Convergence in the Boundary Layer for Nonhomogeneous Linear Singularly Perturbed Systems

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Abstract

Convergence of the solutions of nonhomogeneous linear singularly perturbed systems to that of the corresponding reduced singular system on the half-line \([0, \infty)\) is considered. To include the situation on a neighborhood of initial instant, a boundary layer, a distributional approach to convergence is adopted. An explicit analytical expression for the limit as a distribution is proved.

Keywords. singular system, inconsistent initial condition, singular perturbation, distribution theory

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1 Introduction

A rational motivation to study singular linear system,

\[ E \dot{x} = Ax + Bu \] (1.1)

with singular matrix \(E\), is that it is an evident simplification of the singularly perturbed systems

\[ E(\epsilon) \dot{x} = Ax + Bu \] (1.2)

for a “small” parameter \(\epsilon\) (may be of vector form), where \(E(\epsilon)\) is nonsingular and tends to \(E\) as \(\epsilon \to 0\). The system (1.2) arises naturally from, for example, coupling subsystems with “slowly” and

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"fastly" varying states respectively, optimal linear-quadratic regulator with cheap control, etc. For detail, see [1]–[8]. For a specific system analysis or synthesis problem, the effectiveness of the above simplification relies on “approximate extent” between the solution to the problem for (1.1) and that for (1.2). Partially for characterizing “approximate extent” in the singular perturbation analysis, some interesting topologies are introduced. See, for example, [8], [9] and the references therein.

In this paper, we are interested in the following singularly perturbed initial value problem

\[
N(\epsilon)\dot{x}(t) = x(t) + f(t), \quad t \geq 0 \\
x(0) = x_0,
\]

and the corresponding reduced one

\[
N\dot{x}(t) = x(t) + f(t), \quad t \geq 0 \\
x(0) = x_0,
\]

Here \(N(\epsilon) \in \mathbb{R}^{n \times n}\) is nonsingular for \(\epsilon \neq 0\) and tends to \(N\), a nilpotent matrix, as \(\epsilon \to 0\). The index of nilpotency of \(N\) is denoted by \(q\), i.e.,

\[
q = \min\{k : k \geq 1, N^k = 0\}.
\]

The nonhomogeneous term \(f\) is a \(q - 1\) times continuously differentiable function mapping \(\mathbb{R}_+ = [0, +\infty)\) to \(\mathbb{R}^n\). Under a regularity assumption, the singular system (1.1) can be transformed into two subsystems through Weierstrass decomposition [10]. One has the form of the normal linear system which has trivial relationship to the corresponding perturbed ones, and another is of the form (1.4). For more detail of background, see [2]. For general initial conditions (“inconsistent initial conditions”), the problem (1.4) has no solution in the sense of classical differentiable function, and the corresponding physical system exhibits impulsive behavior [1]. Thus some generalized solutions are adopted for the problem (1.4). Recently [11]–[13], an explicit distributional solution of (1.4),

\[
x(t) = -\sum_{i=0}^{q-1} N^i f^{(i)}(t) - \sum_{k=1}^{q-1} \delta^{(k-1)}(t)N^k \left\{ x_0 + \sum_{i=0}^{q-1} N^i f^{(i)}(0) \right\}, \quad t \geq 0,
\]

is obtained by Laplace transform. So in what sense and whether the solution of (1.3) given by

\[
x_\epsilon(t) = \exp((N(\epsilon))^{-1}t)x_0 + \int_0^t \exp((N(\epsilon))^{-1}(t - \tau))(N(\epsilon))^{-1}f(\tau)d\tau, \quad t \geq 0,
\]

a classical function mapping \(\mathbb{R}_+\) to \(\mathbb{R}^n\), can be approximated by the distribution (1.6) becomes interesting.
Works [6] and [7] have the same concern, but they only considered natural response (i.e., the solution for \( f = 0 \)). The forced response (i.e., the solution for \( x_0 = 0 \)) of (1.4) also contains impulse term at initial instant according to (1.6). So the convergence in a neighborhood of \( t = 0 \), a "boundary layer" (region of nonuniform convergence, see [6], [14]), for the forced response also appeal to a distributional approach. This motivates a generalization to the results in [7] to include the nonhomogeneous case. For other related works, see [3], [4], [14], [15] and the references therein.

2 Notations and Definitions

We review some notations and definitions in distribution theory [16]. Let \( \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n) \) be the space of infinitely differentiable functions from \( \mathbb{R} \) to \( \mathbb{R}^n \) with compact support. There is a topology on it [16], and then the distribution space is defined as the dual space \( \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n)' \). So a distribution \( w \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n)' \) is a linear continuous functional on \( \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n) \). The value, a real number, of \( w \) on \( \lambda \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n) \) will be denoted by \( \langle w, \lambda \rangle \). The Dirac delta distribution \( \delta \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})' \) is defined by \( \langle \delta, \lambda \rangle = \lambda(0) \) for all \( \lambda \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}) \). For any distribution \( w \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})' \), its \( k \)-th order distributional derivative \( D_d^{(k)} w \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})' \) is defined by

\[
\langle D_d^{(k)} w, \lambda \rangle = (-1)^k \langle w, \lambda^{(k)} \rangle
\]

for all \( \lambda \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}) \), where \( \lambda^{(k)} \) denotes the \( k \)-th order usual derivative. Let \( \mathcal{L}_{loc}(\mathbb{R}, \mathbb{R}^n) \) denote the set of all locally Lebesgue integrable functions from \( \mathbb{R} \) to \( \mathbb{R}^n \). The embedding map \( \mathcal{E} : \mathcal{L}_{loc}(\mathbb{R}, \mathbb{R}^n) \to \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n)' \) is defined by \( \langle \mathcal{E} \lambda, \lambda \rangle = \int_{-\infty}^{+\infty} z(t)^T \lambda(t) dt \) for all \( z \in \mathcal{L}_{loc}(\mathbb{R}, \mathbb{R}^n) \) and all \( \lambda \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n) \). Here \( z(t)^T \) represents the transpose of \( z(t) \), and the integral is in the sense of Lebesgue. We do not distinguish \( z \) and \( \mathcal{E} z \) in following. Lastly, let \( \mathcal{C}^k(\mathbb{R}_+, \mathbb{R}^n) \) denote the set of all \( k \)-times continuously differentiable functions from \( \mathbb{R}_+ \) to \( \mathbb{R}^n \), which can be seen as a subset of \( \mathcal{L}_{loc}(\mathbb{R}, \mathbb{R}^n) \) naturally.

Now we cite the definition of convergence of distribution sequence [16].

**Definition 2.1** Given sequence \( \{z_i\}_{i=1}^\infty \subset \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n)' \) and \( z \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n)' \), then \( \{z_i\}_{i=1}^\infty \) is said to converge to \( z \) in \( \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n)' \), denoted by \( \lim_{i \to \infty} z_i = z \), if for every \( \lambda \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n) \),

\[
\lim_{i \to \infty} \langle z_i, \lambda \rangle = \langle z, \lambda \rangle .
\]

For convenience and without loss of generality, we consider discrete perturbations

\[
\begin{align*}
N_i \dot{x}(t) &= x(t) + f(t), \quad t \geq 0 \\
x(0) &= x_0,
\end{align*}
\]
where $N_i$ is nonsingular, and
\begin{equation}
\lim_{i \to \infty} N_i = N.
\end{equation}
(2.3)

The solution of (2.2) is
\begin{equation}
x_i(t) = \exp\{N_i^{-1}t\}x_0 + \int_0^t \exp\{N_i^{-1}(t - \tau)\}N_i^{-1}f(\tau)d\tau, \ t \geq 0.
\end{equation}
(2.4)

Then we need to explore, in the sense of Definition 2.1, the convergence of the solution sequence $\{x_i\}_{i=1}^\infty$ to the solution (1.6).

In following, except $\delta^{(k)}$, the $k$-th order derivative notation $z^{(k)}$ will always be in the ordinary sense according to pointwise differentiation. In the case $z \in C^k(\mathbb{R}_+, \mathbb{R}^n)$, notation $z^{(k)}(0)$ is understood as that from right hand. We always assume $f \in C^{q-1}(\mathbb{R}_+, \mathbb{R}^n)$ in this paper, where $q$ is the nilpotency index of $N$, to guarantee the distributional solution having the expression (1.6).

3 Uniqueness

For a perturbation manner given by $\{N_i\}_{i=1}^\infty$, the solution sequence $\{x_i(t)\}_{i=1}^\infty$ may not converge. But we will prove that if it does, then the limit must be the solution (1.6) of the reduced system (1.4), not dependent of the perturbation manner. This generalizes Theorem 2 in [7].

Lemma 3.1 [16, p. 21] Let $z \in C^k(\mathbb{R}_+, \mathbb{R}^n)$. Then we have
\begin{equation}
D_d^{(k)}z = z^{(k)} + \sum_{j=0}^{k-1} \delta^{(j)}z^{(k-j)}(0).
\end{equation}
(3.1)

Note that, according to the convention in Section 2, the precise meaning of (3.1) is
\begin{equation}
D_d^{(k)}\mathcal{E}(z) = \mathcal{E}(z^{(k)}) + \sum_{j=0}^{k-1} \delta^{(j)}z^{(k-j)}(0).
\end{equation}

Lemma 3.2 $x_i(t) \in C^q(\mathbb{R}_+, \mathbb{R}^n)$ and for $m = 1, 2, \ldots, q$,
\begin{equation}
x_i^{(m)}(t) = N_i^{-m}x_i(t) + \sum_{l=0}^{m} N_i^{-l}f^{(m-l)}(t), \ t \geq 0.
\end{equation}
(3.2)

Proof. Firstly, we prove the case $m = 1$.
\begin{align*}
x_i^{(1)}(t) &= N_i^{-t}e^{N_i^{-1}t}x_0 + \left(e^{N_i^{-1}t} \int_0^t e^{-N_i^{-1}\tau}N_i^{-1}f(\tau)d\tau\right)' \\
&= N_i^{-t}\left(e^{N_i^{-1}t}x_0 + e^{N_i^{-1}t} \int_0^t e^{-N_i^{-1}\tau}N_i^{-1}f(\tau)d\tau\right) + e^{N_i^{-1}t}e^{-N_i^{-1}t}N_i^{-1}f(t) \\
&= N_i^{-t}x_i(t) + N_i^{-1}f(t).
\end{align*}
(3.3)
Secondly, supposing that the case $m$ holds, we prove the case $m + 1$. Differentiating two sides of (3.2) gives
\[ x_i^{(m+1)}(t) = N_i^{-m} x_i^{(1)}(t) + \sum_{l=1}^{m} N_i^{-l} f^{(m+1-l)}(t). \] (3.4)
Substituting (3.3) in (3.4) gives the result immediately. ■

Lemma 3.3 For $k = 1, 2, \ldots, q$, we have
\[ \mathcal{D}_d^{(k)} x_i = N_i^{-k} x_i + \sum_{l=1}^{k} N_i^{-l} f^{(k-l)} + \sum_{j=0}^{k-1} \delta^{(j)} \left( N_i^{-(k-1-j)} x_0 + \sum_{l=1}^{k-1-j} N_i^{-l} f^{(k-1-j-l)}(0) \right). \] (3.5)

Lemma 3.4 [16, p.28] Let $\{z_i\}_{i=1}^{\infty} \subseteq \mathcal{C}_C^{\infty}(\mathbb{R}, \mathbb{R}^n)'$ and $z \in \mathcal{C}_C^{\infty}(\mathbb{R}, \mathbb{R}^n)'$. If $\lim_{i \to \infty} z_i = z$, then for every $k \geq 1$,
\[ \lim_{i \to \infty} \mathcal{D}_d^{(k)} z_i = \mathcal{D}_d^{(k)} z. \]

Theorem 3.1 If $\{x_i\}_{i=1}^{\infty}$ converges, then $\lim_{i \to \infty} x_i = x$.

Proof. Let $k = q$, the index of nilpotency of $N$, in (3.5). Multiplying two sides from left by $N_i^q$ gives
\[
N_i^q \mathcal{D}_d^{(q)} x_i = x_i + \sum_{l=0}^{q} N_i^{-(q-l)} f^{(l)} + \sum_{l=0}^{q-1} \delta^{(j)} \left( x_0 + \sum_{l=1}^{q-1-j} N_i^{-l} f^{(q-1-j-l)}(0) \right)
\]
\[ = x_i + \sum_{l=0}^{q-1} N_i^l f^{(l)} + \sum_{j=0}^{q-1} \delta^{(j)} N_i^{j+1} \left( x_0 + \sum_{m=0}^{q-2-j} N_i^m f^{(m)}(0) \right). \] (3.6)

Letting $i \to \infty$ and noting that $N_i \to N$, we obtain
\[ N^q \lim_{i \to \infty} \mathcal{D}_d^{(q)} x_i = \lim_{i \to \infty} x_i + \sum_{l=0}^{q-1} N^l f^{(l)} + \sum_{j=0}^{q-1} \delta^{(j)} N^{j+1} \left( x_0 + \sum_{m=0}^{q-2-j} N^m f^{(m)}(0) \right) \]
from Lemma 3.4. Noting that $N^q = 0$ and $N^{j+1} \sum_{m=q-2-j+1}^{q-1} N^m = 0$, we have
\[ \lim_{i \to \infty} x_i = \sum_{l=0}^{q-1} N^l f^{(l)} - \sum_{j=0}^{q-2} \delta^{(j)} N^{j+1} \left( x_0 + \sum_{m=0}^{q-1} N^m f^{(m)}(0) \right) = \sum_{l=0}^{q-1} N^l f^{(l)} - \sum_{k=1}^{q-1} \delta^{(k-1)} N^k \left( x_0 + \sum_{m=0}^{q-1} N^m f^{(m)}(0) \right). \]

This completes the proof. ■

4 Convergence

In this section, we will give a condition on perturbation to guarantee convergence. An example satisfying the condition shows the existence of convergent perturbation. This gives a generalization to Theorem 1 in [7].
Lemma 4.1 If the number sequence \( \{ \int_0^+ ||N_i^{k+1} e^{N_i^{-1}t}|| \, dt, \, i = 1, 2, \ldots \} \) is bounded for some \( k \geq 0 \), and \( f \in C^{q+k}(\mathbb{R}_+, \mathbb{R}^n) \cap L^1(\mathbb{R}_+, \mathbb{R}^n) \) then \( \{x_i\}_{i=1}^\infty \) converges.

Proof. Under the boundedness assumption, the sequence \( \{N_i^{q+1+k} e^{N_i^{-1}t}x_0\}_{i=1}^\infty \) converges to 0 in the sense of Definition 2.1 by Lemma 1 in [7]. Let \( h \in C_c^\infty(\mathbb{R}, \mathbb{R}^n) \) with ||h(t)|| \( \leq C \) for \( \forall t \in \mathbb{R} \). Since

\[
\left| \left\langle N_i^{q+1+k} \int_0^t e^{N_i^{-1}(t-\tau)} N_i^{-1} f(\tau) d\tau, \, h \right\rangle \right| \\
\leq \int_0^+ ||h(t)|| \cdot \left( \int_0^t ||N_i^{q+k} e^{N_i^{-1}(t-\tau)}|| \cdot ||f(\tau)|| d\tau \right) dt \\
\leq \int_0^+ ||f(\tau)|| \left[ \int_0^+ ||N_i^{q+k} e^{N_i^{-1}(t-\tau)}|| \cdot ||h(t)|| d\tau \right] d\tau \\
\leq \int_0^+ ||f(\tau)|| d\tau \cdot ||N_i^q|| C \int_0^+ ||N_i^{k+1} e^{N_i^{-1}t}|| dt \\
\rightarrow 0
\]

by the assumptions (note that \( ||N_i^q|| \rightarrow ||N_i^q|| = 0 \)), the sequence \( \{N_i^{q+1+k} \int_0^t e^{N_i^{-1}(t-\tau)} N_i^{-1} f(\tau) d\tau : i = 1, 2, \ldots \} \) converges to 0 in \( C_c^\infty(\mathbb{R}, \mathbb{R}^n)' \) also. So we have

\[
N_i^{q+1+k} x_i(t) = N_i^{q+1+k} e^{N_i^{-1}t} x_0 + N_i^{q+1+k} \int_0^t e^{N_i^{-1}(t-\tau)} N_i^{-1} f(\tau) d\tau, \, t \geq 0
\]

converges to 0 in \( C_c^\infty(\mathbb{R}, \mathbb{R}^n)' \). By Lemma 3.4, we have

\[
\lim_{i \to \infty} D_d^{(q+1+k)}(N_i^{q+1+k} x_i) = \lim_{i \to \infty} N_i^{q+1+k} D_d^{(q+1+k)} x_i = 0. \tag{4.1}
\]

On the other hand, since \( f \in C^{q+k}(\mathbb{R}_+, \mathbb{R}^n) \), we have

\[
N_i^{q+1+k} D_d^{(q+1+k)} x_i = x_i + \sum_{l=0}^{(q+1+k)-1} N_i^l f(l) \\
+ \sum_{j=0}^{(q+1+k)-1} \delta^{(j)} N_i^{j+1} \left(x_0 + \sum_{m=0}^{(q+1+k)-2-j} N_i^m f(m)(0)\right) \tag{4.2}
\]

like (3.6). From (4.1) and (4.2) we see the existence of \( \lim_{i \to \infty} x_i \) and

\[
\lim_{i \to \infty} x_i = - \sum_{l=0}^{(q+1+k)-1} N_i^l f(l) - \sum_{j=0}^{(q+1+k)-1} \delta^{(j)} N_i^{j+1} \left(x_0 + \sum_{m=0}^{(q+1+k)-2-j} N_i^m f(m)(0)\right).
\]

Noting that \( N_i^q = 0 \), we see that it equals \( x \) by (1.6).

We intend to weaken the higher differentiability requirement for \( f \in C^{q+k}(\mathbb{R}_+, \mathbb{R}^n) \) in Lemma 4.1. Again, we note that \( f \) is always assumed in \( C^{q-1}(\mathbb{R}_+, \mathbb{R}^n) \).

Lemma 4.2 Suppose \( f \in L^1(\mathbb{R}_+, \mathbb{R}^n) \). If the number sequence \( \{ \int_0^+ ||N_i^k e^{N_i^{-1}t}|| \, dt, \, i = 1, 2, \ldots \} \) is bounded for some \( k \geq 0 \), then \( \{x_i\}_{i=1}^\infty \) converges.
Proof. We only prove the result in the case $k = 0$. That for $k \geq 1$ can be proved by some slight modification. Note that $f \in C^{q-1}(\mathbb{R}_+, \mathbb{R}^n)$ but maybe $f \notin C^{q+0}(\mathbb{R}_+, \mathbb{R}^n) = C^q(\mathbb{R}_+, \mathbb{R}^n)$.

Differentiating two sides of (3.6) gives

$$N^q_d(i^{q+1})_x = D_d x_i = D_d x_i + \sum \delta^j (j+1) N^{j+1} (x_0 + \sum_m N^m f(m)(0)) - \delta N^{q-1} f(q-1)(0). \tag{4.3}$$

Noting that $x_i \in C^1(\mathbb{R}_+, \mathbb{R}^n)$ and $f(l) \in C^1(\mathbb{R}_+, \mathbb{R}^n)$ for $l = 0, 1, \ldots, q - 2$, it follows from Lemma 3.1 that

$$D_d x_i = \dot{x}_i + \delta \cdot x_0$$

and

$$D_d f(l) = f(l+1) + \delta \cdot f(l)(0)$$

for $l = 0, 1, \ldots, q - 2$. Substituting in (4.3) gives

$$N^q_d(i^{q+1})_x = \dot{x}_i + \delta \cdot x_0 + \sum \delta^j (j+1) N^{j+1} (x_0 + \sum_m N^m f(m)(0)) - \delta N^{q-1} f(q-1)(0).$$

Then we have

$$N^{q+1}_d(i^{q+1})_x = x_i + \sum \delta^j (j+1) N^{j+1} (x_0 + \sum_m N^m f(m)(0)) - \delta N^{q-1} f(q-1)(0).$$

Noting that

$$\lim_{i \to \infty} N^q_d f(q-1) = N^q_d f(q-1) = 0,$$

the remainder thing is similar to the proof of Lemma 4.1. ■

We need to weaken the integrability requirement $f \in L^1(\mathbb{R}_+, \mathbb{R}^n)$.

Lemma 4.3 For any $b > 0$, there exists $f_b \in C^{q-1}(\mathbb{R}_+, \mathbb{R}^n) \cap L^1(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$f_b(t) = f(t), \; \forall t \leq b. \tag{4.4}$$
Proof. One can construct a (unique) polynomial $P(t)$ of degree $(2q - 1)$ such that

$$P^{(k)}(b) = f^{(k)}(b), \quad P^{(k)}(b + 1) = 0$$

for $k = 0, 1, \ldots, q - 1$ (see [17, p. 88]). Then we define

$$f_b(t) = \begin{cases} f(t), & \text{if } 0 \leq t \leq b, \\ P(t), & \text{if } b < t \leq b + 1, \\ 0, & \text{if } t > b + 1, \end{cases}$$

which satisfies the requirement. □

Theorem 4.1 If the number sequence $\left\{ \int_{0}^{+\infty} ||N_t^k e^{N_{i-1}^{-1}t}||dt, i = 1, 2, \ldots \right\}$ is bounded for some $k \geq 0$, then $\{x_i\}_{i=1}^{\infty}$ converges.

Proof. Arbitrarily choose $h \in C_0^\infty(\mathbb{R}, \mathbb{R}^n)$. Then we have

$$h(t) = 0, \forall t \geq b$$

for some $b > 0$. Let $f_b \in C^{q-1}(\mathbb{R}_+, \mathbb{R}^n) \cap L^1(\mathbb{R}_+, \mathbb{R}^n)$ with (4.4). Then by Lemma 4.2, the sequence

$$y_i(t) = e^{N_{i-1}^{-1}t}x_0 + \int_{0}^{t} e^{N_{i-1}^{-1}(t-\tau)} N_i^{-1} f_b(\tau)d\tau, \quad t \geq 0$$

converges to

$$y(t) = -\sum_{i=0}^{q-1} N_i f_b^{(i)}(t) - \sum_{k=1}^{q-1} \delta^{(k-1)}(t)N^k \left\{ x_0 + \sum_{i=0}^{q-1} N_i f_b^{(i)}(0) \right\}, \quad t \geq 0$$

in the sense of Definition 2.1. By direct computation we can get

$$\langle x_i, h \rangle = \langle y_i, h \rangle, \quad i = 1, 2, \ldots$$

and

$$\langle x, h \rangle = \langle y, h \rangle.$$ 

Therefore $\lim_{i \to \infty} \langle x_i, h \rangle = \langle x, h \rangle$, and this completes the proof. □

Example 4.1 Set $N_i = N - \frac{1}{t} I, \quad i = 1, 2, \ldots$. Then $\left\{ \int_{0}^{+\infty} ||N_t^k e^{N_{i-1}^{-1}t}||dt, i = 1, 2, \ldots \right\}$ is bounded for some $k \geq 0$ (see Lemma 2 in [7]). So according to this perturbation manner, Theorem 4.1 guarantees that the solution sequence $\{x_i\}_{i=1}^{\infty}$ of the perturbed systems (2.2) converges to the solution $x$ of the singular system (1.4).
5 Conclusions

As an idealized model, the nonhomogeneous singular system can approximate some singularly perturbed systems well in a sense of distribution theory. A future work is to give some condition easy to verify on perturbations to guarantee convergence.

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