A Minimal Deductive System for RDFS with Negative Statements

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Abstract

The triple language RDFS is designed to represent and reason with positive statements only (e.g., “antipyretics are drugs”). In this paper, we extend RDFS to deal with various forms of negative statements under the Open World Assumption (OWA). To do so, we consider pdf, a minimal, but significant RDFS fragment that covers all essential features of RDFS, and then extend it to pdf⊥, allowing express also statements such as “radio therapies are non drug treatments”, “Ebola has no treatment”, or “opioids and antipyretics are disjoint classes”. The main features of our proposal are: (i) pdf⊥ remains syntactically a triple language by extending pdf with new symbols with specific semantics and there is no need to revert to the reification method to represent negative triples; (ii) the logic is defined in such a way that any RDFS reasoner/store may handle the new predicates as ordinary terms if it does not want to take account of the extra capabilities; (iii) despite negated statements, every pdf⊥ knowledge base is satisfiable; (iv) the pdf⊥ entailment decision procedure is obtained from pdf via additional inference rules favouring a potential implementation; and (v) deciding entailment in pdf⊥ ranges from P to NP.

1 Introduction

The Resource Description Framework (RDF)¹ and its extension RDF Schema (RDFS)² are both W3C standards, and nowadays quite popular knowledge representation languages. Essentially, a statement in RDF is a triple of the form (s, p, o), allowing to state that subject s is related to object o via the property p. For instance, (fever, hasTreatment, paracetamol) is such a triple whose intended meaning is “fever can be treated via paracetamol”. RDFS is an extension of RDF providing mechanisms for describing groups of related terms and the relationships between these terms via a specific vocabulary of predicates. So, e.g., the RDFS triple (paracetamol, type, antipyretic) express that “paracetamol is an antipyretic” (here type is the predicate for class membership specification), while (antipyretic, sc, drug) asserts that “antipyretic is a subclass of drug” (sc is the predicate for sub-class specification).

As both languages have been designed to represent and reason with positive statements only, they can not properly deal with negative statements such as

“opioids and antipyretics are disjoint classes”; (1)
“radio therapies are non drug treatments”; and (2)
“Ebola has no treatment”. (3)

In particular, we may not infer that “paracetamol is not a treatment for Ebola”. Such a statement could only be inferred with the major assumption that the Knowledge Base (KB) is complete — the so-called Closed-World Assumption (CWA) (Reiter 1978), which however is not realistic to be assumed in many cases. For instance, in medicine, it is important to distinguish between knowing about the absence of a biochemical reaction between substances, and not knowing about its existence at all, which rises then the need for explicitly stating salient negative statements (see, e.g., (Arnaout et al. 2021b) for a recent work about it). This is particularly true in the case in which the information about the represented world is assumed to be incomplete, — the so-called Open World Assumption (OWA).

Contribution. In this paper we show how to extend RDFS to express and reason with various forms of negative statements under the OWA. To do so, we start from rdf (Muñoz, Pérez, and Gutierrez 2009), a minimal, but significant RDFS fragment that covers all essential features of RDFS, and then extend it to pdf⊥, allowing express also negative statements via a specific expressions involving negated classes/properties, disjointness relationships, and no-value existence. So, for instance, the pdf⊥ triple (opioid, ⊥c, antipyretic) expresses (1) (⊥c is the vocabulary predicate for class disjointness specification), (radiotherapy, sc, ¬drugTreatment) expresses (2) (here, essentially we introduce class complements via the ¬ operator), while (ebola, ¬hasTreatment, *treatment) is meant to encode (3) (here, besides property complement, we also allow the * operator, which is the place holder for an universally quantified variable over domain c).³

The main and, to the best of our knowledge, unique features of our proposal are (cf., related work below): (i)

³We refer the reader to Table 1 for an informal First-Order Logic (FOL) reading of some types of pdf⊥ triples.
$pdf_d^s$ remains syntactically a triple language by extending $pdf$ with new symbols with specific semantics and there is no need to revert to the reification method to represent negative triples; (ii) the logic is defined in such a way that any RDFS reasoner/store may handle the new predicates as ordinary terms if it does not want to take account of the extra capabilities; (iii) despite negated statements, every $pdf_d^s$ knowledge base is satisfiable, which is obtained via an intentional like four-valued semantics; (iv) the $pdf_d^s$ entailment decision procedure is obtained from $pdf$ via additional inference rules favouring a potential implementation; and (v) deciding entailment in $pdf_d^s$ ranges from P to NP.

**Related Work.** There have been various works in the past about extending RDFS with negative statements, or applications that would like or require to have such a feature, which we briefly summarise below and indeed inspired our work.

In (Arnaout et al. 2021b) and related works (Arnaout, Razniewski, and Weikum 2020; Arnaout et al. 2021a; Arnaout et al. 2021c), two types of negative statements are considered: (i) grounded negative statements of the form $¬(s, p, o)$, with informal FOL reading $¬p(s, o)$; and (ii) universally negative statements of the form $¬\exists x.p(s, x)$, meaning in FOL terms $¬\exists x.p(s, x)$. The former type of triples have been proposed in (Analyti et al. 2004) (and subsequent works, see below), while the latter has been addressed in (Darari, Prasojo, and Nutt 2015). In (Arnaout et al. 2021b) essentially a statistical inference method is proposed to extract useful negative statements of this form, such as “Leonardo DiCaprio has never been married” and “United Kingdom is not the citizenship of Jimmy Hendrix”. It also publishes datasets that contain useful negative statements about entities in Wikidata. Reasoning has not been addressed (and was not the focus of these works). Both types of negative statements are covered by $pdf_d^s$ and, thus, our work is complementary to (Arnaout et al. 2021b) in the sense that we describe how then to reason with such information.

In (Darari, Prasojo, and Nutt 2015) the problem on how to express the non-existence of information is addressed, which has the form $\forall x.(p_1(s_1, o_1) \land \ldots \land p_n(s_n, o_n))$, with informal FOL reading $\neg\exists x.(p_1(s_1, o_1) \land \ldots \land p_n(s_n, o_n))$, or equivalently, $\forall x.(\neg p_1(s_1, o_1) \lor \ldots \lor \neg p_n(s_n, o_n))$, where $x$ are the variables occurring the triples. It shows how to represent it via the reification method and incorporate it into SPARQL query answering. Reasoning is not addressed however. We consider here only the case $n = 1$ via the expression $(s, \neg p, \star s)$ as the general case $n \geq 2$ would introduce a disjunction, which we would like to avoid for computational reasons.

In (Analyti et al. 2008) and related works (Analyti, Damásio, and Antoniou 2015; Analyti, Antoniou, and Damásio 2008; Analyti, Antoniou, and Damásio 2009; Analyti et al. 2011; Analyti et al. 2013; Analyti et al. 2005; Analyti et al. 2004; Damásio, Analyti, and Antoniou 2010) the authors deal with Extended RDF (ERDF), a non-monotonic logic, where an ERDF ontology consists of two parts: an ERDF graph and an ERDF logic program. An ERDF graph allows negated RDF triples of the form $¬(s, p, o)$, informally in FOL terms $¬p(s, o)$, while in the body of rules all the classical connectives $\neg, \lor, \land, \forall, \exists$ plus the weak negation (negation-as-failure) $\sim$ are allowed. Various “stable model” semantics have been proposed. From a computational complexity point view, decision problems in ERDF are non-polynomial (Analyti, Damásio, and Antoniou 2015). E.g., deciding model existence and, thus, model existence is not guaranteed, ranges from NP to PSPACE, while query answering goes from co-NP to PSPACE, depending on the setting. In comparison, $pdf_d^s$ does not use a rule layer, the triple language is more expressive, model existence is guaranteed and the computational complexity ranges between P and NP. Of course, all inference rules for $pdf_d^s$ can be implemented in the rule layer of ERDF (and in Datalog in general).

Eventually, (Casini and Straccia 2020) considers $pdf_d^s$ on top of which to develop a non-monotonic RDF logic based on Rational Closure (Lehmann and Magidor 1992). $pdf_d^s$ extends $pdf$ allowing to express disjointness among (positive) classes and relations.

In summary, our work aims at putting all together within RDFS to deal with expressions of the form e.g., (1)–(3) in a generalised way.

We proceed as follows. As next we introduce the basic notions about $pdf$ we will rely on. Section 3 is the main part of this paper in which we extend $pdf$ to $pdf_d^s$. The paper concludes with a summary of the contributions and addresses some topics for future work.

## 2 Preliminaries

For the sake of our purposes, we will rely on a minimal, but significant RDFS fragment, called $pdf$ (Muñoz, Pérez, and Gutierrez 2009; Muñoz, Pérez, and Gutiérrez 2007), that covers the essential features of RDFS. In fact, $pdf$ may be considered as the logic behind RDFS and suffices to illustrate the main concepts and algorithms we will consider. $pdf$ is defined as the following subset of the RDFS vocabulary:

$$pdf = \{sp, sc, type, dom, range\}.$$  

(4)

Informally, (i) $(p, sp, q)$ means that property $p$ is a sub property of property $q$; (ii) $(c, sc, d)$ means that class $c$ is a sub class of class $d$; (iii) $(a, type, b)$ means that $a$ is of type $b$; (iv) $(p, dom, c)$ means that the domain of property $p$ is $c$; (v) $(p, range, c)$ means that the range of property $p$ is $c$.

**Syntax.** Assume pairwise disjoint alphabets $U$ (RDF URI references), $B$ (Blank nodes), and $L$ (Literals). We assume $U$, $B$, and $L$ fixed, and for simplicity we will denote unions

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4 Optionally, triples may be annotated with a degree such as e.g., “The Sultan Resort has no parking facility to degree 0.97”. See e.g., (Zimmermann et al. 2012) for a general framework to deal with annotated triples.

5https://github.com/HibaArnaout/usefulnegations

6https://www.wikidata.org

7http://www.w3.org/TR/sparql11-query/
of these sets simply concatenating their names. We call elements in UBL terms (denoted \(a, b, \ldots, u\)), and elements in B variables (denoted \(x, y, z\)). A vocabulary is a subset of UL and we assume that U contains the rdf vocabulary -see Eq. (4). A triple is of the form
\[
(s, p, o) \in \text{UBL} \times \text{U} \times \text{UBL}
\]
where \(s, o \notin \text{rdf}\). We call \(s\) the subject, \(p\) the predicate, and \(o\) the object. A graph (or RDF Knowledge Base) \(G\) is a set of triples \(\tau\). A subgraph is a subset of a graph. The universe of \(G\), denoted \(\text{uni}(G)\), is the set of terms in UBL that occur in the triples of \(G\). The vocabulary of \(G\), denoted by \(\text{voc}(G)\) is the set \(\text{uni}(G) \cap \text{UL}\). A graph is ground if it has no blank nodes, i.e., variables. A map (or variable assignment) is as a function \(\mu : \text{UBL} \rightarrow \text{UBL}\) preserving URIs and literals, i.e., \(\mu(t) = t\), for all \(t \in \text{UL}\). Given a graph \(G\), we define
\[
\mu(G) = \{(\mu(s), \mu(p), \mu(o)) \mid (s, p, o) \in G\}.
\]
We speak of a map \(\mu\) from \(G_1\) to \(G_2\), and write \(\mu : G_1 \rightarrow G_2\), if \(\mu(s)\) is such that \(\mu(G_1) \subseteq G_2\).

Example 1 (Running example). The following is a rdf graph:
\[
G = \{(\text{paracetamol}.\text{type}, \text{antipyretic}),
(\text{antipyretic}, \text{sc}, \text{drugTreatment}),
(\text{morphine}.\text{type}, \text{opioid}), (\text{opioid, sc, drugTreatment}),
(\text{drugTreatment}, \text{sc}, \text{treatment}),
(\text{brainTumour}.\text{type}, \text{tumour}),
(\text{hasDrugTreatment}, \text{sp}, \text{hasTreatment}),
(\text{hasDrugTreatment}, \text{dom}, \text{illness}),
(\text{hasDrugTreatment}, \text{range}, \text{treatment}),
(\text{hasDrugTreatment}, \text{range}, \text{drugTreatment}),
(\text{fever, hasDrugTreatment, paracetamol}),
(\text{brainTumour, hasDrugTreatment, morphine})\}
\]

Semantics. A rdf interpretation \(I\) over a vocabulary \(V\) is a tuple
\[
I = (\Delta_R, \Delta_P, \Delta_C, \Delta_L, P[\cdot], C[\cdot], I)\,
\]
where the finite non-empty sets \(\Delta_R\), \(\Delta_P\), \(\Delta_C\), \(\Delta_L\) are the interpretation domains of \(I\) and \(P[\cdot], C[\cdot], I\) are the interpretation functions of \(I\). They have to satisfy:

1. \(\Delta_R\) are the resources;
2. \(\Delta_P\) are property names;
3. \(\Delta_C \subseteq \Delta_R\) are the classes;
4. \(\Delta_L \subseteq \Delta_R\) are the literal values and contains all the literals in \(L \cap V\);
5. \(P[\cdot]\) is a function \(P[\cdot] : \Delta_P \rightarrow 2^{\Delta_R \times \Delta_R}\);
6. \(C[\cdot]\) is a function \(C[\cdot] : \Delta_C \rightarrow 2^{\Delta_R}\);

\(\Delta\) maps each \(t \in \text{UL} \cap V\) into a value \(t^\Delta \in \Delta_R \cup \Delta_P\), where \(\Delta\) is the identity for literals; and

8. \(\Delta\) maps each variable \(x \in \text{B}\) into a value \(x^\Delta \in \Delta_R\).

As next, for space reasons and without loosing the substantial ingredients, we illustrate the so-called reflexive-relaxed rdf semantics (Muñoz, Pérez, and Gutierrez 2009, Definition 12), in which the predicates sc and sp are not assumed to be reflexive. Informally, the notion entailment is defined using the idea of satisfaction of a graph under certain interpretation. Intuitively a ground triple \((s, p, o)\) in an rdf graph \(G\) will be true under the interpretation \(I\) if \(p\) is interpreted as a property name, \(s\) and \(o\) are interpreted as resources, and the interpretation of the pair \((s, o)\) belongs to the extension of the property assigned to \(p\). Moreover, blank nodes, i.e., variables, work as existential variables. Intuitively the triple \((x, p, o)\) with \(x \in \text{B}\) will be true under \(I\) if \(I\) maps \(x\) into a resource \(s\) such that the pair \((s, o)\) belongs to the extension of the property assigned to \(p\).

Definition 1 (Model/Satisfaction/Entailment \(\models\)). A rdf interpretation \(I\) is a model of a rdf graph, denoted \(I \models G\), if and only if \(I\) is an interpretation over the vocabulary rdf \(\cup \text{uni}(G)\) such that:

Simple:
1. for each \((s, p, o) \in G\), \(p^\Delta \in \Delta_P\) and \((s^\Delta, o^\Delta) \in P[p^\Delta];\)

Subproperty:
1. \((p \geq p^\Delta)\) is transitive over \(\Delta_P;\)
2. if \((p, q) \in P[p^\Delta]\) then \(q, p \in \Delta_P\) and \(P[p] \subseteq P[q];\)

Subclass:
1. \((p \geq p^\Delta)\) is transitive over \(\Delta_C;\)
2. if \((c, d) \in P[p^\Delta]\) then \(c, d \in \Delta_C\) and \(C[c] \subseteq C[d];\)

Typing I:
1. \(x \in C[c] \text{ if and only if } (x, c) \in P[type^\Delta];\)
2. if \((p, c) \in P[dom^\Delta] \text{ and } (x, y) \in P[p]\) then \(x \in C[c];\)
3. if \((p, c) \in P[range^\Delta] \text{ and } (x, y) \in P[p]\) then \(y \in C[c];\)

Typing II:
1. for each \(e \in \text{rdf}, e^\Delta \in \Delta_P;\)
2. if \((p, c) \in P[dom^\Delta] \text{ then } p \in \Delta_P \text{ and } c \in \Delta_C;\)
3. if \((p, c) \in P[range^\Delta] \text{ then } p \in \Delta_P \text{ and } c \in \Delta_C;\)
4. if \((x, c) \in P[type^\Delta] \text{ then } c \in \Delta_C;\)

A graph \(G\) is satisfiable if it has a model \(I\). Moreover, given two graphs \(G\) and \(H\), \(G\) entails \(H\), denoted \(G \models H\), if and only if every model of \(G\) is also a model of \(H\).

Example 2. From \(G\) in Example 1, it can be shown that \(G \models \{\text{fever, hasTreatment, x}, (x, \text{type, drugTreatment})\}\).

Deductive System for rdf. We recap the sound and complete deductive system for rdf (Muñoz, Pérez, and Gutierrez 2009). In every rule, \(A, B, C, D, E, X\) and \(Y\) stand for meta-variables to be replaced by actual terms. An instantiation of a rule is obtained by replacing all meta-variables with terms such that all triples after the replacement are rdf triples.
Definition 2 (Deductive rules for $\rho_{df}$). The deductive rules for $\rho_{df}$ are the following:

1. Simple:
   \[(a) \frac{G'}{\mu : G' \rightarrow G} \quad (b) \frac{G'}{G' \subseteq G}\]

2. Subproperty:
   \[(a) \frac{(A, sp, B, (B, sp, C))}{(A, sp, C)} \quad (b) \frac{(D, sp, E, (X, D, Y), (X, B, Y))}{(X, D, Y)}\]

3. Subclass:
   \[(a) \frac{(A, sp, B, (B, sp, C))}{(A, sp, C)} \quad (b) \frac{(A, sp, B, (X, type, A))}{(X, type, B)}\]

4. Typing:
   \[(a) \frac{(D, dom, B), (X, D, Y), (X, type, B)}{(D, range, B), (X, D, Y)}\]

5. Implicit Typing:
   \[(a) \frac{(A, dom, B), (D, sp, A), (X, D, Y)}{(X, type, B)}\]

Definition 3 (Derivation $\vdash$). Let $G$ and $H$ be $\rho_{df}$-graphs. $G \vdash H$ if and only if there exists a sequence of graphs $P_1, P_2, \ldots, P_k$ with $P_1 = G$ and $P_k = H$ and for each $j$ ($2 \leq j \leq k$) one of the following cases hold:

- There is a map $\mu : P_j \rightarrow P_{j-1}$ (rule (1a));
- $P_j \subseteq P_{j-1}$ (rule (1b));
- There is an instantiation $R' \subseteq R$ of one of the rules (2)-(5), such that $R \subseteq P_{j-1}$ and $P_j = P_{j-1} \cup R'$.

Such a sequence of graphs is called a proof of $G \vdash H$.

When $G \vdash H$, we say that the graph $H$ is derived from the graph $G$. Each pair $(P_{j-1}, P_j)$, $1 \leq j \leq k$ is called a step of the proof which is labeled by the respective instantiation $R' \subseteq R$ of the rule applied at the step.

Please note that if $G \vdash H$ then $H$ is indeed a graph. Finally, the closure of a graph $G$, denoted $\text{Cl}(G)$, is defined as

$$\text{Cl}(G) = \{\tau \mid G \vdash \tau\}$$

where $\vdash$ is the rule except that rule (1a) is excluded.

Example 3. Consider Example 1. Then it can be verified that $\text{Cl}(G) \supseteq \{\text{morphine, type, drug treatment}, \text{brain tumour, type, illness}\}$.

The following proposition recaps salient results taken from (Gutiérrez et al. 2011; Muñoz, Pérez, and Gutiérrez 2009; ter Horst 2005)

Proposition 1. Every $\rho_{df}$-graph is satisfiable. Moreover, let $G$ and $H$ be $\rho_{df}$-graphs. Then

1. $G \vdash H$ if and only if $G \models H$;
2. If $G \models H$ then there is a proof of $H$ from $G$ where rule (1a) is used at most once and at the end;
3. The closure of $G$ is unique and $|\text{Cl}(G)| \in \Theta(|G|^2)$;
4. Deciding $G \models H$ is an NP-complete problem;
5. If $G$ is ground then $\text{Cl}(G)$ can be determined without using implicit typing rules (5);
6. If $H$ is ground, then $G \models H$ if and only if $H \subseteq \text{Cl}(G)$;
7. There is no triple $\tau$ such that $\emptyset \models \tau$.

Remark 1. Please note that: (i) a proof of NP-completeness of point 4, above can be found in (ter Horst 2005, Proposition 2.19) via a reduction of the $k$-clique problem encoding an undirected graph $G'$ into a pdf graph $G'$ (an edge $(v, w)$ is encoded via two triples $(v, e, w)$ and $(w, e, v)$) and $H'$ consists of the triples $(x, e, y)$, where $x$ and $y$ are distinct variables of new set of $k$ blank nodes. Then, $G$ has a clique of size $\geq k$ iff $G' \models H'$, i.e., there is a map $\mu : H' \rightarrow G'$; and (ii) concerning the size of the closure, the lower bound is determined by triples $(p_1, sp, p_2), \ldots, (p_n, sp, p_{n+1})$, whose closure’s size is $\Omega(n^2)$ via rule (2a). The upper bound follows by an analysis of the rules, where the important point is the propagation of triples $(s, p, o)$ via rule (2b). This gives at most a quadratic upper bound (for triples with fixed predicate, the quadratic bound is trivial).

Note that from Proposition 1 it follows that deciding if, for two ground $\rho_{df}$-graphs $G$ and $H$, $G \models H$ can be done in time $O(|H||G|^2)$ by computing first the closure of $G$ and then check whether $H$ is in that closure. An alternative method not requiring to compute the closure with a computational benefit is illustrated by the following proposition:

Proposition 2 (Muñoz, Pérez, and Gutiérrez 2009). Let $G$ and $H$ be two ground $\rho_{df}$ graphs. Then deciding if $G \models H$ can be done in time $O(|H||G|\log |G|)$. The result holds also in case each triple in $H$ has at most one blank node.

3 Extending $\rho_{df}$ with Negative Statements

In this section, we show how to extend $\rho_{df}$ allowing to represent negative statements.

3.1 Syntax

To start with, consider a new pair of predicates, $\perp$ and $\perp_p$, used to represent disjoint information; e.g., (i) $(c, \perp_c, d)$ indicates that the classes $c$ and $d$ are disjoint; analogously, (ii) $(p, \perp_p, q)$ indicates that the properties $p$ and $q$ are disjoint.

We call $\rho_{df_\perp}$ the vocabulary obtained from $\rho_{df}$ by adding $\perp$ and $\perp_p$, that is,

$$\rho_{df_\perp} = \rho_{df} \cup \{\perp, \perp_p\}.$$  

Like for $\rho_{df}$, we assume that $U$ contains the $\rho_{df_\perp}$ vocabulary. Now we extend the alphabet $U$ in the following way:

1. For each (atomic) resource $r \in U \setminus \rho_{df_\perp}$, we add to $U$ a new negated resource $\neg_r$ of $r$. Let $U'$ be the resulting alphabet. We will use the convention that $\neg_r$ is $r$.
2. Informally, $\neg_r$ is intended to represent the complement of $r$. So, for instance, (paracetamol, type, opioid) may encode “paracetamol is a non opioid treatment”;

For each resource $c \in U' \setminus \rho_{df_\perp}$, we add to $U'$ a new resource of the form $c$. Let $U''$ be the resulting alphabet. Informally, e.g., a triple $(s, p, c)$ represents an universal quantification on the third argument over instance of class $c$, i.e., $(s, p, t)$ is true for all $t \in$
Table 1: Informal FOL reading of some types of $\rho df^{-}_I$-triples.

| $\rho df^{-}_I$       | FOL                                                                 |
|-----------------------|----------------------------------------------------------------------|
| $(s, p, o)$           | $p(s, o)$                                                           |
| $(s, p, x)$           | $\exists x. \neg p(s, x)$                                           |
| $(a, \text{type}, c)$ | $\neg c(a)$                                                         |
| $(c, sc, c)$          | $\forall x. c(x) \rightarrow \neg d(x)$                           |
| $(p, dom, c)$         | $\forall x. \forall y. p(x, y) \rightarrow \neg c(x)$             |
| $(\neg p, \text{range}, d)$ | $\forall x. \forall y. \neg p(x, y) \rightarrow d(y)$             |
| $(c, \perp, c)$       | $\forall x. c(x) \land \neg c(x) \rightarrow \perp$               |
| $(\neg p, \text{range}, q)$ | $\forall x. \forall y. \neg p(x, y) \land q(x, y) \rightarrow \perp$ |
| $(\neg p, \perp, c)$  | $\forall y. c(y) \rightarrow \neg p(s, y)$ (i.e., $\forall y. \neg c(y) \land p(s, y)$) |

3. deciding entailment in $\rho df^{-}_I$ still ranges from $P$ to $NP$.

To do so, we will consider a four-valued logic semantics (Belnap 1977) variant of the semantics for $\rho df$. Specifically, we will have positive extensions of $P$ and $C$ (denoted $P^+$ and $C^+$, respectively) and negative extensions of $P$ and $C$ (denoted $P^-$ and $C^-$, respectively). Roughly, $C^+[c]$ will denote the set of resources known to be instances of class $c$, while $C^-[c]$ will denote the set of resources known not to be instances of class $c$ (for properties the case is similar). Note that positive and negative extensions need not to be the complement of each other; e.g., $r \notin C^+[c]$ does not imply necessarily that $r \in C^-[c]$ as $C^-[c]$ will not enforced to be $e.g., \Delta R \setminus C^+[c]$.

The idea of having separate positive and negative extensions is not new at all and we may find already traces of it back in the mid 80s with the seminal work of Patel-Schneider (Patel-Schneider 1985; Patel-Schneider 1986; Patel-Schneider 1987; Patel-Schneider 1988; Patel-Schneider 1989) in which four-valued variants of Terminological Logics (TLs), viz. the so-called Description Logics (DLs) (Baader et al. 2007) nowadays, have been proposed with the aim to obtain some gain from a computational complexity point of view. Later the works (Straccia 1997a; Straccia 1997b; Straccia 1999; Straccia 2000) have been inspired by the same idea, though also to model some sort of relevance entailment, besides being paraconsistent. More recently, a similar idea has been considered also in the context of RDFS (Analyti et al. 2004; Analyti et al. 2005; Analyti et al. 2013; Analyti, Damásio, and Antoniou 2015), which is also the semantics we start from and are going to adapt and extend to meet the before mentioned objectives.

A $\rho df^{-}_I$ interpretation $\mathcal{I}$ over a vocabulary $V$ is a tuple $\mathcal{I} = \langle \Delta R, \Delta P, \Delta C, \Delta L, P^+[\cdot], P^-[\cdot], C^+[\cdot], C^-[\cdot], \mathcal{I} \rangle$, where the finite non-empty sets $\Delta R, \Delta P, \Delta C, \Delta L$ are the interpretation domains of $\mathcal{I}$ and $P^+[\cdot], P^-[\cdot], C^+[\cdot], C^-[\cdot]$ are the interpretation functions of $\mathcal{I}$. They have to satisfy:

1. $\Delta R$ are the resources;
2. $\Delta P$ are property names;
3. $\Delta C \subseteq \Delta R$ are the classes;
4. for each domain $\Delta R, \Delta P$ and $\Delta C$, for each term $t$ in it, there is an unique designated complement term of $t$, denoted $\neg t$, in it;\footnote{\textsuperscript{13}As for $U$, we will use the convention that $\neg \neg t = t$.}
5. $\Delta L \subseteq \Delta R$ are the literal values and contains all the literals in $L \setminus V$;
6. $P^+[\cdot]$ and $P^-[\cdot]$ are functions $\Delta P \rightarrow 2^{\Delta S \times \Delta S}$ such that $P^+[-p] = P^-[-p]$, for each $p \in \Delta P$;
7. $C^+[\cdot]$ and $C^-[\cdot]$ are functions $\Delta C \rightarrow 2^{\Delta S}$ such that $C^+[-c] = C^-[-c]$, for each $c \in \Delta C$;
8. $\mathcal{I}$ maps each $t \in UL \setminus V$, that is not of the form $\ast_{\cdot} t$, into a value $t^\mathcal{I} \in \Delta R \cup \Delta P$, such that $(\neg t)^\mathcal{I} = \neg t^\mathcal{I}$ and $\mathcal{I}$ is the identity for literals; and
9. $\mathcal{I}$ maps each variable $x \in B$ into a value $x^\mathcal{I} \in \Delta R$.\footnote{\textsuperscript{12}The FOL reading is ‘informal’ as the translation isn’t a bijection, e.g., we have a 4-valued intentional semantics and individuals may be classes, which may be negated.}
In the following, we define

\[ P^+ [p] \uparrow = \{ x \in \Delta_R | (x, y) \in P^+ [p] \} \]
\[ P^+ [p] \downarrow = \{ y \in \Delta_R | (x, y) \in P^+ [p] \} \]
\[ P^- [p] \uparrow = \{ x \in \Delta_R | (x, y) \in P^- [p] \} \]
\[ P^- [p] \downarrow = \{ y \in \Delta_R | (x, y) \in P^- [p] \} \]

as the projections of the property extension functions \( P^+ [\cdot] \) and \( P^- [\cdot] \) on the first and second argument, respectively.

Now, the model/satisfaction/entailment definitions for \( \rho_{df} \) are generalised to \( \rho_{df}^- \) as follows:

**Definition 5 (Model/Satisfaction/Entailment \( \rho_{df}^- \)).** A pdf\( _{-} \) interpretation \( I \) is a pdf\( _{-} \)-model of a pdf\( _{-} \) graph \( G \), denoted \( I \models H \), if and only if \( I \) is a pdf\( _{-} \)-interpretation over the vocabulary pdf\( _{df}^- \cup \undegamma(G) \) such that:

**Simple:**
1. if \( (s, p, o) \in G \) and neither \( s \) nor \( o \) are of the form \( \ast_\alpha \), then \( p^T \in \Delta_{\rho} \) and \( (s^T, o^T) \in P^+ [p^T] \);
2. if \( (s, p, \ast_\alpha) \in G \), then \( p^T \in \Delta_{\rho}, c^T \in \Delta_C \) and \( (s^T, y) \in P^+ [p^T], \) for all \( y \in C^- [c^T] \);
3. if \( (\ast_\alpha, p, s) \in G \), then \( p^T \in \Delta_{\rho}, c^T \in \Delta_C \) and \( (x, s^T) \in P^+ [p^T], \) for all \( x \in C^+ [c^T] \);
4. if \( (s, p, \ast_\alpha) \in G \), then \( p^T \in \Delta_{\rho}, c^T \in \Delta_C \) and \( y \in C^- [c^T], \) for all \( s^T, y \in P^+ [p^T] \);
5. if \( (\ast_\alpha, p, s) \in G \), then \( p^T \in \Delta_{\rho}, c^T \in \Delta_C \) and \( x \in C^+ [c^T], \) for all \( x, s^T \in P^+ [p^T] \);

**Subproperty:**
1. if \( (p, q) \in G \) and \( P^+ [sp^T] \) is transitive over \( \Delta_{\rho} \); then \( P^+ [sp^T] \) if and only if \( \rho_{df}^- \);
2. if \( (p, q) \in P^+ [sp^T] \) then \( p, q \in \Delta_{\rho} \) and \( P^+ [p] \subseteq P^+ [q] \);
3. if \( (p, q) \in P^+ [sp^T] \) if and only if \( \rho_{df}^- \);

**Subclass:**
1. if \( (c, d) \in P^+ [sc^T] \) then \( c, d \in \Delta_{\rho} \) and \( C^+ [c] \subseteq C^+ [d] \);
2. if \( (c, d) \in P^+ [sc^T] \) and \( c, d \in \Delta_{\rho} \) and \( C^+ [c] \subseteq C^+ [d] \);
3. if \( (c, d) \in P^+ [sc^T] \) if and only if \( \rho_{df}^- \);

**Typing I:**
1. if \( x \in C^+ [c] \) if and only if \( \rho_{df}^- \);
2. if \( (p, c) \in P^+ [\text{dom}^T] \) and \( (x, y) \in P^+ [p] \) then \( x \in C^+ [c] \);
3. if \( (p, c) \in P^+ [\text{range}^T] \) and \( (x, y) \in P^+ [p] \) then \( y \in C^+ [c] \);
4. if \( (p, c) \in P^+ [\text{dom}^T] \) and \( x \in C^- [c] \) and \( y \in P^+ [p] \), then \( x \in P^+ [p] \);
5. if \( (p, c) \in P^+ [\text{range}^T] \) and \( x \in C^- [c] \) and \( x \in P^+ [p] \), then \( x \in P^+ [p] \);

**Typing II:**
1. if \( (x, c) \in \rho_{df}^- \) then \( x \in \Delta_{\rho} \);
2. if \( (p, c) \in P^+ [\text{dom}^T] \) then \( p \in \Delta_{\rho} \) and \( c \in \Delta_{C} \);
3. if \( (p, c) \in P^+ [\text{range}^T] \) then \( p \in \Delta_{\rho} \) and \( c \in \Delta_{C} \);
4. if \( (x, c) \in P^+ [\text{type}^T] \) then \( c \in \Delta_{C} \);

**Disjointness I:**
1. if \( (c, d) \in P^+ [\perp^T] \) then \( c, d \in \Delta_{\rho} \);
2. if \( (p, q) \in P^+ [\perp^T] \) then \( p, q \in \Delta_{\rho} \);
3. if \( P^+ [\perp^T] \) is symmetric, sub-transitive and exhaustive over \( \Delta_{\rho} \);
   
   **Symmetry:** if \( (c, d) \in P^+ [\perp^T] \), then \( (d, c) \in P^+ [\perp^T] \);
   
   **Sub-Transitivity:** if \( (c, d) \in P^+ [\perp^T] \) and \( (e, c) \in P^+ [sc^T] \), then \( (e, d) \in P^+ [\perp^T] \);
   
   **Exhaustive:** if \( (c, d) \in P^+ [\perp^T] \) and \( d \in \Delta_{\rho} \) then \( (d, c) \in P^+ [\perp^T] \);

**Disjointness II:**
1. if \( (p, q) \in P^+ [\perp^T] \) then \( (q, p) \in P^+ [\perp^T] \);
2. if \( (p, q) \in P^+ [\perp^T] \) and \( (r, p) \in P^+ [\perp^T] \);
3. if \( (c, d) \in P^+ [\perp^T] \) if and only if \( (d, c) \in P^+ [\perp^T] \);
4. if \( (p, q) \in P^+ [\perp^T] \) if and only if \( (p, \neg q) \in P^+ [\perp^T] \);

A graph \( G \) is pdf\( _{df}^- \)-satisfiable if it has a pdf\( _{df}^- \)-model \( I \). Moreover, given two pdf\( _{df}^- \)-graphs \( G \) and \( H \), we say that \( G \) pdf\( _{df}^- \)-entails \( H \), denoted \( G \models H \), if and only if every pdf\( _{df}^- \)-model of \( G \) is also a pdf\( _{df}^- \)-model of \( H \).

**Remark 2 (About Definition 5).** Note that:

1. the positive extension of a negated class is the negative extension of that class;
2. by construction, we also have that for \( (s, \neg p, o) \in G \), \( (s^T, o^T) \in P^+ [(\neg p)^T] = P^+ [\neg p^T] = P^- [p^T] \). That is, \( (s, \neg p, o) \in G \) states that \( (s, p) \in \Delta_{\rho} \) and \( \neg p \) is a non-p that is an \( \alpha \)’s;
3. by construction, e.g., if \( (c, d) \in P^+ [sc^T] \) then also \( C^- [d] \subset C^- [c] \). Moreover, \( x \in C^- [c] \) if and only if \( (x, \neg c) \in P^+ [\text{type}^T] \);
4. concerning e.g., point 5 in Typing I, the informal reading of \( (p, \text{range}, c) \) is \( \forall x \forall y, p(x, y) \rightarrow c(y) \) and \( \forall x \forall y, p(x, y) \rightarrow \neg c(y) \rightarrow \neg p(x, y) \). In the latter case, the aim is to limit the universal quantification in a reasonable way;
5. the presence of e.g., \( (\text{a, type, b}) \) is \( (\text{a, type, c}) \) in a graph does not preclude its satisfiability. In fact, a pdf\( _{df}^- \) graph will always be satisfiable (see Corollary 1 later on) avoiding, thus, the ex falso quodlibet principle. This is in line with the pdf semantics (Muñoz, Pérez, and Gutierrez 2009).

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6. concerning the exhaustive property in Disjointness 1: classically, c disjoint d means that the intersection of c and d is empty. So, c disjoint c tells us that c is empty, so disjoint from any other class. We preserved this property within our semantics.

Example 5. Consider Example 4. Then, it may be verified that
\[G \models ((\text{brainTumor}, \text{hasTreatment}, x), (x, \text{type}, \neg \text{antipyretic}))\] (6)
\[G \models (\text{ebola}, \neg \text{hasTreatment}, \text{paracetomol})\] (7)
\[G \models (\text{ebola}, \neg \text{hasTreatment}, \text{ebola}).\] (8)

Note that the last entailment does not hold as “ebola is not a treatment”, which instead would hold without the restriction on the domain of the universal quantification.

Remark 3. As anticipated in the related work section, (Darari, Prasojo, and Nutt 2015) considers expressions of the form
\[No\{\{(s_1, p_1, o_1), \ldots, (s_n, p_n, o_n)\}\} \] (9)
in informal FOL terms \(\neg \exists x. (p_1(s_1, o_1) \land \ldots \land p_n(s_n, o_n))\), or equivalently, \(\forall x. (\neg p_1(s_1, o_1) \lor \ldots \lor \neg p_n(s_n, o_n))\), where \(x\) are the variables occurring the triples. For instance,
\[No\{\{(\text{obama}, \text{child}, x), (x, \text{gender}, \text{male})\}\}\] (10)
expresses that “Obama has no son”.

\(\text{pdf}_1\) considers only the case \(n = 1\) in (9) via the expression \((s, \neg p, \_x)\) as the general case would introduce a disjunction, which we would like to avoid for computational reasons. Nevertheless, we may consider the option to use e.g., \((\text{obama}, \neg \text{child}, \_x)\) instead to express (10).

3.3 Deductive System for \(\text{pdf}_1\)

We now present a deductive system for \(\text{pdf}_1\).

Definition 6 Deducitive rules for \(\text{pdf}_1\). The deducitive rules for \(\text{pdf}_1\) are all rules for pdf to which we add the following rules (\(\bar{Z}\) is an additional meta-variable):

2. Subproperty:
\[(c) \quad \frac{(A, sp, B)}{(\neg B, sp, \_A)}\]
\[(d) \quad \frac{(A, \_D, \_C, (D, sp, E))}{(A, \_E, \_C)} \quad (e) \quad \frac{(C, D, A)(D, sp, E)}{(C, E, A)}\]

3. Subclass:
\[(c) \quad \frac{(A, sp, B)}{(\neg B, sc, \_A)}\]
\[(d) \quad \frac{(A, \_D, \_C, (B, sc, C))}{(A, \_D, \_C)} \quad (e) \quad \frac{(C, D, A)(B, sc, C)}{(C, E, A)}\]

4. Typing:
\[(c) \quad \frac{(D, dom, B)(X, type, \_D, B)(Z, D, Y)}{(X, \_D, Y)}\]
\[(d) \quad \frac{(D, range, B)(Y, type, \_D, B)(X, D, Z)}{(X, \_D, Y)}\]
\[(c) \quad \frac{(A, D, \_C, (Y, type, C))}{(A, D, Y)} \quad (f) \quad \frac{(C, D, B)(X, type, C)}{(X, D, B)}\]
\[(g) \quad \frac{(A, D, \_C, (A, \_D, Y, type, C))}{(A, D, Y)} \quad (h) \quad \frac{(C, D, B)(X, \_D, B)}{(X, type, \_C)}\]

6. Conceptual Disjointness:
(a) \(\frac{(A, \_D, B)}{(B, \_D, A)}\)
(b) \(\frac{(A, \_D, B), (C, sc, A)}{(C, \_D, B)}\)
(c) \(\frac{(A, \_D, A)}{(A, \_D, B)}\)
(d) \(\frac{(A, \_D, B)}{(A, sc, \_B)}\)
(e) \(\frac{(A, sc, B)}{(A, \_D, \_B)}\)

7. Predicate Disjointness:
(a) \(\frac{(A, \_D, B)}{(B, \_D, A)}\)
(b) \(\frac{(A, \_D, B), (C, sp, A)}{(C, \_D, B)}\)
(c) \(\frac{(A, \_D, A)}{(A, \_D, B)}\)
(d) \(\frac{(A, \_D, B)}{(A, sp, \_B)}\)
(e) \(\frac{(A, sp, B)}{(A, \_D, \_B)}\)

8. Crossed Disjointness:
(a) \(\frac{(A, dom, C), (B, dom, D), (C, \_D, D)}{(A, \_D, B)}\)
(b) \(\frac{(A, range, C), (B, range, D), (C, \_D, D)}{(A, \_D, B)}\)

Now, the definition of derivation among pdf\_1-graphs \(G\) and \(H\), denoted \(G \models H\), is as for pdf (see Definition 3), except that, of course, we now consider all rules of Definition 6 instead. Similarly, the pdf\_1-closure of a graph \(G\), denoted \(\text{Cl}_{1\_1}(G)\), is defined as
\[\text{Cl}_{1\_1}(G) = \{ \tau \mid G \models \_1\_1 \tau \} ,\]
where \(\_1\_1\) is as \(\_1\_1\) except that rule (1a) is excluded.

Example 6. The following is a simple proof of Eq. (6):
(1) \((\text{opioid}, \_D, \text{antipyretic})\) Rule (1b)
(2) \((\text{opioid}, \_C, \neg \text{antipyretic})\) Rule (6b) \(; (1)\)
(3) \((\text{morphine}, \_D, \text{opioid})\) Rule (1b)
(4) \((\text{morphine}, \_C, \neg \text{antipyretic})\) Rule (3b) \(; (2), (3)\)
(5) \((\text{brainTumor}, \_D, \_\text{DrugTreatment}, \_\text{morphine})\) Rule (1b)
(10) \((\text{brainTumor}, \_D, \_\text{hasTreatment}, \_x)\) Rule (1a) \(; (4), (5)\)

In the following, we will also assume that the definition of entailment \(\models \) is extended naturally to pdf\_1-graphs by considering \(\_D, \_D, \neg \_p, \_x\) as resources without any specific semantic constraint. In a similar way, we assume \(\models \) (and \(\text{Cl}(\_D)\)) to be extended to pdf\_1-graphs by assuming that triples involving \(\_D, \_D, \neg \_p, \_x\) are considered as pdf triples. Then, the following can easily be proved:

Proposition 3. Let \(G\) and \(H\) be two pdf\_1-graphs. Then,
1. if \(G \models H\) then \(G \models H\);  
2. if \(G \models H\) then \(G \models H\);  
3. \(\text{Cl}(G) \subseteq \text{Cl}_{1\_1}(G)\).

Of course, conditions 1.-3. above do not hold in general for the opposite direction. For instance, for \(G = \{(a, \_D, b)\}\) we have \(G \models (a, \_D, \_B)\), \(G \models (a, \_D, \_B)\), and \(a, \_D, \_B \notin \text{Cl}(G)\), but \(G \models (a, \_D, \_B)\) and \(a, \_D, \_B \notin \text{Cl}_{1\_1}(G)\).

The next proposition shows the construction of the canonical model for pdf\_1-graphs and extends the result for pdf (see Proposition 1), i.e., all pdf\_1-graphs \(G\) are satisfiable.
Proposition 4 (pdfΓ Canonical model). Given a pdfΓ-graph G, define a pdfΓ interpretation I_G as a tuple

\[ I_G = (\Delta_R, \Delta_P, \Delta_C, \Delta_L, P^+[\cdot], P^-[\cdot], C^+[\cdot], C^-[\cdot], I_G) \]

such that:

1. \( \Delta_R := \text{uni}(G) \cup \{ r \mid r \in \text{uni}(G) \} \cup \text{pdf} \);
2. \( \Delta_P := \{ p \in \text{uni}(G) \mid \text{either } (s, p, o), (s, p, \ast) \}, (s, p, o), (s, p, q), (q, s, p), (p, dom, c), (p, range, d) \}
   or \( (p, \bot_p, q) \) is in \( C^+_I(G) \} \cup \text{pdf} \);
3. \( \Delta_C := \{ c \in \text{uni}(G) \mid \text{either } (x, type, c), (c, sc, d), (d, sc, c), (p, dom, c), (p, range, d), (s, p, \ast), (s, p, o) \}
   or \( (c, \bot_c, d) \) is in \( C^+_I(G) \} \);
4. \( \Delta_L := \{ \neg c \mid c \in \Delta_C \} \);
5. \( I_G[\bot] = G \).

Proof. We prove that \( I_G \) satisfies the constraints in Definition 5. We have to prove that

\[ (\Delta_R, \Delta_P, \Delta_C, \Delta_L, P^+[\cdot], P^-[\cdot], C^+[\cdot], C^-[\cdot], I_G) \]

such that:

1. \( \Delta_R := \text{uni}(G) \cup \{ r \mid r \in \text{uni}(G) \} \cup \text{pdf} \);
2. \( \Delta_P := \{ p \in \text{uni}(G) \mid \text{either } (s, p, o), (s, p, \ast) \}, (s, p, o), (s, p, q), (q, s, p), (p, dom, c), (p, range, d) \}
   or \( (p, \bot_p, q) \) is in \( C^+_I(G) \} \cup \text{pdf} \);
3. \( \Delta_C := \{ c \in \text{uni}(G) \mid \text{either } (x, type, c), (c, sc, d), (d, sc, c), (p, dom, c), (p, range, d), (s, p, \ast), (s, p, o) \}
   or \( (c, \bot_c, d) \) is in \( C^+_I(G) \} \);
4. \( \Delta_L := \{ \neg c \mid c \in \Delta_C \} \);
5. \( I_G[\bot] = G \).

Typing I:

2. Assume \((p, c) \in P^+[\cdot, \cdot]\) and \((x, y) \in P^+[\cdot]\).
   By construction, both \((p, \text{dom}, c)\) and \((x, y)\) are in \( C^+_I(G) \). Therefore, by rule \(4a\), \((x, \text{type}, c) \in C^+_I(G) \) and, thus, by construction \( x \in C^+[\cdot] \), which concludes.

4. Assume \((p, c) \in P^+[\text{dom}^2\cdot], x \in C^-[\cdot] \) and \( y \in P^+[\cdot,p] \). By construction, we have that \( \{(p, \text{dom}, c), (x, \text{type}, \ast), (z, p, y)\} \subseteq C^+_I(G) \).
   Therefore, by rule \(4e\), \((x, \bot_p, y) \in C^+_I(G) \) and, thus, by construction, \((x, y) \in P^+[\cdot, \cdot, \cdot, \cdot]\) which concludes.

Typing II:

2. Assume \((p, c) \in P^+[\cdot, \cdot]\). Then, by construction \((p, \text{dom}, c) \in C^+_I(G) \) and, thus, \( p \in \Delta_P \subseteq \Delta_P \) and \( c \in \Delta_C \subseteq \Delta_C \), which concludes.

Disjointness I:

3. Symmetry: Assume \((c, d) \in P^+[\cdot, \cdot, \cdot, \cdot]\). By construction \((c, \bot_c, d) \in C^+_I(G) \), and, thus, by rule \(8a\) \((d, \bot_c, c) \in C^+_I(G) \). Therefore, by construction, \((d, c) \in P^+[\cdot, \cdot, \cdot, \cdot] \), which concludes.

Disjointness II:

1. Assume \((p, c) \in P^+[\cdot, \cdot, \cdot, \cdot]\) and \((c, d) \in P^+[\cdot, \cdot, \cdot, \cdot] \).
   Then, by construction, we have that \( \{(p, \text{dom}, c), (d, \text{sc}, \cdot, \cdot)\} \subseteq C^+_I(G) \) and, thus, by rule \(8a\) \((p, \bot_p, q) \in C^+_I(G) \). Therefore, by construction, \((p, q) \in P^+[\cdot, \cdot, \cdot, \cdot] \), which concludes.

3. If \((c, d) \in P^+[\cdot, \cdot, \cdot, \cdot] \) then, by construction, \((c, \bot_c, d) \in C^+_I(G) \) and, thus, by rule \(6d\) \((c, \bot_c, d) \in C^+_I(G) \). Therefore, by construction, \((c, d) \in P^+[\cdot, \cdot, \cdot, \cdot] \). Vice versa, if \((c, d) \in P^+[\cdot, \cdot, \cdot, \cdot] \) then by construction, \((c, \bot_c, d) \in C^+_I(G) \). Therefore, by construction, \((c, d) \in P^+[\cdot, \cdot, \cdot, \cdot] \), which concludes.

By Proposition 4, it follows that

**Corollary 1.** Every pdfΓ-graph is satisfiable.

We prove now soundness and completeness of our deduction system for pdfΓ. The proofs are inspired by the analogous ones in (Muñoz, Pérez, and Gutiérrez 2009) for pdf.

The following proposition is needed for soundness.

**Proposition 5 (Soundness).** Let G and H be pdfΓ-graphs and let one of the following statements hold:

1. there is a map \( \mu : H \rightarrow G \);
2. \( H \subseteq G \);
3. there is an instantiation \( R/R' \) of one of the rules in Definition 6, such that \( R \subseteq G \) and \( H = G \cup R' \).

Then, \( G \models H \).

**Proof.** By Corollary 1 we know that G is satisfiable. So, let

\[ I = (\Delta_R, \Delta_P, \Delta_C, \Delta_L, P^+[\cdot], P^-[\cdot], C^+[\cdot], C^-[\cdot], G) \]

be a model of G, i.e., \( I \models G \). Therefore, I satisfies the constraints in Definition 5. We have to prove that \( I \models H \).

The proof is split in cases depending on rule applications of which we address here only some of them. The other cases can be shown similarly.
Rule (4e). Let \( \{(s, p, \ast_e), (o, \text{type}, c)\} \subseteq R \) for some \( R \subseteq G, R' = \{(s, p, o)\} \), obtained via the application of rule (4e), and \( H = G \cup R' \). As \( \mathcal{I} \models_p G \) and \( R \subseteq G, \mathcal{I} \models_p R \) follows. Therefore, \( \mathcal{I} \models_p (o, \text{type}, c) \) and, thus, \( o' \in C^+ \) follows. But, also \( \mathcal{I} \models_p (s, p, \ast_e) \) and, thus, by condition Simple, case 2. in Definition 5, we have that \( (s', o') \in P^+ \). That is, \( \mathcal{I} \models_p (s, p, o) \). Hence, from \( \mathcal{I} \models_p R', \mathcal{I} \models_p G \) and \( H = G \cup R', \mathcal{I} \models_p H \) follows.

Rule (6d). Let \( (c, \bot, d) \in R \) for some \( R \subseteq G, R' = \{(c, \bot, d)\} \), obtained via the application of rule (6d), and \( H = G \cup R' \). As \( \mathcal{I} \models_p G \) and \( R \subseteq G, \mathcal{I} \models_p R \) follows. Therefore, \( \mathcal{I} \models_p (c, \bot, d) \) and, thus, \( (c', d') \in P^+ \). But, by condition Disjointness, case 3. in Definition 5 we have that \( (c', d') \in P^+ \). Therefore, \( \mathcal{I} \models_p (c, \bot, d) \). Hence, from \( \mathcal{I} \models_p R', \mathcal{I} \models_p G \) and \( H = G \cup R', \mathcal{I} \models_p H \) follows.

Proposition 6. Let \( G \) and \( H \) be \( \rho d f \)-graphs. If \( G \models_p H \) then there is a map \( \mu : H \rightarrow \Theta^+_1(G) \).

Proof. Consider the canonical model \( \mathcal{I}_G = \{(\Delta_R, \Delta_P, \Delta_C, \Delta_L, P^+[], P^-[], C^+, C^-, I^G)\} \) of \( G \) as defined in Proposition 4. As \( G \models_p H \), \( \mathcal{I}_G \models_p H \) follows. Therefore, for each \( (s, p, o) \in H, p^G, o^G \in \Delta_P \) and \( (s'^G, o'^G) \in P^+ P^G \). By construction, \( p^G = p \) and \( P^+ p^G = \{t, t' \mid t, t' \in \Theta^+_1(G)\} \). Finally, since \( (s'^G, o'^G) \in P^+ p^G \), we have that \( (s'^G, o'^G) \in \Theta^+_1(G) \), i.e., \( s'^G, o'^G, (s^G, o^G) \in \Theta^+_1(G) \). Therefore, \( \mathcal{I}_G \) is a map such that \( H \models p^G \subseteq \Theta^+_1(G) \), i.e., a map \( \mathcal{I}_G : H \rightarrow \Theta^+_1(G) \), which concludes.

From Proposition 6 we get the following corollary:

Corollary 2. Let \( G \) and \( H \) be \( \rho d f \)-graphs. If \( G \models_p H \) then there is a proof of \( H \) from \( G \) where rule (1a) is used at most once and at the end.

Eventually, combining previous Propositions 5 and 6, we get soundness and completeness of our deductive system.

Theorem 1 (Soundness & Completeness). Let \( G \) and \( H \) be \( \rho d f \)-graphs. Then \( G \models_p H \) iff \( G \models_p H \).

Proof. Concerning soundness, if \( G \models_p H \) then, by Proposition 5, \( G \models_p H \). Concerning completeness, if \( G \models_p H \) then, by Proposition 6, \( H \) can be obtained from \( \Theta^+_1(G) \) using rule (1a). Therefore, as \( G \models_p \Theta^+_1(G), G \models_p H \) follows which concludes.

Finally, unlike \( \rho d f \) (see Proposition 1), the size of the closure of a \( \rho d f \)-graph \( G \) is \( \Theta(|G|^3) \). The upper bound comes from the fact that in a triple \( (s, p, o) \), for each \( s, p \) and \( o \) we may have at most \( |G| \) terms, while the lower bound is given by the following example:

Example 7. It can easily be verified that for \( 1 \leq i < j \leq n \) and \( 1 \leq l, k, h \leq n \)

\[
\{(a_i, \text{type}, c_i), (a_i, p_1, \ast_e), (p_i, s_p, p_j)\} \models_p (a_k, p_h, a_n),
\]

and, thus, the number of triples in the closure is \( \Omega(|G|^3) \).

Furthermore, it is not difficult to see that if \( \ast_e \) terms do not occur in a \( \rho d f \)-graph \( G \), then the closure of \( G \) remains quadratically upper bounded. As case (i) in Remark 1 also applies to \( \rho d f \), it can be shown that

Proposition 7. Let \( G \) and \( H \) be \( \rho d f \)-graphs. Then

1. the closure of \( G \) is unique and \( |\Theta^+_1(G)| \in \Theta(|G|^3); \)
2. if \( \ast_e \) terms do not occur in \( G \) then \( |\Theta^+_1(G)| \in \Theta(|G|^2); \)
3. deciding \( G \models_p H \) is an NP-complete problem;
4. if \( G \) is ground then \( \Theta^+_1(G) \) can be determined without using implicit typing rules (5);
5. if \( H \) is ground, then \( G \models_p H \) if and only if \( H \subseteq \Theta^+_1(G) \);
6. There is no triple \( \tau \) such that \( \emptyset \models_p \tau \).

Eventually, by Proposition 7 it follows immediately that

Corollary 3. Let \( G \) and \( H \) be two ground \( \rho d f \)-graphs. Then deciding if \( G \models_p H \) can be done in time \( O(|H||G|^3) \) and in time \( O(|H||G|^2) \) if \( \ast_e \) terms do not occur in \( G \).

4 Conclusions

We have addressed the problem to add negative statements of various form considered as relevant for RDFS by the literature. We have presented a sound and complete deductive system that consists of RDFS rules plus some additional rules to deal with the extra type of triples we allow. The design of the semantics has been such that to preserve features such as the canonical model property and computational attractiveness.

As future work, Corollary 3 tells us that there is still some computational complexity gap w.r.t. \( \rho d f \) (see Proposition 2), which we would like to reduce as much as possible. In particular, we want to investigate whether the principles of the method proposed in (Muñoz, Pérez, and Gutierrez 2009) for \( \rho d f \) can be adapted to \( \rho d f \) as well. Additionally, we would like to address query answering, in particular to extend our framework to SPARQL and to verify whether and how it impacts w.r.t. \( \rho d f \) graphs. Eventually, we would like to develop a FOL encoding of our semantics in the spirit of (Franconi et al. 2013), which encodes, e.g., subclass transitivity via \( \forall x, y, z. sc(x, y) \land sc(y, z) \rightarrow sc(x, z) \), and determine whether our semantics can be reformulated in terms of a well-known paraconsistent semantics, such as e.g., Quasi-classical logic (Marquis and Porquet 2001).
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