ON ADMISSIBLE RANK ONE LOCAL SYSTEMS

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Abstract. A rank one local system $\mathcal{L}$ on a smooth complex algebraic variety $M$ is 1-admissible if the dimension of the first cohomology group $H^1(M, \mathcal{L})$ can be computed from the cohomology algebra $H^*(M, \mathbb{C})$ in degrees $\leq 2$. Under the assumption that $M$ is 1-formal, we show that all local systems, except finitely many, on a non-translated irreducible component $W$ of the first characteristic variety $\mathcal{V}_1(M)$ are 1-admissible, see Proposition 3.1. The same result holds for local systems on a translated component $W$, but now $H^*(M, \mathbb{C})$ should be replaced by $H^*(M_0, \mathbb{C})$, where $M_0$ is a Zariski open subset obtained from $M$ by deleting some hypersurfaces determined by the translated component $W$, see Theorem 4.3. One consequence of this result is that the local systems $\mathcal{L}$ where the dimension of $H^1(M, \mathcal{L})$ jumps along a given positive dimensional component of the characteristic variety $\mathcal{V}_1(M)$ have finite order, see Theorem 4.7. Using this, we show in Corollary 4.9 that $\dim H^1(M, \mathcal{L}) = \dim H^1(M, \mathcal{L}^{-1})$ for any rank one local system $\mathcal{L}$ on a smooth complex algebraic variety $M$.

1. Introduction

Let $M$ be a connected finite CW-complex. If $M$ is 1-formal, then the first twisted Betti number of $M$ in low degrees, for rank one complex local systems $\mathcal{L}$ near the trivial local system, see [8], Theorem A, the Tangent Cone Theorem.

In this paper, assuming moreover that $M$ is a connected smooth quasi-projective variety, our aim is to show that (a version of) the above statement is true globally, with finitely many exceptions. In such a situation the exponential mapping (2.1) sends the irreducible components $E$ of the first resonance variety $\mathcal{R}_1(M)$ of $M$ onto the non translated irreducible components $W$ of the first characteristic variety $\mathcal{V}_1(M)$ of $M$.

For $\alpha \in E$, $\alpha \neq 0$ (resp. $\mathcal{L} \in W$, $\mathcal{L} \neq \mathcal{C}_M$), the dimension of the cohomology group $H^1(H^*(M, \mathbb{C}), \alpha \wedge)$ (resp. $H^1(M, \mathcal{L})$) is constant (resp. constant with finitely many exceptions where this dimension may possibly increase). The first result is

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that the 1-formality assumption implies the inequality
\[(1.1) \dim H^1(M, \mathcal{L}) \geq \dim H^1(H^*(M, \mathbb{C}), \alpha \wedge)\]

obtained by Libgober and Yuzvinsky when \(M\) is a hyperplane arrangement complement, see [15], Proposition 4.2.

An 1-admissible local system is a system for which the equality in the inequality (1.1) holds. Various characterization of 1-admissible local systems \(\mathcal{L}\) on a non translated component \(W\) of the first characteristic variety \(V_1(M)\) are given in Proposition 3.1. In particular, we show that all local systems, except finitely many, on a non-translated irreducible component \(W\) are 1-admissible.

The main novelty is the analysis of local systems belonging to a positive dimensional translated component \(W'\) of the first characteristic variety of \(M\), see the last section. Such local systems (at least generically) are not 1-admissible. However, for a generic local system in \(W'\), an equality similar to (1.1) holds but now \(H^*(M, \mathbb{C})\) should be replaced by \(H^*(M_0, \mathbb{C})\), where \(M_0\) is a Zariski open subset obtained from \(M\) by deleting some hypersurfaces determined by the translated component \(W'\), see Theorem 4.3.

One consequence of this result is the fact that the local systems \(\mathcal{L}\) where the dimension of \(H^1(M, \mathcal{L})\) jumps along a given positive dimensional irreducible component of the characteristic variety \(V_1(M)\) are local system of finite order, see Theorem 4.7. Using this, we show in Corollary 4.9 that one has
\[
\dim H^1(M, \mathcal{L}) = \dim H^1(M, \mathcal{L}^{-1})
\]
for any rank one local system \(\mathcal{L}\) on a smooth complex algebraic variety \(M\). In this section the role played by the constructible sheaf point of view introduced in [5] is essential.

2. Admissible and 1-admissible local systems

Let \(M\) be a smooth, irreducible, quasi-projective complex variety and let \(\mathbb{T}(M) = \text{Hom}(\pi_1(M), \mathbb{C}^*)\) be the character variety of \(M\). This is an algebraic group whose identity irreducible component is an algebraic torus \(\mathbb{T}(M)_1 \simeq (\mathbb{C}^*)^{b_1(M)}\). Consider the exponential mapping
\[(2.1) \quad \exp : H^1(M, \mathbb{C}) \to H^1(M, \mathbb{C}^*) = \mathbb{T}(M)\]
induced by the usual exponential function \(\exp : \mathbb{C} \to \mathbb{C}^*\). Clearly \(\exp(H^1(M, \mathbb{C})) = \mathbb{T}(M)_1\).

**Definition 2.1.** A local system \(\mathcal{L} \in \mathbb{T}(M)_1\) is 1-admissible if there is a cohomology class \(\alpha \in H^1(M, \mathbb{C})\) such that \(\exp(\alpha) = \mathcal{L}\) and
\[
\dim H^1(M, \mathcal{L}) = \dim H^1(H^*(M, \mathbb{C}), \alpha \wedge).
\]
If $\mathcal{L} = \mathbb{C}_M$, then we can take $\alpha = 0$ and the equality of dimension in Definition 2.1 is obvious. So in the sequel we consider only the case $\mathcal{L} \neq \mathbb{C}_M$.

**Remark 2.2.** When $M$ is a hyperplane arrangement complement or, more generally, a hypersurface arrangement complement in some projective space $\mathbb{P}^n$, one usually defines the notion of *admissible* local system $\mathcal{L}$ on $M$ in terms of some conditions on the residues of an associated logarithmic connection $\nabla(\alpha)$ on a good compactification of $M$, see for instance [11], [18], [10]. For such an admissible local system $\mathcal{L}$ on $M$ one has

$$\dim H^i(M, \mathcal{L}) = \dim H^i(H^*(M, \mathbb{C}), \alpha \wedge)$$

for all $i$ in the hyperplane arrangement case and for $i = 1$ in the hypersurface arrangement case. For the case of hyperplane arrangement complements, see also [12] and [15]. It is clear that "admissible" implies "1-admissible", which is a simpler, but still rather interesting property as we see below.

One has the following easy result.

**Lemma 2.3.** Any local system $\mathcal{L} \in \mathbb{T}(M)$ is 1-admissible if $\dim M = 1$.

**Proof.** Note that in this case the integral homology group $H_1(M)$ is torsion free and hence $\mathbb{T}(M) = \mathbb{T}(M)_1$. Since $\mathcal{L}$ is not the trivial local system, clearly one has $H^0(M, \mathcal{L}) = 0$.

If $M$ is compact, then by duality, see [4], we get

$$H^2(M, \mathcal{L}) = H^0(M, \mathcal{L}^\vee) = 0$$

and hence

$$\dim H^1(M, \mathcal{L}) = b_1(M) - 2 = -\chi(M).$$

If $M$ is not compact, then $M$ is homotopically equivalent to an 1-dimensional CW-complex, and hence $H^2(M, \mathcal{L}) = 0$. In this case we get $\dim H^1(M, \mathcal{L}) = b_1(M) - 1 = -\chi(M)$. It follows that in both cases one has

$$\dim H^1(M, \mathcal{L}) = -\chi(M) = \dim H^1(H^*(M, \mathbb{C}), \alpha \wedge).$$

(2.2)

Note also that if $\mathcal{L} = \mathbb{C}_M$ only the choice $\alpha = 0$ is good, while in the case $\mathcal{L} \neq \mathbb{C}_M$ any choice for $\alpha$ satisfying $\exp(\alpha) = \mathcal{L}$ is valid. 

To go further, we need the characteristic and resonance varieties, whose definition is recalled below.

The *characteristic varieties* of $M$ are the jumping loci for the cohomology of $M$, with coefficients in rank 1 local systems:

$$V^i_k(M) = \{ \rho \in \mathbb{T}(M) \mid \dim H^i(M, \mathcal{L}_\rho) \geq k \}.$$  

(2.3)

When $i = 1$, we use the simpler notation $V_k(M) = V^1_k(M)$. 
The resonance varieties of $M$ are the jumping loci for the cohomology of the complex $H^*(H^*(M, \mathbb{C}), \alpha \wedge)$, namely:

\[
R^i_k(M) = \{ \alpha \in H^1(M, \mathbb{C}) \mid \dim H^i(H^*(M, \mathbb{C}), \alpha \wedge) \geq k \}.
\]

When $i = 1$, we use the simpler notation $R_k(M) = R^1_k(M)$.

**Example 2.4.** Assume that $\dim M = 1$ and $\chi(M) < 0$. Then it follows from the equality (2.2) that

\[
V_1(M) = \mathbb{T}(M) \quad \text{and} \quad R_1(M) = H^1(M, \mathbb{C}).
\]

The more precise relation between the resonance and characteristic varieties can be summarized as follows, see [8].

**Theorem 2.5.** Assume that $M$ is 1-formal. Then the irreducible components $E$ of the resonance variety $R_1(M)$ are linear subspaces in $H^1(M, \mathbb{C})$ and the exponential mapping (2.1) sends these irreducible components $E$ onto the irreducible components $W$ of $V_1(M)$ with $1 \in W$.

**Remark 2.6.** The fact that $M$ is 1-formal depends only on the fundamental group $\pi_1(M)$, see for details [8]. The class of 1-formal varieties is large enough, as it contains all the projective smooth varieties and any hypersurface complement in $\mathbb{P}^n$, see [8]. In fact, if the Deligne mixed Hodge structure on $H^1(M, \mathbb{Q})$ is pure of weight 2, then the smooth quasi-projective variety $M$ is 1-formal, see [16]. The converse implication is not true, since any smooth quasi-projective curve obtained by deleting $k > 1$ points from a projective curve of genus $g > 0$ is 1-formal, but the corresponding mixed Hodge structure on $H^1(M, \mathbb{Q})$ is not pure. Several examples of smooth quasi-projective varieties with a pure Deligne mixed Hodge structure on $H^1(M, \mathbb{Q})$ are given in [7].

In the sequel we concentrate ourselves on the strictly positive dimensional irreducible components of the first characteristic variety $V_1(M)$. They have the following rather explicit description, given by Arapura [1], see also Theorem 3.6 in [5].

**Theorem 2.7.** Let $W$ be a $d$-dimensional irreducible component of the first characteristic variety $V_1(M)$, with $d > 0$. Then there is a regular morphism $f : M \to S$ onto a smooth curve $S = S_W$ with $b_1(S) = d$ such that the generic fiber $F$ of $f$ is connected, and a torsion character $\rho \in \mathbb{T}(M)$ such that the composition

\[
\pi_1(F) \xrightarrow{i} \pi_1(M) \xrightarrow{\rho} \mathbb{C}^*,
\]

where $i : F \to M$ is the inclusion, is trivial and

\[
W = \rho \cdot f^*(\mathbb{T}(S)).
\]
In addition, \( \dim W = -\chi(S_W) + e \), with \( e = 1 \) if \( S_W \) is affine and \( e = 2 \) if \( S_W \) is proper. If \( L \in W \), then \( \dim H^1(M, \mathcal{L}) \geq -\chi(S_W) \) and equality holds for all such \( L \) with finitely many exceptions when \( 1 \in W \).

If \( 1 \in W \), we say that \( W \) is a non-translated component and then one can take \( \rho = 1 \). If \( 1 \notin W \), we say that \( W \) is a translated component.

The following result was obtained in [15], Proposition 4.2 in the case of hyperplane arrangement complements.

**Proposition 2.8.** Assume that \( M \) is 1-formal and \( \exp(\alpha) = \mathcal{L} \). Then

\[
\dim H^1(M, \mathcal{L}) \geq \dim H^1(H^*(M, \mathbb{C}), \alpha \wedge).
\]

**Proof.** If \( \alpha \notin R_1(M) \) or if \( \mathcal{L} \) is the trivial local system, then there is nothing to prove. Otherwise the result follows directly from Proposition 6.6 in [8].

\( \square \)

**Remark 2.9.** (i) When \( M \) is 1-formal, then if \( \mathcal{L} \) is 1-admissible and \( H^1(M, \mathcal{L}) \neq 0 \), then the cohomology class \( \alpha \) realizing the conditions in Definition 2.1 is necessarily in an irreducible component \( E = T_1W \) of \( R_1(M) \), such that \( \mathcal{L} \) belongs to the non-translated irreducible component \( W = \exp(E) \). For all \( \mathcal{L} \in T(M) \), except finitely many, this component \( W \) is uniquely determined by \( L \), see [17].

(ii) Again when \( M \) is 1-formal, this also shows that all the local systems on a translated component \( W \) of \( V_1(M) \), possibly except finitely many located at the intersections of \( W \) with non-translated components, are not 1-admissible. Indeed, all the examples in [20] suggest that the local systems situated at the intersection of two (or several) irreducible components of \( V_1(M) \) are not 1-admissible.

**Remark 2.10.** Note that Proposition 2.8 implies in particular

\[
\exp(R_k(M)) \subset V_k(M)
\]

for all \( k \). Since the differential of exp at the origin is the identity, this implies

\[
R_k(M) \subset TC_1V_k(M).
\]

Since the other inclusion always holds, see Libgober [11], it follows that the inequality in Proposition 2.8 implies the equality

\[
R_k(M) = TC_1V_k(M).
\]

If \( M \) is not 1-formal, then the tangent cone \( TC_1V_k(M) \) can be strictly contained in \( R_k(M) \), see for instance Examples 5.11 and 9.1 in [8]. It follows that the assumption 1-formal is needed to infer the inequality in Proposition 2.8. In other words, one may have

\[
\dim H^1(M, \mathcal{L}) < \dim H^1(H^*(M, \mathbb{C}), \alpha \wedge).
\]
for some varieties $M$, which shows that the last claim in Proposition 4.5 in \cite{10} fails for $k = 0$ and a general quasi-projective variety $M = Z \setminus D$. For instance, if $M = M_g$ is the surface constructed in Example 5.11 in \cite{8}, one has
\[ 2g - 2 = \dim H^1(M, L) < \dim H^1(H^*(M, \mathbb{C}), \alpha \wedge) = 2g - 1 \]
for all $L \neq C_M$ and $\alpha \neq 0$. So in this case the only 1-admissible local system is the trivial local system $C_M$.

**Corollary 2.11.** If $M$ is 1-formal, then any $L \in \mathbb{T}(M)_1$ with $H^1(M, L) = 0$ is 1-admissible. More precisely, if $L = \exp(\alpha)$, then $H^1(H^*(M, \mathbb{C}), \alpha \wedge) = 0$.

**Proof.** Assume that $L = \exp(\alpha)$ and $H^1(H^*(M, \mathbb{C}), \alpha \wedge) \neq 0$. Then Proposition 2.8 gives a contradiction.

The following result says that $\alpha \in \exp^{-1}(L)$ which occurs in Definition 2.1 cannot be arbitrary in general.

**Proposition 2.12.** Assume that $R_1(M) \neq H^1(M, \mathbb{C})$. Then for any local system $L \in \mathbb{T}(M)_1$ there are infinitely many $\alpha \in \exp^{-1}(L)$ such that $H^1(H^*(M, \mathbb{C}), \alpha \wedge) = 0$.

**Proof.** Since $L \in \mathbb{T}(M)_1$, there is a cohomology class $\alpha_0 \in H^1(M, \mathbb{C})$ such that $\exp(\alpha_0) = L$. Then $\exp^{-1}(L) = \alpha_0 + \ker \exp$. We have to show that the set $(\alpha_0 + \ker \exp) \setminus R_1(M)$ is infinite. The result follows from the following.

**Lemma 2.13.** Consider the lattice $L_n = (2\pi i) \cdot \mathbb{Z}^n \subset \mathbb{C}^n$ for $n \geq 1$. Then, for any point $\alpha \in \mathbb{C}^n$ and any subset $A \subset \alpha + L_n$ such that $(\alpha + L_n) \setminus A$ is finite, the Zariski closure of $A$ is $\mathbb{C}^n$.

**Proof.** It is enough to show that any polynomial $g \in \mathbb{C}[x_1, ..., x_n]$ such that $(\alpha + L_n) \setminus Z(g)$ is finite, where $Z(g)$ is the zero-set of $g$, satisfies $g = 0$.

The case $n = 1$ is obvious. Assume the property is established for $n - 1 \geq 1$ and consider the projection $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$, $(x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1})$. Let $g = p|Z(g) : Z(g) \to \mathbb{C}^{n-1}$. It follows that $q(Z(g))$ contains a subset of $p(\alpha) + L_{n-1}$ with a finite complement, and the induction hypothesis implies that $q$ is a dominant mapping, i.e. the Zariski closure of $q(Z(g))$ is $\mathbb{C}^{n-1}$. If $g \neq 0$, then $Z(g)$ is purely $(n - 1)$-dimensional, and hence the generic fibers of $q$ are 0-dimensional. In other words, it exists a non-zero polynomial $h \in \mathbb{C}[x_1, ..., x_{n-1}]$ such that $\dim q^{-1}(y) > 0$ implies $h(y) = 0$. On the other hand, for any $y_0 = p(\alpha) + v$ where $v \in L_{n-1}$, the fiber $q^{-1}(y_0)$ contains infinitely many points of the form $\alpha + v + 2\pi ise_n$

with $s \in \mathbb{Z}$ and $e_n = (0, ..., 0, 1)$. It follows that $\dim q^{-1}(y_0) > 0$ and hence $h(y_0) = 0$. The induction hypothesis implies that $h = 0$, a contradiction. This ends the proof of this Lemma and hence the proof of Proposition 2.12.
In view of Corollary 2.11, we consider in the sequel only local systems \( \mathcal{L} \neq \mathcal{C}_M \) such that \( \mathcal{L} \in \mathcal{V}_1(M) \cap \mathcal{T}(M)_1 \).

3. Non-translated components and 1-admissible local systems

Let \( M \) be a smooth, quasi-projective complex variety. Let \( W \) be an irreducible component of \( \mathcal{V}_1(M) \) such that \( 1 \in W \) and \( \dim W > 0 \). Let \( f : M \to S \) be the morphism onto a curve described in Theorem 2.7, such that \( W = f^*(\mathcal{T}(S)) \).

Note that \( F := R^0f_*(\mathcal{C}_M) = \mathbb{C}_S \) (since the generic fiber of \( f \) is connected) and set \( G := R^1f_*(\mathcal{C}_M) \).

Proposition 3.1. If \( M \) is 1-formal, then the following three conditions on a local system \( \mathcal{L} = f^{-1}\mathcal{L}' \in W \), are equivalent.

(i) \( \mathcal{L} \) is 1-admissible;

(ii) \( \dim H^1(M, \mathcal{L}) = \min_{\mathcal{L}_i \in W} \dim H^1(M, \mathcal{L}_i) \). (This minimum is called the generic dimension of \( H^1(M, \mathcal{L}) \) along \( W \).)

(iii) the natural morphism \( f^*: H^1(S, \mathcal{L}') \to H^1(M, \mathcal{L}) \) is an isomorphism.

The condition

(iv) \( H^0(S, G \otimes \mathcal{L}') = 0 \)

implies the condition (iii) and they are equivalent when \( S \) is affine. Moreover, all these conditions are fulfilled by all \( \mathcal{L} \in W \) except finitely many.

Proof. The equivalence of (i) and (ii) follows from Proposition 2.8 combined with the fact that (ii) holds for all local systems \( \mathcal{L} \in W \) except finitely many.

For the definition of the morphism \( f^*: H^1(S, \mathcal{L}') \to H^1(M, \mathcal{L}) \), see [4], p. 54 and the references given there. The implication \( (ii) \Rightarrow (iii) \) follows from the exact sequence

\[
0 \to H^1(S, \mathcal{L}') \to H^1(M, \mathcal{L}) \to H^0(S, G \otimes \mathcal{L}')
\]

where the first morphism is precisely \( f^* \) and the last morphism is surjective when \( S \) is affine or \( \mathcal{L}' \in \mathcal{T}(S) \) is generic, see Prop.4.3 in [5]. It follows from Lemma 2.3 that the dimension of \( H^1(S, \mathcal{L}') \) is constant for \( \mathcal{L}' \) non-trivial. Proposition 4.5 in [5] gives the generic vanishing of the group \( H^0(S, G \otimes \mathcal{L}') \). It follows that the minimal value for \( \dim H^1(M, \mathcal{L}) \) is precisely \( \dim H^1(S, \mathcal{L}') \), and in such a case the monomorphism \( f^* \) becomes an isomorphism.

Conversely, assume that (iii) holds. Let \( d = \dim W = b_1(S) \). Since the generic fiber of \( f \) is connected, it follows that

\[
f^*: H^1(S, \mathbb{C}) \to H^1(M, \mathbb{C})
\]
is injective. Let $\mathcal{L}' = \exp(\omega)$ and note that then $\mathcal{L} = \exp(\alpha)$, where $\alpha = f^*(\omega)$. Using now the injectivity \[ (3.2) \] and Lemma \[ 2.3 \] it follows that (3.3)

$$\dim H^1(S, \mathcal{L'}) = \dim \frac{\{ \beta \in H^1(S, \mathbb{C}) \mid \omega \wedge \beta = 0 \}}{\mathbb{C} \cdot \omega} = \dim \frac{\{ \gamma \in E \mid \alpha \wedge \gamma = 0 \}}{\mathbb{C} \cdot \alpha}$$

where $E = f^*(H^1(S, \mathbb{C}))$ is a $d$-dimensional vector subspace in $H^1(M, \mathbb{C})$. In fact, it follows from Theorem \[ 2.5 \] that $E$ is the irreducible component of $R_1(M)$ corresponding to the irreducible component $W$ of $\mathcal{V}_1(M)$.

To show that (i) holds, it is enough to show that (3.4)

$$\{ \gamma \in E \mid \alpha \wedge \gamma = 0 \} = \{ \delta \in H^1(M, \mathbb{C}) \mid \alpha \wedge \delta = 0 \}.$$

Note that $\alpha \in R_s(M)$ exactly when $\dim \{ \delta \in H^1(M, \mathbb{C}) \mid \alpha \wedge \delta = 0 \} \geq s + 1$.

Using \[ 8 \], it follows that $R_s(M) = \bigcup_i R^i$, where the union is over all the irreducible components $R^i$ of $R_1(M)$ such that $\dim R^i > s + p(i)$, with $p(i) = 0$ if the corresponding curve $S_i$ is not compact and $p(i) = 1$ when the corresponding curve $S_i$ is compact.

Case 1. If $S$ is not compact, then clearly $\alpha \in (E \setminus 0) \subset (R_{d-1}(M) \setminus R_d(M))$. It follows that

$$\dim \{ \delta \in H^1(M, \mathbb{C}) \mid \alpha \wedge \delta = 0 \} = d = \dim E$$

hence we get the equality (3.4) in this case.

Case 2. If $S$ is compact, then clearly $\alpha \in (E \setminus 0) \subset (R_{d-2}(M) \setminus R_{d-1}(M))$. It follows that

$$\dim \{ \delta \in H^1(M, \mathbb{C}) \mid \alpha \wedge \delta = 0 \} = d - 1 = \dim \{ \gamma \in E \mid \alpha \wedge \gamma = 0 \}$$

hence we get the equality (3.4) in this case as well.

The last claim follows directly from Proposition 6.6 in \[ 8 \].

\[ \square \]

4. Translated components and 1-admissible local systems

Consider now the case of a translated component $W = \rho \cdot f^*(\mathbb{T}(S))$ and recall the notation from Theorem \[ 2.7 \]. Let $\mathcal{L}_0$ be the local system corresponding to $\rho$. We assume that $1 \notin W$ and this implies that the singular support $\Sigma(\mathcal{F})$ of the constructible sheaf $\mathcal{F} = R^0 f_*(\mathcal{L}_0)$ is non-empty see Corollary 5.9 in \[ 5 \] (and coincides with the set of points $s \in S$ such that the stalk $\mathcal{F}_s$ is trivial, see Lemma 4.2 in \[ 5 \]). We set as above $\mathcal{G} = R^1 f_*(\mathcal{L}_0)$ and recall the exact sequence

\[ (4.1) \quad 0 \to H^1(S, \mathcal{F} \otimes \mathcal{L'}) \to H^1(M, \mathcal{L}_0 \otimes \mathcal{L}) \to H^0(S, \mathcal{G} \otimes \mathcal{L'}) \]

where $\mathcal{L} = f^{-1} \mathcal{L}'$ and the last morphism is surjective when $S$ is affine or $\mathcal{L}' \in \mathbb{T}(S)$ is generic, see Proposition 4.3 in \[ 5 \]. Moreover, one has
(A) $H^0(S, \mathcal{G} \otimes \mathcal{L}') = 0$ except for finitely many $\mathcal{L}' \in \mathcal{T}(S)$, see Proposition 4.5 in [5], and

(B) $\mathcal{F} = Rj_*j^{-1}\mathcal{F}$, where $S_0 = S \setminus \Sigma(\mathcal{F})$ and $j : S_0 \rightarrow S$ is the inclusion, see [6].

The proof of this last claim goes like that. It is known that a point $c \in S$ is in $\Sigma(\mathcal{F})$ if and only if for a small disc $D_c$ centered at $c$, the restriction of the local system $\mathcal{L}_\rho$ to the associated tube $T(F_c) = f^{-1}(D_c)$ about the fiber $F_c$ is non-trivial. Let $T(F_c)' = T(F_c) \setminus F_c$ and note that the inclusion $i : T(F_c)' \rightarrow T(F_c)$ induces an epimorphism at the level of fundamental groups. Hence, if $c \in \Sigma(\mathcal{F})$, then $\mathcal{L}_\rho|_{T(F_c)'}$ is a non-trivial rank one local system. In particular

$$H^0(T(F_c)', \mathcal{L}_\rho) = 0.$$ 

If we apply the Leray spectral sequence to the locally trivial fibration

$$F \rightarrow T(F_c)' \rightarrow D'_c$$

where $D'_c = D_c \setminus \{c\}$, we get

$$H^0(D'_c, \mathcal{F}) = H^0(T(F_c)', \mathcal{L}_\rho) = 0.$$ 

It follows that $\mathcal{F}|_{D'_c}$ is a non-trivial rank one local system. Hence $H^0(D'_c, \mathcal{F}) = H^1(D'_c, \mathcal{F}) = 0$, which proves the isomorphism $\mathcal{F} = Rj_*j^{-1}\mathcal{F}$.

We deduce from (B) that the following more general isomorphism

(C) $\mathcal{F} \otimes \mathcal{L}' = Rj_*j^{-1}(\mathcal{F} \otimes \mathcal{L}')$ for any $\mathcal{L}' \in \mathcal{T}(S)$. In particular

$$H^1(S, \mathcal{F} \otimes \mathcal{L}') = H^1(S, Rj_*j^{-1}(\mathcal{F} \otimes \mathcal{L}')) = H^1(S_0, j^{-1}(\mathcal{F} \otimes \mathcal{L}')),$$

where the last isomorphism comes from Leray Theorem, see [1], p.33.

Let $M_0 = M \setminus f^{-1}(\Sigma(\mathcal{F}))$ and denote by $f_0 : M_0 \rightarrow S_0$ the surjective morphism induced by $f$.

**Lemma 4.1.**

$$\mathcal{L}_0|M_0 \simeq f_0^{-1}(\mathcal{F}|_{S_0}).$$

**Proof.** For any local system $\mathcal{L}_1 \in \mathcal{T}(M)$, there is a canonical adjunction morphism

$$a : f^{-1}f_*\mathcal{L}_1 \rightarrow \mathcal{L}_1$$

see [13], (2.3.4), p. 91. In fact, in our situation, $f$ and $f_0$ are open mappings, so for any point $x \in M_0$ and $B_x$ a small open neighbourhood of $x$, one sees that the restriction morphism

$$a(B_x) : \mathcal{L}_0(f^{-1}(f(B_x))) \rightarrow \mathcal{L}_0(B_x) = \mathbb{C}$$

is an isomorphism. Indeed, by Lemma 4.2 in [5], note that $s \in S_0$ if and only if the restriction $\mathcal{L}_0|T(F_s)$ is trivial, where $T(F_s)$ is a small open tube $f^{-1}(D_s)$ about the fiber $F_s = f^{-1}(s)$, with $D_s$ a small disc centered at $s \in S$. □
The same proof yields the following more general result.

**Corollary 4.2.** For any \( L' \in T(S) \) one has
\[
(\mathcal{L}_0 \otimes \mathcal{L})|_{M_0} \simeq f_0^{-1}((\mathcal{F} \otimes \mathcal{L}')|_{S_0})
\]
with \( \mathcal{L} = f^{-1}(\mathcal{L}') \).

For all local systems \( \mathcal{L}' \in T(S) \) except finitely many, the exact sequence \( (4.1) \) and the equality \( (4.2) \) yield
\[
H^1(M, \mathcal{L}_0 \otimes \mathcal{L}) \simeq H^1(S, \mathcal{F} \otimes \mathcal{L}') \simeq H^1(S_0, \mathcal{L}'')
\]
where \( \mathcal{L}'' = j^{-1}(\mathcal{F} \otimes \mathcal{L}') = (\mathcal{F} \otimes \mathcal{L}')|_{S_0} \) is a rank one local system on \( S_0 \).

Note that the curve \( S \) in Theorem 2.7 satisfies \( \chi(S) \leq 0 \) and hence \( \chi(S_0) = \chi(S) - |\Sigma(\mathcal{F})| < 0 \). It follows by Prop.1.7, Section V in Arapura [1] that \( W_0 = f_0^*(\mathcal{T}(S_0)) \) is an irreducible component in the characteristic variety \( V_1(M_0) \) such that \( 1 \in W_0 \) and \( \dim W_0 = b_1(S_0) \geq 2 \).

With this notation, our main result is the following.

**Theorem 4.3.** Assume that \( M \) is a smooth quasi-projective irreducible complex variety. Let \( W = \rho \cdot f^*(\mathcal{T}(S)) \) be a translated \( d \)-dimensional irreducible component of the first characteristic variety \( V_1(M) \), with \( d > 0 \). Let \( \mathcal{L}_0 \) be the rank one local system on \( M \) corresponding to \( \rho \), \( \mathcal{F} = R^0f_*\mathcal{L}_0 \) and \( \Sigma(\mathcal{F}) \) the singular support of \( \mathcal{F} \). Set \( S_0 = S \setminus \Sigma(\mathcal{F}) \) and \( M_0 = f^{-1}(S_0) \). Assume moreover that \( M \) and \( M_0 \) are 1-formal.

Then there is a non-translated irreducible component \( W_0 \) of \( V_1(M_0) \), such that \( W \subset W_0 \) under the obvious inclusion \( T(M) \rightarrow T(M_0) \). In particular, for any local system \( \mathcal{L}_1 \in W \), except finitely many, there is a 1-form \( \alpha(\mathcal{L}_1) \in H^1(M, \mathcal{C}) \) such that \( \exp(\alpha(\mathcal{L}_1)) = \mathcal{L}_1 \) and \( \dim H^1(H^*(M_0, \mathcal{C}), \alpha_0(\mathcal{L}_1) \wedge) = \dim H^1(M, \mathcal{L}_1) \), where \( \alpha_0(\mathcal{L}_1) = \iota^*(\alpha(\mathcal{L}_1)) \), \( \iota : M_0 \rightarrow M \) being the inclusion.

**Proof.** With the above notation, apply Proposition 3.1 to the restriction \( f_0 : M_0 \rightarrow S_0 \) and to the associated component \( W_0 \). We set \( \mathcal{L}_1 = \mathcal{L}_0 \otimes \mathcal{L} \) and use \( (4.3) \) and Corollary 4.2 to get
\[
\dim H^1(M, \mathcal{L}_1) = \dim H^1(S, \mathcal{F} \otimes \mathcal{L}') = \dim H^1(S_0, \mathcal{L}'') =
\]
\[
= \dim H^1(M_0, \mathcal{L}_1|_{M_0}) = \dim H^1(H^*(M_0, \mathcal{C}), \alpha_0(\mathcal{L}_1) \wedge).
\]

The key point here is that Proposition 3.1 holds for all local systems \( \mathcal{L} \in W \) except finitely many, and not just for a generic local system in the sense of Zariski topology on \( T(M) \).

\[\square\]

**Remark 4.4.** One situation when clearly \( M \) and \( M_0 \) are 1-formal is the following. When \( M \) is a hypersurface arrangement complement \( M(A) \) in some \( \mathbb{P}^n \), one can view \( M_0 \) as a new hypersurface arrangement complement \( M(B) \), where \( B \) is obtained
from $\mathcal{A}$ by adding some additional components $H_W$ corresponding to the fibers in $f^{-1}(\Sigma(\mathcal{F}))$. Conversely, $\mathcal{A}$ is obtained from $\mathcal{B}$ by deleting the hypersurfaces in $H_W$. So any translated component $W$ in $\mathcal{V}_I(M)$ corresponds to a non-translated component $W_0$ in a richer arrangement $\mathcal{B} = \mathcal{A} \cup H_W$. Even if $\mathcal{A}$ is a hyperplane arrangement, we see no reason why the richer arrangement $\mathcal{B}$ should contain only hyperplanes.

Remark 4.5. The dimension of $W_0$ is exactly the generic dimension of $H^1(M, \mathcal{L})$ for $\mathcal{L} \in W$ plus one. Indeed, one has $\dim W_0 = -\chi(S_0) + 1$, since $S_0$ is clearly non-compact, see [5], Thm. 3.6.(i). Moreover, the generic dimension of $H^1(M_0, \mathcal{L}_0 \otimes \mathcal{L})$ is $-\chi(S_0)$, see [5], Thm. 3.6.(iv). On the other hand, since the generic dimension is realized outside a finite number of points on $W_0$, it follows that the generic dimension of $H^1(M_0, \mathcal{L}_0 \otimes \mathcal{L})$ coincides to the generic dimension of $H^1(M, \mathcal{L}_0 \otimes \mathcal{L})$.

Example 4.6. This is a basic example discovered by A. Suciu, see Example 4.1 in [20], the so called deleted $B_3$-arrangement. Consider the line arrangement in $\mathbb{P}^2$ given by the equation

$$xyz(x - y)(x - z)(y - z)(x - y - z)(x - y + z) = 0.$$ 

Then there is a 1-dimensional translated component $W$. In this case the new hypersurface $H_W$ is the line $x + y - z = 0$, and $M_0$ is exactly the complement of the $B_3$-arrangement.

The characteristic variety $\mathcal{V}_I(M_0)$ has a 2-dimensional component $W_0$ denoted by $\Gamma$ in Example 3.3 in [20]. In the notation of loc. cit. one has

$$\Gamma = \{(t, s, (st)^{-2}, s, t, (st)^{-1}, s^2, (st)^{-1}) \mid (s, t) \in (\mathbb{C}^*)^2\}.$$ 

An easy computation shows that $W$ corresponds to the translated 1-dimensional torus inside $W_0 = \Gamma$ given by $st = -1$.

The proof of Theorem 4.3 also gives the following result, which follows in the compact case from Simpson’s work [19] and in the non-proper case from Budur’s recent paper [2]. In both cases one should also use in addition a result in [9], saying that two irreducible components of $\mathcal{V}_I(M)$ intersect at most at finitely many points, all of them torsion points in $\mathbb{T}(M)$. Our proof below is much simpler and purely topological.

Theorem 4.7. Assume that $M$ is a smooth quasi-projective irreducible complex variety. Let $W$ be a $d$-dimensional irreducible component of the first characteristic variety $\mathcal{V}_I(M)$, with $d > 0$. Let $\mathcal{L} \in W$ be a rank one local system on $M$ such that

$$\dim H^1(M, \mathcal{L}) > \min_{\mathcal{L}_1 \in W} \dim H^1(M, \mathcal{L}_1).$$

Then $\mathcal{L}$ is a torsion point of the algebraic group $\mathbb{T}(M)$.
Proof. Consider first the case when $W$ is a non-translated component. Then there is a morphism $f : M \to S$ and a local system $\mathcal{L}' \in \mathcal{T}(S)$ such that $\mathcal{L} = f^{-1}(\mathcal{L}')$. The exact sequence (3.1) implies that $H^0(S, \mathcal{G} \otimes \mathcal{L}'') \neq 0$ where we use the same notation as in (3.1). On the other hand, Proposition 4.5 in [5] implies that

\[
H^0(S_0, \mathcal{G} \otimes \mathcal{L}'') \neq 0
\]

where $S_0 = S \setminus \Sigma(\mathcal{G})$. Choose a finite set of generators $\gamma_1, ..., \gamma_m$ for the group $\pi_1(S_0)$. Then the condition (4.4) implies that for each $j = 1, ..., m$, the monodromy $\lambda_j$ of the local system $\mathcal{L}'$ along the path $\gamma_j$ is the inverse of one of the eigenvalues of the monodromy operator $T_j$ of the geometric local system $\mathcal{G}|S_0$ along the path $\gamma_j$. Since the geometric local system $\mathcal{G}|S_0$ comes from an algebraic morphism, the Monodromy Theorem, see for instance [3], implies that all the eigenvalues of any monodromy operator $T_j$ are roots of unity. Hence the same is true for all $\lambda_j$, which shows that $\mathcal{L}'|S_0$ is a torsion point in $\mathcal{T}(S_0)$. The inclusion $\mathcal{T}(S) \to \mathcal{T}(S_0)$ shows that $\mathcal{L}'$ is a torsion point, and hence the same holds for $\mathcal{L} = f^{-1}(\mathcal{L}')$.

Consider now the case when $W$ is a translated component. It follows from the proof of Theorem 4.3 (for this part we do not need the 1-formality assumptions) that there is a Zariski open subset $M_0 \subset M$ and a non-translated component $W_0$ in $\mathcal{V}_1(M_0)$ such that $W \subset W_0$ under the natural inclusion $\mathcal{T}(M) \subset \mathcal{T}(M_0)$ given by $\mathcal{L} \mapsto \mathcal{L}|M_0$. Moreover, in this case $\mathcal{L} = \mathcal{L}_0 \otimes f^{-1}(\mathcal{L}')$ for some morphism $f : M \to S$ and a local system $\mathcal{L}' \in \mathcal{T}(S)$. The subset $M_0$ depends on the torsion local system $\mathcal{L}_0$ and on $f$, but not on $\mathcal{L}'$. The equality

\[
\dim H^1(M, \mathcal{L}) = \dim H^1(M_0, \mathcal{L}|M_0)
\]

for all such local systems $\mathcal{L}$ obtained by varying $\mathcal{L}'$, and the fact that along any component the jumps in dimension occur only at finitely many points, recall Theorem 2.7 for non-translated components and use Corollary 5.9 in [3] in the translated case, implies that $\mathcal{L}$ is a jumping point for dimension along $W$ if and only if $\mathcal{L}|M_0$ is a jumping point for dimension along $W_0$. The first part of this proof shows that in such a case $\mathcal{L}|M_0$ is a torsion point, and the inclusion $\mathcal{T}(M) \to \mathcal{T}(M_0)$ shows that the same holds for $\mathcal{L}$.

\[\square\]

Corollary 4.8. Let $M$ be a smooth, quasi-projective complex variety which is 1-formal. Let $W$ be an irreducible component of $\mathcal{V}_1(M)$ such that $1 \in W$ and $\dim W > 0$. If $\mathcal{L} \in W$ is not 1-admissible, then $\mathcal{L}$ is a torsion point in $\mathcal{T}(M)$. 

Note that the algebraic group of characters $\mathbb{T}(M)$ has a complex conjugation involution, denoted by $\mathcal{L} \mapsto \overline{\mathcal{L}}$ and satisfying
$$\dim H^k(M, \mathcal{L}) = \dim H^k(M, \overline{\mathcal{L}})$$
for all $k$. This follows simply by noting that the complex of finite dimensional $\mathbb{C}$-vector spaces used to compute the twisted cohomology $H^*(M, \mathcal{L})$ (resp. $H^*(M, \overline{\mathcal{L}})$) comes from a complex of real vector spaces such that the corresponding differentials $d^k(\mathcal{L})$ and $d^k(\overline{\mathcal{L}})$ are complex conjugate for all $k$.

This remark and the above Theorem 4.7 have the following consequence.

**Corollary 4.9.** Let $M$ be a quasi-projective manifold. Then, for any rank one local system $\mathcal{L} \in \mathbb{T}(M)$ one has $\dim H^1(M, \mathcal{L}) = \dim H^1(M, \mathcal{L}^{-1})$.

**Proof.** We can assume that either $\mathcal{L}$ or $\mathcal{L}^{-1}$ is in $\mathcal{V}_1(M)$. Since the situation is symmetric, assume that $\mathcal{L} \in \mathcal{V}_1(M)$. If $\mathcal{L}$ belongs to a strictly positive dimensional component $W$ of $\mathcal{V}_1(M)$, it follows from the description of such components given in [5], Corollary 5.8, that $W^{-1} \subset \mathcal{V}_1(M)$. Moreover, the generic dimension of $H^1(M, \mathcal{L})$ along $W$ and along $W^{-1}$ coincide by [3], Corollary 5.9. If $\mathcal{L}$ or $\mathcal{L}^{-1}$ is a jumping point for this dimension, it follows from Theorem 4.7 that both $\mathcal{L}$ and $\mathcal{L}^{-1}$ are torsion points in the group $\mathbb{T}(M)$. Hence $\mathcal{L}^{-1} = \overline{\mathcal{L}}$ and in this latter case the claim follows from the above remark.

On the other hand, if $\mathcal{L}$ is an isolated point of $\mathcal{V}_1(M)$, it follows from [1] that $\mathcal{L}$ corresponds to a unitary character, and hence again $\mathcal{L}^{-1} = \overline{\mathcal{L}}$ and we conclude as above.

\[\square\]

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