The chiral condensate in a constant electromagnetic field

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We study the shift of the chiral condensate in a constant electromagnetic field in the context of chiral perturbation theory. Using the Schwinger proper-time formalism, we derive a one-loop expression correct to all orders in $m_\pi^2/eH$. Our result correctly reproduces a previously derived “low-energy theorem” for $m_\pi = 0$. We show that it is essential to include corrections due to non-vanishing $m_\pi$ in order for a low energy theorem to have any approximate regime of validity in the physical universe. We generalize these results to systems containing electric fields, and discuss the regime of validity for the results. In particular, we discuss the circumstances in which the method formally breaks down due to pair creation in an electric field.

I. INTRODUCTION

Quantum Chromodynamics (QCD) provides the basis for our understanding of hadronic matter at its most fundamental level. As such, mathematically stable solutions to the exact equations of QCD represent stable physical particles. Because extreme external conditions such as high temperatures, strong chemical potentials, or powerful electromagnetic fields couple to quark and gluon fields, they alter the mathematically stable solutions of QCD. Calculating the corresponding response of the various physical observables of hadronic matter in the presence of these extreme conditions is a central question at the core of modern nuclear physics. Research along these lines has mainly focused on the effects of high temperature or high density (or equivalently chemical potential) on QCD matter. The response of QCD observables to very large electromagnetic fields has undergone less extensive study, but has potentially equally interesting consequences.

Since electromagnetic fields couple directly to quarks and not to gluons, it is natural to focus on intensive observables built from quark fields. It is clearly essential to understand the behavior of the chiral condensate, $\Sigma = \langle \bar{\chi} \chi \rangle$, because as the effective theory for QCD its predictions for large, pure magnetic fields in the assumption that $m_\pi = 0$ may place a severe constraint on these results. While results obtained in this theoretical limit are certainly interesting, it is not necessary to impose such a condition in order to formulate a controlled expansion. Consistent chiral expansions can only be formulated when both $eH$ and $m_\pi^2$ can be treated as low mass scales compared to $\Lambda$. However, nothing in the formulation fixes the ratio $m_\pi^2/eH$ which can be kept arbitrary. The low energy theorem of Eq. (1) must be regarded as the leading term in the full chiral expansion restricted to a regime where $eH \gg m_\pi^2$.

By design, $\chi$PT is universally applicable to all theories with underlying chiral symmetry and spontaneous chiral symmetry breaking.

Thus for fields sufficiently small, the $\chi$PT result $\Sigma \propto eH$ will be the leading order result. The principal results of this paper is the generalization of the “low energy theorem” in Eq. (1). This result has been extended to $\chi$PT at two loops, increasing the accuracy of the theoretical predictions for large, pure magnetic fields in the limit where $m_\pi = 0$. It is important to note that the assumption that $m_\pi = 0$ may place a severe constraint on these results. While results obtained in this theoretical limit are certainly interesting, it is not necessary to impose such a condition in order to formulate a controlled expansion. Consistent chiral expansions can only be formulated when both $eH$ and $m_\pi^2$ can be treated as low mass scales compared to $\Lambda$. However, nothing in the formulation fixes the ratio $m_\pi^2/eH$ which can be kept arbitrary. The low energy theorem of Eq. (1) must be regarded as the leading term in the full chiral expansion restricted to a regime where $eH \gg m_\pi^2$.

Physically, $m_\pi \sim 140$ MeV, and it is an open question whether the result derived at $m_\pi = 0$ accurately describes observable shifts in nature for any given values of $H$. Indeed, it is unclear that there is any domain of validity for Eq. (1), since it is not clear that it is possible to simultaneously have $eH \gg m_\pi^2$ while allowing $eH$ to remain within the regime where chiral perturbation theory to one loop is accurate. Most simply,
the condition for \( \chi PT \) to be applicable is \( p^2 / \Lambda^2 \ll 1 \), where \( p \) is any relevant light scale in the problem—mass, magnetic field, external momenta, etc. In practice, one might generically expect \( m_\pi^2 / \Lambda^2 \sim 1/50 \), which is a small expansion parameter when only one condition is required. Difficulties arise, however, when one artificially imposes the extra condition that \( m_\pi^2 / eH \ll 1 \), which will then in practice require the tight hierarchy \( m_\pi^2 / eH \ll 1/50 \). The region of magnetic fields which satisfy this hierarchy will be at best rather narrow.

We will show by explicit calculation, the general low energy theorem, valid for all values of \( m_\pi^2 / eH \), converges quite slowly to Eq. (1). It seems apparent that if an \( H \) field is large enough to be in the regime of validity of Eq. (1), it would also be sufficiently large as to require the inclusion of higher-order operators in the chiral Lagrangian. As a practical matter it is certainly far more useful to obtain a result valid for \( m_\pi^2 \ll \Lambda^2, eH \ll \Lambda^2 \), and arbitrary \( m_\pi^2 / eH \). Such a generalization greatly extends the regime of validity. Thus, our principal result of providing a “low energy theorem” valid at all orders is \( m_\pi^2 / eH \) is essential for having any useable result for the physical world. Moreover, one expects that the coefficients are likely to be rather unfavorable in this case, given \( 1/N \) considerations which we will discuss briefly in the conclusion.

An analogous situation can be found in the example of “low-energy theorems” for QCD at finite temperature. We present it here to emphasize that the theoretically elegant limit of \( \Sigma(0) = \chi \) is not always relevant in practice. Ref. [7] argues that Eq. (1) is a theorem in exactly the same sense that

\[
\frac{\Sigma(T)}{\Sigma(0)} = 1 - \frac{T^2}{8F_\pi^2} \frac{T^4}{384F_\pi^4} - \ldots \quad (2)
\]

is a low energy theorem for the condensate which holds at low \( T \) in the strict chiral limit [13, 14]. Clearly Eq. (2) is formally valid only when the hierarchy of scales, \( m_\pi \ll T \ll T_c \ll \Lambda_{QCD} \), is satisfied. The range of validity for this hierarchy is similar to the one we find in our problem. Also in analogy, while Eq. (2) is formally correct in the \( m_\pi = 0 \) limit, it is never useful in describing a real system at any temperature: any temperature which is high enough to be much bigger than \( m_\pi \) is also beyond the temperature of the QCD phase transition and thus outside the regime of validity of \( \chi PT \). Since the formula is derived with the assumption that \( m_\pi = 0 \) and \( m_\pi \) does not appear in the result, it must be the case that the expression is only valid for \( m_\pi \ll T \); if it is of the same order as \( T \) or greater, it will begin to play an increasingly important role in the result. Thus, for \( T \) of the order of tens of MeV where [2] would be valid, \( m_\pi \) is comparatively large enough to render (2) invalid. Further, since \( T_c \sim 170 \) MeV, if \( T \gg m_\pi \sim 140 \) MeV then one is clearly in the quark-gluon plasma phase, and outside the range of \( \chi PT \). Explicit calculations with the physical value of \( m_\pi \) [14] show conclusively that the “low energy theorem” of Eq. (2) does not accurately reproduce the shift in the chiral condensate for any temperature. We illustrate this explicitly in Fig. 1.

![Figure 1](image_url)

**FIG. 1:** Shift in the condensate to one loop plotted as a function of temperature in the chiral limit and with a realistic finite value for \( m_\pi \).

The approach to implementing \( \chi PT \) at one-loop (\( \mathcal{O}(p^4) \)) for our problem is straightforward. At \( \mathcal{O}(p^4) \) the pions do not interact, and one uses the appropriate non-interacting propagators for a constant external field. In this circumstance, the Schwinger proper time formalism [15] provides a natural framework to study the QCD chiral condensate in the presence of constant electromagnetic fields.

The final issue we address in this paper is the generalization of the low energy theorem to include electric fields—either as pure electric fields or in situations where both \( E \) and \( H \) are present. The character of the shift of the condensate in the presence of a non-trivial electromagnetic field is fundamentally different from the shift in the presence of a pure magnetic field. This difference is essentially due to the famous Schwinger mechanism, and is made manifestly clear within the context of the proper time formalism. Within this formalism, it is a relatively straightforward exercise to evaluate the effective action for charged matter fields in the presence of a uniform electromagnetic field (to one loop). Poles appear in the effective action in the presence of a uniform electric field, which are conventionally interpreted as corresponding to spontaneous real \( \pi^+ \pi^- \) pair creation out of the vacuum [15]. This in turn implies an inherent local instability in a system containing a constant electric field. Indeed, in light of the Schwinger mechanism, a uniform field in such a system does not remain static, but naturally evolves via back-reactions over time [16, 17]. As a practical matter, the effect of pair creation in a constant electric field means that the field can only be considered as constant over length and time scales limited by the parameters of the problem. In contrast, a system with a constant magnetic field has no such instability, and the \( H \) field can consistently be regarded as constant over time.
II. THE CHIRAL CONDENSATE IN AN EXTERNAL MAGNETIC FIELD

In this section we compute the chiral condensate in an external magnetic field to one loop in chiral perturbation theory and to all orders in \( m_2^2 / eH \). Recall that the chiral expansion is typically an expansion in \( m_2^2 / \Lambda^2 \), where \( p \) is a momentum in the problem. Since the external magnetic field is an isovector, it explicitly breaks chiral symmetry, and the expansion becomes an expansion in \( m_2^2 / eH \). Since the scattering amplitude for pions at \( p^2 = 0, m_2^2 = 0, H = 0 \) is zero at lowest order in the theory, the leading-order \( \chi \)PT result for the shift in the chiral condensate due to a magnetic field is simply the one-loop expression for non-interacting pions in a magnetic field. In conventional \( \chi \)PT counting, one-loop effects occur at \( \mathcal{O}(p^4) \). The effects of pion-pion interactions occur at higher order in the chiral expansion and will accordingly be suppressed by powers of \( m_2^2 / eH \).

The exact expression [13] for the effective Lagrangian for a charged pion in an external field is given by

\[
\mathcal{L}_{\text{eff}} = \frac{1}{16\pi^2} \int_0^\infty ds^{-3} e^{-m_2^2 s} \left( \frac{(e_2^2 G)}{\cosh(e s X) - 1} \right),
\]

where \( F = \frac{\mu^2 - e^2}{2} \) and \( G = \vec{E} \cdot \vec{H} \) and \( X = (\mathcal{F} + iG)^{-1} \).

In an \( H \) field, this can be simplified to

\[
\mathcal{L}_{\text{eff}} = \frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m_2^2 s} \left[ \frac{e H s}{\sinh(e H s)} - 1 \right].
\]

As noted above, this corresponds to the one-loop \( \chi \)PT result. Thus, the one-loop chiral condensate as a function of applied magnetic field is given by

\[
\Delta \Sigma(H) = \frac{\partial \mathcal{L}_{\text{eff}}(H, m_\pi)}{\partial m_\pi}.
\]

At this order, the mass of the pion is related to the quark mass, the chiral condensate at zero field, and the pion decay constant via the Gell-Mann-Oakes-Renner relation: \( (m_u + m_d) \Sigma(0) = F_\pi^2 m_\pi^2 \). Therefore, the expression for the shift in the condensate can be expressed as

\[
\Delta \Sigma(H) = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial m_\pi} = \frac{\log(2) e H \Sigma(0)}{16\pi^2 F_\pi^2} I_H \left( \frac{m_\pi^2}{e H} \right),
\]

\[
I_H(y) = -\frac{1}{\log(2)} \int_0^\infty dz \frac{e^{-yz}}{z^2 \sinh(z) - 1}.
\]

where the parameter \( y \) is the dimensionless ratio \( m_\pi^2 / eH \).

A direct comparison with Eq. [11] makes the physical meaning of \( I_H \left( \frac{m_\pi^2}{eH} \right) \) clear: it is a multiplicative factor which encodes the corrections to Eq. [11] due to a non-zero \( m_\pi \) to all orders in \( m_\pi^2 / eH \). The form of this integral simplifies dramatically in the chiral limit, where \( y \to 0 \). After a routine calculation it is easy to see that \( I_H(0) \) is unity, reproducing the low energy theorem originally found in Ref. [7]. However, for our purposes, it is more interesting to note that there is a closed-form expression for this integral [18]:

\[
I_H(y) = \frac{1}{\log 2} \left( \log(2\pi) + y \log \left( \frac{y}{2} \right) - y - 2 \log \Gamma \left( \frac{1 + y}{2} \right) \right),
\]

yielding an analytic result to one-loop order in \( \chi \)PT valid for any ratio of \( m_\pi^2 \) to \( eH \).

A comment about \( \chi \)PT to one-loop is useful at this stage. Typically, one-loop graphs diverge and are only sensible in the context of a renormalization scheme including counterterms in the \( \mathcal{O}(p^4) \) Lagrangian. The present expression is finite. \textit{A priori} this does not mean that there cannot be a finite contribution from a higher-order operator in the chiral Lagrangian. Direct inspection of the terms in the chiral Lagrangian at \( \mathcal{O}(p^6) \) reveals that no terms contribute to the shift in the condensate in an external field at tree level. As it happens, the first such term which contributes to the chiral condensate at tree level occurs at \( \mathcal{O}(p^8) \). Accordingly, the lowest-order chiral Lagrangian, \( \mathcal{O}(p^2) \), is sufficient to determine the shift in the condensate to the order which we are working.
compared to the small fields and pion masses.

III. THE CHIRAL CONDENSATE IN AN EXTERNAL ELECTRIC FIELD

As noted in the Introduction, the $\chi$PT analysis to date has been restricted to constant magnetic fields. In contrast, the NJL model calculations have been made for constant electric fields\textsuperscript{3}. Formally, these NJL calculations neglect an imaginary part which arose in the evaluation of the chiral condensate. The standard interpretation of the emergence of an imaginary part in the calculation of a purely real quantity is a signal that the state is not the true ground state of the system; the state is regarded as unstable. In the mean-field NJL case the instability is due to pair creation of constituent quarks. Since this is a manifestation of the model’s unphysical lack of confinement, one may argue that this may be neglected. As we shall see below an analogous issue arises in $\chi$PT at one loop, but in this case the imaginary parts are due to an instability in associated pion pair production and are undoubtedly physical.

Suppose that an electric field which is approximately uniform over a large region of space is suddenly turned on. This external electric field will cause both a real shift in the condensate and a local breakdown in the vacuum due to real $\pi^+\pi^-$ pair emission. Provided that the instability due to pair emission occurs over a much shorter time scale than the characteristic time with which the condensate responds, it is sensible to discuss shifts in the chiral condensate in the context of a “constant” electric field. Moreover, for small $E$ fields, the rate of pair production scales with $m_\pi$ as $\exp\left(-\frac{m_\pi^2}{eE}\right)$, and is exponentially suppressed\textsuperscript{13}. Thus, in the regime where $eE \ll m_\pi^2$ time evolution due to the imaginary shift is very slow and the question of how the condensate responds to a spatially and temporally constant electric field remains sensible.

The ratio of the calculated real to imaginary parts of the shift in the condensate is a crude indicator of whether the calculation of the real part is meaningful; we trust the result only if the real part dominates. When the imaginary part becomes comparable to the real part, the instability is significant and the calculation of the real part of the shift is unreliable.

Our goal in this section is to calculate the shift of the condensate in an external electric field to one loop in $\chi$PT. Qualitatively, the electric field case and the magnetic field case are quite different, though they are both derived from Eq. (3). Because the chiral condensate is a Lorentz scalar, shifts in the chiral condensate due to electromagnetic fields can only depend on the Lorentz scalars $\mathcal{F}$ and $\mathcal{G}$. Since $\mathcal{G} = 0$ for both constant electric and magnetic fields, the shift in $\Sigma$ is only a function of $\mathcal{F}$. Due to the exclusive dependence on $\mathcal{F}$, and because it is negative for a constant $E$ field and positive for a constant $B$ field, one can obtain the expression for a constant $E$ field by analytically continuing the expression in Eqs. (9), (7) from positive $\mathcal{F}$ to negative $\mathcal{F}$. This analytic continuation is, in effect, the substitution $H \to iE$.

This substitution $H \to iE$ induces the change in the integrand $1/\sinh(z) \to 1/\sin(z)$. As such, the integral acquires an infinite number of poles along the integration path. Thus, we can write the shift as

$$\Delta \Sigma(E) = \frac{\log(2)}{16\pi^2 F^2} I_E \left( \frac{m_\pi^2}{eE} \right)$$

$$I_E(y) = -\frac{1}{\log(2)} \int_0^\infty \frac{dz}{z^2} e^{-yz} \left[ \frac{z}{\sin(z)} - 1 \right] = -iI_H(iy).$$

The question of how to handle the poles in this integral is intimately related to the boundary conditions imposed on the problem which in turn necessarily reflect the underlying physical circumstances. Here, we will adopt the usual convention for the Schwinger mechanism\textsuperscript{16}: we make the substitution $1/\sin(z) \to 1/(\sin(z) + i\epsilon)$ in the integrand. This renders the expression for the shift mathematically well defined: physically it is the regime associated with pair creation from the electric field.

The integral expression for $I_E$ makes it manifestly clear how instabilities in the system arise in the presence of a uniform electric field. The infinite number of poles with non-trivial residues along the integration path allow the integral to be separated into a real principal value part and a purely imaginary part proportional to the sum of residues of the poles. Physically, these poles, and the associated imaginary shift, indicate an instability of the configuration and ultimately non-trivial evolution of the system over time.

We turn now to the evaluation of the shift from the electric field by analytically continuing the closed form for $I_H$ given in (7) to find a closed form for $I_E$. Unfortunately, the analytic structure of (7) is rather complicated; to obtain the correct analytic continuation one must choose the appropriate branch. Making the stan-
FIG. 3: Real and imaginary parts of $I_E$ defined in Eq. (9) are given in subfigures (a) and (b). Subfigure (c) gives their ratio.

The standard choice yields the result for the real and imaginary parts,

$$\overline{I(I_E)} = \frac{1}{\log 2} \log(1 + e^{-\pi y})$$

$$\overline{R(I_E)} = \frac{1}{\log 2} \left\{ y \log \left( \frac{y}{2} \right) - y + C y + 2 \tan^{-1} y \right\}$$

$$+ 2 \sum_{n=1}^{\infty} \left[ \tan^{-1} \left( \frac{y}{2n+1} \right) - \frac{y}{2n} \right]$$

where $C$ is Euler’s constant. We use these results to find the imaginary and real parts of $I_E$, as well as the ratio of the two parts in Fig. 3. Note that when $eE/m_n^2$ is smaller than unity, the imaginary part is very small, and it is meaningful to consider a shift in the chiral condensate due to a “constant” electric field.

As noted, the analytic structure of $I_E$ is complex and accordingly it is not obvious that the choice of the branch yielding the results in (9) and Fig. 3 is done correctly. In order to ensure that we have made the correct analytic continuation, we evaluate the integral numerically to allow a direct comparison.

Summing the residues from each of these poles we find that this result is analytically equivalent to the imaginary part of $I_E$ in Eq. (9). This confirms our choice of branches in the analytic continuation, Eq. (9), and shows why the imaginary part, associated with an instability resulting from pair creation, is unimportant for small field strengths. Expanding out the log, it is easy to see that, as expected, the imaginary contribution is exponentially suppressed when $eE \ll m_n^2$.

It is also useful to check that the real part of $I_E$ obtained via the principal value part of the integral in Eq. (8) agrees with our analytic continuation. Unfortunately, it is quite difficult to directly evaluate the principal value of the integral analytically. On the other hand, the singular behavior about the poles complicates the numerical evaluation of the principal value of $I_E$. This is neatly circumvented by changing the structure of the integrand to:

$$P(I_E) = \int_0^\infty dz \left\{ \frac{e^{-\pi y}}{z^2} \left[ \frac{z}{\sin(z) + i \epsilon} - 1 \right] \right.$$  

$$- \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \pi} e^{-n \pi y} \left( \frac{1}{z - n \pi + i \epsilon} - \frac{1}{z + n \pi + i \epsilon} \right) \right\}.$$  

The added terms in the integrand are explicitly chosen to both have vanishing principal values and have simple poles which exactly cancel the poles that naturally occur in (8). In this manner, the singular behavior about the poles of $1/\sin(z)$ is eliminated from the integrand, facilitating quick and relatively accurate numerical analysis—while simultaneously leaving the principal value completely unchanged. The numerical results are found to agree with the numerical values given by the formally derived result (9), plotted in Fig. 3 to extremely high precision.
IV. GENERAL CASE: \( \vec{E} \cdot \vec{H} \neq 0 \)

The shifts in the chiral condensate at one loop in \( \chi \)PT due to a pure uniform electric field or a pure constant magnetic field are analytically tractable, and have closed form solutions. This is not true in the case where both \( E \) and \( H \) fields are present. If \( \vec{E} \) and \( \vec{H} \) are orthogonal, one can always boost to a frame in which the field is purely electric or magnetic, use the fact that \( \Sigma \) is a Lorentz scalar, and exploit the previous exact results. However, where the Lorentz invariant \( \vec{E} \cdot \vec{H} \) is non-vanishing, one cannot exploit this trick and a new calculation is needed. In this regime, while an integral expression can be found for \( \Sigma \) at one-loop order in \( \chi \)PT, the integral derived from Eq. (3) appears to be intractable analytically. However, the shifts may be evaluated numerically through an extension of the approach to the cases where \( E \) and \( H \) are orthogonal.

As noted previously, \( \Sigma \) is a Lorentz invariant and because it is shifted by external electromagnetic fields, \( \Delta \Sigma \) must be a function of the two Lorentz scalars \( \mathcal{F} = \frac{\mu^2 - E^2}{2} \) and \( \mathcal{G} = \vec{E} \cdot \vec{H} \). Rather than using these variables directly, it is more convenient to use the covariant variables \( f \) and \( \phi \) defined according to

\[
\mathcal{F} = \frac{f^2 \cos(2\phi)}{2}, \quad \mathcal{G} = \frac{f^2 \sin(2\phi)}{2}
\]

with \( \pi/2 \geq \phi \geq -\pi/2 \). We can reduce this domain to \( 0 \leq \phi \leq \pi/2 \) by making the observation that our result is derived from a parity-invariant effective theory for QCD; since \( E \), and therefore \( \phi \), are parity-odd, our result must be an even function of \( \phi \).

One can always boost to a frame in which \( \vec{E} \parallel \vec{H} \); in such a frame the magnitudes \( E \) and \( H \) fully specify \( \mathcal{F} \) and \( \mathcal{G} \) up to an irrelevant sign, and have a very simple relation to \( f \) and \( \phi \)

\[
H = f \cos(\phi) \quad E = f \sin(\phi). \tag{12}
\]

In such a frame, \( f \) is proportional the energy density, while \( \phi \) dial the system from pure electric field aligned (or anti-aligned) with any infinitesimal magnetic field (\( \phi = \pi/2 \)) to pure magnetic field \( \phi = 0 \).

Using the Schwinger proper time expression for the effective Lagrangian of Eq. (3) for the case of non-orthogonal \( E \) and \( H \) fields and differentiating with respect to \( m^2_\pi \) to obtain the chiral condensate yields

\[
\Delta \Sigma(f, \phi) = \frac{ef \Sigma(0)}{16\pi^2 F^2_\pi} I_{\mathcal{F}H}(f, \phi), \tag{13}
\]

where \( I_{\mathcal{F}H}(f, \phi) \) is defined to be

\[
I_{\mathcal{F}H}(f, \phi) = \frac{1}{\log 2} \int_0^\infty \frac{du}{u^2} \left[ (m^2_\pi/ef) u \left( \frac{u^2 \sin(2\phi)}{2 \sin(u \sin(\phi)) \sinh(u \cos(\phi)) + i \epsilon} - 1 \right) \right], \tag{14}
\]

\[
R(I_{\mathcal{F}H}(f, \phi)) = I_{\mathcal{F}H} - \frac{1}{\log 2} \sum_{n=1}^\infty \frac{\cos(\phi) (-1)^n e^{-n \pi m^2_\pi/(ef \sin(\phi))} \sinh(n \pi / \tan(\phi))}{\sinh(n \pi / \tan(\phi))} \int_0^\infty \left( \frac{du}{u - \frac{n \pi}{\sin(\phi)} + i \epsilon} - \frac{du}{u + \frac{n \pi}{\sin(\phi)} + i \epsilon} \right), \tag{15}
\]

\[
T(I_{\mathcal{F}H}(f, \phi)) = \frac{1}{\log 2} \sum_{n=1}^\infty \frac{\pi \cos(\phi) (-1)^n e^{-n \pi m^2_\pi/(ef \sin(\phi))} \sinh(n \pi / \tan(\phi))}{\sinh(n \pi / \tan(\phi))}. \tag{16}
\]

We find the integral in Eq. (14) to be intractable analytically and thus we evaluate it numerically. Using the same \( i \epsilon \) convention as with \( I_E \), the integral can be divided into a principal value part and a contribution from the residue of the poles. To evaluate the principal value part, we use the same prescription for removing the poles as in Sec. III. The principal value (real part) is given in Eq. (15) and the sum of the residues (imaginary part) is given in Eq. (16).

In Fig. 4 we plot the principal value of this integral as a function of \( \tan(\phi) \) and \( ef \). We see that \( R(I_{\mathcal{F}H}(f, \phi)) \) approaches \( R(I_E) \) at one extreme and \( I_{H\mathcal{F}} \) at the other extreme, as expected. Fig. 5 shows the imaginary part arising from the pole structure for various values of \( E \) and \( H \). We note a smooth variation from no imaginary part for a pure magnetic field to the maximal imaginary part for a pure electric field.

We also see that the imaginary part (and therefore the pair creation from the electric field) is suppressed at high mass in all cases. This means that far from the \( m_\pi = 0 \)
limit, the pair creation mechanism will not play a role and the constant field will be nearly stable over relevant time scales.

For $m^2_\pi/ef$ large enough so that the imaginary part is negligible, we also see that $I_{EH}(f, \phi) \to -I_{EH}(f, \phi)$ under $\phi \to \pi/2 - \phi$, which corresponds to switching the electric and magnetic fields. Physically, this means that at high $m^2_\pi/ef$ where the imaginary part is suppressed, the effect of switching the electric and magnetic fields on the shift in the condensate introduces an overall negative sign. Thus, we see the general effect that while the magnetic field acts to increase chiral symmetry breaking, the electric field suppresses it.

In Fig. 6 we plot the ratio of the imaginary to real parts as a function of $\tan(\phi)$, which, again, corresponds to $E/H$ in the limit of parallel fields, now in the opposite extreme where $m^2_\pi/ef = 0$. The imaginary part grows exponentially with $\tan(\phi)$. It is negligible for $\tan(\phi) \lesssim 0.5$, and becomes significant by $\tan(\phi) \sim 1$.

V. CONCLUDING REMARKS

The general expression for the shift in the chiral condensate in QCD due to electric and magnetic fields is given in Eqs. (13) and (14). The expression is valid mathematically up to power corrections in $m^2_\pi/\Lambda^2$ and $ef/\Lambda^2$ and hence may be regarded as a “low energy theorem” in the same sense used by Shushpanov and Smilga[7]. This expression may appear to be dynamics-dependent, in as much as it depends on the specific Lagrangian of $\chi$PT, but as the $\chi$PT Lagrangian is the only possible Lagrangian consistent with the symmetries of QCD possible at low energies, this is not the case.

A non-trivial uniform electric field, which occurs when...
$\mathcal{G} \neq 0$ or $\mathcal{F} < 0$, causes poles to appear in the expression for the shift in the chiral condensate. As a result, when Schwinger’s boundary conditions are imposed, the shift acquires a non-zero imaginary component, which corresponds to an instability in the vacuum. On a physical level, the instability of the vacuum in a constant electric field restricts the applicability of the result to cases where the instability occurs over a relatively long time scale. This essentially restricts one to the range $eE \ll m_\pi^2$, in the frame where $E|\mathcal{H}$, for which the imaginary part is exponentially suppressed. In general, the integral in Eq. (14) needs to be treated numerically. However, for the special cases of pure electric and pure magnetic fields, the integral may be evaluated analytically, leading to the simple expressions in Eqs. (7) and (9). These results generalize previous work both via the inclusion of an electric field and through the inclusion of a non-zero pion mass. The need to include a non-zero pion mass is critical since the weak field and zero $m_\pi$ limits are non uniform. As a result, the behavior at $ef \sim m_\pi^2$, the principal regime of relevance, in our chiral expansion is typically quite different from the behavior at fixed $f$ but zero $m_\pi$.

Clearly, since our expression is just the leading-order term in the chiral expansion it has a limited range of validity in field strength. To improve the accuracy of the description at somewhat larger field strengths, it is natural to work to higher order in the expansion. Such a calculation has been done for purely magnetic fields at $m_\pi = 0$ [8]; it would certainly be of interest to extend this to the general case of electric and magnetic fields to all orders in the ratio of $m_\pi^2/(ef)$.

Before concluding, it is worth noting that the NJL model calculations [3] find a first order response in a magnetic field and zero $m_\pi$ to be $\propto (eH)^2/\Sigma(0)^4$, which is not consistent with the $\chi$PT result. One might expect that the $\chi$PT result, based on QCD in a model independent way, is obviously more accurate than the NJL result which is clearly model dependent. However, the NJL result is of some interest because the nature of the calculations region of validity differs from that of $\chi$PT. It is derived in the Hartree-Fock approximation, justified by the 1/$N_c$ expansion [10, 11]. In contrast, $\chi$PT is an expansion in small momentum and quark masses, but its $N_c$ dependence is less obvious. The low-energy constants (LECs) which multiply each term in the expansion are independent of momentum, but can have non-trivial $N_c$ dependence. Thus, though higher order terms in the expansion may appear to drop with $N_c$, due to inverse powers of $F_\pi$, this is not actually always the case [9]. One question is whether or not the large $N_c$ and chiral limits commute for $\Sigma(H)$. It would hardly be surprising if the limits do not commute as there are many well-known examples in QCD of non-commuting limits [12]. In any case, the NJL results are useful as a hint that the region of validity for $\chi$PT might not be as clean cut as might be hoped. Higher-order terms in the $\chi$PT expansion which are not 1/$N_c$ suppressed can cause the expansion to converge more slowly than would otherwise be expected.

Thus, another motivation for working to higher order is provided by the non-trivial $N_c$ dependence of the LECs noted above. The calculations done here came entirely from a pion loop, and meson loops are generically suppressed by factors of 1/$N_c$. In contrast, tree diagrams in the chiral Lagrangian can yield leading order results in the 1/$N_c$ expansion [13]. As noted in Sec. II, it is typical for tree graphs from the $O(p^4)$ chiral Lagrangian to contribute at the same order in the chiral expansion as one-pion-loop graphs. However, for the shift in the chiral condensate, there are no terms in the $O(p^4)$ Lagrangian which contribute at tree level. This is not the case at the next order in $\chi$PT where tree graphs in $O(p^6)$ Lagrangian do contribute [8]; it is easy to see these contributions come in at leading order in the 1/$N_c$ expansion. An analysis of the actual impact of these leading-order $N_c$ terms may shed light on whether or not the $N_c$ expansion can indicate a slower convergence of $\chi$PT.

In conclusion, we have numerically examined the shift in the QCD chiral condensate due to an electromagnetic field in the framework of Chiral Perturbation Theory. We find that the low energy theorem in [14] is indeed accurate at large $eH/m_\pi^2$. However, a field which is large enough for this limit to apply is large enough that an expansion in $eH/F_\pi^2$ is no longer valid. These results continue to hold in the presence of a small electric field. We are thus able to demonstrate, in a model independent way, that uniform electric fields tend to suppress $\Sigma_{SB}$. However, when the magnitude of the electric field becomes comparable to that of the magnetic field (in the frame where they are both parallel), the calculations begin to break down due to the imaginary shift which corresponds to local instabilities and charged pair creation.

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