A DISCONTINUOUS GALERKIN METHOD BY PATCH RECONSTRUCTION FOR ELLIPTIC INTERFACE PROBLEM ON UNFITTED MESH

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Abstract. We propose a discontinuous Galerkin (DG) method to approximate the elliptic interface problem on unfitted mesh using a new approximation space. The approximation space is constructed by patch reconstruction with one degree of freedom per element. The optimal error estimates in both $L^2$ norm and DG energy norm are obtained, without the typical constraints for DG method on how the interface intersects to the elements in the mesh. Other than enjoying the advantages of DG method, our method may achieve even better efficiency than the conforming finite element method, as illustrated by numerical examples.

Keywords: Elliptic interface problems, Patch reconstructed, Discontinuous Galerkin method, Unfitted mesh.

1. Introduction

In the last decades, numerical method for the elliptic interface problem has attracted pervasive attention since the pioneering work of Peskin [32], for example, the immersed interface method [21, 20] by LeVeque and Li, Mayo’s method on irregular regions [30], the method in [40] with second-order accuracy in the $L^\infty$ norm. In the finite difference fold, we also refer to [28, 17, 18, 14, 11, 31] for some other interesting methods. Meanwhile, finite element (FE) method is also popular for solving the interface problem. Based on the geometrical relationship between the grid and the interface, FE methods could be classified into two categories: interface-fitted method and interface-unfitted method. The body-fitted grid enforces the mesh to align with the interface to render a high-order accurate approximation [12, 10]. However, generating a fitted mesh with satisfied quality is sometimes a nontrivial and time-consuming task [39, 56]. Therefore, there are some techniques for FE methods based on unfitted grid, too. The unfitted FE method can date back to the [5], which introduced a penalty term to weakly enforce the jump on the interface. Li proposed the immersed FE method in [25], which processes a better approximate solution by modifying the basis functions near interface to capture the jump of the solution. We refer to [3, 27, 2, 38, 10] for some recent works. Let us note that the extended FE method is also a popular discretization method [7].

In 2002, A. Hansbo and P. Hansbo proposed an unfitted FE method with the piecewise linear space and proved an optimal order of convergence [15]. The numerical solution comes from separate solutions defined on each subdomain and the jump conditions are imposed weakly by Nitsche’s method. Wadbro et al. [35] developed a uniformly well-conditioned FE method based on Nitsche’s method. Wu and Xiao [39] presented a $hp$ unfitted FE method, which is extended to the three dimensional case. To achieve high-order accuracy and enjoy additional flexibility, some authors tried to apply DG method to the elliptic interface problem, for example the local DG method in [15], the hybridizable DG method in [20] on fitted mesh, and the $hp$ DG method in [29] on unfitted mesh.
Though high-order accuracy can be obtained, solid difficulties remain for DG methods in solving problems with complex interfaces. To fit curved interfaces, Cangiani et al. [9] introduced elements with curved faces to give an adaptive DG method recently. As one of the latest work on unfitted mesh, Burman and Ern [8] proposed a hybrid high-order method, while an extra assumption on the meshes are required to ensure the mesh cells are cut favorably by the interface [29, 37]. In this paper, we are trying to propose a DG method on unfitted mesh for the interface problem still using Nitsche’s method. The novel point is that we adopt a new approximation space by patch reconstruction with one degree of freedom (DOF) per element following the methodology in [23, 22]. The new space may be regarded as a subspace of the approximation space used in [29]. Thanks to the flexibility in choosing reconstruction patches, we may allow the interface to intersect with elements in a very general manner, in comparison to the methods in [8, 29]. Following the standard DG discretization, the elliptic interface problem is approximated by using a symmetric interior penalty bilinear form with a Nitsche-type penalization at the interface. The optimal error estimate is then derived in both DG energy norm and $L^2$ norm. We note that classical DG methods for elliptic problems were challenged [19, 42] since it may use more DOFs than traditional conforming FE methods. As a new observation, we demonstrate by numerical examples that using our new approximation space, one needs much less DOFs than classical DG methods. For high-order approximations, number of DOFs can be even less than conforming FE methods to achieve the same accuracy.

The rest of this paper is organized as follows. In Section 2, we introduce the reconstruction operator and the new approximation space, and we also give the basic properties of the approximation space. In Section 3, the approximation to the elliptic interface problem is proposed and we derive the optimal error estimate in DG energy norm and $L^2$ norm. In Section 4, we present a lot of numerical examples to verify the error estimate in Section 3. To show the performance of our method in efficiency, we make a comparison of number of DOFs respect to numerical error between different methods. We also solve a problem that admits solutions with low regularities to illustrate the robustness of our method.

2. Approximation Space

Let $\Omega \subset \mathbb{R}^2$ be a convex and polygonal domain with boundary $\partial \Omega$ and let $\Gamma$ be a $C^2$-smooth interface which divides $\Omega$ into two open sets $\Omega_0$ and $\Omega_1$ satisfying $\Omega_0 \cap \Omega_1 = \emptyset$, $\overline{\Omega} = \overline{\Omega_0} \cup \overline{\Omega_1}$ and $\Gamma = \partial \Omega_0 \cap \partial \Omega_1$. We denote by $\mathcal{T}_h$ a partition of $\Omega$ into polygonal elements. Here we do not require the faces of elements in $\mathcal{T}_h$ align with the interface (see Fig. 1). Let $\mathcal{E}_h^\circ$ be the set of all interior faces of $\mathcal{T}_h$, $\mathcal{E}_h^b$ the

![Figure 1. Domain and unfitted mesh](image-url)
set of the faces on \( \partial \Omega \) and then \( \mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^b \). We set
\[
h_K = \text{diam}(K), \quad \forall K \in \mathcal{T}_h, \quad h_e = |e|, \quad \forall e \in \mathcal{E}_h,
\]
and we denote by \( h \) the biggest one among the diameters of all elements in \( \mathcal{T}_h \). We assume that \( \mathcal{T}_h \) is share-regular in the sense of satisfying the conditions introduced in [4], which are: there exist
- two positive numbers \( N \) and \( \sigma \) which are independent of mesh size \( h \);
- a compatible sub-decomposition \( \mathcal{T}_h \) into shape-regular triangles;

such that
- any polygon \( K \in \mathcal{T}_h \) admits a decomposition \( \mathcal{T}_{h|K} \) which has less than \( N \) shape-regular triangles;
- the share-regularity of \( \tilde{K} \in \mathcal{T}_h \) follows [4]: the ratio between \( h_{\tilde{K}} \) and \( \rho_{\tilde{K}} \) is bounded by \( \sigma \): \( h_{\tilde{K}}/\rho_{\tilde{K}} \leq \sigma \) where \( \rho_{\tilde{K}} \) is the radius of the largest ball inscribed in \( \tilde{K} \).

The above regularity requirements could bring some useful consequences which are trivial to verify [4]:

**M1** There exists a positive constant \( \rho_v \) such that \( \rho_v h_K \leq h_e \) for every element \( K \) and every edge \( e \) of \( K \).

**M2** There exists a positive constant \( \rho_s \) such that for every element \( K \) the following holds true
\[
\rho_s \max_{K \in \Delta(K)} h_{\tilde{K}} \leq h_K,
\]
where \( \Delta(K) = \{ K' \in \mathcal{T}_h \mid K' \cap K \neq \emptyset \} \) is the collection of the elements touching \( K \).

**M3** There exists a constant \( \tau \) such that for every element \( K \), there is a disk inscribed in \( K \) with center at a point \( z_K \in K \) and radius \( \tau h_K \).

**M4** [Trace inequality] There exists a constant \( C \) such that
\[
(1) \quad \|v\|_{L^2(\partial K)}^2 \leq C \left( h_{K}^{-1} \|v\|^2_{L^2(K)} + h_K \|\nabla v\|^2_{L^2(K)} \right), \quad \forall v \in H^1(K).
\]

**M5** [Inverse inequality] There exists a constant \( C \) such that
\[
(2) \quad \|\nabla v\|_{L^2(K)}^2 \leq C h_K^{-1} \|v\|_{L^2(K)}^2, \quad \forall v \in \mathbb{P}_m(K),
\]
where \( \mathbb{P}_m(\cdot) \) denotes the polynomial space of degree less than \( m \).

Let us note that throughout the paper, \( C \) and \( \bar{C} \) with a subscript are generic constants that may differ from line to line but are independent of the mesh size. We will use the following notations related to the partition.
\[
e^0 = e \cap \Omega_0, \quad e^1 = e \cap \Omega_1, \quad e \in \mathcal{E}_h, \\
K^0 = K \cap \Omega_0, \quad K^1 = K \cap \Omega_1, \quad K \in \mathcal{T}_h, \\
(\partial K)^0 = \partial K \cap \Omega_0, \quad (\partial K)^1 = \partial K \cap \Omega_1, \quad K \in \mathcal{T}_h, \\
\mathcal{T}_h^0 = \{ K \in \mathcal{T}_h \mid |K^0| > 0 \}, \quad \mathcal{T}_h^1 = \{ K \in \mathcal{T}_h \mid |K^1| > 0 \}, \\
\mathcal{E}_h^0 = \{ e \in \mathcal{E}_h \mid |e^0| > 0 \}, \quad \mathcal{E}_h^1 = \{ e \in \mathcal{E}_h \mid |e^1| > 0 \}.
\]

Furthermore, we denote by \( \mathcal{T}_h^\Gamma = \{ K \in \mathcal{T}_h \mid K \cap \Gamma \neq \emptyset \} \) the set of the elements that are divided by \( \Gamma \) and by \( \mathcal{E}_h^\Gamma = \{ e \in \mathcal{E}_h \mid e \cap \Gamma \neq \emptyset \} \) the set of the faces that are divided by \( \Gamma \). We set \( \mathcal{T}_h^\Gamma = \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma \) and \( \mathcal{E}_h^\Gamma = \mathcal{E}_h \setminus \mathcal{E}_h^\Gamma \). For element \( K \in \mathcal{T}_h^\Gamma \) we denote \( \Gamma_K = K \cap \Gamma \).

We make the following natural assumptions about the mesh, which are actually easy to be fulfilled.

**Assumption 1.** We assume that for each element \( K \in \mathcal{T}_h^\Gamma \) the interface \( \Gamma \) intersects its boundary \( \partial K \) twice and each open face at most once.
Assumption 2. For any element $K \in \mathcal{T}_h^T$, there exist two elements $K^0, K^1 \subset \Delta(K)$ such that $K^0 \subset \Omega^0$ and $K^1 \subset \Omega^1$.

For the given partition $\mathcal{T}_h$, we follow the idea in [23,22] to define a reconstruction operator for solving the elliptic interface problem. Firstly for every element $K \in \mathcal{T}_h$, we specify its barycenter $x_K$ as a sampling node. Secondly for each element $K \in \mathcal{T}_h \backslash \mathcal{T}_h^T (i = 0,1)$, we setup an element patch $S^i(K)$ in a recursive manner. Let $S^0_0(K) = \{K\}$, then we define $S^i_0(K)$ as

$$S^i_0(K) = \bigcup_{\tilde{K} \in \mathcal{T}_h^i, \tilde{K} \in s^i_{t-1}(K)} \tilde{K}, \quad t = 1, 2, \ldots$$

Once $S^i_0(K)$ has collected sufficiently large number of elements, we stop the procedure and let $S^i(K) = S^i_0(K)$. The cardinality of $S^i(K)$ is denoted by $\#S^i(K)$.

For any element $K \in \mathcal{T}_h^i$, we only construct one element patch which satisfies that if $K \in \mathcal{T}_h \backslash \mathcal{T}_h^i$, then $S^i(K) \subset \mathcal{T}_h^i$. For any element $K \in \mathcal{T}_h^i$, we assume that $K \in S^0(K^0)$ and $K \in S^1(K^1)$ where $K^0$ and $K^1$ are defined in Assumption 2. With $\#S^0(K^0)$ and $\#S^1(K^1)$ to be mildly greater than that in [23,22], the assumption can be fulfilled according to the method to build the element patch. Consequently, for each element $K \in \mathcal{T}_h^T$, we have two element patches $S^0(K) = S^0(K^0)$ and $S^1(K) = S^1(K^1)$. Some examples are presented in Appendix A to illustrate the construction of the element patch.

For any element $K \in \mathcal{T}_h$, we denote by $\mathcal{I}_K(i = 0,1)$ the set of sampling nodes located inside $S^i(K)$,

$$\mathcal{I}_K = \left\{ x_{\tilde{K}} \mid \forall \tilde{K} \in S^i(K) \right\}.$$  

For any function $g \in C^0(\Omega)$ and an element $K \in \mathcal{T}_h$, we seek a polynomial $\mathcal{R}_K^i g$ defined on $S^i(K)$ of degree $m$ by solving the following least squares problem:

$$\mathcal{R}_K^i g = \arg\min_{p \in \mathbb{P}_m(S^i(K))} \sum_{x \in \mathcal{I}_K} |p(x) - g(x)|^2. \quad (3)$$

The existence and uniqueness of the solution to (3) are decided by the position of the sampling nodes in $\mathcal{I}_K$. Here we follow [24] to make the following assumption:

Assumption 3. For any element $K \in \mathcal{T}_h$ and $p \in \mathbb{P}_m(S^i(K))$,

$$p|_{\mathcal{I}_K} = 0 \quad \text{implies} \quad p|_{S^i(K)} \equiv 0, \quad i = 0,1.$$  

This assumption actually rules out the situation that all the points in $\mathcal{I}_K$ are located on an algebraic curve of degree $m$. Definitely, this assumption requires the cardinality $\#S^i(K)$ shall be greater than $\dim(\mathbb{P}_m)$. Hereafter, we always require this assumption holds.
Since the solution to (3) is linearly dependent on \(g\), we define two interpolation operators \(R^i\) for:

\[
(R^0 g)|_K = (R^0_K g)|_K, \quad \text{for } K \in T^0_h,
\]
\[
(R^1 g)|_K = (R^1_K g)|_K, \quad \text{for } K \in T^1_h.
\]

Given \(R^i(i = 0, 1)\) and \(g \in C^0(\Omega)\), the function \(g\) is mapped to a piecewise polynomial function of degree \(m\) on \(T^i_h\). We denote by \(V^i_h\) the image of the operator \(R^i\). Define \(w^i_K(x) \in C^0(\Omega)\) that

\[
w^i_K(x) = \begin{cases} 1, & x = x_K, \\ 0, & x \in \bar{K}, \quad \bar{K} \neq K, \end{cases} \quad \forall K \in T^i_h.
\]

Then \(V^i_h = \text{span}\{\lambda^i_K | \lambda^i_K = R^i w^i_K\}\), and one can write the operator \(R^i\) in an explicit way

\[
R^i g = \sum_{K \in T^i_h} g(x_K) \lambda^i_K(x), \quad \forall g \in C^0(\Omega).
\]

In Appendix [5] we present a one dimensional example to show the formation of the basis functions \(\lambda^i_K\) and its implementation details.

The operators \(R^i(i = 0, 1)\) are defined for functions in \(C^0(\Omega)\), while we only concern the case for the functions in \(H^1(\Omega_0 \cup \Omega_1)(t \geq 2)\) to \(H^i(\Omega)\) based on the extension operator [1]. For any function \(w \in H^1(\Omega_0 \cup \Omega_1)\), there exist two extension operators \(E^i : H^i(\Omega) \to H^i(\Omega)\) such that \((E^i w)|_{\Omega_i} = w\) and

\[
\|E^i w\|_{H^s(\Omega)} \leq C\|w\|_{H^s(\Omega_1)}, \quad 0 \leq s \leq t.
\]

Let us study the approximation property of the operator \(R^i\). We define \(\Lambda(m, S^i(K))\) for all element patches as

\[
\Lambda(m, S^i(K)) = \max_{p \in \mathcal{P}_m(S^i(K))} \frac{\max_{x \in S^i(K)} |p(x)|}{\max_{x \in T^i_h} |p(x)|}
\]

We note that under some mild conditions on \(S^i(K)\), \(\Lambda(m, S^i(K))\) has a uniform upper bound \(\Lambda_m\), which is crucial in the convergence analysis. We refer to [21, 22] for the conditions and more discussion about the uniform upper bound. We point out that one of the conditions is the cardinality \(#S^i(K)\) should be greater than \(\dim(\mathcal{P}_m)\). In Section [4] we list the values of \(#S^i(K)\) for different \(m\) in all numerical examples.

With \(\Lambda_m\), we have the local approximation error estimates.

**Theorem 1.** Let \(g \in H^1(\Omega_0 \cup \Omega_1)(t \geq 2)\), there exist constants \(C\) such that for any \(K \in T^i_h(i = 0, 1)\) the following estimates hold true

\[
\|E^i g - R^i(E^i g)\|_{H^q(K)} \leq C\Lambda_m h_K^{s-q} \|E^i g\|_{H^s(S^i(K))}, \quad q = 0, 1,
\]
\[
\|D^q(E^i g - R^i(E^i g))\|_{L^2(\partial K)} \leq C\Lambda_m h_K^{s-q-1/2} \|E^i g\|_{H^s(S^i(K))}, \quad q = 0, 1,
\]

where \(s = \min(t + 1, m)\).

**Proof.** It is a direct consequence of [23, lemma 2.4] or [22, lemma 2.5]. \(\square\)

Finally, we give the definition of our approximation space \(V_h\) by concatenating the two spaces \(V^0_h\) and \(V^1_h\). Let us define a global interpolation operator \(R\): for any function \(w \in H^1(\Omega_0 \cup \Omega_1)\), \(R w\) is piecewise defined by

\[
(R w)|_K \triangleq \begin{cases} (R^0_K E^0 w)|_K, & \text{for } K \in T^0_h \setminus T^1_h, \\ (R^1_K E^1 w)|_K, & \text{for } K \in T^1_h \setminus T^0_h, \\ (R^1_K E^i w)|_K, & \text{for } K \in T^i_h, \quad i = 0, 1.
\]

The image of $\mathcal{R}$ is actually our new approximation space $V_h$. We notice that for any function $w \in H^1(\Omega_0 \cup \Omega_1)$, $\mathcal{R}w$ is a combination of $\mathcal{R}^0w$ and $\mathcal{R}^1w$ that $(\mathcal{R}w)|_{K_i} = (\mathcal{R}^i w)|_{K_i} (i = 0, 1)$, and the approximation error estimates of $\mathcal{R}$ are direct consequence from [5].

3. Approximation to Elliptic Interface Problem

We consider the standard elliptic interface problem: find $u$ in $H^2(\Omega_0 \cup \Omega_1)$ such that

$$
-\nabla \cdot \beta \nabla u = f, \quad x \in \Omega_0 \cup \Omega_1,
$$

$$
u = g, \quad x \in \partial \Omega,
$$

and (6) as below.

$$
[u] = an_r, \quad x \in \Gamma,
$$

$$
[\beta \nabla u \cdot n_r] = bn_r, \quad x \in \Gamma,
$$

where $\beta$ is a positive constant function on $\Omega_i (i = 0, 1)$ but may be discontinuous across the interface $\Gamma$, and $n_r$ denotes the outward unit normal to $\Gamma$. The source term $f$, the Dirichlet data $g$ and the jump term $a, b$ are assumed to be in $L^2(\Omega), H^{3/2}(\Omega), H^{3/2}(\Gamma), H^{1/2}(\Gamma)$, respectively, to ensure [6] has a unique solution. We refer to [33] for more details. In [6], the jump operator $[\cdot]$ takes the standard sense in DG framework. More precisely, we define the jump operator $[\cdot]$ and average operator $\{\cdot\}$ as below.

$$
[v] = \begin{cases}
v|_{K_+}n_{K_+} + v|_{K_-}n_{K_-}, & \text{on } e \in \mathcal{E}_h^1 \setminus \mathcal{E}_h^0, \\
v|_{K_+}n_{K_+} + v|_{K_-}n_{K_-}, & \text{on } e \in \mathcal{E}_h^1 \cap \Omega, \quad i = 0, 1, \\
v|_{K^-}, & \text{on } e \in \mathcal{E}_h^0, \\
(\{v\}|_{K^+} - \{v\}|_{K^-})n_r, & \text{on } \Gamma_K, \quad K \in \mathcal{T}_h^\Gamma. 
\end{cases}
$$

$$
\{v\} = \begin{cases}
\frac{1}{2}(v|_{K^+} + v|_{K^-}), & \text{on } e \in \mathcal{E}_h^0 \setminus \mathcal{E}_h^1, \\
\frac{1}{2}(v|_{K^+} + v|_{K^-}), & \text{on } e \in \mathcal{E}_h^1 \cap \Omega, \quad i = 0, 1, \\
v|_{K^-}, & \text{on } e \in \mathcal{E}_h^0, \\
\frac{1}{2}(v|_{K^+} + v|_{K^-}), & \text{on } \Gamma_K, \quad K \in \mathcal{T}_h^\Gamma.
\end{cases}
$$

Here $v$ is a scalar- or vector-valued function. For $e \in \mathcal{E}_h^0$, we denote two neighbouring elements $K_+$ and $K_-$ that share the common face $e$. $n_{K_+}$ and $n_{K_-}$ are the unit outer normal on $e$ corresponding to $\partial K_+$ and $\partial K_-$, respectively. In the case $e \in \mathcal{E}_h^1$, we let $e$ be a face of the element $K$.

Remark 1. To ensure the stability near the interface, some unfitted methods [29, 30] may require a weighted average $[v] = \kappa_0 v|_{\Omega_0} + \kappa_1 v|_{\Omega_1}$, where $\kappa_0$ and $\kappa_1$ are the cut-dependent parameters like $\kappa_i = |K_i|/|K| (i = 0, 1)$ for elements in $\mathcal{T}_h^\Gamma$. In our method, another advantage is just taking the arithmetic one could also guarantee the stability, which may avoid some complicated calculation near interface. Besides, the analysis can be adapted to their choices without any difficulty.

Now we define the bilinear $b_h(\cdot, \cdot)$ and the linear form $l_h(\cdot)$:

$$
b_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_{K \cap \Gamma^1} \beta \nabla u_h \cdot \nabla v_h \, dx
$$

$$
- \left[ \sum_{e \in \mathcal{E}_h^1 \cap \Omega} \int_{\partial e} + \sum_{K \in \mathcal{T}_h^\Gamma} \int_{\Gamma_K} \right] \left( [u_h] \cdot \{\beta \nabla v_h\} + [v_h] \cdot \{\beta \nabla u\} \right) ds
$$

$$
+ \sum_{e \in \mathcal{E}_h^0 \cap \Omega} \int_{\partial e} \eta [u_h] \cdot [v_h] ds + \sum_{K \in \mathcal{T}_h^\Gamma} \int_{\Gamma_K} \eta [u_h] \cdot [v_h] ds,
$$
for $\forall u_h, v_h \in W_h$, and

$$l_h(v_h) = \sum_{K \in T_h} \int_{K \cap \Omega} f v_h \, dx - \sum_{e \in E_h} \int_e q_n \cdot \{\beta \nabla v_h\} \, ds$$

$$+ \sum_{K \in \Gamma} \int_{\Gamma_K} b(v_h) \, ds - \sum_{K \in \Gamma} \int_{\Gamma_K} a n_K \cdot \{\beta \nabla v_h\} \, ds$$

$$+ \sum_{e \in E_h} \frac{\eta}{h_e} q v_h \, ds + \sum_{K \in \Gamma} \frac{\eta}{h_K} a n_K \cdot \{v_h\} \, ds,$$

for $\forall v_h \in W_h$, where $W_h$ denotes the broken Sobolev space

$$W_h = \left\{ v \in L^2(\Omega) \mid v|_K \in H^2(K), \text{ for } K \in T_h^\Gamma, \right\}$$

$$v|_K \in H^2(K^i), \quad i = 0, 1, \quad \text{for } K \in T_h^\Gamma \right\}.$$}

The penalty parameter $\eta$ is nonnegative and will be specified later on. Let us define the DG energy norm for $v_h \in W_h$:

$$\|v_h\|^2 = \|\nabla v_h\|^2_{L^2(\Omega_0 \cup \Omega_1)} + \|h_e^{-1/2}[v_h]\|^2_{L^2(E_h)} + \|h_K^{-1/2}\{\nabla v_h\}\|^2_{L^2(\Gamma)},$$

$$+ \|h_K^{-1/2}[v_h]\|^2_{L^2(\Gamma)} + \|h_K^{-1/2}\{\nabla v_h\}\|^2_{L^2(\Gamma)},$$

where

$$\|h_e^{-1/2}[v_h]\|^2_{L^2(E_h)} = \sum_{e \in E_h} \int_{e \cap \Omega_1} \frac{1}{h_e} \|[v_h]\|^2 \, ds,$$

$$\|h_K^{-1/2}\{\nabla v_h\}\|^2_{L^2(\Gamma)} = \sum_{K \in \Gamma_h^\Gamma} \int_{\Gamma_K} \frac{1}{h_K} \|\{\nabla v_h\}\|^2 \, ds,$$

for $\forall v_h \in W_h$. The approximation problem to the elliptic interface problem is then defined as:

find $u_h \in V_h$ such that

$$b_h(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h.$$ (8)

An immediate consequence from the definitions of the bilinear $b_h(\cdot, \cdot)$ and $l_h(\cdot)$ is the validity of the Galerkin orthogonality, which plays a key role in the error estimate later on.

**Lemma 1.** For the solution $u \in H^2(\Omega_0 \cup \Omega_1)$ of (6), the Galerkin orthogonality holds true:

$$b_h(u - u_h, v_h) = 0, \quad \forall v_h \in V_h.$$ (9)

We verify the boundedness and coercivity of the bilinear form $b_h(\cdot, \cdot)$ with respect to $\|\cdot\|$ at first. For this purpose, we need to estimate the error on the interface. Here we first give some discrete trace inequalities.

**Lemma 2.** For all $K \in \mathcal{T}_h^\Gamma$, there exists a constant $C$ such that

$$\|v_h\|^2_{L^2(\partial K^i)} \leq Ch_K^{-1/2} \|v_h\|_{L^2(K^i)}, \quad \forall v_h \in V_h, \quad i = 0, 1,$$

where $\partial K^i = (\partial K)^i \cup \Gamma_K$. 


Proof. For $K \in \mathcal{T}_h^+$, the patch $S^i(K')$ is the same as the patch $S^i(K'_0)$. From the
definition of the least squares problem (11), it is clear that the solution to (11) on $S^i(K')$ is the same as the solution to (11) on $S^i(K'_0)$. Particularly, $v_h|_{K'}$ and $v_h|_{K'_0}$ are exactly the same polynomial which is denoted as $\tilde{p}$. Based on M1 and M2, there exists a constant $\tilde{\tau}$ such that $B(z_{K'_0}, \tilde{\tau} h_K) \subset K'_0$, where $B(z, r)$ is a disk with center at $z$ and radius $r$. Owing to M2 and Assumption 2, there exists a constant $\tilde{\tau}$ such that $\partial K \subset B(z_{K'_0}, \tilde{\tau} h_K)$, and we observe that

$$\|\tilde{p}\|_{L^2(\partial K')} \leq |\partial K'|^{\frac{1}{2}} \|\tilde{p}\|_{L^\infty(\partial K')} \leq |\partial K'|^{\frac{1}{2}} \|\tilde{p}\|_{L^\infty(B(z_{K'_0}, \tilde{\tau} h_K))}$$

$$\leq C|\partial K'|^{\frac{1}{2}} |B(z_{K'_0}, \tilde{\tau} h_K)|^{-\frac{1}{2}} \|\tilde{p}\|_{L^2(B(z_{K'_0}, \tilde{\tau} h_K))}$$

$$\leq C|\partial K'|^{\frac{1}{2}} |B(z_{K'_0}, \tilde{\tau} h_K)|^{-\frac{1}{2}} \|\tilde{p}\|_{L^2(K'_0)}$$

$$\leq Ch_K^{-\frac{1}{2}} \|\tilde{p}\|_{L^2(K')}.$$  

The third inequality follows from the inverse inequality $\|\tilde{p}\|_{L^\infty(B(0,1))} \leq C\|\tilde{p}\|_{L^2(B(0,\tilde{\tau} \hat{\tau}))}$ for any $\tilde{\tau} \in \mathbb{P}_m(B(0,1))$ and the pullback using the bijective affine map from $B(z_{K'_0}, \tilde{\tau} h_K)$ to $B(0,1)$. As $\Gamma$ is of class $C^2$, it is easy to show (cf. [12]) $|\Gamma_K| \leq Ch_K$. We complete the proof by observing $|\partial K'| \leq h_K^{1/2}$.

Lemma 3. There exists a positive constant $h_0$ independent of $h$ and the location of the interface, such that for all $h \leq h_0$ and any element $K \in \mathcal{T}_h^+$, the following trace inequality holds:

$$\|w\|^2_{L^2(\Gamma_K)} \leq C \left( h_K^{-1} \|w\|^2_{L^2(K')} + h_K \|\nabla w\|^2_{L^2(K')} \right), \quad \forall w \in H^1(K).$$

See the proof of this lemma in [39, 16, 36].

Now we are ready to claim the continuity and coercivity of bilinear form $b_h(\cdot, \cdot)$.

Theorem 2. Let $b_h(\cdot, \cdot)$ be the bilinear form defined in (11) with sufficiently large $\eta$. Then there exists a positive constant $C$ such that

$$|b_h(u, v)| \leq C\|u\| \|v\|, \quad \forall u, v \in W_h,$$

$$b_h(v_h, v_k) \geq C\|v_k\|^2, \quad \forall v_h, v_k \in V_h.$$

Proof. The boundedness result (12) directly follows from the Cauchy-Schwartz inequality.

To obtain (13), we first define a weaker norm $\|\cdot\|_s$ which is a more natural one for analyzing coercivity

$$\|v_h\|^2_s = \|\nabla v_h\|^2_{L^2(\Omega_{h,0}\cup\Omega_1)} + \|h_e^{1/2}v_h\|^2_{L^2(\Omega_a)} + \|h_K^{1/2}v_h\|^2_{L^2(\Gamma)},$$

for $\forall v_h \in V_h$. From the trace estimate (10), (11) and the inverse inequality (12), we immediately obtain

$$\|h_e^{1/2}\nabla v_h\|^2_{L^2(\partial K')} \leq C \left( h_K^{-1} \|h_e^{1/2}\nabla v_h\|^2_{L^2(K')} + h_K \|h_e^{1/2}\nabla^2 v_h\|^2_{L^2(K')} \right)$$

$$\leq C\|\nabla v_h\|^2_{L^2(K')}, \quad \forall K \in \mathcal{T}_h^+, \quad i = 0, 1.$$

The above inequalities give us

$$\|h_K^{1/2}\{\nabla w_h\}|^2_{L^2(\Gamma')} + \|h_e^{1/2}\{\nabla w_h\}|^2_{L^2(\Omega_a)}$$

$$+ \|h_K^{1/2}\{\nabla w_h\}|^2_{L^2(\Gamma)} \leq C\|\nabla w_h\|^2_{L^2(\Omega_{h,0}\cup\Omega_1)},$$
which indicates $\|w_h\| \leq C\|w_h\|_*$ and the equivalence of $\|\cdot\|$ and $\|\cdot\|_*$ restricted on $V_h$.

Then using the Cauchy-Schwartz equality, for $K \in \mathcal{T}_h^\Gamma$ we obtain

$$-\int_{e} 2[v_h] \cdot \{\beta \nabla v_h\} ds \geq -\frac{1}{h_{e}} \|v_h\|^2 ds - \int_{e} h_{e} \|\{\beta \nabla v_h\}\|^2 ds,$$

for any face $e$ of $K$ and any $\varepsilon > 0$. Similarly, for $K \in \mathcal{T}_h^\Gamma$ and $i = 0, 1$ we have

$$-\int_{e'} 2[v_h] \cdot \{\beta \nabla v_h\} ds \geq -\frac{1}{h_{e'}} \|v_h\|^2 ds - \int_{e'} h_{e'} \|\{\beta \nabla v_h\}\|^2 ds,$$

for any face $e'$ of $K$. Further, for $\Gamma_K \in K$, on $K'(i = 0, 1)$ we have that

$$-\int_{\Gamma_K} 2[v_h] \cdot \{\beta \nabla v_h\} ds \geq -\frac{1}{h_{\Gamma_K}} \|v_h\|^2 ds - \int_{\Gamma_K} h_{\Gamma_K} \|\{\beta \nabla v_h\}\|^2 ds,$$

Combining all above inequalities, we conclude that there exist constant numbers $C_0$, $C_1$ and $C_2$ such that

$$b_h(v_h, v_h) - \|v_h\|_* \geq (1 - C_0\varepsilon)\|\nabla v_h\|_{L^2(\Omega_0 \cup \Omega_1)} + \|(\eta - \frac{C_1}{\varepsilon})h_{\varepsilon}^{-1/2}[v_h]\|_{L^2(\varepsilon_h)}^2
+ \|(\eta - \frac{C_2}{\varepsilon})h_{\varepsilon}^{-1/2}[v_h]\|_{L^2(\Gamma)}^2,$$

for any $\varepsilon > 0$. We let $\varepsilon = 1/C_0$ and select a sufficiently large $\eta$ to ensure $b_h(v_h, v_h) \geq C\|v_h\|_*$, which completes the proof.

Now let us give the approximation error in the DG energy norm $\|\cdot\|$.

**Lemma 4.** Let $u \in H^1(\Omega_0 \cup \Omega_1)$ with $t \geq 2$. There exists a constant $C > 0$ such that

$$\|u - R u\| \leq CA_m h^{s-1}\|u\|_{H^s(\Omega_0 \cup \Omega_1)},$$

where $s = \min(m + 1, t)$.

**Proof.** From (5), it is easy to see

$$\|\nabla(u - R u)\|_{L^2(\Omega_0 \cup \Omega_1)} \leq C A_m h^{s-1}\|u\|_{H^s(\Omega_0 \cup \Omega_1)}.$$

Then using trace inequality (1) and (5), for $K \in \mathcal{T}_h^\Gamma(i = 0, 1)$ we have

$$\|u - R u\|_{L^2(\partial K)} \leq \|E^i u - R(E^i u)\|_{L^2(\partial K)} \leq CA_m h_{K}^{s+1/2}\|E^i u\|_{L^2(S^i(K))},$$

$$\|\nabla (u - R u)\|_{L^2(\partial K')} \leq \|E^i u - R(E^i u)\|_{L^2(\partial K')} \leq CA_m h_{K}^{s+1/2}\|E^i u\|_{L^2(S^i(K))}.$$

From the above two inequalities and (14), we could conclude

$$\|h_{e}^{-1/2}[u - R u]\|_{L^2(\varepsilon_h)} + \|h_{e'}^{-1/2}[u - R u]\|_{L^2(\varepsilon_h')} \leq CA_m h^{s-1}\|u\|_{H^s(\Omega_0 \cup \Omega_1)},$$

$$\|h_{e}^{-1/2}[u - R u]\|_{L^2(\varepsilon_h)} + \|h_{e'}^{-1/2}[u - R u]\|_{L^2(\varepsilon_h')} \leq CA_m h^{s-1}\|u\|_{H^s(\Omega_0 \cup \Omega_1)}.$$
Finally we use (11) to bound the error on the interface. For element $K \in \mathcal{T}_h^\Gamma$, we obtain
\begin{align*}
\|h_K^{-1/2}[u - \mathcal{R}u]\|_{L^2(\mathcal{G}_K)} & \leq C \sum_{i=0,1} (h_K^{-1} \|E^i u - \mathcal{R}(E^i u)\|_{L^2(K)} \\
& \quad + h_K \|\nabla (E^i u - \mathcal{R}(E^i u))\|_{L^2(K)}) \\
& \leq C \Lambda_m h_K^{s-1} (\|E^0 u\|_{H^s(\Omega_0 \cup \Omega_1)} + \|E^1 u\|_{H^s(\Omega_1^3(\Omega_1))}).
\end{align*}
A summation over all $K \in \mathcal{T}_h^\Gamma$ gives us
\begin{align*}
\|h_K^{-1/2}[u - \mathcal{R}u]\|_{L^2(\mathcal{G})} & \leq C \Lambda_m h_K^{s-1}\|u\|_{H^{s}(\Omega_0 \cup \Omega_1)},
\end{align*}
Similarly, we could yield
\begin{align*}
\|h_K^{-1/2}[u - \mathcal{R}u]\|_{L^2(\mathcal{G})} & \leq C \Lambda_m h_K^{s-1}\|u\|_{H^{s}(\Omega_0 \cup \Omega_1)}.
\end{align*}
Combining all the inequalities above gives the error estimate (14), which completes the proof.

We are now ready to prove the priori error estimates.

**Theorem 3.** Let $u_h$ be the solution to (8) and let the exact solution $u$ belong to $H^t(\Omega_0 \cup \Omega_1)$ with $t \geq 2$, then the following error estimates hold:
\begin{align}
\|u - u_h\| & \leq C h^{s-1}\|u\|_{H^{s}(\Omega_0 \cup \Omega_1)}, \label{eq:15} \\
\|u - u_h\|_{L^2(\Omega)} & \leq C h^{s}\|u\|_{H^{s}(\Omega_0 \cup \Omega_1)}, \label{eq:16}
\end{align}
where $s = \min(m + 1, t)$.

**Proof.** Together with the Galerkin orthogonality (9), boundedness (12) and coercivity (13) of the bilinear form $b_h(\cdot, \cdot)$ we could have a bound of $\|u - u_h\|$
\begin{align*}
C_0\|u - u_h\| & \leq b_h(u - u_h, u - u_h) = b_h(u - \mathcal{R}u, u - u_h) \\
& \leq C_1\|u - \mathcal{R}u\| \|u - u_h\|, \\
\|u - u_h\| & \leq C_2\|u - \mathcal{R}u\|,
\end{align*}
Combining (14) immediately gives us the estimate (15).

Finally we obtain the optimal order in $L^2$ norm with the standard duality argument. Let $\phi \in H^2(\Omega)$ be the solution of
\begin{align*}
-\nabla \cdot \beta \nabla \phi = u - u_h, & \quad x \in \Omega_0 \cup \Omega_1, \\
\phi = 0, & \quad x \in \partial \Omega, \\
[\phi] = 0, & \quad x \in \Gamma, \\
[\beta \nabla \phi \cdot n] = 0, & \quad x \in \Gamma,
\end{align*}
and satisfy (5)
\begin{align*}
\|\phi\|_{H^2(\Omega_0 \cup \Omega_1)} & \leq C \|u - u_h\|_{L^2(\Omega)}.
\end{align*}
We denote by $\phi_I = \mathcal{R}\phi$ the interpolant of $\phi$. Then with the Galerkin orthogonality (6) we have
\begin{align*}
\|u - u_h\|_{L^2(\Omega)} & = b_h(\phi, u - u_h) - b_h(\phi - \phi_I, u - u_h) \\
& \leq \|\phi - \phi_I\| \|u - u_h\| \leq Ch^{s}\|\phi\|_{H^2(\Omega_0 \cup \Omega_1)} \|u\|_{H^{s}(\Omega_0 \cup \Omega_1)} \\
& \leq Ch^{s}\|u - u_h\|_{L^2(\Omega)} \|u\|_{H^{s}(\Omega_0 \cup \Omega_1)},
\end{align*}
The estimate (16) is obtained by eliminating $\|u - u_h\|_{L^2(\Omega)}$, which completes the proof. □
Table 1. \#S(K) for 1 ≤ m ≤ 3.

| m  | 1   | 2   | 3   |
|-----|-----|-----|-----|
| \#S(K) | 5   | 9   | 15  |

4. Numerical Experiments

In this section, we present some numerical results by solving some benchmark elliptic interface problems. For each case, the source term $f$, the Dirichlet boundary data $g$ and the jump term $a$, $b$ are given according to the solutions. We construct the spaces of order $1 ≤ m ≤ 3$ to solve each problem. For simplicity, we take the same \#S(K) for all problems and we list a group of reference values of \#S(K) for different $m$ in Table 1. A direct sparse solver is used to solve the resulted sparse linear system.

4.1. Example 1. We first consider the classical interface problem on the square domain $[-1,1] \times [-1,1]$ with a circular interface $r^2 = x^2 + y^2$ [21] (see Fig 3). The exact solution and coefficient are chosen to be

$$u(x,y) = \begin{cases} 
  x^2 + y^2, & r \leq 0.5, \\
  \frac{1}{4} \left( 1 - \frac{1}{m} - \frac{1}{b} \right) + \frac{1}{b} \left( \frac{c^2}{2} + r^2 \right), & r > 0.5, 
\end{cases}$$

$$\beta = \begin{cases} 
  2, & r \leq 0.5, \\
  b, & r > 0.5. 
\end{cases}$$

With $b = 10$, $u$ is continuous over $\Omega$. By using a series of quasi-uniform triangular meshes, the $L^2$ and DG energy norm of the error in the approximation to the exact solution with mesh size $h = 1/5, 1/10, \ldots, 1/80$ are reported in Fig 4. For each fixed $m$, we observe that the errors $\|u - u_h\|_{L^2(\Omega)}$ and $\|u - u_h\|$ converge to zero at the rate $O(h^{m+1})$ and $O(h^m)$ as the mesh is refined, respectively. Such convergence rates are consistent with the theoretical results.

4.2. Example 2. In this example, we consider the same interface and domain as in Example 1. The analytical solution $u(x,y)$ and the coefficient are defined in the same way as in Example 1. But we solve the interface problem based on a sequence of polygonal meshes as shown in Fig 5 which are generated by PolyMesher [34].
The numerically detected convergence orders are displayed in Fig. 6 for both error measurements. It is clear that the orders of convergence in $L^2$ and DG energy norms are $m + 1$ and $m$, respectively, which again are in agreement with the theoretical predicts.

For the Example 3 - 6, the computational domain is $[-1, 1] \times [-1, 1]$ and we solve the test problems on a sequence of triangular meshes with mesh size $h = 1/5, 1/10, \cdots, 1/80$.

4.3. Example 3. In this case, we consider the problem in [1] which contains the strongly discontinuous coefficient $\beta$ to test the robustness of the proposed method. We consider the elliptic problem with an ellipse interface (see Fig. 7)

$$
\left( \frac{x}{18/27} \right)^2 + \left( \frac{y}{10/27} \right)^2 = 1.
$$
The exact solution and the coefficient are given as

\[ u(x, y) = \begin{cases} 
5e^{-x^2-y^2}, & \text{outside } \Gamma, \\
e^x \cos(y), & \text{inside } \Gamma,
\end{cases} \]

\[ \beta = \begin{cases} 
1, & \text{outside } \Gamma, \\
1000, & \text{inside } \Gamma.
\end{cases} \]

There is a large jump in \( \beta \) across the interface \( \Gamma \), which may lead to an ill-

conditioned linear system. We still use the direct sparse solver to solve the resulting sparse linear system and our method shows the robustness for this case. As can be seen from Fig 8, the computed rates of convergence match with the theoretical analysis.
Figure 8. The convergence orders under $L^2$ norm (left) and DG energy norm (right) Example 3.

4.4. Example 4. In this example, we consider solving the elliptic problem with a kidney-shaped interface [20], which is governed by the following level set function

$$\phi(x, y) = \left( 2 \left( (x + 0.5)^2 + y \right) - x - 0.5 \right)^2 - \left( (x + 0.5)^2 + y^2 \right) + 0.1.$$ 

The boundary data and source term are derived from the exact solution and coefficient

$$u(x, y) = \begin{cases} 
0.1 \cos(1 - x^2 - y^2), & \text{outside } \Gamma, \\
\sin(2x^2 + y^2 + 2) + x, & \text{inside } \Gamma, 
\end{cases}$$

$$\beta = \begin{cases} 
10, & \text{outside } \Gamma, \\
1, & \text{inside } \Gamma. 
\end{cases}$$

We present numerical results in Fig. 10 and the predicted convergence rates for both norms are verified.

Figure 9. Triangulation for example 4 with mesh size $h = 0.2$ (left) and $h = 0.1$ (right).
4.5. Example 5. Next, we consider a standard test case with an interface consisting of both concave and convex curve segments [41]. The interface is parametrized with the polar angle $\theta$

$$r = \frac{1}{2} + \frac{\sin \theta}{7}.$$ 

The exact solution is selected to be

$$u(x, y) = \begin{cases} 
  e^{x^2+y^2}, & \text{inside } \Gamma, \\
  0.1(x^2 + y^2)^2 - 0.01\ln(2\sqrt{x^2 + y^2}), & \text{outside } \Gamma, 
\end{cases}$$

$$\beta = \begin{cases} 
  1, & \text{inside } \Gamma, \\
  10, & \text{outside } \Gamma. 
\end{cases}$$

The convergence of the numerical solutions is displayed in Fig. 12. Again we observe optimal rates of convergence for both norms as the mesh size is decreased.
4.6. Example 6. In this case, we investigate the performance of our proposed method when dealing with the problem with low regularities. The interface can be found in [18], which is governed by the following level set function

\[
\phi(x, y) = \begin{cases} 
  y - 2x, & x + y > 0, \\
  y + 0.5x, & x + y \leq 0.
\end{cases}
\]

We note that the interface is only Lipschitz continuous and it has a kink at (0, 0),

![Triangulation for Example 6 with mesh size h = 0.2 (left) and h = 0.1 (right).](image)

The analytical solution \( u(x, y) \) is given by

\[
\begin{align*}
  u(x, y) = \begin{cases} 
    8, & (x, y) \in \Omega_0, \\
    \sin(x + y), & (x, y) \in \Omega_1 \text{ and } x + y \leq 0, \\
    x + y, & (x, y) \in \Omega_1 \text{ and } x + y > 0.
  \end{cases}
\end{align*}
\]
We choose $\beta = 1$ over the domain $[-1,1] \times [-1,1]$. The solution $u(x,y)$ is $C^2$ continuous but not $C^3$ continuous across the line $x + y = 1$. The numerical errors in terms of $L^2$ norm and DG energy norm are gathered in Tab 2. It is observed that when $m = 1, 2$ the numerical solutions converge optimally with $m + 1$ order for $L^2$ norm and $m$ order for DG energy norm, which matches with the fact that the exact solution $u$ belongs to $H^3(\Omega_0 \cup \Omega_1)$. When $m = 3$ the computed orders of convergence in $\| \cdot \|_{L^2(\Omega)}$ and $\| \cdot \|$ are about 3.5 and 2.5, respectively. A possible explanation of the convergence orders can be traced to lack of $H^4$-regularity of the exact solution on the domain $\Omega_1$.

4.7. Efficiency comparison. Hughes et al. [19] point out that the number of unknowns of a discretized problem is a proper indicator for the efficiency of a numerical method. To show the efficiency in DOFs of our method, we make a comparison among the unfitted DG method [29], the unfitted penalty finite element method [39, 36] and our method by solving the elliptic interface problem. The first method adopts the standard discontinuous finite element space, and the second method employs the traditional continuous finite element space. The solution and the partition are taken from Example 1. In Fig 14, we plot the $L^2$ norm of the error of three methods against the number of degrees of freedom with $1 \leq m \leq 3$.

One see that for the low orders of approximation($m = 1$), the penalty FE method is the most efficient method. For $m = 2$, our method shows almost the same efficiency as the penalty FE method. For the high order accuracy($m = 3$), our method performs better than the other methods.

5. Conclusion

We proposed a new discontinuous Galerkin method for elliptic interface problem. The approximation space is constructed by solving the local least squares problem. We proved optimal convergence orders in both $L^2$ norm and DG energy norm. A

| order $m$ | $h$     | $L^2$ error | order | DG error | order |
|-----------|---------|-------------|-------|----------|-------|
|           | 2.00e-1 | 7.661e-3   | -     | 1.835e-1 | -     |
|           | 1.00e-1 | 2.515e-3   | 1.61  | 5.022e-2 | 1.00  |
| $m = 1$   | 5.00e-2 | 6.498e-4   | 1.95  | 2.445e-2 | 1.03  |
|           | 2.50e-2 | 1.653e-4   | 1.97  | 1.199e-2 | 1.02  |
|           | 1.25e-2 | 4.202e-5   | 1.98  | 1.156e-2 | 1.00  |
|           | 2.00e-1 | 4.727e-4   | -     | 9.283e-3 | -     |
|           | 1.00e-1 | 6.423e-5   | 2.85  | 2.393e-3 | 1.95  |
| $m = 2$   | 5.00e-2 | 7.249e-6   | 3.16  | 5.872e-4 | 2.01  |
|           | 2.50e-2 | 9.171e-7   | 2.98  | 1.505e-4 | 1.97  |
|           | 1.25e-2 | 1.126e-7   | 3.02  | 6.401e-5 | 2.02  |
|           | 2.00e-1 | 1.229e-4   | -     | 3.145e-3 | -     |
|           | 1.00e-1 | 1.126e-5   | 3.45  | 3.361e-4 | 2.41  |
| $m = 3$   | 5.00e-2 | 9.603e-7   | 3.55  | 5.721e-5 | 2.53  |
|           | 2.50e-2 | 8.249e-8   | 3.55  | 1.816e-5 | 2.55  |
|           | 1.25e-2 | 6.999e-9   | 3.56  | 3.108e-6 | 2.55  |

Table 2. The convergence orders under $L^2$ norm and DG energy norm Example 6.
series of numerical results confirm our theoretical results and exhibit the flexibility, robustness and efficiency of the proposed method.

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Appendix A. Construction of Element Patch

Here we present some examples of constructing element patches. We consider a circular interface. Let \( \Omega_1 \) be the domain inside the circle and \( \Omega_0 = \Omega \setminus \Omega_1 \). For element \( K \in \mathcal{T}_h^0 \setminus \mathcal{T}_h^1 \), the construction of \( S^0(K) \) is presented in Fig. 15. For element \( K \in \mathcal{T}_h^1 \setminus \mathcal{T}_h^2 \), the construction of \( S^1(K) \) is presented in Fig. 16.
The uniform grid on $[-1, 1]$. 

**Appendix B. 1D Example**

Here we present a one-dimensional example to illustrate our method. We consider the interval $\Omega = [-1, 1]$ which is divided into two parts $\Omega_0 = (-1, -0.2)$ and $\Omega_1 = (-0.2, 1)$. We partition $\Omega$ into 8 elements $\{K_1, K_2, \cdots, K_8\}$ with uniform spacing. $\{x_1, x_2, \cdots, x_8\}$ are the set of collocations where $x_i$ is the midpoint of the element $K_i$. Since $\mathcal{T}_h^\Gamma = \{K_4\}$, we construct element patches for elements in $\mathcal{T}_h^\Gamma$. The element patches could be constructed as

$$S^0(K_1) = \{K_1, K_2\}, \quad S^0(K_2) = \{K_2, K_3\}, \quad S^0(K_3) = \{K_2, K_3, K_4\},$$

$$S^1(K_5) = \{K_4, K_5, K_6\}, \quad S^1(K_6) = \{K_5, K_6, K_7\},$$

$$S^1(K_7) = \{K_6, K_7\}, \quad S^1(K_8) = \{K_7, K_8\}.$$  

Then for element $K_4$ it is clear that $K^0_4 = K_3$ and $K^1_4 = K_5$, and the element patches of $K_4$ are

$$S^0(K_4) = S^0(K_3) = \{K_2, K_3, K_4\},$$

$$S^1(K_4) = S^1(K_5) = \{K_4, K_5, K_6\}.$$  

Then we would solve the least squares problem on every patch. We take $S^0(K_3)$ for an example, for a continuous function $g$ and $m = 1$ the least squares problem is written as

$$\arg \min_{(a, b) \in \mathbb{R}} \sum_{i=2}^{4} |(ax_i + b) - g(x_i)|^2.$$  

It is easy to get the unique solution

$$(a, b)^T = (A^T A)^{-1} A^T q,$$

where

$$A = \begin{bmatrix} 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix}, \quad q = \begin{bmatrix} g(x_2) \\ g(x_3) \\ g(x_4) \end{bmatrix}.$$  

We note that the matrix $(A^T A)^{-1} A^T$ has no relationship to the function $g$ and contains all information of the basis functions on element $K_2$. Hence we store the matrix $(A^T A)^{-1} A^T$ for every element patch to represent the basis functions. It is in the same way when we deal with the high dimensional problem.

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