Cosmological models based on a complex scalar field with a power-law potential associated with a polytropic equation of state

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We construct cosmological models based on a complex scalar field with a power-law potential $V = \frac{k}{\gamma+1} (\frac{\phi}{\phi_0})^{\gamma} |\phi|^2$ associated with a polytropic equation of state $P = K \rho^\gamma$ (the potential associated with an isothermal equation of state $P = nk_B T/m$ is $V = -\frac{n k_B T}{m} \ln (m^* |\phi|^2 / \rho_0 h^2) - 1$ and the potential associated with a logotropic equation of state $P = A \ln (\rho / \rho_0)$ is $V = -A \ln (m^* |\phi|^2 / h^2 \rho_0) + 1$). We consider a fast oscillation regime of "spintessence" where the equations of the problem can be simplified. We study all possible cases with arbitrary (positive and negative) values of the polytropic constant and polytropic index. The $\Lambda$CDM model, the Chaplygin gas model and the Bose-Einstein condensate model are recovered as particular cases of our study corresponding to a constant potential ($\gamma = 0$), an inverse square-law potential ($\gamma = -1$), and a quartic potential ($\gamma = 2$). We also derive the two-fluid representation of the Chaplygin gas model.

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I. INTRODUCTION

Scalar fields (SFs) have been invoked in different domains of particle physics, quantum field theory, astrophysics, and cosmology. In particle physics and string theory, they arise in a natural manner as bosonic spin-0 particles described by the Klein-Gordon (KG) equation [1, 2]. Examples include the Higgs particle, the inflaton, the dilaton field of superstring theory, pseudo Nambu-Goldstone bosons, tachyons etc. SFs also arise in the Kaluza-Klein and Brans-Dicke theories [3] as well as in quantum gravity, supergravity, and superstring models (e.g., hidden sector fields and moduli). In cosmology, SFs were first introduced to explain the phase of inflation in the primordial universe assumed to be driven by vacuum energy [4]. The inflaton, which has its origin in the quantum fluctuations of the vacuum, is usually associated with a nonequilibrium phase transition. In canonical SF models based on a Lagrangian of the form $\mathcal{L} = X - V(\phi)$, where $X = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ is the kinetic term and $V(\phi)$ is the potential term, the physical information on the system is contained in the potential $V(\phi)$ of the SF. In k-inflation models based on a Lagrangian of the form $\mathcal{L} = \mathcal{L}(X)$ [5, 6], the physical information on the system is encapsulated in the nonstandard kinetic term.\textsuperscript{1} After the discovery of the accelerating expansion of the universe [7, 10], SFs have been used to describe dark energy (DE). Different forms of SFs were introduced such as quintessence [11, 13], k-essence [14, 15], phantom (ghost) fields [16, 17], tachyons [18, 23], quintom [24, 20] etc. They usually involve a mass of the order of the current Hubble scale ($m \sim H_0 h^2 c^2 \sim 10^{-33} \text{eV}/c^2$). It has also been proposed that dark matter (DM) and superfluid stars may be described in terms of a SF which represents the wave function of a self-gravitating Bose-Einstein condensate (BEC). This leads to the concept of boson stars [27, 29] and BEC stars [30] that could describe DM stars or the superfluid core of neutron stars. DM halos may also be interpreted as giant self-gravitating BECs. In particular, the fuzzy dark matter (FDM) model is based on the assumption that DM is made of extremely light scalar particles (ultralight axions) with mass $m \sim 10^{-22} \text{eV}/c^2$. At large scales, FDM behaves as CDM but at small scales ($\lesssim 1 \text{kpc}$), the wave (quantum) properties of the bosons manifest themselves and may solve the problems of CDM such as the cusp-core problem [32], the missing satellite problem [34, 36], and the “too big to fail” problem [37].

In cosmology, most models attempting to describe the evolution of the universe by a SF consider a real SF. However, complex SFs should be considered as well because they can be given a physical justification in relation to the Higgs mechanism in particle physics or to the phenomenon of Bose-Einstein condensation in ultracold gases. In addition, complex SFs are potentially more relevant than real SFs because they can form stable DM halos while DM halos made of real SFs are either dynamically unstable [38] or oscillating [39].\textsuperscript{2} This is basically due to the fact that the charge of a complex SF is conserved while real SFs have no conserved charge. Therefore, a complex SF seems more promising than a real SF to account both for the cosmic evolution of the

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\textsuperscript{1} General Lagrangians of the form $\mathcal{L}(X, \bar{\phi})$ have also been considered in Refs. [5, 6].

\textsuperscript{2} These dynamical instabilities occur in the relativistic regime for boson stars. Although DM halos can usually be described by nonrelativistic equations, there are certain situations in which relativity may be important. In addition, bosons described by a complex SF with a global $U(1)$ symmetry associated with a conserved charge (Noether theorem) can form BECs even in the early universe while this is more difficult for a system of bosons described by a real SF like the QCD axion.
The cosmological evolution of a spatially homogeneous complex SF with a quartic self-interaction potential (possibly representing the wave function of a relativistic BEC) was recently studied by Li et al. [10] and Suárez and Chavánis [11]. These authors considered a slow oscillation regime where the SF rolls down the potential without oscillating and a fast oscillation regime of “spintessence” [42] where the direction (phase) of the SF rotates rapidly in the complex plane while its modulus changes slowly (adiabatically). In the slow oscillation regime, there is an interesting situation of “kination” where the kinetic term dominates the potential term and the SF behaves as stiff matter [43]. In the fast oscillation regime, the equations of the problem can be averaged over the oscillations of the SF and simplified. Using a hydrodynamic representation of the KG equation, Suárez and Chavánis [41] showed that the fast oscillation regime of spintessence is equivalent to the Thomas-Fermi (TF) approximation where the quantum potential can be neglected (this amounts to taking \( \hbar = 0 \) in the hydrodynamic equations). They investigated the domain of validity of this regime in detail.

A complex SF with a repulsive \( \frac{\lambda}{4\hbar^2} |\varphi|^4 \) (\( \lambda > 0 \)) self-interaction undergoes, in the fast oscillation regime, a radiationlike era (with an equation of state \( P \sim \epsilon/3 \)) followed by a matterlike era (\( P \approx 0 \)). However, the fast oscillation regime is not valid at very early times. It is preceded, in the primordial universe, by a stiff matter era (\( P = \epsilon \)) which is valid in the slow oscillation regime. Therefore, a complex SF with a repulsive \( |\varphi|^4 \) self-interaction undergoes successively a stiff matter era, a radiationlike era, and a DM era. It is also possible to account for the present acceleration of the universe (DE) by introducing a cosmological constant \( \Lambda \) or by shifting the origin of the SF potential to a small positive value equal to the cosmological energy density \( \epsilon_\Lambda = \rho_\Lambda c^2 \) so that \( V_{\text{tot}} = \rho_\Lambda c^2 + m_\varphi^2 c^2 |\varphi|^2 + \frac{\lambda}{4\hbar^2} |\varphi|^4 \) (see Appendix E of [52]). The radiationlike era only exists for sufficiently large values of the self-interaction parameter [40, 41]. Self-interacting SFs may be more relevant than noninteracting SFs. Indeed, it is found in [40, 41] that the FDM model which assumes that the bosons are noninteracting is not consistent with cosmological observations.

A complex SF with an attractive \( \frac{\lambda}{4\hbar^2} |\varphi|^4 \) (\( \lambda < 0 \)) self-interaction can display, in the fast oscillation regime, two types of behaviors which are associated with two branches of solutions. These solutions start at a nonzero scale factor with a finite energy density. In this sense, the SF emerges “suddenly” in the universe. Initially, there is a very short cosmic stringlike era (\( P \sim -\epsilon/3 \)). On the normal (nonrelativistic) branch, the energy density decreases to zero as \( a \to 3 \) and the SF behaves as pressureless DM (\( P \approx 0 \)). On the peculiar (ultrarelativistic) branch, the energy density slowly decreases and tends to a constant at late times giving rise to a de Sitter evolution. In that case, the SF behaves as DE (\( P \sim -\epsilon \)). On the normal branch, the fast oscillation regime is not valid at early times so that quantum mechanics (\( \hbar \neq 0 \)) must be taken into account. In the very early universe, a complex SF with an attractive self-interaction may undergo an inflation era. If the self-interaction is sufficiently small, the inflation era is followed by a stiff matter era. On the peculiar branch, if the self-interaction is sufficiently large, the fast oscillation regime is valid initially but it ceases to be valid at late times so that quantum mechanics (\( \hbar \neq 0 \)) must be taken into account ultimately. As a result, the de Sitter regime may stop and the SF may eventually enter in a stiff matter era implying that the late universe passes from a phase of acceleration to a phase of deceleration.

In the present paper, we study how the evolution of the universe depends on the form of the SF potential. Many

\begin{itemize}
  \item \footnote{Li et al. [10] focused on a repulsive quartic self-interaction while Suárez and Chavánis [11] considered both repulsive and attractive quartic self-interactions. They also developed a general formalism that is valid for an arbitrary potential of interaction \( V(|\varphi|^2) \).}
  \item The radiationlike era is due to the \( |\varphi|^4 \) self-interaction of the SF. It is different from the standard radiation era due to photons or ultrarelativistic particles like neutrinos.
  \item It is well-known that a noninteracting real SF undergoes a stiff matter era followed by an inflation era and, finally (in the fast oscillation regime), by a matter era [44, 47]. This evolution (\( \text{stiff} \to \text{inflation} \to \text{matter} \)) was also found by Scialom and Jetzer [18, 19] for a self-interacting complex SF. These authors showed that the inflation era occurring in the slow-roll regime is sufficiently long provided that the bosonic charge \( Q \) is close to zero (in that case, the phase of the SF remains approximately constant and thus inflation is essentially driven by one component of the SF as if it were real). On the other hand, for a massive complex SF, inflation is less effective when the self-interaction \( \lambda \) increases. This is because the potential becomes less flat and, as a consequence, the slow-roll approximation needed for inflation to occur is no longer well satisfied. Therefore, inflation takes place only if the charge and the self-interaction of the SF are sufficiently small. Otherwise, the background passes directly from the stiff matter era to the oscillatory phase [10, 11] (there may, however, be an inflation era prior to the stiff matter era [50]).
  \item Arbey et al. [51] also considered (independently from [18, 19]) a self-interacting complex SF. For \( \lambda = 0 \) they found a stiff matter era, followed by an epoch where the SF is constant (like the inflation era reported previously but occurring during the standard radiation era), and a DM era. For \( \lambda \neq 0 \) they found, in the fast oscillation regime, a dark radiation era followed by a DM era. For a complex SF in the fast oscillation regime, only the phase \( \theta \) of the SF changes (spintessence). The modulus \( |\varphi| \) of the SF evolves slowly without oscillating. By contrast, for a real SF in the fast oscillation regime, \( \varphi(t) \) oscillates rapidly by taking positive and negative values. Arbey et al. [51] considered a fast oscillation regime different from spintessence where the complex SF behaves as an effective oscillating real SF (axion). In the present paper, when considering the fast oscillation regime of a complex SF, we shall systematically assume that it corresponds to the spintessence regime.
  \item This intriguing behaviour was first reported by Suárez and Chavánis [41] and further discussed by Carvente et al. [53].
\end{itemize}
types of potentials have been studied in the past in the case of a real SF. The novelty of our approach is to consider the cosmic evolution of a complex SF. Furthermore, we focus on the fast oscillation regime where the SF experiences the process of spintessence. The general equations of the problem, valid for an arbitrary self-interaction potential, are given in [41]. In the present paper, we consider algebraic potentials of the form \( V = \frac{K}{\rho^2} \left( \frac{\rho}{m} \right)^2 \left| \varphi \right|^2 \gamma \) where \( K \) and \( \gamma \) can be positive or negative. These power-law potentials are associated with the polytropic equation of state \( P = K \rho^\gamma \) where \( \rho = \frac{m^2}{\hbar^2} \left| \varphi \right|^2 \) is the pseudo rest-mass density. The \( \Lambda \)CDM model is recovered for a constant potential \( V = \rho \Lambda c^2 \) (\( \gamma = 0 \) and \( K = -\rho \Lambda c^2 \)) corresponding to a constant equation of state \( P = -\rho \Lambda c^2 \). The Chaplygin gas is recovered for an inverse square law potential \( V = \frac{2\pi a}{m^3} \left( \frac{\rho}{m} \right)^2 \left| \varphi \right|^4 \) (\( \gamma = 2 \) and \( K = 2\pi a \hbar^2 / m^3 \)) corresponding to the equation of state \( P = \frac{2\pi a}{m^3} \rho^2 \). We also consider a potential of the form \( V = \frac{m k_B T}{\rho \Lambda c^2} \left| \varphi \right|^4 \ln \left( m^2 \left| \varphi \right|^2 / \rho \Lambda c^2 \right) - 1 \) associated with the isothermal equation of state \( P = \rho k_B T / m \). This is the limit of the polytropic equation of state when \( \gamma \to 1 \). In a companion paper [52], we specifically study a potential of the form \( V = -A \left[ \ln \left( m^2 \left| \varphi \right|^2 / \rho \Lambda c^2 \right) - 1 \right] \) associated with the logotropic equation of state \( P = \rho A n / \rho \Lambda c^2 \). This is the limit of the polytropic equation of state when \( \gamma \to 0 \) and \( K \to +\infty \) with \( A = K \gamma \) constant (see Sec. 3 of [52] and Appendix A of [52] for a precise statement). The logotropic model gives a good agreement with the cosmological observations of our universe and is able to account for the universal value of the surface density \( \Sigma_0 = 141_{-52}^{+83} \) of DM halos without free parameter. In the present paper, we do not necessarily try to construct a realistic model of universe. We consider an arbitrary power-law potential \( V \sim K \left| \varphi \right|^{2\gamma} \) and describe the different types of evolutions that it produces depending on the value of \( \gamma \) and on the sign of \( K \). Our work therefore complements the study of the \( \left| \varphi \right|^4 \) potential performed in [40, 41].

The paper is organized as follows. In Sec. [41] we consider a spatially homogeneous SF in an expanding universe. Following our previous paper [41] we establish the general equations of the problem in the fast oscillation (or TF) regime for an arbitrary potential of interaction \( V(|\varphi|^2) \) and show that the SF behaves as a barotropic fluid described by an equation of state \( P(\rho) \) determined by the potential. In Sec. [41] we show that the energy density \( \epsilon \) can be written as the sum of a rest-mass energy density \( \rho_m c^2 \) and an internal energy density \( u \) which play respectively the roles of DM and DE. We relate the internal energy density \( u \) to the SF potential \( V \). We also introduce a two-fluid model associated with the single dark fluid (or SF) model. In Sec. [41] we consider power-law potentials associated with polytropic and isothermal equations of state. In Sec. [41] we describe all possible cosmological evolutions corresponding to a positive polytropic constant \( K > 0 \). In Sec. [VI] we describe all possible cosmological evolutions corresponding to a negative polytropic constant \( K < 0 \). A summary of our main results is provided in the conclusion. Complements about the general formalism are given in the Appendices.

II. THEORY OF A COMPLEX SF

In this section, we recall the basic equations governing the cosmological evolution of a spatially homogeneous complex SF with an arbitrary self-interaction potential in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe. We also recall how these equations can be simplified in the fast oscillation regime (equivalent to the classical or TF approximation) that is considered in the following sections. We refer to our previous papers [41, 52, 55] and references therein for a more detailed discussion.

A. Spatially homogeneous SF

Let us consider a complex SF \( \varphi \) with a self-interaction potential \( V(|\varphi|^2) \) described by the KG equation. For a spatially homogeneous SF \( \varphi(t) \) evolving in an expanding background, the KG equation takes the form\(^7\)

\[
\frac{1}{c^2} \frac{d^2 \varphi}{dt^2} + \frac{3H}{c^2} \frac{d \varphi}{dt} + \frac{m^2 c^2}{\hbar^2} \varphi + \frac{2}{|\varphi|^2} \frac{dV}{d|\varphi|^2} \varphi = 0, \tag{1}
\]

where \( H = \dot{a} / a \) is the Hubble parameter and \( a(t) \) is the scale factor. The second term in Eq. (1) is the Hubble drag. The rest-mass term (third term) can be written as \( \varphi / \lambda_C^2 \) where \( \lambda_C = h / mc \) is the Compton wavelength \( m \) is the mass of the SF). The total potential including the rest-mass term and the self-interaction term writes

\[
V_{\text{tot}}(|\varphi|^2) = \frac{m^2 c^2}{2 \hbar^2} |\varphi|^2 + V(|\varphi|^2). \tag{2}
\]

The energy density \( \epsilon(t) \) and the pressure \( P(t) \) of the SF are given by (see, e.g., [55] for details)

\[
\epsilon = \frac{1}{2c^2} \left| \frac{d \varphi}{dt} \right|^2 + \frac{m^2 c^2}{2 \hbar^2} |\varphi|^2 + V(|\varphi|^2), \tag{3}
\]

\[
P = \frac{1}{2c^2} \left| \frac{d \varphi}{dt} \right|^2 - \frac{m^2 c^2}{2 \hbar^2} |\varphi|^2 - V(|\varphi|^2). \tag{4}
\]

The equation of state parameter is defined by \( w = P / \epsilon \).

\(^7\) See Appendix A for the general expression of the KG equation valid for a spatially inhomogeneous SF.
The Friedmann equations determining the evolution of the homogeneous background are
\[ \frac{de}{dt} + 3H(e + P) = 0 \]  
and
\[ H^2 = \frac{8\pi G}{3c^2} \epsilon. \]  
Equation (6) is valid in a flat universe \((k = 0)\) without cosmological constant \((\Lambda = 0)\). The Friedmann equations can be derived from the Einstein field equations by using the FLRW metric. The energy conservation equation (5) results from the identity \(D_{\mu}T^{\mu\nu} = 0\), where \(T^{\mu\nu}\) is the energy-momentum tensor. It can also be obtained from the KG equation (1) by using Eqs. (3) and (4) (see Appendix G of [35]). Inversely, the KG equation can be derived from Eqs. (3)-(5). Once the SF potential \(V(|\varphi|^2)\) is given, the Klein-Gordon-Friedmann (KGF) equations provide a complete set of equations that can in principle be solved to obtain the evolution of the universe assuming that the energy density is entirely due to the SF (for simplicity we do not consider here the effect of other species like standard radiation and baryonic matter).

Remark: From Eq. (6) we see that the energy density decreases with the scale factor when \(w > -1\) and increases with the scale factor when \(w < -1\). In the second case, the universe has a phantom behavior (we will see that this regime is not allowed for a complex SF in the fast oscillation regime). On the other hand, when \(k = \Lambda = 0\), the deceleration parameter \(q = -\ddot{a}/\dot{a}^2\) is given by \(q = (1 + 3w)/2\). Therefore, the universe is decelerating when \(w > -1/3\) and accelerating when \(w < -1/3\).

B. Charge of the SF

Writing the complex SF as
\[ \varphi = |\varphi|e^{i\theta}, \]  
where \(|\varphi|\) is the modulus of the SF and \(\theta\) is its phase (angle), inserting this decomposition into the KG equation (1), and separating the real and imaginary parts, we obtain the following pair of equations
\[ \frac{1}{c^2} \left( 2\frac{d|\varphi|}{dt} \frac{d\theta}{dt} + |\varphi| \frac{d^2\theta}{dt^2} \right) + 3H \frac{|\varphi|}{c^2} \frac{d\theta}{dt} = 0, \]  
\[ \frac{1}{c^2} \left( \frac{d^2|\varphi|}{dt^2} - |\varphi| \left( \frac{d\theta}{dt} \right)^2 \right) + 3H \frac{|\varphi|}{c^2} \frac{d\theta}{dt} + \frac{m^2 c^2}{h^2} |\varphi| + 2 \frac{dV}{d|\varphi|^2} |\varphi| = 0. \]  
Equation (8) can be rewritten as a conservation equation
\[ \frac{d}{dt} \left( a^3 |\varphi|^2 \frac{d\theta}{dt} \right) = 0. \]  
Introducing the pulsation \(\omega = -\dot{\theta}\), we get
\[ \omega = \frac{Qhc^2}{a^4 |\varphi|^2}, \]  
where \(Q\) is a constant of integration which represents the charge of the SF [31, 32, 33, 34, 35]. This conservation law results from the identity \(D_{\mu}J^{\mu} = 0\) where \(J^{\mu}\) is the KG current. The conservation of charge is equivalent to the conservation of bosons provided that anti-bosons are counted negatively [58]. Eq. (9) can be rewritten as
\[ \frac{d^2|\varphi|}{dt^2} - \omega^2 |\varphi| + 3H \frac{d|\varphi|}{dt} + \frac{m^2 c^4}{h^2} |\varphi| + 2c^2 \frac{dV}{d|\varphi|^2} |\varphi| = 0. \]  
Substituting Eq. (11) into Eq. (12) we obtain the differential equation
\[ \frac{d^2|\varphi|}{dt^2} + 3H \frac{d|\varphi|}{dt} + \frac{m^2 c^4}{h^2} |\varphi| + 2c^2 \frac{dV}{d|\varphi|^2} |\varphi| - \frac{Q^2 h^2 c^4}{a^8 |\varphi|^4} = 0. \]  
This equation is exact. It determines the evolution of the modulus of the complex SF. It differs from the KG equation of a real SF by the presence of the last term and the fact that \(\varphi\) is replaced by \(|\varphi|\). On the other hand, substituting Eq. (7) into Eqs. (3) and (4) we find that the energy density and the pressure are given by
\[ \epsilon = \frac{1}{2c^2} \left( \frac{d|\varphi|}{dt} \right)^2 + \left( \frac{\omega^2}{2c^2} + \frac{m^2 c^2}{2h^2} \right) |\varphi|^2 + V(|\varphi|^2), \]  
\[ P = \frac{1}{2c^2} \left( \frac{d|\varphi|}{dt} \right)^2 + \left( \frac{\omega^2}{2c^2} - \frac{m^2 c^2}{2h^2} \right) |\varphi|^2 - V(|\varphi|^2). \]  

C. Fast oscillation regime and spintessence

In the fast oscillation regime \(\omega = |d\theta/dt| \gg H = \dot{a}/a\) where the pulsation is high with respect to the Hubble expansion rate, Eq. (12) reduces to\(^8\)
\[ \omega^2 = \frac{m^2 c^4}{h^2} - 2c^2 \frac{dV}{d|\varphi|^2}. \]  
This equation can be interpreted as a condition of equilibrium between the centrifugal force \(\omega^2 |\varphi|\) and the force \(c^2 dV_{tot}/d|\varphi|\) produced by the total SF potential (see Sec. V.A. of [11]). When this condition is satisfied, the direction (phase) of the SF rotates rapidly in the complex plane while its modulus changes slowly (adiabatically). This is what Boyle et al. [40] call “spintessence”. There

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\(^8\) For a free field \((V = 0)\), the pulsation \(\omega\) is proportional to the mass of the SF \((\omega = mc^2/h)\) and the fast oscillation condition reduces to \(mc^2/h \gg H\).
The equation of state parameter is then given by
\begin{equation}
Q^2 \hbar^2 c^4 \frac{d^6 \rho}{|\varphi|^4} = \frac{m^2 c^4}{\hbar^2} + 2 \epsilon \frac{dV}{d|\varphi|^2}.
\end{equation}
This equation relates the modulus $|\varphi|$ of the SF to the scale factor $a$ in the fast oscillation regime. The pulsation $\omega$ of the SF is then given by Eq. (11) or (16).

D. Equation of state in the fast oscillation regime

To establish the equation of state of the SF in the fast oscillation regime, we can proceed as follows [10, 41, 59-62]. Multiplying the KG equation (1) by $\varphi^*$ and averaging over a time interval that is much longer than the field oscillation period $\omega^{-1}$, but much shorter than the Hubble time $H^{-1}$, we obtain
\begin{equation}
\frac{1}{c^2} \left\langle \frac{d\varphi}{dt} \varphi^* \right\rangle^2 = \frac{m^2 c^2}{\hbar^2} \left\langle |\varphi|^2 \right\rangle + 2 \left\langle \frac{dV}{d|\varphi|^2} \right\rangle \left\langle |\varphi|^2 \right\rangle. \quad (18)
\end{equation}
This relation constitutes a sort of virial theorem. On the other hand, for a spatially homogeneous SF, the energy density and the pressure are given by Eqs. (3) and (4). Taking the average value of the energy density and pressure, using Eq. (18), and making the approximation
\begin{equation}
\left\langle \frac{dV}{d|\varphi|^2} \varphi^* \right\rangle \simeq V'(|\varphi|^2), \quad (19)
\end{equation}
we get
\begin{equation}
\langle \epsilon \rangle = \frac{m^2 c^2}{\hbar^2} \left\langle |\varphi|^2 \right\rangle + V'(|\varphi|^2) \left\langle |\varphi|^2 \right\rangle + V(|\varphi|^2), \quad (20)
\end{equation}
\begin{equation}
\langle P \rangle = V'(|\varphi|^2) \left\langle |\varphi|^2 \right\rangle - V(|\varphi|^2). \quad (21)
\end{equation}
The equation of state parameter is then given by
\begin{equation}
w = \frac{P}{\epsilon} = \frac{V'(|\varphi|^2) \left\langle |\varphi|^2 \right\rangle - V(|\varphi|^2)}{\frac{m^2 c^2}{\hbar^2} \left\langle |\varphi|^2 \right\rangle + V'(|\varphi|^2) \left\langle |\varphi|^2 \right\rangle + V(|\varphi|^2)}. \quad (22)
\end{equation}
We note that the averages are not strictly necessary in Eqs. (20)-(22) since the modulus of the SF changes slowly with time. Eqs. (20) and (21) can also be obtained from Eqs. (14) and (15) by using Eq. (16) and neglecting the term $(d|\varphi|/dt)^2$.

E. Hydrodynamic variables and TF approximation

Instead of working with the SF $\varphi(t)$, we can use hydrodynamic variables.\(^9\) We write the SF in the de Broglie form (for a homogeneous SF)
\begin{equation}
\varphi(t) = \frac{\hbar}{m \sqrt{\rho(t)}} e^{i S_{\text{tot}}(t)/\hbar}, \quad (23)
\end{equation}
where \(\rho = \frac{m^2}{\hbar^2} |\varphi|^2\) \(^{10}\) is the pseudo rest-mass density and $S_{\text{tot}} = (1/2)\hbar \ln(|\varphi|/\rho)$ is the total action of the SF. In terms of the pseudo rest-mass density the total potential \(^{12}\) can be written as
\begin{equation}
V_{\text{tot}}(\rho) = \frac{1}{2} \rho c^2 + V(\rho). \quad (25)
\end{equation}
On the other hand, the total energy of the SF (including its rest mass energy $mc^2$) is
\begin{equation}
E_{\text{tot}}(t) = - \frac{dS_{\text{tot}}}{dt}. \quad (26)
\end{equation}
Substituting Eq. (23) into the KG equation (1) and taking the imaginary part, we obtain the conservation equation \(^{11, 57}\)
\begin{equation}
\frac{d}{dt} \left( \rho E_{\text{tot}} a^3 \right) = 0. \quad (27)
\end{equation}
It expresses the conservation of the charge of the SF.\(^11\) It can be integrated into
\begin{equation}
\frac{E_{\text{tot}}}{mc^2} = \frac{Qm}{a^3}, \quad (28)
\end{equation}
where $Q$ is the charge of the SF. These equations are equivalent to Eqs. (10) and (11)\(^12\). Next, substituting Eq. (23) into the KG equation (1), taking the real part, and making the TF approximation $\hbar \to 0$, we obtain the Hamilton-Jacobi (or Bernoulli) equation \(^{11, 57}\)
\begin{equation}
E_{\text{tot}}^2 = m^2 c^2 + 2 m^2 c^2 V'(\rho). \quad (29)
\end{equation}
This equation is equivalent to Eq. (16). It can be rewritten as
\begin{equation}
E_{\text{tot}} = mc^2 \sqrt{1 + \frac{2}{c^2} V'(\rho)}. \quad (30)
\end{equation}
\(^9\) See Appendices A and B and our previous works \(^{41, 65, 57, 63, 64}\), for a general hydrodynamical description of the SF valid for possibly inhomogeneous systems.

\(^{10}\) We stress that it is only in the nonrelativistic limit $c \to +\infty$ that $\rho$ has the interpretation of a rest-mass density (in this limit, we also have $c \sim \rho c^2$). In the relativistic regime, $\rho$ does not have a clear physical interpretation but it can always be introduced as a convenient notation.

\(^{11}\) For a spatially homogeneous SF, the density of charge (or rest-mass density $\rho_m$) is proportional to $\rho E_{\text{tot}}$ (see Sec. III and Appendix B).

\(^{12}\) To make the link between the SF variables and the hydrodynamical variables, we use $|\varphi| = (\hbar/m)\sqrt{\rho}$, $\theta = S_{\text{tot}}/\hbar$ and $\omega = -\dot{\theta} = -\dot{S}_{\text{tot}}/\hbar = E_{\text{tot}}/\hbar$. 

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\(^{9}\) See Appendices A and B and our previous works \(^{41, 65, 57, 63, 64}\), for a general hydrodynamical description of the SF valid for possibly inhomogeneous systems.

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\(^{9}\) See Appendices A and B and our previous works \(^{41, 65, 57, 63, 64}\), for a general hydrodynamical description of the SF valid for possibly inhomogeneous systems.
Note that Eq. (29) requires that

$$V'_\text{tot}(\rho) = 1 + \frac{2}{c^2} V'(\rho) > 0. \quad (31)$$

Combining Eqs. (28) and (30), we obtain

$$\rho \sqrt{1 + \frac{2}{c^2} V'(\rho)} = \frac{Qm}{a^3}, \quad (32)$$

which corresponds to Eq. (17). Finally, writing Eqs. (3) and (4) in terms of hydrodynamic variables, making the TF approximation \( h \to 0 \), and using the Bernoulli equation (29), we get [41, 57]

$$\epsilon = \rho c^2 + V(\rho) + \rho V'(\rho), \quad (33)$$

$$P = \rho V'(\rho) - V(\rho), \quad (34)$$

in agreement with Eqs. (20) and (21). Eq. (34) determine the equation of state \( P(\rho) \) for a given potential \( V(\rho) \). Inversely, for a given equation of state, the potential is given by

$$V(\rho) = \rho \int \frac{P(\rho)}{\rho^2} \, d\rho. \quad (35)$$

The various correspondences between the results of this section and the results of the previous sections show that the fast oscillation regime \( (\omega \gg h) \) is equivalent to the TF or semiclassical approximation \( (h \to 0) \). We note that we cannot directly take \( h = 0 \) in the KG equation (this is why we have to average over the oscillations) while we can take \( h = 0 \) in the hydrodynamic equations (see Refs. [11, 53, 57] for details). This is an interest of the hydrodynamic representation of the SF. It can be shown (see, e.g., [53] and Appendix A) that Eqs. (33) and (34) remain valid for a spatially inhomogeneous SF in the TF limit. They determine the equation of state \( P = P(\epsilon) \) in parametric form. The equation of state parameter can be written as

$$w = \frac{P}{\epsilon} = \frac{\rho V'(\rho) - V(\rho)}{\rho c^2 + V(\rho) + \rho V'(\rho)}, \quad (36)$$

which is equivalent to Eq. (22). We note that the condition from Eq. (31) implies \( w > -1 \) so that a universe described by a complex SF in the fast oscillation regime has never a phantom behavior. On the other hand, the universe is decelerating \( (w > -1/3) \) when \( 4\rho V'(\rho) - 2V(\rho) > -\rho c^2 \) (or equivalently \( 2\rho V'_\text{tot}(\rho) > V'_\text{tot}(\rho) \)) and accelerating in the opposite case. Finally, the pseudo squared speed of sound is

$$c_s^2 = \frac{P'(\rho)}{\epsilon} = \frac{\rho V''(\rho)}{c^2 + 2V'(\rho) + \rho V'(\rho)}, \quad (37)$$

while the true squared speed of sound is

$$c_t^2 = \frac{P'(\epsilon)}{\epsilon} = \frac{\rho V''(\rho)}{c^2 + 2V'(\rho) + \rho V'(\rho)}. \quad (38)$$

F. Cosmological evolution of a spatially homogeneous complex SF in the fast oscillation regime

The differential equation determining the evolution of the scale factor \( a(t) \) of the Universe induced by a spatially homogeneous complex SF in the regime where its oscillations are faster than the Hubble expansion is given by [see Eqs. (6), (32) and (33)]

$$\frac{3}{8\pi G} \left( \frac{\dot{a}}{a} \right)^2 = \rho + \frac{1}{c^2} [V(\rho) + \rho V'(\rho)] \quad (39)$$

with

$$\rho \sqrt{1 + \frac{2}{c^2} V'(\rho)} = \frac{Qm}{a^3}, \quad (40)$$

Actually, it is not convenient to solve the differential equation (39) for the scale factor \( a \) because we have to substitute Eq. (40) in order to express \( \rho \) as a function of \( a \) in the r.h.s. of Eq. (39). It is more convenient to view \( a \) as a function of \( \rho \), given by Eq. (40), and transform Eq. (39) into a differential equation for \( \rho \). Taking the logarithmic derivative of Eq. (40), we get

$$\frac{\dot{a}}{a} = -\frac{1}{3} \frac{\dot{\rho}}{\rho} \left[ 1 + \frac{\rho V''(\rho)}{c^2 + 2V'(\rho)} \right]. \quad (41)$$

Substituting this expression into Eq. (39), we obtain the differential equation

$$\frac{c^2}{24\pi G} \left( \frac{\dot{\rho}}{\rho} \right)^2 = \frac{\rho c^2 + V(\rho) + \rho V'(\rho)}{c^2 + 2V'(\rho) + \rho V'(\rho)} \quad (42)$$

For a given SF potential \( V(\rho) \), this equation can be solved easily as it is just a first order differential equation for \( \rho \). Therefore, \( t(\rho) \) can be expressed in the form of an integral. The temporal evolution of the scale factor \( a(t) \) is then obtained in parametric form \( a = a(\rho) \) and \( t = t(\rho) \) from Eqs. (40) and (42).

III. EFFECTIVE DARK MATTER AND DARK ENERGY

In our model, there is no DM and no DE considered as two different and independent species. There is just
a single SF which can be represented, through the de Broglie transformation, as a single dark fluid. It therefore provides a unified dark matter and dark energy (UDM) model. In this section, we interpret this dark fluid as a barotropic gas at \( T = 0 \) and determine its rest-mass density \( \rho_m \), internal energy density \( u \) and equation of state \( P(\rho_m) \) in terms of the SF potential. This equivalence is valid only in the TF approximation. Following [50, 54, 65], we argue that the rest-mass density \( \rho_m \) plays the role of DM and the internal energy density \( u \) plays the role of DE. This provides a simple physical interpretation of these two mysterious components.

### A. First principle of thermodynamics

The first principle of thermodynamics for a relativistic gas can be written as

\[
\frac{d}{dt} \left( \frac{\epsilon}{\rho_m} \right) = -Pd \left( \frac{1}{\rho_m} \right) + Td \left( \frac{s}{\rho_m} \right),
\]

(43)

where

\[
\epsilon = \rho_m c^2 + u(\rho_m)
\]

(44)

is the energy density including the rest-mass energy density \( \rho_m c^2 \) (where \( \rho_m = nm \) is the rest-mass density) and the internal energy density \( u(\rho_m) \), \( s \) is the entropy density, \( P \) is the pressure, and \( T \) is the temperature. We assume that \( Td(s/\rho_m) = 0 \). This corresponds to cold \(( T = 0) \) or isentropic \(( s/\rho_m = \text{cst}) \) gases. In that case, Eq. (43) reduces to

\[
\frac{d}{dt} \left( \frac{\epsilon}{\rho_m} \right) = -Pd \left( \frac{1}{\rho_m} \right) = \frac{P}{\rho_m^2} d\rho_m.
\]

(45)

This equation can be rewritten as

\[
\frac{d\epsilon}{d\rho_m} = \frac{P + \epsilon}{\rho_m},
\]

(46)

where the term on the right hand side is the enthalpy \( h \) (or the chemical potential \( \mu = mh \)). We have

\[
h = \frac{P + \epsilon}{\rho_m}, \quad h = \frac{d\epsilon}{d\rho_m}, \quad dh = \frac{dP}{\rho_m}.
\]

(47)

Equation (45) can be integrated into

\[
\epsilon = \rho_m c^2 + \rho_m \int \frac{P(\rho_m)}{\rho_m^2} d\rho_m,
\]

(48)

establishing that

\[
u(\rho_m) = \rho_m \int \frac{P(\rho_m)}{\rho_m^2} d\rho_m.
\]

(49)

This equation determines the internal energy density as a function of the equation of state \( P(\rho_m) \). Inversely, the equation of state is determined by the internal energy density \( u(\rho_m) \) through the relation

\[
P(\rho_m) = -\frac{d(e/\rho_m)}{d(1/\rho_m)} = \rho_m^2 \frac{\partial u(\rho_m)}{\rho_m^2} = \rho_m u'(\rho_m) - u(\rho_m).
\]

(50)

We note that

\[
P'(\rho_m) = \rho_m u''(\rho_m).
\]

(51)

### B. Application to cosmology

The previous results are general. We now consider a spatially homogeneous gas in an expanding universe. Combining the energy conservation equation (5) with the first principle of thermodynamics [Eq. (46)], we obtain

\[
\frac{d\rho_m}{dt} + 3H \rho_m = 0.
\]

(52)

This equation expresses the conservation of the particle number (or rest-mass). It can be integrated into \( \rho_m \propto a^{-3} \). Inserting this relation into Eq. (44), we see that \( \rho_m \) plays the role of DM while \( u \) plays the role of DE. This decomposition provides therefore a simple interpretation of DM and DE in terms of a single DF [50, 54, 65]. Owing to this interpretation, we can write

\[
\rho_m c^2 = \frac{\Omega_m \epsilon_0}{a^3}
\]

(53)

and

\[
\epsilon = \frac{\Omega_m \epsilon_0}{a^3} + u \left( \frac{\Omega_m \epsilon_0}{c^2 a^3} \right),
\]

(54)

where \( \epsilon_0 \) is the present energy density of the universe and \( \Omega_{m,0} \) is the present proportion of DM. For given \( P(\rho_m) \) or \( u(\rho_m) \) we can get \( \epsilon(a) \) from Eq. (54). We can then solve the Friedmann equation [6] to obtain the temporal evolution of the scale factor \( a(t) \) [50, 54, 65].

Remark: Eqs. (44) and (50) determine the equation of state \( P = P(\epsilon) \). As a result, we can obtain Eq. (53) directly from Eqs. (44), (50) and the energy conservation equation (5). Indeed, combining these equations we obtain Eq. (52) which integrates to give Eq. (53).

### C. Determination of the rest-mass density and internal energy density

We now relate the rest-mass density \( \rho_m \) (proportional to the charge density) and the internal energy density \( u \) of the barotropic gas to the pseudo rest-mass density \( \rho \) and potential \( V \) of the corresponding SF in the TF approximation. According to Eq. (46), we have

\[
\ln \rho_m = \int \frac{\epsilon'(\rho)}{\epsilon + P} d\rho.
\]

(55)
Using Eqs. (33) and (34), which are valid in the TF approximation, we obtain
\[ \ln \rho_m = \int \frac{c^2 + \rho V''(\rho) + 2V'(\rho)}{\rho c^2 + 2\rho V'(\rho)} \, d\rho. \] (56)

This equation determines the function \( \rho_m(\rho) \). It can be explicitly integrated into
\[ \rho_m = \rho \sqrt{1 + \frac{2}{c^2} V'(\rho)}, \] (57)
where the constant of integration has been determined in order to obtain \( \rho_m = \rho \) in the nonrelativistic limit. This relation can also be established in the manner described in Appendix A [see Eq. (A37)]. We emphasize that Eq. (57) is always valid in the TF approximation, even for an inhomogeneous SF. Eliminating \( \rho \) between \( P(\rho) \) [see Eq. (34)] and \( \rho_m(\rho) \) [see Eq. (57)] we obtain the equation of state of the gas under the form \( P(\rho_m) \). On the other hand, according to Eq. (44), its internal energy density can be obtained from the relation
\[ u = \epsilon - \rho_m c^2. \] (58)

From Eqs. (33) and (57), we get
\[ u = \rho c^2 + \rho V'(\rho) + V(\rho) - \rho c^2 \sqrt{1 + \frac{2}{c^2} V'(\rho)}. \] (59)

where \( \rho = \rho(\rho_m) \) can be obtained from Eq. (57). Therefore, the rest-mass density (DM) is determined by Eq. (57) and the internal energy density (DE) is determined by Eq. (59). Eliminating \( \rho \) between Eqs. (57) and (59) we obtain \( u(\rho_m) \). We can also obtain \( u(\rho_m) \) from \( P(\rho_m) \), or the converse, by using Eqs. (49) and (50). As mentioned above, \( \rho_m \) mimics DM and \( u \) mimics DE. However, in our model, we have just one fluid (or just one complex SF).

For a spatially homogeneous SF in a cosmological context and in the TF approximation, combining the Hamilton-Jacobi (or Bernoulli) equation (30) with Eq. (57), we find that the relation between the rest-mass density \( \rho_m \) and the pseudo rest-mass density \( \rho \) is
\[ \rho_m = \rho \frac{E_{\text{tot}}}{mc^2}. \] (60)

Using Eqs. (57) and (60), Eqs. (28) and (32) can be rewritten as
\[ \rho_m = \frac{Q\rho}{a^3}. \] (61)

The rest-mass density (or the charge density) decreases as \( a^{-3} \). This expresses the conservation of the charge of the SF or, equivalently, the conservation of the boson minus antiboson number. Comparing Eqs. (53) and (61), we establish that
\[ Qmc^2 = \Omega_{m,0} \epsilon_0. \] (62)

We note that the constant \( Qmc^2 \) (which is proportional to the charge of the SF) is equal to the present energy density of DM \( \Omega_{m,0} \epsilon_0 \).

Remark: In the homogeneous case, we can directly deduce the relation
\[ \rho_m = \rho \frac{E_{\text{tot}}}{mc^2} = \rho \sqrt{1 + \frac{2}{c^2} V'(\rho)} \] (63)
from Eqs. (28) and (32), but this derivation is less general than the calculations detailed above [in particular Eq. (57)] which remain valid for inhomogeneous systems.

D. Two-fluid model

As mentioned above, in our model, we have a single dark fluid with an equation of state \( P = P(\rho_m) \). Still, the energy density (44) is the sum of two terms, a rest-mass density term \( \rho_m \) which mimics DM and an internal energy density term \( u(\rho_m) \) which mimics DE. It is interesting to consider a two-fluid model which leads to the same results as our single dark fluid model, at least for what concerns the evolution of the homogeneous background. In this two-fluid model, one fluid corresponds to pressureless DM with an equation of state \( F_m = 0 \) and a density \( \rho_m c^2 = \Omega_{m,0} \epsilon_0 a^3 \) determined by the energy conservation equation for DM, and the other fluid corresponds to DE with an equation of state \( P_{\text{de}}(\epsilon_{\text{de}}) \) and an energy density \( \epsilon_{\text{de}}(a) \) determined by the energy conservation equation for DE. We can obtain the equation of state of DE yielding the same results as the one-fluid model by taking
\[ P_{\text{de}} = P(\rho_m), \quad \epsilon_{\text{de}} = u(\rho_m) \] (64)
or, equivalently,
\[ P_{\text{de}} = P(\rho), \quad \epsilon_{\text{de}} = u(\rho). \] (65)

In other words, the equation of state \( P_{\text{de}}(\epsilon_{\text{de}}) \) of DE in the two-fluid model corresponds to the relation \( P(u) \) in the single fluid (or SF) model. An explicit example of the equivalence between the one and two-fluid models for the homogeneous background is given in Secs. VIII A and VIII B in connection to the (anti-)Chaplygin gas. We note that although the one and two-fluid models are equivalent for the evolution of the homogeneous background, they may differ for what concerns the formation of the large-scale structures of the Universe.

IV. POWER-LAW POTENTIALS

The equations of Secs. III and III are general. We now consider specific potentials \( V(|\vec{\varphi}|^2) \) associated with polytropic and isothermal equations of state.
A. Polytropic equation of state

We first consider a power-law SF potential that we write under the form (see also Appendix I of [41])

\[
V(|\varphi|^2) = \frac{K}{\gamma - 1} \left(\frac{m}{\hbar}\right)^2 |\varphi|^{2\gamma} \quad (\gamma \neq 1). \tag{66}
\]

For the sake of generality, we consider arbitrary values of \(K\) (positive and negative) and arbitrary values of \(\gamma\). The total potential including the rest-mass term is

\[
V_{\text{tot}}(|\varphi|^2) = \frac{1}{2} \frac{m^2 c^2}{\hbar^2} |\varphi|^2 + \frac{K}{\gamma - 1} \left(\frac{m}{\hbar}\right)^2 |\varphi|^{2\gamma}. \tag{67}
\]

Introducing the pseudo rest-mass density defined by Eq. [24], we have

\[
V(\rho) = \frac{K}{\gamma - 1} \rho^\gamma \tag{68}
\]

and

\[
V_{\text{tot}}(\rho) = \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \rho^2 + \frac{K}{\gamma - 1} \rho^\gamma. \tag{69}
\]

Using Eqs. [34], [37] and [68], we find that the derivative of the potential, the pressure and the squared speed of sound are given by

\[
V'(\rho) = \frac{K\gamma}{\gamma - 1} \rho^{\gamma - 1}, \tag{70}
\]

\[
P = K\rho^\gamma, \tag{71}
\]

\[
c_s^2 = K\gamma \rho^{\gamma - 1}. \tag{72}
\]

We see that the equation of state [71] associated with the power-law potential [66] is that of a polytrope with polytropic constant \(K\) and polytropic index \(\gamma = 1 + 1/n\). The pressure is positive when \(K > 0\) and negative when \(K < 0\). Negative pressures play an important role in cosmology. They are necessary to account for the early inflation and the present accelerating expansion of the universe (DE). We note that the potential \(V\) given by Eq. [68] is similar to the Tsallis free energy density \(V = -K s_\gamma\), where the polytropic constant \(K\) plays the role of a generalized temperature and \(s_\gamma = -\frac{1}{\gamma - 1} (\rho^\gamma - \rho)\) is the Tsallis entropy density.

For the power-law potential from Eq. [68], the equations of the problem [Eqs. (30), (32), (33), (36) and (39)] become

\[
\rho \sqrt{1 + \frac{2 K\gamma}{c^2 (\gamma - 1) \rho^{\gamma - 1}}} = \frac{Q_m}{a^3}, \tag{73}
\]

\[
\epsilon = \rho c^2 + \frac{\gamma + 1}{\gamma - 1} K \rho^\gamma, \tag{74}
\]

\[
E_{\text{tot}} = \sqrt{1 + \frac{2 K\gamma}{c^2 (\gamma - 1) \rho^{\gamma - 1}}} - 1, \tag{75}
\]

\[
3H^2 = \frac{3}{8\pi G} \frac{\rho + \gamma + 1}{\gamma - 1} \frac{K}{c^2} \rho^\gamma, \tag{76}
\]

\[
w = \frac{K}{c^2} \rho^{\gamma - 1} \frac{1}{1 + \frac{2 K\gamma}{c^2 (\gamma - 1) \rho^{\gamma - 1}}}. \tag{77}
\]

From Eqs. [71] and [74], we obtain the equation of state \(P(\epsilon)\) of the SF under the inverse form \(\epsilon(P)\) as

\[
\epsilon = (\frac{P}{K})^{1/\gamma} c^2 + \frac{\gamma + 1}{\gamma - 1} P. \tag{78}
\]

On the other hand, the differential equation governing the temporal evolution of the pseudo rest-mass density [see Eq. (42)] is

\[
\frac{c^2}{24\pi G} \left(\frac{\dot{\rho}}{\rho}\right)^2 = \frac{\rho c^2 + K (\gamma + 1)}{c^2 (\gamma - 1) \rho^{\gamma - 1}} \frac{1}{1 + \frac{2 K\gamma}{c^2 (\gamma - 1) \rho^{\gamma - 1}}}. \tag{79}
\]

The rest-mass density (DM) and the internal energy density (DE) of the SF are determined by Eqs. [57] and [59] with Eq. [68]. We get

\[
\rho_m = \rho \sqrt{1 + \frac{2\gamma}{\gamma - 1} \frac{K}{c^2 \rho^{\gamma - 1}}}, \tag{80}
\]

\[
u = \rho c^2 + \frac{\gamma + 1}{\gamma - 1} K \rho^\gamma - \rho c^2 \sqrt{1 + \frac{2\gamma}{\gamma - 1} \frac{K}{c^2 \rho^{\gamma - 1}}}. \tag{81}
\]

We note that the rest-mass density [80] can be read-off directly from Eq. [73]. Eqs. [71], [80] and [81] define \(P(\rho_m)\) and \(\nu(\rho_m)\) in parametric form with parameter \(\rho\).

As we have recalled in Sec. III the rest mass density \(\rho_m\) of the SF mimics DM and the internal energy density \(\nu\) of the SF mimics DE [54] [65].

The foregoing equations determine the cosmological evolution of a spatially homogeneous complex SF described by the potential [66] in the fast oscillation regime for any values of \(K\) and \(\gamma\). Some values of \(\gamma\) are of particular interest.

(i) For \(\gamma = -1\) \((n = -1/2)\), we obtain

\[
V = -\frac{K}{2\rho}. \tag{82}
\]

\[
P = K, \tag{83}
\]

\[
\epsilon = \rho c^2. \tag{84}
\]
leading to
\[ P = \frac{K c^2}{\epsilon}. \] (85)

This is the equation of state of the Chaplygin \((K < 0)\) or anti-Chaplygin \((K > 0)\) gas \([60,69]\). The total SF potential is
\[ V_{\text{tot}} = \frac{1}{2} \rho c^2 - \frac{K}{2 \rho} \] (86)
or, equivalently,
\[ V_{\text{tot}}(|\varphi|^2) = \frac{1}{2} \frac{m^2 c^2}{\hbar^2} |\varphi|^2 - \frac{K}{2} \left( \frac{\hbar}{m} \right)^2 \frac{1}{|\varphi|^2}. \] (87)

It corresponds to an inverse square law self-interaction potential \(V(|\varphi|^2) \sim |\varphi|^{-2}\). The rest-mass density and the internal energy density are explicitly given by
\[ \rho_m c^2 = \sqrt{(\rho c^2)^2 + K c^2}, \] (88)
\[ \rho c^2 = \sqrt{(\rho_m c^2)^2 - K c^2}, \] (89)
\[ P = \frac{K c^2}{\sqrt{(\rho_m c^2)^2 - K c^2}}, \] (90)
\[ u = \rho c^2 - \sqrt{(\rho c^2)^2 + K c^2}, \] (91)
\[ u = \sqrt{(\rho_m c^2)^2 - K c^2 - \rho_m c^2}. \] (92)

In the present context, the Chaplygin gas model is justified from a complex SF theory. Furthermore, the pseudo rest-mass density coincides with the energy density \((\epsilon = \rho c^2)\).

(ii) For \(\gamma = 2\) \((n = 1)\), we obtain
\[ V = K \rho^2, \] (93)
\[ P = K \rho^2, \] (94)
\[ \epsilon = \rho c^2 + 3K \rho^2, \] (95)
leading to
\[ \rho = \frac{-c^2 \pm \sqrt{c^4 + 12K \epsilon}}{6K} \] (96)
and
\[ P = \frac{1}{36K} \left( -c^2 \pm \sqrt{c^4 + 12K \epsilon} \right)^2. \] (97)

We must select the sign + when \(K > 0\) while the two signs ± are allowed when \(K < 0\). Alternatively, we can write the equation of state under the form
\[ \epsilon = \sqrt{\frac{P}{K}} c^2 + 3P. \] (98)

The total SF potential is
\[ V_{\text{tot}} = \frac{1}{2} \rho c^2 + K \rho^2 \] (99)
or, equivalently,
\[ V_{\text{tot}}(|\varphi|^2) = \frac{1}{2} \frac{m^2 c^2}{\hbar^2} |\varphi|^2 + K \left( \frac{m}{\hbar} \right)^4 |\varphi|^4. \] (100)

It corresponds to a quartic self-interaction potential \(V(|\varphi|^2) \sim |\varphi|^4\). This is the standard potential of a relativistic BEC. It takes into account two-body interactions in a weakly interacting microscopic theory of superfluidity. In that case, \(K = 2\pi a_s \hbar^2 / m^3\). The self-interaction is repulsive when \(K > 0\) and attractive when \(K < 0\). The relativistic BEC model has been studied in detail in the context of boson stars \([29,30,70]\) and in cosmology \([40,41]\). For \(\epsilon \to +\infty\) we have \(P \sim \epsilon / 3\) (dark radiation) and for \(\epsilon \to +\infty\) we have \(P \sim K(\epsilon / c^2)^2\) (matter). The rest-mass density and the internal energy density are given by
\[ \rho_m = \rho \sqrt{1 + \frac{4K}{c^2} \rho}, \] (101)
\[ u = \rho c^2 + 3K \rho^2 - \rho c^2 \sqrt{1 + \frac{4K}{c^2} \rho}. \] (102)

With Eq. \([104]\), they define \(P(\rho_m)\) and \(u(\rho_m)\) in parametric form. Actually, Eq. \([101]\) is a cubic equation for \(\rho\) which can be solved by standard means to get \(\rho(\rho_m)\). We can then obtain \(P(\rho_m)\) and \(u(\rho_m)\) explicitly.

(iii) For \(\gamma = 0\) \((n = -1)\), we obtain
\[ V = -K, \] (103)
\[ P = K, \] (104)
\[ \epsilon = \rho c^2 - K. \] (105)

The pressure is constant. This is the equation of state of the ΛCDM \((K < 0)\) or anti-ΛCDM \((K > 0)\) model interpreted as an UDM model \([69,71,72]\). In that case \(K = \mp \rho \Lambda c^2\) where \(\rho_\Lambda\) is the cosmological density. The pressure can be rewritten as \(P = \pm \rho \Lambda c^2\) and the energy density as \(\epsilon = \rho c^2 \pm \rho \Lambda c^2\). In the present case, the pseudo rest-mass density \(\rho\) plays the role of DM and \(\pm \rho \Lambda c^2\) the role of DE. The total SF potential is
\[ V_{\text{tot}} = \frac{1}{2} \rho c^2 - K \] (106)
or, equivalently,
\[ V_{\text{tot}}(|\varphi|^2) = \frac{1}{2} \frac{m^2 c^2}{\hbar^2} |\varphi|^2 - K. \] (107)

It corresponds to a constant self-interaction potential \(V(|\varphi|^2) = -K = \pm \rho \Lambda c^2\) equal to the cosmological energy.
density. The rest-mass density and the internal energy density are explicitly given by

\[ P = K, \quad \rho_m = \rho, \quad u = -K. \quad (108) \]

In the present context, the ΛCDM model is justified from a complex SF theory. Furthermore, the pseudo rest-mass density coincides with the rest-mass density (\( \rho_m = \rho \)).

(iv) For \( \gamma = 3 \) (\( n = 1/2 \)), we obtain

\[ V = \frac{1}{2} K \rho^3, \quad (109) \]

\[ P = K \rho^3, \quad (110) \]

\[ \epsilon = \rho c^2 + 2K \rho^3. \quad (111) \]

Equation (111) is a cubic equation for \( \rho \) which can be solved by standard means to get \( \rho(\epsilon) \). Using Eq. (110), we can then obtain the equation of state \( P(\epsilon) \) explicitly. Alternatively, we can write the equation of state under the form

\[ \epsilon = \left( \frac{P}{K} \right)^{1/3} c^2 + 2P. \quad (112) \]

The total SF potential is

\[ V_{\text{tot}} = \frac{1}{2} \rho c^2 + \frac{1}{2} K \rho^3, \quad (113) \]

or, equivalently,

\[ V_{\text{tot}}(|\varphi|^2) = \frac{1}{2} m^2 c^2 |\varphi|^2 + \frac{1}{2} K \left( \frac{m}{\hbar} \right)^6 |\varphi|^6. \quad (114) \]

It corresponds to a sextic self-interaction potential \( V(|\varphi|^2) \sim |\varphi|^6 \) (see, e.g., [73]). It takes into account three-body interactions in a weakly interacting microscopic theory of superfluidity. It can also describe an exotic DM superfluid with a different interpretation [74]. The rest-mass density and the internal energy density are explicitly given by

\[ \rho_m = \rho \sqrt{1 + \frac{3K}{c^2} \rho^2}, \quad (115) \]

\[ \rho = \left( \frac{c^2}{6K} \pm \frac{c^2}{6K} \sqrt{1 + \frac{12K}{c^2} \rho_m^2} \right)^{1/2}, \quad (116) \]

\[ P = K \left( \frac{c^2}{6K} \pm \frac{c^2}{6K} \sqrt{1 + \frac{12K}{c^2} \rho_m^2} \right)^{3/2}, \quad (117) \]

\[ u = \left( \frac{-c^2}{6K} \pm \frac{c^2}{6K} \sqrt{1 + \frac{12K}{c^2} \rho_m^2} \right)^{1/2} \times \left( \frac{2}{3} \pm \frac{1}{3} \sqrt{1 + \frac{12K}{c^2} \rho_m^2} \right) c^2 - \rho_m c^2. \quad (118) \]

(v) For \( \gamma = 1/2 \) (\( n = -2 \)),\(^ {14} \) we obtain

\[ V = -2K \sqrt{\rho}, \quad (119) \]

\[ P = K \sqrt{\rho}, \quad (120) \]

\[ \epsilon = \rho c^2 - 3K \sqrt{\rho}. \quad (121) \]

Equation (121) can be reversed to give

\[ \rho = \left( \frac{3K}{2c^2} \pm \frac{1}{2c^2} \sqrt{9K^2 + 4c^2 \epsilon} \right)^2 \quad (122) \]

and

\[ P = \frac{3K^2}{2c^2} \pm \frac{K}{2c^2} \sqrt{9K^2 + 4c^2 \epsilon}. \quad (123) \]

Alternatively, we can write the equation of state under the form

\[ \epsilon = \left( \frac{P}{K} \right)^2 c^2 - 3P. \quad (124) \]

The total SF potential is

\[ V_{\text{tot}} = \frac{1}{2} \rho c^2 - 2K \sqrt{\rho}, \quad (125) \]

or, equivalently,

\[ V_{\text{tot}}(|\varphi|^2) = \frac{1}{2} m^2 c^2 |\varphi|^2 - 2K \frac{m}{\hbar} |\varphi|. \quad (126) \]

It corresponds to a linear self-interaction potential \( V(|\varphi|^2) \sim |\varphi| \). The rest-mass density and the internal energy density are given by

\[ \rho_m = \sqrt{\rho^2 - \frac{2K}{c^2} \rho^{3/2}}, \quad (127) \]

\[ u = \rho c^2 - 3K \sqrt{\rho} - \rho_m c^2. \quad (128) \]

With Eq. (120) they define \( P(\rho_m) \) and \( u(\rho_m) \) in parametric form but we cannot have more explicit results.

Remark: We emphasize that the previous equations are valid for a possibly inhomogeneous SF. We also note that the polytropic indices \( \gamma = 0 \) and \( \gamma = -1 \) are the only ones for which the polytropic equation of state \( P = K \rho^\gamma \) yields a polytropic equation of state \( P = K(\epsilon/c^2)^\gamma \).

\(^ {14} \) An interpretation of this index is given in Sec. V1C.
B. Isothermal equation of state

We now consider a potential of the form

\[ V(|\varphi|^2) = \frac{k_B T m}{2\hbar^2} |\varphi|^2 \left[ \ln \left( \frac{m^2 |\varphi|^2}{\rho_* \hbar^2} \right) - 1 \right]. \tag{129} \]

For the sake of generality, we consider arbitrary values of \( T \) (positive and negative). The total potential including the rest-mass term is

\[ V_{\text{tot}}(|\varphi|^2) = \frac{1}{2} m^2 c^2 |\varphi|^2 + \frac{k_B T m}{\hbar^2} |\varphi|^2 \left[ \ln \left( \frac{m^2 |\varphi|^2}{\rho_* \hbar^2} \right) - 1 \right]. \tag{130} \]

Introducing the pseudo rest-mass density defined by Eq. [24], we have

\[ V(\rho) = \frac{k_B T}{m} \rho \left[ \ln \left( \frac{\rho}{\rho_*} \right) - 1 \right] \tag{131} \]

and

\[ V_{\text{tot}}(\rho) = \frac{1}{2} \rho c^2 + \frac{k_B T}{m} \rho \left[ \ln \left( \frac{\rho}{\rho_*} \right) - 1 \right]. \tag{132} \]

Using Eqs. [34], [37] and [131], we find that the derivative of the potential, the pressure and the squared speed of sound are given by

\[ V'(\rho) = \frac{k_B T}{m} \ln \left( \frac{\rho}{\rho_*} \right), \tag{133} \]

\[ P(\rho) = \rho \frac{k_B T}{m}, \tag{134} \]

\[ c_s^2 = \frac{k_B T}{m}. \tag{135} \]

We see that the equation of state [134] associated with the potential [129] is the isothermal equation of state with an effective temperature \( T \). It corresponds to a polytrope of index \( \gamma = 1 \) \((n \to +\infty)\). The pressure is positive when \( T > 0 \) and negative when \( T < 0 \). Negative pressures play an important role in cosmology in relation to the early inflation and the present accelerating expansion of the universe (DE). A positive temperature \((T > 0)\) can account for thermal effects in DM. We note that the potential \( V \) given by Eq. [131] is similar to the Boltzmann free energy density \( V = -T s_B \), where \( T \) is the temperature and \( s_B = -k_B (\rho/m) [\ln(\rho/\rho_*) - 1] \) is the Boltzmann entropy density.

For the potential [129] the equations of the problem [Eqs. [30], [32], [33], [36] and [39]] become

\[ \rho \sqrt{1 + \frac{2k_B T}{mc^2} \ln \left( \frac{\rho}{\rho_*} \right)} = \frac{Qm}{a^3}, \tag{136} \]

\[ \epsilon = \rho c^2 + \frac{2k_B T}{m} \rho \ln \left( \frac{\rho}{\rho_*} \right) - \frac{k_B T}{m} \rho, \tag{137} \]

\[ \frac{E_{\text{tot}}}{mc^2} = \sqrt{1 + \frac{2k_B T}{mc^2} \ln \left( \frac{\rho}{\rho_*} \right)}, \tag{138} \]

\[ \frac{3H^2}{8\pi G} = \rho + \frac{2k_B T}{mc^2} \rho \ln \left( \frac{\rho}{\rho_*} \right) - \frac{k_B T}{mc^2} \rho, \tag{139} \]

\[ w = \frac{\rho}{mc^2} = 1 + \frac{2 k_B T}{mc^2} \rho \ln \left( \frac{\rho}{\rho_*} \right) - 1. \tag{140} \]

From Eqs. [134] and [137], we obtain the equation of state \( P(\epsilon) \) of the SF under the inverse form \( \epsilon(\rho) \) as

\[ \epsilon = \frac{mc^2}{k_B T} P + 2P \ln \left( \frac{mc^2}{k_B T} \rho \right) - P. \tag{141} \]

Finally, the differential equation governing the temporal evolution of the pseudo rest-mass density \([Eqs. [42] \text{ and } [59]]\) with Eq. [131] is

\[ \frac{c^2}{24\pi G} \left( \frac{\ddot{\rho}}{\rho} \right)^2 = \rho c^2 + \frac{2k_B T}{m} \rho \ln \left( \frac{\rho}{\rho_*} \right) - \frac{k_B T}{m} \rho \left[ 1 + \frac{2 k_B T}{mc^2} \rho \ln \left( \frac{\rho}{\rho_*} \right) \right]^{-1}. \tag{142} \]

The rest-mass density (DM) and the internal energy density (DE) of the SF are determined by Eqs. [57] and [59] with Eq. [131]. We get

\[ \rho_m = \rho \sqrt{1 + \frac{2k_B T}{mc^2} \ln \left( \frac{\rho}{\rho_*} \right)}, \tag{143} \]

\[ u = \rho c^2 + \frac{2k_B T}{m} \rho \ln \left( \frac{\rho}{\rho_*} \right) - \frac{k_B T}{m} \rho \]

\[ - \rho c^2 \sqrt{1 + \frac{2k_B T}{mc^2} \ln \left( \frac{\rho}{\rho_*} \right)}. \tag{144} \]

Eqs. [134], [143] and [144] determine \( P(\rho_m) \) and \( u(\rho_m) \) in parametric form with parameter \( \rho \). As we have recalled in Sec. III, the rest mass density \( \rho_m \) of the SF mimics DM and the internal energy density \( u \) of the SF mimics DE [54, 65].

The foregoing equations determine the cosmological evolution of a spatially homogeneous complex SF described by the potential [129] in the fast oscillation regime for any value of \( T \).

Remark: We emphasize that the previous equations are valid for a possibly inhomogeneous SF. The results of this section (isothermal systems) can be recover from the results of Sec. IVA (polytropes) in the limit \( \gamma \to 1 \). For example,

\[ V = \frac{K}{\gamma - 1} \rho^\gamma = \frac{K \rho}{\gamma - 1} e^{(\gamma - 1) \ln \rho} \]

\[ \simeq \frac{K \rho}{\gamma - 1} [1 + (\gamma - 1) \ln \rho] + ... = K \rho \ln \rho + \text{cst.} \tag{145} \]

This corresponds to the passage from the Tsallis to the Boltzmann free energy when \( \gamma \to 1 \).
V. THE CASE $K > 0$

In this section, we consider the case of a positive polytropic constant ($K > 0$), corresponding to a positive pressure. Since $P > 0$, the universe is always decelerating. In the figures, we take $c = Qm = 4\pi G = 1$ and $K = 1$.\(^{15}\)

A. The case $\gamma > 1$

For $\gamma > 1$ the equations determining the pseudo rest-mass density and the energy density as a function of the scale factor can be written as

$$\rho \sqrt{1 + \frac{2}{\alpha^2} \frac{K\gamma}{\gamma - 1} \rho^\gamma} = \frac{Qm}{a^3}, \quad (146)$$

$$\epsilon = \rho c^2 + \frac{\gamma + 1}{\gamma - 1} K \rho^\gamma. \quad (147)$$

When $\rho \to +\infty$, Eqs. (146) and (147) reduce to

$$\rho \sim \left(\frac{Q^2 m^2 c^2}{2} \frac{\gamma - 1}{K\gamma}\right)^{1/(1+\gamma)} \frac{1}{a^{6/(1+\gamma)}}, \quad (148)$$

$$\epsilon \sim \frac{\gamma + 1}{\gamma - 1} K \rho^\gamma, \quad (149)$$

$$\epsilon \sim \frac{\gamma + 1}{\gamma - 1} K \left(\frac{Q^2 m^2 c^2}{2} \frac{\gamma - 1}{K\gamma}\right)^{\gamma/(1+\gamma)} \frac{1}{a^{6\gamma/(1+\gamma)}}, \quad (150)$$

This corresponds to the ultrarelativistic regime valid for $a \to 0$. Starting from $+\infty$ when $a \to 0$, the pseudo rest-mass density and the energy density decrease as $a$ increases. Using Eqs. (71) and (149), we obtain the equation of state

$$P \sim \frac{\gamma - 1}{\gamma + 1} \epsilon. \quad (151)$$

The pressure is a linear function $P \sim \alpha \epsilon$ of the energy density with coefficient $\alpha = (\gamma - 1)/(\gamma + 1)$. For $\gamma = 2$, we recover the equation of state of the dark radiation $P = \epsilon/3$ due to a complex SF with a repulsive $|\varphi|^4$ self-interaction.\(^{40,41}\)

When $\rho \to 0$, Eqs. (146) and (147) reduce to

$$\rho \sim \frac{Qm}{a^3}, \quad (152)$$

$$\epsilon \sim \rho c^2. \quad (153)$$

\(^{15}\)The case $K = 0$ corresponds to pure DM with $P = V = u = 0$ and $\epsilon = \rho c^2 = \rho_m c^2 = \Omega_m \rho_0 a^2 / a^3$.

This corresponds to the nonrelativistic regime $P/\epsilon \ll 1$ (matterlike era) valid for $a \to +\infty$. The pseudo rest-mass density and the energy density decrease to zero as $a$ increases to $+\infty$.

When $K > 0$ and $\gamma > 1$, the universe evolves from an $\alpha$-era ($P \sim \alpha \epsilon$) in the early universe to a matterlike era ($P \sim 0$) in the late universe. The case $\gamma = 2$ and $K > 0$, corresponding to a relativistic BEC with a repulsive $|\varphi|^4$ self-interaction, is treated in detail in Sec. III of \[^{41}\]. In that case, the universe evolves from a dark radiation era to a matterlike era. The curves $\rho(a)$ and $\epsilon(a)$ are plotted in Figs. 1 and 2.

The temporal evolution $a(t)$ of the scale factor is represented in Fig. 3. It is obtained by integrating Eq. (79) numerically. Starting from a singularity at $t = 0$ where $a = 0$ and $\epsilon \to +\infty$ (big bang) the scale factor first grows as $a \propto t^{(\gamma+1)/3\gamma}$ (corresponding to $a \propto t^{3/[3(1+\alpha)]}$ in the $\alpha$-era then as $a \propto t^{2/3}$ in the matterlike era [Einstein-de Sitter (EdS) solution]).

The transition between the two regimes typically occurs at

$$a_t = (Qm)^{1+\gamma}/(\gamma - 1) \left(\frac{2}{Q^2 m^2 c^2} \frac{K\gamma}{\gamma - 1}\right)^{\gamma/(3\gamma - 1)}, \quad (154)$$

$$\epsilon_t = \rho c^2 = \left(\frac{c^2}{2} \frac{\gamma - 1}{K\gamma}\right)^{\gamma/(1+\gamma)} c^2. \quad (155)$$

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Evolution of the pseudo rest-mass density as a function of the scale factor for $\gamma > 1$ (specifically $\gamma = 2$). The pseudo rest-mass density decreases more rapidly in the matterlike era than in the $\alpha$-era.}
\end{figure}

B. The case $\gamma = 1$

The case $\gamma = 1$ must be treated specifically. It corresponds to the SF potential from Eq. (129) associated with the isothermal equation of state.\(^{134}\). In the present case $T > 0$. The equations determining
Using Eqs. \[134\] and \[159\], we obtain the equation of state

$$P \simeq \frac{\epsilon}{2 \ln \epsilon}. \quad (160)$$

The pressure is an approximately linear function of the energy density $P \simeq \alpha \epsilon$ with a logarithmic correction yielding a small effective coefficient $\alpha \sim 1/(2 \ln \epsilon) \ll 1$. For $a \to 0$, we have $P/\epsilon \ll 1$.

Considering now small values of $\rho$, we see that Eq. \[156\] imposes the condition $\rho \geq \rho_{\text{Min}}$ with

$$\rho_{\text{Min}} = \rho_\star e\frac{1}{2} - \frac{m^2}{2k_B T^2}. \quad (161)$$

This happens when the scale factor reaches the value

$$a_{\text{max}} = \left(\frac{Qm}{\rho_\star} \sqrt{\frac{mc^2}{k_B T} e^{\frac{1}{2} - \frac{m^2}{2k_B T}}} \right)^{1/3}. \quad (163)$$

The solution of Eqs. \[156\] and \[157\] is defined only for $a \leq a_{\text{max}}$. The pseudo rest-mass density and the energy density decrease as $a$ increases. When $a = a_{\text{max}}$, the energy density vanishes while the pseudo rest-mass density reaches its minimum accessible value $\rho_{\text{Min}}$. This evolution is illustrated in the following section which presents similar features.

### C. The case $0 < \gamma < 1$

For $0 < \gamma < 1$ the equations determining the pseudo rest-mass density and the energy density as a function of the scale factor can be written as

$$\rho \sqrt{1 + \frac{2k_B T}{mc^2} \ln \left(\frac{\rho}{\rho_\star}\right)} = \frac{Qm}{a^3}, \quad (164)$$

$$\epsilon = \rho c^2 + \frac{2k_B T}{m} \rho \ln \left(\frac{\rho}{\rho_\star}\right) - \frac{k_B T}{m} \rho. \quad (157)$$

When $\rho \to +\infty$, Eqs. \[156\] and \[157\] reduce to

$$\rho \sqrt{1 + \frac{2k_B T}{mc^2} \ln \rho} \sim \frac{Qm}{a^3}, \quad (158)$$

$$\epsilon \sim \frac{2k_B T}{m} \rho \ln \rho. \quad (159)$$

This regime is valid for $a \to 0$. Starting from $+\infty$ when $a \to 0$, the pseudo rest-mass density and the energy density decrease as $a$ increases. This corresponds to a matterlike era ($\epsilon \sim a^{-3}$) modified by logarithmic corrections.
when \( a \to 0 \), the pseudo rest-mass density and the energy density decrease as \( a \) increases.

Considering now small values of \( \rho \) (ultrarelativistic regime), we see that Eq. (164) imposes the condition \( \rho > \rho_{\text{min}} \) with

\[
\rho_{\text{min}} = \left( \frac{2}{c^2} \frac{K \gamma}{1 - \gamma} \right)^{1/(1-\gamma)}.
\]  

(168)

The pseudo rest-mass density decreases as \( a \) increases and tends to \( \rho_{\text{min}} \) when \( a \to +\infty \). However, as in the previous section, the energy density vanishes before \( \rho \) reaches its absolute minimum value. Indeed, \( \epsilon_{\text{min}} = -\frac{1}{2K} \rho_{\text{min}} c^2 < 0 \) when \( \rho = \rho_{\text{min}} \). According to Eq. (169), the energy density vanishes (\( \epsilon = 0 \)) at

\[
\rho_{\text{min}} = \left( \frac{\gamma + 1}{\gamma} \frac{K}{1 - \frac{c}{\sqrt{\gamma}} c^2} \right)^{1/(1-\gamma)}.
\]  

(169)

This happens when the scale factor reaches the value

\[
a_{\text{max}} = (Qm)^{1/3} \left( \frac{1 - \gamma}{\gamma + 1} \right)^{(\gamma+1)/(6(1-\gamma))} \left( \frac{c^2}{K} \right)^{1/(3(1-\gamma))}.
\]  

(170)

The solution of Eqs. (164) and (166) is defined only for \( a \leq a_{\text{max}} \). The pseudo rest-mass density and the energy density decrease as \( a \) increases. When \( a = a_{\text{max}} \), the energy density vanishes while the pseudo rest-mass density reaches its minimum accessible value \( \rho_{\text{min}} \). The curves \( \rho(a) \) and \( \epsilon(a) \) are plotted in Figs. 4 and 5.

The temporal evolution \( a(t) \) of the scale factor is represented in Fig. 6. It is obtained by integrating Eq. (79) numerically. Starting from a singularity at \( t = 0 \) where \( a = 0 \) and \( \epsilon \to +\infty \) (big bang), the scale factor first grows as \( a \propto t^{2/3} \) in the matterlike era (EdS solution) until it reaches a maximum value \( a_{\text{max}} \) at which the energy density vanishes (\( \epsilon = 0 \)). After that moment, the universe collapses and forms a singularity at \( t = t_{bc} \) where \( a = 0 \) and \( \epsilon \to +\infty \) (big crunch). This process repeats itself periodically in time. This solution describes a cyclic universe presenting phases of expansion and contraction separated by critical points where the energy density is either infinite (when \( a = 0 \)) or zero (when \( a = a_{\text{max}} \)). At that point the universe “disappears”.

D. The case \( \gamma = 0 \) (anti-ΛCDM model)

For \( \gamma = 0 \), the equations determining the pseudo rest-mass density and the energy density as a function of the scale factor reduce to

\[
\rho = \frac{Qm}{a^3},
\]  

(171)

\[
\epsilon = \rho c^2 - K.
\]  

(172)

They can be combined to give

\[
\epsilon = \frac{Qmc^2}{a^3} - K.
\]  

(173)
This equation is equivalent to the one obtained in the anti-ΛCDM model which assumes that the universe is filled with pressureless DM \((P = 0)\) and that the cosmological constant is negative \((\Lambda < 0)\) or that it is represented by a fluid with an equation of state \(P_{\text{de}} = -\epsilon_{\text{de}}\) yielding a constant energy density \(\epsilon_{\text{de}} = -\rho_{\Lambda}c^2 < 0\). This agreement is expected since, when \(\gamma = 0\), the pressure is constant \((P = K)\) and we know that a constant positive pressure \(P = \rho_{\Lambda}c^2\) returns the anti-ΛCDM model \([69]\). In the present context, the anti-ΛCDM model is obtained from a complex SF theory with a constant negative potential \(V(|\gamma|^2) = -\rho_{\Lambda}c^2\). For this particular model, we see that the pseudo rest-mass density decreases as \(\rho \propto a^{-3}\) and behaves as DM (see the Remark below).

The anti-ΛCDM model has been studied in Sec. 6 of \([69]\). To make the connection with this study, we set \(K = \rho_{\Lambda}c^2\) and \(a_2 = (Q_{\text{m}}/\rho_{\Lambda})^{1/3}\). Equation (173) can then be rewritten as

\[
\epsilon = \rho_{\Lambda}c^2 \left[ \frac{a_2}{a} \right]^3 - 1. \tag{174}
\]

Starting from \(+\infty\) when \(a \to 0\), the energy density decreases as \(a\) increases and vanishes at

\[
a_{\text{max}} = \left( \frac{Q_{\text{mc}}^2}{K} \right)^{1/3} = a_2. \tag{175}
\]

This is the maximum scale factor. At that point, the pseudo rest-mass density reaches its minimum accessible value

\[
\rho_{\text{min}} = \frac{K}{c^2} = \rho_{\Lambda}. \tag{176}
\]

In the nonrelativistic regime \(a \ll a_2\), corresponding to the matterlike era where \(P/\epsilon \ll 1\), we have

\[
\epsilon \sim \frac{Q_{\text{mc}}^2}{a^2} \sim \rho_{\Lambda}c^2 \left( \frac{a_2}{a} \right)^3. \tag{177}
\]

The temporal evolution of the scale factor is obtained by solving the Friedmann equation \([60]\) with Eq. (174). In that case, the solution can be obtained analytically yielding \([69]\)

\[
\frac{a}{a_2} = \sin^{2/3} \left( \sqrt{6\pi} \frac{t}{t_{\Lambda}} \right), \tag{178}
\]

\[
\epsilon = \frac{\rho_{\Lambda}c^2}{\tan^2 \left( \sqrt{6\pi} \frac{1}{t_{\Lambda}} \right)}, \tag{179}
\]

\[
\rho = \frac{\rho_{\Lambda}c^2}{\sin^2 \left( \sqrt{6\pi} \frac{1}{t_{\Lambda}} \right)}, \tag{180}
\]

where \(t_{\Lambda} = 1/\sqrt{6\pi}\rho_{\Lambda}\) is the cosmological time. The evolution of the universe is similar to the one described in Sec. \(\text{V C}\) with the particularity that \(\rho_{\text{Min}} = 0\). The anti-ΛCDM model is studied in more detail in Sec. 6 of \([69]\).

**Remark:** In Eq. (173) the first term is the rest-mass energy density \(\rho_{\text{mc}}c^2\) [see Eqs. (53) and (61)] and the second term is the internal energy density \(u\) [see Eq. (54)]. As discussed in Sec. \(\text{III E}\) the rest-mass density \(\rho_{\text{m}}\) can be interpreted as DM and the internal energy density \(u\) can be interpreted as DE \([54, 65]\). In the present case, the pseudo rest-mass density coincides with the rest-mass density \((\rho = \rho_{\text{m}})\) and the internal energy density is constant \((u = -K = -\rho_{\Lambda}c^2)\). In the two-fluid model associated with the anti-ΛCDM model (see Sec. \(\text{III D}\)), DM has an equation of state \(P_{\text{m}}(\epsilon_{\text{m}}) = 0\) and DE has an equation of state \(P_{\text{de}}(\epsilon_{\text{de}}) = -\epsilon_{\text{de}}\).

**E. The case \(\gamma < 0\)**

For \(\gamma < 0\) the equations determining the pseudo rest-mass density and the energy density as a function of the scale factor can be written as

\[
\rho \sqrt{1 + \frac{2}{c^2} K|\gamma|} \frac{1}{1 - \gamma} = \frac{Q_{\text{m}}}{a^3}, \tag{181}
\]

\[
\epsilon = \rho c^2 - \frac{\gamma + 1}{1 - \gamma} K \frac{1}{\rho |\gamma|}. \tag{182}
\]

When \(\rho \to +\infty\), Eqs. (181) and (182) reduce to

\[
\rho \sim \frac{Q_{\text{m}}}{a^3}, \tag{183}
\]

\[
\epsilon \sim \rho c^2. \tag{184}
\]

This corresponds to the nonrelativistic regime \(P/\epsilon \ll 1\) (matterlike era) valid for \(a \to 0\). Starting from \(+\infty\) when \(a \to 0\), the pseudo rest-mass density and the energy density decrease as \(a\) increases.

When \(\rho \to 0\) (ultrarelativistic regime), Eqs. (181) and (182) reduce to

\[
\rho \sim \left( \frac{Q_{\text{mc}}^2 c^2 (1 - \gamma)}{2} K|\gamma| \right)^{1/(1+\gamma)} \frac{1}{a^{6/(1+\gamma)}}. \tag{185}
\]

We have to distinguish three subcases:

(i) When \(\gamma > -1\), the asymptotic behavior from Eq. (185) is valid for large values of \(a\). The pseudo rest-mass density decreases as \(a\) increases and tends to zero as \(\rho \propto a^{-6/(1+\gamma)}\) when \(a \to +\infty\). However, the energy vanishes before. According to Eq. (182), it vanishes at

\[
\rho_{\text{min}} = \left( \frac{\gamma + 1}{1 - \gamma} K \right)^{1/(1-\gamma)} \tag{186}
\]

This happens when the scale factor reaches the value

\[
a_{\text{max}} = (Q_{\text{m}})^{1/3} \left( \frac{1 - \gamma}{\gamma + 1} \right)^{1/(6(1 - \gamma))} \left( \frac{c^2}{K} \right)^{1/(3(1 - \gamma))}. \tag{187}
\]
In that case, the evolution of the universe is similar to the one described in Sec. V C with the particularity that \( \rho_{\text{min}} = 0 \).

(ii) The case \( \gamma = -1 \) (anti-Chaplygin gas) is specifically treated in the next subsection.

(iii) When \( \gamma < -1 \), the asymptotic behavior from Eq. (185) is valid for small values of \( a \). The pseudo rest-mass density decreases as \( a \) decreases and tends to zero as \( \rho \propto a^{\delta/[1+\gamma]} \) when \( a \to 0 \). In parallel, the energy density increases as \( a \) decreases and tends to \( +\infty \) as

\[
\epsilon \sim \frac{1}{1-\gamma} \left[ 1 - \frac{\gamma + 1}{\rho^{\gamma}} \right] \frac{1}{a^{\delta/[1+\gamma]}} \tag{188}
\]

when \( a \to 0 \). Using Eqs. (71) and (188), we obtain the equation of state

\[
P = \frac{1 - \gamma}{|\gamma + 1|} \epsilon. \tag{189}
\]

The pressure is a linear function \( P \sim \alpha \epsilon \) of the energy density with coefficient \( \alpha = (1 - \gamma)/|\gamma + 1| \). This determines an \( \alpha \)-era. Strikingly, we have two branches of solutions for sufficiently small values of the scale factor: the nonrelativistic branch from Eqs. (183) and (184) and the ultrarelativistic branch described above. The two branches merge at a maximum scale factor

\[
a_{\text{max}} = (Qm)^{1/3} \left( \frac{\gamma + 1}{1 - \gamma} \right)^{|\gamma+1|/[6(1-\gamma)]} \times \frac{1}{|\gamma|^{1/[3(1-\gamma)]}} \left( \frac{\epsilon^2}{K} \right)^{1/[3(1-\gamma)]}, \tag{190}
\]

corresponding to

\[
\rho_c = \left( \frac{\epsilon^2}{K|\gamma|^{1+|\gamma|}} \right)^{1/(\gamma-1)} \tag{191}
\]

and

\[
\epsilon_c = \frac{1 - \gamma}{|\gamma|} \rho_c \epsilon^2. \tag{192}
\]

The curves \( \rho(a) \) and \( \epsilon(a) \) are plotted in Figs. 7 and 8.

The temporal evolution \( a(t) \) of the scale factor is represented in Fig. 7. It is obtained by integrating Eq. (79) numerically, assuming that the SF follows the curve of Fig. 7 from \( \rho = +\infty \) to \( \rho = 0 \). Starting from a singularity at \( t = 0 \) where \( a = 0 \) and \( \epsilon \to +\infty \) (big bang), the energy density decreases as \( a^{-3} \) and the scale factor grows as \( a \propto t^{2/3} \) in the matter era (EdS solution) until it reaches a maximum value \( a_{\text{max}} \) at which the energy density equals \( \epsilon_c \). This phase of expansion corresponds to the nonrelativistic branch of Fig. 7. After that moment, the universe collapses (\( \alpha \)-era) and forms a singularity at \( t_{\text{bc}} \) where \( a = 0 \) and \( \epsilon \to +\infty \) (big crunch). This phase of contraction corresponds to the ultrarelativistic branch of Fig. 7. The energy density increases as \( a^{-\delta/[1+\gamma]} \) while the scale factor decreases as \( a \propto (t_{\text{bc}} - t)^{(\gamma+1)/3|\gamma|} \) (corresponding to \( a \propto (t_{\text{bc}} - t)^{2/[3(1+\alpha)]} \)) in the \( \alpha \)-era.
This process repeats itself periodically in time. This solution describes an asymmetric cyclic universe presenting phases of expansion (explosion) and contraction (implosion) separated by critical points where the energy density is either infinite (when $a = 0$) or equal to $\epsilon_c$ (when $a = a_{\text{max}}$).

Remark: Other possible evolutions can be contemplated. The SF could follow the curve of Fig. 7 in the reverse sense, from $\rho = 0$ to $\rho = +\infty$. This would lead to an asymptotic cyclic universe where the phase of expansion corresponds to the ultrarelativistic branch ($\alpha$-era) and the phase of contraction corresponds to the nonrelativistic branch (matterlike era). We could also consider the case of a symmetric cyclic universe (like in Sec. V C) where the SF follows the same branch (nonrelativistic or ultrarelativistic) during the phases of expansion and contraction. In that case, there would be two possible evolutions.

F. The case $\gamma = -1$ (anti-Chaplygin gas)

For $\gamma = -1$, the equations determining the pseudo rest-mass density and the energy density as a function of the scale factor reduce to

\[
\rho = \sqrt{\frac{Q^2 m^2 c^4}{a^6} - \frac{K}{c^2}}, \quad (193)
\]

\[
\epsilon = \rho c^2. \quad (194)
\]

They can be combined to give

\[
\epsilon = \sqrt{\frac{Q^2 m^2 c^4}{a^6} - Kc^2}. \quad (195)
\]

This equation is equivalent to the one obtained in the anti-Chaplygin gas model. This agreement is expected because, when $\gamma = -1$, we have $\epsilon = \rho c^2$. Therefore, the equation of state $P = K/\rho$ from Eq. (71) can be written as $P = Kc^2/\epsilon$ which is the equation of state of the anti-Chaplygin gas when $K > 0$. The present context, the anti-Chaplygin gas model is obtained from a complex SF theory with a potential $V(\varphi) = -\frac{1}{2}K(\varphi^2)^{2/3}$. For this particular model, the pseudo rest-mass density coincides with the energy density ($\rho = \epsilon/c^2$).

The anti-Chaplygin gas has been studied in Sec. 4.3 of [69]. To make the connection with this study, we set $\rho_* = \sqrt{K/c^2}$ and $a_* = (Q^2 m^2 c^2/K)^{1/6}$. Eq. (173) can then be rewritten as

\[
\epsilon = \rho_* c^2 \sqrt{\left(\frac{a_*}{a}\right)^6} - 1. \quad (196)
\]

Starting from $+\infty$ when $a \rightarrow 0$, the energy density decreases as $a$ increases and vanishes at

\[
a_{\text{max}} = \left(\frac{Q^2 m^2 c^2}{K}\right)^{1/6} = a_*. \quad (197)
\]

This is the maximum scale factor. In the nonrelativistic regime $a \ll a_*$, corresponding to the matterlike era where $P/\epsilon \ll 1$, we have

\[
\epsilon = \rho c^2 \sim \frac{Q mc^2}{a^3} = \rho_* c^2 \left(\frac{a_*}{a}\right)^3. \quad (198)
\]

The temporal evolution of the scale factor is obtained by solving the Friedmann equation (6) with Eq. (196).

\[
\sqrt{96\pi G \rho_* t} = \int_{(a)}^{+\infty} \frac{dx}{x(x-1)^{3/4}}. \quad (199)
\]

The integral can be calculated explicitly and is given by Eq. (30) of [69]. The evolution of the universe is similar to the one described in Sec. V C with the particularity that $\rho$ vanishes at $a_{\text{max}}$ (i.e. $\rho_{\text{min}} = 0$). The anti-Chaplygin gas model is studied in more detail in Sec. 4.3 of [69].

Remark: We can rewrite Eq. (195) as

\[
\epsilon = \frac{Q mc^2}{a^3} + \left(\sqrt{\frac{Q^2 m^2 c^4}{a^6} - Kc^2} - \frac{Q mc^2}{a^3}\right), \quad (200)
\]

where the first term is the rest-mass energy density $\rho_m c^2$ [see Eqs. (53) and (61)] and the second term is the internal energy density $u$ [see Eqs. (54) and (61)]. As discussed in Sec. III the rest-mass density $\rho_m$ can be interpreted as DM and the internal energy density $u$ can be interpreted as DE [54, 55]. In the two-fluid model associated with the anti-Chaplygin gas (see Sec. III D), DM has an equation of state $P_m(\epsilon_m) = 0$ and DE has an equation of state

\[
P_{de}(\epsilon_{de}) = \frac{2Kc^2 \epsilon_{de}}{\epsilon_{de}^2 - Kc^2}, \quad (201)
\]

which is obtained by eliminating $\rho$ between Eqs. (83) and (91), and by identifying $P(u)$ with $P_{de}(\epsilon_{de})$. Solving the energy conservation equation (5) with the equation of state (201), we recover the expression of DE from Eq. (200).

VI. THE CASE $K < 0$

In this section, we consider the case of a negative polytropic constant ($K < 0$), corresponding to a negative pressure. This allows the universe to experience a phase of accelerating expansion. In the figures, we take $c = Qm = 4\pi G = 1$ and $K = -1$.

A. The case $\gamma > 1$

For $\gamma > 1$ the equations determining the pseudo rest-mass density and the energy density as a function of the
scale factor can be written as
\[ \rho \sqrt{1 - \frac{2}{c^2} \frac{|K|}{|\gamma| - 1} \rho^2} = \frac{Qm}{a^3}, \tag{202} \]
\[ \epsilon = \rho c^2 - \frac{\gamma + 1}{\gamma - 1} |K| \rho^2. \tag{203} \]

When \( \rho \to 0 \), Eqs. (202) and (203) reduce to
\[ \rho \sim \frac{Qm}{a^3}, \tag{204} \]
\[ \epsilon \sim \rho c^2. \tag{205} \]

This corresponds to the nonrelativistic regime \( P/\epsilon \ll 1 \) (matterlike era) valid for \( a \to +\infty \). The pseudo rest-mass density and the energy density decrease to zero as \( a \) increases.

Considering now large values of \( \rho \) (ultrarelativistic regime), we see that Eq. (202) imposes the condition \( \rho \leq \rho_{\text{max}} \) with
\[ \rho_{\text{max}} = \left( \frac{c^2}{2} \frac{|K|}{|\gamma| - 1} \right)^{1/(\gamma - 1)}. \tag{206} \]

The pseudo rest-mass density increases as \( a \) increases and tends to \( \rho_{\text{max}} \) when \( a \to +\infty \). In parallel, the energy density decreases as \( a \) increases and tends to a constant
\[ \epsilon_{\text{min}} = \frac{\gamma - 1}{2\gamma} \rho_{\text{max}} c^2 \tag{207} \]
when \( a \to +\infty \). This leads to a DE era corresponding to a de Sitter evolution.

Therefore, we have two branches of solutions for sufficiently large values of the scale factor: the nonrelativistic branch from Eqs. (204) and (205) and the ultrarelativistic branch described above. The two branches merge at a minimum scale factor
\[ a_{\text{min}} = (Qm)^{1/3} \left( \frac{\gamma + 1}{\gamma - 1} \right)^{(\gamma + 1)/6(\gamma - 1)} \times \gamma^{1/[\delta(\gamma - 1)]} \left( \frac{|K|}{c^2} \right)^{1/[\delta(\gamma - 1)]} \tag{208} \]
corresponding to
\[ \rho_i = \left( \frac{c^2 |K|}{\sigma |\gamma|} \right)^{1/(\gamma - 1)} \tag{209} \]
and
\[ \epsilon_i = \frac{\gamma - 1}{\gamma} \rho_i c^2. \tag{210} \]

The case \( \gamma = 2 \) and \( K < 0 \), corresponding to a relativistic BEC with an attractive \( |\varphi|^4 \) self-interaction, is treated in detail in Sec. IV of [11]. The curves \( \rho(a) \) and \( \epsilon(a) \) are plotted in Figs. 10 and 11.

The temporal evolution \( a(t) \) of the scale factor is represented in Fig. 12. It is obtained by integrating Eq. (79) numerically, assuming that the SF follows the curve of Fig. 10 from \( \rho_{\text{max}} \) to \( \rho = 0 \). Starting from a DE era at \( t = -\infty \) where \( a = +\infty \) and \( \epsilon = \epsilon_{\text{min}} \), the energy density slowly increases while the scale factor decreases exponentially rapidly as \( a \sim e^{-(8\pi G m/3c^2)^{1/2}t} \) (de Sitter) until it reaches a minimum value \( a_{\text{min}} \) at which the energy density equals \( \epsilon_{\text{min}} \). This phase of contraction corresponds to the ultrarelativistic branch of Fig. 10. After that moment, the universe expands as it enters into the matterlike era. This phase of expansion corresponds to the nonrelativistic branch of Fig. 10. The energy density decreases as \( \epsilon \propto a^{-3} \) while the scale factor increases as \( a \propto t^{2/3} \) (EdS). This solution describes an asymmetric bouncing universe presenting a phase of contraction and a phase of expansion. There is no big bang singularity in this model.

**Remark:** Other possible evolutions can be contemplated. The SF could follow the curve of Fig. 10 in the reverse sense, from \( \rho = 0 \) to \( \rho_{\text{max}} \). This would lead to an asymmetric bouncing universe where the phase of contraction corresponds to the nonrelativistic branch (matterlike era) and the phase of expansion corresponds to the ultrarelativistic branch (DE era). We could also consider the case of a purely expanding universe where the SF follows the same branch (nonrelativistic or ultrarelativistic). In that case, the SF suddenly appears at a finite scale factor \( a_{\text{t}} \) and behaves either as DM (normal branch) or as DE (peculiar branch). This corresponds to the two possible evolutions (DM and DE) described in [11].

![Fig. 10](image_url)

**FIG. 10:** Evolution of the pseudo rest-mass density as a function of the scale factor for \( \gamma > 1 \) (specifically \( \gamma = 2 \)).

### B. The case \( \gamma = 1 \)

The case \( \gamma = 1 \) must be treated specifically. It corre-
...
associated with the isothermal equation of state (134). In the present case $T < 0$. The equations determining the pseudo rest-mass density and the energy density as a function of the scale factor are

$$\rho \sqrt{1 - \frac{2k_B|T|}{mc^2} \ln \left( \rho \rho_* \right)} = \frac{Qm}{a^3}, \quad (211)$$

$$\epsilon = \rho c^2 - \frac{2k_B|T|}{m} \rho \ln \left( \rho \rho_* \right) + \frac{k_B|T|}{m} \rho. \quad (212)$$

When $\rho \to 0$, Eqs. (211) and (212) reduce to

$$\rho \sqrt{\frac{2k_B|T|}{mc^2} \ln |\rho|} \sim \frac{Qm}{a^3}, \quad (213)$$

$$\epsilon \sim \frac{2k_B|T|}{m} \rho |\ln \rho|. \quad (214)$$

This regime is valid for $a \to +\infty$. The pseudo rest-mass density and the energy density decrease to zero as $a$ increases. This corresponds to a matterlike era ($\epsilon \simeq a^{-3}$) modified by logarithmic corrections. Using Eqs. (211) and (214), we obtain the equation of state

$$P \simeq -\frac{\epsilon}{2|\ln \epsilon|}. \quad (215)$$

The pressure is an approximately linear function of the energy density $P \simeq \alpha \epsilon$ with a logarithmic correction yielding a small effective coefficient $\alpha \sim -1/(2|\ln \epsilon|) \ll 1$. For $a \to +\infty$, we have $|P|/\epsilon \ll 1$.

Considering now large values of $\rho$, we see that Eq. (211) imposes the condition $\rho \leq \rho_{\max}$ with

$$\rho_{\max} = \rho_* e^{\frac{mc^2}{k_B|T|}}. \quad (216)$$

The pseudo rest-mass density increases as $a$ increases and tends to $\rho_{\max}$ when $a \to +\infty$. In parallel, the energy density decreases as $a$ increases and tends to a constant

$$\epsilon_{\min} = \rho_{\max} \frac{k_B|T|}{m} \quad (217)$$

when $a \to +\infty$.

Therefore, we have two branches of solutions for sufficiently large values of the scale factor: the nonrelativistic branch from Eqs. (213) and (214) and the ultrarelativistic branch described above. The two branches merge at a minimum scale factor

$$a_{\min} = \left( \frac{Qm\sqrt{mc^2}}{k_B|T|} e^{\frac{1}{2} \frac{mc^2}{k_B|T|}} \right)^{1/3}. \quad (218)$$

corresponding to

$$\rho_i = \rho_* e^{\frac{1}{2} \frac{mc^2}{k_B|T|}} \quad (219)$$

and

$$\epsilon_i = \frac{2k_B|T|}{m} \rho_i. \quad (220)$$

This situation is similar to the case described previously.

C. The case $0 < \gamma < 1$

For $0 < \gamma < 1$ the equations determining the pseudo rest-mass density and the energy density as a function of the scale factor can be written as

$$\rho \sqrt{1 + \frac{2}{c^2} \frac{|K|}{1 - \gamma \rho^{1-\gamma}}} = \frac{Qm}{a^3}, \quad (221)$$

$$\epsilon = \rho c^2 + \gamma \rho^{1-\gamma} |K| \rho^\gamma. \quad (222)$$

When $\rho \to +\infty$, Eqs. (221) and (222) reduce to

$$\rho \sim \frac{Qm}{a^3}, \quad (223)$$
\[ \epsilon \sim \rho c^2. \]  

(224)

This corresponds to the nonrelativistic regime \( P/\epsilon \ll 1 \) (matterlike era) valid for \( a \to 0 \). Starting from \( +\infty \) when \( a \to 0 \), the pseudo rest-mass density and the energy density decrease as \( a \) increases.

When \( \rho \to 0 \), Eqs. (221) and (222) reduce to

\[ \rho \sim \left( \frac{Q^2 m^2 c^2}{2} \frac{1 - \gamma}{|K| \gamma} \right)^{1/(1+\gamma)} \frac{1}{a^{6/(1+\gamma)}}, \]

(225)

\[ \epsilon \sim \frac{\gamma + 1}{1 - \gamma} |K| \rho^\gamma, \]

(226)

\[ \epsilon \sim \frac{\gamma + 1}{1 - \gamma} |K| \left( \frac{Q^2 m^2 c^2}{2} \frac{1 - \gamma}{|K| \gamma} \right)^{\gamma/(1+\gamma)} \frac{1}{a^{6\gamma/(1+\gamma)}}. \]

(227)

This corresponds to the ultrarelativistic regime valid for \( a \to +\infty \). The pseudo rest-mass density and the energy density decrease with \( a \) and tend to zero when \( a \to +\infty \).

Using Eqs. (71) and (226) we obtain the equation of state

\[ P = -\frac{1 - \gamma}{\gamma + 1} \epsilon. \]

(228)

The pressure is a linear function \( P \sim \alpha \epsilon \) of the energy density with coefficient \( \alpha = -(1 - \gamma)/(\gamma + 1) \).

When \( K < 0 \) and \( 0 < \gamma < 1 \), the universe evolves from a matterlike era (\( P \approx 0 \)) in the early universe to an \( \alpha \)-era (\( P \sim \alpha \epsilon \)) in the late universe. The curves \( \rho(a) \) and \( \epsilon(a) \) are plotted in Figs. 13 and 14.

The temporal evolution \( a(t) \) of the scale factor is represented in Fig. 15. It is obtained by integrating Eq. (79) numerically. Starting from a singularity at \( t = 0 \) where \( a = 0 \) and \( \epsilon \to +\infty \) (big bang) the scale factor first grows as \( a \propto t^{2/3} \) in the matterlike era (EdS solution) then as \( a \propto t^{(\gamma+1)/3\gamma} \) (corresponding to \( a \propto t^{2/(3(1+\alpha))] \)) in the \( \alpha \)-era.

The transition between the two regimes typically occurs at

\[ a_t = \frac{1}{(Qm)^{\frac{1 - \gamma}{1 + \gamma}}} \left( \frac{Q^2 m^2 c^2}{2} \frac{1 - \gamma}{|K| \gamma} \right)^\frac{1}{\gamma/(1+\gamma)}, \]

(229)

\[ \epsilon_t = \rho c^2 = \left( \frac{2 |K| \gamma}{c^2 (1 - \gamma)} \right)^{\frac{1}{\gamma}} c^2. \]

(230)

The expansion of the universe is decelerating (\( \alpha > -1/3 \)) in the \( \alpha \)-era when \( 1/2 < \gamma < 1 \) and accelerating (\( \alpha < -1/3 \)) when \( 0 < \gamma < 1/2 \). This provides a physical interpretation of the particular index \( \gamma = 1/2 \) considered in Sec. IV A.
D. The case $\gamma = 0$ (LCDM model)

For $\gamma = 0$, the equations determining the pseudo rest-mass density and the energy density as a function of the scale factor reduce to

$$\rho = \frac{Qm}{a^3},$$

(231)

$$\epsilon = \rho c^2 + |K|.$$  

(232)

They can be combined to give

$$\epsilon = \frac{Qmc^2}{a^3} + |K|.$$  

(233)

This equation is equivalent to the one obtained in the LCDM model which assumes that the universe is filled with pressureless DM ($P = 0$) and that the cosmological constant is positive ($\Lambda > 0$) or that it is represented by a fluid with an equation of state $P_{de} = -\epsilon_{de}$ yielding a constant energy density $\epsilon_{de} = \rho\Lambda c^2 > 0$. This agreement is expected since, when $\gamma = 0$, the pressure is constant ($P = K$) and we know that a constant negative pressure $P = -\rho\Lambda c^2$ returns the LCDM model [69, 71, 72]. In the present context, the LCDM model is obtained from a complex SF theory with a constant positive potential $V(|\varphi|^2) = \rho\Lambda c^2$. For this particular model, we see that the pseudo rest-mass density decreases as $\rho \propto a^{-3}$ and behaves as DM (see the Remark below).

The LCDM model has been studied in Sec. 5 of [69]. To make the connection with this study, we set $K = -\rho\Lambda c^2$ and $a_2 = (Qm/\rho\Lambda)^{1/3}$. Eq. (233) can then be rewritten as

$$\epsilon = \rho\Lambda c^2 \left(\frac{a_2}{a}\right)^3 + 1.$$    

(234)

Starting from $+\infty$ when $a \to 0$, the energy density decreases as $a$ increases and tends to a constant value $K = \rho\Lambda c^2$ when $a \to +\infty$. The temporal evolution of the scale factor is obtained by solving the Friedmann equation (6) with Eq. (234). In that case, the solution can be obtained analytically yielding [69]

$$\frac{a}{a_2} = \sinh^{2/3} \left(\sqrt{6\pi} \frac{t}{t_\Lambda}\right),$$

(235)

$$\epsilon = \frac{\rho\Lambda c^2}{\tanh^2 \left(\sqrt{6\pi} \frac{t}{t_\Lambda}\right)},$$

(236)

$$\rho = \frac{\rho\Lambda}{\sinh^2 \left(\sqrt{6\pi} \frac{t}{t_\Lambda}\right)},$$

(237)

where $t_\Lambda = 1/\sqrt{G\rho\Lambda}$ is the cosmological time. The evolution of the universe is similar to the one described in Sec. [1E] below with the particularity that $\rho \to 0$ for $a \to +\infty$. The LCDM model is studied in more detail in Sec. 5 of [69].

Remark: In Eq. (233) the first term is the rest-mass energy density $\rho mc^2$ [see Eqs. (5) and (6)] and the second term is the internal energy density $u$ [see Eq. (54)]. As discussed in Sec. [1I], the rest-mass density $\rho_m$ can be interpreted as DM and the internal energy density $u$ can be interpreted as DE [53, 63]. In the present case, the pseudo rest-mass density coincides with the rest-mass density ($\rho = \rho_m$) and the internal energy density is constant ($u = |K| = \rho\Lambda c^2$). In the two-fluid model associated with the LCDM model (see Sec. [1II]) DM has an equation of state $P_m(\epsilon_m) = 0$ and DE has an equation of state $P_{de}(\epsilon_{de}) = -\epsilon_{de}$.

E. The case $\gamma < 0$

For $\gamma < 0$ the equations determining the pseudo rest-mass density and the energy density as a function of the scale factor can be written as

$$\rho \sqrt{1 - \frac{2}{c^2} \frac{|K||\gamma|}{1 - \gamma} \frac{1}{\rho^{1-\gamma}}} = \frac{Qm}{a^3},$$

(238)

$$\epsilon = \rho c^2 + \frac{\gamma + 1}{1 - \gamma} |K| \frac{1}{\rho^{1-\gamma}}.$$ 

(239)

When $\rho \to +\infty$, Eqs. (238) and (239) reduce to

$$\rho \sim \frac{Qm}{a^3},$$

(240)

$$\epsilon \sim \rho c^2.$$  

(241)

This corresponds to the nonrelativistic regime $|P|/\epsilon \ll 1$ (matterlike era) valid for $a \to 0$. Starting from $+\infty$ when $a \to 0$, the pseudo rest-mass density and the energy density decrease as $a$ increases.

Considering now small values of $\rho$ (ultrarelativistic regime) we see that Eq. (238) imposes the condition $\rho \geq \rho_{\text{min}}$ with

$$\rho_{\text{min}} = \left(\frac{2}{c^2} \frac{|K||\gamma|}{1 - \gamma}\right)^{1/(1-\gamma)}.$$  

(242)

The pseudo rest-mass density decreases as $a$ increases and tends to $\rho_{\text{min}}$ when $a \to +\infty$. Similarly, the energy density decreases as $a$ increases and tends to a constant

$$\epsilon_{\text{min}} = \frac{1}{2|\gamma|} \rho_{\text{min}} c^2,$$

(243)

when $a \to +\infty$. We see that the evolution of the universe in this regime is similar to the one induced by a cosmological constant or by a constant energy density. This corresponds to the DE era. The curves $\rho(a)$ and $\epsilon(a)$ are plotted in Figs. 16 and 17.
The temporal evolution $a(t)$ of the scale factor is represented in Fig. 18. It is obtained by integrating Eq. (70) numerically. Starting from a singularity at $t = 0$ where $a = 0$ and $\epsilon \rightarrow +\infty$ (big bang) the scale factor first grows as $a \propto t^{2/3}$ in the matterlike era (EdS) then as $a \propto e^{(8\pi G\epsilon_{\text{min}}/3c^2)^{1/2}t}$ in the DE era (de Sitter).

The transition between the two regimes typically occurs when $Qmc^2/a^3 \sim \epsilon_{\text{min}}$ yielding

$$a_t = \left(\frac{2|\gamma|Qm}{1 - \gamma}\right)^{1/3} \left(\frac{c^2 (1 - \gamma)}{2 |K||\gamma|}\right)^{1/2}. \tag{244}\label{eq:244}$$

F. The case $\gamma = -1$ (Chaplygin gas)

For $\gamma = -1$, the equations determining the pseudo rest-mass density and the energy density as a function of the scale factor reduce to

$$\rho = \sqrt{\frac{Q^2 m^2}{a^6} + |K| c^2}, \tag{245}\label{eq:245}$$

$$\epsilon = \rho c^2. \tag{246}\label{eq:246}$$

They can be combined to give

$$\epsilon = \sqrt{\frac{Q^2 m^2 c^4}{a^6} + |K| c^2}. \tag{247}\label{eq:247}$$

This equation is equivalent to the one obtained in the Chaplygin gas model. This agreement is expected because, when $\gamma = -1$, we have $\epsilon = \rho c^2$. Therefore, the equation of state $P = K/\rho$ from Eq. (71) can be written as $P = Kc^2/\epsilon$ which is the equation of state of the Chaplygin gas when $K < 0$ [66, 69]. In the present context, the Chaplygin gas model is obtained from a complex SF theory with a potential $V(|\phi|^2) = -\frac{1}{2} K (\frac{h}{m})^2 |\phi|^2$. For this particular model, the pseudo rest-mass density coincides with the energy density ($\rho = \epsilon / c^2$).

The Chaplygin gas has been studied in Sec. 4.4 of [69]. To make the connection with this study, we set $\rho_\star = \sqrt{|K|/c^2}$ and $a_\star = (Q^2 m^2 c^2/|K|)^{1/6}$. Eq. (247) can then be rewritten as

$$\epsilon = \rho_\star c^2 \sqrt{\left(\frac{a_\star}{a}\right)^6 + 1}. \tag{248}\label{eq:248}$$

In the nonrelativistic regime where $\rho, \epsilon \rightarrow +\infty$, we have

$$\epsilon = \rho c^2 \sim \frac{Qmc^2}{a^3} = \rho_\star c^2 \left(\frac{a_\star}{a}\right)^3. \tag{249}\label{eq:249}$$

This corresponds to the matterlike era ($\rho \ll \rho_\star$) where $|P|/\epsilon \ll 1$. When $a \rightarrow +\infty$, the energy density tends to a constant

$$\epsilon_{\text{min}} = \sqrt{|K| c^2} = \rho_\star c^2. \tag{250}\label{eq:250}$$

This corresponds to the DE era ($\rho \gg \rho_\star$). The Chaplygin equation of state can mimic the effect of a cosmological constant at late times. The temporal evolution of the scale factor is obtained by solving the Friedmann equation [6] with Eq. (248). This yields [69]

$$\sqrt{96\pi G \rho_\star} t = \int_{x_\star}^{+\infty} \frac{dx}{x(x + 1)^{1/4}}. \tag{251}\label{eq:251}$$
The integral can be calculated explicitly and is given by Eq. (38) of [69]. The evolution of the universe has been described in the previous section. The Chaplygin gas model is studied in more detail in Sec. 4.4 of [69]. Eq. (38) of [69]. The evolution of the universe has been not consistent with the observations.

Chaplygin gas (see Sec. III D), DM has an equation of state in the fast oscillation regime (equivalent to the complex SF with a polytropic or an isothermal equation of state of type II has been studied in [50]. In the nonrelativistic complex SF, there is a transition between a dark radiation era (due to the SF) and a matter era. This model is consistent with the observations. In that case we study the validity of the fast oscillation regime (equivalent to the TF approximation).

In the case $K < 0$ corresponding to a negative pressure, we have found the following results. The models with $\gamma \geq 1$ describe a bouncing universe dominated by DE when $t < 0$ and by pressureless DM when $t > 0$, or the reverse. At $t = 0$ the universe achieves its minimum radius $a_{\text{min}}$ and its maximum energy density $\epsilon_i$. There is no singularity. This model can also describe a peculiar evolution with two branches [41]. The SF emerges suddenly at some finite scale factor $a_{\text{min}}$ and energy density $\epsilon_i$ and follows either the nonrelativistic branch (DM) or the ultrarelativistic branch (DE). Therefore, we get either an asymmetric bouncing universe or a universe with two branches of solutions. These models are not consistent with the observations. The models with $\gamma < 1$ describe the transition between a pressureless DM era and a DE era. They provide UDM models. For $0 < \gamma < 1$, the energy density increases indefinitely in the DE era (DE-0) like in quintessence models. For $\gamma \leq 0$, the energy density tends to a constant in the DE era like in the presence of a cosmological constant. This gives rise to a de Sitter era at late times. These UDM models include the ACMD model ($\gamma = 0$) and the Chaplygin gas model ($\gamma = -1$). The Chaplygin gas model does not give a good agreement with the observations. Only polytropic models with $\gamma$ sufficiently close to 0, i.e., sufficiently close to the ACMD model are consistent with the observations.

A limitation of our study is that we have not studied in detail (case by case) the validity of the fast oscillation regime (this is done in [41] for the quartic potential). This is partly due to reasons of conciseness and partly due to the fact that most models are not consistent with the observations so it may not be necessary to perform a more detailed study than the one given here. However, in a companion paper [52], we consider the logotropic model which is consistent with the observations. In that case we study the validity of the fast oscillation regime in detail.

In Ref. [55] we have shown that the equation of state of a relativistic barotropic fluid can be specified in different manners depending on whether the pressure is expressed in terms of the energy density $\epsilon$ (model I), the rest-mass density $\rho_m$ (model II), or the pseudo rest-mass density $\rho$ (model III). In the present paper, we have considered the cosmological evolution of fluids described by a polytropic equation of state of type III. The cosmological evolution of fluids described by a polytropic equation of state of type I has been studied in [49, 75, 76] and the cosmological evolution of fluids described by a polytropic equation of state of type II has been studied in [50].

Appendix A: Inhomogeneous relativistic complex SF in a curved spacetime

1. Klein-Gordon-Einstein equations

The evolution of a possibly spatially inhomogeneous relativistic complex SF $\varphi(x^\mu) = \varphi(x, y, z, t)$, which may
represent the wavefunction of a relativistic BEC, is governed by the KG equation

\[ \square \varphi + 2 \frac{dV_{\text{tot}}}{d|\varphi|^2} \varphi = 0, \quad (A1) \]

where \( \square = D_\mu \partial^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \) is the d’Alembertian operator in a curved spacetime. The potential \( V_{\text{tot}}(|\varphi|^2) \) can be decomposed into a rest-mass energy term and a self-interaction energy term as

\[ V_{\text{tot}}(|\varphi|^2) = \frac{m^2 c^2}{2\hbar^2} |\varphi|^2 + V(|\varphi|^2). \quad (A2) \]

The KG equation is coupled to the Einstein field equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (A3) \]

where

\[ T_{\mu\nu} = \frac{1}{2} \left( \partial_\mu \varphi^* \partial_\nu \varphi + \partial_\nu \varphi^* \partial_\mu \varphi \right) \]

\[ - g_{\mu\nu} \left[ \frac{1}{2} g^{\rho\sigma} \partial_\rho \varphi^* \partial_\sigma \varphi - V_{\text{tot}}(|\varphi|^2) \right] \]

is the energy-momentum tensor of the SF. This leads to the Klein-Gordon-Einstein (KGE) equations (the term in brackets corresponds to the Lagrangian density of the SF). The energy-momentum tensor satisfies the equation \( D_\mu T^{\mu\nu} = 0 \) expressing the local conservation of energy and momentum. On the other hand, the quadricurrent

\[ J_\mu = - \frac{m}{2i\hbar} (\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*) \quad (A5) \]

satisfies the equation \( D_\mu J^\mu = 0 \) expressing the local conservation of charge (see, e.g., [55] for details). The charge

\[ Q = \frac{e}{mc} \int J^0 \sqrt{-g} d^3x \quad (A6) \]

of the SF is proportional to the number \( N \) of bosons \((Q = Nc)\) provided that antibosons are counted negatively [58]. Therefore, the equation \( D_\mu J^\mu = 0 \) also expresses the local conservation of the boson number. For a real SF the quadricurrent vanishes implying that the particle number is not conserved.

### 2. The de Broglie transformation

We can write the KG equation [A1] under the form of hydrodynamic equations by making the de Broglie [77]–[79] transformation. To that purpose, we write the SF as

\[ \varphi = \frac{\hbar}{m} \sqrt{\rho} e^{iS_{\text{tot}}/\hbar}, \quad (A7) \]

where \( \rho \) is the pseudo rest-mass density [16] and \( S_{\text{tot}} \) is the action. They are given by

\[ \rho = \frac{m^2}{\hbar^2} |\varphi|^2 \quad \text{and} \quad S_{\text{tot}} = \frac{\hbar}{2i} \ln \left( \frac{\varphi^*}{\varphi} \right). \quad (A8) \]

We also have

\[ V_{\text{tot}}(\rho) = \frac{1}{2} \rho c^2 + V(\rho). \quad (A9) \]

For convenience, we define \( \Theta = S_{\text{tot}}/m \). In that case, Eq. [A7] becomes

\[ \varphi = \frac{\hbar}{m} \sqrt{\rho} e^{im\Theta/\hbar}. \quad (A10) \]

The angle (phase) and the pulsation of the SF are given by \( \Theta = S_{\text{tot}}/h \) and \( \omega = -\dot{\Theta} = -\frac{\dot{S}_{\text{tot}}}{h} = -\frac{m\dot{\Theta}}{h} = \frac{E_{\text{tot}}}{h} \),

where \( E_{\text{tot}} = -\dot{S}_{\text{tot}} \) is the energy.

Substituting the de Broglie transformation from Eq. [A10] into the KG equation [A1], and separating the real and the imaginary parts we get

\[ D_\mu (\rho \partial^\mu \Theta) = 0, \quad (A12) \]

\[ \frac{1}{2} \partial_\mu \partial^\mu \Theta - \frac{\hbar^2}{2m^2 \sqrt{\rho}} \square \sqrt{\rho} - V_{\text{tot}}(\rho) = 0. \quad (A13) \]

Equation [A12] can be interpreted as a continuity equation and Eq. [A13] can be interpreted as a quantum relativistic Hamilton-Jacobi (or Bernoulli) equation with a relativistic covariant quantum potential

\[ Q_{\text{db}} = \frac{\hbar^2}{2m^2 \sqrt{\rho}}. \quad (A14) \]

Introducing the pseudo quadrivelocity [18]

\[ v_\mu = - \frac{\partial_\mu S_{\text{tot}}}{m} = -\partial_\mu \Theta, \quad (A15) \]

we can rewrite Eqs. [A12] and [A13] as

\[ D_\mu (\rho v^\mu) = 0, \quad (A16) \]

\[ \frac{1}{2} m v_\mu v^\mu - Q_{\text{db}} - m V_{\text{tot}}(\rho) = 0. \quad (A17) \]

We stress that \( \rho \) is not the rest-mass density \( \rho_{\text{em}} = nm \) (see below). It is only in the nonrelativistic regime \( c \to +\infty \) that \( \rho \) coincides with the rest-mass density \( \rho_{\text{em}} \).

17 The angle was noted \( \Theta \) instead of \( \Theta \) in Sec. II and in [41].

18 The pseudo quadrivelocity \( v_\mu \) does not satisfy \( v_\mu v^\mu = c^2 \) so it is not guaranteed to be always timelike. Nevertheless, \( v_\mu \) can be introduced as a convenient notation.
Taking the gradient of the quantum Hamilton-Jacobi equation (A17) we obtain
\[ \frac{dv^\mu}{dt} = v^\mu \partial_\mu Q_{\text{IB}} + \partial_\mu V'(\rho), \] (A18)
which can be interpreted as a relativistic quantum Euler equation (with the limitation mentioned in footnote 18). The first term on the right hand side can be interpreted as a quantum force and the second term as a pressure force \((1/\rho) \partial_\rho P\). The pressure \(P(\rho)\) satisfies the relation \((1/\rho) P'(\rho) = h'(\rho) = V''(\rho)\), where \(h(\rho) = V'(\rho)\) is the pseudo enthalpy. Integrating this relation, we get
\[ P(\rho) = \rho h(\rho) - V(\rho) = \rho V'(\rho) - V(\rho) = \rho^2 \left[ \frac{V(\rho)}{\rho} \right]'. \] (A19)

This equation determines the pseudo equation of state \(P(\rho)\) as a function of the potential \(V(\rho)\). Inversely, the potential is determined by the pseudo equation of state according to
\[ V(\rho) = \rho \int \frac{P(\rho)}{\rho^2} d\rho. \] (A20)

Using the de Broglie transformation (A7) the energy-momentum tensor (A4) is given, in the hydrodynamic representation, by
\[ T_{\mu\nu} = \rho \partial_\mu \theta \partial_\nu \theta + \frac{\hbar^2}{4m^2 \rho} \partial_\mu \rho \partial_\nu \rho \\
- g_{\mu\nu} \left[ \frac{1}{2} g^{\rho\sigma} \rho \partial_\rho \theta \partial_\sigma \theta + \frac{\hbar^2}{8m^2 \rho} g^{\rho\sigma} \rho \partial_\rho \rho \partial_\sigma \rho - V_{\text{tot}}(\rho) \right]. \] (A21)

Similarly, the quadricurrent (A5) can be written as
\[ J^\mu = -\rho \partial^\mu \theta = \rho v^\mu. \] (A22)

Therefore, the continuity equation (A12) or (A16) is equivalent to \(D_\mu J^\mu = 0\). It expresses the conservation of the charge \(Q\) of the complex SF (or the conservation of the boson number \(N\))
\[ Q = Ne = -\frac{e}{mc} \int \rho \partial_\theta \sqrt{-g} d^3 x. \] (A23)

In the following we take \(c = 1\) so that \(Q = N\). Assuming \(\partial_\mu \theta \partial^\mu \theta > 0\), we can introduce the fluid quadrivelocity
\[ u^\mu = -\frac{\partial_\mu \theta}{\sqrt{\partial_\mu \theta \partial^\mu \theta}}, \] (A24)

which satisfies the identity
\[ u^\mu u^\mu = c^2. \] (A25)

Using Eqs. (A22) and (A24), we have
\[ J^\mu = \frac{\rho}{c} \sqrt{\partial_\mu \theta \partial^\mu \theta} u^\mu, \] (A26)

and we can write the continuity equation (A12) as
\[ D_\mu \left[ \rho \sqrt{\partial_\mu \theta \partial^\mu \theta} u^\mu \right] = 0. \] (A27)

The rest-mass density \(\rho_m = nm\) (which is proportional to the charge density \(\rho\)) is defined by
\[ J^\mu = \rho_m u^\mu. \] (A28)

The continuity equation \(D_\mu J^\mu = 0\) can be written as
\[ D_\mu (\rho_m u^\mu) = 0. \] (A29)

Comparing the expressions of \(J^\mu\) given by Eqs. (A26) and (A28), we find that the rest-mass density \(\rho_m = nm\) of the SF is given by
\[ \rho_m = \frac{\rho}{c} \sqrt{\partial_\mu \theta \partial^\mu \theta}. \] (A30)

We note that \(\rho_m \neq J^0/c\) in general. Using the Bernoulli equation (A13), we get
\[ \rho_m = \frac{\rho}{c} \sqrt{\frac{h^2}{m^2} \frac{\Box}{\sqrt{\rho}} + 2V_{\text{tot}}'(\rho)}. \] (A31)

3. TF approximation

In the classical limit \((\hbar \to 0)\) or in the TF approximation where the quantum potential can be neglected, the hydrodynamic equations (A12) and (A13) reduce to
\[ D_\mu (\rho \partial^\mu \theta) = 0, \] (A32)

\[ \frac{1}{2} \partial_\mu \theta \partial^\mu \theta - V_{\text{tot}}'(\rho) = 0. \] (A33)

Equation (A32) can be interpreted as a continuity equation and Eq. (A33) can be interpreted as a classical relativistic Hamilton-Jacobi (or Bernoulli) equation.

In order to determine the rest mass density, we can repeat the same procedure as before. Assuming \(V_{\text{tot}} > 0\), and using Eq. (A33), we introduce the fluid quadrivelocity
\[ u^\mu = -\frac{\partial_\mu \theta}{\sqrt{2V_{\text{tot}}'(\rho)}} c, \] (A34)

which satisfies the identity (A25). Using Eqs. (A22) and (A34), we have
\[ J^\mu = \frac{\rho}{c} \sqrt{2V_{\text{tot}}'(\rho)} u^\mu, \] (A35)

and we can write the the continuity equation (A32) as
\[ D_\mu \left[ \rho \sqrt{2V_{\text{tot}}'(\rho)} u^\mu \right] = 0. \] (A36)
Comparing the expressions of $J^\mu$ given by Eqs. (A28) and (A35), we find that the rest-mass density $\rho_m = nm$ is given in the TF approximation, by

$$\rho_m = \frac{\rho}{c^2} \sqrt{2V_{tot}'(\rho)}. \quad (A37)$$

This is the limit form of Eq. (A31) with $Q_{AB} = 0$. We note that $\rho_m \neq J^0/c$ in general. When $V = \rho_0 c^2$ is constant (corresponding to the LCDM model), the rest-mass density coincides with the pseudo rest-mass density ($\rho_m = \rho$). We also have $\rho_m = \rho$ in the nonrelativistic limit $c \to +\infty$.

In the TF approximation, the energy-momentum tensor from Eq. (A21) reduces to

$$T_{\mu\nu} = \rho \partial_\mu \theta \partial_\nu \theta - g_{\mu\nu} \left[ \frac{1}{2} g^{\sigma\rho} \rho_0 \partial_\sigma \theta - V_{tot}(\rho) \right]. \quad (A38)$$

Using Eq. (A34), we get

$$T_{\mu\nu} = 2\rho V_{tot}'(\rho) \frac{u_\mu u_\nu}{c^2} - g_{\mu\nu} [\rho V_{tot}'(\rho) - V_{tot}(\rho)]. \quad (A39)$$

The energy-momentum tensor can be written under the perfect fluid form

$$T_{\mu\nu} = (\epsilon + P) \frac{u_\mu u_\nu}{c^2} - P g_{\mu\nu}, \quad (A40)$$

where $\epsilon$ is the energy density and $P$ is the pressure, provided that we make the identifications

$$\epsilon = \rho V_{tot}'(\rho) + V_{tot}(\rho) = \rho c^2 + \rho V'(\rho) + V(\rho), \quad (A41)$$

$$P = \rho V_{tot}'(\rho) - V_{tot}(\rho) = \rho V'(\rho) - V(\rho), \quad (A42)$$

where we have used Eq. (A9) to get the second equalities. Eliminating $\rho$ between these equations, we obtain the equation of state $P(\epsilon)$. The squared speed of sound is

$$c_s^2 = P'(\epsilon)c^2 = \frac{\rho V''(\rho)c^2}{c^2 + \rho V''(\rho) + 2V'(\rho)}. \quad (A43)$$

**Remark:** We note that the expression of the pressure is the same as in Eq. (A19). It is also the same as the one obtained in the nonrelativistic limit $c \to +\infty$ where the KGE equations reduce to the GPP equations (see below). By contrast, the enthalpy differs from the pseudo enthalpy except in the nonrelativistic limit. Using Eqs. (A41), (A35), (A37), (A41) and (A42) we find that the enthalpy is given by

$$h = \sqrt{2V_{tot}'(\rho)}c = c \sqrt{\partial_\mu \theta \partial^\mu \theta}. \quad (A44)$$

Substituting Eq. (A9) into Eq. (A44), subtracting $c^2$, and taking the nonrelativistic limit $c \to +\infty$, we obtain $h = V'(\rho)$.

## 4. Nonrelativistic limit

To take the nonrelativistic limit, it is convenient to work with the conformal Newtonian gauge which is a perturbed form of the FLRW line element

$$ds^2 = c^2 \left( 1 + 2 \frac{\Phi}{c^2} \right) dt^2 - a(t)^2 \left( 1 - 2 \frac{\Phi}{c^2} \right) \delta_{ij} dx^i dx^j, \quad (A45)$$

where $\Phi(\mathbf{r}, t)/c^2 \ll 1$ represents the gravitational potential of classical Newtonian gravity. Making the Klein transformation

$$\varphi(\mathbf{r}, t) = \frac{\hbar}{m} e^{-imc^2\hbar/\psi(\mathbf{r}, t)}, \quad (A46)$$

in the KGE equations (A1) and (A3), using the relation

$$\rho = |\psi|^2 = \frac{m^2}{\hbar^2} |\varphi|^2, \quad (A47)$$

and taking the nonrelativistic limit $c \to +\infty$, we get the generalized Gross-Pitaevskii-Poisson (GPP) equations in an expanding universe (see, e.g., [57, 63, 64] for details)

$$i \hbar \frac{\partial \psi}{\partial t} + \frac{3}{2} i \hbar H \psi = - \frac{\hbar^2}{2ma^2} \Delta \psi + m \Phi \psi + m \frac{dV}{d|\psi|^2} \psi, \quad (A48)$$

$$\frac{\Delta \Phi}{4\pi Ga^2} = |\psi|^2 - \frac{3H^2}{8\pi G}, \quad (A49)$$

where $3H^2/8\pi G = \rho_0$ is the background density [see Eq. (6)].

Making the Madelung [58] transformation

$$\psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)} e^{iS(\mathbf{r}, t)/\hbar}, \quad (A50)$$

$$\rho = |\psi|^2, \quad v(\mathbf{r}, t) = \nabla S / ma, \quad (A51)$$

where $\rho$ is the mass density, $S$ is the action and $v$ is the velocity field, we obtain the system of hydrodynamic equations [63, 81, 82]:

$$\frac{\partial \rho}{\partial t} + 3H \rho + \frac{1}{a} \nabla \cdot (\rho v) = 0, \quad (A52)$$

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2ma^2} = \frac{\hbar^2}{2ma^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} - m \Phi - mh(\rho), \quad (A53)$$

$$\frac{\partial v}{\partial t} + Hv + \frac{1}{a} (v \cdot \nabla) v = \frac{\hbar^2}{2ma^2a^3} \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{1}{a} \nabla \Phi - \frac{1}{ma} \nabla P, \quad (A54)$$

$$\frac{\Delta \Phi}{4\pi Ga^2} = \rho - \frac{3H^2}{8\pi G}, \quad (A55)$$
where \( h(\rho) = V'(\rho) \) is the enthalpy and \( P(\rho) \) is the pressure defined by the relation \( h'(\rho) = P'(\rho)/\rho \). It is explicitly given by \( P(\rho) = \rho h(\rho) - \int h(\rho) \, d\rho \), i.e.,

\[
P(\rho) = \rho V'(\rho) - V(\rho). \tag{A56}
\]

The squared speed of sound is \( c_s^2 = P'(\rho) = \rho V''(\rho) \).

The hydrodynamic equations (A52)–(A55) have a clear physical interpretation. Equation (A52), corresponding to the imaginary part of the GP equation, is the continuity equation. Equation (A53), corresponding to the real part of the GP equation, is the Bernoulli or Hamilton-Jacobi equation. Equation (A54), obtained by taking the gradient of Eq. (A53), is the Euler (momentum) equation. Equation (A55) is the Poisson equation. It can be written as \( \Delta \Phi = 4\pi Ga^2(\rho - \rho_b) \) where \( \rho_b \) is the background density of the expanding universe.

Remark: For a complex SF, we note that the potential \( V(|\psi|^2) \) that appears in the GP equation (A48) is the same as the potential \( V(|\varphi|^2) \) that appears in the KG equation (A1) (see, e.g., [57, 63, 64]). For a real SF, the two potentials generally differ (see, e.g., [73 83]).

**Appendix B: Homogeneous complex SF in an expanding universe**

In this Appendix, we apply the results of Appendix A to a cosmological context, namely for a homogeneous complex SF in an expanding background, and we recover the results of Sec. II.

### 1. General results

The cosmological evolution of a spatially homogeneous complex SF in an expanding universe is governed by the KGE equations

\[
\frac{1}{c^2} \frac{d^2 \varphi}{dt^2} + 3H \frac{d \varphi}{dt} + 2 \frac{dV_{tot}}{d|\varphi|^2} = 0, \tag{B1}
\]

\[
H^2 = \frac{8\pi G}{3c^2} \rho, \tag{B2}
\]

which can be deduced from the KGE equations (A1) and (A3). The energy density \( \epsilon(t) \) and the pressure \( P(t) \) of the SF are given by

\[
\epsilon = \frac{1}{2c^2} \left| \frac{d\varphi}{dt} \right|^2 + V_{tot}(|\varphi|^2), \tag{B3}
\]

\[
P = \frac{1}{2c^2} \left| \frac{d\varphi}{dt} \right|^2 - V_{tot}(|\varphi|^2). \tag{B4}
\]

In the following, we use the hydrodynamic representation of the SF (see Appendix A2). Making the de Broglie transformation (A10), the energy density and the pressure of the SF can be written as

\[
\epsilon = \frac{1}{2c^2} \rho \dot{\varphi}^2 + \frac{\hbar^2}{8m^2 c^2} \rho^2 + V_{tot}(\rho), \tag{B5}
\]

\[
P = \frac{1}{2c^2} \rho \dot{\varphi}^2 + \frac{\hbar^2}{8m^2 c^2} \rho^2 - V_{tot}(\rho). \tag{B6}
\]

The equation \( D_{\mu} T^{\mu\nu} = 0 \) leads to the energy conservation equation

\[
\frac{d\epsilon}{dt} + 3H(\epsilon + P) = 0. \tag{B7}
\]

This equation can also be obtained from the KG equation (B1) with Eqs. (B3) and (B4). The equation \( D_{\mu} J^{\mu} = 0 \), which is equivalent to the continuity equation (A12), can be written as

\[
\frac{d}{dt} (E_{tot} \rho a^3) = 0, \tag{B8}
\]

where

\[
E_{tot} = -\dot{S}_{tot} = -m \dot{\varphi} = \hbar \omega. \tag{B9}
\]

is the energy of the SF. Eq. (B8) expresses the conservation of the charge of the complex SF (or equivalently the conservation of the boson number). It can be written as

\[
\rho E_{tot} = \frac{Q m^2 c^2}{\alpha^3}, \tag{B10}
\]

where \( Q = Ne \) is a constant of integration representing the charge of the SF (proportional to the boson number \( N \)). [40–42, 51, 56, 57, 84]. This equation can also be directly obtained from Eq. (A25).

The quantum Hamilton-Jacobi (or Bernoulli) equation (A13) takes the form

\[
E_{tot}^2 = \hbar^2 \frac{1}{\sqrt{\rho}} \frac{d^2 \sqrt{\rho}}{dt^2} + 3 \hbar^2 \frac{1}{\sqrt{\rho}} \frac{d\sqrt{\rho}}{dt} + 2m^2 c^2 V'(\rho). \tag{B11}
\]

Finally, we have established in the general case that the rest-mass density of the SF is given by Eq. (A30). For a spatially homogeneous SF in an expanding background, we get

\[
\rho_m = -\frac{\rho}{c} \dot{\varphi} = -\frac{\rho}{c} \frac{\hbar \omega}{mc^2} = \frac{E_{tot}}{mc^2} = -\frac{\dot{S}_{tot}}{mc^2}. \tag{B12}
\]

We note that, in this special case, \( \rho_m = J_0/c \), where \( J_0 = -\rho \dot{\varphi} \dot{S}_{tot} / m \) is the time component of the quadrupole of charge, but this relation is not true for an inhomogeneous SF. Using Eq. (B12), Eqs. (B8) and (B10) can be rewritten as

\[
\frac{d\rho_m}{dt} + 3H \rho_m = 0, \tag{B13}
\]
and

\[ \rho_m = \frac{Qm}{a^3}. \]  

(B14)

Equation (B13) can also be obtained by combining the first law of thermodynamics for a cold fluid with the energy conservation equation (see Sec. III). These equations express the conservation of the particle number.

2. TF approximation

In the TF approximation \((\hbar \rightarrow 0)\), the energy density \(\epsilon\) and the pressure \(P\) of the SF [see Eqs. (B5) and (B6)] reduce to

\[ \epsilon = \frac{1}{2c^2} \dot{\rho}^2 + V_{\text{tot}}(\rho) = \frac{\rho E^2_{\text{tot}}}{2mc^2} + V_{\text{tot}}(\rho), \]

(B15)

\[ P = \frac{1}{2c^2} \dot{\rho}^2 - V_{\text{tot}}(\rho) = \frac{\rho E^2_{\text{tot}}}{2mc^2} - V_{\text{tot}}(\rho), \]

(B16)

where we have used Eq. (B9) to get the second equalities. On the other hand, the quantum Hamilton-Jacobi (or Bernoulli) equation (B11) reduces to

\[ E^2_{\text{tot}} = 2mc^2 \rho V'_{\text{tot}}(\rho). \]

(B17)

Combining Eqs. (B10) and (B17), we obtain

\[ \frac{\rho}{c} \sqrt{2V'_{\text{tot}}(\rho)} = \frac{Qm}{a^3}. \]

(B18)

This equation determines the relation between the pseudo rest-mass density \(\rho\) and the scale factor \(a\).

According to Eqs. (B12) and (B17), the rest-mass density is given by

\[ \rho_m = \frac{\rho}{c} \sqrt{2V'_{\text{tot}}(\rho)}. \]

(B19)

This shows that (B18) is equivalent to the conservation of the rest-mass [see Eq. (B14)]. Note that Eq. (B19) is always true in the TF approximation even for inhomogeneous systems (Appendix A.3).

Finally, inserting the Bernoulli equation (B17) into Eqs. (B15) and (B16), we find that the energy density and the pressure of the SF in the TF approximation are given by

\[ \epsilon = \rho V'_{\text{tot}}(\rho) + V_{\text{tot}}(\rho), \]

(B20)

\[ P = \rho V'^{\prime\prime}_{\text{tot}}(\rho) - V_{\text{tot}}(\rho). \]

(B21)

Note that these relations are always true in the TF approximation even for inhomogeneous systems (Appendix A.3).

Remark: Eqs. (B20) and (B21) determine the equation of state \(P = \hat{P}(\epsilon)\). As a result, we can obtain Eq. (B18) directly from Eqs. (B20), (B21) and the energy conservation equation (B7). Indeed, combining these equations we obtain

\[ [2V'_{\text{tot}}(\rho) + \rho V'^{\prime\prime}_{\text{tot}}(\rho)] \frac{d\rho}{dt} = -6H \rho V'_{\text{tot}}(\rho). \]

(B22)

leading to

\[ \int \frac{2V'_{\text{tot}}(\rho) + \rho V'^{\prime\prime}_{\text{tot}}(\rho)}{\rho V'_{\text{tot}}(\rho)} = -6 \ln a. \]

(B23)

Equation (B23) integrates to give Eq. (B18).

3. Nonrelativistic limit

Combining Eqs. (A46) and (A50) and comparing the resulting expression with Eq. (A7), we see that \(\Sigma_{\text{tot}} = -mc^2 s + S\), leading to \(E_{\text{tot}} = mc^2 + E\) with \(E = -dS/dt\). We can then rewrite the previous equations in terms of \(S\) and \(E\) instead of \(\Sigma_{\text{tot}} = m\theta\) and \(E_{\text{tot}}\). Then, taking the nonrelativistic limit \(c \rightarrow +\infty\), we obtain

\[ \frac{d\rho}{dt} + 3H \rho = 0, \quad E = mV'(\rho), \quad \frac{3H^2}{8\pi G} = \rho. \]

(B24)

These equations can also be obtained from Eqs. (A52)-(A55) by considering a homogeneous SF. They are easily solved to give \(\rho \propto a^{-3}, a \propto t^{2/3}\) and \(\rho = 1/(6\pi G t^2)\). This is the EdS solution. Therefore, in the nonrelativistic limit, the homogeneous SF/BEC behaves as CDM. For the power-law potential (66), we get \(E = Km\sqrt{\gamma}/[(\gamma - 1)/(6\pi G t^2)^{\gamma - 1}].\) In particular, for the Chaplygin gas (\(\gamma = -1\)), we get \(E = 18\pi^2 Km\gamma/(3Gm^2 t^2);\) for the standard BEC (\(\gamma = 2\)), we get \(E = 2a_s h^2/[3Gm^2 t^2];\) for the \(\Lambda\)CDM model (\(\gamma = 0\)), we get \(E = 0;\) for the superfluid (\(\gamma = 3\)), we get \(E = K m/(24\pi G t^4).\) On the other hand, for the potential (129), we get \(E = -k_B T \ln(6\pi G/\rho, t^2).\)

Appendix C: Examples of potentials of self-interaction

Let us consider a complex SF \(\varphi\) governed by the nonlinear KG equation

\[ \Box \varphi + \frac{m^2 c^2}{\hbar^2} \varphi + 2 \frac{dV}{d|\varphi|^2} \varphi = 0 \]

(C1)

with a self-interaction potential \(V(|\varphi|^2)\). In the TF approximation, this potential is associated with a barotropic equation of state \(P(\rho) = \rho V'(\rho) - V(\rho)\) where \(\rho = |\varphi|^2 = (m^2/\hbar^2)|\varphi|^2\) is the pseudo rest-mass density (see Appendix A). In the nonrelativistic limit, the nonlinear KG equation reduces to the generalized GP equation

\[ i\hbar \frac{\partial \psi}{\partial t} + \frac{3}{2} i\hbar H \psi = -\frac{\hbar^2}{2ma^2} \Delta \psi + m \Phi \psi + m \frac{dV}{d|\psi|^2} \psi, \]

(C2)
which involves the same self-interaction potential $V(|\psi|^2)$. Let us give some examples of self-interaction potentials and their corresponding equations of state $P(\rho)$.

The power-law potential (see Sec. [IV A])

$$V(|\varphi|^2) = \frac{K}{\gamma - 1} \left( \frac{m}{\hbar} \right)^2 |\varphi|^{2\gamma} \quad (\gamma \neq 1) \quad (C3)$$

is associated with the polytropic equation of state $P = K\rho^{\gamma}$. The KG equation becomes

$$\Box \varphi + \frac{m^2c^2}{\hbar^2} \varphi + \frac{2K\gamma}{\gamma - 1} \left( \frac{m}{\hbar} \right)^2 |\varphi|^{2(\gamma - 1)} \varphi = 0. \quad (C4)$$

In the nonrelativistic limit, using

$$V(|\psi|^2) = \frac{K}{\gamma - 1} |\psi|^{2\gamma} \quad (\gamma \neq 1), \quad (C5)$$

we obtain the generalized GP equation

$$i\hbar \frac{\partial \psi}{\partial t} + 3 \frac{3}{2} \hbar H \psi = - \frac{\hbar^2}{2ma^2} \Delta \psi + m\Phi \psi + \frac{Km\gamma}{\gamma - 1} |\psi|^{2(\gamma - 1)} \psi. \quad (C6)$$

(i) The index $\gamma = -1$ corresponds to the Chaplygin gas

$$V(|\varphi|^2) = - \frac{K}{2} \left( \frac{\hbar}{m} \right)^2 \frac{1}{|\varphi|^2}, \quad P = \frac{K}{\rho}, \quad (C7)$$

$$\Box \varphi + \frac{m^2c^2}{\hbar^2} \varphi + K \left( \frac{\hbar}{m} \right)^2 \frac{1}{|\varphi|^4} \varphi = 0, \quad (C8)$$

$$V(|\psi|^2) = - \frac{K}{2} \frac{1}{|\psi|^2}, \quad (C9)$$

$$i\hbar \frac{\partial \psi}{\partial t} + 3 \frac{3}{2} \hbar H \psi = - \frac{\hbar^2}{2ma^2} \Delta \psi + m\Phi \psi + \frac{Km}{2} |\psi|^4 \psi. \quad (C10)$$

(ii) The index $\gamma = 2$ corresponds to the standard BEC taking into account two-body interactions

$$V(|\varphi|^2) = \frac{k_B T m}{\hbar^2} |\varphi|^6, \quad P = K \rho, \quad (C11)$$

$$\Box \varphi + \frac{m^2c^2}{\hbar^2} \varphi + 4K \left( \frac{m}{\hbar} \right)^4 |\varphi|^6 \varphi = 0, \quad (C12)$$

$$V(|\psi|^2) = K|\psi|^4, \quad (C13)$$

$$i\hbar \frac{\partial \psi}{\partial t} + 3 \frac{3}{2} \hbar H \psi = - \frac{\hbar^2}{2ma^2} \Delta \psi + m\Phi \psi + 2Km|\psi|^2 \psi. \quad (C14)$$

(iii) The index $\gamma = 0$ corresponds to the ΛCDM model interpreted as an UDM model

$$V(|\varphi|^2) = -K, \quad P = K, \quad (C15)$$

$$\Box \varphi + \frac{m^2c^2}{\hbar^2} \varphi = 0, \quad (C16)$$

$$V(|\psi|^2) = -K, \quad (C17)$$

$$i\hbar \frac{\partial \psi}{\partial t} + 3 \frac{3}{2} \hbar H \psi = - \frac{\hbar^2}{2ma^2} \Delta \psi + m\Phi \psi. \quad (C18)$$

(iv) The index $\gamma = 3$ corresponds to a superfluid taking into account three-body interactions

$$V(|\varphi|^2) = \frac{K}{2} \left( \frac{m}{\hbar} \right)^6 |\varphi|^6, \quad P = K \rho^3, \quad (C19)$$

$$\Box \varphi + \frac{m^2c^2}{\hbar^2} \varphi + 3K \left( \frac{m}{\hbar} \right)^6 |\varphi|^4 \varphi = 0, \quad (C20)$$

$$V(|\psi|^2) = \frac{K}{2} |\psi|^6, \quad (C21)$$

$$i\hbar \frac{\partial \psi}{\partial t} + 3 \frac{3}{2} \hbar H \psi = - \frac{\hbar^2}{2ma^2} \Delta \psi + m\Phi \psi + \frac{3Km}{2} |\psi|^4 \psi. \quad (C22)$$

(v) For $\gamma = 1/2$ we get

$$V(|\varphi|^2) = -2K \frac{m}{\hbar} |\varphi|, \quad P = K \rho^{1/2}, \quad (C23)$$

$$\Box \varphi + \frac{m^2c^2}{\hbar^2} \varphi - 2K \frac{m}{\hbar} |\varphi| \varphi = 0, \quad (C24)$$

$$V(|\psi|^2) = -2K |\psi|, \quad (C25)$$

$$i\hbar \frac{\partial \psi}{\partial t} + 3 \frac{3}{2} \hbar H \psi = - \frac{\hbar^2}{2ma^2} \Delta \psi + m\Phi \psi - Km \frac{1}{|\psi|^2} |\psi|. \quad (C26)$$

The potential (see Sec. [IV B])

$$V(|\varphi|^2) = \frac{k_B T m}{\hbar^2} \ln \left( \frac{m^2|\varphi|^2}{\rho, \hbar^2} \right) - 1 \quad (C27)$$

is associated with the isothermal equation of state $P = \rho k_B T / m$. It can take into account finite temperature effects in DM. The KG equation becomes

$$\Box \varphi + \frac{m^2c^2}{\hbar^2} \varphi + \frac{2k_B T m}{\hbar^2} \ln \left( \frac{m^2|\varphi|^2}{\rho, \hbar^2} \right) \varphi = 0. \quad (C28)$$

In the nonrelativistic limit, using

$$V(|\psi|^2) = \frac{k_B T m}{m} |\psi|^2 \ln \left( \frac{|\psi|^2}{\rho_*, \hbar^2} \right) - 1, \quad (C29)$$

we obtain the generalized GP equation

$$i\hbar \frac{\partial \psi}{\partial t} + 3 \frac{3}{2} \hbar H \psi = - \frac{\hbar^2}{2ma^2} \Delta \psi + m\Phi \psi + k_B T \ln \left( \frac{|\psi|^2}{\rho_*, \hbar^2} \right) \psi. \quad (C30)$$
The logarithmic potential \[ V(|\varphi|^2) = -A \left[ \ln \left( \frac{m^2 |\varphi|^2}{\hbar^2 \rho_P} \right) + 1 \right] \] (C31)
is associated with the logotropic equation of state \( P = A \ln (\rho/\rho_P) \). The KG equation becomes
\[ \Box \varphi + \frac{m^2 c^2}{\hbar^2} \varphi - \frac{2A}{|\varphi|^2} \varphi = 0. \] (C32)

In the nonrelativistic limit, using
\[ V(|\varphi|^2) = -A \left[ \ln \left( \frac{|\varphi|^2}{\rho_P} \right) + 1 \right], \] (C33)
we obtain the generalized GP equation
\[ i\hbar \frac{\partial \psi}{\partial t} + \frac{3}{2} \hbar H \psi = -\frac{\hbar^2}{2ma^2} \Delta \psi + m \Phi \psi - mA \frac{|\psi|^2}{|\varphi|^2} \psi. \] (C34)

This model is studied in detail in \[52\]. Instead of the logarithmic potential we can consider the constant potential
\[ V(|\varphi|^2) = V_0 = \epsilon_A, \] (C35)
mimicking a cosmological constant like in Appendix E of \[52\]. It is associated with the constant equation of state \( P = -\epsilon_A \). It leads to the usual KG and GP equations \[\text{(C16)}\] and \[\text{(C18)}\]. Note that a constant potential has no effect on the KG and GP equations but it adds a constant term \( \epsilon_A \) in the energy density interpreted as DE.

Of course, we can consider a multitude of more general SF models by summing the potentials [like, e.g., Eqs. \[\text{(C3)}, \text{(C27)}, \text{(C31)}, \text{and (C35)}\] or, equivalently, by summing the corresponding pressures (polytropic, isothermal, logotropic and constant). This is a form of Dalton’s law. The mixed equation of state generically writes
\[ P = K\rho^\gamma + \frac{k_B T m}{\hbar} - \rho \epsilon A^2 + A \ln \left( \frac{\rho}{\rho_P} \right), \] (C36)
where we can add several polytropic terms with different index \( \gamma \). This mixed equation of state is associated with a complex SF potential of the form
\[ V(|\varphi|^2) = \frac{K}{\gamma - 1} \left( \frac{m^2}{\hbar^2} \right)^{2\gamma} |\varphi|^{2\gamma} + \frac{k_B T m}{\hbar^2} |\varphi|^2 \ln \left( \frac{m^2 |\varphi|^2}{\hbar^2 \rho^2} \right) - 1 + \rho \epsilon A^2 - A \left[ \ln \left( \frac{m^2 |\varphi|^2}{\hbar^2 \rho_P^2} \right) + 1 \right]. \] (C37)
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