Instantons and Non-Perturbative Dynamics of $N = 2$ Supersymmetric Abelian Gauge Theories in Two Dimensions.

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Abstract

We study $N = 2$ supersymmetric Abelian gauge model with the Fayet-Iliopoulos term and an arbitrary number of chiral matter multiplets in two dimensions. By analyzing the instanton contribution we compute the non-perturbative corrections to the mass spectrum of the theory and the quantum deformation of the classical vacua space. In contrast to known examples the non-perturbative bosonic potential is saturated by the one-instanton contribution and can be directly found within the semiclassical expansion around the one-instanton saddle point.
1 Introduction

The non-perturbative aspects of supersymmetric field theories has been extensively studied and it has become clear that a wide class of their properties can be analyzed exactly. It has been also found that the non-perturbative dynamics of supersymmetric theories is essentially governed by instantons. Thus the instanton based calculations result in determination of the vacuum structure of some $N = 1$ supersymmetric non-Abelian gauge theories in $d = 4$ dimensions [1, 2, 3]. Similar results have been obtained in $d = 2$ supersymmetric sigma models [4, 5]. Another example is the four-dimensional non-Abelian gauge theories with $N = 2$ extended supersymmetry where the exact solution for the quantum moduli space is known which sums up all terms of the instanton expansion [6, 7]. The instantons play an essential role also in dynamics of $d = 3$ supersymmetric gauge models [8, 9, 10].

We consider the effects of the instantons in the $d = 2$, $N = 2$ supersymmetric Abelian gauge theory with the Fayet-Iliopoulos term and an arbitrary number of chiral matter multiplets. The models of this type were considered in the context of Calabi-Yao/Landau-Ginzburg correspondence in ref. [11] where some instanton effects were announced. In the present paper we develop the semiclassical perturbation theory (PT) around the one-instanton saddle point. By analyzing the instanton contribution to a set of Green functions (GFs) we construct the effective action which describes the effects of the instantons at large distance. As usual the instantons violate the non-renormalization theorems and induce the correction to the mass spectrum and the deformation of the classical vacua space of the model. However, in contrast to other supersymmetric theories the non-perturbative bosonic potential which determines the spectrum and the vacuum structure of the model is saturated by the one-instanton contribution and can be directly computed within the PT around the one-instanton solution.
2 The model

The $N = 2$ supersymmetric Abelian gauge model in two dimensions can be obtained by dimensional reduction of the $d = 4$, $N = 1$ supersymmetric QED \[11, 12\]. The model is constructed from the charged chiral multiplets $\Phi^i$ ($i = 1, \ldots, N_f$) and the gauge vector multiplet $V$. The (anti)chiral superfields have the following expansion in terms of the component fields

$$
\Phi^i(x, \theta) = \phi^i(y) + \sqrt{2} \theta \psi^i(y) + \theta \theta F^i(y),
$$

$$
\bar{\Phi}^i(x, \theta) = \bar{\phi}^i(\bar{y}) + \sqrt{2} \theta \bar{\psi}^i(\bar{y}) + \theta \theta \bar{F}^i(\bar{y}),
$$

$$
y_{\mu} = x_{\mu} + i \theta \gamma_{\mu} \theta.
$$

The expansion of the gauge superfield reads (in Wess-Zumino gauge)

$$
V = -\bar{\theta} \gamma^\mu \theta v_\mu - \theta \frac{1 + \gamma_5}{2} \theta \sigma - \bar{\theta} \frac{1 - \gamma_5}{2} \theta \sigma + i \theta \theta \bar{c} \gamma_{\sigma} \lambda_c - i \theta \theta \bar{c} \gamma_{\lambda} \sigma + \frac{1}{2} \theta \theta \bar{c} \gamma_{\lambda} \sigma D.
$$

The (anti)chiral superfields obey the constraints $\bar{D} \Phi^i = 0$, $D \Phi^i = 0$ where $D$ $\bar{D}$ stands for the supercovariant derivative

$$
D = \frac{\partial}{\partial \theta_c} - i \bar{\theta} \gamma_{\mu} \theta_v^\mu,
$$

$$
\bar{D} = -\frac{\partial}{\partial \theta_c} + i \theta \gamma_{\mu} \theta_v^\mu,
$$

where $\bar{\theta} = \gamma^\mu \partial_{\mu}$. In two dimensions it is possible to construct also the twisted chiral multiplet $\Sigma$ by acting with the supercovariant derivatives on the vector multiplet

$$
\Sigma = \frac{1}{\sqrt{2}} \bar{D}_L D_L V(z) = \sigma(z) + i \sqrt{2} \bar{\theta}_L \bar{\lambda}_R(z) - i \sqrt{2} \bar{\theta}_R \lambda_L(z) - \sqrt{2} \theta_L \bar{\theta}_R \left( D(z) + i \epsilon_{\mu \nu} v_{\mu \nu}(z) \right).
$$

\footnote{See Appendix A for spinor algebra notations.}
where \( v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \) and \( z_\mu = x_\mu + i\bar{\theta}\gamma_\mu\gamma_5\theta \). The twisted chiral superfield obeys the constraints \( D_L \Sigma = \overline{D}_R \Sigma = 0 \). The twisted antichiral superfield \( \overline{\Sigma} \) is a complex conjugate of eq. (4) and obeys \( D_R \Sigma = \overline{D}_L \Sigma = 0 \).

In the superspace \((\theta, \bar{\theta}, x)\) supersymmetry is realized by the operators

\[
Q = \frac{\partial}{\partial \theta_c} + i\bar{\theta}\theta_c,
\]

\[
\overline{Q} = -\frac{\partial}{\partial \theta_c} + \bar{\theta}\theta_c,
\]

with the following algebra

\[
\{Q, \overline{Q}\} = -2i\bar{\theta} + Z.
\]

In eq. (6) \( Z \) is a complex central charge corresponding to the momenta in the reduced directions of the \( N = 1 \) algebra in four dimensions [10].

The supersymmetry transformation of the component fields can be directly obtained from eqs. (1, 2, 5). However, it is relatively complicated [11] because the supersymmetry transformations have to be accompanied by gauge transformation to preserve the Wess-Zumino gauge.

The Lagrangian of the model has the form [11, 12]

\[
L = L_{\text{matter}} + L_{\text{gauge}} + L_\zeta,
\]

where the first two terms in the right hand side are the kinetic energy of the chiral fields and the kinetic energy of the gauge field respectively. The third term is a twisted chiral superpotential which incorporates both the Fayet-Iliopoulos term and the theta angle [11]. No chiral superpotential interaction is allowed in our model because we choose all the chiral fields to be of the same gauge charge. In terms of superfields the right hand side of eq. (7) reads

\[
L_{\text{matter}} = \int d^2x d^4\theta \sum_i \overline{\Phi}_i e^{2V}\Phi^i.
\]
\[ L_{\text{gauge}} = -\frac{1}{4e^2} \int d^2x d^4\theta \Sigma \Sigma, \quad (9) \]

\[ L_\zeta = \frac{\zeta}{2\sqrt{2}} \int d^2x d\theta L d\bar{\theta} R \Sigma |_{\theta_R = \bar{\theta}_L = 0} + \text{h.c.}, \quad (10) \]

where \( e \) is a gauge coupling constant of dimension one in mass units and \( \zeta = \eta^2 - i\theta/(2\pi) \) with real \( \eta^2 \) and \( \theta \). The component expansion of the first two terms in eq. (7) is

\[ L_{\text{matter}} = \int d^2x \sum_i \left( -D^\mu \bar{\phi}^i D_\mu \phi^i + i\bar{\psi}^i D^\mu \psi^i + \bar{F}^i F^i - 2\bar{\sigma} \sigma\bar{\phi}^i \phi^i + D\bar{\phi}^i \phi^i - \sqrt{2}\sigma \bar{\psi}^i \psi^i \right), \quad (11) \]

\[ L_{\text{gauge}} = \frac{1}{e^2} \int d^2x \left( -\frac{1}{4} v^{\mu\nu} v_{\mu\nu} - \partial^\mu \bar{\sigma} \sigma \partial_\mu \phi + \frac{1}{2} D^2 \right), \quad (12) \]

where \( D_\mu = \partial_\mu - iv_\mu \). Eq. (10) in components reads

\[ L_\zeta = - \int d^2x \left( \eta^2 D + \frac{\theta}{2\pi} \frac{\varepsilon^{\mu\nu} v_{\mu\nu}}{2} \right) \quad (13) \]

i.e. the constants \( \eta^2 \) and \( \theta \) parameterize the Fayet-Iliopoulos and theta angle terms respectively.

Let us consider some properties of the model. At the classical level it has left- and right-moving \( R \) symmetries [11] which provides us with the standard non-renormalization theorems [14]. There is also a chiral symmetry which rotates \( \psi^i, \sigma \) and \( \lambda \) fields and their complex conjugates so that \( (\psi^i_L, \psi^i_R, \lambda_L, \lambda_R, \sigma) \) have charges \((-1, 1, -1, 1, -2)\). Both the \( R \) and chiral symmetries are anomalous at the quantum level. Classically the chiral invariance of the Lagrangian leads to conservation of the axial current

\[ J_5^\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi - \frac{1}{e^2} \bar{\lambda} \gamma_\mu \gamma_5 \lambda + \frac{1}{e^2} (\bar{\sigma} \partial_\mu \sigma - \sigma \partial_\mu \bar{\sigma}), \quad (14) \]

so \( \partial^\mu J_5^\mu = 0 \). However, non-invariance of the partition function under the chiral transformation results in the “diangle” anomaly in the divergence of the axial current [15]

\[ \partial^\mu J_5^\mu = \frac{1}{2\pi} N_f \varepsilon^{\mu\nu} v_{\mu\nu}. \quad (15) \]
Because of the anomaly the $\theta$ angle transforms under the chiral rotation with the parameter $\alpha_\chi$ as follows

$$\theta \rightarrow \theta - 2N_f\alpha_\chi.$$  

(16)

Thus all values of the $\theta$ angle are equivalent and one can put $\theta = 0$.

The equation of motion for the auxiliary fields can be solved with the result

$$D = -e^2(\sum_i \bar{\phi}^i \phi^i - \eta^2),$$

$$F^{i} = 0.$$  

(17)

After eliminating the auxiliary fields the potential energy takes the form

$$U(\phi_i, \sigma, v_\mu) = \frac{e^2}{2} \left( \sum_i \bar{\phi}^i \phi^i - \eta^2 \right)^2 + (2\sigma\sigma - v^\mu v_\mu) \sum_i \bar{\phi}^i \phi^i.$$  

(18)

Thus for a non-zero value of the parameter $\eta$ at the classical level the Higgs effect takes place while the Coulomb phase is absent. In the case $N_f = 1$ no massless fields survive. The spectrum consists of the massive vector field and the real scalar field and there are two distinct vacua $\phi = \pm \eta$. When $N_f > 1$ vanishing of $U$ requires

$$\sum_i \bar{\phi}^i \phi^i = \eta^2,$$  

(19)

i.e. the classical vacua space of the model is a $CP^{(N_f-1)}$ manifold with the Kähler class $\eta^2$ and the massless fields form a supersymmetric $CP^{(N_f-1)}$ model. Note that for an arbitrary $N_f$ the Witten index $(-1)^F$ is non-zero so spontaneous supersymmetry breaking does not happen even at the quantum level.

3 $N_f = 1$ model in detail
3.1 Mass spectrum and running coupling

When \( N_f = 1 \) and \( \eta \neq 0 \) all the fields get masses via the Higgs mechanism. The mass spectrum saturates the Bogomol’nyi-Prasad-Sommerfield (BPS) bound \[17, 18, 19, 20\]

\[
m \geq \frac{1}{2} |Z|,
\]

so both the gauge and chiral fields are the sum of short multiplets \([10]\). The central term is a combination of \( U(1) \) charges \( Y \) and \( \bar{Y} \)

\[
Z = 2\sqrt{2}e\eta \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix},
\]

where the \( Y \) transformation mixes the components of the vector and chiral multiplets:

\[
\delta \phi \sim \sigma/e, \quad \delta \phi \sim \sigma/e, \quad \delta \psi_R \sim i\lambda_R/e, \quad \delta \psi_L \sim i\lambda_L/e, \ldots,
\]

and the \( \bar{Y} \) transformation is a Hermitian conjugate of \( Y \):

\[
\delta \phi \sim \bar{\sigma}/e, \quad \delta \bar{\phi} \sim \bar{\sigma}/e, \quad \delta \bar{\psi}_R \sim -i\lambda_R/e, \quad \delta \bar{\psi}_L \sim -i\bar{\lambda}_L/e, \ldots.
\]

Using eq. \(21\) we find the masses of the gauge and matter fields to be equal to

\[
m = \sqrt{2}e\eta.
\]

Let us consider the quantum corrections to this classical expression. The PT of the model has a natural dimensionless expansion parameter

\[
g^2 = \frac{1}{4\pi} \sim \frac{e^2}{m^2},
\]

so the PT is valid if \( \eta \) is large enough. In the one loop approximation the vacuum expectation value (VEV) \( \eta \) is renormalized by the scalar tadpole \([11]\)

\[
\eta^2(\mu) = \frac{1}{2\pi} \ln \frac{\mu}{\Lambda},
\]

where \( \mu \) is an ultra-violet cutoff and \( \Lambda \) parameterizes the infrared behavior of \( \eta \). Eq. \(24\) corresponds to the non-zero \( \beta \)-function of \( g \) coupling

\[
\beta = \frac{g^2}{4\pi^2} \beta_0, \quad \beta_0 = 1.
\]
Thus instead of eq. (22) one gets the one-loop corrected expression

\[ m^2(\mu) = \frac{1}{\pi} e^2 \ln \frac{\mu}{\Lambda} \]  

(26)

In fact the superrenormalizability of the model and the standard non-renormalization theorems tell us that eq. (26) is the exact quantum analog of eq. (22) at least within the PT framework \[11\].

It is important that the currents associated with the left- and right-moving \( R \) symmetries are also anomalous because they couple to the chiral charged fermions. Since the \( R \) symmetry is anomalous the non-renormalization theorems do not work in the instanton sector. As a consequence eq. (26) can be modified by instantons.

### 3.2 Instanton solution and instanton measure

The described model is known to support the instanton solutions – the Abrikosov-Nielsen-Olesen vortices \[12, 21\]. The one-instanton solution is spherically symmetric and in the polar coordinates \((\alpha, r)\) can be parameterized as follows

\[ v^I_\mu = \varepsilon_{\mu\nu} \partial_\nu \Psi_v(r), \quad \partial_\nu \Psi_v(0) = 0, \]

\[ \phi_I = e^{i\alpha r} \eta(1 - \Psi_\phi(r)), \quad \Psi_\phi(0) = 1. \]  

(27)

where the functions \( \Psi_\phi(r) \) and \( \Psi_v(r) + \ln(Cer) \) exponentially decay at large \( r \) (the constant \( C \) is a characteristic of the solution). Meanwhile, there is no analytical expression of \( \Psi_\phi(r) \) and \( \Psi_v(r) \) in terms of elementary functions.

The instanton configuration satisfies the first-order Bogomol’nyi equations \[18\]

\[ (D - v_{01}) I = 0, \]

\[ (iD_0 + D_1) \phi_I = 0. \]  

(28)

As a consequence the instanton of the winding number \( N \)

\[ N = \frac{1}{2\pi} \int d^2x v_{01} \]  

(29)

\[ ^2 \text{The analytical continuation from the Minkowski to Euclidean metric is implied (see Appendix A).} \]
has the action $2\pi N\eta^2$ and is neutrally stable with respect to $N$ instantons of the winding number $1$ [18, 22].

In the instanton background the Dirac operator acting on the $\psi$ and $\lambda$ fields has zero modes [23]. It is convenient to arrange the $\psi_L$ and $\lambda_R$ fields into one two-component spinor

$$\chi = \begin{pmatrix} \psi_L \\ \lambda_R \end{pmatrix},$$

so the (Euclidean) Dirac operator on these components takes the form

$$iD = \begin{pmatrix} -D_0 + iD_1 & -i\sqrt{2}\phi \\ i\sqrt{2}\phi & -\partial_0 - i\partial_1 \end{pmatrix}.$$ (31)

Using the Bogomol’nyi equations it is possible to show that zero modes of positive chirality do not exist if the external field configuration has a positive winding number [12]. Then the chiral anomaly tells us that in the one-instanton background there is one complex zero mode of negative chirality [24]. This mode is generated by the (complex) supersymmetry transformation of the instanton field [25]

$$\chi^0 = \begin{pmatrix} \psi^0_L \\ \lambda^0_R / e \end{pmatrix} = \begin{pmatrix} \sqrt{2}(iD_0 - D_1)\phi_I \\ (D + v_0)I / e \end{pmatrix}.$$ (32)

The rest of the supersymmetry generators does not affect the instanton field because of eq. (28).

Now it is straightforward to compute the one-instanton contribution to the partition function – the instanton measure. The contribution of the two real bosonic zero modes associated with the translation invariance of the instanton solution to the measure is [26]

$$S_I \int d^2x_0,$$ (33)

where the collective coordinate $x_0$ corresponds to the position of the instanton and the factor $S_I$ normalizes the zero modes. The part of the measure corresponding to
the one complex fermionic zero mode reads \[23\]

\[ J^{-1} \int d\zeta_0 d\overline{\zeta}_0, \]
\[ (34) \]

where \( \zeta_0 \) is the Berezin integration variable and the factor

\[ J = 4S_f \]
\[ (35) \]

normalizes the zero mode \([32]\) to 1. The contribution from the bosonic and fermionic quadratic functional integration around the instanton solution due to the non-zero modes cancel each other because of supersymmetry \([27]\). As usual, the renormalization scale \( \mu \) appears in the measure as a contribution of the Pauli-Villars regulator fields to the fermionic and bosonic determinants due to the zero modes \([23]\). The power of \( \mu \) is exactly the first coefficient of the \( \beta \)-function \([23]\) \( \beta_0 = 1 \) which measures the difference between the numbers of bosonic and fermionic zero modes.

Multiplied by the exponent of the one-instanton action eqs. \([33, 34]\) give the final expression for the measure

\[ \frac{\Lambda}{4} \int d^2x_0 d\zeta_0 d\overline{\zeta}_0, \]
\[ (36) \]

where

\[ \Lambda = \mu e^{-2\pi \eta^2(\mu)}. \]
\[ (37) \]

Note that the PT corrections to this result exist due to the Yukawa coupling

\[ -\sqrt{2}\sigma \overline{\psi}_R \psi_L. \]
\[ (38) \]

### 3.3 Green functions and effective Lagrangian

Let us consider the following GFs

\[ G_{\chi\overline{\chi}}(x - y) = \langle 0 | \chi(x) \overline{\chi}(y) | 0 \rangle, \]
\[ (39) \]

\[ G_{\sigma}(x) = \langle 0 | \sigma(x) | 0 \rangle. \]
\[ (40) \]
These GFs vanish within the PT and have relevant chiral transformation properties to get the one-instanton contribution.

Using the instanton measure (30) one finds the GF (33) to be saturated by the fermionic zero mode (32)

\[
G_{\chi\chi}(x-y) = \frac{\Lambda}{4} \int d^2x_0 \chi^0(x-x_0)\bar{\chi}^0(y-x_0). \tag{41}
\]

From eqs. (28, 32, 41) one obtains the non-zero condensate of the \(\psi_L\) and \(\lambda_R\) fermionic components

\[
\frac{1}{e^2} \langle 0|\lambda_L\lambda_R|0 \rangle = \langle 0|\bar{\psi}_R\psi_L|0 \rangle = \pi \eta^2 \Lambda. \tag{42}
\]

Eq. (41) implies also that in the one-instanton approximation a non-local diagonal mass matrix for the \(\psi_L\) and \(\bar{\chi}R\) components appears

\[
\delta \hat{m}(x-y) = \int d^2x \mathcal{D}_x^0 G_{\chi\chi}(x-y) \mathcal{D}_y^0, \tag{43}
\]

where \(\mathcal{D}^0\) is the free Dirac operator

\[
i\mathcal{D}^0 = \begin{pmatrix}
-\partial_0 + i\partial_1 & -i\sqrt{2}\eta \\
i\sqrt{2}\eta & -\partial_0 - i\partial_1
\end{pmatrix}. \tag{44}
\]

Near the mass shell eq. (43) can be reduced up to the higher derivatives to the effective local mass matrix

\[
\delta \hat{m} = \int d^2x \hat{m}(x). \tag{45}
\]

Using eqs. (28, 29, 32) in the local limit one finds the effective real masses for the \(\lambda\) and \(\psi\) fields

\[
\delta m_{\chi\bar{\chi}} = \delta m_{\bar{\psi}\psi} = 8\eta^2 \pi^2 \Lambda \equiv \delta m. \tag{46}
\]

The fact that eqs. (42, 46) are equivalent for the \(\psi\) and \(\lambda\) fields is a consequence of the non-anomalous \(Y\) (\(\bar{Y}\)) symmetry. Because of the standard non-renormalization theorem \([4]\) the PT corrections to the two-point GF of the gauge fermion and therefore to the parameter \(\delta m_{\chi\bar{\chi}}\) are absent. On the other hand, the PT corrections to the
two-point GF of the matter fermion are possible due to the Yukawa coupling \( (38) \). However, taking into account the \( Y (\bar{Y}) \) symmetry one concludes that these corrections are also absent unless the symmetry is broken.

The calculation of the instanton contribution to the GF \( (10) \) is a little bit more complicated. It vanishes at the tree level because of the fermionic zero mode so one has to take into account the loop corrections to the partition function. Then one gets:

\[
G_\sigma(x) = 4\sqrt{2}\eta^2\pi^2\Lambda \equiv \sigma_0. \tag{47}
\]

Thus the instantons lead to the condensation of the lowest component of the vector superfield. A non-zero VEV of the \( \sigma \) field violates the \( Y (\bar{Y}) \) symmetry and induces the corrections to the two-point GF of the matter fermion field. We will consider these corrections below.

The contribution of the anti-instanton solution with a negative winding number to the partition function is a Hermitian conjugate of the instanton contribution. For example, the anti-instantons saturate the condensates and real mass terms for the \( \lambda_R \) and \( \overline{\psi}_L \) fermionic components.

Now we are able to write down the effective Lagrangian to describe the effects induced by the (anti-)instantons at large distance. This Lagrangian has to reproduce eqs. \( (46, 47) \) and respect all non-anomalous symmetries of the fundamental Lagrangian \( (7) \). These constraints are satisfied with the Lagrangian of the following unique form

\[
\delta L = \int d^2x d\theta^4 \left( \overline{\Phi} e^{2(V - \tilde{V})} \Phi - \overline{\Phi} e^{2\tilde{V}} \Phi \right) + \frac{\delta m}{4e^2} \left( \int d^2x d\theta_L d\overline{\theta}_R \right. \left. \sum^2 \right|_{\theta_R = \overline{\theta}_L = 0} + h.c. \right), \tag{48}
\]

where \( \tilde{V} \) is an auxiliary external vector superfield with the non-zero scalar component.

\(^3\)See Appendix B.
\( \bar{\sigma} = \sigma = \sigma_0 \). In components eq. (48) reads (in the Minkowski metric)

\[
\delta L = \int d^2x \left( -2(\sigma - \bar{\sigma})(\sigma - \bar{\sigma})\bar{\phi}\phi + 2\sigma\bar{\sigma}\phi\phi + \sqrt{2}\bar{\sigma}\psi_L\psi_R + \sqrt{2}\bar{\sigma}\psi_R\psi_L \right) -
\]

\[
- \frac{\delta m}{e^2} \left( \int d^2x \left( \frac{\sqrt{2}}{2} \sigma (D - i\nu_0) + \bar{\lambda}_R\lambda_L \right) + h.c. \right). \tag{49}
\]

The potential energy now is of the following form

\[
U(\phi, \sigma, v_\mu) = \frac{e^2}{2} \left( \phi\phi - \frac{\sqrt{2}}{2} \frac{\delta m}{e^2} (\sigma + \bar{\sigma}) - \eta^2 \right)^2 + \left( 2(\sigma - \bar{\sigma})(\sigma - \bar{\sigma}) - v^\mu v_\mu \right) \phi\phi. \tag{50}
\]

Eq. (50) determines the mass spectrum and the vacuum configuration in the one-instanton approximation. The VEV of the \( \sigma \) field is given by eq. (47). The VEV of the \( \phi \) field can be written in the gauge invariant form

\[
\langle 0 | \phi\phi | 0 \rangle = \eta^2 + \sqrt{2} \frac{\delta m \sigma_0}{e^2}. \tag{51}
\]

Then the instanton contribution leads to the mass splitting: a half of the states is of the mass \( m + \delta m/2 \) while another states are of the mass \( m - \delta m/2 \) (where we keep only the leading order term in \( \Lambda \)). Note that the central charge (21) is modified by eq. (48).

These results need some comments.

i) As it was pointed the expressions (46, 47) for the parameters \( \sigma_0 \) and \( \delta m \) are exact within the PT around the one-instanton solution. Thus the perturbative corrections to the partition function (36) are reduced to the condensation of the \( \sigma \) field. Meanwhile, the effective Lagrangian (49) can be modified by multi-instanton contributions.

ii) Taking into account the real mass term in eq. (49) one can reproduce within the effective theory the value of the gauge fermion condensate (42).

iii) Formally, the first part of eq. (49) contains the real mass term for the \( \psi \) field with mass being equal to \( \sqrt{2}\sigma_0 \). This term is a local approximation of the non-local mass term implied by the contribution of the fermionic zero mode to the two-point GF (39).
(see eqs. [46, 47]). However, when the correct vacuum configuration \( \sigma = \sigma_0 \) is used this mass term disappears. By the same reason the condensate of the matter fermion vanishes. In this way the PT corrections to the two-point GF of the matter fermion field (eqs. [42, 46]) are taken into account within the effective theory framework.

iv) Eq. (49) includes a bosonic part and in particular contains the non-diagonal mass term which mixes the bosonic components of the vector and chiral superfields. The effective coupling in front of the bosonic term is of order \( \Lambda \sim e^{\xi I} \) and therefore is saturated by the one-instanton contribution. This term of the effective Lagrangian can be directly found by analyzing the one-instanton contribution to the corresponding bosonic correlator. In contrast to the considered model in all known examples the non-perturbative bosonic potential is suppressed by an extra power of \( \Lambda \). In this case it can be only reconstructed by supersymmetry transformation from the fermionic part of the effective action while the dynamical calculations are not available.

v) Though eq. (51) is obtained by minimizing the effective potential in the one-instanton approximation there is no instanton contribution to this quantity because of the chiral selection rule. The non-perturbative variation of the VEV of the \( \phi \) field is of order \( \Lambda^2 \) so it corresponds to the contribution of the instanton–anti-instanton pair rather then the contribution of the single (anti-)instanton.

vi) Naively, the effective potential in the one-instanton approximation has an additional vacuum state with

\[
\phi = 0, \\
\sigma = -\frac{e^2 \eta^2}{\sqrt{2} \delta m}.
\]

This, however, contradicts to the constraint on the number of the bosonic vacuum states imposed by the Witten index analysis. The reason of this inconsistency is that eq. (49) is valid only for the region \( \sigma \sim 0 \) and \( \phi \sim \eta \) where the PT around the one-instanton solution is applicable. For the large \( \sigma \) small \( \phi \) region the anomalous
twisted superpotential is also known \[11, 28\]

\[
\frac{1}{4\sqrt{2\pi}} \int d^2 x d\theta_L d\theta_R \Sigma \ln (\Sigma/\mu)|_{\theta_R=\theta_L=0} + h.c. = 
\]

\[
= -\frac{1}{4\pi} \int d^2 x \left( (\ln (\sigma/\mu) + 1)(D - iv_{01}) + \sqrt{2} \frac{x_R \lambda_L}{\sigma} \right) - h.c. \quad (53)
\]

In particular for \(\eta^2 > 0\) the effective potential has no additional vacuum states so eq. (52) is a spurious vacuum. However this is an interesting and non-trivial problem to find the exact solution which interpolates between eq. (49) and eq. (53).

4 Generalization to \(N_f > 1\)

If \(N_f > 1\) the presence of the massless particles leads to the infrared divergence of the PT. To regularize the divergence we add to the Lagrangian a mass term of the form

\[
\int d^2 x d\theta^4 \sum_{i=2}^{N_f} \Phi^i \left( e^{-\left(m_r \bar{\theta}_R \theta_L + m_r \bar{\theta}_L \theta_R \right)/\sqrt{2}} - 1 \right) \Phi^i = 
\]

\[
= -\sum_{i=2}^{N_f} \int d^2 x \left( m_r^* m_r \phi^i \bar{\phi}^i + m_r^* \psi^i_R \psi^i_L + m_r \psi^i_L \psi^i_R \right), \quad (54)
\]

where \(m_r\) is a (complex) parameter. Thus \(\phi^1\) becomes the only field with a non-zero VEV. This term breaks \(SU(N_f)\) flavor symmetry of the original Lagrangian to \(SU(N_f - 1)\). The chiral symmetry is not broken by eq. (54) if we assume \(m_r\) to be of the chiral charge \(-2\). Then eq.(24) transforms to

\[
\eta^2(\mu) = \frac{1}{2\pi} \ln \left( \frac{\mu^{N_f}}{\Lambda |m_r|^{N_f-1}} \right), \quad (55)
\]

so \(\beta_0 = N_f\). The instanton solution for \(N_f > 1\) is identical to one of the \(N_f = 1\) model: the gauge field and the scalar field \(\phi^1\) are given by eq. (27) while \(\phi^i (i = 2, \ldots, N_f)\) are equal to zero\(^4\). However, the instanton measure is modified. At \(m_r = 0\) the Dirac

\(^4\)If some of the chiral superfields have opposite gauge charges the theory has a non-trivial moduli space and the structure of the instanton solution becomes much more complicated \[29\].
operator in the one-instanton background in addition to eq. (32) has $N_f - 1$ extra massless fermionic zero modes

$$\psi^i_L = e^{\Psi^v}, \quad \psi^i_R = 0, \quad i = 2, \ldots, N_f.$$  \hfill (56)

These modes cannot be obtained by a supersymmetry transformation of the instanton field. On the other hand they are related by the supersymmetry transformation to the additional bosonic zero modes

$$\phi^i = e^{\Psi^v}, \quad i = 2, \ldots, N_f.$$  \hfill (57)

Though these modes are non-normalizable they do contribute to the instanton measure because the numbers of the bosonic and fermionic modes are equal \cite{30}. At small $m_r$ the contribution of the zero modes (56) to the fermionic determinant is $m_r^{(N_f-1)}$ up to the higher order corrections in $m_r$. Taking into account the contribution of the regulator fields we find the total contribution of the fermionic zero modes (56) to the instanton measure to be

$$\left(\frac{m_r}{\mu}\right)^{(N_f-1)}.$$  \hfill (58)

Similarly the bosonic zero modes (57) give the factor

$$\left(\frac{\mu^2}{m_r^*m_r}\right)^{(N_f-1)}.$$  \hfill (59)

The contribution of the translation and supersymmetry zero modes is the same as in the $N_f = 1$ case. So the total power of $\mu$ is equal to $\beta_0$. Putting all the factors together and taking into account eq. (53) we found that the instanton measure for an arbitrary $N_f$ is the same as in the $N_f = 1$ model and is given by eq. (56) up to the factor $e^{i(N_f-1)\alpha_m}$ where $\alpha_m$ is a phase of $m_r$. This phase, however, can be absorbed by the redefinition of the $\theta$ angle which enters the instanton measure as the factor $e^{i\theta}$ so that under the chiral group it transforms according to eq. (16) (another contribution to eq. (16) comes from the supersymmetric fermionic zero mode). The only difference
between the models with different \( N_f \) is in the expression of the running coupling \( \eta^2 \)
so we can extend the results of the previous section to an arbitrary \( N_f \). In particular
we find that the instanton contribution results in the non-perturbative variation \( 11 \)
of the Kähler class of the \( CP(N_f) \) model describing the light sector of the model.

The limit of a vanishing \( m_r \) needs a comment. In this limit one expects the phase
transition since Goldstone bosons do not exist in two dimensions \([31]\). Moreover,
the \( CP(N_f) \) model of the massless fields at \( m_r = 0 \) supports the “light” instanton
solutions which involve only the light degrees of freedom \([32]\). We, however, do not
consider these effects and are rather interested in the calculation of the Wilsonian
effective action for the massless fields by integrating out the “heavy” instantons of
the fundamental theory \([11]\). As we see the instanton measure does not depend on \( m_r \)
explicitly. Then the chiral transformation properties of the instanton measure (of the
fermionic determinant) is not changed in the limit \( m_r = 0 \) since \( N_f - 1 \) zero modes
eq \( 56 \) appear instead of the factor \( m_r^{(N_f-1)} \). Thus the parameter \( m_r \) in eq. \( 55 \)
can be considered as some infrared cutoff \( \sim \Lambda \) of the Wilsonian effective action without
specifying a regularization procedure.

5 Conclusion

We have analyzed the non-perturbative properties of the \( N = 2 \) supersymmetric
Abelian gauge theory in two dimensions. We have developed the systematic semi-
classical expansion around the one-instanton saddle point and presented the set of
the bosonic and fermionic GFs saturated by the one-instanton contribution. The
fermionic GFs are determined by the zero modes of the Dirac operator in the in-
stanton field background while one has to take into account the loop corrections to
the partition function to compute the bosonic field correlators. We also constructed
the effective action which describes the instanton induced effects and explicitly re-
produces these GFs at large distance. It includes the non-perturbative contribution to the Kähler potential and the non-perturbative twisted superpotential. In contrast to all known models the effective coupling constant parameterizing the bosonic part of this action turns out to be of the first order in $\Lambda \sim e^{S_{\text{inst}}}$ and is saturated by the one-instanton contribution. This potential determines the corrections to the BPS saturated mass spectrum of the model and to the classical vacuum configuration of the scalar fields. In particular it implies the non-vanishing VEV of the scalar component of the vector superfield. Then the instanton contribution results in the non-perturbative variation of the Kähler class of the $CP^{(N_f-1)}$ manifold of the classical vacua space.

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6 Appendix A

We use the following conventions in two dimensions. The metric tensor is $g_{\mu\nu} = \text{diag}(1, -1)$ and the antisymmetric tensor $\varepsilon_{\mu\nu}$ is defined so that $\varepsilon_{01} = -\varepsilon^{01} = 1$. The gamma matrices are in the following representation

$$
\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = -\gamma_0\gamma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

(60)

so a spinor $\theta$ and its Dirac conjugate $\bar{\theta} = \theta^\dagger\gamma_0$ are of the form

$$
\theta = \begin{pmatrix} \theta_L \\ \theta_R \end{pmatrix}, \quad \bar{\theta} = \begin{pmatrix} \bar{\theta}_R & \bar{\theta}_L \end{pmatrix}.
$$

(61)

where $\bar{\theta}_R = \theta^*_R$, $\bar{\theta}_L = \theta^*_L$. The charge conjugated spinors are defined as follows

$$
\theta_c = \gamma_5 \bar{\theta}^T
$$
$$\overline{\theta}_c = \theta^T \gamma_c$$

where $\gamma_c = \gamma_1$ is the charge conjugation matrix $\gamma_c\gamma_\mu\gamma_c = -\gamma_\mu^T$.

To study the instanton solution we make the analytic continuation from the Minkowski to Euclidean metric $x_0 \rightarrow -ix_0$. The Euclidean Dirac matrices are

$$\gamma_0^E = -i\gamma_0, \quad \gamma_1^E = \gamma_1,$$

$$\gamma_5^E = -i\gamma_0 \gamma_1^E = \gamma_5, \quad \gamma_c^E = \gamma_c.$$  (63)

The representation (61) is still valid but in Euclidean space one has to substitute the Dirac conjugation by the Hermitian one $\overline{\theta} = \theta^\dagger$ so $\overline{\theta} R = \theta^\dagger L, \overline{\theta} L = \theta^\dagger R$.

7 Appendix B

GF (40) can be computed within the standard PT around the one-instanton solution [3] (note that the fermionic zero mode is cancelled by the Yukawa coupling (38)). In this way, however, one needs to know the propagator of the $\sigma$ field in the one-instanton background [3]. Fortunately, it is possible to bypass this dynamical problem by the following trick. Using the functional integral representation of the partition function it is straightforward to prove a “low energy theorem” (see, for example, [33])

$$\frac{d}{d \zeta} G_\sigma(x) = - \int d^2 y \langle 0 | \sigma(x) (D(y) + v_{01}(y)) | 0 \rangle. $$

(64)

On the other hand, the bosonic correlator

$$\langle 0 | \sigma(x) (D(y) + v_{01}(y)) | 0 \rangle$$

is related to the function

$$\langle 0 | \overline{\lambda}_R(y) \lambda_L(x) | 0 \rangle$$

by the supersymmetric Ward-Takahashi identity [1, 3]. Indeed, using the transformation law

$$[\overline{\sigma}_L, \sigma] = -i \sqrt{2} \lambda_L, \quad \{\overline{\sigma}_L, \lambda_R\} = -i (D + v_{01})$$

(67)
and the fact that the vacuum state is annihilated by the supersymmetry generators because supersymmetry is not spontaneously broken one finds

\begin{equation}
\langle 0 | \sigma(x) (D(y) + v_0(y)) | 0 \rangle = -\sqrt{2} \langle 0 | \lambda_L(x) \overline{\lambda}_R(y) | 0 \rangle.
\end{equation}

(68)

Substituting the bosonic correlator by the fermionic one in eq. (64) according to this equation and using eqs. (28, 29, 32, 41) one finds

\begin{equation}
\frac{d}{d\zeta} G_\sigma(x) = 4 \sqrt{2} \pi^2 \Lambda.
\end{equation}

(69)

Integrating this equation for \( \theta = 0 \) one obtains

\begin{equation}
G_\sigma(x) = 4 \sqrt{2} \eta^2 \pi^2 \Lambda + C'.
\end{equation}

(70)

where \( C' \) is an integration constant. To fix this constant one needs some boundary condition. In the limit \( \eta^2 \to 0 \) the theory has no Higgs phase and the instantons smoothly transform to the pure gauge. So one can suppose all instanton saturated quantities to vanish in this limit. Then \( C' = 0 \) and one gets eq. (47). The value \( C' = 0 \) is also consistent with the Witten index analysis. Indeed, if \( C' \neq 0 \) the real mass term (46) of the \( \psi \) field cannot be reabsorbed by the shift of the \( \sigma \) field to its VEV (see the comment after eq. (49)). Then the term

\begin{equation}
- (2C'^2 \phi \phi + \sqrt{2} C' \overline{\psi}_R \psi_L + \sqrt{2} C' \overline{\psi}_L \psi_R)
\end{equation}

(71)

appears in effective Lagrangian (49) and leads to spontaneous supersymmetry breaking.

Because of the non-renormalization property of the two-point GF of the gauge fermion field the result (47) is exact within the PT around the one-instanton solution.

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