Static stability investigation of a tapered asymmetric sandwich beam supported on variable Pasternak foundation

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Abstract The investigation to analyse a sandwich beam’s static stability with asymmetric configuration, tapered along the thickness, placing on a Pasternak foundation having linearly varying stiffness and influenced by an alive axial load is executed for several boundary conditions employing computational method. Use of Hamilton’s principle results in the equations of motion and related boundary conditions. Hill’s equations are achieved from the non-dimensionalized equations of motion with the use of Galerkin’s method. Then, the effects of various parameters on the static stability for different boundary conditions are obtained and are showcased in a sequence of graphs using the appropriate MATLAB program.

Keywords: Static stability; Taper parameter; Core loss factor; Pasternak foundation.

1. Introduction

To achieve superior characteristics like greater stiffness-to-weight ratio, sandwich beams are extensively developed for different engineering structures such as aerospace, helicopter blades, etc. The beams on different foundations are also important and impact of foundations must be considered during the design of beams. The beams can be economical with the variation of cross-section configuration. Kar and Sujata [1] investigated the stability of a non-uniform configuration beam subjected to temperature gradient. The same authors [2] evaluated the stability of a cantilever type sandwich beam under the effect of periodic load having symmetric configuration. They witnessed that the geometric and shear parameter enhances the system’s stability, while the taper parameter had a detrimental effect on stability. They realized that the beam's taper profile, thermal gradient, and elastic foundation stiffness affected the stability. Asnani and Nakra [3] established the equations of motion for a multi-layered sandwich beam and acquired the vibration damping features of beams with 15 layers and simply supported at the ends for cases such as constant weight, constant size, and flexural rigidity. Ray and Kar [4] inspected a sandwich beam with 3-layers and symmetric configuration for several boundary conditions. They detected that the core's loss factor, along with the shear parameter, improved the beam’s stability. Chand et al. [5] examined the stability of a rotational beam with a parabolic-tapered profile and variable temperature grade. The dynamic along with static stability of a tapered sandwich beam lying on a Pasternak foundation in temperature environment is investigated by Pradhan et al.[6]. Pradhan et al. [7] examined a tapered symmetric sandwich beam's stability condition, which is on a variable Pasternak foundation. Pradhan et al. [8] examined a tapered sandwich beam's stability condition, which is on a variable Pasternak foundation. Chand et al. [9] examined the stability of a rotational beam with a parabolic-tapered profile and variable temperature grade. Pradhan and Dash [10] inspected the stability of a non-uniform sandwich beam and viscoelastic support with variable temperature gradient.

The literature assessment informs that selective study has been performed for the stability of non-uniform beams with various conditions. Nevertheless, no research has been implemented before to analyse the static stability investigation of an asymmetric sandwich beam tapered along thickness placing on variable Pasternak foundation. This research work investigates the above-suggested configuration.
2. Formulation of the Problem

An exponentially taper 3-layer asymmetric sandwich beam and length \( l \) resting on a Pasternak foundation with variable stiffness and an alive axial load, \( P_a(t) = P_{0a} + P_{1a} \cos(\omega t) \) about the undeformed axis, is presented in figure 1. \( P_{0a} \) and \( P_{1a} \), represent the static load amplitude and dynamic load amplitude. The elastic layers are with young’s modulus \( E_i \) at top and \( E_j \) at the bottom while the thickness are \((2h_1)\) and \((2h_3)\) respectively. The middle viscoelastic core material is having shear modulus \( G_j \) and thickness \((2h_2)\). At the end, the dimension of the layers are \((2h_1)\), \((2h_3)\), and \((2h_j)\) respectively. The Pasternak foundation consists of number of narrowly and evenly placed vertical springs having varying spring constant \( k(x) \) and a shear layer having thickness \( d \).

The assumptions to formulate equations of motion are the same as in [7].

\[\text{Fig 1: configuration diagram}\]

In this, Potential energy is expressed as

\[ V = \frac{1}{2} \int_0^l \left( E_1 A_1 U_{1,x}^2 + E_3 A_3 U_{3,x}^2 \right) dx + \frac{1}{2} \int_0^l \left( E_i I_i + E_j I_j \right) w_{i,x}^2 dx + \frac{1}{2} \int_0^l G_j A_j \gamma_2^2 dx + \frac{1}{2} \int_0^l G_S B d \gamma_2^2 dx + \frac{1}{2} \int_0^l B d k(x) \gamma_2^2 dx \]

The Kinetic energy is expressed as

\[ T = \frac{1}{2} m \int_0^l w_j^2 dx \]

Work done is expressed as

\[ W_P = \frac{1}{2} \int_0^l P(t) w_j^2 dx \]

Where \( U_1 \) and \( U_3 \) denotes the axial displacements in the elastic layers, \( w_{i,x} = \frac{\partial w_i}{\partial x} \), \( w_{i,t} = \frac{\partial w_i}{\partial t} \) and \( \gamma_2 \) denotes the shear strain in the middle layer, presented by \( \gamma_2 = \frac{U_1 - U_3}{2h_2} - \frac{C w_{1,x}}{2h_2} \). Subsequent complete mathematical modelling equations are reached with the aid of Hamilton’s principle.
\[
\delta \int_{t_i}^{t_f} \left( T - V + W_p \right) dt = 0
\]

\[
\bar{w}_b \bar{w}_{xx} + (1 + Y_b) \bar{w}_{xxxx} - \left[ \frac{3}{2} \frac{G_{sb}}{E_{sb}} \int_{t_i}^{t_f} \left( \frac{b}{l + E_{31b}h_{31b}} \right) \bar{P}_b(T) dt \right]_{0}^{1} \bar{w}_{xx} = 0
\]  

(1)

\[
\frac{3}{2} \frac{K_s(x)}{E_{sb}} \frac{\left( \frac{b}{l + E_{31b}h_{31b}} \right)}{C_b} + \frac{2(h_2)_0}{C_b} \gamma_{2b,xx} = 0
\]

(2)

Here, \( C_b = (h_1)_b + (2h_2)_b + (3h_3)_b \)

\[
t_{ob} = \left[ \frac{\rho \alpha (h_1)_0}{E_{sb} (l)_0} \right]^{1/2}, \bar{p}_{ob} = \frac{P_{ob}^2}{E_{sb}(l)_0}, \bar{p}_{ib} = \frac{P_{ib}^2}{E_{sb}(l)_0}, \bar{p}_b = \frac{P_{ob} + P_{ib} \cos (\bar{w}_T)}{C_b}, \bar{w}_{ob} = \omega T_0
\]

\[
\bar{M}_b = \frac{\rho}{(2b_h)_0} \left( \frac{2(h_2)_0}{(h_1)_0} + \left( (h_1)_0 - (h_1)_0 e^{-\beta_1T} \right) + (h_3)_0 - (h_3)_0 e^{-\beta_2T} \right) \frac{2(h_1)_0}{2(h_1)_0}
\]

(3)

\[
S_b = \frac{G_{sb}^*}{4E_{sb}^2h_{21b}h_{31b}^2} \left( 1 + E_{31b}h_{31b} \right)
\]

(4)

\[
Y_b = \frac{3 \left( 1 - \alpha a_{1b} l \right) \left( 1 + 2 \beta + h_1 \right)}{(1 + \alpha_a) \left( 1 - \alpha a_{1b} l \right)^3 + \frac{E_3}{E_1} h_{31} \left( 1 - \alpha a_{3b} l \right)^3}
\]

(5)

Where, \( \alpha_{1b} = \left[ 1 - \frac{(h_2)_0}{(h_1)_0} \right], \alpha_{3b} = \left[ 1 - \frac{(h_3)_0}{(h_1)_0} \right], \beta = \frac{(h_2)_0}{(h_1)_0}, \mu_1 = \frac{(2)}{(h_1)_0}, \mu_2 = \frac{(h_1)_0}{(h_1)_0}
\]

The connected boundary conditions are satisfied at both ends

\[
(1 + Y_b) \bar{w}_{xxxx} + \frac{2(h_2)_0}{C_b} \gamma_{2b,xx} = 0
\]

(6)

Or

\[
\bar{w}_{ob} = 0
\]

(7)

\[
(1 + Y_b) \bar{w}_{xx} + \frac{2(h_2)_0}{C_b} \gamma_{2b,x} = 0
\]

(8)

Or

\[
\bar{w}_{ob} = 0
\]

(9)

\[
Y_b \frac{2(h_2)_0}{C_b} \gamma_{2b,x} + \bar{w}_{ob} = 0
\]

(10)

Or

\[
\gamma_{2b} = 0
\]

(11)

2.1. Approximation Solution

For (1)-(2), the approximate solutions are assumed to be as follows
Where \( f_j \) and \( f_i \), \( w_j \) and \( \gamma_i \) are used as the time shape function and co-ordinate shape functions respectively. These are considered in such a manner so that it will satisfy maximum number of boundary conditions and the equations of motion [5]. By using Galerkin method and putting the solutions achieved from (12)-(13) in dimensionless form of equation of motion (1)-(2), the consequent equations are obtained in matrix form. As in [3], the shape function used for the considered boundary condition.

\[
\begin{bmatrix}
Q_{1b} \\
Q_{2b}
\end{bmatrix} = \begin{bmatrix} f_{ib} & \cdots \cdots & f_{0b} \\
\end{bmatrix}^T \\
\begin{bmatrix} Q_{1b} \\
Q_{2b}
\end{bmatrix} = \begin{bmatrix} f_{(p+1)b} & \cdots \cdots & f_{2pb} \\
\end{bmatrix}^T
\]

(14)

(15)

Where, \( M_{1ijb} = \frac{1}{E_{ib}}w_{ib}w_{jb}d\vec{x} \)

(18)

\( k_{11bij} = (1+Y_b)\int_0^1 w_{ib}'' w_{jb}'' d\vec{x} + \phi_{ib} \int_0^1 w_{ib}w_{jb} d\vec{x} + \int_0^1 \phi_{ib}'' (\vec{T}) \int_0^1 w_{ib}'' w_{jb}'' d\vec{x} \)

(19)

\( k_{12bjk} = Y_b \int_0^1 w_{jb}'' u_{kb}'' d\vec{x} \)

(20)

\( k_{22bkl} = Y_b \int_0^1 u_{kb}'' u_{lb}'' d\vec{x} + g_{b}^2 Y_b \int_0^1 u_{kb}u_{lb} d\vec{x} \)

(21)

In the above, \( u_{kb} = \frac{2h_{2b}}{C_b} \gamma_{bk}, u_{lb} = \frac{2h_{2b}}{C_b} \gamma_{bl} \) and \( w_{ib}'' = \frac{\partial^2 w_{ib}}{\partial \vec{x}^2} \)

\( \lambda_{sb} = \left( \frac{k_{ib}}{E_1} \right), \phi_{sb} = \frac{3 \lambda_{sb} (h_{1b})^3}{2(1 + E_{3ib} h_{1b}^2)} \), \( \gamma_{sb} = \frac{3G_{sb} d_p (l_{1b})^3}{2E_{1b} (1 + E_{3ib} h_{31b}^3)} \)

(22)

(23)

The equations (14) and (15) are further simplified to

\[
[m_b](\dot{Q}_{1b}) + [k_{11b}](Q_{1b}) + [k_{12b}](Q_{2b}) = \{0\}
\]

(24)

Where, \( \dot{Q}_{1b} = [k_{ib}] - [k_{12b}] [k_{22b}]^{-1} [k_{12b}]^T \)

(25)

\( H_{ijb} = \int_0^1 w_{ib}'' w_{jb}'' d\vec{x} \)

(26)

And

\[
[k_{ib}]^{-1} = (1+Y_b)\int_0^1 w_{ib}'' w_{jb}'' d\vec{x} + \phi_{ib} \int_0^1 w_{ib}w_{jb} d\vec{x} + \Psi_{ib} \int_0^1 w_{ib}'' w_{jb}'' d\vec{x}
\]

(27)

2.2. Critical Buckling Loads
Substituting $\overline{P}_1 = 0$ and $\{\hat{Q}_1\} = 0$ in equation (24) results in eigenvalue problem

$[k_b]^{-1}[H_b]\{Q_{lb}\} = \frac{1}{P_{lb}}\{Q_{lb}\}$. The static critical buckling loads ($P_{lb\text{crit}}$) are acquired by taking the real parts of the inverse of the eigenvalues of $[k_b]^{-1}[H_b]$.

3. Result Analysis

The model of sandwich beam is explored for various boundary conditions like fixed-fixed and hinged-hinged. Numerical values are determined for various parameters like temperature grade parameter, width taper parameter, and depth taper parameters. The given values of parameters are taken for the considered problem

$\eta = 0.01, \alpha = 1, \overline{P}_0 = 0.5, \delta_1 = 0.1, \delta_2 = 0.5$

The temperature at any distance $\xi$ from the beam end is supposed to be $\psi = \psi_0 (1 - \xi)$. By choosing the temperature as $\psi = \psi_0$ at the end and taking the reference temperature at $\xi = 1$, the change in the beam’s young’s modulus is indicated by [2]

$E(\xi) = E_0\left[1 - \lambda \psi_0 (1 - \xi)\right], \quad 0 \leq \lambda \psi_0 < 1$

$= E_0 T(\xi)$

Here we are considering, $\delta_z = \lambda T(\xi)$ is the temperature gradient while $T_0(\xi) = \left[1 - \delta_0 T(1 - \xi)\right]$

$\delta_z$ represents the variable temperature gradient while $\delta_{\delta i} = \delta_i / e^{-\alpha_i \xi}, i = 1, 2$

Here, $\alpha_i = \frac{E_{ibc} A_i}{E_{ibc} A_i[1 - \delta_{ibc} / e^{-\alpha_i \xi}]}$,

$= \frac{E_{ibc} A_i[1 - \delta_{ibc} / e^{-\alpha_i \xi}]}{E_{ibc} A_i[1 - \delta_{ibc} / e^{-\alpha_i \xi}]}$

Where $\delta_i$ and $\delta_{\delta i}$ represents the temperature gradient in the 1st and 3rd layer respectively.

For relevant parameter values by considering equal thickness for the top and bottom elastic layers and considering zero temperature gradient, the equation of motion can be found to be matching with Kar and Sujata [1]. Using MATLAB, the present code is validated for clamped-free condition at the ends with the results in Kar and Sujata [1] and good agreement was attained.

**Results Validation**

Considering the relevant parameters of the sandwich beam presented in the work of Kar and Sujata [1], the critical buckling loads of the sandwich beam having equal thickness of the face layers, in the absence of temperature gradient and Pasternak foundation have been obtained. The results are in good agreement with Kar and Sujata [1].

| Condition at the ends of the sandwich beam | Critical buckling loads |
|------------------------------------------|-------------------------|
| Mode No. | Present result | Figures 3 and 4 of Kar and Sujata [1] |
| Clamped-free (results for geometric parameter with the values $\eta = 0.3, Y = 5, g = 0.1$) | 1 | 7.82 | 7.80 |
|  | 2 | 65.20 | 65.10 |
|  | 3 | 112.60 | 112.50 |
| Clamped-free (results for taper parameter with the values $Y = 50, g = 0.1, \alpha = 0.2$) | 1 | 20.70 | 20.50 |
|  | 2 | 60.15 | 60.00 |
|  | 3 | 108.60 | 108.50 |

Table 1: Result validation of critical buckling loads with Kar and Sujata [1]

3.1. Static Stability Analysis
The following figures represent the influence of several parameters on the static stability for pinned-pinned boundary condition. Figures 2 to 7 (a) indicate the dependency of buckling loads for the static condition of the system upon various parameters for hinged-hinged and Figures 2 to 7 (b) for fixed-fixed end conditions.

**Fig 2**: Static buckling load varying with $\delta_1$

Figure 2 presents the impact of $\delta_1$ on the critical buckling load. With the rise in $\delta_1$, the critical buckling load decreases. A similar effect is also observed due to $\delta_2$ (figure not shown).

**Fig 3**: Static buckling load varying with $\alpha_3$

Figure 3 showcases the consequence of $\alpha_3$ on the static buckling load. Due to the increment in $\alpha_3$, the critical buckling load decreases. A similar effect is observed for $\alpha_1$, so it is not shown here.
The influence of \( \frac{d}{l} \) on the critical buckling load is displayed by figure 4. The critical buckling load enhances with the rise in \( \frac{d}{l} \) values.

Figure 5 displays the impact of \( \frac{G_S}{E_1} \) on the critical buckling load. Surge in the \( \frac{G_S}{E_1} \) values, enhances the critical buckling load.
Figure 6 demonstrates the impact of the $K_0$ on the critical buckling load. With the rise in the $K_0$, the critical buckling load enhances.

Figure 7 displays the result of shear parameter $g$ on the critical buckling load. With the increment in $g$, the static buckling loads increases for both hinged-hinged and fixed-fixed conditions.

4. Concluding Remarks

This paper investigated the static stability of an asymmetric sandwich beam with exponential taper layers supported by a varying Pasternak foundation. An axial pulsating load is acting with temperature gradient for pinned-pinned and clamped-clamped boundary condition. The program is developed by using MATLAB.

Form result and discussion part we can conclude that, for both hinged-hinged and clamped-clamped boundary condition with increase in $\delta_1$, $d/l$, $G_2/E_1$, $K_0$ and $g$ the static buckling loads increases that is it is safe to operate at higher value of these parameters. For $\alpha_1$, $\alpha_3$ and $\delta_1$, the static buckling load decreases for both the end conditions.

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