Renormalization and mixing in lattice QCD:
The case of the chromomagnetic operator

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Abstract. We study matrix elements of the “chromomagnetic” operator on the lattice. This operator is contained in the strangeness-changing effective Hamiltonian which describes electroweak effects in the Standard Model and beyond. Having dimension 5, the chromomagnetic operator is characterized by a rich pattern of mixing with other operators of equal and lower dimensionality, including also non gauge invariant quantities; it is thus quite a challenge to extract from lattice simulations a clear signal for the hadronic matrix elements of this operator. We compute all relevant mixing coefficients to one loop in lattice perturbation theory; this necessitates calculating both 2-point (quark-antiquark) and 3-point (gluon-quark-antiquark) Green’s functions at nonzero quark masses. We use the twisted mass lattice formulation, with Symanzik improved gluon action. We also provide a nonperturbative method to compute mixing with lower dimensional operators; we use this method in numerical simulations, and extract the mixing with the 3-dimensional scalar density, finding good agreement with one-loop results.

1. Introduction

The electroweak effective Hamiltonian describing strangeness changing (∆S = 1) processes, in the Standard Model (SM) and beyond, contains four “magnetic” operators of dimension 5:

\[ H^{\Delta S=1, \, d=5} = \sum_{i=\pm} (C_i^\gamma Q_i^\gamma + C_i^g Q_i^g) + \text{h.c.} \] (1)

The coefficients \( C_i^\gamma \) (\( C_i^g \)), multiplying the electromagnetic–EMO (chromomagnetic–CMO) operators, may be calculated perturbatively via the OPE; they are suppressed within the SM, but become more pronounced beyond the SM, e.g. through penguin diagrams in SUSY.

The mixing \( \Delta S = \pm 1 \) processes, in particular the \( K^0 \to \pi^+ \pi^- \) decay, is parameterized by the relative sizes of the mixing strengths. To leading order in \( \chi PT \), the mixing is described by the mixing matrix

\[ \langle \pi^0 | Q_g^- | K^0 \rangle = \frac{11}{32\pi^2} \frac{M_K^2 M_F^2}{f_\pi (m_s + m_d)} B_g^1 \] (2)

\[ \langle \pi^+ \pi^- | Q_g^+ | K^0 \rangle = \frac{11}{32\pi^2} \frac{M_K^2 M_F^2}{f_\pi (m_s + m_d)} B_g^2 \]

These matrix elements are relevant for the study of \( K^0 \to K^0 \) mixing, \( \epsilon'/\epsilon \), the \( \Delta I = 1/2 \) rule, and \( K \to 3\pi \) decays. To leading order in \( \chi PT \), the \( B \)-parameters are all related; thus, a lattice study of, say, Eq. (2), provides information for Eqs. (3) as well.
2. Operator Mixing – Lattice Action – Symmetries

A formidable issue in the study of the CMO is the fact that it mixes with a large number of other operators under renormalization. Even in dimensional regularization (DR), which has the simplest mixing pattern, the CMO ($\mathcal{O}_{CM} \equiv \mathcal{O}_1$) mixes with a total of 9 other operators ($\mathcal{O}_2 - \mathcal{O}_{10}$), forming a basis of dimension-5, Lorentz scalar operators with the same flavor content as the CMO. Among them, there are also gauge noninvariant operators ($\mathcal{O}_9, \mathcal{O}_{10}$); these are BRST invariant and vanish by the equations of motion, as required by renormalization theory.

We calculated the bare Green’s functions of Eq. (7), first in DR and then in the far more complicated case of the lattice. The purpose of the calculation in DR is twofold: First, it provides the coefficients $Z^{DR,MSS}_{ii}$; second, and most important, it leads to the MSS-renormalized Green’s functions, which are then necessary for extracting the real quantities of interest: $Z^{L,MSS}_{ii}$.

We have adopted the twisted mass action for sea quarks and the Osterwalder-Seiler action for other operators under renormalization. Even in dimensional regularization (DR), which has many as possible of these candidates, a judicious choice of lattice action is imperative. To obtain the simplest mixing pattern, the CMO ($\mathcal{O}_{CM} \equiv \mathcal{O}_1$), we only need the first row of

\[
\begin{align*}
\mathcal{O}_1 &= g \bar{\psi}_s \sigma_{\mu\nu} G_{\mu\nu} \psi_d \\
\mathcal{O}_2 &= (m_d^2 + m_s^2) \bar{\psi}_s \psi_d \\
\mathcal{O}_3 &= m_s m_d \bar{\psi}_s \psi_d \\
\mathcal{O}_4 &= \bar{\psi}_s (\not{D} + m_d) \psi_d \\
\mathcal{O}_5 &= \bar{\psi}_s (\not{D} + m_s) \psi_d \\
\mathcal{O}_6 &= \bar{\psi}_s (\not{D} + m_s) \psi_d \\
\mathcal{O}_7 &= m_s \bar{\psi}_s (\not{D} + m_d) \psi_d + m_d \bar{\psi}_s (\not{D} + m_s) \psi_d \\
\mathcal{O}_8 &= m_d \bar{\psi}_s (\not{D} + m_d) \psi_d + m_s \bar{\psi}_s (\not{D} + m_s) \psi_d \\
\mathcal{O}_9 &= \bar{\psi}_s \not{D} \psi_d \\
\mathcal{O}_{10} &= \bar{\psi}_s (\not{D} + m_d) \psi_d - \bar{\psi}_s (\not{D} + m_s) \not{D} \psi_d
\end{align*}
\]

On the lattice, where certain symmetries are violated, mixing can become considerably more complicated; there can be mixing with additional operators of dimension ≤ 5. To exclude as many as possible of these candidates, a judicious choice of lattice action is imperative.

We have adopted the twisted mass action for sea quarks and the Osterwalder-Seiler action for valence quarks. For gluons we have used the Symanzik improved action with several standard choices of values for the corresponding Symanzik coefficients. For our choice of action there appear only 3 additional operators, compared to DR, and they all have dimension < 5:

\[
\begin{align*}
\mathcal{O}_{11} &= i \bar{\psi}_s [r_d \gamma_5 (\not{D} + m_d) + r_s (\not{D} + m_s) \gamma_5] \psi_d \\
\mathcal{O}_{12} &= i (r_d m_d + r_s m_s) \bar{\psi}_s \gamma_5 \psi_d \\
\mathcal{O}_{13} &= \bar{\psi}_s \psi_d
\end{align*}
\]

3. Renormalization Matrix

Renormalized operators $\mathcal{O}_i^{R}$ are related to bare ones $\mathcal{O}_i$ via a 13×13 renormalization matrix $Z$:

\[
\mathcal{O}_i = \sum_{j=1}^{13} Z_{ij} \mathcal{O}_j^R \quad (O = Z \mathcal{O}^R, \quad \mathcal{O}^R = Z^{-1} \mathcal{O})
\]

where $Z_{ij} = Z^{XY}_{ij}$ depend both on the regularization $X = L$ (lattice), $DR$ (dimensional), etc.) and on the renormalization scheme $Y = MSS, \text{RI}'$, etc.). At tree level: $Z = 1$. For $O_1$, we only need the first row of $Z^{-1}$ (and thus, to one loop, only the first row or $Z_1$).

To obtain $Z_{ii}$, we have calculated, to one loop, and in an arbitrary covariant gauge, the 2-point (quark-antiquark) and 3-point (quark-antiquark-gluon) bare amputated Green’s functions of $\mathcal{O}_1$; these are related to the corresponding renormalized Green’s functions through:

\[
\langle \psi^R \mathcal{O}_1^{R} \psi^R \rangle_{\text{amp}} = Z_{\psi} \sum_{i=1}^{13} (Z^{-1})_{1i} \langle \psi \mathcal{O}_i \psi \rangle_{\text{amp}} , \quad \psi = Z^{1/2}_\psi \psi^R
\]

\[
\langle \psi^R \mathcal{O}_1^{R} \psi^R A^R \rangle_{\text{amp}} = Z_{\psi} Z_{A}^{1/2} \sum_{i=1}^{13} (Z^{-1})_{1i} \langle \psi \mathcal{O}_i \psi A \rangle_{\text{amp}} , \quad A = Z^{1/2}_A A^R
\]

4. Results and Future Work

We calculated the bare Green’s functions of Eq. (7), first in DR and then in the far more complicated case of the lattice. The purpose of the calculation in DR is twofold: First, it provides the coefficients $Z^{DR,MSS}_{ii}$; second, and most important, it leads to the MSS-renormalized Green’s functions, which are then necessary for extracting the real quantities of interest: $Z^{L,MSS}_{ii}$.

An immediate, well-known by-product of $Z^{DR,MSS}_{ii}$ is the anomalous dimension $\tilde{\gamma}_{CM}$ for the operator $\tilde{\mathcal{O}}_{CM} = m \mathcal{O}_{CM}$ : $\tilde{\gamma}_{CM} = g^2/(16 \pi^2) \cdot (4 N_c - 8/N_c)$. We find that the mixing coefficients for the gauge noninvariant operators $\mathcal{O}_9, \mathcal{O}_{10}$ do not vanish.

The lattice relations are formally again Eqs. (7); however all $Z$’s now stand for $Z^{L,MSS}$, and the bare Green’s functions must be calculated using the lattice regularization.
It is a highly nontrivial task to show that the lattice bare Green’s functions, once renormalized, will coincide with those coming from DR: The $\overline{\text{MS}}$-renormalized 3-point function has already an extremely complicated dependence on momenta and masses (involving Spence functions even for $m = 0$), while the lattice bare 3-point function contains $\sim 10^3$ loop integrals depending on masses and external momenta. We have shown this equality in the most general case.

Our results, in the case of the Iwasaki gluon action, read ($\tilde{g}^2 \equiv g^2/(16\pi^2)$):

\[
Z_{1,1}^{L,\overline{\text{MS}}} = 1 + g^2 (N_c (-7.9438 + (1/2) \log(2\mu^2)) + N_c^{-1} (4.4851 - (5/2) \log(2\mu^2))), \quad (8)
\]

\[
Z_{1,2}^{L,\overline{\text{MS}}} = \tilde{g}^2 C_F (4.5370 + 6 \log(2\mu^2), \quad Z_{1,3}^{L,\overline{\text{MS}}} = O(\tilde{g}^4), \quad Z_{1,4}^{L,\overline{\text{MS}}} = O(\tilde{g}^4),
\]

\[
Z_{1,5}^{L,\overline{\text{MS}}} = \tilde{g}^2 (N_c (4.2758 - (3/2) \log(2\mu^2)) + N_c^{-1} (-3.7777 + 3 \log(2\mu^2))), \quad Z_{1,6}^{L,\overline{\text{MS}}} = O(\tilde{g}^4),
\]

\[
Z_{1,7}^{L,\overline{\text{MS}}} = -Z_{1,9}^{L,\overline{\text{MS}}} = -Z_5^{L,\overline{\text{MS}}}/2, \quad Z_{1,8}^{L,\overline{\text{MS}}} = \tilde{g}^2 C_F (-3.7760), \quad Z_{1,10}^{L,\overline{\text{MS}}} = \tilde{g}^2 C_F (3.7777 - 3 \log(2\mu^2)),
\]

\[
Z_{1,11}^{L,\overline{\text{MS}}} = a^{-1} \tilde{g}^2 C_F (-3.2020) = -Z_{1,12}^{L,\overline{\text{MS}}}, \quad Z_{1,13}^{L,\overline{\text{MS}}} = a^{-2} \tilde{g}^2 C_F (36.0613)
\]

Preliminary non-perturbative results: In the calculation of on-shell matrix elements, by virtue of the equations of motion, some of the operators $O_1 - O_13$ will not appear. The remaining ones: $O_1, O_2, O_3, O_4, O_12, O_13$ will be present, and it is crucial to have a stringent estimate of their mixing coefficients. For operators of the same dimensionality as the CMO, i.e. $O_1 - O_4$, our one-loop results are expected to provide satisfactory accuracy; however, for operators of lower dimensionality ($O_{12}, O_{13}$), given that their coefficients are power divergent, perturbation theory is expected to provide only a ballpark estimate at best. Fortunately, it is precisely for the coefficients of these latter operators that we can have best access to non-perturbative estimates.

Imposing conditions such as:

\[
\lim_{m_s, m_d \to 0} \langle \pi(0)|O_1^{\text{sub}}|K(0)\rangle = \lim_{m_s, m_d \to 0} \langle \pi(0)|O_1 + a^{-2} c_{13} O_{13}|K(0)\rangle = 0 \quad (9)
\]

\[
\langle 0|O_1^{\text{sub}}|K(0)\rangle_{m_s, m_d} = \langle 0|O_1 + a^{-2} c_{13} O_{13} + a^{-1} c_{12} O_{12}|K(0)\rangle_{m_s, m_d} = 0 \quad (10)
\]

we can fit the values of $c_{13}(g_0), c_{12}(g_0)$ to data from simulations with varying quark masses.

In a preliminary series of simulations, we have extracted $c_{13}$ at different values of the coupling ($\beta = 6/g_0^2 = 1.90, 1.95, 2.10$).

The results for $c_{13}$ exhibit a linear plus quadratic dependence on $\tilde{g}^2$, corresponding to a dominant one-loop effect (very close to the perturbative result) plus a subleading two-loop contribution, namely:

\[
Z_{1,13}^{\text{non-pert}} \sim a^{-2} \tilde{g}^2 C_F 35.8 (1.0 - 3.0 \tilde{g}^2) \quad Z_{1,13}^{\text{pert}} = a^{-2} \tilde{g}^2 C_F (36.0613) \quad (11)
\]

Besides a series of controls which we have applied to our results, some further ones may be applied: (i) A calculation of 4-point Green’s functions will provide important consistency checks, but no new information, on $Z_{14}$. (On the other hand, $n$-point functions ($n > 4$) are irrelevant: Being superficially convergent, they have a straightforward continuum limit.) (ii) Non-perturbative estimates of all mixing coefficients would be very important cross checks.

A further extension of the present work would be to apply methods of improved perturbation theory (“boosted” coupling, “cactus” diagrams, etc.) to our results.

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