Gravitational Lensing in the Strong Field Limit for Kar’s Metric

Carlos A. Benavides$^1$ · Alejandro Cárdenas-Avendaño$^2$ · Alexis Larrañaga$^1$

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Abstract In this paper we calculate the strong field limit deflection angle for a light ray passing near a scalar charged spherically symmetric object, described by a metric which comes from the low-energy limit of heterotic string theory. Then, we compare the expansion parameters of our results with those obtained in the Einstein’s canonical frame, obtained by a conformal transformation, and we show that, at least at first order, the results do not agree.

Keywords Physics of black holes · Strong lensing · Photon sphere

1 Introduction

The General Theory of Relativity (GR) is the best theory of gravitation available. It establishes a new conception of space and time and describes how curvature acts on matter, to manifest itself as gravity, and how energy and momentum influence spacetime to create curvature: *space tells matter how to move; matter tells space how to curve* [1]. It has also passed several experimental tests in the weak field limit; however, it has not been yet tested in the strong gravitational field regime [2, 3].

GR has important astrophysical implications among which the deflection of light is one of the most important [4]. Light deflection in weak gravitational field is well-known since

Carlos A. Benavides
cabenavidesg@unal.edu.co

Alejandro Cárdenas-Avendaño
alejandro.cardenasa@konradlorenz.edu.co

Alexis Larrañaga
ealarranaga@unal.edu.co

1 Universidad Nacional de Colombia, Av. Carrera 30 # 45-03. Cód. postal 111321, Bogotá, Colombia

2 Fundacion Universitaria Konrad Lorenz, Carrera 9 Bis No. 62 - 43, Bogotá, Colombia
1919, when Eddington observed the light deflection by the sun [4], in lensing of quasars by foreground galaxies [5], in the formation of giant arcs in galaxy clusters [6] and more recently in galactic microlensing [7]. Now it is an important phenomenon in the panorama of astronomical observations [8].

The first study regarding light deflection in the strong regime was made by Darwin in Refs. [9, 10] where he initiated a theoretical research on gravitational lensing resulting from large deflection of light in the vicinity of the photon sphere of Schwarzschild spacetime. These studies were extended to a general static spherically symmetric spacetime by Atkinson in Ref. [11]. Nevertheless, the subject of strong gravitational field lensing remained in a quiet stage for two important reasons: 1) Darwin’s calculations showed that the images are very difficult to be observed with the available observational facilities at that time, 2) the known gravitational lens equation was not adequate for the study of lensing due to large deflection of light [12]. However, as astronomical techniques are improving fast, the possibility of testing the nature of astrophysical black hole candidates with current and future observations has recently become an active research field, since in the following years the technique of very long baseline interferometry (VLBI) and, in a longer term, the (sub)millimeter VLBI “Event Horizon Telescope” will produce images of the Galactic center emission capable to see the silhouette predicted by general relativistic lensing [13, 14]. Therefore, testing the gravitational field in the vicinity of a compact massive object, such as a black hole or a neutron star, could be a possible avenue for such investigations. In this sense, the importance of gravitational lensing in strong fields is highlighted by the possibility of testing the full GR in a regime where the differences with non-standard theories would be manifest, helping the discriminations among the various theories of gravitation [15–17]. For this reason, the scientific community has been interested in the lensing properties near the photon sphere i.e. strong field limit.

The strong field limit lensing regime was first defined in Ref. [18], where the authors studied the strong gravitational lensing due to a Schwarzschild black hole showing that, apart from the primary and the secondary images, there exists a sequence of images on both sides of the optic axis; named relativistic images. One of the most important contributions of that paper was a lens equation that allows, for a large bending of light near a black hole, to model the Galactic supermassive black hole as a Schwarzschild lens and to study point source lensing in the strong gravitational field region [18]. Later on, Virbhadra and Ellis in 2002 modeled massive dark objects in the galactic nuclei as spherically symmetric static naked singularities in the Einstein Massless Scalar (EMS) field theory [19]. In the same year Bozza, in Ref [8], provided a general method to extend the strong field limit to a generic static spherically symmetric spacetime inspired by the previous works [17, 19]. Bozza expanded the deflection angle near the photon sphere and showed that the divergence is logarithmic for all spherically symmetric metrics. According to the author, this method can be applied to any spacetime in any theory of gravitation, as long as the photons satisfy the geodesic equation [8]. Using this method it is possible to discriminate among different types of black holes on the grounds of their strong field gravitational lensing properties e.g., studying the properties of the relativistic images it may be possible to investigate the regions immediately outside of the event horizon because the parameters of the strong field limit expansion are directly connected with observables [8]. For this reason, the strong field limit has been used to estimate the deflection angle and the position of relativistic images produced by different types of black holes (see for instance, Refs. [20–22] and references therein, where gravitational lensing is not conceived as a weak field phenomenon).
In this paper, using the formalism presented in Refs. [8, 19], we calculate the strong field limit deflection angle for a light ray passing near a scalar charged spherically symmetric object, described by the metric proposed by Kar in Ref. [23], which comes from the low-energy limit of the heterotic string theory in the Jordan frame. The metric used in this paper is considered equivalent to the Janis-Newman-Winicour (JNW) metric\(^1\) [29] under a conformal transformation, since it is possible to rewrite it in the Einstein canonical frame (as written in Ref. [25]) by employing the standard relations between the two metrics \(g_{\mu\nu}^{st} = e^{2\phi}g_{\mu\nu}^{E}\) [23]. This result allows us to compare the frames from the strong field limit point of view, since Bozza in Ref. [8] calculated the same angle for the JNW’s metric and obtained different results. At least at first order the difference agrees with the ideas presented by Alvarez and Conde in Ref. [30], where they argued that the equivalence of the frames, for the description of the gravitational effects, is only obtained when all functions involved are smooth, condition which is not satisfied by the method used here, since the deflection angle in the strong field limit diverges around the photon sphere.

This paper is organized as follows. In Section 2 we review briefly the method used, write down the relevant equations and then apply the strong field limit expansion to Kar’s metric and in Section 3 for the JNW’s metric. In Section 4 we compare the results obtained in the Jordan frame with those obtained by Bozza in the Einstein frame and we calculate, as an example, the tangential magnification for the case of the Galactic supermassive ”black hole”. Finally, Section 5 contains the discussion of the results. The technical part of the paper is presented as an Appendix A, which has a detailed description of the calculations to obtain the deflection angle for Kar’s metric in the strong field limit.

2 Strong Field Expansion for Kar’s Metric

A general spherical symmetric spacetime is described by the line element [31]

\[
ds^2 = A(x)dt^2 - B(x)dx^2 - C(x)(d\theta^2 + \sin^2\theta d\phi^2).
\]

In order to obtain the deflection of a light beam it is necessary to consider the motion of a freely falling photon in a static isotropic gravitational field. From the geodesic equation and the line element (1) it is possible to obtain, as a function of the closest distance of approach \(x_0\), the following expression for the deflection angle [31]

\[
\hat{\alpha}(x_0) = 2 \int_{x_0}^{\infty} \sqrt{\frac{B(x)}{C(x)}} \left[ \frac{C(x)}{C(x_0)} \frac{A(x_0)}{A(x)} - 1 \right]^{-\frac{1}{2}} dx - \pi.
\]

According to (2), a photon coming from infinity with some impact parameter \(u = \sqrt{\frac{C(x_0)}{A(x_0)}}\) [31] will be deviated when it is approaching the black hole, it will reach \(x_0\) and then emerge in another direction. By decreasing \(x_0\) the deflection angle increases until it exceeds \(2\pi\), when the photon gives a complete loop around the black hole. By decreasing the impact parameter \(u\) the photon will wind several times before emerging. Finally, for \(x_0\) equal to the photon’s sphere radius (\(x_m\)), the deflection angle diverges and the photon is captured.

\(^1\) This metric was obtained independently by Wyman in 1981 in Ref. [24], but its equivalence with the JNW’s metric was not known until 1997 by Virbhadra in Ref. [25]. See Refs. [26–28] for further details regarding the gravitational lensing in this metric and classification of its singularities.
The formalism presented in Refs. [8, 19] is used to calculate the deflection angle in the strong field limit for a generic static spherically and symmetric spacetime taking the photon sphere as the starting point. The photon sphere is the region of spacetime where gravity is strong enough that photons are forced to travel in orbits [19]. This means that Einstein’s bending angle of a light ray with the closest distance of approach $x_0$ becomes unboundedly large as it tends to $x_m$. In this sense, the method requires that the photon sphere equation [32]

$$C'(x)A(x) = A'(x)C(x)$$

has at least one positive solution. Here a prime represents the derivative with respect to $x$. In general, (3) has several solutions; however, we will take the largest root as the radius of the photon sphere and denote it by $x_m$, as it is defined in [8]. To obtain the deflection angle in the strong field limit, the (2) is written in the following convenient form

$$\hat{\alpha}(x_0) = \int_0^1 R(z, x_0) f(z, x_0) dz - \pi = I(x_0) - \pi$$

where,

$$R(z, x_0) = \frac{2\sqrt{B y}}{CA'}(1 - y_0)\sqrt{C_0},$$

$$f(z, x_0) = \frac{1}{\sqrt{y_0 - [(1 - y_0)z + y_0]C_0^0}}.$$

All functions with the subscript 0 are evaluated at $x = x_0$ and without it are evaluated at $x = A^{-1}[(1 - y_0)z + y_0]$. The prime ' is the derivative with respect to $x$. Equation (2) expressed in this form makes possible to identify the function which contains the divergence. According to (5) and (6), the function $R(z, x_0)$ is regular for values of $z$ and $x_0$. However, the function $f(z, x_0)$ diverges for $z \to 0$. In this sense, to obtain the order of divergence of the integrand it is necessary to expand the argument of the square root in $f(z, x_0)$ to the second order in $z$. Therefore, for $z \to 0$ the function $f(z, x_0)$ can be approximate to

$$f_0(z, x_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}},$$

where $\alpha$ and $\beta$ are expressed by

$$\alpha = \frac{1 - y_0}{C_0A'_0} (C_0'y_0 - C_0A'_0)$$

and

$$\beta = \frac{(1 - y_0)^2}{2C_0^2A_0'^3} [2C_0C_0'A_0'^2 + (C_0C_0'' - 2C_0'^2)y_0A_0' - C_0C_0'y_0A_0''].$$

When $\alpha$ is not zero, the leading order of the divergence in $f_0$ is $z^{-\frac{1}{2}}$, which can be integrate to give a finite result. When $\alpha$ vanishes, the divergence is $z^{-1}$, which makes the integral diverge [8]. If we examine the form $\alpha$, we see that it vanish at $x_0 = x_m$. Each photon having $x_0 < x_m$ is captured by the central object and can not emerge back.

Having found the function which contains the divergence, the next step is to split $I(x_0)$ into two pieces: one part containing the divergence, $I_D(x_0)$, and the other being the original

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2By making $y = A(x)$ and $z = \frac{y - y_0}{1 - y_0}$. 

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integral with the divergence subtracted, $I_R(x_0)$

$$I(x_0) = \int_0^1 R(z, x_0) f(z, x_0) dz = ID(x_0) + I_R(x_0).$$

(10)

Using this idea, in Ref. [8] was shown that the divergence of (2) is logarithmic for all spherically symmetric metrics and has the form,

$$\hat{a}(\theta) = -\tilde{a} \ln \left[ \frac{\theta D_{OL}}{u_m} - 1 \right] + \tilde{b},$$

(11)

where

$$\tilde{a} = \frac{R(0, x_m)}{2 \sqrt{b_m}},$$

$$\tilde{b} = b_R + \tilde{a} \ln \left[ \frac{2 b_m}{\gamma m} \right] - \pi$$

(12)

and

$$u_m = \frac{\sqrt{C(x_m)}}{A(x_m)}.$$  

(13)

As we can see $\tilde{a}, \tilde{b}, u_m$ depend on the metric functions evaluated at $x_m$, $D_{OL}$ is the distance between the lens and the observer and

$$b_R = I_R(x_m).$$

(14)

By the procedure described above we calculated the deflection angle in the strong field limit for the metric proposed by Kar in Ref. [23], which has the form

$$ds^2_{str} = \left( 1 - \frac{2 m}{r} \right)^{\frac{m+\sigma}{\eta}} dt^2 - \left( 1 - \frac{2 \eta}{r} \right)^{\frac{(\sigma-m)}{m}} dr^2 - \left( 1 - \frac{2 \eta}{r} \right)^{1+\frac{\sigma-m}{m}} r^2 d\Omega^2,$$

(15)

where $m$ is the mass, $\sigma$ is the scalar charge and $\eta$ is given by $\eta^2 = m^2 + \sigma^2$. For $\sigma = 0$ this solution reduces to the Schwarzschild solution. Using geometrized units (the gravitational constant $G = 1$ and the speed of light in vacuum $c = 1$) and introducing a radial distance defined as $x = r^2 \eta$ and $x_0 = \frac{r_0}{2 \eta}$, (15) takes the form

$$ds^2_{str} = \left( 1 - \frac{1}{x} \right)^{m+\sigma \eta} dt^2 - \left( 1 - \frac{1}{x} \right)^{(\sigma-m) \eta} dx^2 - \left( 1 - \frac{1}{x} \right)^{1+\frac{\sigma-m}{m} \eta} x^2 d\Omega^2.$$

(16)

Nevertheless, in order to discuss gravitational lensing in the strong field limit for (16), it is more convenient to express it in terms of a single parameter. Using the relation $\eta^2 = m^2 + \sigma^2$, we defined $\xi = \frac{\sigma}{\eta}$ and $\gamma = \frac{m}{\eta}$ (which is the JNW’s parameter used Ref. [8]) so that $\gamma^2 + \xi^2 = 1$. If we choose $\xi$ as the parameter, the Kar’s metric can be expressed as

$$ds^2_{str} = \left( 1 - \frac{1}{x} \right)^{\xi+\sqrt{1-\xi^2}} dt^2 - \left( 1 - \frac{1}{x} \right)^{\xi-\sqrt{1-\xi^2}} dx^2 - \left( 1 - \frac{1}{x} \right)^{1+\xi-\sqrt{1-\xi^2}} x^2 d\Omega^2,$$

(17)

were $\gamma = \sqrt{1-\xi^2}$. Note that for $\xi = 0$ the metric (17) reduces to Schwarzschild. Therefore, in Kar’s metric (17) the functions for a spherically symmetric metric are

$$A(x) = \left( 1 - \frac{1}{x} \right)^{\xi+\sqrt{1-\xi^2}},$$

(18)
\[ B(x) = \left( 1 - \frac{1}{x} \right)^{1-\sqrt{1-\zeta^2}} \]  

and

\[ C(x) = \left( 1 - \frac{1}{x} \right)^{1+\zeta-\sqrt{1-\zeta^2}} x^2. \]  

From (8) with \( \alpha(x_m) = 0 \), the radius of the photon sphere as a function of \( \zeta \) is

\[ x_m = \sqrt{1 - \zeta^2} + \frac{1}{2}. \]  

For \( \zeta = 0 \) (21) reduces to \( x_m = \frac{3}{2} \), which is the value of the Schwarzschild’s photon sphere radius [18].

From (21) and (8) we obtain that

\[ R(0, x_m) = \frac{2\sqrt{1 - \xi^2} + 1}{\sqrt{1 - \xi^2} + \xi} \left\{ \frac{\left( \frac{2\sqrt{1 - \xi^2} - 1}{2\sqrt{1 - \xi^2} + 1} \right)^{1+\zeta-\sqrt{1-\zeta^2}} - \left( \frac{2\sqrt{1 - \xi^2} - 1}{2\sqrt{1 - \xi^2} + 1} \right)^{1+3\zeta-\sqrt{1-\zeta^2}}}{\sqrt{1-\xi^2+\xi}} \right\}, \]

where

\[ y_m = \left( \frac{2\sqrt{1 - \xi^2} - 1}{2\sqrt{1 - \xi^2} + 1} \right)^{\sqrt{1-\xi^2+\xi}} \]

and

\[ \beta_m = \frac{1}{4} \left[ \left( \frac{2\sqrt{1 - \xi^2} + 1}{\sqrt{1 - \xi^2} + \xi} \right)^{\xi+\sqrt{1-\xi^2}} - \left( \frac{2\sqrt{1 - \xi^2} + 1}{\sqrt{1 - \xi^2} + \xi} \right)^{\xi+\sqrt{1-\xi^2}} \right]^2. \]  

Thus, from (12) we obtain that \( \tilde{a} = 1 \), which is the same value obtained for JNW’s metric [8].

In order to obtain \( b_R \) we expand (14) up to first order in \( \xi \), around \( \xi = 0 \), obtaining that

\[ b_R = 2 \ln 6(2 - \sqrt{3}) - 0.226\xi \]  

and using (12)

\[ \tilde{b} = 2 \ln(6(2 - \sqrt{3})) - 0.226\xi - \pi + \ln \left[ \frac{2\beta_m}{y_m} \right]. \]

Making \( x_0 = x_m \) in \( u = \sqrt{\frac{C(x_0)}{A(x_0)}} \), we obtain that the impact parameter is

\[ u_m = \frac{\sqrt{1 - \xi^2} - 1}{\xi+\sqrt{1-\xi^2}}. \]

Finally, using (11), the deflection angle in the strong field limit for Kar’s metric in terms of the parameter \( \zeta \) is

\[ \hat{\alpha} = -\ln \left[ \frac{u}{u_m} - 1 \right] + 2 \ln(6(2 - \sqrt{3})) - 0.226\xi - \pi + \ln \left[ \frac{2\beta_m}{y_m} \right], \]

where \( u \) is the impact parameter. The last equation can be written in terms of the angular separation of the image \( \theta \) by the relation \( \theta = \frac{u}{D_{OL}} \), where \( D_{OL} \) is the distances between the lens and the observer (see (65) for details).
3 Strong Field Expansion for JNW’s Metric

The JNW’s metric [25]

\[ ds^2 = \left( 1 - \frac{1}{x} \right)^\gamma dt^2 - \left( 1 - \frac{1}{x} \right)^{-\gamma} dx^2 - \left( 1 - \frac{1}{x} \right)^{1-\gamma}, \]  

(26)

has a photon sphere, \( x_m \), which now expressed as a function of \( \gamma \) is [8]

\[ x_m = \frac{2\gamma + 1}{2}, \]  

(27)

which implies that the photon sphere exists only for \( \frac{1}{2} < \gamma \leq 1 \) [32]. For this interval, a photon coming from infinity is deflected through an unboundedly large angle; this means, that the photon passes many times around the singularity as the closest distance of approach tends to \( x_m \).

The deflection angle in the strong field limit obtained for JNW’s metric in Ref. [8] is

\[ \tilde{\alpha}(\theta) = \overline{a} \ln \left( \frac{\theta D_{\Omega L}}{u_m} - 1 \right) + \overline{b}, \]  

(28)

where \( \overline{a} = 1 \) and

\[ \overline{b} = 2 \ln[6(2 - \sqrt{3})] - 0.1199(\gamma - 1) - \pi + 2 \ln \left( \frac{(2\gamma + 1)[(2\gamma + 1)\gamma - (2\gamma - 1)\gamma]^2}{2\gamma^2(2\gamma - 1)^2\gamma^{-1}} \right). \]  

(29)

However, note that the values of \( \beta_m \) and, in consequence, the value of \( \overline{b} \) (see (12)) differs from (72) and (75) of Ref. [8]. Our calculations show that

\[ \beta_m = \frac{[(2\gamma + 1)\gamma - (2\gamma - 1)\gamma]^2}{4\gamma^2(4\gamma^2 - 1)\gamma^{-1}} \]  

(30)

thus,

\[ \overline{b} = 2 \ln[6(2-\sqrt{3})] - 0.1199(\gamma - 1) - \pi + \ln \left( \frac{(2\gamma + 1)[(2\gamma + 1)\gamma - (2\gamma - 1)\gamma]^2}{2\gamma^2(2\gamma - 1)^2\gamma^{-1}} \right) \]  

(31)

which differs from a factor of two in the logarithmic. For \( \gamma = 1 \), (30) and (31) reduce to those of Schwarzschild [8].

In order to discuss gravitational lensing in the strong field limit for Kar’s metric (17) and compared it with (26), it is necessary to find the values of \( \zeta \) where (21) has a solution. Using the analysis made in [32] we found that the photon sphere equation has solution only for \( 0 \leq \zeta < \frac{\sqrt{3}}{2} \), i.e., for \( 0 \leq \sigma^2 < 3m^2 \). It is easy to obtain the same interval for \( \zeta \) using \( \frac{1}{2} < \gamma \leq 1 \) (the interval for JNW’s metric) and recalling that \( \gamma = \sqrt{1 - \zeta^2} \). The behaviors of the photon sphere for Kar, JNW and Reissner–Nordström (RN), also presented in Ref. [8], metrics as a function of \( \zeta \), \( \gamma \) and \( q \) (which is the charge in the RN metric), respectively, are plotted in Fig. 1.

In Fig. 2 is plotted the values of \( u_m \), \( \bar{a} \) and \( \bar{b} \) in terms of \( \zeta \) for Kar’s metric. From the figure we see that \( \bar{a} = 1 \) is constant and has the same value of that of JNW’s metric [8]. Note that \( \bar{b} \) has a minimum value at \( z = 0 \), increases as \( \zeta \) increases, and for \( \zeta = \sqrt{3}/2 \) the value of \( \bar{b} \) diverges. In the same figure, the impact parameter \( u_m \) decreases as \( \zeta \) increases until the value \( \zeta = \sqrt{3}/2 \) is reached, where \( u_m \) diverges. Finally, it is clear that each parameter reduces to those of Schwarzschild for \( \zeta = 0 \).
In Fig. 3 is plotted the behavior of the deflection angles as a function of $\zeta$. Once we fix $u = u_m + 0.003$ we see that the deflection angle has a minimum value at $\zeta = 0$, increases as the value of $\zeta$ increases, and for $\sqrt{3}/2$ it diverges. For $\zeta = 0$ the deflection angle reduces to $-\ln\left(\frac{0.006}{3\sqrt{3}}\right) + \ln(216(7 - 4\sqrt{3})) - \pi \approx 6.364$, which is the Schwarzschild deflection angle in the strong field limit when $u = u_m + 0.003$ [8].

4 Analysis

It is well known that before the general theory of relativity was proposed, scalar field has been conjectured to give rise to the long range gravitational fields. In this sense, several theories involving scalar fields have been suggested. One of the most important theories was proposed by C. Brans and R. H. Dicke in [33]. In that paper, the authors discuss the role of the Mach’s principle in physics in relation with the equivalence principle, and the difficulties to incorporate the former into general theory of relativity. As a consequence, the authors develop a new relativistic theory of gravitation (a generalization of general relativity) which
is compatible with Mach’s principle. This theory is not completely geometrical because gravitational effects are described by a scalar field. Therefore, the gravitational effects are in part geometrical and in part due to a scalar interaction. According to the authors there is a connection between this theory and that of Jordan [34] because the two metric tensor are connected by a conformal transformation. Thus, there are two formulations of the Brans-Dicke theory; the so called Jordan conformal frame (JF) and the Einstein conformal frame (EF). The equivalence between this two frame of the Brans-Dicke (BD) theory of gravity under conformal transformations

\[ g^J_{\mu\nu} = e^{2\phi} g^E_{\mu\nu}, \]  

has been discussed in the literature since long ago [35] and, even today, the scientific community is divided into two points of view: one in which the two frames are equivalent and other in which they are not. In this sense, the question of whether or not the two frames are equivalent is very important because there are many applications of scalar-tensor theories and of conformal transformation techniques to the physics of early universe and to astrophysics [36–39].

The theoretical predictions to be compared with the observations depend on the conformal frame adopted [40] but the discussion is still open. For example, in Ref. [41] is argued that the JF is unphysical, while the EF is physical and reveals the physical contents of the theory, in Ref. [40] the authors discussed the question of which conformal frame is physical by providing an example based on gravitational waves and they favoured the EF as the physical one, in Refs. [42, 43] the author showed that the motion of test masses in the field of a scalar gravitational wave is different in the two frames and in Ref. [44] is explained an alternative interpretation of the conformal transformations of the metric according to which the latter can be viewed as a mapping among Riemannian and Weyl-integrable spaces, suggesting that these transformations relate complementary geometrical pictures of a same physical reality and the question about which is the physical conformal frame, does not arise. In this sense, the problem of whether the two formulations of a scalar-tensor theory in the two conformal frames are equivalent or not is not yet settled, and often is the source of confusion in the technical literature [40, 42, 43].

Here, in order to compare both frames, we have also calculated the JNW’s coefficients and deflection angle (see (30) and (31)) in terms of \( \zeta \) as shown in Figs. 4 and 3. In Fig. 4 we plot \( \bar{\alpha}, \bar{b}, \mu_m \) for JNW’s metric as a function of \( \zeta \) by making \( \gamma = \sqrt{1 - \zeta^2} \). It can be seen that the behavior of \( \mu_m \) and \( \bar{\alpha} \) are the same in each frame, when we compare with Fig. 2. Nevertheless, the behavior of \( \bar{b} \) is quite different for each frame (see once again Fig. 2). While \( \bar{b} \) for Kar has a minimum at \( \zeta = 0 \), \( \bar{b} \) for JNW’s metric has a minimum near 0.2. This difference between Kar and JNW’s metrics may arise because equations for \( \bar{b} \) in each
Fig. 4 Behavior of the expansion parameters, $u_m$, $\bar{a}$, and $\bar{b}$ for JNW’s metric in terms of $\zeta$. The parameters with a ‘s’ subscript are the values for Schwarzschild metric are not smooth at $\zeta = \frac{\sqrt{3}}{2}$ (see (12)) and $\gamma = 0.5$ respectively in agreement with the ideas presented in [30]. On the other hand, the photon sphere equation in terms of $\zeta$ for Kar’s metric is (21)

$$x_m = \sqrt{1 - \zeta^2 + \frac{1}{2}}.$$  

(33)

At a first glance (27) and (21) seems to be different; however it is possible to obtain one from the other by using $\gamma^2 + \zeta^2 = 1$. Note that for $\zeta = 0$ ($\gamma = 1$) both reduce to $x_m = \frac{3}{2}$, which is the photon sphere for Schwarzschild, and for $\zeta = \frac{\sqrt{3}}{2}$ ($\gamma = \frac{1}{2}$) both diverge.

Finally in Fig. 3, we plot the deflection angle for JNW’s and Kar’s metrics. Although, both frames reduce to Schwarzschild for $\zeta = 0$ and diverge for $\zeta = \frac{\sqrt{3}}{2}$, the behavior of the deflection angle is very different in each frame.

4.1 Magnification

In Ref. [12] is defined that the magnification of an image formed due to Gravitational lens (GL) is the ratio of the flux of the image to the flux of the unlensed source. Nevertheless, according to Liouville’s theorem, the surface brightness is preserved in GL. Therefore, the magnification $\mu$ of an image turns out to be the ratio of the solid angles of the image and of the unlensed source made at the observer. In this sense, for a circularly symmetric gravitational lensing, the magnification of the images can be expressed as [12]

$$\mu = \mu_r \mu_t,$$  

(34)

where

$$\mu_t = \left(\frac{\sin \delta}{\sin \theta}\right)^{-1}, \quad \mu_r = \left(\frac{d\delta}{d\theta}\right)^{-1},$$  

(35)

where $\mu_r$ and $\mu_t$ are the radial and tangential magnification, respectively.

Tangential critical curves (TCCs) and radial critical curves (RCCs) are, respectively, given by singularities in $\mu_t$ and $\mu_r$ in the lens plane. However, their corresponding values in the source plane are, respectively, termed tangential caustic (TC) and radial caustics (RCs). The parity of an image is called positive if $\mu > 0$ and negative if $\mu < 0$. If the angular source position $\delta = 0$ (i.e., when the source, the lens, and the observer are aligned),
there may be ring shaped image(s), which are called Einstein ring(s) [12]. The tangential magnification is defined as [8]

\[ \mu_{t}^{n} = \left( \frac{\delta}{\theta_{n}^{0}} \right)^{-1}, \tag{36} \]

where

\[ \theta_{n}^{0} = \frac{\mu_{m}}{D_{OL}} \left( 1 + e^{\delta - 2n\pi} \right). \tag{37} \]

To study the behavior of the magnification as a function of \( \delta \) (the source’s position), and compare the behavior for Kar’s and JNW’s metric, we have chosen for this analysis, as the lens, the galactic “black hole” with mass \( M = 2.8 \times 10^{6} M_{\odot} \) and \( D_{OL} = 8.5kpc \) so that \( \frac{M}{D_{OL}} \approx 1.57 \times 10^{-11} \) (geometrized units Cfr. [12]). Therefore, using (23), (37) and (52) we obtain that

\[ \mu_{t}^{n} = \frac{3\sqrt{3}}{2 \times 6.37 \times 10^{10} \delta} \left( 1 + e^{2 \ln(6(2-\sqrt{3})) - 0.226\zeta - \pi + \ln\left[ \frac{2\mu_{m}}{\gamma_{m}} \right] - 2n\pi} \right). \tag{38} \]

In a similar way, the tangential magnification for JNW’s metric, using (31) and (36), is

\[ \mu_{t}^{n} = \frac{3\sqrt{3}}{2 \times 6.37 \times 10^{10} \delta} \left( 1 + e^{2 \ln(6(2-\sqrt{3})) - 0.1199(\gamma - 1) - \pi + \ln\left( \frac{(2\gamma + 1)(2\gamma + 1)^{\gamma} - (2\gamma - 1)^{\gamma}}{2\gamma(2\gamma - 1)^{\gamma - 1}} \right) - 2n\pi} \right). \tag{39} \]

In Fig. 5 we plot the tangential magnification as function of \( \delta \) for \( \zeta \approx \sqrt{3}/2 \) and \( n = 1 \) for the two metrics. We have chosen that value of \( \zeta \) in order to enhance the difference, since at the photon sphere the smoothness of the parameters is lost.

5 Discussion

According to the authors in Ref. [30] “there is no doubt of the equivalence of all frames for the description of the gravitational effects of the string theories at a basic level, at least when all the functions involved are smooth”. This final statement opens the possibility to compare both frames from the strong lensing point of view, since the deflection angle diverges near the photon sphere and makes possible not only to differentiate both frames,
but also to contribute, in some way, to the discussion about the equivalence between EF and JF.

The controversy on conformal frames could appear a purely technical one [42, 43]. However, in this paper we have shown explicitly, at least at first order, that there is a difference in the deflection angle between Jordan and Einstein frames, even when some parameters, i.e., $a$, $u_m$ and $x_m$, of the strong field expansion in each frame are the same. Nevertheless, considering the results presented in Ref. [30], the discrepancies arise due to the strong field expansion, which is not smooth near the photon sphere. Our results support the idea that direct evidence from observations, in this case from gravitational lensing, could distinguish, if there is, a physical difference between the frames. However, it is definitively challenging to reach the required relativistic images, see Fig. 5, to discriminate the frames in the near future, even with new second-generation Very Large Telescope Interferometer (VLTI), as GRAVITY [45, 46] for instance.

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Appendix A: Finding $\hat{\alpha}(\theta)$ for Kar’s Metric

In order to calculate the deflection angle in the strong field limit, we used Kar’s metric in the form proposed in (17), i.e.,

$$ds^2_{str} = \left(1 - \frac{1}{x} \right)^{\zeta + \sqrt{1 - \zeta^2}} A(x) dt^2 - \left(1 - \frac{1}{x} \right)^{\zeta - \sqrt{1 - \zeta^2}} B(x) dx^2 - \left(1 - \frac{1}{x} \right)^{1 + \zeta - \sqrt{1 - \zeta^2}} C(x) \frac{d\Omega^2}{x^2}. \tag{40}$$

The deflection angle is calculated using the strong field expansion [8, 19]

$$\hat{\alpha}(\theta) = \bar{a} \ln \left( \frac{\theta D_{OL}}{u_m} - 1 \right) + \bar{b}, \tag{41}$$

where

$$\bar{a} = \frac{R(0, x_m)}{2 \sqrt{\beta_m}},$$

$$\bar{b} = b_R + \bar{a} \ln \left[ \frac{2 \beta_m}{y_m} \right] - \pi \tag{42}$$

and

$$u_m = \sqrt{\frac{C(x_m)}{A(x_m)}}. \tag{43}$$
Calculation of $\bar{a}$:

Using (5), (6), (9), (21) and (40) we obtain that

$$R(z, x_m) = \sqrt[1]{} \frac{2\sqrt{1 - \zeta^2 + 1}}{\sqrt[1]{} \zeta^2 + \zeta} \left\{ \frac{2\sqrt{1 - \zeta^2 - 1}}{2\sqrt{1 - \zeta^2 + 1}} \right\}^{1+\zeta - \sqrt{1 - \zeta^2}} - \left\{ \frac{2\sqrt{1 - \zeta^2 - 1}}{2\sqrt{1 - \zeta^2 + 1}} \right\}^{3\zeta - \sqrt{1 - \zeta^2 + 1}} \frac{\zeta}{\left( 1 - y_m \right) z + y_m \sqrt[1]{} \zeta^2 + \zeta} \right\} \right\} .$$

(Equation 44)

Evaluating the last expression at $z = 0$ we obtain

$$R(0, x_m) = \frac{2\sqrt{1 - \zeta^2 + 1}}{\sqrt[1]{} \zeta^2 + \zeta} \left\{ \frac{2\sqrt{1 - \zeta^2 - 1}}{2\sqrt{1 - \zeta^2 + 1}} \right\}^{1+\zeta - \sqrt{1 - \zeta^2}} - \left\{ \frac{2\sqrt{1 - \zeta^2 - 1}}{2\sqrt{1 - \zeta^2 + 1}} \right\}^{3\zeta - \sqrt{1 - \zeta^2 + 1}} \frac{\zeta}{y_m \sqrt[1]{} \zeta^2 + \zeta} \right\} \right\} \right\} \right\} .$$

(Equation 45)

where

$$y_m = \left( \frac{2\sqrt{1 - \zeta - 1}}{2\sqrt{1 - \zeta^2 + 1}} \right) \sqrt[1]{} \zeta^2 + \zeta \right\} .$$

(Equation 46)

From (9) we obtain that

$$\beta_m = \frac{1}{4} \left\{ \frac{2\sqrt{1 - \zeta^2 + 1}}{\sqrt[1]{} \zeta^2 + \zeta} \right\}^{1+\zeta - \sqrt{1 - \zeta^2}} - \left\{ \frac{2\sqrt{1 - \zeta^2 - 1}}{2\sqrt{1 - \zeta^2 + 1}} \right\}^{3\zeta - \sqrt{1 - \zeta^2 + 1}} \frac{\zeta}{y_m \sqrt[1]{} \zeta^2 + \zeta} \right\} \right\} .$$

(Equation 47)

and taking into account (21) it is possible to express $R(0, x_m)$ and $\beta_m$ and terms of the photon sphere as,

$$R(0, x_m) = \frac{2x_m}{k} \left( \frac{1 - \frac{1}{x_m}}{\frac{1}{\frac{1}{x_m}}} - \left( 1 - \frac{1}{x_m} \right) \right) .$$

(Equation 48)

$$\beta_m = \frac{(x_m^k - (x_m - 1)^k)^2}{k^2(x_m - 1)^{k-1}x_m} \right\} \right\} .$$

(Equation 49)

then, from (12)

$$\bar{a} = \frac{x_m[(x_m^{k-1} - (x_m - 1)^k)x_m - 1]}{x_m^k - (x_m - 1)^k} = 1 .$$

(Equation 50)

Calculation of $u_m$:

From (13) the impact parameter is calculated as,

$$u_m = \sqrt{\frac{C(x_m)}{A(x_m)}} .$$

(Equation 51)

then, using (40), (21) and taking into account that $A(x_m) = y_m$

$$u_m = \frac{(x_m - 1)\frac{1}{\frac{1}{x_m}} + \sqrt{1 - \zeta^2}}{x_m - \frac{1}{\frac{1}{x_m}} - \sqrt{1 - \zeta^2}}} = \frac{1}{2} (2\sqrt{1 - \zeta^2 - 1}) \frac{\frac{1}{\frac{1}{x_m}} - \sqrt{1 - \zeta^2}}{2(2\sqrt{1 - \zeta^2 + 1})} .$$

(Equation 52)
Calculation of $b_R$:

The term $b_R$ is calculated by (14), which corresponds to the regular part of the integral (10), i.e.,

$$b_R = I_R(x_m) = \int_0^1 g(z, x_m) dz = \int_0^1 [R(z, x_m) f(z, x_m) - R(0, x_m) f_0(z, x_m)] dz. \quad (53)$$

However, as is pointed out in Ref. [8], $b_R$ can not be calculated analytically but by an expansion of $I_R(x_m)$ in powers of some metric's parameter. In this sense, to calculate the regular term for Kar’s metric (17), we made an expansion of $I_R(x_m)$ in powers of $\zeta$ around $\zeta = 0$ and used the first order expansion as our $b_R$. Mathematically this idea is

$$b_R = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n I_R(x_m)}{d\zeta^n} (\zeta - 0)^n \quad (54)$$

and taking terms up to first order in $\zeta$ we have that

$$I_R(x_m) = I_R(x_m)_{\zeta=0} + \left[ \frac{d I_R(x_m)}{d\zeta} \right]_{\zeta=0} \zeta. \quad (55)$$

In this sense, in order to calculate the regular term $b_R$, it is necessary to find the functional form of $R(z, x_m)$, $f(z, x_m)$, $R(0, x_m)$ and $f_0(z, x_m)$. The form of $R(z, x_m)$ and $R(0, x_m)$ are already shown in (44) and (45). From (6), (7) and (22) we have that

$$f(z, x_m) = \frac{1}{\sqrt{y_m - [(1 - y_m)z + y_m] C_m}} \quad (56)$$

and

$$f_0(z, x_m) = \frac{1}{\sqrt{p_m |z|}} = \frac{2(\sqrt{1 - \zeta^2} + \zeta)(3 - 4\zeta^2) \sqrt{1-\zeta^2+\zeta-1} \zeta}{(2\sqrt{1 - \zeta^2} + 1)\sqrt{1-\zeta^2} - (2\sqrt{1 - \zeta^2} - 1)\sqrt{1-\zeta^2} |z|}, \quad (57)$$

where

$$C_m = \left[ \frac{2\sqrt{1 - \zeta^2} + 1}{2} \right]^2 \left[ \frac{2\sqrt{1 - \zeta^2} - 1}{2\sqrt{1 - \zeta^2} + 1} \right]^{1+\zeta-\sqrt{1-\zeta^2}}. \quad (58)$$

$$C = \frac{[(1 - y_m)z + y_m]^{1+\zeta-\sqrt{1-\zeta^2}}}{[1 - [(1 - y_m)z + y_m]^{\zeta+\sqrt{1-\zeta^2}}]^{\zeta+\sqrt{1-\zeta^2}}} \quad (59)$$

As (55) shows, the value of $I_R(x_m)$ reduces to that of Schwarzschild for $\zeta = 0$. Therefore, for $0 \leq z \leq 1$ ($|z| = z$) we obtain

---

3 For $\zeta = 0$ all the expressions reduce to those of Schwarzschild.

4 The symbol $\left[ \frac{d I_R(x_m)}{d\zeta} \right]_{\zeta=0}$ means the derivative evaluated at $\zeta = 0$. 

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\[ I_R(x_m)_{\zeta=0} = 2 \int_0^1 \left[ \frac{1}{|z| \sqrt{1 - \frac{3}{2} z}} - \frac{1}{|z|} \right] dz = 2 \ln 6(2 - \sqrt{3}) = 0.9496. \quad (60) \]

On the other hand,

\[
\left[ \frac{dI_R(x_m)}{d\zeta} \right]_{\zeta=0} = \int_0^1 \left[ \frac{d}{d\zeta} \left( R(z, x_m) f(z, x_m) \right) - \frac{d}{d\zeta} \left( R(0, x_m) f_0(z, x_m) \right) \right]_{\zeta=0} dz
\]

\[
= \left[ f(z, x_m) \frac{d}{d\zeta} R(z, x_m) + R(z, x_m) \frac{d}{d\zeta} f(z, x_m) \right. \\
\left. - f_0(z, x_m) \frac{d}{d\zeta} R(0, x_m) - R(0, x_m) \frac{d}{d\zeta} f_0(z, x_m) \right]_{\zeta=0} dz
\]

\[
= \int_0^1 \left[ f_S(z, x_m) \left[ \frac{d}{d\zeta} R(z, x_m) \right]_{\zeta=0} + 2 \left[ \frac{d}{d\zeta} f(z, x_m) \right]_{\zeta=0} \right. \\
\left. - f_0S(z, x_m) \frac{d}{d\zeta} R(0, x_m) - 2 \left[ \frac{d}{d\zeta} f_0(z, x_m) \right]_{\zeta=0} \right] dz,
\]

where \( f_S(z, x_m), f_0S(z, x_m) \) are those of Schwarzschild (Cfr. [8]). Therefore,

\[
\left[ \frac{dI_R(x_m)}{d\zeta} \right]_{\zeta=0} = \int_0^1 \left[ \frac{d}{d\zeta} R(z, x_m) \right]_{\zeta=0} \left[ \frac{d}{d\zeta} R(0, x_m) \right]_{\zeta=0} \left[ \frac{d}{d\zeta} f(z, x_m) \right]_{\zeta=0} \left[ \frac{d}{d\zeta} f_0(z, x_m) \right]_{\zeta=0} dz,
\]

where all derivatives, evaluated at \( \zeta = 0 \), are:

\[
\left[ \frac{d}{d\zeta} R(z, x_m) \right]_{\zeta=0} = -2 - 2 \ln \left( \frac{2}{3} z + \frac{1}{3} \right)
\]

\[
\left[ \frac{d}{d\zeta} R(0, x_m) \right]_{\zeta=0} = -2 + 2 \ln(3)
\]

\[
\left[ \frac{d}{d\zeta} f_0(z, x_m) \right]_{\zeta=0} = \frac{\ln(3) - 1}{|z|}
\]

\[
\left[ \frac{d}{d\zeta} f(z, x_m) \right]_{\zeta=0} = -\frac{1}{2} \ln \left( \frac{7}{3} z^3 - 2 z^2 \right) + \frac{z(2z + 1)(1 - z) \ln(2z + 1)}{z^3 (1 - \frac{2}{3} z)^{\frac{3}{2}}}. \quad (61)
\]
Thus,
\[
\left[ \frac{dI_R(x_m)}{d\zeta} \right]_{\zeta=0} = \int_0^1 \left[ \frac{2 - 2 \ln(3)}{z} - \frac{2 + 2 \ln \left( \frac{2}{3} z + \frac{1}{3} \right)}{z \sqrt{1 - \frac{2}{3} z}} \right] dz \\
+ \int_0^1 \left[ \frac{\ln(3) \left[ \frac{2}{3} z^2 - 2 z^2 \right] + z(2z + 1)(1 - z) \ln(2z + 1)}{z^3(1 - \frac{2}{3} z)^2} + \frac{2 \ln(3) - 2}{z} \right] dz \\
= \int_0^1 \left[ \frac{2 - 2 \ln(3)}{z} - \frac{2 + 2 \ln \left( \frac{2}{3} z + \frac{1}{3} \right)}{z \sqrt{1 - \frac{2}{3} z}} \right] dz \\
+ \int_0^1 \left[ \frac{-2 \ln(3) z^2 + z(2z + 1)(1 - z) \ln(2z + 1)}{z^3(1 - \frac{2}{3} z)^2} + \frac{2 \ln(3) - 2}{z} \right] dz \\
+ \frac{7 \ln(3)}{3} \int_0^1 \frac{dz}{\left( 1 - \frac{2}{3} z \right)^\frac{3}{2}}.
\]

(62)

The latter integrals were calculated numerically as
\[
\int_0^1 \left[ \frac{-2 \ln(3) z^2 + z(2z + 1)(1 - z) \ln(2z + 1)}{z^3(1 - \frac{2}{3} z)^2} + \frac{2 \ln(3) - 2}{z} \right] dz = -2.3980,
\]
\[
\int_0^1 \left[ \frac{2 - 2 \ln(3)}{z} - \frac{2 + 2 \ln \left( \frac{2}{3} z + \frac{1}{3} \right)}{z \sqrt{1 - \frac{2}{3} z}} \right] dz = -3.457723875,
\]
\[
\frac{7 \ln(3)}{3} \int_0^1 \frac{dz}{\left( 1 - \frac{2}{3} z \right)^\frac{3}{2}} = \frac{14 \sqrt{3} \ln(3)}{(3 + \sqrt{3})}.
\]

(63)

Therefore, \( b_R \) for Kar’s metric is
\[
b_R = 2 \ln(6(2 - \sqrt{3})) - 0.226\zeta.
\]

(64)

Finally, the deflection angle for Kar’s metric is
\[
\hat{\alpha} = -\ln \left[ \frac{\theta D_{OL}}{u_m} - 1 \right] + 2 \ln(6(2 - \sqrt{3})) - 0.226\zeta - \pi + \ln \left[ \frac{2 \beta_m}{y_m} \right],
\]

(65)

where \( u_m, \beta_m \) and \( y_m \) are given by (52), (22) and (46) respectively.

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