$L^p$-Boundedness of Wave Operators for the Three-Dimensional Multi-Centre Point Interaction

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Abstract. We prove that, for arbitrary centres and strengths, the wave operators for three-dimensional Schrödinger operators with multi-centre local point interactions are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < 3$ and unbounded otherwise.

1. Introduction and Main Results

Models of quantum particles in $d$ dimensions which scatter freely in space except for the presence of a number of extremely localised impurities require the definition of a Hamiltonian that acts precisely as the free Hamiltonian on wave functions supported away from the scattering centres and that induces a non-trivial interaction essentially supported on a discrete collection of points. This naturally leads to consider ‘singular’ perturbations of the free Schrödinger operator, which can be thought of as delta-like potentials centred at fixed points, a picture that dates back to the celebrated model of Kronig and Penney [24] for a quantum particle in a one-dimensional array of delta potentials.

One can make sense in various conceptually alternative ways of the formal Hamiltonian

\[ " - \Delta_x + \sum_{j=1}^{N} \mu_j \delta(x - y_j) " \]  

for a quantum particle in $\mathbb{R}^d$ subject to singular interactions centred at the points $y_1, \ldots, y_N \in \mathbb{R}^d$ and of magnitude, respectively, $\mu_1, \ldots, \mu_N$. One is to realise the Hamiltonian as a self-adjoint extension of the restriction of $-\Delta$
to smooth functions supported away from the $y_j$'s, another is to obtain it as the limit of a Schrödinger operator with actual potentials $V_\varepsilon^{(j)}(x - y_j)$ each of which, as $\varepsilon \to 0$, spikes up to a delta-like profile, the support shrinking to the point $\{y_j\}$, and yet another way is to realise (1.1) as the self-adjoint operator of a closed and semi-bounded energy (quadratic) form that consists of a free (gradient) term plus suitable boundary terms at the centres $y_j$'s.

In the mathematical literature, the study of the self-adjoint realisations of (1.1) has a long history, deeply connected with that of the physical systems for which such a model has provided a realistic description. In this work, we focus on the $d = 3$ case: we thus consider the collection

$$Y := \{y_1, \ldots, y_N\}$$

of $N$ distinct points in $\mathbb{R}^3$ and, correspondingly, the operator

$$\tilde{H}_Y := -\Delta |_{C_0^\infty(\mathbb{R}^3 \setminus Y)}$$

in the Hilbert space $L^2(\mathbb{R}^3)$. $\tilde{H}_Y$ is densely defined, real symmetric, and non-negative, with deficiency indices $(N, N)$, and hence, it admits a $N^2$-parameter family of self-adjoint extensions.

The most relevant subclass of them is the $N$-parameter family

$$\{H_{\alpha,Y} | \alpha \equiv (\alpha_1, \ldots, \alpha_N) \in (-\infty, \infty)^N\}$$

of so-called local extensions, namely extensions of $\tilde{H}_Y$ whose domain of self-adjointness is only qualified by certain local boundary conditions at the singularity centres. More precisely, as we shall recall in detail in Sect. 2, the domain $\mathcal{D}(H_{\alpha,Y})$ of $H_{\alpha,Y}$ consists of functions $u$ satisfying the asymptotics

$$\lim_{r_j \downarrow 0} \left( \frac{\partial(r_j u)}{\partial r_j} - 4\pi \alpha_j r_j u \right) = 0, \quad r_j := |x - y_j|, \quad j \in \{1, \ldots, N\}.$$ (1.4)

In fact, the condition

$$u(x) \underset{x \to y_j}{\sim} \frac{1}{|x - y_j|} - \frac{1}{a_j}, \quad a_j := -(4\pi \alpha_j)^{-1}$$ (1.5)

implied by (1.4) is typical for the low-energy behaviour of an eigenstate of the Schrödinger equation for a quantum particle subject to a potential of extremely short, virtually zero, range centred at the point $y_j$ and with $s$-wave scattering length $a_j$, a fact that was noted first by Bethe and Peierls [8,9] [whence the name of Bethe–Peierls contact condition for asymptotics (1.5)].

If for some $j \in \{1, \ldots, N\}$ one has $\alpha_j = \infty$, then no actual interaction is present at the point $y_j$ (no boundary condition as $x \to y_j$) and in practice things are as if one discards the point $y_j$. In particular, the extension $H_{\alpha,Y}$ corresponding to $\alpha = \infty$ is the Friedrichs extensions of $\tilde{H}_Y$, namely the self-adjoint negative Laplacian on $L^2(\mathbb{R}^3)$. We shall also denote it by $H_0$, the free Hamiltonian. It is precisely the extension with no interactions at all.

The operator $H_{\alpha,Y}$ was rigorously studied for the first time by Albeverio et al. [2] and subsequently characterised by Zorbas [37], Grossmaann et al. [19,20], and Dąbrowski and Grosse [10]. A thorough discussion of its features
can be found in [3, Sect. II.1.1]. We shall recall the main properties of \( H_{\alpha,Y} \) in Sect. 2.

Fundamental information about the dynamics generated by \( H_{\alpha,Y} \) through the Schrödinger equation \( i\partial_t u = H_{\alpha,Y} u \) is encoded in the wave operators for the pair \((H_{\alpha,Y}, H_0)\), which are defined by the strong limits

\[
W_{\alpha,Y}^\pm = \lim_{t \to \pm \infty} e^{itH_{\alpha,Y}} e^{-itH_0}.
\]

Since the resolvent difference \((H_{\alpha,Y} - z^2 \mathbb{1})^{-1} - (H_0 - z^2 \mathbb{1})^{-1}\) is of finite rank, as we shall recall in Theorem 2.1 (see (2.3)), standard arguments from scattering theory [27] guarantee that the wave operators \( W_{\alpha,Y}^+ \) and \( W_{\alpha,Y}^- \) exist in \( L^2(\mathbb{R}^3) \) and are complete, meaning that

\[
\text{ran} \ W_{\alpha,Y}^\pm = L^2_{\text{ac}}(H_{\alpha,Y}) = P_{\text{ac}}(H_{\alpha,Y})L^2(\mathbb{R}^3),
\]

where \( L^2_{\text{ac}}(H_{\alpha,Y}) \) denotes the absolutely continuous spectral subspace of \( L^2(\mathbb{R}^3) \) for \( H_{\alpha,Y} \), and \( P_{\text{ac}}(H_{\alpha,Y}) \) denotes the orthogonal projection onto \( L^2_{\text{ac}}(H_{\alpha,Y}) \). In particular, the absolutely continuous part of \( H_{\alpha,Y} \), namely the operator \( H_{\alpha,Y} P_{\text{ac}}(H_{\alpha,Y}) \), is unitarily equivalent to \( H_0 \). Moreover, the singular continuous spectrum is absent from \( H_{\alpha,Y} \) and the point spectrum consists of at most \( N \) negative eigenvalues, whereas non-negative eigenvalues are absent (see Theorem 2.1).

Wave operators are of paramount importance for the study of the scattering governed by an interaction Hamiltonian in comparison with a free (reference) Hamiltonian [25,27]. Owing to their completeness, \( W_{\alpha,Y}^+ \) and \( W_{\alpha,Y}^- \) are unitary from \( L^2(\mathbb{R}^3) \) onto \( L^2_{\text{ac}}(H_{\alpha,Y}) \); moreover, they intertwine \( H_{\alpha,Y} P_{\text{ac}}(H_{\alpha,Y}) \) and \( H_0 \), viz., for any Borel function \( f \) on \( \mathbb{R} \) one has the identity

\[
f(H_{\alpha,Y}) P_{\text{ac}}(H_{\alpha,Y}) = W_{\alpha,Y}^+ f(H_0) W_{\alpha,Y}^- \ast.
\]

Through this intertwining, mapping properties of \( f(H_{\alpha,Y}) P_{\text{ac}}(H_{\alpha,Y}) \) can be deduced from those of \( f(H_0) \) (which, upon Fourier transform, is the multiplication by \( f(\xi^2) \)), provided that the corresponding ones of \( W_{\alpha,Y}^\pm \) are known. Thus, the \( L^p \to L^p \) boundedness of \( f(H_{\alpha,Y}) P_{\text{ac}}(H_{\alpha,Y}) \) follows from the \( L^p \to L^p \) boundedness of \( W_{\alpha,Y}^\pm \); more precisely, if \( W_{\alpha,Y}^\pm \in \mathcal{B}(L^p(\mathbb{R}^d)) \) for some \( p \in [1, \infty] \), then \( (W_{\alpha,Y}^\pm \ast) \in \mathcal{B}(L^{p'}(\mathbb{R}^d)) \) and hence

\[
\| f(H_{\alpha,Y}) P_{\text{ac}}(H_{\alpha,Y}) \|_{\mathcal{B}(L^p, L^p)} \leq C_p \| f(H_0) \|_{\mathcal{B}(L^{p'}, L^{p'})},
\]

the constant \( C_p \) being independent of \( f \). (Here and henceforth, \( p' \) will denote the conjugate of \( p \) via \( p^{-1} + p'^{-1} = 1 \).)

The literature on the \( L^p \)-boundedness of wave operators relative to actual Schrödinger operators of the form \(-\Delta + V\), for sufficiently regular \( V: \mathbb{R}^d \to \mathbb{R} \) vanishing at spatial infinity, is vast [4–7,11,15,22,23,30,32,33,35,36] and the problem is well known to depend crucially on the spectral properties of \(-\Delta + V\) at the bottom of the absolutely continuous spectrum, that is, at energy zero.

For singular perturbations of the Schrödinger operators, the picture is much less developed and is essentially limited to the one-dimensional case.
Analogously to \((1.3)\), the restriction \(\tilde{\mathcal{H}}_Y := -\Delta \rvert C_0^\infty(\mathbb{R}\setminus Y)\) admits a \(N^2\)-parameter family of self-adjoint extensions in \(L^2(\mathbb{R})\) \([3, \text{Sect. II.2.1}]\), among which those extensions with so-called separated boundary condition of \(\delta\)-type—the analogue of \(H_{\alpha,Y}\) considered above. For the latter subfamily of Hamiltonians, Duchêne et al. \([14]\) constructed the corresponding wave operators \(W_{\alpha,Y}^\pm\) relative to the couple \((H_{\alpha,Y}, H_0)\) and proved that \(W_{\alpha,Y}^\pm \in \mathcal{B}(W^{1,p}(\mathbb{R}))\) for \(1 < p < \infty\). Their proof is built on a detailed decomposition of \(W_{\alpha,Y}^\pm\) essentially upon the high-frequency vs low-frequency behaviour of the Jost solutions, an eminently one-dimensional treatment that is hard to export to higher dimensions.

In this work, we study \(L^p\)-bounds for the wave operators \(W_{\alpha,Y}^\pm\) of the three-dimensional multi-centre point interaction Hamiltonian. We provide a manageable formula for (the integral kernel of) \(W_{\alpha,Y}^\pm\), which we obtain by manipulating the resolvent difference \((H_{\alpha,Y} - z^2 \mathbb{1})^{-1} - (H_0 - z^2 \mathbb{1})^{-1}: since this difference is an explicitly known finite rank operator for any dimensions \(d = 1, 2, 3\), our derivation can be naturally exported also to lower dimensions.

Based on our representation of \(W_{\alpha,Y}^\pm\), we then establish our main result:

**Theorem 1.1.** For any \(y_1, \ldots, y_N \in \mathbb{R}^3\) and \(\alpha_1, \ldots, \alpha_N \in \mathbb{R}\), the wave operators

\[
W_{\alpha,Y}^\pm = \lim_{t\to \pm \infty} e^{itH_{\alpha,Y}} e^{-itH_0}
\]

for the pair \((H_{\alpha,Y}, H_0)\) exist and are complete in \(L^2(\mathbb{R}^3)\), and they are bounded in \(L^p(\mathbb{R}^3)\) for \(1 < p < 3\), and unbounded for \(p = 1\) and for \(p \geq 3\).

**Remark 1.2.** The fact that \(L^p\)-boundedness holds only for \(p \in (1,3)\) is consistent with the analogous result for actual Schrödinger operators. Indeed, it is well known \([35,36]\) that the wave operators for three-dimensional Schrödinger operators \(-\Delta + V\) admitting a zero-energy resonance are \(L^p\)-bounded if and only if \(p \in (1,3)\), and moreover, it is also well known \([3, \text{Theorem II.1.2.1}]\) that \(H_{\alpha,Y}\) is actually the strong resolvent limit in \(L^2(\mathbb{R}^3)\), as \(\varepsilon \downarrow 0\), of Schrödinger operators of the form

\[
H(\varepsilon) = -\Delta + \varepsilon^{-2} \sum_{j=1}^N \lambda_j(\varepsilon) V_j \left( \frac{x - y_j}{\varepsilon} \right)
\]

for suitable real-analytic \(\lambda_j(\varepsilon)\)'s with \(\lambda(0) = 1\) and real potentials \(V_j\) of finite Rollnik norm such that \(-\Delta + V_j\) has a zero-energy resonance for each \(j \in \{1, \ldots, N\}\). To fully substantiate such a parallelism between singular and regular Schrödinger operators, it would be of great interest to monitor the convergence, as bounded operators in \(L^p(\mathbb{R}^3)\) for \(p \in (1,3)\), of the wave operators for the pair \((H_\varepsilon, H_0)\) to the wave operator \(W_{\alpha,Y}^\pm\). Along this line, in Sect. 7 we present the proof of this result in the special case \(N = 1, \alpha = 0\).

As a direct consequence of Theorem 1.1 and of bound \((1.9)\), the dispersive properties for the free propagator \(e^{-itH_0}\), encoded in the estimates

\[
\|e^{-itH_0}u\|_p \leq (4\pi|t|)^{-3(\frac{1}{2} - \frac{1}{p})}\|u\|_{p'}, \quad p \in [2, +\infty], \quad t \neq 0,
\]

\(1.12\)
lift to analogous estimates for the Schrödinger dynamics generated by $H_{\alpha,Y}$, albeit for an unavoidably smaller range of $p$’s than in (1.12). Thus, we find:

**Corollary 1.3.** There is a constant $C > 0$ such that, for each $p \in [2, 3)$,

$$\|e^{-itH_{\alpha,Y}} P_{ac}(H_{\alpha,Y})u\|_p \leq C |t|^{-3(\frac{1}{2} - \frac{1}{p})} \|u\|_{p'}, \quad t \neq 0. \quad (1.13)$$

In turn, by means of a well-known argument [16,31], the dispersive estimates (1.13) imply Strichartz estimates for $H_{\alpha,Y}$ for the same range of $p$. We shall call a pair of exponents $(p, q)$ admissible for $H_{\alpha,Y}$ if

$$p \in [2, 3) \quad \text{and} \quad 0 \leq \frac{2}{q} = 3 \left(\frac{1}{2} - \frac{1}{p}\right) < \frac{1}{2}, \quad (1.14)$$

that is, $q = \frac{4p}{3(p-2)} \in (4, +\infty]$.

**Corollary 1.4.** Let $(p, q)$ and $(r, s)$ be two admissible pairs for $H_{\alpha,Y}$. Then, for a constant $C > 0$,

$$\|e^{-itH_{\alpha,Y}} P_{ac}(H_{\alpha,Y})u\|_{L^q(\mathbb{R}^3, L^r(\mathbb{R}^3))} \leq C \|u\|_{L^2(\mathbb{R}^3)} \quad (1.15)$$

and

$$\left\| \int_0^t e^{-i(t-s)H_{\alpha,Y}} P_{ac}(H_{\alpha,Y}) u(s) \, ds \right\|_{L^q(\mathbb{R}^3, L^r(\mathbb{R}^3))} \leq C \|u\|_{L^{r'}(\mathbb{R}^3, L^{s'}(\mathbb{R}^3))}. \quad (1.16)$$

Under the *additional* assumption that the matrix $\Gamma_{\alpha,Y}(\lambda)$ that we define in equation (2.2) in Sect. 2 be invertible for all $\lambda \in [0, +\infty)$, with locally bounded inverse, suitably weighted dispersive estimates for the propagator $e^{-itH_{\alpha,Y}}$ were obtained by D’Ancona, Pierfelice, and Teta [12] in the form

$$\|w^{-1}e^{-itH_{\alpha,Y}} P_{ac}(H_{\alpha,Y})u\|_{L^\infty} \leq C |t|^{-3/2} \|wu\|_1, \quad t \neq 0, \quad (1.17)$$

for the weight function

$$w(x) := \sum_{j=1}^N \left(1 + \frac{1}{|x - y_j|}\right). \quad (1.18)$$

The restriction on $\Gamma_{\alpha,Y}(\lambda)$ is in practice the requirement that zero is not a resonance for $H_{\alpha,Y}$; thus, for $N = 1$, (1.17) was proved for $\alpha \neq 0$ and it was replaced by a slower dispersion rate $|t|^{-1/2}$ in the resonant case $N = 1$, $\alpha = 0$. We also observe that by interpolation (1.17) can be turned into the weighted dispersive estimate

$$\|w^{-(1-\frac{2}{p})}e^{-itH_{\alpha,Y}} P_{ac}(H_{\alpha,Y})u\|_p \leq C_p |t|^{-3(\frac{1}{2} - \frac{1}{p})} \|w^{\frac{2}{p'}}u\|_{p'} \quad (1.19)$$

for the whole range $p \in [2, +\infty]$.

As opposite to (1.19), our Corollary 1.3 removes both the weight and the assumption on $\Gamma_{\alpha,Y}(\lambda)$ in the regime $p \in [2, 3)$. In fact, we can also improve the weight in (1.19) for $p \in [3, +\infty]$ by interpolating between (1.13) of our Corollary 1.3 and (1.17) given by [12].

We also highlight that in the parallel work [21] by one of us in collaboration with Iandoli, the non-weighted dispersive estimate (1.13) is recovered.
by simpler and more direct arguments (i.e. without using any result from the scattering theory for $H_{\alpha,Y}$) in the special case $N = 1$.

The first key ingredient of our analysis is a fairly explicit resolvent formula for $H_{\alpha,Y}$, which is well known to be a rank-$N$ perturbation (in the resolvent sense) of the free Hamiltonian. This is in a way the same spirit as in the above-mentioned work [12] for generic $N$, except that the main difficulty therein was to produce reliable estimates on the propagator $e^{-itH_{\alpha,Y}}$ in the lack of an explicit representation of its kernel (instead, when $N = 1$ the dispersive estimate (1.17) was obtained in [12] directly from the explicit kernel of the propagator $e^{-itH_{\alpha,Y}}$, a kernel found by Scarlatti and Teta [28] and by Albeverio et al. [1]).

In our case, we aim at representing the (kernel of the) wave operators $W_{\alpha,Y}^\pm$ in the first place, based on the explicit resolvent difference $(H_{\alpha,Y} - z^2\mathbb{1})^{-1} - (H_0 - z^2\mathbb{1})^{-1}$. Then, as a second key ingredient, for the $L^p \to L^p$ estimate of $W_{\alpha,Y}^\pm$ we appeal to a large extent to some tool from harmonic analysis, the Calderón–Zygmund operators and the Muckenhoupt weighted inequalities.

We organised the material as follows: In Sect. 2, we recall the precise definition of $H_{\alpha,Y}$ and we collect several technical results needed in the proof of Theorem 1.1, including in particular properties of Calderón–Zygmund operators and the Muckenhoupt weighted inequalities. In Sect. 3, we produce the explicit stationary representation of the wave operators $W_{\alpha,Y}^\pm$ which the proof of Theorem 1.1 is based on. The $L^p$-boundedness part of Theorem 1.1 is proved in Sect. 4 for the single-centre case and in Sect. 5 for the multi-centre case. The $L^p$-unboundedness part is proved in Sect. 6. Last, in Sect. 7 we discuss the convergence of the wave operators relative to the family of Hamiltonians (1.11) to the wave operators $W_{\alpha,Y}^\pm$ (limit of shrinking potentials).

2. Preliminaries and notation

In this section, we recall the precise definition of $H_{\alpha,Y}$ and its basic properties from [3, Sect. II.1.1] and [26] (see also [12, 13]). Here and henceforth, the number $N \in \mathbb{N}$ and the $N$-point set $Y = \{y_1, \ldots, y_N\}$ introduced in (1.2) are fixed, and the multidimensional parameter $\alpha \equiv (\alpha_1, \ldots, \alpha_N)$ is assumed to run over $(-\infty, +\infty)^N$.

We begin with a few remarks on our notation. We write $\mathbb{C}$ for the complex plane and $\mathbb{C}^+$ for the open upper half-plane. By $\delta_{j,\ell}$, we denote the Kronecker delta, namely the quantity 1 for $j = \ell$ and 0 otherwise. As customary, $\langle \lambda \rangle \equiv (1 + \lambda^2)^{1/2}$ for $\lambda \in \mathbb{R}$. The representation of any point $x \in \mathbb{R}^3$ in polar coordinates will be $x = r\omega$, where $r \equiv |x| \geq 0$ and $\omega \in S^2$. For $u, v \in L^2(\mathbb{R}^3)$, $u \otimes v$ denotes the rank-1 operator $f \mapsto u(\langle v, f \rangle)$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{R}^3)$, anti-linear in the first entry and linear in the second. For the Fourier transform in $\mathbb{R}^d$, we use the convention

$$\langle F f \rangle(\xi) \equiv \hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) \, dx.$$
We often write $f \leq |g|$ when $|f| \leq |g|$. $E^{(T)}(d\lambda)$ denotes the spectral measure of the self-adjoint operator $T$. When not specified otherwise, $C$ denotes a universal positive constant and $\mathbb{1}$ is the identity operator on the space that is clear from the context. (This includes also the case of the $N \times N$ identity matrix.)

For $z \in \mathbb{C}$ and $x, y, y' \in \mathbb{R}^3$, we set

$$G_z(x) := \frac{e^{iz|x|}}{4\pi|x|}, \quad G_z^y(x) := \frac{e^{iz|x-y|}}{4\pi|x-y|} = G_z(x-y),$$

and

$$G_z^{yy'} := \begin{cases} \frac{e^{iz|y-y'|}}{4\pi|y-y'|} & \text{if } y' \neq y \\ 0 & \text{if } y' = y, \end{cases} \quad (2.1)$$

and

$$\Gamma_{\alpha,Y}(z) := \left( (\alpha_j - \frac{i z}{4\pi} \delta_{j,\ell} - G_z^{yy'})_{j,\ell=1,...,N} \right).$$

Thus, the function $z \mapsto \Gamma_{\alpha,Y}(z)$ has values in the $N \times N$ symmetric matrices and is clearly entire, and $z \mapsto \Gamma_{\alpha,Y}(z)^{-1}$ is meromorphic in $z \in \mathbb{C}$.

It is known that $\Gamma_{\alpha,Y}(z)^{-1}$ has at most $N$ poles in the closed upper half-plane $\mathbb{C}^+ \cup \mathbb{R}$, which are all located along the positive imaginary semi-axis [3, Theorem II.1.1.4]. We denote by $\mathcal{E}$ the set of such poles.

The following facts are known.

**Theorem 2.1.**

(i) For $z \in \mathbb{C}^+ \setminus \mathcal{E}$ the identity

$$(H_{\alpha,Y} - z^2 \mathbb{1})^{-1} - (H_0 - z^2 \mathbb{1})^{-1} = \sum_{j,k=1}^N (\Gamma_{\alpha,Y}(z)^{-1})_{jk} G_z^{y_j} \otimes G_z^{y_k},$$

defines the resolvent of a self-adjoint operator $H_{\alpha,Y}$ in $L^2(\mathbb{R}^3)$. $H_{\alpha,Y}$ is an extension of the operator $\tilde{H}_Y = -\Delta + C_0^\infty(\mathbb{R}^3 \setminus Y)$ defined in (1.3).

(ii) The domain of $H_{\alpha,Y}$ has the following representation, for any $z \in \mathbb{C}^+ \setminus \mathcal{E}$:

$$\mathcal{D}(H_{\alpha,Y}) = \left\{ \psi = \phi_z + \sum_{j,k=1}^N (\Gamma_{\alpha,Y}(z)^{-1})_{jk} \phi_z(y_k) G_z^{y_j} \bigg| \phi_z \in H^2(\mathbb{R}^3) \right\}.$$  \hspace{1cm} (2.4)

The summands in the decomposition of each $\psi \in \mathcal{D}(H_{\alpha,Y})$ depend on the chosen $z$; however, $\mathcal{D}(H_{\alpha,Y})$ does not. Equivalently, for any $z \in \mathbb{C}^+ \setminus \mathcal{E}$,

$$\mathcal{D}(H_{\alpha,Y}) = \left\{ \psi = \phi_z + \sum_{j=1}^N q_j G_z^{y_j} \bigg| \phi_z \in H^2(\mathbb{R}^3), (q_1, \ldots, q_N) \in \mathbb{C}^N \right\}.$$  \hspace{1cm} (2.5)

At fixed $z$, the decompositions above are unique.
(iii) With respect to decompositions (2.4)–(2.5), one has
\[ (H_{\alpha,Y} - z^2 \mathbb{1}) \psi = (H_0 - z^2 \mathbb{1}) \phi_z. \] (2.6)
Moreover, \( H_{\alpha,Y} \) has the following locality property: if \( \psi \in \mathcal{D}(H_{\alpha,Y}) \) is such that \( \psi|_\mathcal{U} \equiv 0 \) for some open \( \mathcal{U} \subset \mathbb{R}^3 \), then \( (H_{\alpha,Y}\psi)|_\mathcal{U} \equiv 0 \).

(iv) The spectrum \( \sigma(H_{\alpha,Y}) \) of \( H_{\alpha,Y} \) consists of at most \( N \) strictly negative eigenvalues and the absolutely continuous part \( \sigma_{ac}(H_{\alpha,Y}) = [0,\infty) \). Non-negative eigenvalues and the singular continuous spectrum are absent. There is a one-to-one correspondence between the poles \( \lambda \) of \( \Gamma_{\alpha,Y}(z) \) in \( \mathbb{C}^+ \) and the negative eigenvalues \( -\lambda^2 \) of \( H_{\alpha,Y} \), counting the multiplicity. The eigenfunctions that belong to the eigenvalue \( E_0 = -\lambda_0^2 < 0 \) of \( H_{\alpha,Y} \) have the form
\[ \psi_0 = \sum_{j=1}^N c_j G_{i\lambda_0}^{y_j}, \]
where \((c_0, \ldots, c_N)\) are eigenvectors with eigenvalue zero of \( \Gamma_{\alpha,Y}(i\lambda_0) \). The ground state, if it exists, is non-degenerate.

Part (i) of Theorem 2.1 was first proved in [19, 20]—see also the discussion in [3, equation (II.1.1.33)]. Parts (ii) and (iii) originate from [20] and are discussed in [3, Theorem II.1.1.3], in particular (2.5) is highlighted in [13]. Part (iv) is an extension, proved in [3, Theorem II.1.1.4], of some of the corresponding results established in [20].

By exploiting the boundary condition (2.5) between the regular and the singular part of a generic \( \psi \in \mathcal{D}(H_{\alpha,Y}) \), it is straightforward to see that
\[ \lim_{r_j \downarrow 0} \left( \frac{\partial (r_j \psi)}{\partial r_j} - 4\pi \alpha_j r_j \psi \right) = 0, \quad r_j := |x - y_j|, \quad j \in \{1, \ldots, N\}, \] (2.7)
whence also
\[ \lim_{x \to y_j} \left( \psi(x) - \frac{q_j}{4\pi |x - y_j|} - \alpha_j q_j \right) = 0, \quad j \in \{1, \ldots, N\}. \] (2.8)
Thus, the elements of \( \mathcal{D}(H_{\alpha,Y}) \) satisfy the ‘physical’ (Bethe–Peierls) boundary condition
\[ \psi(x) \sim_{x \to y_j} \frac{q_j}{4\pi} \left(\frac{1}{|x - y_j|} - \frac{1}{a_j}\right), \quad a_j := -(4\pi \alpha_j)^{-1} \] (2.9)
at each centre of the point interaction (see (1.5) in Introduction). In fact, \( \mathcal{D}(H_{\alpha,Y}) \) is nothing but the space of those \( L^2 \)-functions \( \psi \) such that the distribution \( \Delta \psi \) belongs to \( L^2(\mathbb{R}^3 \setminus Y) \) and the boundary condition (2.7) is satisfied.

We also record two simple consequences of Theorem 2.1, which will turn out to be useful in our discussion.

**Lemma 2.2.** The operator \( H_{\alpha,Y} \) is a real self-adjoint operator, that is, for a real-valued function \( \psi \in \mathcal{D}(H_{\alpha,Y}) \), \( H_{\alpha,Y} \psi \) is also real-valued.

**Proof.** Let \( z = i\lambda, \lambda > 0 \), be such that \( i\lambda \notin \mathcal{E} \) and let \( \psi \) be a real-valued function in \( \mathcal{D}(H_{\alpha,Y}) \). Then, with the notation of decomposition (2.5) of \( \psi \), asymptotics (2.8) show that the coefficients \( q_1, \ldots, q_N \) are all real. The entries
of $\Gamma_{\alpha,Y}(i\lambda)$ are real too, because $\Re z > 0$. Then (2.5) implies that $\phi_z$ is real-valued and so must be $H_{\alpha,Y}\psi + \lambda^2\psi$, owing to (2.6). □

**Lemma 2.3.** If $z = 0$ is a pole of $\Gamma_{\alpha,Y}(z)^{-1}$, then it is a pole of first order, and in a neighbourhood of $z = 0$ one has

$$\Gamma_{\alpha,Y}(z)^{-1} = \frac{\Theta}{z} + \Gamma_{\alpha,Y}^{(\text{reg})}(z)$$

for some constant matrix $\Theta$ and some analytic matrix-valued function $\Gamma_{\alpha,Y}^{(\text{reg})}(z)$.

**Proof.** We recall first that for a generic self-adjoint operator $T$ in a Hilbert space $\mathcal{H}$ for which zero is not an eigenvalue, one has

$$\lim_{\kappa \downarrow 0} \| i\kappa^2 (T + i\kappa^2)^{-1} u \|^2 = \lim_{\kappa \downarrow 0} \int_{\mathbb{R}} \left| \frac{i\kappa^2}{\lambda + i\kappa^2} \right|^2 \langle u, E^{(T)}(d\lambda)u \rangle = \| E^{(T)}(\{0\}) u \|^2 = 0 \quad \forall u \in \mathcal{H}.$$  

In our case, neither $H_{\alpha,Y}$ nor $H_0$ have zero eigenvalue: therefore, applying the above fact to the resolvent identity (2.3), one finds

$$\lim_{z \rightarrow 0} \sum_{j,k=1}^N z^2 (\Gamma_{\alpha,Y}(z)^{-1})_{jk} \langle u, G^{y_j}_z \rangle \langle G^{y_k}_{-iz}, v \rangle = 0$$

for any $u, v \in C_0^\infty(\mathbb{R}^3)$. Suppose that $\Gamma_{\alpha,Y}(z)^{-1}$ has a pole of order $\geq 2$ at $z = 0$ with matrix residue $\tilde{\Theta}$: then the identity above implies

$$\sum_{j,k=1}^N \tilde{\Theta}_{jk} \langle u, G^{y_j}_{0} \rangle \langle G^{y_k}_{0}, v \rangle = 0 \quad \forall u, v \in C_0^\infty(\mathbb{R}^3).$$

It follows that

$$\sum_{j,k=1}^N \frac{\tilde{\Theta}_{jk}}{|x - y_j||y - y_k|} = 0, \quad x, y \in \mathbb{R}^3.$$  

Since $\tilde{\Theta}$ is a symmetric matrix, this implies $\tilde{\Theta} = 0$ and the pole must be of first order. □

Last, we collect in the remaining part of this section some results from one-dimensional harmonic analysis, which we shall make crucial use of in the course of our discussion. For the definition of Calderón–Zygmund operators, we refer to [17, Definitions 7.4.1, 7.4.2] and to [18, Definitions 4.1.2 and 4.1.8], whereas for the definition of $A_p$ Muckenhoupt weights we refer to [17, Definitions 7.1.3]. We shall use interchangeably the same symbol for a Calderón–Zygmund operator and for its integral kernel.

The following properties are known.
Theorem 2.4.

(i) The convolution operator on \( \mathbb{R} \) with a function \( L(x) \) is a Calderón–Zygmund operator if \( \hat{L}(\xi) \) is bounded and, for a constant \( C > 0 \), one has
\[
|L(x)| \leq C|x|^{-1} \quad \text{and} \quad \left| \frac{dL}{dx}(x) \right| \leq C|x|^{-2} \quad \text{for } x \neq 0.
\]
(ii) If \( L \) is a Calderón–Zygmund operator and \( w \) is an \( A_p \)-weight for some \( p \in (1, \infty) \), then \( L \) is bounded in \( L^p(\mathbb{R}, w(x)dx) \) in the sense that
\[
\int_{\mathbb{R}} |(Lu)(x)|^p w(x) \, dx \leq \int_{\mathbb{R}} |u(x)|^p w(x) \, dx \quad \forall u \in C_0^\infty(\mathbb{R}). \tag{2.10}
\]
(iii) If \( w \) is an \( A_p \)-weight for some \( p \in (1, \infty) \) and
\[
(M(u))(x) := \sup_{r>0} \frac{1}{2r} \int_{|x-y|<r} |u(y)| \, dy \tag{2.11}
\]
is the Hardy–Littlewood maximal function of some \( u \in C_0^\infty(\mathbb{R}) \), then
\[
\int_{\mathbb{R}} |(M(u))(x)|^p w(x) \, dx \leq \int_{\mathbb{R}} |u(x)|^p w(x) \, dx. \tag{2.12}
\]
If, for some function \( L(x) \) one has \( |L(x)| \leq A(x) \) in \( \mathbb{R} \) for some \( A \in L^1(\mathbb{R}) \) which is bounded, non-negative, even, and non-increasing on \((0, +\infty)\), then \( |(L*u)(x)| \leq C(M(u))(x) \), and hence, the convolution operator on \( \mathbb{R} \) with the function \( L(x) \) is bounded in \( L^p(\mathbb{R}, w(x)dx) \).
(iv) The function \( |x|^a \) is an \( A_p \)-weight on \( \mathbb{R} \) if and only if \( a \in (-1, p - 1) \).

Concerning part (i), we refer to [18, Remark 4.1.1]. Part (ii) is a corollary of [17, Theorem 7.4.6]. The first and second statements of part (iii) are, respectively [29, Theorem 1, Sect. V.3] and the Proposition in page 57 of [29, Sect. II.2.1]. For part (iv), we refer to [17, Example 7.1.7].

3. Stationary Representation of Wave Operators

Following a standard procedure [25], in order to prove the \( L^p \)-boundedness of \( W_{\alpha,Y}^+ \) we want to represent \( W_{\alpha,Y}^+ \) by means of the boundary values attained by the resolvents of \( H_{\alpha,Y} \) and \( H_0 \) on the reals.

To this aim, we introduce the operators \( \Omega_{jk}, j, k \in \{1, \ldots, N\} \), acting on \( L^2(\mathbb{R}^3) \), defined by
\[
(\Omega_{jk}f)(x) := \lim_{\delta \downarrow 0} \frac{1}{\pi^2} \int_0^{+\infty} d\lambda \, \lambda e^{-\delta \lambda} \times \left( \int_{\mathbb{R}^3} (T_{\alpha,Y}(-\lambda)^{-1})_{jk} \mathcal{G}_{-\lambda}(x)(\mathcal{G}_{\lambda}(y) - \mathcal{G}_{-\lambda}(y)) f(y) \, dy \right). \tag{3.1}
\]
and we also introduce the translation operators \( T_{x_0} : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3), x_0 \in \mathbb{R}^3 \), defined by
\[
(T_{x_0}f)(x) := f(x - x_0). \tag{3.2}
\]
First of all, we show that the $\Omega_{ij}$’s are well-defined. It is convenient to rewrite $\Omega_{jk}$ by using the spherical mean $M_u$ of a given function $u$, namely

$$M_u(r) := \frac{1}{4\pi} \int_{S^2} u(r\omega) \, d\omega, \quad r \in \mathbb{R}. \quad (3.3)$$

Observe that $\mathbb{R} \ni r \mapsto M_u(r)$ is even. It is also convenient to define the matrix-valued function $\lambda \mapsto F(\lambda) := (F_{jk}(\lambda))_{jk}$ by

$$F_{jk}(\lambda) := 1_{(0, +\infty)}(\lambda) \lambda (\Gamma_{\alpha,Y}(-\lambda)^{-1})_{jk}, \quad j, k \in \{1, \ldots, N\}, \quad (3.4)$$

where $1_{\Lambda}$ denotes the characteristic function of the set $\Lambda$.

Lemma 3.1.

(i) The function $\lambda \mapsto F(\lambda)$ of (3.4) is smooth and uniformly bounded on $\mathbb{R}$, and

$$\lim_{\lambda \to +\infty} F(\lambda) = -4\pi i \mathds{1}. \quad (3.5)$$

(ii) Limit (3.1) exists in $L^2(\mathbb{R}^3)$, and $\Omega_{jk}$ may be written in the form

$$(\Omega_{jk} u)(x) = \frac{1}{i(2\pi)^{\frac{3}{2}}|x|} \int_{\mathbb{R}} e^{-i|\lambda|x|} F_{jk}(\lambda)(rM_u)(-\lambda) \, d\lambda. \quad (3.6)$$

If we introduce the distributional Fourier transform of $F_{jk}(\lambda)$ as

$$L_{jk}(\rho) := \frac{1}{\sqrt{2\pi}} \lim_{\delta \downarrow 0} \int_{0}^{+\infty} d\lambda \, e^{-\delta \lambda} e^{-i\lambda \rho} F_{jk}(\lambda), \quad (3.7)$$

it follows from (3.6) that

$$(\Omega_{jk} u)(x) = \frac{1}{i(2\pi)^{\frac{3}{2}}|x|} (L_{jk} * rM_u)(|x|). \quad (3.8)$$

Proof of Lemma 3.1. (i) Recall from Theorem 2.1(iv) that $\mathbb{C} \ni z \mapsto \Gamma_{\alpha,Y}^{-1}(z)$ is a meromorphic function, whose poles in the complex upper half-plane are all located on the positive imaginary axis. In particular, the only pole on the real line can be $z = 0$, in which case it is a pole of order one, owing to Lemma 2.3. This implies that $\lambda \mapsto \lambda \Gamma_{\alpha,Y}^{-1}(-\lambda)$ is smooth and bounded on compact sets of $(0, +\infty)$, and so is $\lambda \mapsto F(\lambda)$ on compact sets of $\mathbb{R}$. Concerning the behaviour as $\lambda \to +\infty$, we see from (2.2) that

$$\Gamma_{\alpha,Y}(-\lambda) = -(4\pi i)^{-1} \lambda \mathds{1} + R(\lambda)$$

for some symmetric matrix $R(\lambda)$ that is uniformly bounded for $\lambda \in (0, \infty)$. Thus, as $\lambda \to +\infty$,

$$\frac{\Gamma_{\alpha,Y}(-\lambda)}{\lambda} = -(4\pi i)^{-1} \mathds{1} + \frac{R(\lambda)}{\lambda} \to -(4\pi i)^{-1} \mathds{1},$$

which proves (3.5).

(ii) Let $u \in C_0^\infty(\mathbb{R}^3)$. Then, for $\lambda \in \mathbb{R}$,

$$\int_{\mathbb{R}^3} G_\lambda(y) u(y) \, dy = \int_{\mathbb{R}^3} \frac{e^{i|\lambda|y}}{4\pi |y|} u(y) \, dy = \int_0^{+\infty} e^{i\lambda r} rM_u(r) \, dr.$$
Since $\mathbb{R} \ni r \mapsto M_u(r)$ is even, the identity above yields
\[
\int_{\mathbb{R}^3} \left( G_\lambda(y) - G_{-\lambda}(y) \right) u(y) \, dy = \int_{\mathbb{R}} e^{i\lambda r} r M_u(r) \, dr = \sqrt{2\pi} \widehat{(r M_u)(-\lambda)} \] (3.9)
and (3.1) may be rewritten as
\[
(\Omega_{jk}u)(x) = \lim_{\delta \downarrow 0} \frac{1}{(2\pi)^{3/2}} \int_0^{+\infty} e^{-\delta \lambda} F_{jk}(\lambda) e^{-i\lambda|x|} \widehat{(r M_u)(-\lambda)} \, d\lambda. \tag{3.10}
\]
Here, $\widehat{(r M_u)(-\lambda)}$ is a square integrable function of $\lambda \in \mathbb{R}$ because Parseval’s identity and Hölder’s inequality yield
\[
\left\| \widehat{(r M_u)(-\lambda)} \right\|_{L^2(\mathbb{R})} = \left\| r M_u \right\|_{L^2(\mathbb{R})} \leq \left( \sqrt{\pi} \right)^{-1} \left\| u \right\|_{L^2(\mathbb{R}^3)}.
\]
Since $F_{jk}(\lambda)$ is bounded, the Fourier inversion formula implies that the limit $\delta \downarrow 0$ in (3.10) exists in $L^2(\mathbb{R}^3)$ and (3.6) follows.

The main result of this section is the following representation formula for the wave operator.

**Proposition 3.2.** Let $u, v \in L^2(\mathbb{R}^3)$. Then,
\[
\langle W^+_{\alpha,y} u, v \rangle = \langle u, v \rangle + \sum_{j,k=1}^N \langle T_{y_j} \Omega_{jk} T_{y_k}^* u, v \rangle. \tag{3.11}
\]

**Proof.** It suffices to prove (3.11) for $u, v \in C^\infty_0(\mathbb{R}^3)$.

Limit (1.10) when $t \to +\infty$ equals its Abel limit, and thus, we rewrite
\[
\langle W^+_{\alpha,y} u, v \rangle = \lim_{\varepsilon \downarrow 0} \frac{2\varepsilon}{\pi} \int_0^{+\infty} \langle e^{-it(H_0-\varepsilon\mathbb{1})} u, e^{-it(H_{\alpha,y}-\varepsilon\mathbb{1})} v \rangle \, dt. \tag{3.12}
\]
Let now $\mu \in \mathbb{R}$. Exploiting the Fourier transform
\[
(H_0 - (\mu + i\varepsilon) \mathbb{1})^{-1} = i \int_0^{+\infty} e^{i\mu t} e^{-it(H_0-\varepsilon\mathbb{1})} \, dt \quad (\varepsilon > 0)
\]
(and the analogue for $H_{\alpha,y}$), Parseval’s formula in the r.h.s. of (3.12) yields
\[
\langle W^+_{\alpha,y} u, v \rangle = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle R_0(\lambda + i\varepsilon) u, R_{\alpha,y}(\lambda + i\varepsilon) v \rangle \, d\lambda. \tag{3.13}
\]
Here and henceforth
\[
R_0(\mu) := (H_0 - \mu \mathbb{1})^{-1} \quad \mu \in \mathbb{C} \setminus [0, +\infty),
\]
\[
R_{\alpha,y}(\mu) := (H_{\alpha,y} - \mu \mathbb{1})^{-1} \quad \mu \in \mathbb{C} \setminus \sigma(H_{\alpha,y}), \tag{3.14}
\]
that is, the resolvents of the operators $H_0$ and $H_{\alpha,y}$.
Substituting $R_{\alpha,Y}(\lambda+i\varepsilon)$ in the r.h.s. of (3.13) with the resolvent identity (2.3), one obtains

$$
\langle W_{\alpha,Y}^+ u, v \rangle = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle R_0(\lambda + i\varepsilon) u, R_0(\lambda + i\varepsilon) v \rangle \, d\lambda \\
+ \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \sum_{j,k=1}^{N} \int_{\mathbb{R}} (\Gamma_{\alpha,Y}(\sqrt{\lambda + i\varepsilon})^{-1})_{jk} \\
\times \langle R_0(\lambda + i\varepsilon) u, \mathcal{G}_{\sqrt{\lambda + i\varepsilon}}^{y_j} \otimes \mathcal{G}_{\sqrt{\lambda + i\varepsilon}}^{y_k} v \rangle \, d\lambda.
$$

The first summand in the r.h.s. of (3.15) gives

$$
\frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle R_0(\lambda + i\varepsilon) u, R_0(\lambda + i\varepsilon) v \rangle \, d\lambda = \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle u, R_0(\lambda + i\varepsilon) R_0(\lambda + i\varepsilon) v \rangle \, d\lambda \\
= \frac{\varepsilon}{\pi} \int_{\mathbb{R}} d\lambda \int_{\sigma(H_0)} \langle u, E(H_0)(dh)v \rangle \frac{1}{(h - \lambda)^2 + \varepsilon^2} \\
= \int_{\sigma(H_0)} \langle u, E(H_0)(dh)v \rangle \frac{\varepsilon}{\pi} \int_{\mathbb{R}} d\lambda \frac{\varepsilon}{(h - \lambda)^2 + \varepsilon^2} = \langle u, v \rangle,
$$

thus (3.15) reads

$$
\langle W_{\alpha,Y}^+ u, v \rangle = \langle u, v \rangle + \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \sum_{j,k=1}^{N} \int_{\mathbb{R}} (\Gamma_{\alpha,Y}(\sqrt{\lambda + i\varepsilon})^{-1})_{jk} \\
\times \langle u, R_0(\lambda + i\varepsilon) \mathcal{G}_{\sqrt{\lambda + i\varepsilon}}^{y_j} \otimes \mathcal{G}_{\sqrt{\lambda + i\varepsilon}}^{y_k} v \rangle \, d\lambda.
$$

We recall that $\sqrt{z}$ is chosen in the upper complex half-plane and, for $z \in \mathbb{C} \setminus [0, \infty),

$$
\mathcal{G}_{\sqrt{\lambda + i\varepsilon}}^{y_j}(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ip(x-y_j)}}{p^2 - z} \, dp \left( \equiv \lim_{L \to \infty} \frac{1}{(2\pi)^3} \int_{|p|<L} \frac{e^{ip(x-y_j)}}{p^2 - z} \, dp \right).
$$

Thus, for $z \equiv \lambda + i\varepsilon$, both $\sqrt{\lambda + i\varepsilon}$ and $\sqrt{\lambda - i\varepsilon}$ belong to $\mathbb{C}^+$, and we compute

$$
\frac{\varepsilon}{\pi} R_0(\lambda - i\varepsilon) \mathcal{G}_{\sqrt{\lambda + i\varepsilon}}^{y_j}(x) = \frac{1}{(2\pi)^3} \frac{\varepsilon}{\pi} \int_{\mathbb{R}^3} \frac{e^{ip(x-y_j)}}{(p^2 - \lambda + i\varepsilon)(p^2 - \lambda - i\varepsilon)} \, dp \\
= \frac{1}{(2\pi)^3} \frac{1}{2\pi i} \int_{\mathbb{R}^3} e^{ip(x-y_j)} \left( \frac{1}{(p^2 - \lambda - i\varepsilon)} - \frac{1}{(p^2 - \lambda + i\varepsilon)} \right) \, dp \\
= \frac{1}{2\pi i} \left( \mathcal{G}_{\sqrt{\lambda + i\varepsilon}}^{y_j}(x) - \mathcal{G}_{\sqrt{\lambda - i\varepsilon}}^{y_j}(x) \right).
$$

The second summand in the r.h.s. of (3.16) can be then written as

$$
\lim_{\varepsilon \downarrow 0} \sum_{j,k=1}^{N} \frac{1}{2\pi i} \int_{\mathbb{R}} d\lambda \left( \int_{\mathbb{R}^3} dy \, \overline{u(y)} \left( \mathcal{G}_{\sqrt{\lambda - i\varepsilon}}^{y_j}(y) - \mathcal{G}_{\sqrt{\lambda - i\varepsilon}}^{y_j}(y) \right) \\
\times (\Gamma_{\alpha,Y}(\sqrt{\lambda - i\varepsilon})^{-1})_{jk} \left( \int_{\mathbb{R}^3} dx \, \mathcal{G}_{\sqrt{\lambda + i\varepsilon}}^{y_j}(x) v(x) \right) \right).
$$
Because \(u\) and \(v\) are smooth and with compact support, an integration by parts shows that both the \(d\xi\)-integral and the \(dy\)-integral in (3.19) above are bounded by \(C(\lambda)^{-\frac{1}{2}}\) uniformly in \(\varepsilon\). Moreover, as established in Lemma 2.3, the matrix \(\Gamma_{\alpha,Y}(\sqrt{\lambda + i\varepsilon})^{-1}\) has the singularity \((\sqrt{\lambda + i\varepsilon})^{-1}\) near \(\lambda = 0\) (in the limit \(\varepsilon \downarrow 0\)) if \(H_{\alpha,Y}\) has a zero-energy resonance, whereas it is bounded otherwise, with \(|\Gamma_{\alpha,Y}(\sqrt{\lambda + i\varepsilon})^{-1}| \leq C(\lambda)^{-\frac{1}{2}}\). Therefore, the \(\lambda\)-integrands is uniformly bounded by \(C\lambda^{-\frac{1}{2}}(\lambda)^{-1}\), dominated convergence is applicable in (3.19) above, the \(d\lambda\)-integration and the \(\varepsilon \downarrow 0\)-limit can be exchanged, and (3.19) becomes

\[
\sum_{j,k=1}^{N} \frac{1}{2\pi^1} \int_{\mathbb{R}} d\lambda \left( \int_{\mathbb{R}^3} dy \, u(y) \left( G^{y_j}_{\sqrt{\lambda+10}}(y) - G^{y_j}_{\sqrt{\lambda-10}}(y) \right) \right) \times (\Gamma_{\alpha,Y}(\sqrt{\lambda+10})^{-1})_{jk} \left( \int_{\mathbb{R}^3} dx \, G^{y_k}_{\sqrt{\lambda+10}}(x) \, v(x) \right). \tag{3.20}
\]

Owing to the difference \(G^{y_j}_{\sqrt{\lambda+10}} - G^{y_j}_{\sqrt{\lambda-10}}\), we see that the \(\lambda\)-integration in (3.20) is only effective when \(\lambda \geq 0\). Indeed, if \(\lambda < 0\), then \(\sqrt{\lambda \pm i\delta} = i\sqrt{|\lambda|}\) and the integrand vanishes. We then consider (3.20) only with \(\lambda \in [0, +\infty)\), and with the change of variable \(\lambda \mapsto \lambda^2\) we obtain

second summand in the r.h.s. of (3.16) =

\[
= \sum_{j,k=1}^{N} \frac{1}{2\pi^1} \int_{0}^{+\infty} d\lambda \lambda \left( \int_{\mathbb{R}^3} dy \, u(y) \left( G^{y_j}_\lambda(y) - G^{y_j}_{-\lambda}(y) \right) \right) \times (\Gamma_{\alpha,Y}(\lambda)^{-1})_{jk} \left( \int_{\mathbb{R}^3} dx \, G^{y_k}_\lambda(x) \, v(x) \right) \\
= \lim_{\delta \downarrow 0} \sum_{j,k=1}^{N} \frac{1}{2\pi^1} \int_{0}^{+\infty} d\lambda \lambda \exp(-\delta \lambda) \left( \int_{\mathbb{R}^3} dy \, u(y + y_k) \left( G_{\lambda}(y) - G_{-\lambda}(y) \right) \right) \times (\Gamma_{\alpha,Y}(\lambda)^{-1})_{jk} \left( \int_{\mathbb{R}^3} dx \, G^{y_j}_{\lambda}(x) \, v(x) \right) \\
= \lim_{\delta \downarrow 0} \int_{\mathbb{R}^3} dx \, v(x) \sum_{j,k=1}^{N} \int_{0}^{+\infty} d\lambda \lambda \exp(-\delta \lambda) \int_{\mathbb{R}^3} dy \, \\
\times \left( \frac{1}{2\pi^1} (\Gamma_{\alpha,Y}(-\lambda)^{-1})_{jk} G^{y_j}_{-\lambda}(x) \left( G_{\lambda}(y) - G_{-\lambda}(y) \right) u(y + y_k) \right). \tag{3.21}
\]

In the first step of (3.21) above, we used the fact that \(\sqrt{\lambda^2 ± i\delta} = \pm \lambda\) for \(\lambda > 0\). In the second step, the insertion of the exponential cut-off \(\exp(-\delta \lambda)\) is justified by the fact that the \(\lambda\)-integrands is uniformly bounded by \(C(\lambda)^{-\frac{3}{2}}\), as discussed above; we also exchanged \(j \leftrightarrow k\), using the fact that \(\Gamma_{\alpha,Y}(\lambda)^{-1}\) is symmetric, and made the change of variable \(y \mapsto y + y_k\), using (2.1). In the third step, we used the properties \(G_{\lambda}(x) = G_{-\lambda}(x)\) and \(\Gamma_{\alpha,Y}(\lambda)^{-1} = \Gamma_{\alpha,Y}(-\lambda)^{-1}\) that follow, respectively, from (2.1) and (2.2). Identity (3.11) then follows immediately from (3.21). \(\square\)
Summarising so far, we produced representation (3.1)–(3.11) of the kernel of the wave operator $W^\pm_{\alpha,Y}$. Because of the obvious $L^p$-boundedness of $T_x$, in order to prove Theorem 1.1 it suffices to study the $L^p$-boundedness or unboundedness of each $\Omega_{jk}$, that is, to consider the quantities

$$
\|\Omega_{jk} u\|^p_{L^p(\mathbb{R}^3)} = \frac{4\pi}{(2\pi)^{3p/2}} \int_0^{+\infty} |(L_{jk} \ast \rho M_u)(\rho)|^p \rho^{2-p} \, d\rho,
$$

(3.22)

whose expression follows from (3.8).

For a more compact notation, it is convenient to introduce the matrix functions

$L(\rho) := (L_{jk}(\rho))_{jk}$, $\Omega(\rho) := (\Omega_{jk}(\rho))_{jk}$,

(3.23)

in terms of which

$$(\Omega u)(x) = \frac{1}{i(2\pi)^{3/2}|x|} \int_0^{+\infty} e^{-i\lambda|x|} F(\lambda)(rM_u)(-\lambda) \, d\lambda$$

(3.24)

and

$$(\Omega u)(x) = \frac{1}{i(2\pi)^{3/2}|x|} (L \ast rM_u)(|x|).$$

(3.25)

The additional formulas (3.24)/(3.25) have the virtue of reducing the problem to the estimate of singular integral operators in one dimensions and will play an important role in our next arguments—although in certain steps we need to go back to the more complicated, but more flexible expression (3.1).

4. $L^p$-bounds for the Single-Centre Case

In this section and in the two following ones, we present the proof of Theorem 1.1. In fact, only the statements concerning the boundedness and the unboundedness of $W^\pm_{\alpha,Y}$ need be proved, because the existence of $W^\pm_{\alpha,Y}$ in $L^2(\mathbb{R}^3)$ and their completeness follow at once from the Birman–Kato–Pearson theorem [27], due to the fact (Theorem 2.1(i), identity (2.3)) that the resolvent difference $R_{\alpha,Y}(z) - R_0(z)$ is a rank-$N$ operator.

We also observe that, by virtue of Lemma 2.2, the complex conjugation $u \mapsto \bar{C}u := \bar{u}$ reverses the direction of time, i.e.

$$C^{-1} e^{-itH_{\alpha,Y}} C = e^{itH_{\alpha,Y}}, \quad C^{-1} e^{-itH_0} C = e^{itH_0},$$

(4.1)

whence

$$W^-_{\alpha,Y} = C^{-1} W^+_{\alpha,Y} C.$$  

(4.2)

Thus, once the $L^p$-boundedness is proved for $W^+_{\alpha,Y}$ and all $p \in (1,3)$, the same result follows for $W^-_{\alpha,Y}$ via (4.2). Analogously, it suffices to prove the $L^p$-unboundedness of $W^+_{\alpha,Y}$, for $p = 1$ and $p \in [3,\infty)$, in order to have same result for $W^-_{\alpha,Y}$.

We start with the proof of the boundedness part of Theorem 1.1 in the special case of $N = 1$ centre. This case is simpler, for the oscillating terms $G^y_{y_j} y_k$ are now absent; nevertheless, it retains most of the essential ideas needed in the proof of the general case, which is the object of the following Sect. 5.
We shall control the two regimes $p \in (1, \frac{3}{2})$ and $p \in (\frac{3}{2}, 3)$ separately. Then the overall $L^p$-boundedness for $p \in (1, 3)$ follows by interpolation.

4.1. $L^p$-boundedness of $W^+_{\alpha,Y}$ for $N = 1$ and $p \in (\frac{3}{2}, 3)$

In this regime, the proof is based on Theorem 2.4 and on the following fact.

**Lemma 4.1.** Suppose that $[0, +\infty) \ni \lambda \mapsto W(\lambda)$ is a smooth and bounded function such that $\lambda \mapsto W'(\lambda)$ and $\lambda \mapsto \lambda W''(\lambda)$ are both integrable. Let $Z(\rho)$, $\rho \in \mathbb{R}$, be the Fourier transform of $W(\lambda)$, in the sense of distributions, defined by

$$Z(\rho) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} d\lambda e^{-i\lambda \rho} W(\lambda).$$

Then, the convolution operator with $Z(\rho)$ is a Calderón–Zygmund operator on $\mathbb{R}$. In particular, the operator $u \mapsto L \ast u$, where $L$ is defined in (3.7) for the case $N = 1$, is of Calderón–Zygmund type.

**Proof.** The operator of convolution with $Z$ is bounded in $L^2(\mathbb{R})$ because $Z$ is the Fourier transform of a bounded function $W$. Integration by parts, using $e^{-i\lambda \rho} = i\rho^{-1}\partial_\lambda e^{-i\lambda \rho}$, yields

$$Z(\rho) = \frac{i}{\rho \sqrt{2\pi}} W(0) - \frac{i}{\rho \sqrt{2\pi}} \int_{0}^{+\infty} d\lambda e^{-i\lambda \rho} W'(\lambda) \lesssim |\cdot|\frac{C}{|\rho|}, \quad \rho \neq 0,$$

and differentiating further in $\rho$ yields

$$Z'(\rho) = -\frac{i W(0)}{\rho^2 \sqrt{2\pi}} + \frac{i}{\rho^2 \sqrt{2\pi}} \int_{0}^{+\infty} d\lambda e^{-i\lambda \rho} W'(\lambda)$$

$$- \frac{1}{\rho \sqrt{2\pi}} \int_{0}^{+\infty} d\lambda e^{-i\lambda \rho} \lambda W'(\lambda).$$

The first two summands in the r.h.s. above are obviously bounded in absolute value by $C|\rho|^{-2}$ for $\rho \neq 0$; so too is the third summand, as follows from integration by parts:

$$\left| \frac{-1}{\rho \sqrt{2\pi}} \int_{0}^{+\infty} d\lambda e^{-i\lambda \rho} \lambda W'(\lambda) \right| = \frac{i}{\rho^2 \sqrt{2\pi}} \int_{0}^{+\infty} d\lambda e^{-i\lambda \rho} (W'(\lambda) + \lambda W''(\lambda))$$

$$\lesssim |\cdot|\frac{C}{\rho^2}, \quad \rho \neq 0.$$

Thus, we conclude from Theorem 2.4(i) that $u \mapsto Z \ast u$ is a Calderón–Zygmund operator on $\mathbb{R}$. Concerning the second statement of the thesis, we see that in the case $N = 1$ (3.4) reads

$$F(\lambda) = \lambda \left( \alpha + \frac{i\lambda}{4\pi} \right)^{-1}.$$  \hspace{1cm} (4.3)

$F$ is therefore bounded and smooth on $[0, +\infty)$ and both $F'(\lambda)$ and $\lambda F''(\lambda)$ are integrable, whence the conclusion for the operator of convolution by $L$ defined in (3.7). \hfill \Box
The proof of the $L^p$-boundedness of $W^{+}_{\alpha,Y}$ for $N = 1$ and $p \in (\frac{3}{2}, 3)$ then becomes particularly simple. First, we recall from (3.22) that
\[
\|\Omega u\|_{L^p(\mathbb{R}^3)}^p = \frac{4\pi}{(2\pi)^{3p/2}} \int_0^{+\infty} |(L * \rho M_u)(\rho)|^p \rho^{2-p} d\rho,
\]
where $\rho^{2-p}$ is an $A_p$-weight for $p \in (\frac{3}{2}, 3)$ (Theorem 2.4(iv)) and the convolution with $L$ is a Calderón–Zygmund operator on $\mathbb{R}$ (Lemma 4.1). Then it follows from Theorem 2.4(ii) that
\[
\|\Omega u\|_{L^p(\mathbb{R}^3)}^p \leq \int_0^{+\infty} |(\rho M_u)(\rho)|^p \rho^{2-p} d\rho = \int_0^{+\infty} |M_u(\rho)|^p \rho^2 d\rho
\]
(4.4)
for some constant $C_p > 0$, whence the conclusion.

4.2. $L^p$-boundedness of $W^{+}_{\alpha,Y}$ for $N = 1$ and $p \in (1, \frac{3}{2})$

In the regime $p \in (1, \frac{3}{2})$, the general harmonic analysis treatment provided by Theorem 2.4 only allows us to find an $L^p$-bound to part of the function (see (3.6) above)
\[
(\Omega u)(x) = \frac{1}{i(2\pi)^{\frac{3}{2}}|x|} \int_0^{+\infty} e^{-i\lambda|x|} F(\lambda)(\rho M_u)(-\lambda) d\lambda,
\]
whereas for the remaining part we need to produce further analysis.

Integrating by parts the above expression of $\Omega u$, using $e^{-i\lambda \rho} = i \rho^{-1} \partial_\lambda e^{-i\lambda \rho}$, yields
\[
\Omega u = \Omega_1 u + \Omega_2 u,
\]
(4.5)
where
\[
(\Omega_1 u)(x) := \frac{-i}{(2\pi)^{\frac{3}{2}}|x|^2} \int_0^{+\infty} e^{-i\lambda|x|} F(\lambda)(\rho^2 M_u)(-\lambda) d\lambda,
\]
(4.6)
\[
(\Omega_2 u)(x) := \frac{-1}{(2\pi)^{\frac{3}{2}}|x|^2} \int_0^{+\infty} e^{-i\lambda|x|} F'(\lambda)(\rho M_u)(-\lambda) d\lambda.
\]

Now, concerning $\Omega_1 u$, we rewrite
\[
(\Omega_1 u)(x) = \frac{-i}{(2\pi)^{\frac{3}{2}}|x|^2} (L * \rho^2 M_u)(|x|)
\]
(4.7)
with $L$ given by (3.7). Owing to Lemma 4.1, $u \mapsto L * u$ is a Calderón–Zygmund operator, and owing to Theorem 2.4(iv), $|x|^{2-2p}$ is an $A_p$-weight on $\mathbb{R}$ for $p \in (1, \frac{3}{2})$. Therefore,
\[
\|\Omega_1 u\|_{L^p(\mathbb{R}^3)}^p = \frac{4\pi}{(2\pi)^{3p/2}} \int_0^{+\infty} |(L * \rho^2 M_u)(\rho)|^p \rho^{2-2p} d\rho
\]
\[
\leq \int_0^{+\infty} |\rho^2 M_u(\rho)|^p \rho^{2-2p} d\rho = \int_0^{+\infty} |M_u(\rho)|^p \rho^2 d\rho
\]
\leq C_p \|u\|_{L^p(\mathbb{R}^3)}^p
\]
(4.8)
for some constant $C_\rho > 0$, where in the second step we applied Theorem 2.4(ii). This proves the $L^p$-boundedness of $\Omega_1$.

Concerning $\Omega_2 u$, instead, we rewrite

$$
(\Omega_2 u)(x) = -\frac{1}{(2\pi)^{\frac{d}{2}} \rho^2} (\mathcal{L} * r M_u)(\rho),
$$

where $\mathcal{L}$ is the Fourier transform of the function $1_{(0, \infty)} F'(\lambda)$, and

$$
F'(\lambda) = \alpha \left( \alpha + \frac{i \lambda}{4\pi} \right)^{-2}.
$$

Thus, in the non-trivial case $\alpha \neq 0$, $F'$ is smooth and bounded, and correspondingly both $F''$ and $\lambda F'''$ are integrable. This implies, through Lemma 4.1, that $u \mapsto \mathcal{L} \ast u$ is a Calderón–Zygmund operator on $\mathbb{R}$. Since $|x|^{2-2p}$ is an $A_p$-weight on $\mathbb{R}$ for $p \in (1, \frac{3}{2})$ (Theorem 2.4(iv)), then Theorem 2.4(ii) yields

$$
\|\Omega_2 u\|_{L^p(\mathbb{R}^3)}^p = \frac{4\pi}{(2\pi)^{3p/2}} \int_0^{+\infty} |(\mathcal{L} \ast \rho \cdot M_u)(\rho)| \rho^{2-2p} \, d\rho
\leq \int_0^{+\infty} |\rho \cdot M_u(\rho)|^p \rho^{2-2p} \, d\rho \leq C \int_{\mathbb{R}^3} \frac{|u(x)|^p}{|x|^3} \, dx
$$

for some constant $C > 0$. This shows that

$$
\|\Omega_2 1_{\{|x| \geq 1\}} u\|_{L^p(\mathbb{R}^3)}^p \leq C \|1_{\{|x| \geq 1\}} u\|_{L^p(\mathbb{R}^3)}^p.
$$

For $L^p$-functions supported on $|x| \leq 1$, a further argument is needed. In other words, so far from (4.8) and (4.12) we have

$$
\|\Omega u\|_{L^p(\mathbb{R}^3)}^p \leq 2 \|\Omega_1 u\|_{L^p(\mathbb{R}^3)}^p + 2 \|\Omega_2 1_{\{|x| \geq 1\}} u\|_{L^p(\mathbb{R}^3)}^p
\leq C_\rho \|u\|_{L^p(\mathbb{R}^3)}^p + C \|1_{\{|x| \geq 1\}} u\|_{L^p(\mathbb{R}^3)}^p
\leq C_\rho \|1_{\{|x| \leq 1\}} u\|_{L^p(\mathbb{R}^3)}^p,
$$

and we are left with producing the estimate

$$
\|\Omega_2 1_{\{|x| \leq 1\}} u\|_{L^p(\mathbb{R}^3)} \leq C_\rho \|1_{\{|x| \leq 1\}} u\|_{L^p(\mathbb{R}^3)}.
$$

To this aim, let us establish first the following result.

**Lemma 4.2.** Suppose that $[0, +\infty) \ni y \mapsto Y(y)$ is a bounded $C^1$-function such that $\lambda \mapsto \lambda^\theta Y(\lambda)$ and $\lambda \mapsto (1 + \lambda)^\theta Y'(\lambda)$ are both integrable for all $\theta \in (0, 1)$, and let

$$
T(x, y) := \frac{1}{|x|^2} \int_0^{+\infty} \left( e^{-i\lambda(|x|+|y|)} - e^{-i\lambda(|x|-|y|)} \right) Y(\lambda) \, d\lambda
$$

for $x, y \in \mathbb{R}^3$. Then, for any $R > 0$ and $p \in (1, \frac{3}{2})$, the integral operator $T$ on $\mathbb{R}^3$ with the integral kernel $T(x, y)$ is $L^p(\Lambda_R) \to L^p(\mathbb{R}^3)$ bounded, with $\Lambda_R := \{ x \in \mathbb{R}^3 \mid |x| \leq R \}$. 


Proof. We only consider the case $R = 1$, the proof for generic $R$ is similar. Let us deal with the region $|x| \leq 10$ first. Since 
\[ |e^{-i\lambda(|x|-|y|)} - e^{-i\lambda(|x|+|y|)}| \leq 2(\lambda|y|)^{1-\theta} \] for any $\theta \in (0, 1)$, and since $\lambda^{1-\theta}Y \in L^1(0, +\infty)$, then
\[ |T(x, y)| \leq \frac{1}{2\pi|x|^2|y|^0} \int_0^{+\infty} |Y(\lambda)| \lambda^{1-\theta} d\lambda \leq \frac{C_0}{|x|^2|y|^0}, \quad |x| \leq 10, \]
for some constant $C_0 > 0$. For fixed $p$ in $(1, \frac{3}{2})$, we take $\theta \in (0, 1)$ such that $p'\theta < 3$, where $p' = \frac{p}{p-1}$ as usual. With this choice, $|y|^{-\theta} \in L^{p'}(\Lambda_1)$ and $|x|^{-2} \in L^p(\Lambda_{10})$, with $\Lambda_R = \{ x \in \mathbb{R}^3 ||x| \leq R \}$ as in the statement of the Lemma. For each $f \in L^p(\Lambda_1)$, H"older’s inequality and the above bound for $|T(x, y)|$ then imply
\[ \|Tf\|_{L^p(\Lambda_{10})} \leq C_0 \|x|^{-2}\|_{L^{p'}(\Lambda_{10})} \cdot \|f\|_{L^p(\Lambda_1)} \cdot \|y|^{-\theta}\|_{L^{p'}(\Lambda_1)} = \kappa_p^{-} \|f\|_{L^p(\Lambda_1)} \]
for some constant $\kappa_p^{-} > 0$. Next, let us consider the region $|x| \geq 10$. Integration by parts gives
\[ T(x, y) = \frac{1}{4\pi|x|^2|y|^0} \int_0^{+\infty} \partial\lambda \left( \frac{e^{-i\lambda(|x|-|y|)}}{-i(|x| - |y|)} - \frac{e^{-i\lambda(|x|+|y|)}}{-i(|x| + |y|)} \right) Y(\lambda) d\lambda \]
\[ = \frac{1}{4\pi i|x|^2|y|^0} \int_0^{+\infty} \left( \frac{e^{-i\lambda(|x|-|y|)}}{|x| - |y|} - \frac{e^{-i\lambda(|x|+|y|)}}{|x| + |y|} \right) Y'(\lambda) d\lambda. \tag{I} \]
\[ + \frac{1}{4\pi i|x|^2|y|^0} \int_0^{+\infty} \left( \frac{e^{-i\lambda(|x|-|y|)}}{|x| - |y|} - \frac{e^{-i\lambda(|x|+|y|)}}{|x| + |y|} \right) Y''(\lambda) d\lambda. \tag{II} \]
Since $|x| \pm |y| \geq \frac{9}{10}|x| \geq 9$ whenever $|x| \geq 10$ and $|y| \leq 1$, and since $Y$ is bounded, then clearly
\[ |(I)| \leq \frac{C}{|x|^4} \leq \frac{C}{|x|^3|y|^0} \]
for some constant $C > 0$ and any $\theta \in (0, 1)$. As for the summand (II), since
\[ \frac{e^{-i\lambda(|x|-|y|)}}{|x| - |y|} - \frac{e^{-i\lambda(|x|+|y|)}}{|x| + |y|} \leq |\cdot| \frac{2|y|}{|x|^2 - |y|^2} + \frac{2\lambda|y|^{1-\theta}}{|x| + |y|} \]
\[ \leq C \left( \frac{|y|}{|x|^2} + \frac{(\lambda y)^{1-\theta}}{|x|} \right) \]
for some constant $C > 0$ and any $\theta \in (0, 1)$, and since $(1 + \lambda)^{1-\theta} Y'' \in L^1(0, +\infty)$, then
\[ |(II)| \leq \frac{C}{|x|^2|y|^0} \left( \frac{|y|}{|x|^2} + \frac{|y|^{1-\theta}}{|x|} \right) = C \left( \frac{1}{|x|^4} + \frac{1}{|x|^3|y|^0} \right) \leq \frac{2C}{|x|^3|y|^0} \]
and hence also
\[ |T(x, y)| \leq \frac{C}{|x|^3|y|^0}, \quad |x| \geq 10, \]
for some constant $C > 0$ and any $\theta \in (0, 1)$. For fixed $p \in (1, \frac{3}{2})$, we take $\theta \in (0, 1)$ such that $p'\theta < 3$ and $f \in L^p(\Lambda_1)$: with this choice, H"older’s inequality yields
\[ \| Tf \|_{L^p(\mathbb{R}^3 \setminus \Lambda_{10})} \leq C_\theta \| |x|^{-3} \|_{L^p(\mathbb{R}^3 \setminus \Lambda_{10})} \cdot \| f \|_{L^p(\Lambda_1)} \cdot \| |y|^{-\theta} \|_{L^{p'}(\Lambda_1)} \]

for some constant \( \kappa_p^+ > 0 \). Combining the above bounds yields the boundedness of \( T \) as a map from \( L^p(\Lambda_1) \) to \( L^p(\mathbb{R}^3) \). \( \square \)

Let us now complete the proof of the \( L^p \)-boundedness of \( W_{\alpha,Y}^+ \) for \( N = 1 \) and \( p \in (1, \frac{3}{2}) \). We only need to show (4.14). Upon rewriting the second equation in (4.6) by means of (3.9), that is,

\[ (\Omega_2 u)(x) = \frac{-1}{(2\pi)^2 |x|^2} \int_0^{+\infty} e^{-i\lambda|x|} F'(\lambda) \left( \int_{\mathbb{R}^3} \frac{e^{i|y|} - e^{-i\lambda|y|}}{4\pi |y|} u(y) \, dy \right) d\lambda, \]

it is immediate to recognise that

\[ \Omega_2 u = -(2\pi)^{-2} Tu, \]

where \( T \) is the integral operator given by (4.15) with \( Y \equiv F' \), and \( F' \) does satisfy the assumptions of Lemma 4.2. From this, we conclude (4.14) at once.

5. \( L^p \)-bounds for the General Multi-Centre case

The additional complication in the case \( N \geq 2 \) is due to the presence, in the function \( F \) defined in (3.4) and (3.23), of the terms \( G_{\lambda}^{y_jy_k} \) (definitions (2.1)–(2.2)), which are oscillatory in \( \lambda \).

Let us start the discussion by rewriting

\[ F(\lambda) = \lambda \Gamma_{\alpha,Y}(-\lambda)^{-1} = \lambda \left( A + \frac{i\lambda}{4\pi} - \tilde{G}(-\lambda) \right)^{-1}, \quad \lambda > 0, \quad (5.1) \]

with

\[ A := \text{diag}(\alpha_1, \ldots, \alpha_N), \quad (5.2) \]

\[ \tilde{G}(\lambda) := (G_{\lambda}^{y_jy_k})_{j,k=1,\ldots,N}. \quad (5.3) \]

We decompose \( F(\lambda) \) into a small-\( \lambda \) and a large-\( \lambda \) part by means of two cut-off functions \( \omega_< \) and \( \omega_> \) such that

\[ \omega_\in C^\infty_0(\mathbb{R}), \quad \omega_>(\lambda) := 1 - \omega_<(\lambda), \]

\[ \omega_<(\lambda) = \begin{cases} 1 & \text{if } |\lambda| \leq \gamma \\ 0 & \text{if } |\lambda| \geq 2\gamma, \end{cases} \quad (5.4) \]

where \( \gamma > 0 \) is a sufficiently large number so that,

\[ \| A - \tilde{G}(-\lambda) \| < |\lambda|(16\pi)^{-1}, \quad |\lambda| \geq \gamma \quad (5.5) \]

(\( \| E \| \) being the operator norm of the matrix \( E \) as an operator on \( \mathbb{C}^N \)), and the r.h.s. of (5.1) is invertible. Explicitly,

\[ F = F^< + F^>, \quad F^< := \omega_< F, \quad F^> := \omega_> F. \quad (5.6) \]
From (5.1) and (5.5), we expand
\[
F^>(\lambda) = -4\pi i \omega_>(\lambda) \left\{ 1 - \frac{4\pi i}{\lambda} (A - \tilde{G}(-\lambda)) + \left(\frac{4\pi i}{\lambda} (A - \tilde{G}(-\lambda))\right)^2 \right\} 
- 4\pi i \omega_>(\lambda) \left( \frac{4\pi i}{\lambda} (A - \tilde{G}(-\lambda)) \right)^3 \left( 1 - \frac{4\pi i}{\lambda} (A - \tilde{G}(-\lambda)) \right)^{-1}.
\]
(5.7)

We collect all terms that do not contain \(\tilde{G}(-\lambda)\) or for which the oscillation of \(\tilde{G}(-\lambda)\) is harmless into the quantity
\[
F(0)(\lambda) := F^>(\lambda) - 4\pi i \omega_>(\lambda) \left\{ 1 - \frac{4\pi i}{\lambda} A - \frac{16\pi^2}{\lambda^2} A^2 \right\}
- 4\pi i \omega_>(\lambda) \left( \frac{4\pi i}{\lambda} (A - \tilde{G}(-\lambda)) \right)^3 \left( 1 - \frac{4\pi i}{\lambda} (A - \tilde{G}(-\lambda)) \right)^{-1},
\]
(5.8)

whereas
\[
F(1)(\lambda) := 4\pi i \omega_>(\lambda) \times \left\{ - \frac{4\pi i}{\lambda} \tilde{G}(-\lambda) - \frac{16\pi^2}{\lambda^2} (A \tilde{G}(-\lambda) + \tilde{G}(-\lambda) A) + \frac{16\pi^2}{\lambda^2} \tilde{G}(-\lambda)^2 \right\}
\]
(5.9)
contains the oscillations explicitly.

Thus,
\[
F = F(0) + F(1) \quad \text{and} \quad \Omega = \Omega(0) + \Omega(1),
\]
(5.10)
where, by means of (3.24),
\[
(\Omega^{(\ell)} u)(x) := \frac{1}{i(2\pi)^{\frac{3}{2}} |x|} \int_{0}^{+\infty} e^{-i\lambda |x|} F^{(\ell)}(\lambda) (r M u)(-\lambda) d\lambda,
\]
(5.11)

5.1. \(L^p\)-boundedness of \(\Omega^{(0)}\)

The \(L^p\)-boundedness of the map \(u \mapsto \Omega^{(0)} u\) can be established via a straightforward adaptation of the arguments of Sect. 4, of course understanding that this is done for each component \(\Omega^{(0)}_{jk}\), and this is possible precisely thanks to the lack of relevant oscillations in \(\Omega^{(0)}\).

This means that first we write, in analogy to (3.25),
\[
(\Omega^{(0)} u)(x) = \frac{1}{i(2\pi)^{\frac{3}{2}} |x|} (F^{(0)} \ast r M u)(|x|),
\]
(5.12)
and it is easy to check that \(F^{(0)}\) satisfies the properties of the function \(W\) in Lemma 4.1, from which, reasoning as in (4.4),
\[
\|\Omega^{(0)} u\|_{L^p(\mathbb{R}^3)} \leq C_p \|u\|_{L^p(\mathbb{R}^3)}, \quad p \in \left( \frac{3}{2}, 3 \right),
\]
(5.13)
for some constant \(C_p > 0\).

Then, in analogy to (4.5), (4.6), (4.7), (4.9), and (4.16), we split
\[
\Omega^{(0)} u = \Omega^{(0)}_1 u + \Omega^{(0)}_2 u
\]
(5.14)
with
\[(\Omega_1^{(0)}u)(x) := -\frac{i}{(2\pi)^{\frac{3}{2}}|x|^2} \int_0^{+\infty} e^{-i\lambda|x|} F^{(0)}(\lambda) (\mathcal{R}_u M_u)(-\lambda) \, d\lambda, \tag{5.15}\]
and
\[(\Omega_2^{(0)}u)(x) := -\frac{1}{(2\pi)^{\frac{3}{2}}|x|^2} \int_0^{+\infty} e^{-i\lambda|x|} F^{(0)'}(\lambda) (\mathcal{L}_u M_u)(\rho) \, d\lambda \]
\[\quad = -\frac{1}{(2\pi)^{\frac{3}{2}}\rho^2} (\mathcal{L}^{(0)} \ast rM_u)(\rho) \]
\[\quad = -\frac{1}{(2\pi)^{\frac{3}{2}}|x|^2} \int_0^{+\infty} e^{-i\lambda|x|} F^{(0)'}(\lambda) \left( \frac{e^{i\lambda|y|} - e^{-i\lambda|y|}}{4\pi|y|} \right) u(y) \, dy \, d\lambda, \tag{5.16}\]
where \(\mathcal{L}^{(0)}\) is the Fourier transform of the function \(1_{(0,\infty)} F^{(0)'}\).

Since, as observed already, \(F^{(0)}\) behaves like \(W\) in Lemma 4.1, we have, reasoning as in (4.8),
\[\|\Omega_1^{(0)} u\|_{L_p^2} \leq C_p \|u\|_{L_p^2}, \quad p \in (1, \frac{3}{2}), \tag{5.17}\]
and since \(1_{(0,\infty)} F^{(0)'}\) too satisfies the properties of the function \(W\) in Lemma 4.1, we have, using the second line in the r.h.s. of (5.16) and reasoning as in (4.11)–(4.12),
\[\|\Omega_2^{(0)} \chi_{\{|x| \geq 1\}} u\|_{L_p^2} \leq C_p \|\chi_{\{|x| \geq 1\}} u\|_{L_p^2}, \quad p \in (1, \frac{3}{2}), \tag{5.18}\]
for some constant \(C_p > 0\). Last, since \(1_{(0,\infty)} F^{(0)'}\) satisfies the properties of the function \(Y\) in Lemma 4.2, we have, using the third line in the r.h.s. of (5.16) and reasoning as in (4.16)–(4.17) and (4.14),
\[\|\Omega_2^{(0)} \chi_{\{|x| \leq 1\}} u\|_{L_p^2} \leq C_p \|\chi_{\{|x| \leq 1\}} u\|_{L_p^2}, \quad p \in (1, \frac{3}{2}). \tag{5.19}\]

Combining together bounds (5.13), (5.17), (5.18), and (5.19), plus interpolation so as to cover also the case \(p = \frac{3}{2}\), yields finally
\[\|\Omega_1^{(0)} u\|_{L_p^2} \leq C_p \|u\|_{L_p^2}, \quad p \in (1, 3), \tag{5.20}\]
for some constant \(C_p > 0\).

### 5.2. \(L^p\)-boundedness of \(\Omega^{(1)}\)

The proof of the \(L^p\)-boundedness of the map \(u \mapsto \Omega^{(1)} u\) is somewhat more involved; however, the basic idea of the proof is similar to that for \(\Omega_1^{(0)}\). First we rewrite (5.11) in analogy to (3.24) and (5.12) as
\[(\Omega^{(1)} u)(x) = \frac{1}{i(2\pi)^{\frac{3}{2}}|x|} \widehat{(F^{(1)} \ast rM_u)(|x|)}. \tag{5.21}\]
Owing to (5.9), the matrix elements of $F^{(1)}(\lambda)$ entering (5.11) and (5.21) above are of the form

$$\frac{\omega_>(\lambda)}{\lambda} |y_j - y_k|, \quad \frac{\omega_>(\lambda)}{\lambda^2} |y_j - y_k|^2, \quad \frac{\omega_>(\lambda)}{\lambda^2} |y_j - y_k| |y_r - y_s|$$

(observe that the $\lambda$-dependence of the matrix elements of $\hat{G}$ in (5.9) is $\hat{G}(-\lambda)$).

This means that denoting by $a > 0$ any of the numbers $|y_j - y_k|$ or $|y_j - y_k| + |y_r - y_s|$ and by $X(\lambda)$ the function $\lambda^{-1}\omega_>(\lambda)$ or $\lambda^{-2}\omega_>(\lambda)$, formulas (5.11) and (5.21) imply that $\Omega^{(1)}u$ is a linear combination of terms of the form

$$(\Xi u)(x) := \frac{1}{i|x|} \int_0^{+\infty} e^{-i\lambda(|x|+a)} X(\lambda) (r M u)(-\lambda) \, d\lambda$$

(5.22)

and we need to prove the $L^p$-boundedness of the map $u \mapsto \Xi u$. In fact, we shall establish it for each of the two terms of the bound

$$\|\Xi u\|_{L^p(\mathbb{R}^3)} \leq \|1_{\{|x| \geq R\}} \Xi u\|_{L^p(\mathbb{R}^3)} + \|1_{\{|x| \leq R\}} \Xi u\|_{L^p(\mathbb{R}^3)}$$

(5.23)

for a suitable $R > 0$.

Let us cast the discussion of such two terms into the following two Lemmas. The combination of (5.23) above with (5.24) and (5.25) below will then complete the proof of the $L^p$-boundedness of $\Omega^{(1)}$.

**Lemma 5.1.** For any $p \in (1, 3)$ and $R > a$ there exists a constant $C_p > 0$ such that

$$\|1_{\{|x| \geq R\}} \Xi u\|_{L^p(\mathbb{R}^3)} \leq C_p \|u\|_{L^p(\mathbb{R}^3)}$$

(5.24)

for all $u \in L^p(\mathbb{R}^3)$, where $\Xi u$ is defined in (5.22).

**Proof.** We consider first the case $p \in (\frac{3}{2}, 3)$. From (5.22) and from the fact that $\rho \geq R + a$ implies $\frac{1}{2}\rho \leq \rho - a \leq \rho$,

$$\|1_{\{|x| \geq R\}} \Xi u\|_{L^p(\mathbb{R}^3)}^p = 4\pi \int_R^{+\infty} \rho^{2-p} |(\tilde{X} * r M u)(\rho + a)|^p \, d\rho$$

$$= 4\pi \int_{R+a}^{+\infty} (\rho - a)^{2-p} |(\tilde{X} * r M u)(\rho)|^p \, d\rho$$

$$\leq C_p \int_0^{+\infty} |(\tilde{X} * r M u)(\rho)|^p \rho^{2-p} \, d\rho.$$

Now, $\rho^{2-p}$ is an $A_p$-weight on $\mathbb{R}$ because $p \in (\frac{3}{2}, 3)$ (Theorem 2.4(iv)) and the convolution with $\tilde{X}$ is a Calderón–Zygmund operator on $\mathbb{R}$ because the function $X$ obviously satisfies the properties of the function $W$ in Lemma 4.1. Then it follows from Theorem 2.4(ii) that

$$\|1_{\{|x| \geq R\}} \Xi u\|_{L^p(\mathbb{R}^3)}^p \leq C_p \int_0^{+\infty} |(\rho M u)(\rho)|^p \rho^{2-p} \, d\rho \leq C_p' \|u\|_{L^p(\mathbb{R}^3)}^p$$

for suitable $C_p' > 0$. Lemma is then proved in the case $p \in (\frac{3}{2}, 3)$.  

Next we consider the case \( p \in (1, \frac{3}{2}) \). Integration by parts in (5.22), using 
\[ e^{-i\lambda(\rho + a)} = i(\rho + a)^{-1}\partial_\lambda e^{-i\lambda(\rho + a)}, \]
yields
\[ \Xi u = \Xi_1 u + \Xi_2 u \]
with
\[ (\Xi_1 u)(x) := \frac{-i}{|x|(|x| + a)} \int_0^{+\infty} e^{-i\lambda(|x| + a)} X(\lambda)(r^2 M_u)(-\lambda) \, d\lambda \]
\[ = \frac{-i}{|x|(|x| + a)} (\hat{X} \ast r^2 M_u)(|x| + a) \]
and
\[ (\Xi_2 u)(x) := \frac{-1}{|x|(|x| + a)} \int_0^{+\infty} e^{-i\lambda(|x| + a)} X'(\lambda)(r M_u)(-\lambda) \, d\lambda \]
\[ = \frac{-1}{|x|(|x| + a)} (\hat{X}' \ast r M_u)(|x| + a). \]
Up to a change of variable, the quantity \( \|1_{\{|x| \geq R\}} \Xi_1 u\|_{L^p(\mathbb{R}^3)} \) is estimated precisely as the quantity \( \|\Omega_1 u\|_{L^p(\mathbb{R}^3)} \) in Sect. 4.2—see (4.8) above. Indeed,
\[ \|1_{\{|x| \geq R\}} \Xi_1 u\|_{L^p(\mathbb{R}^3)}^p = \int_{R}^{+\infty} \frac{4\pi \rho^2}{\rho^p(\rho + a)^p} |(\hat{X} \ast r^2 M_u)(\rho + a)|^p \, d\rho \]
\[ = \int_{R+a}^{+\infty} 4\pi (\rho - a)^{2-p} \rho^{-p} |(\hat{X} \ast r^2 M_u)(\rho)|^p \, d\rho \]
\[ \leq C \int_{0}^{+\infty} |(\hat{X} \ast r^2 M_u)(\rho)|^p \rho^{2-p} \, d\rho \]
\[ \leq C \int_{0}^{+\infty} |M_u(\rho)|^p \rho^2 \, d\rho \leq C_p \|u\|_{L^p(\mathbb{R}^3)}^p \]
for some constants \( C, C_p > 0 \), having used \( \frac{1}{2} \rho \leq \rho - a \leq \rho \) in the third step and Theorem 2.4(ii) in the fourth step. This was possible because \( \rho^{2-2p} \) is an \( A_p \)-weight on \( \mathbb{R} \) for \( p \in (1, \frac{3}{2}) \) (Theorem 2.4(iv)) and because \( f \mapsto \hat{X} \ast f \) is a Calderón–Zygmund operator on \( \mathbb{R} \) (the function \( X \) does satisfy the assumptions on the function \( W \) in Lemma 4.1).

It remains to estimate the quantity \( \|1_{\{|x| \geq R\}} \Xi_2 u\|_{L^p(\mathbb{R}^3)} \) in the regime \( p \in (1, \frac{3}{2}) \), and we proceed by splitting
\[ \|1_{\{|x| \geq R\}} \Xi_2 u\|_{L^p(\mathbb{R}^3)}^p = \|1_{\{|x| \geq R\}} \Xi_2 1_{\{|x| \geq R\}} u\|_{L^p(\mathbb{R}^3)}^p \]
\[ + \|1_{\{|x| \geq R\}} \Xi_2 1_{\{|x| \leq R\}} u\|_{L^p(\mathbb{R}^3)}^p. \]
For estimating \( \|1_{\{|x| \geq R\}} \Xi_2 1_{\{|x| \geq R\}} u\|_{L^p(\mathbb{R}^3)} \), we observe that
\[ \|1_{\{|x| \geq R\}} \Xi_2 u\|_{L^p(\mathbb{R}^3)}^p = \int_{R}^{+\infty} \frac{4\pi \rho^2}{\rho^p(\rho + a)^p} |(\hat{X}' \ast r M_u)(\rho + a)|^p \, d\rho \]
\[ = \int_{R+a}^{+\infty} 4\pi (\rho - a)^{2-p} \rho^{-p} |(\hat{X}' \ast r M_u)(\rho)|^p \, d\rho \]
\[ \leq C \int_{0}^{+\infty} |(\hat{X}' \ast r M_u)(\rho)|^p \rho^{2-p} \, d\rho \]
\[ \leq C \int_{0}^{+\infty} |M_u(\rho)|^p \rho^{2-2p} \, d\rho \]
\[ \leq C \int_{0}^{+\infty} |M_u(\rho)|^p \rho^2 \, d\rho \leq C_p \|u\|_{L^p(\mathbb{R}^3)}^p. \]
\[ \leq C \int_{0}^{+\infty} |(\hat{X}' \ast rM_u)(\rho)|^p \rho^{2-2p} \, d\rho \]

for some constant \( C > 0 \), where we used again \( \frac{1}{2} \rho < \rho - a \leq \rho \). Then we can proceed exactly as in (4.11)–(4.12), because \( \rho^{2-2p} \) is an \( A_p \)-weight on \( \mathbb{R} \) for \( p \in (1, \frac{3}{2}) \) and \( f \mapsto \hat{X}' \ast f \) is a Calderón–Zygmund operator on \( \mathbb{R} \); the conclusion is the same as in (4.12), that is,

\[ \|1_{\{|x| \geq R\}} \Xi_2 1_{\{|x| \geq R\}} u\|_{L^p(\mathbb{R}^3)}^p \leq C_p \|1_{\{|x| \geq R\}} u\|_{L^p(\mathbb{R}^3)}^p \]

for some constant \( C_p > 0 \). We also observe from (*) that

\[ \|1_{\{|x| \geq R\}} \Xi_2 u\|_{L^p(\mathbb{R}^3)}^p \leq \int_{\mathbb{R}^3} dx \left| \frac{1}{|x|^2} \int_{0}^{+\infty} e^{-i\lambda|x|} X'(\lambda) \widehat{(rM_u)(-\lambda)} \, d\lambda \right|^p = \|\widehat{\Xi_2 u}\|_{L^p(\mathbb{R}^3)}^p, \]

where \( \widehat{\Xi_2 u} \) has precisely the same structure as \( \Omega_2 u \) in (4.6) with the function \( X' \) here in place of the function \( F' \) therein. Therefore, as argued in (4.16)–(4.17), since \( X' \) satisfies the assumptions on the function \( Y \) in Lemma 4.2, the conclusion is the same as in (4.14), that is,

\[ \|1_{\{|x| \geq R\}} \Xi_2 1_{\{|x| \leq R\}} u\|_{L^p(\mathbb{R}^3)}^p \leq C_p \|1_{\{|x| \leq R\}} u\|_{L^p(\mathbb{R}^3)}^p \]

for some constant \( C_p > 0 \). Therefore,

\[ \|1_{\{|x| \geq R\}} \Xi_2 u\|_{L^p(\mathbb{R}^3)}^p \leq C_p \|u\|_{L^p(\mathbb{R}^3)}^p \]

and Lemma is then proved in the case \( p \in (1, \frac{3}{2}) \).

Last, by interpolation the Lemma is also proved in the case \( p = \frac{3}{2} \). \( \square \)

**Lemma 5.2.** For any \( p \in (1, 3) \) and \( R > 100a \), there exists a constant \( C_p > 0 \) such that

\[ \|1_{\{|x| \leq R\}} \Xi u\|_{L^p(\mathbb{R}^3)} \leq C_p \|u\|_{L^p(\mathbb{R}^3)} \quad (5.25) \]

for all \( u \in L^p(\mathbb{R}^3) \), where \( \Xi u \) is defined in (5.22).

**Proof.** By means of (3.9), we see that the map \( u \mapsto \Xi u \) defined in (5.22) is an integral operator with kernel \( \frac{1}{4\pi} K_\Xi(x, y) \) given by

\[ K_\Xi(x, y) := \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-i\lambda|x| + a} (e^{-i\lambda|y|} - e^{i\lambda|y|}) \lambda X(\lambda) \, d\lambda. \quad (5.26) \]

Since \( X(\lambda) = \lambda^{-1} \omega_\geq(\lambda) \) or \( \lambda^{-2} \omega_\geq(\lambda) \), obviously \( \rho \mapsto \hat{X}(\rho) \) is smooth for \( \rho \neq 0 \) and with rapid decrease as \( \rho \to +\infty \). Moreover, since \( X \in L^q(\mathbb{R}) \) for any \( q > 1 \), \( \hat{X} \in L^p(\mathbb{R}) \) for any \( p \in [2, \infty) \), owing to the Hausdorff–Young inequality. Thus, \( \hat{X} \in L^p(\mathbb{R}) \) for any \( p \in [1, \infty) \).

We shall prove Lemma by splitting

\[ \|1_{\{|x| \leq R\}} \Xi u\|_{L^p(\mathbb{R}^3)}^p = \|1_{\{|x| \leq R\}} \Xi 1_{\{|x| \geq 10R\}} u\|_{L^p(\mathbb{R}^3)}^p + \|1_{\{|x| \leq R\}} \Xi 1_{\{|x| \leq 10R\}} u\|_{L^p(\mathbb{R}^3)}^p \quad (5.27) \]

and estimating separately the two summands in the r.h.s. above.
When \( R > 100a \), \(|x| \leq R\), and \(|y| \geq 10R\), one has \( |\hat{X}(|x| + |y| + a)| \leq C_n(y)^{-n} \) for any \( n \in \mathbb{N} \) and suitable constants \( C_n > 0 \), which follows from the rapid decrease of \( \hat{X} \). Then, the identity

\[
K_{\Xi}(x, y) = \frac{\hat{X}(|x| + a + |y|) - \hat{X}(|x| + a - |y|)}{|x||y|}
\]

shows that in this regime \( |K_{\Xi}(x, y)| \leq 2C_n|x|^{-1}|y|^{-1} \langle y \rangle^{-n} \). Therefore, for any \( p \in (1, 3) \) and corresponding \( n \) large enough,

\[
\|K_{\Xi}(x, \cdot)1_{\{|x| \geq 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \leq \left( \int_{|y| \geq 10R} \frac{1}{|y|^{p'} \langle y \rangle} \right)^{1/p'} \leq \frac{C_p}{|x|}
\]

for some constant \( C_p > 0 \). The latter bound and Hölder’s inequality then yield, for any \( p \in (1, 3) \),

\[
\begin{align*}
\|1_{\{|x| \leq R\}} & \equiv 1_{\{|x| \geq 10R\}}u \|_{L^p(\mathbb{R}^3)}^p \\
& \leq \int_{\mathbb{R}^3} dx 1_{\{|x| \leq R\}}(x) \int_{\mathbb{R}^3} dy K_{\Xi}(x, y)1_{\{|y| \geq 10R\}}(y) u(y) \\
& \leq C_p \left\| 1_{\{|x| \leq R\}} \right\|_L^p \left\| 1_{\{|y| \geq 10R\}}u \right\|_{L^p(\mathbb{R}^3)}^p \\
& = C'_p \left\| 1_{\{|y| \geq 10R\}}u \right\|_{L^p(\mathbb{R}^3)}^p
\end{align*}
\]

for some constant \( C'_p > 0 \).

This provides the first partial estimate for the proof of (5.25): the proof is completed when we show in addition that

\[
\|1_{\{|x| \leq R\}} \equiv 1_{\{|y| \leq 10R\}}u \|_{L^p(\mathbb{R}^3)}^p \leq C_p \|1_{\{|x| \leq R\}}u \|_{L^p(\mathbb{R}^3)}^p
\]

for any \( p \in (1, 3) \) and suitable constant \( C_p > 0 \). We shall establish (5.30) above in three separate regimes: \( p \in (2, 3) \), \( p \in \left(2, \frac{3}{2}\right) \), and \( p \in (1, \frac{3}{2}) \). By interpolation, also the cases \( p = \frac{3}{2} \) and \( p = 2 \) will then be covered.

From (5.28), we estimate

\[
\|K_{\Xi}(x, \cdot)1_{\{|y| \geq 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \leq \frac{(4\pi)^{3/2}}{|x|} \sum_{\pm} \left( \int_0^{10R} d\rho \rho^{2-p'} |\hat{X}(|x| + a \pm \rho)|^{p'} \right)^{1/p'}.
\]

When \( p \in (2, 3) \), and hence \( p' \in \left(\frac{3}{2}, 2\right) \), we have \( \rho^{2-p'} \leq (10R)^{2-p'} \) for every \( \rho \in [0, 10R] \), and (5.31) then yields

\[
\|K_{\Xi}(x, \cdot)1_{\{|y| \leq 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \leq C \frac{\|\hat{X}\|_{L^{p'}(\mathbb{R})}}{|x|}.
\]

When instead \( p \in \left(\frac{3}{2}, 2\right) \), and hence \( p' \in (2, 3) \), the r.h.s. of (5.31) is estimated with Hölder’s inequality, with weights \( q = \frac{p'-1}{2(p'-2)} \) and \( q' = \frac{p'-1}{3-p'} \), as

\[
\|K_{\Xi}(x, \cdot)1_{\{|y| \leq 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \leq \frac{C}{|x|} \left( \int_0^{10R} \frac{d\rho}{\rho^{p'-1}} \right)^{\frac{2(2-p)}{p'}} \|\hat{X}\|_{L^{\frac{p'(p'-1)}{3-p'}}(\mathbb{R})}.
\]
In order to obtain analogous estimates to (5.32)–(5.33) in the remaining regime $p \in (1, \frac{3}{2})$, it is convenient to integrate by parts in (5.26), using
\[
e^{-i\lambda(|x|+a)} = i(|x|+a)^{-1}\partial_\lambda e^{-i\lambda(|x|+a)},
\]
so as to split
\[
K_\Xi(x, y) = K_\Xi^{(1)}(x, y) + K_\Xi^{(2)}(x, y)
\]
(5.34) with
\[
K_\Xi^{(1)}(x, y) := \frac{-1}{\sqrt{2\pi}|x|(|x|+a)} \int_0^{+\infty} \left( e^{-i\lambda(|x|+a+|y|)} + e^{i\lambda(|x|+a-|y|)} \right) X(\lambda) \, d\lambda
\]
\[
= \frac{-1}{|x|(|x|+a)} (\hat{X}(|x|+a+|y|) + \hat{X}(|x|+a-|y|))
\]
(5.35) and
\[
K_\Xi^{(2)}(x, y) := \frac{-i}{\sqrt{2\pi}|x|(|x|+a)} \int_0^{+\infty} e^{-i\lambda(|x|+a)} \left( e^{-i\lambda|y|} - e^{i\lambda|y|} \right) X'(\lambda) \, d\lambda.
\]
(5.36)

Using (5.35), we get
\[
\|K_\Xi^{(1)}(x, \cdot)1_{\{|\cdot| \leq 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \\
\leq \frac{(4\pi)^{\frac{1}{p'}}}{|x|(|x|+a)} \sum \left( \int_0^{10R} \rho^2 |\hat{X}(|x|+a+\rho)|^{p'} \right)^{1/p'}
\leq \frac{2 (4\pi)^{\frac{1}{p'}} (10R)^2}{|x|(|x|+a)} \|\hat{X}\|_{L^{p'}(\mathbb{R})},
\]
(5.37)

As for $K_\Xi^{(2)}$, we exploit (5.36) using the bound $|X'(\lambda)| \leq C\langle\lambda\rangle^{-2}$ for some $C > 0$, which follows from the fact that $X(\lambda) = \lambda^{-1}\omega_>(\lambda)$ or $\lambda^{-2}\omega_>(\lambda)$, and the bound $|e^{-i\lambda|y|} - e^{i\lambda|y|}| \leq 2(\lambda|y|)^{1-\theta}$, $\forall \theta \in (0, 1)$. Thus,
\[
|K_\Xi^{(2)}(x, y)| \leq \frac{1}{|x|(|x|+a)|y|} \int_0^{+\infty} 2(\lambda|y|)^{1-\theta} |X'(\lambda)| \, d\lambda \leq \frac{C}{|x|(|x|+a)|y|^\theta},
\]
whence
\[
\|K_\Xi^{(2)}(x, \cdot)1_{\{|\cdot| \leq 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \leq \frac{C'}{|x|(|x|+a)} \left\| \frac{1_{\{|\cdot| \leq 10R\}}}{|y|^\theta} \right\|_{L^{p'}(\mathbb{R}^3)},
\]
(5.38)
for suitable constants $C, C' > 0$, where the $L^{p'}$-norm in the r.h.s. is finite whenever $\theta p' < 3$.

Estimates (5.32), (5.33), (5.34), (5.37), and (5.38) together then imply that, for some constant $C_p > 0$,
\[
\|K_\Xi(x, \cdot)1_{\{|\cdot| \leq 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \leq \frac{C_p}{|x|}, \quad p \in (1, \frac{3}{2}) \cup (\frac{3}{2}, 2) \cup (2, 3).
\]
(5.39)
Then (5.39) and Hölder’s inequality yield
\[
\|1_{\{|x| \leq R\}} \Xi 1_{\{|y| \leq 10R\}} u\|_{L^p(\mathbb{R}^3)}^p \\
\leq \int_{\mathbb{R}^3} dx 1_{\{|x| \leq R\}}(x) \int_{\mathbb{R}^3} dy K\Xi(x, y) 1_{\{|y| \leq 10R\}}(y) u(y) \| \|^p \\
\leq C_p \left\| 1_{\{|x| \leq R\}} \right\|_{L^p(\mathbb{R}^3)}^p \left\| 1_{\{|y| \leq 10R\}} u\right\|_{L^p(\mathbb{R}^3)}^p \\
= C'_p \left\| 1_{\{|y| \leq 10R\}} u\right\|_{L^p(\mathbb{R}^3)}^p
\]
for some constant $C'_p > 0$.

We have thus obtained precisely the desired estimate (5.30). This completes the proof because, as commented already, (5.29) and (5.30) together give (5.25). \qed

6. Unboundedness in $L^1(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$, $p \geq 3$.

In this section, we complete the proof of Theorem 1.1 as far as the unboundedness part is concerned, hence showing that the wave operators $W_{\alpha,Y}^\pm$ are unbounded in $L^p(\mathbb{R}^3)$ whenever $p \in \{1\} \cup [3, +\infty]$. As commented already at the beginning of Sect. 4, it is enough to prove this property for $W_{\alpha,Y}^+$: then, the same conclusion follows for $W_{\alpha,Y}^-$.\[\]

6.1. Unboundedness of $W_{\alpha,Y}^+$ in $L^p(\mathbb{R}^3)$ for $p \in [3, +\infty]$.

Because of the $L^p$-boundedness of $W_{\alpha,Y}^+$ for $p \in (1, 3)$, it is clear that we only need to prove that $W_{\alpha,Y}^+$ is unbounded in $L^3(\mathbb{R}^3)$, for any $L^p$-boundedness for $p > 3$ would then contradict, by interpolation, the unboundedness when $p = 3$.

Let us assume for contradiction that $W_{\alpha,Y}^+$ is bounded in $L^3(\mathbb{R}^3)$, which by duality implies also that $(W_{\alpha,Y}^+)^*$ is bounded in $L^{3/2}(\mathbb{R}^3)$.

Theorem 2.1(iv) guarantees that we may choose $c > 0$ sufficiently large so as to make the matrix $\Gamma_{\alpha,Y}(ic)$ non-singular. Correspondingly, $R_0(-c^2) = (H_0 + c^2 I)^{-1}$ maps continuously $L^{3/2}(\mathbb{R}^3)$ into $L^{2,3/2}(\mathbb{R}^3)$ and hence also $L^{3/2}(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ for any $q \in [3^2, \infty)$, owing to a Sobolev embedding.

Thus, the $L^{3/2}$-boundedness of $(W_{\alpha,Y}^+)^*$, the $L^{3/2} \to L^3$-boundedness of $R_0(-c^2)$, and the $L^3$-boundedness of $W_{\alpha,Y}^+$ imply, by means of the intertwining property (1.8), that also the operator
\[
R_{\alpha,Y}(-c^2) P_{ac}(H_{\alpha,Y}) = W_{\alpha,Y}^+ R_0(-c^2) (W_{\alpha,Y}^+)^*
\]
is continuous from $L^{3/2}(\mathbb{R}^3)$ to $L^3(\mathbb{R}^3)$. As a consequence, we read out from the resolvent identity (2.3) that for any $u \in L^2_{ac}(H_{\alpha,Y}) \cap L^{3/2}(\mathbb{R}^3)$ the function
\[
R_{\alpha,Y}(-c^2) u - R_0(-c^2) u
\]
\[
= \sum_{j,k=1}^{N} (\Gamma_{\alpha,Y}(ic))^{-1}_{jk} G_{ic}^{\alpha,j}(x) \int_{\mathbb{R}^3} G_{ic}^{\alpha,k}(y) u(y) dy \tag{*}
\]
must belong to $L^3(\mathbb{R}^3)$.\[\]
Let us make now a choice of $u$ for which the r.h.s. of (*) above fails instead to belong to $L^3(\mathbb{R}^3)$. Since $u \in L^2_{ac}(H_{\alpha,Y})$, then $u$ is orthogonal to all the eigenfunctions of $H_{\alpha,Y}$, that is, owing to Theorem 2.1(iv), $u$ is orthogonal to an (at most) $N$-dimensional subspace spanned by suitable linear combinations of $G_{\alpha_k}^{y_1}, \ldots, G_{\alpha_k}^{y_N}$ for $k \in \{1, \ldots, N\}$, where $-\lambda_1^2, \ldots, -\lambda_N^2$ are the eigenvalues of $H_{\alpha,Y}$. Because of our choice of $c$, in such an orthogonal complement there is surely $u$ which is not orthogonal to the $G_{ic}^{y_k}$, namely
\[
\int_{\mathbb{R}^3} G_{ic}^{y_k}(y) u(y) dy \neq 0 \quad \forall k \in \{1, \ldots, N\}.
\]
(In fact, such a $u$ can also be found in $C^\infty_0(\mathbb{R}^3) \cap L^2_{ac}(H_{\alpha,Y})$: indeed, the point spectral subspace of $H_{\alpha,Y}$ is at most $N$-dimensional, whereas the set of $u$’s that satisfy the non-vanishing condition above is open in the topology of the space of test functions.) For such $u$, because of the invertibility of the matrix $\Gamma_{\alpha,Y}(ic)$, the expression
\[
\sum_{j,k=1}^N (\Gamma_{\alpha,Y}(ic)^{-1})_{jk} G_{ic}^{y_j}(x) \int_{\mathbb{R}^3} G_{ic}^{y_k}(y) u(y) dy
\]
is a linear combination of the $G_{ic}^{y_j}$’s with at least one nonzero coefficient, say, the one for $j = j_0$. Therefore, in a sufficiently small neighbourhood of $y_{j_0}$ (so small as not to contain any of the $y_j$’s of $Y$, for $j \neq j_0$) the latter function must be of the form $c_{j_0}|x-y_{j_0}|^{-1}+R(x)$ for some constant $c_{j_0} \neq 0$ and some bounded (in fact, smooth) function $R(x)$. This would mean that in the considered neighbourhood of $y_{j_0}$ $R_{\alpha,Y}(-c^2)u - R_0(-c^2)u$ is not a $L^3$-function, a contradiction.

### 6.2. Unboundedness of $W^+_{\alpha,Y}$ in $L^1(\mathbb{R}^3)$

For this case, the following preliminary observation is going to be useful.

**Remark 6.1.** Let $g \in C^\infty_0(\mathbb{R})$. Then
\[
\frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-i\lambda \rho} \hat{g}(-\lambda) d\lambda = g(\rho) - i(\mathcal{H}g)(\rho),
\]
where $g \mapsto \mathcal{H}g$ denotes the Hilbert transform, defined as
\[
(\mathcal{H}g)(\rho) := \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{+\infty} \frac{g(\tau)}{\rho - \tau} d\tau.
\]
Indeed, following from the fact [17, Eq. (5.1.13)] that the Hilbert transform is the Fourier multiplier
\[
\widehat{(\mathcal{H}g)}(\lambda) = -i \text{sgn}(\lambda) \hat{g}(\lambda),
\]
one has
\[
g(\rho) - i(\mathcal{H}g)(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\rho \lambda} (1 - \text{sgn}(\lambda)) \hat{g}(\lambda) d\lambda = 2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\rho \lambda} \hat{g}(\lambda) d\lambda = 2 \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-i\lambda \rho} \hat{g}(-\lambda) d\lambda.
\]
Let us now prove the fact that the wave operator $W_{\alpha,Y}^+$ is unbounded in $L^1(\mathbb{R}^3)$. We may assume without loss of generality to take the set $Y = \{y_1, \ldots, y_N\}$ of interaction centres so that $y_1 = 0$.

Let $u \in C_0^\infty (\mathbb{R}^3)$ be rotationally invariant, and we write $u(x) = f(|x|)$ for some $f : [0, +\infty) \to \mathbb{C}$ which is smooth and compactly supported. We extend $f$ to an even function on the whole $\mathbb{R}$. By construction, $f(r) = M_u(r)$, the spherical mean of $u$.

Our starting point is the stationary representation (3.11) for $W_{\alpha,Y}^+ u$, that is,

$$W_{\alpha,Y}^+ u = u + \sum_{j,k=1}^N T_{y_j} \Omega_{jk} T_{-y_k} u,$$  

(6.3)

and for each $j, k \in \{1, \ldots, N\}$ we set $K_{jk} u := T_{y_j} \Omega_{jk} T_{-y_k} u$. Explicitly,

$$(K_{jk} u)(x) = \frac{1}{i\pi} \int_{\mathbb{R}^3} dy \, u(y) \int_0^{+\infty} d\lambda \, F_{jk}(\lambda) \frac{e^{-i\lambda|x-y_j|}}{4\pi|x-y_j|} \frac{e^{i\lambda|y-y_k|} - e^{-i\lambda|y-y_k|}}{4\pi|y-y_k|},$$

(6.4)

where we used (3.1) and (3.4).

We now proceed by re-scaling $u$ and $f$ as

$$u_{\epsilon}(x) := \epsilon^{-3} u(\epsilon^{-1} x), \quad f_{\epsilon}(r) := \epsilon^{-3} f(\epsilon^{-1} r), \quad \epsilon > 0,$$  

(6.5)

which makes the norms

$$4\pi \|r^2 f_{\epsilon}\|_{L^1(0, +\infty)} = \|[u_{\epsilon}]\|_{L^1(\mathbb{R}^3)} = \|u\|_{L^1(\mathbb{R}^3)} = 4\pi \|r^2 f\|_{L^1(0, +\infty)}$$

(6.6)

$\epsilon$-independent. This re-scaling is devised so as to make all interaction centres but $y_1$ ineffective, because $u_{\epsilon}$ is only bumped around the origin, and then to reduce the question to the unboundedness of the wave operator relative to a single-centre point interaction Hamiltonian, for which the answer will then come by direct inspection.

From (6.4) and (6.6),

$$(K_{jk} u_{\epsilon})(x) = \frac{1}{i\pi \epsilon^2} \int_{\mathbb{R}^3} dy \, u(y) \int_0^{+\infty} d\lambda \, F_{jk}(\lambda) \frac{e^{-i\lambda|x-y_j|}}{4\pi|x-y_j|} \frac{e^{i\lambda|y-y_k|} - e^{-i\lambda|y-y_k|}}{4\pi|y-y_k|},$$

(6.7)

having made the changes of variables $y \to \epsilon y$ and $\lambda \to \epsilon^{-1} \lambda$ in the integrations. If we now define, for arbitrary $v \in C_0^\infty (\mathbb{R}^3)$,

$$(K_{jk}^{(\epsilon)} v)(x) = \frac{1}{i\pi} \int_{\mathbb{R}^3} dy \, v(y) \int_0^{+\infty} d\lambda \, F_{jk}(\lambda) \frac{e^{-i\lambda|x-y_j|}}{4\pi|x-y_j|} \frac{e^{i\lambda|y-y_k|} - e^{-i\lambda|y-y_k|}}{4\pi|y-y_k|},$$

(6.8)

then for the considered $u$ and its re-scaled $u_{\epsilon}$, we have

$$\left\| \sum_{j,k=1}^N K_{jk} u_{\epsilon} \right\|_{L^1(\mathbb{R}^3)} = \left\| \sum_{j,k=1}^N K_{jk}^{(\epsilon)} u \right\|_{L^1(\mathbb{R}^3)},$$

(6.9)
which follows by making the change of variable $x \mapsto \varepsilon x$ in the integration on the l.h.s.

We now want to study the contribution of each term $K^{(\varepsilon)}_{jk} u$ as $\varepsilon \downarrow 0$. We shall establish the following limits

$$
\lim_{\varepsilon \downarrow 0} \left( K^{(\varepsilon)}_{11} u \right)(x) = -\sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \frac{e^{-i\lambda |x|}}{|x|} (\hat{r}f)(-\lambda) \, d\lambda,
$$

$$
\lim_{\varepsilon \downarrow 0} \left( K^{(\varepsilon)}_{jk} u \right)(x) = 0, \quad (j, k) \neq (1, 1),
$$

pointwise for a.e. $x \in \mathbb{R}^3$.

To this aim, we first find the bound

$$
\int_{\mathbb{R}^3} \frac{e^{i\lambda |y - \frac{y_k}{\varepsilon}|} - e^{-i\lambda |y - \frac{y_k}{\varepsilon}|}}{4\pi |y - \frac{y_k}{\varepsilon}|} u(y) \, dy \leq \| \cdot \|_{C \mathcal{L}^{-2}} \int_{\text{supp } u} \frac{dy}{|y - \varepsilon^{-1} y_k|} \tag{6.11}
$$

for some constant $C_u > 0$ depending on $u$, but not on $\varepsilon$. (6.11) is obvious for small $\lambda$’s, since $u$ is compactly supported, whereas for large $\lambda$’s we apply the distributional identity

$$
(-\Delta_y - \lambda^2) \left( \frac{e^{\pm i\lambda |y - \frac{y_k}{\varepsilon}|}}{4\pi |y - \frac{y_k}{\varepsilon}|} \right) = \delta(y - \frac{y_k}{\varepsilon}),
$$

and integrating by parts we find

$$
\int_{\mathbb{R}^3} \frac{e^{i\lambda |y - \frac{y_k}{\varepsilon}|} - e^{-i\lambda |y - \frac{y_k}{\varepsilon}|}}{4\pi |y - \frac{y_k}{\varepsilon}|} u(y) \, dy = \lambda^{-2} \int_{\mathbb{R}^3} \frac{e^{i\lambda |y - \frac{y_k}{\varepsilon}|} - e^{-i\lambda |y - \frac{y_k}{\varepsilon}|}}{4\pi |y - \frac{y_k}{\varepsilon}|} (-\Delta) u(y) \, dy
$$

$$
\leq \| \cdot \|_{C \mathcal{L}^{-2}} \int_{\text{supp } u} \frac{dy}{|y - \varepsilon^{-1} y_k|},
$$

thus, (6.11) is proved.

Next, in order to prove the first of limits (6.10) by taking $\varepsilon \downarrow 0$ in (6.8), we use the asymptotics (3.5), namely

$$
\lim_{\varepsilon \downarrow 0} F_{11}(\varepsilon^{-1}\lambda) = -4\pi i,
$$

and we also recognise that the asymptotics as $\varepsilon \downarrow 0$ of the $y$-integration of (6.8) is precisely the quantity

$$
\int_{\mathbb{R}^3} \frac{e^{i\lambda |y|} - e^{-i\lambda |y|}}{4\pi |y|} u(y) \, dy = \sqrt{2\pi} \left( \hat{r}M_u \right)(-\lambda) = \sqrt{2\pi} \left( \hat{r}f \right)(-\lambda)
$$

discussed in (3.9). The limit $\varepsilon \downarrow 0$ can be exchanged with the integrations in $\lambda$ and in $y$ by dominated convergence, because $F_{11}(\frac{\lambda}{\varepsilon})$ is uniformly bounded (see
Lemma 3.1(i)) and (6.11) provides a majorant that is integrable in \( \lambda \). Thus,

\[
\lim_{\varepsilon \downarrow 0} (K_{jk}^{(\varepsilon)} u)(x) = \frac{1}{i\pi} (-4\pi i) \int_{0}^{+\infty} \frac{e^{-i\lambda|x|}}{4\pi|x|} \sqrt{2\pi} (\hat{f})(-\lambda) \, d\lambda
\]

\[
= -\sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \frac{e^{-i\lambda|x|}}{|x|} (\hat{f})(-\lambda) \, d\lambda, \quad x \neq 0,
\]

and the first limit of (6.10) is proved.

Concerning now (6.10) when \((j, k) \neq (1, 1)\), from our estimate (6.11) we deduce

\[
|(K_{jk}^{(\varepsilon)} u)(x)| \leq C_u \frac{1}{|x - \frac{y_j}{\varepsilon}|} \left( \int_{0}^{+\infty} |F_{jk}(\frac{\lambda}{\varepsilon})|^2 \, d\lambda \right) \left( \int_{\text{supp } u} \frac{dy}{|y - \frac{y_k}{\varepsilon}|} \right)
\]

(6.12)

for some new constant \(C_u' > 0\). Since at least one among \(y_j\) and \(y_k\) does not coincide with the origin, and since \(u\) is compactly supported, we conclude at once that

\[
\lim_{\varepsilon \downarrow 0} (K_{jk}^{(\varepsilon)} u)(x) = 0, \quad x \neq 0 \quad \text{if} \quad j = 0.
\]

The proof of (6.10) is thus completed, and in turn (6.10) implies

\[
\lim_{\varepsilon \downarrow 0} \sum_{j, k=1}^{N} (K_{jk}^{(\varepsilon)} u)(x) = -\frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} \frac{e^{-i\lambda|x|}}{|x|} (\hat{f})(-\lambda) \, d\lambda
\]

(6.13)

pointwise for a.e. \(x \in \mathbb{R}^3\).

This latter fact allows us to take the limit \(\varepsilon \downarrow 0\) in the r.h.s. of (6.9), provided that the \(L^1\)-norm is taken on compacts of \(\mathbb{R}^3\). Indeed, for fixed \(R > 0\) and any sufficiently small \(\varepsilon > 0\) such that \(|x - \frac{y_j}{\varepsilon}| \geq |x|\) for any \(x \in \{x \, | \, x| \leq R\} \cup \text{supp } u\) and \(j = 1, \ldots, N\), estimate (6.12) implies (1\(R \equiv \) the characteristic function of the ball \(|x| \leq R\))

\[
1_R(x) \sum_{j, k=1}^{N} |(K_{jk}^{(\varepsilon)} u)(x)| \leq N^2 C_{u, R} \frac{C_{u, R}}{|x|} \int_{\text{supp } u} \frac{dy}{|y|} \leq N^2 C_{u, R} \frac{C_{u, R}}{|x|}
\]

for suitable constants \(C_{u, R}, C_{u, R}' > 0\), which gives a majorant in \(L^1(\mathbb{R}^3)\). Then, by (6.13) and dominated convergence,

\[
\lim_{\varepsilon \downarrow 0} \int_{|x| \leq R} \left| \sum_{j, k=1}^{N} (K_{jk}^{(\varepsilon)} u)(x) \right| \, dx
\]

\[
= \frac{2}{\sqrt{2\pi}} \int_{|x| \leq R} \, dx \left| \int_{0}^{+\infty} \frac{e^{-i\lambda|x|}}{|x|} (\hat{f})(-\lambda) \, d\lambda \right|
\]

\[
= \sqrt{32\pi} \int_{0}^{R} d\rho \left| \int_{0}^{\infty} \rho e^{-i\lambda \rho} (\hat{f})(-\lambda) \, d\lambda \right|. \quad (6.14)
\]
An integration by parts and formula (6.1) in Remark 6.1 yield
\[
\sqrt{32\pi} \int_0^\infty \rho e^{-\lambda \rho} (\hat{r} f)(-\lambda) d\lambda = \sqrt{32\pi} \int_0^\infty e^{-\lambda \rho} (\hat{r}^2 f)(-\lambda) d\lambda = 4\pi (r^2 f)(\rho) - i(\mathcal{H} r^2 f)(\rho).
\] (6.15)

In the integration by parts, the boundary term does not appear because \(r \mapsto rf(r)\) is an odd function and \((\hat{r} f)(0) = 0\). The conclusion from (6.14) and (6.15) is therefore
\[
\lim_{\varepsilon \to 0} \left\| \mathbf{1}_R \sum_{j,k=1}^N K_{j,k}^{(\varepsilon)} u \right\|_{L^1(\mathbb{R}^3)} = 4\pi \int_0^R \left| (1 - i\mathcal{H})(r^2 f)(\rho) \right| d\rho.
\] (6.16)

The proof of the \(L^1\)-unboundedness of \(W_{\alpha,Y}^+\) is completed as follows. Suppose for contradiction that \(W_{\alpha,Y}^+\) is instead \(L^1\)-bounded. Then, for arbitrary \(R > 0\),
\[
4\pi \int_0^R \left| (1 - i\mathcal{H})(r^2 f)(\rho) \right| d\rho = \lim_{\varepsilon \to 0} \left\| \mathbf{1}_R \sum_{j,k=1}^N K_{j,k}^{(\varepsilon)} u \right\|_{L^1(\mathbb{R}^3)} \leq \liminf_{\varepsilon \to 0} \left\| \sum_{j,k=1}^N K_{j,k}^{(\varepsilon)} u \right\|_{L^1(\mathbb{R}^3)} = \liminf_{\varepsilon \to 0} \left\| (W_{\alpha,Y}^+ - \mathbb{1}) u \varepsilon \right\|_{L^1(\mathbb{R}^3)} \leq (1 + \|W_{\alpha,Y}^+\| \mathcal{B}(L^1(\mathbb{R}^3))) \| u \varepsilon \|_{L^1(\mathbb{R}^3)} \leq (1 + \|W_{\alpha,Y}^+\| \mathcal{B}(L^1(\mathbb{R}^3))) \| r^2 f \|_{L^1(0,\infty)},
\]
where we applied (6.16) in the first step, (6.9) in the third step, (6.3) in the fourth step, the assumption of \(L^1\)-boundedness in the fifth step, and the scale invariance (6.6) in the last step. Moreover, due to the arbitrariness of \(R\), the estimate above also implies
\[
4\pi \left\| (1 - i\mathcal{H})(r^2 f) \right\|_{L^1(0,\infty)} \leq (1 + \|W_{\alpha,Y}\| \mathcal{B}(L^1(\mathbb{R}^3))) \| r^2 f \|_{L^1(0,\infty)}.
\]
However, the inequality (*) can be surely violated. Indeed, it is well known that the Hilbert transform on \(\mathbb{R}\) maps even functions into odd functions, but fails to map even (and compactly supported) \(L^1\)-functions into \(L^1\)-functions, as one may see with (a suitable mollification, so as to make it \(C_0^\infty\) and even, of) the function \(f_0(r) = (r^2 + 1)^{-1}\), the Hilbert transform of which is \((\mathcal{H} f_0)(r) = r(r^2 + 1)^{-1}\). Therefore, (*) is a contradiction. The conclusion is that \(W_{\alpha,Y}^+\) is necessarily unbounded on \(L^1(\mathbb{R}^3)\).
7. $L^p$-convergence of Wave Operators

In this concluding section, we establish a result of $L^p$-convergence of wave operators in the limit when a regular Schrödinger Hamiltonian converges to a singular point interaction Hamiltonian. This is part of the general picture outlined in Remark 1.2 concerning the connection between two completely analogous results, on the one hand our main result (Theorem 1.1) of $L^p$-boundedness for $p \in (1, 3)$ and $L^p$-unboundedness for $p \in \{1\} \cup [3, \infty)$ of the wave operators relative to the point interaction Hamiltonian $H_{\alpha,Y}$, and on the other hand the analogous results available in the previous literature, precisely in the same regimes of $p$, for wave operators relative to Schrödinger Hamiltonians of the form $-\Delta + V$.

For concreteness, we restrict our attention to the case $N = 1$ and $\alpha = 0$, thus taking without loss of generality $Y = \{0\}$.

Besides the corresponding point interaction Hamiltonian $H_{\alpha,Y}$ and wave operators $W^\pm_{\alpha,Y}$, let us consider the Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^3)$, where $V$ is a real measurable potential such that $|V(x)| \leq C|x|^{-\delta}$ for some $\delta > 5/2$. It is well known [25] that $H$ has a unique self-adjoint realisation on $L^2(\mathbb{R}^3)$ and that the wave operators

$$W^\pm := \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} \quad (7.1)$$

relative to the pair $(H, H_0)$ exist and are complete in $L^2(\mathbb{R}^3)$; $W^\pm$ extend to bounded operators on $L^p(\mathbb{R}^3)$ in the following regimes: for all $p \in [1, +\infty]$ if zero is neither a resonance nor eigenvalue of $H$ [6], and only for $p \in (1, 3)$ if zero is a resonance [35] (see Proposition 7.1).

Parallel to that, and with the same $V$, we consider now the re-scaled version $H(\varepsilon)$ of $H$ obtained by ‘shrinking’ the potential $V$ at a scale $\varepsilon^{-1}$, more precisely, the self-adjoint operator

$$H(\varepsilon) := -\Delta + \frac{1}{\varepsilon^2} V \left( \frac{x}{\varepsilon} \right), \quad \varepsilon > 0, \quad (7.2)$$

as well as the wave operators relative to the pair $(H(\varepsilon), H_0)$, defined in analogy to (7.1) as

$$W^\pm_\varepsilon := \lim_{t \to \pm \infty} e^{itH(\varepsilon)} e^{-itH_0}. \quad (7.3)$$

Choice (7.2) for the scaling is driven by the fact [3, Theorem I.1.2.5] that under suitable spectral properties of $H$ one has $H(\varepsilon) \to H_{\alpha,Y}|_{\alpha=0,Y=\{0\}}$ as $\varepsilon \downarrow 0$ in the norm resolvent sense of operators on $L^2(\mathbb{R}^3)$, and this in turn motivates us to investigate the relation between $W^\pm_\varepsilon$ and $W^\pm_{\alpha,Y}$ when $\varepsilon \downarrow 0$, as bounded operators on $L^p(\mathbb{R}^3)$ for $p \in (1, 3)$. Our result is the following.

**Proposition 7.1.** Suppose that $V$ is a real measurable potential such that $|V(x)| \leq C|x|^{-\delta}$ for some $\delta > 7$. Then, for any $p \in (1, 3)$ the wave operators $W^\pm_\varepsilon$ extend to bounded operators on $L^p(\mathbb{R}^3)$. If zero is a resonance but not an eigenvalue for the self-adjoint operator $H = -\Delta + V$ on $L^2(\mathbb{R}^3)$, then in the
weak topology of \( L^p(\mathbb{R}^3) \) with \( p \in (1, 3) \), and hence also in the strong topology of \( L^2(\mathbb{R}^3) \),

\[
\lim_{\varepsilon \to 0} W^\pm_\varepsilon u = W^\pm_{\alpha,Y} u, \quad u \in L^p(\mathbb{R}^3). \tag{7.4}
\]

**Proof.** The statement on the \( L^p \)-boundedness of \( W^\pm_\varepsilon \) follows directly from [35]. Concerning limit (7.4), we shall prove it for \( W^+_\varepsilon \), the argument for \( W^-_\varepsilon \) being completely analogous.

Let us consider the scaling operator \( u \mapsto U_\varepsilon u \) defined by

\[
(U_\varepsilon u)(x) := \frac{1}{\varepsilon^{3/2}} u \left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \ x \in \mathbb{R}^3. \tag{7.5}
\]

For any \( \varepsilon > 0 \) and \( p \in [1, +\infty] \) the operator \( U_\varepsilon \) is a bounded bijection on \( L^p(\mathbb{R}^3) \) with norm

\[
\|U_\varepsilon\|_{\mathcal{B}(L^p(\mathbb{R}^3))} = \varepsilon^{3(\frac{1}{p} - \frac{1}{2})} \tag{7.6}
\]

and inverse

\[
(U_\varepsilon)^{-1} = U_{\varepsilon^{-1}}. \tag{7.7}
\]

In particular, \( U_\varepsilon \) is unitary on \( L^2(\mathbb{R}^3) \), and it induces the unitary equivalence

\[
H^{(\varepsilon)} = U_\varepsilon (\varepsilon^{-2} H) U_\varepsilon^*. \tag{7.8}
\]

As a consequence, \( W^+_\varepsilon \) and \( W^+ \) are unitarily equivalent too as operators on \( L^2(\mathbb{R}^3) \), for

\[
W^+_\varepsilon = s\text{-lim}_{t \to +\infty} e^{i t H^{(\varepsilon)}} e^{-i t H_0} = U_\varepsilon s\text{-lim}_{t \to +\infty} e^{i t \varepsilon^{-2} H} e^{-i t \varepsilon^{-2} H_0} U_\varepsilon^* = U_\varepsilon W^+ U_\varepsilon^*. \tag{7.9}
\]

Moreover,

\[
\|W^+_\varepsilon\|_{\mathcal{B}(L^p(\mathbb{R}^3))} = \|W^+\|_{\mathcal{B}(L^p(\mathbb{R}^3))} < +\infty \tag{7.10}
\]

for any \( p \in (1, 3) \), as follows by combining (7.6), (7.7), and (7.9).

For the proof of (7.4), it suffices to show that, when \( \alpha = 0 \) and \( Y = \{0\} \),

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} (W^+_\varepsilon u)(x) v(x) \, dx = \int_{\mathbb{R}^3} (W_{\alpha,Y} u)(x) v(x) \, dx \tag{7.11}
\]

for any \( u \) and \( v \) in

\[
\mathcal{D} := \{ u \in \mathcal{S}(\mathbb{R}^3) \mid \hat{u} \in C^\infty_0(\mathbb{R}^3) \} \tag{7.12}
\]

which is dense in \( L^p(\mathbb{R}^3) \) for any \( 1 < p < \infty \). Indeed by means of a straightforward density argument, applicable because of the uniform norm-boundedness (7.10), result (7.11) can then be lifted to any \( u \in L^p(\mathbb{R}^d) \) and \( v \in L^p(\mathbb{R}^d) \), whence the conclusion. Moreover, with choice (7.12) we can equivalently rewrite (7.11) in Hilbert scalar product notation as

\[
\lim_{\varepsilon \to 0} \langle W^+_\varepsilon u, v \rangle = \langle W_{\alpha,Y} u, v \rangle. \tag{7.13}
\]

Aimed at establishing (7.13), let us fix \( u, v \in \mathcal{D} \). Then there is \( R > 0 \) such that \( \hat{u}(\xi) = 0 \) for \( |\xi| > R \), and also

\[
(U^*_\varepsilon u)(\xi) = \frac{1}{\varepsilon^{3/2}} \hat{u} \left(\frac{\xi}{\varepsilon}\right), \quad (U^*_\varepsilon u)(\xi) = 0 \quad \text{for} \ |\xi| > R\varepsilon. \tag{7.14}
\]
We shall make crucial use of the well-known fact from the stationary scattering theory [25] that
\[
W^+ = 1 - \frac{1}{i\pi} \int_0^{+\infty} G_0(-\lambda) V (\mathbb{1} + G_0(-\lambda)V)^{-1} (G_0(\lambda) - G_0(-\lambda)) \lambda \, d\lambda, \quad (7.15)
\]
where
\[
G_0(\pm\lambda) := \lim_{\eta \downarrow 0} (H_0 - (\lambda^2 \pm i\eta)\mathbb{1})^{-1} = \lim_{\eta \downarrow 0} R_0(\lambda^2 \pm i\eta), \quad \lambda \geq 0. \quad (7.16)
\]
Then (7.9) and (7.15), together with \[G_0(\pm\lambda)^* = G_0(\mp\lambda)\] yield
\[
\langle W^+ \varepsilon u, v \rangle - \langle u, v \rangle = \frac{1}{i\pi} \int_0^{+\infty} \left( (1 + G_0(-\lambda)V)^{-1} (G_0(\lambda) - G_0(-\lambda)) U^*_\varepsilon u , V G_0(\lambda) U^*_\varepsilon v \right) \lambda \, d\lambda.
\]
\[
\quad (7.17)
\]
In fact, the \[\lambda\]-integration in (7.17) is only effective for \[\lambda < R\varepsilon\]. To see this, we compute the Fourier transform
\[
\left( (G_0(\lambda) - G_0(-\lambda)) U^*_\varepsilon u \right)(\xi) = \lim_{\eta \downarrow 0} \left( (\xi^2 - \lambda^2 + i\eta) \right)^{-1} (\hat{U}^*_\varepsilon u)(\xi)
\]
\[
= \lim_{\eta \downarrow 0} \frac{2i\eta}{(\xi^2 - \lambda^2)^2 + \eta^2} (\hat{U}^*_\varepsilon u)(\xi)
\]
and we argue that the function in (7.18) surely vanishes when \[|\xi| > R\varepsilon\], owing to (7.14), and when in addition \[\lambda > R\varepsilon\] such function also vanishes when \[|\xi| \leq R\varepsilon\], because in this case \((\xi^2 - \lambda^2)^2 > 0\) and the above limit in \(\eta\) is zero. Thus,
\[
(G_0(\lambda) - G_0(-\lambda)) U^*_\varepsilon u \equiv 0 \quad \text{when } \lambda > R\varepsilon. \quad (7.19)
\]
By exploiting the scaling in \(\varepsilon\) in (7.17), we obtain
\[
\langle W^+_{\varepsilon} u, v \rangle - \langle u, v \rangle = \varepsilon^2 \int_0^{+\infty} \left( (1 + G_0(-\varepsilon\lambda)V)^{-1} (G_0(\varepsilon\lambda) - G_0(-\varepsilon\lambda)) u \right) \lambda \, d\lambda,
\]
\[
\quad (7.20)
\]
where it has to be remembered that, owing to (7.19), the integration actually only takes place when \(\lambda \in [0, R]\).

Next, in order to compute the limit \(\varepsilon \downarrow 0\) in (7.20), we consider separately the behaviour of the operators
\[\varepsilon^{\frac{3}{2}} G_0(\varepsilon\lambda) U^*_\varepsilon\] and \[\varepsilon (1 + G_0(-\varepsilon\lambda)V)^{-1}\]. Indeed, we shall see that they do converge strongly in a suitable Banach space. A weak-type Hölder’s inequality implies that
\[
\langle \varepsilon^{\frac{3}{2}} G_0(\varepsilon\lambda) U^*_\varepsilon u \rangle(x) = \int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|} u(y) \, dy
\]
is bounded by a constant (by \((4\pi)^{-1}\||x|^{-1}\|_{L^3,\infty}\|u\|_{L^\frac{3}{2},1}\) in terms of Lorentz norms); therefore, uniformly for \(\lambda \in [0, R]\) and \(x\) in compact sets,

\[
\lim_{\varepsilon \downarrow 0} \left( \varepsilon \frac{1}{2} G_0(\pm \varepsilon \lambda) U_{\varepsilon}^* u \right)(x) = \int_{\mathbb{R}^3} \frac{e^{\pm i\lambda |y|}}{4\pi |y|} u(y) \, dy = \langle \overline{G_{\pm \lambda}}, u \rangle.
\]

As a consequence, we deduce that

\[
\lim_{\varepsilon \downarrow 0} \left\| \varepsilon \frac{1}{2} G_0(\pm \varepsilon \lambda) U_{\varepsilon}^* u - \langle \overline{G_{\pm \lambda}}, u \rangle \mathbf{1} \right\|_{L^2_{-\beta}(\mathbb{R}^3)} = 0 \tag{7.21}
\]

for \(\beta > \frac{3}{2}\), where \(L^2_{-\beta}(\mathbb{R}^3) \equiv L^2(\mathbb{R}^3, \langle x \rangle^{-2\beta}dx)\) and \(\mathbf{1}\) denotes the function \(\mathbf{1}(x) = 1 \ \forall x \in \mathbb{R}^3\). Moreover, owing to the spectral and decay assumptions on \(V\), it is a standard fact \([34, \text{Theorem 4.8}]\) that

\[
\lim_{\varepsilon \downarrow 0} \left\| \varepsilon (1 + G_0(-\varepsilon \lambda)V)^{-1} - \frac{4\pi i}{\lambda a^2} |\varphi\rangle \langle V \varphi| \right\|_{\mathcal{B}(L^2_{-\beta}(\mathbb{R}^3))} = 0 \tag{7.22}
\]

for \(\lambda > 0\) and \(\beta \in (\frac{3}{2}, \delta - \frac{1}{2})\), where \(\varphi\) is the so-called resonance function relative to \(V\) (thus, a distributional solution to \(H\varphi = (-\Delta + V)\varphi = 0\)), uniquely identified by the conditions \(\int_{\mathbb{R}^3} |V\varphi|^2 dx = -1\) and \(\int_{\mathbb{R}^3} V\varphi \, dx > 0\), and where \(a := \int_{\mathbb{R}^3} V\varphi \, dx\).

If we now and henceforth restrict \(\beta\) to the regime \(\beta \in (\frac{3}{2}, \frac{\delta}{2})\), and then, (7.21) and (7.22) are still valid, and in addition the multiplication by \(V\) is a \(L^2_{-\beta}(\mathbb{R}^3) \to L^2_{-\beta}(\mathbb{R}^3)\) continuous map. Thus, the \(L^2\)-scalar product appearing in the r.h.s. of (7.20) can be also regarded as a \(L^2_{-\beta}-L^2_{-\beta}\) duality product. Using this fact, and by means of (7.21) and (7.22), which are applicable because the \(\lambda\)-integration in (7.20) is actually only effective for \(\lambda \in [0, R]\), we find

\[
\lim_{\varepsilon \downarrow 0} \langle \text{r.h.s. of (7.20)} \rangle = \frac{1}{1\pi} \int_0^{+\infty} \langle 4\pi i \frac{|\varphi\rangle \langle V \varphi| \overline{G_{\lambda}(u)} - \overline{G_{-\lambda}(u)} \mathbf{1}, V \overline{G_{\lambda}(v)} \mathbf{1} \rangle_{L^2_{-\beta}, L^2_{-\beta}} \lambda \, d\lambda
\]

\[= -4 \int_0^{+\infty} d\lambda \left( \int_{\mathbb{R}^3} dy \overline{u(y)} \left( G_{-\lambda}(y) - G_{\lambda}(y) \right) \right) \left( \int_{\mathbb{R}^3} dx \overline{G_{\lambda}(x)} \left( v(x) \right) \right).
\]

Summarising, we have found

\[
\lim_{\varepsilon \downarrow 0} \langle W_{\varepsilon}^* u, v \rangle = \langle u, v \rangle + 4 \int_0^{+\infty} d\lambda \left( \int_{\mathbb{R}^3} dy \overline{u(y)} \left( G_{\lambda}(y) - G_{-\lambda}(y) \right) \right) \left( \int_{\mathbb{R}^3} dx \overline{G_{\lambda}(x)} \left( v(x) \right) \right). \tag{7.23}
\]

Since the r.h.s. above is precisely the quantity \(\langle W_{\alpha, Y} u, v \rangle\) that we obtained in (3.1) in the special case \(N = 1, \alpha = 0\), the limit \(\langle W_{\varepsilon}^* u, v \rangle \to \langle W_{\alpha, Y} u, v \rangle\) of (7.13) is then established and, as already argued, this completes the proof. \(\Box\)
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