On level set type methods for elliptic Cauchy problems

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Abstract: Two methods of level set type are proposed for solving the Cauchy problem for an elliptic equation. Convergence and stability results for both methods are proven, characterizing the iterative methods as regularization methods for this ill-posed problem. Some numerical experiments are presented, showing the efficiency of our approaches and verifying the convergence results.

1 Introduction

We start by introducing the inverse problem under consideration. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open bounded set with piecewise Lipschitz boundary $\partial\Omega$. Further, we assume that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i$ are two open connected disjoint parts of $\partial\Omega$. We denote by $P$ the elliptic operator defined in $\Omega$ by

$$P(u) := -\sum_{i,j=1}^{d} D_i(a_{i,j} D_j u),$$

where the real functions $a_{i,j} \in L^\infty(\Omega)$ are such that the matrix $A(x) := (a_{i,j})_{i,j=1}^{d}$ satisfies $\xi^t A(x) \xi > \alpha ||\xi||^2$, for all $\xi \in \mathbb{R}^d$ and for a.e. $x \in \Omega$, where $\alpha > 0$.

We denote by elliptic Cauchy problem the following boundary value problem (BVP)

$$(CP) \quad \left\{ \begin{array}{l} Pu = f, \ \text{in} \ \Omega \\ u = g_1, \ \text{at} \ \Gamma_1 \\ u_\nu = g_2, \ \text{at} \ \Gamma_1 \end{array} \right.$$

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The functions $g_1, g_2 : \Gamma_1 \to \mathbb{R}$ are given Cauchy data, and $f : \Omega \to \mathbb{R}$ is a known source term in the model.

As a solution of the elliptic Cauchy problem (CP) we consider every $H^m(\Omega)$-distribution, which solves the weak formulation of the elliptic equation in $\Omega$ and also satisfies the Cauchy data at $\Gamma_1$ in the sense of the trace operator ($m \in \mathbb{N}$ still has to be chosen). Notice that, if we know either the Neumann or the Dirichlet trace of $u$ at $\Gamma_2$, then $u$ can be computed as a solution of a mixed BVP in a stable way. Therefore, it is enough to consider the task of determining a trace (Dirichlet or Neumann) of $u$ at $\Gamma_2$.

It is well known that elliptic Cauchy problems are not well posed in the sense of Hadamard. A famous example given by Hadamard in the early 20’s [16, 23] shows that one cannot expect the solution of (CP) to depend continuously on the data. For Lipschitz bounded domains $\Omega \subset \mathbb{R}^2$, the severely ill-posedness of (CP) was recently investigated in [3] using a Steklov-Poincaré approach. Existence of solutions for arbitrary Cauchy data $(g_1, g_2)$ cannot be assured as a direct argumentation with the Schwartz reflection principle shows [15] (the Cauchy data $(g_1, g_2)$ are called consistent if the corresponding problem (CP) has a solution). It has been recently shown [2] that in the case $m = 1$, there exists a dense subset $M$ of $H^{1/2}(\Gamma_1) \times [H^{1/2}_0(\Gamma_1)]'$ such that (CP) has a $H^1(\Omega)$ solution for Cauchy data $(g_1, g_2) \in M$. What concerns uniqueness of solution for (CP), it is possible to extend the Cauchy–Kowalewsky and Holmgren theorems to the $H^m$-context and prove uniqueness of weak solutions (see, e.g., [12] for the case $m = 1$). For classical uniqueness results we refer the reader to [8]. A weak uniqueness result for nonlinear Cauchy problems can be found in [21].

A variety of numerical methods for solving (CP) can be found in the literature. An optimization approach based on least squares and Tikhonov regularization was used in [13]. In [20] Mazya et al proposed an iterative algorithm based on the successive solutions of well posed mixed BVPs. A generalization of this method, based on fixed point theory, was derived in [23]. A further generalization for nonlinear elliptic Cauchy problems can be found in [21]. In [22, 18] the Backus-Gilbert method was used to solve (CP). A Mann iterative regularization method was proposed for (CP) in [12]. In [17] a method of conjugate gradient type was investigated. Finite element approximations, based on an optimal control formulation of (CP), are discussed in [9]. An application of the quasi-reversibility method for (CP) is considered in [4, 5]. In [2] Ben Abda et al introduce an energy functional, which depends on both unknown traces of the solution of (CP) at $\Gamma_2$. 

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Our main goal in this article is to apply level set methods for obtaining stable approximations of the solution of (CP). The first application of level set methods to inverse problems was proposed by Santosa [26]. These methods can be used in identification problems where the unknown parameter is a piecewise constant function assuming one of only two possible values. The methods considered here are adequate to solve (CP) in the special case where the Neumann trace of $u$ at $\Gamma_2$ is known \textit{a priori} to satisfy $u_\nu|_{\Gamma_2} = \chi_D$, for some $D \subset \Gamma_2$.

A related application corresponds to the inverse problem in corrosion detection [7, 19]. This problem consists in determining information about corrosion occurring on the inaccessible boundary part ($\Gamma_2$) of a specimen. The data for this inverse problem correspond to prescribed current flux ($g_2$) and voltage measurements ($g_1$) on the accessible boundary part ($\Gamma_1$) and the model is the Laplace equation with no source term ($P = \Delta$, $f = 0$). For simplicity one assumes the specimen to be a thin plate ($\Omega \subset \mathbb{R}^2$) and $\partial \Omega = \Gamma_1 \cup \Gamma_2$. Moreover, the unknown corrosion damage $\gamma$ is assumed to be the characteristic function of some $D \subset \Gamma_2$, corresponding to the boundary condition $u_\nu + \gamma u = 0$ at $\Gamma_2$.

The numerical methods analyzed in this article can be extended in a straightforward way to arbitrary elliptic Cauchy problems possessing a solution with similar structure, i.e. whenever the assumption that $u_\nu|_{\Gamma_2}$ is a piecewise constant function assuming one of only two possible values (not necessarily zero and one) is valid. The proposed methods are inspired from the approaches followed in [14, 6], and relate to evolution flows of Hamilton-Jacobi type.

The manuscript is outlined as follows: In Section 2 we write the elliptic Cauchy problem in the functional analytical framework of an (ill-posed) operator equation. This is the starting point for the level set approaches derived in the two subsequent sections. In Section 3 we investigate a level set method for (CP) based on the approach proposed in [6]. We prove convergence and stability of the proposed method, as well as a monotonicity result analog to the one known for the asymptotic regularization method [27]. In Section 4 we derive a second level set method for (CP) based on the ideas presented in [14]. First we prove existence of minima for a least square functional related to (CP). In the sequel we prove convergence and stability results for our regularization strategy. The corresponding level set method is derived from an explicit Euler method for solving the evolution equation related to the first order optimality condition of the least square functional. Section 5 is devoted to numerics. Three different experiments are provided,
in order to illustrate the effectiveness of the level set method considered in Section 4.

2 Formulation of the inverse problem

In this section we rewrite the elliptic Cauchy problem (CP) in the functional analytical framework of an operator equation. This is the starting point for the level set approaches derived in the Sections 3 and 4.

The functional analytical framework established in this section is similar to the one derived in [23]. The difference is that, instead of looking for a fixed point operator, we follow an optimal control approach proposed by Lions [25]. We begin by defining the auxiliary problem:

\[
\begin{align*}
    P v &= f, \text{ in } \Omega \\
    v &= g_1, \text{ at } \Gamma_1 \\
    v_\nu &= \varphi, \text{ at } \Gamma_2
\end{align*}
\]  

(2)

This mixed BVP defines the operator \( T : \varphi \mapsto v_\nu|_{\Gamma_1} \). Notice that, if \( \varphi = u_\nu|_{\Gamma_2} \), where \( u \) is the solution of (CP), then it would follow \( T(\varphi) = g_2 \).

Thus, a simple least square approach [9, 19] for (CP) consists in solving the optimization problem.

\[ \|T(\varphi) - g_2\|^2 \rightarrow \min. \]

Due to the superposition principle for linear elliptic BVPs [15], one can split the solution of (2) in

\[
\begin{align*}
    P v_a &= 0, \text{ in } \Omega \\
    v_a &= 0, \text{ at } \Gamma_1 \\
    (v_a)_\nu &= \varphi, \text{ at } \Gamma_2 ;
\end{align*}
\]  

(3)

and from

\[
\begin{align*}
    P v_b &= f, \text{ in } \Omega \\
    v_b &= g_1, \text{ at } \Gamma_1 \\
    (v_b)_\nu &= 0, \text{ at } \Gamma_2 .
\end{align*}
\]  

(4)

From (3) we can define the linear operator

\[ L : \varphi \mapsto (v_a)_\nu|_{\Gamma_1} \],

(5)

and from (4) we define the function \( z := (v_b)_\nu|_{\Gamma_1} \). Since \( T(\varphi) = L \varphi + z \), the Cauchy problem (CP) can be written in the form of the operator equation

\[ L \varphi = g_2 - z, \]

where the constant term \( z \) depends only on \( g_1, f \) and \( P \). Therefore, it can be computed \emph{a priori}.

In order to derive our first level set approach (Section 3) for solving (6), we have to formulate (CP) in such a way that \( L \) is continuous with respect to the \( L^2 \)-norm. For the second level set approach (Section 4), we state (CP) such that \( L \) is continuous with respect to the \( L^1 \)-norm. In the sequel we present these two possible formulations of (CP).
2.1 Framework for the first level set approach

We consider (CP) in the form of equation \((6)\). Further we assume the Cauchy data to satisfy
\[
(g_1, g_2) \in H^{1/2}(\Gamma_1) \times [H^{1/2}_0(\Gamma_1)]',
\]
and the source term \(f\) to be a \(L^2(\Omega)\) distribution.

From this choice of \(g_1\) and \(f\), it follows that the mixed BVP in \((4)\) has a unique solution \(v_b \in H^1(\Omega)\) \([15, 23]\). Therefore, \(z := (v_b)_{\nu}|_{\Gamma_1} \in [H^{1/2}_0(\Gamma_1)]'\) and the distribution \(g_2 - z\) on the right hand side of \((6)\) is in \([H^{1/2}_0(\Gamma_1)]'\).

Notice that, if we choose \(\varphi \in L^2(\Gamma_2) \subset [H^{1/2}_0(\Gamma_2)]'\), the mixed BVP in \((3)\) has a unique solution \(v_a \in H^1(\Omega)\) and the linear operator \(L\) in \((5)\) is well defined from \(L^2(\Gamma_2)\) into \([H^{1/2}_0(\Gamma_1)]'\). Indeed, this assertion follows from

**Proposition 2.1** Let the domain \(\Omega \subset \mathbb{R}^d\) with \(d = 2, 3\) and the operator \(P\) be defined as in Section \([7]\). Then, the linear operator defined in \((5)\) is an injective bounded map \(L : L^2(\Gamma_2) \to [H^{1/2}_0(\Gamma_1)]'\).

Proof. Since \(d = 2, 3\), the boundary part \(\Gamma_2\) is either a 1D or a 2D Lipschitz manifold and the embedding \(L^2(\Gamma_2) \subset [H^{1/2}_0(\Gamma_2)]'\) is continuous. Therefore, given \(\varphi \in L^2(\Gamma_2)\), the mixed BVP in \((3)\) has a unique solution \(v_a \in H^1(\Omega)\) satisfying the \textit{a priori} estimate
\[
\|v_a\|_{H^1(\Omega)} \leq C_1 \|\varphi\|_{[H^{1/2}_0(\Gamma_2)]'},
\]
for some positive constant \(C_1\) (depending on \(P, \Omega\) and \(\Gamma_2\)). Now, from the continuity of the Neumann trace operator \(\gamma_{N,1} : H^1(\Omega) \ni v \mapsto v_{\nu}|_{\Gamma_1} \in [H^{1/2}_0(\Gamma_1)]'\), follows
\[
\|L\varphi\|_{[H^{1/2}_0(\Gamma_1)]'} \leq C_2 \|v_a\|_{H^1(\Omega)} \leq C_3 \|\varphi\|_{L^2(\Gamma_2)},
\]
and the continuity of \(L\) follows. It remains to prove the injectivity of \(L\). Notice that, if \(L\varphi = 0\), the function \(v_a\) in \((3)\) satisfies: \(Pv_a = 0\) in \(\Omega\), \(v_a = (v_a)_{\nu} = 0\) at \(\Gamma_1\). Then, \(\varphi = 0\) follows from the uniqueness of weak solution for (CP) \([12]\). \(\Box\)

Summarizing, this setup allow us to state (CP) in the form of the operator equation \((6)\), where \(L\) is the linear continuous operator
\[
L : L^2(\Gamma_2) \to [H^{1/2}_0(\Gamma_1)]'
\]
defined in \((5)\).
2.2 Framework for the second level set approach

In the sequel we shall assume \( \Omega \subset \mathbb{R}^d \) with \( d = 2, 3 \) and define yet another functional analytical framework to analyze (6). The Cauchy data is assumed to satisfy
\[(g_1, g_2) \in [H^{1/2}_{00}(\Gamma_1)]' \times [H^{3/2}_{00}(\Gamma_1)]' \tag{9}\]
and the source term \( f \) to be a \( H^{-1}(\Omega) \) distribution.

Due to the choice of \( g_1 \) and \( f \) above, the elliptic theory allow us to conclude that the mixed BVP in (4) has a unique solution \( v_b \in L^2(\Omega) \) \([10, 15]\). Therefore, \( z := (v_b)_\nu|_{\Gamma_1} \in [H^{3/2}_{00}(\Gamma_1)]' \) and the term \( g_2 - z \) on the right hand side of (6) is a distribution in \( [H^{3/2}_{00}(\Gamma_1)]' \).

In the next proposition we prove that the linear operator \( L \) in (5) is well defined continuous and injective from \( L^1(\Gamma_2) \) to \( [H^{3/2}_{00}(\Gamma_1)]' \).

**Proposition 2.2** Let the domain \( \Omega \subset \mathbb{R}^d \) with \( d = 2, 3 \) and the operator \( P \) be defined as in Section 1. Then, the linear operator defined in (5) is an injective bounded map \( L : L^1(\Gamma_2) \to [H^{3/2}_{00}(\Gamma_1)]' \).

**Proof.** If \( d = 2 \), the boundary part \( \Gamma_2 \) is a 1D manifold and the Sobolev embedding theorem \([1, 15]\) implies \( H^s_0(\Gamma_2) \subset L^\infty(\Gamma_2) \) for \( s > 1/2 \). If \( d = 3 \), the embedding above still holds, however only for \( s > 1 \). In either case, we can take \( s = 3/2 \) and conclude that \( L^1(\Gamma_2) \subset [L^\infty(\Gamma_2)]' \subset H^{-3/2}(\Gamma_2) \). This result together with \( H^{-3/2}(\Gamma_2) \subset [H^{3/2}_{00}(\Gamma_2)]' \) imply the continuity of the embedding \( L^1(\Gamma_2) \subset [H^{3/2}_{00}(\Gamma_2)]' \). Thus, from the elliptic theory we conclude that given \( \varphi \in L^1(\Gamma_2) \), the mixed BVP in (3) has a unique solution \( v_a \in L^2(\Omega) \) satisfying the \textit{a priori} estimate
\[\|v_a\|_{L^2(\Omega)} \leq C_1\|\varphi\|_{[H^{3/2}_{00}(\Gamma_2)]'},\]
for some positive constant \( C_1 \) (depending on \( P \), \( \Omega \) and \( \Gamma_2 \)). Now, from the continuity of the Neumann trace operator \( \gamma_{N,1} : L^2(\Omega) \ni v \mapsto v_\nu|_{\Gamma_1} \in [H^{3/2}_{00}(\Gamma_1)]' \), follows
\[\|L\varphi\|_{[H^{3/2}_{00}(\Gamma_1)]'} \leq C_2\|v_a\|_{L^2(\Omega)} \leq C_3\|\varphi\|_{L^1(\Gamma_2)},\]
and the continuity of \( L \) is proven. In order to prove the injectivity of \( L \), one argues analogously as in the last part of the proof of Proposition 2.1.

Summarizing, this setup allow us to state (CP) in the form of equation (6), where \( L \) is the linear continuous operator
\[L : L^1(\Gamma_2) \to [H^{3/2}_{00}(\Gamma_1)]' \tag{10}\]
defined in (5).
2.3 A remark on the dimension of $\Omega \subset \mathbb{R}^d$

In the approach presented in Subsection 2.1, the key argument to prove the desired regularity of the operator $L$ (see (8)) was the continuity of the embedding $L^2(\Gamma_2) \subset [H^{1/2}_{00}(\Gamma_2)]'$. Since the continuity of this embedding does not depend on $d$, Proposition 2.1 actually holds for every $d \in \mathbb{N}$.

In the case of the approach presented in Subsection 2.2, the situation is different: Proposition 2.2 only holds for $d = 2, 3$. Indeed, the desired regularity of $L$ in (10) depends on the continuity of the embedding $L^1(\Gamma_2) \subset [H^{3/2}_{00}(\Gamma_2)]'$, which follows from the Sobolev embedding theorem:

$$H^s_0(\Gamma_2) \subset C^0(\Gamma_2) \subset L^\infty(\Gamma_2), \text{ for } s > (d - 1)/2$$

(notice that $\Gamma_2 \in \mathbb{R}^{d-1}$). Therefore, if $\Omega \in \mathbb{R}^d$ for $d \geq 4$, we would have to choose $s \geq 2$ in the proof of Proposition 2.2. The proof would no longer hold, since the mixed BVP in (3) would not have a solution $v_a$ on a Sobolev space of non-negative index.

This is not actually a disadvantage of the second level set approach, since almost all real life applications are related to either two or three-dimensional domains $\Omega$.

2.4 A remark on noisy Cauchy data

The second remark concerns the investigation of (CP) for noisy data. If only corrupted noisy data $(g_1^\delta, g_2^\delta)$ are available, we assume the existence of a consistent pair of Cauchy data $(g_1, g_2)$ such that

$$\|g_1 - g_1^\delta\|_{L^2(\Gamma_1)} + \|g_2 - g_2^\delta\|_{L^2(\Gamma_1)} \leq \delta. \quad (11)$$

For clarity of the presentation, we discuss separately the two frameworks introduced above in this section:

Noisy data within the framework of Subsection 2.2
Let the noisy data be given as in (11) and the exact Cauchy data satisfy (9). Since $z$ in (6) depends continuously on $g_1$ in the $[H^{1/2}_{00}(\Gamma_1)]'$ topology, we can solve the mixed BVP in (4) using $g_1^\delta$ as data, and obtain a corresponding $z^\delta \in [H^{3/2}_{00}(\Gamma_1)]'$ such that

$$\|(g_2 - z) - (g_2^\delta - z^\delta)\|_{[H^{3/2}_{00}(\Gamma_1)]'} \leq C\delta, \quad (12)$$

where the constant $C$ depends on $\Omega$, $P$, $\Gamma_1$ and $f$. Summarizing, we have
Lemma 2.3  Within the framework of Subsection 2.2, the Cauchy problem (CP) with noisy data satisfying (11) reduces to equation

\[ L \phi = g^\delta_2 - z^\delta, \]

where \( L \) satisfies (10) and the right hand side \((g^\delta_2 - z^\delta)\) satisfies (13).

Noisy data within the framework of Subsection 2.1

Let the noisy data be given as in (11) and the exact Cauchy data satisfy (7). Since \( z \) in (6) depends continuously on \( g^1 \) in the \( H^{1/2}(\Gamma_1) \) topology, a natural question arises:

Q) Is it possible to obtain from measured data \((g^\delta_1, g^\delta_2)\) satisfying (11), a corresponding \( z^\delta \in [H^{1/2}(\Gamma_1)]' \) such that \( \| z - z^\delta \|_{[H^{1/2}(\Gamma_1)]'} \leq \delta \)?

We claim that such \( z^\delta \) can be obtained under the a priori assumption \( g^1 \in H^s(\Gamma_1) \), for some \( s > 1/2 \). Indeed, under this assumption, \cite{12, Lemma 8} guarantees the existence of a smoothing operator \( S : L^2(\Gamma_1) \to H^{1/2}(\Gamma_1) \), and of a positive function \( \mu \) with \( \lim_{t \to 0} \mu(t) = 0 \), such that for \( \delta > 0 \) and \( g^\delta_1 \in L^2(\Gamma_1) \) with \( \| g_1 - g^\delta_1 \|_{L^2(\Gamma_1)} \leq \delta \), we have \( \| g_1 - S(g^\delta_1) \|_{H^{1/2}(\Gamma_1)} \leq \mu(\delta) \). Thus, after smoothing the noisy data \( g^\delta_1 \), we obtain \( \hat{g}_1 := S(g^\delta_1) \in H^{1/2}(\Gamma_1) \). Next, we solve the mixed BVP in (4) using \( \hat{g}_1 \) as data, and obtain a corresponding \( z^\delta \in [H^{1/2}(\Gamma_1)]' \) with \( \| z - z^\delta \|_{[H^{1/2}(\Gamma_1)]'} \leq C \mu(\delta) \), with \( C \) as in \cite{12}.

Once we are able to give an affirmative answer to question Q), it follows from (11) that

\[ \| (g_2 - z) - (g^\delta_2 - z^\delta) \|_{[H^{1/2}(\Gamma_1)]'} \leq (1 + C) \max\{ \delta, \mu(\delta) \}. \]  

(13)

Summarizing, we have

Lemma 2.4  Consider the framework of Subsection 2.1. Assume the noisy Cauchy data satisfy (11), where \( g_1 \in H^s(\Gamma_1) \) for some \( s > 1/2 \). Then (CP) reduces to equation

\[ L \phi = g^\delta_2 - z^\delta, \]

where \( L \) satisfies (8) and the right hand side \((g^\delta_2 - z^\delta)\) satisfies (13).

3  A first level set approach

In this section we investigate a level set method for (CP) based on the approach introduced in \cite{6}. For this purpose, we consider the functional analytical framework derived in Subsection 2.1 and summarized in Lemma 2.4.
The starting point of this approach is the assumption that the solution \( \varphi \) of (6) is the characteristic function \( \chi_D \) of a subdomain \( D \subset \subset \Gamma_2 \). A function \( \phi : \Gamma_2 \times \mathbb{R}^+ \to \mathbb{R} \) is introduced, allowing the definition of the level sets \( D(t) = \{ \phi(\cdot, t) \geq 0 \} \). The function \( \phi \) should be chosen such that
\[
\varphi(\cdot, t) := \chi_D(t) \to \chi_D = \varphi \quad \text{as} \quad t \to \infty.
\]
as \( t \to \infty \). The level set method corresponds to a continuous evolution for an artificial time \( t \), where the level set function \( \phi \) is defined by a Hamilton-Jacobi equation of the form
\[
\frac{\partial \phi}{\partial t} + V \cdot \nabla \phi = 0,
\]
with initial value \( \phi(x, 0) = \phi_0(x) \), where \( \phi_0 \) is an appropriate indicator function of a measurable set \( D_0 \subset \subset \Gamma_2 \). The function \( V : \mathbb{R}^{d-1} \times \mathbb{R}^+ \to \mathbb{R}^{d-1} \) describes the velocity of the level sets of \( \phi \). Following [26], \( V \) is chosen in the normal direction of the level set curves of \( \phi \), i.e. \( V = v \nabla \phi / |\nabla \phi| \), for \( v : \mathbb{R}^{d-1} \times \mathbb{R}^+ \to \mathbb{R} \).

Here, the guideline for the choice of the velocity function \( V \) is a property of the asymptotic regularization method [27]. In this method the approximations \( \varphi(\cdot, t) \) for the solution \( \varphi \) of (6) satisfy
\[
\frac{d}{dt} \| \varphi(\cdot, t) - \varphi \|^2 = -2 \| L(\varphi(\cdot, t)) - (g_2 - z) \|^2.
\]
As we shall see, a level set method satisfying (16) can be analyzed in a similar way to asymptotic regularization[7]. Our first goal is to determine how to choose the velocity \( V \) such that, if \( \phi \) solves (15) then the function \( \varphi \) defined in (14) satisfies (16).

**Proposition 3.1** Let the function \( h \) be defined by
\[
h(x, t) = (-1 + 2\varphi(x, t)) \cdot \text{div} \, V(x, t), \quad x \in \Gamma_2, \quad t \in \mathbb{R}^+,
\]
for some \( V \in L^\infty(0, T, L^2(\mathbb{R}^{d-1})) \) with \( \text{div} \, V \in L^1(0, T, L^\infty(\mathbb{R}^{d-1})) \cap L^\infty(0, T, L^2(\mathbb{R}^{d-1})) \). Moreover, let the level set function \( \phi \) satisfy (15) and \( \varphi \) be defined by (14). Then
\[
\frac{d}{dt} \| \varphi(\cdot, t) - \varphi \|_{L^2(\Gamma_2)}^2 = \int_{\Gamma_2} (\varphi(\cdot, t) - \varphi) \cdot h(x, t) dx,
\]
for all \( t \in (0, T) \).

The method of asymptotic regularization can not be used directly to construct piecewise approximations for \( \varphi \), since it uses the time derivative of \( \varphi(\cdot, t) \) which is not defined in \( L^2(\Gamma_2) \).
Proof. See [6, Proposition 3.3].

Remark 3.2 If \( V \) satisfies the assumptions of Proposition 3.1 for each \( T > 0 \), then (17) is satisfied for all \( t \in \mathbb{R}^+ \).

Since we are assuming the operator \( L \) to satisfy (8), we conclude from Proposition 3.1 that relation (16) is satisfied for the velocity \( V \) satisfying

\[
- \text{div} \ V = 2(-1 + 2\varphi)^{-1}L^*(L \varphi - (g_2 - z)) \text{ in } \Gamma_2 \times \mathbb{R}^+.
\]

(18)

Observe that, if the normal derivative \( V_\nu(\cdot, t) \) vanishes on \( \partial \Gamma_2 \) for \( t \in \mathbb{R}^+ \) then the support of \( \varphi(\cdot, t) \) remains a subset of \( \Omega \) during the evolution, what is a desirable property. It is worth noticing that a solution \( V \) of (18) with homogeneous Neumann boundary condition on \( \partial \Gamma_2 \) always exists. Indeed, it is enough to choose \( V = \nabla \psi \), where \( \psi \) solves

\[
-\Delta \psi = 2(-1 + 2\varphi)^{-1}L^*(L \varphi - (g_2 - z)), \text{ on } \Gamma_2, \quad \psi = 0, \text{ at } \partial \Gamma_2.
\]

In the sequel we derive a convergence analysis for the level set method defined by (14) (15), with the choice of velocity in (18). We consider noisy Cauchy data as in Lemma 2.4. Moreover, we define the stopping time \( T(\delta, g^\delta_1, g^\delta_2) \) by the generalized discrepancy principle [11]

\[
T(\delta, g^\delta_1, g^\delta_2) := \inf \{ t \in \mathbb{R}^+; \|L(\varphi(\cdot, t)) - (g^\delta_2 - z^\delta)\| \leq \tau \delta \},
\]

(19)

for some \( \tau > 1 \).

The next theorem summarizes the main convergence and stability results for this level set method

Theorem 3.3 (Convergence analysis) Let \( V \) satisfy (18) and \( \varphi, \phi \) be defined by (14) (15).

i) Monotonicity: For noisy Cauchy data and \( \tau > 1 \), the iteration error is strictly monotone decreasing, i.e.,

\[
\frac{d}{dt}\|\varphi(\cdot, t) - \phi\|^2 < 0,
\]

for all \( t > 0 \) with \( \|L(\varphi(\cdot, t)) - (g^\delta_2 - z^\delta)\| > \tau \delta \). Moreover, for exact Cauchy data (i.e., \( \delta = 0 \)), we have the inequality

\[
\int_0^\infty \|L(\varphi(\cdot, t)) - L(\phi)\|^2 dt < \infty;
\]
ii) **Convergence:** If the Cauchy data is exact, then \( \varphi(\cdot, t) \to \overline{\varphi} \) in \( L^2(\Gamma_2) \) as \( t \to \infty \), where \( \overline{\varphi} \) is the solution of (6) corresponding to the consistent data \((g_1, g_2)\);

iii) **Stability:** For noisy Cauchy data, the stopping time \( T(\delta, g^\delta_1, g^\delta_2) =: T_\delta \) defined by (19) with \( \tau > 1 \) is finite. Moreover, given a sequence \( \delta_k \to 0 \) and \( \{(g^\delta_1, g^\delta_2)\}_k \) corresponding noisy data satisfying (14) for some consistent data pair \((g_1, g_2)\), then the approximations \( \varphi(\cdot, T_\delta) \) converge to \( \overline{\varphi} \) in \( L^2(\Gamma_2) \) as \( \delta_k \to 0 \), where \( \overline{\varphi} \) is the solution of (6) corresponding to \((g_1, g_2)\).

**Proof.** Item i): The monotonicity result is a consequence of (16). The inequality in the second statement is a well known property of the asymptotic regularization and its proof is analog to [6, Proposition 4.1].

Item ii): This statement concerning convergence for exact data is also known to hold for the Landweber iteration as well as for asymptotic regularization. The proof follow the lines of [6, Theorem 4.4].

Item iii): This convergence result for noisy data has a counterpart in the asymptotic regularization. The proof carries over from [27, Theorem 4].

**Remark 3.4** Under the assumptions of Theorem 3.3 one can prove, analogous as in item i), that the residual function \( t \mapsto \|L(\varphi(\cdot, t)) - (g^\delta_2 - z^\delta)\| \) is monotonically non-increasing.

### 4 A second level set approach

In the sequel we investigate a second level set method for (CP). Our approach is based on [14]. In what follows we shall consider the functional analytical framework for (CP) discussed in Subsection 2.2 and summarized in Lemma 2.3.

Let functions \( \varphi \) and \( \phi \) be defined as in Section 3. For simplicity we adopt the notation \( Y := H^{1/2}_{00}(\Gamma_1)' \). If we denote by \( H \) the heavy-side projector\(^2\) then the Cauchy problem (6) can be written in the form of the constrained optimization Problem

\[
\min \|L \varphi - (g^\delta_2 - z^\delta)\|_Y^2, \quad \text{s.t.} \quad \varphi = H(\phi).
\]

\(^2\)The projector \( H \) and its approximation \( H_\varepsilon \) are defined by

\[
H(t) := \begin{cases} 
0 & \text{for } t < 0, \\
1 & \text{for } t \geq 0.
\end{cases}
\]

\[
H_\varepsilon(t) := \begin{cases} 
0 & \text{for } t < -\varepsilon, \\
1 + t/\varepsilon & \text{for } -\varepsilon \leq t \leq 0, \\
1 & \text{for } t \geq 0.
\end{cases}
\]
Alternatively, we can minimize
\[
\min \| L(\phi) - (g^\delta_2 - z^\delta) \|^2_Y ,
\] (20)
over \( \phi \in H^1(\Gamma_2) \). Tikhonov regularization for (20) using \( TV - H^1 \) penalization consists in the minimization of the cost functional
\[
F_\alpha(\phi) := \| L(\phi) - (g^\delta_2 - z^\delta) \|^2_Y + \alpha [\beta |H(\phi)|_{BV} + \| \phi - \phi_0 \|^2_{H^1}] ,
\] (21)
where \( \alpha > 0 \) plays the role of a regularization parameter and \( \beta > 0 \) is a scaling factor [24, 14]. Since \( H \) is a discontinuous operator, one cannot prove that the Tikhonov functional in (21) attains a minimizer.

In order to guarantee existence of a minimizer of \( F_\alpha \), we use the concept of generalized minimizers in [14, Lemma 2.2]. \( F_\alpha \) is no longer considered as a functional on \( H^1 \), but as a functional defined on the \( w \)-closure of the graph of \( H \), contained in \( BV \times H^1(\Gamma_2) \). A generalized minimizer of \( F_\alpha(\phi) \) is defined as a minimizer of
\[
F_\alpha(\xi, \phi) := \| L(\phi) - (g^\delta_2 - z^\delta) \|^2_Y + \alpha \rho(\xi, \phi)
\] (22)
on the set of admissible pairs
\[
Ad := \{ (\xi, \phi) \in L^\infty(\Gamma_2) \times H^1(\Gamma_2) ; \exists \{ \phi_k \} \in H^1 \text{ and } \{ \varepsilon_k \} \in \mathbb{R}^+ \text{ s.t.} \lim_{k \to \infty} \varepsilon_k = 0 , \lim_{k \to \infty} \| \phi_k - \phi \|_{L^2} = 0, \lim_{k \to \infty} \| H_{\varepsilon_k}(\phi_k) - \xi \|_{L^1} = 0 \} ,
\]
where \( \rho(\xi, \phi) := \inf_{\{ \phi_k \}, \{ \varepsilon_k \} \to 0} \liminf_{k \to \infty} \{ 2\beta |H_{\varepsilon_k}(\phi_k)|_{BV} + \| \phi_k - \phi_0 \|^2_{H^1} \} \).

As a consequence of this definition, the penalization term in (21) can be interpreted as a functional \( \rho : Ad \to \mathbb{R}^+ \). In order to prove coerciveness and weak lower semi-continuity of \( \rho \), the assumption that \( L \) is a continuous operator on a \( L^1 \) space is crucial (see Proposition 2.2). These properties of \( \rho \) are the main arguments needed to prove existence of a generalized minimizer \( (\xi_\alpha, \phi_\alpha) \) of \( F_\alpha \) in \( Ad \) [14, Theorem 2.9].

The classical analysis of Tikhonov type regularization methods [11] do apply to functional \( F_\alpha \), as we shall see next.

**Theorem 4.1 (Convergence)** Let \( \alpha : \mathbb{R}^+ \to \mathbb{R}^+ \) be a function satisfying \( \lim_{\delta \to 0} \alpha(\delta) = 0 \) and \( \lim_{\delta \to 0} \delta^2 \alpha^{-1}(\delta) = 0 \). Given a sequence \( \delta_k \to 0 \) and \( \{ (g^k_1, g^k_2) \} \) corresponding noisy data satisfying (11) for some consistent data pair \( (g_1, g_2) \), then the minimizers \( (\xi_k, \phi_k) \) of \( F_\alpha(\delta_k) \) converge in \( L^1(\Gamma_2) \times L^2(\Gamma_2) \) to a minimizer \( (\bar{\xi}_\alpha, \bar{\phi}_\alpha) \) of \( F_\alpha \) in (22).

**Proof.** See [14] Section 2.3. \[ \square \]
Numerical realization

What concerns numerical approximations to the solution of (6), the functional $F_\alpha$ in (21) has an interesting property. Namely, its generalized minimizers can be approximated by minimizers of the stabilized functional $2F_{\alpha,\epsilon}(\phi) := \|L(H_\epsilon(\phi)) - (g_2 - z)^2\|_Y^2 + \alpha[\beta|H_\epsilon(\phi)|_{BV} + \|\phi - \phi_0\|_{H^1}^2].$ (23)

In other words: Let $\phi_{\alpha,\epsilon}$ be a minimizer of $F_{\alpha,\epsilon}$; given a sequence $\epsilon_k \to 0^+$, we can find a subsequence $(H(\phi_{\alpha,\epsilon}), \phi_{\alpha,\epsilon})$ converging in $L^1(\Gamma_2) \times L^2(\Gamma_2)$ and the limit minimizes $F_\alpha$ in $Ad$.

The existence of minimizers of $F_{\alpha,\epsilon}$ in $H^1(\Gamma_2)$ still has to be cleared: Since $H_\epsilon$ is continuous and the operator $L$ is linear, continuous, satisfying [10], the existence of minimizers for $F_{\alpha,\epsilon}$ follows directly from [14, Lemma 3.1].

This relation between the minimizers of $F_\alpha$ and $F_{\alpha,\epsilon}$ is the starting point for the derivation of a numerical method. We can formally write the optimality condition for $F_{\alpha,\epsilon}$ as

$$\alpha(I - \Delta)(\phi - \phi_0) = R_{\alpha,\epsilon}(\phi),$$

where

$$R_{\alpha,\epsilon}(\phi) := -H'_\epsilon(\phi)L'(H_\epsilon(\phi))^*[L(H_\epsilon(\phi)) - (g_2^\delta - z^\delta)] + \beta \alpha H'_\epsilon(\phi) \nabla : (\nabla H_\epsilon(\phi)/|\nabla H_\epsilon(\phi)|).$$

Identifying $\alpha = 1/\Delta t$, $\phi(0) = \phi_0$, $\phi(\Delta t) = \phi$, we find

$$(I - \Delta) \left( \frac{\phi(\Delta t) - \phi(0)}{\Delta t} \right) = R_{1/\Delta t,\epsilon}(\phi(\Delta t)).$$

Considering $\Delta t$ as a time discretization, we find that (in a formal sense) the iterative regularized solution $\phi(\Delta t)$ is a solution of an implicit time step for the dynamic system

$$(I - \Delta) \left( \frac{\partial \phi(t)}{\partial t} \right) = R_{1/\Delta t,\epsilon}(\phi(t)).$$ (24)

Our second level set method is based on the solution of the dynamic system (24). In algorithmic form we have:

0. Choose $\phi_0 \in H^1(\Gamma_2)$ and set $k = 0$; Compute $z^\delta = u_\nu|_{\Gamma_1}$, where

$$\Delta u = f, \text{ in } \Omega, \quad u|_{\Gamma_1} = g_1^\delta, \quad u_\nu|_{\Gamma_2} = 0;$$
1. Evaluate the residual \( r_k := L(H_\varepsilon(\phi_k)) - (g_2^\delta - z^\delta) \). Notice that 

\[
L(H_\varepsilon(\phi_k)) = (u_k)_\nu|_{\Gamma_1},
\]

where 

\[
\Delta u_k = 0, \text{ in } \Omega, \quad u_k|_{\Gamma_1} = 0, \quad (u_k)_\nu|_{\Gamma_2} = H_\varepsilon(\phi_k);
\]

2. Evaluate \( V_k := L'(H_\varepsilon(\phi_k))^\ast(r_k) \). Notice that \((L')^\ast(r_k) = -v_k|_{\Gamma_2}\), where 

\[
\Delta v_k = 0, \text{ in } \Omega; \quad v_k|_{\Gamma_1} = r_k, \quad (v_k)_\nu|_{\Gamma_2} = 0;
\]

3. Evaluate the velocity \( w_k \) by solving 

\[
(I - \Delta)w_k = H_\varepsilon'(\phi_k) \left[ -v_k + \beta \nabla \cdot \left( \frac{\nabla H_\varepsilon(\phi_k)}{\nabla H_\varepsilon(\phi_k)} \right) \right], \text{ in } \Gamma_2;
\]

\[
(w_k)_\nu|_{\partial \Gamma_2} = 0.
\]

4. Update the level set function \( \phi_{k+1} = \phi_k + \frac{1}{\alpha} w_k; \)

Notice that Steps 1 and 2 above involve the solution of a mixed BVP in \( \Omega \subset \mathbb{R}^d \). On the third step, the computation of the velocity function for the level set method requires a solution of a Neumann BVP at \( \Gamma_2 \).

In the next section we present some numerical experiments, which where implemented using the above algorithm.

5 Numerical experiments

In what follows we present three numerical experiments for the level set method analyzed in Section 4. In Subsection 5.1 we consider a problem with exact Cauchy data, where the solution is the characteristic function of a non-connected set. In Subsection 5.2 we investigate how the degree of ill-posedness of an elliptic Cauchy problem affects the performance of the level set method. In Subsection 5.3 we consider a problem with noisy Cauchy data and test the stability of our method.

5.1 A Cauchy problem with non-connected solution

One of the main advantages of the level set approach is the fact that no \textit{a priori} assumption on the topology of the solution set is needed [26]. In this first numerical experiment we exploit this feature of the method by solving a Cauchy problem with non-connected solution. Let \( \Omega = (0,1) \times (0,0.5) \) with
Figure 1: Framework for the first numerical experiment: On the left hand side, the exact solution $\varphi$ of the Cauchy problem (dotted blue line) and the initial guess for the level set method (solid red line). The solution $u$ of the elliptic BVP corresponding to (25) is depicted on the right hand side. Notice that $u_\nu = g_2$ and $u = 0$ at $\Gamma_1$ (lower edge). Moreover, $u_\nu = \varphi$ at $\Gamma_2$ (top edge).

$\Gamma_1 := (0, 1) \times \{0\}$, $\Gamma_2 := (0, 1) \times \{0.5\}$ and $\Gamma_3 := \partial \Omega / \{\Gamma_1 \cup \Gamma_2\}$. Consider the Cauchy problem

$$\Delta u = 0, \quad \text{in } \Omega \quad u = 0, \quad \text{at } \Gamma_1 \quad u_\nu = g_2, \quad \text{at } \Gamma_1 \quad u_\nu = 0, \quad \text{at } \Gamma_3. \tag{25}$$

The Cauchy data $(0, g_2)$ is chosen in such a way that the solution $\varphi$ of (25) is the indicator function of a non-connected subset of $\Gamma_2$ (see Figure 1).

In Figure 2 we show the level set iterations for the Cauchy problem (25) and in Figure 3 the corresponding evolution of level set function. Notice the large number of steps required to obtain a precise approximation. As already observed when applying level set methods to other models [14, 26], the splitting of the level sets happens only after a large number of iterations (over 10000 in this experiment). Nevertheless, the a priori required precision could always be reached (in this experiment, $\|\varphi_k - \varphi\|_{L^2(\Gamma_2)} < 10^{-2}$ was reached after 15000 steps). It is worth noticing that, in our experiments, the total number of steps needed to reach a pre-specified precision does not depend from the initial guess for the level set method.

### 5.2 Degree of ill-posedness affecting convergence precision

In this experiment we analyze the effect of the ill-posedness degree of the Cauchy problem on the performance of the level set method introduced in Section 4. For this purpose we introduce the domains $\Omega_1 := (0, 1) \times (0, 1)$ and $\Omega_2 := (0, 1) \times (0, 0.5)$. The boundary part $\Gamma_1 := (0, 1) \times \{0\}$ is the same for both Domains. Moreover we define $\Gamma_{2,1} := (0, 1) \times \{1\}$, $\Gamma_{2,2} := (0, 1) \times \{0.5\}$ and $\Gamma_{3,i} := \partial \Omega_i / \{\Gamma_1 \cup \Gamma_{2,i}\}$, $i = 1, 2$. 

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We compare the performance of our method for the Cauchy problems (CP)\(_i\), \(i = 1, 2\), defined by

\[
\Delta u_i = 0, \quad \text{in } \Omega_i \quad u_i = 0, \quad \text{at } \Gamma_1 \quad (u_i)_\nu = g_{2,i}, \quad \text{at } \Gamma_1 \quad (u_i)_\nu = 0, \quad \text{at } \Gamma_3.
\]

The Cauchy data \((0, g_{2,i})\) is chosen such that problems (CP)\(_i\) have the same solution, i.e. \((u_1)_\nu |_{\Gamma_{2,1}} = (u_2)_\nu |_{\Gamma_{2,2}}\) (see Figures 4, 5).

It is known from the literature that the ill-posedness degree of (CP) increases with the distance between the boundary parts \(\Gamma_1\) and \(\Gamma_2\) \([23]\). Since we are considering simple domains \(\Omega_i\), it is possible to compute the eigenvalues \(\{\lambda_{i,j}\}_j\) of the operators \(L_i\) in (5) corresponding to (CP)\(_i\)

\[
\lambda_{i,j} = \sinh(j/i)^{-1}, \quad j = 1, 2, \ldots, \quad i = 1, 2.
\]

This gives us a measure how ill-posed (CP)\(_1\) is when compared with (CP)\(_2\). In the sequel we compare the performance of the level set method for both problems (exact Cauchy data is used). In Figures 5 and 6 we see the level set iterations for problems (CP)\(_1\) and (CP)\(_2\) respectively. Due to the difference between the ill-posedness degree of both problems, the method converges faster for problem (CP)\(_2\). Nevertheless, the same accuracy can be reached.
for both problems if one iterates long enough. We conclude that the degree of ill-posedness only affects the amount of computational effort needed to obtain an approximate solution with a priori defined precision, and not the quality of the final approximation. Our experiments indicate that the number of iteration steps needed by the level set method (to reach a desired accuracy) increases exponentially with the distance between $\Gamma_1$ and $\Gamma_2$, the same way the degree of ill-posedness also does.

### 5.3 An experiment with noisy Cauchy data

For the next numerical experiment we consider once more problem (CP)$_2$ in Subsection 5.2. This time however, we perturbed the Cauchy data $g_{2,2}$ with 10% random noise (Figure 7). The initial guess for the level set method is the same used in the second experiment (see Figure 6).

The performance of the level set method for exact and corrupted data can be compared in Figures 7 and 6. Notice that for the noisy data the best possible approximation is obtained after 300 steps. We iterated further (until 500 steps) and observed that, although the level set function oscillates, the corresponding level sets remain almost unchanged.
Figure 4: Framework for the second numerical experiment: The solutions $u_i$ of the elliptic BVPs corresponding to (CP)$_i$, $i = 1, 2$, are depicted on the left and on the right hand side respectively. The Cauchy data $g_{2,i}$ correspond to $(u_1)_\nu$ at $\Gamma_1$ (lower edge). Notice also that the Dirichlet boundary condition $u_1 = u_2 = 0$ at $\Gamma_1$ holds. Both Cauchy problems share the same solution $(u_1)_\nu|_{\Gamma_2,1} = \nabla = (u_2)_\nu|_{\Gamma_2,2}$ (top edge).

Since we assume the Cauchy data to satisfy $g_1 = 0$ in our experiments, it seems natural not to introduce noise in this component of the data. We conjecture this is the main reason for obtaining stable reconstruction results for this exponentially ill-posed problem in the presence of high levels of noise (in [12] only 5% noise is used; moreover it is not white noise but corresponds to an eigenfunction of $L$).

6 Conclusions

In this article two possible level set approaches for solving elliptic Cauchy problems are considered. For each one of them a corresponding framework is established and a convergence analysis is provided (monotony, convergence, stability results). Further we discuss the numerical realization of the second level set approach. Three different numerical experiments illustrate relevant features of this level set method: rates of convergence, adaptability to identify non-connected inclusions, robustness with respect to noise.

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Figure 5: Second numerical experiment: On the top left, the exact solution $\phi$ of the Cauchy problems (dotted blue line) and the initial guess for the level set method (solid red line). The other pictures show the evolution of the level set method for problem (CP)$_1$ after 100, 500, 1000, 2000 and 3000 iterative steps. The solid (red) line represent the iteration and the dotted (blue) line the exact solution.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975

[2] S. Andrieux, T.N. Baranger and A. Ben Abda, *Solving Cauchy problems by minimizing an energy-like functional*, Inverse Problems 22 (2006), 115–133

[3] F.B. Belgacem, *Why is the Cauchy problem severely ill-posed? Inverse Problems* 23 (2007), 823–836

[4] L. Bourgeois, *A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace’s equation*, Inverse Problems 21 (2005), 1087–1104

[5] L. Bourgeois, *Convergence rates for the quasi-reversibility method to solve the Cauchy problem for Laplace’s equation*, Inverse Problems 22 (2006), 413–430
Figure 6: Second numerical experiment: Evolution of the level set method for problem (CP)\textsubscript{2}. Plots of the initial guess and after 30, 100, 300, 400 and 500 iterative steps. The solid (red) line represent the iteration and the dotted (blue) line the exact solution.

[6] M. Burger, *A level set method for inverse problems*, Inverse Problems 17 (2001), 1327–1355

[7] F. Cakoni and R. Kress, *Integral equations for inverse problems in corrosion detection from partial Cauchy data*, Inverse Problems and Imaging 1 (2007), 229–245.

[8] A.-P. Calderón, *Uniqueness in the Cauchy problem for partial differential equations*, Amer. J. Math. 80 (1958), 16–36

[9] A. Chakib and A. Nachaoui, *Convergence analysis for finite element method approximation to an inverse Cauchy problem*, Inverse Problems 22 (2006), 1191–1206

[10] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 2, Springer, New York, 1988

[11] H. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996

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Figure 7: Third numerical experiment: On the top left, the exact Cauchy data $g_{2,2}$ (dotted blue line) and the data corrupted with 10% random noise (solid red line). The other pictures show the evolution of the level set method for problem (CP)$_2$ with noisy data after 30, 200, 300, 400 and 500 iterative steps. The solid (red) line represent the iteration and the dotted (blue) line the exact solution.

[12] H. Engl and A. Leitão, A Mann iterative regularization method for elliptic Cauchy problems, Numer. Funct. Anal. Optim. 22 (2001), 861–864

[13] R.S. Falk and P.B. Monk, Logarithmic convexity of discrete harmonic functions and the approximation of the Cauchy problem for Poisson's equation, Math. Comput. 47 (1986), 135-149.

[14] F. Frühauf, O. Scherzer and A. Leitão, Analysis of regularization methods for the solution of ill-posed problems involving discontinuous operators, SIAM Journal of Numerical Analysis 43 (2005), 767–786

[15] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer, Berlin, 1977

[16] J. Hadamard, Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques, Hermann, Paris, 1932
[17] D.N. Hao and D. Lesnic, *The Cauchy problem for Laplace’s equation via the conjugate gradient method*, IMA J. Appl. Math. 65 (2000), 199–217

[18] Y.C. Hon and T. Wei, *Backus-Gilbert algorithm for the Cauchy problem of the Laplace equation*, Inverse Problems 17 (2001), 261–271

[19] G. Inglese, *An inverse problem in corrosion detection*, Inverse Problems 13 (1997), 977–994

[20] V.A. Kozlov, V.G. May’za and A.V. Fomin, *An iterative method for solving the Cauchy problem for elliptic equations*, Comput. Maths. Phys. 1 (1991), 45–52

[21] P. Kügler and A. Leitão, *Mean value iterations for nonlinear elliptic Cauchy problems*, Numerische Mathematik 96 (2003), 269–293

[22] A. Leitão, *Applications of the Backus–Gilbert method to linear and some nonlinear equations*, Inverse Problems 14 (1998), 1285–1297

[23] A. Leitão, *An iterative method for solving elliptic Cauchy problems*, Numer. Funct. Anal. Optim. 21 (2000), 715–742

[24] A. Leitão and O. Scherzer, *On the relation between constraint regularization, level sets, and shape optimization*, Inverse Problems 19 (2003), L1–L11

[25] J.-L. Lions, *Optimal control of systems governed by partial differential equations*, Springer, New York, 1971

[26] F. Santosa, *A level set approach for inverse problems involving obstacles*, ESAIM: Control, optimization and Calculus of Variations 1 (1996), 17–33

[27] U. Tautenhahn, *On the asymptotical regularization of nonlinear ill-posed problems*, Inverse Problems 10 (1994), 1405–1418