The Multistage Homotopy Perturbation method for solving Hyperchaotic Chen system

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Abstract. Finding accurate and efficient methods for solving nonlinear hyper chaotic problems has long been an active research undertaking. Like the well-known Adomian decomposition method (ADM) and based on observations, it is hopeless to find HPM solutions of hyper chaotic systems valid globally in time. To overcome this shortcoming, we employ the multistage homotopy-perturbation method (MHPM) to the nonlinear hyperchaotic Chen system. Based on the cases investigated, MHPM is more stable for a longer time span than the standard HPM. Comparisons with the standard HPM and the well-known purely numerical fourth-order Runge-Kutta method (RK4) suggest that the MHPM is a powerful alternative for nonlinear hyper chaotic system. The new algorithm and the new technique for choosing the initial approximations were shown to yield rapidly convergent series solutions.

1. Introduction

Nonlinear problems modelled by differential equations play a prominent role in various fields such as physics, chemistry, biology, mathematics, engineering and other disciplines. There are few phenomena in different fields of science occurring linearly. Most problems are essentially nonlinear and are described by nonlinear differential equations. If the considered problem is linear, the corresponding set of differential equations is also linear and can be solved without mathematical difficulties; if the researched problem is nonlinear, the obtained set of differential equations is generally nonlinear. Up to now, there are few effective numerical methods to solve this kind of nonlinear differential equations with a large number of unknowns and strong nonlinearity. Generally, iterative techniques are used to solve these nonlinear problems; however, the convergence of iterations seems sometimes just luck, especially when the nonlinearity of the considered problems is strong. Chaos is very interesting nonlinear phenomenon and has been intensively studied in the last three decades. The dynamical systems that exhibit chaotic behavior are sensitive to initial conditions. Chaotic behavior can be found in a variety of systems such as electrical circuits, lasers, fluid dynamics, mechanical devices, time evolution of the magnetic field of celestial bodies, population growth in ecology, the dynamics of
molecular vibrations, and not forgetting the weather [1,2]. In this paper, we consider the hyperchaotic Chen system [3] defined as
\[
\begin{align*}
\dot{x} &= a(y - x) + w, \\
\dot{y} &= dx - xz + cy, \\
\dot{z} &= xy - bz, \\
\dot{w} &= yz + rw.
\end{align*}
\] (1)

The above system is hyperchaotic for the parameters of \( a = 35, b = 3, c = 12, d = 7, \) and \( r = 0.5. \)

Hyperchaotic system was first discovered by O.E. Rossler in 1979 [4]. Details of other hyperchaotic systems can be found in many references [5-9, 36-37].

The Homotopy technique has been applied in many nonlinear problems in engineering field for the reason that homotopy method deforms continuously from a difficult into a simpler problem which can be easily solved. Ji-huan He, founder of Homotopy Perturbation method in 1998 [10, 11], in his paper showed that the Homotopy Perturbation Method is more powerful than Homotopy Analysis Method by Liao which is can be categorized into a generalized Taylor expansion method [12]. HPM was further developed and improved by He [13-15]. Changbum and Rathinasamy [16] solved the linear and nonlinear two-point boundary value problems using HPM and compared the result with extended Adomian Decomposition Method (EADM) and the shooting method. As a result, HPM yields more accurate results with rapid convergence than other methods. Recently, Chowdhury et al. [17], Chowdhury and Hashim [18], Hashim and Chowdhury [19] and Hashim et al. [20] were the first to successfully apply the multistage homotopy-perturbation method (MHPM) to the chaotic Lorenz system, chaotic Chen system and a class of ODEs. Other successfully solved problems by HPM can be found in references [21-24].

Inspired from the previous works done by others, in this present paper, we are interested to solve hyperchaotic Chen systems using multistage homotopy perturbation method (MHPM). Based on the past researches, chaotic Lorenz system has been solved using Adomian decomposition method (ADM) [25-27] and differential quadrature method (DQM) [28]. Fractional Lorenz system also has been solved using ADM [29] and homotopy analysis method (HAM) [30]. Meanwhile for chaotic Chen system, a lot of work has been done to find its solution using numerical method such as homotopy perturbation method (HPM) [31], variational iteration method (VIM) [32], solutions by HAM [33] and also by ADM [27]. Differential transformation method (DTM) is used by Alomari and Mossa et al. for the fractional Chen system [34, 35].
2. The Solution Procedure

Consider a general system of first-order ODEs:

\[
\begin{align*}
\frac{du_1}{dt} + g_1(t,u_1,u_2,\ldots,u_m) &= f_1(t), \\
\frac{du_2}{dt} + g_2(t,u_1,u_2,\ldots,u_m) &= f_2(t), \\
&\vdots \\
\frac{du_m}{dt} + g_m(t,u_1,u_2,\ldots,u_m) &= f_m(t),
\end{align*}
\]

subject to the initial conditions

\[
\begin{align*}
u_1(t_0) &= c_1, \quad u_2(t_0) = c_2, \quad \ldots, \quad u_m(t_0) = c_m.
\end{align*}
\]

First we write the system (3) in operator form

\[
\begin{align*}
L(u_1) + N_1(u_1,u_2,\ldots,u_m) - f_1 &= 0, \\
L(u_2) + N_2(u_1,u_2,\ldots,u_m) - f_2 &= 0, \\
&\vdots \\
L(u_m) + N_m(u_1,u_2,\ldots,u_m) - f_m &= 0.
\end{align*}
\]

Subject to the initial conditions (4), where \( L = \frac{d}{dt} \) is a linear operator and \( N_1, N_2, \ldots, N_m \) are the nonlinear operators. We will next present the solution approaches for (4) based on standard HPM and MHPM separately.

2.1 Solution by HPM

According to HPM, we construct the homotopy for (4) which satisfies the following relations:

\[
\begin{align*}
L(u_1) - L(v_1) + pL(v_1) + p[N_1(u_1,u_2,\ldots,u_m) - f_1] &= 0, \\
L(u_2) - L(v_2) + pL(v_2) + p[N_2(u_1,u_2,\ldots,u_m) - f_2] &= 0, \\
&\vdots \\
L(u_m) - L(v_m) + pL(v_m) + p[N_m(u_1,u_2,\ldots,u_m) - f_m] &= 0.
\end{align*}
\]

Where \( p \in [0,1] \) is an embedding parameter and \( v_1,v_2,\ldots,v_m \) are initial approximations satisfying the given conditions. It is obvious that when the perturbation parameter \( p = 0 \), Eqs. (5) become a linear system and when \( p = 1 \) we get the original nonlinear system. Let take the approximations as follows:

\[
\begin{align*}
u_1(t) &= u_{1,0}(t) + pu_{1,1}(t) + p^2u_{1,2}(t) + p^3u_{1,3}(t) + \ldots, \\
u_2(t) &= u_{2,0}(t) + pu_{2,1}(t) + p^2u_{2,2}(t) + p^3u_{2,3}(t) + \ldots, \\
&\vdots \\
u_m(t) &= u_{m,0}(t) + pu_{m,1}(t) + p^2u_{m,2}(t) + p^3u_{m,3}(t) + \ldots,
\end{align*}
\]

and

\[
\begin{align*}
u_{1,0}(t) &= v_1(t) = u_1(t_0) = c_1, \\
u_{2,0}(t) &= v_2(t) = u_2(t_0) = c_2, \\
&\vdots \\
u_{m,0}(t) &= v_m(t) = u_m(t_0) = c_m,
\end{align*}
\]

where \( u_{i,j}, \quad (i = 1,2,\ldots,m; \quad j = 1,2,\ldots) \) are functions yet to be determined. Substituting (6)-(7) into (5) and arranging the coefficient of same powers of \( p \), we get
\[ L(u_{1,1}) + L(v_1) + N_1(u_{1,0}, u_{2,0}, \ldots, u_{m,0}) - f_1 = 0, \]
\[ u_{1,1}(t_0) = 0, \]
\[ L(u_{2,1}) + L(v_2) + N_2(u_{1,0}, u_{2,0}, \ldots, u_{m,0}) - f_2 = 0, \]
\[ u_{2,1}(t_0) = 0, \]
\[ \vdots \]
\[ L(u_{m,1}) + L(v_m) + N_m(u_{1,0}, u_{2,0}, \ldots, u_{m,0}) - f_m = 0, \]
\[ u_{m,1}(t_0) = 0, \]
\[ L(u_{1,2}) + N_1(u_{1,1}, u_{2,1}, \ldots, u_{m,1}) = 0, \]
\[ u_{1,2}(t_0) = 0, \]
\[ L(u_{2,2}) + N_2(u_{1,1}, u_{2,1}, \ldots, u_{m,1}) = 0, \]
\[ u_{2,2}(t_0) = 0, \]
\[ \vdots \]
\[ L(u_{m,2}) + N_m(u_{1,1}, u_{2,1}, \ldots, u_{m,1}) = 0, \]
\[ u_{m,2}(t_0) = 0, \]

etc. solve the above systems of equations for the unknown \( u_{i,j} (i = 1, 2, \ldots, m; j = 1, 2, \ldots) \) by applying the inverse linear operator

\[ L^{-1}(\cdot) = \int_{t_i}^{t} \cdot \, dt. \] (9)

Therefore, according to HPM the n-term approximations for the solutions of (4) can be expressed as

\[ \phi_{1,i}(t) = u_i(t) = \lim_{p \to 1} u_i(t) = \sum_{k=0}^{n-1} u_{1,k}(t), \]
\[ \phi_{2,i}(t) = u_2(t) = \lim_{p \to 1} u_2(t) = \sum_{k=0}^{n-1} u_{2,k}(t), \] (10)
\[ \vdots \]
\[ \phi_{m,i}(t) = u_m(t) = \lim_{p \to 1} u_m(t) = \sum_{k=0}^{n-1} u_{m,k}(t). \]

2.2 Solution by MHPM
The solution by HPM is not valid for large \( t \). A simple way of ensuring validity of the approximations for large \( t \) is to treat the algorithm of HPM in a sequence of intervals choosing the initial approximations as

\[ u_{1,0}(t) = v_1(t) = u_1(t^*) = c_1^w, \]
\[ u_{2,0}(t) = v_2(t) = u_2(t^*) = c_2^w, \] (11)
\[ \vdots \]
\[ u_{m,0}(t) = v_m(t) = u_m(t^*) = c_m^w, \]

where \( t^* \) is the left-end point of each subinterval. Then solve the (8) for the unknowns \( u_{i,j} (i = 1, 2, \ldots, m; j = 1, 2, \ldots) \) by applying the inverse linear operator
\[ L^{-1}(\cdot) = \int_{t^*}^{-} (\cdot) \, dt. \]  

(12)

In order to carry the iterations in every subinterval of equal length \( \Delta t, [t_i, t_{i+1}) \), we need to know the values of

\[ u_{1,0}^*(t) = u_1(t^*), \quad u_{2,0}^*(t) = u_2(t^*), \ldots, \quad u_{m,0}^*(t) = u_m(t^*). \]  

(13)

But in general, we do not have this information on our clearance except at the initial point \( t^* = t_0 \). A simple way for obtaining necessary values could be my means of the previous \( n \)-terms of approximations \( \phi_{1,n}, \phi_{2,n}, \ldots, \phi_{m,n} \) of the preceding subintervals given by Eqs. (11), i.e.

\[ u_{1,0}^* \approx \phi_{1,n}(t^*), \quad u_{2,0}^* \approx \phi_{2,n}(t^*), \ldots, \quad u_{m,0}^* \approx \phi_{m,n}(t^*). \]  

(14)

3. Application

In this section, we will study the hyperchaotic Chen system (1) subject to the initial conditions:

\[ x(t_0) = c_1, \quad y(t_0) = c_2, \quad z(t_0) = c_3, \quad w(t_0) = c_4. \]  

(15)

According to HPM, we can construct a homotopy which satisfies the following relations

\[ \dot{v}_1 - \dot{x}_0 + p(\dot{x}_0 - av_2 + av_1 - v_2) = 0, \]
\[ \dot{v}_2 - \dot{y}_0 + p(\dot{y}_0 + v_1v_3 - cv_2) = 0, \]
\[ \dot{v}_3 - \dot{z}_0 + p(\dot{z}_0 - v_1v_2 + bv_3) = 0, \]
\[ \dot{v}_4 - \dot{w}_0 + p(w_0 - v_1v_3 - rv_4) = 0. \]  

(16)

We take the initial approximations as

\[ v_{1,0}(t) = x_0(t) = x(t^*) = c_1, \]
\[ v_{2,0}(t) = y_0(t) = y(t^*) = c_2, \]
\[ v_{3,0}(t) = z_0(t) = z(t^*) = c_3, \]
\[ v_{4,0}(t) = w_0(t) = w(t^*) = c_4. \]  

(17)

and

\[ \dot{v}_1(t) = v_{1,0}(t) + pv_{1,1}(t) + p^2v_{1,2}(t) + p^3v_{1,3}(t) + \cdots, \]
\[ \dot{v}_2(t) = v_{2,0}(t) + pv_{2,1}(t) + p^2v_{2,2}(t) + p^3v_{2,3}(t) + \cdots, \]
\[ \dot{v}_3(t) = v_{3,0}(t) + pv_{3,1}(t) + p^2v_{3,2}(t) + p^3v_{3,3}(t) + \cdots, \]
\[ \dot{v}_4(t) = v_{4,0}(t) + pv_{4,1}(t) + p^2v_{4,2}(t) + p^3v_{4,3}(t) + \cdots. \]  

(18)

where \( u_{i,j}(t), i = j = 1, 2, 3, \ldots \) are functions yet to be determined. Substitute (17)-(18) into (16) and arrange the terms with the same powers of \( p \), we have:

\[ \dot{v}_{1,1} - av_{2,0} + av_{1,0} - v_{4,0} = 0, \]
\[ \dot{v}_{1,2} - av_{2,1} + av_{1,1} - v_{4,1} = 0, \]
\[ \dot{v}_{1,3} - av_{2,2} + av_{1,2} - v_{4,2} = 0. \]
\[ \dot{v}_{2,1} - d v_{1,0} + v_{1,0} v_{3,0} - c v_{2,0} = 0, \]
\[ \dot{v}_{2,2} - d v_{1,1} + v_{1,0} v_{3,1} + v_{1,1} v_{3,0} - c v_{2,1} = 0, \]
\[ \dot{v}_{2,3} - d v_{1,2} + v_{1,0} v_{3,2} + v_{1,1} v_{3,1} + v_{1,2} v_{3,0} - c v_{2,2} = 0, \]
\[ \dot{v}_{3,1} - v_{1,0} v_{2,0} + b v_{3,0} = 0, \]
\[ \dot{v}_{3,2} - v_{1,0} v_{2,1} - v_{1,1} v_{2,0} + b v_{3,1} = 0, \]
\[ \dot{v}_{3,3} - v_{1,0} v_{2,2} - v_{1,1} v_{2,1} - v_{1,2} v_{2,0} + b v_{3,2} = 0, \]
\[ \dot{v}_{4,1} - v_{2,0} v_{3,0} + r v_{4,0} = 0, \]
\[ \dot{v}_{4,2} - v_{2,0} v_{3,1} - v_{2,1} v_{3,0} + r v_{4,1} = 0, \]
\[ \dot{v}_{4,3} - v_{2,0} v_{3,2} - v_{2,1} v_{3,1} - v_{2,2} v_{3,0} + r v_{4,2} = 0. \]

Solve the unknowns of the above Eqs. (19) by taking the initial conditions as \( u_{i,j} = 0, i, j = 1, 2, 3 \)

\[ v_{1,1}(t) = [ac_2 - ac_1 + c_4](t - t^*) \]
\[ v_{1,2}(t) = [dc_1 - c_1 c_3 + cc_2](t - t^*) \]
\[ v_{1,3}(t) = [c_1 c_2 - b c_3](t - t^*) \]
\[ v_{2,1}(t) = \frac{1}{2} \left\{ a (dc_1 - c_1 c_3 + cc_2) - a (ac_2 - ac_1 + c_4) + c_2 c_3 + re_4 \right\} (t - t^*)^2 \]
\[ v_{2,2}(t) = \frac{1}{2} \left\{ d (ac_2 - ac_1 + c_4) - (c_1 c_2 - b c_3) - c_3 (ac_2 - ac_1 + c_4) \right\} + c (dc_1 - c_1 c_3 + cc_2) \right\} (t - t^*)^2 \]
\[ v_{2,3}(t) = \frac{1}{2} \left\{ c_2 (c_2 c_2 - b c_3) + c_3 (dc_1 - c_1 c_3 + cc_2) + r (c_2 c_3 + re_4) \right\} (t - t^*)^2 \]
\[ v_{3,1}(t) = \frac{1}{6} \left\{ a (d (ac_2 - ac_1 + c_4) - c_1 (c_2 c_2 - b c_3) - c_3 (ac_2 - ac_1 + c_4) \right\} + c (dc_1 - c_1 c_3 + cc_2) \right\} (t - t^*)^2 \]
\[ v_{3,2}(t) = \frac{1}{2} \left\{ c_2 (c_2 c_2 - b c_3) + c_3 (dc_1 - c_1 c_3 + cc_2) + r (c_2 c_3 + re_4) \right\} (t - t^*)^3 \]
\[ v_{3,3}(t) = \frac{1}{6} \left\{ c_1 (d (ac_2 - ac_1 + c_4) - c_1 (c_2 c_2 - b c_3) - c_3 (ac_2 - ac_1 + c_4) \right\} + c (dc_1 - c_1 c_3 + cc_2) \right\} (t - t^*)^3 \]

(19)

(20)
\[ v_{4,3}(t) = \frac{1}{6} [c_2(c_1 - c_1 c_3 + c c_2) + c_2(a c_2 - a c_1 + c_4) - b(c_1 c_2 - b c_3)] + 2(d c_1 - c_1 c_3 + c c_2)(c_1 c_2 - b c_3) + (d(a c_2 - a c_1 + c_4) - c_1)(c_1 c_2 - b c_3) - c_3(a c_2 - a c_1 + c_4 + c(d c_1 - c_1 c_3 + c c_2)) + r(c_2(c_1 c_2 - b c_3) + c_3(d c_1 - c_1 c_3 + c c_2) + r(c_2 c_3 + r c_4))(t-t^*)^3. \]

Thus, the solution of system (1) is

\[
\begin{align*}
x(t) &= \sum_{k=0}^{\infty} v_{1,k}(t), \\
y(t) &= \sum_{k=0}^{\infty} v_{2,k}(t), \\
z(t) &= \sum_{k=0}^{\infty} v_{3,k}(t), \\
w(t) &= \sum_{k=0}^{\infty} v_{4,k}(t).
\end{align*}
\]

(21)

To carry out the iterations on every subinterval of equal length \( \Delta t \), we need to know the values of the following initial conditions:

\[ c_1 = x(t^*), \quad c_2 = y(t^*), \quad c_3 = z(t^*), \quad c_4 = w(t^*). \]

(22)

In general, we do not have these information at our clearance except at the initial point \( t^* = t_0 = 0 \), but we can obtain these values following the MHPM as discussed earlier. We note that the 15-term approximation of \( x, y, z \) and \( w \) are denoted as

\[
\begin{align*}
x(t) &\approx \phi_{15}(t) = \sum_{i=0}^{14} v_{1,i}, \quad y(t) \approx \phi_{15}(t) = \sum_{i=0}^{14} v_{2,i}, \quad z(t) \approx \phi_{15}(t) = \sum_{i=0}^{14} v_{3,i}, \\
w(t) &\approx \phi_{15}(t) = \sum_{i=0}^{14} v_{4,i}.
\end{align*}
\]

4. Results and Discussions

The system (1) is hyperchaotic for the parameters of \( a = 35, b = 3, c = 12, d = 7, \) and \( r = 0.5 \) with the initial conditions as \( x(0) = -20, y(0) = 0, z(0) = 0 \) and \( w(0) = 15 \). The series solutions for 15-term

HPM for hyperchaotic Chen system (1) are obtained as :

\[
\begin{align*}
x(t) &= -20.0 + 715.0t - 14958.7500t^2 + 193915.2083t^3 - 1862311.641t^4 + 11648014.04t^5 - 782382.0271t^6 - 1397601846t^7 + 27028163520t^8 - 334919921200t^9 + 25862825040t^{10} + 3406869902000t^{11} - 581768028700000t^{12} + 13372041570000000t^{13} - 212029685400000000t^{14} \\
y(t) &= -140.0t + 1662.500000t^2 - 18920.41667t^3 - 196909.6354t^4 + 11464051.41t^5 - 279184644t^6 + 4764117419t^7 - 58999397170t^8 + 406793603700t^9 +
\end{align*}
\]
There are 3 methods used in this paper to solve the hyperchaotic Chen which are HPM, MHPM and the well-established RK4 as the reference. The algorithm discussed is coded in the computer algebra package Maple together with the Maple’s built-in fourth-order Runge-Kutta. The Maple environment variable Digits controls the number of significant digits to 16 in all calculations done in this paper. Also, we fixed the term used to 15 and step size to $\Delta t = 0.001$ by comparing the solution of RK4 at different step sizes i.e. $\Delta t = 0.01$, $\Delta t = 0.001$ and $\Delta t = 0.0001$. From the results presented in table 1 for hyperchaotic Chen, the absolute errors between $\Delta t = 0.01$ with $\Delta t = 0.001$ are still large comparing to $\Delta t = 0.001$ with $\Delta t = 0.0001$ which the minimum difference recorded is to the value of $10^{-5}$. Therefore, the step size of $\Delta t = 0.001$ is chosen for this study because it gives small error and computationally costly as the time taken is reasonable. The time range in this work is taken to be from $t=0$ to $t=10$.

### Table 1: Hyperchaotic Chen system, differences between RK4 solutions on $\Delta t = 0.01$, $\Delta t = 0.001$ and $\Delta t = 0.0001$

| $t$ | $\Delta t = 0.01$ | $\Delta t = 0.001$ | $\Delta t = 0.0001$ |
|-----|------------------|------------------|------------------|
| 1   | 6.956E-09        | 1.711E-08        | 6.184E-08        |
| 2   | 2.405E-07        | 1.535E-07        | 4.548E-07        |
| 3   | 1.228E-07        | 1.982E-07        | 7.074E-07        |
| 4   | 1.573E-08        | 3.831E-08        | 2.091E-07        |
| 5   | 3.710E-07        | 4.749E-07        | 1.235E-07        |
| 6   | 2.775E-06        | 5.468E-06        | 3.047E-06        |
| 7   | 2.962E-06        | 3.717E-06        | 3.362E-07        |
| 8   | 7.866E-07        | 1.589E-06        | 3.345E-06        |
| 9   | 8.932E-07        | 2.363E-07        | 1.016E-05        |
| 10  | 8.919E-06        | 8.120E-06        | 1.256E-05        |
Table 2: Differences between 15-term HPM and 15-term MHPM with RK4 solutions on \( \Delta t=0.001 \).

| \( t \)  | \( \Delta x \) | \( \Delta y \) | \( \Delta z \) | \( \Delta w \)  |
|-----|-----|-----|-----|-----|
| 1   | 1.992E+17 | 8.317E+17 | 5.285E+17 | 5.756E+16  |
| 2   | 3.367E+21 | 1.446E+22 | 8.706E+21 | 1.068E+21  |
| 3   | 9.931E+23 | 4.186E+24 | 2.545E+24 | 3.247E+23  |
| 4   | 5.603E+25 | 2.365E+26 | 1.429E+26 | 1.859E+25  |
| 5   | 1.278E+27 | 5.398E+27 | 3.251E+27 | 4.279E+26  |
| 6   | 1.644E+28 | 6.949E+28 | 4.175E+28 | 5.538E+27  |
| 7   | 1.425E+29 | 6.025E+29 | 3.614E+29 | 4.820E+28  |
| 8   | 9.252E+29 | 3.913E+30 | 2.344E+30 | 3.139E+29  |
| 9   | 4.817E+31 | 2.037E+31 | 1.219E+31 | 1.638E+30  |
| 10  | 2.107E+31 | 8.914E+31 | 5.330E+31 | 7.180E+30  |

According to the data in table 2 for hyperchaotic Chen system, we see that the 15-term MHPM solutions at \( \Delta t=0.001 \) highly agree with the solutions of RK4 to at least 5 decimal places while HPM solutions are not valid even at \( t=1 \). This demonstrates that the hyperchaotic Chen is solved accurately by MHPM.

Figure 1: The comparison between 15-term HPM, MHPM and RK4 on \( \Delta t=0.001 \). Solid line represents HPM, dash line is RK4 while circle is for MHPM.

In Figure 1, when plotting solutions of HPM, MHPM and RK4 together, we find that the HPM solution deviate rapidly from RK4 and MHPM solutions at \( t<0.1s \). It is obvious that HPM is not valid for the solution of hyperchaotic systems as time goes by. But, solution of MHPM and RK4 agreed to each other as time increases. Next, the \( x-y-z, x-y-w, x-z-w \) and \( y-z-w \) phase portraits of the hyperchaotic Chen system using 15-term MHPM solution on \( \Delta t=0.001 \) are illustrated in figure 2.
5. Conclusion
In this study, HPM and MHPM have been successfully employed to obtain the approximate analytical solutions for hyper chaotic system. The MHPM has the advantage of giving an analytical form of the solution within each time interval which is not possible in purely numerical techniques like RK4. The present technique offers an explicit time-marching algorithm that works accurately over such a bigger time span and fewer terms than the standard HPM. The results presented in this paper suggest that MHPM is also readily applicable to more complex nonlinear hyper chaotic systems.

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