Form factors for principal chiral field model with Wess-Zumino-Novikov-Witten term.

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Abstract. We construct the form factors of the trace of energy-momentum tensor for the massless model described by $SU(2)$ principal chiral field model with WZNW term on level 1. We explain how this construction can be generalized to a class of integrable massless models including the flow from tricritical to critical Ising model.

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From F. Smirnov. During several months I worked with Pierre Mejean. After his premature decease which deeply affected everybody who new him I decided to collect and to publish the results which we obtained together.

1 Introduction

There is a wide class of integrable models which describe the flows between two different models of Conformal Field Theory (CFT) in UV and IR regions [1]. To describe these models the factorized scattering of massless particles was proposed in [2]. In spite of general difficulties arising from the fact that the scattering of massless particles is not properly defined in 2D the application of this method provided very good results for several particular models. Certainly, the physical results are related to possibility of extracting the off-shell information from these S-matrices. Similarly to the massive case there are two ways to do that.

The first way is TBA approach. The TBA equations can be written for the massless scattering which allow to calculate the effective central charge and to show that the latter really interpolates between corresponding UV and IR values [2].

The second way consists in generalization proposed in [3] of the form factor bootstrap approach (which is originally formulated for the massive models [4]) to the massless flows. One must distinguish between the form factor bootstrap in massive and massless cases. In the massive case the form factor bootstrap stays on the solid ground because the space of states is well defined as the Fock space of particles. In the massless case this very definition is doubtful and one has to consider the form factor bootstrap rather as intuitive than rigorous procedure. Indeed for certain operators (as, for example, the order-disorder operators for the flow from tricritical to critical Ising model) the straightforward application of the method leads to divergent series for the Green functions [3]. However, the situation can be improved even for those operators, furthermore there are operators for which the series converge providing spectacular examples of interpolation between UV and IR limits [3]. So, the form factor bootstrap is a method which works in massless case in spite of all the problems of general character.

It must be noticed, however, that the complete construction of all the form factors is not given in the paper [3] even for the simplest model describing the flow from tricritical to critical Ising model considered there. In the present paper we shall give the complete construction for more complicated model: the Principal Chiral Field with Wess-Zumino-Novikov-Witten term. We shall explain briefly that our construction can be generalized to a wide class of models including the one considered in [3].

2 Formulation of the problem.

On of the most beautiful examples of the massless flows for which the S-matrix is known is given by the Principal Chiral Field with Wess-Zumino-Novikov-Witten (WZNW) term on level 1 (PCM1):

\[ S = \frac{1}{2\lambda^2} \int tr(g^{-1} \partial_\mu g)(g^{-1} \partial_\mu g)d^2x + i\Gamma(g) \]

where the WZNW term \( \Gamma(g) \) is defined by means of continuation of \( g \) to 3D manifold \( B \) for which the 2D space-time is the boundary:

\[ \Gamma(g) = \int_B \epsilon^{\mu\nu\lambda} tr(g^{-1} \partial_\mu g)(g^{-1} \partial_\nu g)(g^{-1} \partial_\lambda g)d^3x \]

In the UV limit the pure PCM action dominates the central charge being equal to 3. In the IR region the flow is attracted by the fixed point \( \lambda^2 = 8\pi \) which corresponds to the conformal WZNW model with the central charge equal to 1. It is explained in [2] that the flow arrives at the IR point along the direction defined by the irrelevant operator \( TT \) composed of the right and left components of the energy-momentum tensor.

In the IR region the theory is conformal, so, two charialities essentially decouples. One describes corresponding left and right level-1 WZNW models in terms of massless particles. This is exactly left-left and right-right scattering which seems to be doubtful in 2D. The prescription of the paper [2] for the definition of this scattering can be understood in the following way. We know that the level-1 WZNW model coincides with the UV limit of the massive \( SU(2) \)-invariant Thirring model. The local operators for the latter
model are defined via their form factors [4]. For the operators chiral in the limit (as chiral components of the energy-momentum tensor) the limiting values of the correlation functions are obtained by replacing the massive dispersion by the massless ones. On the other hand these limiting correlation functions coincide with the conformal ones. So, we get the representation of the conformal correlation functions for chiral operators in terms of the form factor series with massless particles. The form factors are defined through the S-matrix for the original massive theory which is considered now as S-matrix of massless particles. It is proposed to use this representations of the correlators as a starting point of the description of massless flows.

More precisely, left and right particles are parametrized by the rapidities $\beta$ and $\beta'$ such that the energy-momentum are respectively

$$e = -p = M e^{-\beta}, \quad e = p = M e^{\beta'}$$

where $M$ is the mass scale which can be chosen arbitrary on this stage. The theory possesses $SU(2)_L \otimes SU(2)_R$ symmetry, the left (right) movers are doublets with respect to $SU(2)_L$ ($SU(2)_R$). The factorizable S-matrices which describe the left and right CFT are given by

$$S_{LL}(\beta_1, \beta_2) = S^Y(\beta_1 - \beta_2), \quad S_{RR}(\beta'_1, \beta'_2) = S^Y(\beta'_1 - \beta'_2)$$

where $S^Y(\beta)$ is the Yangian S-matrix for the scattering of spin-$\frac{1}{2}$ particles [3]:

$$S^Y(\beta) = \frac{\Gamma(\frac{1}{2} + \frac{\beta}{2\pi i}) \Gamma(-\frac{\beta}{2\pi i})}{\Gamma(\frac{1}{2} - \frac{\beta}{2\pi i}) \Gamma(\frac{\beta}{2\pi i})} \left( \beta I - \pi i P \right)$$

where $I$ and $P$ are respectively unit and permutation operators acting in the tensor product of two-dimensional isotopic spaces.

The crucial point is in introducing the non-trivial left-right and right-left S-matrices. Contrary to $S_{LL}$ and $S_{RR}$ whose definition is rather formal the S-matrices $S_{LR}$ and $S_{RL}$ allow quite rigorous interpretation. For the PCM the proposal of [3] is

$$S_{RL}(\beta' - \beta) = \frac{1}{S_{LR}(\beta - \beta')} = U(\beta' - \beta), \quad U(\beta) = \tanh \frac{1}{2} \left( \beta - \frac{\pi i}{2} \right)$$

The scale normalization $M$ is fixed by the requirement that the zero of this S-matrix is situated exactly at $\beta = \frac{\pi i}{2}$. It is quite amusing that the IR limit corresponds to $\beta - \beta' = \log \Lambda, \Lambda \to \infty$, indeed in this limit the s-variable goes to zero. This fact is very interesting because in the massive case infinite rapidities are always related to UV behaviour of the form factors which has been investigated in several cases in [4], so, we can use the familiar methods for solving absolutely different problems.

Let us describe the form factor bootstrap approach to massless flows as it is formulated in [3]. Consider the matrix element of certain local operator $\mathcal{O}$ taken between the vacuum and the state containing The left and right particles with rapidities $\beta_1, \ldots, \beta_l$ and $\beta'_1, \ldots, \beta'_k$ respectively:

$$f_{\mathcal{O}}(\beta_1, \ldots, \beta_l | \beta'_1, \ldots, \beta'_k)$$

It is very convenient to collect all the rapidities together into the set $\theta_1, \ldots, \theta_{k+l} = \beta_1, \ldots, \beta_l, \beta'_1, \ldots, \beta'_k$ and to introduce index $a_i = L, R$ which distinguish the left and right particles. The first requirement of the form factors is that of symmetry:

$$f_{\mathcal{O}}(\cdots, \theta_i, \theta_{i+1}, \cdots, a_{i}, a_{i+1}, \cdots, a_{i+l}, a_{i+l+1}) = f_{\mathcal{O}}(\cdots, \theta_{i+1}, \theta_i, \cdots, a_{i+1}, a_i, \cdots, a_{i+l}, a_{i+l+1})$$

(2)

If $a_i \neq a_j$ this equation has to be considered as definition which allows to construct the form factor with arbitrary placed left and right particles starting from [3].

The second requirement is

$$f_{\mathcal{O}}(\theta_1, \cdots, \theta_{k+l-1}, \theta_{k+l} + 2\pi i)_{a_1, \ldots, a_{k+l-1}, a_{k+l}} = f_{\mathcal{O}}(\theta_{k+l}, \theta_1, \cdots, \theta_{k+l-1})_{a_{k+l}, a_1, \ldots, a_{k+l-1}} =$$

$$= f_{\mathcal{O}}(\theta_1, \cdots, \theta_{k+l-1}, \theta_{k+l})_{a_1, \cdots, a_{k+l-1}, a_{k+l}} =$$

$$= f_{\mathcal{O}}(\theta_1, \cdots, \theta_{k+l-2}, \theta_{k+l-1}, \theta_{k+l})_{a_1, \cdots, a_{k+l-2}, a_{k+l-1}, a_{k+l}} =$$

$$= \delta_{a_{k+l-1}, a_{k+l}} f_{\mathcal{O}}(\theta_1, \cdots, \theta_{k+l-2})_{a_1, \cdots, a_{k+l-2}} \otimes c_{k+l-1,k+l}$$

$$\times (1 - S_{a_{k+l-1}, a_1} (\theta_{k+l-1} - \theta_1) \cdots S_{a_{k+l-2}, a_{k+l-1}} (\theta_{k+l-1} - \theta_{k+l-2}))$$

(3)

Since we do not have bound states in the theory the form factor $f_{\mathcal{O}}(\theta_1, \cdots, \theta_{k+l})$ is supposed to be a meromorphic function of $\theta_{k+l}$ in the strip $0 < \theta_{k+l} < 2\pi$ whose only singularities are the simple poles at the points $\theta_{k+l} = \theta_j + \pi i$. It is important that these singularities appear only in left-left and right-right channels. The residue at $\theta_{k+l} = \theta_j + \pi i$ is given by

$$2\pi i \text{res} f_{\mathcal{O}}(\theta_1, \cdots, \theta_{k+l-2}, \theta_{k+l-1}, \theta_{k+l})_{a_1, \cdots, a_{k+l-2}, a_{k+l-1}, a_{k+l}} =$$

$$= \delta_{a_{k+l-1}, a_{k+l}} f_{\mathcal{O}}(\theta_1, \cdots, \theta_{k+l-2})_{a_1, \cdots, a_{k+l-2}} \otimes c_{k+l-1,k+l}$$

$$\times (1 - S_{a_{k+l-1}, a_1} (\theta_{k+l-1} - \theta_1) \cdots S_{a_{k+l-2}, a_{k+l-1}} (\theta_{k+l-1} - \theta_{k+l-2}))$$

(4)
here \( c_{k+l-1,k+l} \) is a vector in the tensor product of two isotopic spaces constructed from the charge conjugation matrix; in our case it is the singlet vector in the tensor product of two spin-\( \frac{1}{2} \) representations of \( SU(2) \).

These requirements on the massless form factors do not differ too much from the form factor axioms of [4]. However, the physical situation is quite different and the solutions to these equation can not be found in [4].

### 3 Form factors of the energy-momentum tensor.

It the present paper we are going to construct the form factors of the trace of energy-momentum tensor (\( \Theta \)) for PCM. Our methods are applicable to other operators, but we are considering this particular one because of its nice properties and physical importance. Since the symmetry under \( SU(2)_L \otimes SU(2)_R \) is not broken by the perturbation the form factors have to be singlets with respect to both isotopic groups. That is why \( l = 2n \) and \( k = 2m \). The form factors of \( \Theta \) satisfy general conditions (3, 5, 6) and additional requirements following from the fact that we consider this particular operator.

1. The energy-momentum conservation implies that

\[
\frac{2\pi i \text{ res}_{\beta_2=\beta_1+\pi i} \sum e^{-\beta_j} f(\beta_1, \ldots, \beta_{2n} | \beta_1', \beta_2')}{\text{res}_{\beta_2=\beta_1+\pi i} \sum e^{-\beta_j} f(\beta_1, \ldots, \beta_{2n} | \beta_1', \beta_2')}
\]

where for \( n > 1 \) and \( m > 1 \) the function \( f \) does not other singularities that of \( f_{\Theta} \), for \( n = 1(m = 1) \) it has simple poles at \( \beta_2 = \beta_1 + \pi i (\beta_2' = \beta_1' + \pi i) \) which are cancelled by \( e^{-\beta_1} + e^{-\beta_2} (e^{\beta_1'} + e^{\beta_2'}) \).

2. The lowest form factor of \( \Theta \) is that corresponding to 2+2 particles. However by the conservation law we can construct from \( f_{\Theta} \) the form factors of the left and right components of the energy momentum tensor \( T \) and \( \overline{T} \) whose lowest form factors are of the type \( 2n+0 \) and \( 0+2m \) respectively. These lowest form factors must coincide with the form factors of pure \( k = 1 \) WZNW model i.e. with those of \( SU(2) \)-invariant Thirring model. One easily finds that it implies:

\[
2\pi i \text{ res}_{\beta_2=\beta_1+\pi i} \sum e^{-\beta_j} f(\beta_1, \ldots, \beta_{2n} | \beta_1', \beta_2') =
\]

\[
\left( \frac{1}{e^{\beta_1'} + e^{-\beta_2'}} \right) \left( \frac{1}{e^{\beta_1} + e^{-\beta_2}} \right)
\]

\[
= \tilde{f}_T(\beta_1, \ldots, \beta_{2n} \otimes c_{1,2} (1 - S_{LR}(\beta_1' - \beta_1) \cdots S_{LR}(\beta_1' - \beta_1)),
\]

\[
2\pi i \text{ res}_{\beta_2=\beta_1+\pi i} \sum e^{-\beta_j} f(\beta_1, \beta_2 | \beta_1', \beta_2') =
\]

\[
= \tilde{f}_{\overline{T}}(\beta_1', \ldots, \beta_{2m} \otimes c_{1,2} (1 - S_{RL}(\beta_1 - \beta_1') \cdots S_{RL}(\beta_1 - \beta_1') \cdots )
\]

3. The IR limit corresponds to \( \beta_i - \beta_j' \approx \log A \) and \( \Lambda \to \infty \). In this limit one has to reproduce the operator \( T \overline{T} \) which defines the direction of the flow in the IR region. So, we must have

\[
f_{\Theta}(\beta_1 + \log A, \ldots, \beta_{2n} + \log A | \beta_1', \ldots, \beta_{2m}') = (M \Lambda)^{-2} \tilde{f}_T(\beta_1, \ldots, \beta_{2n}) \tilde{f}_{\overline{T}}(\beta_1', \ldots, \beta_{2m}')
\]

Let us try to satisfy all this requirement. The simple form of the left-right S-matrix allows to exclude it from the equations (3, 5). Consider the function \( g \) defined as follows:

\[
f_{\Theta}(\beta_1, \ldots, \beta_{2n} | \beta_1', \ldots, \beta_{2m}) = \prod \psi(\beta, \beta') g(\beta_1, \ldots, \beta_{2n} | \beta_1', \ldots, \beta_{2m})
\]

where

\[
\psi(\beta, \beta') = 2^{-\frac{3}{2}} \left( \frac{1}{4} \left( \beta + \beta' \right) - \int_0^\infty \frac{2 \sin^2 \frac{1}{2} (\beta - \beta' + \pi i) k + \sin^2 \frac{\pi k}{2} \cosh \frac{\pi k}{2} dk}{2 k \sinh \pi k \cosh \frac{\pi k}{2}} \right)
\]

The function \( \psi(\beta, \beta') \) satisfies the equations

\[
\psi(\beta, \beta' + 2\pi i) = \psi(\beta, \beta') S_{RL}(\beta' - \beta), \quad \psi(\beta + 2\pi i, \beta') = \psi(\beta, \beta') S_{LR}(\beta' - \beta)
\]

\[
\psi(\beta, \beta' + \pi i) \psi(\beta, \beta') = 1 \quad \psi(\beta + \pi i, \beta') \psi(\beta, \beta') = 1
\]

It is clear that the equation (3) rewritten in terms of \( g \) does not contain the left-right S-matrices which means that the function \( g \) must be of the form

\[
g(\beta_1, \ldots, \beta_{2n} | \beta_1', \ldots, \beta_{2m}) = \sum c_{K,L}(\beta_1, \ldots, \beta_{2n} | \beta_1', \ldots, \beta_{2m}) \tilde{f}_K(\beta_1, \ldots, \beta_{2n}) \tilde{f}_{\overline{L}}(\beta_1', \ldots, \beta_{2m})
\]
where \( \hat{f}^K(\beta_1, \ldots, \beta_{2n}) \) and \( \hat{f}^L(\beta'_1, \ldots, \beta'_m) \) are different single solutions (counted by \( K \) and \( L \) whose nature will be explained later) of the equations

\[
\hat{f}^K(\beta_1, \ldots, \beta_{i+1}, \beta_{i+2}, \ldots) = \hat{f}^K(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots), \\
\hat{f}^K(\beta_1, \ldots, \beta_{2n-1}, \beta_{2n} + 2\pi i) = \hat{f}^K(\beta_2, \beta_1, \ldots, \beta_{2n-1}), \\
\hat{f}^L(\beta'_1, \beta'_2, \ldots, \beta'_{i+1}, \beta'_{i+2}, \ldots) = \hat{f}^L(\beta'_1, \beta'_2, \ldots, \beta'_i, \ldots), \\
\hat{f}^L(\beta'_1, \beta'_{2n-1}, \beta'_{2n} + 2\pi i) = \hat{f}^L(\beta'_2, \beta'_1, \ldots, \beta'_{2n-1}),
\]

(11)

The functions \( c_{K,L}(\beta_1, \ldots, \beta_{2n} \mid \beta'_1, \ldots, \beta'_m) \) are quasiconstants: \( 2\pi i \)-periodical symmetrical with respect to \( \beta_1, \ldots, \beta_{2n} \) and \( \beta'_1, \ldots, \beta'_m \), functions with possible singularities only at \( \beta_i, \beta'_i = \pm \infty \). The equations for left and right parts are the same, so, let us concentrate for the moment only on the left one.

It is well known \( [4, 5, 8] \) that the solutions to the equations (11) are counted by the functions \( K(A_1, \ldots, A_{n-1} | B_1, \ldots, B_{2n}) \) which are antisymmetrical polynomials of \( A_1, \ldots, A_{n-1} \) such that \( 1 \leq \deg A_i(K) \leq 2n-1, \forall i \) and symmetrical Laurent polynomials of \( B_1, \ldots, B_{2n} \). The solutions are given by the formula

\[
\hat{f}^K(\beta_1, \ldots, \beta_{2n}) = d^n \exp \left( \frac{N}{2} \sum \beta_j \right) \prod \zeta(\beta_i - \beta_j)
\]

\[ \times \int \int \cdots \int \prod \varphi(\alpha_i, \beta_j) \langle \Delta_n(0) \rangle_n(\alpha_1, \ldots, \alpha_{n-1} | \beta_1, \ldots, \beta_{2n}) K(e^{\alpha_1}, \ldots, e^{\alpha_{n-1}} | e^{\beta_1}, \ldots, e^{\beta_{2n}}) \]

where

\[
\varphi(\alpha_i, \beta_j) = e^{-\frac{1}{2} (\alpha + \beta)} \Gamma\left( \frac{1}{4} + \frac{\alpha - \beta}{2\pi i} \right) \Gamma\left( \frac{1}{4} - \frac{\alpha - \beta}{2\pi i} \right)
\]

We do not give here the formulae for \( \langle \Delta_n(0) \rangle_n(\alpha_1, \ldots, \alpha_{n-1} | \beta_1, \ldots, \beta_{2n}) \) which is a rational function of all variables with values in the tensor product of the isotopic spaces, for \( \zeta(\beta) \) which is certain transcendental function and for the constant \( d \): these formulae can be found in the book \([4, 5, 7]\) (Chapter 7).

It has to be emphasized that the integral (12) vanishes for the two kinds of function \( K \) \([4, 5, 7]\): 

\[
K(A_1, \ldots, A_{n-1} | B_1, \ldots, B_{2n}) = \sum_{k=1}^{n-1} (-1)^k (P(A_k) - P(-A_k)) K'(A_1, \ldots, A_{k-1} | B_1, \ldots, B_{2n}), \\
K(A_1, \ldots, A_{n-1} | B_1, \ldots, B_{2n}) = \sum_{k<\ell} (-1)^{k+\ell} C(A_k, A_\ell) K''(A_1, \ldots, A_{k-1} | B_1, \ldots, B_{2n})
\]

(13)

where \( K', K'' \) are some polynomials of the less number of variables with the same properties as \( K \), \( P(A) = \prod \langle A_k + iB_j \rangle \) and

\[
C(A_1, A_2) = \frac{1}{A_1 A_2} \left\{ \frac{A_1 - A_2}{A_1 + A_2} P(A_1) P(A_2) - P(-A_1) P(-A_2) + (P(-A_1) P(A_2) - P(A_1) P(-A_2)) \right\}
\]

So, the polynomials \( K \) are defined modulo the polynomials of the kind (13), the fact that has been used in \([4, 5, 7]\) to show that we have correct number of solutions to (11).

Combining \([8, 10, 11]\) and (12) we find that the from factors satisfying (2) and (3) are of the form

\[
\hat{f}_0(\beta_1, \ldots, \beta_{2n} \mid \beta'_1, \ldots, \beta'_m) = M^2 \prod \psi(\beta_i, \beta'_j) \prod \zeta(\beta_i - \beta_j) \prod \zeta(\beta'_i - \beta'_j)
\]

\[ \times \int \int \int \int \int \cdots \int \cdots \int \int \cdots \int \prod \varphi(\alpha_i, \beta_j) \langle \Delta_n(0) \rangle_n(\alpha_1, \ldots, \alpha_{n-1} | \beta_1, \ldots, \beta_{2n}) \langle \Delta_n(0) \rangle_n(\alpha'_1, \ldots, \alpha'_{m-1} | \beta'_1, \ldots, \beta'_{2m}) \]

\[ \times M_{n,m}(e^{\alpha_1}, \ldots, e^{\alpha_{n-1}} | e^{\beta_1}, \ldots, e^{\beta_{2n}}) \]

(14)

where \( M_{n,m}(A_1, \ldots, A_{n-1} | B_1, \ldots, B_{2n}) \) is a antisymmetrical polynomial of \( A_1, \ldots, A_{n-1} (A'_1, \ldots, A'_{m-1}) \) whose degree with respect to every variable is from 1 to \( 2n-1 \) (from 1 to \( 2m-1 \)) and symmetrical Laurent polynomial of \( B_1, \ldots, B_{2n} (B'_1, \ldots, B'_{2m}) \).

(5)
Now we have to satisfy the rest of our requirements on the form factors. In the paper [4] there is a general prescription for handling the residue condition (14) for the integrals of the form (12). Applying this prescription to our situation and using the equations (15) one finds that the residue condition (14) is satisfied if and only if the function $M_{n,m}$ possesses the properties:

First,

$$M_{n,m}(A_1, \ldots, A_{n-1}|A_1', \ldots, A_{m-1}|B_1, \ldots, B_{2n-2}, B, -B|B'_1, \ldots, B'_{2m}) =$$

$$= \sum_{k=1}^{n-1} (-1)^k \prod_{p \neq k} (A_p^2 + B^2) M_{n-1,m}^k(A_1, \ldots, A_{n-1}|A_1', \ldots, A_{m-1}|B_1, \ldots, B_{2n-2}|B|B'_1, \ldots, B'_{2m})$$

$$M_{n,m}(A_1, \ldots, A_{n-1}|A_1', \ldots, A_{m-1}|B_1, \ldots, B_{2n}|B'_1, \ldots, B'_{2m-2}, B', -B') =$$

$$= \sum_{k=1}^{m-1} (-1)^k \prod_{p \neq k} ((A_p')^2 + (B')^2) M_{m-1,n}^k(A_1, \ldots, A_{n-1}|A_1', \ldots, A_{m-1}|B_1, \ldots, B_{2n}|B'_1, \ldots, B'_{2m-2}|B')$$

where $M_{n-1,m}^k$ and $M_{m-1,n}^k$ are some polynomials in $A_i$ and $A_i'$.

Second,

$$M_{n-1,m}^k(A_1, \ldots, A_{k-1} \pm iB, A_{k-1}, \ldots, A_{n-1}|A_1', \ldots, A_{m-1}|B_1, \ldots, B_{2n-2}|B|B'_1, \ldots, B'_{2m}) =$$

$$= \pm B \prod_{j=1}^{2m} (B \mp iB_j) M_{n-1,m}^k(A_1, \ldots, A_{k-1} \pm iB, A_{k-1}, \ldots, A_{n-1}|A_1', \ldots, A_{m-1}|B_1, \ldots, B_{2n-2}|B|B'_1, \ldots, B'_{2m})$$

$$M_{m-1,n}^k(A_1, \ldots, A_{n-1}|A_1', \ldots, A_{k-1} \pm iB', A_{k-1}, \ldots, A_{m-1}|B_1, \ldots, B_{2n}|B'_1, \ldots, B'_{2m-2}|B') =$$

$$= \pm B' \prod_{j=1}^{2m} (B' \pm iB_j) M_{m-1,n}^k(A_1, \ldots, A_{n-1}|A_1', \ldots, A_{k-1} \pm iB', A_{k-1}, \ldots, A_{m-1}|B_1, \ldots, B_{2n}|B'_1, \ldots, B'_{2m-2})$$

These equations are necessary and sufficient for the formula (14) to define form factors of a local operator. Certainly, they have infinitely many solutions. We shall give only one of these solutions corresponding to the operator $\Theta$. Let us introduce the notations for the sets of integers: $S = \{1, \ldots, 2n\}$, $S' = \{1, \ldots, 2m\}$ then

$$M_{n,m}(A_1, \ldots, A_{n-1}|A_1', \ldots, A_{m-1}|B_1, \ldots, B_{2n}|B'_1, \ldots, B'_{2m}) = \prod_{i<j} (A_i - A_j) \prod_{i<j} (A_i' - A_j')$$

$$\times \prod_{j=1}^{2n} B_j^{-1} \prod_{i=1}^{n-1} A_i^2 \prod_{i=1}^{m-1} A_i' \sum_{T \subseteq \delta \subseteq \delta} \sum_{\delta \subseteq \delta'} \prod_{j \in T} B_j \prod_{j \in T'} (A_i + iB_j) \prod_{j \in T} (A_i' + iB_j')$$

$$\times \prod_{i,j \in T} (B_i + B_j) \prod_{i,j \in T'} (B_i' + B_j') \prod_{i,j \in T} \left( \prod_{i \neq j} B_i - B_j \prod_{j \in T'} \left( \prod_{i \neq j} B_i' - B_j' \right) \right)$$

$$\times \prod_{i,j \in T} (B_i + iB_j') \prod_{i,j \in T'} (B_i - iB_j') X_{T,T'}(B_1, \ldots, B_{2n}|B'_1, \ldots, B'_{2m})$$

(17)

where $T = S \setminus T$, $T' = S' \setminus T'$,

$$X_{T,T'}(B_1, \ldots, B_{2n}|B'_1, \ldots, B'_{2m}) = \sum_{i_1, i_2 \in T} \prod_{p=1}^{2} \left( \prod_{j \in T} \frac{(B_{i_1} + B_j)}{(B_{i_2} + B_j)} \prod_{j \in T'} \frac{(B_{i_1} + iB_j')}{(B_{i_2} + iB_j')} \right)$$

The polynomial $X_{T,T'}(B_1, \ldots, B_{2n}|B'_1, \ldots, B'_{2m})$ is in fact quite symmetric with respect to replacement $B \leftrightarrow B'$.

Let us show that $M_{n,m}$ satisfies all necessary requirements. The relations (15) are easily checked using the formula

$$\prod_{i=1}^{n-1} \prod_{j=1}^{n-1} (A_i + iB_j) \prod_{i<j} (A_i - A_j) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} (A_i^2 + B_j^2) \prod_{i<j} \frac{1}{B_i - B_j} \det \left( \frac{1}{A_i - iB_j} \right)$$

6
So, $M_{n,m}$ really defines a local operators. We have to show that the additional conditions formulated at the beginning of this section are satisfied in order to show that this local operator is indeed the trace of the energy-momentum tensor.

Obviously, $M_{n,m}$ is a homogeneous function of all its variables $(A, B, A', B')$ of total degree $(m + n)^2 - 2m - 2n$. Considering the formula (14) one realizes that this fact provides that the operator defined by $M_{n,m}$ is Lorentz scalar i.e. its form factors are invariant under simultaneous shift of all the rapidities. Let us consider now the conditions 1-3 formulated earlier.

We start form the condition 3. One finds that

$$
\psi(\beta + \log \Lambda, \beta') \rightarrow \Lambda^{-\frac{1}{2}} e^{-\frac{1}{2} \beta}
$$

The integrals with respect to $\alpha_i$ in (14) are concentrated near the points $\beta_j$, so when $\beta_j$ become of order $\log \Lambda$ the integration variables $\alpha_i$ must be of the same order. One finds that when $\log \Lambda \rightarrow \infty$

$$
M_{n,m}(\Lambda A_1, \cdots, \Lambda A_{n-1}| A_1', \cdots, A_{m-1}'| \Lambda B_1, \cdots, \Lambda B_{2n}| B_1', \cdots, B_{2m}') \rightarrow \Lambda^{2mn+n^2-2n-2} \prod_{j=1}^{2n} B_j^{2m-1} \prod_{j=1}^{2n} B_j^{-1} \prod_{i=1}^{n-1} A_i^3 \prod_{i<j} (A_i^2 - A_j^2) \sum_{j=1}^{2m} B_j^{2m-2} \prod_{i=1}^{m-1} A_i' \prod_{i<j} (A_i'^2 - A_j'^2)
$$

This formula is equivalent to (12) because the form factors of the energy-momentum tensor of $SU(2)$-invariant Thirring model $\tilde{f}_T(\beta_1, \cdots, \beta_{2n})$ are given by the formulae on the type (12) with the polynomial $K$ equal respectively to

$$
M^2 \prod_{j=1}^{2n} B_j^{-1} \prod_{j=1}^{2n} B_j^{-1} \prod_{i=1}^{n-1} A_i^3 \prod_{i<j} (A_i^2 - A_j^2) \quad \text{and} \quad M^2 \sum_{j=1}^{2m} B_j' \prod_{i=1}^{m-1} A_i' \prod_{i<j} (A_i'^2 - A_j'^2)
$$

(18)

Let us consider the condition 1. One finds that

$$
\frac{1}{B_1' + B_2'} M_{n,m}(A_1, \cdots, A_{n-1}| \emptyset | B_1, \cdots, B_{2n}| B_1', B_2') \bigg|_{B_2' = -B_1'} = \prod_{j=1}^{2n} B_j^{-1} \prod_{i=1}^{n-1} A_i^3 \prod_{i<j} (A_i^2 - A_j^2) \left( \prod_{B_j' + iB_j - \prod_{B_j' - iB_j} \right)
$$

which together with (13) gives the first equation from (12), the second relation is proven similarly.

The condition (12) is the most complicated to prove. Naively it has to be equivalent to the fact that $M_{n,m}$ is divisible by $\sum B_j^{-1}$ and $\sum B_j'$, but that is not the case: the function $M_{n,m}$ has to be substituted into the integral hence it is defined modulo the functions of the type (13) (and similar functions of $A_j'$). Thus the divisibility has to be proven modulo these null-polynomials. We have checked this fact for many particular examples, but still we lack a general proof. However, the calculations in particular cases go so nicely that we have no doubt that the relation (12) is satisfied generally.

4 Some generalizations.

The model considered in this paper provides a special case of wide class of massless flows. Consider the massless flow $\tilde{f}_T$ between the UV coset model $su(2)_{k+1} \otimes su(2)_{2k+1}$ and the IR coset model $su(2)_{k} \otimes su(2)_{1}/su(2)_{k+1}$, the latter model is nothing but the minimal model $M_{k+2}$. This flow is defined in UV by the relevant operator of dimension $1 - 2/(2k + 3)$, it arrives at IR region along $T\bar{T}$. The massless S-matrices for these flows are written in terms of RSOS restriction of the sine-Gordon (SG) S-matrix $S^\pi(\beta)$ ($\pi$ is SG coupling constant defined as in (14)). Namely (3),

$$
S_{LL}(\beta_1, \beta_2) = S^\pi_{RSOS}(\beta_1 - \beta_2), \quad S_{RR}(\beta_1', \beta_2') = S^\pi_{RSOS}(\beta_1' - \beta_2')
$$

The left-right S-matrix is independent of $k$, it is the same as above. When $k = \infty$ the model coincides with PCM1. Another extreme case is $k = 1$ when the model describes the flow between tricritical and critical Ising models. It is well known that the RSOS-restriction for $\pi = 3\pi$ effectively reduces soliton to one-component particle with free scattering:

$$
S^\pi_{RSOS}(\beta) = -1
$$
Thus we find the following formula for the form factors of $\Theta$ for this model:

$$K(A_1, \cdots, A_{n-1}) = \prod A_i^2 \prod_{i<j} (A_i^2 - A_j^2)$$

The value of the integral for this kind of polynomial (taking in account the functions $\zeta$ also) is

$$\prod \tanh \frac{1}{2} (\beta_i - \beta_j) \exp \left( \frac{1}{2} \sum \beta_j \right)$$

Consider the formula (17). We have to take the functions

$$\prod (A_i - A_j) \prod_{i=1}^{n-1} A_i^2 \prod_{i=1}^{n-1} (A_i + iB_j)$$

and

$$\prod (A_i' - A_j') \prod_{i=1}^{m-1} A_i' \prod_{i=1}^{m-1} (A_i' + iB_j'),$$

to decompose them with respect to antisymmetrical polynomials of $A_i$ and $A_i'$ corresponding to different partitions and to find the coefficients with which enter the polynomials $\prod A_i^2 \prod_{i<j} (A_i^2 - A_j^2)$ and $\prod (A_i')^2 \prod_{i<j} ((A_i')^2 - (A_j')^2)$. These coefficients are

$$\prod_{j \in T} B_j \prod_{i,j \in T'} (B_i + B_j) \quad \text{and} \quad \prod_{i,j \in T'} (B_i + B_j')$$

Thus we find the following formula for the form factors of $\Theta$ for this model

$$f_\Theta(\beta_1, \cdots, \beta_{2n} | \beta'_1, \cdots, \beta'_{2m}) = M^2 \prod \psi(\beta_i, \beta'_j) \prod \tanh \frac{1}{2} (\beta_i - \beta_j) \prod \tanh \frac{1}{2} (\beta_i - \beta_j)$$

$$\times \exp \left( \frac{1}{2} \sum \beta_j + \frac{1}{2} \sum \beta'_j \right) Q_{n,m}(e^{\beta_1}, \cdots, e^{\beta_{2n}} | e^{\beta'_1}, \cdots, e^{\beta'_{2m}})$$

where

$$Q_{n,m}(B_1, \cdots, B_{2n} | B'_1, \cdots, B'_{2m}) =$$

$$= \prod_{j=1}^{2n} B_j^{-1} \sum_{T \subset S} \sum_{T' \subset S'} \prod_{i \in T} B_i^2 \prod_{i \in T'} (B_i + B_j) \prod_{i \in T'} (B_i' + B_j')$$

$$\times \prod_{i,j \in T} (B_i + B_j) \prod_{i,j \in T'} (B_i' + B_j') \prod_{j \in T} \frac{1}{B_i - B_j} \prod_{j \in T'} \frac{1}{B_i' - B_j'}$$

$$\times \prod_{i \in T} (B_i + iB_j') \prod_{i \in T'} (B_i - iB_j') X_{T,T'}(B_1, \cdots, B_{2n} | B'_1, \cdots, B'_{2m})$$

(19)

one can write a formula for this polynomial in determinant form, but we think that (13) shows quite transparently how all the required properties of this polynomial are satisfied.

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