Couplings between Majorana bound states mediated by topologically trivial chains

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We study the coupling between Majorana bound states (CMBS) mediated by a topologically trivial chain in the absence/presence of pairing couplings at the boundaries. Both situations of short-range and long-range interactions in the chain are considered. Our results show that CMBS can be enhanced by the pairing coupling and long-range interaction in the trivial chain. With a driving field applied to the chain, we calculate the dependence of CMBS on the frequency and amplitude of the driving field. Discussion on the application of the tunable CMBS in quantum computation is made.

I. INTRODUCTION

Topological quantum computation\textsuperscript{[1, 2]} is immune to certain types of noise. It has attracted much attention since it was proposed and becomes active again in recent years due to the progress in experiments. The gates used for quantum computation can be conducted by creating quasi-particles, braiding them and measuring their states. Two well-known types of quasi-particles are Fibonacci anyons and Ising anyons (Majorana bound states). The former are capable of offering universal topological quantum computation, while the latter can not form an universal set of gates by braiding operation only. So the non-topologically protected gates have to be introduced in the computation. This type of gate always requires coupling between Majorana bound states (CMBS). Although there is great progress in theories \textsuperscript{[3–10]} and experiments \textsuperscript{[11, 16]}, the question how to couple two MBSs is barely explored.

Recently, Schmidt and his co-workers \textsuperscript{[17, 18]} presented proposals to couple MBSs by putting the system into a microwave cavity, where the microwave field can effectively drive population transfer between MBSs. The authors found that if the microwave frequency approaches the band gap of the topologically trivial region, the coupling between the MBSs is exponentially enhanced. In other words, by modulating microwave frequency or changing number of photons in the cavity, the CMBS can be controlled. Inspiring by this idea, in this work, we study how the central chain affects CMBS. We realize the universal Majorana qubit rotation (UMQR) by controlling the tunable local gate voltage (TLGV).

The paper is organized as follows. In Sec. \textsuperscript{II} we first briefly review the Kitaev model and calculate CMBS in presence of pairing coupling of the central chain itself. In Sec. \textsuperscript{III} we explore the performance of CMBS when there exist pairing couplings at the boundaries. In Sec. \textsuperscript{IV} we extend our calculations to the central chain with long-range interactions. In Sec. \textsuperscript{VI} we develop a scheme to control CMBS by modulating the amplitude or the frequency of the driving field. Sec. \textsuperscript{VII} is devoted to discussions and conclusions.

II. COUPLING INDUCED BY A TOPOLOGICALLY TRIVIAL CHAIN

We here consider an inhomogeneous Kitaev chain \textsuperscript{[19]} which can be divided into three homogeneous parts. The total Hamiltonian of the whole chain reads

\[ H_{total} = H_l + H_c + H_r + H_{lc} + H_{rc}, \]

where $H_l$ ($H_r$) denotes the Hamiltonian of left (right) chain with sites from $-N_1$ ($N + 1$) to 0 ($N_2$), and $H_c$ represents the Hamiltonian of central chain with sites from 0 to $N$. $H_{lc}$ ($H_{rc}$) denotes the Hamiltonian of the coupling between the left (right) chain and the central chain. To be specific the Hamiltonian of the three homogeneous Kitaev chains can be expressed as

\[ H_\nu = \sum_{n=M_\nu}^{M_\nu'} \mu_\nu a_n^\dagger a_n - \sum_{n=M_\nu}^{M_\nu'-1} (t_\nu/2) a_n^\dagger a_{n+1} \]

\[ + \Delta_\nu e^{i\phi_\nu} a_n^\dagger a_{n+1} \frac{1}{2} + h.c., \]

where $a_n$ and $a_n^\dagger$ are the spinless fermion creation and annihilation operators at site $n$ with chemical potential $\mu_\nu$ ($\nu = l, c, r$). $M_\nu = \{-N_1, 0, N + 1\}$ and $M_\nu' = \{-1, N, N_2\}$ label the beginning and end sites of the left, central, and right chain, respectively. $t_\nu$ and $\Delta_\nu e^{i\phi_\nu}$ are the hopping and pairing amplitudes, respectively. Since the phase of the pairing amplitude can be removed from the Hamiltonian by a gauge transformation, we here and hereafter assume both $t_\nu$ and $\Delta_\nu$ to be real.

Physically, the Kitaev model can be exactly mapped into a spin-$\frac{1}{2}$ chain with XY interactions by Jordan-Wigner transformation \textsuperscript{[20]} and can be realized in the semiconductor nanowire proximity coupling to a s-wave superconductor \textsuperscript{[21, 22]}. This paradigm model shows

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plentiful fascinating topological properties and there are two distinct phases, i.e., the topologically trivial and non-trivial phases. The critical points lie at $\mu_i/t_v = 1$ and $\Delta_r = 0$.

By choosing $\mu_l = \mu_r = 0$ (the chemical potential can be modulated by TLGV $V_i$ shown in Fig. 1) and $\Delta_l = \Delta_r = t_l = t_r$ for the left and right chains, both chains are in the topologically non-trivial phase. As a result there exist MBSs at the ends of both chains, depicted by the stars in Fig. 1. One can define Majorana operators
\[ \gamma_n = a_n + a_n^\dagger, \quad \gamma'_n = i(a_n^\dagger - a_n), \quad n = -N_1, ..., N_2. \]  
After substituting them into Eq. (2), the Hamiltonian of left (right) chain becomes
\[ H_\nu = \frac{i\nu}{2} \sum_{n=\nu}^{\nu+M'-1} \gamma_n \gamma'_{n+1}, \quad \nu = l, r. \]  
Clearly, the Majorana operators $\gamma_n$ and $\gamma'_{n+1}$ are coupled with different sites. In particular the Majorana operators $\gamma'_{N_1} = i(a^\dagger_{N_1} - a_{-N_1})$, $\gamma_{-1} = a_{-1} + a_{-1}^\dagger$, $\gamma'_{N_2} = i(a^\dagger_{N_2} - a_{N_2+1})$, and $\gamma_{N_2} = a_{N_2} + a_{N_2}^\dagger$ do not appear in Eq. (4), which indicate there are four MBSs in our system. Since the four MBSs are decoupled to the remaining sites and only locate at the end of both left and right chains, we can ignore the Hamiltonian of the remaining sites when we study CMBS.

![Fig. 1: Schematic illustration for realizing universal Majorana qubit rotation in the Kitaev chain. There is a direct relationship between the Kitaev model and the solid state system in the low-density limit (for details, see Refs. 13-23). The braiding operation can be realized in the T-junction on the left chain by adiabatically controlling TLGV $V_i$, and the CMBS can be modulated by the voltage $V_c$.](image)

Note that to couple MBSs one should choose the parameters \{$\mu_c, t_c, \Delta_c$\} to make sure the central chain is in the topologically trivial phase. Otherwise the MBSs $\gamma_{-1}$ and $\gamma'_{N+1}$ would disappear in this inhomogeneous chain. As we just consider the nearest-neighbour interactions in Eq. (2), only the MBSs $\gamma_{-1}$ and $\gamma'_{N+1}$ can be coupled to the central chain. So the Hamiltonian of interest is given by
\[ H_i = H_c + H_{ic} + H_{rc}, \]
\[ H_{ic} = -\frac{t_1}{2} \gamma_{-1} \gamma^\dagger_{0}, \]
\[ H_{rc} = -\frac{it_2}{2} (a_N + a_N^\dagger) \gamma'_{N+1}. \]

In order to obtain the energy spectrum of the central chain, it is convenient to carry out Fourier transform $a_n = \frac{i}{\sqrt{N+1}} \sum_k a_k e^{ikn}$ by imposing periodic boundary condition (we have set the lattice spacing to be unit and Fourier transform will be precise when the number of sites is very large, $N \sim \infty$). The validity of Fourier transform is also verified by the numerical calculations in Fig. 2. The Hamiltonian of the central chain becomes
\[ H_c = \sum_k \left( \mu_c - t_c \cos k \right) a_k^\dagger a_k - i \Delta_c \sin k a_k^\dagger a_{-k}^\dagger + h.c. \]  
One can rewrite the above Hamiltonian in a normal Bogoliubov-de Gennes (BdG) form by defining a two component operator $A^\dagger_k = [a_k^\dagger, a_{-k}^\dagger]$, i.e.,
\[ H_c = \sum_k A_k^\dagger H_k A_k, \quad H_k = \begin{pmatrix} h_z & -ih_y \\ ih_y & -h_z \end{pmatrix}, \]  
where $h_z = \mu_c - t_c \cos k$ and $h_y = \Delta_c \sin k$. Consequently the Hamiltonian is simplified as (up to a constant)
\[ H_c = \sum_k E_k b_k^\dagger b_k, \]
where the quasi-particle operator $b_k$ is defined as $b_k = u_k a_k - v_k a_{-k}^\dagger$ and the energy spectrum of the central chain is given by $E_k = \sqrt{(\mu_c - t_c \cos k)^2 + \Delta_c^2 \sin^2 k}$, $|u_k| = \sqrt{\frac{1}{2} (1 + \frac{\mu_c - t_c \cos k}{E_k})}$, $|v_k| = \sqrt{\frac{1}{2} (1 - \frac{\mu_c - t_c \cos k}{E_k})}$. Using the above notations, we can rewrite the Hamiltonian of interest more explicitly as
\[ H_i = \sum_k E_k b_k^\dagger b_k - \sum_k \frac{t_1}{2} \gamma_{-1} [(u_k + v_k) b_k^\dagger - (u_k^* + v_k^*) b_k] - \sum_k \frac{t_2}{2} \left[ e^{-ikN} (u_k - v_k) b_k^\dagger + e^{ikN} (u_k^* - v_k^*) b_k \right] \gamma'_{N+1}. \]

Since we are particularly interested in the low-energy dynamics of system, we apply Schrieffer-Wolff transformation \[25\] to eliminate high-energy spectrum. We denote the first term in Eq. (10) by $H_0$ and the remaining terms by $H_1$. By choosing the unitary transformation $S$ to satisfy the relation $[H_0, S] = H_1$, one has
\[ S = \sum_k \frac{t_1 (u_k + v_k) \gamma_{-1} b_k - t_1 (u_k^* + v_k^*) b_k^\dagger \gamma_{-1}}{2E_k} + \sum_k \frac{t_2 e^{-ikN} (u_k - v_k) \gamma'_{N+1} b_k - t_2 e^{ikN} (u_k^* - v_k^*) b_k^\dagger \gamma'_{N+1}}{2E_k}. \]  
Making use of the Baker-Hausdorff formula
\[ e^{-S} H_c e^S = H_i + [H_i, S] + \frac{1}{2} [[H_i, S], S] + \cdots, \]
and keeping the terms up to first order, we obtain an effective Hamiltonian between the adjacent MBSs $\gamma_{-1}$ and $\gamma'_{N+1}$, i.e.,

$$H_{0N} = i\epsilon \gamma_{-1} \gamma'_{N+1}, \quad \epsilon = \frac{t_1 t_2}{\sqrt{\mu_c^2 + \Delta_c^2}} e^{-N/\xi_0}. \quad (12)$$

Here $\epsilon_0$ is the coherence length and $\epsilon_0^{-1} = \ln \frac{\Delta_c^2 + t_c^2 - \mu_c}{\Delta_c^2 - t_c^2 - \mu_c}$. Eq. (12) demonstrates that the effective CMBS can be induced by the central chain. Physically, $\epsilon$ represents an amplitude for the MBSs to tunnel across the central chain. This is a virtual co-tunneling process since there exists an energy gap in the central chain and real electrons and holes cannot tunnel from one MBS to another. In addition, the coupling strength $\epsilon$ depends exponentially on the length $N$, which is the reason why the MBSs are topologically protected. In Fig. 2 we plot the coupling strength $\epsilon$ versus the chemical potential $\mu_c$ and the pairing amplitude $\Delta_c$ of the central chain. One can find that the analytical solutions are in agreement with the numerical solutions when the chemical potential is much larger than the hopping amplitude. It is also readily observed that the coupling strength $\epsilon$ decreases with the increasing of chemical potential $\mu_c$ and is enhanced by the pairing amplitude $\Delta_c$ of the central chain, reminiscent of the results where the Majorana qubit setup is placed in a microwave cavity [18]. Physically, it originates from the fact that the chemical potential associates with the energy gap, and the pairing amplitude is related to the feature of the energy spectrum (cf. $E_k$ in Eq. (5)). As we know, a larger chemical potential $\mu_c$ gives rise to a larger energy gap. The larger energy gap would prohibit the electron co-tunneling, leading to a weak CMBS. On the other hand, a larger pairing amplitude $\Delta_c$ gives rise to a wider energy spectrum, making more quasi-particles participate in the cotunneling process. As a result the coherence length $\xi_0$ also be large, which can be found in Fig. 2(d). Therefore the CMBS gets strong with large pairing amplitude $\Delta_c$.

III. COUPLING INDUCED BY THE TRIVIAL CHAIN WITH PAIRING COUPLING AT THE BOUNDARIES

In this section we will study CMBS when the central chain exists the pairing coupling at the boundaries. The Hamiltonian of interest then becomes

$$H_i = H_c + H_{lc} + H_{rc},$$

$$H_{lc} = -\frac{t_1}{2} \gamma_{-1}(a_0 - a_0^\dagger) - \frac{\Delta_1}{2}(a_{-1}^\dagger a_0^\dagger - a_{-1} a_0),$$

$$H_{rc} = -\frac{t_2}{2} (a_N + a_N^\dagger) \gamma'_{N+1} - \frac{\Delta_2}{2} (a_N^\dagger a_{N+1} - a_N a_{N+1}),$$

(13)

where $H_c$ is the same as in Eq. (5). By using the Majorana operators representation in Eq. (5), the Hamiltonian can be written as,

$$H_{lc} = -\frac{t_1 + \Delta_1}{2} \gamma_{-1}(a_0 - a_0^\dagger) + \frac{i \Delta_1}{2} \gamma'_{-1}(a_0 + a_0^\dagger),$$

$$H_{rc} = -\frac{i(t_2 + \Delta_2)}{2} (a_N + a_N^\dagger) \gamma'_{N+1} + \frac{\Delta_2}{2} (a_N - a_N^\dagger) \gamma_{N+1}.$$  

(14)

The second term in the Hamiltonian $H_{lc}$ ($H_{rc}$) can be ignored if we only consider CMBS, but this term affects the spatial distribution of the MBSs $\gamma_{-1}$ and $\gamma'_{N+1}$. It is interesting to find that the pairing coupling is equivalent to adjust the effective hopping coupling between the central chain and the left (right) chain with an effective hopping $\frac{t_1 + \Delta_1}{2}$ ($\frac{t_2 + \Delta_2}{2}$).

Fig. 3 demonstrates the coupling strength $\epsilon$ as a function of pairing amplitude in the central chain. It can be observed that the analytical solutions are in agreement with the numerical solutions when $\mu_c$ is large enough, and the coupling strength $\epsilon$ can reach an relatively high value even though the central chain is long. Note that CMBS would in turn modify the spatial distribution of the MBSs $\gamma_{-1}$ and $\gamma'_{N+1}$. Especially, the stronger the coupling strength is, the larger modification the spatial distribution will be. So the coupling strength should not be very large for this consideration. Noting that it would cost long time to perform the operation $U_{0N}(t)$ (cf. Eq. (20)) when the coupling strength is too weak, one should trade off those factors to choose a reasonable coupling strength to implement universal Majorana
qubit rotation (UMQR).

![Image](image.png)

**FIG. 3:** The coupling strength $\epsilon$ as a function of the pairing amplitude $\Delta_c$ with the different number of sites. (a) $N = 5$. (b) $N = 10$. (c) $N = 15$. The solid line and dot line denote the exact numerical solutions (without periodic boundary condition) and the approximate analytical solutions, respectively. The hopping amplitude $t_1t_2$ are revised as $(t_1 + \Delta_1)(t_2 + \Delta_2)$. All parameters are chosen in units of hopping amplitude in the central chain. $\Delta_1 = \Delta_2 = 5$. (d) The exact numerical solutions of coupling strength $\epsilon$ versus the pairing amplitude $\Delta_c$ when $\mu_c = 2$. $\Delta_1 = \Delta_2 = 1$.

**IV. COUPLING INDUCED BY THE TRIVIAL CHAIN WITH LONG-RANGE INTERACTIONS**

In this section we turn to study CMBS mediated by trivial chains with long-range interactions. The Hamiltonian is described by

$$H_c = \sum_{n=0}^{N} \mu_c a_n^\dagger a_n - \sum_{m=1}^{2} \sum_{n=0}^{N-m} \left( \frac{t_{cm}}{2} a_n^\dagger a_{n+m} + \frac{\Delta_{cm}}{2} a_n^\dagger a_{n+m} + h.c. \right).$$

That is, the next-to-nearest neighbor interactions are added in the central chain. With the help of Raman laser, this model can be realized by the system of trapped fermi atoms in optical lattice with the zigzag structure coupling to a 3D BEC reservoir, as shown in Fig. 4. The relative strength of hopping amplitudes $t_{c1}$ and $t_{c2}$ can be modulated by changing the zigzag geometry. For details, we refer the readers to [26, 27].

The Hamiltonian of the hopping coupling between the left (right) chain and the central chain becomes

$$H_{c} = -\frac{t_1}{2} \gamma_{-1}(a_0 - a_0^\dagger) - \frac{t_2}{2} \gamma_{1}(a_1 - a_1^\dagger),$$

$$H_{rc} = -\frac{it_2}{2} (a_N + a_N^\dagger) \gamma_{N+1} - \frac{it_2}{2} (a_{N-1} + a_{N-1}^\dagger) \gamma_{N+1}.$$  (16)

Following the similar derivation procedures in Eqs. [5, 12], one can estimate the effective Hamiltonian of CMBS,

$$H'_{0N} = i\epsilon \gamma_{-1} \gamma_{N+1}$$

$$\epsilon = \sum_{k} \left| \lim_{z \rightarrow z_k} 2(t_1 z + t'_1)(t_2 z + t'_2)z^{N-1} \prod_{i=1}^{4} (z - z_i) \right|,$$

$$z_k \in \left\{ z_i \mid |z_i| < 1, \ i = 1, 2, 3, 4 \right\}.$$  (17)

where $z_i (i = 1, 2, 3, 4)$ is the root of the quartic equation $(\Delta_{c2} - t_{c2})z^4 + (\Delta_{c1} - t_{c1})z^3 + 2\mu_c z^2 - (\Delta_{c1} + t_{c1})z - (\Delta_{c2} + t_{c2}) = 0$. Obviously, the Hamiltonian returns to Eq. (12) if one sets $t_{c2} = \Delta_{c2} = 0$. Fig. 5 depicts the relation between the coupling strength and the parameters of the central chain. One observes that the analytical solutions are in well agreement with the numerical solutions when $\mu_c \gg t_c$, i.e., $E_k \gg t_{1,2}$. In the presence of long-range interactions, the coupling strength $\epsilon$ is also enhanced since the value of coupling strength $\epsilon$ takes minimum when $t_{c2} = \Delta_{c2} = 0$, as shown in Fig. 3(b). It is not surprising since there are multiple channels for electrons co-tunneling in the long-range interactions, i.e., the next-to-nearest hopping amplitude also makes contributions to the electrons co-tunneling process.

**V. MODULATION OF CMBS BY PERIODIC DRIVING**

As we have shown, CMBS is closely related to the chemical potential, the hopping amplitude, and the pairing amplitude of the central chain (cf. Eq. [12]). This implies the coupling strength $\epsilon$ can be manipulated through controlling those parameters. Considering the hopping amplitude and the pairing amplitude cannot be easily manipulated in practice, we first explore the relation between the coupling strength $\epsilon$ and the chemical potential $\mu_c$. We do not plot directly the dependence of $\epsilon$
on \( \mu_c \). Instead, we show the dependence of Rabi oscillation between two MBSs on \( \mu_c \). This Rabi oscillation can show not only how the coupling strength changes with the chemical potential (since distinct coupling strengths can reflect on distinct Rabi frequencies, cf. Eq. (25)), but also how well the Majorana qubit works.

To show the Rabi oscillation, we define the BdG Hamiltonian \( \mathcal{H}_{\text{total}} \) of our system as follows. \( \mathcal{H}_{\text{total}} = \frac{1}{2} A^\dagger \mathcal{H}_{\text{total}} A \), where \( A = [a_{-N_1}, ..., a_{-N_2}, a_{N_1}^\dagger, ..., a_{N_2}^\dagger]^T \). By this definition, we can solve the following equation

\[
 i \frac{d|\Psi(t)\rangle}{dt} = \mathcal{H}_{\text{total}}|\Psi(t)\rangle,
\]

with the initial condition \( |\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \). The label \( |\uparrow\rangle \) denotes the vector (in basis \( A \)) with components \( |\uparrow\rangle_j = \delta_{1,j} \) for \( j \in \{−N_1, ..., 2N_2\} \), and so on. \( N \) denotes the total number of sites. In the Heisenberg picture, \( |\Psi(0)\rangle \) stands for the initial operator \( \frac{1}{\sqrt{2}}(a_{-1} + a_{1}^\dagger) \), which is the left MBS in vicinity of the central chain. With these notations, \( |\Psi(t)\rangle \) clearly represents the operator evolving at time \( t \), and \( \langle \Psi(0)|\Psi(t)\rangle^2 \) defines the distance between \( |\Psi(t)\rangle \) and \( |\Psi(0)\rangle \). Fig. 6 shows the distance as a function of evolution time. If there is no CMBS, the Rabi oscillation can not appear in the system, as depicted in Fig. 6a. In Fig. 6b-d, different chemical potentials \( \mu_c \) result in different Rabi frequencies. When the chemical potential is large, the coupling strength would be small, leading to a small Rabi frequency. That is, the Rabi frequency increases with the decreasing of the chemical potential. Similarly, the coupling strength can be manipulated by changing the pairing amplitude, as the blue-dark lines show in Fig. 4. Besides, the coupling strength can be modulated purely by the phase of pairing amplitude, e.g., Refs. [23, 29, 31].

Since the pairing amplitude is inherently determined by the property of superconductors, it may be difficult to directly modulate the pairing amplitude with the current techniques. In following we show that this goal can be reached by driving the central chain with periodic field. The Hamiltonian of the periodic driving field reads

\[
 H_\mu(t) = \mu_0 \cos \omega t \sum_{n=0}^{N} a_n^\dagger a_n,
\]

where \( \mu_0 \) and \( \omega \) are the amplitude and the frequency of the driving field, respectively. In an realistic situation, this driving field can be achieved by applying an external ac electric potential to TLGV, since the on-site chemical
potential can be modulated by TLGV.

To get an effective Hamiltonian, we work in an rotation frame defined by the unitary transformation, \( U(t) = e^{-i \omega t} \). The effective Hamiltonian of the whole chain becomes,

\[
H'_{\text{total}} = U^\dagger(t)[H_{\text{total}} + H_\mu(t)]U(t) - iU^\dagger(t)\dot{U}(t),
\]

\[
= H_l + H_r + H'_c + H'_{lc} + H'_{rc},
\]

\[
H'_c = \sum_{n=0}^N \mu_n a_n^\dagger a_n - \sum_{n=0}^{N-1} \left( \frac{t_c}{2} a_n^\dagger a_{n+1} + \Delta_c \sin(\omega t) a_n^\dagger a_n + h.c., \right)
\]

\[
+ e^{i\omega t} \frac{2a_0}{\omega} \left( a_0^\dagger a_0 + h.c. \right),
\]

\[
H'_{lc} = -\frac{t_l}{2} J_0(\frac{2a_0}{\omega}) a_0^\dagger a_0 + h.c.,
\]

\[
H'_{rc} = -\frac{t_r}{2} J_0(\frac{2a_0}{\omega}) a_{N+1}^\dagger a_{N+1} + h.c.,
\]

(20)

where \( H_l \) and \( H_r \) are invariant under this rotation. By making use of the identity

\[
e^{ix \sin \omega t} = \sum_{n=-\infty}^{\infty} J_n(x) e^{i\omega nt},
\]

with the \( n \)-order Bessel function \( J_n(x) \), the time-dependent effective Hamiltonian in the high-frequency limit (i.e., \( \omega \gg \mu_c, \mu_r \)) becomes

\[
H'_c = \sum_{n=0}^N \mu_n a_n^\dagger a_n - \sum_{n=0}^{N-1} \left( \frac{t_c}{2} a_n^\dagger a_{n+1} + \Delta_c \sin(\omega t) a_n^\dagger a_n + h.c., \right)
\]

\[
+ \frac{\Delta_c}{2} J_0(\frac{2a_0}{\omega}) a_0^\dagger a_0 + h.c.,
\]

\[
H'_{lc} = -\frac{t_l}{2} J_0(\frac{2a_0}{\omega}) a_0^\dagger a_0 + h.c.,
\]

\[
H'_{rc} = -\frac{t_r}{2} J_0(\frac{2a_0}{\omega}) a_{N+1}^\dagger a_{N+1} + h.c.,
\]

(22)

Under this approximation, the pairing amplitude of the central chain is modulated by the amplitude and the frequency of the driving field through the zero-order Bessel function, i.e., \( \Delta_{\text{eff}} = \frac{\Delta_c}{\omega} J_0(\frac{2a_0}{\omega}) \). We plot the time evolution of the MBSSs by driving field in Fig. 7. The green-grey lines show that the Rabi frequency in the presence of the driving field is a bit different from the exact results. This originates from the fact that the effective hopping amplitude is also changed by the driving field (see \( H'_{lc} \) and \( H'_{rc} \) in Eq. (22)), rendering the correction of Rabi frequency. We would like to address that, this correction does not make difference for UMQR as it only change the time to complete the operation. Interestingly, we can also modulate the effective hopping amplitude at the boundaries by the driving field to control CMBS. Fig. 8(a) shows that the adjacent MBSSs \( \gamma_{-1} \) and \( \gamma'_{N+1} \) can be decoupled by making the effective hopping amplitude at boundaries \( H'_{lc} \) and \( H'_{rc} \) vanish, i.e., setting \( J_0(\frac{2a_0}{\omega}) = 0 \). Otherwise, CMBS can be really induced, manifesting in the Rabi oscillation shown in Fig. 8(b)-(d).

VI. DISCUSSION AND CONCLUSION

As well known, an ordinary spinless fermion can be used to encode a logical qubit because it can span a two-dimensional Hilbert space (occupy or empty). However, it is not true for MBSSs since the operators satisfy \( \gamma_1 = \gamma_1^\dagger \) and \( \gamma_2 = 1 \). By recombining the operators \( \gamma_i \), one can use two MBSSs to construct a Dirac fermion, e.g., \( d_1 = \frac{1}{2}(\gamma_{N_1}^\dagger + i\gamma_{-1}) \) and \( d_2 = \frac{1}{2}(\gamma_{N+1}^\dagger + i\gamma_{N+1}) \) in Fig. 11. It seems that a logical qubit can be encoded by two MBSSs. Nevertheless, for the system with parity conservation (calculated through the Dirac fermions formed by MBSSs), the coherent superposition of MBSSs with different parities is prohibited. Therefore it cannot encode a logical qubit by two MBSSs in the parity conservation system. To guarantee the two computational bases having the same parity, four MBSSs (see \( \gamma_{N_1}, \gamma_{-1}, \gamma_{N+1}^\dagger, \) and \( \gamma_{N+1} \) in Fig. 11) is necessary to construct a logical qubit. E. g., we can construct the Majorana-based qubit in the odd parity subspace,

\[
|1_10_2\rangle = d_1^\dagger|0_10_2\rangle, |0_11_2\rangle = d_2^\dagger|0_10_2\rangle,
\]

where \(|0_10_2\rangle\) is the vacuum state of the Dirac fermions. The topologically protected single qubit operations are achieved by exchanging spatial positions of the MBSSs \( \gamma_{N_1} \) and \( \gamma_{-1} \), which exhibit the non-Abelian statistics. This braiding operation, i.e., the \( 1\pi \) phase gate, can be represented by the following unitary operation

\[
U_{N_1,0} = e^{i\pi \sigma_z},
\]

where \( \sigma_z = |1_10_2\rangle\langle1_10_2| - |0_11_2\rangle\langle0_11_2| \). It can be achieved in the one-dimensional semiconducting wires with T-
junction by controlling TLGV adiabatically \cite{23}. Note that the braiding operation is insufficient for realizing universal quantum computation since it is not able to perform arbitrary single qubit rotations \cite{2, 32}, which are usually not topologically protected (rotation angle \( \theta \neq \frac{\pi}{n} \), \( n \) is integer).

When there exists CMBS in the system (it has been also investigated in the continuous model instead of lattice model \cite{34, 35}), the effective Hamiltonian takes \( H_{0N} = \frac{c}{2} \gamma_1 \gamma_{N+1} \). We can obtain the following unitary operation for a fixed evolution time

\[
U_{0N}(t) = e^{-\frac{i}{\hbar} \sigma_x}, \tag{25}
\]

where \( \sigma_x = |1_1 0_2 \rangle \langle 0_1 1_2 | + |0_1 1_2 \rangle \langle 1_1 0_2 | \). Together with the braiding operation \( U_{N,0} \), one can implement UMQR via the successive operations \( U' = U_{0N}(t_2)U_{N,0}U_{0N}(t_1) \) \cite{18}.

To realize the operation \( U' \), exact control over CMBS with a fixed evolution time is required. This is crucial since the procedure is usually not topologically protected.

The other consideration is that the MBSs should decouple instantaneously from each other after (before) the coupling of the adjacent MBSs. As shown in Figs. 6(a) - 6(c), the two considerations are sufficiently solved since the modulation of the amplitude or frequency of the electric potential on TLGV can be controlled with high precision. As an example, in order to manipulate the chemical potential, we adopt two distinct voltages, e.g., \( \mu_{c1} = 10 \) and \( \mu_{c2} = 2.5 \) as shown in Fig. 6(a) and Fig. 6(c). We first apply a low voltage \( \mu_{c2} \) for periods of time \( t_1 \) to realize the operation \( U_{0N}(t_1) \). Then we switch it to a high voltage \( \mu_{c1} \) to realize the braiding operation \( U_{N,0} \). Next we change the gate to the low voltage \( \mu_{c2} \) for periods of time \( t_2 \) to realize the operation \( U_{0N}(t_2) \). Finally we back to the high voltage \( \mu_{c1} \) to cancel CMBS. This suggests that the UMQR can be realized by changing TLGV in a square-wave form, composed by \( \mu_{c1} \) and \( \mu_{c2} \). In fact, the result is the same as in the case where the driving frequency is switched to \( \omega_1 \) or \( \omega_2 \) in sequence, see Fig. 6(a) and Fig. 6(c).

In conclusion, we have presented a proposal to couple Majorana bound states mediated by a topologically trivial chain. Both cases of short-range and long-range interactions in the chain are considered. We demonstrate that CMBS depends sharply on the pairing amplitude and the chemical potential of the central chain. Particularly, CMBS can be enhanced when the pairing amplitude of the chain is large. When a driving field is applied to the chain, we explore the dependence of the CMBS on the frequency and amplitude of the driving field. These results suggest that there are many ways to change CMBS. One way is to manipulate the chemical potential of the central chain. The other way is to modulate the effective pairing amplitude of the central chain, which can be realized by changing the frequency of periodic driving field. Finally, we have demonstrated the application of tunable CMBS into the realization of quantum gates, i.e., implementing the unitary rotation operator \( U \) for quantum computation.

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