THE NUMBER OF ARROWS IN THE QUIVER OF
TILTING MODULES OVER A PATH ALGEBRA OF TYPE
A AND D

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Abstract. Happel and Unger defined a partial order on the set of basic
tilting modules. The tilting quiver is the Hasse diagram of the poset of
basic tilting modules. We determine the number of arrows in the tilting
quiver over a path algebra of type A or D.

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Introduction

In this paper we use the following notations. Let $A$ be a finite dimensional
algebra over an algebraically closed field $k$, and let \text{mod-}$A$ be the category
of finite dimensional right $A$-modules. For $M \in \text{mod-}$A we denote by $\text{pd}_A M$
the projective dimension of $M$, and by add $M$ the full subcategory of direct
sums of direct summands of $M$. Let $Q = (Q_0, Q_1)$ be a finite connected
quiver without loops and cycles, and $Q_0$ (resp.$Q_1$) be the set of vertices
(resp.arrows) of $Q$ (we use this notation for an arbitrary quiver). We denote
by $kQ$ the path algebra of $Q$ over $k$, and by $\text{rep}$Q the category of finite
dimensional representations of the quiver $Q$ which is category equivalent to
mod-$kQ$. For $M \in \text{rep}$Q, denote by $M_a$ the vector space of $M$ associated
to a vertex $a$, and denote by $M_{a \rightarrow b}$ the linear map $M_a \rightarrow M_b$ of $M$. For a
vertex $a$ of $Q$, let $\sigma_a Q$ be the quiver obtained from $Q$ by reversing all arrows
starting at $a$ or ending at $a$. A module $T \in \text{mod-}$A is called a tilting module
provided the following three conditions are satisfied:

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(a) \( \text{pd} \, T < \infty \),
(b) \( \text{Ext}^i(T,T) = 0 \) for all \( i > 0 \),
(c) there exists an exact an sequence
\[
0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0 \quad (T_i \in \text{add} \, T)
\]
in \( \text{mod}-A \). In the hereditary case the tilting condition above is equivalent to the following:
(a) \( \text{Ext}^1(T,T) = 0 \),
(b) the number of indecomposable direct summands of \( T \) (up to isomorphism) is equal to the number of simple modules.

In section 1, following \[8\], \[9\], \[10\], \[16\], we define a partial order on the set \( \text{Tilt}(A) \) of all basic tilting modules (up to isomorphism) over \( A \) and define the quiver of tilting modules \( \vec{K}(A) \). In Section 2, we explain results from \[11\]. In Section 3, we first show that the number of arrows of \( \vec{K}(kQ) \) is equal to the number of arrows of \( \vec{K}(kQ') \) if \( Q \) and \( Q' \) share the same underlying graph by applying the results from Section 2. Then we determine the number of arrows of \( \vec{K}(kQ) \) for any Dynkin quiver \( Q \) of type \( A \) or \( D \). Note that the underlying graph of \( \vec{K}(kQ) \) may be embedded into the exchange graph, or the cluster complex, of the corresponding cluster algebra of finite type: the tilting modules of \( kQ \) correspond to positive clusters \[3\] and \[12\]. The number of positive clusters when the orientation is alternating is given in \[6\, \text{prop. 3.9}\]. However, according to experts, the number of edges of this subdiagram of positive clusters is not known in the cluster tilting theory. Note also that if we consider the similar problem for the exchange graph, it is not interesting, because the number of edges is \( \frac{n}{2} \times \) (the number of vertices), and the number of vertices is given in \[6\, \text{prop. 3.8}\]. The following is known.

\[
\# \vec{K}(kQ)_0 = \begin{cases} 
\frac{1}{n+1} \binom{2n}{n} & \text{if } Q \text{ is a Dynkin quiver of type } A_n, \\
\frac{3n-4}{2n} \binom{2(n-1)}{n-1} & \text{if } Q \text{ is a Dynkin quiver of type } D_n.
\end{cases}
\]

The main result of this paper is as follows.

**Theorem 0.1.** (1) Let \( Q \) be a quiver without loops and cycles. Then \( \# \vec{K}(kQ)_1 \) is independent of the orientation.

(2):

\[
\# \vec{K}(kQ)_1 = \begin{cases} 
\frac{(2n-1)}{n+1} & \text{if } Q \text{ is a Dynkin quiver of type } A_n, \\
(3n-4) \binom{2(n-2)}{n-3} & \text{if } Q \text{ is a Dynkin quiver of type } D_n.
\end{cases}
\]

As a corollary of our theorem, we also have the following. The exchange graph here is defined in a purely combinatorial manner by using the notion of compatibility degree.

**Theorem 0.2.** We consider the root system \( \Phi \) (resp. the positive root system \( \Phi_{>0} \)) of type \( A_n \) or \( D_n \). Let \( E(\Phi) \) be the exchange graph of \( \Phi \) (see \[6\, \text{def.}


1.14) and $E(\Phi_{>0})$ the subgraph of $E(\Phi)$ whose vertices are positive clusters. Then the number of edges in $E(\Phi_{>0})$ is

$$
\begin{cases}
\begin{aligned}
\binom{2n-1}{n+1} & \text{if } \Phi \text{ is of type } A_n, \\
(3n-4)(\binom{2(n-2)}{n-3}) & \text{if } \Phi \text{ is of type } D_n.
\end{aligned}
\end{cases}
$$

For the proof, let $Q$ be the alternating Dynkin quiver associated to $\Phi$. Then (by [12, cor. 4.12]) $E(\Phi_{>0})$ coincides with the underlying graph of $\tilde{K}(kQ)$.

1. Preliminaries

In this section we define a partial order on tilting modules. First, for a tilting module $T$, we define the right perpendicular category

$$
T^\perp = \{ X \in \text{mod-}A \mid \text{Ext}_{A}^{>0}(T, X) = 0 \}.
$$

Lemma 1.1. (cf. [9, lemma2.1 (a)]) For tilting modules $T, T'$ the following conditions are equivalent:

1. $T^\perp \subset T'^\perp$,
2. $T \in T'^\perp$.

Recall that $\text{Tilt}(A)$ is the set of basic tilting modules of $A$.

Definition 1.2. We define a partial order on $\text{Tilt}(A)$ by

$$
T \leq T' \iff T^\perp \subset T'^\perp \iff T \in T'^\perp,
$$

for $T, T' \in \text{Tilt}(A)$.

By definition, $A_A$ is the unique maximal element of $(\text{Tilt}(A), \leq)$. On the other hand, $(\text{Tilt}(A), \leq)$ does not always admit a minimal element.

Definition 1.3. (cf. [16]) Let $C$ be a full subcategory of $\text{mod-}A$ which is closed under direct sums, direct summands and isomorphisms. The subcategory $C$ is called contravariantly finite in $\text{mod-}A$, if every $X \in \text{mod-}A$ has a right $C$-approximation, i.e. there is a morphism $F_X \rightarrow X$ with $F_X \in C$ s.t. the induced morphism $\text{Hom}_A(C, F_X) \rightarrow \text{Hom}_A(C, X)$ is surjective for all $C \in C$.

Theorem 1.4. (cf. [8, thm 3.3 and cor 3.4]) $(\text{Tilt}(A), \leq)$ has a minimal element if and only if $\mathcal{P}^{<\infty}$ (the full subcategory of $\text{mod-}A$ of modules with finite projective dimension) is contravariantly finite in $\text{mod-}A$. Moreover the minimal element is unique if it exists.

Next we define the tilting quiver $\tilde{K}(A)$, and recall its some properties. Let $\text{ind}A$ be a category of indecomposable modules in $\text{mod-}A$. 

Definition 1.5. The tilting quiver $\bar{\mathcal{K}}(A) = (\bar{\mathcal{K}}(A)_0, \bar{\mathcal{K}}(A)_1)$ is defined as follows.

(1) $\bar{\mathcal{K}}(A)_0 = \text{Tilt}(A)$,
(2) $T' \to T$ in $\bar{\mathcal{K}}(A)$, for $T, T' \in \text{Tilt}(A)$, if $T' = M \oplus X$, $T = M \oplus Y$ with $X, Y \in \text{ind} A$ and there is a non-split short exact sequence

$$0 \to X \to \tilde{M} \to Y \to 0$$

with $\tilde{M} \in \text{add} M$.

Theorem 1.6. (cf. [S] thm 2.1) $\bar{\mathcal{K}}(A)$ is the Hasse-diagram of $(\text{Tilt}(A), \leq)$ (i.e. if $T \to T' \in \bar{\mathcal{K}}(A)_1$ and $T \geq T'' \geq T'$ then $T'' = T$ or $T'' = T'$).

Proposition 1.7. (cf. [S] cor 2.2) If $\bar{\mathcal{K}}(A)$ has a finite component $\mathcal{C}$, then $\bar{\mathcal{K}}(A) = \mathcal{C}$.

Corollary 1.8. $\bar{\mathcal{K}}(A)$ contains at most one sink. Moreover it contains a sink if and only if $\mathcal{P}^{<\infty}(A)$ is contravariantly finite in mod$A$.$^*$

Proposition 1.9. (cf. [S] cor 3.6) Let $T \in \text{Tilt}(A)$ s.t. $\text{End}_A(T)$ is representation finite. Then $\mathcal{P}^{<\infty}(A)$ is contravariantly finite.

Let $Q = (Q_0, Q_1)$ be a quiver without loops and cycles and $A = kQ$. For $T \in \text{Tilt}(A)$, let

$$s(T) = \# \{ T' \in \text{Tilt}(A) \mid T \to T' \text{ in } \mathcal{K}(kQ) \}$$
$$e(T) = \# \{ T' \in \text{Tilt}(A) \mid T' \to T \text{ in } \mathcal{K}(kQ) \}$$

and define $\delta(T) = s(T) + e(T)$.

Proposition 1.10. (cf. [10] prop 3.2) $\delta(T) = n - \# \{ a \in Q_0 \mid (\dim T)_a = 1 \}$, where $n = \# Q_0$.

2. A theorem of Ladkani

In this section, we review [11]. Let $Q$ be a quiver without loops and cycles and let $x$ be a source of $Q$. Let $\text{Tilt}(Q) := \text{Tilt}(kQ)$ and define

$$\text{Tilt}(Q)^x := \{ T \in \text{Tilt}(Q) \mid S(x) \mid T \}$$

where $S(x)$ is the simple module associated to $x$.

Definition 2.1. Let $(X, \leq_X), (Y, \leq_Y)$ be posets and $f : X \to Y$ an order-preserving function. Then we define the preorder $\leq^f_+, \leq^f_-$ of $X \sqcup Y$ as follows.

$$a \leq^f_+ b \iff \begin{cases} a \leq_X b & \text{if } a, b \in X, \\ a \leq_Y b & \text{if } a, b \in Y, \\ f(a) \leq_Y b & \text{if } a \in X \text{ and } b \in Y. \end{cases}$$

$$a \leq^f_- b \iff \begin{cases} a \leq_X b & \text{if } a, b \in X, \\ a \leq_Y b & \text{if } a, b \in Y, \\ a \leq_Y f(b) & \text{if } a \in Y \text{ and } b \in X. \end{cases}$$
Lemma 2.2. Define the functors
\[ j^{-1} : \text{rep } Q \longrightarrow \text{rep}(Q \setminus \{x\}) \]
and
\[ j_* : \text{rep}(Q \setminus \{x\}) \longrightarrow \text{rep } Q, \]
by
\[ (j^{-1} M)_a = M_a \quad (j^{-1} M)_{a \to b} = M_{a \to b} \]
and
\[ (j_* N)_a = \begin{cases} N_a & (a \neq x) \\ \oplus_{x \to y} N(y) & (a = x) \end{cases}, \quad (j_* N)_{a \to b} = \begin{cases} N_{a \to b} & (a \neq x) \\ (j_* N)_x \text{ projection} & (a = x). \end{cases} \]
Then \( j^{-1} \) and \( j_* \) are exact and \( j_* \) is right adjoint to \( j^{-1} \).

Denote by \( D^b(Q) \) the bounded derived category \( D^b(\text{rep } Q) \).

Lemma 2.3. The functors \( j^{-1} \) and \( j_* \) induce functors
\[ j^{-1} : D^b(Q) \longrightarrow D^b(Q \setminus \{x\}), \quad j_* : D^b(Q \setminus \{x\}) \longrightarrow D^b(Q) \]
with
\[ \text{Hom}_{D^b(Q \setminus \{x\})}(j^{-1} M, N) \cong \text{Hom}_{D^b(Q)}(M, j_* N), \]
for all \( M \in D^b(Q), \ N \in D^b(Q \setminus \{x\}) \).

Lemma 2.4. The functors \( j^{-1} \) and \( j_* \) identify \( \text{rep}(Q \setminus \{x\}) \) with the right perpendicular subcategory
\[ S(x)^\perp = \{ M \in \text{rep } Q \mid \text{Ext}^i(S(x), M) = 0 \text{ for all } i \geq 0 \} \]
of \( \text{rep } Q \).

Lemma 2.5. The functor \( j_* \) takes indecomposables of \( \text{rep}(Q \setminus \{x\}) \) to indecomposables of \( \text{rep } Q \).

Proposition 2.6. Let \( T \) be a tilting module in \( \text{rep } Q \). Then \( j^{-1} T \) is a tilting module in \( \text{rep}(Q \setminus \{x\}) \).

For \( M = \oplus_{i=1}^m N_i^{r_i} \) (where \( N_i \in \text{ind } Q, r_i > 0 \)), let \( \text{basic}(M) = \oplus_{i=1}^m N_i \).

Corollary 2.7. The map \( \pi_x : T \mapsto \text{basic}(j^{-1} T) \) is an order-preserving function
\[ (\text{Tilt}(Q), \leq) \rightarrow (\text{Tilt}(Q \setminus \{x\}), \leq). \]

Proposition 2.8. Let \( T \in \text{Tilt}(Q \setminus \{x\}) \). Then \( S(x) \oplus j_* T \in \text{Tilt}(Q) \).

Corollary 2.9. The map \( \iota_x : T \mapsto S(x) \oplus j_* T \) is an order-preserving function
\[ (\text{Tilt}(Q \setminus \{x\}), \leq) \rightarrow (\text{Tilt}(Q), \leq). \]
Proposition 2.10. We have

$$\pi_x \iota_x(T) = T,$$

for all $T \in \text{Tilt}(Q \setminus \{x\})$. In addition,

$$T \geq \iota_x \pi_x(T),$$

for all $T \in \text{Tilt}(Q)$, with equality if and only if $T \in \text{Tilt}(Q)^x$.

In particular, $\pi_x$ and $\iota_x$ induce an isomorphism of posets between $\text{Tilt}(Q)^x$ and $\text{Tilt}(Q \setminus \{x\})$.

Corollary 2.11. Let $X = \text{Tilt}(Q) \setminus \text{Tilt}(Q)^x$ and $Y = \text{Tilt}(Q)^x$. Define $f : X \to Y$ by $f = \iota_x \pi_x$. Then

$$\text{Tilt}(Q) \simeq (X \sqcup Y, \leq_f).$$

Now let $Q' = \sigma_x Q$. Then $x$ is a sink of $Q'$ and, by arguing in the similar way, we obtain the dual results by replacing

$$(j^{-1}, j, \pi_x, \iota_x, X, Y, f, \leq_f)$$

with

$$(i^{-1}, i, \pi'_x, \iota'_x, X', Y', f', \leq'_{+, +}).$$

In particular we get

$$\text{Tilt}(Q')^x \simeq \text{Tilt}(Q \setminus \{x\}),$$

and

$$\text{Tilt}(Q') \simeq (X' \sqcup Y', \leq'_{+, +}),$$

where $X' = \text{Tilt}(Q') \setminus \text{Tilt}(Q')^x$ and $Y' = \text{Tilt}(Q')^x$.

Theorem 2.12. There exists an isomorphism of posets

$$\rho : \text{Tilt}(Q) \setminus \text{Tilt}(Q)^x \to \text{Tilt}(Q') \setminus \text{Tilt}(Q')^x$$

such that the following diagram commutes.

$$\begin{array}{ccc}
\text{Tilt}(Q) \setminus \text{Tilt}(Q)^x & \xrightarrow{\rho_x} & \text{Tilt}(Q) \setminus \text{Tilt}(Q')^x \\
\sim & & \sim \\
\text{Tilt}(Q)^x & \xrightarrow{\pi_x} & \text{Tilt}(Q \setminus \{x\})
\end{array}$$

$$\begin{array}{ccc}
\text{Tilt}(Q) \setminus \text{Tilt}(Q')^x & \xrightarrow{\iota'_x} & \text{Tilt}(Q')^x \\
\sim & & \sim \\
\text{Tilt}(Q) & \xrightarrow{\iota_x} & \text{Tilt}(Q \setminus \{x\})
\end{array}$$

Corollary 2.13. $\# \text{Tilt}(Q) = \# \text{Tilt}(Q')$.

Remark 2.14. In [11] the partial order on $\text{Tilt}(A)$ is defined by

$$T \geq T' \iff T' \subseteq T\top \quad (\text{opposite to our definition}).$$
3. Main results

In this section we determine the number of arrows of $\tilde{K}(kQ)$ in the case $Q$ is a Dynkin quiver of type $A$ or $D$. Let

$$\text{Gen}(M) := \{ N \in \mod \cdot A \mid M' \xrightarrow{\text{surjection}} N \text{ for some } M' \in \text{add } M \}$$

$$\text{Cogen}(M) := \{ N \in \mod \cdot A \mid N \xleftarrow{\text{injection}} M' \text{ for some } M' \in \text{add } M \} .$$

**Lemma 3.1.** (cf.[5, prop 1.3]) Let $A$ be hereditary, $T = M \oplus Y \in \text{Tilt}(A)$ with $Y \in \text{ind } A$. If $Y \in \text{Gen}(M)$, then there exists a unique (up to isomorphism) indecomposable module $X$ which is not isomorphic to $Y$ s.t. $M \oplus X \in \text{Tilt}(A)$ and there exists an exact sequence

$$0 \rightarrow X \rightarrow E \rightarrow Y \rightarrow 0$$

with $E \in \text{add } M$.

Dually, if $Y \in \text{Cogen}(M)$ then there exists a unique (up to isomorphism) indecomposable module $X$ which is not isomorphic to $Y$ s.t. $M \oplus Y \in \text{Tilt}(A)$ and there exists an exact sequence

$$0 \rightarrow Y \rightarrow E \rightarrow X \rightarrow 0$$

with $E \in \text{add } M$.

**Lemma 3.2.** Let $Q$ be a quiver without loops and cycles. If $x$ is a sink, then for all $T = M \oplus S(x) \in \text{Tilt}(Q)$, $S(x)$ is in $\text{Cogen}(M)$. If $x$ is a source, then for all $T = M \oplus S(x) \in \text{Tilt}(Q)$, $S(x)$ is in $\text{Gen}(M)$.

**Proof.** By Proposition 2.10

$$\text{Tilt}(Q \setminus \{x\}) \xleftarrow{1:1} \text{Tilt}(Q)^x : T \mapsto F(T) \oplus S(x)$$

where $F(T) \in \mod \cdot kQ$ is defined by

$$F(T)_a = \begin{cases} T_a & \text{if } a \neq x, \\ \oplus_{y \rightarrow x} T_y & \text{if } a = x \text{ and } x \text{ is a sink}, \\ \oplus_{x \rightarrow y} T_y & \text{if } a = x \text{ and } x \text{ is a source}. \end{cases}$$

$$F(T)_{a \rightarrow b} = \begin{cases} T_{a \rightarrow b} & \text{if } a, b \neq x, \\ T_y \xrightarrow{\text{injection}} \oplus_{y \rightarrow x} T_y & \text{if } a = y \text{ with } y \rightarrow x \text{ and if } b = x \text{ and } x \text{ is a sink}, \\ \oplus_{x \rightarrow y} T_y \xrightarrow{\text{projection}} T_y & \text{if } b = y \text{ with } x \rightarrow y \text{ and if } a = x \text{ and } x \text{ is a source}. \end{cases}$$

Now if $x$ is a sink then

$$S(x) \in \text{Cogen}(M) \iff M_x \neq 0,$$

and if $x$ is a source then

$$S(x) \in \text{Gen}(M) \iff M_x \neq 0.$$

So the lemma follows from the fact that if $T \in \text{Tilt}(Q)$ then $(\dim T)_a \geq 1$, for all $a$. ⊰
Lemma 3.3. If $x$ is a sink then
\[ \{ \alpha \in \tilde{\mathcal{K}}(Q)_1 \mid s(\alpha) \in \text{Tilt}(Q)^x, t(\alpha) \in \text{Tilt}(Q) \setminus \text{Tilt}(Q)^x \} \xrightarrow{1:1} \text{Tilt}(Q)^x. \]
If $x$ is a source then
\[ \{ \alpha \in \tilde{\mathcal{K}}(Q)_1 \mid t(\alpha) \in \text{Tilt}(Q)^x, s(\alpha) \in \text{Tilt}(Q) \setminus \text{Tilt}(Q)^x \} \xrightarrow{1:1} \text{Tilt}(Q)^x. \]

Where, for $T \xrightarrow{\alpha} T'$, $s(\alpha) = T$ and $t(\alpha) = T'$.

Proof. Suppose $x$ is a sink, and let $T \in \text{Tilt}(Q)^x$. Then there exists a unique $T' \in \text{Tilt}(Q) \setminus \text{Tilt}(Q)^x$ s.t. $T \rightarrow T'$ in $\tilde{\mathcal{K}}(Q)$ (by lemma 3.1, 3.2).

On the other hand, let $T' \in \text{Tilt}(Q) \setminus \text{Tilt}(Q)^x$ and suppose that there exists $T_1, T_2 \in \text{Tilt}(Q)^x$ s.t. $T_1 \rightarrow T', T_2 \rightarrow T'$, for $T' \in \text{Tilt}(Q) \setminus \text{Tilt}(Q)^x$, in $\tilde{\mathcal{K}}(Q)$. Write $T_i = M \oplus S(x) \oplus Y_i$ with $Y_i \in \text{ind}kQ(i = 1, 2)$ then $Y_i \mid T'$; $\text{Ext}(Y_i, Y_j) = 0$ $(i, j = 1, 2)$. Thus $\text{Ext}(T_1 \oplus Y_2, T_1 \oplus Y_2) = 0$ and $Y_1 = Y_2$ follows.

Corollary 3.4.
\[ \#\tilde{\mathcal{K}}(Q)_1 = \#\tilde{\mathcal{K}}(\sigma_x Q)_1. \]

In particular, if $Q$ is a Dynkin quiver then $\#\tilde{\mathcal{K}}(Q)_1$ depends only on the underlying graph of $Q$.

Proof. By corollary 2.11 and lemma 3.3
\[ \#\tilde{\mathcal{K}}(Q)_1 = \#\tilde{\mathcal{K}}(Q \setminus \{x\})_1 + \#\tilde{\mathcal{K}}(\text{Tilt}(Q) \setminus \text{Tilt}(Q)^x)_1 + \#\text{Tilt}(Q)^x = \#\tilde{\mathcal{K}}(\sigma_x Q)_1. \]

3.1. case A. In this subsection we consider the quiver,
\[ Q = \overset{1}{\circ} \overset{2}{\circ} \cdots \overset{n}{\circ}. \]

By Gabriel’s theorem, $\text{ind}kQ = \{ L(i, j) \mid 0 \leq i < j \leq n \}$ where
\[ L(i, j) = \begin{cases} k & (i < a \leq j), \\ 0 & \text{otherwise}, \end{cases} \quad L(i, j)_{a \rightarrow b} = \begin{cases} 1 & (i < a, b \leq j), \\ 0 & \text{otherwise}. \end{cases} \]

And
\[ \tau L(i, j) = \begin{cases} L(i + 1, j + 1) & (j < n), \\ 0 & (j = n), \end{cases} \]
where $\tau$ is a Auslander-Reiten translation.

Definition 3.5. A pair of intervals $[[i, j], [i', j']]$ is compatible if
\[ [i, j] \cap [i', j'] = \emptyset \text{ or } [i, j] \subset [i', j'] \text{ or } [i', j'] \subset [i, j]. \]

Applying Auslander-Reiten duality,
\[ \text{DExt}(M, N) \cong \text{Hom}(N, \tau M) (\text{D} = \text{Hom}_k(-, k)) \]
we get the following lemma.
Lemma 3.6. We have
\[ \text{Ext}(L(i,j),L(i',j')) = 0 = \text{Ext}(L(i',j'),L(i,j)) \]
if and only if \([i,j],[i',j']\) is compatible.

Proof. It is obvious that \(\text{Hom}(L(i,j),L(i',j')) \neq 0\) if and only if \(i' \leq i \leq j' \leq j\). So the lemma follows from this fact and the AR-duality. \(\square\)

Lemma 3.7. For any \(T \in \text{Tilt}(Q)\), we get
\[ \delta(T) = n - 1. \]

Proof. Let \(T \in \text{Tilt}(Q)\) then the projective-injective module \(L(0,n)\) is a direct summand of \(T\). From this fact, we get \(\delta(T) < n\).

Denote by \(X\) the set of indecomposable direct summands of \(T\) not isomorphic to \(L(0,n)\) and define
\[ a := \begin{cases} \max\{i \mid L(0,i) \in X\} & \text{if } L(0,i) \in X \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases} \]

Then, by lemma 3.6, we get
\[ \text{Ext}(T,L(a+1,n)) = 0 = \text{Ext}(L(a+1,n),T). \]

By \(\text{Ext} = 0\) condition, we can see \(L(a+1,n)\) is a direct summand of \(T\). In particular
\[ (\dim T)_i = 1 \iff i = a + 1. \]

The lemma follows from this fact and proposition 1.10. \(\square\)

Now it is easy to check the number of arrows in \(\tilde{\mathcal{K}}(Q)\), because it is equal to
\[ \frac{1}{2} \sum_{T \in \text{Tilt}(Q)} \delta(T) \]

Corollary 3.8. \(\# \tilde{\mathcal{K}}(Q)_1 = \frac{n-1}{2(n+1)} \binom{2n}{n} = \binom{2n-1}{n-2}.\)

3.2. case \(D\). Through this subsection, we consider the quiver
\[ Q = Q_n = \begin{array}{c}
1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n^+ \\
\vdots \\
\vdots \\
\vdots \\
1 \longrightarrow \cdots \longrightarrow n^- \end{array} \]

Then \(\text{ind } kQ = \{L(a,b) \mid 0 \leq a < b \leq n-1\} \cup \{L^\pm(a,n) \mid 0 \leq a \leq n-1\} \cup \{M(a,b) \mid 0 \leq a < b \leq n-1\}\)
where

\[
L(a, b)_i = \begin{cases} k & \text{if } a < i \leq b, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
L(a, b)_{i \rightarrow j} = \begin{cases} 1 & \text{if } a < i < b, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
L(a, n)_{i}^{\pm} = \begin{cases} k & \text{if } a < i \leq n - 1 \text{ or } i = n^{\pm}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
L(a, n)_{i \rightarrow j}^{\pm} = \begin{cases} 1 & \text{if } a < i < n - 1 \text{ or } i = n - 1, j = n^{\pm}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
M(a, b)_i = \begin{cases} k & \text{if } a < i \leq b \text{ or } i = n^{\pm}, \\ k^2 & \text{if } b < i \leq n - 1, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
M(a, b)_{i \rightarrow j} = \begin{cases} 1 & \text{if } a < i < b, \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } i = b, \\ (1, 0) & \text{if } i = n - 1, j = n^{+}, \\ (0, 1) & \text{if } i = n - 1, j = n^{-}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } b < i < n - 1, \\ 0 & \text{otherwise}. \end{cases}
\]

Then

\[
\tau L(a, b) = \begin{cases} L(a + 1, b + 1) & \text{if } b < n - 1, \\ M(0, a + 1) & \text{if } b = n - 1, \end{cases}
\]

\[
\tau L^{+}(a, n) = L^{-}(a + 1, n),
\]

\[
\tau L^{-}(a, n) = L^{+}(a + 1, n),
\]

\[
\tau M(a, b) = \begin{cases} M(a + 1, b + 1) & \text{if } b < n - 1, \\ 0 & \text{if } b = n - 1. \end{cases}
\]
Lemma 3.9.

(1) \( \text{Ext}(L(a,b),L(a',b')) = 0 = \text{Ext}(L(a',b'), L(a,b)) \)
\( \iff ([a,b], [a',b']) : \text{compatible}. \)

(2) \( \text{Ext}(L(a,b),L^+(a',n)) = 0 = \text{Ext}(L^+(a',n), L(a,b)) \)
\( \iff ([a,b], [a',n]) : \text{compatible}. \)

(3) \( \text{Ext}(L(a,b), M(a',b')) = 0 = \text{Ext}(M(a',b'), L(a,b)) \)
\( \iff ([a,b], [a',b']) : \text{compatible}. \)

(4) \( \text{Ext}(M(a,b), L^+(a',n)) = 0 = \text{Ext}(L^+(a',n), M(a,b)) \)
\( \iff a \leq a' \leq b. \)

(5) \( \text{Ext}(L^+(a,n), L^+(a',n)) = 0 = \text{Ext}(L^+(a',n), L^+(a,n)) \) for all \( a, a' \).

(6) \( \text{Ext}(L^+(a,n), L^-(a',n)) = 0 = \text{Ext}(L^-(a',n), L^+(a,n)) \)
\( \iff a = a'. \)

(7) \( \text{Ext}(M(a,b), M(a',b')) = 0 = \text{Ext}(M(a',b'), M(a,b)) \)
\( \iff [a,b] \subset [a',b'] \lor [a',b'] \subset [a,b]. \)

Proof. (1) and (2) follow from the case \( A \) and (5),(6) are obvious.

(3): (case \( b < a' \)) It is obvious that
\( \text{Ext}(L(a,b), M(a',b')) = 0 = \text{Ext}(M(a',b'), L(a,b)). \)

(cas\( e a < a' \leq b < b' \)) In this case we claim that
\( \text{Hom}(M(a',b'), \tau L(a,b)) \neq 0. \)

In fact \( 0 \neq f = (f_i)_i \in \text{Hom}(M(a',b'), \tau L(a,b)) \) where
\[
 f_i = \begin{cases} 
 1 & \text{if } a' < i \leq b + 1, \\
 0 & \text{otherwise}. 
\end{cases}
\]

(cas\( e a < a' < b' \leq b < n - 1 \)) In this case we claim that
\( \text{Hom}(M(a',b'), \tau L(a,b)) \neq 0. \)

In fact \( 0 \neq f = (f_i)_i \in \text{Hom}(M(a',b'), \tau L(a,b)) \) where
\[
 f_i = \begin{cases} 
 1 & \text{if } a' < i \leq b', \\
 (0,1) & \text{if } b' < i \leq b, \\
 0 & \text{otherwise}. 
\end{cases}
\]

(cas\( e a < a' < b' \leq b = n - 1 \)) In this case we also claim that
\( \text{Hom}(M(a',b'), \tau L(a,b)) \neq 0. \)
In fact $0 \neq f = (f_i)_i \in \text{Hom}(M(a', b'), \tau L(a, n-1))$ where

$$f_i = \begin{cases} 
(1) & \text{if } a' < i \leq b', \\
1 & \text{if } b' < i \leq n-1 \text{ or } i = n^\pm, \\
0 & \text{otherwise}.
\end{cases}$$

(case $a' \leq a < b < b' < n-1$) In this case we claim that

$$\text{Hom}(M(a', b'), \tau L(a, b)) = 0 = \text{Hom}(L(a, b), \tau M(a', b')).$$

Let $f = (f_i)_i \in \text{Hom}(M(a', b'), \tau L(a, b))$. Then $f_i = 0$ if $i \leq a + 1$ or $b + 1 < i \leq n - 1$ or $i = n^\pm$ and

$$f_{a+2} = f_{a+3} = \cdots = f_{b+1}.$$

Now the commutative square for $f_{a+1}, f_{a+2}$ shows $f_{a+2} = 0$. So

$$\text{Hom}(M(a', b'), \tau L(a, b)) = 0.$$

And similarly

$$\text{Hom}(L(a, b), \tau M(a', b')) = 0.$$

(case $a' \leq a < b < b' = n - 1$) It is obvious that

$$\text{Hom}(M(a', n-1), \tau L(a, b)) = 0$$

and, since $M(a', n-1)$ is projective, we have

$$\text{Hom}(L(a, b), \tau M(a', b')) = 0.$$

(case $a' \leq a < b' \leq b < n-1$) In this case we claim that

$$\text{Hom}(M(a', b'), \tau L(a, b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \text{Hom}(M(a', b'), \tau L(a, b))$ where

$$f_i = \begin{cases} 
(1, -1) & \text{if } b' < i \leq b+1, \\
0 & \text{otherwise}.
\end{cases}$$

(case $a' \leq a < b' \leq b = n - 1$) In this case we also claim that

$$\text{Hom}(M(a', b'), \tau L(a, b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \text{Hom}(M(a', b'), \tau L(a, n-1))$ where

$$f_i = \begin{cases} 
1 & \text{if } a' < i \leq a + 1 \text{ or } i = n^\pm, \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } a + 1 < i \leq b', \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } b' < i \leq n-1, \\
0 & \text{otherwise}.
\end{cases}$$
(case \( b' \leq a \)) Similar to the case \( a' \leq a < b < b' \), we get
\[
\text{Hom}(M(a', b'), \tau L(a, b)) = 0 = \text{Hom}(L(a, b), \tau M(a', b')).
\]
So we have proved (3).

(4),(7): The proofs are similar to (3).

\( \square \)

Lemma 3.10. \ Let \( T \in \text{Tilt}(Q) \).

(1) \( L(0, n - 1) \mid T \) implies \( L^\pm(0, n) \mid T \).

(2) If \( L^+(0, n) \mid T \) (resp. \( L^-(0, n) \mid T \)) and all indecomposable direct summands of \( T \) are insincere, then
\[
L^-(0, n) \mid T \ (\text{resp.} \ L^+(0, n) \mid T).
\]

Proof. (1) : Suppose \( L(0, n - 1) \mid T \). Then
\[
\text{Ext}(T, L(0, n - 1)) = 0 = \text{Ext}(L(0, n - 1), T)
\]
and there exists an injection
\[
\tau L^\pm(0, n) \rightarrow \tau L(0, n - 1).
\]
So we get
\[
\text{Ext}(L^\pm(0, n), T) \simeq \text{Hom}(T, \tau L^\pm(0, n)) = 0.
\]
Since \( L^\pm(0, n) \) is injective, we also get
\[
\text{Ext}(T, L^\pm(0, n)) = 0.
\]
Therefore, \( L^\pm(0, n) \mid T \).

(2): Suppose \( L^+(0, n) \mid T \) and that all indecomposable direct summands of \( T \) are insincere. Now \((\text{dim} T)_n^- \neq 0\), so there exists some indecomposable direct summand \( N \) s.t.
\[
(\text{dim} N)_n^- \neq 0.
\]
If \( N = M(a, b) \) then \( \text{Ext}(M(a, b), L^+(0, n)) = 0 = \text{Ext}(L^+(0, n), M(a, b)) \) so \( a = 0 \) and \( N \) is sincere. This is a contradiction. So \( N = L^-(a, n) \) and \( a = 0 \) by \( L^+(0, n) \mid T \). \( \square \)

Lemma 3.11. \ For all \( T \in \text{Tilt}(Q) \) there exists some indecomposable direct summand \( N \) of \( T \) s.t.
\[
(\text{dim} N)_i \geq 1, \text{ for all } i \leq n - 1.
\]
Thus, \( N = L(0, n - 1), L^\pm(0, n) \) or \( M(0, b) \), for some \( b \).

Proof. For an indecomposable direct summand \( N \) of \( T \) s.t. \((\text{dim} N)_1 = 1\), define
\[
a(N) \overset{\text{def}}{=} \sup\{i \mid 1 \leq i \leq n - 1, (\text{dim} N)_i \geq 1\}.
\]
Suppose that \( \sup a(N) = a < n - 1 \), then \( L(0, a) \mid T \). So indecomposable direct summands of \( T \) are of the following form
\[
L(a', b') \text{ for } b' \leq a \text{ or } a + 1 \leq a', \\
L^+(a', n) \text{ for } a + 1 \leq a', \\
M(a', b') \text{ for } a + 1 \leq a'.
\]
So \((\dim T)_{a+1} = 0\). This is a contradiction. \(\square\)

**Lemma 3.12.** We have

\[ \#\{ i \mid 1 \leq i \leq n - 1, \ (\dim T)_i = 1 \} \leq 1. \]

In particular, \(\delta(T) \geq n - 2\).

**Proof.** Let \(i \neq n^\pm\) s.t. \((\dim T)_i = 1\). Then we claim that

\[ L(0, i - 1) \mid T. \]

By lemma \(3.11\) there exists a unique indecomposable direct summand \(N\) of \(T\) s.t.

\[ (\dim N)_j \geq 1 \text{ for all } j \leq n - 1. \]

So, by lemma \(3.10\) \(N = M(0, j)\) for some \(i \leq j \leq n - 1\) and any indecomposable direct summand of \(T\) not isomorphic to \(N\) is one of the following,

\[
\begin{align*}
L(a, b) & \quad \text{for } b \leq i - 1 \text{ or } i \leq a, \\
L^\pm(a, n) & \quad \text{for } i \leq a, \\
M(a, b) & \quad \text{for } i \leq a.
\end{align*}
\]

It implies

\[ \Ext(T, L(0, i - 1)) = 0 = \Ext(L(0, i - 1), T), \]

so that

\[ L(0, i - 1) \mid T. \]

\(\square\)

**Corollary 3.13.** Let \(T \in Tilt(Q)\) then \(\delta(T) \geq n - 1\), and \(\delta(T) = n - 1\) if and only if \(L^\pm(0, n) \mid T\) and other indecomposable direct summands of \(T\) have the form \(L(a, b)(0 \leq a < b \leq n - 1)\). In particular,

\[ \#\{ T \in Tilt(kQ) \mid \delta(T) = n - 1 \} = \frac{1}{n} \left( \begin{array}{c} 2(n-1) \\ n-1 \end{array} \right) = \frac{1}{n-1} \left( \begin{array}{c} 2(n-1) \\ n-2 \end{array} \right). \]

**Proof.** Suppose that all indecomposable direct summands of \(T\) are insincere. Then, by lemma \(3.10\) and \(3.11\) \(L^\pm(0, n)\) and \(L^-(0, n)\) are both direct summands of \(T\). So (dimT)\(_i\) = 1 if and only if \(i = n^\pm\). So we have \(\delta(T) \geq n - 1\). If the equality holds then indecomposable direct summands of \(T\) not isomorphic to \(L^\pm(0, n)\) are of the form \(L(a, b)\).

Next we suppose there is a sincere indecomposable direct summand \(N\) of \(T\). If \(\delta(T) = n - 2\) then, by lemma \(3.12\) there is a unique \(i \leq n - 1\) s.t.

\[ (\dim T)_i = (\dim T)_{n^\pm} = 1. \]

So all indecomposable direct summands of \(T\) not isomorphic to \(N\) are of the form \(L(a, b)\) (\(b < i\) or \(i \leq a\)). As their direct sum may be viewed as a rigid module in type \(A_{i-1} \times A_{n-i-1}\), we get

\[ \#\{ L(a, b) \mid L(a, b) \mid T \} \leq (i - 1) + (n - 1 - i) = n - 2, \]

which is a contradiction. Next we consider the case \(\delta(T) = n - 1\).
(a) \((\dim T)_i = (\dim T)_{n+1} = 1\), for a unique \(i(\leq n - 1)\). Then indecomposable direct summands of \(T\) not isomorphic to \(N\) are of the following form:
\[
L(a, b) \quad \text{for } b < i \text{ or } i \leq a,
\]
\[
L^{-}(a, n) \quad \text{for } i \leq a.
\]
We get by the same argument that
\[
\#\{L \in \text{ind} kQ \mid L \mid T, L \neq N\} \leq (i - 1) + (n - i) = n - 1,
\]
which is a contradiction.

(b) \((\dim T)_i = (\dim T)_{n-1} = 1\), for a unique \(i(\leq n - 1)\). Then, similar to (a), we reach a contradiction.

(c) \((\dim T)_{n \pm} = 1\). Then indecomposable direct summands of \(T\) not isomorphic to \(N\) are of the form \(L_{a, b}\). Thus
\[
\#\{L(a, b) \mid L(a, b) \mid T\} \leq n - 1.
\]
It is a contradiction. So we get \(\delta(T) \geq n\) and \(\delta(T) = n - 1\) does not occur in this case.

Thus we have proved that if \(\delta(T) = n - 1\) then \(L_{0, n} \mid T\) and the other indecomposable direct summands of \(T\) has the form \(L(a, b)\). The converse implication is clear. \(\square\)

Now we define subsets of \(\text{Tilt}(Q)\) by
\[
T_{0} := \left\{ T \in \text{Tilt}(Q) \mid \delta(T) = n + 1\right\},
\]
\[
T_{1} := \left\{ T \in \text{Tilt}(Q) \mid \delta(T) = n\right\},
\]
\[
T_{2} := \left\{ T \in \text{Tilt}(Q) \mid \delta(T) = n - 1\right\}.
\]

**Lemma 3.14.** Fix \(1 \leq i \leq n - 1\), then
\[
\{T \in T_{1} \mid (\dim T)_{i} = 1\} \xleftarrow{1:1} T_{\text{tilt}}(\circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ^{i-1}) \times \{T \in \text{Tilt}(Q_{n-i+1}) \mid (\dim T)_{1} = 1, \delta(T) = n - i + 1\}.
\]

**Proof.** Let \(T \in T_{1}\) s.t. \((\dim T)_{1} = 1\), for a unique \(i(\leq n - 1)\). By lemma 3.10 and 3.11 there exists a unique \(j = j(T)(\geq i)\) s.t. \(M(0, j) \mid T\). Now let
\[
X(T) = \left\{ L(a, b) \mid L(a, b) \mid T, b < i\right\}
\]
and
\[
Y(T) = \left\{ N \in \text{ind} kQ \mid N \mid T\right\} \setminus \left\{ X(T) \cup \{M(0, j)\}\right\}.
\]
We define the maps
\[
\varphi_{T} : X(T) \longrightarrow \text{ind} k(\circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ^{i-1})
\]
and
\[
\psi_{T} : Y(T) \longrightarrow \text{ind} kQ_{n-i+1},
\]
by
\[
(\varphi_{T}(N))_{a} = (N)_{a} \quad (1 \leq a < i),
\]
\[
(\psi_{T}(N))_{a} = (N)_{a+i-1} \quad \text{(let } (n - i + 1)^{\pm} + i - 1 = n^{\pm}).
\]
Then

\[ T \mapsto \left( \bigoplus_{x \in X(T)} \varphi_T(x), \bigoplus_{y \in Y(T)} \psi_T(y) \bigoplus M(0, j(T) - i + 1) \right) \]

induces a bijection between

\[ \{ T \in \mathcal{T}_1 \mid (\text{dim}T)_i = 1 \} \]

and

\[ \text{Tilt}(\circ \to \circ \to \cdots \to \circ^{i-1}) \times \{ T \in \text{Tilt}(Q) \mid (\text{dim}T)_1 = 1, \delta(T) = n - i + 1 \}. \]

\[ \square \]

Let us define the following subsets of \( \mathcal{T}_1 \):

\[ \mathcal{A}_\pm := \left\{ T \in \mathcal{T}_1 \mid \text{all indecomposable direct summands of } T \text{ are not sincere} \right\}, \]

\[ \mathcal{B}_\pm := \{ T \in \mathcal{T}_1 \mid (\text{dim}T)_{n^\pm} = 1, \text{there exists some } j \text{ s.t. } M(0, j) \mid T \}, \]

\[ \mathcal{B}_\pm(j) := \{ T \in \mathcal{B}_\pm \mid M(0, j) \mid T \}, \]

\[ \mathcal{C} := \{ T \in \mathcal{T}_1 \mid (\text{dim}T)_1 = 1 \}, \]

\[ \mathcal{C}(j) := \{ T \in \mathcal{C} \mid M(0, j) \mid T \}. \]

**Theorem 3.15.** (1) \( \mathcal{A}_\pm = \emptyset \).

(2) \( \mathcal{B}_\pm(j) \xrightarrow{1:1} \{ T' \in \text{Tilt}(\circ \to \cdots \to \circ^n) \mid \min\{ j' \mid L(j', n - 1) \mid T' \} = j \} \).

In particular,

\[ \mathcal{B}_\pm \xleftarrow{1:1} \text{Tilt}(\circ \to \cdots \to \circ^n) \setminus \{ T' \in \text{Tilt}(\circ \to \cdots \to \circ^n) \mid L(0, n - 1) \mid T' \}, \]

and we have

\[ \#\mathcal{B}_\pm = \frac{1}{n + 1} \left( \binom{2n}{n} - \frac{1}{n} \binom{2(n - 1)}{n - 1} \right). \]

(3) \( \mathcal{C}(j) \xrightarrow{1:1} \{ T' \in \text{Tilt}(Q_{n-1}) \mid j'(T') = j - 1 \} \)

where

\[ j'(T') = \sup\{ b \mid L^+(b, n - 1) \text{ or } L^-(b, n - 1) \text{ or } M(a, b) \mid T' \text{ for some } a \}. \]

In particular,

\[ \mathcal{C} \xleftarrow{1:1} \text{Tilt}(Q_{n-1}), \]

and we have

\[ \#\mathcal{C} = \frac{3n - 4}{2n} \left( \frac{2(n - 1)}{n - 1} \right). \]

**Proof.** (1) Suppose that there exists some \( T \in \mathcal{A}_+ \). Then, by lemma 3.11, we have \( L^\pm(0, n) \mid T \). Now there exists some indecomposable direct summand \( N \) of \( T \) not isomorphic to \( L^-(0, n) \) s.t. \( (\text{dim}N)_{n^-} = 1 \).

If \( N = M(a, b) \) or \( L^-(a, n) \) then \( a = 0 \). This is a contradiction because \( L^\pm(0, n) \mid T \). So \( \mathcal{A}_+ = \emptyset \) and similarly we have \( \mathcal{A}_- = \emptyset \).
(2) : Define the maps
\[ \varphi : \{ L(a, b) \mid 0 \leq a < b \leq n - 1 \} \cup \{ L^-(a, n) \mid 0 \leq a \leq n - 1 \} \rightarrow \text{ind}\ k(\circ \rightarrow \cdots \rightarrow \circ) \]
and
\[ \psi : \text{ind}\ k(\circ \rightarrow \cdots \rightarrow \circ) \rightarrow \{ L(a, b) \mid 0 \leq a < b \leq n - 1 \} \cup \{ L^-(a, n) \mid 0 \leq a \leq n - 1 \} \]
by
\[ (\varphi(L))_a = \begin{cases} L_a & \text{if } 0 \leq a \leq n - 1, \\ L_n^- & \text{if } a = n, \end{cases} \]
\[ (\psi(L'))_a = \begin{cases} L'_a & \text{if } 0 \leq a \leq n - 1, \\ L'_n & \text{if } a = n^-, \\ 0 & \text{if } a = n^+. \end{cases} \]

Then \( \varphi \circ \psi = 1 = \psi \circ \varphi \). For \( T \in B_+(j) \) and \( T' \in \text{Tilt}(\circ \rightarrow \cdots \rightarrow \circ) \), define
\[ Z(T) := \{ N \in \text{ind}\ kQ \mid N \mid T, N \nless M(0, j) \} \]
and
\[ Y(T') := \{ N \in \text{ind}\ k(\circ \rightarrow \cdots \rightarrow \circ) \mid N \mid T' \}. \]

Then it is easy to see that the maps induce a bijection
\[ B_+(j) \leftrightarrow \{ T' \in \text{Tilt}(\circ \rightarrow \cdots \rightarrow \circ) \mid \text{min}\{ j' \mid L(j', n - 1) \mid T' \} = j \} \]
by
\[ T \mapsto \bigoplus_{L \in Z(T)} \varphi(L). \]

The inverse map is
\[ T' \mapsto \left( \bigoplus_{L' \in Y(T')} \psi(L') \right) \oplus M(0, j). \]

In fact, if \( T \in B_+(j) \) then all indecomposable direct summands of \( T \) not isomorphic to \( M(0, j) \) are either
\[ L(a, b) \ (a \geq j \text{ or } b < j) \text{ or } L^-(a, n) \ (a \leq j), \]
which implies \( L(j, n - 1), L^-(j, n) \mid T \). It follows
\[ \text{min}\{ j' \mid L(j', n - 1) \mid \bigoplus_{L \in Z(T)} \varphi(L) \} = j. \]

Conversely, if
\[ T' \in \{ T' \in \text{Tilt}(\circ \rightarrow \cdots \rightarrow \circ) \mid \text{min}\{ j' \mid L(j', n - 1) \mid T' \} = j \} \]
then \( \left( \bigoplus_{L' \in Y(T')} \psi(L') \right) \oplus M(0, j) \in B_+(j) \).

(3) : Define the maps
\[ \varphi : \{ N \in \text{ind}\ kQ_n \mid (\dim N)_1 = 0 \} \rightarrow \text{ind}\ kQ_{n-1} \]
and 
\[ \psi : \text{ind } kQ_{n-1} \longrightarrow \{ N \in \text{ind } kQ_n \mid (\dim N)_1 = 0 \} \]
by the obvious way. Then \( \varphi \circ \psi = 1 = \psi \circ \varphi \). For \( T \in \mathcal{C}(j) \) and \( T' \in Tilt(Q_{n-1}) \), define
\[ Z(T) := \{ N \in \text{ind } kQ_n \mid N \mid T, N \not\cong M(0, j) \} \]
and
\[ Y(T') := \{ N \in \text{ind } kQ_{n-1} \mid N \mid T' \}. \]

Then they induce a bijection
\[ \mathcal{C}(j) \overset{1:1}{\longleftrightarrow} \{ T' \in Tilt(Q_{n-1}) \mid j'(T') = j - 1 \} \]
by
\[ T \longmapsto \bigoplus_{N \in Z(T)} \psi(N). \]
The inverse map is
\[ T' \longmapsto \left( \bigoplus_{N' \in Y(T')} \psi(N') \right) \oplus M(0, j). \]

In fact, if \( T \in \mathcal{C}(j) \) then
\[ Z(T) \subset \{ L(a, b) \mid 1 \leq a < b < j \} \cup \{ L^\pm(b, n) \mid 1 \leq b \leq j \} \]
\[ \cup \{ M(a, b) \mid 1 \leq a < b \leq j \}. \]

It implies \( M(1, j) \mid T \) and \( j' \left( \bigoplus_{N \in Z(T)} \varphi(N) \right) = j - 1 \). Conversely, if \( j = j'(T') + 1 \) then
\[ Y(T') \subset \{ L_{n-1}(a, b) \mid b \leq j - 2 \text{ or } a \geq j - 1 \} \cup \{ L^\pm_n(b, n - 1) \mid 0 \leq b \leq j - 1 \} \]
\[ \cup \{ M_{n-1}(a, b) \mid 0 \leq a < b \leq j - 1 \}, \]
and
\[ \left( \dim \bigoplus_{N' \in Y(T')} \psi(N') \right) \begin{cases} \geq 1 \quad (a \geq 2) \\ = 0 \quad (a = 1). \end{cases} \]

It implies
\[ \left( \bigoplus_{N' \in Y(T')} \psi(N') \right) \oplus M(0, j) \in \mathcal{C}(j). \]

\[ \square \]

**Corollary 3.16.**
\[ \# \mathcal{T}_1 = 3^{2(n - 1)} \binom{2(n - 1)}{n - 2}. \]
Proof. First we claim that

\[
\sum_{i=1}^{n} \frac{1}{i(n+1-i)} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i} = \frac{1}{n+1} \binom{2n}{n}.
\]

This follows from the fact that

\[Tilt(\circ \to \cdots \to n \circ) = \bigsqcup \{T \in Tilt(\circ \to \cdots \to n \circ) \mid \min\{i' \mid L(i', n) \mid T, i' > 0\} = i\}.
\]

Thus, by lemma 3.14 and theorem 3.15, \(\#T_1\) is equal to

\[
2 \left( \frac{1}{n+1} \binom{2n}{n} - \frac{1}{n} \binom{2(n-1)}{n-1} \right) + \sum_{i=1}^{n-1} \frac{3(n-i) - 1}{2i(n-i+1)} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i}
\]

\[
= 2 \left\{ \left( \frac{1}{n+1} \binom{2n}{n} - \frac{1}{n} \binom{2(n-1)}{n-1} \right) - \sum_{i=1}^{n-1} \frac{1}{i(n-i+1)} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i} \right\}
\]

\[
+ \sum_{i=1}^{n-1} \frac{3}{2i} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i}
\]

\[
= \frac{3}{2} \sum_{i=1}^{n-1} \frac{1}{i} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i}.
\]

Now let

\[a_n = \sum_{i=1}^{n} \frac{1}{i} \binom{2(i-1)}{i-1} \binom{2(n+1-i)}{n+1-i},
\]

and

\[f(X) = \left( \sum_{i=1}^{n} \frac{1}{i} \binom{2(i-1)}{i-1} X^i \right)^2.
\]

Then the coefficient of \(X^{n+1}\) in \(f'(X)\) is equal to

\[2a_n - 2 \binom{2n}{n}.
\]
On the other hand, the coefficient of $X^{n+2}$ in $f(X)$ is equal to
\[
\sum_{i=1}^{n+1} \frac{1}{i(n-i+2)} \binom{2(i-1)}{i-1} \binom{2(n-i+1)}{n-i+1} - \frac{2}{n+1} \binom{2n}{n}
\]
\[= \frac{1}{n+2} \binom{2(n+1)}{n+1} - \frac{2}{n+1} \binom{2n}{n}.\]

So
\[2a_n = \binom{2(n+1)}{n+1} - \frac{2}{n+1} \binom{2n}{n} = 4 \binom{2n}{n-1}.\]

We conclude that
\[\#\mathcal{T}_1 = \frac{3}{2} a_{n-1} = 3 \binom{2(n-1)}{n-2}.\]

\[\square\]

**Corollary 3.17.** We have
\[\#\mathcal{T}_0 = \frac{3(n-1)}{n+1} \binom{2(n-1)}{n-2}.\]

**Proof.** In fact,
\[\#\mathcal{T}_0 = \frac{3n-1}{2(n+1)} \binom{2n}{n} - 3 \binom{2(n-1)}{n-2} - \frac{1}{n} \binom{2(n-1)}{n-1}
\]
\[= \frac{3(n-1)}{n+1} \binom{2(n-1)}{n-2}.\]

\[\square\]

**Theorem 3.18.**
\[\#\mathcal{K}(Q)_1 = (3n-1) \binom{2(n-1)}{n-2}.\]

**Proof.** In fact, $\#\mathcal{K}(Q)_1$ is equal to
\[\frac{1}{2} \left\{ \frac{n-1}{n-1} \binom{2(n-1)}{n-2} + 3n \binom{2(n-1)}{n-2} + 3(n-1) \binom{2(n-1)}{n-2} \right\}
\]
\[=(3n-1) \binom{2(n-1)}{n-2}.\]

\[\square\]
Example 3.19. (n=3)

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