Quantum sensing of control errors in three-level systems by coherent control techniques

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The quantum coherent control of a quantum system with high fidelity is rather important in quantum computation and quantum information processing. Many control techniques are used to reach these targets, such as resonant excitation, adiabatic passages, shortcuts to adiabaticity, and composite pulses. However, for a single pulse to realize population transfer, a tiny external error has a slight influence on the final population. The repeated application of the same pulse will greatly amplify the error effect, making it easy to be detected. Here, we propose to measure small control errors in three-level quantum systems through a coherent amplification of their effects using several coherent control techniques. For the two types of Hamiltonian with an SU(2) dynamic symmetry, we analyze how the fidelity of the population transfer is affected by the Rabi frequency error and static detuning deviation based on the pulse sequence with alternating and same phases, respectively. The results show that the sensitivity of detecting these errors can be effectively amplified by control pulse sequences. Furthermore, we discuss the efficiency of sensing the two errors with the control techniques by comparing the full width at half maximum of the population profiles. The results provide an accurate and reliable way for detecting the weak error in three-level quantum systems by repeatedly applying the coherent control pulse.

atom lasers, sensor, multiphoton processes, Raman lasers

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1 Introduction

The quantum coherent manipulation of a quantum state is a fundamental prerequisite in atomic and molecular physics [1], optics [2, 3], chemistry [4], quantum information, and computation[5, 6]. In particular, in recent years, it has had many practical applications in the field of quantum information science, including the preparation of quantum gates and quantum states [7], molecular chirality resolution [8], quantum batteries [9], and detection of parity violation [10].

Moreover, research related to this topic has been expanded from closed systems to open systems [11-13]. To reach universal quantum computation or information, the prepared quantum gate or quantum state should have an ultra-high fidelity (typically with an error in the range of \(10^{-2}-10^{-4}\)) under the influence of the parameter fluctuations of systems or external environmental noises.

Several protocols have been proposed to achieve an accurate and robust quantum coherent control with ultra-high fidelity, such as resonant pulses [14], adiabatic passages [15-19], shortcuts to adiabaticity (STA) [20-31], optimal control theory [18, 32-34], dressed-state driving [35-37], and...
composite pulses [38-40]. Recently, these protocols have attracted much attention in theory and experiment [41-46] and have application prospects for quantum communication [47, 48]. In 2015, Barredo et al. [42] investigated coherent excitation hopping in a spin chain of three Rydberg atoms through an experiment. In 2015, Nöbauer et al. [43] experimentally achieved tailored robustness against qubit inhomogeneities and control errors using the quantum optimal control technique. In 2017, Damme et al. [44] established time- and energy-minimum optimal control strategies for the robust and precise state control of two-level quantum systems. In 2021, Wu et al. [46] proposed a fast mixed-state control scheme, which is robust to multiple decoherence noises through the mixed-state inverse engineering scheme.

In general, the preparation of quantum states and gates is sensitive to the external environmental disturbance and parameter deviation of Hamiltonian [49, 50]. Moreover, some control errors originating from specific experimental implementations and conditions cannot be avoided in the experiment. Therefore, finding a simple and reliable method to detect such unwanted experimental errors is necessary. Recently, some protocols measure small parameter errors in quantum systems through the coherent amplification of their effect for the quantum sensing of weak electric and magnetic fields [51-55]. For instance, in 2015, Ivanov et al. [51] introduced a quantum sensing protocol for detecting very small forces assisted by a symmetry-breaking adiabatic transition in the quantum Rabi model. In 2018, Ivanov et al. [53] proposed a quantum sensing scheme for measuring phase-space displacement parameters using a single trapped ion. In 2021, Zhang et al. [54] presented a quantum sensing model using color detuning dynamics with dressed-state driving based on the stimulated Raman adiabatic passage. In 2021, Vitanov [55] showed a quantum sensing of weak electric and magnetic fields through a coherent amplification of the effects of small energy-level shifts.

In this paper, we propose a quantum sensing method to detect the error effects originating from the control parameter variation of Hamiltonian in three-level quantum systems using some coherent control pulses. The error effects are mapped on the populations of the quantum states by repeatedly applying quantum gate operations. In principle, the sensitivity of sensing the errors of a single pulse is usually very small, and the error effect can be coherently amplified by repeatedly applying the same pulse in a specific composite pulse sequence. For the Rabi frequency error and static detuning deviation, we employ the pulse sequence with alternating and similar phases to examine the sensitivity of a population transfer, respectively. The results show that the sensitivity of sensing the two errors can be greatly improved. Furthermore, we provide a detailed comparison of the error effects in different coherent control techniques and accordingly find that a different sequential order exists in these control techniques in terms of different errors.

The remainder of the paper is organized as follows: In sect. 2, we briefly introduce the three-level Hamiltonian model and the two different pulse sequences. In sect. 3, we analyze the three-level Hamiltonian in the case of off-resonance and use two pulse sequences to detect two kinds of errors: Rabi frequency error and static detuning deviation. In sect. 4, we examine the error effect from the Rabi frequency error in the one-photon resonance case. In sect. 5, we finally show the discussion and summary.

2 Preliminaries

A general A-type three-level system driven by external coherent fields can be described by the Schrödinger equation:

\[
\frac{i\hbar}{\partial t}\psi(t) = H_0(t)\psi(t),
\]

where \(\psi(t) = [c_1(t), c_2(t), c_3(t)]^T\) is a column vector with the probability amplitudes of the three bare states \(|1\rangle, |2\rangle, \text{and} |3\rangle\). The Hamiltonian, under the rotating wave approximation, on the basis of the bare states \(|1\rangle, |2\rangle, \text{and} |3\rangle\) is

\[
H_0 = \frac{\hbar}{2} \begin{pmatrix}
0 & \Omega_pe^{i\alpha} & 0 \\
\Omega_pe^{-i\alpha} & 2\Delta_0 & \Omega_se^{i\beta} \\
0 & \Omega_se^{-i\beta} & 2\Delta_1
\end{pmatrix},
\]

where \(\Omega_p\) and \(\Omega_s\) are the Rabi frequencies of the pump \((P)\) and Stokes \((S)\) pulses or electromagnetic fields and \(\alpha\) and \(\beta\) are the phases of the pulses, respectively, as shown in Figure 1(a). The detunings, respectively, are defined as \(\Delta_0 = (E_2 - E_1)/\hbar - \omega_p\), \(\Delta_s = (E_2 - E_3)/\hbar - \omega_s\), and \(\Delta_1 = \Delta_p - \Delta_s\), where \(\omega_p\) and \(\omega_s\) are the drive frequencies of pulses. \(E_n\) \((n = 1, 2, 3)\) are the energies of the bare state. Levels \(|1\rangle\) and \(|2\rangle\) are coupled by the \(P\) pulse, and levels \(|2\rangle\) and \(|3\rangle\) are coupled by the \(S\) pulse. The three-level system has many useful physical effects, such as coherent population trapping, electromagnetically induced transparency [56, 57], losing without inversion, and stimulated Raman adiabatic passage (STIRAP) [58]. On two-photon resonance \((\Delta_1 = 0)\), the STIRAP can be used to realize a high-fidelity population transfer from the state \(|1\rangle\) to the state \(|3\rangle\) along the adiabatic dark state based on the counterintuitive order of pulses. We assume that \(\alpha = \beta\) in the context. By tuning the control parameters of the Hamiltonian in eq. (2), we can obtain two families of the Hamiltonian with the SU(2) dynamic symmetry. CASE (I): off-resonance Hamiltonian with the Majorana decomposition when \(\Omega = \Omega_p = \Omega_s\) and \(\Delta = \Delta_p = -\Delta_s\); CASE (II): one-photon resonance Hamiltonian when \(\Delta_p = \Delta_s = 0\).
In principle, both of them can be reduced to the effective two-level Hamiltonian, while they show different performances on the control of quantum states. In the following, we shall discuss the two cases separately.

In the ideal case, many techniques with different pulses, including adiabatic and nonadiabatic ones, can be used to drive the evolution of a quantum system to achieve a complete population transfer. However, if there exists a weak error or perturbation on the control of the system, such as the Rabi frequency and detuning errors, the efficiency and fidelity of the transfer will be reduced. Therefore, we can distinguish the existence of the error by measuring the populations of states. However, for a single pulse, the experimental error usually has a small effect on the populations, so the use of composite pulse schemes to amplify the error effects is effective.

To sense the experimental errors originating from specific experimental implementations and conditions, i.e., Rabi frequency error and static detuning deviation with the SU(2) dynamic symmetry Hamiltonian, we examine how the population transfer fidelity is affected by these errors based on two simple composite pulse sequences (i.e., pulse sequences with the alternating and similar phases). In the pulse sequence with alternating phases, the same pulse is applied N times, but the phase by \( \pi \) is flipped from pulse to pulse (see Figure 1(b) top). That is, a minus sign is simultaneously applied N times, where \( N \) is used to represent the total number of pulses (see Figure 1(b) bottom).

### 3 CASE (I): off-resonance

Here, we study the case that \( \Omega = \Omega_p = \Omega_s \) and \( \Delta = \Delta_p = -\Delta_s \), and the Hamiltonian \( H_0 \) is reduced to the Hamiltonian \( H_1 \), which possesses the SU(2) dynamic symmetry. In this case, the Hamiltonian of the system can be written as:

\[
H_1 = \frac{\hbar}{2} \begin{pmatrix}
-2\Delta & \Omega e^{i\sigma} & 0 \\
\Omega e^{-i\sigma} & 0 & \Omega e^{i\sigma} \\
0 & \Omega e^{-i\sigma} & 2\Delta \\
\end{pmatrix},
\]

which can also be constructed by Heisenberg interacting-spin models [24] and trapped-ion system [59]. Moreover, the three-level Hamiltonian \( H_1 \) can be expressed as:

\[
H_1 = \hbar \left[ \Omega \cos(\alpha) J_x / \sqrt{2} + \Omega \sin(\alpha) J_y / \sqrt{2} + \Delta J_z \right],
\]

where

\[
J_x = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix},
\]

\[
J_y = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & i & 0 \\
-i & 0 & i \\
0 & -i & 0 \\
\end{pmatrix},
\]

\[
J_z = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

are the angular momentum operators of a spin-1 system [27], which satisfy the following commutator relation:

\[
[J_i, J_j] = i\varepsilon_{ijk} J_k,
\]

where \( \varepsilon_{ijk} \) is the Levi-Civita tensor. As the spin-1/2 operator and spin-1 operator possess the same SU(2) symmetry, we can construct a Hamiltonian \( H \) as:

\[
H = \hbar \left[ \Omega \cos(\alpha) S_x / \sqrt{2} + \Omega \sin(\alpha) S_y / \sqrt{2} + \Delta S_z \right],
\]

by replacing \( J_x, J_y, \) and \( J_z \) in eq. (4) with \( S_x, S_y, \) and \( S_z, \) respectively, where

\[
S_x = \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix},
\]

\[
S_y = \frac{1}{2} \begin{pmatrix}
0 & i & 0 \\
-i & 0 & i \\
0 & i & 0 \\
\end{pmatrix},
\]

\[
S_z = \frac{1}{2} \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

are the angular momentum operators of a spin-1/2 system. They fulfill a similar structure of the commutator relation:

\[
[S_i, S_j] = i\varepsilon_{ijk} S_k,
\]

with \( S_i = \sigma_i/2 \) \( (i = x, y, z) \). Clearly, the Hamiltonian \( H_1 \) possesses the same form as \( H \), so the propagator of \( H_1 \) can be derived in a similar way from the propagator of \( H \) by the Majorana decomposition \([60, 61]\). For the Hamiltonian \( H \), its propagator is an SU(2) matrix, which can be expressed in terms of the Cayley-Klein parameters \( a \) and \( b \) as:

\[
U = \frac{1}{2} \begin{pmatrix}
a & b \\
-b^* & a^* \\
\end{pmatrix}
\]

(10)
By mapping it onto the three-level system, the propagator for \( H_1 \) is

\[
U_1 = \begin{pmatrix}
a^2 & \sqrt{2}ab & b^2 \\
-\sqrt{2}ab^* & |a|^2 - |b|^2 & \sqrt{2}a^*b \\
-b^2 & -\sqrt{2}a^*b^* & a^2
\end{pmatrix}.
\] (11)

The populations of the energy levels \( |1\rangle \) and \( |3\rangle \) in the three-level system are the square of those of \( |g\rangle \) and \( |e\rangle \) in the two-level system, i.e.,

\[
P_1 = |a|^4 + p_g^2, \quad P_3 = |b|^4 + p_e^2.
\] (12)

where \( P_n (n = 1, 2, 3) \) are the level populations of \( |n\rangle \) in the three-level systems with \( H_1 \) and \( p_k \) \((k = g, e)\) are the level populations of \( |k\rangle \) in the two-level systems with \( H \). In the following, we study the quantum sensing of the change of the electric and magnetic fields by amplifying the population shifts in two different ways: Rabi frequency error and static detuning deviation.

### 3.1 Rabi frequency error

When the Rabi frequency errors occur on the \( P \) and \( S \) pulses in this three-level system, the Rabi frequency \( \Omega \) in \( H_1 \) has an error. We use a small time-independent and dimensionless parameter \( \lambda \) to represent the error-inducing Rabi frequency variation, i.e., \( \Omega \to (1 + \lambda)\Omega \). To begin with, let us consider the feasibility of the pulse sequence with alternating phases. For the Hamiltonian \( H \) in two-level systems, the propagator of a pulse with the \( \pi \) phase is

\[
U_\pi = \begin{pmatrix}a & -b \\ b^* & a^*\end{pmatrix}.
\] (13)

When the parameters of the Hamiltonian satisfy the following symmetry:

\[
\Delta(t) = -\Delta(t_f - t), \quad \Omega(t) = \Omega(t_f - t),
\] (14)

where \( t_f \) is the final time for a pulse. The parameter of the propagator of a single pulse \( a \) is a real number [62]. As a result, we have

\[
U = \begin{pmatrix}a & b \\ -b^* & a\end{pmatrix}, \quad U_\pi = \begin{pmatrix}a & -b \\ b^* & a\end{pmatrix}.
\] (15)

\( UU_\pi = U \), \( UU_\pi = I \), implying that the whole pulse sequence is acting as an identity matrix if we use the even number of the pulses in the sequence and the odd number of the pulses in the sequence is acting as a single pulse. Thus, the use of a pulse sequence with alternating phases to improve the sensitivity to the Rabi frequency error is meaningless.

Now, let us consider the pulse sequence with the same phases using four different pulse techniques. First, we use the simplest model, \( \pi \) pulses, to sense weak electromagnetic fields by amplifying the effects of the Rabi frequency error. The parameters of the Hamiltonian in eq. (7) can be chosen as:

\[
\Delta(t) = 0, \quad \int_0^t \Omega(t)/\sqrt{2}dt = \pi,
\] (16)

and the propagator of a single pulse can be analytically expressed as:

\[
U = \begin{pmatrix}-\sin \lambda \pi/2 & -i \cos \lambda \pi \\
-i \cos \lambda \pi/2 & -\sin \lambda \pi/2\end{pmatrix}.
\] (17)

The total propagator of any number of pulses \( N \) can be expressed as:

\[
U^{2n} = \begin{pmatrix}(-1)^n \cos n\lambda \pi & i(-1)^{n+1} \sin n\lambda \pi \\
-i(-1)^{n+1} \sin n\lambda \pi & (-1)^n \cos n\lambda \pi\end{pmatrix},
\] (18)

and

\[
U^{2n+1} = \begin{pmatrix}(-1)^{n+1} \cos \frac{2n + 1}{2} \lambda \pi & i(-1)^{n+1} \cos \frac{2n + 1}{2} \lambda \pi \\
i(-1)^{n+1} \cos \frac{2n + 1}{2} \lambda \pi & (-1)^{n+1} \sin \frac{2n + 1}{2} \lambda \pi\end{pmatrix},
\] (19)

where \( N = 2n \) or \( N = 2n + 1 \) indicates that the number of pulses is even or odd, respectively. Therefore, for the three-level system with the Hamiltonian \( H_1 \), the population transfer from \( |1\rangle \) to \( |3\rangle \) can be realized by applying odd pulses, and it gives

\[
P_3 = \cos^4(N\lambda \pi/2).
\] (20)

When even pulses are applied, the population will remain in the energy level \( |1\rangle \) with

\[
P_1 = \cos^4(N\lambda \pi/2).
\] (21)

With the increase in the number of pulses \( N \), the curve periods of the populations \( P_1 \) and \( P_3 \) decrease in Figure 2, and the change direction of the population increases near \( \lambda = 0 \); that is, the sensitivity with respect to the Rabi frequency increases. In Figure 2, we plot the population \( P_3 \) with the change in the Rabi frequency error for \( N = 5, 7, \) and \( 9 \) based on the \( \pi \) pulse sequence with alternating and same phases, respectively. The pulse sequence with alternating phases cannot improve the sensitivity of sensing the Rabi frequency error. For the single pulse, the profile of the population is much broader than the sequences of pulses of the same phases with \( N = 5, 7, \) and \( 9 \) near \( \lambda = 0 \). Moreover, the narrow features produced by the pulse sequence with the same phases become narrower as the number of pulses increases.
Next, we use Gaussian pulses with linear chirp [63] and Allen-Eberly (AE) adiabatic passages, which is a special case of the level-crossing Demkov-Kunike model [64] to demonstrate how the population is affected by the Rabi frequency error. For Gaussian pulses with a linear chirp. The parameters of the Hamiltonian are

\[ \Delta(t) = \sqrt{2}t/T^2, \quad \Omega(t) = 2\sqrt{2}e^{-t^2/T^2}/T, \] (22)

where \( T \) is the pulse width. For the AE adiabatic passages, the Hamiltonian parameters are set as:

\[ \Delta(t) = \sqrt{2}\tanh(t/T)/T, \quad \Omega(t) = \sqrt{6}\sech(t/T)/T. \] (23)

In Figure 3, we plot the population \( P_3 \) with the change in the Rabi frequency error for \( N = 5, 7, \) and 9 based on the Gaussian pulse and AE adiabatic pulse sequence with the same phases, respectively. The sensitivity of sensing the Rabi frequency error increases as the number of pulses increases. Furthermore, at around \( \lambda = 0 \), the full width of half maximum (FWHM) of the population for AE adiabatic passages is larger than that for Gaussian pulses, suggesting that Gaussian pulses are more sensitive to sensing the Rabi frequency error than AE adiabatic passages. In fact, the adiabaticity of AE adiabatic passages is better than that of Gaussian pulses. Thus, AE adiabatic passages are more stable than the latter in the presence of control errors.

Finally, we use the STA technique to analyze the sensitivity of a population transfer via the Rabi frequency error. For the Hamiltonian \( H(t) \), there is an invariant [20]

\[ I = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\gamma} \\ \sin \theta e^{i\gamma} & -\cos \theta \end{pmatrix}, \] (24)

which satisfies the dynamic equation

\[ \frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{\hbar} [I, H] = 0. \] (25)
By solving this equation, we obtain the constraint conditions:
\[ \dot{\theta} = -\Omega \sin \gamma / \sqrt{2}, \quad \dot{\gamma} = -\Omega \cot \theta \cos \gamma / \sqrt{2} - \Delta. \]  
(26)

Using the eigenstates \(|\varphi_i(t)\rangle\) of the invariant \(I\), the solution of Schrödinger equation \(i\hbar \partial / \partial t = H(t) |\varphi(t)\rangle\) [20] can be written as:
\[ |\varphi(t)\rangle = \sum_j C_j e^{i\varepsilon_j(t)} |\varphi_j(t)\rangle, \]
(27)
where \(C_j\) are constants and \(\varepsilon_j(t)\) are the LR phases with
\[ \varepsilon_i(t) - \varepsilon_-(t) = \frac{\theta \cos \gamma}{2 \sin \theta \sin \gamma}. \]
(28)

To realize population inversion along the eigenstate \(|\varphi_-(t)\rangle\), we can choose \(\theta(t)\) as:
\[ \theta(t) = \pi \gamma / T. \]
(29)

Here, we try the Fourier series type of Ansatz:
\[ \varepsilon_i(t) = -\theta - n \sin(2\theta), \]
(30)
where \(n\) is a freely chosen parameter. Using eq. (28), we obtain
\[ \gamma(t) = -\arccot[2(1 + 2n \cdot \cos 2\theta) \sin \theta]. \]
(31)

Based on the perturbation expansion [22], we can define the error sensitivity \(q_\lambda\) with respect to the Rabi frequency error \(\lambda\) as:
\[ q_\lambda = \frac{-\partial^2 P(t_f)}{\partial \lambda^2} \bigg|_{\lambda=0}, \]
(32)
where \(P(t_f)\) is the transition probability at the final time. \(q_\lambda\) is determined by selecting \(n\). The larger the value of \(q_\lambda\), the more sensitive it is. Here, we choose \(n = 0.5\), and this gives a relatively large value of \(q_\lambda\) with 1.91. In Figure 4, we plot \(P_\lambda\) with the change in the Rabi frequency error for \(N = 5, 7,\) and 9 based on the STA pulse sequence with the same phases. The number of pulses in the STA pulse sequence can affect the efficiency of the population transfer, and the sensitivity increases with the increase in the number of pulses.

Finally, we compare the effect of sensing the Rabi frequency error in these different schemes with the same phases by comparing the FWHM change curve near \(\lambda = 0\) based on the same numbers of pulses, as shown in Table 1. The smaller the value of the FWHM, the higher the sensitivity of the scheme with the Rabi frequency error. For the same number of pulses, the sensitivity to the Rabi frequency error of these schemes shows a sequential order: \(\pi\) pulse, Gaussian pulse, STA pulse, and AE adiabatic pulse sequences.

### Table 1

| Number of pulses | \(\pi\)-S | Gaussian-S | AE-S | STA-S |
|------------------|---------|-----------|------|-------|
| \(N = 5\)       | 0.146   | 0.153     | 0.233| 0.165 |
| \(N = 7\)       | 0.104   | 0.109     | 0.162| 0.117 |
| \(N = 9\)       | 0.082   | 0.084     | 0.123| 0.091 |

### 3.2 Static detuning deviation

Now, let us consider the case that a small static detuning deviation exists on the Hamiltonian \(H_1\), i.e., \(\Delta \rightarrow \Delta + \delta\), where \(\delta\) is static detuning. This may be caused by deviating the frequencies of the laser, microwave, radio-frequency generator, or static magnetic field. Due to the existence of static detuning, the Hamiltonian parameters do not satisfy symmetry in eq. (14), and the Cayle-Klein parameter \(a\) is no longer a real number. Its propagator is described by the most general form in eq. (10).

For the pulse sequence with alternating phases, the propagator of the two-level system after using two pulses is
\[ U_\pi^2 = U_\pi U = \begin{pmatrix} a^2 + |b|^2 & 2ib\alpha \; 2ia\alpha \; 2ib^2a \; a^2 + |b|^2 \end{pmatrix}. \]
(33)

Thus, we can write the propagator of any even number of pulses \(N = 2n\) with alternating phases in terms of
\[ U_\pi^{2n} = (U_\pi^2)^n = \begin{pmatrix} \cos n\Theta + 2ia\omega E & 2ib\omega E \\ 2ib^*\alpha E & \cos n\Theta - 2ia\omega E \end{pmatrix}, \]
(34)
where $\Theta = \arccos(1 - 2a^2)$ and $E = \sin n\Theta / \sin \Theta$. In addition, the propagator of any odd number of pulses $N = 2n + 1$ with alternating phases can be expressed as:

$$U_{z+1} = U_{z}^{2n} = \left( \begin{array}{cc} aF + 2iaE & bF \\ -b^*F & aF - 2iaE \end{array} \right),$$

(35)

where $F = \cos(n + 1/2)\Theta / \cos(\Theta/2)$. Therefore, the populations of $|1\rangle$ when $N$ is even and $|3\rangle$ when $N$ is odd are, respectively,

$$P_1 = |b|^4 \cos^4(N\Theta/2)/\cos^4(\Theta/2), \quad N = \text{even};$$

$$P_3 = |b|^4 \cos^4(N\Theta/2)/\cos^4(\Theta/2), \quad N = \text{odd}.$$  

(36)

Meanwhile, when we consider the pulse sequence with the same phases in this disturbed two-level system, the total propagator is the multiplication of $N$ individual propagators $U$, that is

$$U^N = \left( \begin{array}{cc} \cos N\theta + ia_0 D & bD \\ -b^*D & \cos N\theta - ia_0 D \end{array} \right),$$

(37)

where $a_0$ and $a_i$ are the real and imaginary parts of $a$, $\theta = \arccos(a_0)$, and $D = \sin N\theta / \sin \theta$, respectively. The population of the excited state $|e\rangle$ in the two-level system after using the pulse sequence with the same phases is

$$P_e = |b|^2 \sin^2 N\theta / \sin^2 \theta.$$  

(38)

Therefore, in the three-level system, the corresponding populations of the states $|3\rangle$ and $|1\rangle$, respectively, are

$$P_3 = p^3(e) = |b|^4 \sin^4 N\theta / \sin^4 \theta,$$  

(39)

and

$$P_1 = \left| 1 - |b|^2 \sin^2 N\theta / \sin^2 \theta \right|^2.$$  

(40)

In the ideal case, using a single pulse, the population inversion in the two-level system is achieved, i.e., $a = 0$. This leads to $\theta = \pi/2$ and $\Theta = 0$. Now, we employ the series expansion to simplify the equations in eqs. (36), (39), and (40) by ignoring the second higher-order term of $a_i$ and $a_i$. Consequently, for the pulse sequence with alternating phases, we have

$$P_1 \approx 1 - 2N^2 a^2, \quad N = \text{even};$$

$$P_3 \approx 1 - 2N^2 a^2, \quad N = \text{odd},$$

(41)

and for the pulse sequence with the same phases, we have

$$P_1 \approx 1 - 2N^2 a^2, \quad N = \text{even};$$

$$P_3 \approx 1 - 2N^2 a^2 - 2a^2, \quad N = \text{odd}.$$  

(42)

For both types of pulse sequences, the populations of $|1\rangle$ and $|3\rangle$ decrease as the number of pulses $N$ increases, so it is easier to detect smaller static detuning when $N$ is larger. However, the two pulse sequences increase their sensitivity in different ways: the pulse sequence with alternating phases is dependent on the imaginary part of the Cayley-Klein parameter $a_i$, whereas the pulse sequence with the same phases is dependent on its real part, $a_r$. Therefore, we can choose the proper pulse sequence to sense the effect of the static detuning deviation in terms of the size of the real and imaginary parts of the Cayley-Klein parameter. In the following, we use flat $\pi$ and Gaussian pulses to examine how the effectiveness of the population transfer is affected by the static detuning deviation.

The first model is the flat $\pi$ pulse model, whose Hamiltonian parameters are

$$\Delta(t) = 0, \quad \Omega(t) = \sqrt{2}\pi/T.$$  

(43)

Its propagator has an analytical form, and the corresponding Cayley-Klein parameter $a$ can be expressed as:

$$a = \cos \left( \frac{\sqrt{\pi^2 + \delta^2}}{2} \right) + \frac{i\delta}{\sqrt{\pi^2 + \delta^2}} \sin \left( \frac{\sqrt{\pi^2 + \delta^2}}{2} \right).$$  

(44)

Its series expansion can be written as:

$$a \approx \frac{\delta}{\pi} - \frac{\delta^2}{4\pi} + O(\delta^3).$$  

(45)

The real ($a_r$) and imaginary ($a_i$) parts of $a$ are obtained as:

$$a_r \approx -\frac{\delta^2}{4\pi^2}, \quad a_i \approx \frac{\delta}{\pi}.$$  

(46)

$a_i$ and $a_r$ are a first-order and second-order small quantities with $\delta$, respectively. Hence, the reduction of population caused by static detuning deviation is greater using a pulse sequence with alternating phases than that using a pulse sequence with the same phases. In Figure 5(a) and (b), we plot the population $P_3$ with the change in the static detuning deviation for $N = 5, 7$, and $9$ based on a flat $\pi$ pulse sequence with alternating and similar phases, respectively. Under the same number of pulses $N$, the pulse sequence with alternating phases is more sensitive than the pulse sequence with the same phases with respect to the static detuning error.

The second model is the Gaussian pulse model, whose Hamiltonian parameters can be set as:

$$\Delta(t) = 0, \quad \Omega(t) = \sqrt{2}\pi e^{-\left(\delta/T\right)^2}/T.$$  

(47)

In Figure 5(c) and (d), we plot the population vs. static detuning deviation for $N = 5, 7$, and $9$ based on the Gaussian pulse sequence with alternating and similar phases, respectively. Around $\delta = 0$, each of the four cases in Figure 5 has a narrow feature, and it gets narrower as the number of pulses
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1. Static detuning $\delta$

Figure 5 (Color online) Population of the level $|3\rangle$ vs. static detuning deviation $\delta$ for the sequences of $N = 5$, 7, and 9 identical pulses, where $N = 5$ (blue, solid line), $N = 7$ (red, dashed line), and $N = 9$ (black, dotted line). (a) Pulse sequence with alternating phases using flat $\pi$ pulses; (b) pulse sequence with the same phases using flat $\pi$ pulses; (c) pulse sequence with alternating phases using Gaussian pulses; (d) pulse sequence with the same phases using Gaussian pulses.

$N$ increases. This narrow feature is much narrower for pulse sequences of alternating phases than that for pulse sequences of the same phases. Moreover, the feature in the Gaussian pulses near $\delta = 0$ is much more narrow than that in flat $\pi$ pulses for the same $N$. Finally, for a clear comparison, we present the FWHM change profile with respect to the static detuning error near $\delta = 0$ in four different schemes in Table 2. The FWHM of the flat $\pi$ pulses is more than three times those of the Gaussian pulses for the pulse sequences of the alternating phases, whereas they are more than twice those of the Gaussian pulses for the pulse sequences of the same phases.

Table 2  FWHM in different schemes to sense static detuning deviation. We use -A and -S to indicate the pulse sequences with alternating and similar phases, respectively

| Number of pulses | Flat $\pi$-A | Flat $\pi$-S | Gaussian-A | Gaussian-S |
|------------------|--------------|--------------|------------|------------|
| $N = 5$          | 0.72         | 2.21         | 0.32       | 0.68       |
| $N = 7$          | 0.52         | 1.91         | 0.22       | 0.58       |
| $N = 9$          | 0.40         | 1.71         | 0.17       | 0.52       |

i.e.,

$\Omega_i \rightarrow (1 + \eta)\Omega_i$,  \hspace{1cm} (49)

where $i = s, p$. We will call $\Omega_i$ as the standard pulse and $\Omega'_i = (1 + \eta)\Omega_i$ as the error pulse. First, when $\Omega_s$ and $\Omega_p$ are constants, the propagator of the system can be expressed as:

$U_2 = \begin{pmatrix} U_{211} & U_{212} & U_{213} \\ U_{221} & U_{222} & U_{223} \\ U_{231} & U_{232} & U_{233} \end{pmatrix}$,  \hspace{1cm} (50)

where

$[U_{211}] = (\Omega^2 + \Omega^2_p \cos A)T^2/A^2$,  
$[U_{212}] = \cos A$,  
$[U_{213}] = (\Omega^2_p + \Omega^2 \cos A)T^2/A^2$,  
$[U_{221}] = [U_{212}]$,  
$[U_{222}] = [U_{213}]$,  
$[U_{223}] = -i\Omega_p T \sin A/A$,  
$[U_{231}] = [U_{213}] = \Omega_p \Omega_s (\cos A - 1)T^2/A^2$.  

4 Case (II): one-photon resonance

Here, in the case of the one-photon resonance where $\Delta_p = \Delta_s = 0$, the Hamiltonian $H_0$ can be simplified as $H_2$ with

$H_2 = \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_p e^{i\vartheta} & 0 \\ \Omega_p e^{-i\vartheta} & 0 & \Omega_s e^{i\vartheta} \\ 0 & \Omega_s e^{-i\vartheta} & 0 \end{pmatrix}$.

In the following, the Rabi frequency error may occur on either the $P$ or $S$ pulse alone. If the $S$ or $P$ pulse has an error,
Here, $A = \sqrt{\Omega_p^2 T^2 + \Omega_s^2 T^2}/2$. The solution to achieve a complete population transfer without error is

$$\Omega_p = \Omega_s = \sqrt{\frac{2}{\pi T}},$$

(52)

where $T$ is the pulse width.

We denote this scheme as the CDS. In particular, the error pulse can be applied to one of $P$ and $S$ all the time (fixed-error case) or applied alternatively to the pulses between $P$ and $S$ (alternating error case). Specifically, the alternating error uses the error pulse $\Omega'_s$ in the first CDS, $\Omega'_p$ in the second CDS, and $\Omega'_p$ in the third CDS.

Here, we study how the population of the level $|1\rangle$ is affected by the Rabi frequency error for even-numbered pulses in the CDS for four cases: (i) the pulse sequence with alternating phases for fixed errors; (ii) pulse sequence with the same phases for fixed errors; (iii) pulse sequence with alternating phases for alternating errors; (iv) pulse sequence with the same phases for alternating errors. For case (i), we can find that

$$U_2(\eta, 0)U_2(\eta, 0) = U_2(0, \eta)U_2(0, \eta) = I,$$

(53)

where $U_2(\eta, 0)$ and $U_2(\eta, 0)$ represent the propagator that there is a Rabi frequency error in an $P$ pulse with zero and $\pi$ phases, respectively. $U_2(0, \eta)$ and $U_2(0, \eta)$ represent the propagator that there is a Rabi frequency error in a $S$ pulse with zero and $\pi$ phases, respectively. This finding implies that the sensitivity to the Rabi frequency error will not increase with the increase in the number of pulses in the case of the pulse sequence with alternating phases for fixed errors, as shown in Figure 6(a). In Figure 6(b)-(d), we plot the performance of the population with respect to $\eta$ for the other three cases. The sensitivity increases with the increase in the number of pulses in cases (ii)-(iv). Among them, the pulse sequence with the same phases for alternating error shows the highest sensitivity.

As a comparison, we use the STIRAP to study the population change with respect to the Rabi frequency error. In the traditional STIRAP, the $P$ and $S$ pulses are usually identical symmetric functions of time but implemented in a counterintuitive order, i.e.,

$$\Omega_p = \Omega_0 f(t), \quad \Omega_s = \Omega_0 f(t - \tau),$$

(54)

where $\Omega_0$ and $\tau$ are the peak Rabi frequency and pulse delay, respectively. The $S$ pulse is implemented earlier than the $P$ pulse. If a STIRAP, which has the $P$ and $S$ pulses, can achieve a complete population transfer of $|1\rangle \rightarrow |3\rangle$, then one

![Figure 6](Color online) Population of the level $|1\rangle$ vs. the Rabi frequency error $\eta$ for the sequences of $N = 4$, 6, and 8 identical pulses based on the constant-driven scheme (CDS), where $N = 4$ (blue, solid line), $N = 6$ (red, dashed line), and $N = 8$ (black, dotted line). (a) Pulse sequence with alternating phases for the fixed error; (b) pulse sequence with the same phases for the fixed error; (c) pulse sequence with alternating phases for alternating errors; (d) pulse sequence with the same phases for alternating errors.
can change the time order of the $P$ and $S$ pulses to achieve the transfer of $|3\rangle$ back to $|1\rangle$. However, adiabatic protocols usually require a high pulse peak to achieve a high-fidelity population transfer. If we adopt the adiabatic pulse schemes with a low pulse peak, we usually cannot achieve the above population transfer process. Here, by adopting the adiabatic pulse method with a low peak value, we consider the pulse sequence with alternating phases for the following reasons: without the Rabi frequency error, we can completely drive the system to return to the initial state $|1\rangle$ after executing a pair of STIRAPs with opposite phases. It usually cannot completely return to $|1\rangle$ in the presence of this error. Hence, we can detect this error by measuring the population change of $|1\rangle$. Moreover, the fixed-error case is not considered here because of the following condition:

$$U_2(\eta, 0)\tilde{U}_2(\eta, 0) = U_2(0, \eta)\tilde{U}_2(0, \eta) = 1,$$

(55)

where the tilde indicates the propagator that the time order of $P$ and $S$ is exchanged. For the STIRAP, we make use of the pulse sequence with alternating phases for an alternating error to sense the Rabi frequency error by applying several common pulses: Gaussian pulses, hyperbolic secant (sech) pulses, and sinusoidal pulses.

The Rabi frequencies of the Gaussian pulses are

$$\Omega_p = \Omega_0 e^{-\eta^2/(T^2)}, \quad \Omega_s = \Omega_0 e^{-\eta^2/(T^2-2)},$$

(56)

and the Rabi frequencies of the hyperbolic secant pulses are

$$\Omega_p = \Omega_0 \text{sech}(\eta/T), \quad \Omega_s = \Omega_0 \text{sech}(\eta/T - 5),$$

(57)

respectively. Here, the amplitude of the Rabi frequency is set as $\Omega_0 = \sqrt{2}\pi/T$. In Figure 7, we plot the population of the level $|1\rangle$ with respect to the Rabi frequency error for $N = 4, 6, \text{ and } 8$ pulses in the pulse sequence with alternating phases for alternating errors using Gaussian and sech pulses, respectively. Their population profiles are largely influenced by the Rabi frequency error. Each of them forms a narrow feature near $\eta = 0$, which gets narrower with the increase in the number of pulses $N$. Furthermore, the feature of the sech pulse sequences is narrower than that of the Gaussian pulse sequences near $\eta = 0$.

Finally, we consider two types of sinusoidal pulses for comparison. The Rabi frequencies of the first sinusoidal pulses are

$$\Omega_p = \Omega_0 \sin(\pi T/T), \quad \Omega_s = \Omega_0 \cos(\pi T/T).$$

(58)

The Rabi frequencies of the second sinusoidal pulses take the form of

$$\Omega_p = \Omega_0 \sin^2(\pi T/T), \quad \Omega_s = \Omega_0 \cos^2(\pi T/T),$$

(59)

where $\Omega_0 = \sqrt{2}\pi/T$. In Figure 8, we also plot the population of the level $|1\rangle$ with respect to the Rabi frequency error for $N = 4, 6, \text{ and } 8$ pulses in the pulse sequence with alternating phases for alternating errors using the two sinusoidal pulses, respectively. The sensitivity with the Rabi frequency error increases with the increase in the number of pulses, and the sensitivity of the sin pulse sequence is higher than that of the sin$^2$ pulse sequence.

At the end of this section, we summarize the performances of the different schemes by comparing the FWHM change curve with respect to the Rabi frequency error near $\eta = 0$ when the number of pulses is the same, as shown in Table 3. Among them, the sech pulse sequence with alternating phases for alternating errors shows the highest sensitivity, followed by the Gaussian pulses and then the CDS protocol. The sinusoidal pulse methods take the highest values of the FWHM of the population change curve.

**Figure 7** (Color online) Population of level $|1\rangle$ vs. the Rabi frequency error coefficient $\eta$ for the sequences of $N = 4, 6, \text{ and } 8$ identical pulses, where $N = 4$ (blue, solid line), $N = 6$ (red, dashed line), and $N = 8$ (black, dotted line). (a) Pulse sequence with alternating phases for the alternating error using Gaussian pulses; (b) pulse sequence with alternating phases for the alternating error using sech pulses.
Figure 8  (Color online) Population of level $|1\rangle$ vs. the Rabi frequency error coefficient $\eta$ for the sequences of $N = 4$, 6, and 8 identical pulses, where $N = 4$ (blue, solid line), $N = 6$ (red, dashed line), and $N = 8$ (black, dotted line). (a) Pulse sequence with alternating phases for alternating errors using sin pulses; (b) pulse sequence with alternating phases for alternating errors using $\sin^2$ pulses.

Table 3  FWHM in different schemes to sense the Rabi frequency error. We use -SF, -AA, and -SA to indicate the pulse sequence with similar phases for fixed errors, alternating phases for alternating errors, and similar phases for alternating errors, respectively

| Number of pulses | CDS-SF | CDS-AA | CDS-SA | Gaussian-AA | sech-AA | sin-AA | $\sin^2$-AA |
|------------------|-------|-------|-------|-------------|--------|-------|------------|
| $N = 4$          | 0.369 | 0.406 | 0.267 | 0.206       | 0.120  | 0.561 | 0.720      |
| $N = 6$          | 0.225 | 0.264 | 0.178 | 0.137       | 0.080  | 0.375 | 0.482      |
| $N = 8$          | 0.184 | 0.198 | 0.184 | 0.103       | 0.060  | 0.282 | 0.362      |

5 Conclusion

In conclusion, we investigate the sensitivity of composite pulse sequences to parameter variation in two families of Hamiltonian with the SU(2) dynamic symmetry, namely, the off-resonance case and one-photon resonance case, using some popular coherent control techniques. All of the control pulses can realize a high-fidelity population transfer in three-level quantum systems. The repeated application of the same gate-generating pulse greatly amplifies the effect of the experimental errors originating from the imperfection of parameter control, and the error effect is mapped onto the change of populations. The sensitivity with respect to these errors is greatly improved with the increase in the number of pulses. In the first case, the pulse sequence with similar phases is more sensitive to the Rabi frequency error than that with alternating ones, and the $\pi$ pulse sequence is the most sensitive among the considered coherent control schemes. For static detuning deviation, the pulse sequence with alternating phases is more sensitive than that with similar ones, and the Gaussian pulse sequence is more sensitive than the flat $\pi$ pulse sequence. In the second case, we use the pulse sequence with alternating phases to study the sensitivity with respect to the Rabi frequency error. Based on the comparison of the FWHM of the population change in the different control schemes, the sech pulse sequence shows the highest sensitivity, followed by the Gaussian pulses, then the CDS protocol, and finally, the sinusoidal pulses.

There is still much room for promotion in the work of using composite pulse techniques to realize the desired quantum sensing. For instance, one can design an optimal composite pulse sequence that is highly sensitive to the specific parameter in physical systems and can suppress the influence of other parameters at the same time. Moreover, only two kinds of composite pulse sequences are considered in this study, and there may still be more complex composite pulse sequences that are more sensitive to experimental errors. Finally, the composite pulse schemes in different quantum systems show different performances. Thus, one can design a simple and suitable pulse sequence to improve the error sensitivity in terms of the properties of these systems.

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