Train Tracks on Graphs of Groups and Outer Automorphisms of Hyperbolic Groups

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Abstract

Stallings remarked that an outer automorphism of a free group may be thought of as a subdivision of a graph followed by a sequence of folds. In this thesis, we prove that automorphisms of fundamental groups of graphs of groups satisfying this condition may be represented by irreducible train track maps in the sense of Bestvina–Handel (we allow collapsing invariant subgraphs). Of course, we construct relative train track maps as well. Along the way, we give a new exposition of the Bass–Serre theory of groups acting on trees, morphisms of graphs of groups, and foldings thereof. We produce normal forms for automorphisms of free products and extend an argument of Qing–Rafi to show that they are not quasi-geodesic. As an application, we answer affirmatively a question of Paulin: outer automorphisms of finitely-generated word hyperbolic groups satisfy a dynamical trichotomy generalizing the Nielsen–Thurston “periodic, reducible or pseudo-Anosov.” At the end of the thesis we collect some open problems we find interesting.
To my parents, who always believed in me and challenged me to pursue excellence.

To Jeffrey, Jeremy, Kay and Mara; it is a gift to be seen and heard.

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Train Tracks on Graphs of Groups and Outer Automorphisms of Hyperbolic Groups
Introduction

The Nielsen–Thurston classification of elements of the mapping class group of a surface has spurred a massive development in the theory of mapping class groups, and by analogy, outer automorphisms of free groups. In 1992, Bestvina and Handel [BH92] constructed relative train track maps representing each outer automorphism of a free group.

Relative train track maps play a similar role to Thurston’s normal form for mapping classes, and led Paulin [Pau91] to ask whether all outer automorphisms of finitely-generated word hyperbolic groups satisfy a dynamical trichotomy generalizing the Nielsen–Thurston “periodic, reducible, or pseudo-Anosov.”

The main contribution of this thesis is the construction of relative train track maps representing those automorphisms of a group that may be realized on some graph of groups splitting as a subdivision followed by a sequence of folds. As an application we answer Paulin’s question in the affirmative.

We think it useful to digress to discuss the source of this project; we will discuss the results of this thesis in more detail in the following section of this introduction, to which the impatient reader is invited to skip ahead.

This thesis, and the author’s graduate career, begins with the outer automorphism groups Out($W_n$), where

$$W_n = \underbrace{C_2 \ast \cdots \ast C_2}_{n \text{ factors}} = \langle a_1, \ldots, a_n \mid a_1^2 = \cdots = a_n^2 = 1 \rangle,$$

that is, $W_n$ is the free product of $n$ copies of the cyclic group of order 2. As usual, the group Out($W_n$) is the quotient

$$\text{Out}(W_n) = \text{Aut}(W_n)/\text{Inn}(W_n),$$

where Inn($W_n$) denotes the group of inner automorphisms, arising from the conjugation action of $W_n$ on itself. The groups $W_n$ are the simplest infinite Coxeter groups.

The groups Out($W_n$) share some of the properties of the braid groups and some of the properties of Out($F_{n-1}$), the outer automorphism group of a free group of rank $n - 1$. Indeed, Out($W_n$) contains the quotient of the $n$-strand braid group by its (infinite cyclic) center as a subgroup (of infinite index once $n \geq 4$), and Out($W_n$) is isomorphic to the quotient of a subgroup of Out($F_{n-1}$) by a finite normal subgroup.
In particular $\text{Out}(W_1)$ is trivial, $\text{Out}(W_2)$ is cyclic of order 2, and $\text{Out}(W_3)$ is isomorphic to the projective linear group $\text{PGL}_2(\mathbb{Z})$. A particularly striking connection is the sporadic isomorphism

$$\text{Aut}(B_4) \cong \text{Aut}(F_2) \cong \text{Aut}(W_3),$$

where $B_4$ denotes the 4-strand braid group. For $n \geq 4$, the groups $\text{Out}(W_n)$, and outer automorphisms of free products more generally are much less well-understood than either the braid groups or $\text{Out}(F_{n-1})$.

Brady and McCammond [BM10] construct a contractible metric simplicial complex on which the $n$-strand braid group acts geometrically. For $n \leq 6$, this metric is known to satisfy Gromov’s non-positive curvature condition CAT(0), and conjecturally the condition is satisfied for all $n$. On the other hand, for $n \geq 3$, $\text{Out}(F_n)$ cannot act geometrically on a CAT(0) metric space. This suggests the following question of Kim Ruane.

**Question 1.** For $n \geq 4$, does $\text{Out}(W_n)$ act geometrically on a CAT(0) metric space?

Charles Cunningham in his thesis [Cun15] showed that a candidate simplicial complex constructed by McCullough and Miller in [MM96] equipped with a simplicial action of $\text{Out}(W_n)$ with finite stabilizers and finite quotient does not support an $\text{Out}(W_n)$-equivariant CAT(0) metric. The result is reminiscent of Bridson’s thesis, where the same result is shown for the spine of Culler–Vogtmann’s Outer Space.

Gersten [Ger94] gave the first proof that $\text{Out}(F_n)$ cannot act geometrically on a CAT(0) metric space when $n \geq 4$ by constructing a group $G_\Psi$ which cannot be a subgroup of a group acting geometrically on a CAT(0) metric space, but is a subgroup of $\text{Aut}(F_n)$ for $n \geq 3$ and of $\text{Out}(F_n)$ when $n \geq 4$.

The group $G_\Psi$ fits in a short exact sequence

$$1 \longrightarrow F_3 \longrightarrow G_\Psi \longrightarrow \mathbb{Z} \longrightarrow 1,$$

and is thus called a free-by-cyclic group. The sequence splits; $G_\Psi$ decomposes as a semidirect product $F_3 \rtimes_\Psi \mathbb{Z}$, where abusing notation we identify $\Psi: \mathbb{Z} \to \text{Aut}(F_3)$ with the image $\Psi: F_3 \to F_3$ of the generator 1. Automorphisms of free groups are determined by their action on a free basis; writing $F_3 = \langle a, b, c \rangle$, Gersten’s automorphism $\Psi$ is described by its action as

$$\Psi \begin{cases} 
    a \mapsto a \\
    b \mapsto ba \\
    c \mapsto caa.
\end{cases}$$

A simple computation reveals that word lengths of elements of $F_3$ grow at most linearly under repeated application of $\Psi$: we say $\Psi$ is polynomially-growing. To answer Question 1 in the negative, the author set out to find a Gersten-type example $W_n \rtimes_\Phi \mathbb{Z}$. Surprisingly, the following theorem implies no example exists.
Theorem A (Lyman ’19). Suppose $\Phi: W_n \to W_n$ is a polynomially-growing automorphism. There exists $k \geq 1$ such that $W_n \rtimes_{\phi^k} \mathbb{Z}$ acts geometrically on a CAT(0) 2-complex. The power $k$ is bounded by Landau’s function $g(n) < n!$.

(In fact, the theorem holds for more classes of automorphisms of free products. Details appear in [Lym].) Automorphisms of $W_n$ are closely related to automorphisms of $F_n$ which send generators of some fixed free basis $x_1, \ldots, x_n$ to palindromes in the $x_i$—words spelled the same forwards and backwards. Let $\iota: F_n \to F_n$ denote the automorphism inverting each element of our fixed free basis. It is a pleasant exercise to show that palindromic automorphisms $\Phi: F_n \to F_n$ are precisely those automorphisms centralizing $\iota$ [GJ00, Section 2].

Corollary 1 (Lyman ’19). If $\Phi: F_n \to F_n$ is a polynomially-growing palindromic automorphism, there exists an integer $k \geq 1$ such that $F_n \rtimes_{\phi^k} \mathbb{Z}$ acts geometrically on a CAT(0) 2-complex. The power $k$ is bounded by Landau’s function $g(n + 1) < (n + 1)!$.

A direct construction proving the corollary is easiest to give for those palindromic automorphisms $\Phi$ of $F_n = \langle x_1, \ldots, x_n \rangle$ which are upper-triangular, in the sense that for each $i$, the element $\Phi(x_i)$ may be written as a word using only those $x_j$ satisfying $j \leq i$. In fact, the corollary is proven by arguing that in fact every polynomially-growing, palindromic automorphism $\Phi: F_n \to F_n$ has a power $k \geq 1$ for which $\Phi^k$ is upper-triangular with respect to some free basis.

The demonstration of this latter statement requires some understanding of automorphisms $\Phi: F_n \to F_n$—or better, their outer classes $\varphi \in \text{Out}(F_n)$—independent from their expression in a given basis. The main tools for this understanding are relative train track maps, originally introduced by Bestvina and Handel in [BH92] to prove a conjecture by Scott that the rank of the fixed subgroup
\[
\{ x \in F_n : \Phi(x) = x \}
\]
is at most $n$ for all automorphisms $\Phi: F_n \to F_n$.

As we learn in algebraic topology, free groups are fundamental groups of graphs. Since every graph $\Gamma$ is an Eilenberg–Mac Lane space, every automorphism of a free group may be represented by a basepoint-preserving homotopy equivalence of a graph. The (free) homotopy class of a homotopy equivalence determines an outer automorphism of the fundamental group. A relative train track map is a homotopy equivalence of a graph with specified extra structure that allows for exactly the kind of basis-independent reasoning we hoped for.

Thus to complete the proof of the corollary, one would hope for a “palindromic” relative train track map of some kind. It is a theorem of Culler [Cul84] building on earlier work of Karrass, Pietrowski and Solitar [KPS73] that finite groups of (outer) automorphisms of free groups may be realized as homeomorphisms of graphs, and Bass–Serre theory allows one to consider a kind of “orbifold quotient” of the action [Bas93]—this is a graph of groups (with, as it turns out, fundamental group $W_n$ in the case of $\iota$). One way to guarantee that a relative train track map respects the action of the outer automorphism would be to work directly in the quotient graph of groups. This is the work of this thesis.
Statement of Results

We briefly describe the contents of this thesis, referring the curious reader to the appropriate section for more details.

Chapter 1: Graphs of Groups. We give an exposition of Bass–Serre theory. All the results in the chapter are contained in or follow easily from [Ser03] or [Bas93], but the proofs are new. The central theorem is Theorem 1.2.6 establishing a correspondence between graphs of groups \((\Gamma, \mathcal{G})\) and actions of the associated fundamental group \(\pi_1(\Gamma, \mathcal{G}, p)\) without inversion on a tree \(\tilde{\Gamma}\).

Our proof of Theorem 1.2.6 first gives a geometric construction of the tree \(\tilde{\Gamma}\) from the data of the graph of groups \((\Gamma, \mathcal{G})\). Our definition of the fundamental group \(\pi_1(\Gamma, \mathcal{G}, p)\) avoids the use of the path group, preferring instead to directly develop a homotopy theory of edge paths in graphs of groups. This approach eases the transition in Chapter 3 to a more directly topological approach.

Graphs of groups form a category, in particular there is a notion, essentially due to Bass, of a morphism between graphs of groups. Let us give the brief explanation. Each graph \(\Gamma\) determines a small category, also called \(\Gamma\), with (i) objects the vertices of the first barycentric subdivision of the graph and (ii) arrows from barycenters of edges to the vertices incident to the edge.

A graph of groups \((\Gamma, \mathcal{G})\) is a connected graph \(\Gamma\) and a functor \(\mathcal{G}: \Gamma \to \text{Group}^{\text{mono}}\) to a subcategory of the category of groups. A morphism of graphs \(f: \Lambda \to \Gamma\) is a functor of the associated small categories, and thus a morphism of graphs of groups \(f: (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G})\) ought to be a morphism of functors—i.e. a natural transformation \(f: \mathcal{L} \to \mathcal{G}\).

We develop this perspective beginning in Section 1.8. It turns out that the right notion of a morphism is instead a pseudonatural transformation; certain diagrams only commute up to conjugation by specified elements of the target vertex groups. This is essentially the definition given by Bass in [Bas93] except for two small differences, the first of which is that our morphisms may collapse edges to vertices. In Proposition 1.10.2 and Proposition 1.10.3, we prove a correspondence between morphisms of graphs of groups

\[ f: (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G}) \]

and equivariant morphisms of Bass–Serre trees

\[ \tilde{f}: \tilde{\Lambda} \to \tilde{\Gamma} \]

preserving the distinguished lift of the basepoint. It is a consequence of the second difference in our definitions that Bass’s definition ignores basepoints. Bass’s definition is aimed at understanding the group of equivariant automorphisms of the tree \(\tilde{\Gamma}\), where it is convenient to allow inner automorphisms, but seems inappropriate for most other applications.

Let us remark that the correspondence just mentioned allows one to understand the pseudonaturality condition: the identification of a group \(G\) acting without inversions on a tree \(T\) with the fundamental group of the corresponding graph of groups...
involves a choice of fundamental domain for the action of \( G \) on \( T \). The conjugating elements in the definition of \( f: (Λ, L) \rightarrow (Γ, G) \) arise to measure the difference between the image of the fundamental domain for \( Λ \) and the fundamental domain for \( Γ \).

Bass’s paper leaves implicit the classification of covering spaces of a graph of groups. The result follows quickly from Proposition 1.10.2 and Proposition 1.10.3; we prove it as Theorem 1.12.6.

Chapter 2: Folding. Stallings foldings, introduced in [Sta83] are a fundamental tool in studying outer automorphisms of free groups and maps of graphs more generally. Stallings subsequently gave an extension in [Sta91] to morphisms between Bass–Serre trees, and the theory was later refined and extended in [BF91, Dun98, KWM05]. Most of the first four sections are expository, giving a uniform treatment of results in the previously mentioned papers. Our one contribution in these early sections is a study of two kinds of collapsing that can occur: collapsing of subgraphs, where the fundamental group \( π_1(Γ, G) \) is preserved but the topology of \( Γ \) changes, and collapsing of stabilizers, where the graph \( Γ \) is preserved, but the graph of groups structure \( G \) is changed. The result is Proposition 2.2.1, which serves both as a preliminary step for folding a morphism \( f: (Λ, L) \rightarrow (Γ, G) \) and, as Dunwoody observed, an intermediary step for folding in the case where \( π_1(Λ, L) \) and \( π_1(Γ, G) \) differ.

In Section 2.5, we use folding to study automorphism groups of free products. Stallings remarked that an automorphism of a free group may be thought of as a subdivision of a graph followed by a sequence of folds. For us, a topological realization of an endomorphism \( π_1(Γ, G) \rightarrow π_1(Γ, G) \) is a continuous map \( f: (Γ, G) \rightarrow (Γ, G) \) which is a subdivision of the domain followed by a morphism (in the sense of the previous chapter) from the subdivided graph of groups to \( (Γ, G) \). Topologically realizable automorphisms are exactly those for which Stallings’s observation applies, and folding may be used to study them.

We give a conceptually simple proof in Theorem 2.5.1 of a result of Fouxe-Rabinovitch finding a generating set for the automorphism group of a free product. Actually, Fouxe-Rabinovitch gives a presentation; we do not. By carefully analyzing the process of folding, we give as Corollary 2.5.2 a method for determining whether a topologically realizable endomorphism \( Φ: π_1(Γ, G) \rightarrow π_1(Γ, G) \) is an automorphism, and if so give a normal form for \( Φ \) in the Fouxe-Rabinovitch generators. Every automorphism of a free product is topologically realizable on a (Grushko) splitting we call the “thistle.”

The method given above is algorithmic in principle, although is not in full generality a true algorithm because the factors in the free product may be arbitrary.

In the case of a free group \( F_n \), the methods in the previous paragraph are well-known, but their extension to free products appears to be new. In the case of the free group, Qing and Rafi show in [QR18] that any normal forms that rely solely on folding are not quasi-geodesic in the Cayley graph of \( \text{Aut}(F_n) \) once the rank of the free group is at least 3. We show in Proposition 2.5.3 that our normal forms are
likewise not quasi-geodesic once the Kurosh rank of the free product $A_1 * \cdots * A_n * F_k$ satisfies $n + k \geq 4$ or $k \geq 3$, where the $A_i$ are freely indecomposable and not infinite cyclic.

Let us remark that folding techniques may be used to study automorphisms of virtually free groups. It would be interesting, for instance, to know whether a uniform method can be given for computing a presentation for the subgroup of $\text{Aut}(G)$ or $\text{Out}(G)$ that can be topologically realized on a given splitting. We collect a number of such questions at the end of this thesis.

**Chapter 3: Train Track Maps** In [BH92], Bestvina and Handel construct relative train track maps for outer automorphisms of free groups. A train track map is an efficient topological realization of a given outer automorphism; it minimizes a certain algebraic integer $\lambda \geq 1$ called its stretch factor. The idea of the proof is to associate a nonnegative integral matrix to a topological realization and use folding and some auxiliary moves to minimize the associated Perron–Frobenius eigenvalue.

A relative train track map preserves a filtration

$$\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma$$

of the graph of groups $(\Gamma, \mathcal{G})$ into subgraphs such that within each stratum

$$H_k = \Gamma_k \setminus \Gamma_{k-1},$$

our topological realization $f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ “looks like” a train track map. We make this precise in Proposition 3.9.2 by showing that if $H_k$ is an irreducible stratum, then the topological realization obtained by first restricting to $\Gamma_k$ and then collapsing each component of $\Gamma_{k-1}$ satisfies the train track property (although a priori the resulting graph of groups is disconnected.)

The main result of the chapter is an extension of Bestvina–Handel’s methods to topological realizations of automorphisms of fundamental groups of graphs of groups. In Theorem 3.2.1, we show that if an automorphism $\Phi : G \to G$ can be topologically realized, then it can be realized by a train track map. In the proof, we collapse any invariant subgraphs that occur, which explains why we get a train track map and not a relative train track map.

If one repeats the argument on any invariant subgraphs that occur, the result is a hierarchy of irreducible train track maps,

$$f_1 : (\Lambda_1, \mathcal{L}_1) \to (\Lambda_1, \mathcal{L}_1), \ldots, f_m : (\Lambda_m, \mathcal{L}_m) \to (\Lambda_m, \mathcal{L}_m),$$

where each $(\Lambda_i, \mathcal{L}_i)$ corresponds to a vertex of $(\Lambda_j, \mathcal{L}_j)$ with vertex group $\pi_1(\Lambda_i, \mathcal{L}_i)$ for some $j > i$. In the case where each graph of groups has trivial edge groups, we prove a partial converse to Proposition 3.9.2 as Theorem 3.9.6, namely that one can graft the $(\Lambda_i, \mathcal{L}_i)$ together to yield a topological realization $f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ that has no valence-one or valence-two vertices, only irreducible strata, and each irreducible stratum satisfies a weakening of the relative train track properties due to Gaboriau, Jaeger, Levitt and Lustig [GJLL98] called a partial train track property.
Of course, one may use techniques of Bestvina–Handel to modify $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$, yielding a relative train track map with the same exponentially-growing strata, which we record as Theorem 3.9.3.

As an application of Theorem 3.9.3, we answer affirmatively as Theorem 3.8.1 a question of Paulin, who asks whether each automorphism of a hyperbolic group satisfies a generalization of the Nielsen–Thurston “periodic, reducible, or pseudo-Anosov” trichotomy.

We expect Theorem 3.2.1 and a future strengthening of it constructing the completely split relative train track maps of [FH11] on graphs of groups to be a useful tool in studying outer automorphisms of free products, and subgroups of outer automorphisms of free groups preserving the conjugacy classes of certain free factors. For example, we have used Theorem 3.2.1 in the case of $W_n$, the free product of $n$ copies of the cyclic group of order 2, to show that fully irreducible elements $\varphi \in \text{Out}(W_n)$ are either realizable as pseudo-Anosov homeomorphisms of once-punctured orbifolds, or $W_n \rtimes \phi Z$ is word-hyperbolic.

Perhaps most noticeably absent from this thesis is the connection to deformation spaces of trees, where a train track map corresponds to an axis for the outer automorphism in the (asymmetric) Lipschitz metric. This is how Francaviglia and Martino [FM15] prove the existence of relative train track maps for outer automorphisms of free products. We collect some questions and directions for future work at the end of this thesis.
Chapter 1

Graphs of Groups

Bass–Serre theory [Ser03] studies the structure of groups acting on trees. Suppose a group $G$ acts on a tree $T$. The idea is to treat the quotient $G \backslash T$ as a kind of “orbifold,” in the sense that understanding the topology of the quotient space should be in some way equivalent to understanding the action of $G$ on $T$. The original definition is combinatorial. Scott and Wall [SW79] gave a topological interpretation by introducing graphs of spaces. In this thesis we need both the combinatorial clarity of graphs of groups and the ability to reason about our objects using the tools of one-dimensional topology.

A reader familiar with Bass–Serre theory may skim this chapter up until Section 1.8. Our exposition differs from that in [Ser03] only in its preference for combinatorial topology. We construct the tree first, then analyze the resulting action of the fundamental group on it.

This decision pays off in the latter part of the chapter as it makes clear the mysterious appearance of a pseudonaturality condition appearing in the definition of a morphism of a graph of groups. Many authors mention casually that vertex and edge groups of a graph of groups are really only well defined “up to inner automorphism.” Indeed, a careful reading of Example 1.2.3 below already shows why this might be the case. One way to anticipate the definition is to imagine how a different choice of fundamental domain for the action of a group on a tree might yield a different, yet isomorphic graph of groups structure on the quotient.

Beginning with Section 1.8, we develop the covering space theory of graphs of groups. Our main source is Bass [Bas93]. Bass’s ultimate goal appears to be understanding the group of $\pi_1$-equivariant isomorphisms of the Bass–Serre tree $T$ in [BJ96]. Our goal is to develop Stallings foldings and to understand the subgroups of $\text{Aut}(\pi_1)$ or $\text{Out}(\pi_1)$ which may be realized as homotopy equivalences of a given graph of groups. These goals are at odds with each other, hence our definitions and treatment diverge somewhat from those of Bass.
1.1 Two Notions of Graphs

For us, a graph is a 1-dimensional CW complex. Its 0-cells are vertices, and its 1-cells are edges. We think of an edge $e$ as coming with a choice of orientation, and write $\bar{e}$ for $e$ with its orientation reversed.

There is a convenient combinatorial definition of graphs due to Gersten. In this definition a graph is a set $\Gamma$ with an involution $e \mapsto \bar{e}$ on $\Gamma$, and a retraction $\tau: \Gamma \to \Gamma$ onto the fixed point set $V(\Gamma)$ of the involution. Thus for all $e \in \Gamma$ we have $\tau(e) \in V(\Gamma)$, and $\tau^2 = \tau$. In this definition the fixed point set $V(\Gamma)$ is the set of vertices of $\Gamma$, and its complement $E(\Gamma) = \Gamma - V(\Gamma)$ is the set of oriented edges, with the involution $e \mapsto \bar{e}$ reversing orientations. The map $\tau$ sends an edge to its terminal vertex. One recovers the topological definition by constructing a CW complex with 0-skeleton $V(\Gamma)$ and a 1-cell for each orbit $\{e, \bar{e}\}$ in $E(\Gamma)$, which is attached to $\{\tau(\bar{e}), \tau(e)\}$.

The advantage of this definition is that there is an obvious category of graphs, with objects triples $(\Gamma, e \mapsto \bar{e}, \tau)$ as above. Let $\Gamma$ and $\Gamma'$ be graphs. A morphism $f: \Gamma \to \Gamma'$ is a map of the underlying sets compatible with the involution and retraction in the following sense. Abusing notation by writing $e \mapsto \bar{e}$ and $\tau$ for the involution and retraction, respectively, of both $\Gamma$ and $\Gamma'$, a map $f: \Gamma \to \Gamma'$ is a morphism if the following diagrams commute

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{f} & \Gamma' \\
\downarrow{\tau} & & \downarrow{\tau} \\
\Gamma & \xrightarrow{f} & \Gamma'
\end{array}
\quad
\begin{array}{ccc}
\Gamma & \xrightarrow{f} & \Gamma' \\
\downarrow{\tau} & & \downarrow{\tau} \\
\Gamma & \xrightarrow{f} & \Gamma'
\end{array}
$$

Topologically, morphisms define maps of graphs that send vertices to vertices, and send edges either homeomorphically to edges or collapse them to vertices.

We will pass back and forth continuously between the topological and combinatorial notions of graphs. By an “edge,” we will always mean a 1-cell; we will write “oriented edge” and $E(\Gamma)$ when we need to consider $e$ to be distinct from $\bar{e}$.

A simply-connected graph is a forest, and a forest is called a tree if it is connected. We remark that trees are also naturally simplicial complexes.

1.2 Graphs of Groups

Here is the abstract nonsense. A graph $\Gamma$ determines a small category (let us also call it $\Gamma'$) whose objects are the vertices and edges of $\Gamma$, with morphisms recording the incidence structure of vertices and edges. Thus if $e$ is an edge, there are morphisms $e \to \tau(e)$ and $e \to \tau(\bar{e})$. We have $e$ and $\bar{e}$ equal as objects in the small category $\Gamma$.

A graph of groups structure on $\Gamma$ is a diagram of groups and monomorphisms in the shape of $\Gamma$. That is, it is a functor $\mathcal{G}: \Gamma \to \text{Grp}^{\text{mono}}$ from the small category $\Gamma$ to the category of groups with arrows monomorphisms.

Let us be more concrete.
Definition 1.2.1. A graph of groups is a pair \((\Gamma, \mathcal{G})\), namely a connected graph \(\Gamma\), together with a diagram \(\mathcal{G}\) of groups and homomorphisms between them. For each vertex \(v\) and edge \(e\), \(\mathcal{G}\) contains groups \(G_v\) and \(G_e\), which are called vertex groups and edge groups, respectively. For each edge \(e\), there are two monomorphisms, \(\iota_e: G_e \to G_{\tau(e)}\) and \(\iota_e: G_e \to G_{\tau(e)}\).

Example 1.2.2. Every graph is naturally a graph of groups by declaring each graph in \(\mathcal{G}\) to be the trivial group.

Here is the fundamental example capturing the situation we would like to understand. Suppose \(G\) is a group acting (on the left) by automorphisms on a tree \(T\). We assume further that the action is without inversions in edges. That is, if \(e\) is an edge of \(T\), then no element of \(G\) sends \(e\) to \(\bar{e}\). Topologically this requirement says that if an element of \(G\) sends an edge to itself, then it fixes each incident vertex. The advantage of this definition is that the graph structure on \(T\) naturally defines a graph of groups structure on the quotient \(G\backslash T\). An action of a group \(G\) on a tree \(T\) by automorphisms always yields an action on the barycentric subdivision of \(T\) without inversions in edges.

Example 1.2.3. Suppose \(G\) acts on a tree \(T\) without inversions in edges. We say \(T\) is a \(G\)-tree. The quotient \(G\backslash T\) is a graph, which will serve as \(\Gamma\). Choose a maximal subtree (called a spanning tree) \(T_0 \subset \Gamma\) and a lift \(\bar{T}_0\) that is sent bijectively to \(T_0\) under the natural projection \(\pi: T \to G\backslash T\).

We define the graph of groups structure \(\mathcal{G}\) first on \(T_0\). For a vertex \(v\) or an edge \(e\) of \(T_0\), write \(\bar{v}\) or \(\bar{e}\), respectively, for its preimage in \(\bar{T}_0\). Define the groups \(G_v\) and \(G_e\) to be the stabilizers of \(\bar{v}\) and \(\bar{e}\) under the action of \(G\), respectively. If \(e\) is an edge of \(T_0\), the assumption that \(G\) acts on \(T\) without inversions in edges says that the edge group \(G_e\) is naturally a subgroup of the vertex groups \(G_{\tau(e)}\) and \(G_{\tau(\bar{e})}\). These inclusions define the required monomorphisms.

Suppose now that \(e'\) is an edge of the complement \(\Gamma \setminus T_0\). Let \(\bar{e}'\) be a lift of \(e'\) to \(T\) such that \(\tau(\bar{e}') \in \bar{T}_0\). As before, let \(G_{e'}\) be the stabilizer of \(\bar{e}'\), which is naturally a subgroup of \(G_{\tau(\bar{e}')}\) as above. We now describe the monomorphism \(G_{e'} \to G_{\tau(\bar{e}')}\). Let \(\bar{v}\) be the lift of \(\tau(\bar{e}')\) incident to \(e'\). Since \(T_0\) is a spanning tree of \(\Gamma\), \(\bar{T}_0\) contains a vertex \(\bar{v}'\) in the \(G\)-orbit of \(\bar{v}\). Let \(g\) be an element of \(G\) with \(g.\bar{w} = \bar{v}'\). Observe that if \(h\) stabilizes \(\bar{e}'\), then \(ghg^{-1}\) stabilizes \(\bar{v}'\). Therefore define the corresponding monomorphism \(\iota_{e'}: G_{e'} \to G_{\tau(\bar{e}')}\) by \(\iota_{e'}(h) = ghg^{-1}\).

This collection of groups and monomorphisms defines a graph of groups structure \(\mathcal{G}\) on \(\Gamma\). We say that \(\mathcal{G}\) is induced by the action of \(G\) on \(T\).

In the course of the above example, we built a closed, connected subtree \(F \subset T\) containing \(\bar{T}_0\) and a lift \(\bar{e}'\) for each edge \(e'\) of \(\Gamma \setminus T_0\). The orbit \(GF\) is all of \(T\), and the natural projection \(\pi: T \to G\backslash T\) restricts to an embedding on the interior of \(F\). Such a subset \(F\) is a fundamental domain for the action of \(G\) on \(T\). It is compact if \(G\backslash T\) is compact.

Remark 1.2.4. The graph of groups structure defined above is not quite unique. Different choices of spanning tree, for instance, or different choices of the conjugat-
ing elements $g$ could change the description slightly while not changing $\Gamma$ or the isomorphism type of any of the groups $G_v$ and $G_e$.

**Remark 1.2.5.** The local structure of $T$ is faithfully reflected in the induced graph of groups structure. If $v$ is a vertex of a graph $\Gamma$, define the *star* of $v$, written $\text{st}(v)$, to be the set of oriented edges incident to $v$:

$$\text{st}(v) = \tau^{-1}(v) = \{ e \in E(\Gamma) : \tau(e) = v \}.$$

The *valence* of $v$ is the cardinality of $\text{st}(v)$.

In the above example, if $v$ is a vertex of $\Gamma$ and $\tilde{v} \in \pi^{-1}(v)$, the projection induces a surjection $\pi_{\tilde{v}} : \text{st}(\tilde{v}) \to \text{st}(v)$. The stabilizer of $\tilde{v}$ is conjugate to $G_v$ as a subgroup of $G$, so there is an action of $G_v$ on $\text{st}(\tilde{v})$ which respects $\pi_{\tilde{v}}$. The action is well-defined up to an inner automorphism of $G_v$. If $e$ is an oriented edge of $\text{st}(v)$, the action of $G_v$ on $\text{st}(\tilde{v})$ puts the fiber $\pi^{-1}(e)$ in $G_v$-equivariant bijection with the set of cosets $G_v/\iota_e(G_e)$.

The following theorem states that this latter observation allows one to rebuild both $T$ and $G$ up to equivariant isomorphism from the graph of groups structure $(\Gamma, \mathcal{G})$.

**Theorem 1.2.6** (I.5.4, Theorem 13 [Ser03]). If $(\Gamma, \mathcal{G})$ is a graph of groups, then there exists a tree $T$ and a group $G$ acting on $T$ without inversions in edges such that $G \backslash T \cong \Gamma$, and the action of $G$ on $T$ induces the graph of groups structure $\mathcal{G}$ on $\Gamma$. If $G'$ and $T'$ are another such group and tree inducing the graph of groups $(\Gamma, \mathcal{G})$, then $G \cong G'$, and there is a $G$-equivariant isomorphism $T \to T'$.

The proof of Theorem 1.2.6 occupies the next several sections until Section 1.8. We will adopt the notation of the theorem statement until the completion of the proof.

### 1.3 Construction of the Bass–Serre Tree

In the construction we will give, the group $G$ is called the *fundamental group of the graph of groups* $\pi_1(\Gamma, \mathcal{G})$, for reasons that will become clear—in particular there is a choice of basepoint we are temporarily suppressing. The tree $T$ is called the *universal cover* of the graph of groups $(\Gamma, \mathcal{G})$, although more commonly it is called the *Bass–Serre tree*.

**The case of $\Gamma$ a tree.** Suppose $(\Gamma, \mathcal{G})$ is a graph of groups where $\Gamma$ is a tree. We will build the tree $T$ inductively from copies of $\Gamma$, which we will call *puzzle pieces*. Each puzzle piece has a natural simplicial homeomorphism onto $\Gamma$, and the projection $\pi : T \to \Gamma$ will be assembled from these homeomorphisms.

We can build $T$ inductively. Begin with $T$ a single puzzle piece and $\pi : T \to \Gamma$ a homeomorphism. We would like to make the observation in Remark 1.2.5 hold. Namely, given a vertex $v \in \Gamma$ and a vertex $\tilde{v}$ in the fiber $\pi^{-1}(v)$, the projection
π: T → Γ induces a map π̂: st(̂v) → st(v), which we would like to induce a $G_v$-equivariant bijection

$$\text{st}(\hat{v}) \cong \bigsqcup_{e \in \text{st}(v)} (G_v/\iota_e(G_e) \times \{e\}).$$

For the inductive step, suppose the above statement does not hold. If $G_e$ were trivial for each edge $e \in \text{st}(v)$, we could form a tree $T_{\hat{v}}$ as the one-point union

$$T_{\hat{v}} = (G_v \times \Gamma)/(G_v \times \{v\}),$$

and then form a new tree $T'$ from $T \cup T_{\hat{v}}$ by identifying $\{1\} \times \Gamma$ in $T_{\hat{v}}$ with the (unique, by induction) puzzle piece containing $\hat{v}$ in $T$. The projection of $T_{\hat{v}}$ onto the $\Gamma$ factor allows us to extend the projection $\pi: T \to \Gamma$ to a projection $T' \to \Gamma$. The induction continues on $T'$.

If some $G_e$ is nontrivial, the outline is the same, but a more sophisticated construction of the tree $T_{\hat{v}}$ is required. Observe that since $\Gamma$ is a tree, each edge in $\text{st}(v)$ determines a unique component of $\Gamma \setminus \{v\}$, whose closure is a subtree denoted $\Gamma_e \subset \Gamma$. The vertex $v$ is a leaf of $\Gamma_e$. The tree $T_{\hat{v}}$ in this case is formed from the disjoint union

$$\bigsqcup_{e \in \text{st}(v)} (G_v/\iota_e(G_e)) \times \Gamma_e$$

by identifying $(g\iota_e(G_e), v)$ with $(h\iota_e'(G_e'), v)$ for each pair of group elements $g$ and $h \in G_v$, and each pair of oriented edge $e$ and $e' \in \text{st}(v)$. Note that for each $g \in G_v$, the image in $T_{\hat{v}}$ of the disjoint union

$$\bigsqcup_{e \in \text{st}(v)} \{g\iota_e(G_e)\} \times \Gamma_e$$

is homeomorphic to $\Gamma$. These are the puzzle pieces contained in $T_{\hat{v}}$. The action of $G_e$ on the disjoint union by permuting the labels descends to an action on $T_{\hat{v}}$ permuting the puzzle pieces, which may intersect. The inductive step completes as in the previous paragraph.

Continue this process inductively, perhaps transfinitely, and pass to the direct limit of the trees constructed. The limiting tree is the Bass–Serre tree $T$. Each puzzle piece in $T$ is sent homeomorphically to $\Gamma$ under the projection $\pi: T \to \Gamma$.

**The general case.** If we drop the assumption that $\Gamma$ is a tree we can build $T$ in two stages. In the first stage, we build the universal cover $\tilde{\Gamma}$ of $\Gamma$ as an ordinary graph. Let $p: \tilde{\Gamma} \to \Gamma$ be the covering map. We define a graph of groups structure $\mathcal{G}$ on $\tilde{\Gamma}$. If $\tilde{v}$ is a vertex of $\tilde{\Gamma}$, set $G_{\tilde{v}} \cong G_{p(\tilde{v})}$, and do the same for edges. A monomorphism $G_e \to G_{\tau(\tilde{e})}$ induces a monomorphism $G_{\tilde{e}} \to G_{\tau(\tilde{e})}$ for each edge $\tilde{e} \in p^{-1}(e)$; these will be the monomorphisms of $\mathcal{G}$.

Now follow the preceding construction to produce the Bass–Serre tree $T$ and projection $p': T \to \tilde{\Gamma}$ for the graph of groups $(\tilde{\Gamma}, \mathcal{G})$. The tree $T$ will also be the Bass–Serre tree for $(\Gamma, \mathcal{G})$, and the projection $\pi$ will be $p'p: T \to \Gamma$. 

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1.4 Fundamental Group of a Graph of Groups

To construct the group $G$ acting on $T$, we introduce $\pi_1(\Gamma, \mathscr{G}, p)$, the fundamental group of the graph of groups based at the point $p \in \Gamma$, and define an action of $\pi_1(\Gamma, \mathscr{G}, p)$ on $T$ inducing the graph of groups structure.

**Paths and edge paths.** Since the tree $T$ is simply connected, any two maps $\gamma, \gamma' : [0, 1] \to T$ are homotopic rel endpoints if and only if their endpoints are equal. It will be more convenient to work with paths that are normalized in the following way.

By a path in $T$ we mean an edge path, a finite sequence $e_1 \cdots e_k$ of oriented edges such that $\tau(e_i) = \tau(e_{i+1})$ for $1 \leq i \leq k - 1$. More generally, we allow paths to be points (i.e. $k = 0$), or for $e_1$ and $e_k$ to be segments—connected, closed subsets of edges, which must contain the appropriate vertex if $k > 1$. For any path there is a map $\gamma : [0, 1] \to T$ which is an embedding on each edge or segment of an edge, and whose image is the specified edge path. Since every map $[0, 1] \to T$ is homotopic rel endpoints to such a path $\gamma$, we will work only with edge paths. A path is tight if it is a point, or if $e_i \neq e_{i+1}$ for $1 \leq i \leq k$. Paths may be tightened by a homotopy rel endpoints to produce tight paths by deleting subpaths of the form $\sigma \tilde{\sigma}$. If $\gamma$ is a path, let $\tilde{\gamma}$ denote $\gamma$ with its orientation reversed.

Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be paths in $T$. We would like a notion of paths $\gamma$ and $\gamma'$ in $(\Gamma, \mathscr{G})$ and a notion of homotopy such that $\gamma$ and $\gamma'$ are homotopic rel endpoints only if their lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ are.

Recall from our construction that $T$ is built from puzzle pieces, which are copies of $\Gamma$ or its universal cover. Vertices $\tilde{v} \in T$ come equipped with actions of their associated vertex groups $G_{\pi(v)}$ permuting the set of puzzle pieces containing $\tilde{v}$. Whenever a path $\tilde{\gamma}$ meets a lift of a vertex with nontrivial vertex group, it has the opportunity to cross into a new puzzle piece. That is, suppose $ee'$ is a path with $e$ and $e'$ contained in the puzzle pieces $P$ and $P'$, respectively. If $v$ is the common vertex of $e$ and $e'$, then there is a group element $g \in G_{\pi(v)}$ taking $P$ to $P'$. The group element $g$ is uniquely determined if and only if $G_{\pi(v)}$ acts effectively on the set of fundamental domains at $v$.

Thus we define a path $\gamma$ in $(\Gamma, \mathscr{G})$ to be a finite sequence

$$\gamma = e'_1 g_1 e_2 g_2 \cdots g_{k-1} e'_k,$$

where $e'_1 e_2 \cdots e_{k-1} e'_k$ is an edge path in $\Gamma$, denoted $|\gamma|$, and for each $i$ satisfying $1 \leq i \leq k - 1$ we have $g_i \in G_{\tau(e_i)} = G_{\tau(e_{i+1})}$. We allow the terminal and initial segments $e'_1$ and $e'_k$ to be empty, in which case they should be dropped from the notation. To reverse the orientation of $\gamma$, replace it by

$$\tilde{\gamma} = e'_k g_{k-1}^{-1} \cdots g_2^{-1} e_2 g_1^{-1} e'_1.$$

A path $\gamma$ is a loop if $|\gamma|$ is a loop in $\Gamma$, i.e. if its endpoints are equal. A loop is based at its endpoint.
Lifting and homotopy. Let \( \gamma = e_1'g_1e_2g_2\cdots g_{k-1}e_k' \) be a path in \((\Gamma, \mathcal{G})\) with initial endpoint \( p \). For any lift of the initial endpoint \( \tilde{p} \in \pi^{-1}(p) \) and puzzle piece \( P \) containing \( \tilde{p} \), we can construct a path \( \tilde{\gamma} : [0, 1] \to T \) with \( \pi \tilde{\gamma} = |\gamma| \). We proceed inductively. Let \( \tilde{e}_1' \) be the lift of \( e_1' \) contained in \( P \). Write \( P_1 = P \) and \( v_1 = \tau(\tilde{e}_1') \). Inductively, let \( P_{i+1} = g_iP_i \) under the action of \( G_{\pi(v_i)} \) on the set of puzzle pieces in \( T \) containing \( v_i \). Then \( \tilde{e}_{i+1} \) is the lift of \( e_{i+1} \) contained in \( P_{i+1} \), and \( v_{i+1} = \tau(e_{i+1}) \).

In the end we construct a path

\[
\tilde{\gamma} = \tilde{e}_1'\tilde{e}_2\cdots\tilde{e}_{k-1}\tilde{e}_k'.
\]

Write \( P(\tilde{\gamma}) = P_k \). This is the ending piece of \( \gamma \), or more properly its lift at \((\tilde{p}, P)\).

It is not hard to see that the following operations on \( \gamma \) do not change the homotopy class of its lift to \( T \):

1. Inserting or removing subpaths of the form \( \sigma\bar{\sigma} \)

2. For \( h \in G_e, g \in G_{\tau(e)} \) and \( g' \in G_{\tau(e)} \), replacing subpaths of the form \( ge\sigma e(h)g' \) with \( g\tau(\tilde{e}_1)h\tau(\tilde{e}_1')g' \).

Two paths \( \gamma \) and \( \eta \) in \((\Gamma, \mathcal{G})\) are homotopic rel endpoints if \( \gamma \) can be transformed into \( \eta \) in a finite number of moves as above. The homotopy class of \( \gamma \) rel endpoints is denoted \([\gamma]\), and if \( \gamma \) and \( \eta \) are homotopic rel endpoints, we write \( \gamma \simeq \eta \).

Definition 1.4.1. Let \((\Gamma, \mathcal{G})\) be a graph of groups, and \( p \) a point of \( \Gamma \). The fundamental group of the graph of groups \( \pi_1(\Gamma, \mathcal{G}, p) \) is the group of homotopy classes of loops in \((\Gamma, \mathcal{G})\) based at \( p \) under the operation of concatenation.

It is clear, as in the case of the ordinary fundamental group, that if \( \gamma \) is a path in \((\Gamma, \mathcal{G})\) from \( p \) to \( q \), then the map \( \pi_1(\Gamma, \mathcal{G}, p) \to \pi_1(\Gamma, \mathcal{G}, q) \) defined by \( [\sigma] \mapsto [\gamma\sigma\gamma] \) is an isomorphism. If \( p = q \), then \( \gamma \) is a loop, and this isomorphism is an inner automorphism.

Lemma 1.4.2. Given \( p \in \Gamma \), a choice of lift \( \tilde{p} \in \pi^{-1}(p) \) and puzzle piece \( P \) containing \( \tilde{p} \) determine an action of \( \pi_1(\Gamma, \mathcal{G}, p) \) on \( T \) by automorphisms of the projection \( \pi : T \to \Gamma \). In particular, the action is without inversions in edges. Different choices of \((\tilde{p}, P)\) alter the action by an inner automorphism of \( \pi_1(\Gamma, \mathcal{G}, p) \).

Proof. Given \( \tilde{p} \) and \( P \) as in the statement and \( g \in \pi_1(\Gamma, \mathcal{G}, p) \), we define an automorphism of the projection and show that these automorphisms define an action of \( \pi_1(\Gamma, \mathcal{G}, p) \) on \( T \). Suppose \( g \) is represented by a loop \( \gamma \in (\Gamma, \mathcal{G}) \) based at \( p \). We constructed a lift \( \tilde{\gamma} \) of \( \gamma \) to \( T \) corresponding to the choice \((\tilde{p}, P)\). Associated to \( \tilde{\gamma} \) was its ending piece \( P(\tilde{\gamma}) \) containing \( \tilde{\gamma}(1) \). Both the terminal endpoint \( \tilde{\gamma}(1) \) and ending piece depend only on the homotopy class \( g \) and not on the choice of representative path \( \gamma \).

1 Following Haefliger [BH99, III.\$3.2\$], it might be better to think of the second operation as equivalence and reserve homotopy for the first. We have no obvious need for such a distinction, so will not draw one.
Path lifting respects composition in $\pi_1(\Gamma, \mathcal{G}, p)$ in the following sense. If $\gamma$ and $\eta$ are loops representing elements of $\pi_1(\Gamma, \mathcal{G}, p)$, then we see that the path $\tilde{\gamma}\tilde{\eta}$ is a lift of the concatenation $[\gamma\eta] \in \pi_1(\Gamma, \mathcal{G}, p)$, where $\tilde{\gamma}\tilde{\eta}$ is obtained by first lifting $\gamma$ at $(\tilde{p}, P)$ to $\tilde{\gamma}$, then lifting $\eta$ at $(\tilde{\gamma}(1), P(\tilde{\gamma}))$ to $\tilde{\eta}$ and $P(\tilde{\gamma}\tilde{\eta})$ and finally concatenating.

This lifting allows us to construct an automorphism of the projection $\pi: T \to \Gamma$ corresponding to $g \in \pi_1(\Gamma, \mathcal{G}, p)$. Namely, define $g.\tilde{p}$ and $g.P$ to be $\tilde{\gamma}(1)$ and $P(\tilde{\gamma})$ coming from lifting some (and hence any) loop $\gamma$ representing $g$ at $(\tilde{p}, P)$. More generally, if $x \in T$, there is a path $\tilde{\sigma}$ in $T$ from $\tilde{p}$ to $x$. There is a path $\sigma$ which lifts at $(\tilde{p}, P)$ to $\tilde{\sigma}$. Take $g.\tilde{\sigma}$ to be the lift of $\sigma$ at $(g.\tilde{p}, g.P)$, and $g.x$ to be $g.\tilde{\sigma}(1)$. It is clear that $\pi(g.x) = \pi(x)$, and not hard to see that $x \mapsto g.x$ is a graph automorphism. The argument in the previous paragraph shows that this defines an action of $\pi_1(\Gamma, \mathcal{G}, p)$ on $T$.

If $(\tilde{p}', P')$ are another choice of lift and fundamental domain, there is a loop $\gamma$ based at $p$ whose lift at $(\tilde{p}, P)$ terminates at $(\tilde{p}', P')$. The loop $\gamma$ determines an element $g \in \pi_1(\Gamma, \mathcal{G}, p)$, and the conjugation $h \mapsto ghg^{-1}$ is the required inner automorphism.

Observe that the action of $\pi_1(\Gamma, \mathcal{G}, p)$ on $T$ sends puzzle pieces to puzzle pieces, and furthermore that the action of an element $g$ is determined by the data $(g.\tilde{p}, g.P)$.

**Remark 1.4.3.** It should be noted that homotopy for edge paths in $(\Gamma, \mathcal{G})$ is possibly a finer invariant than homotopy of their lifts to $T$. This is the case when the action of $\pi_1(\Gamma, \mathcal{G}, p)$ on $T$ is not effective. We will describe the kernel below.

**Corollary 1.4.4.** The action of $\pi_1(\Gamma, \mathcal{G}, p)$ on $T$ induces a homeomorphism

$$\pi_1(\Gamma, \mathcal{G}, p) \backslash T \cong \Gamma.$$  

The action admits a fundamental domain.

**Proof.** In the proof of the previous lemma, we saw that $\pi_1(\Gamma, \mathcal{G}, p)$ acts transitively on the set of pairs $(\tilde{q}, Q)$, where $q \in \pi^{-1}(p)$, and $Q$ is a puzzle piece containing $q$. Since the action is by automorphisms of the projection $\pi: T \to \Gamma$, this implies that the quotient is homeomorphic to $\Gamma$.

Recall that the puzzle piece $P$ is the universal cover of $\Gamma$ as an ordinary graph. The action of the ordinary fundamental group of $\Gamma$ (based at $p$) on $P$ admits a fundamental domain $F$. If we view $F$ as a connected subtree of $P \subset T$, we see that $F$ is a fundamental domain for the action of $\pi_1(\Gamma, \mathcal{G}, p)$. Indeed, $F$ contains one edge for each edge of $\Gamma$, so the restriction of $\pi: T \to \Gamma$ is an embedding on the interior of $F$. It follows from the above that the translates of $F$ cover $T$. $\square$

### 1.5 A Presentation of the Fundamental Group

From our description of homotopy of loops, we can give a presentation for $\pi_1(\Gamma, \mathcal{G}, p)$. Let $T_0 \subset \Gamma$ be a spanning tree of $\Gamma$ containing the basepoint $p$.  

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Theorem 1.5.1. The group $\pi_1(\Gamma, \mathcal{G}, p)$ is isomorphic to a quotient of the free product of the groups $G_v$ for each vertex $v \in \Gamma$ with the free group on the set $E(\Gamma)$ of oriented edges in $\Gamma$

\[(\ast_{v \in V(\Gamma)} G_v) \ast F(E(\Gamma))/R\]

Where $R$ is the normal subgroup imposing the following relations.

1. $\iota_{\bar{e}}(h)e = e \iota_e(h)$, where $e$ is an oriented edge of $\Gamma$, and $h \in G_e$.
2. $\bar{e} = e^{-1}$, where $e$ is an oriented edge of $\Gamma$.
3. $e = 1$ if $e$ is an oriented edge of $T_0$.

We remark that the theorem implies that each $G_v$ is naturally a subgroup of $\pi_1(\Gamma, \mathcal{G}, p)$. It follows that different choices of spanning tree $T_0$ yield differing but isomorphic presentations.

Proof. Write $\pi(\mathcal{G})$ for the free product in the statement,

\[\pi(\mathcal{G}) = (\ast_{v \in V(\Gamma)} G_v) \ast F(E(\Gamma)).\]

The group $\pi(\mathcal{G})/R'$, where $R'$ imposes the first two relations in the statement is called the path group and fills a convenient role in other expositions of the theory. Write $G = \pi(\mathcal{G})/R$ for the group described in the statement of the theorem. We will define a map $f$ sending words in $\pi(\mathcal{G})$ to paths in $(\Gamma, \mathcal{G})$. Let $\gamma_v$ denote the unique tight path in $\Gamma$ (as an ordinary graph) from $p$ to $v$ contained in $T_0$. We define

\[f(g) = \gamma_v g \gamma_v \quad \text{for } g \in G_v, \quad \text{and} \quad f(e) = \gamma_{\tau(e)} e \gamma_{\tau(e)} \quad \text{for } e \in E(\Gamma).\]

Then define $f$ on the word $x_1 \cdots x_n$ by $f(x_1) \cdots f(x_n)$. One checks that the following statements hold.

1. If $w$ and $w'$ represent the same element of $\pi(\mathcal{G})$, then $f(w) \simeq f(w')$.

2. $f(\iota_{\bar{e}}(h)e) \simeq f(e \iota_e(h))$, where $e \in E(\Gamma)$ and $h \in G_e$.

3. $f(\bar{e}) = \overline{f(e)}$.

4. $f(e)$ is null-homotopic (i.e. homotopic to the constant path at $p$) if $e$ is an oriented edge belonging to $T_0$.

The map $f$ defines a homomorphism $f_\sharp: G \to \pi_1(\Gamma, \mathcal{G}, p)$. If $g \in G$ is represented by a word $w$ in $\pi(\mathcal{G})$, the homotopy class of $f(w)$ depends only on $g$, so define $f_\sharp(g) = [f(w)]$.

We claim that $f_\sharp$ is an isomorphism. First observe that if $\gamma = e'_1 g_1 e_2 \cdots g_{k-1} e'_k$ is an edge path loop based at $p$, then we have that

\[f(g_1 e_2 \cdots e_{k-1} g_{k-1}) \simeq \gamma,\]
so $f_2$ is surjective. On the other hand, suppose $g$ and $g'$ are elements of $G$ with $f_2(g) = f_2(g')$. If we represent $g$ and $g'$ by words $w$ and $w'$ in $\pi(\mathcal{G})$ respectively, then

$$f(w) \simeq f(w')$$

by assumption. But observe that the edge path loop $f(w)$ may be thought of as a word in $\pi(\mathcal{G})$ representing $g$, since each $\gamma_v$ (perhaps dropping the initial segment) is a sequence of elements of $E(\Gamma)$ whose image in $G$ is trivial. Performing a homotopy from $f(w)$ to $f(w')$ amounts to applying a finite number of defining relations in $G$. Thus the images of $f(w)$ and $f(w')$ as words in $\pi(\mathcal{G})$ are equal in $G$, implying that $g = g'$, so $f_2$ is injective.

We continue to write $G$ for the group described in the statement of Theorem 1.5.1. The injection of $G_v$ into the free product, combined with the isomorphism in the proof exhibits monomorphisms $f_v: G_v \to \pi_1(\Gamma, \mathcal{G}, p)$, defined as $f_v(g) = [\gamma_v g \gamma_v^{-1}]$.

**Corollary 1.5.2.** Let $\gamma$ be an edge loop representing an element of $\pi_1(\Gamma, \mathcal{G}, p)$. The forgetful map $\gamma \mapsto |\gamma|$ induces a surjection $|\cdot|: \pi_1(\Gamma, \mathcal{G}, p) \to \pi_1(\Gamma, p)$ onto the ordinary fundamental group of $\Gamma$. The map $|\cdot|$ admits a section.

The kernel of the map $|\cdot|$ is the subgroup normally generated by $f_v(G_v)$ for each vertex $v \in \Gamma$.

**Proof.** Consider an element $g$ of the ordinary fundamental group of $\Gamma$ based at $p$. Represent $g$ by an edge path $\gamma = e_1' e_2' \cdots e_{k-1}' e_k'$. Inserting the identity element of every vertex group present in $\gamma$, we may view $\gamma$ as a loop in $(\Gamma, \mathcal{G})$. The homotopy class of $\gamma$ in $\pi_1(\Gamma, \mathcal{G}, p)$ is independent of the choice of $\gamma$. This shows that the map $g \mapsto [\gamma]$ defines a section of $|\cdot|$, which is thus surjective.

It is clear from the proof of Theorem 1.5.1 that each $f_v(G_v)$ is contained in the kernel of $|\cdot|$. Conversely, consider an element $h \in \pi_1(\Gamma, \mathcal{G}, p)$ in the kernel of $|\cdot|$. Let $\eta = e_1' g_1 e_2' \cdots g_{k-1}' e_k'$ be an edge path loop representing $h$. We will use the map $f$ from the proof of Theorem 1.5.1. Write

$$\eta' = f(g_1) f(e_2) \cdots f(g_{k-1}).$$

As we saw, $\eta' \simeq \eta$. For $2 \leq i \leq k-1$, write $\gamma_i = f(e_2 \cdots e_i)$, and observe that

$$\eta' \simeq f(g_1) \gamma_2 f(g_2) \gamma_3 \cdots \gamma_{k-2} f(g_{k-2}) \gamma_{k-1} f(g_{k-1}).$$

Observe that the path $\gamma_{k-1}$ satisfies $\gamma_{k-1}$ satisfies $\gamma$ satisfies $\gamma$ satisfies $\gamma \simeq |\eta|$ in $(\Gamma, \mathcal{G})$, and is thus null-homotopic. Removing it from the above concatenation via a homotopy exhibits $h$ as an element of the subgroup normally generated by the subgroups $f_v(G_v)$.

In the terms of the surjection $\pi(\mathcal{G}) \to G$, the content of the above corollary is that the subgroup of $G$ containing the image of the free group on $E(\Gamma)$ is isomorphic to the ordinary fundamental group $\pi_1(\Gamma, p)$, and there is a retraction from $G$ to this subgroup sending each $G_v$ to 1.
1.6 Proof of the Main Theorem

We collect a few lemmas needed to complete the proof of Theorem 1.2.6. Fixing a choice \((\tilde{p}, P)\) in \(T\) and a choice of spanning tree \(T_0 \subset \Gamma\) containing \(p\), using the isomorphism \(G \cong \pi_1(\Gamma, \mathcal{F}, p)\) from Theorem 1.5.1, we will work with the action of \(G\) on \(T\). By Corollary 1.4.4, there is a fundamental domain \(F_0 \subset P \subset T_0\) containing \(\tilde{p}\), and a subtree \(\tilde{T}_0 \subset F_0\) mapped homeomorphically to \(T_0\) under the projection \(\pi: T \to \Gamma\). If \(v\) is a vertex of \(\Gamma\), write \(\tilde{v}\) for the lift of \(v\) contained in \(\tilde{T}_0\). If \(e\) is an edge of \(\Gamma\), write \(\tilde{e}\) for the lift of \(e\) contained in \(F_0\). We will frequently confuse edges of \(\Gamma\) and vertex groups \(G_v\) with their images in \(G\).

Description of the action of \(G\) on \(T\). If \(g \in G_v\), recall that the image of \(g\) in \(G\) corresponds to the homotopy class of \(\gamma_v g \bar{\gamma}_v\), where \(\gamma_v\) is the unique tight path in \(T_0\) (as an ordinary tree) from \(p\) to \(v\). It follows from the description of the action of \(\pi_1(\Gamma, \mathcal{F}, p)\) in Lemma 1.4.2 that \(G_v \leq G\) stabilizes \(\tilde{v}\) and its action on the set of puzzle pieces containing \(\tilde{v}\) agrees with the action described in the construction of \(T\).

If \(e\) is an edge of \(\Gamma\) not contained in \(T_0\) orient \(e\) so that \(\tau(\tilde{e})\) does not lie in \(\tilde{T}_0\). Write \(v = \tau(e)\). The action of \(e\) (as an element of \(G\)) on \(T\) preserves \(P\) and sends \(\tilde{v}\) to \(\tau(\tilde{e})\).

Lemma 1.6.1. The kernel of the action of \(G\) on \(T\) is the largest normal subgroup contained in the intersection

\[ \bigcap_{e \in \mathcal{E}(\Gamma)} \iota_e(G_{e^*}). \]

Proof. The kernel of the action is clearly normal. An element \(h \in G\) fixes the tree \(T\) pointwise if and only if it sends the pair \((\tilde{p}, P)\) to itself. If \(\eta\) is a loop representing \(h\), it follows that \(\tilde{\eta}, the lift of \(\eta\) at \((\tilde{p}, P)\) is null-homotopic, and thus contains a subpath of the form \(\tilde{\sigma}\bar{\sigma}\). Such a subpath is a backtrack. In \(\eta\), a backtrack of \(\tilde{\eta}\) corresponds to a subpath of the form \(\tilde{\sigma}\bar{\sigma}\). If \(e\) is the last edge appearing in the edge path decomposition of \(\sigma\), it follows that \(g\) belongs to the subgroup \(\iota_e(G_{e^*})\) of \(G_{\tau(e)}\), say \(g = \iota_e(g')\). Perform a homotopy replacing \(eg \simeq \iota_e(g')e\) and removing \(ee\). By iterating the above argument, we see that we may shorten any backtrack contained in \(\eta\) that contains a full edge.

Thus we have that

\[ \eta \simeq e'_1g'e'_1, \text{ for some } g' \in G_{\tau(e_1)}, \]

where \(e'_1\) is a (possibly empty) segment of an edge containing \(p\). This shows that \(h\) belongs to \(G_{\tau(e_1)}\). By assumption, \(g'\) sends \(P\) to itself, so we have that

\[ g' \in \bigcap_{e \in \mathcal{st}(\tau(e_1))} \iota_e(G_e). \]

Given an oriented edge \(e \in \mathcal{st}(\tau(e_1))\), if we write \(g' = \iota_e(k')\), we have

\[ \eta \simeq e'_1e\iota_e(k')\bar{e}, \]

where \(e'_1\) is a (possibly empty) segment of an edge containing \(p\). This shows that \(h\) belongs to \(G_{\tau(e_1)}\). By assumption, \(g'\) sends \(P\) to itself, so we have that

\[ g' \in \bigcap_{e \in \mathcal{st}(\tau(e_1))} \iota_e(G_e). \]

Given an oriented edge \(e \in \mathcal{st}(\tau(e_1))\), if we write \(g' = \iota_e(k')\), we have

\[ \eta \simeq e'_1e\iota_e(k')\bar{e}, \]

where \(e'_1\) is a (possibly empty) segment of an edge containing \(p\). This shows that \(h\) belongs to \(G_{\tau(e_1)}\). By assumption, \(g'\) sends \(P\) to itself, so we have that

\[ g' \in \bigcap_{e \in \mathcal{st}(\tau(e_1))} \iota_e(G_e). \]

Given an oriented edge \(e \in \mathcal{st}(\tau(e_1))\), if we write \(g' = \iota_e(k')\), we have

\[ \eta \simeq e'_1e\iota_e(k')\bar{e}, \]
so $h$ belongs to $G_{\tau(\bar{e})}$, and a similar argument to the above shows that for any vertex $v$,

$$h \in \bigcap_{e \in \text{st}(v)} \iota_e(G_e),$$

so we conclude that $h$ belongs to $\iota_e(G_e)$ for all $e \in E(\Gamma)$.

Now suppose $N$ is a normal subgroup of $G$ satisfying $N \leq \iota_e(G_e)$ for all $e \in E(\Gamma)$. In particular, if $v$ is a vertex, we have

$$N \leq \bigcap_{e \in \text{st}(v)} \iota_e(G_e).$$

Thus if $h \in N \leq G_v$, it follows from the action of $G_v$ on the set of puzzle pieces containing $\bar{v}$ that $h$ sends $P$ to itself. Furthermore, $N$ is contained within the kernel of $G \to \pi_1(\Gamma, p)$ by Corollary 1.5.2, so it follows that $N$ also stabilizes $\bar{p}$. Thus $N$ acts trivially on $T$.

**Corollary 1.6.2.** The action of $G$ on $T$ induces the graph of groups structure $(\Gamma, \mathcal{G})$.

*Proof.* We claim that $G_v$ is the stabilizer of $\bar{v}$. Indeed, that $G_v$ stabilizes $\bar{v}$ is clear. If $h$ stabilizes $\bar{v}$, its action on the set of puzzle pieces containing $v$ agrees with some $g \in G_v$. Thus $g^{-1}h$ sends $(\bar{p}, P)$ to itself, and the claim follows from the previous lemma.

It follows that if $e$ is an edge of $\Gamma$ with $\tau(\bar{e}) \in \bar{T}_0$, then $\iota_e(G_e)$ is the stabilizer of $\bar{e}$. If $\tau(\bar{e})$ is not contained in $\bar{T}_0$, then $\iota_e(G_e) = e\iota_e(G_e)e^{-1}$ is the stabilizer of $\bar{e}$. □

**Remark 1.6.3.** The tree $T$ admits an alternate description: by the orbit-stabilizer theorem, vertices of $T$ lifting $v$ correspond to left cosets of $G_v$ in $G$. Edges lifting $e$ correspond to cosets of $\iota_e(G_e)$ in $G$, and the edge corresponding to $ge\iota_e(G_e)$ has initial vertex corresponding to the coset $gG_\tau(\bar{e})$ and terminal vertex corresponding to $geG_\tau(\bar{e})$. Fix an orientation for the edges of $\Gamma$ such that $\tau(\bar{e})$ is not in $\bar{T}_0$ when $e$ is not in $\bar{T}_0$. Combinatorially this is a subset $O \subset E(\Gamma)$ containing one element of each orbit of the involution $e \mapsto \bar{e}$. The tree $T$ is the quotient of the disjoint union

$$\bigsqcup_{e \in O} G/\iota_e(G_e) \times \{e\}$$

by identifying vertices as above. This is the construction of $T$ given in [Ser03].

*Proof of Theorem 1.2.6.* Much of the statement of the theorem follows from the previous lemmas. To complete the proof, suppose that we have a group $G'$ acting on a tree $T'$ without inversions such that $G' \setminus T'$ is homeomorphic to $\Gamma$, and that the induced graph of groups structure is isomorphic to $(\Gamma, \mathcal{G})$ in the following sense.

By assumption, there is a fundamental domain $F'_0 \subset T'$ for the action of $G'$ on $T'$ such that $F'_0$ and $F_0$ are homeomorphic via a homeomorphism such that the following diagram commutes

$$\begin{array}{ccc}
F_0 & \xrightarrow{\cong} & F'_0 \\
\downarrow & & \downarrow \\
\Gamma & \xrightarrow{\cong} & G' \setminus T',
\end{array}$$

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where the vertical maps are the restriction of the natural projections. Similarly there is a lift \( \tilde{T}' \) of \( T_0 \). For each vertex \( v \) and edge \( e \), write \( \bar{v}' \) for the lift of \( v \) in \( \tilde{T}_0' \) and \( \bar{e}' \) for the lift of \( e \) in \( F_0' \), and let \( G_{v'} \) and \( G_{e'} \) denote their stabilizers in \( \tilde{G}' \), respectively. By assumption, there are isomorphisms \( \hat{f}_v : G_v \to G_{v'} \). If \( \tau(\bar{e}') \in \tilde{T}_0' \), there is an isomorphism \( f_e : G_e \to G_{e'} \) such that the following diagram commutes

\[
\begin{array}{ccc}
G_e & \xrightarrow{f_e} & G_{e'} \\
\downarrow{\iota_e} & & \downarrow{\iota_{e'}} \\
G_v & \xrightarrow{f_v} & G_{v'}
\end{array}
\]

If \( \tau(\bar{e}') \) is not in \( \tilde{T}_0' \) and \( v = \tau(e) \), then by assumption there exists \( g_e \in G' \) with \( g_e.\bar{v}' = \tau(\bar{e}') \), such that for \( h \in G_e \), we have \( g_e^{-1}f_e(h)g_e = f_v(\iota_e(h)) \). Define \( g_e = g_e^{-1} \).

The maps \( f_v : G_v \to G_{v'} \subseteq G' \), \( e \mapsto 1 \) if \( \bar{e}' \in \tilde{T}_0' \) and \( e \mapsto g_e \) otherwise define a homomorphism

\[
\hat{f} : \left( \ast_{v \in \Gamma} G_v \right) \ast F(E(\Gamma)) \to G'.
\]

One checks that the relations in the statement of Theorem 1.5.1 are satisfied in \( G' \), so \( \hat{f} \) induces a homomorphism \( f : G \to G' \).

We will show that \( f \) is an isomorphism. Since \( G' \) admits \( F_0' \) as a fundamental domain, a classical result of Macbeath [Mac64] says that \( G' \) is generated by the set

\[
S = \{ g \in G' : gF_0' \cap F_0' \neq \emptyset \}.
\]

If \( g \in S \), then either \( g \) stabilizes some vertex of \( \tilde{T}_0' \), or there is an edge \( e \) in \( \Gamma \) such that \( g_e^{-1}g \) stabilizes a vertex of \( \tilde{T}_0' \). It follows that \( f : G \to G' \) is surjective.

We define a map \( f_T : T \to T' \). Define \( f_T \) on \( F_0 \) to be the isomorphism taking \( \bar{\bar{e}} \) to \( \bar{e}' \) for each edge \( e \) of \( \Gamma \). For a point \( x \in T \), choose a lift \( \pi(x) \) of \( \pi(x) \) in \( F_0 \) in the following way. If \( x \) is not a vertex, there is a unique lift. If \( x \) is a vertex, choose the lift lying in \( \tilde{T}_0 \). There exists \( g_x \in G \) such that \( g_x.\pi(x) = x \). Extend \( f_T \) to all of \( T \) by defining

\[
f_T(x) = f(g_x).f_T(\pi(x)).
\]

The map \( f_T \) is well-defined. Indeed, if \( h \in G \) satisfies \( h.\pi(x) = x \), then \( h^{-1}g_x.\pi(x) = \pi(x) \), so \( f(h^{-1}g_x) \in S \), from which it follows that

\[
f(h).f_T(\pi(x)) = f(g_x).f_T(\pi(x)).
\]

Since \( f \) is surjective, \( f_T \) is a \( G \)-equivariant surjective morphism of graphs.

We will show that the induced maps \( f_{\tilde{v}} : st(\tilde{v}) \to st(\tilde{v}') \) are injective, from which it follows from \( G \)-equivariance and simple connectivity of \( T' \) that \( f_T \) is an isomorphism, and thus \( f \) is also an isomorphism. Indeed, the map \( f \) restricts to an isomorphism from the stabilizer of \( \bar{v} \) to the stabilizer of \( \bar{v}' \), and similarly for \( \bar{e} \) and \( \bar{e}' \).

By the orbit stabilizer theorem, both \( st(\bar{v}) \) and \( st(\bar{v}') \) are in \( G_v \)-equivariant bijection as sets with the set

\[
\prod_{e \in st(\bar{v})} G_v/\iota_e(G_e),
\]

and this bijection is respected by \( f_\bar{v} \). This completes the proof. \( \Box \)
1.7 Examples

Example 1.7.1. Let $A, B$ and $C$ be groups with monomorphisms $\iota_A : C \to A$ and $\iota_B : C \to B$. The free product of $A$ and $B$ amalgamated along $C$, written $A \ast_C B$ is the fundamental group of the graph of groups $(\Gamma, \mathcal{G})$, and where $\Gamma$ is the graph with one edge $e$ and two vertices $v_1$ and $v_2$, and $\mathcal{G}$ is the graph of groups structure $G_{v_1} = A$, $G_{v_2} = B$, $G_e = C$, and the monomorphisms are $\iota_A$ and $\iota_B$. It has the following presentation

$$A \ast_C B = \langle A, B \mid \iota_A(c) = \iota_B(c), \ c \in C \rangle.$$ 

An example of a fundamental group of a graph of groups whose action on the Bass–Serre tree is not effective comes from the classical isomorphism

$$\text{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \ast_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}.$$ 

The tree in this example is the Serre tree dual to the Farey ideal tessellation of the hyperbolic plane. It has two orbits of vertices, with valence 2 and 3, respectively. The edge group $\mathbb{Z}/2\mathbb{Z}$ corresponds to the hyperelliptic involution $-I$ in $\text{SL}_2(\mathbb{Z})$, which acts trivially on the Farey tessellation, and thus trivially on the tree. Thus the action factors through the quotient map $\text{SL}_2(\mathbb{Z}) \to \text{PSL}_2(\mathbb{Z})$, which acts effectively.

Example 1.7.2. Let $A$ and $C$ be groups, with two monomorphisms $\iota_1 : C \to A$, $\iota_2 : C \to A$. The HNN extension of $A$ with associated subgroups $\iota_1(C)$ and $\iota_2(C)$ is the fundamental group of the graph of groups $(\Gamma, \mathcal{G})$, where $\Gamma$ is the graph with one edge $e$ and one vertex $v$, and $\mathcal{G}$ is the graph of groups structure $G_v = A$, $G_e = C$, and the monomorphisms are $\iota_1$ and $\iota_2$. The name stands for Higman, Neumann and Neumann, who gave the first construction. The standard notation is $A \ast_C$. It has the following presentation

$$A \ast_C = \langle A, t \mid t\iota_1(c)t^{-1} = \iota_2(c), \ c \in C \rangle.$$ 

In the special case where $C = A$, $\iota_1 = 1_A$ is the identity and $\iota_2 \in \text{Aut}(A)$, the HNN extension is isomorphic to the semidirect product $A \rtimes_{\iota_2} \mathbb{Z}$, and the Bass–Serre tree is homeomorphic to $\mathbb{R}$. $A$ acts trivially on the tree and $\mathbb{Z}$ acts by a unit translation of the graph structure.

1.8 The Category of Graphs of Groups

Thinking of a graph of groups $(\Gamma, \mathcal{G})$ as a graph $\Gamma$ and a functor $\mathcal{G}$, a morphism $f : (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G})$ is a morphism of graphs $f : \Lambda \to \Gamma$ and a pseudonatural transformation of functors as below left. Concretely, for each edge $e$ of $\Lambda$ and vertex $v = \tau(e)$, there are homomorphisms $f_e : \mathcal{L}_e \to \mathcal{G}_{f(e)}$ and $f_v : \mathcal{L}_v \to \mathcal{G}_{f(v)}$ of edge
groups and vertex groups. Additionally, for each oriented edge e, there is an element \( \delta_e \in G_{f(v)} \) such that the diagram below right commutes

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{f} & \text{Grp}_\text{mono} \\
\downarrow & & \downarrow \\
\mathbb{L} & \xrightarrow{\iota_e} & \mathcal{G}_{f(v)} \\
\downarrow & & \downarrow \text{ad}(\delta_e)\iota_{f(e)} \\
\mathbb{L}_v & \xrightarrow{f_e} & \mathcal{G}_{f(v)},
\end{array}
\]

where \( \text{ad}(\delta_e) : \mathcal{G}_{f(v)} \to \mathcal{G}_{f(v)} \) is the inner automorphism \( g \mapsto \delta_e g \delta_e^{-1} \). Note that if the graph morphism \( f \) collapses the edge \( e \), the monomorphism \( \iota_{f(e)} \) should be replaced by the identity of \( \mathcal{G}_{f(e)} = \mathcal{G}_{f(v)} \). In other words, if the morphism \( f \) collapses the edge \( e \), then as a functor of small categories, \( f : \Lambda \to \Gamma \) must send the morphism \( \iota_e \) to the identity of \( f(\tau(e)) \).

Bass [Bas93, 2.1] also defines a notion of a morphism of a graph of groups. In the case where \( f \) does not collapse edges, Bass’s definition is equivalent to our definition followed by an inner automorphism of \( \pi_1(\Gamma, \mathcal{G}, p) \); ours is more like his morphism \( \delta \Phi \) [Bas93, 2.9]. It is a straightforward exercise to check that the composition of two morphisms is a morphism, and that the evident “identity” morphism really behaves as the identity.

### 1.9 The Action on Paths

A morphism \( f : (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G}) \) acts on edge paths

\[
\gamma = e'_1 g_1 e_2 g_2 \cdots g_{k-1} e'_k \in (\Lambda, \mathcal{L})
\]

with each \( g_i \in \mathcal{L}_{v_i} \) by acting as \( f_{v_i} \) on \( \mathcal{L}_{v_i} \) and sending \( e_i \) to \( \delta_{e_i} f(e_i) \delta_{e_i}^{-1} \). Thus \( f \) sends \( \gamma \) to the edge path

\[
f(\gamma) = f(e'_1) \delta_{e_1}^{-1} f_{v_1}(g_1) \delta_{e_2} f(e_2) \delta_{e_2}^{-1} f_{v_2}(g_2) \cdots f_{v_{k-1}}(g_{k-1}) \delta_{e_{k}} f(e_{k}) \in (\Gamma, \mathcal{G}).
\]

Observe that the pseudonaturality condition implies that if \( e \) is an edge of \( \Gamma \) with \( h \in G_e \), then we have that

\[
f(e_{\tau(e)(h)}) = \delta_e f(e) \cdot \delta_e^{-1} f_{\tau(e)}(\iota_e(h)) = \delta_e f(e) \cdot \iota_{f(e)}(f_e(h)) \delta_e^{-1} \\
\simeq \delta_{e_{\tau(e)(f_e(h))}} f(e) \cdot \delta_e^{-1} = f_{\tau(e)}(\iota_e(h)) \delta_e \cdot f(e) \delta_e^{-1} = f(\iota_e(h) e),
\]

from which it follows that \( f \) preserves homotopy classes of paths in \( (\Lambda, \mathcal{L}) \). Thus if \( p \) is a point in \( \Lambda \), \( f \) induces a homomorphism

\[
f_p : \pi_1(\Lambda, \mathcal{L}, p) \to \pi_1(\Gamma, \mathcal{G}, f(p)).
\]

**Example 1.9.1.** Let \( \gamma \) be a path in \( (\Gamma, \mathcal{G}) \) with endpoints at vertices, say

\[
\gamma = g_1 e_1 \cdots e_k g_{k+1}.
\]
The path $\gamma$ determines a morphism $\gamma: I \to (\Gamma, \mathcal{G})$, where $I$ is an interval of $\mathbb{R}$. To wit: give $I$ a graph structure with $k+1$ vertices and $k$ edges $E_1, \ldots, E_k$ and the trivial graph of groups structure. Orient the $E_i$ so they point towards the terminal endpoint of $I$. There is an obvious graph morphism $\gamma: I \to \Gamma$ sending $E_i$ to $e_i$. To define $\gamma: I \to (\Gamma, \mathcal{G})$, we need to choose $\delta_E \in G_{\tau(f(E))}$ for each oriented edge $E$ of $I$. So set $\delta_{E_i} = g_i$ for $1 \leq i \leq k$. For $1 \leq i \leq k-1$, define $\delta_{E_i} = 1$, and define $\delta_{E_k} = g_k^{-1}$. Since each edge and vertex group of $I$ is trivial, the pseudonaturality condition is trivially satisfied, and observe that

$$\gamma(E_1 \cdots E_k) = \delta_{E_1} \cdots \delta_{E_k} = g_1 \cdots g_k. $$

**A normal form for paths.** For each oriented edge $e \in E(\Gamma)$, fix $S_e$ a set of left coset representatives for $G_{\tau(e)}/\tau_e(G_e)$. We require that $1 \in G_{\tau(e)}$ belongs to $S_e$. A path $\gamma$ in $(\Gamma, \mathcal{G})$ with endpoints at vertices is in *normal form* if

$$\gamma = s_1 e_1 s_2 \cdots s_{k-1} e_k,$$

where $s_i \in S_{e_i}$ and $g_k \in G_{\tau(e_k)}$. Furthermore, for those $s_i$ with $s_i = 1$, we require that $e_{i-1} \neq e_i$. Any path is homotopic rel endpoints to a unique path in normal form, and one may inductively put a path into normal form in the following way. Suppose $\gamma$ is a path satisfying

$$\gamma = \eta g_{i-1} e_i g_i \cdots e_k g_k,$$

where $\eta = s_1 e_1 \cdots s_{i-2} e_{i-1}$ is in normal form. Then $g_{i-1} = s_{i-1} \tau_{e_i}(h)$ for some $s_{i-1} \in S_{e_{i-1}}$, and we may replace $\gamma$ with

$$\gamma' = \eta s_{i-1} e_i \tau_{e_i}(h) g_i \cdots e_k g_k.$$

We have that $\gamma \simeq \gamma'$, and $\eta s_{i-1} e_i$ is in normal form unless $s_{i-1} = 1$ and $e_{i-1} = e_i$, in which case we may replace $e_{i-1}$ and $e_i$ via a homotopy. We have either increased the portion of $\gamma$ which is in normal form or shortened the length of $|\gamma|$ by two, so iterating this process will terminate in finite time.

### 1.10 Morphisms and the Bass–Serre Tree

The most important example of a morphism between graphs of groups is the natural projection $\pi: T \to (\Gamma, \mathcal{G})$. Since $T$ is an ordinary graph, its vertex groups and edge groups are all trivial. For $\pi$ to define a morphism of graphs of groups, we need only to define $\delta_{\tilde{e}}$ for each oriented edge $\tilde{e}$ of $T$.

Our definition depends on a choice of fundamental domain $F \subset T$. Choose a basepoint $p \in \Gamma$, a lift $\tilde{p} \in \pi^{-1}(p)$, and a fundamental domain $F$ containing $\tilde{p}$. In this way we fix an action of $\pi_1(\Gamma, \mathcal{G}, \tilde{p})$ on $T$. Observe that for every vertex $\tilde{v}$ of $T$, there is a unique path $\gamma$ in $(\Gamma, \mathcal{G})$ in the normal form

$$\gamma = e_1 s_1 e_2 \cdots s_{k-1} e_k.$$
which lifts to a tight path $\tilde{\gamma}$ from $\tilde{p}$ to $\tilde{v}$. If $\tilde{e}$ is an oriented edge of $T$, let $\tilde{\gamma}$ be the unique tight path in $T$ connecting $\tilde{p}$ to $\tau(\tilde{e})$. The path $\tilde{\gamma}$ corresponds to a unique path $\gamma$ in $(\Gamma, \mathcal{G})$ of the above form. If $\pi(\tilde{e}) = e_k$ is the last edge appearing in $\gamma$, define $\delta_\gamma = s_{k-1}$. Otherwise define $\delta_\gamma = 1$. Observe that under this definition, $\pi(\tilde{\gamma}) = \gamma$ as a path in $(\Gamma, \mathcal{G})$.

Recall that for each vertex $v \in \Gamma$ and each lift $\tilde{v} \in \pi^{-1}(v)$, there is a $G_v$-equivariant bijection

$$\text{st}(\tilde{v}) \cong \coprod_{e \in \text{st}(v)} G_v/\iota_e(G_e) \times \{e\}.$$ 

In fact, the morphism $\pi$ induces this bijection: if $\pi(\tilde{e}) = \delta_e \delta_\gamma^{-1}$, the above map is $\tilde{e} \mapsto ([\delta_e], e)$, where $[\delta_e]$ denotes the left coset of $\iota_e(G_e)$ in $G_v$ represented by $\delta_e$.

**Remark 1.10.1.** Observe that $\pi_1(\Gamma, \mathcal{G}, p)$ does not act as a group of automorphisms of $\pi: T \to (\Gamma, \mathcal{G})$ as a morphism of graphs of groups: if $g \in \pi_1(\Gamma, \mathcal{G}, p)$, the equality $\delta_{g\tilde{e}} = \delta_{\tilde{e}}$ only holds when $\tau(\tilde{e}) \neq \tilde{p}$. This rigidity of the projection will turn out to be useful.

More generally let $f: (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G})$ be a morphism of graphs of groups. If the graph morphism $f$ does not collapse any edges, define a map

$$f_{\text{st}(v)}: \coprod_{e \in \text{st}(v)} \mathcal{L}_v/\iota_e(\mathcal{L}_e) \times \{e\} \to \coprod_{e' \in \text{st}(f(v))} \mathcal{G}_{f(v)}/\iota_{e'}(\mathcal{G}_{e'}) \times \{e'\}$$

$$f_{\text{st}(v)}([g], e) = ([f_e(g)\delta_e], f(e)).$$

The pseudonaturality condition ensures that $f_{\text{st}(v)}$ is well-defined: since

$$f_v(g\iota_e(h))\delta_e = f_v(g)f_v(\iota_e(h))\delta_e = f_v(g)\delta_{f(e)}(e),$$

we conclude different choices of representative determine the same coset of $\iota_{f(e)}(\mathcal{G}_{f(e)})$ under $f_{\text{st}(v)}$.

**Proposition 1.10.2.** Let $f: (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G})$ be a morphism, $p \in \Lambda$ and $q = f(p) \in \Gamma$. Write $\tilde{\Lambda}$ and $\tilde{\Gamma}$ for the Bass–Serre trees of $(\Lambda, \mathcal{L})$ and $(\Gamma, \mathcal{G})$, respectively. There is a unique morphism $\tilde{f}: \tilde{\Lambda} \to \tilde{\Gamma}$ such that the following diagram commutes

$$\begin{array}{ccc}
(\tilde{\Lambda}, \tilde{p}) & \stackrel{f}{\longrightarrow} & (\tilde{\Gamma}, \tilde{q}) \\
\downarrow_{\pi_\Lambda} & & \downarrow_{\pi_\Gamma} \\
(\Lambda, \mathcal{L}, p) & \stackrel{f}{\longrightarrow} & (\Gamma, \mathcal{G}, q),
\end{array}$$

where $\pi_\Lambda$ and $\pi_\Gamma$ are the covering projections, and $\tilde{p}$ and $\tilde{q}$ are the distinguished lifts of $p$ and $q$, respectively.

**Proof.** If $x$ is a point of $\tilde{\Lambda}$, there is a unique tight path $\gamma$ from $\tilde{p}$ to $x$. The path $f\pi_\Lambda(\gamma)$ has (unique) lift $\tilde{\gamma}$ to $\tilde{\Gamma}$ beginning at $\tilde{q}$ such that $\pi_\Gamma(\tilde{\gamma})$ differs from $f\pi_\Lambda(\gamma)$ only possibly by an element of $\mathcal{G}_{f\pi_\Lambda(x)}$ when $x$ is a vertex. In particular, the terminal
endpoint $\tilde{\gamma}(1)$ is well-defined, and depends only on the homotopy class rel endpoints of $f\pi_\Lambda(\gamma)$. Define $\tilde{f}(x) = \tilde{\gamma}(1)$. It is easy to see that $\tilde{f} : \tilde{\Lambda} \to \tilde{\Gamma}$ defines a morphism, and that the diagram commutes. For uniqueness, observe that any morphism $\tilde{f}'$ making the above diagram commute must satisfy

$$\pi_\Gamma \tilde{f}'(\tilde{\eta}) = f\pi_\Lambda(\tilde{\eta})$$

for any path $\tilde{\eta}$ in $(\tilde{\Lambda})$, so specializing to paths connecting $\tilde{p}$ to $x$, we see that $\tilde{f}'(x) = f(x)$ for all $x \in \Lambda$.

Notice that the map $\tilde{f}$ is $f_\sharp$-equivariant, in the sense that if $h \in \pi_1(\Lambda, L, p)$, then for all $x \in \tilde{\Lambda}$ we have

$$\tilde{f}(h.x) = f_\sharp(h).\tilde{f}(x).$$

The converse is also true.

**Proposition 1.10.3** (cf. 4.1–4.5 of [Bas93]). Let $f_\sharp : \pi_1(\Lambda, L, p) \to \pi_1(\Gamma, G, q)$ be a homomorphism, and let $\tilde{f} : (\tilde{\Lambda}, \tilde{p}) \to (\tilde{\Gamma}, \tilde{q})$ be an $f_\sharp$-equivariant morphism of trees in the sense above. There is a morphism $f : (\Lambda, L, p) \to (\Gamma, G, q)$ which induces $f_\sharp$ and $\tilde{f}$ and such that the following diagram commutes

$$(\Lambda, L, p) \xrightarrow{\tilde{f}} (\Gamma, G, q).$$

**Proof.** As a morphism of graphs $f$ is easy to describe. By $f_\sharp$-equivariance, the map $\pi_\Gamma \tilde{f}$ yields a well-defined map on $\pi_1(\Lambda, L, p)$-orbits; this is the map $f : \Lambda \to \Gamma$ as a morphism of graphs.

Identify the graph of groups structures $L$ and $G$ with those induced by the actions of the respective fundamental groups on $\tilde{\Lambda}$ and $\tilde{\Gamma}$. For $\tilde{\Lambda}$ this involves a choice of fundamental domain $F \subset \tilde{\Lambda}$ containing $\tilde{p}$. Each edge $e \in \Lambda$ has a single preimage $\tilde{e} \in F$, and as a morphism of graphs of groups, $\pi_\Lambda$ satisfies

$$(\pi_\Lambda)_{st(\tau(\tilde{e}))}(\tilde{e}) = ([1], e).$$

If $e$ is not collapsed by $f$, define $\delta_e$ for the morphism $f$ as $\delta_{\tilde{f}(\tilde{e})}$ for the morphism $\pi_\Gamma$. Thus, for $\tilde{\gamma}$ a path in $F$, $\pi_\Gamma \tilde{f}(\tilde{\gamma}) = f\pi_\Lambda(\tilde{\gamma})$. If $e$ is collapsed, define $\delta_e = 1$.

Let $v$ be a vertex of $\Lambda$ and write $w = f(v)$. To define $f_v : L_v \to G_w$, recall that under the identification, $L_v$ is the stabilizer of a vertex $\tilde{v} \in \pi_\Lambda^{-1}(v) \cap F$. Similarly, $G_{f(v)}$ is identified with the stabilizer of some vertex $\tilde{w}$ in $\tilde{\Gamma}$. The stabilizers of $\tilde{f}(\tilde{v})$ and $w$ are conjugate in $\pi_1(\Gamma, G, q)$ via some element $g_w$ such that $g_w.\tilde{f}(\tilde{v}) = w$. Furthermore recall that there is a preferred translate of the fundamental domain for the action of $\pi_1(\Gamma, G, q)$ containing $\tilde{f}(\tilde{v})$ and another preferred translate containing $\tilde{w}$. Namely, in the language we have been developing, the translate containing the edges in the preimage of

$$\{(([1], e) : e \in st(w)\}$$
under the maps \((\pi_1)_s t (f(\bar{v}))\) and \((\pi_1)_s t (\bar{v})\), respectively. We require \(g_w\) to take the
preferred translate for \(f(\bar{v})\) to the preferred translate for \(\bar{w}\). The restriction of

\[
\begin{align*}
  h \mapsto g_w f_{\ddagger}(h)g_w^{-1}
\end{align*}
\]

to the stabilizer of \(\bar{v}\) defines a homomorphism \(f_{\ddagger} : L_v \rightarrow G_w\).

Now let \(e\) be an edge of \(\Lambda\) and write \(a = f(e)\). The story is similar: the stabilizer of
the preimage \(\tilde{e}\) in \(F\) is identified with \(L_e\), and some element \(g_a \in \pi_1(\Gamma, G, q)\) takes
\(\bar{f}(\tilde{e})\) to the preferred preimage \(\bar{a}\). If \(a\) is a vertex, we again require \(g_a\) to match up
preferred fundamental domains. The homomorphism \(f_e : L_e \rightarrow G_a\) is

\[
\begin{align*}
  h \mapsto g_a f_{\ddagger}(h)g_a^{-1}.
\end{align*}
\]

Whether \(e\) is an edge of \(\Gamma\) or \(\Lambda\), the monomorphisms \(\iota_e\) have a uniform description.
If \(v = \tau(e)\), and \(\tau(\tilde{e}) = \bar{v}\), then the monomorphism \(\iota_e\) is the inclusion \(\text{stab}(\tilde{e}) \hookrightarrow \text{stab}(\bar{v})\). If not, there is some element \(t_e\) with \(t_e.\tau(\tilde{e}) = \bar{v}\) and matching up preferred
fundamental domains. Then \(\iota_e\) is

\[
\begin{align*}
  h \mapsto t_e h t_e^{-1}.
\end{align*}
\]

In the former case, write \(t_e = 1\). In the case where \(f\) collapses \(e\) to a vertex \(a\), let \(\iota_a = \iota_g a\) and \(t_a = 1\). Tracing an element \(h \in L_e\) around the pseudonaturality square, we have

\[
\begin{align*}
  \xymatrix{ h \ar[r]^{f_e} & \ar[d]^{\iota_e} g_a f_{\ddagger}(h)g_a^{-1} \ar[d]^{\text{ad}(\delta_e)\iota_a} \\
  t_e h t_e^{-1} \ar[r]_{f_e} & \ar@{=}[r] g_w f_{\ddagger}(t_e h t_e^{-1})g_w^{-1} = \delta_e t_a g_a f_{\ddagger}(h)g_a^{-1} t_a^{-1} \delta_e^{-1}.}
\end{align*}
\]

Equality holds: to see this, we only need to check the case where \(e\) is not collapsed.
In this case, note that \(\delta_e \in G_w\) was defined to take \((t_a g_a) \cdot \bar{f}(\tilde{e})\), whose image under \((\pi_1)_s t (w)\) is \([1, e]\), to \((g_w f_{\ddagger}(t_e)) \cdot \bar{f}(\tilde{e})\). This completes the definition of \(f\) as a
morphism of graphs of groups.

With this description of \(f\), it is easy to see that \(f\) induces \(f_{\ddagger}\). Choose a spanning
tree \(T \subset \Lambda\). For \(v\) a vertex of \(\Lambda\), let \(\gamma_v\) be a path in \(T\) (as a graph) between \(p\) and \(v\). A set of generators of \(\pi_1(\Lambda, L, p)\) is given by

\[
\{ \gamma_e g^{\gamma_v}, \gamma_{\tau(\tilde{e})} e^{\gamma_{\tau(e)}} : g \in L_v, e \text{ an edge of } \Lambda \setminus T \}.
\]

One checks that \(f(\gamma_v) f_{\ddagger}(g) f(\gamma_v) = f_{\ddagger}(g) f_{\ddagger}(g)\) by observing that both elements
stabilize \(\bar{f}(\tilde{v})\) and act as the same element of \(G_{f(\tilde{v})}\) on the set of fundamental
domains containing \(\bar{f}(\tilde{v})\). Similarly, observe that \(f(\gamma_{\tau(e)}) e^{\gamma_{\tau(e)}} = f_{\ddagger}(\gamma_{\tau(e)}) e^{\gamma_{\tau(e)}}\) since
if \(\tau(e) = v\), both elements take \(\bar{f}(\tilde{v})\) to \(\bar{f}(\tau(\tilde{v}))\) and match fundamental domains
identically. From this it follows from \(f_{\ddagger}\)-equivariance that \(f\) also induces \(\bar{f}\).
1.11 Equivalence of Morphisms

In the notation of the previous proposition, it is not the case that \( f \) is uniquely determined by \( f_1 \) and \( f \). Here is a simple example. Let \( \Gamma \) be the graph with one edge \( e \) and two vertices, \( v_1 \) and \( v_2 \), and let \( G \) be the graph of groups structure \( G \) with vertex groups \( G_1 \) and \( G_2 \) and trivial edge group. Orient \( e \) so that \( \tau(e) = v_1 \). Let \( f: (\Gamma, G) \to (\Gamma, G) \) be the morphism which is the identity on each vertex and edge group, and set \( \delta_e = z \) for some element \( z \) in the center of \( G_1 \), and set \( \delta_e = 1 \). At any basepoint other than \( v_1 \), the morphism \( f \) has \( f_1 \) and \( \tilde{f} \) equal to the identity, while at \( v_1 \) the homomorphism \( f_z: \pi_1(\Gamma, G, v_1) \to \pi_1(\Gamma, G, v_1) \) is the automorphism \( \mathrm{ad}(z) \), and \( \tilde{f} \) is equal to the action of \( z \) on the Bass–Serre tree \( T \), viewed as an element of \( \mathrm{stab}(\tilde{v}_1) \).

More generally, if \( f: (\Lambda, \mathcal{L}, p) \to (\Gamma, G, q) \) is a morphism of graphs of groups, the following operations do not change \( f_1 \) or \( \tilde{f} \).

1. If \( v \neq p \) is a vertex of \( \Lambda \) and \( g \in \mathcal{G}_{f(v)} \), replace \( f_v \) with \( \mathrm{ad}(g) \circ f_v \) and replace \( \delta_e \) with \( g \delta_e \) for each oriented edge \( e \in \mathrm{st}(v) \).

2. If \( e \) is an edge of \( \Lambda \) not containing \( p \) and \( g \in \mathcal{G}_{f(e)} \), replace \( \delta_e \) and \( \delta_e \) with \( \delta_{e, f(e)}(g) \) and \( \delta_{e, f(e)}(g) \), respectively.

A simple calculation reveals that neither operation affects \( f_1 \) nor \( \tilde{f} \). If \( f \) can be transformed into \( f' \) by a finite sequence of the above operations, we say \( f \) and \( f' \) are equivalent fixing \( p \). More generally, if one ignores the stipulations regarding the basepoint, we will say two such morphisms are equivalent. Equivalence classes of morphisms will become relevant for representing outer automorphisms of \( \pi_1(\Lambda, \mathcal{L}) \).

Twist automorphisms. We would like to draw attention to a particular family of automorphisms of \((\Lambda, \mathcal{L})\). For \( \nu \) a vertex of \( \Lambda \), \( e \) an oriented edge of \( \mathrm{st}(\nu) \) and \( g \in \mathcal{L}_{\nu} \), define the twist of \( e \) by \( g \),

\[
t_{ge}: (\Lambda, \mathcal{L}) \to (\Lambda, \mathcal{L})
\]

by setting \( (t_{ge})_x \) equal to the identity of \( \mathcal{G}_x \) for \( x \) an edge or vertex of \( \Lambda \), setting \( \delta_e = g \), and setting \( \delta_{e'} = 1 \) for each other edge of \( \Lambda \). In the Bass–Serre tree \( \tilde{\Lambda} \), the lift \( \tilde{t}_{ge} \) corresponds to changing the fundamental domain \( F \) for the action of \( \pi_1(\Lambda, \mathcal{L}, p) \). Since \( F \) is a tree, removing the interior of \( \tilde{F} \) separates \( F \) into two components. Call the component containing \( \tilde{p} \) \( F_0 \), call \( F_1 \) the closure of \( F \setminus F_0 \). The lift \( \tilde{t}_{ge} \) acts by fixing \( F_0 \) and sending \( F_1 \) to \( g \cdot F_1 \).

Cohen and Lustig [CL95] define a similar notion which they call a Dehn twist automorphism. Our twists are Dehn twists when \( g \) lies in the image of the center of the edge group \( Z(\mathcal{L}_e) \) under \( t_e \). Recall that an essential simple closed curve \( \gamma \) on a surface \( S \) determines a splitting of \( \pi_1(S) \) as a one-edge graph of groups.

---

2 One could argue for saying homotopy equivalent instead. We see no particular reason to prefer one terminology over the other, save that neither \( f_1 \) nor \( \tilde{f} \) see any change under this form of equivalence.
The action of the Dehn twist about \( \gamma \) on \( \pi_1(S) \) may be realized as a Dehn twist automorphism of \((\Gamma, \mathcal{G})\). More generally, Dehn twist automorphisms on some splitting of \( \pi_1(S) \) as a graph of groups with cyclic edge groups correspond to products of commuting Dehn twists about some multi-curve on \( S \).

## 1.12 Covering Spaces

Let \( f \) be a morphism that does not collapse edges. The map \( f_{st(v)} \) captures the local behavior of \( f \) at \( v \). We say that \( f \) is locally injective or an immersion if (i) each homomorphism \( f_e \) and \( f_v \) is a monomorphism, and (ii) each \( f_{st(v)} \) is injective. An immersion is an embedding if additionally \( f \) is injective as a morphism of graphs. We say an immersion \( f \) is a covering map if \( f \) is surjective as a morphism of graphs and each \( f_{st(v)} \) is a bijection. The composition of two immersions is clearly an immersion, and the composition of two covering maps is a covering map. (Recall that we require graphs of groups to be connected.)

**Lemma 1.12.1.** If \( f: (\Lambda, \mathcal{L}, p) \to (\Gamma, \mathcal{G}, f(p)) \) is an immersion, the induced homomorphism

\[
f_{\ast}: \pi_1(\Lambda, \mathcal{L}, p) \to \pi_1(\Gamma, \mathcal{G}, f(p))
\]

is a monomorphism.

**Proof.** Suppose to the contrary that \( g \in \pi_1(\Lambda, \mathcal{L}, p) \) is a nontrivial element in the kernel of \( f_{\ast} \), and represent \( g \) by an immersion \( \gamma: I \to (\Lambda, \mathcal{L}) \).\(^3\) Since \( f(\gamma) \) is null-homotopic, (the image of) \( \gamma \) has a subpath \( e\overline{e}' \) such that \( f(e) = f(\overline{e}') \) and

\[
\delta_e^{-1} f_v(g) \delta_{e'} = \iota_{f(e)}(h),
\]

where \( v = \tau(e) \) and \( h \in G_{f(e)} \). Thus \( [\delta_e] = [f_v(g) \delta_{e'}] \), so we conclude that

\[
f_{st(v)}([1], e) = f_{st(v)}([g], \overline{e}').
\]

But because \( \gamma: I \to (\Lambda, \mathcal{L}) \) is an immersion, either \( e \neq \overline{e}' \) or \( [g] \neq [1] \), so this contradicts the assumption that \( f \) is an immersion. \( \square \)

Put another way, the above proof shows that an immersion \( f: (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G}) \) sends immersed paths in \( (\Lambda, \mathcal{L}) \) to immersed paths in \( (\Gamma, \mathcal{G}) \).

**Lemma 1.12.2** (Corollary 4.6 of [Bas93]). Let \( f: (\Lambda, \mathcal{L}, p) \to (\Gamma, \mathcal{G}, q) \) be a morphism. The corresponding morphism of Bass–Serre trees \( \tilde{f}: \tilde{\Lambda} \to \tilde{\Gamma} \) is an embedding if and only if \( f \) is an immersion, and an isomorphism if and only if \( f \) is a covering map. \( \square \)

We saw in the previous section that the projection \( \pi: T \to (\Gamma, \mathcal{G}) \) from the Bass–Serre tree \( T \) is a covering map. Let \( H \) be a subgroup of \( \pi_1(\Gamma, \mathcal{G}, p) \). The action of \( H \) on \( T \) gives \( H \backslash T \) the structure of a graph of groups, with covering map \( \rho: T \to H \backslash T \).

\(^3\) Technically our combinatorial definition of morphism requires \( p \) to be a vertex of \( \Lambda \). This can be accomplished by declaring \( p \) to be a vertex and altering \( \mathcal{L} \) in the obvious way.
Corollary 1.12.3. With notation as above, the map $\rho$ fits in a commutative diagram of covering maps of graphs of groups as below

\[
\begin{array}{ccc}
T & \xrightarrow{\rho} & H \backslash T \\
\downarrow \pi & & \downarrow r \\
(\Gamma, \mathcal{G}).
\end{array}
\]

Proof. Apply Proposition 1.10.3 to the following diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\rho} & T \\
\downarrow \pi & & \downarrow r \\
H \backslash T & \longrightarrow & (\Gamma, \mathcal{G}).
\end{array}
\]

Proposition 1.12.4. Suppose $f : (\Lambda, \mathcal{L}, p) \to (\Gamma, \mathcal{G}, q)$ is a morphism of graphs of groups and that $\rho : (\Gamma, \mathcal{G}, \bar{q}) \to (\Gamma, \mathcal{G}, q)$ is a covering map. There exists a morphism $\tilde{f}$ making the following diagram commute

\[
\begin{array}{ccc}
(\bar{\Gamma}, \bar{\mathcal{G}}, \bar{q}) & \xrightarrow{\tilde{f}} & (\bar{\Lambda}, \bar{\mathcal{L}}, \bar{p}) \\
\downarrow \rho & & \downarrow f \\
(\Gamma, \mathcal{G}, q) & \xrightarrow{f} & (\Lambda, \mathcal{L}, p)
\end{array}
\]

if and only if $f_{\sharp}(\pi_1(\Lambda, \mathcal{L}, p)) \leq \rho_{\sharp}(\pi_1(\bar{\Gamma}, \bar{\mathcal{G}}, \bar{q}))$.

Proof. That the condition is necessary is obvious. For sufficiency observe that the previous corollary implies that Proposition 1.10.3 applies to the diagram

\[
\begin{array}{ccc}
(\bar{\Lambda}, \bar{p}) & \xrightarrow{\tilde{f}} & (\bar{\Gamma}, \bar{q}) \\
\downarrow \pi_{\Lambda} & & \downarrow \pi_{\bar{f}} \\
(\Lambda, \mathcal{L}, p) & \longrightarrow & (\bar{\Gamma}, \bar{\mathcal{G}}, \bar{q}).
\end{array}
\]

The proposition immediately implies the following corollary.

Corollary 1.12.5. If $\rho_1 : (\Lambda_1, \mathcal{L}_1, p_1) \to (\Gamma, \mathcal{G}, q)$ and $\rho_2 : (\Lambda_2, \mathcal{L}_2, p_2) \to (\Gamma, \mathcal{G}, q)$ are covering maps, and $(\rho_1)_{\sharp}(\pi_1(\Lambda_1, \mathcal{L}_1, p_1)) = (\rho_2)_{\sharp}(\pi_1(\Lambda_2, \mathcal{L}_2, p_2))$, then there is an isomorphism $f : (\Lambda_1, \mathcal{L}_1, p_1) \to (\Lambda_2, \mathcal{L}_2, p_2)$ such that $\rho_2 f = \rho_1$.

In fact, this observation proves the following classification of covering spaces.

Theorem 1.12.6. There is a Galois correspondence between (connected) pointed covering maps $\rho : (\Lambda, \mathcal{L}, q) \to (\Gamma, \mathcal{G}, p)$ and subgroups $H \leq \pi_1(\Gamma, \mathcal{G}, p)$. The correspondence sends a covering map $\rho$ to the subgroup $\rho_{\sharp}(\pi_1(\Lambda, \mathcal{L}, q))$ and sends a subgroup $H$ to the covering map $r : (H \backslash T, q) \to (\Gamma, \mathcal{G}, p)$ constructed above, where $q$ is the image of $\bar{p}$ under the covering projection $T \to H \backslash T$. 

\[\square\]
Chapter 2
Folding

In the seminal paper [Sta83], Stallings introduced the notion of folding morphisms of graphs and used it to give conceptually powerful new proofs of many classical results about free groups. He later observed [Sta91] that the result of equivariantly folding a $G$-tree is again a $G$-tree, so the theory extends to morphisms of $G$-trees. Extending Dunwoody’s work on accessibility, Bestvina and Feighn [BF91] analyzed the effects of folding a $G$-tree on the quotient graph of groups to bound the complexity of small $G$-trees when $G$ is finitely presented. Recall that a group is small if it does not contain a free subgroup of rank two. A $G$-tree is small if each edge stabilizer is small. Accessibility of finitely presented groups is equivalent to a finite bound (depending on $G$) for the complexity of $G$-trees with finite edge stabilizers. Dunwoody [Dun98] extended folding to equivariant morphisms of trees to show that finitely generated inaccessible groups contain infinite torsion subgroups. Kapovich, Weidmann and Miasnikov [KWM05] used foldings of graphs of groups to study the membership problem.

The idea of Stallings folding. Here is the setup. Suppose $G$ is the fundamental group of a graph $\Gamma$ (based at a vertex $p \in \Gamma$). By Theorem 1.12.6, a subgroup $H \leq G$ corresponds to a covering map $\Gamma_H \to \Gamma$. Even if $H$ is finitely generated, if it is not of finite index in $G$, the graph $\Gamma_H$ will be infinite. In this case $\Gamma_H$ contains a compact core $K_H$ where the fundamental group is concentrated, and the complement of $K_H$ in $\Gamma_H$ is a forest. The restriction of the covering map $\Gamma_H \to \Gamma$ to $K_H$ is no longer a covering map, but it remains an immersion, and the image in $\pi_1(\Gamma)$ is $H$.

If one knew how to describe the core $K_H$, a whole host of useful information about $H$ would follow: the rank of $H$, a nice free basis for $H$, whether $H$ has finite index in $G$, and so forth. If on the other hand one were just given a generating set $S$ for $H$, recovering this information purely algebraically is not straightforward.

On the other hand, $S$ does give a representation of $H$ as a morphism of graphs with codomain $\Gamma$: for each $s \in S$, represent $s$ by an edge path loop $\gamma_s$ in $\Gamma$ based at $p$. This yields a map from the circle $\gamma_s: S^1 \to \Gamma$. Subdividing the circle yields a basepointed morphism of graphs. Taking the wedge sum at the basepoint $*$ yields a graph $\Lambda_H$ and a morphism $f: (\Lambda_H, *) \to (\Gamma, p)$ with $f_\#(\pi_1(\Lambda_H, *)) = H$. Since $H$ is finitely generated, $\Lambda_H$ is a finite graph.
The main insight of Stallings is that $f$ may be promoted to an immersion in a finite and algorithmic fashion as we now explain. If $f$ is not already an immersion, then it identifies a pair of edges incident to a common vertex. We say $f$ folds these edges, and $f$ factors through the quotient map $\Lambda_H \to \Lambda'$ which identifies them. The quotient map is called a fold. If the resulting morphism $f' : \Lambda' \to \Gamma$ is not an immersion, we continue to fold. At each step, the number of edges in the graph decreases, so at some finite stage, the resulting morphism $\bar{f} : \bar{\Lambda} \to \Gamma$ must be an immersion—in fact, it is not hard to see that the core of $\bar{\Lambda}$ and $K_H$ are isomorphic!

In principle, if $H$ is not finitely generated (or more precisely, if the set $S$ is not finite), nothing untoward happens: one just folds “forever,” and passing to the direct limit yields an immersion. Although the process is no longer algorithmic in any strict sense, the constructive nature of the process remains valuable in applications.

Our interest in folding will be primarily to study automorphisms of groups acting on trees. We take up this study at the end of the chapter.

### 2.1 Folding Bass–Serre Trees

Let us begin making the foregoing discussion precise. Call a morphism $f : (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G})$ of graphs of groups Stallings if it (i) does not collapse edges and (ii) each group homomorphism $f_v : \mathcal{L}_v \to \mathcal{G}_v$ and $f_e : \mathcal{L}_e \to \mathcal{G}_e$ is a monomorphism. By Proposition 1.10.2, once we fix a basepoint, there is an equivariant morphism $\tilde{f} : \tilde{\Lambda} \to \tilde{\Gamma}$ of Bass–Serre trees lifting $f$. If $f$ is not an immersion, neither is $\tilde{f}$, so there is some vertex $\tilde{v} \in \tilde{\Lambda}$ such that $\tilde{f}_{st}(\tilde{v})$ is not injective, i.e. there are distinct edges $\tilde{e}_1$ and $\tilde{e}_2$ with $\tau(\tilde{e}_1) = \tau(\tilde{e}_2) = \tilde{v}$, and $\tilde{f}(\tilde{e}_1) = \tilde{f}(\tilde{e}_2)$.

Define an equivalence relation on the $\tilde{\Lambda}$ as follows:

1. $\tilde{e}_1 \sim \tilde{e}_2$ and $\tau(\tilde{e}_1) \sim \tau(\tilde{e}_2)$.

2. Extend item 1. equivariantly: if $x \sim y$ and $g \in \pi_1(\Lambda, \mathcal{L})$, then $g.x \sim g.y$.

Let $\tilde{\Lambda}/[\tilde{e}_1=\tilde{e}_2]$ be the quotient of $\tilde{\Lambda}$ by this equivalence relation. Stallings [Sta83, Section 3] notes that this is an example of a pushout construction. An observation of Chiswell [Sta91, Theorem 4] says that $\tilde{\Lambda}/[\tilde{e}_1=\tilde{e}_2]$ is a tree, $\pi_1(\Lambda, \mathcal{L})$ acts without inversions, and the morphism $\tilde{f}$ factors as below

$$
\begin{array}{ccc}
\tilde{\Lambda} & \xrightarrow{\tilde{f}} & \tilde{\Gamma} \\
\downarrow & & \downarrow \\
\tilde{\Lambda}/[\tilde{e}_1=\tilde{e}_2] & \xrightarrow{\tilde{f}} & \tilde{\Gamma}.
\end{array}
$$

The morphism $\tilde{\Lambda} \to \tilde{\Lambda}/[\tilde{e}_1=\tilde{e}_2]$ is called a fold. In fact, by Proposition 1.10.3, there is
also a factorization of $f$ such that the following diagram commutes

$$
\begin{array}{ccc}
\tilde{\Lambda} & \xrightarrow{j} & \tilde{\Gamma} \\
\downarrow{\pi_{\Lambda}} & & \downarrow{\pi_{\Gamma}} \\
(\Lambda, \mathcal{L}) & \xrightarrow{f} & (\Gamma, \mathcal{G}) \end{array}
$$

where $e_i = \pi_{\Lambda}(\tilde{e}_i)$, where we define

$$(\Lambda, \mathcal{L})/\llbracket e_1 = e_2 \rrbracket = \pi_1(\Lambda, \mathcal{L}) \setminus \tilde{\Lambda}/\llbracket \tilde{e}_1 = \tilde{e}_2 \rrbracket,$$

and where the morphism $\tilde{\Lambda}/\llbracket \tilde{e}_1 = \tilde{e}_2 \rrbracket \to (\Lambda, \mathcal{L})/\llbracket e_1 = e_2 \rrbracket$ is the covering projection.

Bestvina and Feighn analyze the combinatorial possibilities for the resulting morphism $(\Lambda, \mathcal{L}) \to (\Lambda, \mathcal{L})/\llbracket e_1 = e_2 \rrbracket$, which is Stallings, and which we will also call a fold. We discuss these possibilities below.

Ideally, one would like to say that if $f: (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G})$ is a Stallings morphism and $\Lambda$ is a finite graph, then inductively $f$ factors as a finite sequence of folds followed by an immersion. Unfortunately, this is just not true; here is the problem. In the notation above, let $\tilde{v}_1$ and $\tilde{v}_2$ denote the initial vertices of $\tilde{e}_1$ and $\tilde{e}_2$ respectively, and let $\tilde{w}$ be their common image in $\tilde{\Lambda}/\llbracket \tilde{e}_1 = \tilde{e}_2 \rrbracket$. Note that

$$\langle \text{stab}(\tilde{v}_1), \text{stab}(\tilde{v}_2) \rangle \leq \text{stab}(\tilde{w}),$$

but that the left-hand subgroup is typically much larger than

$$\langle f_\sharp(\text{stab}(\tilde{v}_1)), f_\sharp(\text{stab}(\tilde{v}_2)) \rangle \leq \text{stab}(f(\tilde{v}_1)).$$

Thus the morphism $(\Lambda, \mathcal{L})/\llbracket e_1 = e_2 \rrbracket \to (\Gamma, \mathcal{G})$ may not be Stallings. On the level of group theory, observe that $(\Lambda, \mathcal{L})$ and $(\Lambda, \mathcal{L})/\llbracket e_1 = e_2 \rrbracket$ have the same fundamental group. Thus a necessary condition for a factorization into folds would be that $f_\sharp: \pi_1(\Lambda, \mathcal{L}) \to \pi_1(\Gamma, \mathcal{G})$ is injective. This is already too restrictive a condition when studying subgroups of free groups, since we begin with only a generating set and not a free basis.

Dunwoody [Dun98, Section 2] remedies this problem by introducing what he calls vertex morphisms—we prefer vertex collapses—effectively replacing $\text{stab}(\tilde{w})$ with its image in $\text{stab}(f(\tilde{v}_1))$. Dunwoody’s vertex morphisms have to do with collapsing. We will pause to develop tools for collapsing graphs of groups, and then return to folding.

### 2.2 Collapsing Graphs of Groups

There are two kinds of collapsing we shall be interested in, corresponding to the two ways in which a morphism of graphs of groups can fail to be Stallings. The former is collapsing of subgraphs. In terms of the graph of groups $(\Gamma, \mathcal{G})$, this operation
preserves \( \pi_1(\Gamma, \mathcal{G}) \) while altering the topology of \( \Gamma \). The latter is collapsing of stabilizers. This operation preserves \( \Gamma \), but alters the graph of groups structure \( \mathcal{G} \), which has the effect of replacing \( \pi_1(\Gamma, \mathcal{G}) \) with its image under \( f_* \). In each case the quotient morphism \( f : (\Gamma, \mathcal{G}) \to (\Gamma', \mathcal{G}') \) induces surjective maps \( \tilde{f} \) and \( f_* \).

The results of this section are summarized as follows.

**Proposition 2.2.1.** Every morphism \( f : (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G}) \) factors into a composition of collapse moves followed by a Stallings morphism. The collapse moves proceed by collapsing each component of each preimage \( f^{-1}(v) \) as \( v \) varies over the vertices of \( \Gamma \), followed by collapsing the resulting morphism to \( (\Gamma, \mathcal{G}) \) onto its image. \( \square \)

We remark that if \( \Lambda \) is not a finite graph, then the first sequence of collapses need not be finite. The collapses nonetheless form a directed system, and one could define a direct limit of these collapses. We are primarily interested in morphisms where \( \Lambda \) is finite.

**Collapsing subgraphs.** Let \( (\Gamma, \mathcal{G}) \) be a graph of groups. A **subgraph of groups** is a pair \( (\Lambda, \mathcal{G}|_{\Lambda}) \), where \( \Lambda \) is a (nonempty) connected subgraph of \( \Gamma \) and \( \mathcal{G}|_{\Lambda} \) is the restriction of the graph of groups structure \( \mathcal{G} \) to \( \Lambda \). In terms of abstract nonsense, the functor \( \mathcal{G}|_{\Lambda} \) is the restriction of \( \mathcal{G} \) to the full subcategory of \( \Gamma \) determined by \( \Lambda \). The inclusion \( (\Lambda, \mathcal{G}|_{\Lambda}) \to (\Gamma, \mathcal{G}) \) is an embedding. For convenience choose a basepoint in \( \Lambda \). Thus \( \pi_1(\Lambda, \mathcal{G}|_{\Lambda}) \) becomes the subgroup of \( \pi_1(\Gamma, \mathcal{G}) \) consisting of homotopy classes of loops contained in \( (\Lambda, \mathcal{G}|_{\Lambda}) \). The graph of groups obtained by **collapsing the subgraph** \( \Lambda \) is \( (\Gamma/\Lambda, \mathcal{G}/\Lambda) \), where the graph of groups structure \( \mathcal{G}/\Lambda \) is obtained from \( \mathcal{G} \) in the following way. For \( x \) a vertex or an edge not contained in \( \Lambda \), set \( (\mathcal{G}/\Lambda)_x = \mathcal{G}_x \). The vertex \( \Lambda \) of \( \Gamma/\Lambda \) has vertex group \( (\mathcal{G}/\Lambda)_\Lambda = \pi_1(\Lambda, \mathcal{G}|_{\Lambda}) \).

If an oriented edge \( e \) and its terminal vertex \( \tau(e) \) both lie outside \( \Lambda \), the monomorphism \( \iota_e \) remains the same in \( \mathcal{G}/\Lambda \). If \( \tau(e) \in \Lambda \), then \( \iota_e \) becomes the composition \( \mathcal{G}_e \xrightarrow{\iota_e} \mathcal{G}_{\tau(e)} \xrightarrow{\pi_1(\Lambda, \mathcal{G}|_{\Lambda})} \).

Alternatively, this process can be seen in the Bass–Serre tree. By Proposition 1.10.2, the inclusion \( (\Lambda, \mathcal{G}|_{\Lambda}) \to (\Gamma, \mathcal{G}) \) induces an embedding of Bass–Serre trees \( \tilde{\Lambda} \to \tilde{\Gamma} \). The image of \( \tilde{\Lambda} \) is the component of the full preimage of \( \Lambda \) in \( \tilde{\Gamma} \) containing the lift of the basepoint. Let \( \tilde{\Gamma}/\tilde{\Lambda} \) be the graph obtained by collapsing each component of the \( \pi_1(\Gamma, \mathcal{G}) \)-orbit of \( \tilde{\Lambda} \) to a vertex. Thus \( \tilde{\Gamma}/\tilde{\Lambda} \) naturally inherits an action of \( \pi_1(\Gamma, \mathcal{G}) \), and \( \pi_1(\Lambda, \mathcal{G}|_{\Lambda}) \) stabilizes the image of \( \Lambda \).

In fact, these two descriptions are equivalent.

**Proposition 2.2.2.** In the notation above, \( \pi_1(\Gamma/\Lambda, \mathcal{G}/\Lambda) = \pi_1(\Gamma, \mathcal{G}) \), and \( \tilde{\Gamma}/\tilde{\Lambda} \) is the Bass–Serre tree, and the following diagram of covering projections and quotient maps described above commutes.

\[
\begin{array}{ccc}
\tilde{\Gamma} & \longrightarrow & \tilde{\Gamma}/\tilde{\Lambda} \\
\downarrow & & \downarrow \\
(\Gamma, \mathcal{G}) & \longrightarrow & (\Gamma/\Lambda, \mathcal{G}/\Lambda).
\end{array}
\]
Proof. Stallings’s neat homological proof [Sta91, Theorem 4] of Chiswell’s result above applies to show that $\tilde{\Gamma}/\tilde{\Lambda}$ is also a tree. The idea is that a tree is a graph with the homology of a point, so there is a short exact sequence of reduced simplicial chain groups for $\tilde{\Gamma}$ (which are $\pi_1(\Gamma, G)$-modules). The $\pi_1(\Gamma, G)$-modules generated by the images of 1- and 0-chains of $\Lambda$ in $\tilde{\Gamma}$ are isomorphic and each equal to the corresponding kernel of the chain map induced by the quotient $\tilde{\Gamma} \to \tilde{\Gamma}/\tilde{\Lambda}$. That $\tilde{\Gamma}/\tilde{\Lambda}$ is a tree follows from the nine lemma.

By definition, an element $g \in \pi_1(\Gamma, G)$ stabilizes the image of $\tilde{\Lambda}$ in $\tilde{\Gamma}$ if and only if $g$ preserves $\tilde{\Lambda}$ in its action on $\tilde{\Gamma}$. But then $g$ has a representative loop entirely contained in $(\Lambda, G|\Lambda)$, so $g$ belongs to the image of $\pi_1(\Lambda, G|\Lambda)$. It follows that $\tilde{\Gamma}/\tilde{\Lambda}$ is the Bass–Serre tree for $(\Gamma/\Lambda, G/\Lambda)$.

More generally one can collapse a disconnected subgraph of groups by collapsing each component.

Collapsing stabilizers. Suppose $f: (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G})$ is a morphism which does not collapse edges. We can produce a new graph of groups structure on $\Lambda$ by collapsing $\mathcal{L}$ to its image $f(\mathcal{L})$ in the following way. If $x$ is a vertex or an edge of $\Lambda$, define $f(\mathcal{L})_x = f_x(\mathcal{L}_x) \leq \mathcal{G}_{f(x)}$. Let $e$ be an edge of $\Lambda$ with $\tau(e) = v$. We define the monomorphism $\iota_e: f(\mathcal{L})_e \to f(\mathcal{L})_v$ as

$$f(\mathcal{L})_e \xrightarrow{\iota_e} \mathcal{G}_{f(e)} \xrightarrow{\iota_{f(e)}} \mathcal{G}_{f(v)} \xrightarrow{\text{ad}(\delta_e)} \mathcal{G}_{f(v)},$$

where $\delta_e \in \mathcal{G}_{f(v)}$ is part of the data of $f$. Observe that the pseudonaturality condition of Section 1.8 guarantees that the image of the above monomorphism lies in $f(\mathcal{G})_v$.

The morphism $f$ factors as follows

$$\begin{array}{ccc}
(\Lambda, \mathcal{L}) & \xrightarrow{f} & (\Gamma, \mathcal{G}) \\
\downarrow & & \downarrow \\
(\Lambda, f(\mathcal{L})) & \xrightarrow{f_*} & (\Lambda, f(\mathcal{L}))
\end{array}$$

where the morphism $f_*: (\Lambda, \mathcal{L}) \to (\Lambda, f(\mathcal{L}))$ is the identity on $\Lambda$ and acts as $f$ on vertex and edge groups. By definition, if $e$ is an edge with $\tau(e) = v$, we have $\iota_e(f_*)_e = (f_*)_e \iota_e$, so the conjugating element $\delta_e$ for $f_*$ is trivial. The morphism $(\Lambda, f(\mathcal{L})) \to (\Gamma, \mathcal{G})$ is Stallings. It acts as the natural inclusion on vertex and edge groups, as $f$ on $\Lambda$, and has the same conjugating elements $\delta_e$ as $f$. The fundamental group $\pi_1(\Lambda, f(\mathcal{L}))$ is a quotient of $\pi_1(\Lambda, \mathcal{L})$.

The Bass–Serre tree $\tilde{\Lambda}/f$ for $(\Lambda, f(\mathcal{L}))$ is built from the same “puzzle piece,” in the language of Section 1.3, as $\tilde{\Lambda}$, only the attaching differs. Recall that the stabilizer of a vertex $\tilde{v} \in \tilde{\Lambda}$ is canonically isomorphic to $\mathcal{L}_u$ and permutes the set of puzzle pieces containing $\tilde{v}$. Every vertex has a preferred puzzle piece corresponding to the identity element of its stabilizer. One can give an inductive description of the morphism $\tilde{f}_*: \tilde{\Lambda} \to \tilde{\Lambda}/f$ as follows: the preferred puzzle piece containing the lift of the basepoint in $\tilde{\Lambda}$ is sent via an isomorphism to the preferred puzzle piece in $\tilde{\Lambda}/f$ containing the lift of the basepoint. If $\tilde{f}_*(\tilde{v})$ is already defined, extend $\tilde{f}_*$ to the set
of puzzle pieces containing \( \tilde{v} \) by sending the preferred puzzle piece \( P \) for \( \tilde{v} \) to the preferred puzzle piece for \( \tilde{f}_*(\tilde{v}) \), and then using the identification of the stabilizer of \( \tilde{v} \) with \( L_{\tilde{v}} \), sending \( g.P \) to \( (f_*(g)).\tilde{f}_*(P) \) for \( g \in \text{stab}(\tilde{v}) \).

Dunwoody’s vertex morphisms, which we will call vertex collapses to avoid confusion are a special case of collapsing a graph of groups structure onto its image that arises when folding. Let \( v \) be a vertex of \( \Lambda \), and let \( K_v \) be a normal subgroup of \( L_v \) such that for each oriented edge \( e \in \text{st}(v) \), \( \iota_e(L_e) \cap K_v = 1 \). We may define a new graph of groups structure \( L/K_v \) on \( \Lambda \) by setting \( (L/K_v)_v = L_v/K_v \), and setting \( (L/K_v)_x = \mathcal{L}_x \) otherwise. By assumption, the morphism

\[
\mathcal{L}_e \xrightarrow{\iota_e} \mathcal{L}_v \longrightarrow \mathcal{L}_v/K_v
\]

is a monomorphism for each oriented edge \( e \) in \( \text{st}(v) \). The evident morphism \( f : (\Lambda, \mathcal{L}) \rightarrow (\Lambda, \mathcal{L}/K_v) \), is equal to the result of collapsing \( \mathcal{L} \) to its image under \( f \). We call \( f \) a vertex collapse at \( v \).

### 2.3 Folding Graphs of Groups

We return to our description of folding. Let \( (\Lambda, \mathcal{L}) \) be a graph of groups and let \( \Lambda/[\tilde{e}_1 = \tilde{e}_2] \) be the result of collapsing a pair of edges \( \tilde{e}_1 \) and \( \tilde{e}_2 \) in \( \Lambda \). Let \( \tilde{v}_i \) be the initial vertex of \( \tilde{e}_i \), and let \( e_i \) and \( v_i \) be the corresponding edge and vertex of \( \Lambda \). Let \( e \) and \( v \) be the images of the \( e_i \) and \( v_i \), respectively, in \( \Lambda/[e_1 = e_2] \). Bestvina and Feighn [BF91, Section 2] distinguish several types of folds by their effect on \( (\Lambda, \mathcal{L}) \). There are three types. Bestvina and Feighn distinguish these further into subtypes based primarily on whether \( \tau(e_1) \) is distinct from the \( v_i \). We will not draw such a distinction.

Throughout, let \( f : (\Lambda, \mathcal{L}) \rightarrow (\Gamma, \mathcal{G}) \) be a Stallings morphism that is not an immersion at \( v \). Recall that this means that the map

\[
f_{\text{st}(v)} : \coprod_{e \in \text{st}(v)} \mathcal{L}_v/\iota_e(\mathcal{L}_e) \times \{e\} \rightarrow \coprod_{e' \in \text{st}(f(v))} \mathcal{G}_{f(v)}/\iota_{e'}(\mathcal{G}_{e'}) \times \{e'\}
\]

\[
([g], e) \mapsto ([f_v(g)\delta_e], f(e))
\]

is not injective.

#### Type I
In this case, \( e_1 \neq e_2 \), and \( v_1 \neq v_2 \). In other words, the orbits of the \( \tilde{e}_i \) and \( \tilde{v}_i \) are distinct. The stabilizer of the image of \( \tilde{e}_i \) is the subgroup generated by the stabilizers of \( \tilde{e}_1 \) and \( \tilde{e}_2 \), and similarly for the stabilizer of the image of \( \tilde{v}_1 \). The situation in the quotient graph of groups is described in Figure 2.1. By \( \langle \mathcal{L}_{v_1}, \mathcal{L}_{v_2} \rangle \), for instance, we mean the subgroup of \( \pi_1(\Lambda, \mathcal{L}) \) generated by \( \mathcal{L}_{v_1} \) and \( \mathcal{L}_{v_2} \), under their identification with the stabilizers of \( \tilde{v}_1 \) and \( \tilde{v}_2 \). Note that we do not assume that \( w = \tau(e_1) \) is distinct from \( v_1 \) or \( v_2 \), even though we have drawn it as distinct in the figure.
We briefly describe the resulting morphism \( \hat{f} : (\Lambda, \mathcal{L}) /_{\{e_1 = e_2\}} \rightarrow (\Gamma, \mathcal{F}) \). Certainly as a morphism of graphs, it is clear how to define \( \hat{f} \). If \( x \) is a vertex or edge of \( \Lambda /_{\{e_1 = e_2\}} \), define \( \hat{f}_x = f_x \). The morphism \( \hat{f}_v : (\mathcal{L}_{v_1}, \mathcal{L}_{v_2}) \rightarrow \mathcal{F}_{f(v)} \) must be the restriction of \( f_x \) to \( (\mathcal{L}_{v_1}, \mathcal{L}_{v_2}) \). Suppose first that as part of the data of \( f \), we have \( \delta_{e_1} = \delta_{e_2} \) and \( \delta_{e_1} = \delta_{e_2} \). Observe that the pseudonaturality condition says that if \( h \in \mathcal{L}_v \) for \( i = 1, 2 \), we have that

\[
\delta_{e_i}^{-1} f_v t_{e_i}(h) \delta_{e_i} \in t_{f(e_i)}(\mathcal{F}_{f(e_i)}).
\]

So if \( \delta_{e_1} = \delta_{e_2} \), we may use the above to define \( \hat{f}_v \), and check that the pseudonaturality condition holds with \( \delta_e = \delta_{e_1} \) and \( \delta_{e} = \delta_{e_2} \).

If it is not the case that \( \delta_{e_1} = \delta_{e_2} \) and \( \delta_{e_1} = \delta_{e_2} \), we alter the situation. If \( \delta_{e_1} \neq \delta_{e_2} \), that \( f_{g(v)} \) is not injective implies that there exist \( g \in \mathcal{L}_v \) and \( h \in \mathcal{F}_{f(e)} \) such that \( f_v(g) \delta_{e_2} t_{f(e)}(h) = \delta_{e_1} \). We may pass from \( f \) to an equivalent morphism as in Section 1.11 so that \( \delta_{e_2} = \delta_{e_2} t_{f(e)}(h) \). Finally, we may twist \( e_2 \) by \( g \) so that \( \delta''_2 = \delta_{e_1} \). If now \( \delta_{e_1} \neq \delta_{e_2} \), observe that at least one of the \( v_i \) is distinct from \( w \), so we may pass from \( f \) to an equivalent morphism so that \( \delta_{e_1} = \delta_{e_2} \). All of these operations are demonstrated in the examples in the following section.

**Figure 2.1: The effect of a Type I fold.**

**Type II.** In this case, \( e_1 = e_2 \), so \( v_1 = v_2 \). In other words, there is some \( g \in \pi_1(\Lambda, \mathcal{L}) \) with \( g \tilde{e}_1 = \tilde{e}_2 \). Thus \( g \) belongs to \( \mathcal{L}_w \), where \( w = \tau(e_1) = \tau(e_2) \). After folding, \( g \) will stabilize the image of \( \tilde{e}_1 \), and thus also stabilize the image of \( \tilde{v}_1 \). The situation in the quotient graph of groups is described in Figure 2.2. Again, we do not assume that \( w \) is distinct from \( v_1 \).

More generally, one might allow Type II folds to encompass the more general situation where \( H \) is a subgroup of \( \mathcal{L}_w \) and we fold \( \tilde{e}_1 \) with \( h.\tilde{e}_1 \) for all \( h \in H \). If \( H \) happens to be finitely generated, this is a finite composition of Type II folds in the stricter sense.

**Type III.** In this case, \( e_1 \neq e_2 \), but \( v_1 = v_2 \). In other words, there is some \( g \in \pi_1(\Lambda, \mathcal{L}) \) with \( g \tilde{v}_1 = \tilde{v}_2 \), but \( \tilde{e}_1 \) and \( \tilde{e}_2 \) are in distinct orbits. After folding, \( g \) will stabilize the image of \( \tilde{v}_1 \).
The factoring of the morphism \( f \) is broadly the same as in a Type I fold. However, since \( v_1 = v_2 \), we cannot always arrange that \( \delta_{e_1} = \delta_{e_2} \). So set \( \delta_{e} = \delta_{e_1} \). The image of \( g \) in \( G \) will be the difference \( \delta_{e_1}^{-1}\delta_{e_2} \).

Remark 2.3.1. Bestvina and Feighn define an additional type of fold, Type IIIC, which occurs when there exists \( g \in \pi_1(\Lambda, L) \) with \( g\tilde{e}_1 = \tilde{e}_2 \). In this case, \( e_1 = e_2 \) is a loop, and \( g \) acts as an inversion of \( \tilde{\Lambda}/[\tilde{e}_1 = \tilde{e}_2] \). This type of fold does not occur in factorizations of a Stallings morphism \( f: (\Lambda, L) \to (\Gamma, G) \). The reason is that in this case \( f_\sharp(g) \) would act as an inversion on \( \tilde{\Gamma} \), which we do not allow.

The best unconditional result is the following proposition, due to Stallings [Sta91, Theorem 5], Bestvina–Feighn [BF91, Proposition, Section 2] and Dunwoody [Dun98, Theorem 2.1].

Proposition 2.3.2. Let \( f: (\Lambda, L) \to (\Gamma, G) \) be a Stallings morphism. If \( f \) is not an immersion, then it admits a fold followed by a vertex collapse. There is a factorization

\[
\begin{array}{ccc}
(\Lambda, L) & \xrightarrow{e} & (\Gamma, G), \\
\downarrow m & & \downarrow \pi_1(H, \mathcal{H}) \\
(H, \mathcal{H}) & \xrightarrow{f} & (\Gamma, G),
\end{array}
\]

where \( m \) is an immersion and each of \( e: \Lambda \to H \), \( \tilde{e}: \tilde{\Lambda} \to \tilde{H} \) and \( e_1: \pi_1(\Lambda, L) \to \pi_1(H, \mathcal{H}) \) are surjective.

Proof. Observe that if \( f: (\Lambda, L) \to (\Gamma, G) \) is not an immersion, then it factors through some fold \( (\Lambda, L)/[e_1 = e_2] \). All the vertex and edge groups of \( L/[e_1 = e_2] \) are equal to those of \( L \) save one vertex group and one edge group. Call the new edge group \( H_e \) and the new vertex group \( H_v \). Observe that the subgroups that combine to form \( H_e \) belong to some vertex group \( L_w \). Dunwoody’s insight is that thus the resulting morphism \( f': (\Lambda, L)/[e_1 = e_2] \to (\Gamma, G) \) is Stallings everywhere except possibly at \( H_v \), so we may perform a vertex collapse.
Nothing prevents us from iterating this process. Unfortunately, without further assumptions, nothing can prevent the process from continuing indefinitely. Stallings observes that one may continue to fold transfinitely and pass to direct limits when necessary. At some possibly transfinite stage, the resulting graph of groups \((H, \mathcal{H})\) admits no further folds, so the map \(m: (H, \mathcal{H}) \to (\Gamma, \mathcal{G})\) is an immersion. Each fold and vertex collapse induces surjective maps of graphs, Bass–Serre trees and fundamental groups. Thus the same holds true for the resulting map \(e: (\Lambda, \mathcal{L}) \to (H, \mathcal{H})\).

Note that if \((\Lambda, \mathcal{L})/\{e_1 = e_2\}\) is obtained from the finite graph of groups \((\Lambda, \mathcal{L})\) by a Type I or Type III fold, then \(\Lambda/\{e_1 = e_2\}\) has fewer edges than \(\Lambda\). Thus, if the process of foldings and vertex collapses does not terminate, after some finite stage, all further folds are of Type II.

Kapovich, Weidmann and Miasnikov give a simple example [KWM05, Example 5.10] of an HNN extension \(G = F_2 \ast F_2\) corresponding to an injective non-surjective endomorphism of a free group of rank two, and a two-generated subgroup \(H \leq G\) for which the sequence of Type II folds for the associated morphism of graphs of groups does not terminate.

Type II folds replace some edge group \(L_e\), which we may think of via \(f\) as a subgroup of \(G_{f(e)}\), with some strictly larger subgroup. Thus to guarantee that the process terminates after finitely many iterations, we need to argue that each edge group will only be increased finitely many times.

An obvious sufficient condition is that for each edge, the homomorphism

\[ f_e: \mathcal{L}_e \to \mathcal{G}_{f(e)} \]

is an isomorphism—in fact, this would guarantee that only Type I and Type III folds occur. More useful for our purposes would be the condition that \(f_2\) is surjective and that the edge groups \(\mathcal{G}_{f(e)}\) are finitely generated. This is the condition that Dunwoody uses in [Dun98, Theorem 2.1], modeled on the analogous condition in [BF91, Proposition, Section 2]. Another possible sufficient condition would be that the edge groups of \((\Gamma, \mathcal{G})\) are Noetherian, that is, every ascending chain

\[ G_0 \leq G_1 \leq G_2 \leq \cdots \]

of subgroups stabilizes. For groups, the Noetherian property is clearly equivalent to requiring that every subgroup is finitely generated. This latter property of a group is sometimes called slender.

The main family of examples of Noetherian groups are virtually polycyclic groups. A polycyclic group is a group \(G\) with a (finite) subnormal series

\[ 1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k = G \]

whose factor groups \(G_i/G_{i-1}\) are cyclic. Examples of virtually polycyclic groups include finite groups, finitely generated abelian groups, and virtually solvable subgroups of \(\text{GL}_n(\mathbb{Z})\). Auslander [Aus67] and Swan [Swa67] show that in fact any virtually polycyclic group is a virtually solvable subgroup of \(\text{GL}_n(\mathbb{Z})\) for some \(n\).
As such, virtually polycyclic groups have good computational properties. For this reason, Kapovich, Weidmann and Miasnikov restrict their algorithms to graphs of groups \((\Gamma, \mathcal{G})\) all of whose edge groups are virtually polycyclic.

Thus we have the following corollary of Proposition 2.3.2.

**Corollary 2.3.3.** If \(f : (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G})\) is a Stallings morphism where \(\Lambda\) is a finite graph and either (i) each edge group of \((\Gamma, \mathcal{G})\) is Noetherian, or (ii) each edge group of \(\mathcal{G}\) is finitely generated and \(f_* : \pi_1(\Lambda, \mathcal{L}) \to \pi_1(\Gamma, \mathcal{G})\) is surjective, then \(f\) factors into a finite sequence of folds and vertex collapses followed by an immersion.  

2.4 Examples

Let \(C_n\) denote the cyclic group of order \(n\). Consider the following group

\[
G = \langle a, b, c, t \mid [a, b] = a^2 = b^2 = c^3 = 1, \ tbt^{-1} = a \rangle = (C_2 * C_2) * C_3.
\]

We have \(\langle a, b \rangle \cong C_2^2\), and \(\langle c \rangle \cong C_3\). The group \(G\) is the fundamental group of a graph of groups \((\Gamma, \mathcal{G})\) as described in Figure 2.4. Here \(x\) denotes the generator of \(\mathcal{G}_{e_1} = C_2\).

![Figure 2.4: The graph \(\Gamma\) and graph of groups structure \(\mathcal{G}\).](image)

Base the fundamental group at \(v_1\). Thus \(t \in G\) corresponds to the loop \(e_1\). We will demonstrate folding by showing how to represent the subgroup

\[
H = \langle a, ctk, ctk^{-1} \rangle
\]

as an immersion. We will define a graph \(\Lambda\) and a morphism \(f : \Lambda \to (\Gamma, \mathcal{G})\) so that \(f_* \pi_1(\Lambda, *) = H\). Since our generating set for \(H\) has three elements, \(\Lambda\) will be a subdivision of a rose with three petals. Note that because \(v_1\) has nontrivial stabilizer, some elements of \(G\) are represented by loops of length zero. We choose loops \(\gamma_1, \gamma_2\) and \(\gamma_3\) based at \(v_1\) corresponding to the displayed generating set for \(H\).

\[
\gamma_i = \begin{cases} 
  a & i = 1 \\
  e_2c\bar{e}_2e_1b & i = 2 \\
  e_2\bar{e}_2e_1e_2c^{-1} & i = 3.
\end{cases}
\]

To better see what is happening, we will first subdivide each edge of \(\Lambda\) into three edges, and then replace the middle edges with the paths \(\gamma_i\), see Figure 2.5.
In Figure 2.5 we have described the morphism \( f : \Lambda \to (\Gamma, \mathcal{G}) \). The basepoint (starred) and the unlabeled edges are collapsed to \( v_1 \). The preferred orientation of each edge \( e \) is indicated by the arrowhead. The edge \( e \) is labeled by its image \( \delta_e f(e) \delta_e^{-1} \) in the preferred orientation, with \( \delta_e = 1 \) omitted. \( \Lambda \) has fundamental group \( F_3 \).

The morphism \( f \) is not Stallings, so we collapse a subgraph, through which \( f \) factors as \( f_1 : (\Lambda^1, \mathcal{L}^1) \to (\Gamma, \mathcal{G}) \). In \( \Lambda_1 \), the basepoint \( \ast \) has vertex group \( \mathcal{L}_\ast = \mathbb{Z} \), so \( f_1 \) is not Stallings. We perform a vertex collapse at \( \ast \). The resulting morphism \( f_2 : (\Lambda^2, \mathcal{L}^2) \to (\Gamma, \mathcal{G}) \) is Stallings. See Figure 2.6.

The morphism \( f_2 \) is not an immersion at \( \ast \). We perform three Type I folds, folding the left loop over the right loop. In order to perform the third fold, we first replace \( f \) by an equivalent morphism so that the path around the outer loop reads

\[ e_2 c \bar{e}_2 e_1 b^{-1} e_2 c^{-1} \bar{e}_2. \]

The resulting morphism \( f_4 \) is an immersion. See Figure 2.7. We see that \( H \cong F_2 * C_2 \).
Suppose now that we begin with $H' = \langle a, ctc^{-1}, cta \rangle$. The beginning steps of the folding process are essentially identical, so we omit them, but the fourth map is no longer an immersion, since $a^{-1}e_2$ and $e_2$ differ by an element of $L''_4$. We may twist the edge labeled $a^{-1}e_2$ by $a$—or, more properly speaking, by the element of $L''_4$ mapping to $a$—and then perform a Type III fold followed by a vertex collapse, yielding the Stallings morphism $f'_5$. See Figure 2.8.

From here there are two more Type I folds, and finally a Type II fold, pulling the generator of $C_2$ across the loop labeled $e_1a$, yielding $f'_6: (\Lambda^6, L'^6)$ as in Figure 2.8. Performing a final vertex collapse yields an isomorphism.
2.5 Applications to Automorphism Groups

Stallings remarks [Sta83] that an automorphism of a free group can be thought of as a subdivision of a graph followed by a sequence of folds. Bestvina and Handel [BH92] developed this insight to great effect, giving an analogue of the Nielsen–Thurston classification of surface diffeomorphisms for homotopy equivalences of graphs. We build on this work in the following chapter.

In this section we will use folding to give a conceptually simple proof of a theorem of Fouxe-Rabinovitch [FR41].

**Theorem 2.5.1.** Let \( G = A_1 \star \cdots \star A_n \star F_k \) be a finitely generated free product, where the \( A_i \) are freely indecomposable and not infinite cyclic, and \( F_k \) is a free group of rank \( k \). The group \( \text{Aut}(G) \) is generated by partial conjugations, transvections, the \( \text{Aut}(A_i) \), and a finite group \( \Sigma_G \). In particular, \( \text{Aut}(G) \) is finitely generated if each \( \text{Aut}(A_i) \) is.

Actually, Fouxe-Rabinovitch, building on earlier work of Nielsen [Nie24], gave a full presentation of \( \text{Aut}(G) \). This is a finite presentation if each \( A_i \) and each \( \text{Aut}(A_i) \) is finitely presented.

Let \( G \) be a free product as in the statement, and fix a free basis \( F_k = \langle x_1, \ldots, x_k \rangle \).

For convenience, we write \( A_{n+i} = \langle x_i \rangle \) for \( 1 \leq i \leq k \). Let \( i \) and \( j \) be distinct integers with \( 1 \leq i, j \leq n+k \), and choose \( g_i \in A_i \). The *partial conjugation of \( A_j \) by \( g_i \) is the automorphism determined by its action on the free factors as*

\[
\chi_{g_i,j} \begin{cases} 
g_j \mapsto g_i^{-1} g_j g_i & g_j \in A_j 
g_\ell \mapsto g_\ell & g_\ell \in A_\ell, \ \ell \neq j, \ 1 \leq \ell \leq n+k.
\end{cases}
\]

Now let \( i \) and \( j \) be integers with \( 1 \leq i \leq n+k \) and \( 1 \leq j \leq k \) such that \( i \neq n+j \). Choose \( g_i \in A_i \). The *left and right transvections of \( x_j \) by \( g_i \) are the automorphisms determined by their actions on the free factors as*

\[
\lambda_{g_i,j} \begin{cases} 
x_j \mapsto g_i x_j 
g_\ell \mapsto g_\ell & g_\ell \in A_\ell, \ \ell \neq n+j, \ 1 \leq \ell \leq n+k,
\end{cases}
\]

and

\[
\rho_{g_i,j} \begin{cases} 
x_j \mapsto x_j g_i 
g_\ell \mapsto g_\ell & g_\ell \in A_\ell, \ \ell \neq n+j, \ 1 \leq \ell \leq n+k,
\end{cases}
\]

respectively. In the case of the free group \( G = F_k \), the left and right transvections \( \lambda_{x_i,j} \) and \( \rho_{x_i,j} \) are called *Nielsen transformations*. These, together with a finite group \( \Sigma_{F_k} \) that permutes and inverts the \( x_i \), are the standard generators for \( \text{Aut}(F_k) \). In general, the group \( \Sigma_G \) is the product of \( \Sigma_{F_k} \) with a finite group that permutes those \( A_i \) that are isomorphic. Obviously this requires fixing some choice of isomorphism between them.

Our method of proof appears to be well-known in the case where \( G = F_k \) is a free group; we learned it from course notes of Bestvina [Bes01]. The idea is to follow up
on Stallings’s remark: given an automorphism $\Phi: F_k \to F_k$, construct a topological realization $f: \Gamma \to \Gamma$ on a graph $\Gamma$ with $\pi_1(\Gamma) = F_k$ as a subdivision of a graph followed by a sequence of folds. The result follows by carefully keeping track of the fundamental group.

In fact, we show that this method works in complete generality so long as one can carry out the first step. That is, if $G$ splits as $G = \pi_1(\Gamma, \mathcal{G})$, folding can be used to analyze any automorphism $\Phi: G \to G$ that can be topologically realized on $(\Gamma, \mathcal{G})$.

We return to the case of $G$ a free product. In the case where each $A_i$ is finitely generated, a choice of finite generating set yields a finite set of partial conjugations and transvections that suffice to generate the rest. The subgroup of $\text{Aut}(G)$ generated by partial conjugations and transvections is finitely presented if each $A_i$ is. Call this subgroup $\text{Aut}^0(G)$, and the generating set described above a set of Fouxe-Rabinovitch generators for $\text{Aut}^0(G)$. The algorithmic nature of folding yields the following useful corollary.

**Corollary 2.5.2.** There is a normal form for $\text{Aut}^0(G)$, and a method that takes as input an $(n + k)$-tuple $(g_i)$ of elements of $G$ and determines whether $\text{ad}(g_i^{-1})(A_i)$ together with $\{g_{n+1}, \ldots, g_{n+k}\}$ generate $G$, and if so produces a normal form for an automorphism $\Phi \in \text{Aut}^0(G)$ in the Fouxe-Rabinovitch generators taking the $A_i$ and a free basis of $F_k$ to $\text{ad}(g_i)(A_i)$ and $\{g_{n+1}, \ldots, g_{n+k}\}$.

In principle—but not in actual fact in general—our method is algorithmic. The problem is that arbitrary finitely-generated groups, as is well-known, are not algorithmically well-behaved. In many cases of interest, for instance when the $A_i$ are small finite groups, the method is simple enough that fairly complicated examples can be worked easily by hand or programmed into a computer. Of course, if the $\text{Aut}(A_i)$ are also amenable to algorithmic computation, the corollary gives a method for putting any automorphism of $G$ in a normal form by first folding, then performing factor automorphisms, and finally permuting isomorphic factors.

One particularly useful kind of combing would associate to a group element a quasi-geodesic normal form. Briefly, given a group $G$ and finite generating set $S$, there is an associated Cayley graph, with vertices the elements of $G$ and an edge connecting $h$ to $g$ when their difference $h^{-1}g$ belongs to $S$. Any (connected) graph can be given a geodesic metric by declaring each (open) edge isometric to an (open) unit interval and setting the distance between two points equal to the infimal length of a path connecting them. The fine structure of a Cayley graph depends very much on the choice of generating set, but the large-scale geometry turns out to be an invariant of the group. A geodesic (shortest path) from $h$ to $g$ corresponds to a way of writing $h^{-1}g$ as a product $s_1 \cdots s_\ell$ of elements of $S$ with $\ell$ as small as possible.

Let $K$ and $C$ be real numbers with $K \geq 1$ and $C > 0$, and let $(X, d_X)$ be a geodesic metric space. A $(K, C)$-quasi-geodesic $\gamma: \mathbb{R}_+ \to X$ is a ray—that is, a singly infinite path, which we assume to be parametrized by arc length—such that for all $x, y \in \mathbb{R}_+$,

$$\frac{1}{K} |y - x| - C \leq d_X(\gamma(x), \gamma(y)) \leq K |y - x| + C.$$
In other words, the true distance in $X$ between points in the image of $\gamma$ differs from their distance along $\gamma$ by a bounded multiplicative and additive error. A set of quasi-geodesic normal forms associates to each $g \in G$ a path from 1 to $g$ in the Cayley graph, i.e. a finite sequence of elements of $S$ whose product is $g$, which is $(K,C)$-quasi-geodesic in the true length of $g$. Such a set of normal forms would remain quasi-geodesic in any generating set, and could prove useful for studying the large-scale geometry of $\text{Aut}(G)$.

Unfortunately, our normal forms do not enjoy this property.

**Proposition 2.5.3.** As long as $n + k \geq 4$ or $n = 0$ and $k = 3$, the combing constructed in Corollary 2.5.2 is not quasi-geodesic.

In the case of $G$ a free group, the above proposition is due to Qing and Rafi [QR18, Theorem A]. We prove the proposition by showing that Qing and Rafi’s example occurs in free products more generally. We thank Kasra Rafi for pointing out their example to us and suggesting the proposition. The proof of the proposition amounts to working an enlightening example, so we give the proof of it first, and then the theorem.

**Proof of Proposition 2.5.3.** The case $n = 0$ and $k = 3$ is Theorem A of [QR18]. It suffices to prove the cases $n + k = 4$ where $k \in \{0, 1, 2\}$. Let $G = A * B * C * D$, where $A, B, C, D$ are nontrivial finitely generated groups, the last $k$ of which are infinite cyclic. Choose nontrivial elements $a \in A$, $b \in B$ and $c \in C$. For $m$ and $n$ positive integers, consider the following automorphisms

\[
\Phi_{0,n,m} \begin{cases} 
  x \mapsto x & x \in A * B * C \\
  d \mapsto ((ab)^{-m}c(ab)^{m}a)^{-n}d((ab)^{-m}c(ab)^{m}a)^n & d \in D,
\end{cases}
\]

\[
\Phi_{1,n,m} \begin{cases} 
  x \mapsto x & x \in A * B * C \\
  d \mapsto d((ab)^{-m}c(ab)^{m}a)^n & D = \langle d \rangle
\end{cases}
\]

and

\[
\Phi_{2,n,m} \begin{cases} 
  x \mapsto x & x \in A * B * C \\
  d \mapsto d(c(ab)^{m})^n & D = \langle d \rangle
\end{cases}
\]

**Topological realization.** We will focus on $\Phi_{1,n,m}$; the other cases are entirely analogous. Consider the graph of groups $T_{3,1}(G)$ described in Figure 2.9 below. Identify $G = \pi_1(T_{3,1}(G), \ast)$ by choosing a spanning tree and orientations as in Theorem 1.5.1 so that $d$, the generator of $D$, is represented by the loop around $e_4$ in the positive direction. A similar graph of groups $T_{n,k}(G)$ exists for $G$ an arbitrary free product. We call it the thistle with $n$ prickles and $k$ petals. It interpolates between the thistle of [Lym19] (no petals), and the rose (no prickles) commonly used to study the free group.

Let $\sigma$ be the loop $\bar{e}_1ae_1\bar{e}_2be_2$ based at $\ast$, and let $\gamma_m$ be the loop $\bar{\sigma}_m\bar{e}_3ce_3\sigma_m\bar{e}_1ae_1$, where $\sigma_m$ is the $m$-fold concatenation of $\sigma$ with itself. Note that $\Phi_{1,m,n}(d) \in G$ is represented by the loop $e_4\gamma_m^n$. A topological realization of $\Phi$ on $T_{3,1}(G)$ is given by
the map \( f : \mathcal{T}_{3,1}(G) \to \mathcal{T}_{3,1}(G) \) which is the identity on the complement of \( e_4 \) and sends \( e_4 \) to the path \( e_4 \gamma_m^n \). If we declare each point \( e_4 \) which is mapped to a vertex to be a vertex (with trivial vertex group), the map \( f \) becomes a morphism, thus it has a factorization into folds, all of which must be of Type I. Because \( e_4 \gamma_m^n \) is a tight path, \( f \) is an immersion away from the basepoint \( \star \).

**Folding Automorphisms.** As in Theorem 1.5.1, we will choose a spanning tree to identify the fundamental group of the subdivided thistle with \( G \). The obvious choice is the spanning tree containing all the edges of the subdivided thistle, save for the one mapping to \( e_4 \). Thus \( d \) is represented by the loop that runs once around the subdivision of \( e_4 \).

The last two edges of \( \gamma_m^n \) read \( \bar{e}_1ae_1 \). Thus we may fold the two edges labeled \( e_1 \) together. Before folding again, we need to twist the edge labeled \( \bar{e}_1a \) by \( a^{-1} \). This sends the loop representing \( d \) in the previous identification to the loop now representing \( da \). The twisting performed the right transvection of \( d \) by \( a \), which we will write as \( \rho_{a,d} \). We record this as the first automorphism in our normal form.

Iterating this process, we see that folding produces the normal forms

\[
\Phi_{0,n,m} = \left( (\chi_{a,d})^{-m} \chi_{c,d} \chi_{a,d} \chi_{b,d}^m \chi_{a,d} \right)^n, \\
\Phi_{1,n,m} = \left( (\rho_{a,d})^{-m} \rho_{c,d} \rho_{a,d} \rho_{b,d} \rho_{a,d} \right)^m \rho_{a,d}^n, \\
\Phi_{2,n,m} = \left( \rho_{c,d} \rho_{a,d} \rho_{b,d} \right)^m, \\
\Phi_{3,n,m} = \left( \rho_{a,d} \rho_{b,d} \right)^m, \\
\Phi_{4,n,m} = \left( \rho_{b,d} \right)^m, \\
\Phi_{5,n,m} = \left( \rho_{a,d} \right)^m, \\
\Phi_{6,n,m} = \left( \rho_{b,d} \right)^m, \\
\Phi_{7,n,m} = \left( \rho_{a,d} \right)^m, \\
\Phi_{8,n,m} = \left( \rho_{b,d} \right)^m, \\
\Phi_{9,n,m} = \left( \rho_{a,d} \right)^m.
\]
whose lengths in the Fouxe-Rabinovitch generators are $n(4m + 2)$, $n(4m + 2)$ and $n(2m + 1)$, respectively. (Automorphisms are composed as functions.)

On the other hand, one checks that

\[
\Phi_{0,n,m} = (\chi_{a,c}\chi_{b,c})^m(\chi_{c,d}\chi_{a,d})^n(\chi_{a,c}\chi_{b,c})^{-m},
\]

and

\[
\Phi_{1,n,m} = (\chi_{a,c}\chi_{b,c})^m(\rho_{c,d}\rho_{a,d})^n(\chi_{a,c}\chi_{b,c})^{-m},
\]

whose lengths in the Fouxe-Rabinovitch generators are $4m + 2$, $4m + 2$ and $4m + n$, respectively. As $n$, and $m$ go to infinity, this shows that our normal forms cannot be $(K,C)$-quasi-geodesic for any choice of $(K,C)$.

**Proof of Theorem 2.5.1.** All the ideas are already present in the proof above. A famous theorem of Grushko says that if $G = A_1 \ast \cdots \ast A_n \ast F_k$ and $G = B_1 \ast \cdots \ast B_m \ast F_\ell$, where the $A_i$ are freely indecomposable and not infinite cyclic, then $m = n$, $k = \ell$ and after reordering, $B_i$ is conjugate to $A_i$, and no element of $F_\ell$ is conjugate into any $A_i$. Thus if $\Phi: G \rightarrow G$ is an automorphism, it must permute the conjugacy classes of the $A_i$.

We conclude that $\Phi$ can be topologically realized by $f: T_{n,k}(G) \rightarrow T_{n,k}(G)$ using exactly the same process as above. Since $\Phi$ composed with the surjection of $G$ onto $F_k$ is a surjection, we conclude that $f$ is a homotopy equivalence of the underlying graph. Thus after subdividing we may factor $f$ into a product of folds, followed by an isomorphism. The possible isomorphisms of $T_{n,k}(G)$ include elements of $\text{Aut}(A_i)$ and the finite group $\Sigma_G$.

At each step, whenever possible, we fold edges belonging to the maximal tree, possibly passing to equivalent automorphisms, performing twist automorphisms or changing the maximal tree in order to keep folding. Both of the latter operations result in automorphisms of $G$, but the former does not.

In the previous proposition, we analyzed the effect of twisting an edge $e$ by $g^{-1}$. It may happen that a single twist induces a commuting product of transvections and partial conjugations by $g$ or $g^{-1}$, based on whether the loop corresponding to a generator traverses $e$ once or twice, and in which direction.

Consider changing the maximal tree $T$ by adding an edge $e$ that is not a loop and removing $e'$. The edges of the loop $\gamma_{\tau(e)}e\gamma_{\tau(e)}$ determine a cycle in the homology of the graph, we may choose any oriented edge appearing with positive exponent in this cycle as our $e'$. If the loop $\gamma_{\tau(e)}e\gamma_{\tau(e)}$ represented the generator $x_i$, changing the maximal tree corresponds to a commuting product of transvections and partial conjugations of generators not equal to $x_k$ by $x_k$ and $x_k^{-1}$, according to whether the loop corresponding to a generator traverses $e'$ once or twice, and in which direction. A similar statement holds for folding an edge $e'$ belonging to the maximal tree across a loop edge $e$. Writing down the automorphisms in the order they occur, we can write any element of $\text{Aut}(G)$ in the Fouxe-Rabinovitch generators.

**Proof of Corollary 2.5.2.** To make the foregoing discussion algorithmic, we need only stipulate how we fold, and place the generators that occur in a commuting
product in some order. For this step we assume that there is a normal form for elements of each $A_i$ and an algorithm to put elements of $A_i$ in normal form. For the former question, we will perform all folds that do not result in automorphisms first. Then, we perform the automorphism where the acting letter, the $i$ in $\chi_{g_{i,j}}$ or $\rho_{g_{i,j}}$, is smallest. When we change the maximal tree, choose the edge $e'$ in the cycle whose initial vertex is closest to the terminal vertex of $e$. Order the commuting product so that the smallest $j$ appears first.

To determine whether an $(n + k)$-tuple $(g_i)$ of elements of $G$ corresponds, as in the statement, to a generating set, choose tight paths $\gamma_i$ in $T_{n,k}(G)$ representing them, and form the map $f: T_{n,k}(G) \to T_{n,k}(G)$ with $f(e_i) = \gamma_i$. Factor $f$ into a product of folds. If the last morphism is not an isomorphism, the tuple does not determine a generating set. Otherwise we have a normal form for it as above.

Remark 2.5.4. As we remarked earlier, nothing about our method of proof is special to $G$ a free product, except perhaps the guarantee that every automorphism $\Phi: G \to G$ may be topologically realized on $T_{n,k}(G)$, and our good understanding of automorphisms of $T_{n,k}(G)$.
Chapter 3

Train Track Maps

As we have already seen, representing an automorphism of $G$ as a map on a graph of groups with fundamental group $G$ is a fruitful way to analyze its structure. Ideally, one wants such a representative to reveal aspects of the dynamics of the action of the automorphism on the fundamental group, similar to the Jordan normal form of a linear transformation. In a seminal paper, Bestvina and Handel [BH92] showed that every outer automorphism $\varphi \in \text{Out}(F_n)$ has such a representative, called a relative train track map.

The inspiration for their construction was the Nielsen–Thurston classification of surface diffeomorphisms. Let $S$ be an orientable surface with negative Euler characteristic. The mapping class group of $S$, Mod($S$), also sometimes called the Teichmüller modular group of $S$, is the group of homotopy classes of orientation-preserving diffeomorphisms of $S$. Thurston [Thu88] introduced pseudo-Anosov diffeomorphisms $f : S \rightarrow S$ and showed that for each element $\varphi \in \text{Mod}(S)$, either

1. $\varphi$ has finite order,

2. $\varphi$ is reducible, meaning it preserves some finite set of homotopy classes of essential, simple closed curves on $S$, or

3. $\varphi$ is represented by a pseudo-Anosov diffeomorphism.

In the reducible case, one may cut $S$ along the preserved collection of curves and continue classifying. Thurston proves that pseudo-Anosov representatives are “efficient” representatives: they are unique within their homotopy class and minimize a quantity called topological entropy.

A train track map (see below for a definition) is analogous to a pseudo-Anosov diffeomorphism in that it is an efficient representative for an element of Out($F_n$). Although train track maps are typically not unique, it is still true that they minimize a form of topological entropy.

Just as not all elements of Mod($S$) are represented by pseudo-Anosov diffeomorphisms, not all elements of Out($F_n$) are represented by train track maps. A relative train track map parallels the reducible case above: rather than cutting a surface, however, the metaphor is collapsing a subgraph. The purpose of this chapter is to
extend Bestvina and Handel’s construction to automorphisms representable on a graph of groups.

As an application, we answer affirmatively a question of Paulin [Pau91], who asked whether, for a word-hyperbolic group \( G \), there exists a trichotomy for outer automorphisms \( \varphi \in \text{Out}(G) \). Paulin’s trichotomy is a common generalization of the situation for \( G \) a free group and for \( G \) the fundamental group of a closed surface with negative Euler characteristic. See Section 3.8 below for details.

### 3.1 Train Track Maps on Graphs

Let \( \Gamma \) be a graph. A homotopy equivalence \( f: \Gamma \to \Gamma \) is a train track map if for all \( k \geq 0 \), the restriction of \( f^k \) to each edge is an immersion. The definition is due to Thurston. Now, in general \( f \) itself cannot be an immersion: since \( f_\sharp: \pi_1(\Gamma) \to \pi_1(\Gamma) \) is an isomorphism, if \( f \) were an immersion, the results of the previous chapter would imply that \( f \) is an automorphism of \( \Gamma \), and would therefore have finite order.

**An application of Thurston.** Let us illustrate one useful consequence of having a train track map for an outer automorphism \( \varphi \in \text{Out}(F_n) \). The following is due to Thurston [BH92, Remark 1.8]. We may associate a nonnegative matrix \( M \) with integer entries to any self-map of a finite graph sending edges to edge paths, once we fix an ordering of the edges. The \( ij \)th entry of the matrix \( M \) records the number of times the image of the \( j \)th edge crosses the \( i \)th edge in either direction. If \( f: \Gamma \to \Gamma \) is a train track map, then the number of edges crossed by \( f^k(e_j) \) is the sum of the entries in the \( j \)th column of \( M^k \).

Let us make the additional assumption that the transition matrix \( M \) is irreducible (we give the definition below). Every irreducible nonnegative integer matrix \( M \) has a Perron–Frobenius eigenvalue \( \lambda \geq 1 \), and a corresponding eigenvector \( \vec{v} \) with positive entries [Sen81]. If we metrize \( \Gamma \) by assigning each edge \( e_j \) a length of \( \vec{v}_j \), then \( f \) expands each edge by a factor of \( \lambda \).

Suppose our train track map \( f: \Gamma \to \Gamma \) represents \( \varphi \in \text{Out}(F_n) \). Although outer automorphisms do not act on elements of \( F_n \), they do act on their conjugacy classes. If \( c \) is such a conjugacy class, let \( |c| \) denote the minimum word length of a representative in some fixed finite generating set of \( F_n \). The exponential growth rate of \( c \) under \( \varphi \) is

\[
\text{EGR}(\varphi, c) = \limsup_{n \to \infty} \frac{\log |\varphi^n(c)|}{n}.
\]

A quick calculation reveals that distorting the word length on \( F_n \) by a quasi-isometry does not alter \( \text{EGR}(\varphi, c) \). In particular, the length of an immersed loop representing \( \varphi(c) \) in the metric above will do, and we conclude that \( \text{EGR}(\varphi, c) \leq \log \lambda \), with equality when \( f \) restricted to the immersed loop determined by \( c \) is an immersion.

In fact, it turns out that for any conjugacy class \( c \), \( \text{EGR}(\varphi, c) \) is either \( \log \lambda \) or 0, with the latter occurring only when \( \varphi \) acts periodically on \( c \). The key tool in completing the proof is Thurston’s Bounded Cancellation Lemma [Coo87]. We will
complete the argument below (see Remark 3.6.4) once we are equipped with more vocabulary.

Now suppose that \( f : \Gamma \to \Gamma \) takes edges to edge paths, but that \( f \) is not a train track map. In this case, the Perron–Frobenius eigenvalue of the associated matrix \( M \) overestimates the exponential growth rate. If one could show that in this situation \( f \) may be altered so that the eigenvalue \( \lambda \) decreases, it follows that \( f \) is a train track map when \( \lambda \) reaches a minimum. Furthermore, one could hope to find a train track map algorithmically by applying certain moves to decrease the eigenvalue, and arguing that a minimum must be reached after iterating this process finitely many times. In fact, this is exactly what Bestvina and Handel do.

### 3.2 Train Track Maps on Graphs of Groups

Notice that if \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) is a topological realization of an outer automorphism of \( \pi_1(\Gamma, \mathcal{G}) \), it makes sense to ask whether \( f \) is a train track map. As foreshadowed in the previous chapter, it turns out that the whole theory goes through so long as one can carry out the first step. The first main result of this chapter is the following theorem.

**Theorem 3.2.1.** Let \( G \) be a finitely generated group. If \( \varphi \in \text{Out}(G) \) can be realized on some nontrivial splitting of \( G \) with finitely generated edge groups, then there exists a nontrivial splitting \( G = \pi_1(\Gamma, \mathcal{G}) \) with finitely generated edge groups and an irreducible train track map \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) realizing \( \varphi \).

**Remark 3.2.2.** If in the statement above, \( \varphi \) was originally realized on the splitting \( (\Lambda, \mathcal{L}) \), then, modulo appropriate subdivisions, there is a morphism \( (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G}) \) inducing an isomorphism of fundamental groups, but there may not be any such morphism in the other direction. In the above situation we say that \( (\Lambda, \mathcal{L}) \succeq (\Gamma, \mathcal{G}) \), but we may not have \( (\Lambda, \mathcal{L}) \cong (\Gamma, \mathcal{G}) \). In terms of the \( G \)-action on the Bass–Serre tree, every vertex stabilizer of \( \tilde{\Lambda} \) stabilizes some vertex of \( \tilde{\Gamma} \), but some elements may stabilize vertices of \( \tilde{\Gamma} \) while acting freely on \( \tilde{\Lambda} \).

In Bestvina and Handel’s original formulation, not every \( \varphi \in \text{Out}(F_n) \) is represented by a train track map. The obstruction to promoting \( f : \Gamma \to \Gamma \) to an irreducible train track map is finding a noncontractible \( f \)-invariant subgraph. In this case, Bestvina and Handel find a stratification of \( \Gamma \) into subgraphs and a relative train track map, which preserves the stratification, and resembles a train track map on each stratum. We will give a more precise definition below.

In our proof of Theorem 3.2.1, \( f \)-invariant subgraphs are collapsed, thus we end up with an irreducible train track map. If \( (\Lambda, \mathcal{L}) \) is an \( f \)-invariant subgraph of groups, it makes sense to restrict \( f \) to \( (\Lambda, \mathcal{L}) \), and we can apply Theorem 3.2.1 again. If we began with a finite graph, the process terminates in finitely many steps, and we end up with a stratified collection of train track maps, where train track maps appearing lower in the stratification happen within vertex groups of some higher-stratum train track map.
Thus one way to prove the existence of relative train track maps from the existence of train track maps is to carefully “blow up” the vertex groups of the higher strata to yield a realization on a stratified graph of groups. Another way is to adapt the original Bestvina–Handel argument. This is the second main result of the chapter.

**Theorem 3.2.3.** Let $G$ be a finitely generated group. If $\varphi \in \text{Out}(G)$ is realized by $f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$, for some splitting $G = \pi_1(\Gamma, \mathcal{G})$ with finitely generated vertex and edge groups, then there is a relative train track map $f' : (\Gamma', \mathcal{G}') \to (\Gamma', \mathcal{G}')$ realizing $\varphi$ such that $(\Gamma, \mathcal{G}) \simeq (\Gamma', \mathcal{G}')$.

A satisfying consequence of the thorough understanding of morphisms of graphs of groups we developed in previous chapters is that adapting the original proof poses relatively few new difficulties.

**Definition 3.2.4.** The collection of $\varphi \in \text{Out}(G)$ realizable on a given splitting $G = \pi_1(\Gamma, \mathcal{G})$ form a subgroup, which we will call the modular group of $(\Gamma, \mathcal{G})$, $\text{Mod}(\Gamma, \mathcal{G})$.

Because every homotopy equivalence of a surface $S$ that restricts to a homeomorphism on the boundary of $S$ is homotopic to a diffeomorphism, $\text{Mod}(S)$ may be defined as the group of homotopy classes of orientation-preserving homotopy equivalences $f : S \to S$. This parsimony of definitions is the source of our term.

There are other candidates for a name. Sykiotis [Syk04] calls such outer automorphisms symmetric. In some situations, $\text{Mod}(\Gamma, \mathcal{G})$ coincides with the subgroup of $\text{Out}(G)$ preserving the deformation space of $G$-trees $\mathfrak{T}$ containing $\tilde{\Gamma}$, which Guirardel and Levitt refer to as $\text{Out}_{\mathfrak{T}}(G)$ [GL07a]. Symmetric automorphisms of free groups already refer to a particular $\text{Mod}(\Gamma, \mathcal{G})$ on the one hand, and in certain cases, e.g. the Baumslag–Solitar group $BS(2, 4)$, the modular group of a graph of groups is much smaller than $\text{Out}_{\mathfrak{T}}(G)$.

If $G$ is a free product $A_1 \ast \cdots \ast A_n \ast F_k$, where the $A_i$ are freely indecomposable and not infinite cyclic, the argument in the previous chapter shows that $\text{Out}(G) = \text{Mod}(\mathcal{T}_{n,k}(G))$. The existence of relative train track maps for free products has been proven several times, first by Collins and Turner [CT94], and later by Sykiotis [Syk04] (who allows certain finite edge groups), Francaviglia and Martino [FM15], and most recently by the present author [Lym19]. This chapter generalizes the early sections of [Lym19] and comprises parts of a future revision of that paper.

Recently Feighn and Handel [FH11], building on earlier work of Bestvina, Feighn and Handel [BFH00], developed an improvement of relative train track maps they call completely split relative train track maps, CTs for short. We plan to show how to construct CTs on graphs of groups, but leave that to the future.

**Hierarchical structure and a question.** Before we proceed, let us note an interesting connection we are presently unable to tease out in full. Let $S$ be an orientable surface. The curve complex of $S$, denoted $\mathcal{C}(S)$, is a simplicial complex with vertices isotopy classes of essential, simple closed curves on $S$, and with an
edge between two vertices when the associated curves may be realized disjointly. The complex is flag; a higher dimensional simplex is present if and only if its 1-skeleton is.\footnote{Warren Dicks: “every non-simplex contains a non-edge.”} \(\text{Mod}(S)\) acts on \(\mathcal{C}(S)\), and a celebrated theorem of Masur and Minsky [MM99] says that \(\mathcal{C}(S)\) is \textit{Gromov-hyperbolic}, that is, negatively curved in a large-scale sense.

A mapping class \(\varphi \in \text{Mod}(S)\) is pseudo-Anosov if and only if it has no periodic orbit in \(\mathcal{C}(S)\), and reducible if and only if \(\varphi\) fixes a simplex \(\sigma\). In the latter case, \(\varphi\) acts on the subcomplex whose vertices are all adjacent to each vertex of \(\sigma\). This turns out to be the curve complex of the surface obtained by cutting along the curves making up the vertices of \(\sigma\). Thus one may understand elements of \(\text{Mod}(S)\) via their action on a hierarchy of hyperbolic spaces. This situation has been codified by Behrstock, Hagen and Sisto [BHS17] into the study of \textit{hierarchically hyperbolic spaces}.

One interesting aspect of Theorem 3.2.1 is how it parallels the hierarchical situation in the mapping class group. Indeed, in the case where \(G\) is a free product, there are a number of Gromov-hyperbolic complexes that \(\text{Out}(G)\) acts on that could play the role of curve complexes.

However, there must be some subtlety: in [BHS19], Behrstock, Hagen and Sisto adapt an argument of Bowditch to show that hierarchically hyperbolic groups—very roughly, those groups with a nice action on a hierarchically hyperbolic space—satisfy a \textit{quadratic isoperimetric inequality}. On the other hand, Handel and Mosher [HM13] and Bridson and Vogtmann [BV12] showed that for \(n \geq 3\), the optimal isoperimetric inequality—also called the \textit{Dehn function}—for \(\text{Out}(F_n)\) is exponential.

Notably, this argument does not show \textit{where} the hierarchically hyperbolic machinery fails. It would be interesting to know for which splittings \((\Gamma, \mathcal{G})\), if any, the hierarchical structure gestured at here shows that \(\text{Mod}(\Gamma, \mathcal{G})\) is a hierarchically hyperbolic group. Such a question likely amounts to asking for the Dehn function of \(\text{Mod}(\Gamma, \mathcal{G})\), which remains unknown even for \(G\) a free product that is not a free group.

### 3.3 Topology of Finite Graphs of Groups

In the preceding chapters, we have been carefully describing graphs of groups and morphisms between them in an essentially algebraic or combinatorial way. Indeed, in principle everything that follows could be expressed this way. We find it clarifying to instead adopt the usual language of topology.

Let \((\Lambda, \mathcal{L})\) and \((\Gamma, \mathcal{G})\) be graphs of groups with fundamental group isomorphic to \(G\). Let \(f : \Lambda \to \bar{\Gamma}\) be a \(G\)-equivariant continuous map. Each edge in the fundamental domain of \(\Lambda\) is sent to a path (a continuous map of the interval) in \(\bar{\Gamma}\), which is homotopic rel endpoints to a piecewise-linear embedding with respect to some metric or to a constant map—a \textit{tight edge path}, in other words. Additionally, each \(G\)-equivariant map is equivariantly homotopic to a map that sends vertices to vertices.
In sum, every $G$-equivariant map $f: \tilde{\Lambda} \to \tilde{\Gamma}$ is equivariantly homotopic to a map $f': \tilde{\Lambda} \to \tilde{\Gamma}$ which satisfies the following conditions.

1. After subdividing each edge of $\tilde{\Lambda}$ into finitely many edges, $f': \tilde{\Lambda} \to \tilde{\Gamma}$ becomes a morphism,

2. $f'$ sends vertices to vertices, and

3. $f'$ sends edges to tight edge paths.

In fact, the same is true of the induced map $\left(\Lambda, \mathcal{L} \right) \to \left(\Gamma, G\right)$. We specialize this discussion to self-maps.

**Definition 3.3.1.** Let $\varphi \in \text{Out}(G)$ be an outer automorphism, and $(\Gamma, \mathcal{I})$ a graph of groups with $\pi_1(\Gamma, \mathcal{I}, \ast) = G$. A map $f: (\Gamma, \mathcal{I}) \to (\Gamma, \mathcal{I})$ is a topological realization of $\varphi$ if it satisfies the three conditions above, does not collapse edges, and the induced outer action of $f$ on $(\Gamma, \mathcal{I}, \ast)$ is $\varphi$. (Note that we do not require $f$ to fix the basepoint.)

Thus up to homotopy, every $G$-equivariant map of $G$-trees may be represented on the quotient graph of groups. There is however an additional point of subtlety: the translation between the two depends on a fixed choice of fundamental domain in each $G$-tree. In the quotient graph of groups, this is the same data as fixing an identification of the fundamental group with $G$.

### 3.4 Markings

In what follows, we will begin with an element $\varphi \in \text{Mod}(\Gamma, \mathcal{I})^2$ and a topological realization $f: (\Gamma, \mathcal{I}) \to (\Gamma, \mathcal{I})$. To promote $f$ to a train track map, we will perform a series of operations which change $(\Gamma, \mathcal{I})$. For $f$ to remain a topological realization of $\varphi$, we need a way of keeping track of the fundamental group.

To do this, fix once and for all a reference graph of groups $G$, a vertex $\ast \in G$, and an identification $\pi_1(G, \ast)$ with $G$. For example, if $G$ is a free product, we may take $G$ to be the thistle $T_{n,k}(G)$, and $\ast$ to be the vertex of $T_{n,k}(G)$.

**Definition 3.4.1.** A marked graph of groups is a graph of groups $(\Gamma, \mathcal{I})$ together with a homotopy equivalence $\tau: G \to (\Gamma, \mathcal{I})$. That is, $\tau$ is a (continuous) map, and there exists a (not unique) map $\sigma: (\Gamma, \mathcal{I}) \to G$ such that $\tau \sigma$ and $\sigma \tau$ are each homotopic to the identity. Thus $(\Gamma, \mathcal{I}) \simeq G$.

To prove Theorem 3.2.1, it may happen that we need to pass to a graph of groups $(\Gamma', \mathcal{I}')$ such that $G \preceq (\Gamma', \mathcal{I}')$. The induced map $\tau': G \to (\Gamma', \mathcal{I}')$ will fail to admit a homotopy inverse, essentially because it collapses certain edges. Nonetheless, by the results of the previous chapter, we may collapse subgraphs to yield $G'$ and a homotopy equivalence $\tau'': G' \to (\Gamma', \mathcal{I}')$, which will play the role of a marking. We will typically leave this implicit in what follows.

---

2 We are tempted to call it a mapping class.
3.5 Irreducibility

Fix $\varphi \in \text{Mod}(G)$, and a topological realization $f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ of $\varphi$. Label the edges of $\Gamma$ as $e_1, \ldots, e_m$.

**Definition 3.5.1.** The transition matrix $M$ of $f$ is an $m \times m$ matrix with $ij$th entry $m_{ij}$ equal to the number of times the edge path $f(e_j)$ contains $e_i$ in either orientation. A matrix with nonnegative integer entries is irreducible if for each pair $i$ and $j$, there exists some positive integer $k$ such that the $ij$th entry of $M^k$ is positive.

**Remark 3.5.2.** Here is a simple criterion for checking whether such a matrix $M$ is irreducible. Create a directed graph $\Gamma_M$ with vertices $v_1, \ldots v_m$ in one-to-one correspondence with the columns of $M$. Draw $m_{ij}$ directed edges from $v_j$ to $v_i$. $M$ is irreducible if and only if $\Gamma_M$ is strongly connected. This means that for each vertex $v \in \Gamma_M$, there exists directed paths connecting $v$ to every vertex of $\Gamma_M$ (including $v$ itself).

**Example 3.5.3.** Consider the free product

$$G = C_2 * C_2 * C_2 * C_2 = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1 \rangle$$

and an automorphism $\Phi : G \to G$ defined by its action on generators as

$$\Phi = \begin{cases} a \mapsto b \\ b \mapsto c \\ c \mapsto d \\ d \mapsto cbdadbc. \end{cases}$$

(Notice that, e.g. $c^{-1} = c$.) Let $\mathcal{T}_4 = \mathcal{T}_{4,0}(G)$. A topological realization $f : \mathcal{T}_4 \to \mathcal{T}_4$ of $\Phi$ is described in Figure 3.1. The transition matrix of $f$ is

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$ 

One checks that $M$ is irreducible.

The following theorem is the main tool for the proof of Theorem 3.2.1.

**Theorem 3.5.4** (Perron–Frobenius [Sen81]). Let $M$ be an irreducible, nonnegative integral matrix. Then there is a unique positive eigenvector $\vec{v}$ with norm 1 whose associated eigenvalue $\lambda$ satisfies $\lambda \geq 1$. If $\lambda = 1$, then $M$ is a transitive permutation matrix. If $\vec{w}$ is a positive vector and $\mu > 0$ satisfies $(M\vec{w})_i \leq \mu \vec{w}_i$ for all $i$ and $(M\vec{w})_j < \mu \vec{w}_j$ for some $j$, then $\lambda < \mu$. 

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If the transition matrix of a topological realization $f: (\Gamma, \mathcal{G}) \to (\Gamma', \mathcal{G}')$ is irreducible, we say $f$ is irreducible, and we call the eigenvalue $\lambda$ in the statement above the Perron–Frobenius eigenvalue of $f$.

For the topological realization $f: T_4 \to T_4$ in Example 3.5.3, the Perron–Frobenius eigenvalue $\lambda$ is the largest root of the polynomial $x^4 - 2x^3 - 2x^2 - 2x - 1$; we have $\lambda \approx 2.948$.

**Remark 3.5.5.** Note that the theorem implies that if $M$ and $M'$ are irreducible matrices with $m_{ij} \leq m'_{ij}$ for all pairs $i$ and $j$ and with strict inequality in at least one pair, then the associated Perron–Frobenius eigenvalues $\lambda$ and $\lambda'$ satisfy $\lambda < \lambda'$.

**Collapsing revisited.** If $f: (\Gamma, \mathcal{G}) \to (\Gamma', \mathcal{G}')$ is not irreducible, then the graph associated to the transition matrix for $f$ is not strongly connected. This means there is some edge of $\Gamma$ whose forward orbit under $f$ is contained within some (possibly disconnected) proper $f$-invariant subgraph of $\Gamma$.

One possibility is that $\tilde{\Gamma}$ is not minimal, i.e. $\Gamma$ contains a valence-one vertex $v$ with incident edge $e$ such that $\iota_e: \mathcal{G}_e \to \mathcal{G}_v$ is an isomorphism. We will show that such valence-one vertices may be safely removed.

Here is the other possibility. Suppose $\Gamma_0$ is a proper $f$-invariant subgraph of $\Gamma$ which is maximal with respect to inclusion (among proper subgraphs, naturally). Let $(\Gamma_1, \mathcal{G}_1)$ be the graph of groups obtained by collapsing each component of $\Gamma_0$ and let $p: (\Gamma, \mathcal{G}) \to (\Gamma_1, \mathcal{G}_1)$ be the quotient map. By Proposition 2.2.1, $f$ yields a map $f_1: (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1)$ such that the following diagram commutes

$$
\begin{array}{ccc}
(\Gamma, \mathcal{G}) & \xrightarrow{f} & (\Gamma, \mathcal{G}) \\
\downarrow{p} & & \downarrow{p} \\
(\Gamma_1, \mathcal{G}_1) & \xrightarrow{f_1} & (\Gamma_1, \mathcal{G}_1).
\end{array}
$$

If $\tau: \mathcal{G} \to (\Gamma, \mathcal{G})$ is the original marking, then $p\tau$ is the new marking. Because $f$ sends edges to tight paths, for each edge $e$ that is not collapsed, any maximal segment of $f(e)$ contained in $\Gamma_0$ is not null-homotopic rel endpoints. It follows that $f_1$ sends edges to tight paths. Since $\Gamma_0$ was assumed to be maximal, $f_1$ does not collapse edges and is thus a topological realization. Since $\Gamma_0$ was a maximal invariant
subgraph, if $\tilde{\Gamma}_1$ is minimal, then $\Gamma_1$ has no proper $f_1$-invariant subgraph, so $f_1$ is irreducible. The transition matrix for $f_1$ is obtained from that for $f$ by removing the rows and columns corresponding to the edges of $\Gamma_0$.

We clearly have $(\Gamma, \mathcal{G}) \succeq (\Gamma_1, \mathcal{G}_1)$. In order to have $(\Gamma, \mathcal{G}) \simeq (\Gamma_1, \mathcal{G}_1)$, each $g \in G$ must fix a point of $\tilde{\Gamma}_1$ if and only if $g$ fixes a point of $\tilde{\Gamma}$. This happens if and only for each component $(\Lambda, \mathcal{L})$ of $\Gamma_0$, each element of $\pi_1(\Lambda, \mathcal{L})$ fixes a point of $\tilde{\Lambda}$. It follows by finite generation of $G$ that there is a point of $\tilde{\Lambda}$ fixed by all of $\pi_1(\Lambda, \mathcal{L})$; see for instance [Tit77]. $\tilde{\Lambda}$ is equivariantly contractible to this fixed point. Thus $\Lambda$ must be a tree, and some vertex $v \in \Lambda$ satisfies $\mathcal{L}_v \simeq \pi_1(\Lambda, \mathcal{L})$.

In this case we say $(\Lambda, \mathcal{L})$ is a contractible tree in $(\Gamma, \mathcal{G})$. It is nontrivial if it contains an edge. (This definition is standard for train tracks, but nonstandard for graph theory.) If each component of a subgraph $\Gamma_0 \subseteq \Gamma$ determines a contractible tree in $(\Gamma, \mathcal{G})$, we say $\Gamma_0$ is a contractible forest. It is nontrivial if some component is nontrivial. A (necessarily contractible) forest is pretrivial for a homotopy equivalence $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ if each edge contained in the forest is eventually mapped to a point by some iterate of $f$. If $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ only fails to be a topological realization because it collapses edges, collapsing a maximal pretrivial forest yields a topological realization.

In what follows, whenever we collapse invariant subgraphs, we will always be working with a marked graph of groups $(\Gamma, \mathcal{G})$ with $\tilde{\Gamma}$ minimal. This avoids the silly case where collapsing a maximal invariant subgraph causes $G$ to act with global fixed point on the resulting Bass–Serre tree.

**Definition 3.5.6.** We say that $\varphi \in \text{Mod}(G)$ is irreducible if for every topological realization $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ with $G \simeq (\Gamma, \mathcal{G})$ such that $\tilde{\Gamma}$ is minimal and $\Gamma$ contains no nontrivial $f$-invariant contractible forests, we have that $f$ is irreducible.

In general, if $\varphi$ belongs to $\text{Mod}(G)$, where the underlying graph of $G$ is finite, then $\varphi$ is irreducible in some $\text{Mod}(G')$, where $G \succeq G'$—if only because every topological realization on a graph of groups with one edge is homotopic to an automorphism.

### 3.6 Operations on Topological Realizations

Suppose a topological realization $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ is not a train track map. Thus there is some edge $e$ of $\Gamma$ and $k > 1$ such that $f^k|_e$ is not an immersion. By the results of the previous chapter, $f^k|_e$ factors through a fold. After performing that fold, we may be able to tighten $f$, thus decreasing the associated Perron–Frobenius eigenvalue.

**Definition 3.6.1.** A turn based at a vertex $v$ in a marked graph of groups $(\Gamma, \mathcal{G})$ is an unordered pair of oriented edges $\{ \tilde{e}, \tilde{e}' \}$ in $\text{st}(\tilde{v})$, where $\tilde{v}$ is a lift of $v$ to $\tilde{\Gamma}$. Equivalently, a turn is a pair of elements $\{ ([g], e), ([g'], e') \}$ of the set

$$\prod_{e \in \text{st}(\tilde{v})} \mathcal{G}_e / \iota_e(\mathcal{G}_e) \times \{ e \}.$$
We will use both definitions interchangeably. A turn is nondegenerate if \( \hat{e} \neq \hat{e}' \), otherwise it is degenerate.

**Remark 3.6.2.** Our definition of \( \text{st}(v) \) as those oriented edges with terminal vertex \( v \) means that our orientation convention for turns is the opposite of the \( \Out(F_n) \) literature.

A topological realization \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) yields a self-map \( Df \) of the set of turns in \( (\Gamma, \mathcal{G}) \). If \( \hat{f} \) is the morphism determined by subdividing \( f \), we have

\[
Df\{([g], e), ([g'], e')\} = \{\hat{f}_{\text{st}(v)}([g], e), \hat{f}_{\text{st}(v)}([g'], e')\},
\]

where \( \hat{f}_{\text{st}(v)} \) is the map defined in Section 1.10. If \( f(eg) = g_i e_1 \cdots e_k g_{k+1} \), we have \( \hat{f}_{\text{st}(v)}([g], e) = ([g_{k+1}], e_k) \).

In Example 3.5.3, \( * \) is mapped to itself by \( f \), and the restriction of \( Df \) to \( * \) is determined by the dynamical system \( e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \).

**Definition 3.6.3.** A turn is illegal if its image under some iterate of \( Df \) is degenerate. It is legal otherwise.

Observe that \( \mathcal{G}_v \) acts on the set of turns based at \( v \) by left multiplication on the cosets, and \( Df \) respects this action. Thus legality and illegality is a property of a turn’s orbit under \( \mathcal{G}_v \).

In Example 3.5.3, a turn \( \{e_i, e_j\} \) based at \( \ast \) is illegal if \( i \equiv j \mod 2 \), and is legal otherwise.

Consider the edge path

\[
\gamma = g_1 e_1 \cdots e_k g_{k+1}.
\]

We say \( \gamma \) crosses or takes the turns \( \{([1], e_i), ([g_{i+1}], \bar{e}_{i+1})\} \) for \( 1 \leq i \leq k \). A path \( \gamma \) is legal if it takes only legal turns, and is illegal otherwise. Thus a topological realization \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) is a train track map if and only if \( f \) sends each edge of \( \Gamma \) to a legal path. In Example 3.5.3, \( f \) is not a train track map because the \( f \)-image of \( e_4 \) takes the illegal turn \( \{\bar{e}_4, \bar{e}_2\} \).

**Remark 3.6.4.** We can now finish the argument of Thurston. Suppose \( f : \Gamma \to \Gamma \) is an irreducible train track map representing \( \varphi \in \Out(F_n) \) with Perron–Frobenius eigenvalue \( \lambda > 1 \), and that \( c \) is a conjugacy class in \( F_n \). We may represent \( c \) by an immersion \( \sigma : S^1 \to \Gamma \). As we argued above, there exists a metric on \( \Gamma \) with respect to which \( f \) expands lengths of legal paths by a factor of \( \lambda \). For \( \gamma : S^1 \to \Gamma \), let \( |\gamma| \) denote the length of the unique immersed circle homotopic to \( \gamma \) in this metric.

Since \( f \) sends legal paths to legal paths, for all \( k \geq 0 \), \( f^k(\sigma) \) has a finite number of number of illegal turns bounded independently of \( k \). At each illegal turn, there may be nontrivial cancellation, so the length of an immersed circle homotopic to \( f^k(\sigma) \) is a priori not greater than \( \lambda^k(\sigma) \). On the other hand, the Bounded Cancellation Lemma [Coo87] implies that the length of cancellation at any illegal turn is bounded independently of \( k \) and \( \sigma \). Thus we have

\[
\lambda^k|\sigma| - K \leq |f^k\sigma| \leq \lambda^k|\sigma|,
\]

for some constant \( K \). It follows that either \( |f^k(\sigma)| \) is bounded independent of \( k \), or \( \text{EGR}(\varphi, c) = \log \lambda \).
Example 3.5.3 continued. Let us fold $f: T_4 \to T_4$ at the illegal turn $\{e_4, e_2\}$. To do this, subdivide $e_4$ at the preimage of the vertex with vertex group $\langle c \rangle$ into the edge path $e'_4e''_4$, and fold $e''_4$ with $e_2$. The action of the resulting map $f': (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1)$ on edges is obtained from $f$ by replacing instances of $e_4$ with $e'_4e_2$. Therefore

$$f'(e'_4) = e_1\bar{e}_2e'_4de'_4e_2\bar{e}_2\bar{e}_3c.$$ 

We may tighten $f'$ by a homotopy with support on $e'_4$ to remove $e_2\bar{e}_2$, yielding an irreducible topological realization $f_1: (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1)$. See Figure 3.2.

![Figure 3.2: The topological realization $f_1: (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1)$.](image)

The Perron–Frobenius eigenvalue $\lambda_1$ for $f_1: (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1)$ is the largest root of the polynomial $x^4 - 2x^3 - 2x^2 + x - 1$ and satisfies $\lambda_1 \approx 2.663$; thus $\lambda_1 < \lambda$.

However, $f_1$ is still not a train track map: $Df_1$ sends the turn $\{(1, e_4), (b, \bar{e}_2)\}$ crossed by $f_1(e'_4)$ to $\{(c, \bar{e}_3), (c, \bar{e}_3)\}$, which is thus illegal. As explained in the previous chapter, in order to fold again, we need to twist the edge $e'_4$ by $b^{-1} = b$. This changes the marking by replacing $\langle d \rangle$ with $\langle bdb \rangle$. This also replaces $f_1(e_3)$ with $e'_4b\bar{e}_2$ and replaces $f_1(e'_4)$ with $e_1\bar{e}_2b\bar{e}'_4de'_4\bar{e}_2\bar{e}_3c$. Then we fold $e'_4$ and $\bar{e}_2$, producing a new marked graph $(\Gamma_2, \mathcal{G}_2)$ which is abstractly isomorphic to $T_4$, but with a different marking. The action of the resulting map $f'': (\Gamma_2, \mathcal{G}_2) \to (\Gamma_2, \mathcal{G}_2)$ on edges is obtained by replacing instances of $e'_4$ with $e''_4\bar{e}_2$. Thus we have

$$f''(e''_4) = e_1\bar{e}_2be'_4b\bar{e}'_4\bar{e}_2\bar{e}_2e_2,$$

and we may tighten to produce an irreducible topological realization $f_2: (\Gamma_2, \mathcal{G}_2) \to (\Gamma_2, \mathcal{G}_2)$. See Figure 3.3.

The Perron–Frobenius eigenvalue $\lambda_2$ is the largest root of $x^4 - 2x^3 - 2x^2 + 2x - 1$ and satisfies $\lambda_2 \approx 2.539$; thus $\lambda_2 < \lambda_1$. The restriction of $Df_2$ to turns incident to $\star$ is determined by the dynamical system $e_1 \mapsto e_2 \leftrightarrow e_3, e_4 \mapsto e_4$. The only nondegenerate illegal turn in $(\Gamma_2, \mathcal{G}_2)$ is $\{e_1, e_3\}$, which is not crossed by the $f_2$-image of any edge, so $f_2: (\Gamma_2, \mathcal{G}_2) \to (\Gamma_2, \mathcal{G}_2)$ is a train track map.

The name “train track map” asks the reader to imagine drawing the edges incident to $\star$ in such a way that the illegal turn $e_1$ and $e_3$ make a very sharp corner,
while each other pair of edges makes a much looser one. A legal path, then, is one that does not make any sharp turns—it is the kind of path a train could make as it moves along its tracks. In a setting with nontrivial vertex stabilizers, one should perhaps imagine this picture in the Bass–Serre tree or in an orbifold sense.

The broad-strokes outline of the proof of Theorem 3.2.1 is much the same as in the previous example. By folding at illegal turns, we often produce nontrivial tightening, which decreases the Perron–Frobenius eigenvalue $\lambda$. By controlling the presence of valence-one and valence-two vertices, we may argue that the transition matrix lies in a finite set of matrices, thus the Perron–Frobenius eigenvalue $\lambda$ may only be decreased finitely many times. In the remainder of the section, we make this precise by recalling Bestvina and Handel’s original analysis. The proofs are identical to the original, so we omit them.

**Subdivision.** Given a topological realization, $f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$, if $p$ is a point in the interior of an edge $e$ such that $f(p)$ is a vertex, we may give $(\Gamma, \mathcal{G})$ a new graph of groups structure by declaring $p$ to be a vertex, with vertex group equal to $\mathcal{G}_e$.

**Lemma 3.6.5 (Lemma 1.10 of [BH92]).** If $f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ is a topological realization of $\varphi \in \text{Mod}(\mathcal{G})$, and $f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1)$ is obtained by subdivision, then $f_1$ is a topological realization of $\varphi \in \text{Mod}(\mathcal{G})$. If $f$ is irreducible, then $f_1$ is too, and the associated Perron–Frobenius eigenvalues are equal. \hfill \qed

**Valence-One Homotopy.** We distinguish two kinds of valence-one vertex of $(\Gamma, \mathcal{G})$. A valence-one vertex with incident edge $e$ is *inessential* if the monomorphism $\iota_e : \mathcal{G}_e \to \mathcal{G}_v$ is an isomorphism. A valence-one vertex is *essential* if it is not inessential. The Bass–Serre tree is minimal if and only if $(\Gamma, \mathcal{G})$ contains no inessential valence-one vertices.

If $v$ is an inessential valence-one vertex with incident edge $e$, then $e$ is a contractible subtree. Let $(\Gamma_1, \mathcal{G}_1)$ denote the subgraph of groups determined by $\Gamma_1 \setminus \{e, v\}$, and let $\pi : (\Gamma, \mathcal{G}) \to (\Gamma_1, \mathcal{G}_1)$ be the collapsing map. Let $f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1)$ be the topological realization obtained from $\pi f|_{(\Gamma_1, \mathcal{G}_1)}$ by tightening and

![Figure 3.3: The topological realization $f_2 : (\Gamma_2, \mathcal{G}_2) \to (\Gamma_2, \mathcal{G}_2)$.](image)
collapsing a maximal pretrivial forest. We say \( f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1) \) is obtained from \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) by a valence-one homotopy.

**Lemma 3.6.6** (Lemma 1.11 of [BH92]). If \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) is an irreducible topological realization with Perron–Frobenius eigenvalue \( \lambda \) and \( f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1) \) is obtained from \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) by performing valence-one homotopies on all inessential valence-one vertices of \((\Gamma, \mathcal{G})\) followed by the collapse of a maximal invariant subgraph, then \( f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1) \) is irreducible, and the associated Perron–Frobenius eigenvalue \( \lambda_1 \) satisfies \( \lambda_1 < \lambda \). \( \square \)

**Valence-Two Homotopy.** We likewise distinguish two kinds of valence-two vertex of \((\Gamma, \mathcal{G})\). A valence-two vertex \( v \) with incident edges \( e_i \) and \( e_j \) is inessential if at least one of the monomorphisms \( \iota_{e_i} : \mathcal{G}_{e_i} \to \mathcal{G}_v \) and \( \iota_{e_j} : \mathcal{G}_{e_j} \to \mathcal{G}_v \) is an isomorphism, say \( \iota_{e_j} : \mathcal{G}_{e_j} \to \mathcal{G}_v \). Again, it is essential if it is not inessential. (For convenience, if \((\Gamma, \mathcal{G})\) has one vertex, that vertex is essential regardless of its valence.) Let \( \pi \) be the map that collapses \( e_j \) to a point and expands \( e_i \) over \( e_j \). Define a map \( f' : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) by tightening \( \pi f \). Observe that no vertex of \((\Gamma, \mathcal{G})\) is mapped to \( v \). Thus we may define a new graph of groups structure \((\Gamma', \mathcal{G}')\) by removing \( v \) from the set of vertices. Thus the edge path \( e_i e_j \) is now an edge, which we will call \( e_i \). Let \( f'' : (\Gamma', \mathcal{G}') \to (\Gamma, \mathcal{G}) \) be the map obtained by tightening \( f''(e_i) = f'(e_i e_j) \). Finally, let \( f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1) \) be the topological realization obtained by collapsing a maximal pretrivial forest. We say that \( f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1) \) is obtained by a valence-two homotopy of \( v \) across \( e_j \).

**Lemma 3.6.7** (Lemma 1.12 of [BH92]). Let \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) be an irreducible topological realization, and suppose \((\Gamma, \mathcal{G})\) has no inessential valence-one vertices. Suppose \( f_2 : (\Gamma_2, \mathcal{G}_2) \to (\Gamma_2, \mathcal{G}_2) \) is the irreducible topological realization obtained by performing a valence-two homotopy of \( v \) across \( e_j \) followed by the collapse of a maximal invariant subgraph. Let \( M \) be the transition matrix of \( f \) and choose a positive eigenvector \( \vec{w} \) with \( M \vec{w} = \lambda \vec{w} \). If \( \vec{w}_i < \vec{w}_j \), then \( \lambda_2 < \lambda \); if \( \vec{w}_i > \vec{w}_j \), then \( \lambda_2 > \lambda \). \( \square \)

**Remark 3.6.8.** The statement of the lemma hides a problem: if we cannot freely choose which edge incident to an inessential valence-two vertex to collapse via a valence-two homotopy, we may be forced to increase \( \lambda \). Fortunately, such valence-two vertices are not produced in the proof of Theorem 3.2.1.

**Folding.** We have already seen folding in the previous chapter. We may fold edges \( e \) and \( e' \) who share a common terminal vertex and whose images under \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) are equal. More generally, we may subdivide \( e \) and \( e' \) and fold their maximal common terminal segment.

**Lemma 3.6.9** (Lemma 1.15 of [BH92]). Suppose \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) is an irreducible topological realization and that \( f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1) \) is obtained by folding a pair of edges. If \( f_1 \) is a topological realization, then it is irreducible, and the associated Perron–Frobenius eigenvalues satisfy \( \lambda_1 = \lambda \). Otherwise, let \( f_2 : (\Gamma_2, \mathcal{G}_2) \to (\Gamma_2, \mathcal{G}_2) \) be the map that collapses \( e_j \) to a point and expands \( e_i \) over \( e_j \). Define a map \( f' : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) by tightening \( \pi f \). Observe that no vertex of \((\Gamma, \mathcal{G})\) is mapped to \( v \). Thus we may define a new graph of groups structure \((\Gamma', \mathcal{G}')\) by removing \( v \) from the set of vertices. Thus the edge path \( e_i e_j \) is now an edge, which we will call \( e_i \). Let \( f'' : (\Gamma', \mathcal{G}') \to (\Gamma, \mathcal{G}) \) be the map obtained by tightening \( f''(e_i) = f'(e_i e_j) \). Finally, let \( f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1) \) be the topological realization obtained by collapsing a maximal pretrivial forest. We say that \( f_1 : (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1) \) is obtained by a valence-two homotopy of \( v \) across \( e_j \).
be the irreducible topological representative obtained by tightening, collapsing a maximal pretrivial forest, and collapsing a maximal invariant subgraph. Then the associated Perron–Frobenius eigenvalues satisfy $\lambda_2 < \lambda$.

\[ \square \]

### 3.7 Proof of the Main Theorem

We begin with $\varphi \in \text{Mod}(G)$, and a topological realization $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$. By performing preliminary valence-one and valence-two homotopies, we may assume that $(\Gamma, \mathcal{G})$ has no inessential valence-one and valence-two vertices. Indeed, we may also assume that $G$ does not have inessential valence-one and valence-two vertices. Likewise, by collapsing a maximal invariant subgraph, we may assume that $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ is irreducible.

**Lemma 3.7.1.** Given a graph of groups $(\Gamma, \mathcal{G})$, let $\eta(\Gamma, \mathcal{G})$ be the number of vertices $v$ of $\Gamma$ such that for each oriented edge $e \in \text{st}(v)$, the monomorphism $\iota_e: \mathcal{G}_e \to \mathcal{G}_v$ is not surjective. For the moment we call these vertices marked. Let $\beta(\Gamma)$ be the first Betti number of $\Gamma$. If $G \succeq (\Gamma, \mathcal{G})$ and $(\Gamma, \mathcal{G})$ has no inessential valence-one or valence-two vertices, then $\Gamma$ has at most $2\eta(G) + 3\beta(G) - 3$ edges.

**Remark 3.7.2.** The statement above has a corner case that is worth mentioning. This occurs when $\beta(G) = 1$ and $\eta(G) = 0$. In this case, $G$ is homotopy equivalent to a graph of groups $(\Gamma, \mathcal{G})$ with one vertex and one edge $e$ where at least one of the monomorphisms $\iota_e$ and $\iota_\bar{e}$ is surjective. In this case every topological realization $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ is homotopic to an automorphism of $(\Gamma, \mathcal{G})$, so Theorem 3.2.1 holds after passing to this graph with one vertex by repeatedly removing inessential valence-one and valence-two vertices via homotopies.

**Proof.** If $(\Gamma, \mathcal{G}) \simeq G$, a homotopy equivalence between them sends marked vertices of $G$ to marked vertices of $(\Gamma, \mathcal{G})$ and vice versa, so $\eta(\Gamma, \mathcal{G}) = \eta(G)$. Only marked vertices of $\Gamma$ may have valence less than three. Similarly $\beta(\Gamma) = \beta(G)$. Choose a cyclic ordering of the marked vertices of $\Gamma$ and form a new graph $\Gamma'$ by attaching an edge from each marked vertex of $\Gamma$ to its successor in the cyclic order. We have thus added $\eta(G)$ edges to $\Gamma$. The graph $\Gamma'$ has no valence-one nor valence-two vertices and first Betti number $\eta(G) + \beta(G)$. The lemma follows in the case $(\Gamma, \mathcal{G}) \simeq G$ by a simple Euler characteristic argument.

If instead $G \succeq (\Gamma, \mathcal{G})$, the map $\sigma: G \to (\Gamma, \mathcal{G})$ identifying $G$ with $\pi_1(\Gamma, \mathcal{G})$ sends marked vertices of $G$ to marked vertices of $(\Gamma, \mathcal{G})$. If a vertex of $(\Gamma, \mathcal{G})$ is marked but not in the image of any marked vertex of $G$, then it is obtained by collapsing some subgraph of $G$ with nontrivial first Betti number. In other words, we have

$$\eta(\Gamma, \mathcal{G}) + \beta(\Gamma) \leq \eta(G) + \beta(G) \quad \text{and} \quad \beta(\Gamma) \leq \beta(G),$$

and at least one inequality is strict. The lemma follows by the argument above. \[ \square \]

We now turn the proof of the main theorem. The argument is essentially due to Bestvina and Handel [BH92, Theorem 1.7].
Proof of Theorem 3.2.1. We keep the notation as above. Suppose $\lambda = 1$. Then $f$ permutes the edges of $\Gamma$ and is thus an automorphism of $(\Gamma, \mathcal{G})$. In particular, $f$ is a train track map.

So assume $\lambda > 1$. We will show that if our irreducible topological realization $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ is not a train track map, then there is an irreducible topological realization $f_1: (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1)$ without inessential valence-one or valence-two vertices such that the associated Perron–Frobenius eigenvalues satisfy $\lambda_1 < \lambda$.

The previous lemma shows the size of the transition matrix of $f_1: (\Gamma_1, \mathcal{G}_1) \to (\Gamma_1, \mathcal{G}_1)$ is uniformly bounded depending only on $\mathcal{G}$. Furthermore, if $M$ is an irreducible matrix, its Perron–Frobenius eigenvalue $\lambda$ is bounded below by the minimum sum of the entries of a row of $M$. To see this, let $\vec{w}$ be the associated positive eigenvector. If $w_j$ is the smallest entry of $\vec{w}$, $\lambda w_j = (M\vec{w})_j$ is greater than $w_j$ times the sum of the entries of the $j$th row of $M$.

Thus if we iterate, reducing the Perron–Frobenius eigenvalue, there are only finitely many irreducible transition matrices that can occur, so at some finite stage the Perron–Frobenius eigenvalue will reach a minimum. At this point, we must have a train track map.

To complete the proof, we turn to the question of decreasing $\lambda$. Suppose $f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G})$ is not a train track map. Then there exists a point $p$ in the interior of an edge such that $f(p)$ is a vertex, and $f^k$ is not locally injective at $p$ for some $k > 1$. We assume that topological realizations act linearly on edges with respect to some metric on $\Gamma$. Since $\lambda > 1$, this means the set of points of $\Gamma$ eventually mapped to a vertex is dense. Thus we can choose a neighborhood $U$ of $p$ so small that it satisfies the following conditions.

1. The boundary $\partial U$ is a two point set $\{s, t\}$, where $f^\ell(s)$ and $f^\ell(t)$ are vertices for some $\ell \geq 1$.
2. $f^i|_U$ is injective for $1 \leq i \leq k - 1$.
3. $f^k$ is two-to-one on $U \setminus \{p\}$, and $f^k(U)$ is contained within a single edge.
4. $p \notin f^i(U)$, for $1 \leq i \leq k$.

First we subdivide at $p$. Then we subdivide at $f^i(s)$ and $f^i(t)$ for $1 \leq i \leq \ell - 1$ (in reverse order so that subdivision is allowed). The vertex $p$ has valence two; denote the incident edges by $e$ and $e'$. Observe that $f^{k-1}(e)$ and $f^{k-1}(e')$ are each single edges that are identified by $f$. Thus we may fold. The resulting map $f': (\Gamma', \mathcal{G}') \to (\Gamma', \mathcal{G}')$ may be a topological realization, in which case $\lambda' = \lambda$. In this case, $f'^{k-2}(e)$ and $f'^{k-2}(e')$ are single edges that are identified by $f'$. In the contrary case, nontrivial tightening occurs. After collapsing a maximal pretrivial forest and a maximal invariant subgraph, the resulting irreducible topological realization $f'': (\Gamma'', \mathcal{G}'') \to (\Gamma'', \mathcal{G}'')$ satisfies $\lambda'' < \lambda$.

Repeating this dichotomy $k$ times if necessary, we have either decreased $\lambda$, or we have folded $e$ and $e'$ so that $p$ is now an inessential vertex of valence one.

We remove inessential valence-one and valence-two vertices by the appropriate homotopies. Since valence-one homotopy always decreases the Perron–Frobenius
eigenvalue, the resulting irreducible topological realization \( f_1: (\Gamma_1, \mathcal{F}_1) \to (\Gamma_1, \mathcal{F}_1) \) satisfies \( \lambda_1 < \lambda \).

**Remark 3.7.3.** As in the original, the proof of Theorem 3.2.1 provides in outline an algorithm that takes as input a topological realization of an automorphism and returns a train track map.

### 3.8 On a Question of Paulin

In this section, we establish the following alternative for outer automorphisms of word hyperbolic groups. It answers affirmatively two questions of Paulin [Pau91, page 333] [Pau91, page 150].

A group is **word hyperbolic** if its word metric for some (and hence any) finite generating set satisfies a certain large-scale negative curvature condition called **Gromov-hyperbolicity** or \( \delta \)-**hyperbolicity**. Free groups, free products of finite groups, and fundamental groups of surfaces with negative Euler characteristic are all examples of word hyperbolic groups.

**Theorem 3.8.1.** Let \( G \) be a finitely-generated word hyperbolic group. Each \( \Phi \in \text{Aut}(G) \) satisfies one of the following conditions.

1. **Periodic:** the outer class of \( \Phi \) has finite order in \( \text{Out}(G) \).

2. **Reducible:** there exists a splitting \( G = \pi_1(\Gamma, \mathcal{F}) \) with virtually cyclic edge groups and an automorphism \( \Phi' \) in the outer class of \( \Phi \) such that \( \Phi \) is an automorphism of \( (\Gamma, \mathcal{F}) \).

3. **Irreducible:** There exists a small isometric action of \( G \) on a real tree \( T \) and a homothety \( H: T \to T \) with stretch factor an algebraic integer \( \lambda > 1 \) such that

\[
H(g.x) = \Phi'(g).H(x)
\]

for all \( x \in T \) and \( g \in G \). The homothety \( H \) has a fixed point.

Clearly the third bullet point needs some explanation. A **real tree** (or an \( \mathbb{R} \)-**tree**) \( T \) is a metric space where every pair of distinct points is connected by a unique arc, by which we mean an embedding of an interval of \( \mathbb{R} \). Furthermore, this arc is a **geodesic**, i.e. its length realizes the distance between the pair of points. A group is **small** if it does not contain a free group of rank 2. An action of a group \( G \) on a real tree \( T \) is **small** if each arc stabilizer is small. A **homothety** of a metric space \((X, d)\) is a map \( h: X \to X \) together with a **stretch factor** \( \lambda > 0 \) such that

\[
d(h(x), h(y)) = \lambda d(x, y)
\]

for all \( x, y \in X \).

A real number is an **algebraic integer** if it is the root of a monic polynomial with integer coefficients. Eigenvalues of integral matrices are algebraic integers.

For the reader familiar with Thurston’s theory, the trichotomy in the statement really is a true generalization of Thurston’s dynamical trichotomy, as we now briefly explain.
A simple closed curve $\gamma$ on a surface $S$ determines a splitting of $\pi_1(S)$ as an amalgamated free product or HNN extension of free groups with cyclic edge group generated by the loop determined by $\gamma$ in $\pi_1(S)$ according to whether $\gamma$ separates or does not separate $S$, respectively. If a mapping class $\varphi \in \text{Mod}(S)$ fixes a simple closed curve up to isotopy, it has a representative $\Phi' : S \to S$ that fixes the curve (and the basepoint). Thus $\Phi'$ acts as an automorphism of the corresponding splitting.

Associated to a pseudo-Anosov diffeomorphism $\Phi : S \to S$ there is an attracting geodesic lamination. The precise definition is perhaps not so important, but one may think of points of the lamination as accumulation points of geodesic representatives for the curves $\Phi^k(c)$ for $k \geq 0$ in some fixed hyperbolic metric on $S$, where $c$ is an arbitrary essential simple closed curve. Dual to the lift of the attracting lamination to the universal cover of $S$ is an $\mathbb{R}$-tree $T$ equipped with a small action of $\pi_1(S)$. Since the pseudo-Anosov diffeomorphism $\Phi$ preserves the leaves of the lamination, it acts on the dual tree. The stretch factor $\lambda$ is the stretch factor of $\Phi$, hence is an algebraic integer satisfying $\lambda > 1$.

The proof of Theorem 3.8.1 follows rather quickly from Theorem 3.2.1, building on deep theorems of Dunwoody, Bowditch, Bestvina-Feighn and Paulin. The method of passing from a (relative) train track to an invariant $\mathbb{R}$-tree appears to have been discovered simultaneously by many people. We sketch the construction given by Gaboriau–Jaeger–Levitt–Lustig [GJLL98].

Paulin [Pau97] has a partial result in the vein of Theorem 3.8.1. Namely, he shows the existence of the invariant tree $T$ and homothety $H$. Our proof shows that the stretch factor of $H$ is always an algebraic integer, and that when $H$ has stretch factor 1, $\Phi$ is reducible.

Proof. Fix a finitely-generated word hyperbolic group $G$. Bestvina–Feighn [BF95] show that $\text{Out}(G)$ is finite when $G$ does not admit a splitting with virtually cyclic edge groups. In this case, each automorphism $\Phi : G \to G$ is periodic.

So assume $G$ admits a splitting with virtually cyclic edge groups. Finitely-generated word hyperbolic groups are finitely presented, so $G$ is accessible in the sense of Dunwoody [Dun85]. Thus $G$ admits a maximal splitting as a finite graph of groups with finite edge groups (and without inessential valence-one or valence-two vertices). In this case each vertex group is either finite or a one-ended word hyperbolic group. Bowditch [Bow98] constructs a canonical splitting called the JSJ splitting for one-ended word hyperbolic groups that do not act geometrically on the hyperbolic plane. It is a finite graph of groups with virtually cyclic edge groups. We may “blow up” Dunwoody’s accessible splitting by grafting in Bowditch’s JSJ splitting for every vertex with one-ended vertex group that does not act geometrically on the hyperbolic plane. As explained above, in the case where $G$ does act geometrically on the hyperbolic plane, the trichotomy is due to Thurston; thus we may assume that $G$ does not act geometrically on the hyperbolic plane, and we may leave such vertices alone in the splitting. Call the resulting graph of groups $\mathcal{G}$.

Since the conjugacy classes of vertex groups of $\mathcal{G}$ are determined by $G$, $\text{Mod}(\mathcal{G}) = \text{Out}(G)$. By Theorem 3.2.1, if $\varphi \in \text{Out}(G)$ is the outer class of $\Phi$, there exists an irreducible train track map $f : (\Gamma, \mathcal{F}) \to (\Gamma, \mathcal{F})$ realizing $\varphi$ with $\mathcal{G} \succeq (\Gamma, \mathcal{F})$, which
thus has virtually cyclic edge groups. If the associated Perron–Frobenius eigenvalue \( \lambda \) is equal to 1, then \( f \) is an automorphism of \( (\Gamma, \mathcal{G}) \). To choose \( \Phi' \), one need only choose a basepoint \( \star \) in \( (\Gamma, \mathcal{G}) \) and a path in the marking spanning tree of \( \Gamma \) from \( \star \) to \( f(\star) \).

If \( \lambda > 1 \), we use the construction of a real tree \( T \) associated to a train track map from [GJLL98]. In particular we refer the reader to them for the choice of \( \Phi' \) with fixed point in \( T \). The idea is the following. Assign each edge of \( \Gamma \) a length equal to the corresponding entry of the positive eigenvector \( \vec{w} \) with norm 1. Assume \( f \) linearly expands edges by a factor of \( \lambda \). Fix some lift \( \tilde{f} \) to the Bass–Serre tree \( \tilde{\Gamma} \).

As above, this involves some choices. We may equip \( \tilde{\Gamma} \) with the lifted metric. We have \( \tilde{f}(g.x) = \Phi'(g).f(x) \), and \( d(\tilde{f}(x), \tilde{f}(y)) \leq \lambda d(x, y) \) for all \( g \in G \) and \( x, y \in \tilde{\Gamma} \).

Define a new distance function on \( T \) as the limit of the non-increasing sequence

\[
d_{\infty}(x, y) = \lim_{k \to \infty} \frac{d(\tilde{f}^k(x), \tilde{f}^k(y))}{\lambda^k}.
\]

One checks that the \( G \) action preserves \( d_{\infty} \), and that \( d_{\infty}(\tilde{f}(x), \tilde{f}(y)) = \lambda d_{\infty}(x, y) \). The distance \( d_{\infty} \) fails to be a metric on \( \tilde{\Gamma} \) only because there are a priori distinct points \( x \) and \( y \) with \( d_{\infty}(x, y) = 0 \). The real tree \( T \) is obtained from \( (\tilde{\Gamma}, d_{\infty}) \) by identifying those points \( x, y \) of \( \tilde{\Gamma} \) with \( d_{\infty}(x, y) = 0 \).

### 3.9 Relative and Partial Train Track Maps

In the previous sections, we constructed irreducible train track maps from topological realizations by using the machinery of graphs of groups to collapse any invariant subgraphs that occurred. While this gives a conceptually simple result, a train track map for every element of \( \text{Mod}(G) \), it hides much of the dynamical complexity possible.

A finer tool is a relative train track map. Here is the idea. A filtration of a graph \( \Gamma \) is a nested sequence of subgraphs

\[
\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma.
\]

The subgraphs need not be connected. A relative train track map is a topological realization respecting the filtration that looks like, in a sense which we make precise below, a train track map on each \( \Gamma_k \) once we collapse \( \Gamma_{k-1} \).

Let us make the forgoing more precise. The \( k \)th stratum of a filtration \( \emptyset \subset \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma \) is the subgraph

\[
H_k = \Gamma_k \setminus \Gamma_{k-1}.
\]

Suppose a topological realization \( f: (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) preserves the filtration, in the sense that \( f(\Gamma_i) \subset \Gamma_i \). Then the \( k \)th stratum has a transition matrix \( M_k \), which has rows and columns for the edges of \( H_k \), and \( ij \)th entry equal to, as usual, the number of times the \( f \)-image of the \( j \)th edge crosses the \( i \)th edge in either direction. The \( k \)th stratum is irreducible if \( M_k \) is an irreducible matrix. We say a filtration is maximal
if each stratum is either irreducible or has transition matrix identically equal to zero. The latter are zero strata. Every filtration can be refined to a maximal filtration. We always assume that \( f \) preserves a given filtration, and that it is maximal.

Each irreducible stratum has a Perron–Frobenius eigenvalue \( \lambda_k \). We say the \( k \)th stratum is exponentially-growing if \( \lambda_k > 1 \). If \( \lambda_k = 1 \), then \( H_k \) is non-exponentially-growing. A stratum \( H_k \) contains the turn \( \{ ([g], e), ([g'], e') \} \) if both \( e \) and \( e' \) belong to \( H_k \).

**Definition 3.9.1.** A topological realization \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) is a relative train track map if there is a maximal filtration \( \emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma \) preserved by \( f \) such that for each exponentially-growing stratum \( H_k \), the map \( f \) satisfies the following three properties.

(RTT-i) The map \( Df \) maps the set of turns in \( H_k \) into itself.

(RTT-ii) If \( \gamma \subset \Gamma_{k-1} \) is a nontrivial path with endpoints in \( H_k \cap \Gamma_{k-1} \), then \( f(\gamma) \) is a homotopically nontrivial path with endpoints in \( H_k \cap \Gamma_{k-1} \).

(RTT-iii) For each legal path \( \gamma \) in \( H_k \), \( f(\gamma) \) is a path that does not contain any illegal turns in \( H_k \).

**Proposition 3.9.2.** Suppose \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) is a relative train track map with irreducible stratum \( H_k \). Let \( (\Gamma'_k, \mathcal{G}'_k) \) denote the marked graph of groups obtained from \( \Gamma_k \) by collapsing each component of \( \Gamma_{k-1} \). The restriction of \( f \) to \( \Gamma_k \) factors through the collapse, and the resulting map \( f_k : (\Gamma'_k, \mathcal{G}'_k) \to (\Gamma'_k, \mathcal{G}'_k) \), is an irreducible train track map.

**Proof.** If \( H_k \) is non-exponentially-growing, then \( f_k \) is an automorphism of \( (\Gamma'_k, \mathcal{G}'_k) \). Therefore suppose \( H_k \) is exponentially-growing.

Let \( e \) be an edge of \( H_k \). By (RTT-i) and (RTT-iii), we have

\[
    f(e) = \alpha_1 \gamma_1 \cdots \gamma_\ell \alpha_\ell,
\]

where each \( \alpha_i \) is a legal path in \( H_k \), and each \( \gamma_i \) is contained in \( \Gamma_{k-1} \). After collapsing, each \( \gamma_i \) determines a vertex group element \( g_i \). Thus

\[
    f_k(e) = \alpha_1 g_1 \cdots g_\ell \alpha_\ell,
\]

which is a legal path except possibly at the turns where \( \alpha_i \) and \( \alpha_{i+1} \) meet. Suppose \( T = \{ ([g], e), ([g'], e') \} \) is such a turn. Because \( f \) sends edges to immersed paths, the only case where \( T \) may be illegal occurs if \( e \) and \( e' \) do not form a turn in \( \Gamma_k \) because there we have \( \tau(e) \neq \tau(e') \), but we have \( f^i(\tau(e)) = f^i(\tau(e')) \) for some \( i \geq 1 \).

Write \( v = f^{i-1}(\tau(e)) \) and \( w = f^{i-1}(\tau(e')) \). In this case, by considering a homotopy inverse for \( f \), we see that there is some path \( \sigma \) connecting \( v \) to \( w \) such that \( f(\sigma) \) is homotopically trivial. (cf. Remark 2.8 of [FH11].) This contradicts (RTT-ii), so we conclude that \( f_k(e) \) is a legal path, from which the statement follows. \( \square \)
With the tools developed so far, the existence of relative train track maps as proven by Bestvina and Handel in [BH92, Section 5] may be adapted straightforwardly, using the complexity bound in Lemma 3.7.1 in place of Bestvina–Handel’s $3n - 3$. We leave the adaptation to the reader, and record the following result.

**Theorem 3.9.3.** Every $\varphi \in \text{Mod}(G)$ may be realized by a relative train track map $f : (\Gamma, \mathcal{F}) \to (\Gamma, \mathcal{F})$ where $(\Gamma, \mathcal{F}) \simeq G$.

**Remark 3.9.4.** As shown in [FH18], the construction of relative train track maps may be made algorithmic—in principle, although not in actual fact if our vertex and edge groups are allowed to be arbitrary.

Another way to prove that every element $\varphi \in \text{Mod}(G)$ may be realized by a relative train track map $f : (\Gamma, \mathcal{F}) \to (\Gamma, \mathcal{F})$ with $(\Gamma, \mathcal{F}) \simeq G$ would be to prove a kind of converse to Proposition 3.9.2. That is, if one could “blow up” subgraphs that were collapsed by carefully stitching in the associated train track maps, one could assemble a topological realization from a hierarchy of train track maps. In fact, what one constructs from this process is not quite yet a relative train track map but a partial train track map in the sense of [GJLL98]. Bestvina and Handel describe two operations called core subdivision and collapsing inessential connecting paths in [BH92, Lemmas 5.13 and 5.14], which may be adapted to allow one to modify our partial train track maps into relative train track maps. The details will appear elsewhere. For the remainder of this section we content ourselves to describing the “blowing up” in the special case where all edge groups are trivial.

We give a slightly different, stronger definition than Gaboriau, Jaeger, Levitt and Lustig. Their definition is tuned to the construction of an $\mathbb{R}$-tree, and is thus only concerned with the top stratum.

**Definition 3.9.5.** A topological realization $f : (\Gamma, \mathcal{F}) \to (\Gamma, \mathcal{F})$ is a partial train track map with respect to an $f$-invariant maximal filtration $\emptyset \subset \Gamma_0 \subset \cdots \subset \Gamma_m = \Gamma$ if it satisfies the following conditions:

1. $(\Gamma, \mathcal{F})$ has no inessential valence-one or valence-two vertices.
2. Each stratum $H_k$ is irreducible.
3. If $p$ and $q$ are points connected by a path $\sigma$ contained in the interior of an edge $e$ in $H_k$ such that for some $\ell > 1$, $f^\ell(p) = f^\ell(q)$ and $f^\ell(\sigma)$ is null-homotopic, then $f^\ell(\sigma) \subset \Gamma_{k-1}$.

**Theorem 3.9.6.** Let $G$ be a free product. Every outer automorphism $\varphi \in \text{Out}(G)$ may be topologically realized by a partial train track map $f : (\Gamma, \mathcal{F}) \to (\Gamma, \mathcal{F})$, where the graph of groups $(\Gamma, \mathcal{F})$ has trivial edge groups.

**Remark 3.9.7.** Of course, if $\varphi \in \text{Mod}(G)$, where $G$ is any splitting of $G$ with trivial edge groups, the proof shows that $(\Gamma, \mathcal{F}) \simeq G$. 68
Proof. We begin with a topological realization of $\varphi$ on a graph of groups $G$ with trivial edge groups. (One exists, by the results of the previous chapter.) We use the method in the proof of Theorem 3.2.1 to construct a hierarchy of train track maps in the following way. Suppose during the process of constructing the initial train track map, we encounter an invariant subgraph $(\Lambda, L)$ that is not a contractible forest. Since $(\Lambda, L)$ is $f$-invariant, it makes sense to restrict $f$ to $(\Lambda, L)$. In fact, we shall restrict to those components of $(\Lambda, L)$ which are not contractible trees. Abusing notation, we will call the union of these components $(\Lambda, L)$. Note that in the strictest sense, $(\Lambda, L)$ is not a graph of groups, and $f$ is not a topological realization, simply because a priori $\Lambda$ may not be connected. Nonetheless, the proof of Theorem 3.2.1 does not rely on the connectivity of $\Lambda$, so we may apply it to $(\Lambda, L)$.

In Lemma 3.7.1, we gave a bound on the number of edges contained in a marked graph of groups without inessential valence-one or valence-two vertices. For a Grushko decomposition of $G$ as

$$G = A_1 \ast \cdots \ast A_n \ast F_k,$$

where the $A_i$ are freely indecomposable and not infinite cyclic and $F_k$ is free of rank $k$, if $(\Gamma, \varphi)$ is a graph of groups with trivial edge groups and fundamental group $G$ this bound is $2n + 3k - 3$, discounting the exceptional cases $(n, k) = (1, 0)$ or $(0, 1)$, where the bounds are respectively 0 and 1. This bound strictly decreases both when collapsing an invariant subgraph that is not a contractible forest, and when passing to such an invariant subgraph. Thus the process of passing to invariant subgraphs terminates after finitely many iterations. As a result, we have a nested sequence of train track maps

$$f_1: (\Lambda_1, L_1) \to (\Lambda_1, L_1), \ldots, f_m: (\Lambda_m, L_m) \to (\Lambda_m, L_m),$$

where for each $i$ there is some $j$ satisfying $i < j$ such that each component of $\Lambda_i$ corresponds to a vertex of $\Lambda_j$ with vertex group equal to the fundamental group of that component.

We will inductively construct a new sequence of graphs of groups and topological realizations of $\varphi$

$$f'_m: (\Lambda'_m, L'_m) \to (\Lambda'_m, L'_m), \ldots, f'_1: (\Lambda'_1, L'_1) \to (\Lambda'_1, L'_1)$$

beginning with $f'_m$ and proceeding downwards. The marked graph of groups $(\Lambda'_1, L'_1)$ will satisfy $(\Lambda'_1, L'_1) \cong G$. Moreover, each component of $(\Lambda_i, L_i)$ will correspond to a vertex of $(\Lambda'_i, L'_i)$ as above. Begin by setting $(\Lambda'_m, L'_m) = (\Lambda_m, L_m)$.

Assume that each $f'_i: (\Lambda'_i, L'_i) \to (\Lambda'_i, L'_i)$ has been constructed for $k \leq i \leq m$. The marking on each component $(C, E)$ of $(\Lambda_{k-1}, L_{k-1})$ includes a choice of basepoint $p_C$ (which we may assume to be a vertex), and a choice of spanning tree. Let $v_C$ be the vertex of $\Lambda_k$ corresponding to $(C, E)$. The graph of groups $(\Lambda'_{k-1}, L'_{k-1})$ is obtained from the disjoint union of $(\Lambda_{k-1}, L_{k-1})$ and $(\Lambda'_k, L'_k)$ by removing each $v_C$ and reattaching the incident edges to $p_C$. 

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The foregoing is the step we are currently unable to take while allowing nontrivial edge groups. If all edge groups are finitely generated, then for each oriented edge $e \in \text{st}(v_C)$, we have that $\ell_e((L_k')_e)$ stabilizes some vertex of the Bass–Serre tree $\tilde{\Lambda}_{k-1}$. However, if the edge groups differ, it appears that it may happen that the marking on $(\Lambda_{k-1}, L_{k-1})$ does not allow us to choose a fundamental domain in $\tilde{\Lambda}_{k-1}$ containing each of these vertices.

We will define $f'_{k-1}$ from $f_k$ and $f_{k-1}$. To avoid notational complexity, we will describe the process assuming that $\Lambda_{k-1}$ is connected with basepoint $p$ and that $(\Lambda_{k-1}, \tilde{\Lambda}_{k-1})$ corresponds to the vertex $v$ in $\Lambda'_k$. The general case is entirely analogous.

We define $f'_{k-1}: (\Lambda'_k, L'_k) \rightarrow (\Lambda'_k, L'_k)$ in the following way. On the subgraph $(\Lambda_{k-1}, L_{k-1})$, set $f'_{k-1} = f_{k-1}$. If $e$ is an edge of $\Lambda'_k$, its image $f'_k(e)$ is of the form

$$f'_k(e) = \sigma_1 g_1 \cdots \sigma_{\ell-1} g_{\ell-1} \sigma_{\ell},$$

where each $g_i$ belongs to $(L'_k)_v = \pi_1(\Lambda_{k-1}, L_{k-1}, p)$ and where each $\sigma_i$ is a tight path in $(\Lambda'_k, L'_k)$ whose interior does not meet $v$. Assume firstly that $\sigma_1$ and $\sigma_{\ell}$ are nontrivial paths, i.e. that $f'_k(e)$ neither begins nor ends at $v$. In this case define

$$f'_{k-1}(e) = \sigma_1 \gamma_1 \cdots \sigma_{\ell-1} \gamma_{\ell-1} \sigma_{\ell-1}.$$

To complete the definition, we have to address a technicality. We have $(L'_k)_v = \pi_1(\Lambda_{k-1}, L_{k-1}, p)$, but the vertex group map $(f'_k)_v: (L'_k)_v \rightarrow (L'_k)_v$ is an automorphism, not an outer automorphism. Meanwhile, the basepoint $p$ need not be fixed by $f_{k-1}$, so the train track map $f_{k-1}: (\Lambda_{k-1}, K_{k-1}) \rightarrow (\Lambda_{k-1}, L_{k-1})$ need not yet induce a well-defined automorphism of $\pi_1(\Lambda_{k-1}, L_{k-1}, p)$, much less $(f'_k)_v$. To remedy this, choose some tight path $\eta_p$ from $p$ to $f(p)$ such that the map

$$\gamma \mapsto \eta_p f_{k-1}(\gamma) \bar{\eta}_p$$

of loops in $(\Lambda_{k-1}, L_{k-1})$ based at $p$ induces the automorphism $(f'_k)_v$. In the notation of the previous paragraph, if $f'_k(e)$ begins or ends at $v$, the corresponding path $\sigma_1$ or $\sigma_{\ell}$ is trivial, and it should be replaced in the definition of $f'_{k-1}(e)$ by $\bar{\eta}_p$ or $\eta_p$ respectively and then tightening. That is, if $f'_k(e)$ both begins and ends at $v$, we define $f'_{k-1}(e)$ by tightening the path

$$\bar{\eta}_p \gamma_1 \sigma_2 \cdots \sigma_{\ell-1} \gamma_{\ell-1} \eta_p.$$

In the general case where $\Lambda_{k-1}$ is not connected, the foregoing discussion applies mutatis mutandis. For instance, the path $\eta_C$ for the component $C$ should be defined as the appropriate tight path from $p_{C'}$ to $f_{k-1}(p_C)$, where $C'$ is the component of $\Lambda_{k-1}$ containing $f(p_C)$. The rest of the construction follows by induction.

Note that each $f'_k: (\Lambda'_k, L'_k) \rightarrow (\Lambda'_k, L'_k)$ is a topological realization, and $(\Lambda'_1, L'_1)$ satisfies $(\Lambda'_1, L'_1) \simeq G$. Each edge of $\Lambda_i$ determines an edge of $\Lambda'_i$. Set $\Gamma_i$ to be the union of the edges of $\Lambda'_i$ coming from each $\Lambda_j$ for $j \leq i$. This defines a maximal filtration

$$\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Lambda'_1.$$

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preserved by $f$.

The desired partial train track map is $f'_1: (\Lambda'_1, \mathcal{L}'_1) \to (\Lambda'_1, \mathcal{L}'_1)$. It is clear from the construction that $(\Lambda'_1, \mathcal{L}'_1)$ has no inessential valence-one nor valence-two vertices, simply because each $(\Lambda_i, \mathcal{L}_i)$ satisfy this property. Likewise each stratum is irreducible. Finally the train track property for each $f_i: (\Lambda_i, \mathcal{L}_i) \to (\Lambda_i, \mathcal{L}_i)$ implies that if $p$ and $q$ are points of $H_k$ connected by a path $\sigma$ that lies in the interior of a single edge $e$, then if $f'_1(p) = f'_1(q)$ and $f'_1(\sigma)$ is null-homotopic for some $\ell > 1$, then $f'_1(\sigma) \subset \Gamma_{k-1}$. \qed
Questions and Problems

We collect here fifteen questions we find interesting for which the techniques in this thesis might prove useful. No attempt has been made at being comprehensive. We often default to asking about Out($W_n$), where $W_n$ is the free product of $n$ copies of the cyclic group of order 2. Many of these questions could just as well be asked about the outer automorphism group of a free product or virtually free group, or even Mod($G$) as defined in the previous chapter.

Recent Work. Automorphism and outer automorphism groups of free products have attracted a lot of interest recently. Let us mention a few results. Das [Das18] showed that when $G$ is a free product of $n$ finite groups satisfying $n \geq 4$, Out$(G)$ is thick in the sense of [BDM09], and thus not relatively hyperbolic.

On the other hand, Healy [Bur16] showed that Out($W_n$) is acylindrically hyperbolic, a property also enjoyed by Out($F_n$) and the mapping class group of a closed surface. Genevois and Horbez [GH20] showed that Aut$(G)$ is acylindrically hyperbolic when $G$ is finitely generated and has infinitely many ends. They remark that in general the following question is open.

**Question 1.** If $G = A_1 \ast \cdots \ast A_n$ is finitely generated and $n$ satisfies $n \geq 3$, is Out$(G)$ acylindrically hyperbolic?

Varghese [Var19] showed that Aut$(W_n)$ does not have Kazhdan’s property (T) as soon as it is infinite. Her argument exhibits a kind of “strand-forgetting” homomorphism similar to that defined on the pure braid group. Thus her argument implicitly shows that Aut$(G)$ and Out$(G)$ do not have Kazhdan’s property (T) when $G$ is a free product of $n$ finite groups.

Finally, in a striking parallel with the case of the outer automorphism group of the braid group, Guerch [Gue20] showed that Out(Out($W_n$)) is trivial for $n \geq 5$ and a cyclic group of order 2 when $n = 4$.

Connections to CAT(0) Groups

Let us recall the following question from the Introduction. Write $W_n$ for the free product of $n$ copies of the cyclic group of order 2. We say a group $G$ is CAT(0) when there exists a geometric action of $G$ on a metric space $X$ satisfying Gromov’s non-positive curvature condition CAT(0).
**Question 2** (Ruane). Is $\text{Out}(W_n)$ a CAT(0) group when $n \geq 4$?

The thrust of this question asks whether $\text{Out}(W_n)$ is more similar to the outer automorphism group of a free group, which is not a CAT(0) group, or more similar to the quotient of the braid group by its center, which is a CAT(0) group when the number of strands is at most 6. There are a number of ways to address this question, all of which are interesting in their own right.

Cunningham in his thesis [Cun15] shows that McCullough–Miller space [MM96], a contractible simplicial complex of dimension $n - 2$ on which $\text{Out}(W_n)$ acts simplicially with finite stabilizers and finite quotient, cannot support an $\text{Out}(W_n)$-equivariant CAT(0) metric. We would like to highlight two directions this result may be pushed further. Firstly, McCullough–Miller space is an equivariant deformation retract of the spine of Guirardel–Levitt’s Outer Space for $W_n$, which is also a contractible simplicial complex of dimension $n - 2$ on which $\text{Out}(W_n)$ acts with finite stabilizers and finite quotient.

**Question 3.** Does Cunningham’s result extend to the spine of Guirardel–Levitt Outer Space?

Both McCullough–Miller space and the spine of Guirardel–Levitt Outer Space are defined for free products more generally, and the action of $\text{Out}(G)$ on each space has finite stabilizers when $G$ is a free product of $n$ finite groups.

**Question 4.** Does Cunningham’s result extend to McCullough–Miller space for $\text{Out}(G)$, where $G$ is a free product of $n$ finite groups and $n \geq 4$?

A weaker question than Question 2 asks whether $\text{Out}(W_n)$ (or $\text{Out}(G)$) acts geometrically on a CAT(0) cube complex. A result of Huang, Jankiewicz and Przytycki [HJP16] in the case $n = 4$ and extended to all $n \geq 4$ by Haettel [Hae15] says that the $n$-strand braid group cannot act geometrically on a CAT(0) cube complex. Thus one expects a negative answer to the following question.

**Question 5.** For $n \geq 4$, if $G$ is the free product of $n$ finite (or infinite cyclic) groups, does $\text{Out}(G)$ act geometrically on a CAT(0) cube complex?

As discussed in the introduction, the present author shows in [Lym] that a natural class of free-by-cyclic groups that might provide a negative answer to Question 2 are in fact (virtually) CAT(0) groups.

We say a free-by-cyclic group $F_n \rtimes \mathbb{Z}$ is of Gersten type if it contains elements $g$ and $h$ such that

1. the subgroup $\langle g, h \rangle$ is free abelian of rank two and
2. the elements $h$, $gh$ and $g^2h$ are all conjugate in $F_n \rtimes \mathbb{Z}$.

A Gersten-type free-by-cyclic group cannot be a subgroup of a CAT(0) group.

The previously mentioned result of the author suggests the following question.
**Question 6.** If a free-by-cyclic group $G$ is not of Gersten type, is $G$ (virtually) a CAT(0) group? In particular, are all $W_n \rtimes \mathbb{Z}$ groups virtually CAT(0)?

Ignat Soroko has informed the author that he and Brady ask whether a free-by-cyclic group $G$ is CAT(0) if and only if it is virtually special, that is, if $G$ acts geometrically on a CAT(0) cube complex $X$ whose geometry is particularly well-behaved. (A famous result of Agol and Wise says that fundamental groups of hyperbolic 3-manifolds are virtually special.) A weaker, strictly group-theoretic form of this question appears as Question 1 in [BS19].

The CAT(0) 2-complexes constructed by the present author do not admit a cubical structure in general. Since every $W_n \rtimes \mathbb{Z}$ group is virtually free-by-cyclic, one avenue for pursuing Brady–Soroko’s question would be to answer the following.

**Question 7.** Are polynomially-growing $W_n \rtimes \mathbb{Z}$ groups virtually special?

Finally, Rodenhausen and Wade [RW15] compute a presentation for the centralizers in $\text{Aut}(F_n)$ of Dehn-twist automorphisms of the free group $F_n$, a class originally defined by Cohen and Lustig [CL95]. Using this computation for a generator of $\text{Aut}(F_n)$ called a Nielsen transformation, they prove very strong restrictions for actions of $\text{Aut}(F_n)$ on complete CAT(0) metric spaces.

**Question 8.** Do centralizers of Dehn-twist automorphisms of $W_n$ exhibit the same behavior as those of $\text{Aut}(F_n)$?

Similar behavior is exhibited by Dehn twists in the mapping class group of a closed surface, but is absent in the braid groups.

### Topology and Geometry of Outer Spaces

By analogy with the Teichmüller space of a surface, Culler and Vogtmann define in [CV86] a contractible space now called Outer Space, $CV_n$ on which $\text{Out}(F_n)$ acts properly discontinuously. Guirardel–Levitt generalize this construction to the outer automorphism group of a free product in [GL07b], and Krstić–Vogtmann study fixed-point sets of finite group actions on $CV_n$ in [KV93].

The topology of $CV_n$ is fairly well-understood, but its geometry remains mysterious. In particular, the most natural choice of metric on Outer Space, called the Lipschitz metric is not symmetric! Nevertheless, one can realize an irreducible train track map with Perron–Frobenius eigenvalue $\lambda > 1$ as defining an axis in the Lipschitz metric for the action of the associated outer automorphism $\varphi \in \text{Out}(F_n)$ on $CV_n$. Francaviglia and Martino prove the existence of relative train track maps for outer automorphisms of free products by finding such axes in [FM15]. The results of this thesis imply, for instance, that if $\varphi \in \text{Out}(F_n)$ has an irreducible train track representative with Perron–Frobenius eigenvalue $\lambda > 1$ and $\varphi$ centralizes a finite subgroup $H \leq \text{Out}(F_n)$, then $\varphi$ has an axis entirely contained within the fixed-point set of $H$. It would be interesting to know if the following holds.
Question 9. Are fixed-point sets of finite subgroups $H \leq \Out(F_n)$ convex in the Lipschitz metric?

The spine of Culler–Vogtmann and Guirardel–Levitt Outer Space is a simplicial complex to which the corresponding Outer Space equivariantly deformation retracts. We may give the spine a metric by setting each edge to have unit length. Let $G$ be a free product of finitely generated groups. Given $\varphi \in \Out(G)$ and an automorphism $\Phi: G \to G$ representing it, the normal form given in Corollary 2.5.2 defines a preferred path in the 1-skeleton of the spine of Guirardel–Levitt Outer Space from $\varphi.v$ to $v$, for a fixed base vertex $v$. Although we saw in Proposition 2.5.3 that these paths are not quasi-geodesic in any choice of $\Out(G)$-equivariant metric on the spine, a good understanding of homotopies between such paths combined with a counting argument could provide upper bounds on the Dehn function of $\Out(G)$, which is a kind of optimal isoperimetric inequality for $\Out(G)$ and any space on which it acts geometrically.

Question 10. Do the paths in Corollary 2.5.2 satisfy an exponential isoperimetric inequality? What is the Dehn function of $\Out(G)$?

For $\Out(F_n)$, the optimal isoperimetric inequality is exponential, but both upper and lower bounds remain unknown for $\Out(G)$. A positive answer to Question 2 would yield a quadratic Dehn function for $\Out(W_n)$. Let us also mention in connection with this question the question mentioned in Chapter 3.

Question 11. For which graphs of groups $G$ is the group $\Mod(G)$ a hierarchically hyperbolic group? In particular, is $\Out(W_n)$ a hierarchically hyperbolic group?

In [Vog95], Vogtmann studies paths in the 1-skeleton of the spine of $CV_n$. By showing how to push loops and homotopies between loops outside of compact sets, she shows that the spine of $CV_n$ is simply-connected at infinity when $n \geq 5$.

Question 12. For $n \geq 6$, is $\Out(W_n)$ simply-connected at infinity?

Connectivity at infinity of $\Out(W_n)$ would allow one to prove that $\Out(W_n)$ is a virtual duality group in the sense of Bieri–Eckmann. For this in the case of $\Out(W_n)$, one needs $(n-4)$-connectivity.

Question 13. Is $\Out(W_n)$ $(n-4)$-connected at infinity?

Dynamics of Train Track Maps

In [Thu14], Thurston shows that every algebraic integer $\lambda \geq 1$ occurs as the Perron–Frobenius eigenvalue of an irreducible train track map representing an element $\varphi \in \Out(F_n)$ as $n$ is allowed to vary. By comparison, Perron–Frobenius eigenvalues attached to pseudo-Anosov diffeomorphisms of surfaces must satisfy the stronger condition of being bi-Perron numbers. It would be interesting to know the answer to the following question.
**Question 14.** Does every algebraic integer $\lambda \geq 1$ occur as the Perron–Frobenius eigenvalue of a train track map representing an element $\varphi \in \text{Out}(W_n)$ as $n$ varies?

For each fixed $n$, by contrast, there is a smallest $\lambda > 1$ that can be a Perron–Frobenius eigenvalue for a train track map representing an element $\varphi \in \text{Out}(W_n)$. This $\lambda$—or rather its logarithm—is the \textit{least dilatation} of $\text{Out}(W_n)$, $L(W_n)$. Since $W_n$ contains the quotient of the $n$-strand braid group modulo its center, and every train track map for $\varphi \in \text{Out}(W_n)$ determines a train track map for $\hat{\varphi} \in \text{Out}(F_{n-1})$ with identical Perron–Frobenius eigenvalue, we have

$$L(F_{n-1}) \leq L(W_n) < L(B_n/Z),$$

where $B_n/Z$ denotes the $n$-strand braid group modulo its center. The author has examples showing that the right hand inequality is in fact a strict inequality.

**Question 15.** Does $L(F_{n-1})$ satisfy $L(F_{n-1}) < L(W_n)$? Describe the asymptotics of

$$\frac{L(F_{n-1})}{L(W_n)} \quad \text{and} \quad \frac{L(W_n)}{L(B_n/Z)}$$

as $n$ tends to infinity.
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