Common origin of $\theta_{13}$ and dark matter within the flavor symmetric scoto-seesaw framework

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ABSTRACT: To understand the observed pattern of neutrino masses and mixing as well as to account for the dark matter we propose a hybrid scoto-seesaw model based on the $A_4$ discrete flavor symmetry. In this setup, including at least two heavy right-handed neutrinos is essential to employ the discrete flavor symmetry that mimics once popular tribimaximal neutrino mixing at the leading order via type-I seesaw. The scotogenic contribution then acts as a critical deviation to reproduce the observed value of the reactor mixing angle $\theta_{13}$ (within the trimaximal mixing scheme) and to accommodate potential dark matter candidates, pointing towards a common origin of $\theta_{13}$ and dark matter. The model predicts the atmospheric angle to be in the upper octant, excludes some regions on the Dirac CP phase, and restricts the Majorana phases too. Further, normal and inverted mass hierarchies can be distinguished for specific values of the relative phases associated with the complex light neutrino mass matrix. Owing to the considered flavor symmetry, contributions coming from the scotogenic mechanism towards the lepton flavor violating decays such as $\mu \to e\gamma$, $\tau \to e\gamma$ vanish, and a lower limit on the second right-handed neutrino mass can be obtained. Prediction for the effective mass parameter appearing in the neutrinoless double beta decay falls within the sensitivity of future experiments such as LEGEND-1k and nEXO.

KEYWORDS: Flavour Symmetries, Theories of Flavour, CP Violation

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1 Introduction

The discovery of neutrino oscillation [1–5] suggests that at least two neutrinos are massive but having very small masses with respect to the charged leptons and quarks. This situation opens up a window for interpretations which go beyond the Standard Model (SM) of particle physics. On the other hand, it is also well established that the neutrino flavor mixing is significantly large compared to the quark mixing which also demands an extension of the SM (either by particle content or symmetry extension). Another extraordinary problem in particle physics as of today is the nature of dark matter (DM), whose relic abundance is precisely measured by the WMAP [6] and PLANCK [7] satellite experiments and the existence of such DM is strongly supported by the gravitational lensing, galactic rotation curve and large scale structure of the Universe [8] as well. However, the SM of particle physics fails to provide an appropriate candidate for DM. Now, from a theoretical perspective, the origin of tiny neutrino masses can be well understood within the framework of various seesaw mechanisms. The simplest one is the type-I seesaw mechanism [9–13] where usually three singlet right-handed neutrinos are added to the SM; then three left-handed neutrinos can be massive. This “complete” sequential seesaw scenario is denoted by (#ν_L,#N_R)=(3,3) where #ν_L, #N_R denote the number of generations of left-handed and
right-handed neutrinos respectively. However, as first pointed out in [13], the number of right-handed neutrinos added to the SM is not fixed as they do not carry any anomaly [14]. A simpler version of the type-I seesaw is the minimal seesaw which further reduces the number of free parameters [15–17]. In the (3,2) seesaw mechanism only two right-handed neutrinos are introduced to obtain viable neutrino masses. However, the flavor structure of the relevant lepton mass matrices still remains undetermined.

The flavor structures of the lepton mass matrices and hence the observed non-trivial pattern of the lepton flavor mixing can be examined by incorporating non-Abelian discrete flavor symmetries to the SM. For this purpose discrete symmetry groups such as $A_4, S_4, A_5, \Delta(27)$ are often used [18]. For a general overview and implementation of discrete symmetries in neutrino physics see [19–25]. Such discrete flavor symmetries can explain various fixed mixing schemes such as bi-maximal (BM), golden ratio (GR) and hexagonal (HG) [26–30] mixings, with the most popular one being the tri-bimaximal (TBM) mixing [31, 32]. In the TBM mixing scheme the solar and atmospheric mixing angles take values $\sin^2 \theta_{12} = 1/3$ and $\sin^2 \theta_{23} = 1/2$ respectively, whereas the reactor mixing angle is fixed at $\sin^2 \theta_{13} = 0$. Therefore, a discrete flavor symmetric construction with two right-handed neutrinos is a very economical scenario to explain neutrino masses and TBM mixing simultaneously [33–35]. Neutrino model building with two right-handed neutrinos in various limiting cases can be found in [36]. Among various discrete groups employed for this purpose, $A_4$ is the most popular one [37–42]. This symmetry was initially proposed as an underlying family symmetry for the quark sector. It is a discrete group of even permutations of four objects with three inequivalent one-dimensional representations (1, 1′, and 1″) and a three-dimensional representation (3). Interestingly, the three generations (or flavors) of right-handed charged lepton singlets can fit into three inequivalent one-dimensional representations. Conversely, $A_4$ is the smallest group with a three-dimensional irreducible representation. Then three SM lepton doublets can transform together as a triplet under $A_4$ [39–41]. So far, so good; however, in the last decade, the reactor neutrino mixing angle $\theta_{13}$ is decisively measured [43–47] to be adequately large ($\sim 9^\circ$) and hence the era of fixed patterns (such as BM, TBM, GR, HG) of the lepton mixing matrix is over. To generate non-zero $\theta_{13}$ various approaches are considered either by additional contributions to the neutrino sector or considering additional corrections from the charged lepton sector or including corrections to vacuum alignments of the flavons etc. [48–93]. As consequence various descendants of the fixed mixing schemes have emerged. For example, even if the TBM mixing is obsolete now, two of its successors are still compatible with data. These mixing schemes are known as trimaximal mixing (TM) mixings [30, 62, 94, 95] which preserves the first (second) column of the TBM mixing matrix and are called TM1 (TM2) mixing, respectively.

In this work, we consider a scotogenic contribution [96] to the underlying TBM mixing scheme establishing a common origin of the nonzero $\theta_{13}$ and cosmological DM. Many radiative models account for the tiny neutrino masses [97–101] and perhaps the simplest one is the scotogenic model which also naturally accommodates potential candidate for dark matter [96]. In the present article, we explore the idea of combining seesaw and scotogenic (termed as scoto-seesaw) model [102] to explain neutrino mass and mixing in a consistent way with well-motivated $A_4$ discrete flavor symmetry. In our proposal, the nature of cos-
mological dark matter and reactor mixing angle $\theta_{13}$ share a unified origin. Our full model comprises of (3,2) seesaw and then combines it with the scotogenic mechanism, which predicts three neutrinos to be massive. The Yukawa structure of the scotogenic contribution is formulated by adding flavon fields which transform non-trivially under the flavor symmetry. With this, we show that our model can successfully explain the lepton mixing with non-zero reactor angle $\theta_{13}$ and includes leptonic CP violation. Due to the flavor symmetric construction, the model is extremely predictive in nature and offers many interesting results involving neutrino mass hierarchy, octant of the atmospheric mixing angle $\theta_{23}$ and restricts the Dirac CP phase. In addition, we constrain the absolute neutrino masses and Majorana CP phases. On the other hand, the model can be falsified by null results of future neutrinoless double beta decay experiments. Interestingly, due to the specific flavor structure of the Yukawa couplings as a consequence of the flavor symmetry, the scotogenic part does not contribute in the lepton flavor violating decays such as $\mu \rightarrow e\gamma$ and only right-handed neutrinos contribute in such decays. We begin with a minimal type-I seesaw assisted by the $A_4$ discrete flavor symmetry, which helps reproduce the TBM mixing. The model also contains additional $Z_N$ symmetries to forbid unwanted contributions in the lepton sector, and an inherent $Z_2$ symmetry also ensures the stability of the dark matter. Thanks to the considered symmetry, the charged lepton mass matrix as well as the heavy right-handed Majorana neutrino mass matrix, are diagonal to start with. Therefore, the structure of the $3 \times 2$ Dirac Yukawa matrix turns out to be solely responsible for generating the TBM mixing with two right-handed neutrinos. Now, the inclusion of the scotogenic contribution to the neutrino mass helps in reproducing the TM2 mixing and generates the observed value of the reactor mixing angle $\theta_{13}$. It also naturally incorporates dark matter candidates (three potential dark matter candidates, such as the dark fermion and real and imaginary components of the scalar field involved in the scotogenic contribution) into the picture.

The rest of the paper is organized as follows. In section 2 we first describe the minimal scoto-seesaw model. Then in section 3 we present the $A_4$ flavor symmetric scoto-seesaw model with two right-handed neutrinos and describe the construction of the model based on the symmetries of the framework. In section 4 we present the correlation among the parameters involved in our analysis. We carry out the complete analysis for various limiting and general cases and present their predictions in section 5. Then in section 6 we mention various phenomenological implications of undertaken analysis and finally conclude in section 7. We also included in the appendix a short note on $A_4$ multiplication rules used in our analysis.

2 Minimal scoto-seesaw model

In ref. [102], the (3,1) scenario of seesaw mechanism [12, 13, 103] and the scotogenic model [96] are combined to propose a minimal scoto-seesaw model. This model consists of only one right-handed neutrino $N_R$, one singlet fermion $f$, and one extra scalar doublet $\eta$. In addition to these particles, one $Z_2$ symmetry is introduced, which is responsible for the stability of the dark matter. All the standard model fields, $N_R$, are even under the $Z_2$ symmetry, while the dark sector consists of one fermion $f$ and scalar field $\eta$ which are odd under
This proposal generates the atmospheric neutrino mass scale at the tree level with the conventional \((3,1)\) seesaw term with \(N_R\), and the solar neutrino mass scale is generated at a one-loop level, as a result, the hierarchy between the atmospheric and solar scale is maintained.\(^1\) With this field content, the lepton Yukawa and mass terms can be written as

\[
\mathcal{L} = -Y_N^k \bar{L}^k i\sigma_2 H^* N_R + \frac{1}{2} M_R \bar{N}_R N_R + Y_f^k \bar{L}^k i\sigma_2 \eta^* f + \frac{1}{2} M_f \bar{f} f + h.c.,
\]

(2.1)

where \(L^k\) are the lepton doublets. The scalars \(H = (H^+, H^0)^T\) and \(\eta = (\eta^+, \eta^0)\) are the SU(2) doublets. \(Y_N\) and \(Y_f\) are complex \(3 \times 1\) Yukawa coupling matrices, and \(M_{R,f}\) are the mass matrices for \(N_R\) and \(f\). The total neutrino mass reads\(^{102}\)

\[
M_{\nu}^{ij} = -\frac{\nu^2}{M_N} Y_N^i Y_N^j + \mathcal{F}(m_{\eta R}, m_{\eta I}, M_f) Y_f^i Y_f^j.
\]

(2.2)

Here the first term is due to the tree-level seesaw mechanism while the second term originates from the scotogenic correction with

\[
\mathcal{F}(m_{\eta R}, m_{\eta I}, M_f) = \frac{1}{32\pi^2} \left[ \frac{m_{\eta R}^2 \log \left( \frac{M_f^2}{m_{\eta R}^2} \right)}{M_f^2 - m_{\eta R}^2} - \frac{m_{\eta I}^2 \log \left( \frac{M_f^2}{m_{\eta I}^2} \right)}{M_f^2 - m_{\eta I}^2} \right],
\]

(2.3)

where \(m_{\eta R}\) and \(m_{\eta I}\) are the masses of the neutral component of \(\eta\)\(^{108}\). However, this model predicts one massless neutrino and demands extension to explain all the neutrino oscillation data. Hence, to understand the observed pattern of neutrino masses and mixing, in the next section we will present a modified scoto-seesaw model where two right-handed neutrinos and \(A_4\) discrete flavor symmetry will reproduce non-zero \(\theta_{13}\) mixing angle and will account for the dark matter content. In ref. \(^{109}\), the authors already discussed the model with two right-handed neutrinos in the context of the scoto-seesaw scenario but with the \(Z_8\) symmetry. As we will see, the set-up based on the \(A_4\) flavor symmetry is substantially different in construction from the \(Z_8\) scenario, resulting in a completely different texture of the Yukawa couplings and mass matrices. As mentioned earlier, the \(A_4\) flavor symmetry is well motivated in reproducing the TBM mixing scheme and such symmetry can arise in various ways, such as starting from a continuous group\(^{110–115}\) or superstring theory in compactified extra dimensions\(^{40, 41, 116–125}\). We will show that due to the presence of the \(A_4\) symmetry with diagonal structure of the charged leptons and heavy right-handed neutrinos, the TBM mixing can be generated at the leading order in the context of the minimal type-I seesaw. Subsequently, the scotogenic contribution acts as a crucial deviation from TBM mixing to generate non-zero \(\theta_{13}\) (reproducing the TM2 mixing scheme) as well as providing essential dark matter candidates, thus unifying the origin of \(\theta_{13}\) and dark matter. The \(A_4\) symmetry assists us to obtain analytic expressions for neutrino masses and mixing angles as well as yields interesting correlations among the

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\(^{1}\)Earlier such hierarchy of the atmospheric and solar neutrino mass scales (and associated mixing) was explained with a type-I seesaw mechanism where the right-handed neutrinos contribute hierarchically and implemented within the frameworks of sequential dominance\(^ {16, 17, 36, 104–106}\) and constrained sequential dominance\(^ {59, 107}\).
oscillation parameters with distinctive predictions; see section 4. In [109], the CP symmetry is spontaneously broken by the complex vacuum expectation value of the singlet field whereas in our analysis the source of CP violation is due to the complex couplings and relative values of CP phases determine the hierarchy of the masses.

3 Scoto-seesaw with flavor $A_4$ symmetry: the FSS model

The model which we propose is a hybrid scoto-seesaw framework with usual scotogenic fermion $f$ and scalar doublet $\eta$, supported additionally by the $A_4$ discrete flavor symmetry and two right-handed neutrinos $N_{R_1,2}$. To obtain the flavor structure of the Yukawa couplings the flavons $\phi_s$, $\phi_a$, $\phi_T$, $\xi$ are introduced. The inclusion of flavon fields (SM gauge singlets) is a characteristic feature of models with discrete flavor symmetries [19–25]. In a similar manner, we also incorporate additional $Z_N$ discrete symmetries which forbid the exchange of flavon fields eliminating unwanted terms [20–25, 126–128]. In what follows, we will call the whole framework the Flavor-Scoto-Seesaw (FSS) model where each element of the FSS model’s construction is well motivated towards understanding a common origin of $\theta_{13}$ and DM. As we employ the $A_4$ discrete symmetry, compared to ref. [109] and the $Z_8$ choice, we have a fuller symmetry with larger particle content. This is the price we pay to predict the $TM_2$ structure of neutrino masses and mixing. The role of each of $Z_N$ auxiliary symmetries will be explained in detail as we proceed. Interestingly, the model contains an intrinsic $Z_2$ symmetry under which both $f$ and $\eta$ are odd. The stability of the dark matter is ensured by this $Z_2$ symmetry. In table 1, we present transformation properties of all the fields content of our model under the complete discrete $A_4$ flavor symmetry. The desired mass matrix structure will be obtained when the flavons get a vacuum expectation value (VEV) in a suitable direction. The VEV alignment considered here is widely used [35, 41, 107] and can be realized in a natural way by analyzing the complete scalar potential [20, 74, 107, 129, 130]. Here, the low energy scalar potential is identical to the potential presented in [102], and for brevity we omit it here. With the fields content of table 1, the charged lepton Lagrangian can be described by

$$\mathcal{L}_l = \frac{y_e}{\Lambda} (L\phi_T) H e_R + \frac{y_\mu}{\Lambda} (L\phi_T) H \mu_R + \frac{y_\tau}{\Lambda} (L\phi_T) H \tau_R + h.c., \quad (3.1)$$

to the leading order, where $\Lambda$ is the cut-off scale of our model and $y_e$, $y_\mu$ and $y_\tau$ are coupling constants. As the SM lepton doublet, $L$ transforms as a triplet under $A_4$, the

| Fields | $e_{R_1}, \mu_{R_1}, \tau_{R_1}$ | $L_\alpha$ | $H$ | $N_{R_1}$ | $N_{R_2}$ | $f$ | $\eta$ | $\phi_s$ | $\phi_a$ | $\phi_T$ | $\xi$ |
|--------|---------------------------------|------------|-----|-----------|-----------|-----|-------|--------|--------|--------|------|
| $A_4$  | $1, 1''$ | $1$        | $1$ | $1$       | $1$       | $1$ | $1$   | $3$    | $3$    | $3$    | $1''$|
| $Z_4$  | $-i$   | $-i$       | $1$ | $-1$      | $1$       | $1$ | $1$   | $i$    | $-i$   | $1$    | $-1$ |
| $Z_3$  | $\omega$ | $\omega$ | $\omega^2$ | $1$       | $1$       | $\omega^2$ | $1$ | $1$    | $-1$   | $-1$   | $1$    |
| $Z_2$  | $-1$    | $1$        | $1$ | $-1$      | $-1$      | $1$ | $1$   | $1$    | $-1$   | $-1$   | $-1$ |
involvement of another triplet is essential as seen in the above Lagrangian. Terms in the first parenthesis of eq. (3.1) represent the product of two $A_4$ triplets which results a true singlet after contracting with $A_4$ singlets $e_R$, $\mu_R$ and $\tau_R$ (charged as $1$, $1''$ and $1'$ respectively). The multiplication rule of $A_4$ symmetry is summarized in the appendix and a detailed discussion on $A_4$ symmetry can be found in [19, 20]. Now, when the flavon $\phi_T$ gets VEV in the direction $\langle \phi_T \rangle = (v_T, 0, 0)^T$ [41] and also the Higgs field gets VEV $\langle H \rangle = v$, the charged lepton mass matrix will be a diagonal form

\[
M_i = \frac{v_T}{\Lambda} v \begin{pmatrix}
y_e & 0 & 0 \\
0 & y_\mu & 0 \\
0 & 0 & y_\tau
\end{pmatrix}.
\] (3.2)

The Lagrangian in the neutrino sector constitutes two parts: a type-I seesaw contribution with two right-handed neutrinos $N_{R1}$ and $N_{R2}$ and another is a scotogenic contribution with a scalar field $\eta$ and a fermionic field $f$. The Lagrangian that generates neutrino mass at a tree level by the type-I seesaw mechanism in our model can be written as

\[
\mathcal{L}_N = \frac{y_{N1}}{\Lambda} (\bar{L}\phi_s) \tilde{H} N_{R1} + \frac{y_{N2}}{\Lambda} (\bar{L}\phi_a) \tilde{H} N_{R2} + \frac{1}{2} M_{N1} \tilde{N}_{R1} N_{R1} + \frac{1}{2} M_{N2} \tilde{N}_{R2} N_{R2} + h.c.,
\] (3.3)

where $y_{N1,2}$ are the corresponding couplings and $M_{N1,2}$ are the Majorana masses of right-handed neutrinos. To get the flavor structure, we assume that the flavon fields get VEVs along $\langle \phi_s \rangle = (0, v_s, -v_s)$, $\langle \phi_a \rangle = (v_a, v_a, v_a)$ [59, 107]. With these flavon vevs, the Dirac mass matrix will appear from the first two terms of eq. (3.3) while the Majorana matrix which follows from the next two terms of the Lagrangian of eq. (3.3) can be found as follows

\[
M_D = \frac{v}{\Lambda} \begin{pmatrix}
0 & y_{N2} v_a \\
y_{N1} v_s & y_{N2} v_a \\
y_{N1} v_s & y_{N2} v_a
\end{pmatrix} = vY_N, \quad M_R = \begin{pmatrix}
M_{N1} & 0 \\
0 & M_{N2}
\end{pmatrix}.
\] (3.4)

The $Z_4$ symmetry in table 1, under which $N_{R1}$ is odd and $N_{R2}$ even and the $Z_2$ symmetry under which $N_{R1}$ is even and $N_{R2}$ odd ensures the diagonal structure of $M_R$ as obtained in eq. 3.4. The VEV alignment considered here is widely used in the context of form dominance [131], sequential dominance [106], constrained sequential dominance [59] etc., to obtain the textures of the Dirac and Majorana mass matrices. Now using the type-I seesaw formula the light neutrino mass matrix at the leading order can be written as

\[
(M_\nu)_{\text{TREE}} = -M_D M_R^{-1} M_D^T.
\] (3.5)

With the structure of $M_D$ and $M_R$ obtained in eq. (3.4), the light neutrino mass matrix is given by

\[
(M_\nu)_{\text{TREE}} = -\begin{pmatrix}
B & B & B \\
B & A + B & -A + B \\
B & -A + B & A + B
\end{pmatrix}, \quad A = \frac{v^2 v_s^2 y_{N1}^2}{\Lambda^2 M_{N1}}, \quad B = \frac{v^2 v_a^2 y_{N2}^2}{\Lambda^2 M_{N2}}.
\] (3.6)
The above mass matrix for light neutrinos obtained from type-I seesaw is incapable to generate non-zero $\theta_{13}$ (charged lepton mass matrix being diagonal). As neutrino oscillation data established adequately large $\theta_{13}$, we include a scotogenic contribution to our model to explain correct neutrino mixing which also naturally incorporates few potential DM candidates. The scotogenic contribution in our model with the fermion $f$ and scalar field $\eta$ can be written as

$$\mathcal{L}_S = \frac{y_s}{\Lambda^2} (\bar{L}_i \sigma_2 \eta^c f + \frac{1}{2} M_f \bar{f}^c f + h.c.) ,$$

where $y_s$ is the coupling and $M_f$ is the mass of $f$. Owing to the considered symmetry in table 1, the leading order contribution $\bar{L}_i \sigma_2 \eta^c f$ is disallowed, as it is not invariant under $A_4$ (and $Z_4, Z_2$) symmetry. The SM lepton doublet being a triplet under $A_4$, just like charged lepton sector, involvement of another $A_4$ triplet (here $\phi_s$) is essential. As a consequence of $Z_{4,2}$ symmetry involvement of the $A_4$ flavons $\phi_s$ and $\xi$ is necessary in the first term of eq. (3.7). The VEV of $\phi_s$ (mentioned below) and the non-trivial $A_4$ singlet $\xi$ (provides appropriate $A_4$ contraction) crucially dictates the structure of the scotogenic contribution and helps in breaking of TBM mixing [71, 74, 92, 132, 133]. Therefore the contribution in the effective neutrino mass matrix originated from the scotogenic radiative corrections is given by [96, 102, 109]

$$(M_\nu)^{\text{LOOP}} = \mathcal{F}(m_{\eta R}, m_{\eta I}, M_f) M_f Y_f^T Y_f^T.$$ (3.8)

Once the flavons $\phi_s$ and $\xi$ acquire VEVs in the direction $\langle \phi_s \rangle = (0, v_s, -v_s)$ and $\langle \xi \rangle = v_\xi$ respectively the associated couplings can be written as

$$Y_F = (Y_F^e, Y_F^\mu, Y_F^\tau)^T = (y_s \frac{v_s v_\xi}{\Lambda}, 0, -y_s \frac{v_s v_\xi}{\Lambda})^T.$$ (3.9)

Therefore, the corresponding mass matrix takes the form

$$(M_\nu)^{\text{LOOP}} = C \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad C = \mathcal{F}(m_{\eta R}, m_{\eta I}, M_f) y_s^2 \frac{v_s^2 v_\xi^2}{\Lambda^2}.$$ (3.10)

Here $\mathcal{F}(m_{\eta R}, m_{\eta I}, M_f)$ is the loop function given in eq. (2.3). Finally, combining the seesaw and scotogenic contributions, the effective light neutrino mass matrix is the addition of the two mass matrices given in eq. (3.6) and eq. (3.10) and reads

$$M_\nu = (M_\nu)^{\text{TREE}} + (M_\nu)^{\text{LOOP}}$$

$$= \begin{pmatrix} -B + C & -B & -B - C \\ -B & -(A + B) & A - B \\ -B - C & A - B & -A - B + C \end{pmatrix}.$$ (3.11)

In the present context, neutrino masses are obtained through a combination of the type-I seesaw and scotogenic mechanisms. Now, there could also be operators like $LHLH/\Lambda$, which can also contribute to the light neutrino mass. In our model, this term is not
invariant under the $Z_4$ symmetry mentioned in table 1. Any contributions coming from $LHHL(\phi_a, \phi_s, \phi_T, \xi)/\Lambda^2$ is also disallowed due to the considered discrete symmetries $Z_4, Z_3$ and $Z_2$. For the scotogenic contribution, the coupling $\bar{L}_f \sigma^a H$ is allowed only at $1/\Lambda^2$ level with the involvement of the flavons $\phi_s, \xi$, see eq. (3.7). In this sector, any higher-order contributions in the bare mass term of $f$ can be absorbed in the leading order contribution. On the other hand for charged lepton sector, the leading contribution only appears at dimension-5 due to the considered $A_4$ symmetry. There are next-to-leading order corrections present in this sector coming from $(\bar{L}_f \phi_s \phi_a)h_{\alpha R}/\Lambda^2$, where $\alpha$ is the associated right-handed charged lepton. Thanks to the VEV alignment of the flavons $\phi_s$ and $\phi_a$ this term essentially vanishes following the $A_4$ multiplication rules mentioned in the appendix. For right-handed Majorana neutrinos, the non-vanishing next-to-leading order corrections up to $O(1/\Lambda^2)$ in the mass matrix arise from the following terms:

$$
\begin{align*}
\delta \mathcal{L}_{MR} &= \frac{1}{\Lambda} \left( \tilde{N}^c_{R_1} N_{R_1} + \tilde{N}^c_{R_2} N_{R_2} \right) \left( \phi^a_a \phi_a + \phi^s_s \phi_s \right) \\
&+ \frac{1}{\Lambda} (\tilde{N}^c_{R_1} N_{R_2} + \tilde{N}^c_{R_2} N_{R_1}) (\phi^s_s \phi_a + \phi^a_a \phi_s + \phi^s_s \phi^a_a) + \frac{1}{\Lambda^2} (\tilde{N}^c_{R_1} N_{R_2} + \tilde{N}^c_{R_2} N_{R_1}) \xi^3.
\end{align*}
$$

Here in eq. (3.12), the first term represents a correction to the diagonal entry which can be absorbed in the leading order $M_R$. The second term represents off-diagonal entries of the right-handed neutrino mass matrix at $O(1/\Lambda)$ which also vanishes due to the specific VEV direction of $\phi_s$ and $\phi_a$. The last term in eq. (3.12) represents off-diagonal entries at $O(1/\Lambda^2)$. Although this contribution is very small compared to the leading order contribution, it can also be forbidden by considering another $Z'_2$ symmetry under which charged leptons, $f$ and the flavons $\phi_T, \xi'$ are odd (while all other particles are even). The Dirac Yukawa coupling is allowed at dimension-5 as given in eq. (3.3). Here the next-to-leading order contribution at $O(1/\Lambda)$ can be written as $(\bar{L}_f \phi^a_a \phi_T) \tilde{H} N_{R_1}/\Lambda^2$ and $(\bar{L}_f \phi^s_s \phi_T) \tilde{H} N_{R_2}/\Lambda^2$ respectively. These terms are however forbidden owing to the $Z_3$ symmetry mentioned in table 1. Therefore, from table 1, it is clear that along with the $A_4$ symmetry the auxiliary discrete symmetries crucially dictate allowed structures of the fermionic mass matrices, and such symmetries are an integral part of the flavor symmetric approach to understand neutrino mixing [20, 37, 40, 41, 134–136].

4 Neutrino masses and mixing in the FSS model

From the previous discussion, we find that the effective neutrino mass matrix consists of two parts, one of them is coming from the type-I seesaw mechanism given by eq. (3.6) and another one originates from the scotogenic contribution given by eq. (3.10). Now the mass matrix originating from type-I seesaw given in eq. (3.6) can be diagonalized by the TBM mixing matrix ($U_{TB}$) via

$$
U_{TB}^T(M_\nu)_\text{TREE} U_{TB} = 
\begin{pmatrix}
0 & 0 & 0 \\
0 & -3B & 0 \\
0 & 0 & -2A
\end{pmatrix},
$$

(4.1)
where

\[
U_{TB} = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\] (4.2)

Clearly, a pure type-I seesaw contribution in the present set-up predicts \(\theta_{13} = 0\). However, thanks to the scotogenic contribution, we can obtain a deviation from \(\theta_{13} = 0\) to be consistent with the observed experimental value \([43–45]\). Therefore considering the effective light neutrino mass matrix given in eq. (3.11) and rotating it by \(U_{TB}\), \(M_\nu\) takes the form

\[
M'_\nu = U_{TB}^T M_\nu U_{TB}
\]

\[
= \frac{1}{2} \begin{pmatrix}
3C & 0 & -\sqrt{3}C \\
0 & -6B & 0 \\
-\sqrt{3}C & 0 & -4A + C
\end{pmatrix},
\] (4.3)

Here we find that the mass matrix in the tri-bimaximal basis is block diagonalized. Therefore a further rotation by a unitary matrix \(U_{13}\) in the 13 plane via \(M_\nu^{\text{diag}} = U_{13}^T M'_\nu U_{13}\) takes \(M'_\nu\) to a diagonal one. This unitary matrix \(U_{13}\) can be parametrized as

\[
U_{13} = \begin{pmatrix}
\cos \theta & 0 & \sin \theta e^{-i\phi} \\
0 & 1 & 0 \\
-\sin \theta e^{i\phi} & 0 & \cos \theta
\end{pmatrix},
\] (4.4)

where \(\theta\) is the rotation angle and \(\phi\) is the associated phase factor. The full diagonalization relation of the mass matrix \(M_\nu\) can be written as

\[
(U_{TB} U_{13})^T M_\nu U_{13} U_{TB} = \text{diag}(m_1 e^{i\gamma_1}, m_2 e^{i\gamma_2}, m_3 e^{i\gamma_3}),
\] (4.5)

where \(m_1, m_2, m_3\) are the real and positive mass eigenvalues and \(\gamma_1, \gamma_2\) and \(\gamma_3\) are the phases extracted from the corresponding complex eigenvalues. We are now in a position to evaluate the neutrino mixing matrix \(U_\nu\) such that \(U_\nu^T M_\nu U_\nu = \text{diag}(m_1, m_2, m_3)\). Thus \(U_\nu\) becomes \(U_\nu = U_{TB} U_{13} U_m\), where \(U_m = \text{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2})\) is the Majorana phase matrix with \(\alpha_{21} = \gamma_1 - \gamma_2\) and \(\alpha_{31} = \gamma_1 - \gamma_3\), one common phase being irrelevant. Using eq. (4.2) and eq. (4.4), the \(U_\nu\) mixing matrix in its explicit form can be written as

\[
U_\nu = \begin{pmatrix}
\sqrt{\frac{2}{3}} \cos \theta & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} e^{i\phi} \sin \theta \\
-\frac{\cos \theta}{\sqrt{6}} - i \frac{\sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\cos \theta - \frac{e^{i\phi} \sin \theta}{\sqrt{6}} \\
-\frac{\cos \theta}{\sqrt{6}} + i \frac{\sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\cos \theta + \frac{e^{i\phi} \sin \theta}{\sqrt{6}}
\end{pmatrix} U_m.
\] (4.6)

Such deviation from the TBM mixing is well known and this particular pattern of \(U_\nu\) is called TM\(_2\) mixing as described earlier. This lepton mixing matrix \(U_\nu\) can now be compared with \(U_{PMNS}\) which in its standard parametrization is given by \([137]\)

\[
U_{PMNS} = \begin{pmatrix}
c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{CP}} \\
-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{CP}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{CP}} & s_{23} c_{13} \\
s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{CP}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{CP}} & c_{23} c_{13}
\end{pmatrix} U_m.
\] (4.7)
where \( \theta_{12}, \theta_{13} \) and \( \theta_{23} \) are three mixing angles, \( \delta_{\text{CP}} \) is the CP violating Dirac phase and \( \alpha_{21}, \alpha_{31} \) are the Majorana phases.

The parameters \( A, B \) and \( C \) appearing in eq. (4.3) are in general complex, and without loss of generality we can write \( A = |A|e^{i\phi_A}, B = |B|e^{i\phi_B}, C = |C|e^{i\phi_C} \). Now, for calculation purpose, let us define \( \alpha = |A|/|C| \) and \( \beta = |B|/|C| \), and phase differences \( \phi_{AC} = \phi_A - \phi_C \) and \( \phi_{BC} = \phi_B - \phi_C \). As \( U_{13} \) diagonalizes \( M'_\nu \) of eq. (4.3), \( \theta \) and \( \phi \) can be expressed in terms of model parameters as

\[
\tan \phi = \frac{\alpha \sin \phi_{AC}}{1 - \alpha \cos \phi_{AC}}, \quad \tan 2\theta = \frac{\sqrt{3}}{\cos \phi + 2\alpha \cos(\phi_{AC} + \phi)}. \tag{4.8}
\]

Further comparing \( U_\nu = U_{TB}U_{13}U_m \) as given in eq. (4.6) with \( U_{PMNS} \) as in eq. (4.7), we find the following relations for mixing angles and \( \delta_{\text{CP}} \) as a function of \( \theta \) and \( \phi \) as [92, 138–141]

\[
\sin \theta_{13}e^{-i\delta_{\text{CP}}} = \sqrt{\frac{2}{3}}e^{-i\phi} \sin \theta, \quad \tan^2 \theta_{12} = \frac{1}{2 - 3\sin^2 \theta_{13}} \tag{4.9}
\]

\[
\tan^2 \theta_{23} = \left( \frac{1 + \frac{\sin \theta_{13} \cos \phi}{\sqrt{2 - 3\sin^2 \theta_{13}}} + \frac{\sin^2 \theta_{13} \sin^2 \phi}{(2 - 3\sin^2 \theta_{13})}}{1 - \frac{\sin \theta_{13} \cos \phi}{\sqrt{2 - 3\sin^2 \theta_{13}}} + \frac{\sin^2 \theta_{13} \sin^2 \phi}{(2 - 3\sin^2 \theta_{13})}} \right)^2 \tag{4.10}
\]

The above relations show that the mixing angles are correlated which is a characteristic feature of the considered \( A_4 \) discrete flavor symmetry. For \( \sin \theta > 0 \), the relation of \( \delta_{\text{CP}} \) implies that \( \delta_{\text{CP}} = \phi \) and for \( \sin \theta < 0 \), the same relation implies that \( \delta_{\text{CP}} = \phi \pm \pi \). Hence, for both cases, we have \( \tan \delta_{\text{CP}} = \tan \phi \).

Now, using eq. (3.11), the complex mass eigenvalues are calculated to be

\[
m^c_{1,3} = -A + C \pm \sqrt{A^2 + AC + C^2}, \tag{4.11}
\]

\[
m^c_2 = -3B. \tag{4.12}
\]

The real and positive eigenvalues can be written as

\[
m_1 = |C|(1 - \alpha \cos \phi_{AC} - P)^2 + (Q + \alpha \sin \phi_{AC})^2 \tag{4.13}
\]

\[
m_2 = |C|\beta, \tag{4.14}
\]

\[
m_3 = |C|(1 - \alpha \cos \phi_{AC} + P)^2 + (Q - \alpha \sin \phi_{AC})^2 \tag{4.15}
\]

where

\[
p^2 = \frac{M \pm \sqrt{M^2 + N^2}}{2}, \quad Q^2 = \frac{-M \pm \sqrt{M^2 + N^2}}{2}, \tag{4.16}
\]

\[
M = 1 + \alpha \cos \phi_{AC} + \alpha^2 \cos 2\phi_{AC}, \quad N = \alpha \sin \phi_{AC} + \alpha^2 \sin 2\phi_{AC}. \tag{4.17}
\]

Following eq. (4.11) and eq. (4.12), the phase associated with complex mass eigenvalues \( m^c_{1,2,3} \) can be written as \( \gamma_i = \phi_C + \phi_i \), where \( \phi_i \) are

\[
\phi_1 = \tan^{-1} \left( \frac{Q + \alpha \sin \phi_{AC}}{1 - \alpha \cos \phi_{AC} - P} \right), \quad \phi_2 = \phi_{BC}, \quad \phi_3 = \tan^{-1} \left( \frac{Q - \alpha \sin \phi_{AC}}{1 - \alpha \cos \phi_{AC} + P} \right). \tag{4.18}
\]
| parameters | best-fit | 3σ range  |
|-----------|---------|-----------|
| $\frac{\Delta m_{21}^2}{10^{-3} \text{eV}^2}$ | 7.50 | 6.94–8.14 |
| $\frac{|\Delta m_{31}^2|}{10^{-3} \text{eV}^2}$ (NH) | 2.55 | 2.47–2.63 |
| $\frac{|\Delta m_{31}^2|}{10^{-3} \text{eV}^2}$ (IH) | 2.45 | 2.37–2.53 |
| $\sin^2 \theta_{12}/10^{-1}$ | 3.18 | 2.71–3.69 |
| $\sin^2 \theta_{13}/10^{-2}$ (NH) | 2.200 | 2.000–2.405 |
| $\sin^2 \theta_{13}/10^{-2}$ (IH) | 2.225 | 2.018–2.424 |
| $\sin^2 \theta_{23}/10^{-1}$ (NH) | 5.74 | 4.34–6.10 |
| $\sin^2 \theta_{23}/10^{-1}$ (IH) | 5.78 | 4.33–6.08 |

Table 2. Global fits of three active-neutrino oscillation data taken for ref. [5] for NH and IH, used in our analysis.

The two Majorana phases in $U_m$ (see eq. (4.6)) therefore can be derived as

$$\alpha_{21} = \tan^{-1} \left( \frac{Q + \alpha \sin \phi_{AC}}{1 - \alpha \cos \phi_{AC} - P} \right) - \phi_{BC}, \quad (4.19)$$

$$\alpha_{31} = \tan^{-1} \left( \frac{Q + \alpha \sin \phi_{AC}}{1 - \alpha \cos \phi_{AC} - P} \right) - \tan^{-1} \left( \frac{Q - \alpha \sin \phi_{AC}}{1 - \alpha \cos \phi_{AC} + P} \right). \quad (4.20)$$

The overall phase factor $\phi_C$ appearing in $\gamma_i$ has no physical significance in computing the Majorana phases. The mixing angles and the phases depend on the parameters $\alpha, \phi_{AC}, \phi_{BC}$ whereas the light neutrino masses depend on these parameters as well as on $\beta$ and $|C|$ as observed in eq. (4.8)–(4.20). In the next section, we constrain these parameters using experimental data for neutrino mixing angles and masses.

5 Numerical analysis of the FSS model

In order to constrain the parameters involved in our analysis, using eq. (4.13)–(4.15), we can define a ratio $r$ as

$$r = \frac{\Delta m_{21}^2}{|\Delta m_{31}^2|}, \quad (5.1)$$

where $\Delta m_{21}^2 = m_2^2 - m_1^2$ and $|\Delta m_{31}^2| = |m_3^2 - m_1^2|$ are the solar and atmospheric mass squared differences. From the expressions for the mixing angles (namely, $\theta_{13}, \theta_{12}$ and $\theta_{23}$) as well as the absolute neutrino masses ($m_{1,2,3}$), their sum ($\sum m_j$) and the ratio $r$ defined in eq. (5.1) all depend on the variables $\alpha, \beta, \phi_{AC}$ and $\phi_{BC}$ as discussed in section 4. Over the last two decades neutrino oscillation parameters have been measured with incredible accuracy [5, 142, 143]. Therefore using the precisely determined neutrino oscillation data on $\theta_{13}, \theta_{12}, \theta_{23}, \alpha_{21}, \alpha_{31}$, the sum of the absolute masses of the three light neutrinos
Figure 1. $\sin^2 \theta_{12}$ is plotted against $\sin^2 \theta_{13}$ for both NH and IH of neutrino masses. The vertical grid-lines are $3\sigma$ allowed for $\sin^2 \theta_{13}$ and horizontal grid-lines represent corresponding restriction on $\sin^2 \theta_{12}$ in our analysis.

Before we proceed further, let us point out that the correlation between the mixing angles $\theta_{13}$ and $\theta_{12}$ given in eq. (4.9) is a feature of the TM$_2$ mixing mentioned above [139, 141]. In figure 1, we have plotted this $\sin^2 \theta_{12} - \sin^2 \theta_{13}$ correlation for TM$_2$ mixing and we find that $\sin^2 \theta_{12}$ is restricted within a narrow range (between $0.3401 \leq \sin^2 \theta_{12} \leq 0.3415$) corresponding to the $3\sigma$ ranges of $\sin^2 \theta_{13}$. Now, in order to evaluate absolute neutrino masses we also need to find the overall factor $|C|$ appearing in the mass eigenvalues given in eq. (4.13)–(4.15). Although this common factor $|C|$ cancels out when we calculate $r$ but $|C|$ can be calculated by fitting the solar (or atmospheric) mass-squared differences knowing the model parameters. After $|C|$ is evaluated, we can get the estimation of absolute neutrino masses $m_{1,2,3}$ and their sum $\sum m_i$. Similarly, substituting the estimations for $\alpha, \beta, \phi_{AC}$ and $\phi_{BC}$ in eq. (4.19) and eq. (4.20) we can also quantify the Majorana phases. Knowing the neutrino mixing angles, masses, and associated CP phases, finally in our analysis, we will have a prediction on the effective neutrino mass parameter $m_{\beta\beta}$ characterizing the neutrinoless double beta decay. The effective mass parameter $m_{\beta\beta}$ can be described as a function of the lightest neutrino mass ($m_1$ for NH and $m_3$ for IH respectively) and can be written as\footnote{The expression for the effective mass parameter $m_{\beta\beta}$ can also be written in a symmetrical form where only two Majorana phases [144] appear instead of three phases appearing in the standard PDG parametrization [137].}

$$
\text{NH: } m_{\beta\beta} = \left| m_1 c_{12} c_{13}^2 + \sqrt{m_3^2 + \Delta m_{21}^2 s_{12}^2 e^{i\alpha_{21}}} + \sqrt{m_1^2 + \Delta m_{31}^2 s_{13}^2 e^{i(\alpha_{31} - 2\delta_{CP})}} \right|, \quad (5.2)
$$

$$
\text{IH: } m_{\beta\beta} = \left| \sqrt{m_3^2 + \Delta m_{31}^2 c_{12}^2 c_{13}^2} + \sqrt{m_3^2 + \Delta m_{32}^2 s_{12}^2 c_{13}^2 e^{i\alpha_{21}}} + m_3 s_{13}^2 e^{i(\alpha_{31} - 2\delta_{CP})} \right|. \quad (5.3)
$$

Now, for a better understanding of the behaviour of the parameters involved and the model predictions, we can divide our numerical analysis into some special cases by taking some particular values of the relative phases $\phi_{AC}$ and $\phi_{BC}$. All mixing angles in eq. (4.9), (4.10)
Figure 2. Left Panel: plot for $\sin \theta_{13}$ vs $\alpha$. Horizontal grid lines are the 3$\sigma$ allowed range for $\sin \theta_{13}$ (for NH and IH) whereas the vertical grid lines represent the corresponding allowed range for $\alpha$ (1.6557–1.8391 and 1.6465–1.8754 for NH and IH respectively). Right Panel: contour plot for $r = 0.03$ in $\alpha$–$\beta$ plane. The vertical grid lines represent the allowed regions for $\alpha$ from the left panel and the horizontal grid lines are the corresponding allowed regions for $\beta$.

and neutrino mass eigenvalues in eq. (4.13)–(4.15) and the Majorana phase $\alpha_{31}$ depend on one relative phase, namely, $\phi_{AC}$. In contrast, the other Majorana phase $\alpha_{21}$ as well as $m_{\beta\beta}$ depend on both $\phi_{AC}$ and $\phi_{BC}$. In the following, we choose five simple special cases depending on the values of these phases, namely: (i) Case I: $\phi_{AC} = 0$, $\phi_{BC} = 0$, (ii) Case II $\phi_{AC} = 0$, (iii) Case III $\phi_{AC} = \phi_{BC}$, (iv) Case IV $\phi_{BC} = 0$ and subsequently in (v) Case V we present the general scenario where both $\phi_{AC}$ and $\phi_{BC}$ vary between $0 - 2\pi$.

Below, we have explored these cases and as we proceed it will be clear that some of these cases have the potential to distinguish the normal and inverted hierarchy of light neutrino masses and produce interesting predictions on neutrino parameters.

5.1 Case I: $\phi_{AC} = \phi_{BC} = 0$

Here we make the simplest choice for the relative phases, i.e., $\phi_{AC} = \phi_{BC} = 0$. With this value, the eq. (4.8) and (4.9) have the simple form

$$\tan 2\theta = \frac{\sqrt{3}}{1 + 2\alpha}, \quad \sin \theta_{13} = \sqrt{\frac{2}{3}} |\sin \theta|. \tag{5.4}$$

with $\tan \delta_{CP} = 0$. Clearly, $\sin \theta_{13}$ only depends on $\alpha$ and in figure 2 left panel, we have plotted $\sin \theta_{13}$ as a function of $\alpha$ using eq. (5.4). The 3$\sigma$ allowed range for $\sin \theta_{13}$ (given by the area between horizontal lines) restricts $\alpha$ within 1.6465–1.8754 (1.6557–1.8391) for IH (NH) given by the dashed (continuous) vertical lines. With $\phi_{AC} = \phi_{BC} = 0$ the real positive mass eigenvalues given in eq. (4.13)–(4.15) can be expressed as

$$m_1 = |C|(1 - \alpha - \sqrt{1 + \alpha + \alpha^2}), \quad m_2 = |C|3\beta, \quad m_3 = |C|(1 - \alpha + \sqrt{1 + \alpha + \alpha^2}). \tag{5.5}$$
With the above mass eigenvalues, one can write the ratio of solar to atmospheric mass-squared differences as defined in eq. (5.1) as

\[ r = \pm \frac{9\beta^2 - (1 - \alpha - \sqrt{1 + \alpha + \alpha^2})^2}{(1 - \alpha + \sqrt{1 + \alpha + \alpha^2})^2 - (1 - \alpha - \sqrt{1 + \alpha + \alpha^2})^2}, \]  

(5.8)

where the ± signs are for NH and IH respectively. When \( \phi_{AC} = \phi_{BC} = 0 \), as a consequence of the considered discrete flavor symmetry, NH of light neutrino masses can not be realized with eq. (5.5)–(5.8). Hence only IH of light neutrino masses is allowed. From eq. (5.8), we notice that \( r \) depends on both \( \alpha \) and \( \beta \) and in the right panel of figure 2, we have plotted this dependence in this \( \alpha - \beta \) plane for the best fit value of the ratio \( r = 0.03 \) [5] only for IH. For the 3σ allowed range for \( \alpha \) (fitting \( \sin \theta_{13} \)), obtained from the left panel of figure 2, \( \beta \) found to be in the range 0.997 – 1.14 for IH. The contour plot for \( r \) yields a one-to-one correspondence between \( \alpha \) and \( \beta \) as evident from the right panel of figure 2. For example, the best fit value of \( \sin \theta_{13} \) and \( r \) fixes \( \alpha : 1.752, \beta : 1.067 \) for IH. With the known sets of \( (\alpha, \beta) \) corresponding to 3σ range of \( \sin \theta_{13} \), we can calculate \( |C| \) by fitting the best fit value of the solar mass-squared difference (given in table 2) as \( |C| = (2.04 - 1.68) \times 10^{-2} \text{eV} \). Then, for the allowed sets of \( (\alpha, \beta, |C|) \), we can estimate the absolute neutrino masses \( m_{1,2,3} \) and their sum \( \sum m_i \). In figure 3, we have plotted the individual neutrino masses \( m_{1,2,3} \) (by blue dashed, red and orange lines respectively) and their sum \( \sum m_i \) (green line) against \( \alpha \) as obtained from the left panel of figure 2. Now for IH, cosmology sets an upper limit on the sum of the masses of three light neutrinos as \( \sum m_i \leq 0.15 \text{eV} \) [5] as given by the horizontal cyan shaded region in the left panel of figure 3. Thus all \( (\alpha, \beta) \) found from the right panel of figure 2 are not allowed. The vertical dashed line in the left panel of figure 3 thus represents a further restriction on \( \alpha \) (and hence on \( \beta \)) and the lower bound on \( \alpha \) shifted from 1.6567 to 1.74. As a result the corresponding values of \( \beta \) and \( |C| \) will also be shifted. The final allowed values of the model parameters are summarized in table 3. Substituting \( \phi_{AC} = 0 \) in eq. (4.18) we obtain \( \phi_1 = \phi_3 = 0 \) and with \( \phi_{BC} = 0 \) we get \( \phi_2 = 0 \). Altogether substituting these in eq. (4.19) and eq. (4.20), the Majorana phases are found to be zero. Thus for vanishing values of the relative phase \( \phi_{AC} = 0 \) and \( \phi_{BC} = 0 \), the Dirac and Majorana phases also vanishes. With these values of the phases and known

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**Figure 3.** Left Panel: absolute neutrino masses \( m_3 \) (orange line), \( m_2 \) (blue dashed line), \( m_3 \) (red line) and their sum \( \sum m_i \) (green line) plotted against \( \alpha \). Right Panel: the prediction on the effective mass parameter \( m_{\beta\beta} \). In both cases we have considered \( \phi_{AC} = \phi_{BC} = 0 \).
| Parameters | Allowed ranges |
|------------|----------------|
| $\alpha$ | 1.74-1.875 |
| $\beta$ | 1.059-1.147 |
| $|C|$ (eV) | $(1.87-1.68) \times 10^{-2}$ |
| $\sum m_i$ (eV) | 0.1496-0.1408 |
| $m_{\beta\beta}$ (eV) | 0.0568-0.0585 |

**Table 3**. The allowed ranges for $\alpha$, $\beta$, $|C|$, $\sum m_i$ and $m_{\beta\beta}$ when $\phi_{AC} = \phi_{BC} = 0$.

| $\phi_{BC}$ | $\pi/6$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $5\pi/6$ | $\pi$ |
|--------------|----------|----------|----------|-----------|----------|--------|
| $m_{\beta\beta}$ (eV) | 0.057-0.055 | 0.051-0.050 | 0.043-0.042 | 0.034-0.033 | 0.024-0.023 | 0.019-0.018 |

**Table 4**. Prediction on $m_{\beta\beta}$ depending on different values of $\phi_{BC}$ in the Case II with $\phi_{AC} = 0$.

sets of $(\alpha, \beta, |C|)$ we can now finally estimate the effective mass parameter $m_{\beta\beta}$ appearing in the neutrinoless double beta decay. In the right panel of figure 3, we have plotted the prediction for $m_{\beta\beta}$ as a function of $\alpha$ with $\phi_{AC} = \phi_{BC} = 0$. The finding for the sum of the absolute neutrino masses and the effective mass parameter as obtained from figure 3 are also summarized in table 3.

### 5.2 Case II: $\phi_{AC} = 0$

In the second case, we consider $\phi_{AC} = 0$ but do not make any choice of $\phi_{BC}$. We already know that neutrino mixing angles and neutrino masses both depend on the phase $\phi_{AC}$ alone, which can be understood from the general expressions of eq. (4.8)–(4.10) and eq. (4.13)–(4.15). The Majorana phases $\alpha_{21}$ and hence neutrinoless double beta decay effective mass parameter $m_{\beta\beta}$ depend on both $\phi_{AC}$ and $\phi_{BC}$. Now, with the choice of $\phi_{AC} = 0$, the simplified expressions for $\theta$ and $\sin \theta_{13}$ are same as in eq. (5.4). The real positive mass eigenvalues and the ratio of the mass-squared differences $r$ will take the form as in eqs. (5.5)–(5.8) respectively. Hence, with these expressions, the limits on $\alpha$, $\beta$, $|C|$, $m_i$ and $\Sigma m_i$ will same as in Case I ($\phi_{AC} = \phi_{BC} = 0$) which are summarized in table 3. Thus in this case too only IH of light neutrino masses is allowed. The only difference between Case I and Case II is that in the latter case $\phi_{BC}$ is free. As $\phi_{BC}$ only appears for Majorana phases, which can be understood just by looking at the eq. (4.18), the prediction on Majorana phases will be different from the previous case and hence prediction on $m_{\beta\beta}$ will change accordingly. For $\phi_{AC} = 0$, $\phi_1$ and $\phi_3$ are zero (obtained from eq. (4.18)) as mentioned in Case I. Hence, one of the Majorana phase $\alpha_{31}$ vanishes but due to non-zero $\phi_2$, the other Majorana phase ($\alpha_{21}$) found to be $\alpha_{21} = -\phi_{BC}$. Hence, we will calculate the neutrinoless double beta decay effective mass parameter $m_{\beta\beta}$ using eq. (5.2) by choosing different values for $\phi_{BC}$. In table 4 where we have summarized the ranges of $m_{\beta\beta}$ for different values of $\phi_{BC}$ such as $\phi_{BC} = \pi/6$, $\pi/3$, $\pi/2$, $2\pi/3$, $5\pi/6$ and $\pi$. Prediction on $m_{\beta\beta}$ in this case for $\phi_{BC} = 0 - \pi$ recurs the same value again for $\phi_{BC} = \pi - 2\pi$. From Case I and Case II, one can say that only IH is allowed with $\phi_{AC} = 0$ (with $\phi_{BC} = 0$ or arbitrary).
Figure 4. The contour plot for $\delta_{\text{CP}} = 0.78$ (blue dots) and best fit of $\sin \theta_{13}$ (red) for NH of neutrino masses. The region between red dots represents the $3\sigma$ region of $\sin \theta_{13}$ value of neutrino oscillation data for NH of neutrino masses. The intersection point represents the value of $\alpha$ and $\cos \phi_x$.

5.3 Case III: $\phi_{AC} = \phi_{BC} = \phi_x$

In this case, we consider the scenario when the relative phases $\phi_{AC}$ and $\phi_{BC}$ both are equal, say, $\phi_{AC} = \phi_{BC} = \phi_x$. Hence, the general expressions for the rotation angle $\theta$ and associated phase $\phi$ appearing in the unitary rotation matrix $U_{13}$ as given in eq. (4.8) can be rewritten as

$$
\tan \phi = \frac{\alpha \sin \phi_x}{1 - \alpha \cos \phi_x}, \quad \tan 2\theta = \frac{\sqrt{3}}{\cos \phi + 2\alpha \cos(\phi_x + \phi)}.
$$

(5.9)

Thus we can substitute $\phi$ in the second equation above to evaluate $\sin \theta_{13}$ using eq. (4.9). Furthermore as $\tan \delta_{\text{CP}} = \tan \phi$, we find that a particular value of $\delta_{\text{CP}}$, $\sin \theta_{13}$ both depends on $\phi_x$ and $\alpha$. In figure 4 we provide contour plots for $\sin \theta_{13} = 0.148$ and $\delta_{\text{CP}} = 0.541$ (or $31^\circ$) denoted by purple and blue dotted lines respectively. The intersection between $\sin \theta_{13}$ and $\delta_{\text{CP}}$ contours indicate the simultaneous satisfaction of them and figure 4 it is indicated by a black dot with which a pair of $\cos \phi_x$ and $\alpha$ are attached. Similar intersections of the green dot-dashed lines ($\delta_{\text{CP}} \leq 0.541$) with the purple line represent other pairs of $\cos \phi_x$ and $\alpha$. In this case the mass eigenvalues $m_1$, $m_2$ and $m_3$ can be written as

$$
m_1 = |C|[(1 - \alpha \cos \phi_x - P)^2 + (Q + \alpha \sin \phi_x)^2]^{1/2},
$$

(5.10)

$$
m_2 = |C|3\beta,
$$

(5.11)

$$
m_3 = |C|[(1 - \alpha \cos \phi_x + P)^2 + (Q - \alpha \sin \phi_x)^2]^{1/2},
$$

(5.12)

and using these equations we can also evaluate the ratio of the mass-squared differences $r$ using eq. (5.1). One can then compute $\beta$ with $r = 0.03$ and the corresponding value for $|C|$ for each pair of $\alpha$ and $\cos \phi_x$ obtained from intersecting points in figure 4. With this flavor structure of the mass eigenvalues given in eq. (5.10)–(5.12), the inverted hierarchy of neutrino mass is not possible with $\phi_{AC} = \phi_{BC}$. For each set of $\alpha$, $\beta$, $\phi_x$ and $|C|$ estimated above, we can predict the sum of the absolute neutrino mass $\sum m_i$ and in table 5 we present few such representative values. From eq. (4.18)–(4.20), we find that the Majorana phases $\alpha_{21}$ and $\alpha_{31}$ are defined as $\alpha_{21} = \phi_1 - \phi_2$ and $\alpha_{31} = \phi_1 - \phi_3$ where $\phi_1$, $\phi_2$ and $\phi_3$ in the
Table 5. The allowed ranges of parameters satisfying neutrino data for various values of $\delta_{CP}$ with $\phi_{AC} = \phi_{BC}$. The last column represents prediction for $m_{\beta\beta}$ for $\phi_{AC} = 0 \div 2\pi$ and $\phi_{BC} = 0$.

| $\delta_{CP}$ | $\alpha$ | $\cos \phi_x$ | $\beta$ | $|C|$ (eV) | $\sum m_i$ (eV) | $m_{\beta\beta}$ (eV) | $m_{\beta\beta}\mid_{\phi_{BC}=0}$ (eV) |
|----------------|---------|---------------|-------|---------|-------------|-----------------|-----------------------------|
| 0.087          | 2.765   | -0.9848       | 0.5721| 0.0084  | 0.0773      | 0.00589         | 0.0128                      |
| 0.174          | 2.765   | -0.9659       | 0.5895| 0.00858 | 0.0792      | 0.0077         | 0.0148                      |
| 0.262          | 2.765   | -0.9327       | 0.6356| 0.008956| 0.0804      | 0.0100         | 0.0141                      |
| 0.349          | 2.759   | -0.8853       | 0.7089| 0.0096  | 0.0922      | 0.0144         | 0.0108                      |
| 0.436          | 2.754   | -0.8332       | 0.7849| 0.0104  | 0.1026      | 0.0188         | 0.0188                      |
| 0.523          | 2.754   | -0.7669       | 0.872 | 0.0117  | 0.1181      | 0.0252         | 0.0259                      |
| 0.541          | 2.754   | -0.748        | 0.894 | 0.0122  | 0.123       | 0.0257         | 0.0251                      |

The present case can be written as

$$\phi_1 = \tan^{-1}\left(\frac{Q + \alpha \sin \phi_x}{1 - \alpha \cos \phi_x - P}\right), \quad \phi_2 = \phi_x, \quad \phi_3 = \tan^{-1}\left(\frac{Q - \alpha \sin \phi_x}{1 - \alpha \cos \phi_x + P}\right). \quad (5.13)$$

Substituting $\alpha$ and $\cos \phi_x$ obtained from figure 4 in the above equations we can compute the Majorana phases and estimate the effective mass parameter appearing in the neutrinoless double beta decay as defined in eq. (5.2). These predictions for $m_{\beta\beta}$ are listed in the last column of table 5. From the results obtained form figure 4 and table 5, we find that all values for $\delta_{CP}$ are not compatible with the cosmological bound on the sum of neutrino masses $\sum m_i$ (satisfying correct neutrino oscillation data). The region $\delta_{CP} \leq 0.54 \,(31^\circ)$ given by the cyan shaded region is disallowed due to the cosmological upper bound $\sum m_i \leq 0.12\text{eV}$ for NH [5]. Thus the green dot-dashed region in figure 4 with $0 < \delta_{CP} \leq \pi/6$ satisfy both neutrino oscillation data and the cosmological bound on the sum of neutrino masses.

In table 5 we have summarized constraints on $\alpha$, $\beta$, $\cos \phi_x$ and $|C|$ for a specific value of $\delta_{CP}$ as well as predictions on $\sum m_i$ and $m_{\beta\beta}$. Here we also find that predictions for the parameters given in table 5 repeats again for $\pi < \delta_{CP} \leq 7\pi/6$. From the results summarized in the table 5 one can find that with increase of $(\beta, \cos \phi_x, |C|)$, $\alpha$ decreases whereas both $\sum m_i$, $m_{\beta\beta}$ increases with the increase of $\delta_{CP}$.

5.4 Case IV: $\phi_{BC} = 0$

In this case we consider $\phi_{BC} = 0$ while $\phi_{AC}$ is varied arbitrarily between $0 - 2\pi$. This scenario is very similar to the previous case, with $\phi_{AC} = \phi_{BC} = \phi_x$. From eq. (4.18)–(4.20) we find that $\phi_{BC} = 0$ only effects the Majorana phases and hence the effective mass parameter appearing in the neutrinoless double beta decay. The conclusion drawn in Case III for absolute values of light neutrino masses and their hierarchy will be identical. For simplicity and resembles to the previous case, we consider $\phi_{AC} = \phi_x$. With this, the expressions for rotation angle $\theta$ and the phase $\phi$ appearing in $U_{13}$ are already given in eq. (5.9). Using eq. (4.9) we also find that $\delta_{CP}$ and $\sin \theta_{13}$ are both function of $\alpha$ and $\phi_x$. For a particular value of $\delta_{CP}$ we can again compute the pairs of $\alpha$ and $\phi_x$ from the intersections of contour plots for the chosen value of $\delta_{CP}$ and best-fit value of $\sin \theta_{13}$ as
given in figure 4. Similar to Case III, by fitting $r = 0.03$, we can get $\beta$ (and subsequently $|C|$), where the expression of $r$ in eq. (5.1) involves neutrino masses those are given in eq. (5.10)–(5.12). Therefore the results of table 5 will be same for this case up to the sixth column for $\sum m_i$. The change will occur in the seventh column of table 5 as here in Case IV, we have $\phi_{BC} = 0$. Although, with this choice, the expression for $\phi_1$ and $\phi_3$ will be the same as eq. (5.13) but unlike the previous case we have $\phi_2 = 0$ here. The two Majorana phases follow the relation $\alpha_{21} = \phi_1 - \phi_2$, $\alpha_{31} = \phi_1 - \phi_3$ hence $\alpha_{31}$ will be identical and $\alpha_{21}$ will be different from Case III. As a result, $m_{\beta\beta}$ will be different compared to the previous scenario. Hence in the last column of table 5 we append the prediction of $m_{\beta\beta}$ for a few allowed values for $\delta_{\text{CP}}$. Note that here also the allowed ranges of the Dirac CP phase are $0 < \delta_{\text{CP}} \leq \pi/6$ and $\pi < \delta_{\text{CP}} \leq 7\pi/6$ respectively which satisfy both neutrino oscillation data and the cosmological bound on the sum of absolute neutrino masses. A point to remember is that this case only satisfies NH of neutrino mass and IH is not allowed.

5.5 General case

In the above cases, we have analyzed neutrino mixing for various limiting values for the relative phases associated with our study. Now, we will carry out a full numerical analysis for the most general case where we vary both $\phi_{AC}$ and $\phi_{BC}$ to their entire range from 0 to $2\pi$. Then, eq. (4.8)–(4.10) will be used for the calculation of mixing angles. On the other hand, the ratio of the mass-squared differences $r$ defined in eq. (5.1) can also be calculated using the general expression for the mass eigenvalues of eq. (4.13)–(4.15). As explained earlier, in order to evaluate the absolute neutrino masses, we also need to evaluate the common factor $|C|$ associated with each mass eigenvalue. Here we obtain $|C|$ by fitting the solar mass-squared difference taken from [5]. In our analysis for this most general case, we have also included the bound coming from cosmological observations on the sum of absolute neutrino masses as $\sum m_i < 0.12 \text{eV}$ for NH and $\sum m_i < 0.15 \text{eV}$ for IH [5]. Using the $3\sigma$ allowed range for the neutrino oscillation data [5] given in table 2, we vary both $\phi_{AC}$ and $\phi_{BC}$ between 0 to $2\pi$. In figure 5 we have plotted the allowed region in the $\alpha - \beta$ plane for NH (left panel, light red shaded region) and IH (right panel, blue shaded region) respectively. Here we find that for NH (IH), the allowed ranges for $\alpha$ vary between $2.33 \leq \alpha \leq 2.9$...
Figure 6. The allowed ranges for $\phi_{AC}$ and $\phi_{BC}$ for the correct value of $\sin^2 \theta_{13}$ and $r$ along with for both NH (left panel) and IH (right panel) of neutrino masses.

Figure 7. $\sin^2 \theta_{23} - \delta_{\text{CP}}$ correlation for NH (left panel) and IH (right panel). The cyan patch represents the disallowed region coming from the constraints on light neutrino masses.

$(1.65 \leq \alpha \leq 2.8)$. On the other hand, allowed ranges for $\beta$ are restricted by $0.3 \leq \beta \leq 1$ for NH whereas for IH we have $1.9 \geq \beta \geq 1$. This implies $\beta$ values less than 1 are favored for NH whereas values greater than 1 are favored for IH. Similarly, in figure 6 we have plotted the allowed region in $\phi_{AC} - \phi_{BC}$ plane for NH (left panel, light red shaded region) an IH (right panel, blue shaded region) respectively. Here $\phi_{BC}$ between $0 - 2\pi$ is compatible for both of the hierarchies, however, two distinct regions for $\phi_{AC}$ are allowed, namely, $2.61 \leq \phi_{AC} \leq 5.38$ for NH and $\phi_{AC} \leq 2 \& 6 \leq \phi_{AC} \leq 2\pi$ for IH respectively. Clearly, the full range of $\phi_{BC}$ is allowed because it is not sensitive to low energy masses, and mixing and only appears in one of the Majorana phase $\alpha_{21}$. Thus the values of $\phi_{AC}$ crucially dictate the hierarchy of light neutrino masses. As an artifact of the considered flavor symmetry, combining the results from figure 5 and 6, we can conclude that with $\beta \leq 1$ and $2.61 \leq \phi_{AC} \leq 5.38$ one can reproduce NH whereas to obtain IH we need $\beta \geq 1$ and $0 \leq \phi_{AC} \leq 2$ (or $6 \leq \phi_{AC} \leq 2\pi$).

Now with the allowed values for $\alpha, \beta, \phi_{AC}$ and $\phi_{BC}$ obtained from figure 5 and 6, we are now equipped to study the correlation between neutrino mixing parameters and predictions associated with the phases and masses. Due to the presence of the $A_4$ discrete flavor symmetry, this model yields an interesting correlation among the observables appearing in neutrino mixing. Following eq. (4.8)–(4.10), we find one such important correlation between the atmospheric mixing angle $\theta_{23}$ and Dirac CP phase $\delta_{\text{CP}}$. This correlation is very crucial because still there are some unsettled issues with the measurement of these two oscillation
Figure 8. Sum of absolute neutrino masses $\sum m_i$ plotted against lightest neutrino mass for both NH (left panel, light red shaded region) and IH (right panel, blue shaded region). The area between the dashed lines represents $3\sigma$ allowed range and the cyan patch represents the area excluded by cosmology.

parameters such as (a) octant of $\theta_{23}$, i.e., $\theta_{23} < 45^\circ$ (lower octant, LO) or $\theta_{23} > 45^\circ$ (higher octant, HO) and (b) magnitude of Dirac CP phase $\delta_{CP}$. The $\delta_{CP} - \theta_{23}$ correlation obtained here is plotted in figure 7 and given by light red (blue) shaded region for NH (IH) in the left (right) panel and shades some light on the above mentioned unsettled issues. It is evident from figure 7 that for both hierarchies only higher octant of $\theta_{23}$ is favoured (i.e. $\theta_{23} \geq 45^\circ$) in our analysis. Furthermore, the cyan patch in both of the panels represents the disallowed region for $\delta_{CP}$ in order to satisfy the limits on light neutrino masses [5]. From figure 7 the allowed regions for Dirac CP phase $\delta_{CP}$ are given by $-1.57 \leq \delta_{CP} \leq 1.37$ and $1.4 \leq \delta_{CP} \leq 1.57$ for NH, whereas for IH the predictions are $-1.57 \leq \delta_{CP} \leq 0.5$ and $0.86 \leq \delta_{CP} \leq 1.57$. The disallowed region for $\delta_{CP}$ is small in IH of neutrino masses compared to the NH as cosmology puts a tighter constraint on NH compared to IH [5].

Again, with the permitted values of $\alpha$, $\beta$, $\phi_{AC}$ and $\phi_{BC}$, we have the predictions on light neutrino masses which can be understood from the correlation plot of $\sum m_i$ vs lightest neutrino ($m_1$ for NH and $m_3$ for IH) mass for both hierarchies as given in figure 8. Here also the allowed regions are given by light red (blue) shaded region for NH (IH) in the left (right) panel. The horizontal cyan region in each plot represents the disallowed regions mentioned earlier in this subsection. Clearly, this framework predicts that the lightest neutrino mass can take smaller values for NH ($m_{\text{lightest}} \geq 0.0012$ eV), compared to the IH scenario ($m_{\text{lightest}} \geq 0.014$ eV). In figure 7, we showed the $\sin^2 \theta_{23} - \delta_{CP}$ correlation which is generic feature for the TM$_2$ mixing. Now to elucidate the additional predictions which go beyond TM$_2$ mixing in FSS, we present a few additional correlations among neutrino masses and mixing. Hence in figure 9 we plot the correlation between the sum of absolute neutrino masses $\sum m_i$ and other observables such as $\delta_{CP}$, $\sin^2 \theta_{23}$, $\sin^2 \theta_{12}$ and $\sin^2 \theta_{13}$ in the FSS framework. Here the upper panel with light red shaded regions represents the allowed parameter space for NH whereas the lower panel with blue shaded regions represents the allowed parameter space for IH and the cyan patch represents the area excluded by cosmology. Here from the $\sum m_i - \delta_{CP}$ (first column of figure 9) it is clear that the bound on the absolute neutrino masses disallows some regions of the Dirac CP phase and $\sum m_i$, $\sin^2 \theta_{23}$, $\sin^2 \theta_{12}$ correlations are characteristics signature of this model. From the third
Figure 9. The correlation plot between $\sum m_i$ and $\delta_{\text{CP}}$, $\sin^2 \theta_{23}$, $\sin^2 \theta_{12}$, $\sin^2 \theta_{13}$. The upper panels are schematics for NH whereas the lower panel is for IH and the cyan patch represents the area excluded by cosmology.

Figure 10. The correlation plot between two Majorana phases $\alpha_{21}$ and $\alpha_{31}$ for NH (left panel) and IH (right panel) for $3\sigma$ allowed range of neutrino oscillation data [5].

column of figure 9 we find that $\sin^2 \theta_{12}$ in FSS is restricted within a narrow range (the plot corresponds to the $3\sigma$ allowed range). The Majorana phases in our analysis, for the most general case, can be evaluated using the expressions given in eq. (4.19) and (4.20) with the allowed regions of $\alpha$, $\phi_{AC}$, $\phi_{BC}$ given in figure 5 and 6. Thus we can constrain the Majorana phases using the low-energy neutrino oscillation data. In figure 10, present a correlation plot in the $\alpha_{21} - \alpha_{31}$ plane for NH (left panel, light red shaded region) and IH (right panel, blue shaded region) respectively with $3\sigma$ allowed ranges of neutrino oscillation data [5].

Finally, with the estimation for neutrino masses and phases in hand, we are now able to plot the effective mass parameter characterizing neutrinoless double beta decay ($m_{\beta\beta}$) given in eq. (5.2) and eq. (5.3). In figure 11, we have plotted $m_{\beta\beta}$ against the lightest neutrino mass for both NH ($m_1$) and IH ($m_3$) respectively by light red and blue shaded regions respectively. The predictions for $m_{\beta\beta}$ are $1-30$ meV for NH and 16-60 meV for IH. Here the green and magenta shaded regions represent $3\sigma$ allowed regions for the $m_{\beta\beta}$ predictions for NH and IH respectively. The vertical cyan-shaded regions represent the cosmological upper limit on the sum of absolute neutrino masses ($\sum m_i$). The gray shaded region represents the upper limit for $m_{\beta\beta}$ by combined analysis of KamLAND-Zen [145] and GERDA [146].
Figure 11. Neutrinoless double beta decay effective mass parameter $m_{\beta\beta}$ is plotted as a function of the lightest neutrino mass for the case of both NH (light red shaded region) and IH (blue shaded region). Green and magenta regions are three neutrino-allowed regions when all the parameters are varied within their 3$\sigma$ range. The gray-shaded region, brown dashed, and black dotted-dash lines stand for experimental limits by KamLAND-Zen+GERDA, LEGEND-1k, and nEXO respectively. The vertical cyan area represents a disallowed region by cosmology.

Figure 12. The correlation plot between $\sum m_i$ and $\delta_{CP}$, $\sin^2 \theta_{23}$, $\alpha_{21}$, $\alpha_{31}$. The upper panels are schematics for NH whereas the lower panel is for IH.

Experiments and predictions for $m_{\beta\beta}$ in our model fall within this upper limit. In this plot the brown dashed and black dotted-dash lines stand for future sensitivities of the LEGEND-1k [147] and nEXO [148] experiments respectively. Thus these near-future experiments have the potential to almost entirely falsify the IH prediction and probe a major part prediction for $m_{\beta\beta}$ for NH of light neutrino mass. Guided by the symmetry construction, the model also sets a lower limit on the effective mass parameter as $m_{\beta\beta} \geq 1$ meV for NH. Similar to figure 9, to obtain additional predictive correlations among neutrino masses and mixing, we plot a few more schematics. Thus in figure 12 we present the dependence of $m_{\beta\beta}$ on $\sin^2 \theta_{23}, \delta_{CP}, \alpha_{21}$ and $\alpha_{31}$ for the FSS framework. Here the upper panel with light red shaded regions represents the allowed parameter space for NH. The lower panel with blue shaded regions represents the allowed parameter space for IH. Together with figure 9, the correlations between $\sum m_{\beta\beta}, \delta_{CP}, \sin^2 \theta_{23}, \alpha_{21}, \alpha_{31}$ presented in figure 12 are typical features of FSS. The fate of the present model crucially depends on these correlations.
Table 6. Summary of the different cases of our numerical analysis. The \( \checkmark \) and \( \times \) symbols stand for allowed and disallowed regimes. The numbers in the parenthesis represent the allowed ranges in each scenario.

| Cases       | NH | IH | \( \delta_{\text{CP}} \) | \( \alpha_{21} \) | \( \alpha_{31} \) | \( \sum m_i \) (eV) | \( m_{\beta\beta} \) (eV) |
|-------------|----|----|-----------------|---------------|----------------|---------------|----------------|----------------|
| Case I      |    | \( \checkmark \) | 0,\( \pi \)     | 0             | 0             | (0.1408, 0.1496) | (0.057,0.059) |
| Case II     |    | \( \checkmark \) | 0,\( \pi \)     | (0.2\( \pi \)) | 0             | (0.1408-0.1496) | (0.019-0.054)  |
| Case III    | \( \checkmark \) | \( \times \) | (0-0.78)        | (-2.362,-1.26) | (0.61,1.20)   | (0.0773,0.12)  | (0.0059,0.026) |
| Case IV     | \( \checkmark \) | \( \times \) | (0.78)          | (0.60,1.84)   | (0.61,1.20)   | (0.0773,0.12)  | (0.013,0.025)  |
| General Case| \( \checkmark \) | \( \times \) | (-1.5,0.6)      | (-3)          | (-2.85,1.13)  | (0.06,0.12)    | (0.001,0.03)   |
| General Case| \( \times \) | \( \checkmark \) | (-1.5,0.6)      | (-3)          | (-1.5,-0.01)  | (0.115,0.15)   | (0.016,0.06)   |

To sum up the full numerical analysis, we present table 6, where we give a summary of all the results including both special cases and general cases. In our analysis, we divided our special case into four categories depending on the values of the input relative phases \( \phi_{AC} \) and \( \phi_{BC} \). In Case I, where \( \phi_{AC} = \phi_{BC} = 0 \), only IH of neutrino masses are allowed and \( \delta_{\text{CP}} = 0 \) or \( \pi \), and two Majorana phases are coming out to be zero. In case II, where \( \phi_{AC} = 0 \) but \( \phi_{BC} \neq 0 \), IH is predicted and the main difference occurs in the prediction of Majorana phases. As a result, the prediction on \( m_{\beta\beta} \) is different from Case I. In both Case III and Case IV where \( \phi_{AC} = \phi_{BC} \neq 0 \) or \( \phi_{AC} \neq 0, \phi_{BC} = 0 \), respectively, NH of neutrino masses are predicted. The allowed regions of \( \delta_{\text{CP}} \) are given in table 6 which are the same for both these cases. The prediction on the Majorana phase \( \alpha_{31} \) is the same for both of these cases whereas the prediction on the \( \alpha_{21} \) and hence \( m_{\beta\beta} \) are different in both cases. Finally, as a most general case study, in Case V we vary \( \phi_{AC} \) and \( \phi_{BC} \) arbitrarily within its full range. The analysis constrains \( \delta_{\text{CP}} \) into two particular regions. We also find the values of the parameters \( \beta \) and the phase \( \phi_{AC} \) plays a crucial role in determining the neutrino mass hierarchy with distinct limits on neutrino masses for each hierarchy.

6 Phenomenological implications for the FSS model

Owing to the flavor symmetry of the model, charged lepton sector’s Yukawa couplings are diagonal so the flavors are conserved. But there are sources of the lepton flavor violation arising outside the charged lepton sector from both the Yukawa couplings \( y_N \) and \( y_s \) associated with the seesaw and scotogenic contributions, respectively. These Yukawa interactions lead to lepton flavor violating processes such as \( l_\alpha \rightarrow l_\beta \gamma, l_\alpha \rightarrow 3l_\beta \) (\( \alpha, \beta = e, \mu, \tau \)) etc. For related studies on lepton flavor violation in a pure scotogenic model see [149–151]. Studies of such lepton flavor-violating processes in our framework depend heavily on the proposed symmetry configuration as described below.
In our framework, the branching ratios of the $l_\alpha \to l_\beta \gamma$ decays for the scotogenic contribution can be written as [102, 149]

$$
\text{Br}(l_\alpha \to l_\beta \gamma) \approx \frac{3\pi\bar{\alpha}}{64G_F} |Y_F^{\beta\gamma}|^2 \left( \frac{1}{m_{\eta^+}} \right)^2 \left( F \left( \frac{M^2_j}{m_{\eta^+}^2} \right) \right)^2 \text{Br}(l_\alpha \to l_\beta \nu_\alpha \bar{\nu}_\beta). \tag{6.1}
$$

Here $G_F$ is the Fermi constant, $\bar{\alpha} = e^2/4\pi$ is the fine structure constant. $Y_F$ is the Yukawa coupling matrix coming from the scotogenic contribution given in eq. (3.9). The expression for the function $F$ is given by

$$
F(x) = \frac{1 - 6x - 3x^2 + 2x^3 - 6x^2\log x}{6(1 - x)^4}. \tag{6.2}
$$

In our discussion, considered discrete symmetries dictate the structure of the associated Yukawa couplings. Due to the specific VEV alignment of the $A_4$ triplet flavon $\phi_s$ and its contraction (following the multiplication rules given in the appendix) with the non-trivial $A_4$ singlet $\xi$ (charged as 1'), we find $Y_F^{\beta\gamma} = 0$ as given in eq. (3.9). Therefore owing to the $A_4$ symmetry, the scotogenic part alone yields a vanishing contribution in the lepton flavor violating decays for $\mu \to e\gamma$ and $\tau \to \mu\gamma$. The only non-vanishing contribution arising in the $l_\alpha \to l_\beta \gamma$ decays originates from the $\tau \to e\gamma$ decay and the branching fraction can be written as

$$
\text{Br}(\tau \to e\gamma) \approx \frac{3\pi\bar{\alpha}}{64G_F} |y_s y_s^* \epsilon|^2 \left( \frac{1}{m_{\eta^+}} \right)^2 \left( F \left( \frac{M^2_j}{m_{\eta^+}^2} \right) \right)^2 \text{Br}(\tau \to e\nu_\tau \bar{\nu}_e), \tag{6.3}
$$

$$
= \frac{3\pi\bar{\alpha}}{64G_F} \left( \frac{|C|}{F(m_{\eta^+}, m_{\eta^+}, M_f)} \right)^2 \left( \frac{1}{m_{\eta^+}} \right)^2 \left( F \left( \frac{M^2_j}{m_{\eta^+}^2} \right) \right)^2 \text{Br}(\tau \to e\nu_\tau \bar{\nu}_e). \tag{6.4}
$$

In the above $\epsilon = v_f/\Lambda$ where we assume flavons VEVs to be equal, i.e., $v_\xi = v_{s,a} = v_f$. There is also possible another type of the flavor violating decay $l_\alpha \to 3l_\beta$ ($l_\alpha \to l_\beta l_\beta$) and the corresponding branching ratio is given by [149]

$$
\text{Br}(l_\alpha \to 3l_\beta) \approx \frac{3\pi\bar{\alpha}^2}{512G_F^2} |Y_F^{\beta\gamma}|^2 \left( \frac{1}{m_{\eta^+}} \right)^2 \left( G \left( \frac{m_\alpha}{m_\beta} \right) \right)^2 \left( F \left( \frac{M^2_j}{m_{\eta^+}^2} \right) \right)^2 \text{Br}(l_\alpha \to l_\beta \nu_\alpha \bar{\nu}_\beta). \tag{6.5}
$$

where

$$
G \left( \frac{m_\alpha}{m_\beta} \right) = \left( \frac{16}{3} \log \left( \frac{m_\alpha}{m_\beta} \right) - \frac{22}{3} \right). \tag{6.6}
$$

Since in FSS we have $Y_F^{\mu\nu} = 0$, then branching fractions for $\mu \to 3e$ and $\tau \to 3\mu$ decays coming through the scotogenic contribution also vanishes. The only non-vanishing contribution originates from the $\tau \to 3e$ decay, and the branching fraction can be written as

$$
\text{Br}(\tau \to 3e) \approx \frac{3\pi\bar{\alpha}^2}{512G_F^2} |y_s y_s^* \epsilon|^2 \left( \frac{1}{m_{\eta^+}} \right)^2 \left( G \left( \frac{m_\tau}{m_e} \right) \right)^2 \left( F \left( \frac{M^2_j}{m_{\eta^+}^2} \right) \right)^2 \text{Br}(\tau \to e\nu_\tau \bar{\nu}_e). \tag{6.7}
$$

Clearly, from eq. (6.3) and (6.7) we find with fixed values of the mass parameters that $\text{Br}(\tau \to e\gamma)$ and $\text{Br}(\tau \to 3e)$ depend upon $y_s$ as well as $\epsilon$ (the ratio of flavon VEVs $v_f$.
to the cut-off scale $\Lambda$). Hence in figure 13 we present contour plots for the corresponding branching fractions in the $y_s-\epsilon$ plane considering $m_{\eta^+} = 600$ GeV and $M_f$ = 10 TeV. The near future sensitivity of these two branching ratios is of the order of $10^{-9}$ [152]. Therefore we have plotted contours for the branching fraction of $\tau \to e\gamma$ (left panel) and $\tau \to 3e$ (right panel) fixed at $10^{-9}$, $10^{-12}$ and $10^{-15}$ given by the dashed, dotted and continuous lines respectively. The $y_s-\epsilon$ correlation in figure 13 also helps us to estimate the ratio $\epsilon$, and we find $\epsilon \leq 1$ since it is suppressed by the cut-off scale of the theory.

Now, for the type-I seesaw contribution in the lepton flavor violating decays, the decay of the form of $l_\alpha \to l_\beta \gamma$ will put a constraint on the FSS parameters. The branching ratio for such type of decay in our framework can be written as [153–157]

\[
\begin{align*}
\text{Br}(l_\alpha \to l_\beta \gamma) & \approx \frac{3\tilde{\alpha}v^4}{8\pi} \left| \sum_{i=1}^{2} K_{i\alpha} K_{i\beta} f \left( \frac{M_N^2}{M_W^2} \right) \right|^2, \\
& = \sum_{i=1}^{2} \frac{3\tilde{\alpha}v^4}{8\pi M_N} \left| (Y_N)_{i\beta} (Y_N^{\dagger})_{i\alpha} f \left( \frac{M_N^2}{M_W^2} \right) \right|^2
\end{align*}
\]
where $\mathcal{K} = Y_N^\dagger (M_R^{-1})^* Y_N = M_D/v$, as obtained from eq. (3.4). Similarly to the scotogenic contribution, the VEV alignment of the $A_4$ flavon $\phi_s$ once again plays a crucial role in obtaining the estimation for the branching ratio $l_\alpha \rightarrow l_\beta \gamma$ decays. The VEV configuration of $\phi_s$ is such that it gives rise to $(Y_N)_{e1} = 0$ (see eq. (3.4)) and the contribution associated with $N_1$ essentially vanishes for $\mu \rightarrow e\gamma$ and $\tau \rightarrow e\gamma$. Therefore only surviving contribution in these decays originates from $N_2$. Again due to the $A_4$ flavor symmetry we find $(Y_N)_{e2} = (Y_N)_{\mu 2} = (Y_N)_{\tau 2}$ as written in the second column of $M_D$, see eq. (3.4). As a result, the expression for the branching fraction for the above two decays will be the same i.e., $\text{Br}(\mu \rightarrow e\gamma) = \text{Br}(\tau \rightarrow e\gamma)$. Out of these two decays, the most stringent constraint comes from the $\mu \rightarrow e\gamma$ decay and in the following, we discuss the numerical analysis for the same.

Following eq. (6.9), in our framework, the branching ratio of $\mu \rightarrow e\gamma$ can be written as

$$\text{Br}(\mu \rightarrow e\gamma) = \frac{3\tilde{\alpha} v^4}{8\pi M_{N_2}^4} \left| (y_{N_2} y_{N_2}^*)^2 f\left(\frac{M_{N_2}^2}{M_W^2}\right) \right|^2 = \frac{3\tilde{\alpha}}{8\pi M_{N_2}^2} \beta^2 |C|^2 \left( f\left(\frac{M_{N_2}^2}{M_W^2}\right) \right)^2. \quad (6.10)$$

where we have used $B = v^2 y_{N_2}^2 \epsilon^2 / M_{N_2}$ from eq. (3.6) and the definition $\beta = |B|/|C|$ to obtain eq. (6.10). The loop function $f(x)$ in eq. (6.10) can be written as

$$f(x) = \frac{x(2x^3 + 3x^2 - 6x - 6x^2\log(x) + 1)}{2(1-x)^4}. \quad (6.11)$$

From eq. (6.10) we find that the contribution in $\text{Br}(\mu \rightarrow e\gamma)$ coming from the seesaw mechanism depends on the mass of the heavy right-handed neutrino $N_2$, $\beta$ and $|C|$. The parameters $\beta$ and $|C|$ are already fixed by neutrino data, as discussed in section 5. As $\text{Br}(\mu \rightarrow e\gamma) \propto \beta^2 |C|^2$, in figure 14 left panel we have shown the variation of $\beta^2 |C|^2$ with $|C|$ for the most general case of our analysis. In this plot, the light red and blue shaded regions represent an estimation for $\beta^2 |C|^2$ as a function of $|C|$ for NH and IH light neutrino masses, respectively. Furthermore, figure 14 left panel also depicts that $\beta^2 |C|^2$ acquires higher values for IH compared to NH. This is because for NH $\beta$ is smaller ($\beta \leq 1$) compared to IH ($\beta \geq 1$) for similar values of $|C|$. With this when we plot $\text{Br}(\mu \rightarrow e\gamma)$ for seesaw contribution in our scenario as a function of $M_{N_2}$ in the right panel of figure 14. Here the light red and blue shaded region represent prediction for $\text{Br}(\mu \rightarrow e\gamma)$ for NH and IH of light neutrino mass. As the $\beta^2 |C|^2$ takes higher values for IH compared to NH, the branching ratio for the $\mu \rightarrow e\gamma$ decay is higher for IH compared to NH as seen in this figure. The horizontal orange shaded region represents current experimental limit $\text{Br}(\mu \rightarrow e\gamma) \leq 4.2 \times 10^{-13}$ [158] and the purple dashed stands for the future reach $\text{Br}(\mu \rightarrow e\gamma) \leq 6 \times 10^{-14}$ [159] which puts a lower limit on the mass of $N_2$ in the range $0.2$ TeV $\leq M_{N_2} \leq 1.6$ TeV ($1.7$ TeV $\leq M_{N_2} \leq 3$ TeV) for NH (IH) of light neutrino mass. Again, following eq. (6.9), the branching ratio for $\tau \rightarrow \mu\gamma$ in case of the seesaw contribution can be expressed as

$$\text{Br}(\tau \rightarrow \mu\gamma) = \sum_{i=1}^2 \frac{3\tilde{\alpha} v^4}{8\pi M_{N_i}^4} \left| (Y_N)^{\tau i}(Y_N^\dagger)^{\mu i} f\left(\frac{M_{N_i}^2}{M_W^2}\right) \right|^2, \quad (6.12)$$

$$= \frac{3\tilde{\alpha}}{8\pi M_{N_1}^2} \alpha^2 |C|^2 \left( f\left(\frac{M_{N_1}^2}{M_W^2}\right) \right)^2 + \frac{3\tilde{\alpha}}{8\pi M_{N_2}^2} \beta^2 |C|^2 \left( f\left(\frac{M_{N_2}^2}{M_W^2}\right) \right)^2. \quad (6.13)$$
Figure 15. Contour plot for branching fraction of $\mu \to e\gamma$ (fixed at $10^{-13}$, $10^{-14}$, $10^{-15}$, $10^{-16}$ respectively) in the $M_{N_1}$-$M_{N_2}$ plane. The left and right panels represent allowed regions (obtained from figure 5) for NH and IH respectively.

where we have used the relations $A = v^2 y^2 N_1 \epsilon^2 / M_{N_1}$ and $\alpha = |A| / |C|$. Thus the contribution to $Br(\tau \to \mu\gamma)$ coming from the seesaw mechanism depends on the masses of the heavy right-handed neutrinos $N_{1,2}$ as well as $\alpha, \beta$ and $|C|$. The parameters $\alpha, \beta$, and $|C|$ are already fixed to satisfy correct neutrino oscillation data as discussed in section 5. Hence in figure 15 we have plotted different contours for $Br(\tau \to \mu\gamma)$ in the $M_{N_1}$-$M_{N_2}$ plane, corresponding to the $3\sigma$ allowed range of neutrino data for both NH and IH. In two panels, we have plotted the contours showing the $M_{N_1}$-$M_{N_2}$ correlations with branching fractions fixed at $10^{-13}$, $10^{-14}$, $10^{-15}$ and $10^{-16}$ (given by yellow, magenta, blue and green regions respectively). Here it is worth mentioning that among $l_\alpha \to l_\beta \gamma$ decays, as a consequence of the flavor symmetric construction, only the seesaw part contributes to the $\mu \to e\gamma$ and $\tau \to \mu\gamma$ decays. Whereas in the branching fraction of the decay $\tau \to e\gamma$, both scotogenic and seesaw parts contribute. To understand the relative magnitude of these two contributions involved in the $\tau \to e\gamma$ decay, we define a ratio $R$ as

$$R = \frac{Br(\tau \to e\gamma)_{\text{scoto}}}{Br(\tau \to e\gamma)_{\text{seesaw}}}.$$  

(6.14)

Now following eq. (6.3) and eq. (6.10) along with eq. (3.10), in the FSS framework we find that the ratio $R$ is proportional to $1/\beta^2$ for specific values of the scoto-seesaw mass parameters. In figure 16, we have plotted $R$ considering $m_{\eta^+} = 600$ GeV, $m_{\eta_R} = 650$ GeV, $M_f = 10^5$ GeV, and $M_{N_2} = 10^6$ GeV for both NH (left panel) and IH (right panel) respectively. In the panels, the brown dashed, blue continuous, and green dotted lines represent the estimation for $R$ with $m_{\eta_R} = 655$ GeV, $660$ GeV, and $665$ GeV respectively. These lines correspond to the allowed range for $\beta$ obtained earlier. Since $R \geq 10^3$ ($R \geq 2 \times 10^2$) for NH (IH), we can conclude that the scotogenic part dominates the seesaw contribution in the lepton flavor violating decay such as $\tau \to e\gamma$.

The scotogenic contribution in our analysis offers us the opportunity to explain the nature of dark matter. In this model, there is an inherent dark $\mathbb{Z}_2$ symmetry that ensures the stability of the lightest dark particle, and three feasible dark matter candidates exist. These are, namely, the fermionic dark matter $f$ and scalar dark matter, which are real and imaginary components of $\eta$, given by $\eta_R$ and $\eta_I$, respectively. When the lightest dark
Figure 16. The ratio ($R$) of the branching fractions for the scoto and seesaw contributions versus $\beta$ for both NH (left panel) and IH (right panel).

particle originates from $\eta$, it resembles the inert Higgs doublet model [160]. Considering $\eta_R$ as the DM candidate, there exists several annihilation and co-annihilation channels in this model which involve annihilation to quarks and leptons, SM gauge bosons, and the Higgs boson such as $\eta_R\eta_R \rightarrow ZZ, W^+W^-, hh, q\bar{q}$ etc. Collectively they all contribute to the relic abundance of $\eta_R$. Present dark matter abundance is often expressed in terms of the relic density parameter $\Omega^2 h$ and reported to be $\Omega^2 h = 0.120 \pm 0.001$ at 68\% CL [161]. In a minimal scito-seesaw framework [108], it has been argued that correct relic density can be obtained for three different mass ranges for $\eta_R$. These are respectively $m_{\eta_R} < 50$ GeV, $70$ GeV $< m_{\eta_R} < 100$ GeV and $m_{\eta_R} > 550$ GeV. The dark matter mass in the range $m_{\eta_R} < 50$ is disallowed as it is in conflict with the LHC Higgs invisible decay limit [162]. The intermediate region $70$ GeV $< m_{\eta_R} < 100$ GeV is not completely ruled out by the LHC and LEP data and dark matter mass in the range $m_{\eta_R} > 550$ GeV is not affected by the collider constraints. The phenomenology of fermionic dark matter is worth exploring and is beyond the scope of the current study.

7 Conclusions

We have proposed the flavor-scoto-seesaw (FSS) model that establishes a common origin of the nonzero $\theta_{13}$ and cosmological dark matter. The framework is based on the $A_4$ flavor symmetry where both type-I seesaw and scotogenic mechanisms contribute to the effective light neutrino mass. FSS explains observed neutrino masses and mixing angles provides rich phenomenology and accommodates potential dark matter candidates. Guided by the discrete symmetry, we show that the minimal type-I seesaw first reproduces the widely popular TBM mixing, a first-order approximation of the lepton mixing matrix. Subsequently, the scotogenic contribution acts as a requisite deviation to the TBM mixing and addresses the issue of the nature of dark matter. We also demonstrate that the neutrino mixing pattern exhibits TM$_2$ mixing scheme (a viable descendent of the TBM mixing pattern) within this scoto-seesaw scenario.

The model which we construct here is highly predictive in nature. Using the current experimental observation on neutrino oscillation and other cosmological limits, we found that the allowed parameter space in the FSS model restricts some of the key observables associated with neutrinos (the atmospheric mixing angle, Dirac and Majorana CP phases,
effective mass parameter appearing in the neutrinoless double beta decay) and crucially dictates the lepton flavor violating decays. To understand the behaviours of the parameters involved (namely, $\alpha, \beta, \phi_{AC}, \phi_{BC}$), we divide the numerical analysis into a few specific cases based on the choice of the associated phases ($\phi_{AC}, \phi_{BC}$) and then carried out the complete general numerical analysis. These limiting cases can easily distinguish between light neutrino masses’ normal and inverted hierarchy. For example, when the relative phase between the seesaw contribution associated with right-handed neutrino $N_1$ and the scotogenic contribution is considered zero (i.e., $\phi_{AC} = 0$ for Case I and II), only inverted hierarchy is allowed. On the other hand, when $\phi_{AC} \neq 0$ (Case III and IV), only normal hierarchy is allowed. Considered discrete flavor symmetries play an instrumental role in producing such distinctive constraints. Subsequently, for the most general case, we carried out the numerical analysis for all possible choices of the parameters involved and found that normal hierarchy can be realized only with $\beta \leq 1$ and $2.61 \leq \phi_{AC} \leq 5.38$ whereas to realize inverted hierarchy we need $\beta \geq 1$ and $0 \leq \phi_{AC} \leq 2$ (or $6 \leq \phi_{AC} \leq 2\pi$).

This analysis also predicts the atmospheric mixing angle lying in the upper octant, in good agreement with the latest global fit neutrino oscillation data, and restricts Dirac CP phase $\delta_{CP}$ within $-1.57 \leq \delta_{CP} \leq 1.37$ (and $1.4 \leq \delta_{CP} \leq 1.57$) for normal hierarchy and $-1.57 \leq \delta_{CP} \leq 0.5$ (and $0.86 \leq \delta_{CP} \leq 1.57$) for inverted hierarchy. Along with the Dirac CP phase $\delta_{CP}$, the Majorana phases also get restricted in our analysis. Furthermore, we obtain a lower limit on the lightest neutrino mass as $m_{\text{lightest}} \geq 0.0012$ eV for normal hierarchy and $m_{\text{lightest}} \geq 0.014$ eV for inverted hierarchy. We have also estimated the prediction for the effective mass parameter $m_{\beta\beta}$ characterizing the neutrinoless double beta decay and found it to be in the range $1 - 30$ meV for normal hierarchy and $16 - 60$ meV for inverted hierarchy, respectively. These values are within the reach of future neutrinoless double beta decay experiments. The unique prediction on the correlation among the masses and mixing such as $m_{\beta\beta} - \delta_{CP}$, $\sin^2 2\theta_{23}$, $\alpha_{21}$, $\alpha_{31}$ as well as $\sum m_i - \delta_{CP}$, $\sin^2 2\theta_{23}$, etc. are typical features of the discussed FSS model. For example, the sum of absolute neutrino masses crucially dictates the allowed ranges for $\delta_{CP}$ mentioned above. In the end, we also comment on the phenomenological implications, such as lepton flavor violation and the prospects of dark matter candidates in such a scenario. As a consequence of the flavor structure, the scotogenic part of the model does not contribute to the lepton flavor violating decays such as $\mu \rightarrow e\gamma$ and $\tau \rightarrow \mu\gamma$, and a lower limit on the mass of the heavy right-handed neutrinos can be obtained from the constraints on the branching ratios involving such decays. On the other hand, both scoto and seesaw parts contribute to the decay $\tau \rightarrow e\gamma$. However, the scotogenic part dominates this rare decay and a constraint on the ratio of flavon VEVs to the cut-off scale of the theory can be obtained. A detailed discussion of the phenomenological aspects in this direction is dedicated to future investigations.

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A $A_4$ symmetry

$A_4$ is a discrete group of even permutations of four objects. Geometrically, it is an invariance group of a tetrahedron. It has 12 elements which can be generated by two basic objects $S$ and $T$ which obey the following relation

$$S^2 = T^3 = (ST)^3 = 1$$  \hspace{1cm} (A.1)

The $A_4$ group has three one-dimensional irreducible representations $1$, $1'$ and $1''$ and one three dimensional irreducible representation $3$. Products of the singlets and triplets are given by

$$1 \otimes 1 = 1; \quad 1' \otimes 1'' = 1, \quad 1'' \otimes 1' = 1', \quad 3 \otimes 3 = 1 + 1' + 1'' + 3_s + 3_a,$$  \hspace{1cm} (A.2)

where the subscripts “$s$” and “$a$” denote symmetric and antisymmetric part respectively. Writing two triplets as $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ respectively, their products are given by

$$1 \sim x_1 y_1 + x_2 y_3 + x_3 y_2,$$  \hspace{1cm} (A.5)

$$1' \sim x_3 y_3 + x_1 y_2 + x_2 y_1,$$  \hspace{1cm} (A.6)

$$1'' \sim x_2 y_2 + x_1 y_3 + x_3 y_1,$$  \hspace{1cm} (A.7)

$$3_s \sim \begin{pmatrix} 2x_1 y_1 - x_2 y_3 - x_3 y_2 \\ 2x_3 y_3 - x_1 y_2 - x_2 y_1 \\ 2x_2 y_2 - x_1 y_3 - x_3 y_1 \end{pmatrix},$$  \hspace{1cm} (A.8)

$$3_a \sim \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_1 y_2 - x_2 y_1 \\ x_3 y_1 - x_1 y_3 \end{pmatrix}.$$  \hspace{1cm} (A.9)

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³For a detailed discussion on $A_4$ see refs. [19, 20].
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