FUNDAMENTAL THEOREM FOR SUBMANIFOLDS IN GENERAL AMBIENT SPACES

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Abstract. In this paper, we extend the fundamental theorem for submanifolds to general ambient spaces by viewing it as a higher codimensional Cartan-Ambrose-Hicks theorem. The key ingredient in obtaining this is a generalization of development of curves in the positive codimensional case. One advantage of our results is that it also provide a geometric construction of the isometric immersion when the isometric immersion exists.

1. Introduction

Let $(M^n, g)$ be a Riemannian submanifold of $(\tilde{M}^{n+s}, \tilde{g})$. Then, the curvature tensors $R$ and $\tilde{R}$ for $M$ and $\tilde{M}$ respectively are related by the following three equations:

\begin{align}
(1.1) \quad & R(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + \langle h(X,W), h(Y,Z) \rangle - \langle h(X,Z), h(Y,W) \rangle, \\
(1.2) \quad & \langle (\nabla^\bot_X h)(Y,Z) - (\nabla^\bot_Y h)(X,Z), \xi \rangle = \tilde{R}(Z,\xi,X,Y), \\
(1.3) \quad & R^\bot(\xi,\eta,X,Y) = \tilde{R}(\xi,\eta,X,Y) + \langle A_\xi(Y), A_\eta(X) \rangle - \langle A_\eta(Y), A_\xi(X) \rangle
\end{align}

for any $X,Y,Z,W \in \Gamma(TM)$ and $\xi,\eta \in \Gamma(T^\bot M)$, where

\begin{equation}
(1.4) \quad h(X,Y) = (\nabla_X Y)^\bot \quad (\forall X,Y \in \Gamma(TM))
\end{equation}

is the second fundamental form with $\nabla$ the Levi-Civita connection for $\tilde{g}$, $\nabla^\bot$ is the normal connection, $R^\bot$ is the curvature tensor for the normal connection, and $A_\xi(X)$ is the Weingarten operator defined by

\begin{equation}
(1.5) \quad \langle A_\xi(X), Y \rangle = \langle h(X,Y), \xi \rangle \quad (\forall X,Y \in \Gamma(TM), \xi \in \Gamma(T^\bot M)).
\end{equation}

The equation (1.1), (1.2) and (1.3) are called the Gauss equation, Codazzi equation and Ricci equation respectively. The classical fundamental theorem for submanifolds tracing back to the work of Bonnet [2] says that the converse of the above is true when the ambient space $(\tilde{M}, \tilde{g})$ is a space form. For the precise statement of the theorem, please see [5]. The reason one requires that

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the ambient space \((\tilde{M}, \tilde{g})\) is a space form is that the curvature \(\tilde{R}\) is intrinsic in this case. More precisely, when \(\tilde{M}\) is a space form of sectional curvature \(K\), one has

\[
\tilde{R}(X, Y, Z, W) = K(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle)
\]

for any tangent vectors \(X, Y, Z, W\) of \(\tilde{M}\). There are some efforts to extend the fundamental theorem for submanifolds to more general ambient spaces such as products of space forms, symmetric spaces in the past decades. See for examples [6, 7, 11, 12, 13] and references there in.

In this paper, we extend the fundamental theorem for submanifolds to general ambient spaces by viewing it as a higher codimensional Cartan-Ambrose-Hicks theorem. The classical Cartan-Ambrose-Hicks theorem (See [1, 3, 4, 9]) can be viewed as the converse of the fact that local isometries between two Riemannian manifolds will preserve curvature tensors. So, the classical Cartan-Ambrose-Hicks theorem can be viewed as a fundamental theorem for submanifolds in general ambient space with zero codimension. This is a key observation of the work.

In [16], by observing the fact that local isometries will also preserve developments of curves. We obtained an alternative form and proof of the Cartan-Ambrose-Hicks theorem by using developments of curves. Let’s recall the notion of developments of curves first. This notion is presented in the language of principle bundles in [10].

**Definition 1.1.** Let \((M, g)\) be a Riemannian manifold and \(v : [0, T] \to T_pM\) be a curve in \(T_pM\). The development of \(v\) is a curve \(\tilde{\gamma} : [0, T] \to \tilde{M}\) such that

\[
\tilde{\gamma}(0) = \tilde{p} \quad \text{and} \quad \tilde{\gamma}'(t) = P^\tilde{t}_0(\tilde{\gamma})(v(t)) \quad \text{for any} \quad t \in [0, T],
\]

where \(P^\tilde{t}_0(\tilde{\gamma})\) is the parallel displacement from \(\tilde{\gamma}(t_1)\) to \(\tilde{\gamma}(t_2)\) along \(\tilde{\gamma}\).

A proof of the uniqueness and local existence for developments of curves can be found in [16]. We will denote the development of \(v\) as \(\text{dev}(p, v)\) when it exists. The alternative form of the Cartan-Ambrose-Hicks theorem we obtained in [16] is as follows.

**Theorem 1.1** (An alternative form of Cartan-Ambrose-Hicks theorem in [16]). Let \((M^n, g)\) and \((\tilde{M}^n, \tilde{g})\) be two Riemannian manifolds. Let \(p \in M\), \(\tilde{p} \in \tilde{M}\) and \(\varphi : T_pM \to T_{\tilde{p}}\tilde{M}\) be a linear isometry. Suppose that \(M\) is simply connected and for any smooth curve \(\gamma : [0, 1] \to M\) with \(\gamma(0) = p\), the development \(\tilde{\gamma}\) of \(\varphi(v_{\gamma})\) exists in \(\tilde{M}\) where

\[
v_{\gamma}(t) = P^0_0(\gamma)(\gamma'(t))
\]

for \(t \in [0, 1]\). Moreover, suppose that

\[
R = \tau_{\gamma}^\ast \tilde{R}
\]

for any smooth curve \(\gamma : [0, 1] \to M\) where

\[
\tau_{\gamma} = P^1_0(\tilde{\gamma}) \circ \varphi \circ P^0_1(\gamma) : T_{\gamma(1)}M \to T_{\tilde{\gamma}(1)}\tilde{M},
\]
for any $c \in \mathbb{R}$ and $\bar{R}$ are the curvature tensors of $(M, g)$ and $(\bar{M}, \bar{g})$ respectively. Then, the map $f(\gamma(1)) = \bar{\gamma}(1)$ from $M$ to $\bar{M}$ is well defined and $f$ is the local isometry from $M$ to $\bar{M}$ with $f(p) = \bar{p}$ and $f_*p = \varphi$.

Motivated by Theorem 1.1, one can use $\tau, \bar{R}$ to deal with the problem that $\bar{R}$ is not intrinsic when considering fundamental theorem for submanifolds in general ambient spaces. This is another key observation of the work.

The final key step and most difficult part of this work is to extend the notion of developments of curves to the positive codimensional case. Note that the original definition for developments of curves does not work for the positive codimensional case because developments of curves in a proper submanifold is different with that in the ambient space unless the submanifold is totally geodesic. Intuitively, for developments of curves in positive codimensional case, we want to recover parallel displacements on submanifolds and on the normal vector bundles intrinsically by just using the second fundamental form. Due to this consideration, we define the generalized development of curves as follows.

**Definition 1.2.** Let $(\bar{M}^{n+s}, \bar{g})$ be a Riemannian manifold and $\bar{p} \in \bar{M}$. Let $T_{\bar{p}}\bar{M} = T^n \oplus N^s$ be an orthogonal decomposition of $T_{\bar{p}}\bar{M}$, and $\bar{h}(t) : [0, b] \to \text{Hom}(T \circ T, N)$ and $\bar{\nu} : [0, b] \to T$ be smooth maps. Let $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ be an orthonormal basis of $T$ and let $\bar{e}_{n+1}, \ldots, \bar{e}_{n+s}$ be an orthonormal basis of $N$. A curve $\bar{\gamma} : [0, b] \to \bar{M}$ is called a generalized development of $\bar{\nu}$ if there exists a moving frame $\{\bar{E}_A | A = 1, 2, \ldots, n+s\}$ along $\bar{\gamma}$ satisfying the following equations:

$$
\begin{align*}
\bar{\nabla}_{\bar{\nu}(t)} \bar{E}_a &= \sum_{\alpha = n+1}^{n+s} \left\langle \bar{h}(t)(\bar{\nu}(t), \bar{e}_\alpha), \bar{e}_\alpha \right\rangle \bar{E}_a & a = 1, 2, \ldots, n \\
\bar{\nabla}_{\bar{\nu}(t)} \bar{E}_a &= -\sum_{\alpha = 1}^{n} \left\langle \bar{h}(t)(\bar{\nu}(t), \bar{e}_\alpha), \bar{e}_\alpha \right\rangle \bar{E}_a & a = n + 1, \ldots, n+s \\
\bar{\gamma}'(t) &= \sum_{a=1}^{n} \langle \bar{\nu}(t), \bar{e}_a \rangle \bar{E}_a \\
\bar{\gamma}(0) &= \bar{p} \\
\bar{E}_A(0) &= \bar{e}_A & A = 1, 2, \ldots, n+s
\end{align*}
$$

Here $T \circ T$ means the symmetric product of $T$. Moreover, define the map $D^t(\bar{\gamma})$ as

$$
D^t(\bar{\gamma}) : T_{\bar{\gamma}(t_1)} \bar{M} \to T_{\bar{\gamma}(t_2)} \bar{M}, \sum_{A=1}^{n+s} c_A \bar{E}_A(t_1) \mapsto \sum_{A=1}^{n+s} c_A \bar{E}_A(t_2)
$$

for any $c_1, c_2, \ldots, c_{n+s} \in \mathbb{R}$.

It is not hard to see that the definition above is independent of the choices of the orthonormal basis $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n+s}$. When $\bar{h} = 0$, one can see that the generalized developments of curves are just the same as the classical developments of curves in Definition 1.1 and in this case, $D^0(\bar{\gamma}) = P^0(\bar{\gamma})$. By a similar
argument as in [14], one can derive the equations for generalized development of curves and show the local existence and uniqueness of the generalized developments (See Theorem 2.1). We will then denote the curve \( \tilde{\gamma} \) in Definition 1.2 as \( \text{dev}(p, \tilde{v}, \tilde{h}) \) when it exists.

Before stating the existence part of the fundamental theorem for submanifolds in general ambient spaces, we first fix some notations that will be used. Let \((M^n, g)\) be a Riemannian manifold, \((\tilde{M}^{n+\ast}, \tilde{g})\) be two Riemannian manifolds. Let \((V^s, h, D)\) be a Riemannian vector bundle with Riemannian metric \( h \) and compatible connection \( D \), and \( \tilde{h} \in \Gamma(\text{Hom}(TM \otimes TM, V)) \). For any \( \eta \in V_p \), we will define \( A_\eta : T_p M \to T_p \tilde{M} \) by

\[
(A_\eta(X), Y)_g = (h(X, Y), \eta)_h
\]

for any tangent vectors \( X, Y \in T_p M \). We are now ready to state the first main result of this paper, the existence part of the fundamental theorem for submanifolds in general ambient spaces.

**Theorem 1.2.** Let \((M^n, g)\) and \((\tilde{M}^{n+\ast}, \tilde{g})\) be two Riemannian manifolds. Let \((V^s, h, D)\) be a Riemannian vector bundle with Riemannian metric \( h \) and compatible connection \( D \), and \( \tilde{h} \in \Gamma(\text{Hom}(TM \otimes TM, V)) \). Let \( p \in \tilde{M} \), \( \phi : T_p \tilde{M} \oplus V_p \to T_p \tilde{M} \) be a linear isometry, and let \( T := \phi(T_p M) \) and \( N := \phi(V_p) \).

Suppose that \( \tilde{M} \) is simply connected and for any smooth curve \( \gamma : [0, 1] \to \tilde{M} \) with \( \gamma(0) = p \), the generalized development \( \tilde{\gamma} \) of \( \tilde{v} \) and \( \tilde{h} \) exists in \( \tilde{M} \). Here

\[
\tilde{v}(t) = \phi(P^0_t(\gamma)(\gamma'(t)))
\]

and

\[
\tilde{h}(t) = (\phi^{-1})^*P^0_t(\gamma)h
\]

for \( t \in [0, 1] \). Moreover, for any smooth curve \( \gamma : [0, 1] \to \tilde{M} \) with \( \gamma(0) = p \), suppose that

(1) for any \( X, Y, Z, W \in T_{\gamma(1)}M \),

\[
R(X, Y, Z, W) = (\tau^*_\gamma \tilde{R})(X, Y, Z, W) + (h(X, W), h(Y, Z)) - (h(X, Z), h(Y, W)),
\]

where \( R \) and \( \tilde{R} \) are curvature tensors of \( M \) and \( \tilde{M} \) respectively;

(2) for any tangent vectors \( X, Y, Z \in T_{\gamma(1)}M \) and \( \xi \in V_{\gamma(1)} \),

\[
\langle (D_X h)(Y, Z) - (D_Y h)(X, Z), \xi \rangle = (\tau^*_\gamma \tilde{R})(Z, \xi, X, Y);
\]

(3) for any \( X, Y \in T_{\gamma(1)}M \) and \( \xi, \eta \in V_{\gamma(1)} \),

\[
R^V(\xi, \eta, X, Y) = (\tau^*_\gamma \tilde{R})(\xi, \eta, X, Y) + \langle A_\xi(Y), A_\eta(X) \rangle - \langle A_\eta(Y), A_\xi(X) \rangle,
\]

where \( R^V \) is the curvature tensor of the vector bundle \( V \). Here

\[
\tau_\gamma = D^1_0(\gamma) \circ \phi \circ P^0_1(\gamma) : T_{\gamma(1)}M \oplus V_{\gamma(1)} \to T_{\gamma(1)}\tilde{M}.
\]

Then, the map \( f(\gamma(1)) = \tilde{\gamma}(1) \) from \( M \) to \( \tilde{M} \) and the map \( \tilde{f} : V \to T^\perp M \) with \( \tilde{f}|_{\gamma(1)} = \tau_\gamma|_{V_{\gamma(1)}} \) are well defined. Moreover \( f \) is an isometric immersion from \( M \) to \( \tilde{M} \) and \( f(p) = \tilde{p} \) and \( f_* = \phi|_{T_p M} \), and \( \tilde{f} \) is a local linear isometry of Riemannian vector bundles preserving connections such that \( \tilde{f}|_{V_p} = \phi|_{V_p} \) and
\tilde{\gamma}^* h_{\tilde{M}} = h \text{ where } h_{\tilde{M}} \text{ is the second fundamental form of the isometric immersion } f : M \to \tilde{M}.

When the ambient space \((\tilde{M}, \tilde{g})\) in Theorem 1.2 is a space form with constant sectional curvature \(K\),
\[
\tau_{\gamma}^* \tilde{R}(X, Y, Z, W) = K \{ \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \}
\]
because \(\tau_{\gamma}\) is a linear isometry (See Proposition 2.1). It is then clear that Theorem 1.2 contains the existence part of the fundamental theorem for submanifolds in space forms ([2, 14, 15]) because the requirements (1), (2) and (3) in Theorem 1.2 are independent of the curve \(\gamma\) in this case. One advantage of Theorem 1.2 is that it also provide a geometric construction of the isometric immersion when it exists. Moreover, when \(s = 0\), Theorem 1.2 will reduce to Theorem 1.1. So, Theorem 1.2 is in fact a Cartan-Ambrose-Hicks theorem for isometric immersions.

The second main result of this paper is to extend Theorem 1.1 to the case that based on a submanifold which can give us the uniqueness part of fundamental theorem for submanifolds when the ambient space is a space form.

**Theorem 1.3.** Let \((M^n, g)\) and \((\tilde{M}^n, \tilde{g})\) be two Riemannian manifolds, and let \(S^r\) and \(\tilde{S}^r\) be embedded submanifolds of \(M\) and \(\tilde{M}\) respectively. Let \(\varphi : S \to \tilde{S}\) be a local isometry and \(\psi : T^\perp S \to T^\perp \tilde{S}\) be a bundle map along \(\varphi\) preserving metrics, second fundamental forms and normal connections. Suppose that \(S\) is connected and \(\pi_1(M, S)\) is trivial, and for any smooth curve \(\gamma : [0, 1] \to M\) with \(\gamma(0) \in S\), the development \(\tilde{\gamma}\) of \(\tilde{v}\) exists in \(\tilde{M}\). Here
\[
\tilde{v} = (\varphi_* + \psi)(P^1_t(\gamma)(\gamma'(t)))
\]
for \(t \in [0, 1]\). Moreover, suppose that
\[
R = \tau_{\gamma}^* \tilde{R}
\]
for any smooth curve \(\gamma : [0, 1] \to M\) with \(\gamma(0) \in S\) where
\[
\tau_{\gamma} = P^1_0(\gamma) \circ (\varphi_* + \psi) \circ P^0_1(\gamma) : T_{\gamma(1)}M \to T_{\tilde{\gamma}(1)}\tilde{M},
\]
and \(R\) and \(\tilde{R}\) are the curvature tensors of \((M, g)\) and \((\tilde{M}, \tilde{g})\) respectively. Then, the map \(f(\gamma(1)) = \tilde{\gamma}(1)\) from \(M\) to \(\tilde{M}\) is well defined and \(f\) is the local isometry from \(M\) to \(\tilde{M}\) with \(f|_S = \varphi\) and \(f_*|_{T^\perp S} = \psi\).

Note that \(\pi_1(M, S)\) is trivial if and only if for any two curves \(\gamma_0, \gamma_1 : [0, 1] \to M\) with \(\gamma_0(0), \gamma_1(0) \in S\) and \(\gamma_0(1) = \gamma_1(1)\), there is a homotopy \(\Phi : [0, 1] \times [0, 1] \to M\) with
\[
\begin{align*}
\Phi(u, 0) &\in S & \forall u \in [0, 1] \\
\Phi(0, t) &\gamma_0(t) & \forall t \in [0, 1] \\
\Phi(1, t) &\gamma_1(t) & \forall t \in [0, 1] \\
\Phi(u, 1) &\gamma_0(1) = \gamma_1(1) & \forall u \in [0, 1].
\end{align*}
\]
We will also say that $M$ is relatively simply connected with respect to $S$ when $\pi_1(M, S)$ is trivial. When $M$ itself is simply connected and $S$ is connected, by the long exact sequence connecting homotopy groups and relative homotopy groups (see [3, P. 344]), we know that $M$ is also relatively simply connected with respect to $S$. So, one of the advantages in considering Cartan-Ambrose-Hicks theorem based on submanifolds is that one can relax the topological assumption of the source manifold from simply connectedness to relatively simply connectedness. Moreover, when taking $M = M$ as space forms, Theorem 1.3 will give us the uniqueness part of the fundamental theorem for submanifolds (see [2, 14, 15]) since $R = \tau_\ast \bar{R}$ is clearly true in this case. It is clear that Theorem 1.3 reduces to Theorem 1.1 when $r = 0$.

Finally, note that Theorem 1.2 and Theorem 1.3 can be viewed as a Cartan-Ambrose-Hicks theorem for isometric immersions and a Cartan-Ambrose-Hicks theorem based on submanifolds respectively. So, it is natural to consider their combination, a Cartan-Ambrose-Hicks theorem for isometric immersions based on submanifolds. Before stating the result, we first fix some notations that will be used. Let $(M^n, g)$ be a Riemannian manifold, $(V^s, h, D)$ be a Riemannian vector bundle with Riemannian metric $h$ and compatible connection $D$, and $h \in \Gamma(\text{Hom}(TM \oplus TM, V))$. Let $S^r$ be an embedded submanifold of $M$. We will equip with the vector bundle $T^\perp S \oplus V|_S$ the natural direct-sum metric and the connection $\tilde{D}$ defined as follows:

\begin{align}
\tilde{D}X\xi &= \nabla^\perp_X \xi + h(X, \xi) \quad (\forall \, X \in \Gamma(TS), \, \xi \in \Gamma(T^\perp(S))) \\
\tilde{D}X\eta &= -(A_h(X))^\perp + D_X\eta \quad (\forall \, X \in \Gamma(TS), \, \eta \in \Gamma(V)).
\end{align}

Here $\nabla^\perp$ is the normal connection on $T^\perp S$. It is not hard to see that $\tilde{D}$ is compatible with the natural direct-sum metric on $T^\perp S \oplus V|_S$. We are now ready to state the result, a Cartan-Ambrose-Hicks theorem for isometric immersions based on submanifolds.

**Theorem 1.4.** Let $(M^n, g)$ and $(\bar{M}^{n+s}, \bar{g})$ be two Riemannian manifolds. Let $(V^s, h, D)$ be a Riemannian vector bundle with Riemannian metric $h$ and compatible connection $D$, and $h \in \Gamma(\text{Hom}(TM \oplus TM, V))$. Let $S^r$ and $\bar{S}^r$ be embedded submanifolds of $M$ and $\bar{M}$ with second fundamental forms $\sigma$ and $\bar{\sigma}$ respectively. Let $\varphi : S \to \bar{S}$ be a local isometry and $\psi : T^\perp S \oplus V|_S \to T^\perp \bar{S}$ be a bundle map along $\varphi$ that preserves metrics, connections and satisfies

\begin{equation}
\psi^\ast \bar{\sigma} = \sigma + h|_S.
\end{equation}

Suppose that $S$ is connected and $\pi_1(M, S)$ is trivial, and for any smooth curve $\gamma : [0, 1] \to M$ with $\gamma(0) \in S$, the generalized development $\tilde{\gamma}$ of $\tilde{\nu}$ and $\tilde{h}$ exists in $\bar{M}$. Here

\begin{equation}
\tilde{\nu}(t) = \tilde{\psi}\left(P^0_t(\gamma)(\gamma'(t))\right)
\end{equation}

for any $t \in [0, 1]$ and

\begin{equation}
\tilde{h}(t)(x, y) = \psi((P^0_t(\gamma)h)(\tilde{\nu}^{-1}(x), \tilde{\nu}^{-1}(y)))
\end{equation}
for \( t \in [0, 1] \) and any \( x, y \in T \) where \( T := \tilde{\psi}(T_{\gamma(0)}M) \) and
\[
\tilde{\psi} = \varphi_{\gamma(0)} + \psi|_{T_{\gamma(0)}M}.
\]
Moreover, for any smooth curve \( \gamma : [0, 1] \to M \) with \( \gamma(0) \in S \), suppose that
\( (1) \) for any \( X, Y, Z, W \in T_{\gamma(1)}M \),
\[
R(X, Y, Z, W) = \langle (\tau^*_{\gamma} \tilde{R})(X, Y, Z, W) + \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle, \]
where \( R \) and \( \tilde{R} \) are the curvature tensors of \( M \) and \( \tilde{M} \) respectively;
\( (2) \) for any tangent vectors \( X, Y, Z, W \in T_{\gamma(1)}M \) and \( \xi \in V_{\gamma(1)} \),
\[
\langle (DXh)(Y, Z) - (DYh)(X, Z), \xi \rangle = \langle \tau^*_{\gamma} \tilde{R}(Z, \xi, X, Y), \rangle;
\]
\( (3) \) for any \( X, Y \in T_{\gamma(1)}M \) and \( \xi, \eta \in V_{\gamma(1)} \),
\[
R^V(\xi, \eta, X, Y) = \langle \tau^*_{\gamma} \tilde{R}(\xi, \eta, X, Y) + \langle A_\xi(Y), A_\eta(X) \rangle - \langle A_\eta(Y), A_\xi(X) \rangle, \rangle,
\]
where \( R^V \) is the curvature tensor of the vector bundle \( V \). Here
\[
\tau_{\gamma} = D_0(\gamma) \circ (\varphi_{\gamma} + \psi) \circ P_1^0(\gamma) : T_{\gamma(1)}M + V_{\gamma(1)} \to T_{\gamma(1)}\tilde{M}.
\]
Then, the map \( f(\gamma(1)) = \tilde{\gamma}(1) \) from \( M \) to \( \tilde{M} \) and the map \( \tilde{f} : V \to T^\perp \tilde{M} \) with \( \tilde{f}|_{\gamma(1)} = \tau_{\gamma}|_{V_{\gamma(1)}} \) are well defined. Moreover \( f \) is an isometric immersion from \( M \) to \( \tilde{M} \) with \( f|_S = \varphi \) and \( f_s|_S = \varphi_{\gamma} + \psi|_{T^\perp S} \), and \( \tilde{f} \) is a local linear isometry of Riemannian vector bundles preserving connections such that \( \tilde{f}|_S = \psi|(\gamma|_S) \) and \( \tilde{f}^*h_{\tilde{M}} = h \) where \( h_{\tilde{M}} \) is the second fundamental form of the isometric immersion \( f : M \to \tilde{M} \).

The strategy for the proofs of all the main results are similar to the proof of Theorem 1.3 in [16] using the equation for variation fields of variations for developments of curves. This idea can be traced back to the proof of Cartan’s lemma using the Jacobi field equation. The new ingredient here is the generalized development of curves in Definition 1.2. Although Theorem 1.2 and Theorem 1.3 are special cases of Theorem when \( r = 0 \) and when \( s = 0 \) respectively, we will not prove Theorem 1.4 directly and leave Theorem 1.2 and Theorem 1.3 as direct corollaries, because Theorem 1.2 and Theorem 1.3 looks much simpler and more direct for applications, and the ideas of their proofs are easier to catch.

Finally, although Cartan’s lemma (See [1]) is a local version of Ambrose’s result (See [11]), the curvature assumption of Cartan’s lemma is less restrictive because one only need the curvature condition \( R = \tau^*_{\gamma} \tilde{R} \) on geodesics starting at the base point. So, it may be useful to mention the corresponding Cartan’s lemma of the main results in this paper. Because the proof of the result is similar to and simpler than the proof of Theorem 1.4 we will omit the proof for simplicity.

**Theorem 1.5** (Cartan’s lemma for isometric immersions based on submanifolds). Let \((M^n, g)\) and \((\tilde{M}^{n+s}, \tilde{g})\) be two Riemannian manifolds with \( \tilde{M} \) complete. Let \((V^s, h, D)\) be a Riemannian vector bundle with Riemannian metric
where $R$ and $\tilde{R}$ are embedded submanifolds of $M$ and $\tilde{M}$ with second fundamental forms $\sigma$ and $\tilde{\sigma}$ respectively. Let $\varphi : S \to \tilde{S}$ be a local isometry and $\psi : T^\perp S \oplus V|_S \to T^\perp \tilde{S}$ be a bundle map along $\varphi$ that preserves metrics, connections and satisfies

(1.12) \[ \psi^* \sigma = \sigma + h|_S. \]

Let $\Omega$ be an open neighborhood of $S$ such that for each $x \in \Omega$ there is a unique geodesic $\gamma : [0,1] \to \Omega$ with $\gamma(0) \in S$, $\gamma'(0) \perp S$ and $\gamma(1) = x$. For each geodesic $\gamma : [0,1] \to \Omega$ with $\gamma(0) \in S$, $\gamma'(0) \perp S$, let $\tilde{\gamma}$ be the generalized development of $\tilde{\gamma}$ and $h$ where

(1.13) \[ \tilde{\gamma}(t) = \psi(\gamma'(0)) \]

for any $t \in [0,1]$, and

\[ \tilde{h}(t)(x,y) = \psi((P^0_\tilde{\gamma}(\gamma)h)(\tilde{\psi}^{-1}(x),\tilde{\psi}^{-1}(y))) \]

for $t \in [0,1]$ and any $x, y \in T$ where $T = \tilde{\psi}(T_{\gamma(0)}M)$ and

\[ \tilde{\psi} = \varphi \circ \psi|_{T_{\gamma(0)}M}. \]

Moreover, for any geodesic $\gamma : [0,1] \to \Omega$ with $\gamma(0) \in S$, $\gamma'(0) \perp S$, suppose that:

1. For any $X,Y,Z,W \in T_{\gamma(1)}M$,

\[ R(X,Y,Z,W) = (\tau^*_{\gamma} \tilde{R})(X,Y,Z,W) + \langle h(X,W), h(Y,Z) \rangle - \langle h(X,Z), h(Y,W) \rangle, \]

where $R$ and $\tilde{R}$ are the curvature tensors of $M$ and $\tilde{M}$ respectively;

2. For any tangent vectors $X,Y,Z \in T_{\gamma(1)}M$ and $\xi \in V_{\gamma(1)}$,

\[ \langle (D_X h)(Y,Z) - (D_Y h)(X,Z), \xi \rangle = (\tau^*_{\gamma} \tilde{R})(Z,\xi,X,Y); \]

3. For any $X,Y \in T_{\gamma(1)}M$ and $\xi, \eta \in V_{\gamma(1)}$,

\[ R^\gamma(\xi,\eta,X,Y) = (\tau^*_{\gamma} \tilde{R})(\xi,\eta,X,Y) + \langle A_\xi(Y), A_\eta(X) \rangle - \langle A_\eta(Y), A_\xi(X) \rangle, \]

where $R^\gamma$ is the curvature tensor of the vector bundle $V$. Here

\[ \tau^*_{\gamma} = D^0_\tilde{\gamma}(\gamma) \circ (\varphi \circ \psi) \circ P^0_\gamma(\gamma) : T_{\gamma(1)}M \oplus V_{\gamma(1)} \to T_{\tilde{\gamma}(1)}\tilde{M}. \]

Then, the map $f(\gamma(1)) = \tilde{\gamma}(1)$ from $\Omega$ to $\tilde{M}$ is an isometric immersion with $f|_S = \varphi$ and $f_*|_S = \varphi_* + \psi|_{T^\perp S}$. Moreover, the map $\tilde{f} : V|_\Omega \to T^\perp \tilde{M}$ with $\tilde{f}_{|_{\gamma(1)}} = \tau_{\gamma|_{V_{\gamma(1)}}}$ is a local linear isometry of Riemannian vector bundles preserving connections such that $\tilde{f}|_S = \psi|_{(V|_S)}$ and $\tilde{f}^* h_{\tilde{M}} = h$ where $h_{\tilde{M}}$ is the second fundamental form of the isometric immersion $f : \Omega \to \tilde{M}$.

Note that when $r = s = 0$, the last result reduces to the classical Cartan’s lemma. When $r = 0$, it is the Cartan’s lemma corresponding to Theorem 1.2. When $s = 0$, it is the Cartan’s lemma corresponding to Theorem 1.3.

The rest of the paper is organized as follows. In section 2, we show the uniqueness and local existence of generalized developments and prove Theorem 1.2. In Section 3, we will prove Theorem 1.3 and in Section 4, we will prove
Theorem 1.4. For purpose of simplicity, we will adopt the Einstein summation convention for indices and the following conventions of notations throughout the paper:

1. denote indices in \( \{1, 2, \cdots, n + s\} \) as \( A, B, C, D \);
2. denote indices in \( \{1, 2, \cdots, n\} \) as \( a, b, c, d \);
3. denote indices in \( \{1, 2, \cdots, r\} \) as \( i, j \);
4. denote indices in \( \{r + 1, r + 2, \cdots, n\} \) as \( \mu, \nu \);
5. denote indices in \( \{n + 1, n + 2, \cdots, n + s\} \) as \( \alpha, \beta \);
6. the symbol \( \prime \) means taking derivative with respect to \( t \).

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2. Generalized developments of curves and Cartan-Ambrose-Hicks Theorem for isometric immersions

In this section, we will show the local existence and uniqueness of the generalized developments of curves, derive the equation for the variation field of a variation of generalized developments of curves, and finally give the proof of Theorem 1.2.

Theorem 2.1. Let the notations be the same as in Definition 1.2. Then, the generalized development is unique and exists for a short time. Moreover, if the Riemannian manifold \( (\tilde{M}, \tilde{g}) \) is complete, then \( \tilde{\gamma} \) will exist all over \( [0, b] \).

Proof. Let \( (x_1, x_2, \cdots, x_{n+s}) \) be a local coordinate at \( \tilde{p} \) with \( x_A(\tilde{p}) = 0 \) for \( A = 1, 2, \cdots, n + s \), and

\[
\frac{\partial}{\partial x_A}(p) = \tilde{e}_A.
\]

Suppose that \( \tilde{\gamma}(t) = (x_1(t), x_2(t), \cdots, x_{n+s}(t)) \),

\[
\tilde{E}_A = x_{AB} \frac{\partial}{\partial x_B}
\]

and

\[
\tilde{v} = v_a \tilde{e}_a.
\]

Moreover, suppose that

\[
h_{ab}^\alpha(t) = \left< h(t)(\tilde{e}_a, \tilde{e}_b), \tilde{e}_\alpha \right>.
\]

Substituting all the above into (1.7), we have

\[
\begin{cases}
x'_{aA} + x_{aB}x_{bC}v_b \tilde{\Gamma}_{BC}^A - h_{ab}^\alpha v_b x_{aA} = 0 \\
x'_{aA} + x_{aB}x_{bC}v_b \Gamma_{BC}^A + h_{ab}^\alpha v_b x_{aA} = 0 \\
x'_{A} - v_a x_{aA} = 0 \\
x_A(0) = 0 \\
x_{AB}(0) = \delta_{AB}.
\end{cases}
\]

Here \( \tilde{\Gamma}_{AB}^C \)'s are the Christoffel symbols for \( \tilde{M} \). By standard theory for ODEs, we get the local existence and uniqueness for the solution of the equation.
Moreover, when $\tilde{M}$ is complete, by standard extension argument, we get the global existence of $\tilde{\gamma}$. \hfill \Box

Next, we want to show that $D_{t_i}^2(\tilde{\gamma})$ is a linear isometry.

**Proposition 2.1.** Let the notations be the same as in Definition 1.2. Then,

\[
(2.4) \quad \langle \tilde{E}_A, \tilde{E}_B \rangle = \delta_{AB}
\]

for any $A, B = 1, 2, \cdots, n + s$. So $D_{t_i}^2(\tilde{\gamma})$ is an isometry.

**Proof.** Let $X_{AB} = \langle \tilde{E}_A, \tilde{E}_B \rangle - \delta_{AB}$. Then, $X_{AB}(0) = 0$ for any $A, B = 1, 2, \cdots, n + s$. Moreover, by (1.7),

\[
(2.5) \quad X'_{aa} = \frac{d}{dt} \langle \tilde{E}_a, \tilde{E}_a \rangle
\]

and similarly,

\[
(2.6) \quad X'_{ab} = \langle \tilde{h}(t)(\tilde{v}(t), \tilde{e}_a), \tilde{e}_b \rangle X_{ab} + \langle \tilde{h}(t)(\tilde{v}(t), \tilde{e}_b), \tilde{e}_a \rangle X_{aa}
\]

and

\[
(2.7) \quad X'_{a\beta} = -\langle \tilde{h}(t)(\tilde{v}(t), \tilde{e}_a), \tilde{e}_\beta \rangle X_{a\beta} - \langle \tilde{h}(t)(\tilde{v}(t), \tilde{e}_\beta), \tilde{e}_a \rangle X_{aa}.
\]

So, $X_{AB}$’s satisfy a first order homogeneous linear system of ODEs with initial data $X_{AB}(0) = 0$. This implies that $X_{AB}(t) = 0$ for any $t$ and completes the proof of the proposition. \hfill \Box

Next, we come to derive the equation for the variation field of a variation of generalized developments of curves.

**Theorem 2.2.** Let $(\tilde{M}^{n+s}, \tilde{\gamma})$ be a Riemannian manifold and $\tilde{\gamma} \in \tilde{M}$. Let $T_p\tilde{M} = T^n \oplus N^s$ be an orthogonal decomposition of $T_p\tilde{M}$ and $\tilde{v}(u, t) : [0, 1] \times [0, 1] \to T$ and $\tilde{h}(u, t) : [0, 1] \times [0, 1] \to Hom(T \odot T, N)$ be smooth maps. Let

\[
(2.8) \quad \tilde{\Phi}(u, t) = \tilde{\gamma}_0(t) = \text{dev}(\tilde{p}, \tilde{v}(u, \cdot), \tilde{h}(u, \cdot))(t),
\]

$\tilde{e}_1, \cdots, \tilde{e}_n$ be an orthonormal basis for $T$, and $\tilde{e}_{n+1}, \cdots, \tilde{e}_{n+s}$ be an orthonormal basis for $N$. Moreover, let $\tilde{E}_A(u, t) = D_0^k(\tilde{\gamma}_u)(\tilde{e}_A)$ for $A = 1, 2, \cdots, n + s$. Suppose that

\[
(2.9) \quad \tilde{v}(u, t) = v_a(u, t)\tilde{e}_a
\]

and

\[
(2.10) \quad \tilde{h}(u, t)(\tilde{e}_a, \tilde{e}_b) = h_{ab}^0(u, t)\tilde{e}_a.
\]

Let

\[
(2.11) \quad \frac{\partial}{\partial u} := \frac{\partial \tilde{\Phi}}{\partial u} = \tilde{U}_A \tilde{E}_A.
\]
Moreover, by (1.7), we have

\[ \nabla_{\partial_t} E_A = \tilde{X}_{AB} E_B. \]

Then, \( \tilde{X}_{AB} = -\tilde{X}_{BA} \), and

\[
\begin{align*}
\tilde{U}_a'' &= 2\tilde{U}_a h_{ab}^\alpha v_b + \tilde{U}_a \partial_t (h_{ab}^\alpha v_b) + \tilde{U}_c h_{cd}^\alpha h_{ab}^\gamma v_a v_b + \tilde{R}_{bacA} \tilde{U}_a v_b v_c \\
&\quad + \partial_a \tilde{v}_a + (\partial_t v_b) \tilde{X}_{ba} - v_b v_c h_{bc}^\alpha \tilde{X}_{aa} \\
\tilde{U}_a' &= -2\tilde{U}_a h_{ab}^\alpha v_b - \tilde{U}_a \partial_t (h_{ab}^\alpha v_b) + \tilde{U}_b h_{bc}^\beta h_{ab}^\gamma v_a v_c + \tilde{R}_{baaA} \tilde{U}_a v_b \\
&\quad + (\partial_t v_a) \tilde{X}_{aa} + \partial_a (v_a v_b h_{ab}^\gamma) + v_a v_b h_{ab}^\beta \tilde{X}_\beta \\
\tilde{X}_{ab} &= \tilde{X}_{ab} h_{bc}^\alpha v_c - h_{ac}^\alpha v_c \tilde{X}_{ba} + \tilde{R}_{abcA} \tilde{U}_a v_b \\
\tilde{X}_{aa} &= -\tilde{X}_{ab} h_{bc}^\alpha v_c + \partial_a (h_{ab}^\alpha v_b) + h_{ab}^\beta v_b \tilde{X}_\beta + \tilde{R}_{aabA} \tilde{U}_a v_b \\
\tilde{X}_{\alpha\beta} &= \tilde{X}_{ab} h_{bc}^\alpha v_b - \tilde{X}_{\alpha\beta} h_{ab}^\alpha v_b + \tilde{R}_{\alpha\beta} \tilde{U}_a v_b \\
\tilde{U}_A(u,0) &= \tilde{X}_{AB}(u,0) = \tilde{U}_a(u,0) = 0 \\
\tilde{U}_a'(u,0) &= \partial_a \tilde{v}_a(u,0).
\end{align*}
\]

Here \( \tilde{R}_{ABCD} = \tilde{R}(\tilde{E}_A, \tilde{E}_B, \tilde{E}_C, \tilde{E}_D) \) with \( \tilde{R} \) the curvature tensor of \( (\tilde{M}, \tilde{g}) \).

**Proof.** By Proposition 2.1.

\[
\tilde{X}_{AB} = \langle \nabla_{\partial_t} E_A, \tilde{E}_B \rangle = -\langle \tilde{E}_A, \nabla_{\partial_t} E_B \rangle = -\tilde{X}_{BA}.
\]

Moreover, by (1.7), we have

\[
\frac{\partial}{\partial t} = \frac{\partial \tilde{\phi}}{\partial t} = \tilde{\gamma}_a = v_a(u,t) \tilde{E}_a(u,t).
\]

So,

\[
\nabla_{\partial t} \frac{\partial}{\partial t} = \partial_t v_a(u,t) \tilde{E}_a(u,t) + v_a(u,t) \nabla_{\partial t} \tilde{E}_a(u,t) = \partial_t \tilde{v}_a \tilde{E}_a + v_a v_b h_{ab}^\alpha \tilde{E}_\alpha,
\]

\[
\nabla_{\partial u} \frac{\partial}{\partial u} = \tilde{U}_a' \tilde{E}_A + \tilde{U}_a \nabla_{\partial t} \tilde{E}_a + \tilde{U}_a \nabla_{\partial t} \tilde{E}_a = \tilde{U}_a' \tilde{E}_A + \tilde{U}_a h_{ab}^\alpha v_b \tilde{E}_\alpha - \tilde{U}_a h_{ab}^\alpha v_b \tilde{E}_a,
\]

and

\[
\nabla_{\partial u} \nabla_{\partial u} \frac{\partial}{\partial u} = \tilde{U}_a'' \tilde{E}_A + 2\tilde{U}_a h_{ab}^\alpha v_b \tilde{E}_\alpha - 2\tilde{U}_a h_{ab}^\alpha v_b \tilde{E}_a
\]

\[
\quad + [\tilde{U}_c \partial_t (h_{bc}^\alpha v_b) - \tilde{U}_\beta h_{bc}^\beta h_{ab}^\gamma v_a v_c] \tilde{E}_\alpha - [\tilde{U}_\alpha \partial_t (h_{ab}^\alpha v_b) + \tilde{U}_c h_{cd}^\beta h_{ab}^\gamma v_d v_b] \tilde{E}_a.
\]
Similarly, by computing
\[ \tilde{\nabla}_a \tilde{\nabla}_a \frac{\partial}{\partial t} = \tilde{\nabla}_a \tilde{\nabla}_a \frac{\partial}{\partial t} = \tilde{\nabla}_a \tilde{\nabla}_a \frac{\partial}{\partial t} + \tilde{R} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial u} \right) \frac{\partial}{\partial t} \] 

Comparing (2.18) and (2.19), and by that \( \tilde{X}_{AB} = -\tilde{X}_{BA} \), we have
\[ \tilde{U}''_a = 2\tilde{U}'_a h^a_{ab} v_b + \tilde{U}_a \partial_t (h^a_{ab} v_b) + \tilde{U}_c h^a_{cd} h^c_{ab} v_d v_b \]
\[ + \tilde{R}_{bacA} \tilde{U}_A v_b v_c + \partial_a \partial_t v_a + (\partial_t v_b) \tilde{X}_{ba} - v_b v_c h^c_{bc} \tilde{X}_{aa} \]

Moreover,
\[ \tilde{\nabla}_a \tilde{\nabla}_a \tilde{E}_a = \tilde{\nabla}_a \tilde{\nabla}_a (\tilde{X}_{aa} \tilde{E}_a) + \tilde{\nabla}_a \tilde{\nabla}_a (\tilde{X}_{ab} \tilde{E}_b) \]
\[ = (\tilde{X}'_{aa} + \tilde{X}_{ab} h^a_{bc} v_c) \tilde{E}_a + (\tilde{X}'_{ab} - \tilde{X}_{aa} h^a_{bc} v_c) \tilde{E}_b \]

and on the other hand,
\[ \tilde{\nabla}_a \tilde{\nabla}_a \tilde{E}_a = \tilde{\nabla}_a \tilde{\nabla}_a \tilde{E}_a + \tilde{R} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial u} \right) \tilde{E}_a \]
\[ = (h^a_{ac} v_c \tilde{X}_{ab} + \tilde{R}_{abcA} \tilde{U}_A v_c) \tilde{E}_b + [\partial_a (h^a_{ab} v_b) + h^\beta_{ab} v_b \tilde{X}_{\beta a} + \tilde{R}_{aabA} \tilde{U}_A v_b] \tilde{E}_a. \]

Hence, by comparing (2.22) and (2.23), and by that \( \tilde{X}_{AB} = -\tilde{X}_{BA} \),
\[ \tilde{X}'_{ab} = \tilde{X}_{aa} h^a_{bc} v_c - h^a_{ac} v_c \tilde{X}_{ba} + \tilde{R}_{abcA} \tilde{U}_A v_c \]

and
\[ \tilde{X}'_{aa} = -\tilde{X}_{ab} h^a_{bc} v_c + \partial_a (h^a_{ab} v_b) + h^\beta_{ab} v_b \tilde{X}_{\beta a} + \tilde{R}_{aabA} \tilde{U}_A v_b. \]

Similarly, by computing \( \tilde{\nabla}_a \tilde{\nabla}_a \tilde{E}_a \) as the above, one has
\[ \tilde{X}'_{\alpha \beta} = \tilde{X}_{aa} h^a_{ab} v_b - \tilde{X}_{a\beta} h^a_{ab} v_b + \tilde{R}_{\alpha \beta a} v_a \tilde{U}_A. \]
Finally, $\tilde{U}_A(u, 0) = 0$ because $\Phi(u, 0) = \tilde{p}$, and $X_{AB}(u, 0) = 0$ because $\tilde{E}_A(u, 0) = \tilde{e}_A$. Moreover, by

$$\tilde{U}'_A(u, 0)\tilde{e}_A = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial u}} \bigg|_{t=0}$$

(2.27)

$$= \lim_{t \to 0^+} \nabla_{\frac{\partial}{\partial u}} (v_a \tilde{E}_a)$$

$$= (\partial_u v_a(u, 0))\tilde{e}_a,$$

we know that $\tilde{U}'_a(u, 0) = \partial_u v_a(u, 0)$ and $\tilde{U}'_a(u, 0) = 0$. \hfill \Box$

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $e_1, e_2, \cdots, e_n$ and $e_{n+1}, e_{n+2}, \cdots, e_{n+s}$ be orthonormal basis of $T_p M$ and $V_p$ respectively, and let $\tilde{e}_A = \varphi(e_A)$ for $A = 1, 2, \cdots, n + r$. For $x \in M$, let $\gamma_0, \gamma_1 : [0, 1] \to M$ be two smooth curves joining $p$ to $x$. Since $M$ is simply connected, there is a smooth map $\Phi : [0, 1] \times [0, 1] \to M$ such that

$$\begin{cases}
\Phi(0, t) = \gamma_0(t) & \text{for } t \in [0, 1] \\
\Phi(1, t) = \gamma_1(t) & \text{for } t \in [0, 1] \\
\Phi(u, 0) = p & \text{for } u \in [0, 1] \\
\Phi(u, 1) = x & \text{for } u \in [0, 1].
\end{cases}
$$

(2.28)

Let $\gamma_u(t) = \Phi(u, t)$ for any $u \in [0, 1]$ and

$$v(u, t) = P^0_t(\gamma_u)(\gamma'_u(t)).$$

(2.29)

Then $\gamma_u$ is the development of $v(u, \cdot)$. Let $\tilde{v} = \varphi(v)$, and

$$\tilde{h}(u, t) = (\varphi^{-1})^*P^0_t(\gamma_u)h(u, t).$$

Moreover, let

$$\tilde{\Phi}(u, t) = \tilde{\gamma}_u(t) = \text{dev}(\tilde{p}, \tilde{v}(u, \cdot), \tilde{h}(u, \cdot))(t),$$

$$\tilde{E}_A(u, t) = D^0_t(\tilde{\gamma}_u)(\tilde{e}_A)$$

and

$$E_A(u, t) = P^t_0(\gamma_u)(e_A)$$

for $A = 1, 2, \cdots, n + s$. Suppose that

$$v = v_a e_a$$

and

$$h(u, t)(e_a, e_b) = h^a_{ab}(u, t)e_a.$$  

(2.30)

Then, it is clear that

$$\tilde{v} = v_a \tilde{e}_a$$

and

$$\tilde{h}(\tilde{e}_a, \tilde{e}_b) = h^a_{ab} \tilde{e}_a.$$  

(2.31)
Suppose that

\[
\frac{\partial \Phi}{\partial u} = U_a E_a, \tag{2.34}
\]

\[
\nabla \frac{\partial}{\partial u} E_a = X_{ab} E_b \tag{2.35}
\]

and

\[
D \frac{\partial}{\partial u} E_\alpha = X_{\alpha\beta} E_\beta. \tag{2.36}
\]

Then,

\[
X'_{\alpha\beta} E_\beta = \left( D \frac{\partial}{\partial t} D \frac{\partial}{\partial u} E_\alpha + R^V \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial u} \right) \right) E_\alpha = R^V_{\alpha\beta ab} v_a U_b E_\beta. \tag{2.37}
\]

Combining this and Lemma 2.2 in [16], one has

\[
\left\{
\begin{array}{l}
U''_a = R_{cadb} v_c v_d U_b + \partial_u \partial_t v_a + \partial_t v_b X_{ba} \\
X'_{ab} = R_{abdc} v_d U_c \\
X'_{\alpha\beta} = R^V_{\alpha\beta ab} v_a U_b \\
X_{ab}(u, 0) = 0 \\
X_{\alpha\beta}(u, 0) = 0 \\
U_a(u, 0) = 0 \\
U'_a(u, 0) = \partial_u v_a(u, 0),
\end{array}\right. \tag{2.38}
\]

where \(R_{abcd} = R(E_a, E_b, E_c, E_d)\) and \(R^V_{\alpha\beta ab} = R^V(E_\alpha, E_\beta, E_a, E_b)\).

Furthermore, suppose that

\[
\frac{\partial \tilde{\Phi}}{\partial u} = \tilde{U}_A E_A, \tag{2.39}
\]

and

\[
\nabla \frac{\partial}{\partial u} \tilde{E}_A = \tilde{X}_{AB} \tilde{E}_B. \tag{2.40}
\]

We claim that

\[
\left\{
\begin{array}{l}
\tilde{U}_a = U_a \\
\tilde{U}_\alpha = 0 \\
\tilde{X}_{ab} = X_{ab} \\
\tilde{X}_{\alpha\beta} = X_{\alpha\beta} \\
\tilde{X}_{\alpha\alpha} = h_{ab} U_b.
\end{array}\right. \tag{2.41}
\]

By Theorem 2.2, we only need to verify that the \(\tilde{U}_A\)’s and \(\tilde{X}_{AB}\)’s defined above satisfy the Cauchy problem (2.13).

The initial data in (2.13) are clearly satisfied in (2.41) by the initial data in (2.38).
Moreover, by (2.38) and assumption (1) in the statement of Theorem 1.2

\[
\tilde{U}_a'' = U_a'' \\
= R_{cadb} v_c v_d U_b + \partial_a \partial_v u_a + \partial_t v_b X_{ba}
\]

(2.42)

\[
= (\tilde{R}_{cadb} + h^\alpha_{cb} h^\alpha_{ad} - h^\alpha_{cd} h^\alpha_{ab}) v_c v_d U_b + \partial_a \partial_v u_a + \partial_t v_b X_{ba}
\]

\[
= 2 \tilde{U}_a h_{ab} v_b + \tilde{U}_a \partial_t (h_{ab} v_b) + \tilde{U}_c h^\alpha_{cd} h^\alpha_{ab} v_d v_b + \tilde{R}_{bacd} \tilde{U}_d v_c \\
+ \partial_a \partial_t v_a + (\partial_t v_b) \tilde{X}_{ba} - v_b v_c h_{bc} \tilde{X}_{aa}.
\]

So, the first equation in (2.13) is satisfied.

By assumption (2) in the statement of Theorem 1.2

(2.43)

\[
\partial_a (v_a v_b h^\alpha_{ab})
\]

\[
= \partial_a \left( \left\langle h \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) , E_\alpha \right\rangle \right) \\
= \left\langle \frac{D_{\partial v}}{\partial v} \left( \left\langle h \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) , E_\alpha \right\rangle \right) , E_\alpha \right\rangle + \left\langle h \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) , D_{\partial v} E_\alpha \right\rangle \\
= \left\langle \left( \frac{D_{\partial v}}{\partial v} h \right) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) , E_\alpha \right\rangle + 2 \left\langle h \left( \nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial t} \right) , E_\alpha \right\rangle + \left\langle h \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) , D_{\frac{\partial}{\partial v}} E_\alpha \right\rangle \\
= \left\langle \left( \nabla_{\frac{\partial}{\partial v}} h \right) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) , E_\alpha \right\rangle + \tilde{R}_{bacd} v_a v_b U_c \\
+ 2 \left\langle h \left( \nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial t} \right) , E_\alpha \right\rangle + \left\langle h \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) , D_{\frac{\partial}{\partial v}} E_\alpha \right\rangle \\
= (\partial_t h^\alpha_{ab}) U_a v_b + \tilde{R}_{bacd} v_a v_b \tilde{U}_d + 2 U_a h^\alpha_{ab} v_b + h^\beta_{ab} v_a v_b X_{\alpha\beta}.
\]

So, by the last equation,

(2.44)

\[
-2 \tilde{U}_a h^\alpha_{ab} v_b - \tilde{U}_c \partial_t (h^\alpha_{bc} v_b) + \tilde{U}_\beta h^\beta_{bc} h^\alpha_{ab} v_a v_c \\
+ \tilde{R}_{bac} \tilde{U}_d v_a v_b + (\partial_t v_a) \tilde{X}_{aa} + \partial_a (v_a v_b h^\alpha_{ab}) + v_a v_b h^\beta_{ab} \tilde{X}_{\beta a} \\
= -2 U_a h^\alpha_{ab} v_b - U_c \partial_t (h^\alpha_{bc} v_b) + \tilde{R}_{bac} \tilde{U}_d v_a v_b \\
+ (\partial_t v_a) h^\alpha_{ab} U_b + [(\partial_t h^\alpha_{ab}) U_a v_b + \tilde{R}_{bac} v_a v_b \tilde{U}_d + 2 U_a h^\alpha_{ab} v_b + h^\beta_{ab} v_a v_b X_{\alpha\beta}] + v_a v_b h^\beta_{ab} X_{\beta a} \\
= 0 \\
= \tilde{U}_a''.
\]

Hence, the second equation in (2.13) is satisfied.
By (2.45) and assumption (1) in the statement of Theorem 1.2,
\[
\tilde{X}_{ab} = X_{ab} = R_{abcd} v_c U_d = (\tilde{R}_{abcd} + h_{ad}^{\alpha} h_{bc}^{\alpha} - h_{ac}^{\alpha} h_{bd}^{\alpha}) v_c U_d = \tilde{X}_{aa} h_{bc}^{\alpha} v_c - h_{ac}^{\alpha} v_c \tilde{X}_{ba} + \tilde{R}_{abcA} \tilde{U}_A v_c.
\]

By assumption (2) in the statement of Theorem 1.2,
\[
\tilde{X}_{a\alpha} = \partial_t (h_{a\alpha} U_b)
\]
\[
= \partial_t \left( h \left( \frac{\partial}{\partial u}, E_a \right), E_\alpha \right) + \left( D_{a\alpha} h \left( \frac{\partial}{\partial u}, E_a \right), E_\alpha \right) + \left( D_{\alpha\alpha} h \left( \frac{\partial}{\partial u}, E_a \right), E_\alpha \right) + \tilde{R}_{ab\alpha} U_c v_b + \tilde{R}_{ab\alpha} U_c v_b + \tilde{X}_{aa} h_{bc}^{\alpha} v_c - h_{ac}^{\alpha} v_c \tilde{X}_{ba} + \tilde{R}_{ab\alpha} U_c v_b.
\]
Finally, by assumption (3) in the statement of Theorem 1.2 and (2.38),
\[
\tilde{X}_{a\beta} = X_{a\beta} = R_{\alpha\beta ab} v_a U_b = (\tilde{R}_{\alpha\beta ab} + h_{bc}^{\alpha} h_{ac}^{\beta} - h_{ac}^{\alpha} h_{bc}^{\beta}) v_a U_b = \tilde{X}_{aa} h_{bc}^{\alpha} v_c - h_{ac}^{\alpha} v_c \tilde{X}_{ba} + \tilde{R}_{aa\alpha A} v_a \tilde{U}_A.
\]
So, all the equations in (2.13) are satisfied by (2.41). This completes the proof the claim.

By the claim, we know that \( \tilde{U}_A(u, 1) = 0 \) for any \( u \) and \( A = 1, 2, \ldots, n + s \). This implies that \( \gamma_0(1) = \gamma_1(1) \). So, \( f \) is well defined. Moreover, note that \( f_* (\frac{\partial \Phi}{\partial u}) = \frac{\partial \Phi}{\partial u} \) and
\[
\left\| \frac{\partial \Phi}{\partial u} \right\| = \sqrt{\sum_{a=1}^{n} U_a^2} = \sqrt{\sum_{a=1}^{A} \tilde{U}_a^2} = \left\| \frac{\partial \tilde{\Phi}}{\partial u} \right\|,
\]
so \( f \) is a local isometry. By that \( \tilde{X}_{\alpha\beta} = X_{\alpha\beta} \), we know that \( \tilde{f} \) is well-defined. The other properties of \( f \) and \( \tilde{f} \) are not hard to be verified by noting that \( f_*E_i = \tilde{E}_i \) and \( \tilde{f}E_\alpha = \tilde{E}_\alpha \). This completes the proof of the theorem. \( \square \)

3. Cartan-Ambrose-Hicks theorem based on submanifolds

In this section, we will prove Theorem 1.3. We first derive the equation for the variation field of a variation of developments starting from a submanifold. The only difference with the case we discussed before in [16, Lemma 2.2] is the initial data.

**Lemma 3.1.** Let \((M^n, g)\) be a Riemannian manifold, \(S^r\) be an embedded submanifold of \(M\) and \(\sigma\) be its second fundamental form. Let \(\theta(u) : [0, 1] \to S\) be a curve in \(S\) and \(v(u, t) : [0, 1] \times [0, 1] \to TM\) be a smooth map such that \(v(u, t) \in T_{\theta(u)}M\). Let

\[
\Phi(u, t) = \gamma_u(t) = \text{dev}(\theta(u), v(u, \cdot))(t).
\]

Moreover, let \(e_1, e_2, \ldots, e_r\) be an orthonormal frame parallel along \(\theta\) on \(S\) and \(e_{r+1}, e_{r+2}, \ldots, e_n\) be an orthonormal frame parallel along \(\theta\) on the normal bundle \(T^\perp S\) of \(S\). Let

\[
E_a(u, t) = P^t_0(\gamma_u)(e_a(u)).
\]

Suppose that

\[
\frac{\partial}{\partial u} := \frac{\partial \Phi}{\partial u} = U_a E_a
\]

and

\[
\nabla_a E_a = X_{ab} E_b.
\]

Moreover, suppose that \(v = v_a e_a, \theta' = \theta_i e_i\) and

\[
\sigma'_{ij} = \langle \sigma(e_i, e_j), e_\mu \rangle.
\]

Then, \(X_{ab} = -X_{ba}\), and

\[
\begin{align*}
U_a'' &= R_{abcd}v_b v_c U_d + \partial_u \partial_t v_a + \partial_t v_b X_{ba} \\
X_{ab}' &= R_{abcd} v_c U_d \\
U_i(u, 0) &= \theta_i(u) \\
U_\mu(u, 0) &= X_{ij}(u, 0) = X_{\mu\nu}(u, 0) = 0 \\
X_{ij}(u, 0) &= \sigma_{ij}' \theta_j(u) \\
U_i'(u, 0) &= \partial_u v_i(u, 0) - v_\mu(u, 0) \sigma_{ij}' \theta_j(u) \\
U_\mu'(u, 0) &= \partial_u v_\mu(u, 0) + v_i(u, 0) \sigma_{ij}' \theta_j(u).
\end{align*}
\]

Here \(R_{abcd} = R(E_a, E_b, E_c, E_d)\) with \(R\) is curvature tensor of \((M, g)\).

**Proof.** The proof of the equations is the same as in [16]. We only need to show the initial data. Note that

\[
\begin{align*}
\frac{\partial}{\partial u} \bigg|_{t=0} &= \theta'.
\end{align*}
\]
So, \( U_i(u, 0) = \theta_i(u) \) and \( U_\mu(u, 0) = 0 \). Because \( e_i \) is parallel along \( \theta \) on \( S \), we have

\[
\nabla_{\frac{\partial}{\partial u}} E_i \bigg|_{t=0} = \nabla_{\theta'} e_i = h(\theta', e_i) = \sigma_{ij}^\mu \theta_j e_\mu.
\]

So, \( X_{ij}(u, 0) = 0 \) and \( X_{i\mu}(u, 0) = \sigma_{ij}^\mu \theta_j(u) \). Similarly, because \( e_\mu \) is parallel along \( \theta \) on \( T^\perp S \), we have

\[
\nabla_{\frac{\partial}{\partial u}} E_\mu \bigg|_{t=0} = \nabla_{\theta'} e_\mu = -\sigma_{ij}^\mu \theta_j e_i.
\]

So, \( X_{\mu\nu}(u, 0) = 0 \). Moreover,

\[
\begin{align*}
\nabla_{\frac{\partial}{\partial u}} & \frac{\partial}{\partial t} \bigg|_{t=0} \\
= & \nabla_{\frac{\partial}{\partial u}} (v_a E_a) \bigg|_{t=0} \\
= & (\partial_a v_a(u, 0) + v_b(u, 0) X_{ba}(u, 0)) e_a.
\end{align*}
\]

So,

\[
\begin{align*}
U'_i(u, 0) & = \partial_a v_i(u, 0) - v_\mu X_{i\mu}(u, 0) = \partial_a v_i(u, 0) - v_\mu(u, 0) \sigma_{ij}^\mu \theta_j(u) \\
\text{and} \\
U'_\mu(u, 0) & = \partial_a v_\mu(u, 0) + v_i X_{i\mu}(u, 0) = \partial_a v_\mu(u, 0) + v_i(u, 0) \sigma_{ij}^\mu \theta_j(u).
\end{align*}
\]

These complete the proof of the lemma.

We are now ready to prove Theorem \[1.3\].

Proof of Theorem \[1.3\]: For each \( x \in M \), let \( \gamma_0, \gamma_1 : [0, 1] \to M \) be two smooth curves with \( \gamma_0(0), \gamma_1(0) \in S \) and \( \gamma_0(1) = \gamma_1(1) = x \). Because \( \pi_1(M, S) \) is trivial, there is a smooth homotopy \( \Phi : [0, 1] \times [0, 1] \to M \) such that

\[
\begin{align*}
\Phi(u, 0) & \in S \quad \forall u \in [0, 1] \\
\Phi(0, t) & = \gamma_0(t) \quad \forall t \in [0, 1] \\
\Phi(1, t) & = \gamma_1(t) \quad \forall t \in [0, 1] \\
\Phi(u, 1) & = x \quad \forall u \in [0, 1].
\end{align*}
\]

Let \( \gamma_u(t) = \Phi(u, t), \theta(u) = \Phi(u, 0) \) and

\[
v(u, t) = P_t^0(\gamma_u(\gamma'_u(t))).
\]

Then

\[
\gamma_u = \text{dev}(\theta(u), v(u, \cdot)).
\]

Let \( \bar{v} = \psi(v), \bar{\theta}(u) = \varphi(\theta(u)) \) and

\[
\Phi(u, t) = \bar{\gamma}_u(t) = \text{dev}(\bar{\theta}(u), \bar{v}(u, \cdot))(t).
\]

(3.9)
Moreover, let $e_1, e_2, \ldots, e_r$ be an orthonormal frame parallel along $\theta$ on $S$ and $e_{r+1}, e_{r+2}, \ldots, e_n$ be an orthonormal frame parallel along $\theta$ on $T^\perp S$. Let $\tilde{e}_i = \varphi_*(e_i)$ and $\tilde{e}_\mu = \psi(e_\mu)$. Then, $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_r$ is an orthonormal frame parallel along $\tilde{\theta}$ on $\tilde{S}$ since $\varphi$ is a local isometry, and $\tilde{e}_{r+1}, \tilde{e}_{r+2}, \ldots, \tilde{e}_n$ is an orthonormal frame parallel along $\tilde{\theta}$ on $T^\perp \tilde{S}$ since $\psi$ preserves metrics and normal connections.

Suppose that
\begin{align*}
(3.10) \quad & v = u_a e_a, \\
(3.11) \quad & \theta' = \theta_i e_i,
\end{align*}
and
\begin{equation}
(3.12) \quad \sigma^\mu_{ij} = \langle \sigma(e_i, e_j), e_\mu \rangle
\end{equation}
where $\sigma$ is the second fundamental form of $S$. It is clear that
\begin{align*}
(3.13) \quad & \tilde{v} = v_a \tilde{e}_a, \\
(3.14) \quad & \tilde{\theta}' = \theta_i \tilde{e}_i,
\end{align*}
and
\begin{equation}
(3.15) \quad \langle \tilde{\sigma}(\tilde{e}_i, \tilde{e}_j), \tilde{e}_\mu \rangle = \sigma^\mu_{ij}
\end{equation}
where $\tilde{\sigma}$ is the second fundamental form of $\tilde{S}$, since that $\psi$ preserves second fundamentals and $\varphi$ is a local isometry.

Furthermore, let $E_a(u, t) = P_0^t(\gamma_u)(e_a(u))$ and $\tilde{E}_a(u, t) = P_0^t(\tilde{\gamma}_u)(\tilde{e}_a(u))$. Suppose that
\begin{equation}
(3.16) \quad \frac{\partial \Phi}{\partial u} = U_a E_a
\end{equation}
and
\begin{equation}
(3.17) \quad \frac{\partial \tilde{\Phi}}{\partial u} = \tilde{U}_a \tilde{E}_a.
\end{equation}
Note that from the curvature assumption, we have
\begin{equation}
(3.18) \quad R_{abcd} = \tilde{R}_{abcd}.
\end{equation}
So, by Lemma 3.1 and the above, $U_a$'s and $\tilde{U}_a$'s will satisfy the same Cauchy problems for ODEs. Therefore
\begin{equation}
(3.19) \quad U_a(u, t) = \tilde{U}_a(u, t).
\end{equation}
In particular, $\tilde{U}_a(u, 1) = U_a(u, 1) = 0$ and hence $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$ which implies that $f$ is well defined. It is then not hard to verify that $f$ is the local isometry satisfying the properties $f|_S = \varphi$ and $f_*|_{T^\perp S} = \psi$.
\hfill \Box
4. Cartan-Ambrose-Hicks theorem based on submanifolds for isometric immersions

In this section, we will prove Theorem 1.4. Similar as before, we first come to derive the equation for the variation field of a variation of generalized developments of curves starting from a submanifold.

**Lemma 4.1.** Let \((M^n, g)\) and \((\tilde{M}^{n+s}, \tilde{g})\) be two Riemannian manifolds. Let \((V^s, \tilde{h}, D)\) be a Riemannian vector bundle with Riemannian metric \(\tilde{h}\) and compatible connection \(D\), and \(h \in \Gamma(\text{Hom}(TM \otimes TM, V))\). Let \(S^r\) and \(\tilde{S}^r\) be embedded submanifolds of \(M\) and \(\tilde{M}\) with second fundamental forms \(\sigma\) and \(\tilde{\sigma}\) respectively. Let \(\varphi : S \to \tilde{S}\) be a local isometry and \(\psi : T^1S \oplus V_{\mid S} \to T^1\tilde{S}\) be a bundle map along \(\varphi\) that preserves metrics, connections and satisfies

\[
(4.1) \quad \psi^* \tilde{\sigma} = \sigma + h_{\mid S}.
\]

Let \(\tilde{\psi} : TM_{\mid S} \to T\tilde{M}_{\mid S}\) be the bundle map along \(\varphi\) given by

\[
\tilde{\psi} = \varphi_* + \psi_{\mid T^1S}.
\]

Let \(\theta : [0, 1] \to S\) be a smooth curve on \(S\) and \(\tilde{\theta} = \varphi \circ \theta\), and let \(v(u, t) : [0, 1] \times [0, 1] \to TM\) be a smooth map with \(v(u, t) \in T_{\theta(u)}M\) and \(\tilde{v} = \tilde{\psi}(v)\). Let

\[
T(u) = \tilde{\psi}(T_{\theta(u)}M) \subset T_{\tilde{\theta}(u)}\tilde{M}
\]

and

\[
N(u) = \psi(V_{\mid \theta(u)}) \subset T_{\tilde{\theta}(u)}\tilde{M}.
\]

Let

\[
\tilde{h}(u, t) : [0, 1] \times [0, 1] \to \text{Hom}(T \otimes T, N)
\]

be a smooth map with \(\tilde{h}(u, t) \in \text{Hom}(T(u) \otimes T(u), N(u))\) defined by

\[
\tilde{h}(u, t)(x, y) = \psi((P^0_\theta(\gamma_u)(h)) (\tilde{\psi}^{-1}(x), \tilde{\psi}^{-1}(y)))
\]

for any \(x, y \in T(u)\) where \(\gamma_u = \text{dev}(\theta(u), v(u, \cdot))\). Moreover, let

\[
(4.2) \quad \tilde{\Phi}(u, t) = \tilde{\gamma}_u(t) = \text{dev}(\tilde{\theta}(u), \tilde{v}(u, \cdot), \tilde{h}(u, \cdot))(t),
\]

\(e_1, \ldots, e_r\) be an orthonormal frame parallel along \(\theta\) on \(S\), \(e_{r+1}, e_{r+2}, \ldots, e_n\) be an orthonormal frame parallel along \(\theta\) on \(T^1S\), and \(e_{n+1}, \ldots, e_{n+s}\) be an orthonormal frame parallel along \(\theta\) for \(V\). Let \(\tilde{e}_A = (\varphi_* + \psi)(e_A), \tilde{E}_A(u, t) = P^0_\theta(\gamma_u)(e_A(u))\) and \(\tilde{E}_A(u, t) = D^0_0(\tilde{\gamma}_u)(\tilde{e}_A(u))\). Suppose that

\[
(4.3) \quad \theta'(u) = \theta_i(u)e_i(u),
\]

\[
(4.4) \quad \sigma(e_i, e_j) = \sigma^\mu_{ij}e_\mu,
\]

\[
(4.5) \quad v(u, t) = v_a(u, t)e_a(u)
\]

and

\[
(4.6) \quad h(E_a(u, t), E_b(u, t)) = h^\alpha_{ab}(u, t)E_a(u, t).
\]
Let
\[
(4.7) \quad \frac{\partial}{\partial u} := \frac{\partial \Phi}{\partial u} = \tilde{U}_A \tilde{E}_A
\]
and
\[
(4.8) \quad \tilde{\nabla}_{\frac{\partial}{\partial u}} \tilde{E}_A = \tilde{X}_{AB} \tilde{E}_B.
\]

Then, \( \tilde{X}_{AB} = -\tilde{X}_{BA} \) and
\[
(4.9) \quad \begin{aligned}
\tilde{U}_a'' &= 2\tilde{U}_a' h_{ab}^\alpha v_b + \tilde{U}_a \partial_i (h_{ab}^\alpha v_b) + \tilde{U}_b h_{bc}^\alpha h_{ad}^\beta v_d + \tilde{R}_{bacA} \tilde{U}_a v_b v_c \\
&\quad + \partial_a \partial_i v_a + (\partial_i v_b) \tilde{X}_b a - v_b v_c h_{bc}^\alpha \tilde{X}_a a \\
\tilde{U}_a' &= -2\tilde{U}_a' h_{ab}^\alpha v_b - \tilde{U}_a \partial_i (h_{ab}^\alpha v_b) + \tilde{U}_b h_{bc}^\beta h_{ad} v_a v_c + \tilde{R}_{aabA} \tilde{U}_a v_b v_c \\
&\quad + (\partial_i v_a) \tilde{X}_a a + \partial_a (v_a v_b h_{ab}^\alpha) + v_a v_b h_{ab}^\beta \tilde{X}_a a \\
\tilde{X}_{i\alpha} &= \tilde{X}_{a\alpha} h_{bc}^\alpha v_c - h_{ai}^\alpha v_c \tilde{X}_b a + \tilde{R}_{abcA} \tilde{U}_a v_c \\
\tilde{X}_{i\beta} &= \tilde{X}_{a\beta} h_{bc}^\beta v_b - h_{ai}^\beta v_b \tilde{X}_b a + \tilde{R}_{aabA} \tilde{U}_a v_b \\
\tilde{U}_i (u, 0) &= t_i (u) \\
\tilde{U}_\mu (u, 0) &= \tilde{U}_\alpha (u, 0) = \tilde{U}_i' (u, 0) = \tilde{X}_{i\alpha} (u, 0) = \tilde{X}_{i\beta} (u, 0) = 0 \\
\tilde{X}_{i\mu} (u, 0) &= \sigma_{ij}^\mu \theta_j (u) \\
\tilde{X}_{i\alpha} (u, 0) &= h_{ai}^\alpha \theta_i (u) \\
\tilde{U}_i' (u, 0) &= \partial_a v_i (u, 0) - v_\mu (u, 0) \sigma_{ij}^\mu \theta_j (u) \\
\tilde{U}_\mu' (u, 0) &= \partial_a v_\mu (u, 0) + v_i (u, 0) \sigma_{ij}^\mu \theta_j (u).
\end{aligned}
\]

Here \( \tilde{R}_{ABCD} = \tilde{R} (\tilde{E}_A, \tilde{E}_B, \tilde{E}_C, \tilde{E}_D) \) with \( \tilde{R} \) the curvature tensor of \((\tilde{M}, \tilde{g})\).

**Proof.** The proofs of the ODEs for \( \tilde{U}_A \) and \( \tilde{X}_{AB} \) in (4.9) are the same as the proof of (2.13). We only need to verify the initial data.

Note that
\[
(4.10) \quad \frac{\partial}{\partial u} \bigg|_{t=0} = \tilde{\theta}' (u) = \theta_i (u) \tilde{e}_i.
\]

So, \( \tilde{U}_i (u, 0) = \theta_i (u) \) and \( \tilde{U}_\mu (u, 0) = \tilde{U}_\alpha (u, 0) = 0 \). By that \( \tilde{e}_i \) is parallel along \( \tilde{\theta} \) on \( \tilde{S} \) and (4.11),
\[
(4.11) \quad \tilde{\nabla}_{\frac{\partial}{\partial u}} \tilde{E}_i \bigg|_{t=0} = \tilde{\nabla}_{\tilde{\theta}'} \tilde{e}_i = \tilde{\sigma} (\tilde{\theta}', \tilde{e}_i) = \sigma_{ij}^\mu \theta_j \tilde{e}_\mu + h_{ij}^\alpha \theta_j \tilde{e}_\alpha,
\]
we have $\tilde{X}_{ij}(u, 0) = 0$, $\tilde{X}_{i\mu}(u, 0) = \sigma_{ij}^\mu \theta_j(u)$, and $\tilde{X}_{i\alpha}(u, 0) = h_{ij}^\alpha \theta_j(u)$. Moreover, by that $\psi$ preserves connections,

$$\tilde{X}_{\mu\nu}(u, 0) = \left\langle \tilde{\nabla}_{\partial t} \frac{\partial}{\partial u} \tilde{E}_\mu \bigg|_{t=0}, \tilde{e}_\nu \right\rangle$$

$$= \left\langle \tilde{\nabla}_{\partial t} \tilde{e}_\mu, \tilde{e}_\nu \right\rangle$$

$$= \left\langle \tilde{\nabla}_\theta \tilde{e}_\mu, \tilde{e}_\nu \right\rangle$$

$$= \left\langle \tilde{D}_\theta e_\mu, e_\nu \right\rangle$$

$$= \left\langle \nabla_\theta e_\mu + h(\theta', e_\mu), e_\nu \right\rangle$$

$$= 0$$

(4.12)

since $e_\mu$ is parallel along $S$ with respect to the normal connection on $T^\perp S$. By the same reason,

(4.13)

$$\tilde{X}_{\mu\alpha}(u, 0) = \left\langle \tilde{\nabla}_{\partial t} \tilde{e}_\mu, \tilde{e}_\alpha \right\rangle = \left\langle \tilde{\nabla}_\theta \tilde{e}_\mu, \tilde{e}_\alpha \right\rangle = \left\langle \tilde{D}_\theta e_\mu, e_\alpha \right\rangle = \left\langle \nabla_\theta e_\mu + h(\theta', e_\mu), e_\alpha \right\rangle = h_{\mu}^\alpha \theta_i(u).$$

Similarly, by that $\psi$ preserves connections and $e_\alpha$ is parallel along $\theta$,

(4.14)

$$\tilde{X}_{\alpha\beta}(u, 0) = \left\langle \tilde{\nabla}_{\partial t} \tilde{e}_\alpha, \tilde{e}_\beta \right\rangle = \left\langle \tilde{\nabla}_\theta \tilde{e}_\alpha, \tilde{e}_\beta \right\rangle = \left\langle \tilde{D}_\theta e_\alpha, e_\beta \right\rangle = \left\langle D_\theta e_\alpha - A_\alpha(\theta'), e_\beta \right\rangle = 0.$$  

Finally, by that

(4.15)

$$\tilde{\nabla}_{\partial t} \frac{\partial}{\partial u} \bigg|_{t=0} = \tilde{\nabla}_{\partial t} \big( \tilde{U}_A \tilde{E}_A \big) \bigg|_{t=0}$$

$$= \tilde{U}'_A(u, 0) \tilde{e}_A + \tilde{U}_A(u, 0) \tilde{\nabla}_{\partial t} \tilde{E}_A \bigg|_{t=0}$$

$$= \tilde{U}'_A(u, 0) \tilde{e}_A + \tilde{U}_i(u, 0) \tilde{\nabla}_{\partial t} \tilde{E}_i \bigg|_{t=0}$$

$$= \tilde{U}'_A(u, 0) \tilde{e}_A + \theta_i(u) h_{ia}^\alpha v_a(u, 0) \tilde{e}_\alpha$$

where we have used (1.7) in the last equality, and

(4.16)

$$\tilde{\nabla}_{\partial t} \frac{\partial}{\partial u} \bigg|_{t=0} = \tilde{\nabla}_{\partial t} \bigg|_{t=0}$$

$$= \tilde{\nabla}_{\partial t} \big( v_a \tilde{E}_a \big) \bigg|_{t=0}$$

$$= \partial_a v_a(u, 0) \tilde{e}_a + v_a(u, 0) \tilde{X}_{A}(u, 0) \tilde{e}_A,$$
we have
\[
\tilde{U}'(u, 0) = \partial_a v_i(u, 0) + v_a(u, 0) \tilde{X}_{ai}(u, 0)
\]
(4.17)
\[
= \partial_a v_i(u, 0) + v_{\mu}(u, 0) \tilde{X}_{\mu i}(u, 0)
\]
\[
= \partial_a v_i(u, 0) - \sigma^a_{ij} \theta_j(u) v_{\mu}(u, 0),
\]
\[
\tilde{U}'(u, 0) = \partial_a v_{\mu}(u, 0) + v_a(u, 0) \tilde{X}_{a\mu}(u, 0)
\]
(4.18)
\[
= \partial_a v_{\mu}(u, 0) + v_i(u, 0) \tilde{X}_{i\mu}(u, 0)
\]
\[
= \partial_a v_{\mu}(u, 0) + \sigma^a_{ij} \theta_j(u) v_i(u, 0)
\]
(4.19)
where we have used \(\tilde{X}_{ij}(u, 0) = \sigma^a_{ij} \theta_a(u)\) and \(\tilde{X}_{aa}(u, 0) = h^a_{ai} \theta_i(u)\) which have been obtained before. This completes the proof of the lemma.

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** For each \(x \in M\), let \(\gamma_0, \gamma_1 : [0, 1] \to M\) be two smooth curves with \(\gamma_0(0), \gamma_1(0) \in S\) and \(\gamma_0(1) = \gamma_1(1) = x\). Because \(\pi_1(M, S)\) is trivial, there is a smooth homotopy \(\Phi : [0, 1] \times [0, 1] \to M\) such that
\[
\begin{align*}
\Phi(u, 0) &\in S & \forall u \in [0, 1] \\
\Phi(0, t) &\equiv \gamma_0(t) & \forall t \in [0, 1] \\
\Phi(1, t) &\equiv \gamma_1(t) & \forall t \in [0, 1] \\
\Phi(u, 1) &\equiv x & \forall u \in [0, 1].
\end{align*}
\]
Let \(\gamma_u(t) = \Phi(u, t), \theta(u) = \Phi(u, 0)\) and
\[
v(u, t) = P_t^0(\gamma_u)(\gamma'_u(t)).
\]
Then
\[
\gamma_u = \text{dev}(\theta(u), v(u, \cdot)).
\]
Let the other notations be the same as in Lemma 4.1. Suppose that
\[
\frac{\partial \Phi}{\partial u} = U_a E_a.
\]
(4.20)
\[
\nabla_a E_a = X_{ab} E_b,
\]
and
\[
D_a E_{\alpha} = X_{\alpha\beta} E_{\beta}.
\]
(4.22)
By comparing the initial data in Lemma 4.1 and Lemma 3.1 and by the same argument as in the proof of Theorem 1.2, we know that

\[
\begin{align*}
\tilde{U}_a &= U_a \\
\tilde{U}_\alpha &= 0 \\
\tilde{X}_{ab} &= X_{ab} \\
\tilde{X}_{\alpha\beta} &= X_{\alpha\beta} \\
\tilde{X}_{\alpha\alpha} &= h_{ab} U_b.
\end{align*}
\]

The same as in the proof of Theorem 1.2 we complete the proof of the theorem.

\[\square\]

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