Geometric and arithmetic relations concerning origami

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Abstract. We present a formalization of geometric instruments that considers separately geometric and arithmetic aspects of them. We introduce the concept of tool, which formalizes a physical instrument as a set of axioms representing its geometric capabilities. We also define a map as a tool together with a set of points and curves as an initial reference. We rewrite known results using this new approach and give new relations between origami and other instruments, some obtained considering them as tools and others considering them as maps.

1. Introduction

The determination of constructible numbers with origami is a problem with an interesting development, that was completely solved only after the axiomatization proposed by Huzita-Justin (cf. [7], [8]). The work by Alpering-Lang ([2]) proved that the list of possible one-fold axioms was complete and settled a new scenario, where the role of new axioms was still emphasized. The axiomatic viewpoint seems a natural perspective for the study of other geometrical instruments.

In this work we present a general purpose formal language for the axiomatization of geometrical instruments. Our formalization takes into account both the geometric and arithmetic properties of the instruments: the concept of tool formalizes an instrument as a set of axioms, while the concept of map formalizes the constructible points and curves of the instrument. Our formalization provides a natural frame to express well-known results, but also leads to new relations between instruments.

The key concepts of our language are introduced in section 2. In section 3, we define geometric equivalence and virtual equivalence of tools and prove relations between the tools described in this work. In section 4 we present an equivalence relation between maps and define an arithmetical equivalence of tools. We conclude with an arithmetic classification of the tools described.

A more extended version of this work can be found in [10] as an evolution of a previous work (cf. [11]).

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2. Definitions

2.1. Axioms.

Definition 2.1. A construction axiom $C$ is an elementary geometric process that generates a finite ordered set of curves from a non empty, finite and ordered set of points and curves.

An intersection axiom $I$ is an elementary geometric process that generates a finite ordered set of points from a non empty, finite and ordered set of curves and points.

The notation we will use for axioms is $O_1, O_2, \ldots, O_r = \text{AxiomName}(I_1, I_2, \ldots, I_s)$ where the $I_j$ are the given curves and points and the $O_i$ are the elements generated by the axiom.

We collect in the Annex a list of basic axioms which formalize the common tasks performed by the geometric instruments that will be considered in this work (mainly ruler, compass, origami). A more comprehensive list of axioms can be found in [10].

2.2. Tools. We formalize geometric instruments in terms of their capabilities, i.e., by means of the axioms they can perform:

Definition 2.2. A tool $T$ is a couple $\langle C, I \rangle$, where $C$ is a non empty finite set of construction axioms and $I$ is a non empty finite set of intersection axioms.

Some basic examples of tools are:

- Ruler
  \[ R := \langle \{\text{Line}\}, \{\text{LineIntersect}\} \rangle \]
- Compass
  \[ C := \langle \{\text{Circle}, \text{RadiusCircle}\}, \{\text{CircleIntersect}\} \rangle \]
- Ruler and compass
  \[ RC := \langle \{\text{Line, Circle, RadiusCircle}\}, \{\text{LineIntersect, CircleIntersect, LineCircleIntersect}\} \rangle. \]
- Euclidean compass
  \[ EC := \langle \{\text{Circle}\}, \{\text{CircleIntersect}\} \rangle. \]
- Ruler and Euclidean compass
  \[ REC := \langle \{\text{Line, Circle}\}, \{\text{LineIntersect, CircleIntersect, LineCircleIntersect}\} \rangle. \]
- Origami
  \[ O := \langle \{\text{Line, PerpendicularBisector, Bisector, Perpendicular, Tangent, CommonTangent, PerpendicularTangent}\}, \{\text{LineIntersect}\} \rangle. \]
- Thalian Origami ([11])
  \[ TO := \langle \{\text{Line, PerpendicularBisector}\}, \{\text{LineIntersect}\} \rangle. \]
- Pythagorean origami ([11])
  \[ PO := \langle \{\text{Line, PerpendicularBisector, Bisector}\}, \{\text{LineIntersect}\} \rangle. \]
- Conics ([12])
  \[ CO := \langle \{\text{Line, Circle, RadiusCircle, Conic}\}, \{\text{LineIntersect, CircleIntersect, LineCircleIntersect, ConicLineIntersect, ConicCircleIntersect, ConicIntersect}\} \rangle. \]
2.3. Constructions.

Definition 2.3. A construction of a set of points and lines $V$ from $U_0$ is a finite sequence

$$C(U_0; V) = \{O_1 = A_1(U_1), \ldots, O_n = A_n(U_n)\},$$

where

- $U_0$ is an initial ordered non empty set of points and curves;
- $A_1, \ldots, A_n$ are axioms;
- Every $U_k$ is a subset of $U_0 \cup \cdots \cup U_{k-1} \cup O_1 \cup \cdots \cup O_{k-1}$;
- $V \subseteq O_1 \cup \cdots \cup O_n$, but $V \not\subseteq O_1 \cup \cdots \cup O_{n-1}$.

We say that $C(U_0; V)$ is a construction with the tool $E = (C, I)$ if $A_1, \ldots, A_k \in C \cup I$, and we write $C(U_0; V) \in E$ in this case.

Notation 1. To simplify the notation, we enumerate the elements of the sets $U_0, V$ in a single list, using a semicolon to separate the last element of $U_0$ and the first of $V$.

Example 2.4. Given a line $\ell$ and a point $P$ not on $\ell$, the construction Parallel generates the parallel $\ell_2$ to the line $\ell$ passing through the point $P$:

$$\text{Parallel}(\ell; \ell_2) = \{\ell_1 = \text{Perpendicular}(\ell, P), \ell_2 = \text{Perpendicular}(\ell_1, P)\}.$$

Clearly $\text{Parallel}(\ell; \ell_2) \in O$.

A wide catalog of constructions is given in [10].

2.4. Maps.

Definition 2.5. A map is a pair $M = (E, U_0)$ composed by a tool $E$ and a non empty finite initial set $U_0$ of points and curves.

Notation 2. Given a set $U$ of points and curves, we will write $U = [C, P]$ to specify its subset $C$ of curves and its subset $P$ of points.

Definition 2.6. Let $M = (E, U_0)$ be a map with $E = (C, I)$. The sequence of layers $U_n = \{[C_n, \Psi_n]_{n \in \mathbb{N}}\}$ is the sequence defined by:

i) $U_0 = [C_0, \Psi_0]$;
ii) $C_n$ is the union of $C_{n-1}$ with the set of curves obtained applying all construction axioms from $C$ in all possible ways to the elements of $U_{n-1}$;
iii) $\Psi_n$ is the union of $\Psi_{n-1}$ with the set of points that is obtained applying all intersection axioms from $I$ in all possible ways to the elements of $[C_n, \Psi_{n-1}]$.

We write $U^M = [C^M, \Psi^M] := \cup_{n=0}^{\infty} U_n$ to denote the set of constructible points and curves with $M$. A map is infinite if $U^M$ is infinite.

Table 1 describes the set of constructible points of the tools introduced in previous section. As usual, $\mathbb{P}$ denotes the Pythagorean closure of $\mathbb{Q}$ (i.e., the smallest extension of $\mathbb{Q}$ where every sum of two squares is a square); the field of Euclidean numbers (i.e., the smallest subfield of $\mathbb{Q}$ closed under square roots) is denoted by $\mathcal{E}$, and $\mathcal{O}$ is the field of origami numbers (i.e., the smallest subfield of $\mathbb{Q}$ which is closed under the operations of taking square roots, cubic roots and complex conjugation).

For a description of the set of constructible points of the Thalian origami map, see [1] Theorem 3.3.
Since we have defined tools in terms of their capabilities, it is natural to classify them according to this philosophy. We will consider the arithmetic capabilities in the next section, and we concentrate now on the geometric capabilities. It seems reasonable to say that two tools are equivalent if they can solve the same problems, but this has to be carefully defined. For instance, the second problem of Euclid is normally formulated as it follows: given three points $A, B, C$ in general position, one has to determine a point $D$ such that the segments $AD$ and $BC$ are congruent. Clearly, the point $D$ is not uniquely determined, and indeed, there exist several different constructions generating different solution points. This kind of situation leads to introduce the following definition:

**Definition 3.1.** We say that the constructions $C(U_0; V)$ and $C'(U_0; V')$ are equivalent, $C(U_0; V) \sim C'(U'_0; V')$, if $V$ and $V'$ have the same geometric links with $U_0$.

A problem $P(U_0; V)$ is an equivalence class of constructions under this relation.

A solution of the problem $P(U_0; V)$ is any representant of this equivalence class, that is, any construction $C(U_0; V)$ of $V$ from $U_0$.

A problem can be solved with $E$ if it has a solution $C(U_0; V) \in E$.

**Definition 3.2.** The tool $E$ generates the tool $E'$, $E \leftarrow E'$ if any problem that can be solved with $E'$ can also be solved with $E$. Two tools $E$ and $E'$ are geometrically equivalent, $E \leftarrow E'$ if they solve the same problems.

Obviously, if the set of axioms of a tool $E'$ is a subset of the axioms of the tool $E$, then $E \leftarrow E'$. Hence we have the following relations

$$CO \leftarrow RC \leftarrow REC \leftarrow R;$$
$$O \leftarrow PO \leftarrow TO \leftarrow R.$$

**Proposition 1.** The tool $TO$ does not generate the tools $PO, O$.

**Proof.** It is enough to see that $TO$ does not generate $PO$. Let us suppose we have two perpendicular lines and their point of intersection constructed. With the tool $PO$ and the use of axiom **Bisector** we can construct the bisectors of this couple of lines. However with the tool $TO$ we cannot construct neither a new line, nor a new point. □

While it is evident that the compass $C$ does not generate the ruler and compass $RC$ since it cannot construct lines, both tools generate the same points from the initial set $\{0, 1\}$. To describe this situation in a more general setting we introduce the following definitions:

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**Table 1.** Sets $\Psi^M$ for different maps

| Map                     | Initial set | $\Psi^M$   | References |
|-------------------------|-------------|------------|------------|
| Ruler                   | $\{1, 2, i, 2i\}$ | $\mathbb{Q}(i)$ | [9] page 79 |
| Compass                 | $\{0, 1\}$  | $\mathbb{C}$ | [9] chap. 3 |
| Ruler and compass       | $\{0, 1\}$  | $\mathbb{C}$ | [5] page 261 |
| Euclidean compass       | $\{0, 1\}$  | $\mathbb{C}$ | [9] page 7  |
| Ruler and euclidean compass | $\{0, 1\}$  | $\mathbb{C}$ | [9] page 7  |
| Origami                 | $\{0, 1\}$  | $\mathbb{O}$ | [1]        |
| Pythagorean Origami     | $\{0, 1\}$  | $\mathbb{P}(i)$ | [1] Theorem 3.3 |
| Conics                  | $\{0, 1\}$  | $\mathbb{O}$ | [12]       |
Definition 3.3. A construction of points with points (CPP) is a construction $CPP(U_0; V)$ where $U_0$ and $V$ contain only points.

Definition 3.4. The tool $\mathcal{E}$ generates virtually the tool $\mathcal{E}'$, $\mathcal{E} \Rightarrow \mathcal{E}'$, if any CPP with $\mathcal{E}'$ is equivalent to a construction with $\mathcal{E}$. The tools $\mathcal{E}$ and $\mathcal{E}'$ are virtually equivalents, $\mathcal{E} \Rightarrow \mathcal{E}'$ if $\mathcal{E} \Rightarrow \mathcal{E}'$ and $\mathcal{E}' \Rightarrow \mathcal{E}$.

Theorem 3.5. The Origami tool $O$ generates virtually the ruler and Euclidean compass tool $REC$.

Proof. It is enough to describe constructions of the intersection of a line with a circle and of the intersection of two circles with $O$. We can find them in [4].

As a more advanced example of construction, we describe the intersection of a circle and a line in our formal language. The construction $\text{LineCircleIntersectOrigami}(A, B, C, D; E, F)$ generates the intersection points of the line through the points $A$ and $B$ with the circle with center $C$ passing through $D$.

This construction requires the point $C$ to be exterior to the parabola with focus $D$ and directrix the line through $A$ and $B$.

$\text{LineCircleIntersectOrigami}(A, B, C, D; E, F) = \{ \ell_1 = \text{Line}(A, B), \ell_2, \ell_3 = \text{Tangent}(\ell_1, D, C), \ell_4 = \text{Perpendicular}(\ell_2, D), \ell_5 = \text{Perpendicular}(\ell_3, D), E = \text{LineIntersect}(\ell_1, \ell_4), F = \text{LineIntersect}(\ell_1, \ell_5) \}$.

Figure 1 summarizes the main relations between tools.

![Figure 1. Geometric relations between tools](image-url)
4. Arithmetic relations between maps and tools

DEFINITION 4.1. The maps $M$ and $M'$ are equivalent if $\mathfrak{P}^M = \mathfrak{P}^{M'}$.

Figure 2 shows the relations between sets of points of maps of Table 1.

DEFINITION 4.2. Two tools $E, E'$ are arithmetically equivalents, $E \leftrightarrow E'$, if there exists finite sets of points $U_0, U'_0$ such that:

i) The maps $M = (E, U_0)$ and $M' = (E', U'_0)$ are infinite and equivalent.

ii) The construction of the set $U^M$ (resp. $U'^M$) needs the application of all the axioms of $E$ (resp. $E'$).

In order to determine whether two given tools are arithmetically equivalent two different approaches can be taken: one can consider the geometric properties of the tools or one can associate particular maps to the tools and relate them. The following results illustrate these ideas.

THEOREM 4.3 (Mohr-Mascheroni). The tools $C$ and $RC$ are arithmetically equivalent.

PROOF. The first step consists in proving that we obtain the same points with $RC$ than doing inversions of points respect to circles. Then we have to prove that any point obtained from an inversion respect to a circle can be constructed with $C$ in a finite, arbitrary high, number of steps. The details can be found in [10]. □

THEOREM 4.4 (Poncelet-Steiner). The maps $RC$ and $RP := (\{Line\}, \{LineIntersect, LineUnitCircleIntersect\}), U_0 = \{0, 2, 2i, X^2 + Y^2 = 1\}$ are equivalent.

PROOF. Of course, a totally geometric proof can be given ([6 page 192]), but we present here an arithmetic proof in our language, following [9] page 98. It is clear that $1, i \in \mathbb{R}^P$, and thus $\mathbb{Q}(i) = \mathbb{R} \subset \mathbb{R}^P \subset \mathbb{F}^C = \mathbb{C}$. Since $\mathbb{C}$ is the smallest extension of $\mathbb{Q}$ closed under square roots, it is sufficient to see that $\mathbb{R}^P$ is closed under square roots. The construction of the square roots of a complex number reduces to the construction of
the bisector of two lines through the origin and the construction of the square root of a positive real number.

Given two points $A, B \in \mathbb{R}^2$ on the unit circle, let $C$ be the second point of intersection of the diameter through $A$ with the unit circle. Then the angle $AOB$ is twice the angle $OCB$. The equality

$$\sqrt{r} = \left(\frac{r + 1}{2}\right) \sqrt{1 - \left(\frac{r - 1}{r + 1}\right)^2}$$

shows that we only need the construction of the square root of numbers of the form $1 - c^2$ with $c \in (-1, 1)$, and this consists in constructing a point on the circle having $1 - c^2$ as a $x$-coordinate. This can be done constructing the perpendicular to the $x$-axis through point $1 - c^2$. An example of the construction of this perpendicular is described in [9, pages 79–80].

Finally, using the sets of constructible points of maps in Table 1 we can deduce the following relations:

**Theorem 4.5. Arithmetical classification of tools:**

i) $C \xleftrightarrow{\text{ar}} RC \xleftrightarrow{\text{ar}} EC \xleftrightarrow{\text{ar}} RC$.

ii) $O \xleftrightarrow{\text{ar}} CO$.

5. Conclusions

We have introduced a new formal language and illustrated it with the most common geometric instruments, even though a more extended study can be found in [10]. Some of the instruments presented there, such as the marked ruler or the marked ruler and compass, require a precise description of the *neusis* process and lead to interesting axioms, involving curves as the conchoid of Nicomedes or the Limaçon of Pascal.

The language proposed has the advantage that it is open, in the sense that other instruments different that those we have considered can be studied and formalized in this way: it suffices to analyze the axioms they can perform and define them using curves and points. After that, the geometric and arithmetic relations between this instrument and other existing instruments can be studied.

Finally, another significant advantage of this language is the possibility of introducing virtual tools, that is, tools not necessarily attached to any physical instrument: we can choose some existing axioms and combine them to create a virtual tool. These kind of tools can be both interesting on their own and useful as auxiliary resources for studying known instruments.
### Annex

| Axiom | Description |
|-------|-------------|
| $\ell = \text{Line}(A, B)$ | Line through points $A$, $B$. |
| $c = \text{Circle}(A, B)$ | Circle with center $A$ through $B$. |
| $c = \text{RadiusCircle}(A, B, C)$ | Circle with center $A$ and radius the distance $BC$. |
| $\ell_1, \ell_2 = \text{Bisector}(\ell, \ell')$ | Bisectors of the angle formed by the lines $\ell$ and $\ell'$. |
| $\ell = \text{PerpendicularBisector}(A, B)$ | Perpendicular bisector of the segment $AB$. |
| $\ell' = \text{Perpendicular}(\ell, P)$ | Line perpendicular to line $\ell$ passing through point $P$. |
| $\ell = \text{PointPerpendicular}(A, B, C)$ | Line perpendicular to the segment $AB$ through $C$. |
| $\ell_1, \ell_2 = \text{Tangent}(\ell, F, A)$ | Tangents through $A$ to the parabola with directrix $\ell$ and focus $F$. |
| $\ell_1, \ell_2, \ell_3 = \text{CommonTangent}(\ell, F, \ell', F')$ | Common tangents to the parabola with directrix $\ell$ and focus $F$ and the parabola with directrix $\ell'$ and focus $F'$. |
| $\ell = \text{PerpendicularTangent}(\ell_1, F, \ell_2)$ | The tangent line to the parabola with directrix $\ell_1$ and focus $F$ which is perpendicular to the line $\ell_2$. |
| $c = \text{Conic}(\ell, F, A, B)$ | Conic with directrix $\ell$, focus $F$ and excentricity the distance between $A$ and $B$. |

| $P = \text{LineIntersect}(\ell, \ell')$ | Intersection point of lines $\ell$ and $\ell'$. |
| $P_1, P_2 = \text{CircleIntersect}(c, c')$ | Intersection points of circles $c$ and $c'$. |
| $P_1, P_2 = \text{LineCircleIntersect}(\ell, c)$ | Intersection points of line $\ell$ with circle $c$. |
| $P_1, \ldots, P_4 = \text{ConicIntersect}(c, c')$ | Intersection points of conics $c$, $c'$. |
| $P_1, P_2 = \text{LineConicIntersect}(\ell, c)$ | Intersection points of line $\ell$ with conic $c$. |
| $P_1, \ldots, P_4 = \text{CircleConicIntersect}(c, c')$ | Intersection points of circle $c$ with conic $c'$. |
For the axioms generating several objects, one has to specify an ordering to properly identify each of them. When there is no natural ordering, we use a radial sweep, a common technique in computational geometry. It consists in sweeping counterclockwise the plane with a given half-line; the points are ordered in the order they are met. For the axioms described in this table we propose the following ordering:

**Bisector**: Line $\ell_1$ bisects the oriented angle $\widehat{\ell\ell'}$ and $\ell_2$ bisects $\widehat{\ell'\ell}$.

**Tangent**: The lines $\ell_1$ and $\ell_2$ are ordered following a radial sweep with center $A$ and half-line $AF$.

**CommonTangent**: The lines $\ell_1$, $\ell_2$, $\ell_3$ are ordered according to the order of their contact points with the first parabola in a radial sweep with center $F$ and half-line $FF'$.

**CircleIntersect**: Let $C$ and $C'$ be the centers of circles $c$ and $c'$ respectively. The order of points $P_1$ and $P_2$ is given by a radial sweep with center $C$ and half-line $CC'$. We use the same criterion to order the output of axioms **ConicIntersect** and **CircleConicIntersect**, taking as the center of a conic the midpoint of the segment defined by the focus.

**LineCircleIntersect**: If $\ell$ is not a diameter of circle $c$, we order $P_1$ and $P_2$ to assure that the angle $P_2CP_1$ is positive. If the line is a diameter, we order points $P_1$, $P_2$ as points on the line $\ell$, using a radial sweep with center $C$ and half-line $CO$. We use the same criterion to order the output of the axiom **LineConicIntersect**.
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