Uniqueness of two-dimensional tomography with unknown projection directions

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Abstract. We consider uniqueness of two-dimensional parallel beam tomography in which both the object being imaged and the projection directions are unknown. This problem occurs in certain practical applications. For example, in magnetic resonance imaging there may be uncertainty in the projection directions due to the involuntary motion of the patient. The three-dimensional version of this problem occurs in cryo electron microscopy of viral particles, where the projection directions may be completely unknown due to the random orientations of the particles being imaged. We show that the problem is related to some algebraic geometric properties of a certain system of homogeneous polynomials. We also show that for sufficiently asymmetric objects, the object is uniquely determined up to an orthogonal transformation by the projection data from unknown directions.

1. Introduction

It is a well known fact from the theory of the Radon transform that an integrable function in the plane is uniquely determined by its line integral projections [1]. It follows that in the standard formulation of parallel beam tomography, the problem admits a unique solution, i.e. the object being imaged is uniquely determined by its projections from all directions. In this standard formulation it is assumed that the projection directions, i.e. the view angles at which the projections are acquired, are known.

In this paper we consider the uniqueness of two-dimensional parallel beam tomography in which both the object and the projection directions are unknown. In practical applications the projection directions may be unknown or known only approximately due to the uncontrollable motion of the object being imaged, for example. For instance, this can happen in magnetic resonance imaging due to the involuntary motion of the patient [2].

In three dimensions the problem of tomography with unknown projection directions is also of considerable interest. There the problem arises in cryo electron microscopy of viral particles, where one obtains projection data from a large number of identical particles at unknown orientations [3, 4]. This results in a data set equivalent to many projections of a single object at unknown directions. Many of the results in three dimensions are based on the so-called projection or Fourier slice theorem [5], which cannot be applied to the two-dimensional case.

For the two-dimensional case, methods of recovering unknown view angles from the projection data have been proposed in [6, 7, 8] and [9]. Of these, only in [9] rigorous analysis of the uniqueness of the solution to the problem is given. In that paper, it is shown that given a finite but large enough set of view angles, these view angles are uniquely determined for objects in
a generic set. The generic set depends on the given view angles, and does not allow a simple characterization to determine whether an object belongs to the set or not. For the definition of a generic set, we refer to [9].

In a recent paper of the authors [10], we proved that for sufficiently asymmetric objects, the object is uniquely determined by projections at unknown directions. The class of objects for which the result holds, i.e. the objects with the required asymmetry, was characterized by a simple condition in terms of the object’s geometric moments. For this class of objects, also a necessary and sufficient condition was given under which the view angles are uniquely determined by the projections.

This paper reviews the result about the uniqueness of the object [10]. Also, the following extension to the result is given. We show that the result holds if the different projections are not related necessarily by an unknown rotation, but by an unknown orthogonal transformation, i.e. also reflections are allowed. In addition, we elaborate further on the conditions for the required asymmetry. Namely, we show that the class of objects for which the uniqueness result applies is a large subset of the set of all objects in a topological sense.

2. Parallel beam tomography

An object is, by definition, a function in $L^1(B)$, the space of real integrable functions in the closed unit disk $B \subset \mathbb{R}^2$. By assumption all objects admit the zero extension outside of $B$, hence the domain of every object is $\mathbb{R}^2$. If $\xi$ is an object and $U \in O(2)$, the group of real orthogonal $2 \times 2$-matrices, the transformed object $\xi_U$ is defined by

$$\xi_U(x) = \xi(U^T x) \quad x \in \mathbb{R}^2,$$

where $U^T$ is the transpose of $U$.

The (parallel-beam line integral) projection of object $\xi$ at orientation $U \in O(2)$ is the function $P(\xi; U) \in L^1(\mathbb{R})$ defined almost everywhere by

$$P(\xi; U)(x_1) = \int_{\mathbb{R}} \xi_U(x) \, dx_2 \quad x_1 \in \mathbb{R}.$$

If $U$ above corresponds to a rotation, i.e.

$$U = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

for some $\theta \in \mathbb{R}$, then the angle $\theta$ is called the view angle of the projection. However, the projection is defined above for arbitrary orthogonal $U$, not just for the rotations.

It is clear from the definition of projection that

$$P(\xi_S; U) = P(\xi; US)$$

for any object $\xi$ and $U, S \in O(2)$. It follows that projections at unknown orientations determine the object at most up to the following equivalence.

**Definition 2.1.** Two objects $\xi$ and $\eta$ are equivalent, denoted $\xi \sim \eta$, if there exists a matrix $U \in O(2)$ such that $\eta = \xi_U$.

The object is determined at most up to the above equivalence also in the case that the projections are from unknown view angles, i.e. in the case that they are related by an unknown rotation [10].
Our main instrument in studying the uniqueness of objects are the geometric moments (henceforth abbreviated as moments) of objects. For any \( d = 0, 1, 2, \ldots \), the \( d \)-th order moments of the object \( \xi \) are defined by

\[
m_k(d; \xi) = \int_{\mathbb{R}^2} x_1^{d-k} x_2^k \xi(x) \, dx \quad k = 0, 1, \ldots, d.
\]

The connection between moments and tomography comes from the fact that the moments \( m_0(d; \xi_U) \) are determined by the projection of \( \xi \) at orientation \( U \). Indeed, by Fubini’s theorem

\[
m_0(d; \xi_U) = \int_{\mathbb{R}^2} x_1^d \left( \int_{\mathbb{R}} \xi_U(x) \, dx_2 \right) \, dx_1 = \int_{\mathbb{R}} x_1^d P(\xi_U)(x_1) \, dx_1.
\] (2)

On the other hand, by the change of variables we have for \( U = (u_{ij}) \in O(2) \)

\[
m_0(d; \xi_U) = \int_{\mathbb{R}^2} x_1^d \xi(U^T x) \, dx = \int_{\mathbb{R}^2} (U x_1)^d \xi(x) \, dx = \sum_{n=0}^d \binom{d}{n} m_n(d; \xi) u_{11}^{d-n} u_{12}^n.
\] (3)

In the case the matrix \( U \) is a rotation by \( \theta \), it follows that the view angle \( \theta \) satisfies the trigonometric polynomial

\[
\sum_{n=0}^d \binom{d}{n} m_n(d; \xi) \cos^{d-n} \theta \cdot \sin^n \theta = \int_{\mathbb{R}} x_1^d P(\xi; U)(x_1) \, dx_1.
\] (4)

Given projections of an unknown object \( \xi \) at unknown view angles, the projection data gives rise to a system of trigonometric polynomials with unknown coefficients \( \binom{d}{n} m_n(d; \xi) \) as in (4). The idea of using this system to recover the unknown view angles has been proposed in [7], and was also used in [9, 11].

In [10] we gave a necessary and sufficient condition for a class of objects admitting so-called \((2, d)-asymmetric moments\) under which the above method of recovering view angles admits a unique solution (up to a constant rotation and reflection in the view angles). This is the same class of objects for which the result about the uniqueness of the object holds. The required asymmetry is characterized in terms of the object’s moments as follows.

**Definition 2.2.** Let \( d \geq 1 \) be an odd integer. The object \( \xi \) admits \((2, d)-asymmetric moments\), if the following two conditions are fulfilled: (i) the second and \(d\)-th order moments of \( \xi \) and \( \xi_U \) coincide only if \( U \in O(2) \) is the identity, and (ii) the second order moments of \( \xi_U \) are not independent of \( U \in O(2) \).

The first condition implies, among others, that the object is not symmetric with respect to any nontrivial orthogonal transformation. The objects that fail to satisfy the second condition are said to admit spherical inertia, that is, as far as their second order moments are concerned, they transform as a ball or a sphere under orthogonal transformations.

Let us denote the class of objects admitting \((2, d)-asymmetric moments\) by \( G_d \subset L^1(B) \). The following theorem shows that \( G_d \) is a large set in \( L^1(B) \) in a topological sense. The topology of \( L^1(B) \) is the standard topology induced by the integral norm.

**Theorem 2.3.** The complement of \( G_d \) in \( L^1(B) \) is a nowhere dense set.

**Proof.** For any object \( \xi \), define the vectors

\[
\Lambda_d(\xi) = (m_0(d; \xi), m_2(d; \xi), \ldots, m_{d-1}(d; \xi))
\] (5)

\[
\underline{\Lambda}_d(\xi) = (m_d(d; \xi), m_{d-2}(d; \xi), \ldots, m_1(d; \xi))
\] (6)
and a linear mapping \( L : L^1(B) \to \mathbb{R}^{4+d} \) by

\[
L \xi = (m_0(2; \xi), m_1(2; \xi), m_2(2; \xi), \Lambda_d(\xi), \hat{\Lambda}_d(\xi)).
\]

We show that \( L \) is an open mapping.

The continuity of \( L \) is obvious, so by the open mapping theorem it suffices to show that \( L \) is onto. By linearity it is enough to find \( \xi \) such that \( L \xi = e_i \), where \( e_i \) is the \( i \)-th vector of the standard basis of \( \mathbb{R}^{4+d} \).

Define polynomials \( p_{ij} \in L^2(B) \) by \( p_{ij}(x) = x_1^i x_2^j \). Then \( m_i(k; \xi) = \langle p_{k-i,l}, \xi \rangle \) for any object \( \xi \in L^2(B) \subset L^1(B) \) and \( k \geq l \geq 0 \), where \( \langle \cdot, \cdot \rangle \) is the standard inner product of \( L^2(B) \).

Fix \( i \in \{1, 2, \ldots, 4+d\} \) and choose \( k \in \{2, d\} \) and \( l \in \{0, 1, \ldots, k\} \) such that

\[
(\xi)_i = \langle p_{k-l,i}, \xi \rangle
\]

for all \( \xi \in L^2(B) \). Since the polynomials \( p_{ij} \), \( i + j = 2, d \) are linearly independent, there exists an object \( \xi \in L^2(B) \) in the orthogonal complement of \( \{p_{ij} : i + j = 2, d, (i, j) \neq (k-l, l)\} \) such that \( \langle p_{k-l,i}, \xi \rangle = 1 \). It follows that \( L \xi = e_i \).

Let \( F \) denote the complement of \( G_d \) in \( L^1(B) \) and suppose that \( U \) is an open subset of \( L^1(B) \) such that \( U \subset \overline{F} \) (the bar denotes the closure). By [10, Lemma 4.3] \( LF \) is a nowhere dense set in \( \mathbb{R}^{4+d} \). Since \( LU \) is open and \( LF \subset LF \), it follows that

\[
LU \subset \text{int}(LF) = \emptyset,
\]

which proves that \( U = \emptyset \) and \( F \) is nowhere dense in \( L^1(B) \).

The following lemma gives an alternative characterization of \((2,d)\)-asymmetry of moments in terms of a particular representative of the equivalence class of an object (under the equivalence \( \sim \)). The vectors \( \Lambda_d(\xi) \) and \( \hat{\Lambda}_d(\xi) \) below are defined by the equations (5) and (6).

**Lemma 2.4 ([10]).** Let \( \xi \) be an object and \( d \geq 1 \) an odd integer. There exists an object \( \xi' \sim \xi \) such that

\[
m_0(2; \xi') \geq m_2(2; \xi') \quad \text{and} \quad m_1(2; \xi') = 0,
\]

and

\[
\text{the first nonzero elements of } \Lambda_d(\xi') \text{ and } \hat{\Lambda}_d(\xi') \text{ are positive.}
\]

Furthermore, if \( \xi \) and \( \xi' \) are as above, then \( \xi \) admits \((2,d)\)-asymmetric moments if and only if strict inequality occurs in (7) and both of the vectors \( \Lambda_d(\xi') \) and \( \hat{\Lambda}_d(\xi') \) are nonzero.

### 3. Uniqueness of objects in tomography at unknown orientations

In this section we will relate the uniqueness of the object in tomography at unknown orientations to properties of the zero set of a certain system of homogeneous polynomials. Let us first recall the concept of the projective space.

Define an equivalence relation \( \| \) on \( \mathbb{C}^{k+1} \setminus \{0\} \) by setting \( x \| y \) if and only if there exists a nonzero \( \lambda \in \mathbb{C} \) such that \( x = \lambda y \). The projective space \( \mathbb{P}^k \) is the set of equivalence classes of \( \| \) on \( \mathbb{C}^{k+1} \setminus \{0\} \).

We will use the same notation for a nonzero vector \( x \in \mathbb{C}^{k+1} \) and the equivalence class of \( x \) in \( \mathbb{P}^k \). Furthermore, we will denote a point \([x_1, x_2, y_1, y_2]^T \in \mathbb{P}^3\) also by \((x, y)\), where \( x, y \in \mathbb{C}^2 \) and at least one of them is nonzero.

If \( x, y \in \mathbb{C}^{k+1} \) and \( x = \lambda y \) for some \( \lambda \in \mathbb{C} \), then \( f(x) = \lambda^m f(y) \) for any homogeneous polynomial \( f \) of total degree \( m \). It follows that \( f(x) = 0 \) for \( x \in \mathbb{C}^{k+1} \setminus \{0\} \) if and only if \( f(y) = 0 \) for all \( y \in \mathbb{C}^{k+1} \setminus \{0\} \), such that \( x = y \) in \( \mathbb{P}^k \). Hence the set of zeros in \( \mathbb{P}^k \) of a homogeneous polynomial is well defined.

The following is our main result.
Theorem 3.1. Let $\xi \in L^1(B)$ admit $(2,d)$-asymmetric moments, for some odd $d \geq 1$, and let $U_i \in O(2)$, $i \in \mathbb{N}$, be distinct. If $\eta$ is an object and $Q_i$ are orthogonal matrices such that $P(\eta; Q_i) = P(\xi; U_i)$ for all $i \in \mathbb{N}$, then $\eta \sim \xi$.

Note that it is not a priori assumed that the object $\eta$ admits $(2,d)$-asymmetric moments nor that the matrices $Q_i$ are distinct. It is also not assumed that the matrices $U_i$ or $Q_i$ are rotations, but any orthogonal matrices. Furthermore, note that an infinite number of projections is necessary for the uniqueness of the object even if the orientations $U_i$ are known [5].

Let $\xi, \eta, U_i = (u_{kl}(i))$ and $Q_i = (q_{kl}(i))$ be as in the statement of Theorem 3.1. Let us fix the following notation for the rest of this section. The moments of $\xi$ and $\eta$ are denoted by $m_1(k) = m_1(k; \xi)$ and $\hat{m}_1(k) = m_1(k; \eta)$, and $u(i) = [u_{11}(i) \ u_{12}(i)]$ and $q(i) = [q_{11}(i) \ q_{12}(i)]$ denote the first rows of $U_i$ and $Q_i$, respectively. Without loss of generality, as the following lemma shows, we shall assume the following facts about the objects $\xi$ and $\eta$ and the orientations $U_i$:

\begin{align}
\xi \text{ and } \eta \text{ satisfy conditions (7) and (8)},
\end{align}

and

\begin{align}
\text{the matrices } U_i \text{ are such that } u(i) \neq \pm u(j) \text{ for all } i \neq j.
\end{align}

Lemma 3.2. If Theorem 3.1 holds with the assumptions (9) and (10), it holds also without them.

Proof. Suppose the theorem holds under the additional assumptions. Let $\xi, \eta, U_i$ and $Q_i$ be any objects and matrices as in the statement of Theorem 3.1, i.e. not necessarily satisfying the assumptions (9) and (10).

Define equivalence relation $\equiv$ on $\{U_i\}$ by setting $U_i \equiv U_j$ if and only if $u_i = \pm u_j$. Since $u_i$ determines the second row of $U_i$ up to the sign and the matrices $U_i$ are distinct, it is easy to see that the equivalence class of $U_i$ under the equivalence $\equiv$ consists of at most four matrices. Pick one matrix from each equivalence class of the quotient set of $\{U_i\}$ by $\equiv$ to form a subsequence $\{U_{in}\}$. Then $u(i_n) \neq \pm u(i_m)$ if $n \neq m$.

By Lemma 2.4 there exists objects $\xi' \sim \xi$ and $\eta' \sim \eta$ that satisfy (7) and (8). Then $\xi'$ admits $(2,d)$-asymmetric moments and there exists orthogonal matrices $U$ and $Q$ such that $\xi = \xi'_U$ and $\eta = \eta'_Q$. By (1)

\begin{align}
P(\eta'; Q_{in}Q) = P(\eta; Q_{in}) = P(\xi; U_{in}) = P(\xi'; U_{in}U),
\end{align}

where $U_{in}U \in O(2)$ are distinct and satisfy (10). By assumption the above implies $\eta' \sim \xi'$, which in turn implies $\eta \sim \xi$.

The vectors $u(i)$ and $q(i)$ satisfy

\begin{align}
u_1(i)^2 + u_2(i)^2 = ||u(i)||^2 = 1 = ||q(i)||^2 = q_1(i)^2 + q_2(i)^2
\end{align}

for all $i = 1, 2, \ldots$. By (2) and (3) we also have

\begin{align}
\sum_{n=0}^{k} \binom{k}{n} m_1(k) u_1(i)^{k-n} u_2(i)^n = \int_{\mathbb{R}} x_1^k P(\xi; U_i)(x_1) \, dx_1
\end{align}

\begin{align}
= \int_{\mathbb{R}} x_1^k P(\eta; Q_i)(x_1) \, dx_1
\end{align}

\begin{align}
= \sum_{n=0}^{k} \binom{k}{n} \hat{m}_1(k) q_1(i)^{k-n} q_2(i)^n
\end{align}
for $i = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$ By (11), (12) and (9), each point $(u, q) = (u(i), q(i)) \in \mathbb{P}^3$ is a zero of the following system of homogeneous polynomials in $\mathbb{P}^3$

$$u_1^2 + u_2^2 - q_1^2 - q_2^2 = 0$$
$$m_0(2)u_1^2 + m_2(2)u_2^2 - \hat{m}_0(2)q_1^2 - \hat{m}_2(2)q_2^2 = 0$$
$$\sum_{n=0}^{d} \binom{d}{n} m_n(d)u_1^{d-n}u_2^n - \sum_{n=0}^{d} \binom{d}{n} \hat{m}_n(d)q_1^{d-n}q_2^n = 0.$$  

The assumptions (9) and (10) imply that the points $(u(i), q(i))$ are distinct in $\mathbb{P}^3$, and that the coefficients $m_i(k)$ and $\hat{m}_i(k)$ satisfy

$$m_0(2) > m_2(2) \text{ and } \hat{m}_0(2) \geq \hat{m}_2(2),$$

and

$$\Lambda_d(\xi) \neq 0 \neq \hat{\Lambda}_d(\xi), \text{ and the first nonzero elements of } \Lambda_d(\xi), \hat{\Lambda}_d(\xi), \Lambda_d(\eta) \text{ and } \hat{\Lambda}_d(\eta) \text{ are positive.}$$

The equations (13)–(15) are the projective space counterpart to the trigonometric polynomial (4), with the additional condition that they are written for particular representatives of the equivalence class of the objects under consideration, i.e. for those objects that satisfy (7) and (8). For the detailed analysis of the properties of the above system, we refer to [10]. The following theorem is the key for the proof of Theorem 3.1.

**Theorem 3.3 ([10]).** If the system (13)–(15) of homogeneous polynomials in $\mathbb{P}^3$ with real coefficients satisfying the assumptions (16) and (17) admits the zeros $(u, q) = (x(i), y(i))$, $i = 1, 2, \ldots$, where $x(i), y(i) \in \mathbb{R}^2 \setminus \{0\}$ and $x(i)$ are distinct in $\mathbb{P}^1$, then $x(i) \neq y(i)$ for at most finitely many $i$.

**Proof of Theorem 3.1.** By Theorem 3.3, there exists an infinite subset $I \subset \mathbb{N}$ of the indices such that $u(i) = q(i)$ for all $i \in I$. For a non-negative integer $k$, by (12),

$$\sum_{n=0}^{k} \binom{k}{n} (m_n(k) - \hat{m}_n(k))u_1^{k-n}u_2^n = 0$$

for all $i \in I$. In $\mathbb{P}^1$ a nontrivial homogeneous polynomial of total degree $k$ admits at most $k$ distinct zeros [12, Chapter 8]. Since the vectors $u(i)$ are distinct in $\mathbb{P}^1$ by (10), the polynomial (18) is trivial, and consequently $m_n(k) = \hat{m}_n(k)$ for $n = 0, 1, \ldots, k$.

The above implies that all of the moments of $\xi$ and $\eta$ agree, and consequently $\int_B p(\xi - \eta) d\mathbf{x} = 0$ for all polynomials $p : B \rightarrow \mathbb{R}$. Since the support of $\xi$ and $\eta$ is in $B$, approximation of continuous functions by polynomials together with the above yields $\int_{\mathbb{R}^2} f(\xi - \eta) d\mathbf{x} = 0$ for all continuous functions $f$. It follows that $\eta = \xi$ a.e.

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