On the well formulation of the Initial Value Problem of metric–affine \( f(R) \)-gravity

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We study the well formulation of the initial value problem of \( f(R) \)-gravity in the metric-affine formalism. The problem is discussed in vacuo and in presence of perfect-fluid matter, Klein-Gordon and Yang-Mills fields. Adopting Gaussian normal coordinates, it can be shown that the problem is always well-formulated. Our results refute some criticisms to the viability of \( f(R) \)-gravity recently appeared in literature.

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I. INTRODUCTION

Every theory of physics is "physically" viable if an appropriate initial value problem is suitably formulated. This means that, starting from the assignment of suitable initial data, the subsequent dynamical evolution of the physical system is uniquely determined. In this case, the problem is said well-formulated. For example, in classical mechanics, given the initial positions and velocities of the particles (or of the constituents of a physical system), if the system evolves without external interferences, the dynamical evolution is determined. This is true also for field theories as, for example, the Electromagnetism. However, also if the initial value problem is well-formulated, we need further properties that a viable theory has to satisfy. First of all, small changes and perturbations in the initial data have to produce small perturbations in the subsequent dynamics over all the space-time where it is defined. This means that the theory should be "stable" in order to be "predictive". Besides, changes in the initial data region have to preserve the causal structure of the theory. If both these requirements are satisfied, the initial value problem of the theory is also well-posed.

It can be shown that General Relativity has a well-formulated and well-posed initial value problem but, as for other relativistic field theories, we need initial value constraints and gauge choices in order to make Einstein’s field equations suitable to correctly formulate the Cauchy problem. The consequence of this well-posedness is that General Relativity is a "stable" theory with a robust causal structure where singularities can be classified. A detailed discussion of these topics can be found in \([2, 3]\). In this paper, we focus attention on the well formulation of the initial value problem of metric–affine \( f(R) \)-theories of gravity, where \( f(R) \) is a generic function of the Ricci scalar \( R \). The aim is to prove that the Cauchy problem is well formulated in different and physically important cases.

The possible modifications of Einstein’s theory has a long history which reaches back to the early times of General Relativity \([4, 5, 6, 7, 8, 9]\). The early extensions and modifications were aimed at a unification of gravity with other fundamental interactions (in particular the Electromagnetism). The recent interest in such modifications comes from cosmological observations. For comprehensive reviews, see for example \([10, 11, 12, 13]\). These observations usually lead to the introduction of additional ad-hoc concepts like dark energy/matter, if interpreted within Einstein’s theory. On the other hand, the emergence of such stopgap measures in a cosmological context could be interpreted as a first signal of a breakdown of General Relativity on these scales \([14, 15]\), and led to the proposal of many alternative modifications of the underlying gravity theory. In particular, more recent works focused on the cosmological implications of \( f(R) \)-gravity since such models may lead to an alternative explanation of the acceleration effect observed in cosmology \([16, 17, 18, 19, 20]\) and to the explanation of the missing matter puzzle observed at astrophysical scales \([15, 21]\).

While it is very natural to extend Einstein’s gravity theories with additional geometric degrees of freedom, see for example \([22, 23, 24]\) for some general surveys on this subject as well as \([25]\) for a list of works in a cosmological context, recent attempts focused on the old idea of modifying the gravitational Lagrangian in a purely metric framework, leading to higher-order field equations. Due to the increased complexity of the field equations in this framework, the main body of works dealt with some formally equivalent theories, in which a reduction of the order of the field equations was achieved by considering the metric and the connection as independent objects \([26]\). In addition, many authors exploited the formal relationship to scalar-tensor theories to make some statements about the weak field regime, which was already worked out for scalar-tensor theories \([27]\).

However, a concern which comes with generic higher-order gravity theory is linked to the initial value problem. It is, up to now, unclear if the prolongation of standard methods can be used in order to tackle this problem in every theory. Hence it is doubtful that the full Cauchy problem can be properly addressed, if one takes into account the results already obtained in fourth-order theories stemming from a quadratic Lagrangian \([28, 29]\).

On the other hand, being \( f(R) \)-gravity, like General Relativity, a gauge theory, the initial value formulation depends on suitable constraints and on suitable "gauge choices" that mean a choice of coordinates so that the Cauchy problem results well-
formulated and possibly well-posed. In [28, 30], the initial value problem was studied for quadratic Lagrangians in the metric approach with the conclusion that it is well-posed. On the other hand, in [31], the Cauchy problem for generic $f(R)$-models has been studied in metric and Palatini approaches using the dynamical equivalence between these theories and the Brans-Dicke gravity. The result was that the problem is well-formulated for metric approach in presence of matter and well-posed in vacuo. For the Palatini approach, instead, the Chauchy problem is not well-formulated even in vacuo since, considering the gravity. The result was that the problem is well-formulated for metric approach in presence of matter and well-posed in vacuo. For the Palatini approach, instead, the Chauchy problem is not well-formulated even in vacuo since, considering the gravity. The reason of the apparent contradiction with respect to the results in [31] lies on the above mentioned gauge choice. Following [2], we adopt Gaussian normal coordinates. Such a choice, introducing further constraints on the Cauchy data surface, results more suitable to set the initial value problem in such a way that the well-posedness of the Cauchy problem has to be shown too. This topic is not dealt with in the present paper, being still an open problem under investigation. Anyway, here the important point is that, up to now and in our opinion, no objections to the viability of metric–affine $f(R)$-gravity based on the ill–formulation of the initial value problem can be made.

The layout of the paper is the following. In Sec.II, following [2], the initial value formulation of General Relativity is recalled. Sec. III is devoted to set the problem for $f(R)$-gravity in metric-affine formalism in vacuo. In this case, the problem is well-formulated and well-posed since dynamics reduces to the Einstein theory plus cosmological constant. $f(R)$-gravity in presence of matter is discussed in Sec.IV. As paradigmatic examples we discuss the coupling with perfect-fluid matter and Yang-Mills fields, showing that in both the cases the initial value problem results well-formulated.

In Sec.V, the problem in presence of a Klein-Gordon scalar field is discussed. In all these cases, the Gaussian normal coordinates allow the well-formulation of the initial value problem. It is important to stress that further constraint equations can emerge on the initial surface and this fact could reduce the set of admissible initial data. Discussion and conclusions are given in Sec.VI. The Appendix A is devoted to the demonstration of two useful propositions related to the effective stress-energy tensor and the Bianchi Identities.

## II. WELL FORMULATION OF THE CAUCHY PROBLEM FOR GENERAL RELATIVITY

Before starting with our considerations for $f(R)$-gravity, let us recall the initial value formulation of General Relativity (i.e. $f(R) = R$) where it is well-formulated (and also well-posed as shown in [3]). We adopt the formalism developed in [2].

Let us consider a system of Gaussian normal coordinates [3] where the Latin indexes $i, j$ run from 1 to 4 and the Greek indexes $\alpha$ run from 1 to 3. In these coordinates, the time components of metric tensor are $g_{44} = -1$ and $g_{4\alpha} = 0$ with the signature $(+ + + -)$. These are particularly useful to split the spatial hypersurface $S_3$ from the orthogonal time-geodesics in a given space-time $M$.

After, given a second rank symmetric tensor $W_{ij}$, defined on the globally hyperbolic space-time manifold $(M, g_{ij})$, it is possible to define the symmetric conjugate tensor $W'_{ij}$ as $W'_{ij} = W_{ij} - \frac{1}{2}W_{44}g_{ij}$, where $W := W^\alpha g_{\alpha\beta}$ is the trace of $W_{ij}$. Furthermore, if $S_3$ is a space-time domain in $M$ where $g_{44} \neq 0$ and $S_3$ is the 3-surface given by the equation $x^4 = 0$, then the following statements are equivalent:

a) $W_{ij} = 0$ in $S_4$.

b) $W^\alpha_{\alpha} = 0$ and $W_{4j} = 0$ in $S_4$.

c) $W^\alpha_{\alpha} = 0$ and $W^i_{ij} = 0$ in $S_4$ with $W_{4j} = 0$ in $S_3$.

Let us take into account now the Einstein equations in the form

$$G_{ij} = -k T_{ij}$$

with the Bianchi identities

$$T_{ij}^{\alpha |j} = 0,$$

where $G_{ij} = R_{ij} - \frac{1}{2}g_{ij} R$ is the Einstein tensor. We can define the tensor

$$W_{ij} := G_{ij} + k T_{ij}.$$

The conjugate tensor is

$$W^*_{ij} = R_{ij} + k T^*_{ij}.$$

(1)

(2)

(3)
and then the Einstein equations becomes

\[ W_{ij} = 0. \]  

(4)

These are 10 equations for 20 unknown functions \(g_{ij}\) and \(T_{ij}\). Let us assign now the 10 functions \(g_{ij}\) and \(T_{ij}\). The remaining 10 functions \(g_{\alpha\beta}\) and \(T_{i4}\), can be determined by Eqs. (4). Using the above results, these functions can be expressed in the equivalent form

\[ R_{\alpha\beta} + k T_{\alpha\beta}^* = 0, \quad W_{j|i} = T_{j|i}^* = 0, \]  

(5)

with the condition

\[ G_{k4} + k T_{4k} = 0 \]  

on the surface \(x^4 = 0\).

(6)

Eqs. (5) can be rewritten in the form

\[ g_{\alpha\beta,44} = 2 \bar{R}_{\alpha\beta} - \frac{1}{2} A g_{\alpha\beta,4} + g^{\mu\nu} g_{\alpha\mu,4} g_{\beta\nu,4} + 2k T_{\alpha\beta}^*, \]  

(7a)

\[ T_{4j,4} = - T_{j,4}^* = T_{j,4}^\alpha + \Gamma_{i\alpha}^j T_{j,4} - \Gamma_{ij}^\alpha T_{\alpha,4}, \]  

(7b)

where \( \bar{R}_{\alpha\beta} \) is the intrinsic Ricci tensor defined on the surface \(x^4 = 0\), \( \Gamma_{ij}^k \) is the Levi-Civita connection related to the metric \(g_{ij}\), the coma is the partial derivative and

\[ A := g^{\mu\nu} g_{\mu\nu,4}. \]  

(8)

In the same way, the constraint equations (6) become

\[ A_{,\alpha} - D^\sigma g_{\alpha\sigma,4} + 2k T_{3\alpha} = 0, \]  

(9a)

\[ \bar{R} - \frac{1}{4} A^2 + \frac{1}{4} B + 2k T_{44} = 0, \]  

(9b)

where \( \bar{R} \) is the intrinsic curvature scalar of the surface \(x^4 = 0\), \( D_{,\alpha} \) denotes the covariant derivatives on the surface \(x^4 = 0\) associated to the Levi-Civita connection of the intrinsic metric \(g_{\alpha\beta}|_{x^4=0}\) and

\[ B = g^{\mu\nu} g^{\rho\sigma} g_{\mu\rho,4} g_{\nu\sigma,4}. \]  

(10)

Let us assign now the set of Cauchy data on the surface \(x^4 = 0\)

\[ g_{\alpha\beta}, \quad g_{\alpha\beta,4}, \quad T_{i4}. \]  

(11)

Such data have to satisfy the constraint equations (9). Eqs. (7) explicitly give the values of the quantities

\[ g_{\alpha\beta,44}, \quad T_{3j,4}, \]  

as a function of the Cauchy data. By deriving Eqs. (7), it is straightforward to obtain the time-derivatives of further order as a function of the Cauchy data. This procedure allows to reconstruct the solution of the field equations as a power-law series of the time variable \(x^4\).

In other words, this means that the 3-surface \(S_3\), given by the equation \(x^4 = 0\), is a Cauchy surface of the globally hyperbolic space-time \((M, g_{ij})\) and that the initial value formulation (the Cauchy problem) is well-formulated in General Relativity. Our task is now to extend these results to \(f(R)\)-gravity in metric-affine formalism.

### III. THE CAUCHY PROBLEM FOR \(f(R)\)-GRAVITY IN METRIC-AFFINE FORMALISM IN EMPTY SPACE

In the metric-affine formulation of \(f(R)\)-gravity, the dynamical fields are given by the couple of functions \((g, \Gamma)\) where \(g\) is the metric and \(\Gamma\) is the linear connection. In vacuo, the field equations are obtained by varying with respect to the metric and the connection the following action

\[ A(g, \Gamma) = \int \sqrt{|g|} f(R) \, ds \]  

(13)
where \( f(R) \) is a real function, \( R(g, \Gamma) = g^{ij} R_{ij} \) (with \( R_{ij} := R^h_{\ ihj} \)) is the scalar curvature associated to the dynamical connection \( \Gamma \).

More precisely, in the approach with torsion, one can ask for a metric connection \( \Gamma \) with torsion different from zero while, in the Palatini approach, the \( \Gamma \) is non-metric but torsion is null [32].

In vacuo, the field equations for \( f(R) \)-gravity with torsion are [32, 33, 34]

\[
f'(R)R_{ij} - \frac{1}{2} f(R)g_{ij} = 0 ,
\]

(14a)

\[
T_{ij}^h = - \frac{1}{2f'(R)} \frac{\partial f'(R)}{\partial x^p} \left( \delta^p_i \delta^h_j - \delta^p_j \delta^h_i \right) ,
\]

(14b)

while the field equations for \( f(R) \)-gravity à la Palatini are [26, 35, 36, 37]

\[
f'(R)R_{ij} - \frac{1}{2} f(R)g_{ij} = 0 ,
\]

(15a)

\[
\nabla_k (f'(R)g_{ij}) = 0 .
\]

(15b)

In both cases, considering the trace of Einstein-like field equations (14a) e (15a), one gets

\[
f'(R)R - 2 f(R) = 0 .
\]

(16)

IV. THE CAUCHY PROBLEM IN PRESENCE OF MATTER

Let us now take into account the presence of perfect-fluid matter in the formulation of the Cauchy problem for \( f(R) \)-gravity. In order to deal simultaneously with the Palatini approach and the torsion, we will consider the connection not coupled with matter. In other words, we will assume that the matter Lagrangian does not explicitly depend on the dynamical connection. With this working hypothesis, the field equations are

\[
f'(R)R_{ij} - \frac{1}{2} f(R)g_{ij} = \Sigma_{ij} ,
\]

(17a)

with

\[
T_{ij}^h = - \frac{1}{2f'(R)} \frac{\partial f'(R)}{\partial x^p} \left( \delta^p_i \delta^h_j - \delta^p_j \delta^h_i \right)
\]

(17b)

in the case of \( f(R) \)-gravity with torsion, and

\[
f'(R)R_{ij} - \frac{1}{2} f(R)g_{ij} = \Sigma_{ij} ,
\]

(18a)

\[
\nabla_k (f'(R)g_{ij}) = 0 ,
\]

(18b)

in the case of \( f(R) \)-gravity in the Palatini approach. In Eqs. (17a) and (18a), \( \Sigma_{ij} := - \frac{1}{\sqrt{|g|}} \frac{\delta L_m}{\delta g^{ij}} \) is the stress-energy tensor.

Considering the trace of Eqs. (17a) and (18a), \( \Sigma := g^{ij} \Sigma_{ij} \). We have

\[
f'(R)R - 2 f(R) = \Sigma .
\]

(19)

It is worth noticing that any time that \( \Sigma = \text{const.} \) the theory reduces to the General Relativity with cosmological constant and the initial value problem is identical to the above empty-space case.
By the hypotheses that the relation (19) is invertible and \( \Sigma \neq \text{const.} \), the curvature scalar \( R \) can be expressed as a function of \( \Sigma \), that is

\[
R = F(\Sigma).
\]

Starting from this fact, it is easy to show that the Einstein-like equations of both Palatini and metric-affine theory with torsion can be expressed in the same form \([32, 33, 37]\), that is

\[
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = \frac{1}{\varphi} \Sigma_{ij} + \frac{1}{\varphi^2} \left( -3 \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} + \varphi \tilde{\nabla}_j \frac{\partial \varphi}{\partial x^i} + \frac{3}{4} \frac{\partial \varphi}{\partial x^h} \frac{\partial \varphi}{\partial x^k} g^{hk} g_{ij} \right) - \frac{\varphi}{\partial x^i} g_{ij} - V(\varphi) g_{ij},
\]

where we have defined the effective potential

\[
V(\varphi) := \frac{1}{4} \left[ \varphi F^{-1}(f'^{-1}(\varphi)) + \varphi^2 (f')^{-1}(\varphi) \right],
\]

for the scalar field

\[
\varphi := f'(F(\Sigma)).
\]

Introducing the conformal transformation \( \bar{g}_{ij} = \varphi g_{ij} \), Eq. (21) assume the simpler form (see for example \([32, 37, 38]\))

\[
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} \bar{g}_{ij} = \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^3} V(\varphi) \bar{g}_{ij},
\]

where \( \tilde{R}_{ij} \) and \( \tilde{R} \) are respectively the Ricci tensor and the curvature scalar derived from the conformal metric \( \bar{g}_{ij} \). It is worth noticing that the conformal transformation is working if the trace \( \Sigma \) of the stress-energy tensor is independent of the metric \( g_{ij} \). In such a case, Eqs. (24) depend exclusively on the conformal metric \( \bar{g}_{ij} \) and the other matter fields.

Furthermore, we have to stress that the connection \( \Gamma \), solution of the field equations with torsion, is given by

\[
\Gamma_{ij}^h = \tilde{\Gamma}_{ij}^h + \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^3} \delta_i^h - \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^p} g^{ph} g_{ij},
\]

where \( \tilde{\Gamma}_{ij}^h \) is the Levi-Civita connection induced by the metric \( g_{ij} \), while the connection \( \tilde{\Gamma} \), solution of the Palatini field equations, coincides with the Levi-Civita connection associated to the conformal metric \( \bar{g}_{ij} \). Both connections, \( \Gamma \) and \( \tilde{\Gamma} \), satisfy the relation

\[
\tilde{\Gamma}_{ij}^h = \Gamma_{ij}^h + \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^i} \delta_j^h.
\]

The identities

\[
\tilde{\Gamma}_{ij}^h = \tilde{\Gamma}_{ij}^h + \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^3} \delta_i^h - \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^p} g^{ph} g_{ij} + \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^i} \delta_j^h,
\]

hold. This result shows the relation between the Levi-Civita connections induced by the metrics \( g_{ij} \) and \( \bar{g}_{ij} \).

In general, the Einstein-like Eqs. (21) have to be considered together with the matter field equations. To this purpose, we have to keep in mind that the conservation equations for both the metric-affine theories (with torsion and \( \text{a la Palatini} \)) coincide with the standard conservation laws of General Relativity (see the Appendix A). This means

\[
\tilde{\nabla}_j \Sigma^{ij} = 0.
\]

It is straightforward to show that Eqs. (28) are equivalent to the conservation laws

\[
\tilde{\nabla}_j T^{ij} = 0 \quad \text{where} \quad T_{ij} = \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^3} V(\varphi) \bar{g}_{ij},
\]

for the conformally transformed theories (24). In fact, by an explicit calculation of the divergence \( \tilde{\nabla}_j T^{ij} \) where the relations (27) have been used, we obtain the equations

\[
\tilde{\nabla}_j T_{ij} = \frac{1}{\varphi^2} \tilde{\nabla}_j \Sigma_{ij} + \frac{1}{\varphi^2} \frac{\partial \varphi}{\partial x^i} \left( -\frac{1}{2} \Sigma + \frac{3}{2} V(\varphi) - V'(\varphi) \right).
\]
The constraint equations \(28\) and \(29\) are then mathematically equivalent in view of the relation

\[
\Sigma - \frac{6}{\varphi} V(\varphi) + 2V'(\varphi) = 0, \tag{31}
\]

which is equivalent to the definition \(\varphi = f'(F(\Sigma)) \) \[32\].

With these results in mind, the Cauchy problem for Eqs. \(21\) and the related matter field equations can be faced by discussing the equivalent initial value problem of the conformally transformed theories. Considering as in the above General Relativity case, Gaussian normal coordinates and starting from Eqs. \(24\) and \(29\), it is easy to conclude that the Cauchy problem is well formulated also in this case.

It is worth pointing out that, in general, the matter field equations imply the Levi-Civita connection derived from the metric \(g_{ij}\) and not the connection induced from the conformal metric \(\tilde{g}_{ij}\). In view of the relation \(27\), this is not a problem since the connection \(\tilde{\Gamma}\) can be expressed as a function of the connection \(\Gamma\) and of the scalar field \(\varphi\), which, on the other hand, is a function of the source matter fields. As a result, we could obtain slightly more complicated equations implying further constraints on the initial data but, in any case, the same equations can be always recast in "normal form" with respect to the maximal order time derivatives of matter fields allowing a well-formulated Cauchy problem \[3\].

As an example, let us examine in detail the perfect-fluid case with equation of state \(p = p(\rho)\). The corresponding stress-energy tensor is

\[
\Sigma_{ij} = pV_iV_j + p(V_iV_j + g_{ij}) , \tag{32}
\]

the matter field equations are given by Eqs. \(28\) with the further condition

\[
g_{ij}V^iV^j = -1 , \tag{33}
\]

where \(V^j\) are 4-velocities. Specifically, Eqs. \(28\) give the field equations

\[
(\rho + pV^j)_{ij}V_i + (\rho + p)V_{ij}V^j + \frac{\partial p}{\partial x^i} = 0 , \tag{34}
\]

where \(\partial\) denotes the covariant derivative with respect to the Levi-Civita connection induced by \(g_{ij}\). By saturating with \(V^i\), one gets

\[
(\rho V^j)_{ij} = -pV_{ij} , \tag{35a}
\]

while by substituting Eq. \(35a\) into Eqs. \(34\) for \(\alpha = 1, 2, 3\), we have

\[
(\rho + p)V^jV^\alpha_{ij} = -\frac{\partial p}{\partial x^j} (V^\alpha V^j + g^{\alpha j}) . \tag{35b}
\]

As already pointed out, Eqs. \(33\) and \(35\) involve the metric \(g_{ij}\) and its first derivatives. By using the relations \(27\), we can rewrite these equations in the conformal metric \(\tilde{g}_{ij}\), the scalar field function (of the matter density) \(\varphi = \varphi(\rho)\) and their first derivatives. Immediately we get

\[
\frac{1}{\varphi} g_{ij} V^iV^j = -1 , \tag{36a}
\]

\[
\frac{\partial}{\partial x^j} (\rho V^j) + \tilde{\Gamma}_{js}^i \rho V^s - \frac{2}{\varphi} \frac{\partial \varphi}{\partial x^s} \rho V^s = -p \left( \frac{\partial V^j}{\partial x^j} + \tilde{\Gamma}_{js}^i V^s - \frac{2}{\varphi} \frac{\partial \varphi}{\partial x^s} V^s \right) , \tag{36b}
\]

\[
(\rho + p)V^j \left[ \frac{\partial V^\alpha}{\partial x^j} + \tilde{\Gamma}_{js}^i V^s + \frac{1}{2\varphi} \left( -\frac{\partial \varphi}{\partial x^s} \delta^\alpha_j + \frac{\partial \varphi}{\partial x^j} \delta^\alpha_s - \frac{\partial \varphi}{\partial x^p} g^{ps} \delta^\alpha_j \right) \right] = -\frac{\partial p}{\partial x^j} (V^\alpha V^j + g^{\alpha j}) . \tag{36c}
\]

In the Gaussian normal coordinates, where \(\tilde{g}_{\alpha \pm} = \pm 1\) (depending on the sign of \(\varphi\)) and \(\tilde{g}_{\alpha 4} = 0\), we obtain, from Eq. \(36a\), the expression of \(V^4\) in terms of the remaining \(V^\alpha\). Eqs. \(36a\) and \(36b\) can be considered as linear equations for the functions \(\frac{\partial V^\alpha}{\partial x^j}\) and \(\frac{\partial p}{\partial x^j}\). The explicit resolution of these equations, in terms of the unknown functions, could give rise to further constraints on the initial data and on the form of the \(f(R)\)-function. In any case, from Eqs. \(36a\) and \(36b\) and in
Gaussian normal coordinates, one can derive the quantities \( \partial^\alpha V^\alpha / \partial x^4 \) and \( \partial \rho / \partial x^4 \) as functions of the initial data \( \tilde{g}_{\alpha \beta}, \partial \tilde{g}_{\alpha \beta} / \partial x^4, V_\alpha \) and \( \rho \) allowing to put the matter-field equations in normal form. This means that the Cauchy problem is well-formulated.

As another important example, let us consider the initial value formulation of \( f(R) \)-gravity coupled with Yang-Mills fields, in particular with Electromagnetic field. Also in this case, the problem is well-formulated. In fact, the stress-energy tensor of a Yang-Mills field has null trace. From Eq. (12), it is easy to prove that the curvature scalar \( R \) is constant and then the Cauchy problem is well-formulated (this last result is immediate for the Electromagnetic field). Regarding the well-posed initial formulation, the above results work for any theory where the trace of stress-energy tensor is a constant and then the Cauchy problem is well-formulated (this last result is immediate for the Electromagnetic field). Regarding the well-posed initial formulation, the above results work for any theory where the trace of stress-energy tensor is a constant since, as above, the \( f(R) \)-gravity reduces to Einstein gravity plus cosmological constant.

V. THE CAUCHY PROBLEM IN THE CASE OF COUPLING WITH A SCALAR FIELD

Let us consider now the case of coupling with a Klein-Gordon scalar field \( \psi \) with self-interacting potential \( U(\psi) = \frac{1}{2} m^2 \psi^2 \). The stress-energy tensor is given by

\[
\Sigma_{ij} = \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} - \frac{1}{2} g^{ij} \left( \frac{\partial \psi}{\partial x^p} \frac{\partial \psi}{\partial x^q} g_{pq} + m^2 \psi^2 \right). \tag{37}
\]

The trace of tensor (37) is

\[
\Sigma = -\frac{\partial \psi}{\partial x^p} \frac{\partial \psi}{\partial x^q} g_{pq} - 2m^2 \psi^2. \tag{38}
\]

In the relation (38), the metric \( g_{pq} \) explicitly appears. This means that the conformal transformation procedure of the previous section cannot be applied in this case and then the well-formulation of the Cauchy problem has to be directly shown starting from the field Eqs. (21) and the Klein-Gordon equation (see also [40]), that is

\[
\tilde{\nabla}_j \frac{\partial \psi}{\partial x^j} g^{ij} = m^2 \psi. \tag{39}
\]

As standard, we take into account Gaussian normal coordinates where \( g_{4\alpha} = 0 \) and \( g_{44} = -1 \). It is easy to show that \( \tilde{\Gamma}^{4}_{4\alpha} = 0 \) and then Eq. (39), suitably developed, can be rewritten in the form

\[
\frac{\partial^2 \psi}{\partial x^4} = \frac{\partial \psi}{\partial x^4} \left( \frac{\partial^2 \psi}{\partial x^4} - \tilde{\Gamma}^{4}_{4\alpha} \frac{\partial \psi}{\partial x^\alpha} \right) - m^2 \psi, \tag{40}
\]

where

\[
\tilde{\Gamma}^{4}_{4\alpha} \frac{\partial \psi}{\partial x^\alpha} = \frac{1}{2} g^{\alpha \gamma} \left( \frac{\partial g_{\alpha \gamma}}{\partial x^\beta} + \frac{\partial g_{\alpha \gamma}}{\partial x^\alpha} \frac{\partial g_{\alpha \beta}}{\partial x^\gamma} - \frac{\partial g_{\alpha \beta}}{\partial x^\alpha} \frac{\partial g_{\alpha \gamma}}{\partial x^\gamma} \right) \frac{\partial \psi}{\partial x^\lambda} + \frac{1}{2} \frac{\partial g_{\alpha \beta}}{\partial x^4} \frac{\partial \psi}{\partial x^\alpha}. \tag{41}
\]

The Einstein-like equations are exactly of the form (21) but now they depend also on \( \psi \)

\[
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = \frac{1}{\varphi} \Sigma_{ij} + \frac{1}{\varphi^2} \left( -3 \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} + \varphi \tilde{\nabla}_j \frac{\partial \varphi}{\partial x^i} + 3 \frac{\partial \varphi}{\partial x^4} \frac{\partial \varphi}{\partial x^h} \tilde{g}^{hk} g_{ij} - \varphi \tilde{\nabla}_h \frac{\partial \varphi}{\partial x^h} g_{ij} - V(\varphi) g_{ij} \right). \tag{42}
\]

Adopting the notation of Sec.II, we can rewrite Eqs. (42) in the form

\[
W_{ij} := \varphi \tilde{G}_{ij} + T_{ij} = 0, \tag{43}
\]

where

\[
-T_{ij} = \Sigma_{ij} + \frac{1}{\varphi} \left( -3 \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} + \varphi \tilde{\nabla}_j \frac{\partial \varphi}{\partial x^i} + 3 \frac{\partial \varphi}{\partial x^4} \frac{\partial \varphi}{\partial x^h} \tilde{g}^{hk} g_{ij} - \varphi \tilde{\nabla}_h \frac{\partial \varphi}{\partial x^h} g_{ij} - V(\varphi) g_{ij} \right). \tag{44}
\]
We have now all the ingredients to discuss the Cauchy problem for the system (39) and (43). As first result, it has to be pointed out that Eqs. (39) imply the conservation laws

\[ \nabla_j \Sigma^{ij} = 0. \]  

(45)

In Appendix A, it is shown that Eqs. (45) are equivalent to the conservation laws

\[ \nabla^j W_{ij} = 0, \]  

(46)

see also [39]. In this case, it is useful to remember that both metric-affine theories (à la Palatini and with torsion) are equivalent to a metric scalar-tensor theory with a Brans-Dicke parameter \( \omega = -\frac{3}{2} \) [32, 35, 36].

For the the Cauchy problem, it is important to point out that the second of Eqs. (5) is given by the Klein-Gordon Eq. (39). Regarding the constraint (6), from the field Eqs. (42), it is easy to show that it involves only first order derivatives with respect to the time variable \( x^4 \). In order to show this point, let us observe that the r.h.s. of Eqs. (42) contains the second derivatives of the scalar field \( \varphi = f'(F(\Sigma)) \) and then implies the second derivatives of the metric \( g_{ij} \) and the third derivatives of the Klein-Gordon field \( \psi \). In more details, the second derivatives of \( \varphi \) are present in the terms

\[ \nabla^j \partial \varphi = \frac{\partial^2 \varphi}{\partial x^j \partial x^s} - \Gamma^s_{ji} \partial \varphi, \]  

(47a)

and

\[ \nabla^h \partial \varphi g_{ij} = \left( g^{hk} \frac{\partial^2 \varphi}{\partial x^k \partial x^i} - g^{hk} \hat{\Gamma}^k_{ih} \partial \varphi \right) g_{ij}. \]  

(47b)

Assuming as before Gaussian normal coordinates, which means \( g_{44} = g^{44} = -1 \) and \( g_{\alpha \alpha} = g^{\alpha \alpha} = 0 \), it is easy to show that in the equations \( \check{G}_{4i} = -T_{4i} \), there are no second order time derivatives \( \frac{\partial^2 \varphi}{(\partial x^4)^2} \), and the only time derivatives are \( \frac{\partial \varphi}{\partial x^4} \) and \( \frac{\partial^2 \varphi}{\partial x^4 \partial x^\alpha} \). On the other hand, it is useful to observe that these terms involve also second order time derivatives of the Klein-Gordon field \( \frac{\partial^2 \psi}{(\partial x^4)^2} \), and \( \frac{\partial^3 \psi}{\partial x^\alpha (\partial x^4)^2} \), the second order spatial derivatives of the time derivative \( \frac{\partial^4 \psi}{\partial x^\alpha \partial x^5 \partial x^4} \), and the derivatives of the metric \( \frac{\partial g_{i\beta}}{\partial x^4} \) and \( \frac{\partial^2 g_{i\beta}}{\partial x^5 \partial x^4} \).

At this point, it is sufficient to observe that the Klein-Gordon Eq. (40), evaluated on the initial surface \( x^4 = 0 \), allows to express the second order time derivative \( \frac{\partial^2 \psi}{(\partial x^4)^2} \) and its spatial derivatives \( \frac{\partial^3 \psi}{\partial x^\alpha (\partial x^4)^2} \) on the same surface as a function of the Cauchy data \( (g_{\alpha \beta}, \frac{\partial g_{\alpha \beta}}{\partial x^4}, \psi, \frac{\partial \psi}{\partial x^4}) \) and of their spatial derivatives. The remaining quantities \( \frac{\partial^3 \psi}{\partial x^\alpha (\partial x^4)^2}, \frac{\partial g_{\alpha \beta}}{\partial x^4}, \frac{\partial^2 g_{\alpha \beta}}{\partial x^4} \) and \( \frac{\partial^2 g_{\alpha \beta}}{\partial x^5 \partial x^4} \), defined on \( x^4 = 0 \), can be directly calculated as functions of the same Cauchy data.

In conclusion, in the equations corresponding to Eqs. (5), only second order time derivatives of the metric appear. Obviously, the initial value problem is well-formulated if these equations are in normal form with respect to \( \frac{\partial^2 g_{\alpha \beta}}{\partial (x^4)^2} \) at least on \( x^4 = 0 \). As in the perfect-fluid case, such a request could impose further constraints on the initial data and, possibly, on the form of the \( f(R) \)-function, but, in general, it works and the corresponding Cauchy problem is well-formulated.
VI. DISCUSSION AND CONCLUSIONS

In this paper, we have shown that the initial value problem for metric-affine $f(R)$-gravity is well-formulated. This means that there are no objections to the viability of metric–affine $f(R)$-theories of gravity based on the well formulation of the initial value problem. However, also the well–posed problem is necessary in order to achieve a complete control of dynamics but, in this case, the role of source fields has to be carefully discussed. This topic will be discussed in a forthcoming paper.

Since $f(R)$-gravity is a gauge theory, like General Relativity, it is crucial the choice of suitable coordinates in order to correctly formulate the problem. We have adopted Gaussian normal coordinates which can be defined any time that the derivative operator $\nabla_i$ arises from a metric $g_{ij}$. They are also called synchronous coordinates and are particularly useful for calculations on a given non-null surface $S_3$, i.e. an 3-dimensional embedded sub-manifold of the 4-dimensional manifold $M$. They allow to define uniquely orthogonal geodesics to $S_3$ and then to correctly formulate the conditions of validity for the Cauchy-Kowalewski theorem [3].

In metric-affine formalism, it is always possible to show that a given $f(R)$-theory, in vacuo, can be recast in Einstein’s gravity plus a cosmological constant. This means that the initial value problem is always well-constructed and well-posed. The same conclusion holds in the case of matter coupling every time that the trace of the stress-energy tensor is a constant.

As shown in [32, 33, 34], by introducing matter fields in the Palatini and in the metric-affine approach with torsion, one can define, in two steps $R = F(\Sigma)$ and $\varphi := f'(F(\Sigma))$, a suitable scalar field that allows: i) to reduce the theory to scalar-tensor theory; ii) to relate the form of $f(R)$ to the trace of the matter (field) stress-energy tensor. In this case, it is always possible a well–formulation of the Cauchy problem avoiding the singularities which could emerge with other gauge choices [31]. Besides, matter fields could induce further constraints on the Cauchy surface $x^i$ which, if suitably defined, lead to the normal form of the matter-field equations. This is one of the main requests to obtain a well-formulated value problem. However different fields, acting as sources of the field equations, like perfect fluids, Yang-Mills, and Klein-Gordon fields, could generate different constraints on $x^i$. Such constraints could imply also restrictions on the possible forms of $f(R)$. In conclusion, as in General Relativity, the gauge choice is essential for a correct formulation of the initial value problem while the source fields have to be carefully discussed to obtain a well-posed problem. In a forthcoming paper, we will study specific problems and $f(R)$-models where both the issues (i.e. well-formulation and well-posedness) could be achieved.

APPENDIX A: THE EFFECTIVE STRESS-ENERGY TENSOR AND THE CONSERVATION LAWS

Proposition A.1. Eqs. (21), (22) and (23) imply the usual conservation laws $\tilde{\nabla}^i \Sigma_{ij} = 0$

PROOF. First of all, we recall that Eq. (22) and (23) is equivalent to the relation

$$\Sigma - \frac{6}{\varphi} V(\varphi) + 2V'(\varphi) = 0 \quad (A1)$$

(see [32] for the proof). After that, taking the trace of Eq. (21) into account, we get

$$\tilde{R} + \frac{3}{2} \frac{1}{\varphi^2} \varphi_i \varphi^i - 3 \frac{\tilde{\nabla} i \varphi^i}{\varphi} + \frac{2}{\varphi^2} V(\varphi) - \frac{2}{\varphi} V'(\varphi) = 0 \quad (A2)$$

where for simplicity we have defined $\varphi_i := \frac{\partial \varphi}{\partial x^i}$. Substituting Eq. (A2) in Eq. (A1), we obtain

$$\tilde{R} + \frac{3}{2} \frac{1}{\varphi^2} \varphi_i \varphi^i - 3 \frac{\tilde{\nabla} i \varphi^i}{\varphi} + \frac{2}{\varphi^2} V(\varphi) - \frac{2}{\varphi} V'(\varphi) = 0 \quad (A3)$$

We rewrite Eq. (21) in the form

$$\varphi \tilde{R}_{ij} - \frac{\varphi}{2} \tilde{R} g_{ij} = \Sigma_{ij} + \frac{1}{\varphi} \left( -\frac{3}{2} \varphi_i \varphi_j + \varphi \tilde{\nabla} j \varphi_i + \frac{3}{4} \varphi_h \varphi^h g_{ij} + \right. \left. -\varphi \tilde{\nabla} h \varphi_h g_{ij} - V(\varphi) g_{ij} \right) \quad (A4)$$

The covariant divergence of (A4) yields

$$(\tilde{\nabla}^i \varphi) \tilde{R}_{ij} + \varphi \tilde{\nabla}^i \tilde{G}_{ij} = \left( \frac{1}{\varphi} \left( -\frac{3}{2} \varphi_i \varphi_j + \frac{3}{4} \varphi_h \varphi^h g_{ij} - V(\varphi) g_{ij} \right) \right) \varphi + \tilde{\nabla}^i \left[ \frac{1}{\varphi} \left( -\frac{3}{2} \varphi_i \varphi_j + \frac{3}{4} \varphi_h \varphi^h g_{ij} - V(\varphi) g_{ij} \right) \right] \quad (A5)$$
By definition, the Einstein and the Ricci tensors satisfy \( \tilde{\nabla}^j \tilde{G}_{ij} = 0 \) and \((\tilde{\nabla}^j \varphi) \tilde{R}_{ij} = (\tilde{\nabla}^j \tilde{\nabla}^i \tilde{\nabla}^j - \tilde{\nabla}^i \tilde{\nabla}^j \tilde{\nabla}^i) \varphi \). Then Eq. (A5) reduces to

\[
- \frac{1}{2} \tilde{R} \tilde{\nabla} \varphi = \tilde{\nabla}^j \Sigma_{ij} + \tilde{\nabla}^j \left[ \frac{1}{\varphi} \left( - \frac{3}{2} \varphi_i \varphi_j + \frac{3}{4} \varphi h^h \varphi i - V(\varphi) g_{ij} \right) \right]
\]  
(A6)

Finally, making use of Eq. (A3) it is easily seen that

\[
- \frac{1}{2} \tilde{R} \tilde{\nabla} \varphi = \tilde{\nabla}^j \left[ \frac{1}{\varphi} \left( - \frac{3}{2} \varphi_i \varphi_j + \frac{3}{4} \varphi h^h \varphi i - V(\varphi) g_{ij} \right) \right]
\]  
(A7)

from which the conclusion \( \tilde{\nabla}^j \Sigma_{ij} = 0 \) follows. \( \square \)

**Proposition A.2.** Given the Levi-Civita connection

\[
\tilde{\Gamma}^i_{\, jk} = \Gamma^i_{\, jk} + \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^j} \delta^i_k - \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^k} \delta^i_j + \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^i} \delta^j_k
\]  
(A8)

associated with the conformal metric tensor \( \tilde{g} = \varphi g \), and given the effective energy–impulse tensor

\[
T_{ij} = \frac{1}{\varphi^3} V(\varphi) \tilde{g}_{ij}
\]  
(A9)

the condition \( \tilde{\nabla}^j T_{ij} = 0 \) is equivalent to the condition \( \tilde{\nabla}^j \Sigma_{ij} = 0 \)

**Proof.**

\[
\tilde{\nabla}^j T_{ij} = \frac{1}{\varphi} g^{sj} \tilde{\nabla} T_{ij} = \frac{1}{\varphi} g^{sj} \tilde{\nabla} \left( \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^2} V(\varphi) g_{ij} \right) = \frac{1}{\varphi} \tilde{\nabla}^i \Sigma_{ij} + \frac{1}{\varphi^2} \tilde{\nabla} T_{ij} + \frac{1}{\varphi} \tilde{\nabla}^j V(\varphi) g_{ij} = \frac{1}{\varphi} \tilde{\nabla}^j \Sigma_{ij} + \frac{1}{\varphi^2} \tilde{\nabla} T_{ij} + \frac{1}{\varphi} V(\varphi) g_{ij}
\]  
(A10)

We have separately

\[
\frac{1}{\varphi} g^{sj} \tilde{\nabla} T_{ij} = \frac{1}{\varphi} g^{sj} \tilde{\nabla} \left( \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^2} V(\varphi) g_{ij} \right) = \frac{1}{\varphi} \tilde{\nabla}^i \Sigma_{ij} + \frac{1}{\varphi^2} \tilde{\nabla} T_{ij} + \frac{1}{\varphi} V(\varphi) g_{ij}
\]  
(A11)

\[
\frac{1}{\varphi} g^{sj} \tilde{\nabla} T_{ij} = \frac{1}{\varphi} g^{sj} \tilde{\nabla} \left( \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^2} V(\varphi) g_{ij} \right) = \frac{1}{\varphi} \tilde{\nabla}^i \Sigma_{ij} + \frac{1}{\varphi^2} \tilde{\nabla} T_{ij} + \frac{1}{\varphi} V(\varphi) g_{ij}
\]  
(A12)

\[
\frac{1}{\varphi} g^{sj} \tilde{\nabla} T_{ij} = \frac{1}{\varphi} g^{sj} \tilde{\nabla} \left( \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^2} V(\varphi) g_{ij} \right) = \frac{1}{\varphi} \tilde{\nabla}^i \Sigma_{ij} + \frac{1}{\varphi^2} \tilde{\nabla} T_{ij} + \frac{1}{\varphi} V(\varphi) g_{ij}
\]  
(A13)
Collecting Eqs. (A11), (A12) and (A13) we have then

\[ \bar{\nabla} j T_{ij} = \frac{1}{\varphi^2} \bar{\nabla} j \Sigma_{ij} + \frac{1}{\varphi^2} \frac{\partial \varphi}{\partial x^i} \left[ -\frac{1}{2} \Sigma + \frac{3}{\varphi} V(\varphi) - V'(\varphi) \right] = \frac{1}{\varphi^2} \bar{\nabla} j \Sigma_{ij} \quad (A14) \]

because the identity \(-\frac{1}{2} \Sigma + \frac{3}{\varphi} V(\varphi) - V'(\varphi) = 0\) holds identically, being equivalent to the definition \(\varphi = f'(F(\Sigma))\) \[32\]. □

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