Spin-1 gravitational waves. 
Theoretical and experimental aspects*

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Abstract
Exact solutions of Einstein field equations invariant for a non-Abelian 2-dimensional Lie algebra of Killing fields are described. Physical properties of these gravitational fields are studied, their wave character is checked by making use of covariant criteria and the observable effects of such waves are outlined. The possibility of detection of these waves with modern detectors, spherical resonant antennas in particular, is sketched.

Introduction
Gravitational waves, that is a propagating warpage of space time generated from compact concentrations of energy, like neutron stars and black holes, have not yet been detected directly, although their indirect influence has been seen and measured with great accuracy. Presently there are, worldwide, many efforts to detect gravitational radiation, not only because a direct confirmation of their existence is interesting per se but also because new insights on the nature of gravity and of the Universe itself could be gained. For these reasons exact solutions of the Einstein field equations deserve special attention when they are of propagative nature. The need of taking into full account the nonlinearity of Einstein’s equations when studying the generation of gravitational waves from strong sources is generally recognized [47]. Moreover, despite the great distance of the sources from Earth (where most of the experimental devices, laser interferometers and resonant antennas, are located) there are situations where the non linear effects cannot be neglected. This is the case when the source is a binary coalescence: indeed it has been shown [II] that a secondary wave, called the Christodoulou memory is generated via the non linearity of Einstein’s field equations. The memory seems to be too weak to be detected

*The article is dedicated, in the occasion of his 60th birthday, to Raphael Sorkin who first introduced gravitational particles with spin different from 2.
from the present generation of interferometers [46] (even if its frequency is in
the optimal band for the LIGO/VIRGO interferometers) but of the same order
as the linear effects related to the same source, thus stressing the relevance of
the nonlinearity of the Einstein’s equations also (soon) from an experimental
point of view.

On the theoretical side, starting from the seventy’s new powerful mathematical
methods have been developed to deal with nonlinear evolution equations.
For instance, a suitable generalization of the \textit{Inverse Scattering Transform}
allows to integrate [7] Einstein field equations for a metric of the form
\[ g = f(z,t) \left( dt^2 - dz^2 \right) + h_{11}(z,t) dx^2 + h_{22}(z,t) dy^2 + 2h_{12}(z,t) dx dy. \]

Indeed, the corresponding vacuum Einstein field equations reduce essentially\(^1\) to
\[ (\alpha H^{-1}H_{\xi} \eta)_{\eta} + (\alpha H^{-1}H_{\eta})_{\xi} = 0, \]
where \( H \equiv \|h_{ab}\| \), \( \xi = (t+z)/\sqrt{2} \), \( \eta = (t-z)/\sqrt{2} \), \( \alpha = \sqrt{\det H} \). This
is a non-linear differential equation whose form is typical of two-dimensional
integrable systems and can be integrated [7] by using a suitable generalization
of the \textit{Inverse Scattering Transform}, yielding \textit{solitary waves solutions}.

A geometric inspection of the metric above shows that it is invariant under
translations along the \( x,y \)-axes, \( i.e. \) it admits two Killing fields, \( \partial_x \) and \( \partial_y \), closing on an Abelian two-dimensional Lie algebra \( A_2 \). Moreover, the distribution \( D \), generated by \( \partial_x \) and \( \partial_y \), is 2-dimensional and the distribution \( D^\perp \) orthogonal to \( D \) is integrable and transversal to \( D \).

Thus, it has been natural to consider [43] the general problem of characteriz-
ing all gravitational fields \( g \) admitting a Lie algebra \( \mathcal{G} \) of Killing fields such
that:

\begin{enumerate}
  \item the distribution \( D \), generated by vector fields of \( \mathcal{G} \), is 2-dimensional;
  \item the distribution \( D^\perp \), orthogonal to \( D \) is integrable and transversal to \( D \).
\end{enumerate}

According to whether \( \dim \mathcal{G} \) is 2 or 3, two qualitatively different cases occur.
A 2-dimensional \( \mathcal{G} \), is either Abelian (\( A_2 \)) or non-Abelian (\( G_2 \)). A metric \( g \)
satisfying I and II, with \( \mathcal{G} = A_2 \) or \( G_2 \), will be called \( \mathcal{G} \)-integrable.

The study of \( A_2 \)-integrable Einstein metrics goes back to Einstein and Rosen
[20], Rosen [39], Kompaneets [24], Geroch [21], Belinsky and Khalatnikov [6].
Recent results can be found in [10].

The greater rigidity of \( G_2 \)-integrable metrics, for which some partial results
can be found in [22, 2, 12], allows an exhaustive analysis. It will be shown that
they are parameterized by solutions of a linear second order differential equation
on the plane which, in its turn, depends linearly on the choice of a \( j \)-\textit{harmonic}
function (see later). Thus, this class of solutions has a \textit{bilinear structure} and,
as such, admits two \textit{superposition laws}.

\(^1\)The function \( f \) can be obtained by quadratures in terms of the matrix \( H \).
When \( \dim \mathcal{G} = 3 \), assumption II follows from I and the local structure of this class of Einstein metrics can be explicitly described. Some well known exact solutions \cite{36, 41}, e.g. Schwarzschild, belong to this class.

Besides the new local \( \mathcal{G}_2 \)-integrable solutions, a procedure to construct new global solutions, suitable for all such \( \mathcal{G} \)-integrable metrics, will be also described.

The paper is organized as follows. In section 1 gravitational fields invariant for a two dimensional Lie algebra are characterized. In section 2 the Einstein equations for such metrics are reduced, by using the symmetry, to the so called \( \mu \)-deformed Laplace equation. Harmonic coordinates are also introduced. Section 3 is devoted to the analysis of the wave-like character of the solutions through the Zel’manov and the Pirani criterions. In section 4 the canonical and the Landau energy-momentum pseudo-tensors are introduced and a comparison with the linearised theory is performed. In section 5 realistic sources for such gravitational fields are described. Eventually, section 6 is devoted to the analysis of the polarization of the waves.

In the following, \( \text{Kil}(g) \) will denote the Lie algebra of all Killing fields of a metric \( g \) while \( \text{Killing algebra} \) will denote a sub-algebra of \( \text{Kil}(g) \).

Moreover, an integral (two-dimensional) submanifold of \( \mathcal{D} \) will be called a \textit{Killing leaf}, and an integral (2-dimensional) submanifold of \( \mathcal{D}^\perp \) \textit{orthogonal leaf}.

\section{Geometrical aspects}

- **Semiadapted coordinates.**

Let \( g \) be a metric on the space-time \( M \) (a connected smooth manifold) and \( \mathcal{G}_2 \) one of its Killing algebras whose generators \( X, Y \) satisfy \([X, Y] = sY, \ s = 0, 1\)

The Frobenius distribution \( \mathcal{D} \) generated by \( \mathcal{G}_2 \) is 2-dimensional and a chart \((x^1, x^2, x^3, x^4)\) exists such that

\[
X = \frac{\partial}{\partial x^3}, \quad Y = (\exp sx^3) \frac{\partial}{\partial x^4}
\]

From now on such a chart will be called \textit{semiadapted} (to the Killing fields).

- **Invariant metrics**

It can be easily verified \cite{43, 44, 45} that in a semiadapted chart \( g \) has the form

\[
g \ = \ g_{ij} dx^i dx^j + 2 \left( l_i + sm_i x^4 \right) dx^i dx^3 - 2 m_i dx^i dx^4 + \\
\left( s^2 \lambda \left( x^4 \right)^2 - 2 s \mu x^4 + \nu \right) dx^3 dx^3 + \\
2 \left( \mu - s \lambda x^4 \right) dx^3 dx^4 + \lambda dx^4 dx^4, \quad i = 1, 2; j = 1, 2
\]

with \( g_{ij}, m_i, l_i, \lambda, \mu, \nu \) arbitrary functions of \((x^1, x^2)\).
• **Killing leaves.**

Condition II allows to construct semi-adapted charts, with new coordinates \((x, y, x^3, x^4)\), such that the fields \(e_1 = \partial/\partial x, e_2 = \partial/\partial y\), belong to \(D^\perp\). In such a chart, called from now on adapted, the components \(l_i's\) and \(m_i's\) vanish.

We will call **Killing leaf** an integral (2-dimensional) submanifold of \(D\) and **orthogonal leaf** an integral (2-dimensional) submanifold of \(D^\perp\). Since \(D^\perp\) is transversal to \(D\), the restriction of \(g\) to any Killing leaf, \(S\), is non-degenerate. Thus, \((S, g|_S)\) is a homogeneous 2-dimensional Riemannian manifold. Then, the Gauss curvature \(K(S)\) of the Killing leaves is constant (depending on the leave). In the chart \((p = x^3|_S, q = x^4|_S)\) one has

\[
g|_S = \left(s^2\lambda q^2 - 2s\bar{\mu}q + \bar{\nu}\right)dp^2 + 2\left(\bar{\mu} - s\bar{\lambda}q\right)dpdq + \bar{\lambda}dq^2,
\]

where \(\bar{\lambda}, \bar{\mu}, \bar{\nu}\), being the restrictions to \(S\) of \(\lambda, \mu, \nu\), are constants, and

\[
K(S) = \bar{\lambda}s^2\left(\bar{\mu}^2 - \bar{\lambda}\bar{\nu}\right)^{-1}.
\]

### 1.1 Einstein metrics when \(g(Y, Y) \neq 0\).

In the considered class of metrics, vacuum Einstein equations, \(R_{\mu\nu} = 0\), can be completely solved \[43\]. If the Killing field \(Y\) is not of light type, i.e. \(g(Y, Y) \neq 0\), then in the adapted coordinates \((x, y, p, q)\) the general solution is

\[
g = f(dx^2 \pm dy^2) + \beta^2[(s^2k^2q^2 - 2slq + m)dp^2 + 2(l - skq)dmdq + kdq^2] \tag{1}
\]

where \(f = -\Delta \pm \beta^2/2s^2k\), and \(\beta(x, y)\) is a solution of the tortoise equation

\[
\beta + A \ln |\beta - A| = u(x, y),
\]

where \(A\) is a constant and the function \(u\) is a solution either of Laplace or d’Alembert equation, \(\Delta u = 0, \Delta = \partial^2_{xx} \pm \partial^2_{yy}\), such that \((\partial_x u)^2 \pm (\partial_y u)^2 \neq 0\). The constants \(k, l, m\) are constrained by \(km - l^2 = \mp 1, k \neq 0\) for Lorentzian metrics or by \(km - l^2 = \pm 1, k \neq 0\) for Kleinian metrics (if \(f > 0\)). Ricci flat manifolds of Kleinian signature appear in the no boundary proposal of Hartle and Hawking \[23\] in which the idea is suggested that the signature of the spacetime metric may have changed in the early universe. Some other examples of Kleinian geometry in physics occur in the theory of heterotic \(N = 2\) string (see \[33\] and \[4\]) for which the target space is four dimensional

### 1.1.1 Canonical form of metrics when \(g(Y, Y) \neq 0\)

The gauge freedom of the above solution, allowed by the function \(u\), can be locally eliminated by introducing the coordinates \((u, v, p, q)\), the function \(v(x, y)\) being conjugate to \(u(x, y)\), i.e. \(\Delta \pm v = 0\) and \(u_x = v_y, u_y = \mp v_x\). In these coordinates the metric \(g\) takes the form (local "Birkhoff’s theorem")
\[ g = \exp \frac{u - \beta}{2s^2k^2\beta} (du^2 \pm dv^2) + \beta^2 [(s^2k^2q^2 - 2slq + m)dp^2 + 2(l - skq)d(pdq + kdq^2)] \]

with \( \beta(u) \) a solution of \( \beta + A \ln |\beta - A| = u \).

### 1.1.2 Normal form of metrics when \( g(Y,Y) \neq 0 \).

In geographic coordinates \((\vartheta, \phi)\) along Killing leaves one has

\[ g|_S = \beta^2 [d\vartheta^2 + F(\vartheta) d\phi^2] , \]

where \( F(\vartheta) \) is equal either to \( \sinh^2 \vartheta \) or \( -\cosh^2 \vartheta \), depending on the signature of the metric. Thus, in the normal coordinates, \((r = 2s^2k^2, \tau = v, \vartheta, \phi)\), the metric takes the form

\[ g = \varepsilon_1 \left( \left[ 1 - \frac{A}{r} \right] d\tau^2 \pm \left[ 1 - \frac{A}{r} \right]^{-1} dr^2 \right) + \varepsilon_2 r^2 \left[ d\vartheta^2 + F(\vartheta) d\phi^2 \right] \quad (2) \]

where \( \varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1 \) with a choice coherent with the required signature 2.

The geometric reason for this form is that, when \( g(Y,Y) \neq 0 \), a third Killing field exists which together with \( X \) and \( Y \) constitute a basis of \( so(2,1) \). The larger symmetry implies that the geodesic equations describe a non-commutatively integrable system \[^{12}\] and the corresponding geodesic flow projects on the geodesic flow of the metric restricted to the Killing leaves. The above local form does not allow, however, to treat properly the singularities appearing inevitably in global solutions. The metrics \[^{11}\], although they all are locally diffeomorphic to \[^{12}\], play a relevant role in the construction of new global solutions as described later.

### 1.2 Einstein metrics when \( g(Y,Y) = 0 \).

If the Killing field \( Y \) is of light type, then the general Lorentzian solution of vacuum Einstein equations, in the adapted coordinates \((x, y, p, q)\), is given by

\[ g = 2f(dx^2 + dy^2) + \mu((w(x,y) - 2sq)dp^2 + 2d(pdq], \quad (3) \]

where \( \mu = A\Phi + B \) with \( A,B \in \mathbb{R}, \Phi \) is a non constant harmonic function of \( x \) and \( y \), \( f = (\nabla \Phi)^2 \sqrt{\mu}\)/\( \mu \), and \( w(x,y) \) is solution of the \( \mu \)-deformed Laplace equation:

\[ \mu \Delta w + \nabla \mu \cdot \nabla w = 0. \]

Metrics \[^{3}\] are Lorentzian if the orthogonal leaves are conformally Euclidean, i.e. the positive sign is chosen, and Kleinian if not. Only the Lorentzian case will be analyzed and these metrics will be called of \((\mathcal{G}_2, 2)\) isotropic type.
In the particular case $s = 1$, $f = 1/2$ and $\mu = 1$, the above metrics are locally diffeomorphic to a subclass of the vacuum Peres solutions \[34\], that for later purpose we rewrite in the form

$$g = dx^2 + dy^2 + 2dudv + 2(\varphi_x dx + \varphi_y dy)du,$$

where

$$u = e^p, \quad v = qe^{-p} + \varphi(x, y, u),$$

with $\varphi(x, y, u)$ a harmonic function of $x$ and $y$ arbitrarily dependent on $u$.

In the case $\mu = \text{const}$, the $\mu$-deformed Laplace equation reduces to the Laplace equation; for $\mu = 1$, in the harmonic coordinates system $(x, y, z, t)$:

$$z = \left[(2q - w(x, y))e^{-p} + e^p\right]/2$$

$$t = \left[(2q - w(x, y))e^{-p} - e^p\right]/2,$$

the Einstein metrics \[3\] take \[16\] the particularly simple form,

$$g = 2f(dx^2 + dy^2) + dz^2 - dt^2 + d(w) d(\ln|z - t|).$$

This shows that, when $w$ is constant, the Einstein metrics given by Eq. \[5\] are static and, under the further assumption $\Phi = x\sqrt{2}$, they reduce to the Minkowski one. Moreover, when $w$ is not constant, gravitational fields \[5\] look like a disturbance propagating at light velocity along the $z$ direction on the Killing leaves (integral two-dimensional submanifolds of $\mathbb{D}$).

2 Physical properties

The wave character of gravitational fields \[5\] has been checked by using covariant criteria. In the following we will shortly review the most important properties of these waves.

2.1 Asymptotic flatness

A first step toward a physical interpretation of metrics \[5\] is to characterize those which are spatially asymptotically flat. For the metrics \[5\], in the vacuum case, we will consider (spatially) asymptotic flat a metric approaching the Minkowski metric for $x^2 + y^2 \to \infty$. In terms of the functions $f, \mu$ and $w$, this asymptotic flatness condition reads:

$$x^2 + y^2 \to \infty \implies f \to \text{const}, \quad \mu \to \text{const}, \quad w \to c_1 x + c_2 y + c_3,$$

where $c_1, c_2$ and $c_3$ are arbitrary constants and the behavior of $w$ can be easily recognized by looking at the Riemann tensor. In order for metrics \[5\] to be spatially asymptotically flat $\mu$ must be constant \[16\]. For $\mu = 1$, the equation for $w$ reduces to a two-dimensional Laplace equation, then in the vacuum case
the only possible choice is \( w = \text{const} \), because the Laplace equation does not have solutions tending to a constant value.

Let’s consider the non-vacuum case. The simplest source for metrics (8) is dust with density \( \rho \) and velocity \( U^\mu \) with an energy-momentum tensor \( T_{\mu\nu} = \rho U_\mu U_\nu \) [14]. When \( U^\mu \) is a light-like vector field, this tensor can describe the energy and momentum of null electromagnetic waves. This is not surprising; in fact Peres himself [34, 35] indicated this as a possible source for his metrics (PP-waves) which, as we know, are diffeomorphic to a subclass of solutions found in [13, 44, 45]. With this stress-energy tensor we can depict realistic astrophysical sources as Gamma Ray Bursts or Cosmic Strings\(^2\); the symmetries of the vacuum solution is preserved.

Being the time coordinate in the Killing leaves, the dust will be chosen to move parallel to the light-like Killing field \( Y \), i.e., with velocity \( U^\mu = \delta^\mu q \). The non vacuum Einstein equations with the energy-momentum tensor

\[
T_{\mu\nu} = \mu^2 \rho \delta_{\mu p} \delta_{\nu p}
\]

are fulfilled, with \( f \) and \( \mu \) the same as in the vacuum case and \( w \) solution of the \( \mu \)-deformed Poisson equation

\[
\mu \Delta w + \nabla \mu \cdot \nabla w = 2f \mu^2 \rho,
\]

with \( c = 1, 8\pi G = 1 \). From the asymptotic flatness condition with \( \mu = 1 \), the equation for \( w \) reduces to a two-dimensional Poisson equation

\[
\Delta w = \rho.
\]

It is well known that, if \( \rho \) goes to zero fast enough, it is possible to find non trivial everywhere regular solutions \( w \) tending to a constant value. The function \( f \) satisfies the equation

\[
f \Delta f - (\nabla f)^2 = 0,
\]

and this implies that the function

\[
\psi = \ln |f|
\]

is harmonic. Thus, in order to have everywhere regular spatially asymptotically flat solutions, \( f \) and \( \mu \) must be constant functions and the fluid density \( \rho \) must tend to zero fast enough.

However, if we admit \( \delta \)-like singularity in the \((x,y)\) plane (i.e., string-like singularity, by taking into account the third spatial dimension), spatially asymptotically flat vacuum solutions with \( f \neq \text{const} \) and \( w \neq \text{const} \) can exist. In this limiting case in which \( \rho (x,y) \to \delta (x,y) \), the energy-momentum tensor becomes the one usually employed to describe the gravitational effects of cosmic strings.

\(^2\)Possible observations of Cosmic Strings have been reported recently [40].
As far as Gamma Ray Bursts sources are concerned, they are usually modelled with null-like hypersurfaces\(^3\). In several papers (i.e. \([13, 30]\)) the study of exact solutions, describing impulsive gravitational waves, have been posed starting from a metric tensor of a PP-wave and extended to quadratic curvature gravity \([31, 32]\) and a rough estimation for the intensity of such waves is provided. Assuming typical parameters for Gamma Ray Bursts like an energy flux \(\sim 10^{-2} \text{erg cm}^{-2} \text{s}^{-1}\), a temporal extension \(\delta t \sim 10s\) and an energy density \(\rho_0 \approx E \frac{E}{4\pi z^2 c^2}\, z\) being the distance from the source and \(E\) the total energy emitted, the maximal amplitude for a gravitational wave signal arriving on heart would be of the order \(2\pi \rho_0 G \delta t^2 \sim 10^{-38}\), far below the sensitivity of modern detectors.

### 2.2 Zelmanov’s and Pirani’s criteria

To check the propagative nature of gravitational fields described by metrics \((3)\) several covariant criteria have been employed. In the general case when \(f\) and \(\mu\) are not constant functions the Zel’manov criterion \([50]\) is satisfied \([16]\). Moreover, when \(f\) is a constant function, the only non-vanishing components of the Riemann tensor field reduce to

\[
R_{txzx} = \frac{w_{xx}}{2(z-t)^2}, \quad R_{txzy} = \frac{w_{xy}}{2(z-t)^2}, \quad R_{txzy} = \frac{w_{yy}}{2(z-t)^2}
\]

which, \(w(x, y)\) being a harmonic function, are all harmonic functions of \(x, y\). As a consequence, the generalized Zel’manov criterion is still satisfied \([16]\).

Besides the Zel’manov-Zakharov criterion, the Pirani algebraic criterion is also satisfied. In light-cone coordinates \((u = (z-t)/\sqrt{2}, \, v = (z+t)/\sqrt{2})\), where the metrics given by Eq.\((5)\) read

\[
g = 2f(dx^2 + dy^2) + 2dudv + d\ln |u|,
\]

the vector fields \(\partial_u\) and \(\partial_v\) are both isotropic. Moreover, the only non vanishing components of the Riemann tensor are

\[
R_{uki} = \pm \frac{1}{2u^2} \partial_i^2 w
\]

and this corresponds to a \(\text{type-N}\) Riemann tensor in the Petrov classification. From the natural interpretation of the Pirani criterion \([50]\) it follows that the gravitational wave propagates along the null vector field \(\partial_u\), that’s to say the gravitational wave \([5]\) propagates along the \(z\)-axis with the light velocity \(c = 1\).

### 2.3 The energy-momentum pseudo-tensors

The exact gravitational wave

\[
g = dx^2 + dy^2 + dz^2 - dt^2 + d(w) \, d(\ln |z-t|),
\]

\(^3\)Null-like frequently considered as models for astrophysical processes involving relativistic jets and sudden acceleration of huge quantity of mass \([5]\)
given by Eq. 5 for \( \mu = 1, f = 1/2 \), has the physically interesting form of a perturbed Minkowski metric with \( h = dw d\ln|z-t| \). Moreover, besides being an exact solution of the Einstein equations, it is a solution of the linearized Einstein equations on a flat background too:

\[
\begin{align*}
\eta^{\mu\nu} \partial_\mu \partial_\nu h &= 0 \\
\eta^{\mu\nu} (2 h_{\mu\rho,\nu} - h_{\mu\nu,\rho}) &= 0
\end{align*}
\]

To study its energy and polarization, the standard tools of the linearized theory, and in particular the canonical energy-momentum pseudo-tensor, can be used [19, 48].

With \( h = dw (w) d(\ln|z-t|) \) the \( \tau_0^0 \) component of the canonical energy-momentum tensor vanishes since \( h \) has only one index in the plane transversal to the propagation direction because the components of the tensor \( h \) cannot be expressed in the transverse-traceless gauge.

The non vanishing components of the 4-momentum density tensor \( p^\mu \equiv \tau^\mu_0, \tau^\rho\kappa \) denoting the Landau-Lifshitz energy-momentum pseudo-tensor [25], are \( p^0 \) and \( p^3 \).

\[
\begin{align*}
p^0 &= 4 (t-z)^{-2} [C_1 (w,xx)^2 + C_2 (w,xy)^2] + 4 (t-z)^{-4} C_3 \nabla \cdot [\nabla w |\nabla w|^2 \nabla w], \\
p^1 &= 0, \\
p^2 &= 0, \\
p^3 &= 4 (t-z)^{-2} [C_1 (w,xx)^2 + C_2 (w,xy)^2] + 4 (t-z)^{-4} C_3 \nabla \cdot [\nabla w |\nabla w|^2 \nabla w].
\end{align*}
\]

where \( C_i \) are some positive numerical constants, \( \nabla = (\partial_x, \partial_y) \) and the harmonicity condition for \( w \) has been used [15, 16].

The use of the Bel’s superenergy tensor \( \mathbf{T}^{\alpha\beta\gamma\delta} \)

\[
\mathbf{T}^{\alpha\beta\gamma\delta} = \frac{1}{2} \left( R^{\alpha\beta\gamma\delta} R^{\mu\nu} - * R^{\alpha\beta\gamma\delta} * R^{\mu\nu} \right),
\]

where the symbol \( * \) denotes the volume dual, leads to the same result. Indeed, the only non vanishing independent components of the covariant Riemann tensor \( R_{\alpha\beta\gamma\delta} = g_{\alpha\rho} R^\rho_{\beta\gamma\delta} \) are

\[
R_{1313} = -w_{,11}; \quad R_{1323} = -w_{,12}; \quad R_{2323} = -w_{,22}.
\]

It follows that the density energy represented by the Bel’s scalar

\[
W = T_{\alpha\beta\gamma\delta} U^\alpha U^\beta U^\gamma U^\delta,
\]

the \( U^\alpha \)'s denoting the components of a time-like unit vector field, depends on the squares of \( w_{,\alpha\beta} \).

Thus, both the Landau-Lifshitz pseudo-tensor and the Bel superenergy tensor single out the same physical degrees of freedom. In particular, we can take the components \( h_{tx} \) and \( h_{ty} \) as fundamental degrees of freedom for the gravitational wave [3].

Since \( p^0 = p^3 \), these waves move at light velocity, according with the result obtained by the Pirani criterion.
2.4 Polarization

Even more controversial than for the energy and momentum, the definition of spin or polarization for a theory, such as general relativity, which is non-linear and possesses a much bigger invariance than just the Poincaré one, deserves a careful analysis.

It is well known that the concept of particle together with its degrees of freedom like the spin may be only introduced for linear theories (for example for the Yang-Mills theories, which are non-linear, it is necessary to perform a perturbative expansion around the linearized theory). In these theories, when Poincaré invariant, the particles are classified in terms of the eigenvalues of two Casimir operators of the Poincaré group, $P^2$ and $W^2$ where $P_\mu$ are the translation generators and $W_\mu = \frac{1}{2}\epsilon_{\mu\rho\sigma} P^{\rho\sigma}$ is the Pauli-Ljubanski polarization vector with $M^{\mu\nu}$ Lorentz generators. Then, the total angular momentum $J = L + S$ is defined in terms of the generators $M_{\mu\nu}$ as $J^i = \frac{i}{2}\epsilon^{0ijk} M_{jk}$. The generators $P_\mu$ and $M_{\mu\nu}$ span the Poincaré algebra, $ISO(3,1)$

\[
\begin{align*}
[M_{\mu\nu}, M_{\rho\sigma}] & = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \\
[M_{\mu\nu}, P_\rho] & = i(\eta_{\nu\rho} P_\mu - \eta_{\nu\mu} P_\rho) \\
[P_\mu, P_\nu] & = 0.
\end{align*}
\]

(9)

Let us briefly recall a few details about the representation theory of this algebra. The Pauli-Ljubanski operator is a translational invariant Lorentz vector, that is $[P_\mu, W_\nu] = 0$, $[M_{\mu\nu}, W_\rho] = i(\eta_{\nu\rho} W_\mu - \eta_{\nu\mu} W_\rho)$. In addition it satisfies the equation

\[ W_\mu P^{\mu} = 0. \]

(10)

The unitary (infinite-dimensional) representations of the Poincaré group fall mainly into three different classes:

- $P^2 = m^2 > 0$, $W^2 = -m^2 s(s + 1)$, where $s = 0, \frac{1}{2}, 1, ...$ denotes the spin. From Eq. (10) we deduce that in the rest frame the zero component of the Pauli-Ljubanski vector vanishes and its space components are given by $W_i = \frac{1}{2}\epsilon_{i0jk} P^0 S^k$ so that $W^2 = -m^2 S^2$. This representation is labelled by the mass $m$ and the spin $s$.

- $P^2 = 0$, $W^2 = 0$. In this case $W$ and $P$ are linearly dependent

\[ W_\mu = \lambda P_\mu; \]

the constant of proportionality is called helicity and it is equal to $\pm s$ . The time component of $W$ is $W^0 = \overrightarrow{P} \cdot \overrightarrow{J}$, so that

\[ \lambda = \frac{\overrightarrow{P} \cdot \overrightarrow{J}}{P_0} \]

which is the definition of helicity for massless particles like photons.
• \( P^2 = 0, \ W^2 = -\rho^2 \), where \( \rho \) is a continuous parameter. This type of representation, which describes particles with zero rest mass and an infinite number of polarization states labeled by \( \rho \), does not seem to be realized in nature.

Let us turn now to the gravitational fields represented by Eq. (8). As it has been shown, they represent gravitational waves moving at the velocity of light, that is, in the would be quantized theory, particles with zero rest mass. Thus, if a classification in terms of Poincaré group invariants could be performed, these waves would belong to the class of unitary (infinite-dimensional) representations of the Poincaré group characterized by \( P^2 = 0, \ W^2 = 0 \). Recall that, in order for such a classification to be meaningful \( P^2 \) and \( W^2 \) have to be invariants of the theory. This is not the case for general relativity, unless we restrict to a subset of transformations selected for example by some physical criterion or by experimental constraints. For the solutions of the linearized vacuum Einstein equations the choice of the harmonic gauge does the job \( [18] \). There, the residual gauge freedom corresponds to the sole Lorentz transformations.

The polarization of gravitational fields represented by Eq. (7) can be estimated by looking at the transformation properties of the two independent physical components of this metrics under a rotation in the plane \((x, y)\) orthogonal to the propagation direction. This physical components, \( h_{tx} \) and \( h_{ty} \), have only one index in the \((x, y)\) plane orthogonal to the propagation direction \( \partial_u \). Under the infinitesimal rotation \( R \) in the \((x, y)\) -plane they transform as a vector.

Applied to any vector \((v_1, v_2)\) the infinitesimal rotation generator \( R \), has the effect

\[
Rv_1 = v_2, \quad Rv_2 = -v_1,
\]

from which

\[
R^2v_i = -v_i \quad i = 1, 2,
\]

so that \( iR \) has the eigenvalues \( \pm 1 \). Thus, the components of \( h_{\mu\nu} \) that contribute to the energy correspond to spin-1 fields. The reason why it’s commonly believed that spin-1 do not exist is that, in treating with the linearized theory, solutions are implicitly assumed to be square integrable: this is not the case for solution like metrics \( [7] \).

These solutions are interesting for (at least) two reasons. First, they are asymptotically flat (at least with a \( \delta \)-like singularity) in the plane transversal to the propagation direction. Second, they are both solutions of the linearized theory and of the exact theory, so that the spin-1 result cannot be attributed to the first order approximation.

### 2.5 Raychaudhuri equation

A possible approach to the analysis of physical effects of gravitational waves is based on the Raychaudhuri equation. This approach has the important advantage to be covariant so that it is not needed anymore to care about the choice
of a coordinate system. The evolution of a beam (congruence) of non light-like curves (i.e., trajectories of test masses which one would like to observe) on curved space-time is constrained by the Raychaudhuri equation:

\[ \nabla_A \theta = -R_{cb} A^c A^b + 2 \omega^2 - 2 \sigma^2 - \frac{\theta^2}{3} + \nabla_b (\nabla_A A)^b, \]  
\[ \omega_{ab} \equiv \nabla_{[a} A_{b]}, \quad \sigma_{ab} \equiv \nabla_{(a} A_{b)} - \theta (g_{ab} \pm A_a A_b) / 3 + (\nabla_A A)_a A_b \]  
\[ \nabla_A \equiv A^c \nabla_c, \quad \theta \equiv \nabla_b A^b, \quad A_c A^c = \mp 1 \]

where \( R_{cb} \) are the components of the Ricci tensor, \( A^c \) denotes the vector field tangent to the curve, the signature of \( g \) is \((+, -, -, -)\), the upper and the lower signs in Eq. (12) correspond respectively to the space-like and time-like case. The functions \( \theta, \sigma_{ab}, \omega_{ab} \) are called the expansion, shear and twist of the congruence.

In our case it is easier to perform computations in the "Peres system of coordinates" where the metrics read:

\[ ds^2 = dx^2 + dy^2 + \varphi(x, y, u) du^2 + 2 dudv, \quad \Delta_{(x, y)} \varphi = 0 \]  

and, as a timelike curve describing the motion, it is convenient to choose

\[ A^a = \left(0, 0, \frac{1}{\sqrt{\varphi - 2}}, -\frac{1}{\sqrt{\varphi - 2}}\right), \]

so that

\[ \theta = -\frac{\partial_u \varphi}{2 (\sqrt{\varphi - 2})^3}, \]

\[ 2 \omega^2 - 2 \sigma^2 = \frac{(\partial_x \varphi)^2 + (\partial_y \varphi)^2}{(\varphi - 2)^2} - \frac{(\partial_u \varphi)^2}{3 (\varphi - 2)^3}, \]

\[ \dot{\theta} = -\frac{5 (\partial_u \varphi)^2}{12 (\varphi - 2)^3} + \frac{(\partial_x \varphi)^2 + (\partial_y \varphi)^2}{(\varphi - 2)^2}, \]

in which, assuming that \( \varphi - 2 > 0 \), it is possible to isolate the physical effects of the spin-1 gravitational waves described by Eq. (14). It is clear that, due to the non trivial dependence of \( \dot{\theta} \) on the transverse coordinates, spin-1 gravitational waves have distinguishing features with respect to spin-2 gravitational waves. These effects are manifest, for example, in a distribution of test particles that, in the case of spin-1 waves will experience a permanent displacement that is, to say, a memory effects [11].

3 Detection of gravitational waves

The observable effect of a gravitational wave acting on two nearby test masses in free fall is mathematically described by the Jacobi geodesics deviation equation.
If the separation four-vector between two test masses is $Z^\mu$ the equation can be written in the following form:

$$\frac{D^2 Z^\mu}{D^2 s^2} = R^\mu_{\nu\rho\sigma} Z^\nu u^\rho u^\sigma$$

where $D$ denotes covariant derivation, $s$ is the proper time along the reference geodesic, $u^\nu = dx^\nu / ds$ is the velocity four-vector, $R^\mu_{\nu\rho\sigma}$ is the Riemann tensor.

It’s commonly believed that reasonable sources of gravitational waves are so far from earth that it’s always possible to consider the weak field approximation of Einstein’s field equations. In this approximation a gravitational wave is described by the perturbation $h_{\mu\nu}$ to the Minkowsky metric. Moreover, according with the standard textbook analysis \[29\], it is possible to choose a TT-gauge and express the geodesics deviation equation \[15\] in the following form:

$$\frac{d^2 X_i}{dt^2} = 1 \frac{1}{2} \delta_{ij} \frac{\partial^2 h_{TT}^{jk}}{\partial t^2} X^k,$$

where the geodesic deviation four-vector $Z^\mu$ has been chosen to be \((0, \mathbf{X})\)

where $\mathbf{X}$ denotes the position vector of one of two particles in the comoving reference frame of the other. The r.h.s. of equation \[16\] appears as an effective Newtonian force acting on test masses. An essential feature of these equation is the possibility of factorizing the time dependence (see \[19\] and \[20\] later).

Now let’s consider Eq. \[15\] for metrics \[8\] in weak field and small velocities \((ds \sim dt)\) limits. We get:

$$\frac{d^2 X_i}{dt^2} = \frac{\eta^{ij}}{(z - t)^2} \partial_k \partial_j w, \quad i = 1, 2, 3,$$  \[17\]

Equations \[16\] and \[17\] show that both spin-1 and spin-2 waves are transversal to the propagation direction. Equations \[17\] are less trivial than Eq.s \[16\]. They cannot be integrated to give a general solution. Moreover, the dependence on the time variable cannot be factorized. A reasonable expectation is a permanent deformation on the initial distribution of test masses, as in the case of the Christodoulou memory \[11\], a well known effect due to the non-linearity of the gravitational field. Similar effects involved in physical process generated by burst sources are called burst with memory (BWM) \[8\].

More in general, Eq. \[15\], for metric \[14\] and $Z^\mu = (0, \mathbf{X})$, reads:

$$\frac{d^2 X_i}{ds^2} = -g^{ij} X^j \partial_i \partial_t \varphi.$$

(18)

The above equation shows that one can have either attraction or repulsion according to the choice of the function $\varphi(x, y, u)$ which is constrained, outside the matter source, only by the condition to be a harmonic function of $x$ and $y$. For example, the choice $\varphi = \rho(x, y) \sigma(u)$, with $\rho$ a harmonic function of $x$ and $y$ and $\sigma$ a decreasing function of $u$, will give repulsion. This is not surprising because it is known from QFT that spin-odd bosons generate repulsion between particles of the same charge, the charge, in this case, corresponding to the mass.
3.1 Experimental devices

In this section the possibility of detection of spin-1 gravitational waves from the experimental point of view is considered. Nowadays there are two kind of ground based instruments able to investigate gravitational waves in the high frequency band ($1 \text{Hz} \div 10^4 \text{Hz}$): laser interferometers (IFOs) and resonant mass detectors \cite{47}. In spite of the high complexity of any experimental device (whether IFOs or resonant antennas), the detection principle is very simple. It essentially consists in a measurement of the displacement of test masses described by \eqref{15} and \eqref{16}. In the following we will review some basic features of the interaction mechanism between such instruments and gravitational waves.

In the case of IFOs the test masses are suspended mirrors and the displacement induced by gravitational waves is measured by laser interferometry. At present time the most sensible IFOs are the two LIGO (with coherent antenna patterns) and VIRGO (with antenna pattern coherent with the one of GEO). Despite of their higher sensitivity, IFOs have non isotropic antenna patterns, i.e. their sensitivity strongly depends on the relative orientation of the incoming wave and the plane of the IFO’s arms. Moreover, a single IFOs cannot perform spin measurements. We will not consider them further.

In the case of resonant antennas the detector is considered as an elastic body bathed by gravitational waves. The response of the detector can be studied by making use of the classical theory of elasticity \cite{26}: the generic infinitesimal mass element constituting the detector can be considered as a test mass and the relative displacement produced by gravitational waves is measured via the normal modes of oscillation. At present, cylindrical detectors (or Weber bars) are worldwide spread. They are generally three meters long and two tons heavy cylindrical objects made of aluminium. The most important resonant bars are those belonging to the IGEC network: ALLEGRO, AURIGA, EXPLORER, NAUTILUS, NIOBE \cite{1}. Like an IFOs a single Weber bar cannot perform spin measurements and have non-isotropic sensitivity. One could imagine a combined use of several suitably oriented antennas to obtain some information about spin. At present time this is not feasible because they are oriented with coherent antenna patterns to reduce the false alarm probability. Therefore, from the point of view of spin measurement, the whole array is equivalent to a single bar. In this contest we will focus our attention on spherical detectors because, unlike other detectors, in principle they are able to determinate the polarization of any incoming gravitational wave (and in a wider sense, to distinguish between different metric theories of gravitation \cite{49}) so they seem to be the most natural instruments to investigate the spin-1 gravitational waves.

In order to describe the effect of spin-2 gravitational waves on an infinitesimal mass element of the detector located at position $\vec{r} = (x^1, x^2, x^3)$ in a reference frame whose origin is at the center of the detector, it will be useful (also for later purpose) to denote by $f_{s=2}^{ij}(\vec{r}, t)$ the force in Eq. \eqref{19}:

$$f_{s=2}^{ij}(\vec{r}, t) = \frac{1}{2} \delta^{ij} \frac{\partial^2 h_{jk}(t)}{\partial t^2} x^k.$$  \hspace{1cm} (19)
Factorizing the radial and the angular dependence in \( f_{s=2}(\vec{r}, t) \) one obtains a result depending on spherical harmonics with \( l = 2 \) only:

\[
f_{s=2}(\vec{r}, t) = \frac{\partial}{\partial x^j} \sqrt{\frac{\pi}{15}} \delta_{ij} r^2 \sum_m h_m(t) Y_{2m}. \tag{20}
\]

Let \( \vec{u}(\vec{r}, t) \) be the displacement vector of an infinitesimal mass element, located at position \( \vec{r} \) with respect to the center of mass of the initially unperturbed solid with constant density \( \rho \) and Lamé coefficients \( \lambda \) and \( \mu \). When a force \( \vec{f}(\vec{r}, t) \) is acting on the body, the induced displacement vector \( \vec{u}(\vec{r}, t) \) field is solution of the following system of partial differential equations\(^4\):

\[
\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \vec{u}) + \mu \nabla^2 \vec{u} + \vec{f}(\vec{r}, t). \tag{21}
\]

In the following we will describe by external force \( \vec{f}(\vec{r}, t) \) the effects coming from the geodesic deviations\(^5\), whose explicit expression is given by (19) or equivalently (20). A generic solution of Eq. (21) is expressed as linear combination of eigenfunctions, with time dependence appearing in the coefficients \( a_m(t) \) only:

\[
\vec{u}(\vec{r}, t) = \sum_m a_m(t) \vec{u}_m(\vec{r}),
\]

where \( \vec{u}_m(\vec{r}) \) is eigenfunction of the equation

\[
-\rho \omega_m^2 \vec{u}_m = (\lambda + \mu) \nabla (\nabla \cdot \vec{u}_m) + \mu \nabla^2 \vec{u}_m \tag{22}
\]

and describes a free oscillation with frequency \( \omega_m \). This model applies to any resonant mass detector once we impose the boundary conditions determined by the detector’s shape, i.e. to cylindrical antennas in three dimensions\(^6\) or in one dimensional approximation\(^7\), to spherical detectors\(^8\,9\,10\).

The boundary conditions for sphere’s surface free from stress and strain are expressed by the following relation:

\[
\sigma_{ij} n_j = 0 \text{ at } r := |\vec{r}| = R, \tag{23}
\]

\( R \) being the radius of the sphere, \( n_j \) the components of the outgoing surface normal vector, \( \sigma_{ij} = \lambda u_{kk} \delta_{ij} + 2 \mu u_{ij} \) the stress tensor and \( u_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \) the strain tensor. Then, the time-dependent normal mode amplitude \( a_m(t) \) is solution of a driven harmonic oscillator equation:

\[
\ddot{a}_m(t) + \omega_m^2 a_m(t) = \frac{1}{\rho N_m} \int \vec{u}_m(\vec{r}) \cdot \vec{f}(\vec{r}, t) d^3 r, \tag{24}
\]

\( ^4 \) Dissipation terms can be easily included in realistic cases

\( ^5 \) In the context of resonant antennas \( \vec{f}(\vec{r}, t) \) generally contains two contributions: one from geodesic deviations and one describing the forces between the surface of the elastic body and other objects eventually matched with it (i.e. resonant transducers suitably tuned to the natural frequencies of oscillation).
with \( N_m \) normalization constants. All the interaction with external world (only gravitation in this case) can be included in the r.h.s. as additional terms of effective force acting on every normal mode. With some straightforward calculations Eq. (22) with boundary conditions (23) can be solved giving the following solutions:

\[
\vec{u}(\vec{r}) = \frac{C_0}{k_{irr}^2} \nabla \varphi(\vec{r}) + \frac{iC_1}{k_{\text{div-free}}^2} \vec{L} \psi(\vec{r}) + \frac{iC_2}{k_{\text{div-free}}^2} \nabla \times \vec{L} \psi(\vec{r}), \quad (25)
\]

with wave numbers \( k_{irr} = \frac{\rho \omega^2}{\lambda + 2 \mu} \), \( k_{\text{div-free}} = \frac{\rho \omega^2}{\mu} \), where \( \varphi(\vec{r}) \) and \( \psi(\vec{r}) \) are scalar functions solutions of Helmholtz’s equation, \( \vec{L} \) is the angular momentum operator \( \vec{L} = -i \vec{r} \times \nabla \), \( C_0, C_1, \) and \( C_2 \) are constants, their numerical value depending on the boundary conditions (23) in the specific case. To get regular solutions in \( r = 0 \) the scalar functions \( \varphi(\vec{r}) \) and \( \psi(\vec{r}) \) must take the form:

\[
\varphi(\vec{r}) = j_l(qr) Y_{lm}(\vec{\theta}), \quad \psi(\vec{r}) = j_l(kr) Y_{lm}(\vec{\theta}),
\]

where \( j_l \) is a spherical Bessel function and \( Y_{lm} \) a spherical harmonic.

Normal modes of oscillation can be divided in two family: Thoroidal modes \( (C_0 = C_2 = 0) \) and Spheroidal modes \( (C_1 = 0) \). The latter can be expressed in the form:

\[
\vec{u}_{nlm}(\vec{r}) = A_{nl}(r) Y_{lm}(\vec{\theta}) - B_{nl}(r) R \nabla Y_{lm}(\vec{\theta}), \quad (26)
\]

with \( A_{nl}(r) \) and \( B_{nl}(r) \) combinations of Bessel functions.

Let’s go back to the effective force (24) acting on every normal mode. If we chose for \( \vec{f}(\vec{r}, t) \) the expression (20), the only non-zero integrals on the r.h.s. of Eq. (24) will be those containing the eigenfunctions \( \vec{u}_{nlm} \) with spherical harmonics \( Y_{2m}(\theta, \phi) \):

\[
\vec{u}_{lm} = \left[ \alpha_l(r) \vec{u} + \beta_l(r) R \nabla \right] Y_{lm}(\theta, \phi).
\]

Functions \( \alpha_l(r) \) and \( \beta_l(r) \) are combinations of spherical Bessel function of order 2 and determinate the motion in radial and tangential direction respectively. According with this standard analysis, only few modes of the spheres, the five quadrupolar modes \( (l = 2) \), will be coupled to gravity. Moreover, several studies \[28\] show that a finite number of resonators opportunely tuned to this mode’s frequency, will suffice to completely solve the problem of deconvolution of the signal revealed by spheres.

Suppressing the index \( l \) we can write the r.h.s. of Eq. (24) as:

\[
F_m(t) = \int_{\text{sphere}} \vec{u}_{lm} \cdot \vec{f}_{s=2} d^3 r = \frac{1}{2} h_m(t) \gamma MR, \quad (27)
\]

the constant \( \gamma \) depending on the physical properties of the elastic medium.

In the case of spin-1 gravitational waves relation (20) cannot be used. If we want to take into account the whole interaction of spin-1 waves with spherical
detectors, we will be compelled to solve the integral in the r.h.s. of Eq. (24) making use of geodesic deviation (17) and (18) whose r.h.s. will be denoted by \( f'_{s=1}(\vec{r}, t) \). Let us consider the most simple case choosing the \( l = 0 \) normal mode (25). For \( l = 0 \) toroidal modes are absent while the non vanishing contribution to spheroidal modes (26) comes from the term \( A_{nl}(r)Y_{lm}(\hat{\vec{n}}) \vec{n} \). Even if it cannot be explicitly evaluated, there are no reasons for integral on the r.h.s. of (24) to vanish

\[
F_0(t) = \int_{\text{sphere}} \vec{n}_{n0} \cdot \vec{f}_{s=1} d^3r \neq 0
\]

This computation, which heavily depends on the choice of the harmonic function \( w \), does not lead to a general solution as in the standard spin-2 case, but stresses the interaction between an acoustic detector and a spin-1 gravitational wave. Moreover, since \( F_0 \) is non vanishing, spherical modes with \( l = 0 \) are activated even in General Relativity too. Thus, the standard identification of normal mode’s index \( l \) with the spin component of the driving gravitational wave is not ensured. A further step in this direction would be to test the coupling between normal modes and spin components in more general cases.

4 Conclusions

It is still deep-seated the belief that spin-1 gravitational waves cannot be present, not only in General Relativity but in every metric theory of gravitation. It has been shown that this erroneous belief derives from the implicit assumption of considering only square integrable solutions of the linearized Einstein equations: indeed it is not true that it is always possible to reduce to TT-gauge and to remove all the non spin-2 components by a gauge transformation.

Once we accept that gravitational waves may have spin-1 and may be emitted by reasonable sources, it becomes important to define the experimental conditions necessary to observe their spin. The first concrete possibility could be the use of spherical detectors. Due to their resonant spectrum, described by Eq. (25), spherical detectors are the ideal instruments for studying the polarization. According to the standard theory, spherical devices are prepared to observe only spin-0 and spin-2 waves. In such an analysis the \((x, y)\)–coordinates dependence is not taken into account. Clearly it will be difficult to detect spin-1 gravitational waves with instruments having dimensions smaller than typical length scale of spatial variation of the waves. This does not imply that spin-1 waves do not exist at all or that it is not possible to conceive new experimental apparatus capable to measure their spin.

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