Analytic $m$-isometries without the wandering subspace property

Akash Anand, Sameer Chavan and Shailesh Trivedi

Abstract. The wandering subspace problem for an analytic norm-increasing $m$-isometry $T$ on a Hilbert space $\mathcal{H}$ asks whether every $T$-invariant subspace of $\mathcal{H}$ can be generated by a wandering subspace. An affirmative solution to this problem for $m = 1$ is ascribed to Beurling-Lax-Halmos, while that for $m = 2$ is due to Richter. In this paper, we capitalize on the idea of weighted shift on one-circuit directed graph to construct a family of analytic cyclic 3-isometries, which do not admit the wandering subspace property and which are norm-increasing on the orthogonal complement of a one-dimensional space. Further, on this one dimensional space, their norms can be made arbitrarily close to 1. We also show that if the wandering subspace property fails for an analytic norm-increasing $m$-isometry, then it fails miserably in the sense that the smallest $T$-invariant subspace generated by the wandering subspace is of infinite codimension.

1. Introduction

The structural theory of $z$-invariant subspaces in the Hilbert spaces of analytic functions led many mathematicians to develop the theory of non-isometric Hilbert space operators. For instance, the structural theory of $z$-invariant subspaces in Dirichlet space led Richter [24] to study the class of 2-isometries. A similar quest inspired Aleman [3] to study the class of Dirichlet-type operators. Further, the structural theory of $z$-invariant subspaces in the Bergman space motivated Shimorin [26] to study the class of Bergman-type operators. The basic problem here is to see whether or not the given analytic operator admits the wandering subspace property. In this terminology (attributed to Halmos), a celebrated result of Beurling [6] says that the unweighted shift operator in the Hardy space of the unit disc admits the wandering subspace property. A counter-part of Beurling’s Theorem for the Bergman shift was a long-standing open problem. Indeed, a similar result had been deemed virtually impossible by many analysts in view of the huge lattice of its invariant subspaces (see [5, Corollaries 3.3 and 3.4]). Finally in [4], the trio Aleman-Richter-Sundberg settled this problem affirmatively.

2000 Mathematics Subject Classification. Primary 47B37; Secondary 47A15, 05C20.

Key words and phrases. wandering subspace property, Wold-type decomposition, weighted shift, one-circuit directed graphs.

The work of the third author is supported through the Inspire Faculty Fellowship DST/INSPIRE/04/2018/000338.
Later in the influential paper [26], Shimorin not only obtained an alternate proof of their theorem but at the same time developed an axiomatic approach to the wandering subspace problem (see, for instance, [16 Section 6.3]).

All the Hilbert spaces occurring below are complex, infinite-dimensional and separable. Let $\mathcal{H}$ denote a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the unital $C^*$-algebra of bounded linear operators on $\mathcal{H}$. The Hilbert space adjoint of $T \in \mathcal{B}(\mathcal{H})$ is denoted by $T^*$. The kernel of $T$ is denoted by $\ker T$, whereas the range of $T$ is denoted by $\text{ran} T$. An operator $T \in \mathcal{B}(\mathcal{H})$ is left-invertible if there exists $L \in \mathcal{B}(\mathcal{H})$ (a left-inverse) such that $LT = I$, where $I$ denotes the identity operator on $\mathcal{H}$. If $T$ is left-invertible, then $T^*T$ is invertible. This fact provides a canonical choice of left-inverse $L := T^*$ for any left-invertible operator $T$, where $T' := T(T^*T)^{-1}$. The operator $T'$ is referred to as the Cauchy dual of $T$, a notion coined and studied by Shimorin [26]. For an operator $T \in \mathcal{B}(\mathcal{H})$, the hyper-range of $T$ is given by

$$T^\infty(\mathcal{H}) := \bigcup_{n=0}^{\infty} T^n \mathcal{H}.$$  

If $T$ is left-invertible, then $T^\infty(\mathcal{H})$ is a closed subspace of $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is analytic if $T^\infty(\mathcal{H}) = \{0\}$. Note that an analytic operator on a non-zero Hilbert space is never invertible. We say that $T \in \mathcal{B}(\mathcal{H})$ admits the wandering subspace property if $[\ker T^*]_T = \mathcal{H}$, where

$$[\ker T^*]_T := \bigvee \{T^n h : n \in \mathbb{N}, h \in \ker T^*\},$$  

(1.1)

where $\mathbb{N}$ denotes the set of non-negative integers. Here $\ker T^*$ is the wandering subspace in the sense that

$$\ker T^* \perp T^n(\ker T^*), \quad n \geq 1.$$  

Following [26], we say that a left-invertible operator $T \in \mathcal{B}(\mathcal{H})$ admits Wold-type decomposition if

$$T = U \oplus A \text{ on } \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_a,$$  

where $\mathcal{H}_u$ is the hyper-range of $T$, $U$ is unitary on $\mathcal{H}_u$, $A$ is analytic on $\mathcal{H}_a$, and $A$ admits the wandering subspace property. It turns out that for a left-invertible operator $T \in \mathcal{B}(\mathcal{H})$, $T$ is analytic if and only if the Cauchy dual $T'$ of $T$ admits wandering subspace property (see [26 Proposition 2.7]). In particular, $T$ admits Wold-type decomposition if and only if $T'$ admits Wold-type decomposition.

Given a positive integer $m$ and $T \in \mathcal{B}(\mathcal{H})$, set

$$B_m(T) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k}T^k.$$  

We say that an operator $T$ is norm-increasing or expansive (resp. isometry) if $B_1(T) \leq 0$ (resp. $B_1(T) = 0$). For $m \geq 2$, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an $m$-isometry (resp. $m$-concave) if $B_m(T) = 0$ (resp. $(-1)^m B_m(T) \leq 0$). A 2-concave operator is usually known as concave operator. It turns out that the approximate point spectrum of any $m$-isometric operator is contained in the unit circle and hence its spectrum is contained in the closed unit disc (see [2 Lemma 1.21]). For the basic theory of $m$-isometries, the reader is referred to [2].
Remark 1.1. Let $T \in \mathcal{B}(\mathcal{H})$ be an $m$-concave operator. It follows from the definition that the approximate point spectrum of $T$ is contained in the unit circle. Since the approximate point spectrum contains the boundary of the spectrum, the spectrum of $T$ is contained in the closed unit disc. In particular, the Cauchy dual operator of $T$ is well-defined.

This paper is partly motivated by the following problem addressed by Shimorin in [26, Pg 185]:

Question 1.2. If $T \in \mathcal{B}(\mathcal{H})$ is a norm-increasing $m$-isometry (or norm-increasing $m$-concave), then whether $T$ admits Wold-type decomposition?

The above question has affirmative answer in cases $m = 1$ (isometries) [6, 21, 15] and $m = 2$ (concave operators) [24] (cf. [23, Corollary 2.1]). In case $m > 2$, the best known result till date, due to Shimorin, is as follows (cf. [25, Corollary 1.6]):

Theorem 1.3. [26, Theorem 3.8] Assume that $T \in \mathcal{B}(\mathcal{H})$ is norm-increasing and satisfies the inequality

$$T^* T^2 - 3T^* T + 3I - T^* T' - P_{\text{ker}T^*} \leq 0,$$  \hspace{1cm} (1.2)

where $P_{\text{ker}T^*}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\ker T^*$. Then $T$ is a $3$-concave operator, which admits Wold-type decomposition.

Unlike the condition of $3$-concavity, the condition (1.2) is not stable with respect to the process of taking the restriction to an invariant subspace. This is one of the obstructions in deriving the wandering subspace property for norm-increasing operators satisfying (1.2). Further, as pointed out in [26], the main difficulty in Question 1.2 is whether the analyticity implies the wandering subspace property. To see this deduction, note that $\mathcal{H}_u := T^\infty(\mathcal{H})$ is $T$-invariant and $T|_{\mathcal{H}_u}$ is invertible. Since the restriction of a norm-increasing $m$-isometry to an invariant subspace is again a norm-increasing $m$-isometry, by [26, Proposition 3.4], $T|_{\mathcal{H}_u}$ is unitary and $\mathcal{H}_u$ is reducing for $T$. The assumption that $T$ is norm-increasing in Question 1.2 is essential since there exist invertible (and hence non-analytic) cyclic $3$-isometries on an infinite-dimensional Hilbert space, which are not unitary (refer to [2, Section 4]). However, we are not aware of any known example of an analytic $m$-isometry, which does not admit the wandering subspace property. One of the purposes of this paper is to settle the wandering subspace problem for analytic $m$-isometries in the negative. This is achieved by constructing a family of analytic cyclic $3$-isometries, which do not admit the wandering subspace property (the reader is referred to [19, Corollary 1], [26, Pg 186], [17, Proposition 2], [11, Corollary 6.2], [10, Theorem 2.2], [22, Corollary 4.1], [25, Theorem 1.5] for a variety of results pertaining to the phenomenon of failure of wandering subspace property in Hardy, Bergman and Dirichlet-type spaces). Further, these $3$-isometries are norm-increasing except on a one-dimensional space, where their norms can be made arbitrarily close to 1. Needless to say, these examples shed light on the role of the expansivity property in Question 1.2. Our construction capitalizes on the idea of weighted shift operator on directed graph recently boosted in the realm of graph-theoretic operator theory (refer to [18, 12, 8, 13]).
Here is the outline of the paper. In Section 2, we present some preparatory results including a characterization of a class of analytic weighted shifts on a locally finite one-circuit directed graph (see Theorem 2.1). Section 3 includes the construction of analytic 3-isometries without the wandering subspace property (see Example 3.1). In Section 4, we prove that for analytic norm-increasing m-concave operators, either the wandering subspace property holds or it fails miserably. This result applies to the class of analytic weighted shifts on a locally finite one-circuit directed graph as discussed in Section 2. In the remaining part, we collect preliminaries related to weighted shift on directed graphs mostly from [18] and [8].

1.1. Weighted shifts on one-circuit directed graphs. The notion of adjacency operator on a directed graph first appeared in [14], while that of weighted shift on a (one-circuit) directed graph, central to the present investigations, has been formulated and investigated in [8] (refer to [8], Section 3) for its connection with weighted composition operators on discrete measure spaces. The reader is referred to [18], Chapter 2] for an excellent exposition on directed graphs and all relevant notions including weighted shift on directed trees (see, in particular, the discussion on the parent \( \text{par}(\cdot) \) as a partial function). It is worth mentioning that \( \text{par}(\cdot) \) becomes a function in case the underlying directed graph is a one-circuit directed graph.

**Definition 1.4.** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root \( \text{root} \). For \( l \in \mathbb{N} \), let \( C_l := \emptyset \) if \( l = 0 \) and \( C_l := \{w_1, \ldots, w_l\} \) otherwise. A one-circuit directed graph associated with the rooted directed tree \( \mathcal{T} = (V, E) \) is a directed graph \( \mathcal{G} = (W, F) \) given by

\[
W := V \sqcup C_l, \\
F := \begin{cases} 
E \cup \{(\text{root, root})\} & \text{if } l = 0, \\
E \cup \{(\text{root, } w_1), (w_1, w_2), \ldots, (w_{l-1}, w_l), (w_l, \text{root})\} & \text{otherwise.}
\end{cases}
\]

![Figure 1](image-url)  

**Figure 1.** A one-circuit directed graph with one branching vertex

We refer to \( l \) as the length of the circuit. For a subset \( \{\lambda_w : w \in W\} \) of \( \mathbb{C} \), consider the weight system \( \Lambda : W \to \mathbb{C} \) given by \( \lambda(w) := \lambda_w, w \in W \). The weighted shift operator \( S_{\lambda, \mathcal{G}} \) on \( \mathcal{G} \) is defined by

\[
\mathcal{D}(S_{\lambda, \mathcal{G}}) := \{f \in l^2(W) : A_\mathcal{G} f \in l^2(W)\}, \\
S_{\lambda, \mathcal{G}} f := \Lambda g f, \quad f \in \mathcal{D}(S_{\lambda, \mathcal{G}}),
\]

where \( A_\mathcal{G} \) is the mapping defined on complex functions \( f \) on \( V \) by

\[
(A_\mathcal{G} f)(w) := \lambda_w \cdot f(\text{par}(w)), \quad w \in W.
\]
Remark 1.5. Assume that $S_{\lambda, G}$ belongs to $\mathcal{B}(\ell^2(W))$. It is well-known that the above notion is closely related to the notions of weighted shift on rooted directed trees and composition operator on discrete measure spaces (refer to [14, 18, 8 Theorem 3.2.1]). This is summarized as follows:

(i) Consider the weighted shift operator $S_{\lambda, G}$ with weight system $\lambda|_{V \setminus \{\text{root}\}}$ on the rooted directed tree $G$. Note that $\ell^2(V \setminus \{\text{root}\})$ is an invariant subspace for $S_{\lambda, G}$ and

$$S_{\lambda, G}|_{\ell^2(V \setminus \{\text{root}\})} = S_{\lambda, G}|_{\ell^2(V \setminus \{\text{root}\})},$$

$$S_{\lambda, G}e_{\text{root}} = \begin{cases} S_{\lambda, G}e_{\text{root}} + \lambda e_{\text{root}} & \text{if } l = 0, \\ S_{\lambda, G}e_{\text{root}} + \lambda w_1 e_{w_1} & \text{otherwise}. \end{cases}$$

(ii) Consider the map $\phi : W \to W$ given by

$$\phi(w) = \text{par}(w), \quad w \in W.$$ 

Then $S_{\lambda, G}$ can be identified with the weighted composition operator $C_{\phi, \lambda}$ given by

$$(C_{\phi, \lambda}f)(w) = \lambda_w f(\phi(w)), \quad f \in \ell^2(W), \quad w \in W.$$ 

Let $G = (W, F)$ be a directed graph and let $U$ be a subset of $W$. Set

$$\text{Ch}_G(U) := \bigcup_{u \in U} \{w \in W : (u, w) \in F\}.$$ 

We define inductively $\text{Ch}_G^{(n)}(U)$ for $n \in \mathbb{N}$ as follows:

$$\text{Ch}_G^{(n)}(U) := \begin{cases} U & \text{if } n = 0, \\ \text{Ch}_G(\text{Ch}_G^{(n-1)}(U)) & \text{if } n \geq 1. \end{cases}$$

Given $w \in W$, we set $\text{Ch}_G(w) := \text{Ch}_G(\{w\})$.

Assumption. Let $G = (W, F)$ be a one-circuit directed graph associated with the rooted directed tree. Unless stated otherwise, $G$ is assumed to be countably infinite (that is, $W$ is countably infinite) and leafless (that is, $\text{Ch}_G(w) \neq \emptyset$ for every $w \in W$). Further, we assume that $\lambda$ consists of positive numbers and $S_{\lambda, G}$ belongs to $\mathcal{B}(\ell^2(W))$.

In what follows, we need the following elementary fact in the sequel.

Lemma 1.6. Let $l \in \mathbb{N}$ and let $G = (W, F)$ be a one-circuit directed graph associated with the rooted directed tree $G = (V, E)$ and the circuit $\{w_1, \ldots, w_{l+1}\}$, where $w_{l+1} := \text{root}$. Then $W$ can be partitioned as follows:

$$W = \left( \bigcup_{k=1}^{l+1} \text{Ch}_G^{(k)}(\text{root}) \right) \sqcup \left( \bigcup_{k=1}^{l+1} \bigcup_{m=1}^{\infty} \text{Ch}_G^{(l+1)\text{root}}(\text{root}) \right),$$

where $\sqcup$ denotes the disjoint sum.

Proof. Clearly, the right hand side of (2.1) is contained in $W$. To see the inclusion other way round, first note that $W = V \sqcup \{w_1, \ldots, w_l\}$ and

$$\text{Ch}_G^{(k)}(\text{root}) = \text{Ch}_G^{(k)}(\text{root}) \sqcup \{w_k\}, \quad k = 1, \ldots, l + 1. \quad (1.4)$$

Let $w \in W$. Then either $w \in V \setminus \{\text{root}\}$ or $w = w_j$ for some $j = 1, \ldots, l + 1$. If $w = w_j$ for some $j = 1, \ldots, l + 1$, then, as noted above, $w \in \text{Ch}_G^{(j)}(\text{root})$. 

Let \( w \in V \setminus \{ \text{root} \} \). Then by [18, Corollary 2.1.5], there exists a positive integer \( n \) such that \( w \in \text{Chi}^{(n)}_{\mathcal{G}}(\text{root}) \). By division algorithm, there exists \( m \in \mathbb{N} \) such that \( n = m(l+1) + k \) for some \( k = 0, \ldots, l \). It is now easy to see that \( w \) belongs to the right hand side of (1.3).

\[ \square \]

### 2. Analyticity of weighted shifts on one-circuit directed graphs

The main result of this section characterizes a class of left-invertible analytic weighted shifts on a locally finite one-circuit directed graph (see Theorem 2.1 and Remark 2.2). Unlike weighted shifts on rooted directed trees (see [12, Lemma 3.3]), these shifts need not be analytic. Indeed, they may admit non-zero eigenvalues (cf. [9, Theorem 2.1]).

**Theorem 2.1.** Let \( l \in \mathbb{N} \) and let \( \mathcal{G} = (W, F) \) be a one-circuit directed graph associated with the locally finite rooted directed tree \( \mathcal{T} = (V, E) \) with root \( \text{root} \) and the circuit \( \{ w_1, \ldots, w_{l+1} \} \), where \( w_{l+1} := \text{root} \). Let \( S_{\mathcal{G}} \) be a weighted shift on \( \mathcal{G} \) such that \( \inf \lambda > 0 \) and let

\[
\lambda^{(k)} := \lambda(\lambda \circ \text{par}) \cdots (\lambda \circ \text{par}^{k-1}), \quad k \geq 1.
\]

Then \( S_{\mathcal{G}} \) is analytic if and only if for each \( k = 1, \ldots, l+1 \), the following series diverges:

\[
\sum_{v \in \text{Chi}^{(k)}_{\mathcal{G}}(\text{root})} \left( \frac{\lambda^{(k)}(v)}{\lambda^{(k)}(w_k)} \right)^2 + \sum_{m=1}^{\infty} \sum_{v \in \text{Chi}^{((l+1)m+k)}_{\mathcal{G}}(\text{root})} \left( \frac{\lambda^{((l+1)m+k)}(v)}{\lambda^{((l+1)m+k)}(w_k)} \right)^2.
\]

**Remark 2.2.** Note that the set \( \{ S_{\mathcal{G}}e_w \}_{w \in W} \) is orthogonal in \( \ell^2(W) \). It is now easy to see that the assumption \( \inf \lambda > 0 \) implies that \( S_{\mathcal{G}} \) is left-invertible, and hence the hyper-range of \( S_{\mathcal{G}} \) is closed in \( \ell^2(W) \). Finally, note that since \( \mathcal{G} \) is locally finite, the first sum in (2.2) is finite.

In the proof of Theorem 2.1, we need a couple of lemmas. We begin with the following variant of a known fact pertaining to weighted composition operators (see [20, Theorem 2.1.1] and [7, Remark 45]).

**Lemma 2.3.** Let \( \mathcal{G} = (W, F) \) be a one-circuit directed graph associated with a locally finite rooted directed tree. Let \( S_{\mathcal{G}} \) be a weighted shift on \( \mathcal{G} \) such that \( \inf \lambda > 0 \). Then, for any positive integer \( k \), the range of \( S_{\mathcal{G}}^k \) is given by

\[
\left\{ f \in \ell^2(W) : \frac{f}{\lambda^{(k)}} \big|_{\text{Chi}^{(k)}_{\mathcal{G}}(w)} \text{ is constant for each } w \in W \right\},
\]

where \( \lambda^{(k)} \) is given by (2.1).

**Proof.** Suppose that \( f \in \text{ran } S_{\mathcal{G}}^k \). Thus there exists \( g \in \ell^2(W) \) such that \( f(w) = \lambda^{(k)}(w)g(\text{par}^k(w)), \ w \in W \). Let \( v \in W \) be fixed and let \( u, w \in \text{Chi}^{(k)}_{\mathcal{G}}(v) \). Then

\[
\frac{f(u)}{\lambda^{(k)}(u)} = g(\text{par}^k(u)) = g(v) = g(\text{par}^k(w)) = \frac{f(w)}{\lambda^{(k)}(w)}.
\]
Thus \( \frac{f}{\lambda^{(k)}} |_{\text{Chi}^{(k)}_{g}(w)} \) is constant. Conversely, suppose that \( f \in \ell^{2}(W) \) is such that \( \frac{f}{\lambda^{(k)}} |_{\text{Chi}^{(k)}_{g}(w)} \) is constant for each \( w \in W \). This allows us to define \( g : W \rightarrow \mathbb{C} \) by

\[
g(w) = \frac{f(v)}{\lambda^{(k)}(v)} , \quad v \in \text{Chi}^{(k)}_{g}(w).
\]

Since \( \inf \lambda > 0 \) and \( f \in \ell^{2}(W) \), \( g \in \ell^{2}(W) \). Further,

\[
(S^{k}_{\lambda, g}g)(w) = \lambda^{(k)}(w)g(\text{par}^{k}(w)) = f(w), \quad w \in W.
\]

Thus \( f \in \text{ran} S^{k}_{\lambda, g} \) and the proof is over. \( \square \)

We next analyze the hyper-range of weighted shifts on a one-circuit directed graph associated with the locally finite rooted directed tree.

**Lemma 2.4.** Let \( l \in \mathbb{N} \) and let \( \mathcal{G} = (W, F) \) be a one-circuit directed graph associated with the locally finite rooted directed tree \( \mathcal{T} = (V, E) \) with root \( \text{root} \) and the circuit \( \{w_{1}, \ldots, w_{l+1}\} \), where \( w_{l+1} = \text{root} \). Let \( S_{\lambda, g} \) be a weighted shift on \( \mathcal{G} \) such that \( \inf \lambda > 0 \) and let

\[
g_{k} := \sum_{w \in \text{Chi}^{(k)}_{g}(\text{root})} \frac{\lambda^{(k)}(w)}{\lambda^{(k)}(w)} e_{w} + \sum_{m=1}^{\infty} \sum_{w \in \text{Chi}^{(l+1)m+k}_{g}(\text{root})} \frac{\lambda^{(l+1)m+k}(w)}{\lambda^{(l+1)m+k}(w_{k})} e_{w},
\]

\( k = 1, \ldots, l + 1. \) \hspace{1cm} (2.3)

Then the following statements hold:

(i) The hyper-range of \( S_{\lambda, g} \) is spanned by \( h_{k} : W \rightarrow [0, \infty) \), \( k = 1, \ldots, l + 1 \), given by

\[
h_{k} := \begin{cases} g_{k} & \text{if the series in (2.2) converges}, \\ 0 & \text{otherwise}. \end{cases}
\]

(ii) The hyper-range of \( S_{\lambda, g} \) is given by

\[
\left\{ f \in \ell^{2}(W) : \left. \frac{f}{\lambda^{(k)}} \right|_{\text{Chi}^{(k)}_{g}(\text{root})} \text{ is constant for each } k \geq 1 \right\}.
\]

**Proof.** Let \( f \) belong to the hyper-range of \( S_{\lambda, g} \). By (1.3), \( f \) can be decomposed as

\[
f = \sum_{k=1}^{l+1} \sum_{w \in \text{Chi}^{(k)}_{g}(\text{root})} f(w) e_{w} + \sum_{k=1}^{l+1} \sum_{m=1}^{\infty} \sum_{w \in \text{Chi}^{(l+1)m+k}_{g}(\text{root})} f(w) e_{w}, \hspace{1cm} (2.4)
\]

Further, Lemma 2.3 combined with the fact that \( w_{k} \in \text{Chi}^{(l+1)m+k}_{g}(\text{root}) \) yields the following identities:

\[
f(w) = f(w_{k}) \frac{\lambda^{(l+1)m+k}(w)}{\lambda^{(l+1)m+k}(w_{k})}, \quad w \in \text{Chi}^{(l+1)m+k}_{g}(\text{root}),
\]

\( m \in \mathbb{N}, \ k = 1, \ldots, l + 1. \)

Substituting this into (2.4), we obtain \( f = \sum_{k=1}^{l+1} f(w_{k}) g_{k} \), where \( g_{k}, k = 1, \ldots, l + 1, \) is given by (2.3). Since \( f \in \ell^{2}(W) \) and supports of \( g_{1}, \ldots, g_{l+1} \) are disjoint, if \( g_{k} \notin \ell^{2}(W) \) for some \( k = 1, \ldots, l + 1, \) then \( f(w_{k}) = 0 \). This completes the proof of (i).
To see (ii), note that by Lemma 2.3, \( f \) is in the hyper-range of \( S_{\lambda,g} \) if and only if
\[
\frac{f}{\lambda^{(k)}(w)} \big|_{\text{Chi}_{\lambda,g}^{(k)}(w)} \text{ is constant for each } w \in W \text{ and } k \geq 1. \tag{2.5}
\]

We claim that (2.5) is equivalent to
\[
\frac{f}{\lambda^{(k)}(\text{root})} \big|_{\text{Chi}_{\lambda,g}^{(k)}(\text{root})} \text{ is constant for each } k \geq 1. \tag{2.6}
\]
Clearly, (2.5) implies (2.6) by taking \( w = \text{root} \). Suppose that (2.6) holds. Let \( w \in W \). By (1.3), there exists \( n \in \mathbb{N} \) such that \( w \in \text{Chi}_{\lambda,g}^{(n)}(\text{root}) \). Now for any \( k \in \mathbb{N} \),
\[
\text{Chi}_{\lambda,g}^{(k)}(w) \subseteq \text{Chi}_{\lambda,g}^{(n+k)}(\text{root}).
\]

Let \( u, v \in \text{Chi}_{\lambda,g}^{(k)}(w) \). Then
\[
\frac{f(u)}{\lambda^{(k)}(u)} \quad \overset{2.4}{=} \quad \frac{f(u)}{\lambda^{(n+k)}(u)}(\lambda_w \cdots \lambda_{par^{n-1}(w)}) \quad \overset{2.5}{=} \quad \frac{f(v)}{\lambda^{(n+k)}(v)}(\lambda_w \cdots \lambda_{par^{n-1}(v)}) \quad = \quad \frac{f(v)}{\lambda^{(k)}(v)}.
\]
This establishes (2.6) and hence completes the proof. \( \square \)

We now complete the proof of Theorem 2.1

**Proof of Theorem 2.1.** If, for each \( k = 1, \ldots, l+1 \), the series in (2.2) diverges, then by Lemma 2.4(i), \( S_{\lambda,g} \) is analytic. Conversely, suppose that for some \( k = 1, \ldots, l+1 \), the series in (2.2) converges. In particular, \( g_k \), as given by (2.3), belongs to \( \ell^2(W) \). We check that \( g_k \) belongs to the hyper-range of \( S_{\lambda,g} \). In view of Lemma 2.4(ii), we only need to verify that \( f := g_k \) satisfies (2.6). To see that, let \( n \geq 1 \) be an integer and write \( n = (l+1)p + r \) for some \( p \in \mathbb{N} \) and \( 0 \leq r < l+1 \). It now follows from (1.4) that
\[
\text{Chi}_{\lambda,g}^{(n)}(\text{root}) = \begin{cases} \text{Chi}_{\lambda,g}^{(n)}(\text{root}) \cup \{w_r\} & \text{if } r > 0, \\ \text{Chi}_{\lambda,g}^{(n)}(\text{root}) \cup \{\text{root}\} & \text{otherwise}. \end{cases}
\]
Thus by (2.3), we have
\[
\frac{g_k}{\lambda^{(n)}(w_k)} \big|_{\text{Chi}_{\lambda,g}^{(n)}(\text{root})} = \begin{cases} \frac{1}{\lambda^{(n)}(w_k)} & \text{if } r = k, \\ \frac{1}{\lambda^{(n)}(\text{root})} & \text{if } r = 0 \text{ and } k = l + 1, \\ 0 & \text{otherwise}. \end{cases}
\]
This completes the proof of the theorem. \( \square \)
3. Analytic 3-isometries without wandering subspace property

We now construct a family of cyclic analytic 3-isometries, which do not admit the wandering subspace property and which are norm-increasing on the orthogonal complement of a one-dimensional space.

**Example 3.1.** Consider the directed tree $\mathcal{T}$ with the set of vertices $V := \mathbb{N}$ and root $= 0$. We further require that $\operatorname{Chi}_{\mathcal{T}}(n) = \{ n+1 \}$ for all $n \in \mathbb{N}$ (see (6.2.10)). Let $\mathcal{G} = (W, F)$ be the following one-circuit directed graph associated with the rooted directed tree $\mathcal{T}$:

For positive numbers $a, b$, consider the degree 2 polynomial $p_{a,b} : \mathbb{N} \to (0, \infty)$ given by $p_{a,b}(n) = 1 + an + bn^2$, $n \in \mathbb{N}$. By the Archimedean property, there exists $\epsilon_0 \in (0, 1)$ such that $\epsilon + b(2/\epsilon - 1) > a$ for every $\epsilon \in (0, \epsilon_0)$. For $\epsilon \in (0, \epsilon_0)$, let $\lambda_w = \{ \lambda_w : w \in W \} = \{ \lambda_n : n \in \mathbb{N} \}$ denote the weight system of positive real numbers given by

$$
\begin{align*}
\lambda_0 := \sqrt{1 - \epsilon}, & \quad \lambda_1 := \sqrt{\frac{\epsilon}{\lambda_0}} \quad \text{with} \quad K_\epsilon := \epsilon^2 - \epsilon(a + b) + 2b > 0, \\
\lambda_n := \sqrt{\frac{p_{a,b}(n-1)}{p_{a,b}(n-2)}}, & \quad n \geq 2.
\end{align*}
$$

(3.1)

Let $S_{\lambda_w, \mathcal{G}}$ (resp. $S_{\lambda_w, \mathcal{T}}$) be a weighted shift on $\mathcal{G}$ (resp. $\mathcal{T}$). An inductive argument shows that $e_w$ belongs to the linear span of $\{ S_{\lambda_n, \mathcal{G}} e_0 : n \in \mathbb{N} \}$ for every $w \in W$, and hence $S_{\lambda_w, \mathcal{G}}$ is cyclic with cyclic vector $e_0$:

$$
\bigvee \{ S_{\lambda_n, \mathcal{G}} e_0 : n \in \mathbb{N} \} = \ell^2(W).
$$

By (3.1) and Remark 2.2, $S_{\lambda_w, \mathcal{G}} \in \mathcal{B}(\ell^2(W))$ is left-invertible. Note that Cauchy dual operator $S_{\lambda_w, \mathcal{G}}'$ is the weighted shift $S_{\lambda_w, \mathcal{G}}$ on $\mathcal{G}$ with weight system $\lambda'_w = \{ \lambda'_w : w \in W \}$ given by

$$
\begin{align*}
\lambda'_0 = \frac{\lambda_0}{\lambda^2_0 + \lambda^2_1}, & \quad \lambda'_1 = \frac{\lambda_1}{\lambda_0 + \lambda_1^2}, & \quad \lambda'_n = \sqrt{\frac{p_{a,b}(n-2)}{p_{a,b}(n-1)}}, \quad n \geq 2.
\end{align*}
$$

(3.2)

Then we have the following statements:

(i) $S_{\lambda_w, \mathcal{G}}$ is an analytic 3-isometry.

(ii) $S_{\lambda_w, \mathcal{G}}$ is norm-increasing if and only if $K_\epsilon \leq \epsilon^2$.

(iii) $S_{\lambda_w, \mathcal{G}}$ admits the wandering subspace property if and only if

$$
K_\epsilon < \frac{\epsilon^3}{\sqrt{1 - \epsilon(1 - \sqrt{1 - \epsilon})}}.
$$

(3.3)

(iv) $S_{\lambda_w, \mathcal{G}}'$ admits Wold-type decomposition if and only if (3.3) holds.

We verify the above statements as follows. Since $\{ p_{a,b}(n) \}_{n \in \mathbb{N}}$ is an increasing sequence, $S_{\lambda_w, \mathcal{T}}$ is a norm-increasing weighted shift on $\mathcal{T}$. Further, since $\| S_{\lambda_n, \mathcal{T}} e_1 \|^2 = p_{a,b}(k)$, $k \in \mathbb{N}$, is a degree 2 polynomial, by [1] Theorem 2.1, $S_{\lambda_n, \mathcal{T}}$ is a 3-isometry. Since for any positive integer $k$, the sequence
It is easy to see that (3.4) is immediate from (3.1). Thus $S$ is analytic. This completes verifications of (i) and (ii). The above argument only if 

\[
K_\epsilon \leq \epsilon^2, \\
1 - 3\lambda_0^2 - 3\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_0^2(3 - 3\lambda_0^2 - 3\lambda_1^2 + \lambda_2^2 + \lambda_3^2). 
\]

(3.4)

It is easy to see that (3.4) is immediate from (3.1). Thus $S_{\lambda, g}$ is a 3-isometry, which is norm-increasing if and only if $K_\epsilon \leq \epsilon^2$. On the other hand, by Theorem 2.1, $S_{\lambda, g}$ is analytic if and only if

\[
1 + \left(\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\right)^2 + \sum_{m=1}^{\infty} \left(\frac{\lambda_m^{(m+1)}(m+1)}{\lambda_m^{(m)}(0)}\right)^2 = \infty.
\]

A routine inductive argument using (2.1) shows however that for $m \in \mathbb{N},$

\[
\lambda_m^{(m+1)}(m+1) = \prod_{j=1}^{m+1} \lambda_j \quad \text{and} \quad \lambda_m^{(m+1)}(0) = \lambda_0^{m+1}.
\]

Thus $p_{a,b}(m) \geq 1$ for every $m \in \mathbb{N}$, we have

\[
\sum_{m=0}^{\infty} \left(\frac{\lambda_m^{(m+1)}(m+1)}{\lambda_m^{(m+1)}(0)}\right)^2 = \sum_{m=0}^{\infty} \frac{\lambda_1 p_{a,b}(m)}{\lambda_0^{2(m+1)}} \geq \lambda_1 \sum_{m=0}^{\infty} \frac{1}{\lambda_0^{2(m+1)}}.
\]

Also, since $\lambda_0 < 1$, the series on the right hand side diverges, and hence $S_{\lambda, g}$ is analytic. This completes verifications of (i) and (ii). The above argument applied to the Cauchy dual operator $S_{\lambda, g}$ together with (3.2) yields that $S_{\lambda, g}$ is analytic if and only if the following series diverges:

\[
\sum_{m=0}^{\infty} \left(\frac{\lambda_0^2 + \lambda_1^2}{\lambda_0^2 + \lambda_1^2}\right)^{2(m+1)} \lambda_0^{2(m+1)} p_{a,b}(m) = \sum_{m=0}^{\infty} \frac{(\lambda_0^2 + \lambda_1^2)^{2(m+1)}}{\lambda_0^{2(m+1)}} \lambda_0^{2(m+1)} p_{a,b}(m).
\]

Since $p_{a,b}$ is a degree 2 polynomial with positive coefficients, the above series diverges if and only if $\lambda_0^2 + \lambda_1^2 > \lambda_0$. On the other hand, by (3.1), we obtain

\[
\lambda_0^2 + \lambda_1^2 > \lambda_0 \iff 1 - \epsilon + \frac{\epsilon^3}{K_\epsilon} > \sqrt{1 - \epsilon} \iff (3.3) \quad \text{holds.}
\]

Thus $S_{\lambda, g}$ is analytic if and only (3.3) holds. However, by Proposition 3.4, $S_{\lambda, g}$ admits the wandering subspace property if and only if $S_{\lambda, g}$ is analytic. This yields (iii). To see (iv), recall that $S_{\lambda, g}$ admits Wold-type decomposition if and only if $S_{\lambda, g}$ admits Wold-type decomposition (see Proposition 3.4). Further, by (i) and (iii), $S_{\lambda, g}$ admits Wold-type decomposition if and only if (3.3) holds. Combining last two observations,
we obtain (iv). It is interesting to note that in case (3.3) does not hold, then the hyper-range of $S_{\lambda, \mathcal{G}}$ is spanned by
\[ g = e_0 + \sum_{m=1}^{\infty} \frac{\lambda^{(m)}_{\epsilon}(m)}{\lambda^{(m)}_{\epsilon}(0)} e_m. \]
In particular, the space spanned by $g$ is not invariant under $S^*_{\lambda, \mathcal{G}}$ (since $\lambda'_{\epsilon}$ is not eventually constant in view of (3.2)).

Let us discuss some consequences of (i)-(iv). Firstly, since $0 < \epsilon < 1$, we obtain $\frac{\epsilon^2}{\sqrt{1 - \epsilon(1 - \sqrt{1 - \epsilon})}}$, and hence it follows from (ii) and (iii) that if $S_{\lambda, \mathcal{G}}$ is norm-increasing, then $S_{\lambda, \mathcal{G}}$ necessarily admits the wandering subspace property. Secondly, since
\[ \lim_{\epsilon \to 0^+} K_{\epsilon} = 2b > 0, \quad \lim_{\epsilon \to 0^+} \frac{\epsilon^3}{\sqrt{1 - \epsilon(1 - \sqrt{1 - \epsilon})}} = 0, \]
we may conclude from (i) and (iii) that there exists $\epsilon_1 \in (0, \epsilon_0)$ such that $S_{\lambda, \mathcal{G}}$ is an analytic 3-isometry without the wandering subspace property for every $\epsilon \in (0, \epsilon_1)$. It is worth noting that $S_{\lambda, \mathcal{G}}$ is always norm-increasing on the orthogonal complement of the space spanned by $e_0$. Although $S_{\lambda, \mathcal{G}}$ is not norm-increasing, we have
\[ \|S_{\lambda, \mathcal{G}} e_0\|^2 = 1 - \epsilon + \frac{\epsilon^3}{K_{\epsilon}} \to 1^- \text{ as } \epsilon \to 0^+. \]
In particular, Question 1.2 has a negative answer if we replace the assumption $\|S_{\lambda, \mathcal{G}} e_0\| \geq 1$ by $\|S_{\lambda, \mathcal{G}} e_0\| \geq 1 - \delta$ for arbitrarily small $\delta > 0$.

**Remark 3.2.** Let $l \in \mathbb{N}$ and let $\mathcal{G} = (W, F)$ be a one-circuit directed graph associated with the rooted directed tree (as in the preceding example) and the circuit $\{w_1, \ldots, w_{l+1}\}$, where $w_{l+1} := \text{root}$ is the only branching vertex of $\mathcal{G}$. It is not difficult to see that there are no norm-increasing analytic 3-isometry weighted shifts $S_{\lambda, \mathcal{G}}$ without the wandering subspace property. Indeed, if
\[ \lambda_{w_j} \geq 1, \quad j = 2, \ldots, l+1, \quad \lambda_{w_1}^2 + \lambda_1^2 \geq 1, \quad \frac{\lambda_{w_1}}{\lambda_{w_1}^2 + \lambda_1^2} > \prod_{j=2}^{l+1} \lambda_{w_j} \]
(with the convention that product over an empty set is 1), then $\lambda_{w_1} \geq 1$, and hence $\lambda_{w_1} \geq \lambda_{w_1}^2 + \lambda_1^2$, which is not possible.

4. A dichotomy

As noted above, if $S_{\lambda, \mathcal{G}}$ is a norm-increasing analytic 3-isometry weighted shift on a one-circuit directed graph associated with a rooted directed tree having one branching vertex, then it must admit the wandering subspace property. In view of this, it is tempting to explore the wandering subspace problem for the class of weighted shifts on a one-circuit directed graph associated with a rooted directed tree having more than one branching vertex. However, it turns out that by incorporating more than one branching vertex in the underlying rooted directed tree does not yield an answer to Shimorin’s question. Indeed, we have the following general fact.
Lemma 4.1. Let \( T \in \mathcal{B}(\mathcal{H}) \) be an analytic norm-increasing operator with spectrum contained in the closed unit disc and let \([\ker T^*]_T\) be the \( T \)-invariant closed subspace generated by \( \ker T^* \) (see (1.1)). Then \( \mathcal{H} \oplus [\ker T^*]_T \) is either zero or infinite-dimensional.

Proof. Assume that \( \mathcal{H} \oplus [\ker T^*]_T \) is finite-dimensional. Note that the hyper-range \( \mathcal{H}'_u \) of \( T' \) is a closed invariant subspace for \( T' \) and let \( S := T'|_{\mathcal{H}'_u} \). By [20] Proposition 3.4, \( \mathcal{H}'_u = \mathcal{H} \oplus [\ker T^*]_T \), which is finite-dimensional. It now suffices to check that \( \mathcal{H}'_u = \{0\} \). Assume that \( \mathcal{H}'_u \) is non-zero. Thus \( S \) is an invertible finite-dimensional operator on \( \mathcal{H}'_u \), and hence there exists a non-zero \( g \in \mathcal{H}'_u \) such that \( Sg = \lambda g \) for some \( \lambda \in \mathbb{C} \setminus \{0\} \). It follows that \( T'g = \lambda g \). However, \( T' \) is a contraction (since \( T \) is an expansion), and hence \( |\lambda| \leq 1 \). As \( T^*T' = I \), we must have \( \lambda T^*g = g \). Since the spectrum of \( T \) is contained in the closed unit disc, \( |\lambda| = 1 \). Recall the fact that the fixed points of a contraction and its adjoint are same (see [27] Proposition 3.1). Applying this fact to the contractive operator \( \lambda T' \) shows that \( T^*g = \overline{\lambda}g \). It follows that \( T^*T'g = g \). Now observe that
\[
Tg = T'(T^*T')^{-1}g = T'g = \lambda g.
\]
It is immediate that \( g \) belongs to the hyper-range of \( T \), which is \( \{0\} \) by the analyticity of \( T \). This contradicts the fact that \( g \) is non-zero. Hence \( \mathcal{H}'_u \) must be zero. \( \square \)

Remark 4.2. An examination of Example 3.1 shows that one may construct analytic norm-increasing weighted shifts by letting constant weights with value bigger than 1 for which \( \mathcal{H} \oplus [\ker T^*]_T \) is non-zero and finite dimensional. The above lemma is not applicable in this case, since these operators do not have spectrum contained in the closed unit disc.

The previous lemma together with Remark 1.1 yields the following:

Theorem 4.3. Let \( T \in \mathcal{B}(\mathcal{H}) \) be an analytic norm-increasing \( m \)-concave operator and let \([\ker T^*]_T\) be the \( T \)-invariant closed subspace generated by \( \ker T^* \) (see (1.1)). Then either \( T \) admits the wandering subspace property or \( \mathcal{H} \oplus [\ker T^*]_T \) is of infinite dimension.

Combining the last theorem with Lemma 2.4(i), we obtain the following:

Corollary 4.4. Let \( \mathcal{G} = (W, F) \) be a one-circuit directed graph associated with the locally finite rooted directed tree \( T = (V, E) \) with circuit of length 0. Let \( S_{\lambda, \mathcal{G}} \) be a weighted shift on \( \mathcal{G} \) such that \( \inf \lambda > 0 \). If \( S_{\lambda, \mathcal{G}} \) is an analytic norm-increasing \( m \)-concave operator, then \( S_{\lambda, \mathcal{G}} \) admits the wandering subspace property.

As a future direction, it is natural to incorporate more than one circuit in a rooted directed tree. In this case, the associated weighted shifts are no more composition operators on discrete measure spaces (note that the proof of Lemma 2.3 relies heavily on the fact that these weighted shifts are composition operators). The problem of characterizing analytic weighted shifts within this class, an important step in the above problem, is of course of independent interest.

Acknowledgment. The authors convey their sincere thanks to Zenon Jabłoński and Jan Stochel for several useful comments.
References

[1] B. Abdullah and T. Le, The structure of $m$-isometric weighted shift operators, Oper. Matrices, 10 (2016), 319-334.
[2] J. Agler and M. Stankus, $m$-isometric transformations of Hilbert spaces, I, II, III, Integr. Equ. Oper. Theory 21, 23, 24 (1995, 1995, 1996), 383-429, 1-48, 379-421.
[3] A. Aleman, The multiplication operators on Hilbert spaces of analytic functions, Habilitationsschrift, Fernuniversitat Hagen, 1993.
[4] A. Aleman, S. Richter and C. Sundberg, Beurling’s theorem for the Bergman space, Acta Math. 177 (1996), 275-310.
[5] C. Apostol, H. Bercovici, C. Foias and C. Pearcy, Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. I. J. Funct. Anal. 63 (1985), 369-404.
[6] A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math. 81 (1949), 239-255.
[7] P. Budzyński, Z. Jabłoński, Il Bong Jung and J. Stochel, Unbounded subnormal composition operators in $L^2$-spaces, J. Funct. Anal. 269 (2015), 2110-2164.
[8] P. Budzyński, Z. Jabłoński, Il Bong Jung and J. Stochel, Subnormality of unbounded composition operators over one-circuit directed graphs: exotic examples, Adv. Math. 310 (2017), 484-556.
[9] J. Carlson, The spectra and commutants of some weighted composition operators, Trans. Amer. Math. Soc. 317 (1990), 631-654.
[10] B. J. Carswell, Univalent mappings and invariant subspaces of the Bergman and Hardy spaces, Proc. Amer. Math. Soc. 131 (2003), 1233-1241.
[11] B. J. Carswell, P. L. Duren and M. I. Stessin, Multiplication invariant subspaces of the Bergman space, Indiana Univ. Math. J. 51 (2002), 931-961.
[12] S. Chavan and S. Trivedi, An analytic model for left-invertible weighted shifts on directed trees, J. London Math. Soc. 94 (2016), 253-279.
[13] P. Dymek, A. Planeta, and M. Ptak, Generalized multipliers for left-invertible analytic operators and their applications to commutant and reflexivity, J. Funct. Anal. 276 (2019), 1244-1275.
[14] M. Fujii, H. Sasaoka and Y. Watatani, Adjacency operators of infinite directed graphs, Math. Japon. 34 (1989), 727-735.
[15] P. Halmos, Shifts on Hilbert spaces, J. reine angew. Math. 208 (1961), 102-112.
[16] H. Hedenmalm, B. Korenblum and K. Zhu, Theory of Bergman spaces, Springer-Verlag, New York 2000.
[17] H. Hedenmalm and K. Zhu, On the failure of optimal factorization for certain weighted Bergman spaces, Complex Variables Theory Appl. 19 (1992), 165-176.
[18] Z. Jabłoński, Il Bong Jung and J. Stochel, Weighted shifts on directed trees, J. London Math. Soc. 94 (2016), 253-279.
[19] D. Khavinson, T. L. Lance and M. I. Stessin, Wandering property in the Hardy space, Michigan Math. J. 44 (1997), 597-606.
[20] P. Kumar, A study of composition operators on $\ell^p$ spaces, Thesis (Ph.D.), Banaras Hindu University, Varanasi, 2011.
[21] P. D. Lax, Translation invariant spaces, Acta Math. 101 (1959), 163-178.
[22] M. Nowak, R. Rososzczuk and M. Wołoszkiewicz-Cył, Extremal functions in weighted Bergman spaces, Complex Variables and Elliptic Equations, 62 (2017), 98-109.
[23] A. Olofsson, Wandering subspace theorems, Integr. Equ. Oper. Theory 51 (2005), 395-409.
[24] S. Richter, Invariant subspaces of the Dirichlet shift, J. Reine Angew. Math. 386 (1988), 205-220.
[25] D. Seco, A $z^k$-invariant subspace without the wandering property, Journal Math. Anal. Appl. 472 (2019), 1377-1400.
[26] S. Shimorin, Wold-type decompositions and wandering subspaces for operators close to isometries, J. Reine Angew. Math. 531 (2001), 147-189.
[27] B. Sz.-Nagy, C. Foias, H. Bercovici and L. Kérchy, Harmonic analysis of operators on Hilbert space, Second edition. Revised and enlarged edition. Universitext. Springer, New York, 2010. xiv+474 pp.
DEPARTMENT OF MATHEMATICS AND STATISTICS, INDIAN INSTITUTE OF TECHNOLOGY KANPUR, INDIA

E-mail address: akasha@iitk.ac.in
E-mail address: chavan@iitk.ac.in
E-mail address: shailtr@iitk.ac.in