Kummer Covers with Many Points

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Introduction

Let $p$ be a prime and $\mathbb{F}_q$ be a finite field with $q = p^m$ elements and let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. In this paper we present a method for constructing curves over finite fields with many points which are Kummer covers of $\mathbb{P}^1$, or of other suitable base curves. For this we look at rational functions $f \in \mathbb{F}_q(x)$ such that $f$ assumes a fixed value $a \in \mathbb{F}_q$ on a (preferably large) set $\mathcal{P} \subseteq \mathbb{P}^1(\mathbb{F}_q)$. To take a concrete example we set $a = 1$. Then the algebraic curve which is the Kummer covering of $\mathbb{P}^1$ given by the equation

$$y^{q-1} = f(x),$$

has fibres with many rational points and judicious choices of $f$ lead to improvements and extensions of the tables [2] of curves with many points. The methods we employed in the past were mostly based on Artin-Schreier covers of the projective line.

In Section 1 we sketch the method and describe a way to obtain good rational functions as above. This is based on an appropriate splitting $f = f_1 + f_2$ of a linearized polynomial $f$ having as zero set a linear subspace $L$ of $\mathbb{F}_q$. In the following section we treat the case where the linear space $L$ is the full space $\mathbb{F}_q$. We find curves $C_m$ defined over $\mathbb{F}_q$ for which the ratio $\#C_m(\mathbb{F}_q)/g(C_m)$ of the number of rational points by the genus exceeds $\sqrt{q}$ for $m$ even and $2\sqrt{pq}/(p+1)$ for $m$ odd. For $g$ large compared to $q$ the only way known so far to prove the existence of curves with a comparable ratio is by class field theory, which is less explicit (cf. [1]). Note that the result of Drinfeld-Vladut, that $\limsup_{g \to \infty} \#C(\mathbb{F}_q)/g(C) \leq \sqrt{q} - 1$, shows that for given $q$ there are only finitely many isomorphism classes of curves $C$ of over $\mathbb{F}_q$ whose ratio $\#C(\mathbb{F}_q)/g(C)$ exceeds $\sqrt{q}$ for $m$ even.

In section 3 we consider the case where the linear subspace is of codimension 1 in $\mathbb{F}_q$ and we complement this note with a section with variations. We illustrate the sections with numerous examples and thus obtain a number of improvements of the existing tables. In many cases the methods also give a relatively easy way to construct for certain pairs $(q, g)$ a curve realizing the lower entry of the interval in the tables [2]. We conclude the article with two tables summarizing the new results from the examples.
§1. The Method

We consider the non-singular complete irreducible Kummer curve \( C \) over \( \mathbb{F}_q \) defined by the affine equation

\[
y^{q-1} = f(x),
\]

where the rational function \( f(x) \in \mathbb{F}_q(x) \) satisfies the following conditions.

(1.1) Conditions.

i) \( f \) is not the \( d \)-th power of an element \( g \in \bar{\mathbb{F}}_q(x) \) for any divisor \( d > 1 \) of \( q - 1 \);

ii) \( f(x) = 1 \) on a substantial subset \( \mathcal{P} \) of \( \mathbb{P}^1(\bar{\mathbb{F}}_q) \);

iii) \( f(x) \) has many multiple zeros and poles.

By i) the curve \( C \) is a cyclic cover of \( \mathbb{P}^1 \) of degree \( q - 1 \), by ii) the curve \( C \) has at least \((q - 1)\#\mathcal{P} \) rational points and condition iii) keeps the genus of \( C \) within bounds.

The Hurwitz-Zeuthen formula gives the genus of \( C \) (cf. e.g. [3]):

(1.2) Proposition. If the divisor of \( f \) is \((f) = \sum_{i=1}^{\ell} d_i P_i \) with distinct \( P_i \in \mathbb{P}^1(\bar{\mathbb{F}}_q) \) then the genus \( g(C) \) of \( C \) is given by

\[
2g(C) - 2 = (\ell - 2)(q - 1) - \sum_{i=1}^{\ell} \gcd(q - 1, |d_i|).
\]

Note that a small value of \( \ell \) and the greatest common divisors influence the genus in a favourable way for our game.

Rational functions which satisfy Conditions (1.1) arise for instance in the following way.

Let \( L \) be an \( r \)-dimensional subspace of the \( \mathbb{F}_p \)-vector space \( \mathbb{F}_q = p^m \) with \( r \geq 2 \). Then the polynomial

\[
R = \prod_{c \in L} (x - c)
\]

is a \( p \)-linearized polynomial, i.e. of the form

\[
R(x) = \sum_{i=0}^{r} a_i x^{p^i} \in \mathbb{F}_q[x]
\]

and moreover satisfies \( a_0 a_r \neq 0 \).

Now we split \( R \) as

\[
R(x) = R_1(x) + R_2(x)
\]

such that \( R_1(x) = \sum_{i=s}^{r} b_i x^{p^i} \in \mathbb{F}_q[x] \) and \( R_2(x) = \sum_{i=0}^{t} c_i x^{p^i} \) with \( 0 < s < r, t \leq s, b_s b_r \neq 0 \) and \( c_0 c_t \neq 0 \). We denote the zero sets of \( R_1 \) (resp. \( R_2 \)) by \( L_1 \) (resp. \( L_2 \)) with \( \#L_1 = p^{r-s} \) (resp. \( \#L_2 = p^t \)). Furthermore, in connection with Condition (1.1) i) we require that \( L_1 \neq L_2 \).
It is obvious that
\[
f(x) = \frac{R_1(x)}{R_2(x)} = -\left(\frac{\sum_{i=0}^{t} c_i x^i}{\sum_{i=s}^{r} b_i x^{p^{-1} i}}\right)^{p^s}
\]  
(3)
satisfies \( f(x) = 1 \) for \( x \in L - (L_1 \cup L_2) \). From (2) it follows that \( L \cap (L_1 \cup L_2) = L_1 \cap L_2 \), which means that \( L - (L_1 \cup L_2) = L - (L_1 \cap L_2) \). Moreover, the zeros of \( R_1 \) and the pole \( \infty \) have multiplicities > 1. Hence \( f \) satisfies the Conditions (1.1).

(1.3) Proposition. The Kummer cover \( C \) of \( P^1 \) defined by the equation \( y^{q-1} = f(x) \) with \( f(x) = -R_1/R_2 \) as in (3) has genus
\[
g = \left\{(p^r - s + p^t - \delta - 1)(q - 2) - \delta p^{\gcd(m,s)} - p^{\gcd(m,r-t)} + 2\delta + 2\right\}/2
\]  
(4)
and the number of \( \mathbb{F}_q \)-rational points on \( C \) satisfies
\[
\#C(\mathbb{F}_q) \geq (p^r - \delta)(q - 1),
\]  
(5)
where \( \delta = \#(L_1 \cap L_2) \).

Proof. By the assumption \( L_1 \neq L_2 \) it follows that the function \( f \) satisfies (1.1) i). The divisor of \( f \) is
\[
(f) = \sum_{P \in L_1 \cap L_2} (p^s - 1)P + \sum_{P \in L_1 - (L_1 \cap L_2)} p^s P - \sum_{P \in L_2 - (L_1 \cap L_2)} P - (p^r - p^t)P_\infty.
\]
The number \( \ell \) of distinct zeros and poles of \( f \) is
\[
\#L_1 + \#L_2 - \#(L_1 \cap L_2) + 1 = p^r - s + p^t - \delta + 1.
\]
According to Prop. (1.2) the genus satisfies
\[
2g(C) - 2 = (p^r - s + p^t - \delta - 1)(q - 1) - \delta p^{\gcd(m,s)} - (p^r - s - \delta) - (p^{\gcd(m,r-t)} - 1)
\]
and we obtain (4). For \( x \in L - (L_1 \cup L_2) \) we have \( f(x) = 1 \) and thus over each \( y \in \mathbb{F}_q^* \) we find \( p^r - \delta \) rational points on \( C \). Other rational points could come from the branch points of \( C \). The set of branch points is \( L_1 \cup L_2 \cup \infty \) and they contribute rational points if the ramification points over such branch points happen to be rational. This yields the required estimate (5). 

(1.4) Example. Take \( \mathbb{F}_{16} \) with \( L = \mathbb{F}_{16} \). Then \( R = x^{16} + x \) and we split \( R \) as \( R = R_1 + R_2 \) with \( R_1 = x^{16} + x^2 \) and \( R_2 = x^2 + x \). In this case \( r = 4, s = t = 1, L_1 = \mathbb{F}_8, L_2 = \mathbb{F}_2 \) and \( \delta = 2 \). From Prop. (1.3) we see that the curve \( C \) defined over \( \mathbb{F}_{16} \) by
\[
y^{15} = (x^6 + x^2)/(x^2 + x) = x^{14} + x^{13} + \ldots + x
\]
has genus \( g(C) = 49 \) and \( \#C(\mathbb{F}_{16}) = 14 \times 15 + 3 = 213 \) since the ramification points over the branch points in \( \mathbb{F}_2 \cup \infty \) are rational. This provides a new entry for the tables in [2].
We remark that the ratio $\#C(F_q)/g(C)$ for the curves that appear in Proposition (1.3) exceeds $2p^r/(p^{r-s} + p^s)$, which is optimal for $s = t = \lfloor r/2 \rfloor$. For that choice

$$\#C(F_q)/g(C) > \begin{cases} \sqrt{p^r} & \text{for } r \text{ even}, \\ 2\sqrt{p^{r+1}/(p+1)} & \text{for } r \text{ odd}. \end{cases}$$  

(6)

From (6) it follows that the case $L = F_{p^m}$ with $R = x^{p^m} - x$ is of special interest.

§2. The Case $L = F_q$

In this Section we consider the case where $L$ equals the full vector space $F_q$. For odd $m$ we write

$$x^{p^m} - x = R_1 + R_2 = (x^{p^m} - ax^{(m-1)/2}) + (ax^{(m-1)/2} - x),$$

with $a \in F_q^*$, i.e., we look at the case $s = t = \lfloor m/2 \rfloor$. Since

$$\gcd(x^{p^m} - ax^{(m-1)/2}, ax^{(m-1)/2} - x) = \gcd(x^{p^m} - x, ax^{(m-1)/2} - x)$$

we have for $u \in F_q^*$

$$u \in L_1 \cap L_2 \iff u^{(m-1)/2-1} = 1/a.$$  

This equation has no solutions in $F_q^*$ if $a$ is not a $(p^{(m-1)/2} - 1)$-th power in $F_q^*$ and the number of solutions in $F_q^*$ is $\gcd(p^{(m-1)/2} - 1, p^m - 1) = p - 1$ if $a$ is a $(p^{(m-1)/2} - 1)$-th power in $F_q^*$. The latter holds always if $p = 2$.

First we consider the case that $a$ is a $(p^{(m-1)/2} - 1)$-th power in $F_q^*$. Often we shall write $a \in (F_q^*)^d$ to indicate that $a$ is a $d$-th power in $F_q^*$.

(2.1) Proposition. For odd $m \geq 3$ the curve $C_m$ defined over $F_{q=p^m}$ by the equation

$$y^{q-1} = -\frac{(x^{p^{(m+1)/2}} - ap^{(m+1)/2}/x)^{p^{(m-1)/2}}}{ax^{p^{(m-1)/2}} - x}$$

with $a \in (F_q^*)^{p^{(m-1)/2} - 1}$ has genus

$$g(C_m) = \{(p^{(m+1)/2} + p^{(m-1)/2} - p - 1)(q - 2) - p^2 + p + 2)/2$$

and has the following number of rational points

$$\#C_m(F_q) = \begin{cases} (q - 1)/(q - p) & \text{for odd } p, \\ (q - 1)/(q - p) + 3 & \text{for } p = 2. \end{cases}$$

Proof. The degree of $\gcd(R_1, R_2) = \gcd(x^{p^{(m+1)/2}} - ap^{(m+1)/2}/x, ax^{p^{(m-1)/2}} - x)$ is the cardinality of the $F_p$-vector space $L_1 \cap L_2$. The condition that $a$ is $p^{(m-1)/2} - 1$-th power implies $\delta = p$. We have $s = t = (m - 1)/2$ and the expression for the genus follows now directly by substitution in Prop. (1.3). Over each $y \in F_q^*$ we have $p^m - \delta = p^m - p$ rational
points on $C_m$. The only branch points which possibly contribute rational points to $C_m$ are the branch points in $\mathbb{F}_p \cup \infty$. Over each point of $\mathbb{F}_p \cup \infty$ there lie $p-1$ ramification points on $C_m$. These are rational if and only if $-a$ is a $(p-1)$-th power in $\mathbb{F}_q$. This holds for pairs $(p, m)$ with $pm$ even which implies our formula for $\#C_m(\mathbb{F}_q)$. ●

(2.2) Example. As an illustration of Proposition (2.1) we take $p = 3, m = 3$ and get the curve $C$ over $\mathbb{F}_{27}$ given by

$$-y^{26} = x^{24} + x^{22} + \ldots + x^2$$

with $g(C) = 98$ and $\#C(\mathbb{F}_{27}) = 624$. In this case the Oesterlé upper bound is $b = 745$, so $C$ satisfies our qualification criterion $\#C(F_{27}) \geq [b/\sqrt{2}]$ for the tables in [2].

For another example we take $\mathbb{F}_{32}$. Then the curve $C$ with affine equation

$$y^{31} = (x^8 + x)^4/(x^4 + x)$$

has genus $g(C) = 135$ and $\#C(\mathbb{F}_{32}) = 31 \times 30 + 3 = 933$. The Oesterlé upper bound in this case is 1098.

For $q = 3^5$ we obtain from Prop. (2.1) a curve $C$ of genus $g(C) = 3854$ and $\#C(\mathbb{F}_{243}) = 58080$. The Oesterlé upper bound is 81835.

For a not a $(p^{(m-1)/2} - 1)$-th power in $\mathbb{F}_q^*$ we have a similar proposition.

(2.3) Proposition. For odd $m \geq 3$ the curve $C_m$ over $\mathbb{F}_{q^m}$ defined by

$$y^{q-1} = -\frac{(xp^{(m+1)/2} - ap^{(m+1)/2})p^{(m-1)/2}}{a xp^{(m-1)/2} - x}$$

with $a \not\in (\mathbb{F}_q^*)^{p^{(m-1)/2}-1}$ has genus

$$g(C_m) = \{(p^{(m+1)/2} + p^{(m-1)/2} - 2)(q-2) - 2p + 4\} / 2$$

and has the following number of rational points

$$\#C_m(\mathbb{F}_q) = \begin{cases} (q - 1)^2 & \text{if } -a \not\in (\mathbb{F}_q^*)^{p-1}, \\
(q - 1)^2 + 2(p - 1) & \text{if } -a \in (\mathbb{F}_q^*)^{p-1}. \end{cases}$$

Proof. The proof is similar to that of Prop. (2.1), with the following modifications. In this case $\gcd(R_1, R_2)$ has degree 1 which means that $\delta = \#(L_1 \cap L_2) = 1$ and over each $y \in \mathbb{F}_q^*$ we have $p^m - \delta = p^m - 1$ rational points on $C_m$. The branch points which possibly contribute rational points on $C_m$ are 0 and $\infty$. Over these points there are $p-1$ ramification points on $C_m$ which are rational points if and only if $-a$ is a $(p-1)$-th power in $\mathbb{F}_q$. This gives the formula for the number of rational points. ●

(2.4) Examples. For $p$ odd we take $a = -1$ since $-1$ is not a $(p^{(m-1)/2} - 1)$-th power in $\mathbb{F}_q^*$. Over $\mathbb{F}_{27}$ the curve $C_m$ has genus $g(C_m) = 124$ and $\#C_m(\mathbb{F}_{27}) = 680$ while the Oesterlé upper bound is 901. Over $\mathbb{F}_{3^5}$ we find $g(C_m) = 4096$ and $\#C_m(\mathbb{F}_{3^5}) = 58568$. The Oesterlé upper bound is here 86441.

For $q = p^m$ with $m$ even the splitting

$$x^q - x = (x^q - ax\sqrt{q}) + (ax\sqrt{q} - x),$$

where $a \in \mathbb{F}_q^*$ is such that $a \not\in (\mathbb{F}_q^*)^{\sqrt{q}-1}$ yields very good curves.
(2.5) Proposition. If \( q = p^m \) with \( m \) even then the curve \( C_m \) defined over \( \mathbb{F}_q = p^m \) by the equation

\[
y^{q-1} = -x^q - ax^{\sqrt{q}} \quad \text{with} \quad a \in \mathbb{F}_q^*, a \not\in (\mathbb{F}_q^*)^{\sqrt{q}-1}
\]
has genus \( g(C_m) = (\sqrt{q} - 1)(q - 2) - \sqrt{q} + 2 \) and \( \#C_m(\mathbb{F}_q) = (q - 1)^2 \).

Proof. In this situation we have \( s = t = m/2 \) and the condition \( a\sqrt{q} + 1 \neq 1 \) implies \( L_1 \cap L_2 = \{0\} \), so \( \delta = 1 \). The formula for \( g(C_m) \) follows from Prop. (1.3). Over each \( y \in \mathbb{F}_q^* \) there are \( p^m - \delta \) rational points on \( C_m \). The only branch points which possibly give rise to rational points on \( C_m \) are \( 0 \) and \( \infty \). The ramification points over \( 0 \) (resp. \( \infty \)) on \( C_m \) are rational iff the equation \( w\sqrt{q} - 1 = -a \) (resp. \( w\sqrt{q} - 1 = -1/a \)) is solvable in \( \mathbb{F}_q \). Since \( a\sqrt{q} + 1 \neq 1 \) these equations have no solutions in \( \mathbb{F}_q \) and consequently we have \( \#C_m(\mathbb{F}_q) = (q - 1)^2 \).

Note that in this case \( \#C_m(\mathbb{F}_q)/g(C_m) > \sqrt{q} + 1 \).

(2.6) Examples. For \( q = 9 \) we find \( g(C_m) = 13 \) and \( \#C_m(\mathbb{F}_9) = 64 \). This is very close to the Oesterlé upper bound \( 66 \) and might well be optimal (i.e. equal to the actual maximum number \( N_q(g) \), cf. [2]). For \( q = 16 \) we find \( g(C_m) = 40, \#C_m(\mathbb{F}_{16}) = 225 \); the Oesterlé upper bound is \( 244 \). For \( q = 64 \) we find \( g(C_m) = 428 \) and \( \#C_m(\mathbb{F}_{64}) = 3969 \) with Oesterlé upper bound \( 4786 \). For \( q = 81 \) we find \( g(C_m) = 625 \) and \( \#C_m(\mathbb{F}_{81}) = 6400 \), still reasonable compared with the Oesterlé upper bound \( 7824 \).

§3. Subspaces of Codimension 1

We take as subspace of \( \mathbb{F}_q \) the \((m - 1)\)-dimensional subspace

\[
L = \{ x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x) = 0 \}, \quad \text{where} \quad \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x) = x^{p^{m-1}} + \ldots + x^p + x,
\]
and put \( R(x) = \sum_{i=0}^{m-1} x^{p^i} \). Note that by a transformation \( x \mapsto ax \) on \( \mathbb{F}_q \) with \( a \in \mathbb{F}_q^* \) we can transform any codimension 1 space into this subspace \( L \). We split the polynomial \( R \) as \( R_1 + R_2 \) with \( R_1 = \sum_{i=s}^{m-1} x^{p^i} \) and \( R_2 = \sum_{i=0}^{s-1} x^{p^i} \). The corresponding curve \( C_m \) over \( \mathbb{F}_q \) is defined by

\[
y^{q-1} = -(x^{p^{m-1-s}} + \ldots + x)^p/(x^{p^{s-1}} + \ldots + x) \quad (7)
\]

Applying Prop. (1.3) to this situation gives the following result.

(3.1) Proposition. For \( m \geq 3 \) and \( 0 < s < m - 1 \) such that \( \gcd(m, s) = 1 \) the curve \( C_m \) given by (7) has genus

\[
g(C_m) = \{(p^{m-1-s} + p^{s-1} - 2)(q - 2) - 2p + 4\}/2
\]

and

\[
\#C_m(\mathbb{F}_q) = \begin{cases} 
(p^{m-1} - 1)(q - 1) & \text{if } pm \text{ odd and } p \nmid s(m - s), \\
(p^{m-1} - 1)(q - 1) + (p - 1) & \text{if } pm \text{ odd and } p | s(m - s), \\
(p^{m-1} - 1)(q - 1) + 2(p - 1) & \text{if } pm \text{ even and } p \nmid s(m - s), \\
(p^{m-1} - 1)(q - 1) + 3(p - 1) & \text{if } pm \text{ even and } p | s(m - s).
\end{cases}
\]
Proof. In the notation used in Section 1 we find

\[ L_1 = \{ x \in \mathbb{F}_{p^{m-s}} : \text{Tr}_{\mathbb{F}_{p^{m-s}}/\mathbb{F}_p}(x) = 0 \} \]

and

\[ L_2 = \{ x \in \mathbb{F}_{p^s} : \text{Tr}_{\mathbb{F}_{p^s}/\mathbb{F}_p}(x) = 0 \}. \]

Then \( L_1 \cap L_2 \subset \mathbb{F}_{p^{m-s}} \cap \mathbb{F}_{p^s} = \mathbb{F}_{p^{\gcd(m,s)}} = \mathbb{F}_p \). Combining \( \gcd(m,s) = 1 \) with the condition on the traces gives \( L_1 \cap L_2 = \{0\} \), hence \( \delta = 1 \). If \( p \not| s(m-s) \) the ramification points over \( L_1 - \{0\} \) and \( L_2 - \{0\} \) are not rational. On the other hand, if \( p|s(m-s) \) then \( \mathbb{F}_q^s \cap (L_1 \cup L_2) = \mathbb{F}_p^s \) and the ramification points over \( \mathbb{F}_p^s \) are rational. Over the branch point 0 (resp. \( \infty \)) which has multiplicity \( p^s - 1 \) (resp. \( p^{m-1} - p^{s-1} \)) there lie \( p-1 \) ramification points on \( C_m \). These are rational if and only if \( -1 \) is a \((p-1)\)-th power in \( \mathbb{F}_q \) which holds if and only if \( pm \) is even. The formulas now follow from Prop. (1.3). 

(3.2) Examples. Take \( \mathbb{F}_{27} \) then \( L = \{ x \in \mathbb{F}_{27} : \text{Tr}_{\mathbb{F}_{27}/\mathbb{F}_3}(x) = 0 \} \) is given by \( R(x) = x^9 + x^3 + x \) and we can consider the curve

\[ C : \quad y^{26} = -(x^8 + x^2). \]

It follows from Prop. (3.1) that \( g(C) = 24 \) and \( \#C(\mathbb{F}_{27}) = 208 \) which improves [2].

For \( \mathbb{F}_{32} \) with (7) of the form \( y^{31} = (x^4 + x^2 + x)^4/(x^2 + x) \) we obtain according to Prop. (3.1) a curve \( C \) of genus 60 and \( \#C(\mathbb{F}_{32}) = 468 \). The Oesterlé upper bound is 542.

(3.3) Example. Finally we consider an example where \( \gcd(m,s) \neq 1 \). Take \( \mathbb{F}_{64} \) and

\[ f(x) = \frac{x^{32} + x^{16}}{x^8 + x^4 + x^2 + x} = \frac{(x^2 + x)^{15}}{(x^4 + x + 1)(x^2 + x + 1)}. \]

For the curve \( C \) given by \( y^{63} = f(x) \) Prop. (1.2) implies that \( 2g-2 = 7 \times 63 - 3 \times 3 - 6 \times 1 \), hence \( g(C) = 214 \). Each of the branch points 0, 1 and \( \infty \) induces 3 rational ramification points on \( C \), the zeros of \( x^2 + x + 1 \) induce 1 ramification point each, while the ramification points from the zeros of \( x^4 + x + 1 \) are not rational. The number of rational points on \( C \) is thus \( \#C(\mathbb{F}_{64}) = (32 - 2) \times 63 + 11 = 1901 \). The Oesterlé upper bound is 2553.

(3.4) Remark. For even \( m \) and \( L = \{ x \in \mathbb{F}_{q^{m-p}} : \text{Tr}_{\mathbb{F}_{q^{m-p}}/\mathbb{F}_p}(x) = 0 \} \) the splitting

\[ \sum_{i=0}^{m-1} x^{p^i} = R_1 + R_2 = \sum_{i=m/2}^{m-1} x^{p^i} + \sum_{i=0}^{(m/2)-1} x^{p^i} \]

does not satisfy condition i). The corresponding equation \( y^{q-1} = -R_1/R_2 = -R_2^{\sqrt{q}-1} \) leads to the curve

\[ C : \quad y^{\sqrt{q}+1} = aR_2 = a(x^{p^{(m/2)-1}} + x^{p^{(m/2)-2}} + \ldots + x) \]
where \( a \in F_q^* \) is such that \( a \sqrt[q]{q} + a = 0 \).

To determine \( g(C) \) and \( \#C(F_q) \) we consider the \( F_p \)-linear map \( \phi \) on \( L \) defined by \( \phi(x) = aR_2(x) \). The kernel of \( \phi \) is

\[ \ker(\phi) = \{ x \in F_{\sqrt[q]{q}} : \text{Tr}_{F_{\sqrt[q]{q}}/F_p}(x) = 0 \} \quad \text{and} \quad \phi(L) = F_{\sqrt[q]{q}}. \]

For \( y \in F_q^* \) we have \( y\sqrt[q]{q+1} \in F_{\sqrt[q]{q}}^* \) so over each \( y \in F_q^* \) there are \( \# \ker(\phi) = \sqrt[q]{q}/p \) rational points on \( C \). The set of branch points is \( \ker(\phi) \cup \infty \) and each branch point induces 1 rational point on \( C \). Hence

\[ \#C(F_q) = (q-1)\sqrt[q]{q}/p + \sqrt[q]{q}/p + 1 = (q\sqrt[q]{q}/p) + 1. \]

From Prop. (1.2) we find \( g(C) = (q-p\sqrt[q]{q})/2p \). We thus get explicit maximal curves:

\[ (3.5) \text{ Proposition.} \quad \text{The curve } C \text{ over } F_q \text{ given by } (8) \text{ with } g(C) = (q-p\sqrt[q]{q})/2p \text{ and } \#C(F_q) = (q\sqrt[q]{q}/p) + 1 \text{ is a maximal curve, i.e. it attains the Hasse-Weil upper bound.} \]

By the substitution \( x \mapsto z^p - z \) in (8) we obtain the equation for the Hermitian curve \( \sqrt[q]{q+1} = a(z\sqrt[q]{q} - z) \). So the curve \( C \) figuring in Prop. (3.5) is a quotient of the Hermitian curve.

\section*{§ 4. Variations}

To find curves with many points with this method it is not necessary to depart from a linearized polynomial. This is illustrated by the following example where we take a curve of the form

\[ y^{q-1} = x f(x)^p \]

with \( f(x) \in F_q[x] \).

\[ (4.1) \text{ Example.} \quad \text{Take } F_{16} \text{ and consider the irreducible complete non-singular curve } C \text{ given by the affine equation} \]

\[ y^{15} = x(x^2 + x + 1)^2. \]

Remark that \( x(x^2 + x + 1)^2 = x^5 + x^3 + x \) satisfies Conditions (1.1). According to (1) the curve \( C \) has genus \( g(C) = 12 \) and the number of points is \( \#C(F_{16}) = 15 \times 5 + 8 = 83 \), where the branch point \( \infty \) contributes 5 rational points and the branch points in \( F_4 - \{1\} \) each contribute 1 rational point. This example provides a new entry for the tables [2], where the interval [68–97] is given.

An advantage of our method is that we can also find good curves \( C \) such that only a few fibres over \( P^1(F_q) \) contribute to the rational points on \( C \), but these then do so substantially, as in the preceding example. We can use this for instance to construct Artin-Schreier covers of \( C \) given by

\[ z^p - z = h(x), \]
where in order to obtain good curves one has to impose the condition \( \text{Tr}(h(x)) = 0 \) for a few values \( x \) only.

**Example (4.2).** Take the field \( \mathbb{F}_{32} \) and consider the curve \( C \) defined by
\[
y^{31} = x^5 + x^3.
\]
The polynomial \( x^5 + x^3 + 1 \) is irreducible over \( \mathbb{F}_2 \), so it has 5 zeros in \( \mathbb{F}_{32} \). There are three ramification points \( P_0, P_1 \) and \( P_\infty \) lying over 0, 1 and \( \infty \).

We find
\[
g(C) = 15, \quad \#C(\mathbb{F}_{32}) = 158,
\]
which comes up to the best value known for \( (q, g) = (32, 15) \) in [2].

We immediately see that the zeros \( x \in \mathbb{F}_{32} \) of \( x^5 + x^3 + 1 \) satisfy \( \text{Tr}(x) = 0 \). The divisor of \( x \) is
\[
(x) = 31P_0 - 31P_\infty.
\]
The Artin-Schreier cover \( \tilde{C} \) of \( C \) given by
\[
z^2 + z = x
\]
has 2 rational points over each of the 155 points \( (x, y) \) of \( C(\mathbb{F}_{32}) \) with \( y \in \mathbb{F}_{32}^* \). We thus find
\[
\#\tilde{C}(\mathbb{F}_{32}) = 2 \times 155 + 1 + 2 = 313,
\]
and \( g(\tilde{C}) = 45 \). (See [3] for formulas for the genus.) This improves [2], where the interval is \([302–428]\).

As a variation on this theme we take \( \mathbb{F}_{16} \) with the curve \( C \) given by
\[
y^{15} = x^4 + x^3.
\]
This has genus \( g(C) = 6 \) with 65 rational points. The Artin-Schreier cover \( \tilde{C} \) of \( C \) defined by \( z^2 + z = 1/x \) yields a curve of genus \( g(\tilde{C}) = 20 \) with \( \#\tilde{C}(\mathbb{F}_{16}) = 127 \).

If one has a curve \( C \) with many points then often a curve \( C' \) obtained as the image under a \( \mathbb{F}_q \)-morphism \( C \rightarrow C' \) is also a good curve because the set of eigenvalues of Frobenius for \( C' \) is a subset of those for \( C \). In the cases dealt with in the preceding sections where the curve is of the form
\[
y^{q-1} = f(x^{p-1}),
\]
we can consider the curves \( y^s = f(x^t) \) for any divisor \( s \) of \( q - 1 \) and \( t \) of \( p - 1 \).

**Example (4.3).** From the curve \( C \) over \( \mathbb{F}_{27} \) given in Example (2.2) we obtain the curve \( C' \)
\[
-y^{13} = x^{24} + x^{22} + \ldots + x^2 \quad \text{with} \quad g(C') = 48 \quad \text{and} \quad \#C'(\mathbb{F}_{27}) = 316,
\]
where the tables give \([325–402]\), and
\[
-y^{26} = x^{12} + x^{11} + \ldots + x \quad \text{with} \quad g(C') = 49 \quad \text{and} \quad \#C'(\mathbb{F}_{27}) = 314,
\]
a new entry in the tables.

Of course, the methods can be varied in several ways. For example, one can replace \( y^{q-1} \) by \( y^t \) for \( t \) a divisor of \( q - 1 \) and take a function \( f \) which assumes for many \( x \) a \( t \)-th power in \( F_q \). We now give an example of this.

**Example (4.4).** Take \( F_{81} \) and consider the curve given by the equation

\[
y^{10} = x^2 + x.
\]

For \( y \in F_{81} \) we have \( y^{10} \in F_9 \), so the equation \( x^2 + x = y^{10} \) has always solutions \( x \in F_{81} \). The curve \( C \) has genus \( g(C) = 4 \) and \( \#C(F_{81}) = 154 \). Consider the double cover \( \tilde{C} \) of \( C \) given by

\[
z^2 = x^2 + x + 2.
\]

Over each \( (x, y) \in C(F_{81}) \) with \( y \in F_{81}^* \) the curve \( \tilde{C} \) has rational points since \( x^2 + x + 2 \in F_9 \). A computation of the genus and the number of points yields

\[
g(\tilde{C}) = 17, \quad \#\tilde{C}(F_{81}) = 288.
\]

This is a new entry for the tables [2].

We can also apply the methods to a base curve different from \( P^1 \) as the following examples show.

**Example (4.5).** Take \( F_8 \) and consider the curve \( C \) of genus 1 defined by

\[
y^2 + y = x + \frac{1}{x} + 1.
\]

It has 14 rational points, namely the two ramification points \( P_0 \) and \( P_\infty \), and six pairs of points \( P_\zeta, P'_\zeta \), one over each 7-th root \( \zeta \neq 1 \) of 1. Consider now the cover \( \tilde{C} \) of \( C \) defined by

\[
z^7 = x(x^6 + 1)/(x + 1),
\]

cf. Prop. (2.1). It has branch points \( P_0 \) and \( P_\infty \) and \( P_x, P'_x \) for \( x \) a third root of unity. Then the genus \( g(\tilde{C}) \) satisfies \( 2g(\tilde{C}) - 2 = 7 \times 0 + 8 \times 6 = 48 \), hence \( g(\tilde{C}) = 25 \). The rational points come from 12 fibres of order 7 over \( P_\zeta \) and \( P'_\zeta \), and from the two ramification points over \( P_0 \) and \( P_\infty \), giving \( \#\tilde{C}(F_8) = 86 \) which improves the entry [84–97] of the tables.

**Example (4.6).** Take \( F_8 \) and consider the Klein curve \( C \) of genus 3 defined by \( y^3 + x^3 y + x = 0 \). It has 24 rational points. Consider then the cover \( \tilde{C} \) given by \( z^7 = x(x^6 + 1)/(x + 1) \). The branch points on \( C \) are the points lying over \( x = 0, x = \) a third root of unity and \( x = \infty \). We find \( g(\tilde{C}) = 51 \) and \( \#\tilde{C}(F_8) = 132 \). The Oesterlé upper bound is 173.

**Example (4.7).** Take \( F_9 = F_3[i] \) with \( i^2 = -1 \) and consider the curve \( C \) of genus 1 defined by

\[
y^2 = x^3 + x.
\]

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It has 16 rational points over $\mathbb{F}_9$, the 4 ramification points $P_0$, $P_\infty$, $P_i$ and $P_{-i}$, and six pairs $P_x, P'_x$ for $x \in \mathbb{F}_9 - \{0, \pm i\}$. Take the function $f = x/y$ with divisor $(f) = P_0 + P_\infty - P_i - P_{-i}$ and consider the cover $\tilde{C}$ of $C$ defined by

$$z^4 = f^3 + f.$$ 

Observe that for $u \in \mathbb{F}_9^*$ the expression $u^3 + u$ is a 4-th power in $\mathbb{F}_9^*$. One has $(f(f^2 + 1)) = P_0 + P_\infty + 2P_1 + 2P'_1 - 3P_i - 3P_{-i}$. The curve $\tilde{C}$ has genus $g(\tilde{C}) = 9$ and has $10 \times 4 + 4 + 4 = 48$ rational points. Here the points $P_0, P_\infty, P_i, P_{-i}$ are branch points with total ramification, while the branch points $P_1, P'_1$ each contribute 2 rational points. This comes up to the best known curve and is very close to the Oesterlé upper bound 51.

**Example.** Take $\mathbb{F}_9 = \mathbb{F}_3[i]$ with $i^2 = -1$ and consider the curve $C$ of genus 2 defined by

$$z^2 = x(x^4 + x^2 + 2).$$

It has 18 rational points over $\mathbb{F}_9$. We denote them by $P_0, P_\infty$ and by $P_x, P'_x$ in the fibre over $x$ for each $x \in \mathbb{F}_9^*$. According to Prop. (2.5) the Kummer cover $D$ of $\mathbb{P}^1$ defined by

$$y^8 = -\frac{(x^9 - ax^3)}{(ax^3 - x)} = -\frac{x^2(x^2 - a^3)^3}{ax^2 - 1}$$

with $a$ such that $a^2 + a + 2 = 0$ (9)

has fibres consisting of 8 rational points over each $x \in \mathbb{F}_9^*$. We consider the curve $\tilde{C}$ which is the cover of $C$ defined by (9). The branch points on $C$ are the four points $P_0, P_\infty, P_\xi, P'_\xi$ with $\xi^2 = a^3$ and the four points $P_\eta, P'_\eta$ with $\eta^2 = 1/a$. The divisor of the function $f$ given by the right hand side of (9) is

$$(f) = 4P_0 + 6P_\xi + 6P_{-\xi} - P_\eta - P'_\eta - P_{-\eta} - P'_{-\eta} - 12P_\infty.$$ 

By Hurwitz-Zeuthen the genus is 33. Over each $x \in \mathbb{F}_9^*$ we find 16 rational points on $\tilde{C}$ giving $\#\tilde{C}(\mathbb{F}_9) = 128$, a significant improvement of the entry [109–133] in the tables [2].

If we take here instead of the base curve $C$ the curve $C'$ of genus 2 with 18 rational points defined by

$$z^2 = x(x^4 + x^3 + x^2 + x + 1)$$

then (9) defines a Kummer cover $\tilde{C}'$ of $C'$ of genus 41 with 128 rational points.

By employing the methods in a systematic way we expect more improvements and supplements to the tables in [2].
§5. Tables

We now summarize the new results from our examples for tables of curves with many points.

Table $p = 2$.

| $q$ | $g(C)$ | new entry | old entry |
|-----|--------|-----------|-----------|
| 8   | 25     | [86–97]   | [84-97]   |
| 8   | 51     | [132–173] |           |
| 16  | 12     | [83–97]   | [68-97]   |
| 16  | 20     | [127–140] | [121–140] |
| 16  | 40     | [225–244] | [197–244] |
| 16  | 49     | [213–286] |           |
| 32  | 45     | [313–428] | [304–428] |
| 32  | 60     | [468–542] |           |
| 32  | 135    | [933–1098]|           |
| 64  | 214    | [1901–2553]|         |
| 64  | 428    | [3969–4786]|         |

Table $p = 3$.

| $q$ | $g(C)$ | new entry | old entry |
|-----|--------|-----------|-----------|
| 9   | 13     | [64–66]   | [60–66]   |
| 9   | 33     | [128–133] | [109–133] |
| 9   | 41     | [128–158] | [119–158] |
| 27  | 24     | [208–235] | [190–235] |
| 27  | 49     | [314–409] |           |
| 27  | 98     | [624–745] |           |
| 27  | 124    | [680–901] |           |
| 81  | 17     | [288–387] |           |
| 81  | 625    | [6400–7824]|         |
| 243 | 3854   | [58080–81835]|      |
| 243 | 4096   | [58568–86441]|      |
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