Finite sample inference for generic autoregressive models

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Abstract

Autoregressive stationary processes are fundamental modeling tools in time series analysis. To conduct inference for such models usually requires asymptotic limit theorems. We establish finite sample-valid tools for hypothesis testing and confidence set construction in such settings. Further results are established in the always-valid and sequential inference framework.

1 Introduction

Let $X = (X_t)_{t \in \mathbb{Z}}$ be a stationary stochastic process, where $X_t \in \mathbb{X} \subseteq \mathbb{R}^d$ for each $t \in \mathbb{Z}$ ($d \in \mathbb{N}$), and let $\mathcal{F}_t = \sigma (X_t, X_{t-1}, \ldots)$ be the $\sigma$-algebra generated by $X_t, X_{t-1}, \ldots$. We assume that $X$ arises from a generic $p$th order ($p \in \mathbb{N}$) parametric autoregressive process, in the sense that for each $A \subseteq \mathbb{X}$,

$$\Pr (X_t \in A | \mathcal{F}_{t-1}) = \int_A f_\theta (x_t | X_{t-p}^{t-1}) \, dx_t.$$  

Here, $X_t^s = (X_r, X_{r+1}, \ldots, X_s)$ ($r \leq s$) and $f_\theta (x_t | X_{t-p}^{t-1})$ is a probability density function of $x_t$, conditional on the event $X_{t-p}^{t-1} = x_{t-p}^{t-1}$, characterized by the parameter $\theta \in \Theta$.

Suppose that we observe a subsequence $X_T = (X_t)_{t \in [T]}$ ($[T] = \{1, \ldots, T\}$; $T \in \mathbb{N}$), arising from a data generating process (DGP) characterized by an unknown parameter $\theta^*$, but with known conditional PDF $f_\theta$. Typically, we then wish to use $X_T$ to estimate $\theta^*$, to construct confidence sets with specified probabilities of containing $\theta^*$, and to test hypotheses regarding the value of $\theta^*$.

Although difficult, the problem of parametric estimation is largely resolvable via optimization of well-constructed functions of the data; see, for example, the comprehensive treatment of De Gooijer (2017, Ch. 6). The problems of confidence set construction and hypothesis testing are generally addressed via limit theorems or resampling methods for dependent processes, as described in Davidson (1994) and Potscher & Prucha (1997), and Politis et al. (1999) and Lahiri (2003), respectively.

In some simple cases, concentration inequality are applicable for the derivation of finite sample inference tools. For example, in the simple case of the univariate first order autoregressive model with normal noise, finite sample results have been derived in Vovk (2007), Bercu et al. (2015, Sec. 4.1), and Bercu & Touati (2019).

In Wasserman et al. (2020), the authors consider the construction of estimator agnostic and finite sample valid confidence sets and hypothesis tests for independent data, using the martingale property of the generalized likelihood ratio statistic (cf. de la Pena et al., 2009, Sec. 17.1). Following a note regarding conditional likelihoods, we extend the results of Wasserman et al. (2020) to address the problem of estimator agnostic and finite sample valid inference for data arising from generic autoregressive DGPs. This therefore contributes to the developing literature regarding finite sample inference for time series models.

We present two sets of results. The first set uses Markov’s inequality in order to construct confidence sets and tests via a split data setup. The second set of results uses the maximal inequality of Ville (cf. Howard et al., 2020a, Lem. 1), in order to construct always-valid sequential confidence sets and tests. These constructions are closely related to the test martingales of Shafer et al. (2011) and the recently popular notion of e-values (see, e.g. Shafer & Vovk, Ch. 10 and Vovk, 2020).

The paper proceeds as follows. In Sections 2 and 3, we present split data inference tools and always-valid inference tools, respectively. Proofs of main results are provided in Section 4. Further technical results are provided in the Appendix.

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2 Split data inference

Let $T = T_1 + T_2$, where $T_1, T_2 \in \mathbb{N}$ and $T_1 \geq p$. Suppose that $X$ arises from a DGP, characterized by an unknown parameter $\theta^* \in \Theta$, and let $\hat{\theta}_{T_1}$ be an arbitrary estimator of $\theta^*$, obtained from $X_{T_1}$. For arbitrary $\theta$, define the conditional likelihood based on $X_{T_1+1}^T$ by

$$L_{T_1+1}(\theta) = \prod_{t=T_1+1}^T f_\theta(X_t|X_{t-1}^{t-p}).$$

Using (1), we define the split data conditional likelihood ratio statistic (CLRS) by

$$U_{T_1+1}^T(\theta) = \frac{L_{T_1+1}(\hat{\theta}_{T_1})}{L_{T_1+1}(\theta)}.$$

We shall firstly consider confidence sets of the form

$$C_{T_1+1}(\alpha) = \{\theta \in \Theta : U_{T_1+1}^T(\theta) \leq 1/\alpha \},$$

for $\alpha \in [0,1]$. Let $E_{\theta^*}$ and $Pr_{\theta^*}$ denote the expectation and probability operators, evaluated under the assumption that the DGP of $X$ is characterized by the parameter of value $\theta^*$. The following result establishes the finite sample validity of such confidence constructions.

**Proposition 1.** For each $\alpha \in [0,1]$, $T_1 \geq p$ and $T_2 \in \mathbb{N}$, $C_{T_1+1}(\alpha)$ is finite sample valid 100 $(1 - \alpha)$% confidence set, in the sense that $Pr_{\theta^*}(\theta^* \in C_{T_1+1}(\alpha)) \geq 1 - \alpha$.

Next, we shall consider the problem of testing hypotheses of the form:

$$H_0 : \theta^* \in \Theta_0 \text{ and } H_1 : \theta^* \notin \Theta_0,$$

where $H_0$ and $H_1$ denote the null and alternative hypotheses, respectively, and $\Theta_0 \subset \Theta$ is a composite null set.

Define the maximum conditional likelihood estimator based on $X_{T_1+1}^T$, under $H_0$, by

$$\hat{\theta}_{T_1+1}^T \in \arg \max_{\theta \in \Theta_0} L_{T_1+1}(\theta),$$

and test (2) via the split data conditional likelihood ratio test (CLRT) rule:

$$\text{Reject } H_0 \text{ if } V_{T_1+1}^T > 1/\alpha,$$

where

$$V_{T_1+1}^T = L_{T_1+1}(\hat{\theta}_{T_1}) / L_{T_1+1}(\hat{\theta}_{T_1+1}^T).$$

We have the following result regarding the finite sample validity of the split data CLRT.

**Proposition 2.** For each $\alpha \in [0,1]$, $T_1 \geq p$ and $T_2 \in \mathbb{N}$, the split data CLRT, defined by (4), controls the Type I error at the significance level $\alpha$, in the sense that $\sup_{\theta^* \in \Theta_0} Pr_{\theta^*}(V_{T_1+1}^T > 1/\alpha) \leq \alpha$.

**Remark 1.** Instead of (4), we can also use the duality between confidence sets and tests (see, e.g., Thm 2.3 of Hochberg & Tamhane, 1987, Appendix 1) to test the hypotheses (2). That is, we can test hypotheses (2) using the rule:

$$\text{Reject } H_0 \text{ if } \Theta_0 \cap C_{T_1+1}(\alpha) = \emptyset.$$

Rule (5) replaces the optimization problem of computing (3) by potential complications regarding the derivation of the set intersect $\Theta_0 \cap C_{T_1+1}(\alpha)$. Since both tests correctly control the Type I error, the choice between the alternatives is a matter of practicality.
3 Always-valid inference

We now consider the sampling of the elements of the subsequence $X_T$, characterized by the DGP parameterized by $\theta^* \in \Theta$, one at a time. For $T > p$, let $\tilde{\theta}_T$ be a non-anticipatory estimator of $\theta^*$ (i.e., $\tilde{\theta}_T$ is only dependent on $X_T$).

We wish to use the sequence of estimators $(\tilde{\theta}_T)_{T>p}$ to sequentially test the hypotheses (2) and construct confidence sets for $\theta^*$. At any time $T > p$, define the running CLRT by the rule:

\[ \text{Reject } H_0 \text{ if } M_T > 1/\alpha, \quad (6) \]

where

\[ M_T = \frac{\prod_{t=p+1}^{T} f_{\tilde{\theta}_{t-1}} (X_t | X_{t-p}^T)}{\prod_{t=p+1}^{T} f_{\tilde{\theta}_T} (X_t | X_{t-p}^T)}, \]

and

\[ \tilde{\theta}_T = \arg \max_{\theta \in \Theta_0} \prod_{t=p+1}^{T} f_{\theta} (X_t | X_{t-p}^T). \]

We shall also define $M_T = 1$, for all $0 \leq T \leq p$.

Let $\tau_{\theta^*}$ denote the time at which the sequence of tests stops, under rejection rule (6), when the DGP of $X$ is characterized by the parameter $\theta^*$. The following result establishes that $\tau_{\theta^*}$ is finite with probability no greater than $\alpha$.

**Proposition 3.** The running CLRT, defined by (6), has Type I error at most $\alpha$. That is, $\sup_{\theta^* \in \Theta_0} \Pr_{\theta^*} (\tau_{\theta^*} < \infty) \leq \alpha$, for each $\alpha \in [0, 1]$.

Let $P_T = 1/M_T$ and $\tilde{P}_T = \min_{T \leq t} \{1/M_T\}$ be $p$-values for the test (2), and let $T \in \mathbb{N}$ be a random variable. Then, the randomly indexed $p$-values $P_T$ and $\tilde{P}_T$ are both valid.

**Proposition 4.** For any random $T \in \mathbb{N}$, not necessarily a stopping time, $P_T$ and $\tilde{P}_T$ are valid, in the sense that $\sup_{\theta^* \in \Theta_0} \Pr_{\theta^*} (P_T \leq \alpha) \leq \alpha$, and $\sup_{\theta^* \in \Theta_0} \Pr_{\theta^*} (\tilde{P}_T \leq \alpha) \leq \alpha$, for all $\alpha \in [0, 1]$.

Let

\[ D_T (\alpha) = \{ \theta \in \Theta : R_T (\theta) \leq 1/\alpha \}, \]

where

\[ R_T (\theta) = \frac{\prod_{t=p+1}^{T} f_{\tilde{\theta}_{t-1}} (X_t | X_{t-p}^T)}{\prod_{t=p+1}^{T} f_{\tilde{\theta}_T} (X_t | X_{t-p}^T)}, \]

for $T > p$ and $R_T (\theta) = 1$, for $T \leq p$. Further, let $\tilde{D}_T (\alpha) = \bigcap_{t=1}^{T} D_t^\alpha$. We have the fact that $(D_T (\alpha))_{T \in \mathbb{N}}$ and $(\tilde{D}_T (\alpha))_{T \in \mathbb{N}}$ are sequences of confidence sets that are all simultaneously valid.

**Proposition 5.** For any $\alpha \in [0, 1]$, the confidence sequences $(D_T (\alpha))_{T \in \mathbb{N}}$ and $(\tilde{D}_T (\alpha))_{T \in \mathbb{N}}$ are valid, in the sense that $\Pr_{\theta^*} (\forall T \in \mathbb{N} : \theta^* \in D_T (\alpha)) \geq 1 - \alpha$ and $\Pr_{\theta^*} (\forall T \in \mathbb{N} : \theta^* \in \tilde{D}_T (\alpha)) \geq 1 - \alpha$.

4 Proofs of main results

4.1 Split data inference

The following lemma provides the basis for the proofs of Propositions 3 and 2.

**Lemma 1.** For each $T_1 \geq p$ and $T_2 \in \mathbb{N}$, $E_{\theta^*} \left[ U_{T_1+1}^T (\theta^*) \right] = 1$. 

3
Proof. Let $X_{t-1}^{t-p} = x_{t-1}^{t-p}$, for $t > p$, and $X_{t-}^{t-p} = (X_{t-p}, \ldots, X_0, x_1, \ldots, x_{t-1})$, for $t \leq p$. Write

$$
E_{\theta^*}[U_{T_1+1}^{T} (\theta^*) | \mathcal{F}_{T_1}] = \int_{\mathcal{X}^{T_2}} \prod_{t=T_1+1}^{T} f_{\theta_{T_1}} (x_t | X_{t-1}^{t-p}) \prod_{t=T_1+1}^{T} f_{\theta^*} (x_t | X_{t-1}^{t-p}) \, dx_{T_1+1}^{T}
$$

$$
= \int_{\mathcal{X}^{T_2}} \prod_{t=T_1+1}^{T} f_{\theta_{T_1}} (x_t | X_{t-1}^{t-p}) \, dx_{T_1+1}^{T}
$$

$$
= (i) \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} f_{\hat{\theta}_{T_1}} (x_{T_1+1} | X_{T_1}^{T_1+1-p}) \, dx_{T_1+1}^{T} \cdots f_{\hat{\theta}_{T_3}} (x_T | X_{T_1-1}^{T}) \, dx_{T}
$$

where (i) is due to Tonelli’s Theorem and (ii) is due to the definition of a conditional PDF. Then, we apply the law of iterated expectations to obtain the desired result:

$$
E_{\theta^*}[U_{T_1+1}^{T} (\theta^*)] = E_{\theta^*}[E_{\theta^*}[U_{T_1+1}^{T} (\theta^*) | \mathcal{F}_{T_1}]] = 1.
$$

\[ \square \]

4.1.1 Proof of Proposition 1

For any $\theta^* \in \Theta$, we have

$$
\Pr_{\theta^*} (\theta^* \notin C_{T_1+1}^{T} (\alpha)) = \Pr_{\theta^*} (U_{T_1+1}^{T} (\theta^*) > 1/\alpha)
$$

\[ (i) \leq \alpha E_{\theta^*} [U_{T_1+1}^{T} (\theta^*)] \leq \alpha, \]

where (i) is due to Markov’s inequality and (ii) is due to Lemma 1.

4.1.2 Proof of Proposition 2

For any $\theta^* \in \Theta_0$, we have

$$
\Pr_{\theta^*} (V_{T_1+1}^{T} > 1/\alpha) \leq \alpha E_{\theta^*} [V_{T_1+1}^{T}]
$$

\[ (i) \leq \alpha E_{\theta^*} [U_{T_1+1}^{T} (\theta^*)] \leq \alpha, \]

where (i) is due to Markov’s inequality, and (ii) is due to the fact that

$$
L_{T_1+1}^{T} (\hat{\theta}_{T_1+1}^{T}) \geq L_{T_1+1}^{T} (\theta^*),
$$

by definition of (3), for all $\theta^* \in \Theta_0$. Finally, (iii) is obtained by Lemma 1.

4.2 Always-valid inference

Let

$$
M_T = \prod_{t=p+1}^{T} f_{\theta_{t-1}} (X_t | X_{t-p}) / \prod_{t=p+1}^{T} f_{\theta^*} (X_t | X_{t-p}),
$$

where we define $M_0 = 1$, for each $0 \leq T \leq p$. Firstly, we wish to establish that $(M_T)_{T \in \mathbb{N} \cup \{0\}}$ is a martingale, adapted to the natural filtration $(\mathcal{F}_T^t)_{T \in \mathbb{N} \cup \{0\}}$, where $\mathcal{F}_T = \sigma (X_T, \ldots, X_1)$.

Lemma 2. For each $T \in \mathbb{N}$, $E_{\theta^*} [M_T | \mathcal{F}_{T-1}] = M_{T-1}^*$. 

4
Proof. We firstly prove the result for $T > p + 1$. Write

$$E_{\theta^*}[M_T^*|F_{T-1}] = \int_X \prod_{t=p+1}^{T-1} f_{\theta_{t-1}}(X_t|X_{t-1}^{t-p}) f_{\theta^*}(x_T|X_{T-1}^{T-p}) \, dx_T$$

$$= \frac{\prod_{t=p+1}^{T-1} f_{\theta_{t-1}}(X_t|X_{t-1}^{t-p})}{\prod_{t=p+1}^{T-1} f_{\theta^*}(X_t|X_{t-1}^{t-p})} \int_X f_{\theta_{T-1}}(x_T|X_{T-1}^{T-p}) \, dx_T \quad (i)$$

where (i) is due to the definition of a conditional PDF. By definition of $M_T^*$, the result also holds for $T \leq p + 1$, as required.

4.2.1 Proof of Proposition 3

By Lemmas 2 and 3, for any $\alpha > 0$, we have

$$\Pr_{\theta^*}(\exists T \in \mathbb{N} : M_T^* \geq 1/\alpha) \leq \alpha M_0^*.$$  

Since

$$\{\tau_{\theta^*} = \infty\} = \{\forall T \in \mathbb{N} : M_T^* < 1/\alpha\},$$

we have

$$\Pr_{\theta^*}(\tau_{\theta^*} < \infty) = \Pr_{\theta^*}(\exists T \in \mathbb{N} : M_T^* \geq 1/\alpha) \leq \Pr_{\theta^*}(\exists T \in \mathbb{N} : M_T^* \geq 1/\alpha) \leq \alpha M_0^* \quad (ii)$$

where (i) is due to the fact that

$$\prod_{t=p+1}^{T} f_{\theta_t}(X_t|X_{t-1}^{t-p}) \geq \prod_{t=p+1}^{T} f_{\theta^*}(X_t|X_{t-1}^{t-p}),$$

for every $\theta^* \in \Theta_0$, and (ii) is by definition of $M_0^*$.

4.2.2 Proof of Proposition 4

For the case of $P_T$, we apply Lemma 4 together with the fact that

$$\{\exists T \in \mathbb{N} : M_T \geq 1/\alpha\} = \{\exists T \in \mathbb{N} : P_T \leq \alpha\} = \bigcup_{T=1}^{\infty} \{P_T \leq \alpha\}.$$  

Then, we obtain the result for $\hat{P}_T$ using the fact that

$$\{\hat{P}_T \leq \alpha\} = \bigcup_{t=1}^{T} \{P_t \leq \alpha\},$$

which implies

$$\bigcup_{T=1}^{\infty} \{\hat{P}_T \leq \alpha\} = \bigcup_{T=1}^{\infty} \bigcup_{t=1}^{T} \{P_t \leq \alpha\} = \bigcup_{T=1}^{\infty} \{P_t \leq \alpha\}.$$
First, note that $R_T(\theta^*) = M_T^*$. Then,

$$
\Pr_{\theta^*}(\exists T \in \mathbb{N} : \theta \notin D_T(\alpha)) = \Pr_{\theta^*}(\exists T \in \mathbb{N} : R_T(\theta^*) > 1/\alpha) \\
\leq \Pr_{\theta^*}(\exists T \in \mathbb{N} : M_T^* \geq 1/\alpha) \overset{\text{(i)}}{\leq} \alpha,
$$

where (i) is due to Lemma 2. Thus, $(D_T(\alpha))_{T \in \mathbb{N}}$ is valid.

Next, note that

$$
\{\theta^* \notin \bar{D}_T(\alpha)\} = \left\{\theta^* \notin \bigcap_{t=1}^{T} D_t(\alpha)\right\} = \bigcup_{t=1}^{T} \{\theta^* \notin D_t(\alpha)\}.
$$

Then, we obtain the validity of $(\bar{D}_T(\alpha))_{T \in \mathbb{N}}$, due to

$$
\{\exists T \in \mathbb{N} : \theta^* \notin \bar{D}_T(\alpha)\} = \bigcup_{T=1}^{\infty} \{\theta^* \notin \bar{D}_T(\alpha)\} \\
= \bigcup_{T=1}^{\infty} \bigcup_{t=1}^{T} \{\theta^* \notin D_t(\alpha)\} \\
= \bigcup_{T=1}^{\infty} \{\theta^* \notin D_t(\alpha)\} = \{\exists T \in \mathbb{N} : \theta^* \notin D_T(\alpha)\}.
$$

5 Technical results

We state some technical results that are required throughout the text. References for unproved results are provided at the end of the section.

**Lemma 3** (Ville’s Inequality). If $(Y_T)_{T \in \mathbb{N} \cup \{0\}}$ is a non-negative supermartingale, adapted to the filtration $(\mathcal{F}_T)_{T \in \mathbb{N} \cup \{0\}}$. Then, for any $\alpha > 0$, we have

$$
\Pr(\exists T \in \mathbb{N} : Y_T \geq 1/\alpha) \leq \alpha Y_0.
$$

**Lemma 4.** Let $(A_T)_{T \in \mathbb{N}}$ be a sequence of events in some filtered probability space, and let $A_\infty = \limsup_{T \to \infty} A_T$. If $\alpha \in [0,1]$, then the following statements are equivalent: (a) $\Pr(\bigcup_{T=1}^{\infty} A_T) \leq \alpha$, (b) $\Pr(A_\tau) \leq \alpha$ for all random (potentially not stopping times) $\tau$, (c) $\Pr(A_\tau) \leq \alpha$ for all stopping times $\tau$ (possibly infinite).

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