Efficient algorithms for computing the Euler-Poincaré characteristic of symmetric semi-algebraic sets

Saugata Basu and Cordian Riener

Abstract. Let \( R \) be a real closed field and \( D \subset R \) an ordered domain. We consider the algorithmic problem of computing the generalized Euler-Poincaré characteristic of real algebraic as well as semi-algebraic subsets of \( R^k \), which are defined by symmetric polynomials with coefficients in \( D \). We give algorithms for computing the generalized Euler-Poincaré characteristic of such sets, whose complexities measured by the number the number of arithmetic operations in \( D \), are polynomially bounded in terms of \( k \) and the number of polynomials in the input, assuming that the degrees of the input polynomials are bounded by a constant. This is in contrast to the best complexity of the known algorithms for the same problems in the non-symmetric situation, which are singly exponential. This singly exponential complexity for the latter problem is unlikely to be improved because of hardness result (\( \#P \)-hardness) coming from discrete complexity theory.

1. Introduction

Let \( R \) be a real closed field which is fixed for the remainder of the paper, and let \( C \) denote the algebraic closure of \( R \). Given a semi-algebraic set \( S \subset R^k \), i.e., a set defined by unions and intersections of polynomial inequalities, it is a fundamental question of computational algebraic geometry to compute topological information about \( S \). This problem of designing efficient algorithms for computing topological invariants – such as the Betti numbers as well as the Euler-Poincaré characteristic – has a long history. The first algorithms \cite{19} used the technique of cylindrical algebraic decomposition and consequently had doubly exponential complexity. Algorithms for computing the zeroth Betti number (i.e. the number of semi-algebraically connected components) of semi-algebraic sets using the critical points method were discovered later \cite{11, 14, 15, 5} and improving this complexity bound remains an active area of research even today. Later, algorithms with singly exponential complexity for computing the first Betti number \cite{7}, as well as the first few Betti numbers \cite{2} were discovered. Algorithms with singly exponential complexity for computing the Euler-Poincaré characteristic are also known \cite{6}. It remains an open problem to design an algorithm with singly exponential complexity for computing all the Betti numbers of a given semi-algebraic set. Algorithms

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with polynomially bounded complexity for computing the Betti numbers of semi-algebraic sets are known in a few cases – for example, for sets defined by a few (i.e. any constant number of) quadratic inequalities \[3, 4\]. Also note that the problem of expressing the Euler-Poincaré characteristic of real algebraic varieties in terms of certain algebraic invariants of the polynomials defining the variety has been considered by several other authors (see for example \[13\] and \[20\]). But these studies do not take into account the computational complexity aspect of the problem.

In this article we will restrict to the case when \(S\) is defined by symmetric polynomial inequalities whose degree is at most \(d\in\mathbb{N}\), which we will think of as a fixed constant. It is known \[21, 16\] that in this particular setup one can decide emptiness of \(S\) in a time which is polynomial in \(k\) - the number of variables. Despite being a rather basic property, it is known to be a \(\text{NP}\)-hard problem (in the Blum-Shub-Smale model) to decide if a given real algebraic variety \(V\subset\mathbb{R}^k\) defined by one polynomial equation of degree at most 4 is empty or not \[10\]. Following this notable difference it is natural to ask, if in general symmetric semi-algebraic sets are algorithmically more tractable than general semi-algebraic sets and if it is possible to obtain polynomial time (for fixed degree \(d\)) algorithms for computing topological invariants of such sets. In this article we answer this in the affirmative for the problem of computing the generalized Euler-Poincaré characteristic (both the ordinary as well as the equivariant versions) of symmetric semi-algebraic sets.

The problem of computing the generalized Euler-Poincaré characteristic is important in several applications both theoretical and practical. For example, such an algorithm is a key ingredient in computing the integral (with respect to the Euler-Poincaré measure) of constructible functions, and this latter problem has been of recent interest in several applications \[1\].

Before proceeding further we first fix some notation.

**Notation 1.** For \(P \in \mathbb{R}[X_1,\ldots,X_k]\) (respectively \(P \in \mathbb{C}[X_1,\ldots,X_k]\)) we denote by \(\text{Zer}(P,\mathbb{R}^k)\) (respectively, \(\text{Zer}(P,\mathbb{C}^k)\)) the set of zeros of \(P\) in \(\mathbb{R}^k\) (respectively, \(\mathbb{C}^k\)). More generally, for any finite set \(P \subset \mathbb{R}[X_1,\ldots,X_k]\) (respectively, \(P \subset \mathbb{C}[X_1,\ldots,X_k]\)), we denote by \(\text{Zer}(P,\mathbb{R}^k)\) \(\text{Zer}(P,\mathbb{C}^k)\) (respectively, \(\text{Zer}(P,\mathbb{C}^k)\)) the set of common zeros of \(P\) in \(\mathbb{R}^k\) (respectively, \(\mathbb{C}^k\)).

**Notation 2.** Let \(P \subset \mathbb{R}[X_1,\ldots,X_k]\) be a finite family of polynomials.

1. We call any Boolean formula \(\Phi\) with atoms, \(P \sim 0, P \in P\), where \(\sim\) is one of \(=, >,\) or \(<\), to be a \(P\)-formula. We call the realization of \(\Phi\), namely the semi-algebraic set
   \[
   \text{Reali}(\Phi,\mathbb{R}^k) = \{x \in \mathbb{R}^k \mid \Phi(x)\}
   \]
   a \(P\)-semi-algebraic set.

2. We call an element \(\sigma \in \{0, 1, -1\}^P\), a sign condition on \(P\). For any semi-algebraic set \(Z \subset \mathbb{R}^k\), and a sign condition \(\sigma \in \{0, 1, -1\}^P\), we denote by \(\text{Reali}(\sigma, Z)\) the semi-algebraic set defined by
   \[
   \{x \in Z \mid \text{sign}(P(x)) = \sigma(P), P \in P\},
   \]
   and call it the realization of \(\sigma\) on \(Z\).

3. We call a Boolean formula without negations, and with atoms \(P \sim 0, P \in P\), and \(\sim\) one of \(\leq, \geq\), to be a \(P\)-closed formula, and we call the realization, \(\text{Reali}(\Phi,\mathbb{R}^k)\), a \(P\)-closed semi-algebraic set.
(4) For any finite family of polynomials $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$, we call an element $\sigma \in \{0, 1, -1\}^\mathcal{P}$, a sign condition on $\mathcal{P}$. For any semi-algebraic set $Z \subset \mathbb{R}^k$, and a sign condition $\sigma \in \{0, 1, -1\}^\mathcal{P}$, we denote by $\text{Reali}(\sigma, Z)$ the semi-algebraic set defined by

$$\{x \in Z \mid \text{sign}(P(x)) = \sigma(P), P \in \mathcal{P}\},$$

and call it the realization of $\sigma$ on $Z$.

(5) We denote by $\text{SIGN}(\mathcal{P}) \subset \{0, 1, -1\}^\mathcal{P}$ the set of all sign conditions $\sigma$ on $\mathcal{P}$ such that $\text{Reali}(\sigma, \mathbb{R}^k) \neq \emptyset$. We call $\text{SIGN}(\mathcal{P})$ the set of realizable sign conditions of $\mathcal{P}$.

**Notation 3.** For any semi-algebraic set $X$, and a field of coefficients $F$, we will denote by $H_i(X, F)$ the $i$-th homology group of $X$ with coefficients in $F$, by $b_i(X, F) = \dim F H_i(X, F)$.

Note here that we work over any real closed field. Therefore the definition of homology groups is a little bit more delicate, in particular because $\mathbb{R}$ might be non-archimedean. In case of a closed and bounded semi-algebraic set, $S$ the homology $H_i(S, F)$ can be defined as the $i$-th simplicial homology group associated to a semi-algebraic triangulation of $S$. The general case then is taken care of by constructing to a general semi-algebraic set $S$ a semi-algebraic set $S'$, which is closed, bounded, and furthermore semi-algebraically homotopy equivalent to $S$. We refer the reader to [8, Chapter 6] for details of this construction. The topological Euler-Poincaré characteristic of a semi-algebraic set $S \subset \mathbb{R}^k$ is the alternating sum of the Betti numbers of $S$. More precisely,

$$\chi^{\text{top}}(S, F) = \sum_i (-1)^i \dim F H_i(S, F).$$

For various applications (such as in motivic integration [12] and other applications of Euler integration [17, 18, 23, 1]) the generalized Euler-Poincaré characteristic has proven to be more useful than the ordinary Euler-Poincaré characteristic. The main reason behind the usefulness of the generalized Euler-Poincaré characteristic of a semi-algebraic set is its additivity property, which is not satisfied by the topological Euler-Poincaré characteristic. The generalized Euler-Poincaré characteristic agrees with the topological Euler-Poincaré characteristic for compact semi-algebraic sets, but can be different for non-compact ones (see Example 1). Nevertheless, the generalized Euler-Poincaré characteristic is intrinsically important because of the following reason.

The Grothendieck group $K_0(\text{sa}_\mathbb{R})$ of semi-algebraic isomorphic classes of semi-algebraic sets (two semi-algebraic sets being isomorphic if there is a continuous semi-algebraic bijection between them) (see for example [12, Proposition 1.2.1]) is isomorphic to $\mathbb{Z}$, and the generalized Euler-Poincaré characteristic of a semi-algebraic set can be identified with its image under the isomorphism that takes the class of a point (or any closed disk) to 1.

**Definition 1.** The generalized Euler-Poincaré characteristic, $\chi^{\text{gen}}(S)$, of a semi-algebraic set $S$ is uniquely defined by the following properties [22]:

1. $\chi^{\text{gen}}$ is invariant under semi-algebraic homeomorphisms.
(2) $\chi^{\text{gen}}$ is multiplicative, i.e. $\chi^{\text{gen}}(A \times B) = \chi^{\text{gen}}(A) \cdot \chi^{\text{gen}}(B)$.

(3) $\chi^{\text{gen}}$ is additive, i.e. $\chi^{\text{gen}}(A \cup B) = \chi^{\text{gen}}(A) + \chi^{\text{gen}}(B) - \chi^{\text{gen}}(A \cap B)$.

(4) $\chi^{\text{gen}}([0, 1]) = 1$.

The following examples are illustrative.

Example 1. For every $n \geq 0$,

$\chi^{\text{gen}}([0, 1]^n) = \chi^{\text{gen}}([0, 1])^n = 1$,

$\chi^{\text{top}}([0, 1]^n) = 1$,

$\chi^{\text{gen}}((0, 1)^n) = (\chi^{\text{gen}}(0, 1))^n = (\chi^{\text{gen}}([0, 1]) - \chi^{\text{gen}}(0) - \chi^{\text{gen}}(1))^n = (-1)^n$,

$\chi^{\text{top}}((0, 1)^n) = 1$.

Let $\mathfrak{S}_k$ denote the symmetric group on $k$-letters. Throughout the article we will consider the more general case of products of symmetric groups and we fix the following notation.

Notation 4. For $\omega \in \mathbb{N}$ and $k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{>0}^\omega$, with $k = \sum_{i=1}^\omega k_i$ denote $\mathfrak{S}_k = \mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_\omega}$. If $\omega = 1$, then $k = k_1$, and we will denote $\mathfrak{S}_k$ simply by $\mathfrak{S}_k$.

A set $X \subset \mathbb{R}^k$ is said to be symmetric, if it is closed under the action of $\mathfrak{S}_k$. For such a set we will denote by $X / \mathfrak{S}_k$ the orbit space of this action.

Notation 5. For any $\mathfrak{S}_k$ symmetric semi-algebraic subset $S \subset \mathbb{R}^k$ with $k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{>0}^\omega$, with $k = \sum_{i=1}^\omega k_i$, and any field $\mathbb{F}$, we denote

1. $\chi^{\text{top}}(S, \mathbb{F}) = \sum_{i \geq 0} (-1)^i b_i(S, \mathbb{F})$,

2. $\chi_{\mathfrak{S}_k}(S, \mathbb{F}) = \sum_{i \geq 0} (-1)^i b^i_{\mathfrak{S}_k}(S, \mathbb{F})$,

3. $\chi^{\text{gen}}(S) = \chi^{\text{gen}}(S / \mathfrak{S}_k) = \chi^{\text{gen}}(\phi_k(S))$.

1.1. Main result. We describe new algorithms for computing the generalized Euler-Poincaré characteristic (see Definition 1) of semi-algebraic sets defined in terms of symmetric polynomials. The algorithms we give here have complexity which is polynomial (for fixed degrees and the number of blocks) in the number of symmetric variables. Since for systems of equations with a finite set of solutions, the generalized Euler-Poincaré characteristic of the set of solutions coincides with its cardinality, it is easily seen that that computing the generalized Euler-Poincaré characteristic of the set of solutions of a polynomial system with a fixed degree bound is a $\#\mathbf{P}$-hard problem in general (i.e., in the non-symmetric situation). Thus, this problem is believed to be unlikely to admit a polynomial time solution.

We prove the following theorems.

Theorem 1. Let $D$ be an ordered domain contained in a real closed field $\mathbb{R}$. Then, there exists an algorithm that takes as input:

1. a tuple $k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{>0}^\omega$, with $k = \sum_{i=1}^\omega k_i$;

2. a polynomial $P \in D[X^{(1)}, \ldots, X^{(\omega)}]$, where each $X^{(i)}$ is a block of $k_i$ variables, and $P$ is symmetric in each block of variables $X^{(i)}$;

and computes the generalized Euler-Poincaré characteristics

$\chi^{\text{gen}}(\text{Zer} \ (P, \mathbb{R}^k)), \chi^{\text{gen}}_{\mathfrak{S}_k}(\text{Zer} \ (P, \mathbb{R}^k))$.

The complexity of the algorithm measured by the number of arithmetic operations in the ring $D$ (including comparisons) is bounded by $(\omega d)^{O(D)}$, where $d = \deg(P)$ and $D = \sum_{i=1}^\omega \min(k_i, 2d)$. 

Notice that in case, $\omega = 1$ and $k = (k)$, the complexity is polynomial in $k$ for fixed $d$.

We have the following result in the semi-algebraic case.

**Theorem 2.** Let $D$ be an ordered domain contained in a real closed field $R$. Then, there exists an algorithm that takes as input:

1. A tuple $k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{\geq 0}^\omega$, with $k = \sum_{i=1}^\omega k_i$;
2. A set of $s$ polynomials $P = \{P_1, \ldots, P_s\} \subset D[X^{(1)}, \ldots, X^{(\omega)}]$, where each $X^{(i)}$ is a block of $k_i$ variables, and each polynomial in $P$ is symmetric in each block of variables $X^{(i)}$ and of degree at most $d$;
3. A $P$-semi-algebraic set $S$, described by

$$S = \bigcup_{\sigma \in \Sigma} \text{Reali} (\sigma, R^k),$$

where $\Sigma \subset \{0, 1, -1\}^P$;

and computes the generalized Euler-Poincaré characteristics $\chi_{\text{gen}}(S), \chi_{\text{gen}}^{\omega}(S)$. The complexity of the algorithm measured by the number of arithmetic operations in the ring $D$ (including comparisons) is bounded by

$$\text{card}(\Sigma)^{O(1)} + sD' k^d d^{O(D'D'')} + sD' d^{O(D'')} (k\omega D)^{O(D'')},$$

where $D = d(D'' \log d + D' \log s)$, $D' = \sum_{i=1}^\omega \min(k_i, d)$, $D'' = \sum_{i=1}^\omega \min(k_i, d)$, and $D''' = \sum_{i=1}^\omega \min(k_i, 2D)$.

The algorithm also involves the inversion matrices of size $sD' d^{O(D''')}$ with integer coefficients.

Notice that the complexity in the semi-algebraic case is still polynomial in $k$ for fixed $d$ and $s$ in the special case when $\omega = 1$, and $k = (k)$. Also note that, as a consequence of Proposition 8 below, the number of sign conditions with non-empty realizations in $\Sigma$ is bounded by $sD' d^{O(D''')}$. 

**Remark 1.** An important point to note is that we give algorithms for computing both the ordinary as well as the equivariant generalized Euler-Poincaré characteristics. For varieties or semi-algebraic sets defined by symmetric polynomials with degrees bounded by a constant, the ordinary generalized Euler-Poincaré characteristic can be exponentially large in the dimension $k$. Nevertheless, our algorithms for computing it have complexities which are bounded polynomially in $k$ for fixed degree.

### 1.2. Outline of the main techniques.

Efficient algorithms (with singly exponential complexity) for computing the Euler-Poincaré characteristics of semi-algebraic sets [8, Chapter 13] usually proceed by first making a deformation to a set defined by one inequality with smooth boundary and non-degenerate critical points with respect to some affine function. Furthermore, the new set is homotopy equivalent to the given variety and the Euler-Poincaré characteristic of this new set can be computed from certain local data in the neighborhood of each critical point (see [8, Chapter 13] for more detail). Since the number of critical points is at most singly exponential in number, such algorithms have a singly exponential complexity.

The approach used in this paper for computing the Euler-Poincaré characteristics for symmetric semi-algebraic sets is similar – but differs on two important
points. Firstly, unlike in the general case, we are aiming here for an algorithm with polynomial complexity (for fixed \(d\)). This requires that the perturbation, as well as the Morse function both need to be equivariant. The choices are more restrictive (see Proposition 6).

Secondly, the topological changes at the Morse critical points need to be analyzed more carefully (see Lemmas 2 and 3). The main technical tool that makes the good dependence on the degree \(d\) of the polynomial possible is the so called “half-degree principle” [16, 21] (see Proposition 7), and this is what we use rather than the Bezout bound to bound the number of (orbits of) critical points. The proofs of these results appear in [9], where they are used to prove bounds on the equivariant Betti numbers of semi-algebraic sets.

Using these results, we prove exact formulas for the ordinary as well as the equivariant Euler-Poincaré characteristic of symmetric varieties (see (2) and (3) in Theorem 3), which form the basis of the algorithms described in this paper.

We adapt several non-equivariant algorithms from [8] to the equivariant setting. The proofs of correctness of the algorithms described for computing the ordinary as well as the equivariant (generalized) Euler-Poincaré characteristics of algebraic as well as semi-algebraic sets (Algorithms 2, 5 and 6) follow from the equivariant Morse lemmas (Lemmas 2 and 3). The complexity analysis follows from the complexities of similar algorithms in the non-equivariant case [8], but using the half-degree principle referred to above. In the design of Algorithms 2, 5 and 6 we need to use several subsidiary algorithms which are closely adapted from the corresponding algorithms in the non-equivariant situation described in [8]. In particular, one of them, an algorithm for computing the set of realizable sign conditions of a family of symmetric polynomial (Algorithm 3), whose complexity is polynomial in the dimension for fixed degree could be of independent interest.

The rest of the paper is organized as follows. In §2, we recall certain facts from real algebraic geometry and topology that are needed in the algorithms described in the paper. These include definitions of certain real closed extensions of the ground field \(R\) consisting of algebraic Puiseux series with coefficients in \(R\). We also recall some basic additivity properties of the Euler-Poincaré characteristic. In §3, we define certain equivariant deformations of symmetric varieties and state some topological properties of these deformations, that mirror similar ones in the non-equivariant case. The proofs of these properties appear in [9] and we give appropriate pointers where they can be found in that paper. In §4 we describe the algorithms for computing the Euler-Poincaré characteristics of symmetric semi-algebraic sets proving Theorems 1 and 2.

2. Mathematical Preliminaries

In this section we recall some basic facts about real closed fields and real closed extensions.

2.1. Real closed extensions and Puiseux series. We will need some properties of Puiseux series with coefficients in a real closed field. We refer the reader to [8] for further details.

Notation 6. For \(R\) a real closed field we denote by \(R(\varepsilon)\) the real closed field of algebraic Puiseux series in \(\varepsilon\) with coefficients in \(R\). We use the notation
R \langle \varepsilon_1, \ldots, \varepsilon_m \rangle$ to denote the real closed field $R \langle \varepsilon_1 \rangle \langle \varepsilon_2 \rangle \cdots \langle \varepsilon_m \rangle$. Note that in the unique ordering of the field $R \langle \varepsilon_1, \ldots, \varepsilon_m \rangle$, $0 < \varepsilon_m \ll \varepsilon_{m-1} \ll \cdots \ll \varepsilon_1 \ll 1$.

**Notation 7.** For elements $x \in R \langle \varepsilon \rangle$ which are bounded over $R$ we denote by $\lim_\varepsilon x$ to be the image in $R$ under the usual map that sets $\varepsilon$ to 0 in the Puiseux series $x$.

**Notation 8.** If $R'$ is a real closed extension of a real closed field $R$, and $S \subset R^k$ is a semi-algebraic set defined by a first-order formula with coefficients in $R$, then we will denote by $\text{Ext}(S, R') \subset \mathbb{R}^k$ the semi-algebraic subset of $R^k$ defined by the same formula. It is well-known that $\text{Ext}(S, R')$ does not depend on the choice of the formula defining $S$ [8].

**Notation 9.** For $x \in R^k$ and $r \in \mathbb{R}$, $r > 0$, we will denote by $B_k(x, r)$ the open Euclidean ball centered at $x$ of radius $r$, and we denote by $S^{k-1}(x, r)$ the sphere of radius $r$ centered at $x$. If $R'$ is a real closed extension of the real closed field $R$ and when the context is clear, we will continue to denote by $B_k(x, r)$ (respectively, $S^{k-1}(x, r)$) the extension $\text{Ext}(B_k(x, r), R')$ (respectively, $\text{Ext}(S^{k-1}(x, r), R')$). This should not cause any confusion.

### 2.2. Tarski-Seidenberg transfer principle.

In some proofs that involve Morse theory (see for example the proof of Lemma 3), where integration of gradient flows is used in an essential way, we first restrict to the case $R = \mathbb{R}$. After having proved the result over $\mathbb{R}$, we use the Tarski-Seidenberg transfer theorem to extend the result to all real closed fields. We refer the reader to [8, Chapter 2] for an exposition of the Tarski-Seidenberg transfer principle.

### 2.3. Additivity property of the Euler-Poincaré characteristics.

We need the following additivity property of the Euler-Poincaré characteristics that follow from the Mayer-Vietoris exact sequence.

**Proposition 1.** If $S_1, S_2$ are closed semi-algebraic sets, then for any field $\mathbb{F}$ and every $i \geq 0$,

\[ \chi^{\text{top}}(S_1 \cup S_2, \mathbb{F}) = \chi^{\text{top}}(S_1, \mathbb{F}) + \chi^{\text{top}}(S_2, \mathbb{F}) - \chi^{\text{top}}(S_1 \cap S_2, \mathbb{F}). \]

**Proof.** See for example [8, Proposition 6.36].

We also recall the definition of the Borel-Moore homology groups of locally closed semi-algebraic sets and some of its properties.

### 2.4. Borel-Moore homology groups.

**Definition 2.** Let $S \subset \mathbb{R}^k$ be a locally closed semi-algebraic set and let $S_r = S \cap B_k(0, r)$. The $p$-th Borel-Moore homology group of $S$ with coefficients in a field $\mathbb{F}$, denoted by $H_p^{BM}(S, \mathbb{F})$, is defined to be the $p$-th simplicial homology group of the pair $(\overline{S_r}, \overline{S_r} \setminus S_r)$ with coefficients in $\mathbb{F}$, for large enough $r > 0$.

**Notation 10.** For any locally closed semi-algebraic set $S$ we denote

\[ \chi^{BM}(S, \mathbb{F}) = \sum_{i \geq 0} (-1)^i \dim_\mathbb{F} H_i^{BM}(S, \mathbb{F}). \]

It follows immediately from the exact sequence of the homology of the pair $(\overline{S_r}, \overline{S_r} \setminus S_r)$ that
Proposition 2. If $S$ is a locally closed semi-algebraic set then for all $r > 0$ large enough
\[
\chi_{\text{BM}}(S, Q) = \chi_{\text{top}}(S_r, Q) - \chi_{\text{top}}(S \cap S^{k-1}(0, r), Q).
\]

It follows from the fact that $\chi_{\text{BM}}(\cdot, Q)$ is additive for locally closed semi-algebraic sets (cf. [8, Proposition 6.60]), and the uniqueness of the valuation $\chi_{\text{gen}}(\cdot)$ that:

Proposition 3. If $S$ is a locally closed semi-algebraic set, then
\[
\chi_{\text{gen}}(S) = \chi_{\text{BM}}(S, Q).
\]

Moreover, if $S$ is a closed and bounded semi-algebraic set then,
\[
\chi_{\text{gen}}(S) = \chi_{\text{BM}}(S, Q) = \chi_{\text{top}}(S, Q).
\]

The following proposition is an immediate consequence of Definition 2, Notation 10 and Propositions 2 and 3.

Proposition 4. Let $S \subset \mathbb{R}^k$ be a closed semi-algebraic set. Then,
\[
\chi_{\text{gen}}(S) = \chi_{\text{gen}}\left(S \cap B_k(0, r)\right) - \chi_{\text{gen}}(S \cap S^{k-1}(0, r))
\]
for all large enough $r > 0$.

Proof. By the theorem on conic structure of semi-algebraic sets at infinity (see [8, Proposition 5.49]) we have that $S$ is semi-algebraically homeomorphic to $S \cap B_k(0, r)$ for all large enough $r > 0$. Also, note that $S \cap B_k(0, r)$ is a disjoint union of $S \cap B_k(0, r)$ and $S \cap S^{k-1}(0, r)$. The proposition follows from the additivity of $\chi_{\text{gen}}(\cdot)$. \qed

Corollary 1. Let $S \subset \mathbb{R}^k$ be a $\mathcal{P}$-closed semi-algebraic set. Let $\Gamma \subset \{0, 1, -1\}^\mathcal{P}$ be the set of realizable sign conditions $\gamma$ on $\mathcal{P}$ such that $\text{Reali}(\gamma, \mathbb{R}^k) \subset S$. Then,
\[
\chi_{\text{gen}}(S) = \sum_{\gamma \in \Gamma} \chi_{\text{gen}}\left(\text{Reali}(\gamma, \mathbb{R}^k)\right).
\]

Proof. Clear from the definition of the generalized Euler-Poincaré characteristic (Definition 1). \qed

3. Equivariant deformation

In this section we recall the definition of certain equivariant deformations of symmetric real algebraic varieties that were introduced in [9]. These are adapted from the non-equivariant case (see for example [8]), but keeping everything equivariant requires additional effort.

Notation 11. For $i \in \mathbb{N}$ let $p^{(k)}_i := \sum_{j=1}^{(k)} X_j$ denote the $i$-th Newton sum and for any $P \in \mathbb{R}[X_1, \ldots, X_k]$ we denote
\[
\text{Def}(P, \zeta, d) = P - \zeta \left(1 + p^{(k)}_d\right),
\]
where $\zeta$ is a new variable.

Notice that if $P$ is symmetric in $X_1, \ldots, X_k$, so is $\text{Def}(P, \zeta, d)$.
3.1. Properties of Def($P, \zeta, d$). We now state some key properties of the deformed polynomial Def($P, \zeta, d$) that will be important in proving the correctness, as well as the complexity analysis, of the algorithms presented later in the paper. Most of these properties, with the exception of the key Theorem 3, have been proved in [9] and we refer the reader to that paper for the proofs. We reproduce the statements below for ease of reading and completeness of the current paper.

**Proposition 5.** [9, Proposition 3] Let $d \geq 0$ be even, $\mathbf{k} = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{\geq 0}^\omega$, with $k = \sum_{i=1}^\omega k_i$, and $P \in R[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\omega)}]_{\leq d}$, where each $\mathbf{X}^{(i)}$ is a block of $k_i$ variables, such that $P$ is non-negative and symmetric in each block of variable $\mathbf{X}^{(i)}$. Also suppose that $V = \text{Zer}(P, R^k)$ is bounded. Then, Ext$(V, R(\zeta)^k)$ is a semi-algebraic deformation retract of the (symmetric) semi-algebraic subset $S$ of $R(\zeta)^k$, consisting of the union of the semi-algebraically connected components of the semi-algebraic set defined by the inequality Def($P, \zeta, d$) $\leq 0$, which are bounded over $R$. Hence, Ext$(V, R(\zeta))$ is semi-algebraically homotopy equivalent to $S$. Moreover, $\phi_k(\text{Ext}(V, R(\zeta)^k))$ is semi-algebraically homotopy equivalent to $\phi_k(S)$.

**Proposition 6.** [9, Proposition 4] Let $P \in R[X_1, \ldots, X_k]$, and $d$ be an even number with $\deg(P) < d = p + 1$, with $p$ a prime. Let $F = p_1^{(k)}(X_1, \ldots, X_k)$. Let $V_\zeta = \text{Zer} (\text{Def}(P, \zeta, d), R(\zeta)^k)$.

Suppose also that $\text{gcd}(p, k) = 1$. Then, the critical points of $F$ restricted to $V_\zeta$ are finite in number, and each critical point is non-degenerate.

**Notation 12.** For any pair $(\mathbf{k}, \mathbf{\ell})$, where $\mathbf{k} = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{\geq 0}^\omega$, $k = \sum_{i=1}^\omega k_i$, and $\mathbf{\ell} = (\ell_1, \ldots, \ell_\omega)$, with $1 \leq \ell_i \leq k_i$, we denote by $A_{\mathbf{k}}^\mathbf{\ell}$ the subset of $R^{k}$ defined by

$$A_{\mathbf{k}}^\mathbf{\ell} = \left\{ x = (x^{(1)}, \ldots, x^{(\omega)}) \mid \text{card} \left( \bigcup_{j=1}^{k_i} \{x^{(i)}_j\} \right) = \ell_i \right\}.$$

**Proposition 7.** [9, Proposition 5] Let $\mathbf{k} = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{\geq 0}^\omega$, with $k = \sum_{i=1}^\omega k_i$, and

$$P \in R[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\omega)}],$$

where each $\mathbf{X}^{(i)}$ is a block of $k_i$ variables, such that $P$ is non-negative and symmetric in each block of variable $\mathbf{X}^{(i)}$ and $\deg(P) \leq d$. Let $(X_1, \ldots, X_k)$ denote the set of variables $(\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\omega)})$ and let $F = p_1^{(k)}(X_1, \ldots, X_k)$. Suppose that the critical points of $F$ restricted to $V = \text{Zer} (P, R^k)$ are isolated. Then, each critical point of $F$ restricted to $V$ is contained in $A_{\mathbf{k}}^\mathbf{\ell}$ for some $\mathbf{\ell} = (\ell_1, \ldots, \ell_\omega)$ with each $\ell_i \leq d$.

With the same notation as in Proposition 7:

**Corollary 2.** Let $\mathcal{P} \subset R[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(k_\omega)}]$ be a finite set of polynomials, such that for each $P \in \mathcal{P}$, $P$ is non-negative and symmetric in each block of variable $\mathbf{X}^{(i)}$, and $\deg(P) \leq d$. Let $C$ be a bounded semi-algebraically connected component of $\text{Zer}(\mathcal{P}, R^k)$. Then, $C \cap A_{\mathbf{k}}^\mathbf{\ell} \neq \emptyset$, for some $\mathbf{\ell} = (\ell_1, \ldots, \ell_\omega)$, where for each $i, 1 \leq i \leq \omega$, $1 \leq \ell_i \leq 2d$.

**Proof.** Let $d'$ be the least even number such that $d' > d$ and such that $d' - 1$ is prime. By Bertrand’s postulate we have that $d' \leq 2d$. Now, if $p$ divides $k$, replace each $P \in \mathcal{P}$ by the polynomial

$$P + X_{k+1}^2.$$
and let \( \omega' = \omega + 1, k' = k + 1 \), and \( k' = (k, 0) \). In either case, we have that \( \gcd(p, k') = 1 \), and \( k' \leq k + 1 \).

Let \( Q = \sum_{p \in P} P \), and let \( V_\zeta = \text{Zer}(\text{Def}(Q, \zeta, d'), R(\zeta)^{k'}) \). Then for every bounded semi-algebraically connected component \( C \) of \( \text{Zer}(Q, R^{k'}) \), there exists a semi-algebraically connected component of \( C_\zeta \) of \( V_\zeta \) bounded over \( R \), such that \( \lim_\zeta C_\zeta \subset C \) (see [8, Proposition 12.51]). Now every bounded semi-algebraically connected component \( C_\zeta \) of \( V_\zeta \) contains at least two critical points of the polynomial \( e_1^{(k)} \) restricted to \( V_\zeta \), and they are isolated by Proposition 6. The corollary now follows from Proposition 7. \( \square \)

The next theorem which gives an exact expression for both \( \chi(S, F) \) as well as \( \chi(S/G_k, F) \) (where \( S \) is as in Proposition 5) is the key result needed for the algorithms in the paper. We defer its proof to the appendix.

Before stating the theorem we need to introduce a few more notation.

**Notation 13. (Partitions)** We denote by \( \Pi_k \) the set of partitions of \( k \), where each partition \( \pi = (\pi_1, \pi_2, \ldots, \pi_\ell) \in \Pi_k \), where \( \pi_1 \geq \pi_2 \geq \cdots \geq \pi_\ell \geq 1 \), and \( \pi_1 + \pi_2 + \cdots + \pi_\ell = k \). We call \( \ell \) the length of the partition \( \pi \), and denote \( \text{length}(\pi) = \ell \).

More generally, for any tuple \( k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{\geq 0}^\omega \), we will denote by \( \Pi_k = \prod_{k_{i_1}} \times \cdots \times \prod_{k_{i_\omega}} \), and for each \( \pi = (\pi^{(1)}, \ldots, \pi^{(\omega)}) \in \Pi_k \), we denote by \( \text{length}(\pi) = \sum_{i=1}^\omega \text{length}(\pi^{(i)}) \). We also denote for each \( \ell = (\ell_1, \ldots, \ell_\omega) \in \mathbb{Z}_{\geq 0}^\omega \),

\[
|\ell| = \ell_1 + \cdots + \ell_\omega.
\]

**Notation 14.** Let \( \pi \in \Pi_k \) where \( k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{\geq 0}^\omega \), with \( k = \sum_{i=1}^\omega k_i \).

For \( 1 \leq i \leq \omega \), and \( 1 \leq j \leq \text{length}(\pi^{(i)}) \), let \( L_{\pi_i^{(j)}} \subset \mathbb{R}^k \) be defined by the equations

\[
X^{(i)}_{\pi^{(i)}_1 + \cdots + \pi^{(i)}_{j-1} + 1} = \cdots = X^{(i)}_{\pi^{(i)}_1 + \cdots + \pi^{(i)}_j},
\]

and let

\[
L_\pi = \bigcap_{1 \leq i \leq \omega} \bigcap_{1 \leq j \leq \text{length}(\pi^{(i)})} L_{\pi_i^{(j)}}.
\]

**Notation 15.** Let \( L \subset \mathbb{R}^k \) be the subspace defined by \( \sum_i X_i = 0 \), and \( \pi = (\pi^{(1)}, \ldots, \pi^{(\omega)}) \in \Pi_k \). Let for each \( i \), \( 1 \leq i \leq \omega \), \( \pi^{(i)} = (\pi^{(i)}_1, \ldots, \pi^{(i)}_{\ell_i}) \), and for each \( j \), \( 1 \leq j \leq \ell_i \), let \( L_{\pi_j^{(i)}} \) denote the subspace \( L \cap L_{\pi_j^{(i)}} \) of \( L \), and \( M_{\pi_j^{(i)}} \) the orthogonal complement of \( L_{\pi_j^{(i)}} \) in \( L \). We denote

\[
L_{\text{fixed}} = L \cap L_\pi.
\]

We have the following theorem which gives an exact expression for the Euler-Poincaré characteristic of a symmetric semi-algebraic set defined by one polynomial inequality satisfying the same conditions as in Lemmas 2 and 3 above. The proof of the theorem which depends on the properties of \( \text{Def}(P, \zeta, d) \) stated above is given in \( \S 5 \).

**Theorem 3.** Let \( k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{\geq 0}^\omega \), with \( k = \sum_{i=1}^\omega k_i \), and let \( S \subset \mathbb{R}^k \) be a bounded symmetric basic semi-algebraic set defined by \( P \leq 0 \), where \( P \in \mathbb{R}[X^{(1)}, \ldots, X^{(\omega)}]^{\oplus k} \). where each \( X^{(i)} \) is a block of \( k_i \) variables.
Let \( W = \text{Zer}(P, \mathbb{R}^k) \) be non-singular and bounded. Let \((X_1, \ldots, X_k)\) denote the variables \((X^{(1)}, \ldots, X^{(\omega)})\) and suppose that \( F = P^{(k)}(X_1, \ldots, X_k) \) restricted to \( W \) has a finite number of critical points, all of which are non-degenerate. Let \( C \) be the finite set of critical points \( \mathbf{x} \) of \( F \) restricted to \( W \) such that \( \sum_{1 \leq i \leq k} \frac{\partial F}{\partial X_i}(\mathbf{x}) < 0 \), and let \( \text{Hess}(\mathbf{x}) \) denote the Hessian of \( F \) restricted to \( W \) at \( \mathbf{x} \). Then, for any field of coefficients \( \mathbb{F} \),

\[
\chi^{\text{top}}(S, \mathbb{F}) = \sum_{\pi = (\pi^{(1)}, \ldots, \pi^{(\omega)}) \in \Pi_k} \sum_{\mathbf{x} \in C \cap L_{\mathbf{x}}} (-1)^{\text{ind}^-(\text{Hess}(\mathbf{x}))} \left( \binom{k}{\pi^{(1)}} \cdots \binom{k}{\pi^{(\omega)}} \right),
\]

(2)

\[
\chi^{\text{top}}(S, \mathbb{F}) = \sum_{\pi \in \Pi_k} \sum_{\mathbf{x} \in C \cap L_{\mathbf{x}}, L^{-}(\mathbf{x}) \subseteq L_{\text{fixed}}} (-1)^{\text{ind}^-(\text{Hess}(\mathbf{x}))},
\]

(3)

(where for \( \pi = (\pi^{(1)}, \ldots, \pi^{(\ell)}) \in \Pi_k \), \( \binom{k}{\pi^{(1)}, \ldots, \pi^{(\ell)}} \) denotes the multinomial coefficient \( \binom{k}{\pi^{(1)}, \ldots, \pi^{(\ell)}} \)).

**Proof.** See \( \S 5 \) (Appendix).

**Theorem 3** is illustrated by the following simple example.

**Example 2.** In this example, the number of blocks \( \omega = 1 \), and \( k = k_1 = 2 \). Consider the polynomial

\[
P = (X_1^2 - 1)^2 + (X_2^2 - 1)^2 - \varepsilon,
\]

for some small \( \varepsilon > 0 \). The sets \( \text{Zer}(P, \mathbb{R}^2) \), and \( S = \{ x \in \mathbb{R}(\zeta)^2 \mid \bar{P} \leq 0 \} \), where \( \bar{P} = \text{Def}(P, \zeta, 6) \) is shown in the Figure 1.

The polynomial \( \bar{P}^{(2)}(X_1, X_2) = X_1 + X_2 \) has 16 critical points, corresponding to 12 critical values, \( v_1 < \cdots < v_{12} \), on \( \text{Zer}(\bar{P}, \mathbb{R}(\zeta)^2) \) of which \( v_2 \) and \( v_9 \) are indicated in Figure 1 using dotted lines. The corresponding indices of the critical points, the number of critical points for each critical value, the sign of the polynomial \( \frac{\partial \bar{P}}{\partial X_1} + \frac{\partial \bar{P}}{\partial X_2} \) at these critical points, and the partition \( \pi \in \Pi_2 \) such that the corresponding critical points belong to \( L_{\pi} \) are shown in Table 1. The critical points corresponding to the shaded rows are then the critical points where \( \left( \frac{\partial \bar{P}}{\partial X_1} + \frac{\partial \bar{P}}{\partial X_2} \right) < 0 \), and these are the critical points which contribute to the sums in Eqns. (2) and (3).

| Critical values | Index | SIGN \( \left( \frac{\partial \bar{P}}{\partial X_1} + \frac{\partial \bar{P}}{\partial X_2} \right) \) | \( \pi \) | \( L^{-}(p) \) | \( L_{\text{fixed}} \) | \( L^{-}(p) < L_{\text{fixed}} \) |
|-----------------|------|-------------------------------------------------|--------|-------------|-----------------|-----------------|
| \( v_2 \)       | 0    | -1                                              | \( -1 \) | 0           | 0               | yes             |
| \( v_3 \)       | 0    | 1                                               | \( 2 \) | 0           | 0               | no              |
| \( v_4 \)       | 0    | -1                                              | \( 2 \) | 0           | 0               | no              |
| \( v_5 \)       | 1    | 1                                               | \( 2 \) | \( L \)     | 0               | yes             |
| \( v_6 \)       | 0    | -1                                              | \( 1 \) | \( L \)     | 0               | yes             |
| \( v_7 \)       | 0    | 1                                               | \( 1 \) | \( L \)     | 0               | yes             |
| \( v_8 \)       | 1    | 1                                               | \( 1 \) | \( L \)     | 0               | yes             |
| \( v_9 \)       | 0    | -1                                              | \( 2 \) | \( L \)     | 0               | no              |
| \( v_{10} \)    | 0    | 1                                               | \( 2 \) | \( L \)     | 0               | yes             |
| \( v_{11} \)    | 1    | -1                                              | \( 2 \) | \( L \)     | 0               | no              |
| \( v_{12} \)    | 1    | 1                                               | \( 2 \) | \( L \)     | 0               | no              |

**Table 1.**
It is now easy to verify using Eqns. (2) and (3) that,
\[
\chi_{\text{top}}(\text{Zer}(P, R^k), Q) = \chi_{\text{top}}(S, Q) = \chi_{S^2}(S, Q)\]
\[
= (-1)^0 \left(\frac{2}{2}\right) + (-1)^1 \left(\frac{2}{2}\right) + (-1)^0 \left(\frac{2}{1,1}\right)
+ (-1)^1 \left(\frac{2}{1,1}\right) = 2.
\]

4. Algorithms and the proofs of the main theorems

In this section we describe new algorithms for computing the (generalized) Euler-Poincaré characteristic of symmetric semi-algebraic subsets of $R^k$, prove their
correctness and analyze their complexities. As a consequence we prove Theorems 1 and 2.

We first recall some basic algorithms from [8] which we will need as subroutines in our algorithms.

4.1. Algorithmic Preliminaries. In this section we recall the input, output and complexities of some basic algorithms and also some notations from the book [8]. These algorithms will be the building blocks of our main algorithms described later.

Definition 3. Let \( P \in \mathbb{R}[X] \) and \( \sigma \in \{0, 1, -1\}^{\text{Der}(P)} \), a sign condition on the set \( \text{Der}(P) \) of derivatives of \( P \). The sign condition \( \sigma \) is a Thom encoding of \( x \in \mathbb{R} \) if \( \sigma(P) = 0 \) and \( \text{Reali}(\sigma) = \{x\} \), i.e. \( \sigma \) is the sign condition taken by the set \( \text{Der}(P) \) at \( x \).

Notation 16. A \( k \)-univariate representation \( u \) is a \( k+2 \)-tuple of polynomials in \( \mathbb{R}[T] \),

\[
\begin{align*}
  u &= (f(T), g(T)), \text{ with } g = (g_0(T), g_1(T), \ldots, g_k(T)),
\end{align*}
\]

such that \( f \) and \( g_0 \) are co-prime. Note that \( g_0(t) \neq 0 \) if \( t \in \mathbb{C} \) is a root of \( f(T) \).

The points associated to a univariate representation \( u \) are the points

\[
(4) \quad x_u(t) = \left( \frac{g_1(t)}{g_0(t)}, \ldots, \frac{g_k(t)}{g_0(t)} \right) \in \mathbb{C}^k
\]

where \( t \in \mathbb{C} \) is a root of \( f(T) \).

Let \( P \subset \mathbb{R}[X_1, \ldots, X_k] \) be a finite set of polynomials such that \( \text{Zer}(P, \mathbb{C}^k) \) is finite. The \( k+2 \)-tuple \( u = (f(T), g(T)) \), represents \( \text{Zer}(P, \mathbb{C}^k) \) if \( u \) is a univariate representation and

\[
\text{Zer}(P, \mathbb{C}^k) = \{ x \in \mathbb{C}^k | \exists t \in \text{Zer}(\{f, g\}), x = x_u(t) \}.
\]

A real \( k \)-univariate representation is a pair \( u, \sigma \) where \( u \) is a \( k \)-univariate representation and \( \sigma \) is the Thom encoding of a root of \( f \), \( t_\sigma \in \mathbb{R} \). The point associated to the real univariate representation \( u, \sigma \) is the point

\[
(5) \quad x_u(t_\sigma) = \left( \frac{g_1(t_\sigma)}{g_0(t_\sigma)}, \ldots, \frac{g_k(t_\sigma)}{g_0(t_\sigma)} \right) \in \mathbb{R}^k.
\]

For the rest of this section we fix an ordered domain \( D \) contained in the real closed field \( \mathbb{R} \). By complexity of an algorithm whose input consists of polynomials with coefficients in \( D \), we will mean (following [8]) the maximum number of arithmetic operations in \( D \) (including comparisons) used by the algorithm for an input of a certain size.

We will use four algorithms from the book [8]: namely, Algorithm 10.98 (Univariate Sign Determination), Algorithm 12.64 (Algebraic Sampling), Algorithm 12.46 (Limit of Bounded Points), and Algorithm 10.83 (Adapted Matrix). We refer the reader to [8] for the descriptions of these algorithms and their complexity analysis.
4.2. Computing the generalized Euler-Poincaré characteristic of symmetric real algebraic sets. We now describe our algorithm for computing the generalized Euler-Poincaré characteristic for real varieties, starting as usual with the bounded case. Note that using Proposition 3, for a closed and bounded semi-algebraic set $S$,

$$\chi_{\text{gen}}(S) = \chi_{\text{top}}(S, \mathbb{Q}).$$

Algorithm 1 (Generalized Euler-Poincaré characteristic for bounded symmetric algebraic sets)

**Input:**
1. A tuple $k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{\geq 0}^\omega$, with $k = \sum_{i=1}^\omega k_i$.
2. A polynomial $P \in \mathbb{D}[X^{(1)}, \ldots, X^{(\omega)}]$, where each $X^{(i)}$ is a block of $k_i$ variables, and $P$ is non-negative, symmetric in each block of variables $X^{(i)}$, and such that $\text{Zer}(P, \mathbb{R}^k)$ is bounded and of degree at most $d$.

**Output:**
$\chi_{\text{gen}}(\text{Zer}(P, \mathbb{R}^k))$ and $\chi_{\text{gen}}^S(\text{Zer}(P, \mathbb{R}^k))$.

**Procedure:**
1. Pick $d'$, such that $d < d' \leq 2d$, $d'$ even, and $d' - 1$ a prime number, by sieving through all possibilities, and testing for primality using the naive primality testing algorithm (i.e. testing for divisibility using the Euclidean division algorithm for each possible divisor).
2. if $d' - 1 \mid k$ then
3. $P \leftarrow P + X_i^{2(k_i^2 + 1)}$, $k \leftarrow k' = (k, 1)$, and $k \leftarrow k + 1$.
4. end if
5. $Q \leftarrow \text{Def}(P, \zeta, d')$.
6. $\chi_{\text{gen}}^k \leftarrow 0$, $\chi_{\text{gen}}^S \leftarrow 0$.
7. for each $\ell = (\ell_1, \ldots, \ell_\omega), 1 \leq \ell_i \leq \min(k_i, d')$, and each $\pi = (\pi^{(1)}, \ldots, \pi^{(\omega)}) \in \pi_{k, \ell}$
8. $I \leftarrow \{(i, j) \mid 1 \leq i \leq \omega, 1 \leq j \leq \ell_i\}$.
9. Let $Z^{(1)}, \ldots, Z^{(\omega)}$ be new blocks of variables, where each $Z^{(i)} = (Z^{(i)}_1, \ldots, Z^{(i)}_{\ell_i})$ is a block of $\ell_i$ variables, and $Q_\pi \in \mathbb{D}(\zeta)[Z^{(1)}, \ldots, Z^{(\omega)}]$ be the polynomial obtained from $Q$ by substituting in $Q$ for each $(i, j) \in I$ the variables

$$X^{(i)}_{\pi_1^{(i)} + \cdots + \pi_{\ell_i - 1}^{(i)} + 1}, \ldots, X^{(i)}_{\pi_1^{(i)} + \cdots + \pi_{\ell_i}^{(i)}}$$

by $Z^{(i)}_j$.
10. $Q_\pi \leftarrow Q_\pi^2 + \sum_{(i, j), (i', j') \in I} \left( \pi_j^{(i)} \frac{\partial Q_\pi}{\partial Z^{(i)}_j} - \pi_{j'}^{(i')} \frac{\partial Q_\pi}{\partial Z^{(i')}_{j'}} \right)^2$.
11. Using Algorithm 12.64 (Algebraic Sampling) from [8] compute a set $U_\pi$ of real univariate representations representing the finite set of points

$$C = \text{Zer}(Q_\pi, \mathbb{R}(\zeta)^k).$$
Let $\text{Hess}_\pi(Z^{(1)}, \ldots, Z^{(ω)})$ be the symmetric matrix obtained by substituting for each $(i, j) \in I$ the variables

$$X^{π−1} + \cdots + X^{π−j} − 1 + 1, \ldots, X^{π−1} + \cdots + X^{π−j},$$

by $Z^{(i)}$ in the $(k − 1) \times (k − 1)$ matrix $H$ whose rows and columns are indexed by $[2, k]$, and which is defined by:

$$H_{i,j} = \frac{∂^2 Q}{∂(X_1 + X_i)∂(X_1 + X_j)}, 2 ≤ i, j ≤ k.$$

for each point $z \in C$ represented by $u_z \in U_π$ do compute using Algorithm 10.98 (Univariate Sign Determination) in [8], the sign of the polynomial

$$\sum_{(i,j) \in I} \frac{∂Q_π}{∂Z^{(i)}_j},$$

as well as the index, $\text{ind}^−(\text{Hess}_π)$, at the point $z$.

Using Gauss-Jordan elimination (over the real univariate representation $u_z$), and Algorithm 10.98 (Univariate Sign Determination) from [8], determine if the negative eigenspace, $L^−(\text{Hess}_π(z))$ of the symmetric matrix $\text{Hess}_π(z)$ is contained in the subspace $L$ defined by

$$\sum_{i=1}^k X_i = 0.$$

if $\sum_{(i,j) \in I} \frac{∂Q_π}{∂Z^{(i)}_j}(z) < 0$ then

$$\chi_{\text{gen}}^z \leftarrow \chi_{\text{gen}}^z + (-1)^{\text{ind}^−(\text{Hess}_π(z))} \prod_{i=1}^ω \left(\begin{array}{c} k_i \\ \pi_{i}^{(ω)} \end{array}\right).$$

end if

if $\sum_{(i,j) \in I} \frac{∂Q_π}{∂Z^{(i)}_j}(z) < 0$, and $L^−(\text{Hess}_π(z)) \subset L$ then

$$\chi_{\text{gen}_{k}}^z \leftarrow \chi_{\text{gen}_{k}}^z + (-1)^{\text{ind}^−(\text{Hess}_π(z))}.$$

end if

Output

$$\chi_{\text{gen}}(\text{Zer}(P, R^k)) = \chi_{\text{gen}}^{\text{gen}},$$

$$\chi_{\text{gen}_{k}}(\text{Zer}(P, R^k)) = \chi_{\text{gen}_{k}}^{\text{gen}}.$$

end for

Proof of correctness. The correctness of the algorithm follows from Propositions 5, 7, 6, Theorem 3, as well as the correctness of Algorithms 12.64 (Algebraic Sampling) and Algorithm 10.98 (Univariate Sign Determination) in [8].

Complexity analysis. The complexity of Step 1 is bounded by $O(1)$. The complexities of Steps 3, 5, 9 are all bounded by $(ωk)^O(d)$. Using the complexity analysis of Algorithm 12.64 (Algebraic Sampling) in [8], the complexity of Step 11 is bounded by $(\text{length}(\pi))d^O(\text{length}(\pi))$. The number and the degrees of the real
univariate representations output in Step 11 are bounded by $d^{O(length(\pi))}$. The complexity of Step 13 is bounded by $d^{O(length(\pi))}$ using the complexity analysis of Algorithm 10.98 (Univariate Sign Determination) in [8]. Each arithmetic operation in the Gauss-Jordan elimination in Step 15 occurs in a ring $D[\mathbb{C}[T]/(f(T))]$ (where $u_z = (f, g_0, \ldots, g_{length(\pi)}, \rho_z)$ with $deg_{T, \mathbb{C}}(f) = d^{O(length(\pi))}$). The number of such operations in the ring $D[\mathbb{C}[T]/(f(T))]$ is bounded by $(length(\pi) + k)^{O(1)}$. Thus, the total number of arithmetic operations in the ring $D$ performed in Step 15 is bounded by $(length(\pi)kd)^{O(length(\pi))}$.

The number of iterations of Step 7 is bounded by the number of partitions $\pi \in \Pi_{k, \ell}$ with $\ell = (\ell_1, \ldots, \ell_\omega), 1 \leq \ell_i \leq min(k_i, d')$, $1 \leq i \leq \omega$, which is bounded by

$$\sum_{\ell=(\ell_1, \ldots, \ell_\omega), 1 \leq \ell_i \leq min(k_i, d')} p(k, \ell) = k^{O(D)},$$

where $D = \sum_{i=1}^\omega min(k_i, 2d)$. Thus, the total complexity of the algorithm measured by the number of arithmetic operations (including comparisons) in the ring $D$ is bounded by $(\omega kd)^{O(D)}$.

**Algorithm 2** (Generalized Euler-Poincaré characteristic for symmetric algebraic sets)

**Input:**
1. A tuple $k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{>0}^\omega$, with $k = \sum_{i=1}^\omega k_i$.
2. A polynomial $P \in D[X^{(1)}, \ldots, X^{(\omega)}]$, where each $X^{(i)}$ is a block of $k_i$ variables, and $P$ is symmetric in each block of variables $X^{(i)}$, with $deg(P) = d$.

**Output:**
$\chi^{gen}(\text{Zer}(P, R^k))$ and $\chi^{gen}_{\mathbb{R}}(\text{Zer}(P, R^k))$.

**Procedure:**
1: $P_1 \leftarrow P^2 + \left( X_{k+1}^2 + \sum_{i=1}^k X_i^2 - \Omega^2 \right)^2,$
$P_2 = P^2 + \left( \sum_{i=1}^k X_i^2 - \Omega^2 \right)^2.$

2: Using Algorithm 1 with $P_1$ and $P_2$ as input compute:

$\chi^{(1)} = \chi^{gen}(\text{Zer}(P_1, R(1/\Omega)^{k+1})),$
$\chi^{(1)}_{\mathbb{R}} = \chi^{gen}_{\mathbb{R}}(\text{Zer}(P_1, R(1/\Omega)^{k+1})),$
$\chi^{(2)} = \chi^{gen}(\text{Zer}(P_2, R(1/\Omega)^k)),$
$\chi^{(2)}_{\mathbb{R}} = \chi^{gen}_{\mathbb{R}}(\text{Zer}(P_2, R(1/\Omega)^k)).$

where $k' = (k, 1)$.

3: Output

$\chi^{gen}(\text{Zer}(P, R^k)) = \frac{1}{2}(\chi^{(1)} - \chi^{(2)}),$

$\chi^{gen}_{\mathbb{R}}(\text{Zer}(P, R^k)) = \frac{1}{2}(\chi^{(1)}_{\mathbb{R}} - \chi^{(2)}_{\mathbb{R}}).$
Proof of correctness. Since $V = \text{Zer}(P, R^{m+k})$ is closed, by Proposition 4 we have that
\[ \chi_{\text{gen}}^{\text{top}}(V) = \chi_{\text{BM}}(V, Q) = \chi_{\text{top}}(\text{Ext}(V, R/(1/\Omega)) \cap B_k(0, \Omega)) - \chi_{\text{top}}(\text{Ext}(V, R/(1/\Omega)) \cap S^{k-1}(0, \Omega)) \]
(6)
\[ = \chi_{\text{top}}(\text{Ext}(V, R/(1/\Omega)) \cap B_k(0, \Omega)) - \chi^{(2)}. \]
Now $\text{Zer}(P_1, R/(1/\Omega)^{k+1})$ is semi-algebraically homeomorphic to two copies of $\text{Ext}(V, R/(1/\Omega)) \cap B_k(0, \Omega)$,
\[ \text{glued along a semi-algebraically homeomorphic copy of} \]
$\text{Ext}(V, R/(1/\Omega)) \cap S^{k-1}(0, \Omega) = \text{Zer}(P_2, R/(1/\Omega)^k)$. It follows that,
\[ \chi^{(1)} = \chi_{\text{gen}}(\text{Zer}(P_1, R/(1/\Omega)^{k+1})) = 2\chi_{\text{top}}(\text{Ext}(V, R/(1/\Omega)) \cap B_k(0, \Omega)) - \chi_{\text{gen}}(\text{Zer}(P_2, R/(1/\Omega)^k)) \]
\[ = 2\chi_{\text{top}}(\text{Ext}(V, R/(1/\Omega)) \cap B_k(0, \Omega)) - \chi^{(2)}, \]
and hence
\[ \chi_{\text{top}}(\text{Ext}(V, R/(1/\Omega)) \cap B_k(0, \Omega)) = \frac{1}{2}(\chi^{(1)} + \chi^{(2)}). \]
(7)
It follows from Eqns. (6) and (7) that
\[ \chi_{\text{gen}}^{\text{top}}(V) = \frac{1}{2}(\chi^{(1)} + \chi^{(2)}) - \chi^{(2)} = \frac{1}{2}(\chi^{(1)} - \chi^{(2)}). \]
The proof for the correctness of the computation of $\chi_{\text{gen}}^{\text{top}}(V)$ is similar and omitted.

Complexity analysis. The complexity of the algorithm measured by the number of arithmetic operations (including comparisons) in the ring D is bounded by $(\omega kd)^{O(D)}$, where $D = \sum_{i=1}^{\omega} \min(k_i, 2d)$. This follows directly from the complexity analysis of Algorithm 1.

Proof of Theorem 1. The correctness and the complexity analysis of Algorithm 2 prove Theorem 1. \qed

4.3. Computing the generalized Euler-Poincaré characteristic of symmetric semi-algebraic sets. We now consider the problem of computing the (generalized) Euler-Poincaré characteristic of semi-algebraic sets. We reduce the problem to computing the generalized Euler-Poincaré characteristic of certain symmetric algebraic sets for which we already have an efficient algorithm described in the last section. This reduction process follows very closely the spirit of a similar reduction that is used in an algorithm for computing the generalized Euler-Poincaré characteristic of the realizations of all realizable sign conditions of a family of polynomials given in [6] (see also [8]).

We first need an efficient algorithm for computing the set of realizable sign conditions of a family of symmetric polynomials which will be used later. The following algorithm can be considered as an equivariant version of a very similar
Algorithm 3 (Computing Realizable Sign Conditions of Symmetric Polynomials)

**Input:**
(1) A tuple $k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_{\geq 0}^\omega$, with $k = \sum_{i=1}^\omega k_i$.
(2) A set of $s$ polynomials $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbb{D}[X^{(1)}, \ldots, X^{(\omega)}]$, where each $X^{(i)}$ is a block of $k_i$ variables, and each polynomial in $\mathcal{P}$ is symmetric in each block of variables $X^{(i)}$ and of degree at most $d$.

**Output:**
$\text{SIGN}(\mathcal{P})$.

**Procedure:**
1. for each $i, 1 \leq i \leq s$ do $\mathcal{P}_i^* \leftarrow \{P_i \pm \gamma_1, P_i \pm \delta_1\}$.
2. end for
3. for every choice of $j \leq D' = \sum_{i=1}^\omega \min(k_i, d)$ polynomials $Q_{i1} \in \mathcal{P}_{i1}^*, \ldots, Q_{ij} \in \mathcal{P}_{ij}^*$, with $1 \leq i_1 < \cdots < i_j \leq s$ do
4. $Q_1 \leftarrow Q_{i1}^2 + \cdots + Q_{ij}^2$,
5. $Q_2 \leftarrow Q_{i1}^2 + \cdots + Q_{ij}^2 + \left(\pm (|X^{(1)}|^2 + \cdots + |X^{(\omega)}|^2) - 1\right)^2$.
6. for each $\ell = (\ell_1, \ldots, \ell_\omega), 1 \leq \ell_i \leq \min(k_i, 4d)$, and each partition $\pi = (\pi^{(1)}, \ldots, \pi^{(\omega)}) \in \pi_{\ell_1, \ell}$ do
7. $I \leftarrow \{(i, j) \mid 1 \leq i \leq \omega, 1 \leq j \leq \ell_i\}$
8. Let $Z^{(1)}, \ldots, Z^{(\omega)}$ be new blocks of variables, where each $Z^{(i)} = (Z_{\ell}^{(i)}(\pi^{(i)}), \ldots, Z_{\ell}^{(i)}(\pi^{(i)}))$ is a block of $\ell_i$ variables, and $Q_1, Q_2, \pi$ be the polynomials obtained from $Q_1, Q_2$ respectively, by substituting for each $(i, j) \in I$ the variables $X_{\pi^{(i)}_{\ell}+\cdots+\pi^{(i)}_{\ell}+1, \ldots, \pi^{(i)}_{\ell}+\cdots+\pi^{(i)}_{\ell}}$ by $Z_{\ell}^{(i)}$.
9. Using Algorithm 12.64 (Algebraic Sampling) from [8], compute a set $U_{\pi, \ell}, i = 1, 2$ of real univariate representations representing the finite set of points $C_1, C_2 \subset \text{Zer}(Q_1, \pi, R^{\text{length}(\pi)})$
10. where $R = R(\varepsilon, \delta_{i_1}, \ldots, \delta_{i_j}, \gamma, \zeta)$.
11. Apply the limit, using Algorithm 12.46 (Limit of Bounded Points) in [8], to the points in $C_1, C_2$ which are bounded over $R(\varepsilon, \delta_{i_1}, \ldots, \delta_{i_j})$, and obtain a set of real univariate representations $(u, \sigma)$ with
12. $u = (f(T), g_0(T), \ldots, g_{\text{length}(\pi)}(T)) \in D[\varepsilon, \delta_{i_1}, \ldots, \delta_{i_j}]^{\text{length}(\pi)+2}$
13. Add these real univariate representations to $U_{\pi, \ell}$.
14. for each $u \in U_{\pi, \ell}$ do
15. Compute the signs of $P_\pi$ for each $P \in \mathcal{P}$ at the points $z$, associated to $u$ using Algorithm 10.98 (Univariate Sign Determination) from [8], where $P_\pi \in R[Z^{(1)}, \ldots, Z^{(\omega)}]$ is the polynomial obtained from $P$ by substituting in $P$ for each $(i, j) \in I$ the variables $X_{\pi^{(i)}_{\ell}+\cdots+\pi^{(i)}_{\ell}+1, \ldots, \pi^{(i)}_{\ell}+\cdots+\pi^{(i)}_{\ell}}$ by $Z_{\ell}^{(i)}$.
Let \( \sigma \in \{0, 1, -1\}^P \) be the sign vector defined by \( \sigma(P) = \text{sign}(P_n(\sigma)) \).

\[
\text{SIGN} := \text{SIGN} \cup \{\sigma\}.
\]

end for

end for

end for

Output \( \text{SIGN}(P) = \text{SIGN} \).

Proof of correctness. We first need a lemma whose proof can be found in [9].

Definition 4. For any finite family \( P \subset \mathbb{R}[X_1, \ldots, X_k] \) and \( \ell \geq 0 \), we say that \( P \) is in \( \ell \)-general position with respect to a semi-algebraic set \( V \subset \mathbb{R}^k \) if for any subset \( P' \subset P \), with \( \text{card}(P') > \ell \), \( \text{Zer}(P', V) = \emptyset \).

Let \( k = (k_1, \ldots, k_\omega) \) with \( k = \sum_{i=1}^\omega k_i \), and

\[
P = \{P_1, \ldots, P_s\} \subset \mathbb{R}[X^{(1)}, \ldots, X^{(\omega)}]^{\mathfrak{S}_k}
\]

be a fixed finite set of polynomials where \( X^{(i)} \) is a block of \( k_i \) variables. Let \( \text{deg}(P_i) \leq d \) for \( 1 \leq i \leq s \). Let \( \bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_s) \) be a tuple of new variables, and let \( P_{\bar{\varepsilon}} = \bigcup_{1 \leq i \leq s} \{P_i \pm \varepsilon_i\} \).

The following lemma appears in [9].

Lemma 1. [9, Lemma 7] Let

\[
D' = \sum_{i=1}^\omega \min(k_i, d).
\]

The set of polynomials \( P_{\bar{\varepsilon}} \subset \mathbb{R}'[X^{(1)}, \ldots, X^{(\omega)}] \) is in \( D' \)-general position for any semi-algebraic subset \( Z \subset \mathbb{R}^k \) stable under the action of \( \mathfrak{S}_k \), where \( R' = R(\bar{\varepsilon}) \).

Now observe that Lemma 1 implies that the set \( \bigcup_{1 \leq i \leq s} P_{i,\bar{\varepsilon}} \) is in \( D' \)-general position. Propositions 13.1 and 13.7 in [8] together imply that the image under the \( \lim_\gamma \) map of any finite set of points meeting every bounded semi-algebraically connected component of each algebraic set defined by polynomials

\[
Q_{i_1} \in P_{i_1}^*, \ldots, Q_{i_j} \in P_{i_j}^*,
\]

where \( 1 \leq i_1 < \cdots < i_j \leq s, 1 \leq j \leq D' \), and \( Q_0 = |\varepsilon|(|X^{(1)}|^2 + \cdots + |X^{(\omega)}|^2) - 1 \), will intersect every semi-algebraically connected component of \( \text{Reali}(\sigma, \mathbb{R}^k) \) for every \( \sigma \in \text{SIGN}(P) \).

Moreover, noticing that the degrees of the polynomials \( Q_{ij} \) above are bounded by \( 2d \), it follows from Corollary 2 that each semi-algebraically connected component of the algebraic sets listed above has a non-empty intersection with \( A_{k'}^{\ell_i} \), for some \( \ell = (\ell_1, \ldots, \ell_\omega) \), and \( 1 \leq \ell_1 \leq \min(k_i, 4d), 1 \leq i \leq \omega \).

The correctness of the algorithm now follows from the correctness of Algorithm 12.64 (Algebraic Sampling) and Algorithm 10.98 (Univariate Sign Determination) in [8].
COMPLEXITY ANALYSIS. The complexity of Step 3 measured by the number of
arithmetic operations in the ring $D[\delta_1, \ldots, \delta_s, \gamma]$ is bounded by

$$O \left( D' \left( \binom{k+d}{k} \right) \right),$$

where $D' = \sum_{i=1}^{\omega} \min(k_i, d)$.

It follows from the complexity analysis of Algorithm 12.64 (Algebraic Sampling)
in [8] that each call to Algorithm 12.64 (Algebraic Sampling) in Step 8 requires
$d^{O(\text{length}(\pi))}$ arithmetic operations in the ring $D[\varepsilon, \delta_1, \ldots, \delta_s, \gamma]$. The number and
degrees of the real univariate representations $u_{\pi, i}$ output in Step 8 is bounded by

$$d^{O(\text{length}(\pi))}.$$

Using the complexity analysis of Algorithm 12.46 (Limit of Bounded
Points) in [8], each call to Algorithm 12.46 (Limit of Bounded Points) in Step 9 re-
quires $d^{O(\text{length}(\pi))}$ arithmetic operations in the ring $D[\varepsilon, \delta_1, \ldots, \delta_s, \gamma]$, and thus the
total complexity of this step in the whole algorithm across all iterations measured by
the number of arithmetic operations in the ring $D[\varepsilon, \delta_1, \ldots, \delta_s]$ is bounded by

$$D' \sum_{j=1}^{s} 2^j \binom{s}{j} \left( d^{O(D'')} + O \left( D' \left( \binom{k+d}{k} \right) \right) \right),$$

where $D'' = \sum_{i=1}^{\omega} \min(k_i, 4d)$, noting that $\text{length}(\pi) \leq D''$.

Similarly, using the complexity analysis of Algorithm 10.98 (Univariate Sign
Determination) in [8], each call to Algorithm 10.98 (Univariate Sign Determination)
in Step 11 requires $d^{O(\text{length}(\pi))}$ arithmetic operations in the ring $D[\varepsilon, \delta_1, \ldots, \delta_s]$, and thus the total complexity of this step in the whole algorithm across all iterations measured by
the number of arithmetic operations in the ring $D[\varepsilon, \delta_1, \ldots, \delta_s]$ is bounded by

$$D' \sum_{j=1}^{s} 2^j \binom{s}{j} \left( d^{O(D'')} + O \left( D' \left( \binom{k+d}{k} \right) \right) \right).$$

However, notice that in each call to Algorithm 12.64 (Algebraic Sampling) from [8]
in Step 8, to Algorithm 12.46 (Limits of Bounded Points) in [8] in Step 9, as well as
and also in the calls to Algorithm 10.98 (Univariate Sign Determination) from [8]
in Step 11, the arithmetic is done in a ring $D$ adjoined with $O(D')$ infinitesimals.
Hence, the total number of arithmetic operations in $D$ is bounded by

$$\sum_{j=1}^{s} 2^j \binom{s}{j} \left( d^{O(D'')} + O \left( D' \left( \binom{k+d}{k} \right) \right) \right) = sD' k^d d^{O(D'')}.$$

The total number of real univariate representations produced in Step 8 is bounded by

$$\sum_{j=1}^{s} 2^j \binom{s}{j} d^{O(D'')} = sD' d^{O(D'')}.$$

Their degrees are bounded by $d^{O(D'')}$. Thus, the total number of real points asso-
ciated to these univariate representations, and hence also

$$\text{card}(\text{SIGN}(\mathcal{P})) = sD' d^{O(D'')}.$$
The complexity analysis of Algorithm 3 yields the following purely mathematical result.

**Proposition 8.** Let \( k = (k_1, \ldots, k_\omega) \in \mathbb{Z}^\omega_{>0} \), with \( k = \sum_{i=1}^\omega k_i \), and let \( \mathcal{P} = \{P_1, \ldots, P_s\} \subset R[X^{(1)}, \ldots, X^{(\omega)}] \) be a finite set of polynomials, where each \( X^{(i)} \) is a block of \( k_i \) variables, and each polynomial in \( \mathcal{P} \) is symmetric in each block of variables \( X^{(i)} \). Let \( \text{card}(\mathcal{P}) = s \), and \( \max_{P \in \mathcal{P}} \deg(P) = d \). Then,

\[
\text{card}(\text{SIGN}(\mathcal{P})) = s^{D'}d^{O(D'')},
\]

where \( D' = \sum_{i=1}^\omega \min(k_i, d) \), and \( D'' = \sum_{i=1}^\omega \min(k_i, 4d) \).

In particular, if for each \( i, 1 \leq i \leq \omega, d \leq k_i \), then \( \text{card}(\text{SIGN}(\mathcal{P})) \) can be bounded independent of \( k \).

**Notation 17.** Given \( P \in R[X_1, \ldots, X_k] \), we denote

\[
\text{Reali}(P=0,S) = \{x \in S \mid P(x) = 0\},
\]

\[
\text{Reali}(P>0,S) = \{x \in S \mid P(x) > 0\},
\]

\[
\text{Reali}(P>0,S) = \{x \in S \mid P(x) < 0\},
\]

and \( \chi^\text{gen}(P=0,S), \chi^\text{gen}(P>0,S), \chi^\text{gen}(P<0,S) \) the Euler-Poincaré characteristics of the corresponding sets. The Euler-Poincaré-query of \( P \) for \( S \) is

\[
\text{EuQ}(P,S) = \chi^\text{gen}(P>0,S) - \chi^\text{gen}(P<0,S).
\]

If \( P \) and \( S \) are symmetric we denote by

\[
\chi^\text{gen}_{\mathcal{G}_k}(P=0,S), \chi^\text{gen}_{\mathcal{G}_k}(P>0,S), \chi^\text{gen}_{\mathcal{G}_k}(P<0,S)
\]

the Euler-Poincaré characteristics of the corresponding sets. The equivariant Euler-Poincaré-query of \( P \) for \( S \) is

\[
\text{EuQ}_{\mathcal{G}_k}(P,S) = \chi^\text{gen}_{\mathcal{G}_k}(P>0,S) - \chi^\text{gen}_{\mathcal{G}_k}(P<0,S).
\]

Let \( \mathcal{P} = P_1, \ldots, P_s \) be a finite list of polynomials in \( R[X_1, \ldots, X_k] \).

Let \( \sigma \) be a sign condition on \( \mathcal{P} \). The realization of the sign condition \( \sigma \) over \( S \) is defined by

\[
\text{Reali}(\sigma,S) = \{x \in S \mid \bigwedge_{P \in \mathcal{P}} \text{sign}(P(x)) = \sigma(P)\},
\]

and its generalized Euler-Poincaré characteristic is denoted

\[
\chi^\text{gen}(\sigma,S).
\]

Similarly, if \( P \) and \( S \) are symmetric with respect to \( \mathcal{G}_k \) for some \( k = (k_1, \ldots, k_\omega) \in \mathbb{Z}^\omega_{>0} \), the equivariant Euler-Poincaré characteristic of \( \text{Reali}(\sigma,S) \) is denoted

\[
\chi^\text{gen}_{\mathcal{G}_k}(\sigma,S) := \chi^\text{gen}_{\mathcal{G}_k}(\phi_k(\text{Reali}(\sigma,S)), \mathbb{Q}).
\]

**Notation 18.** Given a finite family \( \mathcal{P} \subset R[X_1, \ldots, X_k] \) we denote by \( \chi^\text{gen}(\mathcal{P}) \) the list of generalized Euler-Poincaré characteristics

\[
\chi^\text{gen}(\sigma) = \chi^\text{gen}(\text{Reali}(\sigma,R^k))
\]

for \( \sigma \in \text{SIGN}(\mathcal{P}) \).
Given $\alpha \in \{0, 1, 2\}^P$ and $\sigma \in \{0, 1, -1\}^P$, we denote

$$\sigma^\alpha = \prod_{P \in P} \sigma(P)^{\alpha(P)},$$

and

$$P^\alpha = \prod_{P \in P} P^{\alpha(P)}.$$

When $\text{Reali}(\sigma, Z) \neq \emptyset$, the sign of $P^\alpha$ is fixed on $\text{Reali}(\sigma, Z)$ and is equal to $\sigma^\alpha$ with the understanding that $0^0 = 1$.

We order the elements of $P$ so that $P = \{P_1, \ldots, P_s\}$. We order $\{0, 1, 2\}^P$ lexicographically. We also order $\{0, 1, -1\}^P$ lexicographically (with $0 < 1 < -1$).

Given $A = \alpha_1, \ldots, \alpha_m$, a list of elements of $\{0, 1, 2\}^P$ with $\alpha_1 <_{\text{lex}} \ldots <_{\text{lex}} \alpha_m$, we define

$$P^A = P^{\alpha_1}, \ldots, P^{\alpha_m},$$

$$\text{EuQ}(P^A, S) = \text{EuQ}(P^{\alpha_1}, S), \ldots, \text{EuQ}(P^{\alpha_m}, S).$$

Given $\Sigma = \sigma_1, \ldots, \sigma_n$, a list of elements of $\{0, 1, -1\}^P$, with $\sigma_1 <_{\text{lex}} \ldots <_{\text{lex}} \sigma_n$, we define

$$\text{Reali}(\Sigma, S) = \text{Reali}(\sigma_1, Z), \ldots, \text{Reali}(\sigma_n, Z),$$

$$\chi_{\text{gen}}(\Sigma, S) = \chi_{\text{gen}}(\sigma_1, Z), \ldots, \chi_{\text{gen}}(\sigma_n, Z).$$

We denote by $\text{Mat}(A, \Sigma)$ the $m \times s$ matrix of signs of $P^A$ on $\Sigma$ defined by

$$\text{Mat}(A, \Sigma)_{i,j} = \sigma_j^{\alpha_i}.$$

**Proposition 9.** If $\cup_{\sigma \in \Sigma} \text{Reali}(\sigma, S) = S$, then

$$\text{Mat}(A, \Sigma) \cdot \chi_{\text{gen}}(\Sigma, S) = \text{EuQ}(P^A, S).$$

**Proof.** See [8, Proposition 13.44].

We consider a list $A$ of elements in $\{0, 1, 2\}^P$ adapted to sign determination for $P$ (cf. [8, Definition 10.72]), i.e. such that the matrix of signs of $P^A$ over $\text{SIGN}(P)$ is invertible. If $P = P_1, \ldots, P_s$, let $P_i = P_1, \ldots, P_s$, for $0 \leq i \leq s$. A method for determining a list $A(P)$ of elements in $\{0, 1, 2\}^P$ adapted to sign determination for $P$ from $\text{SIGN}(P)$ is given in Algorithm 10.83 (Adapted Matrix) in [8].

We are ready for describing the algorithm computing the generalized Euler-Poincaré characteristic. We start with an algorithm for the Euler-Poincaré-query.
Algorithm 4 (Euler-Poincaré-query)

**Input:**
1. A tuple \( k = (k_1, \ldots, k_\omega) \in \mathbb{Z}_+^\omega \), with \( k = \sum_{i=1}^{\omega} k_i \).
2. Polynomials \( P, Q \in D[X^{(1)}, \ldots, X^{(\omega)}] \), where each \( X^{(i)} \) is a block of \( k_i \) variables, and \( P, Q \) are symmetric in each block of variables \( X^{(i)} \), and of degree at most \( d \).

**Output:**
The Euler-Poincaré-queries

\[
\begin{align*}
\text{EuQ}(P, Z) &= \chi^\text{gen}(P > 0, Z) - \chi^\text{gen}(P < 0, Z), \\
\text{EuQ}_{\mathcal{E}_k}(P, Z) &= \chi^\text{gen}(S_k(P > 0, Z)) - \chi^\text{gen}(S_k(P < 0, Z)), \\
\end{align*}
\]
where \( Z = \text{Zer}(Q, R^k) \).

**Procedure:**
1. Introduce a new variable \( X_{k+1} \), and let
   \[
   \begin{align*}
   Q^+ &= Q^2 + (P - X_{k+1}^2)^2, \\
   Q^- &= Q^2 + (P + X_{k+1}^2)^2.
   \end{align*}
   \]
2. Using Algorithm 2 compute
   \[
   \begin{align*}
   \chi^\text{gen}(\text{Zer}(Q^+, R^{k+1})), \chi^\text{gen}(\text{Zer}(Q^-, R^{k+1})),
   \end{align*}
   \]
   and
   \[
   \begin{align*}
   \chi^\text{gen}(S_k(\text{Zer}(Q^+, R^{k+1}))), \chi^\text{gen}(S_k(\text{Zer}(Q^-, R^{k+1}))).
   \end{align*}
   \]
3. Output
   \[
   \begin{align*}
   \frac{(\chi^\text{gen}(\text{Zer}(Q^+, R^{k+1})) - \chi^\text{gen}(\text{Zer}(Q^-, R^{k+1})))/2,}
   \frac{(\chi^\text{gen}(S_k(\text{Zer}(Q^+, R^{k+1}))) - \chi^\text{gen}(S_k(\text{Zer}(Q^-, R^{k+1}))))/2}.
   \end{align*}
   \]

**Proof of Correctness.** The algebraic set \( \text{Zer}(Q^+, R^{k+1}) \) is semi-algebraically homeomorphic to the disjoint union of two copies of the semi-algebraic set defined by \( (P > 0) \land (Q = 0) \), and the algebraic set defined by \( (P = 0) \land (Q = 0) \). Hence, using Corollary 1, we have that

\[
\begin{align*}
2\chi^\text{gen}(P > 0, Z) &= \chi^\text{gen}(\text{Zer}(Q^+, R^{k+1})) - \chi^\text{gen}(\text{Zer}((Q, P), R^k)), \\
2\chi^\text{gen}_{\mathcal{E}_k}(P > 0, Z) &= \chi^\text{gen}_{\mathcal{E}_k}(\text{Zer}(Q^+, R^{k+1})) - \chi^\text{gen}_{\mathcal{E}_k}(\text{Zer}((Q, P), R^k)).
\end{align*}
\]

Similarly, we have that

\[
\begin{align*}
2\chi^\text{gen}(P < 0, Z) &= \chi^\text{gen}(\text{Zer}(Q^-, R^{k+1})) - \chi^\text{gen}(\text{Zer}((Q, P), R^k)), \\
2\chi^\text{gen}_{\mathcal{E}_k}(P < 0, Z) &= \chi^\text{gen}_{\mathcal{E}_k}(\text{Zer}(Q^-, R^{k+1})) - \chi^\text{gen}_{\mathcal{E}_k}(\text{Zer}((Q, P), R^k)).
\end{align*}
\]

**Complexity Analysis.** The complexity of the algorithm is \( (\omega kd)^{O(D'')} \), where \( D'' = \sum_{i=1}^{\omega} \min(k_i, 4d) \), using the complexity analysis of Algorithm 2.

We are now ready to describe our algorithm for computing the Euler-Poincaré characteristic of the realizations of sign conditions.
Algorithm 5 (Generalized Euler-Poincaré Characteristic of Sign Conditions)

Input:
1. A tuple \( \mathbf{k} = (k_1, \ldots, k_\omega) \in \mathbb{Z}^\omega_{\geq 0} \), with \( k = \sum_{i=1}^{\omega} k_i \).
2. A set of \( s \) polynomials \( \mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbb{D}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\omega)}] \), where each \( \mathbf{X}^{(i)} \) is a block of \( k_i \) variables, and each polynomial in \( \mathcal{P} \) is symmetric in each block of variables \( \mathbf{X}^{(i)} \) and of degree at most \( d \).

Output:
The lists \( \chi^{gen}(\mathcal{P}), \chi^{gen}_{\mathbf{e}_k}(\mathcal{P}) \).

Procedure:
1. \( \mathcal{P} \leftarrow \{P_1, \ldots, P_s\} \), where \( \mathcal{P}_i = \{P_1, \ldots, P_i\} \). Compute \( \text{SIGN}(\mathcal{P}) \) using Algorithm 3 (Sampling).
2. Determine a list \( A(\mathcal{P}) \) adapted to sign determination for \( \mathcal{P} \) on \( \mathbb{Z} \) using Algorithm 13.12 (Adapted Matrix) in [8].
3. Define \( A = A(\mathcal{P}), M = M(\mathcal{P}^A, \text{SIGN}(\mathcal{P})). \)
4. Compute \( \text{EuQ}(\mathcal{P}^A), \text{EuQ}_{\mathbf{e}_k}(\mathcal{P}^A) \) using repeatedly Algorithm 4 (Euler-Poincaré-query).
5. Using
   \[
   M \cdot \chi^{\text{gen}}(\mathcal{P}, \mathcal{Q}) = \text{EuQ}(\mathcal{P}^A),
   M \cdot \chi^{\text{gen}}_{\mathbf{e}_k}(\mathcal{P}, \mathcal{Q}) = \text{EuQ}_{\mathbf{e}_k}(\mathcal{P}^A).
   \]
   and the fact that \( M \) is invertible, compute \( \chi^{\text{gen}}(\mathcal{P}, \chi^{\text{gen}}_{\mathbf{e}_k}(\mathcal{P})) \).

Proof of correctness. The correctness follows from the correctness of Algorithm 3 and the proof of correctness of the corresponding algorithm (Algorithm 13.12) in [8].

Complexity analysis. The complexity analysis is very similar to that of Algorithm 13.12 in [8]. The only difference is the use of the bound on \( \text{card}(\mathcal{P}) \) afforded by Proposition 8 in the symmetric situation instead of the usual non-symmetric bound. By Proposition 8

\[
\text{card}(\text{SIGN}(\mathcal{P})) \leq s^{D'}d^{O(D'^{\prime\prime})},
\]
where \( D' = \sum_{i=1}^{\omega} \min(k_i, d) \), and \( D'' = \sum_{i=1}^{\omega} \min(k_i, 4d) \). The number of calls to Algorithm 4 (Euler-Poincaré-query) is equal to \( \text{card}(\text{SIGN}(\mathcal{P})) \). The calls to Algorithm 4 (Euler-Poincaré-query) are done for polynomials which are products of at most

\[
\log(\text{card}(\text{SIGN}(\mathcal{P}))) = O(D'' \log d + D' \log s)
\]
products of polynomials of the form \( P \) or \( P^2 \), \( P \in \mathcal{P} \) by Proposition 10.84 in [8], hence of degree bounded by \( D = O(d(D'' \log d + D' \log s)) \). Using the complexity analysis of Algorithm 3 (Sampling) and the complexity analysis of Algorithm 4 (Euler-Poincaré-query), the number of arithmetic operations is bounded by

\[
s^{D'}d^{O(D'^{\prime\prime})} + s^{D'}d^{O(D'^{\prime\prime})}(k \omega d)^{O(D'^{\prime\prime\prime})},
\]
where \( D = d(D'' \log d + D' \log s) \), \( D' = \sum_{i=1}^{\omega} \min(k_i, d) \), \( D'' = \sum_{i=1}^{\omega} \min(k_i, d) \), and \( D'''' = \sum_{i=1}^{\omega} \min(k_i, 2D) \).

The algorithm also involves the inversion matrices of size \( s^{D'}d^{O(D'^{\prime\prime})} \) with integer coefficients.
Algorithm 6 (Computing generalized Euler-Poincaré characteristic of symmetric semi-algebraic sets)

**Input:**
1. A tuple $k = (k_1, \ldots, k_\omega) \in \mathbb{Z}^\omega_{>0}$, with $k = \sum_{i=1}^\omega k_i$.
2. A set of $s$ polynomials $P = \{P_1, \ldots, P_s\} \subset D[X^{(1)}, \ldots, X^{(\omega)}]$, where each $X^{(i)}$ is a block of $k_i$ variables, and each polynomial in $P$ is symmetric in each block of variables $X^{(i)}$, and of degree at most $d$.
3. A $P$-semi-algebraic set $S$, described by $S = \bigcup_{\sigma \in \Sigma} \text{Reali}(\sigma, R^k)$, where $\Sigma \subset \{0, 1, -1\}$ is the set of sign conditions on $P$.

**Output:**
$\chi_{\text{gen}}^k(S)$ and $\chi_{\text{gen}}^k(S)$.

**Procedure:**
1. Compute using Algorithm 3 the set $\text{SIGN}(P)$.
2. Identify $\Gamma = \text{SIGN}(P) \cap \Sigma$.
3. Compute using Algorithm 2, $\chi_{\text{gen}}^k(P)$, $\chi_{\text{gen}}^k(S)$, and $\Gamma$.
4. Compute using $\chi_{\text{gen}}^k(P)$, $\chi_{\text{gen}}^k(S)$, and $\Gamma$,
   $$\chi_{\text{gen}}^k(S) = \sum_{\sigma \in \Sigma} \chi_{\text{gen}}^k(\sigma),$$
   $$\chi_{\text{gen}}^k(S) = \sum_{\sigma \in \Sigma} \chi_{\text{gen}}^k(\sigma).$$

**Proof of correctness.** The correctness of Algorithm 6 follows from the correctness of Algorithms 3 and 5, and the additive property of the generalized Euler-Poincaré characteristic (see Definition 1).

**Complexity analysis.** The complexity is dominated by Step 3, and is thus bounded by
$$\text{card}(\Sigma)^{O(1)} + sD'k^dD' + sD'dO(D')k\omega D'kD''O(D''),$$
where $D = d(D'' \log d + D' \log s)$, $D' = \sum_{i=1}^\omega \min(k_i, d)$, $D'' = \sum_{i=1}^\omega \min(k_i, 2D_i)$, and $D''' = \sum_{i=1}^\omega \min(k_i, 2D_i)$.

The algorithm also involves the inversion matrices of size $sD'dO(D'')$ with integer coefficients.

**Proof of Theorem 2.** The correctness and the complexity analysis of Algorithm 6 prove Theorem 2. 

5. Appendix

**Notation 19.** For $x \in \mathbb{R}^k$ or $\mathbb{C}^k$, let $G_x$ be the isotropy subgroup of $x$ with respect to the action of $\mathfrak{S}_k$ on $\mathbb{R}^k$ or $\mathbb{C}^k$ permuting coordinates. Then, it is easy to verify that
$$G_x \cong \mathfrak{S}_{\ell_1} \times \cdots \times \mathfrak{S}_{\ell_m},$$
where $k \geq \ell_1 \geq \ell_2 \geq \cdots \geq \ell_m > 0$, $\sum_i \ell_i = k$, and $\ell_1, \ldots, \ell_m$ are the cardinalities of the sets
$$\{i \mid 1 \leq i \leq k, x_i = x\}, x \in \bigcup_{i=1}^k \{x_i\}$$
in non-decreasing order. We denote by $\pi(x)$ the partition $(\ell_1, \ldots, \ell_m) \in \Pi_k$.

More generally, for $k = (k_1, \ldots, k_w) \in \mathbb{Z}_{\geq 0}^w$, with $k = \sum_{i=1}^w k_i$, and $x = (x^{(1)}, \ldots, x^{(\omega)}) \in \mathbb{R}^k$, where each $x^{(i)} \in \mathbb{R}^{k_i}$, we denote

$$\pi(x) = (\pi(x^{(1)}), \ldots, \pi(x^{(\omega)})) \in \Pi_k.$$

In the following proposition we use Notation 15.

**Proposition 10.** [[9, Proposition 7]] Let $L'_{\text{fixed}} \subset L_{\text{fixed}}$ any subspace of $L_{\text{fixed}}$, and $I \subset \{(i, j) \mid 1 \leq i \leq \omega, 1 \leq j \leq \ell_i\}$. Then the following hold.

A. The dimension of $L'_{\text{fixed}}$ is equal to $\sum_{i=1}^w \ell_i - 1 = \text{length}(\pi) - 1$.

B. The product over $i \in [1, \omega]$ of the subgroups $\mathfrak{S}_{\pi(i)} \times \mathfrak{S}_{\pi(i)} \times \cdots \times \mathfrak{S}_{\pi(i)}$ acts trivially on $L'_{\text{fixed}}$.

C. For each $i, j, 1 \leq i \leq \omega, 1 \leq j \leq \ell_i$, $M^{(i)}_j$ is an irreducible representation of $\mathfrak{S}_{\pi(i)}$, and the action of $\mathfrak{S}_{\pi(i)}$ on $M^{(i)}_j$ is trivial if $(i, j) \neq (i', j')$.

D. There is a direct decomposition $L = L_{\text{fixed}} \oplus \left( \bigoplus_{1 \leq i \leq \omega, 1 \leq j \leq \ell_i} M^{(i)}_j \right)$.

E. Let $D$ denote the unit disc in the subspace $L'_{\text{fixed}} \oplus \left( \bigoplus_{(i, j) \in I} M^{(i)}_j \right)$. Then, the space of orbits of the pair $(D, \partial D)$ under the action of $\mathfrak{S}_k$ is homotopy equivalent to $(\ast, \ast)$ if $I \neq \emptyset$. Otherwise, the space of orbits of the pair $(D, \partial D)$ under the action of $\mathfrak{S}_k$ is homeomorphic to $(D, \partial D)$.

**Lemma 2.** [[9, Lemma 5]] Then, for $1 \leq i < N$, and for each $c \in [c_i, c_{i+1})$, $\phi_k(S_{\leq c})$ is semi-algebraically homotopy equivalent to $\phi_k(S_{\leq c_i})$.

Let $L^+(x) \subset L$ and $L^-(x) \subset L$ denote the positive and negative eigenspaces of the Hessian of the function $p_1^{(k)}$ restricted to $W$ at $x$. Let $\text{ind}^{-1}(x) = \dim L^{-}(x)$.

The proof of the following lemma follows closely the proof of a similar result (Lemma 6) in [9].

**Lemma 3.** Let $G_c$ denote a set of representatives of orbits of critical points $x$ of $F$ restricted to $W$ with $F(x) = c$. Then, for all small enough $t > 0$,

$$\chi^{\text{top}}(\phi_k(S_{\leq c}), F) = \chi^{\text{top}}(\phi_k(S_{\leq c-t}), F) + \sum_x (-1)^{\text{ind}^{-1}(x)},$$

where the sum is taken over all $x \in G_c$ with $\sum_{1 \leq i \leq k} \frac{\partial P}{\partial x_i}(x) < 0$.

**Proof.** We first prove the proposition for $R = \mathbb{R}$. We will also assume that the function $F$ takes distinct values on the distinct orbits of the critical points of $F$ restricted to $W$ for ease of exposition of the proof. Since the topological changes at the critical values are local near the critical points which are assumed to be isolated, the general case follows easily using a standard partition of unity argument. Also, note that the value of $\text{ind}^{-1}(x)$ (respectively, $\text{sign}(\sum_{1 \leq i \leq k} \frac{\partial P}{\partial x_i}(x))$) are equal for all critical points $x$ belonging to one orbit.

Suppose that for each critical point $x \in W$, with $F(x) = c$,

$$\sum_{1 \leq i \leq k} \frac{\partial P}{\partial x_i}(x) > 0.$$
We prove that in this case, for all small enough \( t > 0 \),
\begin{equation}
\chi^{\text{top}}(\phi_k(S_{\leq c}), \mathcal{F}) = \chi^{\text{top}}(\phi_k(S_{\leq c-\epsilon}), \mathcal{F}).
\end{equation}

If
\[
\sum_{1 \leq i \leq k} \frac{\partial P}{\partial X_i}(x) > 0,
\]
then \( S_{\leq c} \) retracts \( \mathfrak{S}_k \) equivariantly to a space \( S_{\leq c-\epsilon} \cup_B A \) where the pair \((A, B) = \prod_{i} (A_{x_i}, B_{x_i})\), and where the disjoint union is taken over the set critical points \( x \) with \( F(x) = c \), and each pair \((A_{x_i}, B_{x_i})\) is homeomorphic to the pair \((D^i \times [0, 1], \partial D^i \times [0, 1] \cup D^i \times \{1\})\), where \( i \) is the dimension of the negative eigenspace of the Hessian of the function \( c_1^{(k)} \) restricted to \( W \) at \( x \). This follows from the basic Morse theory (see [8, Proposition 7.21]). Since the pair \((D^i \times [0, 1], \partial D^i \times [0, 1] \cup D^i \times \{1\})\) is homotopy equivalent to \((*, *)\), \( S_{\leq c} \) is homotopy equivalent to \((S_{\leq c-\epsilon} \cup_B A)\), and each pair \((A_{x_i}, B_{x_i})\) is homotopy equivalent to \( \phi_k(S_{\leq c-\epsilon}) \) as well, because of the fact that retraction of \( S_{\leq c} \) to \( S_{\leq c-\epsilon} \cup_B A \) is chosen to be equivariant. The equality (9) then follows immediately.

We now consider the case when for each critical point \( x \in W \), with \( F(x) = c \),
\[
\sum_{1 \leq i \leq k} \frac{\partial P}{\partial X_i}(x) < 0.
\]
Let \( T_x W \) be the tangent space of \( W \) at \( x \). The translation of \( T_x W \) to the origin is then the linear subspace \( L \subset \mathbb{R}^k \) defined by \( \sum_i X_i = 0 \). Let \( \pi \in \pi_k \), where for each \( i, 1 \leq i \leq \omega, \pi^{(i)} = (\pi_1^{(i)}, \ldots, \pi_{\ell_i}^{(i)}) \in \Pi_{k_i} \). The subspaces \( L^+(x), L^-(x) \) are stable under the natural action of the subgroup \( \prod_{1 \leq i \leq \omega, 1 \leq j \leq \ell_i} \mathfrak{S}_{\pi_j^{(i)}} \) of \( \mathfrak{S}_k \). For \( 1 \leq i \leq \omega, 1 \leq j \leq \ell_i \), let \( L_j^{(i)} \) denote the subspace \( L \cap L_{\pi_j^{(i)}} \) of \( L \), and \( M_j^{(i)} \) the orthogonal complement of \( L_j^{(i)} \) in \( L \). Let \( L_{\text{fixed}} = L \cap L_{\pi} \). It follows from Parts (B), (C), and (D) of Proposition 10 that:

i. For each \( i, j, 1 \leq i \leq \omega, 1 \leq j \leq \ell_i \), \( M_j^{(i)} \) is an irreducible representation of \( \mathfrak{S}_{\pi_j^{(i)}} \), and the action of \( \mathfrak{S}_{\pi_j^{(i)}} \) on \( M_j^{(i)} \) is trivial if \( (i, j) \neq (i', j') \). Hence, for each \( i, j, 1 \leq i \leq \omega, 1 \leq j \leq \ell_i \), \( \mathcal{L} \cap \mathcal{P} \cap M_j^{(i)} = 0 \) or \( M_j^{(i)} \).

ii. The subgroup \( \prod_{1 \leq i \leq \omega, 1 \leq j \leq \ell_i} \mathfrak{S}_{\pi_j^{(i)}} \) of \( \mathfrak{S}_k \) acts trivially on \( L_{\text{fixed}} \).

iii. There is an orthogonal decomposition \( L = L_{\text{fixed}} \oplus \left( \bigoplus_{1 \leq i \leq \omega, 1 \leq j \leq \ell_i} M_j^{(i)} \right) \).

It follows that
\[
\mathcal{L} \cap \mathcal{P} \cap L_{\text{fixed}} \oplus \left( \bigoplus_{1 \leq i \leq \omega, 1 \leq j \leq \ell_i} M_j^{(i)} \right),
\]
where \( L_{\text{fixed}} \) is some subspace of \( L_{\text{fixed}} \) and \( I \subset \{(i, j) \mid 1 \leq i \leq \omega, 1 \leq j \leq \ell_i \} \).

It follows from the proof of Proposition 7.21 in [8] that for all sufficiently small \( t > 0 \) then \( S_{\leq c} \) retracts \( \mathfrak{S}_k \)-equivariantly to a space \( S_{\leq c-\epsilon} \cup_B A \) where the pair \((A, B) = \prod_{i} (A_{x_i}, B_{x_i})\), and the disjoint union is taken over the set critical points \( x \) with \( F(x) = c \), and each pair \((A_{x_i}, B_{x_i})\) is homeomorphic to the pair \((D^{\text{ind}^{-}}(x)), \partial D^{\text{ind}^{-}}(x))\). It follows from the fact that the retraction mentioned above
is equivariant that \( \phi_k(S_{c-e}) \) retracts to a space obtained from \( \phi_k(S_{c-e}) \) by gluing orbit \( \mathcal{S}_A \) along orbit \( \mathcal{S}_B \). Now there are the following cases to consider:

(a) \( \text{ind}^-(x) = 0 \). In this case

\[
\text{orbit}_{\mathcal{S}_A} \left( \prod_x A_x, \prod_x B_x \right)
\]

is homotopy equivalent to \((*, \emptyset)\).

(b) \( L^-(x) \subset L_{\text{fixed}} \) (i.e. \( I = \emptyset \) in this case). In this case

\[
\text{orbit}_{\mathcal{S}_A} \left( \prod_x A_x, \prod_x B_x \right)
\]

is homeomorphic to \((\mathbf{D}^{\text{ind}^-(x)}, \partial \mathbf{D}^{\text{ind}^-(x)})\) by Part (E) of Proposition 10.

(c) Otherwise, there is a non-trivial action on \( L^-(x) \) of the group

\[
\prod_{(i,j) \in I} \mathcal{S}_{x}^{(i,j)},
\]

and it follows from Part (E) of Proposition 10 that in this case

\[
\text{orbit}_{\mathcal{S}_A} \left( \prod_x A_x, \prod_x B_x \right)
\]

is homotopy equivalent to \((*, *)\).

The equality (8) follow immediately from (1).

This finishes the proof in case \( R = \mathbb{R} \). The statement over a general real closed field \( R \) now follows by a standard application of the Tarski-Seidenberg transfer principle (see for example the proof of Theorem 7.25 in [8]). \( \square \)

**Proof of Theorem 3.** The proof follows immediately from Lemmas 2 and 3. \( \square \)

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, USA
E-mail address: sbasu@math.purdue.edu

AALTO SCIENCE INSTITUTE, AALTO UNIVERSITY, ESPOO, FINLAND
E-mail address: cordian.riener@aalto.fi