Resummation of Vacuum Bubble Diagrams
in Gaussian Propagator Model

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Abstract

An imaginary part of the false-vacuum energy density in a metastable system, i.e., the decay width due to quantum tunneling, might be reproduced by Borel resummation of vacuum bubble diagrams. We examine the convergence of this prescription in the Gaussian propagator model, in which the analytical expression of vacuum bubbles to the ninth order of loop expansion is available.
Usually, one uses the bounce calculus [1] to estimate the decay width of a metastable system due to quantum tunneling. This is consistent with the conventional picture that the quantum tunneling is a purely non-perturbative phenomenon. However, several methods have been proposed so far [2–4] which might allow us to handle tunneling effects with using a conventional perturbation series. According to these approaches, a certain class of physical systems seems to have in the perturbation series the whole physical information, including higher order corrections to the leading bounce approximation.

In [3], Suzuki and the present author proposed such a perturbative method to reproduce an imaginary part of the false-vacuum energy density. The method is based on Borel resummation [5] of vacuum bubble diagrams and is applicable to super-renormalizable field theoretical models. The validity of the method has been tested in one-dimensional (i.e., quantum mechanical) and two-dimensional $O(N)$ symmetric $\lambda\phi^4$ model [3]. In the former, the imaginary part of the ground state energy was successfully reproduced in a high accuracy. In the latter, unfortunately, a shortage of the perturbation series prevents us to draw a quantitative conclusion on the convergence of the method.

In this note, we test the convergence of our proposal by applying it to the scalar $\phi^4$ theory with a Gaussian propagator [6]. The Euclidean action is given by

$$S[\phi] = \int d^D x \left[ \frac{1}{2} \phi(x)e^{-\Delta \phi(x)} - \frac{g}{4!} \phi^4(x) \right],$$

where $\Delta$ is $D$-dimensional Laplacian. This model has advantages for our purpose: Since the propagator in momentum space is

$$G(p) = e^{-p^2},$$

an evaluation of Feynman diagram is just Gaussian integrations and no UV divergences present. These facts allow the evaluation of Feynman diagrams to a relatively high order in an arbitrary dimensional space [3]. This model also allows to carry out the analytical bounce calculus [3]. Unfortunately, the kinematical meaning of this model is not clear due to the odd-looking “kinetic term,” (actually, the classical equation of motion has an infinite order
time derivative) and thus the meaning of “tunneling” or “decay” is not obvious. Nevertheless, we adopt this model as the testing ground to investigate the convergence of our prescription for Borel resummation of vacuum bubble diagrams.

We shall consider a “metastable case” \((g > 0)\) and study a sum of vacuum bubble diagrams:

\[
E(g) \sim \sum_{n=0}^{\infty} c_n g^n. \tag{3}
\]

The actual numerical values of \(c_n\)s are given in [6] to \(n = 8\) (to nine loops) from \(D = 0\) to \(D = 4\) and we will use those values.

In an analogy of the conventional field theory, we shall call \(E(g)\) “the vacuum energy density.” The imaginary part of the vacuum energy density is also evaluated in the leading order bounce approximation [6]:

\[
\text{Im} E(g)_{\text{bounce}} = \frac{1}{(2\pi)^{D/2}} JM^{-1/2} \exp \left( -\frac{A_0}{g} \right), \tag{4}
\]

where

\[
A_0 = \frac{3}{2} \left( \frac{27\pi}{2} \right)^{D/2}, \quad J = \left( \frac{3\pi^{D/2}}{g} \right)^{D/2},
\]

\[
M = 2 \left[ \left( \frac{2}{3} \right)^{D/2+1} \right]^D \exp \Sigma_2(D), \quad \Sigma_2(D) = \sum_{m=2}^{\infty} \frac{(m + D - 1)!}{m! (D - 1)!} \ln(1 - 3^{1-m}). \tag{5}
\]

This formula tells us the leading large order behavior of the perturbation series \([3][6]\), or equivalently, the position and nature of the Borel singularity nearest to the origin of the Borel plane.

In completely the same reasoning as [3], we thus define the Borel-Leroy transform of the perturbation series \(c_n\):

\[
B(z) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n + (D + 1)/2)} z^n. \tag{6}
\]

The original vacuum energy density is supposed to be reproduced by the Borel integral:

\[
E(g) = \frac{1}{g^{(D+1)/2}} \int_0^{\infty} dz \, e^{-z/g} z^{(D+1)/2-1} B(z). \tag{7}
\]
The leading imaginary part \( (4) \) for \( g \ll 1 \) implies that there exists a singularity of \( B(z) \) at \( z = A_0 \). The argument of the gamma function in \( (3) \) was chosen to sufficiently weaken this singularity: The bounce singularity at \( z = A_0 \) becomes a square root branch point, irrespective of \( D \). The integration contour of \( (7) \), therefore, must avoid the bounce singularity. This deformation develops the imaginary part and we select the contour along the upper side of the real axis.

The singularity at \( z = A_0 \), however, limits the convergence radius of the series \( (6) \). Consequently, in order to carry out the integral \( (7) \) beyond \( z = A_0 \), we have to execute the analytic continuation of \( B(z) \). This continuation is possible only if we completely know the Borel function \( B(z) \) within the convergence circle (i.e., if all \( c_n \)'s are known). However, we have only a finite number of perturbation coefficients in actual.

To avoid this difficulty, we use the conformal mapping technique. Introducing a new variable \( \lambda \) instead of \( z \),

\[
z = 4A_0 \frac{\lambda}{(1 + \lambda)^2},
\]

new integration contour is confined in the convergence circle on \( \lambda \) plane. Indeed, the bounce singularity is transformed to \( \lambda = 1 \) and new integration contour goes along inside the upper arc of the convergence circle.

In this way, we have an imaginary part of the vacuum energy density expressed by the perturbation coefficients \( (3) \):

\[
\left[ \text{Im} \mathcal{E}_0(g) \right]_P = \left( \frac{A_0}{g} \right)^{(D+1)/2} \int_0^\pi d\theta \frac{A_0}{g \cos^2(\theta/2)} \frac{\sin(\theta/2)}{\cos^{D+2}(\theta/2)} \sum_{k=0}^P d_k \sin(k\theta),
\]

with

\[
d_k = \sum_{n=0}^k (-1)^{k-n} \frac{\Gamma(k+n)(4A_0)^n}{(k-n)!\Gamma(2n)\Gamma(n + (D+1)/2)} c_n,
\]

where \( P \) denotes the order of approximation in this approach.
Now we present the result of eq. (9). When choosing the presented range of the coupling constant $g$ in Figures, we selected as the upper bound the point that the effective mass changes its sign. To the one-loop order, we have

$$\Sigma(p)_{p=0} = 1 - \frac{g^2}{2(4\pi)^{D/2}} + \mathcal{O}(g^2).$$

(11)

Thus we choose as the upper bound of $g$, $g = 2(4\pi)^{D/2}$ (numerically, for $D = 1, 2, 3$ and $4$, it is $g = 7.09, 25.1, 89.1$ and $315$, respectively). This upper bound might be regarded as an order of magnitude estimation of the boundary of “tunneling region” [3].

We plot eq. (9) with $P = 4, 5$ and $8$ as the function of $g$. The imaginary part is normalized by the result of the leading bounce approximation (4). For $D = 0$ (Fig. 1), all the perturbative coefficients in (3) can be analytically computed because the partition function is just an ordinary integral. Eq. (9) rapidly converges to a uniform curve as shown in Fig. 1 and our prescription certainly seems to give the correct imaginary part in $D = 0$.

For $D = 1$ (Fig. 2), three curves almost coincide over a wide range of $g$. In particular, the impressive agreement of $P = 5$ and $8$ indicates that the perturbation series to the fifth order is practically sufficient to reproduce the correct imaginary part. For $D = 2$ (Fig. 3), the range in which $P = 5$ and $8$ agree becomes narrower and simultaneously the difference becomes larger. But still eq. (9) displays a rapid convergence for all $g$ and therefore $P = 8$ seems to indicate the correct value. The order of perturbation series required for a plausible evaluation of the imaginary part is higher than the cases of $D = 0$ and $1$.

For $D = 3$ and $4$ (Figs. 4 and 5), there is a wild oscillation in small $g$ region. (For $g = 0+$ the correct value is 1 because the bounce result (4) becomes exact.) In spite of this undesirable situation for the weak coupling, eq. (9) seems converging in the strong coupling region. Therefore, in both cases of $D = 3$ and $4$, the correct imaginary part in the strong coupling region seems to be smaller than the bounce result.

In summary, our method applied to the Gaussian propagator model seems certainly working in lower dimensional cases, exhibiting a good convergence. The results in turn indicate the correct imaginary part of the vacuum energy density is much smaller than that
of the leading bounce approximation. On the other hand, the convergence in the weak
coupling region becomes worse in higher dimensional cases. We guess this tendency holds
in conventional field theories.

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FIGURES

FIG. 1. The imaginary part of the “vacuum energy density” evaluated by our prescription (9) versus $g$: $P = 4$ (boxes), $P = 5$ (circles) and $P = 8$ (filled circles). The imaginary part is normalized by the result of the leading bounce approximation (4).

FIG. 2. Same as Fig. 1 but for $D = 1$.

FIG. 3. Same as Fig. 1 but for $D = 2$.

FIG. 4. Same as Fig. 1 but for $D = 3$.

FIG. 5. Same as Fig. 1 but for $D = 4$. 
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5