Force on proton vortices in superfluid neutron stars

M. E. Gusakov*
Ioffe Institute, Politekhnicheskaya 26, 194021 St.-Petersburg, Russia

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ABSTRACT
Force on proton vortices in superfluid and superconducting matter of neutron stars is calculated at vanishing stellar temperature. Both longitudinal (dissipative) and transverse (Lorentz-type) components of the force are derived in a coherent way and compared in detail with the corresponding expressions available in the literature. This allows us to resolve a controversy about the form of the Lorentz-type force component acting on proton vortices. The calculated force is a key ingredient in magnetohydrodynamics of superconducting neutron stars and is important for modeling the evolution of stellar magnetic field.

Key words: stars: neutron – stars: interiors.

1 INTRODUCTION AND FORMULATION OF THE PROBLEM
The magnetic field in neutron stars (NSs) varies in a very wide range from \(~10^8\) to \(10^{10}\) G in millisecond pulsars and neutron stars in low-mass X-ray binaries to \(10^{12}\) G in ordinary radio pulsars and up to \(10^{15}\) G in magnetars (Kaspi 2010; Viganò et al. 2013). It is a challenge for theorists to explain such diverse objects within a unified theoretical model. The problem is significantly complicated by the fact that NS matter can become superfluid/superconducting at stellar temperatures \(T\sim10^8 - 10^{10}\) K (Page et al. 2013; Gezerlis et al. 2014; Sedrakian & Clark 2018). The magnetic field in such matter is confined to proton vortices, also called Abrikosov vortices or proton flux tubes (Baym, Pethick, & Pines 1969; Sauls 1989). Therefore, to describe the evolution of the magnetic field in NSs it is necessary to understand in detail the vortex dynamics, a very complicated problem, full of controversies in the literature, which has not been fully solved yet (see Haskell & Sedrakian 2017; Sedrakian & Clark 2018 for recent reviews).

Here we would like to focus on one such controversy related to the forces acting on proton vortices in superconducting NSs. In what follows, to make our analysis as simple as possible, we consider a strongly degenerate npe-matter of NS cores composed of superfluid neutrons (n), superconducting protons (p), and electrons (e) (the effect of muons will be discussed in Section 3). For simplicity, we assume that the temperature \(T\) is so small that there are almost no thermal Bogoliubov neutron and proton excitations in the system in the absence of vortices. We also neglect the effects of neutron-proton entrainment, assuming that the off-diagonal elements of the entrainment matrix \(\rho_{k}\) vanish, \(\rho_{np} = \rho_{pn} = 0\) (see, e.g., Andreev & Bashkin 1976 for a definition of \(\rho_{k}\)). In principle, all these simplifying assumptions, except for the assumption \(T = 0\), can be easily relaxed. However, extension of our results to finite temperatures is more intricate, since it requires a detailed understanding of how vortices interact with neutron and proton thermal Bogoliubov excitations – an almost unexplored problem in the context of NS physics (see Kopnin 2002; Sonin 2016 for a general approach to attack it).

Description of the controversy
We shall start with the equation, describing the magnetic field evolution in superconducting NSs (Konenkov & Geppert 2000; Gusakov & Dommes 2016; Dommes & Gusakov 2017; Bransgrove, Levin, & Beloborodov 2018),

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V}_L \times \mathbf{B}),
\]

(1)

where \(\mathbf{B}\) is the stellar magnetic field, averaged over the volume containing many vortices (more precisely, it is the magnetic induction field); \(\mathbf{V}_L\) is the local vortex velocity. This equation simply states, that the magnetic field, confined to proton vortices, is transported with the vortex velocity, \(\mathbf{V}_L\). To solve (1) one needs first to define \(\mathbf{V}_L\), i.e., to express it through available transport particle velocities in the system. For example, for npe-mixture at zero temperature \((T = 0)\), the relevant velocities are the electron velocity, \(\mathbf{u}_e\), and the superfluid neutron and proton velocities, \(\mathbf{V}_{sn}\) and \(\mathbf{V}_{sp}\), respectively (see Section 3 for an accurate definition of these velocities). To express \(\mathbf{V}_L\) through \(\mathbf{u}_e\), \(\mathbf{V}_{sn}\), and \(\mathbf{V}_{sp}\), one should write down a force balance equation for a vortex, which can be customarily presented in the form (e.g.,

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* gusakov@astro.ioffe.ru

1 We assume that protons form a type-II superconductor, see Section 2 for more details.
Glampedakis, Andersson, & Samuelsson (2011):

\[ F_{\text{buoyancy}} + F_{\text{tension}} + F_{\text{npe-v}} = 0. \]  

(2)

In writing this equation we assumed that the mass of the vortex per unit length is negligible, so that the sum of the forces (per unit length) on a vortex must vanish (Donnelly 2005). In equation (2) \( F_{\text{buoyancy}} \) and \( F_{\text{tension}} \) are the buoyancy and tension forces, respectively (their actual form is not important for us here; see, e.g., Dommes & Gusakov 2017 for details; in Section 3 these forces, which do not depend on transport velocities, will be denoted \( F_{\text{ext}} \); \( F_{\text{npe-v}} \) is the total velocity-dependent force on a vortex from npe-matter. Taking into account the so called ‘screening condition’, \( V_{\text{sp}} = u_e \), which should be satisfied in NS bulk, and neglecting entrainment effects, \( F_{\text{npe-v}} \) takes the form (see Section 3 for a detailed derivation),

\[ F_{\text{npe-v}} = -D e_e \times (u_e \times (u_e - V_\text{L})) + D' e_e \times (u_e - V_\text{L}). \]  

(3)

The first term here describes longitudinal (dissipative) part of the force, the second term is the transverse (Lorentz-like) part.

According to Jones (1991, 2006) \( D' = 0 \). His result is based on the following arguments. Superconducting protons act on a vortex with the Magnus force (e.g., Nozières & Vinen 1966; Kopnin 2002; Glampedakis et al. 2011),

\[ F_\text{M} = -\pi \hbar n_p [e_e \times (V_\text{sp} - V_\text{L})] = -\pi \hbar n_p [e_e \times (u_e - V_\text{L})], \]  

(4)

where \( n_p \) is the proton number density and in the second equality we made use of the screening condition, \( V_{\text{sp}} = u_e \). Note that the Magnus force (4) coincides with the Lorentz force on protons, \( F_{\text{Lp}} = (1/e) [ J_p \times \Phi_0] \), where \( J_p = e_p n_p (V_{\text{sp}} - V_\text{L}) \) is the proton current density in the coordinate system in which \( V_\text{L} = 0 \); \( e_p \) is the proton charge; \( \Phi_0 = \Phi_0 e_e \) is the vector directed along \( e_e \), whose absolute value equals the total magnetic flux of a proton vortex, \( \Phi_0 = \pi \hbar/e_p \approx 2.07 \times 10^{-7} \) G cm\(^2\). The fact that protons act on a vortex with the Lorentz force \( F_{\text{Lp}} = (1/e) [ J_p \times \Phi_0] \) may lead to idea that electrons also act on a vortex with the corresponding Lorentz force, \( F_{\text{Le}} = (1/e) [ J_e \times \Phi_0] \), where \( J_e = e_e n_e (u_e - V_\text{L}) \), and \( e_e, n_e \) are the electron charge and number densities, respectively. Sum of these two forces, \( F_{\text{Lp}} + F_{\text{Le}} \), equals zero (Jones 2009) because of the screening and quasi-neutrality conditions (\( n_e = n_0 \)), which allows Jones to conclude that the total transverse force on a vortex vanishes, \( D' = 0 \).

Unlike Jones, Alford & Sedrakian (2010) postulated (without justification) that \( D' = -\pi \hbar n_p \), i.e., the only transverse force on a vortex is the Magnus force, \( F_\text{M} \). In turn, Glampedakis et al. (2011) also assumed that \( D' = -\pi \hbar n_p \), arguing that there are three transverse forces acting on a vortex in npe-matter, namely, the Magnus force \( F_\text{M} \), the (minus) electron Lorentz force, \( -F_{\text{Le}} \), and the (minus) proton Lorentz force, \( -F_{\text{Lp}} \) (the minus sign appears due to the Newton’s third law; for example, electrons are subject to the force \( F_{\text{Le}} \) in the magnetic field of a vortex, thus the force on a vortex is \( -F_{\text{Le}} \)). Sum of these forces, \( F_\text{M} + (-F_{\text{Le}}) + (-F_{\text{Lp}}) \) equals \( F_\text{M} \), hence \( D' = -\pi \hbar n_p \).

Both interpretations of Jones (1991, 2006) and Glampedakis et al. (2011) are not very convincing. The interpretation by Jones have an obvious problem with the Newton’s third law: Electrons act on a vortex with the Lorentz force \( F_{\text{Le}} \) on electrons, which is strange. The interpretation by Glampedakis et al. is also confusing, because it assumes that the Magnus and Lorentz forces \( F_\text{M} \) and \( F_{\text{Lp}} \) are of different origin, although it is, in fact, two different names for the same force (Nozières & Vinen 1966).

So, what is the correct value of \( D' \)? The answer is very important since the vortex velocity \( V_\text{L} \), defined by equation (2), can vary by orders of magnitude depending on the choice of \( D' \). This uncertainty can affect dramatically the typical magnetic field evolution timescales (see equation 1 and compare the evolution timescales, e.g., in Jones 2006; Bransgrove et al. 2018 and in Graber et al. 2015; Ellritz et al. 2016).

The present work is devoted to answering this question. In Section 2 we discuss the basic parameters characterizing proton vortices and length scales that play a role in our problem. In Section 3 we derive a basic expression for the total force on a vortex. Instead of considering separate contributions to the force from each particle species (a way, which apparently leads to contradictory results in the literature), we decided to extract the force on a vortex from the analysis of total magnetic field evolution equation for the system as a whole. This derivation method is inspired by the work of Sonin (1976); Galperin & Sonin (1976); Aronov et al. (1981). Further, in Section 4 we calculate two necessary cross-sections, which determine the coefficients \( D \) and \( D' \). We discuss the obtained force on a vortex and compare it with the results available in the literature in Section 5. Finally, we conclude in Section 6.

2 BASIC PARAMETERS AND HIERARCHY OF LENGTHSCALES

Schematically, the proton vortex consists of the ‘normal’ core\(^3\) with the radius of the order of the coherence length, \( \xi \), surrounded by the more extended region containing the magnetic field (see Fig. 1). The radius of that region is \( \sim \lambda \), where \( \lambda \) is the London penetration depth. The parameters \( \xi \) and \( \lambda \) are given by the formulas (e.g., Landau & Lifshitz 1980; De Gennes 1999)

\[ \frac{\hbar p_{\text{FP}}}{\pi m_p^* \Delta p} \approx 28 \text{ fm} \left( \frac{n_p}{0.18 n_0} \right)^{1/3} \left( \frac{m_p}{m_p^*} \right) \left( \frac{0.456 \text{ MeV}}{\Delta p} \right), \]  

(5)

\[ \lambda = \sqrt{\frac{m_p c^2}{4 \pi e^2 n_p}} \approx 42.4 \text{ fm} \left( \frac{0.18 n_0}{n_p} \right)^{1/2}. \]  

(6)

Here \( p_{\text{FP}}, m_p, m_p^* \) are the proton Fermi momentum, mass, and effective mass, respectively; \( n_0 = 0.16 \text{ fm}^{-3} \) is the

\(^2\) It turns out that there are also no agreement about the value of the coefficient \( D \) in the literature (see Section 5).

\(^3\) Inside the vortex core proton quasiparticles exist even at \( T = 0 \) (Caroli, De Gennes, & Matricon 1964).
nuclear matter density: \( n_{\text{ep}}(T) \) and \( \Delta_p(T) \) are the superfluid proton number density and energy gap, respectively. At \( T = 0 \) one has \( n_{\text{ep}} = n_p \) and \( \Delta_p \approx k_BT_{cp}/0.567 \), where \( T_{cp} \) is the proton critical temperature. In particular, \( \Delta_p \approx 0.456 \text{ MeV} \) for \( T_{cp} = 3 \times 10^9 \text{ K} \).

Note that, generally, \( \xi(T) \) can be comparable to \( \lambda(T) \) for NS conditions, but proton vortices may exist only for type-II superconductors, for which \( \xi(T) < \sqrt{2} \lambda(T) \) (Landau & Lifshitz 1980; De Gennes 1999). This condition can be violated in the deep layers of NS cores (e.g., Sedrakian 2005; Jones 2006; Glampedakis et al. 2011; Gusakov & Dommes 2016).

Following Alpar, Langer, & Sauls (1984), we parametrize the vortex magnetic field \( \mathbf{B}(r) \) as

\[
\mathbf{B} = \mathbf{e}_z \left( \frac{\Phi}{\pi \xi^2} \right) \cdot \begin{cases}
1 - \frac{\xi}{\lambda} K_1 \left( \frac{\xi}{\lambda} \right) I_0 \left( \frac{r}{\lambda} \right), & 0 \leq r < \xi; \\
\frac{\xi}{\lambda} I_1 \left( \frac{\xi}{\lambda} \right) K_0 \left( \frac{r}{\lambda} \right), & r \geq \xi,
\end{cases}
\]

where \( \Phi \) is the magnetic flux associated with the vortex line; for a proton vortex \( \Phi = \Phi_0 \) (see Section 1 for a definition of \( \Phi_0 \); we emphasize that the results obtained in this paper are presented in the form valid for arbitrary magnetic flux of a vortex). The corresponding vector potential \( \mathbf{A}(r) \) is (we work in the cylindrical coordinate system \( r, \theta, z \), defined in Fig. 1)

\[
\mathbf{A} = -\mathbf{e}_z \left( \frac{\Phi}{2\pi r} \right) \cdot \begin{cases}
\frac{r}{\xi} \left[ 2K_1 \left( \frac{\xi}{\lambda} \right) I_0 \left( \frac{r}{\lambda} \right) - K_1 \left( \frac{\xi}{\lambda} \right) I_0 \left( \frac{r}{\lambda} \right) \right], & 0 \leq r < \xi; \\
1 - 2r\xi I_1 \left( \frac{\xi}{\lambda} \right) K_1 \left( \frac{r}{\lambda} \right), & r \geq \xi,
\end{cases}
\]

Further it will be convenient to rewrite it as

\[
\mathbf{A} = A_\theta \mathbf{e}_\theta = \frac{\hbar c}{e_p} \zeta \mathcal{P}(r) \mathbf{e}_\theta,
\]

where

\[
\zeta = \frac{e_p \Phi}{2\pi \hbar c} = \frac{\Phi}{2\Phi_0}
\]

and the function \( \mathcal{P}(r) \) is defined by equation (8) and is related to the magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} \) by the following obvious equation:

\[
\mathbf{B} = \mathbf{e}_z \frac{\hbar c}{e_p} \zeta \frac{d\mathcal{P}(r)}{dr}.
\]

In equations (7)–(9) \( \mathbf{e}_\theta = \mathbf{B}/B \) is the unit vector in the direction of the magnetic field. The important characteristic of magnetized NSs is the average distance between neighbouring vortices, \( d_B \) (e.g., Tinkham 1996; De Gennes 1999)

\[
d_B = \sqrt{\frac{2\Phi}{\sqrt{3}B}} = 4.89 \times 10^3 \text{ fm} \sqrt{\frac{10^{12} G}{B}},
\]

which is defined by specifying the average stellar magnetic induction field \( B \). In this work we assume that \( d_B \) is larger than \( \lambda \), i.e., we consider NSs with \( B \) smaller than

\[
B < \frac{8\pi \Phi e_p^2 n_{\text{ep}}}{\sqrt{3m_p c^2}} \approx 1.33 \times 10^{16} \text{ G} \left( \frac{n_{\text{ep}}}{0.18 n_0} \right) \left( \frac{\Phi}{\Phi_0} \right).
\]

Another important parameter in our problem is the typical electron wavelength (divided by 2\( \pi \))

\[
\frac{1}{k_{Fe}} = 1.05 \text{ fm} \left( \frac{0.18 n_0}{n_e} \right)^{1/3},
\]

where \( k_{Fe} = p_{Fe}/\hbar \) with \( p_{Fe} \) being the electron Fermi momentum. One sees that it is much smaller than \( \xi, \lambda \), and \( d_B \). This enables us to study the forces acting on an isolated vortex within the quasiclassical approximation and ignoring the presence of other vortices, i.e., the collective effects.\(^4\)

\(^4\) Note that the scattering cross-sections, calculated in Section 4, which have a dimension of length, are also much smaller than \( d_B \).
The dimensionless parameters \( k_{\text{F}e} \xi \) and \( k_{\text{F}p} \lambda \) are plotted in Fig. 2 as functions of the baryon number density \( n_s \). Here and below to plot the figures we employ the equation of state IHJ (Heiselberg & Hjorth-Jensen 1999) and assume \( T_{\text{cp}} = 3 \times 10^8 \) K. Finally, the last important parameter that should be mentioned here is the electron mean free path, \( l \). Generally, it is much larger than \( d_B \) (e.g., Schmitt & Shternin 2017) and hence than other typical lengthscales discussed above even for normal (nonsuperfluid and nonsuperconducting) matter. Nucleon superfluidity further increases \( l \). This means that at distances \( r \ll l \) a perturbation of the electron distribution function caused by the vortex can be found from the collisionless kinetic equation for electrons. This property will be used in the next section.

Summarizing, there are five relevant lengthscales, \( \xi, \lambda, d_B, 1/k_{\text{F}e}, \) and \( l \), in the problem of calculation of the force acting on a vortex, and for typical NS conditions they are related by the inequality
\[
l \gg d_B \gg \lambda \gg \xi \gg 1/k_{\text{F}e}.
\]

3 GENERAL EXPRESSION FOR THE FORCE ON A PROTON VORTEX

Let us create a straight proton vortex (also called the Abrikosov vortex or flux tube) in the initially homogeneous system, with the magnetic field directed along the axis \( z \), as shown in Fig. 1. As in Section 1, the vortex velocity is \( V_L \); the velocities of neutrons, protons, and electrons far from the vortex are denoted as \( V_{\text{nn}}, V_{\text{sp}}, \) and \( u_s \), respectively. Because of the screening condition (Jones 1991, 2006; Glamppedakis et al. 2011; Gusakov & Dommes 2016) the electron and proton currents (and hence velocities) must coincide to a very high precision in the bulk of NS superconductor,
\[
V_{\text{sp}} = u_s.
\]

Our aim is to calculate the velocity-dependent force (per unit length), which acts on the vortex from the surrounding matter. Below in Sections 3 and 4 we assume that electrons scatter only on the magnetic field of a vortex and do not scatter on the localized proton excitations in the vortex core. The effect of the latter type of scattering will be discussed in Section 5. Taking into account the condition (16) the force can be, quite generally, written as (e.g., Donnelly 2005; Sonin 2016)
\[
F_{\text{npe} \to \text{V}} = -D[e_z \times [e_z \times (u_s - V_L)]] + D'[e_z \times (u_s - V_L)] + D_2 e_z [e_z \cdot (u_s - V_L)],
\]
where we accounted for the fact that the neutron condensate does not interact with the proton vortex in the absence of entrainment, so that there are no terms in equation (17), depending on \( V_{\text{nn}} - V_L \) (a subsequent calculation confirms this expectation). In equation (17) \( D, D', \) and \( D_2 \) are the kinetic coefficients to be determined below (the coefficients \( D \) and \( D' \) have already been introduced in Section 1). In what follows we assume that the difference \( u_s - V_L \) is sufficiently small and restrict ourselves to calculations valid in linear order in \( u_s - V_L \). In this approximation the coefficients \( D, D', \) and \( D_2 \) are velocity-independent and to determine them we, for simplicity, consider two cases. First, assume that the vector \( u_s - V_L \) is collinear with \( e_z \). Then only the last term \( \propto D_2 \) survives in equation (17) and the corresponding force is directed along the vortex line. But any such force should vanish since the magnetic field cannot scatter electrons traveling along the axis \( z \). We come to conclusion that \( D_2 = 0 \). Assume now that the vector \( u_s - V_L \) lies in the \( xy \)-plane (see Fig. 3). Then equation (17) can be conveniently represented as
\[
F_{\text{npe} \to \text{V}} = (D(u_s - V_L) + D'[u_s \times (u_s - V_L)]).
\]

Our system is assumed to be stationary in the coordinate system moving with the vortex, which means that the force \( F_{\text{npe} \to \text{V}} \) must be balanced by some ‘external’ force, \( F_{\text{ext}} \), acting on a vortex (in NSs it can be, for example, the tension and/or buoyancy forces, which do not depend on particle velocities; see equation (2) in Section 1 and Dommes & Gusakov 2017 for details),
\[
F_{\text{npe} \to \text{V}} + F_{\text{ext}} = 0.
\]

Let us express \( F_{\text{npe} \to \text{V}} \) through the stress tensor of npe-matter far from the vortex. With this aim we make use of the total momentum conservation (see Appendix B for more details)
\[
\frac{\partial G}{\partial t} + \nabla \Pi_{ik} = f_{\text{ext}},
\]
where \( G \) and \( \Pi_{ik} \) are the momentum density and stress tensor for npe-matter (including the electromagnetic field), respectively, and \( f_{\text{ext}} \) is the external force density localized in the vortex core.\(^5\) Now, let us choose a cylinder of unit length with the symmetry axis coinciding with the vortex line, and radius \( r_0 \) satisfying the inequality \( l \gg d_B \gg r_0 \gg \lambda \) (see Fig. 3). Integrating equation (20) over the cylinder volume and using the Gauss theorem, as well as the fact that the system is stationary, \( \partial G/\partial t = 0 \), and that, by definition, \( \int f_{\text{ext}} \, dV = F_{\text{ext}} \), one finds
\[
- \oint \Pi_{ik} n_k dS + F_{\text{ext}} = 0,
\]
\(^5\) In fact, this requirement is not necessary for derivation of equation (22) below.

Figure 3. Top view on the vortex and a cylinder of radius \( r_0 \) used to take the integral (22).
or, taking into account the relation (19),

\[ \mathbf{F}_{\text{nepe-V}} = - \oint \Pi_{ik} n_k \, dS, \]  

(22)

where the integration is performed over the cylinder surface; \( dS \) is the surface element; and \( n_k \) is the outer normal unit vector.

Formula (22) is very useful since it allows us to find the force on the vortex provided that the stress tensor far from the vortex is known (an explicit expression for \( \Pi_{ik} \) is presented in Appendix B). However, far from the vortex (at \( r \gg \lambda \)) the vortex magnetic field and the proton superfluid velocity, generated by the vortex, are exponentially suppressed (see equation 7 and, e.g., De Gennes 1999). As it is shown in Appendix B, in these circumstances only electrons contribute to the integral (22). The electron stress tensor \( \Pi_{ik}^{(e)} \) is given by the standard expression (see, e.g., Landau & Lifshitz 1981),

\[ \Pi_{ik}^{(e)} = \sum_{\mathbf{p}, \mathbf{v}} \rho_{ik}(\mathbf{p}) \rho_{\mathbf{p} \mathbf{v}}, \]  

(23)

where \( \mathbf{p}, \mathbf{v} \), and \( \mathbf{v} = \partial \mathbf{p} / \partial \mathbf{p} \) are the electron (kinetic) momentum, energy, and velocity, respectively; \( \rho_{\mathbf{p} \mathbf{v}} \) is the electron distribution function; and summation is assumed over the electron momenta and spins. If \( \rho_{\mathbf{p} \mathbf{v}} \) does not explicitly depend on spins (our case), one has \( \sum_{\mathbf{p} \mathbf{v}} \equiv 2/(2\pi)^3 \int d^4 \mathbf{p} \).

The problem, therefore, reduces to finding the electron distribution function far from the vortex. At distances \( l \gg r \gg \lambda \) it can be written as a sum of three terms to be discussed below:

\[ \rho_{\mathbf{p} \mathbf{v}} = \rho_{\mathbf{p} \mathbf{v}}^{(e)} + \delta \rho_{\mathbf{p} \mathbf{v}}^{(sc)} + \delta \rho_{\mathbf{p} \mathbf{v}}^{(ind)} , \]  

(24)

The first term here represents the incident flow of electrons with velocity \( \mathbf{u}_i \). In the coordinate system in which \( \mathbf{V}_L = 0 \) it is given by the shifted Fermi-Dirac distribution function,

\[ \rho_{\mathbf{p} \mathbf{v}}^{(e)} = \rho_{\mathbf{p} \mathbf{v}}(\mathbf{p} - \mathbf{u}_i - \mathbf{p}_i[\mathbf{u}_i - \mathbf{V}_L]) , \]  

(25)

where \( \rho_{\mathbf{p} \mathbf{v}}(\mathbf{p}) \equiv 1/[e^{e^{\mathbf{p}^2/2m^2} + 1}] \) and \( \mu_{\mathbf{i}} \) is the electron chemical potential far from the vortex. Clearly, this term does not contribute to the force (22), since it ‘does not know’ about the presence of the vortex line.

The second term in equation (24) describes the electrons scattered by the vortex. The asymptotic expression for \( \delta \rho_{\mathbf{p} \mathbf{v}}^{(sc)} \), valid at large distances from the vortex, has been derived by Sonin (1976); Galperin & Sonin (1976); Aronov et al. (1981), and is given by (see also Appendix A)

\[ \delta \rho_{\mathbf{p} \mathbf{v}}^{(sc)}(r, \theta, p, \mathbf{v}_p) = \frac{d\eta(p)}{dp} \left\{ (\mathbf{u}_i - \mathbf{V}_L)p \sigma_\parallel \right\} + (e_e \times (\mathbf{u}_i - \mathbf{V}_L))p \sigma_\perp \frac{\delta(\theta_p - \theta)}{r} , \]  

(26)

where \( \sigma_\parallel \) is the well-known transport cross-section,

\[ \sigma_\parallel = \int_{-\pi}^{\pi} \sigma(\gamma)(1 - \cos \gamma) \, d\gamma , \]  

(27)

and

\[ \sigma_\perp = \int_{-\pi}^{\pi} \sigma(\gamma) \sin \gamma \, d\gamma , \]  

(28)

6 This statement is not precise; see the text below and Appendix B for a detailed explanation.

In equations (26)–(28) \( r \) and \( \theta \) are, respectively, the cylindrical radius and angle – the coordinates of a point in the cylindrical coordinate system with the centre at the vortex line (see Fig. 1); \( \theta_p \) is the angle coordinate of momentum \( \mathbf{p} \) in the same coordinate system; \( \sigma(\gamma) \) is the effective differential cross-section for scattering of electrons off the vortex line; \( \gamma \) is the scattering angle: \( \gamma = \theta_p - \theta_e \), where \( \mathbf{p} \) and \( \mathbf{p}_i \) are the electron momenta before and after scattering, respectively (see also Fig. 4).7 Because of 2d-character of our scattering problem, the cross-sections (27) and (28) have a dimension of length.

It is easy to understand the general structure of the expression (26). The correction \( \delta \rho_{\mathbf{p} \mathbf{v}}^{(sc)} \), describing scattered electrons, is proportional to \( d\eta(p)/dp \), which means that only electrons close to the Fermi surface can scatter off the vortex line – an expected result for a strongly degenerate matter; far from the vortex \( \delta \rho_{\mathbf{p} \mathbf{v}}^{(sc)} \propto 1/r \), which is natural since the total number of scattered electrons should be conserved; delta-function in equation (26) indicates that scattered electrons move radially from the vortex to the observation point (\( \theta_p = \theta_e \)); finally, the combinations \( (\mathbf{u}_i - \mathbf{V}_L)p \) and \( (e_e \times (\mathbf{u}_i - \mathbf{V}_L))p \) in curly brackets in equation (26) are the only scalars that can be composed of the available vectors in the problem.

Now let us turn to the third term in equation (24). It describes a subtle effect, ignored so far in the literature, and related to the fact that scattered electrons carry a charge. As a result, a weak electric field will appear far from the vortex and this will slightly change the electron distribution function. In addition, this will also modify the proton chemical potential. In Appendix B we show that contribution to \( \mathbf{F}_{\text{nepe-v}} \) from both these effects mutually cancel each other, so we should not care about the last term in equation (24).

Correspondingly, the final expression for the force (22) can be rewritten as

\[ \mathbf{F}_{\text{nepe-V}} = - \oint \delta \Pi_{ik}^{(e, \text{sc})} n_k \, dS , \]  

(29)

where

\[ \delta \Pi_{ik}^{(e, \text{sc})} = \sum_{\mathbf{p}, \mathbf{v}} \rho_{ik}(\mathbf{p}) \delta \rho_{\mathbf{p} \mathbf{v}}^{(sc)} . \]  

(30)

Integrating (29) using equations (26) and (30), one arrives at the expression (18) for \( \mathbf{F}_{\text{nepe-V}} \), in which

\[ D = \frac{1}{2} \sum_{\mathbf{p}} \left( - \frac{d\eta(p)}{dp} \right) p_{\perp}^2 v_{\perp}^2 \sigma_{\perp}(p_{\perp}) , \]  

(31)

\[ D' = \frac{1}{2} \sum_{\mathbf{p}} \left( - \frac{d\eta(p)}{dp} \right) p_{\perp}^2 v_{\perp}^2 \sigma_{\perp}(p_{\perp}) . \]  

(32)

The expressions (31) and (32) were derived long ago by Sonin (1976) (see also Galperin & Sonin 1976; Aronov et al. 1981; Sonin 2016). In equations (31) and (32) \( p_{\perp} \) and \( v_{\perp} \) are, respectively, the projections of the electron momentum and velocity on the plane \( xy \). To determine the coefficients \( D \) and \( D' \) we need to calculate the cross-sections \( \sigma_{\parallel} \) and \( \sigma_{\perp} \),

7 Note that, apparently, Sonin (1976); Galperin & Sonin (1976); Aronov et al. (1981) define \( \gamma \) (and hence \( \sigma_\perp \)) with an opposite sign. As a result, terms depending on \( \sigma_\perp \) in our equations (26) and (32) differ by the sign from the corresponding equations in these references.
which are, generally, the functions of \( p_\perp \). The next section is devoted to such calculation.

4 CROSS-SECTIONS \( \sigma_\parallel \) AND \( \sigma_\perp \) DUE TO ELECTRON SCATTERING OFF THE VORTEX MAGNETIC FIELD

4.1 Simple calculation within the classical scattering theory

Let us first calculate the cross-sections \( \sigma_\parallel \) and \( \sigma_\perp \) assuming that the electrons can be treated classically. This is a justified assumption provided that their wavelength, \( \hbar/p \), is much smaller than the typical lengthscale of the magnetic field variation, \( \lambda \). Indeed, it is well known (Landau & Lifshitz 1980) that in the latter case one can use the standard Boltzmann equation (B6) with the external Lorentz force incorporated, and this result is independent of whether the electrons in the system are degenerate or not. The Boltzmann equation can (in principle) be solved by the characteristics method, and the asymptotic correction \( \delta n_{pp}^{(e)} \) for the distribution function \( n_{pp} \) can be derived, which is equivalent to finding the cross-sections \( \sigma_\parallel \) and \( \sigma_\perp \) (see equation 26). On the other hand, the same Boltzmann equation can be used to find \( \sigma_\parallel \) and \( \sigma_\perp \) for purely classic problem of particle scattering on the magnetic field of a vortex. It is clear, therefore, that any ‘classic’ derivation should give the correct answer for the cross-sections \( \sigma_\parallel \) and \( \sigma_\perp \). This conclusion is additionally verified in Section 4.2 by a more rigorous calculation within the quasiclassical scattering theory.

Assume that the projection \( p_\perp \) of the momentum \( p \) of an incident electron on the \( xy \)-plane is directed along the axis \( y \), while the electron impact parameter is \( b \) (see Fig. 4).

Electrons are fast, so that the transferred momentum \( \delta p = p_\perp - p_\perp = p_\perp - p_\perp \) (due to the action of the Lorentz force) and the scattering angle \( \gamma \) are both small. In these conditions \( \delta p \) is parallel to the axis \( x \) and one may approximately write

\[
\delta p = \int \frac{e_v}{c} v_\perp \times B(r) \, dt \approx \frac{e_v}{c} \int B(\sqrt{b^2 + y^2}) \, dy.
\]

Figure 4. Electron trajectory (schematic) in the magnetic field of a vortex. A projection \( p_\perp \) of an incident electron is directed along the axis \( y \); \( p_\perp \) is the momentum projection of the scattered electron, \( \gamma \) is the scattering angle, \( b \) is the impact parameter.

where in the second equality we use the fact that \( v_\perp \approx (dy/d\gamma)e_y \) (\( e_x \) and \( e_y \) are the unit vectors along the axes \( x \) and \( y \)). The scattering angle \( \gamma(b) \) is now given by

\[
\gamma(b) = \frac{\delta p}{p_\perp} = \frac{1}{p_\perp} \int b \lim_{y \to \pm \infty} B(\sqrt{b^2 + y^2}) \, dy.
\]

The differential cross-section \( \sigma(\gamma) \) is defined by the formula (e.g., Sonin 2016): \( \sigma(\gamma) = dB(\gamma)/d\gamma \). Plugging this definition into equations (27) and (28) and using the fact that \( \gamma \) is small, one arrives at the following expressions for \( \sigma_\parallel \) and \( \sigma_\perp \),

\[
\sigma_\parallel \approx \int_{-\infty}^{+\infty} \frac{\gamma(b)^2}{2} \, db,
\]

\[
\sigma_\perp \approx \int_{-\infty}^{+\infty} \gamma(b) \, db,
\]

where \( \gamma(b) \) is given by equation (34). Formula (36) for \( \sigma_\perp \) can be easily integrated for an arbitrary vortex magnetic field \( B \),

\[
\sigma_\perp = \frac{e_v}{p_\perp c} \approx 3 \text{ fm} \left( \frac{e_v}{\Phi B} \right) \left( \frac{\Phi}{\Phi_0} \right) \left( \frac{1 \text{ fm}^{-1}}{k_\perp} \right),
\]

where \( k_\perp \equiv p_\perp / h \). To obtain equation (36) we make use of the fact that the total magnetic flux carried by the vortex is \( \Phi = \int B \, d\Phi \). The cross-section \( \sigma_\perp \) is negative for electrons because the magnetic field turns them to the left (negative angles \( \gamma \)). In turn, equation (35) for \( \sigma_\parallel \) can be represented in a more ‘quantum-mechanical’ form (see Appendix C for details),

\[
\sigma_\parallel = \frac{e_v^2}{4\pi\Phi_0^2 c^2} \int_{-\infty}^{+\infty} |\Pi(q)|^2 \, dq,
\]

where the form-factor \( \Pi(q) \) equals

\[
\Pi(q) = \int B(\sqrt{b^2 + y^2}) e^{-iqb} \, dbdy.
\]

For the model (7) of the vortex magnetic field, we have \( \Pi(q) = 2\Phi_0 \Pi(q/\xi) / q[\Phi(1 + q^2\lambda^2)] \), so that \( \sigma_\parallel \) can finally be presented as

\[
\sigma_\parallel = \frac{e_v^2\Phi^2}{\pi p_\perp^2 c^2 \xi} G(\lambda/\xi) = \left( \frac{e_v}{e_p} \right) \left( \frac{\Phi}{\Phi_0} \right) \left( \frac{1}{k_\perp\xi} \right) G(\lambda/\xi) \sigma_\perp.
\]

Figure 5. The function \( G(a) \). Solid line: fitting formula (42); circles: numerical calculation using equation (41).
where the function $G(a)$ is plotted in Fig. 5 and is given by
\[
G(a) = \int_{-\infty}^{\infty} \frac{J_1(x)^2}{x^2(1 + x^2a^2)} \, dx. \tag{41}
\]
In two limiting cases, $a \gg 1$ and $a \ll 1$, this function equals, respectively, $\pi/(8a)$ and $8/(3\pi)$. We fit $G(a)$ by the following approximate formula, which reproduces the results of our numerical calculations with the maximum error $\sim 0.3%$,
\[
G(a) = \frac{8 + c_1a + c_2a^2 + c_3a^3 + (3\pi^2/8)c_4a^4}{3\pi(1 + c_5a^3 + c_6a^6)}, \tag{42}
\]
where $c_1 = -0.9411$; $c_2 = -28.679$; $c_3 = -1.7340$; $c_4 = 12.774$; $c_5 = -6.3427$.

4.2 Quasiclassical calculation using the scattering theory

Here our aim is to determine the cross-sections $\sigma_\parallel$ and $\sigma_\perp$ using the standard scattering theory. Because the electron wavelength, $h/p$, is much smaller than other typical length-scales in the problem, the scattering can be considered quasiclassically. For nonrelativistic electrons the problem of quasiclassical electron scattering by a vortex was analyzed long ago by Cleary (1968), who, however, employed a different model of the vortex magnetic field (see also Sonin 2016 for a recent discussion of this problem); in Section 4.2.1 we present a similar analysis. An extension of the results of Section 4.2.1 to the relativistic case is given in Section 4.2.2.

4.2.1 Nonrelativistic electrons

The electron wave function $\Psi(r)$ in the presence of a proton vortex with the magnetic field $B$ can be found from the stationary Schrödinger equation,
\[
E \Psi = \frac{1}{2m_e} \left( -i\hbar \nabla - \frac{e}{c} A \right)^2 \Psi - \mu B \Psi, \tag{43}
\]
where $E$ and $m_e$ are the electron energy and bare mass, respectively; $\mu$ is the electron magnetic momentum operator (see, e.g., Landau & Lifshitz 1977); and $A$ is the vector-potential of the electromagnetic field given by the equation (9).\footnote{To solve equation (47) in the quasiclassical approximation one needs first to transform it to the form of the standard one-dimensional Schrödinger equation by introducing a new function, $\chi(r) \equiv \sqrt{r} Q_l(r)$ (Landau & Lifshitz 1977). In the quasiclassical approximation the main} In what follows we shall ignore the last term in equation (43), describing interaction of the electron spin with the magnetic field. It can be shown (by essentially repeating the derivation that will be presented below) that the contribution from this term to $\sigma_\parallel$ and $\sigma_\perp$ is negligible for scattering of unpolarized quasiclassical electron. To simplify formulas we also assume that the (conserved) $z$-component of the electron momentum vanishes.

We are interested in the solution to equation (43) with the asymptotic behavior
\[
\Psi \approx e^{i|k|y} + \frac{f(\theta)}{\sqrt{r}} e^{i|k|r}, \tag{44}
\]
describing the incident plane wave moving along the axis $y$ (the first term) and the scattered cylindrical wave (the second term) (see, e.g., Cleary 1968; Landau & Lifshitz 1977; Sonin 2016). In equation (44) $k_\perp = p_\perp/\hbar$; $r$ and $\theta$ are the coordinates introduced in Section 2; and $f(\theta)$ is the scattering amplitude, which is related to the differential cross-section $\sigma(\gamma)$ by the formula (e.g., Landau & Lifshitz 1977; note that in this section the scattering angle $\gamma$ coincides with the angular coordinate $\theta$):
\[
\sigma = |f(\theta)|^2. \tag{45}
\]
Generally, $\Psi(r, \theta)$ can be decomposed as
\[
\Psi(r, \theta) = \sum_{l = -\infty}^{\infty} e^{il\theta} Q_l(r). \tag{46}
\]
Plugging this expression into equation (43) (with the last term omitted) and making use of equation (9), one finds
\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dQ_l}{dr} \right) + \left[ k_\perp^2 - \left( \frac{l}{r} - \frac{e}{\hbar c} A_\theta \right)^2 \right] Q_l = 0, \tag{47}
\]
In the system without a vortex ($A_\theta = 0$) a solution to this equation, regular at $r = 0$, is the Bessel function, $J_l(k_\perp r)$. At $k_\perp r \gg 1/|l|$ its asymptote is
\[
J_l(k_\perp r) \approx \sqrt{\frac{2}{\pi k_\perp r}} \cos \left( k_\perp r - \frac{\pi}{2} |l| - \frac{\pi}{4} + \delta_l \right). \tag{48}
\]
On the other hand, at $A_\theta \neq 0$ the function $Q_l$ will have an asymptote which differ from (48) only by a phase shift $\delta_l$ and normalization factor $C_l$, i.e.,
\[
Q_l = C_l \sqrt{\frac{2}{\pi k_\perp r}} \cos \left( k_\perp r - \frac{\pi}{2} |l| - \frac{\pi}{4} + \delta_l \right). \tag{49}
\]
Using equations (46) and (49) and requiring that the asymptotic expression for $\Psi$ takes the asymptotic form (44), one arrives at the following expression for the normalization factor $C_l$ and the scattering amplitude $f(\theta)$ (Cleary 1968; Sonin 2016):
\[
C_l = e^{i\pi |l| + \delta_l}, \tag{50}
\]
\[
f(\theta) = \sqrt{\frac{1}{2\pi i k_\perp}} \sum_{l = -\infty}^{\infty} e^{i\theta} \left( e^{2i\delta_l} - 1 \right). \tag{51}
\]
Now, plugging equations (45) and (51) into the definitions (27) and (28) and recalling that $\gamma = \theta$, one obtains the following expressions for the cross-sections $\sigma_\parallel$ and $\sigma_\perp$, first derived by Cleary (1968) and confirmed subsequently by Olariu & Popescu (1985); Nielsen & Hedegård (1995); Sonin (1997); Shelankov (1998, 2000); Sonin (2016):
\[
\sigma_\parallel = \frac{2}{k_\perp^2} \sum_{l = -\infty}^{\infty} \sin^2 (\delta_{l+1} - \delta_l), \tag{52}
\]
\[
\sigma_\perp = \frac{1}{k_\perp^2} \sum_{l = -\infty}^{\infty} \sin (2\delta_l - 2\delta_{l+1}). \tag{53}
\]
Both these quantities depend on the phase shifts $\delta_l$. To calculate $\delta_l$, we make use of the fact that for electrons with $k_\perp \lambda \gg 1$ one may work in the quasiclassical (WKBJ) approximation. In the quasiclassical approximation the main
contribution to the scattering amplitude $f(\theta)$ and to the differential cross-section $\sigma$ comes from the partial waves with large $l$ (Landau & Lifshitz 1977), i.e., large angular momenta. For such $l$ one can present $\delta_l$ as (Cleary 1968; Landau & Lifshitz 1977)

$$\delta_l \approx \int_{\sqrt{l^2 - k^2}}^{\infty} \frac{e_\omega A_\gamma}{\hbar c r \sqrt{k^2 - l^2/r^2}} dr$$

$$= \frac{e_\omega} {e_\rho} \frac{k} {k_\perp} \int_{\sqrt{l^2 - k^2}}^{\infty} \mathcal{P}(r) \frac{1}{r \sqrt{l^2 - k^2}} dr,$$

where the function $\mathcal{P}(r)$ is defined by equation (9). Using the expression (54) one can easily calculate the cross-sections $\sigma_\perp$ and $\sigma_\parallel$ from the equations (52) and (53). The calculation can be simplified by noticing that $\delta_l$ is a slowly varying function of $l$, so that one may treat $l$ as a continuous variable and write $\delta_{l+1} - \delta_l \approx d\delta_l/dl$. Then equations (52) and (53) can be presented as (Cleary 1968)

$$\sigma_\parallel \approx 2 \frac{k} {k_\perp} \int_{l_\perp}^{\infty} \left( \frac{d\delta_l}{dl} \right)^2 dl,$$

$$\sigma_\perp \approx 2 \frac{k} {k_\perp} \int_{l_\perp}^{\infty} \frac{d\delta_l}{dl} dl.$$  

Although these formulas look different from those obtained in Section 4.1, they are, in fact, completely equivalent to the expressions (35) and (36) for $\sigma_\parallel$ and $\sigma_\perp$, as shown in Appendix D. Correspondingly, in the limit $l_\perp \gg 1$, one should use the cross-sections $\sigma_\parallel$ and $\sigma_\perp$, given by the formulas (37) and (40).

It is interesting to note that $\sigma_\parallel$ and $\sigma_\perp$, calculated in the limit $l_\perp \gg 1$, differ drastically from those obtained in the opposite limit, $l_\perp \ll 1$, corresponding to the classic Aharonov-Bohm effect (when the flux tube can be treated as infinitely thin; Aharonov & Bohm 1959). It is easy to show that in the limit $l_\perp \ll 1$ the phase shifts are given by the formula (e.g., Sonin 2016)

$$\delta_l \sim \left( |l| - l + \frac{e_\omega} {e_\rho} \frac{\pi} {2} \right),$$

and hence from equations (52) and (53) it follows that

$$\sigma_\parallel = \frac{2} {k_\perp} \sin^2 \left( \frac{\pi e_\omega} {e_\rho} \frac{\pi} {2} \right) \sigma_\perp = \frac{2} {k_\perp} \sin^2 \left( \frac{e_\omega} {e_\rho} \frac{\pi} {2} \Phi_0 \right),$$

$$\sigma_\perp = \frac{1} {k_\perp} \sin^2 \left( \frac{2\pi e_\omega} {e_\rho} \frac{\pi} {2} \right) = \frac{1} {k_\perp} \sin \left( \frac{e_\omega} {e_\rho} \Phi_0 \right).$$

The cross-section $\sigma_\perp$ in this limit was (implicitly) calculated, e.g., in Olariu & Popescu (1985); Alford & Sedrakian (2010); in turn, $\sigma_\perp$ was calculated, e.g., in Sonin (1997, 2016) and implicitly considered in Olariu & Popescu (1985); Nielsen & Hedegård (1995); Shelankov (1998, 2000). Note that for a proton vortex $\sigma_\parallel = 2/k_\perp$ and $\sigma_\perp = 0$ because in that case $\Phi = \Phi_0$.

4.2.2 Relativistic generalization

The results of the previous section cannot be used directly since electrons in the internal layers of NSs are ultrarelativistic (e.g., Haensel, Potekhin, & Yakovlev 2007). In order to find the cross-sections $\sigma_\parallel$ and $\sigma_\perp$ for a relativistic electron one needs, in principle, to solve the scattering problem for the Dirac equation. A similar problem was studied long ago by Alford & Wilczek (1989), who calculated the differential cross-section $\sigma$ for the scattering of a relativistic fermion off a vortex (cosmic string). Although these authors were interested in finding $\sigma$ in the limit of infinitely thin vortex, $k_\perp \ll 1$, their approach can also be used in our situation. Namely, Alford & Wilczek (1989) exploited the symmetry of the vortex under $z$ translations. It turns out that for $z$-independent problems, there exists a representation of the Dirac $\gamma$-matrices that allow one to decouple the Dirac equation into two independent first-order differential equations\footnote{One equation is for ‘spin-up’ and one for ‘spin-down’ electron.} for two-component spinors (de Vega 1978). Each of these equations is exactly equivalent to the nonrelativistic Schrödinger equation (47). Therefore, all the consideration of Section 4.2.1 remains unaffected and leads to the same cross-sections, $\sigma_\parallel$ and $\sigma_\perp$, as in the relativistic limit. We refer the interested reader to the work by Alford & Wilczek (1989) for more details.

5 RESULTS AND COMPARISON WITH THE PREVIOUS WORKS

Using equations (37) and (40) we can now calculate the coefficients $D$ and $D'$ from the formulas (31) and (32):

$$D = \frac{\hbar^2 \Phi^2 \rho}{8\pi^2 \hbar^2 c^2} \frac{m_1}{m_2} \frac{m_1}{m_2} \frac{m_1}{m_2} \frac{m_1}{m_2} G(\lambda/\xi),$$

$$= \frac{3\pi}{8} \left( \frac{e_\omega}{e_\rho} \right) \left( \frac{\Phi}{\Phi_0} \right) \left( \frac{1}{k_\rho \xi} \right) G(\lambda/\xi) D',$$

$$D' = \frac{e_\omega}{c} n_e \Phi = \pi n_e \left( \frac{e_\omega}{e_\rho} \right) \left( \Phi/\Phi_0 \right),$$

where $n_e = m_1^3/(3\pi^2 \hbar^2)$. One sees that $D > 0$, which means that the (dissipative) longitudinal force on a vortex, $F_\parallel \equiv D (\mathbf{u}_e - \mathbf{V}_e)$, acts in the direction of the axis $y$ (see equation 18). This is an expected result since the momentum of electrons along the axis $y$ changes in the course of scattering (see Fig. 4). In turn, $D' < 0$, i.e., the transverse force on a vortex, $F_\perp \equiv D' (\mathbf{e}_\perp \times (\mathbf{u}_e - \mathbf{V}_e))$, acts in the direction of the axis $x$. This result is also reasonable, since exactly the same force (Lorentz force) acts on electrons in the opposite direction.

One may note that $F_\perp$ formally coincides with the so-called Magnus force\footnote{This coincidence takes place only in the quasiclassical limit, $k_\rho \lambda \gg 1$.} $\mathbf{F}_M$ (see equation 4 and recall the quasineutrality condition, $n_e = n_p$), which is the transverse force acting on a vortex from superconducting protons and is well-defined for extreme type-II superconductors (when $\xi < \lambda$). To understand this coincidence, assume for a moment that $\xi \ll \lambda$ for our problem and recall that the force $\mathbf{F}_{npe} = \mathbf{F}_M + \mathbf{F}_\perp$, calculated by us above, is the total force from the npe-mixture on a vortex. What are the actual mechanism and an actual particle species participating in transferring the momentum to the vortex core we have not yet discussed. Clearly, this cannot be neutrons, because they in no way interact with the vortex. Also, this cannot be electrons because they, generally, scatter off the magnetic field localized far from the vortex core ($\lambda \gg \xi$); see Appendix E for a more detailed justification of this statement.
This magnetic field is generated and supported by the superconducting proton currents, consequently, scattered electrons transfer their momentum to superconducting proton component, but not to the vortex. We come to conclusion that in this example only protons are able to transfer the momentum directly to the vortex core. How does it happen? The mechanism of the transverse force appearance is essentially the same as in liquid helium-II (e.g., Sonin 1987, 2016). This is because the proton superfluid velocity, generated by the vortex, scales as \(1/r\) at distances \(r\) from the vortex centre, such that \(\xi \ll r \ll \lambda\) (Nohières & Vinen 1966; Landau & Lifshitz 1980; De Gennes 1999). The superfluid velocity near the vortex core in helium-II behaves in exactly the same way and this is known to produce a transverse force on a vortex if a superfluid transport current is applied to the system (e.g., Sonin 1987, 2016). This force can be found from equations (B2) and (B4) in Appendix B by considering a momentum carried by superconducting protons per unit time through the walls of a cylinder of radius \(r\), with the result that it equals \(F_M\) (Nohières & Vinen 1966; Sonin 1987, 2016). Therefore, it is not surprising that \(F_{\perp} = F_M\) if \(\xi \ll \lambda\). But the expression for the force \(F_{\perp}\) does not contain \(\xi\) and/or \(\lambda\), so this result should remain unchanged in the more general case of arbitrary ratio between these parameters.

Note that our results for the transverse force \(F_{\perp}\) agree with the assumptions about the form of the force made, e.g., by Alford & Sedrakian (2010); Glampedakis et al. (2011) and disagree with the conclusions of Jones (1991, 2006), where it is argued that \(F_{\perp} = 0\). The coefficient \(D'\) would vanish only in the limit of large electron wavelength (when \(\sigma_{\perp} = 0\), see equation 59), but this limit is not realized in NSs, for which \(\hbar/p_{pe} \ll \lambda\). For completeness, below we provide the coefficients \(D\) and \(D'\) calculated in the limit \(\hbar/p_{pe} \lambda \ll 1\) (using \(\sigma_{\parallel}\) and \(\sigma_{\perp}\) given, respectively, by equations 58 and 59):

\[
D = 2\hbar n_e \sin^2 \left( \frac{\epsilon_v}{\epsilon_p} \frac{\pi \Phi}{2\Phi_0} \right),
\]

\[
D' = \hbar n_e \sin \left( \frac{\epsilon_v}{\epsilon_p} \frac{\pi \Phi}{2\Phi_0} \right).
\]

Similar expression for the coefficient \(D\) was obtained in Olariu & Popescu (1985); Nielsen & Hedegård (1995); Alford & Sedrakian (2010), while the coefficient \(D'\) in this limit was studied in Olariu & Popescu (1985); Nielsen & Hedegård (1995); Sonin (1997); Shelyakno (1998, 2000); Sonin (2016).

Now, let us discuss in some more detail the longitudinal force \(F_{\parallel}\) on a vortex and compare it with the results available in the literature. The longitudinal force due to electron scattering off the vortex magnetic field was calculated by Jones (1987) (see also Harvey, Ruderman, & Shaham 1986) within classical mechanics. Using the formulas of Section 4.1, it is easy to verify that our result (equation 60) agrees with that of Jones. The force \(F_{\parallel}\) was also calculated by Alpar et al. (1984) (see also Sauls, Stein, & Serene 1982). Strictly speaking, Alpar et al. (1984) considered a bit different problem, namely, the electron scattering off the neutron vortices, which can carry magnetic field due to the entrainment effect (Andreev & Bashkin 1976). However, their solution can easily be applied to our problem (Sedrakian & Sedrakian 1995). Alpar et al. (1984) used a very different method of derivation of \(F_{\parallel}\) and, moreover, worked in the Born approximation.\(^{12}\) Meanwhile, this approximation is unjustified for relatively large magnetic fluxes, associated with the vortex, \(\Phi \sim \Phi_0\), for which it can lead to incorrect results.\(^{13}\) Thus, it is interesting to look whether our force \(F_{\parallel}\) differs from that of Alpar et al. (1984).

Actually, Alpar et al. (1984) calculated the so-called ‘velocity coupling time between the plasma and the core superfluid’, \(\tau_v\). In the limit of \(k_{pe}\xi \gg 1\) it is given by\(^{14}\)

\[
\frac{1}{\tau_v} = \frac{3}{2} \frac{p_{pe}c}{m_e^2\sigma} \frac{1}{\alpha} G(\lambda/\xi),
\]

where \(\alpha = 2p_{pe}\xi/\hbar; \tau_v^{-1} = \pi N_v\Phi^2\); the function \(G(\lambda/\xi)\) is defined by equation (41); and

\[
N_v = \frac{2\pi}{\hbar} n_v \left( \frac{e\hbar}{2m_e c} \right)^2 \left( \frac{m_e c^2}{p_{pe}c} \right)^2 \frac{p_{pe}c}{\pi \hbar c^2} \]

with \(n_v\) being the surface vortex density. As shown, e.g., in Sedrakian & Sedrakian (1995); Andersson et al. (2006); Kantor & Gusakov (2017), this relaxation time is related to the force on a vortex per unit length by the formula:

\[
F_{Alpar \parallel} = \frac{m_e n_v}{n_v \tau_v} (u_\perp - V_L).
\]

Plugging equation (64) into (66) and comparing the coefficient at \((u_\perp - V_L)\) with the expression (60), one verifies that, somewhat unexpectedly, \(F_{Alpar \parallel} = F_{\parallel}\), i.e., the longitudinal force calculated by Alpar et al. (1984) exactly coincides with our result.

Equations (60) and (61) determine the force on a single vortex. However, in astrophysical applications one is usually interested in the force density \(f_{npe-V}\) acting on a system of vortices. Assuming that we have a locally rectilinear array of proton vortices with the surface density \(n_v = n/\Phi\), one can present \(f_{npe-V}\) as (cf. equation 17)

\[
f_{npe-V} = -D'[e_x \times (u_\perp - V_L)] + D'[e_x \times (u_\perp - V_L)],
\]

where

\[
D = n_v D = \frac{3\pi}{8} \frac{e_v}{\epsilon_p} \left( \frac{\Phi}{\Phi_0} \right) \left( \frac{1}{k_{pe}\xi} \right) G(\lambda/\xi) D',
\]

\[
D' = n_v D' = \frac{e_v}{c} n_v T_B.
\]

\(^{12}\) Alpar et al. (1984) did not find the transverse force \(F_{\perp}\), which vanishes in the Born approximation, because in that case \(\sigma(\gamma) = \sigma(\gamma)\) and thus \(\sigma_{\perp} = 0\).

\(^{13}\) It is worth noting that the Born approximation can be inadequate even for \(\Phi \ll \Phi_0\). This is the case for a nonrelativistic problem of electron scattering off an infinitely thin flux tube, if the latter is treated with the Schrödinger equation (see, e.g., Aharonov et al. 1984). However, the same problem, analyzed in the Born approximation making use of the Dirac equation, gives correct asymptotic expression for the differential cross-section, valid at \(\Phi \ll \Phi_0\) (see Vera & Schmidt 1990 for details).

\(^{14}\) This formula follows from equation (30b) of Alpar et al. (1984). Note a misprint in equation (30b): instead of \(x^2 + a^2\) there should be \(x^2/2 + a^2\), as we checked by independent calculation of \(\sigma_{\parallel}\) in the Born approximation.
should be revised (Gusakov et al., in preparation).

One sees that

\[
\frac{D}{D'} = \frac{D}{D'} \approx \begin{cases} 
\frac{3\pi^2}{64} \left( \frac{1}{k_{p\lambda}} \right) \approx 0.46 \left( \frac{1}{k_{p\lambda}} \right), & \lambda \gtrsim \xi, \\
\frac{1}{k_{p\lambda}}, & \lambda \ll \xi. 
\end{cases}
\]

(70)

(The latter case is relevant for neutron vortices and is not interesting for us here.) Generally, \( D \) and \( D' \) are much smaller than, respectively, \( D \) and \( D' \) for typical NS conditions, for which \( k_{p\lambda} \sim (30\,\text{–}\,50) \) (see Fig. 2).\(^{15}\) This is also illustrated in Fig. 6, where the coefficients \( D \) and \( D' \) are plotted as functions of \( n_0 \) for \( B = 10^{12} \, \text{G} \). For comparison, we also present the dissipative coefficient \( D_{\text{Jones}} \) used by Jones (1991, 2006) and, recently, by Bransgrove et al. (2018) in their studies of the magnetic field expulsion timescale from the NS cores. (Note, that these authors completely ignored the effect of electron scattering by the vortex magnetic field, thus assuming \( D = D_{\text{Jones}} \), \( D' = 0 \).) The temperature-dependent coefficient \( D_{\text{Jones}} \) enters the expression for the dissipative force, similar to the first term in equation (67). This force arises due to the electron scattering off the unpaired proton quasiparticles localized in the vortex core. Jones (1991, 2006) and Bransgrove et al. (2018) used the following simple order-of-magnitude estimate for this coefficient (see also Galperin & Sonin 1976):

\[
D_{\text{Jones}} \approx \frac{n_e \rho_p}{c_{\text{ep}}} \left( \frac{B}{2B_{c2}} \right),
\]

(71)

where \( B_{c2} = \Phi/(2\pi \xi^2) \) and \( \tau_{\text{ep}} \) is the typical timescale of electron-proton collisions in the normal (nonsuperfluid and nonsuperconducting) matter. It is given by the formula (see, e.g., Iakovlev & Shalybkov 1991; Yakovlev & Shalybkov 1991; Glampedakis et al. 2011; Gusakov & Dommes 2016), which, recently, by Bransgrove et al. (2018) substantially underestimates the typical timescales of magnetic field evolution in NSs.

Inclusion of muons

If there are muons in the system, they will also scatter off the proton vortices. The corresponding analogue of equation (17) for the force on a vortex in the case of npe-matter is (we already set to zero the component of the force along the axis \( z \)):

\[
F_{npe\mu\rightarrow\nu} = -D'[e_z \times (u_z - V_L)] + D_{\mu} [e_z \times (u_p - V_L)] + D_{\mu} [e_z \times (u_{\mu} - V_L)],
\]

(73)

where \( u_{\mu} \) is the muon velocity far from the vortex; \( D_{\mu} \) and \( D_{\mu}' \) are the muon coefficients similar to, respectively, \( D \) and \( D' \). In order to write the force in the form (73) we have already used the screening condition (Jones 1991, 2006; Glampedakis et al. 2011; Gusakov & Dommes 2016), which, in the presence of muons (and neglecting entrainment), can be written as

\[
e_p n_0 V_{\mu} + e_\mu n_\mu u_\mu + k_{F\mu} n_\mu u_\mu = 0.
\]

Here and below \( e_\mu, n_\mu, \rho_p, \) and \( k_{F\mu} \) are the muon charge, number density, Fermi momentum and Fermi wave number, respectively. Assuming now \( k_{F\mu} \lambda \gg 1 \) (which is a valid assumption sufficiently far from the threshold for the muon appearance), it is straightforward to show that the coefficients \( D_{\mu} \) and \( D_{\mu}' \) are given by the same equations (60) and (61) as for electrons, with the obvious replacements \( e_\mu \approx e_\nu, \rho_{p\mu} \approx \rho_{p\nu}, \) and \( n_\mu \approx n_\nu \). Equations (68) and (69) can be adjusted to allow for muons in a similar way.

6 CONCLUSIONS

We calculated the force acting on a proton vortex from neutron-proton-electron mixture at vanishing stellar temperature and neglecting, for simplicity, entrainment effects between the superfluid neutrons and superconducting protons. It was assumed that far from the vortex the electron...
and proton charge current densities are non-zero and equal to one another because of the screening condition (16). The force is found by analyzing the outgoing momentum flow, generated by the vortex per unit time. This approach has been previously used in application to liquid helium-II and electrons in type-II superconductors, e.g., by Sonin (1976); Galperin & Sonin (1976); Aronov et al. (1981). Our main results are summarized as follows:

- For typical NS conditions the electron wavelength is much smaller than all other relevant lengthscales in the problem, in particular, than the London penetration depth, $\lambda$. This permits us to use a quasiclassical scattering theory in order to determine the electron cross-sections $\sigma_{\parallel}$ and $\sigma_{\perp}$, responsible for the appearance of transverse $F_\perp$ and longitudinal $F_\parallel$ forces on a vortex. In Section 4.1 it is shown that purely classic calculation of $\sigma_{\parallel}$ and $\sigma_{\perp}$ gives the same result. Note that the celebrated differential Aharonov-Bohm cross-section (Aharonov & Bohm 1959) cannot be used to calculate the force on a vortex in our situation (as it is done, e.g., in application to quark matter by Alford & Sedrakian 2010), because it is obtained in the opposite limit, $k_F \lambda \ll 1$.\(^{16}\)

- The calculated transverse force $F_\perp$ coincides (only in the limit $k_F \lambda \gg 1$) with the ordinary Magnus force, discussed in the context of superconductors, e.g., by Nozières & Vinen (1966); Kopnin (2002). It also equals to the (minus) Lorentz force acting on electrons in the magnetic field of a vortex. This result proves that the assumptions made, e.g., in Alford & Sedrakian (2010); Glampedakis et al. (2011) about the form of $F_\perp$ are correct. At the same time, our result disagrees with the conclusion of Jones (1991, 2006) that $F_\perp = 0$.

- The longitudinal force on a proton vortex $F_\parallel$ is, typically, smaller than $F_\perp$ by a factor of $k_F \lambda \sim (30 - 50)$. It coincides with the force calculated by Jones (1987) and (after some straightforward adjustment) with the force on a neutron vortex calculated by Alpar et al. (1984). The latter coincidence is rather surprising since Alpar et al. (1984) worked within the Born approximation, which is not justified in our problem.

- Jones (2006) and Bransgrove et al. (2018) ignored in their analysis electron scattering by the magnetic field of a vortex, thus effectively setting $D = 0$. Instead, they considered a different scattering mechanism, namely, scattering of electrons off the proton localized excitations in the vortex core. As is shown in Section 5, this mechanism leads to a longitudinal force much smaller than our $F_\parallel$ (i.e., the friction coefficient $D_{\text{Jones}}$, suggested by Jones 2006, is much smaller than our coefficient $D$ given by equation 60 and can be ignored). This means that Jones (2006) and Bransgrove et al. (2018) substantially underestimate the typical timescale $\tau_B$ for the magnetic field evolution in the NS core.

- In Section 5 we show how our results should be modified to allow for muons in the system.

The results obtained in this paper confirm the form of the force on a vortex postulated, e.g., by Alford & Sedrakian (2010) and Glampedakis et al. (2011). As a consequence, simple estimates of the (very long) magnetic field evolution timescales made in Graber et al. (2015); Dommes & Gusakov (2017) and numerically found in Elfritz et al. (2016) look more realistic than those obtained in Jones (2006); Bransgrove et al. (2018) (but see footnote 15). These estimates imply, however, that NS matter as a whole is immobile, the assumption that can be incorrect for magnetized NSs (Gusakov, Kantor, & Oelgemöllner 2017; Oelgemöllner & Gusakov 2018). Account for macroscopic fluid motions in the core may dramatically accelerate the magnetic field evolution.

This work can be extended in a number of ways. First, it is straightforward to account for additional particle species in the system (e.g., hyperons) and allow for non-vanishing entrainment between the superfluid baryons. Secondly, the approach developed in the present paper can be directly applied to study forces that act on a neutron vortex. However, we do not expect that the expression for such force will differ noticeably from that already used in the literature (e.g., Glampedakis et al. 2011). Thirdly, it would be very interesting to generalize the results obtained here to the case of finite stellar temperatures. This can be a more difficult task since at finite $T$ one should also account for scattering of neutron and proton thermal Bogoliubov excitations off the quasiparticles localized in the vortex core (Kopnin 2002; Sonin 2016). Whether additional forces appearing due to such scattering play a role in the NS dynamics remains an open question to be investigated in the future.

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APPENDIX A: DERIVATION OF THE ASYMPTOTIC DISTRIBUTION FUNCTION (26) FOR SCATTERED ELECTRONS

Let us first calculate the change $dN_{pp}$ in the number of electrons with momentum $p$ and spin $\sigma$ per unit time due to electron scattering by the vortex in the linear approximation in $u_e - V_L$.\(^{17}\) Since scattering by the magnetic field of a vortex is elastic, the $z$-component $p_z$ of the momentum $p$ and the absolute value $p_z$ of the projection of $p$ on the plane $xy$ are both conserved. Also, as noted in Section 4.2.1 the effect of magnetic field interaction with the electron spins is small and can be ignored; thus, the spin is also

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\(^{16}\) A dissipative force $F_\parallel$ on a color-magnetic flux tube in quark matter can be easily calculated in the limit of small wavelength of scattered particles, $k_F \lambda \gg 1$, following our approach. The resulting expression will be suppressed by a factor of $k_F \lambda$ in comparison to the result presented in Alford & Sedrakian (2010).

\(^{17}\) Note that it is not a real scattering in a statistical sense, since there are no ‘element of chance’ in the problem (Gantmakher & Levinson 1987): the wave function of an electron in the field of a vortex is a well defined quantity that can be found from the Schrödinger (or Dirac) equation.
a conserved quantity. The effect of scattering, therefore, reduces to changing the electron angle coordinate from \( \theta_p \) to \( \theta_{p'} \) (see Fig. A1 showing the geometry of the problem). In these circumstances \( dN_{pp'} \) can be written as

\[
dN_{pp'} = -d^3p d\gamma \int_{\gamma_p}^{\gamma_{p'}} n_{p\perp}(\gamma, p_{\perp}) d\theta_p' + d^3p d\gamma \int_{\gamma_p}^{\gamma_{p'}} n_{p\perp}(\gamma, p_{\perp}') d\theta_p',
\]

Figure A1. Scattering geometry for electrons. \( p_{\perp} \) and \( p'_{\perp} \) are the projections of electron momenta \( p \) and \( p' \) on the \( xy \)-plane. See the text for details.

Further, \( n_{p\perp} \) is the equilibrium distribution function (25), unperturbed by the vortex. It is justifiable to use \( n_{p\perp}^{(eq)} \) in equation (A1) since we work in the linear approximation in the velocities. Substituting

\[
n_{p\perp}^{(eq)} \approx n_{p\perp}(p) = \frac{dn_{p\perp}}{dp} p(u_e - V_L),
\]

\[
n_{p\perp}^{(eq)} \approx n_{p\perp}(p) = \frac{dn_{p\perp}}{dp} p(u_e - V_L) \cos \theta_p,
\]

\[
n_{p\perp}(\gamma, p_{\perp}) \approx n_{p\perp}(p) = \frac{dn_{p\perp}}{dp} p(u_e - V_L) \cos \theta_p',
\]

into (A1), and using the relation \( \theta_{p'} = \gamma + \theta_p \), we arrive at the formula

\[
dN_{pp'} = d^3p d\gamma \int_{\gamma_p}^{\gamma_{p'}} n_{p\perp}(p) \sigma_{\parallel} + [e_z \times (u_e - V_L) \times \sigma_{\perp}],
\]

where the naturally appearing cross-sections \( \sigma_{\parallel} \) and \( \sigma_{\perp} \) are given by equations (27) and (28), respectively. The quantity \( dN_{pp'}/d\gamma \) describes the total number of electrons with momentum \( p \) and spin \( \sigma \) produced in the vicinity of a vortex line per unit time and per unit vortex length due to scattering by the magnetic field of a vortex. It is this quantity which is responsible for a deviation \( \delta n_{p\perp}^{(eq)}(r, \theta, p, \theta_p) \) (see equation (26)) of the electron distribution function from \( n_{p\perp}^{(eq)} \) far from the vortex. To find \( \delta n_{p\perp}^{(eq)} \) one needs to solve the kinetic Boltzmann equation, which has the asymptotic form

\[
v_p \cdot \nabla \delta n_{p\perp}^{(eq)} = St n_{p\perp},
\]

where \( St n_{p\perp} \) is the ‘collision integral’ given by

\[
St n_{p\perp} = \frac{\delta(r)}{2\pi r} \frac{dN_{pp'}}{2\pi d\gamma dz}.
\]

The delta-function here indicates that the scattering occurs in the very vicinity of the vortex (where exactly is not important since we are interested in the asymptotic solution for \( \delta n_{p\perp}^{(eq)} \) at \( r \gg \lambda \)); the normalization factor \( 2\pi r \) in the denominator ensures that the total number of scattered electrons is \( d^3p \int St n_{p\perp} dV = dN_{pp'} \). The equation (A4) can be rewritten as \( \nabla : (n_p \delta n_{p\perp}^{(eq)}) = St n_{p\perp} \); the solution to this equation can be readily obtained and coincides with the expression (26).

**APPENDIX B: STRESS TENSOR FOR NPE-MATTER AND PROOF OF EQUATION (29)**

**B1 Stress tensor**

At \( T = 0 \) the (nonrelativistic) equations of motion for protons far from the vortex consist of the continuity equation,

\[
\frac{\partial n_p}{\partial t} + \div (n_p V_{sp}) = 0,
\]

superfluid equation (see, e.g., Nozières & Vinen 1966; Putterman 1974; Aronov et al. 1981)

\[
\frac{\partial V_{sp}}{\partial t} = -\frac{e_p}{m_p} B,
\]

and the condition (Landau & Lifshitz 1980; De Gennes 1999)

\[
\nabla \times V_{sp} = -\frac{e_p}{m_p} B,
\]

specific to superconducting systems. In equations (B1)–(B3) \( n_p \) and \( m_p \) are the proton number density and mass, respectively; \( \mu_p \) is the proton chemical potential, defined in the coordinate system, in which \( V_{sp} = 0 \). From these equations one can derive the proton momentum conservation equation,

\[
\frac{\partial G_p}{\partial t} = -\partial_k (n_p V_{sp, k} V_{sp}) - n_p \nabla \mu_p + e_p n_p E + \frac{e_p}{c} n_p V_{sp} \times B,
\]

where \( G_p = m_p n_p V_{sp} \) is the proton momentum density. A similar equation can also be written out for neutrons,

\[
\frac{\partial G_n}{\partial t} = -\partial_k (n_n V_{sn, k} V_{sn}) - n_n \nabla \mu_n,
\]

where \( G_n = m_n n_n V_{sn} \); \( n_n, m_n, \) and \( \mu_n \) are the neutron number density, mass, and chemical potential, respectively.

Now we turn to the momentum conservation equation for electrons. Since \( k_{Fp} \lambda \gg 1 \) it can be derived from the standard Boltzmann kinetic equation,

\[
\frac{\partial n_{p\perp}}{\partial t} + v \frac{\partial n_{p\perp}}{\partial r} + e_z \left( E + \frac{1}{c} v \times B \right) \frac{\partial n_{p\perp}}{\partial r} = 0.
\]

Multiplying it by \( p \) and summing over \( p \) and \( \sigma \), one finds

\[
\frac{\partial G_{px}}{\partial t} = -\partial_k \Pi_{kk}^{(x)} + e_z n_e E + \frac{1}{c} j_z \times B,
\]
where $\Pi^{(s)}$ is given by equation (23); $G_s = \sum_{pe} p_n p_{pe}$ is the electron momentum density; and $\delta f = \sum_{pe} c_{pe} \mathbf{u}_{pe} n_{pe}$ is the electron charge current density. Using the Maxwell's equations, the sum of the last two terms in the right-hand sides of equations (B4) and (B7) can be transformed, in a standard way, as

$$
e_p n_p E + \frac{c}{e} n_p \mathbf{V}_{sp} \times \mathbf{B} + c n_e E + \frac{1}{e} j \times \mathbf{B}$$

$$= \frac{1}{4\pi} \partial_t \left[ \mathbf{E}, \mathbf{E}_k - \frac{1}{2} \mathbf{E}^2 \delta_{ik} + B_k B_i - \frac{1}{2} B^2 \delta_{ik} \right]$$

$$- \frac{1}{4\pi c} \partial_t (\mathbf{E} \times \mathbf{B}). \quad (B8)$$

Summing up equations (B4), (B5), (B7), and making use of equation (B8), we finally arrive at the total momentum conservation equation for our system,\(^\text{18}\)

$$\frac{\partial \mathbf{G}}{\partial t} = -\partial_t \Pi_k \quad (B9)$$

with

$$\Pi_k = m_p n_p V_{sp} V_{sp} + m_n n_n V_{sn} V_{sn} + P_{nuc} \delta_{ik} \quad \Pi^{(s)} + \Pi^{(e-m)}, \quad (B10)$$

where $\mathbf{G} = G_p + G_n + G_e + \mathbf{E} \times \mathbf{B} / (4\pi c)$ is the total momentum density; $P_{nuc}$ is the total neutron-proton pressure, such that $\nabla P_{nuc} = (n_0 \nabla \mu_p + n_n \nabla \mu_n)$; and $\Pi^{(e-m)}$ is the electromagnetic stress tensor, given by

$$\Pi^{(e-m)} = -\frac{1}{4\pi} \left[ \mathbf{E}, \mathbf{E}_k - \frac{1}{2} \mathbf{E}^2 \delta_{ik} + B_k B_i - \frac{1}{2} B^2 \delta_{ik} \right]. \quad (B11)$$

### B2 Proof of equation (29)

Now we are able to discuss why the force on a vortex can be calculated from equation (29). Below any thermodynamic quantity $A$ is presented as $A = A_0 + \delta A$, where $A_0$ is its value in the absence of a vortex (note that $\nabla A_0 = 0$) and $\delta A$ is the vortex-related perturbation.

In the system without a vortex the electron distribution function is $n_{pe}^{(eq)}$ (see equation 25). The associated electron charge density, $c_{e, n_{eq}} = \sum_{pe} c_{pe} n_{pe}^{(eq)}$, is neutralized by the background proton charge density, $c_{p, n_{eq}}$. However, when we add a vortex line to the system, an additional contribution $\delta n_{pe}^{(sc)}$ will arise to the electron distribution function due to electron scattering off the vortex line. This contribution generates a non-zero charge and charge current densities, $c_{e, \delta n^{(sc)}}$ and $c_{p, \delta n^{(sc)}}$, even far from the vortex. Indeed, summing up equation (26) over momenta and spins, one obtains

\[ \delta n^{(sc)} = \sum_{pe} \frac{c_{pe}}{v} \delta n_{pe}^{(eq)} \quad (B14) \]

where $E = -\nabla \phi$ and it is assumed that the electrostatic potential $\phi$ vanishes at $r \to \infty$. The corresponding contribution to the electron stress tensor is

\[ \delta \Pi^{(e, \delta n^{(sc)})} = \sum_{pe} \frac{c_{pe}}{v} \delta n_{pe}^{(eq)} - \frac{c_{e, n_{eq}} \phi \delta_{ik}}{v}, \quad (B15) \]

while the related electron density perturbation is

\[ \delta n^{(ind)} = \sum_{pe} \delta n_{pe}^{(ind)} = \alpha_3 c_{e, \delta n^{(sc)}}, \quad (B16) \]

where the actual form of the density-dependent parameter $\alpha_3$ is not important for us.

We turn now to calculation of the pressure perturbation, $\delta P_{nuc}$, and the proton number density perturbation, $\delta n_{pe}^{(sc)}$.

\[ \delta n^{(sc)} = \sum_{pe} \frac{c_{pe}}{v} \delta n_{pe}^{(eq)} \quad (B12) \]

where $\mathbf{e}_r$ is the unit vector along $r$; $\alpha_1$ and $\alpha_2$ are some density-dependent non-vanishing scalars, which can be explicitly calculated from equation (26), but are not important for the subsequent consideration.

The presence of uncompensated electron charge, $c_{e, \delta n^{(sc)}}$, produces an electric field $\mathbf{E}$, which affects the distribution function of incident electrons (so that the total distribution function is given by equation 24) and slightly changes the density of background protons. All these effects should be accounted for self-consistently. In what follows we shall work in the coordinate system, in which vortex is at rest ($V_L = 0$). We assume that ‘transport’ velocities of incident electrons, protons, and neutrons are small in this coordinate system and retain only the terms linear in the velocities in our calculations.

We start with the derivation of the expression for the induced electron distribution function $\delta n_{pe}^{(ind)}$ far from the vortex (at a distance $l \gg r \gg \lambda$). At such $r$ the magnetic field and proton superfluid velocity, generated by the vortex in the absence of transport electron and proton currents are exponentially suppressed and can be neglected. Then the kinetic equation (B6) can be written, in the linearized form, as

\[ \mathbf{v} \cdot \partial \delta n_{pe}^{(ind)} / \partial r + e_p \mathbf{E} \partial \delta n_{pe}^{(ind)} / \partial p = 0, \quad (B13) \]

where we take into account that the system is stationary. Note that, because our problem is linear, the correction $\delta n_{pe}^{(sc)}$ does not appear in this equation (it satisfies an independent equation A4). The solution to equation (B13), vanishing at $r \to \infty$, is

\[ \delta n_{pe}^{(ind)} = \frac{c_{e, \phi}}{v} \delta n_{pe}^{(eq)} \quad (B12) \]

where $\mathbf{E} = -\nabla \phi$ and it is assumed that the electrostatic potential $\phi$ vanishes at $r \to \infty$. The corresponding contribution to the electron stress tensor is

\[ \delta \Pi^{(e, \delta n^{(sc)})} = \sum_{pe} \frac{c_{pe}}{v} \delta n_{pe}^{(eq)} - \frac{c_{e, n_{eq}} \phi \delta_{ik}}, \quad (B15) \]

while the related electron density perturbation is

\[ \delta n^{(ind)} = \sum_{pe} \delta n_{pe}^{(ind)} = \alpha_3 c_{e, \delta n^{(sc)}}, \quad (B16) \]

where the actual form of the density-dependent parameter $\alpha_3$ is not important for us.

Similar expression can also be written out for $c_{e, \delta n^{(sc)}}$. However, it is easy to verify (see the text after equation B26) that the presence of this current does not lead to additional momentum flux through the boundary of a cylinder defined in Fig. 3 (i.e., does not affect the force $\mathbf{F}_{npe \to \gamma}$).

---

\(^{18}\) In contrast to equation (20), this momentum conservation equation does not include the external force density $\mathbf{f}_{ext}$ applied to the vortex. This is justified since in what follows the equation (B9) will be used at $r \gg \lambda$ (i.e., far from the vortex core).
\[ \nabla \mu_p = e_p E, \]  
\[ \nabla n_p = 0. \]  
\[ \nabla P_{\text{muc}} = n_p \nabla \mu_p + n_n \nabla \mu_n, \]

Hence, from the definition \( \nabla P_{\text{muc}} = n_p \nabla \mu_p + n_n \nabla \mu_n \), one has

\[ \nabla P_{\text{muc}} = e_p n_p E, \]

and thus

\[ \delta P_{\text{muc}} = -e_p n_p \delta \phi, \]

where we replaced \( n_n \) with the unperturbed proton number density \( n_{p0} \), which is justifiable in the linear approximation. The perturbation \( \delta n_p \) can be found by noticing that \( \mu_n \) and \( P_{\text{muc}} \) are the functions of \( n_n \) and \( n_p \), hence

\[ \nabla \mu_n = \frac{\partial \mu_n}{\partial n_n} \nabla n_n + \frac{\partial \mu_n}{\partial n_p} \nabla n_p = \frac{\partial \mu_n}{\partial n_n} \nabla \delta n_n + \frac{\partial \mu_n}{\partial n_p} \nabla \delta n_p, \]

\[ \nabla P_{\text{muc}} = \frac{\partial P_{\text{muc}}}{\partial n_n} \nabla n_n + \frac{\partial P_{\text{muc}}}{\partial n_p} \nabla n_p = \frac{\partial P_{\text{muc}}}{\partial n_n} \nabla \delta n_n + \frac{\partial P_{\text{muc}}}{\partial n_p} \nabla \delta n_p, \]

where all the partial derivatives are taken in the unperturbed matter (in the vortex-free matter) and we used the fact that \( \nabla n_n, \delta n_p \) and \( \nabla \delta n_n \) can be found by noticing that \( \mu_n \) and \( P_{\text{muc}} \) are the functions of \( n_n \) and \( n_p \). The solution to this system allows one to relate \( \delta n_p \) with the electrostatic potential \( \phi \),

\[ \delta n_p = a_4 e_p \phi, \]

where \( a_4 \) is a combination of partial derivatives from equations (B21) and (B22) (we are not interested in its exact form). It remains to find \( \phi \) using the Maxwell’s equation,

\[ \nabla \cdot E = -\Delta \phi = 4\pi (e_o \delta n^{(sc)} + e_o \delta n^{(ind)} + e_p \delta n_p). \]

On the right-hand side here we see the total charge density; the last two terms are proportional to \( \phi \) and describe reaction of the system to the perturbation \( e_o \delta n^{(sc)} \) (the first term). One may show that the asymptotic solution to this equation, valid at \( r \to \infty \), corresponds to vanishing right-hand side of (B24), \( e_o \delta n^{(sc)} + e_o \delta n^{(ind)} + e_p \delta n_p = 0 \), that is (see equations B16 and B23)

\[ \phi = -\frac{e_o \delta n^{(sc)} - e_o \delta n^{(ind)} + e_p \delta n_p}{e_o^2 a_4^2 + e_o^2 a_4} + O \left( \frac{1}{r^2} \right). \]

As expected, \( \phi \propto 1/r \) at \( r \to \infty \).

Now we have everything at hand to prove equation (29). Consider a general expression (22) for the force \( F_{\text{npe-v}} \) with \( \Pi_{ik} \) given by equation (B10). Since in the absence of the vortex \( F_{\text{npe-v}} = 0 \), we can rewrite (22) as

\[ F_{\text{npe-v}} = -\int \delta \Pi_{ik} n_k \, dS, \]

where \( \delta \Pi_{ik} \) contains only vortex-related quantities. In the linear approximation the first two velocity-dependent terms in equation (B10) can be omitted\(^{20} \) and \( \delta \Pi_{ik} \) can be presented as

\[ \delta \Pi_{ik} = \delta P_{\text{muc}} \delta n_k + \Pi_{ik}^{(e, sc)} + \Pi_{ik}^{(e, ind)} \]

\[ \sim -\frac{1}{4\pi} \left[ E_k E_e - \frac{1}{2} E_k^2 \delta \Pi_{ik} + B_k B_e - \frac{1}{2} B_k^2 \delta \Pi_{ik} \right] \]

where the electron contributions \( \Pi_{ik}^{(e, sc)} \) and \( \Pi_{ik}^{(e, ind)} \) are given by equations (30) and (B15), respectively, and we used equation (B11) to express the electro-magnetic stress tensor \( \Pi_{ik}^{(e-m)} \). The first and the third terms here cancel each other out in view of equations (B15), (B20) and the quasineutrality condition for unperturbed matter, \( n_{e0} = n_{p0} \). Because \( \phi \sim 1/r \) at large \( r \), the terms depending on the electric field \( E \) make a negligible contribution to \( F_{\text{npe-v}} \) and can also be ignored. Finally, due to the symmetry of the problem, the magnetic field induced by the current \( \delta \gamma^{(sc)} \) of scattered electrons, can only be directed along the axis \( z \), and hence also does not contribute to \( F_{\text{npe-v}} \), which lies in the \( xy \)-plane. Thus, the only non-vanishing contribution to the force comes from the term \( \Pi_{ik}^{(e, sc)} \) (scattered electrons), that is equation (29) is proved.

APPENDIX C: DERIVATION OF EQUATION (38)

Inserting delta-function in equation (35), it can be rewritten as

\[ \sigma_{\parallel} = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \gamma(b) \gamma(\tilde{b}) \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\tilde{b}(b - \tilde{b})} \, \tilde{b} \, d\tilde{b} \, db = \]

\[ = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \gamma(b) e^{-i\tilde{b} b} \, \tilde{b} \, db \, d\sigma \]

Taking into account equation (34), (C1) coincides with equation (38).

APPENDIX D: EQUIVALENCE OF THE CLASSICAL AND QUASICLASSICAL EXPRESSIONS FOR \( \sigma_{\parallel} \) AND \( \sigma_{\perp} \)

Let us show that expressions (55) and (56) are equivalent to, respectively, equations (35) and (36) and thus lead to the same \( \sigma_{\parallel} \) and \( \sigma_{\perp} \). With this aim we note that the angular momentum \( \hbar l \) of a quasiclassical electron is related to its angular momentum \( l \).

\(^{20}\) Recall that we work in the coordinate frame comoving with the vortex. In this frame the proton superfluid velocity at \( r \gg \lambda \) can be presented as \( u_p + \mathbf{V}_p \), where \( u_p \) is the asymptotic proton (and electron) velocity far from the vortex, and \( \mathbf{V}_p = \mathbf{V}_{\text{sp}} + \mathbf{V}_M(\mathbf{u}_s - \mathbf{V}_L) \) is the small perturbation induced by the charge current \( \delta \gamma^{(sc)} \) of scattered electrons (note that the quantity \( \mathbf{V}_p \) and the associated perturbation of the magnetic field can both be expressed through \( \delta \gamma^{(sc)} \) using Ampere’s law and equation B3). Consequently, the contribution from the term \( m_p \sigma_{\parallel} V_{sp} V_{np} \) to (B26) is of the order of \( O(\|u_s - V_L\|^2) \) and indeed can be neglected.
impact parameter $b$ by the formula $bl = \hbar k_{\perp} b$. Then, introducing the coordinate $y = \sqrt{r^2 - b^2}$, one rewrites equation (54) as
\[
\delta_l = -\zeta \frac{e^o_{\perp}}{e_p^o} b \int_0^\infty P(b\sqrt{y^2 + y^2}) \frac{dy}{y^2 + y^2} = -\zeta \frac{e^o_{\perp}}{e_p^o} \int_0^\infty P(y) \frac{dy}{1 + y^2} \sqrt{1 + y^2}.
\]
where in the second equality we changed the variable $y \rightarrow \tilde{y} \equiv y/b$. The derivative $d\delta_l / dl$ can now be calculated as
\[
\frac{d\delta_l}{dl} = \frac{d\delta_l}{dy} \frac{dy}{dl} = -\frac{1}{k_{\perp}} \frac{e^o_{\perp}}{e_p^o} \int_0^\infty \frac{dP(r)}{r} \frac{dy}{\sqrt{1 + y^2}} \sqrt{y^2 + y^2},
\]
where we make use of equation (11) in the last equality. Comparing now equations (D2) and (34) one verifies that $2d\delta_l / dl = \gamma$ (which is an expected result, see, e.g., Landau & Lifshitz 1977) and hence the expressions (55) and (56) coincide with, respectively, expressions (35) and (36).

**APPENDIX E: ANALYSIS OF THE EXPRESSION FOR THE TRANSVERSE FORCE $F_\perp$**

Assume that protons form a strong type-II superconductor ($\xi \ll \lambda$). Our aim here is to further justify that in this case electrons do not act *directly* on the proton vortex with the transverse force $F_\perp$ (see Section 5 for a definition of $F_\perp$).

The force on a vortex $F_{\text{npe} \rightarrow V}$ is given by the general equation (22). The surface integral in this equation can be taken over any closed surface around the vortex, which is sufficiently far from the region where the external force $F_{\text{ext}}$ is applied. In what follows, we consider two cylindrical surfaces of unit length, $S_1$ and $S_{\text{II}}$ (see Fig. E1). Let the cylindrical surface $S_1$ be the same as in Section 3, i.e., a radius $l \gg r_{\text{II}} \gg \lambda$; in turn, assume that the cylindrical surface $S_{\text{II}}$ has a radius $\lambda \gg r_{\text{II}} \gg \xi$. Then $F_{\text{npe} \rightarrow V}$ can be presented in two equivalent ways,
\[
F_{\text{npe} \rightarrow V} = -\int_{S_1} \Pi_{\text{ik}} n_k dS = -\int_{S_{\text{II}}} \Pi_{\text{ik}} n_k dS,
\]
where $\Pi_{\text{ik}}$ is the stress tensor for npe-matter given by equation (B10). It consists of the contributions from the neutrons, protons, electrons, and electromagnetic field. As shown in Section 3, only electrons contribute to the integral over the surface $S_1$ (see equation 29). And what is the electron contribution to the integral over $S_{\text{II}}$? Using equation (B7) (in which $\partial G_{\text{npe}} / \partial t = 0$), one can write
\[
-\int_{S_{\text{II}}} \Pi_{\text{ik}}^{(e)} n_k dS = -\int V \Pi_{\text{ik}}^{(e)} dV = -\int \left( e_n n_e E + \frac{1}{c^2} j_e \times B \right) dV,
\]
where $j_e = \sum_{\text{pe}} e_n v n_{\text{pe}}, n_e = \sum_{\text{pe}} n_{\text{pe}}$, and integration is performed over the cylinder of radius $r_{\text{II}}$. The integral (E2) is much smaller than the corresponding electron contribution to the momentum flux through the surface $S_1$, as it is demonstrated below. Indeed, neglecting the modification of $n_{\text{pe}}, E$, and $B$ caused by the electrons scattered off the vortex (i.e., using the ‘shifted Fermi-sphere’, $n_{\text{pe}}^{(\text{eq})}$, as an electron distribution function, see equation 25), one can easily perform volume integration in equation (E2) and approximately write: 21
\[
-\int_{S_{\text{II}}} \Pi_{\text{ik}}^{(e)} n_k dS \approx -\frac{e_n}{c} \Phi_{S_{\text{II}}} \left[ (u_e - V_L) \times e_z \right],
\]
where $\Phi_{S_{\text{II}}} = \int_0^{r_{\text{II}}} B 2\pi r dr$ (E5) is the magnetic flux enclosed by the cylindrical surface $S_{\text{II}}$. On the other hand, the ‘transverse’ component of the electron momentum flux through the surface $S_1$ is given by (see Section 5 and equation 61 there; see also footnote 21)
\[
F_\perp = D' [e_z \times (u_e - V_L)] = \frac{e_n}{c} \Phi [e_z \times (u_e - V_L)],
\]
(E6)

The ratio of the integrals (E4) and (E6) equals $\Phi_{S_1} / \Phi \sim r_{\text{II}}^2 / \lambda^2 \ll 1$, i.e., the electron contribution to the momentum flux through the surface $S_{\text{II}}$ is $r_{\text{II}}^2 / \lambda^2$ times smaller than through $S_1$. But the total momentum flux through any of these surfaces must be conserved (see equation E1). It is clear, therefore, that the missing momentum flux through $S_{\text{II}}$ should be transferred by protons. This result can be easily obtained from the analysis of the proton superfluid and momentum conservation equations (B2) and (B4) if we note that the proton velocity scales as $1/r$ at distances $r \sim r_{\text{II}}$, i.e., $\lambda \gg r \gg \xi$. Following the same derivation as, e.g., in Sonin (1987), one then finds that the proton contribution to the momentum flux through $S_{\text{II}}$ is the ordinary Magnus force on proton vortices in neutron stars

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21 Similar approximate calculation of the electron momentum flux through the surface $S_1$ would give
\[
-\int_{S_1} \Pi_{\text{ik}}^{(e)} n_k dS \approx -\frac{e_n}{c} \Phi \left[ (u_e - V_L) \times e_z \right].
\]
This expression coincides with the transverse part $F_\perp$ of the force on a vortex (see Section 5 and equation E6). This means that our approximation of unperturbed quantities $n_{\text{pe}}, E$, and $B$ correctly reproduces the transverse force on a vortex. However, to calculate the dissipative longitudinal part, $F_\perp$, one needs to account for small deviations of these quantities caused by the electron scattering off the vortex magnetic field.
force $F_M$ (see equation 4), which coincides with the force $F_\perp$ given by equation (E6) – exactly what we need to restore momentum conservation! We come to conclusion that the momentum flux through the surface $S_II$ is mainly transported by protons, while through the surface $S_I$ – by electrons. In other words, it is the protons (not electrons), which directly act on the vortex with the transverse force $F_\perp$.

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