R^{\alpha\beta\mu\nu} + (g_{\nu;\mu} - g_{\mu;\nu} - g_{\nu;\nu} + g_{\mu;\mu}) s^{\beta}_{\nu;\mu} - (g_{\alpha;\mu} g^{\beta\nu} - g_{\alpha;\nu} g^{\beta\mu}) s^{\tau}_{\nu;\mu} = 2\pi G \left( f^{\alpha\beta}\partial^n - f^{\nu\beta}\partial^n a^{\alpha} \right), \]

which reduces for the Hilbert Lagrangian $L_{R,H}$ from Eq. (79) to the algebraic equation

\[ (g^{\alpha\mu} g^{\nu\tau} - g^{\alpha\nu} g^{\mu\tau}) s^{\beta}_{\nu;\mu} - (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu}) s^{\tau}_{\nu;\mu} = 2\pi G \left( f^{\alpha\beta}\partial^n - f^{\mu\beta}\partial^n a^{\mu} \right). \]

**9. Conclusions**

The minimum set of postulates to work out a theory for the interaction of the spacetime geometry with some source fields is that (i) the theory should be derived from an action principle and that (ii) the theory should be diffeomorphism-invariant, hence be a member of the $\text{Diff}(M)$ symmetry group. The appropriate basis for the formulation of such a theory is given by Noether’s theorem—which directly follows from the action principle: it provides for any symmetry of the given action a pertaining conserved Noether current. From the latter, one can then set up the most general field equation of geometrodynamics for systems with $\text{Diff}(M)$ symmetry. The general recipe to set up this equation is as follows:

(i) Establish the covariant representation of the canonical energy-momentum tensor for the given system Lagrangian $L_0$—which must be a world scalar. This means for a system of a scalar field $\phi$ and a massive vector field $a_{\alpha}$

\[ \theta_{\mu}^{\nu} = \frac{\partial L_0}{\partial (\partial_{\beta}^{\phi})} \frac{\partial \phi}{\partial x^{\nu}} + \frac{\partial L_0}{\partial a_{\alpha;\nu}} a_{\alpha;\mu} - \delta_{\nu}^{\alpha} L_0. \]

The covariant form of the canonical energy-momentum tensor thus contains direct coupling terms of vector field $a_{\alpha}$ and connection $\gamma_{\mu\nu}^{\alpha}$, and thereby also causes a coupling to a torsion of spacetime. For vector fields, which represent the classical limit of massive spin particles, the correct source of gravitation is constituted by the canonical energy-momentum tensor and not by the conventionally used metric (Hilbert) energy-momentum tensor—in agreement with Hehl [8]. The source term changes for systems with additional symmetries—such as a system with additional U(1) symmetry, in which case the metric (Hilbert) energy-momentum tensor turns out to be the appropriate source term [7].
(ii) With the source term and the \( postulated \) Lagrangians for both, the dynamics of the “free” (uncoupled) connection, \( \mathcal{L}_R \), and of the metric, \( \mathcal{L}_g \), the new and most general equation of geometrodynamics is given by

\[
2 \frac{\partial \mathcal{L}_R}{\partial R^\eta_{\alpha\beta\mu}} R^\eta_{\alpha\beta\nu} + \frac{\partial \mathcal{L}_g}{\partial g_{\alpha\beta\mu}} g_{\alpha\beta\nu} - \delta^\eta_{\nu} \mathcal{L}_R = \mathcal{L}_g = -\theta^\eta_{\nu}.
\]

On first sight, \( \mathcal{L}_R \) may be any world scalar formed out of the Riemann tensor \( R^\eta_{\alpha\beta\tau} \) and the metric \( g^{\alpha\beta} \). Correspondingly, \( \mathcal{L}_g \) may be any world scalar formed out of the covariant derivative of the metric and the metric itself. Yet, the choices of \( \mathcal{L}_R \) and \( \mathcal{L}_g \) are restricted by the requirement that the subsequent field equation is consistent with regard to its trace and its covariant divergence.

(iii) The particular case of metric compatibility, hence a covariantly conserved metric, is implemented for \( \mathcal{L}_g = 0 \). The correlation of the connection and the metric is then encountered as

\[
g_{\alpha\beta\nu} = \frac{\partial g_{\alpha\beta}}{\partial x^\nu} - g_{\gamma\beta} g^\tau_{\alpha\nu} - g_{\alpha\tau} g^\nu_{\beta\nu} \equiv 0,
\]

while the correlation of the Riemann tensor \( R^\eta_{\alpha\beta\mu} \) to the source simplifies to the following form of a generic Einstein-type equation

\[
2 \frac{\partial \mathcal{L}_R}{\partial R^\eta_{\alpha\beta\mu}} R^\eta_{\alpha\beta\nu} - \delta^\eta_{\nu} \mathcal{L}_R = \delta^\mu_{\nu} - \theta^\mu_{\nu} \mathcal{L}_0 = \delta^\mu_{\nu} \mathcal{L}_0 - \frac{\partial \mathcal{L}_0}{\partial \phi} \frac{\partial \phi}{\partial x^\nu} - \frac{\partial \mathcal{L}_0}{\partial a_{\alpha\mu}} a_{\alpha\nu}.
\]

The left-hand side can be interpreted as the covariant canonical energy-momentum tensor associated with the Lagrangian \( \mathcal{L}_R \) describing the dynamics of the gravitational field in classical vacuum.

The simplest case for \( \mathcal{L}_R(R, g) \) is given by the Hilbert Lagrangian \( (79) \), which directly yields the Einstein tensor on the left-hand side. This requires the covariant divergence of the energy-momentum tensor to be zero in order for the resulting field equation to be consistent. Correspondingly, for each choice of \( \mathcal{L}_R \) one must make sure that the resulting field equation is consistent.

Summarizing, our generic theory of geometrodynamics generalizes Einstein’s General Relativity as follows:

(i) The description of the dynamics of the “free” gravitational fields is not restricted to the Hilbert Lagrangian. In the case of a quadratic and linear dependence of \( \mathcal{L}_R(R, g) \) on the Riemann tensor, the more general field equation

\[
g_1 \left( R^\eta_{\alpha\beta\mu} R^\eta_{\alpha\beta\nu} - \frac{1}{4} \delta^\eta_{\nu} R^\eta_{\alpha\beta\mu} R^\eta_{\alpha\beta\tau} \right) - \frac{1}{8\pi G} \left( R^\mu_{\nu} - \frac{1}{2} \delta^\mu_{\nu} R + \Lambda \delta^\mu_{\nu} \right) = \theta^\mu_{\nu},
\]

is encountered. It equally complies with the Principle of General Relativity. The additional term proportional to the \textit{dimensionless} coupling constant \( g_1 \) is equally satisfied by the Schwarzschild and the Kerr metric \( (9) \). Yet, it entails a different description of the dynamics of spacetime in the case of a non-vanishing source term \( \theta^\mu_{\nu} \) as compared to the solution based on only the Einstein tensor—which follows setting \( g_1 = 0 \).
The covariant derivative of the field equation
\[ g^1 R^\alpha_\mu \eta \eta^\mu = \theta^\nu_\mu, \]
shows that the generalized theory allows for an exchange of energy and momentum of the source system with the gravitational field. For \( g^1 = 0 \), this reduces to the Einstein case, where the energy-momentum tensor \( \theta^\nu_\mu \) of the source system must be covariantly conserved.

(ii) The generalized theory is not restricted to a covariantly conserved metric, hence to metric compatibility.

(iii) The spacetime is not assumed to be generally torsion-free. Hence, the generalized theory allows for sources of gravity which generate and couple to a torsion of spacetime. This applies in particular to those vector fields, which represent the classical limit of massive spin-1 particles. For this case, the canonical energy-momentum tensor is the appropriate source term. Moreover, one encounters the Poisson-type equation for the spacetime torsion, wherein the vector field acts as the source term
\[ \frac{g^1}{2} R^\eta_\mu \eta^\mu + (g^\eta_\mu g^\nu_\eta - g^\mu_\eta g^\eta_\nu) S^\eta_\mu - \left( g^\eta_\mu g^\eta_\eta - g^\eta_\eta g^\eta_\eta \right) s^\tau_\eta_\mu = 8\pi G \left( \frac{\partial L_0}{\partial a^{\eta_\mu}} a^\eta - \frac{\partial L_0}{\partial a^{\eta_\eta}} a^\eta \right), \]
which reduces to an algebraic equation for the case of the Hilbert Lagrangian, hence for \( g^1 = 0 \).

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Appendix A. Identity for a scalar-valued function \( S \) of an \((n, m)\)-tensor \( T \) and the metric

Proposition 1 Let \( S = S(g, T) \in \mathbb{R} \) be a scalar-valued function constructed from the metric tensor \( g_{\mu \nu} \) and an \((n, m)\)-tensor \( T^{\xi_1...\xi_n}_{\eta_1...\eta_m} \), where \((m - n)/2 \in \mathbb{Z}\). Then the following identity holds:

\[ \frac{\partial S}{\partial g_{\mu \eta}} g_{\nu \eta} + \frac{\partial S}{\partial g_{\mu \epsilon}} g_{\nu \eta_\epsilon} - \frac{\partial S}{\partial T^{\eta_\mu}_{\xi_2...\xi_n}} T^{\eta_\mu}_{\xi_2...\eta_\epsilon} - \cdots - \frac{\partial S}{\partial T^{\eta_\mu}_{\xi_2...\eta_n}} T^{\eta_\mu}_{\xi_2...\eta_n} \equiv 0 \delta^\mu_\nu. \]  

(A.1)

Proof 1 The proof is easily obtained by induction. Let \( S_1 = S U_{a}^\alpha \) be a scalar constructed from the metric tensor \( g_{\mu \nu} \) and an \((n + 1, m + 1)\)-tensor \( T \otimes U \). Provided that Eq. (A.1) holds,
one finds for the scalar $S$

\[
\frac{\partial S}{\partial U_\mu} U_\nu U_\beta - \frac{\partial S}{\partial U_\nu} U_\beta U_\mu = S \frac{\partial U^\alpha}{\partial U_\nu} U_\beta - S \frac{\partial U^\alpha}{\partial U_\beta} U_\mu
\]

\[
= S \left( \delta^\alpha_\beta U_\nu U_\mu - \delta^\alpha_\nu U_\beta U_\mu \right)
\]

\[
= S \left( U_\mu U_\nu - U_\mu U_\nu \right) \equiv 0 \delta^\nu_\nu.
\]

Equation (A.1) thus also holds for $S_1$.

The derivative of a Lagrangian $L$ with respect to the metric $g_{\mu\nu}$ can thus always be replaced by the derivatives with respect to the appertaining tensors $T$ that are made into a scalar by means of the metric. The identity thus provides the correlation of the metric and the canonical energy-momentum tensors of a given system.

**Corollary 1** The contraction of Eq. (A.1) then yields a condition for the scalar $S$:

\[
\frac{\partial S}{\partial g_{\alpha\beta}} g_{\alpha\beta} \equiv \frac{n - m}{2} \frac{\partial S}{\partial T^\xi_1...\xi_n} T^\xi_1...\xi_n - \eta_1...\eta_m.
\]

**Proof 2** Contracting Eq. (A.1) directly yields Eq. (A.2).

**Corollary 2** Let $\tilde{L} = \tilde{L}(g, T_k) \in \mathbb{R}$ be a scalar density (i.e. a relative scalar of weight one) valued function of the (symmetric) metric $g_{\mu\nu}$ and $k$ tensors $T^\xi_1...\xi_n$ of respective rank $(n_k, m_k)$, where $(m_k - n_k)/2 \in \mathbb{Z}$. Then the following identity holds:

\[
2 \frac{\partial \tilde{L}}{\partial g_{\mu\nu}} g_{\beta\gamma} - \frac{\partial \tilde{L}}{\partial T^\xi_1...\xi_n} T^\mu_\xi_1...\xi_n - \eta_1...\eta_m - \frac{\partial \tilde{L}}{\partial T^\xi_1...\xi_n} T^\xi_1...\xi_n - \mu_\nu - \eta_1...\eta_m\equiv \delta^\mu_\nu \tilde{L}.
\]

**Proof 3** Combine Eq. (A.1) with (70).

Equation (A.3) is clearly a tensor calculus representation of Euler’s theorem on homogeneous functions.

**Appendix B. Examples**

**Appendix B.1. Ricci scalar**

The Ricci scalar $R$ is defined as the following contraction of the Riemann tensor

\[
R = R_{\eta\xi\lambda} g^{\eta\xi} g^{\alpha\lambda}.
\]

With $S = R$ and the Tensor $T$ represented by the Riemann tensor, the general Eq. (A.1) takes on the particular form

\[
\frac{\partial R}{\partial g^{\mu\beta}} g^{\nu\beta} + \frac{\partial R}{\partial g^{\nu\delta}} g^{\rho\delta} - \frac{\partial R}{\partial R_{\eta\xi\lambda}} R_{\eta\xi\lambda} - \frac{\partial R}{\partial R_{\eta\xi\lambda}} R_{\eta\xi\lambda} - \frac{\partial R}{\partial R_{\rho\xi\lambda}} R_{\rho\xi\lambda} - \frac{\partial R}{\partial R_{\rho\xi\lambda}} R_{\rho\xi\lambda} \equiv 0.
\]
Without making use of the symmetries of the Riemann tensor and the metric, this identity is actually fulfilled as
\[
\frac{\partial R}{\partial g^{\mu \nu}} g^{\nu \beta} = R_{\nu \alpha \xi \lambda} \left( \delta^\mu_\alpha \delta^\xi_\beta + g^{\xi \eta} \delta^\lambda_\eta \delta^\mu_\beta \right) g^{\beta \mu} = R_{\nu \alpha} + R^\xi_\nu. 
\]

Similarly
\[
\frac{\partial R}{\partial g^{\nu \beta}} g^{\mu \beta} = R^\mu_{\alpha \nu} + R^\xi_\nu. 
\]

The derivative terms of the Riemann tensor are
\[
\frac{\partial R}{\partial R_{\mu \alpha \xi \lambda}} R_{\nu \alpha \xi \lambda} = \delta^\mu_\eta g^{\eta \xi} g^{\alpha \lambda} R_{\nu \alpha \xi \lambda} = R_{\nu \alpha} 
\]
and
\[
\frac{\partial R}{\partial R_{\eta \alpha \xi \mu}} R_{\eta \alpha \xi \mu} = \delta^\eta_\alpha g^{\eta \xi} g^{\alpha \lambda} R_{\eta \alpha \xi \lambda} = R^\xi_\nu 
\]
which obviously cancel the four terms emerging from the derivatives with respect to the metric.

Making now use of the skew-symmetries of the Riemann tensor in its first and second index pair and of the symmetry of the metric, Eq. (B.2) simplifies to
\[
\frac{\partial R}{\partial g^{\nu \beta}} g^{\mu \beta} \equiv \frac{\partial R}{\partial R_{\mu \alpha \xi \lambda}} R_{\nu \alpha \xi \lambda} + \frac{\partial R}{\partial R_{\eta \alpha \xi \mu}} R_{\eta \alpha \xi \nu} \equiv 0. 
\]

For zero torsion, the Riemann tensor has the additional symmetry on exchange of both index pairs. Then
\[
\frac{\partial R}{\partial g^{\nu \beta}} g^{\mu \beta} \equiv 2 \frac{\partial R}{\partial R_{\mu \alpha \xi \lambda}} R_{\nu \alpha \xi \lambda} \quad \Leftrightarrow \quad \frac{\partial R}{\partial g^{\nu \beta}} g^{\mu \beta} \equiv 2 \frac{\partial R}{\partial R_{\alpha \xi \lambda}} R_{\nu \alpha \xi \lambda}. 
\]

Appendix B.2. Ricci tensor squared

The scalar made of the (not necessarily symmetric) Ricci tensor $R_{\eta \alpha}$ is defined by the following contraction with the metric
\[
S = R_{\eta \alpha} g^{\eta \xi} g^{\alpha \lambda}. 
\]
With Eq. (B.5), the general Eq. (A.1) now takes on the particular form
\[
\frac{\partial S}{\partial g^{\nu \beta}} g^{\mu \beta} + \frac{\partial S}{\partial g^{\nu \beta}} g^{\mu \beta} - \frac{\partial S}{\partial R_{\mu \beta}} R_{\nu \beta} - \frac{\partial S}{\partial R_{\beta \mu}} R_{\nu \beta} \equiv 0. 
\]
Without making use of symmetries of the Ricci tensor and the metric, this identity is actually fulfilled as
\[
\frac{\partial S}{\partial g_{\nu\beta}} g^{\mu\beta} = R_{\nu\rho} R_{\rho\xi} \left( \delta^\xi_\rho \delta^\rho_\nu g^{\alpha\lambda} + g^{\alpha\xi} \delta^\rho_\nu \right) g^{\beta\mu} = \left( R_{\nu\rho} R_{\rho\xi} g^{\alpha\lambda} + g^{\alpha\xi} R_{\nu\rho} R_{\rho\xi} \right) g^{\beta\mu} = R_{\nu\rho} R^{\rho\mu} + R^{\rho\mu} R_{\rho\xi}.
\]

Similarly
\[
\frac{\partial S}{\partial g_{\beta\nu}} g^{\rho\mu} = R_{\nu\rho} R^{\rho\mu} + R^{\rho\mu} R_{\rho\xi}.
\]

The derivative terms of the Ricci tensor are
\[
\frac{\partial S}{\partial R^{\mu\beta}_{\nu\rho}} R_{\nu\beta} = \left( \delta^\rho_\mu \delta^\beta_\nu R^{\alpha\lambda}_{\xi\eta} + R_{\rho\xi} \delta^\beta_\nu \delta^\mu_\alpha \right) g^{\xi\eta} R_{\nu\rho} = 2 R_{\nu\beta} R^{\rho\mu}
\]

and
\[
\frac{\partial S}{\partial R^{\beta\nu}_{\rho\mu}} R_{\beta\nu} = \left( \delta^\nu_\beta \delta^\mu_\rho R^{\alpha\lambda}_{\xi\eta} + R_{\nu\xi} \delta^\mu_\rho \delta^\beta_\eta \right) g^{\xi\eta} g^{\alpha\lambda} R_{\beta\nu} = 2 R_{\beta\nu} R^{\rho\mu},
\]

which obviously cancel the four terms emerging from the derivatives with respect to the metric.

For zero torsion, the Ricci tensor is symmetric. Then
\[
\frac{\partial S}{\partial g_{\nu\beta}} g^{\mu\beta} \equiv \frac{\partial S}{\partial R^{\mu\beta}_{\nu\rho}} R^{\rho\mu} \Leftrightarrow \frac{\partial S}{\partial g^{\mu\beta}} \equiv \frac{\partial S}{\partial R^{\mu\beta}_{\nu\rho}} g_{\nu\rho}.
\]

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