Spin operator matrix elements in the quantum Ising chain: fermion approach

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Abstract. Using some modification of the standard fermion technique we derive factorized formulas for spin operator matrix elements (form factors) between general eigenstates of the Hamiltonian of the quantum Ising chain in a transverse field of finite length. The derivation is based on the approach recently used to derive factorized formulas for $Z_N$-spin operator matrix elements between ground eigenstates of the Hamiltonian of the $Z_N$-symmetric superintegrable chiral Potts quantum chain. The obtained factorized formulas for the matrix elements of the Ising chain coincide with the corresponding expressions obtained by the separation of variables method.

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1. Introduction

During the past decade more attention has been drawn to spin matrix elements in finite Ising systems. Bugrij and Lisovyy in [1,2] have pointed out that one may write completely factorized closed expressions for spin matrix elements and a form-factor representation for the correlation function of the two-dimensional Ising model on a finite lattice. The finite volume form factors of the spin field in the Ising field theory was derived by Fonseca and Zamolodchikov [3]. The Bugrij–Lisovyy formula for spin matrix elements on a finite lattice was proved in [4,5] by the separation of variables method. We also note that in a recent paper [6] this formula was derived using an approach based on Clifford algebra symmetry, suggested in [7,8].

Using the separation of variables method the factorized formulas for spin operator matrix elements were obtained for the quantum Ising chain in a transverse field [4,5] and for the XY quantum chain [9]. The quantum Ising chain (QIC) is the special case ($N = 2$) of the $Z_N$-symmetric superintegrable chiral Potts quantum chain (SCPC) [10]. In the framework of extended Onsager algebra [11], Baxter conjectured the SCPC-spin matrix elements for the ground state Onsager sectors [12]. Independently, in the same framework in [13], these matrix elements were re-obtained, generalized to arbitrary Onsager sectors and proved. There it was shown that the matrix elements of spin operators between the eigenvectors of the SCPC Hamiltonian for arbitrary Onsager sectors can be presented in a factorized form with unknown normalization factors $N_{PQ}$ depending only on Onsager sectors but not on particular vectors of these sectors. The derivation does not use the

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information on how these Onsager sectors are included in a large quantum space of the
model. But in order to calculate scalar factors $N_{PQ}$ we need such information. Thus,
probably the derivation of these factors will require information on the Bethe-states of
the related $\tau_3$-model and $sl(2)$-loop algebra symmetries [14]–[18].

In [11] Baxter noted that the calculation of the matrix elements of spin operators
between eigenvectors of the Hamiltonian $H = H_0 + \hbar H_1$ is greatly simplified if one starts
from the calculation of the matrix elements of spin operators between eigenvectors of
the Hamiltonian $H_0$ (that is of $H$ at $\hbar = 0$). In this paper we use a fermion technique
to describe the Onsager sectors of the QIC. To identify the bases of Onsager sectors in [13]
with the fermion bases we diagonalize $H_0$ by means of dual Jordan–Wigner fermion
operators. As in the case of the duality transformation of the 2D Ising model on a
finite lattice [19], the duality transformation for the finite quantum Ising chain requires
consideration of the periodic and antiperiodic boundary conditions in a uniform way
together with a modification of the standard Jordan–Wigner fermions. Then derivation of
the spin matrix elements between the eigenstates of $H_0$ reduces to a Cauchy determinant,
which in turn allows one to extract the factors $N_{PQ} = N_{P_0Q_0}$ in an explicit form.
In order to get the spin matrix elements between the eigenstates of $H$ we use a Bogoliubov
transformation of the fermion operators. This transformation of a pair of fermion
operators with opposite momenta is directly related to rotation in a two-dimensional
tensor component of Onsager sectors. The Bogoliubov transformation leads to a sum over
the different pairs of fermions which can be summed to get a factorized formula (79) for
matrix elements of the spin operator between the eigenvectors of $H$.

This paper is organized as follows: in section 2 we define the Hamiltonian of the QIC
with unified periodic and antiperiodic boundary conditions, duality transformation, as well
as Jordan–Wigner and dual Jordan–Wigner fermions. In section 3 the derivation of matrix
elements of the spin operator between eigenstates of $H_0$ is presented by reducing them to
in a Cauchy determinant. This allows us to find the scalar factors $N_{P_0Q_0}$ (appendix A). In
section 4 we define the Bogoliubov transformation of fermions and find a relation between
the eigenstates of $H_0$ and $H_1$. In section 5 we use the results of sections 3 and 4 to
get factorized matrix elements of the spin operator between the eigenstates of $H$. The
derivation uses a summation formula from appendix B. Section 6 summarizes our results.

2. The quantum Ising chain

2.1. The Hamiltonian of the quantum Ising chain

The quantum Ising chain (QIC) of length $L$ with periodic boundary condition is defined
by the Hamiltonian

$$H^P = H_0^P + \hbar H_1^P = -\sum_{k=1}^{L-1} \sigma_k^x \sigma_{k+1}^x - \sigma_1^x \sigma_L^x - \hbar \sum_{k=1}^{L} \sigma_k^z. \quad (1)$$

The same model with antiperiodic boundary condition is governed by the Hamiltonian

$$H^A = H_0^A + \hbar H_1^A = -\sum_{k=1}^{L-1} \sigma_k^x \sigma_{k+1}^x + \sigma_1^x \sigma_L^x - \hbar \sum_{k=1}^{L} \sigma_k^z. \quad (2)$$
Both Hamiltonians include a term describing interaction with a transverse magnetic field of strength $h$. The space of states of these systems is the $L$-fold tensor product of two-dimensional spaces $V_j, j = 1, \ldots, L$: $V = V_1 \otimes \cdots \otimes V_L$. The spin operators

$$\sigma^\alpha_k = 1 \otimes \cdots \otimes 1 \otimes \sigma^\alpha_{k\text{th}} \otimes 1 \otimes \cdots \otimes 1,$$

where $\alpha = x, y, z, k = 1, \ldots, L$, and $\sigma^\alpha$ are Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

act non-trivially only on the space $V_k$ in the tensor product $V$. In order to calculate the matrix elements of the spin operator $\sigma^\alpha_k$ in this model it is convenient to consider the periodic and antiperiodic boundary conditions in a unified formalism. For this aim we introduce an additional auxiliary two-dimensional space $V_0$. The space of states becomes

$$V = V_0 \otimes V_1 \otimes \cdots \otimes V_L$$

and the unified Hamiltonian is

$$H = H_0 + hH_1,$$

where

$$H_0 = -\sum_{k=1}^{L-1} \sigma^x_k \sigma^x_{k+1} - \sigma^y_k \sigma^y_{k+1} = -\sum_{k=1}^{L} \sigma^x_k \sigma^x_{k+1}, \quad H_1 = -\sum_{k=1}^{L} \sigma^z_k,$$

with the following boundary conditions for spin operators

$$\sigma^x_{L+1} = \sigma^x_1, \quad \sigma^y_{L+1} = \sigma^y_1,$$

which can be extended to a ‘quasiperiodic’ condition

$$\sigma^x_{k+L} = \sigma^x_k, \quad \sigma^y_{k+L} = \sigma^y_k.$$  

In what follows the main object of our consideration is the Hamiltonian (4) acting on the space (3). The space of states $V$ decomposes into a direct sum $V = V^P \oplus V^A$ of two subspaces $V^P$ and $V^A$ with eigenvalues $+1$ and $-1$ of $\sigma^z$, respectively. Since $\sigma^z$ commutes with the Hamiltonian $H$, these two subspaces are invariant under the action of $H$. The restriction of $H$ to these subspaces is $H^P$ and $H^A$, respectively. The boundary conditions (6) become periodic conditions, $\sigma^x_{L+1} = \sigma^x_1, \sigma^y_{L+1} = \sigma^y_1,$ or antiperiodic conditions, $\sigma^x_{L+1} = -\sigma^x_1, \sigma^y_{L+1} = \sigma^y_1,$ with respect to the eigenvalue of $\sigma^z$.

The Hamiltonian $H$ also commutes with the Hermitian operator $W$

$$W = -\sigma^z_0 \prod_{k=1}^{L} \sigma^z_k.$$  

Since $W^2 = 1$, the eigenvalues of $W$ are $w = \pm 1$ and the space $V$ can be decomposed into the direct sum $V = V_{NS} \oplus V_R$ of two subspaces $V_{NS}$ and $V_R$ corresponding to the eigenvalues $w = -1$ and $w = +1$, respectively. The subspace $V_{NS}$ (resp. $V_R$) is called the Neveu–Schwarz sector or NS-sector (resp. Ramond sector or R-sector).

With respect to the eigenvalues of the commuting operators $\sigma^z$ and $W$, the space of states $V$ decomposes into the following direct sum:

$$V = V^A_{NS} \oplus V^P_{NS} \oplus V^A_R \oplus V^P_R,$$
where, for example, if $|\Psi\rangle \in \mathcal{V}_{NS}^A$ we have $\mathcal{W}|\Psi\rangle = -|\Psi\rangle$, $\sigma_b^x|\Psi\rangle = -|\Psi\rangle$. Each of these subspaces has dimension $2^{L-1}$ and is invariant under the action of $H$, $H_0$ and $H_1$ because they commute with $\sigma_b^x$ and $\mathcal{W}$.

Let us define the translation operator on $\mathcal{V}$
\[
T = \frac{1}{2}(1 + \sigma_b^x + \sigma_1^z - \sigma_b^x \sigma_1^z)T_{1,2}T_{2,3} \cdots T_{L-1,L},
\]
where the operator
\[
T_{k,k+1} = \frac{1}{2}(1 + \sigma_b^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_b^x \sigma_k^x)
\]
swaps the vectors in the spaces $V_k$ and $V_{k+1}$. The translation operator $T$ acts on the local spin operators as (see also (6))
\[
T \sigma_k^x = \sigma_{k+1}^x T, \quad T \sigma_k^z = \sigma_{k+1}^z T, \quad T \sigma_b^x = \sigma_b^x T, \quad T \sigma_b^z = \sigma_b^z \sigma_1^z T
\]
and satisfies the relations
\[
TT^+ = 1, \quad T^L = \frac{1}{2}(1 + \sigma_b^x + W - \sigma_b^x W), \quad T^{2L} = 1, \quad TW = WT.
\]
The QIC possesses generalized translation invariance with respect to $T$, i.e. $[H,T] = 0$. On the subspace $\mathcal{V}^0$ the operator $T$ turns into standard translation operator.

### 2.2. Duality transformation

The presence of the matrix $\sigma_b^x$ in the Hamiltonian (4) results in some modification of the standard duality transformation of the QIC. We place the dual Pauli matrices $\tilde{\sigma}_l^\alpha$, $l = 1, \ldots, L$, on the dual chain sites which are located between the neighboring sites of the original chain, and take the following numbering of dual sites: the dual site $l$ is located between the sites $l$ and $l + 1$ of the original chain. We define the dual Pauli matrices by the following relations:
\[
\tilde{\sigma}_l^x = \sigma_b^x \prod_{k=1}^l \sigma_k^z, \quad \tilde{\sigma}_l^z = \sigma_1^z \sigma_{l+1}^z, \quad \tilde{\sigma}_1^z = \sigma_b^x \sigma_1^z, \quad \tilde{\sigma}_L^x = \sigma_b^x \prod_{l=1}^L \sigma_l^z,
\]
\[
\tilde{\sigma}_L^z = \sigma_b^x \sigma_1^z \sigma_L^z, \quad \tilde{\sigma}_1^z = \sigma_b^x \sigma_2^z, \quad \tilde{\sigma}_b^x = \sigma_b^x \sigma_1^z, \quad \tilde{\sigma}_b^z = \prod_{l=1}^L \sigma_l^z.
\]

The inverse duality transformation has the form
\[
\sigma_k^x = \tilde{\sigma}_k^x \prod_{l=k}^L \tilde{\sigma}_l^z, \quad \sigma_k^z = \tilde{\sigma}_k^z \tilde{\sigma}_{k-1}^z, \quad \sigma_L^x = \tilde{\sigma}_L^x \tilde{\sigma}_L^z, \quad \sigma_1^z = \tilde{\sigma}_1^z \prod_{l=1}^L \tilde{\sigma}_l^z,
\]
\[
\sigma_L^z = \tilde{\sigma}_L^z \tilde{\sigma}_L^x, \quad \sigma_1^z = \tilde{\sigma}_1^z \tilde{\sigma}_1^x \tilde{\sigma}_L^z, \quad \sigma_b^x = \tilde{\sigma}_b^x \tilde{\sigma}_L^z, \quad \sigma_b^z = \prod_{l=1}^L \tilde{\sigma}_l^z.
\]

Duality transformation interchanges $H_0$ and $H_1$ and, in terms of the dual Pauli matrices, the Hamiltonian $H$ becomes
\[
H = H_0 + \hbar H_1 = \hbar \left( -\sum_{l=1}^L \tilde{\sigma}_l^x \tilde{\sigma}_{l+1}^x - \tilde{\hbar} \sum_{k=1}^L \tilde{\sigma}_k^z \right) = \hbar \left( \tilde{H}_0 + \tilde{\hbar} \tilde{H}_1 \right) = \hbar \tilde{H},
\]
where $\tilde{\hbar} = \hbar^{-1}$ and $\tilde{H}$ is the dual Hamiltonian with boundary conditions
\[
\tilde{\sigma}_{L+1}^x = \tilde{\sigma}_b^z \tilde{\sigma}_1^z, \quad \tilde{\sigma}_{L+1}^z = \tilde{\sigma}_1^z.
\]
Let us define the dual operator $\tilde{W}$ by
\[
\tilde{W} = -\hat{\sigma}_b^z \prod_{k=1}^L \hat{\sigma}_k^z.
\]
It satisfies the relation $\tilde{W} = W$, that is, $W$ is invariant under the duality transformation.

### 2.3. Jordan–Wigner fermion operators

Usually the first step for diagonalization of the QIC Hamiltonian is the introduction of the Jordan–Wigner (JW) fermion creation and annihilation operators. In our case they are
\[
c_k = \sigma_b^x \cdot \prod_{l=1}^{k-1} \sigma_l^z \cdot \sigma_k^+, \quad c_k^+ = \sigma_b^x \cdot \prod_{l=1}^{k-1} \sigma_l^z \cdot \sigma_k^-,
\]
where $\sigma_k^\pm = (\sigma_k^x \pm i\sigma_k^y)/2$. It is easy to verify that JW operators (16) satisfy:

(a) the standard anticommutation relations
\[
\{c_k, c_l^+\} = \delta_{k,l}, \quad \{c_k, c_l\} = 0, \quad \{c_k^+, c_l^+\} = 0,
\]
(b) the boundary conditions (we take into account (6) and $\sigma_{L+1}^y = \sigma_1^x\sigma_1^y$)
\[
c_{L+1} = WC_1, \quad c_{L+1}^+ = WC_1^+,
\]
(c) the ‘quasiperiodic’ conditions
\[
c_{k+L} = WC_k, \quad c_{k+L}^+ = WC_k^+,
\]
(d) action of the translation operator $T$ consistent with (19)
\[
Tc_k = c_{k+1}T, \quad Tc_k^+ = c_{k+1}^+T.
\]

In terms of the fermion number operator $Q = \sum_{k=1}^L c_k^+c_k$ we have the relations
\[
W = -\hat{\sigma}_b^z \prod_{k=1}^L \hat{\sigma}_k^z = -\hat{\sigma}_b^z \prod_{k=1}^L \exp(\pi c_k^+c_k) = -\sigma_b^z(-1)^Q,
\]
\[
[W, c_k] = 0, \quad [W, c_k^+] = 0.
\]

The operators $H_0$ and $H_1$ from (5) in terms of the JW fermion operators take the form
\[
H_0 = -\sum_{k=1}^L \sigma_k^x\sigma_{k+1}^x = -\sum_{k=1}^L (c_k^+ - c_k)(c_{k+1}^+ + c_{k+1}),
\]
\[
H_1 = -\sum_{k=1}^L \sigma_k^z = 2\sum_{k=1}^L (c_k^+c_k - \frac{1}{2}) = \sum_{k=1}^L (c_k^+ - c_k)(c_k^+ + c_k).
\]

The last equation implies that the JW fermion operators diagonalize $H_1$.

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2.4. Dual Jordan–Wigner fermion operators

As discussed in section 1, in order to calculate the spin operator matrix elements in the QIC we must first solve the auxiliary problem, namely, to calculate the matrix element in the basis of eigenvectors of the Hamiltonian $H_0$, which is $H$ at $h = 0$. In order to diagonalize $H_0$ it is convenient to define the dual JW fermion creation and annihilation operators $c_k^+, a_k$, $k = 1, \ldots, L$, through the relations

$$c_k^+ - c_k = -(a_k^+ - a_k), \quad c_{k+1}^+ + c_k = a_k^+ + a_k$$

interchanging $H_0$ and $H_1$ given by (23) and (24):

$$H_0 = -\sum_{k=1}^{L}(c_k^+ - c_k)(c_{k+1}^+ + c_k) = \sum_{k=1}^{L}(a_k^+ - a_k)(a_k^+ + a_k) = 2\sum_{k=1}^{L}(a_k^+ a_k - \frac{1}{2}),$$

$$H_1 = \sum_{k=1}^{L}(c_k^+ - c_k)(c_k^+ + c_k) = -\sum_{k=1}^{L}(a_k^+ - a_k)(a_k^+ + a_k).$$

From (25), we have the explicit formulas for the dual JW fermion operators

$$a_k = \frac{1}{2}(c_{k+1}^+ + c_{k+1} + (c_k^+ - c_k)), \quad a_k^+ = \frac{1}{2}(c_{k+1}^+ + c_{k+1} - (c_k^+ - c_k)),$$

Also these operators can be written in terms of $\sigma_k^\alpha$

$$a_k = \frac{1}{2}\sigma_k^x \cdot \prod_{l=1}^{k} \sigma_l^z \cdot (\sigma_{k+1}^+ - \sigma_k^-), \quad a_k^+ = \frac{1}{2}\sigma_k^x \cdot \prod_{l=1}^{k} \sigma_l^z \cdot (\sigma_{k+1}^+ + \sigma_k^-)$$

or in terms of the dual operators $\tilde{\sigma}_k^\alpha$

$$a_k = \tilde{\sigma}_k^x \cdot \prod_{l=k+1}^{L} \tilde{\sigma}_l^z \cdot \tilde{\sigma}_k^+, \quad a_k^+ = \tilde{\sigma}_k^x \cdot \prod_{l=k+1}^{L} \tilde{\sigma}_l^z \cdot \tilde{\sigma}_k^-,$$

where $\tilde{\sigma}_k^x = (\sigma_k^x \pm i\sigma_k^y)/2$ and $\tilde{\sigma}_k^y = i\sigma_k^x \sigma_k^z$.

Due to the linearity of the transformation (28), and using the properties (17)–(22) of the JW operators $c_l^+, c_l$ together with (29), a straightforward calculation for the dual JW operators gives:

(a) the anticommutation relations

$$\{a_k, a_l^+\} = \delta_{k,l}, \quad \{a_k, a_l\} = 0, \quad \{a_k^+, a_l^+\} = 0,$$

(b) the boundary conditions

$$a_{L+1} = W a_1, \quad a_{L+1}^+ = W a_1^+,$$

(c) the ‘quasiperiodic’ conditions

$$a_{k+L} = W a_k, \quad a_{k+L}^+ = W a_k^+,$$

(d) the commutativity with the operator $W = W$

$$[W, a_k] = [W, a_k^+] = 0,$$

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(e) the anticommutativity with the operator \( \sigma_b^z \)
\[
\{ \sigma_b^z, a_k \} = 0, \quad \{ \sigma_b^z, a_k^+ \} = 0,
\]

(f) action of the translation operator \( T \) consistent with (33)
\[
Ta_k = a_{k+1}T, \quad Ta_k^+ = a_{k+1}^+T.
\]

From (34) it follows that the subspaces \( \mathcal{V}_{\text{NS}} \) and \( \mathcal{V}_{\text{R}} \) are invariant with respect to the action of fermion algebra (31) having a unique irreducible representation—the standard Fock representation of dimension \( 2^L \). Since the dimensions of \( \mathcal{V}_{\text{NS}} \) and \( \mathcal{V}_{\text{R}} \) are also \( 2^L \), the action of the algebra (31) on each of them is irreducible. We will denote the corresponding vacuum states in the irreducible subspaces \( \mathcal{V}_{\text{NS}} \) and \( \mathcal{V}_{\text{R}} \) by \( |0\rangle_{0,\text{NS}} \) and \( |0\rangle_{0,\text{R}} \), respectively. Using (8) and (29) it is easy to verify that the following two vectors
\[
|0\rangle_{0,\text{NS}} = \frac{1}{2^{(L+1)/2}} \left[ \begin{pmatrix} (1)^{(b)} & (1)^{(1)} & \cdots & (1)^{(L)} \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} (1)^{(b)} & (1)^{(1)} & \cdots & (1)^{(L)} \\ 0 \\ -1 \\ -1 \end{pmatrix} \right], \\
|0\rangle_{0,\text{R}} = \frac{1}{2^{(L+1)/2}} \left[ \begin{pmatrix} (1)^{(b)} & (1)^{(1)} & \cdots & (1)^{(L)} \\ 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} (1)^{(b)} & (1)^{(1)} & \cdots & (1)^{(L)} \\ 0 \\ -1 \\ -1 \end{pmatrix} \right]
\]
belonging to \( \mathcal{V} = \mathcal{V}_{\text{NS}} \oplus \mathcal{V}_{\text{R}} = V_b \otimes V_1 \otimes \cdots \otimes V_L \) satisfy the necessary equations for these vacuum states:
\[
\mathcal{W}|0\rangle_{0,\text{NS}} = -|0\rangle_{0,\text{NS}}, \quad a_l|0\rangle_{0,\text{NS}} = 0, \quad l = 1, \ldots, L, \\
\mathcal{W}|0\rangle_{0,\text{R}} = |0\rangle_{0,\text{R}}, \quad a_l|0\rangle_{0,\text{R}} = 0, \quad l = 1, \ldots, L.
\]
Also we have \( T|0\rangle_{0,\text{NS}} = |0\rangle_{0,\text{NS}} \) and \( T|0\rangle_{0,\text{R}} = |0\rangle_{0,\text{R}} \).

Because of (34) we may restrict operators \( a_l, a_l^+ \), \( l = 1, \ldots, L \), to the Neveu–Schwarz sector \( \mathcal{V}_{\text{NS}} \) (resp. to the Ramond sector \( \mathcal{V}_{\text{R}} \)) where \( \mathcal{W} \) has eigenvalue \( -1 \) (resp. \( +1 \)). The restricted operators will be denoted \( f_l, f_l^+ \)—Neveu–Schwarz fermion operators (resp. \( d_l, d_l^+ \)—Ramond fermion operators) which due to (32) obey antiperiodic (resp. periodic) boundary conditions:
\[
f^+_L = -f_L^+, \quad d^+_L = d_L^+.
\]
The set of the vectors
\[
(f_{1}^{+})^{n_1}(f_{2}^{+})^{n_2} \cdots (f_{L}^{+})^{n_L} |0\rangle_{0,\text{NS}}, \quad (d_{1}^{+})^{n_1}(d_{2}^{+})^{n_2} \cdots (d_{L}^{+})^{n_L} |0\rangle_{0,\text{R}}, \quad n_k \in \{0, 1\},
\]
constitute a basis of \( \mathcal{V} = \mathcal{V}_{\text{NS}} \oplus \mathcal{V}_{\text{R}} \) on which the operator (26) is diagonal.

As was explained in section 2.1, the boundary condition of \( H \) and \( H_0 \) is fixed by the eigenvalues of the operator \( \sigma_b^z \). Using (30), one can relate the dual fermion number operator \( \tilde{Q} \) to \( \sigma_b^z \):
\[
\sigma_b^z = \prod_{l=1}^{L} \tilde{\sigma}_l^z = (-1)^{\tilde{Q}}, \quad \tilde{Q} = \sum_{k=1}^{L} a_k^+ a_k.
\]
Hence the states from (40) with an even number of excitations (including the vacuum states \( |0\rangle_{0,\text{NS}} \) and \( |0\rangle_{0,\text{R}} \)) belong to \( \mathcal{V}^\text{P} \) and the states with an odd number of excitations belong to \( \mathcal{V}^\text{A} \).
Let us define the momentum representation for the dual Jordan–Wigner fermion operators. It is defined by means of the discrete Fourier transformation:

\[ f_q = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} f_l e^{-iq_l}, \quad f_q^+ = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} f_l^* e^{iq_l}, \quad \text{(42)} \]

\[ d_p = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} d_l e^{-ip_l}, \quad d_p^+ = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} d_l^* e^{ip_l}, \quad \text{(43)} \]

where in the NS-sector, due to the antiperiodicity condition for the NS-fermion operators in (39), the momentum \( q \) takes \('half-integer'\) values \( q = 2\pi(k + 1/2)/L, \ k \in \mathbb{Z} \), and in the R-sector, due to the periodicity condition for the R-fermion operators in (39), the momentum \( p \) takes \('integer'\) values \( p = 2\pi k/L, \ k \in \mathbb{Z} \). All the momenta are defined up to a multiple of 2\( \pi \). The set of \('half-integer'\) (resp. \('integer'\)) momenta \( q \) with \(-\pi < q \leq \pi\) will be denoted \( Q \) (resp. \( P \)).

All the momenta enter \( Q \) (resp. \( P \)) in pairs (i.e. two different momenta \( q, -q \in Q \)) except the momenta 0 and \( \pi \). The momentum \( 0 \in P \), the momentum \( \pi \in P \) if the length \( L \) of the chain is even and \( \pi \in Q \) if \( L \) is odd. In what follows we will use the basis

\[ |Q\rangle_{0,\text{NS}} = |q_1, \ldots, q_m\rangle_{0,\text{NS}} = f_{q_1}^+ f_{q_2}^+ \cdots f_{q_m}^+ |0\rangle_{0,\text{NS}} \in \mathcal{V}_{\text{NS}}, \]

\[ |P\rangle_{0,\text{R}} = |p_1, \ldots, p_n\rangle_{0,\text{R}} = d_{p_1}^+ d_{p_2}^+ \cdots d_{p_n}^+ |0\rangle_{0,\text{R}} \in \mathcal{V}_{\text{R}} \]

of \( \mathcal{V} \) instead of basis (40). These states are labeled by the subsets \( Q \subset Q \) and \( P \subset P \) of excited fermion momenta in the NS-sector and R-sector, respectively. We introduce the subsets \( Q_+, Q_-, Q_0 \subset Q \) as

\[ Q_+ = \{ q \in Q | 0 < q < \pi, -q \in Q \}, \quad Q_- = \{ q \in Q | -\pi < q < 0, -q \in Q \}, \]

\[ Q_0 = Q \setminus (Q_+ \cup Q_-) \] and the subsets \( Q_+, Q_-, Q_0 \subset Q = Q \setminus Q \) by similar formulas with the replacement \( Q \rightarrow \bar{Q} \). The cardinality of the sets \( Q, Q_+, \) etc are \( m, m_+, \) etc, respectively.

We also will use the sets \( Q_+ = Q_+ \cup Q_+, Q_0 = Q_0 \cup Q_0, Q_- = Q_- \cup Q_- \). Analogously we introduce the subsets \( P, P_+, \) etc of \( P \) which have the cardinality \( n, n_+, \) etc and the sets \( P_+ = P_+ \cup P_+, P_0 = P_0 \cup P_0, P_- = P_- \cup P_- \). We have the relations

\[ L = m + \bar{m} = n + \bar{n}, \quad m = 2m_+ + m_0, \quad \bar{m} = 2\bar{m}_+ + \bar{m}_0, \]

\[ n = 2n_+ + n_0, \quad \bar{n} = 2\bar{n}_+ + \bar{n}_0. \]

Formulas (42) and (43) imply that the translation operator \( T \) acts diagonally on fermion operators in the momentum representation:

\[ T f_q = e^{i q} f_q T, \quad T f_q^+ = e^{-i q} f_q^+ T, \quad T d_p = e^{i p} d_p T, \quad T d_p^+ = e^{-i p} d_p^+ T. \]

(46)

Using (26) and (27) we can rewrite restrictions of \( H_0 \) and \( H_1 \) on the NS-and R-sectors in terms of the dual Jordan–Wigner fermion operators in the momentum representation:

\[ H_0 = \sum_q (2a_q^+ a_q - 1), \quad H_1 = \sum_q ((1 - 2a_q^+ a_q) \cos q + i(a_q^+ a_{-q}^+ + a_q a_{-q}) \sin q), \]

(47)

where for the NS-sector \( a_q = f_q, \ q \in Q \), and for the R-sector \( a_q = d_q, \ q \in P \). In what follows we will often use the same notations \( H, H_0 \) and \( H_1 \) for the restrictions of the corresponding Hamiltonians (4) and (5) to the subspaces \( \mathcal{V}_{\text{NS}} \) and \( \mathcal{V}_{\text{R}} \).
For convenience, we will use the following ordering of the momenta in the state \( |Q\rangle_{0,NS} = |q_1, q_2, \ldots, q_n\rangle_{0,NS} \): from the beginning the pairs \((-q, q)\), for \( q \in Q_+\), go in sequence and then \( q \in Q_0 \) go in sequence in an order fixed over all this paper. Such an ordering defines the sign of \( |Q\rangle_{0,NS} \) uniquely. Analogously we define the ordering of the momenta for \( |P\rangle_{0,R} \). It is obvious that the sets \( Q_+, Q_0 \) and \( P_+, P_0 \) completely characterize the states \( |Q\rangle_{0,NS} \) and \( |P\rangle_{0,R} \). Hence we can use the following notations:

\[
|Q_+, Q_0\rangle_{0,NS} = |Q\rangle_{0,NS}, \quad |P_+, P_0\rangle_{0,R} = |P\rangle_{0,R}.
\]

The operators \( H_0 \) and \( H_1 \) generate Onsager algebra. From (47) it follows that the space \( \mathcal{V}_{Q_0} \subset \mathcal{V}_{NS} \) of all the states with the same fixed \( Q_0 \) is invariant with respect to the action of \( H_0 \) and \( H_1 \) and in fact it is space of an irreducible representation of this Onsager algebra. The basis vectors in \( \mathcal{V}_{Q_0} \) are labeled by the sets \( Q_+ \) and the dimension of \( \mathcal{V}_{Q_0} \) is \( 2^{Q_+} \).

The basis vectors in \( \mathcal{V}_{Q_0} \) can be identified with the tensor product of \( |Q_+\rangle \) two-dimensional spaces with each tensor component labeled by \( q \in Q_+ \). To vector \( i^{-m+} |Q_+, Q_0\rangle_{0,NS} \) we put in correspondence the tensor product of two-dimensional vectors in each tensor component: \((1, 0)^T \) if \( q \in Q_+ \) and \((0, 1)^T \) if \( q \in Q_- \). Then from (47) it follows that in the basis of vectors \( i^{-m+} |Q_+, Q_0\rangle_{0,NS} \) with the above-mentioned identification of bases the Hamiltonian \( H \) has the form of equation (10) from [13]. We have an analogous situation for Onsager sector \( \mathcal{V}_{R_0} \subset \mathcal{V}_R \).

Finally, we list the number of Onsager sectors: in NS-sector we have \( 3^{L/2} \) (resp. \( 2 \times 3^{(L-1)/2} \)) Onsager sectors for even \( L \) (resp. odd \( L \)), in R-sector we have \( 4 \times 3^{(L-2)/2} \) (resp. \( 2 \times 3^{(L-1)/2} \)) Onsager sectors for even \( L \) (resp. odd \( L \)).

### 3. Spin operator matrix elements between the eigenstates of the Hamiltonian \( H_0 \)

In this section we will find the matrix elements of the spin operator \( \sigma^z_1 \) between eigenstates (44) and (45) of \( H_0 \). The relation \( W \sigma^z_1 = -\sigma^z_1 W \) gives that the action of \( \sigma^z_1 \) swaps two eigenvalues \( \pm 1 \) of \( W \), that is the operator \( \sigma^z_1 \) maps \( \mathcal{V}_{NS} \) to \( \mathcal{V}_R \) and vice versa. The operator \( \sigma^z_1 \) commutes with \( H_0 \) from (5) and therefore, due to (26), it commutes with the dual fermion number operator \( \tilde{Q} \) from (41). Hence the action of \( \sigma^z_1 \) conserves the number of excitations. In particular, it should map \( |0\rangle_{0,NS} \) to a vector proportional to \( |0\rangle_{0,R} \). Using explicit formulas (37) and (38) we get

\[
|0\rangle_{0,R} = \sigma^z_1 |0\rangle_{0,NS}, \quad |0\rangle_{0,NS} = \sigma^z_1 |0\rangle_{0,R}.
\]

Using (29) we get the commutation relations

\[
\begin{align*}
&\sigma^z_1 a_k + a^+_k \sigma_1^z = 0, \quad a_k^+ \sigma_1^z + \sigma_1^z a_k^+ = 0, \quad k = 1, 2, \ldots, L, \\
&\sigma^z_1 d_k \sigma^z_1 = -f_k, \quad \sigma^z_1 d_k^+ \sigma^z_1 = -f_k^+, \quad k = 1, 2, \ldots, L,
\end{align*}
\]

which give after Fourier transformations (42) and (43)

\[
\begin{align*}
\sigma^z_1 d_p \sigma^z_1 &= -\frac{1}{\sqrt{L}} \sum_k f^+_k e^{-ipk} = -\frac{1}{L} \sum_k \sum_q f_q e^{i(q-p)k} = \frac{2}{L} \sum_q f_q \frac{1}{1 - e^{i(p-q)}}, \\
\sigma^z_1 d_p^+ \sigma^z_1 &= -\frac{1}{\sqrt{L}} \sum_k f^+_k e^{ipk} = -\frac{1}{L} \sum_k \sum_q f^+_q e^{-i(q-p)k} = \frac{2}{L} \sum_q f^+_q \frac{1}{1 - e^{-i(p-q)}},
\end{align*}
\]

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allowing the calculation of the matrix elements $0_R \langle P | \sigma_1^T | Q \rangle_{0,NS}$ of the spin operator $\sigma_1^T$ between vectors (44) and (45). Since $\sigma_1^T$ conserves the number of excitations, the non-zero matrix elements will appear only if $n = m$:

$$0_R \langle p_1, \ldots, p_n | \sigma_1^T | q_1, \ldots, q_m \rangle_{0,NS} = 0_R \langle 0 | d_{p_1} \cdots d_{p_n} \sigma_1^T f_1^+ f_2^+ \cdots f_m^+ | 0 \rangle_{0,NS}$$

$$= \frac{2}{L} \sum_{q \in Q} \frac{1}{1 - e^{-i(q - p_1)}} 0_R \langle 0 | d_{p_1} d_{p_{m-1}} \cdots d_{p_2} \sigma_1^T f_1^+ f_2^+ \cdots f_m^+ | 0 \rangle_{0,NS}$$

$$= \delta_{n,m} \frac{2}{L} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{1 - e^{-i(q_k - p_1)}} 0_R \langle 0 | d_{p_1} \cdots d_{p_2} \sigma_1^T f_1^+ f_2^+ \cdots f_{q_m}^+ | 0 \rangle_{0,NS}.$$  

Taking into account $0_R \langle 0 | \sigma_1^T | 0 \rangle_{0,NS} = 1$, which follows from (49), we see that we get a recurrence relation for the expansion of the determinant

$$0_R \langle 0 | d_{p_1} \cdots d_{p_2} \sigma_1^T f_1^+ f_2^+ \cdots f_m^+ | 0 \rangle_{0,NS} = \delta_{n,m} \det \begin{pmatrix} A_{1,1}, & A_{1,2}, & \cdots, & A_{1,m} \\ A_{2,1}, & A_{2,2}, & \cdots, & A_{2,m} \\ \vdots, & \vdots, & \ddots, & \vdots \\ A_{m,1}, & A_{m,2}, & \cdots, & A_{m,m} \end{pmatrix},$$  

(53)

with the matrix elements

$$A_{i,j} = \left( \frac{2}{L} \right) \frac{1}{1 - e^{i(p_i - q_j)}} = \left( \frac{2}{L} \right) \frac{e^{i q_j}}{e^{i q_j} - e^{i p_i}},$$  

(54)

with respect to first row. It is a variant of the Wick theorem with two-particle pairing given by $A_{i,j}$. The determinant (53) can be transformed to the Cauchy determinant, for which we have

$$\det \left( \frac{1}{x_i - y_j} \right)_{1 \leq i,j \leq n} = \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_i - y_j)}{\prod_{i,j=1}^{n} (x_i - y_j)}.$$  

Introducing the variables $x_j = e^{i q_j}$, $y_i = e^{i p_i}$ we get

$$0_R \langle P | \sigma_1^T | Q \rangle_{0,NS} = 0_R \langle p_1, p_2, \ldots, p_m | \sigma_1^T | q_1, q_2, \ldots, q_m \rangle_{0,NS}$$

$$= \delta_{n,m} (-1)^{m(m-1)/2} \left( \frac{2}{L} \right) \prod_{i < j}^{m} \left( \frac{e^{i q_i} - e^{i q_j}}{e^{i q_i} - e^{i p_j}} \right) \prod_{j=1}^{m} e^{i q_j}.$$  

(55)

Note that we use the momenta ordering described at the end of section 2. The final formula for the matrix elements (79) will not depend on this ordering.

Using a quite different approach, starting from Baxter’s spin operator extended Onsager algebra [11], the matrix elements of the spin operator up to unknown constants $N_{P_0,Q_0}$ depending only on Onsager sectors have been obtained in [13]:

$$0_R \langle P | \sigma_1^T | Q \rangle_{0,NS} = 0_R \langle P_+, P_0 | \sigma_1^T | Q_+, Q_0 \rangle_{0,NS} = i^{(m_0 - m_0)/2} \delta_{2L,1} \delta_{L-m_0-m_0} N_{P_0,Q_0}$$

$$\times \frac{(-1)^{n_+}}{\prod_{\alpha \in I} (\cos \alpha + 1)^{\sigma}} \frac{\tan \frac{\alpha}{2}}{\prod_{\beta \in I} (\cos \beta - 1)^{\tau}} \prod_{\alpha \in I} \prod_{\beta \in I} (\cos \alpha - \cos \beta)^{\lambda},$$  

(56)
where

\[ I = Q_+ \cup \bar{P}_+, \quad \bar{I} = P_+ \cup \bar{Q}_+, \quad |I| = m_+ + \bar{n}_+, \]

\[ L = 2m_+ + 2\bar{m}_+ + m_0 + \bar{m}_0, \quad 2|L| - L + \bar{n}_0 + m_0 = m - n, \]

\[ 2\tau = -P_{0,0} + \bar{P}_{0,0} + Q_{0,0} - \bar{Q}_{0,0} + 1, \quad 2\sigma = -Q_{0,\pi} + \bar{Q}_{0,\pi} + P_{0,\pi} - \bar{P}_{0,\pi} + 1, \]

\[ P_{0,0} = \begin{cases} 1, & 0 \in P_0 \\ 0, & 0 \notin P_0, \end{cases} \quad P_{0,\pi} = \begin{cases} 1, & \pi \in P_0 \\ 0, & \pi \notin P_0, \end{cases} \]

\[ \bar{P}_{0,0} = \begin{cases} 1, & 0 \in \bar{P}_0 \\ 0, & 0 \notin \bar{P}_0, \end{cases} \quad \bar{P}_{0,\pi} = \begin{cases} 1, & \bar{\pi} \in \bar{P}_0 \\ 0, & \bar{\pi} \notin \bar{P}_0. \end{cases} \]

The functions \( Q_{0,0}, Q_{0,\pi}, \bar{Q}_{0,0}, \bar{Q}_{0,\pi} \) are defined similarly. Note that we always have \( Q_{0,0} = \bar{Q}_{0,0} = 0, \) but we include them for uniformity. From (57) it follows that \( \sigma, \tau \in \{0, 1\} \). Also we used \( \delta_{m,n} = \delta_{m-a_0,2(n+1)} = \delta_{2|l|,L-m-a_0}. \)

Formula (55) fixes \( N_{P_0,Q_0} \) in (56) uniquely. The identity (A.1) at \( n = m \) from the appendix A has the same form as (56). In appendix A it is shown that the constant \( N_{P_0,Q_0} \) does not depend on the sets \( P_+ \) and \( Q_+ \) (even for \( n \neq m \)). It allows one to find this constant in different equivalent forms and we extract it from (A.1) for corresponding choices of \( P_+ \) and \( Q_+ \). One possible expression for \( N_{P_0,Q_0} \) corresponding to empty \( P_+ \) and \( Q_+ \) is given by

\[
N_{P_0,Q_0} = \delta_{(\text{mod} 2)}^{m_0-m_0,0} \left( \frac{2}{L} \right)^{(m_0+m_0)/2} N^0_+ N_2 N_e A_{P_0,Q_0} \times \prod_{q \in Q_+} \frac{1}{\tan(q/2)} \prod_{q \in Q_+} (\cos q - 1)^{\tau} \prod_{p \in P_+} (\cos p + 1)^{\sigma} \prod_{q \in Q_+} \prod_{q \in Q_+} (\cos p - \cos q),
\]

(59)

where \( \delta_{m-a_0,0} = ((-1)^m-m_0+1)/2, \)

\[
N^0_+ = (-1)^{(m_0-1)/2} (-1)^{(m_0-m_0)(m_0-m_0-2)/8} (-1)^{(L-m_0-a_0)(m_0-m_0)/4},
\]

\[
N_2 = 2^{-(m_0-m_0)/2}, \quad N_e = \prod_{p \in P_0} e^{ip(m_0-m_0)/2} \prod_{q \in Q_0} e^{-iq(m_0-m_0-2)/2},
\]

\[
A_{P_0,Q_0} = \prod_{p<p'} \prod_{q \in Q_0} \left( e^{ip} - e^{ip'} \right) \prod_{q \in Q_0} \left( e^{iq} - e^{iq'} \right) \prod_{p \in P_0} \prod_{q \in Q_0} \left( e^{iq} - e^{iq} \right).
\]

This expression does not explicitly depend on particular vectors of the Onsager sectors. This expression for \( N_{P_0,Q_0} \) will not be used for the calculations in this paper. Another more useful equivalent expression for \( N_{P_0,Q_0} \) is given by (77). Now we can substitute the obtained \( N_{P_0,Q_0} \) into the formulas of sections 4.2 and 4.3 of [13] (they correspond to formula (76) of present paper) and obtain, after some regrouping of factors, the factorized matrix elements of the spin operator between eigenvectors of \( H \). Instead of this direct way we choose a longer, but self-contained presentation of the results and show also that the rotations in representations of the Onsager algebra correspond to Bogoliubov transformations of pairs of fermions with opposite momenta.

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4. Eigenvectors of the quantum Ising chain

Using (47) we can rewrite the initial Hamiltonian $H = H_0 + \hbar H_1$ in terms of the dual Jordan–Wigner fermion operators in the momentum representation:

$$H = \sum_q \left( (2a_q^+ a_q - 1)(1 - \hbar \cos q) + i(a_q^+ a_{-q} + a_q a_{-q})\hbar \sin q \right),$$

where for the NS-sector $a_q = f_q$, $q \in \mathcal{Q}$, and for the R-sector $a_q = d_q$, $q \in \mathcal{P}$. For $0 \leq q \leq \pi$, define new fermion operators, obtained from the previous ones by a Bogoliubov transformation:

$$b_q = c_q a_q + i s_q a_{-q}^+, \quad b_q^+ = c_q a_{-q}^+ - i s_q a_q, \quad b_{-q} = c_q a_{-q} - i s_q a_q^+, \quad b_{-q}^+ = c_q a_q^+ + i s_q a_{-q}.$$

They diagonalize the Hamiltonian (60)

$$H = 2 \sum_q \varepsilon_q (b_q^+ b_q - \frac{1}{2})$$

where in the ferromagnetic regime, $0 \leq \hbar < 1$, we have

$$\varepsilon_q = \sqrt{1 + \hbar^2 - 2\hbar \cos q}, \quad \varepsilon_0 = 1 - \hbar, \quad \varepsilon_\pi = 1 + \hbar,$$

$$c_q = \cos \frac{\theta_q}{2}, \quad s_q = \frac{\sin \frac{\theta_q}{2}}{\sqrt{1 - \hbar \cos q}}, \quad \tan \theta_q = \frac{\hbar \sin q}{1 - \hbar \cos q}.$$

Note that $c_q$ and $s_q$ can be represented in the form

$$c_q(\varepsilon_q, \hbar) = \sqrt{\frac{(\varepsilon_\pi + \varepsilon_q)(\varepsilon_0 + \varepsilon_q)}{4\varepsilon_q}}, \quad s_q(\varepsilon_q, \hbar) = \frac{1}{2\varepsilon_q} \sqrt{(\varepsilon_\pi - \varepsilon_q)(\varepsilon_q - \varepsilon_0)\varepsilon_q}.$$

Formally we have

$$c_q(-\varepsilon_q, \hbar) = s_q(\varepsilon_q, \hbar), \quad s_q(-\varepsilon_q, \hbar) = -c_q(\varepsilon_q, \hbar).$$

The vacuum states $|0\rangle_{NS}$ and $|0\rangle_{R}$ for the Hamiltonian $H$ in the NS- and R-sector are

$$|0\rangle_{NS} = \prod_{0 < q < \pi} (c_q + is_q a_{-q}^+ a_q^+) |0\rangle_{0,NS}, \quad |0\rangle_{R} = \prod_{0 < q < \pi} (c_q + is_q d_{-q}^+ d_q^+) |0\rangle_{0,R},$$

where $|0\rangle_{0,NS}$ and $|0\rangle_{0,R}$ are the vacuum states of the Hamiltonian $H_0$ in the corresponding sectors. It is easy to show that for both sectors the following relations are fulfilled: $b_q|0\rangle = 0$ and

$$b_q^+ (c_q + is_q a_{-q}^+ a_q^+) |0\rangle = a_q^+ |0\rangle, \quad b_{-q}^+ b_{-q}^+ (c_q + is_q a_{-q}^+ a_q^+) |0\rangle = (c_a a_{-q}^+ a_q + is_q)|0\rangle.$$

As in the case of the eigenvectors of $H_0$ we will label the eigenvectors of $H$ by the same type of sets of momenta and use the same ordering. For example, in the NS-sector the vectors are labeled by a set $Q$ of momenta:

$$|Q\rangle_{NS} = |q_1, \ldots, q_m\rangle_{NS} = b_{q_1}^+ b_{q_2}^+ \cdots b_{q_m}^+ |0\rangle_{NS} \in \mathcal{V}_{NS}.$$
Spin operator matrix elements in the quantum Ising chain

or by two sets $Q_+$ and $Q_0$ with the following ordering of the momenta in the state:

$$|q_1, q_2, \ldots, q_m\rangle_{\text{NS}} = |Q\rangle_{\text{NS}} = |Q_+, Q_0\rangle_{\text{NS}} = \prod_{q \in Q_+} (b_q^+ b_q^+) \prod_{q \in Q_0} b_q^+ |0\rangle_{\text{NS}}.$$  

The formulas (46) for the action of the translation operator together with the formulas for the Bogoliubov transformation imply

$$T b_q = e^{i q} T, \quad T b_q^+ = e^{-i q} T$$

and therefore

$$T |Q\rangle_{\text{NS}} = e^{-i \sum_{q \in Q} q} |Q\rangle_{\text{NS}}, \quad |Q\rangle_{\text{NS}} T = |Q\rangle_{\text{NS}} e^{-i \sum_{q \in Q} q}.$$  

The formulas (65) and (66) allow one to find explicit expressions for the eigenvectors of $H$ as linear combinations of the eigenvectors of $H_0$:

$$|Q_+, Q_0\rangle_{\text{NS}} = \prod_{q \in Q_+} (c_q a_{-q}^+ a_q^+ + is_q) \prod_{q \in Q_0} (c_q + is_q a_{-q}^+ a_q^+) \prod_{q \in Q_0} a_q^+ |0, 0\rangle_{\text{NS}}.$$  

It is convenient to introduce the notations: $\varepsilon(q) = \varepsilon_q$ if $q \in Q_+$ and $\varepsilon(q) = -\varepsilon_q$ if $q \in \tilde{Q}_+$,

$$\alpha(q) = \sqrt{\frac{\varepsilon_0 + \varepsilon(q)}{4\varepsilon(q)}}, \quad \beta(q) = \frac{1}{2\varepsilon(q)} \sqrt{(\varepsilon_0 - \varepsilon(q))(\varepsilon(q) - \varepsilon_0)\varepsilon(q)}.$$  

The formulas (64) give

$$\alpha(q) = c_q, \quad \beta(q) = s_q \quad \text{if } q \in Q_+,$$

$$\alpha(q) = s_q, \quad \beta(q) = -c_q \quad \text{if } q \in \tilde{Q}_+,$$

which allow one to present the products over $Q_+$ and $\tilde{Q}_+$ in (69) uniformly:

$$|Q_+, Q_0\rangle_{\text{NS}} = i^{m_+} \prod_{q \in Q_+} (\alpha(q) a_{-q}^+ a_q^+ + i\beta(q)) \prod_{q \in Q_0} a_q^+ |0, 0\rangle_{\text{NS}}$$

$$= i^{m_+} (-1)^{m_+} \sum_{Q'_+ \subseteq Q_+} i^{-m'_+} \prod_{q \in Q'_+} \beta(q) \prod_{q \in Q_0} \alpha(q) |Q'_+, Q_0\rangle_{0, 0, \text{NS}},$$

where

$$|Q'_+, Q_0\rangle_{0, 0, \text{NS}} = \prod_{q \in Q'_+} (a_{-q}^+ a_q^+) \prod_{q \in Q_0} a_q^+ |0, 0\rangle_{\text{NS}}.$$  

It means that the eigenvectors of the Hamiltonian $H$ in a given Onsager sector $\mathcal{V}_0$ can be represented as linear combinations of the eigenvectors of the Hamiltonian $H_0$ from the same Onsager sector.

Similarly to (70) the eigenvectors of the Hamiltonian $H$ from the R-sector are

$$|P_+, P_0\rangle_{\text{R}} = i^{n_+} (-1)^{n_+} \sum_{P'_+ \subseteq P_+} i^{-n'_+} \prod_{p \in P'_+} \beta(p) \prod_{p \in P_0} \alpha(p) |P'_+, P_0\rangle_{0, 0, \text{R}},$$

where

$$|P'_+, P_0\rangle_{0, 0, \text{R}} = \prod_{p \in P'_+} (a_{-p}^+ a_p^+) \prod_{p \in P_0} a_p^+ |0, 0\rangle_{\text{R}}.$$  

\[ \text{doi:10.1088/1742-5468/2011/02/P02028} \]
5. Spin operator matrix elements for the eigenstates of Hamiltonian $H$

For calculation of the matrix element $R \langle P | \sigma_+^x | Q \rangle_{\text{NS}}$ we use the expressions (70) and (71) for the eigenstates of the Hamiltonian $H$ and (56) for the matrix element $0_R \langle P_0, P_+ | \sigma_+^x | Q_+^0, Q_0 \rangle_{0, \text{NS}}$. In order to obtain the factorized formula for the matrix elements $R \langle P | \sigma_+^x | Q \rangle_{\text{NS}}$ we need to make summation over the eigenstates of $H_0$ labeled by the sets $Q_+^0$ and $P_+^0$. For the summation, we will do some algebraic transformations that will allow us to use the summation formula from appendix B:

$$
R \langle P_0, P_+ | \sigma_+^x | Q_+, Q_0 \rangle_{\text{NS}} = i^{m_+ - n_+} (-1)^{n_+ + m_+} \sum_{Q_+^0 \subset Q_+} \sum_{P_+^0 \subset P_+} i^{n'_+ - m'_+} \times \prod_{p \in P_+} \beta(p) \prod_{q \in Q_+^0} \beta(q) \prod_{p' \in P_+^0} \alpha(p) \prod_{q' \in Q_+} \alpha(q) \cdot 0_R \langle P_0, P_+ | \sigma_+^x | Q_+^0, Q_0 \rangle_{0, \text{NS}}
$$

$$
= i^{m_+ - n_+} (-1)^{n_+ + m_+} \sum_{Q_+ \subset \bar{Q}_+} \sum_{P_+ \subset \bar{P}_+} \sum_{\delta_{\bar{Q}_+} \subset \bar{P}_+} \frac{1}{\prod_{x \in I} \prod_{y \in I} (\cos x - \cos y)},
$$

where $I' = Q_+^0 \cup P_+^0$, $\bar{P}_+ = P_+ \cup Q_+^0$,

$$
\mu = \frac{1}{2} (L - m_0 - \bar{m}_0), \quad \bar{\mu} = \frac{1}{2} (L - m_0 - \bar{m}_0).
$$

The following relations will be useful in what follows:

$$
\tau + \mu = \sigma + \bar{\mu}, \quad |I| = \mu + \frac{m - n}{2}, \quad |\bar{I}| = \bar{\mu} + \frac{n - m}{2},
$$

where definitions of $\sigma$ and $\tau$ are given by (57). Using the relations

$$
\left| \tan \frac{p}{2} \right| = \sqrt{\frac{\varepsilon(p)^2 - \varepsilon_0^2}{\varepsilon_\pi^2 - \varepsilon(p)^2}}, \quad \frac{\alpha(p)}{\beta(p)} \left| \tan \frac{p}{2} \right| = \frac{\varepsilon_0 + \varepsilon(p)}{\varepsilon_\pi - \varepsilon(p)}, \quad -\frac{\beta(q)}{\alpha(q)} \left| \tan \frac{q}{2} \right| = \frac{\varepsilon_0 - \varepsilon(q)}{\varepsilon_\pi + \varepsilon(q)}
$$

and

$$
\cos x + 1 = \frac{\varepsilon_\pi^2 - \varepsilon(x)^2}{2h}, \quad \cos x - 1 = \frac{\varepsilon_0^2 - \varepsilon(x)^2}{2h}, \quad \cos x - \cos y = \frac{\varepsilon(y)^2 - \varepsilon(x)^2}{2h},
$$

in the expression for the matrix element we get

$$
R \langle P_0, P_+ | \sigma_+^x | Q_+, Q_0 \rangle_{\text{NS}} = i^{m_+ - n_+} (-1)^{n_+ + m_+ + \bar{\mu}} \sum_{P_+ \subset \bar{P}_+} \prod_{p \in P_+} \beta(p) \prod_{q \in Q_+^0} \alpha(q) \times \sum_{Q_+ \subset \bar{Q}_+} \prod_{p \in P_+} \frac{\varepsilon_0 + \varepsilon(p)}{\varepsilon_\pi - \varepsilon(p)} \prod_{q \in Q_+} \frac{\varepsilon_0 - \varepsilon(q)}{\varepsilon_\pi + \varepsilon(q)}
$$

$$
\times \prod_{x \in I} (\varepsilon_\pi^2 - \varepsilon(x)^2) \prod_{y \in I} (\varepsilon_0^2 - \varepsilon(y)^2) \prod_{x \in I, y \in \bar{I}} (\varepsilon(y)^2 - \varepsilon(x)^2).
$$

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In the terms of the notations
\[ \gamma_p = -\varepsilon(p), \quad \text{if } p \in \mathcal{P}_+, \quad \gamma_q = \varepsilon(q), \quad \text{if } q \in \mathcal{Q}_+, \]
we have
\[
\mathcal{R}(P_0, P_+|\sigma^z |Q_+, Q_0)_{\text{NS}} = \frac{(2h)^{\sigma\mu + \tau\bar{\mu} + \mu\bar{\mu}}}{\prod_{x \in P_+, \gamma \in \mathcal{Q}_+} (\varepsilon^2 - \gamma^2)^\sigma} \prod_{p \in \mathcal{P}_+} \beta(p) \prod_{q \in \mathcal{Q}_+} \alpha(q) \times \sum_{I' \subset \mathcal{P}_+ \cup \mathcal{Q}_+} \delta_{|I|, \mu} \prod_{y \in I'} (\varepsilon - \gamma_y)^\sigma (\varepsilon_0 - \gamma_y)^{1-\tau} \prod_{x \in I', y \in I} (\varepsilon_y - \gamma_y)^{\tau} \cdot \prod_{x \in \mathcal{P}_+} (\varepsilon_x + \gamma_x)^{1-\sigma} (\varepsilon_0 + \gamma_x)^{\tau} \prod_{x \in \mathcal{Q}_+} (\varepsilon_0 + \gamma_x)^{\tau}.
\]
Now we make summation over \( I' \) by means of the following formula from appendix B:
\[
\frac{2^{2\min(\mu, \bar{\mu})}((-1)^{\mu} + (-1)^{\bar{\mu}})/2}{\prod_{x \in \mathcal{P}_+ \cup \mathcal{Q}_+} (\varepsilon_x + \gamma_x)^{1-\sigma} (\varepsilon_0 + \gamma_x)^{\tau}} \prod_{x \in \mathcal{Q}_+} (\varepsilon_0 + \gamma_x)^{\tau} \prod_{x \in \mathcal{P}_+} (\varepsilon_x + \gamma_x)^{\tau},
\]
where \( \min(\mu, \bar{\mu}) = (\mu + \bar{\mu} - |\mu - \bar{\mu}|)/2 = (||I| + |I'| - |\sigma - \tau|)/2 \). As a result we obtain
\[
\mathcal{R}(P_0, P_+|\sigma^z |Q_+, Q_0)_{\text{NS}} = \frac{(2h)^{\sigma\mu + \tau\bar{\mu} + \mu\bar{\mu}}}{\prod_{x \in \mathcal{P}_+ \cup \mathcal{Q}_+} (\varepsilon^2 - \gamma^2)^\sigma} \prod_{p \in \mathcal{P}_+} \beta(p) \prod_{q \in \mathcal{Q}_+} \alpha(q) \times \prod_{\mu \in \mathcal{P}_+} 2^{2(\min(\mu, \bar{\mu}) + 1)/2} \prod_{x \in \mathcal{Q}_+} (\varepsilon_0 + \gamma_x)^{\tau} \prod_{x \in \mathcal{P}_+} (\varepsilon_x + \gamma_x)^{\tau} \prod_{x \in \mathcal{Q}_+} (\varepsilon_0 + \gamma_x)^{\tau},
\]
Using the relations (69), (72), (73) and \( \gamma_\alpha = \varepsilon_\alpha \) for \( \alpha \in I, \gamma_\beta = -\varepsilon_\beta \) for \( \beta \in I \) we get
\[
\mathcal{R}(P_0, P_+|\sigma^z |Q_+, Q_0)_{\text{NS}} = (-1)^{\tilde{m} + \tilde{\mu} + \mu + 1/2} \frac{(2h)^{\sigma\mu + \tau\bar{\mu} + \mu\bar{\mu}}}{\prod_{x \in \mathcal{P}_+ \cup \mathcal{Q}_+} (\varepsilon^2 - \gamma^2)^\sigma} \prod_{p \in \mathcal{P}_+} \beta(p) \prod_{q \in \mathcal{Q}_+} \alpha(q) \times \prod_{\alpha, \beta \in I} 2^{-\sigma/2} \prod_{\alpha \in I} \left( \frac{(\varepsilon_\alpha + \varepsilon_\beta)(\varepsilon_\alpha + \varepsilon_\beta)}{2\varepsilon_\alpha} \right) \prod_{\beta \in I} \left( \frac{(\varepsilon_\beta - \varepsilon_\alpha)(\varepsilon_\beta - \varepsilon_\alpha)}{2\varepsilon_\beta} \right) \prod_{\alpha \in I} (\varepsilon_\alpha + \varepsilon_\alpha)(\varepsilon_\alpha - \varepsilon_\alpha) \prod_{\beta \in I} (\varepsilon_\beta + \varepsilon_\beta) \prod_{\alpha \in I} (\varepsilon_\alpha + \varepsilon_\beta) \prod_{\beta \in I} (\varepsilon_\beta + \varepsilon_\alpha) \prod_{\alpha \in I} (\varepsilon_\alpha - \varepsilon_\beta) \prod_{\beta \in I} (\varepsilon_\beta - \varepsilon_\alpha),
\]
where
\[
M_- = (-1)^{\tilde{m} + \tilde{\mu} + \mu + 1/2} \frac{(2h)^{\sigma\mu + \tau\bar{\mu} + \mu\bar{\mu}}}{\prod_{x \in \mathcal{P}_+ \cup \mathcal{Q}_+} (\varepsilon^2 - \gamma^2)^\sigma} \prod_{p \in \mathcal{P}_+} \beta(p) \prod_{q \in \mathcal{Q}_+} \alpha(q) \times \prod_{\alpha, \beta \in I} 2^{-\sigma/2} \prod_{\alpha \in I} \left( \frac{(\varepsilon_\alpha + \varepsilon_\beta)(\varepsilon_\alpha + \varepsilon_\beta)}{2\varepsilon_\alpha} \right) \prod_{\beta \in I} \left( \frac{(\varepsilon_\beta - \varepsilon_\alpha)(\varepsilon_\beta - \varepsilon_\alpha)}{2\varepsilon_\beta} \right) \prod_{\alpha \in I} (\varepsilon_\alpha + \varepsilon_\alpha)(\varepsilon_\alpha - \varepsilon_\alpha) \prod_{\beta \in I} (\varepsilon_\beta + \varepsilon_\beta) \prod_{\alpha \in I} (\varepsilon_\alpha + \varepsilon_\beta) \prod_{\beta \in I} (\varepsilon_\beta + \varepsilon_\alpha) \prod_{\alpha \in I} (\varepsilon_\alpha - \varepsilon_\beta) \prod_{\beta \in I} (\varepsilon_\beta - \varepsilon_\alpha),
\]
As was discussed before (59), we can choose the coefficient \( N_{P_0, Q_0} \) in many different forms, extracting it from (A.1) of appendix A for different choices of \( P_+ \) and \( Q_+ \) (even for \( n \neq m \)). Here we need such an expression for \( N_{P_0, Q_0} \), which follows from (A.1) with \( P_+ \) and \( Q_+ \) as

\[\text{doi:10.1088/1742-5468/2011/02/P02028}\]
in (76):

\[
N_{P_0,Q_0} = \delta_{m_0-n_0,0}^2 \left[ 1^{m_+ + m_-} 2^{-(n-m)^2/4} \left( \frac{2}{L} \right)^{(n+m)/2} \right] N_0^{P_0} \prod_{p \in P_0} e^{-ip(m-n)/2} \prod_{q \in Q_0} e^{-iq(m-n-2)/2} A_{P,Q} \\
\times (-1)^{m_+} \prod_{\beta \in I} \frac{1}{\tan(\beta/2)} \prod_{\beta \in I} \left( \cos \beta - 1 \right)^{\sigma} \prod_{\alpha \in I} \left( \cos \alpha + 1 \right)^{\tau} \prod_{\alpha \in I, \beta \in I} \left( \cos \alpha - \cos \beta \right),
\]

where

\[
A_{P,Q} = \frac{\prod_{p < p' \in P} (e^{ip} - e^{ip'}) \prod_{q < q' \in Q} (e^{iq} - e^{iq'})}{\prod_{q \in Q} \prod_{p \in P} (e^{ip} - e^{ip'})},
\]

\[
N_0 = (1)^{n_0}(1)^{m_0}/2(1)^{n_0}(1)^{m_0}/2h^{n_0}(1)^{m_0}/2h^{n_0}/2.
\]

Then using (75) and \(2^{-(\sigma-\tau)/2} = 2^{-(1/4)(\varepsilon_0 + \varepsilon_\pi)(1-2\sigma)(1-2\tau)/4} \), which follows from \(\sigma, \tau \in \{0, 1\} \) and \(\varepsilon_0 = 1 - h, \varepsilon_\pi = 1 + h, \varepsilon_\pi \), we get

\[
R(P_0, P_+ | \sigma^+_1 | Q_+, Q_0)_{NS} = \delta_{m_0-n_0,0}^2 \left[ \frac{2}{L} \right]^{(n+m)/2} \left( -1 \right)^{n_0}(1)^{m_0}/2h^{n_0}(1)^{m_0}/2h^{n_0}/2 A_{P,Q} \\
\times \prod_{\beta \in I} (e^{\varepsilon_0 + \varepsilon_\beta})^\tau (e^{\varepsilon_\pi + \varepsilon_\beta})^{1-\sigma} \prod_{\alpha, \beta \in I} (e^{\varepsilon_\alpha + \varepsilon_\beta})^{1/2} \frac{1}{2\varepsilon_\alpha} \frac{1}{2\varepsilon_\beta} \frac{1}{(e^{\varepsilon_\alpha + \varepsilon_\beta})(e^{\varepsilon_\beta + \varepsilon_0})},
\]

where we also extended the products over \(P_0\) and \(Q_0\) of the exponents of momenta to \(P\) and \(Q\), respectively, since it means addition of pairs of opposite momenta. Regrouping factors with respect to the sets \(U = Q + P = Q_+ + Q_0 + P_+ + P_- + P_0\) and \(\bar{U} = P + Q\) we obtain

\[
R(P_0, P_+ | \sigma^+_1 | Q_+, Q_0)_{NS} = \delta_{m_0-n_0,0}^2 \left[ \frac{2}{L} \right]^{(n+m)/2} \left( -1 \right)^{n_0}(1)^{m_0}/2h^{n_0}(1)^{m_0}/2h^{n_0}/2 A_{P,Q} \\
\times \prod_{\beta \in I} (e^{\varepsilon_0 + \varepsilon_\beta})^\tau (e^{\varepsilon_\pi + \varepsilon_\beta})^{1-\sigma} \prod_{\alpha, \beta \in I} (e^{\varepsilon_\alpha + \varepsilon_\beta})^{1/2} \frac{1}{2\varepsilon_\alpha} \frac{1}{2\varepsilon_\beta} \frac{1}{(e^{\varepsilon_\alpha + \varepsilon_\beta})(e^{\varepsilon_\beta + \varepsilon_0})}.
\]

To prove this formula we have to compare the exponents of \((\varepsilon_0 + \varepsilon_\beta)\) for all \(\alpha\) and \(\beta\) in the former and latter expressions and to use definitions (57) of \(\sigma\) and \(\tau\). Now we rewrite the products over the sets \(U\) and \(\bar{U}\) as products over the sets \(P, Q, \bar{P}, \bar{Q}\). Finally we exclude the products over the sets \(\bar{P}, \bar{Q}\) supplementing them to be the products over all the sets of momenta in the first Brillouin zone \(Q = Q \cup \bar{Q}\) for the NS-sector and \(P = P \cup \bar{P}\) for the R-sector. It gives the factorized formula for the matrix element of the spin operator

\[
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\]
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\[ \sigma_\uparrow^+ [4]: \]

\[ R(P|\sigma_\uparrow|Q)_{NS} = R(P_1, p_2, \ldots, p_n|q_1, q_2, \ldots, q_m)_{NS} = \delta_{m-n,0} i^{-(n+m)/2} (-1)^{n(n-1)/2} \prod_{p \in P} e^{-i p/2} \prod_{q \in Q} e^{i q/2} \times \left( \frac{2}{L} \right)^{(m+n)/2} h^{(m-n)/2} e^{i \xi_T} \prod_{q \in Q} \frac{\varepsilon_q^{(q)/2}}{\sqrt{2 \varepsilon_q}} \prod_{p \in P} \frac{\varepsilon_p^{(p)/2}}{\sqrt{2 \varepsilon_p}} \times \frac{\varepsilon_q + \varepsilon_p}{2 \sin(q - p)/2}, \]  

(79)

where \( \varepsilon_\alpha \) is given by (62) and \( \xi = (1 - h^2)^{1/4} \).

Matrix elements with even (resp. odd) \( n \) and \( m \) correspond to the Hamiltonian (1) for the periodic boundary condition (resp. (2) for the antiperiodic boundary condition).

Let us comment on the modification of the formula (79) in the case of the matrix elements of the spin operator \( \sigma_\uparrow^+ \). From (67) and similar formula for the R-sector and also from (11) we obtain

\[ R(P|\sigma_\uparrow|Q)_{NS} = R(P|T^{-k-1}\sigma_\uparrow T^{-k+1}|Q)_{NS} = e^{i(k-1)(\Sigma_{q \in Q} q - \Sigma_{p \in P} p)} R(P|\sigma_\uparrow|Q)_{NS}. \]

6. Conclusions

In [13], using Baxter’s extension of the Onsager algebra, the factorized expressions for the spin operator matrix elements between the eigenstates of Hamiltonians of the finite length superintegrable \( Z_N \)-symmetric chiral Potts quantum chain (SCPS) were found up to unknown scalar factors for any pair of the Onsager sectors. In this paper we have derived the exact expression (59) for these factors for the quantum Ising chain in a transverse field \( (N = 2 \text{ SCPC-model}) \). This derivation uses some modification of the standard fermion technique [20]. In the first stage we diagonalized the Hamiltonian \( H_0 \) by means of the dual Jordan–Wigner fermion operators and calculated the spin matrix elements between eigenstates of \( H_0 \), then after the Bogoliubov transformation we obtained the factorized formulas for the spin matrix elements between eigenvectors of \( H \) using the summation formula (appendix B). It is natural to expect that the fermion technique used in this paper can be applied to calculation of the spin matrix elements in more general free fermion models, for example, in the \( N = 2 \) Baxter–Bazhanov–Stroganov model [21, 5, 22]. We will address this problem in a forthcoming paper.

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Appendix A

In this appendix we prove the following trigonometric identity for the momenta of the quantum Ising chain. We will use the definitions from section 2.4 for different sets of momenta. Supposing that \( m = |Q|, n = |P|, m_0 = |Q_0| \) and \( n_0 = |P_0| \) have equal parity, we claim that

\[
A_{P,Q} := \frac{\prod_{p\leq p' \in P} (e^{ip} - e^{ip'}) \prod_{q\leq q' \in Q} (e^{iq} - e^{iq'})}{\prod_{q \in Q} \prod_{p \in P} (e^{iq} - e^{ip})} = \delta^{(\text{mod} 2)}_{m_0 - n_0, 0} \left( \frac{2}{L} \right)^{(n+m)/2} \times i^{(n_0 - m_0)/2} i^{(m-n)/2} (m-n)^2/4 \prod_{p \in P_0} e^{ip(n-m)/2} \prod_{q \in Q_0} e^{iq(m-n-2)/2} \times N_- N_{P_0, Q_0} \prod_{\alpha \in I} (\cos \alpha + 1)^\sigma \prod_{\beta \in \bar{I}} (\cos \beta - 1)^\tau \prod_{\alpha \in I, \beta \in \bar{I}} (\cos \alpha - \cos \beta),
\]

(A.1)

with \( I = Q_+ \cup \bar{P}_+ \), \( \bar{I} = P_+ \cup \bar{Q}_+ \), \( \delta^{(\text{mod} 2)}_{m_0 - n_0, 0} = ((-1)^{n_0 - m_0} + 1)/2 \),

\[
N_- = (-1)^{n(n-1)/2 + (m-n)(m-n-2)/2 + (L-m_0-n_0)(n-m)/4},
\]

and the coefficient \( N_{P_0, Q_0} \) not depending on \( P_+ \) and \( Q_+ \) (or equivalently not depending on \( I \)). In the main text we use this formula for different choices of \( P_+ \) and \( Q_+ \), namely, as labels of eigenvectors of \( H_0 \) or as labels of eigenvectors of \( H \) with \( P_0 \) and \( Q_0 \) being fixed.

In order to prove (A.1) it is sufficient to show that addition of a pair of opposite momenta to \( P \) or to \( Q \) on the left-hand side of (A.1) does not change the coefficient \( N_{P_0, Q_0} \) on the right-hand side of (A.1). Let us add, for example, a pair of momenta \(-p, p\) to \( P \) and calculate the ratio of left-hand sides of (A.1):

\[
\frac{A_{P\cup\{-p,p\},Q}}{A_{P,Q}} = \frac{\prod_{p' \in P_0} B(p, p')}{\prod_{q' \in Q_0} B(p, q')} \times \frac{\prod_{p' \in P_+} C(p, p')}{\prod_{q' \in Q_+} C(p, q')} \times (e^{-ip} - e^{ip}),
\]

(A.2)

where

\[
B(\alpha, \beta) = (e^{i\alpha} - e^{i\beta})(e^{-i\alpha} - e^{-i\beta}), \quad C(\alpha, \beta) = B(\alpha, \beta) B(\alpha, -\beta).
\]

Note that in (A.1) we used the following ordering of the momenta \( \{q_1, q_2, \ldots, q_m\} \): from the beginning the pairs \((-q, q)\), for \( q \in Q_+ \) go in sequence and then \( q \in Q_0 \) go in sequence in a order fixed in all the paper. Analogously we define the ordering of the momenta in the set \( P = \{p_1, p_2, \ldots, p_n\} \). It order to present the factors of (A.2) in a convenient form we introduce the following functions:

\[
\chi(\alpha, \beta) = (1 - e^{i(\alpha+\beta)})(1 - e^{i(\alpha-\beta)}), \quad \varphi_+(\alpha) = 1 + e^{i\alpha}, \quad \varphi_-(\alpha) = 1 - e^{i\alpha}.
\]
For the products of the functions \( B \) we use the first two of the following identities:

\[
\prod_{p' \in P_0} B(p, p') = L(-1)^{n_0} e^{-i\pi n_0} \prod_{p' \in P_0} e^{i\beta' \phi_-(p)P_{0,0}P_{0,0}^{-1} \phi_+(p)P_{0,0}^{-1}} \prod_{p' \neq p} \chi(p, p'),
\]

(A.3)

\[
\prod_{q' \in Q_0} B(q, q') = 2(-1)^{n_0} e^{-i\pi n_0} \prod_{q' \in Q_0} e^{i\beta' \phi_-(q)Q_{0,0}Q_{0,0}^{-1} \phi_+(q)Q_{0,0}^{-1}} \prod_{q' \neq q} \chi(q, q'),
\]

(A.4)

\[
\prod_{q' \in Q_0} B(q, q') = L(-1)^{n_0} e^{-i\pi n_0} \prod_{q' \in Q_0} e^{i\beta' \phi_-(q)Q_{0,0}Q_{0,0}^{-1} \phi_+(q)Q_{0,0}^{-1}} \prod_{q' \neq q} \chi(q, q'),
\]

(A.5)

\[
\prod_{p' \in P_0} B(p, p') = 2(-1)^{n_0} e^{-i\pi n_0} \prod_{p' \in P_0} e^{i\beta' \phi_-(q)P_{0,0}P_{0,0}^{-1} \phi_+(q)P_{0,0}^{-1}} \prod_{p' \neq p} \chi(q, q').
\]

(A.6)

The functions \( P_{0,0}, P_{0,\pi}, P_{0,\bar{\pi}} \) are defined by (58). The functions \( Q_{0,0}, Q_{0,\pi}, Q_{0,\bar{\pi}} \) are defined similarly. Note that we always have \( Q_{0,0} = \bar{Q}_{0,0} = 0 \), but we include them for uniformity. The identities (A.3)–(A.6) follow from the elementary trigonometric identities:

\[
\prod_{k=1}^{L-1} (1 - \omega^k) = L, \quad \prod_{k=1}^{L} (1 - \omega^{k+1/2}) = 2, \quad \omega = e^{2\pi i/L}.
\]

We also have

\[ C(\alpha, \beta) = e^{-2\alpha} \chi(\alpha, \beta)^2, \quad e^{-i\beta} - e^{i\beta} = e^{-i\beta} \phi_+(p) \phi_-(p). \]

Since \( n_0 \) and \( n_0 \) has the same parity, for the ratio (A.2) we have

\[
\frac{A_{P,\bar{Q},(q,p)} A_{P,Q}}{A_{P,Q}} = \frac{L}{2} e^{i\beta \pi (m-1)} \prod_{p' \in P_0} e^{\beta \alpha} \prod_{q' \in Q_0} e^{-\beta \alpha} \prod_{p', \alpha} \chi(p, \alpha)
\]

where we used (57) and the definition of the sets \( I \) and \( \bar{I} \). Taking into account the identities

\[ e^{-i\alpha} \chi(\alpha, \beta) = 2(\cos \alpha - \cos \beta), \quad e^{-i\alpha} \phi_+^2(\alpha) = 2(\cos \alpha + 1), \]

\[ e^{-i\beta} \phi_+^2(\beta) = 2(\cos \beta - 1), \quad \frac{\phi_-(\alpha)}{\phi_+(\alpha)} = -i \tan(\alpha/2) \]

and the relation \(|\bar{I}|-|I|+\sigma-\tau = n-m\) following from (73) we get

\[
\frac{A_{P,\bar{Q},(q,p)} A_{P,Q}}{A_{P,Q}} = \frac{-iL}{2} 2^{n-m+1} \prod_{p' \in P_0} e^{i\beta} \prod_{q' \in Q_0} e^{-i\alpha} \tan \frac{p}{2} \left( \frac{\cos p + 1}{\cos p - 1} \right) \prod_{\alpha \in I} \left( \frac{\cos p - \cos \alpha}{\cos p - \cos (p - \cos \alpha)} \right).
\]

(A.7)

The corresponding ratio of right-hand sides of (A.1) gives the same result (A.7). For the ratio \( A_{P,Q,\bar{Q},(q,p)}/A_{P,Q} \) the calculation goes in the same way with the use (A.5) and (A.6) for the products of \( B \). It proves the identity (A.1).
Appendix B

The following summation formula over the subsets \( I \in \mathcal{R} \), \( \mathcal{R} = \mathcal{P}_+ \cup \mathcal{Q}_+ \):

\[
\sum_{I \subseteq \mathcal{R}} \delta_{|I|, \mu} \prod_{y \in I} \left( \frac{\left( \varepsilon_\pi - \gamma_y \right)^{\sigma} \varepsilon_0 - \gamma_y}{\left( \varepsilon_\pi + \gamma_y \right)^{1-\sigma} \left( \varepsilon_0 + \gamma_y \right)^{r}} \right) \frac{1}{\prod_{x \in I, y \in I} \left( \gamma_x^2 - \gamma_y^2 \right)} = \frac{q^{\min(\mu, \bar{\mu})} (-1)^{\mu+1/2} (-1)^{\tau(1-\tau)}}{\prod_{y \in \mathcal{R}} \left( \varepsilon_\pi + \gamma_y \right)^{1-\sigma} \left( \varepsilon_0 + \gamma_y \right)^{r} \prod_{x<y \in \mathcal{R}} \left( \gamma_x + \gamma_y \right)}
\]

(B.1)

where \( \min(\mu, \bar{\mu}) = (\mu + \bar{\mu} - |\mu - \bar{\mu}|)/2 \), is valid. In fact the formula contains four subcases depending on the choice of \( \sigma, \tau \in \{0, 1\} \). In this formula \( \mu \) is defined from the relations \( \mu + \tau = \bar{\mu} + \sigma \), \( \mu + \bar{\mu} = |\mathcal{R}| = r \). It is easy to prove this formula by reducing it to the summation formulas from appendix A of [13] for arbitrary variables \( u, v, z_a, a = 1, \ldots, r \) and even and odd \( r \), respectively:

\[
\sum_{I \subseteq \mathcal{R}} \delta_{|I|, r/2} \prod_{a \in I} (z_a + u) \prod_{b \in \mathcal{R} \setminus I} (z_b + v) \prod_{a \in I, b \in \mathcal{R} \setminus I} \left( z_a^2 - z_b^2 \right) = \frac{(-1)^{r(r-2)/8} (u - v)^{r/2}}{\prod_{c \in \mathcal{R}} (z_c + u) (z_c + v) \prod_{c < s} (z_s + z_c)},
\]

(B.2)

\[
\sum_{I \subseteq \mathcal{R}} \delta_{|I|, (r+1)/2} \prod_{a \in I} (z_a + u) \prod_{b \in \mathcal{R} \setminus I} (z_b + v) \prod_{a \in I, b \in \mathcal{R} \setminus I} \left( z_a^2 - z_b^2 \right) = \frac{(-1)^{(r+1)(r-1)/8} (u + v)^{(r-1)/2}}{\prod_{c \in \mathcal{R}} (z_c + u) (z_c + v) \prod_{c < s} (z_s + z_c)},
\]

(B.3)

We may consider the four cases of \( \sigma, \tau \in \{0, 1\} \) one by one.

For example let us prove (B.1) for the case \( \sigma = 1, \tau = 0 \). In this case \( \mu = (r+1)/2 \), \( \bar{\mu} = (r-1)/2 \). The left-hand side of (B.1) is

\[
\sum_{I \subseteq \mathcal{R}} \delta_{|I|, (r+1)/2} \prod_{y \in I} \left( \varepsilon_\pi - \gamma_y \right) \left( \varepsilon_0 - \gamma_y \right) = \prod_{y \in \mathcal{R}} \left( \gamma_y - \varepsilon_\pi \right) \left( \gamma_y - \varepsilon_0 \right) \times \sum_{I \subseteq \mathcal{R}} \delta_{|I|, (r+1)/2} \left( -1 \right)^{\mu \bar{\mu}} \prod_{y \in I} \left( \gamma_y - \varepsilon_\pi \right) \left( \gamma_y - \varepsilon_0 \right) \prod_{x \in I, y \in I} \left( \gamma_x^2 - \gamma_y^2 \right)
\]

\[
= \frac{2^{(r-1)/2} (-1)^{\mu+1/2+\mu-1}}{\prod_{x<y \in \mathcal{R}} \left( \gamma_x + \gamma_y \right)},
\]

where we used (B.3) for \( u = -\varepsilon_0, \ v = -\varepsilon_\pi, \ \{ z_a \} = \{ \gamma_a \} \) and \( u + v = -2 \). Thus we proved (B.1) for the case \( \sigma = 1, \tau = 0 \). The other three cases of \( \sigma \) and \( \tau \) can be considered similarly.

References

[1] Bugrij A and Lisovyy O, 2003 Phys. Lett. A 319 300
[2] Bugrij A and Lisovyy O, 2004 Theor. Math. Phys. 140 987
[3] Fonseca P and Zamolodchikov A A, 2003 J. Stat. Phys. 110 527
[4] von Gehlen G, Jorgov N, Pakuliak S, Shadura V and Tykhby Yu, 2008 J. Phys. A: Math. Theor. 41 095003
[5] von Gehlen G, Jorgov N, Pakuliak S and Shadura V, 2009 J. Phys. A: Math. Theor. 42 287502

doi:10.1088/1742-5468/2011/02/P02028
Spin operator matrix elements in the quantum Ising chain

[6] Iorgov N and Lisovyy O, *Ising correlations and elliptic determinants*, 2010 J. Stat. Phys. at press, arXiv:1012.2856
[7] Palmer J and Hystad G, *Spin matrix for the scaled periodic Ising model*, 2010 arXiv:1008.0352
[8] Hystad G, *Periodic Ising correlations*, 2010 arXiv:1011.2223
[9] Iorgov N, *Form-factors of the finite quantum XY-chain*, 2009 arXiv:0912.4466
[10] von Gehlen G and Rittenberg V, 1985 Nucl. Phys. B 257 351 [FS14]
[11] Baxter R J, 2009 J. Stat. Phys. 137 798
[12] Baxter R J, 2010 J. Phys. A: Math. Theor. 43 145002
[13] Iorgov N, Pakuliak S, Shadura V, Tykhyy Yu and von Gehlen G, 2010 J. Stat. Phys. 139 743
[14] Tarasov V O, 1990 Phys. Lett. A 147 487
[15] Nishino A and Deguchi T, 2006 Phys. Lett. A 356 366
[16] Nishino A and Deguchi T, 2008 J. Stat. Phys. 133 587
[17] Au-Yang H and Perk J H H, 2011 J. Phys. A: Math. Theor. 44 025205
[18] Roan S-s, *Eigenvalues of an arbitrary Onsager sector in superintegrable $\tau$-model and chiral Potts model*, 2010 arXiv:1003.3621
[19] Bugrij A I and Shadura V N, 1997 Phys. Rev. B 55 11045
[20] Lieb E, Schulz T and Mattis D, 1961 Ann. Phys. 16 407
[21] Bugrij A, Iorgov N and Shadura V, 2005 JETP Lett. 82 311
[22] Lisovyy O, 2006 J. Phys. A: Math. Gen. 39 2265

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