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δ-Complement of a Graph

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Abstract: Let \( G(V, X) \) be a finite and simple graph of order \( n \) and size \( m \). The complement of \( G \), denoted by \( \overline{G} \), is the graph obtained by removing the lines of \( G \) and adding the lines that are not in \( G \). A graph is self-complementary if and only if it is isomorphic to its complement. In this paper, we define \( \delta \)-complement and \( \delta' \)-complement of a graph as follows. For any two points \( u \) and \( v \) of \( G \) with \( \deg u = \deg v \) remove the lines between \( u \) and \( v \) in \( G \) and add the lines between \( u \) and \( v \) which are not in \( G \). The graph thus obtained is called \( \delta \)-complement of \( G \). For any two points \( u \) and \( v \) of \( G \) with \( \deg u \neq \deg v \) remove the lines between \( u \) and \( v \) in \( G \) and add the lines between \( u \) and \( v \) that are not in \( G \). The graph thus obtained is called \( \delta' \)-complement of \( G \). The graph \( G \) is \( \delta \) (\( \delta' \))-self-complementary if \( G \cong \overline{G} \) (\( G \cong \overline{G} \)). The graph \( G \) is \( \delta \) (\( \delta' \))-co-self-complementary if \( G_{\delta} \cong \overline{G}_{\delta} \) (\( G_{\delta'} \cong \overline{G}_{\delta'} \)). This paper presents different properties of \( \delta \) and \( \delta' \)-complement of a given graph.

Keywords: \( \delta \)-complement; \( \delta' \)-complement; self-complementary; degree sequence

MSC: 05C07; 05C40; 05C45

1. Introduction

Let \( G \) be a simple, finite and undirected graph. The number of lines in \( G \) is the size of \( G \), denoted by \( m \). The number of lines incident on a point \( v \) is the degree of \( v \) denoted by \( \deg v \). An open neighborhood of a point \( v \), denoted by \( N(v) \) is the set of all points that are adjacent to \( v \). A graph \( G \) is Eulerian if it contains a closed trail that covers all the lines of \( G \). A graph \( G \) is Hamiltonian if it contains a cycle that visits all the points exactly once. The complement of \( G \) is the graph \( \overline{G} \), obtained from \( G \) by removing all the lines of \( G \) and adding the lines between the points that are not in \( G \). The graph \( G \) is said to be self-complementary if and only if \( G \) is isomorphic to \( \overline{G} \). For more information on self-complementary graphs, one can refer [1–3]. For all notations and terminologies we refer to [4,5].

2. Motivation

1. The concept of \( \delta/\delta' \)-complement of a graph can be used in a scenario where the user changes the adjacency of points based on his requirements in order to optimize his end goal. For example, employees working in a company have various skills, qualifications, and strengths.

In comparison to the graph, let graph \( G \) specify a task. Each point in \( G \) represents an employee of the company. A line connecting two points symbolizes a shared competence between two employees. We explore the following two scenarios.

(i) An employee intends to work with other workers having similar skills, with whom he has not previously worked but does not wish to work with the same skilled
employees with whom he has previously worked. This situation is very well described by
the \( \delta \)-complement of a graph.

(ii) Suppose an employee wants to work with workers of different skills with whom
he has not previously worked and wishes to discontinue his work with employees of
different skills with whom he has previously worked. This situation is definitely achieved
via \( \delta' \)-complement of a graph.

2. Let \( v_{ij} \), \( 1 \leq i \leq n \) be the labeling of the points of a graph, where \( j \) represents the
degree of a point. Suppose that each degree signifies a certain activity. The number of
points with the label \( v_{ik} \) represent the number of persons carrying out the \( k \)th activity. The
line connecting two points indicates that the information, resources, or time required for
the execution of the activity has already been shared. Since each member is equally capable,
\( \delta(\delta') \)-complement can be used to determine the output when non-adjacent or non-linked
members of the same(different) activity are made to share information, resources, or time.

For example, The graph \( G \) in Figure 1 consists of five people \( v_1, v_2, v_3, v_4 \) and \( v_5 \) who
are assigned to accomplish three activities, namely, 1, 2, and 3. The person \( v_1 \) is in charge
of activity 1, the persons \( v_3, v_4 \) and \( v_5 \) are in charge of activity 2 and the person \( v_2 \) is in
charge of activity 3. It is observed that \( v_3, v_4 \) and \( v_4, v_5 \) of activity 2 have already shared
information among themselves. Hence, \( v_3 \) and \( v_5 \) share the information and continue to
execute the activity 2 in \( G_\delta \).

![Figure 1. The graphs G and G_\delta.](image-url)

The paper is organized as follows. In Section 2, we define \( \delta \) and \( \delta' \)-complement of a
graph and obtain their characterization. In Sections 3 and 4 we discuss some properties of
\( \delta \) and \( \delta' \)-complement of a graph, respectively.

3. Results

\( \delta/\delta' \)-Complement of a Graph

In this section, we define a new graph complement based on the degree sequence. We
begin by defining the \( \delta \)-complement of a graph.

**Definition 1.** Let \( G(V, X) \) be any graph of order \( n \) and size \( m \). For any two points \( u \) and \( v \) in \( V \)
with \( \deg u = \deg v \) remove the lines between \( u \) and \( v \) in \( G \) and add the lines of \( \overline{G} \) between \( u \) and \( v \).
The graph thus obtained is called \( \delta \)-complement of \( G \) and is denoted by \( G_\delta \).

**Definition 2.** A graph \( G \) is \( \delta \)-self-complementary (\( \delta \)-s.c.) if \( G \cong G_\delta \) and \( G \) is \( \delta \)-co-self-complementary
(\( \delta \)-c.s.c.) if \( G_\delta \cong \overline{G} \), where \( \overline{G} \) is the complement of \( G \).

**Definition 3.** Let \( G(V, X) \) be any graph of order \( n \) and size \( m \). For any two points \( u \) and \( v \) in \( V \)
with \( \deg u \neq \deg v \) remove the lines between \( u \) and \( v \) in \( G \) and add the lines of \( \overline{G} \) between \( u \) and \( v \).
The graph thus obtained is called \( \delta' \)-complement of \( G \) and is denoted by \( G_{\delta'} \).
Definition 4. A graph $G$ is $\delta'$-self-complementary ($\delta'$-s.c.) if $G \cong G_{\delta'}$ and $G$ is $\delta'$-co-self-complementary ($\delta'$-co.s.c.) if $G_{\delta'} \cong \overline{G}$, where $\overline{G}$ is the complement of $G$.

Figure 2 gives $\delta$—self-complementary and $\delta'$—co-self-complementary graphs and Figure 3 gives $\delta'$—co-self-complementary and $\delta'$—self-complementary graphs.

The Table 1 displays the degree sequence and number of lines $m(G_{\delta})$ in $\delta$-complement of various graphs.
Table 1. The degree sequence and number of lines in $\delta'$-complement of various graphs.

| Graph                          | Degree Sequence of $\delta'$-Complement | Number of Lines in $\delta'$-Complement |
|-------------------------------|----------------------------------------|----------------------------------------|
| $r$-regular graph with $n$ points | $\{n - r, 1, n - r - 1, \ldots, n - r - 1\}_{n\text{ times}}$ | $\frac{n(n - r - 1)}{2}$ |
| Path $P_n$                    | $\{2, 2, n - 3, n - 3, n - 5, \ldots, n - 5\}_{(n - 4)\text{ times}}$ | $\frac{n^3 - 7n + 18}{2}$ |
| Wheel graph $W_n$             | $\{n - 1, n - 3, n - 3, \ldots, n - 3\}_{(n - 1)\text{ times}}$ | $\frac{1}{2}(n^2 - 3n + 2)$ |
| Complete bipartite graph $K_{r, r_1, r_2}$, $r_1 \neq r_2$ | $\{r_1 + r_2 - 1, \ldots, r_1 + r_2 - 1\}_{(r_1 + r_2)\text{ times}}$ | $r_1 = r_2$ $\begin{cases} r_1 + \frac{r_2}{2}, & r_1 = r_2 \\ \frac{r_1 + r_2}{2}, & r_1 \neq r_2 \end{cases}$ |
| Friendship graph              | $\{2n, 2n - 1, 2n - 1, \ldots, 2n - 1\}_{(2n)\text{ times}}$ | $2n^2$ |
| Double star $S_{m+1, n+1}$, $m \neq n$ | $\{m + 1, n + 1, m + n, \ldots, m + n\}_{(m + n)\text{ times}}$ | $\begin{cases} 2n^2 + n, & m = n \\ \frac{(m + n)^2 + (m + n + 2)}{2}, & m \neq n \end{cases}$ |
| Cocktail party graph $K_{n \times 2}$ | $\{1, 1, \ldots, 1\}_{(2n)\text{ times}}$ | $n$ |

The degree sequence and number of lines of $\delta'$-complement for different graphs are summarized in Table 2.

Table 2. The degree sequence and number of lines of $\delta'$-complement for different graphs.

| Graph                          | Degree Sequence of $\delta'$-Complement | Number of Lines in $\delta'$-Complement |
|-------------------------------|----------------------------------------|----------------------------------------|
| $r$-regular graph with $n$ points | $\{r, r, \ldots, r\}_{(n)\text{ times}}$ | $\frac{nr}{2}$ |
| Path $P_n$                    | $\{n - 3, n - 3, 2, 2, 4, 4, \ldots, 4\}_{(n - 4)\text{ times}}$ | $3(n - 3)$ |
| Wheel graph $W_n$             | $\{n - 1, 1, 1, \ldots, 1\}_{(n - 1)\text{ times}}$ | $(n - 1)$ |
| Complete bipartite graph $K_{r, r_1, r_2}$, $r_1 \neq r_2$ | $\{0, 0, \ldots, 0\}_{(r_1 + r_2)\text{ times}}$ | $\emptyset$ |
| Friendship graph              | $\{0, 1, 1, \ldots, 1\}_{(2n)\text{ times}}$ | $n$ |
| Double star $S_{m+1, n+1}$, $m \neq n$ | $\{m, n, 1, 1, \ldots, 1\}_{(m + n)\text{ times}}$ | $\begin{cases} 2n + 1, & m = n \\ n + m, & m \neq n \end{cases}$ |
| Cocktail party graph $K_{n \times 2}$ | $\{2n - 2, 2n - 2, \ldots, 2n - 2\}_{(2n)\text{ times}}$ | $n(2n - 2)$ |

Proposition 1. For any graph $G$,

1. $\overline{G'} \cong G'_\delta$ and $\overline{G}_\delta \cong G'_\delta$.

Proof.

1. Let $u, v \in V(G)$.

   Then $u$ is adjacent to $v$ in $\overline{G'}$ if and only if $\deg u = \deg v$ and $u$ is not adjacent to $v$ in $G$ or $\deg u \neq \deg v$ and $u$ is adjacent to $v$ in $G$.

2. Let $u, v \in V(G)$.

   Then $u$ is adjacent to $v$ in $\overline{G}$ if and only if $\deg u \neq \deg v$ and $u$ is not adjacent to $v$ in $G$ or $\deg u = \deg v$ and $u$ is adjacent to $v$ in $G$. 


\[ \iff \ u \text{ is adjacent to } v \text{ in } G' \delta. \]

\[ \square \]

**Corollary 1.** For any graph \( G \),
1. \( G \delta \cong G \) if and only if \( G' \delta \cong \overline{G} \);
2. \( G' \delta \cong G \) if and only if \( G \delta \cong \overline{G} \).

**Proof.**
1. \( G \delta \cong G \iff \overline{G} \delta \cong \overline{G} \iff G' \delta \cong \overline{G} \) from Proposition 1;
2. \( G' \delta \cong G \iff \overline{G'} \delta \cong \overline{G} \iff G \delta \cong \overline{G} \) from Proposition 1.

\[ \square \]

**Corollary 2.** For any graph \( G \),
1. \( (\overline{G})_\delta \cong \overline{G} \) if and only if \( (\overline{G})' \delta \cong G \);
2. \( (G')_\delta \cong G \) if and only if \( (G)' \delta \cong \overline{G} \).

**Proof.**
1. Suppose \( (\overline{G})_\delta \cong \overline{G} \), then
\[ \overline{(G)}_\delta \cong \overline{(G)} \cong G. \] (1)

Let \( \overline{G} = H \). Then Equation (1) implies, \( \overline{H} \delta \cong G \).
From the Proposition 1, \( \overline{H} \delta = H' \delta \). Hence \( H' \delta \cong G \) which implies \( (\overline{G})' \delta \cong G \).
Conversely, suppose \( (\overline{G})' \delta \cong G \), then
\[ (\overline{G})'_\delta \cong \overline{G}. \] (2)

Let \( \overline{G} = H \). Then Equation (2) implies, \( \overline{H'} \delta \cong \overline{G} \).
From the Proposition 1, \( \overline{H'} = H \). Hence \( \overline{H} \delta \cong G \). That is, \( (\overline{G})_\delta \cong \overline{G} \).
Similarly we can prove statement 2.

\[ \square \]

We now characterize \( \delta \)-self-complementary of a graph based on its degree sequence.

**Proposition 2.** The graphs \( G \) and \( G_\delta \) are degree preserving if and only if \( V(G) \) can be partitioned as \( \{ V_1, V_2, \ldots, V_l \} \), \( \delta(G) \leq l \leq \Delta(G) \) such that \( \langle V_i \rangle \) is \( 2k \) regular graph of order \( 4k + 1 \).

**Proof.** Suppose \( \{ V_1, V_2, \ldots, V_l \} \), \( \delta(G) \leq l \leq \Delta(G) \) is a partition of the point set of \( G \) such that \( \langle V_i \rangle \) has \( 4k + 1 \) points with regularity \( 2k \). First, note that every point of each partite is adjacent to all the points of the remaining partites. Otherwise, \( G \) cannot be partitioned into \( l \) partites such that each \( \langle V_i \rangle \) consisting of \( 4k + 1 \) points with regularity \( 2k \). Let \( |V_i| = 4k_i + 1 \), \( 0 \leq k_i \leq \frac{n-1}{4l}, i = 1, 2, \ldots, l \) and let \( G_i = \langle V_i \rangle \), \( 1 \leq i \leq l \). The lines between different partites are unaltered in \( G_\delta \). Also \( \langle V_i \rangle = \overline{G_i}, i = 1, 2, \ldots, l \) with regularity of each point in \( \overline{G_i} \) is \( 2k_i \) in \( G_\delta \). Thus the graphs \( G \) and \( G_\delta \) are degree preserving.

Suppose that the graph \( G \) cannot be partitioned into \( l \) partites such that each \( \langle V_i \rangle \) consists of \( 4k + 1 \) points with regularity \( 2k \). Then the adjacency between the points of different degree is the same in both \( G \) and \( G_\delta \). However, the adjacency between the points of the same degree varies in \( G_\delta \). Thus, the degree sequence does not remain the same in both \( G \) and \( G_\delta \). Hence, \( G \) and \( G_\delta \) are not degree preserving. \( \square \)

**Remark 1.** A graph \( G \) is \( \delta \)-self-complementary if and only if \( V(G) \) can be partitioned as \( \{ V_1, V_2, \ldots, V_l \} \), \( \delta(G) \leq l \leq \Delta(G) \) such that each \( \langle V_i \rangle \) is a self-complementary graph.
Proposition 3. A graph $G$ is $\delta$-co-self-complementary if and only if each point of a given degree is adjacent to exactly half of the number of points of different degrees.

Proof. Suppose each point of a given degree, say $\eta_0$ is adjacent to exactly half of the number of points of different degrees, then in $\overline{G}$ and $G_\delta$ a point of degree $\eta_0$ is adjacent to exactly half of the number of points of different degrees. The adjacency of points of degree $\eta_0$ remains the same in both $\overline{G}$ and $G_\delta$ by definition of $G_\delta$. Since each point of a given degree is adjacent to exactly half of the number of points of different degrees, the structure of both $G_\delta$ and $\overline{G}$ remains the same. Hence $G_\delta \cong \overline{G}$.

Suppose each point of a given degree $\eta_0$ is not adjacent to exactly half of the number of points of different degrees, then in $\overline{G}$ a point $v$ of degree $\eta_0$ is adjacent to either less or more than the number of points of different degrees as in $G$. Also in $G_\delta$, point $v$ is adjacent to the same number of points of different degrees as in $G$. Thus $\deg v$ in $G_\delta \neq \deg v$ in $\overline{G}$ and hence $G_\delta \not\cong \overline{G}$. $\square$

Proposition 4. A graph $G$ is $\delta'$-self-complementary if and only if each point of a given degree is adjacent to exactly half of the number of points of different degrees.

Proof. If each point $v$ of a given degree in $G$ is adjacent to exactly half of the number of points of different degrees, then $v$ is adjacent to half of the remaining points of different degrees in $G_\delta'$ and to the same points of given degree as in $G$. Since every point $v$ of a given degree is adjacent to exactly half of the number of points of different degrees, the adjacency between every pair of points is preserved in $G_\delta'$. Hence $G \cong G_\delta'$.

Suppose each point of a given degree is not adjacent to exactly half of the number of points of different degrees in $G$. Then a point of a given degree is adjacent to either less or more than the number of points of different degrees and to equal number of points of the same degree as in $G$. Thus $G \not\cong G_\delta'$. $\square$

Proposition 5. The graphs $\overline{G}$ and $G_\delta'$ are degree preserving if and only if $V(G)$ can be partitioned as $\{V_1, V_2, \ldots, V_l\}$, $\delta(G) \leq l \leq \Delta(G)$ such that $\langle V_i \rangle$ is $2k$ regular graph of order $4k + 1$.

Proof. Suppose $\{V_1, V_2, \ldots, V_l\}$, $\delta(G) \leq l \leq \Delta(G)$ is a partition of the point set of $G$ such that $\langle V_i \rangle$ has $4k + 1$ points with regularity $2k$. Every point of each partite is adjacent to all the points of the remaining partites. Otherwise, $G$ cannot be partitioned into $l$ partites such that each $\langle V_i \rangle$ consists of $4k + 1$ points with regularity $2k$. Let order of $|V_i| = 4k_l + 1$, $0 \leq k_l \leq \frac{n-1}{2}$, $i = 1, 2, \ldots, l$. There are $k_l k_{l-1} \ldots k_1$ lines between the points of different partites. Then both the graphs $\overline{G}$ and $G_\delta'$ consist of $l$ connected components where each connected component is a $2k_l$ regular graph. Therefore, the graphs $\overline{G}$ and $G_\delta'$ are degree preserving.

Suppose the graph $G$ cannot be partitioned into $l$ partites such that each $\langle V_i \rangle$ consists of $4k + 1$ points with regularity $2k$. Then the adjacency between the points of different degree is the same in both $\overline{G}$ and $G_\delta'$. The adjacency between the points of equal degree remains the same in $G_\delta'$ as in $G$ but it varies in $\overline{G}$. Thus, the degree sequence does not remain the same in both $G_\delta'$ and $\overline{G}$. Hence, $\overline{G}$ and $G_\delta'$ are not degree preserving. $\square$

Remark 2. A graph $G$ is $\delta'$-co-self-complementary if and only if $V(G)$ can be partitioned as $\{V_1, V_2, \ldots, V_l\}$, $\delta(G) \leq l \leq \Delta(G)$ such that each $\langle V_i \rangle$ is a self-complementary graph.

Proposition 6. $G_\delta$ and $G_\delta'$ of a graph $G$ are degree preserving if and only if $G$ is $\frac{n-1}{2}$-regular graph.

Proof. Suppose $G$ is $\frac{n-1}{2}$-regular graph, then both $G_\delta$ and $G_\delta'$ are $\frac{n-1}{2}$-regular graph and hence both $G_\delta$ and $G_\delta'$ are degree preserving graphs. Conversely, from Proposition 2, $G$ and $G_\delta$ are degree preserving if and only if $G$ contains $4k + 1$ points of the same degree and all the points of a given degree are adjacent to exactly $2k$ points of the same degree. Also
from Proposition 4, \( G \cong G'_6 \) if and only if each point of a given degree is adjacent to exactly half of the number of points of different degrees. Thus Proposition 3 and 5 together imply that \( G \) is \( \frac{n-1}{2} \)-regular graph of order \( n \). \( \square \)

4. Some Properties of \( \delta \)-Complement of a Graph

Theorem 1. A graph \( G_\delta \) is a complete graph if and only if \( G \) is a complete multipartite graph with the partition of the point set \( \{ V_1, V_2, \ldots, V_k \} \) with \( |V_i| \neq |V_j| \) for all \( i \neq j \).

Proof. Since the lines between \( V_i \) and \( V_j, i \neq j \) of \( G \) remain in \( G_\delta \) and each \( \langle V_i \rangle \) is isomorphic to complete subgraph, the resultant \( G_\delta \) is a complete graph.

Conversely, suppose that \( G \) is not a complete multipartite graph, then \( G \) may have a line between the points of the same degree or there may exist two non-adjacent points of different degrees in \( G \). If \( G \) has a line between the points \( u \) and \( v \) of same degree, then in \( G_\delta \), there is no line between \( u \) and \( v \) and hence \( G_\delta \) is not complete. If \( G \) has two non-adjacent points of different degrees, then those two points are non-adjacent in \( G_\delta \) also. Hence \( G_\delta \) is not a complete graph. \( \square \)

Remark 3. The \( \delta \)-complement of the star graph of order \( n \) is isomorphic to complete graph \( K_n \).

Theorem 2. If \( G \) is a disconnected graph of order \( n \) with each connected component of \( G \) is \( r \)-regular, then \( G_\delta \) is a connected \( (n - r - 1) \)-regular graph.

Proof. Let \( G_1, G_2, \ldots, G_n \) be connected \( r \)-regular components of \( G \). Then the degree of each point of \( G_\delta \) is \( |V(G_i)| - (r+1) + \sum_{j=1}^{m} |V(G_j)| \), which is \( n - r - 1 \). \( \square \)

Theorem 3. Let \( G \) be a connected graph. The graph \( G_\delta \) is disconnected if

1. \( G \) is a complete graph;
2. \( G \) is a connected \( n - 2 \) regular graph of order \( n \);
3. \( G \) has a point \( v \) of degree \( k \) which is only adjacent to every point of degree \( k \).

Proof. Let \( G \) be a connected graph.

1. If \( G \) is a complete graph then \( G_\delta \) is completely disconnected by the definition of \( G_\delta \); 
2. Suppose \( G \) is a connected \( n - 2 \) regular graph of order \( n \). Then \( n \) must be even and \( G_\delta \) is isomorphic to \( \frac{n}{2} K_2 \); 
3. Suppose \( G \) has a point \( v \) of degree \( k \) that is only adjacent to every point of degree \( k \). Then \( v \) is not adjacent to any of the points in \( G_\delta \). Thus, \( G_\delta \) is disconnected. \( \square \)

Remark 4. The converse of Theorem 3 need not be true. 

For example, consider a regular graph \( C \) in Figure 4. \( C \) is a connected 3-regular graph of order 6, where \( 3 \neq n - 1, n - 2 \). However, the graph \( C_\delta \) is a disconnected 2-regular graph.

![Figure 4. The graph C and its \( \delta \)-complement \( C_\delta \).](image-url)
Consider a non-regular graph of order 8 in Figure 5. Here, no point is adjacent to all the points of the same degree. However, $D_δ$ is disconnected.

![Graph D and its δ-complement D_δ](image)

**Figure 5.** The graph $D$ and its $δ$-complement $D_δ$.

**Theorem 4.** Let $G$ be an Eulerian graph of order $n$. Then $G_δ$ is Eulerian if $G$ has odd number of points of the same degree and every point of a given degree is adjacent to at most half of the points of the same degree and at least half of the points of each different degrees.

**Proof.** Let $G$ be an Eulerian graph of order $n$. $G$ has an odd number of points of the same degree. Consider point $v \in V(G)$ of degree $m$. Either $v$ is adjacent to points of degree $m$ or points of degree other than $m$.

Suppose $v$ is adjacent to an even number of points of degree $m$ and an even number of points of degree other than $m$. As $G$ has an odd number of points of the same degree and by definition of $G_δ$, even number of lines incident on $v$ are removed and an even number of lines that were non-incident on $v$ in $G$ are added. Hence, the degree of $v$ is even in $G_δ$.

Suppose $v$ is adjacent to an odd number of points of degree $m$ and an odd number of points of degree other than $m$. In $G_δ$, $v$ is adjacent to an odd number of points of degree $m$ since $G$ has an odd number of points of the same degree and to an odd number of points of a different degree. Therefore, $v$ is of even degree in $G_δ$.

In addition, it is given that every point of a given degree is adjacent to at most half of the points of the same degree and at least half of the points of each different degrees. In $G_δ$, every point is adjacent to at least half of the points of same degree and at most half of the points of each different degrees. Therefore, the degree of each vertex of $G_δ$ is at least $\frac{n-1}{2}$ and hence it is connected. Therefore, the graph $G_δ$ is Eulerian. □

**Theorem 5.** Let $G$ be a Hamiltonian graph. Then $G_δ$ is Hamiltonian if
1. $G$ is an $r$-regular graph with $n ≥ 3$, $r ≤ \frac{n-2}{2}$;
2. $G$ is a non-regular graph with no point $v$ adjacent to any point with a degree, as that of $v$.

**Proof.** Let $G$ be a Hamiltonian graph.

1. If $G$ is an $r$-regular graph with $n ≥ 3$, $r ≤ \frac{n-2}{2}$, then $G_δ$ is $n - r - 1$ regular graph with $n - r - 1 ≥ \frac{n}{2}$. From this, it follows that $G_δ$ is Hamiltonian;
2. Suppose $G$ is a non-regular graph with no point $v$ adjacent to any point with degree as that of $v$. As every adjacent points of Hamiltonian cycle have different degrees in $G$, we see that the Hamiltonian cycle of $G$ remains in $G_δ$.

□

**Remark 5.** The converse of Theorem 5 need not be true.

For example, consider the graph $K_8 - C_8$ in Figure 6. The graph $K_8 - C_8$ is a 5-regular graph and $5 > \frac{n-2}{2} = 3$. It is a Hamiltonian graph with Hamiltonian cycle $v_1v_2v_3v_8v_4v_5v_7v_1$. The $δ$-complement of the graph $K_8 - C_8$ is $C_8$, which is also a Hamiltonian graph.
Consider a non-regular graph $H$ in Figure 7. The graph $H$ is a Hamiltonian graph with Hamiltonian cycle $v_7v_2v_4v_6v_1v_5v_3$. The point $v_3$ is of degree 4 and it is adjacent to the point $v_4$ of degree 4. However, the graph $\delta H$ is also Hamiltonian with a Hamiltonian cycle $v_2v_5v_3v_1v_6v_4v_2$.

Theorem 6. For any point $u \in V(G)$, $\deg u$ in $G_\delta = \deg u$ in $G$ if and only if $G$ has an $m$ odd number of points of equal degree and $u$ is adjacent to exactly $m - \frac{1}{2}$ points of equal degree.

Proof. Suppose that $G$ has an $m$ odd number of points of equal degree and the point $u$ is adjacent to exactly $m - \frac{1}{2}$ points of equal degree, then in $G_\delta$, $\deg u = \deg u$ in $G - \frac{m - 1}{2} + \frac{m - 1}{2} = \deg u$ in $G$.

Conversely, suppose that $u$ is adjacent to less than $m - \frac{1}{2}$ points of the equal degree, then in $G_\delta$ the point $u$ will be adjacent to more than $m - \frac{1}{2}$ points of the equal degree and the same number of points of unequal degree and hence the degree of $u$ in $G_\delta$ will be more than the degree of $u$ in $G$. A similar argument holds if $u$ is adjacent to more than $m - \frac{1}{2}$ points of the same degree. Thus $\deg u$ in $G_\delta \neq \deg u$ in $G$. □

Theorem 7. $\deg v$ in $G + \deg \delta v$ in $G_\delta = n - 1$ for all $v \in V(G)$ if

1. $G$ is regular or;
2. $G$ is an even order bi-regular graph of degree $r$ and $s$ such that every point of a particular degree is adjacent to half of the points of different degree.

Proof. Suppose that $G$ is $r$–regular. Then $G_\delta$ is a $(n - r - 1)$–regular graph. Thus $\deg v$ in $G + \deg \delta v$ in $G_\delta = n - 1$ for all $v \in V(G)$.

Suppose $G$ is a bi-regular graph of degree $r$ and $s$ such that every point of degree $r$ is adjacent to half of the points of degree $s$. Let $n_1$ and $n_2$ be the number of points of degree $r$ and $s$ respectively. Consider a point $v$ of degree $r$. Suppose $v$ is adjacent to $n_s \frac{r}{2}$ points of degree $s$ and $x$ points of degree $r$. In $G_\delta$ the point $v$ is adjacent to $n_r \frac{s}{2}$ points of degree $s$ and $n_r - x - 1$ points of degree $r$. Thus, $\deg v$ in $G + \deg \delta v$ in $G_\delta = x + n_s \frac{r}{2} + n_1 - x - 1 + n_r \frac{s}{2} = n_1 + n_2 - 1 = n - 1$. □
5. Some Properties of $\delta'$-Complement of a Graph

**Proposition 7.** Let $v \in V(G)$ be the only point of maximum degree $\Delta(G)$. If $G$ is $\delta'$-self-complementary, then $G$ has a point of degree $n - \Delta(G) - 1$.

**Proof.** Let $v \in V(G)$ be the only point of maximum degree $\Delta(G)$. Then in $G_{\delta'}$, $\deg v = n - \Delta(G) - 1$. Since $G$ is $\delta'$-s.c., $G$ must have a point of degree $n - \Delta(G) - 1$. \hfill $\Box$

**Proposition 8.** Let $v \in V(G)$ be the only one point of minimum degree $\delta(G)$. If $G$ is $\delta'$-self-complementary, then $G$ has a point of degree $n - \delta(G) - 1$.

**Proof.** Let $v \in V(G)$ be the only point of degree $\delta(G)$. Then, in $G_{\delta'}$, $\deg v = n - \delta(G) - 1$. Since $G$ is $\delta'$-s.c., $G$ must have a point of degree $n - \delta(G) - 1$. \hfill $\Box$

**Theorem 8.** Let $G$ be a graph with $n$ points. Then $G_{\delta'}$ is disconnected if
1. There exists $v \in V(G)$ such that $N(v) = \{ u \in V(G) \mid \deg v \neq \deg u \}$;
2. Every point of equal degree is adjacent to all the points of different degree;
3. $G$ is a disconnected regular graph.

**Proof.**
1. Suppose there exists a point $v \in V(G)$ such that $N(v) = \{ u \in V(G) \mid \deg v \neq \deg u \}$. Then $v$ is an isolated point in $G_{\delta'}$.
2. Suppose every point of degree $m$ is adjacent to all the points of degree other than $m$. In $G_{\delta'}$, all the points of degree $m$ will not be adjacent to any point of degree other than $m$, which makes it disconnected;
3. $G_{\delta'} \cong G$ if $G$ is a regular graph and is hence disconnected. \hfill $\Box$

**Theorem 9.** Let $G$ be an Eulerian graph of order $n$. Then $G_{\delta'}$ is Eulerian if
1. $G$ is regular;
2. $G$ is a non-regular graph such that the number of points of equal degree are even and every point of a given degree is adjacent to at least half of the points of the same degree and at most half of the points of each different degree.

**Proof.**
1. Let $G$ be an Eulerian graph of order $n$. Then, all the points of $G$ are of even degree. If $G$ is regular, then $G_{\delta'}$ is a regular graph with $G \cong G_{\delta'}$ and hence $G_{\delta'}$ is Eulerian;
2. Suppose that $G$ is non-regular Eulerian graph and the number of points of same degree are even in $G$. Let $v$ be a point of degree $m$ that is adjacent to an even number of points of degree $m$ and an even number of points of degree other than $m$. Then, in $G_{\delta'}$, the point $v$ is adjacent to the same number of points of degree $m$ and an even number of points of degree other than $m$, since there are an even number of points of equal degree. If a point $v$ of degree $m$ is adjacent to an odd number of points of degree $m$ and an odd number of points of degree other than $m$, then in $G_{\delta'}$, the point $v$ is adjacent to the same number of points of degree $m$ and an odd number of points of degree other than $m$, since there is an even number of points of equal degree. Thus in all the cases, the degree of every point of $G_{\delta'}$ is even. In addition, it is given that every point of a given degree is adjacent to at least half of the points of the same degree and at most half of the points of each different degrees. Therefore, in $G_{\delta'}$, every point of a given degree is adjacent to at least half of the points of same degree and at least half of the points of distinct degrees. Then, the degree of each point is at least $\frac{n-1}{2}$ in $G_{\delta'}$. Hence $G_{\delta'}$ is a connected graph with all even degree points. Therefore, $G_{\delta'}$ is Eulerian. \hfill $\Box$

**Remark 6.** Let $G$ be a Hamiltonian graph. The graph $G_{\delta'}$ is Hamiltonian if $G$ is a $r$-regular graph.
Proof. Let $G$ be a Hamiltonian graph. From Table 2, it is clear that $G \cong G_{\delta'}$ if $G$ is regular. Hence, $G_{\delta'}$ is Hamiltonian. □

Remark 7. The converse of Remark 6 need not be true. For example, consider a non-regular graph $J$ in Figure 8. Both the graphs $J$ and $J_{\delta'}$ are Hamiltonian with Hamiltonian cycles $v_1v_2v_3v_4v_5v_6v_1$ and $v_1v_3v_5v_4v_6v_2v_1$ respectively.

Figure 8. The graph $J$ and its $\delta'$-complement $J_{\delta'}$.

6. Conclusions

In this paper, the authors defined the $\delta$ and $\delta'$-complement of a graph and explored properties such as degree preserving, $\delta(\delta')$-self-complementary, $\delta(\delta')$-co-self-complementary, and connectedness. As these complements are based on like/unlike characters, they can be easily applied in social network problems and optimization problems. There is a huge scope to study these complements in comparison with existing self-complementary graphs.

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References
1. Richard, A. Gibbs, Self-complementary graphs. J. Comb. Theory 1974, 16, 106–123.
2. Nair, P.S. Construction of self-complementary graphs. Discret. Math. 1997, 175, 283–287. [CrossRef]
3. Nayak, S.; Dsouza, S.; Gowtham, H.J.; Bhat, P.G. Energy of Partial Complements of a Graph. Proc. Jangjeon Math. Soc. 2020, 22, 369–379.
4. Harary, F. Graph Theory; Addison-Wesley: New York, NY, USA, 1971.
5. West, D.B. Introduction to Graph Theory; Prentice Hall: Upper Saddle River, NJ, USA, 2001.