Solitons in Chern-Simons Theories of Nonrelativistic CP\textsuperscript{N−1} Models: Spin Textures in the Quantum Hall Effect

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Abstract

Topological solitons in CP\textsuperscript{N−1} models coupled with a Chern-Simons gauge field and a Hopf term are studied both analytically and numerically. These models are low-energy effective theories for the quantum Hall effect (QHE) with internal degrees of freedom, like the QHE in double-layer electron systems. These solitons correspond to skyrmions and merons which are charged quasi-excitations in the QHE. We explicitly show that the CP\textsuperscript{N−1} models describe quite well (pseudo-)spin textures in the original Chern-Simons theory of bosonized electrons.

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1 Introduction

Recently, topological solitons in the Chern-Simons (CS) gauge theories of nonlinear $\sigma$-models are studied rather intensively [1]. These solitons generally have both fractional spin and statistics. In the present paper, we shall consider nonrelativistic counterparts of these models, i.e., nonrelativistic $\text{CP}^{N-1}$ (NRCP$^{N-1}$) models coupled with CS gauge field.

In the previous paper [2] (hereafter referred to as I), we studied the fractional quantum Hall effect (FQHE) in double-layer (DL) electron systems in terms of CS gauge theory of bosonized electrons (CSBE). There quasi-excitations are topological solitons with fractional spin and statistics, and we studied them rather intensively. We also showed that a NRCP$^1$ model appears as an effective low-energy field theory and the above solitons can be reinterpreted as solitons in the NRCP$^1$ model [2, 3]. This NRCP$^1$ model couples with a CS gauge field and has a Hopf term. Therefore solitons have fractional spin and statistics.

In this paper, we shall first study topological solitons in the NRCP$^1$ model for the FQHE in DL electron systems. It is important to know how these solitons in the above two different CS theories are related with each other, not only qualitatively but also quantitatively. In the CSBE for the DL electron systems, there are two CS gauge fields generally. In order to obtain the effective theory of nontrivial pseudo-spin textures like merons, i.e., the NRCP$^1$ model, one CS gauge field is integrated out. As a result, a Hopf term appears which contributes to spin and statistics of solitons. The remaining CS gauge field in the NRCP$^1$ model also contributes to spin and statistics. Total spin and statistics of solitons must be consistent with those obtained by the argument in terms of the Aharonov-Bohm (AB) effect in the original CSBE.

Then, we shall consider generalized models, i.e., CS gauge theories of NRCP$^3$ models which are effective field theories for spin textures in the DL QHE with real spin degrees of freedom. Qualitative and quantitative investigations on solitons are given.
This paper is organized as follows. In Sec. 2, we shall briefly review the CSBE and the NRCP\textsuperscript{1} model, and also properties of solitons in the CSBE. In Sec. 3, solitons in the NRCP\textsuperscript{1} model are studied and compared with solitons in the CSBE. Spin and statistics of solitons are obtained. By the numerical calculation, we obtain explicit form of them. These results show that solitons in the above two CS theories are in good agreement with each other not only qualitatively but also quantitatively. In Sec. 4., the NRCP\textsuperscript{3} models coupled with CS gauge fields are introduced. Topological solitons are studied both qualitatively and quantitatively. Section 5 is devoted for conclusion.

## 2 CSBE and NRCP\textsuperscript{N−1} model

In the paper I, we studied the CSBE for DL FQHE, whose Lagrangian is given by,

\[
\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_{CS},
\]

\[
\mathcal{L}_\psi = -\bar{\psi}_\uparrow (\partial_0 - ia_0^+ - ia_0^-) \psi_\uparrow - \bar{\psi}_\downarrow (\partial_0 - ia_0^+ + ia_0^-) \psi_\downarrow - \frac{1}{2M} \sum_{\sigma=\uparrow,\downarrow} |D^\sigma_j \psi_\sigma|^2 - V[\psi_\sigma],
\]

\[
\mathcal{L}_{CS} = \mathcal{L}_{CS}(a_\uparrow^+) + \mathcal{L}_{CS}(a_\downarrow^-)
\]

\[
= -\frac{i}{4} \epsilon_{\mu\nu\lambda} \left( \frac{1}{p} a_\mu^+ \partial_\nu a_\lambda^+ + \frac{1}{q} a_\mu^- \partial_\nu a_\lambda^- \right),
\]

where \( \psi_\sigma (\sigma = 1, 2 \text{ or } \uparrow, \downarrow) \) is the bosonized electron fields in upper and lower layers, respectively, \( M \) is the mass of electrons and \( p \) and \( q \) are parameters. Greek indices take 0, 1 and 2, while roman indices take 1 and 2. \( \epsilon_{\mu\nu\lambda} \) is the antisymmetric tensor and the covariant derivative is defined as

\[
D^\uparrow_j = \partial_j - ia_j^+ \mp ia_j^- + ieA_j,
\]

where external magnetic field is directed to the \( z \)-axis and in the symmetric Coulomb gauge \( A_j = -\frac{B}{2} \epsilon_{jk} x_k \). \( V[\psi_\sigma] \) is interaction term between electrons like the Coulomb
repulsion and short-range four-body interaction, e.g.,

\[ g_1(|\psi_\uparrow|^4 + |\psi_\downarrow|^4) + g_2|\psi_\uparrow \psi_\downarrow|^2, \text{ etc.} \]  \quad (2.4)

Chern-Simons constraints are obtained by differentiating the Lagrangian with respect to \( a_0^\pm \)

\[
\begin{align*}
\epsilon_{ij}\partial_i a_j^+ &= 2p\bar{\psi}\psi \\
\epsilon_{ij}\partial_i a_j^- &= 2q\bar{\psi}\sigma_3\psi \\
\Psi &= (\psi_\uparrow \psi_\downarrow)^t,
\end{align*}
\]  \quad (2.5)

where \( \epsilon_{ij} = \epsilon_{0ij} \). As we explained in the paper I, \((p + q)\) must be an odd integer times \( \pi \) and \((p - q)\) must be an integer times \( \pi \) for the original electrons to be fermionic.

Partition function in the imaginary-time formalism is given as

\[ Z = \int [D\bar{\psi} D\psi Da] \exp\{\int d\tau d^2x (L_\psi + L_{CS})\}. \quad (2.6) \]

The Lagrangian \( L = L_\psi + L_{CS} \) is invariant under \( U(1) \otimes U(1) \) gauge transformation,

\[
\begin{align*}
\Psi &\rightarrow e^{i\theta_1 + i\theta_2 \sigma_3} \Psi \\
a_\mu^+ &\rightarrow a_\mu^+ + \partial_\mu \theta_1 \\
a_\mu^- &\rightarrow a_\mu^- + \partial_\mu \theta_2.
\end{align*}
\]  \quad (2.7)

Ground state for FQHE is given by the following field configuration,

\[
\begin{align*}
\psi_{\uparrow,0} &= \sqrt{\rho_{\uparrow}} = \sqrt{\bar{\rho}/2}, \\
\psi_{\downarrow,0} &= \sqrt{\rho_{\downarrow}} = \sqrt{\bar{\rho}/2}, \\
eB &= \epsilon_{ij}\partial_i a_{j,0}^+ = 2p\bar{\rho}, \quad \epsilon_{ij}\partial_i a_{j,0}^- = 0, \quad a_{j,0}^+ = eA_j, \\
\nu &= \frac{2\pi\bar{\rho}}{eB} = \frac{\pi}{p},
\end{align*}
\]  \quad (2.8)

where \( \bar{\rho} \) is the average electron density. It is easily verified that the above static and uniform configuration is a solution to the field equations if and only if the filling factor \( \nu \) has specific value given by \((2.9)\).
We assume the following specific form of the potential between bosonized electrons;

\[ V[\psi_\sigma] = \frac{p}{M} (\bar{\psi}\Psi)^2 + \frac{q}{M} (\bar{\psi}\sigma_3\Psi)^2. \]  

(2.10)

It is known that the above short-range repulsions represent the Pauli exclusion principle in the bosonization of fermion in the CS method[4].

From (2.10) and the Bogomol’nyi decomposition, the Hamiltonian is given by

\[ H = \frac{1}{2M} \sum_\sigma |(D^\sigma_1 - iD^\sigma_2)\psi_\sigma|^2 + \frac{\omega_c}{2} (\bar{\Psi}\Psi), \]  

(2.11)

where \( \omega_c = eB/M \). Therefore, the lowest-energy configurations satisfy the following “self-dual” equations,

\[ (D^\uparrow_1 - iD^\uparrow_2)\psi_\uparrow = 0, \quad (D^\downarrow_1 - iD^\downarrow_2)\psi_\downarrow = 0. \]  

(2.12)

Topological solitons are solutions to the self-dual equations (2.12) [5]. We consider only spherical symmetric configurations and therefore they are parameterized as follows,

\[ \psi_\uparrow = \sqrt{\rho} \exp \left( w_1(r) + in_1\theta(x) \right), \]
\[ \psi_\downarrow = \sqrt{\rho} \exp \left( w_2(r) + in_2\theta(x) \right), \]  

(2.13)

where \( r = |\vec{x}| \), \( \theta(x) \) is the azimuthal function \( \theta(x) = \arctan \left( \frac{x_2}{x_1} \right) \), and \( n_1 \) and \( n_2 \) are integers which label topological solitons. It is rather straightforward to show that the self-dual equations (2.12) give the following Toda-type equations of \( w_i(r) \) (\( i = 1, 2 \)),

\[ \frac{d^2 w_i}{d\lambda^2} + \frac{1}{\lambda} \frac{dw_i}{d\lambda} + n_i \frac{\delta(\lambda)}{\lambda} + 2 - 2 \sum_j K_{ij} e^{2w_j} = 0, \]  

(2.14)

\[ \lambda = \sqrt{pp} \cdot r = \frac{r}{\sqrt{2}l_0}, \quad K = \frac{1}{p} \begin{pmatrix} p + q & p - q \\ p - q & p + q \end{pmatrix}, \]  

(2.15)

where \( l_0 \) is the magnetic length \( l_0 = 1/\sqrt{eB} \).
It is shown that the integers \( n_i \) must be nonpositive, \( n_i = 0, -1, -2, ... \), for single-valuedness and regularity of solution at the center of soliton. On the other hand, boundary condition at the infinity is given as

\[
\lim_{\lambda \to \infty} e^{w_i(\lambda)} = \frac{1}{\sqrt{2}}.
\]  

(2.16)

In the paper I, we showed that charges, \( Q \) and \( \bar{Q} \), and spin of solitons, \( S(\text{soliton}) \), are given as

\[
Q \equiv \int d^2 x (\bar{\Psi}\Psi - \bar{\Psi}_0\Psi_0) = \frac{\pi}{p} \left( \frac{n_1 + n_2}{2} \right),
\]

\[
\bar{Q} \equiv \int d^2 x (\bar{\Psi}\sigma_3\Psi - \bar{\Psi}_0\sigma_3\Psi_0) = \frac{\pi}{q} \left( \frac{n_1 - n_2}{2} \right),
\]

\[
S(\text{soliton}) = S_+ + S_-,
\]

\[
S_+ = \frac{\pi}{2p} \left( \frac{n_1 + n_2}{2} \right)^2, \quad S_- = \frac{\pi}{2q} \left( \frac{n_1 - n_2}{2} \right)^2.
\]  

(2.17)

On the other hand, statistical parameter is given as follows by the argument of AB effect,

\[
\alpha(\text{soliton}) = \alpha_+ + \alpha_-, \quad \alpha_+ = -pQ^2, \quad \alpha_- = -q\bar{Q}^2.
\]  

(2.18)

Therefore, the spin-statistics relation \( S = -\frac{1}{2\pi}\alpha \) is satisfied.

In the paper I, we solved the self-dual equations (2.12) for various values of \( p, q, n_1 \) and \( n_2 \) by the numerical calculation. Some of them are given in the paper I and Figs.1 and 2 of the present paper. These solutions will be compared with their counterparts in the NRCP\(^1\) model later on. To this end, normalized amplitudes of \( \psi_\sigma \)'s are defined as

\[
\hat{\psi}_\sigma \equiv \frac{|\psi_\sigma|}{\sqrt{|\psi_\uparrow|^2 + |\psi_\downarrow|^2}}.
\]  

(2.19)

In order to obtain an effective field theory for topological excitations from the CSBE, we first parameterize the bosonized electrons as follows;

\[
\psi_\sigma = J_0^{1/2} \phi \ z_\sigma,
\]
\[
\phi = e^{i\tilde{\theta}}\phi_v \in U(1), \quad z = (z_1 \ z_2)^t \in \text{CP}^1,
\]

(2.20)

where \(e^{i\tilde{\theta}}\) is a regular part and \(\phi_v\) is a topologically nontrivial part, which represents vortex degrees of freedom. CP\(^1\) variable \(z_\sigma\) represents pseudo-spin degrees of freedom and satisfies CP\(^1\) condition \(\sum |z_\sigma|^2 = 1\), and \(\psi_\sigma\) is invariant under \(z_\sigma \rightarrow e^{i\alpha}z_\sigma\) and \(\phi \rightarrow e^{-i\alpha}\phi\). After substituting (2.20) into (2.1), we perform duality transformation in the partition function (2.6). Details of the derivation can be seen in the paper I. Final result is given by the following Lagrangian;

\[
\begin{align*}
\mathcal{L}_E &= \mathcal{L}_z + \mathcal{L}_{CS}(-a^-) -ib_i \cdot \hat{J}_i, \\
\mathcal{L}_z &= -\frac{J_0}{2M} \left[ D_j z \cdot D_j z + (\bar{z} \cdot D_j z)^2 \right], \\
D_\mu z &= (\partial_\mu - ia^-_\mu \sigma_3)z,
\end{align*}
\]

(2.21)

where \(\hat{J}_\mu = J^v_\mu + J^S_\mu\) is the sum of topological currents of vortex and pseudo-spin texture and they are explicitly given as

\[
\begin{align*}
J^v_\mu &= \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu (\bar{\phi}_v \frac{\partial_\lambda}{i} \phi_v), \\
J^S_\mu &= \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu (\bar{z} \frac{D_\lambda}{i} z),
\end{align*}
\]

(2.22)

(2.23)

and the vector field \(b_i\) is related with the total electron density \(J_0\) in (2.20) as

\[
J_0 = \frac{1}{2\pi} \epsilon_{ij} \partial_i b_j.
\]

(2.24)

Furthermore, we are interested in only the states in the lowest Landau level (LLL). The LLL condition imposes\(^1\)

\[
J^v_0 + J^S_0 - \nu^{-1} J_0 + \frac{e}{2\pi} \epsilon_{ij} \partial_i A_j = 0.
\]

(2.25)

\(^1\)Here we neglect the potential term \(V[\psi_\sigma]\) in (2.1). Interaction terms in the NRCP\(^1\) model will be discussed in Sec.3.

\(^2\)More precisely as we showed in the paper I, there appear additional higher-derivative terms like \(\epsilon_{ij} \partial_i J^S_j\) in the LLL condition. However total charge of soliton is determined solely by its topological charge (see the discussion in the paper I).
Therefore, the topological charges determine the electric charge of solitons. From (2.24) and (2.25), the last term of $\mathcal{L}_E$ in (2.21) contains the Hopf term, which gives spin and statistics to solitons. It is not so difficult to show that these are $S_+$ and $\alpha_+$ in (2.17) and (2.18), respectively.

We have discussed the case of two-component internal degrees of freedom so far. It is rather straightforward to extend this formalism. For example, when we consider DL electron systems with spin degrees of freedom, internal space has a (approximate) SU(4) symmetry, and nontrivial spin-pseudo-spin textures are described by a NRCP model.

3 Solitons in the NRCP model

In Sec.2, we briefly reviewed the relationship between the CSBE ((2.1) and (2.2)) and the NRCP model ((2.21)). In this section we shall study topological solitons in the NRCP model rather in detail.

The last term of $\mathcal{L}_E$ in (2.21) is rewritten as follows,

$$\int d^2 x \epsilon_{ij} \partial_0 b_i \{ \phi_v \partial_i \phi_v + \bar{z} \cdot \frac{D_j}{i} z \} + \frac{1}{2\pi} \int d^2 x \epsilon_{ij} b_j \{ \partial_j (\phi_v \partial_0 \phi_v) + \partial_j (\bar{z} \cdot D_0 i z) \}.$$  (3.1)

As we consider static configurations in the following discussion, the first two terms on the right-hand side of (3.1) are ignored. From the LLL condition (2.25),

$$J_0 = \frac{1}{2\pi} \epsilon_{ij} \partial_i b_j = \bar{\rho} + \frac{\pi}{p} (J^v_0 + J^S_0).$$  (3.2)

By substituting (3.2) into (3.1), we obtain

$$\mathcal{L}_E = J_0 (\bar{\phi}_v \partial_0 \phi_v) + J_0 (\bar{z} \cdot i D_0 z) - \frac{J_0}{2M} \{ |D_j z|^2 + (\bar{z} \cdot D_j z)^2 \} - \frac{1}{4q} \epsilon_{\mu\nu\lambda} a^-_{\mu} \partial_{\nu} a^\lambda. $$  (3.3)

The Lagrangian $\mathcal{L}_E$ is invariant under local $U(1) \otimes U(1)$ transformation,

$$z \rightarrow e^{i\phi_1 + i\phi_2} z,$$

$$a^-_{\mu} \rightarrow a^-_{\mu} + \partial_{\mu} \phi_2.$$  (3.4)
Field equation of $a_0^-$ gives the CS constraint

$$\frac{1}{2q}\epsilon_{ij}\partial_i a_j^- = J_0(\bar{z} \cdot \sigma_3 z), \quad (3.5)$$

which corresponds to (2.5) in the CSBE.

In the following discussion, we shall put $J_0 = \bar{\rho}$ in the Lagrangian (3.3). This approximation corresponds to neglecting higher-derivative terms of the field $z_j$. Then the Hamiltonian density is obtained as

$$\mathcal{H}^0_E = \frac{\bar{\rho}}{2M} \{ |D_j z|^2 + (\bar{z} \cdot D_j z)^2 \} = \frac{\bar{\rho}}{2M} |\nabla_j \bar{z}|^2, \quad (3.6)$$

where we have defined $\nabla_\mu z = D_\mu z - (\bar{z} \cdot D_\mu z) z$.

In order to make the Hamiltonian Bogomol’nyi type, we add the following potential term to the Hamiltonian (3.6),

$$U[z] = \frac{q}{M} \bar{\rho}^2 (\bar{z} \cdot \sigma_3 z)^2. \quad (3.7)$$

It is obvious that this term corresponds to the second term of the potential $V[\bar{\psi}_\sigma]$ in (2.10). One may conceive that potential corresponding to the first term of $V[\bar{\psi}_\sigma]$ should be also added. However from (3.2),

$$\left( \bar{\Psi} \Psi \right)^2 = J_0^2$$

$$= \left[ \bar{\rho}^2 + \frac{2\bar{\rho}}{p} J_0^v + J_0^S \right] \left\{ \left( \frac{\bar{\pi}}{p} \right)^2 (J_0^v + J_0^S)^2 \right\}.$$

In Eq.(3.8), the term proportional to $J_0^S$ gives just the topological number and it does not contribute to the field equations, and the term $(J_0^S)^2$ is higher derivative of the field $z_j$.

Using the identity,

$$|\nabla_j z|^2 = |(\nabla_1 - i\nabla_2)z|^2 - \{ 2\pi J_0^S + i\epsilon_{kl} \bar{z} \cdot \partial_k \partial_l z + \epsilon_{ij} \partial_i a_j^- (\bar{z} \cdot \sigma_3 z) \}, \quad (3.9)$$
the total Hamiltonian is given as

\[ H_E = H_0^E + U[z] \]

\[ = \bar{\rho} \left( (\nabla_1 - i\nabla_2)z \right)^2 - \frac{\pi \bar{\rho}}{M} J_0^S. \] (3.10)

We have omitted the possible \( \delta \)-function type potential, which may arise from the singularity of \( z \) at the origin, i.e., \( \epsilon_{kl} \ddot{z}(x) \cdot \partial_k \partial_l z(x) \propto \delta(x) \). This is legitimate, for solitons in the original CSBE with both nonvanishing winding numbers \( n_1 \) and \( n_2 \) have vanishing amplitude at the origin, \( J_0(x = 0) = 0 \), and therefore

\[ J_0(x) \cdot \ddot{z}(x) \cdot \epsilon_{kl} \partial_k \partial_l z(x) = 0, \] (3.11)

for arbitrary \( x \).

From (3.10), the lowest-energy configurations satisfy the following self-dual equations;

\[ (\nabla_1 - i\nabla_2)z = 0. \] (3.12)

Since \( z \) is complex field of two component, the equation (3.12) gives four equations. However, it is easily shown that two of them are redundant by the constraint \( \bar{z} \cdot z = 1 \).

“Gound state” of the self-dual equation (3.12) is given as

\[ z_G = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \] (3.13)

up to the \( U(1) \otimes U(1) \) symmetry (3.4). We parameterize configurations as follows;

\[ z = \begin{pmatrix} X(r)e^{in_1\theta} \\ Y(r)e^{in_2\theta} \end{pmatrix}, \] (3.14)

where \( (r, \theta) \) is the polar coordinates as before. We assume that \( X(r) \) and \( Y(r) \) are real functions which depend only on \( r \), and \( n_1 \) and \( n_2 \) are integers which label solitons, as in the CSBE. At sufficiently large distance,

\[ X(r) \to \frac{1}{\sqrt{2}}, \quad Y(r) \to \frac{1}{\sqrt{2}}, \quad \text{as} \ r \to +\infty. \] (3.15)
From (3.14) and (3.15), the CS gauge field must behave as
\[ a_j^- \to \partial_j \theta \cdot \frac{1}{2}(n_1 - n_2), \quad \text{as} \quad r \to +\infty, \]  
for the energy of configuration, which is obtained from (3.10), to be finite. From (3.16), it is obvious that the CS flux of the \((n_1, n_2)\) soliton is given as
\[ \int d^2 x \epsilon_{ij} \partial_i a_j^- = 2\pi \left( \frac{n_1 - n_2}{2} \right). \]  
Therefore from the CS constraint (3.3), the “axial” charge of the \(\mathbb{CP}^1\) soliton is
\[ \bar{Q}_z \equiv \bar{\rho} \int d^2 x (\bar{z} \cdot \sigma_3 z) = \frac{\pi}{q} \left( \frac{n_1 - n_2}{2} \right) = \bar{Q}, \]  
where the last equality comes from (2.17). Then the argument by the AB effect indicates that the CS gauge field \(a_\mu^-\) gives statistical parameter to the soliton as
\[ -q\bar{Q}_z = \alpha_. \]  
We already mentioned that the Hopf term \(b_i \cdot J_i^S\) in (2.21) gives \(\alpha_+\), and therefore the soliton in the NRCP\(^1\) has the same statistical parameter with that of corresponding soliton in the CSBE, as it should be.

Substituting (3.14) into (1.12), we obtain the following two independent field equations;
\[ \partial_k \ln(X^{-2} - 1)^{\frac{1}{2}} = (n_1 - n_2)\epsilon_{kl} \partial_l \theta - 2\epsilon_{kl} a_l^- , \quad k = 1, 2. \]  
Differentiating Eq. (3.20) by \(\partial_k\) and using the constraint (3.3),
\[ \sum_k \partial_k^2 \ln(X^{-2} - 1)^{\frac{1}{2}} + 4q\bar{\rho}(2X^2 - 1) = 2\pi(n_1 - n_2)\delta^{(2)}(x). \]  
It is useful to introduce the function \(u = \frac{1}{2} \ln(X^{-2} - 1)\) and \(\lambda = \sqrt{\bar{\rho}r} = \frac{r}{\sqrt{2}x_0}\), and
\textbf{Eq.(3.21) gives} \[ \left( \frac{1}{\lambda} \frac{d}{d\lambda} \lambda \frac{d}{d\lambda} \right) u(\lambda) + 4 \left( \frac{p}{q} \right) \frac{1 - e^{2u}}{1 + e^{2u}} = (n_1 - n_2) \frac{\delta(\lambda)}{\lambda}. \] \hspace{1cm} (3.22)

In order to remove the $\delta$-function singularity on the right-hand side of (3.22), we furthermore rewrite it in terms of $v(\lambda) = u(\lambda) - \ln \lambda^{(n_1 - n_2)}$. Field equation of $v(\lambda)$ is then given as

\[ \left( \frac{1}{\lambda} \frac{d}{d\lambda} \lambda \frac{d}{d\lambda} \right) v(\lambda) + 4 \left( \frac{p}{q} \right) \left\{ \frac{1 - \lambda^{2(n_1 - n_2)} e^{2v(\lambda)}}{1 + \lambda^{2(n_1 - n_2)} e^{2v(\lambda)}} \right\} = 0. \] \hspace{1cm} (3.23)

In terms of $v(\lambda)$,

\[ X^2 = \frac{1}{1 + \lambda^{2(n_1 - n_2)} e^{2v(\lambda)}}, \quad Y^2 = \frac{1}{1 + \lambda^{-2(n_1 - n_2)} e^{-2v(\lambda)}}. \] \hspace{1cm} (3.24)

From (3.24), it is obvious that at the spatial infinity,

\[ v(\lambda) \sim - (n_1 - n_2) \ln \lambda, \quad \lambda \to \text{large}. \] \hspace{1cm} (3.25)

On the other hand at the origin, $v(\lambda)$ behaves as

\[ v(\lambda) \sim \beta(n_1, n_2) - \frac{\lambda^2}{2}, \] \hspace{1cm} (3.26)

where the parameter $\beta(n_1, n_2)$ depends on the type of soliton and it is determined by the requirement that the solution satisfies the boundary condition at the spatial infinity (3.25) and (3.13).

\textbf{A comment on the solutions to field equation (3.23) is in order.} In the usual case, amplitude of field which has nontrivial winding number must vanish at the origin by the requirement of the uniqueness and the single-valuedness. Therefore one may conceive that topological solitons which have both nonvanishing winding numbers $n_1$ and $n_2$ cannot exist in the present model, for vanishing of both amplitudes, $X(r)$ and

\textbf{3}It should be remarked that the NRCP\textsuperscript{1} model \textit{does not} contain the parameter $p$ and its solitons are independent of the value of $p$ in the original CSBE. Therefore using $\lambda$ is merely for convenience sake, and $\lambda$ measures the distance in the unit of the magnetic length $l_0$.\textsuperscript{12}
$Y(r)$, contradicts the CP$^1$ condition. However as we showed in the paper I and Sec.2 of the present paper, in the original CSBE the sum of electron densities in the upper and lower layers of the above type solutions vanishes at the origin, i.e., $J_0(r = 0) = 0$ for $n_1 \neq 0$ and $n_2 \neq 0$, and therefore the solution is meaningful even if the CP$^1$ part violates the uniqueness condition. Then we do not require the uniqueness condition for the solutions in the NRCP$^1$ model and consider, for example, $(-2, -1)$ soliton, in which at least one of the components of the field $z$ does not vanish at the origin.

It should be compared with the corresponding $(-2, -1)$ solution in the CSBE after normalization.

Let us turn to the results of the numerical calculation. They are given in Figs.1 and 2. For Halperin’s $(m, m, n)$ state, the parameters $p$ and $q$ are given by $p = \frac{\pi}{2}(m + n)$ and $q = \frac{\pi}{2}(m - n)$, as before. Corresponding configurations of $z$ field obtained from the solutions in the CSBE by normalization (2.19) are also shown in the figures. Quantitative agreement of them is obvious. Therefore, we can conclude that the NRCP$^1$ model describes quite well pseudo-spin textures in the original CSBE.

Especially, solitons in the CSBE depend on the parameters $p$ and $q$, whereas those in the NRCP$^1$ model depend on only $q$. That is, solitons in the CSBE’s with same $q$ but with different $p$ correspond to the same soliton in the NRCP$^1$ model.\footnote{Please recall that value of $p$ determines the filling factor as (2.4). Therefore, this result means that a series of solitons at different filling factors in the CSBE have the same form after normalization.} For example, $(-1, 0)$ meron in the $p = \frac{5\pi}{2}$ and $q = \frac{\pi}{2}$ CSBE and that in the $p = \frac{9\pi}{2}$ and $q = \frac{\pi}{2}$ CSBE both correspond to $(-1, 0)$ meron in the $q = \frac{\pi}{2}$ NRCP$^1$. We numerically examined this case and found that after normalization the above two soliton solutions have almost the same form and it coincides with the $(-1, 0)$ meron in the NRCP$^1$ model (see Fig.2). In this sense, the NRCP$^{N-1}$ models are more universal than the CSBE.
4 Solitons in the NRCP$^3$ model

In the previous section, we studied topological solitons in the NRCP$^1$ model both qualitatively and numerically. It was shown that they are in good agreement with those corresponding to them in the CSBE. In this section, we shall consider a generalization of the previous model, i.e., the NRCP$^3$ model coupled with a pair of CS gauge fields. This model can be regarded as an effective low-energy theory which describes nontrivial configurations in the internal space of the DL QHE with real spin degrees of freedom. It is rather straightforward to introduce bosonized electrons of pseudo-spin suffix $\sigma = \uparrow, \downarrow$ and spin suffix $\eta = \uparrow, \downarrow, \psi_{\sigma,\eta}$. Then CP$^3$ field $Z$ is introduced as

$$Z = (z_{\uparrow\uparrow} z_{\uparrow\downarrow} z_{\downarrow\uparrow} z_{\downarrow\downarrow})^t \in \mathbb{CP}^3,$$

$$\psi_{\sigma,\eta} = J_0^{1/2} \phi z_{\sigma,\eta}.$$  

$$\text{(4.1)}$$

We study NRCP$^3$ model which is defined by the following Lagrangian$^5$

$$L_{E}^{CP^3} = J_0 (\bar{Z} \cdot iD_0 Z) - \frac{J_0}{2M} \left\{ |D_j Z|^2 + (\bar{Z} \cdot D_j Z)^2 \right\} - \sum_{l=1,2} \frac{1}{4q(l)} \epsilon_{\mu\nu\lambda} a_{\mu}^{(l)} \partial_{\nu} a_{\lambda}^{(l)},$$  

$$\text{(4.3)}$$

where the coefficients of the CS terms, $q(l)$ ($l = 1, 2$), are parameters which determine the numbers of CS fluxes attaching to the bosonized electrons $\psi_{\sigma,\eta}$ and their values are determined by Coulomb interactions between electrons.$^6$ The covariant derivative in (4.3) is defined as

$$D_\mu Z \equiv (\partial_\mu - ia_{\mu}^{(l)} \Gamma^{(l)})Z,$$  

$$\text{(4.4)}$$

where $(4 \times 4)$ matrices $\Gamma^{(l)}$ are given by

$$\Gamma^{(1)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma^{(2)} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$  

$$\text{(4.5)}$$

$^5$In the present case, we consider only the total spin-singlet and pseudo-spin-singlet quantum Hall state. General cases will be discussed in a future publication.

$^6$In Ref.$^3$, it is shown by numerical calculation that generalized Halperin state $\Psi_{lmn}$ is a good variational wave function for the ground state of DL QHS with real spin degrees of freedom. The parameters $q(l)$ ($l = 1, 2$) in the NRCP$^3$ CS theory are related with $(l, m, n)$ as $q^{(1)} \propto (l - n)$ and $q^{(2)} \propto (l - m)$.
We add the following repulsive interactions as required by the CS bosonization method,

\[ V[Z] = \sum \frac{q^{(l)}}{M} \tilde{\rho}^2 \left( \tilde{Z} \cdot \Gamma^{(l)} Z \right)^2. \]  

(4.6)

Total Hamiltonian is given by

\[ \mathcal{H}_{E}^{CP3} = \frac{\tilde{\rho}}{2M} \left| (\nabla_1 - i \nabla_2) Z \right|^2 - \frac{\tilde{\rho}}{2M} J^S_{0}. \]  

(4.7)

Then lowest-energy configurations satisfy the self-dual equations,

\[ (\nabla_1 - i \nabla_2) Z = 0, \]  

(4.8)

where \( \nabla_\mu Z = D_\mu Z - (\tilde{Z} \cdot D_\mu Z) Z \). The system is invariant under the following gauge transformations,

\[ Z \rightarrow e^{i\alpha_1 \Gamma^{(1)} + i\alpha_2 \Gamma^{(2)}} Z, \]  

\[ a_\mu^{(l)} \rightarrow a_\mu^{(l)} + \partial_\mu \alpha_l, \ l = 1, 2 \]  

and also

\[ Z \rightarrow e^{i\omega} Z. \]  

(4.9)

There is a topological current which is invariant under (4.9),

\[ j^S_\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \left( \tilde{Z} \frac{D_\lambda}{i} Z \right). \]  

(4.10)

As in the previous case, we shall seek topological solitons of the spherical symmetry,

\[ Z = \begin{pmatrix} F(r)e^{in_1 \theta} \\ G(r)e^{in_2 \theta} \\ H(r)e^{in_3 \theta} \\ R(r)e^{in_4 \theta} \end{pmatrix}. \]  

(4.11)

For example, configuration which corresponds to a skyrmion of real spin in the upper layer is described by

\[ n_1 = -2, \ n_2 = n_3 = n_4 = 0. \]  

(4.12)
It is useful to parametrize $F(r), G(r), H(r)$ and $R(r)$ as

\[
F^2 = \frac{e^{2(u_1+u_2)}}{1 + e^{2(u_1+u_2)} + e^{2u_2} + e^{2u_3}},
\]

\[
G^2 = \frac{e^{2u_2}}{1 + e^{2(u_1+u_2)} + e^{2u_2} + e^{2u_3}},
\]

\[
H^2 = \frac{e^{2u_3}}{1 + e^{2(u_1+u_2)} + e^{2u_2} + e^{2u_3}},
\]

\[
R^2 = \frac{1}{1 + e^{2(u_1+u_2)} + e^{2u_2} + e^{2u_3}}.
\]

(4.13)

Substituting (4.11) and (4.13) into the self-dual equation (4.8),

\[
\frac{d^2u_1}{dr^2} + \frac{1}{r} \frac{du_1}{dr} + (n_1 - n_2) \frac{\delta(r)}{r} - 4q^{(2)} \cdot (\bar{Z}\Gamma^{(2)} Z) = 0,
\]

\[
\frac{d^2u_2}{dr^2} + \frac{1}{r} \frac{du_2}{dr} + (n_2 - n_4) \frac{\delta(r)}{r} - 4q^{(1)} \cdot (\bar{Z}\Gamma^{(1)} Z) = 0,
\]

\[
\frac{d^2u_3}{dr^2} + \frac{1}{r} \frac{du_3}{dr} + (n_3 - n_4) \frac{\delta(r)}{r} - 4q^{(2)} \cdot (\bar{Z}\Gamma^{(2)} Z) = 0,
\]

(4.14)

where

\[
\bar{Z}\Gamma^{(1)} Z = F^2 + G^2 - H^2 - R^2,
\]

\[
\bar{Z}\Gamma^{(2)} Z = F^2 - G^2 + H^2 - R^2,
\]

and we have set $\bar{\rho} = 1$ for simplicity.

From (4.14), it is not difficult to get behaviours of $u_i$ ($i = 1, 2, 3$) for $r \sim 0$. For example for soliton ($\{n_i\} = (-1, 0, 0, 0)$

\[
u_1 \sim \ln r + a_1 + q^{(2)} \cdot \left\{\frac{-e^{2a_2} + e^{2a_3} - 1}{1 + e^{2a_2} + e^{2a_3}}\right\} r^2,
\]

\[
u_2 \sim a_2 + q^{(1)} \cdot \left\{\frac{e^{2a_2} - e^{2a_3} - 1}{1 + e^{2a_2} + e^{2a_3}}\right\} r^2,
\]

\[
u_3 \sim a_3 + q^{(2)} \cdot \left\{\frac{-e^{2a_2} + e^{2a_3} - 1}{1 + e^{2a_2} + e^{2a_3}}\right\} r^2,
\]

(4.15)

where $a_i$ ($i = 1, 2, 3$) are parameters which are determined by the boundary condition at the spatial infinity. There are three “interesting” ground states in the present
system;

\[ Z_G = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{4}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \]  

(4.16)

The first one corresponds to the configuration in which the upper layer is filled with electrons with up spin and the lower layer is filled with those with down spin, etc. General “ground state” is given by an arbitrary linear combination of the first two states in (4.16). The parameters \( a_i \) in (4.15) are determined by the requirement that at the spatial infinity the solutions (4.13) approach one of the ground states in (4.16).  

The topological charge in (4.10) depends on the winding number \( n_i \) \((i = 1, 2, 3, 4)\) and also which ground state soliton lives in, i.e. the behaviour at the spatial infinity.

We have studied solitons for various values of \( \{n_i\} \) and for the “ground states” (4.16) by numerical calculation and verified that stable solutions really exist. Some of them are given in Figs.3, 4, 5 and 6. Most of them have similar size with those in the NRCP\(^1\) model. Behaviours of \( F(r) \), \( G(r) \), etc, are different from each other and it is very interesting to observe them by experiment of the bilayer quantum Hall state. We have also observed that there are various (one-parameter family of) stable solutions for fixed \( \{n_i\} \) and the boundary condition at the spatial infinity. For example \((-1, 0, 0, 0)\)-type soliton, behaviours of \( G(r) \) and \( H(r) \) change whereas \( F(r) \) and \( R(r) \) are almost stable in various solutions. Actually, the self-dual equations (4.14) have the following scale-invariant component \( U(r) \equiv u_1(r) - u_3(r) \) whose field equation is

\[ \frac{d^2U}{dr^2} + \frac{1}{r} \frac{dU}{dr} + (n_1 - n_2 - n_3 + n_4) \frac{\delta(r)}{r} = 0. \]  

(4.17)

In real materials, Coulomb interactions between electrons fix this degrees of freedom.

\(^{7}\)More precisely, type of soliton chooses boundary condition at the spatial infinity. For example, the \((-1, 0, 0, 0)\) soliton can exist in the first and the third ground states in (4.16) but not in the second.
5 Conclusion

In this paper, we studied (pseudo-)spin textures in the QHE with internal symmetry. Especially we examined relationship between the CSBE and the low-energy effective field theory, the NRCP\(^{N-1}\) models, and found that the NRCP\(^{N-1}\) models describe quite well topological solitons in the original CSBE. In the NRCP\(^1\) model, the \(\text{CP}^1\) variables couple with CS gauge field and there is the Hopf term, both of which give fractional spin and fractional statistics to solitons. Total spin and statistics of solitons coincide with those of solitons in the CSBE, and numerical calculation shows that forms of these solitons are very close to each other.

We also studied solitons in the CS gauge theory of the NRCP\(^3\) model and obtained topological soliton solutions. Most of them have similar size with those in the NRCP\(^1\) model and are hopefully detectable by experiment.

QHS’s which appear as a result of the condensation of these solitons are under study and their properties will be reported in a future publication[10].
References

[1] C.Kim, C.Lee, P.Ko, B.-H. Lee and H.Min, Phys. Rev. D48(1993)1821;
    G.Nardelli, Phy.Rev.Lett.73(1994)2524;
    Y.M.Cho and K.Kimm, Phys.Rev.D52(1995)7325;
    K.Kimm, K.Lee and T.Lee, Phys.Rev.D53(1996)4436; Phys.Lett.B380(1996)303;
    K.Arthur, D.H.Tchrakian and Y.Yang, Phys.Rev.D54(1996)5245;

[2] I.Ichinose and A.Sekiguchi, `cond-mat/9610054` to appear in Nucl.Phys.B.

[3] T.H.Hansson, A.Karlhede and J.M.Leinaas, Phys.Rev.B54(1996)R11110.

[4] Z.F.Ezawa, M.Hotta and A.Iwazaki, Phys.Rev.B46(1993)7765.

[5] For recent review, see G.Dunne, “Self-Dual Chern-Simons Theories” (Springer
    Verlag, 1995).

[6] D.H.Lee and C.L.Kane, Phys.Rev.Lett.64(1990)1313.

[7] S.L.Sondli, A.Karlhede and S.A.Kivelson, Phys.Rev.B47(1993)16419.

[8] F.Wilczek and A.Zee, Phys.Rev.Lett.51(1983)2250.

[9] T.Nakajima and H.Aoki, Physica B184(1993)91; Phys.Rev.B51(1995)7874.

[10] I.Ichinose, M.Onoda and A.Sekiguchi, paper in preparation.
Figure 1: Comparison between solutions of the effective NRCP\(^1\) theory and normalized solutions of the original CSBE; the former and the latter are drawn in black lines and gray lines, respectively.
Figure 2: Scaling property of (-1,0)-meron solution \( \xi = \sqrt{\frac{q}{p}} \rho r \), \( \eta = \sqrt{\frac{p}{q}} \rho r \). The agreement is obvious.
Figure 3: Numerical solution of $(−1,0,0)$-type soliton $(q^{(1)} = q^{(2)} = \frac{1}{2})$. $\lambda = \sqrt{\frac{2\pi}{\rho r}}$. 

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Figure 4: Numerical solution of \((-1, -1, 0, 0)-\)type soliton \((q^{(1)} = q^{(2)} = \frac{\pi}{2})\).
Figure 5: Numerical solution of \((-2, 0, 0, 0)\)-type soliton \((q^{(1)} = q^{(2)} = \frac{1}{2})\).
Figure 6: Numerical solution of \((-1, 0, 0, -1)-type soliton \((q^{(1)} = q^{(2)} = \frac{\pi}{4})\).