Disformal transformation of cosmological perturbations

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We investigate the gauge-invariant cosmological perturbations in the gravity and matter frames in the general scalar-tensor theory where two frames are related by the disformal transformation. The gravity and matter frames are the extensions of the Einstein and Jordan frames in the scalar-tensor theory where two frames are related by the conformal transformation, respectively. First, it is shown that the curvature perturbation in the comoving gauge to the scalar field is disformally invariant as well as conformally invariant, which gives the predictions from the cosmological model where the scalar field is responsible both for inflation and cosmological perturbations. Second, in case that the disformally coupled matter sector also contributes to curvature perturbations, we derive the evolution equations of the curvature perturbation in the uniform matter energy density gauge from the energy (non)conservation in the matter sector, which are independent of the choice of the gravity sector. While in the matter frame the curvature perturbation in the uniform matter energy density gauge is conserved on superhorizon scales for the vanishing nonadiabatic pressure, in the gravity frame it is not conserved even if the nonadiabatic pressure vanishes. The formula relating two frames gives the amplitude of the curvature perturbation in the matter frame, once it is evaluated in the gravity frame.

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I. INTRODUCTION

The Concordance Model of Cosmology has succeeded in explaining the history of the universe [1]. However, we still do not understand which fundamental physics is behind the elements of the Concordance Model, i.e., inflation, dark matter and dark energy. In the last decades, cosmologists and gravitational physicists have explored the possible modifications of the Einstein gravity on cosmological scales [2], as the alternatives to these elements. A naive modification of the Einstein gravity provides ghostlike degrees of freedom arising from the higher derivative terms as well as inconsistencies with experimental tests on the Einstein gravity. To avoid the appearance of the ghostlike degrees of freedom associated with Ostrogradski’s theorem [3], the equations of motion should be written in terms of the second order differential equations. On the other hand, to pass the experimental tests on the Einstein gravity, a realistic modification of gravity should contain a mechanism to suppress scalar interactions on small scales [4]. After a number of models have been examined, it has turned out that successful models of the modified gravity can be described in terms of a class of Horndeski’s scalar-tensor theory. The theory was originally proposed by Hordenski [5] forty years ago, and recently has been reformulated with the growing interest in applications to cosmological problems [6]. Horndeski’s theory is known as the most general scalar-tensor theory where the equations of motion remain of the second order, despite the existence of the derivative interactions. Horndeski’s theory has been investigated from the various cosmological aspects, e.g., dark energy [7], inflation [8], early universe [9], screening mechanisms [10] and also observational constraints [11].

As the generalization, we may consider the situation that the scalar field is directly coupled to the matter sector. In such a theory, matter does not follow geodesics associated with the gravitational metric $g_{\mu\nu}$ but that associated with the other metric $\bar g_{\mu\nu}$ which differs from $g_{\mu\nu}$ by the contributions of the scalar field. The most familiar and well-studied case with two different metrics for gravity and matter is that the matter frame metric $\bar g_{\mu\nu}$ can be constructed by the gravitational one $g_{\mu\nu}$ and the scalar field $\phi$ itself, but not by the derivatives of the scalar field. In this case, the matter frame metric is conformally related to the gravity frame one by $\bar g_{\mu\nu} = \alpha(\phi)g_{\mu\nu}$ [2]. The gravity and matter frames, $g_{\mu\nu}$ and $\bar g_{\mu\nu}$, are often referred to as the Einstein and Jordan frames, respectively. The conformal transformation does not modify the causal structure of the spacetime. Concerning cosmological perturbations, after a number of studies [12,13], it has been shown that in the case that a single scalar field is responsible for inflation and cosmological perturbations the curvature perturbation in the comoving / uniform energy density gauge, which is conserved on superhorizon scales, is conformally invariant for all orders of perturbations, allowing us to evaluate it in the most convenient Einstein frame, even though the physical frame for matter is the Jordan frame. By the “physical frame”, we mean the frame in which matter minimally couples to its metric and photons propagate on null geodesics associated with it.

As the most general case of the scalar-tensor theory with two different metrics for gravity and matter, however, we may consider the matter frame metric which can also be constructed from the derivatives of the scalar field as well as the gravitational metric and the scalar field it-
self, $\bar{g}_{\mu\nu} = g_{\mu\nu}(g_{\mu\rho}, \phi, \partial_\phi, \partial^2_\phi, \cdots)$. As the next simplest case to the conformal transformation, it would be reasonable to truncate the expansion of $\bar{g}_{\mu\nu}$ with respect to the order of the scalar field derivatives at the first order, by assuming that the effects of the higher derivatives would be suppressed by the higher inverse powers of the cut-off mass scale below which an effective theory description is assumed to be valid. Although the expansion argument does not distinguish operators of $(\partial \phi)^n$ from those of $\partial^n \phi$ ($n \geq 2$), we would keep only the former one, because the latter would give the higher derivative terms in the equations of motion and hence give rise to ghost-like instabilities associated with Ostrogradski’s theorem. Then according to [16], the next simplest matter frame metric involves the first order derivatives of the scalar field as well as the scalar field itself:

$$\bar{g}_{\mu\nu} = \alpha(X, \phi) g_{\mu\nu} + \beta(X, \phi) \phi_\mu \phi_\nu, \quad (1)$$

where $\phi_\mu = \nabla_\mu \phi$ is the covariant derivative of the scalar field associated the gravity frame metric $g_{\mu\nu}$ and $X := -\frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu$. Eq. (1) is often called the disformal transformation. In [16], in the context of the Finsler geometry how in general the gravity and matter frames are related by a single scalar field was argued, and it was shown that its reduction to the Riemannian geometry has to be given by the disformal relation (1). As pointed out in [17], however, the disformal transformation (1) of a class of Horndeski’s theory could give the higher derivative coupling terms which are absent in Horndeski’s theory, and hence the higher derivative terms in the equations of motion. Nevertheless, as argued in [18], the appearance of Ostrogradski’s ghosts may be able to be avoided by the implicit constraints, implying the existence of the more general healthy scalar-tensor theory beyond Horndeski. In this paper, however, we will focus on the simpler case that the disformally-transformed gravitational theory also belongs to a class of Horndeski’s theory [17], namely

$$\bar{g}_{\mu\nu} = \alpha(\phi) g_{\mu\nu} + \beta(\phi) \phi_\mu \phi_\nu. \quad (2)$$

Such a simplification is made, in order to restrict our arguments to Horndeski’s scalar-tensor theory. For $\beta = 0$, (2) reduces to the ordinary conformal transformation, while when $\alpha = 1$, $\beta$ represents the pure disformal transformation. First, we impose that $\alpha > 0$ so that for $\beta = 0$ the conformal transformation is well-defined. In contrast to the conformal transformation, the disformal transformation may change the causal structure of space-time. Consider a null vector field $v^\mu$ for the matter frame metric $g_{\mu\nu}$, $g_{\mu\nu} v^\mu v^\nu = 0$, which can be rewritten as $g_{\mu\nu} v^\mu v^\nu = -\frac{2}{\alpha} (\phi_\mu v^\mu)^2$. Thus for $\beta > 0$, $g_{\mu\nu} v^\mu v^\nu < 0$ and hence $v^\mu$ becomes a timelike vector field for the gravity frame metric $g_{\mu\nu}$, and for $\beta < 0$, $g_{\mu\nu} v^\mu v^\nu > 0$ and hence $v^\mu$ becomes a spacelike vector field for $g_{\mu\nu}$. The matter frame metric with the upper indices is given by

$$\bar{g}^{\mu\nu} = \frac{1}{\alpha} (g^{\mu\nu} - \frac{\beta}{\alpha - 2\beta X} \phi^\mu \phi^\nu). \quad (3)$$

In order to obtain a healthy disformal transformation we should impose the following conditions:

1. In order for $\bar{g}_{\mu\nu}$ to have the Lorentzian signature $\bar{g}_{00} < 0$, $\alpha g_{00} + \beta \phi_0^2 < 0$.
2. In order for $\bar{g}_{\mu\nu}$ to be causal $ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu < 0$,
3. In order for (3) also to be well-defined and have the Lorentzian signature with $\bar{g}^{00} < 0$, $\alpha - 2\beta X > 0$.

These conditions have been argued in [17] (see also [14]). Applications of the disformal transformation to the cosmological problems have been argued, such as to inflation [20], cosmology with varying speed of light [21], dark energy [22], screening mechanism [23–25], MOND [26], dark matter [27] and observational constraints [28].

The purpose of this paper is to clarify how the gauge-invariant cosmological perturbation variables [29, 30] in the gravity and matter frames are related by the disformal transformation (2). To our knowledge, so far there has been no formulation of relating the gauge-invariant perturbation variables in both frames. First, we investigate whether the curvature perturbation in the comoving gauge to the scalar field is invariant under the disformal transformation as well as the conformal transformation [12, 14]. In the simplest case where there is no matter field disformally coupled to gravity and the only scalar field is responsible both for inflation and cosmological perturbations, the comoving curvature perturbation is conserved after the scale of interest crosses the horizon, which gives the final prediction from the given model. Thus if it is disformally invariant, it may be evaluated in any disformally related frame as done in the Einstein frame in the gravitational theory with the nonminimal coupling of the scalar field to the Ricci scalar where two frames are related by the conformal transformation.

Second, we will consider the case where there is the matter sector disformally coupled to the scalar field. In such a case, the curvature perturbation will not be conserved due to the continual conversion of the isocurvature perturbation between the scalar field and matter to the curvature perturbation and even the fluctuations of the disformally coupled matter may be the dominant sources of cosmological perturbations, instead of the scalar field itself. To obtain predictions from such a model, we need to solve the evolution equation of the curvature perturbation on the uniform matter energy density gauge. A model to be of use as a possible reference was considered in Ref. [20]. In this model, the gravity frame experiences a decelerating expansion driven by a canonical kinetic term of the scalar field while the disformally related matter frame experiences a short inflationary expansion in the early phase, and the fluctuations of the

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1 See also [14] for an explicit construction of the scalar-tensor theory beyond Horndeski.
scalar field in the decelerating gravity frame could not produce the realistic scale-invariant spectrum of the curvature perturbation. Instead, the disformally coupled matter quantized in the inflationary matter frame may be able to source the curvature perturbation via a curvaton-like mechanism \[31\]. Although in this paper we will not consider a particular model, such a model may be able to be generalized to the case of a class of Horndeski’s theory. As the first step to investigate the evolution of the curvature perturbations in such a model, it will be important to formulate how the curvature perturbation in the uniform matter energy density gauge evolves in the presence of the disformal coupling. Following \[32\], the evolution equation of the curvature perturbation in the uniform matter energy density gauge will be derived as the consequence of the energy (non)conservation in the matter sector, which will be independent of the choice of the gravity sector. Therefore, we will derive the gauge-invariant evolution equations of the curvature perturbation in the uniform energy density of each component in the gravity and matter frames.

The construction of the paper is as follows. In Section II, we review the covariant equations of motion in the gravity and matter frames in the general scalar-tensor theory with the matter sector disformally coupled to the scalar field. In Section III, we define the gauge-invariant gravitational perturbation variables in the gravity and matter frames and derive the relations between the corresponding perturbation variables in both frames. In Section IV, similarly, we define the gauge-invariant matter perturbation variables in the gravity and matter frames and derive the relations between the corresponding perturbation variables in both frames. In Section V, we derive the evolution equations of the curvature perturbations in the uniform matter energy density gauges in both frames. The paper is closed after giving a brief summary in Section VI.

**II. THE COVARIANT EQUATIONS OF MOTION IN THE GRAVITY AND MATTER FRAMES**

We consider the scalar-tensor theory with matter disformally coupled to the scalar field

\[ S = S_g + S_m; \]

\[ S_g := \int d^4x \sqrt{-g} \mathcal{L}_g[g, \phi], \]

\[ S_m := \int d^4x \sqrt{-g} \mathcal{L}_m[g, \Psi], \tag{4} \]

where \( g_{\mu\nu} \) is the gravity frame metric, \( \mathcal{L}_g \) is the Lagrangian of the gravity sector given by Horndeski’s theory, composed of the four independent combinations of the scalar field operators:

\[ \mathcal{L}_g[g, \phi] = \sum_{i=2}^{5} \mathcal{L}_i[g, \phi], \tag{5} \]

with

\[ \mathcal{L}_2 = P(X, \phi), \quad \mathcal{L}_3 = -G(X, \phi) \Box \phi, \]

\[ \mathcal{L}_4 = G_4(X, \phi) R + G_4 X \left( (\Box \phi)^2 - \phi_{\mu\nu} \phi^{\mu\nu} \right), \]

\[ \mathcal{L}_5 = G_5(X, \phi) \phi_{\mu\nu} \phi^{\mu\nu} - \frac{1}{6} G_5 X \left[ (\Box \phi)^3 - 3(\Box \phi) \phi_{\mu\nu} \phi^{\mu\nu} + 2 \phi_{\mu\nu} \phi^{\alpha\nu} \phi^{\beta}_{\alpha} \right]. \tag{6} \]

where \( P, G, G_4 \) and \( G_5 \) are free functions of both \( X \) and \( \phi, \bar{g}_{\mu\nu} \) is the matter frame metric which is related to the gravity frame metric by the disformal relation \[2\], \( \Psi \) represents matter other than the scalar field, and \( \mathcal{L}_m \) represents the matter Lagrangian. Following \[17\], through the disformal relation \[2\], the gravitational action \( S_g \) in \[4\] can be rewritten in terms of the matter frame metric as

\[ S_g = \int d^4x \sqrt{-g} \mathcal{L}_g[g, \phi], \quad \bar{\mathcal{L}}_g[g, \phi] = \sum_{i=2}^{5} \bar{\mathcal{L}}_i[g, \phi], \tag{7} \]

where \( \bar{\mathcal{L}}_g \) is the Lagrangian of the gravity sector expressed with respect to the matter frame metric \( \bar{g}_{\mu\nu} \) and \( \bar{\mathcal{L}}_i \) is each Lagrangian of Horndeski’s theory with respect to \( \bar{g}_{\mu\nu} \). The relation between \( \mathcal{L}_g \) and \( \bar{\mathcal{L}}_g \) was obtained in \[17\], with the replacements \( A \to \alpha, B \to \beta \) and \( X \to -X \). Readers who are interested in their explicit relation should refer to Appendix C of Ref. \[17\]. The gravity and matter frames are the natural extensions of the Einstein and Jordan frames in the case of the scalar-tensor theory with the nonminimal coupling of the scalar field to the Ricci scalar, respectively, where two frames are related by the conformal transformation. Since the disformal transformation contains two free functions, two more intermediate frames can also be defined as argued in \[17, 25\], which will not be considered in this paper.

In the gravity frame description \[4\], by varying the action with respect to \( g_{\mu\nu} \), the gravitational equations of motion are obtained as

\[ 0 = E^{\mu\nu} + T^{\mu\nu}_{(m)}, \]

\[ E^{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} \mathcal{L}_g, \]

\[ T^{\mu\nu}_{(m)} := \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} (\mathcal{L}_m), \tag{8} \]

where the gravitational tensor \( E^{\mu\nu} \), which reduces to the combination of Einstein tensor and the scalar field energy-momentum in the case of the simplest Einstein-scalar theory, is not divergent free and

\[ - \nabla_{\mu} E^{\mu\nu} = \nabla_{\mu} T^{\mu\nu}_{(m)}, \tag{9} \]

which represents the energy exchange between two sectors through the disformal coupling.

On the other hand, in the matter frame description \[4\], by varying the action with respect to \( \bar{g}_{\mu\nu} \), the gravi-
tional equations of motion are obtained as
\[ 0 = E^\mu{}^\nu + T^\mu{}^\nu_{(m)}, \]
\[ \tilde{E}^\mu{}^\nu := \frac{2}{\sqrt{-g}} \delta(\sqrt{-g} \mathcal{L}_g), \]
\[ \tilde{T}^\mu{}^\nu_{(m)} := \frac{2}{\sqrt{-g}} \delta(\sqrt{-g} \mathcal{L}_m). \] (10)

In contrast to the previous gravity frame description \[ \square \], \( \tilde{E}^\mu{}^\nu \) and \( \tilde{T}^\mu{}^\nu_{(m)} \) are separately conserved as
\[ - \nabla_\mu \tilde{E}^\mu{}^\nu = \nabla_\mu \tilde{T}^\mu{}^\nu_{(m)} = 0, \] (11)
where \( \nabla_\mu \) represents the covariant derivative with respect to the matter frame metric \( \tilde{g}_{\mu\nu} \). The contravariant matter energy-momentum tensors defined in two frames are related by
\[ T^\mu{}^\nu_{(m)} = \sqrt{\frac{\tilde{g}}{g}} g^{\alpha\beta} T_{(m)}^\alpha{}^\beta, \]
\[ \tilde{T}^\mu{}^\nu_{(m)} = \frac{\alpha^3}{\sqrt{-g}} \left( 1 - \frac{2X\beta}{\alpha} \right) T^\mu{}^\nu_{(m)} \mu, \]
\[ T^\mu{}^\nu_{(m)} = \frac{\alpha^2}{\sqrt{-g}} \left( 1 - \frac{2X\beta}{\alpha} \right) \left( \delta^\rho\!_\nu - \frac{\beta\phi_\rho\phi^\rho}{\alpha - 2\beta X} \right) T^\mu{}^\nu_{(m)} \rho, \] (12)

Similarly, the mixed and covariant energy-momentum tensors are related by
\[ T^\mu{}^\nu_{(m)\rho} = \alpha^2 \sqrt{1 - \frac{2X\beta}{\alpha}} \left( \delta^\rho\!_\nu - \frac{\beta\phi_\rho\phi^\rho}{\alpha - 2\beta X} \right) T^\mu{}^\nu_{(m)\rho}, \]
\[ T^\mu{}^\nu_{(m)\rho} = \alpha \left( 1 - \frac{2X\beta}{\alpha} \right) \left( \delta^\rho\!_\nu - \frac{\beta\phi_\rho\phi^\rho}{\alpha - 2\beta X} \right) T^\mu{}^\nu_{(m)\rho}. \] (13)

Oppositely,
\[ \bar{T}^\mu{}^\nu_{(m)} = \frac{1}{\alpha^3 \sqrt{1 - \frac{2X\beta}{\alpha}}} T^\mu{}^\nu_{(m)}, \]
\[ \bar{T}^\mu{}^\nu_{(m)\rho} = \frac{1}{\alpha^3 \sqrt{1 - \frac{2X\beta}{\alpha}}} (\alpha \delta_{\rho\nu}^\rho + \beta\phi_\rho\phi^\rho) T^\mu{}^\nu_{(m)\rho}, \]
\[ \bar{T}^\mu{}^\nu_{(m)\rho} = \frac{1}{\alpha^3 \sqrt{1 - \frac{2X\beta}{\alpha}}} (\alpha \delta_{\rho\nu}^\rho + \beta\phi_\rho\phi^\rho) T^\mu{}^\nu_{(m)\rho}. \] (14)

So far, we have worked in the fixed coordinate system \( x^\mu \). In the homogeneous and isotropic cosmological background, however, the proper time coordinate in the gravity frame is not that in the matter frame (see the next section). Thus, we introduce a new coordinate system \( \hat{x}^\mu \) whose time component gives the proper time coordinate by
\[ d\hat{x}^\mu := \frac{\partial \hat{x}^\mu}{\partial x^\nu} dx^\nu. \] (15)

The components of any tensor defined in the matter frame are related by
\[ \hat{T}^\mu{}^\nu_{\alpha_1...\alpha_n} = T^\mu{}^\nu_{\beta_1...\beta_n}, \]
\[ \times \left( \frac{\partial \hat{x}^\mu}{\partial x^\nu} \frac{\partial \hat{x}^\nu}{\partial x^\rho} \cdots \frac{\partial \hat{x}^\rho}{\partial x^{\sigma}} \right). \] (16)

In the hatted coordinate system, the equations of motion in the matter frame \( \square \) and \( \square \) are rewritten as
\[ 0 = \hat{E}^\mu{}^\nu + \hat{T}^\mu{}^\nu_{(m)}, \]
\[ \hat{E}^\mu{}^\nu := \frac{2}{\sqrt{-g}} \delta(\sqrt{-g} \mathcal{L}_g), \]
\[ \hat{T}^\mu{}^\nu_{(m)} := \frac{2}{\sqrt{-g}} \delta(\sqrt{-g} \mathcal{L}_m), \] (17)
with
\[ - \hat{\nabla}_\mu \hat{E}^\mu{}^\nu = \hat{\nabla}_\mu \hat{T}^\mu{}^\nu_{(m)} = 0, \] (18)
where the covariant derivative \( \hat{\nabla}_\mu \) is associated with \( \hat{g}_{\mu\nu} \).

### III. DISFORMAL TRANSFORMATION OF GRAVITATIONAL PERTURBATIONS

In this section, we present the gauge-invariant gravitational perturbation variables in the gravity and matter frames and derive the relations between the corresponding perturbation variables in both frames.

#### A. Disformal transformation of gravitational perturbations

As the background solution in the gravity frame, we consider the spatially-flat Friedmann-Lemaître-Robertson-Walker metric and the homogeneous scalar field
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \quad \phi = \phi(t), \] (19)
where \( a(t) \) is the scale factor. We then consider the scalar-type perturbations to the above background
\[ ds^2 = -(1 + 2A(t, x^i)) dt^2 + 2a(t) \partial_i B(t, x^i) dt dx^i + a(t)^2 \left[ (1 - 2\psi(t, x^i)) \delta_{ij} + 2\partial_i \partial_j E(t, x^i) \right] dx^i dx^j, \]
(20)
and \( \phi + \delta\phi \), where \( A, B, \psi \) and \( E \) are scalar-type metric perturbation variables. Our notation of cosmological perturbations follows \[ \square \]. \( \psi \) is often referred to as the curvature perturbation, as the intrinsic scalar curvature of the three-dimensional space is given by \[ (3)R = \frac{1}{a^2} \Delta \psi, \] where \( \Delta := \delta^{ij} \partial_i \partial_j \). In this paper, we will focus on the
scalar-type perturbations, as the vector- and tensor-type perturbations are not affected by the disformal transformation \([2]\).

Through the perturbation of the scalar field, two functions in \([2]\) are also perturbed as

\[
\alpha \to \alpha + \alpha' \delta \phi, \quad \beta \to \beta + \beta' \delta \phi, \quad (21)
\]

and the metric in the matter frame is also perturbed. Using \([2]\), then the matter frame metric is given by

\[
d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu
= -(\alpha - \beta \dot{\phi}^2 + 2\tilde{A}(t, x^i)) dt^2 + 2\alpha(t) \partial_i \tilde{B}(t, x^i) dt dx^i
+ a(t)^2 \left[ \left( \alpha - 2\tilde{\psi}(t, x^i) \right) \delta_{ij} + 2\partial_i \partial_j \tilde{E}(t, x^i) \right] dx^i dx^j,
\]

where the metric perturbations in the matter frame are given by

\[
\tilde{A} = \alpha A + \frac{\alpha' \delta \phi}{2} - \frac{\beta' \delta \phi}{2} \dot{\phi}^2 - \beta \dot{\phi} \delta \phi, \quad \tilde{B} = \alpha B + \frac{\beta \dot{\phi}}{a} \delta \phi,
\]

\[
\tilde{\psi} = \alpha \psi - \frac{\alpha' \delta \phi}{2}, \quad \tilde{E} = \alpha E.
\]

Then introducing the hatted coordinate system explained in the previous section by

\[
d\tilde{t} = \sqrt{\alpha - \beta \dot{\phi}^2} dt, \quad \tilde{x}^i = x^i, \quad (24)
\]

the matter frame metric \([22]\) is rewritten as

\[
d\tilde{s}^2 = \hat{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu
= -(1 + 2\tilde{A}(\tilde{t}, \tilde{x}^i)) d\tilde{t}^2 + 2\hat{a}(\tilde{t}) \partial_i \hat{B}(\tilde{t}, \tilde{x}^i) d\tilde{t} d\tilde{x}^i
+ \hat{a}(\tilde{t})^2 \left[ (1 - 2\hat{\psi}(\tilde{t}, \tilde{x}^i)) \delta_{ij} + 2\partial_i \partial_j \hat{E}(\tilde{t}, \tilde{x}^i) \right] d\tilde{x}^i d\tilde{x}^j,
\]

where \(\hat{a} = \sqrt{\alpha} a\) is the scale factor in the matter frame, and the metric perturbations in the matter frames are given by

\[
\hat{A} = \frac{\alpha A + \frac{\alpha' \delta \phi}{2} - \frac{\beta' \delta \phi}{2} \dot{\phi}^2 - \beta \dot{\phi} \delta \phi}{\alpha - \beta \dot{\phi}^2}, \quad \hat{B} = \frac{B + \frac{\beta \dot{\phi}}{a} \delta \phi}{\sqrt{1 - \frac{\beta}{\alpha} \dot{\phi}^2}},
\]

\[
\hat{\psi} = \psi - \frac{\alpha' \delta \phi}{2\alpha}, \quad \hat{E} = E.
\]

Before closing this subsection, we comment on the (in)equivalence of the representative gauge conditions under the disformal transformation. From \([26]\), we find that the transformation of the spatial components of the metric perturbations, \(\psi\) and \(E\), are not affected by the disformal component \(\beta\), while the remaining components, \(A\) and \(B\), are affected by \(\beta\). Thus the synchronous gauges \(A = B = 0\) and \(\hat{A} = \hat{B} = 0\) are not equivalent. Similarly, in contrast to the case of the pure conformal transformation \(\beta = 0\), the longitudinal gauges \(B = E = 0\) and \(\hat{B} = \hat{E} = 0\) are also not equivalent under the disformal transformation. On the other hand, the scalar field perturbation \(\delta \phi\) is invariant under the disformal transformation and hence the comoving gauge to the scalar field \(\delta \phi = 0\) is unique. As observables must be gauge-invariant, in the next section the gauge-invariant gravitational perturbation variables are constructed as \([21, 29]\).

### B. Gauge-invariant gravitational perturbations

Under the gauge transformation \(t \to t + \delta t\) and \(x^i \to x^i + \delta x^i \partial_i \delta x\), the metric and scalar field perturbation variables are transformed as

\[
A \to A - \delta t, \quad B \to B + \frac{1}{\alpha} \delta t - \frac{\dot{\alpha}}{a} \delta x,
\]

\[
\psi \to \psi + \frac{\dot{\alpha}}{\alpha} \delta t, \quad E \to E - \delta x, \quad \delta \phi \to \delta \phi - \dot{\phi} \delta t \quad (27)
\]

\(B = E = 0\) is the longitudinal gauge, \(\psi = 0\) is the spatially-flat gauge and \(\delta \phi = 0\) is the comoving gauge to the scalar field, respectively. Note that while in choosing the longitudinal gauge two gauge transformation functions \(\delta t\) and \(\delta x\) are completely fixed, in choosing the spatially-flat and comoving gauges there is still the remaining gauge degree of freedom of \(\delta x\). Although in this paper we will call them ‘gauges’, they are also often called the spatially-flat and comoving ‘slices’, respectively.

The representative gauge-invariant gravitational perturbation variables constructed in the gravity frame are given by

\[
\Phi = A - \frac{d}{dt} \left[ \frac{1}{a^2} (E - B/a) \right],
\]

\[
\Psi = \psi + \frac{\dot{a}}{a} \left( E - \frac{B}{a} \right),
\]

\[
\mathcal{R}^{(\phi)}_\zeta = \psi + \frac{\dot{\alpha}}{\alpha} \delta \phi, \quad (28)
\]

Their counterparts in the matter frame are given by

\[
\hat{\Phi} = \hat{A} - \frac{d}{d\tilde{t}} \left[ \frac{1}{\hat{a}^2} (\hat{E} - \hat{B}/\hat{a}) \right],
\]

\[
\hat{\Psi} = \hat{\psi} + \frac{\dot{\hat{a}}}{\hat{a}} \left( \hat{E} - \frac{\hat{B}}{\hat{a}} \right),
\]

\[
\hat{\mathcal{R}}^{(\phi)}_\zeta = \hat{\psi} + \frac{\dot{\alpha}}{\alpha} \delta \phi. \quad (29)
\]

Using \([21]\) and \([28]\), we immediately find that the curvature perturbations in the comoving gauge to the scalar field are related by

\[
\hat{\mathcal{R}}^{(\phi)}_\zeta = \mathcal{R}^{(\phi)}_\zeta. \quad (30)
\]

Thus the comoving curvature perturbation to the scalar field is invariant under the disformal transformation at least at the linear order of perturbations, which is a generalization of the well-known conformal invariance of the
same quantity \[12,14\]. In the case that only the scalar field is responsible for inflation and cosmological perturbations and the comoving curvature perturbation is conserved on superhorizon scales \[33,36\], the comoving curvature perturbation evaluated in any convenient disformally related frame during inflation gives the final observables. When there is matter disformally coupled to the scalar field, the curvature perturbation may not be conserved, although the value of \( \mathcal{R}_c^{(i)} \) is still frame-independent.

Similarly, the gauge-invariant metric perturbations in the longitudinal gauges are related by

\[
\hat{\psi} = \Psi - \frac{1}{\alpha - \beta \dot{\phi}^2} \left( \beta \dot{\phi}^2 \frac{\dot{a}}{a} + \frac{\ddot{a}}{2} \right) \dot{\phi},
\]

\[
\hat{\phi} = \frac{1}{\alpha - \beta \dot{\phi}^2} \left\{ \alpha \Phi + \frac{\ddot{a}}{\alpha - \beta \dot{\phi}^2} \left( \ddot{\phi}^2 \beta + 2 \beta \ddot{\phi} \dot{\phi} \right) \right\},
\]

where we have defined the gauge-invariant perturbations of \( Y \) in the longitudinal gauges by

\[
\delta_g Y = \delta Y - a^2 \dot{Y} \left( \dot{E} - \frac{B}{a} \right),
\]

\[
\dot{\delta}_g Y = \delta Y - a^2 \dot{Y} \left( \dot{E} - \frac{B}{a} \right).
\]

The scalar field perturbations in the longitudinal gauges are related by

\[
\delta \phi = \delta \phi - \dot{a}^2 \dot{\phi} \left( \dot{E} - \frac{B}{a} \right) = \frac{\alpha}{\alpha - \beta \dot{\phi}^2} \delta_g \phi.
\]

The scalar field perturbations in the spatially-flat gauges \( \psi = 0 \) and \( \dot{\psi} = 0 \) are defined by

\[
\delta \phi := \delta \phi + \frac{a}{\dot{a}} \dot{\phi} \psi = \frac{\dot{a}}{a} \phi \mathcal{R}_c^{(i)},
\]

\[
\dot{\delta} \phi := \dot{\delta} \phi + \frac{\dot{a}}{a} \dot{\phi} \dot{\phi} \psi = \frac{\ddot{a}}{a} \phi \mathcal{R}_c^{(i)},
\]

which are the primordial sources of cosmological perturbations on subhorizon scales in the single-field inflation models. Using \[36\],

\[
\dot{\delta} \phi = \frac{\ddot{a}}{a} \dot{a} \delta \phi.
\]

Finally, the gauge-invariant combination

\[
\Sigma^{(i)} := A \phi^2 - \dot{\phi} \delta \phi + \ddot{\phi} \delta \phi,
\]

is related to the intrinsic entropy perturbation of the scalar field \( \Gamma^{(i)} := \delta p^{(i)} - \frac{\dot{\rho}^{(i)}}{\rho}, \dot{\rho}^{(i)} \). This combination is disformally transformed as

\[
\hat{\Sigma}^{(i)} := \hat{A} \phi^2 - \dot{\phi} \delta \phi + \ddot{\phi} \delta \phi = \frac{\alpha}{(\alpha - \beta \dot{\phi}^2)^2} \Sigma^{(i)}.
\]

As reviewed in Appendix, in the single-field inflation models, \( \Sigma^{(i)} \) vanishes on superhorizon scales if the comoving curvature perturbation is conserved on superhorizon scales, \( \mathcal{R}_c^{(i)} = 0 \). On the other hand, if matter is disformally coupled, the comoving curvature perturbation is not conserved on superhorizon scales and hence \( \Sigma^{(i)} \) is no longer suppressed. Eqs. \[33\], \[36\] and \[37\] mean that the gauge-invariant scalar field perturbations are frame-independent, up to the rescalings by the background quantities. In the next section, we will construct the gauge-invariant matter perturbation variables.

### IV. DISFORMAL TRANSFORMATION OF MATTER PERTURBATIONS

In this section, we construct the gauge-invariant matter perturbation variables.

#### A. Matter energy-momentum tensor

In this section, we assume that matter is composed of a set of noninteracting fluids

\[
T^{\mu\nu}_{(m)} = \sum_a T^{(a)\mu\nu}, \quad \hat{T}^{\mu\nu}_{(m)} = \sum_a \hat{T}^{(a)\mu\nu}.
\]

According to \[13\] and \[16\] the components of the energy-momentum tensors of the \( (a) \)-th component in both frames are

\[
\hat{T}^{(a)\mu\nu} = \frac{1}{\alpha^3 \sqrt{1 - 2 \frac{\alpha}{\beta}} \frac{\alpha}{\dot{a} \beta}} T^{(a)\mu\nu} \sqrt{\frac{\alpha}{\beta}} \frac{\alpha}{\dot{a} \beta} \frac{\alpha}{\dot{a} \beta},
\]

\[
\hat{T}^{(a)\mu\nu}_{\nu} = \frac{1}{\alpha^3 \sqrt{1 - 2 \frac{\alpha}{\beta}} \frac{\alpha}{\dot{a} \beta}} \left( \alpha \delta^{a \alpha} + \beta \dot{\phi} \phi^{a} \right) T^{(a)\mu\nu} \frac{\alpha}{\dot{a} \beta} \frac{\alpha}{\dot{a} \beta} \frac{\alpha}{\dot{a} \beta} \frac{\alpha}{\dot{a} \beta},
\]

\[
\hat{T}^{(a)\mu\nu}_{\mu} = \frac{1}{\alpha^3 \sqrt{1 - 2 \frac{\alpha}{\beta}} \frac{\alpha}{\dot{a} \beta}} \left( \alpha \delta^{a \alpha} + \beta \dot{\phi} \phi^{a} \right) T^{(a)\mu\nu} \frac{\alpha}{\dot{a} \beta} \frac{\alpha}{\dot{a} \beta} \frac{\alpha}{\dot{a} \beta} \frac{\alpha}{\dot{a} \beta}.
\]

As a further extension, we may consider disformal coupling which depends on the component, \( \hat{g}^{(a)}_{\mu\nu} = \)
\( \alpha^{(a)}(\phi)g_{\mu\nu} + \beta^{(a)}(\phi)\partial_{\mu}\phi_{\nu}. \) In this paper, however, we are interested in how the gauge-invariant perturbations in the gravity and matter frames are related, and do not consider such a generalized case.

We then derive the relation between the perturbed energy-momentum tensors defined in the gravity and matter frames. We assume that in the gravity frame the energy-momentum tensor of the \((a)\)-th component takes the following form

\[
T^{(a)00} = -\rho^{(a)} - \delta\rho^{(a)}, \quad T^{(a)i0} = -\frac{\rho^{(a)} + p^{(a)}}{a} \partial^i \hat{v}^{(a)},
\]

\[
T^{(a)i}_{\ j} = \left( p^{(a)} + \delta p^{(a)} \right) \delta^i_{\ j} + p^{(a)} \left[ \partial^i \partial_j - \frac{1}{3} \delta^i_{\ j} \Delta \right] \Pi^{(a)}, \quad (40)
\]

where \( \rho^{(a)}, p^{(a)} \) are the background energy density and pressure of the \((a)\)-th component, and \( \delta\rho^{(a)}, \delta p^{(a)}, v^{(a)} \) and \( \Pi^{(a)} \) are the perturbed energy density, pressure, velocity and anisotropic stress of the \((a)\)-th component, respectively. The total matter energy-momentum tensor is given by

\[
T^{(m)00} = -\rho - \delta\rho, \quad T^{(m)i0} = -\frac{\rho + p}{a} \partial^i v, \quad T^{(m)i}_{\ j} = \left( p + \delta p \right) \delta^i_{\ j} + p \left[ \partial^i \partial_j - \frac{1}{3} \delta^i_{\ j} \Delta \right] \Pi, \quad (41)
\]

where from \( (38) \)

\[
\rho = \sum_a \rho^{(a)}, \quad p = \sum_a p^{(a)}, \quad (42)
\]

and

\[
\delta\rho = \sum_a \delta\rho^{(a)}, \quad \delta p = \sum_a \delta p^{(a)}, \quad (43)
\]

\[
\rho v = \sum_a \left( \rho^{(a)} + p^{(a)} \right) v^{(a)},
\]

\[
p \Pi = \sum_a \rho^{(a)} \Pi^{(a)}. \quad (44)
\]

Similarly in the matter frame, the energy-momentum tensor of the \((a)\)-th component is expressed by

\[
\hat{T}^{(a)00} = -\hat{\rho}^{(a)} - \delta\hat{\rho}^{(a)}, \quad \hat{T}^{(a)i0} = -\frac{\hat{\rho}^{(a)} + \hat{p}^{(a)}}{a} \partial^i \hat{v}^{(a)},
\]

\[
\hat{T}^{(a)i}_{\ j} = \left( \hat{p}^{(a)} + \delta\hat{p}^{(a)} \right) \delta^i_{\ j} + \hat{p}^{(a)} \left[ \partial^i \partial_j - \frac{1}{3} \delta^i_{\ j} \Delta \right] \hat{\Pi}^{(a)}, \quad (45)
\]

The total matter energy-momentum tensor is also expressed as

\[
\hat{T}^{(m)00} = -\hat{\rho} - \delta\hat{\rho}, \quad \hat{T}^{(m)i0} = -\frac{\hat{\rho} + \hat{p}}{a} \partial^i \hat{v}, \quad \hat{T}^{(m)i}_{\ j} = \left( \hat{p} + \delta\hat{p} \right) \delta^i_{\ j} + \hat{p} \left[ \partial^i \partial_j - \frac{1}{3} \delta^i_{\ j} \Delta \right] \hat{\Pi},
\]

where from \( (38) \)

\[
\hat{\rho} = \sum_a \hat{\rho}^{(a)}, \quad \hat{p} = \sum_a \hat{p}^{(a)}; \quad (46)
\]

\[
\delta\hat{\rho} = \sum_a \delta\hat{\rho}^{(a)}, \quad \delta\hat{p} = \sum_a \delta\hat{p}^{(a)}, \quad \delta\hat{\rho} + \delta\hat{p} = \sum_a \left( \delta\hat{\rho}^{(a)} + \delta\hat{p}^{(a)} \right) \delta\hat{v}^{(a)},
\]

\[
\hat{p} \hat{\Pi} = \sum_a \rho^{(a)} \Pi^{(a)}. \quad (47)
\]

Using Eq. \( (39) \), the background parts of the energy-momentum tensors of the \((a)\)-th component are transformed as

\[
\hat{\rho}^{(a)} = f \rho^{(a)}, \quad \hat{p}^{(a)} = \frac{\alpha}{\alpha - \beta\phi^2} f p^{(a)}, \quad (48)
\]

where \( f := \frac{\sqrt{1 - \beta^2}}{\alpha}. \)

Now we turn to the perturbation parts. The perturbation parts of the energy-momentum tensors of the \((a)\)-th component are related by

\[
\frac{\delta\hat{\rho}^{(a)}}{\hat{\rho}^{(a)}} = \frac{\delta\rho^{(a)}}{\rho^{(a)}} + \frac{5\alpha'\delta\phi}{2\alpha} + \frac{\alpha'\delta\dot{\phi} - 5\beta\dot{\phi}\delta\phi + 2\dot{\phi}\alpha'\phi - 3\beta\dot{\phi}\phi}{2\alpha(\alpha - \beta\phi^2)},
\]

\[
\frac{\delta\hat{p}^{(a)}}{\hat{p}^{(a)}} = \frac{\delta p^{(a)}}{p^{(a)}} + \frac{3\alpha\delta\dot{\phi}}{2\alpha} - \frac{\alpha'\alpha\delta\phi - \dot{\phi}^2\beta\delta\phi + 2\dot{\phi}\alpha'\phi - 3\beta\dot{\phi}\phi}{2(\alpha - \beta\phi^2)},
\]

\[
\hat{\rho} = \frac{\alpha\sqrt{1 - \beta^2}}{\alpha - \beta\phi^2} \left( f \rho^{(a)} + \frac{ap^{(a)}}{\alpha - \beta\phi^2} \right) \left( \rho^{(a)} + p^{(a)} \right) v^{(a)} + \frac{\alpha'}{\alpha - \beta\phi^2} \beta\delta\phi^2 p^{(a)} v^{(a)} - \frac{\beta\phi^2 \rho^{(a)} v^{(a)}}{\alpha - \beta\phi^2} B, \quad (49)
\]

\[
\hat{\Pi} = \Pi^{(a)}.
\]

Thus the difformal transformation does not modify the structure of the perturbed energy-momentum tensor, particularly a perfect fluid in the gravity frame \((\Pi^{(a)} = 0)\) also corresponds to a perfect fluid in the matter frame \((\Pi^{(a)} = 0)\). This is obvious from the fact that the background scalar field \(\phi\) does not break the rotational invariance.

Clearly, from \( (39) \), the uniform density gauges \( \delta\rho^{(a)} = 0 \) and \( \delta\hat{\rho}^{(a)} = 0 \), and the comoving gauges \( v^{(a)} + B = 0 \) and \( \hat{v}^{(a)} + \hat{B} = 0 \), to the \((a)\)-th component, are not equivalent between frames.

**B. Gauge-invariant matter perturbations**

Under the gauge transformation \( t \rightarrow t + \delta t \) and \( x^i \rightarrow x^i + \delta x^i \partial_i \delta x \), the matter perturbation variables are transformed as

\[
\delta\rho^{(a)} \rightarrow \delta\rho^{(a)} + \hat{\rho}^{(a)} \delta t, \quad \delta\hat{p}^{(a)} \rightarrow \delta\hat{p}^{(a)} + \hat{p}^{(a)} \delta t,
\]

\[
v^{(a)} \rightarrow v^{(a)} + a\delta x, \quad \Pi^{(a)} \rightarrow \Pi^{(a)} . \quad (50)
\]
The gauge-invariant combinations of the matter energy-momentum tensor can also be constructed as those of the gravitational perturbations [29, 30]. The curvature perturbations in the uniform energy density gauges of the (a)-th component, which have often appeared in the literature [32, 33], are defined

\[-\zeta^{(a)} := \psi + \dot{a} \frac{\delta \rho^{(a)}}{\rho^{(a)}}, \quad -\dot{\zeta}^{(a)} := \dot{\psi} + \frac{\dot{a}^2}{a} \delta \rho^{(a)} \]  

(51)

Similarly, the curvature perturbations in the comoving gauges to the (a)-th component are given by

\[R_{c}^{(a)} = \psi - \dot{a}(\rho^{(a)} + B), \quad \dot{R}_{c}^{(a)} = \dot{\psi} - \dot{a}, \dot{\zeta}^{(a)} + \dot{B} \]  

(52)

The nonadiabatic pressure perturbations of the (a)-th component are given by

\[\Gamma^{(a)} := \delta p^{(a)} - \frac{\rho^{(a)}}{\rho^{(a)}} \delta \rho^{(a)}, \quad \dot{\Gamma}^{(a)} := \delta \dot{p}^{(a)} - \frac{\rho^{(a)}}{\rho^{(a)}} \delta \dot{\rho}^{(a)} \]  

(53)

Comparing both frames, we find

\[-(\dot{\zeta}^{(a)} - \zeta^{(a)}) = \frac{\dot{\Gamma}^{(a)}}{\rho^{(a)}} - \frac{\dot{\Gamma}^{(a)}}{\rho^{(a)}} \delta \rho^{(a)} + \beta \frac{\dot{\rho}^{(a)}}{\rho^{(a)}} \delta \rho^{(a)}, \quad \dot{\Gamma}(c) = \delta \dot{p}^{(a)} - \frac{\rho^{(a)}}{\rho^{(a)}} \delta \dot{\rho}^{(a)} \]  

(54)

where we have defined the perturbation of Y in the comoving gauges to the scalar field

\[\delta \phi Y := \delta Y - \frac{\dot{Y}}{\phi} \delta \phi, \quad \delta \phi Y := \delta Y - \frac{\dot{Y}}{\phi} \delta \phi. \]  

(55)

and the perturbation of Y in the comoving gauges to the (a)-th component,

\[\delta m^{(a)} Y := \delta Y + a \dot{Y}(v^{(a)} + B), \quad \delta m^{(a)} Y := \delta Y + a \dot{Y} t(\dot{v}^{(a)} + \dot{B}). \]  

(56)

By definition, \(\delta \phi Y = \delta \phi Y\). In the limit of the purely conformal transformation, \(\beta \rightarrow 0\), the dependence on \(\Sigma^{(a)}\) vanishes in [53]. The scalar field perturbations in the comoving gauges to the (a)-th component are related by

\[\delta m^{(a)} \phi = \frac{\alpha(\rho^{(a)} + p^{(a)})}{(\alpha - \beta \dot{\phi}^2) \rho^{(a)} + \alpha p^{(a)}} \delta^{(a)} \phi. \]  

(57)

Thus the scalar field perturbations in the comoving gauges to the (a)-th component are also frame-independent, up to the rescalings by the background dependent quantities, as for the other gauge-invariant constructions of the scalar field perturbations.

The cosmological perturbations in the both frames are related as

\[\frac{\delta m^{(a)}}{\rho^{(a)}} = \frac{\delta m^{(a)}}{\rho^{(a)}} \frac{\delta m^{(a)}}{\rho^{(a)}} \frac{\delta m^{(a)}}{\rho^{(a)}} \]  

(58)

Thus their differences are determined by \(\Sigma^{(a)}\) and the scalar field perturbations in the corresponding gauges.

The isocurvature perturbations between the (a)-th and (b)-th components are given by

\[S^{(ab)} = 3(\zeta^{(a)} - \zeta^{(b)}), \quad \dot{S}^{(ab)} = 3(\dot{\zeta}^{(a)} - \dot{\zeta}^{(b)}), \]  

(59)

and their difference is given by

\[\dot{S}^{(ab)} - S^{(ab)} = -3 \frac{\dot{\zeta}^{(a)}}{\frac{\rho^{(a)}}{\rho^{(a)}} + \frac{\dot{\rho}^{(a)}}{\rho^{(a)}} + \frac{\rho^{(a)}}{\rho^{(a)}}} \frac{\dot{\zeta}^{(b)}}{\frac{\rho^{(b)}}{\rho^{(b)}} - \frac{\dot{\rho}^{(b)}}{\rho^{(b)}}} \frac{\dot{\zeta}^{(a)}}{\frac{\rho^{(a)}}{\rho^{(a)}} - \frac{\dot{\rho}^{(a)}}{\rho^{(a)}}} \frac{\dot{\zeta}^{(b)}}{\frac{\rho^{(b)}}{\rho^{(b)}} - \frac{\dot{\rho}^{(b)}}{\rho^{(b)}}} \Sigma \]  

(60)

The curvature perturbations in the uniform energy density and comoving gauges to total matter are defined by

\[-\zeta := \psi + \frac{\dot{a}}{a} \frac{\delta \rho}{\rho}, \quad \dot{R}_{c} := \psi - \frac{\dot{a}}{a} (\rho + p)(v + B), \]  

\[-\dot{\zeta} := \dot{\psi} + \frac{\dot{a}}{a} \frac{\delta \rho}{\rho}, \quad \dot{R}_{c} := \dot{\psi} - \frac{\dot{a}}{a} (\rho + p)(v + B) \]  

(61)

which from Eqs. [42], [43], [60] and [17] are rewritten as

\[\zeta = \sum_{a} \frac{\rho^{(a)}}{\rho + p} R_{c}^{(a)}, \quad \dot{\zeta} = \sum_{a} \frac{\dot{\rho}^{(a)}}{\rho + p} \dot{R}_{c}^{(a)} \]  

(62)
Similarly, the scalar field perturbation in the comoving gauges to total matter are defined by
\[
\delta_m \phi := \delta \phi + a \dot{\phi} (v + B),
\]
\[
\hat{\delta}_m \phi := \delta \dot{\phi} + \dot{a} \phi (\dot{v} + \dot{B}),
\]
which from (60) are rewritten as
\[
\delta_m \phi = \sum_a \rho^{(a)} + \rho^{(a)} \delta_m \phi, \\
\hat{\delta}_m \phi = \sum_a \rho^{(a)} + \rho^{(a)} \hat{\delta}_m \phi.
\]
The matter cosmological perturbations in the longitudinal gauges are given by
\[
\delta_\gamma \rho = \sum_a \delta_\gamma \rho^{(a)}, \quad \hat{\delta}_\gamma \hat{\rho} = \sum_a \hat{\delta}_\gamma \hat{\rho}^{(a)}.
\]
Finally, the total matter density perturbation in the comoving gauges to total matter is given by
\[
\delta_m \rho := \delta \rho + a(v + B) \dot{\rho}, \quad \hat{\delta}_m \hat{\rho} := \delta \dot{\rho} + \dot{a}(\dot{v} + \dot{B}) \hat{\rho}.
\]
Having the gauge-invariant perturbation variables and their differences between frames, in the next section we will derive the gauge-invariant evolution equations of the curvature perturbations in both frames.

V. EVOLUTION OF CURVATURE PERTURBATIONS IN THE UNIFORM ENERGY DENSITY GAUGES

In this section, we derive the evolution equations of the curvature perturbation in the uniform energy density gauge of each component. Our analysis in this section is based on the energy (non)conservation in the matter sector and hence independent of the choice of the gravity sector. In this section, we will work in the Fourier space and replace the spatial derivatives with the comoving momentum as \( \delta_k \rightarrow ik \) and \( \Delta \rightarrow -k^2 \) in the perturbation equations.

A. Evolution of curvature perturbation in the matter frame

In the matter frame \( \hat{g}_{\mu \nu} \), by definition matter is not directly coupled to the scalar field. Under the assumption that there is no interaction between matter components, the energy-momentum tensor of the \((a)\)-th component of matter is conserved separately \( \nabla_\mu T^{(a)\mu} = 0 \). Thus, the derivation of the evolution equation in the matter frame follows (62). The background part of the energy-momentum conservation law of the \((a)\)-th component is given by
\[
\dot{\rho}^{(a)} + 3 \frac{\dot{a}}{a} (\rho^{(a)} + \dot{\rho}^{(a)}) = 0.
\]
The perturbation part of the energy-momentum conservation law of the \((a)\)-th component, with use of the background relation (67), provides the evolution equation of the curvature perturbation in the uniform energy density gauge of the \((a)\)-th component
\[
\dot{\zeta}^{(a)} = -\frac{1}{\dot{\rho}^{(a)} + \dot{\rho}^{(a)}} \frac{\dot{a}}{a} \Gamma^{(a)} + \frac{1}{3} k^2 \left( \hat{\mathcal{E}}^{(a)} + \hat{\psi}^{(a)} \right),
\]
where the \( k^2 \) terms in (68) are suppressed on superhorizon scales. If we analyze the cosmological dynamics in the matter frame, after solving the evolution equation of the curvature perturbation for each component (68) with the gravitational and scalar field equations of motion, the curvature perturbation in the uniform energy density gauge to total matter \( \hat{\zeta} \) can be obtained from the second relation of (62).

B. Evolution of curvature perturbation in the gravity frame

The evolution of the curvature perturbation is more involved in the gravity frame \( \hat{g}_{\mu \nu} \) than in the matter frame \( \hat{g}_{\mu \nu} \), as the matter energy-momentum tensor is not conserved as (9). Following (25), we first derive the nonconservation of the energy-momentum tensor of matter,
\[
\nabla_\mu E^\mu_\nu = -\nabla_\mu T^\mu_\nu = Q \phi_\nu,
\]
where the coupling strength \( Q \) in the right-hand side can be seen explicitly from the divergence of the gravitational tensor
\[
\nabla_\mu E^\mu_\nu = \left\{ \frac{\partial L_g}{\partial \phi} - \nabla_\mu \left( \frac{\partial L_g}{\partial \phi_\mu} + \nabla_\alpha \nabla_\beta \left( \frac{\partial L_g}{\partial \phi_{\alpha \beta}} \right) \right) \right\} \phi_\nu =: Q \phi_\nu.
\]
As the terms in the curly bracket are equal to \( \frac{\delta L_g}{\delta \phi} \), the equation of motion of the scalar field \( \frac{\delta L_g}{\delta \phi} + \frac{\delta L_g}{\delta \overline{g}} \left( \sqrt{-g} L_m \right) = 0 \) allows us to write the coupling term in terms of the variation of the matter Lagrangian as
\[
Q = -\frac{1}{\sqrt{-g}} \left\{ \frac{\partial}{\partial \phi} \left( \sqrt{-g} L_m \right) - \nabla_\mu \left( \frac{\partial \left( \sqrt{-g} L_m \right)}{\partial \phi_\mu} \right) \right\}.
\]
Note that \( \frac{\partial}{\partial \phi_\mu} \left( \sqrt{-g} L_m [\hat{g}, \Psi] \right) = 0 \) and hence its contribution does not appear in (71), as the matter frame metric \( \hat{g}_{\mu \nu} \) defined in (2) does not contain the second order derivative of the scalar field. Then, using the chain rule, (2) and (5), each term in \( Q \) can further be rewritten
as
\[ \frac{\partial (\sqrt{-g}L_m)}{\partial \phi^\mu} = \frac{\partial (\sqrt{-g}L_m)}{\partial g_{\rho\sigma}} \frac{\partial g_{\rho\sigma}}{\partial g_{\alpha\beta}} \frac{\partial g_{\alpha\beta}}{\partial \phi^\mu}, \]
\[ = \sqrt{-g}T_{(m)}^{\mu\nu} \frac{\beta}{\alpha} \phi^\nu. \]
\[ \frac{\partial (\sqrt{-g}L_m)}{\partial \phi} = \frac{\partial (\sqrt{-g}L_m)}{\partial g_{\rho\sigma}} \frac{\partial g_{\rho\sigma}}{\partial g_{\alpha\beta}} \frac{\partial g_{\alpha\beta}}{\partial \phi} = \frac{1}{2} \sqrt{-g}T_{(m)}^{\mu\nu} \frac{1}{\alpha} (\alpha' g_{\mu\nu} + \beta' \phi \phi^\nu). \quad (72) \]

Substituting (72) into (71),
\[ Q = \nabla_\rho \left( \frac{\beta}{\alpha} T_{(m)}^{\rho\sigma} \phi^\sigma \right) - \frac{1}{2\alpha} T_{(m)}^{\rho\sigma} (\alpha' g_{\rho\sigma} + \beta' \phi \phi^\sigma). \quad (73) \]

From (33), the coupling term can be decomposed into the contributions of each component
\[ Q^{(a)} := \nabla_\rho \left( \frac{\beta}{\alpha} T^{(a)\rho\sigma} \phi^\sigma \right) - \frac{1}{2\alpha} T^{(a)\rho\sigma} (\alpha' g_{\rho\sigma} + \beta' \phi \phi^\sigma). \quad (74) \]

and the divergence of the energy-momentum tensor of the \((a)\)-th component is given by
\[ \nabla_\mu T^{(a)\mu} = -Q^{(a)} \phi_\nu. \quad (75) \]

The background part of the energy-momentum nonconservation in the gravity frame (73) is given by
\[ \rho^{(a)} + \frac{3}{a} (\rho^{(a)} + p^{(a)}) + \frac{\beta}{\alpha} \rho^{(a)} \phi^2 + \frac{3}{a} (\rho^{(a)} + p^{(a)}) \phi \frac{d}{dt} \left( \frac{\beta}{\alpha} \phi \right). \quad (76) \]

The perturbation part is given by
\[ \frac{d}{dt} \delta_\phi \rho^{(a)} + \frac{\dot{\alpha}}{a} \delta_\phi \rho^{(a)} + \frac{\dot{\beta}}{2a^2} \rho^{(a)} \phi^2 - 2 \frac{\Sigma^{(a)} \phi}{\phi^2} \phi + \frac{\delta_\phi \rho^{(a)}}{\phi} \cdot \frac{d}{dt} \left( \frac{\Sigma^{(a)} \phi}{\phi^2} \right) + \frac{d}{dt} \left[ \frac{\beta}{\alpha} \rho^{(a)} \phi \left( \frac{\Sigma^{(a)} \phi}{\phi^2} \phi \phi^2 \right)^2 \phi^2 - \frac{1}{3} \frac{k^2 p^{(a)} \Pi^{(a)} = 0. \quad (77) \]

With the use of the relations between the gauge-invariant perturbations,
\[ \delta_\phi \rho^{(a)} = -\frac{\dot{\alpha}}{a} (\zeta^{(a)} + R_{\phi}^{(a)}), \]
\[ \delta_\phi p^{(a)} = \Gamma^{(a)} - \frac{\dot{\alpha}}{a} (\zeta^{(a)} + R_{\phi}^{(a)}), \]
\[ \delta_\phi \rho^{(a)} = -\frac{\dot{\alpha}}{\phi} \delta_\phi \rho^{(a)} \phi. \quad (78) \]

and the divergence of the energy-momentum tensor of the \((a)\)-th component is given by
\[ \nabla_\mu T^{(a)\mu} = -Q^{(a)} \phi_\nu. \quad (75) \]

The background part of the energy-momentum nonconservation in the gravity frame (73) is given by
\[ \rho^{(a)} + 3 \frac{\dot{\alpha}}{a} (\rho^{(a)} + p^{(a)}) + \frac{\beta}{\alpha} \rho^{(a)} \phi^2 + 3 \frac{\dot{\beta}}{a} \rho^{(a)} \phi^2 + \rho^{(a)} \phi \frac{d}{dt} \left( \frac{\beta}{\alpha} \phi \right). \quad (76) \]

The perturbation part is given by
\[ \frac{d}{dt} \delta_\phi \rho^{(a)} + \frac{\dot{\alpha}}{a} \delta_\phi \rho^{(a)} + \frac{\dot{\beta}}{2a^2} \rho^{(a)} \phi^2 - 2 \frac{\Sigma^{(a)} \phi}{\phi^2} \phi + \frac{\delta_\phi \rho^{(a)}}{\phi} \cdot \frac{d}{dt} \left( \frac{\Sigma^{(a)} \phi}{\phi^2} \phi \phi^2 \right)^2 \phi^2 - \frac{1}{3} \frac{k^2 p^{(a)} \Pi^{(a)} = 0. \quad (77) \]

With the use of the relations between the gauge-invariant perturbations,
\[ \delta_\phi \rho^{(a)} = -\frac{\dot{\alpha}}{a} (\zeta^{(a)} + R_{\phi}^{(a)}), \]
\[ \delta_\phi p^{(a)} = \Gamma^{(a)} - \frac{\dot{\alpha}}{a} (\zeta^{(a)} + R_{\phi}^{(a)}), \]
\[ \delta_\phi \rho^{(a)} = -\frac{\dot{\alpha}}{\phi} \delta_\phi \rho^{(a)} \phi. \quad (78) \]

The divergence of the energy-momentum tensor of the \((a)\)-th component is given by
\[ \nabla_\mu T^{(a)\mu} = -Q^{(a)} \phi_\nu. \quad (75) \]

The background part of the energy-momentum nonconservation in the gravity frame (73) is given by
\[ \rho^{(a)} + 3 \frac{\dot{\alpha}}{a} (\rho^{(a)} + p^{(a)}) + \frac{\beta}{\alpha} \rho^{(a)} \phi^2 + 3 \frac{\dot{\beta}}{a} \rho^{(a)} \phi^2 + \rho^{(a)} \phi \frac{d}{dt} \left( \frac{\beta}{\alpha} \phi \right). \quad (76) \]

The perturbation part is given by
\[ \frac{d}{dt} \delta_\phi \rho^{(a)} + \frac{\dot{\alpha}}{a} \delta_\phi \rho^{(a)} + \frac{\dot{\beta}}{2a^2} \rho^{(a)} \phi^2 - 2 \frac{\Sigma^{(a)} \phi}{\phi^2} \phi + \frac{\delta_\phi \rho^{(a)}}{\phi} \cdot \frac{d}{dt} \left( \frac{\Sigma^{(a)} \phi}{\phi^2} \phi \phi^2 \right)^2 \phi^2 - \frac{1}{3} \frac{k^2 p^{(a)} \Pi^{(a)} = 0. \quad (77) \]

With the use of the relations between the gauge-invariant perturbations,
\[ \delta_\phi \rho^{(a)} = -\frac{\dot{\alpha}}{a} (\zeta^{(a)} + R_{\phi}^{(a)}), \]
\[ \delta_\phi p^{(a)} = \Gamma^{(a)} - \frac{\dot{\alpha}}{a} (\zeta^{(a)} + R_{\phi}^{(a)}), \]
\[ \delta_\phi \rho^{(a)} = -\frac{\dot{\alpha}}{\phi} \delta_\phi \rho^{(a)} \phi. \quad (78) \]

The divergence of the energy-momentum tensor of the \((a)\)-th component is given by
\[ \nabla_\mu T^{(a)\mu} = -Q^{(a)} \phi_\nu. \quad (75) \]

The background part of the energy-momentum nonconservation in the gravity frame (73) is given by
\[ \rho^{(a)} + 3 \frac{\dot{\alpha}}{a} (\rho^{(a)} + p^{(a)}) + \frac{\beta}{\alpha} \rho^{(a)} \phi^2 + 3 \frac{\dot{\beta}}{a} \rho^{(a)} \phi^2 + \rho^{(a)} \phi \frac{d}{dt} \left( \frac{\beta}{\alpha} \phi \right). \quad (76) \]

The perturbation part is given by
\[ \frac{d}{dt} \delta_\phi \rho^{(a)} + \frac{\dot{\alpha}}{a} \delta_\phi \rho^{(a)} + \frac{\dot{\beta}}{2a^2} \rho^{(a)} \phi^2 - 2 \frac{\Sigma^{(a)} \phi}{\phi^2} \phi + \frac{\delta_\phi \rho^{(a)}}{\phi} \cdot \frac{d}{dt} \left( \frac{\Sigma^{(a)} \phi}{\phi^2} \phi \phi^2 \right)^2 \phi^2 - \frac{1}{3} \frac{k^2 p^{(a)} \Pi^{(a)} = 0. \quad (77) \]

With the use of the relations between the gauge-invariant perturbations,
where
\[
C_1^{(a)} = \frac{1}{6a^2(\rho^{(a)}(\alpha - \beta \dot{\phi}^2) + p^{(a)}(\alpha)} \left\{ 2a^2\beta \dot{\phi}^2 \dot{\rho}^{(a)} - 6p^{(a)}\alpha^2(\dot{a}^2 - a\ddot{a}) \right. \\
+ 2a^2 \left(-3\rho^{(a)}(\dot{a}^2 - a\ddot{a}) + a \left(3\dot{a}(\ddot{\rho}^{(a)} + \dot{p}^{(a)}) + a\dot{\rho}^{(a)} \right) \right) \\
+ \alpha \left[ 3a^2\dot{\rho}^{(a)}\dot{\phi} - a^2\dot{\phi}\dot{\rho}^{(a)} + 6\beta \rho^{(a)}\dot{\phi}^2 - 6a\beta a\dot{\rho}^{(a)}\dot{\phi}^2 - 2a^2\beta \rho^{(a)}\dot{\phi}^2 - 2a^2\beta \rho^{(a)}\dot{\phi}\dot{\rho}^{(a)} - 2a^2\beta \rho^{(a)}\dot{\phi}\dot{\rho}^{(a)} \right],
\]
\[
C_2^{(a)} = \frac{1}{3a(\rho^{(a)}(\alpha - \beta \dot{\phi}^2) + p^{(a)}(\alpha)} \left[ 3\dot{p}^{(a)}\dot{a} + (3\dot{\rho}^{(a)} \dot{a} + \dot{p}^{(a)})(\alpha - \beta \dot{\phi}^2) \right],
\]
\[
C_3^{(a)} = \frac{1}{3a}(\rho^{(a)}(\alpha - \beta \dot{\phi}^2) + p^{(a)}(\alpha)} \left[-a\alpha\rho^{(a)} \dot{\beta} + 2\beta \left\{ a\rho^{(a)} \dot{\alpha} - \alpha(3\dot{\rho}^{(a)} \dot{a} + \dot{p}^{(a)}) \right\} \right],
\]
\[
C_4^{(a)} = -\frac{2a\dot{a}^2 + a\ddot{a}}{2a(\rho^{(a)}(\alpha - \beta \dot{\phi}^2) + p^{(a)}(\alpha)}.
\]

The terms in the second line in (80), which are proportional to \(k^2\) are suppressed on superhorizon scales. As a quick check, in the limit of \(\alpha \to 1\) and \(\beta \to 0\) where two frames coincide, with use of the energy conservation law, (80) reduces to (68) without ‘hat’. Because of the disformal coupling of matter to the scalar field, in addition to the nonadiabatic pressure \(\dot{\Gamma}^{(a)}\), \(\delta_{\rho^{(a)}}\dot{\phi}\) and \(\Sigma^{(a)}\) are also sourcing the curvature perturbation on superhorizon scales. If we analyze the cosmological dynamics in the gravity frame, solving (79) for each component of the gravitational and scalar field equations of motion, the curvature perturbation in the uniform matter energy density gauge in the gravity frame \(\zeta\) is obtained from the first relation of (82).

Let us check the adiabaticity and the condition of the conservation of the curvature perturbations on superhorizon scales in both frames. Using (53), we find that when the nonadiabatic pressure of the \((a)\)-th component in the matter frame vanishes, \(\dot{\Gamma}^{(a)} = 0\), and hence the curvature perturbation in the uniform energy density gauge of the \((a)\)-th component in the matter frame is conserved on superhorizon scales, \(\dot{\zeta}^{(a)}_{s,t} = 0\), the nonadiabatic pressure in the gravity frame is given by

\[
\dot{\Gamma}^{(a)} = p^{(a)}(1 + \frac{\dot{\rho}^{(a)}}{\rho^{(a)}}, \frac{\dot{\rho}^{(a)}}{\rho^{(a)}} \frac{\Sigma^{(a)}}{\alpha - \beta \dot{\phi}^2}
- \frac{p^{(a)}}{\rho^{(a)}p^{(a)}} (\frac{\dot{\rho}^{(a)}}{\rho^{(a)}}, \frac{\dot{p}^{(a)}}{\rho^{(a)}}, \frac{\rho^{(a)}}{\dot{p}^{(a)}}, \frac{\dot{p}^{(a)}}{\rho^{(a)}}), \frac{\delta^{(a)} \phi}{\dot{\phi}}, (81)
\]

which does not vanish in general, meaning that the adiabaticity condition for the \((a)\)-th component does not hold in the gravity frame. On superhorizon scales, (79) reduces to

\[
\dot{\zeta}^{(a)} \approx \left[ c_1^{(a)} - \frac{\dot{p}^{(a)}}{\rho^{(a)}}, \frac{\dot{\rho}^{(a)}}{\rho^{(a)}}, \frac{\dot{p}^{(a)}}{\rho^{(a)}}, \frac{\rho^{(a)}}{\dot{p}^{(a)}} \right] c_4^{(a)} \frac{\delta^{(a)} \phi}{\dot{\phi}}
+ C_2^{(a)} \frac{d}{dt} \left( \frac{\delta^{(a)} \phi}{\dot{\phi}} \right)
+ \left[ c_3^{(a)} + \frac{\beta \rho^{(a)}}{\alpha - \beta \dot{\phi}^2} \left( 1 + \frac{\dot{p}^{(a)}}{\rho^{(a)}}, \frac{\dot{\rho}^{(a)}}{\rho^{(a)}}, \frac{\rho^{(a)}}{\dot{p}^{(a)}}, \frac{\dot{p}^{(a)}}{\rho^{(a)}} \right) C_4^{(a)} \right] \Sigma^{(a)},
\]

meaning that the curvature perturbation in the uniform energy density gauge in the gravity frame is not conserved, even if the corresponding curvature perturbation is conserved in the matter frame.

The arguments so far hold for any choice of the gravity sector, as the evolution equations of the curvature perturbations in the uniform energy density gauges (85) and (79) are derived solely from the energy (non)conservation in the matter sector (82). Of course, in order to obtain the closed sets of the equations of motion in each frame, the gravitational and scalar field equations of motion should be derived for a given gravity sector. In evaluating the observables, we need to choose a particular frame where the equations of motion are solved. As it is more convenient to analyze the cosmological dynamics in the Einstein frame than in the Jordan frame in the case of the scalar-tensor theory with the nonminimal coupling of the scalar field to the Ricci scalar, it would also be more convenient to analyze the cosmological dynamics in the gravity frame than in the matter frame. In the gravity frame, the evolution equations of the curvature perturbations (79) are then solved by combining the gravitational and scalar field equations of motion. As in the theories considered in this paper CMB photons follow null geodesics associated with the matter frame, the curvature perturbation in the uniform matter energy density gauge in the matter frame \(\dot{\zeta}\) should be finally obtained. Once the curvature perturbation in the gravity frame \(\dot{\zeta}^{(a)}\) is evaluated, the curvature perturbation in the matter frame...
is evaluated via the frame-transformation rule (51) or equivalently
\[ -\dot{\rho}^{(a)} \zeta^{(a)} = -f \dot{\rho}^{(a)} \zeta^{(a)} + \dot{f} \rho^{(a)} \mathcal{R}_c \phi + f \frac{\dot{\alpha}}{2\alpha} \delta \phi \rho^{(a)} + \frac{\dot{\alpha}}{2\alpha} \sum \phi^{(a)} \omega^{(a)} . \] (83)

The curvature perturbation on the uniform matter energy density gauge in the matter frame is then evaluated by the second relation of (62).

VI. CONCLUSIONS

As the extension of the case of the well-studied conformal transformation, in this paper we have derived the disformal transformation rules of the gauge-invariant cosmological perturbation variables. The disformal transformation has been argued in terms of applications to the specific cosmological problems, and more recently has been focused as the way to transform a class of Horndeski’s theory to another. We have considered a class of the scalar-tensor theory where the gravity and matter frame metrics are related by the disformal transformation (2). We have assumed a class of Horndeski’s theory as the gravity sector and noninteracting fluids disformally coupled to the scalar field as the matter sector. We have also assumed that the disformal coupling to matter is common, in the sense that the form of the coupling is independent of the matter components.

Concerning the perturbations associated with the gravity sector, it is straightforward to confirm that the curvature perturbation in the comoving gauge to the scalar field is invariant under the disformal transformation as well as the conformal transformation. In case that only the scalar field is responsible both for inflation and cosmological perturbations, the frame-independent curvature perturbation comoving to the scalar field is conserved on superhorizon scales and gives the final predictions from inflation. Having the disformally invariant curvature perturbation, it can be evaluated in the most convenient frame, as usually done in the Einstein frame in the case of the scalar-tensor theory with the nonminimal coupling of the scalar field to the Ricci scalar. Since the disformal transformation (2) is induced by the scalar field, the vector- and tensor-type perturbations are invariant under the disformal transformation. Combining with the fact that the comoving curvature perturbation is invariant, the tensor-to-scalar ratio is also invariant in the single-field inflation models.

Concerning the perturbations associated with the matter sector, in the matter frame the curvature perturbation in the uniform matter energy density gauge is conserved on superhorizon scales, if the nonadiabatic pressure defined in the matter frame vanishes. On the other hand, in the gravity frame the curvature perturbation in the uniform matter energy density gauge is not conserved, even if the nonadiabatic pressure defined in the gravity frame vanishes. As the evolution equations of the curvature perturbation were obtained from the energy (non)conservation in the matter sector, the evolution equations of the curvature perturbation hold for any choice of the gravity sector.

Before closing this paper, we would like to mention the related works. First, we emphasize that the results presented in this paper and the disformal invariance of vector and tensor perturbations would apply only to the linear perturbations. Therefore, the analysis beyond the linear order would be very important. There would also be various extensions of the present work: more general disformal transformation (1) which would allow for the healthy scalar-tensor theory beyond Horndeski, component-dependent disformal couplings, disformal transformations by other field species (18) and so on. We hope to come back to these issues in our future publications.

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Appendix A: Intrinsic entropy perturbation in the single field inflation models

In this appendix, we will confirm that the gauge-invariant perturbation \( \Sigma^{(a)} \) is suppressed on superhorizon scales in the general single-field models of inflation in which the gravity action is given by the Einstein gravity, i.e., \( G_4 = \frac{1}{2\kappa^2} \) and \( G_5 = 0 \) in (51).

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + P(X, \phi) - G(X, \phi) \Box \phi \right] . \] (A1)

This theory involves the single-field slow-roll and k-inflation models for \( G = 0 \) and models with the generalized cubic galileon for \( G \neq 0 \). For \( G = 0 \), the suppressed intrinsic entropy perturbation was argued in (34).

Varying the action with respect to the metric \( g_{\mu \nu} \), the gravitational equation of motion is obtained as

\[ G_{\mu \nu} = \kappa^2 T^{(\phi)}_{\mu \nu} , \]
\[ T^{(\phi)}_{\mu \nu} = P g_{\mu \nu} + P X \phi_\nu \phi_\nu - G X \Box \phi_\mu \phi_\nu + g^{\rho \sigma} J_{\rho} \phi_\sigma g_{\mu \nu} - \left( J_{\mu} \phi_\nu + J_{\nu} \phi_\mu \right) , \]
\[ J_{\rho} := \frac{G_{\rho} \phi_\rho - G X g^{\alpha \beta} \phi_\alpha \phi_\beta} . \] (A2)

The background equation of motion of the scalar field is
given by
\[
0 = \dot{\phi} + \left(\frac{3\dot{a}}{a} + \frac{P_{X\phi}\ddot{\phi} + P_{XX}\ddot{\phi}}{P_X}\right)\dot{\phi} - \frac{P_{\theta}}{P_X} \\
+ 2 \left(-G_{\phi} + \frac{3\dot{a}}{a} G_X \phi\right)\dot{\phi} \\
+ 3 \left(G_{X\phi} + G_{XX}\dot{\phi}\right)\ddot{\phi}^2 + 3G_X \left(3\frac{\ddot{\phi}}{a^2} + 3\frac{\dot{\phi}^2}{a^2}\right)\dot{\phi}^2 \\
- \left(G_{\phi\phi}\dot{\phi} + G_X\phi\ddot{\phi} + 6\frac{\dot{a}}{a} G_{\phi}\phi\right)
\] (A3)

The background part of the Laplacian is given by \(\Box\phi = -\ddot{\phi} - 3\frac{\dot{a}}{a} \dot{\phi}\). The perturbation part of the Hamiltonian and momentum constraints obtained from (A2) is given by
\[
\frac{3\dot{a}}{a}(\psi + \frac{\dot{a}}{a} A) + \frac{k^2}{a^2} \left[\psi + a\dot{a} \left(\dot{E} - \frac{B}{a}\right)\right] = -\frac{\kappa^2}{2} \delta \rho^{(\phi)}, \\
\psi + \frac{\dot{a}}{a} A = -\frac{\kappa^2}{2} \delta q^{(\phi)} \delta A
\]
where

\[
\delta \rho^{(\phi)} = -\left(P_X\delta \phi + P_X\delta X\right) + 2XP_X \left(\frac{P_{X\phi}}{P_X} \delta \phi + \frac{P_{XX}}{P_X} \delta X + 2\frac{\ddot{\phi}}{\phi} - 2A\right) \\
- G_X \left(\Box \phi\right) \delta \phi^2 \left(\frac{G_{X\phi}}{G_X} \delta \phi + \frac{G_{XX}}{G_X} \delta X + \frac{\delta (\Box \phi)}{\Box \phi} - 2A + 2\frac{\ddot{\phi}}{\phi}\right) \\
- \dot{\phi}^2 \left(\frac{G_{\phi}}{G_X} \delta \phi + \frac{G_X}{G_X} \delta X - 2A + \frac{\ddot{\phi}}{\phi} + \frac{\ddot{\phi}}{\phi} - \frac{\dot{a}}{a} \right), \\
\delta q^{(\phi)} = -2XP_X \frac{\delta \phi}{\phi} + \left[G_X \left(\Box \phi\right) \delta \phi + \dot{\phi} \left(\frac{G_{\phi} \delta \phi + G_X \phi \left(\delta \phi - A\dot{\phi}\right)\right) + \left(G_{\phi} + G_X \phi\right) \frac{\dot{\phi}}{\phi}\right]
\] (A5)

with \(\delta X = -\ddot{A} \dot{\phi}^2 + \frac{\dot{A}^2}{\phi} + \frac{3\dot{\phi}}{a^2} \frac{\dot{\phi}}{\phi} + 2A \left(\frac{\dot{\phi}}{\phi} + \frac{3\dot{\phi}}{a^2} \frac{\dot{\phi}}{\phi} + \frac{3\dot{\phi}}{a^2} \frac{\dot{\phi}}{\phi} + \frac{3\dot{\phi}}{a^2} \frac{\dot{\phi}}{\phi}\right)\phi + 3\psi + \frac{1}{a^2} \Delta \delta \phi^B.\) Combining these constraints, we obtain \(-\frac{\kappa^2}{a^2} \psi = \frac{\kappa^2}{2} \delta m \rho^{(\phi)}\) where
\[
\delta m \rho^{(\phi)} = \left\{-\left(P_X + 2XP_{XX}\right) - \frac{\dot{\phi}^2}{\phi} G_{\phi\phi} + \frac{3\dot{\phi}}{a^2} G_{XX}\phi\right\} \\
- \frac{6\dot{a}}{a} \frac{\dot{\phi}}{\phi} + 3\frac{\dot{\phi}}{a^2} \phi + \left(2\frac{\dot{\phi}^2}{\phi} + a\ddot{\phi} - a\dot{a}\phi\right) G_X \left(\Box \phi\right) \Sigma^{(\phi)} + 3\frac{\dot{\phi}^3}{a^3} G_X \dot{\Sigma}^{(\phi)} \right\}
\] (A6)

The third line of the right-hand side of (A6) vanishes when the comoving curvature perturbation is conserved \(\dot{\Sigma}^{(\phi)} = 0\) [35]. Thus in this case \(\Sigma^{(\phi)}\) is also suppressed on superhorizon scales.

It is more involved to check whether \(\Sigma^{(\phi)}\) is suppressed for the more general gravitational theory where the gravity action also contains nonminimal couplings to the curvature tensors with \(G_4 \neq \text{const} \) and \(G_5 \neq 0\). But, with the (inverse) disformal transformation \(\gamma_{\mu\nu} = \frac{1}{\alpha} \left(g_{\mu\nu} - \beta \phi_{\mu} \phi_{\nu}\right)\), it is possible to rewrite the Einstein gravity with the form of the disformal coupling

\[
\int d^4x \sqrt{-g} \left[R + \ldots\right] = \int d^4x \sqrt{-\hat{g}} \left[\hat{G}_4 \hat{X}, \phi\right] \hat{R} + \hat{G}_4 \hat{X} \left(\hat{X}, \phi\right) \left((\Box \phi)^2 - (\hat{\nabla}_\mu \hat{\nabla}_\nu \phi)(\hat{\nabla}^\mu \hat{\nabla}^\nu \phi)\right) + \ldots\right. \] (A7)

with \(\hat{G}_4 \hat{X}, \phi := \frac{1}{2} (1 - 2\beta \hat{X}) \frac{\hat{X}}{\alpha} \). This involves the model with the nonminimal coupling \(\hat{G}_4 = \alpha \mu^{(\phi)}\) for \(\beta = 0\) [12–14, 38] and DBI galileon for \(\beta \neq 0\) [25]. As \(\Sigma^{(\phi)}\) in the original frame is suppressed by \(k^2\), through the relation [37] \(\Sigma^{(\phi)}\) is also suppressed on superhorizon scales.

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