A LOWER BOUND FOR THE NUMBER OF NEGATIVE EIGENVALUES OF SCHRODINGER OPERATORS

ALEXANDER GRIGOR’YAN, NIKOLAI NADIRASHVILI, AND YANNICK SIRE

ABSTRACT. We prove a lower bound for the number of negative eigenvalues for a Schrödinger operator on a Riemannian manifold via the integral of the potential.

1. Introduction

Let \((M, g)\) be a compact Riemannian manifold without boundary. Consider the following eigenvalue problem on \(M\):

\[-\Delta u - Vu = \lambda u, \tag{1}\]

where \(\Delta\) is the Laplace-Beltrami operator on \(M\) and \(V \in L^\infty(M)\) is a given potential. It is well-known, that the operator \(-\Delta - V\) has a discrete spectrum. Denote by \(\{\lambda_k(V)\}_{k=1}^{\infty}\) the sequence of all its eigenvalues arranged in increasing order, where the eigenvalues are counted with multiplicity.

Denote by \(N(V)\) the number of negative eigenvalues of (1), that is,

\[N(V) = \text{card}\{k : \lambda_k(V) < 0\}.\]

It is well-known that \(N(V)\) is finite. Upper bounds of \(N(V)\) have received enough attention in the literature, and for that we refer the reader to [2], [5], [12], [11], [15] and references therein.

However, little is known about lower estimates. Our main result is the following theorem. We denote by \(\mu\) the Riemannian measure on \(M\).

**Theorem 1.1.** Set \(\dim M = n\). For any \(V \in L^\infty(M)\) the following inequality is true:

\[N(V) \geq \frac{C}{\mu(M)^{n/2-1}} \left( \int_M V d\mu \right)^{n/2}, \tag{2}\]

where \(C > 0\) is a constant that in the case \(n = 2\) depends only on the genus of \(M\) and in the case \(n > 2\) depends only on the conformal class of \(M\).

In the case \(V \geq 0\) the estimate (2) was proved in [6, Theorems 5.4 and Example 5.12]. Our main contribution is the proof of (2) for signed potentials \(V\) (as it was conjectured in [6]), with the same constant \(C\) as in [6]. In fact, we reduce the case of a signed \(V\) to the case of non-negative \(V\) by considering a certain variational problem for \(V\) and by showing that the solution of this problem is non-negative. The latter method originates from [14].

AG was supported by SFB 701 of German Research Council.

NN was supported by the Alexander von Humboldt Foundation.
In the case \( n = 2 \), inequality (2) takes the form
\[
\mathcal{N}(V) \geq C \int_M V d\mu.
\] (3)

For example, the estimate (3) can be used in the following situation. Let \( M \) be a two-dimensional manifold embedded in \( \mathbb{R}^3 \) and the potential \( V \) be of the form \( V = \alpha K + \beta H \) where \( K \) is the Gauss curvature, \( H \) is the mean curvature, and \( \alpha, \beta \) are real constants (see [8], [4]). In this case (3) yields
\[
\mathcal{N}(V) \geq C (K_{\text{total}} + H_{\text{total}}),
\]
where \( K_{\text{total}} \) is the total Gauss curvature and \( H_{\text{total}} \) is the total mean curvature. We expect in the future many other applications of (2)-(3).

2. A variational problem

Fix positive integers \( k, N \) and consider the following optimization problem: find \( V \in L^\infty(M) \) such that
\[
\int_M V d\mu \rightarrow \max \text{ under restrictions } \lambda_k(V) \geq 0 \text{ and } \| V \|_{L^\infty} \leq N.
\] (4)

Clearly, the functional \( V \mapsto \int_M V d\mu \) is weakly continuous in \( L^\infty(M) \). Since the class of potentials \( V \) satisfying the restrictions in (4) is bounded in \( L^\infty(M) \), it is weakly precompact in \( L^\infty(M) \). In fact, we prove in the next lemma that this class is weakly compact, which will imply the existence of the solution of (4).

**Lemma 2.1.** The class
\[
C_{k,N} = \{ V \in L^\infty(M) : \lambda_k(V) \geq 0 \text{ and } \| V \|_{L^\infty} \leq N \}
\]
is weakly compact in \( L^\infty(M) \). Consequently, the problem (4) has a solution \( V \in L^\infty(M) \).

**Proof.** It was already mentioned that the class \( C_{k,N} \) is weakly precompact in \( L^\infty(M) \). It remains to prove that it is weakly closed, that is, for any sequence \( \{ V_i \} \subset C_{k,N} \) that converges weakly in \( L^\infty(M) \), the limit \( V \) is also in \( C_{k,N} \). The condition \( \| V \|_{L^\infty} \leq N \) is trivially satisfied by the limit potential, so all we need is to prove that \( \lambda_k(V) \geq 0 \). Let us use the minmax principle in the following form:
\[
\lambda_k(V) = \inf_{E \subset W^{1,2}(M) \text{ s.t. } \dim E = k} \sup_{\| u \|_{L^\infty} \leq N} \frac{\int_M |\nabla u|^2 d\mu - \int_M V u^2 d\mu}{\int_M u^2 d\mu},
\]
where \( E \) is a subspace of \( W^{1,2}(M) \) of dimension \( k \). The condition \( \lambda_k(V) \geq 0 \) is equivalent then to the following:
\[
\forall E \subset W^{1,2}(M) \text{ with } \dim E = k \quad \forall \varepsilon > 0 \quad \exists u \in E \setminus \{0\} \quad \text{such that } \int_M |\nabla u|^2 d\mu - \int_M V u^2 d\mu \geq -\varepsilon \int_M u^2 d\mu.
\] (5)

Fix a subspace \( E \subset W^{1,2}(M) \) of dimension \( k \) and some \( \varepsilon > 0 \). Since \( \lambda_k(V_i) \geq 0 \), we obtain that there exists \( u_i \in E \setminus \{0\} \) such that
\[
\int_M |\nabla u_i|^2 d\mu - \int_M V_i u_i^2 d\mu \geq -\varepsilon \int_M u_i^2 d\mu.
\] (6)
Without loss of generality we can assume that \( \| u_i \|_{W^{1,2}(M)} = 1 \). Then the sequence \( \{u_i\} \) lies on the unit sphere in the finite-dimensional space \( E \). Hence, it has a convergent (in \( W^{1,2}(M) \)-norm) subsequence. We can assume that the whole sequence \( \{u_i\} \) converges in \( E \) to some \( u \in E \) with \( \| u \|_{W^{1,2}(M)} = 1 \). It remains to verify that \( u \) satisfies the inequality (5). By construction we have

\[
\int_M |\nabla u_i|^2 \, d\mu \rightarrow \int_M |\nabla u|^2 \, d\mu \quad \text{and} \quad \int_M u_i^2 \, d\mu \rightarrow \int_M u^2 \, d\mu.
\]

Next we have

\[
\left| \int_M V_i u_i^2 \, d\mu - \int_M V u^2 \, d\mu \right| \leq \left| \int_M (V_i u_i^2 - V_i u^2) \, d\mu \right| + \left| \int_M (V_i u_i^2 - V u^2) \, d\mu \right|
\]

\[
\leq N \| u_i - u \|_{L^2}^2 + \left| \int_M (V_i - V) u^2 \, d\mu \right|.
\]

By construction we have \( \| u_i - u \|_{L^2} \rightarrow 0 \) as \( i \rightarrow \infty \). Since \( u^2 \in L^1(M) \), the \( L^\infty \) weak convergence \( V_i \rightarrow V \) implies that

\[
\int_M (V_i - V) u^2 \, d\mu \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty.
\]

Hence, the inequality (5) follows from (4). \( \square \)

**Lemma 2.2.** If \( N \) is large enough (depending on \( k \) and \( M \)) then any solution \( V \) of (4) satisfies \( \lambda_k(V) = 0 \).

**Proof.** Assume that \( \lambda_k(V) > 0 \) and bring this to a contradiction. Consider the family of potentials

\[ V_t = (1 - t)V + tN \quad \text{where} \quad t \in [0,1]. \]

Since \( V_t \geq V \), we have by a well-known property of eigenvalues that \( \lambda_k(V_t) \leq \lambda_k(V) \). By continuity we have, for small enough \( t \), that \( \lambda_k(V_t) > 0 \). Clearly, we have also \( |V_t| \leq N \). Hence, \( V_t \) satisfies the restriction of the problem (4), at least for small \( t \). If \( \mu \{ V < N \} > 0 \) then we have for all \( t > 0 \)

\[
\int_M V_t > \int_M V,
\]

which contradicts the maximality of \( V \). Hence, we should have \( V = N \) a.e.. However, if \( N > \lambda_k(-\Delta) \) then \( \lambda_k(-\Delta - N) < 0 \) and \( V \equiv N \) cannot be a solution of (4). This contradiction finishes the proof. \( \square \)

### 3. Proof of Theorem 1.1

The main part of the proof of Theorem 1.1 is contained in the following lemma.

**Lemma 3.1.** Let \( V_{\max} \) be a maximizer of the variational problem (4). Then \( V_{\max} \) satisfies the inequality

\[ V_{\max} \geq 0 \quad \text{a.e. on} \quad M \]
3.1. **Proof of Theorem 1.1 assuming Lemma 3.1.** Choose $N$ large enough, say $N > \sup_M |V|$.

Set $k = \mathcal{N}(V) + 1$ so that $\lambda_k(V) \geq 0$. For the maximizer $V_{\max}$ of (4) we have

$$\int_M V \, d\mu \leq \int_M V_{\max} \, d\mu.$$ 

On the other hand, since $V_{\max} \geq 0$, we have by (6)

$$\mathcal{N}(V_{\max}) \geq \frac{C}{\mu(M)^{n/2-1}} \left( \int_M V_{\max} \, d\mu \right)^{n/2}.$$

Also, we have

$$\lambda_k(V_{\max}) \geq 0,$$

which implies

$$\mathcal{N}(V_{\max}) \leq k - 1 = \mathcal{N}(V).$$

Hence, we obtain

$$\mathcal{N}(V) \geq \mathcal{N}(V_{\max}) \geq \frac{C}{\mu(M)^{n/2-1}} \left( \int_M V_{\max} \, d\mu \right)^{n/2} \geq \frac{C}{\mu(M)^{n/2-1}} \left( \int_M V \, d\mu \right)^{n/2},$$

which was to be proved.

3.2. **Some auxiliary results.** Before we can prove Lemma 3.1, we need some auxiliary lemmas. The following lemma can be found in [9].

**Lemma 3.2.** Let $V(t, x)$ be a function on $\mathbb{R} \times M$ such that, for any $t \in \mathbb{R}$, $V(t, \cdot) \in L^\infty(M)$ and $\partial_t V(t, \cdot) \in L^\infty(M)$. For any $t \in \mathbb{R}$, consider the Schrödinger operator $L_t = -\Delta - V(t, \cdot)$ on $M$ and denote by $\lambda_k(t)$ the sequence of the eigenvalues of $L_t$ counted with multiplicities and arranged in increasing order. Let $\lambda$ be an eigenvalue of $L_0$ with multiplicity $m$; moreover, let

$$\lambda = \lambda_{k+1}(0) = \ldots = \lambda_{k+m}(0).$$

Let $U_\lambda$ be the eigenspace of $L_0$ that corresponds to the eigenvalue $\lambda$ and $\{u_1, \ldots, u_m\}$ be an orthonormal basis in $U_\lambda$. Set for all $i, j = 1, \ldots, m$

$$Q_{ij} = \int_M \frac{\partial V}{\partial t}\bigg|_{t=0} u_i u_j \, d\mu.$$ 

and denote by $\{\alpha_i\}_{i=1}^m$ the sequence of the eigenvalues of the matrix $\{Q\}_{i,j=1}^m$ counted with multiplicities and arranged in increasing order. Then we have the following asymptotic, for any $i = 1, \ldots, m,$

$$\lambda_{k+i}(t) = \lambda_{k+i}(0) - t\alpha_i + o(t) \quad \text{as} \quad t \to 0.$$

The following lemma is multi-dimensional extension of [14, Lemmas 3.4.3.6]. Given a connected open subset $\Omega$ of $M$ with smooth boundary, the Dirichlet problem

$$\begin{cases}
\Delta u = 0 & \text{in} \ \Omega \\
u|_{\partial \Omega} = f
\end{cases}$$
has for any \( f \in C(\partial \Omega) \) a unique solution that can be represented in the form

\[
    u(y) = \int_{\partial \Omega} Q(x, y) f(x) d\sigma(x)
\]

for any \( y \in \Omega \), where \( Q(x, y) \) is the Poisson kernel of this problem and \( \sigma \) is the surface measure on \( \partial \Omega \). For any \( y \in \Omega \), the function \( q(x) = Q(x, y) \) on \( \partial \Omega \) will be called the Poisson kernel at the source \( y \). Note that \( q(x) \) is continuous, positive and

\[
    \int_{\partial \Omega} q d\sigma = 1.
\]

**Lemma 3.3.** Let \( \Omega \) be a connected open subset of \( M \) with smooth boundary and \( x_0 \) be a point in \( \Omega \). Then, for any constant \( N \geq 1 \), there exists \( \varepsilon = \varepsilon(\Omega, N, x_0) > 0 \) such that for any measurable set \( E \subset \Omega \) with

\[
    \mu(E) \leq \varepsilon
\]

and for any positive solution \( v \in C^2(\Omega) \) of the inequality

\[
    \Delta v + W v \geq 0 \text{ in } \Omega,
\]

where

\[
    W = \begin{cases} 
        N & \text{ in } E, \\
        -\frac{1}{N} & \text{ in } \Omega \setminus E,
    \end{cases}
\]

the following inequality holds

\[
    v(x_0) < \int_{\partial \Omega} v q d\sigma,
\]

where \( q \) is the Poisson kernel of the Laplace operator at the source \( x_0 \).

**Proof.** For any \( \delta > 0 \) denote by \( A_\delta \) the set of points in \( \Omega \) at the distance \( \leq \delta \) from \( \partial \Omega \) (see Fig. 1) and consider the potential \( V_\delta \) in \( \Omega \) defined by

\[
    V_\delta = \begin{cases} 
        N & \text{ in } A_\delta, \\
        -\frac{1}{N} & \text{ in } \Omega \setminus A_\delta.
    \end{cases}
\]

![Figure 1.](image-url)
Since $\|V_\delta^+\|_{L^p(\Omega)}$ can be made sufficiently small by the choice of $\delta > 0$, the following boundary value problem has a unique positive solution:

$$\begin{cases} \\
\Delta w + V_\delta w = 0 \text{ in } \Omega \\
w = f \text{ on } \partial \Omega,
\end{cases} \quad (11)$$

for any positive continuous function $f$ on $\partial \Omega$. Denote by $q_\delta (x)$, $x \in \partial \Omega$, the Poisson kernel of $(11)$ at the source $x_0$. Letting $\delta \to 0$, we obtain that the solution of $(11)$ converges to that of

$$\begin{cases} \\
\Delta w - \frac{1}{N} w = 0 \text{ in } \Omega \\
w = f \text{ on } \partial \Omega.
\end{cases} \quad (12)$$

Denoting by $q_0$ the Poisson kernel of $(12)$ at the source $x_0$, we obtain that $q_\delta \to q_0$ on $\partial \Omega$ as $\delta \to 0$ and, moreover, the convergence is uniform.

Let $q$ be the Poisson kernel of the Laplace operator $\Delta$ in $\Omega$, as in the statement of the theorem. Since any solution of $(12)$ is strictly subharmonic in $\Omega$, we obtain that $q_0 < q$ on $\partial \Omega$. In particular, there is a constant $\eta > 0$ depending only on $\Omega, N, x_0$ such that

$$q_0 < (1 - \eta) q \text{ on } \partial \Omega.$$ 

Since the convergence $q_\delta \to q$ is uniform on $\partial \Omega$, we obtain that, for small enough $\delta$ (depending on $\Omega, N, x_0$),

$$q_\delta < (1 - \eta/2) q \text{ on } \partial \Omega.$$ 

Fix such $\delta$. Consequently, we obtain for the solution $w$ of $(11)$ that

$$w (x_0) < (1 - \eta/2) \int_{\partial \Omega} f q d\sigma. \quad (13)$$

Note that the function $W$ from $(8)$ can be increased without violating $(7)$. Define a new potential $W_\delta$ by

$$W_\delta = \begin{cases} N \text{ in } A_\delta \cup E, \\
-\frac{1}{N} \text{ in } \Omega \setminus A_\delta \setminus E. \end{cases} \quad (14)$$

Observe that, for any $p > 1$

$$\|W_\delta^+\|_{L^p(\Omega)}^p \leq N^p (\mu (A_\delta) + \varepsilon),$$

so that by the choice of $\varepsilon$ and further reducing $\delta$ this norm can be made arbitrarily small. By a well-known fact (see [13]), if $\|W_\delta^+\|_{L^p(\Omega)}$ is sufficiently small, then the operator $-\Delta - W_\delta$ in $\Omega$ with the Dirichlet boundary condition on $\partial \Omega$ is positive definite, provided $p = n/2$ for $n > 2$ and $p > 1$ for $n = 2$.

So, we can assume that the operator $-\Delta - W_\delta$ is positive definite. In particular, the following boundary value problem

$$\begin{cases} \\
\Delta u + W_\delta u = 0 \text{ in } \Omega \\
u|_{\partial \Omega} = v
\end{cases} \quad (15)$$

has a unique positive solution $u$. Comparing this with $(7)$ and using the maximum principle for the operator $\Delta + W_\delta$, we obtain $u \geq v$ in $\Omega$. Since $u = v$ on $\partial \Omega$, the required inequality $(9)$ will follow if we prove that

$$u (x_0) < \int_{\partial \Omega} u q d\sigma. \quad (16)$$
Set \( \Omega_\delta = \Omega \setminus A_\delta \) and prove that
\[
\sup_{\Omega_\delta} u \leq C \int_{\partial \Omega} u d\sigma,
\]
for some constant \( C \) that depends on \( \Omega, N, \delta, n \). By choosing \( \varepsilon \) and \( \delta \) sufficiently small, the norm \( \|W_\delta\|_{L^p} \) can be made arbitrarily small for any \( p \). Hence, function \( u \) satisfies the Harnack inequality
\[
\sup_{\Omega_\delta} u \leq C \int_{\Omega_\delta} u d\mu
\]
where \( C \) depends on \( \Omega, N, \delta \) (see [1], [7]). Let \( h \) be the solution of the following boundary value problem
\[
\begin{cases}
-\Delta h - W_\delta h = 1_{\Omega_\delta} & \text{in } \Omega \\
h = 0 & \text{on } \partial \Omega.
\end{cases}
\]
where \( \Omega_\delta = \Omega \setminus A_\delta \). Since \( \|W_\delta\|_{L^q} \) is bounded for any \( q \), we obtain by the known a priori estimates, that
\[
\|h\|_{W^{2,p}(\Omega)} \leq C\|1_{\Omega_\delta}\|_{L^p(\Omega)},
\]
where \( p > 1 \) is arbitrary and \( C \) depends on \( \Omega, N, \delta, p \) (see [10]). Choose \( p > n \) so that by the Sobolev embedding
\[
\|h\|_{C^1(\Omega)} \leq C \|h\|_{W^{2,p}(\Omega)}.
\]
Since \( \|1_{\Omega_\delta}\|_{L^p(\Omega)} \) is uniformly bounded, we obtain by combining the above estimates that
\[
\|h\|_{C^1(\Omega)} \leq C,
\]
with a constant \( C \) depending on \( \Omega, N, \delta, n \).

Multiplying the equation \(-\Delta h - W_\delta h = 1_{\Omega_\delta}\) by \( u \) and integrating over \( \Omega \), we obtain
\[
\int_{\Omega_\delta} u d\mu = \int_{\partial \Omega} \frac{\partial h}{\partial \nu} u d\sigma \leq C \int_{\partial \Omega} u d\sigma
\]
which together with (18) implies (17).

Let \( w \) be the solution (11) with the boundary condition \( f = u \), that is,
\[
\begin{cases}
\Delta w + V_\delta w = 0 & \text{in } \Omega \\
w = u & \text{on } \partial \Omega.
\end{cases}
\]

Let us consider the difference \( \varphi = u - w \).

Clearly, we have in \( \Omega \)
\[
\Delta \varphi + V_\delta \varphi = (\Delta u + V_\delta u) - (\Delta w + V_\delta w) = (V_\delta - W_\delta) u
\]
and \( \varphi = 0 \) on \( \partial \Omega \). Denoting by \( G_{V_\delta} \) the Green function of the operator \(-\Delta - V_\delta\) in \( \Omega \) with the Dirichlet boundary condition, we obtain
\[
\varphi(x_0) = \int_{\Omega} G_{V_\delta}(x_0, y) (W_\delta - V_\delta) u(y) d\mu(y).
\]
Since we are looking for an upper bound for \( \varphi(x_0) \), we can restrict the integration to the domain \( \{ V_\delta \leq W_\delta \} \). By (14) and (10) we have

\[
\{ V_\delta \leq W_\delta \} = (\Omega \setminus A_\delta) \cap (A_\delta \cup E) = E \setminus A_\delta =: E'
\]

and, moreover, on \( E' \) we have

\[
W_\delta - V_\delta = N + \frac{1}{N} < 2N,
\]

whence it follows that

\[
\varphi(x_0) \leq 2N \int_{E'} G_{V_\delta}(x_0, y) u(y) d\mu(y).
\]

Using (17) to estimate here \( u(y) \), we obtain

\[
\varphi(x_0) \leq 2NC \left( \int_{E'} G_{V_\delta}(x_0, y) d\mu(y) \right) \int_{\partial\Omega} u d\sigma
\]

Since \( \mu(E') \leq \varepsilon \) and the Green function \( G_{V_\delta}(x_0, \cdot) \) is integrable, we see that \( \int_{E'} G_{V_\delta}(x_0, \cdot) d\mu \) can be made arbitrarily small by choosing \( \varepsilon > 0 \) small enough.

Choose \( \varepsilon \) so small that

\[
2NC \int_{E'} G_{V_\delta}(x_0, y) d\mu(y) < \frac{\eta}{2} \inf_{\partial\Omega} q,
\]

which implies that

\[
\varphi(x_0) < \frac{\eta}{2} \int_{\partial\Omega} u q d\sigma.
\]

Since by (13)

\[
w(x_0) < (1 - \eta/2) \int_{\partial\Omega} u q d\sigma,
\]

we obtain

\[
u(x_0) = \varphi(x_0) + w(x_0) < \int_{\partial\Omega} u q d\sigma,
\]

which was to be proved. \( \square \)

Let \( V_{\text{max}} \) be a solution of the problem (11). Denote by \( U \) the eigenspace of \(-\Delta - V_{\text{max}}\) associated with the eigenvalue \( \lambda_k(V_{\text{max}}) = 0 \) assuming that \( N \) is sufficiently large.

**Lemma 3.4.** Fix some \( c > 0 \) and consider the set

\[
F = \{ V_{\text{max}} \leq -c \}.
\]

Then, for any Lebesgue point \( x \in F \), then there exists a non-negative function \( q \in L^\infty(M) \) such that

1. \( \int_M q \, d\mu = 1 \);
2. for any \( u \in U \setminus \{0\} \) we have

\[
u^2(x) < \int_M u^2 q \, d\mu\]  
(19)
Proof. Set $V = V_{\text{max}}$. Any function $u \in U$ satisfies $\Delta u + Vu = 0$, which implies by a simple calculation that the function $v = u^2$ satisfies
\[
\Delta v + 2Vv \geq 0.
\]
Next, we apply Lemma 3.3 with $J = \max(2N, \frac{1}{2c})$. Choose $r$ so small that the density of the set $F$ in $B(x, r)$ is sufficiently close to 1, namely,
\[
\mu(F \cap B(x, r)) > (1 - \epsilon) \mu(B(x, r)),
\]
where $\epsilon = \epsilon(J)$ is given in Lemma 3.3. Since $h \leq 2N \leq J$ in $B(x, r)$ and
\[
\mu \left( \left\{ h > -\frac{1}{J} \right\} \cap B(x, r) \right) \leq \mu(\left\{ h > -2c \right\} \cap B(x, r)) = \mu(\left\{ V > -c \right\} \cap B(x, r)) < \epsilon \mu(B(x, r)),
\]
all the hypotheses of Lemma 3.3 are satisfied. Let $q$ be the function that exists by Lemma 3.3 in some small ball $B(x, r)$. Extending $q$ by setting $q = 0$ outside $B(x, r)$ we obtain a desirable function. □

3.3. Proof of main Lemma 3.1. We can now prove Lemma 3.1 that is, that $V_{\text{max}} \geq 0$. Consider again the set
\[
F = \{ V_{\text{max}} \leq -c \},
\]
where $c > 0$. We want to show that, for any $c > 0$,
\[
\mu(F) = 0,
\]
which will imply the claim. Assume the contrary, that is $\mu(F) > 0$ for some $c > 0$. Denote by $F_L$ the set of Lebesgue points of $F$. For any $x \in F_L$ denote by $q_x$ the function $q$ that is given by Lemma 3.3. For $x \notin F_L$ set $q_x = \delta_x$. Then $x \mapsto q_x$ is a Markov kernel and, for all $x \in M$ and $u \in U$
\[
u^2(x) \leq \int_M u^2 q_x d\mu. \tag{20}
\]
Denote by $\mathcal{M}$ the set of all probability measures on $M$. Define on $\mathcal{M}$ a partial order: $\nu_1 \preceq \nu_2$ if and only if
\[
\int_M u^2 d\nu_1 \leq \int_M u^2 d\nu_2 \text{ for all } u \in U \setminus \{0\}. \tag{21}
\]
Define $\nu_0 \in \mathcal{M}$ by
\[
d\nu_0 = \frac{1}{\mu(F_L)} 1_{F_L} d\mu
\]
and measure $\nu_1 \in \mathcal{M}$ by
\[
\nu_1 = \int_M q_x d\nu_0(x). \nonumber
\]
Since \( \nu_0(F_L) > 0 \), we obtain for any \( u \in U \setminus \{0\} \) that
\[
\int_M u^2 d\nu_1 = \int_M \left( \int_M u^2 q_x d\mu \right) d\nu_0(x) \\
\geq \int_{F_L} \left( \int_M u^2 q_x d\mu \right) d\nu_0(x) + \int_{M\setminus F_L} \left( \int_M u^2 q_x d\mu \right) d\nu_0(x) \\
> \int_{F_L} u^2(x) d\nu_0(x) + \int_{M\setminus F_L} u^2(x) d\nu_0(x) \\
= \int_M u^2 d\nu_0.
\] (22)

In particular, we have \( \nu_0 \preceq \nu_1 \). Consider the following subset of \( \mathcal{M} \):
\[
\mathcal{M}_1 = \{ \nu \in \mathcal{M} : \nu \succeq \nu_1 \}.
\]

Let us prove that \( \mathcal{M}_1 \) has a maximal element. By Zorn’s Lemma, it suffices to show that any chain (=totally ordered subset) \( \mathcal{C} \) of \( \mathcal{M}_1 \) has an upper bound in \( \mathcal{M}_1 \). It follows from \( \dim U < \infty \) that there exists an increasing sequence \( \{\nu_i\}_{i=1}^\infty \) of elements of \( \mathcal{C} \) such that, for all \( u \in U \),
\[
\lim_{i \to \infty} \int_M u^2 d\nu_i \to \sup_{\{\nu \in \mathcal{C}\}} \int_M u^2 d\nu.
\]

The sequence \( \{\nu_i\}_{i=1}^\infty \) of probability measures is \( \text{w}^* \)-compact. Without loss of generality we can assume that this sequence is \( \text{w}^* \)-convergent. It follows that the measure
\[
\nu_C = \text{w}^* \lim \nu_i \in \mathcal{M}_1
\]
is an upper bound for \( \mathcal{C} \).

By Zorn’s Lemma, there exists a maximal element \( \nu \) in \( \mathcal{M}_1 \). Note that the measure \( \nu \) can be alternatively constructed by using a standard balayage procedure (see e.g. [3, Proposition 2.1, p. 250]). Consider first the measure \( \nu' \) defined by \( \nu' = \int_M q_x d\nu(x) \). It follows from (20) that for any \( u \in U \)
\[
\int_M u^2 d\nu' = \int_M \left( \int_M u^2 q_x d\mu \right) d\nu \\
\geq \int_M u^2 d\nu,
\]
that is, \( \nu' \succeq \nu \), in particular, \( \nu' \in \mathcal{M}_1 \). Since \( \nu \) is a maximal element in \( \mathcal{M}_1 \), it follows that \( \nu' = \nu \), which implies the identity
\[
\int_M u^2 d\nu = \int_M \left( \int_M u^2 q_x d\mu \right) d\nu.
\] (23)
Now we can prove that \( \nu(F_L) = 0 \). Assuming from the contrary that \( \nu(F_L) > 0 \), we obtain, for any \( u \in \mathcal{U} \setminus \{0\} \),

\[
\int_M u^2 d\nu = \int_M \left( \int_M u^2 q_x d\mu \right) d\nu(x)
\geq \int_{F_L} \left( \int_M u^2 q_x d\mu \right) d\nu(x) + \int_{M \setminus F_L} \left( \int_M u^2 q_x d\mu \right) d\nu(x)
> \int_{F_L} u^2(x) d\nu(x) + \int_{M \setminus F_L} u^2(x) d\nu(x)
= \int_M u^2 d\nu,
\]

which is a contradiction. Finally, it follows from (22) and \( \nu \in \mathcal{M}_1 \) that, for any \( u \in \mathcal{U} \setminus \{0\} \),

\[
\int_M u^2 d\nu_0 < \int_M u^2 d\nu.
\]

Measure \( \nu \) can be approximated in \( w^* \)-sense by measures with bounded densities sitting in \( M \setminus F_L \). Therefore, there exists a non-negative function \( \varphi \in L^\infty(M) \) that vanishes on \( F_L \) and such that

\[
\int_M \varphi d\mu = 1
\]

and, for any \( u \in \mathcal{U} \setminus \{0\} \),

\[
\int_M u^2 \varphi_0 d\mu < \int_M u^2 d\mu
\]

where \( \varphi_0 = \frac{1}{\mu(F_L)} 1_{F_L} \). Consider now the potential

\[ V_t = V_{\text{max}} + t\varphi_0 - t\varphi. \]

We have for all \( t \)

\[
\int_M V_t d\mu = \int_M V_{\text{max}} d\mu
\]

and for \( t \to 0 \)

\[
\lambda_k(V_t) = \lambda_k(V_{\text{max}}) - t\alpha + o(t),
\]

where \( \alpha \) is the minimal eigenvalue of the quadratic form

\[ Q(u, u) = \int_M u^2 (\varphi_0 - \varphi) d\mu, \]

which by (25) is negative definite. Therefore, \( \alpha < 0 \), which together with \( \lambda_k(V_{\text{max}}) = 0 \) implies that, for all small enough \( t > 0 \)

\[
\lambda_k(V_t) > 0.
\]

Finally, let us show that \( |V_t| \leq N \) a.e. Indeed, on \( F \) we have

\[ V_t \leq -c + t\varphi_0 < N \]

for small enough \( t > 0 \), and on \( M \setminus F_L \) we have

\[ V_t \leq V_{\text{max}} - t\varphi \leq V_{\text{max}} \leq N. \]
Therefore, \( V \leq N \) a.e. for small enough \( t > 0 \). Similarly, we have on \( F_L \)
\[
V_t \geq V_{\text{max}} + t\varphi_0 \geq V_{\text{max}} \geq -N
\]
and on \( M \setminus F \)
\[
V_t \geq -c - t\varphi \geq -N
\]
for small enough \( t > 0 \), which implies that \( |V_t| \leq N \) a.e. for small enough \( t > 0 \).
Hence, we obtain that \( V_t \) is a solution to our optimization problem \([1]\), but it satisfies \( \lambda_k(V_t) > 0 \), which contradicts the optimality of \( V_t \) by Lemma \([2.2]\).

References

[1] Aizenman M., Simon B., Brownian motion and Harnack's inequality for Schrödinger operators, *Comm. Pure Appl. Math.*, 35 (1982) 203-271.
[2] Birman M.Sh., Solomyak M.Z., Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations, *Advances in Soviet Math.*, 7 (1991) 1-55.
[3] Bliedtner J., Hansen W., “Potential theory – an analytic and probabilistic approach to balayage”, Universitext, Springer, Berlin-Heidelberg-New York-Tokyo, 1986.
[4] El Soufi, Ahmad, Isoperimetric inequalities for the eigenvalues of natural Schrödinger operators on surfaces, *Indiana Univ. Math. J.*, 58 (2009) no.1, 335-349.
[5] Grigor’yan A., Nadirashvili N., Negative eigenvalues of two-dimensional Schrdinger equations, arXiv:1112.4986
[6] Grigor’yan A., Netrusov Yu., Yau S.-T., Eigenvalues of elliptic operators and geometric applications, *in: “Eigenvalues of Laplacians and other geometric operators”*, Surveys in Differential Geometry IX, (2004) 147-218.
[7] Hansen W., Harnack inequalities for Schrödinger operators, *Ann. Scuola Norm. Sup. Pisa*, 28 (1999) 413-470.
[8] Harrell II, E.M. On the second eigenvalue of the Laplace operator penalized by curvature, *Diff. Geom. Appl.,* 6 (1996) 397-400.
[9] Kato T., “Perturbation theory for linear operators”, Springer, 1995.
[10] Ladyzenskaja O.A., V.A. Solonnikov, Ural’ceva N.N., “Linear and quasilinear equations of parabolic type”, Providence, Rhode Island, 1968.
[11] Li P., Yau S.-T., On the Schrödinger equation and the eigenvalue problem, *Comm. Math. Phys.*, 88 (1983) 309-318.
[12] Lieb E.H., The number of bound states of one-body Schrödinger operators and the Weyl problem, *Proc. Sym. Pure Math.*, 36 (1980) 241-252.
[13] Lieb E.H., Loss M., “Analysis”, AMS, 2001.
[14] Nadirashvili N., Sire Y., Conformal spectrum and harmonic maps, arXiv:1007.3104
[15] Yang P., Yau S.-T., Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4),* 7 (1980) 55-63.

Department of Mathematics, University of Bielefeld, 33501 Bielefeld, Germany,
GRIGOR@MATH. Uni-Bielefeld.DE

Université Aix-Marseille, CNRS, I2M, Marseille, France, NICOLAS@CMI.UNIV-MRS.FR
Université Aix-Marseille, I2M, UMR 7353, Marseille, France, SIRE@CMI.UNIV-MRS.FR