Inverse antiplane problem on \( n \) uniformly stressed inclusions

Y.A. Antipov
Department of Mathematics, Louisiana State University
Baton Rouge LA 70803, USA

Abstract

The inverse problem of antiplane elasticity on determination of the profiles of \( n \) uniformly stressed inclusions is studied. The inclusions are in ideal contact with the surrounding matrix, the stress field inside the inclusions is uniform, and at infinity the body is subjected to antiplane uniform shear. The exterior of the inclusions, an \( n \)-connected domain, is treated as the image by a conformal map of an \( n \)-connected slit domain with the slits lying in the same line. The inverse problem is solved by quadratures by reducing it to two Riemann-Hilbert problems on a Riemann surface of genus \( n-1 \). Samples of two and three symmetric and non-symmetric uniformly stressed inclusions are reported.

1 Introduction

Inverse boundary value problems for partial differential equations have been of interest for many researchers since the work by Riabuchinsky (1929) on determination of the boundary of a domain if the function is harmonic inside and its values and normal derivative are known on the boundary. An extensive survey of early results on inverse boundary value problems and their applications to the theory of filtration and hydroaerodynamics was given by Aksent’ev et al (1980). Inverse problems of elasticity on determination of the shapes of curvilinear cavities and inclusions with elastic constants different from those of the surrounding matrix when the fields inside the inclusions have prescribed properties have been attracting applied mathematicians, mechanical engineers, and materials scientists since the work by Eshelby (1957). He showed that if the unbounded elastic body is uniformly loaded at infinity, and the body has an elliptic or ellipsoidal inclusion with different elastic constants, then the stress field is uniform inside the inclusion. Eshelby conjectured that there do not exist other shapes of a single inclusion with such a property. Sendeckyj (1970) proved this conjecture in the plane and anti-plane cases. An alternative proof for the antiplane case by the method of conformal mappings was proposed by Ru and Schiavone (1996). The model problem on a plane of finite thickness reinforced by a single inclusion of circular cross-section and subjected to far-field anti-plane shear (mode III) loading (as well as modes I/II loading) was considered by Chaudhuri (2003).

Cherepanov (1974) studied the inverse elasticity problem for a plane uniformly loaded at infinity and having \( n \) holes. The holes boundaries are subjected to constant normal and tangential traction; these boundaries are not prescribed and have to be determined from the condition that the tangential normal stress \( \sigma_t = \sigma = \text{const} \) in all the contours. Cherepanov employed the method of conformal mappings to transform an \( n \)-connected slit domain into the elastic domain and reduced the problem to two homogeneous Schwarz problems of the theory of analytic functions on \( n \) slits. These problems were solved
(Cherepanov, 1974) for the symmetric case of two holes. To solve the Cherepanov problem for any \( n \)-connected domain, Vigdergauz (1976) proposed to employ a circular map from the exterior of \( n \)-circles onto the \( n \)-connected elastic domain, integral equations, and the method of least squares for their numerical solution. An explicit representation in terms of the Weierstrass elliptic function for the profile of an inclusion in the case of a doubly periodic structure was given by Grabovsky and Kohn (1995). Recently, Antipov (2017) developed a method of the Riemann-Hilbert problem on a Riemann surface of genus \( n-1 \) to reconstruct a family of conformal maps for the Cherepanov problem in the general case of two and three equal-strength cavities and for the case \( n \geq 4 \) when the preimage of the plane with holes is a slit domain with the slits lying in the real axis.

Kang et al (2008) analyzed the case of two antiplane inclusions with the Eshelby uniformity property. They employed the Weierstrass zeta function and the Schwarz-Christoffel formula to determine the profile of two symmetric inclusions. A method of Laurent series and a conformal mapping from an annulus to a doubly connected domain to reconstruct the shape of two inclusions with uniform stresses was applied by Wang (2012). Other numerical approaches were applied by Liu (2008) and Dai et al (2017).

In this paper we aim to propose an exact method of conformal mappings and the Riemann-Hilbert problem on a Riemann surface of genus \( n-1 \) for the inverse antiplane problem on \( n \) inclusions. The inclusions may have different shear moduli and are in ideal contact with the surrounding elastic matrix subjected at infinity to uniform antiplane shear \( \tau_{13} = \tau_{1}^{\infty} \) and \( \tau_{23} = \tau_{2}^{\infty} \). The profiles of the inclusions are not prescribed and have to be determined from the condition that the stress field inside all the inclusions is uniform, \( \tau_{13} = \tau_{1} \) and \( \tau_{23} = \tau_{2} \).

In section 2, we assume that such inclusions exist and treat the \( n \)-connected exterior of the inclusions, \( D^e \), as the image of a slit domain \( D^e \) with the slits lying in the real axis. We reduce the problem of determination of the conformal maps to two inhomogeneous Schwarz problems to be solved consecutively. We analyze the particular case \( n = 1 \) in section 3 and show that the profile of the inclusion is an ellipse by employing two maps, a circular and a slit conformal mappings. Section 4 treats the case \( n = 2 \), solves the two Schwarz problems by reducing them to two Riemann-Hilbert problems on an elliptic Riemann surface and derives the associated conformal map in terms of singular and elliptic integrals. In section 5 we analyze the case \( n = 3 \) and derive the conformal mapping in terms of singular and genus-2 hyperelliptic integrals. A class of \( n \)-connected domains \( D^*_e \) which may be considered as images by a conformal map of \( n \) slits lying in the same line is considered in section 6 (for \( n = 1, 2, 3 \), each domain \( D^e \) belongs to this class). A closed-form representation in terms of hyperelliptic integrals for a family of such conformal mappings is constructed by employing the theory of the Riemann-Hilbert problem on a genus-(\( n - 1 \)) Riemann surface.

## 2 Formulation

Consider the following inverse problem of elasticity:

Suppose an infinite isotropic solid contains \( n \) curvilinear inclusions \( D_0, D_1, \ldots, D_{n-1} \). Let the shear moduli of the inclusions \( D_j \) and the matrix \( D^e = \mathbb{R}^2 \setminus \overline{D}, (D = \bigcup_{j=0}^{n-1} D_j) \) be \( \mu_j \) and \( \mu \), respectively. It is assumed that the inclusions are in ideal contact with the matrix, and the solid \( D^e \cup D \) is in a state of antiplane shear due to constant shear stresses applied at infinity, \( \tau_{13} = \tau_{1}^{\infty}, \tau_{23} = \tau_{2}^{\infty} \). It is required to determine the boundaries of the inclusions, \( L_j \), such that the stresses \( \tau_{13} \) and \( \tau_{23} \) are constant in all the inclusions \( D_j \),
\[ \tau_{13} = \tau_1, \tau_{23} = \tau_2, \ j = 0, 1, \ldots, n - 1. \]

Let \( u \) be the \( x_3 \)-component of the displacement vector. Then \( \tau_{j3} = \mu \partial u / \partial x_j \ (j = 1, 2) \), \((x_1, x_2) \in D^e \), and

\[
 u \sim \mu^{-1}(\tau_1^\infty x_1 + \tau_2^\infty x_2) + \text{const}, \ x_1^2 + x_2^2 \to \infty. \tag{2.1}
\]

Since the stresses \( \tau_{12} \) and \( \tau_{13} \) are constant in the inclusions, the \( x_3 \)-displacements \( u_j \) for \((x_1, x_2) \in D_j \) are linear functions

\[
 u_j = \mu_j^{-1}(\tau_1 x_1 + \tau_2 x_2) + d_j', \ (x_1, x_2) \in D_j, \ j = 0, 1, \ldots, n - 1, \tag{2.2}
\]

and \( d_j' \) are real constants. The boundary conditions of ideal contact imply that the traction component \( \tau_{ij3} \) and the \( x_3 \)-component of the displacement are continuous through the contours \( L_j \),

\[
 \mu \frac{\partial u}{\partial \nu} = \mu_j \frac{\partial u_j}{\partial \nu}, \ u = u_j, \ (x_1, x_2) \in L_j, \ j = 0, 1, \ldots, n - 1, \tag{2.3}
\]

where \( \frac{\partial}{\partial \nu} \) is the normal derivative.

The displacement \( u \) and \( u_j \) are harmonic functions in the domains \( D^e \) and \( D_j \), respectively. Let \( v \) and \( v_j \) be their harmonic conjugates and denote \( z = x_1 + i x_2 \). Then the functions \( \phi(z) = u(x_1, x_2) + iv(x_1, x_2) \) and \( \phi_j(z) = u_j(x_1, x_2) + iv_j(x_1, x_2) \) are analytic in the corresponding domains. In terms of these functions the ideal contact boundary conditions can be written as

\[
 \frac{\kappa_j + 1}{2} \phi_j(z) - \frac{\kappa_j - 1}{2} \phi_j(z) = \phi(z) + ib_j, \ z \in L_j, \ j = 0, 1, \ldots, n - 1, \tag{2.4}
\]

where \( \kappa_j = \mu_j / \mu \), and \( b_j \) are real constants. To verify the equivalence of the boundary conditions (2.3) and (2.4), one may use the Cauchy-Riemann conditions \( \frac{\partial u}{\partial \nu} = \frac{\partial u_j}{\partial \nu} \), where \( \frac{\partial}{\partial \nu} \) is the tangential derivative. Since the functions \( u_j \) are known and given by (2.2), due to the Cauchy-Riemann conditions they are defined up to arbitrary constants by

\[
 \phi_j(z) = \frac{\tau z}{\mu_j} + d_j, \ z \in D_j, \ j = 0, 1, \ldots, n - 1, \tag{2.5}
\]

where \( \tau = \tau_1 - i \tau_2, \ d_j = d_j' + i d_j'^{\prime}, \ d_j'' \) are real constants. In view of the relations (2.5), instead of the function \( \phi(z) \), it is convenient to deal with the function \( f(z) = \phi(z) - \tau z / \mu \), \( z \in D^e \). The new function \( f(z) \) is analytic in the domain \( D^e \), satisfies the boundary condition

\[
 f(z) = \frac{1}{\lambda_j} \text{Re}[\tau z] + d_j' + ia_j, \ z \in L_j, \ j = 0, 1, \ldots, n - 1, \tag{2.6}
\]

and, since \( \phi(z) \sim \tau^\infty z / \mu + \text{const}, \ z \to \infty \), the condition at infinity

\[
 f(z) \sim \frac{(\tau^\infty - \tau)z}{\mu} + \text{const}, \ z \to \infty. \tag{2.7}
\]

Here, \( \lambda_j = \mu_j / (1 - \kappa_j), \ a_j = \kappa_j d_j'^{\prime} - b_j \) are real constants, and \( \tau^\infty = \tau_1^\infty - i \tau_2^\infty \).

Let \( z = \omega(\zeta) \) be a conformal map that transforms an \( n \)-connected canonical domain \( D^e \) into the physical domain \( D^p \). An example of the domain \( D^p \) is the exterior of \( n \) circles \( C_j \ (j = 0, 1, \ldots, n - 1) \). By scaling and rotation, it is always possible to achieve that one of the circles say, \( C_0 \), is of unit radius and centered at the origin and, in addition, the center of another circle say, \( C_1 \), falls on the real axis. If the circular map meets the condition
If \( \omega(\infty) = \infty \), then the radius of \( \mathcal{C}_1 \), the complex centers and the radii of the rest \( n - 2 \) circles cannot be selected arbitrarily. If this map is chosen, the problem of determining the function \( f(\omega(\zeta)) \) and the map \( \omega(\zeta) \) itself reduces to two Schwarz problem for the circular domain \( \mathbb{C} \setminus \bigcup_{j=0}^{n-1} K_j \) with \( \partial K_j = C_j \). It may be solved by reducing the problem to two Riemann-Hilbert problems of the theory of automorphic functions (Chibrikova & Silvestrov, 1978; Mityushev & Rogosin, 2000; Antipov & Silvestrov, 2007; Antipov & Crowdy, 2007).

Another choice of the canonical domain \( \mathcal{D}^e \) is the exterior of \( n \) slits \( l_j, j = 0, 1, \ldots, n - 1 \), such that \( L_j \) are the images of the slits \( l_j \) (Antipov & Silvestrov, 2007; Antipov, 2017). The advantage of a slit map over a circular map is that the solution is delivered by quadratures, not in a series form. The disadvantage is that although the slit map method works for any simply-, doubly-, and triply-connected domains, for \( n \geq 4 \), the method is applicable for a certain subclass of domains only, namely when all the slits lie in the same line.

It is known (Keldysh, 1939; Courant, 1950) that there exists an analytic function \( z = \omega(\zeta) \) that conformally maps the extended complex \( \zeta \)-plane \( \mathbb{C} \cup \infty \) cut along \( n \) segments parallel to the real \( \zeta \)-axis onto the \( n \)-connected domain \( \mathcal{D}^e \) in the \( z \)-plane. For this map, the infinite point \( z = \infty \) is the image of a certain point \( \zeta = \zeta_\infty \in \mathcal{D}^e \), and in the vicinity of that point the conformal map \( \omega(\zeta) \) can be represented as

\[
\omega(\zeta) = \frac{c_1 - 1}{\zeta - \zeta_\infty} + c_0 + \sum_{j=1}^{\infty} c_j (\zeta - \zeta_\infty)^j
\]

if \( \zeta_\infty \) is a finite point, and

\[
\omega(\zeta) = c_{-1} \zeta + c_0 + \sum_{j=1}^{\infty} \frac{c_j}{\zeta}
\]

otherwise. Here, \( c_{-1} = c_{-1}' + i c_0'' \).

Denote \( f(\omega(\zeta)) = F(\zeta) \). From the boundary condition \( (2.6) \) regardless of which map is employed we deduce that the functions \( F(\zeta) \) and \( \omega(\zeta) \) satisfy the following two Schwarz problems to be solved consecutively.

Find two functions \( F(\zeta) \) and \( \omega(\zeta) \) analytic in the domain \( \mathcal{D}^e \) and continuous up to the boundary \( l = \bigcup_{j=0}^{n-1} l_j \) such that

\[
\text{Im} F(\zeta) = a_j, \quad \zeta \in l_j, \quad j = 0, 1 \ldots, n - 1,
\]

and

\[
\text{Re}[\bar{\tau} \omega(\zeta)] = \lambda_j \text{Re} F(\zeta) - d_j', \quad \zeta \in l_j, \quad j = 0, 1 \ldots, n - 1.
\]

In the vicinity of the point \( \zeta_\infty \), both of the functions have a simple pole. If the point \( \zeta_\infty \) is a finite point and \( \omega(\zeta) \) admits the representations \( (2.8) \), then

\[
F(\zeta) \sim \frac{\bar{\tau}_\infty - \bar{\tau}}{\mu} \frac{c_{-1}}{\zeta - \zeta_\infty}, \quad \zeta \to \zeta_\infty.
\]

Otherwise, if \( \zeta_\infty = \infty \) and \( (2.9) \) holds, then

\[
F(\zeta) \sim \frac{\bar{\tau}_\infty - \bar{\tau}}{\mu} c_{-1} \zeta, \quad \zeta \to \infty.
\]

In addition, the function \( \omega : l_j \to L_j \ (j = 0, \ldots, n - 1) \) has to be univalent, and the interiors of the images of the contours \( l_j \), the domains \( D_j \), are disjoint sets.

In what follows we apply both maps in the case \( n = 1 \) and proceed with the second choice of the parametric domain \( \mathcal{D}^e \) by employing a slit map to restore the contours \( L_j \ (j = 0, 1 \ldots, n - 1) \) when \( n \geq 2 \).
3 Single inclusion

We start with the simplest case of a single inclusion and derive the solution by exploiting a circular and a slit map.

3.1 \( n = 1 \): a circular map

Without loss of generality, \( C_0 \) is the unit circle centered at the origin, \( \zeta_{\infty} = \infty \), and \( a_0 = 0 \). The solution of the Schwarz problem (2.10), (2.13) for the unit circle \( C_0 \) is given by

\[
F(\zeta) = \beta_0 - i\beta_1 \zeta + i\beta_1 \zeta^{-1}, \tag{3.1}
\]

where

\[
\beta_1 = \beta'_1 + i\beta''_1 = \frac{\zeta_{\infty} - \bar{\tau}}{\mu} - i\kappa_0,
\]

and \( \zeta_{-1} \) and \( \beta_0 \) are real constants. The solution of the second Schwarz problem (2.11), (2.9) can be represented in the form

\[
\bar{\tau}_{\omega}(\zeta) = \gamma_{-1} \zeta^{-1} + \gamma_0 + \gamma_1 \zeta, \tag{3.3}
\]

where \( \gamma_j = \gamma'_j + i\gamma''_j \), \( j = -1, 0, 1 \). On substituting the expressions (3.3) and (3.1) into the boundary condition (2.11) and replacing \( \zeta \) by \( e^{i\varphi} \), \( 0 \leq \varphi \leq 2\pi \), we derive

\[
\gamma'_0 = (\beta_0 - d'_0) \frac{\mu_0}{1 - \kappa_0}, \quad \gamma'_{-1} + \gamma'_1 = \frac{2\beta''_1 \mu_0}{1 - \kappa_0}, \quad \gamma''_{-1} - \gamma''_1 = \frac{2\beta'_1 \mu_0}{1 - \kappa_0}. \tag{3.4}
\]

Finally, by using formula (2.9) and the relations (3.4) we determine the function \( \omega(\zeta) \) up to an additive complex constant \( \gamma \)

\[
\omega(\zeta) = c_{-1} \left( \zeta + \frac{\delta}{\zeta} \right) + \gamma, \tag{3.5}
\]

where

\[
\delta = \frac{2\kappa_0 \tau_{\infty} - (\kappa_0 + 1) \tau}{(1 - \kappa_0)\tau}. \tag{3.6}
\]

Let \( \kappa_0 \neq 1, \tau \neq 0, \) and \( \delta \neq \pm 1 \). Then a point \( z = \omega(\zeta) \) traces an ellipse \( L_0 \) whenever the point \( \zeta \) traverses the unit circle \( C_0 \).

3.2 \( n = 1 \): a slit map

Let \( z = \omega(\zeta) \) be a conformal map that transforms the two-sided segment \( l_0 = l_0^+ \cup l_0^- \), \( l_0^\pm = [-1, 1]^\pm \), into the contour \( L_0 \) such that the point \( \zeta = \infty \) falls into the infinite point of the \( z \)-plane. Such a map is defined up to a real parameter, and it is assumed that \( \omega(\zeta) \sim c_{-1} \zeta + c_0 + \ldots, \zeta \to \infty \), and \( \text{Im} \ c_{-1} = 0 \). Fix a single branch \( q(\zeta) \) of this function in the \( \zeta \)-plane cut along the segment \( [-1, 1] \) by the condition \( q(\zeta) \sim \zeta, \zeta \to \infty \). Let \( \mathbb{C}_1 \) and \( \mathbb{C}_2 \) be two copies of the complex \( \zeta \)-plane with the cut \( l_0 \). The two sheets are glued together and form a genus-0 Riemann surface \( \mathcal{R} \) of the algebraic function \( u^2 = \zeta^2 - 1 \) such that the loop \( l_0 \) is the symmetry line of the surface. Denote by \( (\zeta_*, u_*) = (\zeta, -u(\zeta)) \) the point symmetrical to a point \( (\zeta, u) \), and let in the first sheet, \( (\zeta, u) = (\zeta, q(\zeta)) \). Introduce two functions on the surface \( \mathcal{R} \)

\[
\Phi_1(\zeta, u) = \begin{cases} F(\zeta), & (\zeta, u) \in \mathbb{C}_1, \\ \widehat{F}(\zeta), & (\zeta, u) \in \mathbb{C}_2, \end{cases} \tag{3.7}
\]
and

$$\Phi_2(\zeta, u) = \begin{cases} \frac{i \tau \omega(\zeta)}{\mu}, & (\zeta, u) \in \mathbb{C}_1, \\ -\frac{i \tau \omega(\zeta)}{\mu}, & (\zeta, u) \in \mathbb{C}_2. \end{cases}$$  \hspace{1cm} (3.8)$$

These two functions satisfy the symmetry condition

$$\Phi_j(\zeta, u_*) = \Phi_j(\zeta, u), \quad (\zeta, u) \in \mathcal{R}, \quad j = 1, 2. \hspace{1cm} (3.9)$$

Denote next $\Phi_j^+(\xi, v) = \Phi_j^-(\xi, v)$, $(\xi, v) \in l_0$, where $v = u(\xi)$, $\xi \in l_0$. The two functions are solutions to the following Riemann-Hilbert problems on the contour $l_0$:

$$\Phi_1^+(\xi, v) - \Phi_1^-(\xi, v) = 0, \quad (\xi, v) \in l_0, \hspace{1cm} (3.10)$$

and

$$\Phi_1^+(\xi, v) - \Phi_1^-(\xi, v) = 2i G_0(\xi), \quad (\xi, v) \in l_0, \hspace{1cm} (3.11)$$

Here, $G_0(\xi) = \lambda_0 |\text{Re} F(\xi) - d'_0|$. The general solution of the homogeneous problem (3.9), (3.10) is given by

$$\Phi_1(\zeta, u) = \beta_0 + \beta_1 \zeta + \beta_2 i u(\zeta), \quad (\zeta, u) \in \mathcal{R}, \hspace{1cm} (3.12)$$

where $\beta_0$, $\beta_1$, and $\beta_2$ are real constants. The condition at $\infty$ is satisfied if the constants $\beta_1$ and $\beta_2$ are chosen as

$$\beta_1 = \frac{\tau_0^\infty - \tau_1}{\mu} c_{-1}, \quad \beta_2 = \frac{\tau_2 - \tau_0^\infty}{\mu} c_{-1}. \hspace{1cm} (3.13)$$

The solution of the inhomogeneous Riemann-Hilbert problem (3.9), (3.11) can be represented in terms of a singular integral with the Weierstrass kernel $rac{1}{2}(1 + u/v)/(\xi - \zeta)$

$$\Phi_2(\zeta, u) = \gamma_0 + \gamma_1 \zeta + i \gamma_2 u(\zeta) + \frac{1}{2\pi} \int_{l_0} G_0(\xi) \left( 1 + \frac{u}{v} \right) \frac{d\xi}{\xi - \zeta}, \hspace{1cm} (3.14)$$

where $\gamma_0$, $\gamma_1$, and $\gamma_2$ are arbitrary real constants. The loop $l_0$ is oriented such that the exterior of $l_0$ is on the left. To compute the integral in formula (3.14), we rewrite the density $G_0(\zeta)$ as

$$G_0(\zeta) = \lambda_0 \left| g_0(\zeta) \pm g_1(\xi) \right|, \quad \zeta = \xi \pm i 0 \in l_0^\pm, \hspace{1cm} (3.15)$$

where

$$g_0(\xi) = \beta_1 \xi, \quad g_1(\xi) = -\beta_2 |q(\xi)|. \hspace{1cm} (3.16)$$

By utilizing the relation (3.15) we deduce from formula (3.14)

$$\Phi_2(\zeta, u) = \gamma_0 + \gamma_1 \zeta + i \gamma_2 u(\zeta) + \lambda_0 [\beta_1 u(\zeta) \Lambda_1(\zeta) - \beta_2 \Lambda_2(\zeta)], \hspace{1cm} (3.17)$$

where

$$\Lambda_1(\zeta) = \frac{1}{\pi} \int_{-1}^{1} \frac{\xi d\xi}{\sqrt{1 - \xi^2}(\xi - \zeta)}, \quad \Lambda_2(\zeta) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - \xi^2} d\xi}{\xi - \zeta}. \hspace{1cm} (3.18)$$

These integrals are computed by the theory of residues

$$\Lambda_1(\zeta) = \frac{i \zeta}{q(\zeta)} - i, \quad \Lambda_2(\zeta) = q(\zeta) - \zeta. \hspace{1cm} (3.19)$$
On substituting the expressions (3.19) into (3.17) we determine the solution to the Riemann-Hilbert problem (3.11) explicitly

$$\Phi_2(\zeta, u) = \gamma_0 + \gamma_1 \zeta + i \gamma_2 u(\zeta) + \lambda_0 [\zeta - q(\zeta)] \left[ \beta_2 + \frac{i \beta_1 u(\zeta)}{q(\zeta)} \right]. \quad (3.20)$$

Assuming that $(\zeta, u) \in \mathbb{C}_1$ we derive the formula for the conformal map

$$i \tau \omega(\zeta) = \gamma_0 + \gamma_1 \zeta + i \gamma_2 q(\zeta) + \lambda_0 [\zeta - q(\zeta)] (i \beta_1 + \beta_2), \quad (3.21)$$

where the parameters $\gamma_1$ and $\gamma_2$ are chosen such that the function $\omega(\zeta)$ meets the condition (2.9) at infinity, $\gamma_1 = c_{-1} \tau_2$, $\gamma_2 = c_{-1} \tau_1$. Finally, we substitute the parameters $\gamma_1$, $\gamma_2$, $\beta_1$, and $\beta_2$ into formula (3.21) to deduce

$$\omega(\zeta) = \gamma' + c_{-1} \left( m_1 \zeta + m_2 \sqrt{\zeta^2 - 1} \right), \quad (3.22)$$

where $\gamma' = -i \gamma_0 / \tau$ is an additive constant and

$$m_1 = \frac{1}{\tau} \left[ \frac{\kappa_0}{1 - \kappa_0} (\tau^\infty - \tau) - i \tau_2 \right], \quad m_2 = \frac{1}{\tau} \left[ \frac{\kappa_0}{1 - \kappa_0} (\tau^\infty - \tau) + \tau_1 \right]. \quad (3.23)$$

When the points $\zeta = \xi \pm i0$ traverses the contours $l_0^\pm$, the corresponding points

$$z = c_{-1} \left( m_1 \xi \pm i m_2 \sqrt{1 - \xi^2} \right) \quad (3.24)$$

outline the boundary $L_0$ of a uniformly stressed inclusion $D_0$.

In Fig. 1, we show the profiles of a single inclusion generated by the slit map (line 1) and the circular map (line 2) for $\kappa = 5$, $c_{-1} = 1$, and the loading parameters $\tau_1/\mu = 1$, $\tau_2/\mu = 1$, $\tau_0^\infty/\mu = -1$, and $\tau_2^\infty/\mu = 1$. The contours are cofocal ellipses and coincide if $c_{-1}^\pm = \frac{1}{2} c_{-1}$, where $c_{-1}^\pm$ and $c_{-1}$ are the scaling parameter $c_{-1}$ for the slit and circular maps, respectively.

4 Two inclusions

Every doubly connected domain $D^e$ may be interpreted as the image by a conformal map $z = \omega(\zeta)$ of a slit domain $D^e$, the extended $\zeta$-plane cut along two segments $l_0 = [-1, -k] + \{1, k] -$, and $l_1 = [k, 1] + \{1, k] -$, $0 < k < 1$. Moreover, it is always possible to choose the map such that the preimage $\zeta_\infty$ of the infinite point in the $z$-plane falls into the open segment $(-k, k)$, $\zeta_\infty \in (-k, k)$. We select the single branch $q(\zeta)$ of the function

$$u^2 = (\zeta^2 - 1)(\zeta^2 - k^2) \zeta, \quad \zeta \in \zeta_\infty^\infty \in (-k, k).$$

On the banks of the cuts the function $q(\zeta)$ is pure imaginary, $q(\zeta) = \mp i |q(\zeta)|$, $\zeta \in l_0^\pm = [-1, -k]^\pm$, and $q(\zeta) = \pm i |q(\zeta)|$, $\zeta \in l_1^\pm = [k, 1]^\pm$. It will be convenient to deal with a new function

$$F_0(\zeta) = F(\zeta) - \frac{c}{\zeta - \zeta_\infty}, \quad c = \frac{\tau^\infty - \tau}{\mu} c_{-1}. \quad (4.1)$$

The new function has a removable singularity at the point $\zeta = \zeta_\infty$ and satisfies the boundary condition

$$\text{Im} F_0(\zeta) = a_j - \text{Im} \left( \frac{c}{\zeta - \zeta_\infty} \right), \quad \zeta \in l_j, \quad j = 0, 1. \quad (4.2)$$
Figure 1: Cases $n = 1$ and $n = 2$ when $\tau_1/\mu = \tau_2/\mu = -\tau_1^\infty/\mu = \tau_2^\infty/\mu = 1$, $\gamma' = 0$, and $c_{-1} = 1$. (a): $n = 1$. Line 1 is generated by the slit map, and line 2 is generated by the circular map. (b) – (d): $n = 2$, $k = 0.5$, $\zeta_\infty = 0.3k$, $a_0 = \rho_0 = 0$. (b): $\kappa_0 = \kappa_1 = 5$. (c): $\kappa_0 = 0.5$, $\kappa_1 = 5$. (d): $\kappa_0 = 0.5$, $\kappa_1 = 0.2$.

Let $\mathcal{R}$ be the elliptic surface of the algebraic function $u^2 = (\zeta^2 - 1)(\zeta^2 - k^2)$ symmetric with respect to the contour $l = l_0 \cup l_1$. Similarly to the case $n = 1$ we introduce the function $\Phi_1(\zeta, u)$ on the surface $\mathcal{R}$ by

$$
\Phi_1(\zeta, u) = \begin{cases} 
F_0(\zeta), & (\zeta, u) \in \mathbb{C}_1, \\
F_0(\bar{\zeta}), & (\zeta, u) \in \mathbb{C}_2.
\end{cases} \quad (4.3)
$$

This function is analytic everywhere in the surface $\mathcal{R}$ including the two points with the affix $\zeta = \zeta_\infty$ except for the contours $l_0$ and $l_1$ and is symmetric with respect to these loops. The limit values of the function $\Phi_1(\zeta, u)$ on the cuts $l_j$ satisfy the Riemann-Hilbert boundary condition

$$
\Phi_1^+(\xi, v) - \Phi_1^-(\xi, v) = 2i \left[ a_j - \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \right], \quad (\xi, v) \in l_j, \quad j = 0, 1. \quad (4.4)
$$

The general solution of this problem can be written in the form

$$
\Phi_1(\zeta, u) = \beta_0 + \beta_1 \zeta + \frac{1}{2\pi \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right)} \int_{l_j} \left[ a_j - \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \right] \left( 1 + \frac{u}{v} \right) \frac{d\xi}{\xi - \zeta}, \quad (4.5)
$$

where $u = u(\zeta)$, $v = v(\xi)$, and $\beta_j$ are real constants. The function $\Phi_1(\zeta, u)$ is required to be bounded at the two infinite points $(\infty, \infty)_1$ and $(\infty, \infty)_2$ of the surface $\mathcal{R}$; these two
points are symmetric with respect to the contour $l$. Since the constants $\beta_j$ are real and $v(\xi)$ is pure imaginary in the loops $l_j$, the function $\Phi_1(\zeta, u)$ is bounded at infinity if and only if the following conditions hold:

$$\beta_1 = 0, \quad \sum_{j=0}^{1} \int_{l_j} \left[ a_j - \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \right] \frac{d\xi}{v} = 0. \quad (4.6)$$

The symmetry of the contours $l_0$ and $l_1$ and the second condition in (4.6) imply

$$a_0 - a_1 = \left( \int_{l_j} \frac{d\xi}{|q(\xi)|} \right)^{-1} \left( \int_{l_j} \frac{d\xi}{|q(\xi)|} \right) \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \frac{d\xi}{(\xi - \zeta)v}. \quad (4.7)$$

If the constants $a_0$ and $a_1$ satisfy the condition (4.7), then the function

$$\Phi_1(\zeta, u) = \beta_0 + \frac{u}{2\pi} \sum_{j=0}^{1} \int_{l_j} \left[ a_j - \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \right] \frac{d\xi}{(\xi - \zeta)v}. \quad (4.8)$$

is bounded at infinity and defines the general solution of the problem (4.14). For the next step we need to determine the function $F_0(\zeta)$ on the sides of the loops $l_0$ and $l_1$,

$$\text{Im} F_0(\xi \pm i0) = a_j - \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right), \quad \text{Re} F_0(\xi \pm i0) = \beta_0 \pm (-1)^j g_1(\xi), \quad \xi \pm i0 \in l_j^\pm, \quad j = 0, 1. \quad (4.9)$$

Here,

$$g_1(\xi) = \frac{|q(\xi)|}{\pi} \sum_{j=0}^{1} (-1)^j \int_{l_j^\pm} \left[ a_j - \text{Im} \left( \frac{c}{\eta - \zeta_\infty} \right) \right] \frac{d\eta}{|q(\eta)||\eta - \xi|}. \quad (4.10)$$

Note that the first formula in (4.9) is consistent with the boundary condition (4.2).

Consider now the second Schwarz problem (2.11), (2.8) for the conformal mapping $\omega(\zeta)$. As with the first Schwarz problem, we separate the singular part from the function $\omega(\zeta)$ by writing

$$\omega(\zeta) = \frac{c - 1}{\zeta - \zeta_\infty} + \omega_0(\zeta). \quad (4.11)$$

With this splitting, the second Schwarz problem becomes

$$\text{Im}[i\tau \omega_0(\zeta)] = G_j(\zeta) + \rho_j, \quad \zeta \in l_j, \quad j = 0, 1, \quad (4.12)$$

where

$$G_j(\zeta) = \lambda_j[g_0(\xi) \pm (-1)^j g_1(\xi)], \quad \zeta = \xi \pm i0 \in l_j^\pm,$$

$$\rho_j = \lambda_j \rho_j', \quad \rho_j' = \beta_0 - d_j', \quad g_0(\xi) = \text{Re} \left[ \frac{c - 1}{\xi - \zeta_\infty} \left( \tilde{\tau} - \frac{\xi}{\mu} \right) \right]. \quad (4.13)$$

In order to rewrite the Schwarz problem (4.12) as a Riemann-Hilbert problem, we introduce a new function on the Riemann surface $\mathcal{R}$ by

$$\Phi_2(\zeta, u) = \begin{cases} 
    i\tau \omega_0(\zeta), & (\zeta, u) \in \mathbb{C}_1, \\
    -i\tau \omega_0(\zeta), & (\zeta, u) \in \mathbb{C}_2.
\end{cases} \quad (4.14)$$

The function $\Phi_2(\zeta, u)$ is analytic everywhere on the surface $\mathcal{R}$ (including the points $\zeta_\infty$ and $\infty$) apart from the loops $l_j$, and on its banks it satisfies the boundary condition

$$\Phi_2^+(\xi, v) - \Phi_2^-(\xi, v) = 2i[G_j(\xi) + \rho_j], \quad \xi \in l_j, \quad j = 0, 1. \quad (4.15)$$
The general solution to this problem is
\[ \Phi_2(\zeta, u) = \gamma_0 + \gamma_1 \zeta + \frac{1}{2\pi} \sum_{j=0}^{1} \int_{l_j} [G_j(\eta) + \rho_j] \left(1 + \frac{u}{v}\right) \frac{d\eta}{\eta - \zeta}. \]  
(4.16)

The function \( \Phi_2(\zeta, u) \) is bounded at the two infinite points of the surface if and only if
\[ \gamma_1 = 0, \quad \rho_0 - \rho_1 = -\left(\int_k^1 \frac{d\xi}{|q(\xi)|}\right)^{-1} \left(\lambda_0 \int_{-1}^{-k} - \lambda_1 \int_k^1\right) g_0(\xi) d\xi. \]  
(4.17)

The solution of the Riemann-Hilbert problem \( \text{(4.15)} \) on the first sheet of the surface \( \mathcal{R} \) is the function \( i\tau \omega_0(\zeta) \). By integrating over the contours \( l_j^+ \) and \( l_j^- \) in \( \text{(4.16)} \) and employing the first relation in \( \text{(4.13)} \) we eventually deduce
\[ i\bar{\tau} \omega_0(\zeta) = \gamma_0 + \sum_{j=0}^{1} (-1)^j \lambda_j \int_{l_j^+} \left\{g_1(\eta) + \frac{iq(\zeta)}{|q(\eta)|} g_0(\eta) + \rho_j\right\} \frac{d\eta}{\eta - \zeta}. \]  
(4.18)

The contours \( L_0 \) and \( L_1 \) can be reconstructed by letting a point \( \zeta \) run along the loops \( l_0 \) and \( l_1 \). On using the Sokhotski-Plemelj formulas and the relation \( \text{(4.11)} \) we derive
\[ \omega(\xi \pm i0) = \gamma' + \frac{c_{-1}}{\xi - \zeta_\infty} - \frac{i\mu}{\pi \tau} \left\{ \sum_{j=0}^{1} (-1)^j \tilde{\lambda}_j \int_{l_j^+} \left[g_1(\eta) \pm (-1)^m \frac{|q(\xi)|}{|q(\eta)|} g_0(\eta) + \rho_j\right] \right\} \frac{d\eta}{\eta - \zeta} \]
\[ + \pi i \lambda_m \left[g_0(\xi) + \rho_m^l \pm (-1)^m g_1(\xi)\right], \quad \zeta = \xi \pm i0 \in l_m^\pm, \quad m = 0, 1, \]  
(4.19)

where \( \tilde{\lambda}_j = \kappa_j/(1 - \kappa_j) \), \( \gamma' = -i\gamma_0/\bar{\tau} \) is an arbitrary complex constant.

The conformal map, in addition to the free additive complex constant \( \gamma' \), the scaling parameter \( c_{-1} = c'_{-1} + ic''_{-1} \), and the parameters \( a_0 \) and \( \rho_0 \), is defined up to two free real parameters, \( k \in (-1, 1) \) and \( \zeta_\infty \in (-k, k) \). These parameters have to be chosen such that the contours \( L_0 \) and \( L_1 \) outline two disjoint domains \( D_0 \) and \( D_1 \). Figures 1(b – d) sample contours \( L_0 \) and \( L_1 \) for different choices of the parameters \( \kappa_0 \) and \( \kappa_1 \), while the other parameters of the problem are kept the same.

To reconstruct the profiles of the inclusions, one needs to compute some integrals. The integrals in \( \text{(4.7)} \) and \( \text{(4.17)} \) are calculated by the Gauss quadrature formulas
\[ \int_a^b \frac{h(\xi) d\xi}{\sqrt{(\xi - a)(b - \xi)}} \approx \pi \frac{N}{N} \sum_{j=1}^{N} h(\xi_j), \]  
(4.20)

where
\[ \xi_j = \delta_+ + \delta_- x_j, \quad \delta_\pm = \frac{b \pm a}{2}, \quad x_j = \cos \left(\frac{2j - 1}{2N}\right). \]  
(4.21)

The evaluation of the function \( g_1(\xi) \) and \( \omega(\xi \pm i0) \) \( (\xi \pm i0 \in l_m^\pm) \) by formulas \( \text{(4.10)} \) and \( \text{(4.19)} \), respectively, requires computing singular integrals with the Cauchy kernel of the form
\[ S(\xi) = \int_a^b \frac{h(\eta) d\eta}{\sqrt{(\eta - a)(b - \eta)(\eta - \xi)}}, \quad a < \xi < b. \]  
(4.22)

On making the substitutions
\[ \xi = \delta_+ + \delta_- x, \quad \eta = \delta_+ + \delta_- y, \quad h(\eta) = \tilde{h}(y), \quad -1 < y < 1, \quad -1 < x < 1, \]  
(4.23)
we expand the function $\tilde{h}(y)$ in terms of the Chebyshev polynomials of the first kind

$$\tilde{h}(y) = \sum_{m=0}^{\infty} \alpha_m T_m(y).$$

(4.24)

Here,

$$\alpha_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{\tilde{h}(y)dy}{\sqrt{1 - y^2}}, \quad \alpha_m = \frac{2}{\pi} \int_{-1}^{1} \frac{\tilde{h}(y)T_m(y)dy}{\sqrt{1 - y^2}}, \quad m = 1, 2, \ldots.$$  

(4.25)

Approximately, by the Gauss formula,

$$\alpha_m \approx \frac{2}{N} \sum_{j=1}^{N} \tilde{h}(x_j) \cos \frac{(2j - 1)\pi m}{2N}, \quad m = 1, 2, \ldots.$$  

(4.26)

By substituting the expansion (4.24) into formula (4.22) and employing the relation

$$\int_{-1}^{1} \frac{T_m(y)dy}{\sqrt{1 - y^2}(y - x)} = \begin{cases} 0, & m = 0, \\ \pi U_{m-1}(x), & m = 1, 2, \ldots, \quad -1 < x < 1, \end{cases}$$

(4.27)

where $U_m(x)$ is the Chebyshev polynomial of the second kind, we finally obtain the formula for the singular integral (4.22)

$$S(\xi) = \frac{\pi}{\delta_+} \sum_{m=1}^{\infty} \alpha_m U_{m-1} \left( \frac{\xi - \delta_+}{\delta_-} \right).$$

(4.28)

To reconstruct two uniformly stressed inclusions symmetric with respect to the origin, we choose $\kappa_0 = \kappa_1$ and $\zeta_\infty = 0$. Then

$$a_1 = -a_0 = c''_{-1} \left( \int_{k}^{1} \frac{d\xi}{|q(\xi)|} \right)^{-1} \int_{k}^{1} \frac{d\xi}{\xi |q(\xi)|}.$$  

(4.29)

The functions $g_1(\xi)$ and $g_0(\xi)$ given by (4.10) and (4.13) are both odd and have the form

$$g_0(\xi) = \frac{c_*}{\xi}, \quad c_* = \text{Re} \left[ c_{-1} \left( \frac{\pi^\infty}{\mu} - \frac{\pi}{\mu_0} \right) \right],$$

$$g_1(\xi) = \frac{2\xi |q(\xi)|}{\pi} \int_{k}^{1} \left( \frac{c'}{\eta} - a_1 \right) \frac{d\eta}{|q(\eta)|(|\eta|^2 - \xi^2)}.$$  

(4.30)

This simplifies formula (4.17), and the constants $\rho_0$ and $\rho_1$ have the form

$$\rho_1 = -\rho_0 = -\frac{\mu_0 c_*}{1 - \kappa_0} \left( \int_{k}^{1} \frac{d\xi}{|q(\xi)|} \right)^{-1} \int_{k}^{1} \frac{d\xi}{\xi |q(\xi)|}.$$  

(4.31)

Our next step is to determine the conformal map function $z = \omega(\xi)$ on the boundary of the loops $l_0$ and $l_1$. Without loss we may drop the additive constant $\gamma_0$. Since $g_j(\xi) = -g_j(-\xi)$, $j = 0, 1$, on using the representation (4.11) and the Sokhotski-Plemelj formulas we deduce

$$\omega(-\xi \mp i0) = -\frac{c_{-1}}{\xi} + \frac{i\mu_0 \lambda_0}{\tau} \left\{ \mp ig_1(\xi) + ig_0(\xi) + i\rho_1 \right\}$$

$$+ \frac{1}{\pi} \sum_{j=0}^{1} (-1)^j \int_{k}^{1} \left[ -g_1(\eta) \frac{|q(\xi)|}{|q(\eta)|} (g_0(\eta) + \rho'_1) \right] \frac{d\eta}{\eta - (-1)^j \xi}, \quad -\xi \mp i0 \in l_0^\mp.$$  

(4.32)
Figure 2: Two symmetric inclusions \((\zeta_\infty = 0, a_0 = -a_1, \rho_0 = -\rho_1)\) when \(\tau_1/\mu = -\tau_1^\infty/\mu = 1\) and \(c_{-1} = 1\). (a): \(\tau_2/\mu = \tau_2^\infty/\mu = 1, k = 0.2, \text{ and } \kappa_0 = \kappa_1 = 0.1\). (b): \(\tau_2/\mu = \tau_2^\infty/\mu = 1, k = 0.5, \text{ and } \kappa_0 = \kappa_1 = 2\). (c) – (d): \(\tau_2 = \tau_2^\infty = 0, k = 0.35, \text{ and } \kappa_0 = \kappa_1 = 1000\). (c): the solvability conditions (4.29) and (4.31) are satisfied. (d): \(a_0 = -a_1 = 0\) and \(\rho_0 = -\rho_1 = 0\) – the solvability conditions (4.29) and (4.31) are not satisfied.

Similarly, we let \(\zeta = \xi \pm i0 \in l_1^\pm\) and derive that \(\omega(\xi \pm i0) = -\omega(-\xi \mp i0),\ \xi \pm i0 \in l_1^\pm\). This implies that the two inclusions are symmetric with respect to the origin.

In Figures 2 (a – d) we outline the profiles of two symmetric inclusions. Equations (4.29) and (4.31) fix the parameters \(a_j = \kappa_j d_j' - b_j\) and \(\rho_j = \lambda_j (\beta_0 - d_j')\). These equations are the solvability conditions for the two Schwarz problems (2.10) and (2.11) in the doubly-connected slit domain or, equivalently, the solvability conditions for the inverse problem of elasticity (2.4). If these conditions are violated, then the resulting conformal map generates different inclusions’ profiles. In Figures 2 (c), we show the contours \(L_0\) and \(L_1\) when the parameters \(a_0 = -a_1\) and \(\rho_0 = -\rho_1\) are determined by equations (4.29) and (4.31), while in Figure 2(d), for the same set of the problem parameters, we outline the contours \(L_0\) and \(L_1\) when \(a_0 = -a_1 = 0\) and \(\rho_0 = -\rho_1 = 0\), and the solvability conditions are violated.

5 Three inclusions

If \(D^e\) is any triply connected domain, then there exists a conformal map \(z = \omega(\zeta)\) such that it transforms the exterior \(D^e\) of three two-sided segments \(l_0 = [-1, -k]^+ \cup [-k, -1]^-,\ \ l_1 = [k_1, k_2]^+ \cup [k_2, k_1]^-\), and \(l_2 = [k, 1]^+ \cup [1, k]^-\) into the domain \(D^e\). Here, \(0 < k < 1\)
and \( -k < k_1 < k_2 < k \). The point \( z = \infty \) is the image of a certain point \( \zeta_\infty = \zeta'_\infty + i \zeta''_\infty \) of the parametric \( \zeta \)-plane, and, in general, the parameters \( \zeta'_\infty, \zeta''_\infty, k, k_1, \) and \( k_2 \) cannot be prescribed and should be recovered from some additional conditions.

Denote
\[
p(\zeta) = (\zeta^2 - 1)(\zeta^2 - k^2)(\zeta - k_1)(\zeta - k_2)
\]
and fix a single branch \( q(\zeta) \) of the function \( u^2 = p(\zeta) \) in the \( \zeta \)-plane with the cuts \( l_0, l_1, \) and \( l_2 \) by the condition \( q(\zeta) \sim \zeta^3, \zeta \to \infty \). This branch is pure imaginary on the cuts' sides, \( q(\zeta) = \pm (1)j|q(\zeta)|, \zeta \in l^\pm_j \). As before, we aim to reduce the Schwarz problems \((2.10)\) and \((2.11)\) to two Riemann-Hilbert problems on the Riemann surface \( \mathcal{R} \) of the algebraic function \( u^2 = p(\zeta) \). In the case of three inclusions, \( \mathcal{R} \) is a hyperelliptic genus-2 surface symmetric with respect to the contour \( l = l_0 \cup l_1 \cup l_2 \).

### 5.1 \( \zeta_\infty \) is a finite point

Suppose first that the preimage of the point \( z = \infty \) is a finite point \( \zeta_\infty = \zeta'_\infty + i \zeta''_\infty \). Then the conformal mapping \( \omega(\zeta) \) has a simple pole at the point \( \zeta_\infty \), and its expansion at this point has the form \((2.8)\). The general solution of the corresponding Riemann-Hilbert problem \((4.4)\) for the function \( \Phi_1(\zeta, u) \) introduced in \((4.3)\) is given by
\[
\Phi_1(\zeta, u) = \beta_0 + \beta_1 \zeta + \beta_2 \zeta^2 + \frac{1}{2\pi} \sum_{j=0}^{2} \int_{l^+_j} \left[ a_j - \text{Im} \left( \frac{c}{\zeta - \zeta_\infty} \right) \right] \left( 1 + \frac{u}{v} \right) \frac{d\xi}{\xi - \zeta},
\]
where \( \beta_j \) are real constants. In general, this solution is not bounded at infinity. The necessary and sufficient conditions for the solution to be bounded at the infinite points of the surface \( \mathcal{R} \) are
\[
\beta_1 = \beta_2 = 0, \quad \sum_{j=0}^{2} \int_{l^+_j} \left[ a_j - \text{Im} \left( \frac{c}{\zeta - \zeta_\infty} \right) \right] \frac{\xi^{m-1}d\xi}{v} = 0, \quad m = 1, 2.
\]

Denote the integrals
\[
I_{mj} = \int_{l^+_j} \frac{\xi^{m-1}d\xi}{|q(\xi)|}, \quad j = 0, 1, 2, \quad m = 1, 2.
\]
In terms of these integrals and the constant \( a_0 \) the other constants \( a_1 \) and \( a_2 \) are expressed as
\[
a_1 = \frac{J_2 I_{12} - J_1 I_{22}}{\Delta}, \quad a_2 = \frac{J_2 I_{11} - J_1 I_{21}}{\Delta}.
\]
Here,
\[
J_m = J_{m0} - J_{m1} + J_{m2} - a_0 I_{m0}, \quad \Delta = I_{11} I_{22} - I_{12} I_{21},
\]
\[
J_{mj} = \int_{l^+_j} \text{Im} \left( \frac{c}{\zeta - \zeta_\infty} \right) \frac{\xi^{m-1}d\xi}{|q(\xi)|}.
\]
By utilizing formulas \((4.11), (4.3), (5.2),\) and \((5.3)\) we derive
\[
F(\zeta) = \beta_0 + \frac{c}{\zeta - \zeta_\infty} + \frac{q(\zeta)}{2\pi} \sum_{j=0}^{2} \int_{l^+_j} \left[ a_j - \text{Im} \left( \frac{c}{\zeta - \zeta_\infty} \right) \right] \frac{d\xi}{(\zeta - \zeta_\infty) v}.
\]
As in the case \( n = 2 \) we employ the function \( \Phi_2(\zeta, u) \) introduced in \((4.14)\). The function \( \Phi_2(\zeta, u) \) solves the Riemann-Hilbert problem on the genus-2 Riemann surface
\[
\Phi_2^+(\xi, v) - \Phi_2^-(\xi, v) = 2i[G_j(\xi) + \rho_j], \quad \xi \in l_j, \quad j = 0, 1, 2.
\]
where \( \rho_j = \lambda_j \rho'_j, \) \( \rho'_j = \beta_0 - d'_j, \) and \( G_j(\zeta) \) is given by (4.13) and

\[
g_1(\xi) = \frac{|q(\xi)|}{\pi} \sum_{j=0}^{2} (-1)^j \int_{l_j^+} \left[ a_j - \text{Im} \left( \frac{c}{\eta - \zeta_\infty} \right) \right] \frac{d\eta}{|q(\eta)||\eta - \zeta|}. \tag{5.9}
\]

The solution to the problem (5.8) can be represented in the first sheet \( \mathbb{C}_1 \) as

\[
i\tau \omega_0(\zeta) = \gamma_0 + \frac{1}{\pi} \sum_{j=0}^{2} (-1)^j \lambda_j \int_{l_j^+} \left\{ g_1(\eta) - \frac{i q(\xi)}{|q(\eta)|} (g_0(\eta) + \rho'_j) \right\} \frac{d\eta}{\eta - \zeta}. \tag{5.10}
\]

This function is bounded at infinity if and only if the constants \( \rho_j \) are selected to be

\[
\rho_1 = \frac{K_1 I_{22} - K_2 I_{12}}{\Delta}, \quad \rho_2 = \frac{K_1 I_{21} - K_2 I_{11}}{\Delta},
\]

\[
K_m = K_{m0} - K_{m1} + K_{m2} + \rho_0 I_{m0}, \quad K_{mj} = \lambda_j \int_{l_j^+} \frac{g_0(\xi) \xi^{m-1} d\xi}{|q(\xi)|}, \tag{5.11}
\]

and \( \rho_0 \) is a free parameter.

The inclusions’ profiles \( L_0, L_0, \) and \( L_2 \) are described by the function \( \omega(\zeta) \) when a point \( \zeta \) traverses the loops \( l_0, l_1, \) and \( l_2, \) respectively. On these contours, the function \( \omega(\zeta) \) is determined by

\[
\omega(\xi \pm i0) = \gamma' + \frac{c-1}{\xi - \zeta_\infty} - \frac{i \mu}{\pi^2} \sum_{j=0}^{2} (-1)^j \lambda_j \int_{l_j^\pm} \left[ g_1(\eta) \pm (-1)^m \frac{|q(\xi)|}{|q(\eta)|} \left( g_0(\eta) + \rho'_j \right) \right] \frac{d\eta}{\eta - \xi} + \pi i \lambda_m |g_0(\xi)| |g_0(\eta) + \rho'_j|, \quad \zeta = \xi \pm i0 \in l_m^\pm, \quad m = 0, 1, 2. \tag{5.12}
\]

The conformal mapping \( z = \omega(\zeta) \) described by (5.12) in addition to the scaling parameter \( c_{-1} = c'_{-1} + i c''_{-1} \) has seven real free parameters \( \zeta_\infty', \zeta_\infty'', k, k_1, k_2, a_0, \) and \( \rho_0. \)

Sample profiles of three uniformly stressed inclusions are shown in Figures 3 (a)–(d) in the case when the preimage of the infinite point is a finite point \( \zeta_\infty \) of the parametric \( \zeta \)-plane.

### 5.2 \( \zeta_\infty \) is the infinite point

If the preimage of the infinite point \( z = \infty \) is the infinite point \( \zeta = \infty \), then in a neighborhood of the infinite point the functions \( \omega(\zeta) \) and \( F(\zeta) \) have expansions (2.9) and (2.13), respectively. Instead of the function (4.11) we introduce a new function \( F_0(\zeta) \) by

\[
F_0(\zeta) = F(\zeta) - c\zeta, \quad c = c' + ic''. \tag{5.13}
\]

Then the function \( F_0(\zeta) \) has a removable singularity at the infinite point. My making use of this function similarly to the case when \( \zeta_\infty \) is a finite point we derive the solution to the first Schwarz problem (2.10), (2.13). It is

\[
F(\zeta) = \beta_0 + c\zeta + \frac{q(\zeta)}{2\pi} \sum_{j=0}^{2} \int_{l_j} \left( a_j - c''\xi \right) d\xi. \tag{5.14}
\]

Only one constant say, \( a_0 \), can be chosen arbitrarily. The solution \( F(\zeta) \) is bounded at infinity if and only if the other two constants are chosen as

\[
a_1 = \frac{\hat{J}_2 I_{12} - \hat{J}_1 I_{22}}{\Delta}, \quad a_2 = \frac{\hat{J}_2 I_{11} - \hat{J}_1 I_{21}}{\Delta}, \tag{5.15}
\]

and

\[
\omega(\xi \pm i0) = \gamma' + \frac{c-1}{\xi - \zeta_\infty} - \frac{c_{-1}}{\pi^2} \sum_{j=0}^{2} (-1)^j \lambda_j \int_{l_j^\pm} \left[ g_1(\eta) \pm (-1)^m \frac{|q(\xi)|}{|q(\eta)|} \left( g_0(\eta) + \rho'_j \right) \right] \frac{d\eta}{\eta - \xi} + \pi i \lambda_m |g_0(\xi)| |g_0(\eta) + \rho'_j|, \quad \zeta = \xi \pm i0 \in l_m^\pm, \quad m = 0, 1, 2. \tag{5.12}
\]
Figure 3: Three inclusions when $\zeta_\infty$ is a finite point, $\tau_1/\mu = -\bar{\tau}_1^\infty/\mu = 1$, $k_1 = -0.1$, $a_0 = 0$, $\rho_0 = 0$, $\gamma' = 0$, and $c_{-1} = 1$. (a): $\tau_2 = \tau_1^\infty = 0$, $\kappa = 0.1$, $k = 0.5$, $k_2 = 0.1$, and $\zeta_\infty = 5i$. (b): $\tau_2 = \tau_1^\infty = 0$, $\kappa = 0.2$, $k = 0.5$, $k_2 = 0.1$, and $\zeta_\infty = 5i$. (c): $\tau_2 = \tau_1^\infty = 1$, $\kappa = 0.3$, $k = 0.6$, $k_2 = 0.1$, and $\zeta_\infty = 1 + 2i$. (d): $\tau_2 = \tau_1^\infty = 1$, $\kappa = 0.1$, $k = 0.5$, $k_2 = 0.2$, and $\zeta_\infty = 5 + 5i$.

where

$$\hat{J}_m = c'' \sum_{j=0}^{2} (-1)^j I_{m+1j} - a_0 I_{m0}, \quad I_{sj} = \int_{l_j^+} \frac{\xi^{s-1} d\xi}{|q(\xi)|}.$$ (5.16)

In the same fashion instead of the function (4.11) we introduce the function

$$\omega_0(\zeta) = \omega(\zeta) - c_{-1}\zeta.$$ (5.17)

and reduce the second Schwarz problem (2.9), (2.11) to the following:

$$\text{Im}[i\bar{\tau}\omega_0(\zeta)] = \lambda_j [g_0(\xi) \pm (-1)^j g_1(\xi) + \rho_j'], \quad \zeta = \xi \pm i0 \in l_j^\pm,$$ (5.18)

where

$$g_0(\xi) = c_{sj}\xi, \quad c_{sj} = \text{Re} \left[ c_{-1} \left( \frac{\tau_1^\infty}{\mu} - \frac{\bar{\tau}_1^\infty}{\mu_j} \right) \right],$$

$$g_1(\xi) = \frac{|q(\xi)|}{\pi} \sum_{j=0}^{2} (-1)^j \int_{l_j^+} \frac{(a_j - c''\eta)d\eta}{|q(\eta)|(|\eta - \xi)|}.$$ (5.19)

The solution of the Schwarz problem (5.18) is given by formula (5.10) in the domain $D^e = \mathbb{C} \setminus l$ and formula (5.12) in the contours $l_0$, $l_1$, and $l_2$. In these formulas the
functions \( g_0(\eta) \) and \( g_1(\eta) \) need to be replaced by the ones in (5.19), while the constants \( \rho'_j \) are

\[
\rho'_j = \frac{\rho_j}{\lambda_j}, \quad \rho_1 = \frac{\hat{K}_{122} - \hat{K}_{212}}{\Delta}, \quad \rho_2 = \frac{\hat{K}_{121} - \hat{K}_{211}}{\Delta},
\]

\[
\hat{K}_m = \sum_{j=0}^{2} (-1)^j c_{s_j} \lambda_j I_{m+1j} + \rho_0 I_{m0}, \quad m = 1, 2.
\]  

To reconstruct symmetrically located uniformly stressed inclusions in the case \( \zeta_\infty = \infty \), we take \( \tau_2 = \tau_2^\infty \), \( a_0 = -a_2 \), \( \rho_0 = -\rho_2 \), \( k_1 = -k_2 \), and \( \kappa_0 = \kappa_1 = \kappa_2 \). For \( \zeta_\infty = \infty \), samples of two symmetric and nonsymmetric inclusions are shown in Figures 4 (a), (b) and 4(c), (d), respectively. In Figure 4 (d), the contours intersect each other, and the set of parameters is unacceptable.

6 \( n \geq 4 \) inclusions

Assume that \( n \geq 4 \) and \( \mathcal{D}^e = \mathbb{R}^2 \setminus \mathcal{D} \) is an \( n \)-connected domain \( \mathcal{D} = \bigcup_{j=0}^{n-1} \mathcal{D}_j \) such that there exists a conformal map \( z = \omega(\zeta) \) that transforms a slit domain \( \mathcal{D}^e \) into the domain \( \mathcal{D}^e \), and \( \mathcal{D}^e \) is the exterior of \( n \) slits \( l_j = [k_{2j}, k_{2j+1}]^+ \cup [k_{2j+1}, k_{2j}]^- \) (\( j = 0, 1, \ldots, n-1 \))
lying in the real \( \zeta \)-axis, \( k_0 \prec k_1 \prec \ldots \prec k_{2n-2} \prec k_{2n-1} \), and \( k_0 = -1, k_{2n-1} = 1 \). When \( n \geq 4 \) not for each \( n \)-connected domain \( D^e \) there exists a conformal map that transforms the exterior \( \mathcal{D}^e \) of \( n \) slits into the domain \( D^e \) such that all the slits lie in the same line. In what follows we confine ourselves to the family of domains \( D^e \) when such a map exists. Denote that family \( D^e_\ast \). Notice that every doubly and triply connected domain belongs to the family \( D^e_\ast \). We confine ourselves to the case when the preimage of the point \( z = \infty \) is a finite point \( \zeta_\infty \).

First we fix the branch \( q(\zeta) \) of the function

\[
p^{1/2}(\zeta) = \prod_{j=0}^{2n-1} (\zeta - k_j)^{1/2}
\]

in the domain \( \mathcal{D}^e \) by the condition \( q(\zeta) \sim \zeta^n, \zeta \to \infty \). On the sides of the cuts this branch is pure imaginary and

\[
q(\zeta)|_{\zeta \in \eta_{n-1}} = \pm i (-1)^j |q(\zeta)|, \quad j = 0, 1, \ldots, n-1.
\]

As in the cases \( n = 2 \) and \( n = 3 \) we introduce the functions \( F_0(\zeta) \) and \( \omega_0(\zeta) \) by formulas (4.1) and (4.11), respectively. In terms of the function \( \Phi_1(\zeta, u) \) given by (4.3) on the genus-(\( n-1 \)) Riemann surface \( \mathcal{R} \) of the algebraic function \( u^2 = p(\zeta) \) the first Schwarz problem (2.10), (2.12) can be equivalently written as the Riemann-Hilbert problem

\[
\Phi_1^+(\xi, v) - \Phi_1^-(\xi, v) = 2i \left[ a_j - \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \right], \quad (\xi, v) \in l_j, \quad j = 0, 1, \ldots, n-1.
\]

The general solution of this problem is

\[
\Phi_1(\zeta, u) = \beta_0 + \frac{1}{2\pi} \sum_{j=0}^{n-1} \int_{l_j} \left[ a_j - \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \right] \left( 1 + \frac{u}{v} \right) \frac{d\xi}{\xi - \zeta}.
\]

For the function \( \Phi_1(\zeta, u) \) to be bounded at the two infinite points \( (\infty, \infty)_1 \) and \( (\infty, \infty)_2 \) of the surface \( \mathcal{R} \) it is necessary and sufficient that

\[
\sum_{j=0}^{n-1} \int_{l_j} \left[ a_j - \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \right] \frac{\xi^{m-1} d\xi}{v} = 0, \quad m = 1, 2, \ldots, n-1.
\]

The integrals

\[
A_{mj} = \int_{l_j} \frac{\xi^{m-1} d\xi}{v}, \quad j = 1, \ldots, n-1, \quad m = 1, \ldots, n-1,
\]

form the matrix of the \( A \)-periods of the abelian integrals (Springer, 1956)

\[
\int_{(-1,0)} \frac{\xi^{m-1} d\xi}{v}, \quad m = 1, \ldots, n-1.
\]

Select \( a_0 \) as a free parameter. Then the other constants \( a_j \) are recovered from the system of linear algebraic equations

\[
\sum_{j=1}^{n-1} A_{mj} a_j = b_m, \quad m = 1, \ldots, n-1.
\]
Here,  
\[ b_m = \sum_{j=0}^{n-1} \int_{l_j} \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \frac{\xi^{m-1} d\xi}{v} - A_{m0}a_0. \]  

(6.9)

Since the matrix of the \(A\)-periods of basis abelian integrals is not singular, the solution to the system (4.9) exists and is unique. The function \(v\) is pure imaginary on the sides \(l_j^\pm\) of the loops \(l_j\). Therefore employing the relations (6.2) we represent the system (6.8) as

\[ \sum_{j=1}^{n-1} (-1)^j I_{m_j} a_j = J_m, \quad m = 1, 2, \ldots, n - 1, \]  

(6.10)

where

\[ I_{m_j} = \int_{l_j}^{l_j^+} \frac{\xi^{m-1} d\xi}{|q(\xi)|} > 0, \quad J_m = \sum_{j=0}^{n-1} (-1)^j \int_{l_j}^{l_j^+} \text{Im} \left( \frac{c}{\xi - \zeta_\infty} \right) \frac{\xi^{m-1} d\xi}{|q(\xi)|} - I_{m0}a_0. \]  

(6.11)

Since the coefficients \(I_{m_j}\) and \(J_m\) are real, the constants \(a_j\) \((j = 1, \ldots, n - 1)\) are also real. When \((\zeta, u) \in C_1, \Phi_1(\zeta, u) = F_0(\zeta)\), and therefore the function \(F(\zeta)\) admits the following representation by quadratures:

\[ F(\zeta) = \beta_0 + \frac{c}{\zeta - \zeta_\infty} + \frac{q(\xi)}{2\pi} \sum_{j=0}^{n-1} \int_{l_j}^{l_j^+} a_j - \text{Im} \left( \frac{c}{\eta - \zeta_\infty} \right) \frac{d\eta}{(\eta - \zeta_\infty) v}. \]  

(6.12)

Proceed now to the second Schwarz problem (2.11), (2.8) equivalent to

\[ \text{Im}[i \tau \omega_0(\zeta)] = G_j(\zeta) + \rho_j, \quad \zeta \in l_j, \quad j = 0, 1, \ldots, n - 1, \]  

(6.13)

where \(\rho_j = \lambda_j \rho_j'\), \(\rho_j' = \beta_0 - d_j'\), the functions \(G_j(\zeta)\) and \(g_0(\xi)\) are defined in (4.13), and

\[ g_1(\xi) = \frac{|q(\xi)|}{\pi} \sum_{j=0}^{n-1} (-1)^j \int_{l_j}^{l_j^+} a_j - \text{Im} \left( \frac{c}{\eta - \zeta_\infty} \right) \frac{d\eta}{|q(\eta)||\eta - \zeta_\infty|}. \]  

(6.14)

Similarly to the cases \(n = 2\) and \(n = 3\) we rewrite the Schwarz problem (6.13) as a Riemann-Hilbert problem on the surface \(R\). Ultimately, we derive

\[ i \tau \omega_0(\zeta) = \gamma_0 + \frac{1}{\pi} \sum_{j=0}^{n-1} (-1)^j \lambda_j \int_{l_j}^{l_j^+} \left\{ g_1(\eta) + \frac{(-1)^n i q(\zeta)}{|q(\eta)|} [g_0(\eta) + \rho_j'] \right\} \frac{d\eta}{\eta - \zeta}. \]  

(6.15)

The solution is bounded at infinity if and only if

\[ \sum_{j=0}^{n-1} \int_{l_j} \left[ \rho_j + g(\xi) \right] \frac{\xi^{m-1} d\xi}{v} = 0, \quad m = 1, 2, \ldots, n - 1. \]  

(6.16)

One of the constants say, \(\rho_0\), is free. The others have to be recovered from the following system of linear algebraic equations:

\[ \sum_{j=1}^{n-1} (-1)^j I_{m_j} \rho_j = -K_m, \quad m = 1, 2, \ldots, n - 1. \]  

(6.17)

Here,

\[ K_m = \lambda_j \sum_{j=0}^{n-1} (-1)^j \int_{l_j}^{l_j^+} \frac{\eta^{m-1} g_0(\eta) d\eta}{|q(\eta)|} + I_{m0} \rho_0. \]  

(6.18)
Since the matrix of the system (6.17) is not singular and the coefficients \( I_{m,j} \) and the integrals \( K_m \) are real, the solution to the system (6.17) exists, unique, and real.

When a point \( \zeta \) traverses the loops \( l_m \) the point \( z = \omega(\zeta) \) traverses the contours \( L_m \),

\[
\omega(\xi \pm i0) = \gamma' + \frac{c_{-1}}{\xi - \zeta_\infty} - \frac{i\mu}{\pi\tau} \sum_{j=0}^{n-1} (-1)^j \lambda_j \int_{l_j} |g_1(\eta) \pm (-1)^m g_1(\xi)| \, d\eta
\]

\[
\pm (-1)^m \frac{|q(\xi)|}{|q(\eta)|} \left( g_0(\eta) + \rho_j' \right) \frac{d\eta}{\eta - \xi} + \pi i \lambda_m [g_0(\xi) + \rho_m' \pm (-1)^m g_1(\xi)] ,
\]

\[
\zeta = \xi \pm i0 \in l_m^\pm, \quad m = 0, 1, \ldots, n - 1,
\]

where \( \gamma' = -i\gamma_0/\bar{r} \) is an arbitrary complex constant. The conformal map derived in addition to the free additive complex constant \( \gamma' \) and the scaling parameter \( c_{-1} = c_{-1}' + ic_{-1}' \) is defined up to \( 2n + 2 \) real parameters, \( a_0, \rho_0, \zeta_\infty = \zeta'_\infty + i\zeta''_\infty \), and \( k_1, k_2, \ldots, k_{2n-2} \in (-1, 1) \). These parameters have to be chosen such that the contours \( L_m (m = 0, 1, \ldots, n - 1) \) are not embedded into each other or do not intersect or touch each other.

7 Conclusion

On pursuing the goal of reconstructing the shapes of \( n \) uniformly stressed inclusions in an unbounded elastic body subjected to antiplane uniform shear we advanced the technique of conformal mappings and the Riemann-Hilbert problems on a Riemann surface. The method treats the exterior of the inclusions as the image by a conformal map of an \( n \)-connected slit domain with the slits lying in the real axis. Such a map always exists for simply-, doubly-, and triply-connected domains. In the case \( n \geq 4 \) we considered a family of \( n \)-connected domains for which their preimages lie in the same line. The procedure requires solving consequently two inhomogeneous Schwarz problems of the theory of analytic functions on \( n \) slits or, equivalently, two Riemann-Hilbert problems on a hyperelliptic surface of genus \( n - 1 \). These problems were solved in terms of the Weierstrass integrals, analogs of the Cauchy integrals in a Riemann surface. On satisfying the solvability conditions we derived a nonhomogeneous system of \( n \) linear algebraic equations. The system coefficients are basis abelian integrals, and the system matrix is nonsingular. By deriving the conformal map by quadratures and by letting a point traverse the double-sided slits in the parametric plane we determined the inclusions’ profiles. The algorithm was numerically tested for the cases \( n = 1, n = 2, \) and \( n = 3 \). We managed to recover families of conformal mappings which generate sets of one, two, and three uniformly stressed inclusions. In doubly-connected case, in addition to the complex scaling parameter \( c_{-1} \) the map possesses four real parameters, \( k \in (0, 1), \zeta_\infty \in (-k, k), a_0, \) and \( b_0 \). In the triply-connected case it has seven real parameters, \( k \in (0, 1), k_1, k_2 \) \((-k < k_1 < k_2 < k)), \zeta'_\infty, \zeta''_\infty, a_0, \) and \( b_0 \).

Data accessibility. This work does not have any experimental data.

Competing interests statement. I have no competing interests.

Acknowledgements. This research originated during the author’s visit to Imperial College supported by the LMS.

Funding. This research received no specific grant from any funding agency in the public, commercial or not-for-profit sectors.
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