Loschmidt echo in many-spin systems: contrasting time-scales of local and global measurements.

Pablo R. Zangara, Denise Bendersky, Patricia R. Levstein, and Horacio M. Pastawski

Instituto de Física Enrique Gaviola (CONICET-UNC) and Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, 5000, Córdoba, Argentina

A local excitation in a quantum many-spin system evolves deterministically. A time-reversal procedure, involving the inversion of the signs of every energy and interaction, should produce the excitation revival. This idea, experimentally coined in NMR, embodies the concept of the Loschmidt echo (LE). While such an implementation involves a single spin autocorrelation $M_{1,1}$, i.e. a local LE, theoretical efforts have focused on the study of the recovery probability of a complete many-body state, referred here as global or many-body LE $M_{MB}$. Here, we analyze the relation between these magnitudes, in what concerns to their characteristic time scales and their dependence on the number of spins $N$. We show that the global LE can be understood, to some extent, as the simultaneous occurrence of $N$ independent local LEs, i.e. $M_{MB} \sim (M_{1,1})^{N/4}$. This extensive hypothesis is exact for very short times and confirmed numerically beyond such a regime. Furthermore, we discuss a general picture of the decay of $M_{1,1}$ as a consequence of the interplay between the time scale that characterizes the reversible interactions ($T_2$) and that of the perturbation ($\tau_\Sigma$). Our analysis suggests that the short time decay, characterized by the time scale $\tau_\Sigma$, is greatly enhanced by the complex processes that occur beyond $T_2$. This would ultimately lead to the experimentally observed $T_3$, which was found to be roughly independent of $\tau_\Sigma$ but closely tied to $T_2$. 
I. INTRODUCTION

If an ink drop falls into a pond, the stain diffuses away until no trace of it remains whatsoever. One may naturally say that such a process is in fact irreversible. In the microscopic world, similar phenomena are also ubiquitous. For instance, let us consider a many-spin quantum system in thermal equilibrium where a local polarization excess is injected. Then, this excitation would spread all over as consequence of spin-spin interactions. Such an apparently irreversible process is known as spin diffusion [1, 2] and it can lead the system back to equilibrium. However, this naive picture has its limitations. On the one hand, spreading is not always the rule as there are physical situations where the initial excitation does not vanish. This is the case of Anderson localization [3, 4] or when the excitation remains topologically protected [5]. On the other hand, even in the cases where the system seems to have reached an equilibrium state, the unitarity of quantum dynamics ensures a precise memory of the non-equilibrium initial condition. Then, if some experimental protocol could reverse the many-body dynamics, it would drive the system back to the initial non-equilibrium state [6]. Such a general idea defines the Loschmidt echo (LE), which embodies the various time-reversal procedures implemented in nuclear magnetic resonance (NMR) [7, 8].

The first NMR time-reversal experiment was performed by Hahn in the 1950’s [10]. The procedure, known as spin echo, reverses the precession dynamics of each independent spin around its local magnetic field by inverting the sign of the Zeeman energy. However, the sign of the energy associated to the spin-spin interactions is not inverted and, accordingly, the echo signal is degraded. Such a decay occurs within the time scale $T_2$ that characterizes the spin-spin interactions. Indeed, these interactions determine the survival of a spin excitation at short times as $\sim 1 - (t/T_2)^2$ and its later complex dynamics generating a diffusive spreading. By the early 1970’s, Kesemeier, Rhim, Pines and Waugh implemented the reversal of the dynamics induced by the spin-spin dipolar interaction $[11, 12]$. This results in the “magic echo” which indicates the recovery of a global polarization state. Two decades later, Ernst and collaborators introduced the “polarization echo” $[13]$. There, a local excitation injected in a many-spin system is let to evolve, then time-reversed and finally detected locally at the initial spot. While the success of these time reversal echoes unambiguously evidenced the deterministic nature of spin-dynamics in NMR, it is clear that the reversal is unavoidably degraded by uncontrolled internal or environmental degrees of freedom or by imperfections in the pulse sequences. Furthermore, the degradation seems to occur in a time scale, say $T_3$, much shorter than a naive estimation of the characteristic scale of these perturbations, say $\gamma_2$. Then, the question that arises is whether the complexity inherent to a large number of correlated spins would enhance the fragility of the procedure under perturbations.

A next generation of experiments in organic crystals $[14–16]$ seemed to confirm that the experimental $T_3$ never exceeds a few times $T_2$. In other words, $T_3$ keeps tied to the time scale that characterizes the reversible many-body interaction. This led to postulate that in an infinite many-spin system the complex dynamics could favor the action of any small non-inverted interaction that perturbs the reversal procedure. Thus, reversible interactions become determinant for the irreversibility rate. This constitutes our Central Hypothesis of Irreversibility. Such a wisdom is further reinforced by the natural association of many-body complexity with a form of chaos $[17, 18]$ and the confirmation that quantum dynamics of classically chaotic systems should manifest a dynamical instability $[19]$ which leads to an environment-independent decoherence rate $[20, 21]$.

During the last decade, solid-state NMR kept providing a versatile testing bench to study time reversal in large spin arrays $[22–26]$. As a matter of fact, a standard experiment involves a crystalline sample with an infinitely large number of spins. In contrast, the numerical test of many spin dynamics has to be restricted to strictly finite systems $[27, 28]$. While this appears to be a major limitation, it allows the analysis of a situation that the experiments cannot achieve: moving progressively from small systems to larger ones with a fully controlled perturbation. The expectancy is that a sort of finite size scaling may allow to identify an emergent mechanism that rules reversibility in the thermodynamic limit $[22]$. As in the experiments, the witness for such a transition should be the LE as measured by a single spin autocorrelation function $M_{1,1}$, i.e. a local polarization. For short we call $M_{1,1}$ the local LE. It is not difficult to probe that $\Pi_{n,1} \equiv (M_{1,1} + 1)/2$ is the probability that a given spin, say the $1^{st}$, remains up after the whole procedure. Besides, in a case of a spin excitation in a 1D chain with XY interactions $[30, 32]$, $M_{1,1}$ precisely coincides with the global overlap of two one-body wave functions as defined for semiclassical systems $[20, 33, 54]$. The square of the overlap between the initial and final many-body wave functions, $M_{MB}$, defines a global or many-body LE. It is important to notice that $M_{MB}$ has not been addressed experimentally, but nevertheless it is a natural magnitude in theory $[35, 37]$. Thus, we are left without a precise relation between the object of theoretical studies and experimental ones, i.e. $M_{MB}$ and $M_{1,1}$ respectively. This missing link is the central question we address in this paper.

Here, we consider a system of $N$ spins whose initial state is given by a local excitation injected in an high temperature state. Firstly, we discuss the formal relation between $M_{MB}$ and $M_{1,1}$, which is derived exactly at least for very short times. In particular, we assess how the $N$-dependence or extensivity of $M_{MB}$ is evidenced in the time scales involved. This leads us to hint that the revival of a many-spin state results from the recovery of each individual spin configuration, much as if they were statistically independent events. Since in the initial high temperature state there are $N/2$ spins up, their rough statistical independence would lead to a behavior of the sort of $M_{MB} \sim (\Pi_{1,1})^{N/2} \sim (M_{1,1})^{N/4}$. This
is confirmed by the numerical evaluation of the LE in a specific spin model.

Furthermore, we discuss a general picture beyond the short-time regime, where the decay of $M_{1,1}$ results from the interplay between the time scale that characterizes the reversible interactions ($T_2$) and that of the perturbation ($\tau_2$). This would ultimately lead to the experimentally observed $T_3$. In such a sense, our analysis provides a conceptual hinge between the theoretical and the experimental realms.

The paper is organized as follows. In Sec. II we introduce the LE framework: the initial state and the time-reversal procedure. Here, we define both the local and global LE. In Sec. III we compute the short time expansions for the local LE and its non-local contributions (in particular, the many-body LE). This allows us to discuss a general picture of the LE decay in terms of the times scales that characterize $M_{MB}$ and $M_{1,1}$. The dependence with $N$ is discussed in terms of the extensivity of $M_{MB}$ and statistical independence of the local autocorrelations. In Sec. IV we assess our expectancies by a numerical evaluation of the LE in a spin system. Section V summarizes our main conclusions and some of the important open questions in the field.

II. THE LOSCHMIDT ECHO IN SPIN SYSTEMS.

Let us first specify the initial condition of a “local excitation in many-spin system”. We consider $N$ spins $1/2$ in an infinite temperature state, i.e. completely depolarized mixture, plus a locally injected polarization,

$$\hat{\rho}_0 = \frac{1}{2^N} (\hat{I} + 2\hat{S}^z_1).$$

Here, the spin 1 is polarized while the others are not, i.e. $tr[\hat{S}^z_1 \hat{\rho}_0] = \frac{1}{2} \delta_{1,1}$. Such an initial state can be experimentally implemented not only in NMR [38] but also in cold atoms [39].

As in the early LE experiments [14–16], our numerical evaluation focuses on an imperfect time reversed evolution of the excitation, followed by a local measurement. The procedure is depicted in Fig. 1. A many-spin Hamiltonian $\hat{H}_0$ rules the forward evolution of the system up to a certain time $t_R$. At that moment, an inversion of the sign of $\hat{H}_0$ is performed, leading to a symmetric backward evolution. Nevertheless, there are unavoidable perturbations, denoted by $\Sigma$, that could arise from the incomplete control of the Hamiltonian, acting on both periods. Thus, evolution operators for these $t_R$-periods are $\hat{U}_+ (t_R) = \exp[-\frac{i}{\hbar}(\hat{H}_0 + \Sigma)t_R]$ and $\hat{U}_- (t_R) = \exp[-\frac{i}{\hbar}(-\hat{H}_0 + \Sigma)t_R]$ respectively. It is quite practical to define the LE operator as:

$$\hat{U}_{LE}(2t_R) = \hat{U}_-(t_R)\hat{U}_+(t_R),$$

which produces an imperfect refocusing at time $2t_R$. A local measurement of the polarization, performed at site 1, defines the local LE:

$$M_{1,1}(t) = 2tr[\hat{S}^z_1 \hat{U}_{LE}(t)\hat{\rho}_0 \hat{U}^\dagger_{LE}(t)] = 2tr[\hat{S}^z_1 \hat{\rho}_1].$$

Here, we choose as free variable $t = 2t_R$, the total elapsed time in the presence of the perturbation. The time dependence of $\hat{\rho}_1$ in the Schrödinger picture is,

$$\hat{\rho}_1 = \hat{U}_{LE}(t)\hat{\rho}_0 \hat{U}^\dagger_{LE}(t).$$

Using equation (1), and after some algebraic manipulation, the LE can be explicitly written as a correlation function:

$$M_{1,1}(t) = \frac{1}{2^{N-2}} tr[\hat{U}^\dagger_{LE}(t)\hat{S}_1^z(0)\hat{U}_{LE}(t)\hat{S}_1^z(0)] = \frac{tr[\hat{S}_1^z(t)\hat{S}_1^z(0)]}{tr[\hat{S}_1^z(0)\hat{S}_1^z(0)]}.$$

Here, the time dependence is written according to the Heisenberg picture,

$$\hat{S}_1^z(t) = \hat{U}_{LE}(t)\hat{S}_1^z(0)\hat{U}^\dagger_{LE}(t).$$

Notice that equation (5) is an explicit correlation function at the same site but different times, i.e. an autocorrelation. This kind of a correlation has been recently employed to address localization phenomena in spin systems [4, 10, 41].
and it generalizes the standard one employed to assess spin diffusion \[42\]. In terms of the Hilbert–Schmidt inner product between the initial and the time evolved density matrices, i.e. equations (1) and (4) respectively, the LE can be written as \[15, 43, 44\]:

\[
M_{1,1}(t) = 2^N \text{tr}[\hat{\rho}_0 \hat{p}_i] - 1 = \frac{2 \text{tr}[\hat{\rho}_0 \hat{p}_i]}{\text{tr}[\hat{\rho}_0 \hat{p}_0]} - 1, \quad (7)
\]

which, in the present case, progressively decays from 1 to 0 as it occurs with the statistical overlap between two wave packets in the standard LE definition \[20\].

Equivalent expressions for the LE can be derived decomposing the statistical state in a simpler basis. In order to proceed with the pure state decomposition of \(\hat{\rho}_0\), we consider the computational Ising basis \(\{\beta_i\}\), also known as \(S^z\)-decoupled basis. Additionally, we define the set \(\mathcal{A}\) of indexes \(j\) that label basis states which have the 1st spin pointing up, i.e. \(j \in A \Leftrightarrow \hat{S}_j^z |\beta_j\rangle = +\frac{1}{2} |\beta_j\rangle\). It is straightforward to verify that \(\hat{\rho}_0 = \sum_{j \in \mathcal{A}} 2^{-(N-1)} |\beta_j\rangle \langle \beta_j|\). Then, as introduced in Ref. \[30\],

\[
M_{1,1}(t) = 2 \left[ \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{A}} \frac{1}{2^{N-1}} \left| \langle \beta_j | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2 - \frac{1}{2} \right] = 2 \left[ \Pi_{1,1}(t) - \frac{1}{2} \right]. \quad (8)
\]

Here, \(\Pi_{1,1}(t)\) denotes the probability that the 1st spin keeps pointing up after a time \(t\). After some manipulation,

\[
M_{1,1}(t) = 2 \left[ \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{A}} \frac{1}{2^{N-1}} \left| \langle \beta_j | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2 - \frac{1}{2} \right] = \left[ \sum_{i \in \mathcal{A}} \frac{1}{2^{N-1}} \left( \left| \langle \beta_i | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2 + \right.ight.
\]

\[
\left. + \sum_{j \in \mathcal{B} (j \neq i)} \left| \langle \beta_j | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2 - \sum_{j \in \mathcal{B}} \left| \langle \beta_j | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2 \right]. \quad (9)
\]

Here, \(\mathcal{B}\) stands for the complement of \(\mathcal{A}\), i.e. \(j \in \mathcal{B} \Leftrightarrow \hat{S}_j^z |\beta_j\rangle = -\frac{1}{2} |\beta_j\rangle\). One can naturally identify and define the two terms that contribute to the local polarization \(M_{1,1}(t)\). The first sum in equation (9) stands for the average probability of revival of the many-body states, denoted by \(M_{MB}(t)\).
\[ M_{MB}(t) = \sum_{i \in A} \frac{1}{2^{N-1}} |\langle \beta_i | \hat{U}_{LE}(t) | \beta_i \rangle|^2. \]  

(10)

The second sum in equation (10) represents the average probability of changing the configuration of any spin except the 1st. The third sum stands for the average probability that the 1st spin has actually flipped, i.e. of all those processes that do not contribute to \( M_{1,1}(t) \). Then, the processes that contribute to \( M_{1,1}(t) \) but not to \( M_{MB}(t) \) are denoted as:

\[ M_X(t) = \sum_{i \in A} \frac{1}{2^{N-1}} \left( \sum_{j \in A} \frac{1}{2} \left| \langle \beta_j | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2 - \sum_{j \in B} \left| \langle \beta_j | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2 \right). \]  

(11)

This balance of probabilities leads to the appropriate asymptotic behavior of \( M_{1,1}(t) \) according to the symmetries that constrain the evolution. The identification

\[ M_{1,1}(t) = M_{MB}(t) + M_X(t) \]  

(12)

is a crucial step for the following discussions.

If we use the identity \( \hat{S}_1^z = \hat{S}_1^+ \hat{S}_1^- - \frac{1}{2} \mathbb{I} \) in equation (8), the invariance of the trace under cyclic permutations ensures that \( tr[\hat{S}_1^z(t) \hat{S}_1^z(0)] = tr[\hat{S}_1^z(0) \hat{S}_1^z(t)] - \frac{1}{2} tr[\hat{S}_1^z(t)] \). Since \( tr[\hat{S}_1^z(t)] = tr[\hat{S}_1^z(0)] = 0 \), then:

\[ M_{1,1}(t) = 2 \sum_{i \in A} \frac{1}{2^{N-1}} \langle \beta_i | \hat{S}_1^- (0) \hat{U}_{LE}(t) \hat{S}_1^+ (0) \hat{U}_{LE}(t) \hat{S}_1^z (0) | \beta_i \rangle \]

\[ = 2 \sum_{i \in A} \frac{1}{2^{N-1}} \langle \beta_i | \hat{U}_{LE}(t) \hat{S}_1^z \hat{U}_{LE}(t) | \beta_i \rangle, \]  

(13)

which is indeed an explicit way to rewrite equation (8) in the form of an ensemble average. Remarkably, since \( \hat{S}_1^z \) is a local (“one-body”) operator, its evaluation in equation (13) can be replaced by the expectation value in a single superposition state \[45\],

\[ M_{1,1}(t) = 2 \langle \Psi_{neq} | \hat{U}_{LE}(t) \hat{S}_1^z \hat{U}_{LE}(t) | \Psi_{neq} \rangle, \]  

(14)

where:

\[ |\Psi_{neq}\rangle = \sum_{i \in A} \frac{1}{\sqrt{2^{N-1}}} e^{i \varphi_i} |\beta_i\rangle. \]  

(15)

Here, \( \varphi_i \) is a random phase uniformly distributed in \([0, 2\pi]\). As a matter of fact, the state defined in equation (15) is a random superposition that can successfully mimic the dynamics of ensemble calculations and provides a quadratic speedup of computational efforts \[45\] \[47\].

### III. The Loschmidt Echo Dynamics

#### A. Short time expansions and beyond.

In order to analyze the \( N \)-dependence of the LE and its time scales, we compute here the short time expansion of the magnitudes \( M_{1,1}(t) \), \( M_{MB}(t) \) and \( M_X(t) \). Up to 2nd order in time,

\[ M_{1,1}(t = 2t_R) = 2 \sum_{i \in A} \frac{1}{2^{N-1}} \langle \beta_i | \hat{U}_{LE}(t) \hat{S}_1^z \hat{U}_{LE}(t) | \beta_i \rangle \]

\[ = 1 - (t/\hbar)^2 \sum_{i \in A} \frac{1}{2^{N-1}} \left( \langle \beta_i | \hat{S}^2 | \beta_i \rangle - 2 \langle \beta_i | \hat{S} \hat{S}^+ \hat{S}^+ | \beta_i \rangle \right) + \mathcal{O}(t^3/\hbar^3). \]  

(16)
Similarly, the leading contributions to $M_{MB}(t)$ and $M_X(t)$ are:

$$M_{MB}(t) = \sum_{i\in A} \frac{1}{2N-1} \left| \langle \beta_i | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2$$

$$= 1 - \left( \frac{t}{\hbar} \right)^2 \sum_{i\in A} \frac{1}{2N-1} \left( \langle \beta_i | \hat{\Sigma}^2 | \beta_i \rangle - \langle \beta_i | \hat{\Sigma} | \beta_i \rangle^2 \right) + \mathcal{O} \left( \frac{t}{\hbar} \right)^3,$$

(17)

and

$$M_X(t) = \sum_{i\in A} \frac{1}{2N-1} \left( \sum_{j\in A \setminus \{j\neq i\}} \left| \langle \beta_j | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2 - \sum_{j\in B} \left| \langle \beta_j | \hat{U}_{LE}(t) | \beta_i \rangle \right|^2 \right)$$

$$= \left( \frac{t}{\hbar} \right)^2 \sum_{i\in A} \frac{1}{2N-1} \left( 2 \langle \beta_i | \hat{\Sigma}^2 \hat{\Sigma} | \beta_i \rangle - \langle \beta_i | \hat{\Sigma} | \beta_i \rangle^2 \right) + \mathcal{O} \left( \frac{t}{\hbar} \right)^3.$$

(18)

Let us consider a generic secular (i.e., polarization conserving) perturbation $\hat{\Sigma}$ given by a Hamiltonian with an arbitrary anisotropy $\alpha$,

$$\hat{\Sigma} = \sum_{i,j}^N (J_{\Sigma})_{ij} \left[ 2\alpha \hat{S}_i^z \hat{S}_j^z - \left( \hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y \right) \right].$$

(19)

This is still quite general since even a double quantum perturbation $(\hat{S}_i^+ \hat{S}_j^+ + \hat{S}_i^- \hat{S}_j^-)$, which does not conserve polarization, can be reduced to a secular one by the truncating effects of radiofrequency fields [29]. In addition, we do not consider here the case $[\hat{\Sigma}, \hat{S}_i^z] = 0$ (e.g., pure Ising or on-site diagonal disorder), since in such a condition the first non-trivial order in time is the 4th (see Appendix). Then, the following identities hold:

$$\sum_{i\in A} \frac{1}{2N-1} \langle \beta_i | \hat{\Sigma}^2 | \beta_i \rangle = 2N\sigma^2 \left( \frac{\alpha^2}{4} + \frac{1}{8} \right),$$

(20)

$$\sum_{i\in A} \frac{1}{2N-1} \langle \beta_i | \hat{\Sigma} \hat{S}_i^z \hat{\Sigma} | \beta_i \rangle = 2N\sigma^2 \left( \frac{\alpha^2}{8} + \frac{1}{16} \right) - \frac{1}{2} \sigma^2,$$

(21)

$$\sum_{i\in A} \frac{1}{2N-1} \langle \beta_i | \hat{\Sigma} | \beta_i \rangle^2 = 2N\sigma^2 \alpha^2 \frac{\sigma^2}{4}.$$

(22)

Here, $\sigma^2$ stands for the average local second moment of $\hat{\Sigma}$,

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^N \sigma_i^2 = \frac{1}{N} \sum_{i=1}^N \left[ \sum_{j \neq i}^N \frac{(J_{\Sigma})_{ij}}{2} \right]^2.$$

(23)

Being a perturbation, we notice that $\sigma^2$ is much smaller than the average local second moment $\sigma_0^2$ of the unperturbed Hamiltonian $\hat{H}_0$. In terms of time scales,

$$T_2 = \hbar / \sqrt{\sigma_0^2} \ll \hbar / \sqrt{\sigma^2} = \tau_{\Sigma}.$$  

(24)

The identities in equations (20), (21) and (22) lead to

$$M_{1,1}(t) = 1 - (t/\tau_{\Sigma})^2 + \mathcal{O} \left( \frac{t}{\hbar} \right)^3,$$

(25)

and,

$$M_{MB}(t) = 1 - \frac{1}{4} N \left( t/\tau_{\Sigma} \right)^2 + \mathcal{O} \left( \frac{t}{\hbar} \right)^3,$$

(26)

$$M_X(t) = \left( \frac{N-4}{4} \right) \left( t/\tau_{\Sigma} \right)^2 + \mathcal{O} \left( \frac{t}{\hbar} \right)^3.$$

(27)
These expansions hold for \( t < (\tau_2/N) \). Beyond such a very short time regime, a general term in the expansion of \( M_{1,1}(t) \) will be of the form

\[
c(n,n) t^{n} / (\tau_2^n T_2^{-n-k})
\]

with \( k \geq 2 \) and the coefficient \( c(n,n) \) described by combinatorial numbers of increasing size that depend on the topology of the interactions (e.g. see \( 22, 48 \)). Since the experimental set up corresponds to the limit described by equation \( 24 \), this expansion will be dominated by terms with the lowest possible order in the weak interaction, i.e. \( k = 2 \):

\[
(t/\tau_2)^2 \left[ 1 + \sum_n c(n,n) (t/T_2)^{n-2} \right].
\]

Equation \( 29 \) indicates that beyond the very short time expansion, i.e. \( (\tau_2/N) < t < \tau_2 \), the dependence on \( \tau_2 \) becomes superseded by the diverging terms in the scale \( T_2 \). This could lead to the new time scale \( T_3 \) which was seen experimentally to be tied to \( T_2 \) as

\[
T_2 \lesssim T_3 \ll \tau_2.
\]

In that sense, \( T_3 \) becomes characteristic of the complexity or “chaos” of the many spin system that amplifies the small effect of the perturbation. In addition, it is important to stress that, being an experimental fact, equation \( 30 \) corresponds to a system composed by infinitely many interacting spins. In other words, equation \( 30 \) stands for the relations of time scales in the thermodynamic limit. Quite on the contrary, any numerical simulation involves a finite, very small indeed, number of spins where the irreversibility rate \( T_3 \) would be essentially given by \( \tau_2 \). Then, the LE decay rate evaluated in a finite system would ultimately be perturbation-dependent \( 28 \). Thus, our Central Hypothesis of Irreversibility would mean that equation \( 30 \) is an emergent property. It should rely on the thermodynamic limit, which implies taking the limit \( N \to \infty \) first, and then \( \tau_2 \to \infty \). The non-uniformity of these limits plays a crucial role to yield quantum phase transitions, as discussed in the context of Anderson localization \( 49–51 \).

The physical picture described above is schematically represented in Fig. 2. There, we show the expected interplay between \( M_{MB}(t) \) and \( M_X(t) \) leading to \( M_{1,1}(t) \). Indeed, as stated in equations \( 20 \) and \( 27 \), the very short time dependence of both contributions is extensive in \( N \): \( M_{MB}(t) \) decreases as \( 1 - N\sigma^2 t^2/4 \) and \( M_X(t) \) increases as \( (N - 4)\sigma^2 t^2/4 \). Such a precise balance provides for the short time decay of \( M_{1,1}(t) \) given by equation \( 24 \), i.e.

\[
1 - \sigma^2 t^2.
\]

Notice that there is no reason to assume that the decay of \( M_{MB}(t) \) would remain ruled by \( \tau_2 \). Beyond the very short times, we expect that the time scale \( T_3 \) should also show up as \( T_3/N^\nu \) with \( \nu \sim 1 \) in the decay of \( M_{MB}(t) \). Furthermore, while \( M_{MB}(t) \) goes monotonically to zero, \( M_X(t) \) displays a highly non-trivial behavior. Indeed, \( M_X(t) \) first increases by feeding from the decay of \( M_{MB}(t) \) until it reaches a maximum. This growth indicates a progressive divergence of long-range correlations. Thereafter, \( M_X(t) \) should decay accounting for the fact that the state remains properly normalized. This is precisely what \( M_{1,1}(t) \) measures: a conserved polarization that ultimately distributes uniformly within the spin system. In an isolated finite system this implies the asymptotic plateau \( M_{\infty} \sim 1/N \). As pointed above, the decay of both \( M_{1,1}(t) \) and \( M_X(t) \) occurs in a time scale \( T_3 \), which according to equation \( 29 \), is somewhat longer but close to the “diffusion” time \( T_2 \). This is the regime captured experimentally.

### B. The extensive decay hypothesis

The previous short-time expansions provide a hint on the scaling relation between the local LE, \( M_{1,1}(t) \), and the global one as embodied by \( M_{MB}(t) \). In particular, let us first compare the probability of refocusing the configuration (up or down) of a single spin, i.e. \( \Pi_{1,1}(t) \), and the probability of refocusing a complete many-spin state \( M_{MB}(t) \). If the refocusing of each individual spin could be treated as an independent event, then the scaling between \( \Pi_{1,1} \) and \( M_{MB} \) would be extensive in \( N \),

\[
(\Pi_{1,1}(t))^{N/2} \simeq M_{MB}(t).
\]

Here, the factor 1/2 in the exponent comes from equation \( 11 \), i.e. the initial high temperature state, where basically half of the spins point up, and half of them point down. Then, one can resort to the picture of a lattice gas where
FIG. 2. A pictorial scheme of the time dependence of $M_{1,1}(t)$ (dotted line) and its contributions, i.e. $M_{MB}(t)$ and $M_X(t)$ (solid lines). Their short-time expansions, as stated in equations (25), (26) and (27), are indicated with arrows. In particular, the expansion corresponding to the short-time behavior of $M_{1,1}(t)$ is plotted with a dashed line. (Online version in colour.)

$N/2$ particles jump among $N$ lattice sites. As in the well known Jordan-Wigner transformation [52], a fermion is associated to a spin pointing up and a vacancy corresponds to a spin pointing down. Thus, the microstate of the gas is completely described by the position of $N/2$ particles.

Strictly speaking, the notion of *extensiveness* corresponds to standard thermodynamic quantities such as the entropy of the system. In addition, as discussed in [16], $S = -\ln(M_{1,1}(t))$ is precisely a measure of the entropy. Then, the validity of equation (31) implies an extensivity relation between the entropy per spin and the total entropy of the system.

According to equations (8) and (25),

$$\Pi_{1,1}(t) = 1 - \frac{1}{2} (t/\tau_{\Sigma})^2 + O\left(\frac{t}{\hbar}\right)^3,$$

which in turn, up to 2nd order in time, implies $\Pi_{1,1}(t) \simeq (M_{1,1}(t))^{1/2}$. Thus, equation (31) yields

$$(M_{1,1}(t))^{N/4} \simeq M_{MB}(t).$$

This is precisely the relation verified between equations (25) and (26).

One might expect that beyond the very short-time decay, individual spin autocorrelations deviate from the statistical independence. However, this deviation will still have a local nature and therefore the $N$-extensivity would remain valid. Indeed, we propose

$$(M_{1,1}(t))^\eta \simeq M_{MB}(t),$$

where the exponent $\eta$ would be some appropriate function $\eta = \eta(N, t)$. Our “extensive decay hypothesis” implies that $\eta$ factorizes:

$$\eta(N, t) = N \times f(t),$$

where $f(t)$ stands for a function that encloses information of the correlations originated by the system dynamics. Additionally,

$$\lim_{t \to 0^+} f(t) = \frac{1}{4}$$

is required in order to recover equation (33), i.e. the statistical independence.
IV. A 1D MODEL.

The physical picture described above is discussed here under the light of a specific model. In particular, we assess the validity of equations (33) and (35). We consider a 1-D spin chain with an anisotropic interaction described by:

$$\hat{H}_0 = \sum_{i=1}^{N-1} J_0 \left( \frac{1}{2} S_i^z S_{i+1}^z + S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right),$$

(37)

with periodic boundary conditions, i.e. a ring configuration. Here, $J_0$ stands for the natural units of the spin-spin interaction energy. As a perturbation $\hat{\Sigma}$ we choose a next nearest neighbors interaction described by:

$$\hat{\Sigma} = \sum_{i=1}^{N-2} J_\Sigma \left( \frac{1}{2} S_i^z S_{i+2}^z + S_i^x S_{i+2}^x + S_i^y S_{i+2}^y \right).$$

(38)

Such a perturbation appears naturally when one attempts to build an effective one-body dynamics from linear crystals with dipolar interactions [52]. This is also the case in a regular crystal, when the natural non-secular dipole-dipole terms are truncated by the Zeeman energy of the radiofrequency irradiation, which ultimately leads to effective secular two-body next nearest neighbors interactions [53].

The local second moments $\sigma^2$ and $\sigma^2_0$ of $\Sigma$ and $\hat{H}_0$ respectively can be evaluated as in equation (28):

$$\sigma^2 = \frac{1}{2} (J_\Sigma)^2,$$

(39)

$$\sigma^2_0 = \frac{1}{2} (J_0)^2,$$

(40)

and constitute the main energy scales of our problem.

In Fig. 3 we plot $M_{1,1}(t)$, $M_{MB}(t)$ and $M_X(t)$ for the particular choice $J_\Sigma = 0.1 J_0$. Short-time expansions given in equations (25), (26) and (27) are evaluated according to equation (39). It is observed that $M_{MB}(t)$ vanishes for long times. Actually, a close observation shows that $M_{MB}(t \to \infty) \sim \mathcal{O}(2^{-N})$ (data not shown). In addition, notice that $M_X(t \to \infty) \sim 1/N$. Such an asymptotic contribution provides for the equidistribution of the spin polarization $M_{1,1}(t \to \infty) \sim 1/N$. This long-time saturation corresponds to the equilibration of a finite system.

In contrast to our schematic plot in Fig. 2, here $M_X(t)$ does not get too close to 1 and $M_{MB}(t)$ does not decay much faster than $M_{1,1}(t)$. Since $M_X(t)$ provides for the whole $M_{1,1}(t)$ once that $M_{MB}(t)$ has fully decayed, the contribution of $M_X(t)$ is considerable only at long times. These effects are a consequence of the relatively small size of the system considered. Indeed, the case in Fig. 3 corresponds to $N = 14$ spins, and thus the exponent that relates $M_{1,1}(t)$ and $M_{MB}(t)$ is quite small ($N/4 = 3.5$). The need for larger systems indicates that revealing the dominant orders in equation (29) is a major numerical challenge that may go beyond the state-of-the-art techniques [54].

In order to assess the accuracy of the “extensive decay hypothesis”, in Fig. 4 we address the scaling relation between $M_{1,1}(t)$ and $M_{MB}(t)$ discussed in Sec. IIIB. In particular, we try out the factorization stated in equation (34). By plotting $\log(M_{MB}(t))/\log(M_{1,1}(t)) N$ as a function of time, we observe a unique function which does not depend on $N$ or $J_\Sigma$, but it has a weak dependence on time. Such a unique curve is indeed $f(t)$ as defined in equation (35). This means that the extensivity relation between $M_{1,1}(t)$ and $M_{MB}(t)$ is confirmed. The statistical independence, in turn, fails progressively once $f(t)$ departs from the 1/4 factor of the ideal relation in equations (33) and (36). Since beyond the short-time regime $f(t)$ decreases with time, we conclude that the recovery of a single spin is tied to the recovery of its neighbors. Thus, the spins are positively correlated and the revival probability of the complete $N$-spin state is enhanced. This argument is particularly relevant in 1D systems.

After the onset of the saturation regime, where $M_{1,1} \sim 1/N$ and $M_{MB} \sim \mathcal{O}(2^{-N})$, the universal scaling naturally becomes noisy and the curves for different $N$ and $J_\Sigma$ separate each other. Since the decay is faster for larger perturbations, the appearance of such a spurious behavior is observed to occur first for the largest value of $J_\Sigma$ considered ($J_\Sigma = 0.3 J_0$, plus signs and triangles).

V. CONCLUSION

We presented a detailed analysis of the LE in interacting spin systems. As in the NMR experiments, a local version of the LE, $M_{1,1}$, is defined as a single spin autocorrelation function. Simultaneously, we define a global LE, $M_{MB}$,
FIG. 3. The local LE and its non local contributions. $M_{1,1}(t)$, $M_{MB}(t)$ and $M_X(t)$ correspond to the solid lines as indicated by the labels in the figure. $N = 14$, $J_S = 0.1J_0$. The short-time expansions given in equations (25), (26) and (27) are shown in dashed lines. (Online version in colour.)

FIG. 4. The relation $\log(M_{MB}(t))/(N \log(M_{1,1}(t)))$ as a function of time. For $J_S = 0.1J_0$ the sizes plotted are: $N = 10$ (circles), $N = 12$ (squares), $N = 14$ (diamonds), and $N = 16$ (stars). For $J_S = 0.2J_0$ the sizes plotted are: $N = 10$ (solid line), $N = 12$ (dashed line), $N = 14$ (dash-dot line), and $N = 16$ (dotted line). For $J_S = 0.3J_0$ the sizes plotted are: $N = 10$ (up triangles), $N = 12$ (down triangles), $N = 14$ (plus signs), and $N = 16$ (right triangles). (Online version in colour.)

as the average of the square overlap between many-body wave functions that evolved under perturbed Hamiltonians. While the former constitutes a specific experimental observable, the latter has only been assessed theoretically. Here, we showed the formal relation between both magnitudes, as far as their characteristic time scales and $N$-dependence are concerned.

By analyzing a short-time expansion of $M_{1,1}$ and $M_{MB}$ we derived a precise relation between their time scales. In this regime, the decay of $M_{1,1}$ is given by the average local second moment of the perturbation ($\hbar/\tau_S = \sqrt{\sigma^2}$), and the decay of $M_{MB}$ by $N$ times the local scale ($N\hbar/\tau_S$). This relation hints a scaling law $M_{MB} \sim (M_{1,1})^{N/4}$ that
accounts for the extensivity of $M_{MB}$. In such a case, the recovery of a many-spin state results from the recovery of each individual spin, much as if they were independent events. The numerical evaluation in a specific spin model shows that the exponent slightly diminishes with time, starting from the initial $N/4$. This means that the recovery of a single spin is positively correlated with the probability of recovery of its neighbors, and thus it improves the probability of the revival of the complete $N$-spin state. A precise control of these correlations may hint an experimental access to the global autocorrelation, i.e. $M_{MB}$, just by measuring a single spin (local) autocorrelation $M_{1,1}$. This would require an experimental protocol capable to encode a local excitation into a correlated many-spin state.

In addition, we discussed a general dynamical picture beyond the very short-time regime. There, the decay of $M_{1,1}$ results from the interplay between the time scale that characterizes the reversible interactions ($T_2$) and that of the perturbation ($\tau_\Sigma$). This would ultimately lead to the experimentally observed $T_3$, which was found to be roughly independent of $\tau_\Sigma$ but closely related to $T_2$. The theoretical quest for the emergent $T_3$ time scale remains open and it may be out of the reach of current numerical approaches. Assessing a fair estimate analytically would require a detailed account of the higher order processes that dress the quadratic term in the perturbative expansion.

Notice that our discussion lead us to identify $T_3$, and hence the spin-spin interaction time $T_2$, as the time scales characterizing the complexity or many-spin chaos. As such, they show up not only in the decay of $M_{1,1}$ and $M_{MB}$, but also in the growth of $M_X = M_{1,1} - M_{MB}$. Indeed, in the field of AdS/CFT there is an increasing interest in characterizing the role of chaos in quantum dynamics \cite{57,58}. There, chaos manifests in the growth of four-body correlation functions, following an early suggestion by Larkin and Ovchinnikov \cite{59}. They employed semiclassical arguments to address disordered superconductors and proved that the square dispersion of momentum should grow exponentially in a time scale determined by the collisions with impurities, i.e. with the unperturbed Hamiltonian (in our physical picture, $T_2$). Similarly, our average multispin correlation $M_X$ would ultimately diverge within a time scale $T_3/N$, i.e. independent of the perturbation. This is indeed a measure of the decoherence, and hence of irreversibility, induced by many-spin chaos. Of course, we do not have a precise characterization of this time scale or the specific mathematical dependence on time. Thus, this is a puzzling issue to explore in the field of many-body chaos. Besides the obvious relevance for statistical mechanics and experimental physics, this might also contribute to a possible pathway between quantum mechanics and gravity.

**ACKNOWLEDGMENTS**

This work benefited from discussions with A.D. Dente and F. Pastawski. HMP greatly acknowledges hospitality of A. Kitaev at Caltech, P.A. Lee at MIT and V. Oganesyan at CUNY, where the issues discussed in this paper acquired certain maturity. PRZ acknowledges M.C. Banuls and J.I. Cirac for their kind hospitality at MPQ in Garching. We acknowledge financial support from CONICET, ANPCyT, SeCyT-UNC and MinCyT-Cor. This work used computational resources from CCAD – Universidad Nacional de Córdoba (http://ccad.unc.edu.ar/), in particular the Mendieta Cluster, which is also part of SNCAD – MinCyT, República Argentina.

**VI. APPENDIX**

It is worthy to mention that very short-time expansions in equations \cite{25}, \cite{26} and \cite{27} do not depend on the anisotropy $\alpha$ of the perturbation. In general, it can be proved that if $\left[ \hat{\Sigma}, \hat{S}_1^z \right] = 0$ then:

$$M_{1,1}(t) = 1 - \frac{(t/\hbar)^4}{2^{N+3}} \sum_{i \in A} \left( 2 \langle \beta_i \left| \hat{\Sigma}, \hat{H}_0 \right\rangle \hat{S}_1^z \left[ \hat{\Sigma}, \hat{H}_0 \right] |\beta_i \rangle - \langle \beta_i \left| \hat{\Sigma}, \hat{H}_0 \right\rangle^2 |\beta_i \rangle \right) + \mathcal{O} \left( (t/\hbar)^5 \right).$$

This is precisely the case of a perturbation $\hat{\Sigma}$ enclosing Anderson disorder and Ising interactions \cite{40} or interactions with a fluctuating field \cite{60}.

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