Real splitting and stability index for algebras with involution

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Abstract

For central simple algebras with involution over a formally real field, “splitting” by real closures and maximal ordered subfields is investigated, and another presentation of $H$-signatures of hermitian forms is given. The stability index for central simple algebras with involution is defined and studied.

Key words. Central simple algebra, involution, formally real field, splitting, hermitian form, $H$-signature, stability index

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1 Introduction

It is well-known that any central simple algebra $A$ possesses a splitting field, i.e. a field extension $L$ of its centre $Z$ such that $A \otimes_Z L$ is Brauer equivalent to $L$. For example, any algebraic closure of $Z$ is a splitting field of $A$. By a theorem of Wedderburn, $A$ is isomorphic to a matrix algebra over a central division $Z$-algebra $D$. Any maximal subfield of $D$ is a splitting field of $D$ and thus a splitting field of $A$.

In this paper we study the “real splitting” of an algebra with involution, motivated by the theory of $H$-signatures, developed in [2] and [1]. More precisely: let us assume that $A$ is a simple $F$-algebra, equipped with an $F$-linear involution $\sigma$, where $F$ is a formally real field. Let $\vartheta$ be an involution on $D$ of the same kind as $\sigma$. Let $P$ be an ordering of $F$ and let $F_P$ be a real closure of $F$ at $P$. By a theorem of Frobenius, $A \otimes_F F_P$ is Brauer equivalent to one of $F_P, F_P(\sqrt{-1})$ or $(-1, -1)_{F_P}$ whenever $A \otimes_F F_P$ is simple. In analogy with the first paragraph, it is a natural question to ask if there exists a maximal ordered extension $(L, Q)$ of $(F, P)$, contained in $D$, which behaves like $F_P$ in the sense that $D \otimes_F L$ (and thus $A \otimes_F L$) is Brauer equivalent to one of $L, L(\sqrt{-d})$ or $(-a, -b)_L$ with $a, b, d \in Q$ whenever $D \otimes_F L$ is simple. We answer this question in the affirmative and in addition obtain...
more precise results about the nature of the field $L$, which take into account the involution $\vartheta$.

We use this result to give another presentation of the theory of $H$-signatures and also introduce and study the stability index of algebras with involution.

2 Notation

We give a brief overview of the notation used in this paper and refer to the standard references [8], [9], [10] and [16] as well as [2] and [1] for the details.

For a ring $A$ and an involution $\sigma$ on $A$, we denote the set of symmetric elements of $A$ with respect to $\sigma$ by $\text{Sym}(A, \sigma) = \{ a \in A \mid \sigma(a) = a \}$.

Let $F$ be a formally real field with space of orderings $X_F$ and Witt ring $W(F)$. For an ordering $P \in X_F$ we denote by $F_P$ a real closure of $F$ at $P$. By an $F$-algebra with involution we mean a pair $(A, \sigma)$ where $A$ is a finite-dimensional $F$-algebra with centre $Z(A)$, equipped with an involution $\sigma : A \to A$, such that $F = Z(A) \cap \text{Sym}(A, \sigma)$ and which is assumed to be either simple (if $Z(A)$ is a field) or a direct product of two simple algebras (if $Z(A) = F \times F$). Observe that $\dim_F Z(A) =: \kappa \leq 2$. If $A$ is a division algebra, we call $(A, \sigma)$ an $F$-division algebra with involution. By [9, Prop. 2.14] we may assume that $\sigma$ is the exchange involution when $Z(A) = F \times F$.

When $\kappa = 1$, we say that $\sigma$ is of the first kind. When $\kappa = 2$, we say that $\sigma$ is of the second kind (or of unitary type). Note that $\sigma|_{Z(A)}$ is the non-trivial $F$-automorphism of $Z(A)$ in this case. Assume $\kappa = 1$ and $\dim_F A = m^2 \in \mathbb{N}$. Then $\sigma$ is either of orthogonal type (if $\dim_F \text{Sym}(A, \sigma) = m(m + 1)/2$) or of symplectic type (if $\dim_F \text{Sym}(A, \sigma) = m(m - 1)/2$).

It follows from the structure theory of $F$-algebras with involution that $A$ is isomorphic to a full matrix algebra $M_n(D)$ for a unique $n \in \mathbb{N}$ and an $F$-division algebra $D$ (unique up to isomorphism) which is equipped with an involution $\vartheta$ of the same kind as $\sigma$, cf. [9, Thm. 3.1]. We denote Brauer equivalence by $\sim$.

Let $(A, \sigma)$ and $(B, \tau)$ be $F$-algebras with involution of the same kind. If $A$ and $B$ are Brauer equivalent, then $(A, \sigma)$ and $(B, \tau)$ are Morita equivalent, cf. [5, Ex. 1.4].

For $\varepsilon \in \{-1, 1\}$ we write $W_{\varepsilon}(A, \sigma)$ for the Witt group of Witt equivalence classes of $\varepsilon$-hermitian forms $h : M \times M \to A$, defined on finitely generated right $A$-modules $M$. All forms in this paper are assumed to be non-singular and are identified with their Witt equivalence classes. We write $+$ for the sum in both $W(F)$ and $W_{\varepsilon}(A, \sigma)$. The group $W_{\varepsilon}(A, \sigma)$ is a $W(F)$-module and we denote the product of $q \in W(F)$ and $h \in W_{\varepsilon}(A, \sigma)$ by $q \cdot h$.

Assume that $(A, \sigma)$ and $(B, \tau)$ are Morita equivalent $F$-algebras with involution of the same kind. It follows from [8, Thm. 9.3.5] (for full details, see [6, Chap. 2])
that there exists an isomorphism $W(A, \sigma) \cong W_{\varepsilon_0}(B, \tau)$, where $\varepsilon_0 = 1$ if $\sigma$ and $\tau$ are both orthogonal or both symplectic, and $\varepsilon_0 = -1$ otherwise. If $\sigma$ and $\tau$ are both unitary, then the isomorphism holds for any $\varepsilon_0 \in \{-1, 1\}$, cf. [2, Lemma 2.1(iii)].

In the context of signatures later on (Section 4), we will consider non-trivial morphisms from $W(A \otimes_F F_p, \sigma \otimes \text{id})$ to $\mathbb{Z}$ and therefore need to know when $W(A \otimes_F F_p, \sigma \otimes \text{id})$ is torsion, which motivates the following definition.

**Definition 2.1.** Let $(A, \sigma)$ be an $F$-algebra with involution. We define the set of nil-orderings of $(A, \sigma)$ as follows

\[ \text{Nil}[A, \sigma] := \{ P \in X_F \mid W(A \otimes_F F_p, \sigma \otimes \text{id}) \text{ is torsion} \}. \]

For convenience we also introduce

\[ \tilde{X}_F := X_F \setminus \text{Nil}[A, \sigma], \]

which does not indicate the dependency on $(A, \sigma)$ in order to avoid cumbersome notation.

Let $R$ be a real closed field and let $(A, \sigma)$ be an $R$-algebra with involution. It follows from well-known theorems of Wedderburn and Frobenius and an application of Morita theory that $W(A, \sigma)$ is isomorphic to one of the following Witt groups:

\[
\begin{align*}
W(R, \text{id}) &\cong W_{\pm 1}(R(\sqrt{-1}), -) \cong W((-1, -1)_R, -) \cong \mathbb{Z} \\
W_{-1}(R, \text{id}) &\cong W_{\pm 1}(R \times R, \sim) \cong W_{\pm 1}((-1, -1)_R \times (-1, -1)_R, \sim) \cong 0 \\
W_{-1}((-1, -1)_R, -) &\cong \mathbb{Z}/2\mathbb{Z}
\end{align*}
\]

where $\sim$ denotes conjugation and $\sim$ denotes the exchange involution, cf. [2, Lemma 2.1 and §3.1].

**Proposition 2.2.** Let $(A, \sigma)$ be an $F$-algebra with involution.

1. We have

\[ \text{Nil}[A, \sigma] := \{ P \in X_F \mid W(A \otimes_F F_p, \sigma \otimes \text{id}) \text{ is isomorphic to 0 or } \mathbb{Z}/2\mathbb{Z} \}. \]

2. Let $(B, \tau)$ be an $F$-algebra with involution such that $A \sim B$ and $\sigma$ and $\tau$ are of the same type. Then $\text{Nil}[B, \tau] = \text{Nil}[A, \sigma]$.

3. Let $L$ be an algebraic extension of $F$ and let $P \in X_F$. For every extension $Q$ of $P$ to $L$ we have $Q \in \text{Nil}[A \otimes_F L, \sigma \otimes \text{id}]$ if and only if $P \in \text{Nil}[A, \sigma]$.

4. Let $P \in X_F$. Then $P \in \text{Nil}[A, \sigma]$ if and only if any morphism from $W(A \otimes_F F_p, \sigma \otimes \text{id})$ to $\mathbb{Z}$ is identically zero.
Proof. (1) Let \( P \in X_F \). The statement follows from considering the list (2.1) with \( R = F_P \).

(2) Let \( P \in X_F \). By the assumption and Morita theory, \( W(A \otimes_F F_P, \sigma \otimes \text{id}) \cong W(B \otimes_F F_P, \tau \otimes \text{id}) \).

Statement (3) follows from the observation that \( (A \otimes_F L) \otimes_L L_Q \cong A \otimes_F F_P \) and (4) follows from [11, Thm. 4.1]. \( \square \)

We remark that by Proposition 2.2 our exposition of nil-orderings in this paper is equivalent to those in [2] and [1].

Definition 2.3. Let \((D, \vartheta)\) be an \( F\)-division algebra with involution and let \( P \in X_F \). We say that \((D, \vartheta)\) is \((F, P)\)-real (or simply \( F\)-real in case \( F \) has a unique ordering) if

\[
(D, \vartheta) \in \{(F, \text{id}), (F(\sqrt{-d}), -), ((-a, -b)_F, -)\},
\]

where \( a, b, d \in P \) and \(-\) denotes conjugation.

Lemma 2.4. Let \((D, \vartheta)\) be an \( F\)-division algebra with involution such that \( \deg D \leq 2 \) and let \( P \in X_F \). The following statements are equivalent:

1. \((D, \vartheta)\) is \((F, P)\)-real;
2. \((D \otimes_F F_P, \vartheta \otimes \text{id})\) is \( F_P\)-real;
3. \( P \notin \text{Nil}[D, \vartheta] \).

Proof. (1)\(\Rightarrow\)(2) is clear. (2)\(\Rightarrow\)(3): \( W(D \otimes_F F_P, \vartheta \otimes \text{id}) \) is not torsion (cf. [2, §3.1]) and thus \( P \notin \text{Nil}[D, \vartheta] \). (3)\(\Rightarrow\)(1) follows from an examination of (2.1), Definition 2.3 and Proposition 2.2. \( \square \)

3 Real Splitting and Maximal Ordered Extensions

In this section, we study the “real splitting” behaviour of an \( F\)-algebra with involution \((A, \sigma)\). Up to Brauer equivalence, it suffices to consider the underlying \( F\)-division algebra \( D \), which is equipped with an involution \( \vartheta \) of the same kind as \( \sigma \), as observed in Section 2. Throughout the paper, \( Z \) will denote the centre of \( D \). Note that when \( \vartheta \) is of the first kind, \( Z = F \) and the degree of \( D \), \( \deg D \), is a 2-power, cf. [14] Cor., p. 154].

Lemma 3.1. Let \( D \) be a division algebra with centre \( Z \) and let \( Z \subseteq K \subseteq M \) be subfields of \( D \), where \( M \) is maximal. Then \( D \otimes_Z K \) is Brauer equivalent to a quaternion division algebra over \( K \) if and only if \( [M : K] = 2 \).
Proof. Let $k = [K : Z]$ and $\ell = [M : K]$, then by [14] Chap. 1, §2.9, Prop., p. 139,  
$$D \otimes_Z K \cong C_D(K) \otimes_K M_k(K),$$
where $C_D(K)$ denotes the centralizer of $K$ in $D$. Thus, $\dim_Z D = \dim_K C_D(K) \cdot k^2$. Also, $D \otimes_Z M = M_k(M)$ since $M$ is a splitting field of $D$, cf. [14] Thm. 2, p. 139. Thus, $\dim_Z D = (\ell k)^2$. It follows that $\dim_K C_D(K) = \ell^2$ and thus $C_D(K)$ is a quaternion algebra over $K$ if and only if $\ell = 2$. Since $D \otimes_Z K$ is Brauer equivalent to $C_D(K)$, the result follows. \hfill $\Box$

**Proposition 3.2.** Let $(K, P)$ be an ordered field and $L \supseteq K$ a finite field extension with $[L : K]$ a power of 2 and such that $(K, P)$ has no proper ordered extension in $L$. Then $L$ and $K_P$ are linearly disjoint over $K$. In particular, $L \otimes_K K_P$ is a field and $[L : K] \leq 2$.

Proof. We also use $P$ to denote the set of positive elements of $K_P$. We know that (see for instance [3] Thm. 1.2.2]) $K_P = \bigcup_{\lambda \in \text{ P}} K_P$ where $\lambda$ is an ordinal, $K_0 = K$ and, for each $i < \lambda$,

1. $K_{i+1} = K_i(\alpha_i)$ where $\alpha_i$ is a square root of an element in $K_i \cap P$ or is a root of a polynomial of odd degree with coefficients in $K_i$.

2. If $i$ is a limit ordinal then $K_i = \bigcup_{j < i} K_j$.

The proof will follow this construction of $K_P$ by transfinite induction.

**Fact 3.3.** Let $K \subseteq K' \subseteq K_P$ be such that $L$ and $K'$ are linearly disjoint over $K$. Let $\alpha \in K_P$ be either the square root of an element of $P \cap K'$ or a root of an odd degree polynomial in $K'[X]$. Then every subset of $L$ that is linearly independent over $K'$ is linearly independent over $K' \cap P$. In particular $L$ and $K' \cap P$ are linearly disjoint over $K$.

Proof of the Fact: We only prove the first statement. The second one clearly follows from it since $L$ and $K'$ are linearly disjoint over $K$. We consider two cases, according to $\alpha$.

Case 1: $\alpha = \sqrt{\beta}$ with $\beta \in K' \cap P$. Let $\ell_1, \ldots, \ell_i, L \subseteq L$ be linearly independent over $K'$, and assume $a_1 \ell_1 + \cdots + a_i \ell_i = 0$ for some $a_1, \ldots, a_i \in K'(\alpha)$. Write $a_i = b_i + c_i \sqrt{\beta}$ with $b_i, c_i \in K'$. Then

$$b_1 \ell_1 + \cdots + b_i \ell_i + (c_1 \ell_1 + \cdots + c_i \ell_i) \sqrt{\beta} = 0. \quad (3.1)$$

But $\sqrt{\beta} \not\in L$, otherwise we would have $K(\sqrt{\beta}) \subseteq L$, and $K(\sqrt{\beta})$ would be a proper ordered extension of $(K, P)$ in $L$, contradiction. Since $\sqrt{\beta} \not\in L$, its minimal polynomial has degree at least 2 and (3.1) implies $b_1 \ell_1 + \cdots + b_i \ell_i = 0$ and $c_1 \ell_1 + \cdots + c_i \ell_i = 0$, which yields $b_1 = \cdots = b_i = c_1 = \cdots = c_i = 0$. 

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Case 2: $\alpha$ is a root of some polynomial of odd degree in $K'[X]$. Since $K'$ and $L$ are linearly disjoint over $K$, $L \otimes_K K'$ is a field \([12\text{ Prop. 20.2}]\) and $[L \otimes_K K' : K'] = [L : K]$ is a power of 2. Since $[K'(<\alpha) : K']$ is odd, the field extensions $L \otimes_K K'$ and $K'(<\alpha)$ of $K'$ have relatively prime degrees, so are linearly disjoint over $K'$ \([12\text{ Ex. 20.5}]\). Let $\ell_1, \ldots, \ell_t \in L$ be linearly independent over $K'$. Then $\ell_1, \ldots, \ell_t$, considered as elements of $L \otimes_F K' = LK'$, are linearly independent over $K'$ and by the observation above, are linearly independent over $K'(<\alpha)$. This concludes the proof of the fact.

Using Fact \([3.3]\) we obtain that if $L$ and $K_i$ are linearly disjoint over $K$, then $L$ and $K_{i+1}$ are linearly disjoint over $K$. If $\mu \leq \lambda$ is a limit ordinal, and $L$ and $K_i$ are linearly disjoint over $K$ for every $i < \mu$, then it is immediate that $L$ and $\bigcup_{i<\mu} K_i$ are also linearly disjoint over $K$.

It follows that $L \otimes_K K_P$ is a field, containing $K_P$. Since $K_P$ is real closed, $[L \otimes_K K_P : K_P] \leq 2$ and we conclude that $[L : K] \leq 2$. \(\square\)

**Lemma 3.4.** Let $(D, \vartheta)$ be an $F$-division algebra with involution of the second kind, and let $P \in X_F \setminus \text{Nil}[D, \vartheta]$. Then $P$ does not extend to $Z$.

**Proof.** Assume that $P$ extends to $Z$. Write $Z = F(\sqrt[\mu]{\alpha})$ with $\alpha \in P$. Then the centre of $D \otimes_F F_P$ is isomorphic to $Z \otimes_F F_P$ (cf. \([13\text{ Lemma 12.4c}]\)) which is isomorphic to $F_P \times F_P$. Therefore $D \otimes_F F_P$ is an $F_P$-algebra which is not simple. By \([2\text{ Lemma 2.1(iv)]}) \ W(D \otimes_F F_P, \vartheta_P) = 0$, implying that $P \in \text{Nil}[D, \vartheta]$, a contradiction. \(\square\)

### 3.1 Involutions of the first kind

**Proposition 3.5.** Let $D$ be a central $F$-division algebra whose degree is a power of two. Let $P \in X_F$ and let $(L, Q)$ be a maximal ordered extension of $(F, P)$ in $D$. Then one of the following holds

1. $L$ is a maximal subfield of $D$ and $D \otimes_F L \sim L$;
2. There exists $d \in Q$ such that $L(\sqrt{-d})$ is a proper extension of $L$ and a maximal subfield of $D$. Furthermore, $D \otimes_F L$ is Brauer equivalent to a quaternion division algebra $(-d, -c)_L$ for some $c \in L$.

**Proof.** If $L$ is a maximal subfield of $D$, then $D \otimes_F L$ is Brauer equivalent to $L$, so we assume from now on that $L$ is not a maximal subfield of $D$. Let $L \subseteq M$ with $M$ a maximal subfield of $D$ and let $L_Q$ be a real closure of $L$ at $Q$. Since $[M : L]$ is a power of two, it follows by Proposition \([5.2]\) that $[M : L] = 2$, and thus $M = L(\sqrt{-d})$ for some $d \in Q$. By Lemma \([3.1]\) $D \otimes_F L$ is Brauer equivalent to a quaternion division algebra $H$ over $L$. Now $D \otimes_F L(\sqrt{-d}) \simeq D \otimes_F L \otimes_L L(\sqrt{-d}) \simeq$
Assume that the degree of $D$ is a power of
for some $c \in P$ that
forcing
the Skolem-Noether theorem (cf. [13, p. 230]), there exists
implies that
Let $D$ be an $F$-division algebra with centre $Z$ such that
Lemma 3.7.

$L$ of

$(2)$ Let $X \in X_F$ and a maximal subfield of $D$. Furthermore, $D \otimes F L$ is Brauer equivalent to a quaternion division algebra $(-d, -c)_L$ with $c \in Q$.

Proof. Since $\vartheta$ is of the first kind, deg $D$ is a power of two.

(1) Let $\vartheta$ be orthogonal. Assume that $L$ is not a maximal subfield of $D$. Then $D \otimes_F L \sim (-d, -c)_L$ for some $c \in L$ and $d \in Q$ by Proposition [3.5]. Since $\vartheta$ is orthogonal and $P \notin \text{Nil}[D, \vartheta]$, the algebra $D \otimes_F F_P$ is split, which forces $-c \in Q$. Consider the field extension $L(\sqrt{-c})$, which is a proper extension of $L$ since the quaternion algebra $(-d, -c)_L$ is division. The ordering $Q$ extends to an ordering $R$ of $L(\sqrt{-c})$ since $-c \in Q$. We now observe that $[L(\sqrt{-c}) : F] = [L(\sqrt{-d}) : F] = \deg D$, since $L(\sqrt{-d})$ is a maximal subfield of $D$ (cf. [14, Thm. 2, p. 139]), and that

$$D \otimes_F L(\sqrt{-c}) = (D \otimes_F L) \otimes_L L(\sqrt{-c}) \sim (-d, -c)_L \otimes_L L(\sqrt{-c}) \sim L(\sqrt{-c}).$$

Hence, by [13, Cor. 13.3, p. 241], $L(\sqrt{-c})$ is $F$-isomorphic to a subfield $K$ of $D$. We denote this isomorphism by $f$. Observe that $f(R)$ is an ordering of $K$. By the Skolem-Noether theorem (cf. [13, p. 230]), there exists $a \in D^*$ such that $f|_L = \text{Int}(a)|_L$. It follows that $\text{Int}(a^{-1})(K)$ is a subfield of $D$ and a proper extension of $L$. Furthermore, the ordering $\text{Int}(a^{-1})(f(R))$ of $\text{Int}(a^{-1})(K)$ extends $\text{Int}(a^{-1})(f(Q)) = Q$, contradicting the choice of $(L, Q)$ as a maximal ordered extension of $(F, P)$ in $D$. We conclude that $L$ is a maximal subfield of $D$ and that $D \otimes_F L \sim L$.

(2) Let $\vartheta$ be symplectic. Assume that $L$ is a maximal subfield of $D$. Then $D \otimes_F L \sim L$. Thus $\vartheta \otimes \text{id}$ is a symplectic involution on the split algebra $D \otimes_F F_P$, which implies that $W(D \otimes_F F_P, \vartheta \otimes \text{id}) \simeq W_{-1}(F_P, \text{id})$. It follows from Proposition [2.2.1] that $P \in \text{Nil}[D, \vartheta]$, a contradiction. Thus, by Proposition [3.5], $D \otimes_F L \sim (-d, -c)_L$ for some $c \in L$ and $d \in Q$. As above, since $P \notin \text{Nil}[D, \vartheta]$, $D \otimes_F F_P$ is not split, forcing $c \in Q$. □

3.2 Involutions of the second kind: the 2-power degree case

Lemma 3.7. Let $D$ be an $F$-division algebra with centre $Z$ such that $[Z : F] = 2$. Assume that the degree of $D$ is a power of $2$. Let $P \in X_F$ be such that $P$ does not
extend to $Z$ and let $(L, Q)$ be a maximal ordered extension of $(F, P)$ in $D$. Then $D \otimes_F L$ is Brauer equivalent to $L(\sqrt{-d})$ for some $d \in P$ with $\sqrt{-d} \notin L$.

**Proof.** By hypothesis, $Z = F(\sqrt{-d})$ for some $d \in P$. It follows that $L(\sqrt{-d})$ is a proper field extension of $L$. We claim that $L(\sqrt{-d})$ is a maximal subfield of $D$. Indeed, let $N$ be a maximal subfield of $D$ such that $L(\sqrt{-d}) \subseteq N$. Since $\dim_D Z$ is a power of 2 and $(L, Q)$ has no ordered extension in $N$, we can apply Proposition 3.2, and we obtain $[N : L] = 2$, so $N = L(\sqrt{-d})$ is a maximal subfield of $D$. Therefore, we have $D \otimes_F L \cong (D \otimes_Z Z) \otimes_F L \cong D \otimes_Z (F(\sqrt{-d}) \otimes_F L) \cong D \otimes_Z L(\sqrt{-d}) \sim L(\sqrt{-d})$. □

**Theorem 3.8.** Let $(D, \theta)$ be an $F$-division algebra with involution of the second kind. Assume that the degree of $D$ is a power of 2. Let $P \in X_F \setminus \text{Nil}[D, \theta]$ and let $(L, Q)$ be a maximal ordered extension of $(F, P)$ in $D$. Then $D \otimes_F L$ is Brauer equivalent to $L(\sqrt{-d})$ for some $d \in P$ with $\sqrt{-d} \notin L$.

**Proof.** By Lemma 3.4, $P$ does not extend to the centre of $D$ and we conclude with Lemma 3.7. □

**Proposition 3.9.** Let $D$ be an $F$-division algebra with centre $Z$ such that $[Z : F] = 2$. Assume that the degree of $D$ is a power of 2. Let $P \in X_F$ be such that $P$ extends to an ordering $P'$ of $Z$ and let $(L, Q)$ be a maximal ordered extension of $(Z, P')$ in $D$. Then $D \otimes_F L$ is isomorphic to $M_r(L \times L)$ or to $M_s((a, b)_L \times (a, b)_L)$ for some $r, s \in \mathbb{N}$ and $a, b \in L^x$.

**Proof.** Let $\alpha \in P$ be such that $Z = F(\sqrt{\alpha})$. Let $N$ be a maximal field extension of $L$ in $D$. By Proposition 3.2, $[N : L] = 1$ or 2. Hence $D \otimes_Z L$ is isomorphic to either $M_r(L)$ or $M_s((a, b)_L)$ for some $r, s \in \mathbb{N}$ and $a, b \in L^x$ by Lemma 3.1. Then

\[
D \otimes_F L \cong (D \otimes_Z Z) \otimes_F L \\
\cong D \otimes_Z (Z \otimes_F L) \\
\cong D \otimes_Z (F[x]/(x^2 - \alpha) \otimes_F L) \\
\cong D \otimes_Z L[x]/(x^2 - \alpha) \\
\cong D \otimes_Z (L \times L) \\
\cong (D \otimes_Z L) \times (D \otimes_Z L)
\]

and the result follows. □

**3.3 Involutions of the second kind: the odd degree case**

**Lemma 3.10.** Let $(D, \theta)$ be an $F$-division algebra with involution of the second kind. Let $P \in X_F$ be such that $P$ does not extend to $Z$ and write $Z = F(\sqrt{-d})$
with \( \sqrt{-d} = -\sqrt{-d} \) and \( d \in P \). Let \( L \) be a field extension of \( F \) in \( D \) such that \( \sqrt{-d} \notin L \). Then there is an \( F \)-linear involution \( \sigma : D \to D \) such that \( \sigma \) is the identity on \( L \) and \( \sigma(\sqrt{-d}) = -\sqrt{-d} \).

**Proof.** We consider

\[
\xi : L(\sqrt{-d}) \longrightarrow D, \quad \ell_1 + \ell_2 \sqrt{-d} \longmapsto \vartheta(\ell_1) + \vartheta(\ell_2) \sqrt{-d}.
\]

A direct computation shows that \( \xi \) is a morphism of rings and is \( \mathbb{Z} \)-linear, so a morphism of \( \mathbb{Z} \)-algebras from the simple \( \mathbb{Z} \)-algebra \( L(\sqrt{-d}) \) to the central simple \( \mathbb{Z} \)-algebra \( D \). By the Skolem-Noether theorem (see [13, p. 230]) there is \( a \in D^\times \) such that \( \xi(x) = a^{-1}xa \) for every \( x \in L(\sqrt{-d}) \).

We define \( \sigma := \text{Int}(a^{-1}) \circ \vartheta \). This is an involution on \( D \), and, for \( \ell \in L \),

\[
\sigma(\ell) = a \vartheta(\ell)a^{-1} = a \xi(\ell)a^{-1} = aa^{-1}\ell a a^{-1} = \ell, \quad \text{so} \quad \sigma \text{ is the identity on} \; L.
\]

Finally, \( \sigma(\sqrt{-d}) = a \vartheta(\sqrt{-d})a^{-1} = a(-\sqrt{-d})a^{-1} = -\sqrt{-d} \) (since \( \sqrt{-d} \) is in the centre of \( D \)). \( \square \)

**Theorem 3.11.** Let \((D, \vartheta)\) be an \( F \)-division algebra with involution of the second kind. Assume that the degree of \( D \) is odd. Let \( P = X_F \setminus \text{Nil}[D, \vartheta] \), and write \( Z = F(\sqrt{-d}) \) with \( \vartheta(\sqrt{-d}) = -\sqrt{-d} \) and \( d \in P \) (cf. Lemma 3.10). Let \((L, Q)\) be a maximal ordered extension of \((F, P)\) in \( D \). Then \( L(\sqrt{-d}) \) is a maximal subfield of \( D \) and \( D \otimes_F L \sim L(\sqrt{-d}) \).

**Proof.** Since \((L, Q)\) is an ordered extension of \((F, P)\), we have \( \sqrt{-d} \notin L \), and by Lemma 3.10 there is an \( F \)-involution \( \sigma \) on \( D \) such that \( \sigma \) is the identity on \( L \) and \( \sigma(\sqrt{-d}) = -\sqrt{-d} \). The following is inspired by an argument in the proof of [9, Prop. 11.22]. Let \( M \) be a maximal subfield of \( D \).

Claim: Then there is \( x \in C_D(M) \setminus M \) such that \( \sigma(x) = x \).

Proof of the claim: By construction of \( \sigma \) we have \( \sigma(M) = M \) and it follows easily that \( \sigma(C_D(M)) \subseteq C_D(M) \). Since \( M \) is not maximal, there is \( u \in C_D(M) \setminus M \), cf. [13, p. 236, Cor. b]. Let \( u_1 = (u + \sigma(u))/2 \) and \( u_2 = (u - \sigma(u))/2 \). Since \( \sigma(C_D(M)) \subseteq C_D(M) \) we have \( u_1, u_2 \in C_D(M) \). Obviously \( u = u_1 + u_2 \), with \( \sigma(u_1) = u_1 \) and \( \sigma(u_2) = -u_2 \). If \( u_1 \notin M \), then we can take \( x = u_1 \). If \( u_1 \in M \), then \( u_2 \notin M \), and we can take \( x = u_2 \sqrt{-d} \), which establishes the claim.

Consider the field \( L(x) \). By choice of \( x \) we have \( L \subseteq L(x) \subseteq \text{Sym}(D, \sigma) \). In particular \( \sqrt{-d} \notin L(x) \), and the diagram

\[
\begin{array}{ccc}
L(x) & \xrightarrow{k \; \text{odd}} & Z = F(\sqrt{-d}) \\
\downarrow & & \downarrow \text{2} \\
L(\sqrt{-d}) & \xrightarrow{2} & F
\end{array}
\]

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gives us that \([L(x) : F] = k\) is odd. It follows that \([L(x) : L]\) is odd, and in particular the ordering \(Q\) of \(L\) extends to \(L(x)\), a contradiction to the choice of \((L, Q)\). Thus \(L(\sqrt{-d})\) is a maximal subfield of \(D\) and \(D \otimes_F L \simeq D \otimes_Z (Z \otimes_F L) \simeq D \otimes_Z L(\sqrt{-d}) \sim L(\sqrt{-d})\).

**Proposition 3.12.** Let \((D, \vartheta)\) be an \(F\)-division algebra with involution of the second kind. Assume that the degree of \(D\) is odd. Let \(P \in X_F\) be such that \(P\) extends to \(Z\) and let \(L\) be a maximal subfield of \(D\). Then \(P\) extends to \(L\) and \(D \otimes_F L \simeq M_r(L \times L)\) for some \(r \in \mathbb{N}\).

**Proof.** Since \(P\) extends to \(Z\), we have that 
\[ Z = F(\sqrt{\alpha}) \] 
for some \(\alpha \in P\). Since \([L : Z]\) is odd, \(P\) extends to \(L\). Also, \(D \otimes_Z L \simeq M_r(L)\) for some \(r \in \mathbb{N}\) since \(L\) is maximal in \(D\). Then
\[
D \otimes_F L \simeq (D \otimes_Z Z) \otimes_F L \\
\simeq D \otimes_Z (F(\sqrt{\alpha}) \otimes_F L) \\
\simeq D \otimes_Z (F[x]/(x^2 - \alpha) \otimes_F L) \\
\simeq D \otimes_Z L[x]/(x^2 - \alpha) \\
\simeq D \otimes_Z (L \times L) \\
\simeq M_r(L) \times M_r(L). \]

\[ \square \]

### 3.4 Involutions of the second kind: the arbitrary degree case

**Theorem 3.13.** Let \((D, \vartheta)\) be an \(F\)-division algebra with involution of the second kind. Let \(P \in X_F\) be such that \(P\) does not extend to \(Z\) and write \(Z = F(\sqrt{-d})\) for some \(d \in P\). There exists a maximal ordered extension \((L, Q)\) of \((F, P)\) inside \(D\) such that \(L(\sqrt{-d})\) is a maximal subfield of \(D\) and \(D \otimes_F L \sim L(\sqrt{-d})\).

**Proof.** We may write \(D \simeq D_1 \otimes_Z D_2\) where \(D_1\) is an \(F\)-division algebra of odd degree with centre \(Z\) and \(D_2\) is an \(F\)-division algebra of 2-power degree with centre \(Z\), cf. [13] Primary Decomposition Theorem, p. 261. (Note though that the involution \(\vartheta\) is not necessarily decomposable.)

Let \((L_2, P')\) be a maximal ordered extension of \((F, P)\) inside \(D_2\). By Theorem 3.8 \(L_2(\sqrt{-d})\) is a maximal subfield of \(D_2\) and
\[
D_2 \otimes_F L_2 \simeq D_2 \otimes_Z L_2(\sqrt{-d}) \simeq M_k(L_2(\sqrt{-d}))
\]
for some nonzero integer \(k\). Hence
\[
D \otimes_F L_2 \simeq D_1 \otimes_Z M_k(L_2(\sqrt{-d})) \\
\simeq D_1 \otimes_Z M_k(L_2 \otimes_F F(\sqrt{-d}))
\]

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\[ L_1 \otimes_F L_2 \cong D_1 \otimes_Z M_k(F(\sqrt{-d})) \otimes_F L_2 \]
\[ \cong M_k(D_1) \otimes_F L_2 \]
\[ \cong M_k(D_1 \otimes_F L_2). \]

Since \([L_2(\sqrt{-d}) : Z]\) is coprime to \(d\) and \(D_1 \otimes_F L_2 \cong D_1 \otimes_Z L_2(\sqrt{-d})\), we have that \(D_1 \otimes_F L_2\) is still a division algebra (cf. [13, Prop. 13.4(vi), p. 243]), which is of (odd) degree \(\deg D_1\) over its centre \(L_2(\sqrt{-d})\) (cf. [13, Lemma 12.4c, p. 225]). Since it is Brauer equivalent to \(D \otimes_F L_2\) it possesses an involution \(\tau\) of the second kind with fixed field \(L_2\) (cf. [9, Thm. 3.1]).

Let \((L, Q)\) be a maximal ordered extension of \((L_2, P)\) inside \(D_1 \otimes_F L_2\). By Theorem 3.11, \(L(\sqrt{-d})\) is a maximal subfield of \(D_1 \otimes_F L_2\) and
\[ D_1 \otimes_F L \cong (D_1 \otimes_F L_2) \otimes_{L_2} L \sim L(\sqrt{-d}). \]

Hence
\[ D_1 \otimes_F L \cong (D_1 \otimes_F L_2) \otimes_{L_2} L \]
\[ \cong M_k(D_1 \otimes_F L_2) \otimes_{L_2} L \]
\[ \cong M_k(D_1 \otimes_F L) \]
\[ \sim L(\sqrt{-d}). \]

Note that \(L\) is a subfield of \(D\) since \(L \subseteq D_1 \otimes_F L_2 \cong D_1 \otimes_Z L_2(\sqrt{-d}) \subseteq D_1 \otimes_Z D_2 \cong D\). Thus \(L(\sqrt{-d})\) is a subfield of \(D\) since \(\sqrt{-d} \in D\). Finally,
\[ [L(\sqrt{-d}) : Z] = [L(\sqrt{-d}) : L_2(\sqrt{-d})][L_2(\sqrt{-d}) : Z] = \deg D_1 \cdot \deg D_2 = \deg D \]
since \(L(\sqrt{-d})\) is a maximal subfield of \(D_1 \otimes_F L_2 \cong D_1 \otimes_Z L_2(\sqrt{-d})\) and \(L_2(\sqrt{-d})\) is a maximal subfield of \(D_2\). Hence \(L(\sqrt{-d})\) is a maximal subfield of \(D\) by [14, Cor., p. 139]. It follows that \((L, Q)\) is a maximal ordered extension of \((F, P)\) inside \(D\), for if \((M, R)\) were any proper ordered extension of \((L, Q)\), \(M\) would be maximal and would contain \(Z\), which is a contradiction since \(d \in P\) and \(\sqrt{-d} \in Z\). \(\square\)

**Corollary 3.14.** Let \((D, \vartheta)\) be an \(F\)-division algebra with involution of the second kind. Let \(P \in X_F \setminus \Nil[D, \vartheta]\) and write \(Z = F(\sqrt{-d})\) for some \(d \in P\) (cf. Lemma 3.4). There exists a maximal ordered extension \((L, Q)\) of \((F, P)\) in \(D\) such that \(L(\sqrt{-d})\) is a maximal subfield of \(D\) and \(D \otimes_F L \sim L(\sqrt{-d})\).

**Proof.** By Lemma 3.4, \(P\) does not extend to the centre of \(D\) and we conclude with Theorem 3.13. \(\square\)
Proposition 3.15. Let \((D, \vartheta)\) be an \(F\)-division algebra with involution of the second kind. Let \(P \in X_F\) be such that \(P\) extends to an ordering \(P'\) on \(Z\). There exists a maximal ordered extension \((L, Q)\) of \((Z, P')\) in \(D\) such that \(D \otimes_F L\) is isomorphic to \(M_s((a, b)_L \times (a, b)_L)\) for some \(r, s \in \mathbb{N}\) and \(a, b \in L^\times\).

Proof. We may write \(D = D_1 \otimes_Z D_2\) where \(D_1\) is an \(F\)-division algebra of odd degree with centre \(Z\) and \(D_2\) is an \(F\)-division algebra of 2-power degree with centre \(Z\). Let \(L_1\) be a maximal subfield of \(D_1\). Then \([L_1 : Z]\) is odd. Let \((L_2, Q_2)\) be a maximal ordered field extension of \((Z, P')\) in \(D_2\). By Proposition 3.9 and its proof, \(L_2\) is either a maximal subfield of \(D_2\) or a subfield of index 2 in a maximal subfield of \(D_2\). Since \([L_1 : Z]\) and \([L_2 : Z]\) are relatively prime, \(L_1 \otimes_Z L_2\) is a field and thus a subfield of \(D_1 \otimes_Z D_2\). Since it is an extension of odd degree of \(1 \otimes_Z L_2\), it carries an ordering \(Q_1\) that extends \(Q_2\). Let \((L, Q)\) be a maximal ordered field extension of \((L_1 \otimes_Z L_2, Q_1)\) in \(D\). If \(L_2\) is maximal, then \(L = L_1 \otimes_Z L_2\) is a maximal subfield of \(D\) since \([L : Z] = \deg D\). Otherwise, \(L\) has index at most 2 in a maximal subfield of \(D\). Thus \(D \otimes_Z L \simeq M_s(L)\) (when \(L\) is maximal) or \(D \otimes_Z L \simeq M_s((a, b)_L)\) for certain \(a, b \in L^\times\) (otherwise) by Lemma 3.11 Then

\[
D \otimes_F L \simeq D \otimes_Z (Z \otimes_F L)
\]

\[
\simeq D \otimes_Z (L \times L)
\]

\[
\simeq (D \otimes_Z L) \times (D \otimes_Z L)
\]

and the result follows. \(\square\)

Theorem 3.16. Let \((D, \vartheta)\) be an \(F\)-division algebra with involution of the second kind. Let \(\alpha \in F\) be such that \(Z = F(\sqrt{\alpha})\). Then

\[\text{Nil}[D, \vartheta] = H(\alpha) = \{P \in X_F \mid P\text{ extends to }Z\},\]

where \(H(\alpha)\) denotes the usual Harrison set.

Proof. We will show that \(X_F \setminus \text{Nil}[D, \vartheta] = H(-\alpha)\). By Lemma 3.4 \(P \notin \text{Nil}[D, \vartheta]\) implies \(P \in H(-\alpha)\). Now let \(P \in \text{Nil}[D, \vartheta]\). Assume \(P \in H(\alpha)\). Then \(P\) does not extend to \(Z\). Let \((L, Q)\) be as in Theorem 3.13 and note that \(\alpha \notin Q\). Then \(D \otimes_F L \simeq M_r(L(\sqrt{\alpha}))\) for some \(r \in \mathbb{N}\). Let \(L_Q\) be a real closure of \(L\) at \(Q\). Then \(\sqrt{\alpha} \notin L_Q\) and

\[
D \otimes_F L_Q \simeq (D \otimes_F L) \otimes_L L_Q
\]

\[
\simeq M_r(L(\sqrt{\alpha}) \otimes_L L_Q)
\]

\[
\simeq M_r(L_Q(\sqrt{\alpha}))
\]

\[
\simeq M_r(L_Q(\sqrt{-1})�).
\]

Since \(F_P \simeq L_Q\), it follows that \(D \otimes_F F_P \simeq M_r(F_P(\sqrt{-1}))\) and thus from (2.1) and Proposition 2.2(1) that \(P \notin \text{Nil}[D, \vartheta]\). \(\square\)
Note that therefore Nil[D, θ] is clopen in X_F. This was already proved in a different way in \[2\, \text{Cor. 6.5]\].

We finish this section with the following natural question: Let (D, ϑ) be an F-division algebra with involution of the second kind, let \(P \in X_F\) and let (L, Q) be any maximal ordered extension of (F, P) in D. Do the conclusions of Theorem 3.13 and Proposition 3.15 hold for \(D \otimes_F L\)? We are currently unable to provide an answer.

4 \(H\)-Signatures Revisited

Using results obtained in \[2\] and \[1\] we give a self-contained presentation of \(H\)-signatures of hermitian forms in the following paragraphs. Let \((A, σ)\) be an \(F\)-algebra with involution and let \(P \in \tilde{X}_F\). Using Proposition 2.2(1), Lemma 2.4 and (2.1) we obtain the sequence of group morphisms

\[
W(A, σ) \xrightarrow{r_P} W(A \otimes_F F_P, σ \otimes \text{id}) \xrightarrow{\mu_P} W(D_P, \vartheta_P) \xrightarrow{\rho_P} W(F_P, \text{sign}) \xrightarrow{\delta} \mathbb{Z},
\]

where \(r_P\) is the canonical restriction map, \((D_P, \vartheta_P)\) is an \(F_P\)-real division algebra with involution, \(\mu_P\) is an isomorphism induced by Morita theory, \(\rho_P\) is defined by \(\rho_P(\eta)(x) := \eta(x, x)\) for all \(\eta \in W(D_P, \vartheta_P)\) (cf. \[7\]) and \(\text{sign}\) is the usual signature of quadratic forms at the unique ordering of \(F_P\).

In \[2, \S 3.2\] we showed that the map \(\| \text{sign} \circ \rho_P \circ \mu_P \circ r_P \|\) does not depend on the choice of \(F_P\) and \(\mu_P\). In \[1, \text{Prop. 3.2}\] we showed that there exists a hermitian form \(H_0 \in W(A, σ)\) such that for all \(P \in \tilde{X}_F\),

\[
\text{sign}(\rho_P \circ \mu_P)(H_0 \otimes F_P) \neq 0.
\]

We call \(H_0\) a reference form for \((A, σ)\).

**Definition 4.1.** Let \(H_0 \in W(A, σ)\) be a reference form for \((A, σ)\), let \(h \in W(A, σ)\), let \(P \in \tilde{X}_F\), let \(\ell = \dim_{F_P} D_P\) and let \(\delta_P^{H_0} = \text{sgn}(\text{sign}(\rho_P \circ \mu_P)(H_0 \otimes F_P)) \in \{-1, 1\}\). We define the \(H\)-signature of \(h\) at \(P \in X_F\), \(\text{sign}_P^{H_0} h\), by

\[
\text{sign}_P^{H_0} h := \frac{1}{\ell} \delta_P^{H_0} \text{sign}(\rho_P \circ \mu_P)(h \otimes F_P)
\]

whenever \(P \in \tilde{X}_F\) and \(\text{sign}_P^{H_0} h := 0\) if \(P \in \text{Nil}[A, σ]\), cf. Proposition 2.2(4).

**Remark 4.2.**

(1) We showed in \[2, \S 3.3\] that \(\text{sign}_P^{H_0}\) only depends on \(P\) and \(H_0\) and in \[1, \S 7\] that \(H\)-signatures correspond canonically to a natural class of morphisms from \(W(A, σ)\) to \(\mathbb{Z}\).
(2) If \( L \) is a finite extension of \( F \), it follows from (4.2) that \( H_0 \otimes F \) is a reference form for \((A \otimes F, \sigma \otimes \text{id})\). Moreover, if \( R \) is an ordering on \( L \) that extends \( P \in X_F \), then
\[
\text{sign}_R^{H_0 \otimes F} (h \otimes L) = \text{sign}_P^{H_0} h
\]
for all \( h \in W(A, \sigma) \).

(3) Let \( P \in \overline{X}_F \). If \( H_1 \) is another reference form for \((A, \sigma)\), then easy computations show that
\[
\text{sign}_P^{H_0} h = \delta_P^{H_0} \delta_P^{H_1} \text{sign}_P^{H_1} h
\]
for all \( h \in W(A, \sigma) \) and
\[
\delta_P^{H_0} \delta_P^{H_1} = \text{sgn} \left( \text{sign}_P^{H_1} H_0 \right) = \text{sgn} \left( \text{sign}_P^{H_0} H_1 \right).
\]

**Lemma 4.3.** Let \( P \in X_F \), let \((D, \vartheta)\) be an \( F \)-division algebra with involution which is \((F, P)\)-real and let \( \ell = \dim_F D \). Consider the reference form \( H = \langle 1 \rangle_{\vartheta} \) and the group morphism \( \rho: W(D, \vartheta) \rightarrow W(F) \), where \( \rho(h)(x) := h(x, x) \). Then
\[
\text{sign}_P^H h = \frac{1}{\ell} \text{sign}_P \rho(h).
\]

**Proof.** Observe that \( H \) is indeed a reference form. Let \( F_P \) denote a real closure of \( F \) at \( P \) and note that \( P \notin \text{Nil}[D, \vartheta] \) by Lemma 2.4. Consider the diagram

\[
\begin{array}{ccc}
W(D, \vartheta) & \xrightarrow{r_P} & W(D \otimes_F F_P, \vartheta \otimes \text{id}) \\
\downarrow \rho & & \downarrow \rho_P \\
W(F) & \xrightarrow{r'_P} & W(F_P) \xrightarrow{\text{sign}} \mathbb{Z}
\end{array}
\]

where we used the notation from the sequence (4.1). The square on the left commutes: Let \( h \in W(D, \vartheta) \). Then \( \rho(h) \otimes F_P = \rho_P(h \otimes F_P) \) since
\[
(\rho(h) \otimes F_P)(x \otimes 1) = \rho(h)(x) \otimes 1 = h(x, x) \otimes 1 = (h \otimes F_P)(x \otimes 1) = \rho_P(h \otimes F_P)(x \otimes 1).
\]
It follows that
\[
\text{sign}_P^H h = \frac{1}{\ell} \varepsilon_P \text{sign}(\rho_P \circ \text{id})(h \otimes F_P)
\]
\[
= \frac{1}{\ell} \text{sign} \rho(h) \otimes F_P
\]
\[
= \frac{1}{\ell} \text{sign}_P \rho(h),
\]
where \( \varepsilon_P = \text{sgn} \left( \text{sign}(\rho_P \circ \text{id})(H \otimes F_P) \right) = \text{sgn}(\text{sign}(\ell \times \langle 1 \rangle)) = 1. \)

\[\square\]
Lemma 4.4. Let $P \in \widetilde{X}_F$ and let $(D, \vartheta)$ be an $(F, P)$-real division algebra with involution. Let $H$ be a reference form for $(D, \vartheta)$. The signature map
\[ \text{sign}_P^H : W(D, \vartheta) \rightarrow \mathbb{Z} \]
is surjective.

Proof. Let $h = \langle 1 \rangle_\vartheta$. Then $\text{sign}_P^H h = \pm 1$ by Remark 4.2(3) and Lemma 4.3. \(\square\)

5 Stability Index of Algebras with Involution

In this section, we fix an $F$-algebra with involution $(A, \sigma)$ and a reference form $H$ for $(A, \sigma)$. Let $C(X_F, \mathbb{Z})$ denote the ring of continuous functions from $X_F$ (equipped with the Harrison topology) to $\mathbb{Z}$ (equipped with the discrete topology). For every $h \in W(A, \sigma)$ we denote by $\text{sign}_H^h$ the total signature map from $X_F$ to $\mathbb{Z}$ and remark that it is a continuous map, cf. [2, Thm. 7.2]. If $h \in W(A, \sigma)$, we have $\text{sign}_H^h = 0$ on $\text{Nil}[A, \sigma]$. Therefore it is convenient to introduce the notation
\[ \tilde{C}(X_F, \mathbb{Z}) := \{ f \in C(X_F, \mathbb{Z}) | f = 0 \text{ on } \text{Nil}[A, \sigma] \}. \]
Note that $\tilde{C}(X_F, \mathbb{Z})$ depends on the Brauer class of $A$ and the type of $\sigma$, but indicating this would make the notation cumbersome. Since $(A, \sigma)$ is fixed, no confusion should arise. For $P \in X_F$ and a field extension $L$ of $F$, we define
\[ X_L/P := \{ Q \in X_L | P \subseteq Q \}. \]

Lemma 5.1. Let $P \in \widetilde{X}_F$. Then there exists a hermitian form $h_P \in W(A, \sigma)$ and a positive integer $\ell_P$ such that $\text{sign}_P^H h_P = 2^\ell_P$.

Proof. By Theorem 4.6 and Corollary 3.14 there exists an ordered field extension $(L, Q)$ of $(F, P)$ with $L \subseteq A$ and such that $(A \otimes_F L, \sigma \otimes \text{id})$ is Morita equivalent to an $(L, Q)$-real division algebra with involution. Note that $[L : F]$ is finite. Let $Q_0 \in X_L/P$. By Lemma 4.4 and [1, Thm. 4.2] there exists a hermitian form $h$ over $(A \otimes_F L, \sigma \otimes \text{id})$ such that $\text{sign}_{Q_0}^{H_{Q_0}L} h = 1$. Since $X_L/P$ is finite, there exist $a_1, \ldots, a_r \in L$ such that $\{ Q_0 \} = \{ Q \in X_L/P | a_1, \ldots, a_r \in Q \}$. Using the notation from [2, §5.3], let $h_P := \text{Tr}^r_{A \otimes_{F_L} (\langle a_1, \ldots, a_r \rangle \cdot h)}$. It follows from the Knebusch trace formula [2, Thm. 8.1] that
\[
\text{sign}_P^H h_P = \sum_{Q_0 \in X_L/P} \text{sign}_{Q}^{H_{Q_0}L}(\langle a_1, \ldots, a_r \rangle \cdot h)
= 2^r \text{sign}_{Q_0}^{H_{Q_0}L} h
= 2^r. \] \(\square\)
Lemma 5.2. There exists a hermitian form $h_0 \in W(A, \sigma)$ and a positive integer $k_0$ such that $\text{sign}^H h_0 = 2^{k_0}$ for every $P \in \widetilde{X}_F$.

Proof. By Lemma 5.1, for every $P \in \widetilde{X}_F$ there exists $\ell_P \in \mathbb{N}$, $U_P$ clopen in $\widetilde{X}_F$ containing $P$, and $h_P \in W(A, \sigma)$ such that $\text{sign}^H h_P = 2^{\ell_P}$ on $U_P$ (simply take $U_P = (\text{sign}^H h_P)^{-1}(2^{\ell_P})$).

Therefore $\widetilde{X}_F = \bigcup_{P \in \widetilde{X}_F} U_P = \bigcup_{i=1}^{n} U_P$, since $\widetilde{X}_F$ is compact. By removing the intersections of the sets $U_P$, we obtain $\widetilde{X}_F = \bigcup_{i=1}^{n} C_i$ where each $C_i$ is clopen and $\text{sign}^H \eta_i = 2^{\ell_i}$ on $C_i$ for hermitian forms $\eta_i \in W(A, \sigma)$, $i = 1, \ldots, r$.

Let $q_i \in W(F)$ be such that $\text{sign} q_i$ is equal to $2^{s_i}$ on $C_i$ (for some integer $s_i$) and 0 elsewhere, cf. [10, VIII, Lemma 6.10]. Then $\text{sign}^H (q_i \cdot \eta_i)$ is equal to $2^{s_i + \ell_i}$ on $C_i$ and 0 elsewhere. Taking $k_0 = \max\{s_1 + \ell_1, \ldots, s_r + \ell_r\}$ and multiplying $q_i \cdot \eta_i$ by a suitable power of 2, we obtain a form $h_i \in W(A, \sigma)$ such that $\text{sign}^H h_i = 2^{k_0}$ on $C_i$ and 0 elsewhere. It follows that, for $h_0 := h_1 + \cdots + h_r$, $\text{sign}^H h_0 = 2^{k_0}$ on $\widetilde{X}_F$.

 Proposition 5.3. Let $k_0 \in \mathbb{N}$ and $h_0 \in W(A, \sigma)$ be such that $\text{sign}^H h_0 = 2^{k_0}$ on $\widetilde{X}_F$.

(1) Let $q \in W(F)$. Then there exists $h \in W(A, \sigma)$ such that $\text{sign}^H (2^{k_0} q) = \text{sign}^H (h)$ on $\widetilde{X}_F$.

(2) Let $f \in \widetilde{C}(X_F, \mathbb{Z})$. Then there exists $n \in \mathbb{N}$ such that $2^n f \in \text{Im}(\text{sign}^H)$.

Proof. (1) We take $h = q \cdot h_0$.

(2) We know that there exists $m \in \mathbb{N}$ such that $2^m f = \text{sign}(q)$ for some $q \in W(F)$, cf. [10, VIII, Lemma 6.9]. Then $2^{m+k_0} f = \text{sign}^H (q \cdot h_0)$.

Definition 5.4.

(1) We define $\text{st}^H (A, \sigma)$ to be the smallest nonnegative integer $k$ such that

$$2^k \cdot \widetilde{C}(X_F, \mathbb{Z}) \subseteq \text{Im}(\text{sign}^H)$$

if such an integer exists, and infinity otherwise.

(2) We define $S^H (A, \sigma)$ to be the cokernel of the total signature map

$$\text{sign}^H : W(A, \sigma) \to \widetilde{C}(X_F, \mathbb{Z}).$$

The following corollary follows immediately from Proposition 5.3.

Corollary 5.5. The group $S^H (A, \sigma)$ is 2-primary torsion, and its exponent is $2^{\text{st}(A, \sigma)}$ (with the convention that $2^\infty = \infty$).
Proposition 5.6. Let $H'$ be another reference form for $(A, \sigma)$. Then $S^H(A, \sigma) \simeq S^{H'}(A, \sigma)$. In particular $\text{st}^H(A, \sigma) = \text{st}^{H'}(A, \sigma)$.

Proof. By [1, Prop. 3.3(iii)], there exists $f \in C(X_F, \{-1, 1\})$ such that $\text{sign}^H = f \cdot \text{sign}^{H'}$. Define

$$\xi : \tilde{C}(X_F, \mathbb{Z}) \longrightarrow \tilde{C}(X_F, \mathbb{Z}) / \text{Im sign}^H$$

$g \mapsto f \cdot g + \text{Im sign}^H$

The map $\xi$ is a surjective morphism of groups since $f$ is invertible in $C(X_F, \mathbb{Z})$. Moreover, $g \in \ker \xi$ if and only if $f \cdot g \in \text{Im sign}^H = f \cdot \text{Im sign}^{H'}$, so $\ker \xi = \text{Im sign}^{H'}$ and $\xi$ induces an isomorphism from $S^{H'}(A, \sigma)$ to $S^H(A, \sigma)$. $\square$

Definition 5.7. We call $S(A, \sigma)$ the stability group of $(A, \sigma)$. It is well-defined up to isomorphism by Proposition 5.6. We call $\text{st}(A, \sigma)$ the stability index of $(A, \sigma)$.

Proposition 5.8. Let $h_0 \in W(A, \sigma)$ and $k_0 \in \mathbb{N}$ be as in Lemma 5.2. Then

$$\text{st}(A, \sigma) \leq \text{st}(F) + k_0.$$

Proof. Assume that $\text{st}(F)$ is finite. Let $f \in \tilde{C}(X_F, \mathbb{Z})$. Then there exists $q \in W(F)$ such that $2^{\text{st}(F)}f = \text{sign} q$, and thus $\text{sign}^H(q \cdot h_0) = 2^{\text{st}(F)+k_0}f$. $\square$

Proposition 5.9. Let $(A, \sigma)$ and $(B, \tau)$ be two Morita equivalent $F$-algebras with involution. Then $S(A, \sigma) = S(B, \tau)$ and $\text{st}(A, \sigma) = \text{st}(B, \tau)$.

Proof. It suffices to prove the first part of the statement, but this follows immediately from [1, Thm. 4.2]. $\square$

The following theorem extends a well-known result in quadratic form theory (cf. [4, (1.6)]) to algebras with involution:

Theorem 5.10. Let $(A, \sigma)$ be an $F$-algebra with involution and let $W_i(A, \sigma)$ denote the torsion subgroup of $W(A, \sigma)$. The sequence

$$0 \longrightarrow W_i(A, \sigma) \longrightarrow W(A, \sigma) \xrightarrow{\text{sign}^H} \tilde{C}(X_F, \mathbb{Z}) \longrightarrow S(A, \sigma) \longrightarrow 0$$

is exact. The groups $W_i(A, \sigma)$ and $S(A, \sigma)$ are 2-primary torsion groups.

Proof. This follows from [11, Thm. 4.1], [15, Thm. 5.1] and the definition of $S(A, \sigma)$. $\square$

Examples 5.11. In each of the examples below, $A$ is a quaternion division algebra.

1. Let $F = \mathbb{R}$, $A = (-1, -1)F$ and $\sigma$ quaternion conjugation. Then $\text{Im sign}^H \simeq \mathbb{Z} \simeq \tilde{C}(X_F, \mathbb{Z})$. Hence $S(A, \sigma) \simeq \{0\}$ and $\text{st}(A, \sigma) = 0$. Note that $\text{st}(\mathbb{R}) = 0$. 17
(2) Let \( F = \mathbb{R}, A = (-1, -1)_F \) and \( \sigma \) orthogonal. Then \( \text{Im sign}^H \simeq \{0\} \simeq \tilde{C}(X_F, \mathbb{Z}) \). Hence \( S(A, \sigma) \simeq \{0\} \) and \( \text{st}(A, \sigma) = 0 \).

(3) Let \( F = \mathbb{Q}(\sqrt{2}), A = (-1, -\sqrt{2})_F \) and \( \sigma \) quaternion conjugation. Then \( \text{Im sign}^H \simeq \mathbb{Z} \times \{0\} \simeq \tilde{C}(X_F, \mathbb{Z}) \). Hence \( S(A, \sigma) \simeq \{0\} \) and \( \text{st}(A, \sigma) = 0 \). Note that \( \text{st}(\mathbb{Q}(\sqrt{2})) = 1 \).

(4) Let \( F = \mathbb{R}(x), A = (x, -1)_F \) and \( \sigma \) orthogonal. Then \( \text{Im sign}^H \simeq 2\mathbb{Z} \times \{0\} \) and \( \mathbb{C}(X_F, \mathbb{Z}) \simeq \mathbb{Z} \times \{0\} \). Hence \( S(A, \sigma) \simeq \mathbb{Z}/2\mathbb{Z} \) and \( \text{st}(A, \sigma) = 1 \). Note that \( \text{st}(\mathbb{R}(x)) = 1 \).

We now consider the total signature \( \text{sign}^H \) of a hermitian form \( h \in W(A, \sigma) \). Since this is a continuous function, there exists an integer \( k \) such that \( 2^k \text{ sign}^H(h) \) is the total signature of some quadratic form over \( F \). In the next two results we will show that \( k \) can be chosen independently of \( h \).

**Lemma 5.12.** There exist disjoint clopen subsets \( U_1, \ldots, U_t \) of \( \tilde{X}_F \) and positive integers \( n_1, \ldots, n_t \) such that \( \tilde{X}_F = U_1 \cup \cdots \cup U_t \) and for every \( h \in W(A, \sigma) \) and \( i \in \{1, \ldots, t\} \), there exists \( q_i \in W(F) \) such that \( \text{sign}(q_i) = 2^{n_i} \text{ sign}^H(h) \) on \( U_i \).

**Proof.** By Theorem 3.6 and Corollary 3.14 for every \( P \in \tilde{X}_F \) there exists a finite ordered field extension \( (L_P, R_P) \) of \( (F, P) \) such that \( (A \otimes_F L_P, \sigma \otimes \text{id}) \) is Morita equivalent to an \((L_P, R_P)\)-real division algebra with involution \((D_P, \theta_P)\).

Let \( S_P = \{Q \in \tilde{X}_F \mid Q \text{ extends to } L_P\} \). By [16, Chap. 3, Thm. 4.4] we have \( S_P = \{Q \in \tilde{X}_F \mid \text{sign}_Q(\text{Tr}_{L_P/F}(1)) > 0\} \). The set \( S_P \) is therefore a clopen subset of \( \tilde{X}_F \) containing \( P \), and by compactness there are \( P_1, \ldots, P_t \in \tilde{X}_F \) such that \( \tilde{X}_F = S_{P_1} \cup \cdots \cup S_{P_t} \). It follows that there are disjoint clopen sets \( S_i \subseteq S_{P_i} \) (for \( i = 1, \ldots, t \)) such that \( \tilde{X}_F = S_1 \cup \cdots \cup S_t \).

Let \( i \in \{1, \ldots, t\} \) and write \( S := S_i, L := L_{P_i} \). It suffices to prove the lemma for \( S \) instead of \( \tilde{X}_F \). Let \( Q \in S \) and let \( R \in X_L/Q \). We have morphisms of Witt groups

\[
W(A \otimes_F L, \sigma \otimes \text{id}) \xrightarrow{\mu} W(D, \theta) \xrightarrow{\rho} W(L),
\]

where \((D, \theta)\) is \((L, R)\)-real (note that \( R \notin \text{Nil}[D, \theta] \) since \( Q \notin \text{Nil}[A, \sigma] \) by Proposition 2.2), \( \mu \) is an isomorphism induced by Morita equivalence and \( \rho \) is the map of Lemma 3.3. Let \( H_0 = \langle 1 \rangle_\theta \in W(D, \theta) \) and let \( \delta_R := \text{sgn}(\text{sign}_R^\mu(H_0 \otimes L)) \). Let \( h \in W(A, \sigma) \) be arbitrary. Then, using Remark 4.2, we have

\[
4 \text{ sign}^H(h) = 4 \text{ sign}_R^\mu(H_0 \otimes L) = 4 \text{ sign}_R^\mu(H_0 \otimes L) \mu(h \otimes L)
\]

[by Thm. 4.2]

\[
= 4 \text{ sign}_R^\mu(H_0 \otimes L) \mu(h \otimes L)
\]

\[
= 4 \text{ sign}_R^\mu(H_0 \otimes L) \mu(h \otimes L)
\]

\[
= 4 \delta_R \text{ sign}_R^H(h \otimes L)
\]

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where \( \ell = \text{dim}_L D \in \{1, 2, 4\} \) and \( \varphi := \frac{4}{\ell} \rho(\mu(h \otimes L)) \in W(L) \).

Observe that for every \( R \in X_L/Q \), \( \text{sign}_{\mu(\xi^L)}(H_0) \neq 0 \) since \( H_0 \) is a reference form for \((D, \emptyset)\). Since \( X_L/Q \) is finite, there is a finite tuple \( \bar{a}_Q \in L \) of length \( \ell_Q \) such that \( H(\bar{a}_Q) \cap (X_L/Q) \) contains only one ordering \( R_Q \). In particular

\[
\sum_{R \in X_L/Q} \text{sign}_{\mu(\xi^L)}(\langle \bar{a}_Q \rangle \cdot H_0) = 2^{\ell_Q} \text{sign}_{\mu(\xi^L)} R_Q \cdot H_0 \neq 0.
\]

Define \( \varphi_Q := \langle \bar{a}_Q \rangle \cdot \varphi \). Then, by [16, Chap. 3, Thm. 4.5],

\[
\text{sign}_Q(\text{Tr}^*_L/F \varphi_Q) = \sum_{R \in X_L/Q} \text{sign}_R \varphi_Q = 2^{\ell_Q} \text{sign}_{\mu(\xi^L)} R_Q \cdot \varphi = 2^{\ell_Q + 2} \delta_{R_Q} \text{sign}^H h,
\]

where \( \text{Tr}^*_L/F : W(L) \to W(F) \) denotes the Scharlau transfer. Therefore the clopen subset of \( X_F \)

\[
U_Q := (\text{sign}(\text{Tr}^*_L/F \varphi_Q) - 2^{\ell_Q + 2} \delta_{R_Q} \text{sign}^H h)^{-1}(0)
\]

contains \( Q \), and thus \( S = \bigcup_{Q \in X_F} U_Q \). Since \( S \) is compact we obtain \( S = U_{Q_1} \cup \cdots \cup U_{Q_r} \), for some \( Q_1, \ldots, Q_r \in S \), and for every \( Q \in U_{Q_i} \):

\[
2^{\ell_Q + 2} \delta_{R_Q} \text{sign}^H h = \text{sign}_Q(\text{Tr}^*_L/F \varphi_Q),
\]

and thus

\[
2^{\ell_Q + 2} \text{sign}^H h = \text{sign}_Q(\delta_{R_Q} \text{Tr}^*_L/F \varphi_Q).
\]

Since \( \delta_{R_Q} \text{Tr}^*_L/F \varphi_Q \in W(F) \), the result follows by taking clopen sets \( U_j \subseteq U_{Q_i} \) such that \( S = U_1 \cup \cdots \cup U_r \). \( \square \)

**Theorem 5.13.** There exists \( n_0 \in \mathbb{N} \) such that for every \( h \in W(A, \sigma) \) there exists \( q_h \in W(F) \) with \( 2^{n_0} \text{sign}^H h = \text{sign} q_h \).

**Proof.** We use the terminology of Lemma 5.12. Let \( r := n_1 + \cdots + n_t \). Then for every \( h \in W(A, \sigma) \) and every \( i \in \{1, \ldots, t\} \) there exists \( q_i \in W(F) \) such that \( \text{sign} q_i = 2^r \text{sign}^H h \) on \( U_i \).

Since \( t \) is finite, there exist \( m \in \mathbb{N} \) and \( p_i \in W(F) \) such that \( \text{sign} p_i = 2^m \) on \( U_i \) and \( \text{sign} p_i = 0 \) on \( X_F \setminus U_i \). Therefore \( \text{sign}(p_i q_i) = 2^m \text{sign} q_i = 2^m \text{sign}^H h \) on \( U_i \) and \( \text{sign}(p_i q_i) = 0 \) on \( X_F \setminus U_i \), and we obtain for \( i = 1, \ldots, t \),

\[
\text{sign}(p_1 q_1 + \cdots + p_t q_t) = \text{sign} p_i q_i \text{ on } U_i
\]
and thus
\[
\text{sign}(p_1 q_1 + \cdots + p_t q_t) = 2^{m+r} \text{sign}^H h \text{ on } X_F
\]
(note that by construction the quadratic form \(p_1 q_1 + \cdots + p_t q_t\) has zero signature on \(\text{Nil}[A,\sigma]\)).

\[\square\]

**Definition 5.14.** We define
\[
W_{\text{red}}(A,\sigma) := W(A,\sigma)/W_t(A,\sigma) \cong \text{sign}^H(W(A,\sigma)) \subseteq \widetilde{C}(X_F,\mathbb{Z})
\]
and call this the **reduced Witt group** of \((A,\sigma)\).

In order to compare reduced hermitian forms and reduced quadratic forms, we also introduce
\[
\widetilde{W}_{\text{red}}(F) := \{ q \in W_{\text{red}}(F) \mid \text{sign} q = 0 \text{ on } \text{Nil}[A,\sigma]\}.
\]
Observe that \(W_{\text{red}}(A,\sigma)\) is a \(W_{\text{red}}(F)\)-module and also a \(\widetilde{W}_{\text{red}}(F)\)-module in the natural way.

**Proposition 5.15.** Let \(h_0 \in W(A,\sigma)\) and \(k_0 \in \mathbb{N}\) be such that \(\text{sign}(h_0) = 2^{k_0}\) (cf. Lemma 5.2). With notation as in Theorem 5.13 the maps
\[
\widetilde{W}_{\text{red}}(F) \rightarrow W_{\text{red}}(A,\sigma), \; q \mapsto q h_0
\]
and
\[
W_{\text{red}}(A,\sigma) \rightarrow \widetilde{W}_{\text{red}}(F), \; h \mapsto q_h
\]
are well-defined injective morphisms of \(W_{\text{red}}(F)\)-modules.

**Proof.** Identifying \(W_{\text{red}}(F)\) and \(W_{\text{red}}(A,\sigma)\) with the images of \(\text{sign}\) and \(\text{sign}^H\), we see that the first map is simply multiplication by \(2^{k_0}\).

The second map is well-defined because \(h_1 = h_2\) in \(W_{\text{red}}(A,\sigma)\) is equivalent to \(\text{sign}^H h_1 = \text{sign}^H h_2\), which implies \(\text{sign} q_{h_1} = \text{sign} q_{h_2}\), so \(q_{h_1} = q_{h_2}\) in \(\widetilde{W}_{\text{red}}(F)\). It is also easy to check that it is an injective morphism of \(W_{\text{red}}(F)\)-modules. \(\square\)

We finish this paper by pointing out some difficulties that need to be overcome in order to further the study of the stability index of algebras with involution.

In the quadratic form literature one can find important links between the stability index of the field \(F\) and the powers of the fundamental ideal \(I(F)\), which crucially depend on Pfister forms (see for example [4, Satz 3.17]). Although we can define \(I(A,\sigma)\), the lack of a tensor product of hermitian forms in general is a serious obstacle to the development of analogous concepts and connections for Witt groups of algebras with involution.
Another issue is the following: the quadratic Pfister form $\langle 1, a \rangle$ has signature 2 on $H(a)$ and signature 0 on $H(-a)$, which is a fundamental observation when considering $\text{st}(F)$. In contrast, this behaviour cannot in general be replicated with hermitian forms since the $H$-signature of the form $\langle 1 \rangle_o$ may not be constant and in addition may take values which are not in \{-1, 1\}.

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