CONFORMAL CONTRACTIONS AND LOWER BOUNDS
ON THE DENSITY OF HARMONIC MEASURE

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Abstract. We give a concrete sufficient condition for a simply-connected domain to be the image of the unit disk under a nonexpansive conformal map. This class of domains is also characterized by having sufficiently dense harmonic measure. The relation with the harmonic measure provides a natural higher-dimensional analogue of this problem, which is also addressed.

1. Introduction

The images of the unit disk $\mathbb{D}$ under conformal maps $f$ with the normalization $f(0) = 0 = f'(0) - 1$ have long been understood and characterized in terms of their Green’s function, capacity of the complement, and so on (e.g., the books [5] and [8] expose this circle of ideas). This paper studies the effect of a uniform bound on the derivative of a conformal map: namely, $|f'(z)| \leq 1$ for all $z \in \mathbb{D}$. This condition can be equivalently stated as $|f(z) - f(w)| \leq |z - w|$ for all $z, w \in \mathbb{D}$; such $f$ may be called a conformal contraction. Under the normalization $f(0) = 0$, it follows that the image $f(\mathbb{D})$ must be contained in $\mathbb{D}$. However, not every subdomain of the unit disk is its image under a conformal contraction.

Let us consider a convex domain $\Omega \subset \mathbb{C}$ that contains 0 and has $C^{1,1}$-smooth boundary. With such a domain we associate three radii:

- **outer radius** $R_O$ is the smallest radius of a disk centered at 0 and containing $\Omega$;
- **inner radius** $R_I$ is the largest radius of a disk centered at 0 and contained in $\Omega$;
- **curvature radius** $R_C$ is the minimal radius of curvature of $\partial \Omega$. It is the largest radius $R$ such that $\Omega$ can be written as a union of open disks of radius $R$.

Note that $R_O \geq R_I$ and $R_O \geq R_C$, while there is no general relation between $R_I$ and $R_C$.

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Theorem 1.1. Let $\Omega \subset \mathbb{C}$ be a convex domain that contains $0$ and has $C^{1,1}$-smooth boundary. If the radii $R_O$, $R_I$, and $R_C$ satisfy
\[
(R_O - R_C) \frac{\log R_I - \log R_C}{R_I - R_C} + \frac{1}{2} \log R_C \leq 0
\]
then $\Omega = f(\mathbb{D})$ for some conformal map $f$ such that $f(0) = 0$ and $\sup |f'| \leq 1$. (When $R_I = R_C$, the difference quotient is understood as $1/R_I$.)

We will also consider the harmonic measure of domain $\Omega$ with respect to $0$, denoted $\omega_\Omega(\cdot, 0)$. In the context of Theorem 1.1 of particular interest is the Radon-Nikodym derivative of $\omega_\Omega(\cdot, 0)$ with respect to arclength, which will be called the density of harmonic measure.

The images of $\mathbb{D}$ under conformal contractions fixing $0$ are precisely those domains $\Omega$ for which the density of $\omega_\Omega(\cdot, 0)$ is at least $1/(2\pi)$ everywhere on the boundary. This follows immediately from the conformal invariance of harmonic measure and the fact that its density on the boundary of the unit disk is $1/(2\pi)$. Thus, Theorem 1.1 gives a sufficient condition for $\Omega$ to have harmonic measure with such a lower density bound.

Theorem 1.1 was prompted by a question of J. E. Tener [7] which arose in the following context. When $f$ is a conformal map of $\mathbb{D}$ into itself with $f(0) = 0$, the composition with $f$ is a contraction on the Hardy space $H^2(\mathbb{D})$, see [3, Corollary 3.7]. By the conformal invariance of harmonic measure, this implies that the restriction operator $R: H^2(\mathbb{D}) \to L^2(\partial\Omega, \omega_\Omega(\cdot, 0))$ is a contraction. A lower bound on the density of $\omega_\Omega(\cdot, 0)$ then allows one to estimate the norm of the restriction operator $R: H^2(\mathbb{D}) \to L^2(\partial\Omega)$ where $L^2$ is taken with respect to arclength.

Concerning the structure of condition (1.1) it should be noted that the term
\[
(R_O - R_C) \frac{\log R_I - \log R_C}{R_I - R_C}
\]
is scale-invariant, while the second term, $\frac{1}{2} \log R_C$, tends to $-\infty$ as the domain is scaled down. Thus, for any convex domain $\Omega$ of class $C^{1,1}$ Theorem 1.1 gives an explicit factor $\lambda > 0$ such that the scaled-down domain $\lambda\Omega$ is the image of $\mathbb{D}$ under a conformal contraction. This can be compared to the classical Kellogg-Warschawski theorem [6, Theorem 3.5] which asserts that the conformal map of the disk onto a Dini-smooth Jordan domain $\Omega$ has a uniformly continuous derivative. The latter also implies that $\lambda\Omega$ is the image of $\mathbb{D}$ under a conformal contraction for sufficiently small $\lambda > 0$. However, in contrast to Theorem 1.1 one does not have an explicit suitable value of $\lambda$ in this case.

Examples illustrating and motivating the condition (1.1) are given in [2].

The higher-dimensional version of Theorem 1.1 is stated in terms of the harmonic measure, since there is no longer a rich supply of conformal maps. The desired property of $\Omega$ in this case is having the density of $\omega_\Omega(\cdot, 0)$ at least $1/\sigma_{n-1}$, where $\sigma_{n-1}$ is the surface area of the unit sphere. The quantity
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1/σ_{n-1} is the density of the harmonic measure of the unit ball with respect to its center. We will also use the notation

\[ a^+ = \max(a, 0), \quad \phi(a, b) = \frac{\log a - \log b}{a - b}, \quad \phi(a, a) = 1/a. \]

Theorem 1.2. Let \( \Omega \subset \mathbb{R}^n, n > 2, \) be a convex domain that contains 0 and has \( C^{1,1} \)-smooth boundary. If the radii \( R_0, R_I, \) and \( R_C \) satisfy

\[
R_C R_I^{n-2} e^{n(R_0 - R_C - R_I/2) + \phi(R_I/2, R_C)} \leq \frac{2^{n-2} - 1}{2^{n-1}(n - 2)}
\]

then the density of the harmonic measure of \( \Omega \) with respect to 0 is bounded below by \( \sigma_{n-1}^{-1} \).

The exponential term in (1.2) is scale invariant, while the factor \( R_C R_I^{n-2} \) makes sure that the left hand side of (1.2) tends to 0 as the domain is scaled down.

2. Examples and counterexamples

The sufficient conditions of Theorems 1.1 and 1.2 are not necessary; however, they are reasonably precise. For example, in the special case \( R_0 = R_I = R_C = R \) the hypothesis of Theorem 1.1 is that \( R \leq 1 \), which is both necessary and sufficient in this case. The higher-dimensional estimate is less accurate: the inequality (1.2) simplifies to \( R \leq \frac{1}{2}((2^{n-2} - 1)/(n - 2))^{1/(n-1)} \), where the right hand side is less than 1 but converges to 1 as \( n \to \infty \).

To justify the presence of three radii \( R_0, R_I, R_C \) in Theorems 1.1 and 1.2, let us note that constraining just two of them would not be sufficient for the conclusion. Indeed, a convex polygon has zero density of harmonic measure at the vertices. Slightly rounding the corners, one obtains a domain that fails the conclusion of the theorem, which only the curvature radius detects. To show the necessity of \( R_I \), let \( \Omega \) be the disk of radius 1 centered at the point \( 1 - \epsilon \); the density of \( \omega_\Omega(\cdot, 0) \) is small on most of the boundary. Finally, letting \( \Omega \) be the convex hull of the union of two disks such as \( D(0, 1) \cup D(n, 1) \) shows that the presence of \( R_O \) is also necessary.

![Figure 1. A domain in Theorem 1.1](image-url)
Figure 1 presents a concrete example of a domain that satisfies (1.1), namely a rounded triangle with $R_O = 0.6$, $R_I = 0.5$, and $R_C = 0.4$. A domain satisfying (1.2) could have $n = 3$, $R_O = 3/4$, $R_I = R_C = 1/2$.

3. Preliminaries: Hyperbolic and Quasihyperbolic Metrics

The hyperbolic metric on the unit disk $\mathbb{D}$ is

$$\rho_{\mathbb{D}}(z, w) = \inf \int_{\gamma} \frac{|d\zeta|}{1 - |\zeta|^2},$$

where the infimum is taken over all rectifiable curves $\gamma$ connecting $z$ and $w$. In particular,

$$\rho_{\mathbb{D}}(z, 0) = \int_0^{|z|} \frac{dt}{1 - t^2} = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}.
$$

On other simply-connected domains the hyperbolic metric can be defined by its conformal invariance property: $\rho_{\Omega}(f(z), f(w)) = \rho_{\mathbb{D}}(z, w)$ if $f$ is a conformal map of $\mathbb{D}$ onto $\Omega$. In particular, for a disk $\Omega = D(a, R)$ we have

$$\rho_{D(a, R)}(z, a) = \frac{1}{2} \log \frac{R + |z - a|}{R - |z - a|}.
$$

As a consequence of the Schwarz-Pick lemma, the hyperbolic metric is monotone with respect to domain: if $G$ and $\Omega$ are two simply-connected domains and $z, w \in G \subset \Omega$, then

$$\rho_G(z, w) \geq \rho_{\Omega}(z, w).
$$

The quasihyperbolic metric $\rho^*_{\Omega}$ is defined by

$$\rho^*_\Omega(z, w) = \inf \int_{\gamma} \frac{|d\zeta|}{\text{dist}(\zeta, \partial\Omega)},$$

It is not conformally invariant, but is comparable to $\rho_{\Omega}$ for every simply-connected domain:

$$\frac{1}{4} \rho^*_\Omega(z, w) \leq \rho_{\Omega}(z, w) \leq \rho^*_\Omega(z, w).
$$

See [4, §I.4] or [6, §4.6].

4. Planar Domains: Proof of Theorem 1.1

When the domain $\Omega$ is rescaled by the map $z \mapsto (1 - \epsilon)z$, the left side of (1.1) decreases. Therefore, we may assume that strict inequality holds in (1.1).

Let $f$ be a conformal map of the unit disk $\mathbb{D}$ onto $\Omega$, normalized by $f(0) = 0$. By the Kellogg-Warschawska theorem [6, Theorem 3.5], $f'$ has a continuous extension to $\overline{\mathbb{D}}$, and for $\zeta \in \partial\mathbb{D}$ we have

$$\lim_{z \to \zeta, \ z \in \mathbb{D}} \frac{f(z) - f(\zeta)}{z - \zeta} = f'(\zeta).$$
By the maximum principle, it suffices to show $|f'| \leq 1$ on $\partial \mathbb{D}$. By (1.1) it suffices to show that

\begin{equation}
\lim_{|z| \to 1} \frac{\text{dist}(f(z), \partial \Omega)}{1 - |z|} \leq 1.
\end{equation}

Fix $z \in \mathbb{D}$ and let $d = \text{dist}(f(z), \partial \Omega)$. Since the small values of $d$ are of interest, we may assume $d < R_C$. Our plan is to estimate $\rho_{\Omega}(0, f(z))$ from above, which will yield

\begin{equation}
|z| < 1 - d
\end{equation}

for sufficiently small $d$, thus proving (4.2).

Choose a point $w \in \partial \Omega$ such that $|f(z) - w| = d$. By the definition of $R_C$, there is a disk $D = D(a, R_C)$ that has $w$ on its boundary and is contained in $\Omega$. Observe that $f(z)$ lies on the radius of this disk connecting $a$ to $w$, and therefore $|f(z) - a| = R_C - d$. By (3.3) and (3.2),

\begin{equation}
\rho_{\Omega}(f(z), a) \leq \rho_{D(a, R_C)}(f(z), a) = \frac{1}{2} \log \frac{2R_C - d}{d} \leq \frac{1}{2} \log \frac{2R_C}{d}.
\end{equation}

To estimate $\rho_{\Omega}(a, 0)$ we use the comparison with $\rho_{\Omega}^*$ stated in (3.4). Since $\Omega$ contains $D(0, R_I)$ and $D(a, R_C)$, the convexity of $\Omega$ implies

\begin{equation}
\text{dist}(ta, \partial \Omega) \geq tR_C + (1 - t)R_I, \quad 0 \leq t \leq 1.
\end{equation}

Integration along the line segment from 0 to $a$ yields

\begin{equation}
\rho_{\Omega}^*(a, 0) \leq |a| \int_0^1 \frac{dt}{tR_C + (1 - t)R_I} = |a|\phi(R_I, R_C).
\end{equation}

Since $D(a, R_C) \subset \Omega \subset D(0, R_O)$, we have $|a| \leq R_O - R_C$. In conclusion,

\begin{equation}
\rho_{\Omega}(a, 0) \leq (R_O - R_C)\phi(R_I, R_C).
\end{equation}

Suppose that (4.3) fails, that is, $|z| \geq 1 - d$. From the conformal invariance of hyperbolic metric,

\begin{equation}
\rho_{\Omega}(f(z), 0) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|} \geq \frac{1}{2} \log \frac{2 - d}{d}.
\end{equation}

Combining (4.4), (4.7), and (4.8) we obtain

\[ \frac{1}{2} \log \frac{2 - d}{d} \leq \frac{1}{2} \log \frac{2R_C}{d} + (R_O - R_C)\phi(R_I, R_C), \]

hence

\begin{equation}
\frac{1}{2} \log \left( 1 - \frac{d}{2} \right) \leq \frac{1}{2} \log R_C + (R_O - R_C)\phi(R_I, R_C).
\end{equation}

Since the right hand side of (4.9) is negative, the inequality implies a lower bound on $d$. Therefore, (4.3) hold provided that $d$ is sufficiently small. This proves Theorem 1.1.
5. Higher dimensions: proof of Theorem 1.2

In this section $\Omega$ is a convex domain in $\mathbb{R}^n$, $n > 2$, and $0 \in \Omega$. The density of $\omega_\Omega(\cdot, 0)$ with respect to the surface measure of $\partial \Omega$ is related to Green’s function $g_\Omega$ by

$$\omega_\Omega(E, 0) = \int_E \frac{\partial g_\Omega}{\partial n}. $$

Here the derivative is taken along the interior normal, and $g_\Omega$ is Green’s function with pole at 0, normalized by $g_\Omega(x, 0) = \frac{1}{(n-2)\sigma_{n-1}}|x|^{2-n} + O(1)$ as $x \to 0$.

To this end we use a lower bound for $g_\Omega$ in terms of the quasihyperbolic metric. An estimate of this kind is given in Section 1.2 of [1], namely $g_\Omega(x, 0) \geq \exp(-Ap_\Omega^1(x, 0))$ with unspecified $A$. But we need a more explicit bound, since the presence of $A$ in the exponent does not allow one to conclude with (5.1).

Fix $x \in \Omega$ and let $d = \text{dist}(x, \partial \Omega)$, assuming $d < R_C$. Choose a point $w \in \partial \Omega$ such that $|x - w| = d$. By the definition of $R_C$, there is a ball $B(a, R_C)$ that has $w$ on its boundary and is contained in $\Omega$. Since $|x - a| = R_C - d$, Harnack’s inequality [2, Theorem 1.4.1] yields

$$g_\Omega(x) \geq \left(\frac{R_C - |x - a|}{R_C + |x - a|}\right)^{n-2} g_\Omega(a) = \frac{d}{(2R_C - d)^{n-1}} g_\Omega(a).$$

Since $B(0, R_I) \subset \Omega$, it follows that the restriction of $g_\Omega$ to $B(0, R_I)$ is minorized by Green’s function of this ball: specifically,

$$g_\Omega(x) \geq \frac{1}{(n-2)\sigma_{n-1}}(|x|^{2-n} - R_i^{2-n}), \quad |x| < R_i.$$

In particular, at the point $a' = \frac{R_i}{2} |\nabla u|$, we have

$$g_\Omega(a') \geq \frac{2^{n-2} - 1}{(n-2)\sigma_{n-1}} R_i^{2-n}.$$

As a corollary of Harnack’s inequality [2, Corollary 1.4.2], the gradient of a positive harmonic function on $B(a, r)$ satisfies $|\nabla u(a)| \leq (n/r) u(a)$. Therefore,

$$|\nabla \log g_\Omega(x)| \leq n/\text{dist}(x, \partial \Omega^c)$$

where $\Omega^c = \Omega \setminus \{0\}$. This implies

$$|\log g_\Omega(x) - \log g_\Omega(y)| \leq n\rho_{\Omega^c}(x, y).$$

Case 1: $|a| \geq R_I/2$. Since $|a'| = R_I/2 \leq |a| \leq R_O - R_C$ and $a'$ is a scalar multiple of $a$, it follows that $|a - a'| \leq (R_O - R_C - R_I/2)$. Observe that
the domain $\Omega'$ contains the balls $B(a', R_I/2)$ and $B(a, R_C)$, as well as their convex hull. Integration similar to (4.6) yields

$$\rho_{\Omega'}^*(a, a') \leq |a - a'|\phi(R_I/2, R_C) \leq (R_O - R_C - R_I/2)\phi(R_I/2, R_C).$$

Using (5.5) we obtain

$$g_{\Omega'}(a) \geq \frac{2^{n-2} - 1}{(n-2)\sigma_{n-1}} R_I^{2-n} e^{-n(R_O - R_C - R_I/2)\phi(R_I/2, R_C)}.$$

Case 2: $|a| < R_I/2$. Instead of using $a'$, we have

$$g_{\Omega'}(a) \geq \frac{2^{n-2} - 1}{(n-2)\sigma_{n-1}} R_I^{2-n}$$

as in (5.4).

Thus, in either case

$$g_{\Omega'}(a) \geq \frac{2^{n-2} - 1}{(n-2)\sigma_{n-1}} R_I^{2-n} e^{-n(R_O - R_C - R_I/2)\phi(R_I/2, R_C)},$$

which by virtue of (5.2) implies

$$\frac{\sigma_{n-1} g_{\Omega}(x)}{d} \geq \frac{R_C^{n-2}}{(2R_C - d)^{n-1}} \frac{2^{n-2} - 1}{n-2} R_I^{2-n} e^{-n(R_O - R_C - R_I/2)\phi(R_I/2, R_C)}.$$

As $d \to 0$, the right hand side of (5.6) converges to

$$\frac{2^{n-2} - 1}{2^{n-1}(n-2)} \frac{1}{R_C R_I^{n-2}} e^{-n(R_O - R_C - R_I/2)\phi(R_I/2, R_C)} \geq 1.$$

This proves (5.1) and concludes the proof of Theorem 1.2.

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