EXACT BOUNDARY CONTROLLABILITY FOR THE
BOUSSINESQ EQUATION WITH VARIABLE COEFFICIENTS

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Abstract. In this paper we study the exact boundary controllability for the following Boussinesq equation with variable physical parameters:
\[
\begin{align*}
\rho(x)y_{tt} &= -(\sigma(x)y_{xx})_{xx} + (q(x)y_{x})_x - (y^2)_{xx}, & t > 0, \; x \in (0, l), \\
y(t, 0) &= y_{xx}(t, 0) = y(t, l) = 0, & \sigma(l)y_{xx}(t, l) = u(t) & t > 0,
\end{align*}
\]
where \( l > 0 \), the coefficients \( \rho(x) > 0, \sigma(x) > 0, q(x) \geq 0 \) in \([0, l]\) and \( u \) is the control acting at the end \( x = l \). We prove that the linearized problem is exactly controllable in any time \( T > 0 \). Our approach is essentially based on a detailed spectral analysis together with the moment method. Furthermore, we establish the local exact controllability for the nonlinear problem by fixed point argument. This problem has been studied by Crépeau [Diff. Integ. Equat., 2002] in the case of constant coefficients \( \rho \equiv \sigma \equiv q \equiv 1 \).

1. Introduction. Let \( T > 0 \) and \( l > 0 \). The classical Boussinesq equation on the bounded domain \((0, l)\) is of the form
\[
y_{tt} + \sigma y_{xxxx} - y_{xx} + (y^2)_{xx} = 0, \quad (t, x) \in (0, T) \times (0, l),
\]
where the coefficient \( \sigma \in \mathbb{R} \). This equation was derived by the French mathematician Joseph Boussinesq [5] in 1872 as a model for the propagation of small amplitude of long waves on the surface of water. This was the first to give a scientific explanation of the existence of solitary waves found by Scott Russell's in the 1840's. Depending on whether the coefficient \( \sigma \) in (1) is positive or negative, Equation (1) nowadays known as the “good” or the “bad” Boussinesq equation. In fact, the “bad” Boussinesq equation is linearly unstable and admits the inverse scattering approach [9, 24]. For this reason, we only consider the version of the “good” Boussinesq equation with variable coefficients. These equations arise as a model of nonlinear vibrations along a beam [24], and also for describing electromagnetic waves in nonlinear dielectric materials [23].

From a mathematical point of view, well-posedness and dynamic properties of “good” Boussinesq equations have a huge literature, see the paper by Bona and Sachs [4], see also [12, 17] and references therein.

We are concerned with the exact boundary controllability for the “good” Boussinesq equation with variable coefficients. In this direction, the case of the linear

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“good” Boussinesq equation with constant coefficients has been investigated by Lions in [19]. In that reference, by Hilbert Uniqueness Method “Lions’HUM” (cf. Lions [18, 19]), it was proved that the linearized “good” Boussinesq system

\[
\begin{align*}
y_{tt} &= -y_{xxxx} + y_{xx}, & (t, x) &\in (0, T) \times (0, l), \\
y(t, 0) &= y_{xx}(t, 0) = 0, & t &\in (0, T), \\
y(t, l) &= \tilde{u}(t), & y_{xx}(t, l) &= u(t), & t &\in (0, T), \\
y(0, x) &= y^0, & y_t(0, x) &= y^1, & x &\in (0, l),
\end{align*}
\]

is exactly controllable in any time \(T > 2(l + \frac{1}{\sqrt{\lambda_0}})\), where the two controls \((\tilde{u}, u) \in L^2(0, T) \times H^1(0, T)\) and \(\lambda_0\) denotes the first eigenvalue of the operator \(-\Delta\) with the Dirichlet boundary conditions. Later on, Zhang [25] studied the question of distributed control for the generalized “good” Boussinesq equation with constant coefficients on a periodic domain. A few years after, Crépeau extended in [8] the results obtained in [19]. More precisely, by a detailed spectral analysis and the use of nonharmonic Fourier series, it was shown that System (2)-(5) (for \(\tilde{u} \equiv 0\)) is exactly controllable at any time \(T > 0\), where the control \(u \in L^2(0, T)\).

Furthermore, with the help of the fixed point theorem, it was also proved the local controllability for the nonlinear control problem (1), (3) and (4) (for \(\sigma \equiv 1\) and \(\tilde{u} \equiv 0\)). Concerning the control of the approached “bad” Boussinesq equation, the controllability properties for the so called improved Boussinesq equation have been obtained recently by Cerpa and Crépeau [6].

In the previous studies of controllability, the coefficients of the “good” Boussinesq equation are supposed to be constant. In the present paper, we address the problem of exact boundary controllability for the “good” Boussinesq equation with variable coefficients. More precisely, we consider the following control problem

\[
\left\{
\begin{array}{ll}
\rho(x)y_{tt} = -[(\sigma(x)y_{xx})_{xx} + (q(x)y_x)_x - (y^2)_{xx}], & (t, x) \in (0, T) \times (0, l), \\
y(t, 0) = y_{xx}(t, 0) = y(t, l) = 0, & y_{xx}(t, l) = u(t), & t &\in (0, T), \\
y(0, x) = y^0, & y_t(0, x) = y^1,
\end{array}
\right.
\]

where \(u\) is a control placed at the extremity \(x = l\), and the functions \(y^0, y^1\) are the initial conditions. Here and in what follows, we assume that the coefficients

\[
\rho, \sigma \in H^2(0, l), \quad q \in H^1(0, l),
\]

and there exist constants \(\rho_0, \sigma_0 > 0\), such that

\[
\rho(x) \geq \rho_0, \quad \sigma(x) \geq \sigma_0, \quad q(x) \geq 0, \quad x \in [0, l].
\]

In this paper we prove that the linearized problem

\[
\left\{
\begin{array}{ll}
\rho(x)y_{tt} = -[(\sigma(x)y_{xx})_{xx} + (q(x)y_x)_x], & (t, x) \in (0, T) \times (0, l), \\
y(t, 0) = y_{xx}(t, 0) = y(t, l) = 0, & \sigma(l)y_{xx}(t, l) = u(t), & t &\in (0, T), \\
y(0, x) = y^0, & y_t(0, x) = y^1,
\end{array}
\right.
\]

is exactly controllable in any time \(T > 0\), where the control \(u \in L^2(0, T)\) and the initial conditions \((y^0, y^1)\) taken in \(H_0^1(0, l) \times H^{-1}(0, l)\). Our approach is essentially based on a detailed spectral analysis and the qualitative theory of fourth-order linear differential equations due to Leighton and Nehari [15]. More precisely, we prove that all the eigenfrequencies \(\sqrt{\lambda_{n+1}}\) associated with System (9) (without control) are simple, and by a precise computation of its asymptotics we show that the spectral gap \(|\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n}|^{1/4}\) is of order \(O(n)\). Moreover, we prove that the first derivative of each eigenfunction \(\phi_n, \ n \geq 1\), associated with uncontrolled system does not
vanish at the end \( x = l \). As a consequence of the theory of non-harmonic Fourier series and an extension of Ingham’s Theorem due to Haraux [11], we establish the equivalence between the \( H^1_0 \times H^{-1} \)-norm of the initial data \((\tilde{y}^0, \tilde{y}^1)\) and the quantity \( \int_0^T |\dot{y}_x(t, l)|^2 dt \), where \( \tilde{y} \) is the solution of System (9) without control. Finally, we apply the Lions’ HUM to deduce the exact controllability result for the system (9).

At the end of this paper, we will discuss the local exact controllability for the nonlinear control system (6). To this end, we use some results obtained by Crépeau [8] together with standard fixed-point method (e.g., [7, Chapter 4] and [22]).

The rest of the paper is divided in the following way: In Section 2, we establish the well-posedness of System (9) without control. In Section 3, we prove the simplicity of all the eigenvalues \((\lambda_n)_{n \geq 1}\) and we determinate the asymptotics of the associated spectral gap. In Section 4, we prove the exact controllability result for the linear control problem (9). The last section is devoted to the local controllability for the nonlinear control problem (6).

2. Operator framework and well-posedness. In this section we investigate the well-posedness of the linear homogeneous Boussinesq problem,

\[
\begin{aligned}
\rho(x) y_{tt} &= -(\rho(x) y_{xx})_{xx} + q(x) y_{xx}, \\
y(t,0) &= y_{xx}(t,0) = y(t,l) = y_{xx}(t,l) = 0, \\
y(0,x) &= y^0, \quad y_t(0,x) = y^1,
\end{aligned}
\tag{10}
\]

First of all, let us define by \( L^2_\rho(0,l) \) the space of functions \( f \) such that

\[
\int_0^l |f|^2 \rho(x) dx < \infty,
\]

and we denote by \( H^k(0,l) \) the \( L^2_\rho(0,l) \)-based Sobolev spaces for \( k \geq 0 \). We consider the following Sobolev space

\[
H^2(0,l) \cap H^1_0(0,l)
\tag{11}
\]

endowed with the norm \( \|u\|_{H^2(0,l) \cap H^1_0(0,l)} = \|u''\|_{L^2_\rho(0,l)} \). It is easily seen from Rellich’s theorem that the space \( H^2(0,l) \cap H^1_0(0,l) \) is densely and compactly embedded in the space \( L^2_\rho(0,l) \). In the sequel, we introduce the operator \( A \) defined in \( L^2_\rho(0,l) \) by setting:

\[
A(y) = \rho^{-1} ((\sigma y'')'' - (qy'))',
\tag{12}
\]

on the domain

\[
D(A) = \{ y \in H^4(0,l) \text{ such that } y, y'' \in H^1_0(0,l) \},
\tag{13}
\]

which is dense in \( L^2_\rho(0,l) \).

**Lemma 2.1.** The linear operator \( A \) is positive and self-adjoint such that \( A^{-1} \) is compact. Moreover, \( A^2 \) generates a strongly continuous semi-group on \( L^2_\rho(0,l) \).

**Proof.** Let \( y \in D(A) \), then by integration by parts, we have

\[
\langle Ay, y \rangle_{L^2_\rho(0,l)} = \int_0^l \left( (\sigma(x)y'')'' - (q(x)y')' \right) y dx
\]

\[
= \int_0^l \sigma(x)|y''|^2 dx + q(x)|y'|^2 dx
\tag{14}
\]

since \( \sigma > 0 \) and \( q \geq 0 \) then \( \langle Ay, y \rangle_{L^2_\rho(0,l)} > 0 \), and hence the quadratic form has a positive real values, so the linear operator \( A \) is symmetric. Furthermore, it is easy to show that \( \text{Ran}(A - iId) = L^2_\rho(0,l) \), which implies that \( A \) is selfadjoint. Since
the space $H^2(0, l) \cap H_0^1(0, l)$ is continuously and compactly embedded in the space $L^2_{\rho}(0, l)$, then $A^{-1}$ is compact in $L^2_{\rho}(0, l)$. \hfill \Box

Lemma 2.1 leads to the following corollary.

**Corollary 2.2.** The spectrum of the operator $A$ is discrete. It consists of a sequence of positive eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ tending to $+\infty$:

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \rightarrow_{n \to +\infty} +\infty.$$  

Moreover, the corresponding eigenfunctions $(\Phi_n)_{n \geq 1}$ can be chosen to form an orthonormal basis in $L^2_{\rho}(0, l)$.

We give now a characterization of some fractional powers of the linear operator $A$ which will be useful to give a description of the solutions of Problem (10) in terms of Fourier series. According to Lemma 2.1, the operator $A$ is positive and self-adjoint, and hence it generates a scale of interpolation spaces $H_{\theta}$, $\theta \in \mathbb{R}$. For $\theta \geq 0$, the space $H_{\theta}$ coincides with $D(A^\theta)$ and is equipped with the norm $\|u\|_\theta^2 = \langle A^\theta u, A^\theta u \rangle_{L^2(0, l)}$, and for $\theta < 0$ it is defined as the completion of $L^2_{\rho}(0, l)$ with respect to this norm. Furthermore, we have the following spectral representation of space $H_{\theta}$,

$$H_{\theta} = \{ u(x) = \sum_{n \in \mathbb{N} \setminus \{0\}} c_n \Phi_n(x) : \|u\|_{\theta}^2 = \sum_{n \in \mathbb{N} \setminus \{0\}} \lambda_n^{2\theta} |c_n|^2 < \infty \},$$

where $\theta \in \mathbb{R}$, and the eigenfunctions $(\Phi_n)_{n \geq 1}$ are defined in Corollary 2.2. In particular, $H_{0} = L^2_{\rho}(0, l)$ and $H_{1/2} = H^2(0, l) \cap H^1_0(0, l)$.

Obviously, the linear problem (10) can be rewritten in the abstract form

$$\ddot{y}(t) + A y(t) = 0, \quad (y(0), \dot{y}(0)) = (y^0, y^1), \quad t \geq 0,$$

where $A$ is defined by (12). As a consequence of the spectral decomposition of the operator $A$ and by [14, Theorem 1.1], we have the following existence and uniqueness result for Problem (10) in the spaces $H_{\theta} \times H_{\theta-1/2}$ with $\theta \in \mathbb{R}$.

**Proposition 2.3.** Let $\theta \in \mathbb{R}$ and $(y^0, y^1) \in H_{\theta} \times H_{\theta-1/2}$. Then Problem (10) has a unique solution $y \in C([0, T], H_{\theta}) \cap C^1([0, T], H_{\theta-1/2})$ and is given by the following Fourier series

$$y(t, x) = \sum_{n \in \mathbb{N} \setminus \{0\}} \left( a_n \cos(\sqrt{\lambda_n} t) + \frac{b_n}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n} t) \right) \Phi_n(x), \quad (15)$$

where $y^0 = \sum_{n \in \mathbb{N} \setminus \{0\}} a_n \phi_n$ and $y^1 = \sum_{n \in \mathbb{N} \setminus \{0\}} b_n \phi_n$.

3. **Spectral analysis.** In this section, we shall establish the spectral properties related to System (10). To this end we need some results of the qualitative theory of fourth-order linear differential equations due to Leighton and Nehary [15], see also [3, 1, 2].

We consider the following spectral problem which arises by applying separation of variables to System (10),

$$\sigma(x) \phi''''(x) - (q(x) \phi')' = \lambda \rho(x) \phi, \quad x \in (0, l), \quad (16)$$

$$\phi(0) = \phi''(0) = \phi(l) = 0, \quad (17)$$

$$\phi''(l) = 0. \quad (18)$$

Our first main result in this section is the following:
Theorem 3.1. All the eigenvalues \((\lambda_n)_{n \geq 1}\) of the spectral problem (16)-(18) are simple. Moreover, the corresponding eigenfunctions \((\Phi_n)_{n \geq 1}\) satisfy
\[
\Phi'_n(l)T\Phi_n(l) < 0 \quad \text{for all } n \in \mathbb{N}\setminus\{0\},
\]
where \(T\Phi_n = (\sigma(x)\Phi''_n) - q(x)\Phi'_n\).

In order to prove this theorem, we need the following result.

Lemma 3.2. Let \(\phi\) be a nontrivial solution of the differential equation
\[
(\sigma(x)\phi'')'' - \rho(x)\phi = 0.
\]
If \(\phi, \phi', \phi''\) and \((\sigma\phi'')'\) are nonnegative at \(x = a\) (but not all zero) they are positive for all \(x > a\). If \(\phi, -\phi', \phi''\) and \(-(\sigma\phi'')'\) are nonnegative at \(x = a\) (but not all zero) they are positive for all \(x < a\).

In the case \(q \geq 0\) or if the second-order equation
\[
(\sigma(x)h')' - q(x)h = 0, \quad x \in (0, l),
\]
has a positive solution, they gave a transformation [15, Theorem 12.1] for removing the “middle term” \((q(x)\phi')'\) from Equation (16). Namely, if \(h\) is a positive solution of the equation (21) then the following modified substitution [15, Theorem 12.1]
\[
s(x) := l\omega^{-1}\int_0^x h(t)dt, \quad \omega = \int_0^l h(t)dt,
\]
transform (16) into the equation
\[
(\tilde{\sigma}(s)\tilde{\phi})'' = \lambda\tilde{\rho}(s)\tilde{\phi}, \quad s \in (0, l),
\]
where \(\tilde{\sigma}(s) = (\omega l^{-1}h(x(s)))^3\sigma(x(s)), \tilde{\rho}(s) = l^{-1}\omega h^{-1}(x(s))\rho(x(s))\) and \(\cdot := \frac{d}{ds}\). If \(\phi\) is a nontrivial solution of (16), then \(\varphi \equiv \phi(x(s))\) is a nontrivial solution of (23).

Furthermore, we have the following relations:
\[
\dot{\varphi} = l^{-1}\omega h^{-1}\phi', \quad l^{-2}\omega^2 h^3\dot{\varphi} = h\phi'' - \phi' h', \quad (\tilde{\sigma}\tilde{\phi})'' = T\phi.
\]
We are now ready to prove Theorem 3.1.

Proof. We first prove that the set \(E_\lambda\) of solutions of the following boundary value problem
\[
\begin{cases}
(\sigma(x)\phi'')'' - (q(x)\phi')' = \lambda\rho(x)\phi, & x \in (0, l),
\phi(0) = \phi''(0) = 0, \\
\phi''(l) = 0,
\end{cases}
\]
is one-dimensional subspace. To do this let \(h\) denotes the solution of Equation (21) satisfying the initial conditions
\[
h(0) = 1, \quad h'(0) = 0.
\]
It is known, by Sturm oscillation theorem [16, Chapter 1] that \(h(x) > 0\) on \([0, l]\).
Since
\[
\sigma(x)h'(x) = \int_0^x q(x)h(x)\rho(x)dx,
\]
we have also, \(h'(x) > 0\) on \([0, l]\). Furthermore, by using the transformation (22) and the relations (24), the boundary value problem (25) can be rewritten in the form
\[
(\tilde{\sigma}(s)\tilde{\varphi})'' = \lambda\tilde{\rho}(s)\tilde{\varphi}, \quad s \in (0, l),
\]
\[
\varphi(0) = \tilde{\varphi}(0) = 0,
\]
\[
l^{-2}\omega^2 h^3\dot{\varphi}(l) = -h'\dot{\varphi}(l).
\]
Let $\varphi_1$ and $\varphi_2$ be two linearly independent solutions of (27)-(29). Both $\varphi_1(0)$ and $\varphi_2(0)$ are different from zero since otherwise the first statement of Lemma 3.2 would imply that $\varphi_1'(l) > 0$ and $\varphi_2'(l) > 0$, and this is in contradiction with the boundary condition (29). In view of the assumptions about $\varphi_1$ and $\varphi_2$, the solution $\psi$ defined by

$$\psi(s) = \varphi_2(0)\varphi_1(s) - \varphi_1(0)\varphi_2(s)$$

satisfies $\psi(0) = \dot{\psi}(0) = 0$ and $\dot{\psi}(l) \leq 0$. This again contradicts Lemma 3.2 unless $\psi \equiv 0$, which proves that $\dim \mathcal{E}_\lambda = 1$. Therefore, each eigenvalue $\lambda_n$ ($n \geq 1$) is geometrically simple. On the other hand, by Proposition 2.1, the operator $A$ is self-adjoint in $L^2_{\rho}(0, l)$, and this implies that all the eigenvalues $(\lambda_n)_{n \geq 1}$ are algebraically simple.

Let us now prove (19). Let $\{\lambda_n, \Phi_n\}_{n \geq 1}$ be an eigenpair of Problem (16)-(18), and let $\tilde{h}$ be the solution of Equation (21) satisfying the following initial conditions

$$\tilde{h}(l) = 1, \quad \tilde{h}'(l) = 0.$$  

In a same way as above, by Sturm oscillation theorem, one can prove that $\tilde{h} > 0$ and $\tilde{h}' > 0$ in $[0, l]$. Furthermore, by using the substitution (22), Problem (16)-(18) transforms into

$$\begin{align*}
(\dot{\sigma}(s)\dot{\varphi})^- &= \lambda \rho(s)\varphi, \quad s \in (0, l), \\
\varphi(0) &= 0, \quad l^{-2}\omega^2\tilde{h}^3 \dot{\varphi}(0) = -\tilde{h}'\dot{\varphi}(0), \\
\varphi(l) &= 0, \quad \varphi(l) = 0.
\end{align*}$$

(31) (32) (33)

Suppose that for some $n \geq 1$, $\Phi_n'(l) \mathcal{T}\Phi_n(l) \geq 0$. Then from the relations (24) we have

$$\dot{\varphi}_n(l)(\dot{\sigma}\dot{\varphi}_n)'(l) \geq 0.$$  

It follows from the second statement of Lemma 3.2, that $\varphi_n(0) \neq 0$, but this contradicts the first boundary condition in (32). Hence, $\Phi_n'(l) \mathcal{T}\Phi_n(l) < 0$ for all $n \in \mathbb{N}\setminus\{0\}$. This finalizes the proof of the theorem.

Our second main result in this section establishes the asymptotic behavior of the spectral gap $\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n}$ for large $n$. Namely, we enunciate the following result:

**Theorem 3.3.** The eigenvalues $(\lambda_n)_{n \geq 1}$ of the associated spectral problem (16)-(18) satisfy the following asymptotics:

$$\lambda_n = \left( \frac{n\pi}{\int_0^l \frac{\sqrt{\rho(t)} \dot{\sigma}(t)}{\sigma(t)} \, dt} \right)^4 + \mathcal{O}(n^2),$$

(34)

as $n \to \infty$. Furthermore,

$$\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} = \mathcal{O}(n).$$

(35)

**Proof.** It is known (e.g., [10, Chapter 5, p.235-239] and [21, Chapter 2]) that for $\lambda \in \mathbb{C}$, Equation (16) has four fundamental solutions $\{\phi_i(x, \lambda)\}_{i=1}^4$ satisfying the asymptotic forms

$$\begin{align*}
\phi_i(x, \lambda) &= \left( [\rho(x)]^{\frac{1}{2}} [\sigma(x)]^{\frac{1}{2}} \right)^{\frac{1}{2}} \exp \left\{ \mu \omega_i \int_0^x \frac{\sqrt{\rho(t)} \dot{\sigma}(t)}{\sigma(t)} \, dt \right\} [1], \\
\phi_i^{(k)}(x, \lambda) &= (\mu \omega_i)^{\frac{k}{2}} \left( [\rho(x)]^{\frac{1}{2}} [\sigma(x)]^{\frac{1}{2}} \right)^{\frac{1}{2}} \exp \left\{ \mu \omega_i \int_0^x \frac{\sqrt{\rho(t)} \dot{\sigma}(t)}{\sigma(t)} \, dt \right\} [1].
\end{align*}$$

(36)
where \(\mu^2 = \lambda, \omega_s^4 = 1, \phi^{(k)} := \frac{\partial^k \phi}{\partial x^k}\) for \(k \in \{1, 2, 3\}\), and \([1] = 1 + \mathcal{O}(\mu^{-1})\) uniformly as \(\mu \to \infty\) in a sector \(S_\tau = \{\mu \in \mathbb{C} \text{ such that } 0 \leq \arg(\mu + \tau) \leq \frac{\pi}{4}\}\) where \(\tau\) is any fixed complex number. It is convenient to rewrite these asymptotes in the form

\[
\phi_1(x, \lambda) = \zeta(x) \cos(\mu X)[1],
\phi_2(x, \lambda) = \zeta(x) \cosh(\mu X)[1],
\phi_3(x, \lambda) = \zeta(x) \sin(\mu X)[1],
\phi_4(x, \lambda) = \zeta(x) \sinh(\mu X)[1],
\]

where

\[
\zeta(x) = \left((|\rho(x)|^2 |\sigma(x)|^2)^{\frac{1}{2}}\right)^{\frac{1}{2}} \quad \text{and} \quad X = \int_0^x \sqrt{\rho(t) \sigma(t)} \, dt. \tag{37}
\]

Hence every solution \(\phi(x, \lambda)\) of Equation (16) can be written in the following asymptotic form

\[
\phi(x, \lambda) = \zeta(x) \left(C_1 \cos(\mu X) + C_2 \cosh(\mu X) + C_3 \sin(\mu X) + C_4 \sinh(\mu X)\right)[1]. \tag{38}
\]

and from (36), we have also

\[
\phi^{(k)}(x, \lambda) = \mu^k \zeta(x) \left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{1}{4}} \left(C_1 \cos^{(k)}(\mu X) + C_2 \cosh^{(k)}(\mu X) + C_3 \sin^{(k)}(\mu X) + C_4 \sinh^{(k)}(\mu X)\right)[1], \tag{39}
\]

where \(C_i, i = 1, 2, 3, 4\) are constants. If \(\phi(x, \lambda)\) satisfies the boundary conditions (17), then by the asymptotics (38) and (39), we obtain for large positive \(\mu\) the asymptotic estimate

\[
\phi(x, \lambda) = C_3 \zeta(x) \left(\frac{\sin(\mu X) - \sin(\mu \gamma) \sinh(\mu X)}{\sinh(\mu \gamma)}\right)[1]
= \frac{C_3 \zeta(x)}{\sinh(\mu \gamma)} \left(\sin(\mu X) \sinh(\mu \gamma) - \sin(\mu \gamma) \sinh(\mu X)\right)[1], \tag{40}
\]

and

\[
\phi''(x, \lambda) = -\frac{C_3 \mu^2 \zeta(x)}{\sinh(\mu \gamma)} \left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{1}{4}} \left(\sin(\mu X) \sinh(\mu \gamma) + \sin(\mu \gamma) \sinh(\mu X)\right)[1], \tag{41}
\]

where the constant \(\gamma\) is defined by

\[
\gamma = \int_0^1 \sqrt{\frac{\rho(t)}{\sigma(t)}} \, dt. \tag{42}
\]

It is clear that the eigenvalues \(\lambda_n (n \geq 1)\) are the solutions of the equation \(\phi''(l, \lambda) = 0\). Then by (41), one gets the following asymptotic characteristic equation

\[-2\mu^2 \zeta(l) \left(\frac{\rho(l)}{\sigma(l)}\right)^{1/2} \sin(\mu \gamma) \left[1 + \mathcal{O}(\mu^{-1})\right] = 0,
\]

which can also be rewritten as

\[
\sin(\mu \gamma) + \mathcal{O}(\mu^{-1}) = 0. \tag{43}
\]
Since the solutions of the equation \( \sin(\mu \gamma) = 0 \) are given by \( \tilde{\mu}_n = \frac{n\pi}{\gamma} \), \( n = 0, 1, 2, \ldots \), it follows from Rouché’s theorem that the solutions of (43) satisfy the following asymptotic
\[
\mu_n = \tilde{\mu}_n + \delta_n = \frac{n\pi}{\gamma} + \mathcal{O}(n^{-1}), \text{ as } n \to \infty,
\]
which proves (34). Furthermore, \( \sqrt{\lambda_n} = \left( \frac{n\pi}{\gamma} \right)^2 + \mathcal{O}(1) \), and hence
\[
\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \sim \mathcal{O}(n), \text{ as } n \to \infty.
\]
The theorem is proved. \( \square \)

**Proposition 3.4.** The eigenfunctions \( (\Phi_n)_{n \geq 1} \) of the spectral problem (16)-(18) satisfy the following asymptotic estimates:
\[
\Phi_n(x) = \frac{\sqrt{2} \zeta(x)}{\| \zeta \|_{L^2_0(0,l)}} \sin \left\{ \frac{n\pi}{\gamma} \int_0^x \sqrt{\frac{\rho(t)}{\sigma(t)}} \, dt \right\} + \mathcal{O}(n^{-1}),
\]
Furthermore,
\[
\lim_{n \to \infty} |\lambda_n^{-\frac{1}{2}} \Phi'_n(l)| = \frac{\sqrt{2} \zeta(l)}{\| \zeta \|_{L^2_0(0,l)}} \left( \frac{\rho(l)}{\sigma(l)} \right)^{\frac{1}{4}},
\]
where the function \( \zeta \) is defined by (37).

**Proof.** From (40) and (44), we obtain the following asymptotic estimate for the eigenfunctions \( (\Phi_n)_{n \geq 1} \):
\[
\Phi_n(x) = C \zeta(x) \sin \left\{ \frac{n\pi}{\gamma} \int_0^x \sqrt{\frac{\rho(t)}{\sigma(t)}} \, dt \right\} + \mathcal{O}(n^{-1}) \text{ as } n \to \infty,
\]
where \( C \) is a constant and \( \gamma \) is defined by (42). Taking into account that \( \| \Phi_n \|_{L^2_0(0,l)} = 1 \), a simple computation gives
\[
C = \sqrt{2} \| \zeta \|_{L^2_0(0,l)}^{-1} + \mathcal{O}(n^{-1}),
\]
which proves (45).

In a similar way, from the asymptotics (39), (44) and (47), a straightforward computation yields
\[
\Phi'_n(x) = \sqrt{2} \| \zeta \|_{L^2_0(0,l)}^{-1} \zeta(x) \left( \frac{n\pi}{\gamma} \right) \left( \frac{\rho(x)}{\sigma(x)} \right)^{\frac{1}{4}} \cos \left\{ \frac{n\pi}{\gamma} \int_0^x \sqrt{\frac{\rho(t)}{\sigma(t)}} \, dt \right\} + \mathcal{O}(1),
\]
and hence
\[
| \Phi'_n(l) | \sim \sqrt{2} \| \zeta \|_{L^2_0(0,l)}^{-1} \zeta(l) \left( \frac{n\pi}{\gamma} \right) \left( \frac{\rho(l)}{\sigma(l)} \right)^{\frac{1}{4}}, \text{ as } n \to \infty.
\]
The proof is complete. \( \square \)
4. Exact controllability for the linear problem. The goal of this section is to prove the exact controllability for the linear control problem (9). To this end, we first prove the observability results which are consequence of the spectral properties given in Section 3.

**Proposition 4.1.** Let $T > 0$ and $(y^0, y^1) \in H^1_0(0, l) \times H^{-1}(0, l)$. Then

$$\int_0^T |y_x(t, l)|^2 dt \asymp \|(y^0, y^1)\|_{H^1_0(0, l) \times H^{-1}(0, l)}^2,$$

where $y$ is the solution of Problem (10).

In order to prove Proposition 4.1, we need the following variant of Ingham’s inequality due to Haraux [11].

**Lemma 4.2.** [11] Let $f(t) = \sum_{n \in \mathbb{Z}} c_n e^{-i\lambda_n t}$, where $(\lambda_n)_{n \in \mathbb{Z}}$ is a sequence of real numbers. We assume that there exist $N \in \mathbb{N}$, $\beta > 0$ and $\varrho > 0$ such that

$$|\lambda_{n+1} - \lambda_n| > \varrho, \text{ if } |n| > N,$$

and $|\lambda_{n+1} - \lambda_n| > \beta$, for all $n \in \mathbb{Z}$. Then for any $T > \frac{\pi}{\varrho}$,

$$\int_0^T |f(t)|^2 dt \asymp \sum_{n \in \mathbb{Z}} |c_n|^2,$$

for all sequences of complex numbers $(c_n)_{n \in \mathbb{Z}} \in \ell^2$.

**Proof of Proposition 4.1.** Let us first recall from the spectral representation of the space $H_0$ that

$$\mathcal{H}_{1/4} = \{u(x) = \sum_{n \in \mathbb{N}\setminus\{0\}} c_n \Phi_n(x) : \|u\|_0^2 = \sum_{n \in \mathbb{N}\setminus\{0\}} \lambda_n^{1/2} |c_n|^2 < \infty\}$$

and

$$\mathcal{H}_{-1/4} = \{u(x) = \sum_{n \in \mathbb{N}\setminus\{0\}} c_n \Phi_n(x) : \|u\|_0^2 = \sum_{n \in \mathbb{N}\setminus\{0\}} \lambda_n^{-1/2} |c_n|^2 < \infty\},$$

where the eigenfunctions $(\Phi_n)_{n \geq 1}$ are defined in Corollary 2.2. Since the coefficients $\rho$, $\sigma$ and $q$ are bounded in $[0, l]$, then from the asymptotic estimates (34) and (45), one gets

$$\mathcal{H}_{1/4} = H^1_0(0, l) \quad \text{and} \quad \mathcal{H}_{-1/4} = H^{-1}(0, l).$$

On the other hand, by (15), one has

$$\int_0^T |y_x(t, l)|^2 dt = \int_0^T \left| \sum_{n \in \mathbb{Z}\setminus\{0\}} c_n e^{-i\lambda_n t} \Phi'_n(l) \right|^2 dt,$$

where $\tilde{\lambda}_n = \sqrt{\lambda_n}$, and

$$\begin{cases} 
\tilde{\lambda}_n = -\lambda_n \quad \text{and} \quad \Phi_n = \Phi_{-n}, \quad n \in \mathbb{N}\setminus\{0\}, \\
\tilde{c}_n = \overline{c_n} = \frac{1}{2} \left( a_n - i \frac{b_n}{\sqrt{\lambda_n}} \right), \quad n \in \mathbb{N}\setminus\{0\}. 
\end{cases}$$

(50)

Therefore, in view of the first statement of Theorem 3.1 and the gap condition (35), Lemma 4.2 implies that for every $T > 0$

$$\int_0^T |y_x(t, l)|^2 dt \asymp \sum_{n \in \mathbb{Z}\setminus\{0\}} |c_n \Phi'_n(l)|^2 = \frac{1}{2} \left( \sum_{n \in \mathbb{N}\setminus\{0\}} |a_n \Phi'_n(l)|^2 + \frac{b_n}{\sqrt{\lambda_n}} |\Phi'_n(l)|^2 \right).$$

(51)
Moreover, by the second statement of Theorem 3.1, we have \( \Phi'_n(l) \neq 0 \) for all \( n \in \mathbb{N} \setminus \{0\} \). Thus by (46), there exists \( m, M > 0 \) such that
\[
m \sqrt{\lambda_n} < |\Phi'_n(l)| < M \sqrt{\lambda_n}.
\]
Consequently,
\[
\int_0^T |y_s(t, l)|^2 dt \leq \sum_{n \in \mathbb{N} \setminus \{0\}} \sqrt{\lambda_n}|a_n|^2 + \sum_{n \in \mathbb{N} \setminus \{0\}} \frac{|b_n|^2}{\sqrt{\lambda_n}} = \|y^0, y^1\|_{H^2(0, l) \times H^{-1}(0, l)}^2,
\]
which proves (48). This completes the proof.

Let us now state the existence and uniqueness result for the control system (9). Following [14, Theorem 2.14] and [8], we define a weak solution to the control system (9) using the method of transposition.

**Proposition 4.3.** Let \( T > 0 \), and \( u \in L^2(0, T) \). For any \( (y^0, y^1) \in H^1_0(0, l) \times H^{-1}(0, l) \), there exists a unique weak solution \( y \) of System (9) in the class
\[
(y, y_t) \in C \left([0, T]; H^1_0(0, l) \times H^{-1}(0, l)\right).
\]
Moreover, there exists a constant \( C > 0 \) such that
\[
\|y, y_t\|_{H^1_0(0, l) \times H^{-1}(0, l)} \leq C \left(\|y^0, y^1\|_{H^1_0(0, l) \times H^{-1}(0, l)} + \|u\|_{L^2(0, T)}\right).
\]
This result is basically well known for \( \rho = \sigma = q = 1 \) (see [8, Proposition 10]). The proof can be also easily extended to the variable coefficient case.

We are now ready to state the main result of this section. Notice that, in view of the fact that (9) is linear and reversible in time, this system is exactly controllable if and only if the system is null controllable.

**Theorem 4.4.** Assume that the coefficients \( \rho, \sigma \) and \( q \) satisfy (7) and (8). Given \( T > 0 \) and \((y^0, y^1) \in H^1_0(0, l) \times H^{-1}(0, l)\), there exists a control \( u \in L^2(0, T) \) such that the solution \( y \) of the control problem (9) satisfies
\[
y(T, x) = y_0(T, x) = 0, \quad x \in [0, l].
\]

**Proof.** Following [8, Proposition 11], we apply the Lions’HUM [19], then the control problem is reduced to the obtention of the observability inequalities (48) for the uncontrolled system (10). Therefore, Theorem 4.4 immediately follows from Proposition 4.1.

5. Controllability for the nonlinear problem. In this section we prove the local exact controllability for the nonlinear control problem (6). First of all, we introduce the following space:
\[
\mathcal{H} = \{y \in H^3(0, l) \text{ such that } y(0) = y_{xx}(0) = y(l) = 0\}. \tag{52}
\]

The main result of this section is stated as follows:

**Theorem 5.1.** Let \( T > 0 \) and assume that the coefficients \( \rho, \sigma \) and \( q \) satisfy (7) and (8). Then there exists a constant \( r > 0 \) such that for all initial and final conditions \((y^0, y^1), (y^0_T, y^1_T) \in \mathcal{H} \times H^1_0(0, l)\), with
\[
\|y^0, y^1\|_{\mathcal{H} \times H^1_0(0, l)} < r \quad \text{and} \quad \|y^0_T, y^1_T\|_{\mathcal{H} \times H^1_0(0, l)} < r,
\]
there exists a control \( u \in H^1(0, T) \) such that the solution \( y \in C \left([0, T]; H^3(0, l)\right) \cap C^1 \left([0, T]; H^1_0(0, l)\right) \) of Problem (6) satisfies
\[
y(T, x) = y^0_T, \quad y_0(T, x) = y^1_T, \quad x \in [0, l]. \tag{53}
\]
In order to prove Theorem 5.1, we need the following result whose the proof is similar to that \cite[Proposition 12]{8}.

**Proposition 5.2.** Assume that the coefficients $\rho$, $\sigma$ and $q$ satisfy (7) and (8). Given $T > 0$ and $(\hat{y}_0, \hat{y}_1) \in \mathcal{H} \times H^1_0(0, l)$, then there exists a control $u \in H^1(0, T)$ such that the solution $y \in C([0, T], \mathcal{H}) \cap C^1([0, T]; H^1_0(0, l))$ of the linear control problem (9), satisfies

$$y(T, x) = y_0(T, x) = 0, \quad x \in [0, l].$$

(54)

**Remark 5.3.** In view of the fact that the linearized problem (9) is reversible in time and exactly controllable in the space $\mathcal{H} \times H^1_0(0, l)$. Then there exists a continuous linear map

$$\Gamma : (\hat{y}_0, \hat{y}_1) \in \mathcal{H} \times H^1_0(0, l) \rightarrow u \in H^1(0, T)$$

such that the solution $\hat{y}$ of the problem

$$\begin{cases}
\rho(x)\hat{y}_{tt} + (\sigma(x)\hat{y}_{xx})_{xx} - (q(x)\hat{y}_x)_x = 0, & (t, x) \in (0, T) \times (0, l), \\
\hat{y}(t, 0) = \hat{y}_{xx}(t, 0) = \hat{y}(t, l) = 0, & t \in (0, T), \\
\hat{y}(0, x) = 0, & x \in (0, l),
\end{cases}$$

(55)

satisfies $(\hat{y}(T), \hat{y}_t(T)) = (\hat{y}_0, \hat{y}_1)$. Furthermore there exists a constant $k > 0$ such that

$$\|\Gamma\|_{H^1(0, T)} \leq k \| (\hat{y}_0, \hat{y}_1) \|_{\mathcal{H} \times H^1_0(0, l)}$$

(56)

Now, let us denote by $\psi_0$ and $\psi_1$, the following maps which are linear and continuous by Proposition 5.2:

$$\psi_0 : (\tilde{y}_0, \tilde{y}_1) \in \mathcal{H} \times H^1_0(0, l) \rightarrow \tilde{y} \in C([0, T], \mathcal{H}) \subset L^2(0, T; \mathcal{H}),$$

where $\tilde{y}$ is the solution of the problem

$$\begin{cases}
\rho(x)\tilde{y}_{tt} + (\sigma(x)\tilde{y}_{xx})_{xx} - (q(x)\tilde{y}_x)_x = 0, & (t, x) \in (0, T) \times (0, l), \\
\tilde{y}(t, 0) = \tilde{y}_{xx}(t, 0) = \tilde{y}(t, l) = 0, & t \in (0, T), \\
\tilde{y}(0, x) = \tilde{y}_0, & x \in (0, l),
\end{cases}$$

(57)

and

$$\psi_1 : u \in H^1(0, T) \rightarrow \tilde{y} \in C([0, T], \mathcal{H}) \subset L^2(0, T; \mathcal{H}),$$

where $\tilde{y}$ is the solution of the problem (55). Moreover, there exist two constants $k_0, k_1 > 0$ such that

$$\|\psi_0\|_{L^2(0, T; \mathcal{H})} \leq k_0 \| (\tilde{y}_0, \tilde{y}_1) \|_{\mathcal{H} \times H^1_0(0, l)} \quad \text{and} \quad \|\psi_1\|_{L^2(0, T; H^1)} \leq k_1 \|u\|_{H^1(0, T)}.$$  

(58)

Let us recall also that if $f \in L^1(0, T; H^1(0, l))$, then the following problem

$$\begin{cases}
\rho(x)\tilde{y}_{tt} + (\sigma(x)\tilde{y}_{xx})_{xx} - (q(x)\tilde{y}_x)_x = f, & (t, x) \in (0, T) \times (0, l), \\
\tilde{y}(t, 0) = \tilde{y}_{xx}(t, 0) = \tilde{y}(t, l) = 0, & t \in (0, T), \\
\tilde{y}(0, x) = 0, & \tilde{y}_t(0, x) = 0, & x \in (0, l),
\end{cases}$$

(59)

has a unique weak solution $(\tilde{y}, \tilde{y}_t) \in C([0, T], \mathcal{H} \times H^1(0, l))$. As consequence, the linear map

$$\psi_2 : f \in L^1(0, T; H^1(0, l)) \rightarrow (\tilde{y}, \tilde{y}_t) \in C([0, T], \mathcal{H} \times H^1(0, l))$$

is continuous and there exists a constant $k_2 > 0$, such that

$$\|\psi_2\|_{C([0, T], \mathcal{H} \times H^1(0, l))} \leq k_2 \|f\|_{L^1(0, T; H^1(0, T))}.$$  

(60)

Before we prove Theorem 5.1, we need the following result due to \cite{8}.
Proposition 5.4. [8, Proposition 13] The map
\[ y \in L^2([0, T], \mathcal{H}) \rightarrow (y^2)_{xx} \in L^1([0, T], H^1(0, l)) \]
is well-defined and continuous. Furthermore, there exist a constant \( k_3 > 0 \) such that,
\[ \| (y^2)_{xx} - (z^2)_{xx} \|_{L^1([0, T], H^1(0, l))} \leq k_3 \left( \| y + z \|_{L^2([0, T], \mathcal{H})} \| y - z \|_{L^2([0, T], \mathcal{H})} \right) \]  \hspace{1cm} (61)

We are now ready to prove Theorem 5.1.

Proof. Consider the nonlinear problem (6) with initial data \((y^0, y^1) \in \mathcal{H} \times H^1_0(l, 0, l)\) and a control \( u \in H^1(0, T) \). Let \( y \) and \( \tilde{y} \) be the solutions of (57) and (59), respectively, where \((\tilde{y}^0, \tilde{y}^1) \equiv (y^0, y^1)\) and \( f = (y^2)_{xx}\). Obviously, the solution \( y \) of the nonlinear problem (6), can be written in the form \( y := \hat{y} + \tilde{y} \), where \( \hat{y} \) is the solution (55). Let \( \mathfrak{F} \) be the nonlinear map :
\[ \mathfrak{F} : y \in L^2(0, T; \mathcal{H}) \rightarrow \mathfrak{F}(y) \in L^2(0, T; \mathcal{H}) \]
such that
\[ \mathfrak{F}(y) = \psi_0(y^0, y^1) - \psi_2((y^2)_{xx}) + \psi_1 \circ \Gamma \left( (y_T^0, y_T^1) - (\psi_0(y^0, y^1)(T), \psi_0(y^0, y^1)(T)) \right) \]
\[ - (\psi_2((y^2)_{xx})(T), \psi_2((y^2)_{xx})(T)). \]

To prove the theorem it suffices to show that \( \mathfrak{F} \) has a fixed point. Furthermore, using (56), (58), (60) and (61), it can be shown by a straightforward computation that there exists a constant \( \bar{k} > 0 \) such that
\[ \| \mathfrak{F}(y) \|_{L^2(0, T; \mathcal{H})} \leq \bar{k} \left( \| y \|_{L^2([0, T], \mathcal{H})} + \| (y^0, y^1) \|_{\mathcal{H} \times H^1_0(0, l)} + \| (y_T^0, y_T^1) \|_{\mathcal{H} \times H^1_0(0, l)} \right) \]  \hspace{1cm} (62)
and
\[ \| \mathfrak{F}(y) - \mathfrak{F}(z) \|_{L^2(0, T; \mathcal{H})} \leq \bar{k} \| y - z \|_{L^2([0, T], \mathcal{H})} \left( \| y \|_{L^2([0, T], \mathcal{H})} + \| z \|_{L^2([0, T], \mathcal{H})} \right). \]  \hspace{1cm} (63)
Let \((y^0, y^1), (y_T^0, y_T^1) \in \mathcal{H} \times H^1_0(l, 0, l)\) such that
\[ \| (y^0, y^1) \|_{\mathcal{H} \times H^1_0(0, l)} < r \text{ and } \| (y_T^0, y_T^1) \|_{\mathcal{H} \times H^1_0(0, l)} < r, \]
where \( r > 0 \) is a constant which will be fixed later. For each \( R > 0 \), let us denote the ball of radius \( R \) and centered at the origin by
\[ B(0, R) := \{ y \in L^2(0, T; \mathcal{H}) \mid \| y \|_{L^2([0, T], \mathcal{H})} \leq R \}. \]
Since we aim to use Banach fixed point theorem to the restriction of \( \mathfrak{F} \) to the ball \( \overline{B}(0, R) \), then the constants \( r > 0 \) and \( R > 0 \) can be chosen in (62) and (63), such that \( \bar{k}(2r + R^2) \leq R \) and \( 2R\bar{k} < 1 \). Let \( R = \frac{1}{4\bar{k}} \) and \( r = \frac{3R^2}{2} \), then by (62) and (63), one gets
\[ \mathfrak{F}(B(0, R)) \subset B(0, R), \] \hspace{1cm} (64)
\[ \| \mathfrak{F}(y) - \mathfrak{F}(z) \|_{L^2([0, T], \mathcal{H})} \leq \frac{1}{2} \| y - z \|_{L^2([0, T], \mathcal{H})}, \] \hspace{1cm} (65)
which implies by Banach fixed point theorem that \( \mathfrak{F} \) has a unique fixed point. The proof of the theorem is completed. \( \square \)
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