DS-MLR: Exploiting Double Separability for Scaling up Distributed Multinomial Logistic Regression

Parameswaran Raman
params@ucsc.edu

Sriram Srinivasan
ssriniv9@ucsc.edu

Shin Matsushima
shin_matsushima@mist.i.u-tokyo.ac.jp

Xinhua Zhang
xinhua.zhang.cs@gmail.com

Hyokun Yun
yunhyoku@amazon.com

S.V.N. Vishwanathan
vishy@ucsc.edu

1Department of Computer Science, UC Santa Cruz
2The University of Tokyo, Tokyo
3University of Illinois at Chicago, Chicago
4Amazon Inc., Seattle

Abstract
Scaling multinomial logistic regression to datasets with very large number of data points and classes has not been trivial. This is primarily because one needs to compute the log-partition function on every data point. This makes distributing the computation hard. In this paper, we present a distributed stochastic gradient descent based optimization method (DS-MLR) for scaling up multinomial logistic regression problems to very large data. Our algorithm exploits double-separability, an attractive property we observe in the objective functions of several models in machine learning, that allows us to achieve both data as well as model parallelism simultaneously. In addition to being parallelizable, our algorithm can also easily be made asynchronous. We demonstrate the effectiveness of our method empirically on several real-world datasets, for instance a reddit dataset with data and parameter sizes of 200 GB and 300 GB respectively.

1 Introduction
As the model sizes in machine learning grew from small to medium, data parallel algorithms became fashionable to handle the large data sizes. However, we are at a point now, where increasingly sophisticated models [e.g. [2],[6] are being built that are huge in size. As a result, it has become inevitable to build distributed algorithms which are both data as well as model parallel simultaneously.

In this paper, we are concerned with multinomial logistic regression on large datasets, in the presence of large number of classes. Clearly, this is an important problem and has received significant research attention. The classic paradigm in distributed machine learning is to perform data partitioning, using, for instance, a map reduce style architecture. In other words, the data is distributed across multiple slaves. At the beginning of each iteration, the master distributes a parameter vector to all the slaves, who in turn use this to compute the objective function and gradient values on their part of the data and transmit it back to the master. The master aggregates the results from the slaves and updates the
Table 1: Characteristics of the various algorithms when applied to multinomial logistic regression. (N : # of data points, D : # of features, K : # of classes, P : # of workers).

| Algorithm | Storage per worker | Communication | Data Parallel | Model Parallel |
|-----------|--------------------|---------------|---------------|----------------|
| L-BFGS    | $O(\frac{ND}{P})$ | $O(KD)$       | Yes           | No             |
| ADMM      | $O(\frac{ND}{P})$ | $O(KD + O(\frac{NK}{P})$ | Yes           | No             |
| LC        | $O(ND)$           | $O(\frac{KD}{P}) + O(\frac{N}{P})$ | No            | Yes            |
| DS-MLR    | $O(\frac{ND}{P})$ | $O(\frac{KD}{P}) + O(\frac{K}{P})$ | Yes           | Yes            |

The rest of the paper is organized as follows: Section 2 formally introduces Multinomial Logistic Regression (MLR). Section 3 presents our reformulation (DS-MLR). In section 4, we discuss how our doubly separable objective function can be optimized in a distributed fashion and present synchronous and asynchronous algorithms for it. Section 6 discusses our contributions in the context of related work. Section 7 is devoted to experiments using our asynchronous algorithm in both single-machine and multi-machine settings. Finally, Section 8 concludes the paper.

## 2 Multinomial Logistic Regression

Suppose we are provided training data which consists of $N$ data points $(x_1, y_1), \ldots, (x_N, y_N)$ where $x_i \in \mathbb{R}^d$ is a $d$-dimensional feature vector and $y_i \in \{1, 2, \ldots, K\}$ is a label associated with it; $K$ denotes the number of class labels. Let’s also define an indicator variable $y_{ik} = I(y_i = k)$ denoting the membership of data point $x_i$ to class $k$. The probability that $x_i$ belongs to class $k$ is given by:

$$p(y = k| x_i) = \frac{\exp(w_{ik}^T x_i)}{\sum_{j=1}^{K} \exp(w_{jk}^T x_i)}$$

(1)
where $W = \{w_1, w_2, \ldots, w_K\}$ denotes the parameter vector for each of the $K$ classes. Using the negative log-likelihood of (1) as a loss function, the objective function of MLR can be written as:

$$L_1(W) = \frac{\lambda}{2} \sum_{k=1}^{K} ||w_k||^2 - \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{K} y_{ik} w_k^T x_i + \frac{1}{N} \sum_{i=1}^{N} \log \left( \sum_{k=1}^{K} \exp(w_k^T x_i) \right),$$  \hspace{1cm} (2)$$

where $||w_k||^2$ regularizes the objective, and $\lambda$ is a tradeoff parameter. Optimizing the above objective function (2) when the number of classes $K$ is large, is extremely challenging as computing the log partition function involves summing up over a large number of classes. In addition, it couples the class level parameters $w_k$ together, making it difficult to distribute computation. In this paper, we present an alternative formulation for MLR, to address this challenge.

3 Doubly-Separable Multinomial Logistic Regression (DS-MLR)

In this section, we present a reformulation of the MLR problem, which is closer in spirit to dual-decomposition methods [5]. We begin by first rewriting (2) as,

$$L_1(W) = \frac{\lambda}{2} \sum_{k=1}^{K} ||w_k||^2 - \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{K} y_{ik} w_k^T x_i - \frac{1}{N} \sum_{i=1}^{N} \log \frac{1}{\sum_{k=1}^{K} \exp(w_k^T x_i)},$$  \hspace{1cm} (3)$$

This can be expressed as a constrained optimization problem,

$$L_1(W, A) = \frac{\lambda}{2} \sum_{k=1}^{K} ||w_k||^2 - \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{K} y_{ik} w_k^T x_i - \frac{1}{N} \sum_{i=1}^{N} \log a_i,$$

s.t. $a_i = \frac{1}{\sum_{k=1}^{K} \exp(w_k^T x_i)}$, \hspace{1cm} (4)

where $A = \{a_i\}_{i=1, \ldots, N}$.

Observe that this resembles dual-decomposition methods of the form:

$$\min_{x, z} f(x) + g(z) \text{ s.t. } Ax + Bz = c,$$

where $f$ and $g$ are convex functions. In our objective function (4), the decomposable functions are $f(W)$ and $g(A)$ respectively. Introducing Lagrange multipliers, $\beta_i$, $i = 1, 2 \ldots N$, we obtain the equivalent unconstrained minimax problem,

$$L_2(W, A, \beta) = \frac{\lambda}{2} \sum_{k=1}^{K} ||w_k||^2 - \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{K} y_{ik} w_k^T x_i - \frac{1}{N} \sum_{i=1}^{N} \log a_i + \sum_{i=1}^{N} \sum_{k=1}^{K} \beta_i a_i \exp(w_k^T x_i) - \sum_{i=1}^{N} \beta_i$$

Dual-decomposition methods can reliably find a stationary point, and therefore the solution obtained by our method is also globally optimal. We provide a detailed proof of convergence in section 5. The updates for the primal variables $W$, $A$ and dual variable $\beta$ can be written as follows:

$$W_{k+1}^{t+1} \leftarrow \arg\min_{W_k} L_2(W_k, a^t, \beta^t),$$  \hspace{1cm} (6)$$

$$a_{i+1}^{t+1} \leftarrow \arg\min_{a_i} L_2(W_k^{t+1}, a_i^t, \beta_i^t),$$  \hspace{1cm} (7)$$

$$\beta_i^{t+1} \leftarrow \beta_i^t + \rho \left( a_i^{t+1} \sum_{k=1}^{K} \exp(w_k^{t+1} x_i) - 1 \right)$$

Here, $W_{k+1}^{t+1}$ and $a_{i+1}^{t+1}$ can be obtained by any black-box optimization procedure, while $\beta_i^{t+1}$ is updated via dual-ascent using a step-length $\rho$. Intuitively, the dual-ascent update of $\beta$ penalizes any violation of the constraint in problem (4).

We now make the following interesting observations in these updates:

**Update for $a_i^{t+1}$:** When (7) is solved to optimality, $a_i$ admits an exact closed-form solution given by,

$$a_i = \frac{1}{\sum_{k=1}^{K} \exp(w_k^T x_i)},$$  \hspace{1cm} (9)$$

3
Update for $\beta_i^{t+1}$: As a consequence of the above exact solution for $a_i$, the dual-ascent update for $\beta$ is no longer needed, since the penalty is always zero during such a projection. We therefore set $\beta_i$ to a constant equal to 1.

Update for $W_k^{t+1}$. This is the only update that we need to handle numerically. $L_2(W, A)$ can be first written in this form,

$$L_2(W, B) = \sum_{i=1}^{N} \sum_{k=1}^{K} \left( \frac{\lambda}{2N} \| w_k \|^2 - \frac{1}{N} y_{ik} w_k^T x_i - \frac{1}{N} b_i + \exp(w_k^T x_i + b_i) - \frac{1}{K} \right)$$

(10)

where we denote $b_i = \log(a_i)$ for convenience and $B = \{b_i\}_{i=1,...,N}$. The objective function is now doubly-separable \[18\] since,

$$L_2(w_1, \ldots, w_K, b_1, \ldots, b_N) = \sum_{i=1}^{N} \sum_{k=1}^{K} f_{ki}(w_k, b_i)$$

(11)

Stochastic Optimization: Minimizing $L_2(W, B)$ involves computing the gradients of \[10\] w.r.t. $w_k$ which is often computationally expensive. Instead, one can compute stochastic gradients \[15\] which are computationally cheaper than the exact gradient, and perform stochastic updates as follows:

$$w_k \leftarrow w_k - \eta K \left( \lambda w_k - y_{ik} x_i + \exp(w_k^T x_i + b_i) x_i \right)$$

(12)

where $\eta$ is the learning rate for $w_k$. Being an unbiased stochastic gradient estimator, the standard convergence guarantees of SGD apply here \[10\]. Our formulation of DS-MLR \[10\] offers several key advantages:

- **First**, observe that the objective function $L_2(W, B)$, now splits as summations over $N$ data points and $K$ classes. This means, each term in stochastic updates only depends on one data point $i$ and one class $k$. We exploit this to achieve simultaneous data and model parallelism.
- **Second**, we are able to update the variational parameters $b_i$ in closed-form, avoiding noisy stochastic updates. This improves our overall convergence.
- **Third**, Our formulation lends itself nicely to an asynchronous implementation. Section 4.2 describes this in more detail.

4 Distributing the Computation of DS-MLR

4.1 DS-MLR Sync

We first describe the distributed DS-MLR Synchronous algorithm in Algorithm\[1\]. The data and parameters are distributed among the $P$ processors as illustrated in Figure\[1\] where the row-blocks and column-blocks represent data $X^{(p)}$ and weights $W^{(p)}$ on each local machine respectively. The algorithm proceeds by running $T$ iterations in parallel on each of the $P$ workers arranged in a ring network topology.

Figure 1: $P = 4$ inner-epochs of distributed SGD. Each worker updates mutually-exclusive blocks of data and parameters as shown by the dark colored diagonal blocks \[7\].

Each iteration consist of $2P$ inner-epochs. During the first $P$ inner-epochs, each worker sends/receives its parameters $W^{(p)}$ to/from the adjacent machine and performs stochastic $W^{(p)}$ updates using the block of data $X^{(p)}$ and parameters $W^{(p)}$ that it owns. The second $P$ inner-epochs are used to pass around the $W^{(p)}$ to compute the $b^{(p)}$ exactly using \[9\].
Algorithm 1 DS-MLR Synchronous

1: $K$: # classes, $P$: # workers, $T$: total outer iterations, $t$: outer iteration index, $s$: inner epoch index
2: $W(p)$: weights per worker, $b(p)$: variational parameters per worker
3: Initialize $W(p) = 0, b(p) = \frac{1}{K}$
4: for all $p = 1, 2, \ldots, P$ in parallel do
5:   for all $t = 1, 2, \ldots, T$ do
6:     for all $s = 1, 2, \ldots, P$ do
7:       Send $W(p)$ to worker on the right
8:       Receive $W(p)$ from worker on the left
9:       Update $W(p)$ stochastically using (12)
10: end for
11: for all $s = 1, 2, \ldots, P$ do
12:   Send $W(p)$ to worker on the right
13:   Receive $W(p)$ from worker on the left
14: Compute partial sums
15: end for
16: Update $b(p)$ exactly (9) using the partial sums
17: end for
18: end for

4.2 DS-MLR Async

The performance of DS-MLR can be significantly improved by performing computation and communication in parallel. Based on this observation, we present an asynchronous version of DS-MLR. Due to the double-separable nature of our objective function (10), we can readily apply the NOMAD algorithm proposed in [20]. The entire DS-MLR Async algorithm is described in Algorithm 2.

Algorithm 2 DS-MLR Asynchronous

1: $K$: total # classes, $P$: total # workers, $T$: total outer iterations, $W(p)$: weights per worker
2: $b(p)$: variational parameters per worker, queue[P]: array of $P$ worker queues
3: Initialize $W(p) = 0, b(p) = \frac{1}{K}$ //Initialize parameters
4: for $k \in W(p)$ do
5:   Pick $q$ uniformly at random
6:   queue[q].push((k, w_k)) //Initialize worker queues
7: end for
8: //Start $P$ workers
9: for all $p = 1, 2, \ldots, P$ in parallel do
10:   for all $t = 1, 2, \ldots, T$ do
11:     repeat
12:       $(k, w_k) \leftarrow$ queue[p].pop()
13:       Update $w_k$ stochastically using (12)
14:     Compute partial sums
15:     Compute index of next queue to push to: $\hat{q}$
16:     queue[$\hat{q}$].push((k, w_k))
17:     until # of updates is equal to $K$
18:   Update $b(p)$ exactly (9) using the partial sums
19: end for
20: end for

The algorithm begins by distributing the data and parameters among $P$ workers in the same fashion as in the synchronous version. However, here we also maintain $P$ worker queues. Initially the parameters $W(p)$ are distributed uniformly at random across the queues. The workers subsequently can run their updates in parallel as follows: each one pops a parameter $w_k$ out the queue, updates it stochastically and pushes it into the queue of the next worker. Simultaneously, each worker also records the partial sum that is required for updating the variational parameters. This process repeats
until $K$ updates have been made which is equivalent to saying that each worker has updated every parameter $w_k$. Following this, the worker updates all its variational parameters $b^{(p)}$ exactly using the partial sums $\tilde{f}$. For simplicity of explanation, we restricted Algorithm 2 to $P$ workers on a single-machine. However, in our actual implementation, we follow a Hybrid Architecture. This means that there are multiple threads running on a single machine in addition to multiple machines sharing the load across the network. Therefore, in this setting, each worker (thread) first passes around the parameter $w_k$ across all the threads on its machine. Once this is completed, the parameter is tossed onto the queue of the first thread on the next machine.

![Illustration of the communication pattern in DS-MLR Async algorithm](image)

Figure 2: Illustration of the communication pattern in DS-MLR Async algorithm

## 5 Convergence

Although the semi-stochastic nature of DS-MLR makes it hard to directly apply the existing convergence results, under standard assumptions, it can be shown that it finds $\epsilon$ accurate solutions to the original objective $L_1$ in $T = O(1/\epsilon^2)$ iterations.

**Theorem 1** Suppose all $\|x_i\| \leq r$ for a constant $r > 0$. Let the step size $\eta$ in (12) decay at the rate of $1/\sqrt{t}$. Then, there exists a constant $C$ independent of $N, K, D$ and $P$, such that

$$\min_{t=1, \ldots, T} L_1(W^t) - L_1(W) \leq \frac{C}{\sqrt{t}}, \quad \forall W,$$

where $W^t$ is value of $W$ at the end of the iteration $t$ and $x_i$ denotes the data point. $N, K, D,$ and $P$ denote the number of data points, classes, dimensions and workers respectively.

It is worth noting that this rate of convergence is independent of the size of the problem. In particular, it is invariant to $P$, the number of workers. Therefore, as more workers become available, the computational cost per iteration can be effectively distributed without sacrificing the overall convergence rate, up to the point where communication cost becomes dominant. Detailed proof is relegated to the Appendix A. As a side note, a linear rate of convergence can also be obtained for DS-MLR by following proofs on the lines of SVRG [9].

## 6 Related Work

In this section, we characterize parallel algorithms for machine learning and discuss related work, thereby putting our DS-MLR method in perspective.

**(i) Batch vs. Online:** *Batch* gradient descent methods have several advantages. Firstly, the gradient is less noisy than stochastic updates as it is averaged over a large number of samples; Secondly, it can use optimized matrix operations in the gradient computations. On the downside, it does not scale very well to large number of data points [3]. *Stochastic* gradient descent method (SGD) on the other hand, converges faster as it computes the gradient of one randomly chosen instance per iteration and makes frequent parameter updates [3]. A bigger challenge here is tuning
the learning rate as discussed in [21]. A nice tradeoff between these two approaches is to use Mini-batch stochastic gradient where the gradient is computed over manageable batches of the data. DS-MLR is a stochastic approach. (ii) Exact vs. Inexact: Our work is not the first paper making use of delayed updates for stochastic gradient descent. In [23] delayed updates were explored for SGD in the online setting along with theoretical proofs for their convergence. (iii) Single vs. Multi Machine: Distributed memory approaches offer the potential for much greater improvements than single-machine approaches as they can scale to datasets which struggle to fit on one machine, but they suffer from bandwidth issues arising as a result of communication across machines. Our approach DS-MLR can work in shared, distributed and hybrid settings. Several algorithms for parallelizing SGD have been proposed in the past such as Hogwild [14], Parallel SGD [24], DSGD [7], FPSGD [22] and more recently, Parameter Server [11] and Petuum [17]. Although the importance of data and model parallelism has been recognized in Parameter Server and the Petuum framework [17], to the best of our knowledge this has not been exploited in their specific instantiations such as applications to multinomial logistic regression [16]. We believe this is because [16] does not reformulate the problem like the way DS-MLR does. Several problems in machine learning are not naturally well-suited for data and model parallelism, and therefore such reformulations are essential in identifying a suitable structure. Other doubly-separable methods also exist such as NOMAD [20] for matrix completion and RoBiRank [19] for latent collaborative retrieval. (iv) Synchronous vs. Asynchronous: Synchronous approaches suffer from non-uniform performance distributions of machines where some machines might happen to be very slow at a given time, thus bringing down the performance of the entire algorithm. Asynchronous methods overcome these drawbacks. Parameter Server, HogWild [14], NOMAD are all asynchronous approaches, although they differ in other aspects. DS-MLR has both synchronous and asynchronous variants and the latter is in the spirit of NOMAD.

7 Experiments

In our empirical study, we will focus on DS-MLR Async. We use a wide scale of real-world datasets as listed in Table 2. We implemented DS-MLR in C++ using MPI and Intel TBB. To make the comparison fair, we re-implemented the LC [8] method in C++ and MPI using ALGLIB for the inner optimization. Finally, for the L-BFGS baseline, we used TAO package (from PETSc).

Figure 3: Single-machine Single-core experiments. $\lambda$, $\eta$ denote regularizer and step-size respectively.

**Hardware:** All single-machine experiments were run on a cluster with the configuration of two 8-core Intel Xeon-E5 processors and 32 GB memory per node. For multi-machine multi-core, we used Intel vLab Knights Landing (KNL) cluster with node configuration of Intel Xeon Phi 7250 CPU (272 cores, 200GB memory), connected through Intel Omni-Path (OPA) Fabric.

| Dataset       | # instances | # features | #classes |
|---------------|-------------|------------|----------|
| CLEF          | 10,000      | 80         | 63       |
| NEWS20        | 11,260      | 53,975     | 20       |
| LSHTC1-small  | 4,463       | 51,033     | 1,139    |
| LSHTC1-large  | 93,805      | 347,256    | 12,294   |
| WikipediaLarge| 2,365,436     | 20,000     | 325,056  |
| Reddit-Full   | 302,097,708 | 1,348,192  | 33,000   |

Table 2: Dataset Characteristics
7.1 Single-machine Single-Core

**DS-MLR vs L-BFGS:** L-BFGS is a highly efficient second-order method that has a rapid convergence rate. Even when pitched against such a powerful second order method, DS-MLR performs considerably well in comparison. In fact, on some datasets such as NEWS20, DS-MLR is almost on par with L-BFGS in terms of decreasing the objective and also achieves a better f-score much more quickly. Figure 5 shows the progress of objective function as a function of time for DS-MLR, L-BFGS and LC on NEWS20, CLEF, LSHTC1-small datasets. The corresponding plots showing f-score vs time are available in Appendix [D]. However, L-BFGS loses its applicability when the number of parameters increases beyond what can fit on a single-machine.

**DS-MLR vs LC:** DS-MLR consistently shows a faster decrease in objective compared to LC on all three datasets: NEWS20, LSHTC1-small and CLEF. In fact, LC has a tendency to stall towards the end and progresses very slowly to the optimal objective value. In CLEF dataset, to reach an optimal value of 0.398, DS-MLR takes 1,262 secs while LC takes 21,003 secs. Similarly, in LSHTC1-small, to reach an optimal value of 0.065, DS-MLR takes 1,191 secs while LC takes 32,624 secs.

7.2 Single-machine Multi-Core

**DS-MLR vs LC:** L-BFGS requires all its parameters to fit on one machine and is therefore not suited for model parallelism (even on modestly large datasets such as LSHTC1-large, \( \approx 4.2 \) billion parameters need to be stored demanding \( \approx 34 \) GB). Thus, parallelizing L-BFGS would involve duplicating 34 GB of parameters across all its processors. We ran DS-MLR using 4 nodes \( \times \) 1 mpi task \( \times \) 12 threads. To provide reasonable amount of parallelism to LC, we ran it with 4 nodes \( \times \) 12 mpi tasks each. Figure 4 (left) shows how the objective function changes vs time for DS-MLR and LC on the LSHTC1-large dataset. As can be seen, DS-MLR out performs LC by a wide-margin despite the advantage LC has by duplicating data across all its processors.

7.3 Multi-machine Multi-Core

**Wiki-large:** We ran DS-MLR on wikilarge using 5 nodes \( \times \) 5 mpi tasks \( \times \) 20 threads. The progress in decreasing the objective function value is shown in Figure 4. We ran LC method using a similar configuration with MPI, however after nearly 20 hours it did not complete even a single iteration. In the same amount of time, DS-MLR was able to complete 9 iterations with the objective value changing in second decimals.

**Reddit datasets:** In this sub-section, we demonstrate the capability of DS-MLR to solve a multi-class classification problem of massive scale, using a bag-of-words dataset RedditFull created out of 1.7 billion reddit user comments spanning the period 2007-2015. Our aim is to classify a particular reddit comment (data point) into a suitable sub-reddit (class). The data and model parameters occupy 200 GB and 300 GB respectively. Therefore, both L-BFGS and LC cannot be applied here. The result of running DS-MLR on Reddit-Full is shown in Figure 4.

**Scalability and Predictive Performance:** DS-MLR exhibits near ideal scaling as more workers are supplied to the algorithm. Appendix [B] presents some plots. In Appendix [C] we plot the cumulative distribution of predictive ranks of the test labels. This is a proxy for the precision@k curve (if an algorithm performs well, most of the probability mass in the distribution of its test ranks is covered within the top-k). We find that DS-MLR provides a good accuracy at topk.
8 Conclusion

In this paper, we present a new stochastic optimization algorithm (DS-MLR) to solve multinomial logistic regression problems having large number of examples and classes, by a reformulation that makes it both data and model parallel simultaneously. As a result, DS-MLR can scale to arbitrarily large datasets where to the best of our knowledge, many of the existing distributed algorithms cannot be applied. Our algorithm is distributed, asynchronous and avoids bulk-synchronization overheads. To demonstrate this, we run DS-MLR on a massive dataset with 200 GB data and 300 GB parameters respectively to perform multi-class classification. Our work promises several interesting future directions in areas such as extreme multi-label classification and graphical models.

References

[1] Dimitri P Bertsekas. Nonlinear programming. 1999.
[2] Leon Bottou. Large-scale machine learning with stochastic gradient descent. In Proceedings of COMPSTAT'2010. Springer, 2010.
[3] Leon Bottou and Olivier Bousquet. The tradeoffs of large-scale learning. Optimization for Machine Learning, page 351, 2011.
[4] Stephen Boyd, Neal Parikh, Eric Chu, Borja Peleato, and Jonathan Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. Foundations and Trends® in Machine Learning, 3(1):1–122, 2011.
[5] Stephen Boyd and Lieven Vandenberghe. Convex Optimization. Cambridge University Press, Cambridge, England, 2004.
[6] Trishul Chilimbi, Yutaka Suzue, Johnson Apacible, and Karthik Kalyanaraman. Project adam: Building an efficient and scalable deep learning training system. In OSDI, 2014.
[7] Rainer Gemulla, Erik Nijkamp, Peter J Haas, and Yannis Sismanis. Large-scale matrix factorization with distributed stochastic gradient descent. ACM, 2011.
[8] Siddharth Gopal and Yiming Yang. Distributed training of large-scale logistic models. In Proceedings of the 30th International Conference on Machine Learning (ICML-13), pages 289–297, 2013.
[9] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In Advances in Neural Information Processing Systems, 2013.
[10] Harold Kushner and G George Yin. Stochastic approximation and recursive algorithms and applications, volume 35. Springer Science & Business Media, 2003.
[11] Mu Li, Li Zhou, Zichao Yang, Aaron Li, Fei Xia, David G Andersen, and AJ Smola. Parameter server for distributed machine learning. In Big Learning NIPS Workshop, 2013.
[12] Tomas Mikolov, Ilya Sutskever, Kai Chen, Greg Corrado, and Jeffrey Dean. Distributed representations of words and phrases and their compositionality. In Chris Burges, Leon Bottou, Max Welling, Zoubin Ghahramani, and Kilian Weinberger, editors, Advances in Neural Information Processing Systems 26, 2013.
[13] Jorge Nocedal and Stephen J. Wright. Numerical Optimization. Springer Series in Operations Research. Springer, 2nd edition, 2006.
[14] Benjamin Recht, Christopher Re, Stephen Wright, and Feng Niu. Hogwild: A lock-free approach to parallelizing stochastic gradient descent. In Advances in Neural Information Processing Systems, pages 693–701, 2011.
[15] Herbert E. Robbins and Sutton Monro. A stochastic approximation method. Annals of Mathematical Statistics, 22:400–407, 1951.
[16] Pengtao Xie, Jin Kyu Kim, Yi Zhou, Qirong Ho, Abhimanyu Kumar, Yaoliang Yu, and Eric P. Xing. Distributed machine learning via sufficient factor broadcasting. CoRR, 2015.
[17] Eric P Xing, Qirong Ho, Wei Dai, Jin Kyu Kim, Jiniang Wei, Seunghak Lee, Xun Zheng, Pengtao Xie, Abhimanyu Kumar, and Yaoliang Yu. Petuum: a new platform for distributed machine learning on big data. Big Data, IEEE Transactions on, 2015.
[18] Hyokun Yun. Doubly Separable Models. PhD thesis, Purdue University West Lafayette, 2014.
[19] Hyokun Yun, Parameswaran Raman, and S Vishwanathan. Ranking via robust binary classification. In Advances in Neural Information Processing Systems, 2014.

[20] Hyokun Yun, Hsiang-Fu Yu, Cho-Jui Hsieh, SVN Vishwanathan, and Inderjit Dhillon. Nomad: Non-locking, stochastic multi-machine algorithm for asynchronous and decentralized matrix completion. 2013.

[21] Matthew D. Zeiler. Adadelta: An adaptive learning rate method. CoRR, abs/1212.5701, 2012.

[22] Yong Zhuang, Yu-Chin Juan, and Chih-Jen Lin. A fast parallel stochastic gradient method for matrix factorization in shared memory systems. 2013.

[23] Martin Zinkevich, John Langford, and Alex J. Smola. Slow learners are fast. In Advances in Neural Information Processing Systems 22. 2009.

[24] Martin Zinkevich, Markus Weimer, Lihong Li, and Alex J Smola. Parallelized stochastic gradient descent. In Advances in Neural Information Processing Systems, pages 2595–2603, 2010.
In the following sections, we provide a more detailed proof of convergence for our algorithm, present some scaling plots and and also include additional plots from our empirical study.

A Rates of convergence

First the diameter of $W$ space can be bounded by a universal constant (independent of $N, D, K$) because we can always enforce that $\frac{1}{2\lambda} \|W\|^2 \leq f(0) = \log K$ (ignoring log term). We also assume all $x_i$ are bounded in $L_2$ norm by some constant $r$. We will write $r$ as a constant everywhere. They are not necessarily equal; in fact we may write $r^2$ and $2r$ as $r$. It just stands for some constant that is independent of $\epsilon, D, N$ and $K$.

We index outer iteration by superscript $t$ and inner-epochs within each outer iteration by subscript $k$. So $W^t_k = W^{t+1}_{N+1}$, which we also denote as $W^t$. We consider optimizing the objective

$$L_1(W) = F(W) = \frac{1}{N} \sum_{i=1}^{N} f_i(W),$$

where $f_i(W) = \frac{\lambda}{2} \|W\|^2 - w_{yi}^T x_i + \log \sum_{k=1}^{K} \exp(w_k^T x_i)$. Clearly $f_i$ has a variational representation

$$f_i(W) = \frac{\lambda}{2} \|W\|^2 - w_{yi}^T x_i + \min_{a_i \in \mathbb{R}} \left\{ -a_i + \sum_{k=1}^{K} \exp(w_k^T x_i + a_i) \right\} - 1,$$

where the optimal $a_i$ is attained at $-\log \sum_{k=1}^{K} \exp(w_k^T x_i)$. So given $W$, we can first compute the optimal $a_i$, and then use it to compute the gradient of $f_i$ via the variational form (Danskin’s theorem [11]).

$$\frac{\partial}{\partial w_k} f_i(W) = \lambda w_k - [y_i = k] x_i + \exp(w_k^T x_i + a_i) x_i.$$  

(16)

Here $[\cdot] = 1$ if $\cdot$ is true, and 0 otherwise.

Due to the distributed setting, we are only able to update $a_i$ to their optimal value at the end of each epoch (i.e. based on $W^t$):

$$a^t_i = a_i(W^t) = -\log \sum_{k=1}^{K} \exp(x_i^T w_k^t).$$

(17)

We are not able to compute the optimal $a_i$ for the latest $W$ when incremental gradient is performed through the whole dataset. Fortunately, since $W$ is updated in an epoch by a fixed (small) step size $\eta$, it is conceivable that the $a_i$ computed from $W^t$ will not be too bad as a solution in $[15]$ for $W_k^t, k \in [m]$. In fact, if $\|W_k^t - W^t\|$ is order $O(\eta t)$, then the following Lemma says the gradient computed from $W^t$ using the out-of-date $a_i$ is also $O(\eta t)$ away from the true gradient at $W_k^t$.

**Lemma 2** Denote the approximate gradient of $f_i$ evaluated at $W_k^t$ based on $a_i^t$ as

$$\hat{G}^t_k = (\hat{g}_1, \ldots, \hat{g}_K),$$

(18)

where $\hat{g}_c = \lambda w_{c} - [y_i = c] x_i + \exp(x_i^T w_{k,c}^t + a_i^t) x_i$. Then

$$\|\hat{G}^t_k - \nabla W f_i(W_k^t)\| \leq \frac{r}{\sqrt{K}} \|W_k^t - W^t\|.$$  

**Proof** Unfolding the term $a_i^t$ from (17),

$$\hat{g}_c = \frac{\partial}{\partial w_c} f_i(W_k^t) = \left( \frac{\exp(x_i^T w_{c,k}^t)}{\sum_{c=1}^{K} \exp(x_i^T w_{c}^t)} - \frac{\exp(x_i^T w_{c,k}^t)}{\sum_{c=1}^{K} \exp(x_i^T w_{c,k}^t)} \right) x_i$$

Therefore

$$\|\hat{G} - \nabla W f_i(W_k^t)\| \leq r \sqrt{K} \left| \frac{1}{\sum_{c=1}^{K} \exp(x_i^T w_{c}^t)} - \frac{1}{\sum_{c=1}^{K} \exp(x_i^T w_{c,k}^t)} \right|$$

(11)
So it suffices to upper bound the gradient of $1/\sum_{c=1}^K \exp(x_i^T w_c)$. Since $x_i$ and $w_c$ are bounded, $\exp(x_i^T w_c)$ is lower bounded by a positive universal constant. Now,

$$
\left\| \nabla_w \frac{1}{\sum_{c=1}^K \exp(x_i^T w_c)} \right\| = \frac{1}{(\sum_{c=1}^K \exp(x_i^T w_c))^2} \left\| (\exp(x_i^T w_1)x_i, \ldots, \exp(x_i^T w_K)x_i) \right\|
\leq \frac{\sqrt{K}}{K^2 r}.
$$

Using Lemma 2, we can now show that our algorithm achieves $O(1/\epsilon^2)$ epoch complexity, with no dependency on $m, d, K$. In fact, we just apply Nedic’s algorithm and analysis on $F(W)$. However, we need to adapt their proof a little because they assume the gradients are exact.

First we need to bound some quantities. $\|\nabla f_i(W)\| \leq r$ because $W$ is bounded, and for $K$ numbers $p_1, \ldots, p_K$ on a simplex with $\sum_{c=1}^K p_c = 1$, we have $\sum_{c=1}^K p_c^2 \leq 1$. Without loss of generality, suppose $f_k$ is used for update at step $k$. Then $W_k^t$ is subtracted by $\frac{\eta_t}{m}(\lambda W_k^t x_k \otimes e'_y + \tilde{G}_k^t)$, where $\otimes$ is Kroneker product and $e_c$ is a canonical vector. As long as $\eta_t \leq \frac{1}{K}$, we can recursively apply Lemma 2 and derive bounds

$$
\|W_k^t - W^t\| \leq \frac{k}{m} \eta r, \quad (19)
\|\nabla f_k(W_k^t) - \tilde{G}_k^t\| \leq \eta r, \quad (20)
\|\tilde{G}_k^t\| \leq r, \quad (21)
$$

for all $k$. Now we run Nedic’s proof. Then for any $W$

$$
\|W_{k+1}^t - W\|^2 = \|W_k^t - \frac{\eta_t}{m} \tilde{G}_k^t - W\|^2
= \|W_k^t - W\|^2 - 2 \frac{\eta_t}{m} \langle \tilde{G}_k^t, W_k^t - W \rangle + \frac{\eta_t^2}{m^2} \|\tilde{G}_k^t\|^2
= \|W_k^t - W\|^2 - 2 \frac{\eta_t}{m} \langle \nabla f_k(W_k^t), W_k^t - W \rangle + \langle \tilde{G}_k^t - \nabla f_k(W_k^t), W_k^t - W \rangle
+ \frac{\eta_t^2}{m^2} \|\tilde{G}_k^t\|^2
\leq \|W_k^t - W\|^2 - 2 \frac{\eta_t}{m} \langle f_k(W_k^t) - f_k(W) \rangle - \eta_t r + \frac{\eta_t^2}{m^2} r^2.
$$

Telescoping over $k = 1, \ldots, m$, we obtain that for all $W$ and $t$:

$$
\|W_{t+1}^t - W\|^2 \leq \|W^t - W\|^2 - 2 \frac{\eta_t}{m} \sum_{k=1}^m \langle f_k(W_k^t) - f_k(W) \rangle + \eta_t^2 r
\leq \|W^t - W\|^2 - 2 \eta_t \left( F(W^t) - F(W) + \frac{1}{m} \sum_{k=1}^m \langle f_k(W_k^t) - f_k(W^t) \rangle \right) + \eta_t^2 r.
$$

\footnote{If one is really really meticulous and notes that $\|W\|^2 \leq 2\lambda \log K$ which does involve $K$, one should be appalled that $\exp(\sqrt{\log K}$) is $o(K^{\alpha})$ for any $\alpha > 0$.}
Using the fact that \( \nabla f_k \) is bounded by a universal constant, we further derive

\[
\|W^{t+1} - W\|^2 \leq \|W^t - W\|^2 - 2\eta_t (F(W^t) - F(W)) + 2\eta_t m \sum_{k=1}^m \|W_k^t - W^t\| + \eta_t^2 r \\
\leq \|W^t - W\|^2 - 2\eta_t (F(W^t) - F(W)) + 2\eta_t m \sum_{k=1}^m \frac{k}{m} \eta_t^2 r + \eta_t^2 r \\
= \|W^t - W\|^2 - 2\eta_t (F(W^t) - F(W)) + \eta_t^2 r.
\]

Now use the standard step size of \( O(1/\sqrt{t}) \), we conclude

\[
\min_{t=1, \ldots, T} F(W^t) - F(W) \leq \frac{r}{\sqrt{T}}.
\]

Note the proof has not used the convexity of \( a_i \) in \( [15] \) at all. This is reasonable because it is “optimized out”.

### B Scaling behavior of DS-MLR as a function of the number of workers

In Figure 5, we plot the objective function as well as the f-score as we vary the the number of workers (# mpi tasks \( \times \) # threads). We first increase the number of threads on a single machine (as a single mpi task) as 1, 2, 4, 8, 16, 20. Next, we run two mpi tasks with 10 each and four mpi tasks with 5 threads each. In an ideal scenario with linear scaling, we would expect all the figures to overlap with each other. As can be seen, DS-MLR exhibits near ideal speed up when the number of workers are increased.

![Figure 5: Scaling of DS-MLR on LSHTC1-large as the # of workers are varied](image-url)
C Rank Distribution

In this section, we plot the cumulation distribution of ranks of test labels. This is a proxy for the precision@k curve and gives a more closer indication of the predictive performance of a multinomial classification algorithm. In Figures 6 and 7, we plot the precision obtained after the first 5 iterations (denoted by dashed lines), and after the end of optimization (denoted by solid lines). As seen, DS-MLR performs competitively and in general tends to give a good accuracy within the first 5 iterations.

Figure 6: Cumulative distribution of predictive ranks of the test labels for the three small datasets

Figure 7: Cumulative distribution of predictive ranks of the test labels for the larger datasets: LSHTC1-large
D Additional Plots

Below we show how the macro and micro f-score changes as a function of time on the various datasets. We would like to point out that in some cases (for e.g. in Figures 10 and 11), L-BFGS and LC start off at a high f-score and then gradually come down the final value. We believe this could be a case of overfitting. We observed that the predictive performance of L-BFGS and LC suffers at the points where they seem to peak in their f-score (on LSHTC1-small, we verified this by plotting the predictive rank distribution, like the one we discuss in Appendix C). DS-MLR does not have this problem.

Figure 8: (Left): objective vs time, (Center): test micro F1 vs time, (Right): test macro F1 vs time

Figure 9: (Left): objective vs time, (Center): test micro F1 vs time, (Right): test macro F1 vs time

Figure 10: (Left): objective vs time, (Center): test micro F1 vs time, (Right): test macro F1 vs time
Figure 11: (Left): Objective vs time, (Center): test micro F1 vs time, (Right): test macro F1 vs time

Figure 12: (Left): Objective vs time, (Center): test micro F1 vs time, (Right): test macro F1 vs time