Superspecies and their representations *

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Abstract

Superspecies are introduced to provide the nice constructions of all finite-dimensional superalgebras. All acyclic superspecies, or equivalently all finite-dimensional (gr-basic) gr-hereditary superalgebras, are classified according to their graded representation types. To this end, graded equivalence, graded representation type and graded species are introduced for finite group graded algebras.

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1 Introduction

Clifford algebras, more special Grassmann algebras, and their representations play a quite important role in both mathematics and physics (ref. [31] and some references therein). Here, we shall study more general finite-dimensional superalgebras and their representation theory.

It is well-known that quiver and species play a crucial role in the construction and representation theory of finite-dimensional algebras (ref. [3] [13] [8]). Is there an analog of quiver and species for finite-dimensional superalgebras? In order to answer this question, we shall introduce graded equivalence theory at first (Section 3). Our aim is to reduce the constructions and the representation theory of all finite-dimensional finite group graded algebras to those of all gr-basic ones. We shall show that every finite-dimensional

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finite group graded algebra is graded equivalent to a gr-basic one, which is unique up to congruence (Theorem 3.2). Furthermore, we shall introduce the concept of graded species (Section 4.2), in particular superspecies, which plays a crucial role for finite-dimensional superalgebras as quiver and species for finite-dimensional algebras. In case the underlying field is algebraically closed and of characteristic not equal to 2, we shall obtain the super version of Wedderburn’s principal theorem (Theorem 4.2), and show that each finite-dimensional gr-basic superalgebra is isomorphic to the factor of the graded tensor algebra of a superspecies modulo an admissible graded ideal (Theorem 4.3). In particular, each finite-dimensional gr-basic gr-hereditary superalgebra is graded isomorphic to the graded tensor algebra of an acyclic superspecies.

Next we consider the question: “How to classify acyclic superspecies, or equivalently finite-dimensional (gr-basic) gr-hereditary superalgebras, according to their graded representation types?” In order to define the graded representation types of acyclic superspecies, we shall introduce the graded representation types of finite-dimensional finite group graded algebras at first (Section 5). Then we shall provide the graded version of Drozd’s theorem (Theorem 5.1). After classifying all finite-dimensional gr-division superalgebras over an algebraically closed field, we shall introduce the quiver of a superspecies (Section 6.1), and prove that the category of finite-dimensional representations of a superspecies is equivalent to the category of finite-dimensional representations of its quiver (Theorem 6.1). Furthermore, we shall obtain the super version of Kac’s theorem (Theorem 6.3) and classify all acyclic superspecies according to their graded representation types in terms of their quivers (Theorem 6.2). For a superspecies, we shall also introduce its superquiver (Section 6.3). The construction of the superquiver of a superspecies is easier than that of its quiver. Therefore, we shall classify all acyclic superspecies again according to their graded representation types in terms of their superquivers (Theorem 6.4).

In fact, our original motivation is to realize some Lie superalgebras and their quantized enveloping superalgebras by finite-dimensional gr-hereditary superalgebras, just as the realization of Kac-Moody algebras and their quantized enveloping algebras by finite-dimensional hereditary algebras via Hall algebra approach (ref. [30, 16, 26]). As far as we know, many experts thought or are thinking about this problem. But we think, before this, we should learn much more about finite-dimensional superalgebras and their representation theory.
2 Graded algebras and graded modules

In this section, on one hand, we shall fix some notations and terminologies on graded algebras and graded modules, on the other hand, we shall get some new results on these aspects as well. For the knowledge of graded ring theory, we refer to [24].

2.1 Graded algebras

Throughout we assume that $K$ is a fixed field, $G$ is a finite multiplicative group with identity $e$, and the composition of maps is written from left to right.

A $G$-graded $K$-vector space or graded vector space is a $K$-vector space $V = \oplus_{g \in G} V_g$ where $V_g$’s are $K$-subspaces of $V$. By definition, each $v \in V$ can be uniquely written as $v = \sum_{g \in G} v_g$ where $v_g \in V_g$ for all $g \in G$. A nonzero element $v \in V_g$ is said to be homogeneous of degree $g$, and we write $\deg v = g$. A graded vector space $V$ is said to be trivially graded if $V = V_e$. A graded subspace of a graded vector space $V$ is a $K$-subspace $U$ of $V$ such that $U = \oplus_{g \in G} (U \cap V_g)$. A map of graded vector space is a $K$-linear map $\phi : V \to W$ such that $\phi(V_g) \subseteq W_g$ for all $g \in G$. If $G$ is abelian, it is usually considered as an additive group. In particular, in the case of $G = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, a $G$-graded $K$-vector space is called a super vector space.

A $G$-graded $K$-algebra or graded algebra $A$ is both an associative $K$-algebra with identity and a $G$-graded $K$-vector space $A = \oplus_{g \in G} A_g$ such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. A graded subalgebra (resp. graded ideal) of a graded algebra is both a subalgebra (resp. ideal) and a graded subspace. A graded homomorphism of graded algebras is both an algebra homomorphism and a graded vector space map. In the case of $G = \mathbb{Z}_2$, a $G$-graded $K$-algebra is called a superalgebra.

2.2 Graded modules

A (left) $G$-graded module or graded module $M$ over a graded algebra $A = \oplus_{g \in G} A_g$ is both a (left) $A$-module and a graded vector space $M = \oplus_{g \in G} M_g$ such that $A_g M_h \subseteq M_{gh}$ for all $g, h \in G$. A graded morphism of graded modules is both an $A$-module morphism and a graded vector space map.

All graded $A$-modules and all graded morphisms between them form a Grothendieck category, denoted by $\text{Gr} A$. Denote by $\text{gr} A$ the full subcategory of $\text{Gr} A$ consisting of all finitely generated graded $A$-modules. Moreover, we
2.3 Gr-indecomposable modules

Let $A$ be a graded algebra. A nonzero graded $A$-module $M$ is said to be gr-indecomposable if it is not the direct sum of two nonzero graded modules.

Let $KG^*$ be the dual vector space of $KG$ and $\{p_g|g \in G\}$ its dual basis such that $p_g(x) = a_g \in K$ for all $g \in G$ and $x = \sum_{h \in G} a_h h \in KG$. The smash product $A \# KG^*$ is the $K$-algebra whose underlying $K$-vector space is $A \otimes_K KG^*$ and whose multiplication is given by $(a \# p_g)(b \# p_h) := ab_{gh^{-1}} \# p_h$ where $a \# p_g$ denotes $a \otimes p_g$ (ref. [4, Section 1]). It is well-known that $Gr A \cong Mod A \# KG^*$ (ref. [4, Theorem 2.2]). By this isomorphism and the Krull-Schmidt theorem for finite-dimensional algebras, or using the same strategy as the proof of [27, Corollary 5.3], we obtain the following result:

**Theorem 2.1. (Graded version of Krull-Schmidt theorem)** Let $A$ be a finite-dimensional graded algebra and $M \in gr A$. Then $M$ is gr-indecomposable if and only if $End_{Gr A}(M)$ is local. Moreover, each nonzero graded module in $gr A$ has a unique decomposition of gr-indecomposables up to permutation and graded isomorphism.

A graded module $M$ is said to be gr-simple if 0 and $M$ are the only graded submodules of $M$. A graded module $M$ is said to be gr-semisimple
if $M$ is a direct sum of gr-simple modules. A graded submodule $N$ of a graded module $M$ is called a gr-maximal submodule if $M/N$ is gr-simple. For a graded module $M$, we denote by $J_G(M)$ its graded Jacobson radical, i.e., the intersection of all gr-maximal submodules of $M$. We call $\text{top}_G(M) := M/J_G(M)$ the gr-top of $M$. We say that a graded submodule $N$ of a graded module $M$ is gr-small in $M$ if $N + X = M$ for a graded submodule $X$ of $M$ implies $X = M$. Clearly, $N$ is gr-small in $M$ if and only if $N \subseteq J_G(M)$. Note that $J_G(A)$ is a graded ideal of $A$, called the graded Jacobson radical of $A$, and denoted by $J_G(A)$. Moreover, $J_G(A) \subseteq J(A)$, the Jacobson radical of $A$ (ref. [4, Theorem 4.4(1)]).

The following result is a graded analog of [3, Proposition I.3.5], which can be proved similarly:

**Proposition 2.1.** Let $A$ be a finite-dimensional graded algebra and $M \in \text{gr}A$. Then $J_G(M) = J_G(A)M$.

A graded algebra is called a gr-division algebra if every nonzero homogeneous element is invertible. A graded algebra is said to be gr-local if every homogeneous element is either invertible or nilpotent.

The following result provides a characterization of gr-local algebras:

**Proposition 2.2.** Let $A$ be a finite-dimensional graded algebra. Then the following statements are equivalent:

1. $A$ is gr-local.
2. $A/J_G(A)$ is a gr-division algebra.
3. The initial subalgebra $A_e$ of $A$ is local.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $A$ is gr-local. Let $J$ be the graded ideal of $A$ generated by all homogeneous nilpotent elements in $A$.

**Claim 1.** If $x \in A_g$ is nilpotent then $axb$ is nilpotent for all homogeneous elements $a, b$ in $A$.

**Proof of Claim 1.** First of all, we show that $ax$ is nilpotent. Assume on the contrary that $ax$ is invertible. Since $A$ is graded local, $a$ is either invertible or nilpotent. If $a$ is invertible then so is $x = a^{-1}(ax)$. It is a contradiction. If $a$ is nilpotent then there is $n \geq 2$ such that $a^n = 0$ and $a^{n-1} \neq 0$. On one hand, we have $a^n x = 0$. On the other hand, since $ax$ is invertible, we have $a^n x = a^{n-1}(ax) \neq 0$. It is also a contradiction. Similarly, one can show that $xb$ is nilpotent. Thus $axb$ is nilpotent as well.

**Claim 2.** If $x, y \in A_g$ are nilpotent then $x + y$ is nilpotent.

**Proof of Claim 2.** If $x = 0$ then we need do nothing. If $x \neq 0$ then assume on the contrary that $x + y$ is invertible, i.e., there is $z \in A_{g-1}$ such
that \((x + y)z = 1\). Since \(x\) is nilpotent, there is \(n \geq 2\) such that \(x^n = 0\) and \(x^{n-1} \neq 0\). On one hand, we have \(x^{n-1}(1 - yz) = 0\). On the other hand, by Claim 1, \(yz\) is nilpotent, thus \(1 - yz\) is invertible and \(x^{n-1}(1 - yz) \neq 0\). It is a contradiction.

It follows from Claim 1 and Claim 2 that all homogeneous elements in \(J\) are nilpotent. Thus \(A/J\) is a gr-division algebra and \(J\) is the unique gr-maximal ideal of \(A\). Hence \(J = J_G(A)\) and \(A/J_G(A)\) is a gr-division algebra.

(2) \(\Rightarrow\) (1): Let \(x\) be a homogeneous element of \(A\). We show that \(x\) is either nilpotent or invertible. Since \(J_G(A) \subseteq J(A)\), \(J_G(A)\) is nilpotent. If \(x \in J_G(A)\) then \(x\) is nilpotent. If \(x \not\in J_G(A)\), since \(A/J_G(A)\) is a gr-division algebra, \(\bar{x}\) is invertible in \(A/J_G(A)\). So is \(x\) by [21, Proposition 2.9.1 vi)].

(1) \(\Rightarrow\) (3): It is trivial.

(3) \(\Rightarrow\) (1): Suppose that \(A_e\) is local. For any homogeneous element \(x\) of \(A\), \(x^{[G]} \in A_e\) is either invertible or nilpotent. So is \(x\).

\[\text{Remark 2.1.}\] The conclusion \((1) \iff (3)\) is similar to [15, Theorem 3.1], which says that a \(\mathbb{Z}\)-graded Artin algebra with local initial subring is local. However, the latter does not hold for finite group graded algebras: We consider the \(\mathbb{Z}_2\)-graded algebra \(A = A_0 \oplus A_1\) where \(A_0 := K, A_1 := K\varepsilon\), and \(\varepsilon^2 := 1\). Clearly, the initial subalgebra of \(A\) is local, but \(A\) is not local in the case of \(\text{char} K \neq 2\), since \(\frac{1}{2}(1 + \varepsilon)\) is an idempotent which is neither invertible nor nilpotent.

This example also implies that [15, Theorem 3.2], which says that for a \(\mathbb{Z}\)-graded Artin algebra \(A\), \(M \in \text{gr}A\) is gr-indecomposable if and only if \(F(M)\) is an indecomposable \(A\)-module, does not hold for finite group graded algebras: Indeed, in the case of \(\text{char} K \neq 2\), \(A\) is itself a gr-simple module but \(F(A)\) is decomposable, since forgotten grading \(A \cong K \oplus K\) as algebras.

Now we give another characterization of gr-indecomposable modules.

**Proposition 2.3.** Let \(A\) be a finite-dimensional graded algebra and \(M \in \text{gr}A\). Then \(M\) is gr-indecomposable if and only if \(\text{End}_A(M)\) is gr-local. Moreover, if \(M \in \text{gr}A\) is gr-simple then \(\text{End}_A(M)\) is a gr-division algebra.

**Proof.** By Theorem 2.1 and Proposition 2.2, \(M\) is gr-indecomposable if and only if \(\text{End}_A(M)\) is gr-local.

Let \(M\) be gr-simple. For any \(g \in G\) and \(\phi = F(\psi) \in \text{End}_A(M)_g\) with \(\psi \in \text{Hom}_{\text{Gr}A}(M, S_g(M))\), since \(M\) and \(S_g(M)\) are gr-simple, we have \(\text{Im} \psi = 0\) or \(S_g(M)\), and \(\text{Ker} \psi = 0\) or \(M\). If \(\text{Im} \psi = 0\) or \(\text{Ker} \psi = M\) then \(\psi = 0\). Otherwise, \(\text{Im} \psi = S_g(M)\) and \(\text{Ker} \psi = 0\), i.e., \(\psi\) and thus \(\phi\) is invertible. □
2.4 Gr-projective modules

We say \( F \in \text{Gr}A \) is gr-free if it has an \( A \)-basis consisting of homogeneous elements, equivalently \( F \cong \oplus_{i \in I} S_{g_i}(A) \) with \( g_i \in G \) for all \( i \in I \). A graded module \( P \) is said to be gr-projective if for any graded epimorphism \( \phi : M \to N \) and any graded morphism \( \psi : P \to N \) there is a graded morphism \( \varphi : P \to M \) such that \( \psi = \varphi \phi \).

Proposition 2.4. Let \( A \) be a graded algebra and \( P \in \text{Gr}A \). Then the following assertions are equivalent:

1. \( P \) is gr-projective;
2. \( \text{Hom}_{\text{Gr}A}(P, -) : \text{Gr}A \to \text{Mod}K \) is exact;
3. \( \text{Hom}_A(P, -) : \text{Gr}A \to \text{Mod}K \) is exact;
4. \( P \) is a direct summand of a gr-free module;
5. \( \text{Ext}^1_{\text{Gr}A}(P, N) = 0 \) for all \( N \in \text{Gr}A \);
6. \( \text{Ext}^1_A(P, N) = 0 \) for all \( N \in \text{Gr}A \).

Proof. (1) \( \Leftrightarrow \) (2): By the definition of gr-projectiveness.
(2) \( \Rightarrow \) (4): Applying \( \text{Hom}_{\text{Gr}A}(P, -) \) to a graded epimorphism \( \phi : F \to P \), we obtain \( \phi \) splits.
(4) \( \Rightarrow \) (3): Since \( F(P) \) is projective, the functor \( \text{Hom}_{\text{Mod}A}(F(P), -) : \text{Mod}A \to \text{Mod}K \) is exact. Thus the functor \( \text{Hom}_A(P, -) : \text{Gr}A \to \text{Mod}K \) is exact.
(3) \( \Rightarrow \) (6): By the definition of extension group, we are done.
(6) \( \Rightarrow \) (5): It follows from \( \text{Ext}^n_A(M, N) = \oplus_{g \in G} \text{Ext}^n_A(M, N)_g, \forall M, N \in \text{Gr}A \).
(5) \( \Leftrightarrow \) (2): It is clear by \( \text{Gr}A \cong \text{Mod}A \# KG^* \). \( \square \)

Now we classify all finite-dimensional gr-indecomposable gr-projective modules over a finite-dimensional graded algebra.

Proposition 2.5. Let \( A \) be a finite-dimensional graded algebra. Then, up to graded isomorphism, all finite-dimensional gr-indecomposable gr-projective \( A \)-modules are of form \( S_g(P) \) where \( g \in G \) and \( P \) is a gr-indecomposable direct summand of \( A \).

Proof. Let \( M \) be a finite-dimensional gr-indecomposable gr-projective \( A \)-modules. Since \( M \) is finitely generated, \( M = \sum_{i=1}^n Am_i \) for some homogeneous elements \( m_i \) of degree \( g_i \). Thus \( \phi : \oplus_{i=1}^n S_{g_i-1}(A) \to M, (a_1, ..., a_n) \mapsto \sum_{i=1}^n a_im_i \) is a graded \( A \)-module epimorphism. Since \( M \) is gr-projective, \( M \) is graded isomorphic to a direct summand of \( \oplus_{i=1}^n S_{g_i-1}(A) \). By Theorem 2.1 we are done. \( \square \)
Next we classify all finite-dimensional gr-simple modules over a finite-dimensional graded algebra.

**Proposition 2.6.** Let $A$ be a finite-dimensional graded algebra. Then, up to graded isomorphism, all finite-dimensional gr-simple $A$-modules are of form $S_g(P)/J_G(S_g(P)) \cong S_g(P/J_G(P))$ where $g \in G$ and $P$ is a gr-indecomposable direct summand of $A$.

**Proof.** First of all, we show that for any finite-dimensional gr-indecomposable gr-projective $A$-module $P$, $P/J_G(P)$ is gr-simple. Indeed, there is a natural algebra epimorphism $\text{End}_{\text{Gr}} A(P) \to \text{End}_{\text{Gr}} A(P/J_G(P))$. Since $P$ is gr-indecomposable, $\text{End}_{\text{Gr}} A(P)$ is local. So is $\text{End}_{\text{Gr}} A(P/J_G(P))$. By Theorem 2.1, $P/J_G(P)$ is gr-indecomposable. Since $P/J_G(P)$ is gr-semisimple, it must be gr-simple.

Conversely, for any finite-dimensional gr-simple $A$-module $M$, there is a finite-dimensional gr-projective $A$-module $P$ and a graded $A$-module epimorphism $\phi : P \to M$. Decompose $P$ into the direct sum of gr-indecomposable gr-projective modules, say $P = \bigoplus_{i=1}^n P_i$. Then $P/J_G(P) \cong \bigoplus_{i=1}^n P_i/J_G(P_i)$ and there is a graded $A$-module epimorphism $\psi : \bigoplus_{i=1}^n P_i/J_G(P_i) \to M$. Assume that the restriction of $\psi$ on $P_i/J_G(P_i)$ is nonzero. Since both $P_i/J_G(P_i)$ and $M$ are gr-simple, we have $P_i/J_G(P_i) \cong M$. \hfill \Box

### 2.5 Gr-hereditary algebras

Let $A$ be a finite-dimensional graded algebra. The *gr-projective dimension* of $M \in \text{Gr} A$, denoted by $\text{gr-pd}_A M$, is the least integer $n$ for which there exists an exact sequence $P_n \hookrightarrow \cdots \hookrightarrow P_0 \to M$ in $\text{Gr} A$ where $P_0, \cdots, P_n$ are gr-projective. If no such sequence exists for any $n$ then $\text{gr-pd}_A M := \infty$ (cf. [24, §2.3]). The *gr-global dimension* of $A$ is $\text{gr-gl.dim} A := \sup \{ \text{gr-pd}_A M | M \in \text{Gr} A \}$ (cf. [24, §6.3]).

**Proposition 2.7.** Let $A$ be a finite-dimensional graded algebra and $M \in \text{Gr} A$. Then the following assertions are equivalent:

1. $\text{gr-pd}_A M \leq n$;
2. $\text{Ext}^{n+1}_{\text{Gr} A}(M, N) = 0$ for all $N \in \text{Gr} A$;
3. $\text{Ext}^{n+1}_A(M, N) = 0$ for all $N \in \text{Gr} A$.

**Proof.** (1) $\Rightarrow$ (3) : It follows from $\text{gr-pd}_A M \leq n$ that $\text{pd}_A M \leq n$. Thus $\text{Ext}^{n+1}_{\text{Mod} A}(\mathcal{F}(M), \mathcal{F}(N)) = 0$ and further $\text{Ext}^{n+1}_A(M, N) = 0$ for all $N \in \text{Gr} A$. 

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(3) ⇒ (2): It follows from \( \text{Ext}_A^{n+1}(M, N) = \oplus_{g \in G} \text{Ext}^{n+1}_{\text{Gr}A}(M, N)_g \).

(1) ⇔ (2): It is clear by \( \text{Gr}A \cong \text{Mod}A \# KG^* \).

**Corollary 2.1.** Let \( A \) be a finite-dimensional graded algebra. Then the following assertions are equivalent:

1. \( \text{gr-gl.dim} A \leq n \);
2. \( \text{Ext}_{\text{Gr}A}^{n+1}(M, N) = 0 \) for all \( M, N \in \text{Gr}A \);
3. \( \text{Ext}_A^{n+1}(M, N) = 0 \) for all \( M, N \in \text{Gr}A \).

The same strategy as the proof of [2, Theorem 1] can be used to characterize gr-global dimension as follows:

**Proposition 2.8.** Let \( A \) be a finite-dimensional graded algebra. Then \( \text{gr-gl.dim} A = \sup \{ \text{gr-pd}_A A/I \mid I \text{ is a graded left ideal of } A \} = \text{gr-pd}_A A/J_G(A) \) \( = \sup \{ \text{gr-pd}_A S \mid S \text{ is gr-simple} \} \).

**Proposition 2.9.** Let \( A \) be a finite-dimensional graded algebra. Then the following conditions are equivalent:

1. All graded left ideals of \( A \) are gr-projective;
2. \( J_G(A) \) is gr-projective;
3. \( \text{gr-pd}_A (A/J_G(A)) \leq 1 \);
4. \( \text{gr-gl.dim} A \leq 1 \).

**Proof.** (1) ⇒ (2) ⇒ (3): Trivial.

(3) ⇒ (4): It follows from Proposition 2.8

(4) ⇒ (1): For any graded ideal \( I \) of \( A \), it follows from Corollary 2.1 that \( \text{Ext}_{\text{Gr}A}^2(A/I, N) = 0, \forall N \in \text{Gr}A \). Thus \( \text{Ext}_{\text{Gr}A}^1(I, N) = 0, \forall N \in \text{Gr}A \). By Proposition 2.8 we are done.

**Definition 2.1.** We say a finite-dimensional graded algebra is gr-hereditary if it satisfies the equivalent conditions of Proposition 2.9.

A graded epimorphism \( \varphi : M \to N \) is said to be gr-essential if a graded morphism \( \psi : X \to M \) is a graded epimorphism whenever \( \varphi \psi : X \to N \) is a graded epimorphism.

**Proposition 2.10.** Let \( A \) be a finite-dimensional graded algebra and \( \varphi : M \to N \) a graded epimorphism in gr\(A \). Then the following assertions are equivalent:

1. \( \varphi \) is gr-essential;
2. \( \text{Ker} \varphi \subseteq J_G(M) \);
3. The induced graded epimorphism \( \tilde{\varphi} : M/J_G(M) \to N/J_G(N) \) is a graded isomorphism.
Proof. (1) ⇒ (2): Suppose that ϕ is gr-essential and X is a graded submodule of M such that X + Ker ϕ = M. Denote by λ : X → M the natural graded monomorphism. Then ϕλ : X → N is a graded epimorphism. Thus λ is a graded epimorphism. Hence X = M and Ker ϕ is gr-small. So Ker ϕ ⊆ J_G(M).

(2) ⇒ (3): Since ϕ is a graded epimorphism, one has a natural graded isomorphism ˜ϕ : M/Ker ϕ → N. It follows from Ker ϕ ⊆ J_G(M) that J_G(M/Ker ϕ) = J_G(M)/Ker ϕ. Thus ˜ϕ induces a graded isomorphism ˜ϕ : M/J_G(M) = (M/Ker ϕ)/(J_G(M)/Ker ϕ) → N/J_G(N).

(3) ⇒ (1): Suppose ψ : X → M is a graded morphism such that ϕψ : X → N is a graded epimorphism. Then ˜ϕψ : X/J_G(X) → N/J_G(N) is a graded epimorphism. Since ˜ϕ is a graded isomorphism, ˜ψ is a graded epimorphism. Thus Im ˜ψ + J_G(M) = M. Hence Im ˜ψ = M, i.e., ψ is a graded epimorphism.

A gr-projective cover of a graded module M is a gr-essential graded epimorphism ϕ : P → M with P gr-projective. The following result can be obtained from grA ∼= mod A#KG* and [3, Theorem I.4.2]:

Proposition 2.11. Let A be a finite-dimensional graded algebra. Then each M ∈ grA has a gr-projective cover which is unique up to isomorphism, namely, if ϕ_i : P_i → M, i = 1, 2, are two gr-projective covers of M then there is a graded isomorphism φ : P_1 → P_2 such that ϕ_2φ = ϕ_1.

The following result is an immediate consequence of Proposition 2.10:

Corollary 2.2. A graded epimorphism ϕ : P → M with P gr-projective is a gr-projective cover if and only if the induced graded epimorphism top_G(P) → top_G(M) is a graded isomorphism.

3 Graded equivalence

In this section, we shall introduce graded equivalence theory so that we can reduce the constructions and the representation theory of finite-dimensional finite group graded algebras to those of gr-basic ones. Note that Gordon and Green had introduced the graded equivalence theory for Z-graded Artin algebras (ref. [15, Section 5]). For finite-dimensional finite group graded algebras, it is very similar. However, the introduction of graded natural transformation is new and necessary, which seems to be neglected before.
3.1 Graded equivalences

Let $A$ and $B$ be graded algebras. A functor $\mathcal{U} : \text{Gr}A \to \text{Gr}B$ is called a \textit{graded functor} if it commutes with the shift functors $S_g$ for all $g \in G$.

Let $M, N \in \text{Gr}A$. Then $\text{End}_A(M)$ is a $G$-graded algebra and $\text{Hom}_A(M, N)$ is a graded $\text{End}_A(M)$-modules. For any $\alpha \in \text{Hom}_A(N, L)_e$, $\text{Hom}_A(M, \alpha) : \text{Hom}_A(M, N) \to \text{Hom}_A(M, L)\,$, $(\beta g)_g \mapsto (\beta g\alpha)_g$, is a graded $\text{End}_A(M)$-morphism of degree $e$. Thus we obtain a functor $\text{Hom}_A(M, -) : \text{Gr}A \to \text{Gr}(\text{End}_A(M))$.

**Lemma 3.1.** Let $A$ be a $G$-graded algebra and $M \in \text{Gr}A$. Then $\text{Hom}_A(M, -) : \text{Gr}A \to \text{Gr}(\text{End}_A(M))$ is a graded functor such that $\mathcal{F}\text{Hom}_A(M, -) = \text{Hom}_{\text{Mod}A}(\mathcal{F}(M), -)\mathcal{F}$.

Let $A$ and $B$ be graded algebras and $\mathcal{U}, \mathcal{V} : \text{Gr}A \to \text{Gr}B$ two graded functors. A natural transformation $\eta : \mathcal{U} \to \mathcal{V}$ is said to be \textit{graded} if $\eta_{S g(M)} = S g(\eta_M)$ for all $g \in G$ and $M \in \text{Gr}A$. We say that a graded functor $\mathcal{U} : \text{Gr}A \to \text{Gr}B$ is a \textit{graded equivalence}, if there is a graded functor $\mathcal{V} : \text{Gr}B \to \text{Gr}A$ such that there are graded natural isomorphisms $\mathcal{U} \circ \mathcal{V} \cong 1_{\text{Gr}A}$ and $\mathcal{V} \circ \mathcal{U} \cong 1_{\text{Gr}B}$. We say that $A$ and $B$ are \textit{graded equivalent} if there is a graded equivalence $\text{Gr}A \to \text{Gr}B$.

Let $\mathcal{L} : \text{Mod}A \to \text{Mod}B$ be an equivalence. We call $\mathcal{L}$ a \textit{graded equivalence} if there is a graded functor $\mathcal{U} : \text{Gr}A \to \text{Gr}B$ such that $\mathcal{F}\mathcal{U} = \mathcal{L}\mathcal{F}$. In this case $\mathcal{U}$ is called an \textit{associated graded functor} of $\mathcal{L}$.

**Proposition 3.1.** Let $\mathcal{U}$ be an associated graded functor of a graded equivalence $\mathcal{L} : \text{Mod}A \to \text{Mod}B$, $Q = \mathcal{U}(A)$, and $P = \text{Hom}_B(Q, B)$. Then, as graded algebras, $A \cong \text{End}_B(Q)$ and $B \cong \text{End}_A(P)$. Moreover, the functor $\mathcal{M} := \text{Hom}_{\text{Mod}B}(\mathcal{F}(Q), -)$ is inverse to $\mathcal{L}$, $\mathcal{L} \cong \mathcal{L}^\prime := \text{Hom}_{\text{Mod}A}(\mathcal{F}(P), -)$, $\mathcal{V} := \text{Hom}_B(Q, -)$ is inverse to $\mathcal{U}$, and $\mathcal{U} \cong \mathcal{U}^\prime := \text{Hom}_A(P, -)$.

**Proof.** Clearly, $\mathcal{F}(Q) = \mathcal{F}(\mathcal{U}(A)) = \mathcal{L}(\mathcal{F}(A))$. Since $\mathcal{L}$ is an equivalence, neglecting grading, we have $A \cong \text{End}_{\text{Mod}A}(\mathcal{F}(A)) \cong \text{End}_{\text{Mod}B}(\mathcal{L}(\mathcal{F}(A))) = \text{End}_{\text{Mod}B}(\mathcal{F}(Q)) = \text{End}_B(Q)$. This isomorphism maps $a \in A_g$ to $L(r_a) \in \text{End}_B(Q)_g$, where $r_a \in \text{End}_{\text{Mod}A}(\mathcal{F}(A))$ is the right multiplication by $a$. Thus $A \cong \text{End}_B(Q)$ as graded algebras. Using this isomorphism, we may identify $\text{Gr}((\text{End}_B(Q))$ with $\text{Gr}A$, and $\text{Mod}((\text{End}_B(Q))$ with $\text{Mod}A$. So $\mathcal{M}$ is a functor from $\text{Mod}B$ to $\text{Mod}A$. For any $M \in \text{Mod}A$, we have $M \cong \text{Hom}_{\text{Mod}A}(\mathcal{F}(A), M) \cong \text{Hom}_{\text{Mod}B}(\mathcal{L}(\mathcal{F}(A)), \mathcal{L}(M)) = \mathcal{M}(\mathcal{L}(M))$ as $A$-modules. Thus $\eta_M(m) := \mathcal{L}(r_m)$ for all $M \in \text{Mod}A$ and $m \in M$ defines a natural isomorphism $\eta : 1_{\text{Mod}A} \to \mathcal{M}\mathcal{L} = \text{Hom}_{\text{Mod}A}(\mathcal{L}(\mathcal{F}(A)), \mathcal{L}(-))$. For any $X \in \text{Gr}A$, we have a graded $A$-module isomorphism $X \cong \text{Hom}_{\text{Mod}A}(\mathcal{F}(A), X)$. 

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\[ \mathcal{F}(X) \cong \text{Hom}_{\text{Mod}B}(\mathcal{L}(\mathcal{F}(A)), \mathcal{L}(\mathcal{F}(X))) = \text{Hom}_{\text{Mod}B}(\mathcal{F}(\mathcal{U}(A)), \mathcal{F}(\mathcal{U}(X))) \cong \text{Hom}_B(\mathcal{U}(A), \mathcal{U}(X)) = \mathcal{V}(\mathcal{U}(X)). \]

Thus \( \xi_X(x) := \mathcal{L}(r_x) \) for all \( X \in \text{Gr}A \) and \( x \in X_g \) defines a natural isomorphism \( \xi : 1_{\text{Gr}A} \to \mathcal{V} \mathcal{U} = \text{Hom}_{\text{Mod}B}(\mathcal{L}(\mathcal{F}(A)), \mathcal{L}(\mathcal{F}(-))) \). Since \( \xi_{S_g(x)}(x) = \mathcal{L}(r_x) = \xi_X(x) = S_g(\xi_X)(x) \) for all \( g, h \in G, X \in \text{Gr}A \) and \( x \in X_h \), \( \xi \) is a graded natural isomorphism.

Since \( \mathcal{L} \) is an equivalence, \( \mathcal{F}(Q) \) is a finitely generated projective generator in \( \text{Mod}B \). By [1, Corollary 22.4], the functor \( \mathcal{M} = \text{Hom}_{\text{Mod}B}(\mathcal{F}(Q), -) \) is an equivalence. By Lemma 3.1, \( \mathcal{M} \) is a graded equivalence and \( \mathcal{V} = \text{Hom}_B(Q, -) \) is an associated graded functor of \( \mathcal{M} \). Since \( \mathcal{V}(B) = P \), replacing \( A, B, \mathcal{L}, \mathcal{U}, Q, \mathcal{V}, \mathcal{M} \) with \( B, A, \mathcal{M}, \mathcal{V}, P, \mathcal{U}', \mathcal{L}' \) respectively, by the same argument as above, we obtain \( B \cong \text{End}_A(P) \) as graded algebras and \( \mathcal{U}' \mathcal{V} \cong 1_{\text{Gr}B} \). Therefore \( \mathcal{U}' \cong \mathcal{U}'(\mathcal{V} \mathcal{U}) = (\mathcal{U}' \mathcal{V}) \mathcal{U} \cong \mathcal{U} \) and \( 1_{\text{Gr}B} \cong \mathcal{U} \mathcal{V} \). So \( \mathcal{U} \) is inverse to \( \mathcal{V} \).

Since \( \mathcal{M} \) is an equivalence, \( \mathcal{F}(P) \) is a finitely generated projective generator in \( \text{Mod}A \). Thus \( \mathcal{L}' = \text{Hom}_{\text{Mod}A}(\mathcal{F}(P), -) \) is an equivalence. Furthermore, \( \mathcal{L}' \) is a graded equivalence and \( \mathcal{U}' = \text{Hom}_A(P, -) \) is an associated graded functor. It follows from classical Morita theory that \( 1_{\text{Mod}A} \cong \mathcal{M} \mathcal{L}' \) and \( 1_{\text{Mod}B} \cong \mathcal{L}' \mathcal{M} \). Therefore \( \mathcal{L}' \cong (\mathcal{M} \mathcal{L}') \mathcal{L} \cong \mathcal{L} \) and \( 1_{\text{Mod}B} \cong \mathcal{L} \mathcal{M} \). So \( \mathcal{M} \) is inverse to \( \mathcal{L} \).

**Corollary 3.1.** Let \( A \) and \( B \) be graded algebras. Then any associated graded functor of a graded equivalence \( \text{Mod}A \to \text{Mod}B \) is a graded equivalence.

**Proposition 3.2.** Let \( A \) and \( B \) be graded algebras and \( \mathcal{U} : \text{Gr}A \to \text{Gr}B \) a graded equivalence. Then \( \mathcal{U} \) is isomorphic to an associated graded functor of some graded equivalence \( \text{Mod}A \to \text{Mod}B \).

**Proof.** Let \( \mathcal{V} \) be an inverse of \( \mathcal{U} \) and \( P = \mathcal{V}(B) \). Then \( P \) is a finitely generated gr-projective \( A \)-module. Thus \( \mathcal{F}(P) \) is a finitely generated projective \( A \)-module. Since \( \oplus_{g \in G} S_g(B) \) is a gr-projective generator in \( \text{Gr}B \) and \( \mathcal{V} \) is a graded equivalent functor, \( \oplus_{g \in G} S_g(P) \) is a gr-projective generator in \( \text{Gr}A \). Hence \( A \) is a direct summand of a direct sum of copies of this generator. It follows that \( \mathcal{F}(P) \) is a finitely generated projective generator in \( \text{Mod}A \). Thus the functor \( \text{Hom}_{\text{Mod}A}(\mathcal{F}(P), -) : \text{Mod}A \to \text{Mod}B \) is an equivalence.

Since \( \mathcal{V} \) is a graded equivalent functor, for each \( g \in G \), there is a unique isomorphism \( \phi_g \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}_{\text{Gr}B}(B, S_g(B)) & \xrightarrow{\mathcal{V}} & \text{Hom}_{\text{Gr}A}(P, S_g(P)) \\
\mathcal{F} \downarrow & & \downarrow \mathcal{F} \\
\text{Hom}_B(B, B)_g & \xrightarrow{\phi_g} & \text{Hom}_A(P, P)_g.
\end{array}
\]
Theorem 3.1. The map \( \phi := \bigoplus_{g \in G} \xi_g \) defines an isomorphism of graded algebras \( \text{End}_B(B) \to \text{End}_A(P) \). Using this isomorphism, \( \text{Hom}_A(P, X) \) is a graded \( B \)-module for all \( X \in \text{Gr} A \). By Lemma 3.1, \( \text{Hom}_A(P, -) \) is an associated graded functor of \( \text{Hom}_{\text{Mod} A}(\mathcal{F}(P), -) \). Now it suffices to show that \( U \cong \text{Hom}_A(P, -) \). Since \( U \cong \text{Hom}_B(B, \mathcal{U}(-)) \), it suffices to show \( \text{Hom}_B(B, \mathcal{U}(-)) \cong \text{Hom}_A(P, -) \).

Let \( \eta : \mathcal{U} \to 1_{\text{Gr} A} \) be a graded natural isomorphism. Since \( \mathcal{U} \) and \( \mathcal{V} \) are graded equivalences, for \( X \in \text{Gr} A \) and \( g \in G \), there is a unique isomorphism \( \xi_{X, g} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}_{\text{Gr} B}(B, S_g(\mathcal{U}(X))) & \xrightarrow{\xi_{X, g}} & \text{Hom}_{\text{Gr} A}(P, S_g(X)) \\
\mathcal{F} \downarrow & & \downarrow \mathcal{F} \\
\text{Hom}_{\mathcal{B}}(B, \mathcal{U}(X))_g & & \text{Hom}_A(P, X)_g.
\end{array}
\]

The map \( \xi_X := \bigoplus_{g \in G} \xi_{X, g} : \text{Hom}_B(B, \mathcal{U}(X)) \to \text{Hom}_A(P, X) \) is a graded vector space isomorphism. Now we show that \( \xi_X \) is a graded \( B \)-module morphism, i.e., \( \xi_{X, h g}(b \psi) = b \xi_{X, g}(\psi) \) for \( b \in B_h \) and \( \psi \in \text{Hom}_B(B, \mathcal{U}(X))_g \). Let \( \psi' \in \text{Hom}_{\text{Gr} B}(B, S_g(\mathcal{U}(X))) \) and \( r'_b \in \text{Hom}_{\text{Gr} B}(B, S_h(B)) \) such that \( \mathcal{F}(\psi') = \psi \) and \( \mathcal{F}(r'_b) = r_b \). Then \( \xi_{X, g}(\psi) = \mathcal{F}(\mathcal{V}(\psi')) \mathcal{F}(\eta_{S_g(X)}) \) and \( \xi_{X, h g}(b \psi) = \xi_{X, h g}(r_b \psi) = \mathcal{F}(\mathcal{V}(r'_b \psi')) \mathcal{F}(\eta_{S_h(g(X)}) \mathcal{F}(\eta_{S_g(X)}))(p) = (\mathcal{F}(\mathcal{V}(r'_b \psi')) \mathcal{F}(\eta_{S_h(g(X)}))(p) = \xi_{X, h g}(b \psi) \).

The fifth equality holds since \( \eta \) is a graded natural isomorphism. Therefore, \( \xi \) is a natural isomorphism. Moreover, for all \( g, h \in G, X \in \text{Gr} A \) and \( y \in \mathcal{U}(S_g(X))_h \), i.e., \( y \in \mathcal{U}(X)_{h g} \), we have \( \xi_{S_g(X)}(r_y) = \xi_{S_h(g(X))} h(r_y) = \mathcal{V}(r_y) \eta_{S_h(g(X))} X, h g(r_y) = \xi_{X, h g}(r_y) = S_g(X) \xi_{X}(r_y) = S_g(X) \). Hence \( \xi \) is a graded natural isomorphism. □

Theorem 3.1. Let \( A \) and \( B \) be graded algebras. Then the following statements are equivalent:

1. \( A \) and \( B \) are graded equivalent;
2. There is a graded equivalence \( \text{Mod} A \to \text{Mod} B \);
3. There exists \( P \in \text{Gr} A \) such that \( \mathcal{F}(P) \) is a finitely generated projective generator in \( \text{Mod} A \) and \( \text{End}_A(P) \) is isomorphic to \( B \) as graded algebras.

Proof. (1) \( \Leftrightarrow \) (2): It follows from Proposition 3.1 and Proposition 3.2.

(2) \( \Rightarrow \) (3): It follows from the proof of Proposition 3.1.

(3) \( \Rightarrow \) (2). Assume that there exists an object \( P \) of \( \text{Gr} A \) such that \( \mathcal{F}(P) \) is a finitely generated projective generator in \( \text{Mod} A \) and \( \text{End}_A(P) \) is isomorphic to \( B \) as graded algebras. Then we may identify \( \text{Gr} B \) with \( \text{GrEnd}_A(P) \), and
ModB with ModEnd\(_A(P)\). By Lemma \[3.1\] \(\text{Hom}^\text{Mod}_A(\mathcal{F}(P), -)\) is not only an equivalence but also a graded equivalence.

\[\square\]

### 3.2 Gr-basic algebras

**Lemma 3.2.** Let \(A\) be a finite-dimensional graded algebra and \(M, N \in \text{gr}A\) two gr-indecomposable \(A\)-modules satisfying \(M \ncong S_g(N)\) for all \(g \in G\). Then for all \(\phi \in \text{Hom}_A(M, N)\) and \(\psi \in \text{Hom}_A(N, M)\), we have \(\phi\psi \in J_G(\text{End}_A(M))\).

**Proof.** Let \(\phi = \bigoplus_{g \in G} \phi_g \in \text{Hom}_A(M, N)\) and \(\psi = \bigoplus_{h \in G} \psi_h \in \text{Hom}_A(N, M)\) with \(\phi_g \in \text{Hom}_A(M, N)_g\) and \(\psi_h \in \text{Hom}_A(N, M)_h\). For \(g, h \in G\), by Proposition \[2.3\], \(\phi_g\psi_h \in \text{End}_A(M)\) is either invertible or nilpotent. If it is invertible then \(M \cong S_{gh}(N)\). It is a contradiction. So it is nilpotent. By the proof of Proposition \[2.2\] we have \(\phi\psi \in J_G(\text{End}_A(M))\). \[\square\]

**Lemma 3.3.** Let \(A\) be a finite-dimensional graded algebra and \(P\) a finite-dimensional gr-indecomposable gr-projective module. Then as graded algebras \(\text{End}_A(P)/J_G(\text{End}_A(P)) \cong \text{End}_{A/J_G(A)}(P/J_G(P))\).

**Proof.** For \(f_g \in \text{Hom}_{\text{gr}A}(P, S_g(P))\), by [24, Proposition 2.9.1 iii)], we can define a natural graded morphism \(\bar{f}_g \in \text{Hom}_{\text{gr}A}(P/J_G(P), S_g(P/J_G(P))\). Consequently, we obtain a graded algebra epimorphism \(\phi : \text{End}_A(P) \twoheadrightarrow \text{End}_A(P/J_G(P)), \sum_{g \in G} \mathcal{F}(f_g) \mapsto \sum_{g \in G} \mathcal{F}(\bar{f}_g)\).

By Proposition \[2.6\] we know \(P/J_G(P)\) is gr-simple. It follows from Proposition \[2.3\] that \(\text{End}_{A/J_G(A)}(P/J_G(P)) \cong \text{End}_A(P/J_G(P))\) is gr-division. By Proposition \[2.2\] we have \(\text{Ker}\phi = J_G(\text{End}_A(P))\). Thus, as graded algebras, \(\text{End}_A(P)/J_G(\text{End}_A(P)) \cong \text{End}_{A/J_G(A)}(P/J_G(P))\). \[\square\]

**Proposition 3.3.** Let \(A\) be a finite-dimensional graded algebra. Then the following conditions are equivalent:

1. Any decomposition of \(A\) into gr-indecomposable gr-projective modules \(A = \bigoplus_{i=1}^n P_i\) satisfies \(P_i \ncong \sigma(g)(P_j)\) for all \(g \in G\) and \(i \neq j\).

2. \(A/J_G(A)\) is a direct sum of gr-division algebras.

**Proof.** (1) \(\Rightarrow\) (2): By Proposition \[2.6\] and Proposition \[2.3\] it is enough to show that \(A/J_G(A) \cong \bigoplus_{i=1}^n \text{End}_{A/J_G(A)}(P_i/J_G(P_i))\) as graded algebras.

Let \(E := (E_{ij})_{i,j=1}^n\) where \(E_{ij} := \text{Hom}_A(P_j, P_i)\) for \(i, j = 1, \ldots, n\). Then \(E\) is a finite-dimensional \(G\)-graded algebra with \(E_g := (\text{Hom}_A(P_j, P_i)_g)_{i,j=1}^n\) for all \(g \in G\). Furthermore, \(A \cong E\) as \(G\)-graded algebras.
By Lemma 3.2 and Lemma 3.3 we can show that
\[
J_G(E) = \begin{pmatrix}
J_G(E_{11}) & E_{12} & \cdots & E_{1n} \\
E_{21} & J_G(E_{22}) & \cdots & E_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
E_{n1} & E_{n2} & \cdots & J_G(E_{nn})
\end{pmatrix}
\]
and as graded algebras $A/J_G(A) \cong \text{End}_A(A)/J_G(\text{End}_A(A)) \cong E/J_G(E) \cong \oplus_{i=1}^n \text{End}_A(P_i)/J_G(\text{End}_A(P_i)) \cong \oplus_{i=1}^n \text{End}_A/J_G(A)(P_i/J_G(P_i))$.

(2) $\Rightarrow$ (1) : By Theorem 2.1, up to permutation and graded isomorphism, $A$ can be decomposed uniquely into $A = \oplus_{i=1}^r \oplus_{j=1}^{s_i} S_{g_{ij}}(P_i)$, where $P_i$’s are gr-indecomposable gr-projective $A$-modules such that $P_i \not\cong S_g(P_j)$ for all $g \in G$ and $i \neq j$. Using the same argument as above, we can obtain $A/J_G(A) \cong \oplus_{i=1}^r \text{End}_A(\oplus_{j=1}^{s_i} S_{g_{ij}}(P_i))/J_G(\text{End}_A(\oplus_{j=1}^{s_i} S_{g_{ij}}(P_i)))$. If $s_i > 1$ for some $i$ then it is easy to find a nonzero nilpotent homogeneous element in $A/J_G(A)$. However, this is impossible in the direct sum of gr-division algebras.

\textbf{Definition 3.1.} We say a finite-dimensional graded algebra $A$ is gr-basic if it satisfies the equivalent conditions of Proposition 3.3.

Let $A$ be a finite-dimensional gr-basic graded algebra and $\{P_i|1 \leq i \leq n\}$ a complete set of gr-indecomposable gr-projective $A$-modules up to graded isomorphism and shift. A graded algebra $B$ is called a congruence of $A$ if it is isomorphic to $\text{End}_A(\oplus_{i=1}^n S_{g_{ij}}(P_i))$ for some $(g_1, \ldots, g_n) \in G^n$.

\textbf{Theorem 3.2.} Any finite-dimensional graded algebra $A$ is graded equivalent to a gr-basic algebra $B$, which is unique up to congruence.

\textbf{Proof.} By Theorem 2.1, up to permutation and graded isomorphism, $A$ can be decomposed uniquely into $A = \oplus_{i=1}^r \oplus_{j=1}^{s_i} S_{g_{ij}}(P_i)$, where $P_i$’s are gr-indecomposable gr-projective $A$-modules such that $P_i \not\cong S_g(P_j)$ for all $g \in G$ and $i \neq j$. Let $P = \oplus_{i=1}^n S_{g_{i1}}(P_i)$. Note that $\mathcal{F}(A) = \oplus_{i=1}^r \oplus_{j=1}^{s_i} \mathcal{F}(S_{g_{ij}}(P_i))$ is a finitely generated projective generator in $\text{Mod}A$, so is $\mathcal{F}(P)$. It follows from Theorem 3.1 that $A$ is graded equivalent to $B := \text{End}_A(P)$. Since $B/J_G(B) \cong \oplus_{i=1}^r \text{End}_A/J_G(A)(S_{g_{i1}}(P_i))/J_G(S_{g_{i1}}(P_i))$ as graded algebras, by Proposition 3.3 $B$ is gr-basic.

Assume that $A$ is also graded equivalent to another gr-basic graded algebra $C$. Then $B$ is graded equivalent to $C$. Suppose that $\mathcal{H} : \text{GrB} \to \text{GrC}$ is a graded equivalence, and $B = \oplus_{i=1}^n Q_i$ and $C = \oplus_{j=1}^m R_j$ are the decompositions of $B$ and $C$ into gr-indecomposable gr-projective modules respectively. Then $m = n$. Moreover, by Proposition 2.5 $B \cong \text{End}_B(\oplus_{i=1}^n Q_i) \cong$
End_{C}(\oplus_{i=1}^{n} H(Q_i)) \cong \text{End}_{C}(\oplus_{i=1}^{n} S_{g_i}(R_i)) \text{ for some } (g_1, ..., g_n) \in G^n, \text{ which is a congruence of } \text{End}_{C}(\oplus_{i=1}^{n} R_i) \cong C. \square

4 Constructions of superalgebras

In this section, we shall consider the nice constructions of finite-dimensional superalgebras. Let $Q$ be a quiver with vertex set $Q_0$ and arrow set $Q_1$. Then any map $\text{deg} : Q_1 \to \mathbb{Z}_2$ induces a $\mathbb{Z}_2$-grading on the path algebra $KQ$. Thus $KQ/I$ is a finite-dimensional gr-basic superalgebra for each admissible graded ideal $I$ of $KQ$. It is natural to ask whether all finite-dimensional gr-basic superalgebras can be obtained in this way? The answer is NO! Indeed, exactly all elementary superalgebras can be obtained in this way. For the knowledge of quivers and their representation theory, we refer to [3].

4.1 Elementary superalgebras

An elementary superalgebra $A$ is a finite-dimensional superalgebra such that $A/J_G(A) \cong K \times \cdots \times K$. Clearly, an elementary superalgebra is gr-basic.

A superquiver is an oriented diagram whose vertices are either white vertices $\circ$ or black vertices $\bullet$, and whose arrows are either solid arrows $\longrightarrow$ or dotted arrows $\cdots \cdots$. We say a superquiver is 1-color if all its vertices are of the same color, and 2-color otherwise. An elementary superquiver is just a superquiver with only white vertices. The underlying quiver $\overline{Q}$ of a superquiver $Q$ is the quiver obtained from $Q$ by changing all black vertices into white ones and all dotted arrows into solid ones respectively. The underlying diagram $\overline{Q}$ of a quiver or a superquiver $Q$ is the diagram obtained from $Q$ by changing all black vertices into white ones and all arrows into edges.

For an elementary superquiver $Q$, we define its path superalgebra $KQ$ to be the superalgebra which has all paths as a $K$-basis, whose multiplication is given by the concatenation of paths, and the $\mathbb{Z}_2$-grading on $KQ$ is given by $\text{deg}(\circ) = \text{deg}(\longrightarrow) := 0$ and $\text{deg}(\cdots \cdots) := \overline{1}$.

A set $\{e_1, \cdots, e_n\}$ of idempotents in a $G$-graded algebra $A$ is said to be orthogonal if $e_i e_j = 0$ for all $i \neq j$. A nonzero idempotent $e$ is said to be primitive if $e$ cannot be written as a sum of two nonzero orthogonal idempotents. We say a set $\{e_1, \cdots, e_n\}$ of orthogonal idempotents in $A$ is complete if $1 = e_1 + \cdots + e_n$.

**Proposition 4.1.** Each finite-dimensional graded algebra $A$ has a complete set of degree $e$ primitive orthogonal idempotents.
Proof. Let $A = P_1 \oplus \cdots \oplus P_n$ be a decomposition of $A$ into a direct sum of gr-indecomposable gr-projective modules. Then there exist $e_i \in (P_i)_e$ such that $1 = e_1 + \cdots + e_n$. Thus $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of $A$: Indeed, for any $x_i \in P_i$, we have $x_i = x_i e_i + \cdots + x_i e_i + \cdots + x_i e_n$. Thus $x_i e_j = 0$ for $j \neq i$ and $x_i = x_i e_i$. In particular, we have $e_i \neq 0, e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$. Moreover, $e_i$’s are primitive for all $i$: Assume on the contrary that $e_i = e' + e''$ with $e', e''$ nonzero orthogonal idempotents. Then $P_i = A e_i = A e' \oplus A e''$. It is a contradiction. \[\square\]

Theorem 4.1. An elementary superalgebra $A$ is graded isomorphic to $KQ/I$ where $KQ$ is the path superalgebra of an elementary superquiver $Q$ and $I$ is an admissible graded ideal of $KQ$. If further $A$ is gr-hereditary then $A \cong KQ$ for an acyclic elementary superquiver $Q$. 

Proof. The first part of the theorem can be proved analogous to [3 Theorem III.1.9] by using Proposition [11]. The second part of the theorem can be proved analogous to [3 Proposition III.1.13]. \[\square\]

4.2 Graded species

We have already provided the nice and easy constructions above for elementary superalgebras. To deal with general cases, we introduce graded species and their representations, which are interesting in their own right.

Definition 4.1. A $G$-graded $K$-species or graded species $\mathcal{S} = (D_i, jM_i)_{i,j \in I}$ with $I := \{1, 2, \ldots, n\}$ is a collection of finite-dimensional gr-division algebras $D_i$ and finite-dimensional graded $D_j$-$D_i$-bimodules $jM_i$, such that $K$ operates on $jM_i$ centrally, i.e., $km = mk$ for all $k \in K$ and $m \in jM_i$. A $\mathbb{Z}_2$-graded $K$-species is called a superspecies.

Let $A$ be a finite-dimensional gr-basic graded algebra. Then $A/J_G(A) = \bigoplus_{i=1}^n D_i$ where $D_i$’s are gr-division algebras, and $J_G(A)/J_G(A)^2 = \bigoplus_{i,j \in I} jM_i$ where $jM_i$ is a graded $D_j$-$D_i$-bimodule for all $i,j \in I$. We call $\mathcal{S}_A = (D_i, jM_i)_{i,j \in I}$ the graded species of $A$.

A representation $V = (V_i, j\phi_i)_{i,j \in I}$ of a graded species $\mathcal{S} = (D_i, jM_i)_{i,j \in I}$ is a collection of graded $D_i$-modules $V_i$ and graded $D_j$-module morphisms $j\phi_i : jM_i \otimes_{D_i} V_i \to V_j$ for all $i,j \in I$. We say a representation $V = (V_i, j\phi_i)_{i,j \in I}$ is finite-dimensional if all $V_i$ are finite-dimensional. Let $V = (V_i, j\phi_i)_{i,j \in I}$ and $W = (W_i, j\psi_i)_{i,j \in I}$ be representations of $\mathcal{S}$. Then the direct sum of the representations $V$ and $W$ is the representation
$V \oplus W := (V_i \oplus W_i, \text{diag}\{ j\phi_i, j\psi_i \})_{i,j \in I}$. We say a representation of a graded species is \textit{indecomposable} if it is not the direct sum of two nonzero representations. A \textit{morphism} $\alpha = (\alpha_i)_{i \in I} : V \to W$ is a collection of graded $D_i$-module morphisms $\alpha_i : V_i \to W_j$ such that $(1 \otimes \alpha_i) j\psi_i = j\phi_i \alpha_j$ for all $i, j \in I$. All representations of $\mathcal{S}$ and all morphisms between them form an abelian category $\text{Rep}\mathcal{S}$. We denote by $\text{rep}\mathcal{S}$ the full subcategory of $\text{Rep}\mathcal{S}$ consisting of all finite-dimensional representations.

Let $R$ be a graded algebra and $M$ a graded $R$-$R$-bimodule. With the pair $(R, M)$ we associate a \textit{graded tensor algebra} $T(R, M) = \bigoplus_{i \geq 0} M^\otimes R^i$, where $M^\otimes R^0 = R$, $M^\otimes R^1 = M$, $M^\otimes R^i$ is the $i$-folds tensor product $M \otimes_R \cdots \otimes_R M$, and the multiplication is induced by the tensor product. We say a graded ideal $\mathfrak{i}$ of $T(R, M)$ is \textit{admissible} if $(\bigoplus_{i \geq 1} M^\otimes R^i)^t \subseteq \mathfrak{i} \subseteq (\bigoplus_{i \geq 1} M^\otimes R^i)^{2t}$ for some $t \geq 2$.

With a graded species $\mathcal{S} = (D_i, jM_i)_{i, j \in I}$ we associate a \textit{graded tensor algebra} $T(\mathcal{S}) = T(R, M)$, where $R = \bigoplus_{i \in I} D_i$ and $M = \bigoplus_{i,j \in I} jM_i$. A graded species $\mathcal{S}$ is said to be \textit{acyclic} if its graded tensor algebra $T(\mathcal{S})$ is finite-dimensional.

**Proposition 4.2.** Let $\mathcal{S} = (D_i, jM_i)_{i, j \in I}$ be a graded species. Then $\text{Rep}\mathcal{S}$ and $\text{rep}\mathcal{S}$ are equivalent to $\text{Gr} T(\mathcal{S})$ and $\text{gr} T(\mathcal{S})$ respectively.

**Proof.** It suffices to add “graded” or “gr-” at appropriate places of the proof of [7, Proposition 10.1].

**Remark 4.1.** If $\mathcal{S}$ is an acyclic graded species then by Theorem [27] and Proposition [4.2], $\text{rep}\mathcal{S}$ is a Krull-Schmidt category (ref. [29, p.52]).

### 4.3 Construction of finite-dimensional superalgebras

Let $A$ be a finite-dimensional superalgebra. Then both its opposite algebra $A^\text{op}$ and its enveloping algebra $A^e = A \otimes K A^\text{op}$ are finite-dimensional superalgebras with natural grading.

**Proposition 4.3.** Let $A$ be a finite-dimensional superalgebra and $\phi : A^e \to A$ the graded $A^e$-module epimorphism given by $\phi(x \otimes y^e) = xy$. Then the following conditions are equivalent:

1. $A$ is a gr-projective $A^e$-module;

2. There exists a degree 0 element $\epsilon$ of the graded $A$-$A$-bimodule $A \otimes_K A$ (isomorphic to $A^e$) such that $\phi(\epsilon) = 1$ and $\alpha \epsilon = \epsilon a$ for all $a \in A$.
Proof. (1) $\Rightarrow$ (2): Since $A$ is a gr-projective $A^e$-module, there is a graded $A^e$-module morphism $\psi : A \to A^e$ such that $\psi \phi = \text{id}_A$. Thus $\epsilon := \psi(1_A)$ is as required.

(2) $\Rightarrow$ (1): The map $\psi : A \to A^e, a \mapsto a \epsilon = \epsilon a$, is a graded $A^e$-module morphism and $\psi \phi = \text{id}_A$. \hfill \Box

A finite-dimensional superalgebra is said to be super separable if it satisfies the equivalent conditions in Proposition 4.4.

Lemma 4.1. The direct product of finitely many super separable algebras is super separable.

Proof. If $A_1$ and $A_2$ are super separable then there are degree 0 elements $\epsilon_i \in A_i \otimes_K A_i$ such that $\phi_i(\epsilon_i) = 1_{A_i}$ and $a_i \epsilon_i = \epsilon_i a_i$ for all $a_i \in A_i$ and $i = 1, 2$, where $\phi_i : A^e_i \to A_i, a_i \otimes b^e_i \mapsto a_i b_i$. Note that there is a natural graded $(A_1 \times A_2)^e$-module morphism $\varphi : (A_1 \otimes_K A_1) \times (A_2 \otimes_K A_2) \to (A_1 \times A_2) \otimes_K (A_1 \times A_2), (a_1 \otimes b_1, a_2 \otimes b_2) \mapsto (a_1, 0) \otimes (0, a_2) + (0, b_1) \otimes (0, b_2)$. The element $\epsilon := \varphi(\epsilon_1, \epsilon_2)$ is as required. \hfill \Box

Lemma 4.2. Let $A = \bigoplus_{g \in G} A_g$ be a gr-division algebra. Then $A_g = A_e \epsilon$ for some $\epsilon \in A_g$.

Proof. If $A_g = 0$ then we may take $\epsilon = 0$. If $A_g \neq 0$ then we may take any $\epsilon \in A_g \setminus \{0\}$. Since $\epsilon$ is invertible, we have $\epsilon^{-1} \in A_{g^{-1}}$. Thus $x \epsilon^{-1} \in A_e$ for all $x \in A_g$. Hence $x \in A_e \epsilon$ and $A_g = A_e \epsilon$. \hfill \Box

Proposition 4.4. Let $K$ be an algebraically closed field and $A$ a finite-dimensional gr-division superalgebra. Then $A$ is either $K$ with trivial grading or $D$ with $D_0 = K$, $D_1 = K \epsilon$ and $\epsilon^2 = 1$.

Proof. By Lemma 4.2 we have $A = A_0 \oplus A_0 \epsilon$ for some $\epsilon \in A_1$. Since $A_0$ is a finite-dimensional division algebra and $K$ is algebraically closed, we have $A_0 = K$ by [27, Lemma 3.5]. Thus $A = K \oplus K \epsilon$ with $\epsilon^2 \in K$. If $\epsilon = 0$ then $A = K$. If $\epsilon \neq 0$ then we may assume that $\epsilon^2 = 1$, since $K$ is algebraically closed. \hfill \Box

Proposition 4.5. Let $K$ be an algebraically closed field of char$K \neq 2$ and $A$ a finite-dimensional gr-basic gr-semisimple superalgebra. Then $A$ is super separable.

Proof. By assumption and Proposition 4.4 $A$ is isomorphic to $\bigoplus_{i=1}^n D_i$ where $D_i = K$ or $D$ for all $i$. Clearly, $K$ is super separable. In order to show that
D is super separable, it suffices to take \( \epsilon = \frac{1}{2}(1 \otimes 1 + \varepsilon \otimes \varepsilon) \) which satisfies the second condition in Proposition 4.3. It follows from Lemma 4.1 that \( A \) is super separable.

A square zero super extension of a superalgebra \( R \) by a graded \( R-R \)-bimodule \( M \) is a superalgebra \( E \), together with a superalgebra epimorphism \( \phi : E \to R \) such that \( \ker \phi \) is a square zero graded ideal, and a graded \( R-R \)-bimodule isomorphism of \( M \) with \( \ker \phi \) (cf. [32, §9.3]). We say two square zero super extensions \( E \) and \( E' \) of \( R \) by \( M \) are equivalent if there is a superalgebra isomorphism \( \psi : E \to E' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
M & \to & E \\
\| & \psi & \| \\
M & \to & E'
\end{array}
\]

**Proposition 4.6.** Let \( R \) be a superalgebra and \( M \) a graded \( R-R \)-bimodule. Then the equivalence classes of square zero super extensions of \( R \) by \( M \) are in 1-1 correspondence with the elements of \( \text{Ext}^2_{\text{Gr}^e}(R,M) \).

**Proof.** Analogous to the proof of [32, Theorem 9.3.1]. It suffices to note that all boundary maps in the bar resolution of \( R \) are graded (ref. [23]).

We say that a superalgebra \( R \) is super quasifree if for every square zero super extension \( M \to E \to T \) of a superalgebra \( T \) by a graded \( T-T \)-bimodule \( M \) and every superalgebra homomorphism \( \psi : R \to T \), there exists a superalgebra homomorphism \( \varphi : R \to E \) lifting \( \psi \), i.e., \( \varphi \phi = \psi \).

**Proposition 4.7.** A superalgebra \( R \) is super quasifree if and only if \( \text{Ext}^2_{\text{Gr}^e}(R,M) = 0 \) for all graded \( R-R \)-bimodule \( M \).

**Proof.** Owing to Proposition 4.6, we need do little modification on the proof of [32, Proposition 9.3.3].

**Theorem 4.2.** (Super version of Wedderburn’s principal theorem)
Let \( K \) be an algebraically closed field of \( \text{char} K \neq 2 \) and \( A \) a finite-dimensional gr-basic superalgebra. Then there is a sub-superalgebra \( S \) of \( A \) such that \( A/S \oplus J_{Z_2}(A) \).

**Proof.** Since \( A \) is gr-basic, we know \( R := A/J_{Z_2}(A) = \bigoplus_{i=1}^n D_i \) is a gr-basic gr-semisimple superalgebra. By Proposition 4.5, \( R \) is super separable. Thus \( \text{Ext}^n_{\text{Gr}^e}(R,M) = 0 \) for all \( n \geq 1 \) and \( M \in \text{Gr}^e \). It follows from Proposition 4.7 that \( R \) is super quasifree. Thus the superalgebra identity \( R \to R \) can be lifted successively to \( R \to A/J_{Z_2}^2(A), R \to \)
dimensions we obtain a graded ideal in \( A \). Hence there is a sub-superalgebra \( S = \text{Im} \phi \) of \( A \) such that \( A = S \oplus J_z(A) \).

**Lemma 4.3.** Let \( R \) be a graded algebra, \( M \) a graded \( R-R \)-bimodule, and \( A \) a graded algebra. Suppose \( \phi : R \oplus M \to A \) is a map satisfying:

1. \( \phi|_R : R \to A \) is a graded algebra homomorphism;
2. \( \phi|_M : M \to A \) is a graded \( R-R \)-bimodule morphism when \( A \) is viewed as a graded \( R-R \)-bimodule via \( \phi|_R : R \to A \).

Then there is a unique graded algebra homomorphism \( \tilde{\phi} : T(R, M) \to A \) such that \( \tilde{\phi}|_{R \oplus M} = \phi \).

**Proof.** First of all, we have a unique graded \( R-R \)-bimodule morphism \( \phi_i : M^{\otimes_R i} \to A \) such that \( \phi_i(x_1 \otimes \cdots \otimes x_i) = \phi(x_1) \cdots \phi(x_i) \) for all \( i \) and \( x_1, \ldots, x_i \in M \). Thus the graded algebra homomorphism \( \tilde{\phi} : T(R, M) \to A \) given by \( \tilde{\phi}(\sum_{i=0}^\infty y_i) = \sum_{i=0}^\infty \phi_i(y_i) \) with \( y_i \in M^{\otimes_R i} \) is as required, which is uniquely determined by \( \phi \).

The following lemma is a graded analog of [11, Theorem I].

**Lemma 4.4.** Let \( A \) be a finite-dimensional graded algebra. If \( 1 \subseteq J_G(A)^2 \) is a graded ideal in \( A \) such that \( \text{gr-gl.dim}(A/i) \leq 1 \) then \( i = 0 \).

**Proof.** Clearly, \( 1/J_G(A) \hookrightarrow J_G(A)/J_G(A) \to J_G(A)/i \) is exact in \( \text{Gr}(A/i) \). It follows from \( 1 \subseteq J_G(A)^2 \) that \( \text{top}_G(J_G(A)/J_G(A)) \cong \text{top}_G(J_G(A)/i) \).

Since \( \text{gr-gl.dim}(A/i) \leq 1 \) and \( J_G(A/i) = J_G(A)/i \), we know \( J_G(A)/i \) is a gr-projective \( A/i \)-module. By Corollary 2.2 the lift of the graded morphism \( J_G(A)/i \to \text{top}_G(J_G(A)/i) \cong \text{top}_G(J_G(A)/J_G(A)) \) to \( J_G(A)/i \to J_G(A)/iJ_G(A) \) is a gr-projective cover of \( J_G(A)/iJ_G(A) \). Comparing \( K \)-dimensions we obtain \( J_G(A)/iJ_G(A) = J_G(A)/i, \) i.e., \( 1/J_G(A) = 1 \). Thus \( i = 0 \).

**Theorem 4.3.** Let \( K \) be an algebraically closed field of \( \text{char}K \neq 2 \), \( A \) a finite-dimensional gr-basic superalgebra and \( \mathcal{S}_A = (D_i, jM_i)_{i,j \in I} \) its superspecies. Then \( A \) is graded isomorphic to \( T(\mathcal{S}_A)/i \) where \( i \) is an admissible graded ideal of \( T(\mathcal{S}_A) \). Furthermore, if \( A \) is gr-hereditary then \( \mathcal{S}_A \) is acyclic and \( A \) is graded isomorphic to \( T(\mathcal{S}_A) \).

**Proof.** Due to \( \text{char}K \neq 2 \) and [4, Theorem 4.4], we may let \( J := J_z(A) = J(A), R := A/J, \) and \( M := J/J^2 \). By definition, \( T(\mathcal{S}_A) = T(R, M) \). It follows from Theorem 4.2 that \( A = S \oplus J \) for some sub-superalgebra \( S \) of \( A \). Thus there is a natural superalgebra isomorphism \( \varphi_0 : R = A/J \to S \).


Forgotten grading, $S \cong A/J$ is isomorphic to the direct product of some copies of $K$, so $S$ is separable and $S^e$ is semisimple. Owing to $\text{char } K \neq 2$, applying \cite[Theorem 2.2]{4}, \cite[Theorem 1.3 and Theorem 1.4]{28} or \cite[Theorem 3.5]{3} in turn, we obtain $\text{gr-gl.dim } 2$, applying \cite[Theorem 2.2]{4}, \cite[Theorem 1.3 and Theorem 1.4]{28} or \cite[Theorem 3.5]{3}, so $S^e$ is a gr-semisimple superalgebra. Hence the exact sequence of graded $S$-$S$-bimodules $J^2 \twoheadrightarrow J \twoheadrightarrow J/J^2$ splits. So $J = N \oplus J^2$ for some graded $S$-$S$-bimodule $N$ of $J$. Thus there is a natural graded $R$-$R$-bimodule isomorphism $\varphi_1 : M = J/J^2 \twoheadrightarrow N$, where the graded $R$-$R$-bimodule structure of $M$ is induced by $\varphi_0$. Denote by $N^i$ the set consisting of all finite sum of the products of $i$ elements in $N$. Then $N^i$ is a graded $S$-$S$-bimodule. By induction we obtain $J^i = N^i \oplus J^{i+1}$ for all $i \geq 1$. Suppose $J^{m+1} = 0$ for some $m \geq 0$. Then $A = S \oplus N \oplus N^2 \oplus \cdots \oplus N^m$.

The natural injection $\varphi = \varphi_0 \oplus \varphi_1 : R \oplus M \twoheadrightarrow A$ satisfies the requirements of Lemma \cite[43]{} thus there is a superalgebra homomorphism $\tilde{\varphi} : T(R, M) \twoheadrightarrow A$ such that $\tilde{\varphi}|_{R \oplus M} = \varphi$. Since $A = S \oplus N \oplus N^2 \oplus \cdots \oplus N^m$ and $\varphi_0, \varphi_1$ are graded epimorphisms, $\tilde{\varphi}$ is a graded epimorphism and $A \cong T(R, M)/\text{Ker } \tilde{\varphi}$. Note that $\tilde{\varphi}(M^{\otimes R^i}) \subseteq J^i \subseteq J^2$ for $i \geq 2$ and $\tilde{\varphi}|_{R \oplus M}$ is an isomorphism, so $\text{Ker } \tilde{\varphi} \subseteq \oplus_{i \geq 2} M^{\otimes R^i}$. Since $J^t = 0$ for some $t$, we have $\oplus_{i \geq t} M^{\otimes R^i} \subseteq \text{Ker } \tilde{\varphi}$. Thus $A \cong T(R, M)/\{1\}$ where $1 := \text{Ker } \tilde{\varphi}$ is an admissible graded ideal of $T(\mathcal{S}_A)$.

Now assume that $A$ is a finite-dimensional gr-basic gr-hereditary superalgebra. By above proof we have $A \cong T(\mathcal{S}_A)/\{1\}$ where $1$ is an admissible graded ideal of $T(\mathcal{S}_A)$. Thus $T(\mathcal{S}_A)$ must be finite-dimensional: Indeed, by Proposition \cite[44]{} and its proof, we may assume that $1 = \sum_{i=1}^n e_i$ is a decomposition of $1$ into degree 0 primitive orthogonal idempotents such that $e_i Re_i = D_i$ and $e_j Me_j = J M_i$ for all $i, j \in I = \{1, 2, \ldots, n\}$. If $T(\mathcal{S}_A)$ is infinite-dimensional then there is $s \gg 0$ such that $M^{\otimes R^s} \neq 0$. Thus there are $j_1, \ldots, j_s \in I$ such that $j_r M_{j_{r+1}} \neq 0$ for all $1 \leq r \leq s$ where $j_{s+1} = j_1$. Therefore $e_{j_r} J e_{j_{r+1}} \neq 0$ for all $1 \leq r \leq s$. Suppose $e_{j_r} x e_{j_{r+1}} \in e_{j_r} J e_{j_{r+1}} \backslash \{0\}$ for some $x \in J$ and all $1 \leq r \leq s$. Then the right multiplication by $e_{j_r} x e_{j_{r+1}}$ defines a graded morphism $\lambda_r$ from the gr-indecomposable gr-projective module $A e_{j_r}$ to the gr-indecomposable gr-projective module $A e_{j_{r+1}}$ for each $r$. Since $A$ is gr-hereditary, $\lambda_r$ must be a graded monomorphism. It follows from $x \in J$ that $\lambda_r$ is not surjective. Thus there is a proper injection chain $A e_{j_1} \hookrightarrow A e_{j_2} \hookrightarrow \cdots \hookrightarrow A e_{j_s} \hookrightarrow A e_{j_1}$. It is a contradiction. Hence $T(\mathcal{S}_A)$ is finite-dimensional, i.e., $\mathcal{S}_A$ is acyclic. By Lemma \cite[44]{} we have $1 = 0$ and $T(\mathcal{S}_A)$ is graded isomorphic to $A$.

\begin{remark}
Now we can construct all finite-dimensional superalgebras over
\end{remark}
an algebraically closed field $K$ of char$K \neq 2$.

(1) Firstly, by Theorem 4.3, we can construct all finite-dimensional gr-basic superalgebras by giving a superspecies $\mathcal{S} = (D_i, jM_i)_{i,j \in I}$ and an admissible graded ideal $\mathfrak{i}$ of $T(\mathcal{S})$. Note that $D_i$ is either $K$ or $D$ and $jM_i$ is nothing but a graded $D_j \otimes_K D_i^{\text{op}}$-module. In the case of $D_i = D_j = K$, $jM_i$ is just a super vector space. In the case of $D_i = K$ and $D_j = D$, $jM_i$ is just a left gr-free $D$-module. In the case of $D_i = D$ and $D_j = K$, $jM_i$ is just a right gr-free $D$-module. In the case of $D_i = D_j = D$, $jM_i$ is just a graded $D_e$-module, equivalently, a direct sum of two gr-free $D$-modules, since $D_e = D \otimes_K D^{\text{op}} = D \otimes_K D \cong D \oplus D$. Indeed, the super vector space map $\phi : D \otimes_K D \to D \oplus D$ given by $1 \otimes 1 \mapsto (1, 1), 1 \otimes \varepsilon \mapsto (\varepsilon, \varepsilon), \varepsilon \otimes 1 \mapsto (\varepsilon, -\varepsilon), \varepsilon \otimes \varepsilon \mapsto (1, -1)$, is a superalgebra isomorphism.

(2) Secondly, by Theorem 3.1, we can construct all finite-dimensional superalgebras as $\text{End}_A(P)$, where $A$ is a finite-dimensional gr-basic superalgebra with a decomposition $A = \bigoplus_{i=1}^n P_i$ of gr-indecomposable gr-projectives and $P = \bigoplus_{i=1}^n \bigoplus_{j=1}^{r_i} S_{g_{ij}}(P_i)$ for some $r_i \geq 1$ and $g_{ij} \in \mathbb{Z}_2$.

5 Graded representation type

In order to define the graded representation types of acyclic superspecies, we shall introduce the graded representation types of finite-dimensional finite group graded algebras at first. Then we shall obtain the graded version of Drozd’s theorem. In this section, the underlying field $K$ is assumed to be algebraically closed.

Let $K\langle x, y \rangle$ (resp. $K[x]$) be the free associative algebra in two variables $x, y$ (resp. one variable $x$). Denote by $\text{mod}K\langle x, y \rangle$ the category of finite-dimensional left $K\langle x, y \rangle$-modules. The definition of representation types of a finite-dimensional $K$-algebra is well-known (ref. [5, 10]). Now we define the graded representation types of graded $K$-algebras (compare with [17, 20]).

Definition 5.1. We say that a finite-dimensional $G$-graded $K$-algebra $A$ is gr-representation-finite provided it has only finitely many isomorphism classes of gr-indecomposable $A$-modules.

Definition 5.2. We say that a finite-dimensional $G$-graded $K$-algebra $A$ is gr-tame if for each dimension $d > 0$, there are a finite number of $A$-$K[x]$-bimodules $M_i$ which are $G$-graded as left $A$-modules and free as right $K[x]$-modules such that every gr-indecomposable $A$-module of dimension $d$ is isomorphic to $M_i \otimes_K N$ for some $i$ and some simple $K[x]$-module $N$. 

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Definition 5.3. We say that a finite-dimensional $G$-graded $K$-algebra $A$ is \textbf{gr-wild} if there is a finitely generated $A$-$K\langle x,y \rangle$-bimodule $M$ which is $G$-graded as a left $A$-module and free as a right $K\langle x,y \rangle$-module and such that the functor $M \otimes_{K\langle x,y \rangle} -$ from $\text{mod}K\langle x,y \rangle$ to $\text{gr}A$ preserves indecomposability and isomorphism classes.

Proposition 5.1. A finite-dimensional graded algebra $A$ is \textbf{gr-representation-finite} (resp. \textbf{gr-tame}, \textbf{gr-wild}) if and only if the smash product $A \# KG^*$ is representation-finite (resp. tame, wild).

Proof. Let $M$ be a left $A \# KG^*$-module. Then $M$ is also a $G$-graded left $A$-module defined by $M_g := (1 \# p_g)M$ and $am := (a \# 1)m$ for all $g \in G, a \in A$ and $m \in M$. Conversely, let $M$ be a $G$-graded left $A$-module. Then $M$ is also a left $A \# KG^*$-module defined by $(a \# p_g)m = am_g$ for all $a \in A, g \in G$ and $m \in M$ (ref. [4, Section 2]). By [4, Theorem 2.2], we have that $M$ is an indecomposable $A \# KG^*$-module if and only if it is a gr-indecomposable $A$-module, and $M$ and $N$ are isomorphic as $A \# KG^*$-modules if and only if they are graded isomorphic as graded $A$-modules. Furthermore, the $A$-$K[X]$-bimodules $M_i$ and $A$-$K\langle x,y \rangle$-bimodule $M$ in the definitions of gr-tameness and gr-wildness of $A$ can be viewed as $(A \# KG^*)$-$K[X]$-bimodules $M_i$ and $(A \# KG^*)$-$K\langle x,y \rangle$-bimodule $M$ in the definitions of tameness and wildness of $A \# KG^*$ respectively, and vice versa. Thus the theorem follows. \hfill $\square$

Corollary 5.1. Suppose $\text{char}K \nmid |G|$. Then a finite-dimensional $G$-graded $K$-algebra $A$ is gr-representation-finite (resp. gr-tame, gr-wild) if and only if (forgotten $G$-grading) $A$ is representation-finite (resp. tame, wild).

Proof. It follows from [22, Theorem 4.5] that, in the case of $\text{char}K \nmid |G|$, $A \# KG^*$ and $A$ (forgotten $G$-grading) have the same representation type. By Proposition 5.1 we are done. \hfill $\square$

Theorem 5.1. (Graded version of Drozd’s theorem) A finite-dimensional $G$-graded $K$-algebra $A$ is either graded tame or graded wild, and not both.

Proof. Owing to Proposition 5.1 it suffices to apply Drozd’s Tame-Wild Theorem (ref. [3, Corollary C]) to $A \# KG^*$. \hfill $\square$

Remark 5.1. The condition $\text{char}K \nmid |G|$ in Corollary 5.1 is necessary. Indeed, let $A = KQ/\mathfrak{I}$ be the algebra given by quiver

$$
\begin{array}{c}
1 \\
\overrightarrow{a_3} \\
\overrightarrow{a_4} \\
2 \\
\overrightarrow{a_1} \\
3
\end{array}
$$
with relations $a_1a_2 = a_3a_4$ and $a_3a_2 = a_1a_4$. Then the algebra $A$ is a $\mathbb{Z}_2$-graded algebra defined by $\deg e_i := 0$ for $i = 1, 2, 3$, $\deg a_i := 0$ for $i = 1, 2$, and $\deg a_i := 1$ for $i = 3, 4$. Thus the smash product $A \# K\mathbb{Z}_2^*$ is isomorphic to the algebra $B = KQ'/\imath'$ given by quiver

with all commutative relations. If $\text{char}K = 2$ then, by [14, Section 3], $A$ is wild but $A \# K\mathbb{Z}_2^*$ is tame.

6 Classification of acyclic superspecies

In this section, we shall classify all acyclic superspecies according to their graded representation type in terms of their quivers on one hand and their superquivers on the other hand. From now on, the underlying field $K$ is always assumed to be algebraically closed.

6.1 Quiver of a superspecies

By Proposition 4.4, we know a finite-dimensional gr-division superalgebra is either $K$ or $D$. In $K$, we define $\varepsilon^K_0 := 1$ and $\varepsilon^K_1 := 0$. In $D$, we define $\varepsilon^D_0 := 1$ and $\varepsilon^D_1 := \varepsilon$.

Proposition 6.1. Each graded $D$-module has a degree $\bar{0}$ gr-free basis.

Proof. It follows from [24, Proposition 4.6.1] that each $D$-module $M$ has a homogeneous gr-free basis $\{m_{l_M}|l_M \in I_M\}$. If $\deg m_{l_M} = z_{l_M} \in \mathbb{Z}_2$ then $M = \bigoplus_{l_M \in I_M} Dm_{l_M} = \bigoplus_{l_M \in I_M} D\varepsilon^D_{z_{l_M}} m_{l_M}$. Thus $\{\varepsilon^D_{z_{l_M}} m_{l_M}|l_M \in I_M\}$ is a degree $\bar{0}$ gr-free basis of $M$. \qed

Let $A$ be a finite-dimensional gr-division superalgebra. By Axiom of Choice, we may fix a homogeneous gr-free basis $\{m_{l_M}|l_M \in I_M\}$ for each finitely generated graded $A$-module $M$. It follows from Proposition 6.1 that, in the case of $A = D$, we may fix a degree $\bar{0}$ gr-free basis for each graded $D$-module.

Let $\mathcal{S} = (D_i, jM_i)_{i,j \in I}$ be a superspecies. By Proposition 4.4, we know $D_i = K$ or $D$ for all $i \in I$. Let $\{m_{j,M_i}|l_{j,M_i} \in I_{j,M_i}\}$ be the fixed gr-free basis of the graded $D_j$-module $jM_i$ for all $i, j \in I$, and $z_{j,M_i} := \deg m_{j,M_i}$. Let $I^0_{j,M_i} := \{l_{j,M_i} \in I_{j,M_i}|z_{j,M_i} = z\}$ for all $i, j \in I$ and $z \in \mathbb{Z}_2$. Note that, in the case of $D_j = D$, we have $I^0_{j,M_i} = I_{j,M_i}$ and $I^1_{j,M_i} = \emptyset$. 25
Definition 6.1. The quiver $Q_{\mathcal{J}}$ of a superspecies $\mathcal{J}$ is the quiver $Q = (Q_0, Q_1)$ defined as follows: The vertex set $Q_0 := \{(i, z) | i \in I, z \in \mathbb{Z}_2\}$. We put $\{(i, 1)\} = \emptyset$ in the case of $D_i = D$. The arrow set $Q_1 := \{a^{j}_{i,j}, m_{i,j} | i, j \in I, l_{j,M_i} \in I, z \in \mathbb{Z}_2\}$ where $a^{j}_{i,j} : (i, z) \rightarrow (j, z + z_{i,j,M_i})$ except for $a^{j}_{i,j} : (i, 1) \rightarrow (j, 0)$ in the case of $D_i = K, D_j = D$ and $l_{j,M_i} \in I, z_{i,j,M_i}$.

Theorem 6.1. The categories $\text{rep}(\mathcal{J})$ and $\text{rep}Q_{\mathcal{J}}$ are equivalent.

Proof. First of all, we define a functor $\mathcal{H} : \text{rep}(\mathcal{J}) \rightarrow \text{rep}Q_{\mathcal{J}}$.

Let $(V_i, j\phi_i)_{i \in I}$ be a finite-dimensional representation of $\mathcal{J}$, $\{v_{i,l} | l_i \in I, l \in I, z_i \in \mathbb{Z}_2\}$ the fixed homogeneous gr-free basis of graded $D_i$-module map $V_i$ for all $i \in I$, and $z_i : = \deg v_{i,l}$. Let $I_i := \{l_i \in I, z_i \in \mathbb{Z}_2\}$ for all $i, j \in I$ and $z \in \mathbb{Z}_2$. Note that $I_i^0 = I_i$ and $I_i^1 = \emptyset$ in the case of $D_i = D$. Since $\lambda_{l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}}$ and $\rho_{l_{j,M_i}, l_{j,M_i}}$ the natural injection and projection. Since $j\phi_i$ is a graded $D_j$-module map, we have $j\phi_i = (j\phi_i)_{(l_{j,M_i}, l_{j,M_i})} = \lambda_{l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}}$ where $\lambda_{l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}}$ and $\rho_{l_{j,M_i}, l_{j,M_i}}$ the natural injection and projection. Since $j\phi_i$ is a graded $D_j$-module map, we have $j\phi_i = (j\phi_i)_{(l_{j,M_i}, l_{j,M_i})} = \lambda_{l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}}$, where $c_{l_{j,M_i}, l_{j,M_i}} \in K$. We put $c_{l_{j,M_i}, l_{j,M_i}} = 0$ in the case of $\lambda_{l_{j,M_i}} + \rho_{l_{j,M_i}, l_{j,M_i}} = 0$. We define the representation $\mathcal{H}(V_i, j\phi_i)$ by $\mathcal{H}(V_i, j\phi_i)_{(i, z)} := K_{l_{j,M_i}}^{|l_{j,M_i}|}$ for all $(i, z)$ in $Q_0$ and $\mathcal{H}(V_i, j\phi_i)_{a^{j}_{i,j}} := (c_{l_{j,M_i}, l_{j,M_i}})^{|l_{j,M_i}|} : K_{l_{j,M_i}}^{|l_{j,M_i}|} \rightarrow K_{l_{j,M_i}}^{|l_{j,M_i}|}$ for all $a^{j}_{i,j} : (i, z) \rightarrow (j, z')$ in $Q_1$.

Let $(\alpha_i)_{i \in I} : (V_i, j\phi_i)_{i \in I} \rightarrow (W_i, j\psi_i)_{i \in I}$ be a morphism. Then we have commutative diagrams

\[
\begin{array}{ccc}
JMI \otimes D_i V_i & \xrightarrow{j\phi_i} & V_j \\
\downarrow 1 \otimes \alpha_i & & \downarrow \alpha_j \\
JMI \otimes D_i W_i & \xrightarrow{j\psi_i} & W_j
\end{array}
\]

for all $i, j \in I$. Let $\lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}} = \lambda_{l_{j,M_i}, l_{j,M_i}}$ and $\rho_{l_{j,M_i}, l_{j,M_i}}$, the natural injection and projection. Since $\alpha_i$ is a graded $D_j$-module map, $\alpha_{l_{j,M_i}}(v_{l_{j,M_i}}) = c_{l_{j,M_i}, l_{j,M_i}} = z_{l_{j,M_i}} - z_{l_{j,M_i}}$ where $c_{l_{j,M_i}} \in K$. We put $c_{l_{j,M_i}} = 0$ in
the case of $\varepsilon_{z_{V_i} - z_{W_i}} = 0$. Thus we have commutative diagrams

$$
\bigoplus_{l_j M_i} \bigoplus_{l_{V_i}} D_j m_{l_j M_i} \otimes v_{l_{V_i}} \xrightarrow{(\phi_{l_j M_i}, \psi_{l_{V_i}, l_{W_i}})_{l_j M_i, l_{V_i}, l_{W_i}}} \bigoplus_{l_{V_j}} D_j v_{l_{V_j}} \\
\downarrow (1 \otimes \alpha_{l_{V_i} l_{W_i}})_{l_{V_i}, l_{W_i}} \\
\bigoplus_{l_{W_i}} D_j m_{l_j M_i} \otimes w_{l_{W_i}} \xrightarrow{(\phi_{l_j M_i}, \psi_{l_{W_i}, l_{W_j}})_{l_j M_i, l_{W_i}, l_{W_j}}} \bigoplus_{l_{W_j}} D_j w_{l_{W_j}}
$$

for all $i, j \in I$. We define $\mathcal{H}((\alpha_i)_{i \in I}) := (c_{l_{V_i} l_{W_i}})_{l_{V_i}, l_{W_i}}$ for all $(i, z) \in Q_0$.

Now we show that $\mathcal{H}((\alpha_i)_{i \in I}) := (\mathcal{H}((\alpha_i)_{i \in I})_{(i, z)})_{i, z} \in Q_0$ is a morphism from representation $\mathcal{H}((V_i, j \phi_i)_{i \in I})$ to representation $\mathcal{H}((W_i, j \psi_i)_{i \in I})$.

Since $1 \otimes \alpha_i$ is diagonal on $l_j M_i$, we have commutative diagrams

$$
\bigoplus_{l_{V_i}} D_j m_{l_j M_i} \otimes v_{l_{V_i}} \xrightarrow{(\phi_{l_j M_i}, \psi_{l_{V_i}, l_{W_i}})_{l_j M_i, l_{V_i}, l_{W_i}}} \bigoplus_{l_{V_j}} D_j v_{l_{V_j}} \\
\downarrow (1 \otimes \alpha_{l_{V_i} l_{W_i}})_{l_{V_i}, l_{W_i}} \\
\bigoplus_{l_{W_i}} D_j m_{l_j M_i} \otimes w_{l_{W_i}} \xrightarrow{(\phi_{l_j M_i}, \psi_{l_{W_i}, l_{W_j}})_{l_j M_i, l_{W_i}, l_{W_j}}} \bigoplus_{l_{W_j}} D_j w_{l_{W_j}}
$$

for all $i, j \in I$ and $l_j M_i \in I_j M_i$.

Furthermore, we have commutative diagrams

$$
D_j^{l_{V_i}} \xrightarrow{(c_{l_{V_i} l_{V_j}} \varepsilon_{z_{V_i} - z_{V_j}})_{l_{V_i}, l_{V_j}}} D_j^{l_{V_j}} \\
\downarrow (c_{l_{V_i} l_{W_i}} \varepsilon_{z_{V_i} - z_{V_j}})_{l_{V_i}, l_{W_i}} \\
D_j^{l_{W_i}} \xrightarrow{(c_{l_{W_i} l_{W_j}} \varepsilon_{z_{W_i} - z_{W_j}})_{l_{W_i}, l_{W_j}}} D_j^{l_{W_j}}
$$

denoted by $(\ast)$, for all $i, j \in I$ and $l_j M_i \in I_j M_i$.

**Case $D_1 = D_2 = D$:** Since all $m_{l_j M_i}, v_{l_{V_i}}, v_{l_{V_j}}, w_{l_{W_i}}$ and $w_{l_{W_j}}$ are of degree $0$, we know all $\varepsilon_{z_{V_i} - z_{V_j}}$, $\varepsilon_{z_{V_i} + l_{W_i} - z_{W_j}}$, $\varepsilon_{z_{W_i} - z_{W_j}}$, and $\varepsilon_{z_{V_j} - z_{W_j}}$ in $(\ast)$ are equal to 1. From $(\ast)$ we obtain commutative diagram

$$
D_j^{l_{V_i}} \xrightarrow{(c_{l_{V_i} l_{V_j}})_{l_{V_i}, l_{V_j}}} D_j^{l_{V_j}} \\
\downarrow (c_{l_{V_i} l_{W_i}})_{l_{V_i}, l_{W_i}} \\
D_j^{l_{W_i}} \xrightarrow{(c_{l_{W_i} l_{W_j}})_{l_{W_i}, l_{W_j}}} D_j^{l_{W_j}}
$$

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Furthermore, we have commutative diagrams

\[
\begin{array}{ccc}
K[|I_{V_i}|] & \xrightarrow{(c_{I_{V_i}I_{V_j}})_I} & K[|I_{V_j}|] \\
\downarrow (c_{I_{V_i}I_{W_i}})_I & & \downarrow (c_{I_{V_j}I_{W_j}})_I \\
K[|I_{W_i}|] & \xrightarrow{(c_{I_{W_i}I_{W_j}})_I} & K[|I_{W_j}|]
\end{array}
\]

**Case** \(D_i = D\) and \(D_j = K\): Since all \(v_{I_{V_i}}\) and \(w_{I_{W_j}}\) are of degree 0, we know all \(\varepsilon_{z_{I_{V_i}}-z_{I_{W_j}}} \) in (\(\ast\)) are equal to 1. Moreover, \(\varepsilon_{z_{I_{V_i}}+z_{I_{V_i}}-z_{I_{V_j}}} = 1\) (resp. \(\varepsilon_{z_{I_{V_i}}+z_{I_{V_i}}-z_{I_{V_j}}} = 1\)) precisely when \(z_{I_{V_i}} = z_{I_{V_j}}\) (resp. \(z_{I_{V_i}} = z_{I_{V_j}}, z_{I_{V_i}} = z_{I_{W_j}}\)), and 0 otherwise. From (\(\ast\)) we obtain commutative diagrams

\[
\begin{array}{ccc}
K[|I_{V_i}|] & \xrightarrow{(c_{I_{V_i}I_{V_j}})_I} & K[|I_{V_j}|] \\
\downarrow (c_{I_{V_i}I_{W_i}})_I & & \downarrow (c_{I_{V_j}I_{W_j}})_I \\
K[|I_{W_i}|] & \xrightarrow{(c_{I_{W_i}I_{W_j}})_I} & K[|I_{W_j}|]
\end{array}
\]

for all \(z \in \mathbb{Z}_2\).

**Case** \(D_i = K\) and \(D_j = D\): Since all \(m_{I_{V_i}, I_{V_j}}, v_{I_{V_j}}\) and \(w_{I_{W_j}}\) are of degree 0, we know all \(\varepsilon_{z_{I_{V_i}}-z_{I_{W_j}}} \) in (\(\ast\)) are equal to 1. Moreover, \(\varepsilon_{z_{I_{V_i}}+z_{I_{V_i}}-z_{I_{V_j}}} = 1\) if \(z_{I_{V_i}} = z_{I_{W_i}}\), and 0 otherwise. We also have \(\varepsilon_{z_{I_{V_i}}+z_{I_{V_i}}-z_{I_{V_j}}} = 1\) (resp. \(\varepsilon_{z_{I_{V_i}}+z_{I_{V_i}}-z_{I_{V_j}}} = 1\)) precisely when \(z_{I_{V_i}} = 0\) (resp. \(z_{I_{W_i}} = 0\)), and \(\varepsilon\) otherwise. From (\(\ast\)) we obtain commutative diagrams

\[
\begin{array}{ccc}
D[|I_{V_i}|] & \xrightarrow{(c_{I_{V_i}I_{V_j}})_I} & D[|I_{V_j}|] \\
\downarrow (c_{I_{V_i}I_{W_i}})_I & & \downarrow (c_{I_{V_j}I_{W_j}})_I \\
D[|I_{W_i}|] & \xrightarrow{(c_{I_{W_i}I_{W_j}})_I} & D[|I_{W_j}|]
\end{array}
\]

for all \(z \in \mathbb{Z}_2\). Furthermore, we have commutative diagrams

\[
\begin{array}{ccc}
K[|I_{V_i}|] & \xrightarrow{(c_{I_{V_i}I_{V_j}})_I} & K[|I_{V_j}|] \\
\downarrow (c_{I_{V_i}I_{W_i}})_I & & \downarrow (c_{I_{V_j}I_{W_j}})_I \\
K[|I_{W_i}|] & \xrightarrow{(c_{I_{W_i}I_{W_j}})_I} & K[|I_{W_j}|]
\end{array}
\]
for all $z \in \mathbb{Z}_2$.

**Case $D_i = D_j = K$:** We know $\varepsilon_{z, l_{W_i}} = 1$ (resp. $\varepsilon_{z, l_{W_j}} = 1$, $\varepsilon_{z + z_i, l_{V_i}} = 1$, $\varepsilon_{z + z_j, l_{V_j}} = 1$) in (*) precisely when $z_{l_{V_i}} = z_{l_{W_i}}$ (resp. $z_{l_{V_j}} = z_{l_{W_j}}$), and 0 otherwise. From (*) we obtain commutative diagrams

$$K^{I_{V_i}} \xrightarrow{(c_i^{l_{V_i}, l_{W_i}})_{I_{V_i}, I_{W_i}}} K^{I_{V_j}}$$

for all $z, z' \in \mathbb{Z}_2$.

By the above argument, we always have commutative diagram

$$K^{I_{V_i}} \xrightarrow{(c_i^{l_{V_i}, l_{W_i}})_{I_{V_i}, I_{W_i}}} K^{I_{V_j}} \xrightarrow{(c_i^{l_{V_i}, l_{W_i}})_{I_{V_j}, I_{W_j}}} K^{I_{V_j}}$$

for each arrow $a_{ij}^{l_{M_i}} : (i, z) \to (j, z')$ in $Q_1$. Therefore, $\mathcal{H}((\alpha_i)_{i \in I}) := (\mathcal{H}((\alpha_i)_{i \in I})(i, z))_{i, z} \in \mathcal{Q}$ is a morphism from the representation $\mathcal{H}((V_i, j\phi_i)_{i, j \in I})$ to the representation $\mathcal{H}((W_i, j\psi_i)_{i, j \in I})$.

Clearly, the functor $\mathcal{H}$ is fully faithful. It is also dense: Indeed, for any representation $(V_i, j\phi_i)_{i, j \in I}$ of the quiver $Q$, we may define a representation $(V_i, j\phi_i)_{i, j \in I}$ of $\mathcal{S}$ by $V_i := V_i(0) \oplus V_i(1)$ and $j\phi_i(m_{l_{M_i}} \otimes v_{i, z}) := V_{a_{ij}^{l_{M_i}}}^{l_{M_j}}(v_{i, z})$. Obviously, $\mathcal{H}((V_i, j\phi_i)_{i, j \in I}) \cong (V_i, a_{ij}^{l_{M_i}})$. Thus the functor $\mathcal{H}$ is an equivalence.

### 6.2 Classification of acyclic superspecies, I

An acyclic superspecies $\mathcal{S}$ is said to be $gr$-representation-finite (resp. $gr$-tame, $gr$-wild) if $T(\mathcal{S})$ is. The following result provides a classification of acyclic superspecies in terms of their quivers:
Theorem 6.2. An acyclic superspecies $\mathcal{S}$ is gr-representation-finite (resp. gr-tame) if and only if its quiver $Q_{\mathcal{S}}$ is Dynkin (resp. extended Dykin).

Proof. It follows from [4, Theorem 2.2] that $\text{mod}(\mathcal{S})^\#K^*$ and $\text{gr}(\mathcal{S})$ are equivalent. By Proposition 4.2 we have $\text{gr}(\mathcal{S})$ and $\text{rep}(\mathcal{S})$ are equivalent. According to Theorem 6.1, $\text{rep}(\mathcal{S})$ and $\text{rep}(Q_{\mathcal{S}})$ are equivalent. It follows from [3, Theorem 1.5] that $\text{rep}(Q_{\mathcal{S}})$ and $\text{mod}(KQ_{\mathcal{S}})$ are equivalent. Thus the representation type of $\mathcal{T}(\mathcal{S})^\#K^*$ and $KQ_{\mathcal{S}}$ coincide. By Proposition 5.1 and the well-known results on the representation types of quivers (ref. [12, 9, 25]), we are done.

Let $\mathcal{S}$ be an acyclic superspecies. Then its quiver $Q_{\mathcal{S}}$ corresponds to a unique generalized Cartan matrix, which corresponds to a unique Kac-Moody algebra $g_{\mathcal{S}}$ (ref. [18, 19]). By [18, Theorem 3] and Theorem 6.1, we have the following result:

Theorem 6.3. (Super version of Kac’s Theorem) Let $\mathcal{S}$ be an acyclic superspecies. Then an indecomposable representations of $\mathcal{S}$ corresponds to a positive root of the Kac-Moody algebra $g_{\mathcal{S}}$. Moreover, a positive real root of $g_{\mathcal{S}}$ corresponds to a unique indecomposable representation of $\mathcal{S}$. A positive imaginary root of $g_{\mathcal{S}}$ corresponds to infinitely many indecomposable representations of $\mathcal{S}$.

6.3 Classification of acyclic superspecies, II

Definition 6.2. The superquiver $Q(\mathcal{S})$ of a superspecies $\mathcal{S} = (D_i, jM_i)_{i,j \in I}$ is a superquiver determined by $Q(\mathcal{S})_{0K} = \{i \in I | D_i = K\}$, $Q(\mathcal{S})_{0D} = \{i \in I | D_i = D\}$, $Q(\mathcal{S})_{00} = \{a_{\overline{0}jM_i} : i \rightarrow j | \overline{0}jM_i \in I_{jM_i}^0\}$ and $Q(\mathcal{S})_{11} = \{a_{i\overline{1}jM_i} : i \rightarrow j | \overline{1}iM_i \in I_{jM_i}^1\}$ whose elements correspond to white vertices, black vertices, solid arrows and dotted arrows respectively.

Remark 6.1. Note that not every superquiver is the one of some superspecies: For example, the superquiver $\bullet \overline{0} \circ$ cannot be the superquiver of any superspecies.

The following result provides the classification of all acyclic superspecies according to their representation types in terms of their superquivers:

Theorem 6.4. Let $\mathcal{S} = (D_i, jM_i)_{i,j \in I}$ be an acyclic superspecies over an algebraically closed field $K$. Then
(1) $\mathcal{S}$ is gr-representation-finite if and only if its superquiver is the disjoint union of some superquivers of the following types:

1-color superquiver whose underlying diagram is $A_n, D_n, E_6, E_7$ or $E_8$

- $A(n,1)$:
- $A(1,n)$:
- $A(2,2)$:
- $B(1,n)$:
- $C(n,1)$:
- $F(2,2)$:

(2) $\mathcal{S}$ is gr-tame but not gr-representation-finite if and only if its su-
perquiver is the disjoint union of some superquivers of the following types:

1-color acyclic superquiver whose underlying diagram is $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$

\begin{align*}
A(1,n,1): & \begin{array}{c}
\bullet \quad \circ \quad \cdots \quad \circ \quad \bullet
\end{array} \\
A'(1, n, 1): & \begin{array}{c}
\circ \quad \bullet \quad \circ \quad \cdots \quad \bullet \quad \circ
\end{array} \\
A(3,2): & \begin{array}{c}
\circ \quad \circ \quad \circ \quad \bullet
\end{array} \\
A(2,3): & \begin{array}{c}
\circ \quad \bullet \quad \bullet \quad \bullet
\end{array} \\
B(1,n,1): & \begin{array}{c}
\circ \quad \circ \quad \cdots \quad \circ \quad \bullet
\end{array} \\
B'(1, n, 1): & \begin{array}{c}
\circ \quad \circ \quad \cdots \quad \circ \quad \bullet
\end{array} \\
C(1,n,1): & \begin{array}{c}
\bullet \quad \circ \quad \cdots \quad \circ \quad \bullet
\end{array} \\
C'(1, n, 1): & \begin{array}{c}
\circ \quad \bullet \quad \circ \quad \cdots \quad \bullet \quad \circ
\end{array} \\
F(3,2): & \begin{array}{c}
\circ \quad \circ \quad \cdots \quad \circ \quad \bullet
\end{array} \\
F(2,3): & \begin{array}{c}
\circ \quad \bullet \quad \bullet \quad \bullet
\end{array} \\
D(n,1): & \begin{array}{c}
\circ \quad \bullet \quad \circ \quad \cdots \quad \circ
\end{array} \\
D'(n,1): & \begin{array}{c}
\bullet \quad \circ \quad \circ \quad \cdots \quad \circ \quad \bullet
\end{array} \\
D(1,n): & \begin{array}{c}
\circ \quad \bullet \quad \circ \quad \cdots \quad \bullet \quad \circ
\end{array} \\
D'(1, n): & \begin{array}{c}
\circ \quad \bullet \quad \circ \quad \cdots \quad \bullet \quad \circ
\end{array}
\end{align*}

Proof. The proof of the theorem is based on Theorem 6.1 and the representation type classification of quivers over an algebraically closed field (ref. [12, 9, 25]). For some minimal wild quivers, we refer to [21]. Without loss of generality, we assume that $Q(\mathcal{I})$ is connected.

Claim 1. Let $Q(\mathcal{I})$ be a 1-color superquiver. Then $\mathcal{I}$ is gr-representation-finite (resp. gr-tame but not gr-representation-finite) if and only if $Q(\mathcal{I})$ is
a Dynkin quiver (resp. extended Dynkin quiver).

Proof of Claim 1. It follows from Theorem 6.2 that, if $Q(\mathcal{I})$ is a Dynkin quiver (resp. extended Dynkin quiver, wild quiver) then $\mathcal{I}$ is gr-representation-finite (resp. gr-tame but not gr-representation-finite, gr-wild).

Claim 2. Let $Q(\mathcal{I})$ be a 2-color superquiver with branches, i.e., $Q(\mathcal{I})$ contains vertices which have at least three neighbors. Then $\mathcal{I}$ is not gr-representation-finite, and $\mathcal{I}$ is gr-tame but not gr-representation-finite if and only if $Q(\mathcal{I})$ is of type $D(n, 1)$, $D'(n, 1)$, $D(1, n)$ or $D'(1, n)$.

Proof of Claim 2. In the case of $D = D_j = K$, since $jM_i$ is a $\mathbb{Z}_2$-graded right $D$-module, $m_{ij} := \dim_{D_j} M_i$ is even. Since $Q(\mathcal{I})$ has branches, $Q(\mathcal{I})$ contains a sub-superquiver $Q'$ such that $Q_0' = \{1, 2, 3, 4\}$ and $m_{i1} + m_{1i} \neq 0$ for $i = 2, 3, 4$. If there exist $i, j \in \{2, 3, 4\}$ such that $D_i = D_j \neq D_i$ then $Q(\mathcal{I})$ contains a subquiver of type $T_{1111}$. By Theorem 6.2, $\mathcal{I}$ is gr-wild. On the contrary, we assume that either $D_1 = D_2 = D_3 = K$ or $D_1 = D_2 = D_3 = D$. Thus $Q(\mathcal{I})$ contains a sub-superquiver $Q''$ of type $D(n, 1)$, $D'(n, 1)$, $D(1, n)$ or $D'(1, n)$. If $Q(\mathcal{I}) = Q''$ then $Q(\mathcal{I})$ is of type $D_{2n}$ or $D_{n+1}$. By Theorem 6.2, $\mathcal{I}$ is gr-tame but not gr-representation-finite. If $Q''$ is a proper sub-superquiver of $Q(\mathcal{I})$ then $\mathcal{I}$ is gr-wild.

Claim 3. Let $Q(\mathcal{I})$ be a 2-color superquiver without branches and $m_{ij} \leq 1$ for all $i, j \in I$. Then $\mathcal{I}$ is gr-representation-finite (resp. gr-tame but not gr-representation-finite) if and only if $Q(\mathcal{I})$ is of type $A(n, 1)$, $A(1, n)$, $A(2, 2)$ (resp. $A(1, n, 1)$, $A'(1, n, 1)$, $A(3, 2)$ or $A'(2, 3)$).

Proof of Claim 3. Since $Q(\mathcal{I})$ has no branches, $Q(\mathcal{I})$ is of type $A_n$ or $\tilde{A}_n$. If $Q(\mathcal{I})$ is of type $\tilde{A}_n$ then $Q(\mathcal{I})$ contains a subquiver of type $\tilde{A}_n$. By Theorem 6.2, $\mathcal{I}$ is gr-wild. If $Q(\mathcal{I})$ is of type $A_n$ then $Q(\mathcal{I})$ contains a sub-superquiver of the form $A(n, m) := \cdots \circ \cdots \circ \cdots \circ \cdots \circ \cdots$ with $n$ white vertices and $m$ black vertices. If $Q(\mathcal{I})$ is of type $A(n, 1)$, $A(1, n)$, $A(2, 2)$ (resp. $A(1, n, 1)$, $A'(1, n, 1)$, $A(3, 2)$ or $A'(2, 3)$) then $Q(\mathcal{I})$ is of type $A_{2n+1}, D_{n+2}, E_6$ (resp. $A_{2n+1}, \tilde{D}_{n+3}, E_7, E_6$). By Theorem 6.2, $\mathcal{I}$ is gr-representation-finite (resp. gr-tame but not gr-representation-finite). If $Q(\mathcal{I})$ properly contains a sub-superquiver of type $A(n, 1)$, $A'(1, n, 1)$, $A(3, 2)$ or $A'(2, 3)$ then $\mathcal{I}$ is gr-wild.

Claim 4. Let $Q(\mathcal{I})$ be a 2-color superquiver without branches and $m_{ij} = 2$ for some $i, j \in I$. Then $\mathcal{I}$ is gr-representation-finite (resp. gr-tame but not gr-representation-finite) if and only if $Q(\mathcal{I})$ is of type $B(1, n)$, $C(n, 1)$ or $F(2, 2)$ (resp. $B(1, n, 1)$, $B'(1, n, 1)$, $C(1, n, 1)$, $C'(1, n, 1)$, $F(3, 2)$ or $F(2, 3)$).

Proof of Claim 4. Note that $Q(\mathcal{I})$ is a 2-color superquiver. If $Q(\mathcal{I})$
contains a sub-superquiver of type \( \overrightarrow{e e e e} \) (here a line means either a solid arrow or a dotted arrow, certainly they are of the same orientation), \( \overrightarrow{e e} \) and \( \overrightarrow{e e} \) then \( Q_\mathcal{S} \) contains a subquiver of type \( \tilde{A}_2 \). By Theorem 6.2, \( \mathcal{S} \) is gr-wild. Suppose \( Q(\mathcal{S}) \) contains a sub-superquiver of type \( BC(n, m) := \overrightarrow{e e e e e e e e} \) with \( n \) white vertices and \( m \) black vertices.

If \( Q(\mathcal{S}) \) is of type \( B(1, n), C(n, 1) \) or \( F(2, 2) \) (resp. \( B(1, n, 1), B'(1, n, 1), C(1, n, 1), C'(1, n, 1), F(3, 2) \) or \( F(2, 3) \)) then \( Q_\mathcal{S} \) is of type \( D_{n+2}, A_{2n+1} \) or \( E_6 \) (resp. \( \tilde{D}_{n+3}, \tilde{D}_{n+3}, \tilde{A}_{2n+1}, \tilde{A}_{2n+1}, \tilde{E}_7 \) or \( \tilde{E}_6 \)), and thus \( \mathcal{S} \) is gr-representation-finite (resp. gr-tame but not gr-representation-finite). If \( Q(\mathcal{S}) \) contains a proper sub-superquiver of type \( B(1, n, 1), B'(1, n, 1), C(1, n, 1), C'(1, n, 1), F(3, 2) \) or \( F(2, 3) \) then \( \mathcal{S} \) is gr-wild.

**Claim 5.** Let \( Q(\mathcal{S}) \) be a 2-color superquiver without branches and \( m_{ij} \geq 3 \) for some \( i, j \in I \). Then \( \mathcal{S} \) is gr-wild.

**Proof of Claim 5.** In this case \( Q(\mathcal{S}) \) must contain a sub-superquiver of type \( \overrightarrow{e e e e} \), \( \overrightarrow{e e} \), \( \overrightarrow{e e} \) or \( \overrightarrow{e e e e} \). Thus \( Q_\mathcal{S} \) contains a subquiver of type \( K_3 \). By Theorem 6.2, \( \mathcal{S} \) is gr-wild.

**References**

[1] F.W. Anderson and K.R. Fuller, Rings and categories of modules, Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, New York-Heidelberg, 1974.

[2] M. Auslander, On the dimension of modules and algebras III: Global dimension, Nagoya Math. J. 9 (1955), 67–77.

[3] M. Auslander, I. Reiten and S.O. SmalØ, Representation theory of artin algebras, Cambridge studies in advanced mathematics 36, Cambridge university press, Cambridge, 1995.

[4] M. Cohen and S. Montgomery, Group-graded rings, smash products, and group actions, Trans. Amer. Math. Soc. 282 (1984), 237–258.

[5] W.W. Crawley-Boevey, On tame algebras and bocses, Proc. London Math. Soc. 56 (1988), 451–483.

[6] E.C. Dade, Group-graded rings and modules, Math. Z. 174 (1980), 241–262.

[7] V. Dlab and C.M. Ringel, On algebras of finite representation type, J. Algebra 33 (1975), 306–394.

[8] V. Dlab and C.M. Ringel, Indecomposable representations of graphs and algebras, Mem. Amer. Math. Soc. 173, 1976.

[9] P. Donovan and M.R. Freislich, The representation of finite graphs and associated algebras, Carleton Lecture Notes, vol. 5, 1973.
[10] Yu.A. Drozd, Tame and wild matrix problems, in: Representations and Quadratic Forms, Inst. Math., Acad. Sci. Ukrainian SSR, Kiev, 1979, pp. 39–74; in: Amer. Math. Soc. Transl., vol. 128, 1986, pp. 31–55.

[11] S. Eilenberg and T. Nakayama, On the dimension of modules and algebras V: Dimension of residue rings, Nagoya Math. J. 11 (1957), 9–12.

[12] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972), 71–103.

[13] P. Gabriel, Indecomposable representations II, Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), Academic Press, London, 1973, pp. 81–104.

[14] Ch. Geiss and J.A. de la Peña, An interesting family of algebras, Arch. Math. (Basel) 60 (1993), 25–35.

[15] R. Gordon and E.L. Green, Graded artin algebras, J. Algebra 76 (1982), 111–137.

[16] J.A. Green, Hall algebras, hereditary algebras and quantum groups, Invent. Math. 120 (1995), 361–377.

[17] D.J. Hemmer, J. Kujawa and D.K. Nakano, Representation type of Schur superalgebras, J. Group Theory 9 (2006), 283–306.

[18] V.G. Kac, Infinite root systems, representations of graphs and invariant theory, Invent. Math. 56 (1980), 57–92.

[19] V.G. Kac, Infinite-dimensional Lie algebras, Third edition, Cambridge University Press, Cambridge, 1990.

[20] S. Kasjan and J.A. de la Peña, Galois coverings and the problem of axiomatization of the representation type of algebras, Extracta Math. 20 (2005), 137–150.

[21] O. Kerner, Preprojective components of wild tilted algebras, Manuscripta Math. 61 (1988), 429–445.

[22] G.X. Liu, On the structure of tame graded basic Hopf algebras, J. Algebra 299 (2006), 841–853.

[23] J.L. Loday, Cyclic homology, Grundlehren 301, Springer, Berlin, 1992.

[24] C. Nastasescu and F. Van Oystaeyen, Methods of graded rings, Lecture Notes in Math. 1836, Springer-Verlag, Berlin, 2004.

[25] L.A. Nazarova, Representations of quivers of infinite type, Math. USSR Izv., 7 (1973), 749–792, Izv. Akad. Nauk SSSR Ser. Mat., 37 (1973), 752–791.

[26] L. Peng and J. Xiao, Triangulated categories and Kac-Moody algebras, Invent. Math. 140 (2000), 563–603.

[27] R.S. Pierce, Associative algebras, Graduate Texts in Mathematics 88, Springer-Verlag, New York-Berlin, 1982.

[28] I. Reiten and Ch. Riedtmann, Skew group algebras in the representation theory of Artin algebras, J. Algebra 92 (1985), 224–282.
[29] C.M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math. 1099, Springer-Verlag, 1984.

[30] C.M. Ringel, Hall algebras and quantum groups, Invent. Math. 101 (1990), 583–592.

[31] V.S. Varadarajan, Supersymmetry for mathematicians: an introduction, Courant Lecture Notes in Mathematics, 11, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2004.

[32] C.A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1994.