On One-dimensional Multi-Particle Diffusion Limited Aggregation

Allan Sly *

Abstract

We prove that the one dimensional Multi-Particle Diffusion Limited Aggregation model has linear growth whenever the particle density exceeds 1 answering a question of Kesten and Sidoravicius. As a corollary we prove linear growth in all dimensions $d$ when the particle density is at least 1.

1 Introduction

In the Diffusion Limited Aggregation (DLA) model introduced by Witten and Sanders [5] particles arrive from infinity and adhere to a growing aggregate. It produces beautiful fractal-like pictures of dendritic growth but mathematically it remains poorly understood. We consider a variant, multiparticle DLA, where the aggregate sits in an infinite Poisson cloud of particles which adhere when they hit the aggregate, a model which has been studied in both physics [4] and mathematics [2, 3]. Again one is interested in the growth of the aggregate and its structure.

In the model, initially there is a collection of particles whose locations are given by a mean $K$ Poisson initial density on $\mathbb{Z}^d$. The particles each move independently according to rate 1 continuous time random walks on $\mathbb{Z}^d$. We follow the random evolution of an aggregate $D_t \subset \mathbb{Z}^d$ where at time 0 an aggregate is placed at the origin $D_0 = \{0\}$ to which other particles adhere according the the following rule. When a particle at $v \notin D_t$ attempts to move onto the aggregate $D_t$ at time $t$, it stays in place and instead is added to the aggregate so $D_t = D_{t-} \cup \{v\}$ and the particle no longer moves. Any other particles at $v$ at the time are also frozen in place.

We will mainly focus on the one dimension setting and in Section 5 will discuss how to boost the results to higher dimensions. In this case the aggregate is simply a line segment and the processes on the positive and negative axes are independent so we simply restrict our attention to the rightmost position of the aggregate at time $t$ which we denote $X_t$.

In this case at time $t$ when a particle at $X_{t-} + 1$ attempts to take a step to the left it is incorporated into the aggregate along with any other particles.

It was proved by Kesten and Sidoravicius [2] that $X_t$ grows like $\sqrt{t}$ when $K < 1$. Indeed there simply are not enough particles around for it to grow faster. They conjectured, however, that when $K > 1$ then it should grow linearly. Our main result confirms this conjecture.

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*Princeton University and University of California, Berkeley. Supported by NSF grant DMS-1352013.
Email:asly@math.princeton.edu
Theorem 1 For all $K > 1$ the limit $\lim_{t \to 1} \frac{1}{t} X_t$ exists almost surely and is a positive constant.

We also give a simple extension of these results to higher dimensions and prove the following corollary.

Corollary 2 In all dimensions $d \geq 2$ when $K > 1$ the diameter of the aggregate grows linearly in $t$, that is for some positive constant $\delta > 0$

$$\lim_{t \to 1} \frac{1}{t} \text{Diam}(D_t) > \delta \text{ a.s.}$$

Previously Sidoravicius and Stauffer [3] studied the the case of $d \geq 2$ in a slightly different variant where particles instead perform a simple exclusion process. They showed that for densities close to 1, that there is a positive probability that the aggregate grows with linear speed. Also in Section 5 we describe how for $d \geq 2$ the upper bound on the threshold can be reduced further below 1, for example to $\frac{5}{6}$ when $d = 2$. However, strikingly Eldan [1] conjectured that the critical value is always 0, that is the aggregate grows with linear speed for all $K > 0$. We are inclined to agree with this conjecture but our methods do not suggest a way of reaching the threshold. A better understanding of the growth of the standard DLA seems to be an important starting point.

2 Basic results

We will analyse the function valued process $Y_t$ given by,

$$Y_t(s) := \begin{cases} X_t - X_{t-s} & 0 \leq s \leq t \\ \infty & s > t. \end{cases}$$

(1)

Let $\mathcal{F}_t$ denote the filtration generated by $X_t$. We let $S(t)$ denote the infinitesimal rate at which $X_t$ increases given $\mathcal{F}_t$. Given $\mathcal{F}_t$ the number of particles at $X_t + 1$ is conditionally Poisson with intensity given by the probability that a random walker at $X_t + 1$ at time $t$ was never located in the aggregate. Each of the particles jumps to the left at rate $\frac{1}{2}$ so with $W_t$ denoting an independent continuous time random walk,

$$S(t) = \frac{1}{2} K \mathbb{P}[\max_{0 \leq s \leq t} W_s - Y_t(s) \leq 0 \mid Y_t].$$

Note that $S(t)$ is an increasing as a function of $Y_t$. Indeed we could realise $X_t$ as follows, let $\Pi$ be a Poisson process on $[0, \infty)^2$ and then

$$X_t = \Pi(\{(x, y) : 0 \leq x \leq t, 0 \leq y \leq S(x)\}).$$

Since both $X_t$ and $Y_t$ are increasing functions of $\Pi$ we can make use of the FKG property. Also note that $Y_t$ is stochastically decreasing.

Most of our analysis will involve estimating $S(t)$ and using that to show that $Y_t$ does not become too small for too long. Let $M_t = \max_{0 \leq s \leq t} W_s$ be the maximum process of $W_t$.

Lemma 2.1 For any $i \geq 0$ we have that

$$S(t) \geq \frac{K}{2} \mathbb{P}[M_{2^i} = 0] \prod_{i' = i}^{\infty} \mathbb{P}[M_{2^{i'+1}} \leq Y_t(2^{i'} \mid Y_t]$$
Proof. We have
\[ S(t) \geq K_2 \mathbb{P}[\max_{0 \leq s \leq t} W_s - Y_t(s) \leq 0 \mid Y_t] \]
\[ \geq \frac{K}{2} \mathbb{P}[M_{2i} = 0, \forall i' \geq i \ M_{2i'+1} \leq Y_t(2^{i'}) \mid Y_t] \]
\[ \geq \frac{K}{2} \mathbb{P}[M_{2i} = 0] \prod_{i' \geq i} \mathbb{P}[M_{2i'+1} \leq Y_t(2^{i'}) \mid Y_t] \]
where the final inequality follows from the FKG inequality. \(\square\)

By the reflection principle we have that for any integer \(j \geq 0\),
\[ \mathbb{P}[M_t \geq j] = \mathbb{P}[W_t \geq j] + \mathbb{P}[W_t \geq j + 1]. \]
Thus asymptotically we have that
\[ \mathbb{P}[M_t = 0] \approx \frac{1}{\sqrt{2\pi t}} t^{-1/2} \quad (2) \]

Now let \(T_j\) be the first hitting time of \(j\). Since \(\cosh(s) - 1 \leq s^2\) for \(0 \leq s \leq 1\) we have that for \(t \geq 1\),
\[ \mathbb{E}[e^{\frac{1}{\sqrt{t}} W_t \vee T_j}] \leq \mathbb{E}[e^{\frac{1}{\sqrt{t}} W_t}] = e^{(\cosh(\frac{1}{\sqrt{t}}) - 1)t} \leq e^1, \]
and hence by Markov’s inequality
\[ \mathbb{P}[M_t \geq jt^{1/2}] \leq \mathbb{P}[e^{\frac{1}{\sqrt{t}} W_t \vee T_j} = e^j] \leq e^{-j}. \quad (3) \]

Plugging the above equations into Lemma 2.1 we get the following immediate corollary.

Corollary 2.2 There exists \(i^*\) such that the following holds. Suppose that \(i \geq i^*\) that for all \(i' \geq i\) we have \(j_{i'} = Y_t(2^{i'})2^{-i'/2}\). Then
\[ S(t) \geq K \frac{2^{-i'/2} \prod_{i' = i}^{\infty} (1 - e^{1 - \max\{1,j_{i'}/\sqrt{2}\}}) }{10^{2-i'/2}} \]

Next we check that provided \(S(t)\) remains bounded below during an interval then we get a comparable lower bound on the speed of \(X_t\).

Lemma 2.3 We have that for all \(\rho \in (0,1)\) there exists \(\psi(\rho) > 0\) such that for all \(\Delta > 0\),
\[ \mathbb{P}[\min_{s \in [t, t+\Delta]} S(s) \geq \gamma, X_{t+\Delta} - X_t \leq \rho \Delta \gamma \mid Y_t] \leq \exp(-\psi(\rho) \Delta \gamma) \]
In the case of \(\rho = \frac{1}{2}\) we have \(\psi(\rho) \geq \frac{1}{16}\).

Proof. Using the construction of the process in terms of \(\Pi\) we have that
\[ \mathbb{P}[\min_{s \in [t, t+\Delta]} S(s) \geq \gamma, X_{t+\Delta} - X_t \leq \rho \Delta \gamma \mid Y_t] \leq \mathbb{P}[\Pi([t, t+\Delta] \times [0, \gamma]) \leq \rho \Delta \gamma] \]
\[ = \mathbb{P}[\text{Poisson}(\Delta \gamma) \leq \rho \Delta \gamma] \]
Now if $N \sim \text{Poisson}(\Delta \gamma)$ then $\mathbb{E} e^{-\theta N} = \exp((e^{-\theta} - 1)\Delta \gamma)$ and so by Markov’s inequality
\[
\mathbb{P}[N \leq \rho \Delta \gamma] = \mathbb{P}[e^{-\theta N} \geq e^{-\theta \rho \Delta \gamma}] \leq \frac{\exp((e^{-\theta} - 1)\Delta \gamma)}{\exp(-\theta \rho \Delta \gamma)} = \exp((\theta \rho + e^{-\theta} - 1)\Delta \gamma).
\]

Setting $f_\rho(\theta) = -(\theta \rho + e^{-\theta} - 1)$ and
\[
\psi(\rho) = \sup_{\theta \geq 0} f_\rho(\theta)
\]

it remains to check that $\psi(\rho) > 0$. This follows from the fact that $f_\rho(0) = 0$ and $f'_\rho(0) = 1 - \rho > 0$. Since $f_\rho(\frac{1}{2}) \geq \frac{1}{10}$ we have that $\psi(\frac{1}{2}) \geq \frac{1}{10}$.

\section{Proof of Positive Speed}

To measure our control over $Y_t$ and show that it is moving quickly enough we say that $Y_t$ is permissive at time $t$ and at scale $i$ if $Y_t(2^i) \geq 10i^{2i/2}$. Our approach, will be to consider functions
\[
y_\alpha(s) = \begin{cases} 
0 & s \leq \alpha^{-3/2} \\
\min\{\alpha(s - \alpha^{-3/2}), s^{1/2} \log_2 s\} & s \geq \alpha^{-3/2}.
\end{cases}
\]

and show that if $Y_t(s) \geq y_\alpha(s)$ for increasing values of $\alpha$ with good probability. To measure the speed of the aggregate in an interval of time define events $\mathcal{R}$ as
\[
\mathcal{R}(t, s, \gamma) = \{X_{t+s} - X_t \geq \gamma s\}.
\]

\begin{lemma}
For all $\epsilon > 0$ there exists $0 < \alpha_*(\epsilon) \leq 1$ such that for all $0 < \alpha < \alpha_*$,
\[
\mathbb{P}[\max_{s \geq 0} W_s - y_\alpha((s - \alpha^{-4/3}) \wedge 0) \leq 0] \geq 2(1 - \epsilon)\alpha.
\]
\end{lemma}

\begin{proof}
For small $\alpha_*(\epsilon)$ we have that for $\alpha^{-3/2} \leq s \leq \alpha^{-2},$
\[
\alpha(s - \alpha^{-4/3} - \alpha^{-3/2}) \leq (s - \alpha^{-4/3})^{1/2} \log_2(s - \alpha^{-4/3})
\]

Hence with $\xi = \xi_\alpha = \alpha^{4/3} + \alpha^{-3/2}$ if we set
\[
\mathcal{A} = \{\max_{s \geq 0} W_s - \alpha((s - \xi) \wedge 0) \leq 0\}
\]

and
\[
\mathcal{B} = \{\max_{s \geq \alpha^{-2}} W_s - (s - \alpha^{-4/3})^{1/2} \log_2(s - \alpha^{-4/3}) \leq 0\}
\]

then
\[
\mathbb{P}[\max_{s \geq 0} W_s - y_\alpha((s - \alpha^{-4/3}) \wedge 0) \leq 0] \geq \mathbb{P}[\mathcal{A}, \mathcal{B}] \geq \mathbb{P}[\mathcal{A}]\mathbb{P}[\mathcal{B}],
\]

where the second inequality follows by the FKG inequality since $\mathcal{A}$ and $\mathcal{B}$ are both decreasing events for $W_s$. For large $s$, we have $s^{1/2} \log_2 s \leq 2(s/2)^{1/2} \log_2(s/2)$ and so
\[
\mathbb{P}[\mathcal{B}] \geq \mathbb{P}[\max_{s \geq \alpha^{-2}} W_s - \frac{1}{2}s^{1/2} \log_2(\frac{s}{2}) \leq 0]
\]
\[
\geq \mathbb{P}[\forall i \geq \lceil \log_2(\alpha^{-2}) \rceil M_{2i+1} \leq \frac{1}{2}(i - 1)2^{i/2}]
\]
\[
\geq \prod_{i \geq \lceil \log_2(\alpha^{-2}) \rceil} \mathbb{P}[M_{2i+1} \leq \frac{1}{2}(i - 1)2^{i/2}]
\]
\[
\geq \prod_{i \geq \lceil \log_2(\alpha^{-2}) \rceil} e^{1 - (i - 1)2^{-3/2}}
\]
where the third inequality follows from the FKG inequality and the final inequality is by equation (3). Thus as \( \alpha \to 0 \) we have that \( \mathbb{P}[\mathcal{A}] \to 1 \) so it is sufficient to show that for small enough \( \alpha \) that \( \mathbb{P}[\mathcal{A}] \geq 2\alpha(1 - \epsilon/2) \). By the reflection principle for \( a \leq 1 \),
\[
\mathbb{P}[M_t \geq 1, W_t = a] = \mathbb{P}[M_t \geq 1, W_t = 2 - a] = \mathbb{P}[W_t = a - 2],
\]
and so
\[
\mathbb{P}[M_t \geq 1] = \sum_{a > 1} \mathbb{P}[M_t \geq 1, W_t = a] + \sum_{a \leq 1} \mathbb{P}[M_t \geq 1, W_t = a] = \sum_{a > 1} \mathbb{P}[M_t \geq 1, W_t = a] + \sum_{a \leq 1} \mathbb{P}[W_t = a - 2] = 1 - \mathbb{P}[W_t = 0] - \mathbb{P}[W_t = 1].
\]
Hence by the Local Central Limit Theorem,
\[
\lim_t \sqrt{t} \mathbb{P}[M_t = 0] = \lim_t \sqrt{t} (\mathbb{P}[W_t = 0] + \mathbb{P}[W_t = 1]) = \frac{2}{\sqrt{2\pi}}.
\]
Also we have for \( a \leq 0 \),
\[
\mathbb{P}[W_t = a, M_t = 0] = \mathbb{P}[W_t = a] - \mathbb{P}[W_t = a, M_t \geq 1] = \mathbb{P}[W_t = a] - \mathbb{P}[W_t = a - 2]
\]
and so the law of \( W_t \) conditioned on \( M_t = 0 \) satisfies,
\[
\lim_t \mathbb{P}\left[ \frac{1}{\sqrt{t}} W_t \leq x \mid M_t = 0 \right] = \lim_t \frac{\sum_{a = -\infty}^{x\sqrt{t}} \mathbb{P}[W_t = a] - \mathbb{P}[W_t = a - 2]}{\mathbb{P}[M_t = 0]} = \lim_t \frac{\mathbb{P}[W_t = x\sqrt{t}] + \mathbb{P}[W_t = x\sqrt{t} - 1]}{\mathbb{P}[M_t = 0]} = \lim_t \frac{2\sqrt{2\pi} e^{-x^2/2}}{\sqrt{2\pi}} = e^{-x^2/2}
\]
where \( x \leq 0 \) and hence is the negative of the Rayleigh distribution. Now let \( Z_t = W_t - \alpha t \) and \( U_t = e^{\alpha Z_t} \). Then
\[
\mathbb{E} U_t = \exp((\cosh(\theta) - 1 - \alpha \theta) t).
\]
As \( f_\alpha'(\theta) = \cosh(\theta) - 1 - \alpha \theta \) is strictly convex, it has two roots, one of which is at \( \theta = 0 \). Let \( \theta_\alpha \) be the non-zero root of \( f_\alpha' \). Since
\[
f_\alpha'(\theta) = -\alpha \theta + \frac{1}{2} \theta^2 + O(\theta^4)
\]
for small \( \alpha \) we have that \( \theta_\alpha = 2\alpha + O(\alpha^2) \). Then with \( \theta = \theta_\alpha \) we have that \( U_t = e^{\theta_\alpha Z_t} \) is a martingale. Let \( T = \min_t Z_t > 0 \) and so by the Optional Stopping Theorem,
\[
\mathbb{E}[U_T \mid Z_T = z] = \mathbb{E}[U_0 \mid Z_T = z] = e^{\theta_\alpha z}.
\]
Also since \( U_T \in [0, 1] \) if \( T < \infty \) so
\[
\mathbb{E}[U_T \mid Z_T = z] \geq \mathbb{P}[T < \infty \mid Z_T = z]
\]
so
\[ P[T < \infty \mid Z_T = z] \leq e^{-\theta_\alpha}. \]

Thus we have that as \( \alpha \to 0 \),

\[
P[\mathcal{A}] = P[\max_{s \geq 0} W_s - \alpha((s - \xi) \wedge 0) \leq 0] \\
= \sum_{x = -\infty}^{0} P[M_\xi = 0, W_\xi = x] P[T = \infty \mid Z_T = z] \\
\geq P[M_\xi = 0] \sum_{x = -\infty}^{0} P[W_\xi = x \mid M_\xi = 0](1 - e^{\theta_\alpha x}) \\
\geq \frac{2 + o(1)}{\sqrt{2\pi t}} \sum_{x = -\infty}^{0} P[W_\xi = x \mid M_\xi = 0](-2\alpha x) \\
\rightarrow \frac{4\alpha}{\sqrt{2\pi t}} \sqrt{\frac{\pi}{2}} = 2\alpha,
\]
since the mean of the Rayleigh distribution is \( \sqrt{\frac{\pi}{2}} \). This completes the lemma. \( \square \)

**Lemma 3.2** For all \( K > 1 \) there exists \( i_*(K) \) such that if \( i \geq i_* \) and \( Y_T \) is permissive at all levels \( i \) and above then with

\[ \alpha = \frac{1}{80}2^{-i/2} \]

we have that

\[ P[\inf_{s \geq 0} Y_{T+2^{i'}}(s) - y_\alpha(s) \geq 0 \mid Y_T] \geq 1 - \exp(-2^{i/10}). \]

**Proof.** Since \( Y_T(2^{i'}) \geq 10i'2^{i'/2} \) for all \( i' \geq i \) if we set

\[ \tilde{y}(s) = \begin{cases} 
0 & s < 2^{i+1}, \\
10j2^{j/2} & s \in [2^j,2^{j+2}), j \geq i.
\end{cases} \]

then since \( Y_{T+u}(s) \geq Y_T(s - u) \) then

\[ \inf_{0 \leq u \leq 2^i} \inf_{s \geq 0} Y_{T+u}(s) - \tilde{y}(s) \geq 0. \] (4)

By Corollary 2.2 for all \( t \in [0,2^i] \)

\[ S(t) \geq \frac{1}{10}2^{-(i+1)/2} \prod_{i' = i}^{\infty} (1 - e^{\max\{1.5(i'-1)\}}) \geq \frac{1}{20}2^{-i/2}, \]

where the second inequality holds provided that \( i_*(K) \) is sufficiently large. Defining \( \mathcal{D} \) as the event that \( X_t \) moves at rate at least \( \frac{1}{40}2^{-i/2} \) for each interval \( \ell 2^{2i/3}, (\ell + 1)2^{2i/3} \),

\[ \mathcal{D} = \bigcap_{\ell = 0}^{2^{i/3} - 1} \mathcal{R}(T + \ell 2^{2i/3}, 2^{2i/3}, \frac{1}{40}2^{-i/2}) \]

by Lemma 2.3 we have that

\[ P[\mathcal{D}] \geq 1 - 2^{i/3} \exp(-\frac{1}{10} \cdot 2^{2i/3} \cdot \frac{1}{40}2^{-i/2}) \geq 1 - \exp(-2^{i/10}) \]
where the last inequality holds provided that \( i_*(K) \) is sufficiently large. We claim that on the event \( \mathcal{D} \), we have that \( Y_{T+2^i}(s) \geq y_\alpha(s) \) for all \( s \). For \( s \geq 2^{i+1} \) this holds since by equation (4) we have that

\[
Y_{T+2^i}(s) \geq \tilde{y}(s) \geq s^{1/2} \log_2 s \geq y_\alpha(s).
\]

For \( 0 \leq s < 2^i \), on the event \( \mathcal{D} \),

\[
Y_{T+2^i}(s) \geq [s2^{-2i/3}]2^{2i/3} \frac{1}{40} 2^{-i/2} \geq \max\{0, s - \alpha^{-3/2}\} \frac{1}{40} 2^{-i/2} \geq y_\alpha(s),
\]

and for \( 2^i \leq s \leq 2^{i+1} \)

\[
Y_{T+2^i}(s) \geq Y_{T+2^i}(2^i) \geq 2^i \cdot \frac{1}{40} 2^{-i/2} \geq y_\alpha(2^{i+1}).
\]

Thus for all \( s \geq 0 \), \( Y_{T+2^i}(s) \geq y_\alpha(s) \) which completes the proof. \( \square \)

**Lemma 3.3** For all \( K > 1 \), there exists \( \Delta(K) \) and \( \chi(K) > 0 \) such that if \( 0 \leq \alpha \leq \Delta \) and \( \inf_s Y_T(s) - y_\alpha(s) = 0 \) then

\[
\mathbb{P}\left[ R(T, \alpha^{-4/3}, \frac{\alpha(K + 1)}{2})^c \mid Y_T \right] \leq \exp \left( -\chi(K)\alpha^{-1/3} \right).
\]

**Proof.** With \( \alpha_*(\epsilon) \) defined as in Lemma 3.1 set \( \Delta(K) = \alpha_*(\frac{K-1}{3K}) \). Then for \( 0 \leq s \leq \alpha^{-4/3} \)

\[
S(T + t) = \frac{K}{2} \mathbb{P}\left[ \max_{0 \leq s \leq t} W_s - Y_{T+t}(s) \leq 0 \mid Y_{T+t} \right] \geq \frac{K}{2} \mathbb{P}\left[ \max_{s \geq 0} W_s - y_\alpha((s - \alpha^{-4/3}) \wedge 0) \leq 0 \right] \geq \frac{K}{2} 2^{(1 - \frac{K-1}{3K})\alpha} = \frac{\alpha(2K + 1)}{3}
\]

where the first inequality follows from the fact that

\[
Y_{T+t}(s) \geq Y_T(s - \alpha^{-4/3}) \wedge 0) \geq y_\alpha(s - \alpha^{-4/3}) \wedge 0)
\]

and the second inequality follows from Lemma 3.1. Now take \( \rho = \frac{3K+3}{4K+2} < 1 \) and with \( \psi \) defined in Lemma 2.3 set \( \chi(K) = \psi(\rho) \). Then since

\[
\inf_{0 \leq t \leq \alpha^{-4/3}} S(T + t) \geq \frac{\alpha(2K + 1)}{3} = \rho \frac{\alpha(K + 1)}{2}
\]

by Lemma 2.3 we have that

\[
\mathbb{P}\left[ R(T, \alpha^{-4/3}, \frac{\alpha(K + 1)}{2})^c \mid Y_T \right] \leq \exp \left( -\chi(K)\alpha^{-1/3} \right).
\]

\( \square \)

This result is useful because of the following claim.

**Claim 3.4** For some \( 0 \leq \alpha \leq \frac{1}{2} \) suppose that \( \inf_s Y_T(s) - y_\alpha(s) = 0 \). Then for an \( 0 \leq t \leq \alpha^{-3/2} \) and \( \gamma \geq 1 \) on the event \( \mathcal{R}(T, t, \alpha \gamma) \) we have that \( \inf_s Y_{T+t}(s) - y_\alpha(s) = 0 \).
Lemma 3.5 For all \( y_\alpha(s) = 0 \leq s \leq \alpha^{-3/2} \) it is sufficient to check \( s \geq \alpha^{-3/2} \). Then

\[
Y_{T+t}(s) = Y_T(s-t) + X_{T+t} - X_t
\geq Y_T(s-t) + \alpha \gamma t
\geq y_\alpha(s-t) + \alpha \gamma t
\geq y_\alpha(s) - \alpha t + \alpha \gamma t \geq y_\alpha(t),
\]

where the first inequality is by the event \( \mathcal{R}(T, t, \alpha) \), the second is by assumption and the third is since \( \frac{d}{ds} y_\alpha(s) \) is uniformly bounded above by \( \alpha \).

Proof. Let \( \mathcal{D}_\ell \) denote the event,

\[
\mathcal{D}_\ell = \mathcal{R}(T + \ell \alpha^{-4/3}, \alpha^{-4/3}, \frac{\alpha(K+1)}{2}).
\]

By Claim 3.4 and induction if \( \bigcap_{\ell'=0}^{\ell-1} \mathcal{D}_{\ell'} \) holds then \( \inf_s Y_{T+\ell \alpha^{-4/3}}(s) - y_\alpha(s) = 0 \). Thus by Lemma 3.3 we have that

\[
P\left[ \inf_s Y_{T+\alpha^{-3}}(s) - y_{\alphaK} \bigg| Y_T \right] \leq \alpha^{-5/3} \exp\left( -\chi(K)\alpha^{-1/3} \right).
\]

Lemma 3.5 For all \( K > 1 \), there exists \( \Delta(K) \) and \( \chi(K) > 0 \) such that if \( 0 \leq \alpha \leq \Delta \) and \( \inf_s Y_T(s) - y_\alpha(s) = 0 \) then

\[
P\left[ \inf_s Y_{T+\alpha^{-3}}(s) - y_{\alphaK} \bigg| Y_T \right] \leq \alpha^{-5/3} \exp\left( -\chi(K)\alpha^{-1/3} \right).
\]

Proof. Let \( \mathcal{D}_\ell \) denote the event,

\[
\mathcal{D}_\ell = \mathcal{R}(T + \ell \alpha^{-4/3}, \alpha^{-4/3}, \frac{\alpha(K+1)}{2}).
\]

By Claim 3.4 and induction if \( \bigcap_{\ell'=0}^{\ell-1} \mathcal{D}_{\ell'} \) holds then \( \inf_s Y_{T+\ell \alpha^{-4/3}}(s) - y_\alpha(s) = 0 \). Thus by Lemma 3.3 we have that

\[
P\left[ \mathcal{D}_\ell \bigg| \bigcap_{\ell'=0}^{\ell-1} \mathcal{D}_{\ell'}, Y_T \right] \geq 1 - \exp\left( -\chi(K)\alpha^{-1/3} \right)
\]

and so with \( \mathcal{D}^* = \bigcap_{\ell=0}^{\alpha^{-5/3}-1} \mathcal{D}_\ell \),

\[
P\left[ \mathcal{D}^* \bigg| Y_T \right] \geq 1 - \alpha^{-5/3} \exp\left( -\chi(K)\alpha^{-1/3} \right).
\]

Now suppose that the event \( \mathcal{D}^* \) holds and assume that \( \Delta(K) \) is small enough so that for all \( 0 \leq \alpha \leq \Delta(K) \) the following hold:

- \( \alpha^{-4/3} \leq \left( \frac{\alpha(K+1)}{2} \right)^{-3/2} \),
- \( \frac{K+1}{2} \alpha^{-2} \geq (2 \alpha^{-3})^{1/2} \log_2(2 \alpha^{-3}) \),
- \( \forall s \geq \alpha^{-3}, \min\{\alpha(s - \alpha^{-3/2}), s^{1/2} \log_2 s\} = s^{1/2} \log_2 s \),
- \( \forall s \geq \alpha^{-3}, \min\left\{ \frac{\alpha(K+1)}{2} (s - \left( \frac{\alpha(K+1)}{2} \right)^{-3/2}), s^{1/2} \log_2 s \right\} = s^{1/2} \log_2 s \),
- \( \inf_{s \geq 2 \alpha^{-3}} - s^{1/2} \log_2 s + (s - \alpha^{-3})^{1/2} \log_2 (s - \alpha^{-3}) + \frac{K+1}{2} \alpha^{-2} \geq 0 \).

It is straightforward to check that all of these hold for sufficiently small \( \alpha \). For all \( \left( \frac{\alpha(K+1)}{2} \right)^{-3/2} \leq s \leq \alpha^{-3} \) that

\[
Y_{T+\alpha^{-3}}(s) \geq \left[ 8 \alpha^{-4/3} \right] \alpha^{-4/3} \cdot \frac{\alpha(K+1)}{2} \geq \left( s - \left( \frac{\alpha(K+1)}{2} \right)^{3/2} \right) \frac{\alpha(K+1)}{2} \geq y_{\alphaK}(s).
\]
For $\alpha^{-3} \leq s \leq 2\alpha^{-3}$,
\[
Y_{T+\alpha^{-3}}(s) \geq Y_T(s - \alpha^{-3}) + \frac{K + 1}{2} \alpha^{-2} \\
\geq y_\alpha(s - \alpha^{-3}) + \frac{K + 1}{2} \alpha^{-2} \\
\geq (2\alpha^{-3})^{1/2} \log_2(2\alpha^{-3}) = y_{\alpha(K+1)}(2\alpha^{-3}).
\]

Finally, for $s \geq 2\alpha^{-3}$,
\[
Y_{T+\alpha^{-3}}(s) \geq y_\alpha(s - \alpha^{-3}) + \frac{K + 1}{2} \alpha^{-2} \\
= y_{\alpha(K+1)}(s) - s^{1/2} \log_2 s + (s - \alpha^{-3})^{1/2} \log_2(s - \alpha^{-3}) + \frac{K + 1}{2} \alpha^{-2} \\
\geq y_{\alpha(K+1)}(s).
\]

Combining the previous 3 equations implies that $Y_{T+\alpha^{-3}}(s) \geq y_{\alpha(K+1)}(s)$ for all $s$ and hence
\[
P\left[ \inf_y Y_{T+\alpha^{-3}}(s) - y_{\alpha(K+1)}(s) = 0 \mid Y_T \right] \leq P[\mathcal{D}^*] \leq \alpha^{-5/3} \exp\left(-\chi(K)\alpha^{-1/3}\right).
\]

\[\blacksquare\]

**Lemma 3.6** For all $K > 1$, there exists $i^*(K)$ such that the following holds. If $i \geq i^*$ and $Y_T$ is permissive for all $i' > i$ then
\[
P\left[ \min_{s \in [2^i, 2e^{2i/10}]} Y_{T+s}(2^i) \leq 10i^{2i/2} \mid \mathcal{F}_T \right] \leq 3e^{-2^{i/10}},
\]
that is $Y_{T+s}$ is permissive at scale $i$ for all $s \in [2^i, 2e^{2i/10}]$.

**Proof.** We choose $i^*(K)$ large enough so that,
\[
20i^*K2^{-i^*/2} \leq \Delta(K)
\]
where $\Delta(K)$ was defined in 3.5. Set $t_0 = 2^i$ and $\alpha_0 = \frac{1}{80}2^{-i/2}$. We define $\alpha_\ell = (K+1)^{\ell} \alpha_0$ and $t_\ell = t_{\ell-1} + \alpha_\ell^{-3}$. Define the event $\mathcal{W}_\ell$ as
\[
\mathcal{W}_\ell = \left\{ \inf_y Y_{T+t_\ell}(s) - y_{\alpha_\ell}(s) = 0 \right\}.
\]

By Lemma 3.5, we have that
\[
P[\mathcal{W}_0 \mid \mathcal{F}_T] \geq 1 - \exp(-2^{i/10}),
\]
and by Lemma 3.5, we have that
\[
P\left[ \mathcal{W}_\ell \mid \bigcap_{\ell'=0}^{\ell-1} \mathcal{W}_{\ell'} \mid \mathcal{F}_T \right] \geq 1 - \alpha_\ell^{-5/3} \exp\left(-\chi(K)\alpha_\ell^{-1/3}\right).
\]
Now choose $L$ to be the smallest integer such that $\alpha_L \geq 20i2^{-i/2}$. So $L = \lceil \log(1600i)/\log((K+1)/2) \rceil$ which is bounded above by $i$ provided that $i^*(K)$ is sufficiently large. Thus

$$
\begin{align*}
\mathbb{P}[\mathcal{W}_L \mid \mathcal{F}_T] &\geq 1 - \exp(-2^{i/10}) - \sum_{\ell=0}^{L-1} \alpha_{\ell-1}^{-5/3} \exp \left( - \chi(K) \alpha_{\ell-1}^{-1/3} \right) \\
&\geq 1 - \exp(-2^{i/10}) - i(20i2^{-i/2})^{-5/3} \exp \left( - \chi(K) (20i2^{-i/2})^{-1/3} \right) \\
&\geq 1 - 2 \exp(-2^{i/10})
\end{align*}
$$

where the final inequality holds for $i$ is sufficiently large. Now let $\mathcal{D}_k$ denote the event,

$$
\mathcal{D}_k = \mathcal{B}(T + t_L + k\alpha_L^{-4/3}, \alpha_L^{-4/3}, \alpha_L).
$$

By Claim 3.4 on the event $\mathcal{W}_L$ and $\bigcap_{k'=0}^{k-1} \mathcal{D}_{k'}$ we have

$$
\inf_s Y_{T+t_L+k\alpha_L^{-4/3}}(s) - y\alpha_L(s) = 0.
$$

Thus by Lemma 3.3 we have that

$$
\mathbb{P}[\mathcal{D}_k \mid \mathcal{F}_T, \mathcal{W}_L, \bigcap_{k'=0}^{k-1} \mathcal{D}_{k'}] \geq 1 - \exp \left( - \chi(K) \alpha_L^{-1/3} \right).
$$

Let $\mathcal{D}^*$ be the event

$$
\mathcal{D}^* = \left\{ \mathcal{W}_L, \bigcap_{k'=0}^{e^{2i/10}-1} \mathcal{D}_{k'} \right\}.
$$

Then for $i$ sufficiently large since $\alpha_L \leq 20iK2^{-i/2}$,

$$
\mathbb{P}[\mathcal{D}^* \mid \mathcal{F}_T] \geq 1 - 2 \exp(-2^{i/10}) - \exp \left( 2^{i/10} - \chi(K) \alpha_L^{-1/3} \right) \geq 1 - 3 \exp(-2^{i/10}).
$$

One the event $\mathcal{D}^*$ we have that for all $t_L + 2^i \leq s \leq \alpha_L^{-4/3} e^{2i/10}$ that

$$
Y_{T+s} \geq \alpha_L(2^i - \alpha_L^{-4/3}) \geq 10i2^i/2.
$$

By construction $t_L = 2^i + \sum_{\ell=0}^{L-1} \alpha_{\ell}^{-3} \leq 4^i$ and hence

$$
\mathbb{P} \left[ \min_{s \in [4^i, 2e^{2i/10}]} Y_{T+s}(2^i) \leq 10i2^{i/2} \mid \mathcal{F}_T \right] \leq \mathbb{P}[\mathcal{D}^*^c \mid \mathcal{F}_T] \leq 3e^{-2^{i/10}}.
$$

\[\square\]

**Corollary 3.7** For all $K > 1$, there exists $i^*(K)$ such if $i \geq i^*$ then

$$
\mathbb{P} \left[ \min_{s \in [0, e^{2i/10}]} Y_s(2^i) \leq 10i2^{i/2} \right] \leq 3e^{-2^{i/10}},
$$

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Proof. We can apply Lemma 3.6 to time $T = 0$ since it is permissive at all levels and hence have that
\[ P \left[ \min_{s \in [4^i, 2e^{2i/10}]} Y_s(2^i) \leq 10i2^i/2 \right] \leq 3e^{-2^{i/10}}, \]
Since $Y_t$ is stochastically decreasing in $t$ we have that
\[ P \left[ \min_{0 \leq t \leq e^{2i/10}} Y_t(2^i) \leq 10i2^i/2 - i2^i \right] \leq P \left[ \min_{s \in [4^i, 4^i + e^{2i/10}]} Y_s(2^i) \leq 10i2^i/2 \right] \leq 3e^{-2^{i/10}}, \]
which completes the corollary. □

Lemma 3.8 For all $K > 1$, there exists $i^*(K)$ such that
\[ \inf_t P \left[ \forall i \geq i^*, Y_t(2^i) \geq 10i2^i/2 \right] \geq \frac{1}{2}. \]
Proof. Take $i^*(K)$ as in Lemma 3.6 and suppose that $I \geq i^*$. Let $\mathcal{D}_I$ denote the event that $Y_t$ is permissive for all levels $i \geq I$ and all $t \in [0, e^{2i/10}]$. By Corollary 3.7 we have that
\[ P[\mathcal{D}_I] \leq \sum_{i \geq I} 3e^{-2^{i/10}} \leq 4e^{-2^{i/10}}. \]
Next set $t_0 = \frac{1}{2}e^{2i/10}$ and let $t_k = t_{k-1} + 4^{I-k}$. Let $\mathcal{H}_k$ denote the event that $Y_t$ is permissive at level $I - k$ for all $t \in [t_k, t_k + e^{2(I-k)/10}]$. By Lemma 3.6 then for $0 \leq k \leq I - i^*$,
\[ P[\mathcal{H}_k^c, \cap_{k'=1}^{k-1} \mathcal{H}_{k'}, \mathcal{D}_{i^*}] \leq 3e^{-2(I-k)/10}. \]
Thus, provided $i^*$ is large enough,
\[ P[\cap_{k'=1}^{i^*} \mathcal{H}_{k'}, \mathcal{D}_{i^*}] \geq 1 - 4e^{-2^{i/10}} - \sum_{k'=1}^{I-i^*} 3e^{-2(I-k)/10} \geq \frac{1}{2}. \]
Let $\tau = \tau_I = t_{I-i^*}$. Then for all $I \geq i^*$,
\[ P \left[ \forall i \geq i^*, Y_{\tau_I}(2^i) \geq 10i2^i/2 \right] \geq \delta. \]
since $Y_t$ is stochastically decreasing in $t$ and $\tau_I \to \infty$ and $I \to \infty$,
\[ \inf_t P \left[ \forall i \geq i^*, Y_t(2^i) \geq 10i2^i/2 \right] \geq \frac{1}{2}. \]

Theorem 3.9 For $K > 1$ there exists a random function $Y^*(s)$ such that $Y_t$ converges weakly to $Y^*$ in finite dimensional distributions. Furthermore, with
\[ \alpha^* = \frac{K}{2} \mathbb{E} \left[ \mathbb{P} \left[ \max_{0 \leq s \leq t} W_s - Y^*(s) \leq 0 \mid Y^* \right] \right], \]
we have that $\frac{1}{t}X_t$ converges in probability to $\alpha^* > 0$. 

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Proof. Since $Y_t$ is stochastically decreasing it must converge in distribution to some limit $Y^*$. By Claim 2.2

$$\Pr \left[ \frac{K}{2} \Pr \left[ \max_{0 \leq s \leq t} W_s - Y^*(s) \leq 0 \mid Y^* \right] \geq \frac{K}{10} 2^{-i/2} \prod_{i=i^*}^{\infty} \left( 1 - e^{1 - \max\{1,10^i/\sqrt{2}\}} \right) \right] \geq \frac{1}{2},$$

and so $\alpha^* = \lim_t \mathbb{E}S(t) > 0$. To show convergence in probability fix $\epsilon > 0$. For some large enough $L$,

$$\mathbb{E}\left[ \frac{1}{L} X_L \right] = \frac{1}{L} \int_0^L \mathbb{E}S(t) dt \leq \alpha^* + \epsilon/2.$$

Let $N_k = \mathbb{E}[X_{kL} - X_{(k-1)L} \mid \mathcal{F}_{(k-1)L}]$ and $R_k = X_{kL} - X_{(k-1)L} - N_k$. By monotonicity

$$N_k \leq \mathbb{E}\left[ \frac{1}{L} X_L \right] \leq \alpha^* + \epsilon/2.$$

The sequence $R_k$ are martingale differences with uniformly bounded exponential moments (since it is bounded from below by $-(\alpha^* + \epsilon/2)$ and stochastically dominated by a Poisson with mean $LK$). Thus

$$\lim_n \frac{1}{n} \sum_{k=1}^n R_k = 0 \text{ a.s. }.$$

It follows that almost surely $\limsup_t \frac{1}{t} X_t \leq \alpha^*$. Since $X_t$ is stochastically dominated by Poisson($Kt$) we have that $\mathbb{E}[(\frac{1}{t} X_t)^2] \leq K^2 + K/t$ and so is uniformly bounded. Hence since $\lim \mathbb{E}\frac{1}{t} X_t \to \alpha^*$ it follows that we must have that $\frac{1}{t} X_t$ converges in distribution to $\alpha^*$. □

4 Regeneration Times

In order to establish almost sure convergence to the limit we define a series of regeneration times. We select some small $\alpha(K) > 0$, and say an integer time $t$ is a regeneration time if

1. The function $Y_t$ satisfies $\inf_s Y_t(s) - y_\alpha(s) = 0$.

2. For $J_t$ the set of particles to the right of the aggregate at time $t$, their trajectories $\{\zeta_j(s)\}_{j \in J_t}$ on $(-\infty, t]$ satisfy

$$\inf_s \zeta_j(t - s) - (X_t - y_\alpha(s)) > 0.$$

Let $0 \leq T_1 < T_2 < \ldots$ denote the regeneration times and let $\mathcal{R}$ denote the set of regeneration times.

Lemma 4.1 For all $K > 1$, there exists $\delta(K) > 0$ such that,

$$\inf_{t \in \mathbb{N}} \mathbb{P}[t \in \mathcal{R}] \geq \delta.$$

Proof. Let $\mathcal{R}_t$ be the event that $\inf_s Y_t(s) - y_\alpha(s) = 0$. Provided that $\alpha(K)$ is small enough by Lemmas 3.2 and 3.8 we have that

$$\mathbb{P}[\mathcal{R}_t] \geq \frac{1}{3}.$$
As the density of particles to the right of $X_t$ is increasing in $Y_t$ it is, therefore greatest when $t = 0$ and so $\mathbb{P}[t \in \mathcal{R} | \mathcal{D}_t]$ is minimized at $t = 0$. Let $w_\ell$ be defined as the probability

$$w_\ell = \mathbb{P}[\max_{0 \leq s \leq t} W_s - y_\alpha(s) > \ell]$$

For $0 \leq \ell < \alpha^{-4}$ we simply bound $w_\ell \leq 1$ so let us consider $\ell \geq \alpha^{-4}$. Then

$$w_\ell \leq 1 - \mathbb{P}[M_\ell \leq \ell, \forall i \geq \lceil \log_2(\ell) \rceil : M_{2+i+1} \leq \ell + i2^{i+1}]$$

$$\leq 1 - \mathbb{P}[M_\ell \leq \ell] \prod_{i \geq \lceil \log_2(\ell) \rceil} \mathbb{P}[M_{2+i+1} \leq \ell + i2^{i+1}]$$

$$\leq 1 - (1 - e^{1-\ell^2/2}) \prod_{i \geq \lceil \log_2(\ell) \rceil} (1 - e^{1-i/\sqrt{2}})$$

$$\leq e^{1-\ell^2/2} + \sum_{i \geq \lceil \log_2(\ell) \rceil} e^{1-i/\sqrt{2}}$$

where the third inequality is by the FKG inequality and the final inequality is by equation (3). Then we have that

$$\sum_{\ell \geq \alpha^{-4}} w_\ell \leq \sum_{\ell \geq \alpha^{-4}} e^{1-\ell^2/2} + \sum_{\ell \geq \alpha^{-4}} \sum_{i \geq \lceil \log_2(\ell) \rceil} e^{1-i/\sqrt{2}}$$

$$\leq \sum_{\ell \geq \alpha^{-4}} e^{1-\ell^2/2} + \sum_{i} 2^{i+1} e^{1-i/\sqrt{2}} < \infty,$n

since $2e^{-1/\sqrt{2}} < 1$. Hence $\sum_{\ell=0}^\infty w_\ell < \infty$ and so

$$\mathbb{P}[0 \in \mathcal{R} | \mathcal{D}_0] = \mathbb{P}[\text{Poisson}(K \sum_{\ell=0}^\infty w_\ell) = 0] > 0.$$

Thus there exists $\delta > 0$ such that $\inf_{t \in \mathbb{N}} \mathbb{P}[t \in \mathcal{R}] \geq \delta$. $\square$

We can now establish our main result.

**Proof.** [Theorem 1] By Lemma 4.1 there is a constant density of regeneration times so the expected inter-arrival time is finite. By Theorem 3.9 the process $X_t$ travels at speed $\alpha^*$, at least in probability. By the Strong Law of Large Numbers for renewal-reward processes this convergence must also be almost sure. $\square$

### 5 Higher dimensions

Our approach gives a simple way of proving positive speed in higher dimensions as well although not down to the critical threshold. Simulations for small $K$ in two dimensions produce pictures which look very similar to the classical DLA model. Surprisingly, however, Eldan [11] conjectured that the critical value for $d \geq 2$ is 0! That is to say that despite the simulations there is linear growth in of the aggregate for all densities of particles and that these simulations are just a transitory effect reflecting that we are not looking at large enough times. We are inclined to agree but our techniques will only apply for larger values of $K$. A better understanding of the notoriously difficult classical DLA model may be necessary, for instance that the aggregate has dimension smaller than 2.
Let us now assume that $K > 1$. In the setting of $\mathbb{Z}^d$ it will be convenient for the sake of notation to assume that the particles perform simple random walks with rate $d$ which simply speeds the process be a factor of $d$. The projection of the particles in each co-ordinate is then a rate 1 walk. We let $U_t$ be the location of the rightmost particle in the aggregate (if there are multiple rightmost particles take the first one) at time $t$ and let $X_t$ denote its first coordinate. We then define $Y_t(s)$ according to (1) as before. We call a particle with path $(Z_1(t), \ldots, Z_d(t))$ conforming at time $t$ if $Z_1(s) > X_s$ for all $s \leq t$. By construction conforming particles cannot be part of the aggregate and conditional on $X_t$ form a Poisson process with intensity depending only on the first coordinate.

Let $e_i$ denote the unit vector in coordinate $i$. The intensity of conforming particles at time $t$ at $U_t + e_1$ is then simply

$$K \mathbb{P}[\max_{0 \leq s \leq t} W_s - Y_t(s) \leq 0 \mid Y_t].$$

where $W_s$ is an independent simple random walk. Similarly the rate at which conforming particles move from $U_t + e_1$ to $Y_t$ thus forming a new rightmost particle is

$$S(t) = \frac{1}{2} K \mathbb{P}[\max_{0 \leq s \leq t} W_s - Y_t(s) \leq 0 \mid Y_t],$$

the same as the formula we found in the one dimensional case. Of course by restricting to conforming particles we are restricting ourselves and so the rate at which $X_t$ increments is strictly larger than $S(t)$. Since $S(t)$ is increasing as a function of $X_t$ (through $Y_t$) we can stochastically dominate the one dimensional case by the higher dimensional process which establishes Corollary [2].

Let us now briefly describe how to improve upon $K = 1$. In the argument above we are being wasteful in two regards, first by only considering conforming particles and secondly by considering only a single rightmost particle. If there are two rightmost particles then the rate at which $X_t$ increases doubles. The simplest way to get such a new particle is for a conforming particle at $U_t + e_1 \pm e_i$ to jump first to $U_t \pm e_i$ and then to $U_t$. There are $(2d - 2)$ such location and the first move occurs at rate $S(t)$ and the second at has probability $1/(2d)$ to move in the correct direction and takes time exponential with rate $d$. After this sequence of events the rate at which $X_t$ increments becomes $2S(t)$.

In Lemma [2,3] on which the whole proof effectively rests, we show that for $\rho < 1$ if $S(s) \geq \gamma$ for $s \in [t, t + \Delta]$ then with exponentially high probability $X_{t+\Delta} - X_t \geq \rho \Delta \gamma$ for any $\rho < 1$ which is intuitively obvious since $X_t$ grows at rate $S(s) \geq \gamma$. We can improve our lower bound on $K$ by increasing the range of $\rho$ for which this holds for small values of $\gamma$.

Define the following independent random variables

$$V_1 \sim \text{Exp}(\gamma), V_2 \sim \text{Exp}\left(\frac{2d - 2}{2d} \gamma\right), V_3 \sim \text{Exp}(d), V_4 \sim \text{Exp}(\gamma)$$

where we interpret $V_1$ as the time until the first conforming particle hits $U_t$. We will view $V_2$ as the waiting time for a conforming particle to move from $U_t + \pm e_i + e_1$ to $U_t \pm e_i$ for some $2 \leq i \leq d$ and we further specify that their next step will move directly to $U_t$ which thins the process by a factor $\frac{1}{d^i}$. Let $V_3$ be the time until its next move. On the event $V_2 + V_3 < V_1$ there is an additional rightmost particle before one has been added to the right of $U_t$. Now let $V_4$ be the first time a conforming particle reaches this new rightmost site. So the time for $X_t$ to increase is stochastically dominated by

$$T = \min\{V_1 + V_2 + V_3 + V_4\}.$$
Now using the memoryless property of exponential random variables,

\[ \mathbb{E}T = \mathbb{E}V_1 - \mathbb{E}[(V_1 - (V_2 + V_3 + V_4))I(V_1 \geq V_2 + V_3 + V_4)] = \frac{1}{\gamma}(1 - \mathbb{P}[V_1 \geq V_2 + V_3 + V_4]) \]

and

\[
\mathbb{P}[V_1 \geq V_2 + V_3 + V_4] = \mathbb{P}[V_1 \geq V_2]\mathbb{P}[V_1 \geq V_2 + V_3 | V_1 \geq V_2]\mathbb{P}[V_1 \geq V_2 + V_3 + V_4 | V_1 \geq V_2 + V_3]
\]

\[
= \frac{2d-2}{2d-\gamma} \frac{d}{\gamma + \frac{2d-2}{2d}\gamma + d\frac{d}{2}\gamma}
\]

\[
= \frac{d-1}{2(2d-1)} \frac{d}{\gamma + d}
\]

In the proof we need only to consider the case where \( \gamma \) is close to 0 and

\[
\lim_{\gamma \to 0} \gamma \mathbb{E}T = \frac{3d-1}{4d-2}.
\]

Having \( X_t \) growing at rate \( \gamma \frac{4d-2}{3d-2} \) corresponds in the proof to linear growth provided that \( K > \frac{3d-1}{d-2} \). In the case for \( d = 2 \) this means \( K > \frac{5}{6} \). We are still being wasteful in several ways and expect that a more careful analysis would yield better bounds that tend to 0 as \( d \to \infty \). However, we don’t believe that this approach alone is sufficient to show that the critical value of \( K \) is 0 when \( d \geq 2 \). For that more insight into the local structure is likely needed along with connections to standard DLA.

6 Open Problems

In the one dimensional case the most natural open questions concern the behaviour of \( X_t \) for densities close to 1. Approaching \( K = 1 \) from above one can ask what exponent does the speed of the process satisfy. Perhaps of most interest is what is the exponent of growth for \( X_t \) when \( K = 1 \). Heuristics suggest that it may grow as \( t^{2/3} \).

In higher dimensions the main open problem is to establish Eldan’s conjecture of linear growth for all \( K \). Another natural question is to prove a shape theorem for the aggregate.

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