Reduction of bihamiltonian systems and separation of variables: an example from the Boussinesq hierarchy

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Abstract

We discuss the Boussinesq system with $t_5$ stationary, within a general framework for the analysis of stationary flows of $n$–Gel’fand Dickey hierarchies. We show how a careful use of its bihamiltonian structure can be used to provide a set of separation coordinates for the corresponding Hamilton–Jacobi equations.

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1 Introduction

It is well known that the Boussinesq equation \( u_{tt} = \frac{1}{3}(-u_{xxxx} + 4u^2_x + 4uu_{xx}) \) represented as a first order system of equations in the variables \((u, v)\)

\[
\begin{align*}
  u_t &= -u_x + 2v_x \\
  v_t &= v_x - 2/3 u_{xxx} + 2/3 uu_x
\end{align*}
\]  

is a member of an infinite hierarchy of evolution equations. These equations are defined as follows. In the space \( \mathcal{M}_3 \) of the third–order differential operators \( L = \partial_x^3 + u(x)\partial_x + v(x) \), one considers the third root \( L^\frac{1}{3} \) of \( L \), i.e. a pseudodifferential operator of the form

\[
L^\frac{1}{3} = \partial + \sum_{i=1}^{\infty} q_i(u, v)\partial^{-i}
\]

satisfying \( (L^\frac{1}{3})^3 = L \). By means of the powers of such a third root, one defines an infinite family of commutative flows on \( \mathcal{M}_3 \) as

\[
\frac{\partial}{\partial t_p} L = [L, (L^\frac{1}{3})_+] \]

where \((\cdot)_+\) is the projection on the purely differential part of a pseudodifferential operator. Stationary flows for such a system correspond to that subspace of operators \( L \) satisfying the (non–linear) relation \([L, (L^\frac{1}{3})] = 0 \) for some \( p \).

In this paper we will discuss the case \( p = 5 \). The corresponding equation of the hierarchy reads:

\[
\begin{align*}
  u_5 &= -1/9 u_{xxxxxx} + 5/9 u_xu_{xx} + 5/9 uu_{xxx} + 5/3 u_{xx}v + \\
  &\quad 5/3 u_xv_x - 5/9 u^2 u_x - 10/3 vv_x \\
  v_5 &= -1/9 v_{xxxxxx} + 10/9 u_{xxx}v_x + 20/9 u_{xx}v_x + 5/3 u_xv_{xx} + \\
  &\quad 5/9 uv_{xxx} - 5/3 v_x^2 - 5/3 vv_{xx} - 10/9 vv_x - 5/9 u^2 v_x,
\end{align*}
\]

and one can see that the space \( \mathcal{M}_3^{(5)} \) of the zeroes of \( \frac{\partial}{\partial t_5} \) can parametrized by the space of the Cauchy data of two fifth order ODE in two variables. On such a ten–dimensional manifold we consider the (restriction of the) first four non–trivial flows of the Boussinesq hierarchy

\[
\frac{\partial}{\partial t_p} L = [L, (L^\frac{1}{3})_+] , \quad p = 1, 2, 4, 7.
\]

Our aims are to study the bihamiltonian structure of these equations, and to show how a careful use of such structures explicitly provides coordinates in which the corresponding Hamilton–Jacobi equations are separable.

In the first part we show that on \( \mathcal{M}_3^{(5)} \) one can construct a linear pencil of Poisson brackets \( \{F, G\}_\lambda = \{F, G\}_1 - \lambda\{F, G\}_0 \) with the following properties:

1. they have two Casimir functions \( H(\lambda) = H_1 \lambda + H_2 \) and \( K(\lambda) = H_3 \lambda^3 + H_4 \lambda^2 + H_5 \lambda + H_6 \); the functions \( H_1 \) and \( H_3 \) are Casimir functions of \( \{F, G\}_0 \), the functions \( H_2 \) and \( H_6 \) are Casimir functions of \( \{F, G\}_1 \);
2. the functions $H_2, H_4, H_5$ and $H_6$ are the Hamiltonian functions, with respect to the Poisson bracket $\{F, G\}_0$, of the four vector fields under scrutiny, while the functions $H_1, H_3, H_4, H_5$ are their Hamiltonian functions w.r.t the Poisson bracket $\{F, G\}_1;

3. all these functions are in involution w.r.t. the whole Poisson pencil.

Then we prove that the bihamiltonian structure of the vector fields \([13]\) can be used to integrate the equations of motion according to the following two–step scheme.

At first one considers a level surface $S_0$ of the functions $H_1$ and $H_3$. It is an eight–dimensional symplectic leaf of the Poisson bracket $\{F, G\}_0$. The given vector fields are tangent to $S_0$, so they can be restricted to this manifold. Then one remarks that both Poisson brackets $\{F, G\}_0$ and $\{F, G\}_1$ reduce to $S_0$ by a Marsden–Ratiu reduction procedure. So one can conclude that $S_0$ inherits a Poisson–Nijenhuis structure, i.e., $S_0$ is a manifold endowed with a symplectic form $\omega$ (induced by $\{F, G\}_0$) and with a compatible Nijenhuis tensor $N$ (induced by $\{F, G\}_1$).

The second step is to construct a set of separation variables defined on $S_0$. To this end, we shall use a kind of coordinates introduced in \([18]\) under the name of Darboux–Nijenhuis coordinates. They were exploited in \([20]\), as separation variables for the Hénon–Heiles type systems obtained from the stationary flows of the KdV hierarchy. In this paper, we explicitly compute such coordinates for the stationary Boussinesq flows, and finally we show that they are separation variables for the Hamilton–Jacobi equations associated with the Hamiltonian functions $(H_2, H_4, H_5, H_6)$.

In our opinion these flows provide a good example of the tight connection between separable coordinates and the bihamiltonian structure of soliton equations. We have chosen to study in detail this problem, in order to show that these methods can be effectively used to treat systems that are more involved than the KdV cases (see, e.g., \([1, 2, 5, 13]\)), which fall into the class of bihamiltonian manifold of maximal rank discussed in \([16]\). Accordingly, we will illustrate and display the appropriate computations, rather than proving general propositions. However, we would like to point out that these kind of results are not specific of the Boussinesq equations. Indeed, as it will be clear from our arguments, they have analogues holding, mutatis mutandis, for a wide class of stationary flows of $n$–Gel’fand–Dickey hierarchies. Such a formalization will be the subject of a further publication.

2 The Central System and its reduction

To obtain Lax pairs and Hamiltonian structures for the $t_5$ stationary Boussinesq system (hereinafter the $Bsq_5$ system), we recall the set up for the KP theory discussed in \([7, 14]\).

We consider the space $\mathcal{M}$ of sequences of Laurent series $\{H^{(k)}\}_{k=1,\ldots,\infty}$ (the currents of the theory) having the form

\[ H^{(0)} = 1, \quad H^{(k)} = z^k + \sum_{t \geq 1} H^k_t z^{-t}, \quad (2.1) \]

where $H^k_t$ are scalars. On $\mathcal{M}$ we define a family of vector fields as follows. We associate with a point $\{H^{(k)}\}_{k=0,\ldots,\infty}$ in $\mathcal{M}$ the linear span $\mathcal{H}_+$ of the currents $H^{(k)}$:

\[ \mathcal{H}_+ = \langle 1, H^{(1)}, H^{(2)} \ldots \rangle, \quad (2.2) \]
which is a subspace in the vector space $H$ of Laurent series in the formal variable $z$. Then we consider the following invariance relations:

$$\left( \frac{\partial}{\partial t_j} + H^{(j)} \right) H_+ \subset H_+,$$

(2.3)

as the defining equation for the $j^{th}$ vector field of the family. Explicitly we have:

$$\frac{\partial H^{(k)}}{\partial t_j} = H^{(j+k)} - H^{(j)} H^{(k)} + \sum_{l=1}^{k} H^{j}_{l} H^{(k-l)} + \sum_{l=1}^{j} H^{k}_{l} H^{(j-l)}.$$

(2.4)

which will be called in the sequel the Central System. It has the following properties:

- commutativity: $[\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}] = 0$;
- “exactness”: $\frac{\partial}{\partial t_k} H^{(j)} = \frac{\partial}{\partial t_j} H^{(k)}$.

The connection of the Central System with the usual formulation of the KP theory (see, e.g., [27, 12] and the references quoted therein) as a system of Lax evolution equations for the Sato pseudodifferential operator $Q = \partial + \sum_{i=1}^{\infty} q_i \partial^{-i}$, as well as the proof of the two properties stated above, can be found in [7]. Here we simply recall that reductions of KP to the $n$ Gel’fand–Dickey case are obtained via the constraint

$$H^{(n)} \equiv z^n$$

(2.5)

a requirement equivalent to the usual constraint $(Q^n)_- = 0$. In particular the Boussinesq hierarchy corresponds to the case $n = 3$.

It should be noticed that on the subset $S_n$ of $M$ formed by those points satisfying (2.5) we can read the $n^{th}$ equation of CS as the constraint equation

$$H^{(j+n)} = z^n H^{(j)} - \sum_{l=1}^{n} H^{j}_{l} H^{(n-l)}$$

(2.6)

which allows to recursively define the Laurent coefficients of $H^{(k)}$, $k > n$, in terms of the coefficients of the first $n$ currents. More formally:

**Proposition 2.1** The submanifold $S_n$ is the subset of $H_+$ given by the equation

$$z^n(H_+) \subset H_+,$$

(2.7)

i.e., the set of the points where the operator of multiplication by $\lambda = z^n$ leaves the space $H_+$ invariant. Hence, $S_n$ is generated by the first $n$ currents

$$H^{(0)} \equiv 1, H^{(1)}, \ldots, H^{(n-1)}$$

(2.8)

over the space of polynomials in $\lambda$ with coefficients in the set

$$\{H^{1}_{l_1}, H^{2}_{l_2}, \ldots, H^{n-1}_{l_{n-1}}\}, l_i \in \mathbb{N}.$$  

(2.9)

The reduced equation of motion, $CS_n$ are simply the restriction of CS to $S_n$. 

3
In the case \( n = 3 \), thanks to Proposition (2.1) the degrees of freedom are thus collected into the two series

\[
H^{(1)} \equiv h = z + \sum_{i=1}^{\infty} h_i z^{-i} \quad \text{and} \quad H^{(2)} \equiv k = z^2 + \sum_{i=1}^{\infty} k_i z^{-i}.
\]  

(2.10)

At this level, we have still a dynamical system with an infinite number of degrees of freedom. To obtain finite-dimensional dynamical systems representing the restriction of the flows to the stationary manifolds, one can remark that the set \( Z_p \) of zeroes of the \( p \)th vector field is an invariant submanifold for (2.4) as well, so that one can restrict the flows on intersections of invariant manifolds \( S_n \cap Z_p \). In particular, to study \( B_{5q} \) we shall consider \( M_3^{(5)} = S_3 \cap Z_5 \).

The restricted flows are constructed noticing that the coefficients of the Laurent expansion of the equations

\[
\frac{\partial}{\partial t_5} h = 0; \quad \frac{\partial}{\partial t_5} k = 0
\]

(2.11)

give polynomial equations for the coefficients \((h_i, k_j), i, j \geq 6\) which can be recursively solved in terms of the first ten variables \(\{h_i, k_i, i = 1, \ldots, 5\}\).

Accordingly, upon that substitution, we obtain a system of polynomial vector fields \(X_j\) defined on the phase space \(M_3^{(5)}\) whose first members are displayed in Tables 1 and 2.

**Table 1: The Vector field \(X_1\)**

|               |             |
|---------------|-------------|
| \( \dot{h}_1 \) | \( = k_1 - 2h_2 \), |
| \( \dot{k}_1 \) | \( = -h_3 + h_1^2 - k_2 \), |
| \( \dot{h}_2 \) | \( = -2h_3 - h_1^2 + k_2 \), |
| \( \dot{k}_2 \) | \( = -k_3 + h_2h_1 - h_4 - h_1k_1 \), |
| \( \dot{h}_3 \) | \( = -2h_2h_1 + k_3 - 2h_4 \), |
| \( \dot{k}_3 \) | \( = -k_4 - h_1k_2 - h_5 - k_1h_2 + h_1h_3 \), |
| \( \dot{h}_4 \) | \( = -2h_5 - h_2^2 - 2h_1h_3 + k_4 \), |
| \( \dot{k}_4 \) | \( = -h_1k_3 + h_2h_1^2 - h_1h_4 + h_1^2k_1 - 2k_1h_3 - k_1k_2 - 2k_2h_2 \), |
| \( \dot{h}_5 \) | \( = 3k_5 - 2k_1k_2 - 2k_2h_2 - 2h_2h_3 - 2k_1h_3 + 2h_2h_1^2 + 2h_1^2k_1 - 6h_1h_4 \), |
| \( \dot{k}_5 \) | \( = -3k_2h_3 - k_1h_4 - h_1k_1^2 - h_2k_3 - h_2h_4 + h_2^2h_1 - k_1k_3 \). |

### 3 Lax Representation and Poisson Structures

Another consequence of Proposition (2.1) is that, (we are sticking to the case \( n = 3 \)), for each \( j \neq 3l \) there is a non trivial \(3 \times 3\) matrix \(V^{(j)}(\lambda)\) \textit{polynomially} depending on \( \lambda \) such
Table 2: The Vector field $X_2$

\[
\begin{align*}
\dot{h}_1 &= -h_3 + h_4^2 - k_2, \\
\dot{k}_1 &= h_4 + h_1 k_1 - h_2 h_1 - 2 k_3, \\
\dot{h}_2 &= -k_3 + h_2 h_1 - h_4 - h_1 k_1, \\
\dot{k}_2 &= -k_1^2 - h_2^2 + h_5 - h_1 k_2 - 2 k_4 + 2 k_1 h_2, \\
\dot{h}_3 &= -k_4 - h_1 k_2 - h_5 - k_1 h_2 + h_1 h_3, \\
\dot{k}_3 &= -3 k_5 - k_1 k_2 + k_2 h_2 - h_2 h_1^2 - h_1 k_3 + 2 h_1 h_4 - h_1^2 k_1 + 3 k_1 h_3 - h_2 h_3, \\
\dot{h}_4 &= -h_1 k_3 + h_2 h_1^2 - h_1 h_4 + h_1^2 k_1 - 2 k_1 h_3 - k_1 k_2 - 2 k_2 h_2, \\
\dot{k}_4 &= h_1 k_1^2 - h_2^2 h_1 + h_1 h_5 - 2 h_1 k_4 + 2 k_1 h_4 - 4 k_1 k_3 - k_2 h_3 - k_2^2, \\
\dot{h}_5 &= -3 k_2 h_3 - k_1 k_4 - h_1 k_1^2 - h_2 k_3 - h_2 h_4 + h_2^2 h_1 - k_1 k_3, \\
\dot{k}_5 &= h_2 k_1^2 - h_3^2 + h_2 h_5 - h_2 h_1 k_2 - 2 h_2 k_4 + k_1 h_2^2 - k_1^3 + k_1 h_5 - h_1 k_1 k_2 - 2 k_1 k_4 - 3 k_3 k_2
\end{align*}
\]

that

\[
\left( \frac{\partial}{\partial t_j} + H^{(j)} \right) \begin{bmatrix} 1 \\ h \\ k \end{bmatrix} = \mathcal{V}^{(j)}(\lambda) \cdot \begin{bmatrix} 1 \\ h \\ k \end{bmatrix} \tag{3.1}
\]

The commutativity of CS and the exactness property (2) implies the zero curvature (or Zakharov–Shabat) representation for the restriction of CS to $\mathcal{S}_3$.

\[
\frac{\partial}{\partial t_j} \mathcal{V}^{(k)}(\lambda) - \frac{\partial}{\partial t_k} \mathcal{V}^{(j)}(\lambda) + [\mathcal{V}^{(k)}(\lambda), \mathcal{V}^{(j)}(\lambda)] = 0 \tag{3.2}
\]

The step to get a Lax problem for $B_{sq_5}$ on $\mathcal{M}_3^{(5)}$ is now easy. In fact, on the stationary manifold of $X_5 = \frac{\partial}{\partial t_5}$ we immediately obtain from (3.2) that the Lax matrix $\mathcal{V}^{(5)}(\lambda)$ satisfies the equations

\[
\frac{\partial}{\partial t_k} \mathcal{V}^{(5)}(\lambda) = [\mathcal{V}^{(k)}(\lambda), \mathcal{V}^{(5)}(\lambda)]. \tag{3.3}
\]
In other words, on the matrix $\mathcal{V}^{(5)}$, all the flows of $\mathcal{M}^{(5)}_3$ are Lax evolution equations. Explicitly, the Lax matrix $\mathcal{V}^{(5)}$ (computed using Eq. (2.6)) has the following form:

$$
\mathcal{V}^{(5)}(\lambda) = \lambda^2 \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} + \lambda \begin{bmatrix}
0 & 0 & 1 \\
h_2 & h_1 & 0 \\
-h_3 + k_2 & k_1 - h_2 & -h_1
\end{bmatrix} + 

\begin{bmatrix}
-k_3 & -k_2 & -k_1 \\
L_0^{2,1} & -h_2 h_1 + h_4 & -h_1^2 + h_3 \\
L_0^{3,1} & L_0^{3,2} & h_2 h_1 - h_4 + k_3
\end{bmatrix}
$$

(3.4)

where

$$
L_0^{2,1} = -h_1 k_2 + k_4 - k_1^2 - h_1 h_3 + h_5, \\
L_0^{3,2} = -k_1 h_2 + h_2^2 - h_5 + 2 k_4 - k_1^2, \\
L_0^{3,1} = h_1 k_3 - 2 h_1 h_4 + 3 k_5 + h_2 h_1^2 - 2 k_1 k_2 - 2 k_1 h_3 + h_1 k_2 - 2 k_2 h_2 + h_2 h_3.
$$

(3.5)

One can notice that the map $\{h_1, \ldots, k_5\} \mapsto \mathcal{V}^{(5)}$ is invertible, and thus the Lax equations (3.3) represent faithfully $\text{Bsq}_5$.

The second members of the Lax pair for the flows $X_1$ and $X_2$ are given by

$$
\mathcal{V}^{(1)} = \begin{bmatrix}
0 & 1 & 0 \\
2 h_1 & 0 & 1 \\
\lambda + h_2 + k_1 & h_1 & 0
\end{bmatrix} \quad \mathcal{V}^{(2)} = \begin{bmatrix}
0 & 0 & 1 \\
\lambda + h_2 + k_1 & h_1 & 0 \\
-h_3 + 2 k_2 & \lambda - h_2 + 2 k_1 & -h_1
\end{bmatrix}.
$$

(3.6)

### 3.1 The Hamiltonian structures

Besides their Lax representation, the vector fields of $\text{Bsq}_5$ have another important property: they admit as well a bihamiltonian representation, or a Poisson formulation with parameter. We have seen that the Lax formulation of the problem comes from the Central System. The Poisson formulation can be gotten as follows.

As it is well known [23, 26, 17], on the space $\mathfrak{g}[[\lambda]]$ of Laurent polynomials with values in a Lie algebra $\mathfrak{g}$, there is a family of mutually compatible Poisson tensors, $\mathcal{P}_l$ associated with a family of classical $R$–matrices

$$
R_l(x(\lambda)) = (\lambda^l x(\lambda))^+ - (\lambda^l x(\lambda))^-. 
$$

(3.7)

Some of them (including the values $l = 0, 1$) restrict to the affine manifold

$$
\mathfrak{g}_n^A := \{X \in \mathfrak{g}[[\lambda]] \mid X(\lambda) = \lambda^n A + \sum_{i=0}^{n-1} \lambda^i X_i.\}
$$

(3.8)
where $A$ is a fixed element of $\mathfrak{g}$.

A perhaps less known fact, pointed out in [25], is that $P_0$ and $P_1$ admit further reductions, leading to generalized Mumford systems. Actually we want to show that the Hamiltonian structures of $\text{Bsqs}_5$, gotten via the reduction of the central system discussed in Section 2, are a Marsden–Ratiu bihamiltonian reduction of the pair $P_0$ and $P_1$. This choice of Poisson pair is by no means accidental. In fact a bihamiltonian equation on $\mathfrak{g}_n^A$

$$\dot{X} = (P_1 - \lambda P_0) \nabla F$$

imply that the polynomial $X(\lambda)$ evolves according to a Lax equation

We recall that the MR reduction theorem [19] considers a Poisson manifold $(\mathcal{M}, P)$, a submanifold $S \hookrightarrow \mathcal{M}$ and a distribution $D \subset T\mathcal{M}|_S$ such that $E = D \cap TS$ is a regular foliation with a good quotient $N = S/E$. It states that the Poisson tensor $P$ is reducible to $N$ if the following hold:

1. The functions on $\mathcal{M}$ invariant along $D$ form a Poisson subalgebra of $C^\infty(\mathcal{M})$.
2. $P(D^0) \subset TS + D$, $D^0$ being the annihilator of $D$ in $T^*\mathcal{M}$.

The reduced Poisson tensors can be computed according to the following scheme:

- We choose a covector $v^N_n \in T^*_n N$.
- We choose a point $s \in S$ on the fiber over $n$, and we lift $v^N_n$ to a covector $v^M_s \in T^*_s M$, that is an extension of $\pi^* v^N_n$ lying in the annihilator $D^0$ of the distribution $D$.
- Next we construct the vector field $(P^M)_s v^M_s$ associated with the lifted covector $v^M_s$ through the Poisson tensor of $\mathcal{M}$ at the point $s$. The MR reduction theorem insures that the vector $(P^M)_s v^M_s$ is projectable onto $N$. The projection does not depend on the choice of the particular extension $v^M_s$ or on the point $s$ on the fiber and defines $(P^N)_n v^N_n$.

The bihamiltonian MR reduction theorem [9] considers a manifold $(\mathcal{M}, P_0, P_1)$ endowed with a pair of compatible Poisson structures, i.e. a bihamiltonian manifold. It is a consequence of the MR theorem stemming from the observation that a bihamiltonian manifold admits a kind of double foliation. This suggests to choose $S$ to be a symplectic leaf of $P_0$ and $D = P_1(\text{Ker} P_0)$. Then, provided that $S$ is chosen so that the regularity assumption of the MR theorem is satisfied, points 1) and 2) above are consequences of the compatibility of the Poisson pair $(P_0, P_1)$. Moreover, it states the reducibility of the whole Poisson pencil $P_\lambda = P_1 - \lambda P_0$, and provides $N$ with a bihamiltonian structure which can be computed following the procedure sketched above.

In our $\text{Bsqs}_5$ problem we consider the reduction of the Poisson pencil $P_\lambda = P_1 - \lambda P_0$, associated with the $R$–matrices (3.7), defined on the space $3 \times 3$ traceless matrices of the form $X(\lambda) = \lambda^2 A + \lambda x_1 + x_0$, with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$
Making use of the Killing form on $\mathfrak{sl}(3)$, and of the residue in $\lambda$, we can parameterize tangent and cotangent vectors as matrix–valued (Laurent) polynomials as well:

$$\dot{X} = \lambda \dot{x}_1 + \dot{x}_0; \quad W = \frac{w_0}{\lambda} + \frac{w_1}{\lambda^2}.$$  

The Poisson tensors $P_0$ and $P_1$ have the explicit form [17]:

$$\dot{X} = P_0(W) \Leftrightarrow \begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} [x_1, \cdot] & [A, \cdot] \\ [A, \cdot] & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} [x_1, w_0] + [A, w_1] \\ [A, w_0] \end{pmatrix}$$  \hspace{1cm} (3.10)

and

$$\dot{X} = P_1(W) \Leftrightarrow \begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} -[x_0, \cdot] & 0 \\ 0 & [A, \cdot] \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} [w_0, x_0] \\ [A, w_1] \end{pmatrix}. \quad (3.11)$$

A long but straightforward computation shows that the Lax matrix $V^{(5)}$ fits into the scheme herewith outlined, so that via the bihamiltonian MR reduction process the space $\mathcal{M}_3^{(5)}$ is endowed with a Poisson pencil $P_\lambda = P_1 - \lambda P_0$; in the coordinates $(h_1, k_1, h_2, k_2, \ldots, h_5, k_5)$ $P_0$ and $P_1$ have the form displayed in Tables 3 and 4.

We remark that, in the case of the $t_4$–stationary Boussinesq hierarchy, this procedure yields a pair of Poisson structures which coincide with the ones found in [15].

The hamiltonian functions for the flows can in principle be computed looking for the Casimir function of the Poisson pencil $P_1 - \lambda P_0$, according the Gel’fand–Zakharevich theorem [16]. However, since we are dealing with a Lax problem with spectral parameter, they
are provided as well by the coefficients of the spectral curve associated with the problem, i.e:

\[
0 = \det(\mu - V^{(5)}) = \mu^3 - \mu(H_1\lambda + H_2) - (\lambda^5 + H_3\lambda^3 + H_4\lambda^2 + H_5\lambda + H_6) \tag{3.12}
\]

The explicit expressions of the Hamiltonian functions are:

\[
H_1 = 3k_5 - 3k_1k_2 - 3k_2h_2, \quad H_3 = -3h_1k_2 + 3k_4 - 3k_1^2,
\]

\[
H_2 = -3k_5k_1 - h_5k_2 + h_3k_1^2 - 2h_2h_1h_4 - h_4k_3 + h_3k_2^2 - h_3h_5 - h_1k_3h_1 + h_2h_3h_3 + h_1h_3k_2 + h_1k_2^2 - 3k_1^2k_2 - k_4k_2 + 2h_2h_2k_1 + k_3^2 + h_4^2 - 2h_1^2k_4 + 2h_3k_4 - 2h_3h_1h_2 + 2h_1h_4k_1 + h_1^2h_5. \tag{3.13}
\]

\[
H_4 = -2k_1k_4 - 2k_2h_4 + k_3h_3 - 3h_1h_5 - 2h_2h_5 + 3h_1k_1k_2 + 4h_2h_1k_2 + h_3^2 - k_1^3 - h_1h_2h_3 + h_2k_4 + h_4k_2 - h_1^2k_3 + k_1h_5 + h_1h_3k_1 - k_1h_2^2 + k_3k_2 - h_2h_1^3 - h_1^3k_1 + 2h_1^2h_4.
\]

The functions $H_5$ and $H_6$ are written in Table 5.

| Table 5: The Hamiltonians $H_5$ and $H_6$ |
|------------------------------------------|
| $H_5 = h_1k_4k_2 + 4h_4k_4k_1 + 3k_1k_4h_4 + 2h_2h_1^2k_1 + h_1h_5h_3 - 3h_1k_3k_4 + 3h_1k_1k_5 + h_1h_4k_3 - 2h_2k_1k_4 + k_3k_2h_2 - 3h_2h_1^2h_4 + k_2h_2h_4 - 2h_1k_4h_3 - 2h_2^2h_1k_2 + 3k_1h_2h_4 + 3h_2k_1k_5 + h_2^2h_5 + h_5k_4 - h_5k_1^2 + h_2^2k_2 + h_2^2h_1^3 - h_3k_3^2 - k_2^2h_3 + h_1^3k_3^2 + h_2^2k_4 - k_1h_2h_3^2 - 3h_4k_5 - 4k_1^2k_4 + 2h_1^2k_2^2 + 2h_1h_4^2 + k_1h_2h_5 - h_2h_3k_3 - h_2h_4k_3 + 2k_4^2 + 2k_1^4 - h_5^2 - 5h_2h_1k_2 |
| $H_6 = -k_1^2k_4h_4 - h_1k_4^3k_2 - 3h_2h_1^2k_2^2 - 3k_5k_2h_3 - k_1^2h_2h_1^3 + 2k_1k_2^2h_3 + h_1^2k_1^2k_3 - k_2h_4k_4 + h_1k_2^3h_4 - h_5k_1k_4 + 2k_3^2h_2h_3 - 2k_4h_1^2k_3 + k_1^2h_2h_5 + h_1^4k_2h_2 - k_1^2k_3k_3 - 2h_1k_1h_4^2 + 2k_1h_2^2k_2 + k_1^2h_2^2h_4 - k_1^2h_2k_3 + k_4k_3k_2 + 3h_4k_5 + 2k_4h_2k_3 - 2h_2h_1k_1k_5 - h_1k_2h_2h_5 + h_2h_1k_2h_5 + 2h_1k_1k_2k_4 - 2k_1k_2h_2h_4 + 3h_2^2h_1k_1k_2 + h_2h_1k_3k_1^2 - 2h_2h_1h_4k_3 - h_2h_1^2k_2h_3 + 3h_2h_1^2k_1h_4 - 2h_1k_2h_3k_3 + 2h_1h_3k_1k_4 + 3h_1k_2h_4h_3 - 3h_1^2k_2h_3k_1 - k_1h_2^2k_4 + 3h_1^2k_2k_5 - k_2h_4h_5 + k_3k_2h_5 + h_2h_1^2k_3^2 + h_5k_1^2k_3^2 - h_5k_3h_3 + h_2^2h_3k_3 - h_2h_3k_2 - k_1h_2^2h_5 - k_1h_2^2h_1^3 - h_1k_2^2k_3 + h_1^4k_2k_1 - 2h_1^3k_2h_4 + k_1^2h_2h_4 - h_1k_1h_3k_1^2 + h_1^3k_2k_3 - 2h_1^2k_2k_1 + 3k_1^3k_4 + h_4^2k_3 - h_4k_3^2 + k_1h_5^2 - 2k_1k_4^2 + k_3^3h_2^2 - k_1^4h_2 + h_2h_1k_2k_4 - k_1h_2h_3k_3 - h_1k_1h_5h_3 + h_1h_4k_1k_3 + k_1h_2h_4k_3 - k_1^5 |

A direct computation shows that

**Proposition 3.1** *The coefficients $H_j$ of the characteristic polynomial of the Lax matrix $V^{(5)}(\lambda)$, the vector fields $(X_1, X_2, X_4, X_7)$, and the Poisson tensors $P_0$ and $P_1$ of Tables 3 and 4 fill the bihamiltonian sequences of Figure 1. In particular, the hamiltonian functions $H_j$ are mutually in involution w.r.t. the whole Poisson pencil.*
4 PN manifolds and the Sklyanin procedure

In this Section we show how the bihamiltonian structure of the Bsq hierarchy can be exploited in order to get a set of variables which satisfy Sklyanin’s separation equations. Actually, this kind of variables arise as a set of coordinates which put in “canonical” form a Poisson–Nijenhuis (PN) structure obtained by a further Marsden–Ratiu reduction of the degenerate Poisson pencil \( P_\lambda \).

4.1 The reduction of the Poisson pencil

Let \( M \) be a bihamiltonian manifold with degenerate Poisson tensors \( (P_0, P_1) \) and let \( X_j \) be the vector fields of a bihamiltonian sequence on \( M \). A possible way to analyze the integrability of \( X_j \) is to eliminate the Casimir functions of one Poisson tensor, say \( P_0 \), by fixing the values of its Casimirs. Of course, \( P_0 \) can be restricted to any of its symplectic leaf \( S_0 \) and the \( X_j \), being tangent to \( S_0 \), can be restricted as well to vector fields \( \tilde{X}_j \). These are still Hamiltonian on \( S_0 \), with Hamiltonians \( \tilde{H}_j \) given by the restrictions of the original Hamiltonians to \( S_0 \). However, in general, the bihamiltonian formulation of the restricted problem is lost on \( S_0 \).

Nonetheless, in our situation a specific feature of the pencil allows us to reduce the bihamiltonian structure to any generic leaf \( S_0 \) of \( P_0 \). Indeed, the Poisson pencil \( P_\lambda \) induces on \( S_0 \) a Poisson–Nijenhuis (PN) structure [23, 22]. With respect to such a PN structure, the fields \( \tilde{X}_j \) admit a formulation, to be introduced in Section 5, powerful enough to allow us to find a set of separation variables.

**Lemma 4.1** Let us consider the vector fields

\[
Z_1 = \frac{\partial}{\partial k_5}, \quad Z_2 = \frac{\partial}{\partial k_4} + 2 \frac{\partial}{\partial h_5},
\]

and let \( \mathcal{D} \) be the distribution generated by \( Z_1 \) and \( Z_2 \). Then it holds that:

\[
L_{\phi_1 Z_1 + \phi_2 Z_2} (P_1 - \lambda P_0) = Z_1 \wedge W_1 + Z_2 \wedge W_2;
\]

where \( L_X P \) is the Lie derivative of \( P \) along \( X \), \( \phi_1 \) and \( \phi_2 \) are generic smooth functions and the \( W_i \)’s are suitable vector fields (which depend on \( \phi_1 \) and \( \phi_2 \)).

Figure 1: The Bihamiltonian Sequences
This condition ensures that the space of functions annihilated by $\hat{D}$ is actually a Poisson subalgebra of $C^\infty(M_3^{(3)})$. Moreover $\hat{D}$ is generically transversal to the image of $P_0$. Hence we can apply the MR theorem to state

**Proposition 4.2** The pencil $P_λ$ is reducible to a pencil $\hat{P}_λ$ on $S_0$. Since $\hat{P}_0$ is invertible on $S_0$, that is, $\omega = \hat{P}_0^{-1}$ is a symplectic two-form, we can conclude that $S_0$ admits the structure of a PN manifold, with respect to the pair $(\hat{P}_0, N = \hat{P}_1 \hat{P}_0^{-1})$.

It is convenient to adapt the coordinates on $M$ to the distribution $\hat{D}$ as $\{h_1, \ldots, h_4, g_1 = 2k_4 - h_5, v_1 = H_1, v_2 = H_3\}$. In this way, the reduced Poisson tensors can be obtained from $P_0$ and $P_1$ (written in the new coordinate system) simply deleting the last two rows and columns. They are represented by:

$$\hat{P}_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \\
0 & 1 & 1 & 0 & 0 & 0 & h_1 & 3k_1 - h_2 \\
0 & 0 & 1 & 0 & -h_1 & 0 & -2h_2 & \\
1 & 0 & 0 & 0 & 0 & -3k_1 + h_2 & 2h_2 & 0
\end{bmatrix}, \quad (4.3)$$

$$\hat{P}_1 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & h_1 & k_1 - 2h_2 \\
0 & 0 & 0 & 0 & -k_1 & 0 & -k_2 & \\
0 & 0 & -2h_1 & -k_1 & -h_2 & -k_2 & \\
0 & -k_1 & -k_2 & -k_2 & 0 & \\
0 & 0 & h_3 + h_1^2 & -2h_4 + k_3 + 2h_1k_1 - 2h_2h_1 & \\
* & 0 & h_1k_1 & 3k_1^2 - h_2k_1 & \\
& 0 & g_1 - h_1k_2 - k_1^2 - h_2^2 & \\
& & & & & & & 0
\end{bmatrix}. \quad (4.4)$$
The adjoint of the Nijenhuis tensor is

\[
N^* = \begin{bmatrix}
-k_1 & k_2 & k_2 & 0 & N^{1,5} & 0 & N^{1,7} & 3k_1k_2 - k_2h_2 \\
0 & k_1 & 4k_1 & 0 & -h_1^2 + h_3 & 0 & N^{2,7} & N^{2,8} \\
-h_1 & 0 & -h_2 & k_2 & -h_3 - h_1^2 & 0 & N^{3,7} & N^{3,8} \\
0 & 0 & 2h_1 & k_1 & 0 & 0 & h_3 + h_1^2 & N^{4,8} \\
0 & 0 & 0 & 0 & k_1 & k_2 & k_2 & 0 \\
0 & 0 & 0 & 0 & 0 & k_1 & 0 & k_2 \\
1 & 0 & 0 & 0 & 2h_1 & 0 & h_2 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -h_1 & 2h_2 - k_1 \\
\end{bmatrix},
\]

(4.5)

where

\[
N^{1,5} = -2h_2h_1 + 2h_4 - k_3 - 2h_1k_1, \quad N^{1,7} = -h_2^2 - g_1 + h_1k_2 + k_1^2, \\
N^{2,8} = -6h_2k_1 - h_2^2 + 7k_2^2 - g_1, \quad N^{3,8} = 3h_2^2 - 2h_2k_1 + g_1 - k_1^2, \\
N^{4,8} = -2h_4 + k_3 + 2h_1k_1 - 2h_2h_1, \quad N^{2,7} = 4h_1k_1, \quad N^{3,7} = -2h_2h_1.
\]

The separation of variables (SoV) is an outcome of the spectral analysis of \(N^*\), and relies on the introduction of some special set of coordinates. This will be the subject of the next subsection.

### 4.2 The Darboux–Nijenhuis and the Hankel–Fröbenius coordinates

Let \((M, P_0, N = P_1P_0^{-1})\) be a PN manifold of dimension \(2n\). In [18], it was proved that if \(N\) has \(n\) functionally independent eigenvalues \((\lambda_1, \ldots, \lambda_n)\) (otherwise stated, when \(N\) is maximal) one can introduce a set of coordinates \((\Lambda := (\lambda_1, \ldots, \lambda_n) ; \mu := (\mu_1, \ldots, \mu_n))\) such that \(P_0\) and \(N\) take the form

\[
P_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad N = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix},
\]

(4.6)

where \(I\) is the \(n \times n\) identity matrix and \(\Lambda := \text{diag}(\lambda_1, \ldots, \lambda_n)\). It was also remarked that while the first \(n\) coordinates are the eigenvalues of \(N\) the remaining ones can in general be constructed by quadratures. Since these coordinates are canonical w.r.t. \(P_0\) and diagonalize the Nijenhuis tensor they will be referred to as \textit{Darboux-Nijenhuis} (DN) coordinates.

Another set of coordinates, denoted as \((f := (f_1, \ldots, f_n) ; c := (c_1, \ldots, c_n))\) can be fruitfully introduced. The idea is the following: (minus) the coefficients \(c_j\) of the minimal polynomial of the Nijenhuis tensor

\[
(\text{Det}(\lambda - N))^\frac{1}{2} = \lambda^n - \sum_{i=1}^n c_i \lambda^{n-i}
\]

(4.7)
satisfy the following remarkable recursion property:

\[ N^* dc_1 = dc_2 + c_1 dc_1 \]
\[ N^* dc_2 = dc_3 + c_2 dc_1 \]
\[ \vdots \]
\[ N^* dc_n = c_n dc_1. \]

(4.8)

In complete analogy, we seek for a set of complementary variables \( \{ f_1, \ldots, f_n \} \) such that

1. their differentials generate an \( n \)-dimensional distribution in \( T^* S_0 \), complementary to the one generated by the \( dc_i \)'s;
2. they satisfy the same recursion relation:

\[ N^* df_1 = df_2 + c_1 df_1 \]
\[ N^* df_2 = df_3 + c_2 df_1 \]
\[ \vdots \]
\[ N^* df_n = c_n df_1. \]

(4.9)

A characteristic feature of the coordinates \( (f; c) \) is that \( P_0 \) and \( N^* \) take the matrix form

\[
P_0 = \begin{bmatrix}
0 & H \\
-H^T & 0
\end{bmatrix}, \quad N^* = \begin{bmatrix}
F & 0 \\
0 & F
\end{bmatrix},
\]

(4.10)

where

\[
H = \begin{bmatrix}
0 & 0 & \cdots & 1 \\
0 & \cdots & 1 & -c_1 \\
1 & -c_1 & \cdots & -c_{n-1}
\end{bmatrix}, \quad F = \begin{bmatrix}
c_1 & c_2 & \cdots & c_n \\
1 & 0 & \cdots & 0 \\
0 & \cdots & 1 & 0
\end{bmatrix},
\]

(4.11)

whence the denomination of Hankel–Fröbenius (HF) coordinates.

It is not difficult to show that the relation between the \( (f; c) \) coordinates and the DN coordinates \( (\lambda; \mu) \) is given by the system

\[
\begin{align*}
\lambda^n - \sum_{i=1}^{n} c_i \lambda^{n-i} &= \prod_{i=1}^{n} (\lambda - \lambda_i) \\
\mu_k &= \sum_{i=1}^{n} f_i \lambda_k^{n-i} \quad (k = 1, \ldots, n).
\end{align*}
\]

(4.12)

The advantage of using the Hankel–Fröbenius coordinate system is that, in the cases at hand, they can be computed in an algebraic fashion from the recurrence (4.8).

As a final remark, we note that, owing to the last equations (4.8) and (4.9), the Hamiltonian fields \( X = P_0 dc_1 \) and \( Y = P_0 df_1 \) are Pfaffian quasi-bihamiltonian vector fields in the terminology of [3, 27]. Thus these arguments prove the existence of such fields on any maximal PN manifold.
4.3 Back to Bsq5

By computing the minimal polynomial of $N^*$ of Eq. (4.5) one gets the $c$’s coordinates

\begin{align*}
c_1 &= k_1 + h_2, & c_2 &= 2k_1^2 - h_2^2 - g_1 + 2h_1k_2 - h_2k_1, \\
c_3 &= -3k_1^3 + 2k_1h_2^2 + 2k_1g_1 - 2h_1k_1k_2 - 2h_1h_2k_2 + 2k_2h_4 - k_3k_2 - h_2k_1^2, \\
c_4 &= k_2^2h_3 - g_1k_1^3 + h_2k_1^3 - h_1^2k_2^2 + k_1^4 + 2h_2h_1k_1k_2 - 2k_1h_2h_4 + k_1k_2k_3 - h_2^2k_1^2. \quad (4.13)
\end{align*}

The $f$’s coordinates can be found as follows. Taking into account the homogeneity properties of the theory with respect to the weights $[\lambda] = 3$, $[h_i] = 1 + i$, $[k_i] = 2 + i$, we see that $[P_0] = -8$, and $[c_i] = 3i$. From (4.11) we get that the weight of $f_1$ must be equal to $-4$. Making the ansatz $f_1 = \frac{1}{k_2}$ one finds, using the recursion relation (4.9) that the $f$’s are given by:

\begin{align*}
f_1 &= \frac{1}{k_2}, & f_2 &= -\frac{h_2}{k_2}, & f_3 &= -h_1 + \frac{g_1 + h_2^2 - 2k_1^2}{k_2} \\
f_4 &= k_3 - h_4 + h_1h_2 + \frac{k_1^3 + h_2k_1^2 - k_1h_2^2 - g_1k_1}{k_2}. \quad (4.14)
\end{align*}

The Hankel–Fröbenius coordinates associated with the PN structure allow us to reconstruct the spectral curve (3.14), and the Sklyanin separation equations as follows. In analogy with the corresponding arguments known from the KdV case \[1, 13\], we consider equations (4.12) as defining two fundamental polynomials associated with the problem:

\begin{align*}
Q_1 &= \lambda^4 - \sum_{i=1}^{4} c_i \lambda^{(4-i)}; & Q_2 &= \sum_{i=1}^{4} f_i \lambda^{(4-i)}; \quad (4.15)
\end{align*}

Then, a direct computation shows that on the zeroes $\lambda_j$ of $Q_1$ and the corresponding values $\mu_j = Q_2(\lambda_j)$, the equation of the spectral curve (3.12) is satisfied:

\begin{align*}
\mu_j^3 &= \mu_j(H_1\lambda_j + H_2) + (\lambda_j^5 + H_3\lambda_j^3 + H_4\lambda_j^2 + H_5\lambda_j + H_6). \quad (4.16)
\end{align*}

When we restrict the problem to the leaf $S_0$, we have to consider $H_1 = v_1$ and $H_3 = v_2$ as fixed parameters. Hence, we obtain Sklyanin’s separation equation \[28\], i.e. 4 relations in the canonical coordinates $(\lambda; \mu)$, depending on the 4 commuting Hamiltonians $\tilde{H}_2, \tilde{H}_4, \tilde{H}_5, \tilde{H}_6$. As it is well known, they imply that, in analogy with the classical Stäckel theory of separation of variables for Hamiltonians depending quadratically on the momenta, a complete integral $W = W_1 + W_2 + W_3 + W_4$ of the Hamilton–Jacobi equations

\[ \tilde{H}_i(\lambda, \frac{\partial W}{\partial \lambda}) = \alpha_i, \quad i = 2, 4, 5, 6, \]

can be gotten by solving the four decoupled first–order ODE

\begin{align*}
\left( \frac{dW_j}{d\lambda_j} \right)^3 &= \frac{dW_j}{d\lambda_j} (v_1 \lambda_j + \alpha_2) + (\lambda_j^5 + v_2 \lambda_j^3 + \alpha_4 \lambda_j^2 + \alpha_5 \lambda_j + \alpha_6) & j = 1, \cdots, 4, \quad (4.17)
\end{align*}

a problem which can be treated by means of algebro–geometrical techniques.
5 Nijenhuis chains and Levi–Civita separability

Here we introduce the concept of Nijenhuis chains as a natural extension of the Lenard sequences and provide an alternative and somehow more classical proof of the fact that the Darboux–Nijenhuis coordinates are separation variables for the Hamilton-Jacobi equation corresponding to each Hamiltonian vector field $\hat{X}_j$ of our problem. Remark that, when dealing with algebro–geometrical SoV in Section 4, the original Hamiltonians (3.13) of the ten–dimensional “Gel’fand–Zakharevich” problem of Proposition 3.1 were not involved in the construction of the DN and HF coordinates, and entered the separability proof only at the last stage. The reason was that they, not being invariant under the distribution $\hat{D}$, cannot define a Lenard recursion relation on $S_0$, at least in the classical form enlightened, e.g., in [23, 11]. Here, we will consider their restriction $\hat{H}_j$ to the symplectic leaf $S_0$ in connection with the following criterion for separability due to Benenti [3]:

Proposition 5.1

Let $X = P_0 dK$ a Hamiltonian vector field on a symplectic manifold $(M, \omega = P_0^{-1})$ and $(q; p) (q := (q_1, \ldots, q_n); p := (p_1, \ldots, p_n))$ a set of Darboux coordinates. These coordinates are separation variables for the corresponding Hamilton-Jacobi equation iff $X$ admits $n$ integrals of motion $\{K_i\}_{1 \leq i \leq n}$, such that

$$\det \left( \frac{\partial K_i}{\partial p_j} \right) \neq 0 ,$$

and which are in separable involution w.r.t. the chart $(q; p)$, i.e. satisfy the $n$ relations

$$\{K_i, K_j\} := \frac{\partial K_i}{\partial q_k} \frac{\partial K_j}{\partial p_k} - \frac{\partial K_i}{\partial p_k} \frac{\partial K_j}{\partial q_k} = 0 ,$$

where no summation over $k$ is performed.

In order to exploit this criterion, we introduce the notion of Nijenhuis chain which generalizes the classical Lenard recurrence satisfied by the (normalized) traces of powers of the Nijenhuis tensor $I_k = \frac{1}{2k} \text{Tr} N^k$,

$$N^* dI_k = dI_{k+1} \Rightarrow (N^*)^{k-1} dI_1 = dI_k$$

and the ones satisfied by the HF coordinates $(f; c)$ as a consequence of (4.8, 4.9).

Definition 5.2

Let $(M, P, N)$ a PN manifold and $\{K_j\}_{1 \leq j \leq n}$ $n$ independent functions which enjoy the following property w.r.t. $N^*$

$$dK_j = \sum_{k=1}^{n} a_{jk} (N^*)^{k-1} dK_1 \quad j = 1, \cdots, n ,$$

where $a_{jk}$ are the entries of an invertible matrix-valued function $a$. This recurrence relation (and the corresponding one $X_j = PdK_j$ on the vector fields) will be referred to as a Nijenhuis chain with generator $K_1$ ($X_1 = PdK_1$).

Nijenhuis chains were implicitly considered in [24] where it was proved that the functions $K_j$ admitting the representation (5.3) are in involution w.r.t. the Poisson tensor $P$. Here, we want to state a stronger property of such functions, namely that they are in separable involution w.r.t. the DN coordinates. This can be easily proved, by computing $\{K_i, K_j\}_k$ in the DN coordinates, and recalling that the $\lambda_j$ are eigenvalues of $N^*$. Therefore in virtue of Proposition 5.1 we can state:
Proposition 5.3 The Darboux–Nijenhuis coordinates are separation variables for each function $K_j$ belonging to a Nijenhuis chain.

To prove that $B_{sq}$ fulfills Benenti’s theorem, it is thus sufficient to verify that the restrictions $\tilde{H}_j$, of the Hamiltonians $H_j$ to a generic symplectic leaf $S_0$, belong to a Nijenhuis chain. One can directly check that this is the case: indeed, choosing as generator $K_1$ the Hamiltonian

$$\tilde{H}_2 = k_3^2 - h_4 k_3 + h_3 k_1^2 + h_2^2 h_3 - 2 h_1 k_2^2 - 3 k_1^2 k_2 + h_4^2 - h_1 k_1 k_3 + h_2 h_1 k_3 + h_1 k_2 h_3 - 2 h_1 h_2 h_4 - k_1 k_2 h_2 - 2 k_1 h_2 h_3 + 2 k_1 h_1 h_4 - k_1 v_2 + k_2 g_1 - k_2 v_1 + h_3 g_1 - h_1^2 g_1,$$

upon the identifications $K_2 = \tilde{H}_4, K_3 = \tilde{H}_5, K_4 = \tilde{H}_6$, the matrix $a$ of the Nijenhuis recurrence is given (in the coordinates $(f; c)$) by

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -f_1 c_1 - f_2 & f_1 & 0 & 0 \\ -f_1 c_2 - f_3 & -f_1 c_1 & f_1 & 0 \\ -f_1 c_3 - f_4 & -c_2 f_1 & -f_1 c_1 & f_1 \end{bmatrix} \quad (5.4)$$

6 Summary

In this paper we have discussed, on the ground of the concrete example of the $t_5$ stationary Boussinesq system two related items. In the first part of the paper we have sketched a way to obtain stationary reductions, and their Hamiltonian structures, of the $n$ Gel’fand–Dickey theories in a systematic way. The key points were the study of the Central System (2.4) and of its stationary manifolds, and the zero–curvature and Lax representation (3.2) one naturally gets for the resulting flows. Hamiltonian structures are induced on the stationary manifolds by means of a specific bihamiltonian reduction procedure for a pair of Poisson structures, defined on the space of Lie algebra–valued polynomials in the spectral parameter $\lambda$. This procedure, whose justification rests on the link between Lax and Poisson formulation for Hamiltonian flows, lead us to frame such theories as the Gel’fand–Zakharevich theories of a degenerate pencil of Poisson structures, generically (i.e. for $n > 2$ of non maximal rank (see Section 3)).

In the second part of the paper (Section 4 onwards) we addressed the issue of how one can solve these integrable dynamical systems by the method of separation of variables. Our main theme was to study some of the relations between the classical Hamilton–Jacobi theory (and also in its more recent aspects related to the notion of complete algebraic integrability) and the Hamiltonian structure we got in the first part. To this end, we had to abandon the GZ theory, and, by means of a further (but no more bihamiltonian ) reduction process, make contact with the theory of Poisson–Nijenhuis manifolds. We studied the properties of two remarkable classes of coordinate systems associated with the PN structures: the Darboux–Nijenhuis coordinates and the Hankel–Fröbenius coordinates. By using these coordinates, we proved the separability of the $B_{sq}$ flows in two ways. In the first we made implicitly contact with the “method of the poles of the Baker–Akhiezer function” of the Russian School (see, e. g. [3, 28]). In the second, via the introduction of the notion of Nijenhuis
chain, we proved separability by using an equivalent condition to the classical Levi–Civita separability criterion [24].

As a final comment, we notice that the peculiarity of the second reduction, which still falls in the class of the MR reductions, is given by the fact that two different geometric processes are used simultaneously: the restriction for the vector fields and the projection for the bihamiltonian structure. Due to this fact, we were able to retain the bihamiltonian structure (in the form of a PN structure) but lost the bihamiltonian formulation for the vector fields. This is not a new situation in the Hamiltonian theory of integrable system and happens, for instance, for:

- the integrable Hénon–Heiles system and its multidimensional generalizations obtained by reduction from the stationary flows of the KdV hierarchy [28, 29];
- a large class of potentials recovered from the restricted flows of the coupled KdV systems, whose most representative member is the Garnier system [4].

Such classes of dynamical systems live on a bihamiltonian manifold $M$ of maximal rank, and their Nijenhuis formulation, namely a special case called quasi–bihamiltonian formulation [6, 21] can be gotten after a similar reduction. The stationary flows of the Boussinesq hierarchy (and of the higher $n$ GD theories) are just examples of bihamiltonian structures of non–maximal rank, whose interest may lie in the fact that an analysis similar to the one of [16] for Poisson pencils of maximal rank is not, to the best of our knowledge, at present available in the literature.

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Table 4: The reduced tensor $P_1$. The entries $P^{l,k}$ are the polynomials in the coordinates \{$h_1, \ldots, k_3$\} written below

\[
\begin{array}{cccccccc}
0 & 0 & -3 & 0 & 0 & 0 & -3h_1 & 2h_2 - k_1 & k_1 - 2h_2 & P^{1,10} \\
0 & 0 & 0 & 0 & 3k_1 & 0 & h_3 - h_1^2 + k_2 & 2h_3 - 2h_1^2 - k_2 & P^{2,10} \\
0 & 0 & 6h_1 & 3k_1 & 3h_2 & 2k_2 + 2h_3 + h_1^2 & k_2 + 4h_3 + 2h_1^2 & P^{3,10} \\
0 & 3k_1 & 3k_2 & 3k_2 & & & P^{4,8} & P^{4,9} & P^{4,10} \\
0 & 0 & -3h_1^2 - 3h_3 & & & & P^{5,8} & k_3 - 2h_2h_1 - 2h_4 & P^{5,10} \\
0 & -3h_1k_1 & & & & & P^{6,8} & P^{6,9} & P^{6,10} \\
* & 0 & & & & & P^{7,8} & P^{7,9} & P^{7,10} \\
& 0 & & & & & P^{8,9} & P^{8,10} & \\
& 0 & & & & & P^{9,10} & & 0 \\
\end{array}
\]

\[
\begin{align*}
P^{1,10} &= h_3 - 2k_2 - h_1^2, & P^{2,10} &= -h_1k_1 + h_2h_1 - h_4 + 2k_3, \\
P^{3,10} &= k_3 + h_4 - h_2h_1 + h_1k_1, & P^{4,8} &= k_3 + h_4 - h_2h_1 + h_1k_1, \\
P^{4,9} &= 2k_3 + 2h_4 - 2h_2h_1 + 2h_1k_1, & P^{4,10} &= -2k_1h_2 + k_1^2 + h_1k_2 - h_5 + h_2^2 + 2k_4, \\
P^{5,8} &= -3h_1k_1 + 2h_2h_1 + 2h_4 - k_3, & P^{5,10} &= -h_1h_3 - 5h_1k_2 - 2k_1h_2 - 3k_1^2 + h_5 + k_4 \\
P^{6,8} &= -6k_1^2 + k_1h_2 - 2h_1k_2 + k_4 - h_1h_3 + h_5, & P^{6,9} &= -3k_1^2 - k_1h_2 - 4h_1k_2 + 2k_4 - 2h_1h_3 + 2h_5, \\
P^{6,10} &= -3k_1h_3 - 8k_1k_2 - 4k_2h_2 + h_1k_3 + h_2h_3 - 2h_1h_4 + h_2h_1^2 + 3k_5 + h_1^2k_1, \\
P^{7,8} &= h_2^2 + 2h_1h_3 - k_4 + 2h_5, & P^{7,9} &= -h_2^2 - 3h_1k_2 + 4h_1h_3 - 3k_1^2 + 4k_4 + h_5, \\
P^{7,10} &= h_1k_3 + h_1h_4 - h_2h_1^2 - h_1^2k_1 - 4k_2h_2 - 2k_1k_2 + 2k_1h_3, & P^{8,9} &= -4k_2h_2 - 4h_1h_4 + 2k_1h_3 - 3k_1k_2 + 3k_5 + 2h_1k_3 - 2h_2h_3, \\
P^{8,10} &= -2k_2^2 - h_1h_5 + 2h_1k_4 - 2k_2h_3 - 3k_1h_4 - 2h_1k_1^2 - h_2k_3 - h_2h_4 + 2h_2^2h_1 + 3k_1k_3, \\
P^{9,10} &= -3k_2h_3 - k_1h_4 - h_1k_1^2 - h_2k_3 - h_2h_4 + h_2^2h_1 - k_1k_3 \\
\end{align*}
\]

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