ON GRAVITATIONAL DRESSING OF 2D FIELD THEORIES IN CHIRAL GAUGE

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ABSTRACT

After giving a pedagogical review of the chiral gauge approach to 2D gravity, with particular emphasis on the derivation of the gravitational Ward identities, we discuss in some detail the interpretation of matter correlation functions coupled to gravity in chiral gauge. We argue that in chiral gauge no explicit gravitational dressing factor, analogue to the Liouville exponential in conformal gauge, is necessary for left-right symmetric matter operators. In particular, we examine the gravitationally dressed four-point correlation function of products of left and right fermions. We solve the corresponding gravitational Ward identity exactly: in the presence of gravity this four-point function exhibits a logarithmic short-distance singularity, instead of the power-law singularity in the absence of gravity. This rather surprising effect is non-perturbative in the gravitational coupling and is a sign for logarithms in the gravitationally dressed operator product expansions. We also discuss some perturbative evidence that the chiral Gross-Neveu model may remain integrable when coupled to gravity.

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1. Introduction

What happens to a renormalizable two-dimensional field theory when it is coupled two gravity? In general we don’t know. Of course, during the past years, tremendous progress has been made on a multitude of particular models, either through discrete matrix model techniques or in the continuum using the Liouville theory to describe gravity in the conformal gauge. Often, as is the case for the Ising model in a magnetic field, the coupling to gravity simplifies the theory allowing for an exact solution otherwise not available. Notwithstanding these successes, the continuum methods are mainly restricted to conformal field theories. Also, we only have a very limited knowledge about correlation functions beyond the two- or three-point functions. Recently, by studying general continuum non-conformal field theories coupled to gravity in chiral gauge [1] it was shown that the one-loop $\beta$-function gets affected by a universal gravitational factor.

We think that the chiral gauge approach deserves further exploration. Here we report on some rather surprising results obtained in chiral gauge concerning a four-point function. This note is organized as follows: first, in section 2, we review the chiral gauge approach of Polyakov et al. [2-4] to 2D gravity, hopefully putting it into a pedagogical setting (see also ref. 5). In particular, we show how one obtains the gravitational action and the gravitational Ward identities. Then, in section 3, we discuss in some detail the interpretation and relevance of non-integrated correlation functions in chiral gauge. In particular, we point out why the non-integrated chiral gauge two-point functions give the gravitational scaling dimensions of the integrated conformal gauge two-point functions. We further argue that, in chiral gauge, for left-right symmetric matter operators $O_M$ no explicit gravitational dressing factor, analogue of the Liouville exponential of conformal gauge, is necessary, and that correlators like $\int d^2x_1 \ldots \int d^2x_n \langle O_M(x_1) \ldots O_M(x_n) \rangle$ are well-defined quantities. Then, in section 4, we describe a sample computation of a four-point function using the Ward identities. It is first obtained as a perturbative solution of a partial differential equation. The perturbation series diverges but can easily be resummed. The resulting function, shown to be valid independent of perturbation theory, shows strong non-perturbative effects, namely logarithmic rather than power-type short distance singularities, invalidating the weak-coupling interpretation. Finally, in section 5, we discuss the relevance of this result to the gravitationally dressed operator product expansions, as well as the implications for the gravitationnal dressing of the inte-
2. **A review of 2D gravity in chiral gauge**

2.1. **The matter action**

To start with, consider the action of a Majorana fermion $\chi$ coupled to gravity. Of course, one could consider a more general matter action as well, but let’s be specific.

$$S_M = \frac{1}{\sqrt{2}} \int d^2x \,(\det e) \bar{\chi} \gamma^a e^a_{\mu} \partial_{\mu} \chi.$$  

(2.1)

Here $e_{a\mu}$ is the zweibein and $e^a_{\mu}$ its inverse. Our conventions are fairly standard. One finds

$$S_M = \int d^2x \,(\chi_-(-e_{++}\partial_+ + e_{++}\partial_-)\chi_- + \chi_+(e_{++}\partial_- + e_{--}\partial_+)\chi_-)$$  

$$= \int d^2x \,\left[ \psi_-(\partial_+ - \frac{e_{++}}{e_{++}}\partial_-)\psi_- + \psi_+(\partial_- - \frac{e_{--}}{e_{++}}\partial_+)\psi_+ \right]$$  

(2.2)

where we have rescaled $\psi_+ = \sqrt{-e_{++}}\chi_+$ and $\psi_- = \sqrt{-e_{--}}\chi_-$ (the local Lorentz phase can be chosen such that the square-roots are real). Note that although $\chi_+, \chi_-$ are diffeomorphism scalars, $\psi_+, \psi_-$ behave as half-differentials.

If one now makes the conformal gauge choice $e_{++} = e_{--} = 0$, $e_{++} = e_{--} = -e^0$ so that $g_{++} = g_{--} = -e^{2\phi}$ and $g_{++} = g_{--} = 0$ one obtains the well-known free-fermion action in conformal gauge: $S_M = \int d^2x \,[\psi_+ \partial_+ \psi_+ + \psi_+ \partial_- \psi_+]$. Here, however, we make a different gauge choice leading to the so-called chiral gauge:

$$e_{++} e_{--} = 1, e_{--} = 0, \frac{e_{++}}{e_{--}} = h_{++}$$  

(2.3)

where the last equation is not a gauge choice but just the definition of $h_{++}$. Then the fermion
action becomes
\[ S_{\mathcal{M}} = \int d^2 x \left[ \psi_- (\partial_+ - h_{++} \partial_-) \psi_- + \psi_+ \partial_- \psi_+ \right]. \tag{2.4} \]

Gravity is represented by the field \( h_{++} \) and only the left fermion \( \psi_- \) couples to gravity. It is straightforward to compute the metric tensor in this gauge:
\[ g_{+-} = g_{-+} = g_{+}^+ = g_{-}^- = -1, \quad g_{--} = g_{++} = 0, \quad g_{++} = -2h_{++}, \quad g_{--} = 2h_{++} \tag{2.5} \]

and \( -g \equiv \det g_{\mu \nu} = 1 \). It is also easy to find that the only non-vanishing Christoffel symbols are
\[ \Gamma_{+-}^- = \Gamma_{--}^+ = \partial_- h_{++}, \quad \Gamma_{++}^+ = -\partial_- h_{++}, \quad \Gamma_{++}^- = \partial_+ h_{++} + 2h_{++} \partial_- h_{++}. \tag{2.6} \]

The stress tensor is defined in general as
\[ T_{\mu \nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu \nu}} = \frac{1}{\det e} \epsilon_{\alpha \rho} \frac{\delta S}{\delta e_{\nu}^\alpha} \tag{2.7} \]

and one obtains for the matter part in chiral gauge
\[ T_{+-}^M = \psi_- \partial_- \psi_-, \quad T_{--}^M = T_{++}^M = h_{++} \psi_- \partial_- \psi_-, \quad T_{++}^M = \psi_+ \partial_+ \psi_+ + (h_{++})^2 \psi_- \partial_- \psi_. \tag{2.8} \]

It is straightforward to show that \( T_{\mu \nu}^M \) is classically conserved: \( \nabla^\nu T_{\mu \nu}^M = 0 \). For \( \nu = + \) e.g., using (2.6) this is equivalent to the vanishing of
\[ \partial_+ T_{+-}^M + \partial_- T_{++}^M - (\partial_+ h_{++}) T_{+-}^M - 2(\partial_- h_{++}) T_{++}^M - 2h_{++} \partial_- T_{--}^M, \]

which in turn is shown using the equations of motion
\[ \partial_+ \psi_- = h_{++} \partial_- \psi_- + \frac{1}{2} (\partial_- h_{++}) \psi_- , \quad \partial_- \psi_+ = 0. \tag{2.9} \]

Note also that the stress tensor is traceless: \((T^M)_{\mu}^\mu = 0\).
2.2. Diffeomorphisms

Next, let us consider the effect of diffeomorphisms. Under a general infinitesimal diffeomorphism \( x^\pm \to x'^\pm = x^\pm + \epsilon^\pm(x^+, x^-) \) one has

\[
\delta e_a^\mu(x) \equiv e'^a_\mu(x) - e_a^\mu(x) = \epsilon^\lambda \partial_\lambda e_a^\mu + e_a^\lambda \partial_\mu \epsilon^\lambda
\]

\[
\delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = g_{\nu\rho} \nabla_\mu \epsilon^\rho + g_{\mu\rho} \nabla_\nu \epsilon^\rho
\]

(2.10)

since \( \nabla_\mu g_{\nu\rho} = 0 \). Residual diffeomorphisms preserving chiral gauge must obey \( \delta g_{-+} = \delta g_{+-} = 0 \) which implies (we write \( \epsilon^\mu_R \) for residual diffeomorphisms)

\[
\nabla_\mu \epsilon^\mu_R = \nabla_- \epsilon^+_R = 0 \quad \Leftrightarrow \quad \partial_\mu \epsilon^\mu_R = \partial_- \epsilon^+_R = 0 .
\]

(2.11)

It is a straightforward exercise to check, using \( \delta \psi_\pm = \epsilon^\lambda_R \partial_\lambda \psi_\pm \pm \frac{1}{2} (\partial_\mu \epsilon^\mu_R) \psi_\pm \), \( \delta h_{++} = \epsilon^\lambda_R \partial_\lambda h_{++} + \partial_+ \epsilon^-_R + 2h_{++} \partial_+ \epsilon^+_R \), that the matter action (2.4) is invariant under residual diffeomorphisms. We will also need to consider non-residual diffeomorphisms with general \( \epsilon^- \), but \( \epsilon^+ = 0 \). In this case one has to remember that \( h_{++} \) is defined as \( h_{++} = \frac{e^+}{e^-} \) while \( g_{++} = -2e^+e^- = -2(e^+e^-)h_{++} \). Although \( e^+e^- = 1 \) in chiral gauge, one has \( \delta(e^+e^-) = \partial_- \epsilon^- \) (but still \( \delta e_- = 0 \)) so that \( \delta g_{++} = -2\partial_- \epsilon^- h_{++} - 2\delta h_{++} \). Thus

\[
\delta h_{++} = \partial_+ \epsilon^- + \epsilon^- \partial_- h_{++} - h_{++} \partial_- \epsilon^- \quad \text{for} \quad \epsilon^+ = 0 .
\]

(2.12)

Furthermore, for such diffeomorphisms

\[
\delta \psi_+ = \epsilon^- \partial_- \psi_+ \quad \text{for} \quad \epsilon^+ = 0
\]

\[
\delta \psi_- = \epsilon^- \partial_- \psi_- + \frac{1}{2} (\partial_- \epsilon^-) \psi_- \quad \text{for} \quad \epsilon^+ = 0
\]

(2.13)

so that one can show that the matter action (2.4) in chiral gauge is still invariant under these non-residual diffeomorphisms with \( \epsilon^+ = 0 \).
2.3. The gravitational action

We want to compute the effective action $\Gamma$ in chiral gauge, much along the lines of Friedan’s Les Houches lectures for the conformal gauge [6]. The matter part is rather trivial since the original action is quadratic in the matter fields, but the gravitational part will turn out to be much less trivial. Start with the generating functional

$$Z[\eta_+, \eta_-] = \frac{1}{\Omega_\text{diff}} \int Dg D\psi \exp \left(-S_0^{\text{gr}}(g) - S_M(g, \psi) + \int d^2x \sqrt{-g}(\psi_- \eta_- + \psi_+ \eta_+)\right)$$

(2.14)

where $S_0^{\text{gr}}(g) = \mu_0 \int d^2x \sqrt{-g}$ and $\Omega_\text{diff}$ is the volume of the diffeomorphism group. One has $Dg \equiv Dg_{++} Dg_{--} Dg_{+-} = Dv^+ Dv^- Dh_{++} dm \times J$ where $v^\pm$ are vector fields parametrizing the diffeomorphisms, $m$ stands for the moduli, and the Jacobian $J$ can be represented as usual by an integral over ghosts [3]

$$J \sim \int D\xi_{++} D\xi_{--} D\xi_{+-} \xi_{-} e^{-S_{\text{gh}}}$$

$$S_{\text{gh}} = \int d^2x \left[ \xi_{++} \nabla_- \xi_+ + \xi_{-} (\nabla_+ \xi_- + \nabla_- \xi_+)\right]$$

(2.15)

so that

$$Z[\eta_+, \eta_-] = \int dm Dh_{++} Z[\eta_+, \eta_-, h_{++}, m]$$

$$Z[\eta_+, \eta_-, h_{++}, m] = e^{-S_0^{\text{gr}}} \left( \int Dg h e^{-S_{\text{gh}}} \right) \left( \int D\psi e^{-S_M} \right) \times \exp \left( -\frac{1}{4} \int (\eta_- D_{++}^{-1} \eta_- + \eta_+ \partial_+^{-1} \eta_+) \right)$$

(2.16)

where $D_+ = \partial_+ - h_{++} \partial_-$ and where we shifted as usual $\psi_- \to \psi_- + \frac{1}{2} D_+^{-1} \eta_-$ and $\psi_+ \to \psi_+ + \frac{1}{2} \partial_+^{-1} \eta_+$. Let $\Sigma_{\text{gh}}(g) = - \log \int Dg h e^{-S_{\text{gh}}}$ and $\Sigma_M(g) = - \log \int D\psi e^{-S_M}$. Then the generating functional of connected $\psi$-correlation functions is*

$$W[\eta_+, \eta_-, h_{++}] = \log Z[\eta_+, \eta_-, h_{++}]$$

$$= -S_0^{\text{gr}} - \Sigma_{\text{gh}}(h_{++}) - \Sigma_M(h_{++}) - \frac{1}{4} \int (\eta_- D_{++}^{-1} \eta_- + \eta_+ \partial_+^{-1} \eta_+) .$$

(2.17)

* We do not explicitly write the dependence on the moduli any longer.
To obtain the effective action $\Gamma$ one introduces the classical fields $\hat{\psi}_\pm = \frac{\delta W}{\delta \eta_\pm}$ so that

$$\Gamma[\hat{\psi}_+,\hat{\psi}_-,h++] = \int d^2x \left( \eta_- \hat{\psi}_- + \eta_+ \hat{\psi}_+ \right) - W[\eta_+,\eta_-,h++]$$

$$= S_0^{gr} + \Sigma_{gh}(h++) + \Sigma_M(h++) + \int d^2x \left( \hat{\psi}_- D_+ \hat{\psi}_- + \hat{\psi}_+ D_- \hat{\psi}_+ \right)$$

$$\equiv \Gamma_0[h++] + \Gamma_{\text{excit}}[\hat{\psi}_+,\hat{\psi}_-,h++]$$

(2.18)

which we separate into a (non-trivial) ground-state contribution $\Gamma_0 = S_0^{gr} + \Sigma_{gh} + \Sigma_M$ and a trivial excitation part $\Gamma_{\text{excit}}$ which is formally identical to the original matter action $S_M$. The effective action is to be used to compute correlation functions as

$$\langle \hat{\psi}_\pm(x_1) \ldots \hat{\psi}_\pm(x_n) \rangle = \int dm Dh++ D\hat{\psi}_+ D\hat{\psi}_- \hat{\psi}_\pm(x_1) \ldots \hat{\psi}_\pm(x_n) e^{-\Gamma[\hat{\psi}_+,\hat{\psi}_-,h++]}. \quad (2.19)$$

We have gone through the usual field theoretic formalism to show that the gravitational part $\Gamma_0$ of $\Gamma$ gets contributions from the ghost and matter sectors. Equivalently one could have argued à la David-Distler-Kawai [7] that in (2.16) the measures $Dgh \equiv Dg_{gh}$ and $D\psi \equiv Dg_{\psi}$ are complicated but factorize into $e^{-\Sigma_{gh}} D_{0gh}$ and $e^{-\Sigma_M} D_{0\psi}$ where $D_{0gh}$ and $D_{0\psi}$ now are trivial (flat) measures. Then one would directly obtain (2.19) (except for the replacement $\hat{\psi}_\pm \rightarrow \hat{\psi}_\pm$).

Now, if the Faddeev-Popov procedure of factorizing the volume of the diffeomorphism group is to make sense, then $\Gamma$ must be invariant under general diffeomorphisms, not only residual ones.† To check the invariance of $\Gamma$ properly one should have imposed, instead of (2.3), conditions like $e_+ e_- = \alpha$, $\frac{e_+}{e_-} = \beta$ and the definition $\frac{e_+}{e_-} = h++$, and keep $\alpha$ and $\beta$ as classical non-dynamical background fields. Then under a diffeomorphism (2.10) one has e.g. $\delta_\beta_-|_{\beta_- = 0} = \partial_- \epsilon^+$. The classical action $S_M$ would then read

$$S_M = \int d^2x \left[ \psi_- (\partial_+ - h++ \partial_-) \psi_- + \psi_+ (\partial_- - \beta_- \partial_+) \psi_+ \right] \quad (2.20)$$

which is obviously invariant under diffeomorphisms, since it is just the original invariant matter action written in a particular form. Actually, for our purpose of deriving the gravitational

† One may view this requirement as a condition on the counterterms.
action $\Gamma_0$ is is enough to consider diffeomorphisms with $\epsilon^+ = 0$ and $\epsilon^-$ arbitrary. In this case there is no need to introduce the $\alpha$ and $\beta$ into $S_M$ and one can directly work with $S_M$ as given by (2.4). As already noted earlier, (2.4) is invariant under diffeomorphisms with $\epsilon^+ = 0$. Since $S_M$ is formally identical with $\Gamma_{\text{excit}}$ the same is true for the latter. Thus imposing invariance of $\Gamma$ translates into imposing invariance of $\Gamma_0[g_{\mu\nu}]$:

$$0 = \delta\Gamma_0 = \int d^2 x \delta g^{\mu\nu} \frac{\delta \Gamma_0}{\delta g^{\mu\nu}} = \int d^2 x \sqrt{-g} \epsilon^\nu \nabla^\mu T^\text{grav}_{\mu\nu} = -2 \int d^2 x \sqrt{-g} \epsilon^\nu \nabla^\mu T^\text{grav}_{\mu\nu} \tag{2.21}$$

where

$$T^\text{grav}_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \Gamma_0}{\delta g^{\mu\nu}}. \tag{2.22}$$

Equation (2.21) implies the conservation of the gravitational stress-energy tensor, in particular with $\epsilon^+ = 0$:

$$0 = \nabla^\mu T^\text{grav}_{\mu-} = -\nabla_+ T^\text{grav}_{-+} + 2h_{++} \nabla_- T^\text{grav}_{-+} - \nabla_- T^\text{grav}_{++}$$

$$= \nabla_+ T^\text{grav}_{--} + h_{++} \nabla_- T^\text{grav}_{--} + \frac{1}{2} \nabla_- (T^\text{grav})^\mu_\mu \tag{2.23}$$

or

$$[\partial_+ - h_{++} \partial_- - 2(\partial_- h_{++})] T^\text{grav}_{--} = \frac{1}{2} \nabla_- (T^\text{grav})^\mu_\mu. \tag{2.24}$$

Now one uses the fact that $(T^\text{grav})^\mu_\mu$ is a gravitational scalar of dimension two, and hence must be proportional to the curvature scalar:

$$(T^\text{grav})^\mu_\mu = \frac{\lambda g}{24} R. \tag{2.25}$$

Combining eqs (2.24) and (2.25) gives

$$[\partial_+ - h_{++} \partial_- - 2(\partial_- h_{++})] T^\text{grav}_{--} = \frac{\lambda g}{48} \partial_- R \tag{2.26}$$

which is the central equation for determining the gravitational action $\Gamma_0$. It is straightforward to compute $R$ in chiral gauge:

$$R = 2\partial_+^2 h_{++}. \tag{2.27}$$

Furthermore, since $\sqrt{-g} = 1$ one has $T^\text{grav}_{--} = \frac{1}{2} \frac{\delta \Gamma_0}{\delta h_{++}}$ and (2.26) turns into a functional differ-
ential equation for $\Gamma_0$:

$$[\partial_+ - h_{++}\partial_- - 2(\partial_- h_{++})] \frac{\delta \Gamma_0[h_{++}]}{\delta h_{++}} = \frac{\lambda g}{12} \partial^3 h_{++} .$$

(2.28)

Its solution is

$$\Gamma_0[h_{++}] = \frac{\lambda g}{24} \int d^2x (\partial^2 h_{++}) \frac{1}{\partial_- (\partial_+ - h_{++}\partial_-)} \partial^2 h_{++} + \mu \int d^2x .$$

(2.29)

Indeed, observing that $\left(\partial_+ - h_{++}\partial_- \right)^{-1}$ is a symmetric pseudodifferential operator one immediately finds

$$T_{-\rightarrow}^{\text{grav}} = \frac{1}{2} \frac{\delta \Gamma_0}{\delta h_{++}} = \frac{\lambda g}{24} \partial_+ - h_{++}\partial_- \partial_- h_{++} - \frac{\lambda g}{48} \left( \partial_- \frac{1}{\partial_+ - h_{++}\partial_-} \partial_- h_{++} \right)^2 .$$

(2.30)

Applying then $\partial_+ - h_{++}\partial_-$ to the r.h.s. of this equation yields $\frac{\lambda g}{24} \partial^2 h_{++} + 2\partial_- h_{++} T_{-\rightarrow}^{\text{grav}}$, showing that (2.28) is satisfied. Since $\partial_- (\partial_+ - h_{++}\partial_-)$ is $-1/2$ times the Laplacian on a scalar, one finds of course that $\Gamma_0$ can be written as

$$\Gamma_0 = -\frac{\lambda g}{48} \int d^2x \sqrt{-g} R \frac{1}{\nabla^2} R + \mu \int d^2x \sqrt{-g}$$

(2.31)

which is the famous covariant form written by Polyakov [3], and which, in conformal gauge, reproduces the Liouville action.

### 2.4. Gravitational Ward identities

Note that equation (2.28) can be rewritten, using (2.12), as

$$\int d^2x \delta h_{++} \frac{\delta \Gamma_0}{\delta h_{++}} = -\frac{\lambda g}{12} \int d^2x \epsilon^- \partial^3 h_{++} .$$

(2.32)

This means that $\Gamma_0$, if considered as a functional of $h_{++}$ only, is not (completely) invariant under diffeomorphisms (even with $\epsilon^+ = 0$) but has the anomaly as given by the r.h.s. of this equation. Equation (2.21) however, expresses the diffeomorphism invariance of $\Gamma_0$ as a covariant functional of $g_{++}$ and $g_{+-}$ (and $g_{-\rightarrow}$), where the chiral gauge is only imposed after the variation is performed.
It is now straightforward to derive Ward identities for correlation functions:

\[
\langle \psi(x_1) \ldots \psi(x_n) \rangle = \int \mathcal{D}h_{++} e^{-\Gamma_0[h_{++}]} \int \mathcal{D}\psi \psi(x_1) \ldots \psi(x_n) e^{-\Gamma_{\text{excit}}[h_{++},\psi]} .
\]  

(2.33)

Here \( \psi \) need not be the previous fermion fields, but could be more general matter fields. Changing variables of integration to \( \tilde{h}_{++} = h_{++} + \delta h_{++} \) and \( \tilde{\psi} = \psi + \delta \psi \) such that \( \delta h_{++} \) and \( \delta \psi \) correspond to the diffeomorphisms (2.12) and (2.14), hence \( \delta \Gamma_{\text{excit}} = 0 \), one obtains, using (2.32), the following Ward identity

\[
\sum_{i=1}^{n} \langle \psi(x_1) \ldots \delta \psi(x_i) \ldots \psi(x_n) \rangle + \frac{\lambda^g}{12} \int d^2z \epsilon^{-}(z) (\partial^2 h_{++}(z) \psi(x_1) \ldots \psi(x_n)) = 0 .
\]  

(2.34)

Next, following ref. 2, we will turn this Ward identity into a partial differential equation for \( \langle \psi(x_1) \ldots \psi(x_n) \rangle \). To do so one has to eliminate \( h_{++} \) from the correlation function. Here, we will again concentrate on the above example of fermions. The classical equation of motion

\[
\partial_{-} \psi_{-} = h_{++} \partial_{-} \psi_{-} + \Delta(\partial_{-} h_{++}) \psi_{-}
\]  

(2.35)

with \( \Delta = \frac{1}{2} \) carries over to the quantum case but for two differences: first, the product of fields at the same point needs to be regularized by some normal-ordering prescription which may modify the weight \( \Delta \). Second, when inserted into the functional integral, the equations of motion remain true up to contact terms (e.g. \( \langle \frac{\delta S}{\delta \phi(x)} \phi(y) \rangle = \delta(x-y) \)). Covariance of the quantum equation of motion (2.35) requires that the variation of \( \psi_{-} \) under diffeomorphisms also gets modified: eq. (2.13) gets replaced by

\[
\delta \psi_{-} = \epsilon^{-} \partial_{-} \psi_{-} + \Delta(\partial_{-} \epsilon^{-}) \psi_{-}
\]  

(2.36)

with the same \( \Delta \) as in (2.35). The Ward identity then becomes

\[
\sum_{i=1}^{n} \left( \delta^{(2)}(z - x_i) \frac{\partial}{\partial x_i} - \Delta \frac{\partial}{\partial z} \delta^{(2)}(z - x_i) \right) \langle \psi_{-}(x_1) \ldots \psi_{-}(x_n) \rangle
\]

\[+ \frac{\lambda^g}{12} \frac{\partial^3}{\partial(z^{-})^3} \langle h_{++}(z) \psi_{-}(x_1) \ldots \psi_{-}(x_n) \rangle = 0 .
\]  

(2.37)
Using the identities
\[
\frac{\partial^3}{\partial (z^i)^3} \frac{(z^i - x^i)^2}{z^+ - x^i_\pm} = 4\pi i \delta^{(2)}(z - x^i) ,
\]
\[
\frac{\partial^3}{\partial (z^i)^3} \frac{2(z^i - x^i)}{z^+ - x^i_\pm} = 4\pi i \frac{\partial}{\partial z^i} \delta^{(2)}(z - x^i) ,
\]
eq (2.38)

eq. (2.37) can be integrated as
\[
\sum_{i=1}^n \left( \frac{(z^i - x^i)^2}{z^+ - x^i_\pm} \frac{\partial}{\partial x^i_\pm} - 2\Delta \frac{(z^i - x^i)}{z^+ - x^i_\pm} \right) \langle \psi_-(x_1) \ldots \psi_-(x_n) \rangle
\]
\[
= -\frac{i\pi \lambda g}{3} \langle h_{++}(z) \psi_-(x_1) \ldots \psi_-(x_n) \rangle .
\]
eq (2.39)

Now one uses the quantum equations of motion (2.35) as:
\[
\frac{\partial}{\partial z^+} \langle \psi_-(z) \psi_-(x_2) \ldots \psi_-(x_n) \rangle
\]
\[
= \langle (h_{++}(z) \partial_+ \psi_-(z) + \Delta \partial_+ h_{++}(z) \psi_-(z)) \psi_-(x_2) \ldots \psi_-(x_n) \rangle
\]
\[
= \frac{\partial}{\partial z^-} \langle h_{++}(z) \psi_-(z) \psi_-(x_2) \ldots \psi_-(x_n) \rangle + (\Delta - 1) \langle (\partial_- h_{++}(z)) \psi_-(z) \psi_-(x_2) \ldots \psi_-(x_n) \rangle
\]
\[
= \left\{ \gamma \frac{\partial}{\partial z^+} + \sum_{i=2}^n \left[ \frac{(z^i - x^i)^2}{z^+ - x^i_\pm} \frac{\partial}{\partial z^i} \frac{\partial}{\partial x^i_\pm} + 2\Delta \frac{z^i - x^i}{z^+ - x^i_\pm} \left( \frac{\partial}{\partial x^i_\pm} - \frac{\partial}{\partial z^i} \right) - \frac{2\Delta^2}{z^+ - x^i_\pm} \right] \right\} \langle \psi_-(x_1) \ldots \psi_-(x_n) \rangle = 0
\]
eq (2.40)

(up to contact terms). Using (2.39) and its derivative w.r.t. \( z^- \) one finally arrives at
\[
\left\{ \gamma \frac{\partial}{\partial z^+} + \sum_{i=2}^n \left[ \frac{(z^i - x^i)^2}{z^+ - x^i_\pm} \frac{\partial}{\partial z^i} \frac{\partial}{\partial x^i_\pm} + 2\Delta \frac{z^i - x^i}{z^+ - x^i_\pm} \left( \frac{\partial}{\partial x^i_\pm} - \frac{\partial}{\partial z^i} \right) - \frac{2\Delta^2}{z^+ - x^i_\pm} \right] \right\} \langle \psi_-(x_1) \ldots \psi_-(x_n) \rangle = 0
\]
eq (2.41)

where we set
\[
\gamma = \frac{i\pi \lambda g}{3} .
\]
eq (2.42)

We quote without proof [3] that \( \gamma \) is related to the total central charge \( c \) of the matter coupled to gravity (e.g. \( c = \frac{1}{2} \) for a Majorana fermion) by the relation
\[
\gamma = \frac{1}{12} \left( c - 13 - \sqrt{(c - 1)(c - 25)} \right) .
\]
eq (2.43)

Equation (2.41) was first written in ref. 1.
As an example for the use of the Ward identities, consider the two-point function of $\psi_-$. Perturbation theory \* suggests the ansatz

$$
\langle \psi_-(x)\psi_-(y) \rangle \sim \frac{1}{(x^- - y^-)^{1+2\delta}(x^+ - y^+)^{2\delta}} = \frac{[(x^- - y^-)(x^+ - y^+)]^{-2\delta}}{x^- - y^-}. \tag{2.44}
$$

Gravity only contributes a left-right symmetric factor $[(x^- - y^-)(x^+ - y^+)]^{-2\delta}$. Furthermore, by the usual arguments, $\frac{1}{2} + \delta$ must coincide with the anomalous dimension $\Delta$ of eq. (2.36). Inserting this ansatz into (2.41) for $n = 2$ gives an algebraic equation for $\Delta$:

$$
\Delta - \Delta_0 = \frac{\Delta(\Delta - 1)}{\gamma}. \quad (2.45)
$$

where $\Delta_0 = \frac{1}{2}$. This is the well-known KPZ-equation [4] expressing the anomalous dimension $\Delta$ in the presence of gravity in terms of the dimension $\Delta_0$ without gravity.

### 3. Interpretation and relevance of non-integrated correlation functions

What do correlation functions like (2.44) mean? Since one has integrated over the metrics, i.e. over $h_{++}$, what is the distance between $x$ and $y$? Clearly, these are non-trivial questions. Let us compare with what one does in conformal gauge. In conformal gauge, one usually computes integrated correlation functions at fixed area $A$, like

$$
\left. \left< \int d^2x e^{\alpha \phi(x)} O(x) \int d^2y e^{\alpha \phi(y)} O(y) \right> \right|_{\text{fixed } A} \quad (3.1)
$$

where $O$ is a left-right symmetric matter field of conformal dimensions $(\Delta_0, \Delta_0)$ (e.g. the product of our fermion fields $\psi_-$ and $\psi_+$ with $\Delta_0 = \frac{1}{2}$) and $\phi$ the Liouville field. The constant $\alpha$ is chosen such that the conformal dimension of $e^{\alpha \phi}$ is $(1-\Delta_0, 1-\Delta_0)$, so that the total integrand has conformal dimensions $(1, 1)$, and the integral is invariant under conformal transformations. The area of the surface can be fixed by adjusting the zero-mode of the Liouville field $\phi$.

\* One can do a simple Feynman diagram expansion of the two-point function using the vertices and propagators derived from $\Gamma$. This is a perturbation series in $\frac{1}{\lambda g} \sim \frac{1}{\gamma}$. 

\[11\]
Correlation functions like (3.1) are conformal scalars, i.e., are invariant under the residual diffeomorphisms of conformal gauge and have a well-defined meaning. It has been shown [7] that (3.1) scales with the area as $A^{2-2\Delta}$ where the gravitational scaling dimension $\Delta$ is given by the KPZ formula (2.45). Hence this $\Delta$ coincides with the $\Delta$-exponent characterizing the non-integrated two-point function in chiral gauge:

$$\langle O(x)O(y) \rangle \sim \frac{1}{(x^- - y^-)^{2\Delta}(x^+ - y^+)^{2\Delta - 2\Delta_0}} \times \frac{1}{(x^+ - y^+)^{2\Delta_0}}$$ (3.2)

where e.g. in the case of the fermions the first factor comes from $\psi_-$ and the second from $\psi_+$. One sees that although such non-integrated correlation functions in chiral gauge are not invariant, their singularity structure (exponent $\Delta$) nevertheless has an invariant meaning. Let us try to understand why $\Delta$ as given by (3.2) should coincide with the gravitational scaling dimension of (3.1). In chiral gauge, since $\sqrt{-g} = 1$, the area of a surface is completely independent of the metric $h_{++}$. Whereas in conformal gauge one could choose the range of the coordinates $x^+, x^-$ to be fixed, in chiral gauge their range is relevant to the geometry (and is part of the moduli of the surface). Integrating (3.2) in $x$ and $y$ over the surface then gives $A^{2-2\Delta}$, possibly up to an $A$-independent constant. Thus the $\Delta$ characterizing the power-law behaviour of the non-integrated two-point function directly gives the gravitational scaling dimension without further dressing by some field $f(h_{++})$ that would be the chiral gauge analogue of the $e^{\alpha\phi}$-dressing. To understand why no such extra dressing is required in chiral gauge, let’s go back to the example of the fermion fields, i.e. $O = \psi_+\psi_-$. Under an $\epsilon^-$-diffeomorphism we had

$$\delta\psi_- = \epsilon^-\partial_-\psi_- + \Delta(\partial_-\epsilon^-)\psi_-$$
$$\delta\psi_+ = \epsilon^-\partial_-\psi_+$$ (3.3)

and similarly one finds for an $\epsilon^+$-diffeomorphism

$$\delta\psi_- = \epsilon^+\partial_+\psi_- + (\Delta - \frac{1}{2})(\partial_+\epsilon^+)\psi_-$$
$$\delta\psi_+ = \epsilon^+\partial_+\psi_+ + \frac{1}{2}(\partial_+\epsilon^+)\psi_+$$ (3.4)

which combines into

$$\delta\psi_- = \epsilon^\lambda\partial_\lambda\psi_- + \Delta(\partial_\lambda\epsilon^\lambda)\psi_- - \frac{1}{2}(\partial_+\epsilon^+)\psi_-$$
$$\delta\psi_+ = \epsilon^\lambda\partial_\lambda\psi_+ + \frac{1}{2}(\partial_+\epsilon^+)\psi_+.$$ (3.5)
For residual diffeomorphisms (preserving chiral gauge) one has \( \partial_\lambda \epsilon^\lambda_R = 0 \) and it follows

\[
\delta_R \psi_\pm = \epsilon^\lambda_R \partial_\lambda \psi_\pm \pm \frac{1}{2} (\partial_+ \epsilon^+) \psi_\pm
\]

(3.6)

and as a consequence

\[
\delta_R (\psi_+ \psi_-) = \epsilon^\lambda_R \partial_\lambda (\psi_+ \psi_-) = \partial_\lambda (\epsilon^\lambda_R \psi_+ \psi_-)
\]

(3.7)

which is a total derivative. Hence \( \int d^2 x \; O(x) = \int d^2 x \; \psi_+(x) \psi_-(x) \) is invariant under residual diffeomorphisms, which is the analogue of the \((1,1)\) condition in conformal gauge. We see that no extra gravitational factor \( f(h_{++}) \) is needed to achieve this, explaining why the exponent \( \Delta = \frac{1}{2} + \delta \) of (2.44) directly gives the gravitational scaling dimension.

This result is of course not restricted to fermions. From (3.3)-(3.7) we see that if under a general diffeomorphism

\[
\delta O = \epsilon^\lambda \partial_\lambda O + \Delta (\partial_\lambda \epsilon^\lambda) O
\]

(3.8)

then under a residual diffeomorphism

\[
\delta_R O = \epsilon^\lambda_R \partial_\lambda O = \partial_\lambda (\epsilon^\lambda_R O) \Rightarrow \delta_R \int d^2 x \; O = 0 .
\]

(3.9)

Thus integrated \( n \)-point functions \( \int d^2 x_1 \ldots d^2 x_n \langle O(x_1) \ldots O(x_n) \rangle \) are perfectly well-defined \( \star \) objects. Of course, it is just as meaningful to consider also the non-integrated \( n \)-point functions which are scalar densities w.r.t. residual diffeomorphisms. The invariance of the integrated \( n \)-point functions under residual diffeomorphisms should be expressible as BRST-invariance.\( \dagger \) To our knowledge, the BRST operator for the chiral gauge has not yet been constructed, nor its cohomoly been investigated, but it is clear that correlators like \( \langle \int d^2 x_1 \psi_+(x_1) \psi_-(x_1) \ldots \int d^2 x_n \psi_+(x_n) \psi_-(x_n) \rangle \) should turn out to be BRST-invariant.

\[\star\] Let us insist that well-defined means invariance (or covariance) under residual diffeomorphisms. Since one works with a fixed gauge this is as much as one can demand. Of course, one should be able to compare to another, say conformal gauge, but this is beyond the scope of this note.

\[\dagger\] Recall that in conformal gauge, BRST-invariance is essentially the statement that the integrand is a \((1,1)\) field w.r.t. conformal transformations, i.e. residual diffeomorphisms.
4. The fermion four-point function

As a non-trivial example of how to use the Ward-identity (2.41) we now compute the fermion four-point function. As discussed in the next section, this is also of some relevance to the gravitational dressing of the chiral Gross-Neveu model. Therefore we also add some colour indices $i, j = 1, \ldots, N$, and take as the matter action $N$ copies of (2.4). We want to compute, for $i \neq j$

$$G_4(w, x, y, z) = \langle \psi_i^- (w) \psi_i^+(x) \psi_j^-(y) \psi_j^+(z) \rangle .$$

In analogy with the two-point function we use the following ansatz

$$\langle \psi_i^- (w) \psi_i^+ (x) \psi_j^- (y) \psi_j^+ (z) \rangle = \frac{f(t^-, t^+)}{(w^+ - x^+)(y^+ - z^+)} ,$$

where the anharmonic ratio $t \equiv t^-$ is given as usual by

$$t \equiv t^- = \frac{(w^- - y^-)(x^- - z^-)}{(w^- - z^-)(x^- - y^-)} ,$$

and similarly for $\bar{t} \equiv t^+$. Inserting the ansatz (4.2) into the Ward identity (2.41) with $n = 4$ leads after some algebra to‡

$$\left[ \gamma \bar{t} \partial_t + \frac{1 - t}{1 - \bar{t}} (\bar{t} - t) \partial_t t \partial_{\bar{t}} + (1 - 4\Delta) t \partial_t + 2\Delta^2 \frac{t + 1}{t - 1} \right] f(t, \bar{t}) = 0 ,$$

Note that $\Delta = 1$ reproduces the equation derived in ref. 1 for the four-current correlation function. The ansatz (4.2) is justified by the fact that we obtain an equation for $f(t, \bar{t})$ involving only $t, \bar{t}, \partial_t, \partial_{\bar{t}}$ and not $x, y, z$ or $w$ explicitly.

The partial differential equation (4.4) has many solutions. To pick out the physical ones we have to compare with perturbation theory in $\frac{1}{\gamma}$. Looking at the effective action $\Gamma$ (cf. (2.18), (2.29)) it is clear upon rescaling $\tilde{h}_{++} = \sqrt{\gamma} h_{++}$ that each interaction involving a $h_{++}$
is accompanied by at least one factor of $\frac{1}{\sqrt{\gamma}}$.

Doing a standard perturbative Feynman diagram expansion, we obtain

$$f = 1 - \frac{1}{2\gamma} \log t + O(1/\gamma^2) = 1 - \frac{2\Delta^2}{\gamma} \frac{t + 1}{t - 1} \log t + O(1/\gamma^2)$$

(4.5)

since $\Delta = \frac{1}{2} + O(1/\gamma)$.

In principle, the partial differential equation (4.4) could be solved order by order in a perturbation series in $\frac{1}{\gamma}$. In practice, this leads to very complicated poly-log integrals already at low orders. The main difficulty is the factor $1/(1 - \bar{t})$. If one considers the vicinity of $t = 1$, this difficulty disappears, and (4.4) becomes

$$\left[ \gamma \bar{t} \partial_t + (t - 1) \partial_t + \frac{4\Delta^2}{t - 1} \right] f_1(t, \bar{t}) = 0$$

(4.6)

where the subscript 1 on $f$ is to remind us that $f_1 \sim f$ only in the vicinity of $t = 1$. This equation (4.6) can be solved exactly. Writing the solution as a perturbation series in $\frac{1}{\gamma}$ (matching to (4.5)) gives

$$f_1(t, \bar{t}) \equiv f_1(g) = \sum_{n=0}^{\infty} \frac{[2\Delta]^n}{n!} g^n, \quad g = -\frac{\log \bar{t}}{\gamma(t - 1)}$$

(4.7)

where $a_n \equiv a(a + 1)(a + 2) \ldots (a + n - 1)$. At each order in $\frac{1}{\gamma}$ one can of course replace $\log \bar{t}$ by $\log t\bar{t}$ in $g$, as suggested by (4.5).

§ Then $f_1$ is no longer an exact solution of (4.6) but an exact solution of another equation, differing from (4.6) only by higher order terms in $(t - 1)$.

The series (4.7) has zero radius of convergence, but its Borel transform can be recognized as the hypergeometric function

$$B[f_1](u) = \sum_{n=0}^{\infty} \frac{[2\Delta]^n}{n!} u^n = F(2\Delta, 2\Delta, 1; u).$$

(4.8)

The inverse Borel transform

$$f_1(g) = \int_0^\infty dv \, e^{-v} B[f_1](uv)$$

(4.9)

gives the resummed function $f_1(g)$ in terms of a Whittaker function. Alternatively, one can directly observe that (4.7) coincides up to an overall factor with the asymptotic expansion of
a Whittaker function. Hence [8]

\[ f_1(g) = \left( -\frac{1}{g} \right)^{2\Delta} \psi(2\Delta, 1; -\frac{1}{g}) = \left( -\frac{1}{g} \right)^{2\Delta - 1/2} e^{-1/2g} W_{1/2 - 2\Delta, 0}(\frac{1}{g}). \tag{4.10} \]

Here \( W \) is the Whittaker function and \( \psi \) is a solution to the degenerate hypergeometric equation [8]. Indeed, although the equation (4.6) has many solutions, if one uses an ansatz with \( f_1 \) only depending on \( t \) and \( \bar{t} \) through \( g \), then equation (4.6) becomes

\[
\left \{ \begin{array}{l}
g^2 \frac{d^2}{dg^2} + [(1 + 4\Delta)g - 1] \frac{d}{dg} + 4\Delta^2 \\
\end{array} \right \} f_1(g) = 0 \] \tag{4.11}

Setting \( f_1(g) = (-1/g)^{2\Delta} u(-1/g) \) one sees that \( u(x) \) satisfies the degenerate hypergeometric equation

\[ xu''(x) + (b - x)u'(x) - au(x) = 0 \tag{4.12} \]

with \( b = 1 \) and \( a = 2\Delta \). Perturbation theory has told us which of the two independent solutions to choose, namely \( u(x) = \Psi(2\Delta, 1; x) \). Let us insist that we just showed that (4.10) is a solution to the differential equation (4.6), independent of perturbation theory in \( \frac{1}{\gamma} \).

Having the perfectly non-perturbative expression (4.10) for \( f_1(g) \), we can now investigate its behaviour for large \( g \), which is just the series expansion of \( \psi(2\Delta, 1; x) \) for small \( x \):

\[
f_1(g) = \left( -\frac{1}{g} \right)^{2\Delta} \sum_{k=0}^{\infty} \frac{\Gamma(2\Delta + k)}{k! \Gamma(2\Delta)^2} \left[ 2\psi(k + 1) - \psi(2\Delta + k) - \log \left( -\frac{1}{g} \right) \right] \left( -\frac{1}{g} \right)^k \tag{4.13}
\]

where \( \psi(x) = \Gamma'(x)/\Gamma(x) \). Recall that \( g = -\log \bar{t}/[\gamma(t - 1)] \), hence large \( g \) means \( t \to 1 \) (for fixed \( \bar{t} \)), so this is the limit where \( f \sim f_1 \). Although (4.13) is the exact asymptotic for \( f_1 \), it gives only the leading order for \( f \) (i.e. we can only trust the \( k = 0 \) term):

\[
f(t, \bar{t}) \sim \left( \frac{\gamma(t - 1)}{\log t} \right)^{2\Delta} \frac{1}{\Gamma(2\Delta)} \left[ \psi(1) - \log \left( \frac{\gamma(t - 1)}{\log t} \right) + O((t - 1), (t - 1) \log(t - 1)) \right]. \tag{4.14}
\]

What does this mean for the fermion four-point function (4.1)? Since

\[
t - 1 = \frac{(w^- - x^-)(y^- - z^-)}{(w^- - z^-)(x^- - y^-)}, \tag{4.15}
\]

one has \( t \to 1 \) if either \( w^- \to x^- \) or \( y^- \to z^- \), i.e. when two fermion operators of the same
colour approach each other. Inserting (4.14) into (4.1) and (4.2) then gives

$$G_4(w, x, y, z) \sim \frac{\gamma^{2\Delta}}{\Gamma(2\Delta)} \left[ (w^- - z^-)(x^- - y^-) \right]^{-2\Delta} \left[ \log \bar{t} \right]^{-2\Delta} \left[ (w^+ - x^+)(y^+ - z^+) \right]^{-2\Delta}$$

$$\times \left[ \psi(1) + \log \left( \frac{\log \bar{t}}{\gamma} \right) - \log \left( \frac{(w^- - x^-)(y^- - z^-)}{(w^- - z^-)(x^- - y^-)} \right) \right].$$

(4.16)

It is important to realize that we work in Minkowski space so that we can take $t \to 1$, keeping $\bar{t} \neq 1$ fixed. Rather surprisingly, the four-point function (4.16) no longer contains the perturbative singularity $\sim (w^- - x^-)^{-2\Delta}(y^- - z^-)^{-2\Delta}$, but resumming the series has transformed it into a logarithmic singularity, plus a non-singular part!*

Mathematically, the origin of the logarithm can be traced to the degenerate hypergeometric equation (4.12) satisfied by $u(x) = x^{-2\Delta} f_1(-1/x)$. For generic parameter $b$ it has two independent solutions [8] $\Phi(a, b; x)$ and $x^{1-b}\Phi(a+1-b, 2-b; x)$. Obviously, for $b \to 1$ the second solution generates $\log x \Phi(a, b; x)$, among others. This is a well-known phenomenon in the theory of ordinary linear differential equations.

Physically however, it was quite unexpected that turning on gravity ($\frac{1}{\gamma} \neq 0$), even infinitesimally weakly, completely changes the singularity structure: this is a truly non-perturbative phenomenon, due to the divergence of the perturbative series in $\frac{1}{\gamma}$.

5. Conclusions and Outlook

What do we learn from all these computations of the four-point function? One obvious lesson is - unlike the situation of the two-point function - that we cannot trust the weak-coupling gravitational perturbation theory in $\frac{1}{\gamma} \sim \frac{1}{\lambda g}$. What is the meaning of the (non-perturbative) logarithmic singularity of the four-point function? In section 3, we have argued that integrating correlation functions like (4.16), computed in chiral gauge, leads to well-defined objects, invariant under residual diffeomorphisms. Obviously, we cannot integrate our result (4.16) since it is valid only in the vicinity $t \sim 1$. However, since integrating is

* Naively it looks as if (4.16) now contains a new singularity $\sim (w^- - z^-)^{-2\Delta}(x^- - y^-)^{-2\Delta}$ as $w^- \to z^-$ or $x^- \to y^-$. However, this means $t \to \infty$ which is clearly outside the domain of validity of eq. (4.16). Let us also insist that the contact terms $\sim \delta^{(2)}(x-y), \delta^{(2)}(x-y)$ we dropped above precisely correspond to $t \to \infty$ and are completely irrelevant to the behaviour as $t \to 1$. 

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a “trivial procedure” it certainly makes sense to study the properties of the non-integrated correlation functions as well. The main question that arises is whether the correlator \( (4.16) \) tells us something about the gravitationally dressed operator product expansion. Equation \( (4.16) \) should express the OPE of \( \phi(w) = \psi_-(w)\psi_+(w) \) with \( \phi(x) = \psi_-(x)\psi_+(x) \) as \( w^- \rightarrow x^- \). It looks like

\[
\phi(w)\phi(x) \sim \tilde{O}(x) + \log(w^- - x^-)O(x) .
\] (5.1)

This is a particular example of a more general OPE with logarithmic short-distance behaviour:

\[
\phi(w)\phi(x) \sim \sum_n (w - x)^{-2\Delta_n} \left[ \tilde{O}_n + \ldots + \log(w - x)O_n + \ldots \right] .
\] (5.2)

In conformal gauge, such logarithms have been noticed before in ref. 9 where the WZW model based on the supergroup \( Gl(1,1) \) was discussed. Later a more systematic discussion was given in ref. 10, where the appearance of logarithms in conformal blocks in the \( c = -2 \) and other non-unitary models was studied. There it has been argued that the emergence of logarithms in correlation functions is due to new so-called logarithmic operators, whose OPEs display logarithmic short-distance singularities. These new logarithmic operators have conformal dimensions degenerate with those of the usual primary operators, and it is this degeneracy that is at the origin of the logarithms (cf. our discussion of the degenerate hypergeometric equation in the previous section). As a result one can no longer completely diagonalize the Virasoro operator \( L_0 \), and the new operators, together with the standard ones form the basis of the Jordan-cell for \( L_0 \). In the case of two operators with degenerate conformal dimensions \( \Delta_n \) the operator product expansion precisely takes the form (5.2), while the OPE with the conformal stress-energy tensor is

\[
T(z)\tilde{O}_n(0) \sim \frac{\Delta_n}{z^2} \tilde{O}_n(0) + \frac{1}{z^2} O_n(0) + \frac{1}{z} \partial \tilde{O}_n(0) ,
\] (5.3)

in particular

\[
L_0|O_n\rangle = \Delta_n|O_n\rangle , \quad L_0|\tilde{O}_n\rangle = \Delta_n|\tilde{O}_n\rangle + |O_n\rangle .
\] (5.4)

This makes it possible to have logarithmic terms in the correlation functions without spoiling the conformal invariance.
Finally we would like to comment on the relevance of our present results for the gravitational dressing of a two-dimensional integrable but not conformally invariant field theory, namely the chiral Gross-Neveu model [11]. As is well-known, its action is given by the massless free-fermion action (2.1), where the fermions are $N$ component fields, and an interaction term between two left and two right fermions $\sim \int d^2 x \psi_i^j \psi^{i}_- \psi^i_+ \psi^j_+$. In chiral gauge, only the left fermions $\psi_-$ interact with gravity, cf. eq. (2.4). Without gravity, it is known that this model is completely integrable [12] and exhibits dynamical mass generation [11]. Does the integrability remain once the model is coupled to gravity? A necessary condition of integrability is that the $S$-matrix for the scattering of the physical particles (here the massive fermions) is factorizable and elastic. This actually is a consequence of the factorizability and elasticity of the $S$-matrix for the pseudoparticles (here the original massless fermions).

As the simplest check, we have investigated whether the two-pseudoparticle $S$-matrix remains elastic in the presence of gravity. Here again we face the issue of the interpretation of the $S$-matrix elements in the presence of gravity: the $S$-matrix, e.g. for the scattering of two left fermions, is obtained from the four-point function (4.2) by removing the external propagators and setting the external momenta on-shell ($p_+ = p'_+ = q_+ = q'_+ = 0$ where $p, p'$ and $q, q'$ are the initial and final momenta). According to our discussion of section 3, this does not seem to lead to a well-defined quantity. However, bearing in mind the “experimental” situation for measuring $S$-matrix elements, even in the presence of gravity, we expect that the $S$-matrix should be well-defined at least within a gravitational weak-coupling expansion, provided the latter makes sense. Now for the chiral Gross-Neveu model, there is no scattering of two right pseudoparticles, while the left-right scattering is always elastic, as can be seen simply by combining momentum conservation and the on-shell condition. It remains to consider the scattering of two left fermions which interact due to their coupling to gravity, cf. eq. (4.2). To first order in $\frac{1}{\gamma}$, this $S$-matrix element vanishes. Indeed, from (4.5) one finds upon Fourier transforming that is is given by $\frac{1}{\gamma}(p+p')(q+q')\frac{(p-p')}{(p-p')^2}$ which vanishes on-shell ($p_+ = p'_+ = 0$). We have verified that this remains true at the next order in $\frac{1}{\gamma}$, including “two graviton exchanges”. If this remains true at all orders in $\frac{1}{\gamma}$ and even non-perturbatively, one would have complete elasticity of the two-pseudoparticle $S$-matrix, and this would certainly be a well-defined and gauge-invariant statement. One could then go on and speculate that all $S$-matrix elements for the pseudoparticle scattering remain elastic and factorizable in the presence of gravity and that the same is true for the physical (massive) fermions, in other
words that the integrability of the Gross-Neveu model survives coupling to gravity. However, there is still a long way to go.

Acknowledgements:

We are grateful to D. Gross, I. Klebanov and A. Polyakov for sharing their insights at the earlier stage of this work when both authors were still at Princeton University. One of us (A.B.) wishes to acknowledge the hospitality of the Theory Division at CERN where this work was completed.

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