Revisiting the stochastic differential equations with multiplicative noise

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The standard path increments need a non-Gaussian extension. The new paths agree with the Fokker-Planck equation (FPE) of the “anti-Ito” case, and they include both the mean and the most probable evolution. The latter is determined by the noiseless motion. Possible steady densities thus have a maximum at its attractors, and the global existence of such states depends on a new criterion that generalizes “detailed balance”.

Key words: Stochastic differential equations; multiplicative noise; random paths; Fokker-Planck equation; steady states.

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I. Introduction

The existing theory does not specify the “integration sense”, and applications meet with difficulties like the “Ito-Stratonovich dilemma” [1,2]. Both phenomena are related with the fact that the existing path increments (in $dt$) are Gaussian, in contrast to the corresponding solution of the FPEs. Extending the increments within the order $dt$ results in non-Gaussian ones that agree with the (simple) FPE of the anti-Ito case. They include both the mean and the most probable evolution, and they are unique for each path of the Wiener process. The question of the integration sense does not arise for numerical computations (while two standard Riemannian sums of consecutive increments converge to different integrals, each with an incomplete information).

The most probable evolution is determined by the noiseless motion. This is not only relevant for optimum prediction and control, but it also shows that a possible steady density has a maximum on the attractor of that motion. This complies with the Freidlin theory [3] for weak noise. The new FPE further shows that steady densities for each noise level are also maximum at such an attractor, and it yields a new condition for their existence, related with detailed balance [4], but without use of a time reversal.

II. Background

The continuous Markov process $\tilde{X}(t)$ is assumed to fulfill the stochastic differential equation (SDE) [3,5-8]

$$d\tilde{X}^i = a^i(\tilde{X}) \, dt + b^i(\tilde{X}) \, dW_t$$

or

$$d\tilde{X} = \tilde{a}(\tilde{X}) \, dt + B(\tilde{X}) \, d\tilde{W}$$  \hspace{1cm} (2.1)

with smooth functions $a^i(\tilde{x}), b^i(\tilde{x})$. The drift $\tilde{a}$ is supposed to be independent of the noise. As usual, (2.1) denotes an integral equation, with the “integration sense” specified by $\alpha$ (0 ≤ $\alpha$ ≤ 1; “Ito” for $\alpha = 0$, “Stratonovich” for $\alpha = 1/2$ and “anti-Ito” for $\alpha = 1$).
The Wiener processes $W_k(t)$ are Gaussian distributed, with $<W_k(t) - W_k(0)> = 0$ and
$<[W_k(t) - W_k(0)]^2 = t$; they are independent of each other.

The standard path increments, with given $\bar{X}(t) = \bar{x}$ and $dt \geq 0$, are given by
\[ \bar{X}(t + dt) - \bar{x} = \bar{a}(\bar{x})dt + B(\bar{x})d\bar{W} + \alpha \bar{a}_{sp}(\bar{x})dt + o(dt) , \] (2.2)
where $d\bar{W} := \bar{W}(t + dt) - \bar{W}(t)$, and with the “spurious” drift
\[ a'_{sp}(\bar{x}) := b^{ki}(\bar{x})b^{kj}(\bar{x}) = (B_{ki}B^{kj})^a . \] (2.3)

Note that (2.2) is Gaussian distributed, since the coefficients are taken at the initial $\bar{x}$.

The “diffusion matrix”
\[ D(\bar{x}) := B(\bar{x})B^T(\bar{x}) \] (2.4)
is symmetric and nonnegative.

The relevant $\bar{a}_{sp}$ is determined by the $\bar{x}$-dependence of $D(\bar{x})$:
\[ a'_{sp} = D^{ak,k}/2 . \] (2.5)

This is evident when $B(\bar{x})$ is diagonal (thus in one dimension), and it generally holds
by stochastic equivalence [9].

The density $w(\bar{x},t)$ of $\bar{X}(t)$ (i.e. of the leading points of the random paths) is determined
by the FPE
\[ w_{,t} = \{-a' + (\alpha a'_{sp})w + (1/2)(D^{ik}w)_{,ik} \}. \] (2.6)

By $(D^{ik}w)_{,ik} = D^{ik,k}w + D^{ik}w_{,k}$ and (2.5) it can be rewritten as
\[ w_{,t} = \nabla \cdot \{ -[\bar{a} + (\alpha - 1)\bar{a}_{sp}]w + D\nabla w/2 \} . \] (2.7)

With the probability current
\[ J(\bar{x},t) := [\bar{a} + (\alpha - 1)\bar{a}_{sp}]w - D\nabla w/2 \] (2.8)
it becomes the continuity equation $w_{,t} + \nabla \cdot J = 0$.

The “propagator” is defined as the solution of (2.7) with an initial deltafunction,
asymptotically for small time steps.

III. Revision for multiplicative noise

3.1 Extending the path increments

For simplicity this is first explained in one dimension and without the drift $a(x)$. Letting

$$b(X) \approx b(x) + b'(x) b(x) dW$$

(and observing the order $O(\sqrt{dt})$ of $dW$) results in

$$X(dt) - x = b(x) dW + a_{sp}(x) (dW)^2 + o(dt). \quad (3.1)$$

While the mean of (3.1) agrees with the standard increment for $\alpha = 1$, its most probable value vanishes, since the density of $Y := (dW)^2$ is given by

$$(8\pi dt)^{-1/2} \exp(-y/2dt),$$

which even diverges at $y = 0$.

In higher dimensions the situation is more involved. In

$$b^{i\mu}(\vec{x} + d\vec{x}) = b^{i\mu}(\vec{x}) + b^{i\mu,\lambda}(\vec{x}) \, dx^\lambda \quad \text{one may insert} \quad dx^\lambda = b^{\lambda\nu} dW_\nu \quad \text{to obtain the second-order noise term}$$

$$b^{i\mu,\lambda} b^{\lambda\nu} dW_\mu dW_\nu := q^i. \quad (3.2)$$

**Properties:**

(i) Its propagator has its maximum at the most probable value of (3.3), i.e., $\vec{a}(\vec{x}) dt$.

(ii) The velocity of the probability current is $\vec{a}(\vec{x})$ wherever $\nabla w(\vec{x}, t) = 0$, see (2.8).

3.2 The Fokker-Planck equation

Since the mean of (3.3) corresponds to $\alpha = 1$ (anti-Ito), the FPE is given by

$$w_t = \nabla \cdot (-\vec{a} w + D \nabla w/2). \quad (3.4)$$

**Properties:**

(i) Its propagator has its maximum at the most probable value of (3.3), i.e., $\vec{a}(\vec{x}) dt$.

It is thus not a Gaussian, in view of the well-known mean $(\vec{a} + \vec{a}_{sp}) dt$.

(ii) The velocity of the probability current is $\vec{a}(\vec{x})$ wherever $\nabla w(\vec{x}, t) = 0$, see (2.8).
A revealing example in one dimension

Assume \( a(x) = -x \), as well as \( 0 < b(x) < \infty \), \( b'(x) > 0 \), whence \( a_{sp} > 0 \). The steady density follows by \( J = -xw - b^2w'/2 \equiv 0 \). Its maximum is at \( x = 0 \). There the most probable increment \( a(0)dt \) vanishes, while the mean one \( a_{sp}(0)dt \) is positive (in accordance with the positive mean of the steady density). This agrees with (3.1), while no (Gaussian) increment (2.2) distinguishes these cases. The adequate path increment is indeed given by (3.1).

3.3 The most probable path

The most probable path (starting from some \( \tilde{x}_0 \) is given by adding up the most probable increments \( \tilde{a} \, dt \) in consecutive time steps. This yields the noiseless path \( \tilde{x}_a(t) \) given by \( \dot{\tilde{x}}_a = \tilde{a}(\tilde{x}_a) \), with \( \tilde{x}_a(0) = \tilde{x}_0 \). It follows that the density (starting with a deltafunction at \( \tilde{x}_0 \)) has its maximum on \( \tilde{x}_a(t) \). Mind, however, that a superposition of densities with different starting points (or times) would not comply with the maximum property.

When \( \tilde{a}(\tilde{x}) \) has an attractor, \( \tilde{x}_a(t) \) describes the most probable approach to it, and the steady density thus has a maximum there. This is essential for the new analysis of the steady states in the following Chap. IV.

3.4 On Riemannian sums and their limit

The evaluation of the resolving paths is based on partitioning an interval \([0, t]\) into consecutive steps \( \Delta_i \) (of length \( dt \)) and on the Riemannian sum of the respective conditional increments (3.4). In the limit \( dt \to 0 \) the sum converges since the increments are of the order \( O(dt) \). The essential facts already show up in one dimension and with \( a \equiv 0 \). Summing up the increments \( a_{sp}(dW)^2 \) of each step leads to the anti-Ito integral.

This is easily seen in the special case of a constant nonzero \( a_{sp} \) : the sum of the \( (dW)^2 \) in each step is well-known to converge to \( t \), with variance zero. Mean increments are
correctly described, but the crucial information about the most probable ones is lost (while contained in the FPE). One may also focus on the most probable increment $\alpha \, dt$ in each step, which leads to the Ito integral. Here the information of the mean increment is lost. In both cases the increments become Gaussian, which is insufficient. A more appropriate Riemannian sum remains to be found. Yet it only matters for theoretical purposes, since (3.1) contains the full information and is sufficient for numerical computations.

IV. Steady states

4.1 The new approach

It is understood that a steady solution $w(\bar{x})$ of (3.4) can be written as $N \exp[-\phi(\bar{x})]$ with the “quasipotential” $\phi$. Note that $\nabla w = -w \nabla \phi$. The current $\vec{J} = \vec{a} w - D \nabla w / 2$ thus takes the form

$$\vec{J} = w \vec{a}_c$$

with the “conservative drift” $\vec{a}_c := \vec{a} + D \nabla \phi / 2$, (4.1)

$\vec{a}_c$ being the velocity of the steady current. The FPE (3.4) becomes $0 = \nabla \cdot (w \vec{a}_c) = w \nabla \cdot \vec{a}_c + \vec{a}_c \cdot \nabla w$, which yields the general equation $\nabla \cdot \vec{a}_c - \vec{a}_c \cdot \nabla \phi = 0$ for $\phi(\bar{x})$. This modifies the Freidlin equation [3]

$$(\vec{a} + D \nabla \phi / 2) \cdot \nabla \phi = 0$$

(4.2)

by the divergence of $\vec{a}_c$. Clearly (4.2) is of the first order but nonlinear, and it can be solved e.g. by characteristics [10,11]. It always holds for weak noise [3], but in view of the general equation also for each noise level whenever $\nabla \cdot \vec{a}_c = 0$. That condition

$$\nabla \cdot \vec{a}_c = \nabla \cdot \vec{a} + \nabla \cdot D \nabla \phi / 2 = 0$$

(4.3)

expresses the balance of dissipation ($-\nabla \cdot \vec{a}$) and diffusion at each $\bar{x}$, which is a strict notion of stationarity. Necessary and sufficient conditions on $\vec{a}(\bar{x})$ and $D(\bar{x})$ will be obtained below.
Suppose that \( \tilde{a}(\bar{x}) \) has an attracting point (where \( \bar{a} = \bar{0} \)). There (4.2) yields \( \nabla \phi = \bar{0} \) (thus \( \tilde{\alpha} = \bar{0} = \bar{J} \)), in accordance with the Chap. III; one can also find the local second derivatives of \( \phi \) [12,13]. Going beyond that weak-noise approximation crucially depends on whether the steady current only vanishes at the attractor (the genuine case) or globally. The quasipotentials with \( \nabla \cdot \tilde{a} = 0 \) will be obtained in what follows.

4.2 Solution for \( \phi \) with a vanishing current

The steady current vanishes when \( \tilde{\alpha} = \bar{0} \). This yields the linear equation \( \bar{a} + D\nabla \phi / 2 = \bar{0} \) for \( \phi \). With a non-singular \( D \) it is solved by \( \nabla \phi = -2D^{-1}\bar{a} \), provided that this is a gradient (always in one dimension), i.e. whenever

\[
\nabla \times D^{-1}\bar{a} = \bar{0}.
\]

(4.4)

Every solution \( \phi(\bar{x}) \) holds for each noise level.

Note that multiplying \( \bar{a} + D\nabla \phi / 2 = \bar{0} \) by any function \( f(\bar{x}) \) rescales both \( \bar{a} \) and \( D \) in the same way, without changing \( \phi \) [in one dimension this allows to replace any \( D(x) \) by 1].

For the time-dependent FPE this only applies with a constant \( f \), which rescales the time.

4.3 Solution for \( \phi \) with a persisting current

The Freidlin equation with \( \tilde{\alpha} = \neq \bar{0} \) is nonlinear. It implies that \( \nabla \phi = A \tilde{\alpha} \) with some antisymmetric \( A(\bar{x}) \). Since \( \nabla \phi \) vanishes where \( \bar{a} = \bar{0} \), one may consider the matrix \( H(\bar{x}) \) defined by \( \bar{a} = H\nabla \phi \). Insertion into (4.2) yields \( \nabla \phi \cdot (H + D / 2) \nabla \phi = 0 \), whence the symmetric part of \( H \) is \( -D / 2 \). The antisymmetric part \( A_H(\bar{x}) \) determines the current since \( \tilde{\alpha} = \bar{a} + (D / 2)\nabla \phi = A_H \nabla \phi \). Note that \( A_H = A^{-1} \). The drift is thus decomposed as

\( \bar{a} = (-D / 2 + A_H)\nabla \phi \), where \( A_H(\bar{x}) \) remains to be specified. It is essential that \( \nabla \phi = A \tilde{\alpha} \)
is a gradient. This holds for any source-free $\tilde{a}_c$ whenever $A$ is constant.

The condition for $\tilde{a}$ and $D$

Both $\tilde{a}_c$ and the decomposition of $\tilde{a}$ still involve the unknown $\phi(\tilde{x})$. It is therefore essential to establish a condition for $\nabla \cdot \tilde{a}_c = 0$ with $\tilde{a}$ and $D$ only. This is obtained by writing $\tilde{a}_c$ as $\tilde{a}_c = \tilde{a} + (D/2)A [\tilde{a} + (D/2)\nabla \phi]$. Taking the divergence results in

$$0 = \nabla \cdot \tilde{a} + \nabla \cdot (D/2)A [\tilde{a} + (D/2)\nabla \phi],$$

and with the dissipation $-\nabla \cdot \tilde{a} := \rho$ in

$$\nabla \cdot (D/2)A \tilde{a} + \nabla \cdot (D/2)A(D/2)\nabla \phi = \rho.$$

The matrix $(D/2)A(D/2)$ is antisymmetric, and if $D$ is constant, the second term on the left-hand side vanishes, which leaves

$$\nabla \cdot \left(\frac{D}{2}A\right) \tilde{a} = 2\rho$$

(see the Appendix for the $\tilde{x}$-dependent $D$). This is a necessary and sufficient condition for $\nabla \cdot \tilde{a}_c = 0$, and it does not involve $\phi$.

Comment: An admissible drift has the form $\tilde{a} = (-D/2 + A^{-1})\nabla \phi$. The first term does not contribute in (4.5), and the second one yields $\nabla \cdot D\nabla \phi = 2\rho$, i.e. $\nabla \cdot \tilde{a}_c = 0$.

It remains to determine the constant $A$. This is conveniently done at a point attractor of $\tilde{a}(\tilde{x})$ (where $\tilde{a} = \tilde{0}$). Since $D$ and $A$ are constant, the derivatives in (4.5) only act on $\tilde{a}$, and with the matrix $M$ of the elements $a_{i,k}$ (4.5) reduces to $D^{ik} A_{ki} M^{ij} = tr D A M = tr A M D = 2\rho$. Splitting $M D$ into its symmetric and antisymmetric parts shows that only the latter contributes, and it follows that

$$A = 2\rho (M D)^{-1} \quad \text{where} \quad (M D)^{-1} := [M D - (M D)^T]/2.$$  

Clearly $A$ does not exist when $M D$ is symmetric, i.e. when $\tilde{a}_c = \tilde{0}$. In three and more dimensions there are further solutions since $\nabla \phi$ is orthogonal on all directions of the
hypersurface $\phi = \text{const}$. They are irrelevant in the present context (note that $A$ only relates $\nabla \phi$ with $\tilde{\alpha}$).

**Remark:** A singular $M D$ can have a non-singular $(M D)_{\alpha}$, as for the Klein-Kramers equation [4] $v w_x + [(-\gamma v - U')w + \gamma T w' \gamma] = 0$, where $A = T^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The solution $\phi(x)$

It can either be obtained by integrating

$$\nabla \phi = H^{-1}\tilde{\alpha} = (-D/2 + A^{-1})^{-1}\tilde{\alpha}$$

along any paths, or else by the characteristics of the Freidlin equation [10,11], which yield the exact solution when (4.5) is met with a constant $A$.

4.4 Relation with detailed balance

The drift $\tilde{\alpha}_c$ is reversible, since a change of its sign is immaterial for (4.2); $\tilde{\alpha} - \tilde{\alpha}_c = -D \nabla \phi / 2$ is irreversible. While detailed balance is based on time reversal, this is not used in the present approach. The decomposition of the drift into an irreversible and a reversible part is now a result (rather than an input), and the individual $x^i$ need not have any pertinent symmetries. The last statement indicates a substantial generalization.

A further novelty is the essential distinction between the cases with a vanishing and a non-vanishing current (including an *a priori* criterion, see (4.6)), and in the latter case it forwards the new condition that the determinant of $D(\bar{x})$ must be constant, see the Appendix (no restriction in one dimension).

V. An alternative to the FPE with weak noise

Recall that the maximum of a density starting with a deltafunction follows the noiseless motion $\bar{x}_c(t)$. Small Gaussian deviations from $\bar{x}_c(t)$ obey the time-dependent FPE with
weak noise \((\tilde{a}_{\text{sp}} \ll \tilde{a})\). It is sufficient to know their variance at each \(t > 0\). It is determined by the time-dependent quasipotential \(\Phi(\tilde{x}, t) := -\log w(\tilde{x}, t)\), more precisely by the matrix \(S(t)\) of its second derivatives at the minimum on \(\tilde{x}(t)\). The change of \(S(t)\) in \(dt\) is determined by the Gaussian propagator (without \(\tilde{a}_{\text{sp}}\)). Using the matrix \(M\) with the elements \((a)^{j,k}\) (as above) one can derive from [14] that
\[
S[\tilde{x}_i(t + dt)] = S[\tilde{x}_i(t)] + (SM + M^T S + SDS)dt,
\]
which amounts to the Riccati equation
\[
\dot{S} = SM + M^T S + SDS.
\]
(5.1)

Multiplying from both sides by the inverse \(S^{-1} := Q\) results in the linear equation
\[
\dot{Q} = MQ + QM^T + D,
\]
(5.2)
which is not difficult to solve numerically. The initial \(Q\) is zero, and the final \(\dot{Q} = 0 = \dot{S}\) yields the quadratic approximation of the steady quasipotential at a point attractor of \(\tilde{a}\), see e.g. [12,13]. In one dimension the solution of (5.2) takes a closed form (note that (5.2) is of the first order and ordinary), in contrast with the FPE, especially when \(D' \neq 0\).

**Appendix: Changing the coordinates of the variable space**

Each nonsingular \(D(\tilde{x})\) becomes the unit matrix in new variables \(\tilde{z}(\tilde{x})\) given by
\[
d\tilde{z} = [D(\tilde{x})]^{-1/2} d\tilde{x}.
\]
This is easily checked by the fact that \(D(\tilde{x})\) is a contravariant tensor [4]. The Freidlin equation is covariant when \(\phi\) is a scalar. Unfortunately the condition \(\nabla \cdot \tilde{a} = 0\) is not covariant. It only persists when \(\tilde{z}(\tilde{x})\) preserves the volume element, and this requires a constant determinant of \(D(\tilde{x})\).

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