Motion of a test body in the presence of an external scalar field which respects the weak equivalence principle

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Abstract

It is shown that the main contribution to the rotational curve of a spiral galaxy may be due essentially to the interaction, in the general relativistic spacetime, of the galactic matter with a very light long range scalar field which respects the weak equivalence principle. The comparison of the theoretical results with 23 spiral galaxy rotation curves shows a good agreement between our proposal and observations.

1 Introduction

According to newtonian dynamics and Newton’s inverse-square force law of gravitation, the circular velocity around an isolated body should decrease with the distance, \( r \), to the centre like \( \frac{1}{\sqrt{r}} \). However, observations show that the rotation curves of many spiral galaxies flatten at large distances. The current interpretation consists to invoke the existence of dark matter though its nature and distribution in space are not yet clearly defined [1].

Alternative proposals, which provide explanations for observed galactic rotation curves without the need of dark matter, have been developed by several authors. In that line of thinking, the Modified Newtonian Dynamics (MOND) proposed by
Milgrom [2] introduces a "universal" constant $a_0$ of the dimension of an acceleration such that newtonian dynamics breaks down at acceleration much smaller than $a_0$ (or equivalently Newton’s law of gravitation fails when the magnitude of the potential gradient is much smaller than $a_0$). This acceleration constant turns out to be of the order of $cH_0$ suggesting a cosmological link [3]. However, actually the equality $a_0 = cH_0$ has no theoretical basis in the framework of MOND and it turns out that at the cosmological level MOND fails to satisfy the cosmological principle [4].

Besides, while having some successes at the scale of galaxies [5, 6, 7] and clusters of galaxies [8, 9], the MOND fails in the laboratory\(^1\) [10]. Moreover, at the scale of clusters of galaxies, Gerbal et al. have found that a certain amount of dark matter is required even with MOND [11, 12].

In the framework of the conformal Weyl gravity, Mannheim and Kazanas obtained a complete, exact exterior solution for a static, spherically symmetric source [13]. In addition to the exterior Schwarzschild solution, their solution contains an extra gravitational potential term $\gamma r$ which grows linearly with the distance $r$ to the centre. A cosmological link is also made by these authors who noted intriguingly that their parameter $\gamma$ is roughly the value of the inverse of the Hubble length, $\frac{c}{H_0}$. Their solution is able to interpolate between those of Robertson-Walker and Schwarzschild in a continuous and smooth manner by exploiting the conformal structure of the conformal Weyl gravity. This solution has been applied successfully by Mannheim to the galactic rotation curves of spiral galaxies [14]. However, by matching the exterior solution to an interior one that satisfies the weak energy condition and a regularity condition at the centre, Perlick and Xu [15] have shown that this leads to contradiction of Mannheim and Kazanas’s suggestion. They conclude that the conformal Weyl gravity is not able to give a viable model of the solar system. Be-

\(^1\)This conclusion is rejected by M. Milgrom, since the experiment has been performed on earth where the gravitational pulling is much greater than $a_0$.  

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sides, it seems that the cosmological models derived from the conformal Weyl gravity fails to fulfill simultaneously the observational constraints on present cosmological parameters and on primordial light element abundances [16].

Other authors have suggested that Newton’s law of gravity, which describes so well the motion of moons and planets in the solar system, may break down over distances comparable to the size of a galaxy. So, alternative force laws have been proposed [17, 18, 19]. However, one should bear in mind that newtonian mechanics is the only appropriate approximation of general relativity in the weak field and low velocity limit (see the criticism of D. Lindley, [31]).

In this paper, we propose a new dark matter scheme in the framework of Einstein's general relativity and its weak field and low velocity limit, newtonian gravity. Our new dark matter candidate is a long range neutral massive scalar field directly coupled to matter, unlike the Brans-Dicke theory in which the scalar field does not exert any direct influence on matter (its only role is that of participant in the field equations that determine the geometry of spacetime) [20]. In addition, one requires that the scalar field under consideration, $\phi$, respects the weak equivalence principle. Let us notice that, if one takes into account the gravitational coupling of $\phi$, with the ordinary matter, then the $g_{\alpha\beta}$ will depend not only on the $x^\mu$ but also on $\phi$, that is $g_{\alpha\beta} = g_{\alpha\beta}(x^\mu, \phi)$. The plan of this paper is as follows. In section 2, we establish the equation of motion of a test body in the presence of the light scalar field, $\phi$. In section 3, we determine the potential $V_\phi$ that couples the ordinary matter to the scalar field. In section 4, the results of both sections 2 and 3 are combined and applied to the rotational curves of spiral galaxies. Finally, in section 5, twenty three rotation curves are confronted with the results of the previous sections.
2 The equation of motion in the presence of a long range scalar field

In the absence of any external field, the equation of motion of a test body writes in general relativity

\[(u^\alpha \nabla_\alpha) u^\mu = 0\]  

(1)

where \( u_\alpha = \frac{dx_\alpha}{ds} \) denotes the velocity four-vector of the test body and \( ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \) is the metric and \( \nabla_\alpha \) is the covariant derivatives \( (\alpha, \beta = 0, 1, 2, 3) \).

In the same manner as in the presence of an electromagnetic field, in the presence of a scalar field, \( \phi \), a force term enters in the right-hand side of equation (1). This force term would be \textit{a priori} of the simple form \(-\frac{1}{c^2} \frac{\partial V_\phi}{\partial x^\mu} \) involved by the lagrangian \( L = -\frac{1}{2} g_{\alpha\beta} u^\alpha u^\beta - \frac{V_\phi}{c^2} \), where the potential \( V_\phi \) depends on \( \phi \) \( (V_\phi = 0 \text{ if } \phi = 0) \) and is assumed to be the same for all matter at any point of spacetime to ensure obedience to the weak equivalence principle. Thus one would write

\[(u^\alpha \nabla_\alpha) u^\mu = -\frac{1}{c^2} \frac{\partial V_\phi}{\partial x^\mu}.\]  

(2)

However, equation (2) is not satisfactory, since the unitarity condition \( u^\mu u_\mu = 1 \) implies

\[u^\mu (u^\alpha \nabla_\alpha) u_\mu = 0.\]  

(3)

In order to satisfy relation (3), there should be at least the additional term \( \frac{1}{c^2} \frac{dV_\phi}{ds} u_\mu \) to the right-hand side of equation (2). Finally, the correct equation of motion writes

\[(u^\alpha \nabla_\alpha) u^\mu = -\frac{1}{c^2} \frac{\partial V_\phi}{\partial x^\mu} + \frac{1}{c^2} \frac{dV_\phi}{ds} u_\mu\]  

(4)

or making appear explicitly the Christoffel symbols,

\[\frac{d u^\mu}{ds} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = -\frac{1}{c^2} \partial^\mu V_\phi + \frac{1}{c^2} \frac{dV_\phi}{ds} u^\mu.\]  

(5)
One may derive equation (4, 5) from the lagrangian

$$L = \frac{-e^{-V_\phi/c^2}}{2}(g_{\alpha\beta}u^\alpha u^\beta + 1).$$

(6)

### 2.1 Small speeds and weak fields approximation

For small speeds and weak gravitational fields, with respect to a galilean referential, it comes $\Gamma^{\alpha}_{00} \approx -\frac{1}{2} \partial^\alpha g_{00}$. Now, $g_{\alpha\beta}(x^\mu, \phi) \approx g_{\alpha\beta}(x^\mu, 0) + \left(\frac{\partial g_{\alpha\beta}}{\partial \phi}\right)_{\phi=0} \phi$ since we are interested by the weak $\phi$-field approximation. One may also write $g_{\alpha\beta}(x^\mu, \phi) \approx g_{\alpha\beta}(x^\mu, 0) + \left(\frac{\partial g_{\alpha\beta}}{\partial \phi}\right)_{\phi=0} \phi$ since we are interested by the weak $\phi$-field approximation. For $V_{\phi}$ is proportional to $\phi$ at the first order. Moreover, for small speeds and weak gravitational fields $g_{00}(x^\mu, 0) \approx 1 + 2 \frac{V_{N}}{c^2}$ whence $g_{00}(x^\mu, \phi) \approx 1 + 2 \frac{V_{N}+(1+f)V_{\phi}}{c^2}$, where $V_{N}$ denotes the newtonian gravitational potential and we have set $\left(\frac{\partial g_{00}}{\partial \phi}\right)_{\phi=0} = \frac{2}{c^2}(1 + f)$ (it will be found in a next study on the mass distribution that $f$ is positive and typically in the range $10^{-9} - 10^{-3}$ in spiral galaxies). Consequently equation (5) reduces at the first order to

$$\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}(V_{N} + fV_{\phi}) + \frac{1}{c^2} \frac{dV_{\phi}}{dt} \vec{r}$$

(7)

where $\frac{dV_{\phi}}{dt} = \frac{\partial V_{\phi}}{\partial t} + \frac{\partial V_{\phi}}{\partial r} \frac{dr}{dt} + \frac{\partial V_{\phi}}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial V_{\phi}}{\partial \varphi} \frac{d\varphi}{dt}$ in spherical coordinate $(r, \theta, \varphi)$. Particularly, for planar orbits in plane polar coordinates $(r, \theta)$, one gets from the above equation

$$\frac{dv_r}{dt} - \frac{v^2}{r} = -\frac{\partial(V_{N} + fV_{\phi})}{\partial r} + \frac{1}{c^2} \frac{dV_{\phi}}{dt} v_r$$

(8)

$$\frac{1}{r} \frac{d(v_r \theta)}{dt} = -\frac{1}{r} \frac{\partial(V_{N} + fV_{\phi})}{\partial \theta} + \frac{1}{c^2} \frac{dV_{\phi}}{dt} v_{\theta}$$

(9)

where $v_r = \frac{dr}{dt}$ is the radial velocity, $v_{\theta} = r \frac{d\theta}{dt}$ the tangential velocity.

### 2.2 Static spherically symmetric fields

For static spherically symmetric fields, equations (8) and (9) become

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\[ \frac{dv_r}{dt} - \frac{v_r^2}{r} = -G \frac{m(r)}{r^2} - \frac{f}{r} \frac{\partial V \phi}{\partial r} \]  

(10)

\[ \frac{1}{r} \frac{d}{dt} (\frac{v_r r}{r}) = \frac{1}{c^2} \frac{dV \phi}{dt} v_\theta \]  

(11)

since \( v_r \ll c, \frac{\partial V \phi}{\partial \theta} = \frac{\partial V \phi}{\partial \phi} = 0 \) and \( \frac{\partial V \phi}{\partial r} = G \frac{m(r)}{r^2} \), where \( G \) is the gravitational constant and \( m(r) \) denotes the mass up to radius \( r \). On integrating through equation (11), one gets

\[ v_\theta = J \exp \left( \frac{V \phi}{c^2} \right) \]  

(12)

where \( J \) is a constant which would represent the angular momentum per unit mass if the scalar field \( \phi \) were not present.

### 3 Derivation of the potential \( V \phi \)

In as much as we are interested by the weak field approximation only, the potential \( V \phi \) is simply a linear function of the scalar field, \( \phi \). Now, the equation of a scalar field, \( \phi \), writes in the weak field approximation

\[ \partial^\mu \partial_\mu \phi + \left( \frac{m \phi c}{\hbar} \right)^2 \phi = -V \phi. \]  

(13)

The above equation, as it is well known, follows from the lagrangian density of a real scalar field \( L = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - U(\phi) \), where \( U(\phi) = \frac{1}{2} \left( \frac{m \phi c}{\hbar} \right)^2 \phi^2 + f V \phi d\phi \) denotes the potential energy of the scalar field \( \phi \) and \( m_\phi \) is its mass. Hereafter, we assume a phenomenological effective potential energy of the form

\[ U(\phi) = \frac{1}{2} \left( \frac{m \phi c}{\hbar} \right)^2 \phi^2 + q \phi^p \]  

(14)

where \( p \) and \( q \) are real constants. Of course, whenever \( p \) is an integer, \( p \geq 3 \). In the following we determine only the exponent \( p \), in as much as the potential \( U(\phi) \)
is assumed negligible with respect to the ordinary matter mass-energy. Besides, since we have considered a long range scalar field $\phi$, we may drop the mass term in equation (13) and consider the massless field equation

$$\partial^\mu \partial_\mu \phi = -V \phi.$$  

(15)

Then, the static spherically symmetric solution $\phi = \phi(r)$ satisfies the following equation

$$\frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = V \phi$$  

(16)

and the potential energy reduces to the power law

$$U(\phi) = q\phi^p.$$  

(17)

So, we look for a solution of the form

$$\phi \propto r^k.$$  

(18)

Then replacing $\phi$ by $r^k$ in equation (16) above it comes

$$V = \frac{k(k+1)}{r^2}$$  

(19)

and eliminating $r$ in relation (19) above by replacing $r$ by $\phi^{1/k}$ gives

$$V \propto k(k+1)\phi^{-2/k}$$  

(20)

$$U \propto \int V \phi \, d\phi = \frac{k^2 k + 1}{2 k - 1} \phi^{2(1-1/k)}.$$  

(21)

Therefore, relation (21) together with relation (17) gives
\[ k = -\frac{1}{\frac{p}{2} - 1}. \] (22)

It is natural to expect the exponent \( p \) to be a positive integer. Particularly, we may consider the special case where this exponent is even. Therefore, as \( p \geq 3 \), it comes \( p = 2(n + 1) \) where the integer \( n \) runs from 1 to \( \infty \). This yields \( \phi \propto r^{-1/n} \) and then the following potential

\[ V_\phi = K r^{-1/n} \] (23)

where \( K \) is a real constant the magnitude of which depends on the strength of the interaction of the scalar field \( \phi \) with the ordinary matter in a given region of spacetime.

4 Application to the rotational curves of spiral galaxies

If a long range scalar field such that the one considered in this paper does exist, then as a dark matter candidate it may influence significantly the dynamics in a spiral galaxy and thus modify the shape of its rotation curve. Mostly, this may be done without the need of a great amount of dark matter of this kind. Indeed, setting \( m^{(\phi)}_{\text{dark}}(r) = \int \frac{dV_\phi}{dr} \frac{1}{G} \), equation (9) takes the form

\[ \frac{dv_r}{dt} - \frac{v_\theta^2}{r} = -G m(r) + m^{(\phi)}_{\text{dark}}(r) \] (24)

For \( K < 0 \), the derivative \( \frac{dV_\phi}{dr} = -\frac{K/n}{r^{1+1/n}} \) is positive and thence \( m^{(\phi)}_{\text{dark}}(r) = \int \frac{|K|}{nG} r^{1-1/n} \) mimics a dark matter mass profile which may be important though the energy corresponding to the potential of the scalar field, \( U(\phi) \), is in fact rather negligible with respect to the real matter mass \( m(r) \). It is worth noticing that our interpretation of \( m^{(\phi)}_{\text{dark}}(r) \) as a hidden mass term involves that the integer \( n \) should run from 2
to $\infty$ (instead of 1 to $\infty$) because one will require the positivity of the derivative $\frac{dm_{\text{dark}}(r)}{dr}$ in addition to the positivity of $m_{\text{dark}}(r)$ that is $\frac{dm_{\text{dark}}(r)}{dr} = \frac{n-1}{n} f \frac{|K|}{c^2} r^{-1/n} > 0$ everywhere within a galaxy (except at the boundary). In this section, we wish to prove the validity of relation (12) on the basis of experimental data available on the rotation curves of spiral galaxies. To test relation (12), it is better to rewrite it as follows

$$\ln (rv_\theta) = \ln J + \frac{V_\phi}{c^2}$$

and plot $\ln (rv_\theta)$ versus $r$ for each rotation curve, where $r$ is the distance to the centre. More precisely we use the least-squares fit and search the coefficients $a$ and $b$ such that $\ln (rv_\theta) = ar^{-1/n} + b$, where $a = \frac{K}{c^2}$ and $b = \ln J$. It turns out that the potential $V_\phi$ is indeed well represented by a function of the form $Kr^{-1/n}$, the exponent $n$ being the integer that leads in absolute value to the highest correlation coefficient, $R$, for a given rotation curve. Consequently, the coefficient $a$ should be negative in as much as the interpretation of $m_{\text{dark}}(r)$ as a missing mass term holds. Table 1 below summarizes the numerical results obtained for twenty three spiral galaxies including three dwarf galaxies (DDO 170, NGC 3109 and the dwarf ”regular” UGC 2259) and three giant low surface brightness disk galaxies (Malin 1, NGC 7589 and F 568-6). Clearly, the coefficient $a$ is always negative whereas $b$ is always positive.

5 Fits to individual rotation curves

The following figures show that the rotation curves of spiral galaxies (covering a wide range of mass and size) can be reproduced essentially by a potential, $V_\phi$, resulting from the interaction of a long range scalar field with the usual galactic matter in as much as this interaction, like gravity, respects the weak equivalence principle. It is
found that fairly good fits can be obtained in the first approximation without the need to take into account the photometric properties of the galaxies.

| name [reference] | $n$ | $R^2$  | $a$  | $b$  |
|------------------|-----|--------|------|------|
| DDO 170 [21]     | 3   | 0.999655088 | -7.23 | 9.8  |
| DDO 170* [21]    | 3   | 0.99970403 | -7.1  | 9.7  |
| NGC 3109 [22]    | 10  | 0.998736961 | -30.2 | 26.29 |
| NGC 3109* [22]   | 10  | 0.998784224 | -30.17 | 26.3 |
| UGC 2259 [23]    | 7   | 0.999719756 | -15.14 | 16.89 |
| Malin 1 [24]     | 3   | 0.999950275 | -11.67 | 12.33 |
| NGC 7589 [24]    | 9   | 0.99981229 | -16.11 | 19.72 |
| F 568-6 [24]     | 9   | 0.999709336 | -17.67 | 20.97 |
| NGC 4419 [25]    | 6   | 0.999743147 | -8.645 | 13.486 |
| NGC 1035 [25]    | 5   | 0.99941517 | -8.4  | 12.51 |
| NGC 1325 [25]    | 13  | 0.999646177 | -18.94 | 23.3 |
| NGC 4062 [26]    | 18  | 0.999703876 | -24.49 | 29  |
| NGC 2742 [25]    | 15  | 0.999726171 | -20.7  | 25.2 |
| NGC 3067 [25]    | 5   | 0.99959742 | -7.58 | 12.04 |
| NGC 247 [27]     | 11  | 0.998423849 | -25.3  | 25.16 |
| NGC 3198 [28]    | 5   | 0.999638676 | -6.716 | 11.52 |
| UGC 12810 [25]   | 11  | 0.999759876 | -16.57 | 20.99 |
| NGC 4051 [29]    | 8   | 0.998352026 | -12.56 | 16.61 |
| UGC 3691 [26]    | 7   | 0.999209314 | -12.57 | 16.14 |
| NGC 3593 [25]    | 2   | 0.999652662 | -2.235  | 6.91 |
| NGC 2639 [25]    | 2   | 0.998720197 | -5.77  | 9.92 |
| NGC 4378 [25]    | 6   | 0.999521022 | -7.79 | 13.34 |
| NGC 4448 [25]    | 4   | 0.998699541 | -6.306 | 11.073 |
| NGC 7606 [25]    | 3   | 0.998640104 | -7.4  | 11.3 |
| NGC 4402 [30]    | 13  | 0.99992101 | -27.85 | 29.22 |

**Tab. 1.** The exponent $n$, the square of the correlation coefficient, $R^2$, and the parameters $a$ and $b$ of the least-squares fit to the rotation curves of 23 spirals are tabulated. The asterisk points out that the parameters presented are those of the final rotation curve obtained after a correction for asymmetric drift (caused by a pressure parameter) has been applied to the individual velocity points. As expected, the correlation coefficient increases when the correction for asymmetric drift is applied.
6 Concluding remarks

The work which has been presented in this paper suggests the existence of a neutral scalar particle which may contribute significantly to the rotation curves of spiral galaxies. The results involved by this proposal are found to be in good agreement with the available observational data. However, we have assumed that the referential attached, to the dynamical centre of a given galaxy is galilean. But, spiral galaxies are disks in differential rotation. So, a small correction may be necessary to fit with more accuracy some peculiar rotation curves. Besides, we have fitted the rotation curves of 23 spiral galaxies without the need of photometric data since in our study the main contribution is brought by the scalar field. Nevertheless, this is not necessary problematical because it just brings one more support to the fact that there exists a strong correlation between the spatial distribution of the luminous matter and the spatial distribution of the dark matter in spiral galaxies.

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