Lower-Critical Dimension of the Random-Field XY Model and the Zero-Temperature Critical Line

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The random-field XY model is studied in spatial dimensions $d = 3$ and 4, and in-between, as the limit $q \to \infty$ of the $q$-state clock models, by the exact renormalization-group solution of the hierarchical lattice or, equivalently, the Migdal-Kadanoff approximation to the hypercubic lattices. The lower-critical dimension is determined between $3.81 < d_c < 4$. When the random-field is scaled with $q$, a line segment of zero-temperature criticality is found in $d = 3$. When the random-field is scaled with $q^2$, a universal phase diagram is found at intermediate temperatures in $d = 3$.

I. INTRODUCTION: ISING AND XY LOWER-CRITICAL DIMENSIONS

Quenched randomness strongly affects the occurrence of order at low spatial dimension $d$, reflected as the lower-critical dimension $d_c$ below which no ordering occurs for a given class of systems. In the random-magnetic-field $n = 1$ component spin Ising model, after a strong experimental and theoretical controversy between $d_c = 2$ claims [1,2] and $d_c = 3$ claims [3], the issue was settled for $d_c = 2$ [3-4]. The fact that $d_c$ is not 3 fell in contradiction with the prediction of a dimensional shift of 2 due to random fields coming from all-order field-theoretic expansions from $d = 6$ down to $d = 1$ [5], which indeed is a considerable distance to expand upon for a small-parameter expansion of $\epsilon = 6 - d$. In this study, the logically next model, namely the $n = 2$ components spin XY model under random fields is examined and surprising results are obtained, this time in near-agreement with the dimensional shift of 2, but also with an interesting zero-temperature critical line segment and a universal scaled finite-temperature phase diagram.

Random-field Ising results supporting $d_c = 2$ were obtained [5,6] by the Migdal-Kadanoff [8,9] renormalization-group calculations in $d = 2$ (no random-field order), $d = 2.32$ (random-field order), and $d = 3$ (more random-field order). In the same vein, for the random-field XY model, Migdal-Kadanoff renormalization-group calculations are done here in $d = 3$ and 4, and in between. The Migdal-Kadanoff renormalization-group calculation (Fig. 1) is a highly successful, flexible, and therefore most used today, physically motivated approximation for hypercubic lattices and, simultaneously, an exact calculation for $d$-dimensional hierarchical lattices [10-12]. The hierarchical lattice connection makes the Migdal-Kadanoff procedure a physically realizable approximation. For recent work using hierarchical lattices, see Refs. [14-21]. Migdal-Kadanoff-hierarchical-lattices correctly give the lower-critical dimensions of $d_c = 1$ of the Ising model [8,9], $d_c = 2$ of the XY [22,23] and $(n = 3$ spin components) Heisenberg [24] models in the absence of quenched randomness. For the much more complex system with competing quenched-random interactions, Migdal-Kadanoff gives the non-integer $d_c = 2.46$ for the Ising spin-glass system [25-31]. In addition to giving the lower-critical temperatures, it yields such diverse results as, e.g., the low-temperature algebraic order of the $d = 2$ XY model [22,23], the chaotic nature [32-34] of the ferromagnetic-antiferromagnetic [35] and left-right chiral [36] Ising spin glasses, and the changeover from second- to first-order phase transitions of $q$-state Potts models in $d = 2$ and 3 [37].

II. MODEL AND METHOD

The XY model is approached as the $q \to \infty$ limit of the $q$-state clock models. In the $q$-state clock models, at each site $i$ of the lattice, a planar unit spin $\hat{s}_i$ can
point in one of \( q \) directions in the plane, namely with the angle \( \theta_k = k(2\pi/q) \), where \( k = 0, 1, ..., q - 1 \). A detailed renormalization-group study on the phase transitions and thermodynamics of the \( q \)-state clock models, without quenched randomness, has been done. The currently studied \( q \)-state clock model, with quenched random fields, is defined by the Hamiltonian

\[
-\beta \mathcal{H} = \sum_{\langle ij \rangle} (J \vec{s}_i \cdot \vec{s}_j + \vec{s}_i \cdot \vec{H}_i + \vec{s}_j \cdot \vec{H}_j),
\]

where \( \beta = 1/k_B T \) and sum is over all nearest-neighbor pairs of spins. In each term in the sum, the random-fields \( \vec{H}_i, \vec{H}_j \) have magnitude \( H \) and each randomly points along one of the allowed directions \( \theta_k \).

We solve this model using the Migdal-Kadanoff renormalization group. The local renormalization-group transformation is given in Fig. 1 and is simple to implement in systems without quenched randomness. With our currently studied quenched random-field model, the renormalization-group evolution of quenched random distributions has to be pursued. Initially, 5,000 nearest-neighbor Hamiltonians are created, with 10,000 randomly chosen magnetic field directions as described above. From this distribution, \( b^q \) nearest-neighbor Hamiltonians are randomly chosen, to effect the local Migdal-Kadanoff transformation and obtain a renormalized nearest-neighbor Hamiltonian. This is repeated 5,000 times and the renormalized distribution is obtained. Each nearest-neighbor Hamiltonian in the distribution is exponentiated and thus kept as a transfer matrix. To conserve, in this distribution, the \((ij) \leftrightarrow (ji)\) and the random-field direction symmetries, each transfer matrix is replicated by its transpose and by the simultaneous cyclic permutations of the rows and columns. Of the resulting \( 2q \times 5000 \) matrices, 5,000 are randomly chosen. Thus, the distribution continues as 5,000 \( q \times q \) matrices.

The flows of the distributions determine the phase diagram: Renormalization-group trajectories starting in the ferromagnetic phase flow to the strong-coupling sink of \( J_{ij} \to \infty, H_i = 0 \). Renormalization-group trajectories starting in the disordered phase flow to the decoupled sink of \( J_{ij}, H_i = 0 \). The boundaries between these flow basins are the phase boundaries.

III. \( d = 3 \) DIMENSIONS AND ZERO-TEMPERATURE CRITICALITY SEGMENT

Our calculated phase diagrams for \( q = 7, 10, 20, 50, 100, 150 \)-state random-field clock models in \( d = 3 \) are in Fig. 2, occurring in the figure respectively from high field to low field. Disordered and ferromagnetic phases occur at high temperature-high field and low temperature-low field, respectively. The \( H/J \) values on the vertical axis are multiplied with \( q \), originally for better graphical visibility, but eventually leading to a physical result, as seen here. Firstly, note that the ferromagnetic region under random fields recedes and disappears as \( q \) is increased. This result is even more evident, when we recall that the vertical axis values are amplified by a factor of \( q \) for better pictorial visibility. The ferromagnetic phase, in random field, disappearing as \( q \to \infty \) indicates that no ferromagnetic phase occurs in the random-field XY model at non-zero temperature in \( d = 3 \).

Secondly and quite interestingly, given our choice of vertical axis values, it revealed that the ordered phase extends at very low temperatures, for the high \( q \) to the
universal value of $qH/J = 5.1$. This is more visible in the left inset box of Fig. 2. Thus, at $q \to \infty$, a line segment of zero-temperature critical points occurs between $qH/J = 0$ and $qH/J = 5.1$. Zero-temperature critical segments and multicritical points have been found before, under exact renormalization-group treatment, in the $d = 1$ Blume-Emery-Griffiths model [39].

Thirdly, for high $q$, the zero-field ferromagnetic transition temperature saturates, as also seen in Ref. [38] and in detail in the right inset box in Fig. 2. Furthermore, when the vertical axis is scaled, not by $q$, but by $q^2$, a universal phase diagram emerges above low temperature for high $q$, as seen in Fig. 3.

FIG. 4. Phase diagrams for $(q = 3, 4, 5, 6, 7, 10)$-state random-field clock models in $d = 3.32$, occurring in the figure respectively from high field to low field.

FIG. 5. The critical line segment, at zero temperature, is between $qH/J = 0$ and the $qH/J$ values shown in this figure for each dimension $d$. The values are consistent with a divergence as $d = 4$ is approached.

**IV. $d = 4$ DIMENSIONS AND THE LOWER-CRITICAL DIMENSION**

The phase diagrams for $(q = 3, 4, 5, 6, 7, 10)$-state random-field clock models in $d = 3.32$ are shown in Fig. 4. It is again seen that the ferromagnetic phase, under random fields, recedes and disappears as $q \to \infty$. Thus, no ferromagnetic phase occurs under random fields in the XY model in $d = 3.32$. However, our calculation again gives the zero-temperature critical segment, between $qH/J = 0$ and $qH/J = 7.6$ universally for all $q$ in $d = 3.32$.

The same results are obtained for $d = 3.58$ and $3.81$, with the zero-temperature critical segment expanding, reaching $qH/J = 10.2$ and $13.9$, respectively.

A qualitatively different picture occurs in the phase diagrams for $d = 4$, seen in Fig. 5. Going from $q = 7$ to $q = 10$, the ferromagnetic phase slightly expands in the random field, as opposed to drastically receding as in the lower dimensions. Going from $q = 10$ to $q = 20$, a much larger $q$ interval, the ferromagnetic phase even more slightly expands in the random field. Thus, the ferromagnetic phase occurs, under random fields, for $q \to \infty$ and for the XY model in $d = 4$.

We thus see that the lower-critical dimension for the random-field XY model is between $d = 3.81$ and $d = 4$, namely $3.81 < d_c < 4$.

**V. CONCLUSION**

In order to investigate the random-field XY model, we have studied the random-field $q$-state clock models for increasing $q$, for dimensions $d = 3, 3.32, 3.58, 3.81, 4$. We find that for the random-field XY model, the lower-critical dimension is between $d = 3.81$ and $d = 4$, namely $3.81 < d_c < 4$. At $d < d_c$, we find a zero-temperature segment of criticality, stretching from zero to a value of $qH/J$ that is $q$-independent for large $q$ and that increases as $d_c$ is approached.

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