Independent Components of an Indexed Object with Linear Symmetries

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Abstract. The problem of finding independent components of an indexed object (e.g., a tensor) with arbitrary number of indices and arbitrary linear symmetries is discussed. It is proved that the number of independent components $f(k)$ is a polynomial of degree not greater than the number of indices $n$, $k$ being the dimension of the space. Several algorithms to compute $f(k)$ for arbitrary $k$ are described and discussed. It is shown that in the worst case finding $f(k)$ for arbitrary $k$ requires solving at most $P(n)$ systems of linear equations with at most $(n!)^2$ equations for at most $n!$ unknowns, $P(n)$ being the number of partitions of $n$. As a by-product, an efficient algorithm to parametrize all components of the object through its independent components is found and implemented in Mathematica.

1 Motivation

In this paper we attempt to find an approach to the problem formulated, e.g., in [3]. The problem consists in calculating the number of independent components of an indexed object $A^{i_1i_2...i_n}$ with symmetries (for example, $A^{i_1i_2...i_n} = A^{k_{i_1i_2...i_n}}$). The importance of this problem for computer algebra systems for indicial tensor calculations can be clarified by the following example given in [3]. In many packages for indicial tensorial calculations one can define an object $A^{ijk}$ with 3 indices and the following two symmetry properties:

$$A^{ijk} = A^{jik}$$

and

$$A^{ijk} = -A^{ikj}.$$  \hspace{1cm} (2)

One can easily check that these two symmetry properties imply that all components of $A^{ijk}$ is zero. However, none of the packages available to the author can automatically recognize this fact. In many cases, however, if the software could recognize such and similar situations, it could simplify many kinds of calculations. On the other hand, it is often quite important to find not only the number of independent components, but also the independent components themselves (for example, to store the components of an object in the best possible way). In this paper we will give efficient algorithms for both finding and counting the independent components of an object with symmetries.

2 The Problem and Some Trivial Properties

Here the following problem with linear symmetries is considered. Let $A^{i_1i_2...i_n}$ be an object with $n$ indices. Each index $i_j$ can take values from $i_j \in \{1, 2, \ldots, k\}$, where $k \geq 1$ is the dimensionality of the space where the object is defined. The object has $s$ symmetries defined by equations of the form

$$\sum_{j=1}^{p_l} a_{jl} A^{\pi_{jl}} = b_l, \quad l = 1, 2, \ldots, s$$  \hspace{1cm} (3)

where $a_{jl}$ and $b_l$ are numbers characterizing the $l$-th symmetry, and $\pi_{jl}$ is a permutation of the $n$ indices $i_1i_2...i_n$, $p_l$ is the number of terms in the $l$-th symmetry. Clearly, the maximal number of the terms in one symmetry and the maximal number of linearly independent symmetries are
constrained as (otherwise the individual terms and the whole symmetries linearly depend on the 
other terms and symmetries, respectively)

\[ p_i \leq n! , \quad (4) \]
\[ s \leq n! . \quad (5) \]

In principle, the number of symmetries \( s \) can be larger than \( n! \), but in this case either the additional symmetries are linearly dependent on the first \( n! \) symmetries or at least one of the additional symmetries is incompatible with the first \( n! \) ones and such an object does not exist. A component of \( A^{i_1 i_2 \ldots i_n} \) is called independent if it is not constrained by symmetries (3) to be a number and if it cannot be calculated as a function of other independent components. The set of independent components is obviously not unique (if a symmetry requires \( A^{21} = A^{12} \) any of these two components can be considered as independent). The size of this set is, however, independent of this non-uniqueness, and it is sufficient to find any version of it. The problem consists in computing of the number \( f \) of independent components of \( A^{i_1 i_2 \ldots i_n} \) as a function of \( k \): \( f(k) \). Since the total number of components of \( A^{i_1 i_2 \ldots i_n} \) for a fixed \( k \) is \( k^n \), one concludes

\[ 0 \leq f(k) \leq k^n . \quad (6) \]

### 3 Objects with One and Two Indices

In order to understand the problem more clearly, let us consider first all possible objects with one 
and two indices. The case of \( n = 1 \) is quite trivial. An object \( A^i \) has at most \( k \) components. If the 
object has no symmetries of the form (3), the number of independent components \( f(k) = k \). The 
only possible symmetry reads

\[ a A^i = b , \quad (7) \]

where \( a \) and \( b \) cannot vanish simultaneously (otherwise the object has again no symmetry). Clearly, 
if \( a = 0 \) and \( b \neq 0 \) the symmetry cannot be satisfied. Such symmetries will be called incompatible. 
In case of incompatible symmetries we will symbolically write \( f(k) = 0 \). If \( a \neq 0 \) then \( A^i = b/a \), i.e. all components for any \( k \) are constrained to be number \( b/a \). This can be summarized as follows

\[ f(k) = \begin{cases} 
  k, & \text{no symmetries } (a = b = 0) , \\
  0, & a \neq 0 , \\
  \emptyset, & a = 0 \& b \neq 0 .
\end{cases} \quad (8) \]

The case of \( n = 2 \) is a bit less trivial. For an object \( A^{ij} \) with no symmetries \( f(k) = k^2 \). Each 
of the two possible symmetries is of the form

\[ a_1 A^{ij} + a_2 A^{ji} = b . \quad (9) \]

Let us consider first only one symmetry (9). For \( C^1_k = k \) “diagonal” components with \( j = i \) one 
can write this symmetry as (no implicit summation is assumed here)

\[ (a_1 + a_2) A^{ii} = b , \quad i = 1, 2, \ldots , k . \quad (10) \]

This symmetry can be analyzed in the same way as we did for (7) above. The \( k \) “diagonal” 
components are constrained independently of each other (but clearly in the same way). Therefore, 
it is sufficient to analyze (10) for one fixed \( i \) and then multiply the result (either 0 or 1 independent 
components) by \( k \). Among the other \( k(k-1) \) “off-diagonal” components with \( j \neq i \) there exist 
\( C^2_k \) pairs of components, \( C^b_a = \frac{a!}{b!(a-b)!} \) being the binomial coefficient. It is only the components 
within each of these pairs which could be potentially constrained by (9). Indeed, for any fixed \( j \) and 
\( i \) (\( j \neq i \)) only two combinations of indices, \( ij \) and \( ji \), are connected to each other by a permutation 
and appear in (9). Therefore, it is sufficient to consider any of these pairs and multiply the results 
(one can have 0, 1 or 2 independent components within each pair) by \( C^2_k = k(k-1)/2 \). Therefore, 
the number of independent components of \( A^{ij} \) for an arbitrary \( k \) can always be calculated as
\[ f(k) = f_2 k(k-1)/2 + f_1 k \] with two integers \( 0 \leq f_2 \leq 2 \) and \( 0 \leq f_1 \leq 1 \) depending on the symmetries of the object. Let \( i = 1 \) and \( j = 2 \). Then (9) give two linear equations constraining \( A^{12} \) and \( A^{21} \):

\[
\begin{align*}
  a_1 A^{12} + a_2 A^{21} &= b, \\
  a_1 A^{21} + a_2 A^{12} &= b. 
\end{align*}
\]

Analyzing (10) and (11) one gets the number of independent components of an object with \( n = 2 \) and one symmetry (9):

\[
f(k) = \begin{cases} 
  k^2, & \text{no symmetries} \ (a_1 = a_2 = b = 0), \\
  \frac{1}{2} k (k+1), & a_1 = -a_2 \neq 0 \ & b = 0, \\
  \frac{1}{2} k (k-1), & a_1 = a_2 \neq 0, \\
  0, & |a_1| \neq |a_2|, \\
  0, & a_1 + a_2 = 0 \ & b \neq 0. 
\end{cases}
\]

Clearly, for \( a_1 = -a_2 \neq 0 \ & b = 0 \) the symmetry (9) can be written as \( A^{ij} - A^{ji} = 0 \). This means that the “diagonal” components are not constrained at all, and the “off-diagonal” components are pairwise equal. We have thus a symmetric “matrix”. For \( a_1 = a_2 \neq 0 \) one has \( A^{ij} + A^{ji} = \tilde{b} = b/a_1 \). This means that the “diagonal” components are all equal to \( \tilde{b}/2 \) and the “above-diagonal” components can be computed from the “below-diagonal” ones as \( A^{ij} = -A^{ji} + \tilde{b} \). For \( \tilde{b} = 0 \) we have a skew-symmetric “matrix” here. In case \( |a_1| \neq |a_2| \) all components are equal to the same number: \( A^{ij} = b/(a_1 + a_2) \).

It is easy to see that for objects \( A^{ij} \) with two symmetries

\[
\begin{align*}
  a_{11} A^{ij} + a_{21} A^{ji} &= b_1, \\
  a_{12} A^{ij} + a_{22} A^{ji} &= b_2 
\end{align*}
\]

\( f(k) \) also takes the same 5 possible values as in (12). The ideas found in these examples will allow us to formulate several important theorems and algorithms below.

### 4 The Number of the Independent Components

Let us first leave aside the question whether the symmetries of the object are compatible and consider that \( f(k) \neq 0 \) for any \( k \). This will be further discussed in Section 5. The results of the previous Section suggest to formulate the following

**Theorem 1.** For an object \( A^{i_{12\ldots n}} \) with arbitrary \( n \) and arbitrary compatible linear symmetries the number of independent components \( f(k) \) is a polynomial of the dimensionality \( k \). The degree of this polynomial does not exceed \( n \).

**Proof.** From (6) it is obvious that if \( f(k) \) is a polynomial its degree cannot exceed \( n \).

Let us denote the set of all possible values of each of the indices \( i_a \) as \( N_k = \{1, 2, \ldots, k\} \). Its cardinality is \( |N_k| = k \). Let us also denote the set of all independent components of an object \( A^{i_{12\ldots n}} \) with all \( i_a \in N_k \) as \( E(N_k) \). The cardinality of \( E(N_k) \) is clearly \( f(k) \). It is clear that we can change \( N_k \) to any set \( S \) with the same cardinality \( k \) and number of independent components of \( A^{i_{12\ldots n}} \) with \( i_a \in S \) will be again \( f(k) \). That is, for any \( S \) one has \( |E(S)| = f(|S|) \).

Let \( S^n, 1 \leq a \leq k \) are the \( C_{k-1}^n \) \( k \) subsets of \( N_k \) such that \( |S^n| = k - 1 \). Clearly, \( S^1 \cup S^2 \cup \ldots \cup S^k = N_k \). Any two components can be potentially related to each other according to (3) only if their sets of \( n \) indices are related by a permutation. Thus, if \( k > n \) the set \( E(N_k) \) of independent components of \( A^{i_{12\ldots n}} \) with \( i_a \in N_k \) is the union of the \( k \) sets \( E(S^n) \) of independent components with \( i_a \in S^n \): \( E(N_k) = E(S^1) \cup E(S^2) \cup \ldots \cup E(S^k) \). For the same reason for any two sets \( S_1 \) and \( S_2 \) one has \( E(S_1) \cap E(S_2) = E(S_1 \cap S_2) \). Applying the Inclusion-Exclusion Principle \[2, 7\] to
the \( k \) sets \( E(S^a) \) one gets

\[
|E(N_k)| = |\bigcup_{a=1}^{k} E(S^a)| = \sum_{1 \leq a \leq k} |E(S^a)| \\
- \sum_{1 \leq a < b \leq k} |E(S^a) \cap E(S^b)| \\
+ \sum_{1 \leq a < b < c \leq k} |E(S^a) \cap E(S^b) \cap E(S^c)| - \ldots \\
+ (-1)^{k-1} |E(S^1) \cap \ldots \cap E(S^k)|, \quad k > n. \tag{14}
\]

This formula immediately implies

\[
f(k) = \sum_{j=1}^{k-1} (-1)^{j-1} f(k - j) C^j_k, \quad k > n. \tag{15}
\]

Since this last formula is valid for any \( k > n \) one can compute \( f(k) \) for \( k > n \) from the \( n \) values \( f(i) \) for \( i = 1, \ldots, n \). This means that the \( f(k) \) is a polynomial of \( k \) of degree \( n \) or less.

From the computational point of view it is useful to explicitly compute the above-mentioned representation of \( f(k) \) through the \( n \) values \( f(i) \) for \( i = 1, \ldots, n \).

**Theorem 2.** For an object \( A^{i_1i_2\ldots i_n} \) with arbitrary \( n \) and arbitrary compatible linear symmetries the number of independent components \( f(k) \) for any \( k > n \) can be computed as

\[
f(k) = \sum_{i=1}^{n} (-1)^{n-i} f(i) C^i_k C^{n-i}_{k-1-i}. \tag{16}
\]

**Proof (1).** A straightforward way to prove (16) is by induction. For \( k = n + 1 \) Eq. (16) coincides with (15) and is thus correct. Suppose that Eq. (16) is valid for some \( k \). Let us prove that it is also valid for \( k + 1 \). Applying Eq. (15) one gets

\[
f(k + 1) = \sum_{i=1}^{k} (-1)^{k-i} f(i) C^i_{k+1}. \tag{17}
\]

Since for all \( f(i) \) with \( n + 1 \leq i \leq k \) Eq. (16) is supposed to be correct, one has

\[
f(k + 1) = \sum_{i=1}^{n} (-1)^{k-i} f(i) C^i_{k+1} + \sum_{i=n+1}^{k} (-1)^{k-i} f(i) C^i_{k+1} \\
= \sum_{i=1}^{n} (-1)^{k-i} f(i) C^i_{k+1} + \sum_{j=n+1}^{k} (-1)^{k-j} C^j_{k+1} \sum_{i=1}^{n} (-1)^{n-i} f(i) C^i_j C^{n-i}_{j-1-i} \\
= \sum_{i=1}^{n} (-1)^{n-i} f(i) \left[ (-1)^{k-n} C^i_{k+1} + \sum_{j=n+1}^{k} (-1)^{k-j} C^j_{k+1} C^i_j C^{n-i}_{j-1-i} \right] \\
= \sum_{i=1}^{n} (-1)^{n-i} f(i) C^i_{k+1} C^{n-i}_{k-i}. \tag{18}
\]

Here we used the well-known properties of the binomial coefficients [2, 7]. In particular, we used

\[
\sum_{s=0}^{k-n} (-1)^s C^{n-i+1+s}_{k-i+1} C^s_{n-i+s} = 1, \quad 0 \leq i \leq n \leq k. \tag{19}
\]

Therefore, Eq. (16) is correct for any \( k > n \).
Proof (2). A more elegant proof of Eq. (16) directly follows from Theorem 1: if a function is a polynomial of degree \( n \) or less, one can take its first \( n \) values \( f(i) \) for \( i = 1, 2, \ldots, n \) and construct a polynomial of degree \( n \) having these values for \( i = 1, 2, \ldots, n \) (and \( f(0) = 0 \)) using the Lagrange interpolation formula. Indeed,

\[
(-1)^{n-i} C_k^i C_{k-1}^{n-i} = \frac{(k-0)(k-1) \ldots (k-(i-1))(k-(i+1)) \ldots (k-n)}{(i-0)(i-1) \ldots (i-(i-1))(i-(i+1)) \ldots (i-n)},
\]

(20)

which is exactly the Lagrange form of the coefficients of the interpolation polynomial. \( \square \)

5 On the Compatibility of the Symmetry Properties

Up to now we have ignored the question of compatibility of the symmetry properties (3). One can easily check if the symmetries are compatible for \( k = 1 \).

**Theorem 3.** For an object \( A^{i_1 i_2 \ldots i_n} \) with arbitrary \( n \) its symmetry properties (3) are incompatible for \( k = 1 \) if and only if at least one of the two conditions are met:

1. for at least one symmetry \( b_l \neq 0 \) and \( c_l = 0 \) with

\[
c_l = \sum_{j=1}^{p_l} a_{jl},
\]

(21)

2. there exist at least two symmetries for which \( c_l \neq 0 \) and \( c_l' \neq 0 \), and \( b_l/c_l \neq b_l'/c_l' \).

Proof. For \( k = 1 \) any of the symmetries can be written as

\[
c_l A^{11 \ldots 1} = b_l,
\]

(22)

c\(_l\) being defined by (21). These properties can be analyzed in the same way as we did for (7). The theorem immediately follows from this analysis. \( \square \)

Another two important results are:

**Theorem 4.** If symmetries of an object \( A^{i_1 i_2 \ldots i_n} \) with arbitrary \( n \) are compatible for dimension \( k = K \), they are also compatible for any dimension \( k < K \).

Proof. This is obvious since the whole system of symmetry-induced equations for any \( k < K \) is a part of the corresponding system for \( k = K \). \( \square \)

**Theorem 5.** If symmetries of an object \( A^{i_1 i_2 \ldots i_n} \) with arbitrary \( n \) are compatible for dimension \( k = n \), they are also compatible for any dimension \( k \).

Proof. For \( k < n \) this follows from Theorem 4 while for \( k > n \) from Theorem 2. \( \square \)

The algorithmic usage of Theorem 5 is clear: one has to check if the symmetries are compatible for \( k = n \).

6 Algorithms to Compute the Independent Components

Below three different algorithms to find the independent components of an indexed object are discussed.

6.1 Algorithm A

The first algorithm is a straightforward one. For a fixed \( k \) all possible combinations of the numerical values of all \( n \) indices are substituted into each of the symmetries (3). Thus, one obtains a system of linear equations for all \( k^n \) components of the object. The total number of equations is \( sk^n \), where \( s \) is the number of symmetries \((s \leq n!)\). Each equation involves at most \( n! \) components. Then, the system is solved and the independent components are found explicitly. Counting them allows one to get \( f(k) \) for that fixed \( k \) for which the system was generated. It is clear that if \( n, k \) or \( s \) are large the calculations can be very time-consuming. The only reason to implement such an algorithm is the possibility to check better algorithms described below.
6.2 Algorithm B

From the algorithmic point of view, Eq. (16) allows one to calculate \( f(k) \) for any \( k > n \) as soon as one has calculated \( f(k) \) for all \( 1 \leq k \leq n \). Moreover, having the set of the independent components for \( k = n \) one can count the number of components among them with indices \( i_a \in N_p \)
\( N_p = \{1, \ldots, p\} \) for \( p = 1, \ldots,n-1 \). The number of such components is exactly \( f(p) \), and, therefore, it is sufficient to have the set of independent components for \( k = n \) to compute \( f(k) \) for any \( k \).

Such an algorithm requires solving at most \( s^p n^n \) linear equations with at most \( n! \) unknowns in each equation and at most \( n^n \) unknowns in the whole system (note that each of the \( s \leq n \) symmetries (3) generates at most \( n^n \) linear equations for at most \( n! \) unknowns). Although, this algorithm is better than straightforward calculation of the independent components for some large \( k > n \), it is still quite time-consuming for larger \( n \).

Another point is that this algorithm does not allow to list the independent components and to represent the other components as functions of the independent ones for \( k > n \). It is certainly possible to augment the algorithm in this direction. However, attempts to do so allowed the author to formulate much more efficient algorithm for both computing \( f(k) \) and finding the dependencies in explicit form. This algorithm described in the next Section.

6.3 Algorithm C

Here we suggest a much faster algorithm based on the fact that the sequences of indices of the components which could be potentially constrained by a symmetry of the form (3) are related with each other by a permutation. This obvious fact has been already mentioned and used above. Therefore, we can split the whole set of \( k^n \) components into such subsets within which the sequences of indices are related by a permutation and then generate and solve the corresponding linear equations only for the components within each of these subsets. Moreover, one can drastically reduce the number of the subsets to be considered since many of them are similar to each other (e.g., they can be obtained from each other by changing, say, value 4 for all indices into value 5).

Let us consider a component \( A^{i_1i_2\ldots i_n} \) with some fixed indices \( 1 \leq i_a \leq k \). Any sequence of indices \( i_1i_2\ldots i_n \) can be characterized by sequence \( X = (x_1, \ldots, x_k) \), where each \( x_k \) is the number of such \( i_a \) in \( i_1i_2\ldots i_n \) that \( i_a = b \). Clearly, one has \( x_1 + x_2 + \ldots + x_k = n \) with constrains \( 0 \leq x_\ell \leq n \).

For fixed \( n \) and \( k \) there are \( C_p^m \) different solutions of this equation with these constrains, and, therefore, \( C_{k+n-1}^n \) different sequences \( X \).

It is clear that two components \( A^{i_1i_2\ldots i_n} \) and \( A^{j_1j_2\ldots j_n} \) can be related to each other by a symmetry of the form (3), only if both sequences of indices \( i_1i_2\ldots i_n \) and \( j_1j_2\ldots j_n \) correspond to the one and same sequence \( X \). The number of components \( A^{i_1i_2\ldots i_n} \) corresponding to the same \( X \) is the multinomial coefficient \( n!/(x_1!x_2!\ldots x_k!) \) with \( x_1 + x_2 + \ldots + x_k = n \).

The subsets of the components corresponding to two different \( X_1 \) and \( X_2 \) can be treated in the same way if \( X_1 \) and \( X_2 \) are related to each other by a permutation. Such a permutation corresponds just to renaming all indices having, say, value 1 to, say, 5, and so on. The two subsets corresponding to two such \( X_1 \) and \( X_2 \) have the same number of independent components and the same dependence of the other components on the independent ones. Therefore, it is sufficient to calculate the dependence of the components only for one \( X \) among all of them related to each other by a permutation. We will consider only the sorted version \( Y \) of \( X \): \( Y = (y_1, \ldots, y_k) \) with \( y_1 \geq y_2 \geq \ldots \geq y_k \).

Let \( Y^l \) be the sequence \( Y \) with exactly \( l \) nonzero elements: \( Y^l = (y_1, \ldots, y_l, 0, \ldots, 0) \), \( |Y^l| = k \), \( y_1 \geq y_2 \geq \ldots \geq y_l > 0 \). It is obvious that \( 1 \leq l \leq \min(n, k) \). Each \( Y^l \) corresponds to a partition of \( n \) into \( l \) parts. There exists \( P(n,l) \) different partitions of this kind.

Now, let \( p \leq l \) denote the number of distinct values among \( y_1, y_2, \ldots, y_l \), and \( 1 \leq s_m \leq n, 1 \leq m \leq p \) is how many times the value number \( m \) (which is one of the \( p \) distinct values in \( Y^l \)) appears among \( y_1, y_2, \ldots, y_l \). Then the number of different \( X \) which can be obtained by permutations from \( Y^l \) is \( C_p^m ll/(s_1s_2!\ldots s_p!) \).
Combining all the results discussed above one gets the following formula for the number of independent components

\[ f(k) = \sum_{l=1}^{\min(n,k)} \left[ \sum_{j=1}^{P(n,l)} \frac{l!}{s_1! s_2! \ldots s_p!} \right] C^l_k, \quad (23) \]

where the inner sum goes over all the partitions of \( n \) into \( l \) parts, each such partition corresponds to \( Y_j^l \), the numbers \( s_1^l, s_2^l, \ldots s_p^l \) is calculated for \( Y_j^l \), and \( g(Y_j^l) \) is the number of independent components among \( n!/\prod (y_1! \ldots y_l!) \) components \( A_1^{i_1 j_1 \ldots i_n j_n} \) corresponding to \( Y_j^l \):

\[ 0 \leq g(Y_j^l) \leq \frac{n!}{y_1! \ldots y_l!} \quad (24) \]

Two identities can be used to check the internal consistency of (23) and (24). First, the total number of sequences \( X \) for fixed \( n \) and \( k \) can be calculated from (23) with \( g(Y_j^l) = 1 \) and should be \( C^m_{k+n-1} \) as discussed above. One can see that this is true:

\[ \sum_{l=1}^{\min(n,k)} \left[ \sum_{j=1}^{P(n,l)} \frac{l!}{s_1! s_2! \ldots s_p!} \right] C^l_k = C^n_{k+n-1}. \quad (25) \]

Second, for an object without symmetries \( g(Y_j^l) = n!/\prod (y_1! \ldots y_l!) \), and the total number of components for fixed \( k \) and \( n \) calculated according to (23) should be \( k^n \). Indeed, it is also true:

\[ \sum_{l=1}^{\min(n,k)} \left[ \sum_{j=1}^{P(n,l)} \frac{n!}{y_1! \ldots y_l!} \frac{l!}{s_1! s_2! \ldots s_p!} \right] C^l_k = k^n. \quad (26) \]

Combining (23) and (16) it is clear that in order to calculate \( f(k) \) for arbitrary \( k \) it is sufficient to calculate the number of independent components within the subsets of \( A_1^{i_1 j_1 \ldots i_n j_n} \) corresponding to \( \sum_{l=1}^{n} P(n,l) = P(n) \) different sequences \( Y_j^l \), \( P(n) \) being the total number of partitions of \( n \). The size of the subsets of \( A_1^{i_1 j_1 \ldots i_n j_n} \) does not exceed \( n! \). Therefore, in the worst case one should solve \( (n!)^2 \) linear equations (each symmetry (3) generates at most \( n! \) distinct equations, and there are at most \( n! \) symmetries) with \( n! \) unknowns. This is much better that for the algorithms A and B.

### 6.4 Reduction of the Number of Linear Equations

A simple idea allows one to reduce further the number of equations in the system generated by the symmetries before solving that system of these equation. One can put the equations into a canonical form in which it is trivial to check if any two equations are equivalent (e.g., the numerical coefficient at the lexicographically first component should be equal to unity) and retain only one among the equivalent equations possibly appearing in the set of generated equations. For example, the symmetry \( T^{ij} - T^{ji} = 0 \) for \( i \in \{1, 2\} \) produces two equations \( T^{12} - T^{21} = 0 \) and \( T^{21} - T^{12} \) which are equivalent and can be considered as one equation. On the other hand, the symmetry \( 3T^{ij} + 4T^{ji} = 7 \) for \( i \in \{1, 2\} \) gives two linearly independent equations \( 3T^{12} + 4T^{21} = 7 \) and \( 3T^{21} + 4T^{12} = 7 \) whose canonical forms are different: \( T^{12} + \frac{4}{3}T^{21} = \frac{7}{3} \) and \( T^{12} + \frac{4}{3}T^{21} = \frac{7}{3} \), respectively. How many equations can be eliminated from the system of equations using this simple equivalence test depends on the symmetry properties (note, that even two different symmetry properties can produce equivalent equations). This reduction scheme for the system of linear equations can be used in all three algorithms described above.

### 7 Implementation in Mathematica

In order to check the performance and cross-check the results all three algorithms A, B, and C were implemented in Mathematica with the idea that the best one should be incorporated into the package EinS for calculations with indexed objects [9, 8]. The main parts of the implementation are:
two routines DefObject and DefSymmetries allowing one to define objects with arbitrary symmetries,
(2) routine ConstrainComponents which explicitly generates and solves the linear equations for individual components induced by the symmetries for some fixed \(k\),
(3) routine GuessPolynomial implementing algorithm A by calling ConstrainComponents for a sufficient number of different values of \(k\) to check if a polynomial of degree \(n\) or less can be fitted to the results,
(4) routine CountIndependentComponents implementing algorithm B by calling ConstrainComponents for \(k = n\) and analyzing the resulted independent components to compute \(f(k)\) from (16), and
(5) routine ListIndependentComponents implementing algorithm C and providing for any \(k\) both \(f(k)\) and, if requested, a list for the independent components and the dependence of the other components.

All the routines allow one to control all the steps of the corresponding algorithms and, if desired, provide the user with various additional information. The implementation consists of about 1000 lines of Mathematica top level code and is available from the author upon request.

To give a practical example let us consider the covariant Riemann tensor \(R_{ijkl}\) with \(n = 4\) and with its four symmetry properties
\[
    R_{ijkl} = R_{klij}, \\
    R_{ijkl} = -R_{jikl}, \\
    R_{ijkl} = -R_{ijlk}, \\
    R_{ijkl} + R_{iljk} + R_{klij} = 0.
\] (27)

The well-known result [11] for the covariant Riemann tensor is \(f(k) = \frac{1}{12} k^2 (k^2 - 1)\).

Algorithm A (GuessPolynomial) explicitly computes the independent components subsequently for \(k = 1, k = 2, \ldots, k = 6\) to verify that a polynomial of degree 4 or less really fits the results. For example, for \(k = 6\) this requires solving of a system of \(4 \times 6^4 = 5184\) linear equations (2526 distinct ones) with for 1296 unknowns. This is quite a heavy task even for a modern PC (note that the system is underdetermined and should be solved exactly).

Algorithm B (CountIndependentComponents) requires solving a total of \(4 \times 4^4 = 1024\) linear equations (504 distinct ones) for \(4^4 = 256\) unknowns among which 112 components turn out to be zero and another 124 turn out to be functions of 20 independent components: \(R_{2121}, R_{3121}, R_{3221}, R_{3231}, R_{3321}, R_{4121}, R_{4131}, R_{4141}, R_{4221}, R_{4231}, R_{4232}, R_{4241}, R_{4242}, R_{4243}, R_{4331}, R_{4332}, R_{4341}, R_{4342}, R_{4343}\). This list allows one to conclude that \(f(1) = 0, f(2) = 1, f(3) = 6, f(4) = 20\). These set of values allows one to compute the above-mentioned result for \(f(k)\) directly from (16).

Algorithm C (ListIndependentComponents) required solving of \(P(4) = 5\) systems of linear equations:

(1) a system of 4 equations (only one distinct equation) with 1 unknown (corresponding to \(Y_1^1 = (4, \ldots)\)),
(2) one system of 16 equations (9 distinct ones) with 4 unknowns (corresponding to \(Y_2^1 = (3, 1, \ldots)\)),
(3) a system of 24 equations (10 distinct ones) with 6 unknowns (corresponding to \(Y_2^2 = (2, 2, \ldots)\)),
(4) a system of 48 equations (24 distinct ones) with 12 unknowns (corresponding to \(Y_3^1 = (2, 1, 1, \ldots)\)),
(5) a system of 96 equations (44 distinct ones) with 24 unknowns (corresponding to \(Y_4^1 = (1, 1, 1, 1, \ldots)\)).

Clearly, algorithm C produces the same results for \(f(k)\) as the other two algorithms, but requires much less resources. This demonstrates the efficiency of algorithm C. It is planned to include algorithm C into the next release of EinS.

8 Concluding Remarks

Certainly, algorithm C can be further improved in certain cases if the structure of particular symmetries are taken into account. Up to now the algorithms does not account for any properties
which the symmetries may have. Here one can use the group-theoretic approach for manipulations with indexed objects developed in [14, 5, 6, 13]. Certain further improvement could be achieved if an algorithm to generate only distinct equations could be found. However, it is doubtful that such an algorithm would be computationally cheaper as the currently used algorithm to find and to drop equivalent equations before solving the system of linear equations (see, Section 6.4). It is interesting also to check if the results of Section 5 can be improved so that the incompatibility of symmetries (3) could be seen in an easier way. Another interesting question is whether, for arbitrary symmetries, one can express algorithmically the combinatorial “finger exercises” allowing one to derive \( f(2) = 1, f(3) = 6 \) and \( f(4) = 20 \) for the covariant Riemann tensor as given, for example, in Section 92 of [11].

The considered form (3) of the symmetries does not allow us to consider some important cases. For example, the definition of a symmetric trace-free (STF) tensor required that a contraction of \( A_{i_1 i_2 \ldots i_l} \) with the Kronecker symbol \( \delta_{ia} \) vanishes for any \( a \) and \( b \). Such a symmetry cannot be written in the form (3) and is out of the scope of this paper. On the other hand, STF tensors plays very important role in modern physics [1, 4, 10] and it is important to have efficient algorithms to store them and manipulate with them. It can be demonstrated that the main results of this paper can be also used for symmetries involving contractions with objects each component of which has some numerical value. That is, one can consider symmetries of the form

\[
\sum_{j=1}^{p} a_j \sum_{i_1, i_2, \ldots, i_m=1}^{k} B_{i_1 i_2 \ldots i_m} A^{\pi_j} = b,
\]

where \( a_j \) and \( b \) are numbers, \( i_1, i_2, \ldots, i_m \) are dummy indices over which the contraction is performed, \( B_{i_1 i_2 \ldots i_m} \) is a number for any values of its indices (this can be, e.g. the Kronecker \( \delta_{ij} \) or the fully antisymmetric Levi-Civita symbol \( \varepsilon_{ijk} \), or anything else), and \( \pi_j \) is an arbitrary permutation of \( n \) indices containing \( m \) dummy indices \( i_1, i_2, \ldots, i_m \) and \( n - m \) free ones. This case will be treated in a separate publication.

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