On constrained analysis and diffeomorphism invariance of generalised Proca theories

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Abstract

In this paper we consider generalised Proca theories coupled to any background field and with time-time and time-space components of Hessian of the vector sector are zero, whereas the space-space part is non-degenerate. By using Faddeev-Jackiw analysis, we derive the conditions that these theories have to satisfy in order for the vector sector to have three propagating degrees of freedom. Most of these conditions are trivialised due to diffeomorphism invariance requirements. This leaves only a condition that a complicated combination of terms should not be trivially zero. This condition is therefore easy to be fulfilled. For completeness, we have also investigated on how diffeomorphism invariance helps in simplifying Faddeev-Jackiw brackets.

1 Introduction

General Relativity has been a successful theory giving predictions which accurately agree with observations [1]. There are, however, results which cannot be described by General Relativity. One of these is the late-time accelerated expansion of the universe [2],[3]. As an attempt towards describing the mechanism behind this, one may consider modifying General Relativity. A simple way is to introduce a scalar-tensor theory, in which there is a scalar field introduced into the Lagrangian. By demanding that the extra scalar field has only one degree of freedom, thus is free from Ostrogradsky instabilities [4], up to the second order derivative of this extra scalar field could appear in the Lagrangian. The scalar sector in flat spacetime is a Galilean theory [5]. When generalised to curved spacetime and include the

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gravity sector, the resulting theory whose equations of motion for both scalar and
gravity are of at most second order derivative can be constructed and is known as
the Horndeski theory \[6, 7, 8, 9\]. Further extensions to the Horndeski theory
can actually be made \[10, 11, 12\]. For a review see \[13\].

An alternative direction is to consider a vector-tensor theory with Galilean-like
interactions. It is pointed out by \[14\] that there is a no-go theorem preventing
the inclusion of Galilean-like interactions in the case where the vector is massless.
One then needs to turn to generalising the Proca theory, which is a massive vector
theory.

A generalised Proca theory describes a system of gauge field with derivative self-
interaction. An original construction of generalised Proca theories is given in \[15, 16\]. The idea is to start from a form of Lagrangian of gauge field in flat spacetime
with several constants to be determined. Demanding that the Hessian determinant
vanishes ensures that the theory has constraints, and hence at most three propagat-
ing degrees of freedom. This requirement gives rise to conditions which relate some
of the constants. After the theory in flat spacetime is constructed, an insight from
Horndeski theory is made use to extend the theory to curved spacetime.

The construction is confirmed and extended by \[17\], which systematically con-
structs the derivative self-interactions for the generalised Proca action beyond de-
coupling limit by using antisymmetric properties of Levi-Civita tensors. This finally
gives rise to the Lagrangian of the form

$$
\mathcal{L}_{\text{gen.Proca}} = -\sqrt{-g} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sqrt{-g} \sum_{n=2}^{6} \beta_n \mathcal{L}_n,
$$

(1.1)

where \(\beta_n, n = 2, 3, 4, 5, 6\) are arbitrary constants and \(\mathcal{L}_n, n = 2, 3, 4, 5, 6\) are self-
interactions of the gauge field given by

\[
\begin{align*}
\mathcal{L}_2 &= G_2(A_\mu, F_{\mu\nu}, \tilde{F}_{\mu\nu}), \\
\mathcal{L}_3 &= G_3(Y) \nabla \cdot A, \\
\mathcal{L}_4 &= G_4(Y) R + G'_4(Y) [(\nabla \cdot A)^2 - \nabla_\rho A_\sigma \nabla^\sigma A^\rho], \\
\mathcal{L}_5 &= G_5(Y) G_{\mu\nu} \nabla^\alpha A^\nu - \frac{1}{6} G'_5(Y) [(\nabla \cdot A)^3 - 3(\nabla \cdot A) \nabla_\rho A_\sigma \nabla^\sigma A^\rho \\
&\quad + 2 \nabla_\rho A_\sigma \nabla^\gamma A^\rho \nabla^\sigma A_\gamma] - G'_6(Y) F^{\alpha\beta} \tilde{F}_{\mu\nu} \nabla_\alpha A_\mu, \\
\mathcal{L}_6 &= G_6(Y) L^{\mu\nu\alpha\beta} \nabla_\mu A_\nu \nabla_\alpha A_\beta + \frac{G'_6(Y)}{2} \tilde{F}^{\alpha\beta} \tilde{F}_{\mu\nu} \nabla_\alpha A_\mu \nabla_\beta A_\nu,
\end{align*}
\]

(1.2)

where \(Y \equiv -A_\mu A^\mu/2, \nabla \cdot A \equiv \nabla_\mu A^\mu, \tilde{F}_{\mu\nu} \) is the Hodge dual of \(F_{\mu\nu}\), and

$$
L^{\mu\nu\alpha\beta} = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta}.
$$

(1.3)

Extensions can also be made. In \[18\], other terms can be added. This results in
theories called “beyond generalised Proca theories”. In \[19\], new classes of theories
are found and classified by the full analysis of vector-tensor theories with up to
quadratic order in the first derivatives of the vector field. The construction made
use of ADM decomposition to eliminate an unwanted mode. In \[20, 21\], a system
of multiple Maxwell fields interacting with multiple Proca fields is constructed. This construction is in flat spacetime.

For definiteness, let us keep calling the vector sector of the above theories, as well as their possible other extensions as generalised Proca theories. The construction of the most general generalised Proca theories remains an open question. In principle, a possible way to do this is by following the idea of the original construction of generalised Proca theories, that is by starting from demanding that Hessian is degenerate. As pointed out recently in [22], in principle, there remains further interaction terms to be found.

We would like to work towards this ultimate goal. As an initial step, we take as a hint the usefulness of the condition for degenerate Hessian determinant in determining generalised Proca theories with three propagating degrees of freedom. This condition has been used in literature with great success since the pioneering works [15], [16] on generalised Proca theories. So we conjecture that the condition for degenerate Hessian determinant almost guarantees that the vector sector has three propagating degrees of freedom. This means that we expect that some conditions coming from constrained analysis (to get three propagating degrees of freedom) are greatly simplified by the condition for degenerate Hessian determinant.

In this paper, we will investigate generalised Proca theories which satisfy

\[
\frac{\partial^2 \mathcal{L}}{\partial A_0 \partial A_\mu} = 0, \quad \det \left( \frac{\partial^2 \mathcal{L}}{\partial A_i \partial A_j} \right) \neq 0.
\]

This condition is stronger than the condition that Hessian determinant is degenerate. Nevertheless, large classes of generalised Proca theories, for example the Lagrangian (1.1), satisfies these conditions. The generalised Proca theories that we investigate in this paper can be coupled to background metric as well as to any other background fields. Furthermore, the vector field as well as all the background fields should all be diffeomorphic. We will argue in this paper that diffeomorphism invariance and the condition (1.4) almost guarantee that the vector sector has three propagating degrees of freedom.

This paper is organised as follows. In section 2, we give a quick review of the Dirac method and the Faddeev-Jackiw method for constrained analysis. Then we demonstrate the use of the Faddeev-Jackiw method on the Proca theory. In section 3, we derive useful conditions from diffeomorphism invariance requirements on generalised Proca theories coupled to any background field and subject to the condition (1.4). In section 4, we consider Faddeev-Jackiw analysis of these theories and derive the conditions that they should satisfy in order for the vector sector to have three propagating degrees of freedom. Some these conditions are trivialised by the equations obtained in section 3. As part of cross-checks, we consider explicit examples in section 5 and discuss how they satisfy conditions obtained in section 4. In section 6, we comment on the implication that diffeomorphism invariance has on the Faddeev-Jackiw bracket. We conclude this paper and discuss possible future works in section 7.
2 Methods for analysing constrained systems

Let us give a quick review of two popular methods to analyse constrained systems. These methods are the Dirac method and the Faddeev-Jackiw method. We give emphases on the latter and demonstrate its use on Proca theory.

2.1 The Dirac method

In the context of classical field theory, a constrained system is the system in which the number of dynamical variables is smaller than the number of generalised coordinates. For a generalised coordinate to be a dynamical variable, its equation of motion should be of second order in time derivative. This corresponds to arbitrariness of the initial values of the generalised coordinate and the corresponding generalised velocity. If, however, the equation of motion of a generalised coordinate is at most of first order in time derivative, then this equation restricts the initial values of the generalised coordinate and velocity. This means that the arbitrariness is lost. In this case, the generalised coordinate is not a dynamical variable.

The criteria described above can be formalised, which gives rise to a simple condition to determine whether a system is constrained. In particular, consider a system with the Lagrangian density of the form

\[ L(\phi_a(x), \partial_\mu \phi_a(x)), \]

where \( a = 1, 2, \cdots, N \). If the determinant of the Hessian

\[ \frac{\partial^2 L}{\partial \dot{\phi}_a \partial \dot{\phi}_b} \]

is zero, then the system is a constrained system.

In order to analyse constrained systems, the Dirac method \[23\], \[24\], \[25\] is a well-known method. Starting from the Lagrangian \( L(\phi_a(x), \partial_\mu \phi_a(x)) \), one defines conjugate momenta as

\[ \pi^a = \frac{\partial L}{\partial \dot{\phi}_a}. \]

If the system is not constrained, eq.(2.3) can be inverted to uniquely express \( \dot{\phi}_a \) in terms of \( \pi^a \). But if the system is constrained, this is no longer the case. One may then proceed to extract from eq.(2.3) constrained equations, which are equations containing \( \dot{\phi}_a \) and \( \pi^a \) but without \( \phi_a \). Suppose there are \( k \) constrained equations of the form \( \Phi^\hat{m} = 0 \), for \( \hat{m} = 1, 2, \cdots, k \). The quantity \( \Phi^\hat{m} \) are called constraints. In particular, since they are the initial set of constraints being generated, they are called “primary constraints”.

Next, one requires that the primary constraints should remain constraints even after the time has evolved. This gives the criteria that the time derivative of constraints should remain on constrained surface in phase space. To compute the time derivative, one needs Hamiltonian. One obtains the Hamiltonian density by using Legendre transformation giving

\[ \mathcal{H} = \pi^a \dot{\phi}_a - L - \gamma_\hat{m} \Phi^\hat{m}, \]

(2.4)
where $\gamma_{\hat{m}}$ are Lagrange multipliers. The Poisson bracket of a phase space variable and the Hamiltonian is the time derivative of that variable. Suppose that one follows the criteria and obtain further constraints. These constraints are called “secondary constraints”. Then the criteria can be applied to the secondary constraints, and in case it generate further constraints, these must satisfy the criteria as well. The process should be repeated until there are no further constraints generated.

Next, one needs to reclassify all of the constraints of the system. If all the Poisson brackets of a constraint with other constraints vanish on constrained surfaces, then that constraint (sometimes, it is necessary to redefine the constraints by writing them as linear combinations of all the constraints) is called a “first-class constraint”. The constraints which are not first-class constraints are called second-class constraints. Immediate and important usage from this classification is to obtain the number of degrees of freedom from the formula

$$\text{number of d.o.f.} = \frac{n_{PS} - 2n_1 - n_2}{2},$$

where $n_{PS}$ is the number of phase space variables, $n_1$ is the number of first-class constraints, and $n_2$ is the number of second-class constraints. The counting of degrees of freedom has been useful for example to check whether a proposed theory is free of ghost degrees of freedom.

Each class of constraints also have their important roles. Let us briefly state them. First-class constraints generate gauge transformation. As for second-class constraints, they are used in order to form Dirac bracket, which is the constrained system counterpart of the unconstrained system’s Poisson bracket. As part of the canonical quantisation of constrained system, the Dirac bracket is promoted to commutator.

### 2.2 The Faddeev-Jackiw method

Another method for analysing the constrained system is called the Faddeev-Jackiw method [26], [27], [28], [29]. This method is relatively simpler than Dirac method, for example, one does not need to classify the constraints. Let us give a brief review as follows.

Let us first consider the Lagrangian density in the form of eq.(2.1). Then follow the same discussions as in Dirac method until obtaining eq.(2.4). However, let us redefine $\gamma_{\hat{m}}$ as $\dot{\gamma}_{\hat{m}}$. So eq.(2.4) becomes

$$\mathcal{H} = \pi^a \dot{\phi}_a - L - \dot{\gamma}_{\hat{m}} \Phi^\hat{m}.$$  (2.6)

There is no generality lost in the redefinition of the Lagrange multiplier. Furthermore, it is part of the standard Faddeev-Jackiw algorithm. Next, one defines

$$\mathcal{L}_{FOF} = \pi^a \dot{\phi}_a - \mathcal{H},$$  (2.7)

which is called the first-order form of the Lagrangian. The reason for this name is that, by construction, $\mathcal{L}_{FOF}$ contains terms at most of first order derivative in time.
Then one collects the phase space variables and the constraints into the symplectic variables \( \xi^I = (\phi^a, \pi^a, \gamma^m) \), for \( I = 1, 2, \cdots, 2N + k \). Then one defines

\[
A_{\xi^I} = \frac{\partial L_{\text{FOF}}}{\partial \dot{\xi}^I}.
\]  

(2.8)

Note that we have used the symplectic variables \( \xi^I \) as labels instead of just simply the indices \( I \). Next, by using eq. (2.8), \( L_{\text{FOF}} \) can be written in the form

\[
L_{\text{FOF}} = A_{\xi^I} \dot{\xi}^I + L_v.
\]  

(2.9)

Note that \( A_{\xi^I} \) and \( L_v \) only depend on \( \xi^I \) but not \( \dot{\xi}^I \).

In order to proceed, it is convenient to make use of differential form language. Let us denote the coordinate basis for vector and 1-form in the space of \( \xi^I \) as

\[
\frac{\delta}{\delta \xi^I(t, \vec{x})}, \quad \text{and} \quad \delta \xi^I(t, \vec{x}),
\]

(2.10)

respectively. Let us define the one-form corresponding to \( A_{\xi^I} \) as

\[
A(t) \equiv \int d^3 \vec{x} A_{\xi^I}(t, \vec{x}) \delta \xi^I(t, \vec{x}).
\]

(2.11)

The quantity \( A(t) \) is called the canonical 1-form. Applying the exterior derivative

\[
\delta \equiv \int d^3 \vec{x} \delta \xi^I(t, \vec{x}) \frac{\delta}{\delta \xi^I(t, \vec{x})}
\]

(2.12)

on \( A(t) \) gives

\[
F(t) \equiv \delta A(t) = \frac{1}{2} \int d^3 \vec{x} \int d^3 \vec{y} \left( \frac{\delta A_{\xi^I}(t, \vec{y})}{\delta \xi^I(t, \vec{x})} - \frac{\delta A_{\xi^I}(t, \vec{x})}{\delta \xi^I(t, \vec{y})} \right) \delta \xi^I(t, \vec{x}) \wedge \delta \xi^I(t, \vec{y})
\]

(2.13)

The quantity \( F(t) \) is called the symplectic 2-form. It is an important quantity which is used to determine if the system has additional constraints. For this, let us consider an equation

\[
i_{z(t)} F(t) = 0,
\]

(2.14)

where

\[
z(t) = \int d^3 \vec{x} \ z^{\xi^I}(t, \vec{x}) \frac{\delta}{\delta \xi^I(t, \vec{x})}.
\]

(2.15)

The non-trivial solution \( z^{\xi^I}(t, \vec{x}) \) to eq. (2.14) is just the zero mode of \( F_{\xi^I \xi^J}(t, \vec{x}, \vec{y}) \). If there is no zero mode, then the system has no further constraint. However, if zero mode exists, there might be new constraints. The new constraints are generated from

\[
\Omega(t) = i_{z(t)} \delta \int d^3 \vec{x} \ L_v(t, \vec{x}),
\]

(2.16)
where \( z(t) \) is the non-trivial solution to eq.\((2.14)\). It might turn out that some zero modes do not lead to a new constraint. This case can occur if after substituting these zero modes into eq.\((2.16)\), one obtains some trivial conditions (for example \( \Omega = 0 \)), or constraints dependent on the ones already discovered.

In case there are new constraints generated from the above steps, one needs to repeat the above steps by first modifying the first-order Lagrangian. The iterations should be continued until there is no further constraint obtained. For definiteness, let us call the steps we just discussed as the first iteration. Next, let us discuss how the next iterations should be carried out.

Suppose there are \( k' \) new constraints. Let us denote all constraints so far as \( \Phi^{\hat{m}} \), where the index \( \hat{m} \) is redefined to take values \( \hat{m} = 1, 2, \cdots, k + k' \). The Hamiltonian is then redefined accordingly so that it still takes the form of eq.\((2.6)\), but with the range of \( \hat{m} \) changed to \( 1, 2, \cdots, 2N + k + k' \). One may then follow the steps in the previous paragraph to obtain further constraints. If there is a new one, repeat the steps again and again until there is no further constraint generated.

When the iterations end, the matrix inverse of the symplectic 2–form gives Faddeev-Jackiw bracket \([\cdot, \cdot]_{\text{FJ}}\). That is \([27]\)

\[
[\xi^I(t, \vec{x}), \xi^J(t, \vec{y})]_{\text{FJ}} = (\mathcal{F}^{-1})^{\xi^I \xi^J}(t, \vec{x}, \vec{y}),
\]

(2.17)

where \((\mathcal{F}^{-1})^{\xi^I \xi^J}(t, \vec{x}, \vec{y})\) is the matrix inverse of \( \mathcal{F}^{\xi^I \xi^J}(t, \vec{x}, \vec{y}) \) in the sense that

\[
\int d^3\vec{y} \mathcal{F}^{\xi^I \xi^J}(t, \vec{x}, \vec{y})(\mathcal{F}^{-1})^{\xi^J \xi^K}(t, \vec{y}, \vec{z}) = \delta^K_I \delta^{(3)}(\vec{x} - \vec{z}).
\]

(2.18)

It has been argued in literature \([27]\), that in various theories, Faddeev-Jackiw bracket is equivalent to Dirac bracket. So after canonical quantisation, the Faddeev-Jackiw bracket could be promoted to commutator.

### 2.3 Faddeev-Jackiw analysis of Proca theory

Let us review the analysis of Proca theory using the Faddeev-Jackiw method. This analysis will form a basis for later generalisations performed in this paper.

The Proca action in a four-dimensional flat Minkowski spacetime with signature \((- , + , + , + )\) is given by

\[
S_{\text{Proca}} = \int d^4x \left( - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} m^2 A_{\mu} A^{\mu} \right),
\]

(2.19)

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) with \( \mu, \nu = 0, 1, 2, 3 \). The conjugate momenta to \( A_\mu \) can easily be worked out, and are given by

\[
\pi^\mu = F^{\mu 0}.
\]

(2.20)

From these equations, one obtains the primary constraint

\[
\Omega_1 \equiv \pi^0 = 0.
\]

(2.21)
The Hamiltonian density is given by

\[ \mathcal{H}(A^\mu, \pi^\mu) = \frac{1}{2} \pi^i \dot{\pi}_i + \pi^i (\partial_i A_0) + \frac{1}{2} m^2 A_\mu A^\mu + \frac{1}{4} F_{ij} F^{ij} - \gamma_1 \pi^0, \]  
(2.22)

where \( \gamma_1 \) is a Lagrange multiplier, and \( A^2 = A_\mu A^\mu \). This gives

\[ \mathcal{L}_{FOF} = \pi^0 \dot{A}_0 - \frac{1}{2} \pi^i \dot{\pi}_i - \pi^i (\partial_i A_0) - \frac{1}{2} m^2 A^2 - \frac{1}{4} F_{ij} F^{ij} + \gamma_1 \pi^0. \]  
(2.23)

After putting eq.(2.23) in the form of eq.(2.9), one sees that \( \mathcal{L}_v \) is given by

\[ \mathcal{L}_v = -\frac{1}{2} \pi^i \dot{\pi}_i - \pi^i (\partial_i A_0) - \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{4} F_{ij} F^{ij}. \]  
(2.24)

Let us define the symplectic variables as \( \xi = (A_0, A_i, \pi^0, \pi^i, \gamma_1) \), and compute the canonical momenta using eq.(2.8). The corresponding canonical 1–form is given by

\[ \mathcal{A} = \int d^3 \vec{x} \left( \pi^0 \delta A_0 + \pi^i \delta A_i + \pi^0 \delta \gamma_1 \right). \]  
(2.25)

The symplectic 2–form is then

\[ \mathcal{F} = \int d^3 \vec{x} \left( \delta \pi^0 \wedge \delta A_0 + \delta \pi^i \wedge \delta A_i + \delta \pi^0 \wedge \delta \gamma_1 \right). \]  
(2.26)

This gives

\[ i_z \mathcal{F} = \int d^3 \vec{x} \left( z \pi^0 \delta A_0 - z A_0 \delta \pi^0 + z A_i \delta \pi^i - z \pi^0 \delta A_i + z \pi^0 \delta \gamma_1 - z \gamma_1 \delta \pi^0 \right). \]  
(2.27)

The solution to \( i_z \mathcal{F} = 0 \) is then

\[ z = \int d^3 \vec{x} \left( A_0 \left( \frac{\delta}{\delta A_0} - \frac{\delta}{\delta \gamma_1} \right) \right). \]  
(2.28)

By using eq.(2.16), (2.24), and (2.28), one sees that there is an additional constraint given by

\[ \Omega_2 = \partial_i \pi^i + m^2 A_0. \]  
(2.29)

Due to the presence of the new constraint (2.29), one needs to consider the second iteration by starting from the following first-order form of Lagrangian

\[ \mathcal{L}_{FOF} = \pi^0 \dot{A}_0 - \frac{1}{2} \pi^i \dot{\pi}_i - \pi^i (\partial_i A_0) - \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{4} F_{ij} F^{ij} + \gamma_1 \pi^0 + \gamma_2 (\partial_i \pi^i + m^2 A_0). \]  
(2.30)

Furthermore, the symplectic variables are now \( \xi^I = (A_0, \pi^0, A_i, \pi^i, \gamma_1, \gamma_2) \). The canonical 1–form and the symplectic 2–form are then given by

\[ \mathcal{A} = \int d^3 \vec{x} \left( \pi^0 \delta A_0 + \pi^i \delta A_i + \pi^0 \delta \gamma_1 + (\partial_i \pi^i + m^2 A_0) \delta \gamma_2 \right), \]  
(2.31)

and

\[ \mathcal{F} = \int d^3 \vec{x} \left( \delta \pi^0 \wedge \delta A_0 + \delta \pi^i \wedge \delta A_i + \delta \pi^0 \wedge \delta \gamma_1 + (\partial_i \delta \pi^i + m^2 \delta A_0) \wedge \delta \gamma_2 \right). \]  
(2.32)
Applying interior derivative on eq. (2.32) gives
\[ i_z F = \int d^3 \vec{x} \left( (z^{\pi^0} - m^2 z^{\gamma_2})\delta A_0 + z^{\pi^i} \delta A_i - (z^{A_0} + z^{\pi^0})\delta \pi^0 + (\partial_i z^{\gamma_1} - z^{A_i})\delta \pi^i 
+ z^{\pi^0} \delta \gamma_1 + (\partial_i z^{\pi_1} + m^2 z^{A_0})\delta \gamma_2 \right). \] (2.33)

The equation \( i_z F = 0 \) can be solved step-by-step as follows. One starts from considering coefficients of \( \delta A_i \) and \( \delta \gamma_1 \). This gives \( z^{\pi^0} = 0 \). Next, one considers coefficients of \( \delta A_0 \) and \( \delta \gamma_2 \). This gives \( z^{\gamma_2} = 0 = z^{A_0} \). Finally, one considers coefficients of \( \delta \pi^i \). This gives \( z^{\pi^1} = 0 = z^{A_i} \). Therefore, the only solution is \( z = 0 \), and hence there is no further constraint.

With all the constraints at hand, it is now possible to count the number of degrees of freedom. This is by first noting that the Poisson’s bracket between the two constraints is
\[ \{ \pi^0(\vec{x}), (\partial_i \pi^i + m^2 A_0)(\vec{y}) \} = -m^2 \delta^{(3)}(\vec{x} - \vec{y}). \] (2.34)

Note that we have omitted writing the dependence on \( t \). So for example \( \pi^0(\vec{x}) \) stands for \( \pi^0(t, \vec{x}) \). From eq. (2.34), we see that both constraints are of second-class. The number of degrees of freedom can be obtained from eq. (2.5). In this case, \( n_{PS} = 8, n_1 = 0, \) and \( n_2 = 2 \). Therefore the Proca theory has three degrees of freedom.

The analysis given above is for the case \( m \neq 0 \). The analysis for the case \( m = 0 \) has to be given separately. For this, we may easily follow the steps from eq. (2.19) to eq. (2.33) by simply setting \( m = 0 \). So at this stage, there are in total two constraints: \( \pi^0 \approx 0 \) and \( \partial_i \pi^i \approx 0 \). The zero modes to eq. (2.33) with \( m = 0 \) are given by
\[ z = \int d^3 \vec{x} \left( z^{\pi_0} \left( \frac{\delta}{\delta A_0} - \frac{\delta}{\delta \gamma_1} \right) + z^{\pi_1} \frac{\delta}{\delta \gamma_2} + \partial_i z^{\pi_1} \frac{\delta}{\delta A_i} \right). \] (2.35)

Although there are two zero modes, it can be checked that they do not lead to a new constraint. So the process has to stop. The Poisson’s bracket between the two constraints can then be computed and found that it vanishes:
\[ \{ \pi^0(\vec{x}), (\partial_i \pi^i)(\vec{y}) \} = 0. \] (2.36)

So both of the constraints are of first-class, and hence the theory has two degrees of freedom.

In fact, it can already be seen from the Faddeev-Jackiw iterative process, without computing Poisson’s bracket, that there exists first-class constraints for Maxwell theory but there is no first-class constraint in Proca theory. The criteria is provided by [30], which is summarised as follows. If all of the zero modes in the Faddeev-Jackiw process give rise to independent constraints, then there is no first-class constraint. However, if there are zero modes which do not give rise to new constraints, then there are first-class constraints. In fact, [30] also provides the way of counting number of degrees of freedom from the number of zero modes and of constraints. However, we will not discuss this way of counting in this paper.
Next, let us obtain Faddeev-Jackiw bracket. We first read off the matrix $F_{\xi^I \xi^J}(\vec{x}, \vec{y})$ from eq.(2.32). We have

$$F_{\xi^I \xi^J}(t, \vec{x}, \vec{y}) = \begin{pmatrix} 0 & -\delta^i_\alpha & 0 & m^2 \delta^0_0 \\ \delta^0_\mu & 0 & \delta^0_0 & -\delta^i_\sigma \partial_x^i \\ 0 & -\delta^0_\mu & 0 & 0 \\ -m^2 \delta^0_0 & -\delta^i_\sigma \partial_x^i & 0 & 0 \end{pmatrix} \delta^{(3)}(\vec{x} - \vec{y}), \quad (2.37)$$

where we have used the labels $\xi^I(t, \vec{x}) = (A_\mu(t, \vec{x}), \pi^\sigma(t, \vec{x}), \gamma_1(t, \vec{x}), \gamma_2(t, \vec{x}))$, and

$\xi^J(t, \vec{y}) = (A_\mu(t, \vec{y}), \pi^\sigma(t, \vec{y}), \gamma_1(t, \vec{y}), \gamma_2(t, \vec{y}))$. Inverting eq.(2.37) gives

$$(F^{-1})^{\xi^I \xi^K}(t, \vec{x}, \vec{y}) = \begin{pmatrix} \frac{2}{m^2} \delta^0_0 \delta^i_\rho \partial_x^i \\ -\delta^0_\mu \delta^i_\rho \\ \frac{-m^2}{m^2} \delta^0_0 \delta^i_\rho \\ \frac{-m^2}{m^2} \delta^0_0 \delta^0_0 \\ 0 \\ \frac{m^2}{m^2} \delta^0_0 \delta^0_0 \end{pmatrix} \delta^{(3)}(\vec{y} - \vec{z}). \quad (2.38)$$

So the Faddeev-Jackiw brackets between the canonical variables are

$$[A_\mu(t, \vec{x}), A_\rho(t, \vec{y})]_{FJ} = \frac{2}{m^2} \delta^0_0 \delta^i_\rho \partial_x^i \delta^{(3)}(\vec{x} - \vec{y}), \quad (2.39)$$

$$[A_\mu(t, \vec{x}), \pi^\sigma(t, \vec{y})]_{FJ} = \delta^i_\mu \delta^0_\rho \delta^{(3)}(\vec{x} - \vec{y}), \quad (2.40)$$

$$[\pi^\rho(t, \vec{x}), \pi^\sigma(t, \vec{y})]_{FJ} = 0, \quad (2.41)$$

which, after taking into account the different convention for metric signature, is in exact agreement with Dirac’s bracket given in [31], [32].

## 3 Conditions from diffeomorphism invariance

A generalised Proca Lagrangian which satisfies the requirement (1.4) should take the form

$$L = T(A_\nu, \partial_k A_\nu, \dot{A}_k, g_{\mu\nu}, \partial_{\kappa\lambda} g_{\mu\nu}, \partial_{\kappa\lambda\mu\nu} g_{\rho\sigma}, \cdots, K)$$

$$+ U(A_\nu, \partial_t A_\nu, g_{\mu\nu}, \partial_{\kappa\lambda} g_{\mu\nu}, \partial_{\kappa\lambda\mu\nu} g_{\rho\sigma}, \cdots, K) \dot{A}_0. \quad (3.1)$$

Here, only $T$ is allowed to depend on $\dot{A}_k$, but $U$ should not be. Note that the action (3.1) describes the dynamics of a vector field coupled to the background metric $g_{\mu\nu}$ and their derivatives. Furthermore we also include for completeness, the coupling to other external fields which are, along with their possible derivatives of any order, collectively called $K$.

The Lagrangian (3.1) is free of Ostrogradsky instability [4]. This is because the Lagrangian depends only up to the first order in time derivative of the field $A_\mu$. On the other hand, the appearance of time derivative of the metric and of other external fields in eq.(3.1) need not concern us because the dynamics of these fields are not determined by the theory (3.1). Of course, care must be taken when this Lagrangian is included into the full Lagrangian, in which every field is dynamical. We leave this as a future work. See also section 7.
We also require that the theory (3.1) is invariant under diffeomorphism. This requirement will impose conditions on the form of $T$ and $U$. In practice, one would normally wish to start from a Lagrangian which is already diffeomorphism invariance, and then put it in the form (3.1) and proceed with constrained analysis. So it seems conditions coming from diffeomorphism invariance requirement do not need to be stated at all. It turns out, however, that some conditions coming from diffeomorphism invariance requirement will help simplifying the constrained analysis. So we will derive these useful conditions before working on constrained analysis.

In this section, we are going to consider diffeomorphism transformation on the Lagrangian (3.1) and require that the theory is diffeomorphism invariant. The conditions to be used do not depend explicitly on $g_{\mu\nu}$ and other external fields. However, to demonstrate the calculation, we let $K$ in eq.(3.1) to be a collection $(\partial_{\sigma_1\cdots\sigma_p} k^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} : m, n, p = 0, 1, \cdots)$, where $k^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}$ is an external tensor field of rank $(m,n)$. Of course, the results to be found in this section also hold when $K$ also include external fermion fields.

Consider a diffeomorphism transformation $x^\mu \rightarrow x^\mu - \epsilon^\mu(x)$, where $\epsilon^\mu(x)$ are arbitrary functions of spacetime. Under this transformation, the fields transform as Lie derivative:

$$\delta_x A_\mu = \epsilon^\rho \partial_\rho A_\mu + A_\rho \partial_\rho \epsilon^\mu,$$  \hspace{1cm} (3.2)

$$\delta_x g_{\mu\nu} = \epsilon^\rho \partial_\rho g_{\mu\nu} + 2 g_{\rho(\nu} \partial_{\mu)} \epsilon^\rho, \hspace{1cm} (3.3)$$

$$\delta_x k^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} = \epsilon^\rho \partial_\rho k^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}$$

$$= - k^\mu_{\rho_{\mu_1\cdots\mu_m-1}}_{\nu_1\cdots\nu_n} \partial_\rho \epsilon^{\mu_1} - \cdots - k^\mu_{\rho_{\nu_1\cdots\nu_n-1}}_{\mu_1\cdots\mu_m-1} \partial_\rho \epsilon^{\mu_m}$$

$$+ k^{\mu_1\cdots\mu_m}_{\rho_{\mu_2\cdots\mu_1}} \partial_\rho \epsilon^\mu + \cdots + k^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n-1} \partial_{\nu_n} \epsilon^\rho.$$

The field variation $\delta_x$ satisfies Leibniz rules. Furthermore, it also commutes with partial derivatives. So for example,

$$\delta_x A_\mu = \partial_\mu (\epsilon^\rho \partial_\rho A_\mu + A_\rho \partial_\rho \epsilon^\mu) = \epsilon^\rho \partial_\rho A_\mu + \epsilon^\rho \partial_\rho A_\mu + A_\rho \partial_\rho \epsilon^\mu,$$

$$\delta_x T = \frac{\partial T}{\partial A_\nu} \delta_x A_\nu + \frac{\partial T}{\partial g_{\mu\nu}} \delta_x g_{\mu\nu} + \frac{\partial T}{\partial k} \partial_\sigma \delta_x k + \cdots$$

$$+ \frac{\partial T}{\partial k} \delta_x k + \cdots$$

$$+ \frac{\partial T}{\partial \partial_x (\partial_{\sigma_1 \cdots \sigma_p} k^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n})} \partial_{\sigma_1 \cdots \sigma_p} \delta_x k^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} + \cdots.$$

For the theory to be diffeomorphism invariant, the Lagrangian should transform as

$$\delta_x \mathcal{L} = \epsilon^\mu \partial_\mu \mathcal{L} + \mathcal{L} \partial_\mu \epsilon^\mu.$$

This is the requirement that the Lagrangian (3.1) has to satisfy. After making $\delta_x$ variation on the Lagrangian (3.1), one can see that the coefficient of $\epsilon^\mu$ is already $\partial_\mu \mathcal{L}$. Furthermore, it can be seen that $\delta_x \mathcal{L} - \partial_\mu (\epsilon^\mu \mathcal{L})$ can be given as a quadratic
polynomial in $\dot{A}_0$ (it need not be a polynomial in $\dot{A}_i$). So coefficient of each order in $\dot{A}_0$ should vanish. This gives three equations. Each of them can furthermore be expressed as linear combinations of expressions of the form $\partial_{\sigma_1 \ldots \sigma_p} e^\mu$. So coefficient of each of these expressions should vanish.

Let us consider the coefficient $\dot{A}_0 \dot{A}_0$. It turns out there is only one term contributes to this coefficient. That is

$$\frac{\partial U}{\partial \dot{A}_0} \dot{A}_0 = \delta e^0 \dot{A}_0 \subset \delta \mathcal{L} - \partial_\mu (e^\mu \mathcal{L}).$$

This implies that

$$\frac{\partial U}{\partial \dot{A}_0} = 0.$$  \hfill (3.9)

Let us now turn to the coefficient of $\dot{A}_0$. We have

$$\left( \frac{\partial T}{\partial \dot{A}_0} + \frac{\partial T}{\partial \dot{A}_k} \dot{A}_k + 2 \frac{\partial U}{\partial \dot{A}_0} + U \frac{\partial e^0}{\partial \dot{A}_0} + \delta U \right) \dot{A}_0 = \delta \mathcal{L} - e^\mu \partial_\mu \mathcal{L},$$

where the coefficient of $\dot{A}_0$ on the last term on LHS are obtained from $\delta U$ by setting $\dot{A}_0 = 0$ and discarding terms linear in $e^\mu$. Conditions can be extracted by letting LHS of eq.(3.10) to be zero and demanding the coefficients of $\dot{A}_0$ to vanish. In particular, we are interested in the coefficient of $\partial_\mu e^0$. So we have

$$0 = \frac{\partial T}{\partial \dot{A}_0} + \frac{\partial T}{\partial \dot{A}_k} \dot{A}_k + \partial_0 e^0 + U \frac{\partial e^0}{\partial \dot{A}_0} + \partial U \left|_{\dot{A}_0=0} \right. \dot{A}_0 = \delta \mathcal{L} - e^\mu \partial_\mu \mathcal{L},$$

$$\left( \frac{\partial T}{\partial \dot{A}_0} + \frac{\partial T}{\partial \dot{A}_k} \dot{A}_k + 2 \frac{\partial U}{\partial \dot{A}_0} + U \frac{\partial e^0}{\partial \dot{A}_0} + \delta U \right) \dot{A}_0 = \delta \mathcal{L} - e^\mu \partial_\mu \mathcal{L},$$

where $\cdots$ are terms involving partial derivatives of $U$ with respect to higher order derivatives of $g_{\mu\nu}$ and to external tensor fields and their derivatives. Two useful conditions can readily be extracted. One of them is obtained by taking derivative of eq.(3.10) with respect to $\partial_j \dot{A}_0$, and use eq.(3.9). One obtains

$$\frac{\partial^2 T}{\partial \dot{A}_i \partial \dot{A}_j} + \frac{\partial^2 T}{\partial \dot{A}_i \partial j \dot{A}_0} + \frac{\partial U}{\partial \partial j \dot{A}_i} = 0.$$  \hfill (3.12)

Similarly, the other condition we are interested in can be obtained by taking derivative of eq.(3.10) with respect to $\partial_j \dot{A}_0$, and use eq.(3.9). One obtains

$$\frac{\partial^2 T}{\partial \partial_i \dot{A}_0 \partial \dot{A}_j} + \frac{\partial^2 T}{\partial \partial_i \partial j \dot{A}_0} + \frac{\partial U}{\partial \partial j \dot{A}_i} = 0.$$  \hfill (3.13)

There is no further condition that will be of use for us. So we end the analysis of diffeomorphism invariance here.
4 Conditions from Faddeev-Jackiw constrained analysis of generalised Proca theories

4.1 Deriving the conditions

Let us now proceed by considering the Faddeev-Jackiw constrained analysis of the Lagrangian (3.1). Let us first compute the conjugate momenta. It is found to be of the form

\[ \pi^\mu = \frac{\partial L}{\partial \dot{A}_\mu} = \frac{\partial T}{\partial A_k} \delta^\mu_k + U \delta_0^\mu, \] (4.1)

The zeroth component gives the constraint

\[ \Omega_1 \equiv \pi^0 - U = 0, \] (4.2)

which does not depend on time derivative of the field \( A_\mu \). As for the spatial components for conjugate momenta, they give

\[ \pi^k = \frac{\partial T}{\partial A_k}. \] (4.3)

The inverse of this equation is of the form

\[ \dot{A}_i = \Lambda_i(A_\nu, \partial_k A_\nu, \pi^k, g_{\rho\sigma}, \partial_k g_{\rho\sigma}, \partial_k \gamma_1, \cdots, K). \] (4.4)

So the first-order form of Lagrangian density is given by

\[ L_{FOF} = \pi^0 \dot{A}_0 + \pi^i \dot{A}_i + \mathcal{L}_v + \dot{\gamma}_1 \Omega_1, \] (4.5)

where \( \mathcal{L}_v \) is given by

\[ \mathcal{L}_v = -\pi^k \Lambda_k + \mathcal{T}, \] (4.6)

where \( \mathcal{T} \) is obtained by replacing \( \dot{A}_i \) in \( T \) by \( \Lambda_i \). The canonical variables are \( \xi^I = (A^\mu, \pi_\nu, \gamma_1) \). By using eq.(2.8), one obtains the canonical 1-form as

\[ \mathcal{A} = \int d^3 \vec{x} \left( \pi^0 \delta A_0 + \pi^i \delta A_i + \Omega_1 \delta \gamma_1 \right). \] (4.7)

The symplectic 2-form \( \mathcal{F} = \delta \mathcal{A} \) can then be computed, and is given by

\[ \mathcal{F} = \int d^3 \vec{x} \left( \delta \pi^0 \wedge \delta A_0 + \delta \pi^i \wedge \delta A_i + \frac{\partial \Omega_1}{\partial A_\mu} \delta A_\mu \wedge \delta \gamma_1 \right. \]

\[ \left. + \frac{\partial \Omega_1}{\partial \partial_i A_\mu} \delta \partial_i A_\mu \wedge \delta \gamma_1 + \delta \pi^0 \wedge \delta \gamma_1 \right). \] (4.8)
Then an interior product with a vector \( z \) is given by

\[
i_z F = \int d^3 \vec{x} \left( \left( z^{\pi^0} - \frac{\partial \Omega_1}{\partial A_0} z^{\gamma_1} + \partial_i \left( \frac{\partial \Omega_1}{\partial \partial_i A_0} z^{\gamma_1} \right) \right) \delta A_0 \\
+ \left( z^{\pi^i} - \frac{\partial \Omega_1}{\partial A_i} z^{\gamma_1} + \partial_j \left( \frac{\partial \Omega_1}{\partial \partial_j A_i} z^{\gamma_1} \right) \right) \delta A_i \\
+ \left( -z^{A_0} - z^{\gamma_1} \right) \delta \pi^0 - z^{A_i} \delta \pi^i \\
+ \left( \frac{\partial \Omega_1}{\partial A_\mu} z^{A_\mu} + \frac{\partial \Omega_1}{\partial \partial_i A_\mu} \partial_i z^{A_\mu} + z^{\pi^0} \right) \delta \gamma_1 \right),
\]

where partial derivatives for \( \Omega_1 \) are taken on \( \Omega_1 \) of the form

\[
\Omega_1(A_\mu, \partial_i A_\mu, \pi^\mu, g_{\mu \nu}, \partial_\sigma g_{\mu \nu}, \cdots, K).
\]

We wish to obtain the solution to \( i_z F = 0 \). For this, let us first consider the coefficients of \( \delta A_0 \), \( \delta \pi^0 \) and \( \delta \gamma_1 \). The condition that these coefficients vanish gives

\[
2 \frac{\partial \Omega_1}{\partial \partial_1 A_0} \partial_i z^{A_0} + \partial_i \left( \frac{\partial \Omega_1}{\partial \partial_i A_0} \right) z^{A_0} = 0.
\]

In the analysis so far in this subsection, we still have not used diffeomorphism invariance requirement. In particular, let us impose eq. (3.9). This makes eq. (4.11) identically vanishes, and hence it does not give a restriction on \( z^{A_0} \). Furthermore, the condition \( i_z F = 0 \) can be consistently solved, and the zero mode of \( F \) is found to be

\[
z = \int d^3 \vec{x} \left( - \frac{\partial \Omega_1}{\partial A_0} z^{A_0} \frac{\delta}{\delta \pi^0} - \frac{\partial \Omega_1}{\partial A_i} z^{A_0} \frac{\delta}{\delta \pi^i} \\
+ \partial_j \left( \frac{\partial \Omega_1}{\partial \partial_j A_i} \right) \frac{\delta}{\delta \pi^j} + z^{A_0} \left( \frac{\delta}{\delta A_0} - \frac{\delta}{\delta \gamma_1} \right) \right).
\]

Since the zero mode depend only on one arbitrary function \( z^{A_0} \), there is at most one new constraint. It turns out that indeed there is a further constraint \( \Omega_2 \), which can be obtained from

\[
\int d^3 \vec{x} \Omega_2 z^{A_0} = i_z \int d^3 \vec{x} \delta \mathcal{L}_v.
\]

After a direct calculation, we obtain

\[
\Omega_2 = \frac{\partial T}{\partial A_0} - \partial_i \left( \frac{\partial T}{\partial \partial_i A_0} \right) - \Lambda_i \frac{\partial U}{\partial A_i} - \partial_j \Lambda_i \frac{\partial U}{\partial \partial_j A_i},
\]

where partial derivatives of \( T \) with respect to \( A_0 \) and \( \partial_i A_0 \) are taken with fixed \( A_j \). It is easy to see that the constraint (4.14) is indeed a new constraint. This is because it is independent from \( \pi^0 \), which appears in \( \Omega_1 \).

So we have seen an interesting result that diffeomorphism invariance ensures that the theory (3.1) has more than one constraint. There will be a further result, which we will encounter shortly.
With the introduction of an extra constraint, we need to start the second iteration by first redefining the first-order form of Lagrangian from eq. (4.5) to

\[ \mathcal{L}_{FOF} = \pi^0 A_0 + \pi^i A_i + \mathcal{L}_v + \gamma_1 \Omega_1 + \gamma_2 \Omega_2. \]  

(4.15)

Then the canonical 1–form for the Lagrangian (4.15) is given by

\[ \mathcal{A} = \int d^3 \tilde{x} \left( \pi^0 \delta A_0 + \pi^i \delta A_i + \Omega_1 \delta \gamma_1 + \Omega_2 \delta \gamma_2 \right). \]

(4.16)

Then, we obtain

\[ \mathcal{F} = \int d^3 \tilde{x} \left( \delta \pi^0 \wedge \delta A_0 + \delta \pi^i \wedge \delta A_i + \delta \Omega_1 \wedge \delta \gamma_1 + \delta \Omega_2 \wedge \delta \gamma_2 \right), \]

(4.17)

and hence

\[ i_z \mathcal{F} = \int d^3 \tilde{x} \left( z^{\pi^0} - \frac{\partial \Omega_1}{\partial A_0} z^{\gamma_1} - \frac{\partial \Omega_2}{\partial A_0} z^{\gamma_2} \right. \]

\[ + \partial j \left( \frac{\partial \Omega_2}{\partial j A_0} z^{\gamma_2} \right) - \partial j \partial k \left( \frac{\partial \Omega_2}{\partial j \partial k A_0} z^{\gamma_2} \right) \delta A_0 \]

\[ + \left( z^{\pi^i} - \frac{\partial \Omega_1}{\partial A_i} z^{\gamma_1} + \partial j \left( \frac{\partial \Omega_1}{\partial j A_i} z^{\gamma_1} \right) \right. \]

\[ - \frac{\partial \Omega_2}{\partial A_i} z^{\gamma_2} + \partial j \left( \frac{\partial \Omega_2}{\partial j A_i} z^{\gamma_2} \right) - \partial j \partial k \left( \frac{\partial \Omega_2}{\partial j \partial k A_i} z^{\gamma_2} \right) \delta A_i \]

\[ + \left. \left( - z^{A_0} - z^{\gamma_1} \right) \delta \pi^0 + \left( - z^{A_i} - \frac{\partial \Omega_2}{\partial \pi^i} z^{\gamma_2} + \partial j \left( \frac{\partial \Omega_2}{\partial j \pi^i} z^{\gamma_2} \right) \right) \delta \pi^i \right) \]

\[ + \left. \left( \frac{\partial \Omega_1}{\partial A_\mu} z^{A_\mu} + \frac{\partial \Omega_1}{\partial \pi^i} \partial A_j \partial A_\mu \partial z^{A_\mu} + \frac{\partial \Omega_2}{\partial \pi^i} \partial A_j \partial A_\mu \partial \partial j \partial A_\mu \partial z^{A_\mu} \right) \}

\[ + \left. \frac{\partial \Omega_2}{\partial \pi^i} z^{\pi^i} \right) \delta \gamma_1 \]

\[ + \left. \frac{\partial \Omega_2}{\partial \pi^i} \partial \partial j \partial z^{A_\mu} \right) \delta \gamma_2 \right), \]

(4.18)

where partial derivatives for $\Omega_2$ are taken on $\Omega_2$ of the form

\[ \Omega_2(A_\mu, \partial A_\mu, \partial j A_\mu, \pi^i, \partial j \pi^i, g_{\mu\nu}, \partial_\sigma g_{\mu\nu}, \ldots, K). \]

(4.19)

We require that there is no further constraint. So let us demand that there is only a trivial solution to $i_z \mathcal{F} = 0$. By eliminating $z^{\pi^i}, z^{\gamma_1}, z^{A_i}$ from the coefficients of $\delta \gamma_1, \delta \pi^0, \delta \pi^i$, and substituting into the coefficient of $\delta A_0$, we obtain

\[ 0 = - \left( \frac{\partial \Omega_2}{\partial j \partial k A_0} + \frac{\partial \Omega_1}{\partial j A_i} \partial \partial j \partial k \right) \partial j \partial k z^{\gamma_2} \]

\[ + \left( - \frac{\partial \Omega_1}{\partial A_i} \frac{\partial \Omega_2}{\partial j \pi^i} + \frac{\partial \Omega_1}{\partial j A_i} \partial \partial j A_\mu \frac{\partial \Omega_2}{\partial j \partial A_\mu} - \frac{\partial \Omega_1}{\partial j A_i} \partial \partial j A_\mu \frac{\partial \Omega_2}{\partial j \partial A_\mu} - \frac{\partial \Omega_1}{\partial j A_i} \partial \partial j A_\mu \frac{\partial \Omega_2}{\partial j \partial A_\mu} \right) \]

\[ + \frac{\partial \Omega_2}{\partial \partial j A_0} + 2 \partial k \left( \frac{\partial \Omega_2}{\partial j \partial k A_0} \right) \partial j z^{\gamma_2} + z^{\gamma_2} \left( \int d^3 \tilde{y} \{- \Omega_1, \Omega_2(\tilde{y})\} \right), \]

(4.20)
where we have imposed the condition (3.9) on the Poisson bracket so that its integral reduces to

\[ \int d^3 \vec{y} \{ \Omega_1, \Omega_2(\vec{y}) \} = \frac{\partial \Omega_1}{\partial A_i} \frac{\partial \Omega_2}{\partial \pi^i} - \frac{\partial \Omega_1}{\partial A_i} \frac{\partial \Omega_2}{\partial \pi^i} + \frac{\partial \Omega_1}{\partial A_i} \frac{\partial \Omega_2}{\partial \pi^i} \]

\[ - \frac{\partial \Omega_1}{\partial \pi^i} \partial \pi^j \left( \frac{\partial \Omega_2}{\partial \pi^j} \right) - \frac{\partial \Omega_2}{\partial A_0} + \partial_j \left( \frac{\partial \Omega_2}{\partial \pi^j} \right) + \partial_j \partial_k \left( \frac{\partial \Omega_2}{\partial \pi^j} \right) \]

\[ - \frac{\partial \Omega_1}{\partial \pi^i} \left( \frac{\partial \Omega_2}{\partial \pi^j} \right) \]

\[ = 0, \]

(4.21)

In order for \( F \) to possess no zero mode, we need to demand that the coefficients of \( \partial_j z^{\gamma_2} \) and of \( \partial_j \partial_k z^{\gamma_2} \) in eq.(4.20) vanish. At the same time, the coefficient of \( z^{\gamma_2} \) should be non-vanishing. This gives rise to the requirements

\[ C_{2}^{jk} = \frac{\partial \Omega_2}{\partial \pi^j} \partial \pi^k \]

\[ = 0, \]

(4.22)

\[ C_{1}^{j} = \frac{\partial \Omega_1}{\partial \pi^j} + \frac{\partial \Omega_1}{\partial \pi^j} \frac{\partial \Omega_2}{\partial \pi^j} - \frac{\partial \Omega_1}{\partial \pi^j} \frac{\partial \Omega_2}{\partial \pi^j} \]

\[ - \frac{\partial \Omega_1}{\partial \pi^j} \left( \frac{\partial \Omega_2}{\partial \pi^j} \right) + \frac{\partial \Omega_2}{\partial \pi^j} \left( \frac{\partial \Omega_2}{\partial \pi^j} \right) \]

\[ = 0, \]

(4.23)

\[ \int d^3 \vec{y} \{ \Omega_1, \Omega_2(\vec{y}) \} \neq 0, \]

(4.24)

Imposing these requirements on eq.(4.20), one obtains \( z^{\gamma_2} = 0 \). After substituting this into eq.(4.18) and requiring that the expression vanishes, one eventually sees that, without imposing any further conditions, the only solution to \( i_z F = 0 \) is \( z = 0 \).

So from the analysis, we see that if the theory (3.1) satisfies the conditions (3.9), (4.22)-(4.24), it has two constraints. Furthermore, the process guarantees that the constraints are of second-class. This is because the condition (4.24) requires that the Poisson brackets of \( \Omega_1 \) and \( \Omega_2 \) is non-vanishing. Now, since the theory has two second-class constraints just like the standard Proca theory, the counting of the degrees of freedom suggests that the theory has three degrees of freedom as required.

An alternative way to see that, under the conditions (3.9), (4.22)-(4.24), the theory possesses two second-class constraints is by using the criteria of [30]. That is, since each zero mode of \( F \) in any step leads to an independent constraint, there is no gauge symmetry in the theory, and hence all of the constraints found are of second-class. On the other hand, if we suppose that some of the conditions (4.22)-(4.24) are not satisfied, then there exists further zero modes. In case these zero modes do not lead to any new constraint, the criteria of [30] suggests that some of the constraints already obtained are of first-class. So the number of degrees of freedom is less than three. Alternatively, if the zero modes do lead to new constraints, then the theory possesses at least three constraints. So in the case, the number of degrees of freedom is also less than three.
In fact, the conditions (3.9) and (4.22)-(4.23) are implied by diffeomorphism invariance. We have shown that this is the case for the condition (3.9). So let us show this for the conditions (4.22)-(4.23).

4.2 Triviality of $C_{2}^{ij} = 0$

The condition (4.22) can easily be shown to be automatically satisfied following the diffeomorphism invariance requirement. For this, let us first express $\Omega_2$ in phase space and keep only terms containing $\partial_i \partial_j A_0$ and $\partial_i \pi^i$. So only the relevant terms are

$$\Omega_2 = -\partial_i \left( \frac{\partial T}{\partial \partial_i A_0} \right) - \partial_j A_i \frac{\partial U}{\partial \partial_j A_i} + \text{(terms free from } \partial_i \partial_j A_0 \text{ and } \partial_i \pi^i)$$

$$= -\frac{\partial^2 T}{\partial \partial_i A_0 \partial \partial_j A_0} \partial_i \partial_j A_0 - \left( \frac{\partial^2 T}{\partial \partial_i A_i \partial \partial_j A_0} + \frac{\partial U}{\partial \partial_j A_i} \right) \partial_j A_i$$

+ \text{(terms free from } \partial_i \partial_j A_0 \text{ and } \partial_i \pi^i) \tag{4.25}$$

$$= \frac{\partial^2 T}{\partial \Lambda_i \partial \Lambda_j} \partial_i \partial_j A_0 \left( -\delta_i^k + \frac{\partial \Lambda_i}{\partial \partial_k A_0} \right) + \frac{\partial^2 T}{\partial \Lambda_i \partial \Lambda_j \partial \pi^m} \partial_j \pi^m$$

+ \text{(terms free from } \partial_i \partial_j A_0 \text{ and } \partial_i \pi^i),$$

where in the third step, we used eq.(3.12)-(3.13) with replacement $\partial_0 A_i \rightarrow \Lambda_i$. The expression for $\Omega_2$ in eq.(4.25) can be further simplified. For this, let us compute partial derivatives of $\Lambda_i$ with respect to $\pi^i$ and to $\partial_i A_0$, we consider derivatives of $\pi^i = \partial T / \partial \Lambda_i$ with respect to phase space variables. This gives

$$\frac{\partial^2 T}{\partial \Lambda_i \partial \Lambda_m} \frac{\partial \Lambda_i}{\partial \pi^m} = \delta_i^m, \tag{4.26}$$

$$0 = \frac{\partial^2 T}{\partial \Lambda_i \partial \partial_j A_0} + \frac{\partial^2 T}{\partial \Lambda_i \partial \Lambda_k \partial \partial_j A_0}. \tag{4.27}$$

By using these equations and eq.(3.12)-(3.13) again, we obtain

$$\Omega_2 \supset \partial U / \partial \partial_j A_i \partial \partial_j A_0 + \partial_i \pi^i. \tag{4.28}$$

By substituting this into eq.(4.22), it can be seen that this is automatically satisfied.

So only the conditions (4.24)-(4.23) are actually required for the theory to have three degrees of freedom. Other conditions are already satisfied thanks to the requirement of diffeomorphism invariance.

So from the analysis, we see that if the theory (3.1) satisfies the conditions (4.24)-(4.23), then it has two constraints. Furthermore, the process guarantees that the constraints are of second-class. This is because the condition (4.24) requires that the Poisson brackets of $\Omega_1$ and $\Omega_2$ vanish. Now, since the theory has two second-class constraints just like the standard Proca theory, the counting of the degrees of freedom suggests that the theory has three degrees of freedom as required.

Combining with the result we obtained previously, we may conclude the finding so far as follows. The requirement of diffeomorphism invariance demands the theory
to have more than one constraint. Next, if there is no zero mode of the symplectic 2–form at the second iteration, then the theory is guaranteed to have two constraints with both being of second-class.

4.3 Triviality of \( C_1^i = 0 \)

Let us now show that the condition (4.23) is automatically satisfied by diffeomorphism invariance requirement. To rewrite the condition (4.23), we need to first express \( \Omega_1 \) and \( \Omega_2 \) in terms of \( T, U \) and their derivatives. Expressing \( \Omega_1 = \pi^0 - U \) is a simple task. So we need to express \( \Omega_2 \). In the previous subsection, we have already computed

\[
\frac{\partial \Omega_2}{\partial \partial_i \pi^j}, \quad \frac{\partial \Omega_2}{\partial \partial_i A_0}. \tag{4.29}
\]

So we are left to compute

\[
\frac{\partial \Omega_2}{\partial \pi^i}, \quad \frac{\partial \Omega_2}{\partial \partial_j A_0}. \tag{4.30}
\]

Let us revisit eq.(4.14). It can be rewritten as

\[
\Omega_2 = \frac{\partial T}{\partial A_0} - \partial_j \left( \frac{\partial T}{\partial \partial_j A_0} + \frac{\partial U}{\partial \partial_j A_i} \Lambda_i \right) - \left( \frac{\partial U}{\partial A_i} - \partial_j \left( \frac{\partial U}{\partial \partial_j A_i} \right) \right) \Lambda_i. \tag{4.31}
\]

Consider \( \partial \Omega_2 / \partial \pi^i \). It can be shown that the second term does not depend on \( \pi^i \). To see this, we consider a diffeomorphism condition (3.11). Changing the variables to phase space ones gives

\[
\frac{\partial T}{\partial \partial_k A_0} + \frac{\partial U}{\partial \partial_k A_i} \Lambda_i = -\pi^k - \frac{\partial U}{\partial A_k} A_0 - \frac{\partial U}{\partial \partial_k A_k} \partial_i A_0 - \frac{\partial U}{\partial g_{\mu k}} g_{\mu 0} - \frac{\partial U}{\partial \partial_k g_{\rho \sigma}} \partial_0 g_{\rho \sigma} - 2 \frac{\partial U}{\partial \partial_\mu g_{\rho k}} \partial_\mu g_{\rho 0} + \cdots, \tag{4.32}
\]

where \( \cdots \) are terms involving partial derivatives of \( U \) with respect to higher order derivatives of \( g_{\mu \nu} \) and to external tensor fields and their derivatives. Since \( U \) does not depend on \( \pi^i \), it can easily be seen that after applying the partial derivative \( \partial_k \) on the above equation, the resulting expression does not depend on \( \pi^k \). Similarly, it can easily be seen that the coefficient of \( \Lambda_i \) in the last term of eq.(4.31) do not depend on \( \pi^i \). So we have

\[
\frac{\partial \Omega_2}{\partial \pi^i} = \left( \frac{\partial^2 T}{\partial A_0 \partial A_k} - \frac{\partial U}{\partial A_k} + \partial_j \left( \frac{\partial U}{\partial \partial_j A_k} \right) \right) \frac{\partial \Lambda_i}{\partial \pi^i}. \tag{4.33}
\]
Next, let us compute $\partial \Omega_2/\partial j A_0$. We obtain

$$
\frac{\partial \Omega_2}{\partial j A_0} = \frac{\partial^2 T}{\partial A_0 \partial j A_0} + \left( \frac{\partial^2 T}{\partial A_0 \partial A_k} - \frac{\partial U}{\partial A_0} \frac{\partial A_k}{\partial j A_0} + \partial_m \left( \frac{\partial U}{\partial \partial m A_k} \right) \right) \frac{\partial A_k}{\partial j A_0} + \frac{\partial U}{\partial j A_j}
$$

$$
+ \frac{\partial^2 U}{\partial j A_i \partial A_0} \Lambda_i
$$

$$
+ \frac{\partial^2 U}{\partial A_j \partial A_0} A_0 + \partial_k \left( \frac{\partial U}{\partial j A_k} \right) + \frac{\partial^2 U}{\partial \partial k A_j \partial A_0} \partial_k A_0
$$

$$
+ 2 \frac{\partial^2 U}{\partial j A_m \partial A_0} g_{\mu 0} + \frac{\partial^2 U}{\partial j A_{\rho \sigma} \partial A_0} \partial_0 g_{\rho \sigma} + 2 \frac{\partial^2 U}{\partial j A_0} \partial_0 g_{\rho 0}
$$

$$
\vdots
$$

$$
(4.34)
$$

where the first step is obtained by first applying eq. (4.32) and then taking derivative, and in the second step, we apply eq. (4.32) again.

With these ingredients, we see that $C_1^j$ is given by

$$
C_1^j = \left( -\delta^j_k + \frac{\partial A_k}{\partial j A_0} - \frac{\partial U}{\partial j A_m} \frac{\partial A_k}{\partial \pi^m} \right) \left( \frac{\partial^2 T}{\partial A_0 \partial A_k} - \frac{\partial U}{\partial A_0} \frac{\partial A_k}{\partial j A_0} + \partial_i \left( \frac{\partial U}{\partial j A_k} \right) \right).
$$

(4.35)

It can be further simplified. For this, let us consider

$$
\frac{\partial^2 T}{\partial A_k \partial A_n} \left( -\delta^j_k + \frac{\partial A_k}{\partial j A_0} - \frac{\partial U}{\partial j A_m} \frac{\partial A_k}{\partial \pi^m} \right)
$$

$$
= \left( \frac{\partial^2 T}{\partial A_j \partial A_n} - \frac{\partial^2 T}{\partial A_n \partial A_j} - \frac{\partial U}{\partial A_0} \frac{\partial A_k}{\partial j A_0} \right)
$$

$$
(4.36)
$$

$$
= 0,
$$

(4.37)

where in the first step, we use eq.(4.26)-(4.27), and in the second step, we used eq.(3.13) with change of variables $\partial_0 A_i \rightarrow \Lambda_i$. Next, the expression $\partial^2 T/\partial A_k \partial A_n$ is always invertible. To see this, we note from a requirement in this paper that

$$
\det \left( \frac{\partial^2 L}{\partial A_i \partial A_j} \right) \neq 0.
$$

(4.38)

By noting that $U$ does not depend on $\dot{A}_i$ and by translating this equation to phase space, we obtain the condition that

$$
\det \left( \frac{\partial^2 T}{\partial A_i \partial A_j} \right) \neq 0.
$$

(4.39)

With this condition, eq.(4.36) then reduces to

$$
-\delta^j_k + \frac{\partial A_k}{\partial j A_0} - \frac{\partial U}{\partial j A_m} \frac{\partial A_k}{\partial \pi^m} = 0.
$$

(4.39)
This further simplifies (4.35) to
\[ C^i_1 = 0. \]  
(4.40)

That is, the condition (4.23) is automatically satisfied by diffeomorphism invariance.

5 Example cases

In the previous section, we have shown that a vector theory which satisfies the conditions (1.4) are almost guaranteed to have three propagating degrees of freedom. This is largely due to diffeomorphism invariance requirements.

Suppose that one proceeds directly to count the number of propagating degrees of freedom of a generalised Proca theory using the Faddeev-Jackiw method. If the theory is diffeomorphic invariance and satisfied (1.4), one is going to obtain the following results:

(I) there exists at least two constraints, and that

(II) it is likely that there are only two constraints, and both constraints are of second-class.

We learned from the previous section that these two results for any diffeomorphic invariance generalised Proca theory which satisfies the conditions (1.4) are due to diffeomorphism invariance requirements. In particular, the result (I) is due to diffeomorphism invariance requirements. Specifically eq.(3.9) trivialises eq.(4.11). This in turn points out the existence of the second constraint. As for the result (II), diffeomorphism invariance requirements trivialises eq.(4.22)-(4.23). If the condition (4.24) is also satisfied, the theory is guaranteed to have two constraints, and both constraints are of second-class. Although the condition (4.24) is not trivialised by diffeomorphism invariance requirements, it is easy to be satisfied. This is in the sense that the condition (4.24) demands a complicated expression not to trivially vanishes.

In this section, we demonstrate this by showing simple examples and show by direct calculation (without using the results from the previous section) of the Faddeev-Jackiw method that these examples satisfy both the results (I) and (II).

5.1 A special case in flat spacetime

Let us consider a special case of eq.(1.2) in flat spacetime. We consider the case where \( \beta_6 = 0, G_2 = G_2(A_\mu A^\mu), \tilde{G}_5 = 0. \) Explicitly, in this example, we consider the Lagrangian
\[
\mathcal{L}_{GP} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{n=2}^{5} \alpha_n \mathcal{L}_n,
\]  
(5.1)
where \( \alpha_n, n = 2, 3, 4, 5 \) are arbitrary constants and \( \mathcal{L}_n, n = 2, 3, 4, 5 \) are self-interactions of the gauge field given by

\[
\begin{align*}
\mathcal{L}_2 &= f_2(A^2), \\
\mathcal{L}_3 &= f_3(A^2) \partial \cdot A, \\
\mathcal{L}_4 &= f_4(A^2)[(\partial \cdot A)^2 - \partial_\rho A_\sigma \partial^\sigma A^\rho], \\
\mathcal{L}_5 &= f_5(A^2)[(\partial \cdot A)^3 - 3(\partial \cdot A) \partial_\rho A_\sigma \partial^\sigma A^\rho + 2 \partial_\rho A_\sigma \partial^\sigma A^\rho \partial^\gamma A_\gamma],
\end{align*}
\]

(5.2)

where \( A^2 \equiv A_\mu A^\mu, \partial \cdot A \equiv \partial_\mu A^\mu \).

We have directly followed Faddeev-Jackiw process and obtained two constraints. Furthermore, each zero mode of symplectic 2-form in each iteration leads to a constraint. So the theory has three propagating degrees of freedom as expected. This by itself suggests that the theory eq.(5.1) satisfies the conditions (3.9) and (4.22)-(4.24).

As a cross check of these conditions, we have substituted the constraints

\[
\begin{align*}
\Omega_1 &= \pi^0 + \alpha_3 f_3(A^2) + 2 \alpha_4 f_4(A^2) \vec{\nabla} \cdot \vec{A} + 3 \alpha_5 f_5(A^2)((\vec{\nabla} \cdot \vec{A})^2 - \partial_i A_j \partial_j A_i), \\
\Omega_2 &= \vec{\nabla} \cdot \vec{\pi} - 2 \alpha_2 f_2(A^2) A_0 + 2 \alpha_3 f_3(A^2) (\vec{A} \cdot (\vec{\pi} + \vec{\nabla} A_0) - A_0 \vec{\nabla} \cdot \vec{A}) \\
&\quad + \alpha_4 \left(4 f_4'(A^2) \left(2 A^{[i| (\pi_i + \partial_i A_0) \partial_j A_j - A_0 \partial_j A^\mu \partial_j A_0 \\
&\quad + A_0 \partial_j A_j \partial_j A_i) - 2 f_4(A^2) \nabla^2 A_0 \right) \\
&\quad - 4 \alpha_3 \alpha_4 f_3'(A^2) f_4(A^2) \vec{A} \cdot \vec{\nabla} A_0 + 16 \alpha_4^2 f_4'(A^2) A^{[i| \partial_j A_0 \partial^j A_i} \\
&\quad + 12 \alpha_5 \left(f_5'(A^2) \left(3 A_i (\pi_i + \partial_i A_0) \partial_j A_j A^k + 2 A_0 \partial_j A^\mu \partial_j A_k \partial^k A_0 \\
&\quad - A_0 \partial_i A_j \partial_j A_k \partial^k A_0 \right) + f_5(A^2) \partial_i \left(\partial_j A_0 \partial_i A^j \right) \right) \\
&\quad + 24 \alpha_3 \alpha_5 f_3'(A^2) f_5(A^2) A_i \partial^j A_0 \partial^j A_j \\
&\quad - 24 \alpha_4 \alpha_5 \left(4 f_4'(A^2) f_5(A^2) A^{[i| A_j \partial^j A_k \partial^k A_0 \partial^j A_j} \\
&\quad + 3 f_4(A^2) f_5'(A^2) A_i \partial_i A_0 \partial_j A^j A^k \right) \\
&\quad - 432 \alpha_5^2 f_5(A^2) f_5'(A^2) A^{[i| A_j A_j \partial^j A_k \partial^j A_0 \partial^j A_j} \right)
\end{align*}
\]

into the conditions (3.9) and (4.22)-(4.24) and see that they are indeed satisfied.

### 5.2 \( U = 0 \)

Let us now turn to the case where \( U = 0 \), and the vector field couples to the metric as well as other background fields. This example is in fact simple enough to be directly demonstrated.

In this case, the Lagrangian is

\[
\mathcal{L} = \mathcal{L}(A_\mu, \partial_\mu A_\nu, g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \cdots, K).
\]

(5.5)
If we trade $\partial \mu A_\nu$ for $F_{\mu\nu}$ and $S_{\mu\nu} \equiv \nabla_\mu A_\nu + \nabla_\nu A_\mu$, it can be seen that the Lagrangian should not depend on $S_{\mu\nu}$, otherwise $S_{00} \supset 2\dot{A}_0$ would appear in the Lagrangian, and hence does not agree with our requirement in this example. Therefore,

$$L = L(A_\mu, F_{ij}, F_{0i}, g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \cdots, K).$$

(5.6)

Furthermore, since $U = 0$ eq.(3.1) reduces to

$$L = T.$$  

(5.7)

Computing conjugate momenta, we arrive at the primary constraint

$$\Omega_1 = \pi^0,$$  

(5.8)

and the conditions which can be used to trade between $\pi^i$ and $F_{0i}$:

$$\pi^i = \frac{\partial T}{\partial F_{0i}}.$$  

(5.9)

The inversion of this equation is of the form

$$F_{0i} = \tilde{\Lambda}_i(A_\mu, F_{ij}, \tilde{\Lambda}_i, g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \cdots, K).$$  

(5.10)

The results given in the previous section relies on the quantities $\Lambda_i$ instead of $\tilde{\Lambda}_i$. So when needed to compare with them, we may recover $\Lambda_i$ from

$$\Lambda_i = \tilde{\Lambda}_i + \partial_i A_0.$$  

(5.11)

Next, by substituting eq.(5.10) into eq.(5.7), we obtain

$$\tilde{T} = \tilde{T}(A_\mu, F_{ij}, \tilde{\Lambda}_i, g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \cdots, K).$$  

(5.12)

Furthermore, eq.(5.9) in phase space is

$$\pi^i = \frac{\partial \tilde{T}}{\partial \Lambda_i}.$$  

(5.13)

The secondary constraint can then be worked out to be

$$\Omega_2 = \frac{\partial \tilde{T}}{\partial A_0} + \partial_i \pi^i.$$  

(5.14)

It can be seen that, as expected, this theory satisfies the conditions (3.9), (4.22)-(4.23). The most non-trivial check is on the condition (4.23). This reduces to

$$\frac{\partial \Omega_2}{\partial \partial_j A_0} = 0.$$  

(5.15)

This condition is trivially satisfied since after expressing $\tilde{T}$ in phase space variables (by substituting $\tilde{\Lambda}_i$ from eq.(5.10) into eq.(5.12)), we see that $\tilde{T}$ do not depend on $\partial_i A_0$.  

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As for the condition (4.24), some theories in this example might not satisfy this. We see that it reduces to
\[
\frac{\partial^2 \tilde{T}}{\partial A_0^2} \neq 0,
\]
(5.16)
or after changing the variables to configuration space,
\[
\frac{\partial^2 T}{\partial A_0^2} \neq 0.
\]
(5.17)
This is the only condition that the Lagrangian (5.5) has to satisfy in order for the vector field to have three propagating degrees of freedom.

As an example, a generalised Proca theory with Lagrangian of the form (5.5) is
\[
\mathcal{L} = -\sqrt{-g} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sqrt{-g} G_2 (A_\mu, F_{\mu\nu}, g_{\mu\nu}).
\]
(5.18)
This theory has three propagating degrees of freedom if
\[
\frac{\partial^2 G_2}{\partial A_0^2} \neq 0.
\]
(5.19)

6 On Faddeev-Jackiw brackets

For completeness, let us discuss the process to compute Faddeev-Jackiw brackets of diffeomorphism invariance generalised Proca theories satisfying the requirement (1.4). Furthermore, we focus on the cases which also satisfy the condition (4.24).

In section 4, the Faddeev-Jackiw constrained analysis was performed on these theories. At the final iteration, we obtain the symplectic 2-form as shown in eq. (4.17). Let us write this in matrix form. For this, we first re-express eq. (4.17) in the form
\[
\mathcal{F} = \frac{1}{2} \int d^3 \bar{x} d^3 \bar{y} \mathcal{F}_{\xi^I (\bar{x}, \bar{y})} \delta \xi^I (\bar{x}) \wedge \delta \xi^J (\bar{y}),
\]
(6.1)
where \( \xi^I = (A_\mu, \pi^\nu, \gamma_1, \gamma_2) \), and \( \xi^J = (A_\rho, \pi^\sigma, \gamma_1, \gamma_2) \). The quantities \( \mathcal{F}_{\xi^I \xi^J (\bar{x}, \bar{y})} \) appearing in eq. (6.1) are elements of the matrix
\[
\mathcal{F} (\bar{x}, \bar{y}) = \begin{pmatrix} A (\bar{x}, \bar{y}) & B (\bar{x}, \bar{y}) \\ C (\bar{x}, \bar{y}) & D (\bar{x}, \bar{y}) \end{pmatrix},
\]
(6.2)
where \( A, B, C, \) and \( D \) are block matrices given by
\[
A (\bar{x}, \bar{y}) = \begin{pmatrix} 0 & -\delta^0_\mu \\ \delta^\mu_\rho & 0 \end{pmatrix} \delta^{(3)} (\bar{x} - \bar{y}),
\]
(6.3)
\[
B (\bar{x}, \bar{y}) = \begin{pmatrix} F^\mu (x) + \delta^0_j G^{ji} (y) \partial_{y^i} & M^\mu (x) + N^\mu (y) \partial_{y^i} + P^{\mu ij} (y) \partial_{y^i} \partial_{y^j} \\ \delta^\mu_\rho & Q^\mu_i (x) \partial_{y^i} \end{pmatrix} \delta^{(3)} (\bar{x} - \bar{y}),
\]
(6.4)
\[
C (\bar{x}, \bar{y}) = \begin{pmatrix} -F^\rho (x) - \delta^\rho_j G^{ji} (x) \partial_{x^i} & -\delta^\rho_0 \\ -M^\rho (x) - N^\rho (x) \partial_{x^i} - P^{\rho ij} (x) \partial_{x^i} \partial_{x^j} & -\delta^\rho_0 \sigma \end{pmatrix} \delta^{(3)} (\bar{x} - \bar{y}),
\]
(6.5)
\[
\mathbf{D}(\vec{x}, \vec{y}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

with
\[
F^\mu = \frac{\partial \Phi_1}{\partial A_\mu}, \quad G^{ij} = \frac{\partial \Phi_1}{\partial \partial_i A_j},
\]
\[
M^\mu = \frac{\partial \Phi_2}{\partial A_\mu}, \quad N^{\mu i} = \frac{\partial \Phi_2}{\partial \partial_i A_\mu}, \quad P^{\mu ij} = \frac{\partial \Phi_2}{\partial \partial_i \partial_j A_\mu},
\]
\[
Q_i = \frac{\partial \Phi_2}{\partial \pi^i}.
\]

Note that, we have used the fact that
\[
\frac{\partial \Phi_1}{\partial \pi^\mu} = \delta_\mu^0, \quad \frac{\partial \Phi_2}{\partial \pi^i} = \delta_i^j,
\]

which can easily be obtained from eq.(4.2) and eq.(4.14).

The matrix (6.2) is already in the block form. So its inverse can be obtained from the formula
\[
\mathcal{F}^{-1} = \begin{pmatrix} \mathcal{A}^{-1} + \mathcal{A}^{-1}\mathcal{B}(\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})^{-1}\mathcal{C}^{-1} & -\mathcal{A}^{-1}\mathcal{B}(\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})^{-1} \\
-(\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})^{-1}\mathcal{C}^{-1} \mathcal{A}^{-1} & (\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})^{-1} \end{pmatrix}.
\]

The least straightforward step in the calculation of the matrix \(\mathcal{F}^{-1}\) is to compute \((\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})^{-1}\). For this, one first needs to compute \((\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}\), and then find the inverse. Let us present \((\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}\) without yet imposing the conditions (4.22)-(4.23). We have
\[
(\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}(\vec{x}, \vec{y}) = \begin{pmatrix} (\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}_{11}(\vec{x}, \vec{y}) & (\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}_{12}(\vec{x}, \vec{y}) \\
(\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}_{21}(\vec{x}, \vec{y}) & (\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}_{22}(\vec{x}, \vec{y}) \end{pmatrix},
\]

where
\[
(\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}_{11}(\vec{x}, \vec{y}) = 0,
\]
\[
(\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}_{12}(\vec{x}, \vec{y}) = -\mathcal{C}^{ij}_2(\vec{x})\partial_{x^i}\partial_{x^j} + \mathcal{C}^i_1(\vec{x})\partial_{x^i} - \tilde{\mathcal{M}}^0(\vec{x})\delta^{(3)}(\vec{x} - \vec{y}),
\]
\[
(\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}_{22}(\vec{x}, \vec{y}) = -\mathcal{C}^{jk}_2(\vec{y})\partial_{x^j}\partial_{x^k} - \tilde{\mathcal{M}}^i(\vec{x})\partial_{x^i} + \tilde{\mathcal{M}}^i(\vec{y})\partial_{x^i} - \tilde{\mathcal{M}}^i(\vec{x})\delta^{(3)}(\vec{x} - \vec{y}),
\]

with
\[
\tilde{\mathcal{M}}^0 = \mathcal{M}^0 - G^{ij}\partial_j Q_i - \partial_i N^{0i} + \partial_i P^{\delta ij} - F^i Q_i,
\]
\[
\tilde{\mathcal{M}}^i = \mathcal{M}^i - \partial_j N^{ij} - N^i Q_\sigma + \partial_j P^{\sigma ij} Q_\sigma - P^{\sigma ij} \partial_j Q_\sigma.
\]

We see that there is the presence of \(\mathcal{C}^{ij}_2, \mathcal{C}^i_1\) in the (12)– and (21)–components of \((\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}\). In order to see the significance of the conditions (4.22)-(4.23), which follows from diffeomorphism invariance requirements, let us first suppose that these conditions are not satisfied. When \(\mathcal{C}^{ij}_2\) and \(\mathcal{C}^i_1\) do not simultaneously vanish, the inverse of \((\mathcal{D} - \mathcal{C}^{-1}\mathcal{A}^{-1})\mathcal{B}\) would contain infinite order derivative on delta function.
For example, the $(12)$-component of $(D - CA^{-1}B)^{-1}$ is the inverse of the expression of the form

$$(a^{ij}(\vec{x})\partial_{x^i}\partial_{x^j} + b^i(\vec{x})\partial_{x^i} + c(\vec{x}))\delta^{(3)}(\vec{x} - \vec{y}).$$

(6.18)

The inverse of (6.18) cannot be expressed using linear combinations of finite terms of the form $\partial_{x^{i_1} \ldots x^{i_n}}\delta^{(3)}(\vec{x} - \vec{y})$. To further illustrate the point, consider a one-dimensional toy example

$$\delta(x - y) + \partial_x \delta(x - y).$$

(6.19)

It can easily be worked out that the inverse of (6.19) is

$$\sum_{r=0}^{\infty} (-1)^r \partial_x^r \delta(x - y).$$

(6.20)

So it can be expected that the inverse of the expression of the form (6.18) would surely contain infinite order derivative on delta function. On the other hand, if we now impose the conditions (4.22)-(4.23), then the matrix (6.12) reduces to

$$(D - CA^{-1}B)(\vec{x}, \vec{y}) = \begin{pmatrix} 0 & -\mathcal{M}^0(\vec{x}) \\ \mathcal{M}^0(\vec{y}) - (\mathcal{M}^i(\vec{x}) + \mathcal{M}^i(\vec{y}))\partial_{x^i} & \delta^{(3)}(\vec{x} - \vec{y}) \end{pmatrix}$$

(6.21)

Its inverse now contains finite linear combinations of the expressions of the form $\partial_{x^{i_1} \ldots x^{i_n}}\delta^{(3)}(\vec{x} - \vec{y})$. More explicitly,

$$(D - CA^{-1}B)^{-1}(\vec{x}, \vec{y}) = \begin{pmatrix} 0 & -\mathcal{M}^0(\vec{x}) \\ -W^{ijk}(\vec{x}) + W^{ijk}(\vec{y})\partial_{x^i}\partial_{x^j}\partial_{x^k} - (W^i(\vec{x}) + W^i(\vec{y}))\partial_{x^i} & W^0(\vec{x}) \end{pmatrix} \delta^{(3)}(\vec{x} - \vec{y}),$$

(6.22)

where

$$W^{ijk} \equiv \frac{p^{ijk}}{\mathcal{M}^0}, \quad W^i \equiv \frac{\mathcal{M}^i}{(\mathcal{M}^0)^2} + 3 \frac{\partial_0\mathcal{M}^0}{(\mathcal{M}^0)^2}\partial_j \left( \frac{p^{ijk}}{\mathcal{M}^0} \right), \quad W^0 \equiv \frac{1}{\mathcal{M}^0}.$$  

(6.23)

One may then substitute eq.(6.22) into eq.(6.11) to obtain $\mathcal{F}^{-1}$ and hence the Faddeev-Jackiw bracket.

7 Conclusion

In this paper, we have shown that a generalised Proca theory coupled to any background field and satisfying eq.(1.4) is likely to have three propagating degrees of freedom in the vector sector provided that the theory is diffeomorphism invariant. By using Faddeev-Jackiw analysis, we have arrived at several conditions that the theory should satisfy in order to obtain three propagating degrees of freedom. It turns out that diffeomorphism invariance trivialises almost all the conditions except for the condition (4.24). This condition demands a complicated combination of terms to not be trivially zero. So this condition can easily be fulfilled.

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For completeness, we have also investigated on how diffeomorphism invariance helps in simplifying Faddeev-Jackiw brackets. It turns out that diffeomorphism invariance requires that Faddeev-Jackiw brackets should be expressible using linear combinations of finite terms of partial derivatives on Dirac delta function. It would be interesting to investigate the implication of this result in details later.

Although the analysis in this paper is given in four-dimensional spacetime, the extension to higher dimensional spacetime can easily be done and all of the results we obtained in this paper still apply. This is because in $d-$dimensional spacetime, massive vector field has $d-1$ degrees of freedom. Correspondingly there are two constraints in the theory and both of them are of second-class.

In this paper, we have only focused on the dynamics in the vector sector. When considering the full, a theory whose vector sector passes all the criteria in this paper is still not guaranteed to be free from pathologies. Although it has three propagating degrees of freedom in the vector sector, higher derivatives in background fields could potentially introduce ghost degrees of freedom in other sectors. For example, the analysis of [33] hints that for example in the Lagrangian (5.1) the counter-terms $G_4(Y)R$, $G_5(Y)G_{\mu\nu} \nabla^\mu A^\nu$, and $G_6(Y)L^{\mu\nu\alpha\beta} \nabla_\mu A_\nu \nabla_\alpha A_\beta$ are required otherwise the gravity sector would possess ghost degrees of freedom. The result in our paper partially confirms this suggestion. That is, even without the counter-terms, the Lagrangian (5.1) should have three propagating degrees of freedom as long as the condition (3.9) is satisfied. So by limiting ourselves in the vector sector, we would not be able to see the pathologies in full theory. If the full theory is pathological, then the pathologies should be from outside the vector sector.

So an interesting extension of this paper is to study dynamics of some other sectors as well as the vector one. We anticipate that diffeomorphism invariance might also help to trivialises many conditions to allow us to obtain the required number of degrees of freedom.

An alternative extension would be to relax the condition (3.9), but still demand that the Hessian determinant is degenerate. In this case, we expect that diffeomorphism invariance also trivialises many conditions.

Diffeomorphism invariance has been of great help in degrees of freedom counting in the vector sector of generalised Proca theories. In fact, we have seen that not all the requirements from diffeomorphism invariance are needed. In particular, all we need from diffeomorphism invariance are to make only eq.(3.9) and eq.(3.11) satisfied. Other conditions coming from diffeomorphism invariance requirements are not needed. So we can still consider the situation where diffeomorphism invariance is broken, for example by putting a generalised Proca field on a diffeomorphism broken background, but eq.(3.9) and eq.(3.11) are still satisfied. It is interesting to explicitly construct these theories.

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