REMARKABLE LOCALIZED INTEGRAL IDENTITIES FOR 3D COMPRESSIBLE EULER FLOW AND THE DOUBLE-NULL FRAMEWORK

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Abstract. We derive new, localized geometric integral identities for solutions to the 3D compressible Euler equations under an arbitrary equation of state when the sound speed is positive. The identities are coercive in the first derivatives of the specific vorticity (defined to be vorticity divided by density) and the second derivatives of the entropy, and the error terms exhibit remarkable regularity and null structures. Our framework allows one to simultaneously unleash the full power of the geometric vectorfield method for both the wave- and transport- parts of the flow on compact regions, and our approach reveals fundamental new coordinate invariant structural features of the flow. In particular, the integral identities yield localized control over one additional derivative of the vorticity and entropy compared to standard results, assuming that the initial data enjoy the same gain. Similar results hold for the solution’s higher derivatives. We derive the identities in detail for two classes of spacetime regions that frequently arise in PDE applications: i) compact spacetime regions that are globally hyperbolic with respect to the acoustical metric, where the top and bottom boundaries are acoustically spacelike but not necessarily equal to portions of constant Cartesian-time hypersurfaces; and ii) compact regions covered by double-acoustically null foliations. As we describe in the paper, the results have implications for the geometry and regularity of solutions, the formation of shocks, the structure of the maximal classical development of the data, and for controlling solutions whose state along a pair of intersecting characteristic hypersurfaces is known. Our analysis relies on a recent new formulation of the compressible Euler equations that splits the flow into a geometric wave-part coupled to a div-curl-transport part. The main new contribution of the present article is our analysis of the positive co-dimension, spacelike boundary integrals that arise in the div-curl identities. By exploiting interplay between the elliptic and hyperbolic parts of the new formulation and using careful geometric decompositions, we observe several crucial cancellations, which in total show that after a further integration with respect to an acoustical time function, the boundary integrals have a good sign, up to error terms that can be controlled due to their good null structure and regularity properties.

Keywords: acoustical metric; characteristic initial value problem; double-null foliation; energy identity; Hodge identity; null condition; null structure; shock formation; vectorfield method

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1. Introduction

The compressible Euler equations are the fundamental equations of compressible fluid mechanics. They are arguably on par with Einstein’s equations of general relativity in terms of their mathematical richness, physical relevance, and variety of subtle nonlinear behaviors that solutions can exhibit. The equations remain a fertile source of outstanding mathematical challenges, even though they are among the earliest PDE systems written down (they were formulated by Euler in 1757 [15]). Despite the complexity of the equations, in recent years, the rigorous mathematical theory of solutions has enjoyed dramatic progress, driven by insights and identities that have their origins in Lorentzian geometry, general relativity, the theory of geometric wave equations, and, in some key cases, the remarkable structures exhibited by a new formulation of the equations [26,33] as geometric wave equations coupled to div-curl-transport equations. As examples of progress, we note Christodoulou’s work [5] on shock formation in 3D without symmetry assumptions for irrotational relativistic Euler solutions, his joint extension [9] of this result to the case of the non-relativistic compressible Euler equations in 3D, his subsequent resolution of the restricted shock development problem [7] in both the relativistic and non-relativistic cases, the second author’s joint extension [27] of Christodoulou’s shock formation result to allow for the presence of vorticity in 2D in the barotropic case, the second author’s recent joint work [13] on low regularity solutions with vorticity and entropy in 3D, Wang’s extension of this work [35] to further lower the regularity of the vorticity in the barotropic case, and the existence of initially $C^\infty$ solutions that form an infinite-density singularity in finite time [28,29].

In this paper, we study the 3D compressible Euler equations under an arbitrary physical [4] equation of state in which the pressure $p$ is a given function of the density $\rho$ and entropy $s$. Our main results augment the geometric insights and tools developed in [5,26,27,33] by allowing for a sharp localization of the key structures that are needed to control the “div-curl-transport-part” of the flow, that is, the vorticity and entropy. We now informally and tersely summarize our main results; see Theorem 1.2 for a more precise – but still schematic – statement of the results, and Theorems 7.2, 8.1, 9.10, and 10.6 for precise statements. See also Subsect. 1.5 for a summary of the key ideas in the proofs.

We derive a new family of coercive, localized (i.e., on compact spacetime regions) integral identities that yield spacetime $L^2$-type control over the vorticity and entropy at one derivative level higher compared to standard estimates. By “one derivative level,” we mean that the gain in regularity for the vorticity and entropy holds for all of their Cartesian partial derivatives, not just in the direction of the material derivative vectorfield (see [5]), for which the improved regularity is a standard result but still schematic – statement of the results, and Theorems 7.2, 8.1, 9.10, and 10.6 for precise statements. See also Subsect. 1.5 for a summary of the key ideas in the proofs.

While analogous integral identities are entirely standard in the context of wave equations and are easily derivable by the vectorfield multiplier method (see Subsect. 9.2), the compressible equations are not wave

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1 Barotrophic equations of state are such that the pressure can be expressed a function of the density alone.
2 By a “physical equation of state,” we mean that the speed of sound, defined in [1], is assumed to be positive, at least for an open set of positive density values (which would be a set of density values for which our results hold).
3 The additional regularity for the specific vorticity and entropy gradient in the direction of the material derivative vectorfield is straightforward to derive by using the transport equations [32a] and [32c] for algebraic substitution and showing that the terms on the right-hand sides of [32a] and [32c] have the desired regularity. In fact, one can derive the additional regularity of the material derivative of the specific vorticity and entropy gradient with respect to norms of type $L^\infty(\text{Time})L^2(\text{Space})$, which is a stronger result (at least locally in time) compared to achieving control in a spacetime $L^2$-type norm; see Footnote 15.
equations. That is, it is well-known that solutions exhibit two kinds of propagation phenomena: the propagation of sound waves, which is present even in the simplified setting of irrotational and isentropic solutions, and the transporting of \( s \) and the vorticity \( \omega := \text{curl} v \) (where \( v \) is the velocity), which is present in general solutions and which occurs at a “slower” speed\(^4\) compared to sound waves. More precisely, as we alluded to above, it is more accurate, at least at the top derivative level, to describe the evolution of vorticity and entropy as being driven by “div-curl-transport” equations. That is, \([26, 33]\) showed that the compressible Euler equations can be formulated as geometric wave equations coupled to div-curl-transport equations for the specific vorticity \( \Omega \) (defined below to be \( \omega \) divided by a dimensionless density) and the entropy\(^5\).

We now further explain – still informally – the significance of our main results.

- **(Extending the geometric vectorfield method to the div-curl-transport part on spatially compact regions)** It is precisely the div-curl-transport part of the flow that lies outside of the scope of the traditional geometric vectorfield method (developed for wave and wave-like equations), and our integral identities allow us to handle this part of the flow, notably the difficult boundary integrals that arise when we derive elliptic Hodge-type identities on spatially compact regions. Handling these (positive co-dimension) boundary integrals requires a combination of elliptic, hyperbolic, and geometric techniques that rely on the special structures of the formulation of compressible Euler flow derived in \([26, 33]\), as well as the precise structure of equations \((4a)-(4c)\), which are a standard first-order formulation of compressible Euler flow; see also Remark \([5.1]\).

- **(Regularity).** Specifically, our results yield a large family of energy identities for the first derivatives of the specific vorticity \( \Omega \) and the second derivatives of the entropy \( s \) on the (compact) spacetime regions under consideration, where the error terms involve the up-to-first order derivatives of velocity \( v \), the density \( \rho \), and \( s \). That is, with \( \partial \) denoting the Cartesian coordinate spacetime gradient and \( \eta \) denoting the Cartesian coordinate spatial gradient, our results yield \( L^2 \) control of \( \partial \Omega \) and \( \partial_s s \) in terms of \( L^2 \) norms of \( \partial^{\leq 1} \rho \), \( \partial^{\leq 1} v \), \( \partial^{\leq 1} \eta \), and \( \partial \eta \), where the latter two quantities, discovered in \([26, 33]\) and recalled below in Def. \([2.1]\) are special combinations of fluid solution variables that solve transport equations (see \((33a)-(33c)\)) with source terms exhibiting unexpectedly good regularity and null properties. In turn, under suitable assumptions on the initial data, the quantities \( \partial^{\leq 1} \rho \), \( \partial^{\leq 1} v \), \( \partial^{\leq 1} \eta \), \( \partial \eta \), and \( \partial \eta \) can be shown to have sufficient \( L^2 \) regularity by virtue of standard energy estimates for the wave equations \((31a)-(31c)\) and the transport equations \((33a)-(33c)\) and \((34a)-(34c)\). In particular, our results allow one to locally propagate a gain of one derivative worth of Sobolev regularity for the vorticity and entropy compared to standard estimates, which is important for the study of the maximal development in the context of shock formation; see Subsect. \([1.4]\).

- **(Null structure).** The “error terms” in the integral identities for \( \partial \Omega \) and \( \partial_s s \) have remarkable quasilinear null structures and regularity properties that, as we explain in Subsect. \([1.4]\), are crucial for applications. For example, for regions that have acoustically null hypersurface boundaries, the identities feature error integrals along the null hypersurfaces, but these error terms involve only tangential derivatives of quantities that have sufficient regularity. By “sufficient regularity,” we mean in particular that the error terms can be treated with transport equation estimates or wave equation energy estimates, where the latter, though well-known to be degenerate along null hypersurfaces, yield control over tangential derivatives.

- **(Flexibility of the approach).** In total, the integral identities allow one to simultaneously implement the full power of the geometric vectorfield method for both the wave- and div-curl-transport-parts of the solution on domains that are important for PDE applications, in a manner such that the error terms exhibit good regularity and quasilinear null structures. Moreover, our framework is

\[^4\] Sound waves can propagate along acoustically null hypersurfaces, while \([26]\) and the equations of Theorem \([2.8]\) imply that specific vorticity and entropy are transported along acoustically timelike curves.

\[^5\] More precisely, the div-curl-transport system \((4a)-(4c)\) for the entropy is expressed in terms of the entropy gradient vectorfield, which we denote by \( S \) and define below in \((3c)\).

\[^6\] For example, in the proof of Lemma \([5.5]\) we use equations \((4a)-(4c)\).

\[^7\] In practice, rather than working with the density, we prefer to work with the logarithmic density \( p \), which we define in \((3a)\); since we consider only solutions with strictly positive density, these two variables are essentially equivalent for the purposes of this article. Moreover, rather than working with the second derivatives of \( s \), we prefer to work with \( \partial S \), where \( S \) is the entropy gradient vectorfield defined in \((3c)\).
well-suited to handle the kinds of “custom modifications” that are typically needed in applications. In particular, one could commute the equations with appropriate geometric vectorfields to obtain similar integral identities for the solution’s higher derivatives, one could incorporate weights\(^8\) into the identities, etc. That is, our framework affords flexibility in the ways it can be implemented.

Central to the approach of the present paper are the div-curl-transport systems for the specific vorticity and entropy gradient derived in [26, 33], see equations (33a)-(33b) and (34a)-(34b). These systems are adapted to constant Cartesian time hypersurfaces. From the perspective of analysis, the main new contributions of the present work are as follows.

1. Even though the div-curl systems from [26, 33] (which we recall as (33a)-(33b) and (34a)-(34b)) are PDEs relative to flat hypersurfaces of constant Cartesian time, our results yield coercive integral identities on regions foliated by arbitrary acoustically spacelike hypersurfaces.
2. The work [33] outlined how to derive the integral identities on much simpler spacetime regions of the form \([0, T] \times \Sigma\), where the 3D “spatial manifold” \(\Sigma\) had an empty boundary (for example, \(\Sigma = \mathbb{R}^3\) or \(\Sigma = \mathbb{R} \times T^2\)); see Subsubsect. 1.5.1 for the main ideas. In the present article, we handle the spatial boundary terms that arise in various div-curl identities on spatially compact domains. In particular, we show that the boundary integrals have a compatible amount of regularity and, at the same time, exhibit the remarkable null structures that have proven to be important in applications. As we will explain, our localized results do not hold for typical div-curl-transport systems, especially on the full class of spacetime regions that we treat in this paper; our results are possible only because we have exploited some newly identified structures in the compressible Euler equations.

We close this introduction by highlighting the significance of allowing the equation of state to depend on entropy:

- Entropy is an unavoidable ingredient in fluid models that incorporate thermodynamic effects. Moreover, non-constant entropy is generated past the formation of a shock [5], even if before the shock the solution is smooth, irrotational, and isentropic.
- In the integral identities, the most difficult error terms – by far – involve the second derivatives of the entropy. We devote Subsects. 5.3 and 5.4 to proving that these terms exhibit the remarkable structures that are crucial for our main results. This requires a host of insights about hidden “quasilinear” geometric and analytic structures in the equations.

1.1. Basic notation and a standard first-order formation of the equations. Before further discussing our main results, we set up our study of compressible Euler flow by introducing a standard first-order formulation of the equations, specifically the system (4a)-(4c). We again stress that the first-order formulation is not the one we use to derive our main results; rather, we use the equations of Theorem 2.8, which are consequences of the first-order formulation and which were derived in [33].

1.1.1. Basic notation and conventions. Throughout, \(\{x^\alpha\}_{\alpha=0,1,2,3}\) denotes the standard Cartesian coordinates on \(\mathbb{R}^{1+3}\), where \(x^0 := t\) denotes time (we also refer to \(t\) as the “Cartesian time function”) and \(\{x^\alpha\}_{\alpha=1,2,3}\) are the Cartesian spatial coordinates. We denote the standard partial derivative vectorfields with respect to the Cartesian coordinates by \(\partial_\alpha := \frac{\partial}{\partial x^\alpha}\), and we often use the alternate notation \(\partial_i := \partial_0\). In addition, \(\Sigma_t\) denotes the standard flat three-dimensional hypersurface of constant Cartesian time \(t\). Moreover, lowercase Latin “spatial” indices such as \(\alpha\) vary over 1, 2, 3, lowercase Greek “spacetime” indices such as \(\alpha\) vary over 0, 1, 2, 3, where 0 is the “time” index, and we use Einstein’s summation convention in that repeated indices are summed over their ranges. If \(f\) is a scalar function and \(X\) is a vectorfield, then \(X f := X^\alpha \partial_\alpha f\) denotes the derivative of \(f\) in the direction \(X\). If \(V\) is a \(\Sigma_t\)-tangent vectorfield with Cartesian spatial components \(\{V^\alpha\}_{\alpha=1,2,3}\), then \(\text{curl} \, V := \epsilon_{ijk} \partial^j V^k\) denotes its standard Euclidean curl, where \(\epsilon_{ijk}\) denotes the fully antisymmetric symbol normalized by \(\epsilon_{123} = 1\) and \(\delta^{ij}\) denotes the Kronecker delta. In addition, \(\text{div} \, V := \partial_\alpha V^\alpha\) denotes its Euclidean divergence. Finally, \(\epsilon_{\alpha \beta \gamma \delta}\) denotes the fully antisymmetric symbol normalized by \(\epsilon_{0123} = 1\). See Subsubsect. 2.1.2 and Convention 3.17 for our conventions for lowering and raising indices.

\(^8\)In fact, in our main results, we allow for the presence of an arbitrary weight function, denoted by \(\mathcal{W}\).
1.1.2. Setup and definitions of the basic fluid variables. The compressible equations can be formulated as evolution equations for the density $\rho : \mathbb{R}^{1+3} \to [0, \infty)$, the Cartesian velocity components $v^i : \mathbb{R}^{1+3} \to \mathbb{R}$, $(i = 1, 2, 3)$, and the entropy $s : \mathbb{R}^{1+3} \to \mathbb{R}$. When setting up the equations, authors frequently also use the pressure $p : \mathbb{R}^{1+3} \to [0, \infty)$. The resulting PDE system is under-determined unless one supplies an additional equation. Here we close the system in the standard fashion by assuming an equation of state $p = p(\rho, s)$, that is, a function yielding the pressure in terms of the density and entropy. In this article, we consider only equations of state such that the speed of sound $c$ is positive when the density is positive:

$$c := \sqrt{\left. \frac{\partial p}{\partial \rho} \right|_s} > 0, \quad \text{when } \rho > 0. \quad (1)$$

In $\left[1\right]$, $\left. \frac{\partial p}{\partial \rho} \right|_s$ denotes the derivative of the pressure with respect to the density at fixed entropy. The positivity of $c$ is fundamental for the hyperbolicity of the PDEs.

In what follows, we will refer to the vorticity $\omega$, which is defined to be the $\Sigma_t$-tangent vectorfield with the following Cartesian spatial components, $(i = 1, 2, 3)$:

$$\omega^i := (\text{curl} v)^i = e_{ijk} \delta^j \partial_k v^k. \quad (2)$$

In this article, we study only solutions with strictly positive density. Thus, rather than studying the density and vorticity, we can fix an (arbitrary) “background density” $\bar{\rho} > 0$ and instead study the logarithmic density $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the specific vorticity $\Omega$. The advantage of working with $\rho$ is that some of our equations take a simplified form when expressed in terms of this variable. Our analysis also crucially relies on a $\Sigma_t$-tangent vectorfield $S$ equal to the spatial gradient of $s$. We now precisely define these quantities.

**Definition 1.1 (Logarithmic density, specific vorticity, and entropy gradient).** Relative to the Cartesian coordinates, we define the logarithmic density $\rho$, which is a scalar function, the specific vorticity $\Omega$, which is a $\Sigma_t$-tangent vectorfield, and the entropy gradient $S$, which also is a $\Sigma_t$-tangent vectorfield, as follows, $(i = 1, 2, 3)$:

$$\rho := \ln \left( \frac{\rho}{\bar{\rho}} \right), \quad (3a)$$
$$\Omega^i := \frac{\omega^i}{(\rho/\bar{\rho})} = \frac{\omega^i}{\exp(\rho)}, \quad (3b)$$
$$S^i := \delta^{ia} \partial_a s, \quad (3c)$$

where $\delta^{ij}$ denotes the Kronecker delta.

From now on, we will view $c = c(\rho, s)$. Similarly, we will view $p$ and its partial derivatives with respect to $\rho$ and $s$ to be functions of $(\rho, s)$.

1.1.3. Standard first-order formulation of the equations. Relative to the standard Cartesian coordinates $(t, x^1, x^2, x^3)$ on $\mathbb{R}^{1+3}$, the compressible equations can be expressed in the following standard first-order form, where $i = 1, 2, 3$ (see Subsect. 1.1.1 regarding our index and summation conventions):

$$B \rho = -\text{div} v, \quad (4a)$$
$$B v^i = -c^2 \delta^{ia} \partial_a \rho - \exp(-p) \frac{\partial p}{\partial \rho} \delta^{ia} \partial_a s, \quad (4b)$$
$$B s = 0. \quad (4c)$$

In $\left(4a\right)$-$\left(4c\right)$ and throughout, $\delta^{ij}$ denotes the Kronecker delta and $B$ denotes the *material derivative* vectorfield, defined relative to the Cartesian coordinates by

$$B := \partial_t + v^a \partial_a. \quad (5)$$

We refer readers to $\left[9\right]$ for a discussion of the physical considerations that lead to the system $\left(4a\right)$-$\left(4c\right)$. We clarify that in $\left[9\right]$, the compressible Euler equations are stated in terms of the density $\rho$ rather than the density $\bar{\rho}$. If $c$ is positive only on an open set of positive density values, then our results hold for solutions whose density is contained in that open set.

\footnote{Of course, if $c$ is positive only on an open set of positive density values, then our results hold for solutions whose density is contained in that open set.}
1.2. Informal description of the spacetime regions $\mathcal{M}$. We now give an informal description of the spacetime regions $\mathcal{M}$ on which our integral identities hold. Our assumptions are geometric and refer to the acoustical metric $g$ of Def. 2.2, a fluid-solution-dependent Lorentzian metric on $\mathcal{M}$ that governs the propagation of sound waves.

Roughly, one could say that $\mathcal{M}$ is allowed to be any region on which one can derive a coercive energy identity for solutions to the wave equation $\Box_g \phi = 0$; see (30) for the definition of $\Box_g$.

We now give a brief description of the first of two classes of regions $\mathcal{M}$ on which our integral identities hold; see Subsect. 3.3 for the precise assumptions and Fig. 1 for a schematic depiction of $\mathcal{M}$, where for reasons explained below, $\mathcal{M}_T$ is alternate notation for $\mathcal{M}$. Later in the paper, we will define all of the objects in the figure. For now, we only note that the top boundary $\tilde{\Sigma}_T$ and the bottom boundary $\tilde{\Sigma}_0$ are allowed to be arbitrary acoustically spacelike portions and that the lateral boundary $H$ is allowed to be an arbitrary acoustically spacelike or acoustically null hypersurface portion. Above and throughout, the term “acoustic,” as well as the Lorentzian notions of “causal,” “spacelike,” “null,” etc., refer to aforementioned acoustical metric $g$. On such domains, our integral identities could be combined with standard energy estimates for wave and transport equations to yield a priori estimates for compressible Euler solutions with vorticity and entropy with initial data given along $\tilde{\Sigma}_0$; see Theorem 9.10 for one such result.

![Figure 1](image-url)

**Figure 1.** A spacetime region $\mathcal{M}$ on which the integral identities hold

The second class of regions $\mathcal{M}$ on which our identities hold comprises regions that are double-null foliated, that is, foliated by a pair of acoustic eikonal functions, whose level sets are acoustically null; see Fig. 2. As in the case of the first class of domains, on double-null-foliated domains, our integral identities could be combined with standard energy estimates for wave and transport equations to yield a priori estimates for compressible Euler solutions with vorticity and entropy. Here, by “a priori estimates,” we mean estimates for the solution on $\mathcal{M}$ in terms of the “state of the solution” along a pair of intersecting acoustically null hypersurfaces. This opens the door for the further study of the (acoustically) characteristic initial value problem; see Subsect. 1.4 for further discussion. To derive the integral identities in the case of double-null foliations, we have to modify the approach that we use to treat the domains featured in Fig. 1. This requires additional constructions, which we provide in Sect. 10.

1.3. A schematic overview of the main results. In this subsection, we provide a schematic overview of the new integral identities on spacetime regions $\mathcal{M}$ of the type depicted in Fig. 1. Similar ideas can be

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10By “Lorentzian,” we mean that the $4 \times 4$ matrix $g_{\alpha\beta}$ of Cartesian components has signature $(-, +, +, +)$. 
used to handle double-null foliated regions of the type featured in Fig. 2. For this reason, do not provide an
overview of the double-null case; see Sect. 1.4 for the details.

1.3.1. A slightly more detailed description of $\mathcal{M}$. We now further describe our assumptions on the regions
depicted in Fig. 1; see Subsect. 3.3 for the complete, precise assumptions. We assume that there is a smooth
“acoustical time function” $\tau$ with a past-directed, $g$-timelike gradient such that $\mathcal{M}$ is foliated by the level
sets of $\tau$. Here and throughout, $g$ is the acoustical metric of Def. 2.2. That is, with $\Sigma_{\tau'} := \mathcal{M} \cap \{(t, x) \in \mathbb{R}^{1+3} \mid \tau(t, x) = \tau'\}$, we assume that there is a $T > 0$ such that $\mathcal{M} = \cup_{\tau' \in [0, T]} \Sigma_{\tau'}$ and such that each $\Sigma_{\tau'}$ is $g$-spacelike, that is, spacelike with respect to the acoustical metric. We assume that the $\partial \mathcal{M} = \Sigma_0 \cup \Sigma_T$, where the lateral boundary $\mathcal{H}$ is smooth and either $g$-spacelike or $g$-null at each of its points (see Def. 2.3).

We also assume that for $\tau' \in [0, T]$, $\Sigma_{\tau'}$ intersects $\mathcal{H}$ transversally in a topological sphere $\mathcal{S}_{\tau'}$, that is, that $\mathcal{S}_{\tau'} := \Sigma_{\tau'} \cap \mathcal{H} = \partial \Sigma_{\tau'}$, where $\mathcal{S}_{\tau'}$ is diffeomorphic to $\mathbb{S}^2$. We set $\mathcal{M}_{\tau} := \cup_{\tau' \in [0, \tau]} \Sigma_{\tau'}$ and $\mathcal{H}_{\tau} = \mathcal{H} \cap \mathcal{M}_{\tau} = \cup_{\tau' \in [0, \tau]} \mathcal{S}_{\tau'}$. Note that $\mathcal{M} = \mathcal{M}_{T}$ and $\mathcal{H} = \mathcal{H}_{T}$.

1.3.2. Schematic statement of the main results. For brevity, we will mainly restrict our attention to providing
an overview of the integral identities involving the square integral of the gradient $\partial \partial v$, which denotes the
gradient of the specific vorticity $\Omega$ (see (3b)) with respect to the Cartesian spatial coordinates. According to
1.3.1. overview of the double-null case; see Sect. 10 for the details.

For this reason, do not provide an overview of the double-null case; see Sect. 10 for the details.

The term $\partial \partial v$ is in fact one derivative more regular, that is, as regular as $\rho$. Since transport equation solutions are generally no more regular than their source terms, equation (6) of Remark 1.3.2. Overview of the integral identities involving the square integral of the gradient $\partial \partial v$.

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The terms “Tangential” depend on the fluid variables, on a vectorfield frame tangent to the lateral boundary portion $H^g$, and on the derivatives of some these variables in directions tangent to $H^g$. These terms can be bounded via energy estimates for the wave and transport equations of Theorem 2.8, even in the case that $H^g$ is acoustically null. The crucial point is that in the acoustically null case, the wave energies along $H^g$ degenerate and control only $H^g$-tangential derivatives, that is, compared to the $g$-spacelike case, they lose their full positivity and become only positive semi-definite. Put differently, all of the terms in “Tangential” are controllable by the degenerate wave energies or transport energies. In the case that the $\Sigma^g$ are standard flat hypersurfaces of constant Cartesian time, the degeneracy of the wave energies is precisely captured by the “null-flux” coercivity result (209a).

The integral $\int_{S^g} |\Omega|^2$ on LHS (7) is positive definite with respect to $\Omega$. In the precise identity, namely (165a), the positivity stems from that of the scalar functions $\zeta$, $\tilde{z}_\tau$, and $\Theta$, defined respectively in (49) and (57a)-(57b), whose positivity in turn stems from the basic geometric properties of $M$ with respect to $g$.

Moreover, the following additional results hold:

- One can incorporate a weight function $\Psi$ into the integral identities; see Theorem 8.1.
- An identity similar to (7) holds with the entropy gradient vectorfield $\nabla := \partial S$ in place of $\partial \Omega$ (and thus the LHS of the identity controls the integral of $|\nabla S|^2 = |\partial S|^2$); see (165b).
- Similar results hold on spacetime regions covered by double-null foliations; see Sect. 10 and Fig. 2 in particular.

Remark 1.3 (The results are most interesting when the lateral boundary is $g$-null). When $H^g$ is $g$-spacelike, the tangential derivative structure of the terms “Tangential” is perhaps to be expected. This is because when $H^g$ is $g$-spacelike, the compressible Euler equations (4a)-(4c) roughly have the following algebraic content: some $H^g$-transversal derivative of the solution can be re-expressed as $H^g$-tangential derivatives. Put differently, in the spacelike case, one can use the compressible Euler equations to eliminate transversal derivatives in terms of tangential derivatives. In contrast, when $H^g$ is $g$-null, this heuristic argument is false, and the tangential-differentiation structure of the terms “Tangential” becomes much more interesting and difficult to uncover.

Remark 1.4 (We have avoided some geometry to assist future applications). Some of the error terms in the integral identities involve first derivatives of various vectorfields in geometric directions. We have intentionally chosen to express these error terms relative to the Cartesian coordinates, even though they could be rewritten in a much more geometric fashion by using covariant derivatives and referring to the second fundamental forms of the foliations. We are motivated by the following consideration: covariant derivatives would involve the Cartesian coordinate Christoffel symbols, which, in the context of the study of shocks, can be singular. This is reminiscent of the renormalizations used in the works on impulsive gravitational wave solutions to Einstein’s equations [24, 25], which showed that the quantities with the best analytic structures are generally not the same as the quantities with the most geometric interpretation.

Remark 1.5 (The smoothness assumption could be substantially weakened). In Theorem 1.2, we have assumed that the solution is sufficiently smooth (e.g., assuming $\rho, v, s$ are $C^3$ would suffice). We made this assumption only for convenience, as it facilitates our arguments involving integration by parts; standard techniques could be used to extend our results to solutions of suitable finite Sobolev regularity.

Remark 1.6 (Similar results for the solution’s higher derivatives). For convenience, in this paper, we have exhibited the identities only at the lowest derivative levels, that is, at the level of the undifferentiated equations of Theorem 2.8. However, as our proofs make clear, similar results hold for the higher-order derivatives of the solution, assuming that the initial data enjoy compatible regularity. The reason is that all of the special cancellations that we observe stem from linear factors on the right-hand side of various equations from Theorem 2.8 and/or the equations (4a)-(4c) (which are a standard first-order formulation of...
1.4. Applications and potential applications of the integral identities. We now highlight some of the main applications/potential applications of the integral identities.

I) Sharpened picture of the basic regularity theory for compressible Euler solutions. As the statement of Theorem 1.2 suggests, our results can be used to exhibit a gain in regularity compared to standard estimates, that is, that the specific vorticity \( \Omega \) and entropy gradient \( S \) are exactly as differentiable as the velocity \( v \) and density \( \varrho \). We illustrate this in detail in Theorem 9.10, where as an application, we derive a priori Sobolev estimates at the level of the undifferentiated equations of [33]. These estimates provide a new, sharpened picture of the basic regularity theory for solutions. We highlight the following main points.

For classical solutions whose initial data enjoy the gain in regularity for \( \Omega \) and \( S \), the correct\(^{12} \) regularity space for these fluid variables on compact acoustically globally hyperbolic spacetime regions is such that at the top derivative level, their divergence and curl\(^{13} \) belong to \( L^\infty(\text{Time})L^2(\text{Space}) \) (which are the standard regularity spaces for hyperbolic PDE solutions). In contrast, their spatial gradient is generally less regular\(^{13} \), belonging only to \( L^2(\text{Time})L^2(\text{Space}) \). That is, unless one considers special solutions/regions\(^{14} \), such that the divergence and curl vanish along the lateral boundary, the regularity theory at the top spatial derivative level involves spacetime integrals of \( |\partial\Omega|^2 \) and \( |\partial S|^2 \), where the “.” schematically denotes a top-order derivative that has been commuted through the Euler equations. In the context of the priori estimates of Theorem 9.10, this is captured by the spacetime integral estimate \(^{213a} \).

Relative to Lagrangian coordinates, the gain in differentiability for the vorticity in the barotropic case, achieved via combinations of Hodge estimates and transport equation estimates, has long been known, specifically because it has played a central role in proofs of local well-posedness for the compressible Euler equations for initial data featuring a fluid-vacuum boundary satisfying the “physical vacuum” boundary condition \(^{10,12,17,18} \). By the nature of free-boundary problems, these results are spatially localized. In the context of estimates across all of space (in particular, without the difficult boundary terms that we handle in this paper), the gain in differentiability for the vorticity with respect to arbitrary vectorfield differential operators was first shown in \(^{26} \), while the gain in differentiability for the entropy was first shown in \(^{33} \). The freedom to gain the derivative relative to general vectorfield differential operators is important for the mathematical theory of shock waves without symmetry assumptions. The reason is that Lagrangian coordinates seem to

\(^{12}\)In applications, one would of course have to construct good commutator vectorfields in order to ensure that the commutator terms also have good structure.

\(^{13}\)By “correct regularity space” for \( \Omega \) and \( S \), we mean the function space for which estimates are available and compatible with the regularity of the other solution variables.

\(^{14}\)Although \( \text{curl} S = 0 \) (see \(^{33a} \)), the curl of the derivatives of \( S \) with respect to vectorfields is generally not 0 due to the study of shock waves without symmetry assumptions, where a huge amount of effort is required to construct appropriate vectorfield differential operators and to control commutator terms.

\(^{15}\)We note, however, that the top-order \( L^\infty(\text{Time})L^2(\text{Space}) \) regularity of \( \text{B}\Omega \) and \( \text{B}S \) can easily be derived by using the transport equations \(^{32a} \) and \(^{32c} \) for algebraic substitution and showing that the terms on the right-hand sides enjoy the desired \( L^\infty(\text{Time})L^2(\text{Space}) \) regularity.

\(^{16}\)If the vorticity and entropy are compactly supported in space, then one can deduce a stronger estimate. That is, from the standard Hodge estimate \( \|\partial \xi\|_{L^2(\mathbb{R}^3)} \lesssim \|\text{div} \xi\|_{L^2(\mathbb{R}^3)} + \|\text{curl} \xi\|_{L^2(\mathbb{R}^3)} \) for vectorfields \( \xi \) on \( \mathbb{R}^3 \) (where \( \partial \) denotes gradient with respect to the Cartesian spatial coordinates), we see that \( \partial \xi \in L^\infty(\text{Time})L^2(\text{Space}) \) would in fact follow from knowing that \( \text{div} \xi \in L^\infty(\text{Time})L^2(\text{Space}) \) and \( \text{curl} \xi \in L^\infty(\text{Time})L^2(\text{Space}) \).
be unsuitable for deriving the full structure of the singular set because they are not adapted to the acoustic characteristics whose intersection corresponds to the formation of a shock; we refer readers to \[26,33\] for further discussion of these issues.

II) **Localized analysis of solutions with vorticity and entropy that form shocks and the interaction of shocks.** In 3D incompressible fluid equations, integral identities of the form \((7)\) are of fundamental importance for the localized analysis of stable shock formation in the presence of vorticity and entropy for solutions with distinct characteristic hypersurfaces. Roughly, shocks are singularities such that \(\rho, v,\) and \(s\) remain bounded while some directional derivatives of \(\rho\) and \(v\) blow up in a very particular fashion. The blowup is tied to the intersection of distinct characteristic hypersurfaces. What blows up are derivatives of the solution in directions transversal to the characteristics; the solution and its derivatives in direction tangent to the characteristics remain bounded. This is a multiple-spatial-dimension analog of the singularity formation that occurs in the model case of the 1D Burgers' equation \(\partial_t \Psi + \Psi \partial_x \Psi = 0.\) We refer readers to \[5,14,16,20,31,33\] for background on shock formation in multiple spatial dimensions without symmetry assumptions. Shocks are singularities of a sufficiently mild nature that one is left with hope that it might be possible to uniquely extend the solution in a weak sense past the singularity, subject to suitable admissibility criteria; see Point IV below for further discussion of this issue. In the work \[27\], the second author and J. Luk proved a stable shock formation result in 2D under barotropic equations of state, a setting that is much simpler than the 3D case since the absence of vorticity stretching in 2D allows one to avoid elliptic estimates for the vorticity. In particular, integral identities of the type that we derive in this paper are not needed to close the estimates. In \[26,33\], the second author and J. Luk outlined a proof of stable shock formation in 3D (full details will be presented in a forthcoming paper) in the barotropic case, but the approach described there neither required nor yielded localized information about the specific vorticity and entropy at the top derivative level. The reason is that the elliptic estimates in \[26,33\] involved integrals taken across of space and thus did not involve the difficult boundary terms that we treat here. Put differently, the approach described in \[26,33\] works only for spacetime regions of the form \([0,T] \times \Sigma,\) where the “space manifold” \(\Sigma\) has no boundary (e.g., \(\Sigma = \mathbb{R}^3\)). In contrast, integral identities of the form \((7)\) allow one to approach the problem of shock formation in the presence of vorticity and entropy via analysis on spatially localized regions. The viability of a local approach is desirable in the sense that solutions exhibit finite speed of propagation, and from a physical point of view, one would like to be able to describe the shock formation using only “local information.” Such localized information is expected to be important for studying the interaction of shock waves.

III) **Sharp information about the boundary of the maximal classical development.** Roughly, the maximal classical development is the largest spacetime region on which the solution exists classically and is uniquely determined by the initial data; we refer to \[30,36\] for further discussion. The localized analysis of shock formation described in Point II above is of crucial importance for obtaining information about the maximal classical development of the initial data, including information about the solution (including the vorticity and entropy) up to the boundary. In \[5\], Christodoulou provided a sharp picture of the maximal classical development near shock singularities for irrotational and isentropic solutions to the 3D relativistic Euler equations. He showed that the boundary of the maximal development can have various components enjoying the structure of a smooth manifold with

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\(^{17}\)Here, in the context of shocks, by “singular set,” we roughly mean the portion of the boundary of the maximal classical development on which the first derivatives of the density and velocity blow up; see below for further discussion of the maximal classical development.

\(^{18}\)Acoustic characteristics are hypersurfaces that are null with respect to the acoustical metric \(g\) defined in of Def \[2.2\]. Sound cones emanating from points serve as examples.

\(^{19}\)In \[27\], it was shown that in the barotropic case in 2D, the specific vorticity remains Lipschitz up to the shock, at least for the open set of initial data treated there.

\(^{20}\)For barotropic equations of state in 2D, the absence of vorticity stretching is equivalent to the fact that RHS \((32a)\) is identically 0, a well-known fact that holds only in the 2D case. Due to this vanishing, one can handle the specific vorticity without using the div-curl-transport system \((33a)-(33b)\), which drastically simplifies the regularity theory of the specific vorticity compared to the general 3D case.
respect to an appropriate dynamically constructed coordinate system. For example, the boundary can have singular components, along which some Cartesian partial derivative of the solution blows up, as well as acoustically null hypersurface portions along which no blowup occurs (roughly, these null portions are Cauchy horizons). His proof exploited that in the irrotational and isentropic setting, the equations of motion reduce to a single quasilinear wave equation for a potential function. In particular, Christodoulou did not have to derive div-curl-transport estimates for the fluid variables. See also \[9\] for a similar sharp description of the maximal classical development of shock-forming solutions to the irrotational and isentropic non-relativistic 3D compressible Euler equations. We highlight the following key point.

Integral identities of the form \( \text{(7)} \) are the main new ingredients needed to extend the framework of \([5,26,33]\) to derive sharp information up to the boundary of the maximal classical development for solutions with vorticity and non-constant entropy. The point is that, roughly, by finite speed of propagation, the boundary of the maximal classical development is “locally determined,” which necessitates the use of localized identities/estimates. Such a result would elevate our understanding of such solutions to the same level achieved by Christodoulou \([5]\) in the irrotational and isentropic case.

We also refer to \([1–4]\) for alternate approaches to proving blowup outside of symmetry. The frameworks used in these works allow one to follow solutions up to the constant-time hypersurface of first blowup for an open set of initial data such that the solution’s first singularity is non-degenerate in the constant-hypersurface of first blowup. Roughly, “non-degenerate” means that the first singularity is isolated in the constant-hypersurface of first blowup and that the reciprocal of the singular directional derivative of the solution behaves quadratically (with respect to suitably constructed coordinates) within the constant-hypersurface of first blowup.

IV) Connections to the shock development problem. Although sharp results about the maximal classical development (as described in Point III) are of interest in themselves, they are also essential for properly setting up the “initial” data for the shock development problem. This is the problem of locally solving the compressible Euler equations past the shock singularity in a weak sense (uniquely under appropriate admissibility conditions tied to jump conditions), and, at the same time, constructing the shock hypersurface across which the solution jumps. In \([8]\), Christodoulou–Lisibach solved the problem for the 3D relativistic Euler equations in spherical symmetry, while in the recent breakthrough monograph \([7]\), Christodoulou solved the restricted shock development problem in 3D without symmetry assumptions. Here, the term “restricted” means that Christodoulou considered only irrotational initial data, and he ignored the jump in entropy and vorticity across the shock hypersurface, so that the mathematical problem concerned only irrotational solutions. We stress that the “initial” data for the restricted shock development problem in \([7]\) are provided by the state of the solution on the boundary of its maximal classical development (which was derived by Christodoulou in \([5]\) in the irrotational case, starting from smooth initial conditions along \(\{t = 0\}\), and moreover, the quantitative estimates near the boundary from \([5]\) are crucially used in \([7]\) to implement the iteration scheme that lies at the heart of the solution of the restricted shock development problem. We also highlight that in the (yet unsolved “unrestricted”) shock development problem, vorticity and entropy are generated across the shock hypersurface, even if the initial data are irrotational and isentropic. Thus, we expect that the results of the present article will be useful for properly setting up/studying the shock development problem for general solutions (that are smooth along \(\{t = 0\}\) but form shocks in finite time), in which the vorticity is non-zero and the entropy is dynamic.

V) Characteristic initial value problem. In the theory of hyperbolic PDEs, the characteristic initial value problem is the Cauchy problem with initial data prescribed on a pair of intersecting characteristic hypersurfaces (i.e., acoustically null hypersurfaces in the context of compressible Euler flow). This kind of Cauchy problem has proven to be of immense value for solving important problems in hyperbolic PDE theory; see, for example, Christodoulou’s celebrated proof \([6]\) of the
roughly speaking, show that \( \Omega \) satisfies the right-hand side of the transport equation (6) for \( \Omega \). To this end, one considers equations (33a)-(33b), which, to keep the discussion short, here we caricature the system as:

\[ H \]

or easier error terms. Here and throughout, \( \Sigma \) denotes the standard Euclidean divergence and curl operators on \( \Sigma \). In the rest of the paper, we also use this alternate notation in the null case; see Convention 3.4.

Subsubsect. 1.3.2, which are caused in part by the lack of sufficient regularity in the terms on the right-hand side of the transport equation (7) for \( \Omega \). To this end, one considers equations (33a)-(33b), which, roughly speaking, show that \( \Omega \) satisfies a div-curl-transport system, where the div and curl operators are the standard Euclidean ones. To keep the discussion short, here we caricature the system as:

\[ \text{div}\Omega := F = \Omega \cdot \partial \rho, \quad B\text{curl}\Omega := \partial v \cdot \partial \Omega + \cdots, \]

where div and curl denote the standard Euclidean divergence and curl operators on \( \Sigma \) and \( \cdots \) denotes similar or easier error terms. Here and throughout, \( \Sigma = \{ t \} \times \mathbb{R}^3 \subset \mathbb{R}^{1+3} \) denotes the standard flat hypersurface of constant Cartesian time \( t \). We highlight that, as (33b) shows (see also Footnote 22), all derivative-quadratic terms in \( G \) are in fact null forms with respect to the acoustical metric \( g \). As we explain in Subsect. 2.1.3, the null form structure is important for applications to shock waves, though we downplay the significance of this structure in the present discussion. We also note that to handle the entropy gradient \( S \), one would carry out similar arguments using the div-curl-transport system (34a)-(34b) in place of (33a)-(33b).

To prove Theorem 1.2 for \( \Omega \), one must circumvent the difficulties described at the start of Subsect. 1.5, which are caused in part by the apparent lack of sufficient regularity in the terms on the right-hand side of the transport equation (7) for \( \Omega \). To this end, one considers equations (33a)-(33b), which, roughly speaking, show that \( \Omega \) satisfies a div-curl-transport system, where the div and curl operators are the standard Euclidean ones. To keep the discussion short, here we caricature the system as:

\[ \text{div}\Omega := F = \Omega \cdot \partial \rho, \quad B\text{curl}\Omega := \partial v \cdot \partial \Omega + \cdots, \]

where div and curl denote the standard Euclidean divergence and curl operators on \( \Sigma \) and \( \cdots \) denotes similar or easier error terms. Here and throughout, \( \Sigma = \{ t \} \times \mathbb{R}^3 \subset \mathbb{R}^{1+3} \) denotes the standard flat hypersurface of constant Cartesian time \( t \). We highlight that, as (33b) shows (see also Footnote 22), all derivative-quadratic terms in \( G \) are in fact null forms with respect to the acoustical metric \( g \). As we explain in Subsect. 2.1.3, the null form structure is important for applications to shock waves, though we downplay the significance of this structure in the present discussion. We also note that to handle the entropy gradient \( S \), one would carry out similar arguments using the div-curl-transport system (34a)-(34b) in place of (33a)-(33b).

To prove Theorem 1.2 for \( \Omega \), one must accomplish the following:

1. Derive an integral identity whose left-hand side is comparable to \( \int_{M_\tau} |\partial\Omega|^2 + \int_{S_\tau} |\Omega|^2 \), where \( \Omega \) is the orthogonally projected \( \Omega \) onto \( S_\tau \). We clarify that in Subsect. 1.5, we will not explicitly display (or even define) the volume forms in any of the integrals.

2. We must show that the error terms on the right-hand side of the integral identity enjoy a consistent amount of Sobolev regularity, assuming that the initial data (which we assume to be prescribed along \( \Sigma \)) also enjoy it. Specifically, the spacetime error integrals \( \int_{M_\tau} \cdots \) on RHS (7) are allowed to have arbitrary dependence on the un-differentiated quantities \( \rho, v, s, \Omega, S \), and up-to-quadratic dependence on \( \partial\rho, \partial v, \partial s \), and \( \text{curl}\Omega \) (in reality, one needs to use \( C \) in place of \( \text{curl}\Omega \) – see Footnote 22). Under standard \( C^1 \)-type bootstrap assumptions enjoyed by classical solutions (see Subsect. 9.7.1), these spacetime error integrals can be bounded using energy standard estimates for the wave and transport equations featured in Theorem 2.8, see e.g. the proof of Theorem 9.10. Moreover, these

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22More precisely, equation (33b) involves a modified version of \( \text{curl}\Omega \), denoted by \( C \) and defined below in (21a). In practice, one must work with \( C \) since it satisfies a transport equation whose source terms exhibit improved regularity and other good structures compared to the transport equation satisfied by \( \text{curl}\Omega \). However, to keep the discussion short, in our schematic overview, we will ignore the distinction between \( \text{curl}\Omega \) and \( C \).
spacetime error integrals are allowed to have \textit{linear dependence} on $\partial \Omega$ and $\partial S$ (see Footnote 11 regarding the admissibility of such linear dependence). Below, we further explain the allowed quadratic dependence on curl$\Omega$.

3. On the right-hand side of the integral identity (7), the \textit{null hypersurface} error integrals $\int_{\Sigma^*} \cdots$ are allowed to have arbitrary dependence on the un-differentiated quantities $\rho, v, s, \Omega, S$, curl$\Omega$, and up-to-quadratic dependence on the first-order $\nabla_{\xi}$-tangential derivatives of $\rho$ and $v$. Under bootstrap assumptions of the type mentioned above, these null hypersurface integrals can be also bounded using energy standard estimates for the wave and transport equations featured in Theorem 2.8; see e.g. the proof of Theorem 9.10.

4. On the right-hand side of the integral identity (7), the “data term” $\int_{S_0} |\Omega|^2$ is allowed.

1.5.1. A \textit{warm-up problem: Flat spacetime slabs without spatial boundaries}. We first explain how to prove Theorem 1.2 in the drastically simplified setting of spacetime regions of the type $\mathcal{M}_T = [0, T] \times \mathbb{R}^3$, that is, for spacetime slabs foliated by the constant Cartesian-time hypersurfaces $\Sigma_t$ (without a boundary). The main challenge is to overcome the regularity difficulties described at the beginning of Subsect. 1.5. We will proceed by using the system (8) and applying the divergence theorem to the $\Sigma_t$-tangent vectorfield with the following Cartesian spatial components:

$$ J^i := \Omega^a \partial_a \Omega^i - \Omega^i \text{div}\Omega. \quad (9) $$

Next, we note that straightforward computations yield the Hodge-type identity

$$ |\partial \Omega|^2 = \text{div} J + (\text{div}\Omega)^2 + |\text{curl}\Omega|^2. \quad (10) $$

Thus, we can apply the divergence theorem to $J$ on $\Sigma_t$ (relative to the standard Euclidean metric on $\Sigma_t$ and without boundary terms), to obtain (suppressing the standard integration measures): $\int_{\Sigma_t} |\partial \Omega|^2 = \int_{\Sigma_t} (\text{div} \Omega)^2 + \int_{\Sigma_t} |\text{curl}\Omega|^2$. Then integrating this elliptic identity with respect to Cartesian time and substituting $\text{div}\Omega$ with $F$ from (8), we deduce:

$$ \int_{\mathcal{M}_t} |\partial \Omega|^2 = \int_{\mathcal{M}_t} F^2 + \int_{\mathcal{M}_t} |\text{curl}\Omega|^2. \quad (11) $$

Note that the difficult boundary integrals $\int_{\Sigma^*} \cdots$ (see Convention 1.7) found on RHS (7) are completely absent in the present simplified setting.

\textbf{Remark 1.8.} In the present simplified setting, it is not necessary to integrate the elliptic identity with respect to time to obtain the spacetime integral identity (11). We have carried out the integration in time only because later on, when treating regions with spatial boundaries, we will need to integrate in time to see various special structures; see the integral identity (20) and the discussion surrounding it.

To complete our proof sketch of Theorem 1.2 in the present simplified setting, we will uncover the regularity properties of the terms in the integral identity (11) and explain why they have sufficient regularity for controlling LHS (11), assuming that the initial data enjoy sufficient regularity. From the point of view of regularity, the most difficult term to handle is the source term $G$ in (8), specifically the factor $\partial \Omega$. Thus, for illustration, we will assume that the other factor in $G$, namely $\partial v$, is uniformly bounded by a constant. Under this assumption, we will explain how to prove that there exists a $C > 0$ such that the following estimate holds (note that in the present simplified setting, we have $\tau := t$ and $\mathcal{M}_t = [0, t] \times \mathbb{R}^3$):

$$ \int_{\Sigma_t} |\text{curl}\Omega|^2 + \int_{\mathcal{M}_t} |\partial \Omega|^2 \leq \frac{1}{2} \int_{\mathcal{M}_t} |\partial \Omega|^2 + C(1 + t^2) \int_{\Sigma_t} |\text{curl}\Omega|^2 + \int_{\mathcal{M}_t} F^2 + t \int_{\Sigma_0} |\text{curl}\Omega|^2 + \cdots. \quad (12) $$

We then note that the first term on RHS (12) can be absorbed back into the left and that the remaining terms on RHS (12) either i) are controlled by the initial data on $\Sigma_0$, ii) have a consistent amount of Sobolev regularity in the sense described in point 2 below (8), or iii) can be treated with Gronwall’s inequality (where LHS (12) is the quantity to which one applies Gronwall’s inequality). In total, as a consequence of (12), the
assumptions described above, and Gronwall’s inequality, one could derive an estimate of the following form for $t \in [0, T]$:  
\[ \int_{\Sigma_t} |\text{curl} \Omega|^2 + \int_{M_t} |\text{div} \Omega|^2 \lesssim \exp[C(1 + t^2)] \cdot \text{data}, \]  where “data” denotes an appropriate Sobolev norm of the initial data on $\Sigma_0$; estimates in the spirit of (13) are fundamentally important for applications of the type described in Subsect. 1.4.

It remains for us to explain how to prove (12). In view of (11), we see that we only have to explain how to control the integral $\int_{\Sigma_t} |\text{curl} \Omega|^2$ on LHS (12). In fact, for this term, control along $\Sigma_t$ can be achieved: standard transport equation energy identities (see Prop. 9.3) for the second equation in (8) yield, for $t \in [0, T]$, taking into account our assumed uniform bound on $\partial_t v$, the following energy inequality, which we depict schematically:

\[ \int_{\Sigma_t} |\text{curl} \Omega|^2 \leq \int_{\Sigma_0} |\text{curl} \Omega|^2 + C \int_{M_t} |\text{curl} \Omega \cdot \partial_t| + \cdots. \]  Integrating (14) with respect to time, we obtain

\[ \int_{M_t} |\text{curl} \Omega|^2 \leq t \int_{\Sigma_0} |\text{curl} \Omega|^2 + Ct \int_{M_t} |\text{curl} \Omega \cdot \partial_t| + \cdots. \]  Using (15) to control the last integral on RHS (11), adding the resulting inequality to the identity (14), and using Young’s inequality in the form $ab \lesssim a^2 + b^2$, we deduce that there is a constant $C > 0$ such that (12) holds.

1.5.2. An overview of the analysis on the domains $M$ from our main results. We will now explain how to handle all of the additional complications that arise when deriving the integral identities on domains $M$ featured in Theorem 1.2, specifically domains of the type featured in Fig. 1 and under the null lateral boundary assumption stated in Convention 1.7. We will again focus our attention on deriving the integral identities for the specific vorticity vectorfield $\Omega$.

First, we highlight that $M$ is assumed to be foliated by level sets of an acoustical time function $\tau$, which is not generally equal to the Cartesian time function $t$; see Subsect. 1.3.1. Since the divergence and curl operators in (8) are the standard “flat” operators on the level sets of constant Cartesian time $t$, we must find a way to relate information about the solution on $\Sigma_t$ to information about the solution on $\Sigma_\tau$, where we recall that $\Sigma_\tau$ denotes the portion of the level set of $\tau$ in $M$. Moreover, to detect the remarkable null structures that are present when the lateral boundary $\mathcal{N}_\tau$ is null, it is crucial that we adapt our approach to the “true geometry,” that is, the geometry corresponding to the acoustical metric $g$ of Def. 2.2. To this end, we rely on a collection of geometric tensorfields adapted to $\Sigma_\tau$. Specifically, we let $\tilde{g}$ denote the Riemannian metric induced on $\Sigma_\tau$ by $g$, and we extend $\tilde{g}$ in the standard fashion to a positive semi-definite quadratic form on spacetime tensors that vanishes along the g-normal of $\Sigma_\tau$. We also let $\tilde{\Pi}$ denote $g$-orthogonal projection onto $\Sigma_\tau$; see Subsect. 3.7 for the precise definitions of these tensorfields. In place of the vectorfield $J$ defined in (9), we use the following vectorfield, which we stress is $\Sigma_\tau$-tangent, even though $\Omega$ is $\Sigma_\tau$-tangent:

\[ J^\alpha := \Omega^\beta \tilde{\Pi}_{\beta \gamma} \tilde{\Pi}_{\alpha \gamma} \partial_\gamma \Omega^\gamma - \Omega^\gamma \tilde{\Pi}_{\alpha \gamma} \tilde{\Pi}_{\beta \gamma} \partial_\gamma \Omega^\beta. \]  A key step in the analysis is provided by Lemma 4.7. This lemma yields the following Hodge-type identity, which can be viewed of an analog of (10) that is adapted to $\Sigma_\tau$:

\[ |\text{curl} \Omega|^2 - (K_\alpha K^\alpha)^2 = \tilde{\nabla}_\alpha J^\alpha + \frac{1}{2} |\partial \Omega|^2 + |\text{div} \Omega|^2 \]  

\[ + \gamma^2 (K_\alpha B^\alpha)^2 + 2(K_\alpha K^\alpha) \text{div} \Omega \]  

\[ - 2\gamma (K_\alpha B^\alpha) K_\beta K^\beta - 2\gamma (\text{div} \Omega) K_\alpha B^\alpha + \cdots. \]  In (17), $|\cdot|_{\tilde{g}}$ denotes the pointwise seminorm of a tensor with respect to $\tilde{g}$ (it is a seminorm since $\tilde{g}$ is positive definite only on $\tilde{\Sigma}_\tau$-tangent tensors). Moreover, $K$ is a $\tilde{\Sigma}_\tau$-tangent vectorfield satisfying $|K|_{\tilde{g}} < 1$ (see 96).

More precisely, in obtaining (17), we have set the weight function $\mathcal{W}$ in Lemma 4.7 equal to unity.
and just above (61), which by Cauchy–Schwarz implies that LHS (17) is positive definite in the derivatives of Ω in directions tangent to Στ. Furthermore, we lower and raise Greek indices with g and its inverse, Ωg denotes the g-dual of Ω (i.e., Ωα = gαβΩβ), d denotes exterior derivative, ν > 0 is a function measuring the length of a certain normal to Στ, · · · denotes error terms that are at most linear in ∂Ω, and we stress that the terms divΩ on RHS (17) denote the standard Euclidean divergence of Ω, while ∇αJα denotes the covariant divergence of J with respect to the Levi–Civita connection of ˜g. In addition, by using the transport equation (6), we see that the error terms on RHS (17) featuring a factor of B02 are also of type “· · ·.”

Next, we apply the divergence theorem 24 along Στ to the vectorfield J defined in (17) and use 8 to substitute for the term (∇Ω)2 on RHS (18), thus obtaining the following integral identity, where Z (see Fig. 1) is the g-unit outer normal to Στ in Στ:

\[
\int_{Στ} \{ |∂Ω|^2_Ω - (K_α KΩ^α)^2 \} = \int_{Στ} Z_α J^α + \int_{Στ} F^2 + \frac{1}{2} \int_{Στ} |d(Ω_β)|_g^2 + \int_{Στ} · · · .
\]

As we further describe below, the integrals \( \int_{Στ} \) on RHS (18) can be shown to have sufficient regularity, after an integration with respect to τ. Even this step requires substantial new ideas compared to the simple domains treated in Subsubsection 1.5.1; we will return to this issue below. For now, we focus on how to handle the “Dangerous” Στ integral on RHS (18). Its presence is simply a manifestation of the fact that in the standard approach to elliptic estimates on Στ, one must know the data along the boundary Sτ in order to treat the boundary integrals. That is, the standard elliptic approach does not allow one to control LHS (18) without specifying boundary data for Ω along Sτ. Put differently:

Since Ω is determined 25 by the initial data on Σ0, we cannot “specify” its boundary data on Sτ; the boundary data Sτ is evolutionarily determined by the initial data, and for the identity (18) to be of any use, we must find a way to “access quantitative information about the boundary data.”

Let us describe a natural attempt at how to control the dangerous integral \( \int_{Στ} Z_α J^α \) on RHS (18), which in the end does not work. Specifically, through careful geometric decompositions, in the spirit of those carried out in the proof of Lemma 13, it can be shown that the boundary integrand \( Z_α J^α \) can be re-expressed as terms of the schematic form \( Ω · ∂Ω \), where the operator ∂ is Sτ-tangent. Now formally, we have a fractional integration by parts identity of the schematic form \( \int_{Στ} Ω · ∂Ω = \int_{Στ} φ^{1/2} Ω · φ^{1/2} Ω + · · · , \) and Sobolev trace estimates suggest that \( \int_{Στ} Ω · ∂Ω \) can be controlled in terms of \( \int_{Στ} |∂Ω|^2_Ω \) (plus lower-order terms that we ignore here), consistent with the strength of LHS (18). However, we highlight the following key point:

The Sobolev trace estimate mentioned above is at a critical regularity level, and the “constant” in the trace estimate could be large. This could prevent one from treating the boundary integral \( \int_{Στ} Ω · ∂Ω \) as an error term to be absorbed into LHS (18). In total, this calls into question the usefulness of the Hodge-type identity (18) and suggests that the Στ integral cannot be directly controlled using only elliptic theory, prompting us to seek a new approach.

To overcome the regularity difficulty described above, we adopt a different strategy to handle the boundary integrand \( Z_α J^α \). First, in Lemma 13, without using the compressible Euler equations, we derive the following crucial geometric identity 26 which we restate here in schematic form as follows, ignoring order unity coefficients but respecting the overall sign of the important terms:

\[
Z_α J^α = -\vec{H} \{ |Ω|^2_Ω \} + E^αΩ^β(∂_βΩ_α - ∂_αΩ_β) + \text{Tangential} + · · ·,
\]

where · · · denotes perfect Sτ-divergences (which therefore vanish when inserted into the Στ integral on RHS (18)). In (19), the vectorfield \( \vec{H} \) is g-null, ∇τ-tangent, and normalized by \( \vec{H} τ = 1 \), i.e., it is a null vector.

---

24See Sect. 1 for the definitions of the geometric volume and area forms that we use in our integrals.
25The transport equation (32c) for Ω, and our assumptions on the spacetime region \( M_T \) guarantee that Ω|Sτ and ∂Ω|Sτ are evolutionarily determined by their initial data on Σ0. However, by itself, equation (32a) does not allow us to conclude the desired quantitative Sobolev control over Ω|Sτ and ∂Ω|Sτ that we need in order for (18) to be useful.
26In obtaining (19), we have set the weight function \( W \) from Lemma 4.8 equal to unity.
generator adapted to \( \tau \) (and thus \( \vec{H} = \frac{\partial}{\partial \tau} \) relative to appropriate coordinates on \( \mathcal{N}_\tau \)). Moreover, \( \vec{E} \) (see Def. 1.10) is \( \mathcal{N}_\tau \)-tangent, \( \theta \) is the \( \mathbf{g} \)-orthogonal projection of \( \Omega \) onto \( \Sigma_\tau \) (and thus \( \Omega \) is \( \mathcal{S}_\tau \)-tangent), and the error term Tangential has the good properties described in Theorem 1.2, e.g., it enjoys admissible regularity and the only derivatives of the density and velocity that appear are tangent to \( \mathcal{N}_\tau \). Next, we substitute RHS (19) for the first integrand on RHS (18) and then integrate (18) with respect to \( \tau \) (see Remark 1.8, use the fundamental theorem of calculus-type result \( -\int_{\mathcal{N}_\tau} \vec{E} \{\|\Omega^2\|\} = -\int_{\mathcal{S}} \{\|\Omega^2\| + \int_{\mathcal{S}} \{\|\Omega^2\| + \ldots \} \) (see Lemma 6.3 for the details), and use \( \vec{\Omega} \) to algebraically substitute for \( \vec{B} \cdot \Omega \), thereby obtaining the following identity for \( \tau \in [0,T] \), expressed in schematic form (see (100) for a precise formula for the integrand of the spacetime integral \( \int_{\mathcal{M}_\tau} \cdot \cdot \cdot \) on the left-hand side):

\[
\int_{\mathcal{M}_\tau} \{ |\vec{B}\Omega|^2 + |\vec{\Omega}|^2 - (K_a K \Omega^2)^2 \} + \int_{\mathcal{S}_\tau} |\Omega|^2 = \int_{\mathcal{S}_\tau} |\Omega|^2 + \int_{\mathcal{N}_\tau} \vec{E} \alpha^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) + \int_{\mathcal{N}_\tau} \text{Tangential}
+ \int_{\mathcal{M}_\tau} F^2 + \frac{1}{2} \int_{\mathcal{M}_\tau} |\partial(\Omega_\tau)|^2 + \int_{\mathcal{M}_\tau} |\Omega \cdot \partial v + \vec{B} \cdot \vec{S}|^2 + \int_{\mathcal{M}_\tau} \cdot \cdot \cdot.
\]

Note that \( \int_{\mathcal{S}_\tau} |\Omega|^2 \) is an “initial data term” that we consider known. Since the timelike vectorfield \( \vec{B} \) is transversal to \( \Sigma_\tau \) (see Footnote 37), it follows that the spacetime integrand on LHS (20) is positive definite in \( \partial \Omega \); this explains the positivity of the quadratic form \( \partial(\vec{\Omega}, \vec{\Omega}) \) in Theorem 1.2; see Lemma 4.4 for a proof of the positivity.

All error integrals on RHS (20) except for \( \int_{\mathcal{N}_\tau} \vec{E} \alpha^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) \) can readily be shown to have the desired regularity and structural properties; see, for example, the proof of Theorem 9.10 for the details. Thus, to finish the proof sketch of Theorem 1.2 it remains for us to explain how to handle the null hypersurface integral \( \int_{\mathcal{N}_\tau} \vec{E} \alpha^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) \). This is the most difficult analysis in the paper. While the integrand \( \vec{E} \alpha^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) \) has the desired \( \mathcal{N}_\tau \)-tangential differentiation structure, it is not clear that it has sufficient regularity to be treated as an error term. The difficulty is that the integral involves \( \partial \Omega \) and is along the hypersurface \( \mathcal{N}_\tau \), while LHS (20) is quadratic in \( \partial \Omega \) and is an integral over the spacetime region \( \mathcal{M}_\tau \); thus, for reasons similar to the ones given in the paragraph above (19), \( \int_{\mathcal{N}_\tau} \vec{E} \alpha^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) \) cannot be controlled with hypersurface trace estimates. We dedicate all of Sect. 5 to showing that the integrand \( \vec{E} \alpha^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) \) enjoys the desired structures. We now highlight the main steps in this analysis.

- **First**, in Cor. 5.7 we use the new formulation of compressible Euler flow from Theorem 2.8 to provide a geo-analytic decomposition of the two-form \( \partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha \), i.e., the decomposition holds only for solutions. The main idea is to split the two-form into components tangent to \( \mathcal{N}_\tau \) and components in the direction of \( \vec{B} \), and to separate out the Euclidean curl components \( \partial \Omega \), which by \( \vec{\Omega} \) can be independently controlled along \( \mathcal{N}_\tau \). The “\( \vec{B} \)" components can be treated with the transport equation (32a) together with the simple fact that \( \vec{B} \cdot \Omega = 0 \), since \( \vec{B} \) is \( \mathbf{g} \)-orthogonal to \( \Sigma_\tau \) (see (27)) while \( \Omega \) is \( \Sigma_\tau \)-tangent. In total, this allows us to show that “most pieces” of the term \( \vec{E} \alpha^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) \) exhibit the desired structures. However, there is one remaining “difficult piece” that requires special geometric treatment.

- **The “difficult piece”** mentioned in the previous sentence is in fact generated by the entropy gradient term \( \vec{B} \cdot \vec{S} \) on RHS (6) and thus is absent in the isentropic case \( s = \text{const} \). In the context of Cor. 5.7, the difficult entropy gradient terms re-emerge as the fifth and sixth terms on RHS (13a), specifically \( -\epsilon_{\alpha \beta \gamma \delta} \vec{E} \alpha^\beta (\vec{B} \cdot \vec{S}) S^\gamma + \epsilon_{\alpha \beta \gamma \delta} \vec{E} \alpha^\beta \vec{B} \cdot \vec{S} S^\gamma \partial_\delta v^\nu \), where \( \epsilon_{\alpha \beta \gamma \delta} \) is the fully antisymmetric symbol normalized by \( \epsilon_{0123} = 1 \). The difficult part of the analysis is showing that, after contracting this combination of terms against \( \vec{E} \alpha^\beta \), the resulting expression involves only \( \mathcal{N}_\tau \)-tangential derivatives of the velocity and density; as we have mentioned, \( \mathcal{N}_\tau \)-tangential derivatives of the velocity and density are controllable using standard energy estimates for the wave equations (31a)-(31b). After a standard preliminary geometric decomposition provided by Lemma 5.8, in which we decompose all derivatives in this combination of terms into their \( \mathcal{N}_\tau \)-tangential components and
Remarkable localized integral identities for 3D compressible Euler flow

\textbf{B}-parallel components and exploit the compressible Euler formulation provided by Theorem 2.8 we are left with precisely one product that needs to be carefully treated

\[ \epsilon_{\alpha\beta\gamma\delta} \mathcal{E}^{\alpha} \Omega^{\beta} L^{\gamma}(S^{\delta} + S^{\delta}_{L_\alpha} B^{\delta}) B_\rho, \]

where \( L \) is a null generator of \( \Sigma_{\tau} \), normalized by \( L t = 1 \) (where \( t \) is Cartesian time). The difficulty is that the factor \( B_\rho \) involves a derivative of \( \rho \) in a direction transversal to \( \Sigma_{\tau} \); since the first derivatives of \( \rho \) can be controlled only with the wave equation \( (31b) \), and since wave equation energies along null hypersurfaces do not control transversal derivatives (see e.g. \( (209a) \)), this calls into question whether or not one can control the corresponding null hypersurface error integral

\[ \int_{\Sigma_{\tau}} \epsilon_{\alpha\beta\gamma\delta} \mathcal{E}^{\alpha} \Omega^{\beta} L^{\gamma}(S^{\delta} + S^{\delta}_{L_\alpha} B^{\delta}) B_\rho. \]

- In Prop. 5.11, we use the detailed structure of the vectorfield \( \mathcal{E} \) (see Def. 3.10) to prove that this remaining difficult product \( \epsilon_{\alpha\beta\gamma\delta} \mathcal{E}^{\alpha} \Omega^{\beta} L^{\gamma}(S^{\delta} + S^{\delta}_{L_\alpha} B^{\delta}) B_\rho \) in fact completely vanishes \( \text{28} \) see the identity \( (145) \).

This finishes our proof sketch of Theorem 1.2 for \( \Omega \) on domains \( \mathcal{M} \) of the type depicted in Fig. 1. The analysis in the case of the entropy gradient is similar but much simpler, mostly because the error term \( \mathcal{E}^{\alpha} \Omega^{\beta} L^{\gamma}(\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) \) is much simpler than the error term \( \mathcal{E}^{\alpha} \Omega^{\beta} \left( \partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha \right) \), stemming from the much simpler structure of the identity \( (134b) \) for \( d(S_\tau) \) compared to the identity \( (134a) \) for \( d(\Omega_\tau) \).

In the case of domains \( \mathcal{M} \) covered by double-null foliations, Theorem 1.2 can be proved using modifications of the arguments sketched above; see Sect. 10 for the details.

1.6. Paper outline. The remainder of the paper is organized as follows.

- In Sect. 2 we recall, as Theorem 2.8, the results of \( [33] \), which provide the geometric formulation of compressible Euler flow that we use in proving our main results. The equations of Theorem 2.8 feature “hyperbolic” wave- and transport-parts, as well as “elliptic-hyperbolic” div-curl-transport parts. We also define some geometric tensors that play a fundamental role in the rest of the article.

- In Sect. 3 we state our assumptions on the acoustical time function \( \tau \) that we use to foliate the spacetime regions \( \mathcal{M} \) under study, and we state the assumptions on \( \mathcal{M} \) that we use to prove our main results. We then derive some basic properties of various tensors tied to the geometry of \( \mathcal{M} \).

- In Sect. 4 we first construct the coercive quadratic forms that are featured in our main integral identities, which are provided by Theorem 2.8. We then derive some divergence-form elliptic Hodge-type identities for the specific vorticity \( \Omega \) and the entropy gradient \( S \). The identities are valid along the portions of the level sets of \( \tau \) that are contained in \( \mathcal{M} \); we denote these portions by \( \Sigma_{\tau} \). In the proof of Theorem 2.8, we will integrate these divergence identities along \( \Sigma_{\tau} \), which leads to boundary integrals along the topological spheres \( \Sigma_{\tau} = \partial \Sigma_{\tau} \). The boundary integrals are seemingly dangerous from the point of view of regularity, and it is not apparent that they are controllable. However, Prop. 7.1 and Theorem 7.2 together show that after an integration with respect to \( \tau \) and arguments that exploit the special geo-analytic structures exhibited by the compressible Euler formulation of Theorem 2.8, these dangerous-looking terms can be shown to be equal to error integrals along the lateral hypersurface \( \Sigma_{\tau} \) featuring only \( \Sigma_{\tau} \)-tangential derivatives of quantities that enjoy sufficient regularity; as we described in Subsect. 1.4 these properties are crucial for the study of shocks.

- In Sect. 5 we provide a series of geometric identities tied to the antisymmetric tensors \( \partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha \) and \( \partial_\alpha S_\beta - \partial_\beta S_\alpha \). The results of this section are crucial for the proof of Prop. 7.1 where they are used to show that some difficult error terms involving these antisymmetric tensors are controllable from the point of view of regularity and thanks to their \( \Sigma_{\tau} \)-tangential derivative structure. Many aspects of the analysis in this section rely on the compressible Euler formulation of Theorem 2.8.

- In Sect. 6 we define the geometric volume forms, area forms, and integrals that we use in our main integral identities, and we derive some simple identities tied to them.

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\( ^{27} \) When \( \Sigma_{\tau} \) is spacelike, the formula in Prop. 5.11 has \( \Sigma \) in place of \( L \), where \( \Sigma \) is normal to \( \Sigma_{\tau} \) and normalized by \( \Sigma t = 1 \). When \( \Sigma_{\tau} \) is null, we have \( \Sigma = L \).

\( ^{28} \) When \( \Sigma_{\tau} \) is spacelike, this product does not vanish. In this case, the identity \( (144) \) provided by Prop. 5.11 allows us to re-express it in terms of \( \Sigma_{\tau} \)-tangential derivatives.
• In Sect. 7, we first prove Prop. 7.1 which yields identities for the error integrals along the lateral boundary hypersurface \( H \) mentioned above. We then prove the aforementioned Theorem 7.2 which exhibits the remarkable structure of the integrands.

• In Sect. 8, we prove Theorem 8.1 which provides the main new integral identities verified by the first derivatives of the specific vorticity and the entropy gradient. In Remark 8.2, we highlight some of the key structural properties of the identities, and in Theorem 7.2, we give a precise description of the most important of these properties. The proof of Theorem 8.1 follows in a straightforward fashion from the divergence identities of Sect. 4 and the identities for the lateral boundary hypersurface error integrals from Prop. 7.1.

• Sect. 9 provides an application of the previous results. Specifically, we combine the results of the previous sections with standard applications of the geometric vectorfield method for wave equations to prove Theorem 9.10 which yields a priori estimates for compressible Euler solutions, localized to compact acoustically globally hyperbolic spacetime subsets \( \mathcal{M} \). The estimates exhibit the local gain in regularity for the specific vorticity and entropy gradient described in Point I of Subsect. 1.4. The proof of Theorem 9.10 crucially relies on the special structures of the integral identities provided by Theorem 8.1 and Theorem 7.2, especially in the case that the lateral boundary \( H \) is acoustically null. To shorten the exposition, in Sect. 9, we assume that the acoustical time function \( \tau \) is equal to the Cartesian time function \( t \).

• In Sect. 10, we extend Theorem 8.1 yield analogous integral identities for spacetime regions covered by double-null foliations; as we explained in Subsect. 1.4, double-null foliations form the starting point for the study of the characteristic initial value problem for compressible Euler flow. The main result of this section is Theorem 10.6.

• In Appendix A, we summarize some of the key notation used throughout the article.

2. The geometric formulation of 3D compressible Euler flow

In this section, we provide Theorem 2.8 which recalls the new formulation of compressible Euler flow from [33]. The remarkable structures in this formulation play a fundamental role in the rest of the paper. Before stating the theorem, we first define some non-standard fluid variables that are prominently featured in its statement, specifically modified fluid variables of Def. 2.1. We also define several geometric tensors associated to the flow, notably the acoustical metric \( g \) of Def. 2.2. Moreover, in Subsubsect. 2.1.3, to explain the significance of the null form structures revealed by Theorem 2.8, we recall the definitions and properties of null forms relative to \( g \).

2.1. Additional geometric and analytic quantities associated to the flow. In this subsection, we define various objects of analytic and physical significance that are needed for the formulation of compressible Euler flow provided by Theorem 2.8.

2.1.1. Modified fluid variables. The tensorfields \( C \) and \( D \) in the next definition are modified versions of \( \text{curl}\Omega \) and \( \text{div}S \). They were discovered in [26,33] and play a fundamental role in our analysis. Specifically, the equations of Theorem 2.8 show that \( C \) and \( D \) satisfy transport equations whose source terms i) are one degree more differentiable than expected and ii) exhibit remarkable null structures. We exploit Property i) when deriving \( \text{div-curl-transport} \) estimates to exhibit a gain in differentiability for \( \Omega \) and \( S \); see Theorem 9.10. As is explained in [14,16,26,51,33], Property ii) is crucial for the study of shock formation without symmetry assumptions; see also Subsubsect. 2.1.3.

Definition 2.1 (Modified fluid variables). We define the Cartesian components of the \( \Sigma_t \)-tangent vectorfield \( C \) and the scalar function \( D \) as follows, \((i = 1, 2, 3)\):

\[
C^i := \exp(-\rho)(\text{curl}\Omega)^i + \exp(-3\rho)c^{-2}\frac{P}{\rho}S^a\partial_a v^i - \exp(-3\rho)c^{-2}\frac{P}{\rho}(\text{div}v)S^i, \tag{21a}
\]

\[
D := \exp(-2\rho)\text{div}S - \exp(-2\rho)\bar{S}a\partial_a \rho, \tag{21b}
\]

where \( \text{curl} \) denotes the standard Euclidean curl operator on \( \Sigma_t \) and \( \text{div} \) denotes the standard Euclidean divergence operator on \( \Sigma_t \).
2.1.2. The acoustical metric $g$, basic properties of $g$ and $B$, and classification of vectors and hypersurfaces. The acoustical metric drives the propagation of sound waves and is featured prominently in Theorem 2.8.

**Definition 2.2** (The acoustical metric and its inverse). We define the *acoustical metric* $g$ and the inverse *acoustical metric* $g^{-1}$ relative to the Cartesian coordinates as follows:

$$
ge := -dt \otimes dt + c^{-2} \sum_{a=1}^{3} (dx^a - v^a dt) \otimes (dx^a - v^a dt), \quad (22a)$$

$$
g^{-1} := -B \otimes B + c^2 \sum_{a=1}^{3} \partial_a \otimes \partial_a. \quad (22b)$$

It is straightforward to check that indeed, the $4 \times 4$ matrix with components $(g^{-1})^{\alpha \beta}$ is the inverse of the $4 \times 4$ matrix with components $g_{\alpha \beta}$.

Throughout, if $X$ and $Y$ are vectors, then $g(X, Y) := g_{\alpha \beta} X^\alpha Y^\beta$ denotes their inner product with respect to $g$.

Most of the geometric constructions in this paper are tied to $g$.

Thus, in the rest of the paper, we lower and raise lowercase Greek “spacetime” indices with $g$ and $g^{-1}$, e.g., $B_\alpha := g_{\alpha \beta} B^\beta$.

On a few occasions, we will find it convenient to explicitly distinguish between a vectorfield and its $g$-dual one-form. Specifically, if $X$ is a vectorfield, then we sometimes use the notation $X_\alpha$ to denote the corresponding $g$-dual one-form, i.e.,

$$(X_\alpha)_\beta := g_{\alpha \beta} X^\beta. \quad (23)$$

We now provide the following basic definition, also tied to $g$, which plays a key role in our analysis.

**Definition 2.3** ($g$-spacelike, $g$-timelike, and $g$-null). Vectors $X$ are classified as follows:

$$
g(X, X) < 0 \quad \text{g-timelike}, \quad (24a)$$

$$
g(X, X) = 0 \quad \text{g-null}, \quad (24b)$$

$$
g(X, X) > 0 \quad \text{g-spacelike}. \quad (24c)$$

Hypersurfaces $\mathcal{H}$ of $\mathbb{R}^{1+3}$ are classified as follows, where $H$ denotes its $g$-normal vectorfield:

$$
g(H, H) < 0 \quad \text{at all points in } \mathcal{H} \quad \text{g-spacelike}, \quad (25a)$$

$$
g(H, H) = 0 \quad \text{at all points in } \mathcal{H} \quad \text{g-null}, \quad (25b)$$

$$
g(H, H) > 0 \quad \text{at all points in } \mathcal{H} \quad \text{g-timelike}. \quad (25c)$$

Moreover, if $\mathcal{S}$ is a co-dimension two submanifold of $\mathbb{R}^{1+3}$, we say that $\mathcal{S}$ is $g$-spacelike if at each of its points, all non-zero vectors $Y$ that are tangent to $\mathcal{S}$ verify $g(Y, Y) > 0$.

We close this subsection with a simple lemma that exhibits some basic properties of $B$.

**Lemma 2.4** (Basic properties of $B$). The vectorfield $B$ defined by (15) is $g$-timelike and has $g$-unit-length:

$$
g(B, B) = -1. \quad (26)$$

Moreover, relative to the Cartesian coordinates, we have

$$
B_\alpha = -(dt)_\alpha = -\delta^0_\alpha, \quad (27)$$

where $\delta^0_\alpha$ is the Kronecker delta. Thus, $B$ is $g$-orthogonal to $\Sigma_t$, i.e., $g(B, V) = 0$ for all vectorfields $V$ that are tangent to $\Sigma_t$.

Finally, we have $B^0 = 1$, which implies in particular that $B$ is future-directed.

**Proof.** The lemma follows from straightforward calculations relative to the Cartesian coordinates, based on (5) and (22a). \hfill $\square$

29 Other authors have defined the acoustical metric to be $c^2 g$. We prefer our definition because it implies that $(g^{-1})^{00} = -1$, which simplifies the presentation of many formulas.

30 If $X$ is $g$-timelike or $g$-null, then we say that $X$ is future-directed if its Cartesian component $X^0 = X_t$ is positive.
2.1.3. Null forms relative to \( g \). The statement of Theorem 2.2 refers to “null forms relative to \( g \),” where \( g \) is the acoustical metric from Def. 2.2. We will now briefly describe their importance in the study of shock waves without symmetry assumptions, since this line of investigation is a primary motivating factor for the results of this paper. By definition, null forms relative to \( g \) are linear combinations (with coefficients depending on \( p, v, s, \Omega \), and \( S \) – but not their derivatives) of the standard null forms relative to \( g \), which we define in Def. 2.3. Since Klainerman’s foundational work \[22\], it has been understood that (quadratic) null form nonlinearities are “weak” in the sense that, at least in the setting of wave-like PDEs, they often allow for proofs of small-data global existence. That is, null forms are quadratic terms exhibiting special cancellations that allow for small global solutions. Null forms are also important in other contexts, such as the study of low regularity well-posedness \[19\][21]. We clarify that in fact, there are different classes of null forms, and that Klainerman’s notion of a null form was adapted to the Minkowski metric, that is, to the geometry of special relativity. Perhaps surprisingly, appropriately defined null forms also play a crucial role in the theory of shock formation. By “appropriately defined,” we mean that the null forms in the theory of shocks do not coincide \[31\] with the null forms from Klainerman’s framework \[22\]; see the next paragraph for further discussion. In the context of shock waves, the appropriately defined null forms are also “weak” in the sense that, at least in certain solution regimes, they are not strong enough to prevent the formation of shocks. That is, in compressible fluid mechanics, shocks are singularities driven by “strong” derivative-quadratic Riccati-type terms, and null forms relative to \( g \) do not contain any such Riccati-type interaction; and see \[26\][32][33] for further discussion.

Let us clarify the phrase “appropriately defined” from the previous paragraph. In the study of shock formation without symmetry assumptions, the precise nonlinear structure of the null forms is crucial, in particular more important than it is in the context of proving small-data global existence. That is, proofs of small-data global existence (say, for wave equations on \( \mathbb{R}^{1+3} \)) are typically stable under the addition of higher-order nonlinearities to the equations, such as perturbing a null form by adding derivative-cubic nonlinearities. In contrast, in the context of shock waves, the needed “special cancellations” (which render the null form weak) become visible only when one decomposes the derivative-quadratic terms relative to the characteristic surfaces (i.e., acoustically null hypersurfaces in the context of compressible Euler flow), which, in quasilinear problems, are evolutionarily determined by the solution. Thus, since in the context of compressible Euler flow, acoustically null hypersurfaces are determined by \( g \), the relevant “special cancellations” are inextricably tied to \( g \). For this reason, we speak of “null forms relative to \( g \).” Specifically, the null form \( \Omega^{(8)} \) defined in \[29\] explicitly depends on \( g \). In the context of compressible fluid mechanics, the special cancellations can be described as follows: if \( \Omega(\partial \phi, \partial \tilde{\phi}) \) is a null form relative to \( g \) (in particular \( \Omega \) is derivative-quadratic) and \( \mathcal{N} \) is any hypersurface that is null \[32\] (i.e., characteristic) relative to \( g \), then there holds a decomposition of the schematic form

\[
\Omega(\partial \phi, \partial \tilde{\phi}) = \mathcal{T} \cdot \partial \tilde{\phi} + \mathcal{T} \cdot \partial \phi, \tag{28}
\]

where \( \mathcal{T} \) denotes a derivative in a direction tangent to \( \mathcal{N} \) and \( \partial \phi \) denotes a generic derivative.

The connection between the decomposition of null forms relative to \( g \) highlighted in \[28\] and shock formation is as follows: in all solution regimes for the compressible Euler equations in which stable shock formation without symmetry assumptions has been shown, one can construct a foliation of spacetime by acoustically null hypersurfaces \( \mathcal{N} \) such that the \( \mathcal{T} \)-derivatives of the solution are much less singular than its derivatives in directions transversal to \( \mathcal{N} \). In fact, in all known results, it is precisely the transversal derivatives of the solution that blow up, as in the model case of Burgers’ equation. Thus, null forms are linear in the terms that blow up. Therefore, they are much less singular than generic quadratic terms \( \partial \phi \cdot \partial \tilde{\phi} \), which are what drive the formation of the shock. This explains why null forms represent “weak” nonlinearities in the context of shock formation, and also clarifies why shock formation proofs are typically unstable under perturbing the equations by generic higher-order nonlinearities, such as derivative-cubic ones: if the blowup is driven by derivative-quadratic terms, then generic derivative-cubic (or higher-order) terms would be expected to blow up at an even faster rate, possibly radically altering the nature of the singularity.

\[31\] The antisymmetric null forms defined in \[29\] do appear in Klainerman’s framework \[22\], but the null form \( \Omega^{(8)} \) defined \[29\] does not; the analog of \( \Omega^{(8)} \) in Klainerman’s framework is \( \Omega^{(m)} \), where \( m \) is the Minkowski metric.

\[32\] Acoustically null hypersurfaces have \( g \)-normals \( L \) that are null, i.e., \( g(L, L) = 0 \).
or even preventing it altogether. The importance of null forms relative to \( g \) in the context of proving shock formation is further explained in Refs. 14, 16, 26, 31, 33.

We close our discussion by highlighting a connection between the good properties of null forms relative to \( g \) in the context of shock formation and the results of the present paper:

In our main integral identities, in the case that the lateral boundary is acoustically null (let us refer to it as \( \mathcal{N} \)), the integrals along \( \mathcal{N} \) involve only tangential derivatives \( T \); see (160) for the precise statement. This structure is crucial for controlling these error integrals in the context of shock-forming solutions, in analogy with the way that null forms relative to \( g \) lead to “weak” (i.e., controllable) error terms.

We now define the standard null forms relative to \( g \).

**Definition 2.5** (Standard null forms relative to \( g \)). The standard null forms \( \Omega^{(g)}(\cdot, \cdot) \) (relative to \( g \)) and \( \Omega_{(\alpha\beta)}(\cdot, \cdot) \), \( 0 \leq \alpha < \beta \leq 3 \), act on pairs \((\phi, \phi)\) of scalar-valued functions as follows:

\[
\Omega^{(g)}(\partial\phi, \partial\phi) := (g^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi, \tag{29a}
\]

\[
\Omega_{(\alpha\beta)}(\partial\phi, \partial\phi) := \partial_\alpha \phi \partial_\beta \phi - \partial_\alpha \phi \partial_\beta \phi. \tag{29b}
\]

**2.1.4. Covariant wave operator.** The statement of Theorem 2.8 refers to the covariant wave operator \( \Box_g \) of the acoustical metric \( g \), defined below in Def. 2.6. The main significance of covariant wave operators is that sophisticated geo-analytic technology has been developed for such operators. It allows one to construct commutator and multiplier vectorfields that are dynamically adapted to that technology. It turns out that this technology is crucial for the study of shocks without symmetry assumptions, in particular for deriving energy estimates with controllable error terms both for the solution and its higher derivatives; we refer readers to Refs. 14, 16, 26, 31, 33 for further discussion of these issues. The technology is also important for the study of low-regularity solutions in the context of quasilinear problems; we refer readers to Ref. 13 for further discussion. In the present article, in our derivation of energy identities, we will use only a basic version of the multiplier method, which we review in Subsect. 9.2.

**Definition 2.6** (Covariant wave operator). The covariant wave operator \( \Box_g \) acts on scalar-valued functions \( \phi \) according to the following formula:

\[
\Box_g \phi := \frac{1}{\sqrt{|\det g|}} \partial_\alpha \left\{ \sqrt{|\det g|}(g^{-1})^{\alpha\beta} \partial_\beta \phi \right\}. \tag{30}
\]

**Remark 2.7** (Coordinate invariance of \( \Box_g \)). It is a standard fact that RHS (30) is coordinate invariant.

**2.1.5. Some notation.** We use the following notation in our statement of Theorem 2.8.

**Notation 2.1** (Differentiation with respect to state-space variables via semicolons). If \( f = f(\rho, s) \) is a scalar function, then we use the following notation to denote partial differentiation with respect to \( \rho \) and \( s \):

\[
f_{:\rho} := \frac{\partial f}{\partial \rho} \quad \text{and} \quad f_{:s} := \frac{\partial f}{\partial s}.
\]

Moreover, \( f_{:\rho:s} := \frac{\partial^2 f}{\partial s \partial \rho} \) and we use similar notation for other higher-order partial derivatives of \( f \) with respect to \( \rho \) and \( s \).

**2.2. The geometric wave-transport-divergence-curl formulation of the compressible Euler equations.** Our main results fundamentally rely on the following formulation of the compressible Euler equations, derived in Ref. 33.

**Theorem 2.8.** Let \( \bar{\rho} > 0 \) be any constant background density\(^3\) and assume that \((\rho, v^1, v^2, v^3, s)\) is a \( C^3 \) solution\(^4\) to the compressible Euler equations (4a)–(4c) in three spatial dimensions under an arbitrary equation of state \( p = p(\rho, s) \) with positive sound speed \( c \) (see (1)). Let \( B \) be the material derivative vectorfield defined in (5), let \( g \) be the acoustical metric from Def. 2.2 and let \( C \) and \( D \) be the modified fluid variables

---

\(^3\)Recall that the definition (34) of \( \rho \) depends on \( \bar{\rho} \).

\(^4\)We have made the \( C^3 \) assumption only for convenience, i.e., so that all of the quantities on the left- and right-hand sides of the equations of Theorem 2.8 are at least continuous. In applications, one can make sense of the equations and solutions in a distributional sense under weaker regularity assumptions (for example, in suitable Sobolev spaces).
Then the scalar-valued functions \( \rho \), \( \Omega_i \), \( s \), \( S^i \), \( \text{div} \Omega \), \( C^i \), \( D \), and \((\text{curl} S)^i \), \((i = 1, 2, 3)\), also solve the following equations, where \( \varepsilon_{ijk} \) is the fully antisymmetric symbol normalized by \( \varepsilon_{123} = 1 \) and the Cartesian component functions \( v^i \) are treated as scalar-valued functions under covariant differentiation on LHS (31a):

**Covariant wave equations**

\[
\Box g v^i = -c^2 \exp(2\rho) C^i + \Omega^i_{(v)} + \mathcal{L}^i_{(v)},
\]

\[
\Box g \rho = -\exp(\rho) \frac{\partial s}{\partial \rho} D + \Omega_{(\rho)} + \mathcal{L}_{(\rho)},
\]

\[
\Box g s = c^2 \exp(2\rho) D + \mathcal{L}_{(s)}.
\]

**Transport equations**

\[
B \Omega^i = \mathcal{L}_{(\Omega)}^i,
\]

\[
B s = 0,
\]

\[
B S^i = \mathcal{L}_{(S)}^i.
\]

**Transport-divergence-curl system for the specific vorticity**

\[
\text{div} \Omega = \mathcal{L}_{(\text{div} \Omega)},
\]

\[
B C^i = -2\delta_{jk} \varepsilon_{iab} \exp(-\rho)(\partial_a v^j)\partial_b \Omega^k + \varepsilon_{ijk} \exp(-\rho)(\partial_a v^j)\partial_j \Omega^k + \exp(-3\rho)c^{-2}p_s \left\{ (BS^a)\partial_a v^i - (Bv^a)\partial_a S^a \right\} + \exp(\rho)c^{-2}p_s \Omega^i_{(C)} + \mathcal{L}^i_{(C)}.
\]

**Transport-divergence-curl system for the entropy gradient**

\[
B D = 2 \exp(-2\rho) \left\{ (\partial_a v^a)\partial_b S^b - (\partial_a S^b)\partial_b v^a \right\} + \exp(\rho)\delta_{ab}(\text{curl}\Omega)^a S^b + \Omega_{(D)} + \mathcal{L}_{(D)},
\]

\[
(\text{curl} S)^i = 0.
\]
Above, \( \Omega_{(v)}^i \), \( \Omega_{(\rho)}^i \), \( \Omega_{(c)}^i \), and \( \Omega_{(D)} \) are the null forms relative to \( g \) defined by

\[
\Omega_{(v)}^i := - \left\{ 1 + c^{-1}c_p \right\} (g^{-1})^{\alpha\beta}(\partial_\alpha \rho)\partial_\beta v^i, \tag{35a}
\]
\[
\Omega_{(\rho)} := -3c^{-1}c_p (g^{-1})^{\alpha\beta}(\partial_\alpha \rho)\partial_\beta \rho + \left\{ (\partial_\alpha v^a)\partial_b v^b - (\partial_\alpha v^b)\partial_b v^a \right\}, \tag{35b}
\]
\[
\Omega_{(c)}^i := \exp(-3p)c^{-2}\frac{P_{s}}{\tilde{\theta}} S^i \left\{ (\partial_a v^b)\partial_b v^a - (\partial_a v^a)\partial_b v^b \right\} \tag{35c}
\]
\[
+ \exp(-3p)c^{-2}\frac{P_{s}}{\tilde{\theta}} S^b \left\{ (\partial_a v^b)\partial_b v^a - (\partial_a v^a)\partial_b v^b \right\}
\]
\[
+ 2\exp(-3p)c^{-2}\frac{P_{s}}{\tilde{\theta}} S^a \left\{ (\partial_a \rho)B v^i - (\partial_a v^i)B \rho \right\}
\]
\[
+ 2\exp(-3p)c^{-3}c_p \frac{P_{s}}{\tilde{\theta}} S^a \left\{ (\partial_a \rho)B v^i - (\partial_a v^i)B \rho \right\}
\]
\[
+ \exp(-3p)c^{-2}\frac{P_{s}}{\tilde{\theta}} S^a \left\{ (\partial_a t)v^i - (\partial_a v^i)B \rho \right\}
\]
\[
+ \exp(-3p)c^{-2}\frac{P_{s}}{\tilde{\theta}} S^b \left\{ (\partial_a t)v^i - (\partial_a v^i)B \rho \right\}
\]
\[
+ \exp(-3p)c^{-2}\frac{P_{s}}{\tilde{\theta}} S^i \left\{ (\partial_a \rho)B v^a - (\partial_a v^a)B \rho \right\}
\]
\[
+ \exp(-3p)c^{-3}c_p \frac{P_{s}}{\tilde{\theta}} S^i \left\{ (\partial_a \rho)B v^a - (\partial_a v^a)B \rho \right\},
\]
\[
\Omega_{(D)} := 2\exp(-2p)S^a \left\{ (\partial_a v^b)\partial_b \rho - (\partial_a \rho)\partial_b v^b \right\}. \tag{35d}
\]

In addition, the terms \( \Omega_{(v)}^i \), \( \Omega_{(\rho)} \), \( \Omega_{(s)} \), \( \Omega_{(D)} \), \( \Omega_{(div D)} \), and \( \Omega_{(c)} \), which are at most linear in the derivatives of the unknowns, are defined as follows:

\[
\Omega_{(v)}^i := 2\exp(\rho)\epsilon_{iab}(B v^a)\Omega^b - \frac{P_{s}}{\tilde{\theta}} \epsilon_{iab} \Omega^a S^b \tag{36a}
\]
\[
- \frac{1}{2} \exp(-\rho)\frac{P_{s}}{\tilde{\theta}} S^a \partial_a v^i
\]
\[
- 2\exp(-\rho)\frac{P_{s}}{\tilde{\theta}} \delta_{ab} \rho \partial_b \partial_a v^i
\]
\[
\Omega_{(\rho)} := -\frac{5}{2} \exp(-\rho)\frac{P_{s}}{\tilde{\theta}} \delta_{ab} S^a \partial_a \rho
\]
\[
- \exp(-\rho)\frac{P_{s}}{\tilde{\theta}} \epsilon_{iab} (B v^a) S^b \tag{36b}
\]
\[
\Omega_{(s)} := \Omega^a \partial_a v^i - \exp(-2p)c^{-2}\frac{P_{s}}{\tilde{\theta}} \epsilon_{iab} (B v^a) S^b, \tag{36c}
\]
\[
\Omega_{(D)} := -\Omega^a \partial_a \rho, \tag{36d}
\]
\[
\Omega_{(c)} := 2\exp(-3p)\epsilon_{iab}(B v^a)\delta_{ab} \Omega^b S^b \tag{36e}
\]
\[
- \exp(-3p)\epsilon_{iab}(B v^a)\delta_{ab} \Omega^b S^b \tag{36f}
\]
\[
+ \exp(-3p)c^{-2}\frac{P_{s}}{\tilde{\theta}} \delta_{ab} (B v^a) S^b S^i \tag{36g}
\]

The terms on the first four lines of RHS \( \Omega_{(v)}^i \) and the first product on RHS \( \Omega_{(c)}^i \) are also null forms relative to \( g \). We have explicitly displayed these null forms since in applications, they are more difficult to treat than \( \Omega_{(v)}^i \), \( \Omega_{(\rho)} \), \( \Omega_{(c)}^i \), and \( \Omega_{(D)} \); from the point of view of regularity, the explicitly displayed null forms need to be treated with div-curl-transport identities.
3. THE SPACETIME REGIONS AND THEIR TOPOLOGY AND GEOMETRY

In this section, we state our assumptions on the spacetime region \( M \) on which we will derive our main integral identities, as well as the acoustical time function \( \tau \) that foliates \( M \). We then define a collection of geometric tensors associated to \( M \) and exhibit their basic properties.

3.1. The spacetime region \( M \), the acoustical time function \( \tau \), and related constructions. Until Sect. 10, our results concern compressible Euler solutions on subsets of spacetime, denoted by \( M \) and depicted in Fig. 1. We assume that \( M \) is a compact, connected manifold with corners; the purpose of the latter assumption is that it allows us to use Stokes’ theorem (more precisely, in the context of the present paper, the divergence theorem) on \( M \). We assume that \( \partial M \) (i.e., the boundary of \( M \), viewed as a subset of \( \mathbb{R}^{1+3} \)) can be decomposed\(^{39}\) as

\[
\partial M = \bar{\Sigma}_0 \cup \bar{\Sigma}_T \cup \bar{H},
\]

where \( \bar{\Sigma}_T \) is the top boundary of \( M \), \( \bar{\Sigma}_0 \) is the bottom boundary of \( M \), \( \bar{H} \) is the lateral boundary of \( M \), and just below, we explain the meaning of the subscripts on the symbol “\( \bar{\Sigma} \).” We assume that for some \( T > 0 \), \( M \) is foliated by \( g \)-spacelike hypersurface portions \( \Sigma_\tau \) for \( \tau \in [0,T] \). More precisely, we assume that \( \tau \) is a smooth acoustical time function (not necessarily equal to Cartesian time) on an open, connected subset \( \mathcal{O} \) of \( \mathbb{R}^{1+3} \) containing \( M \). By “acoustical time function,” we mean a time function in the sense of Lorentzian geometry (see \( \cite{34} \) Section 8.2) for background material on time functions, where the Lorentzian metric is the acoustical metric \( g \). Specifically, we assume that on \( \mathcal{O} \), \( \tau \) has non-vanishing gradient and that \( D \tau \) is past-directed\(^{37}\) where \( D \tau \) denotes the gradient one-form of \( \tau \), and \( (g^{-1})^{\alpha \beta} \partial_\alpha \partial_\beta \tau < 0 \). We then set

\[
\Sigma_{\tau'} := M \cap \{(t,x^1,x^2,x^3) \in \mathcal{O} \mid \tau(t,x^1,x^2,x^3) = \tau' \}.
\]

Note that our assumptions imply that\(^{39}\)

\[
M = \bigcup_{\tau \in [0,T]} \Sigma_{\tau}.
\]

We assume that \( \bar{H} \) is the intersection of \( M \) with a smooth, three-dimensional embedded submanifold of \( \mathcal{O} \), and that \( \bar{H} \) is \( g \)-spacelike at all of its points or \( g \)-null at all of its points (in the sense of Def. 2.3). This implies, in particular, that \( \bar{H} \) is transversal\(^{30}\) to the level sets of the acoustical time function \( \tau \). Finally, we assume that for \( \tau \in [0,T] \), \( \Sigma_\tau \) is diffeomorphic to the closed unit ball in \( \mathbb{R}^3 \). This implies, in particular, that the boundary of \( \Sigma_\tau \), viewed as a subset of the \( \tau \)-level set of the acoustical time function, and which we denote by \( \partial \Sigma_\tau \), is diffeomorphic to \( S^2 \).

The following subsets of spacetime, associated to \( M \), will play a fundamental role in the ensuing discussion.

**Definition 3.1** (Subsets of spacetime). For \( 0 \leq \tau \leq T \), we define

\[
\mathcal{S}_\tau := \bar{H} \cap \Sigma_\tau, \quad (40a)
\]

\[
\mathcal{H}_\tau := \bar{H} \cap M_\tau, \quad (40b)
\]

\[
M_\tau := \bigcup_{\tau' \in [0,\tau]} \Sigma_{\tau'}. \quad (40c)
\]

From the above definitions, it follows that \( M = M_T \); see Fig. 1. Moreover, we note that \( \bar{H} = \mathcal{H}_T \) and that for \( \tau \in [0,T] \), we have

\[
\mathcal{H}_\tau = \bigcup_{\tau' \in [0,\tau]} \mathcal{S}_{\tau'}. \quad (41)
\]

We refer to either of \( \mathcal{H} \) or \( \mathcal{H}_\tau \) as the “lateral hypersurface” or the “lateral boundary” of \( M \).

---

\(^{36}\)The union \(^{37}\) is not disjoint since, as we describe below, our assumptions imply that \( \bar{H} \) intersects \( \Sigma_\tau \) in two-dimensional submanifolds.

\(^{37}\)By “past-directed,” we mean that \( B\tau > 0 \), i.e., \( \tau \) increases along the integral curves of \( B \), like Cartesian time does (since \( Bt = 1 \)). By \(^{27}\), this is equivalent to the assumption that the opposite of the \( g \)-dual of the gradient of \( \tau \), namely \(- (g^{-1})^{\alpha \beta} \partial_\alpha \partial_\beta \tau \), is a future-directed vectorfield in the sense of Footnote 30.

\(^{38}\)The assumption \((g^{-1})^{\alpha \beta} \partial_\alpha \partial_\beta \tau < 0 \) implies that \( \Sigma_\tau \) is \( g \)-spacelike in the sense of Def. 2.3.

\(^{39}\)Throughout, we abuse notation by using the symbol “\( \tau \)” to denote both the acoustical time function and the values that it takes on; the precise meaning of the symbol will be clear from context.

\(^{40}\)That is, at every point in \( \bar{H} \cap \Sigma_\tau \), the normal of \( \Sigma_\tau \) is \( g \)-timelike and therefore cannot be parallel to the normal of \( \bar{H} \).
Remarkable localized integral identities for 3D compressible Euler flow

From the above assumptions, in the language of manifolds with corners, it follows that points in \( S_T \cup S_0 \) are index 1, that points in \( \partial M \setminus (S_T \cup S_0) \) are index 1, and that the remaining points in \( M \) (which belong to its interior) are index 0.

We also note that since \( H \) is transversal to the level sets of the acoustical time function, the following identity holds for \( \tau \in [0, T] \):

\[
\partial \tilde{\Sigma}_\tau = S_\tau.
\]

Thus, in view of our assumption that \( \tilde{\Sigma}_\tau \) is diffeomorphic to the closed unit ball in \( \mathbb{R}^3 \), it follows that \( S_\tau \) is diffeomorphic to \( S^2 \).

Finally, we note that the above assumptions imply that for \( \tau \in [0, T] \), we have

\[
\partial M_\tau := \text{the boundary of } M_\tau \text{ in } \mathbb{R}^{1+3} = \tilde{\Sigma}_0 \cup \tilde{\Sigma}_\tau \cup \tilde{H}_\tau.
\]

See Example 3.5 below for a canonical example of a family of spacetime regions that satisfy our assumptions: truncated backwards sound cones.

3.2. The vectorfields \( N, Q, \bar{N}, Z, H, \text{ and } \bar{H} \), and the scalar function \( \iota \). In this subsection, we define a collection of geometric vectorfields associated to \( M \). We also introduce alternate notation that we often use when the lateral boundary \( H \) is g-null.

**Definition 3.2** (The vectorfields \( N, Q, \bar{N}, Z, H, \text{ and } \bar{H} \), and the scalar function \( \iota \)). We define \( N \) to be the \( g \)-orthogonal to \( \tilde{\Sigma}_\tau \) and normalized by

\[
N_t = 1.
\]

Note that \( N \) is \( g \)-timelike since \( \tilde{\Sigma}_\tau \) is \( g \)-spacelike by assumption.

We next define \( Q \) to be the vectorfield that is \( g \)-orthogonal to \( \tilde{\Sigma}_\tau \) (i.e., parallel to \( N \) ) and normalized by

\[
Q_\tau = 1,
\]

where \( \tau \) is the acoustical time function from the beginning of Sect. 3.

We define \( \bar{N} \) to be the \( g \)-normal to \( H \), normalized by

\[
\bar{N}t = 1.
\]

We define \( Z \) to be the \( g \)-unit outer normal to \( S_\tau \) in \( \tilde{\Sigma}_\tau \). In particular, \( Z \) is tangent to \( \tilde{\Sigma}_\tau \), \( g \)-normal to \( S_\tau \), and satisfies

\[
g(Z, Z) = 1.
\]

We define \( H \) to be the vectorfield that is tangent to \( H \), \( g \)-orthogonal to \( S_\tau \), and normalized by

\[
Ht = 1.
\]

Finally, we define \( \iota \) the scalar function

\[
\iota := \frac{1}{H_\tau}
\]

and the vectorfield

\[
\bar{H} := \iota H.
\]

Note that by (27), (44) is equivalent to

\[
g(N, B) = -1,
\]

(46) is equivalent to

\[
g(N, B) = -1,
\]

41By definition, a point of index \( k \) is contained in a subset \( D_k \) of \( M \) such that \( D_k \) is diffeomorphic to a neighborhood of the origin in \( [0, \infty)^k \times \mathbb{R}^{4-k} \); the case \( k = 0 \) corresponds to the standard notion of a differentiable manifold.

42Since \( H \) and the \( g \)-dual of \( -D_\tau \) are both future-directed in the sense of Footnote 30 (the former by 48 and the latter by assumption), it follows that \( H_\tau > 0 \).
and (48) is equivalent to

\[ g(H, B) = -1. \] (53)

Note also that (49)-(50) imply that

\[ \tilde{H} \tau = 1. \] (54)

We also note that since the acoustical time function \( \tau \) is constant along each hypersurface \( \Sigma_\tau \), it follows from (27) and (44) that

\[ N^\alpha = \frac{(g^{-1})^{\alpha\beta} \partial_\beta \tau}{(g^{-1})^{\lambda\delta} \partial_\lambda \partial_\delta \tau} = -\frac{(g^{-1})^{\alpha\beta} \partial_\beta \tau}{B \tau}. \] (55)

**Remark 3.3.** For setting up the geometry, we find it convenient to normalize various vector fields with respect to Cartesian time \( \tau \), as we did in (44), (46), and (48). Nonetheless, our geometric identities will be able to accommodate foliations of spacetime regions with respect to arbitrary smooth acoustical time functions \( \tau \).

**Convention 3.4 (\( N \) vs. \( H \) and \( L \) vs. \( N \)).** If the lateral hypersurface \( H \) is g-null, we often refer to this as the “null case.” In the null case, we often use the alternate notation \( N \) in place of \( H \), \( N_\tau \) in place of \( H_\tau \), etc. Moreover, in the null case, we often use the notation

\[ L \] (56)

in place of \( N \) since in Lorentzian geometry, \( L \) is common notation for an “ingoing” null vector (where \( L \) will be “ingoing” thanks to our assumptions in Subsect. 3.3).

### 3.3. The positivity of \( \tilde{z}, \tilde{h}, \) and \( \perp \) and some consequences

In this subsection, we introduce the scalar functions \( \tilde{z} \) and \( \tilde{h} \), whose assumed positivity, together with the positivity of \( \perp \) (see (58)), is crucial for all of our main results. We also discuss some geometric and topological consequences of the positivity.

Specifically, we make the following assumptions on \( \mathcal{M} \).

- If \( H \) is either g-spacelike or g-null, then we assume that the scalar functions \( \tilde{z} \) and \( \tilde{h} \) are such that

\[ g(Z, N) := -\tilde{z}, \quad \tilde{z} > 0, \] (57a)

\[ g(H, N) := -\tilde{h}, \quad \tilde{h} > 0. \] (57b)

Note that (52) and the fact that \( B \tau > 0 \) (see Footnote 37) together imply that \( N_\tau > 0 \), i.e., along each sphere \( S_\tau = H \cap \Sigma_\tau, N \) (which by definition is g-orthogonal to \( H \)) points to the future of \( \Sigma_\tau \). The assumption (54) is tantamount to the assumption that \( \tilde{H} \) is in fact ingoing to the future in the sense that when \( H \) is g-spacelike, \( N \) points outward to \( \mathcal{M} \). To explain why \( N \) is outward-pointing when \( H \) is g-spacelike, we first note that at a given point \( q \in \tilde{H} \), the set of vectors belonging to the tangent space of spacetime at \( q \) that are not tangent to \( H \) is equal to the disjoint union of two connected components: \( \{ X \mid g(X, N) < 0\} \cup \{ X \mid g(X, N) > 0\} \). One of these components is the set of inward-pointing vectors to \( \mathcal{M} \) at \( q \), and the other is the set of outward-pointing vectors to \( \mathcal{M} \) at \( q \). Since \( N \) is g-timelike by assumption (and thus \( g(N, N) < 0 \)), (57a) guarantees that \( N \) and \( Z \) belong to the same connected component. Thus, since along each sphere \( S_\tau, Z \) points outward to \( \mathcal{M} \) by assumption, we conclude that when \( H \) is g-spacelike, \( N \) also points outward to \( \mathcal{M} \). Similarly, since \( N \) points outward to \( \mathcal{M} \) along each point in \( \Sigma_\tau \), (57b) is tantamount to the assumption that along each sphere \( S_\tau \subset \tilde{\Sigma}_\tau \), the generator \( H \) of \( H \) (which is tangent to \( H \)) also points outward to \( \mathcal{M}_\tau \).

We next observe that (49), (55), (57b), and the assumption \( B \tau > 0 \) (see Footnote 37) imply that

\[ \perp = \frac{1}{\tilde{H} \tau} = -\frac{1}{g(H, N) B \tau} = \frac{1}{h B \tau} > 0. \] (58)

From the perspective of analysis, the positivity of \( \perp, \tilde{z}, \) and \( \tilde{h} \) is important for the coerciveness of some key terms in our integral identities; see, for example, the \( S_\tau \) integrals on LHSs (165a) and (165b).
Example 3.5 (Canonical examples: truncated backwards sound cones). Here we provide canonical examples of spacetime regions $\mathcal{M}$ with $g$-null lateral boundaries to which the results of this paper apply; such regions arise in the study of shock formation. For convenience, we consider the case in which the acoustical time function $\tau$ is equal to the Cartesian time function $t$. Let $S_0$ be any embedded two-dimensional submanifold of $\Sigma_0$ that is differentiable to $\mathbb{S}^2$ (for example, $S_0$ could be equal to $\mathbb{S}^2 \subset \Sigma_0 \subset \mathbb{R}^3$). At each $q \in S_0$, there is a unique vector $L_q \in T_q S_0$ that is $g$-null, future-directed, $g$-orthogonal to $S_0$, and inwards-pointing in the sense that its $g$-orthogonal projection onto $\Sigma_0$ points inwards to $S_0$. Next, for each fixed $q \in S_0$, we construct the null geodesic curve $\gamma_q : I_q \to \mathbb{R}^{1+3}$ with initial data $\gamma_q(0) = q$ and $\dot{\gamma}_q(0) = \ell_q$, where $I_q = [0, A_q]$ is $q$-dependent interval of parameter-time. That is, with $\gamma_q = \gamma_q(\lambda), \dot{\gamma}_q(\lambda) := \frac{d}{d\lambda} \gamma_q(\lambda)$, and with $D$ denoting the Levi–Civita connection of $g$ (see Subsect. 3.8), we solve the geodesic equation $D_q \gamma_q = 0$ with initial conditions $\gamma_q(0) = q$ and $\dot{\gamma}_q(0) = \ell_q$. Assuming that the compressible Euler flow (on which $g$ depends) is smooth, standard existence and uniqueness theory for ODEs with parameter-dependent initial conditions and the compactness of $S_0$ together imply that the interval $I_q$ can be chosen to be uniform over $q$ (let’s refer to the uniform interval as “$T$”), that there is a $T > 0$ such that $\min_{q \in S_0} \max_{\lambda \in I} \gamma_q(\lambda) > T$ (where $\gamma_q(\lambda)$ is the Cartesian time component of the point $\gamma_q(\lambda)$), and such that the set $H := \{\gamma(I) \mid q \in S_0 \cap [0, T] \times \mathbb{R}^3$ is an embedded three-dimensional manifold-with-boundary. Moreover, the results of [13] Section 9 can be used to show that $H$ is $g$-null, and that $H$ is the lateral boundary of a region of $\mathcal{M}$ (in this case a truncated backwards sound cone with a flat top and bottom) satisfying the assumptions stated in Subsect. 3.1 (where for this example, $\tau = t$).

Remark 3.6 (The regions could be extended to substantially more general spacetime regions). The results of this paper could be extended to apply to substantially more general spacetime regions $\mathcal{M}$, and it is only for convenience and concreteness that we have made the precise assumptions stated above. For example, $\mathcal{M}$ need not be compact and could “extend to spatial infinity” (e.g., $\mathcal{M}$ could be the portion of the exterior of an outgoing sound cone that lies in between two constant-time hyperplanes). The most crucial assumptions are that the lateral boundary $H$ is $g$-spacelike or $g$-null, that positivity properties in the spirit of (57a)-(57b) hold (these are needed to ensure the coerciveness of our integral identities), and that $S_0 = H \cap \Sigma_T$ is a closed manifold (this last assumption is helpful in the sense that it guarantees that no boundary terms occur when we integrate by parts over $S_0$ in the first step of the proof of Prop. 7.1).

Remark 3.7 (The regions $\mathcal{M}_T$ are acoustically globally hyperbolic). Although it is not directly needed in the paper, we can now explain why the spacetime regions $\mathcal{M} = \mathcal{M}_T$ that we study are globally hyperbolic with respect to the acoustical metric $g$. That is, we will show that $\Sigma_0$ is a Cauchy hypersurface in $\mathcal{M}_T$. We will consider in detail the case where the lateral boundary $H_T$ is $g$-spacelike; the $g$-null case can be addressed using similar arguments. More precisely, we will show that every past-inextendible future-directed $g$-causal curve $\gamma$ contained in $\mathcal{M}_T$ must intersect $\Sigma_0$; see [34] Chapter 8 for background material on causality, and note that causal curves do not have to be differentiable. We recall that we are considering only smooth fluid solutions (see Remark 1.5), and thus $B$, $g$, etc. are smooth on $\mathcal{M}_T$. We can assume that $\gamma$ is parametrized by $\int$, that is, there exists an interval $I$ such that the domain of $\gamma$ is $I$ and such that for $t \in I$, $\gamma(t) = t$ and $\gamma(t) \in \mathcal{M}_T$. We argue by contradiction, assuming that $\gamma$ has a past endpoint $q \in \mathcal{M}_T$ such that $q \notin \Sigma_0$. It is straightforward to see that $\gamma$ can be continuously extended so that $q = \gamma(a)$, where $a$ is the left-endpoint of the closure of $I$, and that $\gamma(a)$ must be a boundary point of $\mathcal{M}_T$ not lying in $\Sigma_0$, that is, $\gamma(a) \in (H_T \setminus \Sigma_0) \cup \Sigma_T$. From the discussion just below (57a), we see that if $\gamma(a) \in H_T \setminus \Sigma_0$, then $N|\gamma(a)$ points outward to $\mathcal{M}_T$ at $\gamma(a)$. It follows that there is an $\epsilon > 0$ such that we can extend $\gamma$ as a causal curve such that relative to the Cartesian coordinates, $\gamma(t) \equiv N|\gamma(a)$ for $t \in [a - \epsilon, a]$. This contradicts the assumption that $\gamma$ is inextendible. Similarly, from the discussion just below (57a), we see that if $\gamma(a) \in \Sigma_T$, then $N|\gamma(a)$ points outward to $\mathcal{M}_T$ at $\gamma(a)$, and there is an $\epsilon > 0$ such that we can extend $\gamma$ as a causal curve such that relative to the Cartesian coordinates, $\gamma(t) \equiv N|\gamma(a)$ for $t \in [a - \epsilon, a]$, again contradicting the assumption that $\gamma$ is inextendible. In total, we have shown that $\gamma(a) \in \Sigma_0$ as desired.

43More precisely, [13] Section 9 addressed the existence of outgoing $g$-null cones emanating from a point, but the results can readily be extended so as to apply to the present example.

44We can assume this because $\gamma$ is $g$-causal and because [5] and [225] imply that the gradient of $t$ is $g$-timelike.
3.4. Additional geometric quantities associated to $\mathcal{M}$. In this subsection, we define some additional geometric quantities that we use to prove our main results, and we prove a simple lemma that yields some identities.

**Definition 3.8** ($\eta, \ell, \nu, q, \nu, \hat{N},$ and $\hat{N}$). We define $\eta$ to be the following scalar function, which is positive when $\vec{N}$ is $g$-timelike (because in this case $\vec{H}$ is $g$-spacelike and thus $H$ is $g$-spacelike with $g(\vec{H},H) > 0$):

$$\eta := \sqrt{g(\vec{H},\vec{H})}.$$  

Similarly, we define $\ell$ to be the following scalar function, which is positive when $\vec{H}$ is $g$-spacelike:

$$\ell := \sqrt{g(\vec{H},\vec{H})}.$$  

In addition, we define $\nu > 0$ to be the following scalar function, which is positive because $\bar{\Sigma}_\tau$ is $g$-spacelike:

$$\nu := \sqrt{-g(N,N)}.$$  

Similarly, we define $q > 0$ by

$$q := \sqrt{-g(Q,Q)}.$$  

Next, we define $\nu$ to be the following scalar function, which is positive when $N$ is $g$-timelike and vanishing when $N$ is $g$-null:

$$\nu := \sqrt{-g(N,N)}.$$  

In addition, we define $\hat{N}$ to be the following vectorfield ($\hat{N}$ is the $g$-unit future-directed (see Footnote 30) normal to $\bar{\Sigma}_\tau$):

$$\hat{N}^\alpha := N^\alpha \nu.$$  

Finally, when $N$ is $g$-timelike, we define $\hat{N}$ to be the following vectorfield ($\hat{N}$ is the $g$-unit future-directed normal to $\vec{H}$):

$$\hat{N}^\alpha := N^\alpha \nu.$$  

**Lemma 3.9** (Some convenient identities). Assume that $\vec{H}$ is $g$-spacelike, let $\nu > 0$ be the scalar function defined in (61), let $\nu > 0$ be the scalar function defined in (63), let $\hat{N}$ be the vectorfield defined in (64), and let $\hat{N}$ be the vectorfield defined in (65). Then the following identities hold:

$$g(\vec{B},\hat{N}) = -\frac{1}{\nu},$$  

$$g(\vec{B},\vec{\hat{N}}) = -\frac{1}{\nu}.$$  

Moreover, let $\vec{N}, \vec{H},$ and $N$ be the vectorfields from Def 3.2, let $\eta \geq 0$ be the scalar function from Def 3.8, and let $h > 0$ be the scalar function defined in (57b). Then the following identity holds:

$$N = \frac{h}{h + \eta^2} H + \frac{\eta^2}{h + \eta^2} N.$$  

In addition,

$$g(N,N) = -\frac{h^2 + \eta^2 \nu^2}{h + \eta^2}.$$  

Furthermore,

$$\frac{\nu^2}{\eta^2} = \frac{\eta^2 \nu^2 + h^2}{(h + \eta^2)^2}.$$  

\footnote{The identity (71) implies that $\eta = 0$ when $N$ is $g$-null.}
Finally, in the \( g \)-null case (i.e., \( H = \mathcal{N} \) and \( L = \mathcal{N} \)), we have
\[
L = H. \tag{71}
\]

Proof. (68) is a simple consequence of (51) and (61). Similarly, (67) is a simple consequence of (52) and (55).
To prove (68), we first note that \( H \) and \( N \) belong to the \( g \)-orthogonal complement of \( \mathcal{S}_\tau \) and are linearly independent (since \( N \) is \( g \)-timelike while \( H \) is not). It follows that the two-dimensional subspace \( \text{span}\{H, N\} \) is the \( g \)-orthogonal complement of \( \mathcal{S}_\tau \).

Therefore, since \( N \) is also \( g \)-orthogonal to \( \mathcal{S}_\tau \), there are scalar functions \( a_1 \) and \( a_2 \) such that \( N = a_1 H + a_2 N \). Taking the \( g \)-inner product of each side of this equation with respect to \( B \) and using (51)-(53), we find that \( a_1 + a_2 = 1 \). Next, taking the \( g \)-inner product of the identity with respect to \( H \) and using (57b), (59), and the relation \( g(N, H) = 0 \), we find that \( 0 = a_1 \eta^2 - a_2 h \). Solving the two equations for \( a_1 \) and \( a_2 \), we conclude (68).

To prove (69), we simply take the \( g \)-inner product of each side of (68) with respect to \( N \) and use (57b) and (61).

To prove (70), we take the \( g \)-inner product of each side of (68) with respect to itself and use (57b), (59), (61), and (63), and then carry out straightforward algebraic computations.

The identity (71) holds because in the \( g \)-null case, \( L \) is \( g \)-orthogonal to itself (where \( L \) is alternate notation for \( N \)) and is therefore \( \mathcal{S}_\tau \)-tangent; thus, \( L \) satisfies all of the conditions from Def. 3.2 that uniquely define \( H \).

\hfill \Box

3.5. The vectorfields \( \Theta \) and \( E \). The following two vectorfields are featured prominently in the ensuing analysis.

Definition 3.10 (The vectorfields \( \Theta \) and \( E \)). Let \( B \) be the vectorfield defined in (5), let \( Z \) and \( H \) be the vectorfields from Def. 3.2 and let \( \frac{1}{\check{z}} > 0 \) and \( \frac{1}{\check{h}} > 0 \) be the scalar functions defined in (57a)-(57b). We define the vectorfield \( \Theta \) by
\[
\Theta := B - \frac{1}{\check{h}} H - \frac{1}{\check{z}} Z. \tag{72}
\]
and the vectorfield \( E \) by
\[
E := B - \frac{1}{\check{z}} Z. \tag{73}
\]

3.6. Algebraic identities in which the sign matters. The following lemma provides algebraic identities relating various vectorfields tied to the geometry of \( \mathcal{M} \). In order for our main results to useful, it is crucial that the scalar functions \( \frac{1}{\check{h}} \) and \( \frac{1}{\check{z}} \) on RHS (57) are positive (by assumption – see Subsect. 3.3). The positivity is in particular necessary for the coercivity of the boundary terms in our main integral identities (165a)-(165b).

Lemma 3.11 (Properties of \( \Theta \) and connections between \( B \), \( Z \), \( H \), \( E \), and \( \Theta \)). The vectorfield \( \Theta \) from Def. 3.10 is \( \mathcal{S}_\tau \)-tangent, while the vectorfield \( E \) from Def. 3.10 is \( \mathcal{H} \)-tangent. Moreover, the following identities hold along \( \mathcal{H} \):
\[
E = \frac{1}{\check{h}} H + \Theta, \tag{74}
\]
\[
B = \frac{1}{\check{h}} H + \frac{1}{\check{z}} Z + \Theta = E + \frac{1}{\check{z}} Z. \tag{75}
\]

Proof. The identities (74)-(75) are direct consequences of Def. 3.10. Moreover, once we show that \( \Theta \) is \( \mathcal{S}_\tau \)-tangent, the \( \mathcal{H} \)-tangent nature of \( E \) follows trivially from (74).

It remains for us to prove that \( \Theta \) is \( \mathcal{S}_\tau \)-tangent. We first note that because the surfaces \( \tilde{\Sigma}_\tau \) and \( \mathcal{H} \) are transversal by assumption, their \( g \)-normal vectors, which are \( N \) and \( \mathcal{N} \) respectively, cannot be parallel. Moreover, because \( N \) and \( \mathcal{N} \) are \( g \)-orthogonal to \( \mathcal{S}_\tau = \mathcal{H} \cap \tilde{\Sigma}_\tau \), it follows that the two-dimensional subspace \( \text{span}\{N, \mathcal{N}\} \) is the \( g \)-orthogonal complement of \( \mathcal{S}_\tau \). Thus, in view of (72), the \( \mathcal{S}_\tau \)-tangent property of \( \Theta \) will follow once we show that the vectorfield \( B - \frac{1}{\check{h}} H - \frac{1}{\check{z}} Z \) has vanishing \( g \)-inner product with \( N \) and
These inner products are easy to compute using (51)-(52) and (57a)-(57b) as well as the relations
\[ 0 = g(N, H) = g(N, Z). \]

\[ \square \]

3.7. **First fundamental forms and projections.** Having described our assumptions on \( \mathcal{M} \) and its boundary, and having constructed geometric vectorfields adapted to these sets, we now define some additional geometric tensorfields that are adapted to them. Specifically, we will define various first fundamental forms and projection operators. These standard geometric objects will play an important role in the formulation and proof of our main integral identities.

**Definition 3.12** (First fundamental forms and projections). Let \( B \) be the vectorfield defined in (5), let \( Z \) be the vectorfield from Def. 3.2 and let \( N \) and \( \bar{N} \) be the vectorfields from Def. 3.8, where \( \bar{N} \) is defined only when \( \mathcal{H} \) is \( g \)-spacelike. We define the following symmetric type \( (2) \) tensorfields, where \( g \) and \( \bar{g} \) are defined on \( \mathcal{M} \), while \( \tilde{g} \) and \( \check{g} \) are defined on \( \mathcal{H} \), and \( \bar{g} \) is defined only when \( \mathcal{H} \) is \( g \)-spacelike:
\[
\begin{align*}
g_{\alpha\beta} &:= g_{\alpha\beta} + B_\alpha B_\beta, \\
\bar{g}_{\alpha\beta} &:= g_{\alpha\beta} + \bar{N}_\alpha \bar{N}_\beta, \\
\tilde{g}_{\alpha\beta} &:= g_{\alpha\beta} + \hat{N}_\alpha \hat{N}_\beta, \\
\check{g}_{\alpha\beta} &:= g_{\alpha\beta} + \tilde{N}_\alpha \tilde{N}_\beta - Z_\alpha Z_\beta.
\end{align*}
\]

We define the following symmetric type \( (2) \) tensorfields, where \( g^{-1} \) and \( \bar{g}^{-1} \) are defined on \( \mathcal{M} \), while \( \tilde{g}^{-1} \) and \( \check{g}^{-1} \) are defined on \( \mathcal{H} \), and \( \bar{g}^{-1} \) is defined only when \( \mathcal{H} \) is \( g \)-spacelike:
\[
\begin{align*}
(g^{-1})^{\alpha\beta} &:= (g^{-1})^{\alpha\beta} + B^\alpha B^\beta, \\
(\bar{g}^{-1})^{\alpha\beta} &:= (g^{-1})^{\alpha\beta} + \bar{N}^\alpha \bar{N}^\beta, \\
(\tilde{g}^{-1})^{\alpha\beta} &:= (g^{-1})^{\alpha\beta} + \hat{N}^\alpha \hat{N}^\beta, \\
(\check{g}^{-1})^{\alpha\beta} &:= (g^{-1})^{\alpha\beta} + \tilde{N}^\alpha \tilde{N}^\beta - Z^\alpha Z^\beta.
\end{align*}
\]

Finally, we define the following type \( (1) \) tensorfields, where \( \Pi \) and \( \bar{\Pi} \) are defined on \( \mathcal{M} \), while \( \tilde{\Pi} \) and \( \check{\Pi} \) are defined on \( \mathcal{H} \), and \( \bar{\Pi} \) is defined only when \( \mathcal{H} \) is \( g \)-spacelike:
\[
\begin{align*}
\Pi^{\alpha}_\beta &:= \delta^\alpha_\beta + B^\alpha B_\beta, \\
\bar{\Pi}^{\alpha}_\beta &:= \delta^\alpha_\beta + \bar{N}^\alpha \bar{N}_\beta, \\
\tilde{\Pi}^{\alpha}_\beta &:= \delta^\alpha_\beta + \hat{N}^\alpha \hat{N}_\beta, \\
\check{\Pi}^{\alpha}_\beta &:= \delta^\alpha_\beta + \tilde{N}^\alpha \tilde{N}_\beta - Z^\alpha Z_\beta.
\end{align*}
\]

In the following lemma, we record some basic properties of the tensorfields from Def. 3.12. We omit the proof, which is a routine consequence of the definitions.

**Lemma 3.13** (Basic properties of the tensorfields from Definition 3.12). \( g \) is the first fundamental form of \( \Sigma_t \) in the following sense:
\[
\begin{align*}
g(X, Y) &= g(X, Y) & \text{for all } \Sigma_t\text{-tangent vectorfields } X \text{ and } Y, \\
g(B, X) &= 0, & \text{for all vectorfields } X,
\end{align*}
\]
where \( B \) is the vectorfield defined in (5) (it is the unit \( g \)-normal to \( \Sigma_t \)). In particular, \( g \) is a Riemannian metric (i.e., a positive definite quadratic form) on \( \Sigma_t \), and \( g \) is a positive semi-definite quadratic form on all vectorfields.

Similarly, \( \tilde{g} \) is the first fundamental form of \( \tilde{\Sigma}_t \) in the following sense:
\[
\begin{align*}
\tilde{g}(X, Y) &= g(X, Y) & \text{for all } \tilde{\Sigma}_t\text{-tangent vectorfields } X \text{ and } Y, \\
\tilde{g}(N, X) &= 0, & \text{for all vectorfields } X,
\end{align*}
\]
where $N$ is the vector field from Def. 3.2 (it is $g$-normal to $\Sigma_\tau$). In particular, $g$ is a Riemannian metric on $\Sigma_\tau$, and $\tilde{g}$ is a positive semi-definite quadratic form on all vector fields.

Similarly, when $H$ is $g$-spacelike, $\tilde{g}$ is the first fundamental form of $H$ in the following sense:

\[
\tilde{g}(X,Y) = g(X,Y) \quad \text{for all } H\text{-tangent vector fields } X \text{ and } Y, \quad (81a)
\]
\[
\tilde{g}(N,X) = 0, \quad \text{for all vector fields } X, \quad (81b)
\]

where $N$ is the vector field from Def. 3.2 (it is $g$-normal to $H$). In particular, when $H$ is $g$-spacelike, $\tilde{g}$ is a Riemannian metric on $H$, and $\tilde{g}$ is a positive semi-definite quadratic form on all vector fields.

Similarly, $\hat{g}$ is the first fundamental form of $S_\tau$ in the following sense:

\[
\hat{g}(X,Y) = g(X,Y) \quad \text{for all } S_\tau\text{-tangent vector fields } X \text{ and } Y, \quad (82a)
\]
\[
\hat{g}(V,X) = 0 \quad \text{for all vector fields } X \text{ if } V \in \text{span}\{N,Z\}, \quad (82b)
\]

where $Z$ is the vector field from Def. 3.2 (and thus $\text{span}\{N,Z\}$ is the space of vector fields that is $g$-orthogonal to $S_\tau$). In particular, since $S_\tau$ is a submanifold of the $g$-spacelike submanifold $\Sigma_\tau$, $\hat{g}$ is a Riemannian metric on $S_\tau$, and $\hat{g}$ is a positive semi-definite quadratic form on all vector fields.

In addition, $\Pi$ is the $g$-orthogonal projection onto $S_\tau$ in the following sense:

\[
\Pi^\alpha_\beta X^\beta = X^\alpha, \quad \Pi^\alpha_\beta X^\beta = X^\alpha, \quad \text{if } X \text{ is } S_\tau\text{-tangent}, \quad (83a)
\]
\[
\Pi^\alpha_\beta V^\beta = 0, \quad \Pi^\alpha_\beta V^\beta = 0, \quad \text{if } V \in \text{span}\{N,Z\}, \quad (83b)
\]
\[
\Pi^\alpha_\beta \Pi^\beta_\gamma = \Pi^\alpha_\gamma. \quad (83c)
\]

Moreover, $\hat{g}^{-1}$ is the inverse first fundamental form of $S_\tau$ in the sense that $(\hat{g}^{-1})^{\alpha\beta} \hat{g}_{\kappa\beta} = \Pi^\alpha_\beta$. In particular, when restricted to tensors tangent to $S_\tau$, $(\hat{g}^{-1})^{\alpha\beta} \hat{g}_{\kappa\beta}$ is the identity. In an analogous fashion, $g^{-1}$ is the inverse first fundamental form of $\Sigma_\tau$, $\Pi$ is the $g$-orthogonal projection onto $S_\tau$, and $\hat{g}^{-1}$ is the inverse first fundamental form of $S_\tau$, $\Pi$ is the $g$-orthogonal projection onto $\Sigma_\tau$, and, when $H$ is $g$-spacelike, $\hat{g}^{-1}$ is the inverse first fundamental form of $H$ and $\Pi$ is the $g$-orthogonal projection onto $H$.

Our identities involve projections of tensor fields onto $S_\tau$ and $H$, which we now define.

**Definition 3.14** (Projections of tensor fields). If $\xi^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n}$ is a type $\binom{m}{n}$ tensor field, then $\Pi \xi$ denotes the $g$-orthogonal projection of $\xi$ onto $\Sigma_\tau$, defined by $(\Pi \xi)^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n} := \Pi^{\alpha_1}_{\beta_1} \cdots \Pi^{\alpha_m}_{\beta_m} \xi^{\beta_1\cdots\beta_n}_{\gamma_1\cdots\gamma_m}$. Similarly, we denote the $g$-orthogonal projections of $\xi$ onto $\Sigma_\tau$, $H$, and $S_\tau$ by $\Pi \xi$, $\Pi \xi$, and $\Pi \xi$ respectively; these are defined as above, but with $\tilde{\Pi}$, $\Pi$, and $\Pi$ respectively in the role of $\Pi$, and $\Pi \xi$ is defined only when $H$ is $g$-spacelike.

**Definition 3.15** ($\Sigma_\tau$, $\Sigma_\tau$, $H$, $\Sigma_\tau$, and $N$-tangent tensor fields). If $\xi^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n}$ is a type $\binom{m}{n}$ tensor field, then we say that $\xi$ is $\Sigma_\tau$-tangent if $\xi = \Pi \xi$. Similarly, we say that $\xi$ is $\Sigma_\tau$-tangent if $\xi = \Pi \xi$, and we say that $\xi$ is $S_\tau$-tangent if $\xi = \Pi \xi$. Moreover, if $H$ is $g$-spacelike, we say that $\xi$ is $H$-tangent if $\xi = \Pi \xi$.

In the case that $H = N$ is $g$-null, if $q \in S_\tau \subset N$ and $L_q$ denotes $L$ at $q$, then we say that the vector field $X \in T_q M$ is $N$-tangent at $q$ if $X \in \text{span}\{L_q\} \oplus T_q S_\tau$. We typically avoid explicitly referencing the point $q$ when there is no danger of confusion. Similarly, we say that the one-form $\Pi^{\alpha_1}_{\beta_1} \xi$ is $N$-tangent at $q$ if its $g$-dual vector field $X^\alpha := (g^{-1})^{\alpha\beta} \xi^\beta$ is $N$-tangent at $q$.

In addition, we sometimes use the following alternate notation for the $g$-orthogonal projection of a vector field $V$ onto $S_\tau$:

\[
V := \Pi V, \quad (84)
\]
i.e., $V^\alpha := \Pi^\alpha_\beta V^\beta$.

**Convention 3.16** (Restricting $g$, $\hat{g}$, and $\Pi$ to $S_\tau$-tangent vector fields and displaying only spatial indices). In the rest of the article, we will often adopt the point of view that our formulas are statements about the Cartesian components of tensors, even though many of the formulas could be given a coordinate invariant

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46 For any positive integers $m$ and $n$, the definition of $N$-tangent extends in a natural fashion to type $\binom{m}{n}$ tensor fields $\xi$; we do not need the extended definition in the present article.
interpretation; see also Remark 3.4. Moreover, in much of the remaining discussion, we will only have to consider the action of $g$, $\xi$, and $\Pi$ on $\Sigma$-tangent vectorfields. Thus, when working with $\Sigma$-tangent tensors, we typically only display their spatial components, since the 0 component (i.e., the "Cartesian time component") vanishes. For example, we have $g(\Omega, \Omega) = g_{ab} \Omega^a \Omega^b$. As a second example, we note that $\mathbf{S}^a = \Pi_b \mathbf{S}^b$.

**Convention 3.17** (Lowering and raising Cartesian spatial indices with $g$ and $g^{-1}$). In the remainder of the paper, we often lower and raise indices of $\Sigma$-tangent tensors with $g$ and $g^{-1}$. For example, we have $\Omega_a = g_{ab} \Omega^b$. For Greek spacetime indices, we will continue to use the conventions for lowering and raising stated in Subsection 2.1.2. Note that there is no danger of confusion in the sense that (22a) and (27) imply that when $V$ is $\Sigma$-tangent, we have $g_{ab} \nabla^b V^a = g_{ab} \partial^b V^a$. We caution, however, that $\nabla_0$ is generally not equal to 0; see (131).

**Definition 3.18** (Projections of Cartesian coordinate partial derivative vectorfields). For $\alpha = 0, 1, 2, 3$, we define the following vectorfields relative to the Cartesian coordinates:

$$\begin{align*}
\overline{\alpha}_a & := \Pi^b \partial_b, \\
\overline{\alpha}^a & := (\overline{g}^{-1})^{ab} \partial_b, \\
\overline{\alpha} & := \Pi^a \partial_a, \\
\overline{\alpha}^* & := (\overline{g}^{-1})^{\alpha \beta} \partial_\beta, \\
\overline{g} & := (\overline{g}^{-1})^{\alpha \beta} \partial_\alpha \partial_\beta.
\end{align*}$$  

(85a)

where $\overline{\alpha}_a$ and $\overline{\alpha}^a$ are defined only when $H$ is $g$-spacelike.

For future use, we note that (78b), (78d), and (85a) imply the following vectorfield identity:

$$\overline{\alpha}_a = Z_a Z + \overline{\alpha}.$$  

(86)

Moreover, when $H$ is $g$-spacelike, it follows in a straightforward fashion from (59), (78c), (78d), the fact that $H$ is $g$-orthogonal to $S_v$, and (85a) that the following identity holds:

$$\overline{\alpha}_a = \frac{H}{\eta^2} H + \overline{\alpha}.$$  

(87)

3.8. **Levi–Civita connections and $S_v$-divergence.** We refer readers to [34] for basic background on Levi–Civita connections in the context of Lorentzian geometry. Our ensuing discussion will involve the Levi–Civita connection of $g$, which we denote by $\nabla$. It will also involve the Levi–Civita connection of $g$ (viewed as a Riemannian metric on $\Sigma_v$), which we denote by $\nabla$, the Levi–Civita connection of $\overline{g}$ (viewed as a Riemannian metric on $\Sigma_v$), which we denote by $\nabla$, the Levi–Civita connection of $\tilde{g}$ (viewed as a Riemannian metric on $\Sigma_v$), which we denote by $\nabla$, and, in case that $H$ is $g$-spacelike, the Levi–Civita connection of $g$ (viewed as a Riemannian metric on $H$), which we denote by $\nabla$. We recall the following basic facts from differential geometry: if $\xi$ is $\Sigma_v$-tangent, then $\nabla \xi = \Pi(D\xi)$, where $(D\xi)^{\alpha_1 \cdots \alpha_m}_{\beta_1 \cdots \beta_{n+1}} = D_{\beta_1 \cdots \beta_{n+1}}^{\alpha_1 \cdots \alpha_m}$ is the covariant derivative of $\xi$ with respect to $D$; if $\xi$ is $\Sigma_v$-tangent, then $\nabla \xi = \Pi(D\xi)$; if $\xi$ is $S_v$-tangent, then $\nabla \xi = \Pi(D\xi)$; and if $H$ is $g$-spacelike and $\xi$ is $H$-tangent, then $\nabla \xi = \Pi(D\xi)$.

All of our formulas involving indices should be interpreted as formulas relative to the Cartesian coordinates, even though many of them could be re-expressed in a coordinate invariant form. For example, if $V$ is a vectorfield and $\Gamma^\alpha_{\beta \gamma}$ denotes a Christoffel symbol of $g$ relative to the Cartesian coordinates, then

$$\nabla \beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha_{\beta \gamma} V^\gamma,$$

$$\Gamma^\alpha_{\beta \gamma} := \frac{1}{2} (g^{-1})^{\alpha \kappa} \left\{ \partial_\beta g_{\kappa \gamma} + \partial_\gamma g_{\kappa \beta} - \partial_\kappa g_{\beta \gamma} \right\}.$$  

(88a)

We lower and raise the indices of $\Gamma^\alpha_{\beta \gamma}$ with $g$ and $g^{-1}$. For example, $\Gamma^\beta_{\alpha \gamma} := g_{\alpha \kappa} \Gamma^\beta_{\kappa \gamma}$. Similarly, if $V$ is a $\Sigma_v$-tangent vectorfield and $\Gamma^\alpha_{\beta \gamma}$ denotes a Christoffel symbol of $g$ relative to the Cartesian spatial coordinates, then

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha_{\beta \gamma} V^\gamma,$$

$$\Gamma^\alpha_{\beta \gamma} := \frac{1}{2} (g^{-1})^{\alpha \kappa} \left\{ \partial_\beta g_{\kappa \gamma} + \partial_\gamma g_{\kappa \beta} - \partial_\kappa g_{\beta \gamma} \right\}.$$  

(89a)

A few of our formulas involve the $S_v$-divergence, i.e., the divergence of $S_v$-tangent vectorfields with respect to the connection $\nabla$. Relative to arbitrary local coordinates $(\vartheta^1, \vartheta^2)$ on $S_v$, with $Y = Y^A \partial_A$, we have $d\nabla Y = \nabla A Y^A$. 

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3.9. Expressions for the divergence of various vectorfields. For future use, in the next lemma, we provide an expression for the $\nabla$-divergence of $\Sigma_\tau$-tangent vectorfields and an expression for the $\nabla$-divergence of $S_\tau$-tangent vectorfields.

Lemma 3.19 (Expressions for the divergence of various vectorfields). Let $J$ be a $\Sigma_\tau$-tangent vectorfield defined on $\mathcal{M}$. Then relative to the Cartesian coordinates, we have the following identities:

\[ \nabla_\alpha J^\beta = \partial_\alpha J^\beta - \nabla^\beta J^\gamma \partial_\alpha \nabla_\gamma \]  
\[ + \frac{1}{2} (g^{-1})^{\delta\beta} J^\gamma \partial_\alpha g_{\delta\kappa} + \frac{1}{2} \Pi^\gamma_\alpha (g^{-1})^{\delta\beta} J g_{\gamma\delta} - \frac{1}{2} \Pi^\gamma_\alpha J^\gamma \partial_\alpha g_{\delta\kappa}, \]  
\[ \nabla_\alpha J^\alpha = \partial_\alpha J^\alpha - J_\alpha \nabla^\alpha + \frac{1}{2} (g^{-1})^{\alpha\beta} J g_{\alpha\beta}. \]  

Moreover, if $Y$ is an $S_\tau$-tangent vectorfield defined on $\mathcal{M}$, then relative to the Cartesian coordinates, we have the following identities:

\[ \nabla_\alpha Y^\beta = \delta_\alpha Y^\beta - \nabla^\beta Y^\gamma \delta_\alpha \nabla_\gamma + Z^\beta Y^\gamma \partial_\alpha Z_\gamma \]  
\[ + \frac{1}{2} (g^{-1})^{\delta\beta} Y^\gamma \partial_\alpha g_{\delta\kappa} + \frac{1}{2} \Pi^\gamma_\alpha (g^{-1})^{\delta\beta} Y g_{\gamma\delta} - \frac{1}{2} \Pi^\gamma_\alpha Y^\gamma \partial_\alpha g_{\delta\kappa}, \]  
\[ \text{di} Y = \nabla_\alpha Y^\alpha = \delta_\alpha Y^\alpha + \frac{1}{2} (g^{-1})^{\alpha\beta} Y g_{\alpha\beta}. \]

Finally, in the particular case $Y = \phi$, we have the following identity relative to the Cartesian coordinates:

\[ \text{di} \phi = \nabla_\alpha \phi^\alpha - \nabla^\alpha \phi_\alpha Z^\alpha + \frac{1}{2} (g^{-1})^{\alpha\beta} \partial_\alpha g_{\alpha\beta}. \]  

Proof. We first prove (90a). As we mentioned in Subsect. 3.8 (see also [34]), we have $\nabla_\alpha J^\beta = \Pi^\gamma_\alpha \Pi^\beta_\gamma D_\gamma J^\delta$, where $\Pi$ is the $\Sigma_\tau$ projection tensorfield (see (78b)). Thus, in view of (78b), (85a)-(85b), (88a)-(88b) and the assumption that $J$ is $\Sigma_\tau$-tangent (which implies in particular that $J^\alpha N_\alpha = 0$ and $(\partial_\beta J^\alpha) N_\alpha = -J^\alpha \partial_\beta N_\alpha$), we compute that relative to the Cartesian coordinates, we have

\[ \nabla_\alpha J^\beta = \Pi^\gamma_\alpha \Pi^\beta_\gamma \partial_\alpha J^\delta + \frac{1}{2} (g^{-1})^{\delta\beta} \partial_\alpha \nabla^\gamma \gamma_{\delta\kappa} J^\kappa \]  
\[ = \partial_\alpha J^\beta - \nabla^\beta J^\gamma \partial_\alpha \nabla_\gamma \]  
\[ + \frac{1}{2} (g^{-1})^{\delta\beta} J^\gamma \partial_\alpha g_{\delta\kappa} + \frac{1}{2} \Pi^\gamma_\alpha (g^{-1})^{\delta\beta} J g_{\gamma\delta} - \frac{1}{2} \Pi^\gamma_\alpha J^\gamma \partial_\alpha g_{\delta\kappa}, \]  
as desired.

To prove (90b), we trace (90a) over the indices $\alpha$ and $\beta$ and carry out straightforward computations. The identities (91a)-(91b) can be proved using similar arguments that rely on the identity $\nabla_\alpha J^\beta = \Pi^\gamma_\alpha \Pi^\beta_\gamma D_\gamma J^\delta$ and the expression (78d); we omit the details.

(92) follows from (91b) and the following identity for Cartesian components, which follows from (78d) and (85a): $(\phi_\kappa)^\alpha = \delta_\kappa^\alpha + \nabla_\kappa \nabla^\alpha = Z_\alpha Z^\alpha$.

3.10. Gradients, Euclidean metrics, and pointwise norms with respect to various metrics. Our ensuing analysis involves several kinds of gradients of tensorfields and pointwise norms of tensorfields with respect to various Riemannian metrics. In this subsection, we provide the relevant definitions. We also define the standard Euclidean metrics on $\mathbb{R}^{1+3}$ and $\Sigma_\tau$. We also remind the reader that, as is stated in Convention 3.16, we typically only display the Cartesian spatial indices of $\Sigma_\tau$-tangent tensorfields.

3.10.1. Gradients. Let $\xi$ be a type $(m,n)$ tensorfield with Cartesian components $\xi^{m_1\cdots m_m}_{n_1\cdots n_n}$.
- $\partial_\xi$ denotes the type $(m_{n+1})$ tensorfield with Cartesian components $\partial_\beta \xi^{m_1\cdots m_m}_{n_1\cdots n_n}$.
- $\partial_\xi$ denotes the type $(m_{n+1})$ tensorfield with Cartesian components $\partial_\beta \xi^{m_1\cdots m_m}_{n_1\cdots n_n}$, where $\partial_\xi_{n\alpha} = 0, 1, 2, 3$ denotes the $\Sigma_\tau$-tangent vectorfields from (85a).
• When $\mathcal{H}$ is $g$-spacelike, $\partial_t \xi$ denotes the type $(m_{n+1})$ tensorfield with Cartesian components $\partial_{\alpha} \xi_{\beta_1 \cdots \beta_n \gamma}$, where $\{\partial_{\alpha}\}_{\alpha=0,1,2,3}$ denotes the $\mathcal{H}$-tangent vectorfields from [85a].

• $\partial_t \xi$ denotes the type $(m_{n+1})$ tensorfield with Cartesian spatial components $\partial_{\beta} \xi_{\alpha_1 \cdots \alpha_n \gamma}$, where $\{\partial_{\alpha}\}_{\alpha=0,1,2,3}$ denotes the $S_t$-tangent vectorfields from [85a].

• Given any $\Sigma_t$-tangent type $(m)$ tensorfield $\xi$, with Cartesian components $\xi_{\alpha_1 \cdots \alpha_n \gamma}$, $\partial_t \xi$ denotes the $\Sigma_t$-tangent type $(m_{n+1})$ tensorfield $\partial_{\beta} \xi_{\alpha_1 \cdots \alpha_n \gamma}$, with Cartesian spatial components $\partial_{\beta} \xi_{\alpha_1 \cdots \alpha_n \gamma}$.

3.10.2. The Euclidean metrics on $\mathbb{R}^{1+3}$ and $\Sigma_t$. We let $e$ denote the standard Euclidean metric on $\mathbb{R}^{1+3}$, i.e., relative to the Cartesian coordinates, $e_{\alpha\beta} = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker delta. Moreover, $e^{-1}$ denotes the corresponding inverse metric. Similarly, $e_{ij}$ denotes the standard Euclidean metric on $\Sigma_t$, and $e^{-1}$ denotes the corresponding inverse metric.

3.10.3. Pointwise seminorms and norms.

• Given any spacetime tensorfield $\xi_{\alpha_1 \cdots \alpha_n \gamma}$,

$$|\xi|_e := \sqrt{e_{\alpha_1 \gamma} \cdots e_{\alpha_n \gamma} (e^{-1})^{\beta_1 \gamma} \cdots (e^{-1})^{\beta_n \gamma} \xi_{\beta_1 \cdots \beta_n \gamma}}$$

denotes its pointwise norm with respect to $e$.

• Given any $\Sigma_t$-tangent spacetime tensorfield $\xi_{\beta_1 \cdots \beta_n}$,

$$|\xi|_{\Sigma_t} := \sqrt{\xi_{\alpha_1 \gamma} \cdots \xi_{\alpha_n \gamma} (\xi^{-1})^{\beta_1 \gamma} \cdots (\xi^{-1})^{\beta_n \gamma} \xi_{\beta_1 \cdots \beta_n \gamma}}$$

denotes its pointwise norm with respect to $\xi$. Note that $|\cdot|_{\Sigma_t}$ is a norm on $\Sigma_t$-tangent tensorfields.

• Given any type $(m)$ tensorfield $\xi_{\alpha_1 \cdots \alpha_n \gamma}$, when $\mathcal{H}$ is $g$-spacelike, given any type $(m)$ tensorfield $\xi_{\alpha_1 \cdots \alpha_n \gamma}$,

$$|\xi|_{\mathcal{H}} := \sqrt{\xi_{\alpha_1 \gamma} \cdots \xi_{\alpha_n \gamma} (\xi^{-1})^{\beta_1 \gamma} \cdots (\xi^{-1})^{\beta_n \gamma} \xi_{\beta_1 \cdots \beta_n \gamma}}$$

denotes its pointwise norm with respect to $\xi$. Note that $|\cdot|_{\mathcal{H}}$ is a norm on $\mathcal{H}$-tangent tensorfields.

• Given any type $(m)$ tensorfield $\xi_{\alpha_1 \cdots \alpha_n \gamma}$, $\xi$ is an array of $\Sigma_t$-tangent tensorfields, then $|\varphi|_e^2$ denotes the sum of the squares of the norms $|\cdot|_e$ of the elements of the array. Norms of arrays with respect to other metrics are defined in an analogous fashion, e.g., if $\mathcal{H}$ is $g$-spacelike and if $\varphi$ is an array of $\mathcal{H}$-tangent tensorfields, then $|\varphi|_2^2$ denotes the sum of the squares of the seminorms $|\cdot|_2$ of the elements of the array.

4. The coercive quadratic form, the elliptic-hyperbolic divergence identities, and preliminary analysis of the boundary integrands

Our first main goal in this section is to define the coercive quadratic form $\mathcal{Q}$, featured in our main integral identities (i.e., in Theorem 8.1) and to exhibit its coerciveness properties. Our second main result in this section is Lemma 4.7 which provides the elliptic Hodge-type divergence identity along $\Sigma_t$ that forms the starting point for the proof of Theorem 8.1. Our third main result in this section is Lemma 4.13 in which we provide a preliminary analysis of the boundary integrand terms in the elliptic Hodge-type identities. The lemma shows that most boundary integrand terms involve only $\mathcal{H}$-tangential derivatives of terms enjoying a compatible amount of regularity. The remaining boundary terms also enjoy these same good properties,
but the proof requires an integration with respect to \( \tau \) and substantial additional arguments that, unlike the results of the present section, exploit the special properties of the compressible Euler formulation provided by Theorem \ref{thm:compressible-Euler} see Prop. \ref{prop:compressible-Euler} and Theorem \ref{thm:compressible-Euler} for the detailed statements showing that all boundary integrand terms have these good properties.

4.1. A solution-adapted coercive quadratic form.

4.1.1. A \( \Sigma_{\tau} \)-tangent vectorfield that arises in the analysis. The coercive quadratic form that we provide in Def. \ref{def:coercive-quadratic-form} involves the \( \Sigma_{\tau} \)-tangent vectorfield \( K \) from the following definition.

**Definition 4.1** (A rescaled, \( \Sigma_{\tau} \)-projected version of \( B \)). Let \( \nu > 0 \) be the scalar function defined in \((61)\), and let \( \Pi \) be the \( g \)-orthogonal projection onto \( \Sigma_{\tau} \) as defined in \((78b)\). We define \( K \) to be the \( \Sigma_{\tau} \)-tangent vectorfield with the following Cartesian components:

\[
K^\alpha := \nu(\Pi B)^\alpha = \nu\Pi_\beta^\alpha B^\beta.
\]  

(94)

In the next lemma, we derive some simple identities involving \( K \).

**Lemma 4.2** (Identities involving \( K \)). Let \( K \) be vectorfield defined by \((94)\), let \( \nu > 0 \) be the scalar function defined in \((61)\), let \( N \) be the vectorfield from Def. \ref{def:vectorfield-N}, and let \( \bar{N} \) be the vectorfield from Def. \ref{def:vectorfield-N}. Then the following identities hold:

\[
K^\alpha = \nu B^\alpha - \frac{1}{\nu} N^\alpha = \nu B^\alpha - \bar{N}^\alpha,
\]  

(95)

\[
|K|_{g}^2 := \bar{g}(K,K) = 1 - \nu^2 < 1.
\]  

(96)

Moreover, the following identities hold, where \( \bar{g}^{-1} \) is the inverse first fundamental form of \( \Sigma_{\tau} \) from Def. \ref{def:vectorfield-N}, \( \bar{g}^{-1} \) is the inverse first fundamental form of \( \Sigma_{\tau} \) from Def. \ref{def:vectorfield-N}, and \( \Pi \) is the \( g \)-orthogonal projection onto \( \Sigma_{\tau} \) from Def. \ref{def:vectorfield-N}.

\[
(\bar{g}^{-1})^{\alpha\beta} = (g^{-1})^{\alpha\beta} + K^\alpha K^\beta - \nu B^\alpha K^\beta - \nu K^\alpha B^\beta - (1 - \nu^2)B^\alpha B^\beta,
\]  

(97a)

\[
\bar{\Pi}_\beta^\alpha = \Pi_\beta^\alpha + K^\alpha K_\beta - \nu B^\alpha K_\beta - \nu K^\alpha B_\beta - (1 - \nu^2)B^\alpha B_\beta.
\]  

(97b)

Finally, if \( V \) is a \( \Sigma_{\tau} \)-tangent vectorfield defined on \( \mathcal{M} \), then the following identity holds:

\[
\bar{\Pi}_\beta^\alpha \partial_\alpha V^\beta = \partial_\alpha V^\alpha + K_\alpha K^\alpha V^\beta - \nu K_\alpha B V^\alpha.
\]  

(98)

**Proof.** \((95)\) is a simple consequence of definition \((94)\), \((51)\), \((78b)\), and \((64)\). \((96)\) then follows from \((95)\), \((26)\), \((61)\), and \((61)\).

To prove \((97a)\), we first use \((77a)\) and \((77b)\) to express \((\bar{g}^{-1})^{\alpha\beta} = (g^{-1})^{\alpha\beta} - B^\alpha B^\beta + \bar{N}^\alpha \bar{N}_\beta \). We then use \((96)\) to express the two factors of \( \bar{N} \) in terms of \( K \) and \( B \), which in total yields \((97a)\).

\((97b)\) follows from lowering the index \( \beta \) in \((97a)\) with \( g \).

\((98)\) follows from \((97b)\), \((78a)\), and the fact that by \((27)\), \( B_\beta \partial_\alpha V^\beta = 0 \) when \( V \) is \( \Sigma_{\tau} \)-tangent. \( \square \)

4.1.2. A solution-adapted coercive quadratic form. We now define the quadratic form \( \mathcal{Q} \) that we use to control the first derivatives of the specific vorticity and entropy gradient. \( \mathcal{Q} \) is the main integrand factor on the left-hand side of the integral identities provided by Theorem \ref{thm:compressible-Euler}. In Lemma \ref{lem:compressible-Euler} we exhibit the coerciveness of \( \mathcal{Q} \).

**Definition 4.3** (A solution-adapted coercive quadratic form for controlling the first derivatives of \( \Omega \) and \( S \)). We define the quadratic form \( \mathcal{Q}(\cdot, \cdot) \) on type \((1,1)\) tensorfields \( U_\alpha^\beta \) as follows relative to the Cartesian coordinates, where \( \bar{g} \) and \( g^{-1} \) are respectively the first fundamental form and inverse first fundamental form of \( \Sigma_{\tau} \) from Def. \ref{def:vectorfield-N}.

\[
\mathcal{Q}(U,U) := \bar{\Pi} U_\beta^\alpha \bar{g} - (K_\beta K^\alpha U_\beta^\alpha)^2 + \bar{g}_\alpha^\beta (B^\alpha U_\gamma^\alpha)(B^\beta U_\delta^\beta) + ((\bar{g}^{-1})^{\alpha\beta} B_\gamma U_\alpha^\gamma)(B_\delta U_\beta^\delta) + (B_\beta B^\alpha U_\alpha^\beta)^2.
\]  

(99)
4.1.3. The positive definiteness of $\mathcal{L}(\cdot, \cdot)$. In the next lemma, we exhibit the positive definite nature of $\mathcal{L}(\cdot, \cdot)$.

**Lemma 4.4** (Positivity properties of the quadratic form). Recall that $\Sigma_\tau$ is $g$-spacelike by assumption. Then as a consequence, the quadratic form $\mathcal{L}(U, U)$ from Def. 4.3 is positive definite on the space of type $(1, 1)$ tensorfields $U_\alpha^\beta$.

Moreover, if $V$ is a $\Sigma_\tau$-tangent vectorfield, then the following identity holds (i.e., with $\partial_\alpha V^\beta$ in the role of $U^\beta_\alpha$), where $\Pi$ is the $g$-orthogonal projection onto $\Sigma_\tau$ from Def. 3.12 and $\tilde{g}$ the first fundamental form of $\Sigma_\tau$ from Def. 3.12.

$$\mathcal{L}(\nabla V, \nabla V) = |\Pi(\nabla V)|^2_\tilde{g} - (K_\alpha KV^\alpha)^2 + \tilde{g}_{\alpha\beta}(BV^\alpha)(BV^\beta).$$  \hspace{1cm} (100)

**Proof.** (100) is a straightforward consequence of definition (99) and the identity (27), which in particular implies that $B_\beta \partial_\alpha V^\beta = 0$ when $V$ is $\Sigma_\tau$-tangent.

To prove the positivity of $\mathcal{L}$, we first use $\tilde{g}$-Cauchy–Schwarz, (94), and (96) to deduce that the first two terms $|\Pi U|^2_\tilde{g} - (K_\alpha K^\alpha U^\alpha_\alpha)^2$ on RHS (99) are collectively positive semi-definite on the space of type $(1, 1)$ tensorfields $U^\beta_\alpha$ and positive definite on the subspace of such tensorfields that are tangent to $\Sigma_\tau$. Moreover, the last three products on RHS (99) are manifestly positive semi-definite on the space of type $(1, 1)$ tensorfields. In particular, for all $U$, we have $\mathcal{L}(U, U) \geq 0$. Thus, to demonstrate the desired positive definiteness of $\mathcal{L}(U, U)$, it suffices to show that $\mathcal{L}(U, U) = 0 \implies U = 0$.

To proceed, we assume that $\mathcal{L}(U, U) = 0$. From the discussion in the previous paragraph, it follows that $|\Pi U|^2_\tilde{g} - (K_\alpha K^\alpha U^\alpha_\alpha)^2 = 0$ and that $\Pi U = 0$. From this identity, (95), and the $\Sigma_\tau$-tangent property of $K$, we see that contractions of $U$ against $B$ are equal, up to a scalar function multiple, to contractions against $N$. From this fact, the fact that the third term $\tilde{g}_{\alpha\beta}(B^\alpha U^\alpha_\alpha)(B^\beta U^\beta_\beta)$ on RHS (99) must also vanish, and the fact that $\tilde{g}$ is positive definite on the space of $\Sigma_\tau$-tangent vectorfields and the fact that $\tilde{g}$ vanishes when contracted with $N$, we see that the vectorfield with components $B^\alpha U^\alpha_\alpha$ must be proportional to $N^\beta$. From these facts, the fact that the last term $(B_\beta B^\beta U^\alpha_\alpha)^2$ on RHS (99) must also vanish, and the fact that $B_\beta N^\alpha \neq 0$ (see (66)), we find that $N^\alpha U^\beta_\alpha = 0$. Similar reasoning, based on exploiting the positive semi-definiteness of the fourth and last terms on RHS (99), leads to the identity $N^\alpha U^\alpha_\alpha = 0$. We have therefore shown that the $g$-orthogonal projection of $U$ onto $\Sigma_\tau$ vanishes, and that any contraction of $U$ against the unit normal to $\Sigma_\tau$ (namely $N$) vanishes. This implies that $U = 0$, which completes the proof of the lemma.

4.2. **Definition of the elliptic-hyperbolic currents.** The $\Sigma_\tau$-tangent vectorfields $J$ in the next definition play a key role in our analysis. We refer to them as “elliptic-hyperbolic currents.” We motivate this terminology as follows: even though Lemma 4.2 shows that $J$ is tied to elliptic-type identities along $\Sigma_\tau$, the complete set of structures that we need to control the boundary terms become manifest only in Prop. 4.1 and Theorem 4.2 after we integrate the elliptic identities in time and exploit some special structural features found in the “hyperbolic part” of the equations of Theorem 2.8.

**Definition 4.5** (Elliptic-hyperbolic current). Given a $\Sigma_\tau$-tangent vectorfield $V$ (which in our forthcoming applications will be equal to $\Omega$ or $S$), we define $J[V]$ to be the $\Sigma_\tau$-tangent vectorfield with the following Cartesian components, where $\Pi$ is the $\Sigma_\tau$ projection tensorfield defined in (78b):

$$J^\alpha[V] := V^\gamma \Pi^\alpha_\gamma \Pi^\kappa_\kappa \partial_\lambda V^\kappa - V^\gamma \Pi^\alpha_\gamma \Pi^\kappa_\kappa \partial_\lambda V^\lambda = V^\gamma \Pi^\alpha_\gamma \partial_\lambda V^\lambda - V^\gamma \Pi^\alpha_\gamma \partial_\lambda V^\lambda,$$ \hspace{1cm} (101)

where to obtain the last equality in (101), we used (85a).

**Remark 4.6** ($J[V]$ is $\Sigma_\tau$-tangent). We stress that $J[V]$ is $\Sigma_\tau$-tangent, even though $V$ is $\Sigma_\tau$-tangent.

4.3. **The elliptic divergence identity.** In the next lemma, we derive a covariant divergence identity for the elliptic-hyperbolic current $J[V]$. With future applications in mind, we have allowed for the presence of a “weight function” $\mathcal{W}$ in the identity. We have carefully organized the structure of the “main terms” in the divergence identity so that later on, with the help of Lemma 4.4, we will be able to show that the quadratic form $\mathcal{L}(\nabla V, \nabla V)$ (see in particular (100)) can be used to derive coercive, spacetime $L^2$-type control over
Proof. In view of (101) and (106), we see that once we have proven the identity \( (102) \) in the case \( \delta \), then follows from the simple algebraic identity \( (g^{-1})^{\alpha \gamma} g_{\beta \delta} = \delta^\gamma_\delta \) where \( \delta^\gamma_\delta \) is the Kronecker delta:

\[
\partial_\alpha (g^{-1})^{\alpha \gamma} = -(g^{-1})^{\alpha \gamma} (g^{-1})^{\delta \lambda} \partial_\alpha g_{\gamma \delta}.
\]
We now apply $\partial_\alpha$ to the first equality in (101) and use (78a) to compute the following expression involving the Cartesian partial derivative vectorfields $J$:

\[
\partial_\alpha J^\beta = \bar{\Pi}_\beta^\alpha \Pi_\gamma (\partial_\alpha V^\beta) \partial_\gamma V^\gamma - (\bar{\Pi}_\gamma^\alpha \partial_\alpha V^\lambda)^2 + V^\beta \bar{\Pi}_\beta^\alpha (\partial_\alpha \hat{N}^\beta) \partial_\gamma V^\gamma + \hat{N}^\beta \bar{\Pi}_\gamma^\alpha (\partial_\alpha g_{\beta\kappa}) \hat{N}^\kappa V^\beta \partial_\gamma V^\gamma
\]

Next, using that $V^\beta = (g^{-1})^\beta \kappa V_\kappa$ and $V^\gamma = (g^{-1})^\gamma \lambda V_\lambda$, and using the simple identity

\[
(g^{-1})^{\beta\gamma}(g^{-1})^\alpha (\partial_\alpha V_\beta) \partial_\gamma V_\delta = (g^{-1})^{\beta\gamma}(g^{-1})^\alpha (\partial_\alpha V_\beta) \partial_\gamma V_\delta + \frac{1}{2}(g^{-1})^{\beta\gamma}(g^{-1})^\alpha (\partial_\alpha V_\beta - \partial_\beta V_\alpha) \{\partial_\gamma V_\delta - \partial_\delta V_\gamma\}
\]

we rewrite the first product on RHS (108) as follows:

\[
\bar{\Pi}_\beta^\alpha \Pi_\gamma (\partial_\alpha V^\beta) \partial_\gamma V^\gamma = (g^{-1})^{\beta\gamma}(g^{-1})^\alpha (\partial_\alpha V_\beta) \partial_\gamma V_\delta + \frac{1}{2}(g^{-1})^{\beta\gamma}(g^{-1})^\alpha (\partial_\alpha V_\beta - \partial_\beta V_\alpha) \{\partial_\gamma V_\delta - \partial_\delta V_\gamma\}
\]

Next, using that $V_\beta = g_{\beta\kappa} V^\kappa$ and $V_\delta = g_{\delta\lambda} V^\lambda$, we rewrite the first product on RHS (109) as follows:

\[
(g^{-1})^{\beta\gamma}(g^{-1})^\alpha (\partial_\alpha V_\beta) \partial_\gamma V_\delta = (g^{-1})^{\beta\gamma}(g^{-1})^\alpha (\partial_\alpha V_\beta) \partial_\gamma V_\delta + \frac{1}{2}(g^{-1})^{\beta\gamma}(g^{-1})^\alpha (\partial_\alpha V_\beta - \partial_\beta V_\alpha) \{\partial_\gamma V_\delta - \partial_\delta V_\gamma\}
\]

Next, using (98), we rewrite the second product on RHS (108) as follows:

\[
(\bar{\Pi}_\gamma^\alpha \partial_\alpha V^\lambda)^2 = (K_\alpha KV^\alpha)^2 + (\partial_\alpha V^\alpha)^2 + \nu^2(K_\alpha BV^\alpha)^2 + 2(K_\alpha KV^\alpha)\partial_\alpha V^\alpha - 2\nu(K_\alpha BV^\alpha)K_\alpha KV^\alpha - 2\nu(\partial_\alpha V^\alpha)K_\alpha BV^\alpha.
\]

Combining (108)-(111), and also using (99a) as well as (101) and (85a)-(85b), and rearranging terms and relabeling indices, we arrive at the desired identity (102) in the case $\mathcal{V} = 1$ (note that the term $3[\theta\gamma][V,\partial V]$ defined in (106) vanishes in this case), but in place of the expression for the error term $3[\langle V,\partial V\rangle]$ stated in (105), we instead have the following expression involving the Cartesian partial derivative vectorfields $\partial_\alpha$.
and the $\tilde{\Sigma}_{\tau}$-projected vectorfields $\tilde{\partial}_{\alpha}$ defined in (85a):

$$\mathfrak{J}_{(Coff)}[V, \partial V] = -V^\alpha \tilde{N}_\alpha (\tilde{\partial}_\beta \tilde{N}^\gamma) \partial_\gamma V^\beta - V_\alpha (\tilde{\partial}_\beta \tilde{N}^\alpha) \tilde{N} V^\beta$$

(112)

$$- V^\alpha \tilde{N}_\beta (\tilde{\partial}_\gamma \tilde{N}^\gamma) \tilde{\partial}_{\alpha} V^\gamma - V^\alpha g_{\beta\gamma} (\tilde{N} \tilde{N}^\beta) \tilde{\partial}_{\alpha} V^\gamma$$

$$+ V^\alpha \tilde{N}_{\alpha} (\tilde{\partial}_\beta \tilde{N}^\beta) \tilde{\partial}_\gamma V^\gamma + V^\alpha (\tilde{N} \tilde{N}^\alpha) \tilde{\partial}_\beta V^\beta$$

$$+ V^\alpha \tilde{N}_\beta (\tilde{\partial}_\gamma \tilde{N}^\gamma) \partial_\alpha V^\gamma + V^\alpha g_{\beta\gamma} (\tilde{\partial}_{\alpha} \tilde{N}^\beta) \tilde{N} V^\gamma$$

$$+ V^\alpha \tilde{g}_{\beta\gamma} (\tilde{N} \tilde{N}^\beta) \tilde{\partial}_\alpha V^\gamma - V^\alpha g_{\alpha\beta} (\tilde{N} \tilde{N}^\beta) \tilde{\partial}_\gamma V^\gamma$$

To complete the proof, it remains for us to show that RHS (112) = RHS (105). This can be shown through straightforward calculations (which in particular lead to the cancellation of many terms on RHS (112)), based on splitting the vectorfield $\partial_\gamma$ on RHS (112) into its $\tilde{\Sigma}_{\tau}$-tangential and $\tilde{\Sigma}_{\tau}$-orthogonal parts via the identity

$$\partial_\gamma = \tilde{\partial}_\gamma - \tilde{N}_\gamma \tilde{N},$$

(113)

which follows from definition (85a), from using the identity

$$\tilde{N}_\gamma \partial_\alpha \tilde{N}^\gamma = -\frac{1}{2} \tilde{N} \tilde{N} \partial_\alpha (\tilde{N} \tilde{N}^\alpha),$$

(114)

which follows from differentiating the relation $g_{\alpha\lambda} \tilde{N}^\alpha \tilde{N}^\lambda = -1$, from using (76b) to decompose the factors of $\tilde{g}$ in the terms $V^\alpha \tilde{g}_{\beta\gamma} (\tilde{N} \tilde{N}^\beta) \tilde{\partial}_\alpha V^\gamma - V^\alpha g_{\beta\gamma} (\tilde{N} \tilde{N}^\beta) \tilde{\partial}_\gamma V^\gamma$ on RHS (112), and from using (76h) to decompose the factors of $\tilde{\Pi}$ in the terms $V^\alpha \tilde{\Pi}^\gamma_\beta \tilde{N}^\delta (\tilde{\partial}_\alpha \tilde{N}^\alpha) \tilde{\partial}_\gamma V^\gamma - V^\alpha \tilde{\Pi}^\gamma_\alpha \tilde{N}^\delta (\tilde{\partial}_\gamma \tilde{N}^\gamma) \tilde{\partial}_\alpha V^\alpha$ on RHS (112).

4.4. Preliminary analysis of the boundary integrand. When we apply the divergence theorem on $\tilde{\Sigma}_{\tau}$ to the current $J^a[V]$ defined in (101), we will encounter boundary terms on $\tilde{\Sigma}_{\tau}$. In Prop. 7.1 and Theorem 7.2 we show that after integration with respect to $\tau$, the boundary terms involve derivatives of various quantities only in $H$-tangential directions. In the next lemma, namely Lemma 4.8, we perform some preliminary analysis that essentially shows all terms have the desired structure, except for the term $2\mathfrak{M}^a_{\tilde{z}} \mathfrak{E}^a \mathfrak{V}^\beta (\partial_\alpha V^\beta - \partial_\beta V^\alpha)$ on RHS (115), which is much more difficult to handle; we dedicate all of Sect. 5 to understanding the structure of this term. The proof of Lemma 4.8 relies on careful geometric decompositions, but unlike the analysis of Sect. 4 and the proof of Prop. 7.1, it does not rely on the formulation of compressible Euler flow provided by Theorem 2.8.

Lemma 4.8 (Preliminary analysis of the boundary integrand). Let $V$ be a $\Sigma_{\tau}$-tangent vectorfield defined on $M$, let $\mathfrak{V}$ be its $g$-orthogonal projection onto $\mathfrak{S}_{\tau}$ (see Defs. 3.14 and 3.15), let $J[V]$ be the $\Sigma_{\tau}$-tangent vectorfield from Def. 4.5, and let $\mathfrak{W}$ be an arbitrary scalar function. Let $\tilde{E}$ and $\Theta$ be the vectorfields from Def. 7.10 and Lemma 7.11. The following identity holds along $\mathfrak{S}_{\tau}$, where on RHS (115), $\mathfrak{d} \tilde{\Phi}_\beta$ denotes the
\( \nabla \)-divergence of the \( S_\tau \)-tangent vectorfield \( \theta_\beta \) (as in (112)):

\[
\mathcal{W} Z_\alpha J^\alpha [V] = -\tilde{\mathcal{H}} \left\{ \frac{\mathcal{W} z}{\hbar} |V|^2 g_\beta \right\} + 2\mathcal{W} z F^\alpha V^\beta (\partial_\alpha V_\beta - \partial_\beta V_\alpha) \\
+ \left\{ \tilde{\mathcal{H}} \left[ \frac{\mathcal{W} z}{\hbar} (g^{-1})^{\alpha \beta} \right] \right\} V_\alpha V_\beta + \left\{ \Theta \left[ \mathcal{W} z (g^{-1})^{\alpha \beta} \right] \right\} V_\alpha V_\beta - 2\mathcal{W} V_\alpha (\mathcal{W} F^\alpha)
\]

\[
+ \mathcal{W} z |V|^2 \delta \Theta + \mathcal{W} z V_\alpha V^\beta \delta \Theta g_\beta + \mathcal{W} V_\alpha V^\beta Z^\alpha \\
+ \mathcal{W} V^\alpha Z^\beta V g_\alpha + V_\alpha Z^\alpha \mathcal{W} + \\
- \delta \Theta \left\{ \mathcal{W} |V|^2 \right\} - \delta \Theta \left\{ \mathcal{W} Z^\alpha \right\}.
\]

**Proof.** First, using (101) and (86) and the fact that \( \tilde{\Pi} Z = Z \), we compute that

\[
\mathcal{W} Z_\alpha J^\alpha [V] = \mathcal{W} Z_\alpha V^\beta \Omega^\beta \delta \Theta V^\gamma - \mathcal{W} Z_\alpha V^\alpha \Omega^\alpha \delta \Theta V^\lambda.
\]  

(116)

Differentiating by parts on the last product \(-\mathcal{W} Z_\alpha V^\gamma \Omega^\gamma \delta \Theta V^\lambda\) on RHS (116) and using the simple identity \( V = V^\beta \delta \Theta = V^\lambda \Omega^\lambda \delta \Theta \) (which follows from (84), (86c), and definition (86a)), we compute that

\[
\text{RHS} \ (116) = 2\mathcal{W} Z_\gamma V^\beta \Omega^\beta \delta \Theta V^\gamma - \delta \Theta \left\{ \mathcal{W} Z_\alpha V^\alpha \right\} + \mathcal{W} Z_\alpha V^\alpha V^\beta \delta \Theta g_\beta + V^\alpha \mathcal{W}(\mathcal{W} Z_\alpha),
\]

(117)

where we stress that on RHS (117), we are viewing \( \delta \Theta \) as an \( S_\tau \)-tangent vectorfield. Expressing the last factor on RHS (117) as \( Z_\alpha = g_\alpha \partial^\beta \), we compute that

\[
\text{RHS} \ (117) = 2\mathcal{W} V^\beta \Omega^\delta Z^\alpha \delta \Theta g_\alpha - \delta \Theta \left\{ \mathcal{W} Z_\alpha V^\alpha \right\} + \mathcal{W} Z_\alpha V^\alpha V^\beta \delta \Theta g_\beta + \mathcal{W} V_\alpha V^\gamma Z^\alpha \\
+ \mathcal{W} V^\alpha Z^\beta V g_\alpha + V_\alpha Z^\alpha \mathcal{W}.
\]

(118)

Next, we use (75) to substitute for the factor \( Z^\alpha \) in the first product on RHS (118), which allows us to rewrite the factor as follows:

\[
2\mathcal{W} V^\beta \Omega^\delta Z^\alpha \delta \Theta g_\alpha = 2\mathcal{W} z V^\beta \Omega^\delta \delta \Theta V_\alpha - 2\mathcal{W} z V^\beta \Omega^\delta H_\beta \delta \Theta V_\alpha - 2\mathcal{W} z V^\beta \Omega^\delta \Theta^\alpha \delta \Theta V_\alpha.
\]

Next, since \( B \) is \( g \)-orthogonal to \( S_\tau \), we have \( B^\alpha \delta \Theta = 0 \), and by differentiating this relation, we obtain the identity that \( B^\alpha \partial_\alpha V_\alpha = - (\partial_\alpha B^\alpha) V_\alpha \). Using this identity to remove the derivatives off the factor \( V_\alpha \) in the first term on RHS (119), we deduce that

\[
2\mathcal{W} V^\beta \Omega^\delta Z^\alpha \delta \Theta g_\alpha = -2\mathcal{W} z V_\alpha \mathcal{W} B^\alpha - 2\mathcal{W} z V_\alpha \mathcal{W} H_\beta \delta \Theta V_\alpha - 2\mathcal{W} z V_\alpha \mathcal{W} \Theta^\alpha \delta \Theta V_\alpha.
\]

Next, we use straightforward algebraic calculations to rewrite the last two products on RHS (120) as follows, where we take into account (74):

\[
-2\mathcal{W} z V^\beta \Omega^\delta \delta \Theta V_\alpha - 2\mathcal{W} z V^\beta \Omega^\delta \Theta^\alpha \delta \Theta V_\alpha = -2\mathcal{W} z (g^{-1})^{\delta \beta} V_\beta \mathcal{W} V_\delta - 2\mathcal{W} z (g^{-1})^{\beta \delta} V_\beta \mathcal{W} \Theta^\alpha \delta \Theta V_\alpha \\
+ 2\mathcal{W} z E^\alpha V^\beta \{ \partial_\alpha V_\beta - \partial_\beta V_\alpha \}.
\]

Using (50), differentiating by parts on the first two products on RHS (121), and using the simple identity \( |V|^2 = (g^{-1})^{\alpha \beta} V_\alpha V_\beta \), we rewrite (121) as follows:

\[
2\mathcal{W} z V^\beta \Omega^\delta \delta \Theta V_\alpha - 2\mathcal{W} z V^\beta \Omega^\delta \Theta^\alpha \delta \Theta V_\alpha \\
= -\tilde{\mathcal{H}} \left\{ \frac{\mathcal{W} z}{\hbar} |V|^2 g_\beta \right\} - \delta \Theta \left\{ \mathcal{W} |V|^2 \right\}
\]

\[
+ \left\{ \tilde{\mathcal{H}} \left[ \frac{\mathcal{W} z}{\hbar} (g^{-1})^{\alpha \beta} \right] \right\} V_\alpha V_\beta + \left\{ \Theta \left[ \mathcal{W} z (g^{-1})^{\alpha \beta} \right] \right\} V_\alpha V_\beta + \mathcal{W} z |V|^2 \delta \Theta \\
+ 2\mathcal{W} z E^\alpha V^\beta \{ \partial_\alpha V_\beta - \partial_\beta V_\alpha \}.
\]

Finally, combining (116)–(122), we conclude the desired identity (113).
5. **Additional geometric decompositions tied to \( \partial_\alpha V_\beta - \partial_\beta V_\alpha \)**

In order to derive our main results, we need to uncover some subtle structures found in the term \( 2W^\alpha \bar{E}^{\alpha} \dot{V}^3 (\partial_\alpha V_\beta - \partial_\beta V_\alpha) \) on the right-hand side of the boundary term identity (115). Although this term has the desired feature that it involves only \( \mathcal{H} \)-tangential derivatives of \( V \), as written, it appears to have insufficient regularity for applications. The reason is that our forthcoming analysis (specifically the proof of Prop. 7.1—see the second integral on RHS (157)) involves the integral of \( 2W^\alpha \bar{E}^{\alpha} \dot{V}^3 (\partial_\alpha V_\beta - \partial_\beta V_\alpha) \) along the lateral hypersurface \( \mathcal{H} \), and the difficulty is that for \( V \in \{ \Omega, S \} \), we have no control over even \( \mathcal{H} \)-tangential first derivatives of \( V \) in \( L^2 \) along \( \mathcal{H} \); this is consistent with the fact that our main integral identities, which are provided by Theorem 8.1, yield only \( \mathcal{H} \)-tangential derivatives of quantities with sufficient regularity. In this section, we provide various geo-analytic decompositions that in total reveal the structures of interest. The main results in this section are Lemma 5.5, Cor. 5.7, and Prop. 5.11.

**Remark 5.1** (Exploiting the special structure of the compressible Euler equations). Lemma 5.5, Cor. 5.7, and Prop. 5.11 are the main results in the paper in which we crucially exploit the special properties of the compressible Euler formulation provided by Theorem 2.8 and the precise structure of equations (4a)-(4b) (where the latter two equations are part of the standard first-order formulation of compressible Euler flow).

### 5.1. Some preliminary geometric decompositions.

#### 5.1.1. The vectorfield \( P \)

In our decompositions, we will encounter the vectorfield \( P \), which we now define.

**Definition 5.2** (The vectorfield \( P \)). Let \( N \) and \( \mathcal{N} \) be the vectorfields from Def. 3.2, and let \( h > 0 \) be the scalar function from (57b). Along \( \mathcal{H} \), we define \( P \) to be the following vectorfield:

\[
P := \frac{N - \mathcal{N}}{h}.
\]

**Lemma 5.3** (Properties of \( P \)). The vectorfield \( P \) from Def. 5.2, which is defined along \( \mathcal{H} \), is \( \Sigma_t \)-tangent (i.e., \( Pt = 0 \)) and \( g \)-spacelike.

**Proof.** From (44) and (46), we see that \((N - \mathcal{N})t = 1 - 1 = 0\). In view of (123), we conclude that \( P \) is \( \Sigma_t \)-tangent as desired.

\[\square\]

#### 5.1.2. A geometric decomposition of the Cartesian coordinate partial derivative vectorfield \( \partial_\alpha \)

In our forthcoming analysis, we will often decompose the Cartesian partial derivative vectorfield \( \partial_\alpha \) into a part that is parallel to \( \mathcal{B} \) and a part that is \( \mathcal{H} \)-tangential. In the next lemma, we provide the decomposition.

**Lemma 5.4** (Decomposition of \( \partial_\alpha \) into \( \mathcal{B} \)-parallel and \( \mathcal{H} \)-tangential components). Let \( \mathcal{N} \) and \( \mathcal{H} \) be the vectorfields from Def. 3.2, and let \( P \) be the vectorfield from Def. 5.2. For \( \alpha = 0, 1, 2, 3 \), let \( W_\alpha \) be the vectorfield on \( \mathcal{H} \) defined by the following identity relative to the Cartesian coordinates:

\[
\partial_\alpha = -N_\alpha \mathcal{B} + P_\alpha \mathcal{H} + W_\alpha.
\]

Then \( W_\alpha \) is \( \Sigma_t \)-tangent.

**Proof.** We first claim that for every sphere \( \Sigma_t \subset \mathcal{H} \) and every \( q \in \Sigma_t \), \( T_q \mathcal{M} \) (which is the tangent space to \( \mathcal{M} \) at \( q \)) enjoys the direct sum decomposition \( T_q \mathcal{M} = \text{span}\{\mathcal{B}|_q, \mathcal{H}|_q\} \oplus T_q \Sigma_t \) where \( \mathcal{X}|_q \) denotes the vectorfield \( \mathcal{X} \) evaluated at \( q \) (and the spaces in the direct sum are not necessarily \( g \)-orthogonal). To prove the claim, we first note that since \( \mathcal{H} \) is \( \mathcal{H} \)-tangential by construction, since \( \Sigma_t = \mathcal{H} \cap \Sigma_t \), and since \( \mathcal{H} \) is \( g \)-orthogonal to \( \Sigma_t \), it follows that \( T_q \mathcal{H} = \text{span}\{\mathcal{H}|_q\} \oplus T_q \Sigma_t \). Since \( \mathcal{H} \) is transversal to \( \Sigma_t \), it follows that \( \mathcal{H} \) is transversal to \( \Sigma_t \). Moreover, since \( \mathcal{B} \) is \( g \)-timelike while \( \Sigma_t \) is \( g \)-spacelike, it follows that \( \mathcal{B} \) is transversal to \( \Sigma_t \). Moreover, \( \mathcal{B} \) and \( H \) are linearly independent since \( \mathcal{B} \) is \( g \)-timelike and \( \mathcal{H} \) is \( g \)-spacelike or null. We have therefore shown that \( \{\mathcal{B}|_q, \mathcal{H}|_q\} \) is a two-dimensional subspace of \( T_q \mathcal{M} \) that is transversal to \( \Sigma_t \) at \( q \). Since \( \Sigma_t \) is a two-dimensional submanifold of \( \Sigma_t \), we conclude the claim.
From the claim, it follows that there exist scalar functions $a_1$ and $a_2$ and an $\mathcal{S}_r$-tangent vectorfield $W_{(a)}$ such that the following vectorfield identity holds relative to the Cartesian coordinates: $\partial_a = a_1 B + a_2 H + W_{(a)}$. Taking the $g$-inner product of this identity with respect to $N$ and using (52) and the fact that $N$ is $g$-orthogonal to $H$ and $W_{(a)}$, we find that $N_{\alpha} = -a_1$ and thus $a_1 = -\frac{N_{\alpha}}{N}$, as desired. Similarly, taking the $g$-inner product of the identity with respect to $N$ and using (51), (57b), and the fact that $N$ is $g$-orthogonal to $W_{(a)}$, we find that $N_{\alpha} = -a_1 - a_2 h$ and thus $a_2 = -\frac{\omega_\alpha}{\omega}$. In view of (123), we conclude (124). \hfill \Box

In the next lemma, we show that for compressible Euler solutions, the term $\nabla^2 B\rho$, which involves a derivative of $\rho$ in an $\mathcal{H}$-transversal direction, can be expressed in terms of $\mathcal{H}$-tangential derivatives of the solution. The lemma plays a crucial role in the proof of Prop. 5.11, where it allows us to eliminate some $\mathcal{H}$-transversal derivatives found in the term $2\mathcal{W} \cdot \mathcal{E}^a \mathcal{V}^\beta (\partial_\alpha V_\beta - \partial_\beta V_\alpha)$ on the right-hand side of the boundary term identity (115) in the case $V = \Omega$. We stress that the identity proved in the lemma degenerates as $\mathcal{H}$ becomes null, that is, as the factor $\nabla$ on LHS (125) converges to 0.

**Lemma 5.5** (Expression for $\nabla^2 B\rho$ in terms of $\mathcal{H}$-tangential derivatives). Let $N$ and $H$ be the vectorfields from Def. 3.2, let $P$ be the vectorfield from from Def. 5.2, and let $\{ W_{(a)} \}_{a=1,2,3}$ be the $\mathcal{S}_r$-tangent vectorfields from Lemma 5.4. Let $\nabla$ be the scalar function from Def. 3.8. Then for smooth solutions (see Remark 1.5) to the compressible Euler equations (4a)-(4c) on $\mathcal{M}$, the following identities hold along $\mathcal{H}$:

\[
\nabla^2 B\rho = - \left\{ P_a H v^a + N_{\alpha} P^{\alpha} H \rho + W_{(a)} v^a + (g^{-1})^{ab} N_{\alpha} W_{(b)} \rho + \exp(-\rho) \frac{P_{\alpha}}{\bar{\rho}} N_{\alpha} S^a \right\}. \tag{125}
\]

**Proof.** First, using (4a) and (124), we compute that

\[
B\rho = -\partial_a v^a = N_{\alpha} B v^a - P_a H v^a - W_{(a)} v^a. \tag{126}
\]

We then use (4b) and (124) to express the first product on RHS (126) as follows:

\[
N_{\alpha} B v^a = -c^2 N_{\alpha} \partial_a \rho - \exp(-\rho) \frac{P_{\alpha}}{\bar{\rho}} N_{\alpha} S^a \tag{127}
\]

Next, using (22a), the fact that $N^0 = 1$ (i.e., (46)), and the fact that $v$ is $\mathcal{S}_r$-tangent, we compute that

\[
N_{\alpha} = -v_\alpha + c^{-2} N^\alpha = -c^{-2} \partial_a v^a + c^{-2} N^\alpha. \tag{128}
\]

Using (22a), (46), and (128), we compute that relative to the Cartesian coordinates, we have $c^2 N_{\alpha} N_{\alpha} = v^a v_\alpha (N^0)^2 + c^{-2} N^\alpha N^\alpha - 2N^0 v_\alpha N^\alpha = g_{\alpha\beta} N^\alpha N^\beta + 1$. From this identity and (63), we find that

\[
c^2 N_{\alpha} N_{\alpha} = 1 - \nabla^2. \tag{129}
\]

Substituting RHS (129) for the relevant factors in the first product on RHS (127), we express the first product on RHS (126) as follows:

\[
N_{\alpha} B v^a = (1 - \nabla^2) B\rho - c^2 N_{\alpha} P_a H \rho - c^2 N_{\alpha} W_{(a)} \rho - \exp(-\rho) \frac{P_{\alpha}}{\bar{\rho}} N_{\alpha} S^a. \tag{130}
\]

Finally, substituting RHS (130) for the first product on RHS (126), noting that (22b) implies that $(g^{-1})^{ab} = c^2 \delta^{ab}$ (where $\delta^{ab}$ is the standard Kronecker delta), and carrying out straightforward algebraic computations, we conclude (125). \hfill \Box

5.2. **Expressions for $\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha$ and $\partial_\alpha S_\beta - \partial_\beta S_\alpha$.** In this subsection, we derive identities for the antisymmetric gradient tensorfields $\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha$ and $\partial_\alpha S_\beta - \partial_\beta S_\alpha$. The identities hold only for compressible Euler solutions, and their proof relies on the precise structure of the equations of Theorem 2.8. The identities play a crucial role in revealing the good structure of the term $2\mathcal{W} \cdot \mathcal{E}^a \mathcal{V}^\beta (\partial_\alpha V_\beta - \partial_\beta V_\alpha)$ on the right-hand side of the boundary term identity (115).
5.2.1. Simple identities for $\Sigma_i$-tangent vectorfields. We start by deriving some identities for $\Sigma_i$-tangent vectorfields.

Lemma 5.6 (Identities for $\Sigma_i$-tangent vectorfields). Let $V$ be a $\Sigma_i$-tangent vectorfield. Then relative to the Cartesian coordinates, the following identities hold for $\alpha, \beta = 0, 1, 2, 3$ and $i, j = 1, 2, 3$, where $c = c(p, s)$ is the speed of sound, $\epsilon_{ijk}$ denotes the fully antisymmetric symbol normalized by $\epsilon_{123} = 1$, and $\epsilon_{\alpha\beta\gamma\delta}$ denotes the fully antisymmetric symbol normalized by $\epsilon_{0123} = 1$:

$$V_0 = -g_{ab}v^av^b,$$

$$V_i = c^{-2}V^i, \quad (131)$$

$$\partial_iV_j - \partial_jV_i = c^{-2}\epsilon_{ij\alpha}\text{curl}V^\alpha - 2(\partial_i \ln c)V_j + 2(\partial_j \ln c)V_i, \quad (132a)$$

$$B^\alpha\partial_\alpha V_\beta = -V_\beta(\partial_\alpha B^\alpha) = -V_\alpha \partial_\alpha v^\alpha, \quad (132b)$$

$$\partial_\alpha V_\beta - \partial_\beta V_\alpha = 2(\partial_\beta \ln c)\partial_\alpha v^\alpha - 2(\partial_\alpha \ln c)\partial_\beta v^\alpha + \delta_\alpha^0\delta_\beta^0v^a - \delta_\beta^0\delta_\alpha^0v^a \quad (133)$$

Proof. $(131)$-$(132c)$ from straightforward calculations relative to the Cartesian coordinates based on the identity $\partial_iV^j - \partial_jV^i = \epsilon_{ij\alpha}\text{curl}V^\alpha$ for $\Sigma_i$-tangent vectorfields $V$, the fact that $B^\alpha V_\alpha = g(B, V) = 0$ for $\Sigma_i$-tangent vectorfields $V$ (see (27), (22a), the decomposition $\partial_\alpha = B - v^\alpha\partial_\alpha$ (see (4)), and the fact that $g_{ij} = g_{ij} = c^{-2}\delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta (see (22a) and (79a)).

$(133)$ is just a combining of $(132a)$ and $(132c)$ that takes into account the fact that $B = \partial_\alpha + v^\alpha\partial_\alpha$, the identity $V_0 = -v^aV_a$ (which follows from $(131)$ and $(22a)$), and the form $(22a)$ of the acoustical metric $g$ relative to the Cartesian coordinates.

5.2.2. Specializing the identities to compressible Euler solutions. In the next corollary, we specialize the results of Lemma 5.6 to $\Omega$ and $S$. Unlike the proof of the lemma, the proof of the corollary relies on the equations of Theorem 2.8.

Corollary 5.7 (Sharp decomposition of $\partial_\alpha\Omega_\beta - \partial_\beta\Omega_\alpha$ and $\partial_\alpha S_\beta - \partial_\beta S_\alpha$). For smooth solutions (see Remark 1.5) to the compressible Euler equations (1a)-(1c) on $M$, the following identity holds, where $C$ is the $\Sigma_i$-tangent modified fluid variable from (21a), $\Omega_\gamma$ is the $g$-dual one-form of $\Omega$ (see (23)), and $d\Omega_\gamma$ denotes the exterior derivative of $\Omega_\gamma$:

$$(d\Omega_\gamma)_{\alpha\beta} := \partial_\alpha\Omega_\beta - \partial_\beta\Omega_\alpha = 2(\partial_\beta \ln c)\Omega_\alpha - 2(\partial_\alpha \ln c)\Omega_\beta + 2\delta_\alpha^0\Omega_\beta v^a - 2\delta_\beta^0\Omega_\alpha v^a \quad (134a)$$

$$- c^{-4}\exp(-2p)\frac{\partial^a}{\partial^a}\epsilon_{\alpha\beta\gamma\delta}(B\Omega^\gamma)S^\delta$$

$$+ c^{-4}\exp(-2p)\frac{\partial^a}{\partial^a}\epsilon_{\alpha\beta\gamma\delta}B^\gamma[S^\delta(\partial_\alpha v^a) - S^\delta\partial_\alpha v^a]$$

$$+ c^{-2}\exp(p)\epsilon_{\alpha\beta\gamma\delta}B^\gamma C^\delta.$$  

Moreover, the following identity holds:

$$(dS_\gamma)_{\alpha\beta} := \partial_\alpha S_\beta - \partial_\beta S_\alpha = 2(\partial_\beta \ln c)S_\alpha - 2(\partial_\alpha \ln c)S_\beta. \quad (134b)$$

Proof. To prove $(134a)$, we consider $(133)$ with $\Omega$ in the role of $V$. We next note that definition (3b) implies that RHS $36a$ (which is equal to RHS $(32a)$) can alternatively be expressed as $\Omega_\gamma\partial_\alpha v^a + \exp(-2p)c^{-2}\epsilon_{\alpha\beta\gamma\delta}(B\Omega^\gamma)S^\delta = \Omega_\gamma\partial_\alpha v^a - \exp(-2p)c^{-2}\epsilon_{\alpha\beta\gamma\delta}(B\Omega^\gamma)S^\delta$. We now substitute this “alternative” version of RHS $(32a)$ for the term $B\Omega^\gamma$ on RHS $(133)$ when $\gamma = i \in \{1, 2, 3\}$, and in the case $\gamma = 0$, we use the simple identity $B^0 = 0$. Next, we use $(21a)$ to algebraically substitute for the factor $(\text{curl}\Omega)^\delta$ on RHS $(133)$ in terms of remaining terms in $(21a)$ when $\delta = i \in \{1, 2, 3\}$, and in the case $\gamma = 0$, we use the fact
that \((\text{curl}\Omega)^0 = 0\). From these steps, the fact that \(B = \partial_t + v^a\partial_a\), the form (22a) of the acoustical metric \(g\) relative to the Cartesian coordinates, and straightforward algebraic calculations, we arrive at (134a).

A similar argument yields (134b), where we use (32a) for substitution, we observe that definition (3b) implies that RHS (36c) can alternatively be expressed as \(-S_a\partial_t v^a\), and we also use the simple identity \(\text{curl}S = 0\) (see (34b)).

\[\square\]

5.3. Preliminary decomposition of the most subtle term on RHS (115). In the next lemma, namely Lemma 5.8, we provide a preliminary decomposition of the most subtle part of the term \(2\mathcal{W}_\gamma v^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha)\), which is the second term on RHS (115) in the case \(V = \Omega\). Specifically, in the lemma, we decompose the part of \(2\mathcal{W}_\gamma v^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha)\) that corresponds to the terms

\[ -c^{-4}\exp(-2p)\frac{P_s}{\bar{\rho}}\epsilon_{\alpha\beta\gamma\delta}(Bv^\gamma)S^\delta + c^{-4}\exp(-2p)\frac{P_s}{\bar{\rho}}\epsilon_{\alpha\beta\gamma\delta}B^\gamma[S^\delta(\partial_\alpha v^a) - S^a\partial_\alpha v^\delta] \]  

(135)

on RHS (134a); more precisely, in the lemma, we ignore the overall factor of \(c^{-4}\exp(-2p)\frac{P_s}{\bar{\rho}}\) in the previous expression. Later, in Prop. 5.11, with the help of Lemma 5.8, we will show that for compressible Euler solutions, the special combination of terms in (135) can be expressed in terms of \(\mathcal{H}\)-tangential derivatives of the fluid solution variables. The “remaining part” of the term \(2\mathcal{W}_\gamma v^\beta (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha)\), as well as the entirety of the term \(2\mathcal{W}_\gamma v^\beta(\partial_\alpha S_\beta - \partial_\beta S_\alpha)\), will be easy to treat, thanks to the decompositions provided by Cor. 5.7.

**Lemma 5.8** (Preliminary geometric decomposition of the most subtle terms on RHS (134a)). Let \(N\) and \(H\) be the vectorfields from Def. 3.2, let \(E\) be the \(\mathcal{H}\)-tangential vectorfield from (14), let \(P\) be the vectorfield from Def. 5.2, and let \(\{W_\alpha\}_{\alpha = 1,2,3}\) be the \(S_\gamma\)-tangential vectorfields from Lemma 5.3. For smooth solutions (see Remark 1.3) to the compressible Euler equations (14a), (4c) on \(\mathcal{M}\), the following identity holds along \(\mathcal{H}\):

\[ \epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta \left\{-\left(Bv^\gamma\right)S^\delta + \left(B^\gamma S^\delta(\partial_\alpha v^a) - S^a\partial_\alpha v^\delta\right)\right\} \]  

(136)

Proof: First, using (14b), (124), (128), the fact that \(P\) is \(S\)\(_\gamma\)-tangential (see Lemma 5.3), and the form (22a) of the acoustical metric \(g\) relative to the Cartesian coordinates, we compute that

\[ Bv^j = -c^2\partial_j\rho - \exp(-p)\frac{P_s}{\bar{\rho}}S^j \]  

(137)

Next, using (124), (137), and (128), we compute that

\[ \partial_t v^j = -N_i Bv^j + P_j H v^j + W_{\gamma j}v^j \]  

(138)

Contracting (138) against \(S^j\), we deduce that

\[ S^a\partial_a v^j = -S^a N_{\gamma j}Bv^j + S^a P_j H v^j + S^a W_{\gamma j}v^j \]  

(139)

\[ = -S^a N_{\gamma j}Bv^j + c^2 N_{\gamma j}P_j H\rho + c^2 N_{\gamma j}W_{\gamma j}\rho + P_j H v^j + W_{\gamma j}v^j + \exp(-p)\frac{P_s}{\bar{\rho}}N_{\gamma j}S^j. \]
Using \[137\] to substitute for the spatial components of the factor \(Bv^7\) on LHS \[136\], using \[4a\] to replace the factor \(\partial_a v^a\) on LHS \[136\] with \(-Bp\), using \[139\] to substitute for the spatial components of the last term \(S^a \partial_a v^a\) on LHS \[136\], using the identities \(\beta^j = v^j\) and 0 = \(S^0 = P^0 = v^0 = \nabla^0 - B^0\), and taking into account the form \((22a)\) of the acoustical metric \(g\) relative to the Cartesian coordinates, we deduce the following identity:

\[
\begin{align*}
- \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta (Bv^7) S^\delta + \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta B [S^\delta (\partial_a v^a) - S^a \partial_a v^\delta] \\
= \left\{ - \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta (\nabla^\gamma - B^\gamma) S^\delta - \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta S^\delta + \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta S^\gamma S^a \frac{\partial_a}{\nabla}(N^\gamma - B^\gamma) \right\} Bp \\
+ \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta \rho S^\gamma H \frac{\partial}{\nabla} - S^a \frac{\partial}{\nabla} \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta B^\gamma S^\delta + \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta B^\gamma S^a \frac{\partial_a}{\nabla} (N^\gamma - B^\gamma) \\
+ c^2 \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta \frac{\partial}{\nabla} (W_{(a)} \rho) S^\delta - c^2 S^a \frac{\partial}{\nabla} \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta B^\gamma W_{(a)} \rho \\
- S^a \frac{\partial}{\nabla} \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta B^\gamma S^\delta W_{(a)} v^\delta \\
- \exp(-\rho) \frac{\partial}{\nabla} S^a \frac{\partial}{\nabla} \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta B^\gamma S^\delta.
\end{align*}
\]  
(140)

The desired identity \[136\] now follows as a simple algebraic consequence of \[140\].

5.4. **Remarkable geometric structure of the error term** \(- \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta N^\gamma (S^\delta + S^a \frac{\partial_a}{\nabla} B^\gamma) Bp\). Recall that in Lemma \[5.8\] we provided a preliminary decomposition of the most subtle part of the error term \(2 \partial_a E^\alpha \nabla^\beta \rho (\partial_a V_j - \partial_j V_a)\) on RHS \[115\] (see also RHS \[157\]). The subtle part arises in the case \(V = \Omega\), and we provided the decomposition of it in equation \[136\]. All products on RHS \[136\] manifestly involve only \(H\)-tangential derivatives of the fluid solution, except for the first product \(- \epsilon_{\alpha\beta\gamma\delta} E^\alpha \nabla^\beta N^\gamma (S^\delta + S^a \frac{\partial_a}{\nabla} B^\gamma) Bp\). In Prop. \[5.11\] we show that for compressible Euler solutions, this remaining product can be re-expressed in terms of \(H\)-tangential derivatives of the solution. Moreover, in the crucial case in which the lateral boundary \(H\) is \(g\)-null, the product completely vanishes. The results of the proposition are of fundamental importance for our main results. The proof relies on the precise structure of some of the transport equations in Theorem \[2.8\].

5.4.1. **A decomposition of the entropy gradient.** In our proof of Prop. \[5.11\] we will use the simple geometric decomposition of \(S\) provided by the following lemma.

**Lemma 5.9** (Decomposition of the entropy gradient). Let \(U\) be the vectorfield whose Cartesian components \(U^a\) are defined by the following equation, where \(S\) is the \((\Sigma_t\)-tangent\) entropy gradient vectorfield, the vectorfields \(N\) and \(\nabla\) are as in Def. \[3.2\], and \(h > 0\) is the scalar function defined in \[57b\]:

\[
S^a = -S^a \frac{\partial}{\nabla} N^a + \left\{ \frac{S^a {\nabla} N^a}{h} \right\} H^a + U^a.
\]  
(141)

Then \(U\) is \(S_t\)-tangent.

**Proof.** Contracting each side of \[141\] against \(N_a\), using that \(g(N, H) = 0\), and using \[52\], we find that \(g(U, N) = 0\). Next, contracting each side of \[141\] against \(N_a\) and using \[51\] and \[57b\], we find that \(g(U, N) = 0\). Thus, \(U\) is \(g\)-orthogonal to \(\text{span}\{N, \nabla\}\), which is equal to the \(g\)-orthogonal complement of \(S_t\) (as we noted in the proof of Lemma \[3.11\]). Thus, \(U\) is \(S_t\)-tangent as desired.

5.4.2. **An \(S_t\)-tangent vectorfield arising in the analysis.** In our proof of Prop. \[5.11\] we will encounter the vectorfield featured in the following definition.

**Definition 5.10** (An \(S_t\)-tangent vectorfield arising in Prop. \[5.11\]). Let \(\Theta\) be the \(S_t\)-tangent vectorfield from Lemma \[3.11\] let \(U\) be the \(S_t\)-tangent vectorfield from Lemma \[5.9\] let \(N\) and \(\nabla\) be the vectorfields from Def. \[3.2\] and let \(h > 0\) be the scalar function defined in \[57b\]. We define the \(S_t\)-tangent vectorfield \(Y\) as follows:

\[
Y^a := \left\{ \frac{S^a N_a}{h} - \frac{S^a \nabla N_a}{h} \right\} \Theta^a - \frac{1}{h} U^a.
\]  
(142)
5.4.3. The main proposition revealing the structure of the term \( \epsilon_{\alpha\beta\gamma\delta}E^\alpha \Omega^\beta \nabla^\gamma (S^\delta + S^a N_a B^\delta)B_p \) on RHS \((136)\).
In the next proposition, we provide the main result of Subsect. 5.4. We show that for compressible Euler solutions, the first product \( \epsilon_{\alpha\beta\gamma\delta}E^\alpha \Omega^\beta \nabla^\gamma (S^\delta + S^a N_a B^\delta)B_p \) on RHS \((136)\) can be expressed in terms of the \( H \)-tangential derivatives of the fluid solution and moreover, that the product vanishes when \( H \) is \( g \)-null. Lemma \((5.5)\) plays a key role in the proof.

**Proposition 5.11** (The key determinant-product calculation). Let \( N \) and \( H \) be the vectorfields from Def. \((3.2)\), let \( E \) be the \( H \)-tangential vectorfield from \((74)\), let \( P \) be the vectorfield from Def. \((5.2)\), let \( \{E_{(a)}\}_{a=1,2,3} \) be the \( S_\tau \)-tangential vectorfields from Lemma \((5.4)\), and let \( Y \) be the \( S_\tau \)-tangential vectorfield from Def. \((5.10)\). Let \( \eta \geq 0 \) and \( \nu > 0 \) be the scalar functions from Def. \((5.8)\) and let \( h \geq 0 \) be the scalar function defined in \((576)\). Consider a smooth solution (see Remark \((1.5)\)) to the compressible Euler equations \((1a)-(1e)\) on \( M \). We define

\[
\sigma := \text{sgn} \left( \epsilon_{\alpha\beta\gamma\delta}E^\alpha \Omega^\beta \nabla^\gamma (S^\delta + S^a N_a B^\delta) \right),
\]

where \( \text{sgn}(0) := 0 \) and for \( r \in \mathbb{R} \) with \( r \neq 0 \), \( \text{sgn}(r) := \frac{r}{|r|} \). If \( H \) is \( g \)-spacelike, then along \( H \), the following identity holds relative to the Cartesian coordinates:

\[
\epsilon_{\alpha\beta\gamma\delta}E^\alpha \Omega^\beta \nabla^\gamma (S^\delta + S^a N_a B^\delta)B_p = \frac{\hbar + \eta^2}{\sqrt{\hbar^2 + \eta^2 \nu^2}} \sqrt{\text{det} \begin{pmatrix} g(\Omega, \Omega) & g(\Omega, Y) \\ g(\Omega, Y) & g(Y, Y) \end{pmatrix}} \times \left\{ P_\alpha H^\alpha + N_\alpha P^\alpha H_p + W(a) v^a + (g^{-1})^{ab} N_a W(b) \right\}.
\]

In particular, all derivatives of \( \rho \) and \( \{v^a\}_{a=1,2,3} \) on RHS \((144)\) are \( H \)-tangential. Moreover, if \( H \) is \( g \)-null, then we have (see Convention \((3.4)\)):

\[
\epsilon_{\alpha\beta\gamma\delta}E^\alpha \Omega^\beta \nabla^\gamma (S^\delta + S^a N_a B^\delta)B_p := E^\alpha \Omega^\beta L^\gamma (S^\delta + S^a L_a B^\delta)B_p = 0 \quad \text{RHS} \quad (144).
\]

**Proof.** First, using \((74), (141), (142)\), and the fact that \( \epsilon_{\alpha\beta\gamma\delta} \) must vanish when contracted in three or more slots against vectorfields tangent to the two-dimensional submanifold \( S_\tau \), we compute that

\[
\epsilon_{\alpha\beta\gamma\delta}E^\alpha \Omega^\beta \nabla^\gamma (S^\delta + S^a N_a B^\delta) = \epsilon_{\alpha\beta\gamma\delta}N^\alpha H^\beta Y^\gamma \Omega^\delta.
\]

Next we note the following standard fact: for arbitrary sets of four vectorfields \( \{X_1, X_2, X_3, X_4\} \), \( \left\{ \epsilon_{\alpha\beta\gamma\delta}X^\alpha (1) X^\beta (2) X^\gamma (3) X^\delta (4) \right\}^2 \) is equal to \((\text{det} g)^{-1}\) (where the determinant is taken relative to the Cartesian coordinates) times the determinant of the \( 4 \times 4 \) matrix whose \((A, B)\) entry is \( g(X_A, X_B) \). Moreover, using \((22a)\), we compute that \( \det g = -c^{-6} \). From these facts, \((70)\), the relations \( g(N, N) = -c^2 \), \( g(H, H) = \eta^2 \), \( g(N, H) = 0 \), and the fact that \( N \) and \( H \) are \( g \)-orthogonal to the \( S_\tau \)-tangential vectorfields \( \Omega \) and \( Y \), we express the square of RHS \((146)\) as follows:

\[
\left\{ \epsilon_{\alpha\beta\gamma\delta}N^\alpha H^\beta \Omega^\gamma Y^\delta \right\}^2 = c^6 \text{det} \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & c^2 & 0 & 0 \\ 0 & 0 & g(\Omega, \Omega) & g(\Omega, Y) \\ 0 & 0 & g(\Omega, Y) & g(Y, Y) \end{pmatrix}
\]

\[
= c^6 c^4 \frac{(h + \eta^2)^2}{h^2 + \eta^2 \nu^2} \text{det} \begin{pmatrix} g(\Omega, \Omega) & g(\Omega, Y) \\ g(\Omega, Y) & g(Y, Y) \end{pmatrix}.
\]

From \((125), (146), (147)\), and the fact that RHS \((147)\) vanishes when the lateral boundary \( H \) is \( g \)-null (since \( \nu = 0 \) in this case), we conclude the desired relations \((144)\) and \((145)\).

\( \square \)

6. **Volume forms, area forms, and integrals**

In this section, we define the geometric volume forms and area forms featured in our localized energy-flux identities and derive some simple identities tied to them.

6.1. **Volume forms, area forms, and integrals**
6.1.1. Definitions of the volume and area forms.

**Definition 6.1** (Geometric volume forms and area forms). Let $g$ be the acoustical metric of Def. 2.2, let $g$ be the first fundamental form of $\Sigma_t$, let $\tilde{g}$ be the first fundamental form of $\tilde{\Sigma}_t$, let $\tilde{g}$ be the first fundamental form of $\tilde{H}$ when $\tilde{H}$ is $g$-spacelike, and let $g$ be the first fundamental form of $S_\tau$; see Subsect. 3.7 for the definitions and properties of the latter four tensors and the beginning of Sect. 3 for a description of the acoustic time function $\tau$. We define the following volume forms and area forms:

- $d\varpi_g$ denotes the canonical volume form induced by $g$ on $M$.
- $d\varpi_g$ denotes the canonical volume form induced by $g$ on $S_\tau$.
- $d\varpi_{\tilde{g}}$ denotes the canonical volume form induced by $\tilde{g}$ on $\tilde{S}_\tau$.
- $d\varpi_{\tilde{g}}$ denote the canonical area form induced by $\tilde{g}$ on $\tilde{S}_\tau$.

- In the case that $\tilde{H}$ is spacelike, $d\varpi_{\tilde{g}}$ denotes the canonical volume form induced by $g$ on $\tilde{H}$.
- In the case that the lateral boundary $\tilde{H}$ is $g$-null (in which case we use the alternate notation $\tilde{N} = H$), we endow $\tilde{N} = \cup_{t' \in [0,T]} S_{\tau'}$ with the volume form $d\varpi_{\tilde{g}} d\tau'$, where $d\varpi_{\tilde{g}}$ is the area form induced by $\tilde{g}$ on $S_{\tau'}$.

6.1.2. Identities for the volume and area forms. In the next lemma, we provide some identities satisfied by the forms from Def. 6.1.

**Lemma 6.2** (Identities involving the volume forms on $S_\tau$, $\Sigma_t$, and $H$). Let $(t', x^1, x^2, x^3)$ be the Cartesian coordinates. Then the following identities hold:

$$
dx[3] = c^{-3} dx^1 dx^2 dx^3, \quad d\varpi_g = c^{-3} dx^1 dx^2 dx^3 dt' = d\varpi_g = d\varpi_g dt'.
$$

(148)

Moreover, let $\tau'$ be the acoustical time function from the beginning of Sect. 3 and let $\eta > 0$ be the scalar function defined in (62). Then the following identity holds:

$$
d\varpi_g = \eta d\varpi_g d\tau'.
$$

(149)

Finally, if $H$ is $g$-spacelike, then with $\ell > 0$ denoting the scalar function from (60), we have

$$
d\varpi_{\tilde{g}} = \ell d\varpi_{\tilde{g}} d\tau'.
$$

(150)

**Proof.** Relative to arbitrary coordinates on $M = M_{\tau'}$, we have $d\varpi_g = \sqrt{\det g} dy^0 dy^1 dy^2 dy^3$. In the special case of the Cartesian coordinates $(t, x^1, x^2, x^3)$, the desired identity (148) for $d\varpi_g$ follows from a straightforward computation based on (22a). To obtain the desired identity (148) for $d\varpi_g$, we will (consistent with Convention 3.10) use the symbol “$\tilde{g}$” to denote the restriction of the first fundamental form of $\Sigma_t$ to $\Sigma_t$-tangent vectors. Then by (22a), relative to the Cartesian spatial coordinates $(x^1, x^2, x^3)$ on $\Sigma_t$, we have that $g = c^{-2} \sum_{i,j=1}^3 dx^i \otimes dx^j$, and the desired identity for $d\varpi_g = \sqrt{\det g} dx^1 dx^2 dx^3$ follows easily.

To prove (150), we first fix arbitrary local coordinates $(\vartheta^1, \vartheta^2)$ on the initial sphere $S_0$. We will now explain how we propagate these coordinates to all of $H$. Abusing notation, we will also denote the propagated coordinates by $(\vartheta^1, \vartheta^2)$. Specifically, we propagate the coordinates by solving the transport equations $H \vartheta^A = 0$, $(A = 1, 2)$, where the $H$-tangent vectorfield $H$ is defined by (50) and the initial conditions for $\vartheta^A$ are the ones specified on $S_0$; this is possible since (54) implies that $H$ is transversal to the spheres $S_\tau \subset \tilde{\Sigma}_\tau$. Thus, for $\tau \in [0, T]$, the restriction of $(\vartheta^1, \vartheta^2)$ to $\tilde{S}_\tau$ forms a local coordinate system on $\tilde{S}_\tau$. Moreover, relative to the coordinates $(\tau, \vartheta^1, \vartheta^2)$ on $H$, we have (again by (24)) $H = \vartheta^1 \partial_{\tau}$, and the condition (54) plus the fact that $(\partial_{\vartheta^1}, \partial_{\vartheta^2})|_{S_0}$ is tangent to $S_0$ together ensure that $(\partial_{\vartheta^1}, \partial_{\vartheta^2})$ is tangent to $\tilde{S}_\tau$ for $\tau \in [0, T]$ and $A = 1, 2$. We next recall that $H$ is $g$-orthogonal to $S_\tau$ by construction and, by (60), that it verifies $g(H, H) = g(H, H) = \ell^2 > 0$ (since $H$ is spacelike by assumption). In total, it follows that relative to the coordinates $(\tau, \vartheta^1, \vartheta^2)$ on $H$,

---

48 Throughout the article, we blur the distinction between the forms themselves, which are antisymmetric tensors, and the corresponding integration measures they induce on the relevant manifolds; the precise meaning will be clear from context.

49 For example, relative to arbitrary coordinates $(y^a)_{a=0,1,2,3}$ on $M$, we have $d\varpi_g = \sqrt{\det g} dy^0 dy^1 dy^2 dy^3$, while relative to arbitrary coordinates $(\vartheta^A)_{A=1,2}$ on $S_\tau$, we have $d\varpi_{\tilde{g}} = \sqrt{\det g} d\vartheta^0 d\vartheta^1$.

50 By considering “a limit as the spacelike hypersurface $H$ becomes null,” one can infer that the volume form $d\varpi_{\tilde{g}} d\tau'$ on the limiting null hypersurface is the “correct” form for use in the divergence theorem; see the proof of Prop. 9.3 for further discussion.
we have \( q = \xi^2 d\tau \otimes d\tau + g_{AB} d\theta^A \otimes d\theta^B \) and \( q = g_{AB} d\theta^A \otimes d\theta^B \) (here we are viewing \( q \) as a Riemannian metric on \( S_\tau \)), where \( g_{AB} := \frac{\partial}{\partial \tau} (\frac{\partial}{\partial \tau}) \) and we are using Einstein summation convention for the capital Latin indices, which vary over 1, 2. Thus, \( d\omega_q = \sqrt{|\det q|} d\theta^1 d\theta^2 d\tau \), and \( d\omega_q = d\omega_q \). This is the desired identity \((150)\).

The identity \((149)\) can be proved via similar arguments, as we now explain. We first fix arbitrary coordinates \( \{y^1, y^2, y^3\} \) on \( S_0 \) and propagate them to \( M \) by solving the transport equation \( Qy^A = 0 \), where \( Q \) is as in Def. 3.1 and the initial conditions for \( \{y^1, y^2, y^3\} \) are the ones specified on \( S_0 \). We stress that this procedure allows us to extend \( \{y^1, y^2, y^3\} \) to all of \( M \) since for \( \tau \in (0, T) \), every maximally extended integral curve of \( Q \) contained in \( M \) must intersect \( S_0 \) at some point. To see this, we only have to rule out the possibilities that for \( \tau \in (0, T) \), the past endpoint of a maximally extended integral curve of \( Q \) contained in \( M \) lies in its lateral boundary \( \mathcal{H}_\tau \) or in its top boundary \( \Sigma_\tau \). These two possibilities are straightforward to rule out based on the discussion below \((57)\), the fact that \( Q \) is a positive scalar function multiple of \( N \), (this follows from the fact that \( Q \) and \( N \) are parallel, \((62)\), the fact that \( N \) is \( g \)-timelike and future-directed, and the fact that the gradient of \( \tau \) is \( g \)-timelike and past-directed), and the fact that \( N \) is \( g \)-timelike and future-directed, which in total imply that for \( \tau \in (0, T) \), \( Q \) points outward to \( M \) along its the top boundary \( \Sigma_\tau \) and outward to \( M \) along its lateral boundary \( \mathcal{H}_\tau \). Next, using \((45)\), \((62)\), and the fact that \( Q \) is \( g \)-orthogonal to \( \Sigma_\tau \), we find that relative to the coordinates \( \{x, y^1, y^2, y^3\} \) on \( M \), we have \( Q = \frac{\partial}{\partial \tau} \) and \( g = q^2 d\tau \otimes d\tau + g_{AB} d\theta^A \otimes d\theta^B \), where \( g_{AB} := \frac{\partial}{\partial \tau} (\frac{\partial}{\partial \tau}) \) and the capital Latin indices now vary over 1, 2, 3. From this identity for \( g \), the desired identity \((149)\) readily follows. \(\square\)

6.1.3. Integrals with respect to the geometric volume and area forms. Until Sect. 6.1 we define all of our integrals relative to the volume forms of Def. 6.1. For example, if \( f \) is a scalar function defined on the \( g \)-spacelike lateral boundary \( \mathcal{H}_\tau \), \( \tau \) is the acoustical time function introduced at the beginning of Sect. 3 and \((\partial^1, \partial^2)\) are arbitrary local coordinates on \( S_\tau \) (which is diffeomorphic to \( S^2 \)), then

\[
\int_{S_\tau} f d\omega_g = \int_{S^2} f(\tau, \partial^1, \partial^2) \sqrt{|\det g(\tau, \partial^1, \partial^2)|} d\partial^1 d\partial^2,
\]

while by \((150)\), we have

\[
\int_{\mathcal{H}_\tau} f d\omega_q = \int_{\tau=0}^\tau \int_{S^2} \ell(\tau', \partial^1, \partial^2) f(t, \partial^1, \partial^2) \sqrt{|\det g(\tau', \partial^1, \partial^2)|} d\partial^1 d\partial^2 dt'.
\]

6.2. Differential and integral identities involving \( S_\tau \). The following lemma, though standard, plays a crucial role in our proof of Theorem 5.1. Specifically, we use the identity \((152)\) to show that one of the error integrals in our main integral identity has a good sign, up to error terms that are controllable because they depend only on the \( H \)-tangential derivatives of various quantities.

**Lemma 6.3** (Differential and integral identities involving \( S_\tau \)). Let \( H \) be \( g \)-spacelike or \( g \)-null (i.e. \( H = N \)), and let \( f \) be a smooth function defined on \( H \) (on \( N \) in the null case). Let \( \xi > 0 \) be the scalar function defined in \((49)\), and let \( \tilde{H} := iH \) be the \( H \)-tangential vectorfield defined in \((50)\) (and thus \( \tilde{H} = iL \) in the null case by \((71)\)), let \( \mathcal{L}_H \) denote Lie differentiation with respect to \( H \), and let \( g = g_{\alpha\beta} d\theta^\alpha \otimes d\theta^\beta \) the first fundamental form of \( S_\tau \) (see Def. 3.13). Let \( \tau \) be the acoustical time function introduced at the beginning of Subsect. 3. Then the following identity holds for \( \tau \in [0, T] \):

\[
\frac{d}{d\tau} \int_{S_\tau} f d\omega_g = \frac{1}{2} \left\{ \tilde{H} f + \frac{1}{2} f(g^{-1})^{\alpha\beta} \mathcal{L}_H g_{\alpha\beta} \right\} d\omega_g.
\]

In addition, the following identity holds for \( \tau \in [0, T] \):

\[
\int_{\mathcal{H}_\tau} \tilde{H} f d\omega_q d\tau' = -\frac{1}{2} \int_{\mathcal{H}_\tau} f(g^{-1})^{\alpha\beta} \mathcal{L}_H g_{\alpha\beta} d\omega_q d\tau' + \int_{S_\tau} f d\omega_g - \int_{S_\tau} f d\omega_q,
\]

where we note that when \( H \) is \( g \)-spacelike, \((150)\) implies that \( d\omega_g d\tau' = \xi^{-1} d\omega_q \).
Proof. To initiate the proof of (151), we note that the term \((\mathcal{L}_{\theta} \theta_{\alpha \beta})\) on RHS (151) is coordinate invariant and depends only on \(\mathcal{H}\)-tangent tensors. Thus, the term can be evaluated using the local coordinates \((\tau, \vartheta^1, \vartheta^2)\) on \(\mathcal{H}\) from the proof of Lemma 6.2. Specifically, from the computations carried out in the proof of Lemma 6.2 and the standard determinant differentiation identity \(\frac{\partial}{\partial \tau} \det \mathcal{g} = \det \mathcal{g} \mathcal{g}^{-1})^{\alpha \beta} \frac{\partial}{\partial \tau} \mathcal{g}_{\alpha \beta}\), we see that the integrand \(\int_{\mathcal{H}^1} \mathcal{g} / d\vartheta (f(g^{-1})^{\alpha \beta} \mathcal{L}_{\theta} \theta_{\alpha \beta}) d\vartheta\) on RHS (151) can be expressed as \(\frac{\partial}{\partial \tau} \left( f d\vartheta \right) = \frac{\partial}{\partial \tau} \left( f \mathcal{g} / d\vartheta d\vartheta 1 d\vartheta^2\right)\). (151) therefore follows from differentiating under the integral with respect to \(\tau\).

7. The remarkable structure of the boundary error integrals

In this section, we first prove Prop. 7.1 which yields identities for the boundary error integrals appearing in our main integral identities (which are provided by Theorem 8.1). Then, in Theorem 7.2, we closely examine the structure of the terms appearing in Prop. 7.1 and, using compact notation, reveal why they have the remarkable structures that are crucial for various applications.

7.1. Key identity for the boundary error integrals. In Theorem 8.1, we derive our main spacetime integral identities on \(\mathcal{M}_\tau\). The identities feature “boundary error integrals,” that is, integrals along \(\mathcal{H}_\tau\) and \(\mathcal{S}_\tau\); the discussion surrounding equation (175) shows how these error integrals emerge in the proof of the theorem. In the next proposition, we derive identities for these boundary error integrals which, in conjunction with Theorem 7.2, show that they have remarkable structure.

Proposition 7.1 (Key identity for the boundary error integrals). Let \(\mathcal{W}\) be an arbitrary scalar function, and let \(J[\Omega]\) and \(J[S]\) be the \(\mathcal{S}_\tau\)-tangent vectorfields defined in (101). Let \(Z\) be the unit outer normal to \(\mathcal{S}_\tau\) from Def. 3.2. Let \(\xi > 0\) be the scalar function defined in (49) (see also (58)), let \(\varepsilon > 0\) be the scalar function defined in (57a), and let \(\xi > 0\) be the scalar function defined in (57b). For smooth solutions (see Remark 1.3) to the compressible Euler equations (4a)–(4c), the following integral identities hold, where the volume and area forms are as in Def. 6.1:

\[
\begin{align*}
\int_{\mathcal{H}_\tau} \mathcal{W} Z_\alpha J^\alpha |\Omega| d\vartheta_d\tau' + \int_{\mathcal{S}_\tau} \mathcal{W} \frac{\varepsilon}{h} |\Omega|^2_d\vartheta_d\vartheta = & \int_{\mathcal{S}_\tau} \mathcal{W} \frac{\varepsilon}{h} |\Omega|^2_d\vartheta_d\vartheta \\
& + \int_{\mathcal{H}_\tau} \left\{ \mathcal{D}_\xi \mathcal{W} \right\} |\Omega| + \mathcal{W} \mathcal{D}_\xi |\Omega| + \mathcal{W} \mathcal{D}_\xi (1) |\Omega| \right\} d\vartheta_d\tau', \\
\int_{\mathcal{H}_\tau} \mathcal{W} Z_\alpha J^\alpha |S| d\vartheta_d\tau + \int_{\mathcal{S}_\tau} \mathcal{W} \frac{\varepsilon}{h} |S|^2_d\vartheta_d\vartheta = & \int_{\mathcal{S}_\tau} \mathcal{W} \frac{\varepsilon}{h} |S|^2_d\vartheta_d\vartheta \\
& + \int_{\mathcal{H}_\tau} \left\{ \mathcal{D}_\xi \mathcal{W} \right\} |S| + \mathcal{W} \mathcal{D}_\xi |S| + \mathcal{W} \mathcal{D}_\xi (2) |S| \right\} d\vartheta_d\tau'.
\end{align*}
\]

In (153a)–(153b), for \(V \in \{\Omega, S\}\), we have

\[
\begin{align*}
\mathcal{D}_\xi \mathcal{W} |V| := & \frac{\varepsilon}{h} |V|^2_d \mathcal{H} \mathcal{W} + \varepsilon |V|^2_d \Theta \mathcal{W} + V_\alpha Z^\alpha \mathcal{V} \mathcal{W}, \\
\mathcal{W} |V| := & \frac{1}{2} \varepsilon \frac{\varepsilon}{h} \mathcal{W} (g^{-1})^{\alpha \beta} \mathcal{L}_{\theta} \theta_{\alpha \beta} \\
& + \left\{ \mathcal{H} \left[ \frac{\varepsilon}{h} \mathcal{W} (g^{-1})^{\alpha \beta} \right] \right\} V_\alpha V_\beta + \left\{ \Theta \left[ \varepsilon (g^{-1})^{\alpha \beta} \right] \right\} V_\alpha V_\beta - 2 V_\alpha \mathcal{V} B^\alpha \\
& + \varepsilon V_\alpha V_\beta \Theta + Z_\alpha V_\alpha V_\beta \Theta + V_\alpha Z^\beta \mathcal{g}_{\alpha \beta} + V_\alpha \mathcal{V} Z^\alpha,
\end{align*}
\]
\[ \Sigma_{(1)}[\Omega] := 2\sigma_2 c^{-1} \exp(-2p) \frac{P_{\rho}}{\varrho} \left[ \frac{h + \eta^2}{\sqrt{h^2 + \eta^2 v^2}} \right] \sqrt{\det \left( \mathbf{g}(\Omega, \Omega) - \mathbf{g}(\Omega, Y) \right)} \]  
\[ + 42\Omega_0 E^\alpha E^\rho \ln c - 42\Omega_0^2 J^\rho E^\rho \ln c + 42E^\alpha \Omega_0 \Omega^\rho - 42E^\alpha \Omega_0 E^\rho v^\alpha \]  
\[ + 2\sigma c^{-4} \exp(-2p) \frac{P_{\rho}}{\varrho} \epsilon_{\alpha\beta\gamma\delta} E^\alpha \Omega^\beta P^\delta \mathbf{S}^\rho \varrho + 2\sigma c^{-4} \exp(-2p) \frac{P_{\rho}}{\varrho} S^\alpha \epsilon_{\alpha\beta\gamma\delta} E^\alpha \Omega^\beta \mathbf{B}^\gamma P^\delta \mathbf{H}^\rho \]  
\[ + 2\sigma c^{-2} \exp(-2p) \frac{P_{\rho}}{\varrho} \epsilon_{\alpha\beta\gamma\delta} E^\alpha \Omega^\beta (W_{(a)}) S^b - 2\sigma c^{-2} \exp(-2p) \frac{P_{\rho}}{\varrho} S^\alpha \epsilon_{\alpha\beta\gamma\delta} E^\alpha \Omega^\beta \mathbf{B}^\gamma W_{(a)} \]  
\[ - 2\sigma c^{-4} \exp(-3p) \left( \frac{P_{\rho}}{\varrho} \right)^2 S^\alpha \epsilon_{\alpha\beta\gamma\delta} E^\alpha \Omega^\beta \mathbf{B}^\gamma S^\delta, \]  

and

\[ \Sigma_{(2)}[S] := 42S_0 E^\alpha \mathcal{F} \ln c - 42|S|^2 E^\rho \ln c. \]

Above, \( C \) is the \( \Sigma_1 \)-tangent modified fluid variable from Def. 3.1, \( N \) is the vectorfield from Def. 3.2, \( H \) is the \( \Sigma_1 \)-tangent vectorfield from Def. 3.2, \( E \) is the \( H_1 \)-tangent vectorfield from Lemma 3.11, \( \Theta \) is the \( \Sigma_1 \)-tangent vectorfield from Lemma 3.11, \( P \) denotes the \( \Sigma_1 \)-tangent vectorfield from Lemma 3.11, \( \{ W_{(a)} \} \) are the \( \Sigma_1 \)-tangent vectorfields from Lemma 3.11 and the first product on RHS (155) is equal to \(-2\sigma c^{-4} \exp(-2p) \frac{P_{\rho}}{\varrho} \times \) RHS (144). In particular, Prop. 5.11 implies that the first product on RHS (155) vanishes when the lateral boundary \( H \) is \( g \)-null.

Proof. Let \( V \) be a \( \Sigma_1 \)-tangent vectorfield. We integrate the identity (115) with respect to \( d\varpi d\tau \) (see Def.6.1 for the definitions of the volume and area forms) over \( H_1 \) and observe that the integrals of the last two terms on RHS (115) vanish since they are perfect \( \nabla \)-divergences. Using (152) with \(-\frac{\varpi}{2} \nabla [V]^2 \) in the role of \( f \) to substitute for the integral over \( H_1 \) of the first product \(-H \left\{ \frac{\varpi}{2} \nabla [V]^2 \right\} \) on RHS (115), we deduce the identity

\[ \int_{H_1} \mathcal{W} Z_\alpha J^\alpha [V] d\varpi d\tau + \int_{S_0} \mathcal{W} \frac{\varpi}{2} [V]^2 \varrho d\varpi = \int_{S_0} \mathcal{W} \frac{\varpi}{2} [V]^2 \varrho d\varpi \]  
\[ + 2 \int_{H_1} \mathcal{W} z E^\alpha \gamma^\delta (\partial_\alpha V_\beta - \partial_\beta V_\alpha) d\varpi d\tau \]  
\[ + \int_{H_1} \left\{ \Omega_{(\partial \varpi)} [V] + \mathcal{W} \Omega [V] \right\} d\varpi d\tau, \]

where \( \Omega_{(\partial \varpi)} [V] \) is defined in (154a) and \( \Omega [V] \) is defined in (154b).

We then use the identity (157) with \( \Omega \) in the role of \( V \) and use (134a) to substitute for the integrand factor \( \partial_\alpha \Omega^\beta - \partial_\beta \Omega^\alpha \) found in the second integral on RHS (157). The resulting identity features the integral

\[ 2 \int_{H_1} \mathcal{W} \sigma c^{-4} \exp(-2p) \frac{P_{\rho}}{\varrho} \epsilon_{\alpha\beta\gamma\delta} E^\alpha \Omega^\beta \left\{ -(B^\gamma)^S^\delta + B^\gamma (S^\delta \partial_\alpha v^\alpha - S^\alpha \partial_\alpha v^\delta) \right\} d\varpi d\tau \]

coming from the fifth and sixth products on RHS (134a). We rewrite the factors

\[ \epsilon_{\alpha\beta\gamma\delta} E^\alpha \Omega^\beta \left\{ -(B^\gamma)^S^\delta + B^\gamma (S^\delta \partial_\alpha v^\alpha - S^\alpha \partial_\alpha v^\delta) \right\} \]

in (158) by first using the identity (136) for substitution, and then using the key identity (144) (multiplied by \(-1\)) to substitute for the first product on RHS (136) (we use (145) in place of (144) when the lateral
boundary is g-null). We collect all of these terms (except for the common factor of \(\mathcal{W}\)) into the error term \(\mathcal{H}(\alpha)[\Omega]\) defined in (153a). In total, these steps yield (163a).

The proof of (153a) is similar, but we use (134a) in place of the identity (134b) used in the previous paragraph.

\[\square\]

7.2. The remarkable structure of the error terms. Our main result in this subsection is Theorem 7.2 in which we exhibit the remarkable structure of the terms on RHSs (153a)-156).

7.2.1. Additional notation. In this subsubsection, we introduce some notation that will facilitate our presentation of Theorem 7.2. We let \(\bar{\Psi} := \{\rho, v^1, v^2, v^3, s\}\) denote the array of the Cartesian components of the basic fluid variables. If \(X\) is any vectorfield, then \(\bar{X} := \{X^0, X^1, X^2, X^3\}\) denotes the array of its Cartesian components. We omit the component \(X^0\) when \(X\) is \(\Sigma_t\)-tangent, e.g. \(\bar{T} := \{\Omega^1, \Omega^2, \Omega^3\}\).

If \(V\) is a vectorfield, then \(\bar{V}\bar{\Psi} := \{V\rho, Vv^1, Vv^2, Vv^3, Vs\}\) (where, for example, \(V\rho := V^\alpha \partial_\alpha \rho\) and \(\bar{V}\bar{X} := \{\bar{V}^0\bar{X}^0, \bar{V}^1\bar{X}^1, \bar{V}^2\bar{X}^1, \bar{V}^3\bar{X}^3\}\). Similarly, with \(\{\bar{\vartheta}_\alpha\}_{\alpha=0,1,2,3}\) denoting the \(S_t\)-tangent vectorfields from Def. 3.18 we set \(\bar{\vartheta} \bar{\Psi} := \{\bar{\vartheta}_\alpha \rho, \bar{\vartheta}_\alpha v^1, \bar{\vartheta}_\alpha v^2, \bar{\vartheta}_\alpha v^3, \bar{\vartheta}_\alpha s\}\), and \(\bar{\vartheta} \bar{\Psi} := \{\bar{\vartheta}_\beta X^\alpha\}_{\alpha,\beta=0,1,2,3}\). Moreover, with \(\{\bar{\vartheta}_\alpha\}_{\alpha=0,1,2,3}\) denoting \(\bar{H}_t\)-tangent vectorfields from Def. 3.18 we set \(\bar{\vartheta} \bar{\Psi} := \{\bar{\vartheta}_\alpha \rho, \bar{\vartheta}_\alpha v^1, \bar{\vartheta}_\alpha v^2, \bar{\vartheta}_\alpha v^3, \bar{\vartheta}_\alpha s\}\) and \(\bar{\vartheta} \bar{\Psi} := \{\bar{\vartheta}_\beta X^\alpha\}_{\alpha,\beta=0,1,2,3}\).

If \(A\) and \(B\) are scalar functions or arrays of scalar functions, then \(\mathcal{L}(A)[B]\) schematically denotes linear combinations of the elements of \(B\) with coefficients that are continuous functions of the elements of \(A\). For example, since \(N_\alpha = g_\alpha^\beta N^{\beta}\), \(g_\alpha^\beta = \mathcal{L}(\bar{\Psi})\) (see (22a)), and \(\Omega^\alpha \vartheta_\alpha = \Omega^\alpha \vartheta_\alpha\), we have \(\Omega^\alpha \vartheta_\alpha \vartheta_\beta N_\beta = \mathcal{L}(\bar{\Psi}, \bar{\Omega}, 0)\).

7.2.2. The remarkable structure of the error terms. We now state and prove the theorem that exhibits the remarkable structure of the error terms in Prop. 7.1.

**Theorem 7.2** (The remarkable structure of the error terms in Prop. 7.1). Assume that the lateral boundary \(\mathcal{H}\) is g-spacelike. Under the notation of Subsubsect. 7.2.1 the error terms \(\mathcal{S}[\Omega], \mathcal{S}[S], \mathcal{S}_t[\Omega], \) and \(\mathcal{S}_t[S]\) defined in (154b)-156) exhibit the following structure:

\[
\mathcal{S}[\Omega, \mathcal{S}[S], \mathcal{S}_t[\Omega], \mathcal{S}_t[S]] = \mathcal{L}(\bar{\Psi}, \bar{\Omega}, \bar{S}, \bar{H}, \bar{Z}, \vartheta \tau)[\bar{\vartheta} \bar{\Psi}, \bar{\vartheta} \bar{\Psi}, \bar{\vartheta} \bar{\Psi}, \bar{\vartheta} \bar{\Psi}]
+ \mathcal{L}(\bar{\Psi}, \bar{\Omega}, \bar{S}, \bar{H}, \bar{Z}, \vartheta \tau)[\bar{\vartheta} \bar{\Psi}, \bar{\vartheta} \bar{\Psi}, \bar{\vartheta} \bar{\Psi}, \bar{\vartheta} \bar{\Psi}]
(159)
\]

where \(C\) is the \(\Sigma_t\)-tangent modified fluid variable from Def. 2.1 \(\tau\) is the acoustical time function introduced at the beginning of Sect. 3 \(\mathcal{H}\) is the \(\Sigma_t\)-tangent vectorfield from Def. 3.1 and \(Z\) is the \(\Sigma_t\)-tangent vectorfield from Def. 3.2.

Moreover, in the case that the lateral boundary \(\mathcal{H} = \mathcal{N}\) is g-null, we have

\[
\mathcal{S}[\Omega, \mathcal{S}[S], \mathcal{S}_t[\Omega], \mathcal{S}_t[S]] = \mathcal{L}(\bar{\Psi}, \bar{\Omega}, \bar{S}, \bar{L}, \bar{N})[\bar{L} \bar{\Psi}, \bar{L} \vartheta \bar{\Psi}, \bar{L} \vartheta \bar{\Psi}, \bar{L} \vartheta \bar{\Psi}]
+ \mathcal{L}(\bar{\Psi}, \bar{\Omega}, \bar{S}, \bar{L}, \bar{N})[\bar{\vartheta} \bar{\Psi}, \bar{\vartheta} \bar{\Psi}, \bar{\vartheta} \bar{\Psi}, \bar{\vartheta} \bar{\Psi}]
(160)
\]

where \(L = N = H\) (see Convention 3.4 and 71) is the g-null normal to \(\mathcal{N}\) normalized by \(Lt = 1\).

**Remark 7.3** (The most important feature of Theorem 7.2). The most important feature of the theorem is that all derivatives on RHSs (153a)-156) are in directions tangent to \(\mathcal{H}\). Moreover, the most important applications occur when the lateral boundary is g-null, i.e., when \(\mathcal{H} = \mathcal{N}\), as we explained in Subsubsect. 1.4 (see also Remark 7.2), the \(\mathcal{N}\)-tangential nature of the derivatives is crucial for the local regularity theory of solutions (e.g., the proof of Theorem 5.10 in the null case) as well as the study of shocks.

**Proof of Theorem 7.2.** Throughout we use the notation of Subsubsect. 7.2.1. In particular, \(\bar{\Psi} = \{\rho, v^1, v^2, v^3, s\}\) denotes the array of the Cartesian components of the basic fluid variables. We prove only (159) since (160) can be proved using nearly identity arguments that take into account the fact that \(H = L\) in the null case (see (71)).

\[51\text{By "continuous," we mean continuous on an open set of the arguments "A"; all of our results hold for solutions such that A belongs to the open set.}\]
To show that the terms on RHSs \((154b)-(156)\) (where on RHS \((154b)\), we have \(V \in \{\Omega, S\}\)) have the desired structure, we first note that we have the following identities for scalar functions and the Cartesian components of various tensor fields, where throughout the proof, \(f\) schematically denotes a smooth function that is free to vary from line to line:

- \(c = f(\bar{\Psi})\) (see end of Subsubsection 1.1.2)
- \(p_s = f(\bar{\Psi})\) (see end of Subsubsection 1.1.2)
- \(B^\alpha = f(\bar{\Psi})\) (see \(5\))
- \(g_{\alpha\beta} = f(\bar{\Psi})\) (see \(22a\))
- \((g^{-1})^{\alpha\beta} = f(\bar{\Psi})\) (see \(22b\))
- \(g_{\alpha\beta} = f(\bar{\Psi})\) (see \(76a\))
- \((g^{-1})^{\alpha\beta} = f(\bar{\Psi})\) (see \(77a\))
- \(N^\alpha = f(\bar{\Psi}, \partial_\tau)\) (see \(55\))
- \(\nu = f(\bar{\Psi}, \partial_\tau)\) (see \(61\))
- \(\eta = f(\bar{\Psi}, \bar{H})\) (see \(69\))
- \(\xi = f(\bar{H}, \partial_\tau)\) (see \(49\))
- \(\xi = f(\bar{\Psi}, \bar{H}, \bar{Z}, \partial_\tau)\) (see \(57a\)) and the schematic identity for \(N^\alpha\) stated below
- \(\hat{h} = f(\bar{\Psi}, \bar{H}, \partial_\tau)\) (see \(57b\))
- \(\hat{N}^\alpha = f(\bar{\Psi}, \bar{H}, \partial_\tau)\) (see \(68\))
- \(\eta = f(\bar{\Psi}, \bar{H}, \partial_\tau)\) (see \(63\))
- \(\hat{N}^\alpha = f(\bar{\Psi}, \partial_\tau)\) (see \(64\))
- \(\hat{N}^\alpha = f(\bar{\Psi}, \partial_\tau)\) (see \(65\))
- \(\Theta^\alpha = f(\bar{\Psi}, \bar{H}, \bar{Z}, \partial_\tau)\) (see \(72\))
- \(P^\alpha = f(\bar{\Psi}, \bar{H}, \partial_\tau)\) (see \(12a\))
- \(E^\beta = f(\bar{\Psi}, \bar{H}, \bar{Z}, \partial_\tau)\) (see \(73\))
- \(U^\alpha = f(\bar{\Psi}, \bar{S}, \bar{H}, \partial_\tau)\) (see \(141\))
- \(Y^\alpha = f(\bar{\Psi}, \bar{S}, \bar{H}, \bar{Z}, \partial_\tau)\) (see \(142\))
- \(W^\alpha_{(\beta)} = f(\bar{\Psi}, \bar{H}, \partial_\tau)\) (see \(124\))
- \(\bar{g}_{\alpha\beta} = f(\bar{\Psi}, \bar{Z}, \partial_\tau)\) (see \(76d\))
- \((\bar{g}^{-1})^{\alpha\beta} = f(\bar{\Psi}, \bar{Z}, \partial_\tau)\) (see \(77d\))
- \(\bar{\Pi}^\alpha = f(\bar{\Psi}, \bar{Z}, \partial_\tau)\) (see \(78a\))
- \(\bar{\Pi}^\alpha_{(\beta)} = f(\bar{\Psi}, \bar{H}, \partial_\tau)\) (see \(78c\))

For any scalar function \(\varphi\), since \(\Theta\) is \(S_\tau\)-tangent, we have \(\Theta \varphi = \Theta^{\alpha} \partial_\alpha \varphi = \mathcal{L}(\bar{\Psi}, \bar{H}, \bar{Z}, \partial_\tau)[\varphi]\)

Similarly, \(H \varphi = \bar{H}^{\alpha} \partial_\alpha \varphi = \mathcal{L}(\bar{\Psi}, \bar{H})[\varphi]\)

For scalar functions \(\varphi\), we have \(\varphi = \mathcal{L}(\bar{\Psi}, \bar{H})[\varphi]\) (see \(87\))

From the above facts, the desired result \((159)\) follows easily by expanding the terms on RHSs \((154b)-(156)\) (relative to the Cartesian coordinates) using the chain and product rules, and using the following results, which we prove just below:

\[
(\bar{g}^{-1})^{\alpha\beta} \mathcal{L}_{\bar{H}} \bar{g}_{\alpha\beta} = \mathcal{L}(\bar{\Psi}, \bar{H}, \bar{Z}, \partial_\tau)[\partial_\bar{\Psi}, \partial_\bar{H}, \partial_\bar{Z}, \partial_\tau], \tag{161}
\]

\[
\partial_\xi \Theta = \mathcal{L}(\bar{\Psi}, \bar{H}, \bar{Z}, \partial_\tau)[\partial_\bar{\Psi}, \partial_\bar{H}, \partial_\bar{Z}, \partial_\tau], \tag{162}
\]

\[
\partial_\xi \varphi = \mathcal{L}(\bar{\Psi}, \bar{Z}, \partial_\tau)[\partial_\bar{\Psi}, \partial_\bar{Z}, \partial_\tau]. \tag{163}
\]

The identity \((162)\) follows easily from \((91b)\) with \(\Theta\) in the role of \(Y\) and from using the results obtained earlier in the proof. Similarly, \((163)\) follows easily from \((92)\) and from using the results obtained earlier in the proof. Finally, to obtain \((161)\), we compute that relative to the Cartesian coordinates, we have

\[
(\bar{g}^{-1})^{\alpha\beta} \mathcal{L}_{\bar{H}} \bar{g}_{\alpha\beta} = (\bar{g}^{-1})^{\alpha\beta} \bar{H} \bar{g}_{\alpha\beta} + 2(\bar{g}^{-1})^{\alpha\beta} \bar{g}_{\gamma\beta} \partial_\alpha (\bar{H}^\gamma). \tag{164}
\]
Remarkable localized integral identities for 3D compressible Euler flow

Since \((\hat{g}^{-1})^{\alpha\beta} \hat{g}_{\gamma\delta} = \hat{W}^\gamma_{\delta}\) [see the last part of Lemma 3.13], we see that the last product on RHS (164) is equal to \(2\hat{H}^\alpha \hat{\partial}_{\alpha} + 2\hat{\partial}_\alpha \hat{H}^\alpha = 2\hat{\partial}_\alpha \hat{H}^\alpha\), where to obtain the last equality, we used that \(\hat{H}\) is \(g\)-orthogonal to \(S_r\) by construction. Moreover, the results from earlier in the proof imply that \(2\hat{\partial}_\alpha \hat{H}^\alpha = \mathcal{L}(\hat{H}, \hat{\partial}_\tau)|\hat{\partial} \hat{H}\) [as desired. In addition, the results from earlier in the proof imply that the first product on RHS (164) verifies \((\hat{g}^{-1})^{\alpha\beta} \hat{H}_{\alpha\beta} = \mathcal{L}(\hat{H}, \hat{\partial}_\tau)|\hat{\partial} \hat{H}\). In total, these identities yield (161).

\[\square\]

8. The main theorem: Remarkable Hodge-transport-based integral identities for \(\Omega\) and \(S\)

In this section, we state and prove our main theorem, which, for compressible Euler solutions, provides localized, coercive integral identities yielding control over the first derivatives of the specific vorticity and entropy gradient. The identities feature boundary error integrals \(\int_{\mathcal{H}_r} \ldots\) and \(\int_S \ldots\), and in Theorem 7.2, we exhibited the remarkable structures of the error integrand terms in \(\int_{\mathcal{H}_r} \ldots\), with regard to their regularity and to the tangential nature of the derivatives involved. Moreover, the integrals \(\int_S \ldots\), are positive, which is crucial for the coerciveness of the identities. Together, Theorem 8.1, Theorem 7.2, and the demonstrated positivity of the integrals \(\int_S \ldots\) constitute the main new contributions of the paper. As we described in Subsect. 1.4, the structures revealed by Theorem 8.1 and Theorem 7.2 are crucial for applications.

**Theorem 8.1** (The main theorem: Remarkable Hodge-transport integral identities for \(\Omega\) and \(S\)). Let \(\mathcal{M} = \mathcal{M}_T\) be a spacetime region satisfying the conditions stated in Subsects. 3.1 and 3.3 for some \(T > 0\); see Fig. 4. In particular, assume that the lateral boundary \(\mathcal{H} = \mathcal{H}_T\) is \(g\)-spacelike or is \(g\)-null (in the null case, \(\mathcal{H} := \mathcal{N} = \mathcal{N}_r\)). For vectorfields \(X\), let \(\mathcal{L}(\partial X, \partial X)\) be the quadratic form defined by (39), and recall that the positive definite nature of \(\mathcal{L}\) was revealed in Lemma 3.4. Let \(\mathcal{W}\) be an arbitrary scalar function. Let \(q > 0\) be the scalar function defined in (62), let \(\varepsilon > 0\) be the scalar function defined in (49) [see also (48)], and let \(\varepsilon > 0\) and \(\varepsilon > 0\) be the scalar functions from (57a)-(57b) [see also (58)]. Then for smooth solutions (see Remark 1.5) to the compressible Euler equations (4a)-(5b), the following integral identities hold, where the definitions of the volume and area forms are provided in Def. 6.7, and the remarkable structure of the integrals over \(\mathcal{H}_r\) on RHSs (165a)-(165b) was revealed in Theorem 7.3 (and we refer to Appendix A for a table of the notation):

\[
\begin{align*}
\int_{\mathcal{M}_r} \mathcal{W}^{-1} \mathcal{L}(\partial \Omega, \partial \Omega) \, d\varpi_g + \int_{\mathcal{S}_r} \mathcal{W} \frac{\varepsilon}{\varepsilon} |\Omega|_g^2 \, d\varpi_g & = (165a) \\
= \int_{\mathcal{S}_r} \mathcal{W} \frac{\varepsilon}{\varepsilon} |\Omega|_g^2 \, d\varpi_g \\
+ \int_{\mathcal{M}_r} q^{-1} \left\{ \frac{1}{2} |\mathcal{Q}(\Omega)|_g^2 + |\mathcal{B}(\Omega)|_g^2 + \mathcal{C}(\Omega) + \mathcal{D}(\Omega) + \mathcal{J}(\text{coeff})[\Omega, \partial \Omega]\right\} \, d\varpi_g \\
+ \int_{\mathcal{M}_r} q^{-1} \mathcal{J}(\partial \mathcal{W})[\Omega, \partial \Omega] \, d\varpi_g \\
+ \int_{\mathcal{H}_r} \left\{ \mathcal{Q}_1(\partial \mathcal{W})[\Omega] + \mathcal{W}_1[\Omega] + \mathcal{W}_2[\Omega]\right\} \, d\varpi_g \, d\tau' \\
= \int_{\mathcal{M}_r} q^{-1} \mathcal{L}(\partial S, \partial S) \, d\varpi_g + \int_{\mathcal{S}_r} \mathcal{W} \frac{\varepsilon}{\varepsilon} |\Omega|_g^2 \, d\varpi_g & = (165b) \\
= \int_{\mathcal{S}_r} \mathcal{W} \frac{\varepsilon}{\varepsilon} |\Omega|_g^2 \, d\varpi_g \\
+ \int_{\mathcal{M}_r} q^{-1} \left\{ \frac{1}{2} |\mathcal{Q}(S)|_g^2 + |\mathcal{B}(S)|_g^2 + \mathcal{C}(S) + \mathcal{D}(S) + \mathcal{J}(\text{coeff})[S, \partial S]\right\} \, d\varpi_g \\
+ \int_{\mathcal{M}_r} q^{-1} \mathcal{J}(\partial \mathcal{W})[S, \partial S] \, d\varpi_g \\
+ \int_{\mathcal{H}_r} \left\{ \mathcal{Q}_1(\partial \mathcal{W})[S] + \mathcal{W}_1[S] + \mathcal{W}_2[S]\right\} \, d\varpi_g \, d\tau'.
\end{align*}
\]
On RHSs \((165a)-(165b)\), \(\mathfrak{A}^{(t)}\) and \(\mathfrak{A}^{(S)}\) are two-forms with the Cartesian components
\[
\mathfrak{A}^{(t)}_{\alpha\beta} := 2(\partial_{\beta} \ln c)\Omega_{\alpha} - 2(\partial_{\alpha} \ln c)\Omega_{\beta} + 2S_{\alpha}^{\beta} \Omega_{\alpha} \partial_{\beta} v^\alpha - 2S_{\beta}^{\alpha} \Omega_{\alpha} \partial_{\beta} v^\alpha - c^{-4} \exp(-2p) \frac{P_{\beta}}{\rho} \epsilon_{\alpha\beta\gamma\delta}(B^\gamma) S^\delta \\
+ c^{-4} \exp(-2p) \frac{P_{\beta}}{\rho} \epsilon_{\alpha\beta\gamma\delta}(B^\gamma) [S^\delta(\partial_{\alpha} v^\alpha) - S^\alpha \partial_{\alpha} v^\alpha] \\
+ c^{-2} \exp(p) \epsilon_{\alpha\beta\gamma\delta} B^\gamma C^\delta,
\]
\(\mathfrak{A}^{(S)}_{\alpha\beta} := 2(\partial_{\beta} \ln c)S_{\alpha} - 2(\partial_{\alpha} \ln c)S_{\beta},\)
\(\mathfrak{B}^{(t)}\) and \(\mathfrak{B}^{(S)}\) are \(\Sigma_{1}\)-tangent vector fields with the Cartesian spatial components
\[
\mathfrak{B}^{(t)}_{\alpha} := \Omega^\alpha \partial_\alpha v - \exp(-2p) c^{-2} \frac{P_{\alpha}}{\rho} \epsilon_{\alpha\beta\gamma\delta}(B^\beta) S^\delta,
\]
\(\mathfrak{B}^{(S)}_{\alpha} := -S^\alpha \partial_\alpha v + \epsilon_{\alpha\beta\gamma\delta} \exp(p) \Omega^\beta S^\delta,
\]
\(\mathfrak{C}^{(t)}\) and \(\mathfrak{C}^{(S)}\) are scalar functions defined relative to the Cartesian coordinates by
\[
\mathfrak{C}^{(t)} := -2(K_{\alpha} K^\alpha) \Omega^\alpha \partial_\alpha \rho - 2\nu(K_{\alpha} K^\alpha) K_{\beta} \mathfrak{B}^{(t)}_{\beta},
\]
\(\mathfrak{C}^{(S)} := 2(K_{\alpha} K^\alpha) \{\exp(2p) D + S^\alpha \partial_\alpha \rho\} - 2\nu(K_{\alpha} K^\alpha) K_{\beta} \mathfrak{B}^{(S)}_{\beta},
\]
\(\mathfrak{D}^{(t)}\) and \(\mathfrak{D}^{(S)}\) are scalar functions defined relative to the Cartesian coordinates by
\[
\mathfrak{D}^{(t)} := (\Omega^\alpha \partial_\alpha \rho)^2 + (\nu K_{\alpha} \mathfrak{B}^{(t)}_{\alpha})^2 + 2\nu(\Omega^\alpha \partial_\alpha \rho) K_{\beta} \mathfrak{B}^{(t)}_{\beta},
\]
\(\mathfrak{D}^{(S)} := \{\exp(2p) D + S^\alpha \partial_\alpha \rho\}^2 - 2\nu \{\exp(2p) D + S^\alpha \partial_\alpha \rho\} K_{\beta} \mathfrak{B}^{(S)}_{\beta},
\]
for \(V \in \{\Omega, \Sigma\}\), the scalar function \(\mathfrak{J}(\text{Coeff})[V, \partial V]\) is defined relative to the Cartesian coordinates by
\[
\mathfrak{J}(\text{Coeff})[V, \partial V] = V^\alpha g_{\alpha \beta}(\tilde{\partial}_\alpha \tilde{N}^\beta) \tilde{N} V^\gamma - V_\alpha (\tilde{\partial}_\alpha \tilde{N}^\alpha) \tilde{N} V^\beta \\
+ V^\alpha (\tilde{\partial}_\alpha \tilde{N}^\beta) \tilde{\partial}_\gamma V^\gamma - V^\alpha (\tilde{\partial}_\alpha \tilde{N}^\gamma) \tilde{\partial}_\gamma V^\beta \\
+ V^\alpha (\tilde{\partial}_\alpha \tilde{N}^\beta) \tilde{\partial}_\gamma \tilde{V}^\gamma - V^\alpha (\tilde{\partial}_\alpha \tilde{N}^\gamma) \tilde{\partial}_\gamma \tilde{V}^\beta \\
+ V^\alpha (\tilde{\partial}_\alpha \tilde{N}^\beta) \tilde{\partial}_\gamma \tilde{\partial}_\delta \tilde{N} V^\gamma \\
- V^\alpha (\tilde{\partial}_\alpha \tilde{N}^\beta) \tilde{\partial}_\gamma \tilde{\partial}_\delta \tilde{V}^\gamma - V^\alpha \tilde{\partial}_\alpha \tilde{N} \tilde{\partial}_\gamma \tilde{\partial}_\delta \tilde{V}^\gamma \\
+ \frac{1}{2} V^\alpha \tilde{\partial}_\alpha \tilde{N} \tilde{\partial}_\gamma \tilde{N} \tilde{\partial}_\delta \tilde{V}^\gamma - \frac{1}{2} V^\alpha \tilde{\partial}_\alpha \tilde{N} \tilde{\partial}_\gamma \tilde{N} \tilde{\partial}_\delta \tilde{V}^\gamma \\
+ 2V^\alpha (\tilde{\partial}_\alpha \tilde{N} \tilde{\partial}_\delta) \tilde{\partial}_\gamma \tilde{V}^\gamma - 2\tilde{\partial}_\alpha \tilde{N} \tilde{\partial}_\delta \tilde{\partial}_\gamma \tilde{V}^\gamma \\
+ \frac{1}{2} (\tilde{\partial}_\alpha \tilde{N} \tilde{\partial}_\delta) (\tilde{\partial}_\gamma \tilde{\partial}_\delta \tilde{V}^\gamma - \frac{1}{2} (\tilde{\partial}_\gamma \tilde{\partial}_\delta) (\tilde{\partial}_\delta \tilde{V}^\gamma \\
+ V^\alpha \tilde{\partial}_\alpha \tilde{N} \tilde{\partial}_\gamma \tilde{\partial}_\delta \tilde{V}^\gamma - V^\alpha \tilde{\partial}_\alpha \tilde{N} \tilde{\partial}_\gamma \tilde{\partial}_\delta \tilde{V}^\gamma,
\]
for \(V \in \{\Omega, \Sigma\}\), the scalar function \(\mathfrak{J}(\partial \mathcal{F})[V, \partial V]\) is defined relative to the Cartesian coordinates by
\[
\mathfrak{J}(\partial \mathcal{F})[V, \partial V] := -J[V, \mathcal{F}] = V^\alpha (\tilde{\partial}_\alpha \mathcal{F}) \tilde{\partial}_\alpha V^\lambda - V^\alpha (\tilde{\partial}_\alpha \mathcal{F}) \tilde{\partial}_\alpha V^\lambda,
\]
for \(V \in \{\Omega, \Sigma\}\), the scalar functions \(\mathfrak{J}(\partial \mathcal{F})[V, \partial V]\) and \(\mathfrak{J}[V]\) are defined relative to the Cartesian coordinates by
\[
\mathfrak{J}(\partial \mathcal{F})[V] := \frac{\tilde{z}}{k} \zeta \|V\|_2^2 \mathcal{H} \mathcal{F} + \frac{\tilde{z}}{k} \tilde{\zeta} \|V\|_2^2 \Theta \mathcal{F} + V_\alpha Z^\alpha \mathcal{F},
\]
\(\mathfrak{J}[V] := \frac{1}{2} \frac{\tilde{z}}{k} [\zeta (\tilde{\zeta})_{\alpha \beta} L \mathcal{F} \mathcal{F} + \zeta V_\beta + \{\Theta (\tilde{\zeta})_{\alpha \beta}\} V_\alpha V_\beta - 2V_\alpha \mathcal{F} B^\alpha \\
+ \tilde{\zeta} \|V\|_2^2 \partial_\alpha \Theta + Z_\alpha V^\alpha \mathcal{F} \partial_\beta \tilde{\zeta} \\
+ V^\alpha Z^\beta \mathcal{F} \mathcal{F} + V_\alpha \mathcal{F} Z^\alpha,
\]
the scalar function $\mathcal{S}_{(1)}[\Omega]$ is defined relative to the Cartesian coordinates by

$$\mathcal{S}_{(1)}[\Omega] := 2\zeta c^{-1} \exp(-2p) \frac{p_s}{\bar{q}} \frac{h + \eta^2}{h^2 + \eta^2} \sqrt{\det \left( \begin{array}{cc} \phi(\Omega, \Omega) & \phi(\Omega, Y) \\ \phi(Y, \Omega) & \phi(Y, Y) \end{array} \right)}$$

(173)

\[ \times \left\{ p_s H v^a + N_a \epsilon^a \partial^a H \rho + W(a) v^a + (g^{-1})^{ab} N_a W(b) \rho + \exp(-p) \frac{p_s}{\bar{q}} N_a S^a \right\} \]

+ $4\zeta \Omega_\rho \xi^a \Omega \ln c - 4\zeta |\Omega|^2 \xi \ln c + 4\zeta \Omega^0 \Omega \xi \rho^a - 4\zeta \Omega^0 \Omega \xi E v^a$

+ $2\zeta c^{-4} \exp(-2p) \frac{p_s}{\bar{q}} \epsilon_{a\beta\gamma\delta} \epsilon^a \partial^a \Omega \epsilon^\beta \partial^\beta \Omega \rho - 2\zeta c^{-4} \exp(-2p) \frac{p_s}{\bar{q}} S^a N_a \epsilon_{a\beta\gamma\delta} \epsilon^a \partial^a \Omega \epsilon^\beta \partial^\beta \Omega \rho$

+ $2\zeta c^{-2} \exp(-2p) \frac{p_s}{\bar{q}} \epsilon_{a\beta\gamma\delta} \epsilon^a \partial^a \Omega \epsilon^\beta \partial^\beta \Omega \rho - 2\zeta c^{-2} \exp(-2p) \frac{p_s}{\bar{q}} S^a N_a \epsilon_{a\beta\gamma\delta} \epsilon^a \partial^a \Omega \epsilon^\beta \partial^\beta \Omega \rho$

- $2\zeta c^{-4} \exp(-2p) \frac{p_s}{\bar{q}} S^a \epsilon_{a\beta\gamma\delta} \epsilon^a \partial^a \Omega \epsilon^\beta \partial^\beta \Omega \rho$

and the scalar function $\mathcal{S}_{(2)}[S]$ is defined relative to the Cartesian coordinates by

$$\mathcal{S}_{(2)}[S] := 4\zeta S_\rho \xi^a \xi \ln c - 4\zeta |S| \xi^2 \xi \ln c.$$  

(174)

Note also that Prop. 3.11 implies that the first product on RHS $\mathcal{S}_{(1)}$ vanishes when $\mathcal{H}$ is $\mathbf{g}$-null.

Remark 8.2 (Highlighting some of the key structures in Theorem 8.1). Here we emphasize some of the key structures in the equations of Theorem 8.1.

- All derivatives of $(\rho, v, s)$ and $(\mathcal{H}, Z, \partial \tau)$ on RHSs $\mathcal{S}_{(1)}$ and $\mathcal{S}_{(2)}$ are in directions tangent to $\mathcal{H}$.

We now point out two reasons why this structure is crucial for applications in the $\mathbf{g}$-null case ($\mathcal{H} = \mathcal{N}$): i) The wave equation fluxes degenerate along null hypersurfaces, and they control only derivatives in directions tangent to $\mathcal{N}$ (see, for example, (209a) in a model case in which $\tau = t$). Thus, if $\mathcal{N}_\tau$-transversal derivatives were present on RHSs (172), then the integrals $\int_{\mathcal{N}_\tau}$ would be uncontrollable from the point of view of regularity. ii) In applications to shock waves, the $\mathcal{N}_\tau$-tangential derivatives of the solution are less singular than the $\mathcal{N}_\tau$-transversal derivatives. Thus, in the $\mathbf{g}$-null case, the identity (160) signifies the absence of the most singular terms. This is a manifestation of the good “remarkable quasilinear null structures” highlighted in the indented paragraph near the beginning of the article. See Subsubsection 2.1.3 for further discussion on the importance of $\mathcal{N}_\tau$-tangential derivatives in the context of shock formation.

- All terms on RHSs (165a) and (165b) are controllable from the point of view of regularity. We make this statement precise in Theorem 9.10 in a model case in which $\tau = t$.

Proof of Theorem 8.1. We first prove (165a). We start by considering the $\bar{\nabla}_\tau$-divergence identity (102) with $\omega$ in the role of $V$. We add the term $\mathcal{W} g_{a\beta}(\mathbf{B}^a \partial^a)(\mathbf{B}^\beta \partial^\beta)$ to each side of this identity. Considering (103), we see that after this addition, the left-hand side of the resulting identity is equal to $\mathcal{W} \mathcal{J}^a [\mathcal{V}]$. We then integrate the identity over $\Omega$ with respect to the volume form $d\omega_{\bar{g}}$ of $\bar{g}$ and use the divergence theorem to obtain an integral identity, which features the main term $\int_{\bar{\nabla}_\tau} \mathcal{W} \mathcal{J}^a [\mathcal{V}] d\omega_{\bar{g}}$, on the left-hand side and, on the right-hand side, the boundary term $\int_{\partial \Omega} \mathcal{W} \mathcal{J}^a [\mathcal{V}] d\omega_{\bar{g}}$, which comes from the term $\bar{\nabla}_{a\beta}(\mathcal{W} J^a [\mathcal{V}])$ on RHS (102). We then integrate the integral identity with respect to $\tau$ and use Lemma 6.2 to obtain, in
view of (100), the identity
\[
\int_{\mathcal{M}} \mathcal{W} q^{-1} \mathcal{Q}(\partial\Omega, \partial\Omega) d\omega_g = \int_{\mathcal{H}} \mathcal{W} Z_\alpha J^\alpha[\Omega] d\omega_g d\tau'
\]
(175)
\[
+ \int_{\mathcal{M}} \mathcal{W} q^{-1} \left\{ \bar{g}_{\alpha\beta}(\mathcal{B}^\alpha)\mathcal{B}^\beta + \tilde{J}_{\text{(Anti}} \right\} \partial\Omega, \partial\Omega + \tilde{J}_{(Div)} \partial\Omega, \partial\Omega + \tilde{J}_{(Coeff)} \partial\Omega, \partial\Omega \right\} d\omega_g
\]
+ \int_{\mathcal{M}} q^{-1} \tilde{J} \partial\omega_g) \partial\Omega, \partial\Omega \right\} d\omega_g.

To handle the first integral \( \int_{\mathcal{H}} \mathcal{W} Z_\alpha J^\alpha[\Omega] d\omega_g d\tau' \) on RHS (175), we simply use the identity (153a) to substitute for \( \int_{\mathcal{H}} \mathcal{W} Z_\alpha J^\alpha[\Omega] d\omega_g d\tau' \). To handle the integral \( \int_{\mathcal{M}} \mathcal{W} q^{-1} \bar{g}_{\alpha\beta}(\mathcal{B}^\alpha)\mathcal{B}^\beta d\omega_g \), we note that (32a), (36d), (167a), and the identity \( \Omega^0 = 0 \) imply \( \mathcal{W} q^{-1} \bar{g}_{\alpha\beta}(\mathcal{B}^\alpha)\mathcal{B}^\beta = \mathcal{W} q^{-1} \mathcal{B}^\alpha \mathcal{B}^\beta \), which is explicitly featured on RHS (165a). To handle \( \tilde{J} \partial\omega_g) \partial\Omega, \partial\Omega \right\} d\omega_g \), we simply use (103a), (134a), and (166a) to deduce that this integral is equal to the integral \( \frac{1}{2} \int_{\mathcal{M}} \mathcal{W} q^{-1} \mathcal{B}^\alpha \mathcal{B}^\beta d\omega_g \) featured on RHS (165a). To handle \( \int_{\mathcal{M}} \mathcal{W} q^{-1} \tilde{J}_{\text{(Anti}}} \partial\Omega, \partial\Omega \right\} d\omega_g \), we consider equation (104) with \( \Omega \) in the role of \( V \). Using (33a) and (36d), we rewrite all factors of \( \partial_\alpha \Omega^\alpha \) on RHS (104) as \( -\Omega^\alpha \partial_\alpha \rho \). We place those resulting products involving a factor of \( K_\alpha K^\alpha \) on RHS (104a), and we place the remaining products on RHS (106a). In total, we have proved (165a).

The identity (155b) can be proved using nearly identical arguments where, compared to the previous paragraph, we use (153b) in the role of (153a), (32c) in the role of (32a), (134b) in the role of (134a), and the identity \( \partial_\alpha S^\alpha = \exp(2\rho)\mathcal{B} + S^\alpha \partial_\alpha \rho \) (which follows from (21b)) in the role of (33a).

9. An application: Localized a priori estimates via the integral identities

In this section, we provide a basic application of our main results: the derivation of a priori estimates for solutions that exhibit the localized gain of a derivative for the specific vorticity and entropy gradient, as we described in Subsect. 1.3.

To streamline the presentation, throughout this section, we assume that the acoustical time function \( \tau \) from the beginning of Sect. 3 is equal to the Cartesian time function \( t \).

Our main goal is to derive localized energy-flux-elliptic estimates for solutions to the compressible Euler equations, more precisely the formulation provided by Theorem 2.8. See Theorem 9.10 for a precise statement of these estimates, which rely on standard \( C^1 \)-type boundedness assumptions that we state in Subsect. 9.7. The compressible Euler formulation provided by Theorem 2.8 has an “evolution-part” and an “elliptic-part.” The main ingredients needed to control the elliptic-part are the integral identities on the spacetime region \( \mathcal{M} = \mathcal{M}_T \) provided by Theorem 8.1 and the structural features of the lateral error integrals revealed by Theorem 7.2 (which are important when the lateral boundary \( \mathcal{H} \) of \( \mathcal{M} \) is \( g \)-null). In this section, we complement these results with similar, but much simpler results for the evolution-part of the system. Compared to Theorem 8.1, the results of this section are standard, though some aspects of our analysis rely on the detailed structure of the acoustical metric \( g \) of Def. 2.2 and the geometry of \( \mathcal{M} \), which we derived in Sect. 3.

9.1. Various identities specialized to the case \( \tau = t \). In the next proposition, we provide some identities that hold when \( \tau = t \).

**Proposition 9.1** (Various identities that hold when \( \tau = t \)). Assume that the acoustical time function \( \tau \) from the beginning of Sect. 3 is equal to the Cartesian time function \( t \). Then the following identities hold (and we refer to Appendix A for a table of the notation):

\[ N = Q = B, \]
(176)
\[ \nu = q = 1, \]
(177)
Remarkable localized integral identities for 3D compressible Euler flow

$$\ddot{H} = H,$$  \hspace{1cm} \text{(178)}

$$\ell = \eta,$$  \hspace{1cm} \text{(179)}

$$\iota = 1,$$  \hspace{1cm} \text{(180)}

$$\h = 1,$$  \hspace{1cm} \text{(181)}

$$\tilde{g} = g,$$  \hspace{1cm} \text{(182)}

$$\tilde{g}^{-1} = g^{-1},$$

$$\Pi = \tilde{\Pi},$$  \hspace{1cm} \text{(182)}

$$K = 0.$$  \hspace{1cm} \text{(183)}

Moreover, when $H$ is $g$-null, the following identities hold:

$$\tilde{z} = 1,$$  \hspace{1cm} \text{(184)}

$$\Theta = 0,$$  \hspace{1cm} \text{(185)}

$$B = H + Z = L + Z,$$  \hspace{1cm} \text{(186)}

$$(g^{-1})^{\alpha\beta} = -\frac{1}{2}L^\alpha L^\beta - \frac{1}{2}L^\alpha L^\beta + (g^{-1})^{\alpha\beta},$$  \hspace{1cm} \text{(187)}

where

$$L := L + 2Z = B + Z$$  \hspace{1cm} \text{(188)}

is an outgoing $g$-null vector field that is $g$-orthogonal to $S_t$ and that verifies

$$g(L, B) = -1.$$  \hspace{1cm} \text{(189)}

Proof. To prove (176), we first note that $\tilde{\Sigma}_t$ is equal to a portion of $\Sigma_t$ since $\tau = t$. Thus, from [3], (27), and Def. 3.2, it follows that $N, Q,$ and $B$ are parallel and verify $Nt = Qt = Bt = 1$. That is, these three vector fields are equal, as desired. From this fact, (26), and Def. 3.8, we conclude (177). (182) then follows from these results and Def. 3.12.

(178)-(180) then follow as simple consequences of Def. 3.8, (48)-(50), and our assumption $\tau = t$.

(181) follows from (53), (57b), and (176).

(182) follows easily from Def. 4.1, (176), (182), and the fact that $\tilde{\Pi}N = 0$.

Next, from the proof of Lemma 3.9, (71), and (176), it follows that span $\{L, B\}$ is the $g$-orthogonal complement of $S_t$. Since $Z$ is $g$-orthogonal to $S_t$, there exist scalar functions $a_1$ and $a_2$ such that $Z = a_1L + a_2B$. Taking the $g$-inner product of this identity with respect to $B$ and using (26), (52), and Convention 3.4, we find that $a_1 = -a_2$, that is, $Z = a_1(L - B)$. Taking the $g$-inner product of this identity with respect to itself and using that $L$ is $g$-null, (47), (26), (52), and Convention 3.4, we find that $1 = a_1^2$. Also using (57a), we find that $a_1 = -1$, that is, $Z = B - L$. Considering also (71), we conclude (184), (186), and the last equality in (188).

(189) follows easily from (188), (26), (27), and the fact that $Z$ is $\Sigma_t$-tangent since $\tau = t$ (in particular, $g(B, Z) = 0$ in the present context). Considering also (176), we find that $g(L, L) = 0$, as desired.

(185) follows from the first equality in (75), (181), (184), and the first equality in (186).

The proof (187), we first note that Convention 3.3 and the results from earlier in the proof imply that $g(L, L) = g(L, L) = 0$, and $g(L, L) = g(B + Z, B - Z) = -2$, that $\{L, L\}$ spans the $g$-orthogonal complement of $S_t$, and that for any local $g$-orthogonal frame $\{e_{(1)}, e_{(2)}\}$ on $S_t$, the set $\{L, e_{(1)}, e_{(2)}\}$ spans the tangent space of $M$ at each point where it is defined. Using these facts and (82a)-(82b), the identity (187) is straightforward to verify by contracting each side of it against pairs of elements of the frame $\{L, L, e_{(1)}, e_{(2)}\}$. \hfill $\Box$
9.2. **The geometric energy method for wave equations.** To derive energy-flux identities for solutions to the covariant wave equations \((31a)-(31c)\) we will use the standard vectorfield multiplier method, which we review in this subsection. To obtain coercive energies and fluxes, we will use the “quasilinear multiplier” \(B = \partial_t + v^\alpha \partial_\alpha\), which is adapted to the solution. By \((26)\), \(B\) is always \(g\)-timelike, which is the key property that leads to coercive energies and fluxes.

9.2.1. **Energy-momentum tensor, energy current, deformation tensor, and dominant energy condition.** We start by recalling that \(D\) denotes the Levi–Civita connection of \(g\) (see Subsect. 3.3). Let \(\varphi\) be a scalar function (in practice, \(\varphi\) will be a solution to one of the wave equations \((31a)-(31c)\)). We define the energy-momentum tensor associated to \(\varphi\) to be the following symmetric type \((0, 2)\) tensorfield:

\[
Q_{\alpha\beta}[\varphi] := \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} g_{\alpha\beta} (g^{-1})^{\kappa\lambda} \partial_\kappa \varphi \partial_\lambda \varphi. \tag{190}
\]

Given \(\varphi\) and any “multiplier” vectorfield \(X\), we define the corresponding energy current \((X)J^\alpha[\varphi]\) to be the following vectorfield:

\[
(X)J^\alpha[\varphi] := Q^{\alpha\beta}[\varphi] X_\beta. \tag{191}
\]

We define the deformation tensor of \(X\) to be the following symmetric type \((0, 2)\) tensorfield:

\[
(X)\pi_{\alpha\beta} := D_\alpha X_\beta + D_\beta X_\alpha. \tag{192}
\]

The **dominant energy condition** is the following well-known result: \(Q_{\alpha\beta}[\varphi] X^\alpha Y^\beta\) is a positive definite quadratic form in \(\partial_\varphi\) when \(X\) and \(Y\) are both future-directed (see Footnote 30) and \(g\)-timelike, and \(Q_{\alpha\beta}[\varphi] X^\alpha Y^\beta\) is positive semi-definite if \(X\) and \(Y\) are both future-directed, \(X\) is \(g\)-timelike, and \(Y\) is \(g\)-null. These properties are what allow one to construct coercive energies and fluxes for wave equation solutions. For these reasons, we will be particularly interested in the case \(X := B\), which is future-directed and, by \((26)\), always \(g\)-timelike. In this case, relative to the Cartesian coordinates, we have \(B_\alpha = -\delta_\alpha^0\) (see \((27)\)), and it is straightforward to verify the following identity (where \(\Gamma^0_{\alpha\beta}\) are Christoffel symbols of \(g\) relative to the Cartesian coordinates, as in Subsect. 3.3):

\[
(B)\pi_{\alpha\beta} = 2\Gamma^0_{\alpha\beta}. \tag{193}
\]

A straightforward computation yields the following identity, which will form the starting point for our energy-flux identities for the wave equations \((31a)-(31c)\):

\[
D_\alpha (X)J^\alpha[\varphi] = (\Box_g \varphi) X_\varphi + \frac{1}{2} Q^{\alpha\lambda(\varphi)} \pi_{\alpha\lambda}. \tag{194}
\]

9.3. **Definitions of the geometric energies and fluxes and energy-flux identities.** In this subsection, we define the geometric energies and fluxes that we will use to analyze solutions to the equations of Theorem 2.8. We then derive energy-flux identities for these quantities.

9.3.1. **Definitions of the energies and fluxes.** We now define the energies and fluxes. See Lemma 9.9 for quantified statements regarding their coerciveness properties.

**Definition 9.2** (Energies and fluxes). Assume that the acoustical time function \(\tau\) from the beginning of Sect. 3 is equal to the Cartesian time function \(t\). If the lateral boundary \(\mathcal{H}\) is \(g\)-spacelike, then let \(\Sigma_t\) and \(\Sigma_t^\perp\) be the hypersurface portions defined in Subsect. 3.1 and let \(B\) and \(\mathcal{N}\) be, respectively, their future-directed (see Footnote 30) unit normals (see \((5)\), \((26)\), \((46)\), \((65)\), and \((176)\)). Similarly, if the lateral boundary \(\mathcal{H}\) is \(g\)-null, then let \(L\) be the future-directed null normal normalized by \(L_t = 1\) (see \((46)\) and Convention 5.4). Let \(\varphi\) be a scalar function, let \((B)J^\alpha[\varphi]\) be the energy current defined by \((191)\), and recall that our geometric volume and forms are defined in Def. 6.1.

For \(t \in [0, T]\), we define the following “wave” and “transport” energies along \(\Sigma_t^\perp:\nabla \|\nabla\| W_{wave} \|\varphi\| (t) := \int_{\Sigma_t} \left\{ g_{\alpha\beta}(B)J^\alpha[\varphi]B^\beta + \varphi^2 \right\} d\omega_g, \quad E_{(\text{Transport})}[\varphi](t) := \int_{\Sigma_t} \varphi^2 d\omega_g. \tag{195}\nabla \|\nabla\|
When the lateral boundary $\mathcal{H}$ is $\mathbf{g}$-spacelike, we define the following “wave” and “transport” $\mathcal{H}$-fluxes, where the scalar function $\gamma > 0$ is defined by (63):

$$
F_{(\text{wave})}[\varphi](t) := \int_{\mathcal{H}, t} \left\{ \varphi (t) \frac{\partial \mathbf{g}}{\partial \varphi} + \frac{1}{2} \varphi^2 \right\} d\omega_g, \quad F_{(\text{transport})}[\varphi](t) := \int_{\mathcal{H}, t} \varphi^2 d\omega_g. \quad (196a)
$$

Finally, when the lateral boundary $\mathcal{H} := \mathcal{N}$ is $\mathbf{g}$-null, we define the following “wave” and “transport” $\mathcal{N}$-fluxes:

$$
F_{(\text{wave})}[\varphi](t) := \int_{\mathcal{N}, t} \left\{ \varphi (t) \frac{\partial \mathbf{g}}{\partial \varphi} + \frac{1}{2} \varphi^2 \right\} d\omega_g, \quad F_{(\text{transport})}[\varphi](t) := \int_{\mathcal{N}, t} \varphi^2 d\omega_g dt'. \quad (196b)
$$

9.3.2. Energy-flux identities. We now derive energy-flux identities for the quantities from Def. 9.2

**Proposition 9.3** (Energy-flux identities). Under the assumptions stated in Def. 9.2, the following “wave” energy-flux identity holds for $t \in [0, T)$, where the volume forms are defined in Def. 6.1 and $(\mathbf{g})\pi_{\alpha\beta}$ is defined by (102):

$$
E_{(\text{wave})}[\varphi](t) + F_{(\text{wave})}[\varphi](t) = E_{(\text{wave})}[\varphi](0) - \int_{\mathcal{M}_t} (\Box_g \varphi) \mathbf{B} \varphi d\omega_g + 2 \int_{\mathcal{M}_t} (\mathbf{B} \varphi) \varphi^2 d\omega_g \quad (197)
$$

Moreover, the following “transport” energy-flux identity holds:

$$
E_{(\text{transport})}[\varphi](t) + F_{(\text{transport})}[\varphi](t) = E_{(\text{transport})}[\varphi](0) + 2 \int_{\mathcal{M}_t} (\mathbf{B} \varphi) \varphi^2 d\omega_g. \quad (198)
$$

**Proof.** We start by reminding the reader that in this section, the acoustical time function $\tau$ is equal to the Cartesian time function $t$. To prove (197), we consider the energy current $(\mathbf{g})\mathbf{J}^\alpha := (\mathbf{g})\mathbf{J}^\alpha[\varphi] - \varphi^2 \mathbf{B}^\alpha$, (where $(\mathbf{g})\mathbf{J}^\alpha[\varphi]$ is defined by (102) with $\mathbf{B}$ in the role of $\mathbf{X}$). We integrate $\mathbf{D}_\alpha(\mathbf{g})\mathbf{J}^\alpha$ with respect to $d\omega_g$ (see Def. 6.1 for the definitions of the volume and area forms) over the spacetime region $\mathcal{M}_t$ (see (40c)) with respect to $d\omega_g$ and apply the divergence theorem. The relevant unit normals to the boundary surfaces $\Sigma_0$, $\Sigma_t$, and $\mathcal{H}_t$ (for $\mathcal{N}_t$ in the null case) are stated in Def. 9.2. The relevant volume form on $\Sigma_0$ and $\Sigma_t$ is $d\omega_g$, while when $\mathcal{H}_t$ is $\mathbf{g}$-spacelike, the relevant volume form on $\mathcal{H}_t$ is $d\omega_g$ later in the proof. With the help of (29c), (52), and (67), we see that the boundary integrals that arise in the divergence theorem are precisely the wave energies and fluxes from Def. 9.2. Moreover, we re-express the “bulk term” $\int_{\mathcal{M}_t} \mathbf{D}_\alpha(\mathbf{g})\mathbf{J}^\alpha d\omega_g$ using the identity $\mathbf{D}_\alpha(\mathbf{g})\mathbf{J}^\alpha = (\Box_g \varphi) \mathbf{B} \varphi + \frac{1}{2} \mathbf{Q}^{\alpha\beta}(\mathbf{g})\pi_{\alpha\beta} - 2 (\mathbf{B} \varphi) \varphi - \frac{1}{2} \varphi^2 (\mathbf{g})\pi_{\alpha\beta}$, which follows from (102b), (104), and straightforward computations. We clarify that when $\mathcal{H}$ is $\mathbf{g}$-spacelike, $\mathbf{B}\mid_{\Sigma_0}$ points inwards to $\mathcal{M}_t$, $\mathbf{B}\mid_{\Sigma_t}$ points outwards to $\mathcal{M}_t$, and $\mathcal{N}$ points outwards to $\mathcal{M}_t$ (see 65 and Subsect. 3.3), and that due to the Lorentzian nature of $\mathbf{g}$, in the divergence theorem, the bulk integral $\int_{\mathcal{M}_t} \mathbf{D}_\alpha(\mathbf{g})\mathbf{J}^\alpha d\omega_g$ is equal to boundary integrals involving inward pointing normals. This yields (197) when $\mathcal{H}$ is $\mathbf{g}$-spacelike, and in particular explains the signs in (197). Next, we note that the identity (197) in the $\mathbf{g}$-null case can be obtained as an appropriate limit of the $\mathbf{g}$-spacelike case. Specifically, one can use the relations $\nabla^\alpha d\omega_g = \frac{3}{2} \nabla^\alpha d\omega_g d\omega$ (see 65, 150, and 179) and the fact that if we take a limit as $\mathcal{H}$ becomes $\mathbf{g}$-null (i.e., as $\mathcal{N} \rightarrow 0$), then with $\mathcal{L}$ denoting the $\mathbf{g}$-normal (i.e., the null generator) of the limiting null hypersurface, we have $\nabla \rightarrow \mathcal{L}$ (since $\nabla t = L t = 1$ by 46 and Convention 3.4) and $\frac{\nabla}{\mathcal{L}} \rightarrow 1$, where the latter limit follows from (70), the identity (177) for $\varphi$, and (181).

The identity (198) can be proved in a similar but simpler fashion by applying the divergence theorem to the vectorfield $-\varphi^2 \mathbf{B}^\alpha$ on the spacetime region $\mathcal{M}_t$; we omit the details.

9.4. $L^2$-type Controlling quantities. In this section, we combine the previously derived integral identities and use them to derive localized a priori estimates for solutions. Compared to standard results, our estimates yield a gain of one derivative for the specific vorticity and entropy (assuming that the initial data enjoy the same gain), i.e., we exhibit application I described in Subsect. 1.4. The main result of this section is
Theorem 9.10  The theorem is of particular interest in the case that the lateral boundary \( \mathcal{N} \) is \( g \)-null; as we discussed in Subsect. 1.4, the null case is important for applications to shock waves, and to handle the degeneracy of wave energies along \( \mathcal{H} = \mathcal{N}_t \) we must exploit the special structures in the lateral boundary integrals of Prop. 7.1, which we derived in Theorem 7.2. Specifically, we exploit that the integrands involve derivatives only in directions that are tangent to \( \mathcal{N} \).

We state our a priori estimates in terms of the \( L^2 \)-type controlling quantities provided by the following definition.

**Definition 9.4 (The controlling quantities).** Let \( \mathcal{M} = \mathcal{M}_T \) be a spacetime region satisfying the conditions stated in Subsects. 3.1 and 3.3 for some \( T > 0 \); see Fig. 1. In particular, assume that the lateral boundary \( \mathcal{H} = \mathcal{H}_T \) is \( g \)-spacelike or is \( g \)-null (in the null case, \( \mathcal{H} := \mathcal{N} = \mathcal{N}_T \)). Assume further that the acoustical time function \( \tau \) from the beginning of Sect. 3 is equal to the Cartesian time function \( t \). We define the following controlling quantities, where the volume forms are defined in Def. 6.1, and the quadratic form \( \mathcal{D} \) on RHS (199a) is as in Def. 4.3 and Lemma 4.4 and the energies \( E \) and fluxes \( F \) on RHS (199b) are as in Def 9.2:

\[
\mathbb{K}(t) := \int_{\mathcal{M}_t} \mathcal{D}(\theta^{\Omega}, \theta^{\Omega}) \, dv_g + \int_{\mathcal{M}_t} \mathcal{D}(\theta^S, \theta^S) \, dv_g + \int_{\mathcal{S}_t} \|\Omega\|^2_g \, dv_g + \int_{\mathcal{S}_t} \|S\|^2_g \, dv_g, \tag{199a}
\]

\[
Q(t) := \sum_{\varphi \in \{\rho, v, s\}} \mathcal{E}(\text{Wave})[\varphi](t) + \sum_{\varphi \in \{\rho, v, s\}} \mathcal{E}(\text{Transport})[\varphi](t) + \sum_{\varphi \in \{K, 2\}} \mathcal{E}(\text{Transport})[\varphi](t). \tag{199b}
\]

9.5. **Combining the integral identities.** In the next proposition, we set up the derivation of the a priori estimates by combining the integral identities of Theorem 8.1 and Prop. 9.3 and restating them in terms of the controlling quantities of Def. 9.4.

**Proposition 9.5 (Combining the integral identities).** Let \( \mathcal{M} = \mathcal{M}_T \) be a spacetime region satisfying the conditions stated in Subsects. 3.1 and 3.3 for some \( T > 0 \); see Fig. 1. Assume further that the acoustical time function \( \tau \) from the beginning of Sect. 3 is equal to the Cartesian time function \( t \) (see Footnote 53). Then for smooth solutions (see Remark 1.3) to the compressible Euler equations (1a)–(4c) on \( \mathcal{M}_T \), the controlling quantities \( \mathbb{K}(t) \) and \( Q(t) \) from Def. 9.4 verify the following identities for \( t \in [0, T] \), where the volume forms are defined in Def 6.1 and the terms \( \mathcal{D}^{(\Omega)}, \cdots, \mathcal{D}_{(2)}[S] \) on RHS (200a) are defined in (166a)–(174):

\[
\mathbb{K}(t) = \mathbb{K}(0) + \int_{\mathcal{M}_t} \left\{ \frac{1}{2} \mathcal{D}(\Omega^{(\Omega)})^2_g + |B|^{(\Omega)}_g + \mathcal{C}(\Omega) + \mathcal{D}(\Omega) + \mathcal{J}(C_{eff})(\Omega, \partial \Omega) \right\} \, dv_g \tag{200a}
\]

\[
+ \int_{\mathcal{M}_t} \left\{ \frac{1}{2} \mathcal{D}(S)^2_g + |B|^{(S)}_g + \mathcal{C}(S) + \mathcal{D}(S) + \mathcal{J}(C_{eff})(S, \partial S) \right\} \, dv_g \tag{200b}
\]

\[
+ \int_{\mathcal{M}_t} \left\{ \mathcal{D}[\Omega] + \mathcal{D}_{(1)}[\Omega] \right\} \, dv_g dt' + \int_{\mathcal{M}_t} \left\{ \mathcal{D}[S] + \mathcal{D}_{(2)}[S] \right\} \, dv_g dt',
\]

\[
Q(t) = Q(0) - \sum_{\varphi \in \{\rho, v, s\}} \int_{\mathcal{M}_t} (\mathcal{D}_g \mathcal{B} \varphi \, dv_g - \frac{1}{2} \sum_{\varphi \in \{\rho, v, s\}} \int_{\mathcal{M}_t} \mathcal{Q}^{\alpha \beta}_{\varphi} [B] \pi_{\alpha \beta} \, dv_g) + 2 \sum_{\varphi \in \{\rho, v, s\}} \int_{\mathcal{M}_t} (\mathcal{D}_g \mathcal{B} \varphi \, dv_g + \frac{1}{2} \sum_{\varphi \in \{\rho, v, s\}} \int_{\mathcal{M}_t} \varphi^{(B)} \pi_{\alpha \beta} \, dv_g). \tag{200b}
\]

We also note that (150) and (170) imply that when \( \mathcal{H}_t \) is \( g \)-spacelike, we have \( dv_g dt' = \eta^{-1} \, dv_g \), where \( \eta > 0 \) is the scalar function defined in (59).

**Proof.** (200a) follows from definition (199a), Theorem 8.1 with \( \mathcal{W} := 1 \), and the last equality in (170). (200b) follows from definition (199b) and Prop. 9.3.
9.6. Notation regarding constants and the norm $\| \cdot \|_{C(\mathbb{R})}$. In the rest of Sect. 9, $C > 0$ denotes a uniform constant that is free to vary from line to line. $C$ is allowed to depend on the set $\mathcal{R}$ of fluid variable space featured below in equation (203).

For given quantities $A, B \geq 0$, we write $A \lesssim B$ to mean that there exists a $C > 0$ such that $A \leq CB$. We write $A \approx B$ to mean that $A \lesssim B$ and $B \lesssim A$.

Moreover, if $\varphi$ is a continuous scalar-valued function and $\mathcal{R} \subset \mathcal{M}_T$, is any subset then
\[ \| \varphi \|_{C(\mathcal{R})} := \sup_{q \in \mathcal{R}} |\varphi(q)|. \] (201)

9.7. Assumptions on the solution and coerciveness of the controlling quantities. In this subsection, we state some standard $C^1$-type boundedness assumptions on the solution that we will use in our derivation of a priori estimates, i.e., in our proof of Theorem 9.10. Moreover, in Lemma 9.9 we use the assumptions to quantify the coerciveness of the energies and fluxes from Def. 9.2.

9.7.1. Assumptions on the solution. The following definition describes the subset of solution space on which the compressible Euler equations (specifically, the equations of Theorem 2.8) are hyperbolic in a non-degenerate sense.

Definition 9.6 (Regime of hyperbolicity). We define $\mathcal{H}$ as follows:
\[ \mathcal{H} := \{(p, s, v, \Omega, S) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \mid 0 < c(p, s) < \infty \}. \] (202)

We now state our assumptions on the solution.

Remark 9.7 (The pointwise norms $| \cdot |_e$ and $| \cdot |_e$). We refer to Subsubsect. 3.10.3 for the definitions of the pointwise Euclidean norms $| \cdot |_e$ and $| \cdot |_e$.

Assumptions on the solution

1. We assume that for some $T > 0$, $(p, s, v)$ is a smooth solution (see Remark 1.5) to the compressible Euler equations (4a)-(4c) (and thus $(p, s, v, \Omega, S, C, D)$ is a solution to the equations of Theorem 2.8) on a compact subset $\mathcal{M} = \mathcal{M}_T$ of spacetime satisfying the conditions stated in Subsects. 3.1 and 3.3. We also assume that the acoustical time function $\tau$ from the beginning of Sect. 3 is equal to the Cartesian time function $t$ (see Footnote 53).

2. In particular, as is stated in (37), we assume that the boundary of $\mathcal{M}_T$ is the union of a flat portion consisting of a compact subset of $\Sigma_T$ (denoted by $\bar{\Sigma}_T$), of a flat portion consisting of a compact subset of $\Sigma_0$ (denoted by $\bar{\Sigma}_0$), and of lateral boundary consisting of a $g$-spacelike or $g$-null hypersurface $\mathcal{H}_T$, where we use the alternate notation $\mathcal{H}_T = \mathcal{N}_T$ in the null case.

3. Let $\mathcal{H}$ be as in (202). We assume that there is a compact subset $\mathcal{R}$ of $\mathcal{H}$ such that
\[ (p, s, v, \Omega, S)(\mathcal{M}_T) \subset \mathcal{R}. \] (203)

Note that the assumption (203) implies a uniform $L^\infty(\mathcal{M}_T)$ bound for $|(p, s, v, \Omega, S)|_{\text{e}}$, a fact which we will silently use throughout the rest of Sect. 9.

4. Under the notation of Subsubsects. 3.10.1, 3.10.3, we assume that the constant $A = A(\mathcal{M}_T) > 0$ is such that
\[ \| (p, s, v) \|_{C(\mathcal{M}_T)} + \| \theta(p, s, v) \|_{e} \leq A. \] (204)

5. In the case that $\mathcal{H}_T$ is $g$-spacelike, let $\vec{H}$ and $Z$ be the vectorfields from Def. 3.2 and let $\tilde{z} > 0$ be the scalar function defined in (57a). Under the notation of Subsubsects. 3.10.1, 3.10.3 and 7.2.1, we assume that the constant $B = B(\mathcal{H}_T) > 0$ is such that
\[ \| (\vec{H}, Z) \|_{e} \leq C(\mathcal{H}_T) \leq B. \] (205)

6. In the case that $\mathcal{N}_T$ is $g$-null, let $Z$ be the vectorfield from Def. 3.2 and let $\mathcal{L}$ be as in Convention 3.4, Def. 3.8 and (71) (and note that Prop. 9.1 implies that $\Theta = 0$ and $\tilde{z} = 1$ in the present case).
context). Under the notation of Subsubsects. 9.10.3 and 7.2.1 we assume that the constant $B = B(\nu) > 0$ is such that

$$\|L_e\|_{C(\mathcal{M}_T)} + \|\ell \cdot (L_e \cdot \nu)\|_{C(\mathcal{M}_T)} \leq B.$$  \hfill (206)

**Remark 9.8** (Additional assumptions are needed to control the solution’s higher-order derivatives, and the sub-optimality of the assumptions). The assumptions we have stated above are sufficient for deriving a priori energy estimates for solutions to the equations of Theorem 2.8. To obtain $L^2$-type energy estimates for the solutions’ higher-order derivatives, one would need additional norm-boundedness-type assumptions on the derivatives of some of the solution variables. For example, to control the higher-order derivatives of some of the solution’s higher-order derivatives, one would need additional norm-boundedness-type assumptions on the space variable $\nu$ from Point 3 of Subsubsect. 9.7.1 and, when the lateral boundary is $\phi$-null, then the following identities hold for $t \in [0, T]$, where the implicit constants depend on the compact subset $\mathcal{R}$ of solution-variable space from Conventions 3.4 and 7.1:

$$F_{(Wave)}[\varphi](t) = \|\partial \varphi\|_{L^2(\mathcal{M}_t)}^2 + \|\varphi\|_{L^2(\mathcal{M}_t)}^2,$$

$$F_{(Transport)}[\varphi](t) = \|\varphi\|_{L^2(\mathcal{E}_t)}^2.$$  \hfill (206a)

Moreover, if the lateral boundary $\overline{\mathcal{H}}$ is $g$-spacelike, then the following estimates hold for $t \in [0, T]$, where the implicit constants depend on the compact subset $\mathcal{R}$ of solution-variable space from Conventions 3.4 and 7.1 and, when the lateral boundary $\overline{\mathcal{N}}$ is $g$-null, then the following identities hold for $t \in [0, T]$, where the $g$-null vectorfield $L_e$ on RHS (209a) is as in Convention 3.4 and (71):

$$F_{(Wave)}[\varphi](t) = \frac{1}{2} \|L_e \varphi\|_{L^2(\mathcal{M}_t)}^2 + \|\varphi\|_{L^2(\mathcal{M}_t)}^2,$$

$$F_{(Transport)}[\varphi](t) = \|\varphi\|_{L^2(\mathcal{M}_t)}^2.$$  \hfill (206a)

Finally, if the lateral boundary is either $g$-spacelike at each of its points or $g$-null at each of its points, then the following estimates hold for $t \in [0, T]$, where the implicit constants depend on the compact subset $\mathcal{R}$ of solution-variable space from Point 3 of Subsect. 9.7.1 and, when the lateral boundary is $g$-spacelike, the reciprocal of the scalar function $\mathcal{Y}$ from Convention 3.4:

$$\mathcal{E}(t) \approx \int_{\mathcal{M}_t} |\partial \varphi|_{e}^2 \, d\varphi_{e} + \int_{\mathcal{M}_t} |\partial \varphi|_{g}^2 \, d\varphi_{g} + \int_{S_t} |\partial \varphi|_{e}^2 \, d\varphi_{e} + \int_{S_t} |\partial \varphi|_{g}^2 \, d\varphi_{g}.$$  \hfill (210)
Proof. The equations (207b) and (209b) follow directly from the definitions. (208b) follows from definition (196a) and the fact that $y \approx 1$ when $\mathcal{H}$ is $g$-spacelike (see (65)).

To prove (207a), we first use [20], Def. 2.2, (26), and (190) to compute that $2Q_{\alpha\beta}B^\alpha B^\beta = (B\varphi)^2 + |\nabla\varphi|^2 = \langle \partial_\nu \varphi \rangle^2 + \langle \nu^\alpha \delta_{\alpha\beta} \varphi \rangle^2 + 2\langle \partial_\nu \varphi \rangle \langle \nu^\alpha \delta_{\alpha\beta} \varphi \rangle + c^\delta_{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi$. The assumptions of Subsect. 9.7.1 guarantee that there are constants $0 < c_1 \leq c_2$ such that the speed of sound $c = \sup_{\mathcal{M}_T} c \leq c_2$ and a constant $C' > 0$ such that $\|v\|_{L^\infty(\mathcal{M}_T)} \leq C'$. Hence, by the Cauchy–Schwarz and Young’s inequalities, for any $\varepsilon > 0$, we have $2\|\partial_\nu \varphi \| \leq \frac{1}{1+\varepsilon} \|\partial_\nu \varphi \|^2 + \langle \nu^\alpha \delta_{\alpha\beta} \varphi \rangle^2 + \varepsilon(c')^2 |\nabla\varphi|^2$, where $|\nabla\varphi|^2 = c^\delta_{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi$ and $\delta_{\alpha\beta}$ is the Kronecker delta. Choosing $\varepsilon$ such that $\varepsilon(c')^2 = \frac{1}{2}$, we conclude that $2Q_{\alpha\beta}B^\alpha B^\beta \geq \frac{1}{1+\varepsilon} \|\partial_\nu \varphi \|^2 + \frac{1}{2} c^\delta_{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \geq |\varphi|^2$. A similar argument yields that $Q_{\alpha\beta}B^\alpha B^\beta \leq |\varphi|^2$. Thus, taking the $\mathcal{H}$-inner product of both sides of the decomposition with respect to $\mathcal{B}$ and using (26), we find that $2Q_{\alpha\beta}B^\alpha B^\beta = (B\varphi)^2 + |\nabla\varphi|^2$. Using these facts, (225), and (190), we compute that $Q_{\alpha\beta}B^\alpha N^\beta = \frac{1}{2} \sqrt{1 + \beta^2} (B\varphi)^2 + \beta(B\varphi)P\varphi + \frac{1}{2} \sqrt{1 + \beta^2} |\nabla\varphi|^2$. Next, using the Cauchy–Schwarz and Young’s inequalities, we bound the magnitude of the cross term as follows: $|\beta(B\varphi)\tilde{P}\varphi| \leq \frac{\beta^1}{2} (B\varphi)^2 + \frac{\beta^1}{2} |\nabla\varphi|^2$. It follows that $Q_{\alpha\beta}B^\alpha N^\beta \geq \frac{1}{2} \sqrt{1 + \beta^2} |\beta| |\partial_\nu \varphi|^2 + \frac{1}{2} \sqrt{1 + \beta^2} |\beta| |\nabla\varphi|^2$. Moreover, since $\tilde{N}$ is $g$-timelike by assumption, it follows that there exists a constant $C_1 > 0$ such that $\sup_{\mathcal{H}_T} |\beta| \leq C_1$. It follows that on $\mathcal{H}_T$, the two factors of $(\sqrt{1 + \beta^2} - |\beta|)$ are uniformly bounded from above and below by positive constants depending on $C_1$. Using this fact and the bounds on $c$ noted in the previous paragraph, we conclude that $Q_{\alpha\beta}B^\alpha N^\beta \geq |\partial_\nu \varphi|^2$ as desired. A similar but simpler argument yields that $Q_{\alpha\beta}B^\alpha N^\beta \leq |\varphi|^2$. In view of definitions (191) and (196a), we conclude (208a).

We now prove (209a). Recall that $\tilde{L}$ is alternate notation for $\tilde{N}$ in the null case and using (52), (187), and (188) (which implies that $\mathcal{B} = \frac{1}{2}(\tilde{L} + L)$), we compute that $2Q_{\alpha\beta}B^\alpha L^\beta = (\tilde{L}\varphi)^2 + |\nabla\varphi|^2$. In view of definitions (191) and (196b), we conclude (209a).

Finally, we prove (210). From definition (199a), (100), (182), (183), the fact that $g_{ab} = c^{-2} \delta_{ab}$ (see (22a) and (76a)), and the bounds on $c$ noted two paragraphs above, we find that

$$ K(t) \approx \int_{\mathcal{M}_t} \left\{ |\partial\Omega|^2_{\mathcal{E}} + \sum_{\alpha=1}^3 (B\Omega)^2 + |\partial S|^2_{\mathcal{E}} + \sum_{\alpha=1}^3 (B S)^2 \right\} d\omega_g + \int_{\mathcal{S}_t} \left\{ |\Omega|^2_g + |\Theta|^2_{\mathcal{E}} \right\} d\omega_\nu. \tag{211} $$

Using arguments similar to the ones we used in proving (207a) (based on Cauchy–Schwarz and Young’s inequality), we find that $\partial_\nu \varphi \approx |\partial_\nu \varphi|^2 + \sum_{\alpha=1}^3 (B\Omega)^2$ and $|\partial S|^2_{\mathcal{E}} \approx |\partial S|^2_{\mathcal{E}} + \sum_{\alpha=1}^3 (B S)^2$. From these estimates, (57a), and (211), the desired result (210) readily follows. \hfill \Box

9.8. Localized a priori estimates. We now prove the main result of Sect. 9, namely Theorem 9.10, which yields a priori estimates exhibiting the gain of regularity for the specific vorticity and the entropy gradient (as is manifested by (213a)), as we described in the introduction (see in particular Point I of Subsect. 9.4).

We again stress that when the lateral boundary is $g$-null (i.e., $\mathcal{H} = \tilde{N}$), the theorem crucially relies on the precise structures shown in Theorem 7.2 and Theorem 8.1. In particular, in the $g$-null case, these structures are needed to control the error integrals $\int_{\mathcal{L}} \cdots$ on Riemann's (165a)-(165b) (recall that $\tilde{N}_t = \mathcal{H}_T$ in the present context), since (209a) shows that the wave fluxes on $\tilde{N}_t$ control only $\tilde{N}_t$-tangential derivatives in the $g$-null case.

Theorem 9.10 (Localized a priori estimates exhibiting the gain in regularity for $\Omega$ and $S$). Let $\mathcal{M} = \mathcal{M}_T$ be a spacetime region satisfying the conditions stated in Subsects. 3.1 and 3.4 for some $T > 0$; see Fig. 1. In particular, assume that the lateral boundary $\mathcal{H} = \mathcal{H}_T$ is $g$-spacelike or is $g$-null (in the null case,
\( \mathcal{H} := \mathcal{N} = \mathcal{N}_T \). Assume that the acoustical time function \( \tau \) from the beginning of Sect. [3] is equal to \( \tau^0 \) the Cartesian time function \( t \). Consider a smooth solution (see Remark [7.5]) to the compressible Euler equations (4a)-(4c) on \( \mathcal{M}_T \) that satisfies the assumptions stated in Subsubsect. [9.7.1]. Let \( \mathfrak{R} \) be the set from Point 3 of Subsubsect. [9.7.1] let \( \mathcal{B} \) be the assumed bound on the \( C^1 \) norm of \( (\rho, s, v) \) on \( \mathcal{M}_T \) stated in \( \mathfrak{R} \), and let \( \mathcal{B} \) be the assumed bound on \( (\mathcal{H}, Z) \) and some of their \( \mathcal{H}_T \)-tangential first derivatives stated in \( \mathfrak{R} \) (see [206]) for the assumed \( C^1 \) norm bound in the case of a lateral null hypersurface). Then there exists a constant \( C = C(\mathfrak{R}, A, B) \) (which we allow to vary from line to line) such that the controlling quantities \( \mathcal{K}(t) \) and \( \mathcal{Q}(t) \) from Def. [9.4] verify the following inequalities for \( t \in [0, T] \):

\[
\mathcal{K}(t) \leq 2\mathcal{K}(0) + C \int_0^t \mathcal{Q}(t') \, dt' + C\mathcal{Q}(t), \quad (212a)
\]

\[
\mathcal{Q}(t) \leq \mathcal{Q}(0) + \mathcal{K}(0) + C \int_0^t \mathcal{Q}(t') \, dt'. \quad (212b)
\]

Moreover, the following inequalities hold for \( 0 \leq t \leq T \):

\[
\mathcal{K}(t) \leq C \{ \mathcal{Q}(0) + \mathcal{K}(0) \} \exp(Ct), \quad (213a)
\]

\[
\mathcal{Q}(t) \leq \{ \mathcal{Q}(0) + \mathcal{K}(0) \} \exp(Ct). \quad (213b)
\]

**Proof.** First, we use the wave equations (31a)-(31c) to algebraically substitute for the terms \( \Box g \phi \) on the first line of RHS (200b), and we use the transport equations (32a), (32c), (33b), and (34a) to algebraically substitute for the terms \( \mathcal{B} \phi \) on the second line RHS (200b). We then use (32a) and (32c) to algebraically substitute for all factors of \( \mathcal{B} \) and \( \mathcal{B} \) in all of the resulting expressions. After these substitutions, we use the volume form identities of Lemma [6.2] the assumptions stated in Subsubsect. [9.7.1], Theorem [7.2], and the coerciveness estimates of Lemma [9.9] to bound all integrand factors on RHS (200b) by \( C \) times a quadratic term that is controlled by the controlling quantities of Def. [9.4]. Note that we are using the fact that the right-hand side of the identity (193) for the deformation tensor components \( (\mathcal{B}_\alpha)_\beta \) which also appear on RHS (200b) can be expressed as smooth functions of \( (\rho, v^1, v^2, v^3, s) \) times a factor that is linear in \( \rho \). Also using the Cauchy–Schwarz inequality for integrals, we deduce that

\[
\mathcal{Q}(t) \leq \mathcal{Q}(0) + C \int_0^t \mathcal{Q}(t') \, dt' + C \sqrt{\int_0^t \mathcal{Q}(t') \, dt'} \sqrt{\mathcal{K}(t)}. \quad (214)
\]

We clarify that the last product on RHS (214) comes from the integral

\[
2 \sum_{\phi \in \{\mathcal{C}, \mathcal{D}\}_{i=1,2,3}} \int_{\mathcal{M}_t} (\mathcal{B} \phi) \phi \, d\omega_\mathcal{G}
\]

on RHS (200b) in the cases \( \phi \in \{\mathcal{C}, \mathcal{D}\}_{i=1,2,3} \), specifically from the terms on RSHs (33b) and (34a) that depend on the terms \( \partial \mathcal{O} \) and \( \partial S \); by Cauchy–Schwarz, the corresponding integrals are bounded by

\[
C \sqrt{\int_0^t \mathcal{Q}(t') \, dt'} \sqrt{\int_{\mathcal{M}_t} \{ |\partial \mathcal{O}|^2 + |\partial S|^2 \} \, d\omega_\mathcal{K}},
\]

which in turn is bounded by the last product on RHS (214) as desired.

Similarly, using (200a) (without further need to use the equations of Theorem [2.8]), and exploiting the fact (highlighted in Remark [8.2]) that (159)-(160) show that RHS (172b) (with \( \Omega \) and \( S \) in the role of \( V \)) does not involve any \( \mathcal{H} \)-transversal derivatives of \( (\rho, v, s) \) or \( (\mathcal{H}, \Theta, Z) \) and that the same statement holds for RHS (172a) (this remark is trivial since \( \mathcal{H} := 1 \) in the present context) and RSHs (173a)-(174), we find that

\[
\mathcal{K}(t) \leq \mathcal{K}(0) + C \int_0^t \mathcal{Q}(t') \, dt' + C \sqrt{\int_0^t \mathcal{Q}(t') \, dt'} \sqrt{\mathcal{K}(t)} + C\mathcal{Q}(t). \quad (215)
\]

We stress that when the lateral hypersurface is \( g \)-null (i.e., \( \mathcal{H} = \mathcal{N} \), the coerciveness result (200a) shows that \( \mathcal{Q}(t) \) controls \( L^2(\mathcal{N}^*_t) \) the derivatives of \( (\rho, v, s) \) only in the \( \mathcal{N}^*_t \)-tangential directions; this is the reason

\[\footnote{As we mentioned at the beginning of Sect. [9] we make this assumption only to shorten the presentation; the results of Theorem [9.10] generalize in a straightforward fashion to the case of general smooth acoustical time functions.}\]
Remarkable localized integral identities for $3D$ compressible Euler flow

that the absence of the $N$-transversal derivatives of $(p,v,s)$ on RHS (172b)-(174) is critically important in the $g$-null case.

(212a) then follows from (215) and Young’s inequality.

(212b) then follows from (212a), (214), and Young’s inequality.

(213b) follows from (212b) and Gronwall’s inequality. (213a) follows from (212a) and (213b).

10. Remarkable Hodge-transport-based integral identities relative to double-null foliations

In this section, we extend the integral identities of Theorem 8.1 so that they apply to spacetime regions that are double-null foliated, that is, foliated by a pair $u,u$ of acoustical eikonal functions. We provide the main integral identities in Theorem 10.6. We highlight that an analog of Theorem 7.2 also holds in the present context, that is, that the error terms in Theorem 10.6 along $g$-null hypersurfaces involve only tangential derivatives; see Remark 10.7. Before proving the theorem, we set up the double-null foliation and provide analogs of results from the previous sections, modified so as to apply in the present context.

![Figure 2. A spacetime region $M$ that can be covered by a double-null foliation](image)

10.1. Setup of the double-null foliations. We will derive integral identities for compressible Euler solutions on spacetime regions $M$ that are foliated by the level sets of a pair $u,u$ of acoustic eikonal functions, which we assume to be given solutions of the acoustical eikonal equation:

$$\begin{align*}
(g^{-1})^{\alpha\beta} \partial_{\alpha} u \partial_{\beta} u &= 0,
(g^{-1})^{\alpha\beta} \partial_{\alpha} \overline{u} \partial_{\beta} \overline{u} &= 0.
\end{align*}$$

(216)

As before, in (216), $g$ is the acoustical metric of Def.2.2. We let $H_u$ and $H_{\overline{u}}$ respectively denote the level sets of $u$ and $\overline{u}$. We assume that there are constants $U$ and $\overline{U}$ satisfying $1 + U < U$ and $0 < \overline{U}$ such that $u$ and $\overline{u}$ are smooth with non-vanishing, transversal, past-directed gradients on a spacetime region $M$ corresponding to $1 \leq u \leq U$ and $0 \leq \overline{u} \leq \overline{U}$. Then by (216), $H_u$ and $H_{\overline{u}}$ are three-dimensional $g$-null hypersurfaces on $M$ that intersect transversally in two-dimensional $g$-spacelike submanifolds

$$S_{u,\overline{u}} := H_u \cap H_{\overline{u}}. $$

(217)

We assume that all of the $S_{u,\overline{u}}$ are diffeomorphic to $S^2$.

---

54Throughout, we abuse notation by using the symbols “$u$” and “$\overline{u}$” to denote the acoustical eikonal functions and the values that they take on; the precise meaning of the symbols will be clear from context.

55Equivalently, we assume that $B_u > 0$ and $B_{\overline{u}} > 0$.

56The vectorfield $(g^{-1})^{\alpha\beta} \partial_{\beta} u$ is $g$-null and $g$-orthogonal to the level sets of $u$, while the vectorfield $(g^{-1})^{\alpha\beta} \partial_{\beta} \overline{u}$ is $g$-null and $g$-orthogonal to the level sets of $\overline{u}$.
Remark 10.1 (On the width of the regions). Our assumptions on $U$ and $\overline{U}$ imply that the $u$-width of $\mathcal{M}$ is larger than its $\overline{u}$-width. However, this is only for convenience of exposition; our results could readily be generalized to handle the case that the $u$-width of $\mathcal{M}$ is less than or equal to the $\overline{u}$-width.

We define

$$\tau := u + \overline{u}. \quad (218)$$

From [216] and our assumption that the gradients of $u$ and $\overline{u}$ are past-directed, it follows that

$$(g^{-1})^{\alpha\beta} \partial_\alpha \tau \partial_\beta \tau < 0. \quad (219)$$

In particular, the $g$-normal to the level sets of $\tau$ are $g$-timelike, and thus these level sets are $g$-spacelike. That is, $\tau$ is an acoustical time function on the region under study. For $(u', \overline{u}') \in [1, U] \times [0, \overline{U}]$ and $\tau' \in [1, U + \overline{U}]$, we define

$$\tilde{\Sigma}_{\tau'} := \mathcal{M}_{U, \overline{U}} \cap \{\tau = \tau'\}, \quad (220a)$$

$$\mathcal{H}_{u'}(0, u') := \mathcal{H}_u \cap \{0 \leq u \leq u'\}, \quad (220b)$$

$$\tilde{\mathcal{H}}_{\tau'}(1, u') := \tilde{\mathcal{H}}_u \cap \{1 \leq u \leq u'\}, \quad (220c)$$

$$\mathcal{M}_{u', \tau'} := \{1 \leq u \leq u'\} \cap \{0 \leq \tau \leq \tau'\} = \cup_{u'' \in [1, u']} \mathcal{H}_{u''}(0, u') = \cup_{u'' \in [0, \overline{u}]} \mathcal{H}_{u''}(1, u') \quad (220d)$$

$$= \cup_{(u'', \overline{u}'') \in [1, u'] \times [0, \overline{u}]} S_{u'', \overline{u}'''}. \quad (220e)$$

Note that $\mathcal{M} = \mathcal{M}_{U, \overline{U}}$ and that on $\mathcal{M}_{U, \overline{U}}$, we have

$$1 \leq \tau \leq U + \overline{U}. \quad (221)$$

Note also that $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_{U + \overline{U}}$ are degenerate in the sense that they are not three-dimensional submanifolds with boundary, but rather are two-dimensional submanifolds: $\tilde{\Sigma}_1 = \mathcal{S}_0$, $\tilde{\Sigma}_{U + \overline{U}} = \mathcal{S}_{U, \overline{U}}$. Moreover, we note that for $\tau' \in (1, U + \overline{U})$, the boundary of $\tilde{\Sigma}_{\tau'}$ (in the sense of a manifold-with-boundary), which we denote by $\partial \tilde{\Sigma}_{\tau'}$, satisfies:

$$\partial \tilde{\Sigma}_{\tau'} = S_{\tau', 0} \cup \mathcal{S}_{1, \tau' - 1}, \quad \tau' \in (1, U + \overline{U}], \quad (222a)$$

$$\partial \tilde{\Sigma}_{\tau'} = S_{\tau', 0} \cup \mathcal{S}_{1 - U, \tau' - \overline{U} - U}, \quad \tau' \in [1 + U, U], \quad (222b)$$

$$\partial \tilde{\Sigma}_{\tau'} = S_{\tau' - U - U, \tau' - U - \overline{U} - U}, \quad \tau' \in [U, U + \overline{U}). \quad (222c)$$

In each of the disjoint unions on RHSs [222a]-[222c], we refer to the first set as the “inner boundary” of $\tilde{\Sigma}_{\tau'}$ and the second set as the “outer boundary” of $\tilde{\Sigma}_{\tau'}$; see Fig. 2.

For the purpose of deriving the integral identities, we assume that the fluid solution is smooth on $\mathcal{M}_{U, \overline{U}}$. Moreover, for the purpose of interpreting the integral identities, we imagine that the “state of the fluid solution” is prescribed on $\mathcal{H}_1(0, U)$ and $\mathcal{H}_0(1, U)$ (i.e., we view these as null hypersurfaces where “initial data” are posed); as we mentioned in Subsect. 1.4, a full treatment of the characteristic initial value problem will be the subject of a future work.

10.2. Geometric quantities adapted to the double-null foliation.

10.2.1. $g$-null vectorfields and related scalar functions. Associated to $u$ and $\overline{u}$, we define the geodesic vectorfields $\overset{\alpha}{L}(Geo)$ and $\overset{\alpha}{L}(Geo)$ by

$$L_{\alpha(Geo)} := -(g^{-1})^{\alpha\beta} \partial_\beta u, \quad \overset{\alpha}{L}_{(Geo)} := -(g^{-1})^{\alpha\beta} \partial_\beta \overline{u}. \quad (223)$$

From [223], it follows that $L_{(Geo)}$ is $g$-orthogonal to $\mathcal{H}_u$, while $\overset{\alpha}{L}_{(Geo)}$ is $g$-orthogonal to $\overset{\alpha}{H}_\overline{u}$. The equations in [216] imply that $L_{(Geo)}$ and $\overset{\alpha}{L}_{(Geo)}$ are $g$-null:

$$g(L_{(Geo)}, L_{(Geo)}) = g(\overset{\alpha}{L}_{(Geo)}, \overset{\alpha}{L}_{(Geo)}) = 0. \quad (224)$$

[57]More precisely, using [216], it is straightforward to show that $D_{L_{(Geo)}} L_{(Geo)} = D_{\overset{\alpha}{L}_{(Geo)}} \overset{\alpha}{L}_{(Geo)} = 0$, where $D$ is the Levi–Civita connection of $g$. 


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Next, we define the following scalar functions on $\mathcal{M}$:

\[
\begin{align*}
\mu &:= -\frac{1}{(g^{-1})^{\alpha\beta}\partial_\alpha t\partial_\beta u} = \frac{1}{\mathcal{L}^0_{(Geo)}}, \\
\tilde{\mu} &:= -\frac{1}{(g^{-1})^{\alpha\beta}\partial_\alpha t\partial_\beta u} = \frac{1}{\mathcal{L}^0_{(Geo)}}, \\
\lambda &:= -\frac{1}{(g^{-1})^{\alpha\beta}\partial_\alpha u\partial_\beta u} = -\frac{1}{g(L_{(Geo)}; L_{(Geo)})}, \\
\iota &:= \frac{\lambda}{\mu}, \quad \xi := \frac{\lambda}{\mu}.
\end{align*}
\]

The assumptions of Subsect. 10.1 imply that $\mu > 0$, $\tilde{\mu} > 0$, $\lambda > 0$, $\iota > 0$, $\xi > 0$. \hfill (225)

Next, we define the following vector fields, which are rescaled versions of $L_{(Geo)}$ and $\mathcal{L}^0_{(Geo)}$:

\[
\begin{align*}
L^\alpha &:= -\mu(g^{-1})^{\alpha\beta}\partial_\beta u, \\
\tilde{L}^\alpha &:= -\mu(g^{-1})^{\alpha\beta}\partial_\beta u, \\
L^\alpha &:= -\lambda(g^{-1})^{\alpha\beta}\partial_\beta u, \\
\tilde{L}^\alpha &:= -\lambda(g^{-1})^{\alpha\beta}\partial_\beta u.
\end{align*}
\]

The following identities easily follow from the above definitions:

\[
\begin{align*}
g(L, L) &= g(L, \tilde{L}) = g(\tilde{L}, \tilde{L}) = g(\tilde{L}, \tilde{L}) = 0, \\
g(L, L) &= -\frac{\mu\mu}{\lambda} = -\frac{\lambda}{\mu}, \\
g(\tilde{L}, \tilde{L}) &= -\lambda, \\
L u &= \tilde{L} u = 0, \\
L \tilde{u} &= \frac{1}{\iota}, \tilde{L} u &= \frac{1}{\xi}, \\
L \tilde{u} &= \tilde{L} u = 1, \\
\tilde{L} \tau &= \tilde{L} \tau = 1.
\end{align*}
\]

\[
\begin{align*}
\tilde{L} &= \iota L, \\
\tilde{L} &= \iota L.
\end{align*}
\]

Note that by (27), the last two equalities in (230a) are equivalent to

\[
g(L, B) = g(L, B) = -1.
\]

From [46], Convention 3.4 and the last equality in (230a), it follows that the vector field denoted by \"$L$\" in this section has the same properties as the vector field denoted by the same symbol in Sects. 3-9.

10.2.2. Additional geometric vector fields and scalar functions. In this subsubsection, we define some additional vector fields and scalar functions that play a role in the ensuing analysis.

**Definition 10.2** (The vector field $Z$). We define $Z$ to be the following vector field:

\[
Z := \frac{\tilde{L} - \tilde{L}}{\sqrt{2\lambda}} = \frac{\iota L - \iota \tilde{L}}{\sqrt{2\lambda}},
\]

where the second equality follows from (231).

From (218), (233) and (230b), it follows that $Z \tau = 0$, that is, that $Z$ is $\Sigma_{\tau}$-tangent. Since $\tilde{L}$ and $\tilde{L}$ are $g$-orthogonal to $S_{u,u}$, it follows from (233) that $Z$ is also $g$-orthogonal to $S_{u,u}$. We also note that span{\{$\tilde{L}, \tilde{L}$\} is equal to the $g$-orthogonal complement of $S_{u,u}$. Moreover, from the first equality in (233), (228), and (229b), we compute that

\[
g(Z, Z) = 1.
\]
Remark 10.3 (The orientation of $Z$ and the relevance for the divergence theorem). From the above discussion, it follows that the vectorfield denoted by “$Z$” in (233) has the same properties as the vectorfield denoted by the same symbol in Sects. 3.9 (see Def. 3.2). We highlight that $Z$ points outwards to $\tilde{\Sigma}_{\tau}$ at its outer boundary while $\tilde{Z}$ points inwards to $\Sigma_{\tau}$ at its inner boundary; see just below (222a)-(222c) for the definitions of the inner and outer boundaries of $\Sigma_{\tau}$. The precise orientation of $Z$ will be important for the sign of various terms when we apply the divergence theorem on $\Sigma_{\tau}$; see Fig. 2.

Next, we note that straightforward calculations imply that the vectorfields $N$ and $Q$ from Def. 3.2 can be expressed as follows in the present context of double-null foliations:

\[ N = \frac{\tilde{L} + L}{t + \tilde{t}} = \frac{tL + t\tilde{L}}{t + \tilde{t}}, \quad Q = \frac{\tilde{L} + L}{2} = \frac{(t + \tilde{t})N}{2}. \]  
(235)

Moreover, straightforward calculations based on (228) yield that $N$ and $Q$ are $g$-orthogonal to the vectorfield $Z$ defined in (233):

\[ g(N, Z) = g(Q, Z) = 0. \]  
(236)

In addition, using (228), (229b), and (235), we compute that the scalar function $q := \sqrt{-g(Q, Q)} > 0$ defined in (62) can be expressed as follows:

\[ q = \sqrt{\frac{\lambda}{2}}. \]  
(237)

Next, we define the scalar functions $z$, $h$, and $\tilde{h}$ as follows:

\[ z := -g(Z, L) = -\frac{1}{\sqrt{2\lambda}}g(\tilde{L}, L) = -\frac{1}{\sqrt{2\lambda}}g(\tilde{L}, L) = \frac{t}{\sqrt{2}} \frac{\lambda}{2} = \frac{\mu}{\sqrt{2\lambda}} > 0, \]  
(238a)

\[ z := g(Z, L) = -\frac{1}{\sqrt{2\lambda}}g(\tilde{L}, L) = -\frac{1}{\sqrt{2\lambda}}g(\tilde{L}, L) = \frac{t}{\sqrt{2}} \frac{\lambda}{2} = \frac{\mu}{\sqrt{2\lambda}} > 0, \]  
(238b)

\[ h := -g(L, N) = -\frac{1}{\sqrt{2\lambda}}g(\tilde{L}, L) = -\frac{1}{\sqrt{2\lambda}}g(\tilde{L}, L) = \frac{\lambda}{\sqrt{2}} \frac{\mu}{\sqrt{2\lambda}} > 0, \]  
(238c)

\[ h := -g(L, N) = -\frac{1}{\sqrt{2\lambda}}g(\tilde{L}, L) = -\frac{1}{\sqrt{2\lambda}}g(\tilde{L}, L) = \frac{\lambda}{\sqrt{2}} \frac{\mu}{\sqrt{2\lambda}} > 0, \]  
(238d)

where the further equalities in (238a)-(238d) follow from straightforward computations.

Remark 10.4 (On the signs of $z$, $h$, and $\tilde{h}$). We have chosen the signs in (238a)-(238d) so that $z$, $h$, and $\tilde{h}$ are positive. We note that the functions “$z$” and “$h$” defined in (238a) and (238c) have the same properties as the scalar functions denoted by the same symbols in Sects. 3.9 (see (57a)-(57b)).

10.2.3. First fundamental forms and projections. Let $g$, $g^{-1}$, and $\Pi$ be the tensorfields defined by the following equations, where we consider $g$, $\lambda$, $\tilde{L}$, $L$ to have already been defined by (22a), (225c), and (227b), and $\delta_{\alpha\beta}^\gamma$ is the Kronecker delta:

\[ g_{\alpha\beta} = -\frac{1}{\lambda} L_{\alpha} L_{\beta} - \frac{1}{\lambda} L_{\alpha} \tilde{L}_{\beta} + \delta_{\alpha\beta}, \]  
(239a)

\[ (g^{-1})^{\alpha\beta} = -\frac{1}{\lambda} L^\alpha L^\beta - \frac{1}{\lambda} L^\alpha \tilde{L}^\beta + (g^{-1})^{\alpha\beta}, \]  
(239b)

\[ \Pi^\alpha_{\beta} = \delta^\alpha_{\beta} + \frac{1}{\lambda} L^\alpha L_{\beta} + \frac{1}{\lambda} L^\alpha \tilde{L}_{\beta}. \]  
(239c)

Next, we recall that $\text{span}\{\tilde{L}, L\}$ is equal to the $g$-orthogonal complement of $S_{u,u}$. With the help of (229b), it is straightforward to check that $g$ is the first fundamental form of $S_{u,u}$, that $g^{-1}$ is the inverse first fundamental form of $S_{u,u}$, and that $\Pi$ $g$-orthogonal projection onto $S_{u,u}$ in the sense that these three tensorfields have the properties described in Lemma 3.13 (where “$\text{span}\{N, Z\}$” in (82b) and (83b) is equal to “$\text{span}\{\tilde{L}, L\}$”)

In particular, when restricted to $S_{u,u}$, $g$ is the Riemannian metric induced by $g$. 

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10.2.4. **Geometric decompositions of various vectorfields.** We start by defining the following two vectorfields:

\[ E := B - \frac{1}{z} Z, \quad E := B + \frac{1}{z} Z. \quad (240) \]

Note that the vectorfield \( E \) defined in (240) has the same properties as the vectorfield denoted by the same symbol in Sects.3-9 (see (73)).

Next, we define the vectorfields \( \Theta \) and \( \tilde{\Theta} \) by demanding that the following identities hold, where \( h, h, E, \) and \( E \) are defined by (238c), (238d), and (240):

\[ E = \frac{1}{h} L + \Theta, \quad E = \frac{1}{h} L + \tilde{\Theta}. \quad (241) \]

From arguments nearly identical to those used in the proof of Lemma 3.11, we find that \( \Theta \) and \( \tilde{\Theta} \) are both \( S_{u,u} \)-tangent and that the following identities hold:

\[ B = \frac{1}{h} L + \frac{1}{z} Z + \Theta = E + \frac{1}{z} Z, \quad (242a) \]
\[ B = \frac{1}{h} L - \frac{1}{z} Z + \tilde{\Theta} = E - \frac{1}{z} Z. \quad (242b) \]

Note that the vectorfield \( \Theta \) appearing in (241) and (242a) has the same properties as the vectorfield denoted by the same symbol in Sects.3-9 (see (72)).

Next, we define the following two vectorfields:

\[ P := \frac{L - N}{h}, \quad \tilde{P} := \frac{L - N}{h}. \quad (243) \]

Note that the vectorfield \( P \) defined in (243) has the same properties as the vectorfield denoted by the same symbol in Sects.3-9 (see (123)). Moreover, arguments nearly identical to those used in the proof of Lemma 5.3 yield that \( P \) and \( \tilde{P} \) are both \( \Sigma_{\tau} \)-tangent.

Finally, we note that arguments nearly identical to those used in the proof of Lemma 5.4 yield that there exist \( S_{u,u} \)-tangent vectorfields \( W_{(\alpha)} \) and \( \tilde{W}_{(\alpha)} \) such that for \( \alpha = 0, 1, 2, 3 \), the following identities hold:

\[ \partial_{\alpha} = -L_{\alpha} B + P_{\alpha} L + W_{(\alpha)}, \quad (244a) \]
\[ \partial_{\alpha} = -L_{\alpha} B + \tilde{P}_{\alpha} L + \tilde{W}_{(\alpha)}. \quad (244b) \]

Note that the vectorfield \( W_{(\alpha)} \) in (244a) has the same properties as the vectorfield denoted by the same symbol in Sects.3-9 (see (124)).

10.2.5. **Tensorfields with the same definitions as in Sects.3-9.** In the rest of the paper, our convention is that if we refer to a tensorfield that was not explicitly defined or constructed in Subsubsects.10.2.1-10.2.4, then it has the same definition that it had in Sects.3-9 in terms of the tensorfields from Subsubsects.10.2.1-10.2.4. We refer to Appendix A as an aid for quickly referencing the relevant definitions. As an example, we note that the scalar function \( \nu \) is defined by (61), where it is understood that the vectorfield \( N \) on RHS (61) is as in equation (235). Then the vectorfield \( N \) is understood to be the one defined in (64), where the scalar function \( \nu \) on RHS (64) is as above and the vectorfield \( N \) on RHS (64) is as in equation (235). Similarly, the \( \Sigma_{\tau} \) projection tensorfield \( \tilde{\Pi} \) is defined by (78b), where the vectorfield \( \tilde{N} \) on RHS (78b) is as above. As a final example, we note that the vectorfield \( K \) is defined in (94), where the scalar function \( \nu \) on RHS (94) is as above, and the vectorfield \( \tilde{K} \) on RHS (94) is as above, and the vectorfield \( B \) on RHS (94) is the material derivative vectorfield defined in (5).

10.2.6. **Volume and area forms.** In this subsection, we discuss the volume and area forms that play a role in our ensuing analysis.

- As in Sects.3-9 \( d\varpi_{\delta} \) denotes the canonical volume form on \( M_{u,u} \) induced by \( g \), \( d\varpi_{\bar{g}} \) denotes the canonical volume form on \( \bar{\Sigma}_{\tau} \) induced by \( \bar{g} \), and \( d\varpi_{\hat{g}} \) denotes the canonical area form on \( S_{u,u} \) induced by \( \hat{g} \).
- We endow \( H_{u}(u_{1}, u_{2}) \) with the volume form \( d\varpi_{\hat{g}} du' \), where for \( u' \in [u_{1}, u_{2}] \), \( d\varpi_{\hat{g}} \) is the area form on \( S_{u,u'} \).
Similarly, we endow $\mathcal{H}_\alpha(u_1, u_2)$ with the volume form $d\omega_g du'$, where for $u' \in [u_1, u_2]$, $d\omega_g$ is the area form on $S_{u', u}$.

By (218), on $\mathcal{H}_u(u_1, u_2)$, since $u$ is fixed, we have $d\omega_g du' = d\omega_g d\tau'$, where on the RHS, $d\omega_g$ is the area form on $S_{u', \tau - u}$ and $\tau' = u + u'$. Similarly, on $\mathcal{H}_u(u_1, u_2)$, we have $d\omega_g du' = d\omega_g d\tau'$, where on the RHS, $d\omega_g$ is the area form on $S_{\tau' = u - u}$ and $\tau' = u' + u$. We also note that the identity (149) remains valid in the present context, where in the present context, $q > 0$ verifies (237).

10.2.7. Integral identities involving $\mathcal{H}_u$, $\mathcal{H}_u^\perp$, and $S_{u, \underline{u}}$. In this subsection, we provide an analog of Lemma 6.3 for our double-null foliations.

**Lemma 10.5** (Integral identities involving $\mathcal{H}_u$, $\mathcal{H}_u^\perp$, and $S_{u, \underline{u}}$). Let $f$ be a smooth function defined on $\mathcal{M}_{U, \underline{U}}$. Let $u$, $u_1$, $u_2$, $\underline{u}$, $\underline{u}_1$, and $\underline{u}_2$ be real numbers satisfying $1 \leq u \leq U$, $1 \leq u_1 \leq u_2 \leq U$, $0 \leq \underline{u}_1 \leq \underline{u}_2 \leq \underline{U}$, and $0 \leq u_1 \leq u_2 \leq U$. Let $\bar{L}$ be the $\mathcal{H}_u$-tangent vectorfield defined in (227b), and let $\bar{L}$ be the $\mathcal{H}_u^\perp$-tangent vectorfield defined in (227b). Let $\bar{g}$ be the first fundamental form of $S_{u, \underline{u}}$ (see Subsubsection 10.2.5). Then the following identities hold, where $\mathcal{L}_X$ denote Lie differentiation with respect to the vectorfield $X$, the definition of the area form $d\omega_g$ is provided in Subsubsection 10.2.6, and we refer to Subsubsection 10.2.5 regarding the notation:

\[
\begin{align*}
\int_{\mathcal{H}_\alpha(u_1, u_2)} (\bar{L} f) d\omega_g du' &= -\frac{1}{2} \int_{\mathcal{H}_\alpha(u_1, u_2)} \left( [\bar{g}^{-1}]^{\alpha\beta} \mathcal{L}_\alpha \bar{g}_{\alpha\beta} \right) d\omega_g du' + \int_{S_{u_2, \underline{u}}} f d\omega_g - \int_{S_{u_1, \underline{u}}} f d\omega_g, \quad (245a) \\
\int_{\mathcal{H}_\alpha(u_1, u_2)} (\bar{L} f) d\omega_g du' &= -\frac{1}{2} \int_{\mathcal{H}_\alpha(u_1, u_2)} \left( [\bar{g}^{-1}]^{\alpha\beta} \mathcal{L}_\alpha \bar{g}_{\alpha\beta} \right) d\omega_g du' + \int_{S_{u_2, \underline{u}}} f d\omega_g - \int_{S_{u_1, \underline{u}}} f d\omega_g. \quad (245b)
\end{align*}
\]

Proof: In view of the normalization conditions in (230b), we see that the two identities in the lemma can be derived by a straightforward modification of the proof of Lemma 6.3. \hfill \square

10.3. The main theorem for double-null foliations: Remarkable Hodge-transport integral identities for $\Omega$ and $S$. We now state our main theorem, which is an analog of Theorem 8.1 for double-null foliated regions. The proof is located in Subsection 10.5. See Remark 10.7 regarding the structure of the error terms on the g-null hypersurfaces.

**Theorem 10.6** (Double-null foliations: Remarkable Hodge-transport integral identities for $\Omega$ and $S$). Let $1 < U$ and $0 < \underline{U}$, and let $\mathcal{M}_{U, \underline{U}}$ be a spacetime region that is double-null foliated by acoustical eikonal functions $u$ and $\underline{u}$, as is described in Subsection 10.1 (in particular, on $\mathcal{M}_{U, \underline{U}}$, we have $1 \leq u \leq U$ and $0 \leq \underline{u} \leq \underline{U}$). Let $\mathcal{Q}(\partial X, \partial X)$ be the quadratic form defined by (99), and recall that the positive definite nature of $\mathcal{Q}$ was revealed in Subsection 4.4. Let $\mathcal{H}$ be an arbitrary scalar function. Let $q > 0$ be the scalar function defined in (92) (recall also the identity (237)), let $\iota > 0$ and $\ell > 0$ be the scalar functions defined in (225d) (see also (226)), and let $\bar{z} > 0$, $\bar{h} > 0$, $z > 0$, and $h > 0$ be the scalar functions from (238a)–(238d). For smooth solutions (see Remark 1.6 to the compressible Euler equations (4a)–(4c)), the following integral identities hold, where the definitions of the volume and area forms are provided in Subsection 10.2.6, and
we refer to Subsubsect. 10.2.5 regarding the notation:

\[
\int_{\mathcal{U}} \mathcal{W}^{-1} \partial (\partial \Omega, \partial \Omega) \, d\varpi_g + \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |\Omega|^2 \, d\varpi_g
\]

\[
= \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |\Omega|^2 \, d\varpi_g + \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |\Omega|^2 \, d\varpi_g - \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |\Omega|^2 \, d\varpi_g
\]

\[
+ \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |\Omega|^2 \, d\varpi_g - \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |\Omega|^2 \, d\varpi_g
\]

\[
\int_{\mathcal{U}} \mathcal{W}^{-1} \partial (\partial S, \partial S) \, d\varpi_g + \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |S|^2 \, d\varpi_g
\]

\[
= \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |S|^2 \, d\varpi_g + \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |S|^2 \, d\varpi_g - \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |S|^2 \, d\varpi_g
\]

\[
+ \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |S|^2 \, d\varpi_g - \int_{\mathcal{U}} \mathcal{W} \left\{ \frac{z}{h_t} + \frac{\xi}{h_t} \right\} |S|^2 \, d\varpi_g
\]

On RHs (246a)-(246b), \( \Pi^{(l)} \) and \( \Pi^{(s)} \) are two-forms with the Cartesian components

\[
\Pi^{(l)}_{ij} := 2(\partial_3 \ln c) \Omega_{ij} - 2(\partial_3 \ln c) \Omega_{ij} + 2\delta^{0}_{ij} \Omega_\alpha \partial_\beta v^\alpha - 2\delta^{0}_{ij} \Omega_\alpha \partial_\alpha v^\alpha
\]

\[
- e^{-4} \exp(-2\rho) \frac{\rho_s}{\rho} \varepsilon_{\alpha\beta\gamma} (B^\gamma) S^\delta
\]

\[
+ e^{-4} \exp(-2\rho) \frac{\rho_s}{\rho} \varepsilon_{\alpha\beta\gamma} B^\gamma [S^\delta (\partial_\gamma v^\alpha) - S^\alpha \partial_\delta v^\gamma]
\]

\[
+ e^{-2} \exp(\rho) \varepsilon_{\alpha\beta\gamma} B^\gamma C^\delta,
\]

\[
\Pi^{(s)}_{ij} := 2(\partial_3 \ln c) S_{ij} - 2(\partial_3 \ln c) S_{ij},
\]
\( \mathfrak{B}_1^{(\Omega)} \) and \( \mathfrak{B}_1^{(S)} \) are \( \Sigma_t \)-tangent vectorfields with the Cartesian spatial components
\[
\mathfrak{B}_1^{(\Omega)} := \Omega^a \partial_a v^i - \exp(-2p) c^{-2} \frac{p}{\theta} \epsilon_{ab}(\mathfrak{B}_1^{\alpha}) S^b, \\
\mathfrak{B}_1^{(S)} := -S^a \partial_a v^i + \epsilon_{ab} \exp(p) \Omega^a S^b, 
\]
(248a)

\( \mathfrak{C}^{(\Omega)} \) and \( \mathfrak{C}^{(S)} \) are scalar functions defined relative to the Cartesian coordinates by
\[
\mathfrak{C}^{(\Omega)} := -2(K_a \Omega^a) \Omega^b \partial_b \rho - 2\nu(K_a \Omega^a) K_b \mathfrak{B}_2^{(\Omega)}, \\
\mathfrak{C}^{(S)} := 2(K_a \Omega^a) \{ \exp(2p) \mathcal{D} + S^b \partial_b \rho \} - 2\nu(K_a \Omega^a) K_b \mathfrak{B}_2^{(S)}, 
\]
(249a)

\( \mathfrak{D}^{(\Omega)} \) and \( \mathfrak{D}^{(S)} \) are scalar functions defined relative to the Cartesian coordinates by
\[
\mathfrak{D}^{(\Omega)} := (\Omega^a \partial_a \rho)^2 + \nu(K_a \mathfrak{B}_2^{(\Omega)})^2 + 2\nu(\Omega^a \partial_a \rho) K_b \mathfrak{B}_2^{(\Omega)}, \\
\mathfrak{D}^{(S)} := \{ \exp(2p) \mathcal{D} + S^a \partial_a \rho \}^2 - 2\nu \{ \exp(2p) \mathcal{D} + S^a \partial_a \rho \} K_b \mathfrak{B}_2^{(S)}, 
\]
(250a)

for \( V \in \{ \Omega, S \} \), the scalar function \( \mathcal{J}_{(\text{coef})}/V, \mathbf{\partial}V \) is defined relative to the Cartesian coordinates by
\[
\mathcal{J}_{(\text{coef})}/V, \mathbf{\partial}V := V^\alpha \mathbf{g}_{\beta \gamma} (\partial_\gamma \mathbf{N}^\beta) \mathbf{N}^\gamma - V_\alpha (\partial_\gamma \mathbf{N}^\gamma) \mathbf{N}^\beta 
\]
(251)

for \( V \in \{ \Omega, S \} \), the scalar function \( \mathcal{J}_{(\text{eff})}/V, \mathbf{\partial}V \) is defined relative to the Cartesian coordinates by
\[
\mathcal{J}_{(\text{eff})}/V, \mathbf{\partial}V := -J[V, \mathbf{\partial}V] = V^\alpha (\partial_\gamma \mathbf{\omega}_\beta) \partial_\lambda V^\lambda - V_\alpha (\partial_\gamma \mathbf{\omega}_\beta) \partial_\lambda V^\lambda, 
\]
(252)

for \( V \in \{ \Omega, S \} \), the scalar functions \( \mathfrak{H}^{(\mathbf{\omega})}/V, \mathbf{\partial}V, \mathfrak{H}^{(\mathbf{\omega})}/V, \mathbf{\partial}V, \mathfrak{H}^{(\mathbf{\omega})}/V, \mathbf{\partial}V, \), and \( \mathfrak{H}^{(\mathbf{\omega})}/V, \mathbf{\partial}V, \) are defined relative to the Cartesian coordinates by
\[
\mathfrak{H}^{(\mathbf{\omega})}/V, \mathbf{\partial}V := \frac{\bar{z}}{l_{ht}} |V|_g^2 \mathcal{L}_g \mathbf{\partial}_\beta + \frac{\bar{z}}{l_{ht}} |V|_g^2 \mathbf{\partial}_\gamma \mathbf{\partial}_\beta V^\gamma - V_\alpha Z^\alpha V^\beta, 
\]
(253a)

\[
\mathfrak{H}^{(\mathbf{\omega})}/V, \mathbf{\partial}V := \frac{1}{2} \frac{\bar{z}}{l_{ht}} |V|_g^2 (\mathbf{g}^{-1})^{\alpha \beta} \mathcal{L}_g \mathbf{g}^{\alpha \beta} 
\]
(253b)

\[
\mathfrak{H}^{(\mathbf{\omega})}/V, \mathbf{\partial}V := \frac{\bar{z}}{l_{ht}} |V|_g^2 \tilde{L} \mathbf{\partial}_\beta + \frac{\bar{z}}{l_{ht}} |V|_g^2 \mathbf{\partial}_\gamma \mathbf{\partial}_\beta V^\gamma - V_\alpha Z^\alpha V^\beta, 
\]
(254a)

\[
\mathfrak{H}^{(\mathbf{\omega})}/V, \mathbf{\partial}V := \frac{1}{2} \frac{\bar{z}}{l_{ht}} |V|_g^2 (\mathbf{g}^{-1})^{\alpha \beta} \mathcal{L}_g \mathbf{g}^{\alpha \beta} 
\]
(254b)
the scalar functions $\mathcal{S}_1[\Omega]$ and $\mathcal{S}_1[\Omega]$ are defined relative to the Cartesian coordinates by

\begin{align}
\mathcal{S}_1[\Omega] &:= 4z\Omega_{\alpha}\epsilon_{\alpha\beta} E^\alpha \Omega ln c - 4z|\Omega|^2 E^\alpha \Omega + 4zE^{(\alpha} \Omega_{\beta)} - 4z\Omega^0 \Omega_{\alpha} E^{\alpha} + 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} S^\alpha L_a \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma L^\beta L \rho \\
&+ 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} E_a E^\alpha \Omega^\beta P^\gamma S^\beta L^\rho - 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} S^\alpha L_a \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma L^\beta L \rho \\
&- 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} S^\alpha L_a \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma L^\beta L \rho - 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma S^\alpha W(a) v^\delta \\
&+ 2z c^{-4} \exp(-2\rho) \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma C^\delta \\
&- 2z c^{-4} \exp(-3\rho) \left\{ \frac{P_{\beta}}{\rho} \right\}^2 S^\alpha L_a \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma S^\delta,
\end{align}

\begin{align}
\mathcal{S}_1[\Omega] &:= 4z\Omega_{\alpha}\epsilon_{\alpha\beta} E^\alpha \Omega ln c - 4z|\Omega|^2 E^\alpha \Omega + 4zE^{(\alpha} \Omega_{\beta)} - 4z\Omega^0 \Omega_{\alpha} E^{\alpha} + 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} S^\alpha L_a \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma L^\beta L \rho \\
&+ 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} E_a E^\alpha \Omega^\beta P^\gamma S^\beta L^\rho - 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} S^\alpha L_a \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma L^\beta L \rho \\
&- 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} S^\alpha L_a \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma L^\beta L \rho - 2z c^{-4} \exp(-2\rho) \frac{P_{\beta}}{\rho} \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma S^\alpha W(a) v^\delta \\
&+ 2z c^{-4} \exp(-2\rho) \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma C^\delta \\
&- 2z c^{-4} \exp(-3\rho) \left\{ \frac{P_{\beta}}{\rho} \right\}^2 S^\alpha L_a \epsilon_{\alpha\beta} E^\alpha \Omega^\beta B^\gamma S^\delta,
\end{align}

and the scalar functions $\mathcal{S}_2[S]$ and $\mathcal{S}_1[S]$ are defined relative to the Cartesian coordinates by

\begin{align}
\mathcal{S}_2[S] &:= 4zS_a E^\alpha \epsilon_{\alpha\beta} E^\beta S^\alpha L c - 4z|S|^2 E^\alpha \Omega ln c, \\
\mathcal{S}_1[S] &:= 4zS_a E^\alpha \epsilon_{\alpha\beta} E^\beta S^\alpha L c.
\end{align}

**Remark 10.7** (Extension of Theorem 7.2 to double-null foliations). An analog of Theorem 7.2 also holds in the present context of double-null foliations. More precisely, the error integrands on RHSs (246a)-(246b) along the ingoing $g$-null hypersurfaces $\mathcal{H}_<(1, U)$ and $\mathcal{H}_<(1, U)$, enjoy the properties revealed by Theorem 7.2 that is, they involve only tangential derivatives. Analogous results hold for the error integrands on RHSs (246a)-(246b) along the outgoing $g$-null hypersurfaces $\mathcal{H}_<(U, 0)$ and $\mathcal{H}_<(U, 0)$. These properties could be shown by using essentially the same arguments that we used in the proof of Theorem 7.2 thus, for brevity, we do not provide details.

### 10.4. Preliminary identities for the proof of Theorem 10.6

In this subsection, to facilitate the proof of Theorem 10.6 we derive preliminary integral identities for the main null hypersurface error integrals that arise in its proof. The main result is Prop. 10.12 which is a direct analog of Prop. 7.1.

#### 10.4.1. Preliminary analysis of the boundary integrands

We start with the following lemma, which is a direct analog of Lemma 4.8. As before, the purpose of the lemma is to reveal preliminary good structures in the boundary terms that will appear when we apply the divergence theorem on $\Sigma_\tau$ via the divergence identity (102). We highlight that in the present context of double-null foliations, the boundary of $\Sigma_\tau$ has two connected components: an inner sphere and an outer sphere; see Remark 10.3. Moreover, to prove Theorem 10.6 along each of the two spheres, we need to identify good geo-analytic structures adapted to $\mathcal{H}_a$ and good geo-analytic structures adapted to $\mathcal{H}_a$. This explains why Lemma 10.8 features four identities, while Lemma 4.8 features only one.

**Lemma 10.8** (Double-null foliations: Preliminary analysis of the boundary integrands). Let $V$ be a $\Sigma_t$-tangent vectorfield defined on $\mathcal{M}$, and let $E$, $E$, $\Theta$, and $\hat{\Theta}$ be the vectorfields from (240a)-(241). Let $\mathcal{W}$ be an arbitrary scalar function. Under the assumptions of Theorem 10.7 the following identities hold, where we refer to Subsubsection 10.2.3 regarding the notation.
Identities in the "dynamic region". On $S_{u,U}$, for $u \in [1,U]$, we have:  
\[ \mathcal{W} Z_{\alpha} \mathcal{J}^\alpha[V] = -\hat{L} \left\{ \mathcal{W} \frac{\hat{z}}{ht} |V|^2 g \right\} - \delta x \left\{ \mathcal{W} \frac{\hat{x}}{ht} |V|^2 \Theta \right\} - \delta x \left\{ \mathcal{W} Z_{\alpha} \mathcal{V}^\alpha \mathcal{V} \right\} \] (257)  
\[ + 2 \mathcal{W} \frac{E^\alpha V^\beta (\partial_{\alpha} V_{\beta} - \partial_{\beta} V_{\alpha}) + \hat{S}_{\hat{\alpha}}(\mathcal{W})[V] + \mathcal{W} \left\{ \hat{S}_{\hat{\alpha}}[V] - \frac{1}{2} \frac{\hat{z}}{ht} |V|^2 (g^{-1})^{\alpha\beta} \mathcal{L}_L \hat{g}_{\alpha\beta} \right\} , \]  
where $\hat{S}_{\hat{\alpha}}(\mathcal{W})[V]$ and $\hat{S}_{\hat{\alpha}}[V]$ are defined in (253a) - (253b).

Moreover, on $S_{u,0}$, for $u \in [0,U]$, we have:  
\[ -\mathcal{W} Z_{\alpha} \mathcal{J}^\alpha[V] = -\hat{L} \left\{ \mathcal{W} \frac{\hat{z}}{ht} |V|^2 g \right\} - \delta x \left\{ \mathcal{W} \frac{\hat{x}}{ht} |V|^2 \Theta \right\} + \delta x \left\{ \mathcal{W} Z_{\alpha} \mathcal{V}^\alpha \mathcal{V} \right\} \] (258)  
\[ + 2 \mathcal{W} \frac{E^\alpha V^\beta (\partial_{\alpha} V_{\beta} - \partial_{\beta} V_{\alpha}) + \hat{S}_{\hat{\alpha}}(\mathcal{W})[V] + \mathcal{W} \left\{ \hat{S}_{\hat{\alpha}}[V] - \frac{1}{2} \frac{\hat{z}}{ht} |V|^2 (g^{-1})^{\alpha\beta} \mathcal{L}_L \hat{g}_{\alpha\beta} \right\} , \]  
where $\hat{S}_{\hat{\alpha}}(\mathcal{W})[V]$ and $\hat{S}_{\hat{\alpha}}[V]$ are defined in (254a) - (254b).

Identities where the data are specified. On $S_{u,0}$, for $u \in [1,U]$, we have:  
\[ -\mathcal{W} Z_{\alpha} \mathcal{J}^\alpha[V] = \hat{L} \left\{ \mathcal{W} \frac{\hat{z}}{ht} |V|^2 g \right\} + \delta x \left\{ \mathcal{W} \frac{\hat{x}}{ht} |V|^2 \Theta \right\} \] (259)  
\[ - 2 \mathcal{W} \frac{E^\alpha V^\beta (\partial_{\alpha} V_{\beta} - \partial_{\beta} V_{\alpha}) + \hat{S}_{\hat{\alpha}}(\mathcal{W})[V] + \mathcal{W} \left\{ \hat{S}_{\hat{\alpha}}[V] - \frac{1}{2} \frac{\hat{z}}{ht} |V|^2 (g^{-1})^{\alpha\beta} \mathcal{L}_L \hat{g}_{\alpha\beta} \right\} . \]  
Finally, on $S_{1,0}$, for $u \in [0,U]$, we have:  
\[ -\mathcal{W} Z_{\alpha} \mathcal{J}^\alpha[V] = \hat{L} \left\{ \mathcal{W} \frac{\hat{z}}{ht} |V|^2 g \right\} + \delta x \left\{ \mathcal{W} \frac{\hat{x}}{ht} |V|^2 \Theta \right\} - \delta x \left\{ \mathcal{W} Z_{\alpha} \mathcal{V}^\alpha \mathcal{V} \right\} \] (260)  
\[ - 2 \mathcal{W} \frac{E^\alpha V^\beta (\partial_{\alpha} V_{\beta} - \partial_{\beta} V_{\alpha}) + \hat{S}_{\hat{\alpha}}(\mathcal{W})[V] + \mathcal{W} \left\{ \hat{S}_{\hat{\alpha}}[V] - \frac{1}{2} \frac{\hat{z}}{ht} |V|^2 (g^{-1})^{\alpha\beta} \mathcal{L}_L \hat{g}_{\alpha\beta} \right\} . \]

Remark 10.9 (The signs in Lemma 10.8). Our sign choices in Lemma 10.8 are such the left-hand sides of the identities correspond to outward pointing normals. More precisely, the vectorfield $Z$ is outward pointing to $\Sigma_t$ along $S_{u,U}$ and $S_{1,0}$, while its negation $-Z$ is outward pointing to $\Sigma_t$ along $S_{u,U}$ and $S_{u,0}$; see Fig. 4.

Remark 10.10 (A cancellation in the proof of Prop 10.12). In the proof of Prop 10.12, the product $-\frac{1}{2} \mathcal{W} \frac{E^\alpha V^\beta (g^{-1})^{\alpha\beta} \mathcal{L}_L \hat{g}_{\alpha\beta}}{ht}$ found in the last term on RHS (257) will be canceled by a term that arises from an application of Lemma 10.5. This cancellation is the reason that we have expressed the terms in the last braces on RHS (257) in their stated form. Similar remarks apply to the Lie derivative-involving products in the last braces on RHSs (257)-(260).

Proof of Lemma 10.8. The proof of (257) mirrors the proof of (115), where we use (242a) in place of the identity (75) used in the proof of (115). The proof of (258) is identical.

Similarly, proof of (259) mirrors the proof of (115), where we use (242b) in place of the identity (75) used in the proof of (115). The proof of (260) is identical.

\[ \square \]

10.4.2. Geometric decompositions and remarkable cancellations for the most subtle terms on RHS (134a). The next proposition is an analog of the combination of Lemma 5.8 and Prop 5.11. For the same reasons described in Subsects. 5.3 and 5.4, the purpose of the proposition is to reveal remarkable geo-analytic cancellations in the terms $2 \mathcal{W} \frac{E^\alpha V^\beta (\partial_{\alpha} \Omega_{\beta} - \partial_{\beta} \Omega_{\alpha})}{}$ and $2 \mathcal{W} \frac{E^\alpha V^\beta (\partial_{\alpha} \Omega_{\beta} - \partial_{\beta} \Omega_{\alpha})}{}$ on RHSs (257)-(260) in the case $V = \Omega$.

Proposition 10.11 (Double-null foliations: Geometric decompositions and remarkable cancellations for the most subtle terms on RHS (134a)). Let $E$ be the $H_u$-tangent vectorfield from (240), let $E$ be the $H_u$-tangent vectorfield from (240), let $P$ and $\hat{P}$ be the $\Sigma_t$-tangent vectorfields from (243), let $\{W_{(\alpha)}\}_{\alpha=0,1,2,3}$ be the $S_{u,\hat{u}}$-tangent vectorfields from (244a), and let $\{W_{(\alpha)}\}_{\alpha=0,1,2,3}$ be the $S_{u,\hat{u}}$-tangent vectorfield from (244b).
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For smooth solutions (see Remark 1.5) to the compressible Euler equations (4a)-(4c) on $\mathcal{M}$, the following identities hold, where we refer to Subsubsect. 10.2.5 regarding the notation:

\[
\epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta \left\{ -(Bv)^\gamma S^\delta + B^\gamma \left[ S^\delta (\partial_\alpha v^\alpha) - S^\alpha \partial_\alpha v^\delta \right] \right\} = \epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta P^\gamma S^\delta \frac{\partial}{\partial P} - S^\alpha L_a \epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta \gamma^\delta \frac{\partial}{\partial P} + c^2 \epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta (\bar{W}_{(a)} \partial^\delta) S^\beta - c^2 S^\alpha L_a \epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta \gamma^\delta \frac{\partial}{\partial \bar{W}_{(a)}} \partial^\delta \frac{\partial}{\partial \bar{W}_{(a)}},
\]

\[
\epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta \left\{ -(Bv)^\gamma S^\delta + B^\gamma \left[ S^\delta (\partial_\alpha v^\alpha) - S^\alpha \partial_\alpha v^\delta \right] \right\} = \epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta P^\gamma S^\delta \frac{\partial}{\partial P} - S^\alpha L_a \epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta \gamma^\delta \frac{\partial}{\partial P} + c^2 \epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta (\bar{W}_{(a)} \partial^\delta) S^\beta - c^2 S^\alpha L_a \epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta \gamma^\delta \frac{\partial}{\partial \bar{W}_{(a)}} \partial^\delta \frac{\partial}{\partial \bar{W}_{(a)}},
\]

Proof. To prove (261a), we first note that the proof of (136) goes through verbatim with $L$ in the role of both $N$ and $H$ (see Convention 3.4 and 71), where we use the identity (244a) in the role that we used (124) in the proof of (136). Next, we consider the first term on RHS (136) (again with $L$ in the role of $N$), namely $-\epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta L^\gamma (S^\delta + S^\delta L_a \partial_\alpha \partial^\alpha B)$. This term completely vanishes because the proof of the identity (145) (again with $L$ in the role of $N$) goes through in the present context. In total, we have proved (261a).

Similarly, to prove (261b), we use the identity (244b) in the role that (124) was used in the proof of (136), thus concluding that the identity (136) holds in the present context, but with $L$ in the role of both $N$ and $H$, $E$ in the role of $E$, and $W_{(a)}$ in the role of $W_{(a)}$. Moreover, the analog of the first term on RHS (136), namely $-\epsilon_{\alpha\beta\gamma\delta}E^\alpha\Omega^\beta L^\gamma (S^\delta + S^\delta L_a \partial_\alpha \partial^\alpha B)$, again completely vanishes because the identity (145) (now with $E$ in the role of $E$ and $L$ in the role of $N$) holds in the present context. In total, we have proved (261b), which completes the proof of the proposition. \hfill \Box

We now prove the main result of this subsection.

Proposition 10.12 (Double-null foliations: Key identity for the boundary error integrals). Let $u$, $u_1$, $u_2$, $\underline{u}$, $\underline{u}_1$, and $\underline{u}_2$ be real numbers satisfying $\underline{u} \in \{1, U\}$, $1 \leq u_1 \leq u_2 \leq U$, $\underline{u} \in \{0, U\}$, and $0 \leq \underline{u}_1 \leq \underline{u}_2 \leq \underline{U}$. Under the assumptions of Theorem 10.6 for smooth solutions (see Remark 1.5) to the compressible Euler equations (4a)-(4c), the following integral identities hold, where we refer to Subsubsect. 10.2.5 regarding the

\footnote{The four identities stated in the proposition in fact hold for $1 \leq u \leq U$ and $0 \leq \underline{u} \leq \underline{U}$. We have stated the proposition only for the “endpoint values” $u \in \{1, U\}$ and $\underline{u} \in \{0, U\}$ to help the reader navigate the proof of Theorem 10.6 (see Subsect. 10.5); the endpoint values are the only ones that we use in our proof of the theorem.}
notation:
\[ \int_{\mathcal{H}_u(u_1,u_2)} \mathcal{W}_n \omega_j \, d\omega_j \, du' = -\int_{S_{u_1}} \mathcal{W}_n \frac{z}{h_l} |\Omega_j|^2 \, d\omega_j + \int_{S_{u_2}} \mathcal{W}_n \frac{z}{h_l} |\Omega_j|^2 \, d\omega_j \]  
(262a)

\[ + \int_{\mathcal{H}_u(u_1,u_2)} \left\{ \mathcal{D}_t \mathcal{W}_n \right\}[\Omega] + \mathcal{W}_n \mathcal{D}_t[\Omega] + \mathcal{W}_n \mathcal{D}_t[1] \right\} \, d\omega_j \, du', \]  
(262b)

\[ \int_{\mathcal{H}_u(u_1,u_2)} \mathcal{W}_n \omega_j \, d\omega_j \, du' = -\int_{S_{u_1}} \mathcal{W}_n \frac{z}{h_l} |\Omega_j|^2 \, d\omega_j - \int_{S_{u_2}} \mathcal{W}_n \frac{z}{h_l} |\Omega_j|^2 \, d\omega_j \]  
(262c)

\[ + \int_{\mathcal{H}_u(u_1,u_2)} \left\{ \mathcal{D}_t \mathcal{W}_n \right\}[S] + \mathcal{W}_n \mathcal{D}_t[S] + \mathcal{W}_n \mathcal{D}_t[2] \right\} \, d\omega_j \, du', \]  
(262d)

\[ \int_{\mathcal{H}_u(u_1,u_2)} \mathcal{W}_n \omega_j \, d\omega_j \, du' = -\int_{S_{u_1}} \mathcal{W}_n \frac{z}{h_l} |\Omega_j|^2 \, d\omega_j - \int_{S_{u_2}} \mathcal{W}_n \frac{z}{h_l} |\Omega_j|^2 \, d\omega_j \]  
(262e)

\[ + \int_{\mathcal{H}_u(u_1,u_2)} \left\{ \mathcal{D}_t \mathcal{W}_n \right\}[S] + \mathcal{W}_n \mathcal{D}_t[S] + \mathcal{W}_n \mathcal{D}_t[2] \right\} \, d\omega_j \, du'. \]  
(262f)

On RHSs (262a)-(262d), for \( V \in \{\Omega, S\} \), the scalar functions \( \mathcal{D}_t \mathcal{W}_n[V], \mathcal{D}_t[\mathcal{W}_n[V] \), and \( \mathcal{D}_t[V] \) are defined in (253a)-(253b), and the scalar functions \( \mathcal{D}_t[1], \mathcal{D}_t[2] \) are defined in (255a)-(255b).

Proof. Prop. 10.12 follows from integrating the identities provided by Lemma 10.8 in the same way that Prop. 7.1 followed from integrating the identity provided by Lemma 4.8 where we use the identities (245a)-(245c) in place of the identity (152) used in the proof of Prop. 7.1. For clarity, we will provide a few additional details. When one proves (262a), the argument described above, based on (245a) as well as (257), in the case \( u = U \) and (259) in the case \( u = 0 \), where \( \Omega \) is in the role of \( V \) in both (257) and (259), leads to an integral identity analogous to (157), which in particular features the following error integral on the right-hand side: \( 2 \int_{\mathcal{H}_u(u_1,u_2)} \mathcal{W}_n \mathcal{E}_n \mathcal{V}_n \, (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) \, d\omega_j \, du' \). To handle this integral, we use the identity (134a) to substitute for the integrand factor \( \partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha \) in this integral. The resulting identity features an integral that is analogous to (158), namely

\[ 2 \int_{\mathcal{H}_u(u_1,u_2)} \mathcal{W}_n \mathcal{E}_n \mathcal{V}_n \, (\partial_\alpha \Omega_\beta - \partial_\beta \Omega_\alpha) \, d\omega_j \, du' \]  
(263)

which is generated by the fifth and sixth factors on RHS (134a). We then rewrite the integrand factors

\[ \epsilon_{\alpha\beta\gamma\delta} \mathcal{E}_n \mathcal{V}_n \mathcal{W}_n \mathcal{D}_t[1] \right\} \]  

from (263) by using the remarkable identity (261a) for substitution. In total, these steps yield (262a). The identity (261b) can be proved using nearly identical arguments, where we use the identity (134b) in place of the identity (134a) used in the previous paragraph.

The identity (261b) can be proved using arguments similar to the ones used to prove (262a), where we use (245b) in the role of (245a), (258) in the role of (257), and (260) in the role of (259).

Finally, (261c) can be proved using arguments nearly identical to the ones needed to prove (262a), where we use (134b) in the place of the identity (134a) that is needed for the proof of (262b).

10.5. Proof of Theorem 10.6. In this subsection, we use the previously derived results to prove Theorem 10.6.

We first prove (240a). For \( u'' \) small and positive, we set \( \mathcal{M}_1(u'') := \mathcal{M}_U \cap \{1 + u'' \leq \tau \leq 1 + u'\} \). We set \( \mathcal{M}_2 := \mathcal{M}_{U,U} \cap \{1 + U \leq \tau \leq U\} \). For \( u''' \) close to but less than \( U \), we set \( \mathcal{M}_3(u''') := \mathcal{M}_{U,U} \cap \{U \leq \tau \leq u''' + U\} \).
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Note that \( M_{U,U} = M_1(0) \cup M_2 \cup M_3(U) \); see Fig. 3, which can be viewed as a partitioned, “spherically symmetric” caricature of Fig. 2.

![Figure 3. The spacetime regions in the proof of Theorem 10.6](image)

The main step in proving (246a) is deriving the following three integral identities (where \( u'' \) and \( u'' \) are fixed in the identities):

\[
\int_{M_1(u'')} \mathcal{W}^{-1} \mathcal{Q}(\partial \Omega, \partial \Omega) \, d\mathcal{W} \tag{264}
\]

\[
= \int_{S_{1,U}} \mathcal{W}^{-1} \frac{\mathcal{Z}}{\mathcal{H}} |\Omega|^2 \, d\mathcal{W} - \int_{S_{1,u''}} \mathcal{W}^{-1} \frac{\mathcal{Z}}{\mathcal{H}} |\Omega|^2 \, d\mathcal{W} + \int_{S_{1+U^{'},0}} \mathcal{W}^{-1} \frac{\mathcal{Z}}{\mathcal{H}} |\Omega|^2 \, d\mathcal{W}
\]

\[
+ \int_{M_1(u'')} \mathcal{W}^{-1} \left\{ \frac{1}{2} |\mathcal{V}(\Omega)|^2 \, d\mathcal{W} + |\mathcal{B}(\Omega)|^2 \, d\mathcal{W} + \mathcal{S}(\Omega) + \mathcal{J}(\text{Coeff})[\Omega, \partial \partial \partial \Omega] \right\} \, d\mathcal{W}
\]

\[
+ \int_{M_1(u'')} q^{-1} \mathcal{J}(\mathcal{W})[\Omega, \partial \partial \partial \Omega] \, d\mathcal{W}
\]

\[
- \int_{H_0(1+u''\cup 1+U)} \left\{ \mathcal{S}(\Omega) + \mathcal{W}(\Omega) + \mathcal{W}(\text{Coeff})[\Omega] \right\} \, d\mathcal{W} \, du'
\]

\[
- \int_{H_1(u'' \cup U)} \left\{ \mathcal{S}(\Omega) + \mathcal{W}(\Omega) + \mathcal{W}(\text{Coeff})[\Omega] \right\} \, d\mathcal{W} \, du',
\]
\[
\int_{\mathcal{M}_2} \mathcal{W} q^{-1} \mathcal{D}(\partial \Omega, \partial \Omega) \, d\mathcal{W}_g
\]  
\[= \int_{S_{1,0}} \mathcal{W} \frac{z}{\mathcal{H}} |\Omega|_g^2 \, d\mathcal{W}_g - \int S_{1+U,0} \mathcal{W} \frac{z}{\mathcal{H}} |\Omega|_g^2 \, d\mathcal{W}_g
\]
\[+ \int S_{U-\mathcal{U}} \mathcal{W} \frac{z}{\mathcal{H}} |\Omega|_g^2 \, d\mathcal{W}_g + \int S_{U-\mathcal{U}} \mathcal{W} \frac{z}{\mathcal{H}} |\Omega|_g^2 \, d\mathcal{W}_g
\]
\[+ \int \mathcal{W} q^{-1} \left\{ \frac{1}{2} \langle \mathcal{Q}^{(1)} \rangle_\mathcal{H}^g + |\mathcal{B}^{(1)}|_\mathcal{H}^g + \mathcal{C}^{(1)} + \mathcal{D}^{(1)} + \mathcal{J}_{(Coeff)} \rangle_{\Omega, \partial \Omega} \right\} \, d\mathcal{W}_g
\]
\[+ \int \mathcal{W} q^{-1} \mathcal{J}_{(\mathcal{W})} \rangle_{\Omega, \partial \Omega} \, d\mathcal{W}_g
\]
\[\quad - \int H(U(0, U), U') \left\{ \mathcal{Q}_{(1)} + \mathcal{H}_1^{(1)} |\Omega|_g \right\} \, d\mathcal{W}_g \]
\[\quad + \int H(U(0, U), U') \left\{ \mathcal{Q}_0 + \mathcal{H}_0^{(1)} |\Omega|_g \right\} \, d\mathcal{W}_g \]
\[\quad + \int \mathcal{W} q^{-1} \mathcal{J}_{(\mathcal{W})} \rangle_{\Omega, \partial \Omega} \, d\mathcal{W}_g
\]

Then by adding (264)–(266), noting the cancellation of the two integrals over $S_{1+U,0}$ and the two integrals over $S_{U-\mathcal{U}}$ on RHSs (264)–(266), taking the limit as $u'' \downarrow 0$ in (264) and $u'' \uparrow \bar{U}$ in (266), and recalling that $\mathcal{M}_{U_U} = \mathcal{M}_1(0) \cup \mathcal{M}_2 \cup \mathcal{M}_3(\bar{U})$, we arrive at the desired identity (246a).

It remains for us to prove (264)–(266). We first prove (265). The proof is similar to the proof of (165a), so we only sketch the argument. For each fixed $\tau' \in [1 + \bar{U}, U)$, we consider the $\Sigma_{\tau'}$-divergence identity (262) with $\Omega$ in the role of $V$. We integrate this identity over $\Sigma_{\tau'}$ with respect to the volume form $d\mathcal{W}_g$ of $\bar{g}$, and use the divergence theorem to obtain an integral identity. This “spatial” integral identity features two boundary integrals coming from the term $\nabla_\alpha (\mathcal{W} J^\alpha [V])$ on RHS (262) (in the proof of (165a), we encountered only one boundary integral): 
\[-\int_{S_{U'}, \partial} \mathcal{W} Z_{\alpha} J^\alpha [\Omega] \, d\mathcal{W}_g + \int_{S_{\tau''}, \partial} \mathcal{W} Z_{\alpha} J^\alpha [\Omega] \, d\mathcal{W}_g,\]
where $\tau'' = u'' + \bar{U} = \tau'$. We clarify that by definition (233), $Z$ points inward to $\Sigma_{\tau'}$ at $S_{U''\tau',0}$, while $Z$ points outward to $\Sigma_{\tau'}$ at $S_{U''\tau',\bar{U}}$; see also Figures 2 and 3. This explains the different signs of the two boundary integrals; see also Remark 10.3. To account for the last term on RHS (100) (with $\Omega$ in the role of $V$), we add the integral $\int_{\Sigma_{\tau'}} \mathcal{W} g_{\alpha\beta} (\mathcal{B}^\alpha \mathcal{B}^\beta) \, d\mathcal{W}_g$ to each side of the integral identity. We then integrate the resulting integral identity with respect $\tau'$ over the interval $[1 + \bar{U}, U]$. Considering (218), we see that the two aforementioned boundary integrals lead to the following two $g$-null hypersurface integrals: 
\[-\int_{H(U(1 + \bar{U}, U))} \mathcal{W} Z_{\alpha} J^\alpha [\Omega] \, d\mathcal{W}_g \]
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\[ \int_{H_{\alpha}(1,U-U)} \mathcal{W} Z_{\alpha} J^\alpha[\Omega] \, d\bar{\omega} g / du' \]. We then use two applications of (262a) (see Footnote 58) to substitute for these two g-null hypersurface integrals. Also using the arguments given in the discussion surrounding (175) and the identity \( d\bar{\omega} g / du' = q^{-1} d\omega g \) (see (149)), we arrive at the desired identity (265).

The identity (264) can be proved using arguments similar to the ones we used to prove (265), as we now sketch. We argue as before, this time obtaining a “spatial” integral identity (over \( \tilde{\Sigma} \tau' \)) that involves the two boundary integrals

\[ -\int_{S_{u''}} \mathcal{W} Z_{\alpha} J^\alpha[\Omega] \, d\bar{\omega} g / du' \]

\[ + \int_{S_{u''}} \mathcal{W} Z_{\alpha} J^\alpha[\Omega] \, d\bar{\omega} g / du' \],

where \( u'' = 1 + u''', \quad u''' = \tau' \).

Again adding the integral \( \int_{\Sigma_\tau} \mathcal{W} g_{\alpha\beta}(B^\alpha)(B^\beta) d\bar{\omega} \) to each side of the integral identity to account for the last term on RHS (100) and integrating with respect \( \tau' \in [1 + u'', 1 + U] \), we arrive at an integral identity that involves the following two g-null hypersurface integrals:

\[ -\int_{H_{\alpha}(1,U-U)} \mathcal{W} Z_{\alpha} J^\alpha[\Omega] \, d\bar{\omega} g / du' \]

\[ + \int_{H_{\alpha}(U-U,W)} \mathcal{W} Z_{\alpha} J^\alpha[\Omega] \, d\bar{\omega} g / du' \].

Next, we respectively use (262a)-(262b) to substitute for these g-null hypersurface integrals (note that in proving (265), we used only (262a)). Carefully noting the sign differences between the terms on RHS (262a) and RHS (262b), and again using the arguments given in the discussion surrounding (175) as well as the identity \( d\bar{\omega} g / du' = q^{-1} d\omega g \), we arrive at (264).

The identity (266) can be proved using arguments similar to the ones we used to prove (264). The two g-null hypersurface integrals that one encounters are

\[ -\int_{H_{\alpha}(0, U+U-U)} \mathcal{W} Z_{\alpha} J^\alpha[\Omega] \, d\bar{\omega} g / du' \]

\[ + \int_{H_{\alpha}(U-U,W)} \mathcal{W} Z_{\alpha} J^\alpha[\Omega] \, d\bar{\omega} g / du' \],

and we again respectively use (262a)-(262b) to substitute for them. This completes the proof of (246a).

The identity (246b) can be proved using nearly identical arguments, where we use the identities (262c)-(262d) in place of the identities (262a)-(262b) that we used in proving (246a); we omit the details. We have therefore proved Theorem 10.6.

Appendix A. Notation for Sections 1-9

For the reader’s convenience, in this appendix, we have gathered some of the notation from Sects. 1-9 into a table. We caution that some of the symbols defined in Sects. 1-9 have a slightly different – although analogous – definition in Sect. 10; see Subsubsect. 10.2.5 for clarification of this point. We also refer to Subsubsect. 1.1.1 for basic notation.
| Symbol | Description/ Reference |
|--------|------------------------|
| $\mathcal{L}$ | An arbitrary scalar function, fixed throughout |
| $\mathcal{L}^X$ | Lie differentiation with respect to $X$ |
| $\rho, \Omega, S$ | Def. 1.1 |
| $d\Omega, dS$ | Cor. 3.7 |
| $C, D$ | Def. 2.1 |
| $g, g^{-1}$ | Def. 2.2 |
| $B$ | Def. 2.6 |
| $\square_g$ | Subsect. 3.1 |
| $\tau$ | Convention 3.4 |
| $\mathcal{M}, \mathcal{M}_T, \mathcal{H}, \mathcal{H}_T, S_T$ | Subsect. 3.1 |
| $\mathcal{N}, \mathcal{N}_T, L$ | Def. 3.2 |
| $\hat{\mathcal{N}}, \hat{\mathcal{N}}_T, \mathcal{S}, \mathcal{S}_\tau$ | Subsect. 3.10.1 |
| $\|\xi\|_e, \|\xi\|_g, \|\xi\|_{\tilde{g}}, \|\xi\|_{g^\ell}, \|\xi\|_{\tilde{g}^\ell}$ | Subsect. 3.10.3 |
| $\Theta, \Pi$ | Def. 3.10 |
| $K$ | Def. 4.1 |
| $\mathcal{P}$ | Def. 4.3 |
| $J[V]$ | Lemma 4.4 |
| $P$ | Def. 5.2 |
| $W(\alpha)$ | Lemma 5.4 |
| $\mathcal{Y}$ | Lemma 5.9 |
| $\sigma$ | Def. 6.10 |
| $d\omega_g, d\omega_{\tilde{g}}, d\omega_{g^\ell}, d\omega_{\tilde{g}^\ell}$ | Def. 6.1 |
| $Q$ | Def. 9.2 |
| $E_{\text{Wave}}, E_{\text{Transport}}, F_{\text{Wave}}, F_{\text{Transport}}$ | Def. 9.3 |
| $\mathcal{X}_{\Pi_{\alpha\beta}}$ | Def. 9.4 |

**References**

[1] Serge Alinhac, *Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II*, Acta Math. 182 (1999), no. 1, 1–23. MR1687180 (2000d:35148)

[2] ———, *Blowup of small data solutions for a quasilinear wave equation in two space dimensions*, Ann. of Math. (2) 149 (1999), no. 1, 97–127. MR1680539 (2000d:35147)

[3] Tristan Buckmaster, Steve Shkoller, and Vlad Vicol, *Formation of point shocks for 3D compressible Euler*, arXiv e-prints (2019Dec), arXiv:1912.04429, available at 1912.04429.

[4] ———, *Formation of shocks for 2D isentropic compressible Euler*, arXiv e-prints (2019Jul), arXiv:1907.03784, available at 1907.03784.
Demetrios Christodoulou, The formation of shocks in 3-dimensional fluids, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2007. MR2284927 (2008e:76104)

Demetrios Christodoulou, Mathematical problems of general relativity, I, Zürich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR2391586 (2008m:83008)

Demetrios Christodoulou, The shock development problem, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2019. MR3890062

Demetrios Christodoulou and André Lisibach, Shock development in spherical symmetry, Annals of PDE 2 (2016), no. 1, 1–246.

Demetrios Christodoulou and Shuang Miao, Compressible flow and Euler’s equations, Surveys of Modern Mathematics, vol. 9, International Press, Somerville, MA; Higher Education Press, Beijing, 2014. MR3288725

Daniel Coutand, Hans Lindblad, and Steve Shkoller, A priori estimates for the free-boundary 3D compressible Euler equations in physical vacuum, Comm. Math. Phys. 296 (2010), no. 2, 559–587. MR2608125 (2011c:35629)

Daniel Coutand and Steve Shkoller, Well-posedness in smooth function spaces for moving-boundary 1-D compressible Euler equations in physical vacuum, Comm. Pure Appl. Math. 64 (2011), no. 3, 328–366. MR2779087 (2012d:76103)

Daniel Coutand and Steve Shkoller, Well-posedness in smooth function spaces for the moving-boundary three-dimensional compressible Euler equations in physical vacuum, Arch. Ration. Mech. Anal. 206 (2012), no. 2, 515–616. MR2980528

Daniel Coutand and Steve Shkoller, A new formulation of the 3D compressible Euler equations with dynamic entropy: Remarkable null structures, Arch. Ration. Mech. Anal. 234 (2019), no. 3, 1223–1279. MR4011696

Robert M. Wald, General relativity, University of Chicago Press, Chicago, IL, 1984. MR757180 (86a:83001)

Qian Wang, Rough solutions of the 3-d compressible euler equations, 2019.
[36] Willie Wai-Yeung Wong, *A comment on the construction of the maximal globally hyperbolic Cauchy development*, J. Math. Phys. 54 (2013), no. 11, 113511, 8. MR3154377