Global stability analysis of axisymmetric boundary layer over a circular cone

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Abstract. This paper presents linear biglobal stability analysis of axisymmetric boundary layer over a sharp circular cone with zero angle of attack. The base flow velocity profile is fully non-parallel and non-similar. Linearized Navier-Stokes (LNS) equations are derived for disturbance flow quantities using the standard procedure. The LNS equations are discretized using Chebyshev spectral collocation method. The governing equations along with boundary conditions form a general eigenvalues problem. The numerical solution of general eigenvalues problem is obtained using ARPACK subroutine, which uses Arnoldi iterative algorithm. The global temporal modes are computed for the range of Reynolds number and semi-cone angles (α) for the axisymmetric mode (N=0). The flow is found temporally and spatially stable for 1° semi-cone angle and the range of Reynolds numbers considered. However, flow becomes temporally unstable and spatially stable with the increase in semi-cone angle (α). The wave-like behaviour of the disturbances is found at small semi-cone angles (α).

1. Introduction

The laminar-turbulent transition of the boundary layer is important to study in aerodynamics and marine hydrodynamics applications i.e. submarines, aircrafts etc. While it is important to study the physics of the problem as it has practical applications in aerodynamics and marine. The growth of small disturbances is the first step towards the transition process and it is studied through linear stability analysis. The linear stability analysis of parallel flow is well understood from the Orr-Sommerfeld equation and its solutions [1]. Parallel flow assumption is not valid when the base flow is varying in wall normal and streamwise direction. Thus, the mean flow is non-parallel. The stability analysis of such flows is performed using global stability analysis [2].

In the present work, we study the stability analysis of an incompressible boundary layer over a circular cone. The incoming flow is parallel to the axis of the cone. A pressure gradient is generated in the streamwise direction due to the cone angle. The flow is developing in the streamwise direction and is non-parallel. In this case the transverse curvature and pressure gradient play a crucial role in the stability of boundary layer formed. The literature on the stability analysis of incompressible flow over a circular cone is very sparse. The objectives of
the present work are to understand the effect of transverse curvature and pressure gradient on the boundary layer stability.

2. Problem formulation
Linearized Navier-Stokes equations are derived for disturbance flow quantities in spherical coordinates. The equations are normalized by free stream velocity ($U_\infty$) and cone radius ($a$) at the inlet of the domain respectively. The angle of attack is zero and hence the base flow is axisymmetric. Base flow in the boundary layer over a circular cone is axisymmetric and developing in streamwise direction. The recent development in the global stability analysis has demonstrated that wave-like behaviour of the disturbance amplitudes is not only the solution [3]. The Reynolds number is defined based on cone radius at the inlet of domain and free stream velocity.

$$Re = \frac{U_\infty a}{\nu}$$  \hspace{1cm} (1)

We follow the standard procedure for stability analysis with three dimensional perturbation to mean flow. Disturbances are assumed to be in normal mode form with the amplitudes are varying in $r$ and $\theta$ direction.

$$U_r = U_r + u_r, U_\theta = U_\theta + u_\theta, P = P + p$$ \hspace{1cm} (2)

$$q(r, \theta, t) = \hat{q}(r, \theta)e^{i(N\phi-\omega t)}$$ \hspace{1cm} (3)

where,
- $r$ = radial coordinate
- $\theta$ = polar coordinate
- $\phi$ = azimuthal coordinate
- $\omega$ = frequency of waves
- $N$ = azimuthal wavenumber
- $a$ = cone radius at the inlet

The Linearized Navier-Stokes equations for disturbance flow quantities are obtained as,

$$U_r \frac{\partial u_r}{\partial r} + \frac{U_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{U_r}{r} \frac{\partial u_\theta}{\partial r} + \frac{U_\theta}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{2U_\theta}{r} u_\theta + \frac{\partial p}{\partial r} - \frac{1}{Re} \left[ \nabla^2 u_r - \frac{2}{r^2} u_r - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{2 cot \theta}{r^2} u_\theta \right] = i\omega u_r$$ \hspace{1cm} (4)

$$U_r \frac{\partial u_\theta}{\partial r} + \frac{U_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{U_r}{r} \frac{\partial u_r}{\partial r} + \frac{U_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{U_\theta}{r} + \frac{U_r}{r} + \frac{1}{Re} \left[ \nabla^2 u_\theta + \frac{2}{r^2} u_\theta - \frac{u_\theta}{r^2 sin^2 \theta} \right] = i\omega u_\theta$$ \hspace{1cm} (5)

$$\frac{\partial u_r}{\partial r} + \frac{2 u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta cot \theta}{r} = 0$$ \hspace{1cm} (6)

where,

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{cot \theta}{r^2} \frac{\partial}{\partial \theta}$$ \hspace{1cm} (7)

2.1. Boundary conditions
At the solid surface of cone no slip and no penetration conditions are applied to all velocity disturbance amplitudes. At the free stream, far away from a solid wall, it is expected to vanish all velocity and pressure disturbances. The boundary conditions for pressure do not exist physically at the wall. However, compatibility conditions derived from the LNS equations have collocated at the solid wall [2].

$$u_r(r, \theta_{min}) = 0, u_\theta(r, \theta_{min}) = 0,$$ \hspace{1cm} (8)

$$u_r(r, \theta_{max}) = 0, u_\theta(r, \theta_{max}) = 0, p(r, \theta_{max}) = 0$$ \hspace{1cm} (9)
Along with Arnoldi’s algorithm applied to eigenvalues problem become easy. The major part of the spectrum using shift and invert strategy. The computations of Krylov subspace provide the possibility of extracting few selected eigenvalues. The Krylov subspace is always nearby the shift parameter. The good approximation of shift parameter reduces the number of iterations to converge the solution independent from the shift parameter. The convergence of the solution depends on the value of shift parameter. A good approximation of the shift value needs less number of iterations. However, we have taken a maximum number of iterations equal to 300. Hence the convergence of the solution may not be affected by shift parameter. Given the large subspace size of \( k = 250 \), the part of the spectrum for our instability analysis could be recovered in the single computation that took about 4 hours on Intel Xeon(R) CPU E5 26500@2.00GHz × 18.

\[
\frac{\partial p}{\partial r} = \frac{1}{Re} [\nabla^2 u_r - \frac{2}{r^2} \partial u_\theta] - U_r \frac{\partial u_r}{\partial r} - \frac{U_\theta \partial u_r}{r} \frac{\partial \theta}{\partial \theta} \quad (10)
\]

\[
\frac{1}{r} \frac{\partial p}{\partial \theta} = \frac{1}{Re} [\nabla^2 u_\theta + \frac{2}{r^2} \partial u_r] - U_r \frac{\partial u_\theta}{\partial r} - \frac{U_\theta \partial u_\theta}{r} \frac{\partial \theta}{\partial \theta} \quad (11)
\]

At inlet and outlet linearly extrapolated boundary conditions are applied. The disturbances are defined in the form of unknown values of the disturbances within the interior of the domain itself [3, 4].

\[
u_r [r_3 - r_2] - u_{r_2} [r_4 - r_1] + u_{r_3} [r_2 - r_1] = 0 \quad (12)
\]

\[
u_{r_{n-2}} [r_n - r_{n-1}] - u_{r_{n-1}} [r_{n} - r_{n-2}] + u_{r_n} [r_{n-1} - r_{n-2}] = 0 \quad (13)
\]

LNS equations are discretized using Chebyshev spectral collocation method (\( n \) points in the streamwise direction and \( m \) points in the wall normal direction). The clustering of collocation points is employed in wall normal direction near the wall to improve resolution and hence to capture real flow dynamics [5]. In the streamwise direction grid mapping is applied to remove the Gibbs oscillation [6]. The governing equations together with the boundary conditions forms a generalized eigenvalue problem, of the form,

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
u_r \\
u_\theta \\
p
\end{bmatrix}
= i \omega
\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix}
\begin{bmatrix}
u_r \\
u_\theta \\
p
\end{bmatrix}
\quad (14)
\]

where the matrices \( A \) and \( B \) are real and of size \( 3 \times n \times m \), \( i \omega \) is an eigenvalues and \( \phi \) is eigenvector.

### 2.2. Solution of general eigenvalues problem

The eigenvalues problem described above is very large to solve for all the eigenvalues. However, flow becomes unstable due to very few number of eigenmodes with its largest imaginary part. Hence, we considered the few number of dominant eigenvalues and associated eigenvectors only. The QZ algorithm computes the full spectrum of eigenvalues and hence it requires large computational efforts. The iterative method based on Arnoldi’s algorithm is a proper choice to compute few selected eigenvalues. The Krylov subspace provides the possibility of extracting major part of the spectrum using shift and invert strategy. The computations of Krylov subspace along with Arnoldi’s algorithm applied to eigenvalues problem become easy.

\[(A - \lambda B)^{-1} B \phi = \mu \phi \quad \text{where,} \quad \mu = \frac{1}{i \omega - \lambda} \quad (16)\]

Where, \( \lambda \) being the shift parameter and \( \mu \) is the eigenvalues of the converted problem. Sometimes it is also called spectral transformation, which converts generalized eigenvalues problem to a standard eigenvalues problem. The Krylov subspace may be computed by successive resolution of a linear system with matrix \((A-\lambda B)\), using LU decomposition. Full spectrum method is employed for this small subspace to get a good approximate solution to the original general eigenvalues problem [7]. The major part of the spectrum can be extracted with the large size of Krylov subspace. The computed spectrum is always nearby the shift parameter. The good approximation of shift parameter reduces the number of iterations to converge the solution to required accuracy level. However, the larger size of subspace makes the solution almost independent from the shift parameter \( \lambda \). We tested the code for several values of shift parameter. The convergence of the solution depends on the value of shift parameter. A good approximation of the shift value needs less number of iterations. However, we have taken a maximum number of iterations equal to 300. Hence the convergence of the solution may not be affected by shift parameter. Given the large subspace size of \( k = 250 \), the part of the spectrum for our instability analysis could be recovered in the single computation that took about 4 hours on Intel Xeon(R) CPU E5 26500@2.00GHz × 18.
3. Results and discussions
In the present analysis, Reynolds number is varying from 174 to 1047 and semi-cone angle from 1° to 4° for axisymmetric mode (N=0). The stream wise length is normalized by cone radius at the inlet of the domain. The domain length in wall normal direction is taken as $L_\theta = 12^\circ$. The number of collocation points taken in radial and polar directions are $n = 121$ and $m = 121$ respectively. The general eigenvalues problem is solved using Arnoldi’s iterative algorithm. Eigenvalues with largest imaginary part is selected. It has been also checked for spurious mode. The sponging is applied with 90 percentage sponge strength over 20 percentage domain to prevent spurious reflection at the outlet.

Figure 1 shows the eigenspectrum for semi-cone angle $\alpha = 1^\circ$, $Re = 174$ and azimuthal wave number $N=0$. The most unstable oscillatory mode has an eigenvalues, $\omega = 0.0869 - 0.0045i$ and it is temporally stable because $\omega_i < 0$. The computed eigenmodes (all are not shown here) in the range of Reynolds number and $\alpha = 1^\circ$ are temporally stable because all eigenmode have largest $\omega_i < 0$. Figure 2 and 3 shows the global mode structure for radial ($u_r$) and polar ($u_\theta$) disturbances for eigenmode $\omega = 0.0869 - 0.0045i$, as marked by square in Figure 1. The magnitude of the $u_r$ disturbance amplitudes is one order higher than that of $u_\theta$ disturbances. The spatial structure of the disturbances is also elongated in spatial directions. The radial and polar disturbance amplitudes show positive and negative wavelets in the streamwise direction. This proves wave like nature of the disturbance amplitudes in the stream wise direction for $Re = 174$. Figure 4 shows variation of $u_r$ in streamwise direction for $3^\circ$ semi-cone angle. It shows that amplitudes of disturbances decay in streamwise direction towards the downstream. Increase in semi-cone angle makes the flow spatially stable.
Figure 5 shows the eigenspectrum for semi-cone angle $\alpha = 2^\circ$ and $Re=872$. The most unstable oscillatory mode has an eigenvalue $\omega = 0.1460 + 0.0012i$. The largest $\omega_i > 0$, proves that the flow is temporally unstable. Figure 6 and 7 shows the spatial structure for radial ($u_r$) and polar ($u_\theta$) disturbances for most unstable temporal mode as marked by square in figure 5. Figure 6 and 7 shows that the region of contamination is comparatively less for $u_r$ and $u_\theta$ for higher semi-cone angle ($\alpha = 2^\circ$) and higher Reynolds number ($Re=872$).

Figure 9 shows the normalized spatial amplification rate ($A_x$) in the streamwise direction for various Reynolds numbers for $1^\circ$ semi-cone angle. $A_x$ shows overall effect of all the disturbances together in the streamwise direction towards the down stream [8]. The $A_x$ reduces initially...
and then gradually increases for all Reynolds number in the streamwise direction towards the downstream. $A_x$ also increases with the increase in Reynolds number. However, flow is spatially stable for all the Reynolds number as $A_x < 1$. The decay rate reduces with the increase in Reynolds number. The computations for stability analysis are performed for semi-cone angles $\alpha=1^\circ,2^\circ, 3^\circ$ and $4^\circ$. It has been observed that with the increase in semi-cone angle the flow becomes temporally unstable and wave-like behaviour disappears. Figure 10, 11 and 12 shows $A_x$ for semi-cone angle $\alpha=2^\circ, 3^\circ$ and $4^\circ$ for various Reynolds number. Spatial amplification rate reduces with increase in semi-cone angles at given Reynolds number. The flow is spatially stable for the range of Reynolds number for $\alpha=1^\circ,2^\circ,3^\circ$ and $4^\circ$ for all Reynolds number.

The transverse curvature and favourable pressure gradient have overall damping effect on the disturbances [9]. Effect of pressure gradient is comparatively less for small semi-cone ($1^\circ$) angles. $A_x$ initially reduces and then increases in streamwise direction for $\alpha=1^\circ$ due to decrease in transverse curvature. However, $A_x$ reduces in the streamwise direction for $\alpha=2^\circ$ and higher values of $\alpha$ due to the development of larger favourable pressure gradient.

4. Conclusions

Global stability analysis of an incompressible axisymmetric boundary layer over a circular cone is performed. The global temporal modes are computed for different semi-cone angles for the axisymmetric mode. The flow is found temporally stable for small semi-cone angle ($\alpha=1^\circ$), however with the increase in $\alpha$, $\omega_i$ also increases and makes the flow temporally unstable. Spatial amplification rate of the modes are computed. Spatial amplification rate reduces with the increase in semi-cone angle $\alpha$ from $1^\circ$ to $4^\circ$, which makes the flow spatially stable. Overall, the disturbances are found spatially stable in the streamwise direction. This is due to the transverse curvature and favourable pressure gradient present in the boundary layer. The flow is spatially stable for the range of Reynolds number and semi-cone angles $\alpha$ considered. The wave-like nature of the disturbances observed in the streamwise direction for small semi-cone angles and it disappears as the semi-cone angle($\alpha$) increases.

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