About the regularized Navier–Stokes equations

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Abstract

The first goal of this paper is to study the large time behavior of solutions to the Cauchy problem for the 3-dimensional incompressible Navier-Stokes system. The Marcinkiewicz space $L^{3,\infty}$ is used to prove some asymptotic stability results for solutions with infinite energy. Next, this approach is applied to the analysis of two classical “regularized” Navier-Stokes systems. The first one was introduced by J. Leray and consists in “molliﬁying” the nonlinearity. The second one was proposed by J.L. Lions, who added the artificial hyper-viscosity $(-\Delta)^{\ell/2}$, $\ell > 2$, to the model. It is shown in the present paper that, in the whole space, solutions to those modiﬁed models converge as $t \to \infty$ toward solutions of the original Navier-Stokes system.

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1 Introduction

Since the seminal paper by Leray [21], several methods have been developed to prove existence of global-in-time weak solutions of the Cauchy problem for the three-dimensional Navier–Stokes system

\begin{align}
  u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= F, \quad x \in \mathbb{R}^3, \quad t > 0 \\
  \nabla \cdot u &= 0, \\
  u(0) &= u_0.
\end{align}

(1.1)

The usual tool is to consider a regularized problem (containing a parameter \( \kappa > 0 \)) for which one can prove the existence of a unique smooth solution. Next, due to the energy inequality, one can pass to the limit as \( \kappa \to 0 \) and to show that the limit function is a weak solution to problem (2.1)–(2.2). A detailed description of the possible ways used in the literature for modifying the system (1.1)–(1.3) is contained in [14].

This idea was used already by Leray [21], who mollified equation (1.1) replacing the nonlinearity \( \nabla \cdot (u \otimes u) \) by the smoother term \( \nabla \cdot ((u \ast \omega_\kappa) \otimes u) \) with a smooth function \( \omega \) such that \( \int \omega \, dx = 1 \) and \( \omega_\kappa(x) = \kappa^{-3} \omega(x/\kappa) \). On the other hand, J.-L. Lions proposed to replace the Laplacian \( -\Delta \) by the sum \( -\Delta + \kappa(-\Delta)^{\ell/2}, \ell > 2 \) (in a way that is reminiscent of a Taylor expansion), and for such a modified problem considered in a bounded domain, J.-L. Lions was able to prove (cf. [24, Chap. 1, Remarque 6.11]) the existence of a unique regular solution provided \( \ell \geq 5/2 \) (\( \ell \geq (n + 2)/2 \) for the \( n \)-dimensional problem). An analogous result for the whole space \( \mathbb{R}^3 \) is contained e.g. in [17]. Hence, one can say that the mollified nonlinearity as well as the hyperdissipative term in the equation smooth out solutions.

The goal of this paper is to show that, in the whole space \( \mathbb{R}^3 \), such corrections in the model disappear asymptotically as \( t \to \infty \), at least, when small solutions are considered. More precisely, we fix \( \kappa > 0 \) in both models, and we show that their solutions converge in a suitable sense as \( t \to \infty \) toward solutions of the Navier-Stokes system (1.1)–(1.3) corresponding to the same initial conditions and external forces.

Notations. The notations to be used are mostly standard. For \( 1 \leq p \leq \infty \), the \( L^p \)-norm of a Lebesgue measurable real-valued function defined on \( \mathbb{R}^3 \) is denoted by \( \|v\|_p \). On the other hand, the norm of the weak \( L^p \)-space (the Marcinkiewicz space) \( L^{p,\infty} = L^{p,\infty}(\mathbb{R}^3) \) is denoted by \( \|v\|_{p,\infty} \); cf. Section 3 for suitable definitions. We will always denote by \( \|v\|_X \) the norm of any other Banach space \( X \) used in this paper. Here, we study properties of vector-valued solutions \( u = (u_1, u_2, u_3) \) to the Navier-Stokes system (1.1)–(1.3), hence the notation \( u \in X \) should be understood as \( u_i \in X \) for every \( i = 1, 2, 3 \); moreover, by the very definition, \( \|u\|_X = \max\{\|u_1\|_X, \|u_2\|_X, \|u_3\|_X\} \).
2 Results and comments

Let us recall the projection \( P \) of \((L^2)^3\) onto the subspace \( P[(L^2)^3] \) of solenoidal vector fields (i.e. those characterized by the divergence condition (1.2)). It is known that \( P \) is a pseudodifferential operator of order 0. In fact, it can be written as a combination of the Riesz transforms \( R_j \) with symbols \( \frac{\xi_j}{|\xi|} \),

\[
P(v_1, v_2, v_3) = (v_1 - R_1\sigma, v_2 - R_2\sigma, v_3 - R_3\sigma),
\]

where \( \sigma = R_1v_1 + R_2v_2 + R_3v_3 \). This explicit formula allows us to consider \( P \) as the bounded operator on \( L^p = L^p(\mathbb{R}^3), 1 < p < \infty \), as well as on the Marcinkiewicz weak \( L^p \)-spaces recalled in the next section.

Using this projection, one can remove the pressure from the model (1.1)-(1.3) and obtain an equivalent Cauchy problem

\[
\begin{align*}
    u_t - \Delta u + P \nabla \cdot (u \otimes u) &= Pf, & x \in \mathbb{R}^3, t > 0 \\
    u(0) &= u_0. 
\end{align*}
\]  

Our first goal is to study solutions to problem (2.1)-(2.2) rewritten as the integral equation

\[
\begin{align*}
    u(t) &= S(t)u_0 - \int_0^t S(t-\tau)P \nabla \cdot (u \otimes u)(\tau) d\tau \\
    &\quad + \int_0^t S(t-\tau)P F(\tau) d\tau. 
\end{align*}
\]

Here, the heat semigroup on \( \mathbb{R}^3 \), denoted by \( S(t) \), is realized as the convolution with the Gaussian kernel \( p(x,t) = (4\pi t)^{-3/2} \exp(-|x|^2/(4t)) \). Note that (2.3) has the form \( u = y + B(u, u) \), where the bilinear form is defined as

\[
B(u, v)(t) = -\int_0^t S(t-\tau)P \nabla \cdot (u \otimes v)(\tau) d\tau,
\]

and \( y = S(t)u_0 + \int_0^t S(t-\tau)PF(\tau) d\tau \). Hence, using the classical Picard approach, which is based on Lemma 4.1 below, one can easily construct solutions in the space

\[
\mathcal{X}_3 = C_w([0, \infty), L^{3,\infty})
\]

provided initial data and external forces are small in a suitable sense.

Here, it should be emphasized that the Marcinkiewicz space \( L^{p,\infty} \) is not separable and the heat semigroup is not strongly continuous on the space. Hence, in our considerations below, we introduce the space \( C_w([0, \infty), L^{p,\infty}) \) consisting of functions \( u \) with the following two properties

- \( u \) is bounded and continuous from \((0, \infty)\) to \( L^{p,\infty} \) in the norm topology of \( L^{p,\infty} \),
In the Lebesgue space $L^p$, we have

$$\parallel u \parallel_{L^p} \leq \parallel F \parallel_{L^q}$$

for every $3 < q < p$. Since, by Proposition 4.2, $u \in X_3 \cap X_p$, we easily deduce from (2.8) the decay rates of solutions in the Lebesgue space $L^q$:

$$\sup_{t>0} t^{(1-3/q)/2} \parallel u(t) \parallel_q < \infty, \text{ for every } q \in (3, \infty).$$
The following theorem is the new contribution to the theory concerning large time behavior of solutions discussed above.

**Theorem 2.2** Let the assumptions of Theorem 2.1 hold true. Assume that $u$ and $\tilde{u}$ are two solutions of (2.1)–(2.2) constructed in Theorem 2.1 corresponding to the initial conditions $u_0, \tilde{u}_0 \in L^{3,\infty}$ and external forces $F, \tilde{F} \in Y_3$, respectively.

Suppose that
\[
\lim_{t \to \infty} \left\| S(t)(u_0 - \tilde{u}_0) + \int_0^t S(t - \tau) P(F(\tau) - \tilde{F}(\tau)) \, d\tau \right\|_{3,\infty} = 0. \tag{2.9}
\]

Then
\[
\lim_{t \to \infty} \| u(\cdot, t) - \tilde{u}(\cdot, t) \|_{3,\infty} = 0 \tag{2.10}
\]
holds.

As will be proved in Corollary 4.1 below, conditions (2.9) and (2.10) are, in fact, equivalent.

Section 4 contains more results being direct corollaries of Theorem 2.2. In particular, it is shown that under the assumptions of this theorem
\[
\lim_{t \to \infty} t^{(1-3/p)/2} \| u(\cdot, t) - \tilde{u}(\cdot, t) \|_p = 0 \tag{2.11}
\]
for every $p \in (3, \infty)$. First, we show relation (2.11) with the Lebesgue norm replaced by the Marcinkiewicz $L^{p,\infty}$-norm, next, the limit in (2.11) results directly from the imbedding (2.8). Some details are contained in Proposition 4.3, below.

Theorem 2.2 is the counterpart of a result contained in [11] where global-in-time solutions are constructed in the space
\[
\mathcal{PM}^2 \equiv \{ v \in S'(\mathbb{R}^d) : \hat{v} \in L^1_{\text{loc}}(\mathbb{R}^d), \| v \|_{\mathcal{PM}^2} \equiv \sup_{\xi \in \mathbb{R}^d} |\xi|^2 |\hat{v}(\xi)| < \infty \}.
\]

In particular, in that setting, it is possible to study one-point stationary singular solutions to (2.1)–(2.2) (constructed independently by Landau and Tian and Xin) of the following form (cf. [30])
\[
\begin{align*}
 u_1(x) &= 2 c |x|^2 - 2 x_1 |x| + c x_1^2 \frac{|x|(c|x| - x_1)^2}{|x|(c|x| - x_1)^2}, \\
 u_2(x) &= 2 x_2 (c x_1 - |x|) \frac{|x|(c|x| - x_1)^2}{|x|(c|x| - x_1)^2}, \\
 u_3(x) &= 2 x_3 (c x_1 - |x|) \frac{|x|(c|x| - x_1)^2}{|x|(c|x| - x_1)^2}, \\
 p(x) &= 4 c x_1 - |x| \frac{|x|(c|x| - x_1)^2}{|x|(c|x| - x_1)^2}
\end{align*}
\tag{2.12}
\]

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $c$ is an arbitrary constant such that $|c| > 1$. By straightforward calculations, one can check that, indeed, the functions $u_1(x)$, $u_2(x)$, $u_3(x)$, and $p(x)$ given by (2.12) satisfy (2.1)–(2.2) with $F \equiv 0$ in the pointwise sense for every $x \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. On the other hand, if one treats $(u(x), p(x))$ as a distributional or generalized solution to (2.1)–(2.2) in the whole $\mathbb{R}^3$, they
correspond to the very singular external force $F = (b\delta_0, 0, 0)$, where the parameter $b \neq 0$ depends on $c$ and $\delta_0$ stands for the Dirac delta.

Details of this reasoning and relevant references are gathered in [11]. Here, we would like only to emphasize that small solutions of the form (2.12) can be also obtained from Theorem 2.1 because, as it is shown in Lemma 3.4 below, $F = (c_1\delta_0, c_2\delta_0, c_3\delta_0)$ belongs to the space $Y_3$ defined in (2.6).

The main goal of this paper is to compare, for large $t$, properties of solutions of (2.1)–(2.2) with properties of solutions of the following Cauchy problems: for the mollified Navier–Stokes system

$$
v_t - \Delta v + \mathcal{P} \nabla \cdot \left( (v * \omega) \otimes v \right) = \mathcal{P} G,
$$

$$
v(0) = v_0.
$$

where $\omega$ is a nonnegative smooth compactly supported function on $\mathbb{R}^3$ such that $\int_{\mathbb{R}^3} \omega(x) \, dx = 1$; and for the Navier–Stokes system with the hyperdissipative term

$$
w_t - \Delta w + (-\Delta)^{\ell/2} w + \mathcal{P} \nabla \cdot (w \otimes w) = \mathcal{P} H,
$$

$$
w(0) = w_0
$$

with fixed $\ell > 2$.

**Remark 2.1** Note that the constant $\kappa$, mentioned in Introduction, does not appear in both models. In fact, without loss of generality and for simplicity of notation, we put $\kappa = 1$.

It is not surprising that the theories on the existence of global-in-time small solutions to all models, (2.1)–(2.2), (2.13)–(2.14), and (2.15)–(2.16) are completely analogous. Below, in Theorems 5.1 and 6.1, we state this fact more precisely. However, the main result of this paper consists in showing that the mollification of the nonlinearity in model (2.13)–(2.14) as well as the higher order term $(-\Delta)^{\ell/2}$ with $\ell > 2$ in (2.15)–(2.16) are asymptotically negligible for large $t$. Details are contained in Theorem 2.3 and 2.4, below.

First, however, let us recall that if $u_0 \in L^{3,\infty}$ is a homogeneous function of degree $-1$ and if $F$ satisfies

$$
F(x, t) = \lambda^3 F(\lambda x, \lambda^2 t) \quad \text{for all} \quad \lambda > 0,
$$

we already know (cf. e.g. [8]) that the solution to the Navier-Stokes system (2.1)–(2.2) is self-similar, hence of the form

$$
U(x, t) = t^{-1/2} U(x/t^{1/2}, 1).
$$

Obviously, this is not the case of the mollified system (2.13)–(2.14), because it is not invariant under the well-known rescaling $u(x, t) \mapsto \lambda u(\lambda x, \lambda^2 t)$. The goal of our next theorem is to show, however, that, as $t \to \infty$, solutions of (2.13)–(2.14) converge toward suitable self-similar solutions of the Navier-Stokes system (2.1)–(2.2).
Theorem 2.3 Denote by \( u = u(x, t) \) and \( v = v(x, t) \) the solutions to the problems (2.1)–(2.2) and (2.13)–(2.14), respectively, corresponding to the same initial datum \( u_0 \in L^{3, \infty} \) and the external force \( F \in \mathcal{Y}_3 \). Assume that \( \|u_0\|_{3, \infty} + \|F\|_{\mathcal{Y}_3} < \varepsilon \), \( u_0 \) is homogeneous of degree \(-1\), and \( F \) satisfies (2.17). Then for \( p \in (3, \infty) \)

\[
\lim_{t \to \infty} t^{(1-3/p)/2}\|u(\cdot, t) - v(\cdot, t)\|_p = 0. 
\tag{2.19}
\]

To understand the limit relation (2.19), one should remember that the self-similar solution \( u = u(x, t) = t^{-1/2}U(x/\sqrt{t}) \) used in Theorem 2.3 satisfies

\[
t^{(1-3/p)/2}\|u(\cdot, t)\|_p = \|U\|_p = \text{const. for all } t > 0 \text{ and each } p > 3. \]

One can also look at (2.19) in the following way. Let us consider the rescaled function \( u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t) \) for each \( \lambda > 0 \). Note that the self-similar solution \( u \) is invariant under this rescaling. Hence, by a simple change of variables, for every fixed \( t_0 > 0 \), we obtain

\[
\|u(\cdot, t_0) - v(\cdot, t_0)\|_p = \|u_\lambda(\cdot, t_0) - v(\cdot, t_0)\|_p \\
= \lambda^{1-3/p}\|u(\cdot, \lambda^2 t_0) - v(\cdot, \lambda^2 t_0)\|_p \\
= (t/t_0)^{(1-3/p)/2}\|u(\cdot, t) - v(\cdot, t)\|_p
\]

after substituting \( \lambda = \sqrt{t/t_0}, t > 0 \). Hence, due to these calculations and relation (2.19), it follows that under the assumptions of Theorem 2.3, \( v_\lambda(\cdot, t_0) \to u(\cdot, t_0) \) as \( \lambda \to \infty \) in \( L^p, 3 < p < \infty \), for each fixed \( t_0 > 0 \).

The result on the asymptotic stability of the Navier-Stokes system with hypodissipativity is more general and reads as follows.

Theorem 2.4 Denote by \( u(x, t) \) and \( w(x, t) \) the solutions to the problems (2.1)–(2.2) and (2.15)–(2.16), respectively, corresponding to the same initial datum \( u_0 \in L^{3, \infty} \) and the force of the form \( F = \nabla \cdot V \) with \( V \in C_w([0, \infty), L^{3/2, \infty}) \). Assume that \( \|u_0\|_{3, \infty} + \|F\|_{\mathcal{Y}_3} \leq \varepsilon < 1/(4\eta \mathcal{C}_t) \) for the constant \( \eta \) defined in Proposition 4.1 and \( \mathcal{C}_t \) given by equation (6.2). Then

\[
\lim_{t \to \infty} \|u(\cdot, t) - w(\cdot, t)\|_{3, \infty} = 0.
\]

Moreover, for every \( p \in (3, \infty) \),

\[
\lim_{t \to \infty} t^{(1-3/p)/2}\|u(\cdot, t) - w(\cdot, t)\|_p = 0. 
\tag{2.20}
\]

Note that \( u = u(x, t) \) in Theorem 2.4 is not assumed to be self-similar.

Section 5 contains the detailed analysis (including the proof of Theorem 2.3) of the mollified system (2.13)–(2.14). Analogous results on the hyperviscous problem (2.15)–(2.16) are gathered in Section 6. Finally, in Section 7, we describe how to reformulate our results in a framework of abstract functional Banach spaces more general than \( L^p \) and \( L^{p, \infty} \).

A preliminary version of results from this paper was announced without proof in [10]. The Marcinkiewicz spaces appear as well in the study of the Navier-Stokes
system in an exterior domain. Indeed, “physically reasonable” stationary solutions constructed by Finn [13] in the 3-dimensional exterior problem have the infinite energy and decay like \(|x|^{-1}\) as \(|x| \to \infty\), hence, the Marcinkiewicz space \(L^{3,\infty}\) seems to be a natural space containing functions with such a behavior at infinity. This idea motivated to study the exterior problem for the incompressible Navier-Stokes system in the space \(L^{3,\infty}(\Omega)\) (see e.g. [4, 18, 19, 28, 31] and the references given there). Results in this direction were also obtained in the recent paper [3], where ideas from Theorem 2.2 were adapted.

3 Marcinkiewicz spaces

In this paper, we work in the weak Marcinkiewicz \(L^p\)-spaces (1 \(< p \leq \infty\)) denoted as usual by \(L^{p,\infty} = L^{p,\infty}(\mathbb{R}^3)\). They belong to the scale of the Lorentz spaces and contain measurable functions \(f = f(x)\) satisfying the condition

\[
\{|x \in \mathbb{R}^3 : |f(x)| > \lambda\} \leq C\lambda^{-p}
\]  

for all \(\lambda > 0\) and a constant \(C\). One can check that (3.1) is equivalent to

\[
\int_E |f(x)| \, dx \leq C' |E|^{1/q}
\]

for every measurable set \(E\) with a finite measure, another constant \(C'\), and \(1/p + 1/q = 1\). This fact allows us to define the norm in \(L^{p,\infty}\)

\[
\|f\|_{p,\infty} = \sup\{|E|^{-1/q} \int_E |f(x)| \, dx : E \in \mathcal{B}\}
\]

where \(\mathcal{B}\) is the collection of all Borel sets with a finite and positive measure.

Recall the well-known imbedding \(L^p \subset L^{p,\infty}\) being the consequence of the Markov inequality \(\{|x \in \mathbb{R}^n : |f(x)| > \lambda\} \leq \lambda^{-p} \int_{\mathbb{R}^n} |f(x)|^p \, dx\). Moreover, in the Marcinkiewicz spaces, the following inequalities hold true: the weak Hölder inequality:

\[
\|fg\|_{r,\infty} \leq \|f\|_{p,\infty}\|g\|_{q,\infty}
\]  

for every \(1 < p \leq \infty\) (here, \(L^{\infty,\infty} = L^\infty\), \(1 < q < \infty\), and \(1 < r < \infty\) satisfying \(1/r = 1/p + 1/q\), and the weak Young inequality

\[
\|f * g\|_{r,\infty} \leq C\|f\|_{p,\infty}\|g\|_{q,\infty}
\]

for every \(1 < p < \infty\), \(1 < q < \infty\), and \(1 < r < \infty\) satisfying \(1 + 1/r = 1/p + 1/q\).

The classical Young inequality applied to the heat semigroup implies the existence of a constat \(C = C(p, q)\) such that for every \(u_0 \in L^q\)

\[
\|S(t)u_0\|_p \leq Ct^{-(3/2)(1/q - 1/p)}\|u_0\|_q,
\]

provided \(1 \leq q \leq p \leq \infty\). The counterpart for the Marcinkiewicz spaces is also valid

\[
\|S(t)u_0\|_{p,\infty} \leq Ct^{-(3/2)(1/q - 1/p)}\|u_0\|_{q,\infty}
\]
under the additional assumption $q \neq 1$.

In the following, we also use estimates involving the weak $L^p$ spaces which were recently obtained independently by Y. Meyer in [26] and by M. Yamazaki in [31]. For the completeness of the exposition, we recall them in a form most suitable for our applications.

Denote by $K(x,y,t)$, $T > 0$, $x, y \in \mathbb{R}^3$, the kernel fulfilling the following estimate

$$|K(x,y,t)| \leq C t^{-3} (1 + |x - y|/t)^{-4},$$

and for every $t > 0$ define the operator

$$[P(t)h](x) = \int_{\mathbb{R}^3} K(x,y,t) h(y) \, dy.$$  

The main estimate is contained in the following lemma.

**Lemma 3.1** ([26, Ch. 8, Th. 9], [31, Th. 3.1]) There exists a constant $C$ such that for every $f \in C_w((0, \infty), L^{3/2, \infty} (\mathbb{R}^3))$ and $g = g(x)$ defined by the formula

$$g(x) = \int_0^\infty [P(\tau)f](x, \tau) \, d\tau$$

we have

$$\|g\|_{3, \infty} \leq C \sup_{\tau > 0} \|f(\cdot, \tau)\|_{3/2, \infty}.$$  

In his consideration [26], Meyer applied this lemma to a very special function

$$g(\cdot, t) = \int_0^t \mathcal{P} \nabla S(t - \tau) f(\cdot, \tau) \, d\tau$$  \hspace{1cm} (3.6)

where $\mathcal{P}$ is the Leray projection and $S(t)$ is the heat semigroup. It is well-known that $\mathcal{P} \nabla S(t)$ is given as a convolution operator with the Oseen kernel $K(x,y,t) = t^{-2} K((x - y)/\sqrt{t})$ where $|K(x)| \leq C(1 + |x|)^{-4}$. Now, we change the variables $s = t - \tau$ in the integral (3.6) which leads to

$$g(\cdot, t) = \int_0^\infty \mathcal{P} \nabla S(s) Q(\cdot, s) \, ds$$  \hspace{1cm} (3.7)

with $Q(x,s) = f(x,t-s)$ if $0 \leq s \leq t$, and $Q(x,s) = 0$ if $s > t$. Finally, the application of Lemma 3.1 gives an inequality which plays a crucial role in our reasoning below:

$$\|g(\cdot, t)\|_{3, \infty} \leq \eta \sup_{0 \leq s} \|Q(\cdot, s)\|_{3/2, \infty} = \eta \sup_{0 \leq \tau \leq t} \|f(\tau, s)\|_{3/2, \infty},$$  \hspace{1cm} (3.8)

where the constant $\eta$ is independent of $f$, $t$, and $s$.

Here, we also recall a result on the continuity with respect to $t$ of $g = g(x,t)$ defined in (3.6).
Lemma 3.2 [26, Lem. 24], [31, Th. 3.1]) For every $f \in C_{w}(\mathbb{R}^{3})$, the function $g = g(x, t)$ defined in (3.6) satisfies $g \in C(\mathbb{R}^{3})$. \hfill \Box

An improvement of the Meyer-Yamazaki inequality can be found in the recent paper by Terraneo [29, Prop 1.5].

In Section 2 (cf. (2.6)), we have already defined the space $Y_{3}$ of admissible external forces. Here, we would like to present two sufficient conditions for $F$ to belong to $Y_{3}$.

Lemma 3.3 Assume that $F(x, t) = \nabla \cdot V(x, t)$ for the external potential satisfying $v \in C_{w}((0, \infty), L^{3/2, \infty})$. Then $F \in Y_{3}$; moreover, if $\lim_{t \to \infty} \|V(t)\|_{3/2, \infty} = 0$, then

$$\lim_{t \to \infty} \left\| \int_{0}^{t} S(t - \tau) F(\tau) \, d\tau \right\|_{3, \infty} = 0.$$\hspace{1cm} (3.9)

Proof. The first part of this Lemma is a direct consequence of Lemmata 3.1 and 3.2. Assume now that $\lim_{t \to \infty} \|V(t)\|_{3/2, \infty} = 0$. To prove (3.9), it suffices to repeat the reasoning either from the proof of Theorem 2.2 (cf. equation (4.4)–(4.5), below) with $A = \lim_{t \to \infty} \|V(t)\|_{3/2, \infty} = 0$ or from the proof of Lemma 6.2. Let us skip other details. \hfill \Box

The next lemma deals with forces independent of time.

Lemma 3.4 Assume that $F(\cdot, t) = \mu$ where $\mu$ is the Borel measure on $\mathbb{R}^{3}$. Then $F \in Y_{3}$.

Proof. Recall that by the definition of the space $Y_{3}$, we should find an estimate of the norm $\| \int_{0}^{t} \mathcal{P} S(t - \tau) \mu \, d\tau \|_{3, \infty}$ which are uniform with respect to $t$. Note that the Leray projector $\mathcal{P}$ (being the combination of the Riesz transforms) is bounded $L^{3, \infty}$, hence it suffices to study $\| \int_{0}^{t} S(t - \tau) \mu \, d\tau \|_{3, \infty}$. Computing the Fourier transform of the integral $\int_{0}^{t} S(t - \tau) \mu \, d\tau$ we obtain the product

$$\int_{0}^{t} e^{-(t-\tau)|\xi|^{2}} d\tau \hat{\mu}(\xi) = \frac{1 - e^{-t|\xi|^{2}}}{|\xi|^{2}} \hat{\mu}(\xi).$$

Hence, $\int_{0}^{t} S(t - \tau) \mu \, d\tau = E_{3} * ((I - S(t)) \mu)$, where $E_{3}(x) = (4\pi |x|)^{-1}$ is the fundamental solution of the Laplace operator on $\mathbb{R}^{3}$. Since $E_{3} \in L^{3, \infty}$ and convolutions of Borel measures with elements from $L^{3, \infty}$ are well-defined, we obtain

$$\|E_{3} * ((I - S(t)) \mu)\|_{3, \infty} \leq C \|E_{3}\|_{3, \infty} < \infty.$$

We skip the proof of the regularity with respect to $t$ because the reasoning is more or less similar to that used in the proof of Lemma 3.2. \hfill \Box
Regularized Navier–Stokes equations

4 The Navier-Stokes system

As in [5], the proof of our theorem on the existence, uniqueness and stability of solutions to the problem (2.1)–(2.2) is based on the following abstract lemma, whose slightly more general form is taken from [20].

Lemma 4.1 Let $(X, \| \cdot \|_X)$ be a Banach space and $B : X \times X \to X$ a bounded bilinear form satisfying $\|B(x_1, x_2)\|_X \leq \eta \|x_1\|_X \|x_2\|_X$ for all $x_1, x_2 \in X$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < 1/(4\eta)$ and if $y \in X$ such that $\|y\| < \varepsilon$, the equation $x = y + B(x, x)$ has a solution in $X$ such that $\|x\|_X \leq 2\varepsilon$. This solution is the only one in the ball $\bar{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on $y$ in the following sense: if $\|\tilde{y}\|_X \leq \varepsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x})$, and $\|\tilde{x}\|_X \leq 2\varepsilon$, then

$$\|x - \tilde{x}\|_X \leq \frac{1}{1 - 4\eta\varepsilon} \|y - \tilde{y}\|_X.$$  

Proof. Here, one uses the standard Picard iteration technique completed by the Banach fixed point theorem. For other details of the proof, we refer the reader to [20, Th. 13.2].

Our goal is to apply Lemma 4.1 to the integral equation (2.3) in the space $X_3$ defined in (2.5). To continue, we need the estimate of the form $B(\cdot, \cdot)$.

Proposition 4.1 The bilinear form $B(\cdot, \cdot)$ is bounded on the space $X_3$. In other words, there exists a constant $\eta > 0$ such that

$$\|B(u, v)\|_X \leq \eta \|u\|_X \|v\|_X$$

for all $u, v \in X$.

Proof. The proof of this fact, given by Meyer in [26, Ch. 18], results immediately from Lemmata 3.1 and 3.2 because, for all $u, v \in X$, it follows that $u \otimes v \in C_w([0, \infty), L^{3/2, \infty})$. An independent reasoning which leads to this proposition can be also found in the recent paper by M. Yamazaki [31, Th. 3.1].

Proof of Theorem 2.1. Now, the main theorem on the existence of unique small solutions is a consequence of Lemma 4.1 combined with Proposition 4.1.

Remark 4.1 Homogeneity properties of equation (2.1) imply that if $u$ solves the Cauchy problem, then the rescaled function $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$ is also a solution for each $\lambda > 0$. Thus, it is natural to consider solutions which satisfy the scaling invariance property $u_\lambda \equiv u$ for all $\lambda > 0$, i.e. forward self-similar solutions. By the uniqueness property of solutions of the Cauchy problem, they can be obtained directly from Theorem 2.1 by taking $u_0$ homogeneous of degree $-1$ and $F$ satisfying (2.17).

Regularity of solutions constructed in Theorem 2.1 depends essentially on the regularity of external forces. We precise this fact in our next proposition.
Proposition 4.2 Let the assumptions of Theorem 2.1 hold true. Assume that 
$3 < p \leq \infty$ and recall that the space $X_p$ is defined in (2.7). Suppose, moreover, that the external force $F$ satisfies 
$\int_0^t S(t-\tau) P F(\tau) \, d\tau \in X_p$. There exists $\varepsilon_p \in (0, \varepsilon]$ such that the solution $u = u(x,t)$ constructed in Theorem 2.1 belong to the space $X_3 \cap X_p$. 

We skip the proof of this proposition, because it is more or less standard (see e.g. [1, 2, 5, 6, 9, 16, 15, 20, 31], for details). Let us only mention that it is based on Lemma 4.1 applied in the space $X = X_3 \cap X_p$, and the required estimates of the bilinear form $B(\cdot, \cdot)$ defined in (2.4) can be easily obtained combining the well-known inequalities for the heat semigroup (3.4) and its derivatives with the H"older inequality. Here, the crucial role is played by the inequality

$\|\nabla P S(t-\tau)(f \otimes g)\|_p \leq \eta_p (t-\tau)^{-\frac{1+3}{2}} \|f\|_p \|g\|_p$ (4.1)

valid for every $p \in (3, \infty]$ all $0 < \tau < t$ and a constant $\eta_p$, as well as its counterpart in the $L^{p,\infty}$-spaces. Note that (4.1) holds also true for $p \in [2, 3]$ but, in this case, the function $\zeta(\tau) = (t-\tau)^{-\frac{1+3}{p}/2}$ is not integrable near $\tau = t$. All details concerning the proof of Proposition 4.2 are contained in [15, Th. 5.1], [1, Th. 1].

Proof of Theorem 2.2. Several estimates from this proof will be used later on in the analysis of the regularized problems (2.13)–(2.14) and (2.15)–(2.16), hence we shall try to be very detailed.

We begin by recalling that, by Theorem 2.1, we have

$$\sup_{t \geq 0} \|u(t)\|_{3,\infty} \leq 2\varepsilon < \frac{1}{2\eta} \quad \text{and} \quad \sup_{t \geq 0} \|\tilde{u}(t)\|_{3,\infty} \leq 2\varepsilon < \frac{1}{2\eta}. \quad (4.2)$$

We subtract the integral equation (2.3) for $\tilde{u}$ from the analogous expression for $u$. Next, computing the norm $\|\cdot\|_{3,\infty}$ of the resulting equation we obtain the following inequality

$$\|u(t) - \tilde{u}(t)\|_{3,\infty} \leq \left\| S(t)(u_0 - \tilde{u}_0) + \int_0^t S(t-\tau) P(F - \tilde{F})(\tau) \, d\tau \right\|_{3,\infty}$$

$$+ \left\| \int_0^\delta S(t-\tau) P \nabla \cdot [(u - \tilde{u}) \otimes u + \tilde{u} \otimes (u - \tilde{u})] (\tau) \, d\tau \right\|_{3,\infty}$$

$$+ \left\| \int_\delta^t S(t-\tau) P \nabla \cdot [(u - \tilde{u}) \otimes u + \tilde{u} \otimes (u - \tilde{u})] (\tau) \, d\tau \right\|_{3,\infty} \quad (4.3)$$

where the small constant $\delta > 0$ will be chosen later.

In the term on the right-hand side of (4.3) containing the integral $\int_0^\delta ... \, d\tau$, we apply the weak $L^p - L^q$ estimates of the heat semigroup (3.5), the boundedness of
\( P \) on \( L^{3, \infty} \), the weak Hölder inequality (3.2), and (4.2), in order to estimate it by

\[
C \int_0^\delta (t - \tau)^{-1} \| u(\tau) - \tilde{u}(\tau) \|_{3, \infty} \, d\tau \\
\times \left( \sup_{\tau > 0} \| u(\tau) \|_{3, \infty} + \sup_{\tau > 0} \| \tilde{u}(\tau) \|_{3, \infty} \right)
\leq 4 \varepsilon C \int_0^\delta (1 - s)^{-1} \| u(ts) - v(ts) \|_{3, \infty} \, ds.
\] (4.4)

To deal with the term in (4.3) containing \( f_{\delta t}^t \ldots d\tau \), we use Lemma 3.1 (with \( f = (u - \tilde{u}) \ast u + \tilde{u} \ast (u - \tilde{u}) \) for \( \delta t < \tau < t \) and \( f = 0 \) otherwise) combined with the Hölder inequality (3.2) and with (4.2), to bound it directly by

\[
\eta \left( \sup_{\delta t \leq \tau \leq t} \| u(\tau) \|_{3, \infty} + \sup_{\delta t \leq \tau \leq t} \| \tilde{u}(\tau) \|_{3, \infty} \right) \sup_{\delta t \leq \tau \leq t} \| u(\tau) - \tilde{u}(\tau) \|_{3, \infty}
\leq 4 \varepsilon \eta \sup_{\delta t \leq \tau \leq t} \| u(\tau) - \tilde{u}(\tau) \|_{3, \infty}.
\] (4.5)

Now, we denote

\[
g(t) = \left\| S(t)(u_0 - \tilde{u}_0) + \int_0^t S(t - \tau) P(F - \hat{F})(\tau) \, d\tau \right\|_{3, \infty},
\]
and it follows from the assumptions on initial data and external forces that

\[
g \in L^\infty(0, \infty) \quad \text{and} \quad \lim_{t \to \infty} g(t) = 0.
\] (4.6)

Hence, applying (4.4) and (4.5) to (4.3) we arrive at

\[
\| u(t) - \tilde{u}(t) \|_{3, \infty} \leq g(t) + 4 \varepsilon C \int_0^\delta (1 - s)^{-1} \| u(ts) - \tilde{u}(ts) \|_{3, \infty} \, ds
\]
\[
+ 4 \varepsilon \eta \sup_{\delta t \leq \tau \leq t} \| u(\tau) - \tilde{u}(\tau) \|_{3, \infty}
\] (4.7)

for all \( t > 0 \).

Next, we put

\[
A = \limsup_{t \to \infty} \| u(t) - \tilde{u}(t) \|_{3, \infty} \equiv \lim_{k \to \infty} \sup_{t \geq k} \| u(t) - \tilde{u}(t) \|_{3, \infty}.
\]

The number \( A \) is nonnegative and finite because both \( u, \tilde{u} \in L^\infty([0, \infty), L^{3, \infty}) \), and our claim is to show that \( A = 0 \).

First, we apply the Lebesgue dominated convergence theorem to the obvious inequality

\[
\sup_{t \geq k} \int_0^\delta (1 - s)^{-1} \| u(ts) - \tilde{u}(ts) \|_{3, \infty} \, ds \leq \int_0^\delta (1 - s)^{-1} \sup_{t \geq k} \| u(ts) - \tilde{u}(ts) \|_{3, \infty} \, ds,
\]

and we have

\[
\| u(t) - \tilde{u}(t) \|_{3, \infty} \leq g(t) + 4 \varepsilon C \int_0^\delta (1 - s)^{-1} \| u(ts) - \tilde{u}(ts) \|_{3, \infty} \, ds
\]
\[
+ 4 \varepsilon \eta \sup_{\delta t \leq \tau \leq t} \| u(\tau) - \tilde{u}(\tau) \|_{3, \infty}
\] (4.7)
and we obtain
\[
\limsup_{t \to \infty} \int_0^\delta (1-s)^{-1} \| u(ts) - \tilde{u}(ts) \|_{3, \infty} ds \leq A \int_0^\delta (1-s)^{-1} ds = A \log \left( \frac{1}{1-\delta} \right).
\] (4.8)

Moreover, since
\[
\sup_{t \geq k} \sup_{\delta t \leq \tau \leq t} \| u(\tau) - \tilde{u}(\tau) \|_{3, \infty} \leq \sup_{\delta k \leq \tau < \infty} \| u(\tau) - \tilde{u}(\tau) \|_{3, \infty},
\]
we have
\[
\limsup_{t \to \infty} \left( \sup_{\delta t \leq \tau \leq t} \| u(\tau) - v(\tau) \|_{3, \infty} \right) \leq A. \tag{4.9}
\]

Finally, computing \( \limsup_{t \to \infty} \) of the both sides of inequality (4.7), and using (4.6), (4.8), and (4.9) we get
\[
A \leq \left( 4 \varepsilon C \log \left( \frac{1}{1-\delta} \right) + 4 \varepsilon \eta \right) A.
\]

Consequently, it follows that \( A = \limsup_{t \to \infty} \| u(t) - v(t) \|_{3, \infty} = 0 \) because
\[
4 \varepsilon \eta \left( C \log \left( \frac{1}{1-\delta} \right) + 1 \right) < 1,
\]
for \( \delta > 0 \) sufficiently small, by the assumption of Theorem 2.1 saying that \( 0 < \varepsilon < 1/(4\eta) \). This completes the proof of Theorem 2.2. \( \square \)

As a direct consequence the proof of Theorem 2.2, we have also necessary conditions for (2.10) to hold. We formulate this fact in the following corollary.

**Corollary 4.1** Assume that \( u, \tilde{u} \in X_3 \) are solutions to system (2.1)–(2.2) corresponding to initial conditions \( u_0, \tilde{u}_0 \in L^{3, \infty} \) and external forces \( F, \tilde{F} \in Y_3 \), respectively. Suppose that
\[
\lim_{t \to \infty} \| u(t) - \tilde{u}(t) \|_{3, \infty} = 0. \tag{4.10}
\]

Then
\[
\lim_{t \to \infty} \left\| S(t)(u_0 - \tilde{u}_0) + \int_0^t S(t-\tau) P'(F(\tau) - \tilde{F}(\tau)) \ d\tau \right\|_{3, \infty} = 0.
\]

**Proof.** As in the beginning of the proof of Theorem 2.2, we subtract the integral equation (2.3) for \( \tilde{u} \) from the same expression for \( u \). Next, we compute the \( L^{3, \infty} \)-norm and we use inequalities (4.4) and (4.5) to obtain
\[
\left\| S(t)(u_0 - \tilde{u}_0) + \int_0^t S(t-\tau) P'(F(\tau) - \tilde{F}(\tau)) \ d\tau \right\|_{3, \infty} \leq \| u(t) - \tilde{u}(t) \|_{3, \infty}
\]
\[
+ 4 \varepsilon C \int_0^\delta (1-s)^{-1} \| u(ts) - v(ts) \|_{3, \infty} ds + 4 \varepsilon \eta \sup_{\delta t \leq \tau \leq t} \| u(\tau) - \tilde{u}(\tau) \|_{3, \infty}.
\] (4.11)
The first term on the right-hand side of (4.11) tends to zero as $t \to \infty$ by (4.10). To show the decay of the second one, it suffices to repeat calculations from (4.5) and (4.8). Now, however, one should remember that $A = 0$ is assumed. 

An asymptotic stability result holds also true in the $L^{p, \infty}$ and $L^p$-spaces with $p > 3$.

**Proposition 4.3** Under the assumptions of Theorem 2.2

\[
\lim_{t \to \infty} t^{(1-3/p)/2} \|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{p, \infty} = 0
\]

and

\[
\lim_{t \to \infty} t^{(1-3/q)/2} \|u(\cdot, t) - \tilde{u}(\cdot, t)\|_q = 0
\]

for every $q \in (3, p)$. \hfill \Box

We skip the proof of the first relations in the above proposition, because it a standard fact. A completely analogous reasoning can be found in [27, 15, 2] (see also the last section of this paper). Next, the limit in the $L^q$-spaces is a consequence of the imbedding (2.8).

## 5 The mollified Navier-Stokes system

The goal of this section is to formulate and to prove results on the large time behavior of solutions to the mollified problem (2.13)--(2.14) reformulated in the integral form

\[
v(t) = S(t)v_0 + B_\omega(v, v) + \int_0^t S(t-\tau) \mathbb{P}G(\tau) \, d\tau
\]

with the bilinear form

\[
B_\omega(v, \tilde{v})(t) = -\int_0^t S(t-\tau) \mathbb{P} \nabla \cdot \left[(v * \omega) \otimes \tilde{v}\right](\tau) \, d\tau.
\]

The counterpart of Proposition 4.1 reads as follows.

**Proposition 5.1** For every $v, \tilde{v} \in X_3$, we have $B_\omega(v, \tilde{v}) \in X_3$. Moreover, it follows that

\[
\|B_\omega(v, \tilde{v})\|_{X_3} \leq \eta \|v\|_{X_3} \|\tilde{v}\|_{X_3}
\]

with the same constant $\eta$ as in Proposition 4.1.

**Proof.** This is a direct application of Proposition 4.1, since by the Hölder inequality, we have $\|v * \omega\|_{X_3} \leq \|v\|_{X_3} \|\omega\|_{L^1} = \|v\|_{X_3}$. \hfill \Box

Now, as usual, the existence, uniqueness, and regularity of solutions to (2.13)--(2.14) are deduced from Lemma 4.1.
Theorem 5.1 The counterpart of Theorem 2.1 for the mollified problem (2.13)–(2.14) holds true if we replace \( u(x,t), u_0, \) and \( F \) by the solution \( v = v(x,t) \) to (2.13)–(2.14) corresponding to the initial datum \( v_0 \in L^{3,\infty} \) and the external force \( G \in Y_3 \).

Proposition 5.2 Let the assumptions of Theorem 5.1 hold true. Let \( 3 < p \leq \infty \). Suppose, moreover, that the external force \( G \) satisfies

\[
\int_0^t S(t-\tau) IPG(\tau) \, d\tau \in X_p.
\]

There exists \( \varepsilon_p \in (0,\varepsilon) \) such that the solution \( v = v(x,t) \) constructed in Theorem 5.1 belong to the space \( X_3 \cap X_p \).

Here, we have skiped the proofs of Theorem 5.1 and Proposition 5.2 because they are completely analogous to their counterparts from Section 4.

Now, we are in a position to prove the convergence of solutions of the mollified problem (2.13)–(2.14) toward self-similar solutions of (2.1)–(2.2).

Proof of Theorem 2.3. Recall that by Propositions 4.2 and 5.2, \( u \) and \( v \) exist for all \( t > 0 \), they both belong to the ball \( B(0,2\varepsilon_p) \subset X_p \) : 

\[
\sup_{t>0} t^{(1-3/p)/2} \| u(t) \|_p < \infty
\]

for every \( p \in (3,\infty) \) and some \( \varepsilon_p \in (0,1/(4\eta_p)) \) which implies that

\[
\| u \|_p < 2\varepsilon_p t^{(1-3/p)/2} \quad \text{and} \quad \| v \|_p < 2\varepsilon_p t^{(1-3/p)/2}.
\]

Moreover, \( u = u(x,t) \) is the self-similar solution of the form (2.18).

Here, we study again the difference of the integral formulations of both problems (see (2.3) and (5.1)) written in the following form

\[
\begin{align*}
(u(t) - v(t)) &= - \int_0^t \nabla IP S(t-\tau) [(u - u \ast \omega) \otimes u](\tau) \, d\tau \\
&\quad - \int_0^t \nabla IP S(t-\tau) [(u - v) \ast \omega \otimes u](\tau) \, d\tau.
\end{align*}
\]

It follows from the self-similar form of \( u(x,t) \) that

\[
(u \ast \omega)(x,t) = t^{-1/2} \int_{\mathbb{R}^3} U \left( \frac{x-y}{\sqrt{t}} \right) \omega(y) \, dy
\]

\[
= t^{-1/2} \int_{\mathbb{R}^3} U \left( \frac{x}{\sqrt{t}} - z \right) t^{3/2} \omega(y \sqrt{t}) \, dy
\]

\[
= t^{-1/2} (U \ast \omega^t) \left( \frac{x}{\sqrt{t}} \right),
\]

where \( \omega^t(z) = t^{3/2} \omega(z \sqrt{t}) \). One can easily check that \( \omega^t \) is the approximation of the Dirac delta as \( t \to \infty \). Hence, in particular,

\[
\| U - U \ast \omega^t \|_p \to 0 \quad \text{as} \quad t \to \infty
\]

for every \( p \in (3,\infty) \), because \( U \in L^p \) in this range of \( p \).
Now, we compute the $L^p$-norm of (5.3), next, we multiply the resulting inequality by $t^{(1-3/p)/2}$ and, finally, we use inequality (4.1) in order to obtain
\[
E(t^{(1-3/p)/2})\|u(t) - v(t)\|_p \\
\leq \eta_p t^{(1-3/p)/2} \int_0^t (t - \tau)^{-(1+3/p)/2} \|(u - u*\omega)(\tau)\|_p u(\tau)\|_p d\tau \\
\leq \eta_p t^{(1-3/p)/2} \int_0^t (t - \tau)^{-(1+3/p)/2} \|[(u - v) * \omega](\tau)\|_p v(\tau)\|_p d\tau (5.6) \\
= C(t) + D(t).
\]

The first term on the right-hand-side of (5.6) tends to 0 as $t \to \infty$. To see this fact, we use (5.2), (5.4), and the change of variables $\tau = ts$ in order to show that
\[
C(t) \leq 2\varepsilon_p \eta_p t^{(1-3/p)/2} \int_0^t (t - \tau)^{-(1+3/p)/2} \tau^{-(1-3/p)} \|U - U*\omega^\tau\|_p d\tau \\
= 2\varepsilon_p \eta_p \int_0^1 (1 - s)^{-(1+3/p)/2} s^{-(1-3/p)} \|U - U*\omega^s\|_p d\tau.
\]

Now, $\lim_{s \to \infty} C(t) = 0$ by the Lebesgue dominated convergence theorem.

We apply a similar argument involving (5.2) and the change of variables $\tau = ts$ to estimate the second term in (5.6) by
\[
D(t) \leq 2\varepsilon_p \eta_p \int_0^1 (1 - s)^{-(1+3/p)/2} s^{-(1-3/p)} \left((st)^{(1-3/p)/2} \|u(ts) - v(ts)\|_p\right) d\tau. (5.7)
\]

Next, we define the number
\[
A = \limsup_{t \to \infty} t^{(1-3/p)/2}\|u(t) - v(t)\|_p = \lim_{k \to \infty} \sup_{t \geq k} t^{(1-3/p)/2}\|u(t) - v(t)\|_p
\]
which is nonnegative and finite because $u, v \in X_p$, and our claim is to show that $A = 0$. Since $\limsup_{t \to \infty} C(t) = 0$, it follows from (5.6), (5.7), and from the Lebesgue dominated convergence theorem that
\[
A \leq \left(2\varepsilon_p \eta_p \int_0^1 (1 - s)^{-(1+3/p)/2} s^{-(1-3/p)} ds\right) A
\]
(5.8)

The quantity in the parentheses is smaller than 1 provided $\varepsilon_p$ is sufficiently small. Hence, inequality (5.8) implies that $A = 0$ and the proof of Theorem 2.3 is complete.

\[\Box\]

6 The hyperviscous Navier-Stokes system

In the case of the system (2.15)–(2.16), the counterpart of the integral equation (2.3) has the following form
\[
u(t) = S_t u_0 - \int_0^t S_t S(t - \tau) \nabla P \cdot (u \otimes u)(\tau) d\tau. (6.1)
\]
\[
+ \int_0^t S_\ell(t-\tau)S(t-\tau)\mathbb{P}H(\tau) \, d\tau,
\]
where the semigroup generated by the operator \((-\Delta)^{\ell/2}\) is denoted by \(S_\ell(t)\) which is given by the convolution with the kernel
\[
p_\ell(x, t) = \int_{\mathbb{R}^n} e^{-t|\xi|^\ell + ix \cdot \xi} \, d\xi.
\]
Note that \(p_2(x, t)\) corresponds to the Gauss-Weierstrass kernel \(p(x, t)\). Recall that the function
\[
p_\ell(x, t) = \int_{\mathbb{R}^n} e^{-t|\xi|^\ell + ix \cdot \xi} \, d\xi = t^{-n/\ell} p(x/t^{1/\ell}, 1)
\]
is integrable for every \(\ell > 0\) and all \(t > 0\). Moreover, the self-similar form of \(p_\ell\) implies that \(\|p_\ell(\cdot, t)\|_1 = \|p_\ell(\cdot, 1)\|_1\) for every \(t > 0\). In this section, the constant
\[
C_\ell = \|p_\ell(\cdot, 1)\|_1 \quad (6.2)
\]
appears quite often in our calculation because of the inequality
\[
\|S_\ell(t)h\|_{3,\infty} \leq C_\ell \|h\|_{3,\infty} \quad (6.3)
\]
valid for every \(h \in L^{3,\infty}\) (cf. also Proposition 6.1, below).

**Remark 6.1** Since the kernel \(p_\ell(\cdot, 1)\) is integrable for every \(\ell > 0\), the constant \(C_\ell\) is well-defined. However, only for \(0 < \ell \leq 2\), it is a nonnegative function, consequently, in this range of \(\ell\), we have \(C_\ell = \|p_\ell(\cdot, 1)\|_1 = \int_{\mathbb{R}^n} p_t(x, 1) \, dx = \tilde{p}_t(0, 1) = 1\). On the other hand, the kernel \(p_\ell(x, t)\) changes sign for \(\ell > 2\), hence, for those \(\ell\), \(C_\ell > 1\).

We define the bilinear form
\[
B_\ell(w, \tilde{w})(t) = -\int_0^t S_\ell(t-\tau)S(t-\tau)\mathbb{P}\nabla \cdot [w \otimes \tilde{w}](\tau) \, d\tau.
\]

The following proposition plays again an essential role in our proofs of existence of global-in-time solutions to (2.15)–(2.16) as well as in the study of their large time asymptotics.

**Proposition 6.1** For every \(w, \tilde{w} \in X_3\), we have \(B_\ell(w, \tilde{w}) \in X_3\). Moreover, it follows that
\[
\|B_\ell(w, \tilde{w})\|_{X_3} \leq \eta C_\ell \|w\|_{X_3} \|	ilde{w}\|_{X_3}
\]
for \(\eta\) defined in Proposition 4.1 and \(C_\ell\) given by (6.2).

**Proof.** This inequality results immediately from the Meyer-Yamazaki estimate (3.8) applied to the function defined in (3.7) with \(Q(\cdot, \tau) = S_\ell(t-\tau)(w(\tau) \otimes \tilde{w}(\tau))\).
for $0 < \tau < t$ and $Q(\cdot, \tau) = 0$ otherwise. Next, one should use inequalities (6.3) and (3.2) in the following way

\[
\sup_{\tau > 0} \|Q(\cdot, \tau)\|_{3, \infty} = \sup_{0 < \tau < t} \|S_\ell(t - \tau)(w(\tau) \otimes \tilde{w}(\tau))\|_{3, \infty} \leq C_\ell \sup_{0 < \tau < t} \|w(\tau)\|_{3, \infty} \sup_{0 < \tau < t} \|\tilde{w}(\tau)\|_{3, \infty}.
\]

\[\square\]

**Theorem 6.1** The counterpart of Theorem 2.1 holds true if we replace $u(x, t)$, $u_0$, and $F$ by the solution $w = w(x, t)$ to the hyperviscous problem (2.15)–(2.16) corresponding to the initial datum $w_0 \in L^{3, \infty}$ and the external force $H \in Y_3$, and if we impose additional assumption $\varepsilon < 1/(4\eta C_\ell)$.

\[\square\]

**Remark 6.2** As in the case of problems (2.1)-(2.2) and (2.13)-(2.14), the solution constructed in Theorem 6.1 belongs to the space $X_p$ for $p > 3$ under the additional assumption $\int_0^t S_\ell(t - \tau) S(t - \tau) PH(\tau) d\tau \in X_p$. Here, we omit details because the reasoning is completely analogous to that used in Propositions 4.2 and 5.2.

\[\square\]

The crucial lemma in the study of the large time behavior of solutions to (2.15)–(2.16) says that the semigroup generated by the operator $\Delta - (-\Delta)^{\ell/2}$ can be well-approximated in $L^1$ by the heat semigroup $S(t)$.

**Lemma 6.1** Let $\ell > 0$. There exists a constant $C$ independent of $t$ such that

\[\|p_\ell(t) * p(t/2) - p(t/2)\|_1 \leq Ct^{-(1/2 - 1/\ell)}\]

for all $t > 0$.

**Proof.** Let us recall the inequality

\[\left\|f * g(\cdot) - \left(\int_{\mathbb{R}^n} f(x) \, dx \right) g(\cdot)\right\|_1 \leq C\|\nabla g\|_1 \|f\|_{L^1(\mathbb{R}^n, |x| \, dx)}\]  

(6.4)

which is valid for all sufficiently regular $f$ and $g$, and a constant $C$ independent of $f, g$. The proof of (6.4) (based on the Taylor expansion of the function $g$) and its generalizations can be found in [12]. Now, in (6.4), we substitute

\[f(x) = p_\ell(t, x) \quad \text{and} \quad g(x, t) = p(t/2, x)\]

to obtain (recall that $\int_{\mathbb{R}^n} p_\ell(t, x) \, dx = \tilde{p}(0, t) = 1$)

\[\|p_\ell(t) * p(t/2) - p(t/2)\|_1 \leq C\|\nabla p(t/2)\|_1 \|p_\ell(t)\|_{L^1(\mathbb{R}^n, |x| \, dx)} = Ct^{-(1/2 - 1/\ell)}\]

for all $t > 0$.

\[\square\]

The lemma above is used in the proof of our next result.
Lemma 6.2 Assume that \( f \in C_w([0, \infty), L^{3/2, \infty}(\mathbb{R}^3)) \). Then
\[
\left\| \int_0^t \nabla IP S(t-\tau) \left( S_t(t-\tau) - I \right) f(\cdot, \tau) \, d\tau \right\|_{3, \infty} \to 0 \quad \text{as} \quad t \to \infty, \quad (6.5)
\]
where \( I \) denotes the identity operator.

Proof. First note that the quantity in (6.5) is bounded uniformly with respect to \( t > 0 \) in view of inequality (3.8) (cf. also the proof of Proposition 6.1). To show its convergence to 0 we fix \( \gamma \in (0, 1) \) (to be chosen later on) and we decompose the integral in (6.5) as \( \int_0^t \cdots \, d\tau = \int_0^{\gamma t} \cdots \, d\tau + \int_{\gamma t}^t \cdots \, d\tau \). Now, the estimates of the heat semigroup on the Marcinkiewicz spaces (3.5) give
\[
\left\| \int_0^{\gamma t} \nabla IP S(t-\tau) \left( S_t(t-\tau) - I \right) f(\cdot, \tau) \, d\tau \right\|_{3, \infty}
\leq \int_0^{\gamma t} \left( \| \nabla IP S(t-\tau) S_t(t-\tau) f(\cdot, \tau) \|_{3, \infty} + \| \nabla IP S(t-\tau) f(\cdot, \tau) \|_{3, \infty} \right) \, d\tau
\leq C \int_0^{\gamma t} (t-\tau)^{-1} \, d\tau \sup_{\tau > 0} \| f(\cdot, \tau) \|_{3/2, \infty}
\leq C \log \left( \frac{1}{1-\gamma} \right) \sup_{\tau > 0} \| f(\cdot, \tau) \|_{3/2, \infty}. \quad (6.6)
\]

Observe that the right-hand-side of the inequality above can be made arbitrarily small choosing \( \gamma > 0 \) sufficiently small.

We handle the integral over \([\gamma t, t]\) using the Meyer-Yamazaki estimate (3.8) applied to the function
\[
g(\cdot, t) = \int_0^t \nabla IP \left( \frac{t-\tau}{2} \right) Q(\cdot, \tau) \, d\tau
\]
with \( Q(\cdot, \tau) = [S_t(t-\tau) - I] S_t((t-\tau)/2) f(\cdot, \tau) \) for \( \tau \in [\gamma t, t] \) and \( Q(\cdot, \tau) = 0 \) otherwise. First, using inequality (3.8) and next, Lemma 6.1 we obtain
\[
\| g(\cdot, t) \|_{3, \infty} \leq C \sup_{\gamma t \leq \tau \leq t} \| Q(\cdot, \tau) \|_{3/2, \infty}
\leq C \left( \sup_{\gamma t \leq \tau \leq t} (t-\tau)^{-(1/2-1/\ell)} \right) \sup_{\gamma t \leq \tau \leq t} \| f(\cdot, \tau) \|_{3/2, \infty}
\leq C t^{-(1/2-1/\ell)} \sup_{\gamma t \leq \tau \leq t} \| f(\cdot, \tau) \|_{3/2, \infty}
\]
Note now that the right-hand-side of the above inequalities tends to 0 as \( t \to \infty \) for every \( \gamma > 0 \). This completes the proof of Lemma 6.2. \( \square \)

Proof of Theorem 2.4. Note first that the existence of such solutions is provided by Theorems 2.1 and 6.1. In particular, we have
\[
\sup_{t \geq 0} \| u(t) \|_{3, \infty} \leq 2\varepsilon \quad \text{and} \quad \sup_{t \geq 0} \| \tilde{w}(t) \|_{3, \infty} \leq 2\varepsilon. \quad (6.7)
\]
Here, we describe only how to modify the proof of Theorem 2.2. First, we subtract the integral equation (6.1) for \( w \) from equation (2.3) for \( u \) and, next, we compute the \( L^{3,\infty} \)-norm. After elementary calculations, we obtain

\[
\|u(t) - w(t)\|_{3,\infty} \leq \| [S(t) - S(t)S(t)]u_0 \|_{3,\infty} \\
+ \left\| \int_0^t [S(t - \tau) - S(t - \tau)S(t - \tau)] \mathbb{P}F(\tau) \, d\tau \right\|_{3,\infty} \\
+ \left\| \int_0^t \nabla \mathbb{P}S(t - \tau)(u \otimes u - w \otimes w)(\tau) \, d\tau \right\|_{3,\infty} \\
+ \left\| \int_0^t \nabla \mathbb{P}S(t - \tau)(S(t - \tau) - I)(w \otimes w)(\tau) \, d\tau \right\|_{3,\infty}.
\]

It follows from Lemma 6.1 that

\[
\| [S(t)S(t) - S(t)]u_0 \|_{3,\infty} \leq \| p_\varepsilon(t) * p(t/2) - p(t/2) \|_1 \| S(t/2)u_0 \|_{3,\infty} \\
\leq Ct^{-(1/2 - 1/t)} \| u_0 \|_{3,\infty} \to 0 \quad \text{as} \quad t \to \infty.
\]

The third and the fourth term on the right-hand-side of (6.8) tend to 0 as \( t \to \infty \) in view of Lemma 6.2 applied either to the function \( f(\cdot,t) = V(\cdot,t) \) or to \( f(\cdot,t) = (w \otimes w)(\cdot,t) \).

We deal with the second term in (6.8) exactly in the same way as in the proof of Theorem 2.2. Repeating the calculations from (4.4), (4.5), and (4.7) we obtain

\[
\left\| \int_0^t \nabla \mathbb{P}S(t - \tau)(u \otimes u - w \otimes w)(\tau) \, d\tau \right\|_{3,\infty} \\
\leq 4\varepsilon C \int_0^\delta (1 - s)^{-1} \| u(ts) - w(ts) \|_{3,\infty} \, ds + 4\varepsilon \eta \sup_{\delta t \leq \tau \leq t} \| u(\tau) - w(\tau) \|_{3,\infty}.
\]

Now, we define \( A = \lim \sup_{t \to \infty} \| u(t) - w(t) \|_{3,\infty} \). To show that \( A = 0 \), it suffices to pass to the limit as \( t \to \infty \) in inequality (6.8) and to repeat the reasoning given at the end of the proof of Theorem 2.2.

The limit in (2.20) should be shown for the \( L^p \)-norm replaced by the Marcinkiewicz norm, first. Here, one should proceed as in the proof of Theorem 2.3. Next, the proof of (2.20) is completed by the imbedding (2.8).

\[\square\]

7 Asymptotic stability in abstract Banach spaces

The asymptotic stability analysis described in previous sections can be generalized to the case of more general Banach spaces. Below, we formulate such a kind of results. We skip several details of proofs because they can be found either in [15] or in [20, 25].

The idea of constructing solutions and to study their large time behavior is the following. We impose the conditions on the Banach space \( E \) (cf. Definitions 7.1
and 7.2) which guarantee that our Cauchy problems have local-in-time solutions in the space \( C_w([0, T), E) \) for some \( T > 0 \). Next, we show that a scaling property of \( \| \cdot \|_E \) allows us to obtain, moreover, global-in-time solutions for suitably small initial data. To get such results, we introduce a new Banach space of distributions which, roughly speaking, is a homogeneous Besov type space modeled on \( E \). This approach allows us to get solutions for initial data less regular than those from \( E \). In this abstract setting, we also study large-time behavior of constructed solutions.

### 7.1 Definitions of spaces

Our first two definitions are minor modifications of [26, Def. 7 & 8, Sec. 8] and [20, Def. 4.1].

**Definition 7.1** The Banach space \((E, \| \cdot \|_E)\) is said to be functional and translation invariant if the following three conditions are satisfied:

i. \( S \subset E \subset S' \) and the both inclusions are continuous.

ii. either these two imbeddings have a dense range or \( E \) is the dual space \( F^* \) of a functional Banach space \( F \) for which these two imbeddings have a dense range.

iii. The norm \( \| \cdot \|_E \) on \( E \) is translation invariant, i.e.

\[
\text{for all } f \in E \text{ and } y \in \mathbb{R}^n, \quad \| \tau_y f \|_E = \| f \|_E.
\]

**Definition 7.2** We call the space \((E, \| \cdot \|_E)\) adequate to the problem (2.1)–(2.2) if

i. it is a functional translation invariant Banach space;

ii. for all \( f, g \in E \), the product \( f \otimes g \) is well-defined as the tempered distribution, moreover, there exist \( T_0 > 0 \) and a positive function \( \omega \in L^1(0, T_0) \) such that

\[
\| \mathcal{P} \nabla S(\tau) \cdot (f \otimes g) \|_E \leq \omega(\tau) \| f \|_E \| g \|_E \tag{7.1}
\]

for every \( f, g \in E \) and \( \tau \in (0, T_0) \).

Note that inequality (7.1) for the space \( E = L^p \) appeared already in the proof of Proposition 4.2 (cf. (4.1)). Since we are interested in an incompressible flow, we can say that the Banach space \( \mathcal{P} L^p = \{ f \in L^p : \nabla \cdot f = 0 \} \) is adequate to the Navier-Stokes system (2.1)–(2.2) for every \( p \in (3, \infty) \).

We refer the reader to the paper [15] for other examples of Banach spaces adequate to (2.1)–(2.2). Moreover, the well-suited spaces introduced in [5, 7] are functional translation invariant Banach spaces in the sense of our Definition 7.1 having some additional properties. In particular, they satisfy a slightly stronger condition than (7.1), so they are also adequate spaces in the sense of Definition 7.2 (see [7, Lem. 2.1]). Several examples of the well-suited (or adequate) spaces for the Navier-Stokes system (2.1)–(2.2) are also contained in the book [20].
Remark 7.1 Here, it is worth of emphasizing that if \( E \) is a well-suited Banach space (or, more generally, adequate for the problem (2.1)–(2.2)) then for any initial datum \( v_0 \in E \), \( \nabla \cdot v_0 = 0 \), there exists \( T = T(\|v_0\|_E) \) and the unique “mild” solution to the Navier-Stokes equations in the space \( C([0,T); E) \). Details are contained in [7, Theorem 2.1].

In this paper, we use Banach spaces with norms having additional scaling properties. In order to state this fact more precisely, given \( f : \mathbb{R}^n \to \mathbb{R}^n \), we define the rescaled function

\[
    f_\lambda(x) = f(\lambda x)
\]

for each \( \lambda > 0 \). We extend this definition for all \( f \in S' \) in the standard way.

Definition 7.3 Let \( (E, \| \cdot \|_E) \) be a Banach space, which can be imbedded continuously in \( S' \). The norm \( \| \cdot \|_E \) is said to have the scaling degree equal to \( k \), if

\[
    \| f_\lambda \|_E \equiv \lambda^k \| f \|_E
\]

for each \( f \in E \) such that \( f_\lambda \in E \) and for all \( \lambda > 0 \).

It is evident that the usual norms of the spaces \( L^p \), \( L^{p,\infty} \), \( L^{p,q} \) (the Lorentz space), \( \mathcal{M}^p_q \) (the homogeneous Morrey space) have the scaling degree equal to \( -n/p \) (more details on these spaces can be found e.g. in [15]). On the other hand, the standard norm in the homogeneous Sobolev space \( \dot{H}^s = \{ f \in S' : \| \xi|^s f(\xi) \| L^2 \} \) has scaling degree \( s - n/2 \).

Remark 7.2 In our considerations below, we systematically assume that the norms of Banach spaces have the scaling degrees equal to some \( k \in (-1, 0) \). Since the space \( L^p \) is our model example, to simplify the exposition, we shall assume that \( k = -3/p \) with \( p > 3 \). In this work, Banach spaces endowed with norms having this property will be usually denoted by \( E_p \).

Let us fix a Banach space \( E \subset S' \) and introduce a new space of distributions denoted by \( BE^\alpha \) which, loosely speaking, is a homogeneous Besov space modeled on \( E \). The definition we are going to introduce will be an important tool in the next sections, where global-in-time solutions will be constructed (for suitably small initial data) in \( C([0,\infty); BE^\alpha) \).

Definition 7.4 Let \( \alpha \geq 0 \). Given a Banach space \( E \) imbedded continuously in \( S' \), we define

\[
    BE^\alpha = \{ f \in S' : \| f \|_{BE^\alpha} \equiv \sup_{t>0} t^{\alpha/2} \| S(t) f \|_E < \infty \}.
\]

Let \( E = L^p(\mathbb{R}^n) \) for a moment. It follows immediately from the estimates of the heat semigroup

\[
    \| S(t) f \|_{L^p(\mathbb{R}^n)} \leq C(p,q) t^{-n(1/q - 1/p)/2} \| f \|_{L^q(\mathbb{R}^n)}
\]

for each \( 1 \leq q \leq p \leq \infty \), that \( L^q \subset BE^\alpha \) with \( \alpha = n(1/q - 1/p) \). It is easy to obtain the analogous conclusions for the Marcinkiewicz, Lorentz, or Morrey spaces applying appropriate estimates of the heat semigroup mentioned in [15, Section 3].
Moreover, for $E = L^p(\mathbb{R}^n)$, the norm $\| \cdot \|_{BE^\alpha}$ is equivalent to the standard norm of the homogeneous Besov space $B_{p,\infty}$ introduced via a dyadic decomposition.

**Remark 7.3** If $E$ has a norm with scaling degree $k$, then $\| \cdot \|_{BE^\alpha}$ has degree $k - \alpha$. Indeed, first we observe that for any $f \in S'$ and $\lambda > 0$,

$$S(t)f_\lambda = (S(\lambda^2t)f)_\lambda.$$  \hspace{1cm} (7.3)

Hence, the scaling property of the norm on $E$ implies

$$\| f_\lambda \|_{BE^\alpha} = \sup_{t > 0} \lambda^{1-\alpha/2} \| S(t)f \|_E = \lambda^{k-\alpha} \| f \|_{BE^\alpha}.$$ 

$\blacksquare$

### 7.2 The Navier-Stokes system

Now, assume that $E_p$ is the Banach space adequate to the problem (2.1)–(2.2) which norm has the order if scaling equal to $-3/p$ with $p > 3$. For simplicity of the exposition, we suppose, moreover, the special form of the external forces $F = \nabla \cdot V$. It is proved in [15, Th. 5.1] that there exists $\varepsilon > 0$ such that for each $v_0 \in BE_p^{1-3/p}$ and $V(t) \in E_p$ satisfying

$$\| v_0 \|_{BE_p^{1-3/p}} + \sup_{t > 0} t^{(1-3/(2p))} \| V(t) \|_E < \varepsilon$$

the Cauchy problem (2.1)–(2.2) has a solution $v(x,t)$ in the space

$$\mathcal{X} = \mathcal{C}([0, \infty) : BE_p^{1-3/p}) \cap \{ v : (0, \infty) \to E_p : \sup_{t > 0} t^{(1-3/(2p))} \| v(t) \|_E < \infty \}. \hspace{1cm} (7.4)$$

This is the unique solution satisfying the condition $\sup_{t > 0} t^{(1-n/p)/2} \| v(t) \|_E \leq 2\varepsilon$.

In [15], global-in-time solutions to the Cauchy problem (2.1)–(2.2) are obtained using the standard argument involving the integral equation (2.3) and Lemma 4.1. The necessary estimate of the bilinear form $B(\cdot, \cdot)$ are derived directly from inequality (7.1) combined with the scaling property of the norm in $E_p$. Details are gathered in [15].

The main result on the large time behavior is contained in the following theorem.

**Theorem 7.1** ([15, Th. 6.1]) Let the above assumptions remain valid. Assume that $v$ and $\tilde{v}$ are two solutions of (2.1)–(2.2) corresponding to the initial data $v_0, \tilde{v}_0 \in BE_p^{1-3/p}$ and forces $F = \nabla \cdot V, \tilde{F} = \nabla \cdot \tilde{V}$, respectively. Suppose that

$$\lim_{t \to \infty} t^{(1-3/p)/2} \| S(t)(v_0 - \tilde{v}_0) \|_E + t^{(1-3/(2p))} \| V(t) + \tilde{V}(t) \|_E = 0.$$  

Then

$$\lim_{t \to \infty} t^{(1-3/p)/2} \| v(\cdot, t) - \tilde{v}(\cdot, t) \|_E = 0.$$
The following lemma plays an important role in the proof of Theorem 7.1.

**Lemma 7.1** Let \( w \in L^1(0, 1), w \geq 0, \) and \( \int_0^1 w(x) \, dx < 1 \). Assume that \( f \) and \( g \) are two nonnegative, bounded functions such that

\[
f(t) \leq g(t) + \int_0^1 w(\tau) f(\tau t) \, d\tau.
\]

(7.5)

Then \( \lim_{t \to \infty} g(t) = 0 \) implies \( \lim_{t \to \infty} f(t) = 0 \).

We refer the reader to [15] for the elementary proof of this lemma. Now, to show Theorem 7.1, we apply Lemma 7.1 with \( f(t) = t(1-3/p)/2 \|v(t) - \tilde{v}(t)\|_{E_p} \) and \( g(t) = t(1-3/p)/2 \|\mathcal{P}\nabla S(t-\tau) \cdot (V(\tau) - \tilde{V}(\tau))\|_{E_p} \).

Here, the estimates which appear in the proof of global-in-time-solutions to (2.1)–(2.2) play again the crucial role in our reasoning.

### 7.3 The mollified Navier-Stokes system

Here, the Banach space \( E \) is said to be adequate to the mollified problem (2.13)–(2.14) if it satisfies all the conditions from Definitions 7.1 and 7.2. If we recall the inequality

\[
\|\omega * f\|_E \leq \|w\|_1 \|f\|_E = \|f\|_E
\]

valid for every \( f \in E \), we immediately obtain global-in-time solutions to the mollified problem (2.13)–(2.14) in the space \( \mathcal{X} \) defined in (7.4) under suitable smallness assumptions on initial conditions and external forces. Here, it suffices only to repeat the reasoning from the previous subsection.

In the analysis of the large time asymptotics, however, we should impose an additional assumption on the adequate Banach space. To prove a counterpart of Theorem 2.3, we should guarantee that standard approximations of the Dirac delta converge in \( E_p \) (cf. (5.5)). It is well-known that this fact is valid if test functions are dense in \( E_p \).

**Theorem 7.2** Assume that the imbedding \( \mathcal{S} \subset E \) is dense. Let \( u_0 \in BE_p^{1-3/p} \) and \( V(t) \in E_p \) satisfy

\[
\|u_0\|_{BE_p^{1-3/p}} + \sup_{t>0} t^{-3/(2p)} \|V(t)\|_{E_p} < \varepsilon,
\]

where \( \varepsilon \) is sufficiently small constant. Suppose that \( u_0 \) is homogeneous of degree \(-1\) and \( F = \nabla \cdot V \) satisfies (2.17). Denote by \( u(x, t) = t^{-1/2}U(x/\sqrt{t}) \) and \( v(x, t) \) respectively the unique solutions to (2.1)–(2.2) and to (2.13)–(2.14), both corresponding to the same initial datum \( u_0 \) and external force \( F = \nabla \cdot V \). Then

\[
\lim_{t \to \infty} t^{(1-3/p)/2}\|u(\cdot, t) - w(\cdot, t)\|_{E_p} = 0.
\]

\( \square \)
7.4 The Navier-Stokes system with hyperdissipation

Recall first that solutions to the regularized Navier-Stokes system (2.15)–(2.16) satisfy the integral equation (6.1). If \( E \) is a functional translation invariant Banach space (cf. Definition 7.1), we have

\[
\|S_\ell(t)S(t)f\|_E \leq C_\ell\|S(t)f\|_E \tag{7.6}
\]

for all \( f \in E, \ t > 0 \), and a constant \( C_\ell \geq 1 \) defined in (6.2). Hence, every Banach space adequate to the Navier-Stokes system (2.1)–(2.2) is also adequate to the system with hyperdissipation (2.15)–(2.16). In other words, if inequality (7.1) holds true for the heat semigroup \( S(t) \) and a functional Banach space \( E \), it is also true for \( S(\tau) \) replaced by \( S_\ell(\tau)S(\tau) \) and \( \omega(\tau) \) replaced by \( C\omega(\tau) \). This implies that all estimates needed in the analysis of the Navier-Stokes system (2.1)–(2.2) remain true, if we replace the heat semigroup \( S(t) \) by \( S_\ell(t)S(t) \). One should remember, however, that constants in all inequalities may increase in such a new setting.

Now, we would like to compare solutions to the models (2.1)–(2.2) and (2.15)–(2.16) as \( t \to \infty \).

**Theorem 7.3** Assume that \( n\ell/2 < p \). Let \( u_0 \in BE_p^{1-3/p} \) and \( V(t) \in E_p \) satisfy

\[
\|u_0\|_{BE_p^{1-3/p}} + \sup_{t>0} t^{1-3/(2p)}\|V(t)\|_{E_p} < \varepsilon,
\]

where \( \varepsilon \) is sufficiently small constant. Denote by \( u(x,t) \) and \( w(x,t) \) respectively the unique solutions to (2.1)–(2.2) and to (2.15)–(2.16), both corresponding to the same initial datum \( u_0 \) and the external force \( F = \nabla \cdot V \). Then

\[
\lim_{t \to \infty} (1-3/p)/2 \|u(\cdot,t) - w(\cdot,t)\|_{E_p} = 0.
\]

Lemma 6.1, saying that the semigroup generated by the operator \( \Delta - (-\Delta)^{\ell/2} \) can be well-approximated in \( L^1 \) by the heat semigroup \( S(t) \), is again an important tool in the proof of this theorem. Using this fact we are able to derive an integral inequality of the form (7.5) for the function \( f(t) = t^{(1-3/p)/2}\|u(\cdot,t) - w(\cdot,t)\|_{E_p} \).

It is important in computations that the function \( u \) and \( w \) satisfy the integral equations (2.3) and (6.1), respectively. Finally, Lemma 7.1 completes the proof.

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