Global time asymmetry as a consequence of a wave packets theorem

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When \( t \to \infty \) any wave packet in the liouvillian representation of the density matrices, becomes a Hardy class function from below. This fact, in the global frame of Reichenbach diagram, is used to explain the observed global time asymmetry of the universe.

I. INTRODUCTION

Many authors consider time asymmetry as having a global origin [1]. The Reichenbach branched diagram has being used to explain this global phenomenon [2]. Here we make a resume of this idea. Reichenbach diagram is the combination of all the scattering processes within the universe evolution, beginning at the initial global instability of the universe, which is considered as the source of all energy. In each scattering process, the energy of the incoming states is used to produce unstable states that decay, originating the outgoing ones. So, the outgoing lines of the diagram can always be considered as evolutions from an unstable state towards equilibrium. Essentially, the whole diagram can be considered as having only outgoing lines. In fact, the incoming lines in each scattering are outgoing lines of a previous process. Moreover, the incoming lines of a scattering system “S” cannot be considered as spontaneous evolutions, since they just show the pumping of energy from a precedent process, that is really coupled with the scattering system “S”, making these incoming lines representatives of non spontaneous or forced evolutions. On the contrary, spontaneity clearly characterizes outgoing lines. Considered as a whole, Reichenbach diagram symbolize the asymmetrical flow of all the energy within the universe from its initial instability towards a final equilibrium state, resolving this flow as a sum of scattering processes. Then, time asymmetry (which cannot be explained as a consequence of the local time-symmetric physical laws), can be easily explained as a consequence of the time asymmetry of the object-universe.

On the other hand, time asymmetry can be considered as the consequence of the existence of a time-asymmetric space of physical admissible states [3]. For causality reasons, this space was associated with the space of Hardy class functions from below [4]. If all the lines in the Reichenbach diagram would be electromagnetic waves this choice would be natural, since the outgoing waves of electromagnetic scattering can be represented as states belonging to the just mentioned Hardy space [5]. This conclusion can be extended to all hyperbolic scatterings but not to the parabolic equation of non relativistic quantum mechanics. Nevertheless, we will show that far from the scatterer, the outgoing lines belong to a Hardy space if the physical admissible states are wave packets. Then, all the spontaneously evolving states of the universe would belong to such a type of space, that we will call \( \Psi_- \). The unphysical non spontaneous time-inverted states would belong to a space \( \Psi_+ = K\Psi_- \neq \Psi_- \), where \( K \) is Wigner time-reversal operator \([6]\) \([7]\). The arrow of time would be the consequence of this asymmetry.

II. THE BAKER’S TRANSFORMATION

As a dydactical introduction to the subject we will show how spaces \( \Psi_- \) and \( \Psi_+ \) appears in the famous Baker transformation.

Let us consider the unit square \( S = [0, 1] \times [0, 1] \) with its restricted Lebesgue measure \( \mu \) and the Baker’s transformation:

\[ x' = 2x \mod 1, \quad y' = \frac{y}{2} \mod 1 \]

\[ ^1 \] In the usual popularization language \( \Psi_- \) would be the space of the spontaneous evolutions: the sugar lump solving in the coffee, or the elephant breaking the crystal shop. While \( \Psi_+ \) will be the space of impossible (or better non-spontaneous) evolutions: the sugar lump concentrating in the coffee, or the elephant reconstructing the crystal shop.
Let us consider the independent and generating partition of $B$, i.e., the partition of the unit square into its left and right halves or “vertical” rectangles, $P = \{\Delta_1, \Delta_2\}$. “Independence” of the partition $P$ with respect to $B$ means that

$$\mu \left( \bigcap_{n=-m_1}^{m_2} B^n_-(\Delta_{i_n}) \right) = \prod_{n=-m_1}^{m_2} \mu (\Delta_{i_n})$$

where $\Delta_{i_n} \in P$. The “generating” character of the partition $P$ with respect to $B$ means that any Borel measurable set of the unit square can be obtained by forming countable unions and intersections of sets of the form

$$\bigcap_{n=-m_1}^{m_2} B^n_-(\Delta_{i_n}) \text{ with } \Delta_{i_n} \in P$$

Let $U : L^2(S, \mu) \to L^2(S, \mu)$ be the unitary map

$$(U^n \rho) (w) = \rho (B^{-n}(w))$$

where $\rho \in L^2(S, \mu)$, $w \in S$. As is well known, $U$ has a countable uniform Lebesgue spectrum, and therefore there is a system of imprimitivity in $L^2(S, \mu)$ based on $\mathbb{Z}$ for the group $\{U^n : n \in \mathbb{Z}\}$. In other words, there is a spectral measure $E$ defined on $\mathbb{Z}$ and taking its values in the set of the orthogonal projection operators of $L^2(S, \mu)$, such that

$$\forall n, m \in \mathbb{Z} : U^{-n} E_m U = E_{m+n}$$

where $E_m = E(\{m\})$, and $E_{m+n} = E(\{m+n\})$. Then, we can define the Age operator by

$$A = \int_{\mathbb{Z}} n dE = \sum_{n=-\infty}^{+\infty} nE_n$$

As a consequence of eqs. (6) and (8), and taking into account the properties of $E$, we have

$$\forall n \in \mathbb{Z} : U^{-n} A U = A + nI$$

The age operator $A$ has $\mathbb{Z}$ as uniform spectrum with countable multiplicity. In fact, the functions

$$\alpha_0(w) = \begin{cases} +1 & \text{if } w \in \Delta_1 \\ -1 & \text{if } w \in \Delta_2 \end{cases}$$

together with its transformed by $U^n : \alpha_n = U^n(\alpha_0)$, and all their finite products

$$\alpha_F = \alpha_{n_1} ... \alpha_{n_r}$$

$$(F = \{n_1, ..., n_r\} n_j \in \mathbb{Z})$$

constitute an orthonormal eigenbasis $\{\alpha_F : F \subset \mathbb{Z}\}$ of $A$, being each $\alpha_F$ a “state” of age $n = \max \{n_1, ..., n_r\}$, in the sense that

$$A\alpha_F = n\alpha_F$$

(Clearly, there are countably many states corresponding to each eigenvalue or age $n$, showing its multiplicity). Because of eq. (8), $U$ act as a right bilateral shift on this basis:

$$U\alpha_F = \alpha_{F+1}$$

(4) (where $F + 1 := \{n_1 + 1, ..., n_r + 1\}$). The eqs. (3) and (8) show how the age brought about by the dynamical evolution and growing with it matches with the progress of external (or observer’s) time labelling the dynamical group.

Now, we can decompose $L^2(S, \mu)$ as a direct sum of two Hardy functions spaces $H^2_+$ and $H^2_-$ such that:
\[ H^2_+ = \left\{ \rho : \rho \in L^2(S, \mu) \land \rho = \sum_{F \subset \mathbb{Z}^+} a_F \alpha_F \right\} \]  

having only non null Fourier coefficients \( a_F \in \mathbb{C} \) for positive indices, and

\[ H^2_- = \left\{ \rho : \rho \in L^2(S, \mu) \land \rho = \sum_{F \subset \mathbb{Z}_0^-} a_F \alpha_F \right\} \]  

having only non vanishing Fourier coefficients for negative (or zero) indices.

Then, it is obvious that:

\[ U H^2_+ \subset H^2_+ , \ U H^2_- \subset H^2_+ \oplus H^2_- = L^2(S, \mu) \neq H^2_- \]  

and that:

\[ \lim_{n \to +\infty} U^n (H^2_+ \oplus H^2_-) = H^2_+ \]  

We will describe this fact by saying that the states belonging to \( H^2_+ \) are “stable towards the future” under the induced evolution \( U \), while the those belonging to \( H^2_- \) are “unstable” \( \text{[10]} \). In this way, when acted on by \( U^n \), any function belonging to \( L^2(S, \mu) \) will end in space \( H^2_+ \) in the “far future” (precisely when \( n \to +\infty \)).

In the next section we will consider the “quantum version” of what we have said above.

### III. PURE STATES AND THE HAMILTONIAN

Let us begin considering just pure states \( |\psi\rangle \), belonging to a Hilbert space \( \mathcal{H} \), of a quantum system with hamiltonian \( H \), such that:

\[ H|\omega,n\rangle = \omega|\omega,n\rangle \]  

where \( 0 \leq \omega < \infty \) or \( \omega \in \mathbb{R}^+ \) and \( n \) belongs to a set of indices \( N \), which is the same for any \( \omega \) (for didactical reasons we will assume that this set is numerable, and therefore the index \( n \) will be discrete). Thus, \( H \) can be considered as a typical scattering hamiltonian just endowed with an absolutely continuous and uniform energy spectrum. Precisely, there is a nuclear space \( \Phi \) and a rigging of it with \( \mathcal{H} \)

\[ \Phi \subset \mathcal{H} \subset \Phi^\times \]

such that

\[ \{ |\omega,n\rangle : \omega \in \mathbb{R}^+ \land n \in N \} \subset \Phi^\times \]

(we will denote by \( \Phi^\times \) the antidual space of \( \Phi \), composed of all continuous antilinear functionals on \( \Phi \), and by \( \Phi' \) its dual) is a generalized eigenbasis of \( H \) \( \text{[10]} \) \( \text{[11]} \) in the sense that:

\[ \forall \varphi, \psi \in \Phi : \langle \varphi |\psi\rangle = \sum_n \int_0^\infty d\omega \langle \varphi |\omega,n\rangle \langle \omega,n|\psi\rangle \]  

where the l. h. s. means the scalar product in \( \mathcal{H} \) (antilinear in its left factor), while in the r. h. s. \( \langle \varphi |\omega,n\rangle \) means the evaluation of the antilinear functional \( |\omega,n\rangle \) on \( \varphi \), and \( \langle \omega,n|\psi\rangle \) is the evaluation of the linear functional \( \langle \omega,n \rangle \) on \( \psi \). This justifies Dirac’s notation:

\[ |\psi\rangle = \sum_n \int_0^\infty d\omega |\omega,n\rangle \langle \omega,n|\psi\rangle \]  

\[ \text{2Of course, with repect to the inverse evolution } U^{-1} \text{ “towards the past” we must reverse these terms.} \]
Moreover, let us consider that real physical states are wave packets, mathematically modelled by Schwarz functions of $\omega \in \mathbb{R}^+$, for each value of $n$, so:

$$f(\omega) = \langle \omega, n | \psi \rangle \in S^+ = S(\mathbb{R}^+)$$ (16)

($S^+$ is the space of all infinite differentiable complex-valued functions defined on $[0, +\infty)$, such that converge to zero for $\omega \to +\infty$ faster than the inverse of any polynomial).

Taking into account all the values of $n$ we can say that:

$$f(\omega) = \langle \omega, n | \psi \rangle \in \bigoplus_n S^+_n$$ (17)

This mathematical model is adopted for the following reasons:

1.- It is clear that we do not find infinite energies in nature, and so $\langle \omega, n | \psi \rangle$ must somehow go to zero when $\omega \to +\infty$.

2.- In order to use derivatives in our calculations it is not enough to postulate that the states belong to a Hilbert space. They must be representable by differentiable functions. We postulate that they are infinitely differentiable. After all, we cannot find an experimental contradiction to this assumption.

3.- But since these functions must also be square integrable, we can take for granted that they go to zero when $\omega \to +\infty$. We postulate that they go to zero faster that the inverse of any polynomial.

Of course, we are free to choose other spaces instead of $S^+$, but it is evident that $S^+$ is the simplest model endowed with all the usual properties of wave packets (that’s why the same choice is made in [12]).

IV. MIXED STATES AND THE LIOUVILLIAN

We will use the notation of ref. [13]. Then, the Liouville operator reads.

$$L = [H,.] = H \times I - I \times H$$ (18)

Let us consider the space of “density matrices” $\mathcal{L} = \mathcal{H} \otimes \mathcal{H}$, the rigged Hilbert space

$$\Phi \otimes \Phi \subset \mathcal{H} \otimes \mathcal{H} \subset \Phi^x \otimes \Phi'$$

and the generalized basis:

$$\{|\omega, n\rangle \langle \omega', n'| : \omega, \omega' \in \mathbb{R}^+ \wedge n, n' \in \mathbb{N}\}$$ (19)

(where $|\omega, n\rangle \langle \omega', n'| = |\omega, n\rangle \otimes \langle \omega', n'| \in \Phi^x \otimes \Phi'$).

Let us define the Riesz indices [13]:

$$\nu = \omega - \omega', \quad -\infty < \nu < \infty$$

$$\sigma = \frac{1}{2}(\omega + \omega'), \quad \frac{\nu}{2} \leq \sigma < \infty$$ (20)

It will be convenient to label the basis [13] as:

$$|\omega, n\rangle \langle \omega', n'| = |\nu, \sigma, n, n'|$$ (21)

Then:

$$\mathbb{L}|\nu, \sigma, n, n'| = \nu|\nu, \sigma, n, n'|$$ (22)

So

$$\left\{|\nu, \sigma, n, n'| : \nu \in \mathbb{R} \wedge \frac{|\nu|}{2} \leq \sigma < \infty \wedge n, n' \in \mathbb{N}\right\}$$ (23)

is a generalized eigenbasis of the liouvillian, being $\nu$ the corresponding generalized eigenvalue and $\sigma, n, n'$, the “degeneration indices”. From [24] we see that $\nu \in \mathbb{R}$, while $n$ and $n' \in \mathbb{N}$, and $\sigma \in \mathbb{R}^+$, and these spaces have the same cardinality for any $\nu$. So $\mathbb{L}$ has uniform Lebesgue spectrum $\mathbb{R}$. 


In the basis (23) “the $\nu$-wave function” reads:

$$\rho(\nu) = \langle \rho | \nu, \sigma, n, n' \rangle$$  \hspace{1cm} (24)

Following the ideas of the previous section [1], it is physically justified to suppose that these functions are sums of products of functions $f(\omega) \in S(\mathbb{R}^+)$, namely:

$$f(\omega)g(\omega') = f\left(\sigma + \frac{\nu}{2}\right)g\left(\sigma - \frac{\nu}{2}\right)$$  \hspace{1cm} (25)

They will have infinite derivatives with respect to $\nu$, since $f$ and $g$ are infinitely differentiable. Moreover, $\rho(\nu)$ goes to zero when $\nu \to \pm\infty$ faster than the inverse of any polynomial since this is a property of $f$ and $g$. Thus, for any $\sigma, n, n'$:

$$\rho(\nu) = \langle \rho | \nu, \sigma, n, n' \rangle \in S(\mathbb{R})$$  \hspace{1cm} (26)

### V. THE AGE OPERATOR

Since $L$ has uniform Lebesgue spectrum $\mathbb{R}$, there is a system of imprimitivity in $L$ based on $\mathbb{R}$ for the group $\{U_t : t \in \mathbb{R}\}$. In other words, there is a spectral measure $E$ defined on $\mathbb{R}$ and taking its values in the set of the orthogonal projection operators of $L$ [8], such that

$$\forall t, s \in \mathbb{R} : U_t^{-1}E_sU_t = E_{t+s}$$  \hspace{1cm} (27)

where $E_s = E ((-\infty, s])$, and $E_{t+s} = E ((-\infty, t+s])$. Then we can define the Age operator [14] by

$$A = \int_{\mathbb{R}} t \, dE$$  \hspace{1cm} (28)

As a consequence of eqs. (27) and (28), and taking into account the properties of $E$, we have

$$\forall t \in \mathbb{R} : U_t^{-1}AU_t = A + tI$$  \hspace{1cm} (29)

The age operator $A$ has $\mathbb{R}$ as a uniform Lebesgue spectrum. In fact, for the physical states we have:

$$\mathcal{A}\rho(\nu, \sigma, n, n') \equiv i\frac{\partial}{\partial \nu}\rho(\nu, \sigma, n, n')|_{\sigma, n, n'=\text{const.}}$$  \hspace{1cm} (30)

that is equivalent to the commutation relation

$$[\mathcal{A}, L] = i$$  \hspace{1cm} (31)

($\mathcal{A}$ and $L$ have essentially the same commutation relation as position and momentum operators $q$ and $p$). Then $\mathcal{A}\rho(a, \sigma, n, n')$, the Fourier transform in variables $\nu \leftrightarrow a$ of $\rho(\nu, \sigma, n, n')$, is an eigenvector of $\mathcal{A}$, precisely:

$$\mathcal{A}\mathcal{A}\rho(a, \sigma, n, n') = a\mathcal{A}\rho(a, \sigma, n, n')$$  \hspace{1cm} (32)

Moreover $\mathcal{A}\rho(a, \sigma, n, n') \in S(\mathbb{R})$ in the variable $a$ since it is the Fourier transform of $\rho(\nu, \sigma, n, n')$. Then, the time evolution of $\mathcal{A}\rho(a, \sigma, n, n')$ reads:

$$e^{-iLt}\mathcal{A}\rho(a, \sigma, n, n') = e^{iLt}$$

$$= \mathcal{A}\rho(a, \sigma, n, n') + t\frac{\partial}{\partial a}\mathcal{A}\rho(a, \sigma, n, n') + \frac{t^2}{2!}\frac{\partial^2}{\partial a^2}\mathcal{A}\rho(a, \sigma, n, n') + ... =$$

$$= \hat{\rho}(a + t, \sigma, n, n')$$  \hspace{1cm} (33)

\(^3\)And taking into account that we are considering mixed states as “density matrices” identified with tensor products of pure states.
Thus, $L$ is the generator of the time translations, and $\hat{\rho}(a,\sigma,n,n')$ increase its age as $a \to a + t$ becoming $\hat{\rho}(a + t,\sigma,n,n')$. This fact justify the name given to $A$. But eq. (33) tell us that during its time evolution, the wave packet $\hat{\rho}(a,\sigma,n,n') \in \mathcal{S}(\mathbb{R})$ do not change its shape, being merely shifted to the left. So, in the basis (23) all physical states are wave packets at any time, and moreover verify:

$$\lim_{t \to +\infty} \hat{\rho}(a + t,\sigma,n,n') = 0$$  \hspace{1cm} (34)$$

because functions in Schwarz space go to zero when its variable goes towards the infinite.

VI. THE THEOREM

The quantum version of the Baker’s transformation example would be as follows. Let us consider a quantum system whose states $\rho$ are “density matrices” belonging to a Hilbert-Liouville space $\mathcal{L}$. Let $L$ be the Liouville superoperator in $\mathcal{L}$, assumed as having $\mathbb{R}$ as uniform Lebesgue spectrum. This amounts to say that the evolution superoperator $\exp[-iLt]$ is a bilateral shift, closely related with Hardy classes [9] [8].

We can decompose the space of physical states $\Psi = \Phi \otimes \Phi$ as $\Psi = \Psi_+ \oplus \Psi_-$, where

$$\Psi_+ = \left\{ \rho : \left[ \forall a < 0 : \hat{\rho}(a,\sigma,n,n') = 0 \right] \right\}$$

$$\Psi_- = \left\{ \rho : \left[ \forall a > 0 : \hat{\rho}(a,\sigma,n,n') = 0 \right] \right\}$$

are spaces of wave packets that are also Hardy class functions in the $\nu$-variable, from $\{\text{above}\}$ to $\{\text{below}\}$. Then our theorem states that:

**Theorem:** The limit of any physical state, when $t \to +\infty$, belongs to $\Psi_-$.  

**Proof:** We can decompose any $\hat{\rho}(a)$ (abbreviation for $\hat{\rho}(a,\sigma,n,n')$) as:

$$\hat{\rho}(a) = \hat{\rho}_+(a) + \hat{\rho}_-(a)$$  \hspace{1cm} (36)$$

such that:

$$\hat{\rho}_+(a) = \hat{\rho}(a) \text{ for } a > 0, \quad \hat{\rho}_+(a) = 0 \text{ for } a < 0$$

$$\hat{\rho}_-(a) = \hat{\rho}(a) \text{ for } a < 0, \quad \hat{\rho}_-(a) = 0 \text{ for } a > 0$$  \hspace{1cm} (37)$$

From eq. (34) we see that when $t \to +\infty$, then $\hat{\rho}_-(a + t) \to 0$, and therefore $\hat{\rho}(a + t) \to \hat{\rho}_+(a + t)$. So, when $t \to +\infty$ these last two functions are zero for $a < 0$, and therefore, by the Paley-Wiener theorem ( [11], page 47) the Fourier transform of $\hat{\rho}(a)$, namely $\rho(\nu)$ belongs to $H^2$.  \hspace{1cm} \Box$

So, we have proved

$$\lim_{t \to +\infty} e^{-iLt} \Psi = \Psi_-$$  \hspace{1cm} (38)$$

the analog of eq. (12) for the wave packets, as announced.

VII. CONCLUSION

As the typical distance among the scatterers is much bigger than the characteristic dimension of the scatterers itself, most of the states can be considered far from these scatterers. Therefore, most of the physical states do belong to space $\Psi_-$, thus explaining time-asymmetry (see the introduction). Moreover, using the space of physical admissible states $\Psi_-$, most of the irreversible phenomena of nature can be foresee, obtaining the same results as those of other formalisms (as “coarse-graining”, Lindblad, etc. [13]).
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