Cosmography beyond Standard Candles and Rulers

Jun-Qing Xia$^1$, Vincenzo Vitagliano$^{1,2}$, Stefano Liberati$^{1,2}$, Matteo Viel$^{2,3}$

$^1$Scuola Internazionale Superiore di Studi Avanzati, Via Bonomea 265, 34136 Trieste, Italy
$^2$INFN sez. Trieste, Via Valerio 2, 34127 Trieste, Italy and
$^3$INAF-Osservatorio Astronomico di Trieste, Via G.B. Tiepolo 11, I-34131 Trieste, Italy

We perform a cosmographic analysis using several cosmological observables such as the luminosity distance modulus, the volume distance, the angular diameter distance and the Hubble parameter. These quantities are determined using different data sets: Supernovae type Ia and Gamma Ray Bursts, the Baryonic Acoustic Oscillations, the cosmic microwave background power spectrum and the Hubble parameter as measured from surveys of galaxies. This data set allows to put constraints on the cosmographic expansion with unprecedented precision. We also present forecasts for the coefficients of the kinematic expansion using future but realistic data sets: constraints on the coefficients of the expansions are likely to improve by a factor ten with the upcoming large scale structure probes. Finally, we derive the set of the cosmographic parameters for several cosmological models (including ΛCDM) and compare them with our best fit set. While distance measurements are unable to discriminate among these models, we show that the inclusion of the Hubble data set leads to strong constraints on the lowest order coefficients and in particular it is incompatible with ΛCDM at 3-σ confidence level. We discuss the reliability of this determination and suggest further observations which might be of crucial importance for the viability of cosmographic tests in the next future.

PACS numbers: 98.80.-k, 95.30.Sf, 97.60.Bw, 98.70.Rz

I. INTRODUCTION

Cosmology is going through a golden age as we can nowadays start reconstructing the expansion history of the universe with unprecedented precision. The huge range of data sets spans a wide realm of observations with heterogeneous nature, providing us with a much more accurate tool for investigating the evolution of the universe [1]. In particular, most of the observations agree on the evidence that the universe is undergoing an era of positively accelerated expansion, requiring the existence of a (more or less conservative) source able to produce it. Since a whole set of cosmological models explaining the late time acceleration has been investigated, it goes without saying that a sensible test to discriminate among different cosmological evolutions is related to a proper interpretation of high redshift data.

Among these proposed models an important role is played by extended/modified theories of gravity, i.e. by models in which the late time acceleration is a by-product of some modified gravitational dynamics (see Ref.[2] for a review). In this context, it is clear that the development of a “gravitational dynamics independent” reconstruction of the expansion history of our universe does play a crucial role.

Cosmography provides such unbiased test of the cosmological history by assuming just homogeneity and isotropy and then use the so obtained Friedman-Lemaître-Robertson-Walker (FLRW) metric to express the distances$^1$ of the observed objects as power series in a suitable redshift parameter. The coefficients of such powers, casted into a combination of successive weighted derivatives of the scale factor $a(t)$, contain the relevant information for a kinematic description of the universe $^3$$^4$$^5$.

A comment is necessary here: as already stressed in Ref.[6] the ill-behavior at high $z$ (close and higher than $z \approx 1$) of the usual redshift expansions strongly affects the results leading in general to an underestimate of the errors. In order to avoid these problems, as well as to control properly the approximation associated with the truncation of the expansion, it is useful to recast all the involved quantities as functions of an improved parameter $y = z/(1 + z)$ $^6$$^7$$^8$$^9$. In such a way, being $z \in (0, \infty)$ mapped into $y \in (0, 1)$, it becomes possible to retrieve improved convergence

---

$^1$ Note that depending on which physical quantity one is measuring, it could be more convenient to extract from some data set a particular distance indicator than another one. These different quantities have different expressions of the Taylor expansion in redshift, such that it could be more natural to estimate cosmographic parameters, whose expression instead does not depend on the analytic expansion, in one of these particular frameworks. This ambiguity led also to a misconception about the appropriate definition of distance one should investigate. From now on we will refer only to: luminosity distance as the most direct choice in the case of measures of distance for Supernovae Type Ia (SNeIa) and Gamma Ray Bursts (GRBs); volume distance for Baryon Acoustic Oscillations (BAOs); angular diameter distance for the Cosmic Microwave Background (CMB) (see below for their definitions). For a critical discussion about such difficulties, see Ref.[3].
properties of the Taylor series at high redshift [6, 12].

Some recent papers handle the problem of interpreting the data under a cosmographic perspective using different probes [11, 13, 15]. In this paper we are going to explore the whole ensemble of data sets and use it to constrain the parameters appearing in the expansions of the characteristic scales associated to these indicators: Supernovae and Gamma Ray Bursts, Baryon Acoustic Oscillations, Cosmic Microwave Background and Hubble parameter estimates (Hub). It is quite obvious that by adding higher order powers to the redshift expansions of such scales it is possible to improve the data fitting, since more free parameters are involved. However, for a given data set, there will be an upper bound on the order which is statistically significant in the data analysis [11]. In this sense it is crucial to always determine the order of the expansion which maximizes the statistical significance of the fit for a given data set or an ensemble of them. We shall hence determine such order by performing suitable $F$-tests depending on the collection of data sets we shall consider. However, we will show that these statistical arguments can be misleading whenever is the case of a model, favored by the $F$-test, but with error intervals so large to make the last determined parameter completely unconstrained. The parameters obtained within the cosmographic analysis will then finally allow to evaluate, in a dynamic independent way, the viability of any theory aiming to explain the current expansion of the universe.

The plan of the paper is as follows: in Section II we will describe the cosmographic expansion and the theoretical framework (also partly reported in the Appendix), while in Section III we will present the observational probes used for the present analysis. Section IV contains the main results using the fourth and fifth order expansions. In Section V we focus on the forecasts with futuristic data sets while Section VI is devoted to the study and comparison of the cosmographic expansions for different cosmological models. We conclude in Section VII.

II. COSMOGRAPHIC EXPANSIONS

As a pedagogical example, we will discuss first the procedure that has been followed in order to obtain the cosmographic expansion for the luminosity distance. As already pointed out, we will start from the only assumption that the universe is homogeneous and isotropic, so that the metric describing its properties is the FLRW one

$$ds^2 = -c^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right];$$

(1)

using this metric, it is possible to express the luminosity distance $d_L$ as a power expansion in the redshift parameter $z$ (or in term of the $y$-parameter), where the coefficients of the expansion are some functions of the scale factor $a(t)$ and its higher order derivatives.

Following Ref. [14], the relation between the apparent luminosity $l$ of an object and its absolute luminosity $L$ defines the luminosity distance $d_L$

$$l = \frac{L}{4\pi r_1^2 a^2(t_0)(1+z)^2} = \frac{L}{4\pi d_L^2},$$

(2)

where $r_1$ is the comoving radius of the light source emitting at time $t_1$, $t_0$ is the later time an observer in $r = 0$ is catching the photons, and redshift $z$ is, as usual, defined as $1+z = a(t_0)/a(t_1)$. The radial coordinate $r_1$ in a FLRW universe can be written for small distances as

$$r_1 = \int_{t_1}^{t_0} \frac{c}{a(t)} dt - \frac{k}{3!} \left[ \int_{t_1}^{t_0} \frac{c}{a(t)} dt \right]^3 + \mathcal{O} \left( \left[ \int_{t_1}^{t_0} \frac{c}{a(t)} dt \right]^5 \right),$$

(3)

with $k = -1, 0, +1$ respectively for hyperspherical, Euclidean or spherical universe. In such a way, it is possible to recover the expansion of $d_L$ for small $z$

$$d_L(z) = cH_0^{-1} \left\{ z + \frac{1}{2} (1 - q_0) z^2 - \frac{1}{6} \left( 1 - q_0 - 3q_0^2 + j_0 + \frac{k c^2}{H_0^2 a^2(t_0)} \right) z^3 + \mathcal{O}(z^4) \right\},$$

(4)

having defined the cosmographic parameters as

$$H_0 = \left. \frac{1}{a(t)} \frac{da(t)}{dt} \right|_{t = t_0} = \left. \frac{\dot{a}(t)}{a(t)} \right|_{t = t_0}.$$
\[ q_0 = -\frac{1}{H^2} \frac{d^2a(t)}{dt^2}\bigg|_{t=t_0} = -\frac{1}{H^2} \frac{\ddot{a}(t)}{a(t)}\bigg|_{t=t_0}, \]
\[ j_0 = \frac{1}{H^3} \frac{d^3a(t)}{dt^3}\bigg|_{t=t_0} = \frac{1}{H^3} a^{(3)}(t)\bigg|_{t=t_0}. \]

If instead we use the redshift variable \( y = z/(1+z) \), the definition of the cosmographic parameters will not be affected, while now the luminosity distance turns out to be
\[ d_L(y) = \frac{c}{H_0} \left\{ y - \frac{1}{2}(q_0 - 3)y^2 + \frac{1}{6} \left[ 11 - 5q_0 + 3j_0 + \Omega_{ko} \right] y^3 + O(y^4) \right\}, \]

where \( \Omega_{ko} = -\frac{k}{H_0^2 a^2(t_0)} \) is the spatial curvature energy density. For a flat universe, \( \Omega_{ko} = 0 \). Since we are interested in spanning the universe at any redshift, in the following we will use only the formulation of the expansion in the variable \( y \). In our analysis we will put constraints up to the cosmographic parameters \( s_0 \) and \( c_0 \), namely:
\[ s_0 = \frac{1}{H^4} \frac{d^4a(t)}{dt^4}\bigg|_{t=t_0} = \frac{1}{H^4} a^{(4)}(t)\bigg|_{t=t_0}, \]
\[ c_0 = \frac{1}{H^5} \frac{d^5a(t)}{dt^5}\bigg|_{t=t_0} = \frac{1}{H^5} a^{(5)}(t)\bigg|_{t=t_0}. \]

In the Appendix we list the cosmographic series for the physical quantities involved in our analysis.

### III. OBSERVATIONAL DATA SETS

In our calculations, we rely on the following current observational data sets: i) SNIa and GRB distance moduli; ii) BAO in the galaxy power spectra; iii) CMB; iv) Hubble parameter determinations.

The SNIa distance moduli provide the luminosity distance as a function of redshift \( D_L(z) \). In this paper we will use the latest SNIa data sets from the Supernova Cosmology Project, “Union2 Compilation” which consists of 557 samples and spans the redshift range \( 0 \lesssim z \lesssim 1.55 \) \cite{17}. In this data set, they improved the data analysis method by using and refining the approach of their previous work \cite{18}. When comparing with the previous “Union Compilation”, they extended the sample with the supernovae from Refs. \cite{17,19}. The authors also provide the covariance matrix of data with and without systematic errors and, in order to be conservative, we include systematic errors in our calculations.

In addition, we also consider another luminosity distance indicator provided by GRBs, that can potentially be used to measure the luminosity distance out to higher redshift than SNIa. GRBs are not standard candles since their isotropic equivalent energetics and luminosities span 3–4 orders of magnitude. However, similarly to SNIa it has been proposed to use correlations between various properties of the prompt emission and also of the afterglow emission to standardize GRB energetics (e.g. Ref. \cite{20}). Recently, several empirical correlations between GRB observables were reported, and these findings have triggered intensive studies on the possibility of using GRBs as cosmological “standard” candles. However, due to the lack of low-redshift long GRB data to calibrate these relations, in a cosmology-independent way, the parameters of the reported correlations are given assuming an input cosmology and obviously depend on the same cosmological parameters that we would like to constrain. Thus, applying such relations to constrain cosmological parameters leads to biased results. In Ref. \cite{21} this “circular problem” is naturally eliminated by marginalizing over the free parameters involved in the correlations; in addition, some results show that these correlations do not change significantly for a wide range of cosmological parameters \cite{22,23}. Therefore, in this paper we use the 69 GRBs over a redshift range \( z \in [0.17, 6.60] \) presented in Ref. \cite{24}, but we keep into account in our statistical analysis the issues related to the circular problem that are more extensively discussed in Ref. \cite{21} and also the fact that all the correlations used to standardize GRBs have scatter and a poorly understood physics. For a more extensive discussion and for a full presentation of a GRB Hubble Diagram with the same sample that we used we refer the reader to section 4 of Ref. \cite{24}.

In the calculation of the likelihood from SNIa and GRBs, we have marginalized over the absolute magnitude \( M \) which is a nuisance parameter, as done in Refs. \cite{24,25}:
\[ \chi^2 = A - \frac{B^2}{C} + \ln \left( \frac{C}{2\pi} \right), \]
where

\[ A = \sum_i \left( \frac{\mu_{\text{data}}^i - \mu_{\text{th}}^i}{\sigma_i^2} \right)^2, \quad B = \sum_i \left( \frac{\mu_{\text{data}}^i - \mu_{\text{th}}^i}{\sigma_i^2} \right), \quad C = \sum_i \frac{1}{\sigma_i^2}. \tag{9} \]

BAOs have been detected in the current galaxy redshift survey data from the SDSS and the Two-degree Field Galaxy Redshift Survey (2dFGRS) \cite{31,32}. They can directly measure not only the angular diameter distance, \( D_A(z) \), but also the expansion rate of the universe, \( H(z) \), a powerful tool for studying dark energy \cite{34}. Since current BAO data are not accurate enough for extracting the information of \( D_A(z) \) and \( H(z) \) separately \cite{33}, one can only determine an effective “volume” distance \cite{31}

\[ D_V(z) = \left[ (1 + z)^2 D_A^2(z) \frac{c_\theta}{H(z)} \right]^{1/3}. \tag{10} \]

In this paper we use the Gaussian priors on the distance ratio of the volume distances as recently extracted from the SDSS and 2dFGRS surveys \cite{33} at \( z = 0.35 \) and at \( z = 0.2 \) (the two mean redshifts of the surveys)

\[ \frac{D_V(z = 0.35)}{D_V(z = 0.2)} = 1.736 \pm 0.065 \ (1 \sigma). \tag{11} \]

The \( \chi^2 \) of BAO data used in the Monte Carlo Markov Chain analysis will thus be

\[ \chi^2_{\text{BAO}} = \frac{(D_V(z = 0.35)/D_V(z = 0.2) - 1.736)^2}{0.065^2}. \tag{12} \]

It is worth stressing here that actually both the physics and the data of BAOs depend on the content in matter of the universe \( \Omega_m \). Hence, they are \textit{a priori} dependent on a chosen dynamical framework (see also Ref. \cite{57} for a review). This issue is usually ignored in the data analyses performed in the literature. However, such an approximation turns out to be valid if one does not range far away from the typical fiducial model firstly used in the determination of the physical data points. Indeed, the deviation of different models from the fiducial one can be parametrized and estimated by the ratio \( D_V(\text{new model})/D_V(\text{fiducial model}) \). The impact of the spacetime priors on the power spectrum measurement was analyzed in ref. \cite{57} and led to the conclusion that the ratio Eq. (11) is only weakly dependent on dynamical features. Hence, we can safely use BAOs as a further tool to constrain the cosmographic parameters.

Next step in our analysis is the inclusion of the CMB measurement which is sensitive to the distance from the last scattering surface via the locations of peaks and troughs of the acoustic oscillations. This data constrains the curve of the cosmological history at very high redshift, \( z \simeq 1100 \), and hence could be very helpful for discriminating among competing theoretical models for dark energy, as they necessarily have to coincide at \( z \simeq 0 \) – see for example ref. \cite{45}. The sound horizon at decoupling\(^2\), \( r_s(z_*) \), sets a physical scale for the baryon-photon oscillations depending on the baryon density, the photon energy density, and the cold dark matter density. Now, it is known that the angular diameter distance \( D_A(z) \) describes the ratio between the proper size of an object at a certain redshift \( z \) and the correlated observed angular size. The angle \( \theta_A \) under which the sound horizon is observed today is given by

\[ \theta_A = \pi l_A^{-1} = r_s(z_*) / D_A(z_*) = 0.593^\circ \pm 0.001^\circ \ (1 \sigma), \tag{13} \]

where \( l_A \) denotes the location of the first peak in the multipole space. As for BAOs, the dependence on the cosmological density parameters would not allow the use of CMB observables in a fully cosmographic approach. However, following Ref. \cite{58} it is possible to give model-independent cosmological constraints if one specifies some extra physical assumptions to be fulfilled by cosmological models. The CMB power spectrum today (apart from the low multipoles shape) is shared by models having the same primordial perturbation spectra and the same value of \( \Omega_{\text{CDM}} \) and \( \Omega_{\text{baryon}} \). For this reason a basically model-independent approach can be pursued by restricting our analysis to models having a standard physics up to the decoupling era; asking that new physics after decoupling only modifies the small angle spectrum by changing the overall amplitude and \( D_A(z_*) \), and requiring that any multipole-dependent effect at late time remain small. Such assumptions, while is cutting away some models like \( f(R) \) models with no Dark Matter \cite{39} or models with new radiation degrees of freedom, are still general enough to cover most of the cosmological models on the market.

\(^2\) In our calculation, we choose \( z_\ast = 1091.3 \), the best fit value obtained by the WMAP group \cite{36}.\]
Finally, we add the direct determinations of the Hubble parameter $H(z)$ to constrain the cosmographic expansion. Since the Hubble parameter depends on the differential age of the Universe as a function of redshift,

$$H(z) = -\frac{1}{1+z} \frac{dz}{dt},$$

measuring the $dz/dt$ could directly estimate $H(z)$. Ref. [26] used the Sloan Digital Sky Survey data and obtained a measurement of $H(z)$ at the redshift $z \approx 0$. In Ref. [27], the public data of Gemini Deep Survey (GDDS) survey [28] and archival data [29] were used in order to get the differential ages of galaxies. In practice, they selected samples of passively evolving galaxies with high-quality spectroscopy, and then used stellar population models to constrain the age of the oldest stars in these galaxies (we refer to their paper for a more exhaustive explanation of the method used). After that, they computed differential ages at different redshift bins and obtained eight determinations of the Hubble parameter $H(z)$ in the redshift range $z \in [0.1, 1.8]$. We calculate the $\chi^2$ value of this Hubble parameter data by using

$$\chi^2_{\text{Hub}} = \sum_{i=1}^{9} \frac{(H^{\text{th}}(z_i) - H^{\text{obs}}(z_i))^2}{\sigma^2_H(z_i)},$$

where $H^{\text{th}}(z)$ and $H^{\text{obs}}(z)$ are the theoretical and observed values of Hubble parameter, and $\sigma_H$ denotes the error bar of observed data. We also make use of the newly released prior on the Hubble parameter $H_0$, which consists of a measurement of the Hubble parameter obtained by the Near Infrared Camera and Multi-Object Spectrometer (NICMOS) Camera 2 of the Hubble Space Telescope (HST).

These observations fix the parameter $H_0 = 100 h_0$ (km/s)/Mpc by a Gaussian likelihood function centered around $H_0 = 74.2$ (km/s)/Mpc and with a standard deviation $\sigma = 3.8$ (km/s)/Mpc [30]. We stress that all the mentioned methods for determining $H(z)$ are basically “gravitation theory independent”.

An important point must be underlined: the Taylor series of the Hubble parameter already includes into the coefficient of the $n$-th $y$-power the same number of cosmographic parameters of the other series expanded up to the $(n+1)$-th $y$-power (see Appendix). This is due, in comparison with the other distance definitions above, to an extra derivative with respect to time included in the definition of the Hubble parameter (see also Ref. [12]).

For this reason, and for the different nature of the Hubble data, we will initially consider constraints (based on standard candles and rulers) of the expansion coefficients associated to different notions of distances; at the end, we will add the Hubble data using one order less in the $y$-power expansion with respect to the distance data in order to constrain the same set of parameters.

In order to compute the likelihood, we use a Monte Carlo Markov Chain technique as it is usually done in order to explore efficiently a multi-dimensional parameter space in a Bayesian framework. For each Monte Carlo Markov Chain calculation, we run four independent chains that consist of about 300,000 - 500,000 chain elements each. We test the convergence of the chains by using the Gelman and Rubin criterion [40] with $R - 1$ of order 0.01, which is more conservative than the often used and recommended value $R - 1 < 0.1$ for standard cosmological calculations.

IV. DATA ANALYSIS

In this section we present our main results on the constraints for the cosmographic expansion from the current observational data sets.

With the accumulations of new data and the improvements of their quality, it is of great interest to estimate the free parameters in the polynomial terms of highest order. In our previous paper [11] we already showed the inconsistency of the results in the analysis of the cosmographic expansion caused by early truncations of the power series. For these reasons we will now present the results obtained for the most meaningful term of the expansion. In order to find out which is the most viable truncation of the series for a given data set, one can use a test comparing two nested models (in this case, two different truncations of the Taylor series). The $F$-test provides exactly this criterion of comparison, identifying which of two alternatives fits better, and in the more statistically significant way, the data.

In such test, one assumes the correctness of one of the models (the one with less parameters), and then assess the probability for the alternative model to fit the data as well. If this probability is high, then no statistical benefit comes from the extra degrees of freedom associated to the new model. Hence, the smaller the probability, the more significant the data fitting of the second model against the first one will be. Quantitatively, the $F$-ratio among the two polynomials is defined as

$$F = \frac{(\chi^2_1 - \chi^2_2) N - n_2}{n_2 - n_1},$$

where
the reconstruction of constraints on the cosmography parameters as obtained from different data combinations. Their combinations with $\Omega_k$ of the universe. This would be the cosmographic expansion counterpart of the strong sensitivity on $\Omega_k$. Cosmographic analysis might have to include the spatial curvature effect in the reconstruction of the overall history especially at moderate or high redshift, will substantially improve the constraints. It is then possible that future models with one more parameter fits the data better than the other one. We already found in Ref.[11] that variations of the total energy density of the universe $\Omega_0 = 1 - \Omega_{k0}$, with the spatial curvature parameter ranging between -1 and 1, have a negligible effect on the cosmographic constraints. This is basically due to the fact that the error bars for the cosmographic parameters are still quite large in comparison with the best fit values. Nonetheless, it is worth noting that this will not be necessarily the case when future data, especially at moderate or high redshift, will substantially improve the constraints. It is then possible that future cosmographic analysis might have to include the spatial curvature effects in the reconstruction of the overall history of the universe. This would be the cosmographic expansion counterpart of the strong sensitivity on $\Omega_{k0}$ showed by the reconstruction of $w(z)$.

We then assume $\Omega_{k0} = 0$ in our analysis and only present the results for the cosmographic parameters, instead of their combinations with $\Omega_{k0}$, since the effect of curvature can be for the moment safely neglected. Table I shows the constraints on the cosmography parameters as obtained from different data combinations.

We start performing the data analysis with the SNIa data only. We find that already at the fourth order term in the expansion, the minimal $\chi^2$ is 549.59. This is not reduced significantly when compared with the constraint of the third order case, which has $\chi^2_{\text{min}} = 549.69$. Hence, introducing the snap free parameter $s_0$ does not improve the constraints. Indeed, using the $F$-test, we find a $F$-ratio of 0.11 and a $P$-value of 73.93%. Therefore, cosmography up to the fourth order term does not fit the SNIa data significantly better: the cosmographic expansion up to the jerk term $j_0$ (third order) is enough.

After adding the GRB data, the fourth order case could give a better constraint than third order only. When comparing the SNIa+GRB results, the minimal $\chi^2$ has been reduced by about five ($\chi^2 = 628.70$ instead of $\chi^2 = 627.61/624$). We then assume $\Omega_{k0} = 0$ in our analysis and only present the results for the cosmographic parameters, instead of their combinations with $\Omega_{k0}$, since the effect of curvature can be for the moment safely neglected. Table I shows the constraints on the cosmography parameters as obtained from different data combinations.

We start performing the data analysis with the SNIa data only. We find that already at the fourth order term in the expansion, the minimal $\chi^2$ is 549.59. This is not reduced significantly when compared with the constraint of the third order case, which has $\chi^2_{\text{min}} = 549.69$. Hence, introducing the snap free parameter $s_0$ does not improve the constraints. Indeed, using the $F$-test, we find a $F$-ratio of 0.11 and a $P$-value of 73.93%. Therefore, cosmography up to the fourth order term does not fit the SNIa data significantly better: the cosmographic expansion up to the jerk term $j_0$ (third order) is enough.

After adding the GRB data, the fourth order case could give a better constraint than third order only. When comparing the SNIa+GRB results, the minimal $\chi^2$ has been reduced by about five ($\chi^2 = 628.70$ instead of $\chi^2 = 627.61/624$). We then assume $\Omega_{k0} = 0$ in our analysis and only present the results for the cosmographic parameters, instead of their combinations with $\Omega_{k0}$, since the effect of curvature can be for the moment safely neglected. Table I shows the constraints on the cosmography parameters as obtained from different data combinations.

We start performing the data analysis with the SNIa data only. We find that already at the fourth order term in the expansion, the minimal $\chi^2$ is 549.59. This is not reduced significantly when compared with the constraint of the third order case, which has $\chi^2_{\text{min}} = 549.69$. Hence, introducing the snap free parameter $s_0$ does not improve the constraints. Indeed, using the $F$-test, we find a $F$-ratio of 0.11 and a $P$-value of 73.93%. Therefore, cosmography up to the fourth order term does not fit the SNIa data significantly better: the cosmographic expansion up to the jerk term $j_0$ (third order) is enough.

After adding the GRB data, the fourth order case could give a better constraint than third order only. When comparing the SNIa+GRB results, the minimal $\chi^2$ has been reduced by about five ($\chi^2 = 628.70$ instead of $\chi^2 = 627.61/624$). We then assume $\Omega_{k0} = 0$ in our analysis and only present the results for the cosmographic parameters, instead of their combinations with $\Omega_{k0}$, since the effect of curvature can be for the moment safely neglected. Table I shows the constraints on the cosmography parameters as obtained from different data combinations.

We start performing the data analysis with the SNIa data only. We find that already at the fourth order term in the expansion, the minimal $\chi^2$ is 549.59. This is not reduced significantly when compared with the constraint of the third order case, which has $\chi^2_{\text{min}} = 549.69$. Hence, introducing the snap free parameter $s_0$ does not improve the constraints. Indeed, using the $F$-test, we find a $F$-ratio of 0.11 and a $P$-value of 73.93%. Therefore, cosmography up to the fourth order term does not fit the SNIa data significantly better: the cosmographic expansion up to the jerk term $j_0$ (third order) is enough.

After adding the GRB data, the fourth order case could give a better constraint than third order only. When comparing the SNIa+GRB results, the minimal $\chi^2$ has been reduced by about five ($\chi^2 = 628.70$ instead of $\chi^2 = 627.61/624$). We then assume $\Omega_{k0} = 0$ in our analysis and only present the results for the cosmographic parameters, instead of their combinations with $\Omega_{k0}$, since the effect of curvature can be for the moment safely neglected. Table I shows the constraints on the cosmography parameters as obtained from different data combinations.

We start performing the data analysis with the SNIa data only. We find that already at the fourth order term in the expansion, the minimal $\chi^2$ is 549.59. This is not reduced significantly when compared with the constraint of the third order case, which has $\chi^2_{\text{min}} = 549.69$. Hence, introducing the snap free parameter $s_0$ does not improve the constraints. Indeed, using the $F$-test, we find a $F$-ratio of 0.11 and a $P$-value of 73.93%. Therefore, cosmography up to the fourth order term does not fit the SNIa data significantly better: the cosmographic expansion up to the jerk term $j_0$ (third order) is enough.

After adding the GRB data, the fourth order case could give a better constraint than third order only. When comparing the SNIa+GRB results, the minimal $\chi^2$ has been reduced by about five ($\chi^2 = 628.70$ instead of $\chi^2 = 627.61/624$). We then assume $\Omega_{k0} = 0$ in our analysis and only present the results for the cosmographic parameters, instead of their combinations with $\Omega_{k0}$, since the effect of curvature can be for the moment safely neglected. Table I shows the constraints on the cosmography parameters as obtained from different data combinations.

We start performing the data analysis with the SNIa data only. We find that already at the fourth order term in the expansion, the minimal $\chi^2$ is 549.59. This is not reduced significantly when compared with the constraint of the third order case, which has $\chi^2_{\text{min}} = 549.69$. Hence, introducing the snap free parameter $s_0$ does not improve the constraints. Indeed, using the $F$-test, we find a $F$-ratio of 0.11 and a $P$-value of 73.93%. Therefore, cosmography up to the fourth order term does not fit the SNIa data significantly better: the cosmographic expansion up to the jerk term $j_0$ (third order) is enough.

After adding the GRB data, the fourth order case could give a better constraint than third order only. When comparing the SNIa+GRB results, the minimal $\chi^2$ has been reduced by about five ($\chi^2 = 628.70$ instead of $\chi^2 = 627.61/624$). We then assume $\Omega_{k0} = 0$ in our analysis and only present the results for the cosmographic parameters, instead of their combinations with $\Omega_{k0}$, since the effect of curvature can be for the moment safely neglected. Table I shows the constraints on the cosmography parameters as obtained from different data combinations.

We start performing the data analysis with the SNIa data only. We find that already at the fourth order term in the expansion, the minimal $\chi^2$ is 549.59. This is not reduced significantly when compared with the constraint of the third order case, which has $\chi^2_{\text{min}} = 549.69$. Hence, introducing the snap free parameter $s_0$ does not improve the constraints. Indeed, using the $F$-test, we find a $F$-ratio of 0.11 and a $P$-value of 73.93%. Therefore, cosmography up to the fourth order term does not fit the SNIa data significantly better: the cosmographic expansion up to the jerk term $j_0$ (third order) is enough.

After adding the GRB data, the fourth order case could give a better constraint than third order only. When comparing the SNIa+GRB results, the minimal $\chi^2$ has been reduced by about five ($\chi^2 = 628.70$ instead of $\chi^2 = 627.61/624$). We then assume $\Omega_{k0} = 0$ in our analysis and only present the results for the cosmographic parameters, instead of their combinations with $\Omega_{k0}$, since the effect of curvature can be for the moment safely neglected. Table I shows the constraints on the cosmography parameters as obtained from different data combinations.
FIG. 1: One-dimensional likelihood distributions for the parameters $q_0$, $j_0$, $s_0$ and $c_0$ for the data combinations SNIa+GRB+BAO+CMB+Hub.

633.32). Using one more time the $F$-test to contrast third and fourth order expansions, we find $F$-ratio = 4.59, $P$-value = 3.26%. Thus in this case the fourth order term indeed helps to fit the observed data significantly better. The inclusion of GRBs was found to constrain the deceleration parameter $q_0$ as $q_0 = -0.76 \pm 0.26$, so confirming that our universe is undergoing an accelerated expansion with a confidence level which is marginally at $3 \sigma^{11}$. The $F$-test does not suggest to further improve the expansion up to fifth order.

Including the data point related to BAO does not improve significantly the constraints. The constraining power of BAO is rather weak since there is only one BAO data point and its redshift is much lower than those of SN and GRB data. For this reason we will consider directly the data set that includes both BAO and CMB.

When the CMB angle $\theta_A$ defined in Eq.[15] is added into our analysis, the cosmographic curve is constrained at very high redshift, $z_* \simeq 1100$ (namely $y \simeq 1$). Even though the CMB observable is providing just one data point, due to its high redshift it is in principle very helpful for discriminating among competing theoretical models producing late time accelerated expansion, since these necessarily converge to the same cosmological history at small $z$ (see for example Ref.[45]). The large difference between the two $\chi^2_{\text{min}}$ of the fourth and fifth order expansion in powers of $y$ of the distances, implies that the latter is the (statistically) more reliable parametrization ($F_{\text{ratio}} = 5.69$ and $P_{\text{value}} = 0.02\%$), giving a result very close to the $\Lambda$CDM prediction. Sixth order expansions does not give any substantial statistical improvement.

As already stated at the end of the previous section, Hubble data must be added and analyzed cautiously, since they are inhomogeneous with respect to the previous data sets both in nature and mathematical handling. In Table I we present directly the results for the constraints up to the $c_0$ cosmographic parameter, since this truncation turns out to be strongly favored with respect to the previous nested model ($F_{\text{ratio}} = 19.77$ and $P_{\text{value}} < 0.01\%$).

---

3 Note that our best fit here is different from the one reported in Ref.[11] due to our use of the improved SN catalogue “Union2 Compilation”.
The theoretical curves of $\mu(z)$ and $H(z)$ are in good agreement with the observed cosmological data, as shown in figure 2. The constraint on $H_0$ is close to the usually quoted value, namely at 68% confidence level is $H_0 = 71.16 \pm 3.08$ (km/s)/Mpc. One can see that the addition of the Hubble data leads to relevant improvements in the determination of the other cosmographic parameters with the exception of $c_0$, which is still basically unconstrained. We also checked whether the next cosmographic parameter had to be included. We find that, for the richest combination SN+GRB+BAO+CMB+Hub, the new $\chi^2_{\text{min}}$, is extremely close to the value in Table 4. Therefore, we stop our analysis here.

It is here interesting to underline the power of the $y$-expanded series (convergent as long as $y < 1$) allowing us to describe the whole cosmological history with the use of relatively few parameters. This circumstance becomes for example evident in the left panel of figure 2 where the furthest GRB data point reaches the distance of $y \simeq 0.87$ (corresponding to $z \simeq 6.6$).

V. FORECASTING

Since the present data do not give yet very stringent constraints on the parameters of cosmography, especially for the parameter of fifth order term, it is worthwhile discussing whether future data could determine these parameters more effectively. For this purpose in what follows we shall perform new analysis of possible future constraints by choosing, as a fiducial model, the best fit parameter set for the cosmographic expansion up to the fifth order term as fixed by the combination of all the previously considered data sets.

The projected satellite SNAP (Supernova / Acceleration Probe) would be a space based telescope with a one square degree field of view with $10^9$ pixels. It aims at increasing the discovery rate for SNIa to about 200 per year in the redshift range $0.2 < z < 1.7$. In this paper we simulate about 2000 SNIa according to the forecast distribution of the SNAP. For the error, we follow the Ref. 42 which takes the magnitude dispersion 0.15 and the systematic error $\sigma_{\text{sys}} = 0.02 \times z/1.7$. The whole error for each data is given by $\sigma_{\text{mag}}(z_i) = \sqrt{\sigma^2_{\text{sys}}(z_i) + 0.15^2/n_i}$, where $n_i$ is the number of Supernovae of the $i$-th redshift bin. Furthermore, we add as an external data set a mock data set of 400 GRBs, in the redshift range $0.4 < z < 6.4$ with an intrinsic dispersion in the distance modulus of $\sigma_{\mu} = 0.16$ and with a redshift distribution very similar to that of figure 1 of Ref. 43.

Regarding a future BOAO data set, we adopt the predicted performance of the BOSS survey in SDSS III, which will measure the angular diameter distance $d_A(z)$ and the Hubble expansion rate $H(z)$ of the Universe over a broad range of redshifts. The measurement precision for $d_A(z)$ is 1.0%, 1.1%, and 1.5% at $z = 0.35$, 0.6, and 2.5, respectively, and the forecast precision for the $H(z)$ is 1.8%, 1.7%, and 1.2% at the same redshifts [44]. We also impose a Gaussian prior for the current Hubble parameter $H_0$ with the error of 1% provided by a future direct measurement.

Next coming CMB measurement, mainly via the Planck satellite, could give quite accurate constraints on the cosmological parameters. The error bar of $\theta_A$ could be shrunk by a factor of 3, namely, the standard derivation $\sigma_{\theta_A} = 0.0003^\circ$.

Using all these future mock data, we get the standard derivations of cosmographic parameters: $\sigma_{\theta_0} = 0.02$, $\sigma_{\theta_0} =$

![Image of theoretical predictions of distance moduli (left panel) and Hubble parameter (right panel) from the best fit model with the full data combination, together with the observed data sets. We also show the curves obtained in the $\Lambda$CDM framework for comparison (thin black solid lines).]
0.08, $\sigma_{s_0} = 1.95$, $\sigma_{c_0} = 14.20$ and $\sigma_{H_0} = 0.48$, respectively. We hence see that the constraints on the parameters provided by the future mock data can be strongly improved in comparison with the current constraints in Table I.

VI. COSMOGRAPHIC SELECTION OF Viable COSMOLOGICAL MODELS

In the case of the standard flat $\Lambda$CDM model (namely a model described by Cold Dark Matter with the adding of a cosmological parameter) the set of cosmographic parameters results to be (up to fifth order)

$$q_0 = \frac{3}{2} \Omega_{m_0} - 1$$
$$j_0 = 1$$
$$s_0 = 1 - \frac{9}{2} \Omega_{m_0}$$
$$c_0 = 1 + 3\Omega_{m_0} + \frac{27}{2} \Omega_{m_0}^2.$$

(17)

We can use independent probes to constrain the free parameters of the cosmological model, in this case, for example, the WMAP estimates of $\Omega_{m_0}$ for the $\Lambda$CDM model.

The theoretical predictions of the cosmographic parameters in the standard $\Lambda$CDM model are:

$$q_0 = -0.588, j_0 = 1, s_0 = -0.238 \text{ and } c_0 = 2.846,$$

where we set the current matter density to be the best fit value $\Omega_{m_0} = 0.275$ obtained by the WMAP group [46]. Future experiments, in this perspective, will give stricter constraints on the validity of such hypothesis.

Another example is provided by the Dvali-Gabadadze-Porrati (DGP) self-accelerating braneworld model [47]. The presence of the infinite-volume extra dimension modifies the Friedmann equation as:

$$H_0^2 = \Omega_k (1 + z)^2 + \left( \sqrt{\Omega_{r_c} + \sqrt{\Omega_{r_c} + \Omega_{m_0} (1 + z)^3}} \right)^2$$

(18)

with $\Omega_{r_c} = 1/4 r_c^2 H_0^2$ accounting for the fractional contribution of the bulk-induced term with respect to the crossover radius $r_c$. In a spatially flat universe, $\Omega_k = 0$ and $\Omega_{r_c} = (1 - \Omega_{m_0})^2/4$, the previous equation reads

$$H_0^2 = \left[ \frac{1 - \Omega_{m_0}}{2} + \sqrt{\frac{(1 - \Omega_{m_0})^2}{4} + \Omega_{m_0} (1 + z)^3} \right]^2,$$

(19)

so, expanding both the sides of Eq.(19) and equating terms of the same power, we obtain the following expressions for the cosmographic coefficients as functions of the free parameter $\Omega_{m_0}$ (see also Ref.[48])

$$q_0 = \frac{-1 + 2\Omega_{m_0}}{1 + \Omega_{m_0}}$$
$$j_0 = \frac{1 + 3\Omega_{m_0} - 6\Omega_{m_0}^2 + 10\Omega_{m_0}^3}{(1 + \Omega_{m_0})^3}$$
$$s_0 = \frac{1 - 4\Omega_{m_0} + 19\Omega_{m_0}^2 - 134\Omega_{m_0}^3 + 86\Omega_{m_0}^4 - 80\Omega_{m_0}^5}{(1 + \Omega_{m_0})^5}$$
$$c_0 = \frac{1 + 13\Omega_{m_0} - 141\Omega_{m_0}^2 + 1259\Omega_{m_0}^3 - 1996\Omega_{m_0}^4 + 3828\Omega_{m_0}^5 - 1604\Omega_{m_0}^6 + 880\Omega_{m_0}^7}{(1 + \Omega_{m_0})^7}.$$

(20)

4 The estimate of $\Omega_{m_0}$ is of course known within a certain error. From now on, for illustrative purposes, we will retain the best fit values of the free parameters, independently estimated in every single cosmological model, as the fiducial ones, without taking into account the associated errors.
In Ref. [49], the DGP model has been constrained starting from gravitational lensing statistics; considering the fractional amount of matter obtained therein, \( \Omega_{m_0} = 0.30 \), we obtain the following set of values for the previous parameters: \( q_0 = -0.308 \), \( j_0 = 0.742 \), \( s_0 = -0.432 \), \( c_0 = 2.926 \).

We will now take into account the so-called Cardassian cosmology [50], a model whose modification with respect to standard ΛCDM cosmology consists in the introduction of an additional term \( \rho^n \) in the matter source of the Friedmann equation, so that now it can be written in term of the fractional matter density as:

\[
\frac{H^2}{H_0^2} = \Omega_{m_0} (1 + z)^3 + (1 - \Omega_{m_0})(1 + z)^3n.
\]

Performing one more time the expansion of both sides of the equation, the first four cosmographic parameters can now be expressed as functions of the two parameters \( \Omega_{m_0} \) and \( n \)

\[
q_0 = \frac{1}{2} + \frac{3}{2} (1 - n) (\Omega_{m_0} - 1)
\]

\[
j_0 = \frac{1}{2} [2 + 9n (\Omega_{m_0} - 1) + 9n^2 (1 - \Omega_{m_0})]
\]

\[
s_0 = \frac{1}{4} [4 - 18\Omega_{m_0} - 9n (4 - 7\Omega_{m_0} + 3\Omega_{m_0}^2) + 9n^2 (11 - 17\Omega_{m_0} + 6\Omega_{m_0}^2) - 27n^3 (3 - 4\Omega_{m_0} + \Omega_{m_0}^2)]
\]

\[
c_0 = \left(1 + 3\Omega_{m_0} + \frac{117\Omega_{m_0}^2}{2} - \frac{243\Omega_{m_0}^3}{2} + \frac{1215\Omega_{m_0}^4}{16}\right) - \frac{3}{4} (32 - 80\Omega_{m_0} + 291\Omega_{m_0}^2 - 648\Omega_{m_0}^3 + 405\Omega_{m_0}^4) n + \frac{9}{8} (136 - 242\Omega_{m_0} + 349\Omega_{m_0}^2 - 648\Omega_{m_0}^3 + 405\Omega_{m_0}^4) n^2 - \frac{27}{4} (46 - 73\Omega_{m_0} + 540\Omega_{m_0}^2 - 72\Omega_{m_0}^3 + 45\Omega_{m_0}^4) n^3 + \frac{81}{16} (39 - 56\Omega_{m_0} + 26\Omega_{m_0}^2 - 24\Omega_{m_0}^3 + 15\Omega_{m_0}^4) n^4,
\]

and for the referring values \( \Omega_{m_0} = 0.271 \) and \( n = 0.035 \) [51], Eq. (22) reads \( q_0 = -0.555 \), \( j_0 = 0.890 \), \( s_0 = -0.384 \), \( c_0 = 3.660 \).

Finally, we want to show the coefficients in the cosmographic approach of the CPL parametrization [52] for the equation of state of Dark Energy. If we suppose to be in a flat universe, then the Friedmann equation is:

\[
\frac{H^2}{H_0^2} = \Omega_{m_0} (1 + z)^3 + (1 - \Omega_{m_0})(1 + z)^3(1 + w_0 + w_a) e^{-\frac{3(w_0 + w_a)}{1+w_a}},
\]

and the related cosmographic terms result to be (confront also with ref. [53])

\[
q_0 = 1 + \frac{3}{2} w_0 (1 - \Omega_{m_0})
\]

\[
j_0 = 1 + \frac{3}{2} (3w_0 + 3w_a^2 + w_a) (1 - \Omega_{m_0})
\]

\[
s_0 = -\frac{9}{4} \left(1 - \Omega_{m_0}\right) w_a - \frac{9}{4} \left(1 - \Omega_{m_0}\right) [9 + (7 - \Omega_{m_0}) w_a] w_0 - \frac{9}{4} \left(1 - \Omega_{m_0}\right) (16 - 3\Omega_{m_0}) w_a^2 - \frac{27}{4} \left(1 - \Omega_{m_0}\right) (3 - \Omega_{m_0}) w_a^3
\]

\[
c_0 = \frac{1}{4} (70 + 3w_a (-71 + 3w_a (-7 + \Omega_{m_0})) (-1 + \Omega_{m_0})) + \frac{3}{4} (-1 + \Omega_{m_0}) (-163 + 3w_a (-82 + 21\Omega_{m_0})) w_0 + \frac{9}{4} (-1 + \Omega_{m_0}) (-134 - 69w_a + 3 (14 + 11w_a) \Omega_{m_0}) w_a^2 + \frac{1}{4} (1269 - 1917\Omega_{m_0} + 648\Omega_{m_0}^2) w_a^3 + \frac{1}{4} (486 - 810\Omega_{m_0} + 324\Omega_{m_0}^2) w_a^4;
\]
TABLE II: Comparison among cosmographic parameters of different cosmological models. For every model, the evaluation of the cosmographic parameters, for a pedagogical issue, is based on the best fit values of the free parameters introduced in the dynamics and measured with independent probes. However, the value of the jerk parameter for ΛCDM model is an exact value, see also (17). The values of the cosmographic parameters are compared with our best fits for the series truncations studied in the last two lines of Table I. The distances indicators data set includes SNIa, GRB, BAO, CMB. The complete data set is obtained adding Hubble data.

| Parameter                        | $q_0$    | $j_0$    | $s_0$    | $c_0$    |
|----------------------------------|----------|----------|----------|----------|
| ΛCDM                             | -0.588   | 1        | -0.238   | 2.846    |
| DGP                              | -0.308   | 0.742    | -0.432   | 2.926    |
| Cardassian                       | -0.555   | 0.890    | -0.384   | 3.660    |
| CPL Parametrization              | -0.511   | 0.342    | -2.260   | 1.383    |
| Best fit                        |          |          |          |          |
| SN+GRB+BAO+CMB (5th order)       | -0.49 ± 0.29 | -0.50 ± 4.74 | -9.31 ± 42.96 | 126.67 ± 190.15 |
| SN+GRB+BAO+CMB (5th order) + Hub (4th order) | -0.30 ± 0.16 | -4.62 ± 1.74 | -41.05 ± 20.90 | -3.50 ± 105.37 |

assuming the values suggested by the seventh-year-release of WMAP [40] for the three free parameters, $\Omega_m = 0.275$, $\omega_0 = -0.93$ and $\omega_a = -0.41$, we get $q_0 = -0.511$, $j_0 = 0.342$, $s_0 = -2.260$, $c_0 = 1.383$. Table II shows the values of the cosmographic parameters in the different models taken into account.

It is interesting to note a couple of issues in the comparison of cosmological models with our best fits. Firstly, the best fit for the cosmographic parameters of the SN+GRB+BAO+CMB data set is perfectly compatible with the estimates of the cosmological parameters for a broad variety of models. However, it is easy to realize that the currently available data sets would not allow yet to distinguish among the different cosmological models. In fact, Table II shows that the error bars are still too large with respect to the differences among the cosmographic parameters of the cosmological models. Nonetheless, the previously discussed forecasted improvement in the quality and the quantity of such data (ameliorating by at least a factor ten the error bars on the cosmographic parameter) should be able to discriminate among competing models.

On the contrary, the best fit of the widest combination of data (that is with the inclusion of the Hubble parameter determinations via the differential age technique), seems to exclude, at a 3-$\sigma$ around the jerk mean value $j_0$, almost all cosmological models, including ΛCDM (with the only exception of the CPL modelization, that is still marginally compatible). We have already discussed the intrinsic difference of the Hubble data and why their use should be taken cautiously. It seems clear that this data set while being very powerful in reducing the error bars, is simultaneously introducing strong deviations from the mean values determined via standard candles and rulers. This puzzling discrepancy in the results does not seem related to the order of the truncation: we observed a similar behavior even for (statistically not favored) early or late truncations of the series.

However, it is also true that the high redshift measurements of the Hubble parameter are based on fitting galaxy spectra. As such, this data set is strongly dependent on this fitting procedure which may introduce systematic effects. For this reason, we deem estimates based on the Hubble data currently less robust than those based on standard candles and rulers. Nonetheless, their use here serves to show the possible key role these data could play in the future of Cosmography as they appear to be very effective in reducing the error bars and very sensitive tracers of the cosmological history. We hence conclude that our analysis strongly suggests further investigation of this apparent tension between the Hubble data and ΛCDM (and many competing models) via a refinement of the determination methods developed in Refs. [20–27].

VII. DISCUSSION AND CONCLUSION

Reaching the highest possible redshift allowed by data is a fundamental tool to discriminate among competing cosmological models. Given that most of the models are built in order to recover Dark Energy at low redshift, their expansion histories are degenerate at late times. To break such a degeneracy, it is required an improvement on the knowledge of the early universe expansion curve: this aim can be achieved only by an accurate determination of the higher order parameters, and higher terms in the cosmographic expansion can be consistently reached only using (very) high redshift data.

Compared to our previous work we have added Baryonic Acoustic Oscillations (that are distance indicators at $z \sim 0.3$), a compilation of high redshift Hubble parameter measurements and, at least for a wide gamut of cosmological
models, CMB data about the angular size of the sound horizon. This improved data set is helpful in that, apart from better constraining the previously studied cosmographic parameters, it also allows to cast constraints on the next order, so far unbound, expansion coefficient.

The analysis is performed by using Monte Carlo Markov Chains in the multidimensional parameter space to derive the likelihood. As a first step, we consider the most recent catalogs of standard candles, namely Supernovae Type Ia and (properly standardized, see discussion in Section [III]) GRBs. A combination of such data gives constraints up to the $s_0$ parameter in the cosmographic series. We have also used the BAO (albeit the mildly improve the cosmographic series fitting) discussing the reliability of such tools in this context.

Secondly, we add data at higher redshift from different probes to further improve the constraints. The CMB data account for a very stable and well determined scale. It is worth noting here, anyway, that on the contrary of the other probes, CMB data provide the problem of a lack of universality in the cosmographic approach. Unfortunately, the set of parameters extracted from CMB observations is not truly independent from the dynamics of the underlying gravitational theory. Its definition, in fact, strictly depends on the assumption of a cosmological model that behaves as General Relativity plus a content of matter of arbitrary nature. It is hence impossible to use it straightforwardly within a purely cosmographic analysis which wants to apply also to non-standard cosmologies (based on exotic modified gravity theories)\(^5\). In this paper we proposed CMB data constraints on the cosmographic series by restricting the results to a slightly smaller variety of models. A desirable full solution to this problem can be achieved “standardizing” somehow the CMB parameters or alternatively identifying other CMB observables which could be used as standard rulers (at least approximately, as for BAOs). We leave this to future investigations.

We then added the high redshift measurements of the Hubble parameter. We found that thanks to these data and the CMB one, it is possible to ameliorate the knowledge of the cosmographic expansion up to the $c_0$ parameter.

As a completion of our analysis, we have also discussed foreseeable constraints from futuristic data sets provided by projected experiments. We showed that a strong reduction of the typical errors on the parameters estimates is a realistic goal: future surveys, indeed, do have a solid possibility to sufficiently reduce the uncertainties on the lowest order parameters by a factor ten at least, gaining a concrete chance to assess the viability of alternative cosmological models (possibly based on different dynamics).

Finally, we calculated the cosmographic parameter sets for a sample of cosmological models with alternative dynamics (using the so far available best fits for their free parameters). We showed that, while the data set including “standard” distance indicators gives a best fit with which all the cosmological models are still compatible, the inclusion of the Hubble data introduces a tension between the observed cosmographic parameters and the parameters calculated for different models and in particular with ΛCDM which appears to be ruled out at 3-$\sigma$ due to the jerk best fit value. We have discussed the reliability of such observation taking into account the inhomogeneity of the Hubble data set with respect to the distance indicators ones. While there might be systematic uncertainties in this data due to their complex determination method, we stressed that our analysis strongly suggests they might play a key role in reducing the errors on the estimates of the cosmographic parameters and hence in making Cosmography effective in discriminating among competing cosmological models and gravitational theories.

In conclusion, the search for high redshift standard rulers and most of all the improvement of the data coming from galaxy surveys seem to be what could possibly bring cosmographic studies into a mature stage and make them powerful, gravity theory independent, tools for selecting among theoretical scenarios. We hence hope that this work will further strengthen the case for proposed experiments aimed at improving our knowledge of the cosmic evolution of the high redshift universe (e.g. Ref. \([55]\)).

Appendix: Series Expansions

We present here more extensively the expansions used in this work. A flat universe, $k = 0$, is assumed in all the expressions below.

Hubble parameter:

$$H[z(y)] = H_0 \left[1 + (q_0 + 1)y + y^2 \left(\frac{j_0}{2} - \frac{q_0^2}{2} + q_0 + 1\right) + \ldots\right]$$

\(^5\) Of course, CMB observables can be used within a given gravitational dynamics to fix the free variables of a cosmological model \([54]\) and hence calculate the corresponding cosmographic parameters to be confronted with those determined purely on the base of standard candles and rulers, as we also showed as application to the evaluation of cosmographic parameters in several cosmological models.
\[ + \frac{1}{6} y^3 \left( -4 j_0 q_0 + 3 j_0 + 3 q_0^3 - 3 q_0^2 + 6 q_0 - s_0 + 6 \right) + \]
\[ + \frac{1}{24} y^4 \left( -4 j_0^2 + 25 j_0 q_0^2 - 16 j_0 q_0 + 12 j_0 + c_0 - 15 q_0^4 + \right. \]
\[ + 12 q_0^3 - 12 q_0^2 + 7 q_0 s_0 + 24 q_0 - 4 s_0 + 24 \right) + O[y^5] ; \]

(25)

Luminosity distance:
\[
 d_L[z(y)] = \frac{1}{H_0} \left[ y + \left( \frac{3}{2} - \frac{q_0}{2} \right) y^2 + y^3 \left( \frac{q_0^3}{2} - \frac{5 q_0}{6} - \frac{j_0}{6} + \frac{11}{6} \right) + \right. \]
\[ + y^4 \left( \frac{5 j_0 q_0}{12} - \frac{7 j_0}{24} + \frac{5 q_0^3}{8} + \frac{7 q_0^2}{8} - \frac{13 q_0}{12} + \frac{s_0}{24} + \frac{25}{12} \right) + \]
\[ + y^5 \left( \frac{j_0^2}{12} - \frac{7 j_0 q_0^2}{8} + \frac{3 j_0 q_0}{4} - \frac{47 j_0}{120} - \frac{c_0}{120} + \frac{7 q_0^4}{8} - \frac{9 q_0^3}{8} + \frac{47 q_0^2}{40} - \right. \]
\[ - \frac{q_0 s_0}{8} - \frac{77 q_0}{60} + \frac{3 s_0}{40} + \frac{137}{60} \right] + O[y^6] ; \]

(26)

Angular distance:
\[
 d_A[z(y)] = \frac{1}{H_0} \left[ y - \left( \frac{q_0}{2} + \frac{1}{2} \right) y^2 + y^3 \left( \frac{q_0^3}{2} + \frac{q_0}{6} - \frac{j_0}{6} - \frac{1}{6} \right) + \right. \]
\[ + y^4 \left( \frac{5 j_0 q_0}{12} + \frac{j_0}{24} - \frac{5 q_0^3}{8} - \frac{q_0^2}{8} + \frac{q_0}{12} + s_0 - \frac{1}{12} \right) + \]
\[ + y^5 \left( \frac{j_0^2}{12} + \frac{c_0}{120} - \frac{7 j_0 q_0^2}{8} - \frac{j_0 q_0}{12} + \frac{j_0}{40} + \frac{7 q_0^4}{8} + \frac{q_0^3}{8} + \frac{3 q_0^2}{40} - \frac{q_0 s_0}{8} + \right. \]
\[ + \frac{q_0}{20} - \frac{s_0}{120} - \frac{1}{20} \right] + O[y^6] ; \]

(27)

Volume distance:
\[
 d_V[z(y)] = \frac{1}{H_0} \left[ y + \left( \frac{1}{3} - \frac{2 q_0}{3} \right) y^2 + y^3 \left( \frac{29 q_0^2}{36} - \frac{5 q_0}{18} - \frac{5 j_0}{18} + \frac{7}{36} \right) + \right. \]
\[ + y^4 \left( \frac{43 j_0 q_0}{54} - \frac{13 j_0}{108} - \frac{94 q_0^3}{81} + \frac{13 q_0^2}{36} - \frac{19 q_0}{108} + \frac{s_0}{12} + \frac{11}{81} \right) + \]
\[ + y^5 \left( \frac{59 j_0^2}{324} - \frac{605 j_0 q_0^2}{324} + \frac{233 j_0 q_0}{648} - \frac{32 j_0}{405} - \frac{7 c_0}{360} + \frac{707 q_0^4}{3888} + \frac{523 q_0^3}{972} + \right. \]
\[ + \frac{773 q_0^2}{3240} - \frac{5 q_0 s_0}{18} - \frac{623 q_0}{4860} + \frac{13 s_0}{360} + \frac{2017}{19440} \right] + O[y^6]. \]

(28)

Acknowledgments

The authors are really grateful to the referee for crucial suggestions to improve the paper and to B. Bassett and M. Visser for their useful comments. VV wishes to thank warmly L. Izzo and M. Martinelli for fruitful discussions.
MV is supported by PRIN-MIUR, PRIN-INAF 2009, INFN/PD-51, ASI/AAE grant and the European ERC-StG “cosmoIGM” grant.

[1] M. Kunz, A. R. Liddle, D. Parkinson and C. Gao, Phys. Rev. D 80, 083533 (2009); M. Kunz, Phys. Rev. D 80, 123001 (2009); M. Kunz and D. Sapone, Phys. Rev. Lett. 98, 121301 (2007); M. J. Mortonson, W. Hu and D. Huterer, Phys. Rev. D 79, 023004 (2009); M. J. Mortonson, W. Hu and D. Huterer, Phys. Rev. D 81, 063007 (2010).
[2] T. P. Sotiriou, V. Faraoni, Rev. Mod. Phys. 82, 451-497 (2010).
[3] C. Cattoen and M. Visser, Phys. Rev. D 78, 063501 (2008).
[4] T. Chiba and T. Nakamura, Prog. Theor. Phys. 100, 1077 (1998).
[5] V. Sahni, T. D. Saini, A. A. Starobinsky, U. Alam, JETP Lett. 77, 201-206 (2003); U. Alam, V. Sahni, T. D. Saini, A. A. Starobinsky, Mon. Not. Roy. Astron. Soc. 344, 1057 (2003).
[6] C. Cattoen and M. Visser, Class. Quant. Grav. 24, 5985 (2007).
[7] D. Rapetti, S. W. Allen, M. A. Amin and R. D. Blandford, Mon. Not. Roy. Astron. Soc. 375, 1510 (2007).
[8] M. Visser, Class. Quant. Grav. 21, 2603-2616 (2004).
[9] M. Visser, Gen. Rel. Grav. 37, 1541-1548 (2005).
[10] M. Chevallier, D. Polarski, Int. J. Mod. Phys. D 10, 213 (2001); E. V. Linder, Phys. Rev. Lett. 90, 091301 (2003).
[11] V. Vitagliano, J. Q. Xia, S. Liberati and M. Viel, JCAP 03, 005 (2010).
[12] C. Cattoen and M. Visser, arXiv:gr-qc/0703122.
[13] S. Capozziello and L. Izzo, Astron. Astrophys. 519, A73 (2010); L. Xu and Y. Wang, arXiv:1007.4734; A. C. G. Guimaraes and J. A. S. Lima, arXiv:1005.2986; N. Liang, P. Wu, S. N. Zhang, Phys. Rev. D 81, 083518 (2010); H. Gao, N. Liang and Z. H. Zhu, arXiv:1003.5755.
[14] B. Santos, J. C. Carvalho and J. S. Alcaniz, arXiv:1009.2733.
[15] L. Xu and Y. Wang, Phys. Lett. B702 (2011) 114-120.
[16] S. Weinberg, Cosmology, Oxford, UK: Oxford Univ. Pr. (2008).
[17] R. Amanullah et al., Astrophys. J. 716, 712 (2010).
[18] M. Kowalski et al., Astrophys. J. 686, 749 (2008).
[19] R. Amanullah et al., Astron. Astrophys. 486, 375 (2008); J. A. Holtzman et al., Astron. J. 136, 2306 (2008); M. Hicken et al., Astrophys. J. 700, 331 (2009).
[20] G. Ghirlanda, G. Ghisellini, D. Lazzati and C. Firmani, Astrophys. J. 613, L13 (2004).
[21] H. Li, J. Q. Xia, J. Liu, G. B. Zhao, Z. H. Fan and X. Zhang, Astrophys. J. 680, 92 (2008).
[22] C. Firmani, V. Avila-Reese, G. Ghisellini and G. Ghirlanda, Rev. Mex. Astron. Astrofis. 43, 203 (2007).
[23] B. E. Schaefer, Astrophys. J. 660, 16 (2007).
[24] M. Goliath, R. Amanullah, P. Astier, A. Goobar and R. Pain, Astron. Astrophys. 380, 1 (2001).
[25] E. Di Pietro and J. F. Claeskens, Mon. Not. Roy. Astron. Soc. 341, 1299 (2003).
[26] R. Jimenez, L. Verde, T. Treu and D. Stern, Astrophys. J. 593, 622 (2003).
[27] J. Simon, L. Verde and R. Jimenez, Phys. Rev. D 71, 123001 (2005).
[28] R. G. Abraham et al., Astrophys. J. 127, 2455 (2004).
[29] T. Treu, M. Stiavelli, S. Casertano, P. Moller and G. Bertin, Mon. Not. Roy. Astron. Soc. 308, 1037 (1999); T. Treu, M. Stiavelli, P. Moller, S. Casertano and G. Bertin, Mon. Not. Roy. Astron. Soc. 326, 221 (2001); T. Treu, M. Stiavelli, S. Casertano, P. Moller and G. Bertin, Astrophys. J. 564, L13 (2002); J. Dunlop, J. Peacock, H. Spinrad, A. Dey, R. Jimenez, D. Stern and R. Windhorst, Nature 381, 581 (1996); H. Spinrad, A. Dey, D. Stern, J. Dunlop, J. Peacock, R. Jimenez and R. Windhorst, Astrophys. J. 484, 581 (1997); L. A. Nolan, J. S. Dunlop, R. Jimenez and A. F. Heavens, Mon. Not. Roy. Astron. Soc. 341, 464 (2003).
[30] A. G. Riess et al., Astrophys. J. 699, 539 (2009).
[31] D. J. Eisenstein et al., Astrophys. J. 633, 560 (2005).
[32] S. Cole et al., Mon. Not. Roy. Astron. Soc. 362, 505 (2005); G. Huetsi, Astron. Astrophys. 449, 891 (2006); W. J. Percival et al., Mon. Not. Roy. Astron. Soc. 381, 1053 (2007).
[33] W. J. Percival et al., Mon. Not. Roy. Astron. Soc. 401, 2148 (2010).
[34] A. Albrecht et al., [arXiv:astro-ph/0609591].
[35] T. Okumura, T. Matsubara, D. J. Eisenstein, I. Kayo, C. Hikage, A. S. Szalay and D. P. Schneider, Astrophys. J. 676, 889 (2008).
[36] B. A. Bassett, R. Hlozek, Baryon Acoustic Oscillations, in Dark Energy, Ed. P. Ruiz-Lapuente (2010); arXiv:0910.5224.
[37] M. Tegmark et al., Phys. Rev. D 74, 123507 (2006).
[38] M. V. vanlanthen, S. Rasanen, R. Durrer, JCAP 1008 (2010) 023.
[39] S. Capozziello, V. F. Cardone, A. Troisi, Mon. Not. Roy. Astron. Soc. 375 (2007) 1423-1440.
[40] A. Gelman and D. Rubin, Statist. Sci. 7, 457 (1992).
[41] C. Clarkson, M. Cortes, B. A. Bassett, JCAP 0708, 011 (2007).
[42] A. G. Kim, E. V. Linder, R. Miquel and N. Mostek, Mon. Not. Roy. Astron. Soc. 347, 909 (2004).
[43] D. Hooper and S. Dodelson, Astropart. Phys. 27, 113 (2007).
[44] D. Schlegel, M. White and D. Eisenstein, arXiv:0902.4680.
[45] J. Q. Xia and M. Viel, JCAP 0904, 002 (2009).
[46] E. Komatsu et al., [arXiv:1001.4538]
[47] G. R. Dvali, G. Gabadadze, M. Porrati, Phys. Lett. B 485, 208-214 (2000); C. Deffayet, Phys. Lett. B 502, 199-208 (2001).
[48] M. Bouhmadi-Lopez, S. Capozziello, V. F. Cardone, Phys. Rev. D 82, 103526 (2010); F. Y. Wang, Z. G. Dai, S. Qi, Astron. Astrophys. 507, 53-59 (2009).
[49] Z. -H. Zhu, M. Sereno, Astron. Astrophys. 487, 831 (2008).
[50] K. Freese, M. Lewis, Phys. Lett. B 540, 1-8 (2002).
[51] T. Wang, N. Liang, Sci. China G 53, 1720-1725 (2010).
[52] M. Chevallier, D. Polarski, Int. J. Mod. Phys. D 10, 213-224 (2001); E. V. Linder, Phys. Rev. Lett. 90, 091301 (2003).
[53] S. Capozziello and L. Izzo, Astron. Astrophys. 490, 31 (2008).
[54] O. Elgaroy, T. Multamaki, [arXiv:astro-ph/0702343]
[55] J. Liske et al, Mon. Not. Roy. Astron. Soc. 386, 1192 (2008).