Research Article

On a Quasi-Neutral Approximation of the Incompressible Navier-Stokes Equations

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This paper considers a pressureless Euler-Poisson system with viscosity in plasma physics in the torus $T^3$. We give a rigorous justification of its asymptotic limit toward the incompressible Navier Stokes equations via quasi-neutral regime using the modulated energy method.

1. Introduction

We will consider the following system:

$$\begin{align*}
\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon &= \mu \Delta u^\varepsilon + \nabla U^\varepsilon, \\
\partial_t n^\varepsilon + \text{div} (n^\varepsilon u^\varepsilon) &= 0, \\
\Delta U^\varepsilon &= \frac{n^\varepsilon - 1}{\varepsilon^2},
\end{align*}$$

(1.1)

for $x \in T^3$ and $t > 0$, $n^\varepsilon \in \mathbb{R}$, $u^\varepsilon \in \mathbb{R}^2$. $\varepsilon$ is small parameter and $\mu > 0$ is a constant viscosity coefficient. To solve uniquely the Poisson equation, we add the $\int_{T^3} n^\varepsilon dx = 1$. Passing to the limit when $\varepsilon \to 0$, it is easy to see, at least at a very formal level, that $(n^\varepsilon, u^\varepsilon)$ tends to $(n^{NS}, u^{NS})$, where $n^{NS} = 1$ and

$$\begin{align*}
\partial_t u^{NS} + (u^{NS} \cdot \nabla) u^{NS} &= \mu \Delta u^{NS} + \nabla U^{NS}, \\
\text{div} u^{NS} &= 0.
\end{align*}$$

(1.2)
In other words, $u^{NS}$ is a solution of the incompressible Navier-Stokes equations. The aim of this paper is to give a rigorous justification to this formal computation. The Euler-Poisson system with viscosity (1.1) is a physical model involving dissipation see [1], which here could be regarded as a viscous approximation of Euler-Poisson. Formally, it is a kind of new approximation of the incompressible Navier-Stokes equations of viscous fluid in real world.

It should be pointed out that there have been a lot of interesting results about the topic on the quasi-neutral (or called zero-Debye length) limit, for the readers to see [2–5] for isentropic Euler-Poisson system, [6, 7] for nonisentropic Euler-Poisson system, [8–10] for Vlasov-Poisson system, [11, 12] for drift-diffusion system, [13] for Euler-Maxwell equations, and therein references. We also mention that the above limit has been studied in [14, 15]. But in this present paper, the convergence result and the method of its proof is different from that of [14, 15].

The main focus in this paper is on the use of modulated energy techniques and div-curl for studying incompressible fluids. And for that, we assume that $n^\epsilon(x, \cdot)$ has total mass equal to 1 and the mean values of $u^\epsilon$ vanish, that is, $\mathbf{m}(u^\epsilon)=(1/(2\pi)^3)\int_{\mathbb{T}^3}u^\epsilon\,dx=0$. We also restrict ourselves to the case of well-prepared initial data and the case of periodic torus. Indeed, the quasi-neutral limit is much more difficult without these assumptions.

In this note, we will use some inequalities in Sobolev spaces, such as basic Moser-type calculus inequalities, Young inequality, and Gronwall inequality.

The paper is organized as follows. In Section 2 we state our main result. Estimates and proofs are given in Section 3.

2. Main Result

Throughout the paper, we will denote by $C$ a number independent of $\epsilon$, which actually may change from line to line. Moreover $(\cdot, \cdot)$ and $\| \cdot \|$ stand for the usual $L^2$ scalar product and norm, $\| \cdot \|_s$ is the usual $H^s$ Sobolev norm, and $\| \cdot \|_{s, \infty}$ is the usual $W^{s, \infty}$ norm.

The study of the asymptotic behavior of the sequence $(u^\epsilon, n^\epsilon)$, as $\epsilon$ goes to zero, leads to the statement of our main result.

**Theorem 2.1.** Let $u^{NS}$ be a solution of the incompressible Euler equations (1.2) such that $u^{NS} \in ([0, T], H^{s+3}(\mathbb{T}^3))$ and $\int_{\mathbb{T}^3} u^{NS} \, dx = 0$ for $s > (5/2)$. Assume that $(n_0^\epsilon, u_0^\epsilon)$ be a sequence of initial data such that $\int_{\mathbb{T}^3} n_0^\epsilon \, dx = 1$, $\int_{\mathbb{T}^3} u_0^\epsilon \, dx = 0$ and

$$\left\| u_0^\epsilon - u^{NS}_0 \right\|_{s+1} \leq C\epsilon, \quad \left\| n_0^\epsilon - 1 \right\|_s \leq C\epsilon^2,$$

with $u_0^{NS} = u^{NS} |_{t=0}$. Then there is a sequence $(n^\epsilon, u^\epsilon) \in \mathcal{C}([0, T], H^s \times H^{s+1}(\mathbb{T}^3))$ of solutions to (1.1) with initial data $(n_0^\epsilon, u_0^\epsilon)$ belonging to $\mathcal{C}([0, T_\epsilon], H^s \times H^{s+1}(\mathbb{T}^3))$ with $\liminf_{\epsilon \to 0} T_\epsilon \geq T$. Moreover for any $T_1 < T$ and $\epsilon$ small enough,

$$\left\| u^\epsilon(t) - u^{NS}(t) \right\|_s \leq C\epsilon, \quad \left\| n^\epsilon(t) - 1 \right\|_s \leq C\epsilon^2,$$

for any $0 \leq t \leq T_1$. 

3. Proof of the Theorem

If \((u^\epsilon, n^\epsilon)\) is a solution to system (1.1), we introduce

\[
\begin{align*}
    u^\epsilon &= u^{NS} + \epsilon u, \\
    n^\epsilon &= 1 + \epsilon^2 \left( n + \Delta U^{NS} \right), \\
    v^\epsilon &= U^{NS} + U.
\end{align*}
\]

Since the pressure \(U^{NS}\) in the incompressible Navier-Stokes equation is given by

\[
\Delta U^{NS} = \nabla u^{NS} : \nabla u^{NS},
\]

where, \(\nabla u : \nabla v = \sum_{i,j=1}^{3} (\partial_x u / \partial_x) (\partial_x v / \partial_x)\). Then the vector \((u^1, n^1, U^1)\) solves the system

\[
\begin{align*}
    \partial_t u + u^{NS} \cdot \nabla u &= \nabla U / \epsilon - \epsilon (u \cdot \nabla) u - (u \cdot \nabla) u^{NS} + \mu \Delta u, \\
    \partial_t n + u^{NS} \cdot \nabla n &= -\frac{\text{div} u}{\epsilon} - \epsilon \text{div} \left( (n + \Delta U^{NS}) u \right) - \partial_t \Delta U^{NS} - u^{NS} \cdot \nabla \Delta U^{NS}, \\
    \Delta U &= n.
\end{align*}
\]

As in [16], we make the following change of unknowns:

\[
d = \text{div} u, \quad c = \text{curl} u. \tag{3.4}
\]

By using the last equation and taking the curl and the divergence of the first equation in (3.5), we get the following system:

\[
\begin{align*}
    \partial_t d + u^{NS} \cdot \nabla d &= \frac{n}{\epsilon} - \epsilon (u \cdot \nabla) d - \epsilon \nabla u : \nabla u - \nabla u^{NS} + \mu \Delta d, \\
    \partial_t c + u^{NS} \cdot \nabla c &= -\epsilon (u \cdot \nabla) c - \epsilon (c \cdot \nabla) u - (c \cdot \nabla) u^{NS} \\
    &\quad + \text{curl} \left( \nabla u^{NS} \cdot u - u \cdot \nabla u^{NS} \right) - \epsilon dc + \mu \Delta d, \\
    \partial_t n + u^{NS} \cdot \nabla n &= -\frac{d}{\epsilon} - \epsilon (u \cdot \nabla) n - \epsilon \left( n + \Delta U^{NS} \right) d - u \cdot \nabla \Delta U^{NS} - \left( \partial_t u^{NS} \cdot \nabla \right) \Delta U^{NS}.
\end{align*}
\]

This last system can be written as a singular perturbation of a quasilinear symmetrizable hyperbolic system. Setting \(W^\epsilon = (d, c, n)^T\) yields

\[
\partial_t W^\epsilon + A(t, x, \partial_x) W^\epsilon = \frac{1}{\epsilon} K W^\epsilon - \epsilon B(t, x, \partial_x) W^\epsilon + S(\nabla) + \mu N(W^\epsilon) + R, \tag{3.6}
\]
where

\[
A(t, x, \partial_x) = \text{diag}(u^{NS} \cdot \nabla, u^{NS} \cdot \nabla I_3, u^{NS} \cdot \nabla), \quad B(t, x, \partial_x) = \text{diag}(u \cdot \nabla, u \cdot \nabla I_3, u \cdot \nabla),
\]

\[
K = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad N(\mathbb{W}^c) = \begin{pmatrix}
\Delta d \\
\Delta c \\
0
\end{pmatrix}, \quad R = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \quad (7.3)
\]

\[
S(\mathbb{W}^c) = \begin{pmatrix}
-\epsilon \nabla u : \nabla u - \nabla u : \nabla u^{NS} \\
-\epsilon (c \cdot \nabla) u - (c \cdot \nabla) u^{NS} + \text{curl}(\nabla u^{NS} \cdot u - u \cdot \nabla u^{NS}) - \epsilon d c \\
-\epsilon (n + \Delta u^{NS}) d - u \cdot \nabla \Delta u^{NS}
\end{pmatrix}.
\]

For \(|\alpha| \leq s\) with \(s > d/2\), we set

\[
\mathcal{E}_{s}^{1}(t) = \frac{1}{2} \left( \|\hat{\partial}_x^s d\|^2 + \|\hat{\partial}_x^s c\|^2 + \|\hat{\partial}_x^s n\|^2 \right),
\]

\[
\mathcal{E}_{s}^{1}(t) = \sum_{|\alpha| \leq s} \mathcal{E}_{\alpha,s}^{1}(t).
\]

Before performing the energy estimate, we apply the operator \(\hat{\partial}_x^s\) for \(a \in \mathbb{N}^3\) with \(|\alpha| \leq s\) to (3.6), to obtain

\[
\partial_t \hat{\partial}_x^s \mathbb{W}^c + A(t, x, \partial_x) \hat{\partial}_x^s \mathbb{W}^c = \frac{1}{\epsilon} K \hat{\partial}_x^s \mathbb{W}^c - \epsilon B(t, x, \partial_x) \hat{\partial}_x^s \mathbb{W}^c + \hat{\partial}_x^s N(\mathbb{W}^c) + \mu \hat{\partial}_x^s R.
\]

Now, we proceed to perform the energy estimates for (3.9) in a classical way by taking the scalar product of system (3.9) with \(\hat{\partial}_x^s \mathbb{W}^c\). Let us start the estimate of each term. First, since \(A(t, x, \partial_x)\) is symmetric and \(\text{div} u^{NS} = 0\), we have that

\[
(A(t, x, \partial_x) \hat{\partial}_x^s \mathbb{W}^c, \hat{\partial}_x^s \mathbb{W}^c) = -\int_{\Omega_0} \text{div} u^{NS} \left( \|\hat{\partial}_x^s d\|^2 + \|\hat{\partial}_x^s c\|^2 + \|\hat{\partial}_x^s n\|^2 \right) dx = 0.
\]

Next, since \(K\) is skew-symmetric, we have that

\[
\frac{1}{\epsilon} \langle K \hat{\partial}_x^s \mathbb{W}^c, \hat{\partial}_x^s \mathbb{W}^c \rangle = 0.
\]

By integration by parts, we have

\[
-\epsilon (B(t, x, \partial_x) \hat{\partial}_x^s \mathbb{W}^c, \hat{\partial}_x^s \mathbb{W}^c) = \epsilon \int_{\Omega_0} \text{div} u \|\hat{\partial}_x^s \mathbb{W}^c\|^2 dx \leq \|\text{div} u\|_0 \|\mathcal{E}_s^1(t)\| \leq (\mathcal{E}_s^1(t))^{3/2}.
\]

For later estimates in this paper, we recall some results on Moser-type calculus inequalities in Sobolev spaces [17, 18].
Lemma 3.1. Let $s \geq 1$ be an integer. Suppose $u \in H^s(T^3)$, $\nabla u \in L^\infty(T^3)$, and $v \in H^{s-1}(T^3) \cap L^\infty(T^3)$. Then for all multi-indices $|\alpha| \leq s$, one has $(\partial^s_x(\mu v) - u \partial^s_x v) \in L^2(T^3)$ and

$$
\|\partial^s_x(\mu v) - u \partial^s_x v\| \leq C_s \left( \|\nabla u\|_{0,\infty} \|D^{[\alpha]-1}v\| + \|D^{[\alpha]}u\|\|v\|_{0,\infty} \right),
$$

(3.13)

where

$$
\|D^h u\| = \sum_{|\alpha|=h} \|\partial^\alpha_x u\|, \quad \forall h \in \mathbb{N}.
$$

(3.14)

Moreover, if $s \geq 3$, then the embedding $H^{s-1}(T^3) \hookrightarrow L^\infty(T^3)$ is continuous and one has

$$
\|uv\|_{s-1} \leq C_s \|u\|_{s-1} \|v\|_{s-1}, \quad \|\partial^s_x(\mu v) - u \partial^s_x v\| \leq C_s \|u\|_s \|v\|_{s-1}.
$$

(3.15)

By using basic Moser-type calculus inequalities and Sobolev’s lemma, we have

$$
(\partial^s_x S(\nabla \mu), \partial^s_x \nabla \mu) \leq C_\epsilon \mathcal{E}_s(t) + C_\epsilon (\mathcal{E}_s(t))^{3/2}.
$$

(3.16)

After a a direct calculation, one gets

$$
\mu(\partial^s_x N(\nabla \mu), \partial^s_x \nabla \mu) = -\mu \int_{T^3} \left( |\nabla \partial^s_x d|^2 + |\nabla \partial^s_x c|^2 \right) dx.
$$

(3.17)

To estimate the commutator, we have

$$
([\partial^s_x, A(t, x, \partial_x)]\nabla \mu, \partial^s_x \nabla \mu)
= \int \left( [\partial^s_x, u^{NS} \cdot \nabla] d \partial^s_x d + [\partial^s_x, u^{NS} \cdot \nabla] c \partial^s_x c + [\partial^s_x, u^{NS} \cdot \nabla] n \partial^s_x n \right) dx
\leq C \left( \|u^{NS}\|_s \|\nabla d\|_{0,\infty} + \|u^{NS}\|_{0,\infty} \|\nabla \|_{s-1} \|d\|_s 
+ C \left( \|u^{NS}\|_s \|\nabla c\|_{0,\infty} + \|u^{NS}\|_{0,\infty} \|\nabla c\|_{s-1} \|c\|_s 
+ C \left( \|u^{NS}\|_s \|\nabla n\|_{0,\infty} + \|u^{NS}\|_{0,\infty} \|\nabla n\|_{s-1} \|n\|_s 
\leq C \mathcal{E}_s(t).
$$

(3.18)
Also, we have
\[
-\varepsilon([\partial_x^r, B(t, x, \partial_x)]^{\mathcal{W}^e}, \partial_x^{\mathcal{W}^e})
\]
\[
= -\varepsilon \int ([\partial_x, u \cdot \nabla] d\partial_x^s d + [\partial_x, u \cdot \nabla] c \partial_x^s c + [\partial_x, u \cdot \nabla] n\partial_x^s n)dx
\]
\[
\leq C\varepsilon(\|u\|_s \|\nabla d\|_{0, \infty} + \|u\|_{0, \infty} \|\nabla d\|_{s-1})\|d\|_s
\]
\[
+ C\varepsilon(\|u\|_s \|\nabla c\|_{0, \infty} + \|u\|_{0, \infty} \|\nabla c\|_{s-1})\|c\|_s
\]
\[
+ C\varepsilon(\|u\|_s \|\nabla n\|_{0, \infty} + \|u\|_{0, \infty} \|\nabla n\|_{s-1})\|n\|_s
\]
\[
\leq C\varepsilon(\mathcal{E}_s^e(t))^{3/2}.
\]

Here, we have used the inequality
\[
\|u\|_s \leq C\|\nabla u\|_{s-1} \leq C(\|d\|_{s-1} + \|c\|_{s-1}).
\]

Finally, the Young inequality gives
\[
(\partial_x R, \partial_x^{\mathcal{W}^e}) \leq \left\| \partial_t \Delta \mathcal{U}^{\text{NS}} + u^{\text{NS}} \cdot \nabla \Delta \mathcal{U}^{\text{NS}} \right\|_s \|n\|_s \leq C(1 + \mathcal{E}_s^e(t)).
\]

Notice that, to get the last line, we have used (3.2).

Now, we collect all the previous estimates (3.10)–(3.21) and we sum over \( \alpha \) to find
\[
\frac{d}{dt} \mathcal{E}_s^e(t) \leq C \left( 1 + \mathcal{E}_s^e(t) + \varepsilon(\mathcal{E}_s^e)^{3/2}(t) \right).
\]

We can conclude using a standard Gronwall’s lemma, that if the solution \((u^{\text{NS}}, \mathcal{U}^{\text{NS}})\) of Navier-Stokes equations (1.2) is smooth on the time interval \([0, T]\), for any \(T_1 < T\) there exists \(\varepsilon_0\) such that the sequence \((\mathcal{W}^e)_{\varepsilon \in \varepsilon_0}\) is bounded in \(C([0, T_1], H^s(T^3))\). Then we have
\[
\mathcal{W}^e = (\text{div } u, \text{curl } u, n),
\]
\[
u^e = u^{\text{NS}} + \varepsilon u,
\]
\[
n^e = 1 + \varepsilon^2 (n + \Delta \mathcal{U}^{\text{NS}}).
\]

The assumptions that we have made on the initial data imply that \((1/\varepsilon)(u^e - u^{\text{NS}}), (1/\varepsilon^2)(n-1)\) is bounded. This proves Theorem 2.1.

**References**

[1] P. Degond, “Mathematical modelling of microelectronics semiconductor devices,” in Some Current topics on Nonlinear Conservation Laws, vol. 15 of Proceedings of the Morningside Mathematical Center, Beijing, AMS/IP Studies in Advanced Mathematics, pp. 77–110, AMS Society and International Press, Providence, RI, USA, 2000.
[2] E. Grenier, “Pseudo-differential energy estimates of singular perturbations,” Communications on Pure and Applied Mathematics, vol. 50, no. 9, pp. 821–865, 1997.

[3] S. Wang, “Quasineutral limit of Euler-Poisson system with and without viscosity,” Communications in Partial Differential Equations, vol. 29, no. 3-4, pp. 419–456, 2004.

[4] S. Cordier and E. Grenier, “Quasineutral limit of an Euler-Poisson system arising from plasma physics,” Communications in Partial Differential Equations, vol. 25, no. 5-6, pp. 1099–1113, 2000.

[5] Y.-J. Peng and Y.-G. Wang, “Convergence of compressible Euler-Poisson equations to incompressible type Euler equations,” Asymptotic Analysis, vol. 41, no. 2, pp. 141–160, 2005.

[6] Y.-J. Peng, Y.-G. Wang, and W.-A. Yong, “Quasi-neutral limit of the non-isentropic Euler-Poisson system,” Proceedings of the Royal Society of Edinburgh A, vol. 136, no. 5, pp. 1013–1026, 2006.

[7] Y. Li, “Convergence of the nonisentropic Euler-Poisson equations to incompressible type Euler equations,” Journal of Mathematical Analysis and Applications, vol. 342, no. 2, pp. 1107–1125, 2008.

[8] Y. Brenier, “Convergence of the Vlasov-Poisson system to the incompressible Euler system,” Communications in Partial Differential Equations, vol. 25, no. 3-4, pp. 737–754, 2000.

[9] E. Grenier, “Oscillations in quasineutral plasmas,” Communications in Partial Differential Equations, vol. 21, no. 3-4, pp. 363–394, 1996.

[10] N. Masmoudi, “From Vlasov-Poisson system to the incompressible Euler system,” Communications in Partial Differential Equations, vol. 26, no. 9-10, pp. 1913–1928, 2001.

[11] I. Gasser, C. D. Levermore, P. A. Markowich, and C. Schmeiser, “The initial time layer problem and the quasineutral limit in the semiconductor drift-diffusion model,” European Journal of Applied Mathematics, vol. 12, no. 4, pp. 497–512, 2001.

[12] I. Gasser, L. Hsiao, P. A. Markowich, and S. Wang, “Quasi-neutral limit of a nonlinear drift diffusion model for semiconductors models,” Journal of Mathematical Analysis and Applications, vol. 268, no. 1, pp. 184–199, 2002.

[13] Y.-J. Peng and S. Wang, “Convergence of compressible Euler-Maxwell equations to incompressible Euler equations,” Communications in Partial Differential Equations, vol. 33, no. 1–3, pp. 349–376, 2008.

[14] Q. Ju, F. Li, and S. Wang, “Convergence of the Navier-Stokes-Poisson system to the incompressible Navier-Stokes equations,” Journal of Mathematical Physics, vol. 49, no. 7, 2008.

[15] Q. Ju, F. Li, and H. Li, “The quasineutral limit of compressible Navier-Stokes-Poisson system with heat conductivity and general initial data,” Journal of Differential Equations, vol. 247, no. 1, pp. 203–224, 2009.

[16] G. Loeper, “Quasi-neutral limit of the Euler-Poisson and Euler-Monge-Ampère systems,” Communications in Partial Differential Equations, vol. 30, no. 7–9, pp. 1141–1167, 2005.

[17] S. Klainerman and A. Majda, “Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids,” Communications on Pure and Applied Mathematics, vol. 34, no. 4, pp. 481–524, 1981.

[18] A. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, vol. 53, Springer, New York, NY, USA, 1984.
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