Dimensional Analysis and the Time Required to Urinate

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Abstract

According to the recently discovered 'Law of Urination', mammals, ranging in size from mice to elephants, take, on the average, 21s to urinate. We attempt to gain insights into the physical processes responsible for this uniformity using simple dimensional analysis. We assume that the biological apparatus for urination in mammals simply scales with linear size, and consider the scenarios where the driving force is gravity or elasticity, and where the response is dominated by inertia or viscosity. We ask how the time required for urination depends on the length scale, and find that for the time to be independent of body size, the dominant driving force must be elasticity, and the dominant response viscosity. Our note demonstrates that dimensional analysis can indeed readily give insights into complex physical and biological processes.
I. INTRODUCTION

In a recent publication [1], David Hu and coworkers report the discovery of the "Law of Urination", according to which animals empty their bladders in the nearly constant time interval of $21 \pm 13 \text{ s}$ regardless of size. This is indeed a remarkable result, given that body masses of the animals in question, from mice to elephants, range over some five orders of magnitude. The authors go on to say that this feat is made possible by the increasing urethral length of large animals, which 'amplifies gravitational force and flow rate'. Although the dispersion about the mean time is certainly significant, and the range of linear sizes of the animals considered, taken as cube root of the volume, is about one order of magnitude, the Law of Urination is still striking. In order to gain some insights into this phenomenon, we turn to dimensional analysis.

Dimensional analysis is an exceedingly efficient and powerful tool of physics, enabling insights into complex problems with relatively small computational effort. In the hands of expert practitioners, it can be said to rise to the level of art. Two inspirational examples of its use are the estimate of the yield of the Trinity nuclear test by G.I. Taylor [2] and the estimate of the height of mountains of earth by V. Weisskopf [3] and Goldreich et al [4].

Dimensional analysis is based on the notion that the laws of physics must have the same form in any system of units. This implies that the relationship between the physical variables describing the phenomenon under consideration can be expressed in terms of quantities without units. The art of dimensional analysis lies in determining the relevant physical variables, including only, but all, of the essential ones. In general, a number of independent dimensionless groups can be formed; a formal procedure for obtaining these is provided by the Buckingham Π theorem [5]. Here we adopt a minimalist approach, and select only sufficient physical variables to form a single dimensionless group, and use this simple dimensional analysis to examine the role of gravity and other factors in the time needed for mammals to urinate.

II. ANALYSIS

We assume, for the purpose of this analysis, that the relative dimensions of the bladder and urethra do not change from animal to animal, but the entire structure scales with the
linear size of the animal in question. Specifically, we want to determine the time needed to empty a compact bladder, assumed to be spherical, via a straight tube, the urethra, with circular cross-section. A simple illustration is shown in Fig. 1.

![Schematic of bladder and urethra.](image)

**FIG. 1: Schematic of bladder and urethra.**

As pointed out in [1], some authors ascribe the force which is responsible for expelling the urine to bladder pressure [6-8], while others propose a combination of gravity and bladder pressure [9]. For a given driving force and geometry, the flow velocity is determined primarily by fluid inertia and viscosity. In our simple dimensional analysis, we therefore consider, separately, the effects of the driving forces of gravity and muscle contraction, and of the inertial and viscous response.

The essence of our dimensional analysis is to find how the relevant quantities can be combined to form a dimensionless group, or, more specifically, how the parameters describing the dominant driving force and response can be combined to form a quantity with the units of the quantity of interest: the urination time.

**A. Gravity and Inertia**

Here we assume that the dominant driving force for the flow is the force of gravity on the fluid, and the response is dominated by inertia. In this case, the time $t_u$ to urinate, that is, to empty the bladder is a function of the gravitational force per volume $\rho g$ acting on the fluid, where $\rho$ is the mass density of urine and $g$ is the acceleration of gravity, of inertia,
as measured by the mass density $\rho$, and of the various lengths illustrated in Fig.1. There $R$ is the radius of the bladder, $w$ is the thickness of the detrusor muscle, $l$ is the length and $d$ is the diameter of the urethra. Since all lengths in the problem can be written as some dimensionless factor times a characteristic length $L$, we consider $t_u$ to be a function of $\rho g$, $\rho$ and $L$. If we express units of all relevant quantities in the problem in terms of the fundamental dimensional quantities of mass $M$, time $T$, length $L$, we have

$$[\rho g] = \frac{M}{L^2T^2},$$

(1)

where the square brackets $[]$ indicate 'units (dimensions) of',

$$[\rho] = \frac{M}{L^3},$$

(2)

and $[L] = L$. To form an expression with the unit of time, we must determine the exponents $\alpha$, $\beta$ and $\gamma$ which satisfy

$$T = [(\rho g)^\alpha \rho^\beta L^\gamma] = \frac{M^\alpha}{L^{2a}T^{2a}} \frac{M^\beta}{L^{3\beta}} L^\gamma.$$

(3)

Equating exponents on both sides, we get at once $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$ and $\gamma = 1/2$, so

$$t_u = \sqrt{\frac{L}{g} \times C_1},$$

(4)

where $C_1$ is a dimensionless function of the ratios of lengths in Fig. 1. Since $C_1$ does not change if all lengths are scaled, it follows that, if the dominant contributions were gravity and inertia, the time needed to urinate would be independent of the density of urine but proportional to the square root of the characteristic length of the animal (or proportional to the mass to the one-sixth power, in accordance with [1] for large animals); large animals would take somewhat longer to urinate than small ones, and the required time would not be size independent.

**B. Gravity and Viscosity**

Here we assume that the dominant driving force is again gravity, but the response is dominated by viscosity. The units of dynamic viscosity $\mu$ are, in SI units, $Pa - s$, or in terms of fundamental dimensional quantities

$$[\mu] = \frac{M}{LT}.$$

(5)
Proceeding as before, we have

\[ T = [(ρg)^{α} \mu^{β} L^{γ}] = \frac{M^{α}}{L^{2αT^{2α}}} \frac{M^{β}}{L^{3βT^{β}}} L^{γ}, \]  

(6)

ans we find that \( α = -1, \ β = 1 \) and \( γ = -1 \), so

\[ t_u = \frac{μ}{ρgL} \times C_2, \]  

(7)

where \( C_2 \) again is a scale independent dimensionless function of length ratios. Here, the urination time \( t_u \) is inversely proportional to the characteristic length of the animal; so large animals would take much less time to urinate than small ones.

\[ \text{C. Elasticity and Inertia} \]

We next assume that the dominant driving force is detrusor muscle tension, which is characterized by the stress \( Y \), and that the response is inertia dominated. The dimensions of stress are

\[ [Y] = \frac{M}{LT^2}. \]  

(8)

Proceeding as before,

\[ T = [Y^{α} \rho^{β} L^{γ}] = \frac{M^{α}}{L^{αT^{2α}}} \frac{M^{β}}{L^{3βT^{β}}} L^{γ}, \]  

(9)

and we find that \( α = -\frac{1}{2}, \ β = \frac{1}{2} \) and \( γ = 1 \), so

\[ t_u = L\sqrt{\frac{ρ}{Y}} \times C_3. \]  

(10)

The time is thus proportional to the characteristic length; again at variance with observations. In this scenario, large animals would take much longer to urinate than small ones.

\[ \text{D. Elasticity and Viscosity} \]

Lastly, we consider the case when the driving force is detrusor muscle tension, and the response is viscosity dominated. Proceeding again as before,

\[ T = [Y^{α} \mu^{β} L^{γ}] = \frac{M^{α}}{L^{αT^{2α}}} \frac{M^{β}}{L^{3βT^{β}}} L^{γ}, \]  

(11)

and we find that \( α = -1, \ β = 1 \) and \( γ = 0 \), so

\[ t_u = \frac{μ}{Y} \times C_4, \]  

(12)
that is, the time is independent of characteristic length of the animal, in accordance with observations. According to this result, large and small animals take approximately the same time to urinate, regardless of size.

III. DISCUSSION AND CONCLUSIONS

In our simple model, we have assumed that in the animals considered, the geometry of bladder and urethra is unchanged, and all lengths scale with the characteristic length of the animal. We have considered separately the cases when the dominant driving force for urination is gravity and muscle contraction, and the dominant response is inertial and viscous. In three of the four cases considered, the time required for urination depends on the animal size; only in the case of muscle contraction and a viscous response do we find that the time for urination is size independent. This is in agreement with experimental observations. A rough estimate of the urination time, given in the Appendix, is in reasonable agreement with experimental observations. Our simple model therefore suggests that urination depends primarily on muscle contraction and viscosity; gravity and inertia play a less important role. This conclusion is in disaccord with the argument of Hu et al [1], who stresses the importance of gravity in large animals in explaining experimental observations. We note that astronauts apparently urinate without difficulty even in the absence of gravity [8]; and babies pee in the upward direction as well as down. If indeed the dominant factors are muscle contraction and viscosity, as we argue here, then, according to our simple analysis in the Appendix, the time is proportional to the ratio of urethral length to diameter. This suggests that the urination time of females should be significantly less than (about 1/5 of) that of males of the same size. We are currently seeking information and data to test this prediction. The factor of 5 is nearly within the $8 - 34s$ interval cited in [1].

Our estimate of times in the Appendix does not take a key aspect of urination into account: namely, that the urethra is compliant, and its diameter is a nonlinear function of pressure [9]. This is a fascinating aspect, which suggests non-steady flow, and energy transfer to the surrounding tissue. Since these aspects have not been taken into account, the expressions for urination times in the Appendix cannot be accurate, but they do indicate how the geometry enters the dimensionless factor. We note that, even in the case of compliant urethra, our dimensionless analysis is valid, since the modulus of tissue is comparable to
muscle stress, and thus there are no new dimensional quantities entering the problem.

Finally, we note that for small animals, our analysis does not hold, since other factors, such as surface tension, come into play.

In summary, we have shown here that simple dimensional analysis, with little effort can give interesting and useful insights into complex phenomena, in this case, the Law of Urination.

IV. APPENDIX

In this section, we estimate the urination times using physical, rather than dimensional arguments. Since this approach gives the dimensionless multiplicative constant explicitly, it enables rough estimation of the required times. We note that our simple approach ignores the compliance of the urethra.

A. Gravity and Inertia

On equating gravitational potential energy with kinetic energy in the urethra, we get

\[ \rho g R = \frac{1}{2} \rho v_{ua}^2, \]

where \( v_{ua} \) is the average velocity of the fluid in the urethra. The time required to empty the bladder is of the order

\[ t_u = \frac{4 \pi R^3}{\pi (\frac{d}{2})^2 v_{ua}} = \sqrt{\frac{R}{g}} \times \frac{16}{3 \sqrt{2}} \left( \frac{R}{d} \right)^2, \]

which is proportional to the square root of the characteristic length. This is at variance with the observation that the time is length independent. Interestingly, \( t_u \) is independent of the length of the urethra; it only depends on its diameter.

Evaluating this using length estimates for humans gives \( t_u \simeq 118 \text{s}. \) (72s for a cat, and 195s for an elephant.)

B. Gravity and Viscosity

In the Stokes limit for pipe flow with no slip at the boundaries, the local velocity \( v_u \) in the urethra satisfies

\[ \mu \nabla^2 v_u(r) = -\nabla P = -P', \]

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and, assuming cylindrical symmetry,
\[ \mu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) = -P', \]  
(16)
where the pressure gradient \( P' \) is a constant. The parabolic velocity profile
\[ v_u = v_o \left( 1 - \left( \frac{r}{R_o} \right)^2 \right) \]
(17)
satisfies the equation, and substitution gives
\[ v_o = \frac{R_o^2 P'}{4\mu}. \]
(18)
The volume current density \( J \) is just the velocity \( v \). The flux \( f \) (volume/time) is
\[ f = \int JdA = \int_0^{R_o} v2\pi rdr = 2\pi v_o \int_0^{R_o} (1 - \left( \frac{r}{R_o} \right)^2)rdr = \frac{\pi}{2} v_o R_o^2, \]
and substituting for \( v_o \), we get for the flux
\[ f = \frac{\pi R_o^4 P'}{8\mu}. \]
(20)
If \( R_o = d/2 \) and \( P' = P/l \), we get
\[ f = \frac{\pi}{128} \frac{d^4 P}{l \mu}. \]
(21)
Estimating the pressure as
\[ P = \rho g R, \]
(22)
then the required time is
\[ t_u = \frac{\frac{\pi}{3} R^3}{f} = \frac{\frac{4}{3} \pi R^3 l \mu}{128 \rho g R} = \frac{\frac{4}{3} \pi R^3 l \mu}{128 \rho g R} \times \frac{512}{3} \left( \frac{R}{d} \right)^3 \frac{l}{d}. \]
(23)
Evaluating this using estimates for humans gives \( t_u \simeq 92.8s \). (252s for a cat, and 34s for an elephant.)

C. Elasticity and Inertia

We can estimate the time using energy conservation in this case. The pressure due to muscle contraction is
\[ P = \frac{2Yw}{R}, \]
(24)
and energy conservation gives, for the average velocity in the urethra,

\[ v_{ua} = \sqrt{\frac{2P}{\rho}} = \sqrt{\frac{4Yw}{R\rho}}, \quad (25) \]

and the time to urinate is

\[ t_u = \frac{\frac{4}{3}\pi R^3}{\pi\left(\frac{d}{2}\right)^2 v_{ua}} = \frac{\frac{4}{3}\pi R^3}{\pi\left(\frac{d}{2}\right)^2} \sqrt{\frac{R\rho}{4Yw}} = R\sqrt{\frac{\rho Y}{3}} \times \frac{8}{3} \left(\frac{R}{d}\right)^2 \sqrt{\frac{R}{w}}, \quad (26) \]

and evaluating this using length estimates for humans gives \( t_u = 35.1s \). (13s for a cat, and 95s for an elephant.)

D. Elasticity and Viscosity

The flux in the urethra is, again

\[ f = \frac{\pi d^4 P}{128 l \mu}, \quad (27) \]

and the pressure in the bladder is, again,

\[ P = \frac{2Yw}{R}. \quad (28) \]

The time to urinate therefore is

\[ t_u = \frac{\frac{4}{3}\pi R^3}{f} = \frac{128\frac{4}{3}\pi R^4 l \mu}{2\pi d^4 Yw} = \frac{\mu}{Y} \times \frac{256}{3} \left(\frac{R}{d}\right)^4 \frac{l}{w}. \quad (29) \]

Evaluating this using length estimates for humans gives \( t_u = 8.2s \), regardless of size, which is the right order of magnitude. This results has a remarkably strong dependence - fourth power - of the urination time on length ratios. Given the uniformity of experimental times, some other mechanism (such as the dependence of the effective diameter of the urethra on flow velocity [10]) most likely also contributes to the flow regulation in biological systems.

E. Physical Parameters

The estimates below are for humans, and they are as follows:

\[ g = 9.81 m/s^2 \]
\[ R = 6 \times 10^{-2} m \]
\[ d = 3 \times 10^{-3} m \]
\[ l = 0.12m \quad \text{(for a male, } l = 20 cm, \text{ for a female, } l = 4 cm, \text{ 12 cm is the average.)} \]

\[ w = 2 \times 10^{-3} m \]

\[ \rho = 10^3 \text{kg/m}^3 \]

\[ Y = 1 \times 10^5 Pa \]

\[ \mu = 1 \times 10^{-3} Pa \cdot s \]

\[ m = 80 kg \quad \text{(human)} \]

\[ m = 4 kg \quad \text{(cat)} \]

\[ m = 4000 kg \quad \text{(elephant)} \]

We estimate characteristic lengths on basis of body mass. The ratio of the characteristic lengths of an elephant to that of a human is \( \sqrt[3]{4000/80} = 3.684 \), and the ratio of a human to a cat is \( \sqrt[3]{80/4} = 2.71 \). By this measure, range of length scales covered is \( 3.684 \times 2.71 \simeq 10 \).

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