1 Introduction

The partial theta function is the sum of the bivariate series \( \theta(q, x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j \). For any fixed value of the parameter \( q \) from the open unit disk it is an entire function in the complex variable \( x \). (Recall that the Jacobi theta function is the sum of the series \( \Theta(q, x) := \sum_{j=-\infty}^{\infty} q^{j^2} x^j \) whereas \( \theta(q^2, x/q) = \sum_{j=0}^{\infty} q^{2j} x^j \).) The function \( \theta \) finds applications in the theory of (mock) modular forms (see [3]), in Ramanujan type \( q \)-series (see [20]), in asymptotic analysis (see [2]), in statistical physics and combinatorics (see [18]), and also in questions concerning hyperbolic polynomials (i.e. real polynomials with all zeros real, see [16], [6], [15] and [8]). These questions have been considered by Hardy, Petrovitch and Hutchinson (see [4], [5] and [17]). For more facts about \( \theta \), see also [1].

The spectrum of \( \theta \) (defined by B. Z. Shapiro in [15]) is the set of values of \( q \) for which \( \theta(q,.) \) has a multiple zero in \( x \). Properties of \( \theta \) and its spectrum are studied in [7], [8], [9], [10], [11], [13] and [14]. Thus if \( q \in (0, 1), x \in \mathbb{R} \), for the corresponding spectral numbers \( 0 < \tilde{q}_1 < \tilde{q}_2 < \cdots < 1 \) one has \( \tilde{q}_j = 1 - \pi/2j + (log j)/8j^2 + O(1/j^2) \) and the respective double zeros of \( \theta \) are of the form \( y_j = -e^{\pi} e^{-(log j)/4}+O(1/j) \) (see [14]). If \( q \in (-1, 0), x \in \mathbb{R} \), there are the spectral numbers \( -1 < \cdots < \tilde{q}_k+1 < \tilde{q}_k < 0 \), \( \tilde{q}_k = 1 - \pi/8k + o(1/k) \) and the double zeros \( \tilde{y}_k, |\tilde{y}_k| \to e^{\pi/2}, \) see [13]. The spectral number \( \tilde{q} = \tilde{q}_1 := 0.3092493386 \ldots \) is connected with section-hyperbolic polynomials, see [15]. The function \( \theta(q,.) \) has a single double zero at \( -7.5032559833 \ldots \), its other zeros are real negative and simple. We denote by \( \mathbb{D}_a \) a closed disk centered at 0 and of radius \( a > 0 \) (in the \( x \)- or \( q \)-space). Subscripts indicate derivations (e.g. \( \theta_x = \partial \theta/\partial x \)). The zeros \( \xi_j \) of \( \theta \) can be expanded in Laurent series in \( q \), \( \xi_j = -1/q^j + O(q^{j-1}/2) \), convergent for \( q \in \mathbb{D}_{\rho_j}, 0, \rho_j \geq 0.108, \) see [10] and [12]; for \( q \in \mathbb{D}_{0.108}, \theta(q,.) \) has no multiple zeros.

**Theorem 1.** (1) In \( \mathbb{D}_{0.31}, \tilde{q} \) is the only spectral number of \( \theta \).

(2) For any disk \( \mathbb{D}_a, a \in (0,1), \) there exists \( b > 0 \) such that \( \theta \) has no multiple zeros in \( B_{a,b} := \mathbb{D}_a \times (\mathbb{C} \setminus \mathbb{D}_b), \) it has only isolated spectral numbers for \( q \in \mathbb{D}_a, \) and in \( E_{a,b} := \mathbb{D}_a \times \mathbb{D}_b \) it has at most finitely-many multiple zeros.

(3) The sequence \( \rho_j \) tends to 1 as \( j \) tends to \( \infty \).

A statement close to part (2) is likely to appear in a text co-authored by the present author. The statement is written by one of his co-authors and its proof is obtained independently of the one of part (2) of the theorem.
2 Proofs

Lemma 2. There are no spectral numbers for \( q \in \mathbb{D}_{c_0}, \ c_0 := 0.2078750206 \ldots \).

(A result similar to the lemma has been independently announced by A. Sokal and J. Forsgård.)

Proof. Indeed, set \( \phi := 2 \sum_{n=1}^{\infty} |q|^n / 2 \). If \( |x| = |q|^{-k-1/2}, k \in \mathbb{N} \), then in the series of \( \theta \) the term \( L := x \delta q^{k(k+1)/2} \) has the largest modulus (equal to \( |q|^{-k^2/2} \)) and the sum \( S \) of the moduli of all other terms is \( < |q|^{-k^2/2} \phi(|q|) \). One has \( 1 \geq \phi(|q|) \) exactly when \( |q| \leq c_0 \). Hence \( |L| > S \) for \( |q| \leq c_0 \), i.e. the term \( L \) is dominating. Thus for no zero \( \zeta \) of \( \theta \) one has \( |\zeta| = |q|^{-k-1/2} \). For any \( j \) fixed and for \( |q| \) close to one has \( \xi_j \sim q^{-j} \) (see [10]). Hence for \( |q| \leq c_0 \) one has \( |q|^{-j+1/2} < |\xi_j| < |q|^{-j-1/2}, i.e. all zeros are simple. \)

\[ \Box \]

Proposition 3. For \( q \in \mathbb{D}_{1/3} \) one has \( |q|^{-j+1/2} < |\xi_j| < |q|^{-j-1/2}, j \geq 3 \), hence the zeros \( \xi_j \) of \( \theta \), \( j \geq 3 \), remain simple and distinct.

Proof. Indeed, one can suppose (see Lemma [2]) that \( |q| \geq c_0 \). As in the proof of that lemma we show that if \( q \in \mathbb{D}_{1/3} \) and if \( |x| = |q|^{-k-1/2}, k = 2, 3, \ldots \), then \( \theta(q,x) \neq 0 \). Set \( x := |q|^{-k-1/2} \omega, |\omega| = 1, \) and \( \varepsilon := \omega q^{k+1/2}/|q|^{k+1/2} \); thus \( |\varepsilon| = 1 \). Set \( \varepsilon = e^{i\beta} \) and \( \sigma = \omega^{k} e^{i(\beta+1/2)}|q|^{-k(k+1)/2} \).

\[ \theta(q,x) = \sigma \left( 1 + \sum_{l=1}^{k} (e^l + e^{-l}) q^{l^2/2} + \sum_{l=k+1}^{\infty} e^l q^{l^2/2} \right) = \sigma \left( 1 + 2 \sum_{l=1}^{k} \cos(l\beta) q^{l^2/2} + \sum_{l=k+1}^{\infty} e^l q^{l^2/2} \right). \]

Set \( \theta := \sigma \cdot (A + B) \), where \( A := 1 + 2q^{1/2} \cos \beta + 2q^2 \cos(2\beta) \) and \( B := 2 \sum_{l=3}^{k} \cos(l\beta) q^{l^2/2} + \sum_{l=k+1}^{\infty} e^l q^{l^2/2} \). For \( q \in \mathbb{D}_{1/3} \) one has \( |B| < 2 \sum_{l=3}^{\infty} 3^{-l^2/2} = 0.01456 \ldots \). We show that \( |A| \geq 0.0146 \) for \( q \in \mathbb{D}_{1/3} \). Set \( q^{1/2} := |q|^{1/2} e^{i\gamma} \) and

\[ R := \Re(A) = 1 + 2|q|^{1/2} (\cos \beta) (\cos \gamma) + 2|q|^2 (\cos(2\beta)) (\cos(4\gamma)) \]

\[ I := \Im(A) = 2|q|^{1/2} (\cos \beta) (\sin \gamma) + 2|q|^2 (\cos(2\beta)) (\sin(4\gamma)) \]

If \( |\cos \beta| \leq 1/2 \), then \( |2q^{1/2} \cos \beta + 2q^2 \cos(2\beta)| \leq 1/\sqrt{3} + 2/9 < 0.8 \) and \( |A| \geq R > 0.2 \).

If \( |\cos \gamma| \leq 1/2 \), then again \( R \geq 1 - 1/\sqrt{3} - 2/9 > 0.2 \) hence \( |A| > 0.2 \). So suppose that \( |\cos \beta| > 1/2 \) (hence \( |\cos(2\beta)| < |\cos \beta| \)) and \( |\cos \gamma| > 1/2 \).

If \( (\cos \beta)(\cos \gamma) > 0 \), then \( |R| \geq 1 - 2/9 \) and \( |A| > 0.7 \). Suppose that \( (\cos \beta)(\cos \gamma) < 0 \). Set \( g_0 := 0.1329058248 \ldots \). One has \( |\sin(4\gamma)| \leq 4/3 |\sin \gamma| \) and (as \( |q|^{1/2}(1 - 4|q|^{3/2}) \) is decreasing on \([0,1/3])\),

\[ |I| \geq 2|q|^{1/2} |(\sin \gamma)(\cos \beta)(1 - 4|q|^{3/2}) \geq 2 \cdot 3^{-1/2} |(\sin \gamma)(\cos \beta)| \geq g_0 |\sin \gamma| \]

This is \( > 0.0146 \) for \( |\sin \gamma| > 0.1098522207 \ldots \), i.e. for \( |\cos \gamma| < 0.9939479310 \ldots \).

So suppose that \( |\cos \gamma| \geq 0.9939479310 \ldots \). Hence \( \cos(4\gamma) \geq 0.904624914 \ldots \) and for \( 1/2 < |\cos \beta| \leq 1/\sqrt{2} \) one has \(-1/2 \leq \cos(2\beta) \leq 0 \) and

\[ |R| \geq 1 - 2|q|^{1/2} / \sqrt{2} - 2|q|^2 (1/2) \geq 1 - (2/3)^{1/2} - (1/9) > 0.07 \]

If \( 1/\sqrt{2} < |\cos \beta| \leq 0.85 \), then \( 0 < \cos(2\beta) \leq 0.445 \) and

\[ R > 1 - 2 \cdot 0.85 \cdot |q|^{1/2} > 1 - 1.7/\sqrt{3} > 0.018 \].

2
If $0.85 < |\cos \beta| \leq 0.93$, then $0.445 < \cos(2\beta) \leq 0.7298$ and

$$R > 1 - 2 \cdot 0.93 \cdot |q|^{1/2} + 2 \cdot |q|^2 \cdot 0.904624914 \ldots \cdot 0.445 > 0.015.$$  

If $0.93 < |\cos \beta| \leq 0.98$, then $0.7298 < \cos(2\beta) \leq 0.9208$ and

$$R > 1 - 2 \cdot 0.98 \cdot |q|^{1/2} + 2 \cdot |q|^2 \cdot 0.904624914 \ldots \cdot 0.7298 > 0.015.$$  

Finally, if $0.98 < |\cos \beta| \leq 1$, then $0.9208 < \cos(2\beta)$ and

$$R > 1 - 2|q|^{1/2} + 2 \cdot |q|^2 \cdot 0.904624914 \ldots \cdot 0.9208 > 0.03.$$  

\hfill \Box

**Corollary 4.** There are no spectral numbers for $|q| \leq c_1 := 0.2256613757$.

**Proof.** Indeed, for $|q| \in J := (0.16, c_1)$, $|x| = 7.95$, in the series of $\theta$ the term $qx$ is dominating and $\theta \neq 0$ for such $(q,x)$. As $\theta(0.22,\ldots)$ has simple zeros in $(-21,-19)$ and $(-7,-6)$ and $0.22^{-2.5} = 44.04\ldots$, these are $\xi_1$ and $\xi_2$, see the proposition, so for $|q| \in J$ one has $|\xi_1| < 7.95 < |\xi_2|$ and all zeros of $\theta$ are simple. \hfill \Box

**Corollary 5.** For $q_* \in \mathbb{D}_{1/3}$, if $\theta(q_*,\ldots)$ has a multiple zero at $x_*$, then $|x_*| \leq \gamma := 10.28693902\ldots$.

**Proof.** Indeed, it is $\xi_1$ and $\xi_2$ that coalesce, see Theorem 1 of [8] (and its proof therein) and Proposition 3. As $\sum_{j=1}^{\infty} 1/\xi_j = q$ (see [11] and $|\xi_j| \geq |q_*|^{-j+1/2}$, $j \geq 3$, one has $2/|x_*| \geq f(|q_*|) := |q_*| - \sum_{j=3}^{\infty} |q_*|^{-j-1/2} = |q_*| - |q_*|^{5/2}/(1 - |q_*|)$. Hence $|x_*| \leq 2/f(c_1) = \gamma$ ($f$ is increasing on $[0,0.35]$). \hfill \Box

**Proof of Theorem 7.** The proof is based on the idea to use finite truncations of the series of $\theta$ as its approximations. One expects the values of $q$ for which these truncations have multiple zeros in $x$ to be close to spectral numbers of $\theta$. We use approximations which are explicit polynomials. In the estimations below we restrict these polynomials to line segments. The restrictions are polynomials in one variable whose real and imaginary parts are real polynomials in one variable; in the present paper they are of degree at most 10. To prove that the values of such a polynomial $\tilde{P}$, when restricted to a given segment of the real axis, are smaller (or larger) than a given real number $\tilde{a}$, it suffices to prove that the equation $\tilde{P} = \tilde{a}$ has no real solution on the segment (this can be shown using MAPLE or Mathematica) and that for at least one point of the segment one has $\tilde{P} < \tilde{a}$ (or $\tilde{P} > \tilde{a}$).

Set $U := 1 + qx + q^3 x^2 + q^6 x^3 + q^{10} x^4$ and $V(q) := \text{Res}(U, U_x/q, x)/q^{20} = 256q^{10} - 192q^7 - 128q^6 + 288q^5 - 60q^4 - 80q^3 + 52q^2 - 12q + 1$. Set $K := \{ 0 \leq \text{Re,q} \leq 1/3 \} \supset \mathbb{D}_{1/3}$ and $\partial K := ''\text{border of K}''$. The only zero of $V$ in $K$ is $\lambda_0 := 0.309016994374947\ldots$ (it is a double one).

If $\theta(q_*,\ldots)$ has a multiple zero $x_*$, then $U + a_* + U_x/q + b_*$ have a common zero for

$$a_* = \varphi(q_*, x_*) := \sum_{j=5}^{\infty} q_0^{2j(j+1)/2} x_*^j, \quad b_* = \psi(q_*, x_*) := \sum_{j=5}^{\infty} j q_0^{2j(j+1)/2} x_*^{j-1}.$$  

Consider $\tilde{V}_{a,b}(q) := \text{Res}(U + a, U_x/q + b, x)/q^{26}$ for $|a| \leq a_0 := 0.0081\ldots = \varphi(1/3, \gamma)$, $|b| \leq b_0 := 0.0119\ldots = \psi(1/3, \gamma)$ and $|q| = 1/3$, see Corollary 5. One has
\[ V_{a,b} = V + aV_1 + a^2V_2 + a^3V_3 + b^2W_1 + b^3W_2 + b^4W_3 + ab^2W_4, \quad \text{where} \]
\[ V_1 = 768q^{10} - 384q^7 - 256q^6 + 432q^5 - 60q^4 + 34q^3 - 4q - 80q^3, \]
\[ V_2 = 768q^{10} - 192q^7 - 128q^6 + 144q^5 - 27q^4, \quad V_3 = 256q^{10}, \]
\[ W_1 = -1 - 16q^5 + 24q^4 + 7q - 14q^2, \quad W_2 = -q + 4q^2 - 8q^4, \]
\[ W_3 = q^4, \quad W_4 = 6q^4 - 16q^5. \]

We show that \(|V(q)| > |\tilde{V}_{a,b}(q) - V(q)| (A)\) on \(\partial K\). By the Rouché theorem, \(\tilde{V}_{a,b}\) has as many zeros inside \(K\) as \(V\), i.e. two (counted with multiplicity). The real and imaginary parts of the functions \(V, V_1\) etc. (denoted by \(V^R, V^I\) etc.) restricted to each segment \(K^\pm_h := \{\Re q = \pm 1/3, \Im q \in [-1/3,1/3]\}\) or \(K^\pm_h := \{\Im q = \pm 1/3, \Re q \in [-1/3,1/3]\}\) of \(\partial K\) are real univariate polynomials.

On \(K^R_h\) one has
\[
|V^R| + |V^I| > 2, \quad |V^R_1| + |V^I_1| < 61.3, \]
\[
|V^R_k| + |V^I_k| < 15.8, \quad k = 2, 3, \quad |W^R_j| + |W^I_j| < 21.1, \quad j = 1, 2, 3, 4,
\]
so \(\Sigma := (|V^R| + |V^I|)/\sqrt{2} - \sum_{j=1}^{3} ((|V^R_j| + |V^I_j|)(a_0)^j + (|W^R_j| + |W^I_j|)(b_0)^j+1)
\]
\[
- (|W^R_4| + |W^I_4|)a_0(b_0)^2 > 0
\]
from which \((A)\) follows because \(|V| \geq (|V^R| + |V^I|)/\sqrt{2}, |V_j| \leq |V^R_j| + |V^I_j|\) etc. On \(K^\pm_v\) one has:
\[
|V^R| + |V^I| > 20, \quad |V^R_1| + |V^I_1| < 61.3, \]
\[
|V^R_k| + |V^I_k| < 15.8, \quad k = 2, 3, \quad |W^R_j| + |W^I_j| < 21.5, \quad j = 1, 2, 3, 4
\]
and again \(\Sigma > 0\). We consider \(K^\pm_v \cap \{\Im q \in [0,1/3]\}\) instead of \(K^\pm_v\) (because \(\theta\) is real) and we subdivide it into \(K^0 \cup K^1 \cup K^2 \cup K^3\), where
\[
K^0 = K^\pm_v \cap \{\Im q \in [0,0.05]\}, \quad K^1 = K^\pm_v \cap \{\Im q \in [0.05,0.1]\},
\]
\[
K^2 = K^\pm_v \cap \{\Im q \in [0.1,0.2]\}, \quad K^3 = K^\pm_v \cap \{\Im q \in [0.2,1/3]\}.
\]
On \(K^3, K^2, K^1\) and \(K^0\) one has respectively
\[
|V^R| + |V^I| > 0.14, \quad |V^R_1| + |V^I_1| < 6.29, \quad |V^R_k| + |V^I_k| < 3.79, \quad |W^R_j| + |W^I_j| < 1.9, \]
\[
|V^R| + |V^I| > 0.011, \quad |V^R_1| + |V^I_1| < 0.342, \quad |V^R_k| + |V^I_k| < 0.626, \quad |W^R_j| + |W^I_j| < 0.47, \]
\[
|V^R| + |V^I| > 0.00252, \quad |V^R_1| + |V^I_1| < 0.042, \quad |V^R_k| + |V^I_k| < 0.0865, \quad |W^R_j| + |W^I_j| < 0.17, \]
\[
|V^R| + |V^I| > 0.00044, \quad |V^R_1| + |V^I_1| < 0.0299, \quad |V^R_k| + |V^I_k| < 0.0464, \quad |W^R_j| + |W^I_j| < 0.083.
\]
In each case \(\Sigma > 0\). Hence \(\tilde{V}_{a,b}\) has two zeros \(\alpha^{(\nu)} \in \mathbb{D}_{1/3}, \nu = 1, 2\) (counted with multiplicity). Consider the restrictions of \(V^R, V^I, V^R_1, \ldots\) to the segment \(S := [0.29 + 0 \cdot i, 0.29 + i/3]\). We set
\[ S = S^0 \cup S^1 \cup S^2 \cup S^3 \cup S^4 , \quad \text{where} \quad S^0 := S \cap \{ \text{Im} q \in [0, 0.025]\}, \]
\[ S^1 = S \cap \{ \text{Im} q \in [0.025, 0.05]\}, \quad S^2 = S \cap \{ \text{Im} q \in [0.05, 0.1]\}, \]
\[ S^3 = S \cap \{ \text{Im} q \in [0.1, 0.2]\}, \quad S^4 = S \cap \{ \text{Im} q \in [0.2, 1/3]\}. \]

On \( S^4, S^3, S^2, S^1 \) and \( S^0 \) one has respectively

\[
|V^R| + |V^I| > 0.14, \quad |V^R_1| + |V^I_1| < 5.81, \quad |V^R_k| + |V^I_k| < 3.72, \quad |W^R_k| + |W^I_k| < 2.05, \\
|V^R| + |V^I| > 0.015, \quad |V^R_1| + |V^I_1| < 0.49, \quad |V^R_k| + |V^I_k| < 0.42, \quad |W^R_k| + |W^I_k| < 0.47, \\
|V^R| + |V^I| > 0.0041, \quad |V^R_1| + |V^I_1| < 0.083, \quad |V^R_k| + |V^I_k| < 0.089, \quad |W^R_k| + |W^I_k| < 0.2, \\
|V^R| + |V^I| > 0.0019, \quad |V^R_1| + |V^I_1| < 0.0315, \quad |V^R_k| + |V^I_k| < 0.0315, \quad |W^R_k| + |W^I_k| < 0.08, \\
|V^R| + |V^I| > 0.0005, \quad |V^R_1| + |V^I_1| < 0.024, \quad |V^R_k| + |V^I_k| < 0.015, \quad |W^R_k| + |W^I_k| < 0.048. 
\]

In all cases \( \Sigma > 0 \). Hence in the rectangle \( \{ \text{Re} q \in [-1/3, 0.29], \text{Im} q \leq 1/3 \} \) the functions \( \tilde{V}_{a,b} \) and \( V \) have the same number of zeros, i.e. none.

**Remark 6.** Knowing that there are no spectral values for \( |q| \leq 0.29 \) one can repeat the reasoning of the proof of Corollary 4 to show that \( |x_\ast| \leq 2/f(0.29) =: \lambda = 8.841250518 \ldots \).

**Lemma 7.** There are no zeros of \( \theta_{xx} \) for \( 0 < |q| \leq 0.31 \) and \( |x| \leq \lambda \).

**Proof.** Indeed, \( \theta_{xx}/2q^3 = \sum_{j=0}^\infty (j+1)(j+2)q^{2(j+5)/2}x^j/2 \). For \( |q| \leq 0.31, |x| \leq \lambda \), the first term is dominating. \( \square \)

**Lemma 8.** For \( 0 < |q| \leq 0.31 \) and \( |x| \leq \lambda \) the equalities \( \theta_x = \theta_q = 0 \) do not hold simultaneously hence the sets \( \{ \theta = d \} \) are locally smooth.

**Proof.** Indeed, \( (\theta_q/x - \theta_x/q)/q^2x = \sum_{j=1}^\infty j(j+1)q^{(j-1)(j+4)/2}x^{j-1}/2 \) and the first term is dominating. \( \square \)

**Lemma 9.** For \( 0 < |q| \leq 0.31 \) and \( 5.946 \leq |x| \leq \lambda \) one has \( \theta_q \neq 0 \). For \( 0 < |q| \leq 0.31 \), the curves \( \{ \theta = d \} \) and \( \{ \theta_x = 0 \} \) intersect transversally.

**Proof.** Consider the Jacobian \( J := \theta_x \theta_{q_x} - \theta_q \theta_x \). Its restriction to \( \{ \theta_x = 0 \} \) reduces to \( -\theta_{xx} \theta_q \). By Lemma 7, \( \theta_{xx} \neq 0 \). One has \( \theta_q/x = 1 + \sum_{j=2}^\infty j(j+1)q^{(j+1)/2-1}x^{j-1}/2 \). We show that the second term is dominating for \( |q| \leq 0.31 \) and \( 5.946 \leq |x| \leq \lambda \) which implies \( \theta_q \neq 0 \). Set \( y := q^2x, \chi := \sum_{j=3}^\infty j(j+1)q^{(j-1)(j-2)/2}y^{j-1}/2 \). The sum 1 + \chi is maximal for \( |\chi| = 0.31 \). For \( |q| \) fixed the function \( \chi \) is convex in \( |y| \), so it suffices to check for \( |\chi| = 0.31 \) that \( 3y > 1 + \chi \) for \( x = 5.946 \) and \( x = \lambda \) which is true. For \( |x| < 5.946 \), if \( \theta_q = 0 \), then by Lemma 8, \( \theta_x \neq 0 \). \( \square \)

Lemma 7 implies that each zero of \( \theta_x \) being simple, it is a function in \( q \) continuous in \( \Lambda := D_{0.31} \cap \{ \text{Re} q \geq 0.29 \} \) and holomorphic in its interior. Consider its zero \( \zeta(q) \) which equals \(-7.5 \ldots \) for \( q = \bar{q} \). By Lemma 8 the intersections of the graph of the function \( x = \zeta(q) \) with the sets \( \{ \theta = d \} \) are points. For \( d = 0 \) this means that there is only one point in \( \Lambda \) belonging to the spectrum. This proves part (1) of the theorem.

Prove part (2) (from which part (3) follows). The set of multiple zeros of \( \theta \) is a locally analytic subset in the space \( (q,x) \). It cannot be of positive dimension. Indeed, for each \( q \) fixed there are only isolated multiple zeros (if any) and for \( 0 < |q| \leq 0.18 \) there are no multiple zeros. Hence the spectrum of \( \theta \) consists of isolated points. Further we use properties of the Jacobi
theta function $\Theta(q, x) := \sum_{j=-\infty}^{\infty} q^{j^2} x^j$ which imply corollaries about the function $\Theta^*(q, x) = \sum_{j=-\infty}^{\infty} q^{(j+1)^2/2} x^j = \Theta(\sqrt{q}, \sqrt{q}x)$. Namely, one has $\Theta^*(q, x) = qx\Theta^*(q, qx)$ (*). and the zeros of $\Theta^*$ are all the numbers $\mu_k := -q^{-k}$, $k \in \mathbb{Z}$ (see the formula for the zeros of $\Theta$ in [21] or Chapter X of [19]). Further when we write $x \in \Omega_k(\delta)$ we mean $|x - \mu_k| \leq \delta$, $\delta > 0$.

In a neighbourhood of $\mu_k$ we represent the function $\Theta^*$ in the form $\Theta^*(q, x) = M(x + q^{-k})\tau(q, x)$, where $M \in \mathbb{C}[0]$ and the series $\tau = 1 + d_1(x + q^{-k}) + d_2(x + q^{-k})^2 + \cdots$, $d_j \in \mathbb{C}$, is convergent. Observe that $x = x + q^{-k-s} - q^{-k-s} = -q^{-k-s}(1 - q^{k+s}(x + q^{-k-s}))$ and $\tau(q, q^s x) = 1 + d_1 q^s(x + q^{-k-s}) + d_2 q^{2s}(x + q^{-k-s})^2 + \cdots$. Set $X := x + q^{-k-s}$. Using $s$ times equation (*) we represent $\Theta^*$ close to $\mu_k + s$ in the form

$$\Theta^*(q, x) = q^{s(s+1)/2} X^s M q^s X \tau(q, q^s x) = M(-1)^s q^{s(-2k-s+3)/2}(1 - q^{k+s}) X^s X \tau(q, q^s x).$$

Assume that for all $s \in \mathbb{N} \cup 0$ the variable $X$ takes values in one and the same disk $|X| \leq \delta$. Hence as $s \to \infty$ the functions $(1 - q^{k+s})^s$ and $\tau(q, q^s x)$ tend uniformly to 1. Thus for $|X| \leq \delta$ (i.e. for $x \in \Omega_k(\delta)$) one has $\Theta^* = M(-1)^s q^{s(-2k-s+3)/2}(1 + o(1))X$. For any $B > 0$ there exists $k_0 \in \mathbb{N} \cup 0$ such that for $k \geq k_0$ one has $|q^{s(-2k-s+3)/2}| \geq B$ for any $s \in \mathbb{N}$. Hence there exists $\kappa > 0$ such that for $|\eta| \leq \kappa$, for $k \geq k_0$ and for any $s \in \mathbb{N}$ the equation $\Theta^* = \eta$ has a unique solution $X = X(\eta)$ with $|X| \leq \delta$.

Set $\theta(q, x) = \Theta^*(q, x) + \Xi(q, x)$, where $\Xi(q, x) = -\sum_{j=-\infty}^{j=1} q^{j(j+1)/2} x^j$. For any $q \in \mathbb{D}_1$ the series of $\Xi$ and $\Xi'$ converge for $|x| > 1$, and for any $\varepsilon > 0$ there exists $G \geq 1$ such that $|\Xi| \leq \varepsilon$ and $|\Xi'| \leq \varepsilon$ if $|x| \geq G$. (Both series (of $1/x$) are without constant term and the moduli of all coefficients of $\Xi$ are less than 1.) This reasoning remains valid for $q$ replaced by a sufficiently small closed disk $D \subset \mathbb{D}_1$ centered at $q$

For $k \in \mathbb{N}$ sufficiently large the equation $\theta(q, x) = 0$, i.e. $\Theta^*(q, x) = -\Xi(q, x)$ has a unique solution $x = x(q) \in \Omega_k(\delta)$. (For such $k$ and for $x \in \Omega_k(\delta)$ the equation $\Theta^*(q, x) = \eta$ has a solution $x(q, \eta) = \mu_k + O(\eta)$ holomorphic in $\eta$ for $|\eta|$ sufficiently small.) Substituting $-\Xi(q, x)$ for $\eta$ one obtains an equation $x = \mu_k + \Delta(q, x)$; by choosing $G$ and $k_0$ sufficiently large one makes $\max_{x \in \Omega_k(\delta)} |\Delta(q, x)|$ arbitrarily small. Hence this equation has a unique solution in $\Omega_k(\delta)$. This reasoning is valid for $q \in D$.

Suppose that $a > c_0$ (for $a \leq c_0$ see Lemma 2). The closure $\overline{D}$ of $\mathbb{D}_a \setminus \mathbb{D}_{c_0}$ can be covered by finitely-many disks $D$. Hence there exists $j_0 \in \mathbb{N}$ such that for $j \geq j_0$ and for $q \in \overline{D}$ the function $\theta$ has a zero in $\Omega_j(\delta)$. Choosing, if necessary, a larger $j_0$ one can assume that all sets $\Omega_j(\delta)$ are disjoint (for any fixed $q \in \overline{D}$). Hence all zeros $\xi_j$ of $\theta$ with $j \geq j_0$ are simple and distinct, so 1) one can set $b := \min_{q \in D, j \geq j_0} |\xi_j|$; 2) one has $\rho_j \geq a$ for $j > j_0$; 3) for $|q| < a$, $\xi_j$ depends meromorphically on $q$ with a pole only at 0, i. e. its Laurent series is convergent in $\mathbb{D}_a$. Letting $a$ tend to 1 one obtains the proof of part (3).

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