Multiplicity results of fractional $p$-Laplace equations with sign-changing and singular nonlinearity

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ABSTRACT

In this article, we study the following fractional $p$-Laplacian equation with singular nonlinearity:

$\begin{align*}
\left\{ \begin{array}{l}
-2 \int_{\mathbb{R}^n} \frac{|w(y) - w(x)|^{p-2}(w(y) - w(x))}{|y - x|^{n+ps}} \, dy = a(x)w^{-q} + \lambda b(x)w^r \quad \text{in } \Omega, \\
w > 0 \quad \text{in } \Omega, \\
w = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{array} \right.
\end{align*}$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$, $n > ps$, $s \in (0, 1)$, $\lambda > 0$, $0 < q < 1$, $q < p - 1 < r < p^*_s - 1$ with

$p^*_s = \frac{np}{n - ps}$, $a : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $0 < a(x) \in L^{\frac{p^*_s}{p^*_s - 1 + q}}(\Omega)$,

and $b : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a sign-changing function such that $b(x) \in L^{\frac{p^*_s}{p^*_s - 1 - r}}(\Omega)$. Using variational methods, we show existence and multiplicity of positive solutions of $(P_\lambda)$ with respect to the parameter $\lambda$.

1. Introduction

Let $s \in (0, 1)$ and let $0 \in \Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $n > ps$. Then we consider the following problem with singular nonlinearity:

$\begin{align*}
\left\{ \begin{array}{l}
-2 \int_{\mathbb{R}^n} \frac{|w(y) - w(x)|^{p-2}(w(y) - w(x))}{|y - x|^{n+ps}} \, dy = a(x)w^{-q} + \lambda b(x)w^r \quad \text{in } \Omega, \\
w > 0 \quad \text{in } \Omega, \\
w = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{array} \right.
\end{align*}$

We assume the following assumptions on $a$ and $b$:

(a1) $a : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $0 < a \in L^{\frac{p^*_s}{p^*_s - 1 + q}}(\Omega)$,

(b1) $b : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a sign-changing function such that $b^+ \not\equiv 0$ and $b(x) \in L^{\frac{p^*_s}{p^*_s - 1 - r}}(\Omega)$.

Also $\lambda > 0$ is a parameter and $0 < q < 1$, $q < p - 1 < r < p^*_s - 1$, with $p^*_s = \frac{np}{n - ps}$, known as fractional critical Sobolev exponent.

The fractional power of the Laplacian is the infinitesimal generator of a Lévy stable diffusion process and arises in anomalous diffusions in plasma, population dynamics,
geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. For more details, one can see [1,2] and reference therein. Recently, the fractional elliptic equation attracts a lot of interest in nonlinear analysis such as in [3–5,7,8]. Caffarelli and Silvestre [3] gave a new formulation of fractional Laplacian through Dirichlet–Neumann maps. This is commonly used in the literature since it allows us to write a nonlocal problem to a local problem which allows us to use the variational methods to study the existence and uniqueness.

On the other hand, the fractional elliptic problem has been investigated by many authors, for example, [4–6] for the subcritical case, [7–9] for critical case with polynomial type nonlinearities. Moreover, by Nehari manifold and fibering maps, the author obtained the existence of multiple solutions for fractional equations for critical [10] and subcritical cases [11,12] and reference therein. In case of square root of Laplacian, existence and multiplicity results for sublinear and superlinear type of nonlinearity with sign-changing weight functions are studied in [13]. In [13], the author used the idea of Caffarelli and Silvestre [3], which gives a formulation of the fractional Laplacian through Dirichlet–Neumann maps. Also in case of fractional $p$-Laplacian, existence and multiplicity results for polynomial type nonlinearities are studied by many authors see [11,12,14–16] and reference therein. Also eigenvalue problem related to $p$—fractional Laplacian is studied in [17–19].

For $s = 1$, the paper by Crandall, Robinowitz and Tartar [20] is the starting point on semilinear problems with singular nonlinearity. There is a large literature on singular nonlinearity see [20–32] and reference therein. In [33], Chen showed the existence and multiplicity of the following problem

$$
\begin{align*}
-\Delta w - \frac{\lambda}{|x|^s} w &= \frac{f(x)}{w^q} + \mu g(x) w^p \quad \text{in } \Omega \setminus \{0\} \\
0 < q < 1 < p < \frac{n+2}{n-2}, f(x) > 0 \quad \text{and} \quad g \quad \text{is sign-changing continuous function.}
\end{align*}
$$

where $0 \in \Omega$ is a bounded smooth domain of $\mathbb{R}^n$ with smooth boundary, $0 < \lambda < \frac{(n-2)^2}{4}$, $\Omega \setminus \{0\}$. In [34], Fang proved the existence of solution of the following singular problem

$$
(-\Delta)^s w = w^{-p}, \quad w > 0 \text{ in } \Omega, w = 0 \text{ in } \mathbb{R}^n \setminus \Omega
$$

with $0 < p < 1$, using the method of sub and super solution. Recently, in [35], Barrios, Peral and et al. extended the result of [34]. They studied the existence result for the following fractional equation with singular type nonlinearities

$$
\begin{align*}
( -\Delta )^s w &= \frac{\lambda f(x)}{w^p} + M w^p \quad \text{in } \Omega \\
y > 0 \in \Omega, w = 0 \text{ in } \mathbb{R}^n \setminus \Omega
\end{align*}
$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^n$, $n > 2s$, $0 < s < 1$, $M \in \{0, 1\}$, $y > 0$, $\lambda > 0$, $p > 1$ and $f \in L^m(\Omega)$, $m \geq 1$ is a nonnegative function. For $M = 0$, they proved the existence of solution for every $y > 0$ and $\lambda > 0$. For $M = 1$ and $f \equiv 1$, they showed that there exists $\Lambda$ such that it has a solution for every $0 < \lambda < \Lambda$, and have no solution for $\lambda > \Lambda$. 




To the best of our knowledge, there is no work related to fractional $p$-Laplacian with singular and sign-changing nonlinearity. In this work, we studied the multiplicity results for fractional $p$-Laplacian equation with singular nonlinearity and sign-changing weight function with respect to the parameter $\lambda$. This work is motivated by the work of Chen and Chen in [33]. But one cannot directly extend all the results for fractional $p-$Laplacian, due to the non-local behaviour of the operator and the bounded support of the test function is not preserved. Also due to the singularity of the problem, the associated functional is not differentiable in the sense of Gâteaux. The results obtained here are somehow expected but we show how the results arise out of nature of the Nehari manifold.

The paper is organized as follows: Section 2 is devoted to some preliminaries and notations. We also state our main results. In Section 3, we study the decomposition of the Nehari manifold and the associated energy functional is bounded below and coercive. Section 3 contains the existence of a nontrivial solutions in $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$. We will use the following notation throughout this paper: $\|a\|$, $\|b\|$ denote the norm in $L_{p_1}^{s_1} (\Omega_1)$, $L_{p_2}^{s_2} (\Omega_2)$ respectively.

2. Preliminaries

In this section, we give some definitions and functional settings. At the end of this section, we state our main results. For this we define $W^{s,p}(\Omega)$, the usual fractional Sobolev space $W^{s,p}(\Omega) := \left\{ w \in L^p(\Omega); \frac{(w(x) - w(y))}{|x-y|^{s+p}} \in L^p(\Omega \times \Omega) \right\}$ endowed with the norm

$$
\|w\|_{W^{s,p}(\Omega)} = \|w\|_{L^p(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|w(x) - w(y)|^p}{|x-y|^{n+ps}} \text{d}x \text{d}y \right)^{\frac{1}{p}}.
$$

To study fractional Sobolev spaces in details, we refer to [36]. Due to the non-localness of the operator, we define a linear space as follows:

$$
X = \left\{ w \mid w : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, w|_{\Omega} \in L^p(\Omega) \text{ and } \frac{w(x) - w(y)}{|x-y|^{n+ps}} \in L^p(Q) \right\}
$$

where $Q = \mathbb{R}^{2n} \setminus (C\Omega \times C\Omega)$ and $C\Omega := \mathbb{R}^n \setminus \Omega$. In case of $p = 2$, the space $X$ was firstly introduced by Servadei and Valdinoci [4]. The space $X$ is a normed linear space endowed with the norm

$$
\|w\|_X = \|w\|_{L^p(\Omega)} + \left( \int_{Q} \frac{|w(x) - w(y)|^p}{|x-y|^{n+ps}} \text{d}x \text{d}y \right)^{\frac{1}{p}}.
$$

Then we define

$$
X_0 = \{ w \in X : w = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}
$$

with the norm

$$
\|w\| = \left( \int_{Q} \frac{|w(x) - w(y)|^p}{|x-y|^{n+ps}} \text{d}x \text{d}y \right)^{\frac{1}{p}}.
$$
is a reflexive Banach space. We notice that, the norms in (2.1) and (2.2) are not same because $\Omega \times \Omega$ is strictly contained in $Q$. Now we define the space

$$C_{X_0} := \{ w \in C_c^\infty(\mathbb{R}^n) : w = 0 \text{ in } \mathbb{R}^n \setminus \Omega \}.$$ 

Then $C_{X_0}$ is a dense in the space $X_0$.

Define $S := \inf_{w \in X_0} \left\{ \frac{\int_{\mathbb{R}^n} |w(x)-w(y)|^p |x-y|^{-(n+ps)} \, dx \, dy}{(\int_{\mathbb{R}^n} |w|^p \, dx)^{\frac{p'}{p}}} \right\}$.

**Definition 2.1:** A weak solution of the problem $(P_\lambda)$ is a function $w \in X_0$, $w > 0$ in $\Omega$ such that for every $v \in X_0$

$$\int_Q \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(v(x) - v(y))}{|x-y|^{(n+ps)}} \, dx \, dy$$

$$= \int_{\Omega} a(x)(w^{-q}v)(x) \, dx + \lambda \int_{\Omega} b(x)(w^r v)(x) \, dx.$$  

In order to present the existence of positive solution of $(P_\lambda)$, we will consider the following problem

$$(P_\lambda^+) \left\{ \begin{array}{ll}
-2 \int_{\mathbb{R}^n} \frac{|w(x)-w(y)|^p}{|x-y|^{n+ps}} \, dx \, dy = a(x)w^{-q}_+ + \lambda b(x)w^r_+ & \text{in } \Omega \\
w > 0 \text{ in } \Omega, \quad w = 0 \text{ in } \mathbb{R}^n \setminus \Omega.
\end{array} \right.$$  

where $w_+ := \max\{w, 0\}$ denotes the positive part of $w$. Then the function $w \in X_0$, $w > 0$ in $\Omega$ is a weak solution of the problem $(P_\lambda^+)$ if for every $v \in X_0$

$$\int_{\Omega} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(v(x) - v(y))}{|x-y|^{(n+ps)}} \, dx \, dy$$

$$= \int_{\Omega} a(x)(w^{-q}_+ v)(x) \, dx + \lambda \int_{\Omega} b(x)(w^r_+ v)(x) \, dx.$$  

We note that if $w > 0$ is a solution of $(P_\lambda^+)$ then one can easily see that $w$ is also a solution $(P_\lambda)$. To find the solution of $(P_\lambda^+)$, we will use variational approach. So we define the associated functional $J_\lambda : X_0 \to [-\infty, \infty)$ as

$$J_\lambda(w) = \frac{1}{p} \int_Q \frac{|w(x)-w(y)|^p}{|x-y|^{n+ps}} \, dx \, dy - \frac{1}{1-q} \int_{\Omega} a(x)w^{1-q}_+(x) \, dx - \frac{\lambda}{r+1} \int_{\Omega} b(x)w^{r+1}_+(x) \, dx.$$  

Now for $w \in X_0$, we define the fibre map $\phi_w : \mathbb{R}^+ \to \mathbb{R}$ as

$$\phi_w(t) = J_\lambda(tw) = \frac{t^p}{p} \|w\|^p - \frac{1}{1-q} \int_{\Omega} a(x)w^{1-q}_+(x) \, dx - \frac{\lambda t^{r+1}}{r+1} \int_{\Omega} b(x)w^{r+1}_+(x) \, dx.$$  

Also

$$\phi'_w(t) = t^{p-1}\|w\|^p - t^{-q} \int_{\Omega} a(x)w^{1-q}_+(x) \, dx - \lambda t^r \int_{\Omega} b(x)w^{r+1}_+(x) \, dx,$$

$$\phi''_w(t) = (p-1)t^{p-2}\|w\|^p + qt^{-q-1} \int_{\Omega} a(x)w^{1-q}_+(x) \, dx - r\lambda t^{r-1} \int_{\Omega} b(x)w^{r+1}_+(x) \, dx.$$
It is easy to see that the energy functional $J_\lambda$ is not bounded below on the space $X_0$. But we will show that it is bounded below on an appropriate subset of $X_0$ and a minimizer on subsets of this set gives rise to solutions of $(P_\lambda^+)$. In order to obtain the existence results, we define

$$\mathcal{N}_\lambda := \{ w \in X_0 : \phi''_w(1) \geq 0 \}$$

$$= \left\{ w \in X_0 : \|w\|^p = \int_\Omega a(x)w_+^{r+1} + \lambda \int_\Omega b(x)w_+^{r+1} \right\}.$$

Note that $w \in \mathcal{N}_\lambda$ if $w$ is a solution of problem $(P_\lambda^+)$. Also one can easily see that $tw \in \mathcal{N}_\lambda$ if and only if $\phi'_w(t) = 0$. In order to obtain our result, we decompose $\mathcal{N}_\lambda$ with $\mathcal{N}_\lambda^\pm, \mathcal{N}_\lambda^0$ defined as follows:

$$\mathcal{N}_\lambda^\pm := \{ w \in \mathcal{N}_\lambda : \phi''_w(1) \geq 0 \}$$

$$= \left\{ w \in \mathcal{N}_\lambda : (p - 1 + q)\|w\|^p \geq \lambda (r + q) \int_\Omega b(x)w_+^{r+1} \right\}$$

$$\mathcal{N}_\lambda^0 := \{ w \in \mathcal{N}_\lambda : \phi''_w(1) = 0 \}$$

$$= \left\{ w \in \mathcal{N}_\lambda : (p - 1 + q)\|w\|^p = \lambda (r + q) \int_\Omega b(x)w_+^{r+1} \right\}.$$

Our results are as follows:

Inspired by [33], we show how variational methods can be used to established some existence and multiplicity results for $(P_\lambda^+)$:

**Theorem 2.2:** Suppose that $\lambda \in (0, \Lambda)$, where

$$\Lambda := \frac{(p - 1 + q)}{(r + q)} \left( \frac{r - p + 1}{r + q} \right)^{\frac{r-p+1}{r+1+q}} \frac{1}{\|b\|} \left( \frac{S^+}{\|a\|^{r-p+1}} \right)^{\frac{1}{r+1+q}}$$

then the problem $(P_\lambda)$ has at least two solutions $w \in \mathcal{N}_\lambda^+, W \in \mathcal{N}_\lambda^-$ with $\|W\| > \|w\|$.

Next, we obtain the blow up behaviour of the solution $W_\epsilon \in \mathcal{N}_\lambda^-$ of problem $(P_\lambda)$ with $r = p - 1 + \epsilon$ as $\epsilon \to 0^+$.

**Theorem 2.3:** let $W_\epsilon \in \mathcal{N}_\lambda^-$ be the solution of problem $(P_\lambda)$ with $r = p - 1 + \epsilon$, where $\lambda \in (0, \Lambda)$, then

$$\|W_\epsilon\| > C_\epsilon \left( \frac{\Lambda}{\lambda} \right)^{\frac{1}{\epsilon}},$$

where

$$C_\epsilon = \left( 1 + \frac{p - 1 + q}{\epsilon} \right)^{\frac{1}{p+1+q}} \|a\|^{\frac{1}{p+1+q}} \left( \frac{1}{\sqrt{S}} \right)^{\frac{1}{p+1+q}} \to \infty \text{ as } \epsilon \to 0^+.$$

Namely, $W_\epsilon$ blows up faster than exponentially with respect to $\epsilon$. 


Remark: If \( w \) is a positive solution of the following problem
\[
\begin{aligned}
-2 \int_{\mathbb{R}^n} \frac{|w(y) - w(x)|^{p-2}(w(y) - w(x))}{|y|^{n+ps}} \, dy &= a(x)w^{-q} + \lambda b(x)w^r \quad \text{in} \ \Omega \\
0 > w > 0 & \quad \text{in} \ \Omega, \quad w = 0 \quad \text{in} \ \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]
then one can easily see that \( u = \frac{1}{r+1}w \) is a positive solution of the following problem
\[
(Q_\lambda) \quad \begin{aligned}
-2 \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|y|^{n+ps}} \, dy &= \lambda^{\frac{p-1+q}{r-1+q}} a(x)u^{-q} + b(x)u^r \quad \text{in} \ \Omega \\
0 > u > 0 & \quad \text{in} \ \Omega, \quad u = 0 \quad \text{in} \ \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]
That is, the problem \((Q_\lambda)\) has two positive solutions for \( \lambda \in (0, \Lambda) \).

3. Fibering map analysis

In this section, we show that \( N_\lambda^{\pm} \) is nonempty and \( N_\lambda^0 = \{0\} \). Moreover, \( I_\lambda \) is bounded below and coercive.

**Lemma 3.1:** Let \( \lambda \in (0, \Lambda) \). Then for each \( w \in X_0 \) with \( \int_\Omega a(x)w_+^{1-q}(x)dx > 0 \), we have the following:

(i) \( \int_\Omega b(x)w_+^{r+1}(x)dx \leq 0 \), then there exists a unique \( 0 < t_1 < t_{\text{max}} \) such that \( t_1w \in N_\lambda^+ \) and \( I_\lambda(t_1w) = \inf_{t > 0} I_\lambda(tw) \);

(ii) \( \int_\Omega b(x)w_+^{r+1}(x)dx > 0 \), then there exists a unique \( t_1 \) and \( t_2 \) with \( 0 < t_1 < t_{\text{max}} < t_2 \) such that \( t_1w \in N_\lambda^+ \), \( t_2w \in N_\lambda^- \) and \( I_\lambda(t_1w) = \inf_{t \leq t_{\text{max}}} I_\lambda(tw) \), \( I_\lambda(t_2w) = \sup_{t \geq t_1} I_\lambda(tw) \).

**Proof:** For \( t > 0 \), we define
\[
\psi_w(t) = t^{p-1}w_0 - t^{-r-q} \int_\Omega a(x)w_+^{1-q}(x)dx - \lambda \int_\Omega b(x)w_+^{r+1}(x)dx.
\]
One can easily see that \( \psi_w(t) \to -\infty \) as \( t \to 0^+ \). Now
\[
\psi_w'(t) = (p-1-r)t^{p-2-r}w_0 + (r+q)t^{-r-q-1} \int_\Omega a(x)w_+^{1-q}(x)dx.
\]
\[
\psi_w''(t) = (p-1-r)(p-2-r)t^{p-3}w_0 - (r+q)(r+q+1)t^{-r-q-2} \int_\Omega a(x)w_+^{1-q}(x)dx.
\]
Then \( \psi_w'(t) = 0 \) if and only if \( t = t_{\text{max}} := \frac{(r-p+1)w_0}{(r+q) \int_\Omega a(x)w_+^{1-q}(x)dx} \). Also
\[
\psi_w''(t_{\text{max}}) = (p-1-r)(p-2-r) \left[ \frac{(r-p+1)w_0}{(r+q) \int_\Omega a(x)w_+^{1-q}(x)dx} \right]^{\frac{r+p+3}{p+1+q}} - \|w_0\|^p.
\]
\[ \psi_w(t_{\text{max}}) = \frac{(p - 1 + q) (r - p + 1)}{(r + q)} \left( \frac{r - p + 1}{r + q} \right) \left( \frac{\|w\|^p}{\|a\|} \right) \left( \frac{\sqrt{S}}{S} \right)^{1-q} \left\| \left( \frac{\sqrt{S}}{S} \right)^{1-q} \right\| w^{r+1} \]

where

\[ E_\lambda = \left( \frac{p - 1 + q}{r + q} \right) \left( \frac{r - p + 1}{r + q} \right) \left( \frac{\sqrt{S}}{S} \right)^{1-q} \left\| \left( \frac{\sqrt{S}}{S} \right)^{1-q} \right\| w^{r+1} \]

Then we see that \( E_\lambda = 0 \) if and only if \( \lambda = \Lambda \), where

\[ \Lambda := \left( \frac{p - 1 + q}{r + q} \right) \left( \frac{r - p + 1}{r + q} \right) \left( \frac{\sqrt{S}}{S} \right)^{1-q} \left\| \left( \frac{\sqrt{S}}{S} \right)^{1-q} \right\| w^{r+1} \]

Thus for \( \lambda \in (0, \Lambda) \), we have \( E_\lambda > 0 \), and therefore it follows from (3.3) that \( \psi_w(t_{\text{max}}) > 0 \).
Now we show that if $\psi_w(t) = 0$ and $\psi'_w(t) > 0$, then $tw \in N_\lambda^+$.

$$\psi_w(t) = 0 \Rightarrow t^{p-1-r} \|w\|^p - t^{r-q} \int_\Omega a(x)w_+^{1-q}(x)dx - \lambda \int_\Omega b(x)w_+^{r+1}(x)dx = 0$$

$$\Rightarrow \|tw\|^p = \int_\Omega a(x)(tw)_+^{1-q}(x)dx + \lambda \int_\Omega b(x)(tw)_+^{r+1}(x)dx$$

and therefore

$$\psi'_w(t) > 0 \Rightarrow (p-1-r)t^{p-2-r}\|w\|^p - (r-q)t^{r-q-1} \int_\Omega a(x)w_+^{1-q}(x)dx > 0$$

$$\Rightarrow (p-1-r)\|tw\|^p + (r+q) \int_\Omega a(x)(tw)_+^{1-q}(x)dx > 0$$

$$\Rightarrow (p-1-r)\|tw\|^p + (r+q) \left[ \|tw\|^p - \lambda \int_\Omega b(x)(tw)_+^{r+1}(x)dx \right] > 0,$$

since $tw \in N_\lambda^+.$

Similarly one can show that if $\psi_w(t) = 0$ and $\psi'_w(t) < 0$, then $tw \in N_\lambda^-.$

Now $\phi'_w(t) = t^r \psi'_w(t).$ Thus $\phi'_w(t) < 0$ in $(0,t_1)$ and $\phi'_w(t) > 0$ in $(t_1,t_2)$ and $\phi'_w(t) < 0$ in $(t_2,\infty).$ Hence $J_\lambda(t_1w) = \inf_{0 \leq t \leq t_\lambda} J_\lambda(tw)$, $J_\lambda(t_2w) = \sup_{t \geq t_1} J_\lambda(tw).$ Moreover, $t_1w \in N_\lambda^+$ and $t_2w \in N_\lambda^-.$

(ii) If $\int_\Omega b(x)w_+^{r+1}(x)dx < 0$ and $\psi_w(t) \rightarrow -\lambda \int_\Omega b(x)w_+^{r+1}(x)dx > 0$ as $t \to \infty$. Consequently, $\psi_w(t)$ has exactly one point $0 < t_1 < t_\lambda$ such that

$$\psi_w(t_1) = 0 \text{ and } \psi'_w(t_1) > 0.$$

Using $\phi'_w(t) = t^r \psi'_w(t)$, we have $\phi'_w(t) < 0$ in $(0,t_1)$ and $\phi'_w(t) > 0$ in $(t_1,\infty).$ So, $J_\lambda(t_1w) = \inf_{t \geq 0} J_\lambda(tw).$ Hence, it follows that $t_1w \in N_\lambda^+$.

**Corollary 3.2:** Suppose that $\lambda \in (0,\Lambda)$, then $N_\lambda^\pm \neq \emptyset.$

**Proof:** By (a1) and (b1), we can choose $w \in X_0 \setminus \{0\}$ such that $\int_\Omega a(x)w_+^{1-q}(x)dx > 0$ and $\int_\Omega b(x)w_+^{r+1}(x)dx > 0$. By (ii) of Lemma 3.1, there exists unique $t_1$ and $t_2$ such that $t_1w \in N_\lambda^+$, $t_2w \in N_\lambda^-$. In conclusion, $N_\lambda^\pm \neq \emptyset.$

**Lemma 3.3:** For $\lambda \in (0,\Lambda)$, we have $N_\lambda^0 = \{0\}.$
Proof: We prove this by contradiction. Assume that there exists \( 0 \neq w \in N^0_\lambda \). Then it follows from \( w \in N^0_\lambda \) that

\[
(p - 1 + q)\|w\|^p = \lambda(r + q) \int_\Omega b(x)w_+^{r+1}(x)dx
\]

and consequently

\[
0 = \|w\|^p - \int_\Omega a(x)w_+^{1-q}(x)dx - \lambda \int_\Omega b(x)w_+^{r+1}(x)dx
\]

\[
= \|w\|^p - \int_\Omega a(x)w_+^{1-q}(x)dx - \frac{p - 1 + q}{r + q} \|w\|^p
\]

\[
= \left(\frac{r - p + 1}{r + q}\right)\|w\|^p - \int_\Omega a(x)w_+^{1-q}(x)dx.
\]

Therefore, as \( \lambda \in (0, \Lambda) \) and \( w \neq 0 \), we use similar arguments as those in (3.3) to get

\[
0 < E_\lambda \|w\|^{r+1}
\]

\[
\leq \left(\frac{p - 1 + q}{r + q}\right) \left(\frac{r - p + 1}{r + q}\right) \frac{\|w\|^{p(r + q)}}{\|w\|^{p(r + q) - 1}} - \lambda \int_\Omega b(x)w_+^{r+1}(x)dx
\]

\[
= \left(\frac{p - 1 + q}{r + q}\right) \left(\frac{r - p + 1}{r + q}\right) \frac{\|w\|^{p(r + q)}}{\|w\|^{p(r + q) - 1}} - \frac{(p - 1 + q)}{(r + q)} \|w\|^p = 0,
\]

a contradiction. Hence \( w = 0 \). That is, \( N^0_\lambda = \{0\} \).

We note that \( \Lambda \) is also related to a gap structure in \( N_\lambda \): 

Lemma 3.4: Suppose that \( \lambda \in (0, \Lambda) \), then there exists a gap structure in \( N_\lambda \):

\[
\|W\| > A_\lambda > A_0 > \|w\| \text{ for all } w \in N^+_\lambda, W \in N^-_\lambda,
\]

where

\[
A_\lambda = \left[\frac{(p - 1 + q)}{\lambda(r + q)\|b\|} \left(\frac{1}{\sqrt{\xi}}\right)^{r+1}\right]^{\frac{1}{r+1}} \text{ and } A_0 = \left[\frac{(r + q)}{(r - p + 1)} \left(\frac{1}{\sqrt{\xi}}\right)^{1-q}\right]^{\frac{1}{r+q-1}}.
\]

Proof: If \( w \in N^+_\lambda \subset N_\lambda \), then

\[
0 < (p - 1 + q)\|w\|^p - \lambda(r + q) \int_\Omega b(x)w_+^{r+1}(x)dx
\]

\[
= (p - 1 + q)\|w\|^p - (r + q) \left[\|w\|^p - \int_\Omega a(x)w_+^{1-q}(x)dx\right]
\]

\[
= (p - 1 - r)\|w\|^p + (r + q) \int_\Omega a(x)w_+^{1-q}(x)dx.
\]
Hence it follows from (3.1)

\[(r - p + 1)\|w\|^p < (r + q) \int_\Omega a(x)w^{1-q}(x)dx \leq (r + q)\|a\| \left(\frac{\|w\|}{\sqrt[3]{S}}\right)^{1-q}\]

which yields

\[\|w\| < \left[\frac{(r + q)\|a\|}{(r - p + 1)} \left(\frac{1}{\sqrt[3]{S}}\right)^{1-q}\right]^{\frac{1}{r+p+1}} \equiv A_0.\]

If \(W \in \mathcal{N}_\lambda^\perp\), then it follows from (3.2) that

\[(p - 1 + q)\|W\|^p < \lambda(r + q) \int_\Omega b(x)W^{r+1}(x)dx \leq \lambda(r + q)\|b\| \left(\frac{\|W\|}{\sqrt[3]{S}}\right)^{r+1}\]

which yields

\[\|W\| > \left[\frac{(p - 1 + q)}{\lambda(r + q)\|b\|} \left(\sqrt[3]{S}\right)^{r+1}\right]^{ \frac{1}{r+p+1}} \equiv A_\lambda.\]

Now we show that \(A_\lambda = A_0\) if and only if \(\lambda = \Lambda\).

\[
\lambda = \Lambda = \frac{p - 1 + q}{r + q} \left(\frac{r - p + 1}{r + q}\right) \left(\frac{1}{\|a\|}\right) \left(\frac{S^{r+q}}{\|a\|^{r-p+1}}\right)^{\frac{1}{r+p+1}} \\
\Leftrightarrow A_\lambda = \lambda^{-\frac{1}{r+p+1}} \left(\frac{p - 1 + q}{r + q}\right) \left(\frac{1}{\|b\|}\right) \left(\frac{1}{\sqrt[3]{S}}\right)^{\frac{1}{r+p+1}} \\
= \left(\frac{r + q}{r - p + 1}\right) \left(\frac{1}{\|a\|}\right) \left(\frac{1}{\sqrt[3]{S}}\right)^{\frac{1}{r+p+1}} \\
= \frac{(r + q)\|a\|}{(r - p + 1)(\sqrt[3]{S})^{1-q}} \left(\frac{1}{\sqrt[3]{S}}\right)^{\frac{1}{r+p+1}} \equiv A_0.
\]

Thus for all \(\lambda \in (0, \Lambda)\), we can conclude that

\[\|W\| > A_\lambda > A_0 > \|w\| \text{ for all } w \in \mathcal{N}_\lambda^+, W \in \mathcal{N}_\lambda^\perp.\]

This completes the proof of the Lemma.

\[\square\]

**Lemma 3.5:** Suppose that \(\lambda \in (0, \Lambda)\), then \(\mathcal{N}_\lambda^\perp\) is a closed set in \(X_0\)-topology.

**Proof:** Let \(\{W_k\}\) be a sequence in \(\mathcal{N}_\lambda^\perp\) with \(W_k \to W\) in \(X_0\). Then we have

\[
\|W\|^p = \lim_{k \to \infty} \|W_k\|^p \\
= \lim_{k \to \infty} \left[\int_\Omega a(x)(W_k)^{1-q}(x)dx + \lambda \int_\Omega b(x)(W_k)^{r+1}(x)dx\right] \\
= \int_\Omega a(x)W^{1-q}(x)dx + \lambda \int_\Omega b(x)W^{r+1}(x)dx
\]
and
\[
(p - 1 + q)\|W\|^p - \lambda(r + q) \int_\Omega b(x)W_+^{r+1}(x)dx
= \lim_{k \to \infty} \left[ (p - 1 + q)\|W_k\|^p - \lambda(r + q) \times \int_\Omega b(x)(W_k)_+^{r+1}(x)dx \right] \leq 0,
\]
i.e. \( W \in \mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^0 \). Since \( \{W_k\} \subset \mathcal{N}_\lambda^- \), from Lemma 3.4 we have
\[\|W\| = \lim_{k \to \infty} \|W_k\| \geq A_0 > 0,\]
that is, \( W \not= 0 \). It follows from Lemma 3.1, that \( W \not\in \mathcal{N}_\lambda^0 \) for any \( \lambda \in (0, \Lambda) \). Thus \( W \in \mathcal{N}_\lambda^- \). That is, \( \mathcal{N}_\lambda^- \) is a closed set in \( X_0 \)- topology for any \( \lambda \in (0, \Lambda) \).

**Lemma 3.6:** Let \( w \in \mathcal{N}_\lambda^+ \), then for any \( \phi \in C_{X_0} \), there exists a number \( \epsilon > 0 \) and a continuous function \( f : B_\epsilon(0) := \{v \in X_0 : \|v\| < \epsilon\} \to \mathbb{R}^+ \) such that
\[f(v) > 0, f(0) = 1 \text{ and } f(v)(w + v\phi) \in \mathcal{N}_\lambda^\pm \text{ for all } v \in B_\epsilon(0).\]

**Proof:** We give the proof only for the case \( w \in \mathcal{N}_\lambda^+ \), the case \( \mathcal{N}_\lambda^- \) may be preceded exactly. For any \( C_{X_0} \), we define \( F : X_0 \times \mathbb{R}^+ \to \mathbb{R} \) as follows:
\[F(v, t) = t^{p-1+q}\|w + v\phi\|^p - \int_\Omega a(x)(w + v\phi)_+^{1-q}(x)dx - \lambda t^{r+q} \int_\Omega b(x)(w + v\phi)_+^{r+1}(x)dx\]
Since \( w \in \mathcal{N}_\lambda^+ (\subset \mathcal{N}_\lambda^-) \), we have that
\[F(0, 1) = \|w\|^p - \int_\Omega a(x)w_+^{1-q}(x)dx - \lambda \int_\Omega b(x)w_+^{r+1}(x)dx = 0,\]
and
\[\frac{\partial F}{\partial t}(0, 1) = (p - 1 + q)\|w\|^p - \lambda(r + q) \int_\Omega b(x)w_+^{r+1}(x)dx > 0.\]
Applying the implicit function Theorem at the point \( (0, 1) \), we have that there exists \( \tilde{\epsilon} > 0 \) such that for \( \|v\| < \tilde{\epsilon}, v \in X_0 \), the equation \( F(v, t) = 0 \) has a unique continuous solution \( t = f(v) > 0 \). It follows from \( F(0, 1) = 0 \) that \( f(0) = 1 \) and from \( F(v, f(v)) = 0 \) for \( \|v\| < \tilde{\epsilon}, v \in X_0 \) that
\[0 = f^{p-1+q}(v)\|w + v\phi\|^p - \int_\Omega a(x)(w + v\phi)_+^{1-q}(x)dx - \lambda f^{r+q}(v) \int_\Omega b(x)(w + v\phi)_+^{r+1}(x)dx\]
\[= \|f(v)(w + v\phi)\|^p - \int_\Omega a(x)f(v)(w + v\phi)_+^{1-q}(x)dx - \lambda \int_\Omega b(x)f(v)(w + v\phi)_+^{r+1}(x)dx \]
that is,
\[f(v)(w + v\phi) \in \mathcal{N}_\lambda^\pm \text{ for all } v \in X_0, \|v\| < \tilde{\epsilon}.\]
Since \( \frac{\partial F}{\partial t}(0, 1) > 0 \) and

\[
\frac{\partial F}{\partial t}(v, f(v)) = (p - 1 + q)f^{p-2+q}(v)\|w + v\phi\|^p - \lambda(r + q)f^{r+q-1}(v)
\]

\[
\times \int_{\Omega} b(x)(w + v\phi)^{r+1}(x)\,dx
\]

\[
= (p - 1 + q)\|f(v)(w + v\phi)\|^p - \lambda(r + q)\int_{\Omega} b(x)(f(v)(w + v\phi))^{r+1}(x)\,dx
\]

we can take \( \epsilon > 0 \) possibly smaller (\( \epsilon < \bar{\epsilon} \)) such that for any \( v \in X_0, \|v\| < \epsilon \),

\[
(p - 1 + q)\|f(v)(w + v\phi)\|^p - \lambda(r + q)\int_{\Omega} b(x)(f(v)(w + v\phi))^{r+1}(x)\,dx > 0,
\]

that is,

\[ f(v)(w + v\phi) \in \mathcal{N}_{\lambda}^+ \text{ for all } v \in B\epsilon(0). \]

This completes the proof of Lemma.

\( \square \)

**Lemma 3.7:** \( J_{\lambda} \) is bounded below and coercive on \( \mathcal{N}_{\lambda} \).

**Proof:** For \( w \in \mathcal{N}_{\lambda} \), we obtain from (3.1) that

\[
J_{\lambda}(w) = \left( \frac{1}{p} - \frac{1}{r + 1} \right)\|w\|^p - \left( \frac{1}{1 - q} - \frac{1}{r + 1} \right)\int_{\Omega} a(x)w^{1-q}(x)\,dx
\]

\[
\geq \left( \frac{1}{p} - \frac{1}{r + 1} \right)\|w\|^p - \left( \frac{1}{1 - q} - \frac{1}{r + 1} \right)\|a\| \left[ \frac{\|w\|}{\sqrt{\lambda}} \right]^{1-q} \quad (3.4)
\]

Now consider the function \( \rho : \mathbb{R}^+ \to \mathbb{R} \) as \( \rho(t) = \alpha t^p - \beta t^{1-q} \), where \( \alpha, \beta \) are both positive constants. One can easily show that \( \rho \) is convex(\( \rho''(t) > 0 \) for all \( t > 0 \)) with \( \rho(t) \to 0 \) as \( t \to 0 \) and \( \rho(t) \to \infty \) as \( t \to \infty \). \( \rho \) achieves its minimum at \( t_{\min} = \left[ \frac{\beta(1-q)}{p\alpha} \right]^{\frac{1}{p-1+q}} \) and

\[
\rho(t_{\min}) = \alpha \left[ \frac{\beta(1-q)}{p\alpha} \right]^{\frac{p}{p-1+q}} - \beta \left[ \frac{\beta(1-q)}{p\alpha} \right]^{\frac{1-q}{p-1+q}}
\]

\[
= - \frac{(p - 1 + q)}{p} \beta \left( \frac{1 - q}{r + 1} \right)^{\frac{1-q}{p-1+q}}.
\]

Applying \( \rho(t) \) with \( \alpha = \left( \frac{1}{p} - \frac{1}{r + 1} \right) \), \( \beta = \left( \frac{1}{1-q} - \frac{1}{r + 1} \right)\|a\| \left[ \frac{1}{\sqrt{\lambda}} \right]^{1-q} \) and \( t = \|w\|, w \in \mathcal{N}_{\lambda} \), we obtain from (3.4) that

\[
\lim_{\|w\| \to \infty} J_{\lambda}(w) \geq \lim_{t \to \infty} \rho(t) = \infty,
\]

since \( 0 < q < 1 \leq p - 1 \). That is \( J_{\lambda} \) is coercive on \( \mathcal{N}_{\lambda} \). Moreover, it follows from (3.4) that

\[
J_{\lambda}(w) \geq \rho(t) \geq \rho(t_{\min})(\text{a constant}), \quad (3.5)
\]
i.e.

\[
J_\lambda(w) \geq \frac{(p-1+q)}{p} \beta_{p^{-1+q}} \left( \frac{1-q}{p\alpha} \right)^{\frac{1-q}{p-1+q}} \left( \frac{r+q}{p(r+1-p)} \right)^{\frac{p}{p-1+q}}.
\]

Thus \( J_\lambda \) is bounded below on \( \mathcal{N}_\lambda \).

4. Existence of solutions in \( \mathcal{N}_\lambda^\pm \)

Now from Lemma 3.5, \( \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0 \) and \( \mathcal{N}_\lambda^- \) are two closed sets in \( X_0 \) provided \( \lambda \in (0, \Lambda) \). Consequently, the Ekeland variational principle can be applied to the problem of finding the infimum of \( J_\lambda \) on both \( \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0 \) and \( \mathcal{N}_\lambda^- \). First, consider \( \{w_k\} \subset \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0 \) with the following properties:

\[
\begin{align*}
J_\lambda(w_k) < \inf_{w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(w) + \frac{1}{k} \quad \text{(4.1)} \\
J_\lambda(w) \geq J_\lambda(w_k) - \frac{1}{k} \|w - w_k\|, \text{ for all } w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0. \quad \text{(4.2)}
\end{align*}
\]

**Lemma 4.1:** Show that the sequence \( \{w_k\} \) is bounded in \( \mathcal{N}_\lambda \). Moreover, there exists \( 0 \neq w \in X_0 \) such that \( w_k \rightharpoonup w \) weakly in \( X_0 \).

**Proof:** From equations (3.5) and (4.1), we have

\[
\begin{align*}
at^p - br^{1-q} = \rho(t) \leq J_\lambda(w) < \inf_{w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(w) + \frac{1}{k} \leq C_5,
\end{align*}
\]

for sufficiently large \( k \) and a suitable positive constant. Hence putting \( t = w_k \) in the above equation, we obtain \( \{w_k\} \) is bounded.

Let \( \{w_k\} \) is bounded in \( X_0 \). Then, there exists a subsequence of \( \{w_k\}_k \), still denoted by \( \{w_k\}_k \) and \( w \in X_0 \) such that \( w_k \rightharpoonup w \) weakly in \( X_0 \), \( w_k(\cdot) \rightarrow w(\cdot) \) strongly in \( L^r(\Omega) \) for \( 1 \leq r < p^* \) and \( w_k(\cdot) \rightarrow w(\cdot) \) a.e. in \( \Omega \).

For any \( w \in \mathcal{N}_\lambda^+ \), we have from \( 0 < q < 1 \leq p - 1 < r \) that

\[
\begin{align*}
J_\lambda(w) &= \left( \frac{1}{p} - \frac{1}{1-q} \right) \|w\|^p + \left( \frac{1}{1-q} - \frac{1}{r+1} \right) \lambda \int_\Omega b(x) w_r^{r+1}(x) dx \\
&< \left( \frac{1}{p} - \frac{1}{1-q} \right) \|w\|^p + \left( \frac{1}{1-q} - \frac{1}{r+1} \right) \frac{p-1+q}{r+q} \|w\|^p \\
&= \left( \frac{1}{r+1} - \frac{1}{p} \right) \frac{(p-1+q)}{(1-q)} \|w\|^p < 0,
\end{align*}
\]

which means that \( \inf_{\mathcal{N}_\lambda^+} J_\lambda < 0 \). Now for \( \lambda \in (0, \Lambda) \), we know from Lemma 3.1, that \( \mathcal{N}_\lambda^0 = \{0\} \). Together, these imply that \( w_k \in \mathcal{N}_\lambda^+ \) for \( k \) large and

\[
\inf_{w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(w) = \inf_{w \in \mathcal{N}_\lambda^+} J_\lambda(w) < 0.
\]
Therefore, by weak lower semi-continuity of norm,

\[ J_\lambda(w) \leq \liminf_{k \to \infty} J_\lambda(w_k) = \inf_{\mathcal{N}^+_{\lambda} \cup \mathcal{N}^0_{\lambda}} J_\lambda < 0, \]

that is, \( w \not\equiv 0 \) and \( w \in X_0 \).

**Lemma 4.2:** Suppose \( w_k \in \mathcal{N}^+_{\lambda} \) such that \( w_k \rightharpoonup w \) weakly in \( X_0 \). Then for \( \lambda \in (0, \Lambda) \),

\[ (p - 1 + q) \int_\Omega a(x)w^{1-q}_+(x)dx - \lambda(r - p + 1) \int_\Omega b(x)w^{r+1}_+(x)dx > 0. \quad (4.3) \]

Moreover, there exists a constant \( C_2 > 0 \) such that

\[ (p - 1 + q)\|w_k\|^p - \lambda(r + q) \int_\Omega b(x)(w_k)^{r+1}_+(x)dx \geq C_2 > 0. \quad (4.4) \]

**Proof:** For \( \{w_k\} \subset \mathcal{N}^+_{\lambda} \subset \mathcal{N}_{\lambda} \), we have

\[
(p - 1 + q) \int_\Omega a(x)w^{1-q}_+(x)dx - \lambda(r - p + 1) \int_\Omega b(x)w^{r+1}_+(x)dx \\
= \lim_{k \to \infty} \left[ (p - 1 + q) \int_\Omega a(x)(w_k)^{1-q}_+(x)dx - \lambda(r - p + 1) \int_\Omega b(x)(w_k)^{r+1}_+(x)dx \right] \\
= \lim_{k \to \infty} \left[ (p - 1 + q)\|w_k\|^p - \lambda(r + q) \int_\Omega b(x)(w_k)^{r+1}_+(x)dx \right] \geq 0.
\]

Now, we can argue by a contradiction and assume that

\[ (p - 1 + q) \int_\Omega a(x)w^{1-q}_+(x)dx - \lambda(r - p + 1) \int_\Omega b(x)w^{r+1}_+(x)dx = 0. \quad (4.5) \]

Using \( w_k \in \mathcal{N}_{\lambda} \), the weak lower semi-continuity of norm and (4.5) we have that
Thus for any $\lambda \in (0, \Lambda)$ and $w \neq 0$, by similar arguments as those in (3.3) we have that

$$0 < E_\lambda \|w\|^{p+1}$$

$$\leq \frac{(p - 1 + q)}{(r + q)} \left( \frac{r - p + 1}{r + q} \right) \left( \frac{r - p + 1}{p^{(r+q)}} \right) \|w\|^{p(r+q)} - \lambda \int_\Omega b(x)w_+^{r+1}(x)dx$$

$$= \frac{(p - 1 + q)}{(r + q)} \left( \frac{r - p + 1}{r + q} \right) \left( \frac{r - p + 1}{p^{(r+q)}} \right) \|w\|^{p(r+q)} - \frac{(p - 1 + q)}{(r + q)} \|w\|^p = 0,$$

which is clearly impossible. Now by (4.3), we have that

$$(p - 1 + q) \int_\Omega a(x)(w_k)_+^{1-q}(x)dx - \lambda (r - p + 1) \int_\Omega b(x)(w_k)_+^{r+1}(x)dx \geq C_2 \quad (4.6)$$

for sufficiently large $k$ and a suitable positive constant $C_2$. This, together with the fact that $w_k \in \mathcal{N}_\lambda$, we obtain equation (4.4).

Fix $\phi \in C_{\mathcal{X}_0}$ with $\phi \geq 0$. Then we apply Lemma 3.6 with $w = w_k \in \mathcal{N}_\lambda^+$ ($k$ large enough such that $\frac{(1-q)C_1}{k} < C_2$), we obtain a sequence of functions $f_k : B_{\varepsilon_k}(0) \rightarrow \mathbb{R}$ such that $f_k(0) = 1$ and $f_k(w)(w_k + w\phi) \in \mathcal{N}_\lambda^+$ for all $w \in B_{\varepsilon_k}(0)$. It follows from $w_k \in \mathcal{N}_\lambda$ and $f_k(w)(w_k + w\phi) \in \mathcal{N}_\lambda$ that

$$\|w_k\|^p - \int_\Omega a(x)(w_k)_+^{1-q}(x)dx - \lambda \int_\Omega b(x)(w_k)_+^{r+1}(x)dx = 0 \quad (4.7)$$

and

$$f_k^p(w)\|w_k + w\phi\|^p - f_k^{1-q}(w) \int_\Omega a(x)(w_k + w\phi)_+^{1-q}dx$$

$$- \lambda f_k^{r+1}(w) \int_\Omega b(x)(w_k + w\phi)_+^{r+1}dx = 0. \quad (4.8)$$

Choose $0 < \rho < \varepsilon_k$, and $w = \rho v$ with $\|v\| < 1$ then we find $f_k(w)$ such that $f_k(0) = 1$ and $f_k(w)(w_k + w\phi) \in \mathcal{N}_\lambda^+$ for all $w \in B_\rho(0)$. Also we will use the following notation:

$$w_k(x,y) := |w_k(x) - w_k(y)|^{p-2}(w_k(x) - w_k(y))$$

Lemma 4.3: For $\lambda \in (0, \Lambda)$ we have $|f_k'(0), v|$ is finite for every $0 \leq v \in C_{\mathcal{X}_0}$ with $\|v\| \leq 1$. 

Proof: From (4.7) and (4.8), we have that

\[
0 = [f_k^p(w) - 1]||w_k + w\phi||^p + ||w_k + w\phi||^p - ||w_k||^p
- [f_k^{1-q}(w) - 1] \int_\Omega a(x)(w_k + w\phi)^{1-q}_+ dx
- \int_\Omega a(x)[((w_k + w\phi)^{1-q}_+ - (w_k)^{1-q}_+)]dx
- \lambda[f_k^{r+1}(w) - 1] \int_\Omega b(x)(w_k + w\phi)^{r+1}_+ dx
- \lambda \int_\Omega b(x)[((w_k + w\phi)^{r+1}_+ - (w_k)^{r+1}_+)]dx
\]

Since

\[
(w_k + w\phi)^{1-q}_+(x) - (w_k)^{1-q}_+(x) = \begin{cases} 
(w_k + w\phi)^{1-q}_+(x) - (w_k)^{1-q}_+(x) & \text{if } w_k \geq 0 \\
0 & \text{if } w_k \leq 0, w_k + w\phi \leq 0 \\
(w_k + w\phi)^{1-q}_+(x) & \text{if } w_k \leq 0, w_k + w\phi \geq 0,
\end{cases}
\]

we have \(\int_\Omega a(x)[((w_k + w\phi)^{1-q}_+ - (w_k)^{1-q}_+)(x)]dx \geq 0\).

Now dividing by \(\rho > 0\) and passing to the limit \(\rho \to 0\), we derive that

\[
0 \leq p \langle f'_k(0), \nu \rangle ||w_k||^p + p \int_Q \frac{w_k(x,y)((v\phi)(x) - (v\phi)(y))}{|x - y|^{n+ps}} dxdy
- (1 - q) \langle f'_k(0), \nu \rangle \int_\Omega a(x)(w_k)^{1-q}_+ dx
- \lambda(r + 1) \left[ \langle f'_k(0), \nu \rangle \int_\Omega b(x)(w_k)^{r+1}_+ dx + \int_\Omega b(x)(w_k)^{r+1}_+ v\phi dx \right]
= \langle f'_k(0), \nu \rangle \left[ p ||w_k||^p - (1 - q) \int_\Omega a(x)(w_k)^{1-q}_+(x) dx - \lambda(r + 1) \int_\Omega b(x)(w_k)^{r+1}_+(x) dx \right]
+ p \int_Q \frac{w_k(x,y)((v\phi)(x) - (v\phi)(y))}{|x - y|^{n+ps}} dxdy - \lambda(r + 1) \int_\Omega b(x)(w_k)^{r+1}_+ v\phi dx
\]

\[
= \langle f'_k(0), \nu \rangle \left[ (p - 1 + q) ||w_k||^p - \lambda(r + q) \int_\Omega b(x)(w_k)^{r+1}_+(x) dx \right]
- \lambda(r + 1) \int_\Omega b(x)(w_k)^{r}_+ v\phi dx
+ p \int_Q \frac{w_k(x,y)((v\phi)(x) - (v\phi)(y))}{|x - y|^{n+ps}} dxdy. \tag{4.10}
\]
From (4.4) and (4.10), we know immediately that \( \langle f_k'(0), v \rangle \neq -\infty \). Now we show that \( \langle f_k'(0), v \rangle \neq +\infty \). Arguing by contradiction, we assume that \( \langle f_k'(0), v \rangle = +\infty \). Since

\[
|f_k(\rho v) - 1|\|w_k\| + f_k(\rho v)\|\rho v\phi\| \geq \|f_k(\rho v) - 1\|w_k + \rho v f_k(\rho v)\phi\| \\
= \|f_k(\rho v)(w_k + \rho v\phi) - w_k\| 
\]

and

\[
f_k(\rho v) > f_k(0) = 1
\]

for sufficiently large \( k \). From the definition of derivative \( \langle f_k'(0), v \rangle \), applying Equation (4.2) with \( w = f_k(\rho v)(w_k + \rho v\phi) \in \mathcal{N}_\lambda^+ \), we clearly have that

\[
[f_k(\rho v) - 1]\frac{\|w_k\|}{k} + f_k(\rho v)\frac{\|\rho v\phi\|}{k} \\
\geq \frac{1}{k}f_k(\rho v)(w_k + \rho v\phi) - w_k\| \\
\geq J_\lambda(w_k) - J_\lambda(f_k(\rho v)(w_k + \rho v\phi)) \\
= \left(\frac{1}{p} - \frac{1}{1 - q}\right)\|w_k\|^p + \lambda \left(\frac{1}{1 - q} - \frac{1}{r + 1}\right)\int_\Omega b(x)(w_k)^{r+1}_+dx \\
+ \left(\frac{1}{1 - q} - \frac{1}{p}\right)f_k^p(\rho v)\|w_k + \rho v\phi\|^p \\
- \lambda \left(\frac{1}{1 - q} - \frac{1}{r + 1}\right)f_k^{r+1}(\rho v)\int_\Omega b(x)(w_k + \rho v\phi)^{r+1}_+dx \\
= \left(\frac{1}{1 - q} - \frac{1}{p}\right)\|w_k + \rho v\phi\|^p - \|w_k\|^p + \left(\frac{1}{1 - q} - \frac{1}{p}\right)[f_k^p(\rho v) - 1]\|w_k + \rho v\phi\|^p \\
- \lambda \left(\frac{1}{1 - q} - \frac{1}{r + 1}\right)f_k^{r+1}(\rho v)\int_\Omega b(x)[(w_k + \rho v\phi)^{r+1}_+ - (w_k)^{r+1}_+(x)]dx \\
- \lambda \left(\frac{1}{1 - q} - \frac{1}{r + 1}\right)[f_k^{r+1}(\rho v) - 1]\int_\Omega b(x)(w_k)^{r+1}_+dx.
\]

Dividing by \( \rho > 0 \) and passing to the limit as \( \rho \to 0 \), we can obtain that

\[
\langle f_k'(0), v \rangle \frac{\|w_k\|}{k} + \frac{\|\rho v\phi\|}{k} \geq \left(\frac{p - 1 + q}{1 - q}\right)\langle f_k'(0), v \rangle\|w_k\|^p \\
- \lambda \left(\frac{r + q}{1 - q}\right)\int_\Omega b(x)(w_k)^r_+\phi dx \\
+ \left(\frac{p - 1 + q}{1 - q}\right)\int_Q w_k(x, y)((\rho v\phi)(x) - (\rho v\phi)(y)) \frac{dx dy}{|x - y|^{n+ps}} \\
- \lambda \left(\frac{r + q}{1 - q}\right)\langle f_k'(0), v \rangle\int_\Omega b(x)(w_k)^{r+1}_+dx \\
= \frac{\langle f_k'(0), v \rangle}{1 - q} \left[\|w_k\|^p - \lambda(r + q)\int_\Omega b(x)(w_k)^{r+1}_+(x)dx\right]
\]
\[ + \left( \frac{p - 1 + q}{1 - q} \right) \int_Q \frac{w_k(x, y)((v\phi)(x) - (v\phi)(y))}{|x - y|^{n + ps}} \, dx \, dy \]
\[ - \lambda \left( \frac{r + q}{1 - q} \right) \int_{\Omega} b(x)(w_k')_+^r \, v\phi \, dx \]

that is,
\[
\frac{\|v\phi\|}{k} \geq \frac{\langle f_k'(0), v \rangle}{1 - q} \left[ (p - 1 + q)\|w_k\|^p - \lambda(r + q) \int_{\Omega} b(x)(w_k')_+^{r+1}(x) \, dx - \frac{(1 - q)\|w_k\|}{k} \right]
\[ + \left( \frac{p - 1 + q}{1 - q} \right) \int_Q \frac{w_k(x, y)((v\phi)(x) - (v\phi)(y))}{|x - y|^{n + ps}} \, dx \, dy \]
\[ - \lambda \left( \frac{r + q}{1 - q} \right) \int_{\Omega} b(x)(w_k')_+^r \, v\phi \, dx \quad (4.12)\]

which is impossible because \( \langle f_k'(0), v \rangle = +\infty \) and
\[
(p - 1 + q)\|w_k\|^p - \lambda(r + q) \int_{\Omega} b(x)(w_k')_+^{r+1}(x) \, dx - \frac{(1 - q)\|w_k\|}{k} \geq C_2 - \frac{(1 - q)C_1}{k} > 0.
\]

In conclusion, \( |\langle f_k'(0), v \rangle| < +\infty \). Furthermore (4.4) with \( \|w_k\| \leq C_1 \) and two inequalities (4.10) and (4.12) also imply that
\[
|\langle f_k'(0), v \rangle| \leq C_3
\]
for \( k \) sufficiently large and a suitable constant \( C_3 \).

**Lemma 4.4:** For each \( 0 \leq \phi \in C_{X_0} \) and for every \( 0 \leq v \in X_0 \) with \( \|v\| \leq 1 \), we have \( a(x)w_-^{-q}v\phi \in L^1(\Omega) \) and
\[
\int_Q \frac{w(x, y)((v\phi)(x) - (v\phi)(y))}{|x - y|^{n + ps}} \, dx \, dy - \int_{\Omega} a(x)w_-^{-q}v\phi \, dx - \lambda \int_{\Omega} b(x)w_-^r v\phi \, dx \geq 0, \quad (4.13)
\]
where \( w(x, y) = |w(x) - w(y)|^{p-2}(w(x) - w(y)) \).

**Proof:** Applying (4.11) and (4.2) again, we have that
\[
[f_k(\rho\nu) - 1] \frac{\|w_k\|}{k} + f_k(\rho\nu) \frac{\|\rho v\phi\|}{k}
\geq \frac{1}{k} [f_k(\rho\nu)(w_k + \rho v\phi) - w_k]
\geq f_k(\lambda)(w_k) - f_k(\rho\nu)(w_k + \rho v\phi))
= \frac{1}{p} \|w_k\|^p - \frac{1}{1 - q} \int_{\Omega} a(x)(w_k)_+^{1-q} \, dx
\[ - \frac{\lambda}{r + 1} \int_{\Omega} b(x)(w_k')_+^{r+1} \, dx - \frac{1}{p} \|f_k(\rho\nu)(w_k + \rho v\phi)\|^p \, dx
\[ + \frac{1}{1 - q} \int_{\Omega} a(x)(f_k(\rho\nu)(w_k + \rho v\phi))_+^{1-q} \, dx
\[ + \frac{\lambda}{r + 1} \int_{\Omega} b(x)(f_k(\rho\nu)(w_k + \rho v\phi))_+^{r+1} \, dx
\]
\[
\begin{align*}
= & \frac{f_k^p(\rho v) - 1}{p} \|w_k\|^p - \frac{f_k^p(\rho v)}{p} (\|w_k + \rho v\|^p - \|w_k\|^p) \\
+ & \frac{f_k^{1-q}(\rho v) - 1}{1 - q} \int_{\Omega} a(x)(w_k + \rho v\phi)^{1-q}(x) \\
+ & \frac{1}{1 - q} \int_{\Omega} a(x)[((w_k + \rho v\phi)^{1-q} - (w_k)^{1-q})(x)] \\
+ & \lambda \frac{f_k^{r+1}(\rho v) - 1}{r + 1} \int_{\Omega} b(x)(w_k + \rho v\phi)^{r+1}(x) \\
+ & \frac{\lambda}{r + 1} \int_{\Omega} b(x)[((w_k + \rho v\phi)^{r+1} - (w_k)^{r+1})(x)].
\end{align*}
\]

Dividing by \(\rho > 0\) and passing to the limit \(\rho \to 0^+\), we obtain

\[
\begin{align*}
|\langle f'_k(0), v \rangle| \frac{\|w_k\|}{k} + \|v\phi\| \frac{\|w_k\|}{k} \\
\geq & -|\langle f'_k(0), v \rangle| \|w_k\|^p - \int_{Q} \frac{w_k(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} \, dx \, dy \\
+ & \langle f'_k(0), v \rangle \int_{\Omega} a(x)(w_k)^{1-q}(x) \, dx \\
+ & \liminf_{\rho \to 0^+} \frac{1}{1 - q} \int_{\Omega} a(x)[((w_k + \rho v\phi)^{1-q} - (w_k)^{1-q})(x)] \, dx \\
+ & \lambda \langle f'_k(0), v \rangle \int_{\Omega} b(x)(w_k)^{r+1} \, dx + \lambda \int_{\Omega} b(x)(w_k)^{r} v\phi \, dx \\
= & -|\langle f'_k(0), v \rangle| \left[ \|w_k\|^p - \int_{\Omega} a(x)(w_k)^{1-q}(x) \, dx - \lambda \int_{\Omega} b(x)(w_k)^{r+1} \, dx \right] \\
- & \int_{Q} \frac{w_k(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} \, dx \, dy + \lambda \int_{\Omega} b(x)(w_k)^{r} v\phi \, dx \\
+ & \liminf_{\rho \to 0^+} \frac{1}{1 - q} \int_{\Omega} a(x)[((w_k + \rho v\phi)^{1-q} - (w_k)^{1-q})(x)] \, dx \\
= & - \int_{Q} \frac{w_k(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} \, dx \, dy + \lambda \int_{\Omega} b(x)(w_k)^{r} v\phi \, dx \\
+ & \liminf_{\rho \to 0^+} \frac{1}{1 - q} \int_{\Omega} a(x)[((w_k + \rho v\phi)^{1-q} - (w_k)^{1-q})(x)] \, dx.
\end{align*}
\]

Then by above inequality, one can see that

\[
\liminf_{\rho \to 0^+} \int_{\Omega} \frac{a(x)[((w_k + \rho v\phi)^{1-q} - (w_k)^{1-q})(x)]}{\rho} \, dx
\]
is finite. Now, using (4.9), we have \( a(x)((w_k + \rho v\phi)_+^{1-q} - (w_k)_+^{1-q}) \geq 0 \) for all \( x \in \Omega \), for all \( t > 0 \), then by the Fatou Lemma, we have that

\[
\int_{\Omega} a(x)(w_k)_+^{-q}v\phi \, dx \leq \liminf_{\rho \to 0^+} \frac{1}{1-q} \int_{\Omega} a(x)((w_k + \rho v\phi)_+^{1-q} - (w_k)_+^{1-q}) \, dx
\]

\[
\leq \frac{|\langle f'(0), v \rangle|}{\frac{k}{\rho}} \|w_k\| + \|v\| - \lambda \int_{\Omega} b(x)(w_k)_+^r v\phi \, dx + \int_{Q} \frac{w_k(x,y)(\phi(x) - \phi(y))}{|x-y|^{n+ps}} \, dx \, dy
\]

\[
\leq C_1 C_3 \|v\| + \|v\| - \lambda \int_{\Omega} b(x)(w_k)_+^r v\phi \, dx + \int_{Q} \frac{w_k(x,y)(v\phi(x) - v\phi(y))}{|x-y|^{n+ps}} \, dx \, dy
\]

Again using the Fatou Lemma and the above relation, we have

\[
\int_{\Omega} a(x)w_+^{-q}v\phi \, dx \leq \liminf_{k \to \infty} \frac{1}{\|w_k\|^{1-q}} \int_{\Omega} a(x)(w_k)_+^{-q}v\phi \, dx
\]

\[
= \int_{Q} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(v\phi(x) - v\phi(y))}{|x-y|^{n+ps}} \, dx \, dy
\]

\[
- \lambda \int_{\Omega} b(x)w_+^r v\phi \, dx,
\]

which completes the proof of Lemma.

\[\square\]

**Corollary 4.5:** For every \( 0 \leq \phi \in X_0 \), we have \( a(x)w_+^{-q}\phi \in L^1(\Omega) \), \( w_+ > 0 \) in \( \Omega \) and

\[
\int_{Q} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(\phi(x) - \phi(y))}{|x-y|^{n+ps}} \, dx \, dy - \int_{\Omega} aw_+^{-q}\phi \, dx - \lambda \int_{\Omega} bw_+^r \phi \, dx \geq 0. \quad (4.14)
\]

**Proof:** Choosing \( v \in X_0 \) such that \( v \geq 0 \), \( v \equiv l \) in the neighbourhood of support of \( \phi \) and \( \|v\| \leq 1 \), for some \( l > 0 \) is a constant. Then we note that \( \int_{\Omega} a(x)w_+^{-q}\phi \, dx < \infty \), for every \( 0 \leq \phi \in C_0 \), which guarantees that \( w_+ > 0 \) a.e in \( \Omega \). Putting this choice of \( v \) in (4.13), we have for every \( 0 \leq \phi \in C_0 \)

\[
\int_{Q} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(\phi(x) - \phi(y))}{|x-y|^{n+ps}} \, dx \, dy - \int_{\Omega} aw_+^{-q}\phi \, dx - \lambda \int_{\Omega} bw_+^r \phi \, dx \geq 0.
\]

Hence by density argument, (4.14) holds for every \( 0 \leq \phi \in X_0 \), which completes the proof of the Corollary.

**Lemma 4.6:** We show that \( w > 0 \) and \( w \in N_+^\ast \).

**Proof:** Using (4.14) with \( \phi = w_+ \), we obtain that

\[
0 \leq \int_{Q} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(w_-(x) - w_-(y))}{|x-y|^{n+ps}} \, dx \, dy
\]

\[
\leq -\|w_-\|^2 - 2 \int_{Q} \frac{|w(x) - w(y)|^{p-2}w_-(x)w_+(y)}{|x-y|^{n+ps}} \, dx \, dy \leq -\|w_-\|^2 \leq 0.
\]

i.e. \( w_- = 0 \) a.e. So, \( w = w_+ > 0 \) a.e by Corollary 4.5. Hence \( w > 0 \) in \( \Omega \). Now using (4.14) with \( \phi = w \), we obtain that

\[
\|w\|^p \geq \int_{\Omega} a(x)w_+^{1-q}(x) \, dx + \lambda \int_{\Omega} b(x)w_+^{r+1}(x) \, dx.
\]
On the other hand, by the weak lower semi-continuity of the norm, we have that
\[ \|w\|^p \leq \liminf_{k \to \infty} \|w_k\|^p \leq \limsup_{k \to \infty} \|w_k\|^p \]
\[ = \int_\Omega a(x)w_+^{1-q}(x)dx + \lambda \int_\Omega b(x)w_+^{r+1}(x)dx. \]

Thus
\[ \|w\|^p = \int_\Omega a(x)w_+^{1-q}(x)dx + \lambda \int_\Omega b(x)w_+^{r+1}(x)dx. \tag{4.15} \]

Consequently, \( w_k \to w \) in \( X_0 \) and \( w \in \mathcal{N}_\lambda \). Now from (4.3) it follows that
\[ (p - 1 + q)\|w\|^p - \lambda(r + q) \int_\Omega b(x)w_+^{r+1}(x)dx \]
\[ = (p - 1 + q) \int_\Omega a(x)w_+^{1-q}(x)dx \]
\[ - \lambda(r - p + 1) \int_\Omega b(x)w_+^{r+1}(x)dx > 0, \]
that is, \( w \in \mathcal{N}^+_\lambda. \)

**Lemma 4.7:** Show that \( w \) is in fact a positive weak solution of problem \((P_\lambda)\).

**Proof:** Suppose \( \phi \in X_0 \) and \( \epsilon > 0 \), then we define \( \Psi(x) = (w + \epsilon \phi)_+(x) \). Let \( \Omega = \Omega_1 \times \Omega_2 \) with
\[ \Omega_1 := \{ x \in \Omega : w(x) + \epsilon \phi(x) > 0 \} \]
and \( \Omega_2 := \{ x \in \Omega : w(x) + \epsilon \phi(x) \leq 0 \} \).

Then \( \Psi|_{\Omega_1}(x) = (w + \epsilon \phi)(x) \), and \( \Psi|_{\Omega_2}(x) = 0 \). Decompose
\[ Q := (\Omega_1 \times \Omega_2) \cup (\Omega_2 \times \Omega_2) \cup (\Omega_2 \times \Omega_1) \]
\[ \cup (\Omega_2 \times \Omega_1) \cup (\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \Omega_1) \cup (\Omega_2 \times \Omega_2). \]

Let \( M(x,y) = w(x,y)((w + \epsilon \phi) - (w + \epsilon \phi))K(x,y) \), where \( w(x,y) = |w(x) - w(y)|^{p-2}(w(x) - w(y)) \) and \( K(x,y) = \frac{1}{|x-y|^{n+ps}} \). Then we have
\[ (1) \int_{\Omega_1 \times \Omega_1} M(x,y)dx dy = \int_{\Omega_2 \times \Omega_1} M(x,y)dx dy = 0. \]
\[ (2) \int_{\Omega_2 \times \Omega_2} M(x,y)dx dy = \int_{\Omega_2 \times \Omega_1} |w(x)|^{p-2}w(x)(w + \epsilon \phi)(x)K(x,y)dx dy. \]
\[ (3) \int_{\Omega_2 \times \Omega_2} M(x,y)dx dy = \int_{\Omega_1 \times \Omega_2} |w(x)|^{p-2}w(x)(w + \epsilon \phi)(x)K(x,y)dx dy. \]
\[ (4) \int_{\Omega_1 \times \Omega_1} M(x,y)dx dy = \int_{\Omega_2 \times \Omega_2} w(x,y)(w + \epsilon \phi)(x)K(x,y)dx dy. \]
\[ (5) \int_{\Omega_1 \times \Omega_2} M(x,y)dx dy = \int_{\Omega_1 \times \Omega_2} w(x,y)(w + \epsilon \phi)(x)K(x,y)dx dy. \]
\[ (6) \int_{\Omega_1 \times \Omega_1} M(x,y)dx dy = 0. \]
\[ (7) \int_{\Omega_2 \times \Omega_2} M(x,y)dx dy = \int_{\Omega_2 \times \Omega_2} w(x,y)((w + \epsilon \phi)(x) - (w + \epsilon \phi)(y))K(x,y)dx dy. \]

Putting \( \Psi \) into (4.13) and using (4.15), we see that
\[ 0 \leq \int_Q \frac{w(x,y)(\Psi(x) - \Psi(y))}{|x-y|^{n+ps}} dx dy - \int_\Omega a(x)w_+^{1-q}\Psi dx - \lambda \int_\Omega b(x)w_+^{r}\Psi dx \]
\[
\int_{Q} \frac{w(x, y)((w + \epsilon\phi)(x) - (w + \epsilon\phi)(y))}{|x - y|^{n + p}} \text{dxdy} - \int_{\Omega} a(x)w_{+}^{-q}(w + \epsilon\phi)dx \\
- \lambda \int_{\Omega} b(x)w_{+}^{r}(w + \epsilon\phi)dx - \int_{\Omega} a(x)w_{+}^{-q}(w + \epsilon\phi)dx - \lambda \int_{\Omega} b(x)w_{+}^{r}(w + \epsilon\phi)dx \\
+ \int_{Q} \frac{w(x, y)((w + \epsilon\phi)^{-}(x) - (w + \epsilon\phi)^{-}(y))}{|x - y|^{n + p}} \text{dxdy} \\
= \epsilon \left( \int_{Q} \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n + p}} \text{dxdy} - \int_{\Omega} a(x)w_{+}^{-q}\phi dx - \lambda \int_{\Omega} b(x)w_{+}^{r}\phi dx \right) \\
- \int_{\Omega} a(x)w_{+}^{-q}(w + \epsilon\phi)dx \\
+ \int_{Q} \frac{|w(x) - w(y)|^{p}}{|x - y|^{n + p}} \text{dxdy} + \int_{Q} \frac{w(x, y)((w + \epsilon\phi)^{-}(x) - (w + \epsilon\phi)^{-}(y))}{|x - y|^{n + p}} \text{dxdy} \\
+ \int_{\Omega_{2}} a(x)w_{+}^{-q}(w + \epsilon\phi)dx - \lambda \int_{\Omega_{2}} b(x)w_{+}^{r}(w + \epsilon\phi)dx - \lambda \int_{\Omega_{2}} b(x)w_{+}^{r}(w + \epsilon\phi)dx \\
= \epsilon \left( \int_{Q} \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n + p}} \text{dxdy} - \int_{\Omega} a(x)w_{+}^{-q}\phi dx - \lambda \int_{\Omega} b(x)w_{+}^{r}\phi dx \right) \\
+ \int_{\Omega_{2}} a(x)w_{+}^{-q}(w + \epsilon\phi)dx \\
- \lambda \int_{\Omega_{2}} b(x)w_{+}^{r}(w + \epsilon\phi)dx - \left( 2 \int_{\Omega_{2} \times \Omega^{c}} \frac{|w(x)|^{p-2}w(x)(w + \epsilon\phi)(x)}{|x - y|^{n + p}} \text{dxdy} - 2 \int_{\Omega_{2} \times \Omega_{1}} \frac{w(x, y)(w + \epsilon\phi)(x)}{|x - y|^{n + p}} \text{dxdy} \\
- 2 \int_{\Omega_{2} \times \Omega_{2}} \frac{|w(x) - w(y)|^{p}}{|x - y|^{n + p}} \text{dxdy} \\
- \lambda \int_{\Omega_{2}} b(x)w_{+}^{r}(w + \epsilon\phi)dx - \epsilon \left( 2 \int_{\Omega_{2} \times \Omega^{c}} \frac{|w(x)|^{p-2}w(x)\phi(x)}{|x - y|^{n + p}} \text{dxdy} \\
+ 2 \int_{\Omega_{2} \times \Omega_{2}} \frac{w(x, y)\phi(x)}{|x - y|^{n + p}} \text{dxdy} + 2 \int_{\Omega_{2} \times \Omega_{2}} \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n + p}} \text{dxdy} \right) \\
\leq \epsilon \left( \int_{Q} \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n + p}} \text{dxdy} - \int_{\Omega} a(x)w_{+}^{-q}\phi dx - \lambda \int_{\Omega} b(x)w_{+}^{r}\phi dx \right) \\
- \int_{\Omega_{2} \times \Omega_{1}} \frac{w(x, y)w(x)}{|x - y|^{n + p}} \text{dxdy} + \int_{\Omega_{2}} a(x)w_{+}^{-q}(w + \epsilon\phi)dx \\
- 2\epsilon \left( \int_{\Omega_{2} \times \Omega^{c}} \frac{|w(x)|^{p-2}w(x)\phi(x)}{|x - y|^{n + p}} \text{dxdy} \\
+ \int_{\Omega_{2} \times \Omega_{1}} \frac{w(x, y)\phi(x)}{|x - y|^{n + p}} \text{dxdy} + \int_{\Omega_{2} \times \Omega_{2}} \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n + p}} \text{dxdy} \right)
\[-\lambda \int_{\Omega_2} b(x)w^+_+(w + \epsilon \phi) \, dx \leq \epsilon \left( \int_Q \frac{w(x,y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} \, dxdy \right) - \int_{\Omega} a(x)w^{-q}_+ \, dx - \lambda \int_{\Omega} b(x)w^+_+ \phi \, dx \]

\[+ 2\epsilon \left( \int_{\Omega_2 \times \Omega_1} \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} \, dxdy \right)^{\frac{p-1}{p}} \left( \int_{\Omega_2 \times \Omega_1} \frac{|\phi(x)|^p}{|x - y|^{n+ps}} \, dxdy \right)^{\frac{1}{p}} - 2\epsilon \left( \int_{\Omega_2 \times \Omega_2} \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} \, dxdy \right)^{\frac{p-1}{p}} \left( \int_{\Omega_2 \times \Omega_1} \frac{|\phi(x)|^p}{|x - y|^{n+ps}} \, dxdy \right)^{\frac{1}{p}}

\[+ \int_{\Omega_2} |\phi|^{p^*_r} \, dx \right)^{\frac{r+1}{r^*_r}} - \epsilon \lambda \int_{\Omega_2} b(x)(w^+_+ \phi) \, dx \]

Since the measure of the domain of integration \(\Omega_2 = \{x \in \Omega|(w + \epsilon \phi)(x) \leq 0\}\) tends to zero as \(\epsilon \to 0\), it follows that \(\int_{\Omega_2 \times \Omega_1} \frac{|\phi(x)|^p}{|x - y|^{n+ps}} \, dxdy \to 0\) as \(\epsilon \to 0\), and similarly \(\int_{\Omega_2 \times \Omega_2} \frac{|\phi(x)|^p}{|x - y|^{n+ps}} \, dxdy, \int_{\Omega_2 \times \Omega_2} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{n+ps}} \, dxdy, \lambda \int_{\Omega_2} b(x)w^+_+ \phi \, dx\) and \(\lambda \epsilon \|b\| \left[ \left( \int_{\Omega_2} |\phi|^{p^*_r} \, dx \right)^{\frac{r+1}{r^*_r}} \right]\) all are tend to 0 as \(\epsilon \to 0\). Dividing by \(\epsilon\) and letting \(\epsilon \to 0\), we obtain

\[\int_Q \frac{w(x,y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} \, dxdy - \int_{\Omega} a(x)w^{-q}_+ \, dx - \lambda \int_{\Omega} b(x)w^+_+ \phi \, dx \geq 0\]

and since this holds equally well for \(-\phi\), it follows that \(w\) is indeed a positive weak solution of problem \((P^+_\lambda)\) and hence a positive solution of \((P_\lambda)\).

**Lemma 4.8:** There exists a minimizing sequence \(\{W_k\}_k \in \mathcal{N}^-_\lambda\) such that \(W_k \to W\) strongly in \(\mathcal{N}^-_\lambda\). Moreover, \(W\) is a positive weak solution of \((P_\lambda)\).

**Proof:** Using the Ekeland variational principle again, we may find a minimizing sequence \(\{W_k\}_k \subset \mathcal{N}^-_\lambda\) for the minimizing problem \(\inf_{\mathcal{N}^-_\lambda} J_\lambda\) such that for \(W_k \to W\) weakly in \(X_0\) and pointwise a.e. in \(\Omega\). We can repeat the argument used in Lemma 4.2 to derive that when \(\lambda \in (0, \Lambda)\)

\[(p - 1 + q) \int_{\Omega} a(x)W^{1-q}_+(x) \, dx - \lambda (r - p + 1) \int_{\Omega} b(x)W^{r+1}_+(x) \, dx < 0 \quad (4.16)\]

which yields

\[(p - 1 + q) \int_{\Omega} a(x)(W_k)^{1-q}_+(x) \, dx - \lambda (r - p + 1) \int_{\Omega} b(x)(W_k)^{r+1}_+(x) \, dx \leq -C_4\]
for $k$ sufficiently large and a suitable positive constant $C_4$. At this point, we may proceed exactly as in Lemmas 4.3, 4.4, 4.6, 4.7 and corollary 4.5, we conclude that $W > 0$ is the required positive weak solution of problem $(P_\lambda^+)$ In particular $W \in \mathcal{N}_\lambda$. Moreover from (4.16), it follows that

\[(p - 1 + q)\|W\|^p - \lambda(r + q) \int_\Omega b(x) W_{r+1}^p(x) dx \]
\[= (p - 1 + q) \left[ \int_\Omega a(x) W_{1-q}^r(x) dx + \lambda \int_\Omega b(x) W_{r+1}^p(x) dx \right] \\
- \lambda(r + q) \int_\Omega b(x) W_{r+1}^p(x) dx \\
= (p - 1 + q) \int_\Omega a(x) W_{r+1}^p(x) dx - \lambda(r - p + 1) \int_\Omega b(x) W_{r+1}^p(x) dx < 0,
\]

that is $W \in \mathcal{N}_\lambda^-$. □

**Proof of the Theorem 2.2:** From Lemmas 4.7, 4.8 and 3.4, we can conclude that the problem $(P_\lambda)$ has at least two positive weak solutions $w \in \mathcal{N}_\lambda^+$, $W \in \mathcal{N}_\lambda^-$ with $\|W\| > \|w\|$ for any $\lambda \in (0, \Lambda)$. □

**Proof of the Theorem 2.3:** For any $W \in \mathcal{N}_\lambda^-$, it follows from Lemma 3.4 that

\[\|W\| > A_\lambda = \Lambda^{\frac{-1}{r+p+1}} \left( \frac{p-1+q}{r+q} \right) \frac{1}{\|b\|^{\frac{-1}{r+p+1}}} \left( \frac{1}{\sqrt{\mathcal{S}}} \right)^{\frac{r+1}{r+p+1}} \left( \frac{\Lambda}{\lambda} \right)^{\frac{1}{r+p+1}}.
\]

Thus by the definition of $\Lambda$, and using $\frac{p(r+q)}{(p-1+q)(r-p+1)} - \frac{r+1}{r-p+1} = \frac{1-q}{p-1+q}$, we obtain,

\[\|W\| > \left( 1 + \frac{p - 1 + q}{r - p + 1} \right)^{\frac{1}{r+p+1}} \|a\|^{\frac{1}{r+p+1}} \left( \frac{1}{\sqrt{\mathcal{S}}} \right)^{\frac{1-q}{r+p+1}} \left( \frac{\Lambda}{\lambda} \right)^{\frac{1}{r+p+1}}.
\]

Hence, let $W_\varepsilon \in \mathcal{N}_\lambda^-$ be the solution of problem $(P_\lambda)$ with $r = p - 1 + \varepsilon$, where $\lambda \in (0, \Lambda)$, we have

\[\|W\| > C_\varepsilon \left( \frac{\Lambda}{\lambda} \right)^{\frac{1}{\varepsilon}},
\]

where $C_\varepsilon = \left( 1 + \frac{p - 1 + q}{\varepsilon} \right)^{\frac{1}{r+p+1}} \|a\|^{\frac{1}{r+p+1}} \left( \frac{1}{\sqrt{\mathcal{S}}} \right)^{\frac{1-q}{r+p+1}} \varepsilon \to \infty \text{ as } \varepsilon \to 0^+$. This completes the proof. □

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