Local harmonic approaches with approximate cranking operators

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Abstract

Methods of large amplitude collective motion in the adiabatic limit are examined with a special emphasis on conservation laws. We show that the restriction to point transformations, which is a usual assumption of the adiabatic time-dependent mean-field theory, needs to be lifted. In order to facilitate the application of large amplitude collective motion techniques, we examine the possibility of representing the RPA normal-mode coordinates by linear combinations of a limited number of one-body operators. We study the pairing-plus-quadrupole model of Baranger and Kumar as an example, and find that such representations exist in terms of operators that are state-dependent in a characteristic manner.

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I. INTRODUCTION

The selection of collective variables is an important problem in the study of large amplitude collective motion. In the constrained Hartree-Fock (CHF) or Hartree-Fock-Bogoliubov (CHFB) calculations, the collective subspaces are generated by a small number of one-body constraint (also called generalised cranking) operators which are most commonly taken to be multipoles that can be represented as homogeneous polynomials in the coordinates, i.e., $r^L Y_{L,K}$. In realistic calculations of processes such as fission, the number of coordinates needed to describe the full nuclear dynamics can easily become larger than what can be dealt with in satisfactory manner, and a method to determine the optimal combination needs to be devised. Furthermore, there is no a priori reason to limit oneself to multipole operators, and the cranking operators should be determined by the nuclear collective dynamics itself, from amongst all possible one-body operators.

In our past work, we have investigated a theory of adiabatic large amplitude collective motion as a method to generate such self-consistent collective subspaces (see Ref. [1] and references therein). The key ingredient of the method is the self-consistent determination of the constraint operator, and as such it may provide an answer to the selection question discussed above. Using the local harmonic version of the theory [1], the local harmonic approximation (LHA), we have recently embarked on a study of the properties of large amplitude collective motion in systems with pairing. So far we have dealt with two simple models: The first one is a semi-microscopic model of nucleons interacting through a pairing force, coupled to a single harmonic variable [2]. The second [3] is a fully microscopic O(4) model which may be regarded as a simplified version of the pairing-plus-quadrupole (P+Q) Hamiltonian studied in this paper. It has turned out that the self-consistent collective coordinate obtained by the LHA accounts quite well for the exact dynamics of the models. We have also shown that the CHFB calculations using the mass-quadrupole operator as the constraint operator can result in incorrect results [3].

In this paper, which is an attempt to move towards realistic nuclear problems, we will encounter some problems that need to be resolved first: The main problem is the treatment of conservation laws. In studies of large amplitude collective motion, the mean-field states often do not have the symmetries of the Hamiltonian. Then, the spurious (Nambu-Goldstone) degrees of freedom, such as the translation and rotation of nucleus, inevitably appear, which must be properly separated out from the other collective degrees of freedom like nuclear shape change. In the models discussed in Refs. [2,3], the spurious degrees of freedom could be removed by hand thanks to the simplicity of the models, however in realistic nuclear problems, it is almost impossible to do so. The second and more obvious problem is the increase in the number of degrees of freedom. Within the time-dependent Hartree-Bogoliubov (TDHB) approximation, the dynamics of the models of Refs. [2,3] is described by only a modest number of degrees of freedom (4 ∼ 12). For realistic problems in heavy nuclei, on the other hand, we need to deal with at least tens of thousands, and possibly millions of degrees of freedom! These two points, the effect of the conservation laws and the applications to realistic problems, are the main issues discussed in Sec. I.

In the LHA, the collective path (or collective manifold for more than one coordinate) is determined by solving the CHFB problem with a cranking operator which is self-consistently determined by the local random-phase approximation (RPA). It is of course a well-known
fact that the spurious modes decouple from the physical modes in the RPA. However this is true only when the system is at a minimum of the mean-field potential. Since the local RPA needs to be solved at non-equilibrium points as well, we need to study the properties of the LHA equations. It will be shown that the symplectic version of the LHA combined with an extended adiabatic approximation properly takes account of conservation laws. It will also be shown that this formalism naturally explains the role of the chemical potential term $-\lambda N$ in the Hamiltonian of the quasiparticle RPA.

Since the LHA procedure requires one to solve the local RPA at large number of points along the collective path (manifold), the large dimension of the RPA matrix for realistic problems leads to very heavy computational burdens. Thus, instead of solving the local RPA, we would like to preselect a small group of operators, and find the RPA modes as an optimal combination of these operators at each point of the collective surface. This means that we restrict the RPA diagonalisation to this small space, rather than deal with the full millions-by-millions RPA matrix [4]. In this paper, we attempt to find a set of one-body operators which approximate well the self-consistent cranking operator. The pairing-plus-quadrupole (P+Q) Hamiltonian is adopted for the analysis because the RPA calculation is relatively easy due to the separability of the force (some of the results for the Sm isotopes have already been reported in Ref. [5]). Since the P+Q model is known to be able to describe collective phenomena involving both pairing and quadrupole degrees of freedom [6–14], we expect that a similar choice for the set of one-body operators should work for other realistic Hamiltonians as well.

The P+Q model is probably one of the most simple and successful nuclear Hamiltonians which allows us to discuss realistic problems involving pairing and quadrupole degrees of freedom. Baranger and Kumar analysed in great detail the (adiabatic) collective motion in the P+Q model assuming that the collective variables are the mass quadrupole operators [10–14]. Thus, they reduced the large number of two-quasiparticle (2qp) degrees of freedom (for this model of the order of a thousand) to only two “collective” coordinates, $\beta$ and $\gamma$. However, our previous study of the $O(4)$ model [3] suggests that even for such simple Hamiltonians, the self-consistent collective coordinates are often not as simple as one might think. The P+Q model (with an additional quadrupole pairing interaction) has also been studied by Kishimoto and Tamura [15,16] using boson expansion techniques. They found that the coupling to (non-collective) 2qp excitations is very important, which also suggests that the collective coordinates are not just the mass-quadrupole ones. In this paper we show that the normal-mode coordinate of the random-phase approximation is quite different from the mass-quadrupole operator, and how the structure of the self-consistent collective coordinates changes as the nucleus is deformed.

In Sec. II, we give an account of the general theoretical arguments. The problem of the spurious modes and the extension of the RPA formalism are discussed in Sec. II B and II C. In Sec. II D, the method of projection of the RPA equation onto a set of one-body operators is described. Numerical calculations for the RPA and the projected RPA for the P+Q model are given in Sec. III and finally we give the summary and outlook in Sec. IV.
II. LOCAL HARMONIC FORMULATIONS AND CONSERVATION LAWS

In this section, we use a summation convention where the repeated appearance of the same symbol for upper and lower index indicates a sum over this symbol for all allowed values. We also use the convention that a comma in a lower index indicates the derivative with respect to the coordinate, i.e., $F_\alpha = \partial F/\partial \xi^\alpha$.

A. The TDHFB theory and classical constants of motion

Utilising the TDHFB theory we transcribe the original quantum Hamiltonian into the classical Hamiltonian. We follow Ref. [17] to define the canonical variables. We denote with respect to the coordinate, i.e., $\langle \xi, \pi \rangle \equiv \langle \Psi(t)|a_j^\dagger a_i|\Psi(t)\rangle$, where $|\Psi(t)\rangle$ is assumed to be normalised at all times ($\langle \Psi(t)|\Psi(t)\rangle = 1$) and the quasiparticle operators ($a_i, a_i^\dagger$) are defined with respect to a reference state $|\Psi_0\rangle$. Based on a classical Holstein-Primakoff mapping [18,19],

$$\bar{\rho}_{ij} = [\beta \beta^\dagger]_{ij}, \quad \bar{\kappa}_{ij} = [\beta (1 - \beta^\dagger \beta)^{1/2}]_{ij},$$

the density and pair matrices are now mapped onto complex canonical variables $\beta_{ij}$. As usual real canonical variables $\xi$ and $\pi$ are introduced as $\beta = (\xi + i\pi)/\sqrt{2}$. In this canonical parametrisation, the reference state $|\Psi_0\rangle$ corresponds to the origin of the phase space, where $\xi = \pi = 0$.

We shall specifically look at one-body hermitian operators $R$ and anti-hermitian operators $S$,

$$R = R_0 + \sum_{i>j} R(ij)(a_j^\dagger a_i + a_j a_i) + \sum_{ij} R'(ij) a_j^\dagger a_i,$$

$$S = \sum_{i>j} S(ij)(a_j^\dagger a_i^\dagger - a_j a_i) + \sum_{ij} S'(ij) a_j^\dagger a_i,$$

where $R_0 = \langle \Psi_0|R|\Psi_0\rangle$ and all the matrix elements, $R(ij), R'(ij), S(ij)$ and $S'(ij)$, are assumed to be real. Here $R'(ij)$ ($S'(ij)$) are symmetric (anti-symmetric) with respect to permutation of the indices $i$ and $j$. The corresponding classical representations, $R(\xi, \pi)$ and $S(\xi, \pi)$, can be found by replacing the fermion pair operators $a_j^\dagger a_i$ and $a_j a_i$ by the density and pair matrices $\bar{\rho}_{ij}$ and $\bar{\kappa}_{ij}$ of Eq. (2.2). Expanding $\bar{\rho}$ and $\bar{\kappa}$ with respect to $\pi$, we find that $R(\xi, \pi)$ consists of even-order terms in $\pi$ while $S(\xi, \pi)$ contains odd-order terms only,

$$R(\xi, \pi) = R^{(0)}(\xi) + \frac{1}{2} R^{(2)} \xi \pi + \mathcal{O}(\pi^4),$$

$$S(\xi, \pi) = S^{(1)} \xi \pi + \mathcal{O}(\pi^2).$$

Starting from these equations, pairs of two-quasiparticle (2qp) indices $(ij, kl, \cdots)$ are denoted collectively by a single Greek index $(\alpha, \beta, \cdots)$. The 2qp matrix elements with respect to
the reference state, \( R(ij) \) and \( S(ij) \) which are now denoted as \( R(\alpha) \) and \( S(\alpha) \) respectively, are related to the derivatives of the classical representations,

\[
R(\alpha) = \left. \frac{1}{\sqrt{2}} \frac{\partial R}{\partial \xi^\alpha} \right|_{\xi=\pi=0} = \left. \frac{1}{\sqrt{2}} R^{(0)}_{\alpha} \right|_{\xi=0},
\]

\[
S(\alpha) = \left. \frac{i}{\sqrt{2}} \frac{\partial S}{\partial \pi^\alpha} \right|_{\xi=\pi=0} = \left. \frac{i}{\sqrt{2}} S^{(1)}_{\alpha} \right|_{\xi=0}.
\]

The classical Hamiltonian is given by

\[
\mathcal{H}(\xi, \pi) = \langle \Psi(t)|H|\Psi(t) \rangle,
\]

where the right hand side is calculated in terms of the density and pair matrices \([2.2]\). The expansion of \( \mathcal{H}(\xi, \pi) \) in powers of \( \pi \) defines the potential \( V(\xi) = \mathcal{H}(\pi = 0) \) (zeroth order) and the mass parameter \( B^\alpha_{\beta}(\xi) \) (second order).

\[
\mathcal{H} = V(\xi) + \frac{1}{2} B^\alpha_{\beta} \pi^\alpha \pi^\beta + \mathcal{O}(\pi^4),
\]

\[
B^\alpha_{\beta}(\xi) = \left. \frac{\partial^2 \mathcal{H}}{\partial \pi^\alpha \partial \pi^\beta} \right|_{\pi=0}.
\]

The terms of order of \( \pi^4 \) and higher in the Hamiltonian are neglected in the adiabatic time-dependent mean-field theory. The tensor \( B^\alpha_{\beta} \), which is defined as the inverse of \( B^\alpha_{\gamma} B_{\gamma\beta} = \delta^\alpha_{\beta} \), plays the role of metric tensor in the Riemannian formulation of the local harmonic approximation (LHA) as discussed in Sec. II B.

It is well-known that the TDHFB equation preserves the symmetries of the quantum Hamiltonian: If the one-body operator \( P \) commutes with the Hamiltonian, \([P, H] = 0\), the classical representation of \( P \), \( \mathcal{P}(\xi, \pi) = \langle \Psi(t)|P|\Psi(t) \rangle \), is a classical constant of motion. Consequently, the Poisson bracket between \( \mathcal{P} \) and \( \mathcal{H} \) must vanish,

\[
\{ \mathcal{P}, \mathcal{H} \}_{\text{PB}} = 0,
\]

which can be used to show that

\[
\mathcal{P}^{(1)}_{\alpha} V_{\alpha} = 0,
\]

\[
\mathcal{P}^{(0)}_{\alpha} B_{\alpha\beta} - \mathcal{P}^{(2)}_{\alpha\beta} V_{\alpha} = 0,
\]

which are the terms of zeroth and first order in \( \pi \) in Eq. \([2.12]\), respectively. Here, \( \mathcal{P}^{(0)} \), \( \mathcal{P}^{(1)} \) and \( \mathcal{P}^{(2)} \) are defined, according to Eqs. \([2.5]\) and \([2.6]\), as terms of the zeroth, first and second order in \( \pi \), respectively. The equations \([2.13]\) and \([2.14]\) hold at arbitrary points in configuration space.

**B. Riemannian version of the local harmonic approximation (LHA)**

The adiabatic approach to large amplitude collective motion is based on a search for the collective (and non-collective) coordinates \( q^\nu \) by performing a point transformation of the original coordinates \( \xi^\alpha \).
\[ q^\mu = f^\mu(\xi), \quad \xi^\alpha = g^\alpha(q) \quad (\mu, \alpha = 1, \cdots, n). \]  

(2.15)

The conjugate momenta are given by

\[ p_\mu = g^\alpha_{,\mu} \pi_\alpha, \quad \pi_\alpha = f^\mu_{,\alpha} p_\mu. \]  

(2.16)

The adiabatic Hamiltonian (2.10) is transformed into

\[ \bar{H} = \bar{V}(q) + \frac{1}{2} \bar{B}^{\mu\nu} p_\mu p_\nu + \mathcal{O}(p^4), \]  

(2.17)

\[ \bar{V}(q) = V(g(q)), \quad \bar{B}^{\mu\nu} = f^\mu_{,\alpha} B^{\alpha\beta} f^{\nu}_{,\beta}. \]  

(2.18)

The point transformation on the adiabatic Hamiltonian gives the same result as the one obtained by first performing this transformation on the exact Hamiltonian and then taking the adiabatic limit. This is not true for a general canonical transformation, specifically for the extended adiabatic transformation to be discussed in the next section.

We divide the new coordinate set \( q^\mu \) into three subsets, the collective coordinates \( q^1 \), the spurious coordinates \( q^I, I = 2, \cdots, M + 1 \), and the non-collective coordinates \( q^a, a = M + 2, \cdots, n \). For simplicity, we assume here a single collective coordinate. The spurious coordinates are related to the symmetry breaking in the mean field (Nambu-Goldstone modes). The local harmonic approximation consists of two sets of equations, the force condition and the local RPA, which define the collective manifold in configuration space (see Ref. [1] for a complete description). When spurious modes are present, the force condition is modified slightly,

\[ V_{,\alpha} = \lambda_1 f_{,\alpha}^1 + \lambda_I f_{,\alpha}^I, \]  

(2.19)

where the second term in the right-hand side is the additional term. The local RPA equation can be formulated in different ways [1], and in this section we discuss the case of the Riemannian formulation, based on

\[ V_{,\alpha\gamma} B^{\gamma\beta} f_{,\beta}^1 = (h\Omega)^2 f_{,\alpha}^1, \]  

(2.20)

\[ g_{,\beta}^1 V_{,\alpha\gamma} B^{\gamma\beta} = (h\Omega)^2 g_{,\beta}^\beta. \]  

(2.21)

Here the covariant derivative \( V_{,\alpha\gamma} \) is defined by

\[ V_{,\alpha\beta} = V_{,\alpha\beta} - \Gamma_{,\alpha\beta} V_{,\gamma}, \]  

(2.22)

where the affine connection \( \Gamma \) is defined with the help of metric tensor \( B_{,\alpha\beta} \) as

\[ \Gamma_{,\beta\gamma} = \frac{1}{2} B^{\alpha\delta} (B_{,\delta\beta,\gamma} + B_{,\delta\gamma,\beta} - B_{,\beta\gamma,\delta}). \]  

(2.23)

The collective path will be determined by solving Eqs. (2.19) and (2.20) selfconsistently. It consists in finding a series of points where the local RPA eigenvector \( f_{,\alpha}^1 \) satisfies the force condition at the same time.

The question at present is whether the solution of Eqs. (2.20) and (2.21) is orthogonal to the spurious modes. The spurious coordinates are normally linked to the one-body operators \( P_I \) which commute with the Hamiltonian. It is convenient to discuss separately the case in
which the symmetry operators \( P_I \) are hermitian with real matrix elements and the case in which \( P_I \) are anti-hermitian with real matrix elements. First, let us discuss the latter case. As is seen in Eq. (2.6), neglecting the higher-order terms in \( \pi \), the classical representation of the symmetry operator is a classical momentum variable \( p_I \) that is a constant of motion,

\[
p_I = \langle \Psi | P_I | \Psi \rangle = \mathcal{P}^{(1)\alpha} p_{\alpha} + \mathcal{O}(\pi^3).
\]  

(2.24)

From this, we can immediately see that \( g_{I}^{\alpha} = \mathcal{P}^{(1)\alpha}_I \). Differentiating Eq. (2.13) with respect to \( \xi^\beta \), keeping in mind that \( \mathcal{P}^{(1)\alpha}_I \) must be proportional to \( g_{I}^{\alpha} \), we find

\[
V_{\alpha\beta} g_{I}^{\alpha} + V_{\alpha} g_{I}^{\alpha} f_{\beta}^{\mu} = 0.
\]  

(2.25)

Using an identity for the coordinate transformation [1],

\[
\bar{\Gamma}_{\mu\nu}^\lambda + \Gamma_{\beta\gamma}^\alpha g_{\mu}^{\beta} g_{\nu}^{\gamma} - \bar{\Gamma}_{\mu
u}^\lambda g_{\alpha}^{\lambda} = 0,
\]  

(2.26)

this leads to

\[
\bar{V}_{\alpha\gamma} B^{\gamma\beta} = -\bar{V}_{\alpha\beta} \bar{\Gamma}_{\mu\nu}^\lambda \bar{B}_{\mu\nu}^{\lambda} g_{\nu}^{\beta},
\]  

(2.27)

where \( \bar{\Gamma} \) is defined in the same way as \( \Gamma \), (2.23), with obvious change of \( B \) to \( \bar{B} \). At the equilibrium \( V_\alpha = \bar{V}_{\mu} = 0 \), and since the right-hand side of Eq. (2.27) vanishes, the spurious modes are indeed the zero-energy solutions of the RPA equation. Thus, all finite-energy solutions are automatically orthogonal to the spurious modes. However, at non-equilibrium points, the \( g_{I}^{\alpha} \) are no longer RPA eigenmodes and there is no guarantee that the collective mode \( g_{I}^{\alpha} \) obtained from the local RPA equation (2.21) is free from spurious admixture. In Ref. [2], the local RPA equation was modified to insure that \( f_{I}^{\alpha} g_{I}^{\alpha} = g_{I}^{\alpha} f_{I}^{\alpha} = 0 \). However, in this formulation, we need to calculate both \( f_{I}^{\alpha} \) and \( g_{I}^{\alpha} \), which means that we have to solve the Thouless-Valatin equations for the spurious modes [22,23].

In the second case, where the symmetry operator \( P_I \) is hermitian, we have coordinates \( q^I \) which are the constants of motion:

\[
q^I = \langle \Psi | P_I | \Psi \rangle = \mathcal{P}^{(0)I} \xi + \frac{1}{2} \mathcal{P}^{(2)\alpha\beta} \pi^\alpha \pi^\beta + \mathcal{O}(\pi^4).
\]  

(2.28)

If we limit ourselves to point transformations, we find \( \mathcal{P}^{(0)I} \xi = f^I(\xi) \) and we should ignore all terms depending on \( \pi \). At equilibrium Eq. (2.14) implies that the spurious modes \( f_{\alpha}^I \) correspond to zero eigenmode of the mass matrix \( B_{\alpha\beta} f_{\beta}^I = 0 \). Therefore we have a zero-energy mode,

\[
V_{\alpha\gamma} B^{\alpha\gamma} f_{\alpha}^I = 0.
\]  

(2.29)

Away from equilibrium, however, the \( f_{\alpha}^I \) are no longer RPA eigenmodes, due to the second term in (2.14) which is related to the terms we have neglected in Eq. (2.28).

---

1 If \( \bar{B}_{\mu\nu} \) and \( \bar{\Gamma}_{\mu\nu}^\lambda \) are block-diagonal for the spurious and the collective spaces on the collective path, the spurious solutions correspond to finite energy eigenvalues but are orthogonal to the collective mode. This is the case for HF problem in \( ^{28}\text{Si} \) as discussed in Refs. [1][2][3].
In summary, the Riemannian LHA can automatically separate out the spurious modes only at equilibrium \( V_\alpha = 0 \). Since the local RPA at equilibrium is nothing but the conventional RPA, the decoupling of the spurious modes is well-known. However, at non-equilibrium points, the spurious modes can in general mix with the collective one. This undesirable mixing disappears in the symplectic LHA with an extended adiabatic treatment described in the next section.

C. Symplectic LHA and extended adiabatic transformation

The symplectic version of the LHA is formulated with an affine connection

\[
\tilde{\Gamma}^\alpha_{\beta\gamma} \equiv g^\alpha_{\mu \beta} f^\mu_{\beta \gamma},
\]

which can be written in the usual form (2.23) by replacing the metric tensor \( B_{\alpha\beta} \) by

\[
K_{\alpha\beta} \equiv \sum_\mu f^\mu_{\alpha \beta} f^\mu_{\beta \gamma}.
\]

Since this metric depends on the final coordinates \( q^\mu \), we cannot define it from the beginning in contrast to the Riemannian metric \( B_{\alpha\beta} \). In this formulation, the configuration space is assumed to be flat and diagonal \( (K_{\mu\nu} = \delta_{\mu\nu}) \) in the final coordinate system \( q^{\mu} \). The covariant derivative is defined by

\[
\tilde{V}_{\alpha\beta} = V_{\alpha\beta} - \tilde{\Gamma}^\gamma_{\alpha\beta} V_\gamma
\]

\[
= V_{\alpha\beta} - f^\mu_{\alpha\beta} \tilde{V}_\mu.
\]

For the case that the momenta \( p_I \) are constants of motion, we can prove that the spurious modes are the zero-energy solutions of the symplectic local RPA equation. Indeed, differentiating the chain relation \( g^\alpha_{\mu \beta} f^\mu_{\beta \gamma} = \delta^\alpha_{\beta} \) with respect to \( q^\nu \), we obtain

\[
g^\alpha_{\mu \nu} f^\mu_{\beta \gamma} = -g^\alpha_{\mu \nu} g^\gamma_{\beta \rho} f^\rho_{\beta \gamma}.
\]

Utilising this equation and the definition of the covariant derivatives (2.32), Eq. (2.25) can be rewritten as

\[
\tilde{V}_{\alpha\beta} g^\alpha_{,I} = 0.
\]

This is a consequence of the TDHFB equation of motion. Therefore the spurious modes are zero-energy solutions of the symplectic LHA equations, anywhere in the configuration space.

In the case that the coordinates \( q^I \) are constants of motion, we need to lift the restriction to point transformations, based on the power expansion with respect to \( \pi \) keeping the adiabatic assumption. Instead of Eqs. (2.15) and (2.16), we generalise these to

\[
q^\mu = f^\mu (\xi) + \frac{1}{2} f^{(1)\mu\alpha\beta} (\xi) \pi_\alpha \pi_\beta + \mathcal{O}(\pi^4),
\]

\[
\xi^\alpha = g^\alpha (q) + \frac{1}{2} g^{(1)\alpha\mu\nu} (q) p_\mu p_\nu + \mathcal{O}(p^4),
\]

and
\[ p_\mu = g_\mu^{\alpha} \pi_\alpha + \mathcal{O}(p^3), \quad (2.37) \]
\[ \pi_\alpha = f_{\alpha}^{\mu} p_\mu + \mathcal{O}(p^3), \quad (2.38) \]

where the terms cubic in momenta do not play a role in the modification of the theory. The original Hamiltonian \((2.10)\) is transformed to, up to second order in \(p\) only,

\[ H(q, p) = \tilde{V}(q) + \frac{1}{2} \tilde{B}^{\mu\nu} p_\mu p_\nu, \quad (2.39) \]
\[ \tilde{B}^{\mu\nu} = f_{\alpha}^{\mu} B_{\alpha\beta} f_{\nu}^{\beta} + \mathcal{O}(p^3). \quad (2.40) \]

Substituting Eq. \((2.35)\) in Eq. \((2.36)\), using Eq. \((2.38)\), we find the relation

\[ g^{(1)\alpha\mu\nu} g_{\beta}^{\gamma} = -f^{(1)\lambda\beta\gamma} g_{\lambda}. \quad (2.41) \]

From the canonicity condition \(\{q^\mu, q^n\}_{PB} = 0\), we also find

\[ f_{\alpha}^{\mu} f^{(1)\nu\alpha\beta} = f_{\alpha}^{\mu} f^{(1)\mu\alpha\beta}. \quad (2.42) \]

The major difference between the use of the extended adiabatic transformation and a point transformation is the modification of mass parameter,

\[ \tilde{B}^{\alpha\beta} \equiv g_{\mu}^{\alpha} B^{\mu\nu} g_{\nu}^{\beta} = B^{\alpha\beta} - \omega_\mu f^{(1)\mu\alpha\beta}. \quad (2.43) \]

Here we have used the relation \((2.41)\) to obtain the last equation. The local RPA equations have the same forms as the Riemannian RPA equations, \((2.20)\) and \((2.21)\), after replacing \(V_{\alpha\beta}\) by \(\omega_{\alpha\beta}\) and \(B_{\alpha\beta}\) by \(\tilde{B}^{\alpha\beta}\).

Let us return to the spurious modes. For hermitian constants of motion, i.e., coordinates \(q^I\) as in Eq. \((2.28)\), we can identify \(f^I(\xi) = \mathcal{P}_I^{(0)}(\xi)\) and \(f^{(1)I\alpha\beta} = \mathcal{P}_I^{(2)\alpha\beta}\) for the extended adiabatic transformation. Utilising the TDHFB equation \((2.14)\) and the canonicity condition \((2.42)\), we find

\[ \tilde{B}^{\alpha\beta} f_{\nu,\beta} = B^{\alpha\beta} f_{\nu,\beta} - f^{(1)\mu\alpha\beta} \tilde{V}_\mu f_{\nu,\beta} = 0. \quad (2.44) \]

Therefore, anywhere in the space, it is guaranteed that the spurious modes are zero-energy solutions of the symplectic LHA with the extended adiabatic transformation.

It is illustrative to discuss the difference between the Hartree-Fock (HF) and Hartree-Fock-Bogoliubov (HFB) approaches. The HF state is at an equilibrium point of the potential, \(V_{\alpha} = 0\), where the force condition \((2.19)\) is satisfied with \(\lambda_1 = \lambda_I = 0\). We also see \(\omega_{\alpha\beta} = V_{\alpha\beta} = V_{\alpha\beta}\) and \(\tilde{B}^{\alpha\beta} = B^{\alpha\beta}\). Thus, any version of the LHA is equivalent to the conventional RPA. On the other hand, the HFB state is not a real equilibrium. We have a spurious coordinate associated with the particle number \(q^N = \langle \Psi | N | \Psi \rangle\) and the gradient of the potential has a non-zero component along the direction of \(q^N\), \(\tilde{V}_N \neq 0\) (the other
components of the gradient are all zero). The force condition is again satisfied with \( \lambda_1 = 0 \) and \( \lambda_N = \bar{V}_N \), but neither \( \bar{V}_{\alpha\beta} = V_{\alpha\beta} \) nor \( \bar{B}^{\alpha\beta} = B^{\alpha\beta} \) hold any more. Instead, we have

\[
\bar{B}^{\alpha\beta} = B^{\alpha\beta} - \bar{V}_{,N} f^{(1)N\alpha\beta},
\]
\[
\bar{V}_{\alpha\beta} = V_{,\alpha\beta} - \bar{V}_{,N} f_{,\alpha\beta}^N.
\]

Since \( \bar{V}_N \) is nothing but the chemical potential \( \lambda_N \) at the (constrained) HFB minimum, the symplectic version of the LHA with the extended adiabatic approximation is equivalent to the conventional quasiparticle RPA with the constrained Hamiltonian, \( H' = H - \lambda_N N \). This guarantees the separation of zero-energy spurious modes, while this is not the case for the Riemannian version of the LHA. This symplectic formulation also illuminates why the quasiparticle RPA at the HFB state should be performed using the Hamiltonian \( H' \), not the original \( H \).

As we have seen above, the symplectic formulation of LHA with the extended adiabatic transformation has an advantage over its Riemannian version. Unfortunately we do not know a general method to calculate \( f^{(1)\mu\alpha\beta} \). It may be possible to calculate this quantity if the collective coordinate is explicitly given by a combination of elementary one-body operators (see the discussion in Sec. [IV]).

### D. Projected LHA and the spurious modes

One of the main obstacles in applying the LHA techniques described in previous sections to realistic nuclear problems is the fact that we need to diagonalise an RPA matrix of large dimensionality at each point of the collective path. In this section, we describe a method to approximate the RPA eigenvector by taking linear combinations of preselected one-body operators, generalising the method discussed in Ref. [4].

First we select a small number of one-body operators, \( F^{(k)}, k = 1, \cdots, n' \), assuming that the collective coordinate \( q^1 \) can be approximated locally as a linear combination of these operators,

\[
q^1 \approx \langle \Psi | \hat{f} | \Psi \rangle, \quad \hat{f} = \sum_{k=1}^{n'} c_k F^{(k)}.
\]

This means that the RPA eigenmodes \( f,\alpha \) is projected onto the subspace spanned by \( \{ F^{(k)} \} \),

\[
f,\alpha \approx \bar{f},\alpha = \sum_{k=1}^{n'} c_k F^{(k)},
\]

where \( F^{(k)} \) are the classical representations (expectation values) of \( F^{(k)} \). In order to determine the coefficients \( c_k \) in the linear combinations, the local RPA equation is also projected onto the subspace \( \{ F^{(k)} \} \):

\[
M^{kl} c_l = (\hbar \Omega)^2 N^{kl} c_l,
\]

where \( \hbar \Omega \) is a frequency of the projected RPA and
\[ M^{kl} = F^{(k)}_{\alpha} B^{\alpha\beta} V_{\beta\gamma} B^{\gamma\delta} F^{(l)}_{\delta}, \]  
\[ N^{kl} = F^{(k)}_{\alpha} B^{\alpha\beta} F^{(l)}_{\beta}. \]  
(2.50)  
(2.51)

Normally, in order to obtain the RPA eigenmodes and frequencies, we need to diagonalise the RPA matrix \( B^{\alpha\gamma} V_{\gamma\beta} \) whose dimension is equal to the number of active 2qp degrees of freedom. The dimension of matrices \( M^{kl} \) and \( N^{kl} \) is equal to the number of selected one-body operators \( \{ F^{(k)} \} \). Therefore, if we can approximate the RPA eigenvectors by using a small number of operators, it will significantly reduce the computational task.

The equation for the force condition is also projected the same way,

\[ V^k = \lambda_1 N^{kl} c_l, \]  
(2.52)

with

\[ V^k = F^{(k)}_{\alpha} B^{\alpha\beta} V_{\beta}. \]  
(2.53)

Equations (2.49) and (2.52) must be solved self-consistently in order to determine a collective path.

In cases where spurious modes exist, the spurious components should be removed from each operator as

\[ \tilde{F}^{(k)}_{\alpha} = F^{(k)}_{\alpha} - f_{I,\alpha} I_{I,\alpha}, \]  
(2.54)

and the force condition (2.52) should be also modified by adding the terms \( \lambda_I W^{kl} \) in the right-hand side with \( W^{kl} = \tilde{F}^{(k)}_{\alpha} B^{\alpha\beta} f_{\beta I} \).

The removal of spurious modes in Eq. (2.54) is a tedious task because we need to obtain both \( f_{I,\alpha} \) and \( g_{I,\alpha} \) by solving the Thouless-Valatin equation. However, in case of hermitian constants of motion \( q^I \), adopting the extended adiabatic approximation in the previous section, we show that the explicit removal as in Eq. (2.54) is unnecessary. Using the modified mass parameter (2.43), we have \( B^{\alpha\beta} f_{\beta I} = 0 \), Eq. (2.44) showing that the \( f_{I,\alpha} \) correspond to the zero eigenmodes of mass matrix (2.44). Therefore, we find

\[ M^{kl} = F^{(k)}_{\alpha} B^{\alpha\beta} V_{\beta\gamma} B^{\gamma\delta} F^{(l)}_{\delta}, \]  
(2.55)  
\[ N^{kl} = F^{(k)}_{\alpha} B^{\alpha\beta} F^{(l)}_{\beta}, \]  
(2.56)  
\[ V^k = F^{(k)}_{\alpha} B^{\alpha\beta} V_{\beta} = \tilde{F}^{(k)}_{\alpha} B^{\alpha\beta} V_{\beta}, \]  
(2.57)

and \( W^{kl} = 0 \), which means that the spurious components of one-body operators \( F^{(k)} \) do not play any role in the projected LHA formalism. All we need to do is to keep \( q^I \) constant all the way along the collective path. Then, we can use the original set of one-body operators \( \{ F^{(k)} \} \) on which the LHA equations are projected, without modifying Eqs. (2.49) to (2.53).

In the classical theory of the P+Q model discussed in the next section the constants of motion are coordinates \( q^I \), and not momenta \( p_I \). Therefore this is exactly the case where the mixing of spurious components does not play any role. For further details, see the discussion around Eq. (3.21).
III. COLLECTIVE COORDINATES FOR THE P+Q MODEL

In this section, we investigate the structure of the self-consistent collective coordinates (cranking operators) at the Hartree-Bogoliubov (HB) state for the pairing-plus-quadrupole (P+Q) Hamiltonian, as a first step towards the large amplitude collective motion in heavy nuclei. We also try to find a set of one-body operators which can approximate the RPA eigenvectors.

A. The quasiparticle RPA for the P+Q model

We apply the symplectic version of LHA with the extended adiabatic approximation to the P+Q Hamiltonian. At the HB state, as shown in Sec. II C, the force condition can be always satisfied and the local RPA equation is equivalent to the quasiparticle RPA for the constrained Hamiltonian

\[
H' = H - \sum_{\tau=n,p} \lambda_\tau N_\tau, 
\]

\[
H = \sum_k \epsilon_k c_k^\dagger c_k - \sum_{\tau=n,p} \frac{G_\tau}{2} \left( P_\tau^\dagger P_\tau + P_\tau P_\tau^\dagger \right) - \frac{\chi}{2} \sum_{K=-2}^2 Q_{2K}^\dagger Q_{2K},
\]

\[
= \sum_k \epsilon_k c_k^\dagger c_k - \frac{1}{2} \sum_\sigma \kappa_\sigma R_\sigma R_\sigma + \frac{1}{2} \sum_\sigma \kappa_\sigma S_\sigma S_\sigma,
\]

where \( \epsilon_k \) are spherical single-particle energies and \( N_\tau = \sum_{k \in \tau} c_k^\dagger c_k \) are the number operators for neutrons (\( \tau = n \)) and protons (\( \tau = p \)). The operators \( R_\sigma \) and \( S_\sigma \) are the hermitian and anti-hermitian components, respectively, of the pairing operators, \( P_\tau^\dagger = \sum_{k \in \tau, k>0} c_k^\dagger c_\bar{k} \), and the dimensionless quadrupole operators, \( Q_{2K} = b_0^{-2} \sum_{kl} \langle k | r^2 Y_{2K} | l \rangle c_k^\dagger c_l \), where \( b_0 = (\hbar/m\omega_0)^{1/2} \) is the harmonic oscillator length. The Hamiltonian contains five operators of \( R_\sigma \)-type and four of \( S_\sigma \)-type. Together with the corresponding coupling constants \( \kappa_\sigma \), they are given by

\[
R_\sigma = (P_{\sigma+}^{(\pm)})_{n}, (P_{\sigma+}^{(\pm)})_{p}, Q_{20}^{(\pm)}, Q_{21}^{(\pm)}, Q_{22}^{(\pm)};
\]

\[
S_\sigma = (P_{\sigma-}^{(\pm)})_{n}, (P_{\sigma-}^{(\pm)})_{p}, Q_{21}^{(\pm)}, Q_{22}^{(\pm)};
\]

\[
\kappa_\sigma = G_n, G_p, \chi, \chi, \chi,
\]

where

\[
(P_{\tau+}^{(\pm)})_\tau = \frac{1}{\sqrt{2}} (P_\tau \pm P_\tau^\dagger), \quad \text{for} \quad \tau = n, p,
\]

\[
Q_{2K}^{(\pm)} = \frac{1}{\sqrt{2}} (Q_{2K} \pm Q_{2-K}), \quad \text{for} \quad K = 0, 1, 2.
\]

The signs (\( \pm \)) indicate the signature quantum number, \( e^{-i\pi J_x} O^{(\pm)} e^{i\pi J_x} = \pm O^{(\pm)} \). Following the standard formulation of the model, we will neglect the Fock terms, the contributions of the pairing force to the Hartree potential and those of the quadrupole force to the pairing potential [6–14]. After minimising the HB total energy and diagonalising the HB matrix,
we obtain the HB ground state $|\Psi_0\rangle$ and the quasiparticle energies $E_k$. Using $|\Psi_0\rangle$ as the reference state, the classical Hamiltonian is calculated as

$$\mathcal{H}'(\xi, \pi) = \mathcal{H}(\xi, \pi) - \sum_{\tau} \lambda_{\tau} N_{\tau}(\xi, \pi)$$

$$= E_0' + \sum_k E_k \rho_{kk} - \frac{1}{2} \sum_{i,j} \kappa_\sigma \left( \sum_{i>j} R_\sigma(ij)(\bar{\kappa}_{ij}^* + \bar{\kappa}_{ij}) + \sum_{ij} R_\sigma'(ij)\bar{\rho}_{ji} \right)^2$$

$$+ \frac{1}{2} \sum_{i,j} \kappa_\sigma \left( \sum_{i>j} S_\sigma(ij)(\bar{\kappa}_{ij}^* - \bar{\kappa}_{ij}) + \sum_{ij} S_\sigma'(ij)\bar{\rho}_{ji} \right)^2. \quad (3.6)$$

Applying the Holstein-Primakoff mapping (2.2) and taking into account that the $R'(ij)$ are symmetric and $S'(ij)$ are anti-symmetric with respect to the indices $(ij)$, one can see that $\langle \Psi(t)|R_\sigma|\Psi(t)\rangle$ $(\langle \Psi(t)|S_\sigma|\Psi(t)\rangle)$ are real (pure imaginary), and indeed the Hamiltonian (3.6) is real. Differentiating $\mathcal{H}'$ with respect to $\pi$ and $\xi$, the mass and curvature parameters at the HB state, defined by Eqs. (2.45) and (2.46), are given by

$$\tilde{B}^{\alpha\beta}(\xi = \pi = 0) = B^{\alpha\beta} - \sum_{\tau=n,p} \lambda_\tau f^{(1)}_{N_{\tau}\alpha\beta}$$

$$= E_\alpha \delta_{\alpha\beta} - 2 \sum_{\sigma} \kappa_\sigma S_\sigma(\alpha)S_\sigma(\beta), \quad (3.7)$$

$$\tilde{V}_{\alpha\beta}(\xi = \pi = 0) = V_{\alpha\beta} - \sum_{\tau=n,p} \lambda_\tau f^{N_{\tau}}_{\alpha\beta}$$

$$= E_\alpha \delta_{\alpha\beta} - 2 \sum_{\sigma} \kappa_\sigma R_\sigma(\alpha)R_\sigma(\beta). \quad (3.8)$$

Here, again the Greek index indicates a pair of 2qp index, so that $R(\alpha)$ are the 2qp matrix elements $R(\alpha) = R(ij)$ and $E_\alpha$ are the 2qp energies $E_\alpha = E_i + E_j$. We see that the terms $R'(ij)\bar{\rho}_{ji}$ and $S'(ij)\bar{\rho}_{ji}$ in the Hamiltonian do not contribute to the mass and curvature parameters, which is consistent with the fact that the terms proportional to $(a^\dagger a)$ in one-body operators are neglected in the quasiparticle RPA.

Solving the RPA is equivalent to diagonalisation of the matrix $\tilde{B}^{\alpha\gamma}\tilde{V}_{\gamma\beta}$. For separable forces such as the P+Q model, the problem of diagonalisation can be reduced to a root search in a multi-dimensional dispersion equation, which facilitates the numerical calculations for heavy nuclei [24]. If the ground state has an axial symmetry, the RPA matrix is block-diagonal and can be divided according to $K$ quantum numbers (eigenvalues of $J_z$). However, as we will show below, some nuclei have triaxial HB ground states for which $K$ is no longer a good quantum number. Thus, we use the signature quantum number $r$, which is good even for triaxial cases, and the RPA dispersion equation can be divided into positive ($r = +1$) and negative ($r = -1$) signature sectors. The construction of the response functions and the coupled dispersion equations can be found, for example, in Ref. [25].

**B. Numerical results**

We follow the second and third of the series of papers by Baranger and Kumar [11,12] for details of the P+Q model. The model space consists of two major shells of $N_{\text{osc}} = 5,6$
and 4, 5 for neutrons and protons, respectively. Different harmonic oscillator frequencies are used for neutrons and protons in order to make the root mean square radii the same [11]. The single-particle energies are taken from Table 1 in Ref. [12], where they are listed in units of the harmonic oscillator frequency $\hbar \omega_0 = 41.2A^{-1/3}$ MeV. The pairing force strengths are $G_n = 22A^{-1}$ MeV and $G_p = 27A^{-1}$ MeV for neutrons and protons, respectively. The quadrupole force strength is $\chi = 70A^{-1.4}$ MeV. The effective charge in the $E2$ operator is taken as $e_n = 1.5ZA^{-1}$ and $e_p = 1 + e_n$ for neutrons and protons, respectively. Since matrix elements of the quadrupole force in upper shells are known to be too strong [11], we modify the quadrupole operators by multiplying all quadrupole matrix elements by a factor $\mathcal{N}_L = (N_L + \frac{3}{2})/(N_{\text{osc}} + \frac{3}{2})$, where $N_{\text{osc}}$ is the oscillator quantum number of the shell under consideration and $N_L$ is that of the lower major shell. Moreover, due to the difference of harmonic oscillator frequencies for neutrons and protons, the operators are multiplied by factors $\alpha_n^2 = (2N/A)^{2/3}$ and $\alpha_p^2 = (2Z/A)^{2/3}$ for neutrons and protons, respectively. Thus, the dimensionless quadrupole operators in the interaction (3.4) are actually defined by

$$Q_{2K} = (Q_{2K})_n + (Q_{2K})_p,$$

$$(Q_{2K})_n = \alpha_n^2 b_0^{-2}(r^2Y_{2K})_n, \quad (Q_{2K})_p = \alpha_p^2 \zeta b_0^{-2}(r^2Y_{2K})_p.$$  

(3.9)

These operators will be referred to simply as the “quadrupole operators” in the following discussion.

We have performed calculations for even-even nuclei in the rare-earth region ($N = 82 \sim 126$, $Z = 50 \sim 82$) and have investigated the structure of RPA eigenvectors. Since $\beta$ and $\gamma$ vibrations in these nuclei were studied by Baranger and Kumar in Ref. [12], we can compare our microscopic RPA results with their semi-microscopic ones.

In Table 1, results of the HB and the RPA calculations are presented. The equilibrium deformations ($\beta$, $\gamma$, $\Delta_n$, $\Delta_p$) at HB ground states are found to agree with Table 2 in Ref. [12], except $^{190}$Pt which has a small triaxiality ($\gamma = 52.6^\circ$) in our result but has an oblate shape ($\gamma = 60^\circ$) in theirs. The deformation parameters, $\beta$ and $\gamma$, of the HB states are defined by [11,12].

$$\chi \langle \Psi_0 | Q_{20}^{(+)} | \Psi_0 \rangle = \hbar \omega_0 \beta \cos \gamma, \quad \chi \langle \Psi_0 | Q_{22}^{(+)} | \Psi_0 \rangle = \hbar \omega_0 \beta \sin \gamma.$$  

(3.10)

We have solved the RPA in the positive signature sector where the spurious modes, the pairing and spatial rotation, are coordinates (not momenta) and are characterised by zero eigenmodes of the mass matrix, $B^{\alpha \beta} f_{\beta}^I = 0$. For spherical nuclei ($\beta = 0$), the lowest excited states with multipolarity $\lambda = 2$ are selected as quadrupole vibrations and presented in columns 7 and 8. For prolate ($\beta > 0$, $\gamma = 0$) and oblate ($\beta < 0$, $\gamma = 0$) deformed cases, the first excited states (except zero spurious modes) with $K = 0$ and $K = 2$ are taken as $\beta$ and $\gamma$ vibrations, respectively. However, for some cases, the lowest excited states appear to be non-collective and we have selected higher-energy states. These states are indicated with * in the table. For a triaxial ground state, the first excited states are selected and the excitation energies are shown in column 7 while the transition amplitudes are given in columns 8 and 10. The $E2$ amplitudes $M(E2)_K$ are intrinsic values and calculated as

$$M(E2)_K = |\langle n^{(+)} | e_n \zeta (r^2Y_{2K})^{(+)}_n + e_p \zeta (r^2Y_{2K})^{(+)}_p | 0 \rangle|,$$  

(3.11)

where $|0\rangle$ and $|n^{(+)}\rangle$ are the RPA ground and excited states with signature $r = +1$, respectively. For spherical nuclei, states with $K = 0$, 1 and 2 are degenerate in energy and
we select the $K = 0$ one for $|n^{(+)}\rangle$. For triaxial shapes, the state $|n^{(+)}\rangle$ has non-zero $E2$ amplitudes for both $K = 0$ and $K = 2$ which are shown in columns 8 and 10 respectively.

First we compare the RPA frequencies with the harmonic formula given by Baranger and Kumar, Eqs. (13), (14) and (15) in Ref. [12]. Unlike our results, this formula contains a core contribution to the collective mass (which is denoted by $B^c$). Since our RPA calculations are carried out in the same model space, we should compare the RPA frequencies with those calculated by setting $B^c = 0$. These are shown in Fig. 1. Open symbols are the results of the harmonic formula while solid ones are the RPA frequencies. For spherical and triaxial nuclei, the same RPA frequencies are displayed in the figure for $\beta$ and $\gamma$ vibrations. The figure shows that the frequencies obtained by the formula are larger than the RPA frequencies. For $\gamma$ vibrations, the isotope dependence is well reproduced, even though the absolute values are mostly overestimated by up to a factor two. For $\beta$ vibrations, the harmonic formula breaks down, especially in well-deformed nuclei. It fails to reproduce both the isotope dependence and absolute values.

Let us try to understand the origin of the harmonic formula. It can be derived by taking the small-amplitude limit of the adiabatic time-dependent theory of collective motion developed by Baranger and Kumar [13,14] which assumes that the collective coordinates are the expectation values of the mass quadrupole operators (more precisely the deformation parameters of the Nilsson potential). In contrast with the formula, the RPA normal-mode coordinates are self-consistently determined by diagonalising the RPA matrix. Therefore, the difference between open and closed symbols seen in the figure can be attributed primarily to the difference in the collective coordinates.

Although it is not the purpose of this paper to reproduce the experimental data, we also show observed excitation energies of $\beta$ and $\gamma$ bands for axially deformed nuclei ($Z = 60 \sim 74$) in Fig. 1 [28]. Agreement with the experimental data are significantly improved by choosing the collective coordinates properly. One can see that the isotope dependence is roughly reproduced in this model. However, we still overestimate the energies by typically about 500 keV, which is at least partly due to the neglect of anharmonic effects, and may point at the limitations of the simple model.

In Fig. 2, we show $E2$ transition amplitudes for the RPA states. Evidently the $E2$ amplitudes are smaller for $\beta$ vibrations than $\gamma$ vibrations. This is partly because the $\beta$ vibrations are less collective, but also because the collective coordinates for $\beta$ vibrations are more complex than those for $\gamma$ motion (see below).

Now let us analyse the structure of the RPA normal-modes by using the projection techniques described in Sec. II D, using various sets of one-body operators. First we utilise elementary multipole operators $\{F^{(k)}\}$ on which the RPA matrix is projected. We adopt monopole operators (rank-0), quadrupole operators with radial dependence $r^0$ and $r^2$ (rank-2, $K = 0, 2$), those with spin dependence (rank-2, $K = 0, 2$), and hexadecapole operators (rank-4, $K = 0, 2$):

$$F^{(k)} = (P_+)_\tau, (P_-)_\tau, (Q_{20})_\tau, (Q_{22})_\tau, (r^0Y_{20})_\tau, (r^0Y_{22})_\tau,$$

$$[(r^0Y_{20} \times s_{K=0}^{(2)})_\tau], [(r^0Y_{22} \times s_{K=0}^{(2)})_\tau], [(r^2Y_{20} \times s_{K=2}^{(2)})_\tau], [(r^2Y_{22} \times s_{K=2}^{(2)})_\tau],$$

$$[(r^4Y_{40})_\tau], [(r^4Y_{42})_\tau], \tau = n, p,$$

where all operators have the positive signature $r = +1$, as in Eq. (3.3), but the superscript (+) denoting the signature quantum number is suppressed from here on. All matrix elements
of rank-2 operators are multiplied by factors $\zeta$ and $\alpha^2$, just as for the quadrupole operators (3.3). For $\beta$ vibrations, there are fourteen relevant operators of $K = 0$ while there are ten operators of $K = 2$ for $\gamma$. For a spherical case, the number reduces to eight (only rank-2 operators). For a triaxial case, all 24 operators can mix together.

The calculated frequencies $\hbar\Omega$ of the projected RPA equation (2.49) are shown in Fig. 3 as open symbols with dashed lines and are compared with the real RPA frequencies (solid symbols). As is clearly seen, the projection onto the operators (3.12) fails to reproduce the excitation energies both for $\beta$ and $\gamma$ vibrations. Not only the absolute values but also the isotope dependence turns out to be incorrect for all nuclei (except for $^{208}\text{Pb}$). The result indicates that it is very difficult to obtain sensible approximation by using elementary one-body operators. This is mainly due to the fact that the projected RPA eigenvectors have unrealistically large amplitudes for high-lying $2\text{qp}$ components. In Ref. [5], we have demonstrated for a few Sm isotopes that this can be remedied by introducing a cut-off energy for the $2\text{qp}$ matrix elements. For $^{208}\text{Pb}$, which is spherical and has $\Delta_n = \Delta_p = 0$, the projection accidentally works well because the set of available $1\text{p}1\text{h}$ modes is very limited, $\nu(i_{13/2})^{-1}$, $\nu(g_{9/2}(i_{13/2})^{-1})$, $\pi(h_{9/2}(h_{11/2})^{-1})$, $\pi(f_{7/2}(h_{11/2})^{-1})$, all of which have almost the same particle-hole excitation energies.

In order to check the effect of suppressing the high-energy components, we also perform the projected RPA calculation with the state-dependent operators whose $2\text{qp}$ matrix elements are weighted with a factor $(E_{2\text{qp}})^{-2}$. This means that we employ a set of hermitian one-body operators $\{\tilde{F}(k)\}$ defined by

$$\tilde{F} = \sum_{\alpha} \frac{F(\alpha)}{(E_{\alpha})^2} (a^\dagger a)_{\alpha} + \text{h.c.},$$

(3.13)

where $\alpha$ indicates the $2\text{qp}$ index and $F(\alpha) = \langle \alpha | F | 0 \rangle$. This suppression factor $(E_{2\text{qp}})^{-2}$ can be derived analytically as follows if the residual Hamiltonian consists only of a single-mode separable force $H = -\frac{1}{2}\kappa RR$ and the RPA frequency is much smaller than $2\text{qp}$ energies. In the single-mode case, from Eqs. (3.7) and (3.8), the RPA equation becomes

$$\left((E_{\alpha})^2 - (\hbar\Omega)^2\right) f_{\alpha} = 2\kappa R(\alpha) \sum_{\beta} E_\beta R(\beta) f_{\beta}. $$

(3.14)

Thus, we can analytically determine the RPA eigenvectors $f_{\alpha}$ as

$$f_{\alpha} \propto \frac{R(\alpha)}{(E_{\alpha})^2 - (\hbar\Omega)^2}. $$

(3.15)

In the limit of $\hbar\Omega \ll E_{\alpha}$, this leads to the quoted suppression factor $(E_{2\text{qp}})^{-2}$.

In the calculation, we adopt only the operators in the separable interactions:

$$\tilde{F}(k) = (\tilde{P}_+)_{\tau}, (\tilde{P}_-)_{\tau}, (\tilde{Q}_{20})_{\tau}, (\tilde{Q}_{22})_{\tau}, \quad \tau = n, p.$$  

(3.16)

This results in a six dimensional projected RPA matrix for $\beta$ vibrations and in two dimensional one for $\gamma$ vibrations and for spherical cases. Even for triaxial cases, the matrix dimension is thus only eight. The results are shown in Fig. 3 by open symbols with solid lines. The results are dramatically improved from the previous ones and the obtained frequencies are almost comparable to those of RPA for both $\beta$ and $\gamma$ modes of all nuclei. We
can also see the significant improvement over results of the harmonic formula, especially for the \( \beta \) vibrations (Fig. 1). This is because we incorporate the monopole pairing operators in Eq. (3.14) in addition to the quadrupole ones. It is worth noting that the frequency of the projected RPA gives an upper limit to the real RPA frequency because it is obtained by solving the RPA in a restricted space. On the other hand, this is not necessarily true for the harmonic formula of Ref. [12].

Let us examine the eigenvectors of projected RPA equation in more detail. For spherical nuclei, the RPA eigenvector has a good isoscalar character and can be well approximated by

\[
f^Q = c_n(\tilde{Q}_2)_n + c_p(\tilde{Q}_2)_p, \tag{3.17}
\]

with \( c_n \approx c_p \). Here the tilde indicates that the matrix elements include a suppression factor as in Eq. (3.13). A violation of this isoscalar character is seen for \( ^{138}\text{Ba}, ^{140}\text{Ce}, ^{142}\text{Nd} \) and Pb nuclei in which either the neutrons or protons pairing gap vanishes. The component with finite pairing gap is found to be larger than the one with zero gap, so that \( c_n > c_p \) for Pb and \( c_n < c_p \) for \( ^{138}\text{Ba}, ^{140}\text{Ce} \) and \( ^{142}\text{Nd} \). For deformed nuclei, where the collectivity of the vibrational states is smaller than for spherical nuclei and the pairing modes can mix with the quadrupole ones, the situation is more complex. In Tables II and III taking Dy and Os isotopes as examples, we show the coefficients of the approximate eigenvectors, \( \tilde{f} = \sum_k c_k \tilde{F}^{(k)} \), with respect to the set of operators in Eq. (3.16). The \( \gamma \) vibrations approximately have isoscalar character and the collective coordinate can be approximated by

\[
f^{\gamma} = c_n(\tilde{Q}_{22})_n + c_p(\tilde{Q}_{22})_p, \tag{3.18}
\]

with \( c_n \approx c_p \). For the \( \beta \) vibration, no such simple form can be found, since the properties of RPA eigenmodes change from one nucleus to the other. The \( \beta \) vibrations for \( ^{156-162}\text{Dy} \) possess a quadrupole nature, \( c_n(\tilde{Q}_{20})_n + c_p(\tilde{Q}_{20})_p \), with \( c_n > c_p \), although there are substantial contributions from the monopole pairing modes. For \( ^{164}\text{Dy} \), there is a dominant pairing component \( (P_+)_n \). For Os isotopes, \( ^{182-186}\text{Os} \) have dominant proton pairing \( (P_+)_p \), while \( ^{192}\text{Os} \) has a quadrupole nature \( (c_n < c_p) \). For the triaxial nuclei \( ^{188,190}\text{Os} \), although the \( K = 0 \) and \( K = 2 \) components are mixed, the vibrational state is of isoscalar quadrupole nature with \( c_n \approx c_p \). The changes observed in the RPA normal-mode coordinates for the \( \beta \) vibrations are difficult to be systematically summarised and turn out to be very sensitive to details of the underlying shell structure.

In order to confirm the importance of the monopole components for \( \beta \) vibrations, we have performed a calculation using only the quadrupole operators weighted with the factor \( E_{20\text{p}}^{-2}, (\tilde{Q}_{20})_\tau \) and \( (\tilde{Q}_{22})_\tau \) (\( \tau = n, p \)). For spherical nuclei, there is no change from the previous results since the monopole and quadrupole components are exactly decoupled. For deformed nuclei only, the results are shown in Fig. 3 by open symbols with dotted lines. Differences between solid and dotted lines come from the contributions of monopole pairing operators. The figure clearly shows that the normal-mode coordinate for \( \beta \) vibrations has a significant amount of monopole components.

We now wish to quantify the quality of the approximation. One measure is the difference between the real RPA frequencies \( \hbar \Omega \) and the projected RPA \( \hbar \Omega \), but as usual measures based on the eigenvectors are more sensitive. We define two criteria to directly check the closeness between the eigenvectors \( f \) and \( \tilde{f} \). Choosing the normalisation (of \( f \) and \( \tilde{f} \)) \( f_\alpha B^{\alpha \beta} f_\beta = \tilde{f}_\alpha B^{\alpha \beta} \tilde{f}_\beta = 1 \), those are calculated as...
\[ \delta_B = f_{\alpha} B^{\alpha\beta} (f_{\beta} - \bar{f}_{\beta}) = 1 - f_{\alpha} B^{\alpha\beta} \bar{f}_{\beta}, \]  

\[ \delta_1 = 1 - \frac{(f, \bar{f})}{\sqrt{(f, f)(\bar{f}, \bar{f})}}, \]  

where \((f, f') = \sum_{\alpha} f_{\alpha} f'_{\alpha}\). We have \(0 \leq \delta \leq 1\) with \(\delta = 0\) corresponding to the exact projection and \(\delta = 1\) to the case where \(\bar{f}\) is orthogonal to \(f\). The values of \(\delta_B\) and \(\delta_1\) are listed in Tables I and II. From the values of \(\delta_B\), we see that the projection works reasonably well for Dy and Os isotopes. The worst case is the \(\beta\) vibration of \(^{192}\)Os for which \(\delta_B \approx 0.3\). Roughly speaking, the one-body operators (3.16) possess 70\% of overlap with the real eigenvectors for \(^{192}\)Os (\(\delta_B \approx 0.3\)), and more than 90\% for the others (\(\delta_B \lesssim 0.1\)).

The values of \(\delta_1\) for \(\beta\) vibrations are often much larger than the corresponding \(\delta_B\) (see \(^{156,158}\)Dy and triaxial nuclei \(^{188,190}\)Os). This indicates that the \(\bar{f}\) for \(\beta\) contains some spurious components. In the present calculations, the pairing and spatial rotations are the obvious spurious motions which are zero eigenmodes of the mass matrix,

\[ B^{\alpha\beta} f^I_{\beta} = 0, \quad \text{for } f^I = N_n, N_p, J_x. \]  

The measure \(\delta_B\) (3.19) is insensitive to admixture of these spurious components in \(\bar{f}\). On the other hand, the value of \(\delta_1\) will increase by mixing in spurious components. As we have discussed in the last part of Sec. II D, the spurious modes \(f^I\) do not affect the projected LHA equations. Therefore, the large values of \(\delta_1\) should not cause serious problems in application of the projected LHA to the large amplitude collective motion.

In Fig. 4, the values of \(\delta_B\) are shown for the projected RPA using the set of elementary operators (3.12) (represented by dashed lines) and the set of state-dependent operators containing the factor \(E^{-2}_{2\text{qp}}\) (3.16) (solid lines). For \(\gamma\) vibrations, the suppression factor \(E^{-2}_{2\text{qp}}\) can improve the quality of projection by an order of magnitude and we have \(\delta_B \lesssim 0.1\) for all nuclei. For \(\beta\) vibrations, there are some nuclei in the range \(Z = 68 \sim 80\) for which the operators in Eq. (3.16) are not enough to produce the good approximation. This confirms the complexity of the normal-mode coordinates for \(\beta\).

We also see a strong isotope dependence in Fig. 4a. For instance, the Er isotopes show a staggering behaviour in \(\delta_B\). This behaviour is found to be closely related to the collectivity of the states. Since the \(\beta\) vibrations contains the pairing collectivity in addition to the quadrupole one, the \(E2\) amplitudes are not necessarily the best indicator of the collectivity of the states. We show in Fig. 4a a following quantity for the RPA eigenmodes of \(\beta\) vibrations in Er nuclei [26]:

\[ N^\mu_{\text{eff}} = \sum_{\alpha}(h\Omega f^\mu_{\alpha} f^\mu_{\alpha} + (h\Omega)^{-1}g^\alpha_{\mu}g^\alpha_{\mu}) \sum_{\beta} f^\mu_{\beta} g^\beta_{\mu}(h\Omega) f^\mu_{\beta} f^\mu_{\beta} + (h\Omega)^{-1}g^\beta_{\mu}g^\beta_{\mu}), \]  

(no summation with respect to \(\mu\)).  

If \(n\) 2qp components equally contribute to the RPA mode, we have \(N^\mu_{\text{eff}} = n\). Thus, Eq. (3.22) gives an effective number of 2qp excitations of which the RPA eigenmode consists. The figure shows that the collectivity of \(\beta\) vibrations in Er isotopes is weak (compared to the \(\gamma\) vibrations for which \(N^\mu_{\text{eff}} \gtrsim 8\) in most cases), and \(N^\mu_{\text{eff}}\) shows a staggering pattern correlated with that for \(\delta_B\) (while the \(E2\) amplitudes in Fig. 2 show no sign of staggering behaviour). For \(^{164,166}\)Er, there are dominant 2qp components in the RPA eigenmodes, which decrease \(N^\mu_{\text{eff}}\) and increase \(\delta_B\). For those cases where \(\delta_B \gtrsim 0.3\) even with state-dependent operators, the \(\beta\) vibrations are not collective at all, \(N^\mu_{\text{eff}} \lesssim 3\).
IV. CONCLUSION AND DISCUSSION

We have examined different versions of LHA techniques to see whether the conservation laws in the quantum level are satisfied in the classical LHA. It turns out that the restriction to point transformations must be lifted in order that spurious motion leads to the exact zero modes. The symplectic version of the LHA, which includes an extended adiabatic approximation, can guarantee the exact separation of zero-energy spurious modes.

The projected LHA can be used to truncate the local RPA calculations. We have shown that there is no problem with spurious modes for the projection method, at least when the spurious modes are zero eigenmodes of the mass matrix. This is exactly the case for the pairing and spatial rotation in the P+Q model for which we have investigated the structure of the collective coordinates for quadrupole vibrations and examined the possibility of expressing the self-consistent cranking operator in terms of a limited number of one-body operators. It seems very difficult to approximate the normal-mode vectors with only elementary one-body operators. The difficulty disappears, however, when we use a small number of state-dependent one-body operators. At HB equilibrium, the collective coordinates for quadrupole vibrations in spherical nuclei and for \( \gamma \) vibrations (\( K = 2 \)) in axially deformed nuclei can be roughly approximated as \( f \approx \tilde{Q}_{2K} \), where \( \tilde{Q}_{2K} \) is a state-dependent “mass” quadrupole operator defined by Eq. (3.9) weighted by a factor \( E_{2q}^{-2} \) for each 2qp matrix element (3.13). However, for \( \beta \) vibrations in axial deformed nuclei and for all excitations in triaxial nuclei, the monopole pairing components are as important as the quadrupole ones. It is worth emphasising that the collective coordinates are very different from the usual (state-independent) mass quadrupole operator even for \( \gamma \) and spherical cases. This shows the importance of a self-consistent determination of the collective coordinates for large amplitude collective motion, because the coordinates are now found to have a strong state-dependence as well. The structure of the self-consistent cranking operators is clearly changing when we move from spherical to axially deformed and triaxial nuclei. For the study of large amplitude collective motion in heavy nuclei for which the diagonalisation of the RPA matrix becomes too time-consuming, the results of this paper gives a suggestion for a choice of a state-dependent basis of operators. The choice of a limited set of (state-dependent) basis operators provides a practical way to solve the LHA through the projection. With the use of self-consistent cranking operators, the LHA should provide a significant improvement over a conventional CHFB calculation based on fixed cranking operators.

Despite all the above qualities, the symplectic version of LHA has a difficulty, namely the determination of \( f^{\mu}_{\alpha \beta} \), the second derivatives of the final coordinates \( q^{\mu} \) with respect to the original coordinates \( \xi^\alpha \). For the extended adiabatic transformation (2.33), we also need to calculate \( f^{(1)}_{\mu \alpha \beta} \), the terms of second order in \( \pi \). At HB states, we need these quantities for spurious coordinates \( \mu = I \) only, \( f^{I}_{\alpha \beta} \) and \( f^{(1)}_{1 \alpha \beta} \), which can be calculated because we know a priori the one-body operators corresponding to the spurious motions. We have done this for P+Q model. Away from equilibrium, however, in addition to these \( \mu = I \), we need the second derivatives of the collective coordinate, \( \mu = 1 \). This is not trivial because the local RPA equation determines only the first derivatives, \( f^{1}_{1 \alpha \beta} \) and \( g_{1 \alpha}^{\alpha} \). Unfortunately, so far we do not have a general method to determine \( f^{1}_{1 \alpha \beta} \) and \( f^{(1)}_{1 \alpha \beta} \). The projection method may actually provide us with a partial answer (27). If we project the LHA equation onto a set of one-body operators \{ \( F^{(k)} \) \}, the equation provides us with a local representation of collective
coordinate in terms of a linear combination of the $F^{(k)}$'s. Using the explicit form
\[ F^{(k)} = \sum_{i>j} F^{(k)}(ij)(a_i a_j^\dagger + a_j a_i) + \sum_{ij} F^{(k)(ij)} a_i a_j, \]  
we find
\[ f^{1}_{,\alpha} \equiv \frac{\partial q^{1}_{,\alpha}}{\partial \xi} = \sum_{k} c_k F^{(k)}_{,\alpha} = \sqrt{2} \sum_{k} c_k F^{(k)}(\alpha), \]  
where $F^{(k)}(\alpha) = F^{(k)}(ij)$ for 2qp index $\alpha = (ij)$ and we assume that the operators $F^{(k)}$ are hermitian. Here we use a local coordinate system, in which the point under consideration is the origin of the phase space, and the relation (2.7) to obtain Eq. (4.2). Since we know the explicit form of the operators $F^{(k)}$, we are now able to calculate the second derivatives $f^{1}_{,\alpha\beta}$ and second-order terms in momentum $f^{(1)1}_{,\alpha\beta}$.

\[ f^{1}_{,\alpha\beta} \equiv \frac{\partial^2 q^{1}_{,\alpha\beta}}{\partial \xi^\alpha \partial \xi^\beta} \approx \sum_{k} c_k F^{(k)}_{,\alpha\beta} = 4 \sum_{k} c_k G^{(k)}(\alpha\beta), \]  
\[ f^{(1)1}_{,\alpha\beta} \equiv \frac{\partial^2 q^{1}_{,\alpha\beta}}{\partial \pi^\alpha \partial \pi^\beta} \approx \sum_{k} c_k \frac{\partial^2 F^{(k)}}{\partial \pi^\alpha \partial \pi^\beta} = 4 \sum_{k} c_k G^{(k)}(\alpha\beta), \]  
where we neglect the derivative of $c_k$, for simplicity. $G^{(k)}(\alpha\beta)$ are related to $F^{(k)}(ij)$, by
\[ G^{(k)}(\alpha\beta) \equiv F^{(k)(ij)} \delta_{kl}, \text{ for } \alpha = (ik), \beta = (jl). \]  
Although, following this procedure, it is possible to calculate $f^{1}_{,\alpha\beta}$ and $f^{(1)1}_{,\alpha\beta}$, we have to note that, unlike the determination of $f^{1}_{,\alpha}$ by means of the RPA, the determination of these quantities through Eqs. (4.3) and (4.4) does not depend on the dynamics of the system. In this approach, part of the dynamics is hidden in the coordinate dependence of the coefficients $c_k$, which we have ignored for this simple discussion.

In this paper, we have mainly discussed the structure of the self-consistent cranking operators for the P+Q model only at HB minimum points. The structure at points away from the minimum and the further application of the LHA technique will be the subject of a future publication.

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TABLE I. HB equilibrium shapes ($\beta, \gamma$), pairing gaps $\Delta_\tau$, $\tau = n, p$ (in units of MeV), RPA frequencies $\hbar \Omega_K$ (in MeV) and $E2$ transition amplitudes $M(E2)_K$ (in $e fm^2$). The RPA results for spherical nuclei are shown in columns 7 and 8 (columns 9 and 10 are left blank), while for triaxial ones the RPA results are in columns 7, 9 and 10 (column 8 is blank). Entries labelled with asterisks indicate that they do not correspond to the lowest excited states. See the main text for further details.

| $Z$ | $N$ | $\beta$ | $\gamma$ | $\Delta_n$ | $\Delta_p$ | $\hbar \Omega_0$ | $M(E2)_0$ | $\hbar \Omega_2$ | $M(E2)_2$ |
|----|----|-------|-------|-------|-------|-------|-------|-------|-------|
| Ba | 56 | 0.0   | 0.0   | 0.0   | 1.557 | 2.87  | 30.88 |
|    | 82 | 0.0   | 0.0   | 0.913 | 1.528 | 2.24  | 35.21 |
| Ce | 58 | 0.0   | 0.0   | 0.895 | 1.654 | 2.06  | 41.43 |
|    | 84 | 0.0   | 0.0   | 1.198 | 1.624 | 1.34  | 54.11 |
| Nd | 60 | 0.0   | 0.0   | 0.964 | 1.386 | 1.57  | 27.51 | 1.68  | 23.92 |
|    | 84 | 0.0   | 0.0   | 0.878 | 1.734 | 1.91  | 46.91 |
|    | 86 | 0.0   | 0.0   | 1.176 | 1.703 | 1.10  | 64.37 |
|    | 88 | 0.184 | 0.0   | 0.964 | 1.386 | 1.57  | 27.51 | 1.68  | 23.92 |
|    | 90 | 0.236 | 0.0   | 0.981 | 1.229 | 1.59  | 27.58 | 1.75  | 24.52 |
| Sm | 62 | 0.0   | 0.0   | 0.861 | 1.778 | 1.78  | 51.66 |
|    | 84 | 0.0   | 0.0   | 1.155 | 1.746 | 0.88  | 75.68 |
|    | 86 | 0.198 | 0.0   | 0.910 | 1.394 | 1.57  | 25.37 | 1.63  | 24.09 |
|    | 90 | 0.247 | 0.0   | 0.944 | 1.228 | 1.42  | 28.97 | 1.69  | 23.83 |
|    | 92 | 0.312 | 0.0   | 0.902 | 1.007 | 0.66  | 34.58 | 1.90  | 21.71 |
| Gd | 64 | 0.0   | 0.0   | 0.845 | 1.793 | 1.68  | 55.34 |
|    | 84 | 0.0   | 0.0   | 1.134 | 1.761 | 0.71  | 87.45 |
|    | 86 | 0.203 | 0.0   | 0.873 | 1.386 | 1.55  | 23.66 | 1.59  | 24.61 |
|    | 90 | 0.249 | 0.0   | 0.917 | 1.221 | 1.44  | 27.33 | 1.62  | 25.65 |
|    | 92 | 0.299 | 0.0   | 0.904 | 1.056 | 0.89  | 29.62 | 1.69  | 26.74 |
|    | 94 | 0.320 | 0.0   | 0.880 | 0.980 | 1.15  | 22.52 | 1.60  | 27.64 |
|    | 96 | 0.329 | 0.0   | 0.840 | 0.938 | 1.40  | 17.82 | 1.35  | 26.36 |
| Dy | 66 | 0.0   | 0.0   | 0.897 | 1.212 | 1.55  | 24.78 | 1.58  | 27.71 |
|    | 90 | 0.285 | 0.0   | 0.901 | 1.069 | 1.22  | 25.33 | 1.52  | 29.88 |
|    | 94 | 0.308 | 0.0   | 0.873 | 0.988 | 1.39  | 20.09 | 1.45  | 31.75 |
|    | 96 | 0.320 | 0.0   | 0.827 | 0.941 | 1.56  | 13.31 | 1.23  | 30.98 |
|    | 98 | 0.327 | 0.0   | 0.748 | 0.904 | 1.50  | 0.59  | 1.13  | 29.35 |
| Er | 68 | 0.0   | 0.0   | 0.867 | 0.986 | 1.58  | 20.12 | 1.49  | 30.81 |
|    | 94 | 0.310 | 0.0   | 0.817 | 0.927 | 1.59  | 9.18  | 1.28  | 29.50 |
|    | 98 | 0.320 | 0.0   | 0.736 | 0.884 | 1.47  | 1.98  | 1.17  | 27.45 |
|    | 100| 0.326 | 0.0   | 0.680 | 0.849 | 1.36  | 1.03  | 1.51  | 26.37 |
|    | 102| 0.330 | 0.0   | 0.624 | 0.819 | 1.20  | 5.40  | 1.72  | 26.95 |
| Yb | 70 | 0.0   | 0.0   | 0.874 | 1.016 | 1.59  | 20.62 | 1.56  | 27.79 |
|    | 94 | 0.278 | 0.0   | 0.874 | 1.016 | 1.59  | 20.62 | 1.56  | 27.79 |
|    | 96 | 0.296 | 0.0   | 0.817 | 0.946 | 1.58  | 11.28 | 1.40  | 24.94 |
|    | 98 | 0.309 | 0.0   | 0.734 | 0.892 | 1.45  | 4.82  | 1.29  | 21.61 |
|    | 100| 0.316 | 0.0   | 0.679 | 0.852 | *1.66 | *11.33| 1.67  | 17.41 |
|    | 102| 0.320 | 0.0   | 0.632 | 0.820 | 1.18  | 8.21  | *2.27 | *22.36 |

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| 104 | 0.313 | 0.0 | 0.739 | 0.813 | *1.55 | *13.39 | 2.02 | 25.66 |
|-----|-------|-----|-------|-------|-------|--------|------|-------|
| 106 | 0.305 | 0.0 | 0.803 | 0.811 | 1.42  | 18.82  | 1.68 | 27.51 |
| Hf  | 72    | 94  | 0.252 | 0.0 | 0.907 | 1.060 | 1.63 | 18.96 |
|     |       | 96  | 0.274 | 0.0 | 0.842 | 1.004 | 1.65 | 10.49 |
|     |       | 98  | 0.289 | 0.0 | 0.759 | 0.963 | 1.47 | 8.59  |
|     |       | 100 | 0.299 | 0.0 | 0.706 | 0.932 | *1.72| *15.93|
|     |       | 102 | 0.302 | 0.0 | 0.678 | 0.907 | 1.22 | 12.52 |
|     |       | 104 | 0.293 | 0.0 | 0.744 | 0.893 | *1.69| *14.97|
|     |       | 106 | 0.282 | 0.0 | 0.785 | 0.884 | 1.56 | 4.48  |
|     |       | 108 | 0.272 | 0.0 | 0.824 | 0.878 | 1.66 | 0.28  |
|     |       | 110 | 0.257 | 0.0 | 0.920 | 0.883 | 1.54 | 27.21 |
| W   | 74    | 98  | 0.264 | 0.0 | 0.819 | 0.962 | 1.53 | 13.74 |
|     |       | 100 | 0.275 | 0.0 | 0.768 | 0.940 | 1.47 | 13.95 |
|     |       | 102 | 0.277 | 0.0 | 0.752 | 0.918 | 1.31 | 16.48 |
|     |       | 104 | 0.270 | 0.0 | 0.777 | 0.893 | 1.53 | 8.60  |
|     |       | 106 | 0.259 | 0.0 | 0.796 | 0.872 | 1.59 | 0.68  |
|     |       | 108 | 0.247 | 0.0 | 0.838 | 0.856 | 1.63 | 9.29  |
|     |       | 110 | 0.228 | 0.0 | 0.932 | 0.854 | 1.71 | 16.35 |
|     |       | 112 | 0.202 | 0.0 | 0.988 | 0.872 | 1.70 | 21.51 |
| Os  | 76    | 106 | 0.235 | 0.0 | 0.838 | 0.785 | 1.57 | 2.00  |
|     |       | 108 | 0.222 | 0.0 | 0.883 | 0.762 | 1.51 | 7.88  |
|     |       | 110 | 0.201 | 0.0 | 0.966 | 0.758 | 1.47 | 12.34 |
|     |       | 112 | 0.181 | 22.1 | 0.990 | 0.789 | 0.59 | 18.96 |
|     |       | 114 | 0.171 | 46.0 | 0.982 | 0.830 | 0.52 | 44.03 |
|     |       | 116 | −0.156 | 0.0 | 0.928 | 0.844 | 1.64 | 17.81 |
| Pt  | 78    | 106 | 0.205 | 0.0 | 0.930 | 0.754 | *1.66| *17.68|
|     |       | 108 | 0.190 | 0.0 | 0.976 | 0.723 | 1.44 | 6.59  |
|     |       | 110 | 0.171 | 20.1 | 1.046 | 0.671 | 0.72 | 17.05 |
|     |       | 112 | 0.160 | 52.6 | 1.078 | 0.674 | 0.21 | 72.21 |
|     |       | 114 | −0.149 | 0.0 | 1.030 | 0.676 | 1.44 | 6.93  |
|     |       | 116 | −0.138 | 0.0 | 0.944 | 0.676 | 1.50 | 6.76  |
|     |       | 118 | −0.126 | 0.0 | 0.814 | 0.678 | 1.49 | 18.53 |
|     |       | 120 | −0.107 | 0.0 | 0.693 | 0.692 | 0.99 | 26.14 |
| Hg  | 80    | 112 | −0.139 | 0.0 | 1.106 | 0.0 | 1.61 | 17.73 |
|     |       | 114 | −0.131 | 0.0 | 1.045 | 0.0 | 1.69 | 27.36 |
|     |       | 116 | −0.123 | 0.0 | 0.956 | 0.0 | 1.60 | 26.48 |
|     |       | 118 | −0.114 | 0.0 | 0.830 | 0.0 | 1.52 | 18.62 |
|     |       | 120 | −0.089 | 0.0 | 0.745 | 0.349 | 0.43 | 32.92 |
|     |       | 122 | 0.0    | 0.0 | 0.749 | 0.606 | 0.51 | 49.51 |
|     |       | 124 | 0.0    | 0.0 | 0.510 | 0.596 | 0.90 | 29.38 |
| Pb  | 82    | 118 | 0.0    | 0.0 | 1.011 | 0.0 | 0.80 | 36.93 |
|     |       | 120 | 0.0    | 0.0 | 0.894 | 0.0 | 0.89 | 29.92 |
|     |       | 122 | 0.0    | 0.0 | 0.734 | 0.0 | 1.01 | 22.41 |
|     |       | 124 | 0.0    | 0.0 | 0.497 | 0.0 | 1.12 | 14.14 |
|     |       | 126 | 0.0    | 0.0 | 0.000 | 0.0 | 4.71 | 19.86 |
The two possible measures of the quality of projection, $\delta_B$ and $\delta_1$, are shown at the bottom.

| $\beta$ | $\gamma$ | $\beta$ | $\gamma$ | $\beta$ | $\gamma$ | $\beta$ | $\gamma$ | $\beta$ | $\gamma$ |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $\hbar\Omega$ | 1.55 | 1.58 | 1.22 | 1.52 | 1.39 | 1.45 | 1.56 | 1.23 | 1.50 | 1.13 |
| $\hbar\bar{\Omega}$ | 1.77 | 1.77 | 1.34 | 1.65 | 1.53 | 1.54 | 1.85 | 1.30 | 1.94 | 1.20 |
| $(P_+)_n$ | $-0.061$ | 0.0 | $-0.071$ | 0.0 | 0.030 | 0.0 | 0.106 | 0.0 | 0.988 | 0.0 |
| $(\bar{P}_-)_n$ | $-0.494$ | 0.0 | $-0.437$ | 0.0 | $-0.350$ | 0.0 | $-0.312$ | 0.0 | 0.064 | 0.0 |
| $(Q_{20})_n$ | 0.609 | 0.0 | 0.671 | 0.0 | 0.745 | 0.0 | 0.784 | 0.0 | 0.142 | 0.0 |
| $(Q_{22})_n$ | 0.0 | 0.730 | 0.0 | 0.706 | 0.0 | 0.693 | 0.0 | 0.716 | 0.0 | 0.726 |
| $(P_+)_p$ | $-0.325$ | 0.0 | $-0.246$ | 0.0 | $-0.225$ | 0.0 | $-0.240$ | 0.0 | $-0.009$ | 0.0 |
| $(\bar{P}_-)_p$ | $-0.026$ | 0.0 | $-0.024$ | 0.0 | $-0.021$ | 0.0 | $-0.019$ | 0.0 | $-0.001$ | 0.0 |
| $(Q_{20})_p$ | 0.525 | 0.0 | 0.540 | 0.0 | 0.520 | 0.0 | 0.467 | 0.0 | 0.016 | 0.0 |
| $(Q_{22})_p$ | 0.0 | 0.684 | 0.0 | 0.708 | 0.0 | 0.720 | 0.0 | 0.698 | 0.0 | 0.688 |

$\delta_B$ | 0.058 | 0.064 | 0.010 | 0.028 | 0.011 | 0.008 | 0.132 | 0.007 | 0.137 | 0.010 |
$\delta_1$ | 0.410 | 0.059 | 0.406 | 0.025 | 0.140 | 0.004 | 0.176 | 0.005 | 0.115 | 0.008 |

The nuclei $^{188,190}$Os have a triaxial shape in the ground state.

| $^{182}$Os | $^{184}$Os | $^{186}$Os | $^{188}$Os | $^{190}$Os | $^{192}$Os |
|---------|---------|---------|---------|---------|---------|
| $\beta$ | $\gamma$ | $\beta$ | $\gamma$ | $\beta$ | $\gamma$ | $\beta$ | $\gamma$ |
| $\hbar\Omega$ | 1.57 | 1.12 | 1.51 | 0.85 | 1.47 | 0.42 | 0.59 | 0.52 | 1.64 | 0.61 |
| $\hbar\bar{\Omega}$ | 1.85 | 1.15 | 1.78 | 0.86 | 1.73 | 0.43 | 0.61 | 0.53 | 2.02 | 0.62 |
| $(P_+)_n$ | $-0.014$ | 0.0 | 0.265 | 0.0 | 0.174 | 0.0 | $-0.014$ | $-0.125$ | 0.236 | 0.0 |
| $(\bar{P}_-)_n$ | 0.001 | 0.0 | $-0.081$ | 0.0 | $-0.162$ | 0.0 | 0.208 | 0.113 | $-0.082$ | 0.0 |
| $(Q_{20})_n$ | 0.009 | 0.0 | $-0.186$ | 0.0 | $-0.278$ | 0.0 | 0.178 | 0.504 | 0.561 | 0.0 |
| $(Q_{22})_n$ | 0.0 | 0.690 | 0.0 | 0.699 | 0.0 | 0.705 | 0.658 | 0.472 | 0.0 | 0.705 |
| $(P_+)_p$ | 0.999 | 0.0 | 0.905 | 0.0 | 0.847 | 0.0 | $-0.049$ | $-0.047$ | 0.381 | 0.0 |
| $(\bar{P}_-)_p$ | 0.031 | 0.0 | $-0.069$ | 0.0 | $-0.101$ | 0.0 | 0.099 | 0.064 | $-0.117$ | 0.0 |
| $(Q_{20})_p$ | $-0.028$ | 0.0 | $-0.256$ | 0.0 | $-0.372$ | 0.0 | 0.184 | 0.509 | 0.681 | 0.0 |
| $(Q_{22})_p$ | 0.0 | 0.724 | 0.0 | 0.715 | 0.0 | 0.709 | 0.668 | 0.479 | 0.0 | 0.710 |

$\delta_B$ | 0.072 | 0.002 | 0.067 | 0.0005 | 0.068 | 0.00005 | 0.001 | 0.001 | 0.313 | 0.0004 |
$\delta_1$ | 0.064 | 0.001 | 0.087 | 0.0003 | 0.156 | 0.00004 | 0.835 | 0.657 | 0.329 | 0.0003 |
FIG. 1. Calculated excitation energies in units of MeV, of $\beta$ (a) and $\gamma$ vibrations (b) as functions of mass number $A$ for even-even nuclei in the rare-earth region. The closed circles indicate the RPA results while the open symbols are results of the harmonic formula by Baranger and Kumar [12]. The frequency of the harmonic formula for $^{208}$Pb is 10.9 MeV. For cases in which the HB ground states are spherical or triaxial, we cannot distinguish $\beta$ from $\gamma$ vibrations and the same values have been used for both (a) and (b). Different symbols indicate different isotopes; circles for Ba, Dy and Os; squares for Ce, Er and Pt; diamonds for Nd, Yb, Hg; triangles (upwards) for Sm, Hf and Pb; triangles (downward) for Gd and W. The experimental values are represented by crosses for deformed nuclei [28].
FIG. 2. Calculated intrinsic $E2$ transition amplitudes between the excited RPA and the ground state, for $\beta$ (a) and $\gamma$ vibrations (b) in units of $\text{efm}^2$. For spherical nuclei, these values are also the $E2$ amplitudes in laboratory frame. For triaxial cases, a single state has both $K = 0$ and $K = 2$ components of $E2$ amplitudes which are shown in (a) and (b), respectively. See caption of Fig. [1] for symbols and the text for further details.
FIG. 3. Calculated energies in units of MeV, by means of both the projected RPA (open symbols) and the exact RPA (closed) for $\beta$ (a) and $\gamma$ vibrations (b). The results of projected RPA connected with dashed, dotted and solid lines indicate different sets of one-body operators being employed for the projection. See the main text for further details.
FIG. 4. Quality of projection $\delta_B$, Eq. (3.19), for $\beta$ (a) and $\gamma$ vibrations (b). Dashed lines and solid lines denote different sets of one-body operators being employed for the projection. See the main text for further details.
FIG. 5. The measure $\delta_B$ (open squares and scale on the left) and the measure of collectivity $N_{\text{eff}}$, Eq. (3.22), (closed squares and scale on the right) for $\beta$ vibrations in the Er isotopes.