A new bound on Erdős distinct distances problem in the plane over prime fields

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Abstract

In this paper we obtain a new lower bound on the Erdős distinct distances problem in the plane over prime fields. More precisely, we show that for any set \( A \subset \mathbb{F}_p^2 \) with \( |A| \leq p^{7/6} \) and \( p \equiv 3 \mod 4 \), the number of distinct distances determined by pairs of points in \( A \) satisfies

\[
|\Delta(A)| \gtrsim |A|^{1/2 + 149/4214}.
\]

Our result gives a new lower bound of \( |\Delta(A)| \) in the range \( |A| \leq p^{1+149/4065} \).

The main tools in our method are the energy of a set on a paraboloid due to Rudnev and Shkredov, a point-line incidence bound given by Stevens and de Zeeuw, and a lower bound on the number of distinct distances between a line and a set in \( \mathbb{F}_p^2 \). The latter is the new feature that allows us to improve the previous bound due Stevens and de Zeeuw.

1 Introduction

The celebrated Erdős distinct distances problem asks for the minimum number of distinct distances determined by a set of \( n \) points in the plane over the real numbers. The breakthrough work of Guth and Katz [6] shows that a set of \( n \) points in \( \mathbb{R}^2 \) determines at least \( Cn/\log(n) \) distinct distances. The same problem can be considered in the setting of finite fields.

Let \( \mathbb{F}_p \) be the prime field of order \( p \). The “distance” formula between two points \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) in \( \mathbb{F}_p^2 \) is defined by

\[
||x - y|| := (x_1 - y_1)^2 + (x_2 - y_2)^2.
\]

While this is not a distance in the traditional sense, the definition above is a reasonable analog of the Euclidean distance in that it is invariant under orthogonal transformations.

For \( A \subset \mathbb{F}_p^2 \), let

\[
\Delta(A) = \{||x - y|| : x, y \in A\}
\]

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and let $|\Delta(A)|$ denote its size. It has been shown in a remarkable paper of Bourgain, Katz, and Tao \[3\] that if $|A| = p^\alpha$, $0 < \alpha < 2$, then we have

$$|\Delta(A)| \geq |A|^{\frac{1}{2} + \varepsilon},$$

for some $\varepsilon = \varepsilon(\alpha) > 0$.

This result has been quantified and improved over time. The recent work of Stevens and De Zeeuw \[11\] shows that

$$|\Delta(A)| \geq |A|^{\frac{1}{2} + \frac{1}{30}} = |A|^{\frac{8}{15}},$$

(1)

under the condition $|A| \ll p^{\frac{11}{6}}$.

Here and throughout, $X \ll Y$ means that there exists $c_1 > 0$, independent of $p$, such that $X \leq c_1 Y$, $X \gg Y$ means $X \gg (\log Y)^{-c_2} Y$ for some positive constant $c_2$, and $X \sim Y$ means that $c_3 X \leq Y \leq c_4 X$ for some positive constants $c_3$ and $c_4$.

For the case of large sets, Iosevich and Rudnev \[5\] used Fourier analytic methods to prove that for $A \subset \mathbb{F}_q^d$, where $q$ is not necessarily prime, with $|A| \geq q^{\frac{4d+1}{3}}$, we have $\Delta(A) = \mathbb{F}_q$. It was shown in \[7\] that the threshold $q^{\frac{4d+1}{3}}$ cannot in general be improved when $d$ is odd, even if we wish to recover a positive proportion of all the distances in $\mathbb{F}_q$. In prime fields, the question is open in dimension 3 and higher. In two dimensions, Chapman, Erdogan, Koh, Hart and Iosevich (\[4\]) proved that if $|A| \geq p^{\frac{4}{5}}$, $p$ prime, then $|\Delta(A)| \gg p$. In particular, their proof shows that if $Cp \leq |A| \leq p^{\frac{4}{3}}$ for a sufficiently large $C > 0$, then

$$|\Delta(A)| \gg \frac{|A|^{\frac{3}{2}}}{p}.$$ 

(2)

The 4/3 threshold was extended to all (not necessarily prime) fields by Bennett, Hart, Iosevich, Pakianathan and Rudnev (\[2\]). We refer the reader to \[5, 7\] for further details.

The main purpose of this paper is to improve the exponent $\frac{1}{2} + \frac{1}{30} = \frac{8}{15}$ on the magnitude of $\Delta(A)$ when $A$ is a relatively small set in $\mathbb{F}_p^2$ with $p \equiv 3 \pmod{4}$. The main tools in our arguments are the energy of a set on a paraboloid due to Rudnev and Shkredov, a point-line incidence bound given by Stevens and de Zeeuw, and a lower bound on the number of distinct distances between a line and a set in $\mathbb{F}_p^2$. The following is our main result.

**Theorem 1.1.** Let $\mathbb{F}_p$ be a prime field of order $p$ with $p \equiv 3 \pmod{4}$. For $A \subset \mathbb{F}_p^2$ with $|A| \ll p^{\frac{7}{10}}$, we have

$$|\Delta(A)| \gtrsim |A|^{\frac{122}{150}} = |A|^{\frac{4}{5} + \frac{169}{150}}.$$

**Remark 1.1.** The Stevens-de Zeeuw exponent in \[11\] is $0.533...$, whereas our exponent is $0.535358...$. Thus our result is better than that of the Stevens-de Zeeuw in the range $|A| \ll p^{7/6}$. On the other hand, our result is superior to \(2\) in the range $|A| \leq p^{\frac{1}{2} + \frac{1}{60}} = p^{\frac{49}{120}}$. In conclusion, Theorem \[11\] improves the currently known distance results in the range $|A| \ll p^{\frac{4214}{4065}}$.

**Remark 1.2.** While our improvement over the Steven-de Zeeuw estimate is small, we introduce a new idea, namely the count for the number of distances between a line and a set. This should lead to further improvements in the exponent in the future.

The rest of the paper is devoted to prove Theorem \[11\] and we always assume that $p \equiv 3 \pmod{4}$.
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2 Proof of Theorem 1.1

To prove Theorem 1.1 we make use of the following lemmas. The first lemma is a point-line incidence bound due to Stevens and De Zeeuw in [11].

Lemma 2.1 ([11]). Let $P$ be a set of $m$ points in $\mathbb{F}_p^2$ and $L$ be a set of $n$ lines in $\mathbb{F}_p^2$. Suppose that $m^{7/8} \leq n \leq m^{8/7}$ and $m^{-2}n^{13} \ll p^{15}$. Then we have

$$I(P, L) = \# \{(p, ℓ); p \in P, ℓ \in L\} \ll m^{11/15}n^{11/15}.$$

Let $P$ be a paraboloid in $\mathbb{F}_p^3$. For $Q \subset P$, let $E(Q)$ be the additive energy of the set $Q$, namely, the number of tuples $(a, b, c, d) \in Q^4$ such that $a - b = c - d$. Using Pach and Sharir’s argument in [9] and Lemma 2.1, Rudnev and Shkredov [8] derived an upper bound of $E(Q)$ as follows.

Lemma 2.2 ([8]). Let $P$ be a paraboloid in $\mathbb{F}_p^3$. For $Q \subset P$ with $|Q| \ll p^{26/21}$, we have

$$E(Q) \ll |Q|^{17/7}.$$

In the following theorem, we give a lower bound on the number of distinct distances between a set on a line and an arbitrary set in $\mathbb{F}_p^2$. This will be a crucial step in the proof of Theorem 1.1. The precise statement is as follows.

Theorem 2.3. Let $l$ be a line in $\mathbb{F}_p^2$, $P_1$ be a set of points on $l$, and $P_2$ be an arbitrary set in $\mathbb{F}_p^2$. Suppose that $|P_1|^{3/8} < |P_2| \ll p^{5/8}$. Then the number of distinct distances between $P_1$ and $P_2$, denoted by $|\Delta(P_1, P_2)|$, satisfies

$$|\Delta(P_1, P_2)| \gtrsim \min \left\{ |P_1|^{3/8}, |P_2|^{3/8}, |P_1||P_2|^{1/8}, |P_2|^{7/8}, |P_1|^{-1}|P_2|^{8/7} \right\}.$$

We will provide a detailed proof of Theorem 2.3 in Section 3. The following is a direct consequence from Theorem 2.3.

Corollary 2.4. Let $A \subset \mathbb{F}_p^2$ with $|A| \ll p^{7/6}$. Suppose there is a line containing at least $|A|^{17/23} + \epsilon$ points from $A$. Then we have

$$|\Delta(A)| \gtrsim \min \{|A|^{17/23}, |A|^\epsilon, |A|^{17/23} - \epsilon\}.$$

The above corollary shows that the exponent $8/15$ in (1) due to Stevens and De Zeeuw is improved when $A$ contains many points on a line.

We are now ready to prove Theorem 1.1.
Proof of Theorem 1.1: Let $\epsilon > 0$ be a parameter chosen at the end of the proof. Throughout the proof, we assume that that

$$\frac{16}{15} + 2\epsilon < \frac{8}{7},$$

which is equivalent with $\epsilon < 4/105$. If there is a line containing at least $|A|^{7/15+\epsilon}$ points from $A$, then we obtain by Corollary 2.4 that

$$|\Delta(A)| \gg \min\{|A|^\frac{8}{15+2}, |A|^\frac{8+\epsilon}{1+\epsilon}\}.$$  \hspace{1cm} (4)

Now we assume that there is no line supporting more than $|A|^{7/15+\epsilon}$ points from $A$.

For any line $l$ in $\mathbb{F}_p^2$ defined by the equation $ax + by - c = 0$, the vector $(a, b, c)$ is called a vector of parameters of $l$.

We first start with counting the number of triples $(z, x, y) \in A^3$ such that $||z - x|| = ||z - y||$, where $z = (a, b), x = (x_1, x_2), y = (y_1, y_2)$.

It follows from the equation $||z - x|| = ||z - y||$ that

$$(-2a)(x_1 - y_1) + (-2b)(x_2 - y_2) + (x_1^2 + x_2^2) - (y_1^2 + y_2^2) = 0.$$

This equation defines a line in $\mathbb{F}_p^2$ with the parameters

$$(x_1, x_2, x_1^2 + x_2^2) - (y_1, y_2, y_1^2 + y_2^2) = (x_1 - y_1, x_2 - y_2, x_1^2 + x_2^2 - y_1^2 - y_2^2).$$

Let $L$ be the set of these lines. It is clear that $L$ can be a multi-set.

Let $Q$ be the set of points of the form $(x, y, x^2 + y^2)$ with $(x, y) \in A$. We have $Q$ is a set on the paraboloid $z = x^2 + y^2$ and $|Q| = |A|$.

Notice that the number of triples $(z, x, y) \in A^3$ with the property $||z - x|| = ||z - y||$ is equivalent to the number of incidences between lines in $L$ and points in $-2A := \{(-2a_1, -2a_2) : (a_1, a_2) \in A\}$.

For each line $l$ in $L$, let $f(l)$ be the size of $l \cap (-2A)$, and $m(l)$ be the multiplicity of $l$. Let $L_1$ be the set of distinct lines in $L$.

Thus, we have

$$I(-2A, L) = \sum_{l \in L_1} f(l)m(l)$$

$$= \sum_{l \in L_1, f(l) \leq |A|^{7/15-\epsilon}} f(l)m(l) + \sum_{l \in L_1, |A|^{7/15-\epsilon} \leq f(l) \leq |A|^{7/15+\epsilon}} f(l)m(l)$$

$$= I_1 + I_2.$$

We now bound $I_1$ and $I_2$ as follows.

One can check that the size of $L$ is bounded by $|A|^2$, which implies that

$$I_1 \leq |A|^\frac{8}{15-\epsilon}.$$  \hspace{1cm} (3)

Let $L_2$ be the set of distinct lines $l$ in $L_1$ such that $|A|^{\frac{8}{15-\epsilon}} \leq f(l) \leq |A|^{\frac{8}{15+\epsilon}}$.
To bound $I_2$, we consider the following two cases:

**Case 1:** Suppose

$$\sum_{l \in L_2} m(l) \leq |A|^{2-\frac{15}{11}}.$$

We see that

$$I_2 = \sum_{l \in L_2} f(l)m(l) \leq |A|^\frac{37}{11} + |A|^\frac{44}{11},$$

since any line in $L_2$ contains at most $|A|^{7/15+\epsilon}$ points. Thus in this case we obtain that

$$I(-2A, L) = I_1 + I_2 \leq |A|^{\frac{37}{11}} + |A|^{\frac{44}{11}} \ll |A|^{\frac{37}{11} - \frac{4}{11}}. \tag{5}$$

Now, for each $t \in \mathbb{F}_p$, let $\nu(t)$ denote the number of pairs $(x, y) \in A^2$ such that $\|x - y\| = t$. We have

$$\nu^2(t) = \left( \sum_{x, y \in A: \|x - y\| = t} 1 \right)^2 = \left( \sum_{x \in A} 1 \times \left( \sum_{y \in A: \|x - y\| = t} 1 \right) \right)^2.$$

By the Cauchy-Schwarz inequality,

$$\nu^2(t) \leq |A| \sum_{x \in A} \left( \sum_{y \in A: \|x - y\| = t} 1 \right)^2 = |A| \sum_{x, y, z \in A: \|x - y\| = \|x - z\|} 1.$$

Summing over $t \in \mathbb{F}_p$, we obtain

$$\sum_{t \in \mathbb{F}_p} \nu^2(t) \leq |A| \sum_{x, y, z \in A: \|x - y\| = \|x - z\|} 1.$$

By the Cauchy-Schwarz inequality and the above inequality, we get

$$\frac{|A|^4}{|\Delta(A)|} \leq \sum_{t \in \mathbb{F}_p} \nu^2(t) \leq |A| \# \{(x, y, z) \in A^3: \|x - y\| = \|x - z\|\} \ll |A|I(-2A, L).$$

Combining the above inequality with (5), we obtain

$$|\Delta(A)| \gg \|A\|^\frac{37}{11} + \frac{44}{11}. \tag{6}$$

**Case 2:** Suppose

$$\sum_{l \in L_2} m(l) \geq |A|^{2-\frac{15}{11}}.$$

By the Cauchy-Schwarz inequality and Theorem 2.2, we have

$$\# \{(a - b, ||a|| - ||b||): a, b \in A, (a - b, ||a|| - ||b||) \text{ is a vector of parameters of a line in } L_2\} \gg \frac{(\sum_{l \in L_2} m(l))^2}{E(Q)} \gg |A|^\frac{44}{11} - \frac{30}{11}. \tag{7}$$
In the next step, we are going to show that

\[ |L_2| \leq |A|^{1 + \frac{15}{16}}. \]

Indeed, since each line in \( L_2 \) contains at least \(|A|^{7/15-\epsilon}\) points, the size of \( L_2 \) is at most \(|A|^{16/15+2\epsilon} \ll |A|^{8/7}\). The last inequality follows from our assumption (3). Hence, we are able to apply Theorem 2.1 so that we have

\[ |A|^{\frac{11}{15}} |L_2| \leq I(-2A, L_2) \leq |A|^{11/15} |L_2|^{11/15}, \]

which gives us that

\[ |L_2| \ll |A|^{1 + \frac{15}{16}}. \quad (8) \]

For each line \( l \in L_2 \), let \( m'(l) \) be the number of distinct vectors \((a - b, ||a|| - ||b||)\) with \((a, b) \in A^2\) such that \((a - b, ||a|| - ||b||)\) is a vector of parameters of \( l \).

It follows from (7) and (5) that there exists \( l \in L_2 \) such that

\[ m'(l) \gg |A|^\frac{4 - \frac{8}{15} - \frac{15}{16}}{30}. \quad (9) \]

We now claim that \(|\Delta(A)| \gg m'(l)|. Indeed, suppose that \( l \) is determined by \( m'(l) \) distinct vectors \((a_1 - b_1, ||a_1|| - ||b_1||), \ldots, (a_{m'(l)} - b_{m'(l)}, ||a_{m'(l)}|| - ||b_{m'(l)}||)\). Then we have

\[
(a_2 - b_2, ||a_2|| - ||b_2||) = \lambda_2 \cdot (a_1 - b_1, ||a_1|| - ||b_1||), \\
(a_3 - b_3, ||a_3|| - ||b_3||) = \lambda_3 \cdot (a_1 - b_1, ||a_1|| - ||b_1||), \\
\vdots \\

(a_{m'(l)} - b_{m'(l)}, ||a_{m'(l)}|| - ||b_{m'(l)}||) = \lambda_{m'(l)} \cdot (a_1 - b_1, ||a_1|| - ||b_1||),
\]

for some \( \lambda_2, \ldots, \lambda_{m'(l)} \in \mathbb{F}_p \). Since the vectors \((a_1 - b_1, ||a_1|| - ||b_1||), \ldots, (a_{m'(l)} - b_{m'(l)}, ||a_{m'(l)}|| - ||b_{m'(l)}||)\) are distinct, we have \( \lambda_2, \ldots, \lambda_{m'(l)} \) are distinct. On the other hand, we also have

\[ ||a_2 - b_2|| = \lambda_2^2 \cdot ||a_1 - b_1||, \ldots, ||a_{m'(l)} - b_{m'(l)}|| = \lambda_{m'(l)}^2 \cdot ||a_1 - b_1||, \]

which gives us \(|\Delta(A)| \geq \frac{m'(l)-1}{2} \), and the claim is proved.

Hence, it follows from the equation (9) that

\[ |\Delta(A)| \gg |A|^\frac{4 - \frac{8}{15} - \frac{15}{16}}{30}. \quad (10) \]

By (3) of Case 1 and (10) of Case 2, it follows that if no line contains more than \(|A|^{\frac{11}{15}+\epsilon}\) points in \( A \), then

\[ |\Delta(A)| \gg \min \left\{ |A|^{\frac{11}{15}+\frac{15}{16}}, |A|^{\frac{4 - \frac{8}{15} - \frac{15}{16}}{30}} \right\}. \]

Finally, combining this fact with (10) yields that

\[ |\Delta(A)| \gg \min \left\{ |A|^{\frac{11}{15}+\frac{15}{16}}, |A|^{\frac{4 - \frac{8}{15} - \frac{15}{16}}{30}}, |A|^{\frac{4 - \frac{8}{15} - \frac{15}{16}}{30}} \right\}. \]

To deduce the desirable result, we consier the common solutions \((\epsilon, \delta)\) to the system of the following three inequalities:

\[
\frac{8}{15} + \frac{4\epsilon}{11} \geq \delta, \quad \frac{8}{15} + \frac{1}{7} - \epsilon \geq \delta, \quad \frac{4}{7} - \frac{30\epsilon}{11} - \frac{15\epsilon}{4} \geq \delta.
\]
By a direct computation, we can obtain the largest \( \delta = \frac{1128}{2107} \) for \( \epsilon = \frac{176}{31605} \). Thus, choosing \( \epsilon = \frac{176}{31605} \) gives

\[
|\Delta(A)| \gg |A|^\delta = |A|^{\frac{1128}{2107}},
\]

which completes the proof. \( \square \)

3 Distances between a set on a line and an arbitrary set in \( \mathbb{F}_p^2 \)

In this section, we will prove Theorem 2.3. We first start with an observation as follows: if

\[
|\Delta(P_1, P_2)| \gg \min \left\{ |P_2|^{8/7}|P_1|^{-1}, |P_2|^{7/8} \right\},
\]

then we are done. So WLOG, we assume that

\[
|\Delta(P_1, P_2)| \ll \min \left\{ |P_2|^{8/7}|P_1|^{-1}, |P_2|^{7/8} \right\}. \tag{11}
\]

Hence, to prove Theorem 2.3, it is sufficient to show that

\[
|\Delta(P_1, P_2)| \gtrsim \min \left\{ |P_1|^{1/11}|P_2|^{1/8}, |P_1||P_2|^{1/8} \right\}.
\]

Since the distance function is preserved under translations and rotations, we can assume that the line is vertical passing through the origin, i.e. \( P_1 \subset \{0\} \times \mathbb{F}_p \). For the simplicity, we identify each point in \( P_1 \) with its second coordinate. The following lemma on a point-line incidence bound is known as a direct application of the Kővari–Sós–Turán theorem in [1].

Lemma 3.1. Let \( P \) be a set of \( m \) points in \( \mathbb{F}_p^2 \) and \( L \) be a set of \( n \) lines in \( \mathbb{F}_p^2 \). We have

\[
I(P, L) \leq \min \{ m^{1/2}n + m, n^{1/2}m + n \}.
\]

For \( x \in P_1 \) and \( P_2 \subset \mathbb{F}_p^2 \), we define

\[
\mathcal{E}(P_2, x) := \# \left\{ ((a, b), (c, d)) \in P_2^2 : a^2 + (b - x)^2 = c^2 + (d - x)^2 \right\},
\]

as the number of pairs of points in \( P_2 \) with the same distance to \( x \in P_1 \). In the next lemma, we will give an upper bound for \( \sum_{x \in P_1} \mathcal{E}(P_2, x) \).

Lemma 3.2. Let \( P_1, P_2 \) be sets as in Theorem 2.3. Suppose that \( |P_1|^{4/7} < |P_2| \) and \( |P_2| \ll p^{7/6} \). Then we have

\[
\sum_{x \in P_1} \mathcal{E}(P_2, x) \lesssim |P_1|^{7/11}|P_2|^{18/11} + |P_2|^{15/8}.
\]

Proof. For \( x \in P_1 \) and \( \lambda \in \mathbb{F}_p \), let \( r_{P_2}(x, \lambda) \) be the number of points \( (a, b) \) in \( P_2 \) such that \( a^2 + (b - x)^2 = \lambda \). Then we have

\[
T := \sum_{x \in P_1} \mathcal{E}(P_2, x) = \sum_{(x, \lambda) \in P_1 \times \mathbb{F}_p} r_{P_2}(x, \lambda)^2.
\]
Let \( t = |P_2|^{7/11} > 1 \), and let \( R_t \) be the number of pairs \((x, \lambda) \in P_1 \times \mathbb{F}_p\) such that \( r_{P_2}(x, \lambda) \geq t \). We have
\[
T = \sum_{(x, \lambda) \notin R_t} r_{P_2}(x, \lambda)^2 + \sum_{(x, \lambda) \in R_t} r_{P_2}(x, \lambda)^2 = I + II.
\]
Since \( \sum_{(x, \lambda) \notin R_t} r_{P_2}(x, \lambda) \leq |P_1||P_2| \) and \( r_{P_2}(x, \lambda) < t \) for any pair \((x, \lambda) \notin R_t\), we have
\[
I \leq t|P_1||P_2| = |P_1|^{7/11}|P_2|^{18/11}.
\]

In the next step, we will bound \( II \).

From the equation \( \lambda = a^2 + (b - x)^2 \), we have
\[
a^2 + b^2 = 2bx - x^2 + \lambda.
\]
Let \( P \) be the set of points \((b, a^2 + b^2)\) with \((a, b) \in P_2\), and \( L \) be the set of lines defined by \( y = 2ux - u^2 + v \) with \((u, v) \in R_t\). We have \(|L| = |R_t|\) and \(|P| \sim |P_2|\).

With these definitions, we observe that \( II \) can be viewed as the number of pairs of points in \( P \) on lines in \( L \).

We partition \( L \) into at most \( \log(|P|) \) sets of lines \( L_i \) as follows:
\[
L_i = \{l \in L: 2^it \leq |l \cap P| < 2^{i+1}t\},
\]
and let \( II(L_i) \) denote the number of pairs of points in \( P \) on lines in \( L_i \).

For each \( i \), we now consider the following cases:

**Case 1:** \(|P|^{1/2} < |L_i| \leq |P|^{7/8}\). It follows from Lemma 3.1 that
\[
2^i|L_i| \leq I(P, L_i) \leq |P|^{1/2}|L_i| + |P| \ll |P|^{1/2}|L_i|,
\]
which leads to that \( 2^i t \leq |P|^{1/2} \). Thus
\[
II(L_i) \ll |L_i| \left(|P|^{1/2}\right)^2 \ll |P|^{15/8} \sim |P_2|^{15/8}.
\]

**Case 2:** \(|P|^{7/8} \leq |L_i| \leq |P|^{8/7}\). It follows from Lemma 2.1 that
\[
2^i|L_i| \leq I(P, L_i) \leq |L_i|^{11/15}|P|^{11/15}.
\]
This implies that
\[
|L_i| \leq \frac{|P|^{11/4}}{(2^i t)^{13/4}}.
\]
In this case, we have
\[
II(L_i) \leq \frac{|P|^{11/4}}{(2^i t)^{13/4}} \cdot 2^{2i+2t^2} \ll \frac{|P|^{11/4}}{(2^i t)^{7/4}} \sim \frac{|P_2|^{11/4}}{(2^i t)^{7/4}}.
\]
One can check that the condition \( m^{-2}n^{13} \ll p^{15} \) in the Theorem 2.1 is satisfied once \(|P| \leq p^{7/6}\).

**Case 3:** \(|L_i| \leq |P|^{1/2}\). Applying Lemma 3.1 again, we obtain
\[
2^i|L_i| \leq I(P, L_i) \leq |P|^{1/2}|L_i| + |P| \ll |P|.
\]
By the Cauchy-Schwarz inequality, we have
\[ |\Delta(P_1, P_2)| \gg |P_2|^{7/8}, \]
which contradicts to our assumption (11).
Thus, we can assume that \(2t < |P|^{7/8}\). With this condition, we have
\[ H(L_i) \ll |L_i|(2t)^2 \ll 2t \cdot (|L_i|(2t)) \ll |P|^{15/8} \sim |P_2|^{15/8}, \]
where we have used the inequality (12) in the last step.

**Case 4:** \(|L_i| \geq |P|^{8/7}\). In this case, by the pigeon-hole principle, there is a point \(x\) in \(P_1\) that determines at least \(|P_2|^{8/7}/|P_1|\) lines, and each of these lines contains at least one point from \(P\). This implies that
\[ |\Delta(P_1, P_2)| \gg |P_2|^{8/7}|P_1|^{-1}, \]
which contradicts to our assumption (11).

Putting these cases together, and taking the sum over all \(i\), we obtain
\[ T \lesssim |P_1|^{7/11}|P_2|^{18/11} + |P_2|^{15/8}. \]
This completes the proof of the lemma.

We are ready to prove Theorem 2.3.

**Proof of Theorem 2.3:** As in the beginning of this section, if
\[ |\Delta(P_1, P_2)| \gg \min\{|P_2|^{8/7}|P_1|^{-1}, |P_2|^{7/8}\}, \]
then we are done. Thus, we might assume that
\[ |\Delta(P_1, P_2)| \ll \min\{|P_2|^{8/7}|P_1|^{-1}, |P_2|^{7/8}\}. \]
Let \(N\) be the number of quadruples \((p_1, p_2, p'_1, p'_2) \in P_1 \times P_2 \times P_1 \times P_2\) such that
\[ ||p_1 - p_2|| = ||p'_1 - p'_2||. \]
Let \(T\) be the number of triples \((p_1, p_2, p'_2) \in P_1 \times P_2 \times P_2\) such that \(||p_1 - p_2|| = ||p_1 - p'_2||\).

As in the proof of Lemma 3.2, we have
\[ T \lesssim |P_1|^{7/11}|P_2|^{18/11} + |P_2|^{15/8}. \]
By the Cauchy-Schwarz inequality, we have
\[ N \ll |P_1|T \lesssim |P_1|^{18/11}|P_2|^{18/11} + |P_1||P_2|^{15/8}. \]
By the Cauchy-Schwarz inequality again, one can show that \(\frac{|P_2||P_2|}{|\Delta(P_1, P_2)|} \leq N\). Thus we have
\[ |\Delta(P_1, P_2)| \gtrsim \min\{|P_1|^{4/11}|P_2|^{4/11}, |P_1||P_2|^{1/8}\}. \]
This ends the proof of the theorem.
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