NUCLEARITY FOR FOURIER INTEGRAL OPERATORS IN $L^p$-SPACES

DUVÁN CARDONA

ABSTRACT. In this note we study sharp sufficient conditions for the nuclearity of Fourier integral operators on $L^p$-spaces, $1 < p \leq 2$. Our conditions and those presented in Cardona [2] provide a systematic investigation on the subject for all $1 < p < \infty$. MSC 2010. Primary 35S30; Secondary 58J40.

1. Introduction

In this note we are interested in those sharp conditions providing the nuclearity of Fourier integral operators on Lebesgue spaces $L^p$, for $1 < p \leq 2$. We complete our investigation on the subject for all $1 < p < \infty$, with this work, and those conditions proved in Cardona [2] for $2 \leq p < \infty$. Fourier integral operators (FIOs) on $\mathbb{R}^n$, are integral operators of the form

$$Ff(x) := \int_{\mathbb{R}^n} e^{i\phi(x,\xi)}a(x,\xi)\hat{f}(\xi)d\xi,$$

(1.1)

where $\mathcal{F}f := \hat{f}$ is the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$. As it is well known, properties of FIOs on functions spaces are used to study solutions to Cauchy problems (see Hörmander [25, 26, 27] and Duistermaat and Hörmander [18]) for hyperbolic equations.

By following the theory of FIOs developed by Hörmander [25], the phase functions $\phi$ are positively homogeneous of order 1 and they are considered smooth at $\xi \neq 0$, while the symbols are considered satisfying estimates of the form

$$\sup_{(x,y) \in K} |\partial_x^\beta \partial_y^\alpha \phi(x, y, \xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{\kappa - |\alpha|},$$

(1.2)

for every compact subset $K$ of $\mathbb{R}^{2n}$. The action of Fourier integral operators on $L^p$ spaces can be found in the references Hörmander [25], Eskin[20], Seeger, Sogge and Stein[41], Tao[42], Miyachi [30], Peral[31], Asada and Fujiwara[1], Fujiwara[21], Kumano-go[28], Coriasco and Ruzhansky [4, 5], Ruzhansky and Sugimoto [35, 36, 37, 38], Ruzhansky [40], and Ruzhansky and Wirth [39] where the problem has been extensively investigated.

In this note our main goal is to provide sufficient conditions for the $r$-nuclearity of Fourier integral operators on $L^p$-spaces. This problem has been considered in the case of pseudo-differential operators by several authors and for FIOs in Cardona [2]. Now, we present a summary of the works on the subject. Sufficient
conditions guarantying the nuclearity of pseudo-differential operators on $S^1, Z$, arbitrary compact Lie groups and (closed) compact manifolds can be found in the works of Delgado, Ruzhansky, Wong [9, 10, 11, 12, 14, 7] and Cardona [3]; similar conditions on different functions spaces can be found in the works Delgado and Ruzhansky [13, 15, 16]. Finally, the subject was treated for compact manifolds with boundary by Delgado, Ruzhansky, and Tokmagambetov in [17].

Our work is motivated by the recent works of Ghaemi, Jamalpour Birgani, and Wong in [22, 23, 29] for $S^1, Z$ and also for arbitrary compact and Hausdorff groups. In these references the authors have characterized the nuclearity of pseudo-differential operators by showing that symbols associated to nuclear pseudo-differential operators admit a suitable decomposition where the spatial and momentum variables appear separately. In the compact case, the main tool for providing these characterizations is the fact that the unitary dual of compact and Hausdorff groups is a discrete set. This situation is different for the case of operators on $\mathbb{R}^n$ because the unitary dual of $\hat{\mathbb{R}}^n \equiv \{ e^{i2\pi x \cdot \xi} : \xi \in \mathbb{R}^n \}$ is merely continuous. With this in mind, the techniques used in the euclidean case, could be slightly different or as in the proof of our main theorem far from of those used in [22, 23, 29].

In order to present our main result we recall of notion of $r$-nuclearity of Grothendieck. By following [24], a densely defined linear operator $T : D(T) \subset E \to F$ (where $D(T)$ is the domain of $T$, and $E, F$ are Banach spaces) extends to a $r$-nuclear operator from $E$ into $F$, if there exist sequences $(e'_n)_{n \in \mathbb{N}_0}$ in $E'$ (the dual space of $E$) and $(y_n)_{n \in \mathbb{N}_0}$ in $F$ such that, the discrete representation

$$T f = \sum_{n \in \mathbb{N}_0} \langle e'_n, f \rangle y_n, \quad \text{with} \quad \sum_{n \in \mathbb{N}_0} \| e'_n \|_{E'} \| y_n \|_F < \infty,$$

(1.3)

holds true for all $f \in D(T)$. The class of $r$–nuclear operators is usually endowed with the natural semi-norm

$$n_r(T) := \inf \left\{ \sum_{n} \| e'_n \|_{E'}^r \| y_n \|_F^r : T = \sum_{n} e'_n \otimes y_n \right\}^{1/r}.$$  

(1.4)

If $r = 1$, $n_1(\cdot)$ is a norm and we obtain the ideal of nuclear operators. In addition, when $E = F$ is a Hilbert space $H$ and $r = 1$ the definition above agrees with that of trace class operators. For the case of Hilbert spaces $H$, the set of $r$-nuclear operators agrees with the Schatten-von Neumann class of order $r$ (see Pietsch [32, 33]).

An important notion associated to the theory of $r$-nuclear operators is that of trace. If we choose a $r$-nuclear operator $T : E \to E$, $0 < r \leq 1$, with the Banach space $E$ satisfying the Grothendieck approximation property (see Grothendieck [24]), then the nuclear trace of $T$ is (a well-defined functional) given by

$$\text{Tr}(T) = \sum_{n \in \mathbb{N}_0} e'_n(f_n).$$

In order to illustrate our results about the $r$-nuclearity and the nuclear trace of Fourier integral operators, allow us to recall the following criterion, (see Cardona
ON THE NUCLEAR TRACE OF FOURIER INTEGRAL OPERATORS IN $L^p$-SPACES

Throughout this document the phase function $\phi$ will be considered real valued and measurable.

**Theorem 1.1.** Let $0 < r \leq 1$. Let $a(\cdot, \cdot)$ be a symbol such that $a(x, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^n)$, a.e.w., $x \in \mathbb{R}^n$. Let $2 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let $F$ be the Fourier integral operator associated to $a(\cdot, \cdot)$. Then, $F : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is $r$-nuclear, if and only if, the symbol $a(\cdot, \cdot)$ admits a decomposition of the form

$$a(x, \xi) = \frac{1}{e^{i\phi(x, \xi)}} \sum_{k=1}^{\infty} h_k(x)(\mathcal{F}^{-1}g_k)(\xi), \text{ a.e.w., } (x, \xi), \quad (1.5)$$

where $\{g_k\}_{k \in \mathbb{N}}$ and $\{h_k\}_{k \in \mathbb{N}}$ are sequences of functions satisfying

$$\mathbb{E}^r(g, f) := \sum_{k=0}^{\infty} \|g_k\|_{L^{p_1}}^r \|h_k\|_{L^{p_2}}^r < \infty. \quad (1.6)$$

In this case, the nuclear trace of $F$ is given by

$$\text{Tr}(F) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x, \xi)-2\pi i x \cdot \xi} a(x, \xi) d\xi dx.$$

As a complement of the previous result we present our main theorem where we consider the $r$-nuclearity of $F$ from $L^{p_1}$ into $L^{p_2}$ for $1 < p_1 \leq 2$.

**Theorem 1.2.** Let $0 < r \leq 1$. Let $a(\cdot, \cdot)$ be a measurable on $\mathbb{R}^{2n}$. Let $1 < p_1 \leq 2$, $1 \leq p_2 < \infty$, and $F$ be the Fourier integral operator associated to $a(\cdot, \cdot)$. Then, $F : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is $r$-nuclear if and only if the symbol $a(\cdot, \cdot)$ admits a decomposition of the form

$$a(x, \xi) = \frac{1}{e^{i\phi(x, \xi)}} \sum_{k=1}^{\infty} h_k(x)g_k(\xi), \text{ a.e.w., } (x, \xi), \quad (1.7)$$

where $\{g_k\}_{k \in \mathbb{N}}$ and $\{h_k\}_{k \in \mathbb{N}}$ are sequences of functions satisfying

$$\mathbb{E}^r(g, f) := \sum_{k=0}^{\infty} \|g_k\|_{L^{p_1}}^r \|h_k\|_{L^{p_2}}^r < \infty. \quad (1.8)$$

This theorem is sharp in the sense that the previous condition is a necessary and sufficient condition for the $r$-nuclearity of $F$ when $p_1 = 2$.

**Remark 1.3.** Note that from Reinov and Latif [34], the nuclear trace of every $r$-nuclear operator on $L^p(\mathbb{R}^n)$, agrees with its spectral trace provided that $1/r = 1 + |1/p - 1/2|$. If $F$ is a nuclear operator on $L^2(\mathbb{R}^n)$ we have,

$$\text{Tr}(F) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x, \eta)-2\pi i x \cdot \eta} a(x, \eta) d\eta d\xi = \sum_{n=0}^{\infty} \lambda_n(F),$$

where $\lambda_n(F), n \in \mathbb{N}_0$, is the sequence of eigenvalues of $F$ (counting multiplicity).
2. Proof of the main result

In this section we prove our main result for Fourier integral operators $F$ defined as in (1.1). First, let us observe that every FIO $F$ has an integral representation with kernel $1/K(x,y)$. In fact, straightforward computation shows us that

$$\mathcal{F}f(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) d\xi,$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$. In order to analyze the $r$-nuclearity of the Fourier integral operator $F$ we study its kernel $K$, by us-ing as fundamental tool, the following theorem (see J. Delgado [6, 8]).

**Theorem 2.1** (Delgado Theorem). Let us consider $1 \leq p_1, p_2 < \infty$, $0 < r \leq 1$ and let $p'_i$ be such that $\frac{1}{p_i} + \frac{1}{p'_i} = 1$. Let $(X_1, \mu_1)$ and $(X_2, \mu_2)$ be $\sigma$-finite measure spaces. An operator $T : L^{p_1}(X_1, \mu_1) \rightarrow L^{p_2}(X_2, \mu_2)$ is $r$-nuclear if and only if there exist sequences $(h_k)_k$ in $L^{p_2}(\mu_2)$, and $(g_k)$ in $L^{p'_1}(\mu_1)$, such that

$$\sum_k \|h_k\|_{L^{p_2}} \|g_k\|_{L^{p'_1}} < \infty,$$

and $Tf(x) = \int_{X_1} (\sum_k h_k(x)g_k(y))f(y)d\mu_1(y)$, a.e.w. $x$,

for every $f \in L^{p_1}(\mu_1)$. In this case, if $p_1 = p_2$, and $\mu_1 = \mu_2$, (see Section 3 of [6]) the nuclear trace of $T$ is given by

$$\text{Tr}(T) := \int_{X_1} \sum_k g_k(x)h_k(x)d\mu_1(x).$$

Now, curiously, we present the novelty of this paper that is the short proof of our main result.

**Proof of Theorem 1.2.** Let us consider the Fourier integral operator $F$,

$$\mathcal{F}f(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \hat{f}(\xi) d\xi,$$

with associated symbol $a$. The main strategy for the proof will be to analyze the natural factorization of $F$ in terms of the Fourier transform,

$$(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^n} e^{-i2\pi x \cdot \xi} f(x) dx.$$  

(2.5)

Clearly, if we define the operator with kernel (associated to $\sigma = (\varphi, a)$),

$$K_{\sigma}g(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) g(\xi), \quad g \in \mathcal{S}(\mathbb{R}^n), \quad K_{\sigma}(x, \xi) = e^{i\varphi(x,\xi)} a(x,\xi),$$

then $F = K_{\sigma} \circ \mathcal{F}$. Taking into account the Hausdorff-Young inequality,

$$\|\mathcal{F}f\|_{L^{p'_1}(\mathbb{R}^n)} \leq \|f\|_{L^{p_1}(\mathbb{R}^n)},$$

(2.7)

the Fourier transform extends to a bounded operator from $L^{p_1}(\mathbb{R}^n)$ into $L^{p'_1}(\mathbb{R}^n)$. So, if we prove that the condition (1.8) assures the $r$-nuclearity of $K_{\sigma}$ from $L^{p'_1}(\mathbb{R}^n)$
into $L^p_2(\mathbb{R}^n)$, we can deduce the $r$-nuclearity of $F$ from $L^p_1(\mathbb{R}^n)$ into $L^p_2(\mathbb{R}^n)$. Here, we will be using that the class of $r$-nuclear operators is a bilateral ideal on the set of bounded operators between Banach spaces.

Now, from Theorem 2.1, $K_\sigma : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is $r$-nuclear, if and only if, there exist sequences $\{h_k\}, \{g_k\}$ satisfying

$$K_\sigma(x, \xi) = e^{i\phi(x, \xi)} a(x, \xi) = \sum_k h_k(x) g_k(\xi), \quad (2.8)$$

where

$$\sum_k \|h_k\|_{L^p_1} \|g_k\|_{L^p_2} < \infty, \quad \text{and} \quad K_\sigma f(x) = \int_{\mathbb{R}^n} (\sum_k h_k(x)) g(\xi) d\xi, \quad \text{a.e.} \ x, \quad (2.9)$$

for every $g \in L^{p_1}(\mathbb{R}^n)$. Here, we have used that for $1 < p_1 \leq 2$, $L^{p_1}(\mathbb{R}^n) = L^{p_1}(\mathbb{R}^n)$. We end the proof by observing that (2.8) is in turns equivalent to (1.7).

**Remark 2.2** (Sharpness of Theorem 2.1). Let us note, that from Plancherel Theorem, $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ extends to an isomorphism of Hilbert spaces. In the proof of Theorem 2.1, $\mathcal{F}$ is an invertible and bounded operator for $p_1 = 2$. Hence, from the relation $F = K_\sigma \circ \mathcal{F}$, we deduce that $F$ is $r$-nuclear if and only if $K_\sigma = F \circ \mathcal{F}^{-1}$ is also $r$-nuclear. But, $K_\sigma$ is $r$-nuclear from $L^2$ into $L^2$ if and only if (1.7) holds true. So, for $p_1 = 2$, (1.7) characterizes the $r$-nuclearity of $F$ by showing that it is a necessary and sufficient condition for this fact.

**References**

1. Asada, K., D. Fujiwara, D. On some oscillatory integral transformations in $L^2(\mathbb{R}^n)$. Japan. J. Math. (N.S.), 4(2), 299–361, (1978)
2. Cardona, D. On the nuclear trace of Fourier Integral Operators. arXiv:1807.08389
3. Cardona D. Nuclear pseudo-differential operators in Besov spaces on compact Lie groups, J. Fourier Anal. Appl., 23(5), 1238–1262, (2017)
4. Coriasco, S., Ruzhansky, M. On the boundedness of Fourier integral operators on $L^p(\mathbb{R}^n)$. C. R. Math. Acad. Sci. Paris, 348(15–16), 847–851, (2010)
5. Coriasco, S., Ruzhansky, M. Global $L^p$ continuity of Fourier integral operators. Trans. Amer. Math. Soc., 366(5), 2575–2596, (2011)
6. Delgado, J.: A trace formula for nuclear operators on $L^p$, in: Schulze, B.W., Wong, M.W. (eds.) Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations, Operator Theory: Advances and Applications, 205, 181-193. Birkhuser, Basel (2010)
7. Delgado, J., Wong, M.W.: $L^\infty$-nuclear pseudo-differential operators on $\mathbb{Z}$ and $S^1$., Proc. Amer. Math. Soc., 141 (11), 3935–394, (2013)
8. Delgado, J. The trace of nuclear operators on $L^p(\mu)$ for $\sigma$-finite Borel measures on second countable spaces. Integral Equations Operator Theory, 68(1), 61-74, (2010)
9. Delgado, J. Ruzhansky, M.: $L^\infty$-nuclearity, traces, and Grothendieck-Lidskii formula on compact Lie groups., J. Math. Pures Appl. (9), 102(1), 153-172 (2014)
10. Delgado, J. Ruzhansky, M.: Schatten classes on compact manifolds: Kernel conditions. J. Funct. Anal., 267(3), 772–798, (2014)
11. Delgado, J. Ruzhansky, M.: Kernel and symbol criteria for Schatten classes and r-nuclearity on compact manifolds., C. R. Acad. Sci. Paris. Ser. I. 352. 779–784, (2014)
12. Delgado, J. Ruzhansky, M. Fourier multipliers, symbols and nuclearity on compact manifolds, J. Anal. Math., to appear, arXiv:1404.6479
13. Delgado J., Ruzhansky M., The bounded approximation property of variable Lebesgue spaces and nuclearity, Math. Scand., 122, 299–319, (2018)
14. Delgado J., Ruzhansky M., Schatten classes and traces on compact groups, Math. Res. Lett., 24, 979–1003, (2017)
15. Delgado, J. Ruzhansky, M. Wang, B. Approximation property and nuclearity on mixed-norm $L^p$, modulation and Wiener amalgam spaces. J. Lond. Math. Soc. 94, 391–408, (2016)
16. Delgado, J. Ruzhansky, M. Wang, B. Grothendieck-Lidskii trace formula for mixed-norm $L^p$ and variable Lebesgue spaces. to appear in J. Spectr. Theory.
17. Delgado, J. Ruzhansky, M. Tokmagambetov, N. Schatten classes, nuclearity and nonharmonic analysis on compact manifolds with boundary. arXiv:1505.02261
18. Duistermaat, J.J., Hörmander. Fourier integral operators. II. Acta Math., 128(3-4), 183–269, (1972)
19. Duistermaat, J.J. Fourier integral operators, volume 130 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996
20. Eskin, G. I. Degenerate elliptic pseudodifferential equations of principal type. Mat. Sb. (N.S.), 82(124), 585–628, (1970)
21. Fujiwara, A. construction of the fundamental solution for the Schrödinger equations. Proc. Japan Acad. Ser. A Math. Sci., 55(1), 10–14, (1979)
22. Ghaemi, M. B., Jamalpour Birgani, M., Wong, M. W. Characterizations of nuclear pseudo-differential operators on $S^1$ with applications to adjoints and products. J. Pseudo-Differ. Oper. Appl. 8(2), 191-201, (2017)
23. Ghaemi, M. B., Jamalpour Birgani, M., Wong, M. W. Characterization, adjoints and products of nuclear pseudo-differential operators on compact and Hausdorff groups. U.P.B. Sci. Bull., Series A, 79(4), 207–220, (2017)
24. Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires, Memoirs Amer. Math. Soc. 16, Providence, 1955 (Thesis, Nancy, 1953).
25. Hörmander, L. Fourier integral operators. I. Acta Math., 127(1-2), 79–183, (1971)
26. Hörmander, L.: Pseudo-differential Operators and Hypo-elliptic equations Proc. Symposium on Singular Integrals, Amer. Math. Soc. 10, 138-183 (1967)
27. Hörmander, L.: The Analysis of the linear partial differential operators Vol. III, IV. Springer-Verlag, (1985)
28. Kumano-go, H. A calculus of Fourier integral operators on $\mathbb{R}^n$ and the fundamental solution for an operator of hyperbolic type. Comm. Partial Differential Equations, 1(1), 1–44, (1976)
29. Jamalpour Birgani, M., Characterizations of Nuclear Pseudo-differential Operators on $\mathbb{Z}$ with some Applications, Math. Model. of Nat. Phenomena., 13, 13–30, (2018)
30. Miyachi, A. On some estimates for the wave equation in $L^p$ and $H^p$. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 27(2), 331–354, (1988)
31. Peral, J. C., $L^p$-estimates for the wave equation. J. Funct. Anal., 36(1), 114–145, (1980)
32. Pietsch, A. Operator ideals. Mathematische Monographien, 16. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
33. Pietsch, A. History of Banach spaces and linear operators. Birkhäuser Boston, Inc., Boston, MA, 2007.
34. Reinov, O.I., Latif, Q., Grothendieck-Lidskii theorem for subspaces of $L^p$-spaces. Math. Nachr., 286(2-3), 279–282, (2013)
35. Ruzhansky, M., Sugimoto, M. Global $L^2$-boundedness theorems for a class of Fourier integral operators. Comm. Partial Differential Equations, 31(4-6), 547–569, (2006)
36. Ruzhansky, M., Sugimoto, M. A smoothing property of Schrödinger equations in the critical case. Math. Ann., 335(3), 645–673, (2006)
37. Ruzhansky, M., Sugimoto, M. Weighted Sobolev $L^2$ estimates for a class of Fourier integral operators. Math. Nachr., 284(13), 1715–1738, (2011)
38. Ruzhansky M., Sugimoto M., Global regularity properties for a class of Fourier integral operators, arxiv:
39. Ruzhansky, M., Wirth, J. Dispersive type estimates for Fourier integrals and applications to hyperbolic systems. Conference Publications, 2011, 2011 (Special) : 1263-1270. doi: 10.3934/proc.2011.2011.1263
40. Ruzhansky, M. Regularity theory of Fourier integral operators with complex phases and singularities of affine fibrations, volume 131 of CWI Tract. Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 2001.
41. Seeger, A. Sogge, C. D., Stein, E. M. Regularity properties of Fourier integral operators. Ann. of Math. 134(2), 231–251, (1991)
42. Tao, T. The weak-type (1, 1) of Fourier integral operators of order (n1)/2. J. Aust. Math. Soc., 76(1), 1–21, (2004)

Duván Cardona:
Department of Mathematics
Pontificia Universidad Javeriana
Bogotá
Colombia
E-mail address d.cardona@uniandes.edu.co; duvanc306@gmail.com