SENSITIVITY OF ITERATED FUNCTION SYSTEMS

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ABSTRACT. The present work is concerned with sensitivity of iterated function systems (IFSs). First of all, several sufficient conditions for sensitivity of IFSs are presented. Moreover, we introduce the concept of S-transitivity for IFSs which is relevant with sensitive dependence on initial condition. That yields to a different example of non-minimal sensitive system which is not an M-system. Also, some interesting examples are given which provide some facts about the sensitive property of IFSs.

1. Introduction

In deterministic dynamical systems, chaos concludes all random phenomena without any stochastic factors. Also, it is one of the core themes in dynamical systems which its role is undeniable on the development of nonlinear dynamics.

On the study of chaos, the concept of sensitivity is a key ingredient. The first formulate of sensitivity (as far as we know) was given by Guckenheimer on the study of interval maps, [10]. Sensitivity has recently been the subject of considerable interest (e.g. [1][3][4][5][7][11][13][14]).

Now, we recall some recent relevant results about the sensitivity. In [9] Glasner and Weiss (see also Akin, Auslander and Berg [2]) prove that if $X$ is a compact metric space and $(X, f)$ is a non-minimal M-system then $(X, f)$ is sensitive. After that Kontorovich and Megrelishvili [13] generalize this result to C-semigroups. In fact, they prove that for a polish space $(X, d)$ and a C-semigroup $S$ if $(S, X)$ is an M-system, then the system is either minimal and equicontinuous or sensitive.

The aim of the present paper is to discuss the sensitivity property for iterated function systems (IFS). In our results, we establish the sensitivity for non-minimal iterated function systems which yields to a different example of non-minimal sensitive IFS which is not an M-system. Also, we give an example of a sensitive minimal IFS that is not equicontinuous.

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Moreover, we introduce the concepts of strong transitivity and $S$-transitivity for IFSs which are relevant with sensitive dependence on initial.

**The paper is organized as follows:** In Section 2, we recall some standard definitions about iterated function systems. We study sensitive IFSs in Section 3 and show that every strongly transitive non-minimal IFS is sensitive. This result can be proposed by a weaker form of dynamical irreducibility which is called $S$-transitivity. Furthermore, we prove that local expanding property and expanding property in IFSs yield to sensitivity and cofinite sensitivity, respectively. In Section 4, we give several examples of IFSs. The first shows that strong transitivity does not follows by minimality but the second is a non-minimal $S$-transitive IFS which is not strongly transitive. These examples provide two non-minimal sensitive systems which are not $M$-system. Finally, the third example illustrates a non-sensitive topologically transitive IFS with a dense set of periodic points. Notice that for deterministic dynamics there is an interesting result [5]: each topologically transitive continuous map with a dense subset of periodic points has sensitive dependence to initial conditions. However, the third example in this article shows that this statement does not true for IFSs.

2. Preliminaries

A dynamical system in the present paper is a triple $(\Gamma, X, \varphi)$, where $\Gamma$ is a semigroup, $X$ is a compact metric space and

$$\varphi : \Gamma \times X \to X, (\gamma, x) \mapsto \gamma x$$

is a continuous action on $X$ with the property that $\gamma(\eta x) = (\gamma \eta)x$ holds for every triple $(\gamma, \eta, x)$ in $\Gamma \times \Gamma \times X$. Sometimes we write the dynamical system as a pair $(\Gamma, X)$. The orbit of $x$ is the set $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$. A point $x$ is called almost periodic if the subsystem $\Gamma x$ is minimal and compact. We say that a topological semigroup $\Gamma$ is a $C$-semigroup if for every $x \in X$, $\Gamma \setminus \Gamma x$ is relatively compact. By $\overline{A}$ we will denote the closure of a subset $A \subseteq X$.

Now, consider a finite family of continuous maps $\mathcal{F} = \{f_1, \ldots, f_k\}$ defined on a compact metric space $X$. We denote by $\mathcal{F}^+$ the semigroup generated by these maps. If $\Gamma = \mathcal{F}^+$ then the dynamical system $(\Gamma, X)$ is called an iterated function system (IFS) associated to $\mathcal{F}$. We use the usual notations: IFS$(X; \mathcal{F})$ or IFS$(\mathcal{F})$. Roughly speaking, an iterated function system (IFS) can be thought of as a finite collection of maps which can be applied successively in any order.

In particular, if $\Gamma = \{f^n\}_{n \in \mathbb{N}}$ and $f : X \to X$ is a continuous function, then the classical dynamical system $(\Gamma, X)$ is called a cascade and we use the standard notation: $(X, f)$. 
Throughout this paper, we assume that \((X,d)\) is a compact metric space with at least two distinct points and without any isolated point. Also, we assume that IFS(\(F\)) is an iterated function system generated by a finite family \(\{f_1, \ldots, f_k\}\) of homeomorphisms on \(X\).

For the semigroup \(F^+\) and \(x \in M\) the total forward orbit of \(x\) is defined by
\[
O^+_F(x) = \{ h(x) : h \in F^+ \}.
\]
Analogously, the total backward orbit of \(x\) is defined by
\[
O^-_F(x) = \{ h^{-1}(x) : h \in F^+ \}.
\]
We say that \(x \in X\) is a periodic point of IFS(\(F\)) if \(h(x) = x\) for some \(h \in F^+\).

Now, consider an iterated function system IFS(\(F\)) generated by a finite family of homeomorphisms \(F = \{f_1, \ldots, f_k\}\) on \(X\).

(i) IFS(\(F\)) (or \(F^+\)) is called symmetric, if for each \(f \in F\) it holds that \(f^{-1} \in F\).

(ii) IFS(\(F\)) is called forward minimal or \(F^+\) acts minimally on \(X\), if \(O^+_F(x) = X\) for every \(x \in X\). Equivalently, every forward invariant non-empty closed subset of \(X\) is the whole space \(X\), where \(A \subset X\) is forward invariant for IFS(\(X; F\)) if \(f(A) \subseteq A; \forall f \in F\).

(iii) IFS(\(F\)) is called topologically transitive, if for any two non-empty open sets \(U\) and \(V\) in \(X\), there exists \(h \in F^+\) such that \(h(U) \cap V \neq \emptyset\).

(iv) IFS(\(F\)) is called strongly transitive or backward minimal, if IFS(\(F^{-1}\)) is minimal where \(F^{-1} = \{f_1^{-1}, \ldots, f_k^{-1}\}\), i.e. \(O^-_F(x) = X\) for every \(x \in X\).

It is not hard to show that strong transitivity implies topological transitivity.

Symbolic dynamic is a way to represent the elements of \(F^+\). Indeed, consider the product space \(\Sigma^+_k = \{1, \ldots, k\}^\mathbb{N}\). For any sequence \(\omega = (\omega_1 \omega_2 \ldots \omega_n \ldots) \in \Sigma^+_k\), take \(f^0_\omega := Id\) and
\[
f^n_\omega(x) = f^n_{\omega_1 \omega_2 \ldots \omega_n}(x) = f_{\omega_n} \circ f^{-1}_\omega(x); \forall n \in \mathbb{N}.
\]
Obviously, \(f^n_\omega = f^n_{\omega_1 \ldots \omega_n} = f_{\omega_n} \circ f^{-1}_\omega \in F^+, \text{ for every } n \in \mathbb{N}.
\]

3. Sensitive iterated function systems

In this section, we give several sufficient conditions for sensitivity of IFSs. Before to state our results, we introduce the concept of sensitivity for IFSs which is a generalized version of the existing definition for cascades (see \([8,9]\)).

The IFS(\(F\)) is sensitive on \(X\) if there exists \(\delta > 0\) (sensitivity constant) such that for every point \(x \in X\) and every \(r > 0\) there are \(y \in B(x,r), \omega \in \Sigma^+_k\) and \(n \in \mathbb{N}\) with \(d(f^n_\omega(x), f^n_\omega(y)) > \delta\).
Following [6], we know that there exists an IFS of homeomorphisms of the circle that is forward transitive but not backward minimal which leads to the existence of a strongly transitive (backward minimal) IFS of homeomorphisms of the circle which is not forward minimal. The following theorem ensures that such systems are sensitive.

**Theorem 3.1.** If $IFS(F)$ is strongly transitive but not minimal then $IFS(F)$ is sensitive.

**Proof.** Let $y \in X$ with $O_{F}^{+}(y) \neq X$ and $z \in X \setminus O_{F}^{+}(y)$. Take $V = B(z, \delta)$ where $\delta = \frac{1}{4}d(z, O_{F}^{+}(y))$.

Let $x \in X$, and let $U \subset X$ be an arbitrary neighborhood of $X$. Since $IFS(F)$ is strongly transitive, there exist $T_{1}, \ldots, T_{\ell} \in F^{+}$ so that the following holds:

$(H_{1})$ $X \subseteq \bigcup_{i=1}^{\ell} T_{i}(U)$.

Furthermore, for every $i \in \{1, \ldots, \ell\}$, there exist $T_{1}^{(i)}, \ldots, T_{\ell}^{(i)}$ so that

$(H_{2})$ $X \subseteq \bigcup_{j=1}^{\ell} T_{j}^{(i)}(T_{i}(U))$; $\forall 1 \leq i \leq \ell$.

Indeed, given a non-empty open set $U$ let $A = X \setminus \bigcup_{T \in F^{+}} T(U)$. By construction $A$ is a backward invariant closed subset of $X$ and $A \neq X$ (since $U \neq X$) and so by strong transitivity (backward minimality) of $IFS(F)$ we have that $A = \emptyset$. Thus $X = \bigcup_{T \in F^{+}} T(U)$.

Now by compactness of $X$ the condition $(H_{1})$ is satisfied. To verify the condition $(H_{2})$ notice that $T_{i}$ is a homeomorphism, hence $T_{i}(U)$ is a nonempty open subset of $X$. Therefore, $(H_{2})$ follows by $(H_{1})$.

Take $t := \ell + \max \Xi$, where $\Xi = \{|T_{j}^{(i)}| : 1 \leq i \leq \ell \& 1 \leq j \leq \ell_{i}\}$ and $|T|$ is the length of $T$ (we say that the length of $T$ is equal to $n$ and write $|T| = n$ if $T$ is the combination of $n$ elements of the generating set $F$). Choose a neighborhood $W$ around $y$ such that $diam(f_{j_{0}}^{(i)}(W)) < \delta_{i}$ for every $\rho \in \Sigma_{k}^{+}$ and for each $i = 0, 1, \ldots, t$. Moreover, by the choice of $\delta$ we may assume that $d(f_{j_{0}}^{(i)}(W), V) \geq 2\delta$, for $i = 0, 1, \ldots, t$ and for each $\rho \in \Sigma_{k}^{+}$.

The condition $(H_{1})$ ensures that for some $1 \leq s \leq \ell$, $T_{s}(U) \cap W \neq \emptyset$. On the other hand, according to $(H_{2})$, there exists $1 \leq j_{0} \leq \ell_{s}$ so that $T_{j_{0}}^{(s)}(T_{s}(U)) \cap V \neq \emptyset$. Also, one has that $T_{j_{0}}^{(s)}(T_{s}(U)) \cap T_{j_{0}}^{(s)}(W) \neq \emptyset$. By the choice of $W$, we have $d(T_{j_{0}}^{(s)}(W), V) \geq 2\delta$. Thus, $diam(T_{j_{0}}^{(s)}(T_{s}(U))) > \delta$ which completes the proof.}

**Theorem 3.1** can be true by the following weaker assumptions.

**Definition 3.2.** We say that $IFS(F)$ is $S$-transitive if for every open set $U$, there exist $T_{1}, \ldots, T_{\ell} \in F^{+}$ so that $X \subseteq \bigcup_{i=1}^{\ell} T_{i}(U)$.

**Remark 3.3.** Notice that $S$-transitivity follows by strong transitivity. For the proof of Theorem 3.1 it is enough to have the $S$-transitivity condition. Hence, in the assumptions of Theorem 3.1 the strong transitivity condition may be replaced by the $S$-transitivity condition.
Obviously, one can see that

\[
\text{strong transitivity} \implies \text{S-transitivity} \implies \text{transitivity}
\]

In Section 4 we give an example of non-minimal S-transitive IFS which is not strongly transitive. This example is important from another point of view. In fact, it provides a non-minimal sensitive system which is not an M-system, see [13].

By the above remark, the proof of the following corollary is the same as the proof of Theorem 3.1 and so, it is omitted.

**Corollary 3.4.** If IFS(\(\mathcal{F}\)) is S-transitive but not minimal then IFS(\(\mathcal{F}\)) is sensitive.

Let \(f\) be a homeomorphism on a compact metric space \((X, d)\) and \(x \in X\) be a fixed point of \(f\). Then the stable set \(W^s(x)\) and unstable set \(W^u(x)\) are defined, respectively, by

\[
W^s(x) = \{y \in X : f^n(x) \to y, \text{ whenever } n \to +\infty\},
\]

\[
W^u(x) = \{y \in X : f^n(x) \to y, \text{ whenever } n \to -\infty\}.
\]

**Definition 3.5.** We say that \(x\) is an attracting (or repelling) fixed point of \(f\) if the stable set \(W^s(x)\) (or unstable set \(W^u(x)\)) contains an open neighborhood \(B\) of \(x\). Then \(B\) is called the basin of attraction (or repulsion) of \(x\).

**Theorem 3.6.** If IFS(\(\mathcal{F}\)) is strongly transitive and the associated semigroup \(\mathcal{F}^+\) has a map with a repelling fixed point then IFS(\(\mathcal{F}\)) is sensitive on \(X\).

**Proof.** Suppose that IFS(\(\mathcal{F}\)) is strongly transitive and the associated semigroup \(\mathcal{F}^+\) has a map \(h\) with a repelling fixed point \(q\) and the repulsion basin \(B\).

To get a contradiction suppose that IFS(\(\mathcal{F}\)) is not sensitive. Then for each \(n \in \mathbb{N}\), there is a non-empty open subset \(U_n\) of \(X\) such that the following holds:

\[
diam(h^i(U_n)) < \frac{1}{n^i}; \quad \forall \omega \in \Sigma^+; \quad \forall i \in \mathbb{N}.
\]

(1)

Since \(q\) is attractor point, there exists \(n_0 \in \mathbb{N}\) so that \(B_{1/n_0}(q) \subset B\). Take \(n > n_0\). Since IFS(\(X; \mathcal{F}\)) is strongly transitive, there is \(T \in \langle \mathcal{F} \rangle^+\) so that \(T^{-1}(q) \in U_n\). By continuity there is \(\delta > 0\) and less than \(1/n_0\) such that \(B_\delta(q) \subset B\) and \(T^{-1}(B_\delta(q)) \subset U_n\) which implies that \(B_\delta(q) \subset T(U_n)\).

Let us take \(y \in B_\delta(q) \setminus \{q\}, y' = T^{-1}(y)\) and \(q' = T^{-1}(q)\). Then \(y', q' \in T^{-1}(B_\delta(q)) \subset U_n\). So, one can have

\[
d(h^m \circ T(y'), h^m \circ T(q')) = d(h^m(y), q) \geq \frac{1}{n_0} > \frac{1}{n}
\]

for some \(m \in \mathbb{N}\) which is contradiction. \(\square\)
It is a well known fact that each Morse-Smale circle diffeomorphism with a fixed point possesses a repelling fixed point. Consequently, we get the following result.

**Corollary 3.7.** Let $\mathcal{F}$ be a finite family of Morse-Smale diffeomorphisms on the circle $S^1$. If $\text{IFS}(S^1; \mathcal{F})$ is strongly transitive and $\mathcal{F}$ has a map with a fixed point then $\text{IFS}(\mathcal{F})$ is sensitive on $X$.

Suppose that the generators $f_i, i = 1, \ldots, k$, of $\text{IFS}(\mathcal{F})$ are $C^1$ diffeomorphisms defined on a compact smooth manifold $M$. We say that $\text{IFS}(\mathcal{F})$ is **local expanding** if for each $x \in M$ there exist $h \in \mathcal{F}^+$ so that $m(Dh(x)) > 1$, where $m(A) = ||A^{-1}||^{-1}$ denote the co-norm of a linear transformation $A$. Moreover, we say that $\text{IFS}(\mathcal{F})$ is an **expanding** iterated function system if the generators $f_i, i = 1, \ldots, k$, are $C^1$ expanding maps. Obviously, if $\text{IFS}(\mathcal{F})$ is expanding, by compactness of $M$, there exists $0 < \eta < 1$ so that

\[ ||Df_i(x)^{-1}|| < \eta \text{ for every } x \in M, \text{ and } i = 1, \ldots, k. \]  

It is not hard to see that $\text{IFS}(\mathcal{F})$ is local expanding if and only if there are $h_1, \ldots, h_\ell \in \mathcal{F}^+$, open balls $V_1, \ldots, V_\ell$ in $M$ and a constant $\sigma < 1$ such that $M = V_1 \cup \ldots \cup V_\ell$, and

\[ ||Dh_i(x)^{-1}|| < \sigma \text{ for every } x \in V_i, \text{ i = 1, \ldots, } \ell. \]

Fix the mappings $h_1, \ldots, h_\ell \in \mathcal{F}^+$ and the open subsets $V_1, \ldots, V_\ell$ of $M$ that satisfy (3).

Given $x \in M$, we say that the sequence $\omega = (\omega_0 \omega_1 \ldots \omega_\ell \ldots) \in \Sigma^+_\ell$ is an **admissible itinerary** of $x$ if the following holds: $x \in V_{\omega_0}$ and for each $j \geq 1$,

\[ f_{\omega_j} \circ \cdots \circ f_{\omega_0}(x) \in V_{\omega_{j+1}}. \]

Let $\delta$ be the Lebesgue number of the covering $\{V_1, \ldots, V_\ell\}$. Then, for each point $x \in M$, every open neighborhood $U$ of $x$ and each admissible itinerary $\omega$ of $x$, the subset $N_{\mathcal{F}^+}(U, \omega, \delta)$ is nonempty, where

\[ N_{\mathcal{F}^+}(U, \omega, \delta) = \{ n \in \mathbb{N} : \text{ there exist } x, y \in U \text{ such that } d(f_\omega^n(x), f_\omega^n(y)) > \delta \}. \]

IFS($\mathcal{F}$) is called **cofinitely sensitive** if there exists a constant $\delta > 0$ such that for any nonempty open set $U$ of $X$, there exist a branch $\omega \in \Sigma^+_\ell$ and an $N \geq 1$ such that $N_{\mathcal{F}^+}(U, \delta, \omega) \supset [N, +\infty) \cap \mathbb{N}$, while $\delta$ is called a sensitive constant. Hence the next result follows immediately.

**Proposition 3.8.** The following statements hold.

i) Each local expanding IFS is sensitive.

ii) Each expanding IFS is cofinitely sensitive.
4. Examples

It is a well known fact that for ordinary dynamical systems, the minimality of a map $f$ is equivalent to that of $f^{-1}$. Nevertheless this is not the case for iterated function systems as Kleptsyn and Nalskii pointed at [12, pg. 271]. However, they omitted to include any example of forward but not backward minimal IFS. Recently, the authors in [6], illustrated an example of forward but not backward minimal IFS which we explain in below. That lead to an example of backward minimal IFS which is not forward minimal. So, it is sensitive. Since the previous examples of non-minimal sensitive systems (see, [2, 9]) are M-system, this example maybe more valuable. Indeed, our example is non-minimal sensitive system which is not M-system. Recall that $(S,X)$ is M-system if the set of almost periodic points is dense in $X$ (the Bronstein condition) and, in addition, the system is topologically transitive.

**Example 4.1.** Consider a symmetric IFS($\mathcal{F}$) generated by homeomorphisms of the circle. Then, one of the following possibilities holds [15,7]:

i) there is a point $x$ so that $\text{Card}(\mathcal{O}_f^+(x)) < \infty$;

ii) IFS($\mathcal{F}$) is minimal; or

iii) there is a unique invariant minimal Cantor set $K$ for IFS($\mathcal{F}$), that is, $g(K) = K$ for all $g \in \mathcal{F}$ and $K = \overline{\mathcal{O}_f^+(x)}$ for all $x \in K$.

The Cantor set $K$ in the above third conclusion is usually called exceptional minimal set. By [15, Exer. 2.1.5], one can ensure that there exists a symmetric semigroup $\mathcal{F}^+$ generated by homeomorphisms $f_1, \ldots, f_k$ of $S^1$ which admits an exceptional minimal set $K$ such that the orbit of every point of $S^1 \setminus K$ is dense in $S^1$. So, the closed invariant subsets of $S^1$ for IFS($\mathcal{F}$) are $\emptyset$, $K$ and $S^1$.

Now, consider any homeomorphism $h$ of $S^1$ such that $h(K)$ strictly contains $K$. Then the IFS generated by $f_1^{-1}, \ldots, f_k^{-1}, h^{-1}$ is backward minimal but not forward minimal. Therefore, the IFS generated by $f_1^{-1}, \ldots, f_k^{-1}, h^{-1}$ is sensitive. Also, One can easily check that the IFS generated by $f_1^{-1}, \ldots, f_n^{-1}, h^{-1}$ is not an M-system.

Also, in the following, we provide another example of non-minimal sensitive system which is not an M-system.

\footnote{A point $x$ is called almost periodic if the subsystem $\overline{\mathcal{O}_f^+(x)}$ is minimal and compact.}
Example 4.2. Suppose $f$ is a north-south pole $C^1$-diffeomorphism of the circle $S^1$ possessing an attracting fixed point $p$ as a north pole and a repelling fixed point $q$ as a south pole with multipliers $1/2 < f'(p) < 1$ & $1/2 < (f^{-1})'(q) < 1$.

It is observed that $S^1 \setminus \{p, q\}$ composed of exactly two connected pieces $U_1$ and $U_2$. Now, consider two homeomorphisms $h_1$ and $h_2$ of $S^1$ such that

i) $U_1 \subseteq h_1(U_1)$ and $U_2 \subseteq h(U_2)$.

ii) $p$ is a boundary point for $h_1(U_1)$ and $h_2(U_2)$.

Take $\mathcal{F} = \{f, f^{-1}, h_1, h_2\}$. One can check $\overline{\mathcal{O}_\mathcal{F}^+(x)} = S^1$ & $\overline{\mathcal{O}_\mathcal{F}^-(x)} = S^1$, for all $x \in S^1 \setminus \{p\}$ and also $h(p) = p$, for every $h \in \mathcal{F}^+ \cup (\mathcal{F}^{-1})^+$. Therefore, $IFS(S^1; \mathcal{F})$ is neither forward minimal nor strongly transitive (backward minimal). However, there are wo important facts:

i) $IFS(S^1; \mathcal{F})$ is S-transitive,

ii) $IFS(S^1; \mathcal{F})$ is not an $M$-system.

Indeed, $x$ is not almost periodic point when $x \neq p$ i.e. $\overline{\mathcal{O}_\mathcal{F}^+(x)}$ is not minimal for $x \neq p$.

As a consequence of the previous example we get the next result.

Corollary 4.3. There exists any S-transitive non minimal IFS which is sensitive but it is not an $M$-system.

In [5], Banks, Brooks, Cairns, Davis and Stacey prove an elementary but somewhat surprising result. Let $X$ be an infinite metric space and $f : X \to X$ be continuous. If $f$ is topologically transitive and has dense periodic points then $f$ has sensitive dependence on initial conditions. The next counter example shows that this result does not hold for iterated function systems.

Example 4.4. Let $f_1 : S^1 \to S^1$ be an irrational rotation and $f_2 : S^1 \to S^1$ defined by $f_2(x) = -x$. Clearly $f_2(x) = x$ for each $x \in S^1$. Now consider $\mathcal{F} = \{f_1, f_2\}$ and take $IFS(S^1; \mathcal{F})$ the iterated function system generated by $\mathcal{F}$. Then it is not hard to see that $IFS(S^1; \mathcal{F})$ is topologically transitive and has dense periodic points, but it is not sensitive.

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