On analytical solution of stationary two dimensional boundary problem of natural convection

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Abstract

Approximate analytical solution of two dimensional problem for stationary Navier-Stokes, continuity and Fourier-Kirchhoff equations describing free convective heat transfer from isothermal surface of half infinite vertical plate is presented. The problem formulation is based on the typical for natural convection assumptions: the fluid noncompressibility and Boussinesq approximation. We also assume that orthogonal to the plate component of velocity is small. Apart from the basic equations it includes boundary conditions: the constant temperature and zero velocity on the plate. At the starting point of the flow we fix average temperature and vertical component of velocity, as well as basic conservation laws in integral form. The solution of the boundary problem is represented as a Taylor Series in horizontal variable with coefficients depending on vertical variable.

1 Introduction

The results of theoretical and experimental study of free convective flows from heating objects are widely published and they are very useful to determine convective heat losses from apparatus, devices, pipes in industrial or energetic installations, electronic equipment, architectonic objects and so on by engineers and designers.

The problem of convective flow development traditionally is based either on boundary layer theory or on self-similarity theory [1], [2], [3]. Both methods of natural convection heat transfer description use simplifications that allow to transform the basic fundamental equations of Navier-Stokes, mass and Fourier-Kirchhoff equations. As a main point of the methods is an ordinary differential
equation introduction which solution relates to a general problem by some contraction procedure. Such description don’t give a possibility to investigate some details of a flow field for example at vicinity of its starting point. It is known that the starting point influence is important in many aspects of fluid flow development, especially in critical conditions.

There are some problems of convection for which the boundary layer approach fails. One of such cases is an assumption that boundary layer starts on a plate, that contradicts experiment. The other is conceptual for the theory: the fluid flow is restricted by a conditional boundary which also don’t exist in reality. Due to this Prandtl introduced integration across a boundary layer to determine average values of fluid flow (velocity, temperature).

The similarity solution method is based on a specific combination of independent variables introduction that allows to get rid of two-dimensionality of the general problem. The resulting ordinary differential equation gives an information of velocity and temperature profiles that is an advance in comparison with boundary layer theory. However such description still is rather qualitative than quantitative. For example it doesn’t give an answer about flow behavior in the vicinity of the starting point in the flow. We would stress that an interest to the problem still exists [4], [5], [6].

We consider approximate analytical solution of stationary convective fluid flow induced by an isothermal vertical surface with the axis y parallel to gravity force x is used as a horizontal one. The choice of the coordinate system is typical for laminar natural convection simplifications with the following main assumptions: fluid is incompressible, Boussinesq approximation, normal to the surface component of velocity is neglected [3, 7]. The case of isothermal vertical surface is obtained as a limit of horizontal conic with a base angle of the conic $\alpha = 0$ [8, 9].

Our solutions are build without use neither of boundary layer nor self-similarity concepts. We are looking for the solution as Taylor series in x and derive a system of equations for coefficient of the expansion. Each cutting of the series results in polynomial approximation in x and find a closed system of equations in y. We restrict ourselves by upper half plane ($y \geq 0$). Such approach needs a formulation of boundary conditions in the starting point of the flow. We take into account the physical character of the flow: namely its natural convection origin that means average velocity is zero at $y = 0$ level. For the formulation of boundary conditions we base on conservation laws in integral form.

2 The basic equations

Let us consider a two dimensional stationary flow of incompressible fluid in the gravity field. The flow is generated by a convective heat transfer from solid plate to the fluid. The plate is isothermal and vertical. In the Cartesian coordinates
\[ x, y \] the Navier-Stokes (NS) system of equations have the form [1]:

\[
\rho \left( W_x \frac{\partial W_y}{\partial x} + W_y \frac{\partial W_y}{\partial y} \right) = g \rho_\infty b \left( T - T_\infty \right) - \frac{\partial p}{\partial y} + \rho \nu \left( \frac{\partial^2 W_y}{\partial y'^2} + \frac{\partial^2 W_y}{\partial x'^2} \right)
\]

(1)

\[
\rho \left( W_x \frac{\partial W_x}{\partial x} + W_y \frac{\partial W_x}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho \nu \left( \frac{\partial^2 W_x}{\partial y'^2} + \frac{\partial^2 W_x}{\partial x'^2} \right)
\]

(2)

In the above equations the pressure terms are divided in two parts \( \tilde{p} = p_0 + p \).

The first of them is the hydrostatic one that is equal to mass force \(-g \rho_\infty\), where \( \rho = \rho_\infty (1 - b (T - T_\infty)) = \rho_\infty \rho' \) is the density of a liquid at the nondisturbed area where the temperature is \( T_\infty \). The second one is the extra pressure denoted by \(-\nabla p\). The part of gravity force \( gb (T - T_\infty) \) arises from dependence of the extra density on temperature, \( b \) is a coefficient of thermal expansion of the fluid. In the case of gases \( b = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p = \frac{1}{T_\infty} \). The last terms of the above equations represents the friction forces with the kinematic coefficient of viscosity \( \nu \).

The mass continuity equation in the conditions of natural convection of incompressible fluid in the steady state [2] has the form:

\[
\frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} = 0.
\]

(3)

The temperature dynamics is described by the stationary Fourier-Kirchhoff (FK) equation:

\[
W_x \frac{\partial T}{\partial x} + W_y \frac{\partial T}{\partial y} = a \left( \frac{\partial^2 T}{\partial y'^2} + \frac{\partial^2 T}{\partial x'^2} \right)
\]

(4)

where \( W_x \) and \( W_y \) are the components of the fluid velocity \( \vec{W} \) that are shown on the Fig.1; \( T, p \) - temperature and pressure disturbances correspondingly and \( a \) is the thermal diffusivity. From the point of clarity of further transformations we use the same scale \( l \) along both variables \( x \) and \( y \). We will return to the eventual difference between characteristic scales in different directions while the solution analysis to be provided. After introducing nontraditional variables:

\[
x' = x/l, \ y' = y/l, \ T' = (T - T_w)/(T_w - T_\infty), \ p' = p/p_\infty, \ W'_x = W_x/W_o, \ W'_y = W_y/W_o
\]

(5)

we obtain in Boussinesq approximation (in all terms besides of buoyancy one we put \( \rho \approx \rho_\infty \)).

\[
W'_x \frac{\partial W'_y}{\partial x'} + W'_y \frac{\partial W'_y}{\partial y'} = L \left( T' + 1 \right) - \frac{p_\infty}{\rho_\infty W_o^2} \frac{\partial p'}{\partial y'} + \nu' \left( \frac{\partial^2 W_y'}{\partial y'^2} + \frac{\partial^2 W_y'}{\partial x'^2} \right)
\]

(6)

\[
W'_x \frac{\partial W'_x}{\partial x'} + W'_y \frac{\partial W'_x}{\partial y'} = -\frac{p_\infty}{\rho_\infty W_o^2} \frac{\partial p'}{\partial x'} + \nu' \left( \frac{\partial^2 W_x'}{\partial y'^2} + \frac{\partial^2 W_x'}{\partial x'^2} \right)
\]

(7)
and FK equation is written as

$$W_x \frac{\partial T'}{\partial x'} + W_y \frac{\partial T'}{\partial y'} = a' \left( \frac{\partial^2 T'}{\partial y'^2} + \frac{\partial^2 T'}{\partial x'^2} \right)$$

(8)

where $\frac{\nu}{W_0} = \nu'$, $\frac{a}{W_0} = a'$, $\frac{gb(T_w - T_\infty)}{W_0^3} = L$, $l$ is a characteristic linear dimension and $W_0$ is characteristic velocity. If

$$W_o = \frac{\nu}{l},$$

then $a' = Pr$, $\nu' = 1$ and $L = Gr$, where

$$Gr = \frac{gb(T_w - T_\infty)l^3}{\nu^2}.$$  

(10)

(Grashof number) such link will be used in final solution. After cross differentiation of equations (6) and (7) we have:

$$\frac{\partial}{\partial x'} \left[ W_x \frac{\partial W_y'}{\partial x'} + W_y \frac{\partial W_y'}{\partial y'} - Gr' (T' + 1) - \left( \frac{\partial^2 W_y'}{\partial y'^2} + \frac{\partial^2 W_y'}{\partial x'^2} \right) \right] =$$

$$= \frac{\partial}{\partial y'} \left[ W_x \frac{\partial W_x'}{\partial x'} + W_y \frac{\partial W_x'}{\partial y'} - \left( \frac{\partial^2 W_x'}{\partial y'^2} + \frac{\partial^2 W_x'}{\partial x'^2} \right) \right]$$

(11)
The FK equation rescales as

\[ \text{Pr} \left( W_x \partial'^T + W_y \partial'^T \right) = \left( \partial'^2 T + \partial'^2 T \right) \]  

(12)

Next we would formulate the problem of free convection over the heated vertical isothermal plate \( x = 0, y \in [0, l] \), dropping the primes.

3 Method of solution and approximations

Next aim of this paper is the theory application to the standard example of a finite vertical plate. In this case we assume the angle between the plate and a stream line is small that means a possibility to neglect the horizontal component of velocity of fluid, denoting the vertical component as \( W(y, x) \). It means that the number of equations: (11), (12), (3) exceeds the number of variables: \( T \) and \( W \), hence one of the equations should be excluded. Having in mind mass conservation account in integral form, we exclude the equation (5). The mass conservation law introduces boundary conditions as a link between mass law flow on levels \( y = 0 \) and \( y = L \) (see the section on boundary conditions). Consider the power series expansions of the velocity and temperature in Cartesian coordinates:

\[ W(x, y) = \gamma(y)x + \alpha(y)x^2 + \beta(y)x^3 + \kappa(y)x^4, \]  

(13)

\[ T(x, y) = C(y)x + A(y)x^2 + B(y)x^3 + F(y)x^4, \]  

(14)

According to standard boundary conditions on the plate we assume that the both functions tend to zero when \( x \to 0 \), we choose for a calculation convenient the zero value for a nontraditional temperature \( \frac{\partial T}{\partial x} \) outside of the convective flow tends to \( -1 \).

We would like to restrict ourselves by the fourth order approximation for both variables that means we neglect higher order terms starting from fifth one (see the Fig.1 where the area of the approximations validity is marked as dashed one).

As it will be clear from further analysis we should consider the functions: \( \alpha(y), \beta(y), C(y) \) and \( B(y) \) as variables of the first order, while \( \gamma(y) \) and \( F(y) \) to be the second one. From the relations that appear after substitution of (13) and (14) into (8) and (??) it follows that \( A(y) = 0 \).

Substituting expressions (13) and (14) into the equations (??) and (??) taking into account the assumption that \( W_x = 0 \) and \( W_y = W \) yields

\[ \frac{\partial}{\partial x} \left[ W \frac{\partial W}{\partial y} - G_r (T + 1) - \left( \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial x^2} \right) \right] = 0, \]  

(15)

\[ \text{Pr} W \frac{\partial T}{\partial y} = \left( \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} \right). \]  

(16)
From (16) one found that $A(y) = 0$ and $F(y) = 0$. Finally from both equations (15), (16) we obtain the system of equations for the coefficients $B(y)$, $C(y)$, $\alpha(y)$ and $\beta(y)$:

$$6B(y) + \frac{\partial^2 C(y)}{\partial y \partial y} = 0,$$  \hspace{0.5cm} (17)

$$\Pr \alpha(y) \frac{\partial C(y)}{\partial y} - \frac{\partial^2 B(y)}{\partial y \partial y} = 0,$$  \hspace{0.5cm} (18)

$$-6\beta(y) - Gr C(y) = 0,$$  \hspace{0.5cm} (19)

$$\gamma(y) \frac{\partial \gamma(y)}{\partial y} - \frac{\partial^2 \alpha(y)}{\partial y \partial y} = 0.$$  \hspace{0.5cm} (20)

The first two (17), (18) arise from FK equation and the rest of them are from the NS one. The system of equations is closed if $\gamma(y) = const = \gamma$. It means that the number of equations and the number of unknown functions is the same. Finally, in the first approximation the velocity and temperature are expressed as:

$$W(y, x) = \gamma x + \alpha(y) x^2 + \beta(y) x^3, \quad T(y, x) = C(y) x + B(y) x^3.$$  \hspace{0.5cm} (21)

From (20) one has

$$\alpha(y) = C_1 y + C_2.$$  \hspace{0.5cm} (22)

From (17) it follows that $B(y) = -\frac{1}{6} \frac{\partial^4 C(y)}{\partial y^4}$, hence (18) goes to:

$$\frac{1}{6} \frac{\partial^4 C(y)}{\partial y \partial y \partial y \partial y} + \Pr (y C_1 + C_2) \frac{\partial C(y)}{\partial y} = 0.$$  \hspace{0.5cm} (23)

The equation (19) reads:

$$\beta(y) = -\frac{Gr}{6} C(y).$$  \hspace{0.5cm} (24a)

The equation (23) is the ordinary differential equation of the fourth order, therefore its solution needs four constants of integration, denoted as: $C(0)$, $C'(0)$, $C''(0)$ and $C'''(0)$. These constants depend on two parameters $C_1$ and $C_2$, which enter the coefficients of the Eq. (23). The function $C(y)$ defines the rest functions $\beta(y)$ and $B(y)$ via above relations. It means that we have six constants determining the solution of problem and we need also six corresponding boundary conditions at $y = 0$. 

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4 Boundary conditions formulation

At the starting horizontal edge of the vertical plate the bouncy force yet does not act therefore we assume that the mean vertical component of the velocity is negligibly small. Thus we neglect a fine velocity structure of transition area. Similar the temperature values are averaged in the vicinity of the boundary edge point and taken as value -1 (temperature of incoming from the bottom flow). Then let us define the mean value of a function $F(x)$ (velocity or temperature in the interval $x \in [0, 1]$ (Fig.2) in the nontraditional form as: 
$$\bar{F} = \frac{1}{l} \int_0^l F(x) dx.$$ 
In dimensional form the interval of averaging length is $l$ which hence is defined by this integration boundary. Let us remind that scale $l$ is connected with special (local, horizontal) Grashof number $G_r$ \(^{10}\). In natural convection problem theory vertical velocity component and temperature values in surrounding of the heated plate are: $W_y(0, 0) = 0$, $T(0, 0) = -1$ according to the above notations. Taking it into account we put:

$$W(0, x) = \left[ \int_0^1 (\gamma x + \alpha(y) x^2 + \beta(y) x^3) dx \right]_{y \to 0} = \frac{1}{2} \gamma + \frac{1}{3} \alpha(0) + \frac{1}{4} \beta(0) = 0$$

(25)
therefore due to (22) we arrive at the first boundary condition:

\[ \alpha (0) = C_2 = -\frac{3}{4} \beta (0) - \frac{3}{2} \gamma = \frac{L}{8\nu} C (0) - \frac{3}{2} \gamma \] (26)

Nontraditional temperature of the fluid at the lower half plane, according to above, is \(-1\). In the analogy to the condition for velocity we assume that average temperature at the limit \(y = 0\) is also \(-1\), therefore from (21) \(T(y = 0, x) = \int_0^1 (C(0) x + B(0) x^3) \, dx = \frac{1}{4} B(0) + \frac{1}{2} C(0) = -1\). Then the next boundary relation arises:

\[ B(0) + 2C(0) = -4. \] (27)

Plugging \(B(0) = -\frac{1}{6} \left[ \frac{\partial^2 C(y)}{\partial y \partial y} \right]_{y=0} = -\frac{1}{6} C''(0)\) from (??) into (27) gives the second boundary condition:

\[ \frac{1}{6} C''(0) - 2C(0) = 4. \] (28)

The third and forth boundary condition arize on the level \(y = L\), where the fluid lose the contact with the heated plate. At this point we suppose that

\[ \frac{\partial T}{\partial y} \bigg|_{y=L} = 0 \] (29)

and

\[ \frac{\partial W}{\partial y} \bigg|_{y=L} = 0 \] (30)

Both conditions (29) and (30) have a transparent physical meaning: the temperature and velocity does not grow from the level \(y = 0\). The immediate consequence of the equation (29) is

\[ \frac{\partial C(y)}{\partial y} \bigg|_{y=L} = 0 \] (31)

and

\[ \frac{\partial B(y)}{\partial y} \bigg|_{y=L} = -\frac{1}{6} \frac{\partial^3 C(y)}{\partial^3 y} \bigg|_{y=L} = 0 \] (32)

From (30) one can see that

\[ \frac{\partial \alpha(y)}{\partial y} \bigg|_{y=L} = C_1 = 0 \] (33)

and, according to (24a), \(\frac{\partial \beta(y)}{\partial y} \bigg|_{y=L} = -\frac{G_r}{6} \frac{\partial C(y)}{\partial y} \bigg|_{y=L} = 0\) coincide with (31).

Looking for the rest boundary conditions let us apply conservation laws of mass,
momentum and energy for derivation of boundary conditions at \( y = 0 \). The first of them is the conservation of mass in steady state in two dimensions looks as:

\[
\int_{\Sigma} \rho \vec{W} \cdot \vec{n} \, dS = 0 \tag{34}
\]

where \( \Sigma \) is a closed surface of a control volume, which cross-section is shown in Fig.2.

The second and third of them are produced by the law of momentum vector conservation, which is represented by Navier-Stokes equations (6) and (7). Let us denote

\[
\frac{\partial p}{\partial y} = \rho_{\infty} g (T - T_{\infty}) - \rho W_z \frac{\partial W_y}{\partial x} - \rho W_y \frac{\partial W_z}{\partial y} + \rho \nu \left( \frac{\partial^2 W_x}{\partial y^2} + \frac{\partial^2 W_y}{\partial x^2} \right) = E_y
\]

and

\[
\frac{\partial p}{\partial x} = -\rho W_x \frac{\partial W_z}{\partial x} - \rho W_z \frac{\partial W_x}{\partial y} + \rho \nu \left( \frac{\partial^2 W_x}{\partial y^2} + \frac{\partial^2 W_y}{\partial x^2} \right) = E_x,
\]

then one have the equation:

\[
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \text{rot}_z \vec{E} = 0 \tag{35}
\]

that we use as a basic one. The integral version of this equation is obtained by Stokes theorem:

\[
\int_{S} \text{rot}_n \vec{E} \, dS = \int_{s} \vec{E} \, d\vec{s} = 0, \tag{36}
\]

where \( s \) is the boundary curve of the integration area \( S \) (Fig.3).

One of them has the form (36). If we specify the curve of integration \( s \) as the border of the control area \( S \) and denote the tangent projection as \( E_t = \vec{E} \cdot \vec{t} \) we have

\[
\int_{s} E_t \, ds = 0. \tag{37}
\]

The next one is the first equation of Navier-Stokes (6):

\[
W_x \frac{\partial W_y}{\partial y} = G_r (T + 1) - \frac{\rho_{\infty}}{\rho_{\infty} W_0^2} \frac{\partial p}{\partial y} + \left( \frac{\partial^2 W_x}{\partial y^2} + \frac{\partial^2 W_y}{\partial x^2} \right) \tag{38}
\]

The excess pressure is the case of natural heat convection problem to be considered is small and proportional to the excess of the temperature:

\[
p = \left( \frac{dp}{dT} \right)_{\rho = \rho_{\infty}} (T - T_{\infty}) = \rho_{\infty} R' (T - T_{\infty}). \tag{39}
\]

In the case of the gas one can use the ideal gas equation of state in dimensional form \( \bar{p} = \rho R' (T_{\infty} + (T - T_{\infty})) \) where \( p_0 = \rho_{\infty} R' T_{\infty} \) and \( R' = R / \mu \) (\( \mu \) is a molar mass) is gas constant hence \( p = \rho_{\infty} R' (T - T_{\infty}) \). Going to the nontraditional form, plugging (39) to (38) yields

\[
W_x \frac{\partial W_y}{\partial y} = G_r (T + 1) - H \frac{\partial T}{\partial y} + \left( \frac{\partial^2 W_x}{\partial y^2} + \frac{\partial^2 W_y}{\partial x^2} \right) \tag{40}
\]
where we introduced the new nontraditional parameter
\[ H = \frac{(T_w - T_\infty) R'}{W_o^2} = \frac{\Phi W_o}{(g b \nu')^2} G_r^2, \tag{41} \]
where \( T_w - T_\infty = \Phi, W_o, \) and \( G_r \) are defined by (9) and (10). The viscosity coefficient \( \nu' \) here is dimensional.

The next boundary condition is connected with the conservation of energy in a control volume \( V \) (area \( S \) with unit width see Fig.3) arises from FK equation (12) by integration over the volume.

\[ \text{Pr} \int_V \left( W \frac{\partial T}{\partial y} \right) dV = \int_V \left( \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} \right) dV = \int_S \left( \nabla T \right) \overrightarrow{n} dS \tag{42} \]

Let us now formulate the rest four boundary conditions on the base of the conservation laws.

According to mass conservation law (34) the mass flux don’t depend on \( y \). If we recall that the mean density is approximately constant, then
\[ \int_0^1 \rho_\infty W dx \rightarrow 0 \]
0, or
\[ \int_0^1 (\gamma x + \alpha(y) x^2 + \beta(y) x^3) dx \rightarrow 0. \] Integration and differentiation yield the third boundary condition:
\[ \left[ \frac{1}{3} \frac{\partial \alpha(y)}{\partial y} + \frac{1}{4} \frac{\partial \beta(y)}{\partial y} \right]_{y \rightarrow 0} = 0. \tag{43} \]

Plugging the explicit expressions for \( \alpha(y) \) (22) and \( \beta(y) \) (24a) into (43) yields

It gives a link between the constant of integration \( C_1 \) and the boundary value of \( \frac{\partial C_1(y)}{\partial y} = C'(0) \) at the point of \( y = 0 \)

\[ C_1 = \frac{G_r}{8} C'(0). \tag{44} \]

Let us apply the conservations laws in integral form to the control volume \( V \) (see Fig.3) based on the interval \( y \in [0, \varepsilon] \) and \( x \in [0, 1] \). According to our main assumption about two-dimensionality of the stream we neglect a dependence of variables on \( z \) coordinate. Going to the momentum conservation let us substitute the components of the vector \( \vec{E} \) : into (15) or plugging

\[ E_y = \frac{\partial p}{\partial y} = W \frac{\partial W}{\partial y} - G_r (T + 1) - \left( \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial x^2} \right) \]

results in (see Fig.2):
\[ \oint_s E_1 ds = \int_0^\varepsilon E_y |_{x=0} dy = 0, \tag{45} \]
because the rest parts of the integral are zero according to assumption that velocity component $W_x = 0$ is zero $E_x = 0$.

The left side of the equation (45) according to (5) and the fact that the velocity $W$ vanishes on the vertical plate surface has the form:

$$G_r \varepsilon + \int_0^\varepsilon \frac{\partial^2 W}{\partial x^2} \bigg|_{x=0} dy = G_r \varepsilon + \int_0^\varepsilon \frac{\partial^2 (\alpha(y)x^2)}{\partial x^2} \bigg|_{x=0} dy = G_r \varepsilon + 2 \int_0^\varepsilon \alpha(y) dy = 0,$$

therefore $G_r = -\lim_{\varepsilon \to 0} \frac{2\nu}{\varepsilon} \int_0^\varepsilon \alpha(y) dy = -2\alpha(0)$.

After substitutions of $\alpha(y)$ (22) and $C_2$ from (26) one have two relations. First one links the integration constant $C_2$ with parameters of the problem

$$C_2 = -\frac{G_r}{2}, \quad (46)$$

while the second one gives the boundary value of the basic function $C(0)$

$$C(0) = \frac{12}{G_r} \gamma - 4. \quad (47)$$

Plugging the result into (28) gives:

$$C''(0) = \frac{144}{G_r} \gamma - 24. \quad (48)$$

The momentum conservation in differential form is represented by (40) that we use as a direct source for boundary condition formulation. To determine the fifth boundary condition let us evaluate the mean value of the momentum component balance (shown in above equations) that yields the equation:
The last (sixth) boundary condition we found from the energy conservation equation (42) for the control volume $V$ shown in the Fig.3

$$\frac{\Pr}{\varepsilon} \int_V \left( W \frac{\partial T}{\partial y} \right) dV = \frac{1}{\varepsilon} \int_S \text{grad} T \cdot n \, dS. \quad (50)$$

Taking into account that integral with respect to $z$ gives the factor 1, yields

$$\frac{\Pr}{\varepsilon} \int_0^1 \left( \int_0^L \frac{\partial T}{\partial y} \, dx \right) dy = \frac{1}{\varepsilon} \int_0^L \left( -\frac{\partial T}{\partial y} \bigg|_{y=0} + \frac{\partial T}{\partial y} \bigg|_{y=e} \right) dx - \frac{1}{\varepsilon} \int_0^z \frac{\partial T}{\partial x} \bigg|_{x=0} dy$$

After substitutions $W$ and $T$ (21), the left side of (50) gives:

$$\Pr \left( \frac{\gamma}{70} - \frac{G_r}{504} \right) \frac{\partial^2 C(y)}{\partial y^2} + \Pr \left( \frac{G_r}{120} - \frac{\gamma}{15} \right) \frac{\partial C(y)}{\partial y}, \text{while the right side after transition } \varepsilon \to 0 \text{ tends to} \left[ \frac{1}{2} \frac{\partial^2 B(y)}{\partial y^2} + \frac{1}{12} \frac{\partial^2 C(y)}{\partial y^2} \right]_{y=0} = -C(0).$$

Plugging the expression for $B$ (17) and next using the equation (23) for the basic function $C(y)$ gives:

$$\left[ \frac{1}{2} \frac{\partial^2 C(y)}{\partial y^2} - \frac{1}{24} \frac{\partial^3 C(y)}{\partial y^3} \right]_{y=0} = -C(0) = -\frac{G_r \Pr}{8} C'(0) + \frac{60}{G_r} \gamma - 8.$$

The complete equation (50) gives the last boundary relation:

$$\Pr \left( \frac{\gamma}{70} - \frac{G_r}{504} \right) C'''(0) + \Pr \left( \frac{G_r}{120} - \frac{\gamma}{15} \right) C'(0) = -\frac{G_r \Pr}{8} C'(0) + \frac{60}{G_r} \gamma - 8. \quad (51)$$

5 The explicit form of boundary conditions for $C(y)$

Let us solve the conditions (49) and (51) with respect to the first and third derivatives of $C(y)$ at $y = 0$.

The relation (49) yields at $y = 0$:

$$-\frac{1}{24} HC'''(0) + \left( \frac{1}{2} H - \frac{1}{1680} G_r^2 + \frac{13}{3360} G_r \gamma \right) C'(0) + 12 \gamma - 2 G_r = 0.$$

Plugging $C'''(0) = \left( 12 - \frac{1}{10 \gamma} G_r^2 + \frac{13}{12 \gamma} G_r \gamma \right) C'(0) + \frac{288}{H} \gamma - \frac{48}{H} G_r$ into (51) gives the equation for $\left[ \frac{\partial C(y)}{\partial y} \right]_{y=0} = M,$
\[
\left( \frac{\gamma}{70} - \frac{G_r}{504} \right) \left[ \left( 12 - \frac{G_r^2}{70H} + \frac{13G_r}{140H} \right) C'(0) + \frac{288\gamma}{H} - \frac{48G_r}{H} \right] + \left( \frac{G_r}{120} - \frac{\gamma}{70} \right) M + \frac{G_rM}{8} - \frac{60\gamma}{PrG_r} + \frac{8}{Pr} = 0
\]

Solving this linear equation yields:

\[
M = -\frac{8}{Pr} + \frac{24}{H} (2G_r - 12\gamma) \left( \frac{1}{70} \gamma - \frac{1}{504} G_r \right) + \frac{60}{PrG_r} \gamma \left( \frac{1}{2} H - \frac{1}{1680} G_r^2 + \frac{13}{3360} G_r \gamma \right). \tag{52}
\]

where \( H \) is defined by (41). The result allows to evaluate the third derivative at \( y = 0 \). Let us list the boundary conditions for \( C(y) \):

\[
\begin{align*}
C(0) &= \left( \frac{12}{G_r} \gamma - 4 \right), C'(0) = M, C''(0) = \frac{144}{G_r} \gamma - 24, \tag{53} \\
C'''(0) &= \left( 12 - \frac{1}{70H} G_r^2 + \frac{13}{140H} G_r \gamma \right) M + \frac{288}{H} \gamma - \frac{48}{H} G_r.
\end{align*}
\]

6 The solution of the boundary problem for \( C(y) \)

Let us rewrite the equation (23) substituting the expressions for \( C_1 \)\(^{44} \) and \( C_2 \)\(^{61} \) and introducing the Rayleigh number \( R_a = G_r \ Pr \). We have

\[
\frac{\partial^4 C(y)}{\partial y \partial y \partial y \partial y} + 3R_a \left( \frac{M}{4} y - 1 \right) \frac{\partial C(y)}{\partial y} = 0. \tag{54}
\]

As it is seen from estimation of \( M(Gr) \) one can neglect the term \( \frac{4M}{4} y << 1 \) in comparison with unit within the interval \( y \in [0, 10] \) if the Grashof number \( Gr > 500 \). Such approximation yields the equation with constant coefficients with solution as linear combination of exponents:

\[
C(y) = \sum_{i=0}^{3} A_i \exp[k_i y], \tag{55}
\]

where \( k_i \) are roots of the equation

\[
k^4 - 3R_a k = 0. \tag{56}
\]

It means that

\[
k_0 = 0, k_1 = \sqrt[3]{3R_a} \equiv s, k_{2,3} = \left( -\frac{1}{2} \pm i \frac{\sqrt{3}}{2} \right) s. \tag{57}
\]

Plugging (57) into (58), taking into account reality of \( C(y) \), results in:

\[
C(y) = A_0 + A_1 \exp[sy] + \exp[-\frac{sy}{2}] (B_1 \cos[\frac{\sqrt{3}}{2} sy] + B_2 \sin[\frac{\sqrt{3}}{2} sy]). \tag{58}
\]

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The second term $A_1 \exp[sy]$ in the above expression exponentially grows for positive $s$, therefore the value of $A_1$ should be analyze in details especially for large values of $s$.

Boundary conditions (53) subsequently give

$$C(0) = A_0 + A_1 + B_1 = \left(\frac{12}{G_r} \gamma - 4\right), \quad (59)$$

$$C'(0) = s \left(A_1 - B_1 \frac{1}{2} + B_2 \frac{\sqrt{3}}{2}\right) = M, \quad (60)$$

$$C''(0) = s^2 \left(A_1 - B_1 \frac{1}{2} - \frac{\sqrt{3}B_2}{2}\right) = \frac{144}{G_r} \gamma - 24, \quad (61)$$

$$C'''(0) = s^3 (A_1 + B_1) = \left(12 - \frac{1}{70H} G_r^2 + \frac{13}{140H} G_r \gamma\right) M + \frac{288}{H} \gamma - \frac{48}{H} G_r, \quad (62)$$

Solving the system of above equations with respect to $A_i$ and $B_i$ yields

$$B_2 = \frac{1}{3} \sqrt{3} \left(\frac{M}{s} + \frac{24}{s^2} - \frac{144}{G_r s^2 \gamma}\right), \quad (63)$$

$$B_1 = 8 \frac{M}{s^3} - \frac{1}{3} \frac{M}{s} + \frac{8}{s^2} + \frac{192}{H s^3} \gamma - \frac{32}{H s^3} G_r - \frac{48}{s^2} \gamma G_r - \frac{1}{105H} \frac{M}{s^3} G_r^2 + \frac{13}{210H} \frac{M}{s^3} G_r \gamma, \quad (64)$$

$$A_1 = \frac{1}{3} \frac{M}{s} + \frac{4}{3} \frac{M}{s^3} G_r^2 - \frac{16}{H s^3} G_r + \frac{48}{s^2} \frac{M}{s^3} G_r - \frac{1}{210H} \frac{M}{s^3} G_r^2 + \frac{13}{420H} \frac{M}{s^3} G_r \gamma, \quad (65)$$

$$A_0 = \left(\frac{12}{70H} G_r^2 + \frac{13}{140H} G_r \gamma\right) M + \frac{288}{H} \gamma - \frac{48}{H} G_r + \left(\frac{12}{G_r} \gamma - 4\right). \quad (66)$$

The parameter $\gamma$ in all above expressions is still not defined. The physical meaning of it relates to the velocity profile, namely it is the angle of inclination on the plate (21). In the literature the profiles are given either directly from experimental data [4, 6] or from theoretical models [2]. For example in the book [2] the velocity dimensionless counterpart $U = \frac{d\xi}{d\eta}$ (in our paper is denoted as $W$) in appropriate units is plotted against the function $\eta$ for different Prandtl numbers. The velocity $U = \frac{dV}{dx}$ is the function of $y$ which is normal directed to the plate and $x$ is along the plate (opposite to ours notations). Numerically differentiating the expression for $U$ with respect to $y$ on the level $x = l$ and taking into account the expression (21) we derive the link between constant $\gamma$ and the derivative $\frac{dU}{d\eta}|_{\eta=0} = \zeta$:

$$\gamma = \sqrt{2G_r^3/4} \zeta. \quad (67)$$

The alternative approach to the solution of the equation (54) is the power series expansion. To investigate the parameter values case of order one we use the Frobenius method. According to the method let us solve the equation (54) as a power series of $y$:
\[ C(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + \ldots \]  

(68)

The equations (54) and (68) yield the links: 
\[ a_4 = \frac{1}{8} a_1 R_a, \quad a_5 = \frac{1}{20} a_2 R_a - \frac{1}{160} M a_1 R_a, \]
\[ a_6 = \frac{1}{40} a_3 R_a - \frac{1}{240} M a_2 R_a. \]

The boundary conditions give:
\[ a_0 = C(0) = \frac{12}{G_r} \gamma - 4, \]  

(69)
\[ a_1 = \frac{\partial C(y)}{\partial y} |_{y=0} = M, \]  

(70)
\[ a_2 = \frac{1}{2} \frac{\partial^2 C(y)}{\partial y^2} |_{y=0} = \frac{72}{G_r} \gamma - 12, \]  

(71)
\[ a_3 = \frac{1}{6} \frac{\partial^3 C(y)}{\partial y^3} |_{y=0} = 2 - \frac{1}{420 H} G_r^2 + \frac{13}{840 H} G_r \gamma \]  

\[ + \frac{48}{H} \gamma - \frac{8}{H} G_r, \]
\[ a_4 = \frac{1}{8} M R_a, \quad a_5 = \frac{1}{20} \left( \frac{72}{G_r} \gamma - 12 \right) R_a - \frac{1}{160} M^2 R_a, \]
\[ a_6 = \frac{R_a}{10} \left( 1 - \frac{G_r^2}{1680 H} - \frac{3}{G_r} \gamma + \frac{13 G_r}{3360 H} \gamma \right) M + \frac{R_a}{5} \left( \frac{6}{H} \gamma - G_r \right). \]

7 The example of air

7.1 A solution of first type

For representation of results we choose air as a fluid with typical parameters \( T_{av} = 20^\circ C, p = 760 \) Tr, \( Pr = 0.7 \) and assume \( \Phi = 10 K \). The physical table values in such conditions are following: 
\[ a = 2.142 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}, \nu = 1.506 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}, R = 287.06 \times \frac{j}{kg^2 \text{ s}}, b = 3.430 \times 10^{-3} \frac{1}{\text{ K}}. \]

For such values the parameter \( H \) (41) is:
\[ H = 9.606 \times 10^6 G_r^2. \]  

(72)

In the paper [2], an expression of vertical velocity \( u(x, y) \) is derived by the method of similarity solution. For the example of Prandtl Number \( Pr = 0.7 \) (air) the coefficient is approximately \( \zeta = 3/5 \). Hence the parameter \( \gamma \) (67) is expressed as:
\[ \gamma = \frac{3\sqrt{2}}{5} G_r^{3/4} \]  

(73)

Our solution depends on the parameter \( M \) via equation (54) coefficient. The condition in which the coefficient is approximately constant therefore needs the parameter \( M \) estimation. For the estimation let us express the \( M \) (52) as
a function of the only parameter $G_r$ with the rest physical parameter values account.

$$M = -\frac{1.35 \times 10^{11} \sqrt{G_r} - 37634.4 G_r^{4} + 1.045 \times 10^{5} G_r^{14} + 3360 G_r^{12} - 8.59 \times 10^{11}}{1.05 \times 10^{9} G_r + 1.3 \times 10^{9} G_r^{7} - 11.6 G_r^{3} + 33.7 G_r^{7} + G_r^{12}} \tag{74}$$

Fig. 4 The dependence of the parameter $M$ against Grashof number $G_r$.

On the base of the result (see Fig. 4) we choose the Grashof number range $M/4 < 0.1$ for example evaluations of velocity and temperature profiles as a function of $x$ and $y$. It means that approximately $G_r > 200$.

For the given values of local Grashof number equal to $G_r = 200, 500$ and $1000$ the evaluation of the parameter $\gamma$ by the expression (67) gives: $\gamma = 45.127, 89.721$ and $150.89$. Next the calculation of the solution parameters yields the table:

| $G_r$ | $s$          | $M$          | $A_0$          | $A_1$          | $B_1$          | $B_2$          |
|-------|--------------|--------------|----------------|----------------|----------------|----------------|
| 200   | $\sqrt{3} \cdot G_r \cdot 0.7 = 7.4889(200), 10.164(500), 12.806(1000)$ | $0.29705(200), 6.1589 \times 10^{-2}(500), 1.2004 \times 10^{-2}(1000)$ | $-1.3009(200), -1.8475(500), -2.1894(1000)$ | $6.6523 \times 10^{-2}(200), 8.1911 \times 10^{-3}(500), -4.2815 \times 10^{-3}(1000)$ | $-5.8036 \times 10^{-2}(200), -7.4873 \times 10^{-3}(500), 4.3501 \times 10^{-3}(1000)$ | $-6.4522 \times 10^{-2}(200), -6.786 \times 10^{-3}(500), 8.5365 \times 10^{-3}(1000)$ |

Analysis of the dependence of the coefficient $A_1$ on Grashof number shows its change of sign (see the values for $A_1$ for $G_r = 500$ and $1000$). Plugging the expression for $M$ (74) into the formula for $A_1$ (65) and its numerical evaluation allow to plot the dependence $A_1(G_r)$. The correspondent plot for the case of air is given below (Fig. 5) and allow to determine a critical value of the number $G_{r,critical}$ for which the characteristic linear dimension of the problem has the concrete value $l = \left(\frac{G_{r,critical}^2 \cdot v^2}{g \cdot h \cdot \Phi}\right)^{1/3}$. 

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For $A_1 \approx 0$, form the plot we estimate $G_r = 714.275$, and next calculations give

$$\gamma = 150.89, s = 11.447, M = 2.9162 \times 10^{-2},$$

$$A_0 = -2.0307, B_2 = 3.0748 \times 10^{-3}, B_1 = 2.3332 \times 10^{-4}. \quad (75)$$

Plugging the all the values for four cases ($G_r = 200, 500, 714.275$ and $1000$) into the solution \((58)\) we plot the resulting curves (Fig.6).

As we announced in the introduction this study is focused on the problem of the starting point of the stationary flow in the vicinity of the point $x = 0, y = 0$.

For further illustration of the velocity and temperature profiles we choose the critical value of Grashof number $G_{r, cr} = 714.275$. Pugging the parameters values \((75)\) into formulas for the velocity and temperature \((21)\) we obtain the profiles of velocity $W(0, x) = \gamma x + \left(\frac{-G_{r, cr}}{2} \right) x^2 + \left(\frac{-G_{r, cr}}{6} \left(\frac{12}{G_{r, cr}} \gamma - 4\right)\right) x^3$ and $W(1, x) = \gamma x + \left(\frac{G_{r, cr}}{8} My - \frac{G_{r, cr}}{2} \right) x^2 + \left(\frac{-G_{r, cr}}{6} C (1)\right) x^3$ on the Fig.7 and temperature obtained in similar way on Fig.8.
7.2 The solution of the second type for $C(y)$

Let us recall the basic relations that define a solution of Navier-Stokes and Fourier-Kirchhoff equations via the power series in $y$. Plugging expressions for coefficients $\gamma, \alpha(y), \beta(y), C(y)$ and $B(y)$ in velocity and temperature expansions \[21\] one have

$$W(y,x) = x\gamma + x^2 \frac{G_r}{2} \left( \frac{1}{4} My - 1 \right) - x^3 \frac{G_r}{6} C(y)$$  \hspace{1cm} (76)

$$T(y,x) = C(y) x - \frac{1}{6} \frac{\partial^2 C(y)}{\partial y \partial y} x^3$$  \hspace{1cm} (77)

where: $C(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5$ with the coefficients from \[70\] and \[71\]. According to the physical essence of temperature profile the function $C(y)$ should be negative. It means that at least the coefficient $C(1) = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 < 0$. In such condition of $0 < y < 1$ the function $C(y) < 0$. Plugging the expressions for $\gamma, H$ and $M$ to $C(1)$ yields the complicated function of $G_r$. The result is illustrated in Fig.9.
From the plot Fig.9b it follows that the values of $C(1)$ are negative if $G_r > G_{r,0}$. For an illustration we investigate the approximate solution with the local Grashof number value equal to $G_r = 676.5$, for which $C(1) = -0.45529$ that lays inside the interval $(0, -1)$ which guarantee the negative temperature values for all $x \in (0, 1)$. For the plotting we need the following general parameters (74) for $G_r = 676.5$. The values are: $\gamma = 112.56, H = 7.4030 \times 10^5, M = 3.2974 \times 10^{-2}$ and the coefficients (70, 71) of the polynomial $C(y)$: $a_0 = -2.0034, a_1 = M = 3.3 \times 10^{-2}, a_2 = -0.0207, a_3 = 6.5948 \times 10^{-2}, a_4 = \frac{1}{2} MR_a = 1.9519, a_5 = -0.48202$. Substituting these values into (68) gives: $C(y) = -4.5605y^5 + 1.8226y^4 + 5.8325 \times 10^{-2}y^3 - 0.18232y^2 + 2.9162 \times 10^{-2}y - 2.0304$. In similar way we derive the expression of $C(y)$ for the critical Grashof number of the previous solution $G_{r,cr} = 714.275$. Below we plot both curves on the Fig.10.
As one can see the value of $a_2$ change the sign at about $Gr = 672$.

Finally let us substitute the function of $C(y)$ for the $Gr_{cr}, cr = 714.275$ into the expressions for the velocity and temperature and plot them at the levels $y = 0$ and $y = 0.5$ (see Fig.12 and Fig.13).
8 Analysis of the solution and conclusion

According to the order of power series expansion in $x$ the expressions for velocity and temperature in the form of (76), (77) satisfy the N-S and the F-K equations up to the term $x^3$. Hence the solution is valid in a stripe adjacent to the surface. The equation for the coefficient function $C$ (54) may be applied on the interval $y \in [0, \infty)$ but the approximate solutions we study here give satisfactory results in a vicinity of starting point of the flow $y = 0$.

The present theory allows to include higher terms of the expansion hence to obtain results that are valid in more wide range of coordinates. Our calculations were performed for the air case that corresponds to the choice of state equation in ideal gas form and Prandtl number $Pr = 0.7$. There is an obvious possibility to extend our results for other fluids, for example as in [5], [6].

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