Helly EPT graphs on bounded degree trees: forbidden induced subgraphs and efficient recognition

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Abstract

The edge intersection graph of a family of paths in host tree is called an EPT graph. When the host tree has maximum degree $h$, we say that $G$ belongs to the class $[h, 2, 2]$. If, in addition, the family of paths satisfies the Helly property, then $G \in$ Helly $[h, 2, 2]$. The time complexity of the recognition of the classes $[h, 2, 2]$ inside the class EPT is open for every $h > 4$. In [6], Golumbic et al. wonder if the only obstructions for an EPT graph belonging to $[h, 2, 2]$ are the chordless cycles $C_n$ for $n > h$. In the present paper, we give a negative answer to that question, we present a family of EPT graphs which are forbidden induced subgraphs for the classes $[h, 2, 2]$. Using them we obtain a total characterization by induced forbidden subgraphs of the classes Helly $[h, 2, 2]$ for $h \geq 4$ inside the class EPT. As a byproduct, we prove that Helly $\text{EPT} \cap [h, 2, 2] = \text{Helly} [h, 2, 2]$. Following the approach used in [10], we characterize Helly $[h, 2, 2]$ graphs by their atoms in the decomposition by clique separators. We give an efficient algorithm to recognize Helly $[h, 2, 2]$ graphs.

Keywords: intersection graphs, EPT graphs, UE graphs, tolerance graphs.

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1 Introduction

A graph $G$ is called $EPT$ (or $UE$) if it is the edge intersection graph of a family of paths in a tree. $EPT$ graphs are used in network applications, the problem of scheduling undirected calls in a tree network is equivalent to the problem of coloring an $EPT$ graph (see [2]). The class of $EPT$ graphs was first investigated by Golumbic and Jamison [3,4]. In the last decades many papers were devoted to the study of $EPT$ graphs and their generalizations, see [5,8,11]. In [9], the class of graphs that admit an $EPT$ representation on a host tree with maximum degree $h$ is denoted by $[h,2,2]$. Clearly, $[2,2,2]$ is the class of interval graphs. It is known that $[3,2,2]$ is precisely the class of chordal $EPT$ graphs [9], while $[4,2,2]$ is the class of weakly chordal $EPT$ graphs [7]. Notice that the class of $EPT$ graphs is the union of the classes $[h,2,2]$ for $h \geq 2$. A complete hierarchy of related graph classes emerging by imposing different restrictions on the tree representation is published in [6].

On the algorithmic side, the recognition and coloring problems restricted to $EPT$ graphs are NP-complete, whereas the maximum clique and maximum stable set problems are polynomially solvable. See [3].

The time complexity of the recognition of the classes $[h,2,2]$ inside the class $EPT$ is open for $h > 4$, and it is known to be polynomial time solvable for $h \in \{2,3,4\}$. In [10] and [7], Golumbic et al. wonder if the only obstructions for an $EPT$ graph belonging to $[h,2,2]$ are the chordless cycles of size greater than $h$. In [1], we give a negative answer to this question and present a family of forbidden induced subgraphs called prisms.

In this paper, we generalize the class of prisms and present a wider family of $EPT$ graphs called $k$-gates which are forbidden induced subgraphs for the classes $[h,2,2]$ when $h < k$.

A graph is Helly $EPT$ (or $UEH$) if it admits an $EPT$ representation using a path family that satisfies the Helly property. In [10], Monma and Wei characterize $EPT$ and Helly $EPT$ via decomposing the graph by clique separators and prove that the latter class can be recognized efficiently. Finding a characterization by forbidden induced subgraphs of $EPT$ and of Helly $EPT$ graphs are long standing open problems.

Helly $[h,2,2]$ is the class of graphs that admit a Helly $EPT$ representation on a host tree with maximum degree $h$. Clearly, Helly $EPT \cap [h,2,2] \subseteq$ Helly $[h,2,2]$ but the equality not necessary holds.

We obtain a total characterization by induced forbidden subgraphs of the class Helly $[h,2,2]$ inside the class $EPT$ using gates. As a byproduct, we prove that Helly $EPT \cap [h,2,2] = \text{Helly } [h,2,2]$ which means that, in the
way of getting a Helly representation, it is not necessary to increase the maximum degree of the host tree.

In addition, we characterize Helly \([h, 2, 2]\) graphs by their atoms in the decomposition by clique separators. We give an efficient algorithm to recognize Helly \([h, 2, 2]\) graphs.

The paper is organized as follows: in Section 2 we provide basic definitions and known results. In Section 3 we depict the graphs named \(k\)-gates and focus on their main properties; we show that \(k\)-gates are Helly EPT but do not admit an EPT representation on a host tree with maximum degree less than \(k\). In Section 4 we show that a Helly EPT graph \(G\) belongs to the class Helly \([h, 2, 2]\) if and only if \(G\) does not have a \(k\)-gate as induced subgraph for any \(k > h\). Finally, in Section 5 we use the Monma and Wei decomposition by clique separator to obtain an efficient algorithm for the recognition of Helly \([h, 2, 2]\) graphs.

2 Preliminaries and known results

In this paper all graphs are finite and simple. Given a graph \(G\), \(V(G)\) and \(E(G)\) denote the vertex set and the edge set of \(G\), respectively. An EPT representation of \(G\) is a pair \((\mathcal{P}, T)\) where \(\mathcal{P}\) is a family \((P_v)_{v \in V(G)}\) of subpaths of the host tree \(T\) satisfying that two vertices \(v\) and \(w\) of \(G\) are adjacent if and only if \(E(P_v) \cap E(P_w) \neq \emptyset\). When the maximum degree of the host tree \(T\) is \(h\), the EPT representation of \(G\) is called an \((h, 2, 2)\)-representation of \(G\). The class of graphs that admit an \((h, 2, 2)\)-representation is denoted by \([h, 2, 2]\).

A star is any complete bipartite graph \(K_{1,n}\). The only vertex with degree grater than one is called the center of the star. The edges of a star are called spokes. The star \(K_{1,3}\) is named the claw graph. We will say that a path \(P: (v_1, ..., v_l)\) contains a vertex \(v\) if \(v = v_i\) for some \(1 \leq i \leq l\); and that it contains an edge \(e\) if \(e = v_iv_{i+1}\) for some \(1 \leq i \leq l - 1\).
Fig. 2. An EPT representation of the sun \( S_3 \). In this representation, the central triangle \( \{2, 3, 5\} \) is a claw-clique; the other three triangles are edge-cliques.

Golumbic et al. introduced the notion of pie in order to describe EPT representations of chordless cycles. A **pie of size** \( k \) in an EPT representation \( \langle \mathcal{P}, T \rangle \) is a star subgraph of \( T \) with central vertex \( q \) and neighbors \( q_1, \ldots, q_k \) and a subfamily of paths \( P_1, \ldots, P_k \) of \( \mathcal{P} \) such that \( \{q_i, q, q_{i+1}\} \subseteq V(P_i) \), for \( 1 \leq i \leq k \) (addition is assumed to be module \( n \)). See Figure 1.

**Theorem 1** [3] Let \( \langle \mathcal{P}, T \rangle \) be an EPT representation of a graph \( G \). If \( G \) contains a chordless cycle \( C_k \) with \( k \geq 4 \), then \( \langle \mathcal{P}, T \rangle \) contains a pie of size \( k \) whose paths are in one-to-one correspondence with the vertices of \( C_k \).

A set family \( (S_i)_{i \in I} \) satisfies the **Helly property** if any pairwise intersecting subfamily \( (S_i)_{i \in I'} \) with \( \emptyset \neq I' \subseteq I \) has non-empty total intersection, i.e. \( \bigcap_{i \in I'} S_i \neq \emptyset \). A graph \( G \) is **Helly EPT** if it admits an EPT representation \( \langle \mathcal{P}, T \rangle \) such that the set family \( (E(P))_{P \in \mathcal{P}} \) satisfies the Helly property. In an analogous way, we say that \( G \) is **Helly \([h, 2, 2]\)** if it admits an \((h, 2, 2)\)-representation \( \langle \mathcal{P}, T \rangle \) such that the family \( (E(P))_{P \in \mathcal{P}} \) satisfies the Helly property. Clearly, Helly \([h, 2, 2]\) \( \subseteq \) Helly EPT \( \cap \) \([h, 2, 2]\).

A **complete set** of a graph \( G \) is a subset of \( V(G) \) whose elements are pairwise adjacent. A **clique** is a maximal (with respect to the inclusion relation) complete set.

Given an EPT representation \( \langle (P_v)_{v \in V(G)}, T \rangle \) of \( G \), for every edge \( e \) of \( T \), let \( K_e \) be the complete set \( \{v \in V(G) : e \in E(P_v)\} \). For every claw \( Y \) in \( T \), let \( K_Y \) be the complete set \( \{v \in V(G) : P_v \) contains two spokes of \( Y\}\).

**Theorem 2** [4] Let \( \langle \mathcal{P}, T \rangle \) be an EPT representation of \( G \). If \( C \) is a clique of \( G \) then either there is an edge \( e \in E(T) \) such that \( C = K_e \) or there is a claw \( Y \) in \( T \) such that \( C = K_Y \).

In the former case, when there exists \( e \) such that \( C = K_e \), the clique \( C \) is called an **edge-clique**, otherwise \( C \) is called a **claw-clique**. See Figure 2. Notice that the condition of being an edge-clique or a claw-clique depends on the given representation. Clearly, in a Helly EPT representation every clique is an edge-clique. We say that three paths of a given EPT representation \( \langle \mathcal{P}, T \rangle \) **form a claw** if there exists a claw \( Y \) of \( T \) such that every pair of
spokes of \( Y \) is contained by some of the paths. Clearly, there is claw-clique if and only if three paths form a claw.

If \( S \subseteq V(G) \) then \( G - S \) denotes the graph induced in \( G \) by \( V(G) \setminus S \). When \( S \) contains a unique vertex \( v \), we write simply \( G - v \).

3 Gates and multiples

A clear corollary of Theorem 1 is that every chordless cycle \( C_k \) with \( k > h \geq 3 \) is an obstruction for the class \([h, 2, 2]\). In [6], Golubic et al. wonder if besides cycles there are other EPT forbidden induced subgraphs for this class. In [1], answering negatively the previous question, we described for every \( h > 4 \) an EPT graph \( F_h \) which has no induced cycles of size \( k \) for every \( k > h \), but it does not admit an EPT representation on a host tree with maximum degree less than or equal to \( h \). The graphs introduced in the following definition generalize the graphs \( F_h \). In Section 4, we obtain a total characterization of Helly \([h, 2, 2]\) graphs using them.

We say that two graphs \( G \) and \( G' \) are disjoint if \( V(G) \cap V(G') = \emptyset \). The union of \( G \) and \( G' \) is the graph \( H \) with \( V(H) = V(G) \cup V(G') \) and \( E(H) = E(G) \cup E(G') \).

**Definition 3** The following graphs are called gates.

- Every chordless cycle \( C_n \) with \( n \geq 4 \) is a gate;
- If \( G \) is a gate, \( C \) and \( C' \) are disjoint cliques of \( G \), and \( P : (v_1, \ldots, v_l) \) with \( l \geq 2 \) is a chordless path disjoint from \( G \), then the union of \( G \) and \( P \) plus all edges between \( v_1 \) and the vertices of \( C \), and all edges between \( v_l \) and the vertices of \( C' \) is a gate;
- There are no more gates.

If the number of cliques of a gate \( G \) is \( k \) then we say that \( G \) is a \( k \)-gate.

In Figure 3 we offer some examples of gates.

**Lemma 4** If \( G \) is a \( k \)-gate then \( G \in \text{Helly } [k; 2, 2] \). Furthermore, \( G \) admits a Helly \((k; 2, 2)\)-representation on a host tree that is a star.

**Proof.** We proceed by induction. Clearly the statement holds for \( C_k \).

If \( G \) is not a cycle, then \( G \) is obtained from an \( m \)-gate \( H \) using disjoint cliques \( C \) and \( C' \) of \( H \) and a path \( P : (v_1, v_2, \ldots, v_l) \) with \( l \geq 2 \) disjoint from \( H \). Notice that \( m + (l - 1) = k \). Let \( \langle \mathcal{P}, T \rangle \) be a Helly \((m; 2, 2)\)-representation of \( H \) with...
Fig. 3. Some examples of gates. From left to right, the second gate is obtained from the first using the bold cliques $C$ and $C'$ and the path $P : (v_1, v_2, v_3, v_4)$. The third gate is obtained from the second using the bold cliques $Q$ and $Q'$ and the path $W : (w_1, w_2, w_3)$.

Let $T$ be a star. We can assume that $T$ has $m$ spokes. Let $e$ and $e'$ be spokes of $T$ such that $C = K_e$ and $C' = K_{e'}$. Denote by $T'$ the star that is obtained by adding $l - 1$ spokes $e_1, \ldots, e_{l-1}$ to $T$. Let $P_{v_i}$ be the subpath of $T'$ defined by the edges $e$ and $e_i$. For $2 \leq i \leq l - 1$ let $P_{v_i}$ be the subpath of $T'$ defined by the edges $e_{i-1}$ and $e_i$; and let $P_{v_l}$ the one defined by the edges $e_{l-1}$ and $e'$.

Thus $\langle P', T' \rangle$ is a Helly $(k, 2, 2)$-representation of $G$, where $P'$ is the family $P$ plus the paths $P_{v_i}$ for $1 \leq i \leq l$.

**Lemma 5** If $G$ is a gate and $v \in V(G)$, then $v$ belongs to exactly two cliques of $G$. In addition, if $C_1$ and $C_2$ are those cliques then $C_1 \cap C_2 = \{v\}$.

**PROOF.** We proceed by induction. Clearly the statement holds for chordless cycles.

Let $G$ be a gate obtained from another gate $H$, using disjoint cliques $C$ and $C'$ of $H$ and a chordless path $P : (v_1, \ldots, v_l)$ with $l \geq 2$ disjoint from $H$. Notice that the cliques of $G$ are:

- the cliques of $H$ other than $C$ and $C'$;
- the cliques of $P$, i.e. $\{v_i, v_{i+1}\}$ for $1 \leq i \leq l - 1$;
- $C \cup \{v_1\}$; and
- $C' \cup \{v_l\}$.

The proof follows easily from the fact that $H$ satisfies the statement. \hfill \Box

**Lemma 6** Let $v$ be a vertex of a gate $G$, $C_1$ and $C_2$ cliques of $G$ such that $C_1 \cap C_2 = \{v\}$, and $W : (w_1, \ldots, w_t)$ a chordless path disjoint from $G$ with $t \geq 2$. Then, the graph $G'$ union of $G - v$ and $W$ plus all edges between $w_1$ and the vertices of $C_1 - \{v\}$ and all edges between $w_t$ and the vertices of $C_2 - \{v\}$ is a gate.
Fig. 4. An example following the proof of Lemma 6.

**Proof.** We proceed by induction. Clearly the statement holds for chordless cycles.

Assume $G$ is a gate obtained from another gate $H$, using disjoint cliques $C$ and $C'$ of $H$ and a chordless path $P : (v_1, .., v_l)$ with $l \geq 2$ disjoint from $H$. If $v$ is one of the vertices of $P$ then the proof is direct and simple.

If $v$ is a vertex of $C$ (see Figure 4), we can assume that $C_1 = C \cup \{v_1\}$ and $C_2$ is a clique of $G$ different from $C' \cup \{v_l\}$, which means that in $H$ the vertex $v$ is the intersection between the cliques $C$ and $C_2$. Thus, by the inductive hypothesis, the graph $H'$ obtained from the union of $H - v$ and $W$ plus all edges between $w_1$ and the vertices of $C - \{v\} = C_1 - \{v, v_1\}$ and all edges between $w_l$ and the vertices of $C_2 - \{v\}$ is a gate. Since the path $P$ is disjoint from $H'$, and $(C_1 - \{v_1, v\}) \cup \{w_1\}$ and $C'$ are disjoint cliques of $H'$, thus, by the recursive definition of gate, the union of $H'$ and $P$ plus all edges between $v_1$ and the vertices of $(C_1 - \{v_1, v\}) \cup \{w_1\}$, and all edges between $v_l$ and the vertices of $C'$ is a gate. The proof follows from the fact that this is the same graphs $G'$ depicted in the statement of the theorem.

If $v$ is a vertex of $C'$ or if $v \in V(H) - (C \cup C')$ the proof is analogous. □

Golumbic and Jamison proved that (see Theorem 1) chordless cycles admit a unique EPT representation called pie. In what follow, generalizing that
result, we introduce the definition of multipie and prove that also gates admit a unique EPT representation.

**Definition 7** A **multipie of size** \( k \) **in an EPT representation** \( \langle P, T \rangle \) **is a star subgraph of** \( T \) **with central vertex** \( q \) **and neighbors** \( q_1, \ldots, q_k \) **and a subfamily** \( P' \) **of** \( P \) **such that:**

1. if \( P \in P' \) then \( |V(P) \cap \{q_1, q_2, \ldots, q_k\}| = 2 \) (every path contains two spokes of the star);
2. if \( i \neq j \) then \( |\{P \in P' : \{q_i, q_j\} \subseteq V(P)\}| \leq 1 \) (no two paths contain the same two spokes);
3. if \( 1 \leq i \leq k \) then \( |\{P \in P' : \{q, q_i\} \subseteq V(P)\}| \geq 2 \) (every spoke of the star is contained by at least two paths);
4. no three paths of \( P' \) form a claw.

Observe that every pie is a multipie. The following theorem generalizes Theorem 1.

**Theorem 8** Let \( \langle P, T \rangle \) be an EPT representation of \( G \). If \( G \) contains a \( k \)-gate then \( \langle P, T \rangle \) contains a multipie of size \( k \) whose paths are in one-to-one correspondence with the vertices of the gate.

**Proof.** Let \( \langle P, T \rangle \) be an EPT representation of \( G \) with \( P = (P_v)_{v \in V(G)} \). We can assume, without loss of generality, that \( G \) is a \( k \)-gate. We proceed by induction. If \( G \) is a chordless cycle \( C_k \) then, by Theorem 1, \( \langle P, T \rangle \) contains a pie of size \( k \) and the proof follows.

If \( G \) is not a cycle, then \( G \) is obtained from an \( m \)-gate \( H \) using disjoint cliques \( C \) and \( C' \) of \( H \) and a path \( P : (v_1, v_2, \ldots, v_l) \) with \( l \geq 2 \) disjoint from \( H \). Notice that \( m + (l-1) = k \). By inductive hypothesis, \( \langle P, T \rangle \) contains a multipie of size \( m \) formed by a star subgraph \( S \) of \( T \) and the path subfamily \( P' = (P_v)_{v \in V(H)} \).

Let \( S \) be the star with center \( q \) and leaves \( q_1, \ldots, q_m \). By condition (4) in Definition 7, no three paths of \( P' \) form a claw, then there exists a spoke of \( S \), say \( e_1 = qq_1 \), such that \( C \subseteq K_{e_1} \); and there exists another spoke, without loss of generality say \( e_2 = qq_2 \), such that \( C' \subseteq K_{e_2} \). Even more, by condition (2), \( e_1 \) and \( e_2 \) are the only spokes of \( S \) satisfying the described property.

Let \( d \) be the minimum distance in \( H \) between a vertex of \( C \) and a vertex of \( C' \). Clearly, \( d \geq 1 \). Chose vertices \( u \in C \) and \( u' \in C' \) such that the distance between them in \( H \) is \( d \). Let \( (u, u_1, u_2, \ldots, u_{d-1}, u') \) be a shortest path in \( H \) between \( u \) and \( u' \). Notice that \( u, u_1, u_2, \ldots, u_{d-1}, u', v_l, v_{l-1}, \ldots, v_2, v_1 \) induce a cycle in \( G \) of size \( d + l + 1 \geq 4 \). By Theorem 1 in \( \langle P, T \rangle \) there is a pie corresponding to this cycle. Let \( S' \) be the star subgraph of \( T \) used by this pie. Notice that the center of \( S' \) must be the same vertex \( q \) of \( T \). Even more, since
the vertex $v_1$ of $P$ is adjacent to all vertices in $C$, the vertex $v_1$ is adjacent to all vertices in $C'$, and there are no other adjacencies between vertices of $P$ and $H$, we have that $S'$ has $l - 1$ spokes that are not spokes of $S$. The remaining $(d + l + 1) - (l - 1) = d + 2$ spokes of $S'$ are also spokes of $S$. Therefore the union of $S$ and $S'$ is a star subgraph of $T$ with center $q$ and $m + l - 1 = k$ spokes. Now it is not difficult to check that $P$ forms a multipie around the star $S \cup S'$, and the proof follows. \hfill \Box

4 Forbidden induced subgraphs for Helly EPT graphs on bounded degree trees

The goal of this section is Theorem 9 below. We prove that gates are the only subgraphs which force the use a host a tree with large enough degree in every Helly $\text{EPT}$ representation of a graph.

**Theorem 9** Let $G$ be a Helly $\text{EPT}$ graph and $h \geq 3$. Then, $G \notin \text{Helly \[h, 2, 2\]}$ if and only if there exists $k > h$ such that $G$ has a $k$-gate as induced subgraph.

**PROOF.** We will prove the direct implication, the converse follows from Theorem 8 and the fact that Helly $\text{[h, 2, 2]} \subseteq \text{[h, 2, 2]}$.

Assume that $G$ is a Helly $\text{EPT}$ graph which does not admit a Helly $(h, 2, 2)$-representation. Let $d$ be the smallest positive integer such that $G \in \text{Helly \[d, 2, 2\]}$. Clearly, $d > h$. Let $\langle P, T \rangle$ be a Helly $(d, 2, 2)$-representation of $G$ minimizing the number of vertices of the host tree $T$ with degree $d$.

**Claim 10** We can assume that if $q \in V(T)$ is the end vertex of a path $P \in \mathcal{P}$ then $d_T(q) \leq 2$.

**PROOF.** If it is not the case, by subdividing every edge of $T$ (and consequently every edge of every path of $\mathcal{P}$) in three parts, and after that shortening every path of $\mathcal{P}$ by removing its two end vertices, we obtain the desired representation. \hfill \Box

Let $q_0 \in V(T)$ be a vertex with degree $d$ and call $q_1, ..., q_d$ to its neighbors. Denote by $H$ the subgraph of $G$ induced by the vertices $v$ such that $q_0 \in V(P_v)$.

**Claim 11** The subgraph $H$ contains an induced cycle of length at least 4.
PROOF. Let $P = (v_1, ..., v_l)$ be the longest induced path in $H$ and assume, without loss of generality, that $\{q_i, q_0, q_{i+1}\} \subseteq V(P_v)$, for all $i : 1, ..., l$. Notice that $2 \leq l \leq d - 1$.

Suppose, in order to derive a contradiction, that every path of $P$ containing $q_0 q_l$ also contains $q_0 q_l$. Then, we can modify (as explained below) the representation $\langle P, T \rangle$ to obtain a new Helly $(d, 2, 2)$-representation of $G$ on a host tree with less vertices of degree $d$, contrary to our assumption. Indeed, to obtain the new representation do:

subdivide the edge $q_0 q_l$ adding a new vertex $\tilde{q}_l$;

remove the edge $q_0 q_{l+1}$ and do $q_{l+1}$ adjacent to $\tilde{q}_l$;

in the paths of $P$ containing the edge $q_0 q_{l+1}$, replace the vertex $q_0$ and the edges $q_0 q_{l+1}$ and $q_0 q_l$ by the vertex $\tilde{q}_l$ and the edges $\tilde{q}_l q_{l+1}$ and $\tilde{q}_l q_l$, respectively;

no other path is modified except for the fact of subdividing the edge $q_0 q_l$ if necessary.

Therefore, there must exist $1 \leq j \leq d, j \neq l, l + 1$, and a vertex $x$ of $H$ such that $\{q_j, q_0, q_{l+1}\} \subseteq V(P_x)$. Clearly, $x \not\in V(P)$.

If $j > l + 1$, then $V(P) \cup \{x\}$ induces a path of $H$ longer than $P$, which contradicts the election of $P$.

If $j = l - 1$, then $P_x, P_{v_{l-1}}$ and $P_{v_l}$ violate the Helly property, which contradicts the fact that this a Helly $EPT$ representation of $G$.

Thus $j \leq l - 2$. This implies that $H$ contains the cycle induced by the vertices $\{v_j, ..., v_{l-1}, v_l, x\}$, as we wanted to prove. \qed

It follows from the previous Claim \[\text{(i)}\] that $H$ has at least an induced gate. Let $R$ be a biggest induced gate in $H$, say that $R$ is a $k$-gate, and assume without loss of generality, that the multipie corresponding to the vertices of $R$ use the star with edges $\{q_0 q_1, ..., q_0 q_k\}$ (see Lemma \[\text{(ii)}\]).

We will prove that $k = d$. Since $d > h$, the proof follows.

Clearly, $k \leq d$. Suppose, in order to derive a contradiction, that $k < d$.

Since $G$ is connected there must exists a vertex $y$ such that the path $P_y$ uses one of the edges $q_0 q_1, ..., q_0 q_k$ and an edge $q_0 q_i$ for some $k < i \leq d$. Without loss of generality, we can assume that $\{q_k, q_0, q_{k+1}\} \subseteq V(P_y)$. 

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If all paths containing the edge $q_0q_{k+1}$ also contain the edge $q_0q_k$, then (as we did before) we can modify the representation $\langle P, T \rangle$ to obtain a new representation of $G$ on a host tree with fewer vertices of degree $d$, contrary to assumption.

So, there exists a vertex $z$ and $j \neq k, k+1$ such that $\{q_j, q_0, q_{k+1}\} \subseteq V(P_z)$. Notice that $y$ and $z$ are adjacent and do not belong to the gate $R$.

Assume, in order to derive a contradiction, that $j \leq k - 1$. Let $C_k$ and $C_j$ be the cliques of $R$ corresponding to the edges $q_0q_k$ and $q_0q_j$ of $T$, respectively. Notice that $C_k$ and $C_j$ are disjoint, otherwise $P_y, P_z$ and $P_t$ violate the Helly property, where $v$ is a vertex in the intersection. Using cliques $C_k, C_j$ and the path $P : (y, z)$ disjoint from $R$, we obtain a $(k + 1)$-gate induced in $H$, which contradicts the election of $R$. Therefore, $j > k - 1$. Since $j \neq k, k+1$, say $j = k + 2$.

Denote by $A$ the set of vertices $v \in V(H)$ such that $P_v$ contains an edge $q_0q_i$ for some $i \leq k$ and an edge $q_0q_{i'}$ for some $i' > k$. Notices that $y \in A$ and $z \notin A$. Let $G_z$ be the connected component of $G - A$ containing the vertex $z$.

Clearly, if $v \in V(G_z) \cap V(H)$ then there exist $i$ and $i'$, $k + 1 \leq i < i' \leq d$ such that $\{q_i, q_0, q_{i'}\} \subseteq V(P_v)$, thus, without loss of generality, we can assume that there exists $s$, with $k + 2 \leq s \leq d$, such that

\[
V(G_z) \cap V(H) = \bigcup_{k + 1 \leq i < i' \leq s} \{v \in V(G) : \{q_i, q_0, q_{i'}\} \subseteq V(P_v)\};
\]

and for every $k + 1 \leq i \leq s$

there exists $v \in V(G_z) \cap V(H)$ such that $q_0q_i \in E(P_v)$. \hspace{1cm} (1)

**Claim 12** If $y' \in A$ and $P_{y'}$ contains an edge $q_0q_t$ with $k + 1 \leq t \leq s$ then $P_{y'}$ also contains the edge $q_0q_k$.

**PROOF.** Assume, in order to derive a contradiction, that $q_0q_j \in E(P_{y'})$ with $1 \leq j < k$. Since $q_0q_t \in E(P_{y'})$ and $k + 1 \leq t \leq s$, by [1], there exists $z' \in V(G_z) \cap V(H)$ adjacent to $y'$. We chose $z'$ minimizing its distance to $z$ in $G_z$ (it could be $z' = z$). Let $P : (z, z_1, ..., z')$ be a shortest $zz'$-path in $G_z$. It is clear that $y'$ is adjacent to no vertex of $P$ except $z'$. Notice also that $V(P) \cap V(R) = \emptyset$, and no vertex of $P$ is adjacent to a vertex of $R$. So we will deal with the following cases:

(a) $y$ is adjacent to $y'$ (in this case $t$ must be equal to $k + 1$).

(b) $y$ is non adjacent to $y'$ but it is adjacent to some vertex of $P$ besides $z$.

Thus, it must be $z_1$ and $(y, z_1, ..., z', y')$ is a chordless path.
(c) \( y \) is neither adjacent to \( y' \) nor to a vertex of \( P \) besides \( z \). Thus \((y, z, z_1, ..., z', y')\) is a chordless path.

Let \( C_j \) and \( C_k \) be the cliques of \( R \) corresponding to the edges \( q_0q_j \) and \( q_0q_k \), respectively. If \( C_j \) and \( C_k \) are disjoint, then, by Definition 3 a gate bigger than \( R \) can be obtained using these two cliques and the path described above depending on cases (a), (b) or (c). If \( C_j \) and \( C_k \) are non disjoint, then, by Lemma 5 a gate bigger than \( R \) can be obtained also using these two cliques and the path described above depending on cases (a), (b) or (c). It contradicts the fact that \( R \) is the biggest gate. 

Finally, to end the proof of Theorem 9, we will describe below how to obtain a new Helly \((d, 2, 2)\)-representation \( \langle P', T' \rangle \) of \( G \) using a host tree \( T' \) with fewer vertices of degree \( d \). This contradicts the fact that \( \langle P, T \rangle \) is a representation minimizing the number of vertices with degree \( d \) and the proof follows.

To obtain \( T' \) do:

subdivide the edge \( q_0q_k \) of \( T \) adding a new vertex \( \tilde{q}_k \) adjacent to \( q_0 \) and to \( q_k \);

for every \( k + 1 \leq i \leq s \), remove the edge \( q_0q_i \) and add the edge \( \tilde{q}_kq_i \).

To obtain \( P' \) do:

If \( P \in P \) and there exist \( k + 1 \leq i < j \leq s \) such that \( \{q_i, q_0, q_j\} \in V(P) \), then replace \( P \) by the path \( P' \) obtained from \( P \) by removing the vertex \( q_0 \) and the edges \( q_0q_i \) and \( q_0q_j \) and adding the vertex \( \tilde{q}_k \) and the edges \( \tilde{q}_kq_i \) and \( \tilde{q}_kq_j \).

If \( P \in P \) and there exists \( k + 1 \leq i \leq s \) such that \( \{q_i, q_0\} \in V(P) \) and \( P \) is not in the previous case (then, by Claim 12 and the fact that \( G_z \) is a connected component of \( G - A \), we have that \( \{q_0, q_k\} \subseteq V(P) \)), then replace \( P \) by the path \( P' \) obtained from \( P \) by removing the vertex \( q_0 \) and the edges \( q_0q_i \) and \( q_0q_k \) and adding the vertex \( \tilde{q}_k \) and the edges \( \tilde{q}_kq_i \) and \( \tilde{q}_kq_i \).

No other path is modified except for the fact of subdividing the edge \( q_0q_k \) if necessary.

\textbf{Corollary 13} Helly \( EPT \cap [h, 2, 2] = \text{Helly } [h, 2, 2] \) for any \( h \geq 3 \).

\textbf{PROOF.} Clearly, \( \text{Helly } [h, 2, 2] \subseteq \text{Helly } EPT \cap [h, 2, 2] \).

Assume, in order to derive a contradiction, that \( G \in \text{Helly } EPT \cap [h, 2, 2] \) and \( G \notin \text{Helly } [h, 2, 2] \). By 3, \( G \) contains a \( k \)-gate as induced subgraph for some \( k > h \). Thus by Theorem 8, any \( EPT \) representation of \( G \) contains a multipie of size \( k \). This contradicts the fact that \( G \in [h, 2, 2] \).
5 Decomposition by clique separators and Complexity

A clique $C$ of a connected graph $G$ is a separator if $G - C$ (the subgraph induced by $V(G) \setminus C$) is not connected. An atom is a connected graph with no separators. In [10], a graph is progressively decompose by clique separators to obtain a clique decomposition tree with each leaf node being associated with an atom of $G$ and each internal node being associated with a clique separator of $G$. The atoms of $G$ are invariants. The clique decomposition can be computed in polynomial time. Both EPT graph and Helly EPT graphs are characterize by their clique decomposition tree. The characterization leads to an efficient algorithm to recognize Helly EPT graphs but does not to recognize EPT graphs.

Lemma 14 If $H$ is a Helly EPT atom with exactly $k \geq 4$ cliques then $H$ has a $k$-gate as induced subgraph.

PROOF. Assume, in order to derive a contradiction, that $H$ has no $k$-gates, then it has no $t$-gates for any $t \geq k$. Thus, by Theorem 9 there exists $h \leq k - 1$ such that $H \in \text{Helly}[h, 2, 2] - \text{Helly}[h - 1, 2, 2]$. Let $\langle \mathcal{P}, T \rangle$ be a Helly $(h, 2, 2)$-representation of $H$ minimizing the number of edges of the host tree $T$, this implies that $K_e$ is a clique of $H$ for every $e \in E(T)$, moreover $|E(T)| = k$. On the other hand, since $H$ is an atom, $T$ must be a star (otherwise there exists an edge $e$ of the host tree such that $K_e$ is a cut clique). It follows that $h = k$, in contradiction with the fact that $h < k$.

Lemma 15 Let $H$ be an $k$-gate. If $H$ is an induced subgraph of a graph $G$, then $H$ is an induced subgraph of some atom of $G$.

PROOF. It is enough to prove that a gate has no clique separators which follows trivially from the recursive definition of gates.

Theorem 16 Let $G$ be a Helly EPT graph and $h \geq 3$. Then, $G \in \text{Helly}[h, 2, 2]$ if and only if every atom of $G$ has at most $h$ cliques.

PROOF. If $G \in \text{Helly}[h, 2, 2]$ then, by Theorem 9, $G$ has no gates of size grater than $h$ as induced subgraphs. Thus, by Lemma 14 $G$ has no atoms with more than $h$ cliques.

Conversely, assume, in order to obtain a contradiction, that $G \not\in \text{Helly}[h, 2, 2]$. Thus, by Theorem 9, $G$ has a $k$-gate $H$ as induced subgraph, for some $k > h$. By Lemma 15 $H$ is an induced subgraph of some atom of $G$. It implies that the atom has at least $k$ cliques, which contradicts the assumption.
We will consider the following two problems, the first is posed for a given fixed $h \geq 4$.

**RECOGNIZING HELLY $[h, 2, 2]$ GRAPHS**

Input: A connected graph $G$.
Question: Does $G$ belong to Helly $[h, 2, 2]$?

**CHEAPEST REPRESENTATION**

Input: A connected graph $G$.
Goal: Determine the minimum $h \geq 2$ such that $G \in$ Helly $[h, 2, 2]$.

Clearly an efficient solution of the latter implies an efficient solution of the former.

**Theorem 17** The problem CHEAPEST REPRESENTATION is polynomial times solvable.

**Proof.** Using the efficient algorithm described in [10], determine whether the given graph $G$ belongs to Helly EPT or not. If it does then determine for each atom $G_i$ of $G$ its number of cliques, say $k_i$. Notice that it can be done efficiently since the total number of cliques of a Helly EPT graphs $G$ is at most $\left\lfloor \frac{3|V(G)|-4}{2} \right\rfloor$ [10]. Let $k$ be the maximum $k_i$.

If $k \leq 3$, then every atom is chordal which implies $G \in$ Chordal $\cap$ EPT = $[3, 2, 2]$ (see [10] and [4]). Now test whether $G$ is an interval graph or not and answer $h = 2$ in an affirmative case and $h = 3$ otherwise.

If $k \geq 4$, by Theorem [16] $G \in$ Helly $[k, 2, 2]$ and $G \notin$ Helly $[k-1, 2, 2]$, thus let $h = k$. 

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