Projective characterization of higher-order quantum transformations

Timothée Hoffreumon and Ognyan Oreshkov
Centre for Quantum Information and Communication (QuIC), École polytechnique de Bruxelles, CP 165/59, Université libre de Bruxelles, Avenue F. D. Roosevelt 50, 1050 Brussels, Belgium.
(Dated: June 27, 2022)

Transformations of transformations, also called higher-order transformations, is a natural concept in information processing, which has recently attracted significant interest in the study of quantum causal relations. In this work, a framework for characterizing higher-order quantum transformations which relies on the use of superoperator projectors is presented. More precisely, working with projectors in the Choi-Jamiołkowski picture is shown to provide a handy way of defining the characterization constraints on any class of higher-order transformations. The algebraic properties of these projectors are furthermore identified as a model of multiplicative additive linear logic (MALL). The main novelty of this work is the introduction in the algebra of the ‘prec’ connector. It is used for the characterization of maps that are no signaling from input to output or the other way around. This allows to assess the possible signaling structure of any maps characterized within the projective framework. The properties of the prec are moreover shown to yield a canonical form for projective expressions. This provides an unambiguous way to compare different higher-order theories.

I. INTRODUCTION

The same way that a quantum channel describes the most general transformation mapping an input quantum state to an output quantum state [1], a quantum supermap describes the most general transformation mapping an input quantum channel to an output quantum channel [2]. Interpreting the quantum channel as a transformation between states, the supermap is then a transformation of transformations. For that reason, it is called higher-order transformation. Since nothing forbids a priori to nest transformations of transformations, one can consider successive nestings to recursively build a whole hierarchy of higher-order transformations [3–5].

Fragments of a quantum circuit are a concrete instance of the use of a higher-order hierarchy. A fragment of quantum circuit that ‘goes around’ a channel is a supermap: it takes a channel as input and outputs a channel. This supermap can itself be seen as the input for some super-supermap that will output a channel, and so on. This ensuing hierarchy has been defined under the name quantum comb formalism [3], which has proven to be a valuable tool in the field of quantum information theory.

With a different goal than modeling circuit fragments, supermaps with multiple inputs were subsequently studied. First, the quantum switch was proposed as a supermap that takes two channels and outputs them in an order that depends on a control qubit [6]. Soon after, Process Matrices (PM) were proposed as a general framework of supermaps that take a fixed number of quantum instruments [7] and map them to a joint probability for their outcomes [8]. (In the case when the inputs are channels, that probability is 1.) Both concepts led to the identification of Indefinite Causal Order (ICO) as a feature of supermaps.

One can then wonder what differentiates the switch from a comb, or the PM from a comb. Especially, why certain maps and hierarchies of maps may feature ICO while others will not. Motivated by these considerations, the goal of this work is to present a framework that formalizes and characterizes higher-order quantum transformations. The two main questions answered are ‘given an operator on a set of input and output Hilbert spaces, does it represent (the Choi-Jamiołkowski operator of) a higher order object’ and ‘what is the underlying causal structure(s) of such an object?’. This work extends two previous characterizations, one done using type theory [1] [5], and the other using category theory [9]. This extension relies on the use of superoperator projectors [10].

These projectors have a twofold advantage: first, they make the characterization more straightforward, as one can answer the first question simply by applying the projector corresponding to a given higher-order object on the operator. Second, they offer an intuitive explanation of the type-theoretic semantics of higher order: the algebraic rules for composing these projectors correspond to the semantic rules for forming new types.

Below, we concisely reformulate ideas from Ref. [5] to give the reader an overview of the theory upon which this work is based.

a. Types. The formalism of quantum channels and operations [1] describes transformations of quantum states into quantum states, but is unable to effectively describe the transformations whose inputs and outputs are transformation themselves [4]. To overcome this issue, Perinotti [1] and Bisio [5] extended the concept of a supermap [2, 3] into a suitable framework to treat these general transformations of transformations, or higher-order quantum maps.

At the core of their work is the utilization of the Choi-Jamiołkowski (CJ) isomorphism [11, 12]. Define $\mathcal{H}_A$ to be a Hilbert space of dimension $d_A$ associated to system $A$, $\mathcal{L}(\mathcal{H}_A)$ to be the space of linear operators on $\mathcal{H}_A$, and $\mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$ to be the space of linear maps from...
\( \mathcal{L}(\mathcal{H}^A) \) to \( \mathcal{L}(\mathcal{H}^B) \). Let \( |\phi^+\rangle = \sum_{i=0}^{d_A-1} |i\rangle^A \otimes |i\rangle^A \) be a (unnormalized) maximally entangled state on space \( \mathcal{H}^A \otimes \mathcal{H}^A \), with \( A' \) a copy of \( A \), and \( \phi^+ \equiv |\phi^+\rangle \langle \phi^+| \) its density operator representation. Then,

\[
\mathcal{L}(\mathcal{L}(\mathcal{H}^A), \mathcal{L}(\mathcal{H}^B)) \ni M \mapsto M \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B) : \\
M \equiv ([\mathcal{I} \otimes \mathcal{M}] \{\phi^+\})^T ,
\]

is an isomorphic bijective mapping. The correspondence sends linear maps between operator spaces to operators in tensor product spaces. It has the properties that a Hermitian-preserving (HP) map is mapped to a Hermitian operator and a completely positive (CP) map is mapped to a positive semidefinite operator. To recover the action of the map, the ‘reverse direction’ of the CJ correspondence is used:

\[
\mathcal{M}(A) = (\text{Tr}_A [M \cdot (A \otimes \mathbb{I}_B)])^T .
\]

Using this correspondence, maps, maps of maps, and so on, can all be represented as CJ operators on some composite space, and this is how we will describe them.

Then it is possible to define types of maps. One first defines base types, out of which one defines a hierarchy of higher-order types as maps between types of lower order. The state spaces (in density matrix form) of standard quantum systems of a given finite dimension are standardly taken as base types (but different constructions are also possible). Although two different systems \( A \) and \( B \) could have state spaces of the same type (e.g., if they are both qubits), in order to avoid complicated notation, when we describe types instantiated on different systems we will use different letters – by convention the same letters as for the systems – and will indicate by words, if needed, when the types are the same.

More general types are then defined as sets of admissible maps \([13]\) that have the same types of input and output spaces. For example, the above map \( M \) is in type \( A \rightarrow B \), where \( A \) and \( B \) are base types. However, in the same way one could describe maps whose input and output types \( A \) and \( B \) are maps themselves, by representing the input and output maps via their CJ operators (in this case, the input or output ‘state’ space would generally consist, up to a normalization, of a subset of the set of all standard states on the respective Hilbert space).

The formalism is an instance of a type system. This “\( \rightarrow \)” connector, here nicknamed transformation, is the key element of the type theory of higher order transformations: each set can be seen as an abstract type, and new types can be defined out of existing ones using the transformation connector as a semantic rule.

Two special cases are of particular interest: the set of deterministic effects, where the output type is the trivial one-dimensional system \( 1 \), i.e. \( A \rightarrow 1 \equiv A \); and the parallel composition of two types, obtained as the transformation \( (A \rightarrow (B \rightarrow 1)) \rightarrow 1 = A \rightarrow B \equiv A \otimes B \ [14] \). The type \( A \otimes B \) generalizes the idea of a no signaling channel \([14, 15]\) from the joint inputs of \( A \) and \( B \) to their joint outputs.

Thus, starting from some postulates, a trivial type \( 1 \), an elementary type \( A \) (which is usually taken as the set of quantum states in density matrix form), and this connector rule, all the higher-order generalizations of the quantum formalism can be defined using types. These in turn yield the constraints on the operators representing transformations of a given type \([11, 5]\).

For example, let \( A_0 \) and \( A_1 \) be, respectively, the set of input and output quantum states of Alice, who applies some quantum operation in between. Then, one can infer that the set of allowed transformations to which she has access is of type \( A_0 \rightarrow A_1 \). This simple semantic statement is then translated into constraints to apply on the Hilbert space \( \mathcal{L}(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}) \) and yields (the Choi-Jamiołkowski representation of) the set valid quantum channels for Alice. A more complex example is obtained by recovering the set of bipartite process matrices \([3]\), which corresponds to the type normalized on the local quantum instruments of two parties, say Alice and Bob. Knowing that their local instruments sum up to quantum channels, i.e. they belong to types \( (A_0 \rightarrow A_1) \) and \( (B_0 \rightarrow B_1) \), the set of process matrices is the type that takes a composition of these two types as input and outputs a trivial system. In the semantics, this statement corresponds to type \( ((A_0 \rightarrow A_1) \otimes (B_0 \rightarrow B_1)) \rightarrow 1 \), from which the constraints for the characterization directly ensue.

Importantly, the transformation connector \( \rightarrow \) is not associative: \( A \rightarrow (B \rightarrow C) \neq (A \rightarrow B) \rightarrow C \). This semantic fact is what allows to define the notion of an order with respect to (w.r.t.) a base type. Let \( A \) be an instance of the type used as base defined in space \( \mathcal{L}(\mathcal{H}^A) \), and define \( B, C, D \) in the same way. Then, \( A, B, C, D \) are first-order types (trivial transformation) w.r.t. the base, types \( A \rightarrow B \) and \( C \rightarrow D \) are two instances of second order (both the same kind of transformations, but defined on different Hilbert spaces), and type \( (A \rightarrow B) \rightarrow (C \rightarrow D) \) is third order (a transformation of transformation). Notice that while this last expression is third order, an expression like \( (A \rightarrow B) \rightarrow (C \rightarrow D) \) is only second order, as it is but a cascading of second order transformations. With these considerations, it becomes possible to talk about a hierarchy of higher order quantum transformations as in reference \([3]\).

b. Outline and results. In section \([11]\) types are put in correspondence with subsets of operators called state structures that are characterized by a superoperator projector. Looking at these projectors instead of the state structures, and by extension the types they define, is the idea behind projective characterization. It is then proven that this way of characterizing recovers the type theory.

A convenient aspect of projectors is that they form an algebra, so that manipulations of formulae are simplified by using algebraic properties. This point is developed in Section \([111]\). It is then observed that this algebra furthermore constitutes a model of linear logic.

In Section \([11\text{V}]\) a new projector encoding no signaling between two systems is introduced. In addition to its ob-
II. PROJECTIVE CHARACTERIZATION OF THE TYPE THEORY OF HIGHER-ORDER QUANTUM TRANSFORMATIONS

Central in that work will be the notion of a trace-normalized positive operator system in $\mathcal{L}(\mathcal{H}^A)$. This kind of set is the backbone of the characterization as to each type – or “level in the hierarchy of transformations” – corresponds such a set (Appendix C 1 provides a motivating example if needed).

Definition 1 (Operator system [10]). For a given space of operators, an operator system is a self-adjoint subspace that contains the identity.

We will refer to both an operator system and its positive trace-normalized subset by using the calligraphic font of the letter associated with the subsystem it is de

Definition 2 (State structure). A state structure $\mathcal{A} \in \mathcal{L}(\mathcal{H}^A)$ is a positive operator system that is trace-normalized.

An example of state structure is the set of quantum states in density matrix form, characterized by

$$\rho \geq 0, \quad \text{Tr} [\rho] = 1, \quad \mathcal{I} \{\rho\} = \rho.$$ (4a, 4b, 4c)

Here, $\mathcal{I}$ is the identity mapping, defined as

$$\forall A \in \mathcal{L}(\mathcal{H}^A) : \mathcal{I}_A(A) = A.$$

A. Functionals

First, we characterize the type $\overline{A}$, generalizing the notion of a set of deterministic effects. It is the set of operators representing all of the functionals mapping each element of a state structure to the number 1 via the Hilbert-Schmidt inner product $\text{Tr} [\overline{A} \cdot A] = 1$. Note that in the following we will always mean deterministic effect when we use the word ‘effect’ alone.

In a previous work [17], it was noticed that since both the set of states and effects must contain an element proportional to the identity, the two subspaces on which the states and effects are respectively defined must be quasi-orthogonal [18]. This means that operators $A$ and $\overline{A}$ must be orthogonal everywhere but at the span of the identity. If $\mathcal{P}_A$ is the projector for $\overline{A}$, it is related to $\mathcal{P}_A$.
by \( \overline{\mathcal{F}}_A \equiv \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A \). This leads to the rephrasing in terms of projectors of the characterization of \( \mathcal{F} \) from type theory [5, Lemma 4].

**Proposition 1** (Functional). Let \( \mathcal{A} \) be a state structure. Then the set \( \mathcal{A} \) of operators taking each element of \( \mathcal{A} \) to the number 1 through the inner product,

\[
\mathcal{A} \in \mathcal{A} \iff \forall A \in \mathcal{A} : \text{Tr} [\mathcal{A} \cdot A] = 1 ,
\]

is a state structure characterized by the following conditions:

\[ \mathcal{A} \geq 0 , \]

\[ \text{Tr} [\mathcal{A}] = \frac{d_A}{c_A} = c_A , \]

\[ \{ \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A \} (\mathcal{A}) \equiv \mathcal{P}_A \{ \mathcal{A} \} = \overline{\mathcal{A}} . \]

The proof is provided in Appendix [3], see Figure [1] for an illustration. Concrete examples of states structures characterized by Prop. [1] are given in Appendix [3]. This proposition was first obtained as a theorem in our earlier work on Multi-round Process Matrices (MPM) [17].

Quasi-orthogonality implies the following property [19, Theorem 2.37 iii] :

\[ \forall A \in \mathcal{A}, \forall \mathcal{A} \in \mathcal{A} : \text{Tr} [\mathcal{A} \cdot A] = 1 \text{,} \]

As both sets have an element proportional to the identity, combining equations (8) and (10) yields the following.

\[ \text{Tr} [\mathcal{A} \cdot A] = \text{Tr} \left[ \mathcal{A} \cdot \frac{1}{c_A} \right] \text{Tr} \left[ \frac{1}{c_A} \cdot A \right] . \]

Figure [2] provides a graphical interpretation of this relation. This is the physical content of Proposition [1]. The generalized Born rule is independent of a particular choice of deterministic state and deterministic effect: Both the state and the effect can be replaced by the identity – or any other element of their respective state structures – without altering the outcome probability of 1. Put differently, even when replacing either the state or the effect by pure noise, something has to be measured in the end.

Here, the bar in “\( \overline{\mathcal{F}}_A \equiv \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A \)” can be seen as an operation at the level of projectors that was applied on \( \mathcal{P}_A \). As such, the projector \( \overline{\mathcal{F}}_A \) will be called the negation of \( \mathcal{P}_A \) because the operation has properties similar to a logic not. This is developed in more details below and in Appendix [3].

### B. Tensor product

The tensor product of two types as a parallel composition generalizes the notion of no signaling channel [13]. In terms of projectors, the characterization relies on raising the tensor product operation at the level of superoperators in a natural fashion. Let \( \mathcal{P}_A \) and \( \mathcal{P}_B \) be respectively projectors on operator spaces \( A \) and \( B \), then \( \mathcal{P}_A \otimes \mathcal{P}_B \) acts on \( \mathcal{L} (\mathcal{H}_A \otimes \mathcal{H}_B) \) so that

\[
(\mathcal{P}_A \otimes \mathcal{P}_B) \left\{ \sum_i q_i (A_i \otimes B_i) \right\} \equiv \sum_i q_i \left( \mathcal{P}_A \{ A_i \} \otimes \mathcal{P}_B \{ B_i \} \right) ,
\]

\[ \forall q_i \in \mathbb{C} . \]

**Proposition 2** (No signaling (bipartite) composition). Let \( \mathcal{A} \) and \( \mathcal{B} \) be two state structures as in Eqs. (3), their no signaling composition \( \mathcal{A} \otimes \mathcal{B} \subset \mathcal{L} (\mathcal{H}_A \otimes \mathcal{H}_B) \) is the set of all operators \( X \) characterized by the following constraints:

\[ X \in \mathcal{A} \otimes \mathcal{B} : \]

\[ X \geq 0 , \]

\[ \text{Tr} [X] = c_A c_B , \]

\[ (\mathcal{P}_A \otimes \mathcal{P}_B) \{ X \} = X . \]

Consequently, \( \mathcal{A} \otimes \mathcal{B} \) is the affine span of \( \mathcal{A} \) and \( \mathcal{B} \).

See Appendix [3] for the proof. We have just stated that the affine hull of a tensor product of state structures is a state structure itself [5, Lemma 5], this proposition will help us in the characterization of the CJ representation of mappings between state structures below. A diagrammatic depiction is given in Figure [3] and examples of the use of the above can be found in Appendix [3].

### C. Transformations

In the type theory of higher-order transformations, every type is an instance of a transformation, so the previous characterization of generalized effects (type \( \mathcal{A} \)) and bipartite states (type \( \mathcal{A} \otimes \mathcal{B} \)) are but special cases of a more general rule. This rule says that if \( A \) and \( B \) are types, then \( A \rightarrow B \) is itself a type that takes \( A \) to \( B \). A state of type \( A \) is actually a transformation of the trivial type – the number 1 – into \( A \), noted \( 1 \rightarrow A \). Accordingly, the effect of type \( \mathcal{A} \) is the transformation of type \( A \) into a trivial system, \( A \rightarrow 1 \). No signaling composition is the type \( A \otimes \mathcal{B} \equiv A \rightarrow \mathcal{B} \).

As we have seen, the set of elements of a given type corresponds to a state structure, ultimately determined by its projector. Thus, to type \( A \rightarrow B \) corresponds a state structure, noted \( \mathcal{A} \rightarrow \mathcal{B} \), which is obtained by combining state structures \( \mathcal{A} \) and \( \mathcal{B} \) in a certain way. This
structure should then be characterized by a composite
projector built by combining \( P_A \) and \( P_B \) using the two
previously introduced rules.

Following previous approaches \[1, 4, 5, 21\], we want
the transformation – or dynamics – to conserve the
total probability as well as probabilistic mixtures, and it
should also be defined when acting on a tensor factor of
any no signaling composite system \[5, Definition 7\]. In
other words, the dynamical law should remain valid even
when one is coarse-graining or fine-graining the system.
The preservation of probabilistic mixtures, convex linearity,
implies that the maps representing such dynamics are linear
(see Ref. \[1, 4, 5, 21\] for the proof).

**Definition 3** (Structure-preserving map). Let \( \mathcal{M} \) be
a linear map from \( \mathcal{L}(\mathcal{H}^A) \) to \( \mathcal{L}(\mathcal{H}^B) \). This map is
**structure-preserving** between \( \mathcal{A} \) and \( \mathcal{B} \) if it maps
any element of state structure \( \mathcal{A} \) to one in \( \mathcal{B} \), and moreover if
it keeps this property when these state structures are
embedded in larger systems. That is, for any state structure \( \mathcal{C} \),
the map \( \mathcal{M} \otimes I_C \in \mathcal{L}(\mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^C) , \mathcal{L}(\mathcal{H}^B \otimes \mathcal{H}^C)) \)
should map any element of \( \mathcal{A} \otimes \mathcal{C} \) to one of \( \mathcal{B} \otimes \mathcal{C} \).

Moving on to the characterisation of valid dynamics,
let \( \mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}^A), \mathcal{L}(\mathcal{H}^B)) \) be a linear map. We translate
the first condition of Definition 3 into the following conditions
for all \( A \in \mathcal{A} \) : 

\[
\begin{align*}
\mathcal{M}(A) & \geq 0 , \\
\text{Tr} \left[ \mathcal{M}(A) \right] &= c_B , \\
\mathcal{P}_B \circ \mathcal{M} \circ \mathcal{P}_A &= \mathcal{M} \circ \mathcal{P}_A .
\end{align*}
\]

The second condition further constrains the positivity
preserving condition \(\text{14a}\) to \( \mathcal{M} \) being completely posi-
tive (CP) as \( \mathcal{C} \) can in general be defined over the whole
of \( \mathcal{L}(\mathcal{H}^C) \). If we denote by \( M \) the CJ representation of
\( \mathcal{M} \), the set of all these maps \( M \) is a state structure \[3\]
Proposition 1].

**Proposition 3** (Mapping between state structures). Let \( \mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}^A), \mathcal{L}(\mathcal{H}^B)) \) be a structure-preserving
map between state structures \( \mathcal{A} \) and \( \mathcal{B} \). Call \( M \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B) \) the Choi-Jamiołkowski representation
of this map, as in Eq. \[1\]. Then, the set \{\( M \)\} of all such
maps belongs to the type structure \( \mathcal{A} \otimes \mathcal{B} \), which means
that it satisfies the following conditions:

\[
\begin{align*}
M & \geq 0 , \\
\text{Tr}[M] &= c_A c_B , \\
\{\mathcal{P}_A \Rightarrow \mathcal{P}_B\} \{M\} &= \{\mathcal{P}_A \otimes \mathcal{P}_B\} \{M\} = M .
\end{align*}
\]

**Remark.** The above derivation is very close in spirit
to Ref. \[20\] where the projective characterization of
transformations was first made in the context of transformation
between process matrices, but the obtained projector
was missing two terms \[21\]. After verification, it
appears that this omission has not hindered the other
conclusions of this article \[22\].

Concrete examples of the use of this Proposition can be
found in the Appendix \[C4\]. This Appendix also provides
motivation for the ideas introduced in Section \[IV\].

**D. The algebra of projectors recovers the type theory**

We started from the type theory of higher-order trans-
forms in which \( A, A \Rightarrow B, A \otimes B \) are the new types that
one can obtain from a base type \( A \) under the semantic rules \{\( 1, (,), \Rightarrow \}\).
We observed that types define state structures, which are trace-normalized subsets of positive operator systems for which \( \mathcal{A} \to \mathcal{B}, \mathcal{A} \otimes \mathcal{B} \) are the new state structures one can obtain from a base \( \mathcal{A} \). These state structures are the sets of CJ operators representing higher-order maps.

As these state structures are defined on linear subspaces, most of their properties are encoded by the superoperator projector that defines them. \( \mathcal{P}_A \to \mathcal{P}_B; \mathcal{P}_A \otimes \mathcal{P}_B \) are the new projectors that one can obtain from a projector \( \mathcal{P}_A \) according to the algebraic rules \( 1, (\cdot), \to \) (which can be taken as \( \cap, \otimes, \to \) instead).

Therefore, working with projectors is a handy way to define the characterization constraints implied by type theory.

### III. ABSTRACTING THE ALGEBRA OF PROJECTORS

We now make a short pause in the characterization to point out that the algebraic rules obeyed by the projectors are not arbitrary. They actually form an algebra, as well as a model of logic.

#### A. Beyond type theory: algebra of projectors

Type theory is dependent on a base type. In Ref. [3], the first non-trivial type in the hierarchy, called the elementary type, is taken as the set of quantum states as in Eqs. (23). Within the framework of state structures, one can use other elementary types as a base. The question then is how to classify these nonequivalent theories, and how to compare them.

In terms of subspaces, comparing is straightforward as one can consider the overlap between the two subspaces. For this purpose, one needs two new rules. These are the intersection, nicknamed ‘cap’,

\[
P_A \cap P_A' = P_A \circ P_A', \tag{16}
\]

and the union, ‘cup’,

\[
P_A \cup P_A' = P_A + P_A' - P_A \cap P_A'. \tag{17}
\]

This definition assumes that all projectors in the algebra must commute with respect to the cap, \( P_A \cap P_A' = P_A' \cap P_A \), \( P_A \cap P_A' = P_A' \cap P_A \). This is indeed a sufficient condition for \( P_A \cap P_A' \) to be a projector on operator system as well. More details are given in App. [A]{\ref{A}}

With these, one can prove that an operator system is contained within another by showing either of the following:

\[
P_A \cap P_A' = P_A'; \tag{18a}
\]
\[
P_A \cup P_A' = P_A. \tag{18b}
\]

In terms of projectors, these conditions will be concisely noted

\[
P_A' \subset P_A. \tag{19}
\]

Since the projectors are themselves CPTP maps, inclusion of projectors is indeed sufficient to show inclusion of the corresponding state structures (up to normalization).

As it turns out, the projectors form an algebra under the rules \( \{ \cap, \cup \} \) (see Appendix [A]{\ref{A}} for proof). And because every operator system must contain the identity element, every projector in the algebra is contained between the depolarizing and identity projectors,

\[
\mathcal{D} \subset \mathcal{P} \subset \mathcal{I}. \tag{20}
\]

However, the cap and the cup are new rules that cannot be expressed using the \( \to \) connector. So the downside of the algebra is that it is outside of the type-theoretic framework of references [4, 5].

Adding \( \bar{\cdot} \) as an operation in the algebra promotes it into a Boolean algebra, i.e., an algebra of idempotent elements which possess a negation. \( \bar{\cdot} \) acts as the negation since it makes \( \mathcal{P}_A \) complementary to \( \mathcal{P}_A \), \( P_A \cup P_A^\bar{\cdot} = \mathcal{I}_A \)

and it is an involution \( \overline{\overline{P}_A} = P_A \) (see Appendix [A]{\ref{A}} for more details).

#### B. The algebra of projectors is (almost) Linear Logic

With the further additions of the tensor and of the transformations operations, the Boolean algebra of projectors is lifted to an abstract structure that happens to follow the rules of linear logic (LL) [23]. This extends an observation made in Ref. [9] that the logic of higher-order quantum transformations, in our language the projector rules \( \{ \cap, \otimes, \to \} \), form an instance of multiplicative linear logic.

Observe that the transformation between states is equivalent to the reverse transformation between effects,

\[
P_A \to P_B = P_A^\bar{\cdot} \leftrightarrow P_B = P_A \otimes P_B^\bar{\cdot}. \tag{21}
\]

This is indeed a sign of the algebra being an instance of linear logic: it follows the logic rule that if \( A \) implies \( B \), then not \( B \) must imply not \( A \). As such, the \( \to \) corresponds to the linear implication of LL, nicknamed lollipop and noted “\( \Rightarrow \)”. Negating the input of a transformation allows one to treat it as a composition:

\[
\overline{P}_A \to P_B = \overline{P}_A \otimes \overline{P}_B. \tag{22}
\]

This is a “larger” composition than the tensor as \( \overline{P}_A \otimes \overline{P}_B \neq P_A \otimes P_B \) and

\[
P_A \otimes P_B \subset \overline{P}_A \otimes \overline{P}_B. \tag{23}
\]

As such, operation “\( \Rightarrow \to \)’’ corresponds to the par, “\( \times \)”, of LL, which, together with \( \otimes \), respectively constitute the multiplicative disjunction and conjunction. The
additive disjunction and conjunction correspond to the cup and the cap, respectively, and the linear negation is the negation. They happen to be because on the one hand they respect the De Morgan laws

\[ \overline{\overline{P}_A} = P_A ; \]  
\[ \overline{P_A \cap P_A'} = P_A \cup \overline{P_A'} ; \]  
\[ \overline{P_A \otimes P_B} = P_A \rightarrow P_B . \]

On the other hand, there exist a truth and a falsity for both additive and multiplicative rules, and they are the negation of each other. See Appendix A 4 c for the details. The correspondence with LL is made in Table I where the rules we have introduced are put in correspondence with their usual notation of LL.

| Name               | Symbol proj. | Unit proj. | Symbol LL | Unit LL |
|--------------------|--------------|------------|-----------|---------|
| Negation           | \( \neg \)  | \( \perp \) | \( \neg \) | \( \perp \) |
| Additive conjunction | \( \cap \) & | \( I \) | \( \cap \) | \( I \) |
| Additive disjunction | \( \cup \) \( \oplus \) | \( D \) | \( 0 \) |
| Multiplicative conjunction | \( \otimes \) \( \otimes \) | \( 1 \) | \( 1 \) |
| Multiplicative disjunction | \( \rightarrow \) \( \rightarrow \) | \( \vee \) | \( 1 \) | \( \perp \) |
| Linear Implication | \( \rightarrow \) | \( 1 \) | \( / \) |

TABLE I. Correspondence of the algebraic rules of projective characterization (proj.) with linear logic (LL).

Nonetheless, the algebra of projectors is still not a full model of linear logic as the exponential connectors used to relate additive and multiplicative formulae have not been defined. To be precise, it is a model of multiplicative additive linear logic (MALL). In addition to that, remark that the multiplicative truth and falsity are the same thing: the trivial projector onto the trivial system, *i.e.* the number 1. There is also the issue that the additive falsity is not absorbant for multiplicative formulae, *i.e.*, \( P_A \otimes D_B \neq D_{AB} \).

The fact that the algebra of the projectors characterizing quantum higher-order transformations is an instance of MALL can additionally be understood as the consequence of the correspondence between the types of Ref. [5] and *-autonomous categories as in Ref. [9].

IV. NO SIGNALING

In a previous work, the projective characterization was used to prove that the multi-round process matrix is a linear sum of quantum combs [17 Theorem 2]. That is, it allowed to split an MPM, which typically involves several directions of signaling, into a sum of quantum combs, featuring single, fixed directions.

In this section, the notion of no signaling is extended to state structures, in order for this decomposition to be generalized. Quasi-orthogonality will be shown to be the relevant notion for no signaling at the level of projectors, allowing in turn to define the prec, an algebraic rule for composing projectors that encodes a one-way signaling constraint.

A. Definition

No signaling from the input to the output can be imposed on a transformation as an extra condition. This translates the physical idea that, if the output is assumed to be in the causal future of the input, no (deterministic) operation done in the future can influence, or signal to, the past [21].

The subset obeying this constraint will be noted using a “\( \prec \)” symbol, nicknamed the prec. By convention this connector will be seen as a composition, meaning that to represent a transformation, the input has to be negated. Thus, we want to define the set \( A \prec B \subset A \rightarrow B \).

This condition is expressed as the requirement that the input of the transformation should be independent of the particular choice of effect applied at its output:

\[ M \in A \prec B \subset A \rightarrow B : \quad \forall B, B' \in \mathcal{B}, \quad \text{Tr}_B [M \cdot (1 \otimes B)] = \text{Tr}_B \left[ M \cdot \left( 1 \otimes B' \right) \right] . \]  

As the identity matrix \( 1 \) is always both a valid state and effect (with suitable normalization), this requirement is rephrased as the following constraint,

\[ \forall B \in \mathcal{B}, \quad \text{Tr}_B [M \cdot (1 \otimes B)] = \text{Tr}_B \left[ M \cdot \left( 1 \otimes \frac{1}{c_B} \right) \right] . \]

This is depicted in Fig. 5c. When the above is satisfied, we will use it as a definition to say that \( M \) is no signaling from \( B \) to \( A \) [13 15]. The same way, one can define the no input-to-output signaling subset, \( A \succ B \), by requiring the condition drawn in Fig. 5d.

Relation with the usual definition. This no signaling definition recovers the one used in the respective case of quantum channel formalism. Indeed, the validity condition of transformations between quantum states implies that \( A = \{ \rho | \rho \geq 0 \cap \text{Tr} [\rho] = 1 \} \), so that \( A = \{ 1_A \} \) and \( B = \{ 1_B \} \). Observe that because of Eq. 21, a map \( M \in A \rightarrow B \) can be equivalently interpreted as \( B \rightarrow \mathcal{A} \).

Since \( M \) must obey \( \text{Tr}_B [M (1_A \otimes B)] \in \mathcal{B} \) by definition (see Fig. 5a), and since \( \mathcal{A} = \{ 1_A \} \), we attain the usual condition

\[ \text{Tr}_B [M (1_A \otimes 1_B)] = \text{Tr}_B [M] = 1_A , \]  
which is used as both a validity and a causality condition in previous works [9] (Fig. 3a). The reason is that condition 26 becomes tautological in that case (Fig. 5d). However, this equivalence between a general transformation and a no signaling one is only valid for the case of density matrices and hides the subtle difference between the two definitions in the general case for which being a valid transformation does not guarantee that the output cannot be used to signal to the input.
Theorem 1. Let \( \mathcal{A} \) be a positive operator in \( \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B) \), let \( A \) be one in \( \mathcal{A} \) and let \( B \) be one in \( \mathcal{B} \). Then a necessary and sufficient condition for \( \mathcal{A} \) to be no signaling from the output to the input is that its projector gets restricted accordingly. The projector for \( \mathcal{A} \) is thus obtained by taking the intersection of the projectors

\[
\{\mathcal{P}_A \otimes \mathcal{I}_B \} \{M\} = M .
\]

That is to say, that \( M \) belongs to \( \mathcal{A} \otimes \mathcal{L}(\mathcal{H}^B) \).

As the above theorem states that no signaling is equivalent to restricting the operator to a subspace, a necessary and sufficient condition for the state structure \( \mathcal{A} \to \mathcal{B} \) to be no signaling from the output to the input is that its projector gets restricted accordingly. The projector for \( \mathcal{A} \) is thus obtained by taking the intersection of the projectors

\[
\{\mathcal{P}_A \otimes \mathcal{I}_B \} \{M\} = M .
\]

That is to say, that \( M \) belongs to \( \mathcal{A} \otimes \mathcal{L}(\mathcal{H}^B) \).

As a consequence, \( \mathcal{A} \otimes \mathcal{B} \) is a state structure.

Proposition 4 (One-way signaling composition). Let \( \mathcal{A} \) and \( \mathcal{B} \) be two state structures as in Eqs. (31), their one-way signaling composition \( \mathcal{A} \otimes \mathcal{B} \) is the set of all operators \( W \) characterized by the following conditions:

\[
W \geq 0 , \quad Tr[W] = c_{ABC} , \quad \{\mathcal{P}_A \otimes \mathcal{I}_B \} \{W\} = W .
\]

where

\[
\mathcal{P}_A \otimes \mathcal{P}_B \equiv \mathcal{I}_A \otimes \mathcal{P}_B - \mathcal{P}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B .
\]

Proof. Directly from the above discussion, the conditions are obtained by applying the linear constraint \( \mathcal{I}_A \otimes \mathcal{P}_B \) to the set \( \mathcal{A} \to \mathcal{B} = \mathcal{A} \otimes \mathcal{B} = \mathcal{A} \leftarrow \mathcal{B} \) characterized by Proposition 3.

See Fig. 5c for a diagrammatic depiction of the subspace associated with projector (33c). The main rules for characterizing higher-order theories through their projector rules are summarized in Table II.

| Name | Characterization | Proj. rule | Rule nickname |
|------|------------------|------------|---------------|
| State | Def. 2, Eqs. (3) | \( \mathcal{P}_A \otimes \mathcal{P}_B \) | Tensor |
| Effect | Prop. 1, Eqs. (9) | \( \mathcal{P}_A \) | Negation |
| 2-ways Sign. | Prop. 2, Eqs. (13) | \( \mathcal{P}_A \otimes \mathcal{P}_B \) | Transformation |
| 1-way Sign. | Prop. 3, Eqs. (33) | \( \mathcal{P}_A \otimes \mathcal{P}_B \) | Prec |

TABLE II. Summary of the characterization rules for state structure of states, effects, and the three compositions.
C. Relating the compositions

Using the algebra, the three bipartite rules $\otimes$, $\prec$, $\rightarrow$ can be related together by the following relations:

$$\mathcal{T}_A \otimes \mathcal{P}_B = (\mathcal{T}_A \prec \mathcal{P}_B) \cap (\mathcal{T}_A \rightarrow \mathcal{P}_B), \quad (35a)$$

$$\mathcal{T}_A \rightarrow \mathcal{P}_B = (\mathcal{T}_A \prec \mathcal{P}_B) \cup (\mathcal{T}_A \rightarrow \mathcal{P}_B). \quad (35b)$$

See Appendix [A5] for the proof. These have a concrete physical interpretation. In a first time, consider the set $\mathcal{A} \otimes \mathcal{B}$. Its elements are transformations from $\mathcal{A}$ to $\mathcal{B}$ as each $M \in \mathcal{A} \otimes \mathcal{B}$ satisfies

$$\text{Tr}_A [M \cdot (A \otimes 1)] \in \mathcal{B}, \quad (36)$$

because $\mathcal{T}_A \otimes \mathcal{P}_B \subset \mathcal{P}_A \rightarrow \mathcal{P}_B$.

In addition to that, the right-hand side of Eq. [35a] puts two extra conditions on $\mathcal{A} \otimes \mathcal{B}$ that are similar to Eq. [26]: it requires that $(\mathcal{T}_A \otimes \mathcal{P}_B) \{M\} = M \iff \forall A \in \mathcal{A}, \forall B \in \mathcal{B}$,

$$\text{Tr}_A [M \cdot (A \otimes 1)] = \text{Tr}_A \left[ M \cdot \left( \frac{1}{c_A} \otimes 1 \right) \right]; \quad (37a)$$

$$\text{Tr}_B [M \cdot (1 \otimes B)] = \text{Tr}_B \left[ M \cdot \left( 1 \otimes \frac{1}{c_B} \right) \right]. \quad (37b)$$

Therefore, the above pair of conditions reveals the set $\mathcal{A} \otimes \mathcal{B}$ as the set of transformations that are compatible with no signaling from system A to system B and from B to A at the same time; it is no signaling in both directions, hence the name.

The same way, the $\mathcal{A} \rightarrow \mathcal{B}$ set is the set of transformations that respect no signaling from A to B or B to A. Thus, it may allow linear combinations of signaling in both directions and consequently to indefinite causal order.

The sets $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{A} \rightarrow \mathcal{B}$ lie in between, as they permit signaling in only one direction. Their elements indeed obey condition [35a] but only one of the conditions [37].

This discussion underlies the following chain of inclusions:

$$\mathcal{T}_A \otimes \mathcal{P}_B \subset \mathcal{T}_A \prec \mathcal{P}_B \subset \mathcal{T}_A \rightarrow \mathcal{P}_B, \quad (38a)$$

$$\mathcal{T}_A \rightarrow \mathcal{P}_B \subset \mathcal{T}_A \prec \mathcal{P}_B \subset \mathcal{T}_A \rightarrow \mathcal{P}_B, \quad (38b)$$

See Figure 6 for a diagrammatic interpretation.

Defining the tensor product as no signaling. Naturally, Eqs. [37] are also valid for bipartite states: $(\mathcal{P}_A \otimes \mathcal{P}_B) \{W\} = W \iff \forall A \in \mathcal{A}, \forall B \in \mathcal{B}$,

$$\text{Tr}_A \left[ (\mathcal{A} \otimes 1) \cdot W \right] = \text{Tr}_A \left[ \left( \frac{1}{c_A} \otimes 1 \right) \cdot W \right]; \quad (39a)$$

$$\text{Tr}_B \left[ (1 \otimes B) \cdot B \right] = \text{Tr}_B \left[ \left( 1 \otimes \frac{1}{c_B} \right) \cdot W \right]. \quad (39b)$$

It should be noted that the definition of the no signaling composition $\otimes$ can be done solely by requiring these two equations to be valid simultaneously; there is no need to additionally require that $W$ is a transformation (i.e. that it belongs to $\mathcal{A} \rightarrow \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$). This observation have a two corollaries: on the one hand, it provides a ‘transformation-free’ definition of the tensor product based only on the validity of Theorem [1]. On the other hand, it justifies its nickname of ‘no signaling composition’.

Remark. While the interpretation of the transformation (”$\rightarrow$”) in linear logic is the linear implication (“$\rightarrow$”), the “$\prec$” connector has no studied counterpart in linear logic. Still, the fact that it is a multiplicative connector that commutes with negation makes its appearance in the algebra look natural. Exploring the properties of the prec in LL is left for future work.

Categorical perspectives. In terms of categories, higher-order theories of transformations built using two-way signaling composition can be seen as a category with objects in $\mathcal{A}$ and morphisms in $\mathcal{A} \rightarrow \mathcal{B}$, thereby noted $\mathcal{C}(\mathcal{A}, \mathcal{A} \rightarrow \mathcal{B})$. These are $\ast$-autonomous categories, meaning that they possess two monoidal structures ($\otimes$ and $\ast$) and that negation commutes with neither of them [9].

The one-way signaling composition provides a decomposition of these $\ast$-autonomous categories into the union of two closed compact categories, $\mathcal{C}(\mathcal{A}, \mathcal{A} \prec \mathcal{B})$ and $\mathcal{C}(\mathcal{A}, \mathcal{A} \rightarrow \mathcal{B})$. These induce the following chains of wide subcategories:

$$\mathcal{C}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}) \subset \mathcal{C}(\mathcal{A}, \mathcal{A} \prec \mathcal{B}) \subset \mathcal{C}(\mathcal{A}, \mathcal{A} \rightarrow \mathcal{B}), \quad (40)$$

$$\mathcal{C}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}) \subset \mathcal{C}(\mathcal{A}, \mathcal{A} \rightarrow \mathcal{B}) \subset \mathcal{C}(\mathcal{A}, \mathcal{A} \rightarrow \mathcal{B}). \quad (41)$$

Surely, this decomposition calls for a more systematic functorial treatment: the category of higher-order transformations actually features not two but three monoidal structures, with one of them ($\prec$) being compact and closed.
D. The algebra of projectors recovers no signaling

We started from the projective characterization of higher-order transformations in which \( \mathcal{P}_A, \mathcal{P}_A \rightarrow \mathcal{P}_B, \mathcal{P}_A \otimes \mathcal{P}_B \) are the new projectors one can obtain from a projector \( \mathcal{P}_A \) under the algebraic rules \( \{\rightarrow, \otimes, \_\} \).

We observed that the notion of no signaling could be generalized for the case of state structures. In particular, it was noticed that the notion could be encoded by a new rule \( \prec \) in the algebra of projectors, introduced under the name prec.

Bipartite state structures obtained by combining \( \mathcal{P}_A \) and \( \mathcal{P}_B \) fall into four classes, essentially determined by conditions (36), (37a) and (37b). All of these classes must satisfy the first condition, which amounts to being a valid transformation and to which corresponds the projector \( \mathcal{P}_A \rightarrow \mathcal{P}_B \). If in addition condition (37a) (respectively, (37b)) is satisfied, the set is restricted to transformation that is no signaling from the output to the input (resp., from the input to the output), to which corresponds the projector \( \mathcal{P}_A \prec \mathcal{P}_B \) (resp., \( \mathcal{P}_A \succ \mathcal{P}_B \)). If both conditions are satisfied, the set is no signaling, to which corresponds the projector \( \mathcal{P}_A \otimes \mathcal{P}_B \).

Therefore, working with projectors provides a handy way of assessing the signaling structure within a state structure.

V. IMPLICATIONS OF THE FORMALISM

With the formalization of the projective characterization and the introduction of the one-way signaling subsets using the prec, we now have the tools to address signaling in theories of higher-order quantum transformations. In this section, we are interested only in the qualitative aspects of state structures, meaning that we will not be referring to operators, but only sets and their projectors. As a consequence, the calligraphic notation will be dropped for clarity, and capital letters will refer to state structure or projectors depending on the context.

A. Algebraic manipulations using the prec: two birds with one stone

On purely algebraic grounds, the transformation \( \rightarrow \) has two defining properties that make it inconvenient to work with. The first is not commuting with negation:

\[
\overline{A \rightarrow B} \neq A \rightarrow \overline{B}.
\]  

(42)

While in the categorical treatment this is a manifestation of the *-autonomous character of the category of higher-order transformations, it leads to difficulties for relating seemingly similar sets together using the algebra alone. Two expressions involving the same states and effects may be incomparable because one features a negation on several subsystems at once while the other does not. For example, \( \overline{A \rightarrow B} = A \otimes \overline{B} \) cannot be compared with \( \overline{A \rightarrow \overline{B}} = A \otimes \overline{B} \) without explicitly doing computations to show set inclusions.

The second inconvenient property is that \( \rightarrow \) is not associative:

\[
A \rightarrow (B \rightarrow C) \neq (A \rightarrow B) \rightarrow C.
\]  

(43)

As we have seen, this is what allows one to define the notion of an order with respect to a base type in the first place. Let \( A_0, A_1, A_2, A_3, \ldots \) be some base type \( A \) defined on different spaces \( L (\mathcal{H}_0), L (\mathcal{H}_1), \ldots \), then \( A_0 \) is a type of first order, \( A_0 \rightarrow A_1 \) is second order, \( (A_0 \rightarrow A_1) \rightarrow (A_2 \rightarrow A_3) \) is third order and so on... However, a naive application of the notion of order may hide the equivalence between some theories. For example, consider \( ((A_0 \rightarrow A_1) \rightarrow A_2) \rightarrow A_3 \) and \( (A_1 \rightarrow A_2) \rightarrow (A_0 \rightarrow A_3) \). Some computations will reveal that they both feature \( A_0 \otimes A_1 \otimes A_2 \otimes A_3 \) as their no signaling subset (this notion is more precisely defined below), and accordingly they should be comparable. But they are not of the same order: the former is a succession of three transformations of a state \( A \), whereas the latter is a transformation of higher-order state \( A_1 \rightarrow A_2 \rightarrow A_3 \). Looking at the formula alone, one would conclude that there is no use in comparing them because they do not feature the same base state structures. Yet, in some cases they happen to be the same. In this example, if the base types \( A \)'s are the quantum states, then the two expressions are two ways of describing the same thing, the set of quantum 2-combs \[3\].

Surprisingly, the prec does not lead to these issues, as it commutes with the negation:

\[
\overline{A \prec B} = \overline{A} \prec \overline{B}.
\]  

(44)

And, like the tensor, it is associative:

\[
(A \prec B) \prec C = A \prec (B \prec C) = A \prec B \prec C.
\]  

(45)

As the algebra obeys the De Morgan relations (Eqs. (24)), and that the tensor and transformation split into precs (Eqs. (35)), it becomes possible to express any expression \( \Gamma \) involving \( n \) projectors \( A_1, A_2, \ldots, A_n \) composed with \( \otimes \) and \( \rightarrow \) as the union and intersection of types built using the prec.

To make this point precise, we first define the following.

**Definition 4** (No signaling subset). For a projector expression \( \Gamma \) that involves \( n \) projectors \( A_1, A_2, \ldots, A_n \) composed together using \( \otimes \) and \( \rightarrow \), the **no signaling subset** is the subset built using only negations over single projector and tensor products. It has the form

\[
\overline{A_1 \otimes A_2 \otimes \ldots \otimes A_n} \subset \Gamma,
\]  

(46)

where the \( \overline{A_i} \) notation means that projector \( A_i \) is potentially negated, depending on \( \Gamma \).

Using the definition \( 15c \), any \( \rightarrow \) can be turned into a mix of tensor and negation, \( \cdot \rightarrow \cdot = \cdot \otimes \overline{\cdot} \). Then,
because of Eq. [23], one can always find the no signaling subset of an expression by distributing the negations.

A rule of thumb for finding the no signaling subset is to ‘count the number of bars’ above each projectors in an expression. Since negation is an involution, an odd (respectively, even) amount will indicate that the projector is (not) negated in the no signaling subset. For example, to find the no signaling subset of \((A_0 \otimes (A_1 \rightarrow A_2)) \rightarrow A_3\), one first expresses it using negations and tensors products, \((A_0 \otimes (A_1 \rightarrow A_2)) \rightarrow A_3 = A_0 \otimes A_1 \otimes A_2 \otimes A_3\), then by inspection \(A_0\) and \(A_2\) have an odd number of negations above them (one and three, respectively) so they will be negated, whereas \(A_1\) and \(A_3\) have an even number (two and two) so they will not be. On that account, the no signaling subset of \((A_0 \otimes (A_1 \rightarrow A_2)) \rightarrow A_3\) is \(\overline{A_0} \otimes A_1 \otimes \overline{A_2} \otimes A_3\).

Now, in the algebra, the cap is distributive over the cup,

\[
(A \cup A') \cap A'' = (A \cap A') \cup (A' \cap A'') ,
\]

but the converse is not true. Furthermore, the cap and the cup are both distributive over the prec (and in this case the converse holds, see App. A3a for proofs),

\[
(A \cup A') \prec B = (A \prec B) \cup (A' \prec B) ,
\]

\[
A \prec (B \cap B') = (A \prec B) \cap (A \prec B') .
\]

Where the above are also satisfied when caps and cups are switched, \((\cap \leftrightarrow \cup)\). Consequently, there is an order in distributivity; expressions involving these three connectors can always be rewritten so that they are put into a union of intersections of one-way signaling compositions. As these three operations are associative, the intermediate parenthesis can be dropped everywhere.

Finally, by combining Eqs. (35) with the De Morgan rule (24b) and using the fact that prec commutes with negation, it should be clear that in the decomposition, the projectors on each single subsystem are the same as the ones appearing in the no signaling subset.

For these reasons, any expression \(\Gamma\) involving \(n\) projectors \(A_1, A_2, \ldots, A_n\) composed using \(\otimes\) and \(\rightarrow\) can be brought to a canonical form.

**Definition 5 (Canonical form).** Let \(\Gamma\) be a projector expression that involves \(n\) projectors \(A_1, A_2, \ldots, A_n\) composed together using \(\otimes\) and \(\rightarrow\), and whose no signaling subset is given by \(\hat{A}_1 \otimes \hat{A}_2 \otimes \ldots \otimes \hat{A}_n \subset \Gamma\). Then, its canonical form is the decomposition

\[
\Gamma = \bigcup_{i=1}^{x} \left( \bigcap_{j=1}^{y_i} \hat{A}_{\sigma_{ij}(1)} \prec \hat{A}_{\sigma_{ij}(2)} \prec \ldots \prec \hat{A}_{\sigma_{ij}(n)} \right) ,
\]

in which there are \(x\) unions of expressions labeled by index \(i\), and each expression involves \(y_i\) intersections of sub-expressions labelled by index \(j\). \(\sigma_{ij}\) is an element of the permutation group, so that each sub-expression is a permutation of \(A_1 \prec A_2 \prec \ldots \prec A_n\) (indices \(i\) and \(j\) do not necessarily run over the full permutation group).

This implies that any expression can be brought down to the union of several first-order expressions (which further decompose into intersections if needed). This way of rewriting is essentially unique for any state structure, lifting any ambiguity induced by the negation or non-associativity.

In the first example discussed above, the expressions reduce to \(\hat{A} \rightarrow \hat{B} = (A \prec B) \cap (A \succ B)\) and \(\overline{\hat{A}} \rightarrow \overline{\hat{B}} = (A \prec \overline{B}) \cup (A \succ \overline{B})\). Now one can meaningfully compare them: they are both built from a combination of the same two one-way signaling transformations, but one set is the intersection and the other is the union, we thus conclude that \(\overline{\hat{A}} \rightarrow \overline{\hat{B}} \subset \hat{A} \rightarrow \hat{B}\).

Moreover, the prec connector and the decomposition it induces also allows for an identification of the possible signaling directions these two sets may feature. Indeed, since the prec is exactly the no signaling condition, the above canonical form is a breakdown of a general expression into several ordered expressions built using only the prec, then combined first by requiring no signaling between some subsystems, i.e. using cap, and then by allowing signaling in several directions, i.e. using cup.

Summarizing, the algebraic fact that the prec is associative and commutes with the negation allows one to write an unambiguous ‘canonical’ form useful to compare theories. This gives a tool to determine whether two higher-order theories are equivalent by sole inspection of their projectors. In addition, the physical fact that the prec is a one-way signaling composition, combined with the algebraic rules, allows one to split projectors in several terms with a fixed signaling direction between its base state structures. This implies that one can know the possible signaling structure that a higher-order theory may feature by sole inspection of its projector.

### B. When quantum combs are isomorphic to quantum networks

The next part of the results is concerned with the relationship between two different ways of building higher-order objects using the developed formalism of projectors. This can be seen as an example of how to use it in order to swiftly recover results of reference [5].

Following reference [3], a network is defined as the causally ordered (i.e. one-way signaling) succession of ‘nodes’ of the same state structure. A ‘1-network of base A’ will be the set \(A\) itself, thereafter noted \(A_0\) for clarity, then the ‘2-network’ will be the set \(A_0 \prec A_1\), the ‘3-network’ will be \(A_0 \prec A_1 \prec A_2\), etc. A common occurrence of this structure is the quantum network, where the base object is a quantum channel such as \(A \equiv I_{A_0} \rightarrow I_{A_1}\). This quantum network, here associated with some party that will be called Alice, represents the successive operations of that party. If Alice has a single node quantum network, it means that Alice acts once on the system \(A_0\) with a quantum channel and outputs a quantum state in \(A_1\). If she has a network with two nodes, she will act a
first time on $A_0$ and output a first state at $A_1$, then a second time on $A_2$, now potentially using any size of ancillary qudit as a memory register she preserved from her first operation, and output in $A_3$. And so on, as defined recursively.

Another way of building a higher-order state structure is by doing a comb, which consists of recursively transforming into a base type: the ‘1-comb of base $A$’ is again a single direction of signaling. How could it be that they are all equivalent to networks — that is, objects featuring a single direction of signaling? Especially, why is the 1-comb (built using the two-ways signaling transformation) equivalent to the quantum channel (which is causal)? Because of that, a network of quantum channels is then an alternating network of effects and states as associativity can be used.

Eq. (50b) is for example the reason why the single partite process matrix reduces to an effect and a state in tensor product $\otimes$.

b. Equivalence of quantum combs and networks. Now, all the ideas needed to understand why quantum combs are causal networks are in place. Formally, it was proven [4, 5] that:

1) The comb based on quantum states (for clarity, let $A_i = I_{A_i}$):

$\ldots ((A_0 \rightarrow A_1) \rightarrow A_2) \rightarrow \ldots ) \rightarrow A_{2n-1}$,

is equivalent to the quantum comb (notice the reordering of the terms),

$\ldots ((A_{n-1} \rightarrow A_n) \rightarrow (A_{n-2} \rightarrow A_{n+1})) \rightarrow \ldots ) \rightarrow (A_0 \rightarrow A_{2n-1})$.

2) Operators $M$ in these structures obey the causality condition

$$\text{Tr}_{A_{2n+1}} [M] = \frac{1}{d_{A_{2n}}} \text{Tr}_{A_{2n+1} A_{2n}} [M] \otimes I_{A_{2n}} \forall i.$$ (53)

According to the discussion in Section IV A, the causality conditions can be recast into a single projector,

$(A_0 \rightarrow A_1) \prec \ldots \prec (A_{2n-2} \rightarrow A_{2n-1})$.

Because of the isomorphism (50a) and associativity of the prec, this is equivalent to

$A_0 \prec A_1 \prec \ldots \prec A_{2n-2} \rightarrow A_{2n-1}$. (54)

For $n = 1$, these three equations (51), (52) and (54) are trivially equivalent. For $n = 2$, we recover the second example of the last section: Statement 1) implies that the quantum 2-comb ($A_1 \rightarrow A_2$) is equivalent to the quantum-state-based 4-comb ($(A_0 \rightarrow A_1) \rightarrow A_2 \rightarrow A_3$, although, in general, the latter is a subset of the former. Statement 2) implies that, in the quantum case, the two have a fixed signaling direction, although they are built using the 2-way signaling transformation.

With the prec rules and the isomorphisms of equations (50), the proofs of both statements are obtained from algebraic manipulations. First, for the state-based comb, the transformation is isomorphic to a prec, $A_0 \rightarrow A_1 = A_0 \prec A_1$, then $(A_0 \rightarrow A_1) \rightarrow A_2 \rightarrow A_3 = \ldots$.
\((\overline{A_0} < A_1) \times A_2 < A_3 = \overline{A_0} < A_1 < \overline{A_2} < A_3\), because the negation is distributed over the prec, and this connector is associative.

Next, the equivalence between the combs, statement 1), is obtained using the same properties: \((A_1 \to A_2) \to (A_0 \to A_3) = (\overline{A_1} \to A_2) \to (\overline{A_0} \to A_3) = (A_1 \times A_2) \otimes (A_0 \otimes A_3)\), some reordering and the use of the tensor associativity yield \(A_0 \otimes ((A_1 \times A_2) \otimes A_3) = A_0 \otimes (A_1 \times A_2 < \overline{A_3}) = A_0 < A_1 \times \overline{A_2} < A_3\).

Finally, the proofs can be generalized mutatis mutandis for any number of nodes, see Appendix B 6 for the details. We therefore have

\[\ldots ((A_0 \to A_1) \to A_2) \to \ldots) \to A_{2n-1} = \overline{A_0} < A_1 \times \overline{A_2} \times \ldots \times A_{2n-1}, \quad (56)\]

so the statement 2) that this comb has a single signaling direction is now directly apparent as the structure has a single ‘prec chain’ in canonical form.

We also have that

\[\ldots (A_{n-1} \to A_n) \to \ldots) \to (A_0 \to A_{2n-1}) = \overline{A_0} < A_1 < \overline{A_2} \times \ldots \times A_{2n-1}. \quad (57)\]

Hence isomorphisms between \(\prec\) and \(\to\) in the case of theories whose base structure are the quantum states explain this apparent counter‐logical equivalence of a second‐order (with respect to quantum states) theory with a third‐order one.

A different example of the consequences of this result in the case of MPM can be found in Appendix C 5. It also features another example of the use of the canonical form.

VI. DISCUSSION

In this work, a correspondence was developed between several characterizations of the theory of higher‐order quantum transformations. Starting at the most abstract layer, its formulation in terms of types, we added a new layer, its formulation in terms of superoperator projectors, to bridge to a less abstract layer, its formulation in terms of sets of CJ operators – here called state structures. Correspondences between the characterizations can be summed up as follow: Given types \(A\) and \(B\), types \(A, A \otimes B\), and \(A \to B\) can be constructed using the semantic rules \(\{1, (,), \to\}\). To these types correspond projectors: given projectors \(P_A\) and \(P_B\), projectors \(\overline{P_A}, P_A \otimes P_B\) and \(P_A \to P_B\) can be constructed using the algebraic rules \(\{\prec, \otimes\}\) as in equations (9c), (13c), and (15c). To these types and projectors correspond state structures: given state structures \(S\) and \(B\) as in equation (2), the state structures \(\overline{S}, \overline{S} \otimes B,\) and \(\overline{S} \to B\) can be constructed by requiring that, respectively, conditions [8], (39), and (36) hold.

The main novelty is the prec connector, Prop. [4], giving the state structure \(\overline{S} < B\) of maps which are no signaling from the output to the input. This allows to speak about the possible signaling structure of the maps characterized by the projective framework. In addition to that, the properties of the prec give an unambiguous way to compare different theories as they allow to expand them as unions and intersection of expressions involving only this prec connector.

In the mean time, we identified the features of the algebra of projectors. We showed in particular that it was a Boolean algebra under the rules \(\{\prec, \cup, \otimes\}\), which is subsequently extended into a model of linear logic under the rules \(\{\prec, \cap, \cup, \otimes, \to\}\). The introduced prec fits naturally as a multiplicative rule in‐between \(\otimes\) and \(\to\), which moreover has the property of commuting with the negation. It remains to investigate how the role of the prec is understood in terms of linear logic, as well as in terms of categories. To the authors’ knowledge, the prec connector appears as a genuinely new abstract mathematical concept discovered in a physically motivated problem.

Finally, we rederived the proof of equivalence between combs based on quantum states, quantum combs, and networks based on quantum channels [5] as an example of the use of the projective characterization. This equivalence explains in particular why these structures feature a definite signaling direction, which comes with a clear physical implementation. The question of whether these are the only isomorphic constructions is left open for future work. Another related but more general question that can be asked in terms of projectors is to determine for which base state structures a given construction based on the transformation reduces to a construction based on the one‐way signaling composition. In other words, given an abstract way of constructing higher‐order theories, for which base state structures does it reduce to a network?

Several aspects of the theory of higher‐order transformations have not been explored here and remain open for future work. The composition of types in the sense of reference [5] has only been partially addressed. Knowing the projector characterizing a transformation is sufficient to know what will be the projector characterizing its set of outputs. However, if the input state structure is now restricted to a subset with a different projector, is there a rule to apply on the projector of the transformation in order to get the projector of the possibly now restricted output set? For example, an \(n\)‐comb takes \((n-1)\)-combs as inputs and outputs a \(1\)-comb. One can ask what happens when the inputs are restricted to the no signaling subset: when \((n-1)\) 1-combs are plugged into an \(n\)‐comb instead, is the output set still the full set of 1-combs?

The notion of causal separability [8] [17] [23] [26] still is to be defined for general higher‐order transformations. The link between the prec and single signaling direction, as well as the canonical form, should give a basis for the definition. Nonetheless, there remains to tackle the question of dynamical causal orders, which is a probabilistic
notion, and therefore not naturally captured by the formalism of projectors.

The issue of realizability has not been explored. That is, given a state structure, is it possible to realize each of its elements in a lab experiment? Is there a systematic way to relate these abstract mathematical objects to a circuit realization, as is the case for quantum combs [3] and for time-delocalized subsystems [27, 29]? One may expect that the canonical decomposition of general state structures into the union of several one-way signaling structures could prove useful for answering this question.

**Note added:** After completion of this work and during the preparation of the manuscript, we became aware of an independent work by Simmons and Kissinger [29], which also investigates the signaling structure of higher-order quantum transformations. Some of the results presented here were also found in this other work using a different formalism.

**ACKNOWLEDGMENTS**

T.H. is grateful to the organizers and participants of the 2021 Semy Summer Institute for listening to his rambling interventions about nesting supermaps— in particular to Pablo Arnault, Titouan Carette, Natália Móller, Eleftherios Tsentis, and Augustin Vanrietvelde for technical discussions. T. H. is especially indebted to Titouan Carette for pointing out the connection with linear logic. In addition, T.H. would like to thank Esteban Castro-Ruiz and Joseph Cunningham for help and comments, as well as Alexandra Elbakyan for providing access to the scientific literature.

T. H. benefits from the support of the French Community of Belgium within the framework of the financing of a FRIA grant. O. O. is a Research Associate of the Fonds de la Recherche Scientifique (F.R.S.-FNRS). This publication was made possible through the support of the ID# 61466 grant from the John Templeton Foundation, as part of the “The Quantum Information Structure of Spacetime (QISS)” Project (qiss.fr). The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation. This work was supported by the Program of Concerted Research Actions (ARC) of the Université libre de Bruxelles.

Illustrations were drawn using draw.io ([https://www.draw.io](https://www.draw.io)).
Appendix A: Superoperator projectors

A superoperator \( \mathcal{P}_A \) is a projector on a subspace of \( \mathcal{L}(\mathcal{H}^A) \) if it satisfies the condition

\[
\mathcal{P}_A \circ \mathcal{P}_A = \mathcal{P}_A.
\]

An element \( A \) belongs to the subspace defined by \( \mathcal{P}_A \) if and only if

\[
\mathcal{P}_A \{ A \} = A.
\]

\( \mathcal{P}_A \) projects to a self-adjoint subspace if it obeys

\[
\mathcal{P}_A \circ \dagger \mathcal{P}_A = \dagger \circ \mathcal{P}_A,
\]

where \( \dagger \) means ‘taking the adjoint in \( \mathcal{L}(\mathcal{H}^A) \)’.

A projector is orthogonal if it does not increase the norm of operators:

\[
\|\mathcal{P}_A \{ A \}\| \leq \| A \|,
\]

where \( \| A \| = \sqrt{\text{Tr}[A^\dagger \cdot A]} \) is the Hilbert-Schmidt norm. Note that this condition is equivalent to requiring the projector to be self-adjoint with respect to the inner product (see e.g. Ref. [30, Theorem 10.5]).

\[
\text{Tr} \left[ \mathcal{P}_A \{ A' \} ^\dagger \cdot A \right] = \text{Tr} \left[ A'^\dagger \cdot \mathcal{P}_A \{ A \} \right], \quad \forall A, A'.
\]

All projectors in this work are assumed orthogonal and projecting to self-adjoint subspaces. As the projectors should project onto operator systems, which are subspaces that contain the identity, it should be true that their images contain the span of the identity as a subspace. A necessary and sufficient condition for that to be true is

\[
\mathcal{P}_A \circ \mathcal{D}_A = \mathcal{D}_A \circ \mathcal{P}_A = \mathcal{D}_A,
\]

with \( \mathcal{D}_A \) defined as in Eq. (7).
In addition, it is required for the composition of two projectors to be a projector, \((P_A \circ P_A')^2 = P_A \circ P_A'.\) A necessary and sufficient condition for \(P_A \circ P_A'\) to be a projector is that its constituents commute \([30, \text{Theorem } 2.26.3]\). Therefore, the condition

\[
P_A \circ P_A' = P_A' \circ P_A \quad \forall \, P_A, P_A',
\]

will be required on every pair of abstract superoperator projectors \(P_A, P_A'\) defined on the same space. The reason for requiring such a condition comes from the fact that all the introduced rules in this work will preserve commutativity. Hence, any two projectors \(P\) and \(P'\) obtained by relating the same base projectors on the same subsystems using the various algebraic rules defined in the following will be commutative by construction.

Thus, a \textbf{projector onto an operator system} \(\mathcal{A}\) as in Eq. (4c) is defined as a linear superoperator \(P_A \in \mathcal{L}(\mathcal{L}(\mathcal{H}^A), \mathcal{L}(\mathcal{H}^A))\) that obeys conditions (A1), (A3), (A6), and (A5). In addition to that, any two such projectors are assumed to commute, Eq. (A7).

\section{2. Adding cap and cup operations makes an algebra}

Here, we prove that superoperator projectors on operator systems constitute an algebra under certain rules. Assume \(P_A\) and \(P_A'\) to be two arbitrary projectors on operator systems, not necessarily the same. By definition, a projector is \(\mathbb{C}\)-linear:

\[
P_A\left(\sum_i q_i A_i\right) = \sum_i q_i \, P_A \{A_i\}, \quad (A8)
\]

\(q_i \in \mathbb{C}\). This linearity allows us to carry on the addition ‘+’ of \(\mathcal{L}(\mathcal{H}^A)\) at the level of the projectors:

\[
(P_A + P_A')\{A\} \equiv P_A \{A\} + P_A' \{A\}. \quad (A9)
\]

Using linearity again, one can define the negation ‘\(\cap\)’ and scalar multiplication of projectors, thereby defining a vector space over \(\mathbb{C}\). Associativity and commutativity are not hard to prove from there.

Projectors also have a natural ‘conjunction’ operation that can be interpreted as a multiplication or as a logic ‘and’, nicknamed the \textbf{cap}:

\[
P_A \cap P_A' \equiv P_A \circ P_A'. \quad (A10)
\]

It is again straightforward to prove that it fits the definition: it is a binary operation that is distributive under the addition defined above and it is compatible with scalar multiplication. Moreover, it is both associative and commutative, and it inherits the idempotent property from the projector definition:

\[
P_A \cap P_A \equiv P_A \circ P_A \equiv P_A^2 = P_A. \quad (A11)
\]

In the above equation, we have also defined a shorthand notation for ‘squaring’ under the multiplication \(\cap\): \(P_A \cap P_A^2 \equiv P_A^2\).

An issue revealed by squaring is that the operation ‘+’ does not necessarily map projectors to a valid projector:

\[
(P_A + P_A')^2 = P_A^2 + P_A'^2 + 2 \left(P_A \cap P_A'\right). \quad (A12)
\]

As the last term of the right-hand side is not always zero, \((P_A + P_A')^2 \neq P_A + P_A'\) in general, so addition does not preserve idempotency. We wish nonetheless to have operations that keep us in the vector space of projectors, so we redefine the addition to be the union of two projectors (see e.g. Ref. [31]). This is inspired by set theory: as \(P_A\) maps to a subspace \(\mathcal{A} \subset \mathcal{L}(\mathcal{H}^A)\) and \(P_A'\) to \(\mathcal{A}' \subset \mathcal{L}(\mathcal{H}^A)\) we want an addition ‘\(\cup\’ so that \(P_A \cup P_A'\) maps to \(\mathcal{A} \cup \mathcal{A}'\). This requirement is realized by the ‘disjunction’ operation, thereafter nicknamed \textbf{cup} (see also Ref. [17]):

\[
P_A \cup P_A' \equiv P_A + P_A' - P_A \cap P_A', \quad (A13)
\]

which will acts like addition, or logic ‘or’, in the space of projectors.

\(\cap\) and \(\cup\) can indeed be seen as some multiplication and addition of projectors as the former distributes over the latter:

\[
(P_A \cup P_A') \cap P_A'' = (P_A \cap P_A') \cup (P_A' \cap P_A''). \quad (A14)
\]

Indeed,

\[
(P_A \cup P_A') \cap P_A'' = (P_A + P_A' - (P_A \cap P_A')) \cap P_A'' = (P_A \cap P_A') + (P_A \cap P_A'') - (P_A \cap P_A'') \cap P_A''.
\]

These caps and cup operands give a quick way to prove subspace inclusions. A state structure \(\mathcal{A}\) has its subspace embedded in the subspace of another state structure \(\mathcal{A}'\) if and only if

\[
P_A \cap D_A = D_A. \quad (A15a)
\]

\[
P_A' \cap D_A = D_A. \quad (A15b)
\]

As defined in the main text, this is concisely noted as \(P_A' \subset P_A\). Condition (A16) is then recast into

\[
P_A \cap D_A = D_A. \quad (A16)
\]

Any projector built using cap and cup conserve both idempotency, Eq. (A11) and the identity, Eq. (A16). While in the case of cap this directly follows from definition, in the case of the cup we have proven idempotency above, and condition (A16) follows by distributivity:

\[
D_A \cap (P_A \cup P_A') = (D_A \cap P_A) \cup (D_A \cap P_A') = D_A \cup D_A,
\]

thus

\[
D_A \cap (P_A \cup P_A') = D_A. \quad (A17)
\]

With this, projectors acting on \(\mathcal{L}(\mathcal{H})\) form an algebra over \(\mathbb{C}\) since they are closed under \((\cap, \cup)\) operations as
well as scalar multiplication. The special projectors \( \mathcal{I} \) and \( \mathcal{D} \) we define above play a special role in it: one can identify \( \mathcal{I} \) as the multiplicative identity or ‘**unit**’ of the algebra of projectors since for all projectors \( \mathcal{P} \)

\[
\mathcal{I} \cap \mathcal{P} = \mathcal{P} ; \tag{A18}
\]

it is also the additive absorbant \( i.e. \)

\[
\mathcal{I} \cup \mathcal{P} = \mathcal{I} . \tag{A19}
\]

The other way around, \( \mathcal{D} \) is the multiplicative absorbant or ‘**zero**’ of the algebra,

\[
\mathcal{D} \cap \mathcal{P} = \mathcal{D} , \tag{A20}
\]

and the additive identity,

\[
\mathcal{D} \cup \mathcal{P} = \mathcal{P} . \tag{A21}
\]

Because of that property, it should be clear that for any element of the algebra, the following is true:

\[
\mathcal{D} \subset \mathcal{P} \subset \mathcal{I} . \tag{A22}
\]

We are dealing with an algebra of idempotent elements equipped with a partial ordering.

Although referring to the cap and cup as ‘multiplication’ and ‘addition’ provides an intuitive meaning to the operands, in the following we will call them respectively ‘additive disjunction and conjunction’. This choice is made in order to avoid an ambiguity with the multiplicative conjunction and disjunction that will be introduced in a next section.

### 3. Adding negation makes a Boolean algebra

In the algebra of superoperator projectors, the new object that naturally appears in Proposition \[\ref{prop1}\] \( \overline{\mathcal{P}}_A \), can be seen as an operation on the original projector \( \mathcal{P}_A \), whence the ‘bar over \( \mathcal{P}_A \)’ notation. This new operation is defined for any projector, and it promotes the algebra to a **Boolean algebra**.

It is a Boolean algebra if, in addition to the conjunction (‘multiplication’) operation \( \cap \) and the disjunction (‘addition’) operation \( \cup \), any element is idempotent and has a well-defined complement (or logic ‘not’, thereby referred to as ‘negation’ \[\ref{negation}\]) characterized by the condition that the addition of any projector with its negation yields the additive identity, \( i.e. \)

\[
\mathcal{P}_A \cap \overline{\mathcal{P}}_A = \mathcal{I}_A , \tag{A23}
\]

where \( \overline{\mathcal{P}}_A \) is the negation of \( \mathcal{P}_A \). This is exactly what the quasi-orthogonal complement of Proposition \[\ref{prop1}\]is doing. First, compute the cap as \( \mathcal{P}_A \cap \overline{\mathcal{P}}_A = \mathcal{P}_A \cap \mathcal{I}_A - \mathcal{P}_A = \mathcal{P}_A - \mathcal{P}_A + \mathcal{D}_A = \mathcal{P}_A - \mathcal{P}_A + \mathcal{D}_A , \)

so

\[
\mathcal{P}_A \cap \overline{\mathcal{P}}_A = \mathcal{D}_A . \tag{A24}
\]

Then, it is true for the cup because \( \mathcal{P}_A \cup \overline{\mathcal{P}}_A = \mathcal{P}_A + \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A - (\mathcal{P}_A \cap \overline{\mathcal{P}}_A) = \mathcal{I}_A . \) Thus, we define the **negation** of any projector \( \mathcal{P}_A \) as

\[
\overline{\overline{\mathcal{P}}}_A \equiv \overline{\mathcal{P}}_A \equiv (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) . \tag{A25}
\]

In addition to property \[\ref{prop2}\], one can verify from Eq. \[\ref{prop3}\] that the projector and its negation intersect at the zero of the algebra, which in this case is indeed \( \mathcal{D}_A \) (refer to the diagram in Fig. 1).

As it should be, negation is an involution:

\[
\overline{\overline{\mathcal{P}}}_A = \mathcal{P}_A , \tag{A26}
\]

and it is closed so that the negation of any projector is a valid projector. It is the case because the idempotency property is conserved, \( \overline{\mathcal{P}}_A^2 = (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) \cap (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) = \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A - \mathcal{P}_A + \mathcal{P}_A^2 - \mathcal{D}_A + \mathcal{D}_A - \mathcal{D}_A + \mathcal{D}_A = \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A , \) hence

\[
\overline{\mathcal{P}}_A^2 = \mathcal{P}_A . \tag{A27}
\]

The identity is still contained in any projector built using negation:

\[
\mathcal{D}_A \cap \overline{\mathcal{P}}_A = \mathcal{D}_A . \tag{A28}
\]

This follows from \( \mathcal{D}_A \cap \overline{\mathcal{P}}_A = \mathcal{D}_A \cap (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) = \mathcal{D}_A - \mathcal{D}_A + \mathcal{D}_A . \) If we did not have the heuristic of Prop. \[\ref{prop1}\], this could be interpreted as a reason for choosing Eq. \[\ref{prop2}\] as the definition of negation instead of the orthogonal complement \( \mathcal{I}_A - \mathcal{P}_A \). In this latter case, identity does not belong to the negated state structure. It should also be clear from \( \mathcal{P}_A \cap \overline{\mathcal{P}}_A = \mathcal{P}_A \cap (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) = \mathcal{P}_A - \mathcal{P}_A^2 + \mathcal{D}_A = (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) \cap \mathcal{P}_A \) that negated projectors commute with original ones, and by the same reasoning that the negations of two commuting projectors still commute:

\[
\mathcal{P}_A \cap \mathcal{P}_A' = \mathcal{P}_A' \cap \mathcal{P}_A = \overline{\mathcal{P}}_A \cap \overline{\mathcal{P}}_A' = \mathcal{P}_A \cap \overline{\mathcal{P}}_A . \tag{A29}
\]

Moreover, the De Morgan laws are valid for the Boolean algebra of projectors:

\[
\overline{\mathcal{P}}_A \cup \overline{\mathcal{P}}_A' = \mathcal{P}_A \cap \overline{\mathcal{P}}_A , \tag{A30a}
\]

\[
\overline{\mathcal{P}}_A \cap \overline{\mathcal{P}}_A' = \mathcal{P}_A \cup \overline{\mathcal{P}}_A . \tag{A30b}
\]

The proof is more involved: \( \overline{\mathcal{P}}_A \cup \overline{\mathcal{P}}_A' = \mathcal{I}_A - \mathcal{P}_A \cup \mathcal{P}_A' + \mathcal{D}_A = \mathcal{I}_A - (\mathcal{P}_A + \mathcal{P}_A' - (\mathcal{P}_A \cap \overline{\mathcal{P}}_A') + \mathcal{D}_A = \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A + \mathcal{D}_A - \mathcal{P}_A' - \mathcal{D}_A + (\mathcal{P}_A \cap \overline{\mathcal{P}}_A') - \mathcal{D}_A + \mathcal{D}_A = (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) \cap (\mathcal{I} - \mathcal{P}_A + \mathcal{D}_A) = \mathcal{P}_A \cap \mathcal{P}_A . \) The second identity directly ensues.

### 4. Adding tensor and par makes Linear Logic

**a. The tensor operation**

For Proposition \[\ref{prop4}\] we also defined a tensor product at the level of the projectors. Here we now present its algebraic properties so that we can meaningfully consider
$\mathcal{A} \otimes \mathcal{B} \subset \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B)$ when we know that $\mathcal{A}$ (respectively, $\mathcal{B}$) is characterized by a projector $\mathcal{P}_A$ acting on $\mathcal{L}(\mathcal{H}^A)$ ($\mathcal{P}_B$ acting on $\mathcal{L}(\mathcal{H}^B)$). Because of the isomorphism $\mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B) \cong \mathcal{L}(\mathcal{H}^A) \otimes \mathcal{L}(\mathcal{H}^B)$, the definition of **no signaling composition**, nicknamed tensor, is straightforward:

\[
(P_A \otimes P_B) \left\{ \sum_i q_i \ (A_i \otimes B_i) \right\} \equiv \sum_i q_i \ (P_A \{A_i\} \otimes P_B \{B_i\}), \quad (A31)
\]

and it should hold for all $i$ such that $q_i \in \mathbb{C}, A_i \in \mathcal{L}(\mathcal{H}^A)$, $B_i \in \mathcal{L}(\mathcal{H}^B)$ as any operator in $\mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B)$ can be expressed as such.

By the inherited properties of the tensor product at the level of operators, the tensor at the level of super-operators is also associative, and distributive with respect to multiplication,

\[
(P_A \cap P_{A'}) \otimes P_B = (P_A \otimes P_B) \cap (P_{A'} \otimes P_B), \quad (A32)
\]
as well and additions,

\[
(P_A \cup P_{A'}) \otimes P_B = (P_A \otimes P_B) \cup (P_{A'} \otimes P_B). \quad (A33)
\]

This directly follows from idempotency:

\[
(P_A \cap P_{A'}) \otimes P_B = (P_A \cap P_{A'}) \otimes (P_B \cap P_B), \quad (A34)
\]

and that the composition of projectors is party-wise and thus commute with the tensor product. Another way to see it is that composition of superoperators obeys an interchange law with the tensor product of superoperators:

\[
(P_A \cap P_A) \otimes (P_B \cap P_{B'}) = (P_A \otimes P_B) \cap (P_A' \otimes P_B'). \quad (A35)
\]

Expressions built from the tensor product result in valid projectors as they preserve idempotency,

\[
(P_A \otimes P_B)^2 = (P_A \otimes P_B) \cap (P_A \otimes P_B) \quad (A36)
\]

because of interchange law so

\[
(P_A \otimes P_B)^2 = (P_A \otimes P_B). \quad (A37)
\]

Tensor composition also preserve the identity $\mathcal{K}$, again because of the interchange law, $\mathcal{D}_{AB} \cap (\mathcal{P}_A \otimes \mathcal{P}_B) = (\mathcal{D}_A \otimes \mathcal{D}_B) \cap (\mathcal{P}_A \otimes \mathcal{P}_B) = (\mathcal{D}_A \cap \mathcal{P}_A) \otimes (\mathcal{D}_B \cap \mathcal{P}_B) = \mathcal{D}_A \otimes \mathcal{D}_B,$

\[
\mathcal{D}_{AB} \cap (\mathcal{P}_A \otimes \mathcal{P}_B) = \mathcal{D}_{AB}, \quad (A38)
\]

And the tensor product preserves commutation, as if $\mathcal{P}_A \cap \mathcal{P}_A' = \mathcal{P}_A \cap \mathcal{P}_A$ and $\mathcal{P}_B \cap \mathcal{P}_B' = \mathcal{P}_B \cap \mathcal{P}_B$ then

\[
\mathcal{P}_A \otimes \mathcal{P}_B = \mathcal{P}_A \otimes \mathcal{P}_B \iff \mathcal{P}_A = \mathcal{P}_B = \mathcal{I} \text{ or } \mathcal{D}. \quad (A45)
\]
This is proven by rewriting $\overline{\mathcal{P}_A \otimes \mathcal{P}_B}$ into
\[
\overline{\mathcal{P}_A \otimes \mathcal{P}_B} = \mathcal{P}_A \otimes \mathcal{P}_B + \\
(\overline{\mathcal{P}_A \otimes \mathcal{P}_B} - \mathcal{P}_A \otimes D_A - D_A \otimes \overline{\mathcal{P}_B} + D_A \otimes D_B) + \\
(\mathcal{P}_A \otimes \overline{\mathcal{P}_B} - \mathcal{P}_A \otimes D_B - D_A \otimes \overline{\mathcal{P}_B} + D_A \otimes D_B),
\]
(A46)
using the definition of negation and algebraic properties. It can then be understood from Fig. 8 first term of the above is the right quarter of the wheel, second is the top quarter with its boundary removed, and third is the bottom also without boundaries. As the three parts share no intersection, the regular addition ‘+’ is equivalent to a conjunction ‘∪’. Next, more algebraic manipulations lead to
\[
\overline{\mathcal{P}_A \otimes \mathcal{P}_B} = \mathcal{P}_A \otimes \mathcal{P}_B + \\
(\overline{\mathcal{P}_A \otimes \mathcal{P}_B} + \mathcal{P}_A \otimes \overline{\mathcal{P}_B} - I_A \otimes D_B - D_A \otimes I_B),
\]
(A47)
and from this expression it is direct to check that the term in parenthesis vanishes if and only if either of the conditions in Eq. (A45) hold.

b. The transformation operation

The projector appearing in Proposition 3 is built from the projectors characterizing its input and output. The corresponding operation is defined as the transformation, represented by $\rightarrow$ :

\[
\mathcal{P}_A \rightarrow \mathcal{P}_B \equiv \overline{\mathcal{P}_A \otimes \mathcal{P}_B}.
\]
(A48)
This additional operation in the Boolean algebra of projectors is actually secondary, since it can be entirely defined using the negation and causal composition. Therefore it will automatically be a valid projector in the sense that it will yield obey Eqs. (A11) and (A16). Furthermore, as this expression involves both composition and negation, it will not be associative in general. For example, $((\mathcal{P}_A \rightarrow \mathcal{P}_B) \rightarrow \mathcal{P}_C) \neq (\mathcal{P}_A \rightarrow (\mathcal{P}_B \rightarrow \mathcal{P}_C))$, because the uncurrying rule proven in Ref. [4] can be applied to the right hand side:

\[
\mathcal{P}_A \rightarrow (\mathcal{P}_B \rightarrow \mathcal{P}_C) = (\mathcal{P}_A \otimes \mathcal{P}_B) \rightarrow \mathcal{P}_C,
\]
(A49)
which is obviously different than $((\mathcal{P}_A \rightarrow \mathcal{P}_B) \rightarrow \mathcal{P}_C)$.

At the level of Boolean logic, the transformation operation can be understood as a logical implication. Indeed, the transformation is equal to its inverse implication, meaning that it satisfies

\[
\mathcal{P}_A \rightarrow \mathcal{P}_B = \overline{\mathcal{P}_A} \leftarrow \overline{\mathcal{P}_B},
\]
(A50)
which comes from the definition. Additionally, note that it is equivalent to $\overline{\mathcal{P}_B} \rightarrow \overline{\mathcal{P}_A}$ since the order of the systems in the tensor product does not matter as $\mathcal{H}^A \otimes \mathcal{H}^B \cong \mathcal{H}^B \otimes \mathcal{H}^A$, we only keep it fixed to avoid adding confusion. If we define the trivial system as ‘1’, that is to say, the 1-dimensional state structure made of $\{1\}$, one can interpret the measurement (thus the negation of a given state structure) as a transformation into the trivial system. This leads to the identity

\[
\overline{\mathcal{P}_A} = \mathcal{P}_A \rightarrow 1,
\]
(A51)
which justifies the notation $\mathcal{A} \rightarrow 1 \equiv \overline{\mathcal{A}}$. The proof is straightforward from the definition since 1 is 1-dimensional: $\mathcal{P}_A \rightarrow 1 = \mathcal{P}_A \otimes I = \mathcal{P}_A$. In the same way, one can prove that

\[
\mathcal{P}_A = 1 \rightarrow \mathcal{P}_A.
\]
(A52)

Therefore, a state structure can be seen as a transformation from the trivial system to itself and its negation as a transformation from itself to the trivial system. In view of the link between composition and transformation, one may also interpret the former in terms of the latter. A bipartite system which is causally composed, $\mathcal{A} \otimes \mathcal{B}$ for instance, can actually be seen as characterized by projectors

\[
\mathcal{P}_A \otimes \mathcal{P}_B = \mathcal{P}_A \rightarrow \mathcal{P}_B = \mathcal{P}_A \leftarrow \mathcal{P}_B.
\]
(A53)
Meaning that a no signaling composite bipartite system is equivalent to a linear transformation normalized on a transformation from one side to the complement of the other. We see that the direction of the transformation has no influence, which is expected since it is a causal composition.

c. Linear Logic

We now show under which definition that the rules $\{\vee, \wedge, \neg, \rightarrow\}$ form a model of multiplicative additive linear logic (MALL) [23]. If one sees the projectors as the propositions in a sequent calculus, the set of propositions is similar to those of MALL:

1. The set from which we start is made of valid projectors;
2. For every projector $\mathcal{P}$, there exists a projector $\overline{\mathcal{P}}$;
3. For every projectors $\mathcal{P}_A$ and $\mathcal{P}_B$, there exist an additive conjunction $\mathcal{P}_A \cup \mathcal{P}_B$ as well as an additive disjunction $\mathcal{P}_A \cup \mathcal{P}_B$;
4. For every projectors $\mathcal{P}_A$ and $\mathcal{P}_B$, there exist a multiplicative conjunction $\mathcal{P}_A \otimes \mathcal{P}_B$ as well as a multiplicative disjunction $\mathcal{P}_A \rightarrow \mathcal{P}_B$;
5. There are two constants ($I, \mathcal{D}$) that go with each additive binary connectors;
6. There are two constants $(1, 1)$ that go with each multiplicative binary connectors.
And so are the properties:

1. All binary connectors are commutative;
2. Multiplicative connectors distribute over additive ones;
3. All proposition has a negation obeying the De Morgan rules \([A11]\) and \([A16]\). These where proven for each connector in the previous sections. Proposition 5 follows from equations \([A18]\) and \([A21]\). Proposition 6 happens because of the isomorphism \(L(H) \otimes C \cong 1\) so that \(P \otimes 1 = P\) and the same way, \(\overline{P} \otimes 1 = \overline{P}\). Indeed, \(1 \otimes P = \overline{P} = P\).

Property 1 is true from definition. In the case of the multiplicative connectors, \(P_A \otimes P_B\) and \(P_A \otimes P_B\), the isomorphism \(H^A \otimes H^B \cong H^B \otimes H^A\) should of course be used.

Property 2 follows from Eqs. \([A32]\) and \([A33]\) in the case of \(\otimes\) and application of the de Morgan rules \([A30]\) on these two equations can be used to prove the property in the case of \(\rightarrow\). Put differently, the multiplicative conjunction (respectively, disjunction) distributes over the additive conjunction (disjunction)

\[
P_A \otimes (P_B \cap P_B') = (P_A \otimes P_B) \cap (P_A \otimes P_B'); \\
\overline{P_A} \rightarrow (P_B \cup P_B') = (\overline{P_A} \rightarrow P_B) \cup (\overline{P_A} \rightarrow P_B').
\]

(A54a)

(A54b)

But as \(\otimes\) and \(\rightarrow\) are both operations that merge subspaces, the converse obviously does not hold. Indeed an expression like \(P_X \cap (P_A \otimes P_A)\) \(\otimes (P_X \otimes P_A)\) makes no sense as the right-hand side feature a cap between two superoperators defined on different spaces that are not isomorphic in general.

Property 3 was proven at Eq. \([A26]\) for the negation, Eq. \([A30]\) for the additive connectors, and follows from the definition in the case of multiplicative connectors.

Property 4 is the statement \(I = I = I + D = D\), whose converse hold by idempotency or can be proven by the same kind of computation.

Property 5 is the statement \(I = 1 - 1 + 1\), as in the 1-dimensional case \(D = I = 1\).

As pointed out in the main text, there is however some discrepancies with MALL, that we now list

1. The multiplicative disjunction is not associative;
2. The multiplicative units are equivalent;
3. The additive falsity \((i.e.\ the\ unit\ for\ \cup)\) is not absorbant.

Despite these particularities, the connection between the algebra of projectors and MALL is too obvious not to be pointed out.

5. Adding the prec

Define the \(A \prec B\) one-way signaling composition as

\[
P_A \prec P_B \equiv I_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B.
\]

(A55)

Compared to transformation, one-way signaling composition is better behaved in the sense that it is associative

\[
P_A \prec (P_B \prec P_C) = (P_A \prec P_B) \prec P_C.
\]

(A56)

So that \(P_A \prec (P_B \prec P_C) = P_A \prec P_B \prec P_C\) can be written unambiguously. Even more, it is better behaved than both the no signaling (the tensor) and the two-ways signaling (then transformation) compositions as it is distributive with respect to the negation

\[
\overline{P_A} \prec P_B = \overline{P_A} \prec P_B.
\]

(A57)

We first prove commutation with negation by direct computation,

\[
\overline{P_A} \prec P_B = I_A \otimes P_B - P_A \prec P_B + D_A \otimes D_B = (I_A \otimes I_B - I_A \otimes P_B + I_B \otimes D_B - P_A \otimes D_B) - D_A \otimes D_B = (I_A \otimes P_B) - \overline{P_A} \prec P_B.
\]

(A58)

This can be used to prove associativity,

\[
P_A \prec (P_B \prec P_C) = I_A \otimes (P_B \prec P_C) - (I_A \otimes P_B \prec P_C) - (I_A \otimes P_B - P_A \prec P_B) \otimes D_B + D_A \otimes D_B = (I_A \otimes P_B) - (I_A \otimes P_B - P_A \prec P_B) \otimes D_B + D_A \otimes D_B = (I_A \otimes I_B) \otimes P_C - (I_A \otimes P_B - P_A \prec P_B) \otimes D_B + D_A \otimes D_B = (I_A \otimes I_B) \otimes P_C - (I_A \otimes P_B - P_A \prec P_B) \otimes D_B + D_A \otimes D_B = (I_A \otimes I_B) \otimes P_C - (P_A \prec P_B) \otimes D_C + D_A \otimes D_B = P_A \prec P_B \prec P_C.
\]

(A59)

where we have used the definition to go to the last line, and commutation of the prec with the negation to go to the penultimate one.
Like the tensor product, one-way signaling composition with the trivial system amounts to doing nothing since $P_A \triangleleft 1 = I_A \otimes 1 - P_A \otimes 1 + D_A \otimes 1 = P_A \otimes 1$ and $1 \triangleleft P_A = P_A \otimes 1 - D_A \otimes 1 + D_A \otimes 1$ so that

$$P_A \triangleleft 1 = P_A, \quad (A60a)$$
$$1 \triangleleft P_A = P_A. \quad (A60b)$$

One can finally link the transformation (here seen as two-ways signaling composition by negating its input), one-way signaling, and no signaling compositions by noticing that

$$(P_A \triangleleft P_B) \cap (P_A \triangleright P_B) = P_A \otimes P_B, \quad (A61)$$

and

$$(P_A \triangleleft P_B) \cup (P_A \triangleright P_B) = P_A \rightarrow P_B. \quad (A62)$$

First equation is proven by working it. $$(P_A \triangleleft P_B) \cap (P_A \triangleright P_B) = (I_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B) \cap (P_A \otimes I_B - D_A \otimes P_B + D_A \otimes D_B),$$ and noting that the intersection of any two elements but $(I_A \otimes P_B) \cap (P_A \otimes I_B) = P_A \otimes P_B$ is giving $D_A \otimes D_B$. Thus, the expression reduces to $P_A \otimes P_B$ followed by eight occurrences of $D_A \otimes D_B$ alternating between a plus and minus sign, therefore canceling each other. The second equation has a quick proof using the De Morgan rule and the associativity of the negation:

$$(P_A \triangleleft P_B) \cup (P_A \triangleright P_B)$$

$$= (P_A \otimes P_B) \cup (P_A \otimes P_B)$$

$$= (P_A \otimes P_B) \otimes (P_A \otimes P_B)$$

$$= P_A \rightarrow P_B. \quad (A63)$$

a. Distribution properties of the prec

Cap and prec satisfy an interchange law,

$$(P_A \cap P_A') \triangleleft (P_B \cap P_B') = (P_A \triangleleft P_B) \cap (P_A' \triangleleft P_B'). \quad (A64)$$

This can be shown as follows:

$$(P_A \cap P_A') \triangleleft (P_B \cap P_B')$$

$$= I_A \otimes (P_B \cap P_B') - (P_A \cap P_A') \otimes D_B + D_A \otimes D_B$$

$$= I_A \otimes (P_B \cap P_B') - (P_A \cup P_A') \otimes D_B + D_A \otimes D_B$$

$$= I_A \otimes (P_B \cap P_B') - P_A \otimes D_B + D_A \otimes D_B$$

$$= (I_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B)$$

$$\cap (I_A \otimes P_B' - P_A' \otimes D_B + D_A \otimes D_B)$$

$$= (P_A \triangleleft P_B) \cap (P_A' \triangleleft P_B'). \quad (A65)$$

Remark that the grouping at the penultimate line is arbitrary. In other terms, because the cap is commutative we have $(P_A \cap P_A') \triangleleft (P_B \cap P_B') = (P_A \cap P_A') \triangleleft (P_B \cap P_B) = (P_A \cap P_A') \triangleleft (P_A \cap P_B).$ Hence, the exact grouping does not matter as long as the A’s and B’s are on the correct side of the prec connector.

The cup and the prec satisfy an interchange law as well,

$$(P_A \cup P_A') \triangleleft (P_B \cup P_B') = (P_A \triangleleft P_B) \cup (P_A' \triangleleft P_B'). \quad (A66)$$

This can be shown the same way,

$$(P_A \triangleleft P_B) \cup (P_A \triangleleft P_B') =$$

$$= (I_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B)$$

$$\cup (I_A \otimes P_B' - P_A' \otimes D_B + D_A \otimes D_B)$$

$$= (I_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B)$$

$$+ (I_A \otimes P_B' - P_A' \otimes D_B + D_A \otimes D_B)$$

$$= (I_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B)$$

$$\cup (I_A \otimes P_B' - P_A' \otimes D_B + D_A \otimes D_B)$$

$$= I_A \otimes P_B + I_A \otimes P_B' - I_A \otimes (P_B \cap P_B')$$

$$= (P_A \cup P_A') \otimes D_B - D_A \otimes D_B$$

$$= I_A \otimes P_B + I_A \otimes P_B' - I_A \otimes (P_B \cap P_B')$$

$$= (P_A \cup P_A') \otimes D_B + D_A \otimes D_B$$

$$= (P_A \cup P_A') \triangleleft (P_B \cup P_B'). \quad (A67)$$

These two interchange laws imply distributivity as a special case because the elements in the algebra are all idempotent, so that $P = P \cap P = P \cup P$ and we can do the following: $(P_A \cup P_A') \triangleleft P_B = (P_A \triangleleft P_B) \cup (P_A' \triangleleft P_B) = (P_B \cup P_B) \cap (P_A \triangleleft P_B) \cup (P_A' \triangleleft P_B).$ Therefore, the following relations hold:

$$(A \cap A') \triangleleft B = (A \triangleleft B) \cap (A' \triangleleft B), \quad (A68a)$$

$$(A \cup A') \triangleleft B = (A \triangleleft B) \cup (A' \triangleleft B), \quad (A68b)$$

$$A \triangleleft (B \cup B') = (A \triangleleft B) \cup (A \triangleleft B'), \quad (A68c)$$

$$A \triangleleft (B \cup B') = (A \triangleleft B) \cap (A \triangleleft B'). \quad (A68d)$$

b. Isomorphism in the case of quantum theory

The set inclusions at Eqs. can become equivalences in the special cases were the projectors are either identity or depolarizing. These imply set isomorphisms that are not without consequences: subsets with different signaling constraints get accidentally equivalent.

In some cases, the two-ways and one-way signaling compositions coincide. Putting an identity on either side of the $\rightarrow$ gives

$$P_A \rightarrow I_B = P_A \otimes D_B. \quad (A69)$$
and
\[ \mathcal{I}_A \to \mathcal{P}_B = \mathcal{I}_A \otimes \mathcal{P}_B. \]  
(A70)

As it can be shown directly from the definitions. The same way,
\[ \mathcal{P}_A \prec \mathcal{I}_B = \overline{\mathcal{P}}_A \to \mathcal{I}_B, \]  
(A71)

and
\[ \mathcal{I}_A \prec \mathcal{P}_B = \mathcal{I}_A \otimes \mathcal{P}_B. \]  
(A72)

One has then the following identities:
\[ \mathcal{P}_A \prec \mathcal{I}_B = \mathcal{P}_A \to \mathcal{I}_B; \]  
(A73a)

\[ \mathcal{D}_A \prec \mathcal{P}_B = \mathcal{I}_A \to \mathcal{P}_B. \]  
(A73b)

These relations can be concisely recast as
\[ \mathcal{P}_A \prec \mathcal{P}_B = \mathcal{P}_A \to \mathcal{P}_B \iff \mathcal{P}_A = \mathcal{I}_A \text{ or } \mathcal{P}_B = \mathcal{I}_B. \]  
(A74)

In addition to that, one-way signaling composition is equivalent to the no-signaling composition in the following cases:
\[ \mathcal{P}_A \prec \mathcal{D}_B = \mathcal{P}_A \otimes \mathcal{D}_B; \]  
(A75a)

\[ \mathcal{I}_A \prec \mathcal{P}_B = \mathcal{I}_A \otimes \mathcal{P}_B. \]  
(A75b)

Again, this directly follow from the definition and this can be concisely recast as
\[ \mathcal{P}_A \prec \mathcal{P}_B = \mathcal{P}_A \otimes \mathcal{P}_B \iff \mathcal{P}_A = \mathcal{D}_A \text{ or } \mathcal{P}_B = \mathcal{D}_B. \]  
(A76)

See Fig. 9 for a diagrammatic interpretation of these properties in comparison to other composition rules and the general case.

It must be noted that condition (A76) is stronger than (A74). Actually, when both conditions are satisfied at once, one has either of the two identities:
\[ \mathcal{D}_A \prec \mathcal{D}_B = \mathcal{D}_A \otimes \mathcal{D}_B; \]  
(A77a)

\[ \mathcal{I}_A \prec \mathcal{I}_B = \mathcal{I}_A \otimes \mathcal{I}_B. \]  
(A77b)

The reason this is the case comes from isomorphism (A45), which reduces the transformation into a no signaling composition. Indeed,
\[ \mathcal{D}_A \prec \mathcal{D}_B \overset{A45}{=} \mathcal{I}_A \to \mathcal{D}_B \overset{A45}{=} \mathcal{I}_A \otimes \mathcal{D}_B \]  
\[ = \mathcal{I}_A \otimes \mathcal{D}_B \overset{A45}{=} \mathcal{D}_A \otimes \mathcal{D}_B. \]  
(A78)

And the same way,
\[ \mathcal{I}_A \prec \mathcal{I}_B \overset{A74}{=} \mathcal{D}_A \to \mathcal{I}_B \overset{A45}{=} \mathcal{D}_A \otimes \mathcal{I}_B \]  
\[ = \mathcal{D}_A \otimes \mathcal{I}_B \overset{A45}{=} \mathcal{I}_A \otimes \mathcal{I}_B. \]  
(A79)

Therefore, isomorphisms (A45), (A74), and (A76) are giving the conditions for set equivalences in the composition rules.

Appendix B: Proofs

1. Proof of Proposition 1

Positivity follows from the fact that \( c_A/d_A \mathbb{1} \) is a valid element of \( \mathcal{A} \subset \mathcal{L}(\mathcal{H}^A) \). Then \( \langle \bar{A}, c_A/d_A \sum_i |i\rangle\langle i| \rangle = 1 \) and it must be \( \langle \bar{A}, |i\rangle\langle i| \rangle \geq 0 \) due to the linearity of the inner product. Since \( \{ |i\rangle \} \) is an orthonormal basis of \( \mathcal{H}^A \) and that \( \bar{A} \) is arbitrary, this proves positivity of all \( \bar{A} \), condition (9a).

The trace normalization is obtained by computing the above inner product, \( 1 = \langle \bar{A}, c_A/d_A \mathbb{1} \rangle = \text{Tr} \left[ \bar{A} \cdot c_A/d_A \mathbb{1} \right] = c_A/d_A \text{Tr} \left[ \bar{A} \right] \). Since \( \bar{A} \) is arbitrary, this fixes the normalization for all elements of \( \mathcal{A} \): \( c_A = d_A/c_A = \text{Tr} \left[ \bar{A} \right] \), this is condition (9b).

Finally, the projective condition remains. First, \( 1/c_A \mathbb{1} \)
must belong to $\mathcal{A}$ as it is positive, properly normalized, and it respects Tr $[1/c_A 1 \cdot A] = 1$ for all $A \in \mathcal{A}$. Assuming that $\mathcal{A}$ is a deterministic state structure with projector $\mathcal{P}_A = I - P_A + D$, the if part follows: any positive and properly normalized operator $A$ in $\mathcal{L}(\mathcal{H}^A)$ on which the projector yields $I - P_A + D$ is mapped as a projector in our language by the substitution $X, \lambda \rightarrow c_A, \lambda \rightarrow c_A$. The projector $\mathcal{P}_A$ is applied obeys $\langle P_A \{ A \} , A \rangle = \langle (I - P_A) \{ A \} , A \rangle + \langle D\{ A \} , A \rangle$. The first member on the right part of the equality vanishes because it belongs to the orthogonal complement of $\mathcal{A}$; the second member is normalized so that $\langle D\{ A \} , A \rangle = \langle \text{Tr} \{ A \} /d_A 1 , A \rangle = \left\langle \frac{d_A}{c_A d_A} 1 , A \right\rangle = 1/c_A \text{Tr} [ A ] = 1$.

The only if part is proven by a counterexample: assume that there is a positive $X$ with trace norm $c_A = d_A/c_A$ that does not belong to the space characterized by $\mathcal{P}_A$. Then, $\{ I - \mathcal{P}_A \} \{ X \} = X$. Computing the projector yields $\mathcal{I} - \mathcal{P}_A = \mathcal{P}_A - D$. Now, let $A = c_A d_A 1$, we should have $\langle X , A \rangle = \langle X , c_A /d_A 1 \rangle = 1$. Yet, by applying the projector on $X$ it should be that $\langle (P_A - D) \{ X \} , c_A /d_A 1 \rangle = 1$, hence $\langle P \{ A \} , 1 \rangle = 2d_A/c_A$ becomes $\langle X , P \{ 1 \} \rangle = \langle X , 1 \rangle = \text{Tr} [ X ] = 2d_A/c_A$. However, the trace norm of $X$ was assumed to be $c_A = d_A/c_A$, a contradiction has been reached. This concludes the proof.

2. Proof of Proposition 2

Translating the proof in Ref. 3 in our language, we first make the identification of types with state structures. Thus, to a type $A \otimes B$ will correspond a state structure $\mathcal{A} \otimes \mathcal{B}$. Positivity and normalization are directly obtained (under the substitution $\lambda_X \rightarrow c_A, \lambda_Y \rightarrow c_A$). It remains to characterize the ‘$L_b$’ spaces elements in terms of projectors. Realizing that these subspaces actually correspond to the image of some $\mathcal{P} - D$, condition

$$\Delta_{x \otimes y} = (L_c \otimes \Delta_y) \oplus (\Delta_x \otimes \Delta_y) \oplus (\Delta_x \otimes L_c)$$

is mapped as a projector in our language by the substitutions: $\Delta_x \rightarrow \mathcal{P}_A - D_A, L_c \rightarrow D, \otimes \rightarrow \otimes, \oplus \rightarrow \cup$. See Appendix A 2 for the definition of the $\oplus$ symbol. The above equation becomes

$$(P - D)_{A \otimes B} = (D_A \otimes (P_B - D_B))$$

\begin{equation}
\cup ((P_A - D_A) \otimes (P_B - D_B)) \cup ((P_A - D_A) \otimes D_B)
\end{equation}

Reordering, extending, and simplifying,

$$\begin{align*}
(P - D)_{A \otimes B} &= ((P_A - D_A) \otimes (P_B - D_B)) \cup \\
&\cup ((D_A \otimes (P_B - D_B)) \cup ((P_A - D_A) \otimes D_B))
\end{align*}$$

$$\begin{align*}
&= (P_A \otimes P_B - D_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B) \cup \\
&\cup ((D_A \otimes P_B - D_A \otimes D_B) \cup (P_A \otimes D_B - D_A \otimes D_B))
\end{align*}$$

$$\begin{align*}
&= (P_A \otimes P_B - D_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B) \cup \\
&\cup ((D_A \otimes P_B - D_A \otimes D_B) + (P_A \otimes D_B - D_A \otimes D_B)) .
\end{align*}$$

(B3)

Where in the last line the $\cup$ becomes $\cap$ since $(D_A \otimes P_B - D_A \otimes D_B) \cap (D_A \otimes D_B - D_A \otimes D_B) = 0$. Moreover, as $(P_A \otimes P_B - D_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B) \cap (D_A \otimes P_B - D_A \otimes D_B + P_A \otimes D_B - D_A \otimes D_B) = 0$ as well the above becomes

$$\begin{align*}
(P_A \otimes P_B - D_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B) \cup \\
&\cup (D_A \otimes P_B - D_A \otimes D_B + P_A \otimes D_B - D_A \otimes D_B)
\end{align*}$$

$$\begin{align*}
&= (P_A \otimes P_B - D_A \otimes P_B - P_A \otimes D_B + D_A \otimes D_B) + \\
&\cup (D_A \otimes P_B - D_A \otimes D_B + P_A \otimes D_B - D_A \otimes D_B)
\end{align*}$$

$$\begin{align*}
&= P_A \otimes P_B - D_A \otimes D_B .
\end{align*}$$

(B4)

Hence,

$$\begin{align*}
(P - D)_{A \otimes B} &= P_A \otimes P_B - D_A \otimes D_B .
\end{align*}$$

(B5)

which therefore yields the sought form of the projector, $\mathcal{P}_{A \otimes B} = P_A \otimes P_B$.

3. Proof of Proposition 3

Positivity, Eq. (15a), directly follows from $\mathcal{M}$ being CP by Choi-Kraus theorem. For the trace and the projector, note that since $\mathcal{M}(A) \in \mathcal{B}$ it must obey $\langle B, \mathcal{M}(A) \rangle_{\mathcal{L}(\mathcal{H}^B)} = 1$ for all $B \in \mathcal{B}$. Hence,

$$\langle B, \mathcal{M}(A) \rangle_{\mathcal{L}(\mathcal{H}^B)} = \langle M, A \otimes B^T \rangle_{\mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B)} = 1 ,$$

(B6)

because of the cyclic property of the trace, and because $B$ is positive therefore self-adjoint. The above equation is valid for all $B \in \mathcal{B}$ and $A \in \mathcal{A}$. By Proposition 2 $A \otimes B^T \in \mathcal{A} \otimes \mathcal{B}$, so that Proposition 2 implies $\{ M \} \subseteq \mathcal{A} \otimes \mathcal{B}$. The equality of both sets follows by noticing that Eq. (15a) is still valid under affine combinations of different choices of $B$ and $A$. Conditions (15b) and (15c) follow as a consequence.

4. Quasi-orthogonality is local no signaling

Here we generalize equation (10) into party-wise constraints, so that we can prove that the state structure
built using the tensor composition of two state structures is no signaling between both parties.

Translating this statement into mathematical conditions, if Alice chooses her measurement settings depending on a classical random variable $x$ and gets classical outcome $a$ and if Bob does the same with settings $y$ and outcome $b$ we want the following constraints on their correlations:

$$\forall y, y', \sum_b p(a, b|x, y) = \sum_b p(a, b|x, y'); \quad (B7a)$$

$$\forall x, x', \sum_a p(a, b|x, y) = \sum_a p(a, b|x', y). \quad (B7b)$$

This is the standard statement that no local measurement scheme can be used to deterministically gain knowledge of the other party's actions. The first condition, Eq. (B7a), states that Alice’s distribution of outcome $a$ cannot be used to know which setting $y$ Bob has used, thus that the measurement result of Alice cannot be used to guess Bob’s own choice of measurement. These correlations are no-signaling from Bob to Alice. Respectively, the second condition, Eq. (B7b), defines no-signaling from Alice to Bob correlations.

For formulating this statement in the formalism developed in this article, assume Alice’s and Bob’s operators to be causal as in Proposition 3 (otherwise, they can be trivially signaling to one another just by using the non-commutative nature of their operations). Thus, their joint effects are resolving a state structure $\mathcal{F}$ having the inner structure $\mathcal{F} = \mathcal{A} \otimes \mathcal{B}$. The question reduces to finding what kind of states $W$ Alice and Bob can locally measure so that their outcome distributions are no signaling.

Let $W$ be a state shared by Alice and Bob, $W \in \mathcal{F} \cong \mathcal{A} \otimes \mathcal{B}$. A measurement of the part of the state that is under control of Alice is represented by a collection $\{ \tilde{A}_{a|x} \}$ resolving some operator $\tilde{A}_x \in \mathcal{A}^2$. The ‘$x$’ subscript notation is here to emphasise on that $x$ can condition the choice of the resolved operator as well as the choice of how it splits into effects, depending on Alice’s choice of strategy. The same way, Bob’s measurement is is no longer needed, and one can simply consider different $B \in \mathcal{B}$. Rewriting it as

$$\forall \tilde{B}, \tilde{B}', \quad \text{Tr}_A \left[ \tilde{A}_{a|x} \cdot \text{Tr}_B \left[ (1 \otimes B) \cdot W \right] \right] = \text{Tr}_A \left[ \tilde{A}_{a|x} \cdot \text{Tr}_B \left[ (1 \otimes B') \cdot W \right] \right], \quad (B10)$$

one can simplify further by noticing that the possible $\tilde{A}_{a|x}$ actually range over the whole of $\mathcal{L} (\mathcal{H}_A^2)$ (up to trace normalisation of course), so that only the trace over $B$ part is relevant:

$$\forall \tilde{B}, \tilde{B}', \quad \text{Tr}_B \left[ (1 \otimes \tilde{B}) \cdot W \right] = \text{Tr}_B \left[ (1 \otimes \tilde{B}') \cdot W \right]. \quad (B11)$$

Finally, as $1/c_B$ is a valid element of $\overline{\mathcal{F}}$, we can replace $\tilde{B}'$ by $1$ to obtain the shortened

$$\forall \tilde{B}, \quad \text{Tr}_B \left[ (1 \otimes \tilde{B}) \cdot W \right] = \frac{1}{c_B} \text{Tr}_B \left[ (1 \otimes 1) \cdot W \right]. \quad (B12)$$

Using Proposition 1, observe that $1/c_B = \text{Tr}_B [\tilde{B}] / d_B$ because of Eq. (B9b), we can thus make $\text{Tr}_B [\tilde{B}]$ appear in the right hand side and reach the desired formulation. Doing the same reasoning for condition (B7b), we obtain the following rephrasing of Eq. (B7):

$$\forall \tilde{A}, \quad \text{Tr}_A \left[ (\tilde{A} \otimes 1) \cdot W \right] = \frac{\text{Tr}_A [\tilde{A}] \text{Tr}_A [W]}{d_A}. \quad (B13b)$$

Hence, it was proven that no signaling can be recast into the conditions $\text{Tr}_A [\tilde{A}] = 0$, which amounts to requiring quasi-orthogonality for only one of the tensor factors of an operator. Next, we do the projective characterization of these conditions.

Making this into a general statement, let $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $M \in \mathcal{L} (\mathcal{H}_A \otimes \mathcal{H}_B)$ so that $M$ is positive and of non-zero trace. We want the following to hold:

$$\forall A, \quad \text{Tr}_A [M \cdot (A \otimes 1)] = \text{Tr}_A \left[ M \cdot \left( c_A \frac{1}{d_A} \otimes 1 \right) \right]. \quad (B14)$$

5. Proof of Theorem 11

Let $\{ \sigma_i^X \}$ be a basis of $\mathcal{L} (\mathcal{H}_X^2)$ so that

$$\langle \sigma_i^X, \sigma_j^X \rangle = \delta_{i,j}; \quad (B15a)$$

$$\sigma_0^X = 1; \quad (B15b)$$

$$\text{Tr} \left[ \sigma_i^X \sigma_j^X \delta_{0} \right] = 0; \quad (B15c)$$

$$\text{Tr} \left[ \sigma_i^X \sigma_j^X \right] = \delta_{i,j}. \quad (B15d)$$

We expand $M$ and $A$ using this basis, let $M = \sum_{i,j} m_{ij} \sigma_i^X \otimes \sigma_j^Y$; $A = a_0 \sigma_0^X + \sum_{k \in \mathcal{A}} a_k \sigma_k^X$. By direct
computation:
\[
\text{Tr}_A [M] = d_A \sum_j m_{0j} \sigma_j^B ; \quad (B16)
\]
\[
\text{Tr}_A [A] = d_A a_0 ; \quad (B17)
\]
\[
\text{Tr}_A \left[ M \cdot (A \otimes B) \right] = \left( \sum_{i \in \mathcal{A}} \sum_j d_A m_{ij} a_i \sigma_j^B \right) \cdot B. \quad (B18)
\]
Combining Eqs. (B16) and (B17) in the right-hand side of Eq. (29), and (B18) in its left-hand side, the equality is true if
\[
\left( \sum_{i > 0 \in \mathcal{A}} \sum_j m_{ij} a_i \sigma_j^B \right) \cdot B = 0 , \quad (B19)
\]
because \( B \) is arbitrary and both \( M \) and \( A \) are positive, this equation vanish for all \( B \) and \( A \) if and only if \( m_{ij} = 0 \) whenever \( i > 0 \) corresponds to a basis element of \( \mathcal{A} \). This is equivalent to requiring Eq. (31) on \( M \), concluding the proof.

6. When quantum comb equates quantum network

Here we show that in the case where \( \mathcal{P} = \mathcal{I} \), a quantum network based on quantum channels, a quantum comb (based on channels) and a comb base on states are equivalent.

Recall that a quantum channel \( \mathcal{A} \subset \mathcal{L} (\mathcal{H}_A \otimes \mathcal{H}_A) \) is defined by projector \( \mathcal{I}_A \to \mathcal{I}_A \), and a quantum state \( \mathcal{A} \subset \mathcal{L} (\mathcal{H}_A) \) is defined by projector \( \mathcal{I}_A \).

We start by defining the projectors associated with the higher-order contructions, let
\[
\mathcal{P}^{(\text{n-network})}_{\mathcal{A}_{\text{channel}}} \equiv (\mathcal{I}_A \to \mathcal{I}_A) \prec \cdots \prec (\mathcal{I}_{A_{2n-2}} \to \mathcal{I}_{A_{2n-1}}) , \quad (B20)
\]
so that \( \mathcal{P}^{(\text{1-network})}_{\mathcal{A}_{\text{channel}}} \equiv \mathcal{I}_A \to \mathcal{I}_A ; \) let
\[
\mathcal{P}^{(\text{n-comb})}_{\mathcal{A}_{\text{state}}} \equiv (\cdots (\mathcal{I}_A \to \mathcal{I}_A) \to \cdots ) \to \mathcal{I}_{A_{n-1}} , \quad (B21)
\]
so that \( \mathcal{P}^{(\text{1-comb})}_{\mathcal{A}_{\text{state}}} \equiv \mathcal{I}_A ; \) and let
\[
\mathcal{P}^{(\text{n-comb})}_{\mathcal{A}_{\text{channel}}} \equiv (\cdots (\mathcal{I}_{A_{n-1}} \to \mathcal{I}_{A_{n}}) \to \cdots ) \to (\mathcal{I}_A \to (\mathcal{I}_{A_{2n-1}}) , \quad (B22)
\]
so that \( \mathcal{P}^{(\text{1-comb})}_{\mathcal{A}_{\text{channel}}} \equiv \mathcal{I}_A \to \mathcal{I}_A . \quad (A41) \)

As we now prove, projective relations through the isomorphisms \( A45 \), \( A76 \), and \( A74 \) can explain why these comb-like constructions are equivalent the network construction. Note that the claim for the case of combs was first proven in Ref. [E] using type formalism.

Theorem 2. The following equivalence holds:
\[
\mathcal{P}^{(\text{n-network})}_{\mathcal{A}_{\text{channel}}} = \mathcal{P}^{(\text{n-comb})}_{\mathcal{A}_{\text{channel}}} = \mathcal{P}^{(2n-\text{comb})}_{\mathcal{A}_{\text{state}}} . \quad (B23)
\]
Meaning that with suitable normalisation, the state structure of networks of order \( n \) with a base of quantum channels, is equivalent to the one of combs of order \( n \) with a base of quantum channels, and it is also equivalent to the one of combs of order \( 2n \) with a base of quantum states.

Proof. We want to show the following:
\[
(\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \prec \cdots \prec (\mathcal{I}_{A_{2n-2}} \to \mathcal{I}_{A_{2n-1}}) \quad = \mathcal{P}^{(\text{1-network})}_{\mathcal{A}_{\text{channel}}} \quad (B24a)
\]
\[
(\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \prec \cdots \prec (\mathcal{I}_{A_{2n-2}} \to \mathcal{I}_{A_{2n-1}}) \quad = \mathcal{P}^{(\text{2-comb})}_{\mathcal{A}_{\text{state}}} \quad (B24b)
\]
\[
(\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \prec \cdots \prec (\mathcal{I}_{A_{2n-2}} \to \mathcal{I}_{A_{2n-1}}) \quad = \mathcal{P}^{(\text{2-comb})}_{\mathcal{A}_{\text{channel}}} \quad (B24c)
\]
The first equality directly follows from the application of Eq. (A74). This can be used to recursively prove the second one. Indeed, observe that the 1-comb is characterised by \( \mathcal{I}_{A_0} \to \mathcal{I}_{A_1} \), which is equivalent to
\[
\mathcal{P}^{(\text{2-comb})}_{\mathcal{A}_{\text{state}}} \equiv \mathcal{I}_{A_0} \to \mathcal{I}_{A_1} \quad (A74) \]
Then, each iteration of combs satisfies the latter condition of Eq. (A74). We prove it for \( \mathcal{P}^{(\text{4-comb})}_{\mathcal{A}_{\text{state}}} \) below:
\[
\mathcal{P}^{(\text{4-comb})}_{\mathcal{A}_{\text{state}}} \equiv ((\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \to (\mathcal{I}_{A_2} \to \mathcal{I}_{A_3}) \to \mathcal{I}_{A_4}) \quad (A56) \]
\[
\mathcal{P}^{(\text{4-comb})}_{\mathcal{A}_{\text{state}}} \equiv (\mathcal{I}_{A_0} \to \mathcal{I}_{A_1} \to \mathcal{I}_{A_2} \to \mathcal{I}_{A_3}) \quad (A57) \]
Where we used the associativity of one-way signaling composition, Eq. (A56), and distributed negation, Eq. (A57), to simplify in between each step.

The proof for the \( n \)-comb directly follows by induction on the above computation. Suppose it holds for \( n \), \( \mathcal{P}^{(\text{n-network})}_{\mathcal{A}_{\text{channel}}} = \mathcal{P}^{(\text{2n-comb})}_{\mathcal{A}_{\text{state}}} \) then
\[
\mathcal{P}^{(\text{(n+1)-comb})}_{\mathcal{A}_{\text{state}}} \equiv (\mathcal{P}^{(\text{2n-comb})}_{\mathcal{A}_{\text{state}}} \to (\mathcal{I}_{A_{2n}} \to \mathcal{I}_{A_{2n+1}}) \quad = \mathcal{P}^{(\text{2n-comb})}_{\mathcal{A}_{\text{channel}}} \quad (B27)
where the hypothesis was injected in between the antepenultimate and penultimate lines as well as identity \(\mathcal{I}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}} = (\mathcal{I}_{A_{2n}} \rightarrow \mathcal{I}_{A_{2n+1}})\).

Finally, and for completeness we prove the equivalence \(\text{[B24c]}, \text{ although it follows because Eqs. \text{[B24b]} and \text{[B24c]} were proven to be the same elsewhere }\ [53 \text{ Prop. } 6]\). Again we use induction. It holds by definition for the \(n = 1\) case, suppose it holds for \(n\), \(\mathcal{P}(n\text{-comb}) = (\mathcal{I}_{A_0} \prec \mathcal{I}_{A_1} \prec \ldots \prec \mathcal{I}_{A_{2n-2}} \prec \mathcal{I}_{A_{2n-1}})\), and define the relabelling \(\mathcal{P}(n\text{-comb})' = (\mathcal{I}_{A_1} \prec \mathcal{I}_{A_2} \prec \ldots \prec \mathcal{I}_{A_{2n-1}} \prec \mathcal{I}_{A_{2n}})\) where all indices have been incremented by 1. Then,

\[
\mathcal{P}(2(n+1)\text{-comb}) = (\ldots (\mathcal{I}_{A_n} \rightarrow \mathcal{I}_{A_{n+1}}) \rightarrow \ldots) \rightarrow (\mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_{2n+1}})
\]

\[
= \mathcal{P}(n\text{-comb})' \rightarrow (\mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_{2n+1}})
\]

\[\mathcal{A}_{60} (\mathcal{I}_{A_0} \otimes \mathcal{P}_{\text{state}}(n\text{-comb})') \rightarrow \mathcal{I}_{A_{2n+1}}
\]

\[\mathcal{A}_{76} (\mathcal{I}_{A_0} \prec \mathcal{P}(n\text{-comb})') \rightarrow \mathcal{I}_{A_{2n+1}}
\]

\[\mathcal{A}_{74} \mathcal{I}_{A_0} \prec \mathcal{P}(n\text{-comb})' \prec \mathcal{I}_{A_{2n+1}}
\]

\[= \mathcal{I}_{A_0} \prec \mathcal{P}(n\text{-comb})' \prec \mathcal{I}_{A_{2n+1}}
\]

\[= \mathcal{I}_{A_0} \prec \mathcal{I}_{A_1} \prec \ldots \prec \mathcal{I}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}}
\]

\[= \mathcal{I}_{A_0} \prec \mathcal{I}_{A_1} \prec \ldots \prec \mathcal{I}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}}
\]. \(\text{(B28)}\)

\[\square\]

Appendix C: Examples

To compute probabilities, it becomes interesting to consider families of probabilistic effects resolving an effect. By resolving some state structure \(\mathcal{A}\), we mean that a probabilistic effect \(E_i\) must belong to at least one family \(\{E_i\}\) such that \(\sum_i E_i = \hat{A} \in \mathcal{A}\). This generalizes the idea of POVM elements in quantum theory, for which the POVM are resolutions of the identity \(\sum_i E_i = 1\). Formally,

**Definition 6** (Resolution of a state structure). Let \(\mathcal{A}\) be a state structure in \(\mathcal{L}(\mathcal{H}^A)\). The set of operators resolving \(\mathcal{A}\) is the set of all collections of positive operators summing up to an element of \(\mathcal{A}\). That is, a set of operator \(\{E_i\}\) is a resolution \(\mathcal{A}\) if

\[
\forall E_i \in \{E_i\} \subset \mathcal{L}(\mathcal{H}^A), \exists A \in \mathcal{A} : E_i \geq 0,
\]

\[
E_i + \sum_{j \neq i} E_j = A.
\] \(\text{(C1)}\)

1. Motivating example: Outline of quantum theory as a higher-order theory

The purpose of this article is to characterize higher-order quantum theories which can colloquially described as the theory of ‘quantum transformation of quantum transformations’. Up until now, quantum theory was always taken as the lowest order, while quantum channel -the quantum theory of transformations of quantum states- was the next order. Higher-order theories can then be seen as the next orders one can define. Recent developments in the foundations of quantum theory are indeed based on specific instances of such theories. To help the reader grasp this point, this section will recover single partite process matrix formalism \(\text{[S]}\) by postulating the existence of transformations of a quantum state. This will help single out the spirit of the construction of a higher-order theory, which in turn are insightful in the derivation of the general characterization methods this article is about.

In the following the Greek letter \(\rho\) will be used to refer to arbitrary states of the usual quantum theory (of mixed state), whereas \(E_i\) will refer to an effect of a positive operator valued measurement (POVM) elements of index \(i\) (see e.g. Ref. \(\text{[25]}\) Sec. 2.2.6)). This is specifically at odds with the convention of the main text so to single them out from general states and effects. Quantum states, \(\rho\), are positive operators of trace 1:

\[
\text{States} \equiv \{\rho \in \mathcal{L}(\mathcal{H}) | \rho \geq 0, \text{Tr } [\rho] = 1\} . \hspace{1cm} \text{(C2)}
\]

Any general measurement that can be applied to a state is represented by a collection of effects constituting a POVM, \(\{E_i\}\), where each element are being associated with a particular outcome \(i\) of the measurement. These are represented by positive operators that sum up to identity:

\[
\text{Effects} \equiv \left\{E_i \in \mathcal{L}(\mathcal{H}) | E_i \geq 0 \forall i, \sum_i E_i = 1 \right\} . \hspace{1cm} \text{(C3)}
\]

The probability of seeing the measurement outcome \(i\) after measuring state \(\rho\) is obtained as a Hilbert-Schmidt inner product through the Born rule:

\[
p(i|\rho) = \langle E_i, \rho \rangle \equiv \text{Tr } [E_i^\dagger \cdot \rho] . \hspace{1cm} \text{(C4)}
\]

Note that \(i\) does not depend upon which set \(\{E_i\}\) the operator \(E_i\) was taken from: this is non-contextuality (see e.g. Ref. \(\text{[36]}\) Chap. 7)). In other words, albeit the \(E_i\) operator actually belongs to several POVMs, the particular choice of which POVM was implemented as the actual measurement scheme does not influence the Born rule.

A quantum channel is the most general form of dynamics one can have between quantum states \(\text{[H]}\). For a given output space \(B\), the set of channels is the set of CPTP maps from \(A\) to \(B\):

\[
\text{Channels} \equiv \{\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}^A), \mathcal{L}(\mathcal{H}^B)) | \mathcal{M} \text{ CPTP} \} . \hspace{1cm} \text{(C5)}
\]

When \(\mathcal{H}^A \cong \mathcal{H}^B\) so that there is a one-to-one correspondence between the channels and their adjoints, the dynamics induce two equivalent points of view, depending
on whether the map is applied to the state or the effect (respectively named ‘Schrödinger’ and ‘Heisenberg’ point of view [27]). This results in two equivalent probability rules in the presence of a superoperator dynamics:

\[ p(i|\rho, M) = \langle E_i, M(\rho) \rangle = \langle M^\dagger(E_i), \rho \rangle \quad \text{(C6)} \]

Promoting measurements as dynamical processes themselves result in Quantum instrument formalism [27]. Effects are turned into instruments (in the Heisenberg picture) by the rule

\[ E_i = M_i^\dagger (\mathbb{1}) \quad \text{(C7)} \]

so that that the Born rule (C4) is given by

\[ \langle E_i, \rho \rangle = \langle \mathbb{1}, M_i(\rho) \rangle \quad \text{(C8)} \]

Combining Eq. (C7) with the definition of effects (C3), one see that the set \( \{M_i^\dagger\} \) is made of positive preserving and subunital maps, meaning that the \( M_i^\dagger \)'s are positive preserving and trace non-increasing. As the elements of the set must sum up to valid dynamics, the set of instruments has to be CP. This yields the definition of a quantum instrument as a collection of CP trace non-increasing linear maps that sum up to a quantum channel (i.e. a CPTP map).

However, this actually induce an extra degree of freedom in the representation. When summing over all \( i \) for the instrument defined in Eq. (C7), one has

\[ \mathbb{1} = \sum_i M_i^\dagger (\mathbb{1}) = M^\dagger (\mathbb{1}) \quad \text{(C9)} \]

which is a weaker constraint than \( \mathbb{1} = \sum_i E_i \); there are in general more than one family of CP subunital maps \( \{M_i^\dagger\} \) resulting in the same instrument. As a consequence, the definition of a quantum instrument depends upon which CPTP map the instruments sum up to, in contrast to a POVM which effects always sum up to the same operator \( \mathbb{1} \). The generalized Born rule, when depending on the choice of the unit map \( M^\dagger \), is exactly equation (C6). Physically, quantum instruments are defined up to a deterministic transformation of the state: the party measuring the state (or the one preparing it, depending on the adopted picture) is free to apply a deterministic transformation on it beforehand (or afterwards).

This unconventional presentation of operational quantum theory is actually the prototype of how to build a higher-order theory: we are only one step away of promoting regular quantum theory into a higher-order theory. That is, a static theory (i.e. without dynamics, only featuring states and effects) that define transformations of states and effects as ‘higher-order’ states and effects.

Recasting (C6) into the CJ picture, so that \( M_i \) is mapped to \( M_i \) by Eq. (1), the probability rule becomes

\[ p(i|\rho, M) = \langle M_i, \rho \otimes \mathbb{1} \rangle \quad \text{(C10)} \]

an inner product picture emerges again. By the properties of CJ correspondence, we see that the \( M_i \)'s behave like effects: they are a collection of positive operators resolving certain fixed operators \( M \),

\[ \text{Instruments} \equiv \left\{ M_i \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B) \mid M_i \geq 0 \forall i, \exists M \text{ CPTP} : \sum_i M_i = M \right\} \quad \text{(C11)} \]

where these \( M \)'s belong to the CJ representation of the set of CPTP maps. The final step to be taken to arrive at a higher-order theory is to extend the right part of the inner product, \( \rho \otimes \mathbb{1} \), into the corresponding notion of states. That is, we extend it to any linear operator \( W \) on \( \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B) \) that is well behaved with respect to the probability rule (meaning positive itself, more about that in the following or see Sec. II C):

\[ p(i|W, M) = \langle M_i, W \rangle \quad \text{(C12)} \]

By doing so, we have recovered single party process matrix formalism [8]; the probability rule depicts the most general scenario of a single party receiving a quantum state defined in \( \mathcal{H}^A \) as input, acting onto it, and sending away another state defined on \( \mathcal{H}^B \). The \( M_i \)'s represent all the probabilistic operations that this party can make in between, with \( M \) being the deterministic, i.e. on average, operation she performs. \( W \), which in this case can be proven to be exactly the operators of the form \( \rho \otimes \mathbb{1} \), is the ‘space-time’ generalization of a state, that encodes anything outside of the control of the party.

Summarizing, the above construction, which will be nicknamed dynamics-like through this Appendix, amounts to postulating that the base theory admits well-defined dynamics, and that the measurement can be recast as a dynamical process itself. This lifts the theory to the higher-order

\[ p(i|\rho) \rightarrow p(i|W, M) \quad \text{(C13)} \]

An important thing to notice is that the set of admissible states \( \rho \), Eq. (C2), has the same structure of the set of admissible channels in CJ representation, Eq. (C5). The two are sets of positive and trace-normalized operators that possess the identity, this structure is formalized under the name state structure in the main text (Sec. II).

Repeating this construction, i.e. nesting transformations, leads to the approach of higher-order quantum transformations that was pioneered by the Pavia group in Ref. [3–6].

Another, somewhat equivalent (as will be shown at the end of this article) approach is the one of Ref. [8] which consist of imposing physically motivated constraints on the structure of measurements, and then to prove a generalization of Gleason theorem on it in order to derive the ad hoc notion of state. This kind of construction will be nicknamed Gleason-like.


2. Examples of single party theory

Here we present some examples of theories that can be built assuming Proposition 1. In these cases, we have a structure $\mathcal{A}$, which will constitute the states, and to it we associate to it its negation, $\overline{\mathcal{A}}$, interpreted as the need of a state of $\mathcal{A}$ to get a probability of 1, i.e. the deterministic measurement of a state.

a. Example: quantum theory. The first and most obvious example is to build regular quantum theory by letting $\mathcal{A} = \mathcal{L}(\mathcal{H}^A)$, which results in the state to be the regular notion of the quantum state defined in Eq. (1),

$$A \geq 0, \quad \text{Tr}[A] = 1, \quad \mathcal{I}\{A\} = A. \quad (C14a)$$

To it corresponds the regular notion of effect as $\overline{\mathcal{A}} \equiv \{1\}$ because Proposition 1 implies that

$$\overline{A} \geq 0, \quad \text{Tr}[\overline{A}] = d_A, \quad \mathcal{P}\{\overline{A}\} = \overline{A}. \quad (C15a)$$

where each $\overline{A}_i$ belongs to a family $\{\overline{A}_i\}$ resolving $\{1\}$.

To picture it is that the theory allows for a postselection but in a basis that is by construction mutually unbiased with respect to the basis of the state: in the example Alice can in general postselect in any state of the form $|\pm\rangle = \sigma_z |\pm\rangle$, where $p$ real and $p^2 \leq 1$ because of positivity. That is, she can choose to project the state into a mixture $|\pm\rangle$ of projectors $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, but the state is itself built by superposing $|\pm\rangle$ and $|\pm\rangle$, the two mutually unbiased basis w.r.t. the computational. Therefore her postselection cannot be used to distinguish from the maximally mixed state $I/2$. Equivalently, she cannot distinguish between states by choosing $p \neq 0$, which explains the theory is operationally equivalent (sometimes called ‘tomographically indistinguishable’) to a qubit quantum theory with one ‘forbidden’ axis of the Bloch sphere. In other words, it is also guaranteed to be a perfectly well-behaved theory, in the sense that no matter the choice of $A, \overline{A}$ and its resolution $\{\overline{A}_i\}$, no incoherence like over-normalized or negative probabilities can be found in the outcomes.

This one party example is of course trivial, but as soon as more than one party is allowed, the possibility of deterministically sending a signal from one party to another will coincide with this kind of allowed postselection. We already hinted this fact in the motivating example, where the choice of a quantum channel amounts to deterministically induce a bias, akin to steering, in the outcome probability.

3. Examples of no signaling composite theories

Using Proposition 2 we built examples of theories using the tensor in this section.

a. Bipartite quantum theory. From the previously given example of single party quantum theory, we can define a bipartite theory on space $\mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B)$ by requiring a bipartition $\mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B) \cong \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B)$ so that there exist states upon which bipartite measurements are well defined. The one party example has showed that local measurements of say Alice resolve the state structure $\mathcal{A}$ as in Eqs. (C14). Assume Bob has a similarly defined $\overline{B}$ so that the joint measurement of Alice and Bob, that we assume local, are resolving $\mathcal{A} \otimes \overline{B}$. By Proposition 2 this set is characterized by

$$M \geq 0, \quad \text{Tr}_{AB}[M] = 1, \quad (D_A \otimes D_B)\{M\} = M. \quad (C18a)$$

It is not hard to see that also in the no signaling bipartite case, quantum theory effects resolve a single element, $\{M = I\}$. Note however that the effects can now in general be entangled in the sense that there are resolutions $\{M_i\}$ for which there is no possibility to find a decomposition $\{M_i = \sum q_i E_i^A \otimes F_i^B\}$ where $q_i \geq 0$, $\sum_i q_i = 1$ and for all $q_i$ and $i$, with $E_i(F_i)$ a valid effect resolving an element of $\mathcal{A} \otimes \overline{B}$. A Bell measurement is an instance
of such a collection of entangled effects resolving $1$ in the $d_A = d_B = 2$ case.

We now use Proposition 1 to characterize the valid states. An operator $W \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a valid state if it belongs to the set $\mathcal{A} \otimes \mathcal{B}$, i.e.

$$W \geq 0,$$  

(C19a)

$$\text{Tr}_{AB}[W] = 1,$$  

(C19b)

$$\{\mathcal{D}_A \otimes \mathcal{D}_B\}[W] = W.$$  

(C19c)

Remark that property A45 actually applies to the projective constraint, so that it can be simplified into

$$\{\mathcal{I}_A \otimes \mathcal{I}_B\}[W] = W.$$  

(C20)

Meaning that $\mathcal{A} \otimes \mathcal{B} \cong \mathcal{A} \otimes \mathcal{B}$ actually holds for quantum theory, and surprisingly it does so only in this case. In other words, in quantum theory the set of valid states normalized on local measurements is exactly the set of no signaling states: we recover the intuition that it is impossible for parties measuring a part of a shared quantum state to signal to the other one. What is more surprising is that it is actually the only theory having this property. As we will see in the next example, in general a state normalized on a pair of local measurements can be used for signaling.

b. Bipartite biased quantum theory. In the biased case, where $\mathcal{A} = \mathcal{B} = \text{Span}\{1, \sigma_x\}$ the difference between a bipartite no signaling state and a general bipartite one becomes striking. For simplicity, we assume in the following that $d_A = d_B = 2$ and that the states are normalized to 1 so that $c_A = c_B = 1$, meaning that the effects will resolve some positive operators of trace 2. The set of all valid states normalized on the local effects resolving $\mathcal{A} \otimes \mathcal{B}$ is $\mathcal{A} \otimes \mathcal{B}$ which, according to Prop. 1 is made of the following 13 basis elements:

$$\mathcal{A} \otimes \mathcal{B} \subset \text{Span}\left\{1^A \otimes 1^B, 1^A \otimes \sigma_x^B, 1^A \otimes \sigma_y^B, \sigma_x^A \otimes 1^B, \sigma_x^A \otimes \sigma_y^B, \sigma_x^A \otimes \sigma_z^B, \sigma_y^A \otimes 1^B, \sigma_y^A \otimes \sigma_x^B, \sigma_y^A \otimes \sigma_z^B, \sigma_z^A \otimes 1^B, \sigma_z^A \otimes \sigma_x^B, \sigma_z^A \otimes \sigma_y^B, \right\}.$$  

(C21)

On the other hand, its no signaling subset, $\mathcal{A} \otimes \mathcal{B}$, is only made of 9 elements. The four missing elements are

$$\mathcal{A} \otimes \mathcal{B} \setminus \mathcal{A} \otimes \mathcal{B} \subset \text{Span}\left\{\sigma_x^A \otimes \sigma_x^B, \sigma_x^A \otimes \sigma_y^B, \sigma_x^A \otimes \sigma_z^B, \sigma_y^A \otimes \sigma_z^B, \right\},$$  

(C22)

which are exactly the elements that allow signaling between $A$ and $B$ according to Theorem 1. Indeed, these basis elements are those who are quasi-orthogonal globally, meaning that an operator $W$ that contains some of them will verify

$$\text{Tr} \left[(\hat{A} \otimes \hat{B}) \cdot W\right] = \frac{1}{d_A d_B} \text{Tr} \left[(\hat{A} \otimes \hat{B})\right] \text{Tr}[W],$$  

(C23)

because it satisfies Proposition 1 and therefore Eq. 29. But it does not obey quasi-orthogonality with respect to a local measurement, meaning that it will fail to satisfy at least one of Eqs. 27.

The consequence of this observation is that if Alice and Bob share a bipartite no signaling state, they may observe nonlocal entanglement effects on their outcome distributions, but these correlations will obey no signaling constraints in both directions, Eqs. 27. If, however, they share a general bipartite state, they may use it to achieve deterministic signaling. As we will show, they will be able to signal perfectly in one direction, i.e. Alice can perfectly send a message to Bob for certain states, and vice-versa.

For instance, consider the task where Alice receives a classical bit $x$ and she wants to communicate it to Bob, so that his outcome $b$ has the same value, $b = x$. Without any resources, Bob can only guess and thus succeed with $p(b=x)=1/2$. Now if we allow them to measure a shared state $W_{A<B}$ (the notation in the subscript means ‘Alice can signal to Bob’) in $\mathcal{A} \otimes \mathcal{B}$, they can pick the following state:

$$W_{A<B} = \frac{1}{4} \left(1^A \otimes 1^B + \sigma_x^A \otimes \sigma_x^B\right),$$  

(C24)

and choose to do the following: Alice ‘steers’ her measurement towards $|0\rangle$ or $|1\rangle$ depending on $x$,

$$\hat{A}_x = 1^A + (-1)^x \sigma_x^A,$$  

(C25)

while Bob is measuring an unbiased $\hat{B} = 1^B$ resolved into a measurement in the $|\pm\rangle$ basis,

$$\hat{B}_b = \frac{1}{2} \left(1^B + (-1)^b \sigma_x^B\right),$$  

(C26)

where $b = 0, 1$ so that his effects verify $\hat{B}_0 + \hat{B}_1 = 1^B$. One can check that they are effectively properly normalized positive operators belonging to the proper state structures. The measurement yields the following probability distribution:

$$p(b|x) = \text{Tr} \left[(\hat{A}_x \otimes \hat{B}_b) \cdot W_{A<B}\right].$$  

(C27)

Injecting the above expressions in it, it yields

$$p(b|x) = \frac{1}{2} \left(1 + (-1)^{x+b}\right),$$  

(C28)

which gives 0 when $x \neq b$ and 1 when $x = b$; Alice’s setting is perfectly correlated to Bob’s outcome. We conclude that a bit was perfectly sent from $A$ to $B$, or that using the $W_{A<B}$ state as their resource, they obtained $p(x = b) = 1$.

The bottom line of this example is that there exist states in $\mathcal{A} \otimes \mathcal{B}$ which allow to beat one of the no-signaling constraints, Eqs. 27, with a probability of 1. In this regard, quantum theory plays a special role as it is the only theory characterized by the projective methods of this article whose bipartite states normalized on
no-signaling effects are automatically no-signaling. This fact is the deeper meaning of the equation (A.45). In the next example, we will show that the quantum instruments in the CJ pictures are effects resolving a state structure made of (CJ representation of) quantum channels. This structure is a concrete example of a composite state structure (here input and output systems) that fails to satisfy (A.45), meaning that the instruments can be used for signaling, as expected.

4. Examples of theories with transformation

The first example below revisits the introductory example of Section I.C with the language developed in the article. The second example is a recovering of the bipartite PM formalism. These two examples have regular quantum theory as their first order. The third example, however, is a channel theory based on biased quantum theory and it highlight the fact that objects built like a transformation are actually allowing two way signaling.

a. Quantum channel theory and single partite process matrix. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be regular quantum partite state structure characterized by similar constraints as Eqs. (C14). In accordance with the definition, we introduce dynamics from $\mathcal{A}_0$ to $\mathcal{A}_1$, as the set of all CPTP maps $\{M\}$ from $A_0$ to $A_1$; this is step 1 in Figure 10. Call $\mathcal{A}_0 \to \mathcal{A}_1$ the set of all operators $M \in L(\mathcal{H}_0 \otimes \mathcal{H}_1)$ which are the CJ representation of an admissible transformation, that is an $M$ such that for all $A_0 \in \mathcal{A}_0$, there exists an $A_1 \in \mathcal{A}_1$ such that

$$[\text{Tr} \ [M \cdot (A_0 \otimes 1)]]^T = A_1. \quad (C29)$$

By definition, the operators $M$’s are (representing) quantum channels, so by Proposition 3 they must satisfy

$$(C30a) \quad M \geq 0,$$

$$(C30b) \quad \text{Tr} [M] = d_{A_0},$$

$$(C30c) \quad (\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \ {\{M\}} = M,$$

to be valid; this is depicted as step 2 in Fig. 10.

Remark that the last two conditions can be combined in the more common quantum ‘1-comb’ condition,

$$\text{Tr}_{A_1} [M] = 1_{A_0} \quad \text{(see Ref. [17] and the discussion in Section II.C).}$$

It is obtained by explicitly writing the projector

$$(\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \ {\{M\}} = \mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1} \ {\{M\}} = \mathcal{I}_{A_0} \otimes \mathcal{I}_{A_1} - \mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1} + \mathcal{D}_{A_0} \otimes \mathcal{D}_{A_1} \ {\{M\}}, \quad (C31)$$

and noticing that

$$(C32) \quad (\mathcal{D}_{A_0} \otimes \mathcal{D}_{A_1}) \ {\{M\}} = \frac{\text{Tr}_{A_0A_1} [M]}{d_{A_0}d_{A_1}} \mathbb{1}_{A_0A_1}.$$  

An interesting property of quantum channels is that they are automatically no signaling from the input to the output. Seeing the elements of $\mathcal{A}_0 \to \mathcal{A}_1$ as composite systems in $\mathcal{A}_0 \otimes \mathcal{A}_1$, Eq. (A.45) indicate that the $M$ can be interpreted as a composite object built from combining elements of $\mathcal{A}_0$ with $\mathcal{A}_1$. In other words, a transformation can be seen as the composition of a measurement in $L(\mathcal{H}_0)$ with a preparation in $L(\mathcal{H}_1)$. Since the part in $A_1$ is normalized on quantum effects, as in Eqs. (6), there is only one deterministic measurement at the output of a quantum channel. This implies that the no-signaling condition (37b) from the output to the input of a channel $M$, $\text{Tr}_{A_1} [M \cdot (1 \otimes A_1)] = \text{Tr}_{A_1} [M \cdot (1 \otimes A_1)]$, is automatically satisfied since there is only the single element $\{1_{A_1}\}$ in the set $\mathcal{A}_1$. This property is not true for transformations between other theories as we will show in the example of biased theory below.

Once again, the theory of transformations of quantum states stands out from the theories characterized by projective means as it is the only theory that automatically forbids signaling from the output to the input. However, it is not its only particularity, continuing the construction so that we extend the state and effect pair (pink and green objects in the bottom left corner in the figure), which is a subset of $\mathcal{A}_0 \otimes \mathcal{A}_1$ to the full set. That amounts to extending the pair to their entire affine span, effectively obtaining $\mathcal{A}_0 \otimes \mathcal{A}_1$, the (representation of the) entire set of functionals normalized on quantum channels as one would have obtained in a Gleason-like approach (yellow in the figure), one obtains what is by definition a single partite process matrix.

The remarkable thing here is that the affine span of the objects of the form $\rho^{A_0} \otimes 1_{A_1}$ is exactly the spanning set itself: pure tensor products of the pink and green objects are exactly the yellow one. In other words, a single partite process matrix reduces to inputting a quantum state and tracing out the output; the single partite process formalism is exactly the quantum channel theory (see the supplementary material of Ref. [8], alternatively see Ref. [38]). This is an instance of a supermap that simplifies into a map. In general, extending the theory to the full affine span -what is done in the Gleason-like construction-results in nontrivial new dynamics; supermaps do not reduce to maps. The most studied example of which is the process matrix formalism for more than one party that we will study next.

b. Bipartite process matrix theory. Not only bipartite process matrix is a nice example of the nontrivial difference between the pure tensor within a state structure and its full affine span, i.e. a good example of Gleason-like construction of higher-order theory, it is also a perfect illustration of how induced tensoriality on the effects can make the set of states broader. It is a theory that is constructed mixing the two ways: starting with the regular quantum theory, one first considers no signaling at the level of effects to argue for bipartite states, $\rho \in \mathcal{A}_0 \otimes \mathcal{B}_0$. Then, the theory is extended by allowing quantum channels, which are characterized in a dynamics-like construction as the set $(\mathcal{A}_0 \otimes \mathcal{B}_0) \to (\mathcal{A}_1 \otimes \mathcal{B}_1)$. Refer to Fig.
FIG. 10. Dynamics-like construction from quantum states to single partite PM: 1) adding CPTP dynamics to states (in pink) yields channels (in blue), 2) going to the CJ picture and characterising it using Prop. 3, 3) taking the ‘negation’ of the channels to ‘complete’ the states and effects pair (pink and green) into a single-partite PM (in yellow).

FIG. 11. Gleason-like construction of a bipartite PM: starting from a bipartite channel normalized on a state (in blue and pink, top right), 1) Restrict the set of channels to local operations of Alice and Bob subset (in blue, top left); 2) Go to the CJ picture so that the channels are interpreted as two effects composed in tensor (in blue, bottom left); 3) Extend the allowed higher-order ‘state’ to the most general operator normalized on a tensor product of channels, the bipartite PM (in yellow, bottom right).

Next, one considers another requirement of no signalling, at the level of the transformation this time. The heuristic of which is straightforward: if there are local effects, the dynamics may be constrained to local operations as well. Another way to put it: Alice and Bob are each allowed to do any physical transformation on their qudit before measuring, but no global transformation is allowed. Both points of view result in the same conclusion that the set of channels is restricted to its local subset made of pure tensors of local transformations. The operations of Alice and Bob have then the form

\[ M_{A_0}^{A_0 \rightarrow A_1} \otimes M_{B_0}^{B_0 \rightarrow B_1} \in (A_0 \rightarrow A_1) \otimes (B_0 \rightarrow B_1) \]  

for illustration.

For any \( M \) in \( (A_0 \otimes B_0) \rightarrow (A_1 \otimes B_1) \), we see that there cannot be signalling from \( W_{A_0} \) to \( W_{B_0} \) (respectively, \( W_{B_0} \) to \( W_{A_0} \)) if \( \{ M \} = M \) (this is the bottom left situation of Fig. 11).

Using associativity to add some parenthesis and switched the position of subsystems \( A_1 \) and \( B_0 \) to make it more obvious that Eq. (A33) can be used:

\[ (\mathbb{I}_{A_0} \otimes \mathbb{I}_{B_0} \otimes \mathbb{I}_{A_1} \otimes \mathbb{I}_{B_1}) = (\mathbb{I}_{A_0} \otimes \mathbb{D}_{A_1} \otimes \mathbb{I}_{B_0} \otimes \mathbb{D}_{B_1}) \]

(C33)

The right-hand side of the above equation is none other than the projector characterising the set \( (\mathcal{A}_0 \otimes \mathcal{B}_0) \rightarrow (\mathcal{A}_1 \otimes \mathcal{B}_1) \) has a projector which can be simplified as

\[ \mathbb{I}_{A_0} \otimes \mathbb{D}_{A_1} \otimes \mathbb{I}_{B_0} \otimes \mathbb{D}_{B_1} = (\mathbb{I}_{A_0} \otimes \mathbb{D}_{A_1} \otimes \mathbb{I}_{B_0} \otimes \mathbb{D}_{B_1}) \]

(C34)
are actually a very small portion of the full set, which is the affine span of those. This is the actual downside of the projective methods: because they are linear, they cannot tell much about separability nor localisability [14, 15].

Still, one can ask for the most general linear functional normalized on these pure tensors. Following Ref. [8], the reasoning consists of adding the extra constraint that the parties can share possibly entangled ancilla. By doing so, one extends the scope of what the functional is normalized upon to the entire state structure \((A_0 \rightarrow A_1) \otimes (B_0 \rightarrow B_1)\). Hence, the generalized Gleason theorem applies (more details for the case of PM can be found in Ref. [8, 39]), and the representation of the functional is characterized by Proposition 1. Therefore, any \(W \in (A_0 \rightarrow A_1) \otimes (B_0 \rightarrow B_1)\) represents such functional and by definition is a bipartite process matrix.

In projector algebra terms, we restricted the set of bipartite maps to its no-signaling subset, and its negation is exactly what we call a process matrix. As the negation amounts to going to the quasi-orthogonal complement, the set of valid process matrices is less restricted (thus bigger) than the negation of the channels, that is, states and measurements; this is again a consequence of Eq. (A43). The point is that restricting the state structure resolved by the effects allow a bigger state structure for the states.

c. Biased quantum channel theory. Dynamics of the biased theory considered in a previous example can be postulated as map from biased theory to itself. Prop. 3 characterize these as the set \(A \rightarrow B\) where the state structure of the input and output are the same, \(A = B = \text{Span}\{I, \sigma_x, \sigma_y\}\). Here the fact that transformation allows signaling in both directions can be rapidly proven by examples. Suppose Alice and Bob are sharing the channel \(M = \frac{1}{2} (I \otimes I + \sigma_x \otimes \sigma_x)\). Alice can perfectly signal to Bob by encoding her setting \(x\) in the \(\sigma_x\) basis, \(A = 1/2(I + (-1)^x \sigma_x)\), and if Bob measures in the same basis, they effectively have a perfect single bit channel, \(p(b|x) = 1\).

On the other hand, suppose they share the channel \(M = \frac{1}{2} (I \otimes I + \sigma_x \otimes \sigma_x)\). Now it is Bob who can perfectly signal to Alice: Alice has to use an ancilla so that she can keep the \(A'\) part of the bipartite state \(A = \frac{1}{4} \left( I^{A'} \otimes I^A + \sigma_x^{A'} \otimes \sigma_x^A \right)\) and she sends the other part through the channel. Bob can then apply the operator \(B = I + (-1)^y \sigma_x\) at the outcome of the channel. Alice finally measures in the \(\sigma_x\) basis, \(A' = 1/2(I + (-1)^y \sigma_x)\) and she will get perfect correlation with Bob setting.

The fact that the parties can signal forward and backward depending on the channel is not the only peculiar feature of the biased theory. Actually, identity channel cannot exist in this theory! Indeed, the CJ representation of the identity channel is the maximally entangled state, which is Bell-diagonal, hence having a term like \(\sigma_z \otimes \sigma_z\) in its expansion, something that is forbidden. Nonetheless, one only needs the identity to preserve the \(\sigma_x\) and \(\sigma_y\) basis vectors, so a possible way around this could be to formulate an operator like \((I \otimes I + p(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y))\). The issue is that since the Bell states are extremal, the weight \(p\) can never be 1. The best approximate will be \(p = 1/2\), meaning that an identity channel in this theory must depolarize the states half of the time.

5. Introductory example revisited: Dynamics-like construction up to third order

For this example, we will focus on the case of theories built upon quantum mechanics. In particular, we will continue the construction of the quantum instrument formalism from POVM formalism and repeat it until an indefinite causal order (ICO) arises in the theory. The reason we do that is two-fold: on the one hand, we want to present how the characterization techniques work in a concrete case (as we will see, all the objects that will get defined have already been studied in the literature). On the other hand, and in accordance with the discussion of Sec. V.B, we want to stress how tame quantum theory is with respect to other theories that can be characterized using projective means: albeit we will be using the transformation operation \(\rightarrow\) repetitively, i.e. we are nesting operations in a way that allows for bidirectional signaling, it will not result in a theory with ICO before the third order. This is in stark contrast with the example of biased theory we encountered earlier, in which already at the level of biased quantum channel one could have an operator that was made of a superposition of terms allowing the input to signal to the output and vice-versa.

a. Reformulating the introductory example: the quantum channel is a 1-comb

Reformulating the introductory example of Sec. C1 we start with a state structure \(A_0 \subset \mathcal{L}(H^{A_0})\) with projector \(I_{A_0}\) and trace of 1, as in Eqs. C14 and the zeroth-order theory consists of a pair of elements from complementary (i.e. quasi-orthogonal) state structures, \((\rho, I) \in A_0 \times A_0\), linked by the normalization of the probability rule

\[
\mathbb{1} = \langle 1 , \rho \rangle_{\mathcal{L}(H^{A_0})} \equiv \text{Tr} [I \cdot \rho] . \tag{C35}
\]

Probabilistic assignments are obtained by resolving the effect state structure by a collection of positive operators \(\{E_i\}\),

\[
p(i|\rho) = \text{Tr} [E_i \cdot \rho] , \tag{C36}
\]

yielding the POVM formalism and the usual definition of effects.

To go to first order, one postulates some dynamics so that the state structure \(A_0\) is mapped to a similarly defined state structure \(A_1\) by a CPTP map \(M \in \mathcal{L}(\mathcal{L}(H^{A_0}), \mathcal{L}(H^{A_1}))\) so that

\[
\langle 1 , \rho \rangle_{\mathcal{L}(H^{A_0})} \mapsto \langle 1 , M_{A_0 \rightarrow A_1}(\rho) \rangle_{\mathcal{L}(H^{A_1})} . \tag{C37}
\]
As it was shown in example [C4a] in the CJ picture, this results in a new pair: \( \langle \rho^{A_0} \otimes 1^{A_1}, M_{A_0A_1} \rangle \in (\mathcal{A}_0 \otimes \mathcal{A}_1) \times (\mathcal{A}_0 \rightarrow \mathcal{A}_1) \), so that

\[
1 = \text{Tr} \left[ M \cdot (\rho \otimes 1^T) \right]^T = \text{Tr} \left[ M \cdot (\rho \otimes 1) \right].
\]

(C38)

Here \( M \) is a quantum channel, which by definition is a 1-comb. In the example, we finish the construction of first-order theory by requiring that any state \( W \) in \((\mathcal{A}_0 \otimes \mathcal{A}_1)\) to be allowed so that the normalization becomes

\[
\langle M, W \rangle_{L(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1})} = 1,
\]

(C39)

where \( W \) is a single partite PM and \( M \) is a 1-comb, so that both are first-order objects.

Nonetheless, we have seen that single partite PM trivially decomposes into states and measurements. Meaning, in the case of quantum theory, that the first order theory naturally connects with the zeroth order; there is no possibility to have a non-trivial state entangled with a measurement. In other words, there are no states of first order theory that are outside of what can be built assuming zeroth order alone.

Another difference we noticed in the example is that the 1-comb was one-way signaling despite being built by transformation. Now we can see this fact quickly from isomorphism [A74]: observe that the 1-comb is characterized by \( \mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_1} \), which is equivalent to

\[
\mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_1} = \mathcal{I}_{A_0} \prec \mathcal{I}_{A_1}.
\]

(C40)

Hence, we are guaranteed that the theory has a fixed signaling direction, so it cannot show indefinite causal order. If the base theory was anything else than quantum theory, we would not have witnessed such a simplification: the 1-comb projector would have decomposed into two different orderings: \( \mathcal{P}_{A_0} \rightarrow \mathcal{P}_{A_1} = \bar{\mathcal{P}}_{A_0} \prec \mathcal{P}_{A_1} \cup \mathcal{P}_{A_0} \prec \mathcal{P}_{A_1} \).

Resolving state structure \((\mathcal{A}_0 \rightarrow \mathcal{A}_1)\) yields the quantum instrument formalism,

\[
p(j \mid M, \rho) = \langle M_j, \rho \otimes 1 \rangle.
\]

(C41)

As preparation and measurement are special transformations with respectively, input and output trivial systems, we can also promote the destructive measurement at \( A_1 \) into a measurement performed by a second party, so that the probability rule becomes

\[
p(i, j \mid M, \rho) = \langle M_j, \rho \otimes E_i^T \rangle,
\]

(C42)

this is still one-way signaling as it amounts to having a second party making a POVM measurement \( \{E_i\} \) (which is a special case of a quantum instrument) after the quantum instrument \( \{M_j\} \).

\textbf{Remark.} To better stick to our conventions, we could have passed all probabilistic assignments into the left side of the inner product:

\[
p(i, j \mid M, \rho) = \langle E_i \ast M_j, \rho \rangle_{L(\mathcal{H}^{A_0})},
\]

(C43)

where \( E_i \ast M_j \equiv \text{Tr}_{A_1} \left[ (1^{A_0} \otimes E_i^T) \cdot M_j \right] \) is the link product of the two operators [3], which corresponds to the CJ representation of the sequential composition of these two instruments over subsystem \( A_1 \).

\textbf{Remark 2.} For the sake of the argument, in the following we will take the point of view in which the introduced dynamics are the deterministic objects the parties have no influence over, and everything else is locally controlled by some local parties. For the case of Eq. (C52), this amounts to taking the marginal over \( j \),

\[
p(i \mid M, \rho) = \langle M, \rho \otimes E_i^T \rangle.
\]

We do so because this will maximise the number of local parties, which makes ICO more obvious; it would be difficult to talk about the ICO within the whole operator representing the shared dynamics, whereas it can be more easily discussed by looking at the correlations achievable by a group of parties sharing this global object. In this regard, this amounts of taking the dynamics as the state and everything else as local effects.

\textbf{b. The quantum supermap is a 2-comb}

The second order is obtained by assuming the existence of dynamics over the current dynamics—a transformation of the transformation. The same way structure-preserving maps can be nicknamed ‘superoperator’, we are defining a ‘supermap’ \( \mathcal{N} \) as a linear map between two maps with the same state structure [2]. If the map \( M \) of the previous section is defined between subsystems \( A_1 \) and \( A_2 \), we introduce \( \mathcal{N} \) as a CPTP-preserving supermap that send \( M \) to a similar map \( \mathcal{M} \) between subsystems \( A_0 \) and \( A_3 \),

\[
\langle \mathbb{1}, M(\rho) \rangle_{L(\mathcal{H}^{A_1})} \rightarrow \langle \mathbb{1}, [\mathcal{N}(M)](\rho) \rangle_{L(\mathcal{H}^{A_3})}.
\]

(C44)

Going to the CJ picture, the following probability rule is obtained [10]:

\[
p(i, j \mid N, M, \rho) = \langle N, \rho \otimes M_j^{T A_2} \otimes E_i^T \rangle.
\]

(C45)

The quasi-orthogonal pair is of the form

\[
(N, \rho \otimes M \otimes 1) \in (\mathcal{A}_0 \rightarrow \mathcal{A}_2) \times (\mathcal{A}_0 \otimes (\mathcal{A}_1 \rightarrow \mathcal{A}_2) \otimes \mathcal{A}_3).
\]

This way, \( N \) is a 2-comb, an object that transforms 1-combs into 1-combs. We have already seen in the last section that the 1-combs themselves can be interpreted as one-way signaling objects, so that the signaling of the 2-comb can be made explicit by looking at its correspond-
ing projector:

\[(\mathcal{I}_{A_1} \to \mathcal{I}_{A_2}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_3})\]

\[(\mathcal{A}_7)\]

\[\left(\mathcal{I}_{A_1} \times \mathcal{I}_{A_2}\right) \to (\mathcal{I}_{A_0} \times \mathcal{I}_{A_3})\]

\[(\mathcal{A}_6)\]

\[\bigcup (\mathcal{I}_{A_1} \times \mathcal{I}_{A_2}) \to (\mathcal{I}_{A_0} \times \mathcal{I}_{A_3})\]

\[= \mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_0} \times \mathcal{I}_{A_3}\]

So it may be concluded that the most general 2-comb is a superposition of two possible quantum networks: one in which the channel \(M\) is measured, \(\mathcal{I}_{A_1} \prec \mathcal{I}_{A_2}\), then reprepared as \(\mathcal{I}_{A_0} \times \mathcal{I}_{A_3}\) in its causal future and one where it is first \(\bar{M}\) which is prepared.

Yet, there is again an isomorphism at play that will make the interpretation of the supermap \(N\), a second order transformation, equivalent to a succession of first order transformations with a single signaling direction.

\[(\mathcal{I}_{A_1} \to \mathcal{I}_{A_2}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_3})\]

\[(\mathcal{A}_9)\]

\[= (\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_1} \to \mathcal{I}_{A_3})) \to \mathcal{I}_{A_3}\]

\[(\mathcal{A}_7)\]

\[= (\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_1} \prec \mathcal{I}_{A_3})) \prec \mathcal{I}_{A_3}\]

\[(\mathcal{A}_6)\]

\[= \mathcal{I}_{A_0} \times \mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_3} \prec \mathcal{I}_{A_3} \times \mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_3}.\]

This means that \(N\) can equivalently be understood as a succession of two quantum channels \([\mathcal{B}_0, \mathcal{B}_1]\): this is the most peculiar feature of quantum supermaps, and as we will see later of quantum combs in general.

This surprisingly tame behaviour of having a single signaling direction when reduced to a succession of zeroth-order objects will not carry out in the third order, as we will see next. In addition, remark that this provides an explanation on why a single partite PM does not feature ICO, or anything outside of the usual quantum theory. If the systems \(A_0\) and \(A_3\) were trivial, so that \(N\) is defined as (the CJ of) a supermap that takes a quantum channel to a probability, we see that the associated projector \(\mathcal{I}_{A_0} \times \mathcal{I}_{A_1} \prec \mathcal{I}_{A_2} \prec \mathcal{I}_{A_3} \rightarrow 1 \times \mathcal{I}_{A_1} \prec \mathcal{I}_{A_2} \prec 1\), and this can be simplified into \(\mathcal{I}_{A_1} \prec \mathcal{D}_{A_2}\).

c. The quantum super-supermap is an MPM

To go to the third order, the same procedure is applied again: we introduce a ‘super-supermap’ \(W\) so that it maps supermaps like \(N\) to themselves, \(\mathcal{W}(N) = N\):

\[\{1, [N(M)](\rho)\}_{\lambda(\mathcal{H}_{A_3})} \mapsto \{1, [\mathcal{W}(N)(M)](\rho)\}_{\lambda(\mathcal{H}_{A_7})}.\]

\[(\mathcal{C}48)\]

In the CJ representation, the probability rule is

\[p(i, j, k|W, M, N, \rho) = \left\langle W, \rho \otimes N_j^{\lambda T_A \lambda} T_{A_0} \otimes M_i^{\lambda T_A \lambda} \otimes \vec{E}_i^{\lambda T_A \lambda} \right\rangle.\]

\[(\mathcal{C}49)\]

Here, \(M\) is a transformation of \(\rho\) defined in \(\mathcal{A}_5 \to \mathcal{A}_6\); \(N\) is a transformation of \(M, N \in (\mathcal{A}_2 \to \mathcal{A}_3) \to (\mathcal{A}_4 \to \mathcal{A}_6)\), and the \(W\) operator is a transformation of \(N\),

\[W \in [(\mathcal{A}_2 \to \mathcal{A}_5) \to (\mathcal{A}_1 \to \mathcal{A}_6)] \to (\mathcal{A}_3 \to \mathcal{A}_4) \to (\mathcal{A}_5 \to \mathcal{A}_6).\]

\[(\mathcal{C}50)\]

We can now treat \(N\) as a party itself, which physically corresponds to someone acting a first time between node \(A_1\) and \(A_2\), and a second time between \(A_5\) and \(A_6\), using a quantum network.

Again, one can focus on the projector characterising the state structure of \(N\) to extract its signaling structure. First, successive applications of the uncurrying rule \(\mathcal{A}_9\) yield

\[([\mathcal{I}_{A_2} \to \mathcal{I}_{A_5}] \to (\mathcal{I}_{A_1} \to \mathcal{I}_{A_6}))\]

\[(\mathcal{C}49)\]

\[\Rightarrow (\mathcal{I}_{A_3} \to \mathcal{I}_{A_4}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_7})\]

\[\Rightarrow (\mathcal{A}_{10} \otimes (\mathcal{A}_{11} \to \mathcal{I}_{A_7})) \to (\mathcal{A}_{12} \to \mathcal{I}_{A_7})\]

\[= [\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_3} \to \mathcal{I}_{A_6}) \Rightarrow (\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_3} \times \mathcal{I}_{A_4}) \to (\mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_3} \times \mathcal{I}_{A_4}))) \Rightarrow (\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_3} \times \mathcal{I}_{A_4}).\]

\[(\mathcal{C}49)\]

\[\Rightarrow [\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_3} \times \mathcal{I}_{A_4}) \times (\mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_3} \times \mathcal{I}_{A_4}).\]

\[(\mathcal{C}49)\]

Next, Eqs. \(\mathcal{A}_7\) as well as \(\mathcal{C}47\), \(\mathcal{A}_6\), and \(\mathcal{A}7\) are used successively:

\[= [\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_3} \times \mathcal{I}_{A_4}) \Rightarrow (\mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_3} \times \mathcal{I}_{A_4})] \Rightarrow \mathcal{I}_{A_7} \Rightarrow \mathcal{I}_{A_7} \Rightarrow \mathcal{I}_{A_7} \Rightarrow \mathcal{I}_{A_7}.\]

\[\Rightarrow [\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_3} \times \mathcal{I}_{A_4}) \Rightarrow (\mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_3} \times \mathcal{I}_{A_4}).\]

\[(\mathcal{C}49)\]

\[= [\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_3} \times \mathcal{I}_{A_4}) \Rightarrow (\mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \mathcal{I}_{A_3} \times \mathcal{I}_{A_4}).\]

\[(\mathcal{C}49)\]

In the last step, the negation gets distributed over the prec.

If, for simplicity, we assume the subsystems \(A_0\) and \(A_7\) to be of dimension 1, we can already notice in the first line of the previous equation that the state structure of the operator \(W\) features a projector whose expression is involving the negation of a tensor product of two quantum combs:

\[W \in (\mathcal{A}_2 \to \mathcal{A}_5) \to (\mathcal{A}_1 \to \mathcal{A}_6) \otimes (\mathcal{A}_3 \to \mathcal{A}_4).\]

\[(\mathcal{C}53)\]
i.e. it is the functional normalized on the tensor product between a 2-comb and 1-comb. This is by definition a multi-round process matrix (MPM). We proved in a previous work that it can manifest indefinite causal order and even more that it can beat causal inequalities [17]. The third order therefore presents multiple directions of signaling in a non-trivial manner.

The fundamental reason why, as we showed in this previous article [17], is that this last expression defines a set which is the union of 3 set of quantum combs with different signaling orderings,

\[
(\mathcal{A}_2 \rightarrow \mathcal{A}_5) \rightarrow (\mathcal{A}_4 \rightarrow \mathcal{A}_6) \otimes (\mathcal{A}_3 \rightarrow \mathcal{A}_4)
\]

\[
= \left(\left(\left(\left(\mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_5 \rightarrow \mathcal{A}_6 \rightarrow \mathcal{A}_3 \rightarrow \mathcal{A}_4 \right) \rightarrow \mathcal{A}_4 \right) \cup \left(\left(\mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_3 \rightarrow \mathcal{A}_4 \rightarrow \mathcal{A}_6 \right) \rightarrow \mathcal{A}_6 \right) \cup \right)\right) \left(\left(\mathcal{A}_5 \rightarrow \mathcal{A}_4 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_6 \right) \rightarrow \mathcal{A}_6 \right).
\]  
\[\text{(C54)}\]

In other words, \(W\) can be taken as a superposition of all combs that respect the signaling ordering in between the nodes of the 2-comb and the 1-comb that will be plugged into it, but it itself do not assume a global ordering. The above expression thus has 3 possible global orderings, depending whether the operation of the party with the 1-comb (acting between nodes 3 and 4) is before, in between, or after the two operations of the party with the 2-comb.

Back to the general case of a super-supermap, we can use this insight to further simplify the projector. We inject the simplification for the case of a functional into the central part. Distributing over the union, one obtains a canonical form as in Def. 5

\[
[\left(\mathcal{I}_{A_2} \rightarrow \mathcal{I}_{A_5} \right) \rightarrow \left(\mathcal{I}_{A_1} \rightarrow \mathcal{I}_{A_6} \right)]
\]

\[
\rightarrow \left(\left(\mathcal{I}_{A_3} \rightarrow \mathcal{I}_{A_4} \right) \rightarrow \left(\mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_7} \right) \right)
\]

\[
= \mathcal{I}_{A_0} \prec \mathcal{I}_{A_1} \prec \mathcal{I}_{A_2} \prec \mathcal{I}_{A_3} \prec \mathcal{I}_{A_4} \prec \mathcal{I}_{A_5} \prec \mathcal{I}_{A_6} \prec \mathcal{I}_{A_7} \cup
\]

\[
\mathcal{I}_{A_0} \prec \mathcal{I}_{A_1} \prec \mathcal{I}_{A_2} \prec \mathcal{I}_{A_3} \prec \mathcal{I}_{A_4} \prec \mathcal{I}_{A_5} \prec \mathcal{I}_{A_6} \prec \mathcal{I}_{A_7} \cup
\]

\[
\mathcal{I}_{A_0} \prec \mathcal{I}_{A_1} \prec \mathcal{I}_{A_2} \prec \mathcal{I}_{A_3} \prec \mathcal{I}_{A_4} \prec \mathcal{I}_{A_7} \prec \mathcal{I}_{A_5} \prec \mathcal{I}_{A_6} .
\]
\[\text{(C55)}\]

What we have just done is to track the ICO origin as an ambiguity in the decomposition of higher-order into lower order: using the algebraic rules of the projectors, we could express the third level theory in terms of the negation of a no signaling composition in the second level (by that we mean that we expressed the super-supermap as at most a superposition of supermaps). The issue is that the decomposition is not forbidding the existence of operators that belong to the affine hull of the union. That is, operators that are a superposition of several combs with different signaling directions.