Pairs of $k$-free Numbers, consecutive square-full Numbers

Thomas Reuss
Mathematical Institute, University of Oxford
reuss@maths.ox.ac.uk

May 5, 2014

Abstract
We consider the error term of the asymptotic formula for the number of pairs of $k$-free integers up to $x$. Our error term improves results by Heath-Brown, Brandes and Dietmann/Marmon. We then extend our results to $r$-tuples of $k$-free numbers and improve previous results by Tsang. Furthermore, we establish an error term for consecutive square-full integers. Finally, we will show that for all $\theta < 3$ and for almost all $D$, the fundamental solution $\epsilon_D$ associated to the Pell equation $x^2 - Dy^2 = 1$ satisfies $\epsilon_D > D^\theta$. This improves/recovers previous results by Fouvry and Jouve. The main tool of our work is the approximate determinant method.

Acknowledgment
I am very grateful to my supervisor Roger Heath-Brown who introduced me to the determinant method and its applications. I thank him for many constructive and helpful comments on my work.
I would like to thank Étienne Fouvry and Florent Jouve for drawing my attention to their paper [5] which resulted in Theorem 5 below.
I am also very grateful to the EPSRC[4] and to St. Anne’s College, Oxford who are generously funding and supporting my doctorate’s degree and this project.

1 Introduction

Notation
We write $x \sim X$ to say that $X < x < 2X$ and we write $x \asymp X$ to say that there exist positive constants $A$, $B$, independent of $X$, such that $AX \leq |x| \leq BX$. 

---

1DTG reference number: EP/J500495/1
The Main Theorem

In this paper, we will present a generalization of the approximate determinant method developed by Heath-Brown in [12]. More precisely, we will prove the following theorem:

**Theorem 1.** Let $D, E, x > 1$ and $\epsilon > 0$. Let $k, l, h$ be integers such that $1 \leq l < k$ and $h \neq 0$. Suppose $N(x; D, E)$ is the number of elements in the set

$$\{(d, e, u, v) \in \mathbb{N}^4 : d \sim D, e \sim E, u \sim U, v \sim V, e^kv^l - d^ku^l = h\},$$

where

$$U = \frac{x^{1/l}}{D^{k/l}}, \quad \text{and} \quad V = \frac{x^{1/l}}{E^{k/l}}.$$

Let $M > 1$ be defined by

$$\log(M) = \frac{9 \log(DE) \log(UV)}{8 \log x},$$

and suppose that the following conditions are satisfied:

1. $\log(DE) \asymp \log(UV) \asymp \log(x)$.
2. $l \geq 2$, or $DE \gg_{k,l,h} x^{1/k}$.

Then, if $x$ is large enough in terms of $\epsilon$,

$$N(x; D, E) \ll_{\epsilon,k,l,h} x^{\epsilon} \min\{(DEM)^{1/2} + D + E, (UVM)^{1/2} + U + V\}.$$

Theorem 1 gives an upper bound for the number of integer points on the algebraic variety defined by

$$e^kv^l - d^ku^l = h,$$  \hspace{1cm} (1)

where each of the variables $d, e, u, v$ are restricted to certain sizes $D, E, U, V$. It is possible to generalize our result to algebraic varieties described by $f_1(d, e)v^l + f_2(d, e)u^l = h$ where $f_1$ and $f_2$ are homogeneous polynomials of weighted degree. But for our applications, Theorem 1 suffices. The main strategy of the proof of Theorem 1 will be to reduce the problem of counting integer points on the algebraic variety $e^kv^l - d^ku^l = h$ to counting rational points “close” to the curve $t = s^{k/l}$ where $t = v/u$ and $s = d/e$. The fundamental tool we use to tackle this counting problem will be the Determinant Method for which the interested reader should consult Heath-Brown [10]. Indeed, our proof of Theorem 1 is very similar in many stages to Heath-Brown [12] where he derives an asymptotic formula for square-free values of the form $n^2 + 1$ up to $x$, say. In the following paragraphs, we will illustrate our applications of Theorem 1.
Pairs of $k$-free integers

For $k \in \mathbb{Z}_{\geq 2}$, we say that an integer $n$ is $k$-free if there is no prime $p$ such that $p^k \mid n$. By convention, a 2-free integer is called square-free. It is an elementary fact that

$$S_k(x) := \# \{ n \leq x : n \text{ is } k\text{-free} \} = \frac{x}{\zeta(k)} + O(x^{1/k}).$$

We shall now consider the more general problem of deriving an asymptotic formula for pairs of $k$-free integers up to $x$, say. More precisely, for integers $k \geq 2$ and $h \neq 0$, let $N_{k,h}(x)$ be the number of integers $n \leq x$ such that both $n$ and $n + h$ are $k$-free. Then, we have the following theorem:

**Theorem 2.** For all $\epsilon > 0$ and all sufficiently large $x$, we have that

$$N_{k,h}(x) = c_{k,h}x + O_{c,k,h}(x^{\omega(k)+\epsilon}),$$

where

$$c_{k,h} = \prod_p \left( 1 - \frac{\rho_{k,h}(p)}{p^k} \right),$$

with

$$\rho_{k,h}(p) = \begin{cases} 2 & \text{if } p^k \nmid h, \\ 1 & \text{if } p^k \mid h. \end{cases}$$

and

$$\omega(k) = \begin{cases} \frac{26 + \sqrt{433}}{81} \approx 0.578 & \text{if } k = 2 \\ \frac{169}{144} & \text{for } k \geq 3. \end{cases}$$

The problem of estimating $N_{k,h}(x)$ as in Theorem [2] is most interesting when $k$ is fairly small because $S_k(x)/x \to 1$ as $k \to \infty$. The trivial error term for Theorem [2] is $\omega(k) = 2/(k + 1)$ which can be obtained with elementary arguments (see for example Carlitz [2]).

By considering the Dirichlet series of $\zeta(s)/\zeta(ks)$, it is plausible to suspect that the error term of $S_k(x)$ cannot be improved by a proper power of $x$ below $O(x^{1/k})$ without assuming some quasi-Riemann-Hypothesis. By assuming uniformity in the error term of Theorem [2] with respect to $h \ll x$, one can show that an error term $O(x^\theta)$ in Theorem [2] implies an error term $O(x^{\theta})$ in the formula for $S_k(x)$ and it is thus reasonable to assume that the error term of Theorem [2] cannot be improved below $O(x^{1/k})$ by a power of $x$ without radically new ideas.

Heath-Brown [8] considered the problem of Theorem [2] in the case $k = 2$ and $h = 1$. He obtained $\omega(2) = 7/11 \approx 0.636$ improving the trivial value $\omega(2) = 2/3$. Heath-Brown’s approach uses exponential sums and the Square Sieve. It should be noted that Heath-Brown’s method is actually uniform in $h \ll x$. Brandes [11] has generalized Heath-Brown’s method for general $k$. Her argument yields the value $\omega(k) = 14/(7k + 8)$, which is the currently best available result for small $k$. Our error exponent $\omega(k)$ stated in Theorem [2] is such that $\omega(k) \leq 14/(7k + 8)$ for all values of $k$. Dietmann and Marmon
have worked independently on improving the exponent \( \omega(k) \) at the same time this paper was produced. Their method is similar to ours and they obtain the exponent \( \omega(k) = 14/(9k) \), which improves upon Brandes’ bound for \( k \geq 5 \). Our error exponent is smaller than theirs for all \( k \geq 2 \). The proof of Theorem 2 will be an application of Theorem 1 with \( l = 1 \).

An asymptotic formula for \( r \)-tuples of \( k \)-free integers

The second application of Theorem 1 that we will present is the following:

**Theorem 3.** Let \( k \geq 2 \), \( r \geq 2 \) and \( l_i(x) = a_i x + b_i \in \mathbb{Z}[x] \) for \( i = 1, \ldots, r \) such that \( a_i b_j - a_j b_i \neq 0 \) and \( a_i \neq 0 \) for all \( i, j \) with \( 1 \leq i, j \leq r \) and \( i \neq j \). Then define

\[
\rho(p) = \# \{ n (\text{mod } p^k) : p^k \mid l_i(n) \text{ for some } i \},
\]

and let

\[
c = \prod_p \left( 1 - \frac{\rho(p)}{p^k} \right).
\]

If \( N(x) \) is the number of integers \( n \leq x \) such that \( l_1(n), \ldots, l_r(n) \) are all \( k \)-free. Then for any \( \epsilon > 0 \) and any sufficiently large \( x \) we have that

\[
N(x) = cx + O(x^{3/(2k+1)+\epsilon}).
\]

It should be noted that the implied constant in Theorem 3 depends on the choice of the \( l_i \) and that the best error term in the sense of Theorem 3 available in the literature for \( k = 2 \) was \( O(x^{7/11+\epsilon}) \) (See Tsang [13]). Tsang’s proof uses a form of the Rosser-Iwaniec sieve and the version of Theorem 2 due to Heath-Brown. It should be noted that even though Tsang’s error term is weaker than ours, his implied constants are uniform in \( r \) and \( \max_i ||l_i|| \).

Consecutive square-full numbers

We will prove the following theorem about consecutive square-full numbers. Recall that an integer \( n \) is called square-full if, for all primes \( p \mid n \), we have \( p^2 \mid n \).

**Theorem 4.** Let \( N(x) \) be the number of integers \( n \leq x \) such that both \( n \) and \( n + 1 \) are square-full. Then we have for all \( \epsilon > 0 \) and sufficiently large \( x \) that

\[
N(x) \ll e^{x^{29/100+\epsilon}}.
\]

It can be shown that there are indeed infinitely many consecutive square-full numbers, for if \( n \) and \( n + 1 \) are square-full then so are \( 4n(n+1) \) and \( 4n(n+1) + 1 \). However, it follows from a simple application of the abc-conjecture that there are at most finitely many \( n \) such that \( n, n + 1 \) and \( n + 2 \) are all square-full. The proof of Theorem 4 will be an application of Theorem 1 with the variety given by \( e^3 v^2 - d^3 u^2 = 1 \).
Size of the fundamental solution of Pell Equations

Let \( D > 0 \) be an integer which is not a square. Consider the Pell equation

\[
T^2 - DU^2 = 1
\]

as an equation in \((T, U)\). Write a solution of this equation as \( \eta_D = T + U\sqrt{D} \). For each fixed \( D \), the Pell equation \((2)\) has a unique solution called the fundamental solution \( \epsilon_D \) which satisfies \( \epsilon_D > 1 \) and

\[
\{ \eta_D : \eta_D \text{ is a solution of } (2) \} = \{ \pm \epsilon_D^n : n \in \mathbb{Z} \}.
\]

It is strongly believed that \( \epsilon_D \) is almost always much larger than \( D \). In particular, one may study the quantity

\[
S(X, \alpha) = \# \{(D, \eta_D) : D \sim X, D \text{ is not a square}, 1 < \eta_D \leq D^{1/2 + \alpha} \}
\]

for \( \alpha > 1/2 \). One can show that

\[
S(X, \alpha) \ll X^{\min(\alpha + \epsilon, 1)}. \tag{3}
\]

Fouvry and Jouve \[5\] have shown that this estimate can be improved to

\[
S(X, \alpha) \ll \epsilon X^{3/8 + \epsilon}.
\]

Their bound improves \((3)\) in the range \( 7/8 < \alpha < 5/4 \). We will improve their result further. More precisely, we will prove the following result:

**Theorem 5.**

\[
S(X, \alpha) \ll \epsilon X^{\max\left(\frac{9}{16} \alpha + \frac{1}{2}, \frac{1}{2} \min(1, \alpha), \frac{1}{2} + \frac{7}{8}\right)}.
\]

Our result is stronger than \((3)\) in the range \( 5/8 < \alpha < 5/2 \). We obtain the following corollary:

**Corollary 6.** For all \( 0 \leq \theta < 3 \), we have

\[
\# \{D : 1 < D \leq X, D \text{ is not a square}, \epsilon_D \leq D^\theta \} = o_{\theta}(X).
\]

This improves upon the result by Fouvry and Jouve \[5\]. They have shown that one can take \( \theta < 7/4 \) in Corollary \[6\]. Furthermore, Fouvry and Jouve have proved a slightly weaker version of Corollary \[6\] in \[6\]. They showed that for all \( \theta < 3 \), there is a positive proportion of fundamental discriminants \( D \) such that \( \epsilon_D > D^\theta \). For a more thorough discussion of the history and motivation behind Theorem \[5\] and Corollary \[6\] the interested reader should consult their papers \[5\] and \[6\].
2 The Proof of Theorem 1

We will now start to prove Theorem 1. From now on all implied constants may depend on \(k, l\) and \(h\). We may assume without loss of generality that \((ev, du) = 1\), because if \(p \mid (ev, du)\) then \(p \mid h\). As a first step, we will set \(t = v/u\) and \(s = d/e\) so that the equation (1) becomes

\[
t^l = s^k + O\left(\frac{1}{E^kU^l}\right).
\]

Note that

\[
t^l - s^k = (t - s^{k/l}) \sum_{a+b=l-1} t^a(s^{k/l})^b.
\]

Using that \(s \approx D/E\) and \(t \approx V/U = (D/E)^{k/l}\), we may deduce that

\[
\sum_{a+b=l-1} t^a(s^{k/l})^b \approx \left(\frac{V}{U}\right)^{l-1}.
\]

Note that \(E^k = x/V^l\) and hence

\[
t = s^{k/l} + O\left(\frac{1}{xU}\right).
\]

Thus, we have transformed our problem of counting integer points on a three-fold into a problem where we count rational points close to the curve \(t = s^{k/l}\), where the sizes of the numerators and denominators of \(s\) and \(t\) are determined by \(D\) and \(E\). The determinant method seems to be stronger in counting-problems involving varieties of lower dimension so that dealing with a curve rather than a three-fold will provide the key saving in our proof. In section 2.1 we will show how the determinant method allows us to subdivide the range of \(s\) into intervals \(I\) of equal length so that our problem transforms into counting rational points close to the curve \(t = s^{k/l}\) where \(s\) belongs to some particular interval \(I\). In section 2.2 we will then calculate the contribution of one such interval \(I\) to \(N(x; D, E)\) and in section 2.3 we will then add up all the contributions to finish the proof of the estimate in Theorem 1.

2.1 Determinant Method

Note that \(s\) is of exact order \(D/E\). Our plan is to pick an integer \(M\) such that \(\log x \ll \log M \leq \log x\) and split the range of \(s\) into \(O(M)\) intervals \(I = [s_0, s_0(1+M^{-1})]\). For the rest of this section we will fix one such interval \(I\) and consider solutions \((s, t)\) of (4) with \(s \in I\). We label these solutions as \((s_1, t_1), \ldots, (s_J, t_J)\), say. Consider one such \((s_j, t_j)\). We can write \(s_j = s_0(1 + \alpha_j)\) where \(0 < \alpha_j \leq 1/M\). Note that \(s_0^{k/l} \approx V/U\) and hence

\[
t_j = s_j^{k/l} + O\left(\frac{1}{xU}\right) = s_0^{k/l}\left(1 + \alpha_j\right)^{k/l} + O\left(\frac{1}{x}\right).
\]
Hence, after applying Taylor’s Theorem with a suitable degree in $\alpha_j$ to $(1 + \alpha_j)^{k/l}$, we can write

$$s_j = s_0(1 + \alpha_j) \quad \text{with } 0 < \alpha_j \leq \frac{1}{M},$$

$$t_j = s_0^{k/l}(1 + p(\alpha_j) + \beta_j) \quad \text{with } \beta_j \ll \frac{1}{x},$$

where $p(\alpha_j)$ is a polynomial in $\alpha_j$ with no constant coefficient and with coefficients of size $O(1)$. The next step is to choose positive integers $A$ and $B$ and to label the monomials $s_0^a t_0^b$ with $a \leq A$ and $b \leq B$ as $m_1(s, t), \ldots, m_H(s, t)$, where $H = (A + 1)(B + 1)$. Then one considers the $J \times H$ matrix $\mathcal{M}$ whose $(j, h)$-th entry is $m_h(s_j, t_j)$. We will show that the rank of $\mathcal{M}$ is strictly less than $H$ provided we choose $A, B$ and $M$ appropriately. This will enable us to deduce that there is a non-zero vector $c$ such that $\mathcal{M}c = 0$. If we now consider the polynomial

$$C_I(s, t) = C(s, t) = \sum_{h=1}^{H} c_h m_h(s, t),$$

then we can see that $C(s_j, t_j) = 0$ for all our solutions with $s_j \in I$. The vector $c$ can be constructed from subdeterminants of $\mathcal{M}$ which shows that $c \in \mathbb{Q}^H$ has rational entries. Note that the values $s_j$ and $t_j$ have numerators and denominators of size $\ll$ some power of $x$. By clearing out the common denominator of the coefficients of $C$ we may therefore assume that $C$ has integer coefficients of size $\ll x^{\kappa(A, B)}$, say.

To show that $\mathcal{M}$ has rank strictly less than $H$ we can assume that $H \leq J$ since otherwise this gets trivial. Thus, it suffices to show that every $H \times H$ subdeterminant of $\mathcal{M}$ vanishes. Without loss of generality it suffices to show that the determinant $\Delta$ coming from the first $H$ rows and columns of $\mathcal{M}$ vanishes. The $j$-th row of $\mathcal{M}$ has entries with common denominator $e_j^A u_j^B$ which implies that

$$\left( \prod_{j \leq H} e_j^A u_j^B \right) \Delta \in \mathbb{Z}. \quad \text{(5)}$$

Observe that $\prod_{j \leq H} e_j^A u_j^B \ll E^{AH} U^{BH}$ and hence if we can show

$$|\Delta| \ll_{A, B} E^{-AH} U^{-BH},$$

with a suitable implied constant, then the integer in (5) has to be an integer strictly less than 1 which implies $\Delta = 0$. We substitute $s_j = s_0(1 + \alpha_j)$ and $t_j = s_0^{k/l}(1 + p(\alpha_j) + \beta_j)$ so that $\mathcal{M}$ has entries

$$s_0^{a+bk/l}(1 + \alpha_j)^a(1 + p(\alpha_j) + \beta_j)^b.$$  

Hence

$$\Delta = \left( \prod_{a=0}^{A} \prod_{b=0}^{B} s_0^{a+bk/l} \right) \Delta_1 = s_0^{\frac{1}{l}H(A+Bk/l)} \Delta_1,$$

where $\Delta_1$ is the determinant of the generalized Vandermonde matrix with its entries being polynomials in $\alpha_j$ and $\beta_j$ and coefficients of size $O_{A, B}(1)$. Note that we have

$$|\alpha_j| \leq x_1^{-1} \quad \text{and} \quad |\beta_j| \leq x_2^{-1},$$
where $X_1$ is of exact order $M$ and $X_2$ is of exact order $x$. In particular, note that log $X_1$ and log $X_2$ are both of exact order log $x$. We now order the monomials $X_1^{-a}X_2^{-b}$ decreasing in size, $1 = M_0, M_1, \ldots, M_H, \ldots$ say. Then by Lemma 3 in Heath-Brown [11], we may bound $\Delta_1$ and hence $\Delta$ as follows:

$$\Delta = s_0^{H(A+Bk/l)} \Delta_1 \ll_{A,B} (D/E)^{H(A+Bk/l)} \prod_{h=1}^{H} M_h.$$ 

Let $M_H = W^{-1}$. Then $X_1^{-a}X_2^{-b} \geq M_H$ if and only if

$$a \log X_1 + b \log X_2 \leq \log W.$$  \hspace{1cm} (6)

The number of pairs $(a,b)$ which satisfy this inequality is

$$\frac{1}{2} \frac{(\log W)^2}{(\log X_1)(\log X_2)} + O\left(\frac{\log W}{\log x}\right) + O(1).$$

This number must be equal to $H$ which gives

$$|2H(\log X_2)(\log X_1) - (\log W)^2| \ll (\log W)(\log x) + (\log x)^2,$$

and hence

$$|\sqrt{2H(\log X_2)(\log X_1)} - \log W| \ll \left(\log x\right) \frac{\log W + (\log x)}{\sqrt{2H(\log X_2)(\log X_1) + \log W}} \ll \log x,$$

since $(\log X_2)(\log X_1) \asymp (\log x)^2$. Therefore

$$\log W = \sqrt{2H(\log X_2)(\log X_1)} + O(\log x).$$  \hspace{1cm} (7)

Next, observe that

$$\log \prod_{h=1}^{H} M_h = -\sum_{a,b} (a \log X_1 + b \log X_2)$$

$$= -\frac{1}{3} \frac{(\log W)^3}{(\log X_1)(\log X_2)} + O\left(\frac{(\log W)^2}{\log x}\right) + O(\log x),$$

where the summation is subject to the inequality (6). Using the value for log $W$ in (7), we get

$$\log \prod_{h=1}^{H} M_h = -\frac{2\sqrt{2}}{3} H^\frac{3}{2} \sqrt{(\log X_1)(\log X_2)} + O(H \log x).$$

This shows that

$$\log |\Delta| \leq O_{A,B}(1) + \frac{1}{2} H(A + Bk/l) \log \frac{D}{E} - \frac{2\sqrt{2}}{3} H^\frac{3}{2} \sqrt{(\log X_1)(\log X_2)} + O(H \log x).$$
Thus, to get our required bound on $\log |\Delta|$ it suffices to show that

$$A \log E + B \log U + \frac{1}{2}(A + B\frac{k}{l}) \log \frac{D}{E} + C_1(A, B) + C_2 \log x \leq \frac{2\sqrt{2}}{3} H \frac{1}{2} \sqrt{(\log X_1)(\log X_2)},$$

where $C_1(A, B)$ is some constant depending on $A$ and $B$ and $C_2$ is an absolute constant. Note that $AB \leq H$ and

$$A \log E + B \log U + \frac{1}{2}(A + B\frac{k}{l}) \log \frac{D}{E} = \frac{A}{2} \log (DE) + \frac{B}{2} \log (UV).$$

Hence, it suffices to show

$$\frac{1}{2} A \log (DE) + \frac{1}{2} B \log (UV) + C_1(A, B) + C_2 \log x \leq \frac{2\sqrt{2}}{3} (AB)^{\frac{1}{2}} \sqrt{(\log X_1)(\log X_2)}.$$ 

We will optimize the error by choosing $A = \left\lfloor \frac{B \log (UV)}{\log (DE)} \right\rfloor$. The first assumption of Theorem 1 implies that $\nu := \frac{\log (UV)}{\log (DE)} \approx 1$. In particular, if $B \geq 2/\nu$ then we may apply the inequalities $\frac{1}{2} B \nu \leq |B\nu| \leq B\nu$ for $B \geq 2/\nu$ to get that $A \asymp B$. Analyzing Lemma 3 of Heath-Brown [11] which produces our constant $C_1(A, B)$, we can see that $C_1(A, B)$ can be replaced by some $C_3(B)$, say. Now observe that $\frac{1}{2} A \log (DE) + \frac{1}{2} B \log (UV) \leq B \log (UV)$ and that

$$\frac{2\sqrt{2}}{3} (AB)^{\frac{1}{2}} \sqrt{(\log X_1)(\log X_2)} \geq \frac{2\sqrt{2}}{3} B \nu^{\frac{1}{2}} \left(1 - \frac{1}{B\nu}\right)^{\frac{1}{2}} \sqrt{(\log X_1)(\log X_2)}$$

$$\geq \frac{2\sqrt{2}}{3} B \nu^{\frac{1}{2}} \left(1 - \frac{1}{B\nu}\right) \sqrt{(\log X_1)(\log X_2)}$$

$$= \frac{2\sqrt{2}}{3} B \sqrt{\frac{\log (UV)}{\log (DE)}} \sqrt{(\log X_1)(\log X_2)} + O(\log x)$$

since $(\log X_1)(\log X_2) \asymp (\log x)^2$. It therefore suffices if we have

$$B \log (UV) + C_3(B) + C_2 \log x \leq \frac{2\sqrt{2}}{3} B \sqrt{\frac{\log (UV)}{\log (DE)}} \sqrt{(\log X_1)(\log X_2)}.$$ 

Let $\delta > 0$ be arbitrary and pick $X_1$ such that

$$\frac{2\sqrt{2}}{3} \sqrt{\frac{\log (UV)}{\log (DE)}} \sqrt{(\log X_1)(\log X_2)} \geq (1 + \delta) \log (UV),$$

whence it suffices if we have $B \log (UV) + C_3(B) + C_2 \log x \leq B(1 + \delta) \log (UV)$ which holds if and only if $B\delta \log (UV) \geq C_3(B) + C_2 \log x$. Pick

$$B = B(\delta) \geq \frac{2C_2 \log x}{\delta \log (UV)}$$

9[28]
so that \( C_2 \log x \leq \frac{1}{2} \delta B \log(UV) \) and pick \( x \) large enough in terms of \( \delta \) so that \( C_3(B) \leq \frac{4}{3} \delta B \log(UV) \). Thus, we have shown that if we pick \( B \) (and hence \( A \)) and \( x \) large enough in terms of \( \delta \) and if we can pick a suitable \( X_1 \) as above then \( \Delta = 0 \). Observing that \( X_1 \gg M \) and \( X_2 \gg x \), we may rewrite the condition with a redefined \( \delta \) as

\[
\log M \geq \frac{9}{8}(1 + \delta) \frac{\log(UV) \log(DE)}{\log x}.
\]

In summary, we get the following lemma:

**Lemma 7.** Let \( \delta > 0 \), and assume that an integer \( M \) satisfies \( \log x \ll \log M \leq \log x \) and

\[
\log M \geq \frac{9}{8}(1 + \delta) \frac{\log(DE) \log(UV)}{\log x}.
\]

Then for any interval \( I = (s_0, s_0(1 + M^{-1})) \), there exists a non-zero integer polynomial \( C_I(s, t) \) of total degree \( O(1) \) and coefficients of size \( O(x^\kappa(\delta)) \) where \( \kappa = \kappa(\delta) \) and such that \( C_I(s, t) = 0 \) for all solutions of (4) with \( s \in I \).

### 2.2 Counting Solutions

Next, we want to estimate how much a fixed interval \( I \) contributes to \( \mathcal{N}(x; D, E) \). As in [12], we may assume that \( C_I \) is absolutely irreducible. By clearing the denominators of \( C_I(s, t) = 0 \), we may rewrite the equation in the form

\[ F(d, e; v, u) = 0. \tag{8} \]

In summary, we have created an auxiliary polynomial \( F \) which is bi-homogeneous of degree \( (a, b) \), say, has a total degree of size \( O_\delta(1) \) and coefficients of size \( O(x^{\kappa(\delta)}) \) and it vanishes at the solutions \( (s, t) \) of (4) with \( s \in I \). The condition \( s \in I \) gives \( |d - es_0| \ll D/M \). We also have \( |e| \ll E \). It is convenient to define the linear map \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
T(x_1, x_2) = \left( \frac{M}{D}(x_1 - x_2 s_0), \frac{1}{E} x_2 \right).
\]

This linear map defines a lattice

\[ \Lambda = \{ T(x_1, x_2) : (x_1, x_2) \in \mathbb{Z}^2 \} \]

of determinant \( \det(\Lambda) = M/(ED) \). Consider the rectangle

\[ R = \{ (\alpha_1, \alpha_2) : |\alpha_1| \ll 1, |\alpha_2| \ll 1 \}, \]

with suitable implied constants so that \( s = d/e \in I \), implies \( T(d, e) \in \Lambda \cap R \). Thus, we will now count points falling into \( \Lambda \cap R \).

Let \( g^{(1)} \) be the shortest non-zero vector in \( \Lambda \) and let \( g^{(2)} \) be the shortest vector in \( \Lambda \) not parallel to \( g^{(1)} \). Then \( g^{(1)}, g^{(2)} \) will be a basis for \( \Lambda \). Moreover, \( \lambda_1 g^{(1)} + \lambda_2 g^{(2)} \in R \) implies \( |\lambda_1 g^{(1)}| \ll 1 \) and \( |\lambda_2 g^{(2)}| \ll 1 \). By defining \( L_i \) to be a suitable constant times
For example, if \( l \geq \frac{1}{k/l} \) from the above result about Thue equations. If \( k/l = 2 \) then this follows again by Estermann’s result and if \( k/l > 2 \), the estimate follows from Estermann [4]. Thus, after a change of basis we may replace \((d,e)\) by \((\lambda_1, \lambda_2)\) with \(|\lambda_1|, |\lambda_2| \leq L_i\) where \( L_2 \ll L_1 \) and \( L_1L_2 \gg (DE)/M \).

Our value \( s_0 \) is of the form \( s_0 = \frac{g_0 + D}{g_0 - D} \), where \( x_3 \) is an integer of exact order \( M \). Define \( x_4 := \frac{x_3^{k/l} M^{1-k/l}}{x_4} \) and \( t_0 := \frac{t_0 + V}{t_0 - V} \), so that \( s_0^{k/l} = t_0 + O \left( \frac{V}{M} \right) \). Observe that

\[
t = s_0^{k/l} + s_0^{k/l} p(\alpha) + s_0^{k/l} = s_0^{k/l} + O \left( \frac{V}{M} \right) + O \left( \frac{V}{x_4} \right) = t_0 + O \left( \frac{V}{M} \right),
\]

since \( M \leq x \). This leads to the conditions \(|v - ut_0| \ll \frac{V}{M} \) and \(|u| \ll U \) and hence we can analogously to the above argument replace \((v, u)\) by \((\tau_1, \tau_2)\) say where \(|\tau_1| \leq T_i\) with \( T_2 \ll T_1 \) and \( T_1T_2 \gg \frac{V}{M} \). These substitutions convert equation (11) into an equation

\[
G_0(\lambda_1, \lambda_2; \tau_1, \tau_2) = h,
\]

say where \( G_0 \) is bi-homogeneous of degree \((k, l)\). Similarly, equation [5] will turn into an equation of the form

\[
G_1(\lambda_1, \lambda_2; \tau_1, \tau_2) = 0,
\]

where \( G_1 \) is bi-homogeneous of degree \((a, b)\) and satisfies the same conditions as \( F \). Also, from the above argument it is clear that the vectors \((\lambda_1, \lambda_2)\) and \((\tau_1, \tau_2)\) are primitive. For example, if \( p \mid (\lambda_1, \lambda_2) \) then \( p \mid (d_1, e_1) = 1 \).

If \( a = 0 \) then (10) determines \( O_5(1) \) pairs \((\tau_1, \tau_2)\) each of which gives a pair \((u, v)\). The number of pairs \((d, e)\) corresponding to such a \((u, v)\) is

\[
\# \{(d, e) : d \asymp D, e \asymp E, e^{k/l} - d^{k/l} = h\} \ll x^\varepsilon.
\]

This follows since \( e^{k/l} - d^{k/l} = h \) is a Thue equation for \( k \geq 3 \) which has only \( O_{k,l,h}(1) \) solutions (Thue [4]). For \( k = 2 \), the estimate (11) follows from Estermann [4]. Thus, the case \( a = 0 \) contributes \( O_5(x^\varepsilon) \) to \( N(x; D, E) \).

If \( b = 0 \) then (11) determines \( O_4(1) \) pairs \((\lambda_1, \lambda_2)\) each of which gives a pair \((d, e)\). The number of pairs \((u, v)\) corresponding to such a pair \((d, e)\) is

\[
\# \{(u, v) : u \asymp U, v \asymp V, e^{k/l} - d^{k/l} = h\} \ll x^\varepsilon.
\]

If \( l = 2 \) then this follows again by Estermann’s result and if \( l \geq 3 \) the estimate follows from the above result about Thue equations. If \( l = 1 \) then the equation \( e^k - d^k = h \)
is a linear Diophantine equation in \((u, v)\) and thus, for each pair \((d, e)\), the number of solutions \((u, v)\) to (11) is
\[
\ll \frac{V}{D^k} + 1 = \frac{x}{(DE)^k} + 1 \ll 1,
\]
where the last estimate follows from the assumption \(DE \gg x^{1/k}\) of the theorem. Hence, the case \(b = 0\) contributes \(O_\delta(1)\) to \(N(x; D, E)\).

If \(a \geq 2\) then Lemma 2 of Heath-Brown [12] gives us that (10) has \(O_{\epsilon, \delta}(T_1^{1+\epsilon}x^{\kappa})\) solutions. (recall that the coefficients of \(G_1\) are of order \(x^\kappa\)). Picking \(\epsilon\) small enough in terms of \(\delta\), we may assume that this is \(O_\delta(T_1^{1+\delta}x^{\delta})\). Each solution of (10) produces at most one solution of (11). So in the case \(a \geq 2\) the contribution of \(I\) to \(N(x; D, E)\) is \(O_\delta(T_1^{1+\delta}x^{\delta})\). Similarly, if \(b \geq 2\) then the contribution of \(I\) to \(N(x; D, E)\) is \(O_\delta(L_1^{1+\delta}x^{\delta})\).

If \(a = 1\) then (10) can be written as
\[
\lambda_1 G_{11}(\tau_1, \tau_2) + \lambda_2 G_{12}(\tau_1, \tau_2) = 0,
\]
and hence \(q\lambda_1 = G_{12}(\tau_1, \tau_2)\) and \(q\lambda_2 = -G_{11}(\tau_1, \tau_2)\) for some integer \(q\). We define two polynomials \(g_1, g_2 \in \mathbb{Z}[x]\) by \(G_{1i}(\tau_1, \tau_2) = \tau_1^{i-1}g_i(\tau_2/\tau_1)\) for \(i = 1, 2\). Observe that \(g_1\) and \(g_2\) must be coprime since \(G_1\) is absolutely irreducible. Hence, by the Euclid’s Algorithm there exist polynomials \(h_1, h_2 \in \mathbb{Z}[x]\) and an integer \(H\) such that
\[
g_1h_1 + g_2h_2 = H
\]
where \(H = O(x^\kappa)\). Evaluating this equation at \(\tau_2/\tau_1\) we may deduce that \(q \mid H\tau_1^K\) where \(K\) is some integer. and similarly we may conclude that \(q \mid H\tau_2^K\) where \(H = O(x^\kappa)\). But \(\tau_1\) and \(\tau_2\) are coprime. Thus, we may deduce that we have \(O_\delta(x^\delta)\) choices for \(q\). Each value of \(q\) gives us a value of \(\lambda_1\) and \(\lambda_2\) in terms of \(\tau_1\) and \(\tau_2\) which we may substitute into (9) to get a Thue equation of the form \(G_3(\tau_1, \tau_2) = hq^k\), say. This equation gives \(O(T_1)\) possible pairs \(\tau_1, \tau_2\) which shows that the case \(a = 1\) contributes at most \(O_\delta(x^\delta T_1)\) to \(N(x; D, E)\). Similarly, we may deduce that the case \(b = 1\) contributes at most \(O_\delta(x^\delta L_1)\) to \(N(x; D, E)\). In summary, we obtain the following:

**Lemma 8.** For any \(\delta > 0\), the contribution of the solutions \((s, t)\) with \(s \in I\) to \(N(x; D, E)\) is \(O_\delta(x^\delta \min(L_1^{1+\delta}, T_1^{1+\delta}))\).

### 2.3 Completion of the proof of Theorem 1

In the previous section, we calculated the contribution of each interval \(I\) to \(N(x; D, E)\). It remains to sum up the contribution of the various intervals. To proceed, we write \(g^{(1)}\) from the previous section as
\[
g^{(1)} = ((M/D)(x_1 - x_2s_0), (1/E)x_2).
\]
Recall that \(L_1\) was defined to be a suitable multiple of \(|g^{(1)}|^{-1}\). This gives
\[
L_1(x_1 - x_2s_0) \ll D/M, \quad \text{and} \quad L_1x_2 \ll E.
\]
Recall that we produce the intervals $I = (s_0, s_0 + \frac{1}{M} D)$ by taking $s_0 = x_3 \frac{1}{M} E$ for integers $x_3 \asymp M$. Hence, the number of intervals $I$ for which $L < L_1 \leq 2L$ is at most the number of triples $(x_1, x_2, x_3) \in \mathbb{Z}^3$ for which $\gcd(x_1, x_2) = 1$ and

$$x_2 x_3 = \frac{M E}{D} x_1 + O\left(\frac{E}{L}\right), \quad x_2 \ll \frac{E}{L}, \quad x_3 \asymp M.$$ 

Recalling that $L_1 \gg L_2$ and $L_1 L_2 \gg (DE)/M$, we can deduce that $L \gg (DE/M)^{1/2}$.

If $x_2 = 0$ then $x_1 = \pm 1$ since $(x_1, x_2) = 1$. Hence, $L \asymp |g^{(1)}|^{-1} = D/M$. Note that $x_3 \neq 0$. Thus, there are $O(M)$ choices for $x_3$. Hence, the number of intervals $I$ for which $x_2 = 0$ and $L < L_1 \leq 2L$ is $O(M)$. By Lemma 8, each such interval contributes at most $O(x^\delta L^{1+\delta})$ to $N(x; D, E)$ which gives a total contribution of $O(x^\delta D)$ corresponding to these intervals.

Next consider the cases when $x_1 = 0$. In this case $x_2 = \pm 1$ because $(x_1, x_2) = 1$, and hence $L \asymp E/M$. And thus, a similar argument as above shows that the intervals $I$ for which $x_1 = 0$ and $L < L_1 \leq 2L$ contribute a total of $O(x^\delta E)$ to $N(x; D, E)$.

We are therefore left with the case when $x_1$ and $x_2$ are both non-zero. We can now see that $|g^{(1)}| \gg E^{-1}$ which implies that $L \ll E$. The above conditions on $(x_1, x_2, x_3)$ imply that $x_1 \ll D/L$. Thus, there are $O(D/L)$ choices for $x_1$ and each $x_1$ produces $O(E/L)$ choices for the product $x_2 x_3$. And since $x_2 x_3 \neq 0$, a divisor function estimate shows that each value of $x_2 x_3$ arises from at most $O(\delta(x^\delta))$ pairs $(x_2, x_3)$. Thus, there are $O(\delta(x^\delta D E/L^2))$ intervals $I$ so that $L_1$ is of exact order $L$. Each interval contributes $O(\delta(x^\delta L^{1+\delta}))$ by Lemma 8. Note that $L^\delta \ll E^\delta \ll x^\delta$ and hence we get a total contribution of $O(\delta(x^\delta D E/L)$ corresponding to the intervals with $x_1 x_2 \neq 0$. By dyadic subdivision of the range of $L \gg (DE/M)^{1/2}$ we can conclude that

$$N(x; D, E) \ll \delta x^\delta ((DEM)^{1/2} + D + E).$$

Similarly, we may consider the $(x_1, x_2)$ lattice from the previous section corresponding to $(v, u)$ to get $T_1(x_1 - x_2 t_0) \ll V/M$ and $T_1 x_2 \ll U$. Recall that $t_0 = x_4 \frac{1}{M} E$ where $x_4 = \left\lfloor \frac{x_3}{M} \right\rfloor$. We can see that $x_3 \asymp M$ implies $x_4 \asymp M$ for large enough $x$. Hence by a completely analogous argument to the above we can deduce that

$$N(x; D, E) \ll \delta x^\delta ((UV M)^{1/2} + U + V).$$

Thus, we have established the bound for $N(x; D, E)$ as stated in Theorem 4. Now write $DE = x^\psi$ where $\psi > 0$. Then $UV = x^{2l-k\psi/l}$. We pick the integer $M$ such that

$$\log M = 9 \frac{1}{8} (1 + \delta) \psi (2l - k\psi/l).$$

Note that

$$\frac{9}{8} \psi (2l - k\psi/l) \leq \frac{9}{8 kl} < 1.$$ 

So, indeed with our choice of $M$ we have that $\log M \leq \log x$. This finishes the proof of Theorem 4.
3 The Proof of Theorem 2

3.1 Preliminaries

First, we illustrate on how finding an asymptotic formula for $N_{k,h}(x)$ can be reduced to counting points on the algebraic variety $e^k v - d^k u = h$ inside a certain bounded box. In what follows all implied constants may depend on $h$ and $k$. First observe that by dyadic subdivision it is enough to show that

$$N_{k,h}(2x) - N_{k,h}(x) = c_{k,h}x + O(\varepsilon(x^{\omega(k)} + \varepsilon)).$$

Let

$$\xi(n) = \left( \prod_{p^k | n} p \right) \left( \prod_{p^k | n+h} p \right).$$

Then $\xi(n) = 1$ if and only if $n$ and $n + h$ are $k$-free. Thus, we may deduce that

$$N_{k,h}(2x) - N_{k,h}(x) = \sum_{x < n \leq 2x} \sum_{m | \xi(n)} \mu(m) N(x; m),$$

where

$$N(x; m) = \# \{ x < n \leq 2x : \xi(n) \equiv 0 \pmod{m} \}.$$

Observe that the congruence $\xi(n) \equiv 0 \pmod{p}$ has exactly $\rho_{k,h}(p)$ solutions modulo $p^k$. Thus, by an argument similar to the proof of the Chinese remainder theorem, the congruence $\xi(n) \equiv 0 \pmod{m}$ has exactly

$$\rho_{k,h}(m) = \prod_{p | m} \rho_{k,h}(p)$$

solutions modulo $m^k$. Hence

$$N(x; m) = \rho_{k,h}(m) \left( \frac{x}{m^k} + O(1) \right).$$

Note that for square-free $m$, we have that

$$\rho_{k,h}(m) \leq 2^{\omega(m)} \ll m^\varepsilon.$$

Next, we introduce a parameter $y$ with $x^{1/k} \leq y \leq x^{2/(k+1)}$ which we shall pick in a moment. Now, we look at the small terms in the above sum corresponding to the values
of $m$ with $m \leq y$. These terms contribute
\[
\sum_{m \leq y} \mu(m) \rho_{k,h}(m) \left( \frac{x}{m^k} + O(1) \right)
\]
\[
= x \sum_{m \leq y} \frac{\mu(m) \rho_{k,h}(m)}{m^k} + O \left( \sum_{m \leq y} \rho_{k,h}(m) \right)
\]
\[
= x \sum_{m=1}^{\infty} \frac{\mu(m) \rho_{k,h}(m)}{m^k} + O \left( x \sum_{m>y} \frac{\rho_{k,h}(m)}{m^k} + \sum_{m \leq y} \rho_{k,h}(m) \right)
\]
\[
= c_{k,h} x + O(x^{1+\epsilon}y^{1-k}) + O(x^\epsilon y)
\]
\[
= c_{k,h} x + O(x^\epsilon y),
\]
where the last equality follows from $x^{1/k} \leq y$. To minimize the remaining error term, we pick $y = x^{1/k}$. Thus, we can see that the values of $m$ with $m \leq x^{1/k}$ contribute our main term. Hence, we are left to consider the values of $m$ with $m > x^{1/k}$. For each such $m$ we write $m = de$ where $d^k | n$ and $e^k | n+h$. By dyadic subdivision, these values $d, e$ lie in $O((\log x)^2)$ boxes $D/2 < d \leq D, E/2 < e \leq E$ where $D, E \ll x^{1/k}$ and $DE \gg x^{1/k}$. Hence, for one such pair $D, E$ we must have
\[
N_{k,h}(2x) - N_{k,h}(x) = c_{k,h} x + O(x^{1/k+\epsilon}) + O(x^\epsilon N(x; D, E))
\]
where $N(x; D, E)$ is the number of elements in the set
\[
\{(d, e, u, v) \in \mathbb{N}^4 : D/2 < d \leq D, E/2 < e \leq E, x < d^k u + h = e^k v \leq 2x\}.
\]

3.2 The proof of Theorem 2 for $k \geq 3$

We apply Theorem 1 with $l = 1, U = x/D^k$ and $V = x/E^k$ to deduce that
\[
N(x; D, E) \ll x^{\epsilon} x^\epsilon \min\{(DEM)^{1/2} + D + E, (UVM)^{1/2} + U + V\}. \tag{12}
\]
We assume without loss of generality that $D \leq E$ and hence $V \leq U$. Note that $U/V \geq E/D \geq 1$. First we consider the case $M \geq U/V$. In this case, the estimate (12) becomes
\[
N(x; D, E) \ll x^{\epsilon} x^\epsilon M^{1/2} \min(DE, UV)^{1/2}. \tag{13}
\]
We now set $DE = x^\psi$ so that $UV = x^{2-k\psi}$. Then
\[
\frac{\log N(x; D, E)}{\log x} \leq \epsilon + \frac{1}{2} \min(\psi, 2-k\psi) + \frac{9}{16} \psi(2-k\psi) =: f_k(\psi),
\]
say. The function $f(\psi)$ takes its maximal value at $\psi = 2/3$ if $k = 2$ and at $\psi = 13/(9k)$ if $k \geq 3$. Note that $f_2(2/3) = 7/12$ and $f_k(13/(9k)) = 169/(144k)$ for $k \geq 3$. Note that
\[
(DEM)^{1/2} + D + E \ll x^{169/144k} + x^{1/k} \ll x^{169/144k},
\]
no matter if \( M \geq U/V \) does or does not hold. This completes the proof of Theorem 2 for \( k \geq 3 \). However, for \( k = 2 \), we have \( 169/(144k) \geq 7/12 \) and a better argument is required. Thus, we shall employ the following trivial estimate, where we first sum over the pairs \( e, u \) as follows:

\[
N(x; D, E) \ll \sum_{\substack{E \ll e \ll E \\ U \ll u \ll U}} \# \{ d : d \asymp D, e^2v - d^2u = h \}
\ll \sum_{\substack{E \ll e \ll E \\ U \ll u \ll U}} \# \{ d : d \asymp D, d^2 \equiv -hu^{-1}(\text{mod } e^2) \}
\ll \sum_{\substack{E \ll e \ll E \\ U \ll u \ll U}} \left( \frac{D}{e^2} + 1 \right) \# \{ d \text{ mod } e^2, d^2 \equiv -hu^{-1}(\text{mod } e^2) \}
\ll EU \left( \frac{D}{E^2} + 1 \right) x^\epsilon. \tag{14}
\]

Note that

\[ EU \frac{D}{E^2} = \frac{x}{DE} \ll x^{1/2}, \]

Thus we may assume that \( EU \geq x^{1/2} \) and, hence:

\[ \frac{D^3}{E^3} = \frac{x^2}{(EU)^2ED} \ll \frac{x^2}{(x^{1/2})^3} = x^{1/2}. \]

Thus, \( \frac{D}{E} \ll x^{1/6} \) and \( \frac{U}{V} \ll x^{1/3} \). By interchanging \( E \) and \( D \) and doing the same argument, we may deduce that all of the quotients \( \frac{D}{E}, \frac{E}{D}, \frac{V}{U}, \frac{U}{V} \) are \( \ll x^{1/3} \). We may impose the condition \( M \geq x^{1/3} \geq U/V \), so that the bound (13) does indeed hold. This shows that we can take the value \( \omega(2) = 7/12 \) in Theorem 2. But we can do better than this by considering points on lines contained in the three-fold \( e^2v - d^2u = h \). The following section will illustrate this idea.

### 3.3 Counting Points on Lines, Finishing the proof for \( k = 2 \)

In this section, we will conclude the proof of Theorem 2 by considering points on lines contained in the three-fold (11). For convenience, we will illustrate the proof when \( h = 1 \) but with minor changes, the proof can be adapted to general \( h \). We begin by a similar procedure as in the proof of Theorem 1. We will pick \( M \in [x^{1/2}, x] \) and \( A = B = 1 \) as in section 2.4. Note that this will produce a \( 4 \times 4 \) matrix \( M \) where the \( j \)-th row is

\[
(1 \ s_0(1 + \alpha_j) \ s_0^2(1 + 2\alpha_j + \beta_j) \ s_0^3(1 + \alpha_j)(1 + 2\alpha_j + \beta_j)),
\]

with \( \alpha_j \ll M^{-1} \) and \( \beta_j \ll x^{-1} \). (Note that \( \alpha_j^2 \ll M^{-2} \leq x^{-1} \). Performing column operations, we get a matrix with \( j \)-th row

\[ s_0^6 \cdot (1 \ \alpha_j \ \beta_j \ \alpha_j(2\alpha_j + \beta_j)). \]
Recalling that \( s_0 \approx D/E \), we may deduce that \( \Delta \ll \frac{1}{Mx^6} \). As before, we require \( |\Delta| \ll E^{-\delta}U^{-4} \) in order to deduce that \( \Delta = 0 \). This is satisfied if \( M > (xUV)^{1/3+\delta} \)
where \( \delta > 0 \) is arbitrarily small and \( x \) is large enough in terms of \( \delta \). Setting as before \( DE = x^\psi \) and \( UV = x^{2-2\psi} \) we can see that \( x^{1/2} \leq (xUV)^{1/3+\delta} \leq x \). This enables us to pick \( M = (xUV)^{1/3+\delta} \) provided \( \delta \) is small enough and \( x \) is large enough in terms of \( \delta \).

Hence, as before, the determinant method will produce an irreducible auxiliary polynomial \( F(d,e;u,v) \). This polynomial will be bilinear since we picked the monomials in \( M \) accordingly. That is, in section 2.2 the case that will occur is \( a = b = 1 \). Hence, (10) can be written as
\[
dL_1(u,v) + eL_2(u,v) = 0,
\]
where \( L_1(u,v) = c_1u + c_2v \) and \( L_2(u,v) = c_3u + c_4v \) are linear forms with integral coefficients. As in section 2.2 we get \( O(x^\delta) \) choices for an integer \( q \) such that
\[
eq q = -L_1(u,v), \quad dq = L_2(u,v).
\]
Plugging this into the equation \( e^2v - d^2u = 1 \) gives us a Thue equation
\[
G_3(u,v) = (L_1(u,v))^2v - (L_2(u,v))^2u = q^2,
\]
say. When \( G_3 \) is irreducible then this equation will only have a finite number of solutions (see Thue [14]). Thus, the equation will produce \( O(x^\delta) \) solutions for each \( q \) except if \( G_3 \) is splitting into three equal linear factors,
\[
G_3(u,v) = \alpha(\alpha_1v - \alpha_2u)^3,
\]
say where \( \alpha, \alpha_1, \alpha_2 \in \mathbb{Z} \). Our aim is to show that the points under consideration corresponding to this case actually lie on a line contained in the three-fold defined by (11). Comparing coefficients we get the equations
\[
\begin{align*}
\alpha_1^3 = c_3^2 \\
\alpha_1^2 = c_2^2 \\
3\alpha_1^2\alpha_2 = c_4^2 - 2c_1c_2 \\
3\alpha_1\alpha_2^2 = c_1^2 - 2c_3c_4.
\end{align*}
\]
The first two equations give \( \alpha_1 = (c_2^2/\alpha)^{1/3} \) and \( \alpha_2 = (c_3^2/\alpha)^{1/3} \) which turns the third and fourth equation into
\[
\begin{align*}
3c_4^{4/3}c_3^{2/3} = c_4^2 - 2c_1c_2, \quad \text{and} \quad \text{(16)}
\end{align*}
\]
\[
\begin{align*}
3c_2^{2/3}c_3^{4/3} = c_1^2 - 2c_3c_4 \quad \text{(17)}
\end{align*}
\]
respectively. If \( c_2 = 0 \) then we may deduce from these equations that \( c_4 = 0 \) and hence \( c_1 = 0 \). Hence \( L_1 = 0 \) which gives a contradiction since \( qe \neq 0 \). Thus, \( c_2 \neq 0 \)

\[17/28\]
and similarly $c_3 \neq 0$. Using (16) and (17) we may deduce that $c_3^{2/3}(c_1^2 - 2c_1c_2) = c_2^{2/3}(c_1^2 - 2c_3c_4)$ which may be written as

$$(c_3^{1/3}c_4 + c_3^{2/3}c_2^{2/3})^2 = (c_2^{1/3}c_1 + c_2^{2/3}c_3^{2/3})^2.$$  

After taking the square-root, this either lets us deduce directly that $c_3^3c_3 = c_1^3c_2$ or that

$$c_3^{1/3}c_4 + c_3^{2/3}c_1 + 2c_2^{2/3}c_3^{2/3} = 0.$$  

Multiply this equation by $c_3^{2/3}$ and plug in the expression for $c_3c_4$ given by (17) to get $(c_1 + c_2^{1/3}c_3^{2/3})^2 = 0$. Similarly, we may deduce from (16) that $(c_4 + c_3^{1/3}c_2^{2/3})^2 = 0$. This implies that in either case we have $c_1^3c_3 = c_1^3c_2$. Plugging this value of $c_3$ into (16) we get that

$$c_1 = \frac{1}{\kappa}c_3^{2/3}, \quad c_3 = \frac{1}{\kappa^3}c_3^{2/3},$$

where $\kappa \in \{-1, 3\}$. Consider the equation $c_4^3c_3 = c_1^3c_2$. We set $c_1 = h\alpha$ and $c_4 = h\beta$ where $h = (c_1, c_4)$. From this, we can deduce that $c_2 = k\beta^3$ and $c_3 = k\alpha^3$ for some integer $k$. Observe that $(h, k) = 1$ since $(c_1, c_2, c_3, c_4) = 1$ as $F$ is irreducible. Considering the equation $c_1^3c_3 = c_1^3c_2$ we see that $h = k\alpha\beta$ which implies $k = 1$ and

$$c_1 = k\alpha^2\beta, \quad c_2 = \beta^3, \quad c_3 = \alpha^3, \quad c_4 = k\alpha\beta^2.$$  

Our original Thue equation has now turned into

$$v = \frac{\alpha^2}{\beta^2}u + \frac{Q^2}{\beta^2},$$

where $q = Q^3$ for $Q \in \mathbb{Z}$ with $Q > 0$. Substituting this expression for $v$ into (15), we obtain that $(d, e, u, v)$ must lie on the line

$$\ell : (d, e, u, v) = \left(\frac{\kappa\alpha}{Q}, -\frac{\beta}{Q}, 0, \frac{Q^2}{\beta^2}\right) + u \left((1 + \kappa)\frac{\alpha^3}{Q^3}, -(1 + \kappa)\frac{\alpha^2\beta}{Q^2}, 1, \frac{\alpha^2}{\beta^2}\right).$$

Assume that the line does indeed have an integral point counted by $N(x; D, E)$. Pick $u_1$ to be the smallest integer such that $(d_1, e_1, u_1, v_1)$ is on $\ell$ and counted by $N(x; D, E)$. This gives us the equations

$$Q^2 = v_1\beta^2 - u_1\alpha^2$$  

$$(18)$$

$$\alpha\kappa Q^2 + u_1(1 + \kappa)\alpha^3 = d_1 Q^3$$  

$$(19)$$

$$-\beta Q^2 - u_1(1 + \kappa)\alpha^2\beta = e_1 Q^3.$$  

$$(20)$$

We now consider the lines with $\kappa = -1$. In this case, we may deduce from (19) and (20) that $Q = 1$, $d_1 = -\alpha$ and $e_1 = -\beta$. Thus our points under consideration must lie on the line

$$(d, e, u, v) = (d_1, e_1, u_1, v_1) + \lambda(0, 0, \beta^2, \alpha^2),$$
where \( \lambda \in \mathbb{Z} \). Note that there are no parallel lines with \( \kappa = -1 \) having the same \( \alpha \) and \( \beta \) since (18) determines the pair \((u_1, v_1)\) modulo \((\beta^2, \alpha^2)\). The number of distinct lines with \( \kappa = -1 \) is therefore determined by the number of choices for the direction vectors, that is, by the number of choices for \( \alpha \) and \( \beta \). We must have \( \alpha \beta \ll (U V)^{1/2} \ll x^{1/2} \) which shows that the number of distinct lines with \( \kappa = -1 \) is \( O(x^{1/2+\delta}) \). The number of points on each line is \( \ll 1 + U/\beta^2 \ll 1 \) since \( E \asymp e_1 = -\beta \). Thus, the total number of points counted be \( N(x; D, E) \) lying on lines with \( \kappa = -1 \) is \( O(x^{1/2+\delta}) \).

Next, we consider the lines with \( \kappa = 3 \). From (19) and (20) we may deduce that 
\[
(\beta d_1 + \alpha e_1)Q = 2\alpha \beta.
\]
Let \( a = (Q, \alpha) \) and \( b = (Q, \beta) \) so that \( \alpha = aA, \beta = bB \) and \( Q = abQ_1 \) and \((Q_1, AB) = 1 \) say. After canceling \( ab \) from the last equation, we may deduce that \( A \mid d_1 \) and \( B \mid e_1 \). Set \( d_1 = AD_1 \) and \( e_1 = BE_1 \). Finally, (18) implies that \( a^2 \mid v_1 \) and \( b^2 \mid u_1 \). Let \( v_1 = a^2V_1 \) and \( u_1 = b^2U_1 \). Furthermore, we can deduce from (18) that \((U_1, Q_1) = 1 \) and hence (20) shows that \( Q_1^2 \mid 4U_1A^2 \) and therefore \( Q_1^2 \mid 4 \).

The direction vector of the line \( \ell \) is now
\[
\left( \frac{A^3}{b^4} C, -\frac{A^2B}{ab^2} C, 1, \frac{a^2A^2}{b^2B^2} \right),
\]
where \( C = 4/Q_1^3 \in \{4, 1/2\} \). Note that \( a, A, b, B \) are pairwise coprime since \( d_1^2u_1 \) and \( e_1^2v_1 \) as well as \( \alpha \) and \( \beta \) are coprime. Thus, all lines lying on the surface (1) can be written as
\[
(AD_1, BE_1, b^2U_1, a^2V_1) + \lambda(aA^3B^2C, -bA^2B^3C, ab^3B^2, a^3bA^2),
\]
where \( \lambda \) goes through \( \mathbb{Z}, \mathbb{Z}/2 \) or \( 2\mathbb{Z} \) and subject to the conditions
\[
Q_1^2 = V_1B^2 - U_1A^2 \tag{21}
\]
\[
bD_1 + aE_1 = 2Q_1^1 \tag{22}
\]
\[
U_1A^2Q_1C - D_1bQ_1 = -3. \tag{23}
\]

Fix \( Q_1 \). Given a line \( \ell \), the direction vector and hence \( A, B, a, b \) are uniquely determined up to sign. Thus we also fix \( A \) and \( B \) and consider the number of points on lines with our fixed values of \( Q_1, A \) and \( B \).

If \( D_1', E_1', U_1', V_1' \) is one solution of (21), (22), (23) then the other solutions \( D_1, E_1, U_1, V_1 \) are given by
\[
D_1 = D_1' + \mu aA^2B^2
\]
\[
E_1 = E_1' - \mu bA^2B^2
\]
\[
U_1 = U_1' + \mu abB^2/C
\]
\[
V_1 = V_1' + \mu abA^2/C.
\]

Thus, the number of distinct lines with \( \kappa = 3 \) and given \( A, B \) is
\[
\ll \# \{(a, b) : a \ll D, b \ll E, ab^3 \ll U, a^3b \ll V\}
\ll \# \{(a, b) : ab \ll (UV)^{1/4}\} \ll (UV)^{1/4+\delta}.
\]
The number of points on each line is
\[ \ll \min \left\{ \frac{D}{aA^2B^2}, \frac{E}{bA^2B^2} \right\} \ll \frac{\min(D,E)}{A^2B^2}. \]

Thus, the total number of points on lines with \( \kappa = 3 \) counted by \( N(x; D, E) \) is
\[ \sum_{A,B} (UV)^{1/4 + \frac{\min(D,E)}{A^2B^2}} \ll \min(D,E)(UV)^{1/4 + \frac{\delta}{4}} \ll x^{1/2 + \delta}. \]

Let
\[ N_0(x; D, E) = \{(d, e, u, v) \in N(D, E) : (d, e, u, v) \text{ is not on a line contained in } \Pi \}. \]

We have just shown that
\[ N_0(x; D, E) \ll x^\delta M \ll x^{25 + 1 - 2\psi/3}. \]

In section 3.2, we have deduced the bound (13). Hence,
\[ N_0(x; D, E) \leq N(x; D, E) \ll x^{\delta + \min(\psi, 2 - 2\psi) / 2 + \max(1/6, 9\psi(1 - \psi)/8)}, \]

where essentially \( 1/2 \leq \psi \leq 1 \). One can check that the worst value for the exponent occurs if \( \psi \leq 2/3 \) and then that the worst value for \( \psi \) must satisfy
\[ 1 - \frac{2}{3}\psi = \frac{9}{8}\psi(1 - \psi) + \frac{\psi}{2}. \]

Hence the critical value for \( \psi \) is \((55 - \sqrt{433})/54 = 0.6331\ldots\) and we may deduce that
\( N_0(x; D, E) \ll x^{38 + \omega} \) where \( \omega = (26 + \sqrt{433})/81 \leq 0.5779 \leq 7/12 = 0.5833\ldots\). By the above argument, the points counted by \( N(x; D, E) \) but not being elements of \( N_0(x; D, E) \) contribute \( O(x^{1/2 + \delta}) \). This finishes the proof of Theorem 2.

4 The Proof of Theorem 3

We are now turning to the proof of Theorem 3. We define
\[ \xi(n) = \prod_{i=1}^{r} \prod_{p^k || i(n)} p, \]
and for \( w > 1 \) let
\[ \mathcal{P}(w) = \prod_{p < w} p \]
be the product of primes \( p < w \). Furthermore, for \( z > 1 \) define
\[ S(z) = \sum_{x<n\leq 2x \atop (\xi(n), \mathcal{P}(z))=1} 1. \]
That is, \( S(z) \) is the number of integers \( n \) in the interval \((n, 2n]\) so that \( l_1(n), \ldots, l_r(n) \) all do not have any \( k \)-th power prime divisor \( p^k \) with \( p < z \). In particular, \( N(2x) - N(x) = S(O(x^{1/k})) \). We also define

\[
S_d(w) = \sum_{x < n \leq 2x \atop \xi(n) \equiv 0 \pmod{d} \atop (\xi(n), P(w)) = 1} 1.
\]

Next, we will employ the following identity which is essentially Buchstab’s identity. That is, for \( 1 < w < z \) observe that

\[
S(z) = S(w) - \sum_{w \leq p < z} S_p(p).
\]

Applying the identity twice, we obtain for \( w > 1 \):

\[
N(2x) - N(x) = S(O(x^{1/k})) = S(w) - \sum_{w \leq p \leq x^{1/k}} S_p(w) + \sum_{w \leq q < p \leq x^{1/k}} S_{pq}(q).
\]

First, we will estimate the sum \( \sum_p S_p(w) \). We split the sum over \( p \) in two parts, the first range will be \( w \leq p < y \) and for the second range \( y \leq p \ll x^{1/k} \) a trivial estimate will suffice. Similarly to section 3.1 we can deduce that:

\[
N(x; d) = \# \{ x < n \leq 2x : \xi(n) \equiv 0 \pmod{d} \} = \rho(d) \left( \frac{x}{d^k} + O(1) \right).
\] (24)

Thus,

\[
\sum_{y \leq p \leq x^{1/k}} S_p(w) \ll \sum_{y \leq p \leq x^{1/k}} N(x; p) \ll x^\epsilon \sum_{y \leq p \leq x^{1/k}} \left( \frac{x}{p^k} + 1 \right) \ll x^{\epsilon} (xy^{1-k} + x^{1/k}) \ll x^{1/k + \epsilon}.
\]

To estimate the sum over the remaining range \( w \leq p < y \) we will apply the following Fundamental Sieve Lemma due to Heath-Brown [9] which adapted to our purpose states as follows:

**Lemma 9.** For \( z > 1 \) and \( w > 1 \) we have that

\[
\sum_{d \mid (\xi(n), P(w))} \mu(d) = \sum_{d \mid (\xi(n), P(w))} \mu(d) + O \left( \sum_{\substack{z \leq d < 2w \atop z \mid (\xi(n), P(w))}} 1 \right).
\]

Observe that a prime \( p > w > 1 \) and some \( d \mid P(w) \) both divide \( \xi(n) \) if and only if \( pd \) divides \( \xi(n) \). Thus, we pick some parameter \( z_p > 1 \) which we will determine later and
we split \( S_p(w) \) using the Lemma as follows:

\[
S_p(w) = \sum_{x < n \leq 2x \atop \xi(n) \equiv 0 \pmod{p}} \mu(d) \sum_{d \mid (\xi(n), P(w))} \mu(d) + O \left( \sum_{x < n \leq 2x \atop \xi(n) \equiv 0 \pmod{p}} \sum_{d \mid (\xi(n), P(w))} 1 \right)
\]

\[
= \sum_{x < n \leq 2x \atop \xi(n) \equiv 0 \pmod{p}} \mu(d) \sum_{d \mid (\xi(n), P(w))} \mu(d) + O \left( \sum_{d \mid (\xi(n), P(w))} N(x; pd) \right)
\]

\[
= S_1(p) + O(S_2(p)),
\]

say. For \( S_2(p) \) we obtain

\[
S_2(p) \ll x^\varepsilon \sum_{z_p \leq d < z_p w} \left( \frac{x}{d^k p^k} + 1 \right) \ll x^\varepsilon \left( \frac{x}{p^k z_p^k - 1} + z_p w \right).
\]

We will pick \( z_p = p^{-1}(x/w)^{1/k} \) to minimize this error term. Here we set \( y = (x/w)^{1/k} \) to ensure that \( z_p > 1 \). Note that \( y = p z_p \). Using the fact that \( \sum_{p \leq x} p^{-k} \ll x^\varepsilon \) we can deduce that

\[
\sum_{w \leq p < y} S_2(p) \ll x^{1/k + \varepsilon} w^{1-1/k}.
\]

Next, we use the trivial estimate (24) again to conclude that

\[
- \sum_{w \leq p < y} S_1(p) = x \sum_{w \leq p < y} \sum_{d \mid P(w)} \frac{\mu(pd) \rho(pd)}{(pd)^k} + O(yx^\varepsilon).
\]

The double sum equals

\[
x \sum_{d \leq y} \mu(d) \rho(d) d^k = x \sum_{d=1}^\infty \mu(d) \rho(d) d^k + O(x^{1+\varepsilon} y^{1-k}),
\]

(25)

where the \( \sum' \) restricts the sum to those \( d \mid P(x^{1/k}) \) with exactly one exceptional prime divisor \( p \mid d \) such that \( p > w \). Thus, we have shown that

\[
- \sum_{w \leq p < x^{1/k}} S_p(w) = c^{(1)} x + O(x^{1/k+\varepsilon} w^{1-1/k}),
\]

where \( c^{(1)} \) is the constant from (25), the sum over those \( d \) having exactly one large prime divisor \( p > w \). Next, we consider the sum \( S(w) \). Similarly to the above argument, we
may apply Lemma 9 for some \( z > 1 \) to obtain

\[
S(w) = \sum_{d \mid P(w), d < z} \mu(d)N(x; d) + O \left( \sum_{d \mid P(w)} N(x; d) \right). \tag{26}
\]

Again, we use a trivial estimate for the second sum which yields an error term \( O(x^{1/k+\varepsilon}w^{1-1/k}) \) provided we chose \( z = (x/w)^{1/k} \) optimally. The first sum in (26) is

\[
x \sum_{d \mid P(w), d < z} \frac{\mu(d)\rho(d)}{d^k} + O(z).
\]

Thus, we have shown that

\[
S(w) = c^{(0)}x + O(x^{1/k+\varepsilon}w^{1-1/k}),
\]

where

\[
c^{(0)} = \sum_{d \mid P(w)} \frac{\mu(d)\rho(d)}{d^k}
\]

is the sum over those \( d \) having no large prime divisors \( p > w \). Recall that the overall main term is \( cx = x \sum_{d=1}^{\infty} \mu(d)\rho(d)d^{-k} \). The sum \( c - c^{(0)} - c^{(1)} \) is the sum over those \( d \) having at least 2 distinct prime divisors > \( w \) and thus we have

\[
c^{(0)} + c^{(1)} = c + O \left( x^\varepsilon \sum_{d > w^2} \frac{1}{d^k} \right) = c + O(x^\varepsilon w^{2-2k}).
\]

We minimize the error terms by choosing \( w = x^{1/(2k+1)} \) so that both our error terms \( O(x^{1+\varepsilon}w^{2-2k}) \) and \( O(x^{1/k+\varepsilon}w^{1-1/k}) \) become \( O(x^{3/(2k+1)+\varepsilon}) \). Thus, we have shown

\[
S(w) - \sum_{w \leq p \leq x^{1/k}} S_p(w) = cx + O(x^{3/(2k+1)+\varepsilon}),
\]

and it remains to find a bound for the sum \( \sum_{w \leq q < p \leq x^{1/k}} S_{pq}(q) \). First we consider the terms corresponding to those prime pairs \( q < p \) with \( pq \ll x^{1/k} \). A trivial estimate suffices to yield the bound

\[
\sum_{w \leq q < p \leq x^{1/k}} S_{pq}(q) \ll \sum_{w \leq q < p \leq x^{1/k}} N(x; pq) \ll x^\varepsilon \sum_{w \leq q < p \leq x^{1/k}} \left( \frac{x}{p^kq^k} + 1 \right) \ll x^{3/(2k+1)+\varepsilon}.
\]

For the values with \( pq \gg x^{1/k} \) observe that

\[
S_{pq}(q) \ll N(x; pq) = \# \{ x < n \leq 2x : p^k \mid l_i(n), q^k \mid l_j(n) \text{ for some } i \neq j \}.
\]
(The case \( i = j \) cannot occur since \( pq \gg x^{1/k} \) for any suitable implied constant). Thus, we can fix a particular \( i \) and \( j \) and conclude that
\[
\sum_{w \leq q < p \ll x^{1/k}} S_{pq}(q) \ll \#K,
\]
where
\[
K = \{(p, q, u, v) : w \leq q < p \ll x^{1/k}, pq \gg x^{1/k}, p^k u = l_i(n), q^k v = l_j(n)\}.
\]
Without loss of generality we may write \( l_i(n) = a_1 n + b_1 \) and \( l_j(n) = a_2 n + b_2 \) so that we are left with the Diophantine equation \( a_1 q^k v - a_2 p^k u = a_1 b_2 - a_2 b_1 = h \neq 0 \) say. Note that \( (a_1 q^k v, a_2 p^k u) = (a_1 v, a_2 u) \) since neither \( p \) nor \( q \) divide \( h \) since \( h \) is \( O(1) \) and \( p \) and \( q \) are \( \gg x^{1/(2k+1)} \). Each common divisor of \( a_1 v \) and \( a_2 u \) is a divisor of \( h \) and thus is \( O(1) \). Since also \( a_1 \) and \( a_2 \) are \( O(1) \) we can reduce our problem to \( d(h) = O(1) \) equations of the form \( (1) \) with the additional constraint that \((du, ev) = 1 \) and \( de \gg x^{1/k} \). Thus, we have reduced the problem to the case we dealt with in Theorem 2. Thus, this gives an error term \( O(x^{\max(\omega(k)/3/(2k+1)),1+\epsilon}) \) for the asymptotic formula in Theorem 3. Note that \( \omega(k) \leq 3/(2k+1) \) for all \( k \geq 2 \) which concludes the proof.

5 The Proof of Theorem 4

We are now turning to the problem of consecutive square-full integers. Recall that here, \( N(x) \) is the number of integers \( n \leq x \) such that both \( n \) and \( n + 1 \) are square-full. Observe that every square-full integer \( n \) can uniquely be written as \( n = a^2 b^2 \) with \( \mu^2(b) = 1 \). Thus, we have
\[
N(2x) - N(x) \ll \# \left\{ (d, e, u, v) \in \mathbb{N}^4 : x < d^3 u^2 = e^3 v^2 - 1 \leq 2x \right\}.
\]
As in the proof of Theorem 2 we can now split the ranges of \( d \) and \( e \) into \( O(x^\epsilon) \) boxes with \( D/2 < d < D \) and \( E/2 < e \leq E \) where \( D,E \ll x^{1/3} \). For one such box we then get
\[
N(2x) - N(x) \ll x^\epsilon \mathcal{N}(x; D,E),
\]
where
\[
\mathcal{N}(x; D,E) = \# \left\{ (d, e, u, v) \in \mathbb{N}^4 : x < d^3 u^2 = e^3 v^2 - 1 \leq 2x, d \asymp D, e \asymp E \right\}.
\]
Thus, in order to prove Theorem 4 it remains to show that
\[
\mathcal{N}(x; D,E) \ll_{\epsilon} x^{29/100+\epsilon}.
\]
For convenience, we will set \( y = x^{29/100} \). We may now apply Theorem 4 with \( k = 3, l = 2 \) and \( U := x^{1/2} D^{-3/2}, V := x^{1/2} E^{-3/2} \) so that \( v \asymp V \) and \( u \asymp U \). We get the bound
\[
\mathcal{N}(x; D,E) \ll_{\epsilon} x^{\epsilon} \min\{(DEM)^{1/2} + D + E, (UVM)^{1/2} + U + V\},
\] (27)
where the value of $M$ is as stated in Theorem 1. We will further impose the condition $M \geq x^{9/50}$. We will see below that this will not make our bound worse. The situation we are now in is different compared to the situation in section 3.2, where we used the estimate $D + E \ll x^{1/k}$. In our situation here this would produce an estimate $N(x; D, E) \ll x^{1/3+\epsilon}$ which is worse than our anticipated bound. And without further restricting the ranges of $D$ and $E$, it will not be possible to improve this bound. The results by Estermann and Thue allow us to assume further that $\min(D, E, U) \geq y$. This is however still not enough to improve upon the estimate $N(x; D, E) \ll x^{1/3+\epsilon}$.

Thus, we shall employ an estimate similar to (14), namely:

$$N(x; D, E) \ll EU \left(\frac{D}{E^3} + 1\right) x^\epsilon. \quad (28)$$

We will now distinguish two cases depending on whether $DE^{-3} \ll 1$ does or does not hold. If $DE^{-3} \ll 1$ for some suitable constant, then we may assume $EU \geq y$ by the above estimate. Note that by definition of $U$,

$$y^5 \leq (DE)(EU)^4 = \left(\frac{E}{D}\right)^5 x^2,$$

so that

$$\frac{V}{U} = \left(\frac{D}{E}\right)^{3/2} \leq \frac{x^{3/5}}{y^{3/2}} = x^{33/200} < x^{9/50} \leq M.$$

Next, we consider the case $DE^{-3} \gg 1$. Here, the estimate (28) is too weak for our purposes and we need a stronger approach. The idea is to fix a pair $(e, u)$ and estimate the number of solutions $(d, e, u, v)$ counted by $N(x; D, E)$. Let $d_1, \ldots, d_\mu$ be the solutions to the congruence $d^3 \equiv -u^{-2} (\text{mod } e^3)$. In particular, $\mu \ll e^\epsilon \ll x^\epsilon$ and we get the estimate

$$N(x; D, E) \ll \sum_{e \sim E} \sum_{u \sim U} \sum_{i=1}^\mu \#\{(a, v) : v \sim V, a \ll D/E^3, e^3 v^2 - (d_i + ae^3)^3 u^2 = 1\}.$$

Thus, it suffices to estimate the number of points $(a, v)$ that satisfy

$$p(v) - q(a) - 1 = 0, \quad (29)$$

where $p$ is a polynomial of degree 2 and $q$ is a polynomial of degree 3.

We will now prove that that the left-hand side of (29) is absolutely irreducible. It is enough to show that the polynomial $f(S, T) = S^2 - T^3 - 1$ is absolutely irreducible. Assume that $f(S, T) = g(S, T)h(S, T)$ in some finite extension of $\mathbb{Q}$. In particular, $X^6 - Y^6 - 1 = g(X^3, Y^2)h(X^3, Y^2)$. We may homogenize the equation to get

$$X^6 - Y^6 - Z^6 = G(X, Y, Z)H(X, Y, Z) \quad (30)$$

for some polynomials $G, H$. Now let $(X, Y, Z)$ be a nonzero point such that $G(X, Y, Z) = H(X, Y, Z) = 0$. The gradient of the right-hand side of (30) vanishes whereas the
gradient of the left-hand side is \((6X^5, -6Y^5, -6Z^5)\) which implies that \(X = Y = Z = 0\). Contradiction. Thus, the equation \((29)\) must be absolutely irreducible.

Next, we will proceed to apply Theorem 15 of Heath-Brown \([7]\) to get an upper bound for the number of pairs \((v,a)\) satisfying \((29)\). Using the notation of Heath-Brown’s theorem we will set \(n := 2\), \(B_1 = V\) and \(B_2 = DE^{-3} + 1\) so that indeed \(v \leq B_1\) and \(a \leq B_2\) with \(B_1, B_2 \geq 1\). Note that \(T = \max(B_1^3, B_2^3) \geq B_1^3\) so that the points \((v,a)\) satisfying \((29)\) lie on at most \(k \ll \epsilon x^{1/2}\) auxiliary curves. Thus, using Bézout’s Theorem, we may deduce that the number of points \((v,a)\) under consideration is \(\ll \epsilon x^{1/2}B_2^{1/2}\). Thus, we get the estimate

\[ N(D, E) \ll \epsilon x^\delta EU \left( \frac{D}{E^3} + 1 \right)^{1/2} \ll \frac{x^{1/2+\epsilon}}{DE^{1/2}}, \]

where the last estimate is because \(1 \ll D/E^3\). Hence, we may assume that

\[ x^{1/2}D^{-1}E^{-1/2} \geq y. \]

Thus,

\[ y^7 \leq \left( \frac{x^{1/2}}{DE^{1/2}} \right)^4 (DE)^3 = \frac{x^2E}{D}, \]

so that

\[ \frac{D}{E} \leq \frac{x^2}{y^3} \leq 1. \]

Thus, we have shown that in all cases, \(V/U \leq M\). By interchanging the roles of \(D\) and \(E\), we may similarly prove that \(U/V \leq M\) and hence we conclude that

\[ \max(D/E, E/D, U/V, V/U) \leq M. \]

By considering \((27)\) again, we may now deduce that

\[ N(x; D, E) \ll \epsilon x^\delta M^{1/2} \min\{DE, UV\}^{1/2}. \]

We will set \(DE = x^\psi\) so that \(UV = x^{1-3\psi/2}\). Observe that

\[ \log N(x; D, E) \leq \delta + \frac{1}{2} \min(\psi, 1-3\psi/2) + \frac{1}{2} \max \left\{ \frac{9}{8}\psi(1-3\psi/2), \frac{9}{50}\right\} := f(\psi), \]

say. The function \(f(\psi)\) takes its maximum at \(\psi = 2/5\) and \(f(2/5) = 29/100\). This completes the proof of Theorem \(4\).

6 The Proof of Theorem \(5\) and Corollary \(6\)

In this proof, all implied constants may depend on \(\alpha\). Our initial argument is the same as in \([5]\). We want to find \(r = r(\alpha)\) as small as possible so that the estimate \(S(X, \alpha) \ll \epsilon X^{r+\epsilon}\) holds. As illustrated in \([5]\), we may assume that \(\alpha \geq 1/2\). The
estimate \( (4) \) allows us to assume that \( r \geq 1/2 \). As in \( (3) \), we obtain the estimate
\[
S(X, \alpha) \ll \mathcal{N}(X; D, E),
\]
where \( \mathcal{N}(X; D, E) \) is the number of elements in the set
\[
\# \{(d, e, u, v) : d \sim D, e \sim E, u \sim U, v \sim V, e^2 v - d^2 u = h\},
\]
where \( h \in \{\pm 1, \pm 2\} \) and \( D, E, U, V, X \) are positive real numbers such that \( E^2 V \asymp D^2 U \), \( UV \asymp X \) and \( DE \ll X^\alpha \). In particular,
\[
V \asymp X^{1/2}D/E \quad \text{and} \quad U \asymp X^{1/2}E/D. \tag{31}
\]
We want to apply Theorem 1 with \( x = E^2V, \ k = 2 \) and \( l = 1 \). Note the trivial estimate
\[
\mathcal{N}(X; D, E) \ll DE \left( \frac{V}{D^2} + 1 \right) \ll X^{1/2} + DE.
\]
Hence, if \( DE \ll X^r \) then this estimate suffices. Thus, we may assume that \( DE \geq X^r \).
In particular, this shows that \( DE \geq X^{1/2} \) and from \( x^2 \asymp (DE)^2 X \) we can then conclude that \( DE \gg x^{1/2} \). So, Theorem 1 does indeed apply. We get the estimate
\[
\mathcal{N}(X; D, E) \ll \epsilon X^{\epsilon} \min\{(XM)^{1/2} + U + V, (DEM)^{1/2} + D + E\},
\]
where \( D, E, U \) and \( V \) are as in \( (31) \) and
\[
\log M = \frac{9 \log(DE) \log X}{8 \log(DEX^{1/2})},
\]
since \( x = E^2V \asymp DEX^{1/2} \) and \( UV \asymp X \). As in the proof of Theorem 2 we get the estimate
\[
\mathcal{N}(X; D, E) \ll \epsilon X^\epsilon E \left( \frac{D}{E^2} + 1 \right) = X^\epsilon (X^{1/2} + EU).
\]
Note that \( EU = X^{1/2}E^2/D \). Hence we may assume that \( E^2/D \geq X^{r-1/2} \) and similarly we assume that \( D^2/E \geq X^{r-1/2} \). Without loss of generality, let \( D \leq E \) so that \( V \leq U \).
First, we consider the case when \( M < U/V = (E/D)^2 \). In this case,
\[
(XM)^{1/2} + U + V < X^{1/2}E/D + U + V \ll U.
\]
From \( DE \ll X^\alpha \) and \( D^2/E \geq X^{r-1/2} \), it follows that \( (E/D)^3/2 \ll X^{(\alpha+1)/2 - r} \), and hence
\[
\mathcal{N}(X; D, E) \ll \epsilon X^\epsilon U = X^{1/2 + \epsilon - \frac{2}{3} r + \epsilon} \ll X^{r+\epsilon},
\]
provided \( r > 1/2 + \alpha/5 \). This concludes the case \( M < U/V \). So, we may now consider the case \( M \geq U/V \). Note that \( M \geq U/V = (E/D)^2 \geq E/D \) which yields the estimate
\[
\mathcal{N}(X; D, E) \ll \epsilon X^{\epsilon} M^{1/2} \min\{X, DE\}^{1/2}.
\]
We will set \( DE = X^\psi \) so that essentially \( \psi \leq \alpha \). Then
\[
\frac{\log M}{\log X} = \frac{9}{8} \frac{\psi}{\psi + 1/2} =: f(\psi),
\]
where
say. The function $f(\psi)$ is increasing for $\psi > 0$ and hence $f(\psi) \leq f(\alpha)$. This shows that

$$\log \frac{N(X; D, E)}{\log X} \leq O(1) + \frac{1}{2} f(\alpha) + \frac{1}{2} \min(1, \alpha).$$

This concludes the case $M \geq U/V$. We have therefore proved Theorem 5. Corollary 6 is an easy consequence of Theorem 5.

References

[1] J. Brandes, Twins of $s$-free numbers, Diploma Thesis, Institute for Algebra and Number Theory University of Stuttgart (2009).

[2] L. Carlitz, On a problem in additive arithmetic, *Quart. J. Math. Oxford Ser. 3*, 1 (1932), 273–290.

[3] R. Dietmann, O. Marmon, The density of twins of $k$-free numbers, *arXiv:1307.2481*, to appear in Bulletin of the London Mathematical Society.

[4] T. Estermann, Einige Sätze über quadratfreie Zahlen, *Math. Ann.*, 105 (1931), 653–662.

[5] É. Fouvry, F. Jouve, Size of regulators and consecutive squarefree numbers, *Mathematische Zeitschrift*, Springer-Verlag, 0025-5874 (2012), 1–14.

[6] É. Fouvry, F. Jouve, A positive density of fundamental discriminants with large regulator. *Pacific J. Math.*, 262, no. 1 (2013), 81-107.

[7] D.R. Heath-Brown, Counting Rational Points on Algebraic Varieties, *C.I.M.E. lecture notes*, Springer Lecture Notes Vol. 1891.

[8] D.R. Heath-Brown, The Square-Sieve and Consecutive Square-Free Numbers, *Math. Ann.*, 266 (1984), 251–259.

[9] D.R. Heath-Brown, The number of primes in a short interval, *Journal für die reine und angewandte Mathematik 1988*, 389 (1988), 22–63.

[10] D.R. Heath-Brown, The density of rational points on curves and surfaces, *Ann. of Math. (2)*, 155 (2002), 553–595.

[11] D.R. Heath-Brown, Sums and differences of three $k$-th powers, *J. Number Theory*, 129 (2009), 1579–1594.

[12] D.R. Heath-Brown, Square-Free Values of $n^2 + 1$, *arXiv:1010.6217* (2010).

[13] K.-M. Tsang, The distribution of $r$-tuples of square-free numbers, *Mathematika*, 32 (1985), 265–275.

[14] A. Thue, Über Annäherungswerte algebraischer Zahlen, *J. reine angew. Math.*, 135 (1909), 284–305.