MULTI-PEAK SOLUTIONS TO CHERN-SIMONS-SCHRÖDINGER SYSTEMS WITH NON-RADIAL POTENTIAL

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ABSTRACT. In this paper, we consider the existence of static solutions to the nonlinear Chern-Simons-Schrödinger system

\[
\begin{align*}
-ih D_0 \Psi - h^2 (D_1 D_1 + D_2 D_2) \Psi + V \Psi &= |\Psi|^{p-2} \Psi, \\
\partial_0 A_1 - \partial_1 A_0 &= -\frac{1}{2} ih [\nabla D_2 \Psi - \Psi D_2 \Psi], \\
\partial_0 A_2 - \partial_2 A_0 &= \frac{1}{2} ih [\nabla D_1 \Psi - \Psi D_1 \Psi], \\
\partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} |\Psi|^2,
\end{align*}
\]

where \(p > 2\) and non-radial potential \(V(x)\) satisfies some certain conditions. We show that for every positive integer \(k\), there exists \(h_0 > 0\) such that for \(0 < h < h_0\), problem (0.1) has a nontrivial static solution \((\Psi_h, A_{0h}, A_{1h}, A_{2h})\). Moreover, \(\Psi_h\) is a positive non-radial function with \(k\) positive peaks, which approach to the local maximum point of \(V(x)\) as \(h \to 0^+\).

Key words : Chern-Simons-Schrödinger system, variational method, multi-peak solutions.

AMS Subject Classifications: 35J50, 35J10

1. Introduction

In this paper, we investigate the existence of mutil-peak solutions to Chern-Simons-Schrödinger systems. The Schrödinger equation

\[
i h \frac{\partial \Psi(x,t)}{\partial t} = -h^2 \Delta \Psi(x,t) + V(x)\Psi(x,t) - |\Psi(x,t)|^{p-2}\Psi(x,t)
\]

with \(p > 2\) in \(\mathbb{R}^2 \times \mathbb{R}_+\) can be introduced as the Euler-Lagrange equation of the Lagrange density

\[
\mathcal{L} = h Re \{i \Psi(x,t) \frac{\partial \Psi(x,t)}{\partial t} \} - h^2 |\nabla \Psi(x,t)|^2 + V(x)|\Psi(x,t)|^2 - \frac{2}{p} |\Psi(x,t)|^p,
\]

where \(V(x)\) is the external potential, \(h\) is the Plank constant. A static solution, that is a solution \(\Psi(x,t) = u(x)\) of (1.1) which is independent of \(t\), satisfies the semiclassical Schrödinger equation

\[
-\varepsilon^2 \Delta u(x) + V(x)u(x) = |u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^2
\]

with \(\varepsilon = h\).
Taking into account the interaction of the electromagnetic field and the matter field, one includes the Chern-Simons term into the Lagrangian density. The Lagrangian density then becomes

\[
\mathcal{L}_c = \frac{k}{4} \varepsilon^{\mu\alpha\beta} A_\mu F_{\alpha\beta} - \frac{1}{2} \hbar \text{Re} \{ i \bar{\Psi}(x, t) D_0 \Psi(x, t) \} + \frac{\hbar^2}{2} |D\Psi(x, t)|^2 \\
+ \frac{1}{2} V(x)|\Psi(x, t)|^2 - \frac{1}{p} |\Psi(x, t)|^p, \tag{1.4}
\]

where \( \Psi : \mathbb{R}^{2,1} \to \mathbb{C} \) is the complex scalar field, \( A_\mu : \mathbb{R}^{2,1} \to \mathbb{R}, \mu = 0, 1, 2, \) are the gauge field, which obey the Lorentz condition \( \sum_{\mu=0}^{2} \partial_\mu A_\mu = 0. \) By \( D_0 = \partial_t + i \frac{\hbar}{\kappa} A_0 \) and \( D_j = \partial_{x_j} - i \frac{\hbar}{\kappa} A_j, \) \( j = 1, 2, \) for \((x_1, x_2, t) \in \mathbb{R}^{2,1}\) we denote the the covariant derivatives, and we set \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) for \( \mu, \nu = 0, 1, 2. \) Inside the Lagrangian density \( \mathcal{L}_c \) we denote \( i \) the the imaginary unit, and \(-\frac{i}{4} \varepsilon^{\mu\alpha\beta} A_\mu F_{\alpha\beta}\) the Chern-Simons term. The corresponding Euler-Lagrange system of \( \mathcal{L}_c \) is given as follows.

\[
\begin{cases}
- i \hbar D_0 \Psi - h^2 (D_1 D_1 + D_2 D_2) \Psi + V \Psi = |\Psi|^{p-2} \Psi, \\
\partial_0 A_1 - \partial_1 A_0 = -\frac{i}{2} \hbar \{ \bar{\Psi} D_2 \Psi - \Psi D_2 \bar{\Psi} \}, \\
\partial_0 A_2 - \partial_2 A_0 = \frac{i}{2} \hbar \{ \bar{\Psi} D_1 \Psi - \Psi D_1 \bar{\Psi} \}, \\
\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} \hbar |\Psi|^2.
\end{cases}
\tag{1.5}
\]

The systems (1.5) is proposed in [10–13], which describes the dynamics of large number of particles in an electromagnetic field. This model is important for the study of the high-temperature superconductor, fractional quantum Hall effect and Aharovnov-Bohm scattering. System (1.5) is referred to be the Chern-Simons-Schrödinger system (CSS), it is invariant under the following gauge transformation

\[ \phi \to \phi e^{i\chi}, \ A_\mu \to A_\mu - \partial_\mu \chi \]

for arbitrary \( C^\infty \) function \( \chi : \mathbb{R}^{2,1} \to \mathbb{R}. \)

Since system (1.5) is setting in the whole space, a problem of the loss of the compactness is then raised if the variational method applied. In order to avoid such a problem, in [3] a particular form of solutions of (1.5)

\[ \Psi(t, x) = u(|x|)e^{i\omega t}, \ A_0(t, x) = h_1(|x|), \]
\[ A_1(t, x) = \frac{x_2}{|x|^2} h_2(|x|), \ A_2(t, x) = \frac{x_1}{|x|^2} h_2(|x|), \]

is considered with the constant \( V, \) where \( \omega > 0 \) and \( u, h_1, h_2 \) are real value functions depending only on \( |x|. \) Then, solutions are found in the radially symmetric space \( H^1_1(\mathbb{R}^2) \) as critical points of the associated functional

\[ J(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\nabla u|^2 + (\omega + \xi) u^2 + \frac{u^2}{|x|^2} \left( \int_0^{|x|} \frac{s}{2} u^2(s) \, ds \right)^2 \right\} \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx. \]

However, it is quite involved in finding critical points of \( J. \) Actually, such a problem was treated differently in accordance to the range of the exponent \( p. \) Precisely in [3], for \( p > 4 \) it is considered a minimization problem on the Nehari-Pohozaev manifold; while for \( 2 < p < 4, \) minimization problem is constrained in \( L^2 \) sphere. Essentially, it is a nonlinear
eigenvalue problem. For the case $p = 4$, a self-dual solution can be found by Liouville equations. Later on, for $p \in (2, 4)$ Pomponio and Ruiz studied in [16] the nonexistence and multiplicity results for problem (1.5) with constant potentials $V$ by investigating the geometry of the Euler-Lagrange functional. More related results on problem (1.5) with constant potentials can be found in [3, 4, 6, 7, 16, 17] and references therein. Suppose $V$ is radially symmetric, it is studied in [18] the existence, nonexistence and multiplicity of the same type of solutions for (1.5). While for the case $V$ being non-radial, nontrivial solutions are found in [19] under the assumption $p > 4$.

In this paper, we consider the existence and concentration of solutions for (1.6) under the assumption that $V(x)$ is non-radial and $p > 2$. Since $V(x)$ is not radially symmetric, we encounter the difficulty of the loss of compactness. Instead of working in the radially symmetric space $H^1_r(\mathbb{R}^2)$, one has to use the concentration-compactness principle if minimization problem considered. Furthermore, the presence of $V$ brings essential difficulty to find minimizers of associated functional on the Nehari-Pohozaev manifold, it is also the reason hinders the study of the case $p \in (2, 4)$ although there are existence results emerged for the case $p > 4$. Our approach is to use the Lyapunov-Schmidt reduction method. By this method, we can not only construct multi-peak solutions to problem (1.5), but also treat all the case $p > 2$ in a unified way.

In this paper, we study static solutions of (1.5). Hence, the gauge field $(A_1, A_2)$ obeys the Coulomb condition $\partial_1 A_1 + \partial_2 A_2 = 0$. Moreover, a static solution $(u, A_0, A_1, A_2)$ satisfies

$$
\begin{cases}
-\varepsilon^2 \Delta u + V(x)u + A_0 u + (A_1^2 + A_2^2) u = |u|^{p-2} u, \\
\partial_1 A_0 = A_2 u^2, \quad \partial_2 A_0 = -A_1 u^2 \\
\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0,
\end{cases}
$$

where we set $\varepsilon = h > 0$.

Suppose the external potential $V(x)$ satisfies

$(V_1)$ $V(x) \in C(\mathbb{R}^2, \mathbb{R})$, $\inf_{x \in \mathbb{R}^2} V(x) > 0$, and there exist positive constants $L$ and $\theta$ such that $|V(x) - V(y)| \leq L|x - y|^{\theta}$ for all $x, y \in \mathbb{R}^2$;

$(V_2)$ There exist $\delta > 0$ and $x^0 \in \mathbb{R}^2$ such that $V(x) < V(x^0)$ for $x \in B_\delta(x^0) \setminus \{x^0\} \subset \mathbb{R}^2$.

The main result of this paper is as follows.

**Theorem 1.1.** Suppose that $p > 2$ and $V(x)$ satisfies (V1) and (V2). Then for any positive integer $k$, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, problem (1.6) has a nontrivial solution $(u_\varepsilon, A_0^\varepsilon, A_1^\varepsilon, A_2^\varepsilon)$ such that $u_\varepsilon$ is positive non-radial function with $k$ positive peaks, which approach to the local maximum point of $V(x)$ as $\varepsilon \to 0^+$.

The existence of peak solutions and the concentration phenomenon of solutions has been extensively studied for the semiclassical Schrödinger equation (1.3), see for instance [1, 2, 8, 9, 20] for part of results in this direction. In particular, positive solutions with prescribed number of peaks to nonlinear Schrödinger equations were obtained in [14] by using the well-known Lyapunov-Schmidt reduction scheme. This argument was later generalized in [5] to study nonlinear Schrödinger equations with vanishing potentials. However, it seems no multi-peaks solution has been found for problem (1.6) in literatures.
We will find solutions of problem (1.6) by looking for critical points of the associated functional

\[ J_\varepsilon(u, A_0, A_1, A_2) = \frac{1}{2} \int_{\mathbb{R}^2} (\varepsilon^2 |\nabla u|^2 + V(x)|u|^2 + (A_0 + A_1^2 + A_2^2)|u|^2) \, dx \]

\[ + \frac{1}{2} \int_{\mathbb{R}^2} (A_0 F_{12} + A_1 \partial_2 A_0 - A_2 \partial_1 A_0) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx. \]  

(1.7)

Such a problem can be reduced, see section 2 for details, to find critical points of the functional

\[ I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^2} (\varepsilon^2 |\nabla u|^2 + V(x)u^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \left( -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x-y|^2} u^2(y) \, dy \right)^2 u^2(x) \, dx \]

\[ + \frac{1}{4\pi} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x-y|^2} u^2(y) \, dy \right)^2 u^2(x) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx. \]  

(1.8)

Using the unique ground state $U$ of

\[ \begin{cases}
-\Delta u + V(x)u = u^{p-1}, & u > 0, \ x \in \mathbb{R}^2,

u(0) = \max_{\mathbb{R}^N} u(x), \quad u \in H^1(\mathbb{R}^2),
\end{cases} \]  

(1.9)

we build up the approximate critical points for the functional $I_\varepsilon$. It is well-known that $U(x) = U(|x|)$ is non-degenerate and satisfies

\[ U'(r) < 0, \ \lim_{r \to \infty} r^{\frac{N-1}{2}} e^{-r} U(r) = C > 0, \ \lim_{r \to \infty} \frac{U'(r)}{U(r)} = -1. \]  

(1.10)

Let $k$ be any positive integer. Define

\[ D_k^{\varepsilon, \delta} = \left\{ \mathbf{y} = (y^1, \cdots, y^k) \in (\mathbb{R}^2)^k : y^i \in B_{\frac{2}{\varepsilon}}(x^0), \quad \text{and} \right. \]

\[ \left. \frac{|y^i - y^j|}{\varepsilon} \geq |\ln \varepsilon|^{\frac{1}{2}}, \ i \neq j, i, j = 1, 2 \cdots, k \right\}. \]

Let $U_{\varepsilon, y^i}(x) = U(x - \frac{y^i}{\varepsilon})$ and

\[ E := \left\{ \varphi \in H^1(\mathbb{R}^2) : \left( \frac{\partial U_{\varepsilon, y^i}}{\partial y^i} \varphi \right)_{\varepsilon} = 0, \ i = 1, \cdots, k, \ l = 1, 2 \right\}, \]

where

\[ \langle v_1, v_2 \rangle_{\varepsilon} := \varepsilon^2 \int_{\mathbb{R}^2} \nabla v_1 \nabla v_2 \, dx \, dy + \int_{\mathbb{R}^2} v_1(x)v_2(x) \, dx. \]

Fixing $\mathbf{y} \in D_k^{\varepsilon, \delta}$, we set

\[ U_* = \sum_{i=1}^{k} U_{\varepsilon, y^i} = \sum_{i=1}^{k} U(x - \frac{y^i}{\varepsilon}) \]

and

\[ M_k^{\varepsilon} = \left\{ (\mathbf{y}, \varphi) : \mathbf{y} \in D_k^{\varepsilon, \delta}, \ \varphi \in E \right\}. \]
Then, Theorem 1.1 will be proved by the following result.

**Theorem 1.2.** If $V(x)$ satisfies $(V_1)$ and $(V_2)$, then for any positive integer $k$, there is a positive constant $\varepsilon_0$ depending on $k$ such that for each $\varepsilon \in (0, \varepsilon_0]$, the functional $I_\varepsilon$ has a positive critical point of the form

$$u_\varepsilon(x) = U_* + \varphi_\varepsilon = \sum_{i=1}^{k} U_{\varepsilon,y^i} + \varphi_\varepsilon,$$

where $\varphi_\varepsilon \in E$.

To prove Theorem 1.2, we first use Lyapunov-Schmidt reduction scheme to reduce the problem to a variational problem defined on a closed subset of a finite dimensional Euclidean space. Then we prove that the functional achieves its maximum in the interior of that closed subset. Comparing with previous works, see for instance [5, 14], the Chern-Simons term brings new difficulties in employing the reduction method to deal with singularly perturbed problem (1.6), and the feature of Chern-Simons term requires to establish some new delicate estimates on the energy of the approximate solutions.

This paper is organized as follows. After some preparation in section 2, we expand the functional $I_\varepsilon$ at $U_* + \varphi_\varepsilon$ and fundamental estimates are established in section 3. Finally, in section 4, we prove Theorem 1.2 by the reduction method.

### 2. Preliminaries

In this section, we present the variational framework and establish estimates for the energy functional $I_\varepsilon$ at $U_*$. By equation (1.6), $A_0, A_1$ and $A_2$ can be expressed as functions of $u$. First, integrating by part we find

$$J_\varepsilon(u, A_0, A_1, A_2) = \frac{1}{2} \int_{\mathbb{R}^2} (\varepsilon^2 |\nabla u|^2 + V(x)|u|^2 + (A_0 + A_1^2 + A_2^2)|u|^2) \, dx$$

$$+ \int_{\mathbb{R}^2} A_0 F_{12} \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx. \quad (2.1)$$

Equation (1.6) implies

$$\int_{\mathbb{R}^2} A_0 F_{12} \, dx = -\frac{1}{2} \int_{\mathbb{R}^2} A_0 |u|^2 \, dx. \quad (2.2)$$

Hence,

$$J_\varepsilon(u, A_0, A_1, A_2) = \frac{1}{2} \int_{\mathbb{R}^2} (\varepsilon^2 |\nabla u|^2 + V(x)|u|^2 + (A_1^2 + A_2^2)|u|^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx \quad (2.3)$$

Next, the Coulomb condition $\partial_1 A_1 + \partial_2 A_2 = 0$ and the equation $\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} |u|^2$ yield

$$\Delta A_1 = \frac{1}{2} \partial_2 (|u|^2), \quad -\Delta A_2 = \frac{1}{2} \partial_1 (|u|^2). \quad (2.4)$$
Solving equation (2.4) we obtain

\[ A_1 = A_1(u) = \frac{1}{2} K_2 * (\partial_2 |u|^2) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} |u(y)|^2 \, dy, \]  

(2.5)

\[ A_2 = A_2(u) = -\frac{1}{2} K_1 * (\partial_1 |u|^2) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} |u(y)|^2 \, dy, \]  

(2.6)

where \( K_i = \frac{-x_i}{2\pi|x|^2} \), for \( i = 1, 2 \) and * denotes the convolution. Similarly, the equation

\[ \Delta A_0 = \partial_1 (A_2 |u|^2) - \partial_2 (A_1 |u|^2) \]

implies that

\[ A_0 = A_0(u) = K_1 * (A_1 |u|^2) - K_2 * (A_2 |u|^2). \]  

(2.7)

Therefore, the functional \( J_\varepsilon \) can be written as

\[ I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^2} (\varepsilon^2 |\nabla u|^2 + V(x)u^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^2} ( -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} u^2(y) \, dy )^2 u^2(x) \, dx 

+ \frac{1}{2} \int_{\mathbb{R}^2} (\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} u^2(y) \, dy )^2 u^2(x) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx. \]  

(2.8)

We know that \( I_\varepsilon \) is well defined in \( H^1(\mathbb{R}^2) \) and \( I_\varepsilon \in C^4(\mathbb{H}^1(\mathbb{R}^2)) \). If \( u \) is a critical point of \( I_\varepsilon \), we may define \( A_0, A_1, A_2 \) through (2.5), (2.6) and (2.7), then \( (u, A_0, A_1, A_2) \) is a solution of problem (1.6). In the following, we focus on finding critical points of the functional \( I_\varepsilon \). Precisely, we are looking for critical point \( u \) of \( I_\varepsilon \) in the form

\[ u = U_* + \varphi \]

with \( \varphi \in E \). To this purpose, we expand the functional \( I_\varepsilon \) near approximate solutions.

**Proposition 2.1.** There holds,

\[ I_\varepsilon(U_*) = \left( \frac{1}{2} - \frac{1}{p} \right) k \varepsilon^2 \int_{\mathbb{R}^2} U^p \, dx - \frac{1}{2} \sum_{i=1}^{k} \left( V(x^0) - V(y^i) \right) \varepsilon^2 \int_{\mathbb{R}^2} U^2 \, dx - C \varepsilon^2 \sum_{i \neq j} e^{-\frac{|y^i - y^j|}{\varepsilon}} 

+ O(\varepsilon^{2+\theta} + \varepsilon^2 \sum_{i \neq j} e^{-\frac{(1+\sigma)|y^i - y^j|}{\varepsilon}} + \varepsilon^4), \]

where \( C > 0 \) and \( \sigma \) is a small fixed constant.

**Proof.** Since the integral \( \int_{\mathbb{R}^2} \frac{1}{|x-y|} U^2(y) \, dy \) is uniformly bounded in \( x \in \mathbb{R}^2 \), a change of variable yields

\[ \int_{\mathbb{R}^2} \frac{1}{|x-y|} U^2(y) \, dy = \varepsilon \int_{\mathbb{R}^2} \frac{1}{|y - x^i|} U^2(y) \, dy \leq C \varepsilon. \]
This implies
\[
\left| \int_{\mathbb{R}^2} \left( -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_\varepsilon^2(y) dy \right)^2 U_\varepsilon^2(x) dx \right|
\leq C \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{1}{|x - y|^2} \left( \sum_{i=1}^{k} U_{\varepsilon,y_i}(y) \right)^2 \left( \sum_{i=1}^{k} U_{\varepsilon,y_i}(x) \right)^2 dx \right.
\leq C \int_{\mathbb{R}^2} \left( \sum_{i=1}^{k} \int_{\mathbb{R}^2} \frac{1}{|x - y|^2} U_{\varepsilon,y_i}(y)^2 \left( \sum_{i=1}^{k} U_{\varepsilon,y_i}(x) \right)^2 dx \right.
\leq C \varepsilon^2 \int_{\mathbb{R}^2} \left( \sum_{i=1}^{k} U_{\varepsilon,y_i}(x) \right)^2 dx
\leq C \varepsilon^4.
\]

Analogously,
\[
\frac{1}{2} \int_{\mathbb{R}^2} \left( \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} U_\varepsilon^2(y) dy \right)^2 U_\varepsilon^2(x) dx \leq C \varepsilon^4.
\]

Since
\[
\varepsilon^2 \int_{\mathbb{R}^2} \nabla U_{\varepsilon,y_i} \nabla U_{\varepsilon,y_j} dx + V(x^0) \int_{\mathbb{R}^2} U_{\varepsilon,y_i} U_{\varepsilon,y_j} dx = \int_{\mathbb{R}^2} U_{\varepsilon,y_i}^{p-1} U_{\varepsilon,y_j} dx
\]
for any \( i, j = 1, \ldots, k \), we obtain
\[
\frac{1}{2} \int_{\mathbb{R}^2} \left( \varepsilon^2 |\nabla U_\varepsilon|^2 + V(x) U_\varepsilon^2 \right) dx - \frac{1}{p} \int_{\mathbb{R}^2} |U_\varepsilon|^p dx
\]
\[
= \frac{1}{2} \sum_{i,j=1}^{k} \int_{\mathbb{R}^2} \left[ \left( V(x) - V(x^0) \right) U_{\varepsilon,y_i} U_{\varepsilon,y_j} + U_{\varepsilon,y_i}^{p-1} U_{\varepsilon,y_j} \right] dx - \frac{1}{p} \int_{\mathbb{R}^2} |U_\varepsilon|^p dx.
\]

We write
\[
\mathcal{H} := \sum_{i,j=1}^{k} \int_{\mathbb{R}^2} \left( V(x) - V(x^0) \right) U_{\varepsilon,y_i} U_{\varepsilon,y_j}
\]
\[
= \int_{\mathbb{R}^2} \left( V(x) - V(x^0) \right) \left( \sum_{i=1}^{k} U_{\varepsilon,y_i}^2 + \sum_{i \neq j} U_{\varepsilon,y_i} U_{\varepsilon,y_j} \right) dx
\]
\[
= \int_{\mathbb{R}^2} \sum_{i=1}^{k} \left( V(x) - V(y_i) + V(y_i) - V(x^0) \right) U_{\varepsilon,y_i}^2 dx
\]
\[
+ \int_{\mathbb{R}^2} \sum_{i \neq j} \left( V(x) - V(y_i) + V(y_i) - V(x^0) \right) U_{\varepsilon,y_i} U_{\varepsilon,y_j} dx.
\]
Changing variables we obtain

$$
\mathcal{H} = -\varepsilon^2 \int_{\mathbb{R}^2} U^2(y) \sum_{i=1}^{k} (V(x^0) - V(y^i)) + \varepsilon^2 \int_{\mathbb{R}^2} (V(\varepsilon y + y^i) - V(y^i))U^2(y)dy \\
+ \varepsilon^2 \int_{\mathbb{R}^2} \sum_{i \neq j}^k (V(\varepsilon y + y^i) - V(y^i))U(y)U(y - \frac{y^i - y^j}{\varepsilon})dy \\
+ \varepsilon^2 \sum_{i \neq j}^k \int_{\mathbb{R}^2} (V(x^0) - V(y^i))U(y)U(y - \frac{y^i - y^j}{\varepsilon})dy.
$$

By assumption $V_1$ and (1.10),

$$
\mathcal{H} \leq -\varepsilon^2 \int_{\mathbb{R}^2} U^2 \sum_{i=1}^{k} (V(x^0) - V(y^i)) + \varepsilon^2 k \int_{\mathbb{R}^2} |\varepsilon y|^\theta U^2(y) \, dy \\
+ C\varepsilon^2 \sum_{i \neq j}^k |\varepsilon y|^\theta U(y)U(y - \frac{y^i - y^j}{\varepsilon})dy + C\varepsilon^2 \sum_{i \neq j}^k |\varepsilon|^\theta U(y)U(y - \frac{y^i - y^j}{\varepsilon})dy \\
= -\varepsilon^2 \int_{\mathbb{R}^2} U^2 \sum_{i=1}^{k} (V(x^0) - V(y^i)) + O(\varepsilon^{2+\theta} + \varepsilon^{2+\theta} \sum_{i \neq j}^k e^{-|\varepsilon y^i - y^j|}). \quad (2.12)
$$

Note that if $p > 3$,

$$
\int_{\mathbb{R}^2} \sum_{i \neq j}^k U_{\varepsilon, y^i}^{p-2} U_{\varepsilon, y^j}^2 \, dx \\
= C \int_{\mathbb{R}^2} \varepsilon^2 \sum_{i \neq j}^k U^{p-2}(y - \frac{y^i}{\varepsilon})U^2(y - \frac{y^i}{\varepsilon}) \, dy \\
\leq C \varepsilon^2 \sum_{i \neq j}^k e^{-\min\{(p-2, 2)|y^i - y^j|\}} \leq C \varepsilon^2 \sum_{i \neq j}^k e^{-\frac{(1+\sigma)|y^i - y^j|}{\varepsilon}},
$$

and if $2 < p \leq 3$,

$$
\int_{\mathbb{R}^2} \sum_{i \neq j}^k U_{\varepsilon, y^i}^p U_{\varepsilon, y^j}^p \, dx \\
\leq C \varepsilon^2 \sum_{i \neq j}^k \int_{\mathbb{R}^2} U_{\varepsilon, y^i}^p U(y - \frac{y^i - y^j}{\varepsilon})^\frac{p}{2} \, dy \\
\leq C \varepsilon^2 \sum_{i \neq j}^k e^{-\frac{p|y^i - y^j|}{2\varepsilon}} \leq C \varepsilon^2 \sum_{i \neq j}^k e^{-\frac{(1+\sigma)|y^i - y^j|}{\varepsilon}},
$$

and if $p < 2$,
then we have
\[
\int_{\mathbb{R}^2} U_p^p \, dx = \int_{\mathbb{R}^2} \sum_{i=1}^k U_{\varepsilon,y_i}^p \, dx + p \int_{\mathbb{R}^2} \sum_{i \neq j} U_{\varepsilon,y_i}^{p-1} U_{\varepsilon,y_j} \, dx + \begin{cases} O\left( \int_{\mathbb{R}^2} \sum_{i \neq j} U_{\varepsilon,y_i}^{p-2} U_{\varepsilon,y_j}^2 \, dx \right), & (p > 3) \\ O\left( \int_{\mathbb{R}^2} \sum_{i \neq j} U_{\varepsilon,y_i}^p U_{\varepsilon,y_j}^p \right), & (2 < p \leq 3) \end{cases} \tag{2.13}
\]
\[
= k\varepsilon^2 \int_{\mathbb{R}^2} U_p^p + C p \varepsilon^2 \sum_{i \neq j} e^{-\frac{|y_i - y_j|}{\varepsilon}} + O\left( \varepsilon^2 \sum_{i \neq j} e^{-\frac{(1+\sigma)|y_i - y_j|}{\varepsilon}} \right),
\]
\(\sigma\) is a small fixed constant.
Inserting (2.12) and (2.13) into (2.11), we are led to the result. \(\square\)

3. Energy expansion

In this section, we expand the functional
\[
\mathcal{J}_\varepsilon(y, \varphi) = I_\varepsilon(U_\ast + \varphi), \quad (y, \varphi) \in M_k^\varepsilon
\]
and present some basic estimates. The functional \(\mathcal{J}_\varepsilon(y, \varphi)\) is expanded as follows.
\[
\mathcal{J}_\varepsilon(y, \varphi) = J_\varepsilon(y, 0) + \ell_\varepsilon(\varphi) + \frac{1}{2} L_\varepsilon(\varphi) + R_\varepsilon(\varphi), \tag{3.1}
\]
where
\[
\ell_\varepsilon(\varphi) = \int_{\mathbb{R}^2} \left( V(x) - V(x^0) \right) U_\ast \varphi \, dx + \int_{\mathbb{R}^2} \left( \sum_{i=1}^k U_{\varepsilon,y_i}^{p-1} \varphi - \left( \sum_{i=1}^k U_{\varepsilon,y_i} \right)^{p-1} \varphi \right) \, dx \\
+ \int_{\mathbb{R}^2} \left( -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_\ast^2(y) \, dy \right) U_\ast(x) \varphi(x) \, dx \\
+ \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_\ast^2(y) \, dy \cdot \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} U_\ast(z) \varphi(z) \, dz \right) U_\ast^2 \, dx \\
+ \int_{\mathbb{R}^2} \left( \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} U_\ast^2(y) \, dy \right)^{\ast} U_\ast(x) \varphi(x) \, dx \\
+ \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} U_\ast^2(y) \, dy \cdot \int_{\mathbb{R}^2} \frac{x_1 - z_1}{|x - z|^2} U_\ast(z) \varphi(z) \, dz \right) U_\ast^2 \, dx,
\]
\[
L_\varepsilon(\varphi) = \varepsilon^2 \int_{\mathbb{R}^2} |\nabla \varphi|^2 \, dx + \int_{\mathbb{R}^2} V(x) \varphi^2 \, dx - (p - 1) \int_{\mathbb{R}^2} U_\ast^{p-2} \varphi^2 \, dx.
\]
where $L_{1,\varepsilon}(\varphi) = \varepsilon^2 \int_{\mathbb{R}^2} |\nabla \varphi|^2 dx + \int_{\mathbb{R}^2} V(x)\varphi^2 dx - (p - 1) \int_{\mathbb{R}^2} U^p_{*}\varphi^2 dx$, $L_{2,\varepsilon}(\varphi)$ is the rest, and

\[
R_{\varepsilon}(\varphi) = -\frac{1}{p} \int_{\mathbb{R}^2} \left( (U_* + \varphi)^p - p U_*^{p-1} \varphi - \frac{1}{2} (p - 1) p U_*^{p-2} \varphi^2 \right) dx \\
+ \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_*^2(y) dy \cdot \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} U_*(z) \varphi(z) dz U_*^2(x) dx \right) \\
+ \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_*(y) \varphi(y) dy \cdot \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} \varphi(z) dz U_*^2(x) dx \right) \\
+ \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_*^2(y) dy \cdot \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} \varphi(z) dz U_*^2(x) dx \right) \\
+ \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_*(y) \varphi(y) dy \cdot \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} \varphi(z) dz U_*^2(x) dx \right) \\
+ \frac{1}{16\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_*^2(y) dy \cdot \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} \varphi(z) dz U_*^2(x) dx \right) \\
+ \frac{1}{16\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} \varphi(y) dy \cdot \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} \varphi(z) dz U_*^2(x) dx \right) \\
+ \frac{1}{16\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_*^2(y) dy \cdot \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} \varphi(z) dz u_*(x) \varphi(x) dx \right) \\
+ \frac{1}{16\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} \varphi(y) dy \cdot \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} \varphi(z) dz u_*(x) \varphi(x) dx \right). \]
Lemma 3.1. There exists a constant $C > 0$, such that

\[
\| R_{1,\varepsilon}^{(i)}(\varphi) \| \leq C \varepsilon^{-\min\{1,p-2\}} \| \varphi \|_{\varepsilon}^{\min\{3-i,p-i\}}
+ C \left( \varepsilon \| \varphi \|_{\varepsilon}^{3-i} + \| \varphi \|_{\varepsilon}^{1-i} + \varepsilon^{-1} \| \varphi \|_{\varepsilon}^{5-i} + \varepsilon^{-2} \| \varphi \|_{\varepsilon}^{6-i} \right).
\]

where $R_{1,\varepsilon}(\varphi) = -\frac{1}{p} \int_{\mathbb{R}^2} \left( (U_\varphi + \varphi)^p - U_\varphi^p - pU_\varphi^{p-1} \varphi - \frac{1}{2} (p - 1)pU_\varphi^{p-2} \varphi^2 \right) dx$ and $R_{2,\varepsilon}(\varphi) = R_{\varphi}(\varphi) - R_{1,\varepsilon}(\varphi)$.

In order to find a critical point $(\mathbf{y}, \varphi) \in M_k^\varepsilon$ for $J_{\varepsilon}(\mathbf{y}, \varphi)$, we need to estimate each term in expansion (3.1).
**Proof.** First we estimate \( R_{1,\varepsilon} \). Let \( \tilde{\varphi} = \varphi(\varepsilon x) \). We have

\[
\int_{\mathbb{R}^2} |\varphi|^p \, dx = \varepsilon^2 \int_{\mathbb{R}^2} |\tilde{\varphi}|^p \leq C \varepsilon^2 \left( \int_{\mathbb{R}^2} (|\nabla \tilde{\varphi}|^2 + |\tilde{\varphi}|^2) \, dx \right)^{\frac{p}{2}}
\]

\[
= C \varepsilon^2 (\varepsilon^{-2} \int_{\mathbb{R}^2} (\varepsilon^2 |\nabla \varphi|^2 + |\varphi|^2) \, dx)^{\frac{p}{2}}
\]

\[
\leq C \varepsilon^{2-p} \| \varphi \|_{\varepsilon}^p. \tag{3.2}
\]

If \( 2 < p \leq 3 \), we find

\[
\left| R_{1,\varepsilon}(\varphi) \right| \leq C \int_{\mathbb{R}^2} |\varphi|^p \, dx \leq C \varepsilon^{2-p} \| \varphi \|_{\varepsilon}^p,
\]

\[
\left| \langle R'_1(\varphi), \psi \rangle \right| \leq C \int_{\mathbb{R}^2} |\varphi|^{p-1} \psi \, dx
\]

\[
\leq C \left( \int_{\mathbb{R}^2} |\varphi|^p \, dx \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^2} |\psi|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq C \left( \varepsilon^{2-p} \| \varphi \|_{\varepsilon}^p \right)^{\frac{p-1}{p}} \left( \varepsilon^{2-p} \| \psi \|_{\varepsilon}^p \right)^{\frac{1}{p}}
\]

\[
\leq C \varepsilon^{2-p} \| \varphi \|_{\varepsilon}^{p-1} \| \psi \|_{\varepsilon}
\]

and

\[
\left| \langle R''_1(\varphi), \psi, \xi \rangle \right| \leq C \int_{\mathbb{R}^2} |\varphi|^{p-2} \psi \| \xi \| \, dx
\]

\[
\leq C \left( \int_{\mathbb{R}^2} |\varphi|^{p-2} \, dx \right)^{\frac{p-2}{p}} \left( \int_{\mathbb{R}^2} |\psi|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^2} |\xi|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq C \left( \varepsilon^{2-p} \| \varphi \|_{\varepsilon}^p \right)^{\frac{p-2}{p}} \left( \varepsilon^{2-p} \| \psi \|_{\varepsilon}^p \right)^{\frac{1}{p}} \left( \varepsilon^{2-p} \| \xi \|_{\varepsilon}^p \right)^{\frac{1}{p}}
\]

\[
\leq C \varepsilon^{2-p} \| \varphi \|_{\varepsilon}^{p-1} \| \psi \|_{\varepsilon} \| \xi \|_{\varepsilon}
\]

If \( p > 3 \), we estimate

\[
\left| R_{1,\varepsilon}(\varphi) \right| \leq C \int_{\mathbb{R}^2} U_{\varepsilon,\psi}^{p-3} \left| \varphi \right|^3 \leq C \varepsilon^{-1} \| \varphi \|_{\varepsilon}^3,
\]

\[
\left| \langle R'_1(\varphi), \psi \rangle \right| \leq C \varepsilon^{-1} \| \varphi \|_{\varepsilon}^2 \| \psi \|_{\varepsilon}
\]

and

\[
\left| \langle R''_1(\varphi), \psi, \xi \rangle \right| \leq C \varepsilon^{-1} \| \varphi \|_{\varepsilon} \| \psi \|_{\varepsilon} \| \xi \|_{\varepsilon}.
\]
Similarly, we have the following estimates:

\[
\left| \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_\ast(y) \varphi(y) dy \right| \leq \left| \int_{\mathbb{R}^2} \frac{1}{|x - y|} U_\ast(y) \varphi(y) dy \right|
\]

\[
\leq \left( \int_{\mathbb{R}^2} \frac{1}{|x - y|^2} U_\ast^2(y) dy \right)^\frac{1}{2} \left( \int_{\mathbb{R}^2} |\varphi|^2 dy \right)^{\frac{1}{2}}
\]

\[
\leq C \varepsilon^{-\frac{1}{2}} \|\varphi\|_{\varepsilon} \left( \int_{\mathbb{R}^2} \varepsilon^2 \sum_{j} \frac{U_\ast^3(z)}{|x_\ast - (\varepsilon z + y')|^2} dz \right)^{\frac{1}{2}} \tag{3.3}
\]

\[
\leq C \varepsilon^{-\frac{1}{2}} \|\varphi\|_{\varepsilon} \left( \int_{\mathbb{R}^2} \varepsilon^2 \sum_{j} \frac{U_\ast^3(z)}{|z - x_\ast|^2} dz \right)^{\frac{1}{2}}
\]

\[
\leq C \|\varphi\|_{\varepsilon}
\]

and

\[
\left| \int_{\mathbb{R}^2} \frac{x_1 - z_1}{|x - z|^2} \varphi^2 dz \right| \leq \int_{B_\varepsilon(x)} \frac{1}{|x - z|} \varphi^2 dz + \int_{\mathbb{R}^2 \setminus B_\varepsilon(x)} \frac{1}{|x - z|} \varphi^2 dz
\]

\[
\leq C \left( \int_{B_\varepsilon(x)} \frac{1}{|x - z|^2} dz \right)^{\frac{1}{2}} \|\varphi\|_{L^2(\mathbb{R}^2)}^2 + \varepsilon^{-1} \|\varphi\|_{L^2(\mathbb{R}^2)}^2 \tag{3.4}
\]

\[
\leq C \left( \varepsilon^\frac{1}{3} \varepsilon^{-\frac{1}{4}} \|\varphi\|_{\varepsilon}^2 + \varepsilon^{-1} \|\varphi\|_{L^2(\mathbb{R}^2)}^2 \right)
\]

\[
\leq C \varepsilon^{-1} \|\varphi\|_{\varepsilon}^2.
\]

Equations (3.3), (3.4) and Hölder inequality yield

\[
\left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} \varphi^2(y) dy \right) \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} U_\ast(z) \varphi(z) dy \right| U_\ast^2(x) dx \leq C \varepsilon^{-1} \|\varphi\|_{\varepsilon}^2 \|\varphi\|_{\varepsilon} \int_{\mathbb{R}^2} U_\ast^2(x) dx \leq C \varepsilon \|\varphi\|_{\varepsilon}^3
\]

and

\[
\left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_\ast(y) \varphi(y) dy \right)^2 U_\ast(x) \varphi(x) dx \right| \leq C \|\varphi\|_{\varepsilon}^2 \left( \int_{\mathbb{R}^2} U_\ast^2(x) dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^2} \varphi^2 dx \right)^\frac{1}{2} \leq C \varepsilon \|\varphi\|_{\varepsilon}^3.
\]

Similarly, we have the following estimates:

\[
\left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_\ast^2(y) dy \cdot \frac{x_2 - z_2}{|x - z|^2} \varphi^2(z) dz \right) U_\ast(x) \varphi(x) dx \right| \leq C \varepsilon^{-1} \|\varphi\|_{\varepsilon}^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x - y|} U_\ast^2(y) dy U_\ast(y) \varphi(x) dx \leq C \varepsilon \|\varphi\|_{\varepsilon}^3
\]

\[
\left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_\ast(y) dy \cdot \frac{x_2 - z_2}{|x - z|^2} U_\ast(z) \varphi(z) dz \right) \varphi^2(x) dx \right|
\]
Lemma 3.2. There holds

\[ \|\ell\| \leq C\varepsilon^{1+\theta} \|\varphi\| + C \sum_{i=1}^{k} \varepsilon |V(y^i) - V(x^0)| + C\varepsilon^{-\min\{\frac{k-1}{2}, 1\}}\frac{|y^i - y^j|}{\varepsilon} \|\varphi\| \text{.} \]  

(3.5)
Proof. By assumption \((V_1)\), one has

\[
| \int_{\mathbb{R}^2} (V(x) - V(x^0)) U_\epsilon \varphi \, dx | = | \sum_{i=1}^k \int_{\mathbb{R}^2} (V(x) - V(x_i^0)) U_{\epsilon,y^i} \varphi \, dx |
\]

\[
= \sum_{i=1}^k \int_{\mathbb{R}^2} (V(x) - V(y^i)) U_{\epsilon,y^i} \varphi \, dx + \sum_{i=1}^k \int_{\mathbb{R}^2} (V(y^i) - V(x^0)) U_{\epsilon,y^i} \varphi \, dx
\]

\[
\leq \sum_{i=1}^k \left( \int_{\mathbb{R}^2} [(V(x) - V(y^i)) U_{\epsilon,y^i}]^2 \, dx \right)^{1/2} \| \varphi \|_\epsilon
\]

\[
+ \sum_{i=1}^k \left( \int_{\mathbb{R}^2} [(V(y^i) - V(x^0)) U_{\epsilon,y^i}]^2 \, dx \right)^{1/2} \| \varphi \|_\epsilon
\]

\[
= \sum_{i=1}^k \varepsilon \left( \int_{\mathbb{R}^2} [(V(\epsilon x + y^i) - V(y^i)) U(x)]^2 \, dx \right)^{1/2} \| \varphi \|_\epsilon
\]

\[
+ \sum_{i=1}^k \varepsilon |V(y^i) - V(x^0)| \left( \int_{\mathbb{R}^2} U^2(x) \, dx \right)^{1/2} \| \varphi \|_\epsilon
\]

\[
\leq C \varepsilon^{1+\theta} \| \varphi \|_\epsilon + C \sum_{i=1}^k \varepsilon |V(y^i) - V(x^0)|. \tag{3.6}
\]

On the other hand, we have

\[
\left| \int_{\mathbb{R}^2} \sum_{i=1}^k U_{\epsilon,y^i}^{p-1} \varphi - \left( \sum_{i=1}^k U_{\epsilon,y^i} \right)^{p-1} \varphi \, dx \right| = \left\{ \begin{array}{ll}
O \left( \int_{\mathbb{R}^N} \sum_{i \neq j} U_{\epsilon,y^i}^{p-2} U_{\epsilon,y^j} \varphi \, dx \right), & \text{if } p > 3, \\
O \left( \int_{\mathbb{R}^N} \sum_{i \neq j} U_{\epsilon,y^i}^{p-2} U_{\epsilon,y^j}^{p-1} \varphi \, dx \right), & \text{if } 2 < p \leq 3
\end{array} \right.
\]

\[
\leq C \varepsilon e^{-\min \{ \frac{p-1}{2}, 1 \} \frac{|y^i - y^j|}{\varepsilon}} \| \varphi \|_\epsilon. \tag{3.7}
\]

As the proof of Lemma 3.1, we may verify

\[
\left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_\epsilon^2(y) dy \right) U_\epsilon(x) \varphi(x) \, dx \right| + \left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} U_\epsilon^2(y) dy \right) U_\epsilon(x) \varphi(x) \, dx \right|
\]

\[
\leq C \varepsilon^2 \int_{\mathbb{R}^2} U_\epsilon(x) \varphi(x) \, dx \leq C \varepsilon^2 \| \varphi \|_\epsilon, \tag{3.8}
\]

and

\[
\left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U_\epsilon^2(y) dy \right) \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} U_\epsilon(z) \varphi(z) \, dz \, dx \right|
\]

\[
+ \left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} U_\epsilon^2(y) dy \right) \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} U_\epsilon(z) \varphi(z) \, dz \, dx \right|
\]
\[ \leq C \varepsilon \| \varphi \|_\varepsilon \int_{\mathbb{R}^2} U^2_\varepsilon(x) dx \leq C \varepsilon^3 \| \varphi \|_\varepsilon. \]  
(3.9)

So the result follows from (3.6)-(3.9).

Since \( L_\varepsilon \) is bounded and bi-linear operator, the following lemma can be obtained directly.

**Lemma 3.3.** There exists a positive constant, independent of \( \varepsilon \), such that
\[ \| L_\varepsilon v \| \leq C \| v \|, \quad v \in E. \]

Next, we show that \( L_\varepsilon \) is invertible in \( E \).

**Lemma 3.4.** There are positive constants \( \varepsilon_0 \) and \( \mu_0 \), such that for any \( 0 < \varepsilon < \varepsilon_0 \) and \( v \in E \),
\[ \| L_\varepsilon v \| \geq \mu_0 \| v \|. \]  
(3.10)

**Proof.** We argue indirectly. Suppose on the contrary that there exist \( \varepsilon_n \to 0 \), \( v_n \in E_n \) and \( y^n = (y^{1,n}, \ldots, y^{k,n}) \in D^\varepsilon_n,\delta \) such that
\[ \langle L_{\varepsilon_n} v_n, \varphi \rangle = o_n(1) \| v_n \|_{\varepsilon_n} \| \varphi \|_{\varepsilon_n}, \quad \forall \varphi \in E_n. \]  
(3.11)

Without loss of generality, we assume that \( \| v_n \|_{\varepsilon_n} = \varepsilon_n \). For any \( \varphi \in E \), we have
\[
\left| \int_{\mathbb{R}^2} \left( \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U^2_\varepsilon(y) dy \right)^2 v_n \varphi dx \right| \leq C \varepsilon_n^2 \| v_n \|_{\varepsilon_n} \| \varphi \|_{\varepsilon_n}
\]
and
\[
\frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} U^2_\varepsilon(y) dy \right) \left( \int_{\mathbb{R}^2} \frac{x_2 - z_2}{|x - z|^2} U_\varepsilon(z) v_n d\varepsilon \right) U_\varepsilon(x) \varphi(x) dx
\]
\[
\leq C \varepsilon_n \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x - z|} U_\varepsilon(z) |v_n| d\varepsilon U_\varepsilon(x) |\varphi(x)| dx
\]
\[
\leq C \varepsilon_n^2 \| v_n \|_{\varepsilon_n} \int_{\mathbb{R}^2} U_\varepsilon(x) |\varphi(x)| dx
\]
\[
\leq C \varepsilon_n^2 \| v_n \|_{\varepsilon_n} \| \varphi \|_{\varepsilon_n}
\]

Similarly, we get
\[ |\langle L_{2,\varepsilon_n} v_n, \varphi \rangle| \leq C \varepsilon_n^2 \| v_n \|_{\varepsilon_n} \| \varphi \|_{\varepsilon_n}, \]
which implies
\[ \langle L_{1,\varepsilon_n} v_n, \varphi \rangle = o(1) \| v_n \|_{\varepsilon_n} \| \varphi \|_{\varepsilon_n}, \quad \forall \varphi \in E_n. \]  
(3.14)

Fix \( i \in \{1, \ldots, k\} \) and let
\[ \tilde{v}_{n,i}(x) = v_n(\varepsilon_n x + y^{i,n}). \]

We have
\[
\varepsilon_n^2 \int_{\mathbb{R}^2} |\nabla v_{\varepsilon_n}|^2 dx + \int_{\mathbb{R}^2} V(x) v_{\varepsilon_n}^2 dx = O(\varepsilon_n^2),
\]
and
\[
\int_{\mathbb{R}^2} |\tilde{v}_{n,i}|^2 dx + \int_{\mathbb{R}^2} V(\varepsilon x + y^{i,n})\tilde{v}_{n,i} dx \leq C. \tag{3.15}
\]
Then it follows from (3.15) that \(\{\tilde{v}_{n,i}\}\) is bounded in \(H^1(\mathbb{R}^2)\). Thus, there is a subsequence still denote by \(\tilde{v}_{n,i}\), such that for \(n \to \infty\),
\[
\tilde{v}_{n,i} \rightarrow v_i \quad \text{in} \quad H^1(\mathbb{R}^2), \quad \tilde{v}_{n,i} \rightarrow v_i \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}^2), \quad 2 \leq p < +\infty.
\]

Now we claim that \(v_i = 0\). For any \(\psi \in H^1(\mathbb{R}^N)\), we define
\[
P_{\varepsilon_n} \psi = \psi - \sum_{l=1}^{2} \sum_{i=1}^{k} \alpha_{\varepsilon_n,i} \frac{\partial U_{\varepsilon_n,y^i}}{\partial y^j} \in E_n, \tag{3.16}
\]
where
\[
\alpha_{\varepsilon_n,i} = \sum_{m=1}^{2} \sum_{j=1}^{k} C_{\varepsilon_n,i}^{m,j} \left\langle \frac{\partial U_{\varepsilon_n,y^i}}{\partial y^j}, \psi \right\rangle_{\varepsilon_n},
\]
for some constants \(C_{\varepsilon_n,i}^{m,j}\). Let
\[
\beta_{\varepsilon_n,i} = \langle L_{\varepsilon_n,1} v_n, \frac{\partial U_{\varepsilon_n,y^i}}{\partial y^j} \rangle_{\varepsilon_n}.
\]
Then, we have
\[
\langle L_{1,\varepsilon_n} v_n, \psi \rangle_{\varepsilon_n} = \langle L_{\varepsilon_n,1} v_n, P_{\varepsilon_n} \psi \rangle_{\varepsilon_n} + \sum_{l=1}^{2} \sum_{i=1}^{k} \alpha_{\varepsilon_n,i} \beta_{\varepsilon_n,i}
\]
\[
= o(1) \|v_n\|_{\varepsilon_n} \|\psi\|_{\varepsilon_n} + \sum_{l=1}^{2} \sum_{i=1}^{k} \alpha_{\varepsilon_n,i} \beta_{\varepsilon_n,i} \tag{3.17}
\]
\[
= o(1) \|v_n\|_{\varepsilon_n} \|\psi\|_{\varepsilon_n} + \sum_{l=1}^{2} \sum_{i=1}^{k} \gamma_{\varepsilon_n,i} \left\langle \frac{\partial U_{\varepsilon_n,y^i}}{\partial y^j}, \psi \right\rangle_{\varepsilon_n},
\]
where \(\gamma_{\varepsilon_n,i} = \sum_{m=1}^{2} \sum_{j=1}^{k} C_{\varepsilon_n,i}^{m,j} \beta_{\varepsilon_n,m,j}\). Choosing \(\varphi = \frac{\partial U_{\varepsilon_n,y^i}}{\partial y^j}\) in (3.17), we can estimate
\[
\gamma_{\varepsilon_n,i} = o(\varepsilon_n).
\]
Hence, (3.17) becomes
\[
\langle L_{1,\varepsilon_n} v_n, \psi \rangle_{\varepsilon_n} = o(1) \|v_n\|_{\varepsilon_n} \|\psi\|_{\varepsilon_n}, \quad \forall \psi \in H^1(\mathbb{R}^2). \tag{3.18}
\]
Let \(\tilde{\varphi}_n(x) = \varphi(\frac{x-y^{i,n}}{\varepsilon})\) and substitute it into (3.18), we obtain
\[
\int_{\mathbb{R}^2} \nabla \tilde{v}_{n,i} \nabla \tilde{\varphi}_n dx + \int_{\mathbb{R}^2} V(\varepsilon x + y^{i,n})\tilde{v}_{n,i} \tilde{\varphi}_n dx - (p-1) \int_{\mathbb{R}^2} U_*(\varepsilon x + y^{i,n})^{p-2} \tilde{v}_{n,i} dx
\]
\[
= \varepsilon_n^{-2} \left\{ \varepsilon_n^2 \int_{\mathbb{R}^2} \nabla v_n \nabla \tilde{\varphi}_n dx + \int_{\mathbb{R}^2} V(x)v_n \tilde{\varphi}_n dx - (p-1) \int_{\mathbb{R}^2} U_*(p-2)v_n \tilde{\varphi}_n dx \right\}
\]
\[
= \varepsilon_n^{-2} o(\varepsilon_n) \|\tilde{\varphi}_n\|_{\varepsilon_n} = o(1) \|\tilde{\varphi}_n\|_{\varepsilon_n}.
\]
Therefore, \( v_i \) satisfies the equation
\[-\Delta v_i + V(y^i)v_i = (p - 1)U^{p-2}v_i,\]
and the non-degeneracy of the solution \( U \) gives
\[v_i = \sum_{i=1}^{2} c_i \frac{\partial U}{\partial x_i}.\]
Since \( v_n \in E_n \), that is,
\[\langle v_n, \frac{\partial U_{\varepsilon, y^i_l}}{\partial y^i_l} \rangle_{\varepsilon_n} = 0,\]
we deduce that
\[\langle v_i, \frac{\partial U}{\partial x_l} \rangle = 0, l = 1, 2.\]
which gives \( c_1 = c_2 = 0 \), and then \( v_i = 0 \).

\[o_n(1)\varepsilon_n^2 = \langle L_{\varepsilon_n}v_n, v_n \rangle = \langle L_{\varepsilon_n,1}v_n, v_n \rangle + \langle L_{\varepsilon_n,2}v_n, v_n \rangle = \|v_n\|_{\varepsilon_n}^2 - (p - 1)\int_{\mathbb{R}^2} U_\varepsilon^{p-2}v_n^2dx + o(1)\|v_n\|_{\varepsilon_n}^2\]
\[= (1 + o(1))\varepsilon_n^2 - (p - 1)\int_{B_R(0)} U_\varepsilon^{p-2}v_n^2dx - (p - 1)\int_{\mathbb{R}^2 \setminus B_R(0)} U_\varepsilon^{p-2}v_n^2dx\]
\[= (1 + o(1) + o_R(1))\varepsilon_n^2,\]
which is impossible for large \( n \). Hence, the conclusion follows. \( \square \)

### 4. Proof of the main result

In this section, we prove Theorem 1.2 by the reduction method.

**Proposition 4.1.** For \( \varepsilon \) sufficiently small, there is a \( C^1 \) map from \( D^\varepsilon_{k,\delta} \) to \( E \) and
\[J'_\varepsilon(\omega)\big|_E = 0.\]

Moreover, there exists a constant \( C > 0 \) independent of \( \varepsilon \) small enough such that
\[\|\omega\|_\varepsilon \leq C(\varepsilon^{\frac{N}{2} + \min\{\theta, 2\}} + \varepsilon^{\frac{N}{2}} \sum_{i=1}^{k} (V(y^i) - V(x^0)) + \varepsilon^{\frac{N}{2}} \varepsilon^{-\min\{\frac{\varepsilon^2}{2}, 1\}} \sum_{i, j} \frac{|w^i - w^j|}{\varepsilon}).\]

**Proof.** We will use the contraction theorem to prove it. By the Lemma 3.2, \( \ell(\omega) \) is a bounded linear functional in \( E \). The Riesz representation theorem implies that there is an \( \bar{\ell}_\varepsilon \in E \), such that
\[\ell(\omega) = \langle \bar{\ell}_\varepsilon, \omega \rangle.\]
Therefore, finding a critical point for \( J(\omega) \) is equivalent to solving
\[\bar{\ell}_\varepsilon + L_\varepsilon(\omega) + R'_\varepsilon(\omega) = 0. \tag{4.1}\]
By Lemma 3.4, $L_\varepsilon$ is invertible. Thus (4.1) is equivalent to
\[
\omega = A(\omega) := -L_\varepsilon^{-1}(\tilde{\ell}_\varepsilon + R'_\varepsilon(\omega)).
\]

We set
\[
S_\varepsilon := \{\omega \in E : \|\omega\|_\varepsilon \leq \varepsilon^{1+\theta-k} + \varepsilon^{1-k} \sum_{i=1}^{k} (V(y^i) - V(x^0)) + \varepsilon e^{(-\min\{\frac{\kappa}{2},1\}-\kappa) \frac{|y^i-y^j|}{\varepsilon}} \}
\]
for any small $\kappa > 0$.

Now, we verify that $A$ is a contraction mapping from $S_\varepsilon$ to itself. For $\omega \in S_\varepsilon$, by Lemmas 3.1 and 3.2, we obtain
\[
\|A(\omega)\| \leq C(\|\tilde{\ell}_\varepsilon\| + \|R'_\varepsilon(\omega)\|)
\leq C\left(\varepsilon^{1+\theta} + \varepsilon \sum_{i=1}^{k} (V(y^i) - V(x^0)) + \varepsilon e^{(-\min\{\frac{\kappa}{2},1\}-\kappa) \frac{|y^i-y^j|}{\varepsilon}} \right)
\leq \varepsilon^{1+\theta-k} + \varepsilon^{1-k} \sum_{i=1}^{k} (V(y^i) - V(x^0)) + \varepsilon e^{(-\min\{\frac{\kappa}{2},1\}-\kappa) \frac{|y^i-y^j|}{\varepsilon}}.
\]

Then, $A$ maps $S_\varepsilon$ to $S_\varepsilon$. On the other hand, for any $\omega_1, \omega_2 \in S_\varepsilon$,
\[
\|A(\omega_1) - A(\omega_2)\| = \|L_\varepsilon^{-1}R'_\varepsilon(\omega_1) - L_\varepsilon^{-1}R'_\varepsilon(\omega_2)\|
\leq C \|R'_\varepsilon(\omega_1) - R'_\varepsilon(\omega_2)\|
\leq C \|R''_\varepsilon(\theta \omega_1 + (1-\theta)\omega_2)\|_\varepsilon\|\omega_1 - \omega_2\|_\varepsilon
\leq \frac{1}{2}\|\omega_1 - \omega_2\|_\varepsilon.
\]

So $A$ is a contraction map from $S_\varepsilon$ to $S_\varepsilon$. Consequently, applying the contraction mapping theorem and implicit function theorem, the conclusion is completed.

Now, we are ready to prove our main theorem. Let $\varphi_{\varepsilon,y}$ be the map obtained in Proposition 4.1.

Define
\[
F(y) = I(U_* + \varphi_{\varepsilon,y}), \quad \forall \varphi_{\varepsilon} \in S_\varepsilon.
\]
It is well known that if $y$ is a critical point of $F(y)$, then $U_* + \varphi_{\varepsilon,y}$ is a solution of our problem.

**Proof of Theorem 1.2.** By Lemmas 3.4 and 3.1, Propositions 4.1 and 2.1, we find
\[
F(y) = I_\varepsilon(U_*) + \ell_\varepsilon(\varphi_y) + \frac{1}{2}(L_\varepsilon(\varphi_y), \varphi_y) + R_\varepsilon(\varphi_y)
= I_\varepsilon(U_{\varepsilon,y}) + O(\|\tilde{\ell}_\varepsilon\|_\varepsilon\|\varphi_y\| + \|\varphi_y\|^2)
= A\varepsilon^2 - \frac{1}{2} \sum_{i=1}^{k} (V(x^0) - V(y^i))\varepsilon^2 \int_{\mathbb{R}^2} U^2 dx - C_1\varepsilon^2 \sum_{i \neq j}^{k} e^{-|y^i-y^j|/\varepsilon}.
\]
\[ +O(\varepsilon^{2+\theta} + \varepsilon^2 \sum_{i \neq j}^k e^{-(1+\sigma)|y_i-y_j|/\varepsilon} + \varepsilon^4) \]

\[ (4.2) \]

Consider the following maximizing problem
\[ \max_{y \in D_{k,\varepsilon,\delta}} F(y). \]

Since \( F \in C^1 \), we can assume that \( F \) is achieved by some \( y_\varepsilon \) in \( D_{k,\varepsilon,\delta} \). We will prove that \( y_\varepsilon \) is an interior point of \( D_{k,\varepsilon,\delta} \).

Let \( \bar{y}_\varepsilon = x_0 + M\varepsilon|\ln \varepsilon|e_i \), for some constant \( M > 0 \), vectors \( e_1, \ldots, e_k \) with \( |e_i - e_j| = 1 \) for \( i \neq j \). Thus, for \( M > 0 \) large, \( \varepsilon \) small enough and \( \bar{y}_\varepsilon \in D_{k,\varepsilon,\delta} \), we have
\[ A\varepsilon^2 - C_1\varepsilon^{2+\theta}|\ln \varepsilon|^\theta \leq A\varepsilon^2 - C_2\varepsilon^2 \sum_{i=1}^k (V(x^0) - V(y_i)) - C_2\varepsilon^2 \sum_{i \neq j}^k e^{-|y_i-y_j|/\varepsilon}, \]

\[ (4.3) \]

where \( \bar{y}_\varepsilon = (\bar{y}_\varepsilon^1, \ldots, \bar{y}_\varepsilon^k) \).

Employing (4.3), we deduce that for \( \varepsilon \) sufficient small,
\[ A\varepsilon^2 - C_1\varepsilon^{2+\theta}|\ln \varepsilon|^\theta \leq A\varepsilon^2 - C_2\varepsilon^2 \sum_{i=1}^k (V(x^0) - V(y_i)) - C_2\varepsilon^2 \sum_{i \neq j}^k e^{-|y_i-y_j|/\varepsilon}, \]
i.e.
\[ C_2\varepsilon^2 \sum_{i=1}^k (V(x^0) - V(y_i)) + C_2\varepsilon^2 \sum_{i \neq j}^k e^{-|y_i-y_j|/\varepsilon} \leq C_1\varepsilon^{2+\theta}|\ln \varepsilon|^\theta. \]

That is,
\[ \sum_{i=1}^k (V(y_i) - V(x^0)) \leq C\varepsilon^\theta|\ln \varepsilon|, \]
\[ \sum_{i \neq j}^k \frac{|y_i^j - y_j^i|}{\varepsilon} \geq \theta|\ln \varepsilon| \geq |\ln \varepsilon|^{1/2}. \]

This implies that \( y_\varepsilon \) is an interior point of \( D_{k,\varepsilon,\delta} \) and hence is a critical point of \( F(y) \) for \( \varepsilon \) sufficiently small.

Finally, by the standard argument and the strong maximum principle, we obtain that \( u_\varepsilon = U_* + \varphi > 0 \).
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