Transgression and twisted anomaly cancellation formulas on odd dimensional manifolds

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Abstract

We compute the transgressed forms of some modularly invariant characteristic forms, which are related to the twisted elliptic genera. We study the modularity properties of these secondary characteristic forms and relations among them. We also get some twisted anomaly cancellation formulas on some odd dimensional manifolds.

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1 Introduction

In 1983, the physicists Alvarez-Gaumé and Witten [AW] discovered the "miraculous cancellation" formula for gravitational anomaly which reveals a beautiful relation between the top components of the Hirzebruch $\hat{L}$-form and $\hat{A}$-form of a 12-dimensional smooth Riemannian manifold. Kefeng Liu [Li] established higher dimensional "miraculous cancellation" formulas for $(8k + 4)$-dimensional Riemannian manifolds by developing modular invariance properties of characteristic forms. These formulas could be used to deduce some divisibility results. In [HZ1], [HZ2], for each $(8k + 4)$-dimensional smooth Riemannian manifold, a more general cancellation formula that involves a complex line bundle was established. This formula was applied to spin$^c$ manifolds, then an analytic Ochanine congruence formula was derived. For $(8k + 2)$ and $(8k + 6)$-dimensional smooth Riemannian manifolds, F. Han and X. Huang [HH] obtained some cancellation formulas.

On the other hand, motivated by the Chern-Simons theory, in [CH], Qingtao Chen and Fei Han computed the transgressed forms of some modularly invariant characteristic forms, which are related to the elliptic genera. They studied the modularity properties of these secondary characteristic forms and relations among them. They also got an anomaly cancellation formula for 11-dimensional manifold. Thus a nature question is to get some twisted modular forms by transgression and some twisted anomaly cancellation formulas for odd dimensional manifolds. In this paper, we compute the transgressed forms of some modularly invariant characteristic forms, which are related to the "twisted" elliptic genera. We study the modularity properties of these secondary characteristic forms and relations among them. We also get some
twisted anomaly cancellation formulas on some odd dimensional manifolds. We hope that these new geometric invariants of connections with modularity properties obtained here could be applied somewhere.

This paper is organized as follows: In Section 2, we review some knowledge on characteristic forms and modular forms that we are going to use. In Section 3, for \((4k - 1)\) dimensional manifolds, we apply the Chern-Simons transgression to characteristic forms with modularity properties which are related to the "twisted" elliptic genera and obtain some interesting secondary characteristic forms with modularity properties. We also get two twisted cancellation formulas for 11-dimensional manifolds. In Section 4, for \((4k + 1)\) dimensional manifolds, by transgression, we again obtain some interesting secondary characteristic forms with modularity properties. As a corollary, we get a twisted cancellation formula for 9-dimensional manifolds.

2 characteristic forms and modular forms

The purpose of this section is to review the necessary knowledge on characteristic forms and modular forms that we are going to use.

2.1 characteristic forms. Let \(M\) be a Riemannian manifold. Let \(\nabla^{TM}\) be the associated Levi-Civita connection on \(T^M\) and \(R^{TM} = (\nabla^{TM})^2\) be the curvature of \(\nabla^{TM}\). Let \(\hat{A}(TM, \nabla^{TM})\) and \(\hat{L}(TM, \nabla^{TM})\) be the Hirzebruch characteristic forms defined respectively by (cf. [Z])

\[
\hat{A}(TM, \nabla^{TM}) = \det^{\frac{1}{2}} \left( \frac{\sqrt{-1}}{4\pi} R^{TM} \right) \left( \frac{\sqrt{-1}}{4\pi} \sinh \left( \frac{\sqrt{-1}}{4\pi} R^{TM} \right) \right),
\]
\[
\hat{L}(TM, \nabla^{TM}) = \det^{\frac{1}{2}} \left( \frac{\sqrt{-1}}{2\pi} R^{TM} \right) \left( \tanh \left( \frac{\sqrt{-1}}{4\pi} R^{TM} \right) \right).
\]

(2.1)

Let \(E, F\) be two Hermitian vector bundles over \(M\) carrying Hermitian connection \(\nabla^E, \nabla^F\) respectively. Let \(R^E = (\nabla^E)^2\) (resp. \(R^F = (\nabla^F)^2\)) be the curvature of \(\nabla^E\) (resp. \(\nabla^F\)). If we set the formal difference \(G = E - F\), then \(G\) carries an induced Hermitian connection \(\nabla^G\) in an obvious sense. We define the associated Chern character form as

\[
\text{ch}(G, \nabla^G) = \text{tr} \left[ \exp \left( \frac{\sqrt{-1}}{2\pi} R^E \right) \right] - \text{tr} \left[ \exp \left( \frac{\sqrt{-1}}{2\pi} R^F \right) \right].
\]

(2.2)

For any complex number \(t\), let

\[
\wedge_t(E) = C|_M + tE + t^2 \wedge^2(E) + \cdots, \quad S_t(E) = C|_M + tE + t^2 S^2(E) + \cdots
\]

denote respectively the total exterior and symmetric powers of \(E\), which live in \(K(M)[[t]]\). The following relations between these operations hold,

\[
S_t(E) = \frac{1}{\wedge_{-t}(E)}, \quad \wedge_t(E - F) = \frac{\wedge_t(E)}{\wedge_t(F)}.
\]

(2.3)
Moreover, if \(\{\omega_i\}, \{\omega'_j\}\) are formal Chern roots for Hermitian vector bundles \(E, F\) respectively, then
\[
\text{ch}(\wedge_i(E)) = \prod_i (1 + e^{\omega_i t}).
\] (2.4)

Then we have the following formulas for Chern character forms,
\[
\text{ch}(S_i(E)) = \frac{1}{\prod_i (1 - e^{\omega_i t})}, \quad \text{ch}(\wedge_i(E - F)) = \frac{\prod_i (1 + e^{\omega_i t})}{\prod_j (1 + e^{\omega'_j t})}.
\] (2.5)

If \(W\) is a real Euclidean vector bundle over \(M\) carrying a Euclidean connection \(\nabla^W\), then its complexification \(W_C = W \otimes \mathbb{C}\) is a complex vector bundle over \(M\) carrying a canonical induced Hermitian metric from that of \(W\), as well as a Hermitian connection \(\nabla^{W_C}\) induced from \(\nabla^W\). If \(E\) is a vector bundle (complex or real) over \(M\), set \(\bar{E} = E - \dim E\) in \(K(M)\) or \(KO(M)\).

### 2.2 Some properties about the Jacobi theta functions and modular forms

We first recall the four Jacobi theta functions are defined as follows (cf. [Ch]):
\[
\theta(v, \tau) = 2q^\frac{1}{4}\sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1}v q^j})(1 - e^{-2\pi \sqrt{-1}v q^j})],
\] (2.6)
\[
\theta_1(v, \tau) = 2q^\frac{1}{4}\cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi \sqrt{-1}v q^j})(1 + e^{-2\pi \sqrt{-1}v q^j})],
\] (2.7)
\[
\theta_2(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1}v q^j - \frac{1}{2}})(1 - e^{-2\pi \sqrt{-1}v q^j - \frac{1}{2}})],
\] (2.8)
\[
\theta_3(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi \sqrt{-1}v q^j - \frac{1}{2}})(1 + e^{-2\pi \sqrt{-1}v q^j - \frac{1}{2}})],
\] (2.9)

where \(q = e^{2\pi \sqrt{-1}\tau}\) with \(\tau \in \mathbb{H}\), the upper half complex plane. Let
\[
\theta'(0, \tau) = \left. \frac{\partial \theta(v, \tau)}{\partial v} \right|_{v=0}.
\] (2.10)

Then the following Jacobi identity (cf. [Ch]) holds,
\[
\theta'(0, \tau) = \pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau).
\] (2.11)

Denote \(SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}\) the modular group. Let
\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]
be the two generators of \(SL_2(\mathbb{Z})\). They act on \(\mathbb{H}\) by \(S\tau = -\frac{1}{\tau}, \ T\tau = \tau + 1\). One has the following transformation laws of theta functions under the actions of \(S\) and \(T\) (cf. [Ch]):
\[
\theta(v, \tau + 1) = e^{\frac{\pi \sqrt{-1}}{\tau}}\theta(v, \tau), \quad \theta(v, \frac{1}{\tau}) = \frac{1}{\sqrt{-1}} \left( \frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} e^{\pi \sqrt{-1} v^2} \theta(\tau v, \tau);
\] (2.12)
\[ \theta_1(v, \tau + 1) = e^{\frac{\pi i}{4}} \theta_1(v, \tau), \quad \theta_1(v, -\frac{1}{\tau}) = \left( \frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} e^{\pi \sqrt{-1} \tau v^2} \theta_2(v, \tau); \quad (2.13) \]

\[ \theta_2(v, \tau + 1) = \theta_3(v, \tau), \quad \theta_2(v, -\frac{1}{\tau}) = \left( \frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} e^{\pi \sqrt{-1} \tau v^2} \theta_1(v, \tau); \quad (2.14) \]

\[ \theta_3(v, \tau + 1) = \theta_2(v, \tau), \quad \theta_3(v, -\frac{1}{\tau}) = \left( \frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} e^{\pi \sqrt{-1} \tau v^2} \theta_3(v, \tau). \quad (2.15) \]

Differentiating the above transformation formulas, we get that

\[ \theta'(v, \tau + 1) = e^{\frac{\pi i}{4}} \theta'(v, \tau), \]

\[ \theta'(v, -\frac{1}{\tau}) = \frac{1}{\sqrt{-1}} \left( \frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} e^{\pi \sqrt{-1} \tau v^2} (2\pi \sqrt{-1} \tau v \theta(\tau v, \tau) + \tau \theta'(\tau v, \tau)); \]

\[ \theta'_1(v, \tau + 1) = e^{\frac{\pi i}{4}} \theta'_1(v, \tau), \]

\[ \theta'_1(v, -\frac{1}{\tau}) = \left( \frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} e^{\pi \sqrt{-1} \tau v^2} (2\pi \sqrt{-1} \tau v \theta_2(\tau v, \tau) + \tau \theta'_2(\tau v, \tau)); \]

\[ \theta'_2(v, \tau + 1) = \theta'_3(v, \tau), \]

\[ \theta'_2(v, -\frac{1}{\tau}) = \left( \frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} e^{\pi \sqrt{-1} \tau v^2} (2\pi \sqrt{-1} \tau v \theta_1(\tau v, \tau) + \tau \theta'_1(\tau v, \tau)); \]

\[ \theta'_3(v, \tau + 1) = \theta'_2(v, \tau), \]

\[ \theta'_3(v, -\frac{1}{\tau}) = \left( \frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} e^{\pi \sqrt{-1} \tau v^2} (2\pi \sqrt{-1} \tau v \theta_3(\tau v, \tau) + \tau \theta'_3(\tau v, \tau)) \quad (2.16) \]

Therefore

\[ \theta'(0, -\frac{1}{\tau}) = \frac{1}{\sqrt{-1}} \left( \frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} \tau \theta'(0, \tau). \quad (2.17) \]

**Definition 2.1** A modular form over $\Gamma$, a subgroup of $SL_2(\mathbb{Z})$, is a holomorphic function $f(\tau)$ on $\mathbb{H}$ such that

\[ f(g \tau) := f \left( \frac{a \tau + b}{c \tau + d} \right) = \chi(g)(c \tau + d)^k f(\tau), \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad (2.18) \]

where $\chi : \Gamma \to \mathbb{C}^*$ is a character of $\Gamma$. $k$ is called the weight of $f$.

Let

\[ \Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \text{ (mod 2)} \right\}, \]

\[ \Gamma_0^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \text{ (mod 2)} \right\}, \]

\[ \Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ (mod 2)} \right\} \]
be the three modular subgroups of $SL_2(\mathbb{Z})$. It is known that the generators of $\Gamma_0(2)$ are $T$, $ST^2ST$, the generators of $\Gamma^0(2)$ are $STS$, $T^2STS$ and the generators of $\Gamma_\theta$ are $S, T$ (cf.[Ch]).

If $\Gamma$ is a modular subgroup, let $M_{\mathbb{R}}(\Gamma)$ denote the ring of modular forms over $\Gamma$ with real Fourier coefficients. Writing $\theta_j = \theta_j(0, \tau)$, $1 \leq j \leq 3$, we introduce six explicit modular forms (cf. [Li]),

- $\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4)$, $\epsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4$;
- $\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4)$, $\epsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4$;
- $\delta_3(\tau) = \frac{1}{8}(\theta_1^4 - \theta_2^4)$, $\epsilon_3(\tau) = -\frac{1}{16}\theta_1^4\theta_2^4$.

They have the following Fourier expansions in $q^{\frac{1}{2}}$:

- $\delta_1(\tau) = \frac{1}{4} + 6q + \cdots$, $\epsilon_1(\tau) = \frac{1}{16} - q + \cdots$;
- $\delta_2(\tau) = -\frac{1}{8} - 3q^{\frac{3}{2}} + \cdots$, $\epsilon_2(\tau) = q^{\frac{1}{2}} + \cdots$;
- $\delta_3(\tau) = -\frac{1}{8} + 3q^{\frac{3}{2}} + \cdots$, $\epsilon_3(\tau) = -q^{\frac{1}{2}} + \cdots$,

where the "\cdots" terms are the higher degree terms, all of which have integral coefficients. They also satisfy the transformation laws,

- $\delta_2\left(-\frac{1}{\tau}\right) = \tau^2\delta_1(\tau)$;
- $\epsilon_2\left(-\frac{1}{\tau}\right) = \tau^4\epsilon_1(\tau)$;
- $\delta_2(\tau + 1) = \delta_3(\tau)$;
- $\epsilon_2(\tau + 1) = \epsilon_3(\tau)$.

**Lemma 2.2** ([Li]) $\delta_1(\tau)$ (resp. $\epsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\epsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, while $\delta_3(\tau)$ (resp. $\epsilon_3(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_\theta(2)$ and moreover $M_{\mathbb{R}}(\Gamma^0(2)) = \mathbb{R}[\delta_2(\tau), \epsilon_2(\tau)]$.

### 3 Transgressed forms and modularities on 4k − 1 dimensional manifolds

Let $M$ be a 4$k$ − 1 dimensional Riemannian manifold and $\xi$ be a rank two real oriented Euclidean vector bundle over $M$ carrying with a Euclidean connection $\nabla^\xi$. Set

$$\Theta_1(TC^\infty M, \xi_C) = \bigotimes_{n=1}^\infty S_n(TC^\infty M) \otimes \bigotimes_{m=1}^\infty \wedge q^m(TC^\infty M - 2\xi_C) \otimes \bigotimes_{r=1}^\infty \wedge q^{r-\frac{1}{2}}(\xi_C) \otimes \bigotimes_{s=1}^\infty \wedge q^{s-\frac{1}{2}}(\xi_C),$$
\[ \Theta_2(T_C M, \xi_C) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_C M) \otimes \bigotimes_{m=1}^{\infty} \wedge_{\phi - m - \frac{1}{2}} (T_C M - 2\xi C) \otimes \bigotimes_{r=1}^{\infty} \wedge_{\phi - r - \frac{1}{2}} (\xi C) \otimes \bigotimes_{s=1}^{\infty} \wedge_{\phi - s - \frac{1}{2}} (\xi C), \]

\[ \Theta_3(T_C M, \xi_C) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_C M) \otimes \bigotimes_{m=1}^{\infty} \wedge_{\phi - m - \frac{1}{2}} (T_C M - 2\xi C) \otimes \bigotimes_{r=1}^{\infty} \wedge_{\phi - r - \frac{1}{2}} (\xi C) \otimes \bigotimes_{s=1}^{\infty} \wedge_{\phi - s - \frac{1}{2}} (\xi C). \] 

(3.1)

Let \( c = e(\xi, \nabla^\xi) \) be the Euler form of \( \xi \) canonically associated to \( \nabla^\xi \). Set

\[ \Phi_L(\nabla^{TM}, \nabla^\xi, \tau) = \frac{\hat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{\tau}{2})} \text{ch}(\Theta_1(T_C M, \xi_C), \nabla^{\Theta_1(T_C M, \xi_C)}), \]

\[ \Phi_W(\nabla^{TM}, \nabla^\xi, \tau) = \hat{A}(TM, \nabla^{TM}) \text{cosh}(\frac{C}{2}) \text{ch}(\Theta_2(T_C M, \xi_C), \nabla^{\Theta_2(T_C M, \xi_C)}), \]

\[ \Phi'_W(\nabla^{TM}, \nabla^\xi, \tau) = \hat{A}(TM, \nabla^{TM}) \text{cosh}(\frac{C}{2}) \text{ch}(\Theta_3(T_C M, \xi_C), \nabla^{\Theta_3(T_C M, \xi_C)}). \] 

(3.2)

Let \( \{ \pm 2\pi i j | 1 \leq j \leq 2k - 1 \} \) and \( \{ \pm 2\pi \sqrt{-1} u \} \) be the Chern roots of \( T_C M \) and \( \xi_C \) respectively and \( c = 2\pi \sqrt{-1} u \). Through direct computations, we get (cf. [HZ2])

\[ \Phi_L(\nabla^{TM}, \nabla^\xi, \tau) = \sqrt{2}^{4k-1} \left\{ \prod_{j=1}^{2k-1} \frac{\theta^j(0, \tau) \theta_1(x_j, \tau)}{\theta(x_j, \tau) \theta_1(0, \tau)} \right\}; \]

\[ \Phi_W(\nabla^{TM}, \nabla^\xi, \tau) = \left( \prod_{j=1}^{2k-1} \frac{\theta^j(0, \tau) \theta_2(x_j, \tau)}{\theta(x_j, \tau) \theta_2(0, \tau)} \right) \left( \prod_{j=1}^{2k-1} \frac{\theta^j(0, \tau) \theta_1(x_j, \tau)}{\theta(x_j, \tau) \theta_1(0, \tau)} \right) \] 

(3.3)

\[ \Phi'_W(\nabla^{TM}, \nabla^\xi, \tau) = \left( \prod_{j=1}^{2k-1} \frac{\theta^j(0, \tau) \theta_3(x_j, \tau)}{\theta(x_j, \tau) \theta_3(0, \tau)} \right) \left( \prod_{j=1}^{2k-1} \frac{\theta^j(0, \tau) \theta_2(x_j, \tau)}{\theta(x_j, \tau) \theta_2(0, \tau)} \right) \] 

(3.4)

Consider the following function defined on \( \mathbb{C} \times \mathbb{H} \),

\[ f_{\Phi_L}(z, \tau) = \frac{\theta^j(0, \tau) \theta_1(z, \tau)}{\theta(z, \tau) \theta_1(0, \tau)}, \]

\[ f_{\Phi_W}(z, \tau) = \frac{\theta^j(0, \tau) \theta_2(z, \tau)}{\theta(z, \tau) \theta_2(0, \tau)}, \]

\[ f_{\Phi'_W}(z, \tau) = \frac{\theta^j(0, \tau) \theta_3(z, \tau)}{\theta(z, \tau) \theta_3(0, \tau)}. \]

Applying the Chern-Weil theory, we can express \( \Phi_L, \Phi_W, \Phi'_W \) as follows:

\[ \Phi_L(\nabla^{TM}, \nabla^\xi, \tau) = \sqrt{2}^{4k-1} \det \left( \frac{\theta^j(0, \tau)}{\theta(z, \tau)} \right) \left( \frac{\theta^j(0, \tau)}{\theta(z, \tau)} \right); \]

\[ \Phi_W(\nabla^{TM}, \nabla^\xi, \tau) = \det \left( \frac{\theta^j(0, \tau)}{\theta(z, \tau)} \right) \left( \frac{\theta^j(0, \tau)}{\theta(z, \tau)} \right); \]

\[ (3.6) \]

\[ \Phi'_W(\nabla^{TM}, \nabla^\xi, \tau) = \det \left( \frac{\theta^j(0, \tau)}{\theta(z, \tau)} \right) \left( \frac{\theta^j(0, \tau)}{\theta(z, \tau)} \right); \]

\[ (3.7) \]
\[ \Phi_W' (\nabla_{TM}, \nabla^\xi, \tau) = \det \frac{1}{2} \left( f_{\Phi_W} (\frac{R_{TM}}{4\pi^2}, \tau) \right) \det \frac{1}{2} \left( \frac{\theta_3^2 (0, \tau)}{\theta_3 (\frac{R_{TM}}{4\pi^2}, \tau)} \frac{\theta_2 (0, \tau)}{\theta_2 (0, \tau)} \right). \]  

(3.8)

Let \(E\) be a vector bundle and \(f\) be a power series with constant term 1. Let \(\nabla^E_t\) be deformed connection given by \(\nabla^E_t = (1 - t) \nabla^E_0 + t \nabla^E_1\) and \(R^E_t\), \(t \in [0, 1]\), denote the curvature of \(\nabla^E_t\). \(f'(t)\) is the power series obtained from the derivative of \(f(x)\) with respect to \(x\). \(\omega\) is a closed form. Recall the trivial modification of Theorem 2.2 in [CH],

**Lemma 3.1** ([CH])

\[ \det \frac{1}{2} (f'(R^E_t)) \omega - \det \frac{1}{2} (f(R^E_0)) \omega = d \int_0^1 \frac{1}{2} \det \frac{1}{2} (f(R^E_t)) \omega tr \left[ \frac{d\nabla^E_t f'(R^E_t)}{dt} \right] dt. \]  

(3.9)

Now we let \(E = TM\) and \(A = \nabla^{TM}_1 - \nabla^{TM}_0\), then by Lemma 3.1, we have

\[ \Phi_L (\nabla^{TM}_1, \nabla^\xi, \tau) - \Phi_L (\nabla^{TM}_0, \nabla^\xi, \tau) = \frac{1}{8\pi^2} \int_0^1 \Phi_L (\nabla^T_{TM}, \nabla^\xi, \tau) \left[ A \left( \frac{1}{R^{TM}_{4\pi^2}} - \frac{\theta'(R^{TM}_{4\pi^2}, \tau)}{\theta(R^{TM}_{4\pi^2}, \tau)} \frac{\theta'(R^{TM}_{4\pi^2}, \tau)}{\theta_1 (R^{TM}_{4\pi^2}, \tau)} \right) \right] dt. \]  

(3.10)

We define

\[ CS\Phi_L (\nabla^{TM}_0, \nabla^{TM}_1, \nabla^\xi, \tau) := \frac{\sqrt{\tau}}{8\pi^2} \int_0^1 \Phi_L (\nabla^T_{TM}, \nabla^\xi, \tau) \left[ A \left( \frac{1}{R^{TM}_{4\pi^2}} - \frac{\theta'(R^{TM}_{4\pi^2}, \tau)}{\theta(R^{TM}_{4\pi^2}, \tau)} \frac{\theta'(R^{TM}_{4\pi^2}, \tau)}{\theta_1 (R^{TM}_{4\pi^2}, \tau)} \right) \right] dt. \]  

(3.11)

which is in \(\Omega^{odd} (M, C) [[q^{1/2}]]\). Since \(M\) is 4\(k\)–1 dimensional, \(\{ CS\Phi_L (\nabla^{TM}_0, \nabla^{TM}_1, \nabla^\xi, \tau) \}_{(4k-1)}\) represents an element in \(H^{4k-1} (M, C) [[q^{1/2}]]\). Similarly, we can compute the transgressed forms for \(\Phi_W, \Phi_W'\) respectively and define

\[ CS\Phi_W (\nabla^{TM}_0, \nabla^{TM}_1, \nabla^\xi, \tau) := \frac{1}{8\pi^2} \int_0^1 \Phi_W (\nabla^T_{TM}, \nabla^\xi, \tau) \left[ A \left( \frac{1}{R^{TM}_{4\pi^2}} - \frac{\theta'(R^{TM}_{4\pi^2}, \tau)}{\theta(R^{TM}_{4\pi^2}, \tau)} \frac{\theta'(R^{TM}_{4\pi^2}, \tau)}{\theta_1 (R^{TM}_{4\pi^2}, \tau)} \right) \right] dt; \]  

(3.12)

\[ CS\Phi'_W (\nabla^{TM}_0, \nabla^{TM}_1, \nabla^\xi, \tau) := \frac{1}{8\pi^2} \int_0^1 \Phi'_W (\nabla^T_{TM}, \nabla^\xi, \tau) \left[ A \left( \frac{1}{R^{TM}_{4\pi^2}} - \frac{\theta'(R^{TM}_{4\pi^2}, \tau)}{\theta(R^{TM}_{4\pi^2}, \tau)} \frac{\theta'(R^{TM}_{4\pi^2}, \tau)}{\theta_1 (R^{TM}_{4\pi^2}, \tau)} \right) \right] dt, \]  

(3.13)

which also lie in \(\Omega^{odd} (M, C) [[q^{1/2}]]\) and their top components represent elements in \(H^{4k-1} (M, C) [[q^{1/2}]]\). As pointed in [CH], the equality (3.10) and the modular invariance properties of \(\Phi_L (\nabla^{TM}_0, \nabla^\xi, \tau)\) and \(\Phi_L (\nabla^{TM}_1, \nabla^\xi, \tau)\) are not enough to guarantee that \(CS\Phi_L (\nabla^{TM}_0, \nabla^{TM}_1, \nabla^\xi, \tau)\) is a modular form. However we have the following
Theorem 3.2 Let $M$ be a $4k - 1$ dimensional manifold and $\nabla_0^{TM}, \nabla_1^{TM}$ be two connections on $TM$ and $\xi$ be a two dimensional oriented Euclidean real vector bundle with a Euclidean connection $\nabla^\xi$, then we have

1) $\{CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(4k-1)}$ is a modular form of weight $2k$ over $\Gamma_0(2)$;
2) $\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(4k-1)}$ is a modular form of weight $2k$ over $\Gamma^0(2)$;

The following equalities hold,

$$\{CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, -\frac{1}{\tau})\}^{(4k-1)} = (2\tau)^{2k}\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(4k-1)},$$

$$CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau + 1) = CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau).$$

Proof. By (2.12)-(2.17), we have

$$\frac{\theta'(0, -\frac{1}{2}) \theta_1(z, -\frac{1}{2})}{\theta(z, -\frac{1}{2}) \theta_1(0, -\frac{1}{2})} = \frac{\theta'(0, \tau) \theta_2(\tau z, \tau)}{\theta(\tau z, \tau) \theta_2(0, \tau)};$$

$$\frac{\theta'(1, -\frac{1}{2}) \theta_1(z, -\frac{1}{2})}{\theta(z, -\frac{1}{2}) \theta_1(1, -\frac{1}{2})} = \tau \frac{\theta'(\tau z, \tau)}{\theta(\tau z, \tau)} + \frac{\theta'(z, \tau)}{\theta(z, \tau)};$$

$$\frac{\theta_2^2(0, -\frac{1}{2}) \theta_3(u, -\frac{1}{2}) \theta_2(u, -\frac{1}{2}) \theta_2(0, -\frac{1}{2})}{\theta_1^2(u, -\frac{1}{2}) \theta_3(u, -\frac{1}{2}) \theta_3(0, -\frac{1}{2}) \theta_1(0, -1\tau)} = \frac{\theta_2^2(0, \tau) \theta_3(\tau u, \tau) \theta_1(u, \tau)}{\theta_2^2(u, \tau) \theta_3(0, \tau) \theta_1(0, \tau)}.$$ 

Note that we only take $(4k - 1)$-component, so by (3.6)-(3.8),(3.11), (3.12), (3.14)-(3.16), we can get

$$\{CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, -\frac{1}{\tau})\}^{(4k-1)} = (2\tau)^{2k}\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(4k-1)},$$

$$\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau + 1) = CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}.$$ 

Similarly we can show that

$$CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau + 1) = CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau),$$

$$CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau + 1) = CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau),$$

$$\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, -\frac{1}{\tau})\}^{(4k-1)} = (\tau)^{2k}\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(4k-1)},$$

$$CS\Phi_W'(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau + 1) = CS\Phi_W'(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau).$$

From (3.17) and (3.18), we can get $CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(4k-1)}$ is a modular form of weight $2k$ over $\Gamma_0(2)$. Similarly we can prove that $CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(4k-1)}$ is a modular form of weight $2k$ over $\Gamma_0^0(2)$ and $CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(4k-1)}$ is a modular form of weight $2k$ over $\Gamma_0(2)$. □
Let \( M \) be a compact oriented smooth 3-dimensional manifold, then our transgressed forms are same as transgressed forms in the untwisted case which have been computed in [CH]. From Theorem 3.2, we can imply some twisted cancellation formulas for odd dimensional manifolds. For example, let \( M \) be 11 dimensional and \( k = 3 \). We have that \( \{CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(11)} \) is a modular form of weight 6 over \( \Gamma_0(2) \), \( \{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(11)} \) is a modular form of weight 6 over \( \Gamma^0(2) \) and

\[
\{CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, -\frac{1}{\tau})\}^{(11)} = (2\tau)^{6}\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(11)}.
\]

By Lemma 2.2, we have

\[
\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(11)} = z_0(8\delta_2)^3 + z_1(8\delta_2)\varepsilon_2,
\]

(3.19)

and by (2.19) and Theorem 3.2,

\[
\{CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\}^{(11)} = 2^6[z_0(8\delta_1)^3 + z_1(8\delta_1)\varepsilon_1].
\]

(3.20)

By comparing the \( q^\frac{1}{2} \)-expansion coefficients in (3.19), we get

\[
z_0 = -\left\{ \int_0^1 \tilde{A}(TM, \nabla^T_i)\cosh\left(\frac{c}{2}\right)\text{tr} \left[ A \left(\frac{1}{2R_i^{TM}} - \frac{L_i^{TM}}{8\pi\tan\frac{R_i^{TM}}{4\pi}}\right) \right] dt \right\}^{(11)},
\]

(3.21)

\[
z_1 = \left\{ \int_0^1 \tilde{A}(TM, \nabla^T_i)\cosh\left(\frac{c}{2}\right) \left( \text{ch}(T_{CM}, \nabla^{T_{CM}}_i) - 3(e^c + e^{-c} - 2) \right) \right. 
\times \text{tr} \left[ A \left(\frac{1}{2R_i^{TM}} - \frac{L_i^{TM}}{8\pi\tan\frac{R_i^{TM}}{4\pi}}\right) \right] dt + 
\left. \int_0^1 \tilde{A}(TM, \nabla^T_i)\cosh\left(\frac{c}{2}\right)\text{tr} \left[ A \left(\frac{1}{2R_i^{TM}} + 61 \left(\frac{1}{2R_i^{TM}} - \frac{L_i^{TM}}{8\pi\tan\frac{R_i^{TM}}{4\pi}}\right) \right) \right] dt \right\}^{(11)}.
\]

(3.22)

Plugging (3.21) and (3.22) into (3.20) and comparing the constant terms of both sides, we obtain that

\[
\left\{ \int_0^1 \frac{\sqrt{2}L(TM, \nabla^T_i)}{\cosh^2\frac{c}{2}} \text{tr} \left[ A \left(\frac{1}{2R_i^{TM}} - \frac{L_i^{TM}}{4\pi\sin\frac{R_i^{TM}}{2\pi}}\right) \right] \right\}^{(11)} = 2^3(2^6z_0 + z_1),
\]

so we have the following 11-dimensional analogue of the twisted miraculous cancellation formula.

**Corollary 3.3** The following equality holds

\[
\left\{ \int_0^1 \frac{\sqrt{2}L(TM, \nabla^T_i)}{\cosh^2\frac{c}{2}} \text{tr} \left[ A \left(\frac{1}{2R_i^{TM}} - \frac{L_i^{TM}}{4\pi\sin\frac{R_i^{TM}}{2\pi}}\right) \right] \right\}^{(11)}
\]
\[ = 8 \left\{ \int_0^1 \tilde{A}(T, M, \nabla_{i}^T) \cosh \left( \frac{c}{2} \right) \left( \text{ch}(T_{i} M, \nabla_{i}^T) - 3(e^c + e^{-c} - 2) \right) \right. \\
\times \text{tr} \left[ A \left( \frac{1}{2 R_{i}^T} - \frac{1}{8 \pi \tan \frac{R_{i}^T}{4 \pi}} \right) \right] \left. \right\} dt + \]

\[ \int_0^1 \tilde{A}(T, M, \nabla_{i}^T) \cosh \left( \frac{c}{2} \right) \text{tr} \left[ A \left( -\frac{1}{2 \pi} \sin \frac{R_{i}^T}{2 \pi} - 3 \left( \frac{1}{2 R_{i}^T} - \frac{1}{8 \pi \tan \frac{R_{i}^T}{4 \pi}} \right) \right) \right] \left. \right\} \] .

(3.23)

Next we consider the transgression of \( \Phi_{L}(\nabla_{i}^T, \nabla_{i} \xi, \tau) \), \( \Phi_{W}(\nabla_{i}^T, \nabla_{i} \xi, \tau) \), \( \Phi'_{L}(\nabla_{i}^T, \nabla_{i} \xi, \tau) \) about \( \nabla_{i} \xi \). Let \( \nabla_{i}^{1}, \nabla_{i}^{0} \) be two Euclidean connections on \( \xi \) and \( B = \nabla_{i}^{1} - \nabla_{i}^{0} \). By (3.6)-(3.9), we have
\[ \Phi_{L}(\nabla_{i}^T, \nabla_{i}^{1}, \tau) - \Phi_{L}(\nabla_{i}^T, \nabla_{i}^{0}, \tau) \]
\[ = \frac{1}{8 \pi^2} \int_0^1 \Phi_{L}(\nabla_{i}^T, \nabla_{i}^{1}, \tau) \text{tr} \left[ B \left( \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} + \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} - 2 \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} \right) \right] dt .
\]

(3.24)

We define \( \text{CS} \Phi_{L}(\nabla_{i}^T, \nabla_{i}^{0}, \nabla_{i}^{1}, \tau) \)
\[ := \frac{\sqrt{2}}{8 \pi^2} \int_0^1 \Phi_{L}(\nabla_{i}^T, \nabla_{i}^{1}, \tau) \text{tr} \left[ B \left( \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} + \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} - 2 \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} \right) \right] dt .
\]

(3.25)

which is in \( \Omega^{\text{odd}}(M, \mathbb{C})[[q^{\frac{1}{2}}]] \). Since \( M \) is \( 4k-1 \) dimensional, \( \{ \text{CS} \Phi_{L}(\nabla_{i}^T, \nabla_{i}^{0}, \nabla_{i}^{1}, \tau) \}^{(4k-1)} \)
represents an element in \( H^{4k-1}(M, \mathbb{C})[[q^{\frac{1}{2}}]] \). Similarly, we can compute the transgressed forms for \( \Phi_{W}, \Phi'_{W} \) respectively and define
\[ \text{CS} \Phi_{W}(\nabla_{i}^T, \nabla_{i}^{0}, \nabla_{i}^{1}, \tau) \]
\[ := \frac{1}{8 \pi^2} \int_0^1 \Phi_{W}(\nabla_{i}^T, \nabla_{i}^{1}, \tau) \text{tr} \left[ B \left( \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} + \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} - 2 \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} \right) \right] dt .
\]

(3.26)

\[ \text{CS} \Phi'_{W}(\nabla_{i}^T, \nabla_{i}^{0}, \nabla_{i}^{1}, \tau) \]
\[ := \frac{1}{8 \pi^2} \int_0^1 \Phi'_{W}(\nabla_{i}^T, \nabla_{i}^{1}, \tau) \text{tr} \left[ B \left( \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} + \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} - 2 \frac{\theta_{i} \left( \frac{R_{i}^{1}}{4 \pi}, \tau \right)}{\theta_{i} \left( \frac{R_{i}^{0}}{4 \pi}, \tau \right)} \right) \right] dt ,
\]

(3.26)

which also lie in \( \Omega^{\text{odd}}(M, \mathbb{C})[[q^{\frac{1}{2}}]] \) and their top components represent elements in \( H^{4k-1}(M, \mathbb{C})[[q^{\frac{1}{2}}]] \). Similarly we have
Theorem 3.4 Let $M$ be a $4k - 1$ dimensional manifold and $\nabla^{TM}$ be a connection on $TM$ and $\xi$ be a two dimensional oriented Euclidean real vector bundle with two Euclidean connections $\nabla_1^\xi$, $\nabla_0^\xi$, then we have
1) \{$CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)$\}^{(4k-1)} is a modular form of weight $2k$ over $\Gamma_0(2)$;
2) \{$CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)$\}^{(4k-1)} is a modular form of weight $2k$ over $\Gamma^0(2)$;
3) \{$CS\Phi'_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)$\}^{(4k-1)} is a modular form of weight $2k$ over $\Gamma_0(2)$.

Proof. By (3.14),(3.16) and
\[
\theta'_{2}(z, -\frac{1}{\tau}) + \frac{\theta'_{3}(z, -\frac{1}{\tau})}{\theta_{3}(z, -\frac{1}{\tau})} - 2 \frac{\theta'_{1}(z, -\frac{1}{\tau})}{\theta_{1}(z, -\frac{1}{\tau})} = \tau \left( \frac{\theta'_{1}(\tau z, \tau)}{\theta_{1}(\tau z, \tau)} + \frac{\theta'_{3}(\tau z, \tau)}{\theta_{3}(\tau z, \tau)} - 2 \frac{\theta'_{3}(\tau z, \tau)}{\theta_{2}(\tau z, \tau)} \right),
\]
we can get
\[
\{CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(4k-1)} = (2\tau)^{2k} \{CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(4k-1)},
\]
(3.27)
Similarly we can show that
\[
CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau + 1) = CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau),
\]
\[
\{CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(4k-1)} = \frac{(\tau)}{2}^{2k} \{CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(4k-1)},
\]
\[
CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau + 1) = CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau),
\]
\[
\{CS\Phi'_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(4k-1)} = (\tau)^{2k} \{CS\Phi'_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(4k-1)},
\]
\[
CS\Phi'_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau + 1) = CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau). \] (3.29)

From (3.28) and (3.29), we can get \{$CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)$\}^{(4k-1)} is a modular form of weight $2k$ over $\Gamma_0(2)$. Similarly we can prove that \{$CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)$\}^{(4k-1)} is a modular form of weight $2k$ over $\Gamma^0(2)$ and \{$CS\Phi'_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)$\}^{(4k-1)} is a modular form of weight $2k$ over $\Gamma_0(2)$. □

Let $M$ be a compact oriented smooth 3-dimensional manifold, we have
\[
CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau) = \frac{\sqrt{2}}{8\pi^2} \int_0^1 \Phi_L(\nabla^{TM}, \nabla^\xi_1, \tau) \text{tr} \left[ B \left( \frac{\theta'_{2}(\frac{R}{4\theta})}{\theta_{2}(\frac{R}{4\theta}), \tau} + \frac{\theta'_{3}(\frac{R}{4\theta})}{\theta_{3}(\frac{R}{4\theta}), \tau} - 2 \frac{\theta'_{1}(\frac{R}{4\theta}), \tau)}{\theta_{1}(\frac{R}{4\theta}, \tau)} \right) \right] dt
\]
\[
= \frac{1}{2\pi^2} \int_0^1 \text{tr} \left[ B \left( \frac{\theta'_{2}(\frac{R_{1}^2}{4\theta})}{\theta_{2}(\frac{R_{1}^2}{4\theta}), \tau} + \frac{\theta'_{3}(\frac{R_{1}^2}{4\theta})}{\theta_{3}(\frac{R_{1}^2}{4\theta}, \tau)} - 2 \frac{\theta'_{1}(\frac{R_{1}^2}{4\theta}), \tau)}{\theta_{1}(\frac{R{1}^2}{4\theta}, \tau)} \right) \right] dt
\]
\[
= \frac{1}{8\pi^4} \frac{\partial}{\partial z} \left( \frac{\theta'_{2}(z, \tau)}{\theta_{2}(z, \tau)} + \frac{\theta'_{3}(z, \tau)}{\theta_{3}(z, \tau)} - 2 \frac{\theta'_{1}(z, \tau)}{\theta_{1}(z, \tau)} \right) \bigg|_{z=0} \int_0^1 \text{tr}[BR_{1}^2] dt.
\]
By comparing the \( \frac{\partial}{\partial z} \left( \frac{\theta'_2(z, \tau)}{\theta_2(z, \tau)} + \frac{\theta'_3(z, \tau)}{\theta_3(z, \tau)} - 2 \frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)} \right) \big|_{z=0} \) is a modular form of weight 2 over \( \Gamma_0(2) \), then it is a scalar multiple of \( \delta_1(\tau) \). Direct computations show

\[
\frac{\partial}{\partial z} \left( \frac{\theta'_2(z, \tau)}{\theta_2(z, \tau)} + \frac{\theta'_3(z, \tau)}{\theta_3(z, \tau)} - 2 \frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)} \right) \big|_{z=0} = 2\pi^2 + O(q^\frac{1}{2}),
\]

so

\[
\frac{\partial}{\partial z} \left( \frac{\theta'_2(z, \tau)}{\theta_2(z, \tau)} + \frac{\theta'_3(z, \tau)}{\theta_3(z, \tau)} - 2 \frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)} \right) \big|_{z=0} = 8\pi^2 \delta_1(\tau).
\]

By (4.15) in [CH], we have

\[
CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau) = \frac{1}{2\pi^2} \delta_1(\tau) \text{tr} \left[ B[\nabla^\xi_0, \nabla^\xi_1] + \frac{2}{3} B \wedge B \wedge B \right].
\] (3.30)

Similarly, we obtain that

\[
CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau) = \frac{1}{8\pi^2} \delta_2(\tau) \text{tr} \left[ B[\nabla^\xi_0, \nabla^\xi_1] + \frac{2}{3} B \wedge B \wedge B \right],
\] (3.31)

\[
CS\Phi'_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau) = \frac{1}{8\pi^2} \delta_3(\tau) \text{tr} \left[ B[\nabla^\xi_0, \nabla^\xi_1] + \frac{2}{3} B \wedge B \wedge B \right].
\] (3.32)

Let \( M \) be 11 dimensional and \( k = 3 \). We have that \( \{CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(11)} \) is a modular form of weight 6 over \( \Gamma_0(2) \), \( \{CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(11)} \) is a modular form of weight 6 over \( \Gamma^0(2) \) and

\[
\{CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, -\frac{1}{\tau})\}^{(11)} = (2\tau)^6 \{CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(11)}.
\]

By Lemma 2.2, we have

\[
\{CS\Phi_W(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(11)} = z_0(8\delta_2)^3 + z_1(8\delta_2)\varepsilon_2,
\] (3.33)

and by (2.19) and Theorem 3.4,

\[
\{CS\Phi_L(\nabla^{TM}, \nabla^\xi_0, \nabla^\xi_1, \tau)\}^{(11)} = 2^6[z_0(8\delta_1)^3 + z_1(8\delta_1)\varepsilon_1].
\] (3.34)

By comparing the \( q^{\frac{1}{2}} \)-expansion coefficients in (3.33), we get

\[
z_0 = \left\{ \int_0^1 \tilde{A}(TM, \nabla^{TM}) \cos \left( \frac{R^\xi_t}{4\pi} \right) \text{tr} \left[ \frac{B}{8\pi} \tan \frac{R^\xi_t}{4\pi} \right] dt \right\}^{(11)},
\] (3.35)

\[
z_1 = \left\{ \int_0^1 \tilde{A}(TM, \nabla^{TM}) \cos \left( \frac{R^\xi_t}{4\pi} \right) \left( 3\text{ch}(\xi_C, \nabla^\xi_t C) - \text{ch}(T_C M, \nabla^{T_C M}) + 77 \right) \right. \times \left. \text{tr} \left[ \frac{B}{8\pi} \tan \frac{R^\xi_t}{4\pi} \right] dt + \int_0^1 \tilde{A}(TM, \nabla^{TM}) \cos \left( \frac{R^\xi_t}{4\pi} \right) \text{tr} \left[ \frac{3B}{2\pi} \sin^2 \frac{R^\xi_t}{2\pi} \right] dt \right\}^{(11)}.
\] (3.36)
Plugging (3.35) and (3.36) into (3.34) and comparing the constant terms of both sides, we obtain that

Corollary 3.5 The following equality holds

\[
\int_0^1 \frac{L(TM, \nabla^TM)}{\cos^2(\frac{R^\xi t}{4\pi})} \text{tr} \left[ B \tan \frac{R^\xi t}{4\pi} \right] \, dt \\
= 16\sqrt{2}\pi \left\{ \int_0^1 \hat{A}(TM, \nabla^TM) \cos \left( R^\xi t \right) \left( 3\text{ch}(\xi_t, \nabla_t^C) - \text{ch}(TCM, \nabla^{TCM}) + 13 \right) \right. \\
\times \left. \text{tr} \left[ \frac{B}{8\pi \tan \frac{R^\xi t}{4\pi}} \right] \, dt + \int_0^1 \hat{A}(TM, \nabla^TM) \cos \left( R^\xi t \right) \text{tr} \left[ \frac{3B}{2\pi \sin \frac{R^\xi t}{2\pi}} \right] \, dt \right\}^{(11)}. \quad (3.37)
\]

4 Transgressed forms and modularities on 4k + 1 dimensional manifolds

Let M be a 4k + 1 dimensional Riemannian manifold. Set

\[
\Theta_1(TC^M + \xi_C, \xi_C) = \bigotimes_{n=1}^\infty S_{q^n}(TC^M + \xi_C) \otimes \bigotimes_{m=1}^\infty \wedge_{q^m} (TC^M + \xi_C - 2\xi_C)
\]
\[
\otimes \bigotimes_{r=1}^\infty \wedge_{q^{-r}} (\xi_C) \otimes \bigotimes_{s=1}^\infty \wedge_{q^{-s}} (\xi_C),
\]
\[
\Theta_2(TC^M + \xi_C, \xi_C) = \bigotimes_{n=1}^\infty S_{q^n}(TC^M + \xi_C) \otimes \bigotimes_{m=1}^\infty \wedge_{q^m} (TC^M + \xi_C - 2\xi_C)
\]
\[
\otimes \bigotimes_{r=1}^\infty \wedge_{q^{-r}} (\xi_C) \otimes \bigotimes_{s=1}^\infty \wedge_{q^{-s}} (\xi_C),
\]
\[
\Theta_3(TC^M + \xi_C, \xi_C) = \bigotimes_{n=1}^\infty S_{q^n}(TC^M + \xi_C) \otimes \bigotimes_{m=1}^\infty \wedge_{q^m} (TC^M + \xi_C - 2\xi_C)
\]
\[
\otimes \bigotimes_{r=1}^\infty \wedge_{q^{-r}} (\xi_C) \otimes \bigotimes_{s=1}^\infty \wedge_{q^{-s}} (\xi_C).
\]

Define

\[
\tilde{\Phi}_L(\nabla^TM, \nabla^\xi, \tau) = \frac{L(TM, \nabla^TM) \cosh(\frac{\xi}{2})}{\sinh(\frac{\xi}{2})} \left( \text{ch}(\Theta_1(TC^M + \xi_C, C^2)) - \frac{\text{ch}(\Theta_1(TC^M + \xi_C, C^2))}{\cosh^2(\frac{\xi}{2})} \right),
\]
\[
\tilde{\Phi}_W(\nabla^TM, \nabla^\xi, \tau) = \frac{1}{2\sinh(\frac{\xi}{2})} \left( \text{ch}(\Theta_2(TC^M + \xi_C, C^2)) \right).
\]
\[
\tilde{\Phi}_W(\nabla^{TM}, \nabla^\xi, \tau) = A(TM, \nabla^{TM}) \frac{1}{2\sinh(\frac{c}{2})} \left( \text{ch}(\Theta_3(TCM + \xi_C, C^2)) - \cosh(\frac{c}{2}) \right)
\]

Through direct computations, we get (cf. [HH])

\[
\tilde{\Phi}_L(\nabla^{TM}, \nabla^\xi, \tau) = \sqrt{2} \int_{\mathbb{R}} \prod_{j=1}^{2k} \frac{\theta'(0, \tau)}{\theta(0, \tau)} \left( \frac{\phi(x_j, \tau)}{\phi(x_j, \tau)} \right) \theta'(0, \tau) \left( \begin{array}{c}
\theta_1(0, \tau) \\
\theta_2(0, \tau) \\
\theta_3(0, \tau)
\end{array} \right)
\]

Applying the Chern-Weil theory and Lemma 3.1 again, we can transgress \(\tilde{\Phi}_L, \tilde{\Phi}_W, \tilde{\Phi}_W\) about \(\nabla^{TM}\) and define transgressed forms as follows:

\[
CS\tilde{\Phi}_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)
\]

\[
:= \frac{\sqrt{2}}{8\pi^2} \int_0^1 \tilde{\Phi}_L(\nabla_0^{TM}, \nabla^\xi, \tau) \text{tr} \left[ A \left( \frac{1}{R_1^{TM}} - \frac{\theta'(R_1^{TM})}{\theta(R_1^{TM})} \right) \right] dt.
\]

which is in \(\Omega^{\text{odd}}(M, C)[[q^\frac{1}{2}]]\). Since \(M\) is 4k+1 dimensional, \(CS\tilde{\Phi}_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)\) represents an element in \(H^{4k+1}(M, C)[[q^\frac{1}{2}]]\). Similarly, we can define

\[
CS\tilde{\Phi}_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)
\]

\[
:= \frac{1}{8\pi^2} \int_0^1 \tilde{\Phi}_W(\nabla_0^{TM}, \nabla^\xi, \tau) \text{tr} \left[ A \left( \frac{1}{R_1^{TM}} - \frac{\theta'(R_1^{TM})}{\theta(R_1^{TM})} \right) \right] dt;
\]

\[
CS\tilde{\Phi}_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^\xi, \tau)
\]

\[
:= \frac{1}{8\pi^2} \int_0^1 \tilde{\Phi}_W(\nabla_0^{TM}, \nabla^\xi, \tau) \text{tr} \left[ A \left( \frac{1}{R_1^{TM}} - \frac{\theta'(R_1^{TM})}{\theta(R_1^{TM})} \right) \right] dt.
\]

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Using the same discussions as Theorem 3.2, we obtain

**Theorem 4.1** Let $M$ be a $4k + 1$ dimensional manifold and $\nabla^0_{TM}$, $\nabla^1_{TM}$ be two connections on $TM$ and $\xi$ be a two dimensional oriented Euclidean real vector bundle with a Euclidean connection $\nabla^\xi$, then we have

1) $\{CS\Phi_L(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau)\}^{(4k+1)}$ is a modular form of weight $2k + 2$ over $\Gamma_0(2)$; $\{CS\Phi_W(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau)\}^{(4k+1)}$ is a modular form of weight $2k + 2$ over $\Gamma^0(2)$; $\{CS\Phi'_W(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau)\}^{(4k+1)}$ is a modular form of weight $2k + 2$ over $\Gamma_0(2)$.

2) The following equalities hold,

$$\{CS\Phi_L(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, -\frac{1}{2})\}^{(4k+1)} = (2\tau)^{2k+2}\{CS\Phi_W(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau)\}^{(4k+1)},$$

$$CS\Phi_W(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau + 1) = CS\Phi_W(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau).$$

Let $M$ be 9 dimensional and $k = 2$. We have that $\{CS\Phi_L(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau)\}^{(9)}$ is a modular form of weight 6 over $\Gamma_0(2)$; $\{CS\Phi_W(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau)\}^{(9)}$ is a modular form of weight 6 over $\Gamma^0(2)$ and

$$\{CS\Phi_L(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, -\frac{1}{2})\}^{(9)} = (2\tau)^6\{CS\Phi_W(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau)\}^{(9)}.$$  

By Lemma 2.2, we have

$$\{CS\Phi_W(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau)\}^{(9)} = z_0(8\delta_2)^3 + z_1(8\delta_2)\varepsilon_2,$$  

and by (2.19) and Theorem 4.1,

$$\{CS\Phi_L(\nabla^0_{0T}, \nabla^1_{1T}, \nabla^\xi, \tau)\}^{(9)} = 2^6[z_0(8\delta_1)^3 + z_1(8\delta_1)\varepsilon_1].$$

By comparing the $q^\frac{1}{2}$-expansion coefficients in (4.7), we get

$$z_0 = -\left\{ \int_0^1 \frac{\hat{A}(TM, \nabla^0_{TM})}{2\sinh\left(\frac{t}{2}\right)}(1 - \cosh \frac{C}{2})\text{tr}\left[ A \left( \frac{1}{2R^TM} - \frac{1}{8\pi\tan \frac{R^TM}{4\pi}} \right) \right] dt \right\}^{(9)},$$

$$z_1 = \left\{ -\int_0^1 \frac{\hat{A}(TM, \nabla^0_{TM})}{2\sinh\left(\frac{t}{2}\right)}(1 - \cosh \frac{C}{2})\text{tr}\left[ A \left( \frac{1}{2R^TM} - \frac{1}{8\pi\sin \frac{R^TM}{2\pi}} \right) \right] dt 
+ \int_0^1 \frac{\hat{A}(TM, \nabla^0_{TM})}{2\sinh\left(\frac{t}{2}\right)}\text{tr}\left[ A \left( \frac{1}{2R^TM} - \frac{1}{8\pi\tan \frac{R^TM}{4\pi}} \right) \right] 
\cdot \left( 1 - \cosh \frac{C}{2} \right)(\text{ch}(T_{C,M}, \nabla^0_{T_{C,M}}) + 61) + (1 + 2\cosh \frac{C}{2})(e^c + e^{-c} - 2) \right\}^{(9)}.$$
Corollary 4.2 The following equality holds

\[
\left\{ \int_0^1 \sqrt{2L(TM, \nabla^TM_t)} \frac{\sinh c^2}{\cosh c^2} \mathrm{tr} \left[ A \left( \frac{1}{2R_t^{TM}} - \frac{1}{4\pi \sin \frac{R_t^{TM}}{2\pi}} \right) \right] \right\}^{(9)} = 8 \left\{ - \int_0^1 \frac{\hat{A}(TM, \nabla^TM_t)}{2\sinh(\frac{c}{2})} (1 - \cosh \frac{c}{2}) \mathrm{tr} \left[ A \left( \frac{1}{2R_t^{TM}} - \frac{1}{2\pi \sin \frac{R_t^{TM}}{2\pi}} \right) \right] \mathrm{dt} \right. \\
+ \int_0^1 \frac{\hat{A}(TM, \nabla^TM_t)}{2\sinh(\frac{c}{2})} \mathrm{tr} \left[ A \left( \frac{1}{2R_t^{TM}} - \frac{1}{8\pi \tan \frac{R_t^{TM}}{4\pi}} \right) \right] \right. \\
\cdot (1 - \cosh \frac{c}{2})(\mathrm{ch}(T_CM, \nabla^T_{T_CM}) - 3) + (1 + 2\cosh \frac{c}{2})(e^c + e^{-c} - 2) \right\}^{(9)}. \quad (4.11)
\]

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