Current-Carrying Cosmic Strings in Scalar-Tensor Gravities

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November 11, 2018

Abstract

We study the modifications on the metric of an isolated self-gravitating bosonic superconducting cosmic string in a scalar-tensor gravity in the weak-field approximation. These modifications are induced by an arbitrary coupling of a massless scalar field to the usual tensorial field

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in the gravitational Lagrangian. The metric is derived by means of a matching at the string radius with a most general static and cylindrically symmetric solution of the Einstein-Maxwell-scalar field equations. We show that this metric depends on five parameters which are related to the string’s internal structure and to the solution of the scalar field. We compare our results with those obtained in the framework of General Relativity.

1 Introduction

The assumption that gravity may be intermediated by a scalar field (or, more generally, by many scalar fields) in addition to the usual symmetric rank-2 tensor has considerably revived in the recent years. From the theoretical point of view, they seem to be the most natural alternative to General Relativity. Indeed, most attempts to unify gravity with the other interactions predict the existence of one (or many) scalar(s) field(s) with gravitational-strength couplings. If gravity is essentially scalar-tensorial, there will be direct implications for cosmology and experimental tests of the gravitational interaction (we refer the reader to Damour’s recent account on “Experimental Tests of GR” [1]). In particular, any gravitational phenomena will be affected by the variation of the gravitational “constant” $\tilde{G}_0$. At sufficiently high energy scales where gravity becomes scalar-tensor in nature [3], it seems worthwhile to analyse the behaviour of matter in the presence of a scalar-tensorial gravitational field, specially those which originated in the early universe, such as cosmic strings. In this context, some authors have studied
solutions for cosmic strings and domain walls in Brans-Dicke \[3\], in dilaton theory \[4\] and in more general scalar-tensor couplings \[5\].

On the other hand, topological defects are expected to be formed during phase transitions in the early universe. Among them, cosmic strings have been widely studied in cosmology in connection with structure formation. In 1985, Witten showed that in many field theories cosmic strings behave as superconducting tubes and they may generate enormous currents of order \(10^{20}\) A or more \[6\]. This fact has raised interest to current-carrying strings and their eventual explanations to many astrophysical phenomena, such as origin of the primordial magnetic fields \[7\], charged vacuum condensates \[8\] and sources of ultrahigh-energy cosmic rays \[9\], among others.

In ref. \[10\], Sen have considered solutions of a superconducting string in the Brans-Dicke theory. The aim of this paper is to study the implications of a class of more general scalar-tensor gravities for a superconducting, bosonic cosmic string. In particular, we will be interested on the modifications induced on the string metric and their possible observable consequences on the current carried by the string. These modifications come from an arbitrary coupling of a massless scalar field to the tensor field in the gravitational Lagrangian. The action which describes these theories (in the Jordan-Fierz frame) is

\[
\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} \left[ \hat{\Phi} \hat{R} - \frac{\omega(\hat{\Phi})}{\hat{\Phi}} \tilde{g}^{\mu\nu} \partial_\mu \hat{\Phi} \partial_\nu \hat{\Phi} \right] + \mathcal{S}_m[\Psi_m, \tilde{g}_{\mu\nu}], \quad (1)
\]

where \(\tilde{g}_{\mu\nu}\) is the physical metric in this frame, \(\hat{R}\) is the curvature scalar associated to it and \(\mathcal{S}_m\) denotes the action describing the general matter fields \(\Psi_m\). These theories are metric, e.g., matter couples minimally and
universally to $\tilde{g}_{\mu\nu}$ and not to $\tilde{\Phi}$.

The main purpose of this paper is to study the influence of a scalar-tensorial coupling on the gravitational field of a current-carrying cosmic string described by Witten’s model [6]. For this purpose, we need to solve the modified Einstein’s equations having a current-carrying vortex as source of the spacetime. In General Relativity, the gravitational field of superconducting strings has been studied by many authors [11, 12, 13, 14, 15, 16]. In particular, the following technics have been employed to derive the spacetime surrounding superconducting vortex: analytic integration of the Einstein’s equations over the string’s energy-momentum tensor [11]; linearization of the Einstein’s equations using distribution’s functions [14, 16]; numerical integrations of the fields equations (Einstein plus material fields) [12], among others.

In this paper, we will make an adaptation of Linet’s method [14] to our model. That is, we will solve the linearised (modified) Einstein’s equations using distribution’s functions while taking into account the scalar-tensor feature of gravity. This work is outlined as follows. In section 2, we describe the configuration of a superconducting string in scalar-tensor gravities. In section 3, we start by solving the equations for the exterior region. In the subsection 3.2, we solve the linearised equations by applying Linet’s method, introduced in ref. [14]. Then, we match the exterior solution with the internal parameters. In 3.3, we derive the deficit angle associated to the metric found previously. We also compare our results with previous results obtained in the framework of General Relativity. Finally, in section 4, we end with some conclusions and discussions.
2 Superconducting String Configuration in Scalar-Tensor Gravities

In what follows, we will search for a regular solution of a self-gravitating superconducting vortex in the framework of a scalar-tensor gravity. Hence, the simplest bosonic vortex arises from the action of the Abelian-Higgs $U(1) \times U(1)$ model containing two pairs of complex scalar and gauge fields

$$S_m = \int d^4x \sqrt{-\tilde{g}} \left\{ -\frac{1}{2} \tilde{g}^{\mu\nu} D_\mu \varphi D_\nu \varphi^* - \frac{1}{2} \tilde{g}^{\mu\nu} D_\mu \sigma D_\nu \sigma^* - \frac{1}{16\pi} \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} H_{\mu\alpha} H_{\nu\beta} - \frac{1}{16\pi} \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - V(|\varphi|, |\sigma|) \right\} \quad (2)$$

with $D_\mu \varphi \equiv (\partial_\mu + iqC_\mu)\varphi$, $D_\mu \sigma \equiv (\partial_\mu + ieA_\mu)\sigma$ and $F_{\mu\nu}$ and $H_{\mu\nu}$ are the field-strengths associated to the electromagnetic $A_\mu$ and gauge $C_\mu$ fields, respectively. The potential is “Higgs inspired” and contains appropriate $\varphi-\sigma$ interactions so that there occurs a spontaneous symmetry breaking

$$V(|\varphi|, |\sigma|) = \frac{\lambda_\varphi}{4} (|\varphi|^2 - \eta^2)^2 + f |\varphi|^2 |\sigma|^2 + \frac{\lambda_\sigma}{4} |\sigma|^4 - \frac{m^2}{2} |\sigma|^2, \quad (3)$$

with positive $\eta, f, \lambda_\sigma, \lambda_\varphi$ parameters. A vortex configuration arises when the $U(1)$ symmetry associated to the $(\varphi, C_\mu)$ pair is spontaneously broken. The superconducting feature of this vortex is produced when the pair $(\sigma, A_\mu)$, associated to the other $U'(1)$ symmetry of this model, is spontaneously broken in the core of the vortex.

We restrict ourselves to contemplate configurations of an isolated and static vortex in the $z$-axis. In a cylindrical coordinate system $(t, r, \theta, z)$, such that $r \geq 0$ and $0 \leq \theta < 2\pi$, we make the choice
\[ \varphi = R(r)e^{i\theta} \quad \text{and} \quad C_\mu = \frac{1}{q}[P(r) - 1]\delta_\mu^0. \]  

in much the same way as we proceed with ordinary (non-conducting) cosmic strings. The functions \( R, P \) are functions of \( r \) only. We also require that these functions be regular everywhere and that they satisfy the usual boundary conditions for a vortex configuration \[ R(0) = 0 \quad \text{and} \quad P(0) = 1 \]

\[ \lim_{r \to \infty} R(r) = \eta \quad \text{and} \quad \lim_{r \to \infty} P(r) = 0. \]  

The \( \sigma \)-field is responsible for the bosonic current along the string, and the \( A_\mu \) is the gauge field which produces an external magnetic field; their configuration are taken in the form

\[ \sigma = \sigma(r)e^{i\psi(z)} \quad \text{and} \quad A_\mu = \frac{1}{e}[A(r) - \frac{\partial \psi}{\partial z}\delta^z_\mu]. \]  

The pair \( (\sigma, A_\mu) \) is subjected to the following boundary conditions

\[ \frac{d\sigma(0)}{dr} = 0 \quad \text{and} \quad A(0) = \frac{dA(0)}{dr} = 0 \]

\[ \lim_{r \to \infty} \sigma(r) = 0 \quad \text{and} \quad \lim_{r \to \infty} A(r) \neq 0. \]  

With this choice, we can see that \( \sigma \) breaks electromagnetism inside the string and can form a charged scalar condensate in the string core. Outside the string, the \( A_\mu \) field has a non-vanishing component along the \( z \)-axis which indicates that there will be a non-vanishing energy-momentum tensor in the region exterior to the string.
Although action (1) shows explicitly this gravity’s scalar-tensorial character, for technical reasons, we choose to work in the conformal (Einstein) frame in which the kinematic terms of the scalar and the tensor fields do not mix

\[
S = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left[ R - 2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] + S_m[\Psi_m, \Omega^2(\phi) g_{\mu\nu}],
\]  

(8)

where \( g_{\mu\nu} \) is a pure rank-2 tensor in the Einstein frame, \( R \) is the curvature scalar associated to it and \( \Omega(\phi) \) is an arbitrary function of the scalar field.

Action (8) is obtained from (1) by a conformal transformation

\[
\tilde{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu},
\]  

(9)

and by a redefinition of the quantity

\[
G \Omega^2(\phi) = \tilde{\Phi}^{-1}
\]

which makes evident the feature that any gravitational phenomena will be affected by the variation of the gravitation “constant” \( G \) in the scalar-tensorial gravity, and by introducing a new parameter

\[
\alpha^2 \equiv \left( \frac{\partial \ln \Omega(\phi)}{\partial \phi} \right)^2 = [2\omega(\tilde{\Phi}) + 3]^{-1},
\]

which can be interpreted as the (field-dependent) coupling strength between matter and the scalar field. In order to make our calculations as broad as possible, we choose not to specify the factors \( \Omega(\phi) \) and \( \alpha(\phi) \) (the field-dependent coupling strength between matter and the scalar field), leaving them as arbitrary functions of the scalar field.
In the conformal frame, the Einstein equations are modified. A straightforward calculus shows that the “Einstein” equations are

\[
G_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + 8\pi G T_{\mu\nu} - 4\pi G \phi \nabla g_{\mu\nu} = -4\pi G \alpha(\phi) T.
\] (10)

We note that the last equation brings a new information and shows that the matter distribution behaves as a source for \( \phi \) and \( g_{\mu\nu} \) as well. The energy-momentum tensor is defined as usual

\[
T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}},
\] (11)

but in the conformal frame it is no longer conserved \( \nabla_\mu T^\mu_\nu = \alpha(\phi) T_{\nu} \nabla_\nu \phi \). It is clear from transformation (9) that we can relate quantities from both frames in such a way that \( \tilde{T}^{\mu\nu} = \Omega^{-2}(\phi) T^{\mu\nu} \) and \( \tilde{T}^\mu_\nu = \Omega^{-4}(\phi) T^\mu_\nu \).

Guided by the symmetry of the source, we impose that the metric is static and cylindrically symmetric. We choose to work with a general cylindrically symmetric metric written in the form

\[
ds^2 = e^{2(\gamma - \Psi)} (-dt^2 + dr^2) + \beta^2 e^{-2\Psi} d\theta^2 + e^{2\Psi} dz^2,
\] (12)

where the metric functions \( \gamma, \Psi, \) and \( \beta \) are functions of \( r \) only. In addition, the metric functions satisfy the regularity conditions at the axis of symmetry \( r = 0 \)

\[
\gamma = 0, \quad \Psi = 0, \quad \frac{d\gamma}{dr} = 0, \quad \frac{d\Psi}{dr} = 0, \quad \text{and} \quad \frac{d\beta}{dr} = 0.
\] (13)

With metric given by expression (12) we are in a position to write the full equations of motion for the self-gravitating superconducting vortex in
scalar-tensorial gravity. In the conformal frame, these equations are

\[
\begin{align*}
\beta'' &= 8\pi G\beta e^{2(\gamma-\Psi)}[T_t^t + T_r^r] \\
(\beta \Psi')' &= 4\pi G\beta e^{2(\gamma-\Psi)}[T_t^t + T_r^r + T_\theta^\theta - T_z^z] \\
\beta' \gamma' &= \beta(\Psi')^2 - \beta(\phi')^2 + 8\pi Ge^{2(\gamma-\Psi)}T_r^r \\
(\beta \phi')' &= -4\pi G\alpha(\phi)\beta e^{2(\gamma-\Psi)}T_r^r 
\end{align*}
\]

where ('\ ') denotes “derivative with respect to r". The non-vanishing components of the energy-momentum tensor (computed using equation (11)) are

\[
\begin{align*}
T_t^t &= -\frac{1}{2}\Omega^2(\phi)\{e^{2(\Psi-\gamma)}(R^2 + \sigma^2) + \frac{e^{2\Psi}}{\beta^2} R^2 P^2 + e^{-2\Psi} \sigma^2 A^2 \\
&+ \Omega^{-2}(\phi)e^{-2\gamma}\left(\frac{A^2}{4\pi e^2}\right) + \Omega^{-2}(\phi)\frac{e^{2(\Psi-\gamma)}}{\beta^2}\left(\frac{P^2}{4\pi q^2}\right) \bigg) + 2\Omega^2(\phi)V(R, \sigma)\} \\
T_r^r &= \frac{1}{2}\Omega^2(\phi)\{e^{2(\Psi-\gamma)}(R^2 + \sigma^2) - \frac{e^{2\Psi}}{\beta^2} R^2 P^2 - e^{-2\Psi} \sigma^2 A^2 \\
&+ \Omega^{-2}(\phi)e^{-2\gamma}\left(\frac{A^2}{4\pi e^2}\right) + \Omega^{-2}(\phi)\frac{e^{2(\Psi-\gamma)}}{\beta^2}\left(\frac{P^2}{4\pi q^2}\right) \bigg) - 2\Omega^2(\phi)V(R, \sigma)\} \\
T_\theta^\theta &= -\frac{1}{2}\Omega^2(\phi)\{e^{2(\Psi-\gamma)}(R^2 + \sigma^2) - \frac{e^{2\Psi}}{\beta^2} R^2 P^2 + e^{-2\Psi} \sigma^2 A^2 \\
&+ \Omega^{-2}(\phi)e^{-2\gamma}\left(\frac{A^2}{4\pi e^2}\right) - \Omega^{-2}(\phi)\frac{e^{2(\Psi-\gamma)}}{\beta^2}\left(\frac{P^2}{4\pi q^2}\right) \bigg) + 2\Omega^2(\phi)V(R, \sigma)\} \\
T_z^z &= -\frac{1}{2}\Omega^2(\phi)\{e^{2(\Psi-\gamma)}(R^2 + \sigma^2) + \frac{e^{2\Psi}}{\beta^2} R^2 P^2 - e^{-2\Psi} \sigma^2 A^2 \\
&- \Omega^{-2}(\phi)e^{-2\gamma}\left(\frac{A^2}{4\pi e^2}\right) + \Omega^{-2}(\phi)\frac{e^{2(\Psi-\gamma)}}{\beta^2}\left(\frac{P^2}{4\pi q^2}\right) \bigg) + 2\Omega^2(\phi)V(R, \sigma)\} 
\end{align*}
\]

As we said before, the energy-momentum tensor is not conserved in the conformal frame. Instead, the equation

\[
\nabla_\mu T^\mu = \alpha(\phi)T \nabla_\nu \phi,
\]
where $T$ is the trace of the energy-momentum tensor, gives an additional relation between the scalar field $\phi$ and the source.

In the next section, we will attempt to solve the field equations (14). For the purpose of these calculations, we can divide the space into two regions: an exterior region $r > r_0$, where all the fields drop away rapidly and the only survivor is the magnetic field; and an interior region $r \leq r_0$, where all the string’s field contribute to the energy-momentum tensor. Conveniently, $r_0$ has the same order of magnitude of the string radius. Then, we match the exterior and the interior solutions (to first order in $\tilde{G}_0 = G \Omega^2(\phi_0)$, where $\phi_0$ is a constant) providing a relationship between the internal parameters of the string and the spacetime geometry.

3 Superconducting String Solution in Scalar-Tensor Gravities

3.1 The Exterior Solution and the Modified Rainich Algebra:

In this region, $r > r_0$, the electromagnetic field is the only field which contributes to the energy-momentum tensor. Therefore, the energy-momentum tensor has the form\footnote{Just as a reminder, throughout this paper we will work in the conformal frame for the sake of simplicity. Also, for convenience, we work in units such that $\hbar = c = 1$ and keep Newton’s “constant” $G$.}
\[ T^{\mu\nu} = \frac{1}{4\pi} \left[ F^{\mu\alpha} F_{\alpha}^{\nu} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right] \]  

(16)

with the following algebraic properties

\[ T_{\mu}^{\mu} = 0 \quad \text{and} \quad T_{\nu}^{\alpha} T_{\alpha}^{\mu} = \frac{1}{4} \delta_{\nu}^{\mu} (T_{\alpha\beta} T^{\alpha\beta}). \]

which leads expression (15) to take a simple form

\[ T_{t}^{t} = -T_{r}^{r} = T_{\theta}^{\theta} = -T_{z}^{z} = -\frac{1}{2} e^{-2\gamma} \left( \frac{A'^{2}}{4\pi e^{2}} \right). \]  

(17)

Thus, our problem is reduced to solve the modified Einstein’s equations with source given by (16). That is,

\[ \begin{align*}
\beta'' &= 0 \\
(\beta \Psi')' &= 4\pi G \beta e^{2(\gamma - \Psi)} [T_{t}^{t} - T_{z}^{z}] \\
\beta' \gamma' &= 8\pi G \beta e^{2(\gamma - \Psi)} T_{r}^{r} + \beta (\Psi')^{2} - \beta (\phi')^{2} \\
(\beta \phi')' &= 0.
\end{align*} \]

(18)

In General Relativity (i.e, in the absence of the dilaton field), these equations have been previously investigated by many authors [18]. A source of the form (16) leads to some algebraic conditions on the curvature scalar and the Ricci tensor, known as the Rainich conditions:

\[ R \equiv R_{t}^{t} + R_{r}^{r} + R_{\theta}^{\theta} + R_{z}^{z} = 0, \]

and

\[ (R_{t}^{t})^{2} = (R_{r}^{r})^{2} = (R_{\theta}^{\theta})^{2} = (R_{z}^{z})^{2}. \]
The two equations above admit three sets of solutions: the magnetic case, the electric case and a third case which can correspond to either a static electric or a static magnetic field aligned along the z-axis \[18\]. The superconducting string defined in Witten’s model (2) corresponds to the magnetic case:

\[ R_t^t = R_\theta^\theta, \quad R_{\theta}^\theta = R_z^z \quad \text{and} \quad R_t^t = -R_r^r. \] (19)

In a scalar-tensor gravity, we notice however that the Rainich conditions are no longer valid because of the very nature of the modified Einstein’s equations (actually an Einstein-Maxwell-dilaton system). Instead of the algebraic conditions stated above, we have now:

\[ R \equiv R_t^t + R_r^r + R_\theta^\theta + R_z^z = 2(\phi')^2 e^{2(\Psi - \gamma)}, \] (20)

and the analogous to the magnetic case in the scalar-tensor gravity is a solution of the form:

\[ R_t^t = R_\theta^\theta, \quad R_{\theta}^\theta = -R_z^z \quad \text{and} \quad R_t^t = -R_r^r - 2(\phi')^2 e^{2(\Psi - \gamma)}. \] (21)

We are now in a position to solve the modified Einstein’s equations. The first and last equations in (18) can be solved straightforwardly:

\[ \beta(r) = Br \]
\[ \phi(r) = l \ln(r/r_0). \] (22)

The second and third equations in (18) are solved with the help of the algebraic conditions (20) and (21):

\[ \gamma'' + \frac{1}{r} \gamma' = 0, \]
\[ \Psi'' + \frac{1}{r} \Psi' - \Psi^2 = -\frac{n^2}{r^2}. \]

We, thus, find the remaining metric functions:

\[
\begin{align*}
\gamma(r) &= m^2 \ln(r/r_0) \\
\Psi(r) &= n \ln(r/r_0) - \ln \left[ \frac{(r/r_0)^{2n} + \kappa}{(1 + \kappa)} \right],
\end{align*}
\tag{23}
\]

where the constant \( n \) is related to \( l \) and \( m \) through the expression \( n^2 = l^2 + m^2 \).

Therefore, the exterior metric is given by:

\[
ds^2 = \left( \frac{r}{r_0} \right)^{-2n} W^2(r) \left[ \left( \frac{r}{r_0} \right)^{2m^2} (-dt^2 + dr^2) + B^2 r^2 d\theta^2 \right] + \left( \frac{r}{r_0} \right)^{2n} \frac{1}{W^2(r)} dz^2,
\tag{24}
\]

where

\[ W(r) \equiv \frac{(r/r_0)^{2n} + \kappa}{(1 + \kappa)}. \]

Besides, the solution for the scalar field \( \phi(r) \) in the exterior region is given by equation (22). The integration constants \( B, l, n, m \) will be fully determined after the introduction of the matter fields. In the particular case of Brans-Dicke, metric (24) belongs to a class of metrics corresponding to the case 1 in Sen’s paper \[10\], with an appropriate adjustment in the parameters.

### 3.2 The Internal Solution and Matching:

We start by considering the full modified Einstein’s equations (14) with source (15) in the internal region defined by \( r \leq r_0 \). In this region, all fields contribute to the energy-momentum tensor. In what follows, we will consider the solution for the superconducting string to linear order in \( \tilde{G}_0 \).
Therefore, we assume that the metric $g_{\mu\nu}$ and the scalar field $\phi$ can be written as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

$$\phi = \phi_0 + \phi^{(1)},$$

where $\eta_{\mu\nu} = \text{diag}(-,+,+,+)$ is the Minkowski metric tensor and $\phi_0$ is a constant. Thus, our problem reduces to solve the linearised Einstein’s equations

$$\nabla^2 h_{\mu\nu} = -16\pi G \Omega^2(\phi_0) (T^{(0)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T^{(0)}),$$

in a harmonic coordinate system such that $(h_{\nu}^{\mu} - \frac{1}{2} \delta_{\nu}^{\mu} h)_{,\nu} = 0$. $T^{(0)}_{\mu\nu}$ is the string’s energy-momentum tensor to zeroth-order in $\tilde{G}_0 = G \Omega^2(\phi_0)$ (evaluated in flat space) and $T^{(0)}$ its trace. Besides, we also need to solve the linearised equation for the scalar field

$$\nabla^2 \phi^{(1)} = -4\pi G \Omega^2(\phi_0) \alpha(\phi_0) T^{(0)}.$$ 

Then, we proceed with the junction between the internal and external solutions at $r = r_0$, with both solutions evaluated to linear order in $\tilde{G}_0$.

While doing these calculations, we will briefly recall the method of linearization using distribution functions (presented in Linet’s paper [14] and applied later by Peter and Puy in [16], in the framework of General Relativity).

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1To linear order in $\tilde{G}_0$, the modified Einstein’s equations (10) reduce to the usual linearised Einstein’s equations [14], the electromagnetic field being as in Minkowski spacetime.
3.2.1 The Linearised Field Equations:

First of all, let us evaluate the superconducting string’s energy-momentum tensor to zeroth-order in $\tilde{G}_0$ in cartesian coordinates $(t, x, y, z)$. The non-vanishing components of the energy-momentum tensor can now be re-written under the form:

\[
T_{tt}^{(0)} = -\frac{1}{2} \left[ R'' + \sigma'' + \frac{R'' P^2}{r^2} + \sigma^2 A^2 + \left( \frac{A'^2}{4\pi e^2} \right) + \left( \frac{P'^2}{4\pi q^2} \right) + 2V \right]
\]

\[
T_{tx}^{(0)} = (\cos^2 \theta - \frac{1}{2}) \left[ R'' + \sigma'' - \frac{R'' P^2}{r^2} + \left( \frac{A'^2}{4\pi e^2} \right) \right] - \frac{1}{2} \left[ \sigma^2 A^2 - \left( \frac{P'^2}{4\pi q^2} \right) + 2V \right]
\]

\[
T_{ty}^{(0)} = (\sin^2 \theta - \frac{1}{2}) \left[ R'' + \sigma'' - \frac{R'' P^2}{r^2} + \left( \frac{A'^2}{4\pi e^2} \right) \right] - \frac{1}{2} \left[ \sigma^2 A^2 - \left( \frac{P'^2}{4\pi q^2} \right) + 2V \right]
\]

\[
T_{tz}^{(0)} = -\frac{1}{2} \left[ R'' + \sigma'' + \frac{R'' P^2}{r^2} - \sigma^2 A^2 - \left( \frac{A'^2}{4\pi e^2} \right) + \left( \frac{P'^2}{4\pi q^2} \right)^2 + 2V \right]
\]

With the help of the source tensor defined by Thorne [20], we can establish some linear densities which will be very useful in our further analysis. Let

\[
M^\mu_\nu (r) \equiv -2\pi \int_0^r T^\mu_\nu (r') r' dr'
\]

be the source tensor. Let us define its components (to zeroth-order in $\tilde{G}_0$) as follows. The energy per unit length $U$:

\[
U \equiv M^t_t = -2\pi \int_0^{r_0} T^t_t dr;
\]

the tension per unit length $\tau$:

\[
\tau \equiv M^z_z = -2\pi \int_0^{r_0} T^z_z dr;
\]

and the remaining transversal components as:

\[
X \equiv M^r_r = -2\pi \int_0^{r_0} T^r_r dr;
\]

\[
Y \equiv M^\theta_\theta = -2\pi \int_0^{r_0} T^\theta_\theta dr.
\]
Now, in terms of the cartesian components of the energy-momentum tensor we can define a quantity $Z$ such that

$$Z = - \int r dr d\theta T_x^x = - \int r dr d\theta T_y^y.$$  

If we assume that the string is (idealistically) infinitely thin, then its energy-momentum tensor may be described in terms of distribution functions. Namely,

$$T^{\mu\nu} = \text{diag}(U, -Z, -Z, -\tau)\delta(x)\delta(y). \quad (28)$$

Equation (27) represents the string’s energy-momentum tensor with all quantities integrated in the internal region $r \leq r_0$, in the cartesian coordinate system. Let us now evaluate the electromagnetic energy-momentum tensor (16) to zeroth-order in $\hat{G}_0$ in cartesian coordinates. We can find easily that:

$$T^{tt}_{em} = T^{zz}_{em} = \frac{I^2}{2\pi r^2}, \quad T^{ij}_{em} = \frac{I^2}{2\pi r^4} (2x^i x^j - r^2 \delta_{ij}) \quad (29)$$

where $i, j = x, y$. Though the energy-momentum tensor (29) expresses the string’s energy in the exterior region, one can still write it in terms of distributions, taking into account the relations

$$\nabla^2 \left( \ln \frac{r}{r_0} \right)^2 = \frac{2}{r^2} \quad \text{and} \quad \partial_i \partial_j \ln \left( \frac{r}{r_0} \right) = \frac{(r^2 \delta_{ij} - 2x^i x^j)}{r^4}.$$  

Therefore, (29) becomes

$$T^{tt}_{em} = T^{zz}_{em} = \frac{I^2}{4\pi} \nabla^2 \left( \ln \frac{r}{r_0} \right)^2, \quad T^{ij}_{em} = -\frac{I^2}{2\pi \partial_i \partial_j \ln \frac{r}{r_0}}. \quad (30)$$
We are now in a position to calculate the linearised Einstein’s equations (25) with source identified by:

\[ T_{tt}^{(0)} = U\delta(x)\delta(y) + \frac{I^2}{4\pi} \nabla^2 \left( \ln \frac{r}{r_0} \right)^2, \]

\[ T_{zz}^{(0)} = -\tau\delta(x)\delta(y) + \frac{I^2}{4\pi} \nabla^2 \left( \ln \frac{r}{r_0} \right)^2, \]

\[ T_{ij}^{(0)} = I^2 \left[ \delta_{ij}\delta(x)\delta(y) - \frac{\partial_i \partial_j \ln \frac{r}{r_0}}{2\pi} \right], \quad (31) \]

and trace given by:

\[ T_{(0)} = -(U + \tau - I^2)\delta(x)\delta(y). \quad (32) \]

A straightforward calculus lead to the following solution of eq. (25):

\[ h_{00} = -4\tilde{G}_0 \left[ I^2 \ln^2 \frac{r}{r_0} + (U - \tau + I^2) \ln \frac{r}{r_0} \right], \]

\[ h_{zz} = -4\tilde{G}_0 \left[ I^2 \ln^2 \frac{r}{r_0} + (U - \tau - I^2) \ln \frac{r}{r_0} \right], \]

\[ h_{ij} = -4\tilde{G}_0 \left[ \frac{I^2}{2} r^2 \partial_i \partial_j \ln \frac{r}{r_0} + (U + \tau + I^2)\delta_{ij} \ln \frac{r}{r_0} \right]. \quad (33) \]

One can easily verify that the harmonic conditions \((h_{\mu\nu} - \frac{1}{2} h_{\kappa}\delta_{\kappa\nu}) = 0\), with \(h_{\mu\nu}\) given by (33), are identically satisfied. Using expression (32) for the trace of the energy-momentum tensor, we can solve eq. (26) straightforwardly:

\[ \phi^{(1)} = 2\tilde{G}_0 \alpha(\phi_0)(U + \tau - I^2) \ln \frac{r}{r_0}. \quad (34) \]

As expected, since the linearised (modified) Einstein’s equations are the same as in General Relativity, we re-obtained here the same solutions (34) as in refs. [14, 16]. However, the scalar-tensor feature still brings a new information coming from solution (34).
We return now to the original cylindrical coordinates system and obtain:

\[
\begin{align*}
g_{tt} &= -\left\{ 1 + 4\tilde{G}_0 \left[ I^2 \ln^2 \frac{r}{r_0} + (U - \tau + I^2) \ln \frac{r}{r_0} \right] \right\}, \\
g_{zz} &= 1 - 4\tilde{G}_0 \left[ I^2 \ln^2 \frac{r}{r_0} + (U - \tau - I^2) \ln \frac{r}{r_0} \right], \\
g_{rr} &= 1 + 2\tilde{G}_0 I^2 - 4\tilde{G}_0 (U + \tau + I^2) \ln \frac{r}{r_0}, \\
g_{\theta\theta} &= r^2 \left[ 1 - 2\tilde{G}_0 I^2 - 4\tilde{G}_0 (U + \tau + I^2) \ln \frac{r}{r_0} \right].
\end{align*}
\] (35)

In order to preserve our previous assumption that \( g_{tt} = -g_{\rho\rho} \) (corresponding to the particular case of a magnetic solution of the Einstein-Maxwell-dilaton eqs.), we make a change of variable \( r \to \rho \), such that

\[
\rho = r \left[ 1 + \tilde{G}_0 (4U + I^2) - 4\tilde{G}_0 U \ln \frac{r}{r_0} - 2\tilde{G}_0 I^2 \ln^2 \frac{r}{r_0} \right],
\]
and, thus, we have

\[
\begin{align*}
\text{ds}^2 &= \left\{ 1 + 4\tilde{G}_0 \left[ I^2 \ln^2 \frac{\rho}{r_0} + (U - \tau + I^2) \ln \frac{\rho}{r_0} \right] \right\} (-dt^2 + d\rho^2) \\
&\quad + \left\{ 1 - 4\tilde{G}_0 \left[ I^2 \ln^2 \frac{\rho}{r_0} + (U - \tau - I^2) \ln \frac{\rho}{r_0} \right] \right\} dz^2 \\
&\quad + \rho^2 \left[ 1 - 8\tilde{G}_0 (U + \frac{I^2}{2}) + 4\tilde{G}_0 (U - \tau - I^2) \ln \frac{\rho}{r_0} + 4\tilde{G}_0 I^2 \ln^2 \frac{\rho}{r_0} \right] d\theta^2.
\end{align*}
\] (36)

Expressions (34) and (36) represent, respectively, the solutions of the scalar field and an isolated current-carrying string in the conformal frame, as long as the weak-field approximation is valid. Comparison with the external solutions (22) and (24) requires a linearization of these ones since they are exact solutions. Expanding them in power series of the parameters \( m \) and \( n \), we find

\[
g_{\rho\rho} = -g_{tt} = 1 + 2m^2 \ln \frac{\rho}{r_0} + h(\rho)
\]
\begin{align*}
g_{zz} &= \frac{1}{1 + h(\rho)} \\
g_{\theta\theta} &= B^2 \rho^2 [1 + h(\rho)],
\end{align*}

with
\[ h(\rho) = 2n \frac{1 - \kappa}{1 + \kappa} \ln \frac{\rho}{r_0} + 2n^2 \frac{1 + \kappa^2}{(1 + \kappa)^2} \ln^2 \frac{\rho}{r_0}. \]

Making the identification of the coefficients of both linearised metrics, we finally obtain
\begin{align*}
m^2 &= 4 \tilde{G}_0 I^2 \\
B^2 &= 1 - 8 \tilde{G}_0 (U + \frac{I^2}{2}) \\
l &= 2 \tilde{G}_0 \alpha(\phi_0) (U + \tau - I^2) \\
\kappa &= 1 + \tilde{G}_0^{1/2} (U - \tau - I^2). \tag{37}
\end{align*}

Calculating now the deficit angle for metric (36)
\[ \Delta \theta = 2 \pi \left[ 1 - \frac{1}{\sqrt{g_{\rho \rho} d\rho \sqrt{g_{\theta \theta}}}} \right], \]
we finally obtain
\[ \Delta \theta = 4 \pi \tilde{G}_0 (U + \tau + I^2). \tag{38} \]

### 3.3 Bending of Light Rays:

A light ray coming from infinity in the transverse plane has its trajectory deflected, for an observer at infinity, by an angle given by:
\[ \Delta \theta = 2 \int_{\rho_{\text{min}}}^{\infty} d\rho \left[ -\frac{g_{\phi \phi} \rho^{-2}}{g_{\rho \rho} g_{tt}} - \frac{g_{\theta \theta}}{g_{\rho \rho}} \right]^{1/2} - \pi \]
where \( \rho_{\text{min}} \) is the distance of closest approach, given by \( \frac{d\rho}{d\theta} = 0 \):

\[
\frac{g_{\theta\theta}(\rho_{\text{min}})}{g_{tt}(\rho_{\text{min}})} = -p^2
\]

which gives in turn:

\[
\frac{\rho_{\text{min}}}{r_0} = \left( \frac{p}{Br_0} \right)^{1/(1-m^2)}.
\]

We can now evaluate the deficit angle to first order in \( \tilde{G}_0 \). Performing an expansion to linear order in this factor, in much the same way as Peter and Puy \[16\], we find:

\[
\Delta \theta = 2B(1-m^2)\left[ \frac{\pi}{2}(1 + m^2 \ln \frac{p}{B r_0}) - m^2 \nu \right] - \pi,
\]

where we have defined the quantity \( \nu \) as

\[
\nu \equiv -\int_0^1 \frac{\ln s}{\sqrt{1 - s^2}} ds = \frac{\pi}{2} \ln 2,
\]

with \( s \equiv \frac{p}{Br_0} \left( \frac{\rho}{r_0} \right)^{m^2-1} \). Using expressions (37), we have

\[
\Delta \theta = 4\pi \tilde{G}_0 \left[ U + I^2 \left( \frac{3}{2} + \ln \frac{\rho}{r_0} \right) \right] + 8\nu \tilde{G}_0 I^2. \quad (39)
\]

### 4 Conclusion

In this work we studied the modifications induced by a scalar-tensor gravity on the metric of a current-carrying string described by model with action given by eq. (2). For this purpose, we made an adaptation of Linet’s method which consists in linearising the Einstein’s and dilaton’s equations using distribution’s functions while taking into account the scalar-tensor feature of gravity. We found that the metric depends on five parameters which are
related to the string’s internal structure and to the scalar field (dilaton) solution. Concerning the deflection of light, if we compare our results with those obtained in General Relativity, we see that expression (39) does not change substantially, albeit the metric structure is indeed modified with respect to the one in General Relativity.

Now, an interesting investigation that opens up, and we have already initiated to pursue, is the analysis of the properties of the cosmic string generated by the action (2) in the supersymmetrized version, where it is implicit that the scalar-tensor degrees of freedom of the gravity sector are accommodated in a suitable supergravity multiplet. The study of such a model raises the question of understanding the rôle played by the fermionic partners of the bosonic matter and by the gravitino in the configuration of a string. Also, it might be of relevance to analyse the possibility of gaugino and gravitino condensation in this scenario.

Acknowledgements

The authors are grateful to Brandon Carter, Bernard Linet and Patrick Peter for many discussions, suggestions and a critical reading of this manuscript. One of the authors (MEXG) thanks to the Centro Brasileiro de Pesquisas Físicas (in particular, the Departamento de Campos e Partículas) and to the Abdus Salam ICTP-Trieste for hospitality during the preparation of part of this work. CNF thanks to CNPq for a PhD grant.
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