The Smoluchowski-Kramers Limit of Stochastic Differential Equations with Arbitrary State-Dependent Friction

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Abstract: We study a class of systems of stochastic differential equations describing diffusive phenomena. The Smoluchowski-Kramers approximation is used to describe their dynamics in the small mass limit. Our systems have arbitrary state-dependent friction and noise coefficients. We identify the limiting equation and, in particular, the additional drift term that appears in the limit is expressed in terms of the solution to a Lyapunov matrix equation. The proof uses a theory of convergence of stochastic integrals developed by Kurtz and Protter. The result is sufficiently general to include systems driven by both white and Ornstein–Uhlenbeck colored noises. We discuss applications of the main theorem to several physical phenomena, including the experimental study of Brownian motion in a diffusion gradient.

1. Introduction

For an open subset $\mathcal{U} \subset \mathbb{R}^d$, consider the 2$^d$-dimensional stochastic differential equation (SDE):

\[
\begin{align*}
\frac{d x_m^m}{dt} &= v_m^m \quad dt \\
\frac{d v_m^m}{dt} &= \left[ F(x_m^m) - \frac{\gamma(x_m^m)}{m} v_m^m \right] dt + \frac{\sigma(x_m^m)}{m} d W_t \\
x_0^m &= x, \\
v_0^m &= v,
\end{align*}
\]

with $F: \mathcal{U} \to \mathbb{R}^d$, $\gamma: \mathcal{U} \to \mathbb{R}^d \times d$ a $d \times d$ invertible matrix-valued function, $\sigma: \mathcal{U} \to \mathbb{R}^{d \times k}$ and $W$ a $k$-dimensional Wiener process. The above SDE provides a framework to model many physical systems, from colloidal particles in a fluid [19] to a camera tracking an object [22]. For example, the motion of a Brownian particle can be modeled using an SDE where $x$ and $v$ are one-dimensional and $\gamma(x) = \frac{k_B T}{D(x)}$ and $\sigma(x) = \frac{k_B T \sqrt{2}}{\sqrt{D(x)}}$ (see description below in Sect. 4.1). In fact, the original motivation for the present work was to provide a mathematical explanation of the experimental observation of a noise-induced drift in [36]. While in this model the coefficients $\gamma(x)$ and $\sigma(x)$ are constrained by the fluctuation-dissipation relation such that $\gamma(x) \propto \sigma(x)^2$ [35], our main result, Theorem 1,
does not assume it and has a much more general reach including applications in other fields.

Theorem 1 says that, under the assumptions stated in Sect. 2, the $x$-component of the solution of Equation (1) converges in $L^2$, with respect to the topology on $C_U([0, T])$ (i.e. the space of continuous functions from $[0, T]$ to $U$ with the uniform metric), to the solution of the SDE
\[ dx_t = \left[ y^{-1}(x_t) F(x_t) + S(x_t) \right] dt + y^{-1}(x_t) \sigma(x_t) dW_t, \] (2)

with the original initial condition $x_0 = x$, where $S(x_t)$ is the noise-induced drift whose $i$th component equals
\[ S_i(x) = \frac{\partial}{\partial x_l} \left[ (y^{-1})_{ij}(x) \right] J_{ji}(x), \] (3)

where $J$ is the matrix solving the Lyapunov equation
\[ J \gamma^* + \gamma J = \sigma \sigma^*. \] (4)

Throughout the paper we use Einstein summational convention and "*" denotes the transposition of a matrix. The limiting SDE (2) is given in the Itô form, while we provide in Sect. 5 the corresponding Stratonovich form. Note that for $m > 0$ the process $x_t^m$ has bounded variation and thus all definitions of stochastic integral lead to the same form of SDE (1).

The zero-mass limits of equations similar to Eq. (1) have been studied by many authors beginning with Smoluchowski [34] and Kramers [15]. In the case where $F = 0$ and $\gamma$ and $\sigma$ are constant, the solution to equation (1) converges to the solution of equation (2) almost surely [19]. The case including an external force was treated by entirely different methods in [31]. The problem of identifying the limit for position-dependent noise and friction was studied in [10] for the case when the fluctuation-dissipation relation is satisfied and in [30] for the general one-dimensional case (the multidimensional case is also discussed there but without complete proof). The homogenization techniques described in [23,25,31] were used to compute the limiting backward Kolmogorov equation as mass is taken to zero in [12]. In [24] convergence in distribution is proven rigorously for equations of the same type as Eq. (1), under somewhat stronger assumptions than those made here. The rigorous proof of convergence in probability for $\gamma$ constant and $\sigma$ position-dependent is given in [6]. The present paper contains the first rigorous derivation of the zero-mass limit of Eq. (1) for a multidimensional system with general friction and noise coefficients.

Systems with colored noise can also be treated within the above (suitably adapted) framework. For example, the one-dimensional equation driven by an Ornstein–Uhlenbeck (OU) noise with a short correlation time $\tau$
\[ m \ddot{x}_t^m = F(x_t^m) - \gamma(x_t^m) \dot{x}_t^m + \sigma(x_t^m) \eta_t = \] (5)
can be rewritten in the form of Eq. (1), by defining $v_t^m = (v_t^m, \eta_t \tau)^*$, $x_t^m = (x_t^m, \zeta_t \tau)^*$ and $\tau = \tau_0 m$ [25], as
\[
\begin{cases}
  dx_t^m = v_t^m dt \\
  dv_t^m = \left[ F(x_t^m) - \gamma(x_t^m) v_t^m + \sigma(x_t^m) \eta_t \right] dt \\
  d\zeta_t^\tau = \eta_t^\tau dt \\
  d\eta_t^\tau = -\frac{\eta_t^\tau}{\tau} dt + \sqrt{2\lambda} \frac{\eta_t^\tau}{\tau} dW_t.
\end{cases}
\] (6)