NICE ENUMERATIONS OF $\omega$-CATEGORICAL GROUPS

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ABSTRACT

We give an example of an $\omega$-categorical group without AZ-enumerations. We build AZ-enumerations of some $\omega$-categorical central products of $\omega$ copies of a finite 2-step nilpotent group.

Key Words: $\omega$-categorical groups; Nice enumerations.

2000 Mathematics Subject Classification: 03C45; 20F50.

0. Introduction

The following notion has been introduced by Alhbrandt and Ziegler in [2] as a technical tool for quasiﬁnite axiomatizability. An ordering $<$ of type $\omega$ of a countable structure $M$ is called a nice enumeration of $M$ if for any sequence $a_i \in M$, $i \in \omega$, there are $i, j \in \omega$ and some automorphism $\alpha \in Aut(M)$, such that $\alpha(a_i) = a_j$, and $\alpha(a) < a_j$ for all $a < a_i$. This notion has been applied in several places of model theory. The following question is central in the subject:

Is there an $\omega$-categorical structure without a nice enumeration?

In our paper we will study some version of nice enumerations which was introduced by Hrushovski in [7]. An $\omega$-ordering $<$ of a countable structure $M$ is called an AZ-enumeration of $M$ if for each $n$ and any sequence $\bar{a}_i$, $i \in \omega$, of $n$-tuples of $M$ there are $i, j \in \omega$ and some order preserving elementary map $\alpha : M \to M$ such that $\alpha(\bar{a}_i) = \bar{a}_j$. In [3] structures having AZ-enumerations are called geometrically finite.

It is easy to see that every AZ-enumeration is nice. Albert and Chowdhury have asked in [3] whether there is an $\omega$-categorical structure which is not geometrically finite. In particular they have asked if the random graph has an AZ-enumeration. In our paper we study AZ-enumerations in the case of $\omega$-categorical groups. We answer the question from [3] mentioned above by showing that the 2-step nilpotent group with quantifier elimination found in [5] does not have AZ-enumerations. Using a similar idea we also prove that the random graph does not have AZ-enumerations.

In Section 2 we study the same questions for some $\omega$-categorical central products of $\omega$ copies of a finite 2-step nilpotent group (see [4]). It is easy to see that these groups are reducts of smoothly approximable structures. By [4] this implies that they have AZ-enumerations. We prove that the standard enumerations of these groups are already AZ-enumerations in some stronger sense. Although this theorem

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resembles some statements of Section 4.1 from [4], our proof uses different ideas and moreover provides some additional information.

The research is supported by KBN grant 1 P03A 025 28.

1. Fraïssé limits without AZ-enumerations

We start with the construction of a QE-group of nilpotency class 2 given in [5]. Since the group is build as the Fraïssé limit of a class of finite groups, we give some standard preliminaries (see for example [6]).

Let $K$ be a non-empty class of finite structures of some finite language $L$. We assume that $K$ is closed under taking substructures (satisfies HP, the hereditary property), has the joint embedding property (JEP) and the amalgamation property (AP). The latter is defined as follows: for every pair of embeddings $e : A \rightarrow B$ and $f : A \rightarrow C$ with $A, B, C \in K$ there are embeddings $g : B \rightarrow D$ and $h : C \rightarrow D$ with $D \in K$ such that $g \cdot e = h \cdot f$. Fraïssé has proved that under these assumptions there is a countable locally finite $L$-structure $M$ (which is unique up to isomorphism) such that:

(a) $K$ is the age of $M$, i.e. the class of all finite substructures which can be embedded into $M$ and

(b) $M$ is finitely homogeneous (ultrahomogeneous), i.e. every isomorphism between finite substructures of $M$ extends to an automorphism of $M$.

The structure $M$ is called the Fraïssé limit of $K$.

To define a 2-step nilpotent, $\omega$-categorical group without AZ-enumerations we assume that $K$ is the class of all finite groups of exponent four in which all involutions are central. By [5] $K$ satisfies the HP, the JEP and the AP. Let $G$ be the Fraïssé limit of this class. Then $G$ is nilpotent of class two.

We need the notions of free amalgamation and a-indecomposability in $K$. Following [4] we define them through the associated category of quadratic structures. A quadratic structure is a structure $(U, V; Q)$ where $U$ and $V$ are vector spaces over the field $\mathbb{F}_2$ and $Q$ is a nondegenerate quadratic map from $U$ to $V$, i.e. $Q(x) \neq 0$ for all $x \neq 0$ and the function $\gamma(x, y) = Q(x) + Q(y) + Q(x + y)$ is an alternating bilinear map. By $Q$ we denote the category of all quadratic structures with morphisms $(f, g) : (U_1, V_1; Q_1) \rightarrow (U_2, V_2; Q_2)$ given by linear maps $f : U_1 \rightarrow U_2$, $g : V_1 \rightarrow V_2$ respecting the quadratic map: $gQ_1 = Q_2 f$.

For $G \in K$ define $V(G) := \Omega(G)$, the subgroup of all involutions of $G$, and $U(G) := G/V(G)$. Let $Q_G : U(G) \rightarrow V(G)$ be the map induced by squaring in $G$. Then $QS(G) = (U(G), V(G); Q_G)$ is a quadratic structure and the associated map $\gamma(x, y)$ is the one induced by the commutation from $G/V(G) \times G/V(G)$ to $V(G)$. It is shown in Lemma 1 of [4] that this gives a 1-1-correspondence between $K$ and $Q$ up to the equivalence of central extensions $1 \rightarrow V(G) \rightarrow G \rightarrow U(G) \rightarrow 1$ with $G \in K$.

We now consider the amalgamation process in $K$. To any amalgamation diagram in $K$, $G_0 \rightarrow G_1, G_2$ we associate the diagram $QS(G_0) \rightarrow QS(G_1), QS(G_2)$ of the corresponding quadratic structures and (straightforward) morphisms. Let $QS(G_i) = (U_i, V_i; Q_i), i \leq 2$. Let $U^+, V^+$ be the amalgamated direct sums $U_1 \bigoplus_{U_0} U_2, V_1 \bigoplus_{V_0} V_2$ in the category of vector spaces. We define the free amalgam $QS(G_1)$ and $QS(G_2)$ as a quadratic structure $(U, V; Q)$ with $U = U^+$ and $V = V^+ \bigoplus (U_1/U_0) \bigotimes (U_2/U_0)$ (see [4]). The corresponding quadratic map $Q : U \rightarrow V$ is defined by first
Theorem 1.1. Let \( \mathcal{G} \) be the Fraïssé limit of the class \( \mathcal{K} \) of all finite 2-step nilpotent groups of exponent four such that all involutions are central. Then \( \mathcal{G} \) does not have AZ-enumerations.

Proof. Let \( \{ G_d : d \in \omega \} \) be an antichain of a-indecomposable groups as above. We define the rank \( rk(G_d) \) as the minimal size of a generating set of \( G_d \). Since \( \mathcal{G} \) is \( \omega \)-categorical and all \( G_d \) are embeddable into \( \mathcal{G} \), we may assume that \( 2 < rk(G_d) < rk(G_d) \) for every pair \( 0 \leq d < d' \).

Let \( < \) be an ordering of \( \mathcal{G} \). Suppose for a contradiction that \( < \) defines an AZ-enumeration of \( \mathcal{G} \). We will now define a sequence of triples \( a_n < b_n < c_n, n \in \omega \), and a subsequence \( G_{d_n} \), \( n \in \omega \setminus \{0\} \), satisfying the following conditions. Let \( a_n \) be the \( < \)-minimal element of \( \mathcal{G} \) (thus \( a_0 = \ldots = a_n = \ldots \)). For \( n > 0 \) the elements \( b_n \) and \( c_n \) are chosen so that there is a subset \( T_n \) consisting of some \( x_1, x_2, \ldots, x_{n-1} \), such that the set \( T_n \cup \{ c_n \} \) generates a subgroup isomorphic to \( G_{d_n} \). We also demand that for each \( i < n \) there is no subset \( T \subseteq \{ x : x \leq b_n \} \) such that \( T \cup \{ c_n \} \) generates a subgroup isomorphic to \( G_{d_i} \).

The triples \( (a_n, b_n, c_n) \) are defined by induction. Let \( a_0 = b_0 = c_0 \). At step \( n \) we take \( b_n \) as the first element enumerated after \( c_{n-1} \) such that the initial segment \( \mathcal{G}_n = \{ x : x \leq b_n \} \) contains a set \( T_n \) which together with some \( c \in \mathcal{G} \setminus \{ G_n \} \) generates a subgroup isomorphic to some \( G_d \not\in \{ G_{d_1}, \ldots, G_{d_{n-1}} \} \). Let \( d_n \) be the minimal number \( d \) with this condition. To define \( c_n \) consider a group \( U_n \) which is isomorphic to the free amalgam of \( G_{d_n} \) and \( \langle G_n \rangle \) over \( \langle T_n \rangle \) by an isomorphism fixing \( \mathcal{G}_n \) pointwise. Since \( \mathcal{G} \) is the Fraïssé limit of \( \mathcal{K} \) we see that \( U_n \) can be chosen as \( \langle G_n, c \rangle \) for an appropriate \( c \in \mathcal{G} \). Let \( c_n \) be the element of \( \mathcal{G} \) with the minimal number with respect the condition that \( \langle G_n, c_n \rangle \) is isomorphic with \( \langle G_n, c \rangle \) over \( \mathcal{G}_n \) under an isomorphism taking \( c_n \) to \( c \).

Claim. There are no \( i < n \) and a subset \( T \subseteq \mathcal{G}_n \) such that \( T \cup \{ c_n \} \) generates a subgroup isomorphic to \( G_{d_i} \).

Suppose that such \( T \) exists. This defines a copy of \( G_{d_i} \) in the free amalgam of \( G_{d_n} \) and \( \langle G_n \rangle \) over \( \langle T_n \rangle \). By a-indecomposability either \( \langle T \cup \{ c_n \} \rangle \subseteq G_{d_n} \) or \( \langle T \cup \{ c_n \} \rangle \subseteq \langle G_n \rangle \). The first case is impossible because there is no embedding of \( G_{d_i} \) into \( G_{d_n} \). The second condition contradicts the assumption that \( c_n \not\in \langle G_n \rangle \).
To finish the proof of the theorem assume that \( \rho : M \to M \) is an order preserving elementary map taking \((a_i, b_i, c_i)\) to \((a_j, b_j, c_j)\) for some \(0 < i < j\). Since \(\langle T_i \cup \{c_i\}\rangle\) is isomorphic to \(G_{d_i}\), there is a subset \(T \subseteq G_j\) such that \((T \cup \{c_j\})\) is isomorphic to \(G_{d_i}\) (for example let \(T = \rho(T_i)\)). This contradicts the definition of triples \((a_n, b_n, c_n)\), \(n \in \omega\). \(\square\)

We finish this section by a similar argument applied to graphs. Although it does not concern \(\omega\)-categorical groups, we have decided to include it into the paper. Besides the fact that this argument is very similar, it answers a question from [3], which was somehow distinguished in that paper.

Let \(\mathcal{K}_0\) be the class of all finite graphs. The Fraïssé limit \((\Gamma, R)\) of \(\mathcal{K}_0\) is called the random graph. 

**Proposition 1.2.** The random graph does not have AZ-enumerations.

**Proof.** Assume for a contradiction that there is an ordering \(\prec\) which defines an AZ-numeration of the random graph \((\Gamma, R)\). We define an infinite sequence of triples \(a_n < b_n < c_n\), \(n \in \omega \setminus \{0\}\), satisfying the following conditions. All \(a_n\) always denote the (same) \(<\)-minimal element of \(\Gamma\). For \(n > 3\) the elements \(b_n\) and \(c_n\) are chosen so that there are \(x_1, x_2, \ldots, x_n \leq b_n\) such that \(x_1, x_2, \ldots, x_n\) form an \(n\)-cycle, i.e. \((x_i, x_j) \in R \iff ([j - i] = 1) \lor ([i, j] = \{1, n\})\). The \(n\)-cycle \((x_i, x_j)\) belongs to \(\Gamma\) for all \(i < n\).

On the other hand we demand that for each \(i\) with \(2 < i < n\), any \(y_1, y_2, \ldots, y_i \leq b_n\) does not form an \(i\)-cycle \(R\)-connected with \(c_n\) as above.

The triples \((a_n, b_n, c_n)\), \(n \in \omega\), can be defined by induction. At step \(n\) we take \(b_n\) as the first element enumerated after \(c_{n-1}\) such that there are \(x_1, x_2, \ldots, x_n \leq b_n\) forming an \(n\)-cycle: \((x_i, x_j) \in R\) if only if \([i - j] = 1\) or \([i, j] = \{1, n\}\). To find \(c_n\) consider a finite graph \(G\) consisting of vertices \(x_1, x_2, \ldots, x_n\) and some \(c\) with \((x_i, c) \in R\) for \(i \leq n\). Let \(B_n\) be the free amalgam of \(\Gamma_n = \{x : x \leq b_n\}\) and \(G\) over \(\{x_1, \ldots, x_n\}\). Thus \(B_n = \Gamma_n \cup \{c\}\) where \(c\) is not adjacent to any element of \(\Gamma_n \setminus \{x_1, \ldots, x_n\}\). Since \(\Gamma\) is homogeneous, the element \(c\) can be found in \(\Gamma\). Let \(c_n\) be such an element \(c \in \Gamma\) with the minimal number with respect to \(\prec\). It is clear that for any \(i < n\) there is no subset \(T \subseteq \Gamma_n\) which forms an \(i\)-cycle \(R\)-connected with \(c_n\).

Let \(\rho : M \to M\) be an order preserving elementary map taking \((a_i, b_i, c_i)\) to \((a_j, b_j, c_j)\) for some \(i < j\) with \(3 < i\). Since there is an \(i\)-cycle \(x_1, x_2, \ldots, x_i \leq b_i\) such that \((x_i, c_i) \in R\) for all \(i \leq i\), there is an \(i\)-cycle \(T \subseteq \Gamma_j\) such that \(T \cup \{c_j\}\) is isomorphic to the structure defined on \(\{x_1, \ldots, x_i, c_i\}\). This contradicts the definition of triples \((a_n, b_n, c_n)\), \(n \in \omega\). \(\square\)

2. SOME NILPOTENT GROUPS WITH AZ-ENUMERATIONS

The following construction has been studied by Apps in [1]. Suppose that \(G\) is a group, \(K\) is a subgroup of \(Z(G)\), and \(A\) is some indexing set of cardinality \(\lambda\). We define \(G(A; K)\), the (central) product of \(\lambda\) copies of \(G\) amalgamated over \(K\), as follows. We denote by \(G^\lambda\) and \(K^\lambda\) the direct product of \(\lambda\) copies of \(G\) and \(K\) respectively, indexed by \(A\). Let \((K^\lambda)^0 = \{\gamma \in K^\lambda : \pi_i(\gamma) = 1\}\), where \(\pi_i : K^\lambda \to K\) is the projection map corresponding to \(i \in A\), and let \(G(A; K) := G^\lambda/(K^\lambda)^0\). We write \(G(\nu; K)\) and \(G(\omega; K)\) for \(G(A; K)\) when \(|A| = \nu\) and \(\omega\) respectively. Note that if \(i \in A\), then \(G_i\), the \(i\)-th component of the direct product \(G^\lambda\), embeds into \(G(A; K)\) over the identification map \(K \to K^\lambda/(K^\lambda)^0\). The following theorem has been proved in [1] (Theorem A).
Let $G$ be finite, class 2 nilpotent group, and $K$ be a subgroup of $G$ such that $G' \leq K \leq Z(G)$. Then $G(\omega; K)$ is $\omega$-categorical.

We will improve this theorem by the statement that $G(\omega; K)$ is a reduct of a smoothly approximable structure. This implies by [4] that $G(\omega; K)$ has an $AZ$-enumeration.

**Notation.** Let $G$ and $K$ be as in the theorem. Denote $\Gamma = G(\omega; K)$, $\Gamma_n = G(n; K)$. Let $\pi : G^\omega \to \Gamma$ be the quotient map. For $i \in \omega$, let $G_i$ be the $\pi$-image in $\Gamma$ of the $i$-th component of $G^\omega$ (which is also denoted by $G_i$). We have $G_i = G$ for each $i \in \omega$, and $\langle G_i : 0 \leq i < n \rangle$ is naturally isomorphic to $\Gamma_n$ (by the fact that $(K^\omega)^0 \cap G^n = (K^n)^0$). We therefore view $\Gamma_n$ as a subgroup of $\Gamma$.

We now introduce a countable subgroup $\Omega < \text{Aut}(\Gamma)$ generated by some family of automorphisms studied in [1]. Let $\sigma$ be a finitary permutation of $\omega$. Then $\sigma$ induces an automorphism $\hat{\sigma}$ of $G^\omega$ given by $\hat{\sigma}(g_0, g_1, ..., g) = (g_{\sigma(0)}, g_{\sigma(1)}, ...)$. It is easy to see that $\hat{\sigma}(K^\omega)^0 = (K^\omega)^0$. Thus $\hat{\sigma}$ can be considered as an automorphism of $G(\omega; K)$ such that $\hat{\sigma}(G_n) = G_{\sigma(n)}$ for each $n \in \omega$.

Another kind of our automorphisms is defined as follows. Let $M = m + 2$, where $m$ is the exponent of $G$. Let $\alpha : G \to G^M$ be given by $\alpha(g) = (g, ..., |g|, ...)$, the $M$-tuple whose $i$-th entry is 1, and whose other entries are $g$. Define $\alpha : G^M \to G^M$ by $\alpha(g_0, ..., g_{M-1}) = \alpha_1(g_0) \cdot \alpha_M(g_{M-1})$. Let $\beta = \pi \alpha : G^M \to \Gamma_M$. The following lemma has been proved in [1] (Lemma 2.1).

**Lemma 2.1.** The map $\beta : G^M \to \Gamma_M$ is a homomorphism, and it induces an endomorphism $\beta^*$ of $\Gamma_M$ (i.e. $(K^M)^0 < \text{ker} \beta$). Moreover, $\beta^*$ is a self-inverse automorphism of $\Gamma_M$ which fixes every element of $K$.

It is worth noting that $\beta^*(g, 1, ..., 1) = (1, g, ..., g)$. We can consider $\beta^*$ as an automorphism of $\Gamma$ by defining its action trivially for entries with indexes greater than $M - 1$. To see this it suffices to note that by Lemma [2.1] the kernel of the map $G^\omega \to \Gamma$ corresponding to this extension is contained in $(K^\omega)^0$.

**Lemma 2.2.** Let $i_0, j_0 \in \omega$ and $I$ be a finite subset of $\omega$ such that $\{i_0, j_0\} \cap I = \emptyset$ and the exponent of $G$ divides $|I|$. Then there is an automorphism $\alpha_{i_0, j_0} \in \text{Aut}(\Gamma)$ such that for any $(g_0, g_1, ..., g_{j_0})$ from $G^\omega$ with $g_i = 1$ for $i \in I \cup \{j_0\}$, and $g_{i_0} \neq 1$, the automorphism $\alpha_{i_0, j_0}$ sends $(g_i : i < \omega)$ to $(g'_i : i < \omega)$, where $g'_i = g_i$ for $i \notin I$ and $g'_{i_0} = g_{i_0}$ otherwise.

**Proof.** The automorphism $\alpha_{i_0, j_0}$ can be chosen as a composition of automorphisms of the form $\hat{\sigma}$ for $\sigma \in S_{\text{fin}}(\omega)$ and automorphisms as in Lemma [2.1]. □

Let $\Omega$ be the subgroup of $\text{Aut}(\Gamma)$ generated by all automorphisms as in Lemma 2.2 and all automorphisms of the form $\hat{\sigma}$ for $\sigma \in S_{\text{fin}}(\omega)$.

We start our study of $\Gamma$ with the observation that $\Gamma$ is a reduct of a smoothly approximable structure. We remind the reader that a structure $M$ is smooth approximable if it is $\omega$-categorical and every finite subset of $M$ is contained in a finite substructure $N$ such that all 0-definable relations on $M$ induce 0-definable relations on $N$ and any two enumerations $\bar{a}$ and $\bar{b}$ of $N$ have the same type in $N$ if and only if they have the same type in $M$.

**Proposition 2.3.** Let $G$ be a finite nilpotent group of class 2, and $K$ be a subgroup of $G$ such that $G' \leq K \leq Z(G)$. Then the constant expansion of $G(\omega; K)$ by all elements of $K$ is smoothly approximable.
Proof. Consider all subgroups $\Gamma_n < \Gamma = G(\omega; K)$, $n \in \omega$, realizable on the corresponding indexes $0, \ldots, n-1$. We claim that $(\Gamma, a)_{a \in K}$ is approximated by all $(\Gamma_n, a)_{a \in K}$, $n \in \omega$. To see this it suffices to notice that every automorphism of $\Gamma_n$ fixing $K$ pointwise extends to an automorphism of $\Gamma$. Since $\Gamma$ is the central product of $G(\{i \in \omega : i < n\}; K)$ and $G(\{i \in \omega : i > n-1\}; K)$ amalgamated over $K$, we can extend an automorphism $\phi \in Aut(\Gamma_n/K)$ to $\Gamma$ trivially on $G(\{i \in \omega : i > n-1\}; K)$. 

We now build an explicit AZ-enumeration of $G(\omega; K)$. In fact this is the corresponding version of the standard ordering of a basic linear geometry defined in Section 4.1 of [4]. In our context this construction provides an AZ-enumeration with some additional properties. To formulate them consider a subgroup $H$ of the group of all automorphisms of a structure $M$. The closure of $H$ in the space $M^H$ of all functions $M \to M$ consists of some embeddings of $M$ into $M$. Since every element of $H$ is an elementary map, these embeddings are elementary too. We say that an ordering $<$ of the structure $M$ is an AZ-enumeration with respect to $H$ if for any $n$ and any sequence $\bar{a}_i$, $i \in \omega$, of $n$-tuples from $M$ there are $i \neq j$ and some order preserving map from the closure of $H$ which maps $\bar{a}_i$ to $\bar{a}_j$.

Let us define an AZ-enumeration of $G(\omega; K)$. First we enumerate the group $G^\omega$. Fix an ordering of $G$: $g_0, \ldots, g_{m-1}$, where $g_0 = 1$. Then we order $G^\omega$ by the reverse lexicographic ordering: $(a_1, a_2, \ldots) < (b_1, b_2, \ldots)$ if there is $j \in \omega$ such that $a_j < b_j$ and $a_i = b_i$ for all $i > j$.

We now construct an enumeration $\{v_i : i < \omega\}$ of the group $\Gamma = G^\omega / (K^\omega)^0$ by induction. Suppose that $v_0, v_1, \ldots, v_{n-1}$ are already defined. Then let $v_n$ be the $(K^\omega)^0$-coset having a representant which is minimal (with respect to the ordering above) in $G^\omega$ among sequences not representing $v_0, \ldots, v_{n-1}$.

Theorem 2.4. The ordering of the group $G(\omega; K)$ defined as above is an AZ-enumeration with respect to $\Omega$.

Although this theorem corresponds to Lemma 4.1.6 from [4], our proof is based on some different tricks. When we construct a required order-preserving elementary map we explicitly define an approximating sequence from $\Omega$ guaranteeing that the map belongs to the closure of $\Omega$. It is possible that some special analysis of definable subsets of $G(\omega; K)$ can be applied in this space instead of approximating sequences. However we think that our approach is more direct and elegant.

We start with some preliminaries. The following lemma belongs to G.Higman (see Section 4.1 of [4]).

Lemma 2.5. Let $\Sigma$ be a finite set. Define a partial ordering on the set $\Sigma^*$ of $\Sigma$-words by: $w_1 \leq w_2$ if $w_1$ is a subword of $w_2$, i.e. after deleting some members of $w_2$ we are left with $w_1$. Then $(\Sigma^*, \leq)$ is a partial well ordering: for every sequence $\{w_i : i < \omega\}$ from $\Sigma^*$, there are $i < j < \omega$ such that $w_i \leq w_j$.

We now improve this lemma as follows. Consider again the set $\Sigma^*$ of all finite words over $\Sigma$. We say that a word $(a_i : i \leq n)$ is $*$-embedded into a word $(b_i : i \leq m)$ if there is an order preserving injection $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $b_{f(i)} = a_i$ and

$$(\forall i \leq m)(\exists j \leq n)((i \leq f(j)) \land (b_i = b_{f(j)})).$$

It is easy to see that the following relation is a partial ordering on $\Sigma^*$: $w_1 \leq^* w_2$ if $w_1$ is $*$-embedded into $w_2$. 
Lemma 2.6. Let \( \Sigma \) be a finite set. Then \( (\Sigma^*, \leq^*) \) is a partial well ordering.

Proof. Let \( A = \{ w_i : i \in \omega \} \subseteq \Sigma^* \). We assume that all words in \( A \) are composed from the same letters: \( \sigma_1, \ldots, \sigma_k \). Moreover we may also assume that for all \( i \in \omega \) and \( s < t \leq k \), the last appearance of \( \sigma_s \) in \( w_i \) is before the last appearance of \( \sigma_t \) in \( w_i \). We thus view each word \( w_i \) as \( w_{i1}w_{i2} \ldots w_{ik}\sigma_k \), where for \( l > 1 \) the subword \( w_{il} \) is of the form \( \sigma_{l-1} \sigma_{l+1} \sigma_{l+2} \ldots \sigma_k \), with \( \sigma_{l+1}, \sigma_{l+2}, \ldots, \sigma_k \). Let \( l \leq s \). It is enough to prove, that there are \( i < j \), and an order preserving embedding \( f : \{1, \ldots, |w_i|\} \rightarrow \{1, \ldots, |w_j|\} \), which sends \( w_{il} \) to \( w_{jl} \) for all \( l < k \). For each \( i \in \omega \) let \( o_i \) be \( \max(|w_{i1}|, \ldots, |w_{ik}|) \). In order to apply Lemma 2.3 we will code up \( w_i \) in some new alphabet. Let \( \bar{\Sigma} \) be the alphabet of all \( k \)-tuples from \( G \cup \{x\} \). We associate to \( w_i \) the word \( \bar{\tau}^{(i)} = \tau_1 \tau_2 \ldots \tau_{o_i} \), where every \( \tau_l \) is the sequence of \( l \)-th letters appearing in the corresponding \( w_{il} \) (when \( |w_{il}| < t \) the corresponding place in \( \tau_l \) is signed by \( x \)). By Higman’s lemma, there are \( i < j < \omega \) and an embedding \( f' : \bar{\tau}^{(i)} \rightarrow \bar{\tau}^{(j)} \). Then \( f' \) induces an \( * \)-embedding \( f : \{1, \ldots, |w_{i1}|\} \rightarrow \{1, \ldots, |w_{j1}|\} \) of \( w_i \) to \( w_j \). To see this put \( f(x) = y \) if there are \( q \leq k \) and \( r \leq o_i \) such that \( x = |w_{i1}| + |w_{i2}| + \ldots + |w_{iq}| + r, y = |w_{j1}| + |w_{j2}| + \ldots + |w_{jq}| + f'(r) \). The rest is obvious. □

Proof of Theorem 2.4. Let \( \bar{a}_k \in (\Gamma_\omega)^n, k \in \omega \), be an infinite set of \( n \)-tuples. For every \( k \) and every element of \( \bar{a}_k \) we fix some representative of it in \( G^n \) and think of \( \bar{a}_k \) as a matrix with \( n \) semi-infinite rows, and entries from \( G \), such that almost all of them are equal to 1. Thus we may treat elements \( \bar{a}_k \) as semi-infinite sequences \( \bar{g}_{k,i} \).\( i \in \omega \), over the finite alphabet \( G^n \) such that almost all elements of the sequence are equal to 1. Choosing a subset of \( \{ \bar{a}_k : k \in \omega \} \) if necessary, we may assume that all sequences \( \bar{a}_k \) are represented by the same set of tuples \( \bar{g}_{k,i} \). We can also arrange that for every pair \( \bar{a}_k \) and \( \bar{a}_l \) and any \( g \in G^n \) the exponent \( m \) divides the number \( |\{ i : \bar{g}_{k,i} = g \}| - |\{ i : \bar{g}_{l,i} = g \}| \). Let \( \bar{h}_0, \ldots, \bar{h}_r \) be an enumeration of tuples of \( G^n \) occurring in all \( \bar{a}_k \). We may assume that for any \( s < t \leq r \) and \( k \in \omega \) the last appearance of \( \bar{h}_s \) in \( \bar{a}_k \) is after the last appearance of \( \bar{h}_t \). By \( l(\bar{a}_k) \) we denote the maximal \( i \) for which \( \bar{g}_{k,i} \neq \bar{1} \).

By Lemma 2.6 there are \( i < j \) such that \( \bar{a}_j \)-embeds into \( \bar{a}_j \). Let \( f : \{0, \ldots, l(\bar{a}_j)\} \rightarrow \{0, \ldots, l(\bar{a}_j)\} \) realize this embedding. For \( s \leq r \) let \( l_s \) be the greatest \( r \) such that \( \bar{g}_{j,f(s)} = \bar{h}_s \). Then let \( I_s \subseteq \omega \) be the set of all \( l_s \) such that \( \bar{g}_{j,l} = \bar{h}_s \) and \( l \) is not in the image of \( f \). We see that for each \( s \leq r \), the exponent \( m \) divides \( |I_s| \).

To define a required embedding \( \beta : \Gamma \rightarrow \Gamma \) we describe some rules which determine the \( \beta \)-images of elements of \( \Gamma \) of the form \( \{1, \ldots, 1, g, 1, \ldots, 1, \ldots\} \) where \( g \) is the entry with index \( l \) (and thus determine \( \beta \)). When \( l < l(\bar{a}_i) \) and \( \bar{g}_{j,f(l)} \) also appears in \( \bar{a}_j \) later with a greater index, we define the \( \beta \)-image of \( \{1, \ldots, 1, g, 1, \ldots, 1, \ldots\} \) by the shift of \( g \) from the index \( l \) to \( f(l) \). In the case when \( l \leq l(\bar{a}_i) \) has the property that \( \bar{g}_{j,f(l)} \) does not appear with a greater index in \( \bar{a}_j \) (thus \( f(l) \) is one of the \( i \)-s) we take the element as above to \( \{1, \ldots, 1, g, 1, \ldots, 1, g, 1, \ldots, 1, g, 1, \ldots\} \), where the last entry of \( g \) is of the index \( f(l) \) and all other appearances of \( g \) occupy the indexes of the set \( I_s \), where \( I_s \) is defined by \( \bar{g}_{j,f(l)} \) as above. If \( l > l(\bar{a}_i) \), then the \( \beta \)-image of the element above is defined by the shift of \( g \) from the index \( l \) to \( l(\bar{a}_j) - l(\bar{a}_i) - l \). This construction guarantees that \( \beta \) takes \( \bar{a}_i \) to \( \bar{a}_j \).

To see that \( \beta \) belongs to the closure of the group \( \Omega \) in the space \( T^\Gamma \) take sufficiently large \( l > l' > l(\bar{a}_j) \) and consider a permutation \( \sigma \) of \( \{0, \ldots, l\} \) which extends \( f \) and takes every \( t \in \{l(\bar{a}_i) + 1, \ldots, l'\} \) to \( t + l(\bar{a}_j) - l(\bar{a}_i) \). This permutation naturally
extends to the automorphism \( \hat{\sigma} \in \Omega \) defined as above. When we apply \( \hat{\sigma} \) together with the product \( \alpha_{I_0,i_0,l+1} \cdot \alpha_{I_1,i_1,l+1} \cdot \ldots \cdot \alpha_{I_r,i_r,l+1} \) (see Lemma \( \text{2.2} \)) we obtain an automorphism of \( \Gamma \) which coincides with \( \beta \) on elements represented by sequences which are trivial for indexes greater than \( l' \). This shows that \( \beta \) is approximated by automorphisms from \( \Omega \).

It remains to show that \( \beta \) preserves the ordering of \( \Gamma \). Assume \( (h_0, ..., h_t, ...) < (h'_0, ..., h'_t, ...) \) and \( t_0 \) is the maximal index \( t \) such that \( h_t < h'_t \). Let \( t_1 \) be the maximal index where \( \beta(h_0, ..., h_t, ...) \) and \( \beta(h'_0, ..., h'_t, ...) \) have distinct entries. By the definition of \( \beta \), if \( l(\bar{a}_i) \leq t_0 \), then \( t_1 = t_0 + l(\bar{a}_j) - l(\bar{a}_i) \). Since the \( t_0 \)-entries of \( (h_0, ..., h_t, ...) \) and \( (h'_0, ..., h'_t, ...) \) coincide with the \( t_1 \)-entries of their \( \beta \)-images respectively, we see that \( \beta(h_0, ..., h_t, ...) < \beta(h'_0, ..., h'_t, ...) \).

Consider the case when \( t_0 < l(\bar{a}_i) \). By the definition of \( \beta \) the number \( t_1 \) cannot belong to any \( I_s, s \leq r \). Thus \( t_1 \in \text{Rng}(f) \). This implies that \( f(t_0) = t_1 \). We see that the \( t_0 \)-entries of \( (h_0, ..., h_t, ...) \) and \( (h'_0, ..., h'_t, ...) \) coincide with the \( t_1 \)-entries of their \( \beta \)-images respectively and as above we have \( \beta(h_0, ..., h_t, ...) < \beta(h'_0, ..., h'_t, ...) \).

\( \square \)

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