THE TIME CONSTANT AND CRITICAL PROBABILITIES IN PERCOLATION MODELS

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Abstract
We consider a first-passage percolation (FPP) model on a Delaunay triangulation $D$ of the plane. In this model each edge $e$ of $D$ is independently equipped with a nonnegative random variable $\tau_e$, with distribution function $F$, which is interpreted as the time it takes to traverse the edge. Vahidi-Asl and Wierman [9] have shown that, under a suitable moment condition on $F$, the minimum time taken to reach a point $x$ from the origin $0$ is asymptotically $\mu(F)|x|$, where $\mu(F)$ is a nonnegative finite constant. However the exact value of the time constant $\mu(F)$ still a fundamental problem in percolation theory. Here we prove that if $F(0) < 1 - p_c^*$ then $\mu(F) > 0$, where $p_c^*$ is a critical probability for bond percolation on the dual graph $D^*$.

Introduction
First-passage percolation theory on periodic graphs was presented by Hammersley and Welsh [4] to model the spread of a fluid through a porous medium. In this paper we continue a study of planar first-passage percolation models on random graphs, initiated by Vahidi-Asl and Wierman [9], as follows. Let $P$ denote the set of points realized in a two-dimensional homogeneous Poisson point process with intensity $1$. To each $v \in P$ corresponds an open polygonal region $C_v = C_v(P)$, the Voronoi tile at $v$, consisting of the set of points of $\mathbb{R}^2$ which are closer to $v$ than to any other $v' \in P$. Given $x \in \mathbb{R}^2$ we denote by $v_x$ the almost surely unique point in $P$ such that $x \in C_{v_x}$. The collection $\{C_v : v \in P\}$ is called the Voronoi Tiling of the plane based on $P$.

The Delaunay Triangulation $D$ is the graph where the vertex set $D_v$ equals $P$ and the edge set $D_e$ consists of non-oriented pairs $(v, v')$ such that $C_v$ and $C_{v'}$ share a one-dimensional edge (Figure 1). One can see that almost surely each Voronoi tile is a convex and bounded polygon, and the graph $D$ is a triangulation of the plane [7]. The Voronoi Tessellation $V$ is the graph where the vertex set $V_v$ is the set of vertices of the Voronoi tiles and the edge set $V_e$ is the set

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The time constant and critical probabilities

Figure 1: The Delaunay Triangulation and the Voronoi Tessellation.

of edges of the Voronoi tiles. The edges of $V$ are segments of the perpendicular bisectors of the edges of $D$. This establishes duality of $D$ and $V$ as planar graphs: $V = D^\ast$.

To each edge $e \in D$ is independently assigned a nonnegative random variable $\tau_e$ from a common distribution $F$, which is also independent of the Poisson point process that generates $P$. From now on we denote $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space induced by the Poisson point process $P$ and the passage times $(\tau_e)_{e \in D}$. The passage time $t(\gamma)$ of a path $\gamma$ in the Delaunay Triangulation is the sum of the passage times of the edges in $\gamma$. The first-passage time between two vertices $v$ and $v'$ is defined by

$$T(v, v') := \inf\{t(\gamma) ; \gamma \in C(v, v')\},$$

where $C(v, v')$ the set of all paths connecting $v$ to $v'$. Given $x, y \in \mathbb{R}^2$ we define $T(x, y) := T(x_1, y_1)$. To state the main result of this work we require some definitions involving a bond percolation model on the Voronoi Tessellation $V$. Such a model is constructed by choosing each edge of $V$ to be open independently with probability $p$. An open path is a path composed of open edges. We denote $P^p$ the law induced by the Poisson point process and the random state (open or not) of an edge. Given a planar graph $G$ and $A, B \subseteq \mathbb{R}^2$ we say that a self-avoiding path $\gamma = (v_1, ..., v_k)$ is a path connecting $A$ to $B$ if $[v_1, v_2] \cap A \neq \emptyset$ and $[v_{k-1}, v_k] \cap B \neq \emptyset$ ([x, y] denotes the line segment connecting x to y). For $L > 0$ let $A_L$ be the event that there exists an open path $\gamma = (v_j)_{1 \leq j \leq h}$ in $V$, connecting $\{0\} \times [0, L]$ to $\{3L\} \times [0, L]$, and with $v_j \in [0, 3L] \times [0, L]$ for all $j = 2, \ldots, h - 1$. In this case we also say that $\gamma$ crosses the rectangle $[0, 3L] \times [0, L]$. Define the function

$$\eta^p(p) := \liminf_{L \to \infty} \mathbb{P}_p(A_L),$$

and consider the percolation threshold,

$$p^*_c := \inf\{p > 0 : \eta^p(p) = 1\}. \tag{1}$$

We have that $p^*_c \in (0, 1)$, which follows by standard arguments in percolation theory. For more in percolation thresholds on Voronoi tilings we refer to [1, 2, 11].
Theorem 1 If $\mathbb{F}(0) < 1 - p_c^*$ then there exist constants $c_j > 0$ such that for all $n \geq 1$
\[\mathbb{F}(T(0,n) < c_1 n) \leq c_2 \exp(-c_3 n),\]  
(2)
where $0 := (0,0)$ and $n := (n,0)$.

To show the importance of Theorem 1 we recall two fundamental results proved by Vahidi-Asl and Wierman [9, 10]. Consider the growth process
\[B_x(t) := \{y \in \mathbb{R}^2 : y \in c(C_\nu) \text{ with } \nu \in \mathcal{D}_c \text{ and } T(v_x, \nu) \leq t\}.
\]
where $c(C)$ denotes the closure of $C \in \mathbb{R}^2$. Set
\[\mu(\mathbb{F}) := \inf_{n>0} \frac{ET(0,n)}{n} \in [0,\infty].
\]
and let $\tau_1, \tau_2, \tau_3$ be independent random variables with distribution $\mathbb{F}$. If
\[E\left( \min_{j=1,2,3} \{\tau_j\} \right) < \infty
\]
(3)
then $\mu(\mathbb{F}) < \infty$ and for all unit vectors $\vec{x} \in S^1$ ($|\vec{x}| = 1$) $\mathbb{P}$-a.s.
\[\lim_{n\to\infty} \frac{T(0,n\vec{x})}{n} = \lim_{n\to\infty} \frac{ET(0,n)}{n} = \mu(\mathbb{F}).
\]
(4)
Further, if
\[E\left( \min_{j=1,2,3} \{\tau_j\}^2 \right) < \infty
\]
(5)
and $\mu(\mathbb{F}) > 0$ then for all $\epsilon > 0$ $\mathbb{P}$-a.s. there exists $t_0 > 0$ such that for all $t > t_0$
\[(1 - \epsilon) t D(1/\mu) \subseteq B_0(t) \subseteq (1 + \epsilon) t D(1/\mu),
\]
(6)
where $D(r) := \{x \in \mathbb{R}^2 : |x| \leq r\}$.

We note here that the asymptotic shape is an Euclidean ball due to the statistical invariance of the Poisson point process. Unfortunately the exact value of the time constant $\mu(\mathbb{F})$, as a functional of $\mathbb{F}$, still a basic problem in first-passage percolation theory. Our result provides a sufficient condition on $\mathbb{F}$ to ensure $\mu(\mathbb{F}) > 0$.

Corollary 1 Under assumption (3), if $\mathbb{F}(0) < 1 - p_c^*$ then $\mu(\mathbb{F}) \in (0,\infty)$.

Proof. Together with the Borel-Cantelli Lemma, Theorem 1 and (4) imply
\[0 < c_1 \leq \liminf_{n\to\infty} \frac{T(0,n)}{n} = \lim_{n\to\infty} \frac{T(0,n)}{n} = \mu(\mathbb{F}) < \infty,
\]
which is the desired result. \hfill $\square$

For FPP models on the $\mathbb{Z}^2$ lattice Kesten (1986) has shown that $\mathbb{F}(0) < 1/2 = p_c(\mathbb{Z}^2)$ (the critical probability for bond percolation on $\mathbb{Z}^2$) is a sufficient condition to get (2) by using a stronger version of the BK-inequality. Here we follow a different method and we apply a simple renormalization argument to obtain a similar result. We expect that our condition to get (2) is equivalent to
\[\mathbb{F}(0) < p_c := \inf\{p > 0 \ ; \ \theta(p) = 1\},
\]
where $\theta(p)$ is the probability that bond percolation on $\mathcal{D}$ occurs with density $p$, since it is conjectured that $p_c + p_c^* = 1$ (duality) for many planar graphs. In fact, by combining Corollary 1 with (6) we have:
Corollary 2

\[ 1 \leq p_c + p_c^* . \]

**Proof.** To see this assume we have a first-passage percolation model on \( D \) with

\[ P(\tau_e = 0) = 1 - P(\tau_e = 1) = F(0) = 1 - p > p_c^* . \]  

(7)

Then \( \Pr \)-a.s. there exists an infinite cluster \( W \subseteq D \) composed by edges \( e \) with \( \tau_e = 0 \). Denote by \( T(0, W) \) the first-passage time from \( 0 \) to \( W \). Then for all \( t > T(0, W) \) we have that \( B_0(t) \) is an unbounded set. By (6) (since such a distribution satisfies (3) and (5)), this implies that \( \mu(F) = \mu(p) = 0 \) if \( 1 - p > p_c \). On the other hand, by Corollary 1, \( \mu(p) > 0 \) if \( 1 - p < 1 - p_c^* \), and so (2) must hold. \( \Box \)

Other passage times have been considered in the literature such as \( T(0, H_n) \), where \( H_n \) is the hyperplane consisting of points \( x = (x_1, x_2) \) so that \( x_1 = n \), and \( T(0, \partial[-n, n]^2) \). The arguments in this article can be used to prove the analog of Theorem 1 when \( T(0, n) \) is replaced by \( T(0, H_n) \) or \( T(0, \partial[-n, n]^2) \). For site versions of FPP models the method works as well if we change the condition on \( F \) to \( F(0) < 1 - \bar{p}_c \), where now \( \bar{p}_c \) is the critical probability for site percolation. Similarly to Corollary 2, in this case one can also obtain the inequality \( 1/2 \leq \bar{p}_c \). For more details we refer to [8].

1 Renormalization

For the moment we assume that \( F \) is Bernoulli with parameter \( p \). Let \( L \geq 1 \) be a parameter whose value will be specified later. Let \( z = (z^1, z^2) \in \mathbb{Z}^2 \) and

\[ |z|_\infty := \max_{j=1,2} \{ |z^j| \} . \]

Denote \( C_z \) the circuit composed by sites \( z' \in \mathbb{Z}^2 \) with \( |z - z'|_\infty = 2 \). For each \( A \subseteq \mathbb{R}^2 \), we denote by \( \partial A \) its boundary. For each \( z \in \mathbb{Z}^2 \) and \( r \in \{ j/2 : j \in \mathbb{N} \} \) consider the box

\[ B_r^L := Lz + [-rL, rL]^2 . \]

Divide \( B_r^L \) into thirty-six sub-boxes with the same size and declare that \( B_r^L \) is a full box if all these thirty-six sub-boxes contain at least one point of \( P \). Let

\[ H_z^L := \left[ B_r^L \text{ is a full box } \forall z' \in C_z \right] . \]

Let \( C_L \) be the set of all self-avoiding paths \( \gamma = (\nu_j)_{1 \leq j \leq h} \) in \( D \), connecting \( \partial B_z^{L/2} \) to \( \partial B_z^{3L/2} \) and with \( C_{\nu_j} \cap B_z^{L/2} \) for all \( j = 2, \ldots, h - 1 \). Let

\[ G_z^L := \left[ t(\gamma) \geq 1 \forall \gamma \in C_L \right] . \]

We say that \( B_z^{L/2} \) is a good box (or that \( z \) is a good point) if

\[ Y_z^L := I( H_z^L \cap G_z^L ) = 1 , \]

where \( I(E) \) denotes the indicator function of the event \( E \).
Lemma 1 If \( \mathbb{P}(\tau_e = 0) = 1 - p < 1 - p^*_c \) then
\[
\lim_{L \to \infty} \mathbb{P}(Y^L_0 = 1) = 1.
\]

Proof. First notice that
\[
\mathbb{P}(Y^L_0 = 0) \leq \mathbb{P}((H^L_0)^c) + \mathbb{P}((G^L_0)^c). \tag{8}
\]
By the definition of a two-dimensional homogeneous Poisson point process,
\[
\lim_{L \to \infty} \mathbb{P}((H^L_0)^c) = 0. \tag{9}
\]
Now, let \( X_{e^*} := \tau_e \), where \( e^* \) is the edge in \( V_e \) (the Voronoi tessellation) dual to \( e \). Then \( \{X_{e^*} : e^* \in V_e\} \) defines a bond percolation model on \( V \) with law \( P^*_p \). Consider the rectangles
\[
R^1_L := [L/2, 3L/2] \times [-3L/2, 3L/2], R^2_L := [-3L/2, 3L/2] \times [L/2, 3L/2].
\]
We denote by \( A^i_L \) the event \( A_L \) (recall the definition of \( p^*_c \)) but now translate to the rectangle \( R^i_L \), and by \( F_L \) the event that an open circuit \( \sigma^* \) in \( V \) which surrounds \( B^L_{0} \) and lies inside \( B^{3L/2}_{0} \) does not exist. Thus one can easily see that
\[
\bigcap_{i=1}^4 A^i_L \subseteq (F_L)^c.
\]
Notice that if there exists an open circuit \( \sigma^* \) in \( V \) which surrounds \( B^{L/2}_{0} \) and lies inside \( B^{3L/2}_{0} \), then every path \( \gamma \) in \( C_L \) has an edge crossing with \( \sigma^* \) and thus \( t(\gamma) \geq 1 \). Therefore,
\[
\mathbb{P}((G^L_0)^c) \leq \mathbb{P}_p^*(F_L) \leq 4(1 - \mathbb{P}_p^*(A_L)). \tag{10}
\]
Since \( p > p^*_c \), by using (8), (9), (10) and the definition of \( p^*_c \), we get Lemma 1. \( \square \)

To obtain some sort of independence between the random variables \( Y^L_0 \) we shall study some geometrical aspects of Voronoi tilings. Given \( A \subseteq \mathbb{R}^2 \), let \( \mathcal{I}_P(A) \) be the sub-graph of \( \mathcal{D} \) composed of vertices \( v_1 \) in \( \mathcal{D}_e \) and edges \((v_2, v_3)\) in \( \mathcal{D}_e \) so that \( C_{v_i} \cap A \neq \emptyset \) for all \( i = 1, 2, 3 \).
Lemma 2 Let $L > 0$ and $z \in \mathbb{Z}^2$. Assume that $\mathcal{P}$ and $\mathcal{P}'$ are two configurations of points so that $\mathcal{P} \cap B_z^{5L/2} = \mathcal{P}' \cap B_z^{5L/2}$ and that $B_z^{L/2}$ is a full box with respect to $\mathcal{P}$, for all $z' \in C_z$. Then $I_\mathcal{P}(B_z^{3L/2}) = I_\mathcal{P}'(B_z^{3L/2})$.

PROOF. By the definition of the Delaunay Triangulation, Lemma 2 holds if we prove that

$$C_\nu(\mathcal{P}) \cap B_z^{3L/2} \neq \emptyset \Rightarrow C_\nu(\mathcal{P}) = C_\nu(\mathcal{P}') .$$  \hspace{1cm} (11)

To prove this we claim that

$$C_\nu(\mathcal{P}) \cap B_z^{3L/2} \neq \emptyset \Rightarrow C_\nu(\mathcal{P}) \subseteq B_z^{2L} .$$  \hspace{1cm} (12)

If (12) does not hold then there exist $x_1 \in \partial B_z^{3L/2} \cap C_\nu(\mathcal{P})$ and $x_2 \in \partial B_z^{2L} \cap C_\nu(\mathcal{P})$ (by convexity of Voronoi tilings). Since every box $B_z^{L/2}$ with $|z - z'|_\infty = 2$ is a full box, there exist $v_1, v_2 \in \mathcal{P}$ so that

$$|v_1 - x_1| \leq \sqrt{2}L/6 \text{ and } |v_2 - x_2| \leq \sqrt{2}L/6 .$$

Although, $x_1$ and $x_2$ belong to $C_\nu(\mathcal{P})$ and so

$$|v - x_1| \leq |v_1 - x_1| \text{ and } |v - x_2| \leq |v_2 - x_2| .$$

Thus,

$$L/2 \leq |x_1 - x_2| \leq |x_1 - v| + |x_2 - v| \leq \sqrt{2}L/3 ,$$

which leads to a contradiction since $\sqrt{2}/3 < 1/2$. By an analogous argument, one can prove that

$$C_\nu'(\mathcal{P}') \cap (B_z^{5L/2})^c \neq \emptyset \Rightarrow C_\nu'(\mathcal{P}') \subseteq (B_z^{2L})^c .$$  \hspace{1cm} (13)

Now suppose (11) does not hold. Without loss of generality, we may assume that there exists $v \in \mathcal{P}$ with $C_\nu(\mathcal{P}) \cap B_z^{3L/2} \neq \emptyset$ and $x \in C_\nu(\mathcal{P})$ with $x \notin C_\nu(\mathcal{P}')$. So $x \in C_\nu(\mathcal{P}')$ for some $v' \in \mathcal{P}'$. Although, $\mathcal{P} \cap B_z^{5L/2} = \mathcal{P}' \cap B_z^{5L/2}$ and then $v' \in (B_z^{5L/2})^c$, which is a contradiction with (12) and (13). \qed

For each $l \geq 1$, we say that the collection of random variables $\{Y_z : z \in \mathbb{Z}^2\}$ is $l$-dependent if $\{Y_z : z \in \mathcal{A}\}$ and $\{Y_z : z \in \mathcal{B}\}$ are independent whenever

$$l < d_\infty(\mathcal{A}, \mathcal{B}) := \min\{|z - z'|_\infty : z \in \mathcal{A} \text{ and } z' \in \mathcal{B}\} .$$

Combining Lemma 2 with the translation invariance and the independence property of the Poisson point process we obtain:

Lemma 3 For all $L > 0$, $\{Y^L_z : z \in \mathbb{Z}^2\}$ is a $5$-dependent collection of identically distributed Bernoulli random variables.

Denote $Y^L := \{Y^L_z : z \in \mathbb{Z}^2\}$ and let $M_m(Y^L)$ be the maximum number of pairwise disjoint good circuits in $\mathbb{Z}^2$, surrounding the origin and lying inside the box $[-m, m]^2$.

Lemma 4 If $\mathbb{P}(0 < 1 - p_c^*)$ then there exists $L_0 > 0$ and $c_1 = c_1(L_0) > 0$ such that

$$\mathbb{P}(M_m(Y^{L_0}) \leq c_1 m) \leq \exp(-c_2 m) .$$
The connection between the variable $M_m(Y^L)$ and the first-passage time $T(0, n)$ is summarize by the following:

**Lemma 5**

$$\frac{M_{nL-1}}{6} \leq T(0, n).$$

**Proof.** We say that $(B^{L/2}_{\bar{z}^i})_{1 \leq i \leq h}$ is a circuit of good boxes if $(z_j)_{1 \leq j \leq h}$ is a good circuit in $\mathbb{Z}^2$, and that $(B^{L/2}_{\bar{z}^i})_{1 \leq i \leq h}$ and $(B^{L/2}_{\bar{z}'^j})_{1 \leq j \leq h'}$ are $l$-distant if

$$d_\infty((z_j)_{1 \leq j \leq h}, (z'_j)_{1 \leq j \leq h'}) > l.$$

Denote $M_m := M_m(Y^L)$. Notice that there exist at least $(M_{nL-1}/6)$ pairwise 5-distant circuits of good boxes surrounding the origin and lying inside $[-n, n]^2 \subseteq \mathbb{R}^2$. Therefore, every path $\gamma$ between the origin and any point outside $[-n, n]^2$ must cross at least $(M_{nL-1}/6)$ 5-distant circuits of good boxes. We claim this yields

$$\frac{M_{nL-1}}{6} \leq t(\gamma). \quad (14)$$

Indeed, assume we take two 5-distant good boxes, say $B^{L/2}_{\bar{z}^i}$ and $B^{L/2}_{\bar{z}'^j}$, connected by a path $\gamma$ in $D$. Then $\gamma$ must contain two sub-paths in $D$, say $\gamma_i = (v^i_j)_{1 \leq j \leq h_i}$ for $i = 1, 2$, connecting $\partial B^{3L/2}_{\bar{z}^i}$ to $\partial B^{5L/2}_{\bar{z}^i}$ and with $C_{v^i_j} \cap B^{3L/2}_{\bar{z}^i}$ for all $j = 2, ..., h_i - 1$. Since $B^{L/2}_{\bar{z}^i}$ and $B^{L/2}_{\bar{z}'^j}$ are 5-distant good boxes, by Lemma 2, these sub-paths must be edge disjoint. By the definition of a good box, $t(\gamma_1) \geq 1$ and $t(\gamma_2) \geq 1$, which yields

$$2 \leq t(\gamma_1) + t(\gamma_2) \leq t(\gamma).$$

By repeating this argument inductively (on the number of good boxes which are crossed by $\gamma$) one can get $(14)$. Lemma 5 follows directly from $(14)$.

Now we are ready to prove Theorem 1.

**Proof.** Together with Lemma 5, Lemma 4 implies Theorem 1 under $(7)$. For the general case, assume $F(0) = F(\tau_\infty = 0) < 1 - p_1$. Fix $\epsilon > 0$ so that $F(\epsilon) < 1 - p_\epsilon^*$ (we can do so since $F$ is right-continuous). Define the auxiliary process $\tau_\epsilon^* := I(\tau_\infty > \epsilon)$ and denote by $T^\epsilon$ the first-passage time associated to the collection $\{\tau_\epsilon^* : e \in D_\epsilon\}$. Thus $T^\epsilon(0, n) \leq \epsilon^{-1} T(0, n)$. Since $\tau_\epsilon^*$ has a Bernoulli distribution with parameter $\mathbb{P}(\tau_\epsilon^* = 0) = F(\epsilon) < 1 - p_\epsilon^*$, together with the previous case this yields Theorem 1.

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The time constant and critical probabilities

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