INNER METRIC GEOMETRY OF COMPLEX ALGEBRAIC SURFACES WITH ISOLATED SINGULARITIES

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Abstract. We produce examples of complex algebraic surfaces with isolated singularities such that these singularities are not metrically conic, i.e. the germs of the surfaces near singular points are not bi-Lipschitz equivalent, with respect to the inner metric, to cones. The technique used to prove the nonexistence of the metric conic structure is related to a development of Metric Homology. The class of the examples is rather large and it includes some surfaces of Brieskorn.

1. Introduction

An extremely important corollary of the "Triangulation Theorem" of Lojasiewicz [9] is the existence of topologically conic structure near a singular point of an algebraic set (real or complex). Namely, Lojasiewicz proved the following: let $X$ be an algebraic (or semialgebraic) set in $\mathbb{R}^n$ and let $x_0 \in X$. Then there exists a number $\epsilon > 0$ such that the intersection of $X$ with a ball centered at $x_0$ and radius $\epsilon$ is homeomorphic to a cone over the intersection of $X$ with the sphere of the same radius and centered at $x_0$ (the intersection with a small sphere is usually called the link of $X$ at $x_0$). Moreover, he proved that the homeomorphism can be chosen as a semialgebraic map. The same result, but without the conclusion on the semialgebraicity of the corresponding homeomorphism was obtained by Milnor [11], using the integration of the radial vector field, for complex algebraic hypersurfaces.

One can ask the following. Is the same result true in the sense of Metric Geometry? Namely, is the intersection of $X$ with a ball of small radius centered at $x_0$ bi-Lipschitz homeomorphic with respect to the inner metric to a cone over the intersection of $X$ with a small sphere with the same center?

For real algebraic sets the answer is negative. One can consider so-called $\beta$-horn, i.e. the algebraic set defined by

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1^2 + x_2^2)^{\gamma} = x_3^{2p}\}$$

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where \( p > q \) are coprime positive integers and \( \beta = \frac{p}{q} \). In this case, the corresponding tangent cone has the real dimension 1, but the algebraic set itself has the real dimension 2. For complex algebraic sets, this phenomena does not exists, i.e. a tangent cone of complex algebraic set is a complex algebraic set and has the same dimension of the corresponding algebraic set \[15\].

The question of existence of the inner Lipschitz conic structure is also very important in the Intersection Homology Theory and \( L^p \)-cohomology. If all singularities of a complex algebraic set \( X \) satisfy this property, then, using the results of Brasselet, Goresky, MacPherson \[6\] and Youssin \[14\], one can show that these two cohomology theories are isomorphic.

The main goal of this paper is the following. We show that there exists a big class of complex algebraic surfaces, with isolated singularities, such that their singular points do not admit a metrically conic structure. Namely, for any complex algebraic surface \( X \) from this class and for any \( \epsilon > 0 \), the intersection of \( X \) with a ball of the radius \( \epsilon \) and centered at \( x_0 \) is not bi-Lipschitz equivalent to a cone over any Nash manifold (in particular over the intersection with the corresponding sphere).

We consider two possible versions of the question of the existence of ”Lipschitz conic structure”. The first version of the question is the following: given \( X \) an algebraic set in \( \mathbb{R}^n \) and \( x_0 \) a singular point of \( X \), is the intersection of \( X \) with a small ball centered at \( x_0 \) semialgebraically bi-Lipschitz homeomorphic to the cone over the intersection of \( X \) with a small sphere centered at \( x_0 \)? When the answer is positive, we say that \( X \) is strongly metrically conic at \( x_0 \). For weight homogeneous (not homogenous) surfaces in \( \mathbb{C}^3 \), defined by real polynomials, we present a criteria on nonexistence of this structure. In fact, we proved that if the real part of \( X \) has empty intersection with the union of the coordinates hyperplane in \( \mathbb{C}^3 - \{ 0 \} \), then \( X \) is not strongly metrically conic at \( 0 \). In order to show this result, we use the theory of ”Characteristic Exponents” developed in \[4\] and ”Metric Homology Theory” developed in \[2\], \[3\]. If the intersection of real part of \( X \) with the link of \( X \) at \( x_0 \) presents a nontrivial cycle in 1-dimensional homology of this link, we use the methods developed in \[5\] to compute a characteristic exponent of this singularity. The answer, obtained here, is different then the corresponding answer for strongly metrically conic singularity (see \[3\]). If the intersection of real part of \( X \) with the link of \( X \) at \( x_0 \) presents a trivial cycle in 1-dimensional homology of this link, then we create a so-called Cheeger’s cycle in 3-dimensional Local Metric Homology of \( X \). According to \[3\], 3-dimensional Local Metric Homology of strongly metrically conic singularity is generate only by the fundamental cycle of the link. We show that the Cheeger’s cycle and fundamental cycle
are independent. Note that the existence of Cheeger’s cycles shows the Filtration Theorem proved in [2] for 1-dimensional Local Metric Homology is not true for 3-dimensional Local Metric Homology. This fact is important for the further development Metric Homology Theory.

Finally, we consider the second version of the question about ”Lipschitz conic structure”. In this case, we do not suppose that the corresponding homeomorphism is semialgebraic. The singularities of this sort are called metrically conic and not strongly metrically conic. We present a criteria of nonexistence of this structure for weighted homogeneous (no homogeneous) surfaces in $\mathbb{C}^3$ defined by real polynomials. Namely, the image of the real part of the weighted homogeneous surface by the projection of the corresponding Seifert Fibration has to have more than one connected component. We show that the vanishing rate of the ”real cycles” is bigger than one, and we also show that this cannot happen in the conic case.

In the section 6 we show that there exists series of the surfaces of Brieskorn such that the singularity at zero of these surfaces is also not strongly metrically conic. We show that these surfaces have Cheeger’s cycles. Note that all the Brieskorn surfaces do not satisfies the conditions of Theorem 5.3 and the conditions of Theorem 5.4.

Note that the results of this paper have the following application in the Theory of Minimal Surfaces. The results of L. Caffarelli, R. Hardt and L. Simon [8] produce the examples of embedded minimal hypersurfaces in $\mathbb{R}^{n+1}$ ($n \geq 3$) with isolated singularities which are not conic in a direct sense. According to Federer [10], compact parts of complex algebraic sets are area-minimizing, hence our examples are also examples of area-minimizing with isolated singularities which are not conic even in a metric sense.

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2. Definitions and Notations

Let $X \subset \mathbb{R}^n$ be a connected algebraic set. We define an inner metric on $X$ as follows. Let $x, y \in X$. The inner distance $d(x, y)$ is defined as infimum of lengths of rectifiable arcs $\gamma: [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Note that for connected algebraic sets the inner metric is well defined.
Let $X \subset \mathbb{C}^n$ be an algebraic set and let $x_0 \in X$ be a singular point. We say that $x_0$ is a strongly metrically conic singular point if for sufficiently small $\epsilon > 0$, there exists a semialgebraic bi-Lipschitz homeomorphism, with respect the inner metric,

$$h : c[X \cap S_\epsilon(x_0)] \to X \cap B_\epsilon[x_0],$$

where $B_\epsilon[x_0]$ is the closed ball with center $x_0$ and the radius $\epsilon$, $S_\epsilon(x_0)$ is the sphere with center $x_0$ and the radius $\epsilon$, $c[X \cap S_\epsilon(x_0)]$ is a cone over $X \cap S_\epsilon(x_0)$.

The above definition of strongly metrically conic singular point is equivalent to the following. Consider a semialgebraic triangulation of $X$ and consider the stars of $x_0$, according to this triangulation. The point $x_0$ is strongly metrically conic if the intersection $X \cap B_\epsilon[x_0]$ is semialgebraically bi-Lipschitz homeomorphic to the star of $x_0$, considered with the standard metric of the simplicial complex (see [6]). An isolated singular point $x_0 \in X$ is called metrically conic if for sufficiently small $\epsilon > 0$ there exist a Nash manifold $N$ and a bi-Lipschitz homeomorphism

$$h : c[N] \to X \cap B_\epsilon[x_0],$$

where $c[N]$ is a cone over $N$. Note that we do not suppose $N$ to be homeomorphic to $X \cap S_\epsilon(x_0)$.

3. Weighted Homogeneous Surfaces

Let $w_1, w_2, w_3$ be positive integer numbers and $w = (w_1, w_2, w_3)$. Let

$$\alpha_w : \mathbb{C}^* \times \mathbb{C}^3 \to \mathbb{C}^3$$

be defined by

$$\alpha_w(t, x) = (t^{w_1}x_1, t^{w_2}x_2, t^{w_3}x_3);$$

where $x = (x_1, x_2, x_3)$. We say that $X \subset \mathbb{C}^3$ is weighted homogeneous with respect to $w = (w_1, w_2, w_3)$ if $X$ is invariant by $\mathbb{C}^*$-action $\alpha_w$. When $w_1 = w_2 = w_3$, we say that $X$ is homogeneous.

A 2-dimensional complex algebraic subset $X \subset \mathbb{C}^3$ which is weight homogeneous with respect to $w = (w_1, w_2, w_3)$ is called a weight homogeneous algebraic surface in $\mathbb{C}^3$ with respect to $w = (w_1, w_2, w_3)$.

**Example 3.1.** Let $f(X_1, X_2, X_3) \in \mathbb{C}[X_1, X_2, X_3]$ be a nonzero polynomial and let $w_1, w_2, w_3, d$ be positive integer numbers such that:

$$f(t^{w_1}x_1, t^{w_2}x_2, t^{w_3}x_3) = t^d f(x_1, x_2, x_3),$$

$\forall t \in \mathbb{C}^*$ and $\forall (x_1, x_2, x_3) \in \mathbb{C}^3$. Then

$$X = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : f(x_1, x_2, x_3) = 0\}$$

is a weight homogeneous algebraic surface in $\mathbb{C}^3$ with respect to $w = (w_1, w_2, w_3)$. In this case, we say that $X$ is defined by a weight homogeneous polynomial with respect to $w = (w_1, w_2, w_3)$.
In the following, let $X$ be a weight homogeneous algebraic surface in $\mathbb{C}^3$ with respect to $w = (w_1, w_2, w_3)$, where $w_1, w_2, w_3$ are coprime positive integers. Let $\varphi: \mathbb{C}^3 \to \mathbb{C}^3$ be defined by
\[
\varphi(x_1, x_2, x_3) = (x_1^{w_1}, x_2^{w_2}, x_3^{w_3})
\]
and let $X' = \varphi^{-1}(X)$. We call $(\varphi, X')$ the homogeneous modification of $X$.

The homogeneous modification $X'$ of $X$ is a homogeneous algebraic surface in $\mathbb{C}^3$, thus it defines a projective complex algebraic curve
\[
M' = \{[z_1 : z_2 : z_3] \in \mathbb{C}P^2 : (z_1, z_2, z_3) \in X'\}.
\]

Let us consider that 0 is an isolated singular point of $X$. So, let $M$ be defined by $M = (X - \{0\})/\alpha_w$ and let $\pi: X \to M$ be the standard projection. $M$ is a 2-dimensional orbifold (see [12]) and $\pi: X - \{0\} \to M$ is a Seifert Fibration in the sense of Orlik and Wagreich (see [13]). Moreover, there exist a branched covering $\phi: M' \to M$ such that
\[
\pi \circ \varphi(z_1, z_2, z_3) = \phi[z_1 : z_2 : z_3]
\]
for all $(z_1, z_2, z_3) \in X' - \{0\}$. $(\pi, M)$ is called Seifert Fibration of $X - \{0\}$ associated to $w = (w_1, w_2, w_3)$ (see [13]).

**Proposition 3.2.** Let $F = \{[z_1 : z_2 : z_3] \in M' : z_1z_2z_3 = 0\}$. Then, $M - \phi(F)$ admits a holomorphic structure such that
\[
\phi: M' - F \to M - \phi(F)
\]
is locally biholomorphic.

**Proof.** See [13].

4. **Cheeger’s Cycles**

**Theorem 4.1.** Let $X \subset \mathbb{R}^n$ be a $k$-dimensional semialgebraic set and $x_0 \in X$ be an isolated singular point of $X$ with a connected local link. Let $Y \subset X$ be a semialgebraic subset satisfying:

(1) $x_0 \in Y$ and $X - Y$ has exactly two connected components $W_1$ and $W_2$;
(2) $\mu(X, x_0) = \mu(W_1, x_0) = \mu(W_2, x_0) = k$;
(3) $\mu(Y, x_0) > k$.

Then, for $k < \nu < \mu(Y, x_0)$ the space $MH^\nu_{\text{loc},k-1}(X, x_0)$ has a subspace isomorphic to $\mathbb{R}^2$.

**Remark.** According to the results of [3], the existence of such a subset $Y \subset X$ proves that the singularity of $X$ at $x_0$ is not strongly metrically conic.
Proof of Theorem 4.1. Take $\epsilon > 0$ sufficiently small. The set $Y \cap S(x_0, \epsilon)$ divides $X \cap S(x_0, \epsilon)$ by two connected components $V_1$ and $V_2$. Let $\xi$ be the chain constructed by union of $Y \cap B[x_0, \epsilon]$ and $V_1$. Since $\xi = \partial(W_1 \cap B[x_0, \epsilon])$ we obtain that $\xi$ is a cycle. Let us prove that $\xi$ defines a nontrivial element in $MH^\nu_{\text{loc},m-1}(X,0)$. Observe that $\xi$ is admissible because

$$\mu(\text{supp}(\xi), x_0) = \mu(\text{supp}(Y), x_0) > \nu.$$ 

If $\xi = \partial \eta$ for some chain $\eta$, then

$$W_1 \cap B[x_0, \epsilon] \subset \text{supp}(\eta) \subset X$$

and

$$\mu(W_1, x_0) \leq \mu(\text{supp}(\eta), x_0) \leq \mu(X, x_0).$$

Since

$$\mu(X, x_0) = \mu(W_1, x_0) = k,$$

we get $\mu(\text{supp}(\eta), x_0) = k$, i.e. $\eta$ is not an admissible chain. Thus, we conclude that $[\xi] \neq 0$ in $MH^\nu_{\text{loc},m-1}(X,0)$.

Now, let us prove that $c\xi; c \in \mathbb{R}$ is not homologous to the element of $MH^\nu_{\text{loc},m-1}(X,0)$ defined by the fundamental cycle of $X \cap S(x_0, \epsilon)$. We have that $\text{supp}(f - c\xi)$ is union of $Y \cap B[x_0, \epsilon]$ and $V_2$. Let $f - c\xi = \partial \eta$ for some chain $\eta$. Then

$$W_2 \cap B[x_0, \epsilon] \subset \text{supp}(\eta) \subset X$$

and

$$\mu(W_2, x_0) \leq \mu(\text{supp}(\eta), x_0) \leq \mu(X, x_0).$$

Since,

$$\mu(X, x_0) = \mu(W_2, x_0) = k,$$

we get $\mu(\text{supp}(\eta), x_0) = k$, i.e. $\eta$ is not an admissible chain. \qed

The cycle $\xi$ is called Cheeger’s cycle and the set $Y$ is called the base of the Cheeger’s cycle.

Example 4.2. Let $X \subset \mathbb{R}^4$ be defined by $(x_1, x_2, x_3, t) \in \mathbb{R}^4$ such that

$$((x_1 - t)^2 + x_2^2 + x_3^2 - t^2)((x_1 + t)^2 + x_2^2 + x_3^2 - t^2) = t^{4\beta}; t \geq 0$$

where $\beta > 2$ is a rational number. Let $Y$ be defined by $(x_1, x_2, x_3, t) \in X$ such that $x_1 = 0$. Then $Y \subset X \subset \mathbb{R}^4$ are semialgebraic subsets such that

1. The link of $X$ at 0 is homeomorphic to 2-sphere $S^2$;
2. $0 \in Y$ and $X - Y$ has tow connected components $W_1$ and $W_2$;
3. $\mu(X, 0) = \mu(W_1, 0) = \mu(W_2, 0) = 3$;
4. $\mu(Y, 0) = \beta + 1$.

From above theorem, for $3 < \nu < \beta + 1$, $MH^\nu_{\text{loc},2}(X, x_0)$ contains a subspace isomorphic to $\mathbb{R}^2$. Hence, the filtration theorem of [3] is not valid for $k > 1$. 

5. Main Results

We say a subset $X \subset \mathbb{C}^3$ is a surface defined by a real polynomial $f(x, y, z)$ when $f(x, y, z)$ is a polynomial with real coefficients and

$$X = \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = 0\}.$$

In this case, we define

$$X(\mathbb{R}) = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\}.$$ 

In the following, let $X \subset \mathbb{C}^3$ be a surface defined by a real polynomial $f(x, y, z)$ which is weight homogeneous with respect to $w = (w_1, w_2, w_3)$, where $w_1, w_2, w_3$ are coprime positive integers. Let us suppose that 0 is an isolated singular point of $X$. Let $(\varphi, X')$ be the homogeneous modification of $X$ and $(\pi, M)$ be the Seifert Fibration associated to $w = (w_1, w_2, w_3)$. Let $\phi : M' \to M$ be the branched covering such that

$$\pi \circ \varphi (z_1, z_2, z_3) = \phi\left[z_1 : z_2 : z_3\right]$$

for all $(z_1, z_2, z_3) \in X' - \{0\}$ and let

$$F = \left\{[z_1 : z_2 : z_3] \in M' : z_1z_2z_3 = 0\right\}.$$

Proposition 5.1. Let $M' - \phi(F)$ be with the holomorphic structure presented in Proposition 3.2. Then

1. there exists an antiholomorphic involution
   $$j : M - \phi(F) \to M - \phi(F)$$
   such that
   $$\pi(X(\mathbb{R}) - \{0\}) = \{m \in M - \phi(F) : j(m) = m\};$$
2. if $M - \pi(X(\mathbb{R}) - \{0\})$ is not connected, then it has two connected components $M_1$ and $M_2$ such that $M_1 = j(M_2)$.

Proof. Since $X$ is defined by a real polynomial, then the complex conjugation

$$\tau : \mathbb{C}^3 \to \mathbb{C}^3,$$

given by $\tau(x_1, x_2, x_3) = (\overline{x_1}, \overline{x_2}, \overline{x_3})$, defines a complex involution

$$J = \tau_{|X} : X \to X$$
on $X$. We define a map

$$j : M \to M$$
in the following way: given $m \in M$, let $x$ be a point on $X - \{0\}$ such that $\pi(x) = m$, then $j(m) := \pi(J(x))$. It is clear that

$$j : M \to M$$
is well defined and $j$ is an involution. Using the holomorphic coordinates defined in Proposition 3.2, one can show that $j$ is antiholomorphic on $M - \phi(F)$. Now, let us show that

$$\pi(X(\mathbb{R}) - \{0\}) = \{m \in M - \phi(F) : j(m) = m\}.$$
It is clear that
\[ \pi(X(\mathbb{R}) - \{0\}) \subset \{m \in M - \phi(F) : j(m) = m\}. \]

So, let \( m \in M - \phi(F) \) be such that \( j(m) = m \), i.e. \( m = \pi(x_1, x_2, x_3) \);
\( (x_1, x_2, x_3) \in X \) and \( x_1x_2x_3 \neq 0 \). Since \( j(m) = m \), there exists a \( t \in \mathbb{C}^* \)
such that
\[
(\overline{x_1}, \overline{x_2}, \overline{x_3}) = (t^{w_1}x_1, t^{w_2}x_2, t^{w_3}x_3).
\]

Let \( s \in \mathbb{C}^* \) be such that
\[ \overline{s} = ts. \]

Then, for each \( k = 1, 2, 3 \), we have
\[
\frac{s^{-w_k}x_k}{(s)^{-w_k}x_k} = \frac{(s)^{-w_k}x_k}{(ts)^{-w_k}x_k} = s^{-w_k}x_k
\]
i.e. \( s^{-w_k}x_k \in \mathbb{R} \) for all \( k = 1, 2, 3 \) and
\[
\alpha_w(s, (s^{-w_1}x_1, s^{-w_2}x_2, s^{-w_3}x_3)) = (x_1, x_2, x_3),
\]
i.e. \( m \in \pi(X(\mathbb{R}) - \{0\}) \).

Finally, let us suppose that \( M - \pi(X(\mathbb{R}) - \{0\}) \) is not connected. Since
\[
\pi(X(\mathbb{R}) - \{0\}) = \{m \in M - \phi(F) : j(m) = m\},
\]
we can compile the proof of Proposition 5.4.2 of [7] (page 260) to show that
\( M - \pi(X(\mathbb{R}) - \{0\}) \) has two connected components \( M_1 \) and \( M_2 \)
such that \( M_1 = j(M_2) \).

**Corollary 5.2.** If \( M - \pi(X(\mathbb{R}) - \{0\}) \) is not connected, then \( X - \pi^{-1}(\pi(X(\mathbb{R}))) \) has exactly two connected components \( X_1, X_2 \) such that
\( X_1 = \pi(X_2) \), where \( \tau : \mathbb{C}^3 \to \mathbb{C}^3 \) is the complex conjugation \( \tau(x_1, x_2, x_3) = (\overline{x_1}, \overline{x_2}, \overline{x_3}) \).

**Theorem 5.3.** Let \( X \subset \mathbb{C}^3 \) be a irreducible surface defined by a real weighted homogeneous polynomial \( f(x_1, x_2, x_3) \); with respect to \( w = (w_1, w_2, w_3) \), \( w_1, w_2, w_3 \) are coprime positive integers. Suppose that the singularity of \( X \) at \( 0 \in \mathbb{C}^3 \) is isolated. If
\begin{enumerate}
  \item \( 2w_3 < \inf\{w_1, w_2\} \);
  \item \( X(\mathbb{R}) \neq \{0\} \);
  \item \( X(\mathbb{R}) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1x_2x_3 = 0\} = \{0\} \).
\end{enumerate}
Then the singularity of \( X \) at \( 0 \in \mathbb{C}^3 \) is not strongly metrically conic.

**Proof.** Let \( i : X(\mathbb{R}) - \{0\} \to X - \{0\} \) be the embedding induced by inclusion \( X(\mathbb{R}) \subset X \).

**Case 1.** We suppose that there exists a connected component \( C \) of \( X(\mathbb{R}) - \{0\} \) such that \( i(C) \) presents nontrivial element in \( H_1(X - \{0\}) \).
Let \( Y = \overline{C} = C \cup \{0\} \). By the results of [5], \( Y \) is bi-Lipschitz equivalent,
with respect to inner metric, to a \( \beta \)-horn \( H_\beta \), where \( \beta = \frac{\inf\{w_1, w_2\}}{w_3} \) and from hypothesis \( \beta > 1 \). We found a 1-dimensional nontrivial cycle \( \sigma \) in \( H_1(X - \{0\}) \), given by \( \sigma = \partial \eta \) where \( \eta = Y \cap B(0, \epsilon) \). It means that 1-dimensional characteristic exponent of \( X \) at 0 (see [4], [2], [3]) is bigger than or equal to \( \mu(\eta, 0) = \beta + 1 > 2 \). Therefore, by results of [3], \( X \) at 0 is not strongly metrically conic, because otherwise the corresponding exponent must be smaller than or equal to 2.

**Case 2.** We suppose \( i(X(\mathbb{R}) - \{0\}) \) presents nontrivial element in the homology group \( H_1(X - \{0\}) \). Let \( \gamma = \pi(X(\mathbb{R}) - \{0\}) \), where \((\pi, M)\) is the Seifert Fibration of \( X - \{0\} \) associated to \( w = (w_1, w_2, w_3) \). Then, \([\gamma]\) is a trivial element in \( H_1(M) \). Since \( M \) is a 2-dimensional orbifold, we obtain that \( M - \gamma \) is not connected. Using Proposition 5.1, we obtain that \( M - \gamma \) has two connected components \( M_1 \) and \( M_2 \) such that \( M_1 = j(M_2) \). Moreover, by Corollary 5.2 we obtain that \( X - \pi^{-1}(\pi(X(\mathbb{R}))) \) is a union of two connected components \( X_1 \) and \( X_2 \) such that \( X_1 = \tau(X_2) \), where \( \tau \) is the complex conjugation of \( \mathbb{C}^3 \). The set \( Y = X - \pi^{-1}(\pi(X(\mathbb{R}))) \) is obtained by the revolution of \( X(\mathbb{R}) \) by a 1-dimensional subgroup of isometry group of \( \mathbb{C}^3 \). Since \( \mu(X(\mathbb{R}), 0) = \beta + 1 \), where \( \beta = \frac{\inf\{w_1, w_2\}}{w_3} \) (see [3]), we obtain that
\[
\mu(Y, 0) = \mu(X(\mathbb{R}), 0) + 1 = \beta + 2
\]
and, since \( \beta > 2 \), we have \( \mu(Y, 0) > 4 \). From the other hand,
\[
\mu(X, 0) = \mu(X_1, 0) = \mu(X_2, 0) = 4.
\]
Now, since \( X - \{0\} \) is connected one can apply Theorem 4.1 and, by the remark, \( X \) is not strongly metrically conic at the singular point 0.

**Theorem 5.4.** Let \( X \subset \mathbb{C}^3 \) be a irreducible surface defined by a real weighted homogeneous polynomial \( f(x_1, x_2, x_3) \); with respect to \( w = (w_1, w_2, w_3) \), where \( w_1, w_2, w_3 \) are coprime positive integers. Suppose that the singularity of \( X \) at \( 0 \in \mathbb{C}^3 \) is isolated. If
1. \( w_3 < \inf\{w_1, w_2\} \);
2. \( \pi(X(\mathbb{R})) \subset M \) has more than one connected component;
3. \( X(\mathbb{R}) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1x_2x_3 = 0\} = \{0\} \).
Then the singularity of \( X \) at \( 0 \in \mathbb{C}^3 \) is not metrically conic.

In order to show this theorem, we need the following proposition.

**Proposition 5.5.** Let \( Y \) be a connected component of \( X(\mathbb{R}) - \{0\} \) and let \( \xi = Y \cap S(0, \epsilon) \), for sufficiently small \( \epsilon > 0 \). Let \( [\xi] \neq 0 \) in \( H_1(X - \{0\}) \). Then the singularity of \( X \) at 0 is not metrically conic.
Proof. Suppose that there exist a subset
\[ N \subset \{ x \in \mathbb{R}^m : \|x\| = 1 \} \]
and a bi-Lipschitz map-germ
\[ F : (X, 0) \to (C_0N, 0). \]

Given \( r > 0 \), sufficiently small, let \( \xi_r = Y \cap S(0, r) \). By the conditions of the theorem, \( [\xi_r] \neq 0 \) in \( H_1(X - \{0\}) \). Hence \( F_*[\xi_r] \neq 0 \) in \( H_1(C_0N - \{0\}) \). Let us denote by \( Band(k_1, k_2) \) the following subset
\[ Band(k_1, k_2) = \{ x \in \mathbb{R}^m : k_1 \leq \|x\| \leq k_2 \} , \]
where \( k_1 < k_2 \) are bi-Lipschitz constants of \( F \). Let
\[ \theta_r : \mathbb{R}^m - \{0\} \to \mathbb{R}^m - \{0\} \]
be defined by
\[ \theta_r(x) = \frac{1}{r} x. \]

Since \( F \) is a bi-Lipschitz map-germ with bi-Lipschitz constants \( k_1 < k_2 \), we obtain
\[ \theta_r(F(\xi_r)) \subset Band(k_1, k_2) \cap C_0N \]
and
\[ \text{diam}(\theta_r(F(\xi_r))) \leq \frac{1}{r} k_2 \text{diam}(\xi_r). \]

On the other hand, since the germ \((Y, 0)\) is bi-Lipschitz equivalent to a \( \beta \)-horn, where \( \beta = \frac{\inf\{w_1, w_2\}}{w_3} \) (see [4]), we obtain
\[ \text{diam}(\xi_r) \leq \tilde{k} r^\beta \]
for some constant \( \tilde{k} > 0 \). Thus,
\[ \text{diam}(\theta_r(F(\xi_r))) \leq k_2 \tilde{k} r^{\beta - 1} \]
and, in particular, it means that
\[ \lim_{r \to 0} \text{diam}(\theta_r(F(\xi_r))) = 0. \]

Let \( P : Band(k_1, k_2) \cap C_0N \to N \) be a canonical projection
\[ P(x) = \frac{1}{\|x\|} x. \]

Since \( P \) is a Lipschitz map, we have
\[ \lim_{r \to 0} P(\theta_r(F(\xi_r))) = 0. \]

Since \( N \) is a topological manifold, \( P(\theta_r(F(\xi_r))) \) defines a trivial cycle in \( H_1(N) \), for sufficiently small \( r > 0 \). This is a contradiction because \( P_* \) and \( (\theta_r)_* \) are isomorphisms. \( \square \)
Proof of the theorem. Suppose that $\pi(X(\mathbb{R}))$ is trivial on $M$. Since $M - \pi(X(\mathbb{R}))$ has more than one connected component, there exists a component $C \subset \pi(X(\mathbb{R}))$ such that $[C] \neq 0$ in $H_1(M)$. Then $Y = \pi^{-1}(C)$ satisfies the conditions of proposition above. This proves the theorem. 

6. Example: Surfaces of Brieskorn

Theorem 6.1. The singularity at $0 \in \mathbb{C}^3$ of Brieskorn surface $X$ defined as follows:

$$x^2 + y^2 = z^{2k}, \quad k > 2$$

is not strongly metrically conic.

Note that these surfaces do not satisfy the conditions of Theorem 5.3 and the conditions of Theorem 5.4.

Proof. Proof of the theorem. Let $Y \subset X$ be the result of the $\mathbb{C}^*$-action on $X(\mathbb{R})$. Let us show that $Y$ is a base of a Cheeger’s cycle on $X$. Let us show that $X - Y$ has exactly two connected components. Consider the hyperplane section $\hat{X} = X \cap \{z = 1\}$. It is an affine curve given by the equation

$$x^2 + y^2 = 1.$$ 

The set $\hat{X}(\mathbb{R})$ is homeomorphic to $S^1$ and $\hat{X}$ is homeomorphic to a cone over $S^1$, hence the set $\hat{X} - \hat{X}(\mathbb{R})$ contains exactly two connected components, which are conjugated. Let $(x_0, y_0, z_0) \in X - Y$. We are going to show that there is no continuous path connecting $(x_0, y_0, z_0)$ with $(x_0, y_0, z_0)$. Observe that we can suppose that $(x_0, y_0, z_0)$ does not belong to $X \cap \{z = 0\}$, otherwise we can take a nearly point belonging to $X - \{z = 0\}$ connected to this one by a continuous path on $X - Y$.

Suppose that there exists a continuous path $\gamma: [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = \overline{p}$. By a transversality argument, we can suppose that $\gamma$ does not intersect the set $X \cap \{z = 0\}$. Let

$$\rho: X - \{z = 0\} \to \hat{X}$$

defined by $\rho(x, y, z) = (z^{-k}x, z^{-k}y, 1)$. Since the map $\rho$ respects the complex conjugation, we obtain

$$\overline{\rho(x_0, y_0, z_0)} = \rho(\overline{x_0}, \overline{y_0}, \overline{z_0}).$$

Thus $\rho(x_0, y_0, z_0)$ and $\overline{\rho(x_0, y_0, z_0)}$ belong to the different connected components of $\hat{X} - \hat{X}(\mathbb{R})$. Hence the path $\gamma$ must intersect $Y$, because $\rho(Y - z = 0) = \hat{X}(\mathbb{R})$. It means that $X - Y$ is not connected. Moreover $X - Y$ has exactly two connected components $X_1$ and $X_2$ which are conjugated. Now, since $\mu(X(\mathbb{R}), 0) = k + 1$, we get that $\mu(Y, 0) \geq k + 2$, using the same argument as in the proof of Theorem 5.3.

Finally, we obtain that the sets $X$ and $Y$ satisfy the conditions of Theorem 4.1. The set $Y$ is a base of a Cheeger’s cycle and the singular point $\{0\}$ is not strongly metrically conic. \qed
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