Equation of Motion for a Spin Vortex and Geometric Force

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ABSTRACT

The Hamiltonian equation of motion is studied for a vortex occurring in 2-dimensional Heisenberg ferromagnet of anisotropic type by starting with the effective action for the spin field formulated by the Bloch (or spin) coherent state. The resultant equation shows the existence of a geometric force that is analogous to the so-called Magnus force in superfluid. This specific force plays a significant role for a quantum dynamics for a single vortex, e.g., the determination of the bound state of the vortex trapped by a pinning force arising from the interaction of the vortex with an impurity.
1. Introduction

The quantum vortices in superfluids is one of most attractive subjects in condensed matter physics. Among many aspects in vortex phenomena, the dynamics of many vortices has been developed by starting with the relevant assumption on the boson superfluid \cite{1}, which reproduced the well known form for the Hamiltonian equation for the assembly of vortices \cite{2}. The quantization based on the Hamiltonian equation has also been studied. \cite{3} Recently, a refined formulation has been given for the quatum treatment for superfluid vortex in the framework of the generalized Hamiltonian dynamics starting with the Landau-Ginzburg action \cite{4}. Besides the superfluid vortex, the other types of quantum vortex has been interested for some time, typically, the vortex in the Heisenberg ferromagnet (see for example, \cite{5}). The occurrence of vortex in ferromagnet is quite natural, if one notes a close resemblance between the superfluid He4 and the ferromagnet as quantum condensate, especially, in the vicinity of the ground state. Following the procedure developed in the superfluid, the Hamiltonian dynamics has been studied for vortices occurring in the 2-dimensional spin condensates. \cite{6}

Apart from the quantum dynamics of assembly of vortices, there has been long interest in the peculiar behavior of the motion of a single vortex mainly relation to the type II superconductors. \cite{7,8} One of the main object of this is concerning the existence of a specific force called Magnus force. This specific force is known to occur when a vortex moves in the uniform stream and play a role to explain some characteristic features of the type II superconductors, (see e.g. \cite{9}) The Magnus force is also known to play a crucial role in the dissipative processes in the superfluid: the smallness of the critical velocity, the attenuation ratio of the second sound wave in the rotating superfluid He4 and so on. \cite{10}

The purpose of this paper is to put forward the equation of motion for a spin vortex for the ferromagnetic system within the Hamiltonian formulation of quantum vortex previously developed. \cite{6} As a consequence of this, we naturally arrive at a force of Magnus type. Indeed, if one considers the resemblance
between the superfluid He and the ferromagnet as quantum condensates, it is natural to expect a realization of such an analogous force. This force should be called the ”geometric force”, which differs from the ordinary force derived from a potential function. We also show the other type of force called the pinning force, which comes from the interaction between a vortex and an impurity immersed in condensate. It is shown that the effect of the pinning force is realized by the bound state of the vortex trapped in the pinning potential.

2. Spin field Lagrangian

Our starting point is the spin coherent state (or Bloch state):

\[ |{\{z(x)\}}\rangle = \prod_{\vec{n}} |z_{\vec{n}}\rangle \]  

(2.1)

where \( \vec{n} \) means the vector assigning the lattice point. Each component is given by the SU(2) coherent state:

\[ |z\rangle = \frac{1}{\sqrt{1 + |z|^2}} \exp[z \hat{J}_+ |0\rangle \]  

(2.2)

where \( |0\rangle = |J, -J\rangle \) is the lowest state satisfying \( \hat{J}_- |0\rangle = 0 \) and \( \hat{J}_\pm \) are ladder operator and \( z \) takes any complex values. Using (2.1), we have the action function

\[ S = \int \langle \{z(x)\} |i\hbar \frac{\partial}{\partial t} - \hat{H} |\{z(x)\}\rangle dt = \int \mathcal{L} d^2x dt \]  

(2.3)

the Lagrangian density is given as

\[ \mathcal{L} = \frac{i\hbar}{2} \frac{z^* \dot{z} - \dot{z}^* z}{1 + |z|^2} - H(\{z(x)\}, \{z(x)\}^*) \]  

(2.4)

Here note that the first term is a counterpart of the so-called ”geometric phase”,

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which is represented in terms of the overcomplete set: \[ 12 \]
\[ \Gamma = \oint \langle Z | i\hbar \frac{\partial}{\partial t} | Z \rangle dt \tag{2.5} \]

The quantization may be realized by the constructing the propagator
\[ K = \langle z(x) | \exp[-\frac{iH}{\hbar}T] | z(x) \rangle = \int \exp[i \int L dx dt] \prod_{x,t} d\mu [z(x, t)] \tag{2.6} \]

We shall now adopt the Heisenberg spin model for analyzing the vortex dynamics. As the Hamiltonian, we take the continuous version of the nearest neighbour interaction of anisotropic type:
\[ \hat{H} = \frac{g}{2} \{(\nabla \hat{J}_1)^2 + (\nabla \hat{J}_2)^2 + \lambda (\nabla \hat{J}_3)^2 \} \tag{2.7} \]

where the parameter means the degree of anisotropy, which is assumed to be \( 0 < \lambda < 1 \), and this makes the system to favour the planer spin configuration. The expectation value becomes
\[ H = \frac{g}{2} \{(\nabla J_1)^2 + (\nabla J_2)^2 + \lambda (\nabla J_3)^2 \} \tag{2.8} \]

We have the more transparent form if we use the angle variables via the stereographic projection,
\[ z = \tan \frac{\theta}{2} e^{-i\phi} \tag{2.9} \]

with \( 0 \leq \theta \leq \pi \), \( 0 \leq \phi \leq 2\pi \), or the angular expression for the spin variables, then it is also written as
\[ H = \frac{g}{2} J^2 \{(\cos^2 \theta + \lambda \sin^2 \theta)(\nabla \theta)^2 + \sin^2 \theta (\nabla \phi)^2 \} \tag{2.10} \]

The second term may be called the "fluid kinetic energy":
\[ T \equiv \frac{1}{2} g J^2 \int \sin^2 \theta (\nabla \phi)^2 d^2 x \tag{2.11} \]
The variation principle results in the field equation for the angle variable, namely,

$$\delta \int [J \dot{h} (1 - \cos \theta) \dot{\phi} - H(\theta, \phi)]d^2xdt = 0 \quad (2.12)$$

which yields

$$Jh \sin \theta \frac{\partial \theta}{\partial t} = -\frac{\partial H}{\partial \phi},$$
$$Jh \sin \theta \frac{\partial \phi}{\partial t} = \frac{\partial H}{\partial \theta} \quad (2.13)$$

3. Equation of Motion for a Spin Vortex

We consider a static solution exhibiting the vortex, which is given from a static form of the variation equation, namely, the vortex solution is characterized by choosing the azimuthial angle as

$$\phi = \tan^{-1} \frac{y}{x} \quad (3.1)$$

and furthermore the lateral angle $\theta$ can be chosen as a function of the radial variable $r$ only. Hence,

$$H = \int [r(\cos^2 \theta + \lambda \sin^2 \theta) \left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2} \sin^2 \theta]dr \quad (3.2)$$

The equation for the profile function $\theta$ is derived from the condition such that $H$ takes an extremum. This may be solved by imposing the proper boundary condition for $\theta(r)$ such that the $z$-component spin $J_3(x)$ directs upward inside the core (of radius $a$) and vanishes outside the core. Physically speaking, this boundary condition may be considered to be an idealization of the feature that the spin configuration is planer at infinity. Instead solving exactly, it may be
possible to simulate the profile for $\theta(r)$; for example we can choose

$$
\theta(r) = \begin{cases} 
cr & (0 \leq r \leq a \equiv \pi \frac{r}{2c}) \\
\frac{\pi}{2} & (a \leq r)
\end{cases} (3.3)
$$

We shall now turn to the dynamics for a single vortex. We first adopt the polar form for the complex field, namely, the stereographic projection given above. The physical meaning of the angle $\phi$ is crucial, namely, that is assumed to coincide with the polar angle in the coordinate plane $(x, y)$, which is measured from the vortex center coordinate $X(t) = (X(t), Y(t))$:

$$
\phi = \mu \tan^{-1} \frac{y - Y(t)}{x - X(t)} (3.4)
$$

Here the coefficient $\mu$ stands for the vortex strength. Now, the canonical term is expressed as

$$
L_C = \int \frac{Jh}{2} (1 - \cos \theta) \dot{\phi} d^2x (3.5)
$$

By making use of the chain rule,

$$
\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial X} \frac{dX}{dt} (3.6)
$$

together with the fact that the gradient of the phase function gives the velocity field, namely, if we note the phase is given as a function of $x - X$, we get

$$
\frac{\partial \phi}{\partial X} = \nabla \phi = v (3.7)
$$

where $v$ denotes the velocity field coming from the vortex we are concerned.
Hence we get for the canonical term

\[ L_C = \int \frac{J\hbar}{2} (1 - \cos \theta) \nabla \phi \mathbf{X} dx \quad (3.8) \]

or which can be written as the differential one form:

\[ \omega = \int \eta \mathbf{v} \cdot d\mathbf{X} d^2x. \quad (3.9) \]

where the following quantity is defined:

\[ \eta = \frac{J\hbar}{2} (1 - \cos \theta) \quad (3.10) \]

By using this notation, the boundary condition for \( \theta \) assigned in the above takes over to the profile function \( \eta \) in the canonical term: \( \eta \to 0 \) (for \( r = 0 \)) and \( \eta \to \frac{J\hbar}{2} \) (for \( r \to \infty \)). This feature is similar to the vortex for bose fluid. \(^{[4]}\)

Further we note the fact that the velocity field is written as

\[ \mathbf{v} = \mu \mathbf{k} \times \nabla \log |\mathbf{x} - \mathbf{X}| \quad (3.11) \]

**Canonical Term:**

We shall derive the Lagrangian for the motion of a vortex which is described by the vortex center coordinate. It is firstly noted that the contribution from the Hamiltonian, (3.2), becomes constant and not relevant to the dynamics of the vortex. So the term which we need is the canonical term \( L_C \). Using the expression (3.8), we get \(^{[13]}\)

\[ L_C = \int d^2x \eta(\mathbf{x} - \mathbf{X})(\mathbf{v}(\mathbf{x} - \mathbf{X}) \cdot \dot{\mathbf{X}}) \]

\[ = I_1 \dot{X} + mI_2 \dot{Y}, \quad (3.12) \]

where \( I_\alpha \) in (3.12) is given by

\[ I_\alpha = \int \eta \mathbf{v}_\alpha d^2x \quad (3.13) \]

The integrand of \( I_\alpha \) does not decrease enough fast at the large distance and the result depends on how to take the limit, \( r \to \infty \), in the integral boundary.

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may be handled by a sort of "regularization" and some care should be taken to get the correct results. Having carried out the regularization to evaluate the integral, (see the appendix ), the final result for the canonical term follows

$$L_C = \frac{\eta_0 \mu}{2} (Y \dot{X} - X \dot{Y})$$  \hspace{1cm} (3.14)

If we note the boundary condition for $\theta = \frac{\pi}{2}$ at $r \to \infty$, the value of $\eta$ is given by

$$\eta_0 = \frac{Jh}{2}$$  \hspace{1cm} (3.15)

The formula (3.14) is the main result of the present paper.

**Pinning Effect by Impurity:**

Next we consider the role of the impurity effect for a vortex within the effective theoretical approach. It may be plausible that the interaction comes from the magnetic origin. The simplest choice satisfying this criterion may be given as follows:

$$L_{\text{pin}} = G \int d^2 x J_3(x) s_3(x),$$  \hspace{1cm} (3.16)

where $G$ is a coupling constant. Here $s_3(x)$ represents the spin that is carried by a magnetic impurity. Let us consider the case that one magnetic impurity is located at $Y$, and has the well-localized density distribution. Then we can take the delta function approximation for such a spin distribution

$$s_3(x) = s_3 \delta^2(x - Y).$$  \hspace{1cm} (3.17)

where $s_3 = \pm \frac{1}{2}$. With the aid of (3.3), the integral in (3.16) is calculated easily to result in the Lagrangian for the effective interaction between the vortex and impurity:

$$L_{\text{impurity}}(\equiv U_{\text{pin}}) = \begin{cases} 
  G s_3 \cos \left[ \frac{\pi}{2a} |X - Y| \right] & (|X - Y| < a) \\
  0 & (|X - Y| > a)
\end{cases}$$  \hspace{1cm} (3.18)

Here, if the coupling constant $G$ is positive, the sign of spin $s_3$ is chosen such that the potential of the first half of (3.18) becomes attractive.
Hamiltonian Equation of Motion:

Now combining (3.14), (3.18), the effective lagrangian for one vortex in the presence of impurity becomes

\[ L_{\text{eff}} = \frac{\eta_0 \mu}{2} (Y \dot{X} - X \dot{Y}) - H_{\text{eff}} \tag{3.19} \]

Here the second term is the Hamiltonian for the single vortex, which is nothing but the pinning potential; \( H_{\text{eff}} = U_{\text{pin}} \). The equation of motion for it is obtained as the Euler-Lagrange equation:

\[ \dot{X} = \frac{\partial H_{\text{eff}}}{\partial Y}, \quad \dot{Y} = -\frac{\partial H_{\text{eff}}}{\partial X}, \tag{3.20} \]

which leads to

\[ \frac{\mu \eta_0}{2} (k \times \dot{X}) = \frac{\partial U_{\text{pin}}}{\partial X}. \tag{3.21} \]

where the vector \( k \) denotes the unit vector perpendicular to the \((x,y)\) plane.

(3.20) can be regarded as a special case of the canonical equation of motion,\(^{[4,6]}\) where the pair \((X,Y)\) should be regarded as a canonical pair each other.

From the physical point of view, the first term is nothing but the ”geometric force”, while the second term represents the pinning force derived from the pinning potential, which is the usual force derived from the potential function.

**Bound State by Pinning Potential**

The motion of vortex can be quantized if we note the vortex coordinate \((X,Y)\) is a canonical pair, which leads to the bound state spectra formed of the vortex trapped by the pinning potential. We estimate it by using the Bohr-Sommerfeld (BS) quantization. The BS rule is given by

\[ \frac{\mu \eta_0}{2} \oint_C (X dY - Y dX) = 2\pi n \hbar \tag{3.22} \]

Here \( C \) means a loop which is defined as \( U_{\text{pin}} = E \). For simplicity, the center of impurity is assumed to be placed at the origin \( Y = 0 \), and \( Gs_3 < 0 \), so the
energy contour is given by

\[ Gs_3 \cos \frac{\pi}{2a} |X| = E \]  \hspace{1cm} (3.23)

From this form, the loop C becomes a circle; \(|X| = \sqrt{X^2 + Y^2} \equiv \rho\), hence the quantization rule turns out to be

\[ \frac{\mu_0}{2} \pi \rho^2 = 2\pi n \hbar \]  \hspace{1cm} (3.24)

with \(n = \text{integer}\). Thus we get the energy spectrum

\[ E_n = Gs_3 \cos \left[ \frac{\pi}{a} \sqrt{\frac{n \hbar}{\mu_0}} \right] \]  \hspace{1cm} (3.25)

The critical bound state is limited by \(E_n = 0\), which means the inequality;

\[ \frac{\pi}{a} \sqrt{\frac{n \hbar}{\mu_0}} \leq \frac{\pi}{2} \]

This leads to the condition for the quantum number \(n\):

\[ n \hbar \leq \frac{\mu_0 a^2}{4} \]  \hspace{1cm} (3.26)
4. Summary

We have studied a possible occurrence of the geometric force in a magnetic condensate. This force is analogous to the Magnus force in ordinary superfluids. The characteristic property is the nature of the "transversality", so to speak, since the force is perpendicular to the velocity of "particle(vortex)", which suggests that the force does not attribute to the energy dissipation. This feature is a characteristic of the Lorentz force, so the geometric force is a sort of the Lorentz force. However, it should be noted that the analogy with the Magnus force is not complete, since in the magnetic condensate we have no supercurrent as in the case of the superfluids.

From the above derivation, the geometric force is attributed to the canonical term, which arises from the geometric phase. From the formulation point of view, the geometric force may be regarded as a special case of the previous treatment of the many vortex dynamics. However, the effective Lagrangian for the single vortex can naturally incorporate the effect of pinning force, if we include the interaction with the magnetic impurities immersed in the magnetic substance. Indeed, we have shown that by using the Bohr-Sommerfeld quantization the geometric force results in the bound state of a vortex which is captured by a pinning potential. Apart from such a potential problem, the geometric force would play a role for an estimate of an effect of dynamical perturbation acting for the vortex motion. The details of this will be given elsewhere.
APPENDIX

In this appendix, we evaluate the integral which appears in the canonical term,

\[ I_\alpha = \int \eta v_\alpha d^2 x. \quad (A1) \]

We define two discs as

\[ D_A = \{ x ; |x| \leq R_A \}, \]
\[ D_a = \{ x ; |x - X| \leq a \}, \quad (A2) \]

where \( a \) is the size of vortex defined in and \( R_A \) should be taken large enough for \( D_A \) to include \( D_A \). Let’s define \( V(a, A) \) as the region,

\[ V(a, A) = D_A - D_a. \quad (A3) \]

The integral \( I_\alpha \) is indefinite at \( r \to \infty \), and we take the regularization to get the finite result; the integral region is restricted to the finite region \( V(a, A) \) (cut-off), and the infinite limit is taken

\[ I_\alpha = I_\alpha(a) + \lim_{R \to \infty} I_\alpha(R), \quad (A4) \]

where each integral is written as

\[ I_\alpha(a) = \int_{D_a} d^2 x \eta v_\alpha, \]
\[ I_\alpha(R) = \int_{V(a;R)} d^2 x \eta v_\alpha. \quad (A5) \]

By using the polar coordinates around \( x = X \), the integral \( I_\alpha(a) \) can be
evaluated:

\[ I_\alpha(a) = \int_{D_a} r dr d\theta \frac{\eta_0}{a} \frac{\mu \epsilon_{\alpha\beta} x_\beta}{2\pi r^2}, \]

\[ = \frac{\eta_0 \mu}{2\pi a} \epsilon_{\alpha\beta} \int_0^a r dr \int_0^{2\pi} \hat{x}_\beta d\theta = \frac{\eta_0 \mu}{2\pi a} \epsilon_{\alpha\beta} \frac{a^2}{2} \cdot 0 = 0, \]  

(A6)

where \( \hat{x}_\alpha = x_\alpha/r \). With the aid of the Stokes theorem, the integral \( I_\alpha(R) \) can be represented with the line integral:

\[ I_\alpha(R) = -\epsilon_{\alpha\beta} \eta_0 \int_{V(a;R)} d^2 x \partial_\beta G_V(x;X), \]

\[ = -\epsilon_{\alpha\beta} \eta_0 \int_{\partial D_a + \partial D_R} ds G_V(x;X), \]  

(A7)

Furthermore, by using the polar coordinate around \( x = X \), the integral on \( \partial D_a \) becomes

\[ \int_{\partial D_a} ds G_V(x;X) = -\frac{\mu}{2\pi} \log a \int ds \beta = 0, \]  

(A8)

and, for around \( x = 0 \), the integral on \( \partial D_R \) is evaluated to be

\[ \int_{\partial D_R} ds G_V(x;X) = -\frac{\mu}{2\pi} \int ds_\beta \left[ \log R + \frac{(xX)}{R^2} + O(R^{-2}) \right] \]

\[ = -\frac{\mu}{2\pi} \frac{1}{R^2} \int ds_\beta (xX) + O(R^{-2}) \]

\[ = \frac{\mu}{2\pi} X_\alpha \int d\theta \hat{x}_\alpha \hat{x}_\beta + O(R^{-2}) = \frac{\mu}{2} X_\beta + O(R^{-2}). \]  

(A9)

With combining (A8) and (A9), we get the final result for \( I_\alpha^V \):

\[ I_\alpha = \frac{\eta_0 \mu}{2\pi} \epsilon_{\alpha\beta} X_\beta. \]  

(A10)
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