Triangle Estimation using Polylogarithmic Queries

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Abstract

Estimating the number of triangles in a graph is one of the most fundamental problems in sublinear algorithms. In this work, we provide the first approximate triangle counting algorithm using only polylogarithmic queries. Our query oracle Tripartite Independent Set (TIS) takes three disjoint sets of vertices $A$, $B$ and $C$ as input, and answers whether there exists a triangle having one endpoint in each of these three sets. Our query model is inspired by the Bipartite Independent Set (BIS) query oracle of Beame et al. (ITCS, 2018). Their algorithm for edge estimation requires only polylogarithmic BIS queries, where a BIS query takes two disjoint sets $A$ and $B$ as input and answers whether there is an edge with endpoints in $A$ and $B$. We extend the algorithmic framework of Beame et al., with TIS replacing BIS, for triangle counting using ideas from color coding due to Alon et al. (J. ACM, 1995) and a concentration inequality for sums of random variables with bounded dependency due to Janson (Rand. Struct. Alg., 2004).

Keywords. Triangle estimation, query complexity, and sublinear algorithm

1 Introduction

Counting the number of triangles is a fundamental algorithmic problem in the RAM model [AYZ97, BPWZ13, IR78], streaming [TPTT13, BKS02, JG05, CJ17, BFL+06, AGM12, KMSS12, JSP13, PTTW13, ADNK14, KP17] and query model [GRS11, ELRS17]. In this work, we focus on triangle counting in the query model. Lately, two works [ELRS17, ERS18] have obtained almost matching upper and lower bounds for triangle and $k$-clique counting in the standard query model, where the queries on the graphs are (i) degree query: the oracle reports the degree of a vertex; (ii) neighbor query: the oracle reports the $i^{th}$ neighbor of $v$, if it exists; and (iii) edge existence query: the oracle reports whether there exists an edge between a given pair of vertices. Eden et al. [ELRS17, ERS18] gave an algorithm to estimate the number of triangles using $\tilde{O}\left(\frac{n}{t(G)^{1/3}} + \min\{\frac{m^{3/2}}{t(G)}, m\}\right)$ queries, where $n$, $m$ and $t(G)$ denote the number of vertices, edges and triangles in the input graph, respectively. Their algorithmic results aided by an almost matching lower bound of $\Omega\left(\frac{n}{t(G)^{1/3}} + \min\{\frac{m^{3/2}}{t(G)}, m\}\right)$ have almost closed this line of study. A precursor to triangle counting in graphs is edge estimation. The number of edges in a graph can be estimated by using $\tilde{O}\left(n/\sqrt{m}\right)$ number of neighbor queries and $\Omega(n\sqrt{em})$ queries are necessary to estimate the number of edges even if we allow degree and edge existence queries along with neighbor queries [GR08]. This result would almost have closed the edge estimation problem but for having a relook at the problem with stronger query models and hoping for polylogarithmic number of queries. Beame et al. [BHR+18] precisely did that by estimating the number of edges in a graph using $O\left(\epsilon^{-4}\log^{14} n\right)$ bipartite independent set (BIS)

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$\tilde{O}()$ hides a polynomial factor of $\log n$ and $\frac{1}{\epsilon}$, where $\epsilon \in (0, 1)$ is such that $(1 - \epsilon)t \leq \hat{t} \leq (1 + \epsilon)t$; $\hat{t}$ and $t$ denote the estimated and actual number of triangles in $G$, respectively.
queries. Motivated by this result, we explore whether triangle estimation can be solved using only polylogarithmic queries to a query oracle, named `tripartite independent set (TIS)` oracle. Intuitively, TIS is to triangle counting what BIS is to edge estimation. Queries to TIS or BIS oracles are global in nature as opposed to degree, neighbor or edge existence queries that are local in nature. A bone of contention for any newly introduced query oracle is its worth\(^2\). Beame et al. [BHR+18] had given a subjective justification in favor of BIS to establish it as a query oracle. Next, we formally state the query model, the problem description and the main result followed by an analytic justification in favor of stronger query models using some lower bound arguments.

1.1 Notations, the query model, the problem and the result

We denote the set \(\{1, \ldots, n\}\) by \([n]\). Let \(V(G)\), \(E(G)\) and \(T(G)\) denote the set of vertices, edges and triangles in the input graph \(G\), respectively. Let \(t(G) = |T(G)|\). Whenever we say \(A, B, C\) are disjoint, we mean \(A, B, C\) are pairwise disjoint. For three non-empty disjoint sets \(A, B, C \subseteq V(G)\), \(G(A, B, C)\), termed as a `tripartite subgraph` of \(G\), denotes the induced subgraph of \(A \cup B \cup C\) in \(G\) minus the edges having both endpoints in \(A\) or \(B\) or \(C\). \(t(A, B, C)\) denotes the number of triangles in \(G(A, B, C)\). We use the triplet \((a, b, c)\) to denote the triangle having \(a, b, c\) as its vertices. Let \(\Delta(u, v)\) be the number of triangles having \((u, v)\) as one of its edges and \(\Delta = \max_{(u,v) \in E(G)} \Delta(u, v)\). Let \(\Delta_u\) denote the number of triangles having \(u\) as one of its vertices. For a set \(U\), “\(U\) is COLORED with \([n]\)” means that each member of \(U\) is assigned a color out of \([n]\) colors independently and uniformly at random. Let \(\mathbb{E}[X]\) and \(\mathbb{V}[X]\) denote the expectation and variance of a random variable \(X\). For an event \(\mathcal{E}\), \(\mathcal{E}^c\) denotes the complement of \(\mathcal{E}\). The statement “\(a\) is an \(1 \pm \epsilon\) multiplicative approximation of \(b\)” means \(|b-a| \leq \epsilon \cdot b\). Next, we describe the query oracle.

**Tripartite independent set oracle (TIS)** Given three non-empty disjoint subsets \(V_1, V_2, V_3 \subseteq V(G)\) of a graph \(G\), TIS query oracle answers ‘YES’ if and only if \(t(V_1, V_2, V_3) \geq 0\).

The problem definition and our main result are given as follows.

| **Triangle-Estimation** |
|-------------------------|
| **Input:** Set of vertices \(V(G)\), TIS oracle for graph \(G\) and \(\epsilon \in (0, 1)\). |
| **Output:** \(1 \pm \epsilon\) multiplicative approximation of \(t(G)\). |

**Theorem 1.** Let \(G\) be a graph with \(\Delta \leq d\), \(|V(G)| = n \geq 64\). For any \(\epsilon > 0\), **Triangle-Estimation** can be solved using \(O\left(\epsilon^{-12}d^{12}\log^{25} n\right)\) TIS queries with probability \(1 - O(n^{-2})\).

Note that the query complexity stated in Theorem 1 is \(\text{poly}(\log n, \frac{1}{\epsilon})\), even if \(d = O(\log n)\).

1.2 An analytic justification for stronger query oracles

One might wonder whether it is possible to obtain better results for triangle estimation using query oracles much weaker than TIS. Note that the triangle estimation result by Eden et al. [ELRS17] uses degree, neighbor and edge-existence queries. We show that even if their query model is augmented with a `triangle existence query oracle`\(^3\), triangle estimation does not become easier, asymptotically. This is because a triangle existence query can be emulated by three edge existence queries, and therefore, the lower bound of Eden et al. [ELRS17] holds even when degree, neighbor and edge existence queries are aided by a triangle existence query. More importantly, we show that on graphs

\(^2\)See [http://www.wisdom.weizmann.ac.il/~oded/MC/237.html](http://www.wisdom.weizmann.ac.il/~oded/MC/237.html) for a comment on BIS.

\(^3\)A triangle existence query takes three different vertices \(a, b, c \in V(G)\) as input and reports whether \((a, b, c)\) is a triangle in \(G\).
for which $\Delta$ is bounded by $d$, a condition required by our upper bound result, a similar lower bound on the number of queries holds. The formal statement for this lower bound is given as follows.

**Observation 2.** Any multiplicative approximation algorithm that estimates the number of triangles in a graph $G$ such that $\Delta \leq d$, requires $\Omega \left( \frac{n}{t(G)} \frac{d^n}{t(G)} \right)$ queries, where the allowed queries are degree, neighbor, edge existence and triangle existence.

**Proof.** Specifically, we prove that any multiplicative approximation algorithm that estimates the number of triangles in a graph $G$ such that $\Delta \leq d$, requires

(a) $\Omega \left( \frac{n}{t(G)} \right)$ queries if $d \leq 2$;

(b) $\Omega \left( \frac{n}{t(G)} \frac{d^n}{t(G)} \right)$ queries if $1 \leq t(G) \leq \binom{d}{3}$ and $3 \leq d \leq n$; and

(c) $\Omega \left( \frac{d^n}{t(G)} \right)$ queries if $t(G) > \binom{d}{3}$ and $3 \leq d \leq n$;

The proof idea is motivated by [ELRS17]. For every $n$ and every $d$ as above, let $G_1$ be a graph on $n$ nodes having no edges and $G_2$ be a family of graphs on $n$ nodes. Any two graphs in $G_2$ differ only in labeling of the vertices. Note that $t(G_1) = 0$ and we take $G_2$ such that $t(G) = \theta(t)$ for each $G \in G_2$ and for some $t \in \mathbb{N}$. Our strategy is to show that we can not distinguish whether the input is $G_1$ or some graph in $G_2$ unless we make sufficient number of queries. We will design $G_2$ differently for each one of the cases below.

**Proof of (a)** Assume that $\lceil \frac{d}{3} \rceil (d + 2) < n$. Otherwise, the lower bound is trivial. Take $G_2$ to be a family of graphs satisfying the following. In $G_2$, each graph $G$ consists of (see Figure 1 (a))

- $\lceil \frac{d}{3} \rceil$ many vertex disjoint components $H_1, \ldots, H_{\lceil \frac{d}{3} \rceil}$ such that each $H_i$ has $d + 2$ vertices and $d$ many triangles sharing an edge,
- an independent set of $n - \lfloor \frac{d}{3} \rfloor (d + 2)$ vertices.

**Figure 1:** Lower bound construction for Observation 2
Note that the number of vertices participating in any triangle in any $G \in \mathcal{G}_2$ is at most $\left\lfloor \frac{t}{d} \right\rfloor (d + 2)$. Unless we hit such a vertex, we cannot distinguish whether the input is $G_1$ or some graph in $\mathcal{G}_2$. The probability of hitting such a vertex in a graph selected uniformly from $\mathcal{G}_2$ is at most $\frac{\left\lfloor \frac{t}{d} \right\rfloor (d + 2)}{n}$. Hence, the number of queries required to distinguish between two input cases is at least

$$\frac{n}{\left\lfloor \frac{t}{d} \right\rfloor (d + 2)} = \Omega \left( \frac{n}{t} \right) = \Omega \left( \frac{n}{t(G)} \right).$$

Proof of (b) Take $\mathcal{G}_2$ to be the class of graphs where each $G \in \mathcal{G}_2$ contains a clique of size $\left\lfloor \frac{t^{1/3}}{d} \right\rfloor$ and an independent set of size $n - \left\lfloor \frac{t^{1/3}}{d} \right\rfloor$ (see Figure 1 (b)). Observe that $G$ satisfies $t(G) = \Theta(t)$ and $\Delta \leq d$ as $t(G) \leq \left( \frac{d}{3} \right)$. Using a similar argument as in proof of (a), $\Omega \left( \frac{n}{t(G)} \right)$ queries are required to decide whether the input graph is $G_1$ or some graph in $\mathcal{G}_2$.

Proof of (c) Assume that $\left\lfloor \frac{t}{(\frac{d}{3})} \right\rfloor d < n$. Otherwise, the claimed bound trivially holds. Take $\mathcal{G}_2$ to be a class of graph where each graph $G \in \mathcal{G}_2$ consists of (see Figure 1 (c))

- $\left\lfloor \frac{t}{(\frac{d}{3})} \right\rfloor$ many vertex disjoint cliques each of size $d$,
- an independent set of size $n - \left\lfloor \frac{t}{(\frac{d}{3})} \right\rfloor d$.

Using a similar argument as in proof of (a), one can show that the number of queries required to decide whether the input graph is $G_1$ or some graph in $\mathcal{G}_2$ is at least

$$\frac{n}{\left\lfloor \frac{t}{(\frac{d}{3})} \right\rfloor d} = \Omega \left( \frac{d^2 n}{t} \right) = \Omega \left( \frac{d^2 n}{t(G)} \right).$$

The above observation on lower bound tells us that the introduction of triangle existence queries in our case will not yield anything significant. Thus, it is worth a try to look at other powerful queries, like TIS.

1.3 Organization of the paper

We give a broad overview of the algorithm in Section 2. Sections 3, 4 and 5 give the details of sparsification, exact estimation and coarse estimation of the number of triangles, respectively. The final algorithm is given in Section 6.

2 Overview of the algorithm

Our algorithm for TRiANGLE-ESTIMATION using TIS queries is similar to the algorithm for edge estimation using BIS queries [BHR+18]. In Figure 2, we give a flowchart of the algorithm. The basic building blocks of the algorithm are: two kinds of sparsification routines (one for general graph and another for tripartite graph), a coarse estimator, a sampling scheme of the subgraphs and two algorithms for exactly counting the number of triangles (one for general graph and another for tripartite graph) when the number of triangles is not too large. The building blocks of our algorithm
For each tripartite subgraph \( G(A, B, C) \) check whether \( t(A, B, C) < \text{threshold} \); Compute \( t(A, B, C) \) if it is less than the threshold and remove \( G(A, B, C) \).

Is there any tripartite subgraph left?

Sparsify \( G \) such that the sparsified graph \( G' \) is a union of vertex disjoint tripartite subgraphs and a proper scaling of \( t(G') \) approximates \( t(G) \).

Is there any tripartite subgraph left?

For each subgraph \( G(A, B, C) \), use a \textit{coarse estimator} for \( t(A, B, C) \) that is correct upto \( O(\log^3 n) \) factor.

Sample a bounded number of subgraphs such that a proper weighted scaling of the number of triangles in the subgraphs is approximately same as that of the number of triangles in the original set of subgraphs.

Is the number of tripartite subgraphs present large?

For each subgraph \( G(A, B, C) \), \textbf{Sparsify} \( G(A, B, C) \) such that the sparsified graph \( H \) is a union of vertex disjoint tripartite subgraphs and a proper scaling of \( t(H) \) is \( t(A, B, C) \), approximately. Replace \( G(A, B, C) \) by the tripartite subgraphs, in \( H \), formed formed by sparsification.

Figure 2: Flow chart of the algorithm. The highlighted texts indicate the basic building blocks of the algorithm. We also indicate the corresponding lemmas that support the building blocks.
are similar to those of Beame et al. [BHR+18]. We extend their framework to the case of triangle counting using ideas from color coding due to Alon et al. [AYZ95] and a relatively new concentration inequality, due to Janson [Jan04], for sums of random variables with bounded dependency.

We sparsify the given graph $G$ by $V(G)$ being COLORED with $[3k]$ such that

(i) the sparsified graph is a union of a set of vertex disjoint tripartite subgraphs and

(ii) a proper scaling of the number of triangles in the sparsified graph is a good estimate of $t(G)$ with high probability.

The sparsification result is formally stated next; the proof uses the method of averaged bounded differences and Chernoff-Hoeffding type inequality in bounded dependency setting by Janson [Jan04]. The detailed proof is in Section 3.

Lemma 3 (General Sparsification). Let $G$ be a graph with $V(G) = [n]$ and $\Delta \leq d$. Let $V_1, \ldots, V_{3k}$ be a partition of $V(G)$ formed by $V(G)$ being COLORED with $[3k]$. Then, there exists a constant $\kappa_1$ such that

$$\mathbb{P}\left(\left|\frac{9k^2}{2} \sum_{i=1}^{k} t(V_i, V_{k+i}, V_{2k+i}) - t(G)\right| > \kappa_1 dk^2 \sqrt{t(G) \log n}\right) \leq \frac{2}{n^4}.$$

The above tells us that a proper scaling of the number of triangles, in the sparsified graph, approximately estimates $t(G)$, when $t(G)$ is above a threshold. We apply the sparsification corresponding to Lemma 3 only when $t(G)$ is above the threshold to ensure that the relative error is bounded. We can decide whether $t(G)$ is less than the threshold and if it is so, we compute the exact value of $t(G)$, using the following Lemma, whose proof is inspired by color coding ideas and given in Section 4.

Lemma 4 (Exact Counting). Given a graph $G$ and a positive integer $\tau$, there exists an algorithm that determines whether $t(G) < \tau$ using $O(\tau^6 \log n)$ TIS queries with probability $1 - \frac{1}{n\tau^c}$. Moreover, the algorithm finds the exact value of $t(G)$ in case $t(G) < \tau$.

Assume that $t(G)$ is large and $G$ has undergone sparsification. We initiate a data structure with a set of vertex disjoint tripartite graphs that are obtained after the sparsification step. For each tripartite graph $G(A, B, C)$ in the data structure, we check whether $t(A, B, C)$ is less than a threshold using the algorithm corresponding to Lemma 5. If it is less than a threshold, we compute the exact value of $t(A, B, C)$ using Lemma 6 and remove $G(A, B, C)$ from the data structure. The proofs of Lemma 5 and 6 are given in Section 4.

Lemma 5 (Threshold for Tripartite Graph). Given disjoint subsets $A, B, C$ of $V(G)$ and a positive integral threshold $\tau > 0$, there exists a deterministic algorithm that can decide whether $t(A, B, C) \leq \tau$ using $O(\tau \log n)$ TIS queries.

Lemma 6 (Exact Counting in Tripartite Graphs). Given disjoint subsets $A, B, C$ of $V(G)$, there exists a deterministic algorithm that computes the exact value of $t(A, B, C)$ by using $O(t(A, B, C) \log n)$ TIS queries.

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4High probability means that the probability of success is at least $1 - \frac{1}{n^c}$ for some constant $c$.

5The threshold is a fixed polynomial in $d, \log n$ and $\frac{1}{\epsilon}$.

6Large refers to a fixed polynomial in $d, \log n$ and $\frac{1}{\epsilon}$.
Now we are left with some tripartite graphs such that the number of triangles in each graph is more than a threshold. If the number of such graphs is not large, then we sparsify each tripartite graph $G(A, B, C)$ such that (i) the sparsified graph is a disjoint union of some tripartite subgraphs and (ii) a proper constant scaling of the number of triangles in the sparsified graph is approximately same as that of $t(A, B, C)$. This sparsification result is formally stated in the following Lemma, whose proof is similar to the proof of Lemma 3. We replace $G(A, B, C)$ by constant ($k^7$) many tripartite subgraphs formed after sparsification.

Lemma 7 (Sparsification for Tripartite Graphs). Let $G$ be a graph with $V(G) = [n]$ and $\Delta \leq d$. Let $A, B, C \subseteq V(G)$ be disjoint. Let $A_1, \ldots, A_k, B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ are the partitions of $A, B, C$ formed uniformly at random, respectively. Then there exists a constant $\kappa_2$ such that

$$P\left|\sum_{i=1}^{k} t(A_i, B_i, C_i) - t(A, B, C)\right| > \kappa_2 dk^2 \sqrt{t(G) \log n} \leq \frac{1}{n^8}.$$ 

If we have a large number of vertex disjoint tripartite subgraphs of $G$ and each subgraph contains a large number of triangles, then we coarsely estimate the number of triangles in each subgraph which is correct up to $O(\log^3 n)$ factor by using the algorithm corresponding to the following Lemma, whose proof is in Section 5. Our Coarse-Estimate algorithm is similar in structure to the coarse estimation algorithm for edge estimation, but the analysis involves sophisticated calculations.

Lemma 8 (Coarse Estimation). Given disjoint $A, B, C \subseteq V(G)$, there exists an algorithm that returns an estimate for $t(A, B, C)$ such that

$$\frac{t(A, B, C)}{32 \log n} \leq \hat{t} \leq 32 t(A, B, C) \log^3 n$$

with probability $1 - n^{-9}$. Moreover, the query complexity of the algorithm is $O(\log^4 n)$.

After estimating the number of triangles in each subgraph coarsely, we generate a bounded number of samples of the set of subgraphs using a sampling technique given by Beame et al. [BHR+18]. The sampling is such that a proper weighted sum of the number of triangles in the sample is approximately same as that of the number of triangles in the original set of subgraphs before sampling. The Lemma corresponding to sampling is formally stated in Lemma 15 in Section 6. After getting the sample, for each subgraph in the sample, we apply the sparsification algorithm corresponding to Lemma 7.

Now again, for each tripartite graph $G(A, B, C)$, we check whether $t(A, B, C)$ is less than a threshold using the algorithm corresponding to Lemma 5. If yes, then we can compute the exact value of $t(A, B, C)$ using Lemma 6 and remove $G(A, B, C)$ from the data structure. Otherwise, we iterate on all the required steps discussed above as shown in Figure 2. Observe that the query complexity of each iteration is polylogarithmic. Now, note that the number of triangles reduces by a constant factor after each sparsification step. So, the number of iterations is bounded by $O(\log n)$. Hence, the query complexity of our algorithm is polylogarithmic. This completes the high level description of our algorithm.

3 Sparsification Lemma

In this Section, we prove Lemma 3. The proof of Lemma 7 is similar.

7 In our algorithm, $k$ is a constant. However, Lemma 7 holds for any $k \in \mathbb{N}$.

8 Polylogarithmic refers to a polynomial in $d, \log n$ and $\frac{1}{\epsilon}$. 

7
Proof of Lemma 3. $V(G)$ is COLORED with $[3k]$. Let $V_1, \ldots, V_{3k}$ be the resulting partition of $V(G)$. Let $Z_i$ be the random variable that denotes the color assigned to the $i^{th}$ vertex.

Definition 9. A triangle $(a,b,c)$ is said to be properly colored if there exists $i \in [3k]$ such that one vertex of the triangle is colored with color $i$, one with color $1 + ((k + i - 1) \mod 3k)$ and another with $1 + ((2k + 1) \mod 3k)$.

Let $f(Z_1, \ldots, Z_n) = \sum_{i=1}^{k} t(V_i, V_{k+i}, V_{2k+i})$. Note that $f$ is the number of triangles that are properly colored. The probability that a triangle is properly colored is $\frac{2}{9k^2}$. So, $\mathbb{E}[f] = \frac{2t(G)}{9k^2}$.

For $i \in [3k]$, $\pi(i)$ is a set of three colors defined as follows. $\pi(i) = \{i, (1 + (i + k - 1) \mod 3k), (1 + (i + 2k - 1) \mod 3k)\}$.

Let us focus on the instance when vertices 1, ..., $t - 1$ are colored and we are going to color vertex $t$. Let $S_t(S_r)$ be the set of triangles in $G$ having $t$ as one of the vertices and other two vertices are from $[t-1]$ ($[n] \setminus [t]$). $S_r$ be the set of triangles in $G$ such that $t$ is a vertex and the second and third vertices are from $[t-1]$ and $[n] \setminus [t]$, respectively.

Given that the vertex $t$ is colored with color $c \in [3k]$, let $N^c_r, N^c_s, N^c_r$ be the random variables that denote the number of triangles in $S_t, S_r$ and $S_r$ that are properly colored, respectively. Now, we can deduce the following about $E'_f$, the difference in the conditional expectation of the number of triangles that are properly colored whose $t^{th}$ vertex is (possibly) differently colored, by considering the vertices in $S_t, S_r$ and $S_r$ separately.

$$E'_f = \mathbb{E}[f | Z_1, \ldots, Z_{t-1}, Z_t = a_t] - \mathbb{E}[f | Z_1, \ldots, Z_{t-1}, Z_t = a'_t]$$

$$= N^a_t - N^a_t + \mathbb{E}\left[N^{a_t}_{t} - N^{a'_t}_{t}\right] + \mathbb{E}\left[N^{a_t}_{r} - N^{a'_t}_{r}\right]$$

$$\leq N^a_t - N^a_t + \mathbb{E}\left[N^{a_t}_{t} - N^{a'_t}_{t}\right] + \mathbb{E}\left[N^{a_t}_{r} - N^{a'_t}_{r}\right]$$

Now, consider the following claim, which we prove later.

Claim 10. (a) $\mathbb{P}(|N^a_t - N^{a'_t}| < 8\sqrt{d\Delta_t \log n}) \geq 1 - 4n^{-8}$

(b) $\mathbb{E}[|N^a_t - N^{a'_t}|] \leq \sqrt{d\Delta_t}/k$

(c) $\mathbb{E}[|N^a_t - N^{a'_t}|] \leq 6d\Delta_t \log n$

From the above claim, the following is true with probability at least $1 - \frac{4}{n^8}$. Let $c_t = 15d\Delta_t \log n$. Observe that

$$E'_f < 8\sqrt{d\Delta_t \log n} + \frac{\sqrt{d\Delta_t}}{k} + 6d\Delta_t \log n \leq 15d\Delta_t \log n = c_t.$$ 

Let $B$ be the event that there exists $t \in [n]$ such that $E'_f > c_t$.

By the union bound over all $t \in [n]$, $\mathbb{P}(B) \leq \frac{4}{n^7}$. Using the method of averaged bounded difference [DP09] (See Lemma 17 in Section A), we have

$$\mathbb{P}(|f - \mathbb{E}[f]| > \delta + t(G)\mathbb{P}(B)) \leq e^{-\delta^2/\sum_{i=1}^{n} c_i^2} + \mathbb{P}(B).$$

We set $\delta = 60d\sqrt{t(G) \log n}$. Observe that

$$\sum_{t=1}^{n} c_t^2 = 225d^2 \log n \sum_{t=1}^{n} \Delta_t = 675d^2 t(G) \log n.$$
Hence,
\[ \Pr \left( \left| f - \frac{2t(G)}{9k^2} \right| > 60d\sqrt{t(G) \log n} + t(G)\Pr(\mathcal{B}) \right) \leq \frac{1}{n^4} + \frac{1}{n^7}, \]
that is
\[ \Pr \left( \left| \frac{9k^2}{2} f - t(G) \right| > 270dk^2 \sqrt{t(G) \log n} + \frac{9k^2}{2} \cdot \frac{t(G)}{n^7} \right) \leq \frac{1}{n^4} + \frac{1}{n^7}. \]
Since, \( \frac{9k^2}{2} \cdot \frac{t(G)}{n^7} < dk^2 \sqrt{t(G) \log n} \), we get
\[ \Pr \left( \left| \frac{9k^2}{2} f - t(G) \right| > 271dk^2 \sqrt{t(G) \log n} \right) \leq \frac{2}{n^7}. \]

To prove Claim 10, we need the following intermediate result that is stated in a general form.

**Lemma 11.** Let \( \mathcal{X} \) be a set of \( u \) objects COLORED with \([3k]\). Let \( \alpha, \beta \in [3k] \) and \( \alpha \neq \beta \). A pair of objects \( \{a, b\} \) is said to be colored with \( \{\alpha, \beta\} \) if there is a bijection in terms of coloring from \( \{a, b\} \) to \( \{\alpha, \beta\} \). An object \( o \in \mathcal{X} \) is colored with \( \{\alpha, \beta\} \) if \( o \) is colored with \( \alpha \) or \( \beta \). \( \mathcal{F} \) be a set of \( v \) pairs of objects such that an object is present in at most \( d \) \((d \leq v)\) many pairs and \( \mathcal{P} \subseteq \mathcal{X} \) be a set of \( w \) objects. \( \mathcal{F}_{\{\alpha, \beta\}} \subseteq \mathcal{F} \) be a set of pairs of objects that are colored with \( \{\alpha, \beta\} \). \( M_{\{\alpha, \beta\}} = |\mathcal{F}_{\{\alpha, \beta\}}| \). \( \mathcal{P}_{\{\alpha, \beta\}} \subseteq \mathcal{P} \) be the set of objects that are colored with \( \{\alpha, \beta\} \) and \( N_{\{\alpha, \beta\}} = |\mathcal{P}_{\{\alpha, \beta\}}| \). Then, we have

(i) \( \Pr \left( |M_{\{\alpha, \beta\}} - M_{\{\alpha', \beta'\}}| \geq 8\sqrt{d\log u} \right) \leq \frac{4}{u^5} \),

(ii) \( \mathbb{E} \left[ |M_{\{\alpha, \beta\}} - M_{\{\alpha', \beta'\}}| \right] \leq \frac{\sqrt{2d} v}{k} \), and

(iii) \( \Pr \left( |N_{\{\alpha, \beta\}} - N_{\{\alpha', \beta'\}}| \geq 4\sqrt{w\log u} \right) \leq \frac{4}{u^5} \).

**Proof.** (i) Let \( \mathcal{F} = \{\{a_1, b_1\}, \ldots, \{a_v, b_v\}\} \). Let \( X_i \) be the indicator random variable such that \( X_i = 1 \) if and only if \( \{a_i, b_i\} \) is colored with \( \{\alpha, \beta\} \), where \( i \in [v] \). Note that \( M_{\{\alpha, \beta\}} = \sum_{i=1}^{v} X_i \).

Also, \( \mathbb{E}[X_i] = \frac{2v}{9k^2} \), hence \( \mathbb{E}[M_{\{\alpha, \beta\}}] = \frac{2v}{9k^2} \).

The random variables \( X_i \) and \( X_j \) are dependent if and only if \( \{a_i, b_i\} \cap \{a_j, b_j\} \neq \emptyset \). As each object can be random in at most \( d \) many pairs of objects, there are at most \( 2d \) many \( X_j \)'s on which \( X_i \) depends. Now using Chernoff-Hoeffding’s type bound in the bounded dependent setting \cite{DBLP:journals/jcss/DvoretzkyP99} (see Lemma 21 in Section A), we have
\[ \Pr \left( \left| M_{\{\alpha, \beta\}} - \frac{2v}{9k^2} \right| \geq 4\sqrt{d\log u} \right) \leq \frac{2}{u^5}. \]

Similarly, one can also show that \( \Pr \left( \left| M_{\{\alpha', \beta'\}} - \frac{2v}{9k^2} \right| \geq 4\sqrt{d\log u} \right) \leq \frac{2}{u^5} \). Note that
\[ |M_{\{\alpha, \beta\}} - M_{\{\alpha', \beta'\}}| \leq \left| M_{\{\alpha, \beta\}} - \frac{2v}{9k^2} \right| + \left| M_{\{\alpha', \beta'\}} - \frac{2v}{9k^2} \right|. \]

Hence,
\[ \Pr \left( \left| M_{\{\alpha, \beta\}} - M_{\{\alpha', \beta'\}} \right| \geq 8\sqrt{d\log u} \right) \]
\[ \leq \Pr \left( \left| M_{\{\alpha, \beta\}} - \frac{2v}{9k^2} \right| + \left| M_{\{\alpha', \beta'\}} - \frac{2v}{9k^2} \right| \geq 8\sqrt{d\log u} \right) \]
\[ \leq \Pr \left( \left| M_{\{\alpha, \beta\}} - \frac{2v}{9k^2} \right| \geq 4\sqrt{d\log u} \right) + \Pr \left( \left| M_{\{\alpha', \beta'\}} - \frac{2v}{9k^2} \right| \geq 4\sqrt{d\log u} \right) \]
\[ \leq \frac{4}{u^5}. \]
(ii) Let $X_i, i \in [v]$, be the random variable such that $X_i = 1$ if $\{a_i, b_i\}$ is colored with $\{\alpha, \beta\}$; $X_i = -1$ if $\{a_i, b_i\}$ is colored with $\{\alpha', \beta'\}$; $X_i = 0$, otherwise. Let $X = \sum_{i=1}^{v} X_i$. Note that

$$M_{\{\alpha, \beta\}} - M_{\{\alpha', \beta'\}} = X = \sum_{i=1}^{v} X_i.$$ 

So, we need to bound $\mathbb{E}[|X|]$ to prove the claim.

The random variables $X_i$ and $X_j$ are dependent if and only if $\{a_i, b_i\} \cap \{a_j, b_j\} \neq \emptyset$. As each object can be present in at most $d$ many pairs of objects, there are at most $2d$ many $X_j$‘s on which an $X_i$ depends. Observe that

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{2}{9k^2}.$$ 

So, $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = \frac{4}{9k^2}$. If $X_i$ and $X_j$ are independent, then $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i] \cdot \mathbb{E}[X_j] = 0$. If $X_i$ and $X_j$ are dependent, then $\mathbb{E}[X_i X_j] \leq \mathbb{P}(X_i X_j = 1)$

$$\mathbb{P}(X_i X_j = 1) = \mathbb{P}(X_i = 1, X_j = 1) + \mathbb{P}(X_i = -1, X_j = -1)$$

$$= \mathbb{P}(X_i = 1) \cdot \mathbb{P}(X_j = 1 \mid X_i = 1) + \mathbb{P}(X_i = -1) \cdot \mathbb{P}(X_j = -1 \mid X_i = -1)$$

$$= \frac{2}{9k^2} \cdot \frac{1}{3k} + \frac{2}{9k^2} \cdot \frac{1}{3k}$$

$$= \frac{4}{27k^3}$$

Using the expression $\mathbb{E}[X_i^2] = \sum_{i=1}^{v} \mathbb{E}[X_i^2] + 2 \cdot \sum_{1 \leq i < j \leq v} \mathbb{E}[X_i X_j]$ and recalling the fact that each $X_i$ depends on at most $2d$ many other $X_j$‘s, we get

$$\mathbb{E}[X_i^2] \leq v \cdot \frac{4}{9k^2} + 2dv \cdot \frac{4}{27k^3} \leq \frac{8dv}{9k^2}.$$ 

Now, using $\mathbb{E}[|X|] \leq \sqrt{\mathbb{E}[X^2]}$ we get $\mathbb{E}[|X|] < \frac{\sqrt{6v}}{3k}$.

(iii) Let $P = \{o_1, \ldots, o_w\}$ be the set of $w$ objects. Let $X_i, i \in [w]$, be the indicator random variable such that $X_i = 1$ if and only if $o_i$ is colored with $\{\alpha, \beta\}$. Note that $N_{\{\alpha, \beta\}} = \sum_{i=1}^{w} X_i$. Observe that $\mathbb{E}[X_i] = \frac{2w}{3k}$ and hence, $\mathbb{E}[N_{\{\alpha, \beta\}}] = \frac{2w}{3k}$. Note that $X_i$ and $X_j$ are independent. Applying Hoeffding’s inequality (See Lemma 18 in Section A), we get

$$\mathbb{P}\left(\left|N_{\{\alpha, \beta\}} - \frac{2w}{3k}\right| \geq 2\sqrt{w \log u} \right) \leq \frac{2}{u^8}.$$ 

Similarly, we can also show that

$$\mathbb{P}\left(\left|N_{\{\alpha', \beta'\}} - \frac{2w}{3k}\right| \geq 2\sqrt{w \log u} \right) + \mathbb{P}\left(\left|N_{\{\alpha, \beta\}} - \frac{2w}{3k}\right| \geq 2\sqrt{w \log u} \right) \leq \frac{4}{u^8}.$$ 

---

$\mathbb{E}[|X|] = \mathbb{E}[X^2] - \mathbb{E}^2[|X|] \geq 0$, which implies $\mathbb{E}[|X|] \leq \sqrt{\mathbb{E}[X^2]}$. 

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10
Proof of Claim 10. (a) Let \( S_t = \{ (a_1, b_1, t), \ldots, (a_v, b_v, t) \} \). Note that \( v \leq \Delta_t \). As \( \Delta \leq d \), each vertex in \([n]\) can be present in at most \( d \) many pairs of \( S_t \). Now we apply Lemma 11. Set \( \mathcal{X} = [n] \) and \( \mathcal{F} = S_t \) in Lemma 11. Observe that \( N_{\ell t}^{a_t} = M_{\pi(a_i) \setminus \{a_t\}} \) and \( N_{\ell t}^{a_t} = M_{\pi(a_i) \setminus \{a_t\}} \). So, by Lemma 11 (i), \( \mathbb{P} \left( \left| N_{\ell t}^{a_t} - N_{\ell t}^{a_t} \right| \geq 8\sqrt{d\Delta_t \log n} \right) \leq \frac{4}{n^8} \). This implies 
\[ \mathbb{P} \left( \left| N_{\ell t}^{a_t} - N_{\ell t}^{a_t} \right| \geq 8\sqrt{d\Delta_t \log n} \right) \leq \frac{4}{n^8}. \]

(b) Let \( S_r = \{ (t, a_1, b_1), \ldots, (t, a_v, b_v) \} \). Note that \( v \leq \Delta_t \), the number of triangles incident on vertex \( t \). As \( \Delta \leq d \), each vertex in \([n]\) can be present in at most \( d \) many pairs of \( S_r \). Now we apply Lemma 11. Set \( \mathcal{X} = [n] \) and \( \mathcal{F} = S_r \) in Lemma 11. Observe that \( N_{\ell r}^{a_t} = M_{\pi(a_i) \setminus \{a_t\}} \) and \( N_{\ell r}^{a_t} = M_{\pi(a_i) \setminus \{a_t\}} \). Using Lemma 11 (ii),
\[ \mathbb{E} \left[ \left| N_{\ell r}^{a_t} - N_{\ell r}^{a_t} \right| \right] \leq \frac{\sqrt{dv}}{k} \leq \frac{\sqrt{d\Delta_t}}{k}. \]

(c) Let \( S_{tr} = \{ (a_1, t, b_1), \ldots, (a_w, t, b_v) \} \). Without loss of generality, assume that \( a_i \in [t-1] \) and \( b_i \in [n] \setminus [t] \). Note that \( w \leq \Delta_t \). Given that the vertex \( t \) is colored with color \( c \) and we know \( Z_1, \ldots, Z_{t-1} \), define the set \( P_c \) as \( P_c := \{ (a, t, b) \in S_{tr} : t \text{ is colored with } c \text{ and } \mathbb{P}((a, t, b) \text{ is properly colored}) > 0 \} \). Let \( Q_c = |P_c| \).

Observe that for \( (a, t, b) \in S_{tr} \), \( \mathbb{P}((a, t, b) \text{ is properly colored}) > 0 \) if and only if \( a \) is colored with some color in \( \pi(c) \setminus \{c\} \). Now we apply Lemma 11. Set \( \mathcal{X} = [n] \), \( \mathcal{P} = \{ a_1, \ldots, a_w \} \). Observe that \( P_{\pi(a_i) \setminus \{a_t\}} = P_{a_t} \) and \( P_{\pi(a_i) \setminus \{a_t\}} = P_{a_t} \). By (iii) of Lemma 11, \( \mathbb{P} \left( \left| Q_{a_t} - Q_{a_t} \right| \geq 4\sqrt{w \log n} \right) \leq \frac{4}{n^8} \).

Let \( \mathcal{E} \) be the event that \( |Q_{a_t} - Q_{a_t}| \geq 4\sqrt{w \log n} \). So, \( \mathbb{P}(\mathcal{E}) \leq \frac{4}{n^8} \).

Assume that \( \mathcal{E} \) has not occurred. Let \( P = P_{a_t} \cap P_{a_t} = \{ (x_1, t, y_1), \ldots, (x_q, t, y_q) \} \). Note that \( q \leq w \leq \Delta_t \). Recall that \( Z_x \) is the random variable that denotes the color assigned to vertex \( x \in [n] \). Let \( X_i, i \in [q] \), be the random variable such that \( X_i = 1 \) if \( y_i \) is colored with \( \pi(a_i) \setminus \{Z_x, a_t\} \); \( X_i = -1 \) if \( y_i \) is colored with \( \pi(a_t) \setminus \{Z_x, a_t\} \); 0, otherwise. Let \( X = \sum_{i=1}^{q} X_i \).

Observe that \( X_i \) and \( X_j \) are dependent if and only if \( y_i = y_j \). As \( \Delta \leq d \), there can be at most \( d \) many \( y_j \)'s such that \( y_i = y_j \). So, an \( X_i \) depends on at most \( d \) many other \( X_j \)'s.

Observe that 
\[ \mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{3k}. \]

So, \( \mathbb{E}[X_i] = 0 \) and \( \mathbb{E}[X_i^2] = \frac{2}{3k} \). If \( X_i \) and \( X_j \) are independent, then \( \mathbb{E}[X_iX_j] = 0 \). If \( X_i \) and \( X_j \) are dependent, then
\[ \mathbb{E}[X_iX_j] \leq \mathbb{P}(X_i = 1, X_j = 1) + \mathbb{P}(X_1 = -1, X_j = -1) \]
\[ \leq \mathbb{P}(X_i = 1) + \mathbb{P}(X_j = -1) \]
\[ = \frac{2}{3k}. \]

Using the expression \( \mathbb{E}[X_i^2] = \sum_{i=1}^{q} \mathbb{E}[X_i^2] + 2 \cdot \sum_{1 \leq i < j \leq q} \mathbb{E}[X_iX_j] \) and the fact that each \( X_i \) depends on at most \( d \) many other \( X_j \)'s, we get
\[ \mathbb{E}[X_i^2] \leq v \cdot \frac{2}{3k} + dv \cdot \frac{2}{3k} \leq \frac{dv}{k} \leq \frac{d\Delta_t}{k}. \]
Since, \( E[|X|] \leq \sqrt{E[X^2]} \), we get \( E[|X|] \leq \sqrt{\frac{d\Delta_t}{k}} \). Using \( \Delta \leq d \), we have
\[
\mathbb{E} \left[ |N_{\ell r}^{a_i} - N_{\ell r}^{a_j'}| \mid E^c \right] = d \cdot |Q_{a_i} - Q_{a_j'}| + E[|X|] \\
< 4d\Delta_t \log n + \sqrt{\frac{d\Delta_t}{k}} \\
< 5d\Delta_t \log n.
\]

Observe that \( \mathbb{E} \left[ |N_{\ell r}^{a_i} - N_{\ell r}^{a_j'}| \mid E \right] \leq w \leq \Delta_t \). Putting everything together,
\[
\mathbb{E} \left[ |N_{\ell r}^{a_i} - N_{\ell r}^{a_j'}| \right] = \mathbb{P}(E) \cdot \mathbb{E} \left[ |N_{\ell r}^{a_i} - N_{\ell r}^{a_j'}| \mid E \right] + \mathbb{P}(E^c) \cdot \mathbb{E} \left[ |N_{\ell r}^{a_i} - N_{\ell r}^{a_j'}| \mid E^c \right] \\
< \frac{4}{n^3} \cdot \Delta_t + 1 \cdot 5d\Delta_t \log n \\
\leq 6d\Delta_t \log n.
\]

\[\square\]

4 Proof of the Lemmas corresponding to exact estimation

In this Section, we prove Lemmas 4, 5 and 6.

Proof of Lemma 4. We color \( V(G) \) is colored with \([100\tau^2]\) colors. Let \( h : V(G) \to [100\tau^2] \) be the coloring function and \( V_i = \{ v \in V(G) : h(v) = i \} \), i.e., the vertices with color \( i \), where \( i \in [100\tau^2] \). Note that \( V_1, \ldots, V_{100\tau^2} \) forms a partition of \( V(G) \). We make TIS queries with input \( V_i, V_j, V_k \) for each \( 1 \leq i < j < k \leq 100\tau^2 \). Observe that we make \( O(\tau^6) \) TIS queries. We construct a 3-uniform hypergraph \( H \), where \( U(H) = \{V_1, \ldots, V_{100\tau^2}\} \) \(^{10}\) and \( (V_i, V_j, V_k) \in F(H) \) if and only if TIS oracle answers yes with \( V_i, V_j, V_k \) given as input. We repeat the above procedure \( \gamma \) times, where \( \gamma = 50 \log n \). Let \( H_1, \ldots, H_\gamma \) be the set of corresponding hypergraphs and \( h_i \) be the coloring function to form the hypergraph \( H_i \), where \( i \in [\gamma] \). Then we compute \( A = \max\{|F(H_1)|, \ldots, |F(H_\gamma)|\} \). If \( A \geq \tau \), we report \( t(G) \geq \tau \). Otherwise, we report \( A \) as \( t(G) \). Note that the total number of TIS queries is \( O(\tau^6 \log n) \). Now, we analyze the cases \( t(G) \geq \tau \) and \( t(G) < \tau \) separately.

(i) \((t(G) \geq \tau)\) Consider a fixed set \( T \) of \( \tau \) triangles. Let \( T_v \) be the set of vertices that is present in some triangle in \( T \). Observe that \( |T_v| \leq 3\tau \). Let \( E_i \) be the event that the vertices in \( T_v \) are uniquely colored by the function \( h_i \), i.e., \( E_i : h_i(u) = h_i(v) \) if and only if \( u = v \), where \( u, v \in T_v \). First, we prove that \( \mathbb{P}(E) \geq \frac{9}{10} \) by computing \( \mathbb{P}(E_i^c) \).

\[
\mathbb{P}(E_i^c) \leq \sum_{u, v \in T_v} \mathbb{P}(h_i(u) = h_i(v)) \leq \sum_{u, v \in T_v} \frac{1}{100\tau^2} \leq \frac{|T_v|^2}{100\tau^2} < \frac{1}{10}.
\]

Let \( \text{PROP}_i \) be the property that for each triangle \( z \in T \), there is a corresponding hyperedge in \( F(H_i) \), where \( i \in [\gamma] \). Specifically, for each triangle \((a_1, a_2, a_3) \in T \) there exists a hyperedge \((a_1', a_2', a_3') \in F(H_i) \) such that \( h_i(a_j) = h_i(a_j') \) for each \( j \in [3] \). Note that, if \( \text{PROP}_i \) holds, then \(|F(H_i)| \geq |T| \geq \tau \). By the definition of TIS oracle, \( \text{PROP}_i \) holds when the event \( E_i \) occurs.

\(^{10}\) \( U(H) \) and \( F(H) \) denote the set of vertices and hyperedges in a hypergraph \( H \), respectively.
i.e., $\text{PROP}_1$ holds with probability at least $\frac{9}{10}$. This implies, with probability $\frac{9}{10}$, $|\mathcal{F}(H_i)| \geq \tau$. 
Recall that $A = \max\{|\mathcal{F}(H_1)|, \ldots, |\mathcal{F}(H_\gamma)|\}$ and $\gamma = 50\log n$. So,
$$
\mathbb{P}(A < \tau) = \left(1 - \frac{9}{10}\right)^{50\log n} \leq \frac{1}{n^{10}}. 
$$
Hence, if $t(G) \geq \tau$, our algorithm detects it with probability at least $1 - \frac{1}{n^{10}}$.

(ii) $(t(G) < \tau)$ Let $T$ be the set of all $t(G)$ triangles in $G$ and $T_v$ be the set of vertices that is present in some triangle in $T$. Observe that $|T_v| \leq 3 \cdot t(G) < 3\tau$. Let $\mathcal{E}_i$ be the event that the vertices in $T_v$ are uniquely colored by the function $h_i$, i.e., $\mathcal{E}_i : h_i(u) = h_i(v)$ if and only if $u = v$, where $u, v \in T_v$. First we prove that $\mathbb{P}(\mathcal{E}_i) \geq \frac{9}{10}$ by computing $\mathbb{P}(\mathcal{E}_i^c)$.

$$
\mathbb{P}(\mathcal{E}_i^c) \leq \sum_{u,v \in T_v} \mathbb{P}(h_i(u) = h_i(v)) \leq \sum_{u,v \in T_v} \frac{1}{100\tau^2} \leq \frac{|T_v|^2}{100\tau^2} < \frac{1}{10}. 
$$

Let $\text{PROP}_1$ be the property that for each triangle $\zeta \in T$, there is a corresponding hyperedge in $\mathcal{F}(H_i)$, where $i \in [\gamma]$. Specifically, for each triangle $(a_1, a_2, a_3) \in T$ there exists a hyperedge $(a'_1, a'_2, a'_3) \in \mathcal{F}(H_i)$ such that $h_i(a_j) = h_i(a'_j)$ for each $j \in [3]$. Note that, if $\text{PROP}_1$ holds, then $|\mathcal{F}(H_i)| = t(G)$. By the definition of TIS oracle, $\text{PROP}_1$ holds when the event $\mathcal{E}_i$ occurs, i.e., $\text{PROP}_1$ holds with probability at least $\frac{9}{10}$. This implies, with probability $\frac{9}{10}$, $|\mathcal{F}(H_i)| = t(G)$. Recall that $A = \max\{|\mathcal{F}(H_1)|, \ldots, |\mathcal{F}(H_\gamma)|\}$ and $\gamma = 50\log n$. By the construction of $H_i$, $|\mathcal{F}(H_i)| \leq t(G)$. So, $A \leq t(G)$ and
$$
\mathbb{P}(A \neq t(G)) = \mathbb{P}(A < t(G)) \leq \left(1 - \frac{9}{10}\right)^{50\log n} \leq \frac{1}{n^{10}}. 
$$
Hence, if $t(G) < \tau$, our algorithm outputs the exact value of $t(G)$ with probability at least $1 - \frac{1}{n^{10}}$.

\[\square\]

\textit{Proof of Lemma 6} We initialize a tree $T$ with $(A, B, C)$ as the root. We build the tree such that each node is labeled with either 0 or 1. If $t(A, B, C) = 0$, we label the root with 0 and terminate. Otherwise, we label the root with 1 and do the following as long as there is a leaf node $(U, V, W)$ labeled with 1.

(i) If $t(U, V, W) = 0$, then we label $(U, V, W)$ with 0 and go to other leaf node labeled as 1 if any. Otherwise, we label $(U, V, W)$ as 1 and do the following.

(ii) If $|U| = |V| = |W| = 1$, then we add one node $(U, V, W)$ as a child of $(U, V, W)$ and label the new node as 0. Then we go to other leaf node labeled as 1 if any.

(iii) If $|U| = 1, |V| = 1$ and $|W| > 1$, then we partition the set $W$ into $W_1$ and $W_2$ such that $|W_1| = \left\lceil \frac{|W|}{2} \right\rceil$ and $|W_2| = \left\lfloor \frac{|W|}{2} \right\rfloor$; and we add $(U, V, W_1)$ and $(U, V, W_2)$ as two children of $(U, V, W)$. The case $|U| = 1, |V| > 1, |W| = 1$ and $|U| > 1, |V| = 1, |W| = 1$ are handled similarly.

(iv) If $|U| = 1, |V| > 1$ and $|W| > 1$, then we partition the set $V$ into $V_1$ and $V_2$ (similarly, $W$ into $W_1$ and $W_2$) such that $|V_1| = \left\lceil \frac{|V|}{2} \right\rceil$ and $|V_2| = \left\lfloor \frac{|V|}{2} \right\rfloor$ (similarly, $|W_1| = \left\lceil \frac{|W|}{2} \right\rceil$ and $|W_2| = \left\lfloor \frac{|W|}{2} \right\rfloor$); and we add $(U, V_1, W_1)$, $(U, V_1, W_2)$, $(U, V_2, W_1)$ and $(U, V_2, W_2)$ as four children of $(U, V, W)$. The case $|U| > 1, |V| > 1, |W| = 1$ and $|U| > 1, |V| = 1, |W| > 1$ are handled similarly.
We now prove Lemma 8. Algorithm 2 corresponds to Lemma 8. Algorithm 1 is a subroutine in T. The number of nodes in Lemma 12. If \(|U| > 1, |V| > 1\) and \(|W| > 1\), then we partition the sets \(U, V, W\) into \(U_1\) and \(U_2; V_1\) and \(V_2; W_1\) and \(W_2\), respectively, such that \(|U_1| = \lceil \frac{|U|}{2} \rceil\) and \(|U_2| = \lfloor \frac{|U|}{2} \rfloor\); \(|V_1| = \lceil \frac{|V|}{2} \rceil\) and \(|V_2| = \lfloor \frac{|V|}{2} \rfloor\); \(|W_1| = \lceil \frac{|W|}{2} \rceil\) and \(|W_2| = \lfloor \frac{|W|}{2} \rfloor\). We add \((U_1, V_1, W_1), (U_1, V_1, W_2), (U_1, V_2, W_1), (U_1, V_2, W_2)\), \((U_2, V_1, W_1), (U_2, V_1, W_2), (U_2, V_2, W_1)\) and \((U_2, V_2, W_2)\) as eight children of \((U, V, W)\).

Let \(T'\) be the tree after deleting all the leaf nodes in \(T\). Observe that \(t(A, B, C)\) is the number of leaf nodes in \(T'\); and

1. the height of \(T\) is bounded by \(\max\{\log |A|, \log |B|, \log |C|\} + 1 \leq 2 \log n\),
2. the query complexity of the above procedure is bounded by the number of nodes in \(T\) as we make at most one query per node of \(T\).

The number of nodes in \(T'\), the number of internal nodes of \(T\), is bounded by \(2t(A, B, C)\log n\). So, the number of leaf nodes in \(T\) is at most \(16t(A, B, C)\log n\) and hence the total number of nodes in \(T\) is at most \(16t(U, V, W)\log n\). Putting everything together, the required query complexity is \(O(t(A, B, C)\log n)\). □

**Proof of Lemma 8.** The algorithm proceeds similar to the one presented in the Proof of Lemma 6 by initializing a tree \(T\) with \((A, B, C)\) as the root. If \(t(A, B, C) \leq \tau\), then we can find \(t(A, B, C)\) by using \(16t(A, B, C)\log n\) many queries and the number of nodes in \(T\) is bounded by \(16t(U, V, W)\log n\). So, if the number of nodes in \(T\) is more than \(16\tau \log n\) at any instance during the execution of the algorithm, we report \(t(G) > \tau\) and terminate. Hence, the query complexity is bounded by the number of nodes in \(T\), which is \(O(\tau \log n)\). □

### 5 Proof of the Lemma corresponding to coarse estimation

We now prove Lemma 8. Algorithm 2 corresponds to Lemma 8. Algorithm 1 is a subroutine in Algorithm 2. Algorithm 1 determines whether a given estimate \(\hat{t}\) is correct up to a \(O(\log^3 n)\) factor. Lemma 12 and 13 are intermediate results needed to prove Lemma 8.

**Algorithm 1: Verify-Estimate \((A, B, C, \hat{t})\)**

**Input:** Three pairwise disjoint set \(A, B, C \subseteq V(G)\) and \(\hat{t}\).

**Output:** If \(\hat{t}\) is a good estimate, then Accept. Otherwise, Reject.

```
1 begin
2    for \((i = \log n \text{ to } 0)\) do
3        for \((j = \log n \text{ to } 0)\) do
4            Find \(A_{ij} \subseteq A, B_{ij} \subseteq B, C_{ij} \subseteq C\) by sampling each element of \(A\), \(B\) and \(C\), respectively with probability \(\min\{\frac{2^i}{7}, 1\}, \min\{\frac{2^j}{27}\log n, 1\}, \frac{1}{27}\)
5            if \((t(A_{ij}, B_{ij}, C_{ij}) \geq 0)\) then
6                Accept
7            end
8        end
9    end
10 end
```

**Lemma 12.** If \(\hat{t} \geq 32t(A, B, C)\log^3 n\), \(P(\text{Verify-Estimate } (A, B, C, \hat{t}) \text{ accepts}) \leq \frac{1}{20}\).
Proof. Let \( T(A, B, C) \) denote the set of triangles having vertices \( a \in A, \ b \in B \) and \( c \in C \), where \( A, B \) and \( C \) are disjoint subsets of \( V(G) \). For \( (a, b, c) \in T(A, B, C) \) such that \( a \in A, \ b \in B, \ c \in C \), let \( X_{ij}^{(a,b,c)} \) denote the indicator random variable such that \( X_{ij}^{(a,b,c)} = 1 \) if and only if \( (a, b, c) \in T(A_{ij}, B_{ij}, C_{ij}) \) and \( X_{ij} = \sum_{(a,b,c)\in T(A_{ij}, B_{ij}, C_{ij})} X_{ij}^{(a,b,c)} \). Note that \( t(A_{ij}, B_{ij}, C_{ij}) = X_{ij} \). \( (a, b, c) \) is present in \( T(A_{ij}, B_{ij}, C_{ij}) \) if \( a \in A_{ij}, \ b \in B_{ij} \) and \( c \in C_{ij} \). So,

\[
\Pr(X_{ij}^{(a,b,c)} = 1) \leq \frac{2^i}{i} \cdot \frac{2^j}{2^j} \log n \cdot \frac{1}{2^j} = \frac{\log n}{i}
\]

and \( \mathbb{E}[X_{ij}] \leq \frac{t(A,B,C)}{i} \log n \).

As \( X_{ij} \geq 0 \),

\[
\Pr(X_{ij} \neq 0) = \Pr(X_{ij} \geq 1) \leq \mathbb{E}[X_{ij}] \leq \frac{t(A,B,C)}{i} \log n.
\]

Now using the fact that \( \hat{t} \geq 32t(A,B,C) \log^2 n \), we have \( \Pr(X_{ij} \neq 0) \leq \frac{1}{32 \log^2 n} \).

Observe that Verify-Estimate accepts if and only if there exists \( i, j \in \{0, \ldots, \log n\} \) such that \( X_{ij} \neq 0 \). Using the union bound, we get

\[
\Pr(\text{Verify-Estimate accepts}) \leq \sum_{0 \leq i, j \leq \log n} \Pr(X_{ij} \neq 0) \leq \frac{(\log n + 1)^2}{32 \log^2 n} \leq \frac{1}{20}.
\]

\( \Box \)

**Lemma 13.** If \( \hat{t} \leq \frac{t(A,B,C)}{16 \log n} \), \( \Pr(\text{Verify-Estimate} \ (A, B, C, \hat{t}) \text{ accepts}) \geq \frac{1}{5} \).

*Proof.* For \( p \in \{0, \ldots, \log n\} \), let \( A^p \subseteq A \) be the set of vertices such that for each \( a \in A \), the number of triangles of the form \( (a, b, c) \) with \( b \in B \) and \( c \in C \), lies between \( 2^p \) and \( 2^{p+1} - 1 \).

For \( a \in A^p \) and \( q \in \{0, \ldots, \log n\} \), let \( B^{pq}(a) \subseteq B \) be the set of vertices such that for each \( b \in B \), the number of triangles of the form \( (a, b, c) \) with \( c \in C \) lies between \( 2^q \) and \( 2^{q+1} - 1 \). We need the following result to proceed further.

**Claim 14.**

(i) There exists \( p \in \{0, \ldots, \log n\} \) such that \( |A^p| > \frac{t(A,B,C)}{2^{p+1}(\log n + 1)} \).

(ii) For each \( a \in A^p \), there exists \( q \in \{0, \ldots, \log n\} \) such that \( |B^{pq}(a)| > \frac{2^p}{2^{q+1}(\log n + 1)} \).

*Proof.*

(i) Observe that \( t(A, B, C) = \sum_{p=0}^{\log n} t(A^p, B, C) \) as the sum takes into account all incidences of vertices in \( A \). So, there exists \( p \in \{0, \ldots, \log n\} \) such that \( t(A^p, B, C) \geq \frac{t(A,B,C)}{\log n + 1} \). From the definition of \( A^p \), \( t(A^p, B, C) < |A^p| \cdot 2^{p+1} \). Hence, there exists \( p \in \{0, \ldots, \log n\} \) such that

\[
|A^p| > \frac{t(A^p, B, C)}{2^{p+1}} \geq \frac{t(A,B,C)}{2^{p+1}(\log n + 1)}.
\]

(ii) Observe that \( \sum_{q=0}^{\log n} t(\{a\}, B^{pq}(a), C) = t(\{a\}, B, C) \). So, there exists \( q \in \{0, \ldots, \log n\} \) such that \( t(\{a\}, B^{pq}(a), C) \geq \frac{t(\{a\}, B, C)}{\log n + 1} \). From the definition of \( B^{pq}(a) \), \( t(\{a\}, B^{pq}(a), C) < |B^{pq}(a)| \cdot 2^{q+1} \). Hence, there exists \( q \in \{0, \ldots, \log n\} \) such that

\[
|B^{pq}(a)| > \frac{t(\{a\}, B^{pq}(a), C)}{2^{q+1}} \geq \frac{t(\{a\}, B, C)}{2^{q+1}(\log n + 1)} \geq \frac{2^p}{2^{q+1}(\log n + 1)}.
\]

\( \Box \)
We come back to the proof of Lemma 13. We will show that Verify-Estimate accepts with probability at least $1/5$ when loop executes for $i = p$, where $p$ is such that $|A^p| > \frac{t(A,B,C)}{2^{p+1} \log n + 1}$. The existence of such a $p$ is evident from (i) of Claim 14.

Recall that $A_{pq} \subseteq A, B_{pq} \subseteq B$ and $C_{pq} \subseteq C$ are the samples obtained when the loop variables $i$ and $j$ in Algorithm 1 attain values $p$ and $q$, respectively. Observe that

$$\mathbb{P}(A_{pq} \cap A^p = \emptyset) \leq \left(1 - \frac{2p}{t}\right)^{|A^p|} \leq e^{-\frac{2p}{t}|A^p|} \leq e^{-\frac{2p}{t} \frac{t(A,B,C)}{2^{p+1} \log n + 1}} = e^{-\frac{t(A,B,C)}{2(t \log n + 1)}}.$$

Now using the fact that $\hat{t} \leq \frac{t(A,B,C)}{16 \log n}$ and $n \geq 64$, $\mathbb{P}(A_{pq} \cap A^p = \emptyset) \leq \frac{1}{e^3}$.

Assume that $A_{pq} \cap A^p \neq \emptyset$ and $a \in A_{pq} \cap A^p$. By (ii) of Claim 14 there exists $q \in \{0, \ldots, \log n\}$, such that $B^{pq}(a) \geq \frac{2p}{2^{q+1}(\log n + 1)}$. Observe that we will be done, if we can show that Verify-Estimate accepts when loop executes for $i = p$ and $j = q$. Now,

$$\mathbb{P}(B_{pq} \cap B^{pq}(a) = \emptyset \mid A_{pq} \cap A^p \neq \emptyset) \leq \left(1 - \frac{2q}{2p} \log n\right)^{|B^{pq}(a)|} \leq \frac{1}{e^{3/7}}.$$

Assume that $A_{pq} \cap A^p \neq \emptyset$, $B_{pq} \cap B^{pq}(a) \neq \emptyset$ and $b \in B_{pq} \cap B^{pq}(a)$. Let $S$ be the set such that $(a,b,s)$ is a triangle in $G$ for each $s \in S$. Note that $|S| \geq 2^q$. So,

$$\mathbb{P}(C_{pq} \cap S = \emptyset \mid A_{pq} \cap A^p \neq \emptyset \text{ and } B_{pq} \cap B^{pq}(a) \neq \emptyset) \leq \left(1 - \frac{1}{2^q}\right)^{2^q} \leq \frac{1}{e}.$$

Observe that Verify-Estimate accepts if $t(A_{pq}, B_{pq}, C_{pq}) \geq 0$. Also, $t(A_{pq}, B_{pq}, C_{pq}) \geq 0$ if $A_{pq} \cap A^p \neq \emptyset$, $B_{pq} \cap B^{pq}(a) \neq \emptyset$ and $C_{pq} \cap S \neq \emptyset$. Hence,

$$\mathbb{P}(\text{Verify-Estimate accepts}) \geq \mathbb{P}(A_{pq} \cap A^p \neq \emptyset, B_{pq} \cap B^{pq}(a) \neq \emptyset \text{ and } C_{pq} \cap S \neq \emptyset) \geq \mathbb{P}(A_{pq} \cap A^p \neq \emptyset) \cdot \mathbb{P}(B_{pq} \cap B^{pq}(a) \neq \emptyset \mid A_{pq} \cap A^p \neq \emptyset) \cdot \mathbb{P}(C_{pq} \cap S \neq \emptyset \mid A_{pq} \cap A^p \neq \emptyset \text{ and } B_{pq} \cap B^{pq}(a) \neq \emptyset) \geq \left(1 - \frac{1}{e^3}\right) \left(1 - \frac{1}{e^{3/7}}\right) \left(1 - \frac{1}{e}\right) \geq \frac{1}{5}.$$

\[\square\]

**Algorithm 2: Coarse-Estimate** $(A, B, C)$

**Input:** Three pairwise disjoint sets $A, B, C \subseteq V(G)$.

**Output:** An estimation $\hat{t}$ for $t(A, B, C)$.

1. **begin**
2. \hspace{1em} for $(\hat{t} = n^3, n^3/2, \ldots, 1)$ do
3. \hspace{2em} Repeat Verify-Estimate $(A, B, C, \hat{t})$ for $\Gamma = 2000 \log n$ times. If at least $\frac{\Gamma}{10}$ many Verify-Estimate accepts, then output $\hat{t}$.
4. **end**
Proof of Lemma 8. Note that an execution of COARSE-ESTIMATE for a particular \( \hat{t} \), repeats VERIFY-ESTIMATE for \( \Gamma = 2000 \log n \) times and gives output \( \hat{t} \) if at least \( \frac{\Gamma}{10} \) many VERIFY-ESTIMATE accepts. For a particular \( \hat{t} \), let \( X_i \) be the indicator random variable such that \( X_i = 1 \) if and only if the \( i^{th} \) execution of VERIFY-ESTIMATE accepts. Also take \( X = \sum_{i=1}^{\Gamma} X_i \). COARSE-ESTIMATE gives output \( \hat{t} \) if \( X > \frac{\Gamma}{10} \).

Consider the execution of COARSE-ESTIMATE for a particular \( \hat{t} \). If \( \hat{t} \geq 32t(A,B,C) \log^3 n \), we first show that COARSE-ESTIMATE accepts with probability \( 1 - \frac{1}{n^5} \). Recall Lemma 12. If \( \hat{t} \geq 32t(A,B,C) \log^3 n \), \( P(X_i = 1) \leq \frac{1}{10} \) and hence \( E[X] \leq \frac{\Gamma}{20} \). By using Chernoff-Hoeffding’s inequality (See Lemma 19 (i) in Section A),

\[
P \left( X > \frac{\Gamma}{10} \right) = P \left( X > \frac{\Gamma}{20} + \frac{\Gamma}{20} \right) \leq \frac{1}{n^{10}}.
\]

By using the union bound for all \( \hat{t} \), the probability that COARSE-ESTIMATE outputs some \( \hat{t} \geq 16t(A,B,C) \log^3 n \), is at most \( \frac{3 \log n}{n^{10}} \).

Now consider the instance when the for loop in COARSE-ESTIMATE executes for a \( \hat{t} \) such that \( \hat{t} \leq \frac{t(A,B,C)}{16 \log n} \). In this situation, \( P(X_i = 1) \geq \frac{1}{2} \). So, \( E[X] \geq \frac{\Gamma}{5} \). By using Chernoff-Hoeffding’s inequality (See Lemma 19 (ii) in Section A),

\[
P \left( X \leq \frac{\Gamma}{10} \right) \leq P \left( X \leq \frac{3\Gamma}{20} \right) = P \left( X < \frac{\Gamma}{5} - \frac{\Gamma}{20} \right) \leq \frac{1}{n^{10}}.
\]

By using the union bound for all \( \hat{t} \), the probability that COARSE-ESTIMATE outputs some \( \hat{t} \leq \frac{t(A,B,C)}{16 \log n} \), is at most \( \frac{3 \log n}{n^{10}} \).

Observe that, COARSE-ESTIMATE gives output \( \hat{t} \) that satisfies either \( \hat{t} \geq 16t(A,B,C) \log^3 n \) or \( \hat{t} \leq \frac{t(A,B,C)}{16 \log n} \) is at most \( \frac{3 \log n}{n^{10}} + \frac{3 \log n}{n^{10}} \leq \frac{1}{n^9} \).

Putting everything together, COARSE-ESTIMATE gives some \( \hat{t} \) as output with probability at least \( 1 - \frac{1}{n^9} \) satisfying

\[
\frac{t(A,B,C)}{32 \log n} < \hat{t} < 32t(A,B,C) \log^3 n.
\]

From the description of VERIFY-ESTIMATE and COARSE-ESTIMATE, the query complexity of VERIFY-ESTIMATE is \( O(\log^2 n) \) and COARSE-ESTIMATE calls VERIFY-ESTIMATE \( O(\log^2 n) \). Hence, COARSE-ESTIMATE makes \( O(\log^4 n) \) many queries.

\[\square\]

6 The final triangle estimation algorithm: Proof of Theorem 1

Now we design our algorithm for \( 1 \pm \epsilon \) multiplicative approximation of \( t(G) \). If \( \epsilon \leq \frac{d \log^2 n}{n^{1/4}} \), we query for \( t(\{a\}, \{b\}, \{c\}) \) for all distinct \( a, b, c \in V(G) \) and compute the exact value of \( t(G) \). So, we assume that \( \epsilon > \frac{\log^2 n}{n^{1/4}} \).

We build a data structure such that it maintains two things at any point of time.

- An accumulator \( \psi \) for the number of triangles. We initialize \( \psi = 0 \).

- A set of tuples \( (A_1, B_1, C_1, w_1), \ldots, (A_{\zeta}, B_{\zeta}, C_{\zeta}, w_{\zeta}) \), where tuple \( (A_i, B_i, C_i) \) corresponds to the tripartite subgraph \( G(A_i, B_i, C_i) \) and \( w_i \) is the weight associated to \( G(A_i, B_i, C_i) \). Initially, there is no tuple in our data structure.

The algorithm will proceed as follows.
(1) **(Exact Counting)** Fix the threshold $\tau$ as $\frac{36\kappa_1^2d^2\log^4n}{\epsilon^2}$. Decide whether $t(G) \leq \tau$ by using the result of Lemma 4 where $\kappa_1$ is the constant mentioned in Lemma 3. If yes, we terminate by reporting the exact value of $t(G)$. Otherwise, we go to Step-2. The query complexity of Step-1 is $O(\frac{d^2 \log^5 n}{\epsilon^2})$.

(2) **(General Sparsification)** $V(G)$ is COLORED with $[3]$. Let $A, B, C$ be the partition generated by the coloring of $V(G)$. We initialize the data structure by setting $\psi = 0$ and adding the tuple $(A, B, C, 9/2)$ to the data structure. Note that no query is required in this step. The constant $9/2$ is obtained by putting $k = 1$ in Lemma 3.

(3) We repeat steps 4 to 7 until there is any tuple left in the data structure. We maintain an invariant that the number of tuples stored in the data structure, is at most $\frac{10\kappa_3 \log^{16} n}{\epsilon^2}$, where $\kappa_3$ is a constant to be fixed later.

(4) **(Threshold for Tripartite Graph and Exact Counting in Tripartite Graphs)** For each tuple $(A, B, C, w)$ in the data structure, we determine whether $t(A, B, C) \leq \frac{36\kappa_2^2d^4\log^5 n}{\epsilon^2}$, the threshold, by using the deterministic algorithm corresponding to Lemma 4 with $O(\frac{d^2 \log^5 n}{\epsilon^2} \cdot \log n)$ many queries, where $\kappa_2$ is the constant mentioned in Lemma 7. If yes, we find $t(A, B, C)$ using $O(\frac{d^2 \log^5 n}{\epsilon^2})$ many queries and add $w \cdot t(A, B, C)$ to $\psi$. As there are at most $O\left(\frac{\log^{16} n}{\epsilon^2}\right)$ many triples at any time, the number of queries made in each iteration of the algorithm is $O\left(\frac{d^2 \log^5 n \cdot \log^{16} n}{\epsilon^2}\right) = O\left(\frac{d^2 \log^{21} n}{\epsilon^2}\right)$.

(5) Note that each tuple $(A, B, C, w)$ in this step is such that $t(A, B, C) > \frac{36\kappa_2^2d^4\log^5 n}{\epsilon^2}$. Let $(A_1, B_1, C_1, w_1), \ldots, (A_r, B_r, C_r, w_r)$ be the set of tuples stored at the current instant. If $r > \frac{10\kappa_3 \log^{16} n}{\epsilon^2}$, we go to Step 6. Otherwise, we go to Step 7.

(6) **(Coarse Estimation and Sampling)** For each tuple $(A, B, C, w)$ in the data structure, we find an estimate $\hat{t}$ such that $\frac{t(A, B, C)}{32\log^3 n} < \hat{t} < 32t(A, B, C) \log^3 n$. Note that this can be done due to Lemma 8 and the number of queries is $O(\log^4 n)$ per tuple. Before proceeding further, consider the following Lemma that follows from a Lemma by Beame et al. \cite{bhr+15}. The original statement of Beame et al. is given in Lemma 22 in Section A.

**Lemma 15 (BHR+18).** Let $(A_1, B_1, C_1, w_1), \ldots, (A_r, B_r, C_r, w_r)$ be the tuples present in the data structure and $e_i$ be the corresponding coarse estimation for $t(A_i, B_i, C_i), i \in [r]$, such that

(i) $w_i, e_i \geq 1, \forall i \in [r],$

(ii) $\frac{w_i}{\rho} \leq t(A_i, B_i, C_i) \leq e_i \rho$ for some $\rho > 0$ and $\forall i \in [r],$ and

(iii) $\sum_{i=1}^{r} w_i \cdot t(A_i, B_i, C_i) \leq M.$

Note that the exact values $t(A_i, B_i, C_i)$’s are not known to us. Then there exists an algorithm that finds $(A_1', B_1', C_1', w_1'), \ldots, (A_s', B_s', C_s', w_s')$ such that all of the above three conditions hold and

$$\left| \sum_{i=1}^{s} w_i' \cdot t(A_i', B_i', C_i') - \sum_{i=1}^{r} w_i \cdot t(A_i, B_i, C_i) \right| \leq \lambda S$$

with probability $1 - \delta$; where $S = \sum_{i=1}^{r} w_i \cdot t(A_i, B_i, C_i)$ and $\lambda, \delta > 0$. Also,

$$s = O\left(\lambda^{-2} \rho^4 \log M \left( \log \log M + \log \frac{1}{\delta} \right) \right).$$
We use the algorithm corresponding to Lemma 15 with \( \lambda = \frac{\epsilon}{6 \log n} \), \( \rho = 32 \log^3 n \) and \( \delta = \frac{1}{n^{16}} \) to find a new set of tuples \((A'_i, B'_i, C'_i, w'_i)\), \(i \in [1, n] \) such that \(|S - \sum_{i=1}^n w'_i t(A'_i, B'_i, C'_i)| \leq \lambda S\) with probability \( 1 - \frac{1}{n^{16}} \), where \( S = \sum_{i=1}^n w_i t(A_i, B_i, C_i) \) and \( s = \frac{\kappa_3 \log^{16} n}{\epsilon^2} \) for some constant \( \kappa_3 > 0 \). This \( \kappa_3 \) is same as the one mentioned in Step 3. No query is required to execute the algorithm of Lemma 15. Recall that the number of tuples present at any time is \( O\left(\frac{\log^{16} n}{\epsilon^2}\right) \).

Hence, the number of queries in this step in each iteration, is \( O\left(\frac{\log^6 n \cdot \log^4 n}{\epsilon^2}\right) = O\left(\frac{\log^{20} n}{\epsilon^2}\right) \).

(7) (Sparsification for Tripartite Graphs) We partition each of \( A, B \) and \( C \) into 3 parts uniformly at random. Let \( A = U_1 \cup U_2 \cup U_3; V = V_1 \cup V_2 \cup V_3 \) and \( W = W_1 \cup W_2 \cup W_3 \). We delete \((A, B, C, w)\) from the data structure and add \((U_i, V_i, W_i, 9w)\) for each \( i \in [3] \) to our data structure. Note that no query is made in this step.

(8) Report \( \psi \) as the estimate for the number of triangles in \( G \), when no tuples are left.

First, we prove that the above algorithm produces a \((1 \pm \epsilon)\) multiplicative approximation to \( t(G) \) for any \( \epsilon > 0 \) with high probability. If \( t(G) \leq \frac{36 \kappa_3^2 d^2 \log^4 n}{\epsilon^2} \), then the algorithm terminates in Step-1 and reports the exact number of triangles with probability \( 1 - \frac{1}{n^{16}} \) by Lemma 3. Otherwise, the algorithm proceeds to Step-2. In Step-2, the algorithm colors \( V(G) \) using three colors and incurs a multiplicative error of \( 1 \pm \epsilon \) to \( t(G) \), where \( \epsilon_0 = \frac{\kappa_1 \log n}{\sqrt{t(G)}} \). As \( t(G) > \frac{36 \kappa_3^2 d^2 \log^4 n}{\epsilon^2} \) and \( n \geq 64 \), \( \epsilon_0 \leq \lambda = \frac{\epsilon}{6 \log n} \). Note that the algorithm possibly performs Step-4 to Step-7 multiple times, but not too many times.

Let \((A_1, B_1, C_1, w_1), \ldots, (A_\zeta, B_\zeta, C_\zeta, w_\zeta)\) be the set of tuples present in the data structure currently. We define \( \sum_{i=1}^\zeta t(A_i, B_i, C_i) \) as the number of active triangles. Let \( \text{Active}_{i} \) be the number of triangles that are active in the \( i \)th iteration. Note that \( \text{Active}_{i-1} \leq t(G) \leq n^3 \) by Lemma 7 and Step-7, observe that \( \text{Active}_{i+1} \leq \frac{\text{Active}_{i}}{2} \). So, after 3 log \( n \) many iterations there will be at most constant number of active triangles and then we can compute the exact number of active triangles and add it to \( \psi \). In each iteration, there can be a multiplicative error of \( 1 \pm \lambda \) in Step-5 and \( 1 \pm \epsilon_0 \) due to Step-4. So, using the fact that \( \epsilon_0 \leq \lambda \), the multiplicative approximation factor lies between \((1 - \lambda)^{3 \log n + 1}\) and \((1 + \lambda)^{3 \log n + 1}\). As \( \lambda = \frac{\epsilon}{6 \log n} \), the required approximation factor is \( 1 \pm \epsilon \).

The query complexity of Step 1 is \( O\left(\epsilon^{-1} d^{12} \log^{25} n\right) \). The query complexity of Steps 4 to 6 is \( O\left(\epsilon^{-1} \log^{21} n\right) \) in each iteration and the total number of iterations is \( O(\log n) \). Hence, the total query complexity of the algorithm is \( O\left(\epsilon^{-1} d^{12} \log^{25} n\right) \).

Now, we bound the failure probability of the algorithm. The algorithm can fail in Step-1 with probability at most \( \frac{1}{n^{16}} \), Step-2 with probability at most \( \frac{2}{n^{16}} \), Step-6 with probability at most \( \frac{10 \kappa_3 \log^{16} n}{\epsilon^4} \cdot \frac{1}{n^8} + \frac{1}{n^{16}} \), and Step-7 with probability at most \( \frac{10 \kappa_3 \log^{16} n}{\epsilon^4} \cdot \frac{1}{n^8} \). As the algorithm might execute Steps 4 to 6 for 3 log \( n \) times, the total failure probability is bounded by

\[
\frac{1}{n^{10}} + \frac{2}{n^4} + 3 \log n \left( \frac{10 \kappa_3 \log^{16} n}{\epsilon^4} \cdot \frac{1}{n^8} + \frac{10 \kappa_3 \log^{16} n}{\epsilon^4} \cdot \frac{1}{n^9} + \frac{1}{n^{10}} \right) \leq \frac{c}{n^2}.
\]

Note that the above inequality holds because \( \epsilon > \frac{d \log^2 n}{n^{1/4}} \), and \( n \geq 64 \).

A  Some probability results

**Proposition 16.** Let \( X \) be a random variable. Then \( E[X] \leq \sqrt{E[X^2]} \).

**Lemma 17** (Theorem 7.1 in [DP09]). Let \( f \) be a function of \( n \) random variables \( X_1, \ldots, X_n \) such that
(i) Each $X_i$ takes values from a set $A_i$.
(ii) $\mathbb{E}[f]$ is bounded, i.e., $0 \leq \mathbb{E}[f] \leq M$.
(iii) $\mathcal{B}$ be any event satisfying the following for each $i \in [n]$.
\[
|\mathbb{E}[f \mid X_1, \ldots, X_{i-1}, X_i = a_i, \mathcal{B}^c] - \mathbb{E}[f \mid X_1, \ldots, X_{i-1}, X_i = a'_i, \mathcal{B}]| \leq c_i.
\]

Then for any $\delta \geq 0$,
\[
\mathbb{P}(|f - \mathbb{E}[f]| > \delta + M\mathbb{P}(\mathcal{B})) \leq e^{-\delta^2/\sum_{i=1}^{n} c_i^2} + \mathbb{P}(\mathcal{B}).
\]

**Lemma 18** (Hoeffding’s inequality [DP09]). Let $X_1, \ldots, X_n$ be $n$ independent random variables such that $X_i \in [a_i, b_i]$. Then for $X = \sum_{i=1}^{n} X_i$, the following is true for any $\delta > 0$.
\[
\mathbb{P}(|X - \mathbb{E}[X]| \geq \delta) \leq 2e^{-2\delta^2/\sum_{i=1}^{n} (b_i - a_i)^2}.
\]

**Lemma 19** (Chernoff-Hoeffding bound [DP09]). Let $X_1, \ldots, X_n$ be independent random variables such that $X_i \in [0, 1]$. For $X = \sum_{i=1}^{n} X_i$ and $\mu_i \leq \mathbb{E}[X] \leq \mu_h$, the followings hold for any $\delta > 0$.

(i) $\mathbb{P}(X > \mu_h + \delta) \leq e^{-2\delta^2/n}$.
(ii) $\mathbb{P}(X < \mu_l - \delta) \leq e^{-2\delta^2/n}$.

**Lemma 20** (Theorem 3.2 in [DP09]). Let $X_1, \ldots, X_n$ be random variables such that $a_i \leq X_i \leq b_i$ and $X = \sum_{i=1}^{n} X_i$. Let $\mathcal{D}$ be the dependent graph, where $V(\mathcal{D}) = \{X_1, \ldots, X_n\}$ and $E(\mathcal{D}) = \{(X_i, X_j) : X_i \text{ and } X_j \text{ are dependent}\}$. Then for any $\delta > 0$,
\[
\mathbb{P}(|X - \mathbb{E}[X]| \geq \delta) \leq 2e^{-2\delta^2/\chi^*(\mathcal{D}) \sum_{i=1}^{n} (b_i - a_i)^2},
\]
where $\chi^*(\mathcal{D})$ denotes the fractional chromatic number of $\mathcal{D}$.

The following lemma directly follows from Lemma 20.

**Lemma 21.** Let $X_1, \ldots, X_n$ be indicator random variables such that there are at most $d$ many $X_j$’s on which an $X_i$ depends and $X = \sum_{i=1}^{n} X_i$. Then for any $\delta > 0$,
\[
\mathbb{P}(|X - \mathbb{E}[X]| \geq \delta) \leq 2e^{-2\delta^2/(d+1)n}.
\]

**Lemma 22** ([BHR+18]). Let $(D_1, w_1, e_1), \ldots, (D_r, w_r, e_r)$ are the given structures and each $D_i$ has an associated weight $c(D_i)$ satisfying

(i) $w_i, e_i \geq 1, \forall i \in [r]$;
(ii) $\frac{w_i}{\rho} \leq c(D_i) \leq e_i \rho$ for some $\rho > 0$ and all $i \in [r]$; and
(iii) $\sum_{i=1}^{r} w_i \cdot c(D_i) \leq M$.  

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Note that the exact values $c(D_i)$’s are not known to us. Then there exists an algorithm that finds $(D'_1, w'_1, e'_1), \ldots, (D'_s, w'_s, e'_s)$ such that all of the above three conditions hold and
\[
\left| \sum_{i=1}^{t} w'_i \cdot c(D'_i) - \sum_{i=1}^{r} w_i \cdot c(D_i) \right| \leq \lambda S
\]
with probability $1 - \delta$; where $S = \sum_{i=1}^{r} w_i \cdot c(D_i)$ and $\lambda, \delta > 0$. The time complexity of the algorithm is $O(r)$ and $s = O\left( \frac{\rho^4 \log M \left( \log \log M + \log \frac{1}{\delta} \right)}{\lambda^2} \right)$.

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