Blackfolds in Supergravity and String Theory

Roberto Emparan\textsuperscript{a,b}, Troels Harmark\textsuperscript{c}, Vasilis Niarchos\textsuperscript{d}, Niels A. Obers\textsuperscript{e}

\textsuperscript{a}Institució Catalana de Recerca i Estudis Avançats (ICREA)  
Passeig Lluís Companys 23, E-08010 Barcelona, Spain
\textsuperscript{b}Departament de Física Fonamental and Institut de Ciències del Cosmos,  
Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona
\textsuperscript{c}NORDITA, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden
\textsuperscript{d}Crete Centre for Theoretical Physics, Department of Physics, University of Crete, 71003
\textsuperscript{e}The Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark

emparan@ub.edu, harmark@nordita.org, niarchos@physics.uoc.gr, obers@nbi.dk

Abstract

We develop the effective worldvolume theory for the dynamics of black branes with charges of the kind that arise in many supergravities and low-energy limits of string theory. Using this theory, we construct numerous new rotating black holes with charges and dipoles of D-branes, fundamental strings and other branes. In some instances, the black holes can be dynamically stable close enough to extremality. Some of these black holes, such as those based on the D1-D5-P system, have extremal, non-supersymmetric limits with regular horizons of finite area and a wide variety of horizon topologies and geometries.
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1 Introduction

In [1, 2] we have developed an effective worldvolume theory, called the blackfold approach, for the dynamics of black branes on length scales larger than the thickness of the brane. These initial works focused on the simplest case of vacuum black branes, but it is natural to extend the approach to black branes with charges. Such branes play an important role in higher-dimensional supergravity theories, especially in those that arise in the low energy limit of string/M-theory. Of particular interest for string theory are black \( p \)-branes that carry charges of a Ramond-Ramond field strength \( F_{(p+2)} \). The extremal supersymmetric limit of these black branes is associated with the supergravity field of (stacks of) D-branes. Many aspects of the worldvolume dynamics of these (locally) supersymmetric branes are appropriately captured by the Dirac-Born-Infeld action. The fundamental strings (F1) and Neveu-Schwarz five-branes (NS5), as well as the M2 and M5 branes of M-theory often play similar roles. In supergravity it is natural to consider not only the supersymmetric branes with these charges, but also their ‘blackened’ versions, with non-extremal horizons of finite temperature. Thus we seek a theory of the worldvolume dynamics of black D-branes and other charged black objects.

Besides carrying a RR \( p \)-brane charge, \( D_p \)-branes can also carry ‘dissolved’ in their worldvolumes other \( q \)-branes, with \( q \leq p \). The blackfold techniques can naturally accommodate these, by considering black branes with the charges of \( q \)-branes, \( 0 \leq q \leq p \), that are sources of \((q + 2)\)-form gauge field strengths \( F_{(q+2)} \). Then the worldvolume contains a conserved \( q \)-brane number current. Some of these have been studied in [3], which develops the blackfold formalism for branes with only 0-brane and 1-brane charges and constructs many new classes of charged black holes. It must be borne in mind that, when considering blackfolds with a spatially compact worldvolume in a globally flat Minkowski background, the currents that source the field \( F_{(q+2)} \) give rise to a net conserved charge only when \( q = 0 \). When \( q \geq 1 \) the charge is of dipole type. So, for instance, out of a black \( D_p \)-brane we construct black holes with \( D_p \) dipole. It is worth noting that the presence of the latter typically contributes to the stability of the black hole. In particular, close enough to extremality this charge can suppress the Gregory-Laflamme-type instability that afflicts neutral blackfolds.

Quite generally, the blackfold techniques are the appropriate tool for the study of configurations of D-branes in the probe approximation, in the case that the D-brane worldvolume theory has a thermal population of excitations. The blackfold gives a gravitational description of this thermally excited worldvolume, with a horizon that on short scales is like that of the straight black D-brane. Like in the AdS/CFT correspondence, this gravitational description of the worldvolume theory is appropriate when there is a stack of a large number of D-branes (although not so large as to cause a strong backreaction on the background) and the theory is strongly coupled. Refs. [4] applied these methods to study a thermal version of the D3-brane BIon.

Since there is a large array of possible \( q \)-brane charges that a black \( p \)-brane can support, for the sake of clarity we confine ourselves mostly to a selection of simple and potentially relevant
cases. Thus, we begin our analysis in section 2 with the study of black $p$-folds with only $p$-brane charge. This kind of charge is not only relevant for (unsmeared) $Dp$-branes, but it also presents some qualitative differences with lower brane charges. The formalism allows us to obtain, in section 3, solutions for thermal $D$-branes with a worldvolume whose spatial geometry has the shape of products of spheres, and in section 4 to study the conditions for stability under worldvolume perturbations. The extremal limit of the solutions exhibits a number of interesting features which are not captured by the DBI theory.

In section 5 we move on to study the more general case of black $p$-branes with $q$-brane currents, $0 \leq q \leq p$. Here our study of the equations is less detailed and exhaustive, but much of our analysis is quite general and allows for an arbitrary number of currents. Again, we find large classes of solutions on products of spheres. Then we proceed to illustrate the method on specific two-charge systems: $D0$-$Dp$ and $F1$-$Dp$ (section 6) and the extremal $D1$-$D5$-$P$ system (section 7). Our analysis of the latter is concise, to avoid being repetitious, but its importance comes from the fact that it gives new extremal (but non-supersymmetric) black holes with a regular horizon with a wide variety of topologies, and which are potentially stable. Thus they show how the blackfold techniques can uncover new classes of black holes that have many of the properties of the solutions that have become our best laboratory to study black holes in string theory. Aspects of the string microphysics of our solutions are addressed in the concluding section 8.

Our notation follows [2], but there are new features from the inclusion of charge currents that may be worth spelling out:

- $Q_q$ is the local charge density of $q$-branes on the worldvolume of the $p$-brane; $Q_q$ is the total integrated (dipole) charge on the blackfold. Note that $Q_p = Q_p$.
- $\Phi_q$ is the local potential conjugate to $Q_q$ on the brane; $\Phi_H^{(q)}$ is the global potential, conjugate to $Q_q$, for the black hole.
- $h_{ab}^{(q)}$ is the metric on the worldvolume along the $q$-brane current.

## 2 Blackfolds with $p$-brane charge

### 2.1 Perfect fluids with conserved $p$-brane charge

We want to describe the dynamics of a perfect fluid that lives in a $(p+1)$-dimensional worldvolume $W_{p+1}$ and carries a $p$-brane current

$$J = Q_p \hat{V}_{(p+1)}.$$  \hfill (2.1)

Here $\hat{V}_{(p+1)}$ is the volume form on $W_{p+1}$. We assume that this current is conserved (since it sources a gauge field in the target background spacetime of the brane), so the charge must be constant along the worldvolume,

$$\partial_a Q_p = 0.$$  \hfill (2.2)
Thus $Q_p$ is not a collective variable of the fluid: there are no modes in the worldvolume that describe local fluctuations of this charge, so there is no local degree of freedom associated to it.

Then the collective coordinates of the fluid are the same as those for a neutral fluid, namely the local fluid velocity $u$ and the energy density $\varepsilon$. The presence of $Q_p$ manifests itself in the equation of state of the fluid, where it enters as a parameter. In addition to the intrinsic variables $\varepsilon$ and $u^a$, the worldvolume geometry is characterized by the embedding coordinates $X^\mu(\sigma^a)$, which determine the induced metric $\gamma_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$.

The perfect fluid is characterized by its isotropic stress-energy tensor

$$T^{ab} = (\varepsilon + P)u^a u^b + P\gamma^{ab},$$

and the equation of state, which will be specified later. Locally it satisfies the thermodynamic relations

$$d\varepsilon = T ds, \quad \varepsilon + P = Ts,$$

with $T$ the local temperature and $s$ the entropy density. Note that $Q_p$, which as we saw is not a local variable, does not appear here, although we will see later in what sense it can play a role in the thermodynamics.

The general analysis is very similar to that of perfect neutral fluids: the fluid equations $D_a T^{ab} = 0$ decompose into components parallel (timelike) and transverse (spacelike) to $u^a$. The former gives the energy continuity equation, which for a perfect fluid, and using (2.4), is equivalent to the conservation of entropy

$$D_a (su^a) = 0.$$  \hfill (2.5)

The (spacelike) Euler force equations relate the fluid acceleration along the worldvolume, $\dot{u}^a = u^b D_b u^a$, to the pressure gradient and, through the thermodynamic identity $dP = s dT$, to the temperature gradient so that

$$(\gamma^{ab} + u^a u^b)(\dot{u}_b + \partial_b \ln T) = 0.$$  \hfill (2.6)

The worldvolume of the fluid is also dynamical since we allow fluctuations of its embedding $X^\mu(\sigma^a)$ in the background. Its shape is captured by the first fundamental form $h_{\mu\nu} = \gamma^{ab} \partial_a X^\mu \partial_b X^\nu$ and the extrinsic curvature tensor $K_{\mu\nu}^\rho = h_\mu^\lambda h_\nu^\sigma \nabla_\sigma h_\lambda^\rho$, with mean curvature vector $K^\rho = h^{\mu\nu} K_{\mu\nu}^\rho$. This elastic dynamics is captured by Carter’s equations \cite{5}

$$K_{\mu\nu}^\rho T^{\mu\nu} = \frac{1}{(p + 1)!} \perp_{\sigma} \gamma^{\rho \mu_0 ... \mu_p} F^{\mu_0 ... \mu_p \sigma}.$$  \hfill (2.7)

Here $\perp_{\mu} = \delta_{\mu} - h_{\mu}^\rho$ is the projector orthogonal to the worldvolume, and for generality we include a force from the coupling of the current $J$ to a background $(p + 2)$-form field strength

---

\footnote{On a black brane, this charge can fluctuate on the horizon in directions transverse to the worldvolume, on the small sphere $s^{n+1}$, but these are non-hydrodynamic modes of the black brane that are not included here.}
However, in this article we set this background field to zero. Then, for the perfect fluid stress tensor \( \tau_{\mu \nu} \) these equations become

\[
-P K^\rho = \perp^\rho_\mu \, s T u^\mu .
\]  

This equation determines the acceleration of the fluid along directions orthogonal to the worldvolume as an effect of forcing by the extrinsic curvature.

### 2.1.1 Stationary solutions

In order to proceed further, we restrict ourselves to stationary configurations, for which the fluid velocity is aligned with a Killing vector \( k \) along the worldvolume. Moreover, we assume that \( k = k^\mu \partial_\mu \) generates isometries not only of the worldvolume but also of the background spacetime. Thus

\[
\dot{u}_\mu = \partial_\mu \ln |k| , \quad \nabla (\mu k_\nu) = 0 ,
\]  

and \( \dot{u}_\mu = \partial_\mu \ln |k| \). This determines completely the solution to the intrinsic fluid equations, since the local temperature is obtained by simply redshifting the global temperature \( T \), which is uniform on the worldvolume,

\[
T(\sigma^a) = \frac{T}{|k|} .
\]  

Since now \( s T \dot{u}_\mu = -s \partial_\mu T = -\partial_\mu P \), the extrinsic equation \((2.8)\) is

\[
K^\rho = \perp_{\mu} \partial_\mu \ln(-P) .
\]  

\((P \) is negative in the examples we consider\). This equation can be obtained by extremizing the action

\[
\tilde{I} = \int_{W_{p+1}} d^{p+1} \sigma \sqrt{-\gamma} \, P
\]  

for variations of the brane embedding among stationary fluid configurations with the same \( Q_p \).

In order to compute the physical magnitudes of the brane configuration for any given embedding, we must first specify how \( k \) is related to the background Killing vector \( \xi \) that defines unit-time translations at asymptotic infinity. Without loss of generality we write

\[
k = \xi + \Omega \chi
\]  

where \( \chi \) is a spacelike Killing vector, typically a generator of rotations, orthogonal to \( \xi \), and \( \Omega \) a constant that gives the (angular) velocity relative to orbits of \( \xi \). We assume that on the worldvolume \( W_{p+1} \) the vector \( \xi \) is orthogonal to spacelike hypersurfaces \( \mathcal{B}_p \), with unit normal

\[
n^a = \frac{1}{R_0} \xi^a |_{W_{p+1}} .
\]  

\(^2\)It is easy, but slightly cumbersome, to extend this to the case where the normal to \( \mathcal{B}_p \) is not parallel to the generator of asymptotic time translations \([6]\).
The factor $R_0(\sigma) = -n^b \xi_b$ measures the local gravitational redshifts between points on $\mathcal{W}_{p+1}$ and asymptotic infinity (where $\xi^2 \to -1$). It is also convenient to introduce the rapidity $\eta$ of the fluid velocity relative to the worldvolume time generated by $n^a$, such that

$$-n^a u_a = \cosh \eta.$$  \hfill (2.15)

If we denote $R^2 = |\chi^a \chi_a|_{\mathcal{W}_{p+1}}$ then

$$\tanh \eta = \frac{\Omega R}{R_0}.$$  \hfill (2.16)

Integrations on the worldvolume $\mathcal{W}_{p+1}$ reduce, over an interval $\Delta t$ of the Killing time generated by $\xi$, to integrals over $\mathcal{B}_p$ with measure $dV_{(p)}$ as

$$\int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \sqrt{-\gamma} (\ldots) = \Delta t \int_{\mathcal{B}_p} R_0 dV_{(p)} (\ldots).$$  \hfill (2.17)

The mass, angular momentum and entropy are now obtained in a conventional manner as integrals over $\mathcal{B}_p$,

$$M = \int_{\mathcal{B}_p} dV_{(p)} T_{ab} n^a \chi^b, \quad J = -\int_{\mathcal{B}_p} dV_{(p)} T_{ab} n^a \xi^b, \quad S = -\int_{\mathcal{B}_p} dV_{(p)} s u_a n^a.$$  \hfill (2.18)

For the stress tensor (2.3), and using (2.14) and (2.15), it is easy to find the general expressions for the integrands of these magnitudes in terms of $\varepsilon$, $P$, $s$ and $\eta$.

### 2.1.2 Action principles: fixed charge and fixed potential

Contracting (2.3) with $n_a k_b$ and using (2.4) we find

$$T_{ab} n^a (\xi^b + \Omega \chi^b) + Ts n^a u_a = n^a k_a P = -R_0 P.$$  \hfill (2.19)

This can be regarded as a version of (2.4) where we take into account fluid motion and gravitational redshifts. Integrating over $\mathcal{B}_p$ and using (2.17) we obtain the action (2.12) in the form

$$\tilde{I} = -\Delta t (M - \Omega J - TS).$$  \hfill (2.20)

We can rotate to Euclidean time, $it \to \tau$, with periodicity $\Delta \tau = \beta = 1/T$, and find that the Euclidean action, $i\tilde{I} \to -\tilde{I}_E$ is

$$\tilde{I}_E = \beta (M - \Omega J - TS),$$  \hfill (2.21)

i.e., the thermodynamic potential at constant $T$ and $\Omega$, with $Q_p$ fixed.

Since $T$ and $\Omega$ are integration constants, the extrinsic equations (2.11) are equivalent to requiring that

$$dM = T dS + \Omega dJ \quad \text{(fixed $Q_p$)}.$$  \hfill (2.22)

This is, we extremize the action for variations among configurations with the same value of $T$, $\Omega$, and the charge $Q_p$.

---

3For instance this factor is non-trivial for blackfolds in AdS backgrounds [7].
Note that $Q_p$ is not a worldvolume density but a global quantity for the fluid on the brane. Even if there is no local, intrinsic fluid degree of freedom on the worldvolume associated to this charge, it is one of the conserved total charges of the configuration, together with $M$ and $J$. Thus we should be able to consider variations among stationary fluid brane configurations with different charges. This requires the existence, for stationary branes, of a global potential $\Phi_H^{(p)}$ conjugate to $Q_p$. In order to identify it, note that the local equation of state that relates $\varepsilon$ to $s$ does also depend on $Q_p$ as a parameter that specifies the entire fluid, $\varepsilon(s; Q_p)$. Thus on the fluid worldvolume we can introduce a local potential $\Phi_p(\sigma^a)$ as

$$\Phi_p(\sigma^a) = \frac{\partial \varepsilon(s; Q_p)}{\partial Q_p}.$$  (2.23)

Since $Q_p$ couples to the field strength $F_{[p+2]}$ of the background, this $\Phi_p$ actually corresponds to the potential of $F_{[p+2]}$ on the worldvolume (for a black brane, this quantity is well defined without any arbitrary additive constants [8, 9]). We introduce the global potential

$$\Phi_H^{(p)} = \int_{B_\rho} dV(p) R_0 \Phi_p(\sigma^a)$$  (2.24)

by integrating over spatial directions of the worldvolume taking into account the local redshift $R_0$ relative to infinity.

We can now reformulate the variational principle for stationary solutions using this potential. Locally, we introduce the Gibbs free energy density

$$\mathcal{G} = \varepsilon - T s - \Phi_p Q_p = -P - \Phi_p Q_p$$  (2.25)

which, for hydrodynamic fluctuations for which $Q_p$ is necessarily constant along the worldvolume, satisfies

$$d\mathcal{G} = -sdT - Q_p d\Phi_p = -dP - Q_p d\Phi_p.$$  (2.26)

The extrinsic equations (2.11) can now be written as

$$(\mathcal{G} + \Phi_p Q_p) K^\rho = \perp^{\mu\nu} (\partial_\mu \mathcal{G} + Q_p \partial_\mu \Phi_p).$$  (2.27)

Let us consider variations of the worldvolume embedding where, instead of $Q_p$, we keep constant the potential $\Phi_H^{(p)}$ in (2.24). To achieve this, the local potential $\Phi_p(\sigma^a)$ must vary in such a way that

$$\perp^{\mu\nu} \partial_\mu \Phi_p(\sigma^a) = \Phi_p(\sigma^a) K^\rho$$  (2.28)

(see eq. (A.40) of [2]). Then, the extrinsic equations (2.27) take the form

$$K^\rho = \perp^{\mu\nu} \partial_\mu \ln \mathcal{G}$$  (2.29)

\footnote{Note that $T$ is a true independent local variable of the fluid, which characterizes local fluctuations in the energy density of the fluid, but $\Phi_p(\sigma^a)$ is not: it measures the response of the fluid to an external source that \textit{globally} changes the fluid’s $Q_p$, which remains constant over the worldvolume. So, for the intrinsic dynamics of the fluid, this ‘local grand-canonical ensemble’ is a phony one: $\mathcal{G}$ is only a function of $T$ since $\Phi_p$ is determined by the condition that $Q_p$ remains constant.}
which we can derive by extremizing the action

\[ I = - \int_{W_{p+1}} d^{p+1} \alpha \sqrt{-\gamma} \mathcal{G} \]  

(2.30)

for variations of the embedding among stationary fluid configurations where now \( T, \Omega \) and \( \Phi^{(p)}_H \) are kept fixed. By integrating (2.25) and going to Euclidean time we recover the expected Legendre-transform-type relations

\[ I_E[T, \Omega, \Phi^{(p)}_H] = \beta \left( M - \Omega J - TS - \Phi^{(p)}_H Q_p \right) \]

\[ = \tilde{I}_E[T, \Omega, Q_p] - \beta \Phi^{(p)}_H Q_p , \]  

(2.31)

and \( \beta \Phi^{(p)}_H = \partial \tilde{I}_E / \partial Q_p \) which, consistently, justifies the way we have defined \( \Phi^{(p)}_H \) in (2.24) in terms of worldvolume quantities. The extrinsic equations are now equivalent to the complete form of the global first law,

\[ dM = TdS + \Omega dJ + \Phi^{(p)}_H dQ_p . \]  

(2.32)

### 2.2 Blackfold effective fluid

In order to apply the previous formalism to a specific kind of brane we need to know the equation of state of the effective fluid that lives in its worldvolume. Here we consider the charged dilatonic black \( p \)-branes solutions of the action

\[ I = \frac{1}{16 \pi G} \int d^D x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2(p+2)!} e^{\alpha \phi} F_{(p+2)}^2 \right) \]  

(2.33)

with general parameters \( D, p \). Defining

\[ n = D - p - 3 , \quad a^2 = \frac{4}{N} - \frac{2(p+1)n}{D-2} \]  

(2.34)

the flat black \( p \)-brane solution reads

\[ ds^2 = H^{-\frac{Na}{D-2}} \left( - f dt^2 + \sum_{i=1}^{p} dx_i^2 \right) + H^{\frac{N(p+1)}{D-2}} \left( f^{-1} dr^2 + r^2 d\Omega^2_{n+1} \right) , \]

(2.35)

\[ e^{2\phi} = H^{aN} , \quad A_{(p+1)} = \sqrt{N} \coth \alpha (H^{-1} - 1) dt \wedge dx_1 \wedge \cdots \wedge dx_p , \]  

(2.36)

with

\[ H = 1 + \frac{r_0^2 \sinh^2 \alpha}{r_n} , \quad f = 1 - \frac{r_0^2}{r_n} . \]  

(2.37)

Note that since it must be that \( a^2 \geq 0 \) (otherwise the dilaton would be a ghost), the parameter \( N \) cannot be arbitrarily large but is bounded by

\[ N \leq 2 \left( \frac{1}{n} + \frac{1}{p+1} \right) . \]  

(2.38)

In string/M-theory, \( N \) is typically an integer up to 3 (when \( p \geq 1 \)) that corresponds to the number of different types of branes in an intersection. The D/NS-branes in type II string theory
are obtained for $D = 10$, $N = 1$, $p = 0, \ldots, 6$. The M-branes in M-theory arise for $D = 11$, $N = 1$, $p = 2, 5$.

It is straightforward to compute the properties of the effective stress tensor of the fluid as functions of $r_0$ and $\alpha$,

$$
\varepsilon = \frac{\Omega (n+1)}{16\pi G} r_0^n \left(n + 1 + nN \sinh^2 \alpha \right), \quad P = -\frac{\Omega (n+1)}{16\pi G} r_0^n \left(1 + nN \sin^2 \alpha \right), \quad (2.39)
$$

$$
\mathcal{T} = \frac{n}{4\pi r_0 (\cosh \alpha)^N}, \quad s = \frac{\Omega (n+1)}{4G} r_0^{n+1} (\cosh \alpha)^N, \quad (2.40)
$$

and

$$
Q_p = \frac{\Omega (n+1)}{16\pi G} n \sqrt{N} r_0^n \sinh \alpha \cosh \alpha, \quad \Phi_p = \sqrt{N} \tanh \alpha. \quad (2.41)
$$

The potential $\Phi_p$ is measured from the difference between the values of $A_{(p+1)}$ at the horizon and at $r \to \infty$ in (2.36). The Gibbs free energy density takes a particularly simple form

$$
\mathcal{G} = \frac{\Omega (n+1)}{16\pi G} r_0^n. \quad (2.42)
$$

If we eliminate $r_0$ and $\alpha$ from these expressions we obtain the equation of state that relates $\varepsilon$ to $s$ and $Q_p$. However it is convenient to retain $r_0$ and $\alpha$, not only as a useful parametrization, but also for relating to the geometrical properties of the black hole solutions: $r_0$ and $\alpha$ determine the geometry in the region close to the black brane. Thus they give information about the short-scale structure of the black holes.

A number of simple relations can be found among physical quantities which, although not independent of the local thermodynamic relations and the equation of state, are illustrative. The relation

$$
\varepsilon = -(n+1)P - n \Phi_p Q_p, \quad (2.43)
$$

says that the effect of the charge is to provide an additional tension on the worldvolume. This also becomes apparent in the extrinsic equation

$$
K^\rho = \frac{n \mathcal{T} s}{\mathcal{T} s + n \Phi_p Q_p} \perp^\rho \mu \dot{u}^\mu = \frac{n}{1 + nN \sinh^2 \alpha} \perp^\rho \mu \dot{u}^\mu, \quad (2.44)
$$

according to which the acceleration required in order to balance a given extrinsic curvature increases when charge is present.

An equation equivalent to (2.43) gives the Gibbs free energy density as

$$
\mathcal{G} = \frac{1}{n} \mathcal{T} s, \quad (2.45)
$$

which, taking $\beta = 1/T$, implies that

$$
I_E = \frac{1}{n} S. \quad (2.46)
$$

Using relations analogous to (2.19), we can easily prove that

$$
(D - 3)M - (D - 2)(TS + \Omega J) - n \Phi_{H(p)} Q_p = \mathcal{T}_{\text{tot}} \quad (2.47)
$$
where the total tensional energy is

$$\mathcal{T}_{tot} = - \int_{B_p} dV(p) R_0 (\gamma^{ab} + n^a n^b) T_{ab}. \quad (2.48)$$

In the case of a Minkowski background the black hole will be asymptotically flat and it must satisfy a Smarr relation. This is precisely of the form of eq. (2.47) with $\mathcal{T}_{tot} = 0$. This condition is therefore necessary for equilibrium in these backgrounds, and must follow after solving the extrinsic equations. We will see how it can be used for an efficient determination of the equilibrium value of the rapidity $\eta$.

Observe that the effective stress-energy tensor can be written as

$$T_{ab} = T_s \left( u_a u_b - \frac{1}{n} \gamma_{ab} \right) - \Phi_p Q_p \gamma_{ab} \quad (2.49)$$

We will see later that this form is generic for charged blackfold fluids. It suggests that the stress tensor has a brane-tension component $\propto -\Phi_p Q_p$, and a thermal component $\propto T_s$. However, one must bear in mind that this is not a decomposition into ground-state and thermal-excitation energies: the potential $\Phi_p$ changes depending on the temperature of the system. The actual ground state, discussed below, has the uniform value $\Phi_p = \sqrt{N}$ at zero temperature.

Finally, let us recall that the charge $Q_p$ is a dipole charge of the black hole constructed as a blackfold with compact $B_p$. This is not a conserved asymptotic charge, and the definition of its conjugate potential $\Phi^{(p)}_H$ is somewhat subtle. Global issues such as those found in refs. [8, 9] are presumably relevant in our constructions too (e.g., at the center of the $S^p$ in the configurations of sec. 3 below, when $n = 1$), but we do not address them here. The fact that our definition of $\Phi^{(p)}_H$ leads to the expected form of black hole thermodynamic relations gives us confidence that these issues can be satisfactorily handled.

### 2.3 No boundaries

In contrast to neutral blackfolds, and also to blackfolds with lower-form currents, blackfolds with $p$-brane charges do not admit open boundaries: the charge $Q_p$ would not be conserved at them. While it is possible to consider that the blackfold ends on another brane that carries the requisite charge (and if this is a black brane, then the boundary will be one where $R_0 = 0$), we will not consider such situations in this article.

This rules out, in particular, the existence of blackfold disk and ball solutions of the type obtained in [10, 3].

### 2.4 Extremal limits

The black branes in (2.35) have extremal limits in which the horizon becomes degenerate. We can therefore consider blackfolds in which the horizon approaches this limit uniformly over all of $B_p$. This is in contrast with the situation in [10, 3] where the extremal limit was reached locally at the boundaries of $B_p$, while the rest of $B_p$ remained non-extremal. In the present case there can be no boundaries so this possibility is absent.
For most of the black branes in (2.35) the horizon becomes singular in these extremal limits, but these are ‘good singularities’ in the sense explained in [11]: we can approach them as limits of solutions with smooth non-degenerate horizons, so the singularity can be regarded as the result of integrating out short-distance degrees of freedom that can be thermally excited. Thus these solutions have sensible physical interpretations. The blackfold method can be applied to them, even if the intrinsic dynamics is not a fluid dynamics anymore.

2.4.1 Dirac branes

The straightforward limit to an extremal solution is obtained by taking
\[ r_0 \to 0, \quad \alpha \to \infty \]
while keeping the charge \( Q_p \) fixed. In the limit, \( Ts \to 0 \) and \( \Phi_p \to \sqrt{N} \) so
\[ \varepsilon = -P = \sqrt{N}Q_p. \]
Since \( Q_p \) is constant on the worldvolume, it follows that the energy density and pressure must be uniform too. The stress tensor
\[ T_{ab} = P\gamma_{ab} \]
is that of a brane with uniform tension, whose action is proportional to the volume of \( W_{p+1} \), as studied by Dirac. In the cases where the charge is that of a D-brane, this is the same as the Dirac-Born-Infeld action with vanishing Born-Infeld gauge fields. Thus we refer to them as Dirac branes.

Since these branes have local Lorentz invariance on the worldvolume, the velocity field is pure gauge and is not a physical collective mode anymore. The only dynamics is extrinsic, and in the absence of external form fields, is given by the minimal-surface equations
\[ K^{\rho} = 0. \]
The absence of compact embedded minimal surfaces in Euclidean space implies that in a flat Minkowski background there are no solutions to these equations that would correspond to static extremal black holes with \( p \)-brane dipole. This is in the same spirit as the no-dipole-hair theorem of [12], which forbids a large class of non-extremal static dipole black holes.

2.4.2 Null-wave branes

A more interesting limit results if, in addition to (2.50), we scale the velocity field components to infinity in such a way that \( r_0^{n/2}u^a \) remains finite. Let us introduce then a ‘momentum density’ \( \mathcal{K} \) and a vector \( l \) that remain finite in the limit,
\[ \left( \frac{\Omega^{(n+1)}_{n} r^2}{16\pi G n r_0^n} \right)^{1/2} u^a = \mathcal{K}^{1/2}l^a. \]
Thus, when $r_0 \to 0$,

$$l_a l^a = 0$$  \hspace{1cm} (2.55)\

so $l$ is a lightlike vector: the fluid on the brane is boosted to the speed of light. The stress tensor becomes

$$T_{ab} = K l_a l_b - \sqrt{N} Q_p \gamma_{ab}.$$  \hspace{1cm} (2.56)\

Thus, in addition to the tensional, ground state component $\propto Q_p$, the brane supports a null momentum wave with momentum density $K$, without breaking locally the translational invariance along the wave. This stress tensor cannot be obtained from the DBI action. Since $\varepsilon + P = T S$ vanishes in the limit, we are also outside the realm of proper fluid dynamics.

The dynamics of these extremal null-wave branes is in some respects simpler than for non-extremal branes, so we will briefly describe some aspects of it. As is customary when dealing with null congruences generated by a vector $l^a$, we introduce another null vector $\bar{l}^a$ such that

$$\bar{l}_a \bar{l}^a = 0, \quad \bar{l}_a l^a = -1, \quad l_a D_a \bar{l}^b = 0,$$  \hspace{1cm} (2.57)\

and the projector onto the $(p-1)$-dimensional orthogonal space

$$q_{ab} = \gamma_{ab} + l_a \bar{l}_b + \bar{l}_a l_b.$$  \hspace{1cm} (2.58)\

The intrinsic equations $D_a T^{ab} = 0$ are then projected along $l^a$, $\bar{l}^a$, and $q_{ab}$. The equation $l_b D_a T^{ab} = 0$ is automatically satisfied, while $\bar{l}_b D_a T^{ab} = 0$ implies

$$D_a (K l^a) = 0$$  \hspace{1cm} (2.59)\

so $K l^a$ is a conserved momentum current. Finally, $q_{ab} D_a T^{ab} = 0$ gives

$$l^a D_a \bar{l}^b = 0,$$  \hspace{1cm} (2.60)\

so $l^a$ generates geodesics along the worldvolume. However, these need not be geodesics of the background, and indeed the acceleration of $l$ transverse to the worldvolume appears in the extrinsic equations

$$\mathcal{K} \nabla_\mu l^\nu \nabla_\nu l^\mu = \sqrt{N} Q_p K^\rho$$  \hspace{1cm} (2.61)\

as balancing the tension from extrinsic curvature.

Since we are not dealing with a worldvolume fluid, the conditions for absence of dissipative effects in stationary solutions are less clear than in non-extremal blackfolds. In particular, it does not seem to be required that the the null vector $l^a$ is a Killing vector on $W_{p+1}$ — although this is presumably a necessary requirement for horizon regularity when the extremal brane has a horizon with finite area [13].

Nevertheless, there are some general conditions that are satisfied in Minkowski backgrounds, with $R_0 = 1$. As we have seen, stationary equilibrium in these cases requires that the tensional energy [2.48] vanish. Applied to (2.56), and taking $l^a$ to be normalized so that $n^a l_a = -1$, the condition $T_{\text{tot}} = 0$ relates the averaged momentum density to the charge as

$$\frac{1}{V(p)} \int_{B_p} dV(p) \mathcal{K} = p \sqrt{N} Q_p.$$  \hspace{1cm} (2.62)
Furthermore, since the mass (2.18) is
\[ M = \int_{B_p} dV_{(p)} K + \sqrt{N} V_{(p)} Q_p, \] (2.63)
we can derive that
\[ \frac{1}{p+1} M = \sqrt{N} V_{(p)} Q_p = \frac{1}{p} \int_{B_p} dV_{(p)} K. \] (2.64)

These expressions are of the kind previously obtained in [3], and they show how the total energy \( M \) of the brane in equilibrium is virialized between kinetic energy and potential (charge-tensional) energy. For fundamental strings \( (p = 1, N = 1) \), (2.62) says that at equilibrium the momentum and winding numbers must be equal [14].

The angular momentum in the direction of the rotational vector \( \chi^a \) is
\[ J = \int_{B_p} dV_{(p)} R K \] (2.65)
where
\[ R = \chi^a l_a \] (2.66)
has the interpretation of ‘lever-arm’ radius for the momentum. In cases where this radius is constant over all of \( B_p \), like in the round odd-spheres discussed later, we obtain
\[ \frac{1}{p+1} M = \sqrt{N} V_{(p)} Q_p = \frac{1}{p} J \frac{R}{R}. \] (2.67)

Provided that \( \Omega = 1/R \), this also follows from the extremal limit \( TS \rightarrow 0 \) of the Smarr relation (2.47) and (2.46).

Eqs. (2.64) and (2.67) are correct only to leading order in the blackfold expansion and in general receive corrections at the next orders, since the gravitational and gauge self-interaction of the brane gives rise to Newtonian and Coulombian potential energies that renormalize the mass. This effect has been studied in [14, 15] using the exact black ring solutions of [8, 16].

2.5 Stress-energy of excitations. Near-extremal \( p \)-branes.

As we have seen, the ground state corresponds to the extremal limit with stress-energy tensor (2.52). So the stress tensor for excitations above the ground state of a \( p \)-brane with given charge \( Q_p \) is
\[ T_{ab}^{(\text{exc})} = T_{ab} - T_{ab}^{(\text{ground})} = T_{ab} + \sqrt{N} Q_p \gamma_{ab} \]
\[ = T s \left( u_a u_b + \left( \frac{N \Phi_p}{\sqrt{N} + \Phi_p} - \frac{1}{n} \right) \gamma_{ab} \right). \] (2.68)

When the system is near extremality, \( \Phi_p \sim \sqrt{N} \) and
\[ T_{ab}^{(\text{exc})} \sim T s \left( u_a u_b + \left( \frac{N}{2} - \frac{1}{n} \right) \gamma_{ab} \right). \] (2.69)
This can be regarded as the stress-energy of the thermal gas of excitations of the worldvolume theory on a stack of branes, in the regime (typically at strong coupling) where this theory is appropriately described in terms of a gravitational dual. This stress tensor is traceless when \( N/2 - 1/n = 1/(p+1) \), which implies that the dilaton coupling \( a \) vanishes. Well-known instances of this are the D3, M2, M5, all with \( N = 1 \); strings in six dimensions with \( N = 2 \) (e.g., D1-D5); and strings in five dimensions, with \( N = 3 \) (e.g., M5\perp M5\perp M5).

If we extract an energy density and pressure of excitations from this stress-energy tensor, then, near extremality we find that whenever \( Nn > 2 \) we have \( dP^{(exc)}/d\varepsilon^{(exc)} > 0 \) so the speed of sound is real and the fluid is stable. This includes all the non-dilatonic black branes mentioned above. We will return to this issue of stability in more generality below.

### 3 Odd-sphere blackfolds

The simplest solutions to the equations of the previous section that have spatially compact worldvolume are odd-spheres,

\[
\mathcal{B}_p = S^p, \quad p = 2m + 1
\]

so that the horizon topology is \( S^p \times s^{n+1} \). In the Minkowski background the \( S^p \) are embedded in a \( \mathbb{R}^{p+1} \) subspace

\[
ds^2 = dr^2 + r^2 d\Omega^2_{(p)}
\]

as surfaces at \( r = R \). In principle one may also allow spheres \( S^p \) which are not round, but the solutions can be obtained algebraically only when the radius is constant and the angular momentum is aligned along a diagonal of the Cartan subgroup of rotations of \( S^p \). That is, the angular velocities along the Cartan generators \( \partial/\partial\phi_i \) are all equal and the Killing vector \( \chi \) is

\[
\chi = \frac{\partial}{\partial \phi} = \sum_{i=1}^{m+1} \frac{\partial}{\partial \phi_i}.
\]

Then

\[
u = \frac{1}{\sqrt{1 - \Omega^2 R^2}} \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right).
\]

Since in this case \(|k|\) is homogeneous on \( S^p \), so is the entire solution, and in particular \( r_0 \) and \( \alpha \).

The simplest way to solve the extrinsic equations is by requiring that \( \mathcal{T}_{tot} = 0 \): since all fields are constant on \( S^p \) we need only solve the algebraic equation

\[
\left( \gamma^{ab} + \delta^a_i \delta^b_j \right) T_{ab} = 0.
\]

This is particularly simple in terms of the rapidity \( \eta \) introduced in (2.15), so that \( \tanh \eta = \Omega R \). One obtains

\[
sinh^2 \eta = \frac{p}{n} (1 + nN \sinh^2 \alpha) .
\]

In this form it is apparent that the result from the exact dipole ring solution of [8], with \( p = 1 = n \), is correctly reproduced. One easily sees that (3.3) actually solves (2.44), since
\( K^r = -p/R \) and \( \dot{u}^r = -R^{-1} \sinh^2 \eta \). It is only a little more laborious to verify that this solution extremizes either of the actions (2.12), (2.30) with fixed \( Q_p \) and \( \Phi_H^{(p)} \), respectively. The solution determines the radius of the sphere to be

\[
R = \frac{1}{\Omega} \sqrt{\frac{p(1+nN \sinh^2 \alpha)}{n+p(1+nN \sinh^2 \alpha)}}. \tag{3.7}
\]

However, note that when expressed in terms of \( T \) and \( \Phi_H^{(p)} \), the right hand side of this equation depends on \( R \).

Given the homogeneity of the blackfold solution, the integrals (2.18), (2.24) that give \( M, J, S \) and \( \Phi_H^{(p)} \), are obtained by simply multiplying the corresponding energy densities by the volume \( V(p) = R^p \Omega(p) \) of a round \( p \)-sphere with radius \( R \). We present these results in appendix A.

When the angular velocities along the Cartan generators are not all equal, we expect the spheres to be distorted and have longer circumference along the directions of larger rotation velocity. Compared to the neutral case, the additional tension from the charge increases the rigidity of the brane and will oppose this distortion. Presumably there exists a bound on the size of the ratio between different angular velocities.

**Extremal limit.** Eq. (3.6) clearly shows that the extremal limit \( \alpha \to \infty \) for non-trivial brane solutions (with finite non-zero radius) is also a limit in which the local boost becomes lightlike, \( \eta \to \infty \). Thus we obtain branes with a null momentum wave on them, and \( \Omega R = 1 \). In this limit we can use (2.67) and write \( R = R \) as

\[
R = \frac{p+1}{p} \frac{J}{M} \tag{3.8}
\]

and since the spatial volume is \( V(p) = \Omega(p) R^p \), find a relation between \( M, Q_p, J, \)

\[
Q_p = \frac{1}{\sqrt{N(p+1)\Omega(p)}} \left( \frac{p}{p+1} \right)^p \frac{M^{p+1}}{J^p}. \tag{3.9}
\]

For fixed \( M \), this is a curve in the \((Q_p, J)\) plane that determines the upper bounds on the values that the non-extremal solutions can have: it gives the maximum value of \( J \) for given \( M \) and \( Q_p \), and the maximum charge \( Q_p \) for given \( M \) and \( J \) (and the minimum allowed \( M \) for given \( J \) and \( Q_p \)). It is interesting that it depends on \( p \) but not on \( n \). This relation is correctly reproduced for the exact extremal dipole ring solutions \((p = 1)\) of [8] in the limit of large ring radius.

### 3.1 Products of odd-spheres

The previous solutions can easily be generalized to products of round odd-spheres, \( B_p = \prod_{I=1}^\ell S^{p_I} \), with odd \( p_I \) and \( \sum_{I=1}^\ell p_I = p \), embedded as the surfaces \( r_I = R_I \) in

\[
ds^2 = \sum_{I=1}^\ell \left( dr_I^2 + r_I^2 d\Omega_{(p_I)}^2 \right). \tag{3.10}
\]
A particularly simple case are the tori $\mathbb{T}^p$, where $p_I = 1$. Note that the number of spheres is limited by
\[ \ell \leq n + 2. \tag{3.11} \]

The velocity is taken along $\chi = \sum_{I=1}^{\ell} \Omega_I \partial_{\phi_I}$, where $\partial_{\phi_I}$ are generators of diagonals of the Cartan subgroups of $SO(p_I + 1)$. Then
\[ u = \cosh \eta \left( \frac{\partial}{\partial t} + \sum_{I=1}^{\ell} \Omega_I \frac{\partial}{\partial \phi_I} \right) \tag{3.12} \]
where $\eta$ is the total rapidity so that
\[ \tanh^2 \eta = \sum_{I=1}^{\ell} \Omega_I^2 R_I^2. \tag{3.13} \]

The extrinsic equilibrium, \(2.44\), now involves more than one equation so it cannot be obtained by simply setting $T_{\text{tot}} = 0$. Instead we obtain it from
\[ K^{rI} = -\frac{p_I}{R_I}, \quad \dot{u}^{rI} = -\Omega_I^2 R_I \cosh^2 \eta. \tag{3.14} \]

Eqs. \(2.44\) are now solved by
\[ \Omega_I R_I = \sqrt{\frac{p_I}{p}} \tanh \eta \tag{3.15} \]
where $\sinh^2 \eta$ is given by the same expression as in \(3.6\). More explicitly,
\[ R_I = \frac{1}{\Omega_I} \sqrt{\frac{p_I (1 + nN \sinh^2 \alpha)}{n + p(1 + nN \sinh^2 \alpha)}}. \tag{3.16} \]

Note that there is a larger component of the velocity on spheres $S^{p_I}$ of higher dimensionality — intuitively, the velocity must counterbalance the tension in more directions.

In the extremal limit in which the velocity becomes null, we have
\[ \Omega_I R_I = \sqrt{\frac{p_I}{p}} \tag{3.17} \]
and the null vector is
\[ l = \frac{\partial}{\partial t} + \sum_{I=1}^{\ell} \sqrt{\frac{p_I}{p}} \frac{1}{R_I} \frac{\partial}{\partial \phi_I}. \tag{3.18} \]

Given the homogeneity on $B_p$, eq. \(2.62\) becomes
\[ \mathcal{K} = p \sqrt{NQ_p}, \tag{3.19} \]
and the angular momentum along the direction of rotation of each sphere is
\[ J_I = \int_{B_p} dV_{(p)} \mathcal{K} l_a \left( \frac{\partial}{\partial \phi_I} \right)^a = V_{(p)} \mathcal{K} R_I \sqrt{\frac{p_I}{p}}, \tag{3.20} \]
Table 1: A list of horizon topologies for stationary non-extremal black holes in type IIA/IIB string theory based on the singly-charged blackfolds of the theory with worldvolumes curved into products of odd-spheres. The $s^{n+1}$ denotes the ‘small’ sphere in horizon directions orthogonal to the worldvolume. The number $\ell$ of ‘large’ odd-spheres spanned by the worldvolume is limited by (3.11).

| Brane (IIA) | Worldvolume | $\perp$ Sphere |
|-------------|-------------|----------------|
| F1          | $S^1$       | $s^7$          |
| D2          | $T^2$       | $s^6$          |
| D4          | $S^3 \times S^1, T^4$ | $s^4$          |
| NS5         | $S^5, S^3 \times T^2$ | $s^3$          |
| D6          | $S^3 \times S^3, S^5 \times S^1$ | $s^2$          |

| Brane (IIB) | Worldvolume | $\perp$ Sphere |
|-------------|-------------|----------------|
| D1          | $S^1$       | $s^7$          |
| F1          | $S^1$       | $s^7$          |
| D3          | $S^3, T^3$  | $s^5$          |
| D5          | $S^5, S^3 \times T^2$ | $s^3$          |
| NS5         | $S^5, S^3 \times T^2$ | $s^3$          |

Table 2: The analogue of Table 1 in M-theory for M2 and M5 black branes.

| Brane | Worldvolume | $\perp$ Sphere |
|-------|-------------|----------------|
| M2    | $T^2$       | $s^{\ell}$     |
| M5    | $S^5, S^3 \times T^2, T^5$ | $s^4$          |

where now $V_{(p)} = \prod_I \Omega_{(p)} R^{p_I}_I$. Using (2.63) and eliminating $K$ we find

$$M - \sqrt{N}V_{(p)} Q_p = p\sqrt{N}V_{(p)} Q_p = \left( \sum_{I=1}^{\ell} \frac{J^2_I}{R^2_I} \right)^{1/2},$$

(3.21)

and further eliminating the charge,

$$M = \frac{p+1}{p} \left( \sum_{I=1}^{\ell} \frac{J^2_I}{R^2_I} \right)^{1/2},$$

(3.22)

which generalizes (3.8). Unless all radii are equal there is no expression as simple as (3.9).

Tables 1 and 2 exhibit the limited number of allowed possibilities in this restricted class of solutions in (uncompactified) type IIA/B string theory and M-theory respectively.

Note that, geometrically, the ‘large’ odd-spheres of the horizons are round even when higher-order corrections in the blackfold construction are included. However, the ‘small’ spheres $s^{n+1}$ are round only in the leading test-brane approximation, and will be distorted by the corrections from worldvolume curvature.

**Solutions with Kaluza-Klein circles.** In the previous study the background is globally flat Minkowski spacetime and hence all the cycles that the blackfold wraps are contractible. We can easily extend the analysis to situations where there is a number $p_\circ$ of compact Kaluza-Klein
circles, on which the brane is supported by topology without requiring a centrifugal force to balance. Compactifying on a torus $\sum_{A=1}^{p_o} dx^A dx^A$, for the velocity vector we take

$$u = \frac{1}{\sqrt{1 - \tanh^2 \bar{\eta} - v^2}} \left( \frac{\partial}{\partial \ell} + \sum_{I=1}^{\ell} \Omega_I \frac{\partial}{\partial \phi_I} + \sum_{A=1}^{p_o} v_A \frac{\partial}{\partial x^A} \right), \quad (3.23)$$

with

$$\tanh^2 \bar{\eta} = \sum_{I=1}^{\ell} \Omega_I^2 R_I^2, \quad v^2 = \sum_{A=1}^{p_o} v_A^2. \quad (3.24)$$

The acceleration and extrinsic curvature vanish on the internal torus so the extrinsic equations are non-trivial only in the non-compact directions. The equilibrium conditions on odd-spheres are now

$$\Omega_I R_I = \sqrt{\frac{p_I}{p - p_o}} \tanh \bar{\eta}, \quad (3.25)$$

$$\frac{\sinh^2 \bar{\eta}}{1 - v^2 \cosh^2 \bar{\eta}} = \frac{p - p_o}{n} (1 + n N \sinh^2 \alpha). \quad (3.26)$$

The velocity $v_A$ along the internal torus is arbitrary. When $v^2 = 0$ we simply need to replace $p \rightarrow p - p_o$ in (3.16).

4 Stability

The blackfold approach captures efficiently the potential stability or instability of a black hole at long wavelengths. The general analysis is complicated, but it simplifies for perturbations such that the intrinsic and extrinsic equations decouple. We consider first the stability under intrinsic perturbations whose wavelength is much smaller than the extrinsic length scale $R$, so the curvature of the brane worldvolume can be neglected.

4.1 Sound waves, Gregory-Laflamme and correlated stability

Sound wave instabilities of the effective fluid, which correspond to unstable fluctuations of the thickness of the black brane, have been identified in ref. [2] with the classical Gregory-Laflamme instabilities [17]. In addition, ref. [18] has observed that the inclusion of the first dissipative terms (associated to the contribution of shear and bulk viscosities) provides an impressively good agreement with results obtained by direct study of the Einstein equations, an agreement that improves with increasing dimension $n$.

Here we discuss how the addition of $p$-brane charge affects the stability of a black $p$-brane. Since this charge does not add any local degree of freedom to the fluid, it suffices to study the sound modes and the analysis only differs from that of [2, 18] in the equation of state and parameters of the fluid. In particular, ref. [18] finds that for a generic relativistic fluid with bulk and shear viscosities $\zeta$ and $\eta$, the linearized perturbations with wavenumber $k$ and frequency
\( \omega = -i\Omega \) obey the dispersion relation\(^\text{3}\)

\[
\Omega = \sqrt{-c_s^2 k} - \left[ \left( 1 - \frac{1}{p} \right) \frac{\eta}{s} + \frac{\zeta}{2s} \right] \frac{k^2}{T} + O(k^3). \tag{4.1}
\]

\( c_s \) is the speed of sound

\[
c_s^2 = \left( \frac{\partial P}{\partial \varepsilon} \right)_{Q_p} \tag{4.2}
\]

computed at fixed charge \( Q_p \). An unstable mode exists when \( c_s^2 < 0 \).

**Sound-mode instability.** To identify an instability we need only focus on the first term, linear in \( k \), in (4.1) which arises at the perfect fluid level. Inserting the specific data of a charged dilatonic black \( p \)-brane solution (2.39) we find

\[
c_s^2 = -\frac{1}{n+1} \left[ 1 + \frac{1}{1 + \frac{2 - Nn}{n+1}} \right] \sinh^2 \alpha. \tag{4.3}
\]

Since

\[
\sinh^2 \alpha = \frac{1}{N} \frac{\Phi_p Q_p}{T_s} \tag{4.4}
\]

we can interpret the regimes of large or small \( \alpha \) as regimes where the charge-tensional component of the brane \( \Phi_p Q_p \) is dominant, or subdominant, relative to the thermal component \( T_s \). Observe also that given the upper bound on \( N \), (2.38), and since \( n, p \geq 1 \), we always have \( Nn < 2(n+1) \) so the denominator in (4.3) is always positive. Thus stability requires that \((Nn - 2) \sinh^2 \alpha > 1\), and we find two different situations:

(i) \( Nn \leq 2 \). In this case, \( c_s^2 < 0 \) always and there is no regime of stability, no matter how close to extremality the brane is.

(ii) \( 2 < Nn \). Stability requires

\[
\frac{1}{Nn - 2} < \sinh^2 \alpha. \tag{4.5}
\]

This is a regime where the energy of thermal excitations is sufficiently smaller than the charge tension, which is achieved as extremality is approached.

According to this, black \( p \)-branes in string theory \((N = 1, D = 10)\) are always unstable to sound wave perturbations for \( p \geq 5 \). When \( p < 5 \), they become stable when they are close enough to extremality that (4.5) holds.

The disappearance of the marginally unstable GL zero-mode when (4.5) is satisfied has been verified explicitly for several classes of charged black branes in \([19, 20, 21, 22, 23, 24, 25]\). However, it is important to realize that here we are not (yet) describing these zero modes of finite wavelength, but rather identifying an instability of hydrodynamic modes, with small frequency at very long wavelengths.

---

\(^3\)In this section only, we use \( \eta, \zeta, \Omega \) and \( k \) with a different meaning than in the rest of the article.
**Correlated stability.** The general thermodynamic relation

\[ c_s^2 = \left( \frac{\partial P}{\partial \varepsilon} \right)_{Q_p} = s \left( \frac{\partial T}{\partial \varepsilon} \right)_{Q_p} = \frac{s}{c_{Q_p}} \]  

(4.6)

provides a direct relation between the speed of sound and the specific heat at fixed charge \( c_{Q_p} \). This is in accordance with the Correlated Stability Conjecture \[26\]. In more detail, the blackfold formalism shows that local thermodynamical instabilities are in one-to-one correspondence with long-wavelength, hydrodynamical perturbations of black branes. However, the stability under modes that do not have a hydrodynamic limit (i.e., whose frequency does not vanish as the wavelength goes to infinity) need not be correlated with local thermodynamic stability.

**Viscosity and the complete dispersion relation.** We can now consider the effect of the term quadratic in \( k \) in eq. (4.1). This involves the ratios \( \eta/s \) and \( \zeta/s \) which can be computed with a perturbative analysis of the Einstein equations as in \[18\]. Although we will not perform this analysis in this paper, it is reasonable to anticipate that

\[ \frac{\eta}{s} = \frac{1}{4\pi}, \quad \frac{\zeta}{s} = \frac{1}{2\pi} \left( \frac{1}{p} - c_s^2 \right). \]  

(4.7)

The first relation is the well-known universal value of \( \eta/s \) for event horizons in Einstein gravity theories. The second relation is a similar universal relation for the bulk viscosity proposed in \[27\]. This has been checked to hold in the near horizon limit of the black branes we are considering \[28\] and it may not be unreasonable to expect that it remains valid when computed in the full asymptotically flat geometry. Ref. \[18\] verified this expectation for neutral black branes.

We tentatively assume the validity of the equations (4.7) and proceed to explore their implications. One reason why this can be very interesting is that there are very few results for the dispersion relation of GL modes of \( p \)-brane-charged black branes. Moreover the conventional perturbation analyses have to be redone anew, numerically, for each value of \( n, N \), and the charge. In contrast, our methods give very easily analytical results for all values of these parameters.

Inserting (4.7) into the dispersion relation (4.1) we obtain

\[ \Omega = \sqrt{-c_s^2 k - \frac{1}{4\pi} \frac{k^2}{T} + \mathcal{O}(k^3)} \].  

(4.8)

When \( c_s^2 < 0 \), this equation is expected to be a quantitatively good approximation to the Gregory-Laflamme dispersion relation of unstable charged dilatonic branes at small wavenumbers, namely when \( k/T \ll 1 \).

For neutral black branes (\( \alpha = 0 \)) this extrapolation becomes a better and better approximation to the exact result with increasing \( n \) \[18\] since for these branes \(-c_s^2\) is small when \( n \gg 1 \). More generally, for charged black branes we may also expect that whenever \(-c_s^2\) is small, i.e., close to the stability limit, the dispersion relation is increasingly well approximated by

\[ \Omega \to \sqrt{-c_s^2 k - \frac{1}{4\pi} \frac{k^2}{T}}, \]  

(4.9)
since all unstable wavelengths are much longer than the thermal length. Note, incidentally, that in contrast to neutral black branes, if we keep $N$ fixed and make $n$ large this does not make $c_s$ small.

Numerical results (but with low resolution) are available in [17] for the dispersion relation $\Omega(k)$ of five-dimensional dilatonic strings, whose qualitative behavior appears to agree well with our curves.

**Threshold mode and critical behavior.** The curve (4.9) predicts the appearance of a Gregory-Laflamme threshold mode at

$$\frac{k}{T}|_{\text{thr}} = 4\pi \sqrt{-c_s^2}, \quad (4.10)$$

Numerical results for the threshold mode of several black branes have been computed in [19, 21, 22, 23, 24, 25]. While we have not performed a detailed comparison, plots of $k|_{\text{thr}}$ as a function of the charge appear to show good qualitative agreement with (4.10).

In the case of non-dilatonic branes, refs. [22, 24, 25] have observed that near the critical value of the charge $Q_p^{\text{crit}}$ at which $c_s^2 = 0$, the threshold mode behaves like

$$k|_{\text{thr}} \sim (Q_p - Q_p^{\text{crit}})^{0.5}, \quad (4.11)$$

i.e., a power-law critical behavior with a numerically-determined exponent close to the mean-field value $1/2$. Our analytical result (4.10) yields exactly this exponent, in the regime where we expect our approximations to be accurate. Furthermore, it predicts that it applies also to dilatonic branes.

It would be interesting to have more numerical data to perform more detailed quantitative comparisons of the threshold behavior and of the complete dispersion curve.

**4.2 Elastic stability**

Transverse, elastic perturbations of the worldvolume geometry propagate with speed

$$c_T^2 = \frac{P}{\varepsilon}. \quad (4.12)$$

For a charged dilatonic black $p$-brane solution we find

$$c_T^2 = \frac{1 + nN \sinh^2 \alpha}{1 + n + nN \sinh^2 \alpha} \quad (4.13)$$

which is always positive. In fact, as could be expected, the charge adds to the brane tension and makes the brane more rigid, with the elastic waves approaching lightspeed $c_T \to 1$ as extremality is approached.

Thus we see that, as expected, the addition of $p$-brane charge that cannot be redistributed along the worldvolume enhances the stability of the black $p$-brane under perturbations that would make it inhomogeneous.

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6The result is the same irrespective of whether one fixes the mass, temperature, or $r_0$. 
Although we have not performed a complete analysis of coupled intrinsic-extrinsic perturbations, it seems reasonable to expect that generic $p$-brane-charged blackfolds with $Nn \leq 2$, when they are near-extremal in the sense that $\sinh^2 \alpha \gg 1/(Nn - 2)$, correspond to stable black holes.\footnote{Intrinsic modes that are close to being marginally stable might in principle become unstable through coupling to extrinsic perturbations. This is why we require to be well inside the intrinsic stability regime.}

## 5 Blackfolds with brane currents

D-branes in string theory can support worldvolume currents that correspond to charges of strings or other D-branes ‘dissolved’ in their worldvolume. In the DBI description, they are captured as classical solutions of the Born-Infeld worldvolume gauge fields, \textit{i.e.}, as open-string excitations. In the blackfold approach, instead, they appear as closed-string modes, in very much the same manner as in the AdS/CFT correspondence. The local gauge symmetry on the worldvolume is not present, and the black brane supports a global conserved current that sources the background spacetime gauge field. The effective theory of blackfolds allows to describe thermal excitations of the worldvolume of the D-brane with these currents, something that the classical DBI action cannot do. Moreover, in the same manner as in sec. 2.4.2, the blackfold method can also describe some extremal configurations that lie outside the reach of the DBI action.

Such brane currents greatly expand the possible dynamics of blackfolds and the new classes of black holes that can be constructed out of them. In particular, on a spatially compact worldvolume a $q$-brane current with $q \geq 1$ gives rise to a dipole, while a 0-brane (\textit{i.e.}, a particle) current yields a charge.

In this section we consider generic black $p$-branes that carry $q$-brane currents on their worldvolume. At the perfect fluid level much of the analysis can be done for a multi-charge brane with several $q$-brane currents. At a later point we shall focus mostly on configurations where in addition to the $p$-brane charge there is also $q = 0$ or $q = 1$ brane charge.

### 5.1 Blackfold fluids with $q$-brane currents

We consider the dynamics of a fluid in a worldvolume $W_{p+1}$ that supports one or several conserved $q$-brane currents. In this paper we only sketch the main properties of these theories — many details are similar to [3]. Furthermore, we confine ourselves from the outset to the perfect fluid approximation. This is enough for the purposes of constructing stationary configurations.

Each $q$-brane current foliates $W_{p+1}$ into sub-worldvolumes $C_{q+1} \subset W_{p+1}$. On each of these we consider a unit $(q+1)$-form $\hat{V}_{(q+1)}$ so that the current is

$$J_{(q+1)} = Q_q \hat{V}_{(q+1)}$$

with $Q_q$ being the $q$-brane density. The conservation equation $d * J_{(q+1)} = 0$ requires that $*\hat{V}_{(q+1)} \wedge d * \hat{V}_{(q+1)} = 0$, which is ensured when the currents are hypersurface-forming, as we will assume in what follows.
The currents make the perfect fluid anisotropic, and in particular they induce differences between the pressures in directions parallel and transverse to them. These differences are due to the effective tension $\Phi_q Q_q$ along the current, where $\Phi_q$ is the chemical potential conjugate to the brane density $Q_q$, so the thermodynamic relations

$$d\varepsilon = T \, ds + \sum_q \Phi_q dQ_q,$$

are satisfied locally in the fluid, and the stress-energy tensor can be written as

$$T_{ab} = T s u_a u_b - G \, \gamma_{ab} - \sum_q \Phi_q Q_q h^{(q)}_{ab},$$

The prime in $\sum_q'$ indicates that $q = p$ (the case considered in sec. 2) is excluded from the sum, and is instead included in $\sum_q$. In eq. (5.3) we have introduced

$$G = \varepsilon - T s - \sum_q \Phi_q Q_q$$

as the local Gibbs free energy density and $h^{(q)}_{ab}$ as the projector (induced metric) onto $C_{q+1}$. For instance, $h^{(0)}_{ab} = -u_a u_b$, and $h^{(p)}_{ab} = \gamma_{ab}$.

Note that: (i) the brane densities $Q_q(\sigma)$ are only ‘quasi-local’ in the sense that they are constant along $C_{q+1}$ and can vary only in the $p - q$ directions transverse to the current. In particular, when $q = p$ the charge $Q_p = Q_p$ is a non-dynamical global constant and all the remarks to this effect made in Sec. 2 apply. (ii) The brane currents need not be ‘nested’ in the sense that we need not have $C_{q+1} \subseteq C_{q'+1}$ for two currents with $q \leq q'$. For instance we may have two string currents each along a different direction.

A general relation valid for the black branes of IIA/IIB/11D supergravities and their toroidal compactifications, is

$$\varepsilon = \frac{n+1}{n} T s + \sum_q \Phi_q Q_q.$$  

(5.5)

This is the conventional Smarr relation for the $(n+3)$-dimensional static black hole that results if we compactify the black $p$-brane on a $p$-torus, which has as many charges as $q$-brane currents. An expression equivalent to (5.5) is

$$G = \frac{1}{n} T s.$$  

(5.6)

Thus the general form of the stress-energy tensor of charged blackfold fluids is

$$T_{ab} = T s \left( u_a u_b - \frac{1}{n} \gamma_{ab} \right) - \sum_q \Phi_q Q_q h^{(q)}_{ab}.$$  

(5.7)

---

8In the context of AdS/CFT such systems have been studied recently in [29].

9While we do not have a complete proof that this is the universal form for any charged brane, we believe it is generic for large classes of branes and it definitely applies to the black branes of IIB/IIA/11D supergravities and their toroidal compactifications.
The extrinsic equations for the $p$-brane, $K_{ab}{}^\rho T^{ab} = 0$, now take the form
$$ T s \perp^\rho \mu \dot{w}^\mu = \frac{1}{n} T s K^\rho + \perp^\rho \mu \sum_q \Phi_q Q_q K^\mu_{(q)} $$
(5.8)
where
$$ K^\mu_{(q)} = h^{ab}_{(q)} K_{ab}^\mu $$
(5.9)
is the mean curvature vector of the embedding of $C_{q+1}$ in the background spacetime.

Locally, the velocity $u^a$ corresponds to a boost along the $p$-brane. If this boost is along directions orthogonal to the $q$-brane current, then $\dot{V}_{(q+1)}$ and $h_{ab}^{(q)}$ transform non-trivially under the boost. If the boost is parallel to the current, then they are left invariant by it.

**Charges, potentials, and thermodynamics.** The intrinsic equations are the equations of continuity of charge and energy-momentum on $W_{p+1}$. Instead of repeating their analysis, we shall jump directly to the solution for stationary configurations. These have the velocity $u$ aligned with a Killing vector $k$, and their temperature redshifts locally according to (2.10). For these solutions the potential $\Phi_q(\sigma)$ does not depend on the Killing time coordinate and one can also show that it can not depend either on directions transverse to the current \[3\] — this is the familiar equipotential condition for electric equilibrium. Thus, the integration over spatial directions along the current gives a constant
$$ \Phi_H^{(q)} = \int d^3\sigma |h^{(q)}(\sigma^a)|^{1/2} \Phi_q(\sigma) $$
(5.10)
that corresponds to the global $q$-brane potential for the stationary configuration. This integration over the spatial directions of the worldvolume of the $q$-brane is of the same kind as in (2.24).

As before, we assume that $k$ is of the form (2.13), where $\xi$ foliates $W_{p+1}$ into spatial sections $B_p$ with unit timelike normal $n^a$, (2.14). The total mass, angular momenta and entropy are obtained as in (2.18).

In order to obtain the $q$-brane charge we need to consider the spatial sections of the $q$-brane worldvolume $C_{q+1}$ that are orthogonal to $n^a$. On these, the unit $q$-form $\omega^{(q)}$ orthogonal to $n^a$ is
$$ \omega^{(q)} = \frac{-\dot{V}_{(q+1)} \cdot n}{\sqrt{-h_{ab}^{(q)} n^a n^b}}. $$
(5.11)
The total $q$-brane charge $Q_q$ is obtained by integrating its density over the directions transverse to the $q$-brane current,
$$ Q_q = - \int_{B_{p-q}} dV_{(p-q)} |J_{(q+1)} \cdot (n \wedge \omega^{(q)})| = \int_{B_{p-q}} dV_{(p-q)} \sqrt{-h_{ab}^{(q)} n^a n^b} Q_q(\sigma). $$
(5.12)
The global potential (5.10) is\[10\]
$$ \Phi_H^{(q)} = \int dV_q \frac{R_0}{\sqrt{-h_{ab}^{(q)} n^a n^b}} \Phi_q(\sigma). $$
(5.13)
\[10\]These expressions for $Q_q$ and $\Phi_H^{(q)}$ generalize and simplify the ones given in \[3\] for $q = 0, 1.$
If the velocity is aligned with the $q$-brane then we can always write $\hat{V}_{(q+1)} = n \wedge \omega(q)$ and $\sqrt{-h_{ab}^{(q)} n^a n^b} = 1$, and the potential only undergoes a gravitational redshift $R_0$. Otherwise $(5.13)$ includes also a Lorentz-boost redshift factor. For instance, when the boost is orthogonal to the current, $\sqrt{-h_{ab}^{(q)} n^a n^b} = -u^a n_a$.

By a straightforward extension of previous results, it is possible to obtain the extrinsic equations $(5.8)$ for stationary fluid configurations on the brane from the variation of the action

$$I = -\int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \sqrt{-\gamma} \mathcal{G}$$

keeping $T, \Omega, \Phi^{(q)}_H$ constant. Then the global first law

$$dM = T dS + \Omega dJ + \sum_q \Phi^{(q)}_H dQ_q$$

(5.15) for variations of the brane embedding is equivalent to these extrinsic equations.

When $(5.6)$ is integrated over the blackfold worldvolume, it implies $(2.46)$. Using this we can derive the integrated expression

$$(D-3)M - (D-2)(T S + \Omega J) - \sum_q (D-3-q)\Phi^{(q)}_H Q_q = \mathcal{T}_{tot},$$

(5.16) which, in Minkowski backgrounds, reduces to the Smarr relation for asymptotically flat black holes when $\mathcal{T}_{tot} = 0$. Thus the latter must follow as a consequence of the extrinsic equations.

**Intrinsic solution for $q = 0, 1$ currents.** In the previous analysis each current enters independently. Moreover, the presence of $p$-brane charge does not introduce any new degree of freedom. Therefore, when the brane has any 0-branes or strings in the worldvolume the general analysis of stationary solutions with these charges in [3] carries over directly.

When $q = 0$ the complete solution to the intrinsic equations is obtained by taking the 0-brane potential to redshift like the temperature,

$$\Phi_0(\sigma) = \frac{\Phi^{(0)}_H}{|k|},$$

(5.17) with constant $\Phi^{(0)}_H$.

For string currents, $q = 1$, an intrinsic solution in which the geometry along the currents, $h^{(1)}_{ab}$, is fully specified, can be given explicitly when the strings lie along a spatial Killing vector $\psi$ that commutes with $k$. Then, introducing

$$\zeta^a = \psi^a + (\psi^b u_b) u^a$$

(5.18) we have

$$h^{(1)}_{ab} = -u_a u_b + \frac{\zeta_a \zeta_b}{|\zeta|^2}, \quad |h^{(1)}|^{1/2} = |k||\zeta|, \quad K^\rho_{(1)} = -\left(g^{\rho\mu} - h^{(1)}_{ab} \right) \partial_\mu \ln(|k||\zeta|).$$

(5.19)
Assuming that $\psi$ (and hence $\zeta$) has compact orbits of periodicity $2\pi$, the potential is

$$\Phi_1(\sigma) = \frac{\Phi_H^{(1)}}{2\pi |h^{(1)}(\sigma^a)|^{1/2}},$$

with constant $\Phi_H^{(1)}$.

### 5.2 Extremal limits

**Extremal branes with subluminal worldvolume velocity.** At extremality we have $T_s \to 0$. The conventional limit in which the velocity is not scaled to light speed, takes (5.7) into the form

$$T_{ab} = -\sum_q \Phi_q Q_q h^{(q)}_{ab}.$$  

(5.21)

In contrast to the case in sec. 2.3.1 with only $p$-brane charge, if the velocity lies along a direction orthogonal to at least one of the currents, it is a physical mode that cannot be gauged away. The velocity remains subluminal in this limit.

Assuming that at stationarity $u^a$ lies along a Killing direction, the extremal limit $T S \to 0$ of the Smarr relation (with $T_{\text{tot}} = 0$) and the vanishing of the extremal action $I$ allow to derive

$$M = \sum_q (q + 1) \Phi_H^{(q)} Q_q, \quad \Omega J = \sum_q q \Phi_H^{(q)} Q_q.$$  

(5.22)

These generalize expressions found in [3]. When there is only a 0-brane current, as considered in [3], the kinetic energy term $\Omega J$ must vanish. However we see that, in combination with higher-brane currents, 0-brane-charged extremal blackfolds need not be static, in fact the rotation is required in order to balance the tension that these higher currents create.

In configurations of equilibrium in Minkowski backgrounds, the total rapidity $\eta$ is determined by the condition $T_{\text{tot}} = 0$. If we denote by $q_\perp$ and $q_\parallel$ those currents that are orthogonal and parallel to the velocity, we have

$$(\gamma^{ab} + n^a n^b) h^{(q_\perp)}_{ab} = -\sinh^2 \eta + q_\perp, \quad (\gamma^{ab} + n^a n^b) h^{(q_\parallel)}_{ab} = q_\parallel.$$  

(5.23)

Then $T_{\text{tot}} = 0$ can easily be seen to imply that

$$\int_{B_p} dV_{(p)} \sinh^2 \eta \sum_{q_\perp} (\Phi_{q_\perp} Q_{q_\perp}) = \sum_q q \Phi_H^{(q)} Q_q.$$  

(5.24)

In cases for which the rapidity $\eta$ is constant on $B_p$ this equation fixes its value at equilibrium to be

$$\sinh^2 \eta = \frac{\sum_q q \Phi_H^{(q)} Q_q}{\sum_{q_\perp} \Phi_H^{(q_\perp)} Q_{q_\perp}}.$$  

(5.25)
Null-wave branes. The limit where the fluid velocity becomes lightlike requires that we keep
\[ \sqrt{T_s} u^a = \sqrt{\mathcal{K}} \ l^a \] (5.26)
finite, with \( l^a \) approaching a null vector and \( \mathcal{K} \) representing a momentum density. Then
\[ T_{ab} = \mathcal{K} l_a l_b - \sum_q \Phi_q Q_q h^{(q)}_{ab}. \] (5.27)
When \( q = 0 \), for which \( h^{(0)}_{ab} = -u_a u_b \), this limit requires that we also send \( \Phi_0 Q_0 \to 0 \) so the 0-brane charge disappears in the limit. More generally, whenever the boost occurs along directions orthogonal to the \( q \)-brane the metric \( h^{(q)}_{ab} \) degenerates and the \( q \)-brane charge must disappear in the limit. On the other hand, if the boost occurs along the \( q \)-brane, the metric \( h^{(q)}_{ab} \) remains invariant and non-degenerate, and the \( q \)-brane charge survives the limit. Thus, extremal branes with a null-wave only support currents parallel to the direction of the wave.

Again, we can use the condition of absence of tensional energy to obtain general expressions for equilibrium configurations in Minkowski backgrounds, which generalize those found in sec. 2.4.2. Normalizing \( n^a l_a = -1 \) we find
\[ \int_{\mathcal{B}_p} dV(\mathcal{B}) \mathcal{K} = \sum_q q \Phi_H^{(q)} Q_q. \] (5.28)
The mass is
\[ M = \int_{\mathcal{B}_p} dV(\mathcal{B}) \mathcal{K} + \sum_q \Phi_H^{(q)} Q_q = \sum_q (q + 1) \Phi_H^{(q)} Q_q. \] (5.29)
These are analogous to (5.22). When \( \chi^a l_a = \mathcal{R} \) is constant on \( \mathcal{B}_p \) we obtain a generalization of (2.67),
\[ \frac{J}{\mathcal{R}} = \sum_q q \Phi_H^{(q)} Q_q. \] (5.30)
For branes such that at extremality \( \Phi_q \to 1 \), which we believe occurs when they are marginally bound (like in the D0-D4 and D1-D5 systems below), these expressions apply in the form
\[ M = \sum_q (q + 1)V(q)Q_q, \quad \frac{J}{\mathcal{R}} = \sum_q qV(q)Q_q \] (5.31)
with \( V(q) \) the spatial volume along the \( q \)-brane.

Extremality vs. supersymmetry. Although expressions such as (5.22) and (5.31) may be reminiscent of BPS relations, one must bear in mind that we are considering that \( \mathcal{B}_p \) is compact and hence, in Minkowski space, contractible to a point. Therefore \( Q_q \) for \( q \geq 1 \) correspond to dipoles and not to conserved charges, which are the ones entering the BPS bounds. Generically, these configurations are not supersymmetric.

The charge of \( q = 0 \) currents is a net, conserved charge. Then, if there are only 0-brane currents (excluding also a \( p \)-brane current), the first relation in (5.31) is indeed the BPS relation. As discussed in [3], the resulting (singular, static) blackfolds are supersymmetric.
5.3 Products of odd-spheres

The most straightforward solutions to construct are those in which the blackfold wraps a product of odd-spheres, like we did in sec. 3.1. The general analysis, although essentially straightforward, can become very cumbersome, so we shall confine ourselves to a few simple and interesting instances. Many other particular cases can be readily worked out.

\textbf{(0-\textit{p})-brane solutions.} We begin with the case where we have both 0- and \( p \)-brane currents. The velocity field and the extrinsic curvatures for the product of odd-spheres are like in (3.12), (3.14). For the 0-brane (a particle) the extrinsic curvature vector is the acceleration \( K_0^\mu = -\perp_\mu u^\mu \), and we find that the extrinsic equations (5.8) are solved by

\[
\Omega_I R_I = \sqrt{\frac{p_I}{p}} \tanh \eta ,
\]

\[
\sinh^2 \eta = \frac{p \mathcal{T}s + n \Phi_p Q_p}{n \mathcal{T}s + \Phi_0 Q_0} .
\]

This reproduces (3.6) when \( Q_0 = 0 \), and the result in [3] when \( Q_p = 0 \). In the extremal limit in which \( \mathcal{T}s = 0 \) we find

\[
\sinh^2 \eta = \frac{p\Phi_p Q_p}{\Phi_0 Q_0} ,
\]

which is what (5.25) gives when the blackfold is homogeneous and hence \( \Phi_H^{(q)} Q_q = V^{(p)} \Phi_q Q_q \).

\textbf{(1-\textit{p})-brane solutions.} In this case the string current introduces an anisotropy on the world-volume, and there are different configurations depending on how the strings are aligned relative to the velocity field. Taking the string to lie along any spatial Killing vector, all these possibilities can be obtained using the formalism of eqs. (5.18) and (5.19).

For definiteness and simplicity, we consider the configuration in which the string current is parallel to the velocity field (3.12). Hence we choose \( \psi = \sum_I \Omega_I \partial_{\phi_I} \) in (5.18), and obtain

\[
\zeta = \sinh^2 \eta \frac{\partial}{\partial t} + \cosh^2 \eta \sum_I \Omega_I \frac{\partial}{\partial \phi_I} ,
\]

with \( |\zeta| = \sinh \eta \) and \( |\zeta^{(1)}|^{1/2} = \tanh \eta \). The extrinsic equations are now solved by

\[
\Omega_I R_I = \sqrt{\frac{p_I}{p}} \tanh \eta , \quad \sinh^2 \eta = \frac{p \mathcal{T}s + n \Phi_1 Q_1 + p \Phi_p Q_p}{n \mathcal{T}s} .
\]

Again, the results in [3] for \( Q_p = 0 \) are reproduced.

We can describe these configurations as having helical strings, extending along the orbits of \( \psi \), which are smeared on the worldvolume of the \( p \)-brane. When \( p = 1 \) we can indeed obtain the generalization of the helical strings in [10] that carry string charge, and the equilibrium condition for them is

\[
\sinh^2 \eta = \frac{1}{n} + \frac{\Phi_1 Q_1}{\mathcal{T}s} .
\]

\footnote{For simplicity we are not distinguishing if there is one or two types of strings.}
As shown in [10], these helical strings generically break the Cartan subgroup of the $D$-dimensional rotation group, possibly down to a single $U(1)$. However, when $p > 1$ the smearing of the strings over the worldvolume of the $p$-brane (which itself is not helical) restores the entire Cartan subgroup, which then remains unbroken.

Since the strings are parallel to the fluid velocity, in the extremal limit $T_s \to 0$ this velocity becomes null $\eta \to \infty$, with the momentum $K = T_s \sinh \eta \cosh \eta$ remaining finite. In this limit eqs. (5.36) become

$$K = \Phi_1 Q_1 + p \Phi_p Q_p$$

which, when integrated (trivially) over the worldvolume reproduces (5.25). The component of the velocity on the $I$-th sphere is given by (3.17), the null vector is (3.18), and the angular momentum along the direction of rotation of each sphere is (3.20). Together with (5.29), we obtain

$$M - \Phi_H^{(1)} Q_1 - \Phi_H^{(p)} Q_p = \Phi_H^{(1)} Q_1 + p \Phi_H^{(p)} Q_p = \left( \sum I^2 R_I^2 \right)^{1/2}.$$  

(5.39)

It is also possible to construct more general configurations in which the string current is not aligned with the velocity, i.e., $\psi = \sum \varsigma_I \partial \phi_I$ with constant coefficients $\varsigma_I$. However, even in simple cases the expressions one obtains are not too illuminating, so we omit their study (related examples were constructed in [3]).

**Solutions with Kaluza-Klein circles.** In a flat background with a compact torus $T^{p_0}$, we can wrap the brane with velocity vector (3.23).

For $(0-p)$-brane systems, the equilibrium condition on odd-spheres is then

$$\Omega_I R_I = \sqrt{\frac{p_I}{p-p_o}} \tanh \bar{\eta},$$

(5.40)

$$\frac{\sinh^2 \bar{\eta}}{1 - v^2 \cosh^2 \bar{\eta}} = \frac{p - p_o}{n} \frac{T_s + n \Phi_p Q_p}{T_s + \Phi_0 Q_0}.$$  

(5.41)

For $(1-p)$-brane systems with velocity along the 1-brane, so that $\psi = \sum I \Omega_I \partial \phi_I + \sum_A v_a \partial A$, at equilibrium we find eq. (5.40) and $\tanh \bar{\eta}$ determined by solving

$$\frac{\sinh^2 \bar{\eta}}{1 - v^2 \cosh^2 \bar{\eta}} = \frac{p - p_o}{n} + \frac{1}{T_s} \left( \frac{\Phi_1 Q_1}{1 + v^2/\tanh^2 \bar{\eta}} + (p - p_o) \Phi_p Q_p \right).$$

(5.42)

Note that, in our construction, $v \neq 0$ implies that the 1-brane wraps the internal torus as well as the spheres in the non-compact space. When $v^2 = 0$ we simply need to replace $p \to p - p_o$ in (5.32), (5.33) and (5.36).

6 D0-D$p$ and F1-D$p$ blackfolds

Now we turn to some explicit examples of blackfold fluids with lower-brane currents supported on their worldvolume. For concreteness we focus first on two simple cases based on two-charge
branes in ten-dimensional type IIA/IIB string theory, namely blackfolds based on the D0-Dp \((p = 2, 4, 6)\) and F1-Dp \((p \geq 1)\) brane solutions respectively. These correspond to charged \(p\)-branes with 0-brane (particle) charge current or 1-brane (string) charge current on their worldvolumes.

### 6.1 Effective fluids and stability of charge waves

We start by providing the local thermodynamics of the effective blackfold fluids for D0-Dp and F1-Dp branes in ten-dimensional type II string theory (we do not consider smeared Dp-brane charges here, but the considerations are easily generalized to that case). The local thermodynamic quantities are obtained from the corresponding type II supergravity solution (i.e., the metric and gauge potential) of the corresponding two-charge black brane solutions. These are given in appendix B.

While all these fluids satisfy the general relations (5.5) and (5.6), they differ in specific properties of their equations of state. The most salient difference is visible in the sign of the isothermal permittivity of the local charge \((D0\) or F1)

\[
\epsilon_q \equiv \left( \frac{\partial \Phi_q}{\partial Q_q} \right)_{Q_p, T} = \left( \frac{\partial^2 F}{\partial Q_q^2} \right)_{Q_p, T}
\]

(6.1)

where \(F = \epsilon - T s\) is the Helmholtz free energy density. This permittivity measures the stability against fluctuations of the \(q\)-brane density in the fluid (which is a different kind of perturbation than the sound modes associated to the Gregory-Laflamme instability), and is directly related to whether the \(q\)-branes form actual bound states with the \(p\)-branes, or instead are only marginally or unstably bound. We will compute this quantity for each of the fluids discussed below for the general non-extremal brane system (see also Appendix C for details) and comment in connection with this in the extremal limit. Also note that for all the effective fluids considered here, the relation (2.42) for the Gibbs free energy density continues to hold, as seen using (5.6).

#### D0-D2 fluid

The basic thermodynamic quantities are

\[
\epsilon = \frac{\Omega_{(6)}^{(6)}}{16 \pi G} r_0^6 (6 + 5 \sinh^2 \alpha),
\]

(6.2a)

\[
T = \frac{5}{4 \pi r_0 \cosh \alpha}, \quad s = \frac{\Omega_{(6)}^{(6)}}{4 G} r_0^6 \cosh \alpha,
\]

(6.2b)

\[
Q_{D2} = \frac{5 \Omega_{(6)}^{(6)}}{16 \pi G} r_0^5 \sinh \alpha \cosh \alpha \cos \theta, \quad Q_{D0} = Q_{D2} \tan \theta,
\]

(6.2c)

\[
\Phi_{D2} = \tanh \alpha \cos \theta, \quad \Phi_{D0} = \Phi_{D2} \tan \theta.
\]

(6.2d)

The extremal limit takes \(r_0 \rightarrow 0, \alpha \rightarrow \infty\) keeping \(r_0^6 \sinh^2 \alpha\) finite. In this limit the entropy vanishes and \(\epsilon^2 = Q_{D0}^2 + Q_{D2}^2\).
The permittivity of D0 charge density (for fixed D2 charge) is
\[
\varepsilon_{D0} = \sqrt{\frac{-7 + 12x + \sqrt{49 - 24x}}{-7 + 2x + \sqrt{49 - 24x}} \left( \frac{Q_{D2}^2 + Q_{D0}^2}{2} \right)^2 \left( Q_{D0}^2 + 46x - 21 + 3\sqrt{49 - 24x} \right) \left( Q_{D2}^2 + 10x \left( Q_{D0}^2 + Q_{D2}^2 \right) \right)}.
\] (6.3)

Here we have defined the non-extremality parameter
\[
x = 1 - \frac{Q_{D0}^2 + Q_{D2}^2}{\varepsilon^2}.
\] (6.4)

which by definition lies in the range \( x \in [0, 1) \). At extremality (with zero momentum), namely when \( x \to 0 \), the permittivity (6.3) becomes
\[
\varepsilon_{D0} = \frac{Q_{D2}^2}{(Q_{D2}^2 + Q_{D0}^2)^{3/2}} > 0
\] (6.5)

This indicates that the system is stable to fluctuations of the charge: it tends to distribute uniformly on the worldvolume, which is a consequence of the fact that the D0-D2 form a true (1/2 BPS) bound state. Thus the D0-D2 brane can support stable waves of D0 charge in its worldvolume. Their velocity is determined by this permittivity.

Beyond the extremal limit the D0 charge permittivity remains positive. As a function of the non-extremality parameter \( x \) it is positive and monotonic when we vary \( x \) at fixed \( Q_{D0}, Q_{D2} \) and diverges at \( x = 1 \).

**D0-D4 fluid**

The basic thermodynamic quantities are
\[
\varepsilon = \frac{\Omega_{(4)}}{16\pi G} r_0^3 \left( 4 + 3\sinh^2 \alpha_0 + 3\sinh^2 \alpha_4 \right)
\] (6.6)
\[
\mathcal{T} = \frac{3}{4\pi r_0 \cosh \alpha_0 \cosh \alpha_4}, \quad s = \frac{\Omega_{(4)}}{4G} r_0^4 \cosh \alpha_0 \cosh \alpha_4
\] (6.7)
\[
Q_{D_i} = \frac{\Omega_{(4)}}{16\pi G} 3r_0^3 \cosh \alpha_i \sinh \alpha_i, \quad \Phi_{D_i} = \tanh \alpha_i
\] (6.8)

with \( i = 0, 4 \).

In the extremal limit, \( r_0 \to 0, \alpha_i \to \infty \), with \( r_0^3 \sinh^2 \alpha_i \) finite. In this limit the entropy vanishes and \( \varepsilon = Q_{D0} + Q_{D4} \).

The permittivity of D0 charge is
\[
\varepsilon_{D0} = \frac{1}{Q_{D0}^2} \tanh \alpha_0 \left( \frac{\cosh \alpha_4 - 2}{2 - \cosh \alpha_0 + \sinh^2 \alpha_4} \right) .
\] (6.9)

At extremality this permittivity becomes
\[
\varepsilon_{D0} = 0
\] (6.10)

which indicates that the system is marginally stable to fluctuations of the charge. This is a consequence of the fact that the D0-D4 form a (1/4 BPS) marginal bound state. Beyond extremality, say as we vary \( \alpha_0 \) at fixed \( Q_{D0}, Q_{D4} \), we find that \( \varepsilon_{D0} < 0 \) except in a finite range of \( \alpha_0 \) where it is positive (see App. C).
D0-D6 fluid

In this case the basic thermodynamic quantities are

\[ \varepsilon = \frac{\Omega_{(2)}}{16\pi G} r_0 (2 + \sinh^2 \alpha_0 + \sinh^2 \alpha_6) \] (6.11)

\[ T = \frac{1}{4\pi r_0 \cosh \alpha_0 \cosh \alpha_6} \frac{2(\cosh^2 \alpha_0 + \cosh^2 \alpha_6)}{(\cosh^2 \alpha_0 + 1)(\cosh^2 \alpha_6 + 1)} \] (6.12)

\[ s = \frac{\Omega_{(2)}}{4G} r_0^2 \cosh \alpha_0 \cosh \alpha_6 \frac{(\cosh^2 \alpha_0 + 1)(\cosh^2 \alpha_6 + 1)}{2(\cosh^2 \alpha_6 + \cosh^2 \alpha_6)} \] (6.13)

\[ Q_{Di} = \frac{\Omega_{(2)}}{16\pi G} r_0 \cosh \alpha_i \sinh \alpha_i \sqrt{\frac{\cosh^2 \alpha_i + 1}{\cosh^2 \alpha_0 + \cosh^2 \alpha_6}} \] (6.14)

\[ \Phi_{Di} = \tanh \alpha_i \sqrt{\frac{\cosh^2 \alpha_0 + \cosh^2 \alpha_6}{\cosh^2 \alpha_i + 1}} \] (6.15)

with \( i = 0, 6 \).

Among all D0-Dp systems, this one is unique in having finite entropy density in the extremal limit, in which \( r_0 \rightarrow 0, \alpha_i \rightarrow \infty \), with \( r_0 \sinh^2 \alpha_i \) finite, so

\[ s = 16\pi G Q_{D0} Q_{D6} \] (6.16)

and \( \varepsilon^{2/3} = Q_{D6}^{2/3} + Q_{D0}^{2/3} \).

The expression for the D0 charge permittivity is now more complicated and is given in eq. (C.5). At extremality the permittivity is

\[ \epsilon_{D0} = -\frac{1}{2} \frac{Q_{D0}^{4/3} Q_{D6}^{2/3}}{Q_{D6}^{2/3} + Q_{D0}^{2/3}} < 0 \] (6.17)

which indicates that the system is unstable to fluctuations of the charge: it is favorable for the charge to clump. This is a consequence of the fact that the D0-D6 repel each other: the D0 charge tends to clump so that the D0-branes can be expelled. Further analysis of the expression (C.5) shows that close to extremality the permittivity remains negative, but there are certain regions where it can become positive.

F1-Dp fluid

The basic thermodynamic quantities are

\[ \varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n (1 + n \cosh^2 \alpha) \] (6.18)

\[ T = \frac{n}{4\pi r_0 \cosh \alpha} , \quad s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \cosh \alpha \] (6.19)
\[ Q_{Dp} = \frac{\Omega_{(n+1)} n r_0^p \cos \theta \cosh \alpha \sinh \alpha}{16\pi G n q_p}, \quad \Phi_{Dp} = \cos \theta \tanh \alpha \quad (6.20) \]

\[ Q_{F1} = \frac{\Omega_{(n+1)} n r_0^p \sin \theta \cosh \alpha \sinh \alpha}{16\pi G n q_\alpha}, \quad \Phi_{F1} = \sin \theta \tanh \alpha \quad (6.21) \]

Here \( n = 7 - p \).

The permittivity of F1 charge density is

\[ \epsilon_{F1} = \frac{1}{\tanh \alpha} \frac{(n - 2) Q_{Dp}^2 - Q_{Dp} Q_{F1} \sinh^2 \alpha - 1}{(n - 2) \sinh^2 \alpha - 1}. \quad (6.22) \]

At extremality the permittivity reduces to

\[ \epsilon_{F1} = \frac{Q_{Dp}^2}{(Q_{Dp}^2 + Q_{F1}^2)^{3/2}} > 0 \quad (6.23) \]

reflecting the fact that F1-Dp \( (p \geq 1) \) is a 1/2 BPS bound state. Thus for \( p \geq 1 \) the F1-Dp brane supports waves in which the F1 charge fluctuates in directions transverse to the strings. Beyond the extremal limit the precise behavior of \( \epsilon_{F1} \) depends on the value of \( n \). For \( n = 1, 2 \) the F1 charge permittivity is always positive. For \( n > 2 \), \( \epsilon_{F1} > 0 \) except within an interval (see eq. (C.10)) where it is negative.

The extremal limit with a null wave along the F1 is obtained in appendix B. In terms of its parameters \( q_{p,\alpha} \) and \( \theta \) we have

\[ \mathcal{K} = \frac{\Omega_{(n+1)} n q_p}{16\pi G}, \quad (6.24) \]

\[ Q_{Dp} = \frac{\Omega_{(n+1)} n q_\alpha \cos \theta}{16\pi G}, \quad \Phi_{Dp} = \cos \theta, \quad (6.25) \]

\[ Q_{F1} = \frac{\Omega_{(n+1)} n q_\alpha \sin \theta}{16\pi G}, \quad \Phi_{F1} = \sin \theta. \quad (6.26) \]

### 6.2 Solutions for odd-spheres and products of odd-spheres

Since we have the explicit equations of state for several systems, we can insert them into the generic solutions for products of odd-spheres of sec. 5.3. Blackfolds based on D0-Dp black branes with spatially compact worldvolumes give new D0-brane-charged black holes with Dp-brane dipole, whereas blackfolds based on the F1-Dp solutions give dipole black holes with no conserved charge.

Our description of the solutions will be brief. We give the equilibrium radii of the odd-spheres, and with these it is straightforward to compute the thermodynamics of the various blackfold solutions using (2.18), (2.24) for \( M, J, S \) and \( \Phi_H^{(p)} \) and (5.12), (5.13) for \( \Phi_H^{(q)}, Q_q \). These values are not much harder to obtain than those in appendix A or in refs. [10, 3], so we omit them.

Table 3 gives a summary of the types of horizons we obtain in \( D = 10 \) (with no KK torus).
Table 3: A list of the allowed possibilities for blackfolds based on D0-Dp and F1-Dp black branes wrapping products of odd-spheres. The horizon topology of the corresponding black holes is the product of the worldvolume and the transverse sphere $s^{n+1}$. Eq. (3.11) limits the number $\ell$ of odd-spheres along the worldvolume.

| Worldvolume | $\perp$ Sphere |
|-------------|--------------|
| F1-D1       | (helical) $S^1$ | $s^7$  |
| D0-D2       | $T^2$        | $s^6$  |
| F1-D3       | $S^3, T^3$   | $s^5$  |
| D0-D4       | $S^3 \times S^1, T^4$ | $s^4$  |
| F1-D5       | $S^5, S^3 \times T^2$ | $s^3$  |
| D0-D6       | $S^3 \times S^3, S^5 \times S^1$ | $s^2$  |

6.2.1 D0-Dp

For D0-Dp the radii $R_I$ of the wrapped spheres $S^{p_I}$ take the equilibrium values $\sqrt{p_I}$,

$$R_I = \sqrt{\frac{p_I}{p} \tanh \eta \Omega_I}$$

where the rapidity $\eta$ is given by

$$\sinh^2 \eta = \frac{p}{7-p} \frac{1 + (7-p) \sinh^2 \alpha_p}{1 + \sinh^2 \alpha_0}.$$ (6.28)

These solutions interpolate between Dp and smeared D0: for $\alpha_0 = 0$ we recover the Dp blackfolds of Sec. 3 and for $\alpha_p = 0$ we recover the D0-charged blackfolds of [3], describing rotating charged black holes.

In the extremal limit the rapidity (6.28) correctly reproduces (5.34).

Extremal D0-D6 black holes with finite area. The D0-D6 solutions have finite horizon area at extremality, and therefore we obtain extremal, non-singular D0-charged black holes with D6 dipole, with the variety of horizon topologies in table 3.

These are ten-dimensional black holes, but if the D6 wraps Kaluza-Klein circles we find other extremal black holes in any dimension $4 \leq D \leq 10$. All these black holes are rotating and non-supersymmetric, and as we have seen, they are unstable to clumping the D0 charge.

6.2.2 F1-Dp

For the case of F1-Dp, with the fluid velocity along the F1, the equilibrium solution is easily obtained from (6.36). The total rapidity is determined as

$$\sinh^2 \eta = \frac{p}{n} + (\sin^2 \theta + p \cos^2 \theta) \sinh^2 \alpha.$$ (6.29)
This expression becomes simpler when \( p = 1 \), which describes a F1-D1 bound state. The equilibrium condition for this is

\[
\sinh^2 \eta = \frac{1}{n} + \sinh^2 \alpha \quad (6.30)
\]

which is of the same form as (3.6), but now \( \sinh^2 \alpha \) accounts for the tension of the bound state. Helical rings with these charges are straightforward to obtain.

The extremal limit has a null wave parallel to the F1. The relations derived in secs. 5.2 and 5.3 apply directly to these configurations and they allow to easily compute the physical parameters. We will see explicitly how this works in the configurations that we study next.

7 New extremal D1-D5-P black holes

Black branes with D1-D5 charges are particularly prominent in string theory. They are T-dual to the D0-D4 branes of the previous section, but when their worldvolumes are curved to form a blackfold this duality only applies locally, and is globally broken. Thus the black holes that result are physically inequivalent\(^{12}\).

Using our analysis in sec. 5.3 it is easy to construct non-extremal D1-D5 blackfolds on products of odd-spheres, in particular when the worldvolume velocity is aligned with the D1 current. Since this velocity creates a local momentum on the worldvolume, the branes are locally like a D1-D5-P system. Globally, they are black holes with D1 and D5 dipoles and angular momentum. The main novelty that these present is that their extremal limit, keeping the momentum finite (hence null), has a regular horizon with finite area which is not obviously unstable. Thus in the following we focus exclusively on extremal solutions.

The solution for the extremal D1-D5-P system is given in appendix B. This is a brane configuration with a null wave on it, and in terms of the parameters \( q_{1,5,p} \) of the solution we have

\[
\varepsilon = \frac{\pi}{4G}(q_1 + q_5 + q_p) \quad (7.1)
\]

with charges and potentials

\[
Q_{1,5} = \frac{\pi}{4G} q_{1,5}, \quad \Phi_{1,5} = 1, \quad (7.2)
\]

and momentum density

\[
\mathcal{K} = \frac{\pi}{4G} q_p. \quad (7.3)
\]

The stress tensor is

\[
T_{ab} = \mathcal{K} l_a l_b - \sum_{q=1,5} Q_q h_{ab}^{(q)}. \quad (7.4)
\]

The null vector in the geometry (B.26) is \( l = \partial_v \). The entropy density is finite

\[
s = \frac{\pi^2}{2G} \sqrt{q_1 q_5 q_p}. \quad (7.5)
\]

\(^{12}\)This is besides the fact that the T-duality in supergravity smears the D0-D4 system along a transverse direction. Smeared branes are unstable to Gregory-Laflamme perturbations along the smearing direction \(30\).
Applying our general construction of sec. 5.3 to this system, we obtain 10D extremal rotating black holes with D1-D5 dipoles that have regular horizons of topology $S^3 \times S^5$ and $S^3 \times T^2 \times S^3$ — eq. (3.11) forbids more than four odd-sphere factors along this worldvolume. The general result (5.38) tells us that the condition of equilibrium is

$$q_p = q_1 + 5q_5.$$  

(7.6)

The mass is

$$M = \frac{\pi}{2G} V(5)(q_1 + 3q_5)$$ \hspace{1cm} (7.7)

where $V(5)$ is the total volume that the blackfold wraps,

$$V(5) = \begin{cases} \pi^3 R^5 & \text{for } S^5, \\ 8\pi^4 R_S S_{S_1} S_{S_3} & \text{for } T^2 \times S^3. \end{cases}$$ \hspace{1cm} (7.8)

The angular momenta along the rotation directions of the spheres are

$$J = \frac{\pi^4}{4G} R^6 q_p$$ \hspace{1cm} (7.9)

$$J_I = \frac{\pi}{4G} V(5) q_p R_I \sqrt{\frac{p_I}{5}}$$ \hspace{1cm} (7.10)

The angular velocity on $S^5$ is $\Omega = R^{-1}$, and on each sphere of $T^2 \times S^3$, $\Omega_I = R_I^{-1} \sqrt{p_I/5}$.

On $S^5$ the dipole charges and potentials are

$$Q_{D1} = \frac{\pi^3}{8G} R^4 q_1, \hspace{0.5cm} Q_{D5} = \frac{\pi}{4G} q_5, \hspace{0.5cm} \Phi_{H}^{D1} = 2\pi R, \hspace{0.5cm} \Phi_{H}^{D5} = \pi^3 R^5$$ \hspace{1cm} (7.11)

and we can easily verify that

$$dM = \Omega dJ + \Phi_{H}^{D1} dQ_{D1} + \Phi_{H}^{D5} dQ_{D5}.$$ \hspace{1cm} (7.12)

If we wrap some of the directions of the five-brane on compact KK circles, we obtain black holes in dimension $< 10$. In $D = 5$ we recover the conventional static D1-D5-P black hole, but besides this one, all these blackfolds are not supersymmetric. The D1 and the momentum (which we take to be parallel) must wrap all the contractible cycles in the non-compact part of the space in order to achieve equilibrium, but as discussed in sec. 5.3 the KK circles are stabilized by topology and so it is not necessary that the momentum and the D1 wrap them. Thus we get different solutions depending on whether D1-P wrap the KK circles or not.

The new topologies that we can obtain with $10 - D$ compact KK circles are shown in table 4.

In $D = 6$ we get black rings, which can have helical shape.

The worldvolume of these extremal branes is marginally stable to fluctuations of the D1 charge, and Gregory-Laflamme-type instabilities seem to be absent. The brane being a tensile

\[\text{For equal numbers of D1 and D5 branes, they are among the solutions in sec. 3 with } N = 2, \text{ which have also been constructed in 3.}\]
| Dimension (non-compact) | Worldvolume   | Sphere |
|------------------------|---------------|--------|
| $D = 10$               | $S^5, S^3 \times T^2$ | $s^3$  |
| $D = 9$                | $S^3 \times S^1, T^4$ | $s^3$  |
| $D = 8$                | $S^3, T^3$    | $s^3$  |
| $D = 7$                | $T^2$         | $s^3$  |
| $D = 6$                | $S^1$         | $s^3$  |

Table 4: A list of horizon topologies for stationary extremal rotating black holes with $D1$-$D5$ dipoles in a spacetime with $D$ non-compact dimensions and $10 - D$ compact KK circles. We do not distinguish whether the $D1$-$P$ current wraps some of the compact directions, which gives different kinds of black holes.

Object, its purely extrinsic perturbations are also stable. Therefore, there is no obvious instability that afflicts these new extremal black holes. One possible source of an instability might occur if the coupling between intrinsic and extrinsic perturbations turned a marginal mode into an unstable one. If this did not happen, these could be the first stable, asymptotically flat, extremal, non-supersymmetric black holes with non-spherical horizon topology in $D \geq 6$ (in $D = 5$ there are the black rings of [8]). They would also be the first asymptotically flat extremal (supersymmetric or not) stable black holes with horizon topology other than $S^{D-2}$ or $S^1 \times S^2$. The six-dimensional black rings with $D1$-$D5$ dipole appear as the most likely stable candidates for this class.

8 Discussion

Tables 1, 2, 3 and 4 summarize the topologies of the horizons of the new black hole solutions in string/M-theory that we have constructed. These tables are far from being exhaustive lists of all the possibilities, even for a given kind of brane. We have only studied the simplest class of solutions whose worldvolume geometry is a product of round odd-spheres. However, these lists are illustrative of the many new possibilities afforded by our techniques.\(^\text{14}\)

Given that many of our solutions arise in string and M-theory, it is natural to ask what their microscopic interpretation is. The answer emerges directly from the nature of the blackfold construction. Since, locally, the blackfold is well approximated by a flat black brane, it follows that whatever the microscopic interpretation of the latter is, it applies to the blackfold as well, only now with the microscopic brane configuration being curved over a long length scale.\(^\text{15}\) This is adequate to leading order in the blackfold approximation. Corrections to the blackfold solution, from the extrinsic curvature of the brane and the backreaction of the background, must be matched with corrections in the microscopic picture, but in general neither of these is easy.

\(^{14}\)See [31] for other recent work on new charged and/or dipole black rings, and on near-horizon analyses of supersymmetric non-spherical black holes in string/M theory.

\(^{15}\)This can correspond to turning on a source for the corresponding CFT.
to compute. This point of view was first advocated and developed in [8], where five-dimensional black rings were understood in terms of the microscopic interpretation of the black string that they locally approximate, e.g., the Maldacena-Strominger-Witten CFT for the $M5 \perp M5 \perp M5$ system.

Thus, in the present case we can understand the microscopic entropy of our new extremal black holes built by bending D1-D5-P or D0-D6 branes, in terms of the microphysics of the latter as described in [32] and [33] respectively. There is a caveat to this argument, though, which is most apparent in the D1-D5-P system. In order to derive the $1 + 1$ CFT for the D1-D5 bound state one assumes that the D5 directions transverse to the D1 wrap a space much smaller than along the D1, so that oscillations along the former are very massive and suppressed. In our D1-D5-P blackfolds such a reduction occurs in $D = 6$ (and perhaps also when some odd-spheres are much smaller than a given $S^1$), but not in general. In these other cases, even if the horizon entropy is nominally accounted for in terms of the $1 + 1$ degrees of freedom, there is the possibility that the system increase its entropy by exciting light orthogonal modes. It would be interesting to study this for its possible relevance to the dynamical stability problem. In light of this, the six-dimensional rotating black rings (possibly helical) with D1-D5 dipoles are probably the cleanest and best behaved of all the new black holes that we have constructed.

Similar remarks apply to most other configurations, but it may be worth commenting on the extremal singular solutions with the charge of a D$p$-brane when they carry a null momentum wave. It may seem surprising that a D-brane, whose worldvolume is locally Lorentz-invariant, can support longitudinal waves on it. The resolution has been discussed for the similar case of fundamental strings in [8]: the extremal null-wave branes are the macroscopic, coarse-grained description of branes with an ensemble of travelling (transverse) waves along their worldvolume. For a flat brane, the longitudinal momentum wave can be resolved into transverse travelling waves, both in the supergravity description and in the DBI or Nambu-Goto descriptions. Ref. [14] performs a detailed analysis of all aspects of this picture, and its conclusions can be transported to our present context. In particular, it is noted that when the worldvolume is not flat but bent into a curved shape, the travelling waves will emit gravitational radiation. In the coarse-grained version, the extremal brane with a null wave is stationary at the classical level, but quantum effects will make it decay slowly through the emission of superradiant modes. Therefore, our new D1-D5-P extremal black holes, even if they are classically stable, will decay at quantum-mechanical level. This is indeed a generic feature of most extremal non-supersymmetric rotating black holes [34, 35].

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A Physical parameters for odd-sphere blackfold solutions

For reference and as an illustration of how the explicit results are obtained, here we collect the physical parameters for the blackfolds with $p$-brane current on odd-spheres, with $p = 2m + 1$, as built in section 3.

We express the quantities in terms of the parameters $r_0$, $\alpha$, and $R$, and introduce the shorthand notation $s \equiv \sinh \alpha$. Then,

\[
M = \frac{V(p) \Omega(p+1)}{16\pi G} r_0^n (n + p + 1 + Nn(1 + p)s^2),
\]

\[
S = \frac{V(p) \Omega(p+1)}{4G} r_0^{n+1} \sqrt{\frac{(n + p + nps^2)(1 + s^2)^N}{n}},
\]

\[
T = \frac{n}{4\pi} \sqrt{\frac{n}{(n + p + Nnps^2)(1 + s^2)^N}} \frac{1}{r_0},
\]

\[
J = \frac{V(p) \Omega(p+1)}{16\pi G} R r_0^n \sqrt{p(1 + Nns^2)(n + p + Nnps^2)},
\]

\[
\Omega = \sqrt{\frac{p(1 + Nns^2)}{n + p + Nnps^2}} \frac{1}{R},
\]

\[
Q_p = \frac{\Omega(p+1)}{16\pi G} r_0^n \sqrt{Nns \sqrt{1 + s^2}}, \quad \Phi_H^{(p)} = \frac{V(p) \sqrt{Nn}}{\sqrt{s^2 + 1}}.
\]

Here $V(p) = R^n \Omega(p)$ is the volume of a round $p$-sphere with radius $R$. $J$ and $\Omega$ are the angular momentum and velocity along the vector $\partial/\partial \phi$ in (3.3). For this solution $J$ is equally split into components along each of the Cartan generators $\partial/\partial \phi_i$ of the rotations of the sphere, i.e., $J_i = J/(m + 1)$ and $\Omega_i = \Omega$.

As a check, these quantities agree with those derived in [3] for the 5D dipole ring at $n = p = 1$, in the limit $\nu, \mu, \lambda \to 0$ and $R \to \infty$, keeping fixed

\[
r_0 = \nu R, \quad r_0 \sinh^2 \alpha = \mu R, \quad r_0(2 + N \sinh^2 \alpha) = \lambda R
\]

and redefining $Q \to 4GQ_p$, $\Phi \to \Phi_H^{(p)}/(4G)$.
B  Black brane solutions in type II string theory

As an aid to the reader, in this appendix we give the supergravity solutions of the D0-Dp, F1-Dp and extremal D1-D5-P brane configurations in type II string theory. These are used in Sec. 6 to extract the thermodynamic properties of the corresponding effective fluids.

**D0-D2 brane**

String frame metric (see [36, 37]):

\[
d s^2 = H^{-\frac{1}{2}} \left[ -f dt^2 + D \sum_{i=1}^{2} dx_i^2 + H (f^{-1} dr^2 + r^2 d\Omega^2_6) \right]. \tag{B.1}
\]

Dilaton:

\[
e^{2\phi} = DH^{\frac{1}{2}}. \tag{B.2}
\]

NSNS and RR potentials:

\[
B_{12} = \tan \theta [DH^{-1} - 1], \quad A_0 = \coth \alpha \sin \theta [H^{-1} - 1], \quad A_{012} = \cot \alpha \sec \theta [DH^{-1} - 1]. \tag{B.3}
\]

Functions:

\[
f = 1 - \frac{r_0^5}{r^5}, \quad H = 1 + \frac{r_0^5 \sinh^2 \alpha}{r^5}, \quad D = (H^{-1} \sin^2 \theta + \cos^2 \theta)^{-1}. \tag{B.4}
\]

**D0-D4 brane**

String frame metric:

\[
d s^2 = H_0^{-\frac{1}{2}} H_4^{-\frac{1}{2}} \left[ -f dt^2 + H_0 \sum_{i=1}^{4} dx_i^2 + H_0 H_4 (f^{-1} dr^2 + r^2 d\Omega^2_4) \right]. \tag{B.5}
\]

Dilaton:

\[
e^{2\phi} = H_0^{\frac{3}{2}} H_4^{-\frac{1}{2}}. \tag{B.6}
\]

RR potentials:

\[
A_0 = \coth \alpha_0 [H_0^{-1} - 1], \quad A_{01234} = \coth \alpha_6 [H_4^{-1} - 1]. \tag{B.7}
\]

Functions:

\[
f = 1 - \frac{r_0^3}{r^3}, \quad H_i = 1 + \frac{r_0^3 \sinh^2 \alpha_i}{r^3}. \tag{B.8}
\]

for \( i = 0, 4 \).

**D0-D6 brane**

Take the solution of the static dyonic KK black hole (first derived in [38, 39], here we follow the presentation in [40]), uplift on \( T^6 \) to M-theory and then reduce on the \( y \)-direction to type IIA
string theory. In order to parallel the notation above, we also redefine $2m = r_0$, $q = r_0 \cosh^2 \alpha_0$ and $p = r_0 \cosh^2 \alpha_6$. This means

$$2Q = r_0 \cosh \alpha_0 \sinh \alpha_0 \sqrt{\frac{\cosh^2 \alpha_0 + 1}{\cosh^2 \alpha_0 + \cosh^2 \alpha_6}} \equiv q_0$$

(B.9)

$$2P = r_0 \cosh \alpha_6 \sinh \alpha_6 \sqrt{\frac{\cosh^2 \alpha_0 + 1}{\cosh^2 \alpha_0 + \cosh^2 \alpha_6}} \equiv q_6$$

(B.10)

String frame metric:

$$ds^2 = H_0^{-\frac{1}{2}} H_6^{-\frac{1}{2}} \left[ -f dt^2 + H_0 \sum_{i=1}^{6} dx_i^2 + H_0 H_6 (f^{-1} dr^2 + r^2 d\Omega_2^2) \right].$$

(B.11)

Dilaton:

$$e^{2\phi} = H_0^\frac{3}{2} H_6^{-\frac{3}{2}}.$$  

(B.12)

RR potentials:

$$A_0 = -\frac{q_0}{r} \left[ 1 + \frac{r_0}{2r} \sinh^2 \alpha_0 \right] H_0^{-1}, \quad A_0 = -q_6 \cos \theta.$$  

(B.13)

Functions:

$$f = 1 - \frac{r_0}{r}, \quad H_i = 1 + \frac{r_0 \sinh^2 \alpha_i}{r} + \frac{r_0^2 \cosh^2 \alpha_i}{2r^2} \frac{\sinh^2 \alpha_0 \sinh^2 \alpha_6}{\cosh^2 \alpha_0 + \cosh^2 \alpha_6}.$$  

(B.14)

for $i = 0, 6$.

One may compute

$$F_{0r} = \frac{q_0}{r^2} \frac{H_6}{H_0}, \quad F_{\theta\phi} = q_6 \sin \theta$$

(B.15)

We use $\tilde{F}_{\mu_0...\mu_p r} = -\sqrt{-g} e^{\alpha_0} \epsilon_{\nu_{p+2}...\nu_{D-1}} \epsilon_{\mu_0...\mu_p}{\tilde{F}}{}^{\nu_{p+2}...\nu_{D-1}}$ to compute the dual of the magnetic field strength (in Einstein frame) with $a = 3/2$ for the case at hand. This gives

$$\tilde{F}_{01...6r} = \frac{q_6}{r^2} \frac{H_0}{H_6^2}$$

(B.16)

and hence

$$A_{01...6} = -\frac{q_6}{r} \left[ 1 + \frac{r_0}{2r} \sinh^2 \alpha_0 \right] H_6^{-1}$$

(B.17)

Note also that for one of the $\alpha_i = 0$, the results agree with the (smeared on $\mathbb{T}^6$) D0-brane or the D6-brane respectively.

**F1-Dp brane**

We have $n = 7 - p$. The construction of F1-Dp branes was first explained in [37]. The explicit non-extremal solution that we use here was given in [41].

String frame metric:

$$ds^2 = D^{-\frac{1}{2}} H^{-\frac{1}{2}} \left[ -f dt^2 + dx_1^2 \right] + D\frac{3}{2} H^{-\frac{1}{2}} \sum_{i=2}^{p} dx_i^2 + D^{-\frac{1}{2}} H^{\frac{1}{2}} \left[ f^{-1} dr^2 + r^2 d\Omega_{n+1}^2 \right].$$

(B.18)
Dilaton:
\[ e^{2\phi} = D^{\frac{2p}{2}} H^{\frac{2}{2}} \]  
(B.19)

NSNS and RR potentials:
\[ B_{01} = \sin \theta (H^{-1} - 1) \coth \alpha , \quad A_{2p} = \tan \theta (H^{-1} D - 1) , \quad A_{01p} = \cos \theta D (H^{-1} - 1) \coth \alpha \]  
(B.20)

Functions:
\[ f = 1 - \frac{r_0^n}{r^n} , \quad H = 1 + \frac{r_0^n \sin^2 \alpha}{r^n} , \quad D^{-1} = \cos^2 \theta + \sin^2 \theta H^{-1} \]  
(B.21)

In the extremal limit we send \( r_0 \to 0 \) and \( \alpha \to \infty \) keeping finite
\[ q_\alpha = r_0^n \sin^2 \alpha \]  
(B.22)

If at the same time, we boost the system along \( x_1 \) with rapidity \( \eta \to \infty \) and keep finite
\[ q_p = r_0^n \sin^2 \eta \]  
(B.23)

we obtain (in string frame)
\[ ds^2 = D^{-1} H^{-\frac{1}{2}} \left[ dudv + \frac{q_p}{r^n} dv^2 \right] + D^{\frac{1}{2}} H^{-\frac{1}{2}} \sum_{i=2}^p dx_i^2 + D^{-\frac{1}{2}} H^{\frac{1}{2}} \left[ dr^2 + r^2 d\Omega_{n+1}^2 \right] \]  
(B.24)

with \( u = t + x_1 \) and \( v = t - x_1 \) and
\[ H = 1 + \frac{q_\alpha}{r^n} , \quad D^{-1} = \cos^2 \theta + \sin^2 \theta H^{-1} \]  
(B.25)

**Extremal D1-D5-P brane**

String frame metric:
\[ ds^2 = H_5^{-\frac{1}{2}} H_1^{-\frac{1}{2}} \left[ -dudv + \frac{q_p}{r^2} dv^2 + H_1 \sum_{i=1}^4 dx_i^2 + H_5 H_1 (dr^2 + r^2 d\Omega_3^2) \right] \]  
(B.26)

where
\[ H_1 = 1 + \frac{q_1}{r^2} , \quad H_5 = 1 + \frac{q_5}{r^2} \]  
(B.27)

and the lightcone coordinates are \( u = t + z \) and \( v = t - z \).

Dilaton:
\[ e^{2\phi} = H_5^{-1} H_1 \]  
(B.28)

RR potentials:
\[ A_{01234z} = H_5^{-1} - 1 , \quad A_{0z} = H_1^{-1} - 1 \]  
(B.29)

**C  Details on isothermal permittivities**

We collect some of the details of the isothermal permittivities computed for the effective Dp-brane fluids with q-brane charge that were used in Section 6.1. Similar quantities were computed in [42].
D0-D4 fluid

For the D0-D4 fluid the natural non-extremality parameter is

\[ x \equiv 1 - \left( \frac{Q_{D0} + Q_{D4}}{\varepsilon} \right)^2 \]  

(C.1)

which can be expressed in terms of \( \alpha_0 \) as

\[ x = 1 - \frac{9}{4} \left( \frac{1 + \frac{Q_{D4}}{Q_{D0}}}{4 - 3 \frac{Q_{D4}^2}{Q_{D0}^2} + 3 \cosh(2\alpha_0) + 3 \frac{Q_{D4}^2}{Q_{D0}^2} \cosh^2(2\alpha_0)} \right)^2 . \]  

(C.2)

Also in terms of \( \alpha_0 \) the permittivity of D0 charge \((\text{6.9})\) takes the form

\[ \varepsilon_{D0} = \frac{1}{Q_{D0}} \frac{\tanh\alpha_0 \left( -3 + \sqrt{1 + \frac{Q_{D4}^2}{Q_{D0}^2} \sinh^2(2\alpha_0)} \right)}{2(1 - \sinh^2\alpha_0) - (1 + 8 \sinh^2\alpha_0) \left( -1 + \sqrt{1 + \frac{Q_{D4}^2}{Q_{D0}^2} \sinh^2(2\alpha_0)} \right)} . \]  

(C.3)

When we vary \( \alpha_0 \) (and therefore \( x \)) at fixed \( Q_{D0}, Q_{D4} \), we find that \( \varepsilon_{D0} < 0 \) except in a finite range \( \alpha_0 \in (\alpha_-, \alpha_+) \) where \( \varepsilon_{D0} > 0 \). \( \alpha_- \) is the value of \( \alpha_0 \) where the denominator of the expression \((\text{C.3})\) vanishes and

\[ \sinh^2(2\alpha_+) = 8 \left( \frac{Q_{D0}}{Q_{D4}} \right)^2 . \]  

(C.4)

D0-D6 fluid

For the D0-D6 fluid the D0 charge permittivity \( \varepsilon_{D0} \) in terms of \( \alpha_0, \alpha_6 \) is given by

\[ \varepsilon_{D0} = \frac{1}{Q_{D0}} \frac{\sinh\alpha_0 \cosh^2\alpha_6 (\cosh^2\alpha_0 + \cosh^2\alpha_6)}{(1 + \cosh^2\alpha_6) \sqrt{(1 + \cosh^2\alpha_0)(\cosh^2\alpha_0 + \cosh^2\alpha_6)}} \frac{\mathcal{N}(\alpha_0, \alpha_6)}{\mathcal{D}(\alpha_0, \alpha_6)} \]  

(C.5)

where

\[ \mathcal{N}(\alpha_0, \alpha_6) = \cosh(3\alpha_0) (72 - 9 \cosh(2\alpha_6) + \cosh(6\alpha_6)) \]
\[ + \cosh\alpha_0 (412 + 263 \cosh(2\alpha_6) + 28 \cosh(4\alpha_6) + \cosh(6\alpha_6)) \]
\[ -16 \sech\alpha_0 \cosh^2\alpha_6 (-5 + \cosh(2\alpha_6))(3 + \cosh(2\alpha_6)) , \]  

(C.6)

\[ \mathcal{D}(\alpha_0, \alpha_6) = 71 + 112 \cosh(2\alpha_0) + 37 \cosh(4\alpha_0) + 4 \cosh(6\alpha_0) \]
\[ + \cosh(2\alpha_6) (99 + 129 \cosh(2\alpha_0) + 27 \cosh(4\alpha_0) + \cosh(6\alpha_0)) \]
\[ - \cosh(4\alpha_6) (-27 - 13 \cosh(2\alpha_0) + 7 \cosh(4\alpha_0) + \cosh(6\alpha_0)) \]
\[ -4 \sinh^2\alpha_0 (2 + \cosh(2\alpha_0)) \cosh(6\alpha_6) . \]  

(C.7)
F1-Dp fluid

For the F1-Dp fluid the natural non-extremality parameter is (in analogy to (6.4) for the D0-D2 system),

\[ x \equiv 1 - \frac{Q_{F1}^2 + Q_{Dp}^2}{\epsilon^2} \]  \hspace{1cm} (C.8)

so that

\[ \cosh^2 \alpha = \frac{n + 2 - 2x + \sqrt{(n + 2)^2 - 4(n + 1)x}}{2nx} . \]  \hspace{1cm} (C.9)

The permittivity of F1 charge density \([6.22]\) exhibits different behavior depending on the value of \(n\). For \(n = 1, 2\) the F1 charge permittivity is always positive. For \(n > 2\), \(\epsilon_{F1} > 0\) except within the interval

\[ x \in \left( \frac{1 + n(n + 2) \left( 1 + \frac{1}{n-2} \frac{Q_{F1}^2 + Q_{Dp}^2}{Q_{Dp}^2} \right),}{1 + n \left( 1 + \frac{1}{n-2} \frac{Q_{F1}^2 + Q_{Dp}^2}{Q_{Dp}^2} \right)^2}, \frac{(n - 2)(n - 2 + n(n - 1)(n + 2)}{(n^2 - 2)^2} \right) \]  \hspace{1cm} (C.10)

where it is negative.

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