Strong Converse Theorems for Classes of Multimessage Multicast Networks: A Rényi Divergence Approach

Silas L. Fong and Vincent Y. F. Tan

Abstract

This paper establishes that the strong converse holds for some classes of discrete memoryless multimessage multicast networks (DM-MMNs) whose corresponding cut-set bounds are tight, i.e., coincide with the set of achievable rate tuples. The strong converse for these classes of DM-MMNs implies that all sequences of codes with rate tuples belonging to the exterior of the cut-set bound have average error probabilities that necessarily tend to one (and are not simply bounded away from zero). Examples in the classes of DM-MMNs include wireless erasure networks, DM-MMNs consisting of independent discrete memoryless channels (DMCs) as well as single-destination DM-MMNs consisting of independent DMCs with destination feedback. Our elementary proof technique leverages the properties of the Rényi divergence.

Index Terms

Strong converse, Multimessage multicast networks, Rényi divergence, Wireless erasure networks

I. INTRODUCTION

This paper considers multimessage multicast networks (MMNs) [1, Chapter 18] in which the destination nodes want to decode the same set of messages transmitted by the source nodes. A well-known outer bound on the capacity region of the discrete memoryless MMN (DM-MMN) is the cut-set bound, developed by El Gamal in 1981 [2]. This bound states that for any cut $T$ of the network with nodes indexed by $I$, the sum of the achievable rates of messages on one side of the cut is upper bounded by the conditional mutual information of the input variables in $T$ and the output variables in $T^c \triangleq I \setminus T$ given the input variables in $T^c$. The DM-MMN is a generalization of the well-studied discrete memoryless relay channel (DM-RC) [3]. It is known that the cut-set bound is not tight in general [4], but it is tight for several classes of DM-MMNs, including the physically degraded DM-RC [3], the semi-deterministic DM-RC [5], the deterministic relay network with no interference [6], the finite-field linear deterministic network [7], [8] and the wireless erasure network [9].

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One potential drawback of the cut-set bound is the fact if it is tight, i.e., there exists a matching achievable inner bound, this only implies a weak converse for the problem. In other words, it only guarantees that for all rate tuples not belonging to the region prescribed by the cut-set bound, the average error probability in decoding the transmitted messages is bounded away from zero as the block length of any code tends to infinity. In information theory, it is also important to establish strong converses as such definitive statements indicate that there is a sharp phase transition between rate tuples that are achievable and those that are not. A strong converse implies that for all codes with rate tuples that are in the exterior of the region prescribed by the fundamental limit, the error probability must necessarily tend to one. The contrapositive of this statement can be stated as follows: All codes whose error probabilities are no larger than $\epsilon \in [0,1)$ as the block length grows, i.e., $\epsilon$-reliable codes, must have rate tuples belonging to the region prescribed by the fundamental limit (in our case, a looser version of the cut-set bound that is tight for some DM-MMNs). This is clearly a stronger statement than the weak converse which considers codes with vanishing error probabilities.

A. Main Contribution

The main contribution of this work is a self-contained proof of the strong converse for some classes of DM-MMNs in which the cut-set bound is tight. These classes of DM-MMNs include deterministic relay networks with no interference [6], finite-field linear deterministic networks [7], [8] and wireless erasure networks [9]. So for example, for wireless erasure networks studied by Dana, Gowaker, Palanki, Hassibi and Effros [9], all sequences of codes with rates above the capacity have average error probabilities that necessarily tend to one as the block length grows. The authors of [9] proved using Fano’s inequality [10, Section 2.10] that all codes with rates above capacity have average error probabilities that are bounded away from zero. Thus, a consequence of our main result is an important strengthening of the converse in [9, Theorem 2]. In addition, we show, using our main theorem, that the strong converse holds for DM-MMNs consisting of independent discrete memoryless channels (DMCs) and single-destination DM-MMNs consisting of independent DMCs with destination feedback. Our main result implies that for the aforementioned DM-MMNs, rate tuples of $\epsilon$-reliable codes where $\epsilon \in [0,1)$ must belong to the region prescribed by the cut-set bound [2].

The technique that we employ is based on properties of the Rényi divergence [11]–[13]. This is a powerful technique for establishing strong converses in information theory. It has been employed previously to establish strong converses for point-to-point memoryless DMCs with output feedback [11], [14], classical-quantum channels [15] and most recently, entanglement-breaking quantum channels [16]. We were inspired to use the Rényi divergence technique for our strong converse proof because of the similarities of DM-MMNs to channels with full output feedback as shown in the context of sphere-packing bounds on the reliability function for the DM-RC in [17].

B. Related Work

The papers that are most closely related to the present work are the ones by Behboodi and Piantanida who conjectured that the strong converse holds for DM-RCs [18] and general DM multicast networks [19]. Also see
Appendix C in the thesis by Behboodi [20]. It appears to the present authors, however, that some steps in the justifications, which are based on the information spectrum method [21], are incomplete. Therefore, we are motivated to provide a strong converse for some (albeit somewhat restrictive) classes of DM-MMNs using a completely different and elementary method—namely, the Rényi divergence approach [11]–[13]. As mentioned by Polyanskiy and Verdú [11], this approach is arguably the simplest method for proving that memoryless channels with feedback satisfy the strong converse and thus, we are inspired to leverage it to prove the strong converse for some classes of DM-MMNs.

C. Paper Outline

This paper is organized as follows. Section II presents the notation used in this paper. Section III provides the problem formulation of the DM-MMNs and presents our main theorem. Section IV introduces Rényi divergence and discusses its important properties. Section V contains an important lemma concerning simulating distributions which is used in the proof of our main theorem. Section VI presents the proof of our main theorem. In Section VII, we discuss the above-mentioned classes of DM-MMNs whose cut-set bounds are tight, and we use our main theorem to prove the strong converse for them. We conclude our discussion and suggest avenues for future research in Section VIII. Proofs of the more technical auxiliary results are relegated to the appendices.

II. Notation

We use \( \Pr \{ \mathcal{E} \} \) to represent the probability of an event \( \mathcal{E} \), and we let \( 1(\mathcal{E}) \) be the characteristic function of \( \mathcal{E} \). We use a capital letter \( X \) to denote a random variable with alphabet \( \mathcal{X} \), and use the small letter \( x \) to denote a realization of \( X \). We use \( X^n \) to denote a random vector \( [X_1 \ X_2 \ldots \ X_n] \), where the components \( X_k \) have the same alphabet \( \mathcal{X} \). We let \( \p_X \) and \( \p_{Y|X} \) denote the probability mass distribution of \( X \) and the conditional probability mass distribution of \( Y \) given \( X \) respectively for any discrete random variables \( X \) and \( Y \). For any mapping \( g \) whose domain includes \( \mathcal{X} \), we let \( \p_{g(X)} \) denote the probability mass distribution of \( g(X) \) when \( X \) is distributed according to \( \p_X \). We let \( \p_X(x) \triangleq \Pr \{ X = x \} \) and \( \p_{Y|X}(y|x) \triangleq \Pr \{ Y = y | X = x \} \) be the evaluations of \( \p_X \) and \( \p_{Y|X} \) respectively at \( X = x \) and \( Y = y \). We let \( \p_X \p_{Y|X} \) denote the joint distribution of \( (X, Y) \), i.e., \( \p_X \p_{Y|X}(x, y) = \p_X(x) \p_{Y|X}(y|x) \) for all \( x \) and \( y \). If \( X \) and \( Y \) are independent, their joint distribution is simply \( \p_X \p_Y \). For simplicity, we drop the subscript of a notation if there is no ambiguity. We will take all logarithms to base 2, and we will use the convention that \( \log 0 = 0 \) and \( \log \frac{0}{0} = 0 \) throughout this paper. For any discrete random variable \( (X, Y, Z) \) distributed according to \( \p_{X,Y,Z} \), we let \( H_{p_{X,Y,Z}}(X|Z) \) and \( I_{p_{X,Y,Z}}(X;Y|Z) \) be the entropy of \( X \) given \( Z \) and mutual information between \( X \) and \( Y \) given \( Z \) respectively. The \( \ell_1 \)-distance between two distributions \( \p_X \) and \( \p_X \) on the same discrete alphabet \( \mathcal{X} \), denoted by \( \| \p_X - \p_X \|_{\ell_1} \), is defined as \( \| \p_X - \p_X \|_{\ell_1} \triangleq \sum_{x \in \mathcal{X}} |\p_X(x) - \p_X(x)| \). If \( X, Y \) and \( Z \) are distributed according to \( \p_{X,Y,Z} \) and they form a Markov chain, we write \( (X \to Y \to Z)_{p_{X,Y,Z}} \) or more simply, \( (X \to Y \to Z)_{p} \).
III. Problem Formulation and Main Result

We consider a DM-MMN that consists of $N$ nodes. Let

$$\mathcal{I} \triangleq \{1, 2, \ldots, N\}$$

be the index set of the nodes, and let $\mathcal{S} \subseteq \mathcal{I}$ and $\mathcal{D} \subseteq \mathcal{I}$ be the sets of sources and destinations respectively. We call $(\mathcal{S}, \mathcal{D})$ the multicast demand on the network. The sources in $\mathcal{S}$ transmit information to the destinations in $\mathcal{D}$ in $n$ time slots (channel uses) as follows. Node $i$ transmits message

$$W_i \in \{1, 2, \ldots, \left\lfloor 2^n R_i \right\rfloor\}$$

for each $i \in \mathcal{S}$ and node $j$, for each $j \in \mathcal{D}$, wants to decode $\{W_i : i \in \mathcal{S}\}$, where $R_i$ denotes the rate of message $W_i$. We assume that each message $W_i$ is uniformly distributed over $\{1, 2, \ldots, \left\lfloor 2^n R_i \right\rfloor\}$ and all the messages are independent. For each time slot $k \in \{1, 2, \ldots, n\}$ and each $i \in \mathcal{I}$, node $i$ transmits $X_{i,k} \in \mathcal{X}_i$, a function of $(W_i, Y_{i,k-1})$, and receives, from the output of a channel, $Y_{i,k} \in \mathcal{Y}_i$ where $\mathcal{X}_i$ and $\mathcal{Y}_i$ are some alphabets that possibly depend on $i$. After $n$ time slots, node $j$ declares $\hat{W}_{i,j}$ to be the transmitted $W_i$ based on $(W_j, Y_j^n)$ for each $(i, j) \in \mathcal{S} \times \mathcal{D}$.

To simplify notation, we use the following conventions for each $T \subseteq \mathcal{I}$: For any random tuple $(X_1, X_2, \ldots, X_N) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_N$, we let

$$X_T \triangleq (X_i : i \in T)$$

be a subtuple of $(X_1, X_2, \ldots, X_N)$. Similarly, for any $k \in \{1, 2, \ldots, n\}$ and any random tuple $(X_1,k, X_2,k, \ldots, X_{N,k}) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_N$, we let

$$X_{T;k} \triangleq (X_{i,k} : i \in T)$$

be a subtuple of $(X_1,k, X_2,k, \ldots, X_{N,k})$. For any $N^2$-dimensional random tuple $(\hat{W}_{1,1}, \hat{W}_{1,2}, \ldots, \hat{W}_{N,N})$, we let

$$\hat{W}_{T \times T^c} \triangleq (\hat{W}_{i,j} : (i, j) \in T \times T^c)$$

be a subtuple of $(\hat{W}_{1,1}, \hat{W}_{1,2}, \ldots, \hat{W}_{N,N})$.

The following six definitions formally define a DM-MMN and its capacity region.

**Definition 1:** A discrete network consists of $N$ finite input sets $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N$, $N$ finite output sets $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_N$ and a conditional distribution $q_{Y_2|X_2}$. The discrete network is denoted by $(\mathcal{X}_2, \mathcal{Y}_2, q_{Y_2|X_2})$.

**Definition 2:** Let $(\mathcal{X}_2, \mathcal{Y}_2, q_{Y_2|X_2})$ be a discrete network, and let $(\mathcal{S}, \mathcal{D})$ be the multicast demand on the network. An $(n, R_z)$-code, where $R_z$ denotes the tuple of code rates $(R_1, R_2, \ldots, R_N)$, for $n$ uses of the network consists
of the following:

1) A message set

\[ \mathcal{W}_i = \{1, 2, \ldots, [2^{nR_i}]\} \]

at node \( i \) for each \( i \in \mathcal{I} \), where \( R_i = 0 \) for each \( i \in \mathcal{S} \). Message \( W_i \) is uniform on \( \mathcal{W}_i \).

2) An encoding function

\[ \phi_{i,k} : \mathcal{W}_i \times \mathcal{Y}_k^{k-1} \to \mathcal{X}_i \]

for each \( i \in \mathcal{I} \) and each \( k \in \{1, 2, \ldots, n\} \), where \( \phi_{i,k} \) is the encoding function at node \( i \) in the \( k \)-th time slot such that

\[ X_{i,k} = \phi_{i,k}(W_i, Y_{i}^{k-1}). \]

3) A decoding function

\[ \psi_{i,j} : \mathcal{W}_j \times \mathcal{Y}_j^n \to \mathcal{W}_i \]

for each \((i, j) \in \mathcal{S} \times \mathcal{D}\), where \( \psi_{i,j} \) is the decoding function for message \( W_i \) at node \( j \) such that

\[ \hat{W}_{i,j} = \psi_{i,j}(W_j, Y_j^n). \]

Because the encoder \( \phi_{i,k} \) can depend on the “feedback signal” \( Y_{i}^{k-1} \), we are allowing full output feedback for each of the transmitting nodes; cf. Section VII-C.

**Definition 3:** A discrete network \((\mathcal{X}_I, \mathcal{Y}_I, q_{y_I|X_I})\) with multicast demand \((\mathcal{S}, \mathcal{D})\), when used multiple times, is called a *discrete memoryless multimessage multicast network (DM-MMN)* if the following holds for any \((n, R_I)\)-code:

For all \( T \subseteq \mathcal{I} \), we define \( q_{y_{TV}|X_I}(y_{TV}|x_I) \), the marginal distribution of channel \( q_{y_I|X_I}(y_I|x_I) \), as follows:

\[ q_{y_{TV}|X_I}(y_{TV}|x_I) \triangleq \sum_{y_T \in \mathcal{Y}_T} q_{y_I|X_I}(y_I|x_I) \]

for all \( x_I \in \mathcal{X}_I \) and \( y_{TV} \in \mathcal{Y}_{TV} \). Let \( U^{k-1} \triangleq (W_I, X_I^{k-1}, Y_I^{k-1}) \) be the collection of random variables that are generated before the \( k \)-th time slot. Then, for each \( k \in \{1, 2, \ldots, n\} \) and each \( T \subseteq \mathcal{I} \),

\[ \Pr\{U^{k-1} = u^{k-1}, X_{I,k} = x_{I,k}, Y_{TV,k} = y_{TV,k}\} = \Pr\{U^{k-1} = u^{k-1}, X_{I,k} = x_{I,k}\} q_{y_{TV}|X_I}(y_{TV,k}|x_{I,k}) \quad (1) \]

for all \( u^{k-1} \in \mathcal{U}^{k-1}, x_{I,k} \in \mathcal{X}_I \) and \( y_{TV,k} \in \mathcal{Y}_{TV} \).

**Definition 4:** For an \((n, R_I)\)-code defined on the DM-MMN with multicast demand \((\mathcal{S}, \mathcal{D})\), the average probability of decoding error is defined as

\[ \Pr \left\{ \bigcup_{j \in \mathcal{D}} \bigcup_{i \in \mathcal{S}} \{\hat{W}_{i,j} \neq W_i\} \right\}. \]

We call an \((n, R_I)\)-code with average probability of decoding error not exceeding \( \epsilon_n \) an \((n, R_I, \epsilon_n)\)-code.

**Definition 5:** A rate tuple \( R_I \) is \( \epsilon \)-achievable for the DM-MMN with multicast demand \((\mathcal{S}, \mathcal{D})\) if there exists a
sequence of \((n, R, \epsilon_n)\)-codes for the DM-MMN such that

\[
\limsup_{n \to \infty} \epsilon_n \leq \epsilon.
\]

**Definition 6:** The \(\epsilon\)-capacity region (for \(\epsilon \in [0, 1)\)) of the DM-MMN with multicast demand \((S, D)\), denoted by \(C_\epsilon\), is the set consisting of all \(\epsilon\)-achievable rate tuples \(R_I\) with \(R_i = 0\) for all \(i \in S^c\). The capacity region is defined to be the 0-capacity region \(C_0\).

The following theorem is the main result in this paper.

**Theorem 1:** Let \((X_I, Y_I, q_{Y_I | X_I})\) be a DM-MMN with multicast demand \((S, D)\). Define

\[
R_{\text{out}} \triangleq \bigcap_{T \subseteq I : T \cap D \neq \emptyset} \bigcup_{p_{X_I}} \left\{ R_I : \sum_{i \in T} R_i \leq I_{p_{X_I} q_{Y_{I \cap S} | X_I}}(X_T; Y_{T^c} | X_{T^c}), \quad R_i = 0 \text{ for all } i \in S^c \right\}.
\]

Then for each \(\epsilon \in [0, 1)\),

\[
C_\epsilon \subseteq R_{\text{out}}.
\]

Note that \(R_{\text{out}}\) is similar to the usual cut-set bound [2] except that the union and the intersection operations are interchanged. Consequently, \(R_{\text{out}}\) is potentially looser (larger) than the cut-set bound. However, it can be shown that \(R_{\text{out}} \subseteq C_\epsilon\) for some class of networks including the deterministic relay networks with no interference [6], the finite-field linear deterministic networks [7], [8] and the wireless erasure networks [9] (discussed in Section VII-A), the class of DM-MMNs consisting of independent DMCs (discussed in Section VII-B) and the class of single-destination DM-MMNs consisting of independent DMCs with destination feedback (discussed in Section VII-C). Therefore, Theorem 1 implies the strong converses for these networks.

We briefly outline the content in the sections to follow: The proof of Theorem 1 leverages properties of the Rényi divergence, which we discuss in Section IV. In Section V, we construct so-called simulating distributions, which form an important part of the proof of Theorem 1. The details of the proof of Theorem 1 are provided in Section VI. Readers who are only interested in the the application of Theorem 1 to specific channel models may proceed directly to Section VII.

**IV. PROPERTIES OF THE RÉNYI DIVergence**

The following definitions of (conditional) relative entropy and (conditional) Rényi divergence are standard [11]–[13].

**Definition 7:** Let \(p_X\) and \(q_X\) be two probability distributions on \(\mathcal{X}\), and let \(r_Z\) be a probability distribution on \(\mathcal{Z}\). Let

\[
D(p_X \| q_X) \triangleq \sum_{x \in \mathcal{X}} p_X(x) \log \frac{p_X(x)}{q_X(x)}
\]

be the relative entropy between \(p_X\) and \(q_X\), and let

\[
D(p_{X|Z} \| q_{X|Z} | r_Z) \triangleq \sum_{z \in \mathcal{Z}} r_Z(z) D(p_{X|Z=z} \| q_{X|Z=z})
\]
be the conditional relative entropy between $p_{X|Z}$ and $q_{X|Z}$ conditioned on $r_Z$. Then, the Rényi divergence with parameter $\lambda \in [1, \infty)$ between $p_X$ and $q_X$, denoted by $D_\lambda(p_X \parallel q_X)$, is defined as follows:

$$D_\lambda(p_X \parallel q_X) \triangleq \frac{1}{\lambda-1} \log \sum_{x \in \mathcal{X}} \frac{(p_X(x))^\lambda}{(q_X(x))^{\lambda-1}}$$ if $\lambda > 1$,

$$D(p_X \parallel q_X)$$ if $\lambda = 1$.

In addition, the conditional Rényi divergence with parameter $\lambda \in [1, \infty)$ between $p_{X|Z}$ and $q_{X|Z}$ given $r_Z$, denoted by $D_\lambda(p_{X|Z} \parallel q_{X|Z} | r_Z)$, is defined as follows:

$$D_\lambda(p_{X|Z} \parallel q_{X|Z} | r_Z) \triangleq \frac{1}{\lambda-1} \log \sum_{z \in \mathcal{Z}} r_Z(z) \sum_{x \in \mathcal{X}} \frac{(p_{X|Z}(x|z))^{\lambda}}{(q_{X|Z}(x|z))^{\lambda-1}}$$ if $\lambda > 1$,

$$D(p_{X|Z} \parallel q_{X|Z} | r_Z)$$ if $\lambda = 1$.

Note that for $\lambda > 1$, $D_\lambda(p_{X|Z} \parallel q_{X|Z} | r_Z)$ can be expressed in terms of the unconditional Rényi divergence as

$$D_\lambda(p_{X|Z} \parallel q_{X|Z} | r_Z) = \frac{1}{\lambda-1} \log \sum_{z \in \mathcal{Z}} r_Z(z) 2^{(\lambda-1)D_\lambda(p_{X|Z=Z} \parallel q_{X|Z=Z})}.$$

We summarize two important properties of $D_\lambda(p_{X|Z} \parallel q_{X|Z} | r_Z)$ in the following theorem, whose proof can be found in [22, Theorems 5 and 9].

**Theorem 2:** For any $\lambda \in [1, \infty)$, the following statements hold for any two conditional probability distributions $p_{X,Y|Z}$, $q_{X,Y|Z}$ and any probability distribution $r_Z$:

1. (Continuity) $D_\lambda(p_{X|Z} \parallel q_{Y|Z} | r_Z)$ is continuous in $\lambda$.
2. (Data processing inequality (DPI)) $D_\lambda(p_X \parallel q_X) \geq D_\lambda(p_{g(X)} \parallel q_{g(X)})$ for any function $g$ with domain $\mathcal{X}$. In particular, $D_\lambda(p_{X,Y} \parallel q_{X,Y}) \geq D_\lambda(p_X \parallel q_X)$.

Most converse theorems use Fano’s inequality [23, Section 2.10] to obtain a lower bound on the error probability. However, this can only lead to weak converse results. The following proposition, analogous to Fano’s inequality, enables us to prove strong converse results by providing a better lower bound on the error probability. Essentially, we have the freedom to choose any $\lambda \in (1, \infty)$ in the bound in (4) below.

**Proposition 1:** Let $p_{U,V}$ be a probability distribution defined on $\mathcal{W} \times \mathcal{W}$ for some $\mathcal{W}$, and let $p_U$ be the marginal distribution of $p_{U,V}$. In addition, let $q_V$ be a distribution defined on $\mathcal{W}$. Suppose $p_U$ is the uniform distribution, and let

$$\alpha = \Pr\{U \neq V\}$$

be a real number in $[0, 1)$. Then for each $\lambda \in (1, \infty)$,

$$D_\lambda(p_{U,V} \parallel p_U q_V) \geq \log |\mathcal{W}| + \lambda(\lambda - 1)^{-1} \log(1 - \alpha).$$

**Proof:** Fix a $\lambda \in (1, \infty)$ and let $s_{U,V} \triangleq p_U q_V$. Consider the following chain of inequalities:

$$D_\lambda(p_{U,V} \parallel s_{U,V}) \overset{(a)}{=} D_\lambda(p_{1\{U = V\}} \parallel s_{1\{U = V\}})$$

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(b) \[
\frac{1}{\lambda - 1} \log \left( |W|^{\lambda - 1} (1 - \alpha)^\lambda + \left( \frac{|W|}{|W| - 1} \right)^{\lambda - 1} \alpha^\lambda \right)
\]
\[\geq \log |W| + \lambda (\lambda - 1)^{-1} \log(1 - \alpha),\]

where

(a) follows from the DPI in Theorem 2;
(b) follows from Definition 7 and the facts that
\[
\sum_{u,v} p_{U,V}(u,v) 1\{u = v\} = 1 - \alpha \quad (\text{cf. (3)}) \quad \text{and}
\sum_{u,v} s_{U,V}(u,v) 1\{u = v\} = \frac{1}{|W|};
\]
(c) follows from the fact that \((\frac{|W|}{|W| - 1})^{\lambda - 1} \alpha^\lambda \geq 0\).

This completes the proof.

The following proposition enables us to approximate the conditional Rényi divergence \(D_\lambda\) by the conditional relative entropy \(D_1 = D\). Since the proof for the following proposition is straightforward but involves some tedious algebra, we defer it to Appendix A.

**Proposition 2**: Let \(\lambda \in [1, 5/4]\) be a real number, and let \(p_{X,Y,Z}\) be a probability distribution defined on \(X \times Y \times Z\). Then,
\[
D_\lambda(p_{X,Y,Z}||p_{X|Z}p_{Y|Z}|p_Z) \leq D(p_{X,Y,Z}||p_{X|Z}p_{Y|Z}|p_Z) + 8(\lambda - 1)(|X| |Y|)^5.
\]  
(5)

We made no attempt to optimize the remainder term \(8(|X| |Y|)^5\) as the important part of the statement is that this remainder term is uniform in \(p_{X,Y,Z}\) on a sufficiently small interval to the right of \(\lambda = 1\). In fact, it only depends on the product of the sizes of the alphabets \(|X| |Y|\).

V. SIMULATING DISTRIBUTION

Proposition 1 provides a lower bound for the error probability, and the lower bound holds for all \(q_V\). Therefore, we are motivated to choose a simulating distribution \(q_V\) so that the left hand side of (4) can be simplified. Before describing the simulating distribution, we state the following proposition which facilitates to characterize an important property of Markov chains.

**Proposition 3**: Suppose there exist two probability distributions \(r_{X,Y}\) and \(q_{Z|Y}\) such that
\[
p_{X,Y,Z}(x, y, z) = r_{X,Y}(x, y) q_{Z|Y}(z|y)
\]  
(6)

for all \(x, y\) and \(z\) whenever \(p_Y(y) > 0\). Then
\[
(X \rightarrow Y \rightarrow Z)_{p_{X,Y,Z}}
\]  
(7)
forms a Markov chain. In addition,

\[ p_Z|y = q_Z|y. \]  

\( \text{Proof:} \) The proof of (7) is contained [24, Proposition 2.5]. It remains to show (8). Summing \( x \) and then \( z \) on both sides of (6), we have \( p_{Y,Z}(y,z) = r_Y(y)q_{Z|y}(z|y) \) and \( p_Y(y) = r_Y(y) \) for all \( x, y \) and \( z \) whenever \( p_Y(y) > 0 \), which implies (8).

The construction of the simulating distribution is contained in the following lemma. Before stating lemma, we make the following definitions: Given an \((n,R_T,\epsilon_n)\)-code, we let \( p_{W_Z,X^n_Z,Y^n_T,W_{Z|X}} \) be the probability distribution induced by the code according to Definitions 2 and 3. In the following, we drop the subscripts of the DM-MMN defined in Definition 3, we define \( s_{X^n,Z,Y^n_T} \) for \( k = 2, \ldots, n \) based on \( \{s_{X^n,Z,Y^n_T}\}_{i=1}^{k-1} \) as follows: For all \( x_i \in X \) and \( y_{i-1} \in Y \), let

\[ s_{X^n,Z,Y^n_T}(x_i,k) = \frac{\sum_{x_{i-1} \in X} p(x_{i-1}, y_{i-1}) \prod_{\ell=1}^{k-1} \left( p(x_{i-1}, y_{i-1}, x_{i-1}, y_{i-1}) \left( (p(y_{i-1}|x_{i-1}, y_{i-1}))^{\lambda} \right) \right)}{\sum_{x_{i-1} \in X} \prod_{\ell=1}^{k-1} \left( p(x_{i-1}, y_{i-1}, x_{i-1}, y_{i-1}) \left( (p(y_{i-1}|x_{i-1}, y_{i-1}))^{\lambda} \right) \right)} \]  

(9)

and

\[ s_{X^n,Z,Y^n_T}(x_i,k,y_{i-1}) = s_{X^n,Z,Y^n_T}(x_i,k) q_{Y^n_T|X^n_T}(y_{i-1}|x_i,k) \]  

(10)

It can be verified by using (1), (9) and (10) that \( s_{X^n,Z,Y^n_T}^{(1)}(x_i,k) = p_{X^n,Z,Y^n_T} \) and hence \( s_{X^n,Z,Y^n_T}^{(1)} \) can be viewed as a tilted version of \( p_{X^n,Z,Y^n_T} \). More specifically, we can see from (9) that \( s_{X^n,Z,Y^n_T}^{(1)} \) can be viewed as a weighted version of \( p_{X^n,k|X^n_{k-1},Y^n_{k-1}} \) where the weighting distribution is a tilting of \( \prod_{\ell=1}^{k-1} (p_{X^n,Tc|X^n_{\ell},Y^n_{\ell-1},Y^n_{\ell}} q_{Y^n_{\ell-1}|X^n_{\ell}}) \) towards \( \prod_{\ell=1}^{k-1} (p_{X^n,Tc|X^n_{\ell},Y^n_{\ell}} s_{X^n,Z,Y^n_T}(X^n_{\ell},Y^n_{\ell})) \).

**Lemma 4:** Given an \((n,R_T,\epsilon_n)\)-code for the DM-MMN, let \( p_{W_Z,X^n_Z,Y^n_T,W_{Z|x}} \) be the probability distribution induced by the code according to Definitions 2 and 3. Let \( T \) be an arbitrary subset of \( I \) and fix an arbitrary \( \lambda \in [1, \infty) \). Then there exists a probability distribution \( s_{W_Z,X^n_Z,Y^n_T,W_{Z|x}} \) that satisfies the following properties:

(i) \( s_{W_z} = p_{W_z} \).

(ii) \( s_{W_T,x} | W_T, Y^n_T^n = p_{W_T,x} | W_T, Y^n_T^n \).

(iii) For each \( k \in \{1, 2, \ldots, n\} \), \( (W_T, X_{k-1}^n, Y_{k-1}^n) \rightarrow X_{Tc,k} \rightarrow Y_{Tc,k} \) forms a Markov chain.

(iv) For each \( k \in \{1, 2, \ldots, n\} \), \( s_{Y^n_{Tc,k}|X^n_{Tc,k}} = s_{X^n_{Tc,k}|Y^n_{Tc,k}} \), where \( s_{X^n_{Tc,k}|Y^n_{Tc,k}} \) is induced by the joint distribution in (10).

(v) For each \( k \in \{1, 2, \ldots, n\} \), \( p_{X^n_{Tc,k}|W_Z,X_{k-1}^n,Y_{k-1}^n} = s_{X^n_{Tc,k}|X_{k-1}^n,Y_{k-1}^n} \).

We call \( s_{W_Z,X^n_Z,Y^n_T,W_{Z|x}} \) a \( \lambda \)-simulating distribution of \( p_{W_Z,X^n_Z,Y^n_T,W_{Z|x}} \) neglecting \( T \) because \( s_{X^n_{Tc,k},Y^n_{Tc,k}} \) represents a “\( \lambda \)-tilting” of \( p_{X^n_{Tc,k},Y^n_{Tc,k}} \) through Property (iv) and captures all the important properties of \( (X^n_{Tc,k},Y^n_{Tc,k})_T \) when \( (X^n_{Tc,k},Y^n_{Tc,k})_T \) is generated according to the given code distribution \( p_{W_Z,X^n_Z,Y^n_T,W_{Z|x}} \).
Proof: We prove the lemma by first constructing a distribution of \((W_Z, X_T^n, Y_T^n)\) denoted by \(r\). Subsequently, we use \(r\) as a building block to construct a distribution of \((W_Z, X_T^n, Y_T^n, \hat{W}_{T \times T})\). Define

\[
r_{W_Z,X_T^1,Y_T^1} \triangleq p_{W_Z} x_{T,c} p_{X_{T,c:1}|W_{T,c}} s_{Y_{T,c:1}|X_{T,c:1}}^{(\lambda)}.
\]  

(11)

Recursively construct

\[
r_{W_Z,X_T^k,Y_T^k} \triangleq r_{W_Z,X_T^{k-1},Y_T^{k-1}} p_{X_{T,c:k}|W_{T,c},X_T^{k-1},Y_T^{k-1}} s_{Y_{T,c:k}|X_{T,c:k}}^{(\lambda)}
\]  

(12)

for each \(k = 2, 3, \ldots, n\), where \(s_{Y_{T,c:k}|X_{T,c:k}}^{(\lambda)}\) is as defined in (10). Applying (12) recursively from \(k = 2\) to \(k = n\) and using (11), we have

\[
r_{W_Z,X_T^n,Y_T^n} = p_{W_Z} \prod_{k=1}^n \left( p_{X_{T,c:k}|W_{T,c},X_T^{k-1},Y_T^{k-1}} s_{Y_{T,c:k}|X_{T,c:k}}^{(\lambda)} \right).
\]  

(13)

After defining \(r\) through (11), (12) and (13), we are now ready to define \(s\) as follows:

\[
s_{W_Z,X_T^n,Y_T^n,W_{T \times T}} \triangleq p_{X_T^n} p_{W_{T \times T}} r_{W_Z,X_T^n,Y_T^n} p_{W_{T \times T}|W_{T,c},Y_T^n}.
\]  

(14)

In the rest of the proof, we want to show that \(s_{W_Z,X_T^n,Y_T^n,W_{T \times T}}\) satisfies Properties (i), (ii), (iii), (iv) and (v).

Since

\[
\sum_{x_T^n, y_T^n, w_{T \times T}} s_{W_Z,X_T^n,Y_T^n,W_{T \times T}}(w_T, x_T^n, y_T^n, \hat{w}_{T \times T}) \overset{(14)}{=} \sum_{x_T^n, y_T^n} r_{W_Z,X_T^n,Y_T^n}(w_T, x_T^n, y_T^n) \overset{(13)}{=} p_{W_Z}(w_T)
\]  

for all \(w_T\), it follows that Property (i) holds.

In order to prove Property (ii), we write

\[
s_{W_T^n, Y_T^n, \hat{W}_{T \times T}} \overset{(14)}{=} r_{W_T^n, Y_T^n} p_{\hat{W}_{T \times T}|W_{T,c},Y_T^n}
\]  

which implies from Proposition 3 that \(s_{W_{T \times T}|W_{T,c},Y_T^n} = p_{W_{T \times T}|W_{T,c},Y_T^n}\).

In order to prove Properties (iii), (iv) and (v), we write for each \(k \in \{1, 2, \ldots, n\}\)

\[
s_{W_Z,X_T^n, Y_T^n} \overset{(14)}{=} r_{W_Z,X_T^n, Y_T^n} p_{\hat{W}_{T \times T}|W_{T,c},Y_T^n}
\]  

(15)

where (a) follows from marginalizing (13). It then follows from (15) and Proposition 3 that for each \(k \in \{1, 2, \ldots, n\}\),

\[
(W_Z, X_T^{k-1}, Y_T^{k-1}) \rightarrow X_{T,c,k} \rightarrow Y_{T,c,k}
\]  

(16)
forms a Markov chain and

\[ s_{Y_{Te,k}|X_{Te,k}} = s_{Y_{Te,k}|X_{Te,k}}^{(A)} \]

Properties (iii) and (iv) follow from (16) and (17) respectively. In addition, for each \( k \in \{1, 2, \ldots, n\} \),

\[ s_{W_x, X_{Te}, Y_{Te}^{k-1}}^{(A)} = \prod_{m=1}^{k-1} \left( p_{X_{Te,m}|W_{Te}, Y_{Te}^{m-1}} s_{Y_{Te,m}|X_{Te,m}}^{(A)} \right) \]

Then, for each \( k \in \{1, 2, \ldots, n\} \),

\[ p_{X_{Te,k}|W_x, X_{Te}^{k-1}, Y_{Te}^{k-1}}^{(a)} = p_{X_{Te,k}|W_{Te}, X_{Te}^{k-1}, Y_{Te}^{k-1}} \]
\[ p_{X_{Te,k}|W_x, X_{Te}^{k-1}, Y_{Te}^{k-1}}^{(b)} = s_{X_{Te,k}|W_{Te}, X_{Te}^{k-1}, Y_{Te}^{k-1}} \]

where

(a) follows from the fact that \((W_T, X_{Te}^{k-1}) \rightarrow (W_{Te}, Y_{Te}^{k-1}) \rightarrow X_{Te,k}\) forms a Markov chain (cf. Definition 2).
(b) follows from (18) and Proposition 3.

Property (v) follows from (19).

### VI. PROOF OF THEOREM 1

#### A. Lower Bounding the Error Probability in Terms of the Rényi Divergence

Fix an \( \epsilon \in (0, 1) \) and let \( R_x \) be an \( \epsilon \)-achievable rate tuple for the DM-MMN. By Definitions 5 and 6, there exists a number \( \bar{\epsilon} \in (0, 1) \) and a sequence of \((n, R_x, \epsilon_n)\)-codes on the DM-MMN such that for all sufficiently large \( n \),

\[ \epsilon_n \leq \bar{\epsilon}. \]

Fix a sufficiently large \( n \) such that (20) holds, and let \( p_{W_x, X_{T}^{n}, Y_{T}^{n}, W_{T}^{n}, x, x} \) be the probability distribution induced by the \((n, R_x, \epsilon_n)\)-code on the DM-MMN. Fix an arbitrary \( T \subseteq \mathcal{I} \) such that \( T^{c} \cap \mathcal{D} \neq \emptyset \), and choose a node \( d \in T^{c} \cap \mathcal{D} \). Fix an arbitrary \( \lambda \in (1, \infty) \). Let \( s_{W_x, X_{T}^{n}, Y_{T}^{n}, W_{T}^{n}, x, x} \) be a \( \lambda \)-simulating distribution of \( p_{W_x, X_{T}^{n}, Y_{T}^{n}, W_{T}^{n}, x, x} \) neglecting \( T \) such that \( s_{W_x, X_{T}^{n}, Y_{T}^{n}, W_{T}^{n}, x, x} \) satisfies all the properties in Lemma 4. Then, it follows from Proposition 1 and Definition 2 with the identifications \( U \equiv W_T, V \equiv \tilde{W}_{TX(d)}, p_{U,V} \equiv p_{W_T, \tilde{W}_{TX(d)}}, q_{V} \equiv s_{\tilde{W}_{TX(d)}}, |W| \equiv 2^{|U|} \) and \( \alpha \equiv \Pr\{W_T \neq \tilde{W}_{TX(d)}\} \leq \epsilon \) that

\[ D_\lambda(p_{W_T, \tilde{W}_{TX(d)}} || p_{W_T, s_{\tilde{W}_{TX(d)}}}) \geq \sum_{i \in T} nR_i + \lambda(\lambda - 1)^{-1} \log(1 - \alpha) \]
\[ \geq \sum_{i \in T} nR_i + \lambda(\lambda - 1)^{-1} \log(1 - \bar{\epsilon}). \]
B. Using the DPI to Introduce the Channel Input and Output

Let $U_{T_T}^{k-1} \triangleq (W_T, X_{T_T}^{k-1}, Y_{T_T}^{k-1})$ and $V_{T_T}^{k-1} \triangleq (W_{T_T}, X_{T_T}^{k-1}, Y_{T_T}^{k-1})$ be the random variables generated before the $k$th time slot, and consider the following chain of inequalities:

$$D_\lambda(p_{W_T, W_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}})$$

$\leq D_\lambda(p_{U_{T_T}, W_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}}, W_T \times \{d\})$

$= D_\lambda(p_{U_{T_T}, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| W_T \times \{d\})$ [a]

$= D_\lambda(p_{U_{T_T}, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| W_T \times \{d\})$ [b]

$= D_\lambda(p_{U_{T_T}, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| W_T \times \{d\})$ [c]

$\leq D_\lambda(p_{U_{T_T}, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| W_T \times \{d\})$ [d]

$\leq D_\lambda(p_{U_{T_T}, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| W_T \times \{d\})$ [e]

$$ \leq D_\lambda(p_{U_{T_T}, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| W_T \times \{d\})$$ [f]

where

(a) follows from the DPI of $D_\lambda$ by introducing $W_{T_T}$.

(b) follows from Property (i) in Lemma 4.

(c) follows from the fact that $W_T$ and $W_{T_T}$ are independent.

(d) follows from the DPI of $D_\lambda$ by introducing the channel output $Y_{T_T}^n$.

(e) follows from Property (ii) in Lemma 4 and the fact that

$$(W_T \rightarrow (W_{T_T}, Y_{T_T}^n) \rightarrow \tilde{W}_T \times \{d\})_p$$

forms a Markov chain.

(f) follows from the DPI of $D_\lambda$ by introducing the channel input $X_T^n$.

In order to simplify (22), we consider

$$D_\lambda(p_{U_{T_T}, p_{\tilde{W}_T \times \{d\}}} \| p_{U_{T_T}, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}})$$

$$= D_\lambda(p_{U_{T_T}, p_{\tilde{W}_T \times \{d\}}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}})$$ [a]

$$= D_\lambda(p_{U_{T_T}, p_{\tilde{W}_T \times \{d\}}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}})$$ [b]

$$= D_\lambda(p_{U_{T_T}, p_{\tilde{W}_T \times \{d\}}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}} \| p_{W_T, \tilde{W}_T \times \{d\}})$$ [c]
Using (22), (23) and Definition 7 and omitting subscripts of probability distributions to simplify notation, we have

\[ D(\mathcal{L}) = \sum_{k=1}^{n} \left( p_{X,Z,k}(u_{T,c}^{k-1}Q_{Y,T}|X) \right) \]

(a) follows from Property (iii) in Lemma 4.

(b) follows from Property (v) in Lemma 4.

C. Single-Letterizing the Rényi Divergence

Consider the distribution

\[ p_{U_{T,c}^0,T_{D}^0,Y_{T,c}^0} = p_{U_{T,c}^0,T_{D}^0,Y_{T,c}^0} \]

\[ = p_{U_{T,c}^0} \prod_{k=1}^{n} p_{X,Z,k,Y_{T,c},k} | u_{T,c}^{k-1} \]

\[ = p_{U_{T,c}^0} \prod_{k=1}^{n} (p_{X,Z,k} | u_{T,c}^{k-1} p_{Y_{T,c},k} | u_{T,c}^{k-1}, X_{T,c}, k) \]

\[ \overset{(1)}{=} p_{U_{T,c}^0} \prod_{k=1}^{n} (p_{X,Z,k} | u_{T,c}^{k-1} Q_{Y,T,c} | X) \]

(23)

where

Using (22), (23) and Definition 7 and omitting subscripts of probability distributions to simplify notation, we have

\[ D_{\lambda}(p_{W_{T,c},W_{T,c},X|Y}) \]

\[ \leq \frac{1}{\lambda - 1} \log \left( \sum_{u_{T,c}^0,T_{D}^0,Y_{T,c}^0} p(u_{T,c}^0,T_{D}^0,Y_{T,c}^0) \prod_{k=1}^{n} (p(x_{T,c},k) | u_{T,c}^{k-1}) q(y_{T,c},k | x_{T,c},k) \left( \frac{q(y_{T,c},k | x_{T,c},k)}{s(y_{T,c},k | x_{T,c},k)} \right)^{\lambda - 1} \right) \]

\[ \overset{(24)}{=} \frac{1}{\lambda - 1} \log \left( \sum_{u_{T,c}^0,T_{D}^0,Y_{T,c}^0} p(u_{T,c}^0,T_{D}^0,Y_{T,c}^0) \prod_{k=1}^{n} \left( \frac{q(y_{T,c},k | x_{T,c},k)}{s(y_{T,c},k | x_{T,c},k)} \right)^{\lambda - 1} \right) \]

\[ = \frac{1}{\lambda - 1} \log \left( \sum_{x_{T,c}^0,y_{T,c}^0} p(x_{T,c}^0,y_{T,c}^0) \prod_{k=1}^{n} \left( \frac{q(y_{T,c},k | x_{T,c},k)}{s(y_{T,c},k | x_{T,c},k)} \right)^{\lambda - 1} \right) \]

(25)
Following (25), we consider the following chain of equalities:

\[
\sum_{x_{Tc}, y_{Tc}} p(x_{Tc}^n, y_{Tc}^n) \prod_{k=1}^{n} \left( \frac{q(y_{Tc,k} | x_{Tc,k})}{s(y_{Tc,k} | x_{Tc,k})} \right)^{\lambda-1} \\
= \sum_{x_{Tc}, y_{Tc}} \prod_{k=1}^{n} \left( p(x_{Tc,k}, y_{Tc,k} | x_{Tc}^{k-1}, y_{Tc}^{k-1}) \left( \frac{q(y_{Tc,k} | x_{Tc,k})}{s(y_{Tc,k} | x_{Tc,k})} \right)^{\lambda-1} \right) \\
= \sum_{x_{Tc}, y_{Tc}} \prod_{k=1}^{n} \left( \frac{p(x_{Tc,k}, y_{Tc,k} | x_{Tc}^{k-1}, y_{Tc}^{k-1}) (q(y_{Tc,k} | x_{Tc,k}))^{\lambda}}{(s(y_{Tc,k} | x_{Tc,k}))^{\lambda-1}} \right). 
\]  

(26)

Letting \( f_0^{(\lambda)}(x_{Tc}, y_{Tc}) \equiv 1 \) and

\[
f_k^{(\lambda)}(x_{Tc}, y_{Tc}) \equiv \prod_{\ell=1}^{k} \left( p(x_{Tc,\ell}, x_{Tc,\ell}^{-1}, y_{Tc,\ell}^{-1}) (q(y_{Tc,\ell} | x_{Tc,\ell}))^{\lambda} \right) / (s(y_{Tc,\ell} | x_{Tc,\ell}))^{\lambda-1} 
\]  

(27)

for each \( k \in \{1, 2, \ldots, n\} \) and following (26), we consider

\[
\log \sum_{x_{Tc}, y_{Tc}} \prod_{k=1}^{n} \left( p(x_{Tc,k}, y_{Tc,k} | x_{Tc}^{k-1}, y_{Tc}^{k-1}) \right) \left( \frac{q(y_{Tc,k} | x_{Tc,k})}{s(y_{Tc,k} | x_{Tc,k})} \right)^{\lambda-1} \\
= \log \sum_{x_{Tc}, y_{Tc}} f_k^{(\lambda)}(x_{Tc}, y_{Tc}) \\
= \log \sum_{x_{Tc}, y_{Tc}} \frac{\sum_{x_{Tc}, y_{Tc}} f_k^{(\lambda)}(x_{Tc}, y_{Tc})}{\sum_{x_{Tc}, y_{Tc}} f_k^{(\lambda)}(x_{Tc}, y_{Tc})} \\
= \log \left( \frac{\sum_{x_{Tc}, y_{Tc}} f_k^{(\lambda)}(x_{Tc}, y_{Tc})}{\sum_{x_{Tc}, y_{Tc}} f_k^{(\lambda)}(x_{Tc}, y_{Tc})} \right), 
\]  

(28)

where (a) is a telescoping product. For each \( k \in \{1, 2, \ldots, n\} \), define \( P_{X_{Tc,k}}^{(\lambda)} \) to be the following distribution:

\[
P_{X_{Tc,k}}^{(\lambda)}(x_{Tc,k}) \equiv \frac{\sum_{x_{Tc}^{k-1}, y_{Tc}^{k-1}} p(x_{Tc,k}, y_{Tc,k} | x_{Tc}^{k-1}, y_{Tc}^{k-1}) f_k^{(\lambda)}(x_{Tc}, y_{Tc})}{\sum_{x_{Tc}^{k-1}, y_{Tc}^{k-1}} f_k^{(\lambda)}(x_{Tc}, y_{Tc})} 
\]  

(29)

for all \( x_{Tc,k} \). Combining (26), (28) and (29), we obtain

\[
\log \left( \sum_{x_{Tc}, y_{Tc}} p(x_{Tc}, y_{Tc}) \prod_{k=1}^{n} \left( \frac{q(y_{Tc,k} | x_{Tc,k})}{s(y_{Tc,k} | x_{Tc,k})} \right)^{\lambda-1} \right) = \sum_{k=1}^{n} \log \left( \sum_{x_{Tc,k}} \sum_{y_{Tc,k}} (q(y_{Tc,k} | x_{Tc,k})^{\lambda} \right), 
\]

which implies from (25) and Definition 7 that

\[
D_\lambda(p_{W_T, W_{T \times (d)}} \| p_{W_T, s_{W_{T \times (d)}}}) \leq \sum_{k=1}^{n} D_\lambda(q_{Y_{Tc,k} | X_{Tc,k}} \| s_{Y_{Tc,k} | X_{Tc,k}}) P_{X_{Tc,k}}^{(\lambda)} 
\]  

(30)

D. Representing Distributions in the Rényi Divergence by a Single Distribution

Construct a probability distribution \( P_{X_{Tc,k}, Y_{Tc,k}}^{(\lambda)} \) for each \( k \in \{1, 2, \ldots, n\} \) as

\[
P_{X_{Tc,k}, Y_{Tc,k}}^{(\lambda)}(x_{Tc}, y_{Tc}) \equiv P_{X_{Tc,k}}^{(\lambda)}(x_{Tc}) q_{Y_{Tc,k} | X_{Tc,k}}(y_{Tc} | x_{Tc}) 
\]  

(31)
for all \((x_I, y_{T^c})\) (cf. (29)), where \(q_{Y_{T^c}|X_I}\) denotes the channel of the DM-MMN. Combining (27), (29), (31) and Property (iv) in Lemma 4, we have

\[
p_{X_I, k, Y_{T^c}}^{(\lambda)} = s_{X_I, k, Y_{T^c}}^{(\lambda)}
\]

for each \(k \in \{1, 2, \ldots, n\}\). Then, it follows from Property (iv) in Lemma 4, (32) and (30) that

\[
D_\lambda(p_{W_T, W_{T^c}(d)} || p_{W_T, \bar{W}_{T^c}(d)}) \leq D_\lambda(q_{Y_{T^c}|X_I} || p_{Y_{T^c}|X_I}) \cdot P_{X_I, k}^{(\lambda)}.
\]

Using (31) and Proposition 3, we obtain

\[
p_{Y_{T^c}|X_I}^{(\lambda)} = q_{Y_{T^c}|X_I}
\]

for all \(k \in \{1, 2, \ldots, n\}\), which implies from (33) that

\[
D_\lambda(p_{W_T, W_{T^c}(d)} || p_{W_T, \bar{W}_{T^c}(d)}) \leq \sum_{k=1}^{n} D_\lambda(p_{Y_{T^c}|X_I}^{(\lambda)} || p_{Y_{T^c}|X_I}^{(\lambda)}) \cdot P_{X_I, k}^{(\lambda)}.
\]

**E. Introduction of a Time-sharing Random Variable**

Let \(Q_n\) be a random variable uniformly distributed on \(\{1, 2, \ldots, n\}\) and independent of all other random variables. Construct the probability distribution \(p_{Q_n, X_I, Y_{T^c}}^{(\lambda)}\) such that

\[
p_{Q_n, X_I, Y_{T^c}}^{(\lambda)}(k, x_I^n, y_{T^c}^n) = \frac{1}{n} \prod_{h=1}^{n} p_{X_I, h, Y_{T^c}}^{(\lambda)}(x_I, h, y_{T^c, h})
\]

for all \(k \in \{1, 2, \ldots, n\}\), \(x_I^n \in X^n_I\) and \(y_{T^c}^n \in Y^n_{T^c}\). Then, we can calculate the joint distributions \(p_{Q_n, X_I, Q_n, Y_{T^c}, Q_n}^{(\lambda)}\) and \(p_{X_I, Q_n, Y_{T^c}, Q_n}^{(\lambda)}\) as follows:

\[
p_{Q_n, X_I, Q_n, Y_{T^c}, Q_n}^{(\lambda)}(k, x_I, y_{T^c}) = p_{Q_n}^{(\lambda)}(k) p_{X_I, Q_n, Y_{T^c}, Q_n}^{(\lambda)}(x_I, y_{T^c}|k)
\]

\[
= p_{Q_n}^{(\lambda)}(k) p_{X_I, k, Y_{T^c}, Q_n}^{(\lambda)}(x_I, y_{T^c}|k)
\]

\[
= \frac{1}{n} p_{X_I, k, Y_{T^c}}^{(\lambda)}(x_I, y_{T^c})
\]

(36)

and

\[
p_{X_I, Q_n, Y_{T^c}, Q_n}^{(\lambda)}(x_I, y_{T^c}) = \frac{1}{n} \sum_{k=1}^{n} p_{X_I, k, Y_{T^c}}^{(\lambda)}(x_I, y_{T^c})
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} p_{X_I, k}^{(\lambda)}(x_I) q_{Y_{T^c}|X_I}^{(\lambda)}(y_{T^c}|x_I)
\]

\[
= p_{X_I, Q_n}^{(\lambda)}(x_I) q_{Y_{T^c}|X_I}^{(\lambda)}(y_{T^c}|x_I).
\]

(37)

It follows from (37) and Proposition 3 that

\[
p_{X_I, Q_n, Y_{T^c}, Q_n}^{(\lambda)}(x_I, y_{T^c}|k) = p_{X_I, k, Y_{T^c}}^{(\lambda)}(x_I, y_{T^c}),
\]

(39)
Combining (21), (35) and (42), we obtain
\[
(Q_n \rightarrow X_{T^n,Q_n} \rightarrow Y_{T^n,Q_n})_{p^{(\lambda)}}
\] (40)
and
\[
(Q_n \rightarrow X_{T^n,Q_n} \rightarrow Y_{T^n,Q_n})_{p^{(\lambda)}}.
\] (41)

Following (35), consider the following chain of inequalities:
\[
\frac{1}{n} \sum_{k=1}^{n} D_\lambda(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}} \| p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}} | p_{X_{T^n,k}^{(\lambda)}})
\]
\[
= \frac{1}{(\lambda - 1)n} \sum_{k=1}^{n} \log \sum_{y_{T^n}} p_{X_{T^n,Q_n}^{(\lambda)}}(x_{T^n}|y_{T^n}) \sum_{y_{T^n}} \frac{(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}}^{(\lambda)}(y_{T^n}|x_{T^n}))^\lambda}{(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}}^{(\lambda)}(y_{T^n}|x_{T^n,k}))^\lambda}
\]
\[
\leq (a) \frac{1}{(\lambda - 1)} \log \sum_{k=1}^{n} \frac{1}{n} \sum_{x_{T^n}} p_{X_{T^n,Q_n}^{(\lambda)}}(x_{T^n}|y_{T^n}) \sum_{y_{T^n}} \frac{(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}}(y_{T^n}|x_{T^n}))^\lambda}{(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}}(y_{T^n}|x_{T^n,k}))^\lambda}
\]
\[
= (b) \frac{1}{(\lambda - 1)} \log \sum_{k=1}^{n} \sum_{x_{T^n}} p_{X_{T^n,Q_n}^{(\lambda)}}(x_{T^n}|y_{T^n}) \sum_{y_{T^n}} \frac{(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}}(y_{T^n}|x_{T^n}))^\lambda}{(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}}(y_{T^n}|x_{T^n,k}))^\lambda - 1}
\]
\[
= (c) \frac{1}{(\lambda - 1)} \log \sum_{x_{T^n}} p_{X_{T^n,Q_n}^{(\lambda)}}(x_{T^n}) \sum_{y_{T^n}} \frac{(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}}(y_{T^n}|x_{T^n}))^\lambda}{(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}}(y_{T^n}|x_{T^n}))^\lambda - 1}
\]
\[
= D_\lambda(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}} \| p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}} | p_{X_{T^n,Q_n}^{(\lambda)}}),
\] (42)

where

(a) follows from the concavity of \( t \mapsto \log t \) and Jensen’s inequality.
(b) follows from (36) that \( p_{Q_n}^{(\lambda)}(k) = 1/n \) for all \( k \in \{1, 2, \ldots, n\} \).
(c) follows from (40) and (41).

Combining (21), (35) and (42), we obtain
\[
\sum_{i \in T} nR_i + \lambda(\lambda - 1)^{-1} \log(1 - \bar{\epsilon}) \leq nD_\lambda(p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}} \| p_{Y_{T^n,Q_n}^{(\lambda)}|X_{T^n,Q_n}^{(\lambda)}} | p_{X_{T^n,Q_n}^{(\lambda)}})
\] (43)
for all \( \lambda \in (1, \infty) \).

F. Approximating the Rényi Divergence by Conditional Relative Entropy

For each block length \( n \), choose \( \lambda \) to be dependent on \( n \) as follows:
\[
\lambda_n = 1 + \frac{1}{\sqrt{n}}.
\]
It then follows from (43), Proposition 2, and the fact that $|X_T||Y_{T^c}| \leq |X_T||Y_T|$ that
\[
\sum_{i \in T} R_i + \left(\frac{1}{n} + \frac{1}{\sqrt{n}}\right) \log(1 - \epsilon) \leq D_{\lambda_n}(p_{Y_{T^c}, Q_n|X_T, Q_n}^{(\lambda_n)} \| p_{Y_{T^c}, Q_n|X_T, Q_n}^{(\lambda_n)}, X_{T^c}^{(\lambda_n)}) \leq D(p_{Y_{T^c}, Q_n|X_T, Q_n}^{(\lambda_n)} \| p_{Y_{T^c}, Q_n|X_T, Q_n}^{(\lambda_n)}) + \frac{8(|X_T||Y_T|)^5}{\sqrt{n}}
\] (44)
if $n \geq 16$ (i.e., $\lambda_n \leq 5/4$ so Proposition 2 applies). Taking the limit inferior on both sides of (44), we obtain
\[
\sum_{i \in T} R_i \leq \liminf_{n \to \infty} D(p_{Y_{T^c}, Q_n|X_T, Q_n}^{(\lambda_n)} \| p_{Y_{T^c}, Q_n|X_T, Q_n}^{(\lambda_n)}).
\] (45)
Consider each distribution on $(X_T, Y_{T^c})$ as a point in the $|X_T||Y_{T^c}|$-dimensional Euclidean space. Then, by the compactness of the probability simplex, there exists a subsequence of the natural numbers \{1, 2, \ldots\}, say indexed by \{\ell\}_{\ell=1}^\infty, such that \{$(p_{X_T, Q_n, Y_{T^c}, Q_n}^{(\lambda_{n\ell})})_{\ell=1}^\infty$ is convergent with respect to the $L_1$-distance. Let \bar{p}_{X_T, Y_{T^c}} be the limit of the subsequence such that
\[
\bar{p}_{X_T, Y_{T^c}}(x_T, y_{T^c}) = \lim_{\ell \to \infty} p_{X_T, Q_n, Y_{T^c}, Q_n}^{(\lambda_{n\ell})}(x_T, y_{T^c})
\] (46)
for all $(x_T, y_{T^c})$. Combining (38) and (46), we have
\[
\bar{p}_{X_T, Y_{T^c}}(x_T, y_{T^c}) = \bar{p}_{X_T}(x_T)\bar{p}_{Y_{T^c}|X_T}(y_{T^c}|x_T).
\] (47)
Since $D(p_{Y_{T^c}|X_T} \| p_{Y_{T^c}|X_T} \| p_{X_T})$ is a continuous functional of distribution $p_{X_T, Y_{T^c}}$, it follows from (45) and (46) that
\[
\sum_{i \in T} R_i \leq D(\bar{p}_{Y_{T^c}|X_T} \| \bar{p}_{Y_{T^c}|X_T} \| \bar{p}_{X_T})
\]
\[
= \sum_{x_T} \bar{p}_{X_T}(x_T) \sum_{y_{T^c}} \bar{p}_{Y_{T^c}|X_T}(y_{T^c}|x_T) \log \frac{\bar{p}_{Y_{T^c}|X_T}(y_{T^c}|x_T)}{\bar{p}_{Y_{T^c}|X_T}(y_{T^c}|x_T)}
\]
\[
= \sum_{x_T, y_{T^c}} \bar{p}_{X_T, Y_{T^c}}(x_T, y_{T^c}) \log \frac{\bar{p}_{X_T, Y_{T^c}|X_T}(x_T, y_{T^c}|x_T)}{\bar{p}_{X_T}|X_T}(x_T|X_T)\bar{p}_{Y_{T^c}|X_T}(y_{T^c}|x_T)
\]
\[
= I_{\bar{p}_{X_T, Y_{T^c}}}(X_T; Y_{T^c}|X_T^c).
\] (48)

The theorem then follows from (47) and (48).

VII. CLASSES OF MULTIMESSAGE MULTICAST NETWORKS WITH TIGHT CUT-SET BOUND

In this section, we will use Theorem 1 to prove strong converses for some classes of DM-MMNs whose capacity regions are known. Unless specified otherwise, we let $(S, D)$ denote the multicast demand on the networks.

A. Multicast Networks with Maximal Cut-set Distribution

We start this section by stating an achievability result for multimessage multicast networks in the following theorem, which is a specialization of the main result of noisy network coding by Lim, Kim, El Gamal and Chung [25].
Let \( (X_I, Y_I, q_{Y_I|X_I}) \) be a DM-MMN, and let

\[
\begin{align*}
\mathcal{R}_\text{in} & \triangleq \bigcup_{\prod_{i=1}^N p_{X_i}, \ T \subseteq \mathcal{I} : T^c \cap \mathcal{D} \neq \emptyset} \left\{ \left( \sum_{i \in T} R_i \leq I_{p_{X_I}q_{Y_I|X_I}}(X_I; Y_I^*|X_I^*) - H_{p_{X_I}q_{Y_I|X_I}}(Y_I^*|X_I, Y_{I^c}^*) \right) \right\}.
\end{align*}
\]

Then, \( \mathcal{R}_\text{in} \subseteq \mathcal{C}_0. \)

**Proof:** The theorem follows by taking \( \hat{Y} = Y \) in Theorem 1 of [25].

We would like to identify multicast networks whose inner bounds \( \mathcal{R}_\text{in} \) coincides with our outer bound \( \mathcal{R}_\text{out} \) in Theorem 1. Using the following definition and corollary, we can state, in Theorem 4, a sufficient condition for \( \mathcal{R}_\text{in} = \mathcal{R}_\text{out} \) to hold.

**Definition 8:** A DM-MMN \( (X_I, Y_I, q_{Y_I|X_I}) \) is said to be **dominated by a maximal product distribution** if there exists some product distribution \( p_{X_I}^* \triangleq \prod_{i=1}^N p_{X_i}^* \), such that the following statement holds for each \( T \subseteq \mathcal{I} \):

\[
I_{p_{X_I}^*q_{Y_I|X_I}}(X_I; Y_I^*|X_I^*) = \max_{p_{X_I}} \{ I_{p_{X_I}q_{Y_I|X_I}}(X_I; Y_I^*|X_I^*) \}. 
\]

The following corollary is a direct consequence of Theorem 1 and Definition 8, and the proof is deferred to Appendix B.

**Corollary 5:** Let \( (X_I, Y_I, q_{Y_I|X_I}) \) be a DM-MMN, and let

\[
\begin{align*}
\mathcal{R}_\text{out}^* & \triangleq \bigcup_{\prod_{i=1}^N p_{X_i}, \ T \subseteq \mathcal{I} : T^c \cap \mathcal{D} \neq \emptyset} \left\{ \left( \sum_{i \in T} R_i \leq I_{\prod_{i=1}^N p_{X_i}}(X_I; Y_I^*|X_I^*), \right) \right\}. 
\end{align*}
\]

If the DM-MMN is dominated by a maximal product distribution, then \( \mathcal{C}_\epsilon \subseteq \mathcal{R}_\text{out}^* \) for all \( \epsilon \in [0, 1). \)

**Theorem 4:** Let \( (X_I, Y_I, q_{Y_I|X_I}) \) be a DM-MMN. Suppose the DM-MMN satisfies the following two conditions:

1. The DM-MMN is dominated by a maximal product distribution.
2. For all \( T \subseteq \mathcal{I} \) and all \( p_{X_I}, H_{p_{X_I}q_{Y_I|X_I}}(Y_I^*|X_I, Y_{I^c}^*) = 0. \)

Then \( \mathcal{R}_\text{in} = \mathcal{C}_\epsilon = \mathcal{R}_\text{out}^* \) for all \( \epsilon \in [0, 1). \)

**Proof:** Since the DM-MMN is dominated by a maximal product distribution, it follows from Theorem 3 and Corollary 5 that \( \mathcal{R}_\text{in} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}_\epsilon \subseteq \mathcal{R}_\text{out}^* \) for all \( \epsilon \in [0, 1). \) In addition, it follows from (49), (50) and Condition 2 that \( \mathcal{R}_\text{in} = \mathcal{R}_\text{out}^*. \)

Theorem 4 implies the strong converse for the classes of DM-MMNs which satisfy Conditions 1 and 2. Since the deterministic relay networks with no interference [6], the finite-field linear deterministic networks [7], [8] and the wireless erasure networks [9] satisfy both conditions in Theorem 4, the strong converse holds for these networks.

We note that for the class of wireless erasure networks, one assumes that the erasure pattern of the entire network is known to each destination, i.e., \( Y_d \) contains the erasure pattern as side information for each \( d \in \mathcal{D} \) [9, Section III.C], and hence Condition 2 in Theorem 4 is satisfied. In the following subsection, we introduce a DM-MMN connected by independent DMCs and prove the strong converse using Corollary 5 and Theorem 1.
B. DM-MMN Consisting of Independent DMCs

Consider a DM-MMN where a DMC is defined for every link \((i, j) \in \mathcal{I} \times \mathcal{I}\). Let \(X_{i,j}\) and \(Y_{i,j}\) denote the input and output alphabets of the DMC carrying information from node \(i\) to node \(j\) for each \((i, j) \in \mathcal{I} \times \mathcal{I}\), and let \(q_{Y_{i,j}|X_{i,j}}\) denote the DMC. For each \((i, j) \in \mathcal{I} \times \mathcal{I}\), the capacity of channel \(q_{Y_{i,j}|X_{i,j}}\), denoted by \(C_{i,j}\), is attained by some \(\bar{p}_{X_{i,j}}\), i.e.,

\[
C_{i,j} \triangleq \max_{p_{X_{i,j}}} I(p_{X_{i,j}}; q_{Y_{i,j}|X_{i,j}}(X_{i,j}; Y_{i,j})) = I(\bar{p}_{X_{i,j}}; q_{Y_{i,j}|X_{i,j}}(X_{i,j}; Y_{i,j})).
\]

Then, we define the input and output alphabets for each node \(i\) in the following natural way:

\[
X_i \triangleq X_{i,1} \times X_{i,2} \times \ldots \times X_{i,N}
\]

and

\[
Y_i \triangleq Y_{1,i} \times Y_{2,i} \times \ldots \times Y_{N,i}
\]

for each \(i \in \mathcal{I}\), and we let \(q_{Y_{i}|X_{i}}\) denote the channel of the network. In addition, we assume

\[
q_{Y_{i}|X_{i}} = \prod_{(i,j) \in \mathcal{I} \times \mathcal{I}} q_{Y_{i,j}|X_{i,j}}
\]

i.e., the random transformations (noises) from \(X_{i,j}\) to \(Y_{i,j}\) are independent and the overall channel of the network is in a product form. It then follows from (53) and Proposition 3 that

\[
(\{X_{k\ell}\}_{(k,\ell) \neq (i,j)}, \{Y_{k\ell}\}_{(k,\ell) \neq (i,j)}) \to X_{i,j} \to Y_{i,j})_{p_{X_{i,j}} q_{Y_{i,j}|X_{i,j}}}
\]

forms a Markov chain for all \((i, j) \in \mathcal{I} \times \mathcal{I}\). We call the network described above the **DM-MMN consisting of independent DMCs**. One important example of such networks is the **line network** in which \(\mathcal{I} \times \mathcal{I}\) consists of nonzero-capacity links of the form \((i, i+1)\) for all \(i \in \{1, 2, \ldots, N-1\}\) and zero-capacity links for the other node pairs. Define

\[
\mathcal{R}' \triangleq \bigcap_{T \subseteq \mathcal{I}: T^c \cap \mathcal{T} \neq \emptyset} \left\{ R_T \left| \sum_{i \in T} R_i \leq \sum_{(i,j) \in T \times T^c} C_{i,j}, R_i = 0 \text{ for all } i \in S^c \right. \right\}.
\]

Since the DMCs from \(X_{i,j}\) to \(Y_{i,j}\) are all independent and each of the DMC can carry information at a rate arbitrarily close to the capacity, we deduce that \(\mathcal{R}'\) lies in the capacity region of the DM-MMN consisting of independent DMCs, which is formally stated in the following corollary and proved in Appendix C.

**Corollary 6:** \(\mathcal{R}' \subseteq \mathcal{C}_0\).

We use the outer bound \(\mathcal{R}_{\text{out}}\) proved in Theorem 1 (cf. (2)) to prove the following lemma.

**Lemma 7:** \(\mathcal{C}_0 \subseteq \mathcal{R}_{\text{out}} \subseteq \mathcal{R}'\) for all \(\epsilon \in [0, 1)\).

For completeness, the proof is provided in Appendix D. The following theorem is a direct consequence of Corollary 6 and Lemma 7.
In other words, feedback does not enlarge the MMN consisting of independent DMCs as well as the independent DMCs. Thus, the strong converse also holds for the feedback version of this class of DM-MMNs.

C. Single-Destination DM-MMN Consisting of Independent DMCs with Destination Feedback

In this section, we examine a class of DM-MMNs with destination feedback, which is a generalization of the DM-MMN consisting of independent DMCs discussed in the previous section. We assume \(|D| = 1\) and let \(d \in \mathcal{I}\) denote the (single) destination node throughout this section. We define the single-destination DM-MMN consisting of independent DMCs with feedback as follows.

Definition 9: Let \((X_I, Y_I, q_{Y_I|x_I})\) be DM-MMN consisting of independent DMCs with multicast demand \((S, \{d\})\) as defined in the previous section. A single-destination DM-MMN with multicast demand \((S, \{d\})\), denoted by 
\((X_I, \tilde{Y}_I, \tilde{q}_{Y_I|x_I})\), is called the feedback version of \((X_I, Y_I, q_{Y_I|x_I})\) if the following two conditions hold:

1) \(\tilde{Y}_i = Y_i \times Y_d\) for all \(i \in \mathcal{I}\).
2) For any \(p_{X_I}\), if \((X_I, \tilde{Y}_I)\) and \((X_I, Y_I)\) are distributed according to \(p_{X_I,\tilde{q}_{Y_I|x_I}}\) and \(p_{X_I,q_{Y_I|x_I}}\) respectively, then \(\tilde{Y}_i = (Y_i, Y_d)\) for all \(i \in \mathcal{I}\).

Let \((X_I, \tilde{Y}_I, \tilde{q}_{Y_I|x_I})\) be the feedback version of \((X_I, Y_I, q_{Y_I|x_I})\) with multicast demand \((S, \{d\})\). It then follows from Definitions 9 and 2 that for any \((n, R)\)-code on \((X_I, \tilde{Y}_I, \tilde{q}_{Y_I|x_I})\), both \(Y_i^{k-1}\) and \(Y_i^{k-1}\) are available for encoding \(X_{i,k}\) at node \(i\) for all \(i \in \mathcal{I}\). Consequently, the capacity region of \((X_I, Y_I, q_{Y_I|x_I})\) is always a subset of the capacity region of \((X_I, \tilde{Y}_I, \tilde{q}_{Y_I|x_I})\). Shannon showed in [26] that the capacity of any DMC is equal to the capacity of the feedback version, and the strong converse for the feedback version has been shown in [11, Section IV]. Also see [27, Problem 2.5.16(c)] for another proof sketch of the strong converse for a DMC with feedback.

In other words, feedback does not enlarge the \(\epsilon\)-capacity region of any single-destination DM-MMN consisting of independent DMCs as well as the \(\epsilon\)-capacity region of the feedback version for any \(\epsilon \in [0, 1)\). Theorem 6 can be proved similarly to Theorem 5. We provide a concise proof in Appendix E.

VIII. Conclusion and Future Work

In this paper, we proved that the strong converse holds for some classes of DM-MMNs for which the cut-set bound is achievable by leveraging some elementary properties of the conditional Rényi divergence. We suggest three promising avenues for future research. First, the foremost item is to show that all rate tuples that lie in the exterior of the usual cut-set bound for DM-MMNs [2] result in error probabilities tending to one. This seems rather
challenging as we have to assert the existence of a common distribution \( \tilde{p}_{X, Y} \) for all cut-sets \( T \) in (48). This would allow us to swap the intersection and union in Theorem 1. Second, and less ambitiously, we also hope to extend our result to Gaussian networks [1, Chapter 19], which may be tractable if we restrict the models under consideration to the class of Gaussian networks for which the optimum input distribution is a multivariate Gaussian. Finally, it may be fruitful and instructive to focus our attention on smaller DM-MMNs such as the DM-RC.

**APPENDIX A**

**PROOF OF PROPOSITION 2**

**Proof of Proposition 2:** For any random variables \( U \) and \( V \), we let

\[
S_{U|V} \triangleq \{ u \in U : \Pr\{U = u | V = v \} > 0 \}
\]

be the support of \( U \) conditioned on the event \( \{ V = v \} \). If \( V \) is a trivial random variable, i.e., \( V = \emptyset \), then \( S_{U} \) is simply the support of \( U \). If \( \lambda = 1 \), the statement of the proposition is obvious so henceforth, we prove the statement for \( \lambda \in (1, 5/4] \). Suppose \( (X, Y, Z) \) is jointly distributed according to \( p_{X,Y,Z}(x, y, z) \) which we abbreviate as \( p(x, y, z) \) in this proof. Let

\[
g(\lambda) \triangleq \log \sum_{z \in S_{Z}} p(z) \sum_{(x, y) \in S_{X,Y|z}} \frac{(p(x, y|z))^{\lambda}}{(p(x|z)p(y|z))^{\lambda-1}}
\]

be a function of \( \lambda \) defined on \([1, \infty)\). Straightforward calculations involving l’Hôpital’s rule reveal that \( g(1) = 0 \) and \( g'(1) = D(p_{X,Y|Z}\|p_{X|Z}p_{Y|Z}|p_{Z}) \) (cf. Definition 7). Using Taylor’s theorem, we obtain

\[
g(\lambda) = g(1) + (\lambda - 1)g'(1) + (\lambda - 1)^{2}g''(a) \frac{2}{2}
\]

for some \( a \in [1, \lambda] \), which implies that

\[
g(\lambda) = (\lambda - 1)D(p_{X,Y|Z}\|p_{X|Z}p_{Y|Z}|p_{Z}) + (\lambda - 1)^{2}g''(a) \frac{2}{2}.
\]

(56)

Using standard calculus techniques, we obtain

\[
g''(a) = \frac{\sum_{z \in S_{Z}} p(z) \sum_{(x, y) \in S_{X,Y|z}} (p(x, y|z))^{a} \left( \log \frac{p(x, y|z)}{p(x|z)p(y|z)} \right)^{2}}{\sum_{z \in S_{Z}} p(z) \sum_{(x, y) \in S_{X,Y|z}} (p(x, y|z))^{a} \frac{p(x, y|z)^{a-1}}{p(x|z)p(y|z)}}
\]

\[
- \left( \frac{\sum_{z \in S_{Z}} p(z) \sum_{(x, y) \in S_{X,Y|z}} (p(x, y|z))^{a} \log \frac{p(x, y|z)}{p(x|z)p(y|z)}}{\sum_{z \in S_{Z}} p(z) \sum_{(x, y) \in S_{X,Y|z}} (p(x, y|z))^{a} \frac{p(x, y|z)^{a-1}}{p(x|z)p(y|z)}} \right)^{2}.
\]

(57)

In order to obtain an upper bound for \(|g''(a)|\), we will calculate a lower bound for

\[
\sum_{(x, y) \in S_{X,Y|z}} \frac{(p(x, y|z))^{a}}{(p(x|z)p(y|z))^{a-1}}
\]
and upper bounds for

\[
\sum_{(x,y) \in S_{X,Y|z}} \frac{(p(x,y|z))^a}{(p(x|z)p(y|z))^{a-1}} \log \frac{p(x,y|z)}{p(x|z)p(y|z)}
\]

and

\[
\sum_{(x,y) \in S_{X,Y|z}} \frac{(p(x,y|z))^a}{(p(x|z)p(y|z))^{a-1}} \left( \log \frac{p(x,y|z)}{p(x|z)p(y|z)} \right)^2.
\]

Consider the following chain of inequalities:

\[
\sum_{(x,y) \in S_{X,Y|z}} \frac{(p(x,y|z))^a}{(p(x|z)p(y|z))^{a-1}} \geq \sum_{(x,y) \in S_{X,Y|z}} (p(x,y|z))^a
\]

\[
\geq \max_{x,y} (p(x,y|z))^a
\]

\[
\geq (|X||Y|)^{-a}
\]

\[
\geq (|X||Y|)^{-5/4}.
\]

(58)

On the other hand, fix \(x, y\) and \(z\) such that \(p(z) > 0\) and \(p(x,y|z) > 0\), and consider \(\frac{(p(x,y|z))^a}{(p(x|z)p(y|z))^{a-1}} \log \frac{p(x,y|z)}{p(x|z)p(y|z)}\) as well as \(\frac{(p(x,y|z))^a}{(p(x|z)p(y|z))^{a-1}} \left( \log \frac{p(x,y|z)}{p(x|z)p(y|z)} \right)^2\). Since \(\min\{p(x|z), p(y|z)\} \geq p(x,y|z)\), there exist \(0 \leq k_1 \leq 1\) and \(0 \leq k_2 \leq 1\) such that \(p(x|z) = (p(x,y|z))^{k_1}\) and \(p(y|z) = (p(x,y|z))^{k_2}\). Using the facts that \(a \in (1, 5/4]\) and \(0 \leq k_1 + k_2 \leq 2\), we have

\[
|1 - (k_1 + k_2)| \leq 1
\]

and

\[
a - (a - 1)(k_1 + k_2) \geq 1/2
\]

(59)

Then,

\[
\frac{(p(x,y|z))^a}{(p(x|z)p(y|z))^{a-1}} \log \frac{p(x,y|z)}{p(x|z)p(y|z)} = (p(x,y|z))^{a-(a-1)(k_1+k_2)} \log(p(x,y|z))^{1-(k_1+k_2)}
\]

\[
\leq (p(x,y|z))^{1/2} \log(p(x,y|z))^{-1}
\]

\[
= 2(p(x,y|z))^{1/2} \log(p(x,y|z))^{-1/2}
\]

\[
\leq 2e^{-1} \log e
\]

(61)

and

\[
\frac{(p(x,y|z))^a}{(p(x|z)p(y|z))^{a-1}} \left( \log \frac{p(x,y|z)}{p(x|z)p(y|z)} \right)^2 = (p(x,y|z))^{a-(a-1)(k_1+k_2)} (\log(p(x,y|z))^{1-(k_1+k_2)})^2
\]

\[
\leq (p(x,y|z))^{1/2} (\log(p(x,y|z))^{-1})^2
\]

\[
= 4(p(x,y|z))^{1/2} (\log(p(x,y|z))^{-1/2})^2
\]

\[
\leq 16e^{-2} \log e,
\]

(62)
where
(a) follows from (59) and (60);
(b) follows from calculus that \( q \log q^{-1} \leq e^{-1} \log e \) for all \( 0 < q \leq 1 \);
(c) follows from (59) and (60);
(d) follows from calculus that \( q(\log q^{-1})^2 \leq 4e^{-2} \log e \) for all \( 0 < q \leq 1 \).

Combining (56), (57), (58), (61) and (62), we obtain

\[
g(\lambda) \leq (\lambda - 1)\bar{D}(p_{X,Y,Z}\|p_{X,Z|p_Y,Z}p_Z) + 8(\lambda - 1)^2(|X'||Y'|)^5,
\]

which implies that for each \( \lambda \in (1, 5/4] \) (note \( \lambda \neq 1 \) so we can cancel the common factors \( \lambda - 1 \)),

\[
D_{\lambda}(p_{X,Y,Z}\|p_{X,Z|p_Y,Z}p_Z) \leq D(p_{X,Y,Z}\|p_{X,Z|p_Y,Z}p_Z) + 8(\lambda - 1)(|X'||Y'|)^5
\]

and hence (5) follows.

\[\blacksquare\]

**APPENDIX B**

**Proof of Corollary 5**

Suppose the DM-MMN is dominated by some maximal product distribution \( p^*_X \), i.e.,

\[ i = 1^n p^*_X \]

such that for each \( T \subseteq \mathcal{I} \), we have

\[
I_{p^*_X Y_{T^c}|X_z} (X_T; Y_{T^c}|X_{T^c}) = \max_{p_{X,Y}\in \mathcal{C}_T} \left\{ I_{p_{X,Y}\in \mathcal{C}_T} (X_T; Y_{T^c}|X_{T^c}) \right\}.
\]

This then implies from Theorem 1 that for each \( \epsilon \in [0, 1) \),

\[
\mathcal{C}_\epsilon \subseteq \bigcap_{T \subseteq \mathcal{I}: T^c \cap D \neq \emptyset} \left\{ R_i \left|\begin{array}{c}
\sum_{i \in T} R_i \leq I_{p_{X,Y}\in \mathcal{C}_T} (X_T; Y_{T^c}|X_{T^c}), \\
R_i = 0 \text{ for all } i \in S^c
\end{array}\right. \right\} \subseteq R^*_\text{out}.
\]

This completes the proof.

\[\blacksquare\]

**APPENDIX C**

**Proof of Corollary 6**

Construct a counterpart of the channel \((X_z, Y_z, q_{Y_z|X_z})\) as follows: Let \((\tilde{X}_z, \tilde{X}_z, \tilde{q}_{X_z}|X_z)\)
be a noiseless DM-MMN consisting of independent DMCs with multicast demand \((S, D)\) such that for each

\((i, j) \in \mathcal{I} \times \mathcal{I} \), the DMC carrying information from node \( i \) to node \( j \) is an error-free (noiseless) channel, denoted

by \( \tilde{q}_{X_i,j|\tilde{X}_i,j} \), with capacity \( C_{i,j} \) (cf. (51)). To be more precise, \( \tilde{q}_{X_i,j|\tilde{X}_i,j} \) can carry \( \lfloor n C_{i,j} \rfloor \) error-free bits for each \( (i, j) \in \mathcal{I} \times \mathcal{I} \) for \( n \) uses of \((\tilde{X}_z, \tilde{X}_z, \tilde{q}_{X_z}|X_z)\). Let \( \tilde{C} \) denote the capacity region of \((\tilde{X}_z, \tilde{X}_z, \tilde{q}_{X_z}|X_z)\). It has been shown in [25, Section IIA] that \( \tilde{C} = \mathcal{C}' \). Therefore, it remains to show \( \tilde{C} \subseteq \mathcal{C}_0 \). Since every channel \( q_{Y_{i,j}|X_{i,j}} \) can carry information from node \( i \) to node \( j \) at a rate arbitrarily close to \( C_{i,j} \) (the capacity of \( \tilde{q}_{X_i,j|\tilde{X}_i,j} \)), it follows that for every \( R_{\tilde{C}} \) that lies in the interior of \( \tilde{C} \), there exists a sequence of \((n, R_{\tilde{C}}, \epsilon_n)\)-codes for the original channel

\((X_z, Y_z, q_{Y_z|X_z})\) such that \( \lim_{n \to \infty} \epsilon_n = 0 \). Consequently, \( \mathcal{C}' = \tilde{C} \subseteq \mathcal{C}_0 \), which is what was to be proved.

\[\blacksquare\]
Proof of Lemma 7: Since $C_e \subseteq R_{\text{out}}$ for all $\epsilon \in [0, 1)$ by Theorem 1, it remains to show $R_{\text{out}} \subseteq R'$. In order to obtain an outer bound of $R_{\text{out}}$, we consider the following chain of inequalities for each $p_{X_T}$ and each $T \subseteq \mathcal{I}$:

$$I_{p_{X_T} q_{Y_T e} | X_Z} (X_T; Y_T^e | X_T^e)$$

$$= \sum_{j \in T^e} I_{p_{X_T} q_{Y_T e} | X_Z} (X_T; Y_j | X_T^e, Y_{j \in T^e : j < j})$$

$$= \sum_{j \in T^e} \sum_{\ell = 1}^N I_{p_{X_T} q_{Y_T e} | X_Z} (X_T; Y_{\ell,j} | X_T^e, Y_{j \in T^e : j < j}), \{Y_{m,j}\}_{m=1}^{\ell-1}$$

$$= \sum_{j \in T^e} \sum_{\ell \in T} I_{p_{X_T} q_{Y_T e} | X_Z} (X_{\ell,j}; Y_{\ell,j})$$

$$\leq \sum_{(\ell,j) \in T \times T^e} C_{\ell,j}.$$  \hfill (63)

where

(a) follows from the fact that for all $\ell \in T^e$,

$$I_{p_{X_T} q_{Y_T e} | X_Z} (X_T; Y_{\ell,j} | X_T^e, Y_{j \in T^e : j < j}), \{Y_{m,j}\}_{m=1}^{\ell-1}$$

$$= H_{p_{X_T} q_{Y_T e} | X_Z} (Y_{\ell,j} | X_T^e, Y_{j \in T^e : j < j}), \{Y_{m,j}\}_{m=1}^{\ell-1} - H_{p_{X_T} q_{Y_T e} | X_Z} (Y_{\ell,j} | X_T, Y_{j \in T^e : j < j}), \{Y_{m,j}\}_{m=1}^{\ell-1}$$

$$= H_{p_{X_T} q_{Y_T e} | X_Z} (Y_{\ell,j} | X_T^e, Y_{j \in T^e : j < j}) - H_{p_{X_T} q_{Y_T e} | X_Z} (Y_{\ell,j} | X_T, Y_{j \in T^e : j < j})$$

$$= 0;$$

(b) follows from the fact that for all $\ell \in T$,

$$I_{p_{X_T} q_{Y_T e} | X_Z} (X_T; Y_{\ell,j} | X_T^e, Y_{j \in T^e : j < j}), \{Y_{m,j}\}_{m=1}^{\ell-1}$$

$$\leq H_{p_{X_T} q_{Y_T e} | X_Z} (Y_{\ell,j}) - H_{p_{X_T} q_{Y_T e} | X_Z} (Y_{\ell,j} | X_T, Y_{j \in T^e : j < j}), \{Y_{m,j}\}_{m=1}^{\ell-1}$$

$$= H_{p_{X_T} q_{Y_T e} | X_Z} (Y_{\ell,j}) - H_{p_{X_T} q_{Y_T e} | X_Z} (Y_{\ell,j} | X_T)$$

$$= I(X_{\ell,j}; Y_{\ell,j}).$$

Combining (2), (55) and (63), we have $R_{\text{out}} \subseteq R'$.
APPENDIX E
PROOF OF THEOREM 6

Proof of Theorem 6: Fix any \( \epsilon \in [0, 1) \). Since \( C_\epsilon = \mathcal{R}' \) by Theorem 5 and \( C_\epsilon \subseteq \tilde{C}_\epsilon \), it remains to show that \( \tilde{C}_\epsilon \subseteq \mathcal{R}' \). Define
\[
\tilde{R}_{\text{out}} \triangleq \bigcap_{T \subseteq I: T \cap \{d\} \neq \emptyset} \bigcup_{p_X Z} \left\{ R_T \mid \sum_{i \in T} R_i \leq I_{p_X Z \tilde{Y}_{T c}\mid X_Z} (X_T; \tilde{Y}_{T c}\mid X_{T c}), \right. \\
\left. R_i = 0 \text{ for all } i \in S^c \right\}.
\]
(64)

Since \( \tilde{C}_\epsilon \subseteq \tilde{R}_{\text{out}} \) for all \( \epsilon \in [0, 1) \) by Theorem 1 and \( R_{\text{out}} \subseteq \mathcal{R}' \) by Lemma 7, it suffices to show \( \tilde{R}_{\text{out}} = R_{\text{out}} \). To this end, we consider the following chain of equalities for each \( p_X Z \) and each \( T \subseteq I \) such that \( T^c \cap \{d\} \neq \emptyset \):
\[
I_{p_X Z \tilde{Y}_{T c}\mid X_Z} (X_T; \tilde{Y}_{T c}\mid X_{T c}) \overset{(a)}{=} I_{p_X Z \tilde{Y}_{T c}\mid X_Z} (X_T; Y_{T c}, Y_d|X_{T c}) \\
\overset{(b)}{=} I_{p_X Z \tilde{Y}_{T c}\mid X_Z} (X_T; Y_{T c}|X_{T c}).
\]
(65)

where
(a) follows from Condition 2 in Definition 9;
(b) follows from the fact that \( T^c \cap \{d\} \neq \emptyset \).

Combining (2), (64) and (65), we have \( R_{\text{out}} = \tilde{R}_{\text{out}} \).
[12] D. Xu and D. Erdogmuns, “Rényi’s entropy, divergence and their nonparametric estimators,” in Information Theoretic Learning: Rényi’s Entropy and Kernel Perspectives, J. C. Principe, Ed. Springer, 2010, pp. 47–102.

[13] I. Csiszár, “Generalized cutoff rates and Rényi’s information measures,” IEEE Trans. Inf. Theory, vol. 41, no. 1, pp. 26–34, 1995.

[14] S. Arimoto, “On the converse to the coding theorem for discrete memoryless channels,” IEEE Trans. Inf. Theory, vol. 19, no. 5, pp. 357–359, 1973.

[15] T. Ogawa and H. Nagaoka, “Strong converse to the quantum channel coding theorem,” IEEE Trans. Inf. Theory, vol. 45, no. 7, pp. 2486–2489, 1999.

[16] M. M. Wilde, A. Winter, and D. Yang, “Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy,” ArXiv Preprint, 2013, http://arxiv.org/abs/1306.1586.

[17] V. Y. F. Tan, “On the reliability function of the discrete memoryless relay channel,” Submitted to the IEEE Trans. on Inf. Theory (Revised), 2013, http://arxiv.org/abs/1304.3553.

[18] A. Behboodi and P. Piantanida, “On the asymptotic error probability of composite relay channels,” in Proc. of IEEE Intl. Symp. on Inf. Theory, St Petersburg, Russia, 2011.

[19] ——, “On the asymptotic spectrum of the error probability of composite networks,” in Proc. of IEEE Information Theory Workshop (ITW), Lausanne, Switzerland, 2012.

[20] A. Behboodi, “Cooperative networks with channel uncertainty,” Ph.D. dissertation, Department of Telecommunications, Supélec (École Supérieure d’Électricité), 2012, http://tel.archives-ouvertes.fr/tel-00765429.

[21] T. S. Han, Information-Spectrum Methods in Information Theory. Springer Berlin Heidelberg, Feb 2003.

[22] T. van Erven and P. Harremoës, “Rényi divergence and Kullback–Leibler divergence,” IEEE Trans. Inf. Theory, vol. 60, no. 7, pp. 3797–3820, 2014.

[23] T. M. Cover and J. A. Thomas, Elements of Information Theory, 2nd ed. Wiley-Interscience, 2006.

[24] R. W. Yeung, Information Theory and Network Coding. Springer, 2008.

[25] S. H. Lim, Y.-H. Kim, A. El Gamal and S.-Y. Chung, “Noisy network coding,” IEEE Trans. Inf. Theory, vol. 57, no. 5, pp. 3132–3152, 2011.

[26] C. E. Shannon, “The zero error capacity of a noisy channel,” IRE Trans. Inf. Theory, vol. 2, no. 3, pp. 8–19, 1956.

[27] I. Csiszár and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems. Cambridge University Press, 2011.