Several self-adaptive inertial projection algorithms for solving split variational inclusion problems

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Abstract This paper is to analyze the approximation solution of a split variational inclusion problem in the framework of infinite dimensional Hilbert spaces. For this purpose, several inertial hybrid and shrinking projection algorithms are proposed under the effect of self-adaptive stepsizes which does not require information of the norms of the given operators. Some strong convergence properties of the proposed algorithms are obtained under mild constraints. Finally, an experimental application is given to illustrate the performances of proposed methods by comparing existing results.

Keywords Self-adaptive stepsize · Projection algorithm · Inertial technique · Split variational inclusion problem

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1 Introduction

Inspired by the split variational inequality problem proposed by Censor et al. [1], Moudafi [2] introduced a more general form of this problem, that is, the split monotone variational inclusion problem (for short, SMVIP). It is worth noting that an important special case of the split monotone variation inclusion problem is the split variational inclusion problem (for short, SVIP), which is to find a zero of a maximal monotone mapping in one space, and the image of which under a given bounded linear transformation is a zero of

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another maximal monotone mapping in another space. As well as, the split variational inclusion problem
is also a generalized form of many problems, such as the split variational inequality problem, the split
minimization problem, the split equilibrium problem, the split saddle point problem and the split feasibility
problem; see, for instance, [2–6] and the references therein. As applications, these problems are also widely
applied to radiation therapy treatment planning, image recovery and signal recovery. For detail, we refer
to [7–9]. In SVIP, when the two spaces are the same and the given bounded linear operator is an identity
mapping, SVIP is equivalent to the well-known common solution problem, i.e., the common solution of two
variational inclusion problems. Naturally, common solution problems of other aspects can be obtained, such
as the variational inequality problem, the minimization problem and the equilibrium problem. In general,
the above common solution problems can be regarded as the distinguished convex feasibility problem.

In particular, finding the zero of a maximal monotone mapping is known as the variational inclusion
problem (for short, VIP), which is a special case of the SVIP. Furthermore, the resolvent mapping of the
maximal monotone mapping is considered to solve the approximate solution of VIP. With the help of this
resolvent mapping and the attention of a large number of scholars, the variational inclusion problem and the
split variational inclusion problem has obtained quite a few remarkable results; see, e.g., [10–14], etc. On
the other hand, based on the idea of the implicit discretization of a differential system of the second-order
in time, Alvarez and Attouch [15] introduced an inertial proximal point algorithm to approximate a solution
of the VIP. Under the effect of the inertial technique, the iterative sequence of SVIP and other problems
rapidly converges to the approximation solution of the corresponding problems, such as the split variational
inclusion problem [4–6], the split common fixed point problem [7, 16], the monotone inclusion problem
[17–19].

From the existing results of the split variational inclusion problem, we find that it is easy to get the weak
convergence property, and sometimes its strong convergence is proved in the case of other methods, such
as the viscosity method, the Halpern method, the Mann-type method, the hybrid steepest descent method,
and so on, for detail, see [3, 4, 6, 13]. Unfortunately, the stepsize sequences in these existing results often
depends on the norm of the bounded linear operator. Hence, the work of this paper can be summarized in
two aspects. The first one is to construct some new inertial iterative algorithms that converge strongly to a
solution of SVIP. For this purpose, we consider two projection methods in our algorithms, namely hybrid
projection [20] and shrinking projection [21]. The second one is to design a new stepsize sequence which
does not need prior knowledge of the bounded linear operator in our algorithms.

The rest of the article is outlined as follows. Section 2 introduces the split variational inclusion problem
and some preliminaries. Several new iterative algorithms and their convergence theorems for SVIP are
proposed in Section 3. Theoretical applications on other mathematical problems are given in Section 4.
Finally, in Section 5, the validity and authenticity of the convergence behavior of the proposed algorithms
are demonstrated by some applicable numerical examples.
2 State of problem and preliminaries

2.1 Split variational inclusion problem

Let $H_1$ and $H_2$ be Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be single-valued mappings, $A : H_1 \to H_2$ be a bounded linear operator. The split monotone variational inclusion problem is to find a point $x^* \in H_1$ such that

$$ 0 \in f_1(x^*) + B_1(x^*) \quad \text{and} \quad 0 \in f_2(Ax^*) + B_2(Ax^*). $$

(SMVIP)

When $f_1 \equiv 0$ and $f_2 \equiv 0$, SMVIP can be considered as the split variational inclusion problem, which is to find a point $x^* \in H_1$ such that

$$ 0 \in B_1(x^*) \quad \text{and} \quad 0 \in B_2(Ax^*). $$

(SVIP)

The solution set of SVIP is denoted by $\Omega$, i.e., $\Omega := \{x^* \in H_1 : 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*)\}$.

2.2 Preliminaries

To standardize, the notations $\to$ and $\rightharpoonup$ stand for strong convergence and weak convergence, respectively. The symbol $\text{Fix}(S)$ denotes the fixed point set of a mapping $S$. $\omega_n(x_n)$ represents the set of weak cluster point of a sequence $\{x_n\}$. Let $H$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let $B : H \to 2^H$ be a set-valued mapping with domain $\mathcal{D}(B) = \{x \in H : B(x) \neq \emptyset\}$ and graph $\mathcal{G}(B) = \{(x, w) \in H \times H : x \in \mathcal{D}(B), w \in B(x)\}$. Recall that a mapping $B : H \to 2^H$ is monotone if and only if $\langle x - y, w - v \rangle \geq 0$ for any $w \in B(x)$ and $v \in B(y)$. Further, a monotone mapping $B : H \to 2^H$ is maximal, that is, the graph $\mathcal{G}(B)$ is not properly contained in the graph of any other monotone mapping. In this case, $B$ is a maximal monotone mapping if and only if for any $(x, w) \in \mathcal{D}(B)$ and $(y, v) \in H \times H$, $(x - y, w - v) \geq 0$ implies $v \in B(y)$.

**Definition 2.1** The mapping $S : H \to H$ is said to be

(I) nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in H$;

(II) firmly nonexpansive if $\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y \rangle$, $\forall x, y \in H$.

**Remark 2.1** When $S$ is a firmly nonexpansive mapping, it is also nonexpansive and $I - S$ is also a firmly nonexpansive mapping.

**Lemma 2.1** [22, 23] The resolvent mapping $J^B_\beta$ of a maximal monotone mapping $B$ with $\beta > 0$ is defined as $J^B_\beta(x) = (I + \beta B)^{-1}(x)$, $\forall x \in H$. The following properties associated with $J^B_\beta$ hold.

(1) The mapping $J^B_\beta$ is a single-valued and firmly nonexpansive;

(2) The fixed point set of $J^B_\beta$ is equivalent to $B^{-1}(0) = \{x \in \mathcal{D}(B) : 0 \in B(x)\}$.

**Definition 2.2** The notation $P_C$ denotes the metric projection from $H$ onto $C$, that is, $P_Cx = \arg\min_{y \in C} \|x - y\|$, $\forall x \in H$. Naturally, we can know the following equivalent properties of $P_C$:

$$ \langle P_Cx - x, P_Cx - y \rangle \leq 0, \forall y \in C \iff \|y - P_Cx\|^2 + \|x - P_Cx\|^2 \leq \|x - y\|^2. $$
Algorithm 3.1 Given appropriate parameter sequences \( \{a_n\} \), \( \{\beta_n\} \) and \( \{\gamma_n\} \), for any \( x_0, x_1 \in H_1 \), the sequence \( \{x_n\} \) is constructed by the following iterative form.

\[
\begin{align*}
z_n &= x_n + a_n(x_n - x_{n-1}), \\
u_n &= J_{\beta_n}^B \left( z_n - \gamma_n A^* (I - J_{\beta_n}^B) Ax_n \right), \\
C_n &= \{ x \in H_1 : \| u_n - x \|^2 \leq \| z_n - x \|^2 - \theta_n \}, \\
Q_n &= \{ x \in H_1 : \langle x_n - x_1, x_n - x \rangle \leq 0 \}, \\
x_{n+1} &= R_{C_n \cap Q_n} x_1, n \geq 1,
\end{align*}
\]

where

\[
\theta_n = \gamma_n \left( 2 \| (I - J_{\beta_n}^B) Ax_n \|^2 - \gamma_n \| A^* (I - J_{\beta_n}^B) Ax_n \|^2 \right).
\]

Lemma 3.1 Assumed that (C1)-(C2) hold. For any \( \gamma_n > 0, \beta_n > 0 \) and set \( u_n = J_{\beta_n}^B (z_n - \gamma_n A^* (I - J_{\beta_n}^B) Ax_n) \).

Then,

\[
\| u_n - x \|^2 \leq \| z_n - x \|^2 - \gamma_n \left( 2 \| (I - J_{\beta_n}^B) Ax_n \|^2 - \gamma_n \| A^* (I - J_{\beta_n}^B) Ax_n \|^2 \right), \quad \forall x \in \Omega, \quad n \geq 1.
\]
Proof For any $x \in \Omega$, we have $x \in B_{1}^{-1}(0)$ and $Ax \in B_{2}^{-1}(0)$. According to the property of firmly nonexpansive mappings $J_{\beta_{n}}^{B_{1}}$, $J_{\beta_{n}}^{B_{2}}$ and $I - J_{\beta_{n}}^{B_{2}}$, we have

$$\|u_{n} - x\|^{2} = \|J_{\beta_{n}}^{B_{1}}(z_{n} - \gamma_{n}A^{\ast}(I - J_{\beta_{n}}^{B_{2}})Az_{n}) - x\|^{2} \leq \|z_{n} - \gamma_{n}A^{\ast}(I - J_{\beta_{n}}^{B_{2}})Az_{n} - x\|^{2}$$

$$= \|z_{n} - x\|^{2} + \gamma_{n}^{2}\|A^{\ast}(I - J_{\beta_{n}}^{B_{2}})Az_{n}\|^{2} - 2\gamma_{n}\langle z_{n} - x, A^{\ast}(I - J_{\beta_{n}}^{B_{2}})Az_{n} \rangle \leq \|z_{n} - x\|^{2} + \gamma_{n}^{2}\|A^{\ast}(I - J_{\beta_{n}}^{B_{2}})Az_{n}\|^{2} - 2\gamma_{n}\|A^{\ast}(I - J_{\beta_{n}}^{B_{2}})Az_{n} - (I - J_{\beta_{n}}^{B_{2}})Ax\|^{2}$$

$$= \|z_{n} - x\|^{2} - \gamma_{n}\left(2\|A^{\ast}(I - J_{\beta_{n}}^{B_{2}})Az_{n}\|^{2} - \gamma_{n}\|A^{\ast}(I - J_{\beta_{n}}^{B_{2}})Az_{n}\|^{2}\right).$$

Theorem 3.1 Assumed that (C1)-(C2) and (P1)-(P2) hold. If the solution set $\Omega$ is nonempty, the iterative sequence $\{x_{n}\}$ generated by Algorithm 3.1 converges strongly to $x^{\ast} = P_{\Omega}x_{1} \in \Omega$.

Proof Firstly, we show that $P_{C_{n} \cap Q_{n}}$ is well defined and $\Omega \subset C_{n} \cap Q_{n}$.

From the definition of $C_{n}$ and $Q_{n}$, it is obvious that the sets $C_{n}$, $Q_{n}$ are convex and closed, which implies that $P_{C_{n} \cap Q_{n}}$ is well defined. For any $p \in \Omega$, it follows from Lemma 3.1 that $\Omega \subset C_{n}$. In addition, $Q_{1} = \{x \in H_{1} : \langle x_{1} - x_{1} - x \rangle \leq 0\} = H_{1}$, then $\Omega \subset Q_{1}$. Further, suppose $\Omega \subset C_{n-1} \cap Q_{n-1}$, using the property of metric projection and $x_{n} = P_{C_{n-1} \cap Q_{n-1}}x_{1}$, we get

$$\langle x_{n} - x_{1} - x_{n} - x \rangle \leq 0, \forall x \in C_{n-1} \cap Q_{n-1};$$

$$\langle x_{n} - x_{1} - x_{n} - p \rangle \leq 0, \forall p \in \Omega.$$ 

This implies that $\Omega \subset Q_{n}$. Hence, $\Omega \subset C_{n} \cap Q_{n}, n \geq 1$.

Afterwards, we show that iterative sequence $\{x_{n}\}$ is bounded and $\|x_{n+1} - x_{n}\| \to 0$ as $n \to \infty$.

Since $\Omega$ is a nonempty closed convex set, there exists a point $x^{\ast} = P_{\Omega}x_{1} \in \Omega$. Combining $x_{n+1} = P_{C_{n} \cap Q_{n}}x_{1}$ with $\Omega \subset C_{n} \cap Q_{n}$, we have $\|x_{1} - x_{n+1}\| \leq \|x_{1} - x^{\ast}\|$. Accordingly, the sequence $\{|x_{1} - x_{n}|\}$ is bounded, i.e., the sequence $\{x_{n}\}$ is bounded. From the definition of $Q_{n}$ and $x_{n+1} = P_{C_{n} \cap Q_{n}}x_{1} \in Q_{n}$, we get $x_{n} = P_{Q_{n}}x_{1}$ and $\|x_{1} - x_{n}\| \leq \|x_{1} - x_{n+1}\|$. These indicate that $\lim_{n \to \infty} \|x_{1} - x_{n}\|$ exists. Further, it follows from the property of metric projection $P_{Q_{n}}$ that

$$\|x_{n} - x_{n+1}\|^{2} \leq \|x_{1} - x_{n+1}\|^{2} - \|x_{1} - x_{n}\|^{2}.$$ 

This implies $\lim_{n \to \infty} \|x_{n} - x_{n+1}\| = 0$.

Lastly, we prove that the sequence $\{x_{n}\}$ converges strongly to $x^{\ast} = P_{\Omega}x_{1}$.

From the boundedness of $\{x_{n}\}$, there exists a subsequence $\{x_{n_{k}}\}$ of $\{x_{n}\}$ converges weakly to $q$, for any $q \in \omega_{\infty}(x_{n})$. Furthermore, $\|z_{n} - x_{n}\| = \alpha_{n}\|x_{n} - x_{n-1}\| \to 0$, as $n \to \infty$. This implies that $\{z_{n}\}$ is bounded.
and \( z_{n_j} \to q \). From (P2) and Algorithm 3.1, we have \( \|u_n - x_{n+1}\|^2 \leq \|z_n - x_{n+1}\|^2 - \theta_n^2 \leq \|z_n - x_{n+1}\|^2 \). In addition,
\[
\|u_n - z_n\| \leq \|u_n - x_n\| + \|x_n - z_n\| \\
\leq \|u_n - x_{n+1}\| + \|x_n - x_{n+1}\| + \|x_n - z_n\| \\
\leq 2\|z_n - x_n\| + 2\|x_n - x_{n+1}\| \to 0, \quad n \to \infty.
\]

Hence, the sequence \( \{u_n\} \) is bounded. Using Lemma 3.1, for any \( p \in \Omega \),
\[
\theta_n \leq \|z_n - p\|^2 - \|u_n - p\|^2 \\
\leq (\|z_n - p\| - \|u_n - p\|)(\|z_n - p\| + \|u_n - p\|) \\
\leq \|z_n - u_n\|(\|z_n - z\| + \|u_n - p\|) \to 0, \quad n \to \infty.
\]

If \( Az_n \notin B_{2^{-1}}^2 \), from the definition of \( \theta_n \), \( \lim_{n \to \infty} \|(I - J_{B_{2^{-1}}}) Az_n\| = 0 \). On the other hand, from the definition of \( u_n \) and the firmly nonexpansive property of \( J_{B_{2^{-1}}} \), we obtain
\[
\|u_n - J_{B_{2^{-1}}}^2 z_n\| \leq \|\gamma_n A^* (I - J_{B_{2^{-1}}}^2) Az_n\| \leq \gamma_n \|A\| \|(I - J_{B_{2^{-1}}}^2) Az_n\| \to 0, \quad n \to \infty.
\]

Therefore, we also have \( \lim_{n \to \infty} \|z_n - J_{B_{2^{-1}}}^2 z_n\| = 0 \). Since \( A \) is a bounded linear operator, we get \( Az_n \to Aq \).

By Remark 2.1 and Lemma 2.2, it follows that \( q \in \text{Fix}(J_{B_{2^{-1}}}^2) \) and \( Aq \in \text{Fix}(J_{B_{2^{-1}}}^2) \), that is, \( q \in \Omega \). Meanwhile, if \( Az_n \in B_{2^{-1}}^2 \), we can also get the same result. In summary, we have \( \omega_n(x_n) \subset \Omega \) and \( \|x_n - x_1\| \leq \|x^* - x_1\| \).

By virtue of Lemma 2.3, we obtain that \( \{x_n\} \) converges strongly to \( x^* = P_{\Omega} x_1 \).

### 3.2 The strong convergence of inertial shrinking projection algorithms

**Algorithm 3.2** Given appropriate parameter sequences \( \{\alpha_n\} \), \( \{\beta_n\} \) and \( \{\gamma_n\} \), for any \( x_0, x_1 \in H_1 \), the sequence \( \{x_n\} \) is constructed by the following iterative process.
\[
\begin{align*}
z_n &= x_n + \alpha_n (x_n - x_{n-1}) \\
u_n &= J_{\beta_n}^1 \left(z_n - \gamma_n A^* (I - J_{B_{2^{-1}}}^2) Az_n\right) \\
x_{n+1} &= P_{C_{n+1}} x_1, n \geq 1,
\end{align*}
\]

where
\[
C_{n+1} = \left\{ x \in C_n : \|u_n - x\|^2 \leq \|z_n - x\|^2 - \gamma_n \left(2\|(I - J_{B_{2^{-1}}}^2) Az_n\|^2 - \gamma_n \|A^* (I - J_{B_{2^{-1}}}^2) Az_n\|^2\right) \right\}.
\]

**Algorithm 3.3** Given appropriate parameter sequences \( \{\alpha_n\} \), \( \{\beta_n\} \) and \( \{\gamma_n\} \), for any \( x_0, x_1 \in H_1 \), the sequence \( \{x_n\} \) is constructed by the following iterative process.
\[
\begin{align*}
z_n &= x_n + \alpha_n (x_n - x_{n-1}) \\
u_n &= J_{\beta_n}^1 \left(z_n - \gamma_n A^* (I - J_{B_{2^{-1}}}^2) Az_n\right) \\
x_{n+1} &= P_{C_{n+1}} x_n, n \geq 1,
\end{align*}
\]
where
\[ C_{n+1} = \left\{ x \in C_n : \|u_n - x\|^2 \leq \|z_n - x\|^2 - \gamma_n \left( 2\| (I - J_{\beta_n}^B)Az_n \| \right)^2 \right\} \]

**Theorem 3.2** Assumed that (C1)-(C2) and (P1)-(P2) hold. If the solution set Ω is nonempty, the iterative sequence \( \{x_n\} \) generated by Algorithm 3.2 converges strongly to \( x^* = P_\Omega x_1 \).

**Proof** Firstly, it is obvious that the half space \( C_n \) \( (n \geq 1) \) is convex and closed and \( P_{C_n} \) is well defined. By Lemma 3.1, we can easily get that the solution set \( \Omega \subset C_n \). Using \( x_n = P_{C_n} x_1 \), \( x_{n+1} = P_{C_{n+1}} x_1 \) and \( C_{n+1} \subset C_n \), we have \( \|x_n - x_1\| \leq \|x_{n+1} - x_1\| \), which implies that \( \{\|x_n - x_1\|\} \) is nondecreasing. Furthermore, \( \|x_n - x_1\| \leq \|p - x_1\| \), for any \( p \in \Omega \), that is, \( \{x_n\} \) is bounded. These imply that \( \lim_{n \to \infty} \|x_n - x_1\| \) exists. According to the proof in Theorem 3.1, we also prove that the sequence \( \{x_n\} \) converges strongly to \( x^* = P_\Omega x_1 \). \( \Box \)

**Theorem 3.3** Assumed that (C1)-(C2) and (P1)-(P2) hold. If the solution set Ω is nonempty, the iterative sequence \( \{x_n\} \) generated by Algorithm 3.3 converges strongly to \( x^* = P_\Omega x_1 \).

**Proof** Similarly, we obtain that \( C_n \) \( (n \geq 1) \) is convex and closed, \( P_{C_n} \) is well defined and \( \Omega \subset C_n \). By \( x_n = P_{C_n} x_1 \), \( x_{n+1} = P_{C_{n+1}} x_1 \) and \( C_{n+1} \subset C_n \), we have \( \|x_n - x_1\| \leq \|x_{n+1} - x_1\| \) and \( \|x_n - x_1\| \leq \|p - x_1\| \), \( \forall p \in \Omega \). Using the proof in Theorems 3.1 and 3.2, we have that \( \{x_n\} \) converges strongly to \( x^* = P_\Omega x_1 \). \( \Box \)

**4 Theoretical applications**

In this section, we give several interesting special cases of the split variation inclusion problem (SVIP). At the same time, Algorithms 3.1, 3.2 and 3.3 are applied to these problems. Further, the same strong convergence property in Theorems 3.1, 3.2 and 3.3 are proved.

4.1 Split variational inequality problem

Let \( C \) and \( Q \) be nonempty closed convex subsets of Hilbert spaces \( H_1 \) and \( H_2 \), respectively. Let \( F : H_1 \to H_1 \) and \( G : H_2 \to H_2 \) be given operators, \( A : H_1 \to H_2 \) be a bounded linear operator. The split variational inequality problem is to find a point \( x^* \in C \) such that
\[ \langle F(x^*), x - x^* \rangle \geq 0, \ \forall x \in C \ \text{and} \ \langle G(Ax^*), y - Ax^* \rangle \geq 0, \ \forall y \in Q. \]

Especially, when \( H_1 = H_2, F = G \) and \( A = I \), the split variational inequality problem is transformed into the classical variational inequality problem which is to find a point \( x^* \in C \) such that \( \langle F(x^*), x - x^* \rangle \geq 0, \ \forall x \in C. \)

Hence, the solution set of the variational inequality problem is represented by \( VI(F,C) \). Then, the split variational inequality problem is formulated as

\[ \text{find } x^* \in C \text{ such that } x^* \in VI(F,C) \ \text{and} \ Ax^* \in VI(G,Q). \]

To solve the variational inequality problem, the normal cone \( N_C(x) \) of \( C \) at a point \( x \in C \) is defined as follows:
\[ N_C(x) = \{ z \in H : \langle z, v - x \rangle \leq 0, \ \forall v \in C \}. \]
Further, the set valued mapping $S_F$ related to the normal cone $N_C(x)$ is defined by

$$S_F(x) := \begin{cases} F(x) + N_C(x), & x \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In the sense, if $F$ is a $\alpha$-inverse strongly monotone operator (i.e., for any $x, z \in C$, $\langle F(x) - F(z), x - z \rangle \geq \alpha \| F(x) - F(z) \|^2$), then $S_F$ is a maximal monotone mapping. More importantly, $x \in VI(F, C)$ if and only if $0 \in S_F(x)$. Consequently, let $F$ and $G$ be $\alpha$-inverse strongly monotone operators. The set valued mappings $S_F$ and $S_G$ are associated with $F$ and $G$, respectively. In SVIP, when $B_1 = S_F$ and $B_2 = S_G$, we obtain the above split variational inequality problem.

### 4.2 Split saddle point problem

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. A bifunction $L : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{-\infty, \infty\}$ is convex-concave if and only if $L(x, \cdot)$ is convex for any $x \in \mathcal{X}$ and $L(\cdot, y)$ is concave for any $y \in \mathcal{Y}$. The operator $T_L$ is defined as follows:

$$T_L(x, y) = (\partial_1 L(x, y), \partial_2(-L)(x, y)),$$

where $\partial_1$ is the subdifferential of $L$ with respect to $x$ and $\partial_2$ is the subdifferential of $-L$ with respect to $y$. It is worth noting that $T_L$ is maximal monotone if and only if $L$ is closed and proper, for detail, see, [25]. Naturally, the zeros of $T_L$ coincide with the saddle points of $L$. Therefore, let $\mathcal{X}_i(i = 1, 2)$, $\mathcal{Y}_i(i = 1, 2)$ be Hilbert spaces. Let $A : \mathcal{X}_1 \times \mathcal{Y}_1 \to \mathcal{X}_2 \times \mathcal{Y}_2$ be a bounded linear operator with adjoint operator $A^\ast$. Let $L_1$ and $L_2$ be closed proper convex-concave bifunctions. Then, the split saddle point problem is to find a point $(x^\ast, y^\ast) \in \mathcal{X}_1 \times \mathcal{Y}_1$ such that

$$(x^\ast, y^\ast) \in \text{argminmax}_{(x, y) \in \mathcal{X}_1 \times \mathcal{Y}_1} L_1(x, y) \text{ and } A(x^\ast, y^\ast) \in \text{argminmax}_{(z, w) \in \mathcal{X}_2 \times \mathcal{Y}_2} L_2(z, w).$$

In other words, when $H_i = \mathcal{X}_i \times \mathcal{Y}_i (i = 1, 2)$, $B_i = T_{L_i} (i = 1, 2)$, the split variational inclusion problem is reduced to the split saddle point problem.

### 4.3 Split minimization problem

Let $H_1$ and $H_2$ be Hilbert spaces. Let $\phi : H_1 \to \mathbb{R}$ and $\psi : H_2 \to \mathbb{R}$ be lower semi-continuous convex functions, $A : H_1 \to H_2$ be a bounded linear operator. The split minimization problem is to find $x^\ast \in H_1$ such that

$$x^\ast \in \text{argmin}_{x \in H_1} \phi(x) \text{ and } Ax^\ast \in \text{argmin}_{y \in H_2} \psi(y).$$

As we all know, $x^\ast \in \text{argmin}_{x \in H_1} \phi(x)$ if and only if $0 \in \partial \phi(x^\ast)$, where $\partial \phi$ is the subdifferential of $\phi$ defined by $\partial \phi(x^\ast) := \{ \hat{x} \in H_1 : \phi(x^\ast) + \langle z - x^\ast, \hat{x} \rangle \leq \phi(z), \forall z \in H_1 \}$. Recall that the proximal operator $\text{prox}_\phi$ of $\phi$ is as follows:

$$\text{prox}_\phi x = \text{argmin}_{z \in H_1} \left\{ \phi(z) + \frac{1}{2\gamma} \| z - x \|^2 \right\}, \forall \gamma > 0.$$

It is very important that $\text{prox}_\phi(x) = (I + \gamma \partial \phi)^{-1}(x) = \frac{1}{\gamma} \partial \phi(x)$. In addition, $\partial \phi$ is a maximal monotone mapping and $\text{prox}_\phi$ is a firmly nonexpansive mapping. In view of this, when $B_1 = \partial \phi$ and $B_2 = \partial \psi$ in (SVIP), the split variational inclusion problem is transformed into the split minimization problem.
Remark 4.1 Through the above results, the split variational inclusion problem is transformed into other problems, such as the split variational inequality problem, the split saddle point problem and the split minimization problem. Using the same algorithms and techniques in Theorems 3.1, 3.2 and 3.3, the strong convergence property of these problems are obtained under the above corresponding conditions in Subsections 4.1, 4.2 and 4.3.

5 Numerical example

In this section, a numerical example is provided to illustrate the effectiveness and realization of convergence behavior of Algorithms 3.1, 3.2 and 3.3. All codes were written in Matlab R2018b, and ran on a Lenovo ideapad 720S with 1.6 GHz Intel Core i5 processor and 8GB of RAM. Our results compare the existing conclusions below. Firstly, let $H_1$ and $H_2$ be Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator $A^*$. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two set-valued maximal monotone mappings. Many existing conclusions on the split variational inclusion problem have been proven in such an environment as follows.

Theorem 5.1 (Byrne et al. [3, Algorithm 4.4]) For any initial point $x_1 \in H_1$, $\delta_n \in (0, 1)$ and $\beta > 0$, the iterative sequence $\{x_n\}$ is generated by the following iterative scheme

$$x_{n+1} = \delta_n x_1 + (1 - \delta_n)J_{\beta}^{B_1} \left( x_n - \gamma A^*(I - J_{\beta}^{B_2})Ax_n \right), \quad n \geq 1.$$  

If the sequence $\{\delta_n\}$ satisfies $\lim_{n \to \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$, $0 < \gamma < 2/\|A^*A\|$, then the iterative sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$.

Theorem 5.2 (Long et al. [4, Algorithm (49)]) For any initial points $x_0, x_1 \in H_1$ and $\beta > 0$, the iterative sequence $\{x_n\}$ is generated by the following iterative scheme.

$$
\begin{align*}
z_n &= x_n + \alpha_n (x_n - x_{n-1}), \\
u_n &= J_{\beta}^{B_1} \left( z_n - \gamma_n A^*(I - J_{\beta}^{B_2})Az_n \right), \\
x_{n+1} &= \delta_n f(x_n) + (1 - \delta_n)u_n, \quad n \geq 1,
\end{align*}
$$

where $f : H_1 \to H_1$ is a contraction mapping with coefficient $k \in [0, 1)$. $\{\delta_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \to \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$, $0 < a \leq \gamma_n \leq b < 1/\|A\|^2$, $0 \leq \alpha_n \leq \alpha$ and $\lim_{n \to \infty} \alpha_n / \|\delta_n - \delta_{n-1}\| = 0$. The iterative sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$.

Theorem 5.3 (Anh et al. [6, Algorithm (4)]) For any initial points $x_0, x_1 \in H_1$ and $\beta > 0$, the iterative sequence $\{x_n\}$ is generated by the following iterative scheme.

$$
\begin{align*}
z_n &= x_n + \alpha_n (x_n - x_{n-1}), \\
u_n &= J_{\beta}^{B_1} \left( z_n - \gamma_n A^*(I - J_{\beta}^{B_2})Az_n \right), \\
x_{n+1} &= (1 - \delta_n - \theta_n)x_n + \delta_n u_n, \quad n \geq 1,
\end{align*}
$$
where \( \{\theta_n\} \) is a sequence in \((0, 1)\) with \( \lim_{n \to \infty} \theta_n = 0 \), \( \sum_{n=1}^{\infty} \theta_n = \infty \), \( 0 < a \leq \gamma_n \leq b < 1/\|A\|^2 \), \( 0 \leq \alpha_n < \alpha \) for some \( \alpha > 0 \) with \( \lim_{n \to \infty} \frac{\alpha\|x_n - x_{n-1}\|}{\theta_n} = 0 \), \( 0 < c < \delta_n < d < 1 - \theta_n \). The iterative sequence \( \{x_n\} \) converges strongly to a point \( x^* \in \Omega \).

**Example 5.1** Assume that \( A, A_1, A_2 : \mathbb{R}^m \to \mathbb{R}^m \) are created from a normal distribution with mean zero and unit variance. Let \( B_1 : \mathbb{R}^m \to \mathbb{R}^m \) and \( B_2 : \mathbb{R}^m \to \mathbb{R}^m \) be defined by \( B_1(x) = A_1^*A_1x \) and \( B_2(y) = A_2^*A_2y \), respectively. Consider the problem of finding a point \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)^T \in \mathbb{R}^m \) such that \( B_1(\bar{x}) = (0, \ldots, 0)^T \) and \( B_2(A\bar{x}) = (0, \ldots, 0)^T \). It is easy to see that the minimum norm solution of the mentioned above problem is \( x^* = (0, \ldots, 0)^T \). Our parameter settings are as follows. In our algorithms 3.1–3.3, set \( \alpha_n = 0.5 \), \( \beta_n = 1 \) and \( \sigma_n = 1.5 \). Take \( \beta = 1 \), \( \delta_n = \frac{1}{n+1} \), \( \alpha_n = \frac{1}{\|A\|^2} \), \( \gamma_n = \frac{1}{\|A\|^2} \), and \( f(x) = 0.8x \) in Long et al. [4, Algorithm (49)]. In Anh et al. [6, Algorithm (4)], choose \( \beta = 1 \), \( \theta_n = \frac{1}{n+1} \), \( \delta_n = 0.2(1 - \theta_n) \), \( \alpha = 0.5 \), \( \alpha_n = \frac{1}{\|A\|^2} \), and \( \gamma_n = \frac{1}{\|A\|^2} \). We use \( E_n = \|x_n - x^*\| \) to measure the iteration error of all algorithms. The stopping condition is \( E_n < \varepsilon \) or the maximum number of iterations is 300 times. First, choose \( \varepsilon = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5} \). We test the convergence behavior of all algorithms under different stopping conditions. The numerical results are shown in Table 1 and Figure 1. Second, Figure 2 describes the numerical behavior of all algorithms in different dimensions under the same stopping criterion \( \varepsilon = 10^{-4} \).

**Table 1** The number of termination iterations of all algorithms under different stopping criteria

| The Algorithms   | \( \varepsilon = 10^{-2} \) | \( \varepsilon = 10^{-3} \) | \( \varepsilon = 10^{-4} \) | \( \varepsilon = 10^{-5} \) |
|------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
|                  | iter. | iter. | iter. | iter. |
| Our Alg. 3.3     | 13    | 18    | 23    | 28    |
| Our Alg. 3.2     | 248   | 300   | 300   | 300   |
| Our Alg. 3.1     | 300   | 300   | 300   | 300   |
| Byrne et al. Alg. 4.4 | 300   | 300   | 300   | 300   |
| Long et al. Alg. (49) | 26    | 38    | 48    | 54    |
| Anh et al. Alg. (4) | 37    | 80    | 131   | 185   |
Several self-adaptive inertial projection algorithms

Fig. 1 Numerical behavior of all algorithms under different stopping criteria

It can be seen from the above results that our Algorithms 3.1 and 3.2 are efficient and robust. These results are independent of the selection of initial values and dimensions. Moreover, note that our proposed Algorithms 3.1 and 3.2 are oscillating due to the dual reasons of inertial and projection, but the suggested Algorithm 3.3 performs very well.
6 Conclusion

In this paper, our innovation are twofold. One is to provide a self-adaptive step size selection which does not require the norm of the bounded linear operators. The other is to propose two types of projection algorithms (i.e., hybrid projection algorithms and shrinking projection algorithms), which combine inertial technique with the proposed self-adaptive step size. Under mild constraints, the corresponding strong convergence theorems of SVIP are obtained in the framework of Hilbert spaces. At the same time, our results are also extended to the split variational inequality problem, the split saddle point problem and the split minimization problem. In terms of numerical experiments, the effectiveness of our proposed algorithms are showed by comparing with some existing results.

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