ON THE ERDŐS-SZEKERES CONVEX POLYGON PROBLEM

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1. INTRODUCTION

In their classic 1935 paper, Erdős and Szekeres [7] proved that, for every integer \( n \geq 3 \), there is a minimal integer \( ES(n) \), such that any set of \( ES(n) \) points in the plane in general position contains \( n \) points in convex position; that is, they are the vertices of a convex \( n \)-gon.

Erdős and Szekeres gave two proofs of the existence of \( ES(n) \). Their first proof used a quantitative version of Ramsey’s theorem, which gave a very poor upper bound for \( ES(n) \). The second proof was more geometric and showed that \( ES(n) \leq \binom{2n-4}{n-2} + 1 \) (see Theorem 2.2 in the next section). On the other hand, they showed that \( ES(n) \geq 2^{n-2} + 1 \) and conjectured this to be sharp [8].

Small improvements have been made on the upper bound \( \binom{2n-4}{n-2} + 1 \approx \frac{4^n}{\sqrt{n}} \) by various researchers [3,13,17,18,22,23], but no improvement in the order of magnitude has ever been made. The most recent upper bound, due to Norin and Yuditsky [18] and Mojarrad and Vlachos [17], says that

\[
\limsup_{n \to \infty} \frac{ES(n)}{\binom{2n-4}{n-2}} \leq \frac{7}{16}.
\]

In the present paper, we prove the following.

**Theorem 1.1.** For all \( n \geq n_0 \), where \( n_0 \) is a large absolute constant, \( ES(n) \leq 2n + 6n^{2/3} \log n \).

The study of \( ES(n) \) and its variant [5] has generated a lot of research over the past several decades. For a more thorough history on the subject, we refer the interested reader to [2,15,22]. All logarithms are to base 2.

2. NOTATION AND TOOLS

In this section, we recall several results that will be used in the proof of Theorem 1.1. We start with the following simple lemma.

**Lemma 2.1** (see Theorem 1.2.3 in [14]). Let \( X \) be a finite point set in the plane in general position such that every four members in \( X \) are in convex position. Then \( X \) is in convex position.
Figure 1. A 4-cup and a 5-cap.

The next theorem is a well-known result from [7], which is often referred to as the Erdős-Szekeres cups-caps theorem. Let \( X \) be a \( k \)-element point set in the plane in general position. We say that \( X \) forms a \( k \)-cup (\( k \)-cap) if \( X \) is in convex position and its convex hull is bounded above (below) by a single edge. In other words, \( X \) is a cup (cap) if and only if for every point \( p \in X \), there is a line \( L \) passing through it such that all of the other points in \( X \) lie on or above (below) \( L \). See Figure 1.

**Theorem 2.2** ([7]). Let \( f(k, \ell) \) be the smallest integer \( N \) such that any \( N \)-element planar point set in the plane in general position contains a \( k \)-cup or an \( \ell \)-cap. Then

\[
f(k, \ell) = \binom{k + \ell - 4}{k - 2} + 1.
\]

The next theorem is a combinatorial reformulation of Theorem 2.2 observed by Hubard et al. [10] (see also [9,16]). A transitive 2-coloring of the triples of \( \{1, 2, \ldots, N\} \) is a 2-coloring, say with colors red and blue, such that, for \( i_1 < i_2 < i_3 < i_4 \), if triples \((i_1, i_2, i_3)\) and \((i_2, i_3, i_4)\) are red (blue), then \((i_1, i_2, i_4)\) and \((i_1, i_3, i_4)\) are also red (blue).

**Theorem 2.3** ([7]). Let \( g(k, \ell) \) denote the minimum integer \( N \) such that, for every transitive 2-coloring on the triples of \( \{1, 2, \ldots, N\} \), there exists a red clique of size \( k \) or a blue clique of size \( \ell \). Then

\[
g(k, \ell) = f(k, \ell) = \binom{k + \ell - 4}{k - 2} + 1.
\]

The next theorem is due to Pór and Valtr [20] and is often referred to as the positive-fraction Erdős-Szekeres theorem (see also [1,19]). Given a \( k \)-cap (\( k \)-cup) \( X = \{x_1, \ldots, x_k\} \), where the points appear in order from left to right, we define the support of \( X \) to be the collection of open regions \( C = \{T_1, \ldots, T_k\} \), where \( T_i \) is the region outside of \( \text{conv}(X) \) bounded by the segment \( x_i x_{i+1} \) and by the lines \( x_{i-1}x_i, x_{i+1}x_{i+2} \) (where \( x_{k+1} = x_1, x_{k+2} = x_2 \), etc.). See Figure 2.

**Theorem 2.4** (Proof of Theorem 4 in [20]). Let \( k \geq 3 \), and let \( P \) be a finite point set in the plane in general position such that \( |P| \geq 2^{32k} \). Then there is a \( k \)-element subset \( X \subset P \) such that \( X \) is either a \( k \)-cup or a \( k \)-cap, and the regions \( T_1, \ldots, T_{k-1} \) from the support of \( X \) satisfy \( |T_i \cap P| \geq |P|^{1/2k+1} \). In particular, every \((k-1)\)-tuple obtained by selecting one point from each \( T_i \cap P \), \( i = 1, \ldots, k-1 \), is in convex position.

Note that Theorem 2.4 does not say anything about the points inside region \( T_k \). Let us also remark that in the proof of Theorem 2.4 in [20], the authors find a \( 2k \)-element set \( X \subset P \), such that \( k \) of the regions in the support of \( X \) each contain
at least \( \frac{|P|}{32} k \) points from \( P \), and therefore these regions may not be consecutive. However, by appropriately selecting a \( k \)-element subset \( X' \subset X \), we obtain Theorem 2.4.

3. Proof of Theorem 1.1

Let \( P \) be an \( N \)-element planar point set in the plane in general position, where \( N = \lfloor 2^{n+6n^{2/3} \log n} \rfloor \) and \( n \geq n_0 \), where \( n_0 \) is a sufficiently large absolute constant. Set \( k = \lceil n^{2/3} \rceil \). We apply Theorem 2.4 to \( P \) with parameter \( k + 3 \) and obtain a subset \( X = \{x_1, \ldots, x_{k+3}\} \subset P \) such that \( X \) is a cup or a cap, and the points in \( X \) appear in order from left to right. Moreover since \( k = \lceil n^{2/3} \rceil \) is large, regions \( T_1, \ldots, T_{k+2} \) in the support of \( X \) satisfy

\[
|T_i \cap P| \geq \frac{N}{240k}.
\]

Set \( P_i = T_i \cap P \) for \( i = 1, \ldots, k+2 \). We will assume that \( X \) is a cap, since a symmetric argument would apply. We say that the two regions \( T_i \) and \( T_j \) are adjacent if \( i \) and \( j \) are consecutive indices.

Consider the subset \( P_i \subset P \) and the region \( T_i \), for some fixed \( i \in \{2, \ldots, k+1\} \). Let \( B_i \) be the segment \( x_{i-1}x_{i+2} \). See Figure 2. The point set \( P_i \) naturally comes with a partial order \( \prec \), where \( p \prec q \) if \( p \neq q \) and \( q \in \text{conv}(B_i \cup p) \). Set \( \alpha = 3n^{-1/3} \log n \). By Dilworth’s theorem [4], \( P_i \) contains either a chain of size at least \( |P_i|^{1-\alpha} \) or an antichain of size at least \( |P_i|^{\alpha} \) with respect to \( \prec \). The proof now falls into two cases.

![Figure 2. Regions \( T_1, \ldots, T_6 \) in the support of \( X = \{x_1, \ldots, x_6\} \), and segment \( B_3 \).](image-url)
Case 1. Suppose there are \( t = \lceil n^{1/3} \rceil \) parts \( P_i \) in the collection \( \mathcal{F} = \{P_2, P_3, \ldots, P_{k+1}\} \), such that no two of them are in adjacent regions, and each such part contains a subset \( Q_i \) of size at least \( |P_i|^\alpha \) such that \( Q_i \) is an antichain with respect to \( \prec \). Let \( Q_{j_1}, Q_{j_2}, \ldots, Q_{j_t} \) be the selected subsets.

For each \( Q_{j_r}, r \in \{1, \ldots, t\} \), the line spanned by any two points in \( Q_{j_r} \) does not intersect the segment \( B_{j_r} \) and, therefore, does not intersect region \( T_{j_w} \) for \( w \neq r \) (by the non-adjacency property). Since \( n \) is sufficiently large, we have \( 40k < n^{2/3} \log n \), and therefore

\[
|Q_{j_r}| \geq |P_i|^\alpha \geq \left( \frac{N}{240k} \right)^\alpha \geq 2^{3n^{2/3} \log n + 15n^{1/3} \log^2 n} \geq \left( \frac{n + \lceil 2n^{2/3} \rceil - 4}{n - 2} \right) + 1
\]

Theorem 2.2 implies that \( Q_{j_r} \) contains either an \( n \)-cup or a \( \lceil 2n^{2/3} \rceil \)-cap. If we are in the former case for any \( r \in \{1, \ldots, t\} \), then we are done. Therefore we can assume \( Q_{j_r} \) contains a subset \( S_{j_r} \), that is a \( \lceil 2n^{2/3} \rceil \)-cap, for all \( r \in \{1, \ldots, t\} \).

We claim that \( S = S_{j_1} \cup \cdots \cup S_{j_t} \) is a cap, and therefore \( S \) is in convex position. Let \( p \in S_{j_r} \). Since \( |S_{j_r}| \geq 2 \), there is a point \( q \in S_{j_r} \) such that the line \( L \) supported by the segment \( pq \) has the property that all of the other points in \( S_{j_r} \) lie below \( L \). Since \( L \) does not intersect \( B_{j_r} \), all of the points in \( S \setminus \{p, q\} \) must lie below \( L \). Hence, \( S \) is a cap and

\[
|S| = |S_{j_1} \cup \cdots \cup S_{j_t}| \geq \frac{n^{1/3}}{2} \left( 2n^{2/3} \right) = n.
\]

Case 2. Suppose we are not in Case 1. Then there are \( \lceil n^{1/3} \rceil \) consecutive indices \( j, j+1, j+2, \ldots \), such that each such part \( P_{j+r} \) contains a subset \( Q_{j+r} \) such that \( Q_{j+r} \) is a chain of length at least \( |P_{j+r}|^{1-\alpha} \) with respect to \( \prec \). For simplicity, we can relabel these sets \( Q_1, Q_2, Q_3, \ldots \).

Consider the subset \( Q_i \) inside the region \( T_i \), and order the elements in \( Q_i = \{p_1, p_2, p_3, \ldots\} \) with respect to \( \prec \). We say that \( Y \subset Q_i \) is a right-cap if \( x_i \cup Y \) is in convex position, and we say that \( Y \) is a left-cap if \( x_{i+1} \cup Y \) is in convex position. Notice that left-caps and right-caps correspond to the standard notion of cups and caps after applying an appropriate rotation to the plane so that the segment \( \overline{x_i x_{i+1}} \) is vertical. Since \( Q_i \) is a chain with respect to \( \prec \), every triple in \( Q_i \) is either a left-cap or a right-cap, but not both. Moreover, for \( i_1 < i_2 < i_3 < i_4 \), if \((p_{i_1}, p_{i_2}, p_{i_3})\) and \((p_{i_2}, p_{i_3}, p_{i_4})\) are right-caps (left-caps), then \((p_{i_1}, p_{i_2}, p_{i_3})\) and \((p_{i_1}, p_{i_3}, p_{i_4})\) are both right-caps (left-caps). By Theorem 2.3, if \( |Q_i| \geq f(k, \ell) \), then \( Q_i \) contains either a \( k \)-left-cap or an \( \ell \)-right-cap. We make the following observation.

**Observation 3.1.** Consider the (adjacent) sets \( Q_{i-1} \) and \( Q_i \). If \( Q_{i-1} \) contains a \( k \)-left-cap \( Y_{i-1} \), and \( Q_i \) contains an \( \ell \)-right-cap \( Y_i \), then \( Y_{i-1} \cup Y_i \) forms \( k + \ell \) points in convex position.

**Proof.** By Lemma 2.1, it suffices to show every four points in \( Y_{i-1} \cup Y_i \) are in convex position. If all four points lie in \( Y_j \), then they are in convex position. Likewise if they all lie in \( Y_{i-1} \), they are in convex position. Suppose we take two points \( p_1, p_2 \in Y_{i-1} \) and two points \( p_3, p_4 \in Y_i \). Since \( Q_{i-1} \) and \( Q_i \) are both chains with respect to \( \prec \), the line spanned by \( p_1, p_2 \) does not intersect the region \( T_i \), and the line spanned by \( p_3, p_4 \) does not intersect the region \( T_{i-1} \). Hence \( p_1, p_2, p_3, p_4 \) are in convex position. Now suppose we have \( p_1, p_2, p_3 \in Y_{i-1} \) and \( p_4 \in Y_i \). Since the three lines \( L_1, L_2, L_3 \) spanned by \( p_1, p_2, p_3 \) all intersect the segment \( B_{i-1} \), both \( x_i \) and \( p_4 \) lie in the same
region in the arrangement of $L_1 \cup L_2 \cup L_3$. Therefore $p_1, p_2, p_3, p_4$ are in convex position. The same argument follows in the case that $p_1 \in Y_{i-1}$ and $p_2, p_3, p_4 \in Y_i$. See Figure 3.

We have for $i \in \{1, \ldots, \lceil n^{1/3} \rceil \}$,

\begin{equation}
|Q_i| \geq |P_i|(1-\alpha) \geq \left( \frac{N}{240k} \right)^{1-\alpha} \geq 2^{n+2n^{2/3} \log n - 15n^{1/3} \log^2 n}.
\end{equation}

Set $K = \lceil n^{2/3} \rceil$. Since $n$ is sufficiently large, we have

\begin{equation}
|Q_1| \geq \left( \frac{n + K - 4}{K - 2} \right) + 1 = f(K, n),
\end{equation}

which implies that $Q_1$ contains either an $n$-right-cap or a $K$-left-cap. In the former case we are done, so we can assume that $Q_1$ contains a $K$-left-cap. Likewise, $|Q_2| \geq \left( \frac{n + K - 4}{2K - 2} \right) + 1 = f(2K, n - K)$, which implies $Q_2$ contains either an $(n - K)$-right-cap or a $(2K)$-left-cap. In the former case we are done since Observation 3.1 implies that the $K$-left-cap in $Q_1$ and the $(n - K)$-right-cap in $Q_2$ form $n$ points in convex position. Therefore we can assume $Q_2$ contains a $(2K)$-left-cap.

In general, if we know that $Q_{i-1}$ contains an $(iK - K)$-left-cap, then we can conclude that $Q_i$ contains an $(iK)$-left-cap. Indeed, for all $i \leq \lceil n^{1/3} \rceil$ we have

\begin{equation}
\left( \frac{n + K - 4}{iK - 2} \right) \leq 2^{n+\lceil n^{2/3} \rceil - 4}.
\end{equation}

Since $n$ is sufficiently large, (1) and (2) imply that

\begin{equation}
|Q_i| \geq 2^{n+2n^{2/3} \log n - 15n^{1/3} \log^2 n} \geq \left( \frac{n + K - 4}{iK - 2} \right) + 1 = f(iK, n - iK + K).
\end{equation}

Therefore, $Q_i$ contains either an $(n - iK + K)$-right-cap or an $(iK)$-left-cap. In the former case we are done by Observation 3.1 (recall that we assumed $Q_{i-1}$ contains an $(iK - K)$-left-cap), and therefore we can assume $Q_i$ contains an $(iK)$-left-cap.
Hence for $i = \lceil n^{1/3} \rceil$, we can conclude that $Q_{\lceil n^{1/3} \rceil}$ contains an $n$-left-cap. This completes the proof of Theorem 1.1.

4. CONCLUDING REMARKS

Following the initial publication of this work on arXiv, we have learned that Gábor Tardos has improved the lower order term in the exponent, showing that $ES(n) = 2^{n+O(\sqrt{n \log n})}$.

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