Topological Fluid Dynamics II

Estimating topological entropy from the motion of stirring rods

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Abstract

Stirring a two-dimensional viscous fluid with rods is often an effective way to mix. The topological features of periodic rod motions give a lower bound on the topological entropy of the induced flow map, since material lines must ‘catch’ on the rods. But how good is this lower bound? We present examples from numerical simulations and speculate on what affects the ‘gap’ between the lower bound and the measured topological entropy. The key is the sign of the rod motion’s action on first homology of the orientation double cover of the punctured disk.

Keywords: fluid stirring ; topological entropy ; braids

1. Introduction

The paper of Boyland, Aref & Stremler [1] pioneered the study of two-dimensional rod-stirring devices using tools from topological surface dynamics. The central idea is that some rod motions impose a minimal complexity to the fluid trajectories, resulting in good mixing in at least part of the domain. Since then, many studies have followed: these include several papers dealing directly with rod motion [2, 3, 4, 5, 6, 7, 8, 9, 10]; various work on vortices, ‘ghost rods,’ and almost-invariant sets [11, 12, 13, 14, 15, 16, 17]; papers on the topology of chaotic trajectories and random braids [18, 19, 20, 21, 22, 23, 24]; a paper on a extension to three dimensions using stationary rod inserts [25]; and a review [26] and magazine article [27].

Throughout all this, there remains a vexing question, first raised by Phil Boyland: if one studies the rod motion depicted in Fig. 1(a), which is denoted $\sigma_1$ in terms of braid group generators, the growth rate of material lines in the fluid is almost the same as that predicted by the rod motion, which is a lower bound. How do we explain such a small discrepancy (or gap) between the lower bound and the measured value? Here we do not propose a full solution to this problem, but instead offer some observations, based on numerical simulations, of when the lower bound is and isn’t sharp, what this correlates with, and speculate on possible causes. At the heart of the matter is ‘secondary folding,’ or the observation that in some cases material lines fold a lot more than is strictly required by the topology
of the rod motion. This issue was explored in detail by the authors for toral linked twist maps [28, 29]. Here we focus on physical rod-stirring devices, also called rod mixers.

We can also interpret a small gap in terms of ‘taffy pullers’ [10]. Ignore the fluid and consider a plastic ‘strap’ wrapped around the rods. As the rods move, imagine the plastic strap can stretch, but can never shrink. The motions with a small gap described below will lead to a plastic strap that never develops any slack throughout the entire rod motion. The question is to determine what properties of a braid are needed to ensure this.

2. Braid-based rod mixers

We consider rod-stirring devices or mixers that are constructed such that the rod motion is described by braids [30, 31, 32]. In these two-dimensional circular containers, the rods start along a fixed horizontal line and move in accordance with braid generators, $\sigma_i^{\pm 1}$, depicted in Figure 1(b). For example, in a 3-rod mixer given by the braid $\sigma_1 \sigma_2^{-1}$ [11], first the two leftmost rods move halfway around a circle in a clockwise direction clockwise. Immediately after that, the two rightmost rods move halfway around a circle in a counter-clockwise direction (Figure 1(a)). The circular paths are centred directly between the two rods, and have diameter equal to the rod spacing. The speed of the rods is immaterial, since we are only considering Stokes (slow viscous) flow.

More generally, we write the stirring motion for $n$ rods as a braid expressed as a sequence of generators, $\sigma_i$, $i = 1, \ldots, n - 1$. Each generator represents the clockwise interchange of the $i$th and $(i + 1)$th strands or rods. The inverse, $\sigma_i^{-1}$, is a counter-clockwise interchange (Figure 1(b)). Note that the strands are always numbered from left to right, so a given subscript does not always refer to the same rod. By having the rods move in the same way as a specific braid, we can directly and systematically compare the measured topological entropy in the fluid system to the lower bound predicted by the braid (via the isotopy class [33, 34, 35]).

Remark. There are different conventions in the literature: In some papers $\sigma_i$ is defined as the counter-clockwise interchange, which is the opposite of our definition. There are also differing conventions on composition order. We will always write generators from left to right – that is, in the braid $\sigma_1 \sigma_2$, the $\sigma_1$ interchange occurs before the $\sigma_2$ interchange.

Remark. The lower bound on the entropy, based on the braid, is independent of the specific details of the rod motion. However, the measured flow entropy depends in general on the rod radius, rotation, and how near the rods come to each other and to the outer wall of the container during their motion. In our simulations, the rod radii are relatively small and we keep them from coming too close to the wall to avoid extra growth of material lines due to image effects. Our simulations were performed with the computer program Flop, by Matthew D. Finn, Emmanuelle Gouillart, and
The program is based on the complex-variable method described in [2]. We measure the flow topological entropy $h$ from the growth rate of material lines in the flow [36, 37, 38].

### 2.1 Three-rod mixers

We start by looking at devices with three rods. In particular, we will focus on motions based on braids of the form $\sigma_1^k \sigma_2^{-\ell}$. When $k\ell > 0$, we call the braid counter-rotating; when $k\ell < 0$ we call it co-rotating. Braids of this form are pseudo-Anosov if and only if $|2 + k\ell| > 2$. All counter-rotating braids are pseudo-Anosov, but co-rotating braids are only pseudo-Anosov if $k\ell < -4$. Braids that are not pseudo-Anosov are finite order or reducible, according to the Thurston–Nielsen classification theorem [33, 34, 35]. We will not encounter any reducible braids in this paper.

Figure 2 shows an iterated material line for several different braid mixers. The three in the left column are counter-rotating, and the three in the right column are co-rotating. Of the six braid mixers shown in Figure 2, five are pseudo-Anosov and one is not. With only a quick glance, it is not hard to guess that $\sigma_1 \sigma_2$ is the odd one out – in comparison to the others, the material line in that device has hardly stretched at all, even after 9 periods, and pseudo-Anosov braids have an exponential line stretching rate [33, 34, 35].

However, despite the fact that the braid $\sigma_1 \sigma_2$ is not pseudo-Anosov, we still measure a positive topological entropy for the flow in the mixing device. In fact, the braid mixers tend to fall into two categories: those where the flow entropy $h$ is close to the braid entropy $h_{\text{rods}}$ (of the order of 10% difference), and those where $h$ is considerably larger than $h_{\text{rods}}$ (>25% difference). Table 1 shows, for several braids of the form $\sigma_1^k \sigma_2^{-\ell}$, the measured topological entropy in the braid mixer ($h$) and the lower bound obtained from the rod braid ($h_{\text{rods}}$). The last column gives the ‘gap’ between the two values, expressed as a percentage of $h$. The first set of braids is counter-rotating ($k\ell < 0$); the second set co-rotating ($k\ell > 0$). Note that the counter-rotating mixers show a small gap, and the co-rotating ones have a much larger gap.

The penultimate column of Table 1 gives the sign of the dominant eigenvalue (the one with the largest magnitude) of the Burau matrix representation of the braid. The Burau representation [39, 31, 40, 41, 42] arises from an action of the braid on first homology of a double cover of the punctured disk (actually a $\mathbb{Z}$-cover, but we only use the double cover here). Figure 3 depicts the construction of the double cover for a disk with three rods. Notice in Table 1 that for the pseudo-Anosov braids ($h_{\text{rods}} > 0$), all the counter-rotating cases have a negative Burau eigenvalue, while all the co-rotating cases have a positive eigenvalue. For the non-pseudo-Anosov braids (i.e. those of finite order), the eigenvalue of the Burau matrix is always on the unit circle (complex), so we do not record a sign.

For 3-braids, the logarithm of the spectral radius of the Burau matrix agrees with the topological entropy of the braid. For pseudo-Anosov braids this largest eigenvalue is real but can be either positive or negative. A negative eigenvalue corresponds to a ‘flip’ of the homological generators at every application of the braid. For toral linked twist maps, this is associated with ‘kinks’ in the material lines, as shown in [28]. These are what we call ‘secondary folds,’ as depicted for a fluid system in Figure 5 and discussed in Section 3.1. The conjecture is that these kinks lead to additional growth of material lines, thus causing extra entropy above the lower bound. However, this connection has not been yet rigorously demonstrated.

### 2.2 Four-rod mixers

We now look at four-rod mixing devices. With four rods, there is no sense in classifying braids as counter- or co-rotating. Instead, we will focus on the sign of the dominant eigenvalue of the Burau matrix. Since we have more than three rods, the dominant eigenvalue of the Burau matrix is no longer guaranteed to give the topological entropy of the braid – it merely provides a lower bound [40, 41]. Band & Boyland [42] showed the Burau eigenvalue gives the exact topological entropy for a pseudo-Anosov braid if and only if the corresponding foliation has odd-order singularities at all the punctures, and any interior singularities are of even order. One consequence is that the Burau bound is always sharp for 3-braids, a fact we used in Section 2.1 to compute the entropy.

Figure 4 shows material line patterns for some four-rod mixers. Table 2 lists the braid, the measured topological entropy ($h$), the topological entropy of the braid ($h_{\text{rods}}$) given by the Bestvina–Handel algorithm [43, 44], the Burau bound, the sign of the dominant eigenvalue in the Burau matrix, and size of the gap between the two topological entropy values. Observe that again the braids with a positive Burau eigenvalue have small gaps in entropy – less
Fig. 2. Material line patterns for several three-rod braid mixers.
Fig. 3. How to make the orientation double cover. (a) A disk with three rods, the small segments indicating pronged singularities. (b) Shrink the disk’s boundary and the rods to points, so the surface is a topological sphere. Make two cuts between the rods and the boundary (dotted lines). (c) Glue a second copy of the same sphere along the cuts. (d) The resulting surface is a torus, and the singularities now have two prongs (regular points).

Table 1. Measured topological entropy vs. the lower bound for 3-rod braid mixers. The sign listed is that of the dominant eigenvalue in the Burau matrix.

| braid   | $h$   | $h_{\text{rod}}$ | Burau sign | gap  |
|---------|-------|------------------|------------|------|
| $\sigma_1 \sigma_2^{-1}$ | 0.992 | 0.9624           | pos        | 3.0% |
| $\sigma_1 \sigma_2^{-2}$ | 1.380 | 1.3170           | pos        | 4.6% |
| $\sigma_1 \sigma_2^{-3}$ | 1.714 | 1.5668           | pos        | 8.6% |
| $\sigma_1 \sigma_2^{-4}$ | 2.048 | 1.7627           | pos        | 13.9%|
| $\sigma_1 \sigma_2^{-3}$ | 2.112 | 1.9248           | pos        | 8.9% |
| $\sigma_1 \sigma_2^{-2}$ | 1.867 | 1.7627           | pos        | 5.6% |
| $\sigma_1 \sigma_2^{-3}$ | 2.244 | 2.0634           | pos        | 8.0% |
| $\sigma_1 \sigma_2^{-4}$ | 2.612 | 2.2924           | pos        | 12.2%|

than 2%. We will discuss the sources of discrepancy in the next section.

3. Explaining the gap

Our ultimate goal is to predict when the lower bound from the rod motion is close to the measured topological entropy (small gap), and when it is not (large gap). Furthermore, we wish to understand what causes a large gap, that is: what is it about the flow that creates more topological entropy? The easier question to answer, at least partially, is why the lower bound fails. We will address this first. Then we will attempt to explain why it happens.

3.1. Why there is a gap – secondary folding

Recall that the lower bound on entropy arises from the braid giving the rod motion: this braid labels the isotopy class of the period-1 map. Since the pseudo-Anosov representative of the isotopy class is the ‘simplest’ map in the class (the one with the lowest entropy), the isotopy from the flow to the pseudo-Anosov representative has the effect of pulling tight the material lines. In order for the flow to have a higher topological entropy, there must be some part of the material line pattern that is not already pulled-tight. In other words, there must be some extra folding that is not directly due to the rods. We call this secondary folding [28, 29]. Figure 5(a) shows an example of folding due to a rod, and Figure 5(b) shows secondary folding, which is not associated with a rod and could be removed by pulling
Fig. 4. Material line patterns for several four-rod braid mixers.
Table 2. Measured topological entropy vs. the lower bounds for 4-rod braid mixers. The sign listed is that of the dominant eigenvalue in the Burau matrix. The braids in the last set have vanishing Burau bound, so no sign is ascribed, except for the last braid (marked with †) where we found the action on homology to be positive (Section 4).

| braid               | h     | h_rod | Burau bound | Burau sign | gap   |
|---------------------|-------|-------|-------------|------------|-------|
| $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}$ | 1.949 | 1.92485 | 1.92485 | pos | 1.2% |
| $\sigma_1\sigma_2^{-1}\sigma_4\sigma_2^{-1}$ | 0.970 | 0.96242 | 0.96242 | pos | 0.8% |
| $\sigma_1\sigma_3\sigma_2^{-2}$ | 1.319 | 1.31696 | 1.31696 | pos | 0.2% |
| $\sigma_1\sigma_2^{-1}\sigma_4\sigma_3^{-2}$ | 2.940 | 2.88727 | 2.88727 | pos | 1.8% |
| $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}$ | 2.303 | 2.29243 | 2.29243 | pos | 0.5% |
| $\sigma_1\sigma_2\sigma_3\sigma_2^{-1}$ | 1.562 | 1.31696 | 1.31696 | neg | 15.7% |
| $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-2}$ | 1.914 | 1.56686 | 1.56686 | neg | 18.1% |
| $\sigma_1\sigma_2\sigma_3\sigma_2^{-1}$ | 1.270 | 0.96242 | 0.96242 | neg | 24.2% |
| $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}$ | 1.903 | 1.76275 | 1.76275 | neg | 7.4% |
| $\sigma_1\sigma_2\sigma_3\sigma_2^{-1}$ | 0.509 | 0 | 0 | 100% |
| $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}$ | 1.065 | 0.96242 | 0 | 9.6% |
| $\sigma_1\sigma_2\sigma_3^{-1}$ | 0.275 | 0 | 0 | 100% |
| $\sigma_1\sigma_2\sigma_3^{-1}$ | 0.837 | 0.83144 | 0 | pos† | 0.7% |

Fig. 5. (a) Folding caused by a rod moving through the fluid and dragging along the material lines. (b) Secondary folding of the material line. This is not directly associated with folding around a rod and can be "pulled tight" via homotopy.

Having a few extra folds is not necessarily enough to cause higher topological entropy. Recall that for 2D systems topological entropy is related to the exponential stretching rate of material lines [36, 37, 38]. The extra folds must cause a higher line growth rate in order to affect the topological entropy.

Looking back at the three-braid mixers shown in Figure 2, there is visible secondary folding in the $\sigma_1\sigma_2^2$ and $\sigma_1^2\sigma_2^3$ mixers. From Table 1 we see that these had gaps of 40.3% and 25.3% respectively. In contrast, the $\sigma_1\sigma_2^{-1}$, $\sigma_1\sigma_2^{-3}$, and $\sigma_1^2\sigma_2^{-3}$ mixers have no visible secondary folding, and have gaps of 3.0%, 8.9%, and 8.0% respectively. The same can be seen in the four-rod braid mixers of Figure 4. There is visible secondary folding in the $\sigma_1\sigma_2\sigma_3^2$ and $\sigma_1\sigma_2^2\sigma_3\sigma_2$ mixers, and these have gaps of 15.7% and 24.2% respectively. In comparison, for the $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}$, $\sigma_1^2\sigma_2^{-4}\sigma_3$, $\sigma_1\sigma_2\sigma_3^{-1}$, and $\sigma_1\sigma_2\sigma_3^{-1}\sigma_2^{-1}$ mixers there is no visible secondary folding, with gaps of 1.2%, 1.8%, 0.7%, and 0.8% respectively.

3.2. When there is a gap – negative eigenvalues

We would now like to predict when we can expect a gap between the measured topological entropy and the lower bound given by the braid. From the data presented, it is tempting to say that mixers with a braid whose Burau matrix has a negative dominant eigenvalue have a large gap, while those with positive eigenvalues have a small gap. However,
this says nothing about braids for which the Burau bound is zero. Furthermore, the data does not include any braids whose Burau bound is non-zero, but also not equal to the topological entropy of the braid (because of odd interior singularities in the foliation). We discuss why we didn’t include such braids in Section 4. However, it is clear at this point that the sharpness is closely correlated with the sign of the action on first homology of the orientation double cover, as given by the Burau representation in most cases examined here.

4. Discussion

In summary, we have exhibited a number of examples of braid-based rod mixers. These fall in two categories: those for which the rod motion is a good predictor of the flow entropy, and those for which it isn’t. For both three- and four-rod systems, the sign of the Burau eigenvalue correlates well with the two cases: a positive eigenvalue usually means that the bound is sharp.

When the Burau entropy is not sharp, the relevant quantity is the sign of the action of the braid on homology lifted to the orientation double cover. When the sign is negative, then the entire homological chain must ‘flip’ which each action of the braid. The conjecture is that this flip causes secondary folding by promoting ‘slack’ in the material lines. This is evident when examining toral linked twist maps [28, 29]. Unfortunately, this cannot be the whole story, since repeating the rod motion twice will always make the homological eigenvalue positive, but will clearly not make the lower bound any better.

Why is the orientation double cover important? The foliations obtained on disks are always non-orientable, due to the odd-pronged singularities at the rods. The orientation double cover turns the disk foliation into an orientable foliation on a closed surface of some genus (a torus in Figure 3). It is then easy to compute the topological entropy, since the linear action on homology gives the entropy for the case of orientable foliations. However, in order to construct the orientation double cover we need to know a priori the odd-pronged singularities associated with a braid’s isotopy class.

In general we should be able to ascribe a homological sign even for braids that are not Burau-sharp. This is easy to do when a pseudo-Anosov is given in terms of Dehn twists on the double cover [45], but is not so straightforward when starting from braids on the disk; this is a future challenge. For the braid $\sigma_1 \sigma_2 \sigma_1^{-1}$ in Table 2, we were able to determine that the sign is positive by puncturing at the 3-pronged singularity and computing the Burau action of the resulting 5-braid ($([\sigma_1 \sigma_2 \sigma_1] \sigma_1 \sigma_2^{-1})$. Note that both homological signs can always be realised, since the braids giving rise to different signs are related by the deck transformation (involution) of the double cover [46].

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