A NEW PERSPECTIVE ON NEWTON’S LAW OF COOLING IN FRAME OF NEWLY DEFINED FRACTIONAL CONFORMABLE DERIVATIVE

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Abstract. In this paper, Newton’s law of cooling is considered from a different perspective with newly defined fractional conformable. Obtained results are compared with experimental results and found optimal fractional orders which fit better with real data. Results show that Newton’s law of cooling with fractional conformable derivative gives better results to integer order derivative. Results are given comparatively to Newton’s law of cooling with integer order and experimental data and also, fractional conformable derivative’s advantages are supported by numerical illustrations and error analysis.

Keywords: Fractional conformable derivative, Newton’s law of cooling, Analytical solutions.

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1 Introduction

Fractional differential equations idea was firstly suggested by Leibniz on generalizing of integer order derivative three centuries ago. Many problems with real world phenomena were modelled by integer order derivatives. If the variable in the integer order derivative is specified as time, it gives the amount of change according to time and so, approximate amount of change can be calculated by changing time. Fractional derivative definitions involve variable order and hence, it is not hard to estimate to give better approximations to the integer order derivative. However, many fractional derivative definitions have been introduced in recent years. These new definitions have some advantages and disadvantages, for example, fractional differential problems with Riemann-Liouville derivative do not include initial conditions with integer order, but fractional differential problems with Caputo derivative include initial conditions with integer order. Therefore, the Caputo derivative is more useful than the Riemann-Liouville derivative in some engineering and physical problems. Fractional derivatives with exponential and Mittag-Leffler kernel have been introduced by respectively Caputo-Fabrizio \cite{14} and Atangana-Baleanu \cite{3}. These new operators are more useful in some real world problems due to having non-singularity in its kernels.

Conformable derivative definition was firstly given by Khalil et al. \cite{11,13}, and this operator shows similarity to the integer order derivative, differently it includes a shift as $\varepsilon t^{-\alpha}$ in its limit definition, but it is not conformable at $\alpha = 0$, i.e. $\lim_{\alpha \to 0} sD_0^\alpha f \neq f$. More recently, a new fractional version of conformable derivative definition has been introduced by Jarad et al. \cite{5} in Riemann-Liouville and Caputo sense. Lately, Atangana et al. have introduced $\beta-$derivative and motivated by this, Morales et al. \cite{6} have introduced fractional conformable $\beta-$derivative.

Newton’s law of cooling gives the approximate amount of change in temperature of cooling or heating body to ambient temperature. Elizabeth Gieseking \cite{15} tested experimentally Newton’s law of cooling for three beakers of water for 100, 300 and 800 ml volumes. Gieseking measured
temperature of beakers every minute and compared real data to classical Newton’s law of cooling. More recently, Mondol et al. [9] have considered Newton’s law of cooling with Riemann-Liouville and Caputo sense and have tested experimentally for different liquids. They have studied to obtain to fit real data by using fractional derivatives in Riemann-Liouville and Caputo sense. Ortega et al. [10] have considered Newton’s law of cooling with conformable derivative. Almeida et al. [7, 8] have used some new fractional derivatives to obtain better results to fit data for some modelling problems. Also, some physical modelling problems are studied by [1, 2].

In this paper, we consider Newton’s law of cooling from a different perspective with newly defined fractional conformable and $\beta-$derivatives. Firstly, we obtain new analytical solutions for Newton’s law of cooling with newly defined fractional conformable and $\beta-$derivatives, and then we compare these new analytical solutions with experimental results, tested by Gieseking [15], and study to find optimal fractional orders which fit better with real data. Results show that Newton’s law of cooling with fractional conformable derivative gives better results to integer order derivative. Results are given comparatively to Newton’s law of cooling with integer order and experimental data and also, fractional conformable derivative’s advantages are supported by numerical simulations and error analysis.

2 Preliminaries

**Definition 1.** [16] The Riemann-Liouville derivative of order $\alpha$ is defined as

$$RL_aD^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t f(\xi) (t-\xi)^{n-\alpha-1} d\xi, \quad n-1 < \alpha < n.$$  

**Definition 2.** [16] The Liouville-Caputo derivative of order $\alpha$ is defined as

$$C_aD^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{d^n}{d\xi^n} f(\xi) (t-\xi)^{n-\alpha-1} d\xi, \quad n-1 < \alpha < n.$$  

**Definition 3.** [14] The fractional derivative with exponential kernel for $\alpha > 0$ is defined as

$$CFC_aD^\alpha_t f(t) = \frac{M(\alpha)}{n-\alpha} \int_a^t \frac{d^n}{d\xi^n} f(\xi) e^{-\alpha\frac{\xi}{n-\alpha}(t-\xi)} d\xi, \quad n-1 < \alpha < n,$$

where $M(\alpha)$ is a normalization constant that depends on $\alpha$, which satisfies that, $M(0) = M(1) = 1$.

**Definition 4.** [3] The fractional derivative with Mittag-Leffler kernel is defined as

$$ABC_aD^\alpha_t f(t) = \frac{B(\alpha)}{n-\alpha} \int_a^t \frac{d^n}{d\xi^n} f(\xi) E_{\alpha\left(\frac{\alpha}{n-\alpha}(t-\xi)^{\alpha}\right)} d\xi, \quad n-1 < \alpha < n$$

where $B(\alpha)$ is normalization function and $B(0) = B(1) = 1$.  

Definition 5. Let \( f : (a, \infty) \to \mathbb{R} \). The conformable derivative of \( f(t) \) is defined as follows

\[
aD_{t}^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}
\]

for all \( t > 0, \alpha \in (0, 1] \). If \( f(t) \) is \( \alpha \)-differentiable in some \((0, a), a > 0\) and \( \lim_{\varepsilon \to 0^+} f^{(\alpha)}(t) \) exist, then define \( \lim_{t \to 0^+} f^{(\alpha)}(t) = f^{(\alpha)}(0) \).

Definition 6. The left and right conformable integrals are defined as

\[
aI_{t}^{\alpha} f(x) = \int_{a}^{x} (t-a)^{\alpha-1} f(t) \, dt, \quad x \geq a, \quad 0 < \alpha \leq 1 \quad (1)
\]

and

\[
bI_{x}^{\alpha} f(x) = \int_{x}^{b} (b-t)^{\alpha-1} f(t) \, dt, \quad x \leq b.
\]

Definition 7. Fractional conformable integral is defined as

\[
\beta aI_{t}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)}\int_{a}^{x} \left( \frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} \, dt \quad (2)
\]

Theorem 1. Let \( \text{Re}(\beta) \geq 0, n = \lfloor \text{Re}(\beta) \rfloor + 1, f \in C_{\alpha,a}^{n} ([a,b]). \) Then, Riemann-Liouville fractional conformable derivatives are defined as follows,

\[
\beta aD_{t}^{\alpha} f(x) = \frac{aD_{t}^{\alpha} f(t)}{\Gamma(n-\beta)}\int_{a}^{x} \left( \frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{n-\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} \, dt, \quad (3)
\]

and

\[
\beta D_{b}^{\alpha} f(x) = \frac{(-1)^{n} bD_{b}^{\alpha} f(t)}{\Gamma(n-\beta)}\int_{x}^{b} \left( \frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha} \right)^{n-\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} \, dt. \quad (4)
\]

Theorem 2. Let \( \text{Re}(\beta) \geq 0, n = \lfloor \text{Re}(\beta) \rfloor + 1, f \in C_{\alpha,a}^{n} ([a,b]). \) Then Caputo fractional conformable derivatives are given by

\[
C_{a}^{\beta} D_{t}^{\alpha} f(x) = \frac{1}{\Gamma(n-\beta)}\int_{a}^{x} \left( \frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{n-\beta-1} \frac{aD_{t}^{\alpha} f(t)}{(t-a)^{1-\alpha}} \, dt, \quad (5)
\]

and

\[
C_{b}^{\beta} D_{x}^{\alpha} f(x) = \frac{(-1)^{n}}{\Gamma(n-\beta)}\int_{x}^{b} \left( \frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha} \right)^{n-\beta-1} \frac{bD_{x}^{\alpha} f(t)}{(b-t)^{1-\alpha}} \, dt. \quad (6)
\]
Definition 8. [4] Let $f : \left[ -\frac{a}{\Gamma(\alpha)}, \infty \right) \rightarrow \mathbb{R}$, then a different type of conformable derivative of $f(t)$ is defined as

$$A^\alpha D^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f\left( t + \varepsilon \left( t + \frac{a}{\Gamma(\alpha)} \right)^{1-\alpha} \right) - f(t)}{\varepsilon}.$$ 

The different type of left conformable integral is defined as

$$A^\alpha I^\alpha f(x) = \int_0^t \frac{f(x)}{(x + \frac{a}{\Gamma(\alpha)})^{1-\alpha}} \, dt, \quad 0 < \alpha \leq 1.$$ 

Theorem 3. [6] Let $\text{Re} (\beta) \geq 0$, $n = \lfloor \text{Re} (\beta) \rfloor + 1$, $f \in C^\alpha_{\alpha,a} ([a,b])$. Then a different type of Caputo fractional conformable derivatives are defined as follows,

$$A^{AC\beta^\alpha}_a D^\alpha f(x) = \frac{1}{\Gamma(n-\beta)} \int_a^x \left( \frac{(x+\frac{a}{\Gamma(\alpha)})^{\alpha} - (t+\frac{a}{\Gamma(\alpha)})^{\alpha}}{\alpha} \right)^{n-\beta-1} A^\alpha t^n D^\alpha_{t^n} f(t) \left( t + \frac{a}{\Gamma(\alpha)} \right)^{1-\alpha} dt, \quad (7)$$

and

$$A^{AC\beta^\alpha}_b D^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b \left( \frac{\left( \frac{b}{\Gamma(\alpha)} + t \right)^{\alpha} - \left( \frac{b}{\Gamma(\alpha)} + x \right)^{\alpha}}{\alpha} \right)^{n-\beta-1} A^\alpha t^n D^\alpha_{t^n} f(t) \left( \frac{b}{\Gamma(\alpha)} + t \right)^{1-\alpha} dt. \quad (8)$$

Theorem 4. [6] Let $\text{Re} (\beta) \geq 0$, $n = \lfloor \text{Re} (\beta) \rfloor + 1$, $f \in C^\alpha_{\alpha,a} ([a,b])$. Then a different type of Riemann-Liouville fractional conformable derivatives are defined as follows,

$$A^{AR\beta^\alpha}_a D^\alpha f(x) = \frac{A^\alpha}{\Gamma(n-\beta)} \int_a^x \left( \frac{(x+\frac{a}{\Gamma(\alpha)})^{\alpha} - (t+\frac{a}{\Gamma(\alpha)})^{\alpha}}{\alpha} \right)^{n-\beta-1} f(t) \left( t + \frac{a}{\Gamma(\alpha)} \right)^{1-\alpha} dt, \quad (9)$$

and

$$A^{AR\beta^\alpha}_b D^\alpha f(x) = \frac{(-1)^n A^\alpha}{\Gamma(n-\beta)} \int_x^b \left( \frac{\left( \frac{b}{\Gamma(\alpha)} + t \right)^{\alpha} - \left( \frac{b}{\Gamma(\alpha)} + x \right)^{\alpha}}{\alpha} \right)^{n-\beta-1} f(t) \left( \frac{b}{\Gamma(\alpha)} + t \right)^{1-\alpha} dt. \quad (10)$$

Theorem 5. [6] Let $f \in C^\alpha_{\alpha,a} ([a,b])$, $\beta \in \mathbb{C}$. Then the following property is valid,

$$\beta^\alpha A^\alpha \left( A^{AC\beta^\alpha}_a D^\alpha f(t) \right) = f(t) - \sum_{k=0}^{n-1} \frac{k^{\beta} D^\alpha_{t^n} f(a)(t-a)^{\alpha k}}{\alpha^k k!}.$$ 

and

$$\beta^\alpha A^\alpha \left( A^{AC\beta^\alpha}_a D^\alpha f(t) \right) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k k^{\beta} D^\alpha_{t^n} f(b)(b-t)^{\alpha k}}{\alpha^k k!}.$$
3 Main Results

In this section, we find exact analytical solutions of Newton's law of cooling with Caputo fractional conformable and the different type of conformable derivative.

3.1 Newton's Law of Cooling with Newly Defined Fractional Conformable Derivative

Let's consider Caputo fractional conformable derivative defined in (3).

So, we obtain exact analytical solution of the Newton's law of cooling equation. Let’s consider the initial value problem

\[ C^\beta_a D^\alpha_t T(t) = -k(T - T_m), \quad T(a) = T_0. \]  

**Solution.** Using the inverse operator of \( C^\beta_a D^\alpha_t \) to equation (1), we get

\[ \beta_a I^\alpha C^\beta_a D^\alpha_t T(t) = -k(T - T_m). \]

Considering the Theorem 2 and the initial condition (1), we have

\[ T(t) = T(a) - \beta_a I^\alpha_t \{k(T - T_m)\}. \]

Then

\[ T_{n+1}(t) = T_0 - k_0^\beta I^\alpha_t \{(T_n - T_m)\}, \quad n = 0, 1, 2, \ldots \]

For \( n = 0 \), we can write

\[ T_1(t) = T_0 - k_0^\beta I^\alpha_t \{(T_0 - T_m)\} \]

where

\[ \beta_a I^\alpha_t \{(T_0 - T_m)\} = \frac{(T_0 - T_m)}{\Gamma(\beta)} \int_a^t \left( \frac{(t - a)^\alpha - (x - a)^\alpha}{x - a} \right) dx \frac{1}{(x - a)^{1-\alpha}}. \]

Applying the change of variable \( u = \frac{x-a}{t-a} \), we have

\[ I^\alpha_t \{(T_0 - T_m)\} = \frac{(T_0 - T_m)(t-a)^{\alpha\beta}}{\alpha \beta \Gamma(\beta + 1)} \]

Substituting equation (14) into (13), we have

\[ T_1(t) = T_0 - k(T_0 - T_m)(t-a)^{\alpha\beta} \]

For \( n = 1 \), we get

\[ T_2(t) = T_0 - k_0^\beta I^\alpha_t \{T_1(t) - T_m\} \]

\[ = T_0 - k_0^\beta I^\alpha_t \left\{ \left( T_0 - T_m - \frac{k(T_0 - T_m)(t-a)^{\alpha\beta}}{\alpha \beta \Gamma(\beta + 1)} \right) \right\} \]

\[ = T_0 - k_0^\beta I^\alpha_t \{T_0 - T_m\} + \frac{k^2(T_0 - T_m)^\beta}{\alpha \beta \Gamma(\beta + 1)} I^\alpha_t \left\{ (t-a)^{\alpha\beta} \right\}. \]
Therefore, as $n \to \infty$, we can rewrite equation (11):

$$\beta^\alpha I_t^\alpha \left\{ (t-a)^{\alpha \beta} \right\} = \frac{\Gamma(\beta+1)(t-a)^{2\alpha \beta}}{\alpha^\beta \Gamma(2\beta+1)}$$

$$= T_0 - \frac{k (T_0 - T_m) (t-a)^{\alpha \beta}}{\alpha^\beta \Gamma(\beta+1)} + \frac{k^2 (T_0 - T_m) (t-a)^{2\alpha \beta}}{\alpha^{2\beta} \Gamma(2\beta+1)}$$

$$= T_0 \left( 1 - \frac{k (t-a)^{\alpha \beta}}{\alpha^\beta \Gamma(\beta+1)} + \frac{k^2 (t-a)^{2\alpha \beta}}{\alpha^{2\beta} \Gamma(2\beta+1)} \right) + \frac{T_m k (t-a)^{\alpha \beta}}{\alpha^\beta} \left( \frac{1}{\beta \Gamma(\beta)} - \frac{k (t-a)^{\alpha \beta}}{\alpha^{2\beta} \Gamma(2\beta)} \right).$$

Proceeding inductively we have

$$T_n(t) = T_0 \left( 1 - \frac{k (t-a)^{\alpha \beta}}{\alpha^\beta \Gamma(\beta+1)} + \frac{k^2 (t-a)^{2\alpha \beta}}{\alpha^{2\beta} \Gamma(2\beta+1)} - \ldots \right) + \frac{T_m k (t-a)^{\alpha \beta}}{\alpha^\beta} \left( \frac{1}{\beta \Gamma(\beta)} - \frac{k (t-a)^{\alpha \beta}}{\alpha^{2\beta} \Gamma(2\beta)} + \ldots \right)$$

$$= T_0 \sum_{z=0}^{n} \frac{(-1)^z k^z (t-a)^{z \alpha \beta}}{\alpha^{z\beta} \Gamma(z\beta+1)} + \frac{T_m k (t-a)^{\alpha \beta}}{\alpha^\beta} \sum_{z=0}^{\infty} \frac{(-1)^z k^z (t-a)^{z \alpha \beta}}{\alpha^{z\beta} (z+1) \beta \Gamma(z\beta+\beta)}.$$

Therefore, as $n \to \infty$, we find

$$T(t) = T_0 E_{\beta,1} \left( -\frac{k}{\alpha^\beta} (t-a)^{\alpha \beta} \right) + \frac{T_m k (t-a)^{\alpha \beta}}{\alpha^\beta} \sum_{z=0}^{\infty} \frac{(-1)^z k^z (t-a)^{z \alpha \beta}}{\alpha^{z\beta} (z+1) \beta \Gamma(z\beta+\beta)},$$

where $E_{\beta,1}(t)$ is Mittag-Leffler function [16].

Now, let’s consider the different type of Caputo fractional conformable derivatives defined in (7). We obtain the analytical solution of the Newton’s law of cooling. Considering the initial value problem

$$\frac{AC^\beta}{a} D_t^\alpha T(t) = -k(T - T_m),$$

$$T(a) = T_0.$$

**Solution.** If we apply similar arguments used in the proof of problem (11–11), then we have

$$T(t) = T_0 E_{\beta,1} \left( -\frac{k}{\alpha^\beta} \left( t + \frac{a}{\Gamma(\alpha)} \right)^{\alpha \beta} \right) + \frac{T_m k (t + \frac{a}{\Gamma(\alpha)})^{\alpha \beta}}{\alpha^\beta} \sum_{z=0}^{\infty} \frac{(-1)^z k^z (t + \frac{a}{\Gamma(\alpha)})^{z \alpha \beta}}{\alpha^{z\beta} (z+1) \beta \Gamma(z\beta+\beta)}.$$

### 3.2 Comparative Analysis and Discussions

In this section, we use experimental data tested by Gieseking [15]. Gieseking [15] used three beakers of water in volume of 100, 300 and 800 ml and measured temperature every minute for 35 minutes in a constant ambient temperature 23°C. We study to find the optimal fractional orders which fit better with real data. Results show that Newton’s law of cooling with fractional
conformable derivative gives better results to integer order derivative. Results are given comparatively to Newton’s law of cooling with integer order and experimental data and also, fractional conformable derivative’s advantages are supported by numerical simulations and error analysis. We try different fractional orders \((\alpha = 0.9, \beta = 0.9), (\alpha = 0.9, \beta = 0.95), (\alpha = 0.91, \beta = 0.91)\) and \((\alpha = 0.92, \beta = 0.92)\) for finding optimal order to fit real data.

Let’s consider Newton’s law of cooling with integer order derivative and its solution is as follows

\[
T(t) = T_m + (T_0 - T_m)e^{\tilde{k}t},
\]

\[
T_0 = 100^\circ C,
\]

\[
T_m = 23^\circ C,
\]

\[
\tilde{k} = \frac{\ln(T(t)-T_m)}{T_0-T_m}.
\]

The convection coefficient \(\tilde{k}\) for Newton’s law of cooling with integer order can be found analytically, but the convection coefficient \(k\) for Newton’s law of cooling with fractional order can be found approximately. The convection coefficient \(\tilde{k}\) for three beakers of water in volume of 100, 300, and 800 ml is found as respectively \(\tilde{k} = 0.0676\), \(\tilde{k} = 0.0447\), and \(\tilde{k} = 0.0327\).

Assume that temperatures of these three beakers of water in volume of 100, 300, and 800 ml are known as 45\(^\circ\)C, 55\(^\circ\)C, and 63\(^\circ\)C after 20 minutes for finding the approximate value of convection coefficient \(k\), and \(k\) will change for each value \(\alpha\) and order \(\beta\).

If we consider beaker of water in volume of 100 ml, then we observe that the optimal fractional order \(\alpha = 0.9, \beta = 0.9\) to fit real data with error analysis and simulation in Fig.1, Table1 - 2 - 3 - 4.

If we consider beaker of water in volume of 300 ml, then we observe that the optimal fractional order \(\alpha = 0.9, \beta = 0.9\) to fit real data with error analysis and simulation in Fig.2, Table5 - 6 - 7 - 8.

If we consider beaker of water in volume of 800 ml, then we observe that the optimal fractional order \(\alpha = 0.9, \beta = 0.95\) to fit real data with error analysis and simulation in Fig.3, Table9 - 10 - 11 - 12.

Finally, we compare these results similarly under any value \(\alpha\) and any order \(\beta\) in Fig.4, Fig.5 and Fig.6.
Table 1. $V=100\text{ml}, \alpha=0.92, \beta=0.92, k\simeq 0.0951$

| Time (min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental ($^\circ C$) | 100 | 70  | 57  | 50  | 45  | 41  | 38  | 36  |     |
| Classical ($^\circ C$)   | 100 | 77.916 | 62.1659 | 50.9329 | 42.9216 | 37.208 | 33.1331 | 30.2269 |     |
| Fractional ($^\circ C$)  | 100 | 74.2755 | 60.4967 | 51.379 | 45. | 40.3893 | 36.9786 | 34.4092 |     |
| Error (clsc-expr)        | 0   | 0.1015 | 0.0830 | 0.0183 | 0.04842 | 0.1019 | 0.1468 | 0.1909 | 4.6084 |
| Error (frac-expr)        | 0   | 0.0575 | 0.0578 | 0.0268 | 0   | 0.0151 | 0.0276 | 0.0462 | 1.0097 |

SSE: Error Sum of Squares

Table 2. $V=100\text{ml}, \alpha=0.91, \beta=0.91, k=0.1000$

| Time (min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental ($^\circ C$) | 100 | 70  | 57  | 50  | 45  | 41  | 38  | 36  |     |
| Classical ($^\circ C$)   | 100 | 77.916 | 62.1659 | 50.9329 | 42.9216 | 37.208 | 33.1331 | 30.2269 |     |
| Fractional ($^\circ C$)  | 100 | 74.8976 | 60.9808 | 51.6192 | 45. | 40.1844 | 36.6107 | 33.9174 |     |
| Error (clsc-expr)        | 0   | 0.1015 | 0.0830 | 0.0183 | 0.04842 | 0.1019 | 0.1468 | 0.1909 | 4.6084 |
| Error (frac-expr)        | 0   | 0.0653 | 0.0652 | 0.0313 | 0   | 0.0202 | 0.0379 | 0.0614 | 1.3718 |

Table 3. $V=100\text{ml}, \alpha=0.9, \beta=0.95, k=0.0892$

| Time (min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental ($^\circ C$) | 100 | 70  | 57  | 50  | 45  | 41  | 38  | 36  |     |
| Classical ($^\circ C$)   | 100 | 77.916 | 62.1659 | 50.9329 | 42.9216 | 37.208 | 33.1331 | 30.2269 |     |
| Fractional ($^\circ C$)  | 100 | 74.8976 | 60.9808 | 51.6192 | 45. | 40.1844 | 36.6107 | 33.9174 |     |
| Error (clsc-expr)        | 0   | 0.1015 | 0.0830 | 0.0183 | 0.04842 | 0.1019 | 0.1468 | 0.1909 | 4.6084 |
| Error (frac-expr)        | 0   | 0.0653 | 0.0652 | 0.0313 | 0   | 0.0202 | 0.0379 | 0.0614 | 1.3718 |

Fig. 1: Comparison of cooling water $V=100\text{ml}$
| Time(min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental($^\circ$C) | 100 | 70  | 57  | 50  | 45  | 41  | 38  | 36  |     |
| Classical($^\circ$C)    | 100 | 77.916 | 62.1659 | 50.9329 | 42.9216 | 37.208 | 33.1331 | 30.2269 |     |
| Fractional($^\circ$C)   | 100 | 72.9684 | 59.6254 | 50.9897 | 45.50.6766 | 37.4676 | 35.0336 |     |
| Error(clsc-expr)       | 0.1015 | 0.0830 | 0.0183 | 0.04842 | 0.1019 | 0.1468 | 0.1909 | 4.6093 |     |
| Error(frac-expr)       | 0.04068 | 0.0440 | 0.0194 | 0.0079 | 0.0142 | 0.0275 | 0.5022 |     |     |

Table 4: $V=100ml, \alpha=0.9, \beta=0.9, k\geq0.1052$

| Time(min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental($^\circ$C) | 100 | 81  | 70  | 61  | 55  | 51  | 47  | 45  |     |
| Classical($^\circ$C)    | 100 | 84.578 | 72.2449 | 62.3819 | 54.4943 | 48.1864 | 43.142 | 39.1078 |     |
| Fractional($^\circ$C)   | 100 | 81.1929 | 69.7805 | 61.4169 | 55.5.49507 | 45.9091 | 42.6325 |     |
| Error(clsc-expr)       | 0.0423 | 0.0310 | 0.0221 | 0.0092 | 0.0583 | 0.0894 | 0.1506 | 2.153 |     |
| Error(frac-expr)       | 0.0023 | 0.0031 | 0.0067 | 0.0210 | 0.0237 | 0.0555 | 0.2201 |     |     |

Table 5: $V=300ml, \alpha=0.92, \beta=0.92, k\geq0.0651$

| Time(min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental($^\circ$C) | 100 | 81  | 70  | 61  | 55  | 51  | 47  | 45  |     |
| Classical($^\circ$C)    | 100 | 84.578 | 72.2449 | 62.3819 | 54.4943 | 48.1864 | 43.142 | 39.1078 |     |
| Fractional($^\circ$C)   | 100 | 80.7165 | 69.4317 | 61.2478 | 55.50.0939 | 46.1678 | 42.9812 |     |
| Error(clsc-expr)       | 0.0423 | 0.0310 | 0.0221 | 0.0092 | 0.0583 | 0.0894 | 0.1506 | 2.153 |     |
| Error(frac-expr)       | 0.0035 | 0.0081 | 0.0040 | 0.0180 | 0.0180 | 0.0469 | 0.1690 |     |     |

Table 6: $V=300ml, \alpha=0.91, \beta=0.91, k\geq0.0630$
### Table 7. $V=300\text{ml}, \alpha=0.9, \beta=0.95, k\sim 0.0616$

| Time(min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental($^\circ\text{C}$) | 100 | 81  | 70  | 61  | 55  | 51  | 47  | 45  |     |
| Classical($^\circ\text{C}$)     | 100 | 84.578 | 72.2449 | 62.3819 | 54.4943 | 48.1864 | 43.142 | 39.1078 |     |
| Fractional($^\circ\text{C}$)    | 100 | 81.5723 | 70.0951 | 61.5832 | 55 | 49.7909 | 45.6057 | 42.205 |     |
| Error(clsc-expr)                | 0   | 0.0423 | 0.0310 | 0.0221 | 0.0092 | 0.0583 | 0.0894 | 0.1506 | 2.153 |
| Error(frac-expr)                | 0   | 0.0070 | 0.0013 | 0.0040 | 0 | 0.0242 | 0.0305 | 0.0662 | 0.330 |

### Table 8. $V=300\text{ml}, \alpha=0.9, \beta=0.9, k\sim 0.0716$

| Time(min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental($^\circ\text{C}$) | 100 | 81  | 70  | 61  | 55  | 51  | 47  | 45  |     |
| Classical($^\circ\text{C}$)     | 100 | 84.578 | 72.2449 | 62.3819 | 54.4943 | 48.1864 | 43.142 | 39.1078 |     |
| Fractional($^\circ\text{C}$)    | 100 | 80.2362 | 69.0853 | 61.0812 | 55 | 50.2341 | 46.4208 | 43.3225 |     |
| Error(clsc-expr)                | 0   | 0.0423 | 0.0310 | 0.0221 | 0.0092 | 0.0583 | 0.0894 | 0.1506 | 2.153 |
| Error(frac-expr)                | 0   | 0.0095 | 0.0132 | 0.0013 | 0 | 0.0152 | 0.0124 | 0.0387 | 0.143 |

**Fig.3.** Comparison of cooling water $V=800\text{ml}$
### Table 9. $V=800\text{ml}, \alpha=0.92, \beta=0.92, k=0.00479$

| Time(min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental ($^\circ$C) | 100 | 88  | 76  | 70  | 63  | 59  | 54  | 50  |     |
| Classical ($^\circ$C)      | 100 | 88.3858 | 78.5234 | 70.1487 | 63.0371 | 56.9981 | 51.8701 | 47.5155 |     |
| Fractional ($^\circ$C)            | 100 | 85.602 | 76.2156 | 68.9149 | 63 | 58.1028 | 53.9902 | 50.5007 |     |
| Error(clsc-expr)              | 0   | 0.0043 | 0.0321 | 0.0021 | 0.0005 | 0.0351 | 0.0410 | 0.0522 | 0.590 |
| Error(frac-expr)               | 0   | 0.02801 | 0.0028 | 0.0157 | 0   | 0.0154 | 0.0001 | 0.0099 | 0.220 |

### Table 10. $V=800\text{ml}, \alpha=0.91, \beta=0.91, k=0.0502$

| Time(min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental ($^\circ$C) | 100 | 88  | 76  | 70  | 63  | 59  | 54  | 50  |     |
| Classical ($^\circ$C)      | 100 | 88.3858 | 78.5234 | 70.1487 | 63.0371 | 56.9981 | 51.8701 | 47.5155 |     |
| Fractional ($^\circ$C)            | 100 | 85.8602 | 76.4366 | 69.0358 | 63 | 57.9785 | 53.7464 | 50.1462 |     |
| Error(clsc-expr)              | 0   | 0.0043 | 0.0321 | 0.0021 | 0.0005 | 0.0351 | 0.0410 | 0.0522 | 0.590 |
| Error(frac-expr)               | 0   | 0.0249 | 0.0057 | 0.013 | 0   | 0.0176 | 0.0047 | 0.0029 | 0.185 |

### Table 11. $V=800\text{ml}, \alpha=0.9, \beta=0.95, k=0.0456$

| Time(min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental ($^\circ$C) | 100 | 88  | 76  | 70  | 63  | 59  | 54  | 50  |     |
| Classical ($^\circ$C)      | 100 | 88.3858 | 78.5234 | 70.1487 | 63.0371 | 56.9981 | 51.8701 | 47.5155 |     |
| Fractional ($^\circ$C)            | 100 | 84.8678 | 75.6542 | 68.631 | 63 | 58.3632 | 54.4785 | 51.1827 |     |
| Error(clsc-expr)              | 0   | 0.0043 | 0.0321 | 0.0021 | 0.0005 | 0.0351 | 0.0410 | 0.0522 | 0.590 |
| Error(frac-expr)               | 0   | 0.0369 | 0.0045 | 0.0199 | 0   | 0.0109 | 0.0087 | 0.0231 | 0.388 |

### Table 12. $V=800\text{ml}, \alpha=0.9, \beta=0.9, k=0.0525$

| Time(min.) | 0   | 5   | 10  | 15  | 20  | 25  | 30  | 35  | SSE |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Experimental ($^\circ$C) | 100 | 88  | 76  | 70  | 63  | 59  | 54  | 50  |     |
| Classical ($^\circ$C)      | 100 | 88.3858 | 78.5234 | 70.1487 | 63.0371 | 56.9981 | 51.8701 | 47.5155 |     |
| Fractional ($^\circ$C)            | 100 | 84.8678 | 75.6542 | 68.631 | 63 | 58.3632 | 54.4785 | 51.1827 |     |
| Error(clsc-expr)              | 0   | 0.0043 | 0.0321 | 0.0021 | 0.0005 | 0.0351 | 0.0410 | 0.0522 | 0.590 |
| Error(frac-expr)               | 0   | 0.0369 | 0.0045 | 0.0199 | 0   | 0.0109 | 0.0087 | 0.0231 | 0.388 |

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