QCD THERMODYNAMICS:
LATTICE RESULTS CONFRONT MODELS

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We show that lattice results on four flavor QCD at nonzero temperature and baryon
density compare well with the hadron resonance gas model up to $T \approx 0.95T_c$, and
approach a free field behaviour with a reduced effective number of flavor for $T \geq 1.5T_c$;
chiral symmetry and confinement are interrelated, and we note analogies
between the critical line of QCD and that of simple models with the same global
symmetries.

1. QCD Thermodynamics and Imaginary $\mu_B$

In the last four years a few lattice techniques proven successful in QCD thermodynamics for $\mu_B/T < 1$. While waiting for final results in
the scaling limit and with physical values of the parameters, it is very useful
to contrast and compare current lattice results with model calculations and perturbative studies.

The imaginary chemical potential approach to QCD thermodynamics seems to be ideally suited for the interpretation and comparison
with analytic results. In the following we review our results from this
perspective.

QCD at finite quark chemical potential $\mu$ can be simulated with ordinary methods when $\mu$ is purely imaginary. If one were able to determine
thermodynamic observables with infinite accuracy, standard complex analysis arguments would guarantee that the result will be valid within the
entire analytic domain, i.e. everywhere away from phase transitions. In
practical numerical work one has to take into account two sources of er-
rors: first, the analytic form of the fitting function is not known a priori; second, even if it were so, one has to deal with numerical errors on the coefficients. Cross checks among different analysis and guidance from models are thus most useful. Results from an imaginary $\mu$ have been obtained for the critical line of the two, three and two plus one flavor model $^5$, as well as for four flavor $^1$. Thermodynamics results – order parameter, pressure, number density – were obtained for the four flavor model $^2$. The pressure was defined by introducing an integral method at fixed temperature, and the mass dependence was estimated via the Maxwell relations $^{11,2,7}$.

2. The Critical Line, Chiral Symmetry and Confinement

We used the exact results available for the critical line of the Gross Neveu model, and for Random Matrix Models in the appropriate universality class to show that a second order polynomial in $T$ and $\mu$ approximates well the critical line over a large $\mu$ interval$^3$. The nature of the chiral transition itself can be studied with a great accuracy: the correlation between $<\bar{\psi}\psi>$ and Polyakov loop was observed at several values $\mu_I$, either by studying the results as a function of $\beta$, and by following the Monte Carlo histories directly at $\beta_c(\mu_I)$ $^{1,2}$.

3. The Hadronic Phase

The grand canonical partition function of the Hadron Resonance Gas model$^{12}$ has a simple hyperbolic cosine behaviour. This can be framed

![Graph](image)

Figure 1. Hadronic Phase: (a) One Fourier coefficient fit to the particle number, showing that the Hadron Resonance Model is adequate to describe this data. (b) Compilation of the results for the chiral condensate and the particle number as a function of real chemical potential: the lines are cut in correspondence with $\mu_c$, showing the first order character of the phase transition (inferred from the chiral condensate) and the critical density. See our extended writeups for error analysis and details.
in our discussion of the phase diagram in the temperature-imaginary chemical potential plane which suggests to use Fourier analysis in this region, as observables are periodic and continuous there\(^1\).

For observables which are even \((O_e)\) or odd \((O_o)\) under \(\mu \rightarrow -\mu\) the analytic continuation to real chemical potential of the Fourier series read 
\[
O_e[\mu](\mu_1, N_t) = \sum_n a^{(n)}_e \cosh[\sinh(nN_tN_c\mu_1)].
\]
In our Fourier analysis of the chiral condensate \(^1\) and of the number density \(^2\) - even and odd observables, respectively - we limited ourselves to \(n = 0, 1, 2\) and we assessed the validity of the fits via both the value of the \(\chi^2/\text{d.o.f.}\) and the stability of \(a^{(0)}_e\) and \(a^{(1)}_e\) given by one and two cosine [sine] fits: we found that one cosine [sine] fit describes reasonably well the data up to \(T \simeq 0.985T_c\) (see Fig.1a); further terms in the expansion did not modify much the value of the first coefficients and does not particularly improve the \(\chi^2/\text{d.o.f.}\). This means that our data are well approximated by the hadron resonance gas prediction \(\Delta P \propto (\cosh(\mu_B/T) - 1)\) in the broken phase up to \(T \simeq 0.985T_c\).

The analysis of the corrections requires better precision.

The analytic continuation (Fig. 1b) of any observable \(O\) is valid within the analyticity domain, i.e. till \(\mu < \mu_c(T)\), where \(\mu_c(T)\) has to be measured independently. The value of the analytic continuation of \(O\) at \(\mu_c\), \(O(\mu_c)\), defines its critical value. When \(O\) is an order parameter which is zero in the quark gluon plasma phase, the calculation of \(O(\mu_c)\) allows the identification of the order of the phase transition: first, when \(O(\mu_c) \neq 0\), second, when \(O(\mu_c) = 0\) \(^1, 2\).

4. The Hot Phase

The behaviour of the number density (Fig. 2a) approaches the lattice Stephan-Boltzmann prediction, with some residual deviation. We parametrise the deviation from a free field behavior as \(^{13, 14}\)

\[
\Delta P(T, \mu) = f(T, \mu)P_{\text{free}}^{L}(T, \mu) \tag{1}
\]

where \(P_{\text{free}}^{L}(T, \mu)\) is the lattice free result for the pressure. For instance, in the discussion of Ref. \(^{14}\)

\[
f(T, \mu) = 2(1 - 2\alpha_s/\pi) \tag{2}
\]

and the crucial point was that \(\alpha_s\) is \(\mu\) dependent.

We can search for such a non trivial prefactor \(f(T, \mu)\) by taking the ratio between the numerical data and the lattice free field result \(n_{\text{free}}^{L}(\mu_1)\)
at imaginary chemical potential:

\[ R(T, \mu_I) = \frac{n(T, \mu_I)}{n_{\text{free}}(\mu_I)} \]  

(3)

A non-trivial (i.e. not a constant) \( R(T, \mu_I) \) would indicate a non-trivial \( f(T, \mu) \).

In Fig. 2b we plot \( R(T, \mu_I) \) versus \( \mu_I/T \): the results for \( T \geq 1.5T_c \) seem consistent with a free lattice gas, with an fixed effective number of flavors \( N_f^\text{eff}/4 = R(T) \): \( N_f^\text{eff} = 0.92 \times 4 \) for \( T = 3.5T_c \), and \( N_f^\text{eff} = 0.89 \times 4 \) for \( T = 1.5T_c \).

5. Lattice vs. Models:

Open Issues

We give here a partial list of results together with related questions which calls for a more detailed understanding of the interrelations of analytic and numerical results.

The critical line is similar to the one of model theories with the same global symmetries: \( T/T_c^2 = 1 - 0.0021(2)(\mu/T)^2 \) for four flavor, and similarly for two and three flavors \( 3, 5, 6 \). Can we understand the flavor dependence and the (small) coefficient of \( (\mu/T)^2 \) from these model?

We have observed \( T_{c}^{\text{chiral}}(\mu) = T_{c}^{\text{deconfining}}(\mu) \). Can effective Lagrangians\(^{16} \) account for this observation?

In the hadronic phase \( \Delta P \simeq k(1 - \cosh(\mu_B/T)) \). The corrections are not completely negligible, and can be used to estimate the magnitude of further terms in the expansion. Can these corrections be modeled in some simple, intuitive way?
For $T \geq 1.5T_c$ the results are compatible with lattice Stefan Boltzmann with an active fixed number of flavor $4 \times 0.92$ for $T=3.5\ T_c$ and $4 \times 0.89$ for $T = 2.5T_c$. These finite density corrections appear to leave unchanged the free field structure, even at moderately low temperatures. How can we understand this?

Between $T_c$ and $1.5\ T_c$, can we interpret the deviations from this simple behaviour by use of a rigorous perturbative analysis $^{15}$, and / or in the framework of a strongly interacting quark gluon plasma $^{17}$?

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