DISPERSION ESTIMATES FOR ONE-DIMENSIONAL DISCRETE SCHRÖDINGER AND WAVE EQUATIONS

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Abstract. We derive dispersion estimates for solutions of the one-dimensional discrete perturbed Schrödinger and wave equations. In particular, we improve upon previous works and weaken the conditions on the potentials.

1. Introduction

We are concerned with the one-dimensional discrete Schrödinger equation
\[ i\dot{u}(t) = Hu(t) := (-\Delta_L + q) u(t), \quad t \in \mathbb{R}, \]  
and the corresponding discrete wave (resp. Klein–Gordon) equation
\[ \ddot{u}(t) = (\Delta_L - m^2 - q) u(t), \quad t \in \mathbb{R}, \quad m \geq 0. \]
with real potential \( q \). Here \( \Delta_L \) is the discrete Laplacian given by
\[ (\Delta_L u)_n = u_{n+1} - 2u_n + u_{n-1}, \quad n \in \mathbb{Z}. \]

To formulate our results we introduce the weighted spaces \( \ell^p_{\sigma} = \ell^p_{\sigma}(\mathbb{Z}), \sigma \in \mathbb{R} \), associated with the norm
\[
\|u\|_{\ell^p_{\sigma}} = \begin{cases} 
(\sum_{n \in \mathbb{Z}} (1 + |n|)^{\sigma p} |u(n)|^p)^{1/p}, & p \in [1, \infty), \\
\sup_{n \in \mathbb{Z}} (1 + |n|)^{\sigma} |u(n)|, & p = \infty.
\end{cases}
\]
Of course, the case \( \sigma = 0 \) corresponds to the usual \( \ell^p \) spaces without weight.

As our first main result we will prove the following \( \ell^1 \rightarrow \ell^\infty \) decay
\[ \|e^{-itH} P_c\|_{\ell^1 \rightarrow \ell^\infty} = O(t^{-1/3}), \quad t \to \infty \]  
under the assumption \( q \in \ell^1_1 \).

Here \( P_c \) is the orthogonal projection in \( \ell^2 \) onto the continuous spectrum of \( H \).
In this respect we recall that under the condition \( q \in \ell^1_1 \) it is well-known [13] that the spectrum of \( H \) consists of a purely absolutely continuous part covering \( [0, 4] \) plus a finite number of eigenvalues located in \( \mathbb{R} \setminus [0, 4] \). In addition, there could be resonances at the boundary of the continuous spectrum.

The dispersive decay [13] has been established by Pelinovsky and Stefanov [10] under the assumption that there are no resonances and under the more restrictive condition \( |q_n| \leq C(1 + |n|)^{-\beta} \) with \( \beta > 5 \). In this connection we want to emphasize that we do neither make this assumption nor require a stronger decay condition in the presence of resonances (as was e.g. necessary in [3]). Our result is based on a

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simple but useful generalization of the van der Corput lemma (Lemma 3.1) together with the novel fact that the scattering data associated with $H$ are in the Wiener algebra (Lemma 3.3). The latter result being of independent interest in scattering theory.

For our second result we restrict ourselves to the “nonsingular case” in the terminology of [8], when the resolvent of the operator $H$ is bounded at the edge points of the continuous spectrum. In other words, the boundary points of the spectrum $\lambda = 0, 4$ are no resonances for the operator $H$. Under the decay condition

$$|q_n| \leq C(1 + |n|)^{-\beta}, \quad n \in \mathbb{Z},$$

for some $\beta > 3$, we prove that in the nonsingular case the following dispersive decay

$$\|e^{-itH}P_c(H)\|_{\ell^2_\sigma \to \ell^2_{-\sigma}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty,$$

holds for any $\sigma > 5/2$.

The dispersive decay of type (1.5) has been obtained for the first time in [6] for discrete Schrödinger and Klein–Gordon equations with compactly supported potentials. The result has been generalized in [10] to discrete Schrödinger equation with non-compactly supported potentials under the decay condition (1.4) with $\beta > 5$. Here we improve this result by reducing the decay rate to $\beta = 3$. This reduction is based on the different spectral assumption (4.1). We prove that our assumption is “sharp” in the sense that it is necessary and sufficient for the boundedness of truncated resolvent at the edge points of the continuous spectrum.

Finally, we will obtain similar asymptotics hold for the wave (resp. Klein–Gordon) equation (1.2) (except for the $\ell^1 \to \ell^\infty$ asymptotics with $m = 0$).

Asymptotics of type (1.3) and (1.5) play an important role in proving asymptotic stability of solitons in the associated discrete one-dimensional nonlinear equations [7, 9].

2. Free discrete Schrödinger equation

As a warm-up we will first consider the free equation (1.1) with $q = 0$ and denote $H_0 = -\Delta_L$. It is well-known ([13, Sect. 1.3]) that $H_0$ is self-adjoint and the discrete Fourier transform

$$\hat{u}(\theta) = \sum_{n \in \mathbb{Z}} u_n e^{i\theta n}, \quad \theta \in \mathbb{T} := \mathbb{R}/2\pi \mathbb{Z},$$

maps $H_0$ to the operator of multiplication by $\phi(\theta) = 2 - 2 \cos \theta$:$\widehat{-\Delta_L u}(\theta) = \phi(\theta) \hat{u}(\theta)$.

In particular, the spectrum $\text{Spec}(H_0) = [0, 4]$ is purely absolutely continuous.

Adopting the notation $[K]_{n,k}$ for the kernel of an operator $K$, that is,

$$[K]_{n,k} = \sum_{k \in \mathbb{Z}} [K]_{n,k} u_k, \quad n \in \mathbb{Z},$$

the kernel of the resolvent $R_0(\omega) = (H_0 - \omega)^{-1}$ is given by

$$[R_0(\omega)]_{n,k} = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{-i\theta(n-k)}}{\phi(\theta) - \omega} \, d\theta = \frac{e^{-i\phi(\omega)|n-k|}}{2i \sin \phi(\omega)}, \quad \omega \in \Xi := \mathbb{C} \setminus [0,4],$$

$n, k \in \mathbb{Z}$. Here $\phi(\omega)$ is the unique solution of the equation

$$2 - 2 \cos \theta = \omega, \quad \theta \in \Sigma := \{-\pi \leq \text{Re} \theta \leq \pi, \text{Im} \theta < 0\}/2\pi \mathbb{Z}. \quad (2.2)$$
Observe that $\theta \to \omega = 2 - 2 \cos \theta$ is a biholomorphic map from $\Sigma \to \Xi$.

Next we collect some properties obtained in [6]. To this end we denote by $B(\sigma, \sigma') = L(\ell^2_\sigma, \ell^2_{\sigma'})$ the spaces of bounded linear operators from $\ell^2_\sigma$ to $\ell^2_{\sigma'}$.

**P1** The resolvent $R_0(\omega)$ is an analytic function with values in $B(0, 0)$ for $\omega \in \Xi$.

**P2** For $\omega \in (0, 4)$ the limiting absorption principle holds, which is the convergence

$$R_0(\omega \pm i\varepsilon) \to R_0(\omega \pm i0), \quad \varepsilon \to 0+$$

in $B(\sigma, -\sigma)$ with $\sigma > 1/2$.

**P3** At the edge points $\mu_- = 0$ and $\mu_+ = 4$ the following asymptotics hold

$$R_0(\omega) = A_\pm(\omega - \mu_{\pm})^{-1/2} + B_\pm + O(|\omega - \mu_{\pm}|^{1/2}), \quad \omega \to \mu_{\pm}, \quad \omega \in \Xi$$

in $B(\sigma, -\sigma)$ with $\sigma > 5/2$. Here $A_\pm, B_\pm$ are the operators associated with the kernels

$$[A_{\pm}]_{n,k} = \frac{i}{2}(\mp 1)^{n-k+1}, \quad [B_{\pm}]_{n,k} = -\frac{i}{2}|n-k|(\mp 1)^{n-k+1},$$

respectively.

**P4** The asymptotics (2.4) can be differentiated twice with respect to $\omega$:

$$R''_0(\omega) = -\frac{1}{2}A_\pm(\omega - \mu_{\pm})^{-3/2} + O(|\omega - \mu_{\pm}|^{-1/2}), \quad \omega \to \mu_{\pm}, \quad \omega \in \Xi,$$

in $B(\sigma, -\sigma)$ with $\sigma > 5/2$.

Finally, we turn to the dispersive decay estimates. The kernel of the free propagator can be easily computed using the spectral theorem

$$[e^{-i t H_0}]_{n,k} = \frac{1}{2\pi i} \int_{[0,4]} e^{-i \omega} [R_0(\omega + i0) - R_0(\omega - i0)]_{n,k} d\omega$$

$$= \frac{1}{2\pi} \int_{[0,4]} e^{-i \omega} \left( \frac{e^{-i \theta_+\omega}|n-k|}{\sin \theta_+\omega} - \frac{e^{-i \theta_-\omega}|n-k|}{\sin \theta_-\omega} \right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i 2t(2 - 2 \cos \theta)} e^{-i \theta|n-k|} d\theta$$

where $\theta_+(\omega)$ and $\theta_- (\omega) = -\theta_+(\omega)$ are the solution to $2 - 2 \cos \theta = \omega$ from $[-\pi, 0]$ and $[0, \pi]$, respectively. This last integral is Bessel’s integral implying

$$[e^{-i t H_0}]_{n,k} = e^{i(-2t + \frac{\pi}{2}|n-k|)} J_{|n-k|}(2t),$$

where $J_{\nu}(z)$ denotes the Bessel function of order $\nu$. [14].

For the free discrete Schrödinger equation the $\ell^1 \to \ell^\infty$ decay of type [13] holds. However, the $\ell^2_0 \to \ell^2_{1,0}$ decay holds with rate $t^{-1/2}$ only (as in the continuous case). This is caused by the presence of resonances at the edge points $\omega = 0$ and $\omega = 4$.

**Lemma 2.1.** The following asymptotics hold

$$\|e^{-i t H_0}\|_{\ell^1 \to \ell^\infty} = \mathcal{O}(t^{-1/3}), \quad t \to \infty,$$

$$\|e^{-i t H_0}\|_{\ell^2_0 \to \ell^2_{1,0}} = \mathcal{O}(t^{-1/2}), \quad t \to \infty, \quad \sigma > 2/3.$$
Proof. Step i) Consider $t \geq 1$ and set $v := |n - k|/t \geq 0$. We start from

$$[e^{-itH_0}]_{n,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it(2 - 2\cos \theta + v\theta)} d\theta$$

(2.10)

which is an oscillatory integral with the phase function $\phi_v(\theta) = 2 - 2\cos \theta + v\theta$. The stationary points are the solution of the equation $\phi'_v(\theta) = 2\sin \theta + v = 0$. If $v > 2$ the phase function has no stationary points. For any $v < 2$ the phase function has two stationary points $\theta_{1,2}$. The points are nondegenerate, i.e. $\phi''_v(\theta_{1,2}) \neq 0$. In the case $v = 2$ the phase function has a unique degenerate stationary point $\theta = -\pi/2$. We have

$$\phi''_2(-\pi/2) = 0, \quad \phi''_2(-\pi/2) = 2 \neq 0.$$ (2.11)

Then (2.8) follows by the van der Corput lemma [12, page 334].

Step ii) To establish (2.9) we distinguish the two cases $|n - k| \leq t$ and $|n - k| \geq t$. In the region $|n - k| \leq t$ we obtain the bound

$$\sup_{|n-k| \leq t} |[e^{-itH_0}]_{n,k}| \leq Ct^{-1/2}$$

since $v = |n - k|/t \leq 1 < 2$ then. On the other hand, (2.11) implies

$$\sup_{|n-k| \geq t} |[e^{-itH_0}]_{n,k}| \leq Ct^{-1/3} = Ct^{-1/2}\lambda^{1/6} \leq Ct^{-1/2} |n-k|^{1/6}$$

Then (2.9) follows since $[K]_{n,k} = |n - k|^\alpha$ is bounded in $\ell^2 \to \ell^2_\sigma$ if $[\tilde{K}]_{n,k} = (1 + |n|)^{-\sigma} |n - k|^\alpha (1 + |k|)^{-\sigma}$ is Hilbert–Schmidt in $\ell^2$ which is the case if $\sigma > \frac{1}{2} + \alpha$. □

Remark 2.2. The decay rate in (2.8) is “sharp” as can be seen from the following asymptotics of the Bessel function

$$J_t(t) \sim t^{-1/3}, \quad t \to \infty,$$

see [14, Section 8.2].

3. Dispersive decay in $\ell^1 \to \ell^\infty$

Now we are able to prove $\ell^1 \to \ell^\infty$ decay.

We begin with a small variant of the van der Corput lemma. To this end we recall that the Wiener algebra is the set of all integrable functions whose Fourier coefficients are integrable:

$$A = \left\{ f(\theta) = \sum_{m \in \mathbb{Z}} \hat{f}_m e^{im\theta} \| \hat{f} \|_{\ell^1} < \infty \right\}.$$ (3.1)

Lemma 3.1. Consider the oscillatory integral

$$I(t) = \int_a^b e^{it\Phi(\theta)} f(\theta) d\theta, \quad -\pi \leq a < b \leq \pi,$$ (3.2)

where $\Phi(\theta)$ is real-valued. If $|\phi^{(k)}(\theta)| \geq 1$ for some $k \geq 2$ and $f \in A$, then

$$|I(t)| \leq \frac{C_k \|\hat{f}\|_{\ell^1}}{t^{1/k}},$$ (3.3)

where $C_k$ is a universal constant.
Proof. Using

\[ I(t) = \int_a^b e^{it\phi(\theta)} \sum_{m \in \mathbb{Z}} \hat{f}_m e^{im\theta} \, d\theta = \sum_{m \in \mathbb{Z}} \hat{f}_m I_m(t), \quad I_v(t) = \int_a^b e^{it(\phi(\theta)+v\theta)} \, d\theta. \]

By the van der Corput lemma [12, page 332] we have

\[ |I_v(t)| \leq C_k |t|^{-1/k}, \text{ where } C_k \text{ is a universal constant (independent of } v), \]

and the claim follows. \(\square\)

Remark 3.2. The above lemma is usually found for the case when \( f \) is absolutely continuous in the literature (cf. [12, page 333]) — in fact, the proof immediately extends to functions of bounded variation. However, by the Riemann–Lebesgue lemma the Fourier coefficients of an absolutely continuous function must satisfy

\[ \hat{f}_m = o(m^{-1}) \] (for functions of bounded variation one has \( O(m^{-1}) \)) and considering lacunary Fourier coefficients one obtains an element in the Wiener algebra which is not absolutely continuous (of bounded variation). Conversely, since the Fourier coefficients of an integrable function can have arbitrary slow decay, there are absolutely continuous functions which are not in the Wiener algebra. Finally, note that for continuous \( f \) the decay can be arbitrary slow.

Next we recall a few facts from scattering theory. Under the assumption \( q \in \ell^1 \) there exists Jost solutions \( f^\pm(\theta) \) to

\[ (-\Delta_L + q)f = \omega f, \quad \omega \in [0,4] \]

normalized as

\[ f_n^\pm(\theta) \sim e^{\pm in\theta}, \quad n \to \pm \infty, \]

where \( \theta = \theta(\omega) \) is the solution to \( 2 - 2 \cos \theta = \omega \). One can show as in [1] that

\[ |f_n^\pm(\theta)| \leq C(\theta)e^{\pm 1\text{Im}(\theta)n}, \]

(3.4)

where \( C(\theta) \) can be chosen uniformly in compact subsets of \( \Sigma \) avoiding the band edges. If additionally \( q \in \ell^1 \) then

\[ |f_n^\pm(\theta)| \leq C(\theta) \max(1, \mp n)e^{\pm 1\text{Im}(\theta)n}, \]

(3.5)

where \( C(\theta) \) can be chosen uniformly in compact subsets of \( \Sigma \). Moreover, in this case the Jost solutions are given by

\[ f_n^\pm(\theta) = e^{\mp in\theta} h_n^\pm(\theta), \quad h_n^\pm(\theta) = 1 + \sum_{m=1}^{\pm \infty} \sum_{k=n+|m/2|} K_{n,m}^\pm e^{\mp im\theta}, \]

(3.6)

where (see [13 Sect. 10.1])

\[ |K_{n,m}^\pm| \leq C_n^\pm \sum_{k=n+|m/2|} |q_k|, \]

(3.7)

with

\[ C_n^\pm \leq C^\pm, \quad \text{if } n \geq \mp 1. \]

(3.8)

The estimate (3.7) implies

\[ h_n^\pm(\theta), f_n^\pm(\theta) \in A \quad \text{if } q \in \ell^1. \]

(3.9)

Respectively, the Wronskian of Jost solutions

\[ W(\theta) := W(f^+(\theta), f^-(\theta)) = f_0^+(\theta)f_1^-(\theta) - f_1^+(\theta)f_0^-(\theta) \]

(3.10)
also belongs to the Wiener algebra \( \mathcal{A} \) as \( q \in \ell_1^1 \) as far as the Wronskians \( W^\pm(\theta) = W(f^\pm(\theta), f^\pm(-\theta)) \). Moreover, we have the scattering relations

\[
T(\theta)f^\pm_m(\theta) = R^\mp(\theta)f^\mp_m(\theta) + f^\pm_m(-\theta) 
\]  
(3.11)

where the quantities

\[
T(\theta) = \frac{2i\sin \theta}{W(\theta)}, \quad R^\pm(\theta) = \pm \frac{W^\pm(\theta)}{W(\theta)}, 
\]  
(3.12)

which are known as the transmission and reflection coefficients, also belong to this algebra:

**Lemma 3.3.** If \( q \in \ell_1^1 \), then \( T(\theta) \), \( R^\pm(\theta) \in \mathcal{A} \).

*Proof.* The Wronskian \( W(\theta) \) can vanish only at the edges of continuous spectra, i.e. when \( \theta = 0, \pi \), which correspond to the resonant cases. Moreover, since \( |T(\theta)| \leq 1 \) the zeros of the Wronskian at points 0, \( \pi \) can be at most of first order. Since \( W(\theta) \in \mathcal{A} \) by (3.9), then in the case \( W(0)W(\pi) \neq 0 \) we obtain \( W(\theta)^{-1} \in \mathcal{A} \) by Wiener’s lemma. Therefore, \( T, R^\pm \in \mathcal{A} \).

If \( W(0)W(\pi) = 0 \) we need to work a bit harder. Suppose, for example, \( W(0) = 0 \). In [2], Lemma 4.1, formulas (4.12)–(4.14), the following representation is obtained

\[
V^\pm(\theta) := f^\pm_1(\theta)f^\pm_0(0) - f^\pm_0(\theta)f^\pm_1(0) = (1 - e^{i\theta})\Psi^\pm(\theta), 
\]  
(3.13)

where

\[
\Psi^\pm(\theta) = \sum_{l=\pm 1}^{\pm \infty} g^+_m e^{\mp i m \theta}, \quad \text{with} \quad g^\pm \in \ell^1(Z_\pm) \quad \text{if} \quad q \in \ell_1^1. 
\]  
(3.14)

In other words, \( \Psi^\pm(\theta) \in \mathcal{A} \). Since

\[
W(0) = f^+_0(0)f^-_1(0) - f^-_0(\theta)f^+_1(0) = 0 
\]  
(3.15)

we have two possible combinations: (a) \( f^+_0(0)f^-_0(\theta) \neq 0 \) and (b) \( f^-_1(0)f^+_0(0) \neq 0 \). Consider the case (a). By (3.10), (3.13), and (3.15) we get

\[
W(\theta) = f^+_0(\theta)f^-_0(\theta) \left( \frac{V^-(\theta)}{f^-_0(0)f^-_0(\theta)} - \frac{V^+(\theta)}{f^+_0(0)f^+_0(\theta)} \right) = 
\]

\[
= (1 - e^{i\theta}) \left( \frac{f^+_0(\theta)}{f^-_0(0)}\Psi^-(\theta) - \frac{f^-_0(\theta)}{f^+_0(0)}\Psi^+(\theta) \right) = (1 - e^{i\theta})\Phi(\theta), 
\]

where \( \Phi(\theta) \in \mathcal{A} \) by (3.14) and (3.9). We observe that if \( W(\pi) = 0 \) then \( \Phi(\theta) \neq 0 \) for \( \theta \in [0, \pi) \) and if \( W(\pi) \neq 0 \) then \( \Phi(\theta) \neq 0 \) for \( \theta \in [0, \pi] \). The same result follows in a similar fashion in case (b). Since equality \( W(0) = 0 \) implies \( W^\pm(0) = 0 \) then we can also get similarly \( W^\pm(\theta) = (1 - e^{i\theta})\Phi^\pm(\theta) \) with \( \Phi^\pm(\theta) \in \mathcal{A} \).

Similarly, \( W(\pi) = 0 \) implies \( W(\theta) = (1 + e^{i\theta})\Phi(\theta), \quad W^\pm(\theta) = (1 + e^{i\theta})\tilde{\Phi}^\pm(\theta) \) with \( \tilde{\Phi}, \tilde{\Phi}^\pm \in \mathcal{A} \) and \( \tilde{\Phi}(\theta) \neq 0 \) for \( \theta \in [0, \pi] \) if \( W(0) \neq 0 \). Thus if \( W \) vanishes at only one endpoint, this finishes the proof. If \( W \) vanishes at both endpoints, we can use a smooth cut-off function to combine both representations into \( W(\theta) = (1 - e^{2i\theta})\tilde{\Phi}(\theta) \) (respectively, \( W^\pm(\theta) = (1 - e^{2i\theta})\tilde{\Phi}^\pm(\theta) \)) with \( \tilde{\Phi}, \tilde{\Phi}^\pm \in \mathcal{A} \) and \( \tilde{\Phi}(\theta) \neq 0 \) for \( \theta \in [0, \pi] \). \( \Box \)
Finally, given the Jost solutions we can express the kernel of the resolvent $R(\omega) : \ell^2 \to \ell^2$ for $\omega \in \mathbb{C} \setminus \text{spec}(H)$ as (cf. [13 (1.99)])

$$[R(\omega)]_{n,k} = \frac{1}{W(\theta(\omega))} \begin{cases} f_n^+ (\theta(\omega)) f_k^- (\theta(\omega)) & \text{for } n \geq k, \\ f_k^+ (\theta(\omega)) f_n^- (\theta(\omega)) & \text{for } n \leq k. \end{cases}$$  \hspace{1cm} (3.16)

Recall that $\theta \mapsto \omega(\theta)$ is a biholomorphic map $\Sigma \to \Xi$.

The representation (3.16), the fact that $W(\theta)$ does not vanish for $\omega \in (0, 4)$, and the bound (3.3) imply the limiting absorption principle for the perturbed one-dimensional Schrödinger equation.

**Lemma 3.4.** Let $q \in \ell^1$. Then the convergence

$$R(\omega \pm i\varepsilon) \to R(\omega \pm i0), \quad \varepsilon \to 0+, \quad \omega \in (0, 4)$$  \hspace{1cm} (3.17)

holds in $B(\sigma, -\sigma)$ with $\sigma > 1/2$.

**Proof.** For any $\omega \in (0, 4)$ and any $n, k \in \mathbb{Z}$, there exist the pointwise limit

$$[R(\omega \pm i\varepsilon)]_{n,k} \to [R(\omega \pm i0)]_{n,k}, \quad \varepsilon \to 0.$$  

Moreover, the bound (3.3) implies that $|[R(\omega \pm i\varepsilon)]_{n,k}| \leq C(\omega)$. Hence, the Hilbert–Schmidt norm of the difference $R(\omega \pm i\varepsilon) - R(\omega \pm i0)$ converges to zero in $B(\sigma, -\sigma)$ with $\sigma > 1/2$ by the Lebesgue dominated convergence theorem. \hfill $\square$

**Corollary 3.5.** For any $\omega \in (0, 4)$ and any fixed $\sigma > 1/2$, the operators $R^\pm (\omega) := R(\omega \pm i0) : \ell^2_\sigma \to \ell^2_\sigma$ have integral kernels given by

$$[R^\pm (\omega)]_{n,k} = \frac{1}{W(\theta_\pm)} \begin{cases} f_n^+ (\theta_\pm) f_k^- (\theta_\pm) & \text{for } n \geq k, \\ f_k^+ (\theta_\pm) f_n^- (\theta_\pm) & \text{for } n \leq k \end{cases}$$  \hspace{1cm} (3.18)

where $\theta_+ (\omega)$, and $\theta_- (\omega) = -\theta_+ (\omega)$ are the solution to $2 - 2 \cos \theta = \omega$ from $[-\pi, 0]$ and $[0, \pi]$, respectively.

Now we come to our main result in this section.

**Theorem 3.6.** Let $q \in \ell^1$. Then the asymptotics (1.3) hold, i.e.,

$$\|e^{-itH} P_c\|_{\ell^1 \to \ell^\infty} = O(t^{-1/3}), \quad t \to \infty,$$  \hspace{1cm} (3.19)

and

$$\|e^{-itH} P_c\|_{\ell^2 \to \ell^2_\sigma} = O(t^{-1/2}), \quad t \to \infty, \quad \sigma > 2/3.$$  \hspace{1cm} (3.20)

**Proof.** We apply the spectral representation

$$e^{-itH} P_c = \frac{1}{2\pi i} \int_{[0,4]} e^{-it\omega} (R(\omega + i0) - R(\omega - i0)) \, d\omega.$$  \hspace{1cm} (3.21)

Expressing the kernel of the resolvent in terms of the Jost solutions (cf. [13 (1.99)]), the kernel of $e^{-itH} P_c$ reads:

$$[e^{-itH} P_c]_{n,k} = \frac{1}{2\pi i} \int_0^4 e^{-it\omega} \left[ \frac{f_n^+ (\theta_+) f_k^- (\theta_+)}{W(\theta_+)} - \frac{f_k^+ (\theta_-) f_n^- (\theta_-)}{W(\theta_-)} \right] \, d\omega$$  \hspace{1cm} (3.22)

$$= \frac{1}{\pi i} \int_{-\pi}^{\pi} e^{-it(2 - 2 \cos \theta)} \frac{f_k^+ (\theta) f_n^- (\theta)}{W(\theta)} \sin \theta \, d\theta.$$
for \( n \leq k \) and by symmetry \([e^{-itH}P_c]_{n,k} = [e^{-itH}P_c]_{k,n}\) for \( n \geq k \). Hence, for (3.19) it suffices to prove that

\[
[e^{itH}P_c]_{n,k} = O(t^{-1/3}), \quad t \to \infty.
\]

(3.23)

independent of \( n, k \). We suppose \( n \leq k \) for notational simplicity. Then

\[
[e^{-itH}P_c]_{n,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it(2-2\cos \theta + v\theta)} h^+_n(\theta)h^-_k(\theta)T(\theta) d\theta
\]

(3.24)

where \( v = \frac{k-n}{n} \geq 0 \). We observe that the function \( \Phi_{n,k}(\theta) = h^+_n(\theta)h^-_k(\theta)T(\theta) \) belongs to \( A \), moreover, the \( \ell_1 \)-norm of its Fourier coefficients \( \|\hat{\Phi}_{n,k}(\cdot)\| \) can be estimated by a value, which does not depend on \( n \) and \( k \). To this end introduce

\[
1 + \sup_{\pm n > 0} \sum_{m=1}^{\pm \infty} |K_{n,m}^\pm| = \tilde{C}^\pm > 0.
\]

By (3.7)-(3.8) this supremum is finite. Then

\[
\|h^\pm_n(\cdot)\| \leq \tilde{C}^\pm \quad \text{for} \quad \pm n > 0.
\]

(3.25)

Consider now the three possibilities (a) \( n \leq k \leq 0 \), (b) \( 0 \leq n \leq k \) and (c) \( n \leq 0 \leq k \). In the case (c) the estimate \( \|\Phi_{n,k}(\cdot)\| \leq C \) follows immediately from (3.23) and Lemma 3.3. In two other cases we use the scattering relations (3.11) to get the representation

\[
\Phi_{n,k}(\theta) = \begin{cases} 
  h^-_n(\theta)(R^-(\theta)h^-_k(\theta)e^{2ik\theta} + h^-_k(\theta)) & n \leq k \leq 0, \\
  h^+_n(\theta)(R^+(\theta)h^+_k(\theta)e^{-2in\theta} + h^+_k(\theta)) & 0 \leq n \leq k,
\end{cases}
\]

and again apply Lemma 3.3 together with (3.9) and (3.26).

Now, as in the proof of Lemma 2.1 we split the domain of integration into regions where either the second or third derivative of the phase is nonzero and apply Lemma 3.3 together with the estimates from Lemma 3.3.

The second claim follows analogously as in the proof of Lemma 2.1 \( \Box \)

4. Asymptotics for the Resolvent

Now we consider \( R(\omega) \) near the singular points \( \mu_- = 0 \) and \( \mu_+ = 4 \) and introduce a spectral condition provides the asymptotics (1.15).

**Definition 4.1.** For \( \mu \in \{\mu_-, \mu_+\} \) any nonzero function \( u \in \ell^\infty \) satisfying the equation \( (-\Delta_L + q)u = \mu u \) is called a resonance function, and in this case \( \mu \) is called a resonance.

Below we assume the following

**Spectral condition:** The points \( \mu_\pm \) are no resonances. (4.1)

This condition corresponds to the “nonsingular case” \( [8] \) Section 7. The condition is sharp in the sense that it is equivalent to the boundedness of the resolvent \( R(\omega) \) at the edge points of the continuous spectrum (see Corollary 4.4 below). This boundedness provides the asymptotics (1.15). First we prove that (4.1) holds if and only if the Wronskian \( W(\theta) \) does not vanish at \( \theta = 0, \pi \).

**Lemma 4.2.** Let \( q \in \ell_1^+ \). Then \( \mu_- \) (or \( \mu_+ \)) is a resonance if and only if \( W(0) = 0 \) (or \( W(\pi) = 0 \)).
Lemma 4.5. Assume Corollary 4.4.

Introduce another solution $g^+$ satisfying $W(f^+, g^+) = 1$. Making the ansatz $g^+_n = f^+_n v_n$, where $v_n$ is unknown, we obtain $(v_{n+1} - v_n) f^+_{n+1} = 1$ for sufficiently large positive $n_0$. Solving for $v$ shows

$$g^+_n = f^+_n \sum_{j=n_0}^{n-1} \frac{1}{f^+_j f^+_{j+1}} + v_{n_0} f^+_n = n + o(n), \quad n \to +\infty.$$ 

Hence $f^-_n = \alpha f^+_n + \beta g^+_n$ and there is a bounded solution if and only if $\beta = W(f^+, f^-) = 0$.

Lemma 4.3. Let $q \in \ell^1$ and $\sigma > \frac{3}{2}$. If $\mu_- = 0$ is no resonance then $R(\omega)$ is continuous in $B(\sigma, -\sigma)$ for $\omega$ in a neighborhood of $[0, 4]$ away from $\mu_+$ with $R(0) \neq 0$. If $\mu_- = 0$ is a resonance then $\tilde{R}(\omega) = \sqrt{2} R(\omega)$ is continuous in $B(\sigma, -\sigma)$ for $\omega$ in a neighborhood of $[0, 4]$ away from $\mu_+$ with $\tilde{R}(0) \neq 0$. Similarly near $\mu_+$.

Proof. By (3.10), if $W(0) \neq 0$ the claim follows directly from the estimate (3.11) since the kernel $(1+|n|)^{-\sigma} |R(\omega)| n, k (1+|k|)^{-\sigma}$ is continuous in the Hilbert–Schmidt norm by dominated convergence. Otherwise we use additionally that $W(\theta) = W_0 \theta + o(\theta)$ with $W_0 \neq 0$ (Lemma 4.1) and the claim again follows.

As a simple consequence we note:

Corollary 4.4. Let $q \in \ell^1$. Then condition (1.1) is equivalent to the boundedness of the families

$$\{ R(\omega), |\omega - \mu_+| \leq \varepsilon, \omega \in \Xi \}$$

in $B(\sigma, -\sigma)$ with $\sigma > 3/2$ for sufficiently small $\varepsilon > 0$.

Next, we note that in the case (1.1) we can strengthen the above result a bit:

Lemma 4.5. Assume (1.1), suppose (1.4) holds for some $\beta > 2$, and let $\sigma > \frac{1}{2}$. Then $R(\omega) \in B(\sigma, -\sigma)$ for $\omega$ in a neighborhood of $[0, 4]$ provided $\sigma + \varepsilon > 2$.

Proof. This follows from (3.11) and (3.8) which imply

$$|R(\omega)| n, k \leq C \min(|n|, |k|).$$

in a neighborhood of $[0, 4]$.

The Born decomposition formulas

$$R(\omega) = (1 + R_0(\omega) q)^{-1} R_0(\omega), \quad R(\omega) = R_0(\omega) (1 + q R_0(\omega))^{-1}$$

imply

$$(1 + R_0(\omega) q)^{-1} = 1 - R(\omega) q, \quad (1 + q R_0(\omega))^{-1} = 1 - q R(\omega).$$

Hence, since $q \in B(\sigma, \sigma + \beta)$, we obtain from the previous lemma that for any $\sigma \in (1/2, \beta - 1/2)$ the operators $(1 + R_0(\omega) q)^{-1}$ and $(1 + q R_0(\omega))^{-1}$ are bounded in $B(-\sigma, 0)$ and $B(\sigma, 0)$, respectively.

In particular, for $\beta > 3$ we obtain

$$R'(\omega \pm i \varepsilon) \to R'(\omega \pm i \varepsilon), \quad R''(\omega + i \varepsilon) \to R''(\omega + i 0), \quad \varepsilon \to 0+, \quad \omega \in (0, 4).$$

in $B(\sigma, -\sigma)$ for $\sigma > \frac{5}{2}$. Our next task will be to obtain asymptotics of the resolvent $R(\omega)$ at the edge points $\mu_\pm$. We start with the following lemma:
Lemma 4.6. Assume \((1.4)\), suppose \((1.4)\) holds for some \(\beta > 2\), and let \(\sigma \in (3/2, \beta - 1/2)\). Then
\[
\| (1 + R_0(\omega)q)^{-1} \alpha^\pm \|_{\ell^2_{\sigma}} = \mathcal{O}(|\omega - \mu^\pm|^{1/2}), \ \omega \to \mu^\pm, \ \omega \in \Xi, \tag{4.6}
\]
and
\[
\sum_n \alpha^\pm_n [(1 + qR_0(\omega))^{-1} f]_n = \mathcal{O}(|\omega - \mu^\pm|^{1/2}), \ \omega \to \mu^\pm, \ \omega \in \Xi, \tag{4.7}
\]
for any \(f \in \ell^2_{\sigma}\), where \(\alpha^\pm_n = (\mp 1)^n\).

In particular,
\[
(1 + R_0(\omega)q)^{-1} A^\pm(1 + qR_0(\omega))^{-1} = \mathcal{O}(|\omega - \mu^\pm|), \ \omega \to \mu^\pm, \ \omega \in \Xi, \tag{4.8}
\]
in \(B(\sigma, -\sigma)\), where \(A^\pm\) is given in \((2.8)\).

Proof. The asymptotics \((2.4)\) imply
\[
R(\omega) = (1 + R_0(\omega)q)^{-1} R_0(\omega) = (1 + R_0(\omega)q)^{-1} [A^\pm(\omega - \mu^\pm)^{-1/2} + \mathcal{O}(1)],
\]
and the claim follows from the continuity of \(R(\omega)\), \((1 + R_0(\omega)q)^{-1}\), and \((1 + qR_0(\omega))^{-1}\) in \(B(-\sigma, -\sigma)\) and \(B(\sigma, \sigma)\), respectively. The last claim follows since \(A^\pm = \frac{1}{2i} \alpha^\pm \otimes \alpha^\pm\).

□

Lemma 4.7. Suppose \((1.4)\) holds for some \(\beta > 3\) and \((4.1)\) holds. Then we have the following asymptotics in \(B(\sigma, -\sigma)\) with \(\sigma > 5/2\)
\[
R(\omega) = R^\pm + \mathcal{O}(|\omega - \mu^\pm|^{1/2}),
\]
\[
R'(\omega) = \mathcal{O}(|\omega - \mu^\pm|^{-1/2}), \quad \omega \to \mu^\pm, \quad \omega \in \Xi. \tag{4.9}
\]

Proof. We use the following formulas for the derivatives of \(R\) (cf. \([4, 5]\)):
\[
R' = (1 + R_0q)^{-1} R_0'(1 + qR_0)^{-1}, \quad R'' = \left[ (1 + R_0q)^{-1} R_0'' - 2R'qR_0' \right] (1 + qR_0)^{-1}. \tag{4.10}
\]
Asymptotics \((2.4)\), \((4.6)\)–\((4.8)\), and formulas \((4.10)\) imply
\[
R'(\omega) = \mathcal{O}(|\omega - \mu^\pm|^{-1/2}), \quad R''(\omega) = \mathcal{O}(|\omega - \mu^\pm|^{-3/2}), \quad \omega \to \mu^\pm, \quad \omega \in \Xi. \tag{4.11}
\]
in \(B(\sigma, -\sigma)\) with \(\sigma > 5/2\). The asymptotics \((4.11)\) coincide with the asymptotics \((4.4)\) for the derivatives. Asymptotics \((4.4)\) for \(R(\omega)\) can be obtained by integration of asymptotics \((4.9)\) for the first derivative. □

5. Dispersive decay in \(B(\sigma, -\sigma)\)

Theorem 5.1. Let conditions \((1.4)\) with \(\beta > 3\) and \((4.1)\) hold. Then asymptotics \((1.4)\) hold, i.e.
\[
e^{-itH} P_c = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \tag{5.1}
\]
in \(B(\sigma, -\sigma)\) with \(\sigma > 5/2\).
Proof. The asymptotics are based on the formula (cf. [6])

\[ e^{-itP_c} = \frac{1}{2\pi i} \int_{|\omega| = 0.4} e^{-i\omega (R(\omega + i0) - R(\omega - i0))} d\omega = \int_{|\omega| = 0.4} e^{-i\omega} P(\omega) d\omega, \quad (5.2) \]

where \( F(\omega) = \frac{1}{\pi} \text{Im} R(\omega + i0). \) The asymptotic expansion of \( F(\omega) \) at the edge points \( \mu_{\pm} \) can be deduced from (4.9). Thus we obtain

\begin{align*}
F(\omega) &= O(|\omega - \mu_{\pm}|^{1/2}), \\
F'(\omega) &= O(|\omega - \mu_{\pm}|^{-1/2}), \quad \omega \to \mu_{\pm}, \quad \omega \in (0, 4). \\
F''(\omega) &= O(|\omega - \mu_{\pm}|^{-3/2}),
\end{align*}

Hence the desired decay for large \( t \) follows from Lemma 6.2 below. \( \square \)

The following lemma is a special case of [4, Lemma 10.2].

Lemma 5.2 ([4]). Assume \( B \) is a Banach space, \( a > 0 \), and \( F \in C(0, a; B) \) satisfies \( F(0) = F(a) = 0, F'' \in L^1_{\text{loc}}(0, a; B) \), as well as \( F''(\omega) = O(\omega^{-3/2}) \) and \( F''(a - \omega) = O(\omega^{-3/2}) \) as \( \omega \to 0 + \). Then

\[ \int_0^a e^{-it\omega} F(\omega) d\omega = O(t^{-3/2}), \quad t \to \infty. \]

6. Wave equation

Here we extend our main results to the wave equation (1.2).

6.1. Free wave equation. Let us set \( u_n(t) = (u_n(t), \dot{u}_n(t)) \). Then (6.1) with \( q = 0 \) reads

\[ i\ddot{u}(t) = H_0 u(t), \quad t \in \mathbb{R}, \quad (6.1) \]

where

\[ H_0 = \begin{pmatrix} 0 & i \\ -i(\Delta_L - m^2) & 0 \end{pmatrix}. \]

The continuous spectrum of \( H_0 \) coincides with \( \Gamma \), where

\[ \Gamma = (-\sqrt{m^2 + 4}, -m) \cup (m, \sqrt{m^2 + 4}). \]

The resolvent \( R_0(\omega) = (H_0 - \omega)^{-1} \) can be expressed in terms of \( R_0(\omega) = (H_0 - \omega)^{-1} \) (see [3]):

\[ R_0(\omega) = \begin{pmatrix} \omega R_0(\omega^2 - m^2) & iR_0(\omega^2 - m^2) \\ -i(1 + \omega^2 R_0(\omega^2 - m^2)) & \omega R_0(\omega^2 - m^2) \end{pmatrix}. \quad (6.2) \]

Let us denote by \( B(\sigma, \sigma') = L(\ell^2_\sigma \oplus \ell^2_\sigma', \ell^2_\sigma \oplus \ell^2_\sigma') \) the spaces of bounded linear operators from \( \ell^2_\sigma \oplus \ell^2_\sigma \) to \( \ell^2_\sigma \oplus \ell^2_\sigma' \).

Lemma 6.1. Let \( m > 0 \). Then the following asymptotics hold

\[ \|e^{-itH_0}\|_{B(\sigma, -\sigma)} = O(t^{-1/2}), \quad t \to \infty, \quad (6.3) \]

\[ \|e^{-itH_0}\|_{B(\sigma, -\sigma)} = O(t^{-1/2}), \quad t \to \infty, \quad \sigma > 2/3. \quad (6.4) \]
Proof. As well as in the proof of Lemma 2.1 we consider \( t \geq 1 \) and apply the spectral representation:
\[
e^{-itH_0} = \frac{1}{2\pi} \int e^{-it\omega} (R_0(\omega + i0) - R_0(\omega - i0)) \, d\omega.
\]
Due to (2.1) and (6.2) it suffices to consider the operator \( K(t) \) with the kernel
\[
[K(t)]_{n,k} = \int_{\mathbb{R}} e^{-it\omega} \left( \frac{e^{-i\theta_+|n-k|}}{\sin \theta_+} - \frac{e^{-i\theta_-|n-k|}}{\sin \theta_-} \right) d\omega
\]
where \( \theta_+ = \theta_+(\omega^2 - m^2) \in [-\pi, 0] \) is the solution to \( 2 - 2 \cos \theta = \omega^2 - m^2, \theta_- = -\theta_+ \)
and obtain the bounds
\[
\sup_{n,k} |[K(t)]_{n,k}| \leq Ct^{-1/3}, \quad t \geq 1, \quad (6.6)
\]
\[
\sup_{n,k} |[K(t)]_{n,k}| \leq Ct^{-1/2} |n-k|^{1/6}, \quad t \geq 1. \quad (6.7)
\]
Abbreviate \( v := \frac{|n-k|}{t} \geq 0 \) and denote \( \varpi = (2 + m^2 - \sqrt{4m^2 + 4m^4})/2 \). It is easy to check that if \( v \neq \sqrt{\varpi} \) then the phase function
\[
\phi(\theta) = \sqrt{2 - 2 \cos \theta + m^2 + v\theta}
\]
has at most two nondegenerate stationary points. In the case \( v = \sqrt{\varpi} \) there exist two degenerate stationary points \( \theta^\pm_0 = \pm \arccos \varpi \) such that \( \phi''(\theta^\pm_0) = \pm \sqrt{\varpi} \neq 0 \).
Moreover, the function \( g(\theta) = \frac{1}{\sqrt{2-2\cos \theta + m^2}} \) and its derivative \( g'(\theta) \) are bounded at \([-\pi, \pi] \). Hence (6.6) follows from the van der Corput lemma.

To prove the asymptotics (6.7) one considers the two cases \( |n-k| \leq t\sqrt{\varpi}/2 \) and \( |n-k| \geq t\sqrt{\varpi}/2 \) as in the proof of Lemma 2.1(ii).

\[\Box\]

Remark 6.2. The solution of the free wave equation (6.11), corresponding to \( m = 0 \), does not decay as \( t \to \pm \infty \) since \( R_0(\omega) = R_0(\omega^2) \sim \omega^{-1} \) as \( \omega \to 0 \) in this case. Note that the first component of the solution is given by
\[
u_n(t) = \sum_{m \in \mathbb{Z}} c_{n-m}(t)u_m(0) + s_{n-m}(t)\dot{u}_m(0), \quad (6.9)
\]
where
\[
c_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\sqrt{1 - \cos \theta \sqrt{2t}}) e^{i\eta_n} d\theta = J_{2|n|}(2t), \quad (6.10)
\]
\[
s_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(\sqrt{1 - \cos \theta \sqrt{2t}}) e^{i\eta_n} d\theta = \int_0^t c_n(s) ds
\]
\[
= \frac{\Gamma(2|n|+1)}{2|n|!} F_2 \left( \frac{2|n|+1}{2}, \frac{2|n|+3}{2}; \frac{2|n|+1}{2}; -t^2 \right). \quad (6.11)
\]
Here \( J_n(x), \, \, _p F_q \left( \mu; \nu; x \right) \) denote the Bessel and generalized hypergeometric functions, respectively. In particular, while \( c_n(t) = O(t^{-1/2}) \) for fixed \( n \), we have \( s_n(t) = \frac{1}{2} + O(t^{-1/2}) \) for fixed \( n \).
6.2. **Perturbed wave equation.** In matrix form \(1.2\) reads
\[
iu(t) = Hu(t), \quad t \in \mathbb{R},
\] (6.12)
where
\[
H = \begin{pmatrix}
0 & i \\
i(\Delta_L - m^2 - q) & 0
\end{pmatrix}.
\]
The resolvent \(R(\omega) = (H - \omega)^{-1}\) can be expressed in terms of \(R(\omega) = (H - \omega)^{-1}\) (see [6]):
\[
R(\omega) = \begin{pmatrix}
\omega R(\omega^2 - m^2) & iR(\omega^2 - m^2) \\
i(1 + \omega^2 R(\omega^2 - m^2)) & \omega R(\omega^2 - m^2)
\end{pmatrix}.
\] (6.13)
Representation (6.13) and Lemma 3.4 imply the limiting absorption principle for the perturbed resolvent:

**Lemma 6.3.** Suppose \(q \in \ell^1\). Then for \(\omega \in (-\sqrt{m^2 + 4}, -m) \cup (m, \sqrt{m^2 + 4})\) the convergence
\[
R(\omega + i\varepsilon) \to R(\omega + i0), \quad \varepsilon \to 0+,
\]
holds in \(B(\sigma, -\sigma)\) with \(\sigma > 1/2\).

For the dynamical group associated with the perturbed wave equation (6.12) the spectral representation of type (5.2) holds:
\[
e^{-itH}P_c = \frac{1}{2\pi i} \int e^{-it\omega}(R(\omega + i0) - R(\omega - i0)) d\omega.
\] (6.14)

Here \(P_c\) is the projection onto the continuous spectrum of \(H\). Next, we prove asymptotics of type (1.3) for (6.12).

**Theorem 6.4.** Let \(m > 0\) and \(q \in \ell^1\). Then the following asymptotics holds
\[
\|e^{-itH}P_c\|_{\ell^1 \to \ell^\infty} = O(t^{-1/3}), \quad t \to \infty.
\] (6.15)
and
\[
\|e^{-itH}P_c\|_{B(\sigma, -\sigma)} = O(t^{-1/2}), \quad t \to \infty, \quad \sigma > 2/3.
\] (6.16)

**Proof.** Due to the representation (6.14) and formula (3.18) it suffices to consider
\[
[K(t)]_{n,k} = \frac{1}{2\pi i} \int e^{-it\omega} \left[ f^+_{m}(\theta) f^-_{m}(\theta) \right] d\omega
\]
where \(\theta = \theta_\pm (\omega, -m^2)\), and prove that
\[
\sup_{n \leq k} |[K(t)]_{n,k}| \leq Ct^{-1/3}, \quad t \geq 1,
\] (6.17)
\[
\sup_{n \leq k} |[K(t)]_{n,k}| \leq Ct^{-1/2}|n - k|^{1/6}, \quad t \geq 1.
\] (6.18)

Just as in the proof of Theorem 3.6 we consider three different cases \(n \leq 0 \leq k\), \(0 \leq n \leq k\) and \(n \leq k \leq 0\). For instance, in the case \(n \leq 0 \leq k\) we have
\[
[K(t)]_{n,k} = \frac{1}{2\pi i} \int e^{-it\sqrt{2-2\cos \theta + m^2} + i(n-k)\theta} h^+_{k}(\theta) h^-_{n}(\theta) T(\theta) \frac{d\theta}{\sqrt{2 - 2 \cos \theta + m^2}}.
\]
and we obtain an oscillatory integral with the phase function $\phi(\theta)$ defined in (6.8) with $v = k - \frac{n}{2} \geq 0$. Using the properties of the phase function obtained in Lemma 6.1 one can now proceed as in the proof of Theorem 3.6.

Now we suppose that spectral condition (4.1) holds. Then representation (6.13) and Proposition 4.7 imply Lemma 6.5.

Let conditions (1.4) and (4.1) hold, and let $m > 0$. Then, at the edge points $\mu = \pm m; \pm \sqrt{m^2 + \frac{4}{3}}$, the following asymptotics hold
\[
R(\omega) = R_{\mu} + O(|\omega - \mu|^{1/2}),
\]
\[
R'(\omega) = O(|\omega - \mu|^{-1/2}), \quad \omega \to \mu, \quad \omega \in \mathbb{C} \setminus \Gamma,
\]
\[
R''(\omega) = O(|\omega - \mu|^{-3/2}),
\]
in $B(\sigma, -\sigma)$ with $\sigma > 5/2$.

In the case $m = 0$, we have only two edge points $\mu = \pm 2$ where the asymptotic expansions (6.19) hold.

As in the case of the Schrödinger equation, the spectral representation (6.13) and the asymptotics (6.19) imply Theorem 6.6.

Let conditions (1.4) and (4.1) hold. Then the asymptotics hold
\[
e^{-itHP_c} = O(t^{-3/2}), \quad t \to \infty
\]
in $B(\sigma, -\sigma)$ with $\sigma > 5/2$.

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