A CONNECTION BETWEEN UNIQUENESS OF MINIMIZERS IN TIKHONOV-TYPE REGULARIZATION AND MOROZOV-LIKE DISCREPANCY PRINCIPLES

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Abstract. We state sufficient conditions for the uniqueness of minimizers of Tikhonov-type functionals. We further explore a connection between such results and the well-posedness of Morozov-like discrepancy principle. Moreover, we find appropriate conditions to apply such results to the local volatility surface calibration problem.

1. Introduction. In Tikhonov-type regularization, finding a balancing between data misfit and penalization is crucial, since, otherwise reconstructions may reproduce noise or incorporate too many bias. Such balancing is given by the appropriate choice of the regularization parameter, denoted by $\alpha$. In order to cope with this task different techniques were proposed in the literature. The most known ones are based on the so-called $L$-curve, e.g. [21, 24], or Morozov-type discrepancy principles, e.g. [5, 29, 31] and references therein.

In the $L$-curve-based choice, the appropriate regularization parameter must maximize the curve in $\mathbb{R}^2$ given by the log of the data misfit versus the log of the regularizing functional (penalization term without the regularization parameter). The main advantage of this method is that it does not require the knowledge of the noise level, e.g. [20, 21, 24]. However, there are some cases where it fails to provide an appropriate regularization parameter, see [25].

On the other hand, in discrepancy-based techniques, the regularization parameter is chosen whenever its corresponding data misfit has the same order of the noise level. This is interesting since some convergence and convergence-rate results can be obtained without requirements on the behavior of the regularization parameter as a function of the noise level, in contrast to the standard Tikhonov-type regularization approach [20]. However, in many applications, it is difficult to prove that it is always possible to find $\alpha$ such that, the data misfit keeps the same order of the noise level when the former goes to zero, as in standard convergence-rate results [20]. One of the main reasons for that is related to the non-uniqueness of
the corresponding minimizers of the Tikhonov-type functional. This phenomena, in particular, implies that the map that relates regularization parameter to data misfit can be multi-valued. So, to prove convergence and convergence-rate results, sometimes it is necessary to impose an asymptotic decay of the regularization parameter as a function of the noise level, e.g. [22]. Such kind of hypothesis seems ad hoc and restrictive, since, when choosing this parameter through a discrepancy-based principle, it is not clear how it will decay as the noise level goes to zero. In practice, such choices might imply regularization parameters smaller (or larger) than the necessary for the stability of the approximated solutions.

Main Contributions. The main contributions in this paper are:

(i) We show how the uniqueness of Tikhonov’s type minimizers imply the well-posedness of Morozov’s-type discrepancy-based strategies for choosing the regularization parameter.

(ii) Then, we prove the uniqueness of the minimizers of a Tikhonov-type regularization approach using fairly general assumptions on reflexive Banach spaces in the so-called variational framework. In particular, those results generalize the ones in [30] for reflexive Banach spaces.

(iii) As a benchmark problem, we derive a uniqueness result for the minimizers of a Tikhonov-type approach in the nonlinear inverse problem of local volatility calibration [18]. For this task, we use a different approach, based on geometrical techniques derived in [12].

Article Structure. The basic setup is presented in Section 2. In Section 3 a connection between uniqueness results and the well-posedness of discrepancy principles is established. Section 4 is devoted to state some uniqueness results based on variational techniques and to recall some notions presented in [12] that will be used to prove the uniqueness of the minimizers of a Tikhonov-type regularization approach for the local volatility calibration problem in Section 5.

2. Preliminaries.
The Basic Setup. Let $A$ and $U$ be Banach spaces and let $F : \mathcal{D}(F) \subset A \rightarrow U$ be a not necessarily linear operator. The corresponding inverse problem is, given $\hat{u} \in \mathcal{R}(F) \subset U$, where $\mathcal{R}(F)$ the range of the operator $F$, find $a^\dagger \in \mathcal{D}(F)$, such that it solves the operator equation

\begin{equation}
F(a^\dagger) = \hat{u}.
\end{equation}

Indeed, we are mainly interested in the situation where the inverse problem in (1) is ill-posed, i.e., when its solution does not depends continuously on the data, which is common in applications.

Since, there maybe more than one element of $\mathcal{D}(F)$ solving (1), we look for a $f_{a^*}$-minimizing solution, i.e.,

\[ a^\dagger \in \mathcal{L} := \text{argmin}\{f_{a^*}(a) : F(a) = \hat{u}\}, \]

for some given a priori $a^* \in \mathcal{D}(F) \cap \mathcal{D}(f_{a^*})$. In what follows, the functional $f_{a^*}$ is assumed convex and weakly lower semi-continuous. Moreover, we assume that $\mathcal{D}(F) \subset \mathcal{D}(f_{a^*})$. The concept of $f_{a^*}$-minimizing solutions is introduced to restrict the set of all possible solutions, selecting the ones that present some desirable feature or satisfying some a priori information.
Concerning the data, in general, only a noisy measurement of $\tilde{u}$, denoted by $u^\delta \in U$, is available, that we assume to satisfy the inequality

$$\|\tilde{u} - u^\delta\|_U \leq \delta,$$

for some noise level $\delta > 0$.

Given the ill-posedness of the operator equation (1), instead of solving the inverse problem (1), we look for a stable approximation of $a^\dagger$ in $\mathcal{D}(F)$ w.r.t. the noise level $\delta$, obtained by minimizing the Tikhonov-type functional

$$\mathcal{F}_{\alpha, \delta}(a) = \|F(a) - u^\delta\|^p_U + \alpha f_{a^*}(a).$$

In what follows we state one of the main assumptions on this paper.

**Assumption 2.1.**  
(i) The Banach spaces $A$ and $U$ are reflexive with Fréchet differentiable norms, and they are associated to its weak topologies.

(ii) The exponent in (3) satisfies $p > 1$.

(iii) $f_{a^*} : \mathcal{D}(f_{a^*}) \subset A \to [0, +\infty)$ is convex, sequentially weakly lower semi-continuous, weakly sequentially coercive, i.e,

$$\lim \|a_n\| = \infty,$$

then, $\lim f_{a^*}(a_n) = \infty$,

and satisfies the condition:

$$f_{a^*}(a) = 0 \text{ if, and only if, } a = a^*.$$

(iv) As mentioned above, $\mathcal{D}(F) \cap \mathcal{D}(f_{a^*}) \neq \emptyset$.

(v) For each $\alpha > 0$ and $M > 0$ the level sets

$$\mathcal{M}_\alpha(M) = \{a : \mathcal{F}_{\alpha, \delta}(a) \leq M\}$$

are (sequentially) weakly pre-compact.

(vi) For each $\alpha > 0$ and $M > 0$ the set $\mathcal{M}_\alpha(M)$ is (sequentially) weakly closed.

(vii) The operator $F$ is (sequentially) weakly continuous.

Under the general Assumption 2.1, it is possible to state existence and stability of a minimizer of $\mathcal{F}$ in $\mathcal{D}(F)$, which shall be called Tikhonov minimizer and denoted by $a^\delta$. See Theorems 3.22 and 3.23 in [32]. In addition, when considering a sequence $\{\delta_n\}$, such that $\delta_n \to 0$, and consequently $u^\delta \to \tilde{u}$ in norm, if $\alpha = \alpha(\delta)$ satisfies the estimates

$$\lim_{\delta \to 0} \alpha(\delta) = 0 \text{ and } \lim_{\delta \to 0} \frac{\delta^p}{\alpha(\delta)} = 0,$$

then, every sequence of minimizers $\{a^\delta_{\alpha(\delta_n)}\}$ has a weakly convergent subsequence, converging to some element of the set $\mathcal{L}$. See Theorem 3.26 in [32].

Next, we introduce some definitions and notation that will be used in the remaining part of this paper.

**Definition 2.1.** For any convex and weakly lower semi-continuous functional $f : \mathcal{D}(f) \subset A \to [0, +\infty)$ let $\partial f(a) \subset A^*$ denote the sub-differential of $f$ at $a \in \mathcal{D}(f)$ and $\mathcal{D}(\partial f(a))$ the domain of its sub-differential

$$\mathcal{D}(\partial f(a)) = \{a \in \mathcal{D}(f) : \partial f(a) \neq \emptyset\}.$$ 

The Bregman distance with respect to $\xi \in \partial f(a)$ is defined on $\mathcal{D}(f) \times \mathcal{D}(\partial f)$ by

$$D_\xi(b, a) = f(b) - f(a) - \langle \xi, b - a \rangle.$$
Definition 2.2. Let \( 1 \leq q < \infty \) and let \( D_\xi(\cdot, a) \) be the Bregman distance of Definition 2.1. The later is said to be \( q \)-coercive if there exists a constant \( c > 0 \) such that
\[
D_\xi(b, a) \geq c\|b - a\|^q,
\]
for every \( b \in D(f) \).

Under Assumption 2.1, it is also possible to prove convergence rates in terms of \( \delta \) and Bregman distances. For a nice review on Bregman distances and convergence rates, we refer to Chapter 3 in \[32\].

Definition 2.3. For any \( p > 1 \) fixed, define the duality mapping as the set-valued function \( J_p : U \to 2^{U^*} \) such that
\[
J_p(u) = \left\{ u^* \in U^* : \langle u^*, u \rangle = \|u^*\|\|u\| \text{ and } \|u^*\| = \|u\|^{p-1} \right\}.
\]

As a consequence of the definition above, \( \langle u^*, u \rangle = \|u\|^p \) and \( \|u^*\| \leq \|u\|^{p-1} \|v\| \)

By Assumption 2.1, since \( U \) is reflexive and has a Fréchet differentiable norm, \( J_p : U \to U^* \) is continuous and invertible with a continuous inverse satisfying
\[
J_p^{-1} = J_{p^*} : U^* \to U^{**} \cong U,
\]
where \( \frac{1}{p} + \frac{1}{p^*} = 1 \).

If \( f(u) = \frac{1}{p}\|u\|_p^p \), then, its Bregman distance with respect to \( J_p(u) \) is denoted by \( \Delta_p(v, u) \). See Chapter II in \[13\] and \[28\].

3. Uniqueness and discrepancy principles. The goal of the present section is to show that uniqueness of minimizers of the Tikhonov functional (3) implies the well-posed of discrepancy-based choices of the regularization parameter \( \alpha \).

When dealing with noisy data, it is desirable to choose the regularization parameter \( \alpha \) appropriately to avoid that any minimizer of the Tikhonov-type functional (3) reproduces noise. This can be done by means of discrepancy principles, i.e., by choosing \( \alpha \) such that it satisfies
\[
\eta_1 \delta \leq \|F(a_{\alpha}^\delta) - u^\delta\| \leq \eta_2 \delta,
\]
with \( 1 < \eta_1 \leq \eta_2 \), as proposed in \[5\].

In general, Tikhonov-type functionals associated to nonlinear operators are not convex, and then, uniqueness of its minimizers does not necessarily hold. As a consequence, the map \( \alpha \mapsto \|F(a_{\alpha}^\delta) - u^\delta\| \) is set-valued and (5) may not hold. More precisely, the sets
\[
M(\alpha, u^\delta) := \{ a_\alpha^\delta \in D(F) : a_\alpha^\delta \in \text{argmin}_F(\alpha, u^\delta ; a) \},
\]
can be such that
\[
\inf_{a_\alpha^\delta \in M(\alpha, u^\delta)} \|F(a) - u^\delta\| < \eta_1 \delta \leq \eta_2 \delta < \sup_{b \in M(\alpha, u^\delta)} \|F(b) - u^\delta\|.
\]

Next, we present a very simple example where such situation holds.

Example 3.1. Assume that \( A = U = \mathbb{R} \), \( F(a) = \sqrt{2 - a^2} \) with \( D(F) = [-\sqrt{2}, \sqrt{2}] \).
Set the measurements \( u^\delta = 0 \) for some noise level \( 0 < \delta < 1 \). Furthermore, assume that \( f_{a_{\alpha}^\delta}(a) = \beta(a - a_{\alpha}^\delta)^2 \), with \( \beta > 0 \) fixed, and \( a_{\alpha}^\delta = 0 \). Whenever \( \alpha = 1/\beta \), it follows that
\[
F_{\alpha, \delta}(a) = \|F(a) - 0\|^2 + \alpha f_0(a) = 2 - a^2 + \alpha \beta a^2 = 2.
\]
Hence, any \( a \) in \( [-\sqrt{2}, \sqrt{2}] \) minimizes the Tikhonov functional with \( \alpha = 1/\beta \), whereas, the range of the same set by the misfit functional

\[
a \mapsto \|F(a) - 0\|^2 = 2 - a^2
\]
is \([0, 2]\). Therefore, it is clear that, for some \( 0 < \delta < 1 \), and for any \( 0 < \eta_1 \leq \eta_2 \leq 1 \), the inequality (6) is satisfied.

On the other hand, if uniqueness of minimizers holds, then, it is always possible to guarantee the existence of \( \alpha \) satisfying (5). See Remark 4.7 in [33]. One of the consequences of the use of discrepancy-based choices appears in the proof of convergence-rate results, where no assumption on the asymptotic behavior of \( \alpha = \alpha(\delta, u^\delta) \) when \( \delta \to 0 \) is needed. See [5, 6, 4, 3].

The next assumption is an extra property on \( F \) that will be used to prove that an element of \( D(F) \) cannot be simultaneously a Tikhonov minimizer with regularization parameter \( \alpha > 0 \) and a solution of the inverse problem (1).

**Assumption 3.1.** Let \( t \in (0, 1] \) and \( a, b \in D(F) \) be arbitrary but fixed. Assume that the operator \( F \) satisfy the estimate

\[
\lim_{t \to 0} \frac{1}{t} \|F(ta + (1 - t)b) - F(b)\|^p_U = 0.
\]

If, for example, the operator \( F \) is Hölder continuous with parameter \( \beta > 1/p \), then, the Assumption 3.1 is satisfied.

The following lemma states that, for any \( \alpha \) taken as a function of \( \delta \), it must go to zero as \( \delta \) goes to zero, whenever \( \alpha \) satisfies the discrepancy principle (7) below, for a given constant \( \tau_2 > 0 \). Otherwise, we could find an \( \tilde{\alpha} > 0 \) such that \( a_{\tilde{\alpha}} \) is the solution of the inverse problem (1) and a Tikhonov minimizer with \( \delta = 0 \) and regularization parameter \( \tilde{\alpha} \), contradicting the hypothesis that \( \|F(a_{\tilde{\alpha}}) - u\| > \tau_2 \delta \). This is one of the necessary conditions to state the convergence of Tikhonov minimizers to some \( f_{a_{\tilde{\alpha}}} \)-minimizing solution of the inverse problem (1).

**Lemma 3.1.** Let Assumptions 2.1 and 3.1 hold. Let also \( 1 < \tau_1 \leq \tau_2 \) be fixed constants and \( \|u^\delta - F(a_{\alpha})\|_U > \tau_2 \delta \). If, for each \( \delta > 0 \), there exists \( \alpha = \alpha(\delta, u^\delta) > 0 \) such that

\[
\|F(\alpha_{\delta}) - u^\delta\|_U \leq \tau_2 \delta,
\]

then, \( \alpha(\delta, u^\delta) \to 0 \), as \( \delta \to 0 \).

**Proof.** Firstly, it is necessary to justify the existence of some \( \alpha > 0 \) satisfying (7). Since the map \( \alpha \mapsto \|F(\alpha_{\delta}) - u^\delta\| \) is a non-decreasing function, by Proposition 3.8 in [5], there are \( 0 < \alpha_- < \alpha_+ < \infty \), such that

\[
\|F(\alpha_{\alpha_-}) - u^\delta\|_U < \|F(\alpha_{\alpha_+}) - u^\delta\|_U.
\]

As a consequence, there exists an \( \alpha \in (\alpha_-, \alpha_+) \) such that the discrepancy in (7) holds.

Now, we shall proceed the proof by contradiction. Consider a sequence \( \{\delta_n\}_{n \in \mathbb{N}} \) such that \( \delta_n \searrow 0 \). So, it is possible to select sequences \( \{\alpha_n\}_{n \in \mathbb{N}} \), and \( \{\alpha_{\delta_n}^{\alpha_n}\}_{n \in \mathbb{N}} \), with \( \alpha_n \) and \( \alpha_{\delta_n}^{\alpha_n} \) satisfying (7), whenever \( \delta_n \) replaces \( \delta \). Consider a subsequence of \( \{\alpha_n\}_{n \in \mathbb{N}} \), if necessary, such that it converges to \( \tilde{\alpha} \). Since the corresponding subsequence of minimizers \( \{a_{\delta_n}^{\alpha_n}\} \) is bounded in \( A \), it has a weakly convergent subsequence \( \{\delta_n^{\alpha_k}\}_{k \in \mathbb{N}} \), with limit \( \tilde{\alpha} \). So, passing again to a subsequence if necessary, by the
weakly continuity of the operator $F$, the weakly lower semi-continuity of the norm, the discrepancy in (7) and since $\|F(a^1) - u^{\delta_n}\| \to 0$ as $\delta_n \to 0$, it follows that
\[
\|F(\tilde{a}) - F(a^1)\| \leq \liminf_{k \to \infty} \|F(a^{\delta_n_k}) - u^{\delta_n_k}\| \leq \lim_{k \to \infty} (\tau_2 \delta_n) = 0.
\]

Assume by contradiction that $\tilde{a} > 0$. So, the weakly lower semi-continuity of the norm and $f_{a,\alpha}$, and the weakly continuity of $F$ imply that
\[
\tilde{a} f_{a,\alpha}(\tilde{a}) = \mathcal{F}_{\alpha,0}(\tilde{a}) \leq \liminf_{k \to \infty} \mathcal{F}_{\alpha_k,\delta_k}(a_{\alpha_k}^{\delta_k}) \leq \liminf_{k \to \infty} \mathcal{F}_{\alpha_k,\delta_k}(a) = \|F(a) - \tilde{u}\|_U^p + \tilde{a} f_{a,\alpha}(a) =: \mathcal{F}_{\alpha,0}(a),
\]
for any $a \in \mathcal{D}(F)$. Hence, $\tilde{a}$ is a minimizer of the Tikhonov-type functional $\mathcal{F}_{\alpha,0}$.

Defining $a(t) = (1 - t)\tilde{a} + ta_*$, by the convexity of $f_{a,\alpha}$ and the fact $f_{a,\alpha}(a_*) = 0$, it follows that $f_{a,\alpha}(a(t)) \leq (1 - t)f_{a,\alpha}(\tilde{a})$. So,
\[
\tilde{a} f_{a,\alpha}(\tilde{a}) \leq \mathcal{F}_{\alpha,0}(a(t)) \leq \|F(a(t)) - \tilde{u}\|_U^p + \tilde{a} f_{a,\alpha}(\tilde{a})
\]
and
\[
\tilde{a} f_{a,\alpha}(\tilde{a}) \leq \frac{1}{t} \|F(a(t)) - F(\tilde{a})\|_U^p,
\]
the latter goes to zero as $t \to 0$, by Assumption 3.1. So, $\tilde{a} = a_*$. By the hypotheses,
\[
\|F(a_*) - F(a^1)\| \geq \|F(a_*) - u^{\delta_k}\| + \|u^{\delta_k} - \tilde{u}\| > (\tau_2 - 1)\delta_k > 0.
\]

So, $\|F(\tilde{a}) - \tilde{u}\| = \|F(a_*) - F(a^1)\| > 0$, which contradicts the fact proved above that $F(\tilde{a}) = \tilde{u}$. Therefore, $\tilde{a} = 0$, and the assertion follows. \qed

By Lemma 3.4 in [5], the map
\[
H(\alpha) := \|F(a_*^{\alpha}) - u^{\delta}\|_U
\]
is non-decreasing. The set of discontinuities of $H$ is defined by:
\[
A_H := \left\{ \alpha > 0 : \inf_{a \in M_\alpha} \|F(a) - u^{\delta}\|_U < \sup_{a \in M_\alpha} \|F(a) - u^{\delta}\|_U \right\},
\]
where $M_\alpha$ is the set of minimizers of $\mathcal{F}_{\alpha,\delta}$ in $\mathcal{D}(F)$. By Lemma 3.6 in [5], $A_H$ is at most countable. Thus, if uniqueness of Tikhonov minimizer holds for $\alpha \in (0, \overline{\alpha})$ and $\delta \in (0, \overline{\delta})$, with $\overline{\alpha}$ and $\overline{\delta}$ given positive constants, then $(0, \overline{\alpha}) \cap A_H = \emptyset$.

The next proposition is the main result of this section, it states that, whenever uniqueness of Tikhonov minimizers hold, for any noise level $\delta > 0$ sufficiently small, it is always possible to find some $\alpha > 0$ satisfying the discrepancy in (5).

**Proposition 3.1.** Let Assumption 2.1 and 3.1 hold. Let also $\|u^{\delta} - F(a_*)\| > \tau_2 \delta$, and the constants $1 < \tau_1 \leq \tau_2$ be fixed. If for every $\delta \in (0, \overline{\delta})$ and $\alpha \in (0, \overline{\alpha})$ uniqueness of Tikhonov minimizers hold, then, there exists a positive constant $\overline{\delta}$, such that for every $\delta \in (0, \overline{\delta})$, there exists some $\alpha \in (0, \overline{\alpha})$ satisfying the Morozov-type discrepancy principle (5).

**Proof.** Firstly, it is necessary to show that, if $\delta$ is sufficiently small, then, it is possible to find an $\alpha$ in $(0, \overline{\alpha})$ satisfying $\|F(a_\alpha^{\delta n}) - u^{\delta n}\| > \tau_2 \delta$. So, let us exhibit the existence of an upper bound $\overline{\delta}$ for the set of noise levels $\delta$ satisfying the condition above. Let $\alpha \in (0, \overline{\alpha})$ be fixed. Suppose by contradiction that it is possible to find a sequence $\{\delta_n\}_{n \in \mathbb{N}}$ converging to zero, such that
\[
\|F(a_\alpha^{\delta n}) - u^{\delta n}\|_U < \tau_2 \delta_n \text{ for every } n \in \mathbb{N}.
\]
Since $\delta_n \to 0$, Lemma 3.1 implies that for a sufficiently large $n$, the inequality in (8) cannot hold. In other words, for any $\alpha \in (0, \overline{\alpha})$, there exists a $\delta = \delta_\alpha > 0$, such
that, $\|F(a_\alpha^\delta) - u^\delta\| > \tau_2\delta$. Define $\delta = \sup_{\alpha \in (0, \overline{\alpha})} \delta_\alpha$. Hence, for any $\delta \in (0, \delta)$, there exists $\bar{\delta} \in (0, \overline{\delta})$, such that $\|F(a_\alpha^\delta) - u^\delta\|_U > \tau_2\delta$.

Assume with no loss of generality that $\delta < \delta'$. Since the map $H(\alpha) \mapsto \|F(a_\alpha^\delta) - u^\delta\|$ is non-decreasing and continuous in $(0, \overline{\delta})$ (the former is a direct consequence of the uniqueness of Tikhonov minimizers), it follows that there exists $0 < \alpha \leq \bar{\delta}$ such that $\tau_1\delta \leq \|F(a_\alpha^\delta) - u^\delta\|_U \leq \tau_2\delta$.

Since, $\delta > 0$ is arbitrary and the constants $\tau_1$ and $\tau_2$, are fixed, the assertion follows.

Note that, by Proposition 3.1, Lemma 3.1 and Theorem 4.11 in [5], if $\alpha = \alpha(\delta, u^\delta)$ satisfies (5) when $\delta \to 0$, then the limits in (4) hold, and every sequence of minimizers $\{a_\alpha^\delta\}$ has a subsequence converging weakly to some element of the set $\mathcal{L}$. This is one of the main advantages of this a posteriori choice of the regularization parameter $\alpha$, i.e., no further assumption on the asymptotic behavior of $\alpha(\delta, u^\delta)$ when $\delta \to 0$ is needed to state a convergence result. However, to achieve convergence rates it is necessary to take into account some kind of source condition [33] as the one below. So, as an illustration, the remaining part of this section is devoted to a convergence-rate result.

**Assumption 3.2.** Let $a^1 \in \mathcal{L}$. There exist $\beta_1 \in [0, 1)$, $\beta_2 \geq 0$, and a concave function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$, such that

$$\langle \xi, a^1 - a \rangle \leq \beta_1 D_\xi(a, a^1) + \beta_2 \varphi(\|F(a) - F(a^1)\|),$$

for $a \in \mathcal{M}_{\alpha_{\max}}(\rho) := \{b \in D(F) : \mathcal{F}_{\alpha_{\max}, \delta}(b) \leq \rho\}$, where $\alpha_{\max}$ and $\rho$ satisfy the relation $\rho > \alpha_{\max}f_{a_\alpha}(a^1)$, and $\xi \in \partial f_{a_\alpha}(a^1)$.

Under the Assumption 3.2, we can prove the following:

**Proposition 3.2.** Let Assumption 3.2 and the hypotheses of Proposition 3.1 hold. If $\delta \to 0$ and $\alpha = \alpha(\delta, u^\delta)$ satisfies the Morozov’s principle (5), then the convergence rates

$$D_\xi(a_\alpha^\delta, a^1) = \mathcal{O}(\varphi(\delta)) \quad \text{and} \quad \|F(a_\alpha^\delta) - F(a^1)\|_U = \mathcal{O}(\delta)$$

hold.

**Proof.** The rate $\|F(a_\alpha^\delta) - F(a^1)\|_U = \mathcal{O}(\delta)$ comes from the discrepancy principle (5) and the triangle inequality.

Since $\alpha$ satisfies (5), it also follows that $\tau_1\delta \delta = \alpha f_{a_\alpha}(a_\alpha^\delta) \leq \mathcal{F}_{\alpha, \delta}(a_\alpha^\delta) \leq \mathcal{F}_{\alpha, \delta}(a^1) = \delta + \alpha f_{a_\alpha}(a^1)$, i.e., $f_{a_\alpha}(a^1) - f_{a_\alpha}(a_\alpha^\delta) > 0$. Thus, using the definition of Bregman distance with respect to $\xi \in \partial f_{a_\alpha}(a^1)$,

$$D_\xi(a_\alpha^\delta, a^1) = f_{a_\alpha}(a_\alpha^\delta) - f_{a_\alpha}(a^1) - \langle \xi, a_\alpha^\delta - a^1 \rangle \leq \langle \xi, a^1 - a_\alpha^\delta \rangle.$$

So, by Assumption 3.2,

$$D_\xi(a_\alpha^\delta, a^1) \leq \langle \xi, a^1 - a_\alpha^\delta \rangle \leq \beta_1 D_\xi(a_\alpha^\delta, a^1) + \beta_2 \varphi(\|F(a_\alpha^\delta) - F(a^1)\|)$$
By the concavity of $\varphi$ and the fact that $\varphi(0) = 0$, the second term in the right-hand side of the last inequality is bounded by

$$\beta_2 \varphi(\|F(a^\delta_\alpha) - F(a^\dagger)\|) \leq \beta_2 \varphi((1 + \tau_2)\delta) \leq (1 + \tau_2)\varphi(\delta).$$

Therefore, rearranging the terms in the inequality and using the above estimate, it follows that

$$D_\xi(a^\delta_\alpha, a^\dagger) \leq \frac{\beta_2}{1 - \beta_1} (1 + \tau_2)\varphi(\delta).$$

\[ \square \]

4. **Uniqueness of Tikhonov minimizers.** The aim of the present section is to show some results concerning uniqueness of Tikhonov minimizers using different techniques. We start by presenting a uniqueness result based on variational techniques. However, this is not sufficient to meet all the hypotheses of Proposition 3.1. Therefore, in order to solve this task, we recall some notions introduced in [12], where a uniqueness result for the so-called weakly nonlinear or finite curvature inverse problems are proposed.

4.1. **A uniqueness result using variational techniques.** Here, we present a generalization to the case of reflexive Banach spaces of the variational technique introduced in [30]. More precisely, we determine a lower bound to the regularization parameter $\alpha$, such that, whenever $\alpha$ is sufficiently larger, uniqueness of Tikhonov minimizers holds.

Before we start, it is necessary to assume some further smoothness properties of the operator $F$.

**Assumption 4.1.**

(i) $F$ is differentiable at each $a \in \mathcal{D}(F)$ in every direction $h \in A$, such that $a + h \in \mathcal{D}(F)$.

(ii) The derivative of $F$ at $a$ can be extended to a bounded linear operator in the space $A$, denoted by $F'(a)$, i.e.,

\[
\|F'(a)h\|_U \leq C\|h\|_A, \forall a \in \mathcal{D}(F) \text{ and } h \in A.
\]

The extended version of the derivative of $F$ also satisfies the Lipschitz condition

\[
\|F'(a + h) - F'(a)\|_{\mathcal{L}(A,U)} \leq \tilde{C}\|h\|_A,
\]

for every $a + h, a \in \mathcal{D}(F)$.

Let $a^\delta_\alpha$ and $\tilde{a}^\delta_\alpha$ be two different Tikhonov minimizers associated to the same regularization parameter $\alpha$ and data $u^\delta$. Since $a^\dagger \in \mathcal{L}$ is not necessarily a minimizer of the Tikhonov-type functional (3), then,

\[
\mathcal{F}_{\alpha,\delta}(\tilde{a}^\delta_\alpha) = \mathcal{F}_{\alpha,\delta}(a^\delta_\alpha) \leq \delta^2 + \alpha f\alpha_\alpha(a^\dagger).
\]

Assuming further that $a^\dagger$, $a^\delta_\alpha$ and $\tilde{a}^\delta_\alpha$ are interior points of $\mathcal{D}(F)$, it follows that,

\[
\rho F'(\tilde{a}^\delta_\alpha)^* J_p(F(\tilde{a}^\delta_\alpha) - u^\delta) + \alpha \tilde{\xi}^\delta_\alpha = 0,
\]

with $\tilde{\xi}^\delta_\alpha \in \partial f\alpha_\alpha(\tilde{a}^\delta_\alpha)$.

By the equation (14), the fact that $J_p(-u) = -J_p(u)$, and the definition of Bregman distances, it follows that

\[ \text{Inverse Problems and Imaging Volume 13, No. 1 (2019), 211–229} \]
\( p \Delta_p(F(a_\alpha^\delta) - u^\delta, F(\tilde{a}_\alpha^\delta) - u^\delta) + \alpha D_{\xi_\delta}(a_\alpha^\delta, \tilde{a}_\alpha^\delta) \)

\[
\begin{align*}
 &= \mathcal{F}_{\alpha, \delta}(a_\alpha^\delta) - \mathcal{F}_{\alpha, \delta}(\tilde{a}_\alpha^\delta) = p(J_p(F(\tilde{a}_\alpha^\delta) - u^\delta), F(a_\alpha^\delta) - F(\tilde{a}_\alpha^\delta)) - \alpha(\xi_\alpha^\delta, a_\alpha^\delta - \tilde{a}_\alpha^\delta) \\
&= p(J_p(u^\delta - F(\tilde{a}_\alpha^\delta)), F(a_\alpha^\delta) - F(\tilde{a}_\alpha^\delta)) + p(J_p(F(\tilde{a}_\alpha^\delta) - u^\delta), F'(\tilde{a}_\alpha^\delta)(a_\alpha^\delta - \tilde{a}_\alpha^\delta)) \\
&= p(J_p(u^\delta - F(\tilde{a}_\alpha^\delta)), F(a_\alpha^\delta) - F(\tilde{a}_\alpha^\delta) - F'(\tilde{a}_\alpha^\delta)(a_\alpha^\delta - \tilde{a}_\alpha^\delta)) .
\end{align*}
\]

Using again that \( a_\alpha^\delta \) and \( \tilde{a}_\alpha^\delta \) are interior points of \( \mathcal{D}(F) \),

\( F(a_\alpha^\delta) - F(\tilde{a}_\alpha^\delta) - F'(\tilde{a}_\alpha^\delta)(a_\alpha^\delta - \tilde{a}_\alpha^\delta) = \\
\int_0^1 (F'(\tilde{a}_\alpha^\delta + t(a_\alpha^\delta - \tilde{a}_\alpha^\delta)) - F'(\tilde{a}_\alpha^\delta))(a_\alpha^\delta - \tilde{a}_\alpha^\delta) \, dt .
\]

By these estimates, we are able to state:

**Proposition 4.1.** Let the Bregman distance \( \mathcal{D}_\xi(a,b) \) be 2-coercive for any \( a, b \in \mathcal{D}(F) \) and \( \xi \in \partial f_{a^\perp}(b) \). Let also the regularization parameter satisfy the nonlinear inequality

\( \alpha \frac{p-1}{p} - \alpha \mathcal{C} f_{a^\perp}(a^\perp) - \mathcal{C} \delta^p > 0 \)

where \( \frac{p-1}{p} = c^{-1} \mathcal{C} \), with \( \mathcal{C} \) the Lipschitz constant in (12) and \( c \) the constant in the definition of \( q \)-coerciveness (Definition 2.2). Then, the Tikhonov-type functional (3) has a unique minimizer in \( \mathcal{D}(F) \).

**Proof.** Let \( a_\alpha^\delta \) and \( \tilde{a}_\alpha^\delta \) be as above. Then, substituting (16) in (15), by the estimate (13) and the Lipschitz condition (12), it follows that

\( p \Delta_p(F(a_\alpha^\delta) - u^\delta, F(\tilde{a}_\alpha^\delta) - u^\delta) + \alpha D_{\xi_\delta}(a_\alpha^\delta, \tilde{a}_\alpha^\delta) \)

\[
\begin{align*}
&= p(J_p(u^\delta - F(\tilde{a}_\alpha^\delta)), \int_0^1 (F'(a_\alpha^\delta + t(a_\alpha^\delta - \tilde{a}_\alpha^\delta)) - F'(\tilde{a}_\alpha^\delta))(a_\alpha^\delta - \tilde{a}_\alpha^\delta) \, dt) \\
&\leq p \|u^\delta - F(\tilde{a}_\alpha^\delta)\|_U^{p-1} \max_{t \in [0,1]} \|F'(a_\alpha^\delta + t(a_\alpha^\delta - \tilde{a}_\alpha^\delta)) - F'(\tilde{a}_\alpha^\delta)\|_A \|a_\alpha^\delta - \tilde{a}_\alpha^\delta\|_A \\
&\leq p \|u^\delta - F(\tilde{a}_\alpha^\delta)\|_U^{p-1} \mathcal{C} \|a_\alpha^\delta - \tilde{a}_\alpha^\delta\|_A^{\frac{p-1}{p} - 1} \|a_\alpha^\delta - \tilde{a}_\alpha^\delta\|^2_A.
\end{align*}
\]

By the 2-coerciveness of \( D_{\xi_\delta}(a_\alpha^\delta, \tilde{a}_\alpha^\delta) \),

\[
p \Delta_p(F(a_\alpha^\delta) - u^\delta, F(\tilde{a}_\alpha^\delta) - u^\delta) + \alpha D_{\xi_\delta}(a_\alpha^\delta, \tilde{a}_\alpha^\delta) \leq c^{-1} p \mathcal{C} (\delta^p + \alpha f_{a^\perp}(a^\perp))^{\frac{p-1}{p}} D_{\xi_\delta}(a_\alpha^\delta, \tilde{a}_\alpha^\delta)
\]

If \( \alpha \) satisfy (17),

\[
\alpha > \mathcal{C} (\delta^p + \alpha f_{a^\perp}(a^\perp))^{\frac{p-1}{p}} .
\]

This implies that

\[
p \Delta_p(F(a_\alpha^\delta) - u^\delta, F(\tilde{a}_\alpha^\delta) - u^\delta) + \alpha D_{\xi_\delta}(a_\alpha^\delta, \tilde{a}_\alpha^\delta) < \alpha D_{\xi_\delta}(a_\alpha^\delta, \tilde{a}_\alpha^\delta),
\]

which is clearly a contradiction and the assertion follows. \( \square \)
Remark 4.1.  (i) When \( p = 2 \), the inequality is quadratic, and \( \alpha \) must simply satisfy
\[
\frac{2}{C} \alpha > f_{\ast}(a^\dagger) + \left( f_{\ast}(a^\dagger)^2 + \frac{4}{C} \delta^2 \right)^{1/2}.
\]
(ii) If \( \delta \) is sufficiently small, say \( \delta^p \ll f_{\ast}(a^\dagger) \), then, the term \( \frac{2}{C} \delta^p \) can be dropped and the inequality in (17) can be replaced by \( \alpha > \left[ \frac{C}{\alpha} f_{\ast}(a^\dagger) \right]^{p-1} \).
(iii) If \( f_{\ast} \) is the square of the norm we get exactly the lower bound in [30].

Remark 4.2. The condition in Equation (17) does not allow to choose an arbitrary \( \alpha > 0 \), which implies that, there may be no \( \alpha > 0 \) satisfying the discrepancy principle (5), for fixed constants \( 1 < \eta_1 \leq \eta_2 \), and for a sufficiently small and positive \( \delta \).

4.2. Geometrical techniques. In [12], the authors introduced geometrical techniques to state uniqueness of Tikhonov minimizers associated to weakly nonlinear problems. The main strategy was to find conditions for the curvature and the torsion of the image of line segments through \( F \) be parameterizable by arclength. So, the image of a given set in the domain of the direct operator was strictly quasi-convex. This was a sufficient condition for the uniqueness of least-square projections of elements in a neighborhood of the image of such given set onto itself.

In what follows we recall some notation and a result from [12] that states the uniqueness of Tikhonov minimizers for the local volatility problem in the next section.

Firstly, let \( A \) and \( U \) be Hilbert spaces, \( p = 2 \), and set \( f_{\ast} \) as
\[
f_{\ast}(a) = \|a - a^\ast\|^2_A.
\]
Let also the assumption below hold.

Assumption 4.2. Let \( g, h \in A \) and \( a \in \mathcal{D}(F) \) be such that, \( a + h \in \mathcal{D}(F) \) and \( a + g \in \mathcal{D}(F) \). So, the second derivative of \( F \) at \( a \) in the directions \( g \) and \( h \) is well defined as a map \( A \times A \to U \), which is denoted by \( F''(a)(g,h) \). Moreover, there exists a constant \( K > 0 \) such that
\[
\|F''(a)(g,h)\|_U \leq K\|g\|_A\|h\|_A.
\]

Given \( a, b \in \mathcal{D}(F) \), the path \( \overline{P} \) linking \( F(a) \) to \( F(b) \) is defined as
\[
(19) \quad \overline{P} : t \in [0,1] \mapsto \overline{P}(t) = F((1-t)a + tb) \in U.
\]
So, \( \overline{P}'(t) = F'((1-t)a + tb)(b-a) \) and \( \overline{P}''(t) = F''((1-t)a + tb)(b-a, b-a) \).

Definition 4.1 (Definition 2.1 in [12]). A path \( P \) in \( W^{2,\infty}([0,L],U) \) is parameterized by arclength (path p.a.) if \( \|P'(t)\|_U = 1 \) for almost every \( t \in [0,L] \).

Definition 4.2 (Definition 2.2 in [12]). Given a set \( \mathcal{D} \subset U \), let \( \mathcal{P} \) be a collection of paths with range in \( \mathcal{D} \). It is called a collection of paths p.a. for \( \mathcal{D} \) if
(i) \( \mathcal{P} \) consists of paths p.a. \( P : [0,\delta(P)] \to \mathcal{D} \).
(ii) \( \mathcal{P} \) is complete, i.e., for every \( u_1, u_2 \in \mathcal{D} \), there exists \( P \in \mathcal{P} \), such that \( P(0) = u_1 \) and \( P(\delta(P)) = u_2 \).
(iii) \( \mathcal{P} \) is stable with respect to restriction, i.e., for all \( P \in \mathcal{P} \) and all \( t, \tilde{t} \in [0,\delta(P)] \), with \( t < \tilde{t} \), the path \( \tilde{P} : t \in [0,\tilde{t} - t] \mapsto P(t + t) \) belongs to \( \mathcal{P} \).

This pair is denoted by \((\mathcal{D}, \mathcal{P})\).
Definition 4.3. Consider the pair \((\mathcal{D}, P)\), and \(P\) in \(\mathcal{P}\).

(i) The smallest radius of curvature along the path \(P\) is given by

\[
R(P) := \inf_{t \in [0, \delta(P)]} \|P''(t)\|^{-1}.
\]

Then, \(R := \inf_{P \in \mathcal{P}} R(P)\) denotes the lower bound to radii of curvature.

(ii) The deflection of \(P\) between \(t\) and \(\tilde{t}\) is given by

\[
\Theta(t, \tilde{t}) = \arccos \langle P'(t), P'(\tilde{t}) \rangle U \in [0, \pi].
\]

Then, \(\Theta(P) = \sup_{t, \tilde{t} \in [0, \delta(P)]} \Theta(t, \tilde{t})\) is the largest deflection along \(P\).

Note that, \(\Theta(P) \leq \delta(P)/R(P)\), for every \(P \in \mathcal{P}\). See [12]. To prove the uniqueness of Tikhonov minimizers it is necessary to define the upper bound of all deflections along paths in \(\mathcal{P}\)

\[
\Theta := \sup_{P \in \mathcal{P}} \Theta(P).
\]

and the quantity

\[
R_G = \begin{cases} 
R, & \text{if } 0 \leq \Theta \leq \pi/2 \\
R \sin \Theta, & \text{if } \pi/2 \leq \Theta \leq \pi,
\end{cases}
\]

whenever \(\Theta < \pi\).

Since in [12] only weakly nonlinear problems are considered, the assumption below guarantees that \(F\) satisfies this hypothesis.

Assumption 4.3. Let \(\mathcal{D}\) be a closed and convex subset of \(\mathcal{D}(F)\). There exists a constant \(0 < \zeta < \infty\) such that

\[
\|F''((1-t)a + tb))(b-a, b-a)\|/\|F'((1-t)a + tb))(b-a)\|^2 \leq \zeta,
\]

for every \(a, b \in \mathcal{D}\) and \(t \in [0,1], a \neq b\).

The following lemma states a sufficient condition for the deflection along the paths \(P\) be less than \(\pi\).

Lemma 4.1. Let Assumption 2.1 hold and the convex subset \(\mathcal{D} \subset \mathcal{D}(F)\) be such that

\[
diam(\mathcal{D}) := \sup_{a, b \in \mathcal{D}} \|b-a\|_A < \frac{\pi}{\zeta C},
\]

where \(C\) comes from (11). Then, \(\Theta < \pi\).

Proof. By the estimate on \(\|P''\|_U\) in Equation (A.12) in [12],

\[
R(P)^{-1} = \sup_{t \in [0, \delta(P)]} \|P''(t)\|_U,
\]

and \(\|P''(t)\|_U \leq \|P''(t)\|_{U'}/\|P'(t)\|_{U'}^\beta \leq \zeta\). Since \(\Theta(P) = \delta(P)/R(P)\), and by (11), \(\delta(P) = \int_0^1 \|P'(t)\|_U dt \leq C\|b-a\|_A\), it follows that \(\Theta(P) \leq \zeta C\|b-a\|_A\).

To see that the assertion follows, just recall that \(\Theta = \sup_{P \in \mathcal{P}} \Theta(P)\), where \(\mathcal{P}\) is a family of paths p.a. for \(F(\mathcal{D})\).

It is also necessary to introduce the following quantities:

\[
R_\alpha = \left(1 + \frac{\alpha}{C^2}\right) R,
\]

(22) \[
\Theta_\alpha = \left(1 + \frac{\alpha}{C^2}\right)^{-1/2} \Theta,
\]
and
\[ R_{G,\alpha} = \begin{cases} R_{\alpha}, & \text{if } 0 \leq \Theta_{\alpha} \leq \pi/2, \\ R_{\alpha} \sin \Theta_{\alpha}, & \text{if } \pi/2 \leq \Theta_{\alpha} \leq \pi. \end{cases} \]

By defining
\[ \text{Rad} = \sup_{x \in D} \|x - x_*\|_A, \]
Theorem 2.8 in [12] states that, if \( D \) satisfy (21), then the equation
\[ d_{\alpha}^2 + \alpha \text{Rad}^2 = R_{G,\alpha}^2 \]
defines a unique \( d_\alpha > 0 \), whenever \( \alpha \in [0, \pi] \), with
\[ \alpha = \begin{cases} +\infty, & \text{if } C \text{Rad} < 2R_G; \\ 2C^2 \left[ 1 + \left( 1 - \frac{4R_G^2}{C^2 \text{Rad}^2} \right)^{1/2} \right]^{-1} - C^2, & \text{if } C \text{Rad} \geq 2R_G. \end{cases} \]

Theorem 2.8 in [12] also implies the following result:

**Theorem 4.1.** Let \( \overline{\alpha} \) be defined as in (24), \( d_\alpha \) be defined as in (23) and \( \Theta < \pi \) hold. Then, for every \( u^b \in U \) such that \( \text{dist}(u^b, F(D)) < d_\alpha \), there exists a unique \( a^b_\alpha \) minimizing the Tikhonov-type functional (3) in \( D \), whenever \( \alpha \in (0, \overline{\alpha}) \), and \( D \) satisfies (21).

Therefore, if the noise level and the objective set are small enough, then the Tikhonov functional (3) has a unique minimizer.

5. **Uniqueness in local volatility calibration.**

The Direct Problem: An European call option is a contract that gives to its owner the right, but not the obligation, of buying one share of its underlying asset by a fixed price, called strike, at its expiration or maturity. This kind of contract incorporates in some sense what the market practitioners believe will happen with the price of the underlying asset. Such expectation is necessary in different activities, such as the evaluation of more complex derivative contracts.

Local volatility models, introduced by Dupire [18] and Derman and Kani [17], form a well-known class of extensions of the classical Black-Scholes model [8]. It describes the evolution of a market asset price by a diffusion process, where the diffusion coefficient, or local volatility surface, is a deterministic function of time and the asset price. This implies that the randomness of the model is determined uniquely by one Brownian motion. Thus, by no-arbitrage arguments, as in the Black-Scholes model, European call option prices are evaluated, under the risk-neutral measure, by a parabolic partial differential equation related to the Fokker-Planck equation, whose diffusion parameter is the local volatility function.

Also by no-arbitrage conditions, we could use market call prices for different strikes and maturities to find the local volatility surface through Dupire’s formula, fitting the market smile. However, it involves differentiation of data, which is an unstable procedure. Instead, we must apply some regularization technique, replacing the original problem by a more stable one. Following this idea, many different techniques were proposed, see, for example, [1, 7, 9, 14, 19, 26, 27] and references therein, and Tikhonov regularization was recurrently used. Many theoretical as well as practical aspects concerning the use of Tikhonov [20, 32] regularization were addressed. Nevertheless, the uniqueness of regularized solution was not extensively studied. In [10], the authors analyze such feature under simplifying assumptions,
considering separately the time- and space- dependency of the local volatility surface.

Since the operator that maps local volatility surfaces onto European call option prices is nonlinear, the resulting Tikhonov functional is not necessarily convex, which means that local uniqueness does not necessarily hold. In what follows, we briefly recall the basic setup of this model and state a sufficient condition for uniqueness of Tikhonov minimizers to hold.

Let the triple \((\Omega, G, Q)\) be a probability space with filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). \(Q\) is the so-called risk neutral probability measure. Denote by \(S_t = S(t, \omega)\), with \(\omega \in \Omega\) and \(t > 0\), the price at time \(t\) of an asset. So, the price at time \(t\) of an European call option with strike \(K > 0\) and maturity \(T \geq t > 0\), on \(S_t\), is given by
\[
C(t, S_t, T, K) = \mathbb{E}^Q \left[ e^{-r(T-t)} \max \{0, S_T - K\} \right]_{\mathcal{F}_t}.
\]

Now, let the price \(S_t\) satisfy the stochastic differential equation below:
\[
dS_t = rS_t dt + \sigma(t, S_t) dW^Q_t, \quad t > 0,
\]
where \(W^Q\) is a \(Q\)-Brownian motion, \(r\) is the annualized risk-free interest rate, and \(\sigma(t, S_t)\) is the local volatility surface, a deterministic function of time and the asset price. Assume also that, at \(t = 0\), the asset price is known, i.e.,
\[
S(0) = S_0 > 0,
\]
for some deterministic constant \(S_0 > 0\).

In [18] it was shown that by fixing the current time and stock price at \(t = 0\) and \(S(t = 0) = S_0\), the call option price as a function of maturity and strike satisfies a parabolic partial differential equation. So, by making the change of variables \(y = \log(K/S_0)\) and \(\tau = T\), and defining
\[
u(\tau, y) := C(\tau, S_0 e^y) \quad \text{and} \quad a(\tau, y) := \sigma^2(\tau, S_0 e^y)/2.
\]
it follows that \(u = u(\tau, y)\) satisfies the following PDE problem:
\[
\begin{cases}
-\nu + a(\nu_y - y) - \nu = 0, & 0 < \tau \leq \tau_{\text{max}}, y \in \mathbb{R} \\
u(\tau = 0, y) = S_0(1 - e^y)^+, & y \in \mathbb{R} \\
\lim_{y \to -\infty} u(\tau, y) = S_0, & 0 < \tau \leq \tau_{\text{max}}, \\
\lim_{y \to +\infty} u(\tau, y) = 0, & 0 < \tau \leq \tau_{\text{max}}.
\end{cases}
\]

Denote by \(D := [0, \tau_{\text{max}}] \times \mathbb{R}\) the domain where (27) is defined. By Corollary A.1 in [19], there exists \(p^* > 2\) such that, for \(2 \leq p < p^*\), the PDE problem (27) has a unique solution \(u(a)\) in \(W^{1,2}_{p,\text{loc}}(D)\).

Let \(a, \pi\) be constants satisfying \(0 < a \leq \pi < \infty\). Let also \(a_0 = a_0(\tau, y)\) be a continuous and differentiable function, such that \(a \leq a_0(\tau, y) \leq \pi\), and its first derivatives are in \(H^1(D)\). Define the set
\[
Q = \{a \in a_0 + H^{1+\varepsilon}(D) : a \leq a \leq \pi\};
\]
with \(\varepsilon > 0\), and the parameter-to-solution map or direct operator
\[
F : Q \subset H^{1+\varepsilon}(D) \rightarrow W^{1,2}_{p}(D), \\
a \mapsto F(a) = u(a) - u(a_0),
\]
with \(2 \leq p < p^*\).
The following result summarizes some properties of $F$:

**Proposition 5.1.**

(i) $F$ is continuous, compact, weakly continuous and weakly closed.

(ii) $F$ is differentiable at $a \in Q$ in every direction $h \in H^{1+\varepsilon}(D)$, such that $a + h \in Q$.

(iii) The derivative of $F$ at $a$, denoted by $F'(a)$, can be extended to a bounded linear operator in $H^{1+\varepsilon}(D)$, i.e., there exists a constant $c > 0$, $c = c(\bar{a}, g, p, r)$, such that

$$\|F'(a)h\|_{W^{-1,2}(D)} \leq c\|h\|_{H^{1+\varepsilon}(D)},$$

for every $h \in H^{1+\varepsilon}(D)$.

(iv) $F'(a)$ also satisfies the Lipschitz condition

$$\|F'(a + h) - F'(a)\|_{C(H^{1+\varepsilon}(D), W^{-1,2}(D))} \leq C\|h\|_{H^{1+\varepsilon}(D)},$$

for every $a, a + h \in Q$.

(v) $F$ is injective and satisfies the local tangential cone condition

$$\|F(a) - F(\tilde{a}) - F'(\tilde{a})(a - \tilde{a})\|_{W^{1,2}(D)} \leq \eta\|F(a) - F(\tilde{a})\|_{W^{1,2}(D)},$$

for every $a, \tilde{a}$ in the open ball $B_r(\alpha^*) \subset Q$, centered at $\alpha^* \in Q$ and radius $\rho$, where $\eta = \eta(\alpha^*) < 1/2$.

For the proof of the items (i)-(iv) of Proposition 5.1, see Theorem 5 in [15], and Propositions A.3, 4.1 and 4.3 in [19]. For the proof of item (v), see Theorem 5 in [16].

The Inverse Problem: The local volatility surface calibration problem can be stated as follows: *given a surface of call option prices $\tilde{u}$, such that $\tilde{u} - u(a_0)$ is in the range of the forward operator, denoted by $R(F)$, find the local volatility surface $a^1 \in Q$ satisfying*

$$u(a^1) = \tilde{u}.$$  

Since $F$ is injective, equation (32) has a unique solution.

As in Section 2, it is only possible to access a noisy measurement of $\tilde{u}$, denoted by $u^\delta$, with $\delta > 0$, $\tilde{u}$ and $u^\delta$ satisfying (2).

**Remark 5.1.** Since $u(a) = F(a) + u(a_0)$, and $a_0$ is hold fixed, it follows that $u'(a) = F'(a)$ and $u(a) - u(\tilde{a}) = F(a) - F(\tilde{a})$. Thus, with no loss of generality, we replace $F(a)$ by $u(a)$ whenever it is possible in what follows. Note also that, in this section, $U = W^{1,2}(D)$ is the space of functions $u = u(\tau, y)$ satisfying

$$\|u\|_{W^{1,2}(D)} = \|u\|_{L^2(D)} + \|u_\tau\|_{L^2(D)} + \|u_y\|_{L^2(D)} + \|u_{yy}\|_{L^2(D)} < \infty,$$

$A = H^{1+\varepsilon}(D)$ and $\mathcal{D}(F) = Q$.

Proposition 5.1 implies that Assumption 2.1 holds in the present context. So, the existence and stability of Tikhonov minimizers hold. Moreover, by Proposition 5.1 it follows that Proposition 4.1 holds, In other words, if $a$ in the functional (3) is sufficiently large, then there exists a unique minimizer of the functional (3). However, this is not sufficient to state the well-posedness of the discrepancy principle (5), and additional hypotheses are required to prove that Theorems 4.1 and Proposition 3.1 hold for the local volatility calibration problem.
5.1. **Finite curvature in option pricing.** Given \( a \in Q \) and \( h \in H^{1+\varepsilon}(D) \), let \( v = u'(a)h \) be the derivative of \( F \) at \( a \) in the direction \( h \). So, \( v \) is the solution in \( W^2_1(D) \) of the PDE

\[
-\nu + a(v_{yy} - v_y) - rv = h(u_{yy} - u_y), 0 < \tau \leq T, \; y \in \mathbb{R},
\]

with homogeneous boundary conditions.

Consider \( g, h \in H^{1+\varepsilon}(D) \), such that \( a + h, a + g \in Q \). Define \( v^h := u'(a)h \) and \( v^g := u'(a)g \) as above. If \( w \in W^{1,2}_2(D) \) is the solution of

\[
-\nu + a(w_{yy} - w_y) - rw = h(v^g_{yy} - v^g_y) + g(v^h_{yy} - v^h_y), 0 < \tau \leq T, \; y \in \mathbb{R},
\]

with homogeneous boundary conditions, then \( w \) is the second derivative of \( F \) in the directions \( h \) and \( g \). This is the assertion of the next proposition.

**Proposition 5.2.** Let \( a \in Q \) be fixed and \( g, h \in H^{1+\varepsilon}(D) \) be such that \( a + h, a + g \in Q \). Then, the solution of the PDE problem (34) is the second derivative of \( F \) in the directions \( h \) and \( g \). Moreover, this second derivative can be extended to a bounded bilinear operator in \( H^{1+\varepsilon}(D) \times H^{1+\varepsilon}(D) \). In other words, there exists a constant \( c > 0 \), depending only on \( \overline{a}, g, p \) and \( r \), such that

\[
\|F''(a)(g, h)\|_{W^{1,2}_1(D)} \leq c\|h\|_{H^{1+\varepsilon}(D)}\|g\|_{H^{1+\varepsilon}(D)},
\]

for every \( g, h \in H^{1+\varepsilon}(D) \).

**Proof.** Let \( a \in Q \) be fixed and \( g, h \in H^{1+\varepsilon}(D) \) be such that \( a + h, a + g \in Q \). Consider, \( v^h(a) \) and \( v^h(a + \varepsilon g) \) solutions of (33) with the diffusion coefficient \( a \) and \( a + \varepsilon g \), respectively, where \( 0 < \varepsilon < 1 \) is fixed. Consider also the solution \( w \) of the PDE problem (34).

Denoting by \( w^\varepsilon := \varepsilon^{-1}(v^h(a + \varepsilon g) - v^h(a)) \), it follows by the linearity of (33) that \( w^\varepsilon \) is the solution of

\[
-\nu + a(w_{yy}^\varepsilon - w_y^\varepsilon) - rw_{yy}^\varepsilon = h(v^\varepsilon_{yy} - v^\varepsilon_y) - g(v^h(a + \varepsilon g)_{yy} - v^h(a + \varepsilon g)_y),
\]

with homogeneous boundary conditions, where \( v^\varepsilon := \varepsilon^{-1}(u(a + \varepsilon g) - u(a)) \). The existence and uniqueness of solution of (35) in \( W^{1,2}_p(D) \), with \( 2 \leq p < p^* \), follows by Proposition A.1 in [19].

By the linearity of the PDE problems in (35) and (34), it follows that \( \tilde{w} = w^\varepsilon - w \) is the solution of

\[
-\nu + a(\tilde{w}_{yy} - \tilde{w}_y) - r\tilde{w}_y = h((v^\varepsilon - v^g)_{yy} - (v^\varepsilon - v^g)_y) - g((v^h(a + \varepsilon g) - v^h)_y - (v^h(a + \varepsilon g) - v^h)_y),
\]

with homogeneous boundary conditions. Again, the existence and uniqueness of solution of (35) in \( W^{1,2}_p(D) \), with \( 2 \leq p < p^* \), follows by Proposition A.1 in [19]. In addition, there exists a positive constant \( C \) depending on \( p \in [2, p^*) \), \( a, g \) and \( r \), such that

\[
\|w^\varepsilon - w\|_{W^{1,2}_p(D)} = \|\tilde{w}\|_{W^{1,2}_p(D)} \leq C\|h((v^\varepsilon - v^g)_{yy} - (v^\varepsilon - v^g)_y)\|_{L^p(D)} + \|g((v^h(a + \varepsilon g) - v^h)_y - (v^h(a + \varepsilon g) - v^h)_y)\|_{L^p(D)}.
\]

The Hölder inequality (Corollary 2.5 in [2]) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), implies

\[
\|h((v^\varepsilon - v^g)_{yy} - (v^\varepsilon - v^g)_y)\|_{L^p(D)} \leq \|h\|_{L^{p_1}(D)}\|v^\varepsilon - v^g\|_{W^{1,2}_2(D)}.
\]
Similarly, it follows that
\[
\|g((v^h + \epsilon g) - v^h)_{yy} - (v^h(a + \epsilon g) - v^h)_{yy}\|_{L^p(D)} \leq \|g\|_{L^p(D)}\|v^h(a + \epsilon g) - v^h\|_{W^{1,2}_p(D)}.
\]
By the Sobolev embedding (case B of Theorem 4.2 in [2]), there exists a constant \( \tilde{c} > 0 \) such that
\[
\|h\|_{L^p(D)} \leq \tilde{c}\|h\|_{H^{1+\epsilon}(D)} \quad \text{and} \quad \|g\|_{L^p(D)} \leq \tilde{c}\|g\|_{H^{1+\epsilon}(D)}.
\]
The Lipschitz condition in (30) implies that
\[
\|v^h(a + \epsilon g) - v^h\|_{W^{1,2}_p(D)} \to 0 \quad \text{when} \quad \epsilon \to 0.
\]
Since \( F \) is differentiable at \( a \) in the direction \( g \), it follows that
\[
\lim_{\epsilon \to 0} \|v^\epsilon - v^g\|_{W^{1,2}_p(D)} = 0.
\]
Therefore, \( \|w^\epsilon - w\|_{W^{1,2}_p(D)} \to 0 \) when \( \epsilon \to 0 \), i.e., \( w \), solution of (34) is the second derivative of \( F \) at \( a \in q \), in the directions \( h, g \in H^{1+\epsilon}(D) \).

Proposition A.1 in [19] and the Hölder inequality imply that
\[
\|w\|_{W^{1,2}_p(D)} \leq C\|h\|_{L^p(D)}\|v^g\|_{W^{1,2}_p(D)} + \|g\|_{L^p(D)}\|v^h\|_{W^{1,2}_p(D)}.
\]
By the estimate in (29) combined with the Sobolev embedding (case B of Theorem 4.2 in [2]), there exists a constant \( c > 0 \), depending only on \( \overline{\sigma}, \sigma, \rho \) and \( r \), such that
\[
\|w\|_{W^{1,2}_p(D)} \leq c\|h\|_{H^{1+\epsilon}(D)}\|g\|_{H^{1+\epsilon}(D)}.
\]
Now, it is easy to see that the PDE problem (34) is well-posed in \( W^{1,2}_p(D) \), with \( p \in [2, p^*] \), for every \( g, b \in H^{1+\epsilon}(D) \). So, the assertion follows.

Proposition 5.2 implies that Assumption 4.2 holds. It remains to prove that Assumption 4.3 hold in order to apply Theorem 4.1. However, it is not necessarily true if \( D(F) = Q \) and stronger assumptions are needed as we shall see.

**Lemma 5.1.** If the path \( \overline{P}(t) \) is defined as in (19) with \( a, b \in Q \), then there exists a constant \( c = c(a, b) > 0 \) such that
\[
\|\overline{P}(t)\|_{W^{1,2}_p(D)} \geq c > 0,
\]
for every \( t \in [0, 1] \).

**Proof.** By taking \( L^p(D) \) norm on both sides of (33), the left hand side can be dominated by a constant \( C_4 \), that depends only on \( \overline{\sigma}, \rho \) and \( r \), times \( \|v\|_{W^{1,2}_p(D)} \), i.e.,
\[
C_4\|v\|_{W^{1,2}_p(D)} \geq \|h(u_{yy} - u_y)\|_{L^p(D)}.
\]
This implies that
\[
\|\overline{P}(t)\|_{W^{1,2}_p(D)} \geq \frac{1}{C_4} \|(b-a)(u_{yy} - u_y)\|_{L^p(D)},
\]
where \( u \) is the solution of (27) with \( (1-t)b + ta \) in the place of \( a \). Note that, \( \omega := u_{yy} - u_y > 0 \), since it is the fundamental solution of
\[
-\omega + (\partial_y^2 - \partial_y)[((1-t)b + ta)\omega] + r\omega_y = 0
\]
with homogeneous boundary condition. See the proof of Lemma 6 in [15], the proof of Theorem 4.1 in [19], or Corollary 1 in Appendix in [23]. Thus, by the continuity of $\|\overline{v}(t)\|$ on $[0,1]$ the assertion follows.

**Remark 5.2.** Since $y = \log(K/S_0)$, $u_{yy} - u_y = K^2C_{K^2}$, Lemma 5.1 is related to the so-called static no-arbitrage conditions, which imposes some restrictions in the shape of the call option price surface, also implying that we must have $K^2C_{K^2}(\tau, K) > 0$ for $K > 0$. See page R100 in [10] and pages 4-6 in [11].

By the estimate 38,

$$\frac{\|\overline{P}''(t)\|_{W^{1,2}_p(D)}}{\|\overline{P}'(t)\|^2_{W^{1,2}_p(D)}} \leq C_6 \frac{\|(b-a)(v_{yy}-v_y)\|_{L^p(D)}}{\|(b-a)(u_{yy}-u_y)\|_{L^p(D)} \|v\|_{W^{1,2}_p(D)}}$$

with $u = \overline{P}(t)$ and $v = P'(t)$.

Denote by $\overline{Q}$ the set of $a, b \in Q$ satisfying the following Hölder condition with parameter $\beta$:

$$|a(\tau, y) - a(\tau', y)| \leq K(|\tau - \tau'|^{\beta/2} + |y - y'|^{\beta}),$$

for every $(\tau, y), (\tau', y') \in D$. Let $D_0$ be a compact subset of the interior of $D$, and define the set $\Gamma_{D_0}$, of elements $a, b \in \overline{Q}$ such that $a - b$ has support contained in $D_0$. So, by the Corollary 1 in Appendix in [23], there exists a constant $C > 0$, independent of $a$ and $b$, such that $\inf_{(\tau, y) \in D_0} (u_{yy} - u_y) \geq C$, which imply that

$$\|(b-a)(u_{yy} - u_y)\|^2_{L^p(D)} \geq C\|(b-a)\|^2_{L^p(D)}.$$

Hence,

$$\frac{\|(b-a)(v_{yy} - v_y)\|_{L^p(D)}}{\|(b-a)(u_{yy} - u_y)\|_{L^p(D)} \|v\|_{W^{1,2}_p(D)}} \leq \frac{b-a}{C\|b-a\|_{L^p(D)}} \frac{v_{yy} - v_y}{\|v\|_{W^{1,2}_p(D)}} \leq \frac{\|b-a\|_{L^\infty(D_0)}}{C\|b-a\|_{L^p(D)}} \frac{\|v_{yy} - v_y\|_{L^p(D)}}{\|v\|_{W^{1,2}_p(D)}} \leq \frac{\|b-a\|_{L^\infty(D_0)}}{C\|b-a\|_{L^p(D)}}.$$

To prove that Assumption 4.3 holds, it is necessary to show that $\|b-a\|_{L^\infty(D)}/\|b-a\|_{L^p(D)}$ is uniformly bounded in some subset of $\Gamma_{D_0}$. When $p = 2$, it is possible to assume that for every $a, b$, the Fourier transform of $b-a$ has support inside some fixed compact set, implying that $\|b-a\|_{H^{1+\varepsilon}(D)}/\|b-a\|_{L^2(D)}$ is uniformly bounded. Another possibility is by imposing that $b-a$ is in some fixed finite-dimensional subspace of $H^{1+\varepsilon}(D)$. We choose the second option since it has a strong connection with the approach presented in [4, 3], where the authors propose a simultaneous discrepancy-based choice for the level of discretization in the domain and the regularization parameter in Tikhonov-type regularization. Therefore, we can state the following results:

**Proposition 5.3.** Let $D_0$ be a compact subset of the interior of $D$. Consider also the restriction of $F$ to the set $\Gamma_{D_0}$, of elements $a, b \in \overline{Q}$ such that $a - b$ is in some fixed finite-dimensional subspace of $H^{1+\varepsilon}(D)$ and has support contained in $D_0$. Then, Assumption 4.3 hold.

Under the hypotheses of Proposition 5.3, Theorem 4.1 holds, and there exists a unique local volatility surface minimizing the Tikhonov functional (3), whenever the objective set and the noise level in price data are small enough. Another consequence is that, under the same conditions, the discrepancy principle (3) is
well-posed, i.e., Proposition 3.1 holds. Moreover, the discrepancy in (5) can also be used in the simultaneous selection of the discretization level in the domain and the regularization parameter. See [4, 3].

Concerning convergence-rates, note that the operator \( F \) satisfies the approximate source condition

\[
a - a_\ast = F'(a)^* w + r,
\]

with \( w \in L^2(D)^* \) and \( r \in H^{1+\varepsilon}(D)^* \) with \( \|r\| \) arbitrarily small. By Lemma 16 in [15], the variational source condition (9) holds with \( \varphi(x) = x \), so, the convergence rates in Proposition 3.2 are also valid in the present context.

6. Concluding remarks. The aim of this article was to study the connection between the well-posedness of discrepancy principles and the uniqueness of reconstructions in Tikhonov-type regularization. Naturally, we started by presenting variational techniques, and we found a positive lower bound for regularization parameters to uniqueness of Tikhonov minimizers to hold. Unfortunately, it is not sufficient to establish the referred connection. To solve this task, we had to recall the geometrical approach presented in [12], where, under restrictions in the diameter of the objective set and the noise level size, Tikhonov-type functionals associated to weakly nonlinear operators have unique minimizers.

Whenever the uniqueness of minimizers hold, we have proved that the discrepancy principle (5) is well-posed and then, they can be used as a selection criteria of the regularization parameter in Tikhonov-type regularization.

To apply the above results to the local volatility surface calibration problem, we proved sufficient conditions for the corresponding direct map to be weakly nonlinear.

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