LINEAR CONNECTIONS IN NON-COMMUTATIVE GEOMETRY

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ABSTRACT

A construction is proposed for linear connections on non-commutative algebras. The construction relies on a generalisation of the Leibnitz rules of commutative geometry and uses the bimodule structure of $\Omega^1$. A special role is played by the extension to the framework of non-commutative geometry of the permutation of two copies of $\Omega^1$. The construction of the linear connection as well as the definition of torsion and curvature is first proposed in the setting of the derivations based differential calculus of Dubois-Violette and then a generalisation to the framework proposed by Connes as well as other non-commutative differential calculi is suggested. The covariant derivative obtained admits an extension to the tensor product of several copies of $\Omega^1$. These constructions are illustrated with the example of the algebra of $n \times n$ matrices.

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1. Introduction

Non-commutative geometry [1,2] offers a novel and promising framework for the construction of physical theories. The basic idea is to replace the commutative algebra of functions on a manifold by a more general associative algebra. Geometrical objects such as forms and gauge fields are constructed using the algebraic structure which replaces the manifold structure of commutative geometry. The exterior differential calculus and vector bundles have been successfully generalized to the non-commutative context. An analog of integration has also been proposed by Connes [3], it uses the Dixmier trace and the Wodzicki residue.

Non-commutative geometry has been first used for physics by Witten [4,5] as a suitable framework for open string field theory where the exterior derivative is naturally provided by the BRST charge. The action used by Witten is a topological one, it does not make use of a metric or a linear connection. In the spirit of gauge theories on higher dimensions [6], gauge theories were constructed on algebras of the form \( C(M) \otimes A \), where \( C(M) \) is the algebra of functions on four-dimensional space-time and \( A \) is a discrete algebra [7-15]. These theories lead to a natural appearance of the Higgs field with a symmetry breaking potential. The standard model with some constraints on its parameters has been successfully constructed using this strategy [9,10,11]. These constraints cannot, however, be implemented in a renormalization group invariant way [16]. In the same spirit, gravitational theories have been constructed on the same type of algebras [17,19,18,20]. In the particular case where the algebra \( A \) is that of \( n \times n \) matrices the resulting theory is the truncated version of Kaluza-Klein theory with the group \( SU_n \) as an internal manifold [18]. These constructions relied heavily on the generalisation of Cartan’s structure equations. They used a moving basis to define the connection form. A proposal for the construction of a linear connection on \( C(M) \otimes Z_2 \) independently of the metric has been done [19,20] using an axiomatic generalisation of the Leibnitz rule. This rule uses the left module structure of \( \Omega^1 \).

The purpose of this article is to define linear connections on general non-
commutative algebras. Our construction uses the bimodule structure of $\Omega^1$. That is we postulate two Leibnitz rules that allow to calculate the covariant derivative of $f\omega$ and $\omega f$, where $f$ is an element of $\mathcal{A}$ and $\omega$ an element of $\Omega^1$, from the knowledge of the covariant derivative of $\omega$. The situation is thus as in commutative geometry. This construction makes possible the extension of the covariant derivative to the tensor product over $\mathcal{A}$ of several copies of $\Omega^1$. Had we used only the left module structure of $\Omega^1$ this extension could not be possible. Needless to mention, the importance of the linear connection for the formulation of gravitational theories on general non-commutative algebras cannot be overemphasised (see however [22]).

Section 2 fixes the notation and is a brief review of non-commutative differential calculi, that based on derivations proposed by Dubois-Violette and the one pioneered by Connes. In section 3, we review the construction of linear connections in commutative geometry using a formulation that will be appropriate for a non-commutative generalisation which is the subject of section 4. In this section, a definition for a covariant derivative is proposed for the differential calculus of Dubois-Violette. This definition suggests a way of constructing linear connections in the setting of other differential calculi. The construction of linear connections is reduced to the generalisation to the non-commutative setting of a permutation acting on the tensor product of two copies of $\Omega^1$. Section 4 ends with the study of linear connections on matrix algebras. We collect our conclusions in section 5.

2. Non-commutative differential calculi

2.1. The differential calculus of Dubois-Violette

Let $\mathcal{A}$ be an associative *algebra with unity and $\operatorname{Der}$ the set of derivations of $\mathcal{A}$. An element $X$ of $\operatorname{Der}$ is a linear map from $\mathcal{A}$ to itself satisfying the Leibnitz rule:

$$X(fg) = X(f)g + fX(g)$$  \hspace{1cm} (2.1),

where $f$ and $g$ are elements of $\mathcal{A}$. $\operatorname{Der}$ offers a generalisation of the Lie algebra
of vector fields over a manifold. Contrary to the commutative case, Der does not have the structure of an \( A \)-module, that is if \( X \) is a derivation \( fX \) does not represent a derivation unless \( f \) is in the center of \( A \). A 1-form is defined as a linear map from Der to \( A \), the set of 1-forms is denoted by \( \Omega^1_{\text{Der}} \). A p-form is defined to be a skew symmetric multilinear map over \( Z(A) \), the center of \( A \), from \( \text{Der} \otimes Z(A) \cdots \otimes Z(A) \text{Der} \) to \( A \). The wedge product may be defined as in commutative geometry by appropriate antisymmetrisation (for details see [2]). For example, the wedge product of two 1-forms \( \omega \) and \( \omega' \) is given by:

\[
\omega \wedge \omega'(X_1, X_2) = \omega(X_1)\omega'(X_2) - \omega(X_2)\omega'(X_2).
\]  

(2.2)

Note that, in general, \( \omega \wedge \omega' \) and \( \omega' \wedge \omega \) are not simply related.

The exterior derivative of a p-form is a p+1-form defined, as in commutative geometry, by the formula:

\[
d\omega(X_1, \ldots, X_{p+1}) = \sum_i X_i(\omega(X_1, \ldots, X_{i-1}, \hat{X}_i, X_{i+1}, \ldots, X_{p+1})) \\
- \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).
\]  

(2.3)

Here, the notation \( \hat{X} \) means that \( X \) is omitted. The exterior derivative verifies \( d^2 = 0 \) and the graded Leibnitz rule:

\[
d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^{\deg \omega} \omega \wedge d\omega'.
\]  

(2.4)

The set \( \Omega_{\text{Der}} = \oplus \Omega^p_{\text{Der}} \) is thus a graded differential algebra. In fact, Dubois-Violette considers two graded differential algebras [2]: the other one, \( \Omega_{\text{Der}} \) is the smallest graded differential algebra in \( \Omega_{\text{Der}} \) containing \( A \). This distinction will not be important in the following. For further developments concerning this derivation based differential calculus see [21].
2.2. The differential calculus of Connes

The differential calculus of Connes [1,10] does not refer to derivations for the construction of differential forms. In one formulation it relies on a BRST like charge and the exterior derivative is a graded commutator, and in another more general formulation, it uses an analog of the Dirac operator.

The exterior derivative as a graded commutator

The starting point is the imbedding of the algebra $\mathcal{A}$ in the algebra $l(\mathcal{H})$ of operators on a graded Hilbert space $\mathcal{H}$. An odd operator, $Q$, is chosen in $l(\mathcal{H})$ such that its square is in the center of $l(\mathcal{H})$. The exterior derivative is taken to be the graded commutator with $Q$. It satisfies, as it should, the graded Leibnitz rule as well as the relation $d^2 = 0$. The resulting graded differential algebra is denoted by $\Omega^Q$.

Note that the differential calculus used by Witten relies on this construction. In his work, grading is provided by the ghost number and $Q$ is taken to be the BRST charge.

Differential forms with Dirac operators

The method consists in taking the quotient of the universal differential algebra, $\Omega^*$, by some differential ideal $J_D$. The differential ideal is stable under left and right multiplication by elements of $\mathcal{A}$ and under the universal exterior derivative, $\delta$. The construction of $J_D$ uses the following ingredients:

[1] A graded Hilbert space $\mathcal{H}$ such that there exists a representation, $\rho$, of the algebra $\mathcal{A}$ on the algebra of operators on $\mathcal{H}$, $l(\mathcal{H})$.

[2] An odd unbounded operator $\mathcal{D}$ in $l(\mathcal{H})$ such that $[\mathcal{D}, \rho(f)]$ is bounded for all elements $f$ of $\mathcal{A}$. In the commutative case the operator $\mathcal{D}$ is taken to be the Dirac operator on the manifold.
Consider the extension of $\rho$ on $\Omega^*$ defined by

$$\rho^*(f_0 \delta f_1 \ldots \delta f_p) = \rho(f_0)[\mathcal{D}, \rho(f_1)] \ldots [\mathcal{D}, \rho(f_p)].$$

Let $J$ be the kernel of $\rho^*$ and define the graded ideal $J_{gr}$ by $\oplus_n J \cap \Omega^n$ and the graded differential ideal $J_{\mathcal{D}}$ by $J_{gr} + \delta J_{gr}$. The quotient $\Omega^*_{\mathcal{D}} = \Omega^*/J_{\mathcal{D}}$ is the differential algebra advocated by Connes to generalise the de Rahm algebra on a compact Euclidean spin manifold. Note that one can identify $\Omega^p_{\mathcal{D}}$ with $\rho^*(\Omega^p)\big/\big((\rho^*(\delta J_{gr}^{p-1}))$. Suppose the $n^{th}$ eigenvalue of the operator $\mathcal{D}$ grows as $n^{\frac{d}{2}}$ for $n \to \infty$. One may define a scalar product in the Hilbert space completion of $\rho^*(\Omega^p)$ [10]:

$$<A, B> = \text{Tr}_\omega(A^* B |\mathcal{D}|^{-d}) = \lim_{t \to \infty} t^{\frac{d}{2}} \text{Tr}(A^* Be^{-t\mathcal{D}}). \quad (2.5)$$

The Hilbert space completion of $\Omega^p_{\mathcal{D}}$ may then be identified with the orthogonal to $\rho^*(\delta J_{gr}^{p-1})$.

Note that this construction encodes more information than the exterior differential calculus since the Dirac operator on a manifold depends on the metrical properties of the manifold.

### 3. Linear connections in commutative geometry

In this section the construction of the linear connection of commutative geometry is done in a manner which we will find appropriate for a non-commutative generalisation. In this section, we denote by $\Omega^p$ the set of de Rahm p-forms.

The covariant derivative is a linear map from $\Omega^1$ to $\Omega^1 \otimes_{\mathcal{C}(M)} \Omega^1$ verifying the following Leibnitz rule:

$$\nabla (f \omega) = df \otimes \omega + f \nabla \omega. \quad (3.1)$$

Since functions commute with forms we also have the following Leibnitz rule:

$$\nabla (\omega f) = \sigma(\omega \otimes df) + \nabla \omega f, \quad (3.2)$$
where \( \sigma \) is a permutation acting on \( \Omega^1 \otimes_{\mathcal{C}(M)} \Omega^1 \):

\[
\sigma(\omega \otimes \omega') = \omega' \otimes \omega.
\] (3.3)

The two Leibnitz rules (3.1) and (3.2) are equivalent. We will see that in non-commutative geometry this will not be the case.

Suppose \( M \) is parallelisable and let \( b^a \) be a basis of \( \Omega^1 \), then the covariant derivative of a 1-form is uniquely determined by the covariant derivatives of the basis \( b^a \). The linear connection is defined by \( \Gamma^a = -\nabla b^a \). Let \( m \) be the multiplication map from \( \Omega^1 \otimes_{\mathcal{C}(M)} \Omega^1 \) to \( \Omega^2 \). The operator \( T = d - m\nabla \) is a bimodule homomorphism from \( \Omega^1 \) to \( \Omega^2 \). That is, it verifies \( T(f\omega) = fT(\omega) \) and \( T(\omega f) = T(\omega)f \).

Torsion is defined by

\[
T^a \equiv T(b^a) = db^a + m\Gamma^a
\] (3.4)

The covariant derivative may be extended as a linear map from the tensor product over \( \mathcal{C}(M) \) of \( s \) copies of \( \Omega^1 \) to the tensor product of \( s + 1 \) copies of \( \Omega^1 \). This is done recurrently with the following extension of the Leibnitz rule:

\[
\nabla(\omega \otimes \omega') = \nabla(\omega) \otimes \omega' + \sigma_s(\omega \otimes \nabla \omega'),
\] (3.5)

here \( \omega \) is in \( \Omega^1 \), \( \omega' \) is in \( \otimes^{s-1}\Omega^1 \) and \( \sigma_s \) is given by

\[
\sigma_s = \sigma \otimes 1 \otimes 1 \ldots \otimes 1.
\] (3.6)

A more familiar way of writing this Leibnitz rule is provided by the covariant
derivative along the direction $X$,

$$\nabla_X = \iota_X \nabla.$$  \hfill (3.7)

The Leibnitz rule (3.5) reads

$$\nabla_X (\omega \otimes \omega') = \nabla_X \omega \otimes \omega' + \omega \nabla_X \omega'.$$  \hfill (3.8)

The reason we used equation (3.5) is that it is this one that will give rise to the non-commutative formulation.

Another extension of the covariant derivative as a linear map from $\Omega^* \otimes_{\mathcal{C}(M)} \Omega^1$ to itself, $D$, is given with the aid of the Leibnitz rule:

$$D(\nu \otimes \omega) = d\nu \otimes \omega + (-1)^{\text{deg} \nu} \nu \cdot \nabla \omega,$$  \hfill (3.9)

where the product $\cdot$ is defined by

$$\nu \cdot (\omega \otimes \omega') = (\nu \wedge \omega) \otimes \omega'.$$

Note that, when acting on $\Omega^1 \otimes_{\mathcal{C}(M)} \Omega^1$, $D$ and $\nabla$ are related by:

$$D = m \otimes 1 \nabla + T \otimes 1.$$  \hfill (3.10)

The operator $D^2$ is a module homomorphism: it verifies $D^2 (f \omega) = f D^2 \omega$, where $f$ is an arbitrary function and $\omega$ is an arbitrary 1-form. Curvature may be defined by the equation:

$$D^2 b^a = R^a_{\ b} \otimes b^b.$$  \hfill (3.11)

Suppose $M$ equipped with a metric and let $\theta^a$ be a moving basis. The connection form $\omega^a_{\ b}$ is defined by the equation $\nabla \theta^a = -\omega^a_{\ b} \otimes \theta^b$. The compatibility of the connection with the metric, $g$, is translated by the equation $\nabla g = 0$ which imposes the antisymmetry of $\omega^a_{\ b}$ in the indices $a$ and $b$. Cartan’s structure equations are given by (3.4) and (3.11), with $b^a$ replaced by $\theta^a$. 

8
4. Linear connections in non-commutative geometry

In this section we construct linear connections on non-commutative algebras. We first begin in the setting of the differential calculus of Dubois-Violette and then we propose a generalisation to other frameworks.

4.1. Covariant derivative in derivations based differential calculi

In this section we use the notation $\Omega^p$ for $\Omega^p_{\text{Der}}$.

A covariant derivative is defined as a linear map from $\Omega^1$ to $\Omega^1 \otimes A \Omega^1$ satisfying the two Leibnitz rules:

$$\nabla(f \omega) = df \otimes \omega + f \nabla \omega,$$

$$\nabla(\omega f) = \tau(\omega \otimes df) + \nabla f,$$  \hspace{1cm} (4.1)

here $f$ is an element of $A$, $\omega$ is a 1-form. It remains to define $\tau$ which generalises the permutation $\sigma$ of commutative geometry. When calculating $\nabla(f \omega g)$ in two different ways one discovers that $\tau$ must verify

$$\tau(f \omega \otimes \omega') = f \tau(\omega \otimes \omega'),$$  \hspace{1cm} (4.2)

so this rules out the permutation (3.3) as a possible candidate for $\tau$. Another way of expressing the permutation (3.3) is provided by the canonical imbedding, $i$ of $\Omega^2$ in $\Omega^1 \otimes_{\mathcal{C}(M)} \Omega^1$ and the multiplication map $m$:

$$\sigma = 1 - 2i \circ m.$$  \hspace{1cm} (4.3)

The imbedding $i$ admits a generalisation to the non-commutative differential cal-
culus of Dubois-Violette. It is given by

\[ i(\omega \wedge \omega')(X_1, X_2) = \frac{1}{2} \left( \omega(X_1)\omega'(X_2) - \omega(X_2)\omega'(X_1) \right) \]  

(4.5).

We shall note \( m \) the multiplication map from \( \Omega^1 \otimes_A \Omega^1 \) to \( \Omega^2 \) given by

\[ m(\omega \otimes \omega') = \omega \wedge \omega'. \]  

(4.6)

Note that the canonical imbedding, \( i \), is such that the diagram

\[
\begin{array}{ccc}
\Omega^1 \otimes_A \Omega^1 & \xrightarrow{m} & \Omega^2 \\
\downarrow m & & \downarrow i \\
\Omega^2 & \xleftarrow{\text{def}} & \Omega^1 \otimes_A \Omega^1 \\
\end{array}
\]  

(4.7)

is commutatif. Note also that \( i \circ m(f \omega \otimes \omega') = f i \circ m(\omega \otimes \omega') \). So we propose for \( \tau \) the following natural expression:

\[ \tau = 1 - 2i \circ m. \]  

(4.8)

When acting on derivations we have

\[ \tau(\omega \otimes \omega')(X_1, X_2) = \omega(X_2)\omega'(X_1). \]  

(4.9)

This ends up the definition of the covariant derivative for the non-commutative differential calculus based on derivations. Contrary to the commutative case, the two equations (4.1) and (4.2) with \( \tau \) given by (4.8) are not equivalent.

Let \( b^a \) be a set of 1-forms generating \( \Omega^1 \) as an \( A \)-bimodule. That is a minimal set of 1-forms such that an arbitrary 1-form may be written as a sum of terms of the form \( f b^a g \) with \( f \) and \( g \) elements of \( A \). The covariant derivative of a 1-form
is uniquely determined by $\nabla b^a = -\Gamma^a$ which defines the linear connection. This is so because using equations (4.1) and (4.2) one gets:

$$\nabla (f b^a g) = df \otimes b^a g - f\Gamma^a g + f\tau(b^a \otimes dg).$$  \hspace{1cm} (4.10)

Note that in formulations using only the left-module structure of $\Omega^1$, that is postulating only equation (4.1), the covariant derivative is, in general, determined by a bigger set of forms: one needs a set generating $\Omega^1$ as an $\mathcal{A}$-left module. Note also that contrary to the commutative case, a connection 1-form cannot, in general, be defined since $\Gamma^a$ cannot be expressed as $\omega^a_b \otimes b^b$.

Thanks to our two Leibnitz rules the extension of the covariant derivative may be immediately done to the tensor product over $\mathcal{A}$ of $s$ copies of $\Omega^1$. This is accomplished by the rule:

$$\nabla (\omega \otimes \omega') = \nabla \omega \otimes \omega' + \tau_s(\omega \otimes \nabla \omega'),$$  \hspace{1cm} (4.11)

where $\omega$ is in $\Omega^1$, $\omega'$ in $\otimes^{s-1} \mathcal{A} \Omega^1$ and $\tau_s$ is given by a relation analogous to (3.6):

$$\tau_s = \tau \otimes \overbrace{\underbrace{1 \otimes 1 \otimes \ldots \otimes 1}}^{s-1 \text{ times}}.$$  \hspace{1cm} (4.12)

In order to make the Leibnitz rules (4.1) and (4.2) more transparent one can define the covariant derivative along a derivation $X$ as in (3.7). The Leibnitz rules (4.1) and (4.1), for this case, yield:

$$\nabla_X(f\omega) = X(f)\omega + f\nabla_X\omega,$$  \hspace{1cm} (4.13)

$$\nabla_X(\omega f) = \omega X(f) + \nabla_X\omega f.$$  \hspace{1cm} (4.14)

These formulae are a natural requirement for the covariant derivative in the context of the differential calculus of Dubois-Violette. Similarly, the Leibnitz rule (4.11)
may be made more transparent in this case with the aid of $\nabla_X$:

$$\nabla_X(\omega \otimes \omega') = \nabla_X \omega \otimes \omega' + \omega \otimes \nabla_X \omega'.$$  \hfill (4.15)

The other extension, $D$, may be defined in a way similar to the commutative case as a linear map from $\Omega^* \otimes \mathcal{A} \Omega^1$ to itself verifying the Leibnitz rule:

$$D(\nu \otimes \omega) = d\nu \otimes \omega + (-1)^{\text{deg} \nu} \nu \cdot \nabla \omega.$$  \hfill (4.16)

Torsion is defined as in the previous section. The operator $T = d - m\nabla$ verifies $T(f\omega g) = fT(\omega)g$ for all $f$ and $g$ in $\mathcal{A}$ and $\omega$ in $\Omega^1$. Torsion is the 2-form defined by:

$$T^a = T(b^a) = db^a + m\Gamma^a.$$  \hfill (4.17)

Curvature may be defined with the operator $D^2$ which verifies $D^2(f\omega g) = f(D^2\omega)g$. In general $D^2b^a$ cannot be written in the form $R^a_{\ b} \otimes b^b$. So curvature is defined as the element of $\Omega^2 \otimes \mathcal{A} \Omega^1$ given by

$$R^a = D^2b^a.$$  \hfill (4.18)

### 4.2. Linear connections in formalisms without derivations

Derivations were important ingredients for the construction of linear connections in the setting of non-commutative geometry of Dubois-Violette. Apart from their use in the construction of $\Omega^2_{\mathcal{D}_{\text{er}}}$, they permitted us to define the linear map $\tau$ which appears in the Leibnitz rule using the right-module structure of the set of 1-forms. The generalisation of the construction of the previous section to differential calculi that do not use derivations is thus reduced to the definition of $\tau$. If we had a natural imbedding of $\Omega^2$ in $\Omega^1 \otimes \mathcal{A} \Omega^1$ we could use the expression (4.8) for $\tau$. 

12
A natural imbedding exists for the universal differential algebra, in this case
the tensor product $\Omega^1 \otimes_A \Omega^1$ is isomorphic to $\Omega^2$ so that $i \circ m = 1$, $\tau = -1$ and
the covariant derivative coincides with $\delta - T$, $\delta$ is the universal exterior derivative
and $T$ is a bimodule homomorphism defining torsion.

Consider the case of the Connes’ differential calculus with the Dirac operator.
The scalar product (2.5) provides us with a natural imbedding of $\Omega^2_D$ in $\Omega^1_D \otimes_A \Omega^1_D$.
In order to see this let $P$ be the projector on the orthogonal of $\rho^* (\delta J^1_{gr})$. An
element $\nu$ of $\Omega^2_D$ may be identified with an element $\omega = \sum a[[\mathcal{D}, b][\mathcal{D}, c]$ of $\rho^* (\Omega^2)$
verifying $\omega = P \omega$. $\omega$ is uniquely defined. The imbedding $i$ is thus given by:

$$i(\nu) = \sum a[[\mathcal{D}, b] \otimes [\mathcal{D}, c]. \quad (4.19)$$

When $\mathcal{A}$ is the commutative algebra of functions on a compact Euclidian spin
manifold and $\mathcal{D}$ is the corresponding Dirac operator the above definition of $\tau$
reduces to that of the permutation $\sigma$.

In other differential calculi, such as those on the quantum plane [23] or on
quantum groups [24], a natural imbedding may not exist. In this case, we look for
a generalisation of $\tau$ while keeping some minimal properties.

In the previous cases, $\tau$ was a bimodule homomorphism from $\Omega^1 \otimes_A \Omega^1$ to itself.
The consistency of the definition of the covariant derivative, however, imposes on
$\tau$ merely to be a left module homomorphism. Another requirement is for torsion
defined by (3.4) to be a bimodule homomorphism. This imposes the condition

$$m \circ \tau = -m. \quad (4.20)$$

So, the minimal requirements one may ask from $\tau$ are given by (4.3) and (4.20).
4.3. **An example: linear connections on matrix algebras.**

In this section we study the example of \( n \times n \) matrices to illustrate the constructions of the previous section. The derivations based differential calculus on \( M_n(C) \) has been studied in [8]. We will use this setting for the construction of the linear connection.

Derivations on \( M_n(C) \) are all inner. A basis of \( \mathcal{D}er \) as a vector space over \( C \) is given by:

\[
ed_i = ad_{\lambda_i}, \quad i = 1, \ldots, n^2 - 1,
\]

where the \( \lambda_i \) form a basis of selfadjoint traceless matrices. A convenient set of 1-forms is provided by the duals of \( e_i, \theta^i \) defined by

\[
\theta^i(e_i) = \delta^i_i.
\]

They generate \( \Omega^1 \) as a left \( M_n \) module. The \( \theta^i \) commute with matrices,

\[
f \theta^i = \theta^i f \quad \forall f \in M_n,
\]

because of the defining property (4.22). An arbitrary element of \( \Omega^1 \) may be written as \( \omega = \omega_i \theta^i \), where \( \omega_i \) are \( n \times n \) matrices. The exterior derivative of a matrix, \( f \), is given by

\[
df = e_i f \theta^i = [\lambda_i, f] \theta^i.
\]

We denote the linear connection \(-\nabla \theta^i\) by \( \Gamma^i \). As \( \Gamma^i \) belongs to \( \Omega^1 \otimes_{M_n} \Omega^1 \), it may be written as

\[
\Gamma^i = \Gamma^i_{jk} \theta^j \otimes \theta^k,
\]

where the \( \Gamma^i_{jk} \) are matrices. Note that when acting on \( \theta^i \otimes \theta^j \), \( \tau \) is a permutation:

\[
\tau(\theta^i \otimes \theta^j) = \theta^j \otimes \theta^i.
\]

The two Leibnitz rules (4.1) and (4.2) as well as the property (4.23) allow us to
calculate the covariant derivative of a 1-form, \( \omega = \omega_i \theta^i \), in two different ways:

\[
\nabla \omega = e_j (\omega_i) \theta^j \otimes \theta^i - \omega_i \Gamma^i_{jk} \theta^j \otimes \theta^k,
\]

\[
\nabla \omega = e_j (\omega_i) \theta^j \otimes \theta^i - \Gamma^i_{jk} \omega_j \theta^j \otimes \theta^k.
\]

In order for these two expressions to coincide for arbitrary \( \omega \) it is necessary and sufficient that the \( \Gamma^i_{jk} \) belong to the center of \( M_n \):

\[
\Gamma^i_{jk} \in C.
\] (4.27)

Note that the two Leibnitz rules have restricted the possible linear connections. Torsion may be calculated from \( T \theta^i \) which is given by

\[
T \theta^i = d \theta^i + m \Gamma^i = \frac{1}{2} \left( - C^i_{jk} + \Gamma^i_{[jk]} \right) \theta^j \wedge \theta^k,
\] (4.28)

where the \( C^i_{jk} \) are the structure constants of the derivations \( e_i \). Similarly, curvature may be calculated from \( D^2 \theta^i \) which is given by:

\[
D^2 \theta^i = \left( \frac{1}{2} \Gamma^i_{jm} C^j_{lm} - \Gamma^i_{lk} \Gamma^k_{mn} \right) \theta^l \wedge \theta^m \otimes \theta^n.
\] (4.29)

5. Conclusion

We proposed a definition for linear connections on non-commutative algebras. The covariant derivative was defined as a linear map from \( \Omega^1 \) to \( \Omega^1 \otimes_A \Omega^1 \) verifying two Leibnitz rules which exploit the bimodule structure of \( \Omega^1 \). The definition was reduced to that of a map from \( \Omega^1 \otimes_A \Omega^1 \) to itself, \( \tau \), which reduces to a permutation in the commutative case. A natural definition was proposed for this map in the context of the derivations based differential calculus of Dubois-Violette and the example of matrix geometry was examined. The Dixmier trace was used in order to define \( \tau \) in the setting proposed by Connes. It would be interesting to study more examples in order to illustrate the construction we proposed in the framework of the Connes’ differential calculus. We leave this to a future work.
The physical motivation of defining linear connections on non-commutative algebras is the formulation of gravitational theories on what might be more appropriate than manifolds for the description of the small scale structure of space-time. The definition of what would replace the metric is still lacking. One may define it, as in commutative geometry, as an element of $\Omega^1 \otimes_A \Omega^1$ or as mulilinear mapping from $\Omega^1 \otimes_A \Omega^1$ to $A$ [20] satisfying some symmetry requirements. The two definitions are not, in general, equivalent. It is not clear whether these two definitions are the only candidates for a metric. Then one has to impose a metricity-like condition in order to have what replaces the Levi-Civitá connection. Finally, it remains to construct a Ricci scalar from the previous ingredients. A different line of attack has been proposed in [22].

Another direction which is worth further investigations is the construction of the map $\tau$ for general non-commutative differential calculi from some defining properties; the ones we mentioned at the end of section 4.2 being some examples.

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