On the Riemann hypothesis for self-dual weight enumerators of genus three

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Abstract

In this note, we give an equivalent condition for a self-dual weight enumerator of genus three to satisfy the Riemann hypothesis. We also observe the truth and falsehood of the Riemann hypothesis for some families of invariant polynomials.

Key Words: Zeta function for codes; Invariant polynomial ring; Riemann hypothesis.
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1 Introduction

Zeta functions for linear codes were introduced by Iwan Duursma [6] in 1999 and they have attracted attention of many mathematicians:

Definition 1.1 Let $C$ be an $[n,k,d]$-code over $\mathbf{F}_q$ ($q = p^r$, $p$ is a prime) with the Hamming weight enumerator $W_C(x,y)$. Then there exists a unique polynomial $P(T) \in \mathbf{R}[T]$ of degree at most $n-d$ such that

$$
\frac{P(T)}{(1-T)(1-qT)}(y(1-T) + xT)^n = \cdots + \frac{W_C(x,y) - x^n}{q-1} T^{n-d} + \cdots. 
$$

We call $P(T)$ and $Z(T) = P(T)/(1-T)(1-qT)$ the zeta polynomial and the zeta function of $W(x,y)$, respectively.

If $C$ is self-dual, then $P(T)$ satisfies the functional equation (see [7, §2]):

Theorem 1.2 If $C$ is self-dual, then we have

$$
P(T) = P\left(\frac{1}{qT}\right)q^{gT^{-2g}},
$$

where $g = n/2 + 1 - d$.

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The number $g$ is called the *genus* of $C$. It is appropriate to formulate the Riemann hypothesis as follows:

**Definition 1.3** The code $C$ satisfies the Riemann hypothesis if all the zeros of $P(T)$ have the same absolute value $1/\sqrt{q}$.

The reader is referred to [8] and [9] for other results by Duursma.

**Remark.** The definition of the zeta function can be extended to much wider classes of invariant polynomials: let $W(x, y)$ be a polynomial of the form

$$W(x, y) = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i \in \mathbb{C}[x, y] \quad (A_d \neq 0)$$

which satisfy $W^{\sigma_q}(x, y) = \pm W(x, y)$ for some $q \in \mathbb{R}$, $q > 0$, $q \neq 1$, where

$$\sigma_q = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix} \quad (\text{the MacWilliams transform})$$

and the action of $\sigma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ on a polynomial $f(x, y) \in \mathbb{C}[x, y]$ is defined by $f^\sigma(x, y) = f(ax + by, cx + dy)$. Then we can formulate the zeta function and the Riemann hypothesis for $W(x, y)$ in the same way as Definitions 1.1 and 1.3. For the results in this direction, the reader is referred to [1]–[5], for example. We should also note that we must assume $d, d^\perp \geq 2$, where $d^\perp$ is defined by $W^{\sigma_q}(x, y) = B_0 x^n + B_{d^\perp} x^{n-d^\perp} y^{d^\perp} + \cdots$, when considering the zeta function of $W(x, y)$.

We do not know much about the Riemann hypothesis for self-dual weight enumerators, but one of the remarkable results is the following theorem by Nishimura [11, Theorem 1], an equivalent condition for a self-dual weight enumerator of genus one to satisfy the Riemann hypothesis:

**Theorem 1.4 (Nishimura)** A self-dual weight enumerator $W(x, y) = x^{2d} + A_d x^d y^d + \cdots$ satisfies the Riemann hypothesis if and only if

$$\frac{\sqrt{q} - 1}{\sqrt{q} + 1} \left( \frac{2d}{d} \right) \leq A_d \leq \frac{\sqrt{q} + 1}{\sqrt{q} - 1} \left( \frac{2d}{d} \right).$$

Nishimura also deduces the following, the case of genus two ([11, Theorem 2]):

**Theorem 1.5 (Nishimura)** A self-dual weight enumerator $W(x, y) = x^{2d+2} + A_d x^{2d+2} y^d + \cdots$ satisfies the Riemann hypothesis if and only if the both roots of the quadratic polynomial

$$A_d X^2 - \left( (d - q) A_d + \frac{d + 1}{d + 2} A_{d+1} \right) X - (d + 1)(q + 1) \left( A_d + \frac{A_{d+1}}{d + 2} \right) + (q - 1) \left( \frac{2d + 2}{d} \right)$$

are contained in $[-2\sqrt{q}, 2\sqrt{q}]$.

The purpose of this article is to establish an analogous equivalent condition for the case of genus three. Our main result is the following:

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Theorem 1.6 A self-dual weight enumerator \( W(x, y) = x^{2d+4} + A_d x^{d+4} y^d + \cdots \) satisfies the Riemann hypothesis if and only if all the roots of the cubic polynomial

\[
 f_3 X^3 + f_2 X^2 + f_1 X + f_0
\]

are contained in \([-2\sqrt{q}, 2\sqrt{q}]\), where \( f_i \) is defined as follows.

\[
 f_3 = A_d,
\]
\[
 f_2 = (q - d)A_d - \frac{d + 1}{d + 4} A_{d+1},
\]
\[
 f_1 = \frac{1}{2} (d^2 - 2qd + d - 6q) A_d + (d - q + 1) \frac{d + 1}{d + 4} A_{d+1} + \frac{(d + 1)(d + 2)}{(d + 3)(d + 4)} A_{d+2},
\]
\[
 f_0 = \frac{1}{2} (q + 1)(d^2 + 3d - 4q + 2) A_d + (q + 1)(d + 1)(d + 2) \frac{A_{d+1}}{d + 4} + (q + 1) \frac{(d + 1)(d + 2)}{(d + 3)(d + 4)} A_{d+2} - (q - 1) \left( \frac{2d + 4}{d + 4} \right).
\]

By this theorem, we can verify the truth of the Riemann hypothesis of \( W(x, y) \) only by three parameters \( A_d, A_{d+1}, A_{d+2} \) (the number of parameters which are needed coincides with the genus \( g \), see [1]). Moreover, in many cases, we have \( A_{d+1} = 0 \) and the verification of the Riemann hypothesis is simplified.

Theorem 1.6 leads us to the consideration of the truth or falsehood of the Riemann hypothesis as the numbers \( q \) and \( n \) vary. As was mentioned in Remark before, \( q \) can take other numbers than prime powers. In this context, we can notice the tendency that the Riemann hypothesis becomes harder to hold if \( n \) or \( q \) are larger. Some of the results in [3] and [4] also support it. Theorem 1.6 can illustrate this tendency by considering a certain sequence of invariant polynomials, that is

\[
 W_{n,q}(x, y) = (x^2 + (q - 1)y^2)^n. \tag{1.8}
\]

In Section 2, we give a proof of Theorem 1.6. In Section 3, we observe the behavior of \( W_{n,q}(x, y) \), give some theoretical and experimental results, and state a conjecture on their Riemann hypothesis.

## 2 Proof of Theorem 1.6

Let \( W(x, y) = x^{2d+4} + \sum_{i=d}^{2d+4} A_i x^{2d+4-i} y^i \) be a self-dual weight enumerator. Using the functional equation (1.2) (note that \( g = 3 \) in our case), we can assume that the zeta polynomial \( P(T) \) of \( W(x, y) \) is of the form

\[
 P(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + a_2 q T^4 + a_1 q^2 T^5 + a_0 q^3 T^6.
\]

We obtain another expression of \( P(T) \) because \( 1/q \alpha \) is a root of \( P(T) \) if \( P(\alpha) = 0 \):

\[
 P(T) = a_0 q^3 \prod_{i=1}^{3} (T^2 + b_i T + 1/q). \tag{2.1}
\]
Comparing the coefficients, we get

\[ b_1 + b_2 + b_3 = \frac{a_1}{a_0 q}, \]
\[ b_1b_2 + b_2b_3 + b_3b_1 = \frac{(a_2 - 3a_0 q)}{a_0 q^2}, \]
\[ b_1b_2b_3 = \frac{(a_3 - 2a_1 q)}{a_0 q^3}. \]

Thus \( b_i \) is the roots of the following cubic polynomial:

\[ a_0 q^3 X^3 - a_1 q^2 X^2 + (a_2 - 3a_0 q)qX - a_3 + 2a_1 q. \] (2.2)

Considering the distribution of the roots of each factor \( T^2 + b_i T + 1/q \) in (2.1), we can see that a self-dual weight enumerator \( W(x, y) \) of genus three satisfies the Riemann hypothesis if and only if \( b_1, b_2 \) and \( b_3 \) are contained in \([-2/\sqrt{q}, 2/\sqrt{q}]\). By change of variable in (2.2), we get the following:

**Lemma 2.1** \( W(x, y) \) satisfies the Riemann hypothesis if and only if all the roots of the polynomial

\[ a_0 X^3 - a_1 X^2 + (a_2 - 3a_0 q)X - a_3 + 2a_1 q \] (2.3)

are contained in \([-2\sqrt{q}, 2\sqrt{q}]\).

Our next task is to express the coefficients \( a_i \) in (2.3) by way of \( A_i \) in \( W(x, y) \). This can be done by comparing the coefficients of the both sides in (1.1). Our method is similar to that of Nishimura [11]. The result is the following (here, \( \alpha_{d+i} = A_{d+i}/(q - 1)\binom{n}{d+i} \)):

\[ a_0 = \alpha_d, \]
\[ a_1 = (d - q)\alpha_d + \alpha_{d+1}, \]
\[ a_2 = \frac{1}{2}d(d - 2q + 1)\alpha_d + (d - q + 1)\alpha_{d+1} + \alpha_{d+2}, \]
\[ a_3 = \frac{1}{6}d(d+1)(d-3q+2)\alpha_d + \frac{1}{2}(d+1)(d-2q+2)\alpha_{d+1} \\
   + (d - q + 2)\alpha_{d+2} + \alpha_{d+3}. \]

The coefficient \( a_3 \) is expressed by four parameters \( A_d, \ldots, A_{d+3} \). By invoking the binomial moment, the number of parameters is reduced to three. In fact, we have the following:

**Lemma 2.2** Let \( W(x, y) \) be a self-dual weight enumerator of the form (1.3) and we assume \( g = 3 \). Then we have

\[ \sum_{i=d+1}^{d+3} A_i \binom{2d + 4 - i}{d + 1} = q \sum_{i=0}^{d+1} A_i \binom{2d + 4 - i}{d + 3}. \] (2.4)

**Proof.** The equalities satisfied by the binomial moment of \( W(x, y) \) is given by

\[ \sum_{i=0}^{n-j} \binom{n - i}{j} A_i = q^{\frac{2}{2} - j} \sum_{i=0}^{j} \binom{n - i}{n - j} A_i \quad (j = 0, 1, \ldots, n) \] (2.5)
We get (2.4) by putting $n = 2d + 4$ and $j = d + 1$. Using $A_0 = 1$, $A_1 = \cdots = A_{d-1} = 0$, we can see that (2.4) gives a linear relation among $A_d, \cdots, A_{d+3}$, so we can express $A_{d+3}$ by $A_d, A_{d+1}$ and $A_{d+2}$. Thus we get
\[
a_3 = -\frac{1}{2}(d + 1)(dq + d - 2q + 2)\alpha_d - (qd + d + 2)\alpha_{d+1} - (q + 1)\alpha_{d+2} + 1.
\]
Rewriting (2.3) using above $a_i$, we obtain Theorem 1.6.

3 Some examples and observations

We examine the polynomials (1.8), which has essentially only one parameter $q$ and is easy to see the phenomenon. Using Nishimura’s results ($g = 1, 2$) and our theorem ($g = 3$), we can see that the range of $q$ for which the Riemann hypothesis is true are the following:
\[
g = 1: \quad 4 - 2\sqrt{3} (\approx 0.53590) \leq q \leq 4 + 2\sqrt{3} (\approx 7.46410) \quad (q \neq 1),
\]
\[
g = 2: \quad -4 + 2\sqrt{5} (\approx 0.47214) \leq q \leq \alpha^2 (\approx 3.46812) \quad (q \neq 1),
\]
where
\[
\alpha = \frac{1}{6} \left(1 + \sqrt[3]{5(29 + 6\sqrt{6})} + \sqrt[3]{5(29 - 6\sqrt{6})}\right),
\]
and
\[
g = 3: \quad \beta_1 (\approx 0.47448) \leq q \leq \beta_3^2 (\approx 2.47607) \quad (q \neq 1),
\]
where $\beta_1$ is the unique real root of the polynomial
\[
100t^5 + 495t^4 + 2056t^3 - 2928t^2 + 1408t - 256
\]
and $\beta_3$ is the positive root of the polynomial
\[
13t^4 + 4t^3 - 20t^2 - 24t - 8.
\]
The cases $g = 1$ and 2 are not very complicated, but the last case needs some explanation. The relevant coefficients of $W_{4,q}(x, y)$ are
\[
A_d = A_2 = 4(q - 1), \quad A_3 = 0, \quad A_4 = 6(q - 1)^2.
\]
Using these values, we get the explicit form of the polynomial (1.7) as follows:
\[
g(X) := 5X^3 + 5(q - 2)X^2 - 2(11q - 6)X - 7q^2 + 20q - 8.
\]
Let $D_g$ be the discriminant of $g(X)$, $X_1$ and $X_2$ be the roots of $g'(X)$ (we assume $X_1, X_2$ are real and $X_1 \leq X_2$). Then, by Theorem 1.6, $W_{4,q}(x, y)$ satisfies the Riemann hypothesis if and only if
\[
D_g \geq 0, \quad -2\sqrt{q} \leq X_1, \quad X_2 \leq 2\sqrt{q},
\]
\[
g(-2\sqrt{q}) \leq 0, \quad g(2\sqrt{q}) \geq 0.
\]
We have
\[ \frac{D_g^{35}}{35} = 100q^5 + 495q^4 + 2056q^3 - 2928q^2 + 1408q - 256, \]
so \( D_g \geq 0 \) is equivalent to
\[ q \geq \beta_1 \] (3.3)
with the above mentioned \( \beta_1 \). The roots \( X_i \) are given by
\[ X_1 = \frac{-5(q-2) - \sqrt{25q^2 + 230q - 80}}{15}, \]
\[ X_2 = \frac{-5(q-2) + \sqrt{25q^2 + 230q - 80}}{15}. \]
The range of \( q \) satisfying \( -2\sqrt{q} \leq X_1 \) is (note that we also have \( 25q^2 + 230q - 80 \geq 0 \))
\[ \frac{\sqrt{609} - 23}{5} \leq q \leq \beta_2, \] (3.4)
where \( \beta_2 \) is the square of the unique real root of the polynomial
\[ 10t^3 - 19t^2 - 20t - 6 \] (3.5)
(this polynomial comes from the equation \( -2\sqrt{q} = X_1 \)). The explicit value is
\[ \beta_2 = \frac{1}{300} \left( 761 + \sqrt[3]{386669681 + 396000\sqrt{17318}} + \sqrt[3]{386669681 - 396000\sqrt{17318}} \right) \]
\( (\beta_2 \approx 7.38366, \) this expression of \( \beta_2 \) can be obtained by constructing the cubic polynomial having the squares of roots of (3.5) as its roots: \( 100t^3 - 761t^2 + 172t - 36 \). The inequality \( X_2 \leq 2\sqrt{q} \) gives \( (\sqrt{609} - 23)/5 \leq q \). Finally, putting \( \sqrt{q} = t \), we have
\[ g(-2\sqrt{q}) = 13t^4 + 4t^3 - 20t^2 - 24t - 8, \]
\[ g(2\sqrt{q}) = 13t^4 - 4t^3 - 20t^2 + 24t - 8. \]
The inequalities \( g(-2\sqrt{q}) \leq 0 \) and \( g(2\sqrt{q}) \geq 0 \) give
\[ 0 \leq q \leq \beta_3^2 \quad \text{and} \quad q \geq \beta_4^2 \approx 0.356397, \] (3.6)
respectively. Gathering the inequalities (3.3), (3.4) and (3.6), we obtain the estimate (3.1).
We can see from the above estimation that the range of \( q \) for which the Riemann hypothesis is true becomes smaller as \( n \) becomes larger. We show some results of numerical experiment for \( W_{n,q}(x, y) \). In the following table, “RH true” means the range of \( n \) where the Riemann hypothesis for \( W_{n,q}(x, y) \) seems to be true:

| \( q \) | RH true |
|---|---|
| 2 | 2 \leq q \leq 6 |
| \frac{2}{3} | 2 \leq n \leq 8 |
| \frac{13}{10} | 2 \leq n \leq 36 |
| \frac{24}{20} | 2 \leq n \leq 71 |
| \frac{4}{5} | 2 \leq n \leq 29 |
| \frac{1}{2} | 2 \leq n \leq 5 |
These numerical examples also support the above observation. We conclude the manuscript with the following conjecture:

**Conjecture 3.1** For any \( n \geq 2 \), there exists \( q (q \approx 1) \) and \( W_{n,q}(x,y) \) satisfies the Riemann hypothesis.

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