ON THE DISJOINTNESS PROPERTY OF GROUPS AND A CONJECTURE OF FURSTENBERG

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Abstract. In his seminal 1967 paper [F-67] Furstenberg introduced the notion of disjointness of dynamical systems, both topological and measure preserving. In this paper he showed that for actions of the integers the Bernoulli system \( \Omega = \{0, 1\}^\mathbb{Z} \), is disjoint from every minimal system, and that the subring \( R_0 \), over the field \( \mathbb{Z}_2 = \{0, 1\} \), generated by the minimal functions in \( \Omega \), is a proper subset of \( \Omega \). He conjectured that a similar result holds in general and in [GW-83] we confirmed this by showing that the closed subalgebra \( A \) of \( l^\infty(\mathbb{Z}) \), generated by the minimal functions, is a proper subalgebra of \( l^\infty(\mathbb{Z}) \). In this work we generalize these results to a large class of groups. We call a countable group \( G \) a DJ group if for every metrizable minimal action of \( G \) there exists an essentially free minimal action disjoint from it. We show that amenable groups are DJ and that the DJ property is preserved under direct products. We define a simple dynamical condition DDJ on minimal systems, which is a strengthening of the Gottchalk-Hedlund property, and we say that a group \( G \) is DDJ if every minimal \( G \)-system has this property. The DJ property implies DDJ and by means of an intricate construction we show that every finitely generated DDJ group is also DJ. Residually finite, maximally almost periodic and \( C^* \)-simple groups are all DDJ. Finally we show that Furstenberg’s conjecture holds for every DDJ group.

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Introduction

Let $G$ be an arbitrary countable infinite (discrete) group. We denote by $e$ the identity element of $G$. A $G$ dynamical system is a pair $(X, G)$ where $X$ is a compact metric space and the group $G$ acts on $X$ via a homomorphism $\rho$ from $G$ into $\text{Homeo}(X)$, the Polish group of self homeomorphisms of $X$, equipped with the topology of uniform convergence. We usually suppress the letter $\rho$ in our notation and write $gx$ for the application of $\rho(g)$ to $x$. Also we often write $X$ instead of $(X, G)$. The set $Gx = \{gx : g \in G\}$ is the orbit of $x$.

We say that the system is topologically transitive if for any two nonempty open sets $U, V \subset X$ there is some $g \in G$ with $gU \cap V \neq \emptyset$. A point $x \in X$ is a transitive point if $Gx$ is dense in $X$. By Baire’s category theorem a metrizable system is topologically transitive iff the set $X_{tr}$ of transitive points is a dense $G_\delta$ subset of $X$. A point $x \in X$ is recurrent if for every neighborhood $U$ of $x$ there is some $e \neq g \in G$ with $gx \in V$. Note that a transitive point which is not an isolated point is recurrent. The system is called minimal if $Gx = X$ for every $x \in X$.

A nonempty closed and $G$-invariant subset $Y$ of a system $X$ is called a subsystem. Thus a system is minimal iff it admits no proper subsystems. A point $x$ in a system $(X, G)$ is a uniformly recurrent (or a minimal) point if the subsystem $\overline{Gx}$ is minimal. It is well known that $x \in X$ is a minimal point iff $x$ satisfies the Gottschalk-Hedlund property: for every neighborhood $V$ of $x$, the set $S = \{g \in G : gx \in V\}$ is syndetic; i.e. there is a finite set $F$ such that $G = FS$ (see [GH]).

A continuous surjective mapping $\pi : X \to Y$, where $(X, G)$ and $(Y, G)$ are dynamical systems, is called a homomorphism of dynamical system (or a factor map), when it intertwines the $G$ actions on $X$ and $Y$, i.e. it satisfies the conditions $\pi(gx) = g\pi(x)$ for all $g \in G$ and $x \in X$.

We say that the action $(X, G)$ is effective when the homomorphism $\rho$ is injective. It is free if the condition $gx = x$, for some $x \in X$, implies that $g = e$; and that the action is essentially free if there is a dense $G_\delta$ set of points $x \in X$ where the map $g \mapsto gx$, from $G$ into $X$, is injective (we say that such a point is free). In a minimal system $(X, G)$, the existence of one free point implies that the set of free points forms a dense $G_\delta$ subset of $X$.

The concept of disjointness of two dynamical systems, both topological and measure preserving, was introduced by Furstenberg in his seminal paper 1967 paper “Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation” [F-67]. Two topological dynamical systems $(X, G)$ and $(Y, G)$ are disjoint if the only subsystem $W \subset
$X \times Y$ with $\pi_X(W) = X$ and $\pi_Y(W) = Y$ is $W = X \times Y$. We then write $X \perp Y$. Here $\pi_X$ and $\pi_Y$ are the natural projection maps. It is not hard to see that $X \perp Y$ implies that at least one of the two systems is minimal. Also, when $X$ and $Y$ are both minimal then $X \perp Y$ iff $X \times Y$ is minimal.

In [F-67] Furstenberg showed, among many other beautiful results, that for the integers the Bernoulli shift on $\{0, 1\}^\mathbb{Z}$ is disjoint from every minimal system (and then applied this approach to prove his famous Diophantine theorem: If $\Sigma$ is a non-lacunary semigroup of integers and $\alpha$ is an irrational, then $\Sigma\alpha$ is dense in the circle $\mathbb{R}/\mathbb{Z}$). Considering $\{0, 1\}$ as a field $\{0, 1\}^\mathbb{Z}$ forms a ring under pointwise addition and multiplication. Furstenberg used this disjointness of of the Bernoulli shift from all minimal systems to show that the subring of $\{0, 1\}^\mathbb{Z}$ generated by the minimal points in $\{0, 1\}^\mathbb{Z}$ is not the whole ring. He conjectured that a similar result holds for the algebra of bounded real valued functions, and in [GW-83] we confirmed this by showing that the closed subalgebra $\mathfrak{A}$ of $l^\infty(\mathbb{Z})$, generated by the minimal functions, is indeed a proper subalgebra of $l^\infty(\mathbb{Z})$.

For a general topological group $G$ the analogous conjecture is that the $C^*$-algebra $RUC(G)$, of bounded right uniformly continuous complex valued functions on $G$, coincides with the sub-$C^*$-algebra $\mathfrak{A}$ generated by the minimal functions on $G$. When a topological group $G$ is precompact, then $RUC(G) = \mathfrak{A}$ and in [P-98, page 4163] Pestov formulated a general version of Furstenberg’s conjecture as follows.

**Conjecture:** Suppose $G$ is not precompact, then $\mathfrak{A} \subsetneq RUC(G)$.

In the present work we generalize the results mentioned above to a large class of groups which we call DJ groups. A countable group $G$ is a DJ group if for every metrizable minimal action of $G$ there exists an essentially free minimal action disjoint from it. We also define a simple dynamical condition DDJ on minimal systems, which is a strengthening of the Gottchall-Hedlund property, and we say that a group $G$ is DDJ if every minimal $G$-system has this property. The DJ property implies DDJ, and by means of an intricate construction we show in Section 6 that every finitely generated DDJ group is also DJ.

In Section 8 we generalize our results in [GW-83], by showing that Furstenberg’s conjecture holds for all DDJ groups.

In Section 9 we show that amenable groups are DJ and that residually finite groups are DDJ. In Section 10 we show that maximally almost periodic groups (a class which includes the residually finite groups) and $C^*$-simple groups are DDJ. Finally in Section 11, we show that the DJ property is preserved by direct products. We don’t know if there is any group which is not DJ (or not DDJ).

The structure of the paper is outlined in the table of contents.

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1. The basic definitions

Let $\Omega = \Omega(G) = \{0,1\}^G$ be the binary Bernoulli $G$-system, where the action is given by

$$\sigma_g(\omega)(h) = \omega(hg), \quad g, h \in G, \quad \omega \in \Omega.$$ 

A subsystem of $\Omega$ is called a subshift. Given a subshift $X \subset \{0,1\}^G$, a finite set $K \subset G$ and $x \in X$ we write $x_K = x \mid K \in \{0,1\}^K$ and let

$$X_K = \{x_K : x \in X\} \subset \{0,1\}^K.$$ 

Given an element $\alpha \in \{0,1\}^K$, we define the cylinder set

$$[\alpha] = \{x \mid K : x \in X\}.$$ 

Cylinder sets form a basis for the topology of $X$.

1.1. Definition. Let $X \subset \Omega$ be a minimal subshift. For a finite set $F \subset G$ and $x \in X$ let

$$x_F = x \mid F \in \{0,1\}^F.$$ 

Let $X_F = \{x_F : x \in X\} \subset \{0,1\}^F$. If $D \subset G$ is finite we say that a subset $L \subset G$ is $D$-separated if for any $g_1, g_2$ distinct elements of $L$ we have $Dg_1 \cap Dg_2 = \emptyset$.

- A minimal $X \subset \Omega$ satisfies the combinatorial disjointness property if:
  - For every finite subset $D \subset G$ and every cylinder set $[\alpha] \in \{0,1\}^K$, there exists a finite set $F \subset G$ such that
    - (i) $F$ is $D$-separated, and
    - (ii) for every $x \in X$ there is an $f \in F$ with $x \mid Kf = \alpha$.

- A minimal system $(X,G)$ has the dynamical disjointness property (DDJ) if it satisfies the following condition:
  - For every nonempty open subset $V \subset X$ and every finite subset $D \subset G$, there is a subset (either finite or infinite) $L \subset G$ such that:
    - (i) $L$ is $D$-separated,
    - (ii) $L^{-1}V = X$.

We say that a group $G$ has the DDJ property, or that $G$ is a DDJ group, if every minimal $G$-system satisfies the DDJ property. Clearly the DDJ property is inherited by factors.

- A group $G$ has the Bernoulli disjointness property (BDJ) if every minimal $G$-system $(X,G)$ satisfies $X \perp \Omega$.

- A minimal $G$-system $(X,G)$ has the disjointness property (DJ) if there is an essentially free subshift $(Y,G)$ which is disjoint from it. We say that a group $G$ has the
disjointness property (DJ), or that $G$ is a DJ group, if every metrizable minimal system $(X, G)$ has the DJ property.

1.2. Remark. The requirement that $X$ be metrizable in the Definition of the DJ property in Definition 1.1 item (5), is crucial. Every topological group $G$ admits a unique universal minimal system $(M(G), G)$, which has any other minimal $G$-system as a factor. In particular, the system $(M(G), G)$ is never DJ. Usually this universal system is non-metrizable (e.g. this is the case whenever $G$ is a locally compact non-compact group, hence in particular, when $G$ is infinite, countable, and discrete). However, for many “large” topological groups $M(G)$ is metrizable, or even a trivial one point system. Groups $G$ for which $M(G)$ is trivial are said to have the fixed point on compacta property or to be extremely amenable. Such are, for example, the Polish groups $\mathbb{U}(H)$, of unitary operators on a separable infinite dimensional Hilbert space, or $\text{Aut} (\mu)$, of measure preserving automorphisms of the Lebesgue space. On the other hand, for Polish groups like $\mathbb{S}(\mathbb{N})$, of permutations of the natural numbers, or $\text{Homeo} (\mathbb{C})$, of homeomorphisms of the Cantor set, the space $M(G)$ is non-trivial but metrizable and moreover, it admits an explicit description. Obviously for all of these groups the DJ property fails. For more details on $M(G)$ see e.g. [E-60], [G-98], [GW-02], [GW-03], [KPT-05] and [P-06].

2. DDJ is preserved by extensions

2.1. Theorem. Let $\pi : X \rightarrow Y$ be an extensions between the minimal $G$ systems $X$ and $Y$ with $Y$ essentially free. Then if $Y$ is DDJ so is $X$.

Proof. Let $\pi : X \rightarrow Y$ be an open extensions between the minimal $G$ systems $X$ and $Y$. We assume that $Y$ has DDJ. Let $D \subset G$ be a finite set and $V \subset X$ a nonempty open subset.

By minimality $\text{int} \pi(V)$ is nonempty and we can replace $V$ by $\pi^{-1} (\text{int} \pi(V)) \cap V$. We then have (for the new $V$) that $\pi(V) = U \subset Y$ is nonempty and open.

As $Y$ is essentially free we can assume the $U$ is small enough so that the sets \{gU : g \in D^{-1}D\} are pairwise disjoint.

Pick $x_0 \in V$, let $y_0 = \pi(x_0)$ and let $X_{y_0} = \pi^{-1}(y_0)$. By minimality, for each $x \in X_{y_0}$ there is an element $g \in G$ and a positive $r$ such that $gB_r(x) \subset V$. Take a finite number of these balls $B_{r_i}(x_i)$, with elements $a_i \in G$, $i = 1, 2, \ldots, k$, so that $\bigcup_{i=1}^{k} B_{r_i}(x_i) \supset X_{y_0}$ and $\bigcup_{i=1}^{k} a_iB_{r_i}(x_i) \subset V$. Let $r_0$ be small enough so that $\pi^{-1}(B_{r_0}(y_0))$ is contained in $\bigcup_{i=1}^{k} a_iB_{r_i}(x_i)$.

Notice that the set $A = \{a_1, a_2, \ldots, a_k\}$ is $D$ separated. Indeed, if $d_i a_i = d_j a_j$ for some $i \neq j$ and $d_i, d_j \in D$, then $d_i a_i x_i = d_j a_j x_i$, hence $d_j^{-1}d_i (a_i x_i) = a_j x_i$, and therefore $d_j^{-1}d_i (a_i \pi(x_i)) = d_j^{-1}d_i (a_i y_0) = a_j \pi(x_i) = a_j y_0$. But both $a_i y_0$ and $a_j y_0$ are in $U$, and this contradicts our assumption on $U$.
Now let $E = DA$ and use the DDJ property of $Y$, for $E \subset G$ and $U \subset Y$, to find an $E$-separated set $S$ such that $\bigcup_{s \in S} s^{-1}B_{r_0}(y_0) = Y$. We now have that $\bigcup\{s^{-1}a_i^{-1}V : s \in S, 1 \leq i \leq k\}$ is all of $X$. As the set $L = AS$ is $D$-separated this completes the proof. □

2.2. Corollary. If the group $G$ admits a DDJ system which is essentially free then $G$ has DDJ.

Proof. Let $(Y, G)$ be an essentially free $G$-system which is DDJ. Given any minimal system $(X, G)$ let $M$ be a minimal subset of the product system $X \times Y$. Then, by Theorem 2.1, $M$ which admits $Y$ as a factor is DDJ, and, as obviously the DDJ property is inherited by factors, so does $X$. □

3. DJ implies DDJ

3.1. Lemma. DJ implies DDJ.

Proof. Let $(X, G)$ be a minimal system. By assumption there is a free minimal $Y \subset \Omega$ such that $X \perp Y$. Let $V \subset X$ and $D \subset G$ as in the DDJ property be given. Let $U \subset Y$ be a nonempty open subset such that the collection $\{dU : d \in D\}$ is pairwise disjoint (here we use the fact that $Y$ is essentially free). Let $y_0$ be any point in $U$ and let $L = \{g \in G : gy_0 \in U\}$. Then, by the minimality of $X \times Y$, we have

$$\bigcup\{g^{-1}(V \times U) : g \in L\} \supset X \times \{y_0\},$$

whence $L^{-1}V = \bigcup\{g^{-1}V : g \in L\} = X$. Moreover, if $d_1g_1 = d_2g_2$ for $g_1, g_2 \in L$ and $d_1, d_2 \in D$, then $d_1g_1y_0 = d_2g_2y_0 \in d_1U \cap d_2U$, which contradicts the property of $U$. □

3.2. Lemma. DDJ $\iff$ CDJ.

Proof. DDJ $\Rightarrow$ CDJ: Given a minimal system $X \subset \Omega$, a finite set $D \subset G$ and a cylinder set $V = \{x \in X : x \upharpoonright K = \alpha\}$, Let $L \subset G$ be a $D$-separated set with $L^{-1}V = X$, provided by the DDJ property. Let $F \subset L$ be a finite set with $F^{-1}V = X$. Clearly $F$ is $D$-separated and, given $x \in X$ there is $f \in F$ with $fx \in V$, i.e. with $x \upharpoonright Kf = \alpha$.

CDJ $\Rightarrow$ DDJ: We first note that given a minimal system $(X, G)$, a finite subset $D$ of $G$, $V$ a nonempty open subset of $X$, and $L \subset G$ a $D$-separated set then:

(i) if $U$ is an nonempty open subset contained in $V$ with $L^{-1}U = X$ then clearly also $L^{-1}V = X$, and

(ii) if $V$ is clopen and $\pi : X \to Y \subset \Omega$ is the corresponding “name map” (defined by $\pi(x)(g) = 1$ iff $gx \in V$), then $L^{-1}\{y \in Y : y(e) = 1\} = Y$ iff $L^{-1}V = X$.

Now, as every metric minimal system admits a minimal zero-dimensional extension, it suffices to prove the DDJ property for zero-dimensional systems, and therefore, by (i), we can assume that $V$ is clopen. In view of (ii) we only need to show that the DDJ property
is satisfied in the situation where $X \subset \Omega$ and $V$ is a cylinder set. However, as we have seen in the first part of the proof, in this particular case the two properties coincide. □

We also have the following observation.

3.3. Proposition. Suppose the group $G$ admits two essentially free minimal systems $X$ and $Y$ with $X \perp Y$. Then each of these systems has DDJ and consequently $G$ has the BDJ property.

Proof. By Lemma 3.1 both $X$ and $Y$ are DDJ and the latter claim of the proposition follows from Theorem 2.1. □

4. Furstenberg’s theorem for DDJ groups

In this section we obtain the following generalization (of a particular case) of the celebrated Furstenberg $\mathcal{F}$-flows disjointness theorem [F-67, Theorem II.2] which asserts that every minimal $\mathbb{Z}$-system $X$ is disjoint from $(\Omega(\mathbb{Z}), \mathbb{Z})$, or, in our terminology, has the BDJ property.

4.1. Proposition. DDJ implies BDJ.

Proof. Let $(X, G)$ satisfy DDJ. We prove that $X$ is disjoint from $\Omega = \{0, 1\}^G$ - the 2-shift over $G$. Let $W$ be an invariant subset in $X \times \Omega$ and $w = (x, y)$ a point in $W$ such that the orbit of $y$ is dense in $\Omega$. Such a point exists since $W$ projects onto $\Omega$.

Fix some finite set $F \subset G$ and a configuration $\alpha \in \{0, 1\}^F$. Denote by $[\alpha]$ the clopen set defined by $\alpha$. We will show that if

$$A = \{g : gy \in [\alpha]\}$$

then $Ax$ is dense in $X$. This is enough to show that $W = X \times \Omega$.

Fix some open set $V$ in $X$ and and let $B$ be a finite $F$-separated set such that $B^{-1}V = X$. Define a configuration $\beta$ in $\{0, 1\}^{FB}$ such that for each $Fb$, $b \in B$, $\beta$ restricted to $Fb$ equals $\alpha$. This is possible since $B$ is $F$-separated. Now, since the orbit of $y$ is dense, there is some $g \in G$ such that $gy \in [\beta]$. For this $g$ there is some $b \in B$ such that $bgx \in V$ and thus

$$(bgx, bgy) \in V \times [\alpha]$$

by the definition of $\beta$. □
5. A CONSTRUCTION OF MINIMAL SUBSHIFTS FOR A FINITELY GENERATED GROUP

In this section we describe a construction, for a finitely generated infinite group $G$, of a minimal subshift $Y \subset \Omega = \{0,1\}^G$. This is a very general construction with great flexibility. We will demonstrate its usefulness in the following section where we put it to use in showing that for finitely generated groups the dynamical disjointness property implies the disjointness property.

We are given an infinite finitely generated group $G = \langle S \rangle$, where $S$ is a finite symmetric generating set containing the identity $e$. For a positive integer $r$ we let $B_r = S^r$ be the ball of radius $r$ and we define the word metric on $G$ by

$$d(g,h) = \min\{r : g \in B_r h\} = \min\{r : h \in B_r g\}.$$ 

We will say that a set $C \subset G$ is $r$-separated if the sets $\{B_r c : c \in C\}$ are disjoint. We say that $C \subset G$ is $r$-covering (or syndetic) if $B_r C = G$.

In the construction we will repeatedly use the following simple claim.

5.1. Lemma. A maximal $r$-separated set $L$ is $2r$-covering.

Proof. Let $g \in G$ be given. If $g \notin B_{2r} L$ then $B_r g \cap B_r L = \emptyset$, whence $L' = L \cup \{g\}$ is an $r$-separated set which properly contains $L$; a contradiction. Thus $B_{2r} L = G$. 

We are going to construct inductively:

1. An increasing sequence of positive integers $r_i$, $i = 0,1,2,\ldots$ and corresponding balls $D_i = B_{r_i}$, so that $e \in D_0 \subset D_1 \subset D_2 \subset \cdots$ with $\bigcup D_i = G$.
2. A sequence of configurations $\beta_i \in \{0,1\}^{D_i}$ such that $\beta_{i+1} \upharpoonright D_i = \beta_i$ for $i = 0,1,2,\ldots$.
3. A sequence of finite $2r_{i-1}$-separated sets $F_i$ such that $D_{i-1} F_i \subset D_i$, $i = 1,2,\ldots$. 

A triangular array, $L_i^{(j)}$, $1 \leq i \leq j$, of syndetic subsets of $G$:

$$
\begin{array}{ccccccc}
L_0^{(0)} & L_1^{(0)} & \cdots & L_0^{(n-1)} & L_0^{(n)} & \cdots \\
L_1^{(0)} & L_1^{(1)} & \cdots & L_1^{(n-1)} & L_1^{(n)} & \cdots \\
L_2^{(0)} & \cdots & L_2^{(n-1)} & L_2^{(n)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
L_n^{(n)} & \cdots & L_n^{(n)} & \cdots \\
\end{array}
$$

with $F_{n+1} \subset L_n^{(n)}$.

A sequence of points $\{y_0, y_1, y_2, \ldots \} \subset \{0, 1\}^G$, satisfying $y_i \restriction D_il_i = \beta_i$ for every $l_i \in L_i^{(n)}$, and converging to a point $y_\infty \in \{0, 1\}^G$.

We will then define $Y = \overline{Gy_\infty}$ as the required minimal subshift.

**Step 0:**
We start our construction with an arbitrary $r_0$ and $\beta_0 \in \{0, 1\}^{r_0}$ and set $D_0 = B_{r_0}$. Let $s_0 = 2r_0$ and assume that we are given a finite $s_0$-separated set $F_1$ containing $e$. Enlarge $F_1$ to a maximal $s_0$-separated set $L_0^{(0)}$. Next define $y_0 \in \{0, 1\}^G$ by

$$
y_0 \restriction D_0l_0 = \beta_0, \quad \forall l_0 \in L_0^{(0)}$$

$$
y_0(g) = 0, \quad \forall g \notin D_0L_0^{(0)}.
$$

**Step 1:**
Choose $r_1 = 10s_0 + \min\{d(e, f_1) : f_1 \in F_1\}$ and set $D_1 = B_{r_1}$. We proceed to “clean the boundary” of $D_1$; namely we remove from $L_0^{(0)}$ any $l_0$ such that $B_{2s_0}l_0$ intersects both $D_1$ and its complement $D_1^c = G \setminus D_1$. This defines $\tilde{L}_0^{(0)}$ and we put $\tilde{y}_1$ equal to $y_0$ except on $D_0l_0$ where $l_0$ was removed; there $\tilde{y}_1 \restriction D_0l_0 = 0$. Set $\beta_1 = \tilde{y}_1 \restriction D_1$.

Let $s_1 = 2r_1$ and assume that we are given an $s_1$-separated set $F_2$ containing $e$. Enlarge $F_2$ to a maximal $s_1$-separated set $L_1^{(1)}$. We define $y_1$ so that on each $D_1l_1$, $l_1 \in L_1^{(1)}$, $y_1 \restriction D_1l_1 = \beta_1$. 
(i) On $D_1l_1$, $e \neq l_1 \in L_1^{(1)}$, $L_0^{(1)}$ is defined by translating $\tilde{L}_0^{(0)}$ from $D_1$.

(ii) On $G \setminus D_1L_1^{(1)}$, $L_0^{(1)}$ is $\tilde{L}_0^{(0)}$, but all $l_0 \in \tilde{L}_0^{(0)}$ such that $B_{2s_0}l_0 \cap D_1L_1^{(1)} \neq \emptyset$ are removed, and $y_1$ is defined accordingly.

We now check the covering property of $L_0^{(1)}$.

(a) If $d(g, D_1L_1^{(1)}) > 4s_0$ and $g \in B_{2s_0}l_0$ for some $l_0 \in L_0^{(0)}$, then $d(l_0, D_1L_1^{(1)}) > 4s_0 - 2s_0 = 2s_0$, so that $B_{2s_0}l_0 \subset G \setminus D_1L_1^{(1)}$, $l_0 \in L_0^{(1)}$, and $d(g, L_0^{(1)}) < 2s_0$.

(b) If $g \in B_{r_1 - 4s_0}$ and $g \in B_{2s_0}l_0$ for some $l_0 \in L_0^{(0)}$, then $l_0 \in B_{r_1 - 2s_0}$ and $B_{2s_0}l_0 \subset B_{r_1} = D_1$. Therefore $l_0$ was kept in $L_0^{(1)}$. Again we have $d(g, L_0^{(1)}) < 2s_0$.

(c) If $g \in D_1 \setminus B_{r_1 - 4s_0}$, there is some $h \in B_{r_1 - 4s_0}$ with $d(g, h) \leq 4s_0$. By (b) this $h$ is in $B_{2s_0}l_0$ for some $l_0 \in L_0^{(1)}$, so that $g \in B_{6s_0}l_0$ and we only have $d(g, L_0^{(1)}) \leq 6s_0$.

(d) Finally, for $g$ with $0 < d(g, D_1L_1^{(1)}) = d(g, l_1) \leq 4s_0$ for some $l_1 \in L_1^{(1)}$ and there is some $h \in D_1l_1$ with $d(g, h) \leq 4s_0$. As in (c) since $h \in B_{2s_0}l_0$ with $l_0 \in L_0^{(1)}$ and $g \in B_{6s_0}l_0$.

We conclude that $\hat{L}_0^{(1)}$ is $6s_0$-syndetic.

By stage $n$ we will have defined:

At stage $n$, we have defined

1. An increasing sequence of subsets $e \in D_0 \subset D_1 \subset \cdots \subset D_n$ of $G$, where $D_i = B_{r_i}$ for an increasing sequence of positive integers $r_0 < r_1 < \cdots < r_n$. We let $s_i = 2r_i$.

2. A sequence $\beta_0, \beta_1, \ldots, \beta_n$, with $\beta_i \in \{0, 1\}^{D_i}$ and $\beta_i \upharpoonright D_{i-1} = \beta_{i-1}$, $i = 1, 2, \ldots, n$.

3. A sequence of finite $s_i-1$-separated sets $F_i$ such that $D_{i-1}F_i \subset D_i$, $i = 1, 2, \ldots,$.

4. A triangular array, $L_i^{(j)}$, $1 \leq i \leq j$, of syndetic subsets of $G$: with $F_{n+1} \subset L_i^{(n)}$.

5. A sequence of points $\{y_1, y_2, \ldots, y_n\} \subset \{0, 1\}^G$ with $y_i \upharpoonright D_i = \beta_i$, such that $y_i \upharpoonright D_i l_i = \beta_i$ for every $l_i \in L_i^{(i)}$ $0 \leq i \leq n$.

6. We introduce the following notation: Put $\hat{L}_i^{(n)} = L_i^{(n)}$, and for $0 \leq i \leq n$, $\hat{L}_i^{(n)} = \{l \in L_i^{(n)} : D_{n-1}l \not\subset H_i^{(n)}\}$, where $H_i^{(n)} = \bigcup_{j=i+1}^n D_j L_j^{(n)}$.

With these notations we have:

$$\{g \in G : d(g, H_i^{(n)}) > 4s_i\} \subset B_{2s_i} \hat{L}_i^{(n)}.$$

7. In $D_n$, for $i < n$, in $B_{r_n-4s_i}$, defining analogously sets $\hat{L}_i^{(n)}$ and $H_i^{(n)}$ restricted to $D_n$, we have:

$$\{g \in G : d(g, H_i^{(n)}) > 4s_i\} \subset B_{2s_i} \hat{L}_i^{(n)}.$$

**Step** $n + 1$:

Let $r_{n+1} = \max\{d(f, e) : f \in F_{n+1}\} + 10s_n$ and set $D_{n+1} = B_{r_{n+1}}$. We will next describe how to construct the objects of the $n + 1$ stage so that the properties (1) - (7) above are preserved for the larger array. We will have to do this again by an induction on the columns.
of an auxiliary array:

\[ L_{n,n}^{(n)} \]

\[ L_{n-1,n}^{(n)} \quad L_{n-1,n-1}^{(n)} \]

\[ L_{n-2,n}^{(n)} \quad L_{n-2,n-1}^{(n)} \quad L_{n-2,n-2}^{(n)} \]

\[ \vdots \]

\[ L_{i,n}^{(n)} \quad L_{i,n-1}^{(n)} \quad L_{i,n-2}^{(n)} \quad \ldots \quad L_{i,i}^{(n)} \]

\[ \vdots \]

\[ L_{0,n}^{(n)} \quad L_{0,n-1}^{(n)} \quad L_{0,n-2}^{(n)} \quad \ldots \quad L_{0,0}^{(n)} \]

We will then put \( L_i^{(n+1)} = L_i^{(n)} \) in order to create the new column of the main array.

**Step** \( n+1, n \): We want to “clean” the boundary of the new \( D_{n+1} \). We first remove from \( L_n^{(n)} \) these \( l_n \) such that \( B_{4s_n} l_n \) intersects both \( D_{n+1} \) and its complement, namely the set

\[ R_n = \{ l_n \in L_n^n : B_{4s_n} l_n \cap D_{n+1} \neq \emptyset \neq B_{4s_n} l_n \cap D_{n+1}^c \} \]

This defines \( L_n^{(n)} \) in \( B_{r_{n+1} + 4s_n} \). For \( i < n \), \( L_i^{(n)} \) is defined as \( L_i^n \) except for \( B_{r_{n+1} + 4s_n} R_n \) where \( L_i^{(n)} = L_i^{(n-1)} \), \( 0 \leq i \leq n - 1 \).

**Step** \( n+1, n-1 \): Set

\[ R_{n-1} = \{ l_{n-1} \in L_{n-1,n}^{(n)} : B_{4s_n} l_{n-1} \cap D_{n+1} \neq \emptyset \neq B_{4s_n} l_{n-1} \cap D_{n+1}^c \} \]

we set \( L_{n-1,n-1}^{(n)} = L_{n-1,n}^{(n)} \setminus R_{n-1} \), for \( i \leq n - 2 \). Next \( L_{i,n}^{(n-1)} \) is defined as \( L_{i,n}^{(n)} \) except for \( B_{r_{n-1} + 4s_{n-2}} R_{n-1} \) where \( L_{i,n-1}^{(n)} = L_i^{(n-2)} \) for \( i \leq n - 2 \).

This procedure is continued column by column and finally, in \( B_{r_n + 4s_n} \), \( L_i^{(n+1)} \) is defined as \( L_i^{(n)} \), \( 0 \leq i \leq n \). Put \( s_{n+1} = 2r_{n+1} \).

There is now an \( F_{n+1} \) given which is \( s_{n+1} \)-separated and we enlarge it so that we get a maximal \( s_{n+1} \)-separated set which we call \( L_{n+1}^{(n+1)} \).
On $B_{n+1+4s_n} y_{n+1}$ is defined by putting $y_{n+1} \upharpoonright D_i l_i = \beta_i$ for $l_i \in L_i^{(n+1)}$, $0 \leq i \leq n$, and zero elsewhere. We set $\beta_{n+1} = y_{n+1} \upharpoonright D_{n+1}$.

To define $y_{n+1}$ on the rest of $G$, we define $y_{n+1} \upharpoonright D_{n+1} l_{n+1} = \beta_{n+1}$ for all $l_{n+1} \in L_{n+1}^{(n+1)}$ and clean the external $4s_n$ boundary of all these translates of $D_{n+1}$, just like we did in the external boundary of $D_{n+1}$. In all parts of $G$ which are not near these translates of $D_{n+1}$ we have $y_{n+1} = y_n$, and accordingly $L_i^{(n+1)}$ remains the same as $L_i^{(n)}$.

It is now easy to check that all properties (1)-(7) continue to hold.

This completes the inductive construction of the sequence $\{y_n\}_{n=1}^\infty$ and, as in particular $y_n \upharpoonright D_n = \beta_n$ for every $n$, it follows that the limit $\lim y_n = y_\infty \in \Omega$ exists. We define $Y$ to be its orbit closure in $\Omega$. It is now easy to verify the following:

5.2. **Proposition.** The point $y_\infty$ is uniformly recurrent, so that the subshift $Y$ is minimal.

5.3. **Problem.** The word metric provided a useful tool in the construction of the minimal subshift $Y$. However it seems likely that it is not really a necessary tool and that a similar, albeit perhaps more complicated construction, can provide such minimal subshift for any countable, not necessarily finitely generated, group. As we will see in the next section, this will prove the equivalence of the DDJ and DJ properties for all countable groups.

6. **The Equivalence of DJ and DDJ for Finitely Generated Groups**

6.1. **Theorem.** For finitely generated groups DDJ $\Rightarrow$ DJ.

**Proof.** We are given a minimal dynamical system $(X,G)$ with $G$ a finitely generated group, and we have to construct an essentially free minimal system $(Y,G)$ disjoint from $X$. A priori $X$ need not be essentially free. However, the group $G$ always admits some essentially free minimal system $(Z,G)$ \(^1\), and by taking a minimal subset of the product system $X \times Z$, we obtain a free minimal extension of $X$.

Of course a $(Y,G)$ which is disjoint from this extension is also disjoint from $X$. So we will now assume that $X$ has a free point $x_0 \in X$. For convenience we will first assume that $X$ is a subshift and then indicate how the general case can be similarly obtained. Thus our $X$ now is a subset of $\{0,1\}^G$ and $x_0 \in X$ is a free point.

We will use the construction described in Section 5 as follows: As in Proposition 5.2 we construct inductively:

1. An increasing sequence of finite sets $e \in D_0 \subset D_1 \subset D_2 \subset \cdots$ with $\bigcup D_i = G$,
2. a sequence of configurations $\beta_i \in \{0,1\}^{D_i}$ such that $\beta_{i+1} \upharpoonright D_i = \beta_i$ for $i = 0, 1, 2, \ldots$, and

\(^1\)See Appendix A below for a short proof of this fact.
(3) a sequence of points \( \{y_1, y_2, \ldots\} \subset \{0, 1\}^G \) with \( y_i \upharpoonright D_i = \beta_i \), converging to a point \( y_\infty \), such that the pair \((x_0, y_\infty)\) will be a uniformly recurrent point of \( \Omega \times \Omega \) and, with, \( G(x_0, y_\infty) \supset X \times \{y_\infty\} \), so that, \( X \times Y \) is minimal, with \( Y = Gy_\infty \), i.e. \( X \perp Y \).

Thus we use the construction of Section 5 as a template and we only have to choose, at each stage of the construction, the finite sets \( F_n \subset D_n \) in such a way that property (3) will hold. We do this as follows.

At stage \( n \) the finite set \( F_n \) will be a disjoint union \( F_n = B_n \cup C_n \) such that:

(i) The set \( B_n \) is chosen, using the minimality of \( X \), so that \( \{x_0 \upharpoonright D_{n-1}b : b \in B_n\} \) exhausts the set \( X_{D_{n-1}} = \{x \mid D_{n-1}x \in X\} \).

(ii) Using the CDJ property of \( X \), the set \( C_n \) is chosen so that for any \( g \in G \), there is some \( c \in C_n \), with \( x_0 \upharpoonright D_{n-1}cg = \alpha_{n-1} \).

(iii) For each \( f \in F_n = B_n \cup C_n \), \( y_n \upharpoonright D_{n-1}f = \beta_{n-1} \).

It is now easy to check that with these choices property (3) will result. \( \square \)

6.2. Problem. Using transfinite induction we can now deduce the existence, for a finitely generated group \( G \), of a family \( \{X_\alpha : \alpha < \omega_1\} \) of pairwise disjoint minimal metric \( G \)-systems (where \( \omega_1 \) is the first uncountable ordinal). Can an elaborated construction be conjured that will provide such a family with cardinality \( 2^{\aleph_0} \)?

7. Small sets

Let \( G \) be an arbitrary countable infinite (discrete) group. We denote by \( e \) the identity element of \( G \). Fix an increasing sequence of symmetric finite subsets \( A_n \subset G \) with \( A_0 = \{e\} \) and \( \bigcup_{n=0}^\infty A_n = G \).

7.1. Definition. A subset \( B \subset G \) is small if the unique minimal subsystem in the orbit closure \( \overline{G1_B} \subset \Omega = \{0, 1\}^G \) is the singleton \( \{1_0\} \)

7.2. Lemma. A subset \( B \subset G \) is small if and only if for every \( n \in \mathbb{N} \) the set \( K_{A_n,B} = \{g \in G : A_n g \subset B^c\} \) is syndetic; i.e.

\[ \exists F \subset G \text{ finite, such that } FK(A_n,B) = G. \]

\textbf{Proof.} Let \( S = S(G) \) denote the collection of sets \( B \subset G \) which satisfy the condition of the lemma. It is easy to see that \( S \) is invariant under right translations (for \( h \in G \), \( K_{A,hB} = \{g : Ag \subset (Bh)^c\} = \{g : Ag \subset B^c\}h^{-1} = K_{A,B}h^{-1}\) ). Let \( S = \{1_B : B \in S\} \subset \Omega \), the corresponding collection in \( \Omega \). Clearly we have \( 0 = 1_\emptyset \in \overline{Gx} \) for every \( x \in S \).

We will show next that \( S \) is orbit closed; i.e. \( x \in S \Rightarrow \overline{Gx} \subset S \). Suppose then that \( B \in S \) and that there is a sequence \( g_i \in G \) such that, in \( \Omega \) with \( x = 1_B \), \( g_i x = g_i 1_B = 1_{Bg_i^{-1}} \rightarrow y = 1_D \), where \( D \notin S \). Thus, we assume the existence of \( A = A_n \) such that for all finite \( F \subset G \) we have \( FK_{A,D} \neq G \), where \( K_{A,D} = \{g : Ag \subset D^c\} \). But by assumption (as \( B \in S \)
we have $FK_{A,B} = G$, for some finite $F$, whence for each $i$, $FK_{A,Bg_i} = FK_{A,Bg_i} = G$, and passing to the limit we arrive at the contradicting equality $FK_{A,D} = G$.

Together the last two assertions show that $S$ consists of small sets.

On the other hand, if $B \notin S$. Then there is $A = A_{n_0}$ such that for every finite $F \subset G$ we have $FK_{A,B} \neq G$, where $K_{A,B} = \{g : Ag \subset B^c\}$. In particular, for every $n$ we have $A_nK_{A,B} \neq G$ and we pick $g_n$ in the complement of this set; i.e. $A_ng_n \cap K_{A,B} = \emptyset$. Now by symmetry of $A$ it is easy to check that $K_{A,B} = G \setminus AB$ and we then have that, for every $n$, $A_ng_n \subset AB$, or $A_n \subset ABg_n^{-1}$.

Next choose a subsequence $g_n$, such that $g_n1_B \to 1_D$ with $1_D$ minimal. As $A_n \not\nearrow G$ we conclude that $G = AD$. In particular $D \neq \emptyset$ and $1_D \in G1_B$ implies that $B$ is not small. □

7.3. Corollary. $S$ is an ideal; i.e.

(1) $G \notin S$.

(2) $B_1 \in S$ and $B_2 \subset B_1$ implies $B_2 \in S$.

(3) $B_1, B_2 \in S$ implies $B_1 \cup B_2 \in S$.

Proof. The first two properties are clear.

To see (3), assume to the contrary that $B_1 \cup B_2 \notin S$. Then, by definition there is a sequence $g_i$ in $G$ with $g_i1_{B_1 \cup B_2} \to 1_D = x$ such that $x$ is minimal and $D \neq \emptyset$. We can assume that the limits $g_i1_{B_1} \to x_1$ and $g_i1_{B_2} \to x_2$ exist. Next we choose a sequence $h_j$ in $G$ such that the three limits $h_jx \to x'$, $h_jx_1 \to x_1'$ and $h_jx_2 \to x_2'$ exist and all three points $x', x_1'$ and $x_2'$ are minimal. But then $x_1' = x_2' = 0$, whence $0 = x_1' + x_2' = x'$. However $x' \in Gx$, hence cannot be $0$. □

8. Furstenberg’s conjecture verified for BDJ groups

Given a norm closed, translation invariant subalgebra $A$ of $l^\infty(G)$, containing the constant functions (in brief, an algebra), we say that a subset $A \subset G$ is an $A$-interpolation set if every bounded real valued function on $A$ can be extended to a function in $A$. A function $f \in l^\infty(G)$ is a minimal function if there is a minimal dynamical system $(X,G)$, a point $x_0 \in X$, and a continuous complex valued function $F \in C(X)$ such that

$$f(g) = F(gx_0), \quad \forall g \in G.$$ 

We write $\mathcal{I}_A = \mathcal{I}_A(G)$ for the collection of all $A$-interpolation sets. Let $\mathfrak{A}$ be the (closed, $G$-invariant) subalgebra of $l^\infty(G)$ generated by the minimal functions. Furstenberg’s conjecture, extended to any non compact group $G$, is that always $\mathfrak{A} \subset l^\infty(G)$.

In [GW-83, Theorem 1. (1)], building on crucial preliminary results in the fundamental work of Furstenberg [F-67], we have shown that Furstenberg’s conjecture holds in the case where $G = \mathbb{Z}$. We did this by showing that an interpolation set for the algebra $\mathfrak{A}(\mathbb{Z})$ is necessarily small; i.e. that $\mathcal{I}_q(\mathbb{Z}) \subset S(\mathbb{Z})$, the ideal of small sets in $\mathbb{Z}$. Since obviously $\mathcal{I}_{l^\infty}(\mathbb{Z})$
comprises all subsets of \( \mathbb{Z} \), this shows that, in particular \( \mathfrak{A}(\mathbb{Z}) \subseteq l^\infty(\mathbb{Z}) \). In fact, in [GW-83] we have shown that \( \mathfrak{I} = \mathcal{S} \).

8.1. **Remark.** Recall that a \( G \) dynamical system \((X, x_0, G)\), where \( x_0 \) is a distinguished point with dense orbit, is called an *ambit*. Let \( \beta G \) denote the Stone-Čech compactification of the discrete group \( G \). This universal compactification admits a natural semigroup structure with continuous right multiplication, as well as a \( G \) action. The corresponding ambit \((\beta G, e, G)\) is the *universal \( G \) ambit*; i.e. for every \( G \)-ambit \((X, x_0, G)\) there is a factor map \( \pi : (\beta G, e, G) \to (X, x_0, G) \) with \( \pi(e) = x_0 \). The minimal sets of this action coincide with the minimal left ideals and any such minimal set, say \( M \subset \beta G \), is isomorphic to the universal minimal \( G \) dynamical system. Denoting by \( E(M, G) \) the enveloping semigroup of the system \((M, G)\), it is not hard to see that Furstenberg’s conjecture is the same as the assertion that the canonical factor map \((\beta G, e, G) \to E(M, G)\) is not an isomorphism. See Appendix B below and [Aus-88, page 120], [dV-93], [G-98] and [P-98] for more details on this equivalent formulation of the conjecture. In V. Pestov’s paper [P-98], as well as in [G-98] and [BZ-18], Furstenberg’s conjecture, mistakenly, was referred to as Ellis’ problem.

Now, we claim that Furstenberg’s conjecture holds for every BDJ-group, hence also for every DDJ group.

8.2. **Theorem.** Let \( G \) be a BDJ group. Then \( \mathfrak{I}(G) \subset \mathcal{S}(G) \), the ideal of small sets in \( G \). In particular we deduce that \( \mathfrak{A}(G) \subseteq l^\infty(G) \).

It seems very likely that similar arguments to those given in [GW-83] will show that, in fact, for DDJ groups \( \mathfrak{I}(G) = \mathcal{S}(G) \), but for now, we leave this question open.

**Proof.** As above let \( \Omega = \{0, 1\}^G \). Here we consider \( \Omega \) both as a flow and as a compact topological ring under coordinate-wise multiplication and addition modulo 2. The following (definitions) and results are from [F-67], where one of the main tools used in their proofs was the fact that every minimal \( \mathbb{Z} \) system is disjoint from the Bernoulli shift \( \Omega(\mathbb{Z}) \). (This is a special case of Furstenberg’s \( \mathcal{F} \)-flows disjointness theorem, [F-67, Theorem II.2].)

Generalizing, from \( \mathbb{Z} \) to an arbitrary (countable discrete) \( G \), a closed \( G \)-invariant subset \( X \) of \( \Omega \) is restricted if \( X + Y = \Omega \), for closed invariant \( Y \) implies \( Y = \Omega \). For a BDJ group minimal subsets of \( \Omega \) are restricted, it is here that the disjointness of minimal systems from \( \Omega \) is used, see [F-67, Theorem III.1]. Moreover if \( M \) is minimal and \( X \) is restricted, then \( MX \) is restricted, [F-67, Proposition III.4]. Clearly, every finite sum of restricted sets is restricted, and one concludes that \( Z = \sum_{j=1}^m M_{j1}M_{j2} \cdots M_{jkj} \) is restricted whenever the \( M_{ji} \)'s are minimal sets.

Let \( R \) be the union of all restricted subsets of \( \Omega \). Then it is shown in [F-67, Proposition III.5] that a closed invariant subset \( X \subset \Omega \) is restricted iff \( X \subset R \). Let \( R_0 \subset R \) be the
subring of $\Omega$ generated by the minimal functions in $\Omega$. Clearly $R_0 \subset A$. One can check that the proofs of all these results in [F-67] are still valid for every group with the BDJ property. Hence, by Proposition 4.1, also for every DDJ group.

As in [GW-83] we will need the following two lemmas, ([GW-83, Lemma 2.1] and [GW-83, Lemma 2.4]) whose proofs again are valid for any $G$.

**8.3. Lemma.** Suppose $f \in A$ and for a sequence $g_i$ in $G$ we have $h(g) = \lim f(gg_i)$, $\forall g \in G$, then also $h \in A$.

**8.4. Lemma.** Every function in $A$ whose range is in $\{0, 1\}$ is an element of $R_0$.

Next we go over the proof of Theorem 1. (1) in [GW-83] and see how one can easily modify it to work for any $G$ with the BDJ property.

As in Section 7 fix an increasing sequence of symmetric finite subsets $A_n \subset G$ with $A_0 = \{e\}$ and $\bigcup_{n=0}^{\infty} A_n = G$. We assume, by contradiction, that there is a set $A \in \mathfrak{I}(G)$ which is not small. Put $\Omega_* = \{0, \ast\}^G$ and set $\gamma(g) = \begin{cases} \ast & g \in A \\ 0 & g \notin A. \end{cases}$

Since $A$ is not small we can find $\xi \in G\gamma$ such that $\xi$ is minimal and $\xi \neq 0$. There is a sequence $g_n$ in $G$ with $\lim g_n \gamma = \xi$. Let $B_n = \xi \upharpoonright A_n$ and set $k_n = |B_n|$. By the minimality of $\xi$, we can find sequences $m_i \in \mathbb{N}$ and $n_i \in \mathbb{N}$ such that:

1. In $B_{m_i}$ there are $2^{k_{m_{i-1}}}$ disjoint appearances of $B_{m_{i-1}}$.
2. The sets $I_i = g_{n_i}^{-1} A_{m_i}$ are disjoint.
3. $g_{n_i} \gamma \upharpoonright A_{m_i} = B_{m_i} = \xi \upharpoonright A_{m_i}$.

With these notations one proceeds, almost verbatim, as in the proof of Theorem 1. (1) in [GW-83], to obtain a contradiction. For completeness sake we provide the details.

By induction define an element $\eta \in \Omega$ as follows. Let $\eta \upharpoonright I_1$ be identically zero. Suppose $\eta$ has already been defined on $I_{i-1}$; we next define $\eta$ on $I_i$. Consider $B_{m_i}$. We are going to change all $\ast$’s in $B_{m_i}$ into zeros and ones (however, we never change zeros). The central $I_{i-1}$ pattern of $B_{m_i}$, namely $\xi \upharpoonright g_{n_i}^{-1} A_{i-1}$, we change into $\eta \upharpoonright I_{i-1}$. (By induction hypothesis this does not change zeros into ones.)

There are now at least $2^{k_{l_{i-1}}}$ disjoint appearances of $B_{m_{i-1}}$ in $B_{m_i}$. If there are $r_i \ast$’s in $B_{m_i}$ ($r_i \leq |I_{i-1}|$), then there are $2^{r_i}$ possible replacements of stars into zeros and ones; and we put all these replacements in the $2^{k_{l_{i-1}}}$ disjoint appearances of $B_{m_{i-1}}$ in $B_{m_i}$. On the rest of $B_{m_i}$ we replace all $\ast$’s by zeros. Let $\tilde{B}_{m_i}$ be the new block of 0 and 1 thus obtained, then define $\eta \upharpoonright I_i = \tilde{B}_{m_i}$. This defines $\eta$ on $I = \bigcup_{i=1}^{\infty} I_i$. Define $\eta$ on $G \setminus I$ to be identically zero. We clearly have $\lim g_n, \eta = \theta$ for some $\theta \in \Omega$. 

Next we show that $\theta \in \mathfrak{A}$. Since $\mathfrak{A}$ is an $\mathfrak{A}$-interpolation set, there exists $f \in \mathfrak{A}$ with $f \upharpoonright A = \eta \upharpoonright A$. By passing to a subsequence and then relabelling, we can assume that $\lim g_n f = h$ exists and by Lemma 8.3 $h \in \mathfrak{A}$. Put $D = \{ g \in G : \xi(g) = * \}$; then $1_D$ is a minimal function (hence in $\mathfrak{A}$) and recalling that the support of $\eta$, i.e. the set $\{ g \in G : \eta(g) = 1 \}$, is contained in $A$, we have

$$\theta = \lim g_n \eta = \lim g_n (1_A \cdot \eta) = \lim g_n (1_A \cdot f) = 1_D \cdot h.$$  

Hence $\theta \in \mathfrak{A}$. By Lemma 8.4, $\theta \in R_0$ and we conclude that $X = \overline{G \theta}$ is a restricted subset of $\Omega$. Define

$$Y_0 = \{ y \in \Omega : y \upharpoonright D = 0 \},$$

and let $Y$ be the smallest closed invariant subset of $\Omega$ containing $Y_0$. Since $1_D$ is minimal and $D \neq \emptyset$, it is clear that $Y \neq \Omega$. On the other hand, the construction of $\theta$ ensures that for every finite subset $\{ g_1, g_2, \ldots, g_n \}$ of $D$ and a sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, where $\epsilon_i = 0$ or $1$, there exists a $g \in G$ with $g \theta(g_i) = \theta(g;g) = \epsilon_i$, $i = 1, 2, \ldots, n$. It follows that $X + Y = \Omega$. This contradiction shows that $A \in \mathfrak{I}_\mathfrak{A}$ can not be small and the proof is complete.  

8.5. Remark. As in Remark 1.2 above, for some topological groups there is an easy answer to Furstenberg’s conjecture. Clearly this is the case for an extremely amenable group $G$, where $M(G)$ and $E(M(G))$ are trivial, while the universal ambit is a faithful compactification of $G$ and non-metrizable, being the Gelfand space of the (non-separable) $C^*$-algebra $RUC(G)$. For more results along this line, concerning certain automorphism groups of countable first-order structures, see [BZ-18].

9. Examples of DJ actions and DJ groups

9.1. Proposition. The group of integers $\mathbb{Z}$ has the DJ property.

Proof. Let $\{1\} \cup A \subset \mathbb{R}$ be a Hamel basis for the reals $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers. For each $\alpha \in A$ let $R_\alpha$ denote the rotation by $\alpha$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then the collection $\{ (\mathbb{T}, R_\alpha) : \alpha \in A \}$ consists of minimal pairwise disjoint free $\mathbb{Z}$-systems.

Now let $(X, T)$ be an arbitrary metrizable minimal $\mathbb{Z}$-system and let $\pi : (X, T) \rightarrow (Y, S)$ be its maximal equicontinuous factor. Then either $Y$ is a trivial one point system (i.e. $(X, T)$ is weakly mixing), or the system $(Y, S)$ admits an at most countable family $E$ of eigenvalues (see e.g. [G-03, Exercise 1.5.4]). In any case, as the cardinality of $A$ is that of the continuum, there is an element $\alpha \in A$ which is algebraically independent from $E$ over $\mathbb{Q}$. This implies that $(\mathbb{T}, R_\alpha) \perp (Y, S)$ and, by [EGS-76, Theorem 4.2], it follows that $(\mathbb{T}, R_\alpha) \perp (X, T)$. \hfill $\Box$

9.2. Proposition. Every metric minimal equicontinuous system has the DDJ property.
Proof. Let \((X, G)\) be a metric minimal system. For a metric system equicontinuity is equivalent to the existence of an invariant metric that generates the topology on \(X\). So fix such a metric \(d\).

Given a finite subset \(D\) of \(G\) and a nonempty open subset \(V\) of \(X\) we pick a point \(x_0 \in V\) and \(\epsilon > 0\) such that \(B_0 = B_{\epsilon/2}(x_0) \subset B = B_{\epsilon}(x_0) \subset V\), with \(\epsilon > 0\). Let \(A = \{x_0, x_1, x_2, \ldots, x_N\}\) be an \(\epsilon/2\)-net in \(X\), i.e. \(\bigcup_{i=0}^{N} B_{\epsilon/2}(x_i) = X\). Set \(L_0 = \{\epsilon\}\). We proceed by induction to define an increasing sequence of finite \(D\)-separated sets \(L_n \subset G\) with \(L_n^{-1}B_0 \supset \{x_0, x_1, x_2, \ldots, x_n\}\), for \(1 \leq n \leq N\).

Note that, by minimality, there is a finite set \(F \subset G\) with \(FB_0 = X\). Thus, for every \(g \in G\) also \(gFB_0 = X\). If \(K\) is any finite subset of \(G\) then for some \(g \in G\) we have \(gF \subset G \setminus K\) (otherwise, there are \(f_0 \in F\) and \(k_0 \in K\) such that for an infinite sequence \(g_i \in G\) we have \(g_i f_0 = k_0\)).

Suppose \(L_n\) is already defined; we use this observation to find an element \(\ell_{n+1} \in G \setminus D^{-1}DL_n\) such that \(x_{n+1} \in \ell_{n+1}^{-1}B_0\) and let \(L_{n+1} = L_n \cup \{\ell_{n+1}\}\). This completes the inductive step of the construction and we now let \(L = L_N\). Clearly \(L\) is \(D\)-separated and \(L^{-1}B_0 \supset A\). Now for an arbitrary \(x \in X\) there is an \(n\) such that \(d(x, x_n) < \epsilon/2\) and then for some \(\ell \in L\) we have \(x_n \in \ell^{-1}B_0\). Therefore

\[
d(\ell x, x_1) = d(x, \ell^{-1}x_1) < d(x, x_n) + d(x_n, \ell^{-1}x_1) < \epsilon,
\]

whence \(\ell x \in B \subset V\) and \(x \in \ell^{-1}V\). Thus we have shown that \(L^{-1}V = X\). \qed

In view of Theorem 6.1 we conclude that, for a finitely generated group \(G\), every minimal equicontinuous \(G\)-system has the DJ property. But, in fact, using ergodic theory, we can prove this for any group \(G\).

9.3. Proposition. For every countable group \(G\) and every minimal equicontinuous system \((X, G)\) there is an essentially free minimal system \((Y, G)\) which is disjoint from \((X, G)\).

Proof. Let \((Y, G)\) be the universal minimal dynamical system constructed in [W-12]. Among the invariant probability measures on \(Y\) there is one, say \(\mu\), which corresponds to the measure preserving Bernoulli system \((\Omega, \mathcal{B}, P, G)\), where \(\Omega = \{0, 1\}^G\) and \(P\) is the product measure \(\frac{1}{2}(\delta_0 + \delta_1)^G\). In particular the system \((Y, \mu, G)\) is measure theoretically weakly mixing and therefore also topologically weak mixing. This implies that \((Y, G)\) is topologically disjoint from the isometric system \((X, G)\) (this is e.g. a direct corollary of [G-75, Theorem 2.4]). Of course the system \((Y, \mu, G)\) is measure theoretically free, hence \((Y, G)\) is essentially free, and our proof is complete. \qed

9.4. Proposition. Every residually finite group has the DDJ property.

Proof. Given \(X, V\) and \(D\) as in the definition of the property DDJ, let \(H < G\) be a subgroup of finite index such that \(H \cap D^{-1}D = \{e\}\). If the action \((X, H)\) is minimal, we take \(L = H\). Otherwise \(X\) is the disjoint union of a finite number of \(H\)-minimal subsets \(X = \bigcup_{i=0}^{N} Y_i\).
Pick \( \{ e = g_0, g_1, g_2, \ldots, g_N \} \subset G \) with \( Y_i = g_i Y_0, \ i = 0, 1, \ldots, N \). We can assume that \( V_0 = V \cap Y_0 \) is nonempty, and then, \( HV_0 = Y_0 \). Let \( L_0 = F_0 \subset H \) be a finite set with \( F_0^{-1}V_0 = Y_0 \). Clearly \( L_0 \) is \( D \)-separated. Next consider \( V_1 := g_1V_0 \subset Y_1 \). By the minimality of \((Y_1, H)\) there is a finite set \( F_1 \subset H \) such that \( g_1^{-1}F_1 \cap D^{-1}DF_0 = \emptyset \) and \( F_1^{-1}V_1 = Y_1 \). Let \( L_1 = F_0 \cup g_1^{-1}F_1 \). We then have \( L_1^{-1}V_0 = F_0^{-1}V_0 \cup F_1^{-1}g_1V_0 = F_0^{-1}V_0 \cup F_1^{-1}V_1 = Y_0 \cup Y_1 \), and it is easily seen that \( L_1 \) is \( D \)-separated.

Suppose the sets \( L_j = F_0 \cup g_1^{-1}F_1 \cup \cdots \cup g_j^{-1}F_j \) have been already defined for \( 0 \leq j \leq k < N \). Let \( V_{k+1} = g_{k+1}V_0 \subset Y_{k+1} \) and let \( F_{k+1} \subset H \) be a finite set with the following two properties: (i) \( F_{k+1}^{-1}V_{k+1} = Y_k \), and (ii) \( g_{k+1}^{-1}F_{k+1} \cap D^{-1}DL_k = \emptyset \). Set \( L_{k+1} = L_k \cup g_{k+1}^{-1}F_{k+1} \).

One easily checks that \( L_{k+1} \) is \( D \)-separated, and that \( L_{k+1}^{-1}V_0 = Y_1 \cup Y_2 \cup \cdots \cup Y_{k+1} \). By induction this procedure finally defines \( L = L_N \), which is the required set in the definition of the DDJ property.

\[ \square \]

9.5. Proposition. Every amenable group has the DJ property.

Proof. (A sketch; see Appendix C for more details) We will use Ergodic Theory in order to prove this (purely topological) statement. Let \( G \) be an amenable group. Given an ergodic (probability measure preserving) system \( X = (X, \mathcal{X}, \mu, G) \), del Junco shows that the collection \( X^\perp \) of \( G \)-systems in the Polish space \( \mathcal{A} = \mathcal{A}(G) = MPA \) of measure preserving \( G \)-actions (on a canonical probability measure space) which are disjoint from \( X \), forms a \( G_\delta \) subset [dJ-81]. The Rokhlin lemma shows that if this collection is nonempty then it is dense [FW-04].

If \( X \) is a \( K \)-system (e.g. a Bernoulli system) then any 0-entropy ergodic system is disjoint from it. Thus the set of zero entropy systems forms a dense \( G_\delta \) subset of \( \mathcal{A} \) (see Appendix C).

On the other hand, if \( X \) is a 0-entropy system then, again \( X^\perp \) is non empty (e.g. it contains any Bernoulli system) and again we conclude that \( X^\perp \) is a dense \( G_\delta \) subset. Moreover, as the set of 0-entropy actions forms a dense \( G_\delta \) subset of \( \mathcal{A} \), we conclude that there is a 0-entropy system in \( X^\perp \).

For an arbitrary ergodic \( X \) let \( \pi : X \to X_P \) denote its Pinsker factor. Recall that, by definition, \( X \) is a \( K \)-system iff \( X_P \) is trivial. Moreover, in general, the extension \( \pi \) is a c.p.e. extension.

Now assume that \( X_P \), a 0-entropy system, is not trivial. By the above discussion there is an ergodic 0-entropy system \( Y \in X_P^\perp \). The facts that (i) \( Y \) is a 0-entropy system, (ii) \( Y \perp X_P \) and (iii) \( \pi : X \to X_P \) is a c.p.e. extension, imply that also \( Y \perp X \), [GTW-00].

Summing up, we have shown that for every ergodic \( X \in \mathcal{A} \) there is a nontrivial \( Y \in \mathcal{A} \) with \( X \perp Y \).

We now go back to the topological category and fix a compact minimal system \((X, G)\). Because the group \( G \) is amenable there exists a \( G \)-invariant, ergodic, probability measure, say \( \mu \) on \( X \), and, by minimality, it has a full support. Considering \( X = (X, \mathcal{X}, \mu, G) \) as
a measure theoretical system we apply the above discussion to produce an ergodic \( Y = (Y,\mathcal{Y},\nu,G) \in \mathfrak{X} \). Applying Rosenthal’s version of the Jewett-Krieger theorem (see [R-88] and [R-88a]) to \( Y \) we can assume that \((Y,\mu,G)\) is a strictly ergodic \( G \)-system (i.e., \((Y,G)\) is a topological (compact metric) minimal \( G \)-system and \( \nu \) is the unique \( G \)-invariant probability measure on \( Y \)). Now the disjointness of \( \mathfrak{X} \) and \( Y \) and the fact that both \( \mu \) and \( \nu \) have full support, imply the minimality of the product system \((\mathfrak{X} \times Y,G)\). In fact, if \( W \subset \mathfrak{X} \times Y \) is a nonempty closed and invariant subset, then, by the amenability of \( G \), there is a \( G \)-invariant probability measure on \( \mathfrak{X} \times Y \), say \( \lambda \), with \( \pi_X(\lambda) = \mu \) and \( \lambda(W) = 1 \). By the unique ergodicity of \((Y,\nu,G)\) we have also \( \pi_Y(\lambda) = \nu \). Thus \( \lambda \) is a joining of the systems \((\mathfrak{X},\mu,G)\) and \((Y,\nu,G)\) and, by disjointness, we conclude that \( \lambda = \mu \times \nu \). This forces the equality \( W = \mathfrak{X} \times Y \). We have shown that the non trivial minimal system \((Y,G)\) is disjoint from \((\mathfrak{X},G)\) and our proof is complete. \( \square \)

10. Maximally almost periodic groups and \( C^\ast \)-simple groups are DDJ

10.1. Definition. A topological group \( G \) is called \textit{maximally almost periodic (MaxAP)} if the collection of continuous group homomorphisms into compact topological groups \( \phi : G \to K \) separates points and closed sets of \( G \). In other words \( G \) is MaxAP iff the canonical Bohr compactification \( \Phi : G \to B \), is an embedding. We note that when a countable discrete group \( G \) is MaxAP then already a countable collection of homomorphisms \( \phi_i : G \to K_i, \ i = 1,2,\ldots \) separates points, and the Bohr compactification can be replaced by an injective compactification \( \Phi_0 : G \to K \), where \( K \) is a metrizable (or, equivalently, second countable) compact topological group \( K \).

10.2. Definition. Let \( G \) be a discrete group. Recall that the reduced \( C^\ast \)-algebra \( C^\ast_r(G) \) of \( G \) is the norm closure of the algebra of operators on \( l^2(G) \) generated by the left regular representation \( \lambda_G \) of \( G \). The group \( G \) is said to be \( C^\ast \)-simple if \( C^\ast_r(G) \) is simple, meaning that the only norm-closed two-sided ideals in \( C^\ast_r(G) \) are zero and \( C^\ast_r(G) \) itself.

In [KK-17] the authors prove the following:

10.3. Theorem. A discrete group is \( C^\ast \)-simple if and only if its action on the Furstenberg boundary \( \partial_F G \) of \( G \) is free.

For more information on the Furstenberg boundary \( \partial_F G \) and its identification as \( \Pi_s(G) \), the universal minimal strongly proximal \( G \)-system, see [G-76]. The class of \( C^\ast \)-simple groups is rich; the interested reader can find an in depth discussion of its extent in [BKKO-17]. For information about \( C^\ast \)-simplicity and \( C^\ast \)-simple groups see [KK-17] and [BKKO-17].

10.4. Proposition. A \( C^\ast \)-simple group has trivial amenable radical.
Proof. Let $R \lhd G$ denote the amenable radical of $G$. Let $X = \Pi_s(G)$. For $x \in X$ let $G_x = \{ g \in G : gx = x \}$, the stability group at $x$. As $R$ is amenable it admits an invariant probability measure, say $\mu$, on $X$. Since $R$ is normal each translate $g\mu$ is also $R$ invariant. By strong proximality there is a net $g_i \in G$ with $\lim g_i\mu = \delta_x$, for some $x \in X$, and it follows that $hx = x$ for every $h \in R$; i.e. $R \subset G_x$. By minimality this holds for every $x \in G$ and, as $G$ is $C^*$-simple, it follows that $R$ is trivial. □

10.5. Remark. In [LB-17] the author shows that the converse implication does not hold in general. However, by Theorem 1.5 of [BKKO-17] a discrete group with trivial amenable radical having either non-trivial bounded cohomology or non-vanishing $l^2$-Betti numbers is $C^*$-simple.

10.6. Theorem. The following families of (discrete infinite countable) groups have DDJ

1. Residually finite groups.
2. Maximally almost periodic groups.
3. $C^*$-simple groups.

Proof. By Corollary 2.2 it suffices to show that the group in question admits an essentially free DDJ minimal system $(Y,G)$.

1. Take $Y$ to be the universal profinite completion of $G$.
2. Take $Y$ to be the Bohr compactification of $G$ (see e.g. [vN]).
3. Take $Y$ to be the universal Furstenberg boundary (= maximal strongly proximal minimal system) of $G$. □

In [Z-18] Zucker proves a statement similar to part 2 of Theorem 10.6.

10.7. Remark. For an infinite countable group $G$ it is easy to see that $\Pi_s(G)$ is never metrizable. However, it always admits an essentially free metrizable factor. This factor will again be minimal and strongly proximal.

Summing up our results we have the following chain of implications:

- $C^*$-simple
- $\xrightarrow{\text{Amenable}} DJ \xrightarrow{\text{DDJ}} BDJ \xrightarrow{\text{Furstenberg’s conjecture}} \text{MaxAP}$
- $\xrightarrow{\text{RF}}$ (RF stands for residually finite). Moreover, for finitely generated groups DJ = DDJ.
11. THE DIRECT PRODUCT THEOREM

Our main result in this section is the following:

11.1. Theorem. If the groups $H_1$ and $H_2$ have the DJ property then so does their direct product $G = H_1 \times H_2$.

We first prove a structure theorem regarding a normal subgroup $N \trianglelefteq G$ of a general group $G$. If $(X, G)$ is a $G$ dynamical system and $\pi : (X, G) \to (Y, G)$ a factor map, we say that $Y$ is a $G/N$-factor if $N$ acts trivially on $Y$.

Given a dynamical system $(X, G)$ we denote the collection of closed subsets of $X$ by $2^X$ and endow it with its Vietoris topology. With this topology $2^X$ is a compact Hausdorff space which is metrizable iff $X$ is. The action of $G$ on $X$ induces an action on $2^X$. For an element $B \in 2^X$ we let $\text{OC}(B) = \text{cls}\{gB : g \in G\}$ denote its orbit closure in $2^X$.

Let $N \trianglelefteq G$ be a fixed normal subgroup of $G$.

11.2. Proposition. Let $(X, G)$ be a minimal system. Let

$$\mathcal{X} = \text{cls}\{M : M \subset X \text{ is an } N\text{-minimal subsystem}\} \subset 2^X,$$

a $G/N$-system. There is a unique minimal $G/N$-subsystem $\Xi \subset \mathcal{X}$. Moreover we have the following facts:

- Every maximal element of $\mathcal{X}$ (with respect to inclusion) is an element of $\Xi$.
- $\bigcup \Xi = X$ and every element of $\Xi$ is contained in a maximal element.
- The system $\Xi$ is the trivial one point system iff for every $N$-minimal set $M \subset X$ we have $X \in \text{OC}(M)$ in $2^X$.

Proof. We observe that each $B \in \mathcal{X}$ is closed and $N$-invariant. If $B$ is any element of $\mathcal{X}$ and $M$ any $N$-minimal subset of $X$ then, by the minimality of $X$, there is a net $g_i \in G$ with $D := \lim g_i B$ and $D \cap M \neq \emptyset$, whence $D \supset M$.

Let $A \in \mathcal{X}$ be a maximal element (with respect to inclusion). There is a net $M_i$ of $N$-minimal subsets of $X$, with $A = \lim M_i$. If $B$ is any element of $\mathcal{X}$, by the remark above, for every $i$ there is an element $D_i \in \text{OC}(B)$ with $M_i \subset D_i$. Passing to a subnet, if necessary, we have

$$A = \lim M_i \subset \lim D_i := L \in \text{OC}(B).$$

Hence, by the maximality of $A$, we have $L = A \in \text{OC}(B)$. This shows that $\Xi = \text{OC}(A)$ is the unique minimal subset of $\mathcal{X}$. We also conclude that every maximal element of $\mathcal{X}$ is an element of $\Xi$. The two claims in the second item of the proposition are clear. It is also clear now that $\Xi = \{X\}$ iff in $2^X$ we have $\text{OC}(M) = \{X\}$ for every $N$-minimal subset $M$ of $X$.

11.3. Theorem. With $(X, G)$ and $\Xi$ as above set

$$\tilde{X} = \{(x, B) : x \in B \in \Xi\}.$$
(1) The projection map \( p_X : \tilde{X} \to X \) is a proximal extension.

(2) The extension \( p_\Xi : \tilde{X} \to \Xi \) has the property that for every \( B \in \Xi \) and every \( N \)-minimal subset \( \tilde{M} \subset p_\Xi^{-1}(B) \) we have \( p_\Xi^{-1}(B) \in OC(\tilde{M}) \) in \( 2^{\tilde{X}} \).

(3) The \( G \)-system \( \tilde{X} \) is minimal.

(4) The projection map \( p_\Xi : \tilde{X} \to \Xi \) is an open map.

(5) \( \Xi \) is the maximal \( G/N \)-factor of \( \tilde{X} \).

(6) For any minimal \( G/N \)-system \( (\Theta, G/N) \) which is disjoint from \( (\Xi, G/N) \) we also have \( (\tilde{X}, G) \perp (\Theta, G) \). In particular, if \( \Xi \) is the trivial one-point system then \( (\tilde{X}, G) \perp (\Theta, G) \), whence also \( (X, G) \perp (\Theta, G) \), for every minimal \( (\Theta, G/N) \).

(7) When \( X \) is metric then so is \( \tilde{X} \).

Proof. (1) Suppose \( x = p_X(x, A) = p_X(x, B) \) for \( (x, A), (x, B) \in \mathcal{X} \). Let \( L \supset A \) be a maximal element. Now \( A \cap B \) is a nonempty closed \( N \)-invariant subset of \( X \) and thus contains an \( N \)-minimal subset \( M \subset A \cap B \). By Proposition 11.2 there is a net \( g_i \in G \) with \( \lim g_i M = L \). We can assume that the limits below exist and then

\[
L = \lim g_i M \subset \lim g_i (B \cap A),
\]

and therefore, by the maximality of \( L \), we have \( L = \lim g_i M = \lim g_i B = \lim g_i A \). Thus any two points of \( p_X^{-1}(x) \) are proximal.

(2) Let \( \tilde{M} \) be an \( N \)-minimal subset of \( p_\Xi^{-1}(B) \). Clearly \( p_\Xi^{-1}(B) = \{(x, B) : x \in B\} \) and \( \tilde{M} \) has the form \( \tilde{M} = \{(x, B) : x \in M\} \) for some \( N \)-minimal subset \( M \) of \( B \). Let \( A \supset B \) be a maximal element in \( \Xi \). As we have seen above there is a net \( h_i \) in \( G \) with \( \lim h_i M = \lim h_i B = \lim h_i A = A \), and thus, by minimality of \( \Xi \), also a net \( g_i \) in \( G \) with \( \lim g_i M = \lim g_i B = \lim g_i A = B \). It follows that \( \lim g_i \tilde{M} = p_\Xi^{-1}(B) = \{(x, B) : x \in B\} \subset \tilde{X} \), as required.

(3) Since the extension \( p_X \) is a proximal extension it follows that \( \tilde{X} \) contains a unique \( G \)-minima subset. Let \( (x, B) \) be an element of this minimal set. The \( N \) orbit closure \( OC_N(x, B) \) contains an \( N \)-minimal subset \( \tilde{M} \). By part (2) there is a net \( g_i \in G \) with \( \lim g_i \tilde{M} = p_\Xi^{-1}(B) \). We conclude that indeed \( \tilde{X} \) is minimal.

(4) This follows directly from the definition of \( \tilde{X} \) and the fact that \( \tilde{X} \) is minimal.

(5) Suppose \( \pi : \tilde{X} \to Y \) is a factor map with \( Y \) a \( G/N \)-system. For each \( N \)-minimal subset \( \tilde{M} \subset \tilde{X} \), its image \( \pi(\tilde{M}) \) is a singleton in \( Y \). This implies that \( \pi(B) \) is a singleton for every \( B \in \Xi \) and this defines a homomorphism \( \phi : \xi \to Y \) such that the diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & Y \\
\downarrow{p_\Xi} & & \\
\Xi & \xrightarrow{\phi} & Y
\end{array}
\]

commutes.
(6) Let \( (\Theta, G/N) \) be a minimal system disjoint from \( (\Xi, G/N) \). Let \( W \subset \tilde{X} \times \Theta \) be a closed \( G \)-invariant set. Pick some point \( (\tilde{x}_0, \theta_0) \in W \), and let \( \xi_0 := \pi_\Xi(\tilde{x}_0) \). Let

\[
M \subset OC_N(\tilde{x}_0, \theta_0) \subset W \cap (\pi_\Xi^{-1}(\xi_0) \times \{\theta_0\})
\]

be an \( N \)-minimal set. It has the form \( M = \tilde{M} \times \theta_0 \) where \( \tilde{M} \) is a minimal subset of \( \xi_0 \).

By part (2) there is a sequence \( g_i \in G \) with \( \lim g_i \tilde{M} = p_\Xi^{-1}(\xi_0) \). We can assume that the limit \( \lim g_i \theta_0 = \theta \) exists, so that \( \lim g_i M = \lim g_i(\tilde{M} \times \theta_0) = p_\Xi^{-1}(\xi_0) \times \{\theta\} \subset W \). We now use the fact that \( \Xi \perp \Theta \) and the openness of \( p_\Xi \) to deduce that \( W = \tilde{X} \times \Theta \).

Part (7) is clear. \( \square \)

11.4. **Definition.** We will say that a minimal system \( (\tilde{X}, G) \) is \( N \)-standard if it admits a factor \( \pi_\Xi : \tilde{X} \to \Xi \) with the properties (2), (4) and (5) above. In these terms, Theorem 11.3 says that every minimal system \( (X, G) \) admits a canonical \( N \)-standard extension:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{p_\Xi} & \Xi \\
\downarrow{p_X} & & \downarrow{p_\Xi} \\
X & & \Xi
\end{array}
\]

with \( p_X \) a proximal extension and \( p_\Xi \) an open map.

11.5. **Remark.** We say that a group \( G \) satisfies the weak DJ property if, in Definition 1, we omit the requirement of freeness; i.e. if

for every metrizable, minimal system \( (X, G) \) there is a nontrivial minimal \( (Y, G) \) such that \( X \perp Y \).

From part (6) of Theorem 11.3 we deduce that if \( N \triangleleft G \) and \( G/N \) has the weak DJ property, then also \( G \) has it.

**Proof of Theorem 11.1.** Let \( (X, G) \) be a minimal metric system. By Theorem 11.3 we can assume that \( X \) is \( H_1 \)-standard with \( \pi_\Xi : X \to \Xi \) denoting its maximal \( G/H_1 \)-factor. Note that \( (\Xi, G/H_1) \) can be considered as a \( H_2 \)-system.

Now apply the DJ property of \( H_2 \) to choose a minimal essentially free \( H_2 \)-system \( (\Theta_1, H_2) \) such that \( \Xi \perp \Theta_1 \). By Theorem 11.3.(6) we have \( X \perp \Theta_1 \) and we let \( X \) be the minimal \( G \)-system \( X = X \times \Theta_1 \).

Next construct the canonical \( H_2 \)-standard diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{p_\Xi} & \Xi \\
\downarrow{p_X} & & \downarrow{p_\Xi} \\
X & & \Xi
\end{array}
\]

with \( \Xi \) the maximal \( G/H_2 \cong H_1 \) factor of \( \tilde{X} \). Apply the DJ property of \( H_1 \) to choose a minimal essentially free \( H_1 \)-subsystem \( (\Theta_2, H_1) \) such that \( \Xi \perp \Theta_2 \). Again by Theorem
11.3. (6) we also have $\tilde{X} \perp \Theta_2$. Thus the $G$-system $\tilde{X} \times \Theta_2$ is minimal and hence so is its factor $X \times \Theta_1 \times \Theta_2 = X \times \Theta$ where $\Theta = \Theta_1 \times \Theta_2$. Finally observe that the $G$-system $\Theta$ is essentially free to conclude the proof. □

11.6. **Problem.** Is it true that the DJ property is preserved under short exact sequences? More precisely, suppose the group $G$ admits a short exact sequence

$$1 \to N \to G \to G/N \to 1,$$

where both $N \lhd G$ and $G/N$ are DJ, is $G$ necessarily DJ?

12. **Appendix A: Essentially free minimal actions**

12.1. **Theorem.** *Every countable infinite discrete group $G$ admits a minimal essentially free action on a compact metric space.*

**Proof.** Let $e$ denote the unit element of $G$ and let $e \in D_1 \subset D_2 \subset \cdots$ be a sequence of finite symmetric subsets that increase to $G$. Let

$$\Omega = \prod_{k=1}^{\infty} \{0, k\}^G.$$  

We regard $\Omega$ as the countable product of the Bernoulli shifts $\Omega_k = \{0, k\}^G$, $k = 1, 2, \ldots$. Let $\rho_k : \Omega \to \Omega_k$, be the natural projection maps.

For $k = 1, 2, \ldots$, define

$$Z_k = \{z \in \Omega_k : \text{the set } \{g \in G : z(g) = k\} \text{ is } D_k \text{-separated and } D_k^2 \text{-syndetic in } G\}.$$  

It is easy to see that the sets $Z_k$ are closed.

Now define inductively:

$$Y_1 = \{\omega \in \Omega : \rho_1(\omega) \in Z_1\}$$

$$\vdots$$

$$Y_{k+1} = \{\omega \in Y_k : \rho_{k+1}(\omega) \in Z_{k+1} \text{ and } \forall g \in G \ [\rho_{k+1}(\omega)(g) = k + 1 \Rightarrow \rho_k(\omega)(g) = k]\}.$$  

Each $Y_k$ is closed and we claim that they are not empty. To create an element of $Y_{k+1}$ start with a set $A_{k+1} \subset G$ which is maximal $D_{k+1}$-separated, hence $D_{k+1}^2$-syndetic. Next let $A_k \supset A_{k+1}$ be a maximal $D_k$-separated set, and proceed by induction to choose an increasing sequence

$$A_{k+1} \subset A_k \subset \cdots \subset A_1,$$

such that for each $1 \leq i \leq k + 1$, the set $A_i$ is a maximal $D_i$-separated set (hence also $D_i^2$-syndetic). Then, for each $i$, we have $i1_{A_i} \in Z_i$ and any $\omega \in \Omega$ with $\rho_i(\omega) = i1_{A_i}$, $i = 1, 2, \ldots, k + 1$ will be an element of $Y_{k+1}$.  

Set $Y = \bigcap_{k=1}^{\infty} Y_k$; a closed invariant set. For a point $y \in Y$ and $g \in G$, if $\rho_k(y)(g) = k$ then $\rho_i(y)(g) = i$ for all $i < k$. Let $X$ be a minimal subset of $Y$. Define for each $k$ the set

$$E_k = \{x \in X : \rho_k(x)(e) = k\}.$$

Clearly $E_k$ is closed, nonempty and $E_k \subset E_{k-1} \subset \cdots$. Therefore $E = \bigcap_{k=1}^{\infty} E_k \neq \emptyset$. We claim that any point in $E$ is free. In fact, suppose $x \in E$ and $gx = x$ for some $e \neq g \in G$. Then $g \in D_k$ for some $k$, hence

$$\rho_k(gx)(e) = \rho_k(x)(eg) = \rho_k(x)(g) = \rho_k(x)(e) = k.$$

But this contradicts the fact that $\rho_k(x) = k1_A$ with $A$ a $D_k$-separated set. Thus the action of $G$ on $X$ is essentially free.  

12.2. Remark. A similar argument will work for any locally compact, $\sigma$-compact group $G$, where the sequence $e \in D_1 \subset D_2 \subset \cdots$ consists of compact sets with $G = \bigcup_{k=1}^{\infty} D_k$.

13. Appendix B: An enveloping semigroup formulation of Furstenberg’s conjecture

Let $G$ be a topological group. We write $RUC(G)$ for the Banach algebra of right uniformly continuous complex valued bounded functions on $G$. These are the functions which are uniformly continuous with respect to the right uniform structure on $G$. Thus, $f \in RUC(G)$ iff for every $\epsilon > 0$ there exists a neighborhood $V$ of the identity element $e \in G$ such that $\sup_{g \in G} |f(vg) - f(g)| < \epsilon$ for every $v \in V$.

A triple $(X, x_0, G)$ with compact $X$ and a distinguished transitive point $x_0$ is called a pointed dynamical system (or sometimes an ambit). For homomorphisms $\pi : (X, x_0) \to (Y, y_0)$ of pointed systems we require that $\pi(x_0) = y_0$. When such a homomorphism exists it is unique. Given an ambit $(X, x_0, G)$, we associate, with every $F \in C(X)$, the function $j_{x_0}(F) = f \in RUC(G)$ defined by $f(g) = F(gx_0)$. Then the map $j_{x_0}$ is isometric embedding $C(X) \to RUC(G)$ of $C^*$-algebras. Let us denote its image by $j_{x_0}(C(X)) = \mathcal{A}(X, x_0)$. With $gf(\cdot) = f(g\cdot)$ we have $gf = g(j_{x_0}(F)) = j_{x_0}(F \circ g)$. The Gelfand space $|\mathcal{A}(X, x_0)|$ of the algebra $\mathcal{A}(X, x_0)$ is naturally identified with $X$ and in particular the multiplicative functional $\text{eva}_e : f \mapsto f(e)$, is identified with the point $x_0$. Moreover the action of $G$ on $\mathcal{A}(X, x_0)$ by left translations induces an action of $G$ on $|\mathcal{A}(X, x_0)|$ and under this identification the pointed systems $(X, x_0)$ and $(|\mathcal{A}(X, x_0)|, \text{eva}_e)$ are isomorphic.

Conversely, if $\mathcal{A}$ is a $G$-invariant uniformly closed $C^*$-subalgebra of $RUC(G)$, then its Gelfand space $|\mathcal{A}|$ has a structure of a pointed dynamical system $(|\mathcal{A}|, \text{eva}_e)$. If $\pi : (X, x_0) \to (Y, y_0)$ is an ambit homomorphism, then clearly $j_{y_0}(C(Y)) = \mathcal{A}(Y, y_0) \subset j_{x_0}(C(X)) = \mathcal{A}(X, x_0)$.

In particular, we have, corresponding to the algebra $RUC(G)$, the universal ambit $(S(G), \text{eva}_e)$ where we denote the Gelfand space $|RUC(G)|$ by $S(G)$. 


It is easy to check that for any collection \( \{(X_\theta, x_\theta) : \theta \in \Theta\} \) of pointed systems we have
\[
A \left( \bigvee \{(X_\theta, x_\theta) : \theta \in \Theta\} \right) = \bigvee \{A(X_\theta, x_\theta) : \theta \in \Theta\},
\]
where \( \bigvee \{(X_\theta, x_\theta) : \theta \in \Theta\} \) is the orbit closure of the point \( x \) in the product space \( \prod_{\theta \in \Theta} X_\theta \) whose \( \theta \) coordinate is \( x_\theta \). Here the algebra on the right hand side is the closed subalgebra of \( \text{RUC}(G) \) generated by the union of the subalgebras \( A(X_\theta, x_\theta) \).

13.1. **Definition.** We say that a function in \( \text{RUC}(G) \) is **minimal** if it comes from a minimal ambit; i.e. if there is a minimal ambit \( (X, x_0, G) \) and a continuous function \( F \in C(X) \) such that \( f(g) = F(gx_0) \) for every \( g \in G \). In other words, \( f \) is minimal iff it belongs to some \( A(X, x_0) \).

The **enveloping semigroup** \( E = E(X, G) = E(X) \) of a dynamical system \( (X, G) \) is defined as the closure in \( X^X \) (with its compact, usually non-metrizable, pointwise convergence topology) of the set \( \tilde{G} = \{\tilde{g} : X \to X\}_{g \in G} \), considered as a subset of \( X^X \). Here the map \( g \mapsto \tilde{g} \) is the continuous group homomorphism from \( G \) into the Polish group \( \text{Homeo}(X) \) of self homeomorphisms of \( X \), equipped with the topology of uniform convergence.

With the operation of composition of maps \( E(X) \) is a **right topological semigroup** (i.e. for every \( p \in E(X) \) the map \( R_p : q \mapsto qp, R_p : E(X) \to E(X) \) is continuous). Moreover, the map \( i : G \to E(X), g \mapsto \tilde{g} \) is a right topological semigroup compactification of \( G \).

13.2. **Proposition.** The enveloping semigroup of a dynamical system \( (X, G) \) is isomorphic (as a dynamical system) to the pointed product
\[
(E', \omega_0) = \bigvee \{(Gx, x) : x \in X\} \subset X^X,
\]
where \( \omega_0 \) is the point in \( X^X \) defined by \( \omega_0(x) = x \) for every \( x \in X \).

**Proof.** It is easy to see that the map \( p \mapsto p\omega_0 \), \( (E, i(e), G) \to (E', \omega_0, G) \) is an isomorphism of pointed systems. \( \square \)

A key lemma in the study of this algebraic structure is the following \[E-85\]:

13.3. **Lemma** (Ellis-Numakura). Let \( L \) be a compact Hausdorff semigroup in which all maps \( p \mapsto pq \) are continuous. Then \( L \) contains an idempotent; i.e., an element \( v \) with \( v^2 = v \).

We next recall some basic properties of the enveloping semigroup \( E = E(X, G) \). Most of these are easy consequences of the definitions and Lemma 13.3.

13.4. **Proposition.** (1) A subset \( M \) of \( E \) is a minimal left ideal of the semigroup \( E \) if and only if it is a minimal subsystem of \( (E, G) \). In particular a minimal left ideal is closed. We will refer to it simply as a minimal ideal. Minimal ideals \( M \) in \( E \) exist and for each such ideal the set of idempotents in \( M \), denoted by \( J = J(M) \), is non-empty.
Let $M$ be a minimal ideal and $J$ its set of idempotents then:

(a) For $v \in J$ and $p \in M$, $pv = p$.

(b) For each $v \in J$, $vM = \{vp : p \in M\} = \{p \in M : vp = p\}$ is a subgroup of $M$ with identity element $v$. For every $w \in J$ the map $p \mapsto wp$ is a group isomorphism of $vM$ onto $wM$.

(c) $\{vM : v \in J\}$ is a partition of $M$. Thus, if $p \in M$ then there exists a unique $v \in J$ such that $p \in vM$.

(d) Fix an arbitrary idempotent $u \in J \subset M$ and let $G = uM$ be the corresponding subgroup of $M$. Then, every element $p \in M$ has a unique presentation as $p = v\alpha$, with $v \in J$ and $\alpha \in G$.

The map

$$\alpha \mapsto R_\alpha, \quad G \to \text{Aut}(M,G)$$

is an (algebraic) isomorphism of the group $G$ onto the group $\text{Aut}(M,G)$ comprising the continuous automorphisms of the minimal system $(M,G)$.

Let $K, L$, and $M$ be minimal ideals of $E$. Let $v$ be an idempotent in $M$, then there exists a unique idempotent $v'$ in $L$ such that $vv' = v'$ and $v'v = v$. (We write $v \sim v'$ and say that $v'$ is equivalent to $v$.) If $v'' \in K$ is equivalent to $v'$, then $v'' \sim v$. The map $p \mapsto pv'$ of $M$ to $L$ is an isomorphism of $G$-systems.

Recall that a minimal dynamical system $(X, G)$ is called coalescent if every endomorphism of $(X, G)$ is an automorphism.

Next we collect some well known facts concerning these universal objects. For more details see e.g. [E-59], [G-76].

13.5. Proposition. Let $(S(G), e, G)$ be the universal $G$-ambit and $M \subset S(G)$ an arbitrary minimal ideal and $J \subset M$ the collection of idempotents in $M$.

(1) The enveloping semigroup of $(S(G), e, G)$ is canonically isomorphic, as a dynamical system, to $(S(G), e, G)$ itself.

(2) In particular, for every dynamical system $(X, G)$ there is a canonical continuous surjective homomorphism of enveloping semigroups $(S(G), e) \to (E(X, G), e)$.

(3) Each minimal ideal $M \subset S(G)$ is coalescent.

(4) $M$ is a universal minimal $G$-system; i.e. for every minimal $G$ system $(X, G)$ there is a homomorphism $\pi : M \to X$.

(5) Given a minimal system $(X, G)$ and a point $x \in X$, there is a minimal idempotent $v \in J \subset M$ with $vx = x$. Consequently, the homomorphism $\pi : (M, v) \to (X, x)$ is a homomorphism of ambits.

(6) Let $v \in J \subset M$ be a minimal idempotent in $M$, then $\mathcal{A}(v) = j_v(C(M))$ is a maximal subalgebra of $RUC(G)$ consisting of minimal functions. Conversely, any minimal function in $RUC(G)$ belongs to $\mathcal{A}(v)$ for some $v \in J$. 

Let $\mathfrak{A}$ be the (closed, $G$-invariant) subalgebra of $RUC(G)$ generated by the minimal functions. Furstenberg’s conjecture, extended to any non compact group $G$, is that always $\mathfrak{A} \subseteq RUC(G)$.

We are now in a position to state and prove our claim concerning the enveloping semigroup formulation of Furstenberg’s conjecture.

13.6. Proposition. The equality $\mathfrak{A} = RUC(G)$ holds when and only when the canonical ambit map $\pi : (S(G), e, G) \to E(M, G)$ is an isomorphism.

Proof. The homomorphism $\pi$ induces an isometric embedding $A(E(M, G)) \subseteq A(S(G), e) = RUC(G)$, and $\pi$ is an isomorphism iff these algebras are equal.

By Definition 13.1 and Proposition 13.5.(6), we have $\mathfrak{A} = \bigvee_{v \in J} \mathfrak{A}(v)$. By equation (3)

$$E(M, G) = \bigvee \{(M, p) : p \in M\}.$$  

However, by Proposition 13.4.(d), for every $p = v \alpha \in M$ we have $p = R_\alpha(v)$, and as $R_\alpha$ is an automorphism of $(M, G)$, we see that

$$E(M, G) = \bigvee \{(M, v) : v \in J\}.$$  

By equation (2)

$$A(E(M, G)) = A\left(\bigvee \{M, v \in J\}\right) = \bigvee \{A(M, v) : v \in J\} = \bigvee \{\mathfrak{A}(v) : v \in J\} = \mathfrak{A}.$$  

Thus $RUC(G) = \mathfrak{A}$ when and only when $\pi$ is an isomorphism. \qed

14. Appendix C: On the proof of Proposition 9.5

The results we need in the proof of Proposition 9.5 from the ergodic theory of amenable group actions may be found in [FW-04] and [GTW-00]. The result that zero entropy actions are generic is the sub-claim of claim 20 in [FW-04]. The result of del Junco is theorem 21 (ibid). The disjointness between zero entropy and relative c.p.e. can be found in [GTW-00]. Finally the fact that conjugacy classes in $A(G)$ are dense is explained in § 4.1 of [FW-04].

The version of the “Rokhlin Lemma” that is needed is stated as follows:

Let $H$ be an amenable group. If we are given finite subsets $K, F$ of $H$ and a $\delta > 0$, then we say that $F$ is $(K, \delta)$-invariant if

$$\left| \bigcup_{k \in K} kF \triangle F \right| < \delta|F|.$$  

For $K$ with $K = K^{-1}$ and $e \in K$, this definition is equivalent to the one given in [OW-87].
14.1. **Theorem** (Rokhlin’s Lemma). [OW-87] Suppose that \( H \) is an amenable group, and a finite set \( K \) of \( H \) and \( \delta, \epsilon > 0 \) are given. Then there are sets \( \{F_1, \ldots, F_k\} \) that are \((K, \delta)\)-invariant and numbers \( b_1, \ldots, b_k \in [0, 1] \) with \( \sum b_i > 1 - \epsilon \), so that for any free ergodic action of \( H \) on \([0, 1]\) by measure preserving transformations there are sets \( \{B_1, \ldots, B_k\} \) in the unit interval that satisfy:

1. For each \( i \) the sets \( \{fB_i : f \in F_i\} \) are disjoint.
2. The sets \( \{F_iB_i : 1 \leq i \leq k\} \) are pairwise disjoint.
3. The measure of \( F_iB_i \) is \( b_i \).

It is important to note that the sets \( F_i \) depend only on \( K, \delta \) and \( \epsilon \) and not on the specific action. Also the measures of each of the \( F_iB_i \) depend only on \( i \) and not on the action.

The remarks following these added statements there do not adequately explain how to obtain property (3). We take this opportunity to explain this point.

Fix one ergodic free action \((X_0, \mathcal{B}_0, \mu_0, G)\) and let \( \tilde{b}_i \) equal the measures of the corresponding \( F_iB_i \) for this action. Let \( b_i < \tilde{b}_i \) so that \( \sum b_i > 1 - \epsilon \). By the ergodic theorem, as soon as \( F_n \) is a sufficiently large Følner set, we can find a point \( x \in X_0 \) such that the \( F_n \)-name of \( x \) is covered by the \( F_iB_i \) in a proportion which is greater than \( b_i \). Now for an arbitrary ergodic free action \((X, \mathcal{B}, \mu, G)\), we apply the lemma with \( \tilde{F}_i \)'s that are sufficiently invariant (as above) with appropriate \((\tilde{K}, \tilde{\delta}, \tilde{\epsilon})\) and then copy, on each of the \( \tilde{F}_j\tilde{B}_j \) towers, the special \( \tilde{F}_j \)-names coming from the \((X_0, \mathcal{B}_0, \mu_0, G)\) action. We use these to define \( B_i \) in \( X \). These \( B_i \)'s will have measures \( \tilde{b}_i \) that are \( > b_i \) and can be trimmed to get the exact \( b_i \) required values.

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