Nonsingular deformations of singular compactifications, the cosmological constant, and the hierarchy problem

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Abstract

We consider deformations of the singular “global cosmic string” compactifications, known to naturally generate exponentially large scales. The deformations are obtained by allowing a constant curvature metric on the brane and correspond to a choice of integration constant. We show that there exists a unique value of the integration constant that gives rise to a nonsingular solution. The metric on the brane is $dS_4$ with an exponentially small value of expansion parameter. We derive an upper bound on the brane cosmological constant. We find and investigate more general singular solutions—“dilatonic global string” compactifications—and show that they can have nonsingular deformations. We give an embedding of these solutions in type IIB supergravity. There is only one class of supersymmetry-preserving singular dilatonic solutions. We show that they do not have nonsingular deformations of the type considered here.
1 Introduction and summary.

The proposal that we live on a three-dimensional extended object ("brane") embedded in some higher dimensional spacetime [1], [2], [3] has recently attracted a lot of attention. Such a scenario, for example with a pair of millimeter-size extra dimensions, is not excluded by current observations [4], provided the electroweak and strong interactions do not feel the presence of the extra dimension. Furthermore, the required localization of gauge interactions has a very natural realization in string theory provided the extended object is a $D$-brane [4].

Gravity, on the other hand, is free to propagate in the extra dimensions and becomes four dimensional only at distances large compared to their size. The strength $M_{Pl}$ of four-dimensional gravitational interactions is related to the strength $M$ of the higher-dimensional gravitational theory: $M_{Pl}^2 \sim M^4 V_2$, where $V_2$ is the volume of the transverse space and we assume two large dimensions. Remarkably, if $M$ is in the (multi) $TeV$ range, and the size of the extra dimensions is a (sub) millimeter, the above relation yields the correct value of $M_{Pl} \sim 10^{19}GeV$. Thus, in this “large extra dimensions” scenario [1] the hierarchy problem appears as the question: “Why are the extra dimensions so large?”

It is therefore interesting to look for ways to generate exponentially large distances between objects of codimension two. Potentials in two dimensions vary logarithmically and it is natural to expect the appearance of exponentially large scales [5], [6], [7].

The solutions of the coupled Einstein-Maxwell equations outside charged point particles in two spatial dimensions [8] exhibit some relevant properties—particles of charge $g$ are surrounded by a Killing horizon at an exponentially large distance $\sim e^{O(1)g^2/T^2}$ from the particle. In the case of extended objects ("branes") in codimension two, charged under appropriate rank antisymmetric tensor fields, there is instead a mild #1 naked singularity at an exponentially large distance, $Mr_* \sim e^{O(1)g^2/T^2}$. This was first demonstrated in [9] in the context of global cosmic strings in four dimensional spacetime and generalized in [10] to a charged extended object of codimension two in a spacetime of arbitrary dimension.

The singular properties of charged 3-brane solutions of codimension two have been exploited in [10] to suggest an interesting direction for solving the hierarchy problem: the charged 3-brane creates a singularity at an exponentially large distance, of order $e^{O(1)g^2/T^2} M^{-1}$. The assumption that spacetime terminates at the singularity leads to a finite, exponentially large, four-dimensional $M_{Pl}$ without the usual fine tuning of parameters.

It may be that the singularity is indeed harmless and the appropriate boundary conditions—#1As $r$ approaches $r_*$ the curvature scalar is finite, but the square of the Riemann tensor diverges, see eqns. (13, 14).
no flow of conserved quantum numbers through the singularity—are imposed by the fundamental theory of quantum gravity. However, one of the main points we wish to stress is that it is important to study nonsingular deformations of the singular solutions. Of particular interest are deformations that appear as a choice of integration constants rather than parameters in the lagrangian. It appears natural that the “correct”—from the point of view of the fundamental quantum gravity theory—choice of integration constant would be the one that avoids the singularity; the case is even stronger if there is a unique such value.

A class of deformations of the singular solutions of [10], obtained by varying the bulk cosmological constant $\Lambda_6$ was studied in [14]. It was shown that a unique nonsingular solution exists if the bulk cosmological constant is precisely tuned to a negative (corresponding to an $AdS_6$ vacuum) value, whose modulus is bounded above by an exponentially small scale: $|\Lambda_6| < e^{-O(1/g^2)} M$. Thus, the price to pay for obtaining a nonsingular solution in this case is an enormous fine tuning of a parameter in the lagrangian.

In this paper, we study another class of deformations. We allow a constant curvature metric on the 3-brane—$dS_4$, $AdS_4$, or $R_{1,3}$. This class of deformations appears as a choice of integration constant $\alpha_3$. We show, using the methods of [15], [16], that there exists a unique value of the brane cosmological constant $\alpha$ for which the solution is nonsingular. The induced metric on the brane is de Sitter with an exponentially small cosmological constant. The singularity is replaced by a horizon at finite proper distance from the brane.

We study deformations of two classes of singular charged three-brane solutions. In Section 3 we study the deformations of the “global cosmic string” three-brane solution of [10]. In Section 4 we find a more general solution including also a dilaton field with an arbitrary value of the dilaton coupling. This is motivated in part by supersymmetry: for a particular value of the dilaton coupling the dilatonic charged solutions include the singular BPS solutions described in [21].

We investigate the possibility that the nonsingular solutions we find solve both the hierarchy and cosmological constant problems; further comments along these lines are given in the concluding section and in ref. [20]. Our qualitative approach suffices to show that there is a unique value of the brane cosmological constant yielding a nonsingular solution. However, it allows us to obtain only an upper bound on the value of $|\alpha|$. (To obtain a precise value one must proceed numerically.) Even though exponentially small, this bound is still many orders

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#2 In references [11], [12] a similar situation appears, with a (stronger) singularity at finite distance, for a codimension-one brane in five dimensions. An analysis of the nature of such singularities was made in [13].

#3 The resulting spacetime is geodesically incomplete, as is the case in many recently considered “compactifications” on noncompact spaces of finite volume. One has to either impose appropriate boundary conditions at the horizon, see [17], [18], [19], or complete the space by adding “image” charges as in, e.g. [1].
of magnitude larger than the experimental bound on the cosmological constant; a value of the bound closer to observation can be obtained, but at the cost of fine tuning.

In Section 5, we embed the solutions in type IIB supergravity. We show that, within our embedding, the warped solutions preserve no supersymmetry and that the only supersymmetric solution is the unwarped solution of ref. [21]. Using the analysis of Section 4, we show that this singular BPS solution has no nonsingular deformation (supersymmetric or otherwise) of the class considered here.

Finally, even though this paper is devoted to investigating the “global cosmic string” compactification, we note that similar considerations—choosing integration constants to avoid singularities—can be applied to the “local cosmic string” case considered in [19]. There the solution of the 6d Einstein equations around an uncharged (in our terminology) three brane in AdS$_6$ was found. The authors of [19] only considered the case where the brane cosmological constant $\alpha$ vanishes. In fact, it can be shown, using the results of [22], that the choice of vanishing brane cosmological constant is the only one leading to a nonsingular solution. Outside the core, the solution of [19] is a particular case of the general rotationally invariant metric around a three brane in AdS$_6$ found in [22]:

$$ds^2_6 = z^2 ds^2_4 + f^{-1}(z)dz^2 + f(z)d\theta^2,$$

where $f \sim z^2 - a + \frac{b}{z^3}$, where $a$ and $b$ are constants of integration, with $a \sim \alpha$ (the brane cosmological constant, see [22] for more details). Ref. [19] considered the choice of integration constants $\alpha = a = b = 0$, i.e. $f \sim z^2$. The Minkowski three-brane is the only choice where the metric (1) has no singularity at the horizon $z = 0$ (the core of the solution is at $z = 1$, see eqn. (34) of [19]). This follows from the expression for the curvature,

$$R \sim f'' - 8\frac{f'}{z} + 12\frac{f}{z^2},$$

and from similar expressions for the other curvature invariants.

2 Action, ansatz, and equations of motion.

The action we consider is that of gravity in six dimensions, with the usual Einstein action, minimally coupled to a scalar field $\phi$ and a four-form antisymmetric tensor field $C_{(4)}$:

$$S_{bulk} = M^4 \int d^6 y \sqrt{-g} \left( R - \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{1}{2 \times 5!} e^{-a\phi} F_{(5)}^2 \right).$$

The indices $M, N = 1, ..., 6$, while $\mu, \nu = 0, ..., 3$ and $i, j = 1, 2$ span the rest of the space; the metric has signature $(-,+,\ldots,+)$ and $F_{(5)} = dC_{(4)}$ is the field strength of the four form $C_{(4)}$.  


In what follows, we will consider solutions both with \((a \neq 0)\) and without \((\phi = a = 0)\) scalar fields.

We have in mind “matter”, in the form of 3-branes, whose action is proportional to the area of the world surface they sweep in the six-dimensional spacetime and include a topological coupling to the four-form:

\[
S_{\text{matter}} = - f^4 \int d^6 y \int d^4 \sigma \, \delta^6(y - X(\sigma)) \sqrt{-\tilde{g}} e^{2\phi} \left( 1 + \ldots \right) \\
- T^4 \int d^6 y \int d^4 \sigma \, \delta^6(y - X(\sigma)) \frac{1}{4!} C_{M_1 \ldots M_4} \epsilon^{\mu_1 \ldots \mu_4} X^M_{\mu_1} \ldots X^M_{\mu_4},
\]

where \(X^M(\sigma)\) is the embedding of the world surface in space time (a sum over the various branes is needed if there is more than one brane) \(\tilde{g}_{\mu\nu} := X^M_{\mu} X^N_{\nu} g_{MN}(y)\) is the induced metric and \(\tilde{g} := \det \tilde{g}_{\mu\nu}\) (the terms with the dots denote other possible matter constrained to the brane). \(M\) and \(f^4\) are the six-dimensional Planck scale and brane tension, respectively. We note that the fields \(\phi\) and \(C_{(4)}\) in (3) are dimensionless; the charge of the 3-brane under the antisymmetric tensor field is then seen to be proportional to \((T/M)^4\).

A comment is due on the relevance of the “matter” action (4). One way to think about the brane action is as describing the zero-thickness limit of a solitonic solution of a six-dimensional field theory. Generally, however, in codimension two and higher, the zero-thickness limit of defects coupled to gravity—with the energy momentum tensor becoming a distribution—is not well defined; see ref. [23] for a detailed discussion.\[^4\] We will think, instead, of the solutions we investigate as describing the metric outside the core, of size \(r_0 \sim M^{-1}\), of a solitonic solution; we will not deal in detail with matching to the source. The solution of [9], [10] was found as the metric outside the core of such a soliton; the solution where only the four form is nonvanishing \((\phi = a = \alpha = 0)\) is obtained after a duality transformation (in the bulk) of their solution.

We are looking for a solution which is a warped product of a four dimensional space of constant curvature (Minkowski, de Sitter, or Anti de Sitter) and has an \(SO(2)\) isometry acting on the transverse directions. The most general form of the metric, consistent with these symmetries, is:

\[
ds^2 = e^{2A(t)} \tilde{g}^{(4)}_{\mu\nu}(x) \, dx^\mu dx^\nu + e^{2\beta(t)} \left( dt^2 + d\theta^2 \right),
\]

where \(t = \log r\) and we use dimensionless units; all dimensions are restored at the end by inserting appropriate powers of \(M\). The functions \(A\) and \(\beta\) as well as the scalar field \(\phi\) are

\[^4\]The action (3) will be useful in one instance—the BPS dilatonic solution of [2] matches precisely to the distributional source (4) for \(a = 2\), \(T = f\). The equations of motion of all bulk fields (metric, four-form, and dilaton) are satisfied at all values of \(r\), including the origin; in addition because of the no-force condition, the brane equation following from varying (4) with respect to the embedding coordinates is also obeyed. We will discuss the supersymmetry properties of this solution as well as its \(\alpha \neq 0\) deformation in Section 5.
assumed to depend on the radial coordinate $t$ in the transverse dimensions, while the four-dimensional metric $g^{(4)}_{\mu\nu}(x)$ obeys:

$$R^{(4)}_{\mu\nu} - \frac{1}{2}g^{(4)}_{\mu\nu}R^{(4)} = \alpha g^{(4)}_{\mu\nu}. \quad (6)$$

We will not constrain the sign of the parameter $\alpha$ at this point. Our conventions for the Einstein tensor are such that $\alpha < 0$ ($\alpha > 0$) corresponds to (anti) de Sitter space.

The following ansatz, consistent with the symmetries of (5), for the non-vanishing components of the five-form field strength satisfies both the equations of motion and the Bianchi identity:

$$F^{\mu_1\mu_2\mu_3\mu_4\mu_5} = e^{a\phi(t)} \frac{e^{\mu_1\mu_2\mu_3\mu_4\mu_5\theta}}{\sqrt{-g}} \partial_\theta \chi; \quad \chi := d_1 \theta, \quad (7)$$

where $d_1$ is a constant of integration, related to the charge of the “brane” by Gauss’s law. The rest of the equations are:

$$R^{(6)}_{\mu\nu} - \frac{1}{2}g^{(6)}_{\mu\nu}R^{(6)} = -\frac{1}{4}g^{(6)}_{\mu\nu}e^{-2\beta}(\dot{\phi}^2 + 8qe^{a\phi}),$$

$$R^{(6)}_{tt} - \frac{1}{2}g^{(6)}_{tt}R^{(6)} = \frac{1}{4}(\dot{\phi}^2 - 8qe^{a\phi}),$$

$$R^{(6)}_{\theta\theta} - \frac{1}{2}g^{(6)}_{\theta\theta}R^{(6)} = -\frac{1}{4}(\dot{\phi}^2 - 8qe^{a\phi}),$$

$$0 = 4A'\phi' + \phi'' - 4qae^{a\phi}, \quad (8)$$

where the prime refers to differentiation with respect to $t$ and $q := \frac{d_1^2}{8} > 0$ is related to the square of the charge of the solution. The equations of motion in the form (8) will be useful in Section 5 when studying the embedding of our solutions in type IIB supergravity.

In order to study the $\alpha \neq 0$ deformation of the singular solutions, it is convenient to cast the equations of motion in first order form and study the flow of the resulting dynamical system. To this end, we introduce the “time” $\tau$

$$\tau(t) = \int^t dt \frac{e^{\beta(t)}}{e^{A(t)}}. \quad (9)$$

and the variables

$$W = e^{A-\beta} A' \equiv \dot{A},$$

$$V = e^{A-\beta} \beta' \equiv \dot{\beta},$$

$$S = e^{A-\beta} \dot{\phi} \equiv \dot{\phi}, \quad (10)$$
where the dot refers to differentiation with respect to $\tau$. In terms of the above variables the equations become first order:

$$
\begin{align*}
\dot{W} &= -3 W^2 - VW - \alpha , \\
\dot{V} &= -V^2 + 12 W^2 + 5 VW - \frac{1}{2} S^2 + 4\alpha , \\
\dot{S} &= -3 SW - SV - 8a WV - 12a W^2 + \frac{a}{2} S^2 - 4 a \alpha ,
\end{align*}
$$

subject to the constraint, which is consistently propagated in $\tau$ by the equations of motion (11):

$$
3 W^2 + 2 VW - \frac{1}{8} S^2 + \alpha = -q e^{2A - 2\beta + \alpha \phi} .
$$

The equations of motion will allow us to study both the $\alpha = 0$ and $\alpha \neq 0$ solutions in terms of the flow of the dynamical system of $W, V, S$ described by (11). The solutions of interest have, in addition, to respect the constraint (12)—in particular, only the trajectories of the $W, V, S$ dynamical system for which the l.h.s. of (12) is negative solve the full set of the Einstein/scalar field equations.

To study the singularity structure of the solutions of eqns. (11, 12), it is useful to give some of the curvature invariants of the metric (7) on the solutions of the equations of motion. For the scalar curvature, we obtain (for the de Sitter case, $\alpha = -3H^2$; we only need to consider this case, see Section 3):

$$
R = 4e^{-2A} \left(3H^2 + \frac{1}{4} S^2 - 2VW - 3W^2\right) = 4q e^{a\phi - 2\beta} + \frac{1}{2} S^2 e^{-2A} ,
$$

while, for example, the square of the 6-d Riemann tensor is:

$$
R_{ABCD}R^{ABCD} = \frac{248}{3} q^2 e^{2a\phi - 4\beta} - \frac{4}{3} q e^{a\phi - 2A} \left(\frac{7}{2} S^2 - 128VW\right)
+ \frac{4}{3} e^{-4A} \left(80V^2W^2 - 4S^2VW + \frac{7}{33} S^4\right) .
$$

It is useful to note that, similarly, all curvature invariants can, on the solutions of the equations of motion (11, 12), be expressed as functions of $V, W, S$. From (14) it follows that a solution where these variables go to infinity for some values of $\tau$ will be singular, provided $e^{-2A}$ does not vanish fast enough.

More generally, we note that the phase space analysis of Sections 3, 4 shows that either all $W, V, S$ flow to infinity or all stay finite. Thus, trajectories for which $W, V, S$ all flow to infinity can be approximated, for $|W|, |V|, |S| \gg |\alpha|$, by the trajectories of the system (11, 12) with $\alpha$ set to zero. As we we will see shortly, in this case the equations can be solved exactly and give rise to singular solutions. We therefore conclude that for a nonsingular solution $W, V, S$ have to be bounded.
3 Singular nondilatonic charged solution: Cohen-Kaplan solution and its nonsingular deformation.

The Cohen-Kaplan (CK) solution describes the spacetime outside the core of a global “cosmic string” and in our notations corresponds to ignoring the dilaton and setting \( \alpha = 0 \) in (11, 12). The equations can then be integrated to give:

\[
F(V, W) := \frac{1}{W} \left( \frac{-W}{2V + 3W} \right)^{\frac{3}{8}} \exp \left( \frac{-V}{8W} \right) = \text{const.}
\]  

(15)

We can also solve parametrically for \( A \) and \( \beta \) in terms of the variable \( t \):

\[
e^A = \left( 1 - \frac{4}{c} t \right)^{\frac{1}{4}},
\]

\[
e^\beta = \left( 1 - \frac{4}{c} t \right)^{-\frac{3}{8}} \exp \left( -q \left( t^2 - \frac{c}{2} t \right) \right),
\]

(16)

where \( q = \frac{d^2}{s} \) (see eqn. (7)) and \( c > 0 \) is a constant. By comparing (16), (15), the value of the constant on the r.h.s. of (15) is found to be:

\[-3q^{-\frac{3}{8}} e^{\frac{7}{8}} \exp \left( \frac{qc^2 + 3}{16} \right).\]

(17)

The constants \( q, c > 0 \) can be fixed by considering the core of the soliton to which (16) matches, as explained in Section 2; the parameter \( u_0 \) of (11) is related to our parameters as \( c^2 q = \frac{8}{u_0} \). A singularity occurs at an exponentially large distance \( rM = e^t \sim e^{c/4} \). (Recall that we take \( M^{-1} \) as the characteristic size of the “core”).

That a singularity appears is also clear from considering the flow diagram (Fig. 1) of the first-order-form equations (11), (12) in the \( W - V \) plane. To draw it, note that the lines

\[
W = 0; \quad V + 3W = 0
\]

\[
V = \frac{5 \pm \sqrt{73}}{2} W
\]

(18)

correspond to the lines where velocities change sign, i.e. to \( \dot{W} = 0, \dot{V} = 0 \) respectively. The CK solution asymptotes the line \( V + \frac{3}{2} W = 0 \) as \( t \to \frac{c}{4} \). Note also that the constraint implies that the allowed trajectories lie in the region bounded by the lines \( W = 0, V + \frac{3}{2} W = 0 \). In fact in view of the positivity of \( c, q \) the solution obeys \( W < 0, V > 0 \).

Now, in view of the remarks following (14) any solution that flows to infinity will be singular (note also that, in the \( W < 0, V > 0 \) quadrant \( A \) is a decreasing function and the factor \( e^{-2A} \))
Figure 1: The flow diagram of the equivalent dynamical system (1). The thin solid straight lines represent the lines (18) where the velocities \( \dot{W}, \dot{V} \) of the undeformed, \( H = 0 \), system change sign, while the two thicker (undashed) hyperbolae are the corresponding lines (19) of the deformed, \( H \neq 0 \), system. The signs of the velocities \( \dot{W}, \dot{V} \) in the various regions are shown by \((\pm, \pm)\). The four critical points are where the two hyperbolae intersect. A trajectory which satisfies the constraint (12) must lie between the two branches of the invariant hyperboloid (represented by a thick dashed curve). The invariant hyperboloid is also a critical trajectory, repulsive in the \((H, 0)\) and attractive in the \((-H, 0)\) critical point, as indicated by the arrows. The singular CK solution starts in the \((+, -)\) part of the \((W < 0, V > 0)\) quadrant, turns around and goes to infinity in the \(W, V\) plane asymptoting the \(V = -\frac{3}{2}W\) line. Its nonsingular deformation is the attractive trajectory ending in the \((-H, 0)\) critical point. The singular CK trajectory is always to the left of its nonsingular deformation. The singular unwarped \((W = 0)\) BPS trajectory is given by the vertical axis.
is an increasing function of time). A nonsingular solution, if it exists, should correspond to a trajectory that flows to a critical point. First we note that the system (11) has no critical points for $\alpha > 0$ (AdS$_4$ case). In the dS$_4$ case ($\alpha < 0$) there are four critical points:

$$\{ W = V = \pm \sqrt{|\alpha|} \}$$

and

$$\{ V = 0; W = \pm \sqrt{|\alpha|} \frac{1}{3} \}.$$ 

Only the last two obey the constraint. Moreover, as is clear from investigating the flows, only trajectories terminating at the $\{ W = -\sqrt{|\alpha|}; V = 0 \}$ critical point can be nonsingular deformations of the CK solution and in the following we will focus on this critical point.

For $\alpha < 0$, however, it is not easy to integrate the equations (11, 12) any more. Nonetheless we can still draw a flow diagram on the $W - V$ plane (shown on Figure 1). Now $\dot{W} = 0$, $\dot{V} = 0$ correspond to the hyperboloids

$$\frac{73W^2}{4} - \left( V - \frac{5W}{2} \right)^2 = 4|\alpha|$$

which at infinity asymptote the lines (18). The constraint now implies that the allowed trajectories lie in the region between the two branches of the hyperboloid $(W + \frac{V}{3})^2 - \frac{V^2}{9} = |\alpha|/3$.

We now define

$$H := \sqrt{|\alpha|} \frac{1}{3},$$

(20)

where $H$ is the expansion parameter of the dS$_4$ metric on the brane. A linear analysis near the $\{ W = -H; V = 0 \}$ critical point reveals that it is a saddle. The attractive trajectory, which corresponds to the nonsingular solution, approaches the critical point with slope $dV/dW = -8$ (the repulsive trajectory is, on the other hand, simply the invariant hyperboloid of the dynamical system (11) and is given by the constraint equation (12) with $q = 0$).

The metric (and a “Kruskal” extension) of the nonsingular solution can be obtained explicitly in the limit $\tau \to \infty$, i.e. near the critical point. In this way one establishes the existence of a horizon, which replaces the singularity. To see this, note that the metric near the critical point can be found by solving $\dot{A} = -H$, $\dot{\beta} = 0$. Using a closed spatial section parametrization of dS$_4$, we obtain:

$$ds^2|_{\text{crit}} \simeq y^2 \left( -d\eta^2 + \cosh^2 \eta \ d\Omega_3^2 \right) + dy^2 + e^{2\beta_0}d\theta^2,$$

(21)

where $\beta \to \beta_0$ and $y := \frac{1}{H}e^{-H\tau} \to 0$ as the nonsingular trajectory approaches the critical point $\tau \to \infty$. This metric can be written as:

$$ds^2|_{\text{crit}} \simeq -dT^2 + dR^2 + R^2d\Omega_3^2 + e^{2\beta_0}d\theta^2,$$

(22)
by changing coordinates \( T = -y \sinh \eta, R = y \cosh \eta \). Thus, the original spacetime (21) corresponds to the region outside the light cone of (22), \( R^2 > T^2 \), and can be extended through the horizon \( y = 0 \) to \( y < 0 \). The near-horizon metric (22) is approximately that of five-dimensional Minkowski space times a circle of radius \( e^{\beta_0} \).

The precise matching to a core solution, as discussed in Section 2, is a difficult and model-dependent problem. We will only make some qualitative remarks. The initial conditions, given by the core solution, say at \( t \sim 0 \), are given by a one-parameter curve (e.g., for \( |W(0)| = c^{-1} \ll 1 \), by \( W(0)V(0) \simeq -\frac{4}{2} \)) in the \((+,\,-)\) region of the \((W < 0, V > 0)\) quadrant (see Fig. 1), sufficiently far from the origin, so that the solution can still be approximated by the CK trajectory. Now, as one varies \( H \), the attractive trajectory stays strictly to the left of \( H \) (since it lies entirely in the \((+,\,-)\) region, where the \( V \) velocity is positive) and therefore crosses the initial condition line always to the left of \( H \). On the other hand, the attractive trajectory approaches the vertical axis in the \( H = 0 \) limit; the only trajectory that goes through the critical point is then the vertical axis. Thus, by continuity, one can always find a value for \( H \) such that the nonsingular trajectory ending at the critical point passes through the given point on the initial condition line.

While obtaining the precise value of \( H \) that gives rise to a nonsingular solution is only possible numerically, an upper bound on \( H \) can be obtained from the qualitative analysis of the flows. From this analysis, it is clear that the CK solution is always to the left of its nonsingular deformation. In particular, at the point \( \{V_0, W_0\} \) where the CK trajectory intersects the \( \dot{V} = 0 \) line, we have

\[
|W_0| = \frac{2V_0}{\sqrt{73} - 5} > H. \quad (23)
\]

Note that \( H \) has dimensions of mass, since \( \alpha \) has dimensions \((\text{mass})^2\). Plugging the above into (15), (17), we obtain the inequality:

\[
q^\frac{1}{16} c^{-\frac{1}{8}} \exp \left( -\frac{q c^2}{16} \right) > \frac{H}{M}. \quad (24)
\]

To understand the phenomenological significance of this bound, let us reduce the Einstein term in the six-dimensional action

\[
M^4 \int d^6 x \sqrt{g_6} R_6 = M^4 I \int d^4 x \sqrt{g_4} R_4 := M_{P l}^2 \int d^4 x \sqrt{g_4} R_4, \quad (25)
\]

where

\[
I := \frac{1}{M^2} \int dt \ e^{2A + 2B} \quad (26)
\]

At the critical point (horizon), the curvature invariants do not vanish, but are proportional to \( q \), as follows from eqns. (13), (14). The \( q \)-dependence of the metric only comes in through higher orders in the expansion around the critical point, which were neglected in deriving (21).
and $M_{Pl} \sim 10^{19}$ GeV (the long-distance-theory Planck scale), $M \sim 1$ TeV (the short-distance-theory Planck scale). Assuming that $I$ will not differ significantly from its CK value, we obtain:

$$I \sim \frac{1}{M^2} c^4 q^{-\frac{3}{8}} \exp \left( \frac{qc^2}{8} \right) = \frac{M_{Pl}^2}{M^4}$$ (27)

We will only consider the case when the values of $c, q$ are not fine tuned and the hierarchy of scales is explained by the exponential factor in (27) alone. Then eqns. (27, 25) imply:

$$\exp \left( \frac{qc^2}{8} \right) \sim \left( \frac{M_{Pl}}{M} \right)^2$$ (28)

From equations (28, 24) we then obtain the following upper bound on the expansion parameter of the $dS_4$ metric:

$$H < \left( \frac{M}{M_{Pl}} \right)^2 M_{Pl} \sim 10^{-32} M_{Pl}$$ (29)

On the other hand, the experimental bound on the vacuum energy density, $H^2 M_{Pl}^2 < \rho_{crit} \sim 10^{-120} M_{Pl}^4$, gives

$$H < 10^{-60} M_{Pl}$$ (30)

In other words, the experimental bound on $H$ (30) is thirty orders of magnitude stricter than the upper bound we have established in (29). In terms of vacuum energy density the upper bound (29) is set by the fundamental gravity scale, $M$, and is of order $M^4$. We stress that this is only an upper bound on the cosmological constant (recall that we also used the CK trajectory instead of the nonsingular trajectory to obtain the value of $I$ for our bound). To obtain the value of $H$ needed to obtain a nonsingular solution, one would have to proceed with a numerical analysis of the nonsingular deformation of the CK trajectory; we leave this for future work.

4 Singular charged dilatonic solution and its nonsingular deformation.

Let us now include the dilaton in our discussion. We start with $\alpha = 0$ in which case the equations of motion (11) can be integrated. We demand that the solution reduce to the CK one in the $a \to 0$ limit. The result is:

$$A = \frac{1}{4} \log \left( 1 - \frac{4}{c} t \right)$$
$$\beta = \frac{qc^2}{16} + \frac{3}{8} \log \frac{c}{4} + \frac{1}{4} \left( u_0 + \frac{8 - d}{a^2} \right) \log \left( \frac{c}{4} - t \right) + \frac{2}{a^2} \log \left( 1 - \frac{qa^2}{2} \left( \frac{c}{4} - t \right)^4 \right)$$ (31)
$$\phi = \phi_0 + au_0 A - a \beta,$$
where
\[ d := 8 \sqrt{\left(\frac{u_0}{4} + \frac{3}{8}\right) a^2 + 1}, \tag{32} \]

\( u_0 \geq -\frac{3}{2} \) is an integration constant, and the constant part of the dilaton, \( \phi_0 \), is fixed in terms of \( c, q, u_0 \) by the constraint equation (12). The range of \( t \) is
\[ \frac{c}{4} \geq t \geq \frac{c}{4} - \left(\frac{2}{qa^2}\right)^\frac{1}{d}. \tag{33} \]

It is straightforward to check that indeed in the \( a \to 0 \) limit (31) reduces to (16).

The solution (31) is a “dilatonic” generalization of the singular CK solution and is itself singular. Curvature invariants blow up at both limits of the range of the variable \( t \); as in Sect. 3 we imagine that near the lower limit of the range (33) of \( t \) our solution is matched to a smooth solitonic solution to which eqn. (31) is a long-distance approximation.

As in the previous section, we cannot integrate analytically the equations of motion for \( \alpha \neq 0 \). However, one can still analyze the three-dimensional flow diagram in the space \( W, V, S \). Proceeding as before one establishes the existence of a unique nonsingular solution ending at the critical point \( \{V = S = 0; \ W = -\sqrt{\frac{8}{3}}\} \). The critical point corresponds to an approximately \( R^{1,4} \times S^1 \) horizon, exactly as in the non-dilatonic case.

A bound on \( H \) can be established as before. The integral \( I \), defined in (29), can here also be calculated in a closed form, with the assumption that its value for \( \alpha \neq 0 \) is not significantly different from its value for \( \alpha = 0 \). The result of integrating over the whole range (33) is:
\[ I = \frac{1}{d} \left(\frac{c}{4}\right)^\frac{1}{d} \exp\left(\frac{qc^2}{8}\right) \left(\frac{2}{qa^2}\right)^{c_0} B\left(c_0, 1 + \frac{4}{a^2}\right) \tag{34} \]

where
\[ c_0 := \frac{2}{d} \left(\frac{8 - d}{a^2} + u_0 + 3\right). \tag{35} \]

It is easy to check that the above \( I \) reduces to the value for the CK solution (27) in the \( a \to 0 \) limit. The phenomenological analysis of the previous section goes through virtually unchanged. For example, eqn. (24) gets replaced, upon repeating the analysis of the flows for the \( V, W, S \) dynamical system, by:
\[ \frac{H}{M} < c^{-\frac{3}{4}} \exp\left(-\frac{qc^2}{16}\right) x^\frac{2}{a^2} \left(1 - \frac{qa^2}{2}x\right)^{-\frac{2}{a^2}}, \tag{36} \]

where \( x = (\frac{4}{d} - t_s)^{\frac{4}{d}} \), with \( t_s \) given by solving the equation \( \dot{V} = 0 \). Although rather complicated looking once the explicit form of the solution (31) is plugged into \( \dot{V} = 0 \) (see eq. (11)), this
equation determines $x_*$ as an algebraic function of the parameters $q, a, u_0$. Exponentially large or small numbers can appear only upon fine tuning these parameters. Thus, comparing (34), (36) with (29), (28), we see that—with exponential accuracy—the bound on $H$ can not be improved without fine tuning. (The limits of $a \ll 1$ and $a \gg 1$ are easiest to study analytically and support this conclusion.)

In other words, the inclusion of the dilaton cannot help obtaining a stricter upper bound on the cosmological constant without fine tuning. As in Sect. 3, obtaining the actual value of $H$ for the nonsingular spacetime instead of an upper bound is only possible numerically.

5 Embedding in IIB supergravity.

Here we will embed in type $IIB$ supergravity the solutions found in the previous two sections. We show that the embedding does not preserve any supersymmetry. Therefore, if the “global cosmic string” compactifications producing an exponential hierarchy can be fully embedded in string theory (that is, if a string-derived model for the core can be found—we only consider the embedding of the exterior of the string in $IIB$ supergravity) they would correspond to nonsupersymmetric string vacua.

Our strategy is first to consider a background for $IIB$ involving the fields $\chi, \phi, g_{mn}^{(6)}$ of our six-dimensional solution. We then show that on this background the $IIB$ equations reduce to the equations of motion (8). In this Section, we use $M, N$ for the ten-dimensional indices and $\mu, \nu$—for the 4d indices. The fields of $IIB$ supergravity are the graviton $g_{MN}^{(10)}$, the axion $\rho$ and the dilaton $\varphi$, parametrizing an $SL(2, R)/U(1)$ coset space, a pair of two-forms $B_{mn}^{1,2}$ which form an $SL(2, R)$ doublet, a four-form with self-dual field strength $F_{(5)}$ and two complex Weyl fermions: a gravitino and a dilatino.

We are interested in a type $IIB$ background obeying the six-dimensional equations of motion (8). We will take the fields to be independent of the last four coordinates ($I, J = 6, ..., 9$ in what follows). In addition, the ansatz should have the same isometries as our six dimensional ansatz, (3)—the $SO(1, 3) \times SO(2) \times SO(4)$ isometry (for the warped Minkowski case), with the $SO(2)$ acting as a shift of $\theta$ and the $SO(4)$—the isometry of the four extra coordinates.

We will consider an ansatz where only the metric, dilaton, and axion are nonvanishing (one could instead consider a nonvanishing four-form field; this is related to the case of nonvanishing axion by T-duality in the 6...9 directions). In a background where all forms and all fermions are set to zero the equations of motion of the type $IIB$ supergravity fields are (these are the
equations given in [24], after $U(1)$ gauge fixing):

\[ R_{MN}^{(10)} - \frac{1}{2} g_{MN} R^{(10)} = \frac{1}{2} \left( \partial_M \varphi \partial_N \varphi - \frac{1}{2} g_{MN} \partial_K \varphi \partial^K \varphi \right) + \frac{1}{2} \epsilon^{2 \varphi} \left( \partial_M \rho \partial_N \rho - \frac{1}{2} g_{MN} \partial_K \rho \partial^K \rho \right) \]

\[ 0 = \partial_M \left( \sqrt{-g^{(10)}} \partial^M \varphi \right) - \sqrt{-g^{(10)}} e^{2 \varphi} \partial_M \rho \partial^M \rho \]

\[ 0 = \partial_M \left( \sqrt{-g^{(10)}} \partial^M \rho \right) + 2 \sqrt{-g^{(10)}} \partial_M \varphi \partial^M \rho , \]

where $g^{(10)} = \det g^{(10)}$. The most general ansatz for the nonvanishing components of the ten-dimensional metric consistent with the above symmetries is:

\[ g^{(10)}_{MN} = g^{(6)}_{MN}; \ M, N = 0, \ldots, 5 , \]

\[ g^{(10)}_{IJ} = e^{\Psi(t)} \delta_{IJ}; \ I, J = 6, \ldots, 9 , \]

(38)

where $g^{(6)}_{MN}$ is as in equation (3). It is straightforward to show that the equations of motion (37) reduce to equations (8) for

\[ a = 2; \ \Psi = 0; \ \varphi = \phi; \ \rho = d_1 \theta . \]

(39)

It is natural to ask whether the above embedding of our solution preserves any supersymmetry in the case of warped Minkowski space $g^{(4)}_{\mu \nu} = \eta_{\mu \nu}$. The supersymmetry transformations of the dilatino and the gravitino are parametrized by a complex Weyl spinor $\varepsilon$. The conditions for unbroken supersymmetry in the background (39) with all fields but the graviton and the scalars set to zero are:

\[ \Gamma^M \left( \partial_M \varphi + i e^\varphi \partial_M \rho \right) \varepsilon^* = 0 , \]

\[ \left( \partial_M + \frac{1}{2} \omega_{MNL} \Gamma^{NL} + \frac{i}{4} e^\varphi \partial_M \rho \right) \varepsilon = 0 , \]

(40)

where $\Gamma_{MN} = \frac{1}{2} [\Gamma_M, \Gamma_N]$ and $\omega_{MNL}$ is the spin connection:

\[ \omega_{MNL} = \frac{1}{2} \left( e_{MA}(\partial_N e^A_L - \partial_L e^A_N) - e_{LA}(\partial_M e^A_N - \partial_N e^A_M) - e_{NA}(\partial_L e^A_M - \partial_M e^A_L) \right) . \]

The only nonvanishing components of the spin connection are:

\[ \omega_{\mu \nu t} = -\omega_{\mu t \nu} = -\eta_{\mu \nu} e^{2A} A' , \]

\[ \omega_{\theta \theta t} = -\omega_{\theta t \theta} = e^{2 \beta} \beta' . \]

(41)

Vanishing of the dilatino supersymmetry variation requires that the parameter of the unbroken supersymmetry transformation $\varepsilon$ obey (the gamma matrices in (12), (13) have curved indices, i.e. are vielbein-dependent):

\[ \left( \phi^\prime \Gamma_\theta \Gamma^\prime + i e^\phi d_1 \right) \varepsilon^* = 0 , \]

(42)
while the gravitino variation implies the conditions:

\[
\begin{align*}
\partial_I \varepsilon &= 0 ; \quad I = 6 \ldots 9 , \\
\left( \nabla_\theta - \frac{1}{2} B' \Gamma_\theta^t + \frac{i}{4} e^\phi d_1 \right) \varepsilon &= 0 , \\
\partial_t \varepsilon &= 0 , \\
\left( \partial_\mu - \frac{1}{2} A' \Gamma_\mu^t \right) \varepsilon &= 0 ,
\end{align*}
\]  

(43)

where \( \nabla_\theta \varepsilon := \left( \partial_\theta - \frac{1}{2} \Gamma_\theta^t \right) \varepsilon \) is the \( \theta \)-component of the covariant derivative in flat space and \( B \) is related to \( \beta \) from (3) by \( B = \beta - t \); the transverse part of the metric in \((r = e^t, \theta)\) coordinates is \( e^{2B} (dr^2 + r^2 d\theta^2) \). Along all other directions the flat-space covariant derivative reduces to a simple derivative. The integrability condition following from the fourth equation implies that the factor \( A' = 0 \) if any supersymmetry is to be preserved in the Minkowski background. We conclude that our dilatonic solution of Section 4, which has a nontrivial warp factor \( A \), see eqn. (31), does not preserve any supersymmetry.

However, there exists another class of solutions, already known in the literature [21]. These unwarped \((A = \text{const.})\) singular solutions were shown in [21] to obey a “no-force” condition—a probe brane with the same coupling to the dilaton and antisymmetric tensor field as in (4) with \( f = T \) feels no force towards the brane at \( r = 0 \). We will now obtain the explicit form of these unwarped solutions by requiring unbroken supersymmetry. To begin with, note that eqn. (12) imposes a strong constraint on the form of the supersymmetry-preserving solution: a \( t \)-independent \( \varepsilon \) can obey (12) if and only if:

\[
\phi' = k d_1 e^\phi , \quad k^2 = 1 .
\]  

(44)

The last condition on \( k \) follows from rewriting equation (12) as:

\[
\left( 1 - ik \Gamma_\theta \Gamma^t \right) \varepsilon^* = 0 ,
\]  

(45)

and requiring the existence of a nontrivial solution. Note that (15) (for \( k = \pm 1 \)) is just the chirality projection in the \( t, \theta \) directions and half of the \( \varepsilon \) components are projected out. Now let us demand that \( \varepsilon \) be a covariantly constant spinor of flat ten dimensional Minkowski space. Explicitly,

\[
\partial_\mu \varepsilon = \partial_I \varepsilon = \partial_t \varepsilon = \nabla_\theta \varepsilon = 0 .
\]  

(46)

All supersymmetry equations are automatically obeyed except for the second of eqns. (13) which requires, in view of (14), that

\[
\left( 2ikB' \Gamma_\theta \Gamma^t + \phi' \right) \varepsilon = 0 .
\]  

(47)
Agreement with (45) requires (taking a real $i\Gamma_0 \Gamma^i$):

$$B' = -\frac{1}{2}\phi',$$

(48)
giving, together with (44) the solution of (21), for $k = +1$. In view of the comments following (45) it preserves half the supersymmetry of type $IIB$ supergravity, i.e. has sixteen real supercharges. In our notation, as is easy to see, $W = 0$, $V = -\frac{1}{2}S$ is a solution of (11), obeying the constraint (12) with $\alpha = 0$.

The above supersymmetric (unwarped and singular) solution is represented (projected on the $W, V$-plane) on the flow diagram of Fig. 1 by the vertical axis, with the direction of the flow towards negative infinity. The solution does not have a nonsingular deformation with $\alpha \neq 0$. To see this, recall first, as discussed in Section 4, that the $W, V, S$ system has the same two critical points of interest as the $S = a = 0$ system, and that these are the $(\pm H, 0)$ points shown on Fig. 1. The directions of the flow velocities in Fig. 1 indicate that the singular BPS solution can not flow to the $(-H, 0)$ critical point and could only be deformed to end at the $(H, 0)$ critical point. However, it is easy to see that the attractive trajectories at this critical point must lie on the invariant hyperboloid (now a two-dimensional surface in the $W, V, S$ space), as indicated on Fig. 1. The invariant hyperboloid obeys the constraint equation with vanishing charge for all values of $t$ and hence a trajectory that lies on it can not be the deformation of the $q \neq 0$ nonsingular solution. We conclude that the $H \neq 0$ deformations of the BPS trajectory continue to flow to infinity and the solution remains singular.

We note that while this work was being completed, the paper [25] appeared, which discusses the embedding of dilatonic “global cosmic strings” in string theory.

6 Concluding remarks.

We have demonstrated, both for the dilatonic and nondilatonic singular “global cosmic string” compactifications, the existence of a unique value of the $dS_4$ expansion parameter $H$, for which the singularity is absent and have given a strict upper bound on the critical value of $H$. (One’s hope is that the quantum gravity theory chooses the value of the integration constant for which the solution is nonsingular.) We have also shown that the field configuration outside the core of the string can be embedded in type $IIB$ supergravity and that it breaks all supersymmetry.

With exponential accuracy (without fine tuning parameters) the upper bound—derived using a qualitative analysis of the flows—on the corresponding vacuum energy density is of order $M^4$, with $M$–the fundamental gravity scale ($TeV$, say). Whether the actual critical value of $H$ can naturally be small enough to agree with observations can only be decided numerically.
We should note, however, that even if the classical critical value of $H$ is small enough, quantum loops (e.g., “standard model” quantum corrections to the “brane tension”) will probably change its value, as one expects the critical value of $H$ to depend on the “brane” parameters. It is difficult, even adopting a specific model for the core and ignoring bulk loop corrections, to estimate the magnitude of this change, as an analytic expression for the dependence of the relevant quantities—the critical value of the de Sitter expansion parameter $H$ and the volume of transverse space—on the brane tension is lacking.

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