UNIQUENESS OF SOLUTIONS FOR SECOND ORDER
BELLMAN-ISAACS EQUATIONS WITH MIXED BOUNDARY
CONDITIONS

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Abstract. We investigate the uniqueness of solutions for boundary value
problems in bounded and unbounded domains involving nonlinear degenerate
second order Bellman-Isaacs equations and mixed boundary conditions (Dirich-
let, generalized Dirichlet and state constrained conditions). These boundary
value problems arise from exit or stopping time stochastic differential games
or optimal control problems with constraints, such as state and integral con-
straints.

1. Introduction. In this paper we consider the boundary value problem, for a
second order degenerate elliptic Bellman-Isaacs equation

\[ G(x, u, Du, D^2u) = 0 \quad \text{in } \mathcal{O}, \]

and

\[ u = \psi \quad \text{or} \quad G(x, u, Du, D^2u) = 0 \quad \text{on } \Gamma_1, \]

\[ G(x, u, Du, D^2u) \leq 0 \quad \text{on } \Gamma_2, \]

\[ u = \psi \quad \text{on } \Gamma_3. \]

All these conditions have to be interpreted in the viscosity sense. Here \( \mathcal{O} \) is
an open, possibly unbounded subset of \( \mathbb{R}^N \), \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) denote two by two
disjoint, possibly empty subsets of the boundary \( \partial \mathcal{O} \) such that \( \bigcup_{i=1}^{3} \Gamma_i = \partial \mathcal{O} \), and
\( G \) is a second order degenerate elliptic operator of Bellman-Isaacs type. We will
also consider the special case where the generalized Dirichlet boundary condition
assumed on \( \Gamma_1 \) is strengthened in the following way:

\[ u \geq \psi \quad \text{and} \quad G(x, u, Du, D^2u) \leq 0 \quad \text{if} \quad u > \psi \quad \text{on } \Gamma_1. \]

The main goal of this paper is to prove uniqueness of possibly unbounded viscosity
solutions to the above boundary value problems. We first prove a comparison

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theorem on a bounded domain and derive from it a uniqueness result for bounded viscosity solutions, continuous at the boundary $\partial \mathcal{O}$, of the above boundary value problem. Then we extend these results to unbounded domains and unbounded viscosity solutions with a prescribed growth at infinity, generalizing two theorems of Ishii [12] on Dirichlet problems to mixed boundary value problems. We observe that these results can be extended to parabolic problems too. Furthermore, we establish a general existence result by adapting Perron’s method to our problem.

Mixed boundary conditions arise very naturally in several applications. In fact, through a dynamic programming approach, one can deduce that the value function of very different stochastic and deterministic optimization problems is a viscosity solution of a boundary value problem of the kind considered here. In Section 2, indeed, we briefly outline the most natural minimization problem fitting our framework, namely a stochastic exit-time control problem with state constraints, but we also introduce a stopping time control problem with constraints of integral type (integral equalities and inequalities) which is covered by our results. In this direction, interesting applications arise when the optimization problem involves unbounded controls such as, for instance, finite fuel problems. We deal with the uniqueness question for such problems in [19], while in [20] we prove the existence of a continuous solution of a singular nonlinear finite fuel stochastic control problem.

Here we prove comparison and uniqueness theorems for solutions which are continuous at the boundary. Many recent comparison results for boundary value problems with generalized boundary conditions give instead strong comparison between discontinuous functions (see for example [3] and for a discussion on these two different approaches [4]). However, at least for some applications such as, for instance, the finite fuel control problem studied in [20], strong comparison theorems seem to require stronger assumptions than those needed to prove the continuity of the solutions.

In the case of first order equations, mixed boundary value problems with bounded solutions were first treated by Capuzzo Dolcetta, Lions [6] in bounded domains and then by Bardi, Soravia [5] in unbounded domains. To our knowledge, there are no results dealing with mixed boundary conditions and second order fully nonlinear equations. In fact, there is an extensive literature for the state constrained and the generalized Dirichlet problems, mainly in bounded domains. We just mention Ishii, Lions [14], who study them among other boundary value problems, Barles, Burdeau [1], who treat the case of a semilinear equation and, just for the state constrained case, Katsoulakis [17] and Ishii, Loreti [15]. However, the case of three different boundary conditions is not treated, and more particularly the case of a state constrained condition together with a generalized Dirichlet condition on parts of the boundary not necessarily open.

The paper is organized as follows. In Section 2 we give the precise notion of solution to the above boundary value problems, list the basic assumptions and illustrate possible applications. Section 3 is devoted to establishing comparison and uniqueness results for the solutions to the above problems in bounded domains which are extended to unbounded domains in Section 4. In Section 5 we state an existence result. We postpone to the Appendix most of the technical proofs.

NOTATION AND DEFINITIONS. Given a function $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^K$, the upper and lower semicontinuous envelopes of $f$ are defined for any $x \in \overline{E}$ by $f^*(x) \doteq \limsup_{s \to 0^+} \{ f(y) : y \in E, |y - x| \leq s \}$, $f_*(x) \doteq \liminf_{s \to 0^+} \{ f(y) : y \in E, |y - x| \leq s \}$. 

Of course, $f^*$ is upper semicontinuous (for short, usc) and $f_*$ is lower semicontinuous (for short, lsc). Moreover, we introduce $f^+(x) = \max\{f(x), 0\}$ as the positive part of $f$ and $f^-(x) = \max\{-f(x), 0\}$ as the negative part of $f$. $M(K, K')$ will denote the set of the $K \times K'$ real matrices. Let $E \subset \mathbb{R}^K$ and let $S : \Gamma = E \times \mathbb{R} \times \mathbb{R}^K \times M(K, K) \to \mathbb{R} \cup \{\pm \infty\}$. $S$ and the equation $S(x, u(x), Du(x), D^2u(x)) = 0$ are called (degenerate) elliptic if for all $(x, t, p, X) \in \Gamma, Y \in M(K, K)$ one has $S(x, t, p, X + Y) \leq S(x, t, p, X)$ if $Y \geq 0$, where $Y \geq 0$ means that $Y$ is a positively semi–definite matrix. A nonnegative function $\omega : \mathbb{R} \to \mathbb{R}$, continuous at $0$ and such that $\omega(0) = 0$ will be called a modulus. For any set $E \subset \mathbb{R}^K$ and any constant $R > 0$, $B(E, R)$ will denote the open set $\{z : z \in \mathbb{R}^K \text{ s.t. } \text{dist}(z, E) < R\}$. Given $f, g : \mathbb{R} \to \mathbb{R}^+$ we will write $f = O(g)$ if there exist $C, R > 0$ such that $f(x) \leq Cg(x)$ for $x \in B(0, R) \setminus \{0\}$.

2. Statement of the problem and examples. In the sequel we will consider the equation

$$G(x, u, Du, D^2u) = 0 \quad \text{in} \quad \mathcal{O},$$

with the following boundary conditions

$$u = \psi \quad \text{or} \quad G(x, u, Du, D^2u) = 0 \quad \text{on} \quad \Gamma_1,$$

$$G(x, u, Du, D^2u) \leq 0 \quad \text{on} \quad \Gamma_2,$$

$$u = \psi \quad \text{on} \quad \Gamma_3.$$  

We will refer to such a problem as BVP, while BVP1 will be the problem in which equation (1) will be considered together with (3), (4), and the following condition

$$u \geq \psi \quad \text{and} \quad G(x, u, Du, D^2u) \leq 0 \quad \text{if} \quad u > \psi \quad \text{on} \quad \Gamma_1.$$  

**Remark 1.** Throughout the whole paper we will consider also the problems obtained from BVP and BVP1 by replacing the boundary conditions (3) and (5) with

$$G(x, u, Du, D^2u) \geq 0 \quad \text{on} \quad \Gamma_2,$$

and

$$u \leq \psi \quad \text{and} \quad G(x, u, Du, D^2u) \geq 0 \quad \text{if} \quad u < \psi \quad \text{on} \quad \Gamma_1,$$

respectively. Boundary conditions of this type naturally arise when $G$ is a Bellman operator associated to a stochastic minimization problem with state and integral constraints, as in Subsection 2.2.

Before stating the precise assumptions that we shall use throughout the paper, we recall the definition of viscosity solution of boundary value problems with generalized conditions due to Ishii [11].

**Definition 2.1.** Let $\Omega$ be an open set of $\mathbb{R}^K$ for some $K$ and let $\Lambda, \Gamma$ be respectively a relatively closed and a relatively open subset of $\partial \Omega$. Let $g : \Lambda \to \mathbb{R}$ be a continuous function, let $S : \Omega \times \mathbb{R} \times \mathbb{R}^K \times M(K, K) \to \mathbb{R} \cup \{\pm \infty\}$ and $B : \Gamma \times \mathbb{R} \to \mathbb{R} \cup \{\pm \infty\}$ possibly discontinuous maps. We call a function $u : \Omega \cup \Gamma \cup \Lambda \to \mathbb{R}$ a viscosity subsolution [resp., supersolution] of

$$S(x, u, Du, D^2u) = 0 \quad \text{in} \quad \Omega$$

and

$$B(x, u) = 0 \quad \text{or} \quad S(x, u, Du, D^2u) = 0 \quad \text{on} \quad \Gamma,$$

$$u = g \quad \text{on} \quad \Lambda,$$

problem.
if \( u^*(x) < +\infty \) [resp., \( u_*(x) > -\infty \)] for \( x \in \Omega \cup \Gamma \cup \Lambda \) and whenever \( \varphi \in C^2(\mathbb{R}^K) \) and \( u^* - \varphi \) attains a local maximum [resp., minimum] in \( \Omega \cup \Gamma \) at a point \( y \), we have

\[
S_*(y, u^*(y), D\varphi(y), D^2\varphi(y)) \leq 0 \quad \text{if} \quad y \in \Omega
\]

[resp., \( S^*(y, u_*(y), D\varphi(y), D^2\varphi(y)) \geq 0 \quad \text{if} \quad y \in \Omega \)]

and

\[
S_*(y, u^*(y), D\varphi(y), D^2\varphi(y)) \leq 0 \quad \text{or} \quad B_*(y, u^*(y)) \leq 0 \quad \text{if} \quad y \in \Gamma
\]

[resp., \( S^*(y, u_*(y), D\varphi(y), D^2\varphi(y)) \geq 0 \quad \text{or} \quad B^*(y, u_*(y)) \geq 0 \quad \text{if} \quad y \in \Gamma \),

\[
u^*(y) \leq g(y) \quad \text{[resp.,} \quad u_*(y) \geq g(y) \quad \text{if} \quad y \in \Lambda.\]

Often we will write for short that \( u \) verifies

\[
S(x, u, Du, D^2u) \leq 0 \quad \text{[resp.,} \quad \geq 0 \quad \text{]} \quad \text{on} \quad \Omega
\]

and

\[
B(x, u) \leq 0 \quad \text{[resp.,} \quad \geq 0 \quad \text{]} \quad \text{or} \quad S(x, u, Du, D^2u) \leq 0 \quad \text{[resp.,} \quad \geq 0 \quad \text{]} \quad \text{on} \quad \Gamma,
\]

\[
u \leq g \quad \text{[resp.,} \quad u \geq g \quad \text{]} \quad \text{on} \quad \Lambda.
\]

A viscosity solution of (8)–(10) is defined to be a function on \( \Omega \cup \Gamma \cup \Lambda \) which is both a viscosity sub– and supersolution of (8)–(10). Since all the sub– supersolutions and solutions that we consider are meant in the viscosity sense, in the sequel we will refer to them simply as sub– supersolutions and solutions, respectively.

**Remark 2.** Accordingly to Definition 2.1, for instance, a subsolution [resp., supersolution] to BVP is a function \( u : \overline{\Theta} \to \mathbb{R} \) which is a subsolution [resp., supersolution] to (8)–(10) with \( K = N, \Omega = \Theta, S = G, \Gamma = \Gamma_1 \cup \Gamma_2, \Lambda = \Lambda_0, g = \psi \), and \( B(x, u) = u - \psi(x) \), where \( \hat{\psi}(x) \equiv \psi(x) \) if \( x \in \Gamma_1 \) and \( \hat{\psi}(x) \equiv -\infty \) if \( x \in \Gamma_2 \). Notice that at any \( x \in \Gamma_1 \cup \Gamma_2 \) which is an accumulation point for both \( \Gamma_1 \) and \( \Gamma_2 \), \( B_*(x, u) = u - \psi(x) \) and \( B^*(x, u) = +\infty \) for all \( u \). Hence, no matter whether \( x \) belongs to \( \Gamma_1 \) or \( \Gamma_2 \), at such point any subsolution \( u \) to BVP verifies “\( u \leq \psi \) or \( G(x, u, Du, D^2u) \leq 0 \)” while no information is given on supersolutions.

**2.1. Assumptions.** The operator \( G \) is of the form

\[
\begin{align*}
G(x, r, p, S) &\doteq\inf_{a \in A} \sup_{b \in B} \left\{ -\frac{1}{2} \text{Tr} \{ \Sigma(x, a, b)\Sigma^T(x, a, b)S \} - (d(x, a, b), p) \right. \\
&\quad -c(x, a, b) + m(x, a, b)r \},
\end{align*}
\]

(11)

where \( A \subset \mathbb{R}^M \) and \( B \subset \mathbb{R}^r \) are nonempty sets. We will use the following hypotheses on the sets \( \Gamma_1, \Gamma_2 \) and on the functions which appear in (11).

**H0** \( \Gamma_1 \cup \Gamma_2 \) is relatively open in \( \partial \Theta \) and \( \Gamma_1 \cap \Gamma_2 = \emptyset \). There are a constant \( c > 0 \), an open set \( \Theta \supset \Gamma_1 \cup \Gamma_2 \), and a bounded, uniformly continuous function \( \eta : \overline{\Theta} \cap \Theta \to \mathbb{R}^N \) such that

\[
B(x + r\eta(x), cr) \subset \Theta \quad \forall x \in \overline{\Theta} \cap \Theta, \ 0 < r \leq c.
\]

**H1** The functions \( \Sigma : \overline{\Theta} \times A \times B \to M(N, n') \), \( d : \overline{\Theta} \times A \times B \to \mathbb{R}^N \), \( c, m : \overline{\Theta} \times A \times B \to \mathbb{R} \) are continuous. Moreover, for every bounded subset \( C \subset \overline{\Theta} \), there exist \( L, \hat{C} > 0 \) such that

\[
\| \Sigma(x, a, b) - \Sigma(y, a, b) \| \leq L |x - y| \quad \forall x, y \in C, \ \forall a \in A, \ \forall b \in B;
\]

\[
|d(x, a, b) - d(y, a, b)| \leq L |x - y| \quad \forall x, y \in C, \ \forall a \in A, \ \forall b \in B;
\]
for \( l = m, c \)
\[
|l(x, a, b)| \leq C \quad \forall (x, a, b) \in \mathcal{C} \times A \times B;
\]
\[
\lim \sup_{r \to 0} \{|l(x, a, b) - l(y, a, b)| : x, y \in \mathcal{C}, (a, b) \in A \times B, |x - y| \leq r\} = 0.
\]

(H2) There exists \( C > 0 \) such that
\[
\|\Sigma(x, a, b)\| \leq C(1 + |x|), \quad |d(x, a, b)| \leq C(1 + |x|) \quad \forall (x, a, b) \in \mathcal{C} \times A \times B.
\]

(H3) There is a constant \( \lambda_0 > 0 \) such that
\[
\inf \{m(x, a, b) : (x, a, b) \in \mathcal{C} \times A \times B\} = \lambda_0.
\]

Remark 3. We do not require any regularity of the part of the boundary \( \Gamma_3 \) where a classical Dirichlet condition is in force, even near \( \Gamma_1 \) and \( \Gamma_2 \) (as it would happen if we chose \( \Theta = B(\Gamma_1 \cup \Gamma_2, c) \)). Moreover, differently from [5], when \( \mathcal{O} \) is unbounded we do not impose that
\[
\text{for some } R > 0 \quad \text{dist}((\Gamma_1 \cup \Gamma_2) \setminus B(0, R), \Gamma_3 \setminus B(0, R)) > 0,
\]

namely, that \( \Gamma_1 \cup \Gamma_2 \) and \( \Gamma_3 \) “do not touch” at infinity.

Remark 4. All the results that we prove in the case of a bounded domain could be proved for very general Hamiltonians, not necessarily of Bellman-Isaacs type. In fact, it is easy to see that the proofs in Section 3 are still working if we assume that \( G \) is a continuous degenerate elliptic operator verifying the following assumptions:
\[
\forall R > 0 \quad \exists \gamma_R > 0 \quad \text{such that} \quad G(x, r, p, S) \geq G(x, s, p, S) + \gamma_R(r - s)
\]
\[
\forall x \in \mathcal{C}, \forall R \geq r \geq s \geq -R, \forall p \in \mathbb{R}^N, \forall S \in \mathcal{M}(N, N);
\]
\[
|G(x, r, p, S) - G(y, r, p, S)| \leq \omega_R(|x - y|(1 + |p| + \|S\|))
\]
\[
\forall x, y \in \mathcal{O}, \ |r| \leq R, p \in \mathbb{R}^N, \ R > 0, S \in \mathcal{M}(N, N); \text{ and}
\]
\[
|G(x, r, p, S) - G(x, q, W)| \leq \omega_R(|p - q| + \|S - W\|)
\]
\[
\forall x \in B(\partial \mathcal{O}, \epsilon) \cap \mathcal{O}, \ |r| \leq R, p, q \in \mathbb{R}^N \text{ and } S, W \in \mathcal{M}(N, N) \text{ (see e.g. [14]).}
\]

The hypothesis that \( G \) is of Bellman-Isaacs form is however crucial in order to extend these results to an unbounded domain \( \mathcal{O} \) following the idea, borrowed from [12], of considering suitable perturbations of the sub- and supersolutions of BVP in \( \mathcal{O} \) for which a comparison has to be proved essentially just on a bounded subset \( \mathcal{O}_L \) of \( \mathcal{O} \) (see also Section 4). We point out that this approach allows us to assume all constants and moduli in (H1) local with respect to \( x \), differently from most comparison theorems for general Hamiltonian in unbounded domains (see e.g. Theorem 7.1 in [12] and Theorem 1.4 in [5]).

2.2. Stochastic optimization problems. The optimization problems underlying BVP in full generality are problems of stochastic differential games. For the sake of simplicity, however, we limit ourselves to describe a stochastic exit–time control problem with state constraints and a stochastic finite horizon control problem with integral constraints. Hence in both cases we assume that the data in the definition (11) of \( G \) do not depend on the control \( a \in A \). Moreover in the second problem all the data in (11) are assumed to be time dependent (see also the parabolic problem PBVP in Remark 9). We follow the formulation of stochastic optimal control problem used in [8] and [10] (see also [9]).
2.2.1. Exit-time control problems with state constraints. Let $\mathcal{O}_1, \mathcal{O}_2$ be open, possibly unbounded subsets of $\mathbb{R}^N$ and set $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$. Let $\mathcal{O} \neq \emptyset$ and let $\Gamma_1 = \partial \mathcal{O} \cap \partial \mathcal{O}_1$ and $\Gamma_2 = \partial \mathcal{O} \setminus \Gamma_1$. Given $x \in \mathcal{O}$, a control is a term $\beta = (\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}, \{b_t\}, \{W_t\}, \{X_t\})$, where $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, with a right continuous complete filtration $\{\mathcal{F}_t\}$, $\{b_t\}$ is a $B$-valued control for $t \geq 0$ which is $\{\mathcal{F}_t\}$-progressively measurable, $\rho = \rho_\beta$ denotes the first exit-time of $X_t$ from $\mathcal{O}_1$, i.e., $\rho(\omega) = \inf \{t \geq 0 : X_t(\omega) \notin \mathcal{O}_1\}$, $\{X_t\}$ is an $\mathbb{R}^N$-valued, right continuous, $\mathcal{P}$-a.s. continuous process which is $\{\mathcal{F}_t\}$-progressively measurable, such that

$$X_t = x + \int_0^t d(X_r, b_r) \, dr + \int_0^t \Sigma(X_r, b_r) \, dW_r \quad \text{for } t \geq 0,$$

where $\{W_t\}$ is a standard $p$-dimensional $\{\mathcal{F}_t\}$-Brownian motion.

One wishes to minimize a cost of the form

$$J(\beta) = E_{\mathcal{P}} \left[ \int_0^\rho e^{-\int_0^\rho m(X_s, b_s) \, ds} c(X_\rho, b_\rho) \, d\rho + \chi_{\{\rho < +\infty\}} e^{-\int_0^\rho m(X_s, b_s) \, ds} \psi(X_\rho) \right],$$

over the set of controls such that

$$E_{\mathcal{P}} \left[ \int_0^\rho e^{-\int_0^\rho m(X_s, b_s) \, ds} |c(X_\rho, b_\rho)| \, d\rho \right] < +\infty$$

and

$$X_{t \wedge \rho} \in \overline{\mathcal{O}}_2 \quad \text{for } t > 0 \quad \mathcal{P}-\text{a.s..} \quad (12)$$

Let $\mathcal{B}(x)$ denote the set of such controls, to which we will refer as admissible controls. Assuming that $\mathcal{B}(x) \neq \emptyset$, the value function is defined as

$$\mathcal{V}(x) = \inf_{\beta \in \mathcal{B}(x)} J(\beta) \quad \forall x \in \overline{\mathcal{O}}.$$

Under very mild regularity and growth assumptions, Haussmann and Lepeltier in [10] proved that $\mathcal{V}$ satisfies an abstract version of the dynamic programming principle based on the compactification method introduced by [8]. From this principle, one can derive that $\mathcal{V}$ verifies the equation

$$G(x, u, Du, D^2u) = 0 \quad \text{in } \mathcal{O}, \quad (13)$$

where

$$G(x, r, p, S) = \sup_{b \in \mathcal{B}} \left\{ -\frac{1}{2} \text{Tr} \{ \Sigma(x, b) \Sigma^T(x, b) S \} - \langle d(x, b), p \rangle - c(x, b) + m(x, b) r \right\}$$

and the boundary conditions

$$u \leq \psi \quad \text{and} \quad G(x, u, Du, D^2u) \geq 0 \quad \text{if } u < \psi \quad \text{on } \Gamma_1, \quad (14)$$

$$G(x, u, Du, D^2u) \geq 0 \quad \text{on } \Gamma_2 \quad (15)$$

in the viscosity sense. A rigorous proof of these results is rather long, since it requires to introduce the compactification method, hence we skip it. It can be obtained though arguing as in [20], where a special case of the control problem described above is considered. Here we point out that, while one can expect that $\mathcal{V}$ solves equation (13), the boundary conditions (14), (15) are less obvious. But, in the framework of viscosity theory it is well known that exit-time control problems give rise to generalized Dirichlet boundary conditions like (14), while state constraints are associated to supersolution conditions like (15). More precisely, (14) was first introduced for deterministic problems by Ishii in [11] and then extended to a stochastic control problem involving a semilinear equation by Barles, Burdeau in [1]. Condition (15) instead, is due to Soner [21], who considered a infinite horizon
deterministic control problem. It has been extended to stochastic controls by Ishii and Loreti in [15].

Mixed boundary value problems arise also from stochastic optimal control problems subject to other types of constraints, such as equality and inequality constraints, as described in the next example.

2.2.2. Stopping-time control problems with integral constraints. Let $T$, $H$ and $K$ be fixed positive constants. Given $(t, x, h, k) \in \mathbb{R}^N \times [0, H] \times [0, K]$ a control is a term $\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{b_s\}, \{W_s\}, \{X_s\}, \tau)$, where $(\Omega, \mathcal{F}, P)$ is a complete probability space, with a right continuous complete filtration $\{\mathcal{F}_s\}$, $\{b_s\}$ is a $B$-valued control for $t \leq s \leq T$ which is $\{\mathcal{F}_s\}$-progressively measurable, $\{X_s\}$ is an $\mathbb{R}^N$-valued, right continuous, $P$-a.s. continuous process which is $\{\mathcal{F}_s\}$-progressively measurable, such that

$$X_s = x + \int_t^s d(r, X_r, b_r) \, dr + \int_t^s \Sigma(r, X_r, b_r) \, dW_r,$$

where $\{W_s\}$ is a standard $p$-dimensional $\{\mathcal{F}_s\}$-Brownian motion and $t \leq s \leq T$. $\tau$ is a stopping time such that $t \leq \tau \leq T$ $P$-a.s.. One wishes now to minimize a cost of the form

$$\tilde{J}(\beta) = E_P \left[ \int_t^{\tau \wedge T} c(r, X_r, b_r) \, dr + \psi(\tau \wedge T, X_{\tau \wedge T}) \right],$$

over the set of controls such that

$$\int_t^{\tau \wedge T} c_1(r, X_r, b_r) \, dr = h, \quad \int_t^{\tau \wedge T} c_2(r, X_r, b_r) \, dr \leq k \quad P \text{-a.s.,} \quad (16)$$

where $c_1, c_2$ are nonnegative scalar functions. Let $\mathcal{B}(t, x, h, k)$ denote the set of such admissible controls. We assume $\mathcal{B}(t, x, h, k) \neq \emptyset$ and define the value function

$$\nabla(t, x, h, k) = \inf_{\beta \in \mathcal{B}(t, x, h, k)} \tilde{J}(\beta) \quad \forall (t, x, h, k) \in \mathbb{R}^N \times [0, H] \times [0, K].$$

Notice that replacing the process $X_s$ with $\overline{X}_s = (X_s, X_s^{N+1}, X_s^{N+2})$ verifying

$$\overline{X}_s = \overline{x} + \int_t^s d(r, X_r, b_r) \, dr + \int_t^s \overline{\Sigma}(r, X_r, b_r) \, dW_r,$$

where $\overline{x} = (x, x^{N+1}, x^{N+2}) \equiv (x, h, k)$, $\overline{d} \equiv (d, -c_1, -c_2)$ and $\overline{\Sigma} \equiv (\Sigma, 0, 0)$, the integral constraints (16) become $X_s^{N+1} = 0$ and $X_s^{N+2} \geq 0 \quad P \text{-a.s.}$ Therefore the optimization problem above can be equivalently written as the unconstrained optimal stopping problem of minimizing the following cost

$$\hat{J}(\beta) = E_P \left[ \int_t^{\tau \wedge T} c(r, X_r, b_r) \, dr + \hat{\psi}(\tau \wedge T, \overline{X}_{\tau \wedge T}) \right],$$

where $\hat{\psi}(t, \overline{x}) \equiv \psi(t, x)$ if $(t, \overline{x}) = (t, x, x^{N+1}, x^{N+2}) \in \mathbb{R}^N \times \{0\} \times [0, K]$ and $\hat{\psi}(t, \overline{x}) \equiv +\infty$ elsewhere, over all $\{\mathcal{F}_s\}$-progressively measurable, $B$-valued control processes. As in the previous subsection, from the dynamic programming principle proved in [10] one can derive that under suitable regularity and growth assumptions
\[ V = \mathcal{V}(t, \bar{x}) \] is a viscosity solution of the boundary value problem

\[
\begin{aligned}
- \frac{\partial u}{\partial t} + G(t, x, u, \overline{D}u, D^2u) &= 0 & & \text{in } [0, T] \times \mathbb{R}^n \times ]0, H[ \times [0, K[,
\end{aligned}
\]

\[
\begin{aligned}
\frac{\partial u}{\partial t} + G(t, x, u, \overline{D}u, D^2u) &\geq 0 & & \text{if } u < \psi
\end{aligned}
\]

\[
\begin{aligned}
\frac{\partial u}{\partial t} + G(t, x, u, \overline{D}u, D^2u) &\geq 0 & & \text{on } [0, T] \times \mathbb{R}^n \times \{0\} \times [0, K[,
\end{aligned}
\]

\[
\begin{aligned}
\frac{\partial u}{\partial t} + G(t, x, u, \overline{D}u, D^2u) &= 0 & & \text{on } [0, T] \times \mathbb{R}^n \times [0, H[ \times [0, K[,
\end{aligned}
\]

\[
\begin{aligned}
\frac{\partial u}{\partial t} + G(t, x, u, \overline{D}u, D^2u) &= 0 & & \text{on } \{T\} \times \mathbb{R}^n \times [0, H[ \times [0, K[,
\end{aligned}
\]

(17)

where \( \overline{D}u \equiv (Du, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) \) and

\[
\mathcal{G}(t, x, r, p, p_{N+1}, p_{N+2}, S) \equiv \sup_{b \in B} \left\{ -\frac{1}{2} \text{Tr} \{ \Sigma(t, x, b) \Sigma^T(t, x, b) S \} - (d(t, x, b), p) - c(t, x, b) + c_1(t, x, b) p_{N+1} + c_2(t, x, b) p_{N+2} \}. \]

We recall that, at least for deterministic control problems, Barles and Perthame in [2] proved that the value function of an optimal stopping time problem solves, in the viscosity sense, a variational inequality of obstacle type of the form

\[
\max \left\{ -\frac{\partial u}{\partial t} + \mathcal{G}(t, x, u, \overline{D}u, D^2u), u - \psi \right\} = 0. \tag{18}
\]

Taking into account the definition of \( \psi \), one easily deduces that (18) yields the boundary value problem (17). From an heuristic point of view, the first boundary condition in (17) can also be justified observing that equation \( X^N_{t,T} = 0 \) may be seen as a target condition, (with target \( [0, T] \times \mathbb{R}^n \times \{0\} \times [0, K[ \), and therefore a generalized Dirichlet problem in such a part of the boundary is standard in the framework of viscosity solutions. The second boundary condition derives from the state constraint \( X^N_{t,T} \geq 0 \) for a.e. \( t \), that follows from \( X^N_{T,T} \geq 0 \), and this implies a supersolution condition on the viscosity solution of (17) at the boundary \( k = 0 \).

Of course, an analogous procedure can be followed when there is more than one integral equality and inequality constraint. We point out that, up to reversing time, i.e. by considering the boundary value problem associated to \( \mathcal{W}(t, x) = \mathcal{V}(T-t, \bar{x}) \), one obtains a special case of the parabolic problem PBVP that we shall consider in Remark 9. Let us recall that in the deterministic case some results on the boundary value problems associated to exit-time control problems with state constraints have been obtained in [22], while regularity properties and characterization of the value functions of control problems with state and integral constraints have been considered in [18].

The following example illustrates the problem outlined in Subsection 2.2.1 for a deterministic system.

**Example 1.** Let us consider the set \( \mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2 \subset \mathbb{R}^2 \), where

\[
\mathcal{O}_1 \equiv \{ x = (x_1, x_2) : x_2 < 1 + \min \{ x_1, 0 \} \}, \quad \mathcal{O}_2 \equiv \{ x = (x_1, x_2) : x_2 > x_1 \}
\]

and the boundary subsets \( \Gamma_1 \equiv \{ x : x_1 < 0, x_2 = 1 + x_1 \}, \Gamma_2 \equiv \{ x : x_1 < 1, x_2 = x_1 \}, \Gamma_3 \equiv \{ x : 0 \leq x_1 \leq 1, x_2 = 1 \} \). For any \( x \in \overline{\mathcal{O}} \), let us be given the control system

\[
\begin{aligned}
y'(t) &= (y'_1(t), y'_2(t)) = (|y_1(t)|b_1(t), b_2(t)) & & \forall t > 0,
y(0) &= x, \end{aligned}
\]

where \( b(\cdot) = (b_1(\cdot), b_2(\cdot)) \) is a measurable control function on \( [0, +\infty[ \) taking values in \( B \equiv \{(b_1, b_2) : b_1 \geq 0, b_2 \geq 0, b_1 + b_2 \leq 1 \} \) and let us denote by \( y(\cdot) \) the corresponding solution. Moreover, let us set \( \rho = \rho(x) \equiv \inf \{ t \geq 0 : y(t) \notin \mathcal{O}_1 \} \) and let us define the set of admissible controls \( \mathcal{B}(x) \) as the set of controls such
that \( y(t \wedge \rho) \in \overline{\mathcal{O}}_2 \) for all \( t > 0 \). It is easy to see that \( \mathcal{B}(x) \neq \emptyset \) \( \forall x \in \overline{\mathcal{O}} \). Let \( \psi : \Gamma_1 \cup \Gamma_3 \to \mathbb{R} \) be the exit–cost function given by \( \psi(x) = |1 - x| \).

For any \( x \in \overline{\mathcal{O}} \) let us introduce the value function

\[
\mathcal{V}(x) \doteq \inf_{b(\cdot) \in \mathcal{B}(x)} \left[ \int_0^{\rho} e^{-t} dt + e^{-\rho} \psi(x(\rho)) \right].
\]

Heuristics suggests that for \( x \) with \( 0 \leq x_1 \leq 1 \), any control steering \( x \) to \((1,1)\) in minimum time, such as \( e.g. \) the control \( b(t) \equiv (0,1) \) for all \( t \in [0,1-x_2] \) and, if \( x_1 < 1, b(t) \equiv (1,0) \) for all \( t \in [1-x_2,1-x_2 - \log x_1] \) is optimal, while for \( x \) with \( x_1 < 0 \) there is not an optimum control but the optimal behaviour consists in steering \( x \) as near as possible to the axis \( x_1 = 0 \). Therefore one has

\[
\mathcal{V}(x) = \begin{cases} 
1 - x_1 e^{-(1-x_2)} & \text{for } 0 \leq x_1 \leq 1, \\
1 & \text{for } x_1 < 0.
\end{cases}
\]

Denoting by \( \bar{G}(x,r,p) = \sup_{(b_1,b_2) \in B} \{-\langle(x_1|b_1,b_2),p\rangle + r - 1\} \) the boundary value problem associated to \( \mathcal{V} \) is the following

\[
\begin{align*}
\bar{G}(x,\mathcal{V},D\mathcal{V}) &= 0 & & \text{in } \mathcal{O} \\
\mathcal{V} &= \psi & & \text{on } \Gamma_1 \\
\bar{G}(x,\mathcal{V},D\mathcal{V}) &\geq 0 & & \text{on } \Gamma_2, \\
\mathcal{V} &= \psi & & \text{on } \Gamma_3.
\end{align*}
\]

(see Remark 1 for the inequality on \( \Gamma_2 \).) Notice that the Dirichlet condition \( \mathcal{V} = \psi \) is not satisfied on the whole boundary \( \partial\mathcal{O} \cap \partial\mathcal{O}_1 \) but just on \( \Gamma_3 \), while on \( \Gamma_1 \) one has \( \mathcal{V}(x) = 1 < \psi(x) \). It is not difficult to verify that \( \mathcal{V} \) is a solution of the above problem in the sense of Definition 2.1. Since \( \Gamma_1, \Gamma_2 \) are relatively open in \( \partial\mathcal{O} \) and \( \mathcal{V} \) is bounded on \( \overline{\mathcal{O}} \), we can apply ii) of Corollary 2 below (see also Remark 8) to the first order operator \( \bar{G} \) and obtain that \( \mathcal{V} \) is the unique solution to the above boundary value problem among the functions with sublogaritmic growth in \( x \) at infinity. We point out that, because of the linear growth of the dynamics \( \langle x_1|b_1,b_2 \rangle \) in the \( x \) variable, the hypotheses on the Hamiltonian made in the uniqueness Theorem 1.4 in [5] are not fulfilled.

3. Uniqueness results on bounded domains. In this section we prove comparison and uniqueness theorems on bounded domains which extend in some way previous results, \( e.g. \) [5], [6], [12], [14], [7]. The novelty here is that we consider three different boundary conditions on \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \), respectively. To our knowledge, previous results on viscosity solutions of mixed boundary value problems have been established for first order Hamiltonians and only in the case that either \( \Gamma_1 \) or \( \Gamma_2 \) is the empty set (see [6] and [5]). Dealing with the three boundaries together requires some additional care even in the first order case, but the main point here is the extension of these results to second order equations.

Of course, when the set \( \mathcal{O} \) is bounded all the constants in hypothesis (H1) are global and hypothesis (H2) reduces to a boundedness assumption on \( \Sigma \) and \( d \).

Theorem 3.1. Assume that \( \mathcal{O} \) is a nonempty, bounded and open subset of \( \mathbb{R}^N \), \( \Gamma_1, \Gamma_2 \subseteq \partial\mathcal{O}, \Gamma_3 = \partial\mathcal{O} \setminus (\Gamma_1 \cup \Gamma_2) \), and (H0)–(H3) hold. Let \( u : \overline{\mathcal{O}} \to \mathbb{R} \) be bounded and usc in \( \overline{\mathcal{O}} \), let \( v : \overline{\mathcal{O}} \to \mathbb{R} \) be bounded and lsc in \( \overline{\mathcal{O}} \), continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \). If \( u \) and \( v \) are respectively subsolution and
supersolution of (1) in $\mathcal{O}$ and verify
\begin{align}
\ u & \leq v \text{ or } G(x,u,Du,D^2u) \leq 0 \text{ on } \Gamma_1, \\
\ G(x,u,Du,D^2u) & \leq 0 \text{ on } \Gamma_2, \\
\ u & \leq v \text{ on } \Gamma_3,
\end{align}
then $u \leq v$ in $\overline{\mathcal{O}}$. The same conclusion holds if $u$ and $v$ are respectively subsolution and supersolution of (1) in $\mathcal{O}$, $u$ instead of $v$ is assumed to be continuous on $\Gamma_1 \cup \Gamma_2$, uniformly continuous on $\Gamma_1$ and instead of (19) they verify
\begin{align}
\ u & \leq v \text{ or } G(x,v,Dv,D^2v) \geq 0 \text{ on } \Gamma_1, \\
\ G(x,v,Dv,D^2v) & \geq 0 \text{ on } \Gamma_2, \\
\ u & \leq v \text{ on } \Gamma_3,
\end{align}

The first step towards the proof of Theorem 3.1 is the construction of appropriate inf– and sup–convolutions of $u$ and $v$. For $\varepsilon \in [0,1]$ we set
\begin{align}
\ u^\varepsilon(x) & = \sup_{y \in \mathcal{O}} \left\{ u(y) - \frac{1}{\varepsilon} |x - y|^2 \right\} \quad \forall x \in \mathbb{R}^N, \\
\ v^\varepsilon(x) & = \inf_{y \in \mathcal{O}} \left\{ v(y) + \frac{1}{\varepsilon} |x - y|^2 \right\} \quad \forall x \in \mathbb{R}^N.
\end{align}

We recall that the functions $u^\varepsilon$ and $v^\varepsilon$ are bounded, locally Lipschitz, semiconvex and semiconcave respectively. Moreover by setting $C_0 = \max \left\{ \frac{2\sup|u|}{\varepsilon}, \frac{2\sup|v|}{\varepsilon} \right\}^{1/2}$, for every $x \in \mathbb{R}^N$ the sup in (21) [resp. the inf in (22)] can be taken over the $y$’s such that $|y - x| \leq (C_0 + C)\sqrt{\varepsilon}$, if $\text{dist}(x,\overline{\mathcal{O}}) \leq C\sqrt{\varepsilon}$. Finally, if a function $\Phi$ on $\mathbb{R}^N \times \mathbb{R}^N$ satisfies
\[ \Phi(x,y) \geq C_1 \text{ and } \Phi(x,x) \leq C_2, \]
for $x, y \in \mathbb{R}^N$ and $C_1, C_2 \in \mathbb{R}$ and if $u^\varepsilon(x) - v^\varepsilon(y) - \Phi(x,y)$ attains a maximum at $(x_0, y_0) \in \mathbb{R}^N \times \mathbb{R}^N$, then
\begin{align}
\text{dist}(x_0,\overline{\mathcal{O}}), \text{ dist}(y_0,\overline{\mathcal{O}}) & \leq C_3\sqrt{\varepsilon},
\end{align}
where $C_3 = \left[ C_2 - C_1 + 2\sup|u| + 2\sup|v| \right]^{1/2}$ (see [14]).

The second step in the proof of Theorem 3.1 is given by Propositions 1, 2 below. Proposition 1 is a careful adaptation of Proposition VI.1 in [14] to mixed boundary value problems. We have to remark though that the keypoint in order to prove the results of this section is the introduction of the two operators $G^\varepsilon$ and $G^\varepsilon$ in whose definitions are included the boundary conditions.

**Proposition 1.** Let $C > 0$. We set
\begin{align}
\mathcal{O}^\varepsilon & \doteq \{ x \in \mathbb{R}^N : \text{dist}(x,\mathcal{O}) < C\sqrt{\varepsilon} \}.
\end{align}
Let us define for $(y,r) \in \partial\mathcal{O} \times \mathbb{R}$ the function $B_1(y,r) \doteq r - \hat{v}(y)$, where
\[ \hat{v}(y) \doteq \begin{cases} v(y) & \text{if } y \in \Gamma_1 \cup \Gamma_3 \\
-\infty & \text{if } y \in \Gamma_2. \end{cases} \]
Then the function $u^\varepsilon$ is a subsolution of
\[ G^\varepsilon(x,u^\varepsilon,Du^\varepsilon,D^2u^\varepsilon) = 0 \quad \text{in } \mathcal{O}^\varepsilon, \]
where

\[ G_\varepsilon(x, r, p, S) = \inf \{ \inf \{ G(y, r, p, S), B_1(y, r) \} : \ y \in \overline{O}, \ |y - x| \leq (C_0 + C)\sqrt{\varepsilon} \}, \]

with the convention that \( \inf \{ G(y, r, p, S), B_1(y, r) \} = G(y, r, p, S) \) for \( y \in O \).

Similarly, for \((y, r) \in \partial O \times \mathbb{R}\) let us define the function \( B_2(y, r) = r - \hat{u}(y) \), where

\[ \hat{u}(y) = \begin{cases} u(y) & \text{if } y \in \Gamma_3, \\ -\infty & \text{if } y \in \Gamma_1 \cup \Gamma_2. \end{cases} \]

Then \( v_\varepsilon \) is a supersolution of

\[ G^\varepsilon(x, v_\varepsilon, Dv_\varepsilon, D^2v_\varepsilon) = 0 \quad \text{in } O^\varepsilon, \]

where

\[ G^\varepsilon(x, r, p, S) = \sup \{ \sup \{ G(y, r, p, S), B_2(y, r) \} : \ y \in \overline{O}, \ |y - x| \leq (C_0 + C)\sqrt{\varepsilon} \}, \]

with the convention that \( \sup \{ G(y, r, p, S), B_2(y, r) \} = G(y, r, p, S) \) for \( y \in O \).

Moreover, the function \( G_\varepsilon \) is lsc, while \( G^\varepsilon \) is usc.

**Remark 5.** By the definitions of \( O^\varepsilon \) and \( G^\varepsilon \) one easily deduces that the statement:

"\( v_\varepsilon \) is a supersolution of \( G^\varepsilon(x, v_\varepsilon, Dv_\varepsilon, D^2v_\varepsilon) = 0 \) in \( O^\varepsilon \),"

implies the well known statement:

"\( v_\varepsilon \) is a supersolution of \( G^\varepsilon(x, v_\varepsilon, Dv_\varepsilon, D^2v_\varepsilon) = 0 \) in \( O^\varepsilon \),"

where \( G^\varepsilon(x, r, p, S) = \sup \{ G(y, r, p, S), B_2(y, r) \} : \ y \in \overline{O}, \ |y - x| \leq C_0\sqrt{\varepsilon} \) for all \((x, r, p, S) \in O \times \mathbb{R} \times \mathbb{R}^N \times M(N, N)\) and \( O^\varepsilon = \{ x \in O : \ \text{dist}(x, \partial O) > C_0\sqrt{\varepsilon} \} \).

The formulation adopted here however, not only allows us to consider \( u^\varepsilon \) and \( v_\varepsilon \) on the same domain, but also, via suitable modifications, fits the more general situations treated in Theorems 3.2, 3.3 below, where \( v \) may be a supersolution of equation (1) even at some point of \( \partial O \) (see Proposition 3).

The proof of Proposition 1 is postponed to the Appendix.

**Proposition 2.** Let \( \varphi \in C^2(\mathbb{R}^N \times \mathbb{R}^N) \) and assume that there exists a point \((\bar{x}, \bar{y}) \in O^\varepsilon \times O^\varepsilon \) such that \( u^\varepsilon(x) - v_\varepsilon(y) = \varphi(x, y) \) achieves its maximum over \( \overline{O^\varepsilon} \times \overline{O^\varepsilon} \) at \((\bar{x}, \bar{y})\). Then there exist matrices \( X, Y \in M(N, N) \) such that

\[ G_\varepsilon(\bar{x}, u^\varepsilon(\bar{x}), D_x\varphi(\bar{x}, \bar{y}), X) \leq 0, \]

\[ G^\varepsilon(\bar{y}, v_\varepsilon(\bar{y}), -D_y\varphi(\bar{x}, \bar{y}), -Y) \geq 0, \]

\[ -\frac{2}{\varepsilon} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq D^2\varphi(\bar{x}, \bar{y}), \]

where \( D_x\varphi \) [resp. \( D_y\varphi \)] denotes the gradient with respect to the \( x \) [resp. \( y \)] variable of the function \( \varphi \).

The proof of Proposition 2 is completely analogous to the proof of Proposition 5.1 of [12], which is essentially based on a result due to Jensen [16].

**Proof of Theorem 3.1.** We will prove only the first statement, being the proof of the second one very similar.

We assume by contradiction that there is some \( \gamma > 0 \) such that

\[ \max_{\overline{O}} (u - v) = \gamma. \]  \hspace{1cm} (25)

Owing to Theorem II.2 of [14], this maximum is assumed at some point of \( \partial O \), hence, being \( u \leq v \) on \( \Gamma_3 \), it is assumed on \( \Gamma_1 \cup \Gamma_2 \). Let \( w \in \Gamma_1 \cup \Gamma_2 \) be such that

\[ \max_{x \in \overline{O}} (u(x) - v(x)) = \max_{x \in \partial O} (u(x) - v(x)) = u(w) - v(w) \]  \hspace{1cm} (26)
and consider the test function
\[ \Phi(x, y) = \frac{1}{\delta^2} |x - y + \delta \eta(w)|^2 + |x - w|^4, \tag{27} \]
where $\delta \in [0, 1]$. Let $(\tilde{x_\delta}, \tilde{y_\delta})$ be the maximum of $z_\delta(x, y) = u^\delta(x) - v_\delta(y) - \Phi(x, y)$. By (23) it easily follows that $\text{dist}(\tilde{x_\delta}, \Gamma) < C_3 \sqrt{\varepsilon}$, $\text{dist}(\tilde{y_\delta}, \Gamma) < C_3 \sqrt{\varepsilon}$ for some $C_3 > 0$ independent of $\delta, \varepsilon > 0$. Taking subsequences if necessary, the points $(\tilde{x_\delta}, \tilde{y_\delta})$ converge as $\varepsilon$ tends to 0 to a maximum point $(\tilde{x}, \tilde{y}) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ of $z(x, y) = u(x) - v(y) - \Phi(x, y)$ and $(\tilde{x_\delta}, \tilde{y_\delta})$ is such that $\lim_{\delta \to 0} \tilde{x_\delta} = \lim_{\delta \to 0} \tilde{y_\delta} = \tilde{w}$, where $\tilde{w}$ is the unique maximum point of $u(x) - v(x) - |x - w|^4$. Moreover,
\begin{align*}
\lim_{\varepsilon \to 0} u^\varepsilon(\tilde{x_\delta}) &= u(\tilde{w}), \\
\lim_{\varepsilon \to 0} v^\varepsilon(\tilde{y_\delta}) &= v(\tilde{w}). \tag{28}
\end{align*}
\begin{align*}
\lim_{\delta \to 0} u(\tilde{x}) &= u(\tilde{w}) \quad \text{and} \quad \lim_{\delta \to 0} v(\tilde{y}) = v(\tilde{w}). \tag{29}
\end{align*}
Since $\tilde{w} \in \Gamma_1 \cup \Gamma_2$ which is relatively open in $\partial \mathcal{O}$, one has that $\text{dist}(w, \Gamma_3) > \tilde{\gamma}$ for some $\tilde{\gamma} > 0$. Then by choosing first $\delta$ and then $\varepsilon$ small enough, we can make $\text{dist}(\tilde{x}, \Gamma_3), \text{dist}(\tilde{y}, \Gamma_3), \text{dist}(\tilde{x_\delta}, \Gamma_3), \text{dist}(\tilde{y_\delta}, \Gamma_3)$, and also $\text{dist}(x, \Gamma_3)$ and $\text{dist}(y, \Gamma_3) > \tilde{\gamma}$ for all $x, y \in \overline{\mathcal{O}}$ with $|x - \tilde{x_\delta}| \leq (C_0 + C_3) \sqrt{\varepsilon}$, $|y - \tilde{y_\delta}| \leq (C_0 + C_3) \sqrt{\varepsilon}$. If $w$ belongs to the interior of $\Gamma_2$ (relative to $\Gamma_1 \cup \Gamma_2$), say $\text{int}(\Gamma_2)$, the above inequalities hold with $\text{cl}(\Gamma_1 \cup \Gamma_3)$ replacing $\Gamma_3$, where $\text{cl}(\Gamma_1)$ denotes the closure of $\Gamma_1$ relative to $\Gamma_1 \cup \Gamma_2$. In particular, in this case we can make $\tilde{x_\delta}$, $\tilde{x_\delta}$, and any $x \in \overline{\mathcal{O}}$ such that $|x - \tilde{x_\delta}| \leq (C_0 + C_3) \sqrt{\varepsilon}$, belong to $\mathcal{O} \cup \text{int}(\Gamma_2)$. If $w$ belongs instead to $\text{cl}(\Gamma_1)$, because of (28), (29) and of the uniform continuity of $v$ on $\Gamma_1$, we can reduce $\delta$ and $\varepsilon$ in order to have
\[ u^\varepsilon(\tilde{x_\delta}) \geq v(x) + \frac{\gamma}{2} \quad \forall x \in \text{cl}(\Gamma_1) \text{ such that } |x - \tilde{x_\delta}| \leq (C_0 + C_3) \sqrt{\varepsilon}. \tag{30} \]
Moreover, $z(\tilde{x_\delta}, \tilde{y_\delta}) \geq z(w, w + \delta \eta(w))$ since $B(w + \delta \eta(w), \delta \varepsilon) \subset \mathcal{O}$ in view of hypothesis (H0), and hence the continuity of $v$ at $w \in \Gamma_1 \cup \Gamma_2$ yields
\[ \frac{1}{\delta^2} |\tilde{x_\delta} - \tilde{y_\delta} + \delta \eta(w)|^2 = o(1) \quad \text{as } \delta \to 0, \tag{31} \]
which in turn implies
\[ |\tilde{x_\delta} - \tilde{y_\delta}| \leq C \delta. \tag{32} \]
These facts imply that $B(\tilde{y_\delta}, 3c') \subset \mathcal{O}$ for some $c' > 0$ depending on $\delta$. Therefore, if we choose $\varepsilon > 0$ small enough we can also make $B(\tilde{y_\delta}, 2c') \subset \mathcal{O}$, and $B(y, c') \subset \mathcal{O}$ for all $y \in \overline{\mathcal{O}}$ with $|y - \tilde{y_\delta}| \leq (C_0 + C_3) \sqrt{\varepsilon}$. Therefore it is easy to see that for $x = \tilde{y_\delta}$ one has $G^\varepsilon \equiv G$ while for $x = \tilde{x_\delta}$ one has $G_\varepsilon \equiv G$, where $G_\varepsilon$ and $G^\varepsilon$ are the functions introduced in Proposition 1. Owing to Proposition 2 there are $X, Y \in \mathbb{M}(N, N)$ such that
\[ G(\tilde{x}, u^\varepsilon(\tilde{x_\delta}), D_x \Phi(\tilde{x}, \tilde{y_\delta}), X) \leq, \]
\[ G(\tilde{y}, v_\delta(\tilde{y_\delta}), -D_y \Phi(\tilde{x}, \tilde{y_\delta}), -Y) \geq 0, \]
\[ -\frac{c}{2} I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq D^2 \Phi(\tilde{x}, \tilde{y_\delta}) \tag{33} \]
for some $\tilde{x}, \tilde{y} \in \overline{\mathcal{O}}$, where $\tilde{x_\delta} = \tilde{x_\delta}, \tilde{y_\delta} = \tilde{y_\delta}, |\tilde{x} - \tilde{x_\delta}| \leq (C_0 + C_3) \sqrt{\varepsilon}$ and $|\tilde{y} - \tilde{y_\delta}| \leq (C_0 + C_3) \sqrt{\varepsilon}$, Recalling (H3), we get
\[ \lambda_0 (u^\varepsilon(\tilde{x}), v_\delta(\tilde{y_\delta})) \leq \]
\[ G(\tilde{x}, u^\varepsilon(\tilde{x}), D_x \Phi(\tilde{x}, \tilde{y_\delta}), X) - G(\tilde{y}, v_\delta(\tilde{y_\delta}), D_y \Phi(\tilde{x}, \tilde{y_\delta}), X) = \]
\[ (G(\tilde{x}, u^\varepsilon(\tilde{x}), D_x \Phi(\tilde{x}, \tilde{y_\delta}), X) - G(\tilde{y}, v_\delta(\tilde{y_\delta}), -D_y \Phi(\tilde{x}, \tilde{y_\delta}), -Y)) + \]
\[ (-G(\tilde{x}, v_\delta(\tilde{y_\delta}), D_x \Phi(\tilde{x}, \tilde{y_\delta}), X) + G(\tilde{y}, v_\delta(\tilde{y_\delta}), -D_y \Phi(\tilde{x}, \tilde{y_\delta}), -Y)). \]
Now we use the inequalities in (33) together with the ellipticity of $G$. After sending $\varepsilon$ to 0, it is easy to show that there exists $(a, b) \in A \times B$ such that
\[
\lambda_0 (u(x) - v(y)) \leq \frac{1}{2} \langle \nabla \{\tilde{x}(x, a, b) \Sigma \tilde{y}(x, a, b)\} \{O(|\tilde{x} - w|) + \frac{1}{2} L^2 O(|\tilde{x} - \tilde{y}|^2)\} + |d(x, a, b)|O(|\tilde{x} - w|) + |y - w| + L|\tilde{x} - \tilde{y}| [O(|\tilde{x} - w|) + \frac{A}{2} |\tilde{x} - \tilde{y}| + \delta y(x)] + (R + 1) \omega(|\tilde{x} - \tilde{y}|)
\]
where $\tilde{x} \approx \bar{x}$, $\tilde{y} \approx \bar{y}$, $R \approx \text{max}\{\|u\|, \|v\|\}$, $L$ is the Lipschitz constant relative to $\partial \Omega$ of $d$ and $\Sigma$, and $\omega$ is the modulus of continuity in $\partial \Omega$ of $m$ and $c$. Finally, recalling (25), (26), (29), (31), and (32) one gets the required contradiction by sending $\delta$ to 0.

The next two comparison theorems will play an essential role in order to extend the results of this subsection to the case of $\Omega$ unbounded.

**Theorem 3.2. (Comparison for BVP1)** Assume that $\Omega$ is a nonempty, bounded and open subset of $\mathbb{R}^N$, $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \subseteq \partial \Omega$ are such that $\Gamma_3 \cup \Gamma_4 = \partial \Omega \setminus (\Gamma_1 \cup \Gamma_2)$, $\Gamma_3 \cap \Gamma_4 = \emptyset$ and (H0)-(H3) hold. Let $\psi: \Gamma_1 \cup \Gamma_3 \to \mathbb{R}$ be bounded and continuous on $\Gamma_1 \cup \Gamma_3$ and uniformly continuous on $\Gamma_1$. Let $u: \overline{\Omega} \to \mathbb{R}$ be bounded and usc in $\partial \Omega$, let $v: \overline{\Omega} \to \mathbb{R}$ be bounded and lsc in $\partial \Omega$, continuous on $\Gamma_1 \cup \Gamma_2$ and uniformly continuous on $\Gamma_1$. Then if $\Gamma_2$ is a relatively open set in $\partial \Omega$ and $u$ and $v$ verify
\[
\begin{align*}
G(x, u, Du, D^2 u) &\leq 0 \quad \text{in } \Omega \\
u &\leq \psi \quad \text{on } \Gamma_1 \\
G(x, u, Du, D^2 u) &\leq 0 \quad \text{on } \Gamma_2 \\
u &\leq \psi \quad \text{on } \Gamma_3,
\end{align*}
\] (34)

and
\[
\begin{align*}
u &\geq \psi \quad \text{on } \Gamma_1 \cup \Gamma_3,
\end{align*}
\] (35)

one has that $u \leq v$ in $\overline{\Omega}$.

The same conclusion holds if $u$ instead of $v$ is continuous on $\Gamma_1 \cup \Gamma_2$ and uniformly continuous on $\Gamma_1$ and $u$ and $v$ verify (36) and
\[
\begin{align*}
G(x, u, Du, D^2 u) &\leq 0 \quad \text{in } \Omega \\
u &\leq \psi \quad \text{on } \Gamma_1 \cup \Gamma_3,
\end{align*}
\] (37)

and
\[
\begin{align*}
v &\geq \psi \quad \text{on } \Gamma_1 \cup \Gamma_3,
\end{align*}
\] (38)

**Theorem 3.3. (Comparison for BVP)** Assume that $\Omega$ is a nonempty, bounded and open subset of $\mathbb{R}^N$, $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \subseteq \partial \Omega$ are such that $\Gamma_3 \cup \Gamma_4 = \partial \Omega \setminus (\Gamma_1 \cup \Gamma_2)$, $\Gamma_3 \cap \Gamma_4 = \emptyset$ and (H0)-(H3) hold. Let $\psi: \Gamma_1 \cup \Gamma_3 \to \mathbb{R}$ be bounded and continuous on $\Gamma_1 \cup \Gamma_3$ and uniformly continuous on $\Gamma_1$. Let $u: \overline{\Omega} \to \mathbb{R}$ be bounded and usc in $\partial \Omega$, let $v: \overline{\Omega} \to \mathbb{R}$ be bounded and lsc in $\partial \Omega$, let both $u$ and $v$ be continuous on $\Gamma_1 \cup \Gamma_2$ and uniformly continuous on $\Gamma_1$. If both $\Gamma_1$ and $\Gamma_2$ are relatively open sets in $\partial \Omega$ and $u$ and $v$ verify (34), (36), and
\[
\begin{align*}
G(x, v, Dv, D^2 v) &\geq 0 \quad \text{in } \Omega \\
v &\geq \psi \quad \text{on } \Gamma_1 \cup \Gamma_3,
\end{align*}
\] (39)
then $u \leq v$ in $\overline{O}$.

The same conclusion holds if $u$ and $v$ verify (38), (36), and

\[
\begin{aligned}
G(x, u, Du, D^2u) &\leq 0 \quad \text{in } O \\
&\quad u \leq \psi \quad \text{on } \Gamma_1 \\
&\quad u \leq \psi \quad \text{on } \Gamma_3.
\end{aligned}
\]

As for Theorem 3.1, preliminary steps towards the proofs of Theorems 3.2, 3.3 are the sup- and inf- convolutions of $u$ and $v$ defined as in (21), (22) and the following results completely similar to those stated in Propositions 1, 2. Here too we have to stress the importance of the operators $\bar{G}^c$ and $\bar{G}_c$ introduced in Proposition 3 below, whose definition incorporates the boundary conditions.

**Proposition 3.** Let $C > 0$ and for any $\varepsilon > 0$ let $O^\varepsilon$ be the set defined in (24). Let us define for $(y, r) \in \partial O \times \mathbb{R}$ the function $\bar{B}_1(y, r) = r - \psi_1(y)$, where

\[
\psi_1(y) = \begin{cases} 
\psi(y) & \text{if } y \in \Gamma_1 \cup \Gamma_3 \\
-\infty & \text{if } y \in \Gamma_2 \\
u(y) & \text{if } y \in \Gamma_4.
\end{cases}
\]

Then the function $u^c$ is a subsolution of

\[
\bar{G}_c(x, u^c, Du^c, D^2u^c) = 0 \quad \text{in } O^\varepsilon,
\]

where

\[
\bar{G}_c(x, r, p, S) = \inf \{ \inf\{G(y, r, p, S), \bar{B}_1(y, r)\} : y \in \overline{O}, \ |y - x| \leq (C_0 + C)\sqrt{\varepsilon}\},
\]

with the convention that $\inf\{G(y, r, p, S), \bar{B}_1(y, r)\} = G(y, r, p, S)$ for $y \in O$.

Let us define for $(y, r) \in \partial O \times \mathbb{R}$ the function $\bar{B}_2(y, r) = r - \psi_2(y)$, where

\[
\psi_2(y) = \begin{cases} 
\psi(y) & \text{if } y \in \Gamma_1 \cup \Gamma_3 \\
-\infty & \text{if } y \in \Gamma_2 \\
u(y) & \text{if } y \in \Gamma_4.
\end{cases}
\]

Then $v^c$ is a supersolution of

\[
\bar{G}_c(x, v^c, Dv^c, D^2v^c) = 0 \quad \text{in } O^\varepsilon,
\]

where

\[
\bar{G}_c(x, r, p, S) = \sup \{ \sup\{G(y, r, p, S), \bar{B}_2(y, r)\} : y \in \overline{O}, \ |y - x| \leq (C_0 + C)\sqrt{\varepsilon}\},
\]

with the convention that $\sup\{G(y, r, p, S), \bar{B}_2(y, r)\} = G(y, r, p, S)$ for $y \in O$. Finally, the functions $G_c$ and $G^c$ are respectively lsc and usc and the assertion of Proposition 2 with $G_c$ and $G^c$ replaced by $G_c$ and $\bar{G}^c$, respectively, holds true under the above assumptions and notation.

The proof of Proposition 3 is postponed to the Appendix.

**Proof of Theorem 3.2.** We limit ourselves to prove the first statement, the second one being very similar. Notice that, being $\Gamma_2$ relatively open in $\partial O$, $\Gamma_1$ turns out to be closed relatively to $\Gamma_1 \cup \Gamma_2$. Arguing by contradiction as in the proof of Theorem 3.1, by (25) and (26) the maximum point $w$ belongs to $\Gamma_1 \cup \Gamma_2$ and one of the two following situations occurs:

1. $w$ belongs to $\Gamma_2$,
2. $w$ belongs to $\Gamma_1$ and $u(w) > \psi(w) + \gamma$.

Indeed, if $w \in \Gamma_1$ and $u(w) \leq \psi(w)$, by (26) it follows $v(w) \leq \psi(w) - \gamma$, in contradiction with the boundary condition (35). Since now $\text{dist}(w, \Gamma_3 \cup \Gamma_1) > \tilde{\gamma}$ for some $\tilde{\gamma} > 0$, we can proceed as in the proof of Theorem 3.1 just replacing $\Gamma_3$
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with \( \Gamma_3 \cup \Gamma_4 \). As already done in the proof of Theorem 3.1, in case 1., choosing \( \delta \) and \( \varepsilon \) small enough we can exclude that \( \bar{x}_{\delta}, \bar{\bar{x}}_{\varepsilon,\delta} \), and any \( x \in \mathcal{O} \) with \( |x - \bar{x}_{\varepsilon,\delta}| \leq (C_0 + C_3)\sqrt{\varepsilon} \) belong to \( \Gamma_1 \). In case 2., instead, it may happen that \( \bar{x}_{\varepsilon,\delta} \in \Gamma_1 \) and there could be also some other \( x \in \mathcal{O} \) with \( |x - \bar{x}_{\varepsilon,\delta}| \leq (C_0 + C_3)\sqrt{\varepsilon} \) belonging to \( \Gamma_1 \). But now \( u(w) > \psi(w) + \gamma \) by hypothesis, therefore the uniform continuity of \( \psi \) on \( \Gamma_1 \) together with (28) and (29) implies (30) with \( v \) replaced by \( \psi \), that is

\[
u^2(x,\delta) \leq \psi(x) + \frac{\gamma}{2} \quad \forall x \in \Gamma_1 \text{ such that } |x - \bar{x}_{\varepsilon,\delta}| \leq (C_0 + C_3)\sqrt{\varepsilon}\]

for \( \delta \) and \( \varepsilon \) small enough. At this point, we can easily see that for \( x = \bar{\bar{x}}_{\varepsilon,\delta} \) one has \( G^\varepsilon \equiv G \) while for \( x = \bar{x}_{\varepsilon,\delta} \) one has \( G_\varepsilon \equiv G \), where \( G_\varepsilon \) and \( G^\varepsilon \) are the functions introduced in Proposition 3. Therefore applying Proposition 3 we can conclude the proof of cases 1. and 2. by arguing from now on as in the last part of the proof of Theorem 3.1.

**Proof of Theorem 3.3.** We limit ourselves to prove the first statement, the second one being very similar. Let us begin as in the proof of Theorem 3.2. Since \( \Gamma_1 \) and \( \Gamma_2 \) are now assumed to be relatively open in \( \partial \mathcal{O} \), for the maximum point \( w \) we have the same situations 1. and 2. as above, which become in fact simpler, since now once \( w \in \Gamma_1 \) all further reasonings can be pursued in some neighborhood \( \mathcal{N} \) of \( w \) in \( \mathcal{O} \) such that \( \mathcal{N} \subset \mathcal{O} \cup \Gamma_1 \). However, we have to take into account also a third possibility:

3. \( w \in \Gamma_1 \) and \( v(w) < \psi(w) - \gamma \).

In this case, it is sufficient to modify the test function \( \Phi \) given by (27) as follows

\[
\Phi(x,y) \doteq \frac{1}{\delta^2} |x - y - \delta \eta(w)|^2 + |y - w|^4,
\]

and to repeat the proof of case 2. switching \( x \) with \( y \).

As an immediate consequence of Theorems 3.2, 3.3 one obtains the following uniqueness result.

**Corollary 1. (Uniqueness for BVP and BVP1)** Assume that \( \mathcal{O} \) is a nonempty, bounded and open subset of \( \mathbb{R}^N \), \( \Gamma_1, \Gamma_2 \subseteq \partial \mathcal{O}, \Gamma_3 \subseteq \partial \mathcal{O} \setminus (\Gamma_1 \cup \Gamma_2) \), and (H0)--(H3) hold. Let \( \psi : \Gamma_1 \cup \Gamma_3 \rightarrow \mathbb{R} \) be bounded and continuous on \( \Gamma_1 \cup \Gamma_3 \) and uniformly continuous on \( \Gamma_1 \).

i) If \( \Gamma_2 \) is a relatively open set in \( \partial \mathcal{O} \), there is at most one function \( u : \overline{\mathcal{O}} \rightarrow \mathbb{R} \) bounded, continuous on \( \partial \mathcal{O} \) and uniformly continuous on \( \Gamma_1 \) which solves BVP1 and \( u \) turns out to be continuous in \( \overline{\mathcal{O}} \). The same conclusion holds if \( u \) is a solution of BVP1 with the boundary conditions “\( u \geq \psi \) and \( G(x,u,Du,D^2u) \leq 0 \) if \( u > \psi \) on \( \Gamma_1 \)” , and “\( G(x,u,Du,D^2u) \leq 0 \) on \( \Gamma_2 \)” replaced by “\( u \leq \psi \) and \( G(x,u,Du,D^2u) \geq 0 \) if \( u < \psi \) on \( \Gamma_1 \)” , and “\( G(x,u,Du,D^2u) \geq 0 \) on \( \Gamma_2 \)” , respectively.

ii) If both \( \Gamma_1 \) and \( \Gamma_2 \) are relatively open in \( \partial \mathcal{O} \), there is at most one function \( u : \overline{\mathcal{O}} \rightarrow \mathbb{R} \) bounded, continuous on \( \partial \mathcal{O} \) and uniformly continuous on \( \Gamma_1 \) which solves BVP. The same conclusion holds if \( u \) is a solution of BVP with the boundary condition “\( G(x,u,Du,D^2u) \leq 0 \) on \( \Gamma_2 \)” replaced by “\( G(x,u,Du,D^2u) \geq 0 \) on \( \Gamma_2 \)”.

**Remark 6.** The hypotheses of Theorems 3.2, 3.3 can be weakened in several directions. For instance, if \( \Gamma_2 \) is relatively open (but \( \Gamma_1 \) possibly not) one can obtain a comparison result between sub- and supersolutions of BVP if the subsolution condition (3) in BVP, that is, \( G(x,u,Du,D^2u) \leq 0 \) on \( \Gamma_2 \), is strengthened into...
$G(x, u, Du, D^2u) = 0$ at the points of $\Gamma_2$ “near” $\Gamma_1$ in $\Gamma_1 \cup \Gamma_2$. Moreover, the statement of Theorem 3.2 holds true even if we assume the classical Dirichlet condition $v \geq \psi$ just on $\Gamma_3$ and at the points of $\Gamma_1$ “near” $\Gamma_2$ in $\Gamma_1 \cup \Gamma_2$, and consider the generalized Dirichlet condition “$v \geq \psi$ or $G(x, v, Dv, D^2v) \geq 0$” on the remaining part of $\Gamma_1$.

**Remark 7.** The continuity assumptions of $v$ on $\Gamma_1 \cup \Gamma_2$ in Theorem 3.2 could be relaxed. For instance, we might assume the following non tangential lower semicontinuity property: for each $x \in \Gamma_1$ there is a positive constant $c$, a bounded sequence $\{\eta_n\}_n \subset \mathbb{R}^N$ and a sequence $\{r_n\}_n \subset \mathbb{R}$ converging to 0, such that

$$B(y + r_n \eta_n, cr_n) \subset \Omega \quad \forall y \in B(x, c) \cap \overline{\Omega}.$$ 

This (or some similar) relaxation leads to a version of the comparison principle between non-continuous functions which may be useful for the proof of existence of continuous solutions to mixed boundary value problems. An approach of this type to comparison theorems has been used by Ishii in [11] for first order Hamilton-Jacobi equations and by Katsoulakis in [17] for second order equations.

4. Results on unbounded domains. In this section we extend the previous results to the case of an unbounded set $\Omega \subset \mathbb{R}^N$. These results generalize to mixed boundary value problems Theorems 7.3, 7.4 in [12]. The following lemma together with Theorems 3.1–3.3 is a key point in the proof of comparison and uniqueness results.

**Lemma 4.1.** Given an open, possibly unbounded, nonempty set $\Omega \subset \mathbb{R}^N$, let $\Gamma_1$, $\Gamma_2 \subset \partial \Omega$ verify (H0). Let $\mathcal{L} > 0$ be a constant such that $\Omega \cap B(0, \mathcal{L}) \neq \emptyset$. For any $L > \mathcal{L} + c(c + \sup_{\overline{\Omega} \cap \Theta} |\eta|)$, where $c, \eta$ and $\Theta$ are the same as in (H0), let us set $\tilde{L} = L - c(c + \sup_{\overline{\Omega} \cap \Theta} |\eta|)$. Then the set

$$\Omega_L = \left( \Omega \cap B(0, \tilde{L}) \right) \cup \left\{ B(x + t\eta(x), tc) : \ x \in \Theta \cap B(0, \tilde{L}) \cap \overline{\Omega}, \ 0 < t \leq c \right\}$$

is a nonempty, open and bounded subset of $\Omega$ contained in $B(0, L)$ such that the sets

$$\Gamma_{1L} = \Gamma_1 \cap B(0, \tilde{L}), \quad \Gamma_{2L} = \Gamma_2 \cap B(0, \tilde{L})$$

are contained in $\partial \Omega_L$ and verify (H0) with $\Theta \cap B(0, \tilde{L})$ replacing $\Theta$.

**Proof.** Owing to the choices of $L$ and $\tilde{L}$ the set $\Omega_L$ is trivially open, nonempty and contained in $B(0, L)$. By hypothesis (H0) it follows that

$$\left\{ B(x + t\eta(x), tc) : \ x \in \Theta \cap B(0, \tilde{L}) \cap \overline{\Omega}, \ 0 < t \leq c \right\} \subset \Omega.$$ 

Hence $\Omega_L$ is a subset of $B(0, L) \cap \Omega$, the sets $\Gamma_{1L}, \Gamma_{2L}$ turn out to be disjoint subsets of $\partial \Omega_L$, and $\Gamma_{1L} \cup \Gamma_{2L}$ is relatively open in $\partial \Omega_L$. Moreover, since $\Theta \supset \Gamma_1 \cup \Gamma_2$, one has that $\Theta \cap B(0, \tilde{L}) \supset \Gamma_{1L} \cup \Gamma_{2L}$ by definition. To conclude the proof it remains to show that

$$B(x + t\eta(x), tc) \subset \Omega_L \quad \forall x \in (\Theta \cap B(0, \tilde{L})) \cap \overline{\Omega_L}, \ 0 < t \leq c,$$

but this fact is true for all $x \in \Theta \cap B(0, \tilde{L}) \cap \overline{\Omega_L}$ and $0 < t \leq c$ by the very definition of $\Omega_L$. \qed
Theorem 4.2. Given an open, possibly unbounded, nonempty set $\mathcal{O} \subset \mathbb{R}^N$, assume that $\Gamma_1, \Gamma_2 \subseteq \partial \mathcal{O}$, $\Gamma_3 \doteq \partial \mathcal{O} \setminus (\Gamma_1 \cup \Gamma_2)$, and (H0)-(H3) hold. Let $u : \overline{\mathcal{O}} \to \mathbb{R}$ be usc and locally bounded in $\mathcal{O}$, let $v : \overline{\mathcal{O}} \to \mathbb{R}$ be lsc and locally bounded in $\mathcal{O}$, continuous on $\Gamma_1 \cup \Gamma_2$ and uniformly continuous on $\Gamma_1 \cap B(0,L)$ for all $L > 0$. If $\mathcal{O}$ is unbounded, assume that
\[
\lim_{x \in \mathcal{O}, |x| \to \infty} \frac{u^+(x)}{\log |x|} = 0 \quad \text{and} \quad \lim_{x \in \mathcal{O}, |x| \to \infty} \frac{v^-(x)}{\log |x|} = 0.
\]
Hence if $u$ and $v$ are respectively subsolution and supersolution of (1) in $\mathcal{O}$ and verify (19), then $u \leq v$ in $\overline{\mathcal{O}}$. The same conclusion holds if $u$ and $v$ are respectively subsolution and supersolution of (1) in $\mathcal{O}$, verify (20), and $u$ instead of $v$ is assumed to be continuous on $\Gamma_1 \cup \Gamma_2$ and uniformly continuous on $\Gamma_1 \cap B(0,L)$ for all $L > 0$.

Proof. We limit ourselves to prove the first statement, the second one being very similar. Arguing as in the proof of Theorem 7.3 in [12], we set $g(x) \doteq \log(1 + |x|^2)$ and for $\varepsilon, \delta > 0$ we consider the functions
\[
\tilde{u}(x) = u(x) - \varepsilon g(x) - \delta, \quad \text{and} \quad \tilde{v}(x) = v(x) + \varepsilon g(x) + \delta.
\]
By hypothesis (H2), as soon as we take $C > 1$ and set e.g. $\tilde{C} = 3C^2$, it follows that $|\Sigma(x,a,b)\Sigma^T(x,a,b)| \leq \tilde{C}(1 + |x|^2)$, $|d(x,a,b)| \leq \tilde{C}(1 + |x|)$ for all $(x,a,b) \in \overline{\mathcal{O}} \times A \times B$, and we may also assume that $|Dg(x)| \leq \tilde{C}(1 + |x|)^{-1}$, $||D^2g(x)|| \leq \tilde{C}(1 + |x|^2)^{-1}$.

Hence in view of the definition of $G$ it is easy to check that $\tilde{u}$ turns out to be a subsolution of $G(x,u,Du,D^2u) = -\lambda_0 \delta + 2\varepsilon \tilde{C}^2$ in $\mathcal{O}$ and that $\tilde{v}$ is a supersolution of $G(x,u,Du,D^2u) = \lambda_0 \delta - 2\varepsilon \tilde{C}^2$ in $\mathcal{O}$, where $\lambda_0$ is the same as in (H3). Moreover, $\tilde{u}, \tilde{v}$ verify the following boundary conditions:
\[
\begin{align*}
\tilde{u} &\leq \tilde{v} \quad \text{or} \quad G(x,\tilde{u},D\tilde{u},D^2\tilde{u}) \leq -\lambda_0 \delta + 2\varepsilon \tilde{C}^2 \quad \text{on } \Gamma_1, \\
G(x,\tilde{u},D\tilde{u},D^2\tilde{u}) &\leq -\lambda_0 \delta + 2\varepsilon \tilde{C}^2 \quad \text{on } \Gamma_2, \\
\tilde{u} &\leq \tilde{v} \quad \text{on } \Gamma_3.
\end{align*}
\]

Indeed, if either $x \in cl(\Gamma_1)$ and $u(x) \leq v(x)$ or $x \in \Gamma_3$ then $\tilde{u}(x) \leq \tilde{v}(x)$, being $\tilde{u}(x) < u(x) \leq v(x) < \tilde{v}(x)$.

Let us now consider either $x \in cl(\Gamma_1)$ and $G(x,u,Du,D^2u) \leq 0$ or $x \in int(\Gamma_2)$. We notice that for any $\varphi \in C^2(\mathbb{R}^N)$ such that $\tilde{\varphi} - \varphi$ has a local maximum $\tilde{x}$ in $\mathcal{O}$, the function $u - \tilde{\varphi}$, where $\tilde{\varphi} + \varepsilon g + \delta$, has a local maximum at the same point $x$. Hence by (19) one has $G(x,u(x),D\tilde{\varphi}(x),D^2\tilde{\varphi}(x)) \leq 0$, which by straightforward calculations yields
\[
G(x,\tilde{u}(x),D\varphi(x),D^2\varphi(x)) \leq -\lambda_0 \delta + 2\varepsilon \tilde{C}^2,
\]
recalling the expression of $G$ and (H2), (H3). This concludes the proof of (41).

Therefore, choosing $\delta = \frac{2\varepsilon \tilde{C}^2}{\lambda_0}$, $\tilde{u}$ and $\tilde{v}$ turn out to be respectively subsolution and supersolution of (1) in $\mathcal{O}$ and they verify the boundary conditions (19). Let us now fix $L > 0$ sufficiently large so that $\tilde{u} < 0 < \tilde{v}$ in $\overline{\mathcal{O}} \setminus B(0,\tilde{L})$ where $\tilde{L} = L - c(c+\sup_{\Gamma_1 \cup \Theta} \eta)$ and correspondingly let us consider the sets $\mathcal{O}_L, \Gamma_{1L}$, and $\Gamma_{2L}$ defined as in Lemma 4.1 and the boundary subset $\Gamma_{3L} = \partial \mathcal{O}_L \setminus (\Gamma_{1L} \cup \Gamma_{2L})$. Then the set
\[ \Gamma_1 \cup \Gamma_2 \] is relatively open in \( \partial \Omega \), and \( \Omega \), \( \Gamma_1 \), \( \Gamma_2 \) verify (H0) in view of Lemma 4.1. Moreover, one can easily see that \( \tilde{v} \) is continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \), that \( \tilde{u} \) and \( \tilde{v} \) are respectively subsolution and supersolution of (1) in \( \Omega \), and that they verify (19) with \( \Gamma_1 \), \( \Gamma_2 \), replacing \( \Gamma_1 \), \( \Gamma_2 \), and \( \Gamma_3 \), respectively. Then by Theorem 3.1 it follows that \( \tilde{u} \leq \tilde{v} \) in \( \Omega \), and since \( \tilde{u} < 0 < \tilde{v} \) in \( \Omega \setminus B(0, L) \), we obtain \( \tilde{u} \leq \tilde{v} \) in \( \Omega \). Finally, sending \( \varepsilon \) and \( \delta \) to 0 we get \( u \leq v \) in \( \Omega \), which concludes the proof.

In the second comparison result, hypothesis (H2) is replaced by the following stronger assumption.

(H2)’ For any \( \varepsilon > 0 \) there is a constant \( C_\varepsilon > 0 \) such that
\[
||\Sigma(x, a, b)\Sigma^T(x, a, b)|| \leq C_\varepsilon \varepsilon^2 \quad \text{and} \quad |d(x, a, b)| \leq C_\varepsilon \varepsilon \quad \forall (x, a, b) \in \Omega \times A \times B.
\]

**Theorem 4.3.** Given an open, possibly unbounded, nonempty set \( \Omega \subset \mathbb{R}^N \), assume that \( \Gamma_1, \Gamma_2 \subset \partial \Omega \) are given, \( \Gamma_3 \equiv \partial \Omega \setminus (\Gamma_1 \cup \Gamma_2) \) and that (H0), (H1), (H2)’ and (H3) hold. Let \( u : \bar{\Omega} \to \mathbb{R} \) be lsc and locally bounded in \( \Omega \), let \( v : \bar{\Omega} \to \mathbb{R} \) be lsc and locally bounded in \( \Omega \), continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \). If \( \Omega \) is unbounded, assume that there exists an integer \( r \) such that
\[
\sup_{x \in \bar{\Omega}} \frac{u(x)}{(1 + |x|)^r} < +\infty \quad \text{and} \quad \inf_{x \in \bar{\Omega}} \frac{v(x)}{(1 + |x|)^r} > -\infty.
\]

Hence if \( u \) and \( v \) are respectively subsolution and supersolution of (1) in \( \Omega \) and verify (19), then \( u \leq v \) in \( \Omega \). The same conclusion holds if \( u \) and \( v \) are respectively subsolution and supersolution of (1) in \( \Omega \), verify (20), and \( u \) instead of \( v \) is assumed to be continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \).

**Proof.** The proof is analogous to the previous one once, as in the proof of Theorem 7.4 in [12], we substitute the function \( x \mapsto (1 + |x|^2)^{-\frac{r}{r+1}} \) to the function \( g \).

The main results of this section are the following comparison theorems for the boundary value problems BVP and BVP1.

**Theorem 4.4.** (Comparison for BVP and BVP1, I.) Given an open, possibly unbounded, nonempty set \( \Omega \subset \mathbb{R}^N \), assume that \( \Gamma_1, \Gamma_2 \subset \partial \Omega \) are given, \( \Gamma_3 \equiv \partial \Omega \setminus (\Gamma_1 \cup \Gamma_2) \) and (H0)–(H3) hold. Let \( \psi : \Gamma_1 \cup \Gamma_3 \to \mathbb{R} \) be continuous on \( \Gamma_1 \cup \Gamma_3 \), uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \), and such that
\[
\lim_{x \in \Gamma_1 \cup \Gamma_3, |x| \to \infty} \frac{\log |x|}{|\psi(x)|} = 0.
\]

Let \( u : \bar{\Omega} \to \mathbb{R} \) be locally bounded and lsc in \( \Omega \), let \( v : \bar{\Omega} \to \mathbb{R} \) be locally bounded and lsc in \( \Omega \) and assume that
\[
\lim_{x \in \bar{\Omega}, |x| \to \infty} \frac{u^+(x)}{\log |x|} = 0 \quad \text{and} \quad \lim_{x \in \bar{\Omega}, |x| \to \infty} \frac{v^-(x)}{\log |x|} = 0 \quad (42)
\]
i) If \( \Gamma_2 \) is a relatively open set in \( \partial \Omega \), \( v \) is continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \) and if \( u \) and \( v \) verify (34), (35), then \( u \leq v \) in \( \Omega \). The same conclusion holds if \( u \) instead of \( v \) is continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \), and \( u \) and \( v \) verify (37), (38).
ii) If both \( \Gamma_1 \) and \( \Gamma_2 \) are relatively open sets in \( \partial \mathcal{O} \) and \( u \) and \( v \) are both continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \), if \( u \) and \( v \) verify (34) and (39), then \( u \leq v \) in \( \overline{\mathcal{O}} \). The same conclusion holds if \( u \) and \( v \) verify (38) and (40).

Proof. i) In order to prove that \( u \leq v \) in \( \overline{\mathcal{O}} \), we consider the same functions \( \tilde{u} \) and \( \tilde{v} \) introduced in the proof of Theorem 4.2 and a constant \( L > 0 \) such that \( \tilde{u} < 0 < \tilde{v} \) in \( \overline{\mathcal{O}} \setminus B(0, \tilde{L}) \), where \( \tilde{L} \) is defined as in Lemma 4.1. Then arguing similarly to the proof of Theorem 4.2 one can show that for \( \delta = \frac{2cG^2}{\lambda_0} \) the functions \( \tilde{u}, \tilde{v} \) solve

\[
\begin{cases}
G(x, \tilde{u}, D\tilde{u}, D^2\tilde{u}) \leq 0 & \text{in } \mathcal{O}_L \\
\tilde{u} \leq \psi & \text{on } \Gamma_{1L} \\
G(x, \tilde{u}, D\tilde{u}, D^2\tilde{u}) \leq 0 & \text{on } \Gamma_{2L} \\
\tilde{u} \leq \psi & \text{on } \Gamma_{3L},
\end{cases}
\]

and

\[ \tilde{u} \leq \tilde{v} \quad \text{on } \Gamma_{4L}, \]

where \( \mathcal{O}_L, \Gamma_{1L}, \) and \( \Gamma_{2L} \) are the same as in Lemma 4.1, while \( \Gamma_{3L} = \Gamma_3 \cap B(0, \tilde{L}) \) and \( \Gamma_{4L} = \partial \mathcal{O}_L \setminus (\Gamma_{1L} \cup \Gamma_{2L} \cup \Gamma_{3L}) \). It is now easy to see that Theorem 3.2 can be applied to the sets \( \mathcal{O}_L, \Gamma_{1L}, \Gamma_{2L}, \Gamma_{3L}, \) and \( \Gamma_{4L} \) and to the functions \( \tilde{u}, \tilde{v} \). Therefore we obtain \( \tilde{u} \leq \tilde{v} \) in \( \overline{\mathcal{O}}_L \), and since \( \tilde{u} < 0 < \tilde{v} \) in \( \overline{\mathcal{O}} \setminus B(0, \tilde{L}) \), we obtain \( \tilde{u} \leq \tilde{v} \) in \( \overline{\mathcal{O}} \). Sending \( \varepsilon \) and \( \delta \) to 0 we get \( u \leq v \) in \( \overline{\mathcal{O}} \), as in the proof Theorem 4.2.

We omit the proof in case ii), since it can be deduced from the proof of Theorem 4.2 arguing as above but applying Theorem 3.3 instead of Theorem 3.2 in \( \mathcal{O}_L \). \( \square \)

Theorem 4.5. (Comparison for BVP and BVP1, II.) Given an open, possibly unbounded, nonempty set \( \mathcal{O} \subset \mathbb{R}^N \), assume that \( \Gamma_1, \Gamma_2 \subseteq \partial \mathcal{O} \) are given, \( \Gamma_3 \subseteq \partial \mathcal{O} \setminus (\Gamma_1 \cup \Gamma_2) \) and (H0), (H1), (H2)' and (H3) hold. Let \( \psi : \Gamma_1 \cup \Gamma_3 \to \mathbb{R} \) be continuous on \( \Gamma_1 \cup \Gamma_3 \), uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \), and such that

\[
\sup_{x \in \Gamma_1 \cup \Gamma_3} \frac{|\psi(x)|}{(1 + |x|)^r} < +\infty
\]

for some integer \( r \). Let \( u : \overline{\mathcal{O}} \to \mathbb{R} \) be locally bounded and usc in \( \overline{\mathcal{O}} \), let \( v : \overline{\mathcal{O}} \to \mathbb{R} \) be locally bounded and lsc in \( \overline{\mathcal{O}} \) and assume that

\[
\sup_{x \in \overline{\mathcal{O}}} \frac{u(x)}{(1 + |x|)^r} < +\infty \quad \text{and} \quad \inf_{x \in \overline{\mathcal{O}}} \frac{v(x)}{(1 + |x|)^r} > -\infty.
\]

i) If \( \Gamma_2 \) is a relatively open set in \( \partial \mathcal{O} \), \( v \) is continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \), and if \( u \) and \( v \) verify (34), (35), then \( u \leq v \) in \( \overline{\mathcal{O}} \). The same conclusion holds if \( u \) instead of \( v \) is continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \), and \( u \) and \( v \) verify (37), (38).

ii) If both \( \Gamma_1 \) and \( \Gamma_2 \) are relatively open sets in \( \partial \mathcal{O} \) and \( u \) and \( v \) are both continuous on \( \Gamma_1 \cup \Gamma_2 \) and uniformly continuous on \( \Gamma_1 \cap B(0, L) \) for all \( L > 0 \), if \( u \) and \( v \) verify (34) and (39), then \( u \leq v \) in \( \overline{\mathcal{O}} \). The same conclusion holds if \( u \) and \( v \) verify (38) and (40).
We omit the proof, since via obvious changes it is analogous to the proof of Theorem 4.4.

The following uniqueness results are now straightforward.

**Corollary 2. (Uniqueness for BVP and BVP1)** Let the same assumptions of Theorem 4.4 [resp., of Theorem 4.5] on \(\mathcal{O}, \Gamma_1, \Gamma_2, \Gamma_3, \) and \(\psi\) hold.

i) If \(\Gamma_2\) is a relatively open set in \(\partial \mathcal{O}\), then in the class of functions \(z : \overline{\mathcal{O}} \to \mathbb{R}\) locally bounded, continuous on \(\partial \mathcal{O}\), and verifying

\[
\lim_{x \to \partial \mathcal{O}, |x| \to +\infty} \frac{|z(x)|}{\log |x|} = 0
\]

[resp.,

\[
\sup_{x \in \overline{\mathcal{O}}} \frac{|z(x)|}{(1 + |x|)^r} < +\infty
\]

for the same \(r\) for which \(\psi\) verifies (43)], there is at most one solution \(u\) to BVP and \(u\) turns out to be continuous in \(\overline{\mathcal{O}}\). The same conclusion holds if \(u\) solves BVP1 with the boundary conditions “\(u < \psi\) and \(G(x, u, Du, D^2u) \geq 0\) if \(u < \psi\) on \(\Gamma_1\)” , and “\(G(x, u, Du, D^2u) \geq 0\) on \(\Gamma_2\)” in place of “\(u \geq \psi\) and \(G(x, u, Du, D^2u) \leq 0\) if \(u > \psi\) on \(\Gamma_1\)” , and “\(G(x, u, Du, D^2u) \leq 0\) on \(\Gamma_2\)” , respectively.

ii) If both \(\Gamma_1\) and \(\Gamma_2\) are relatively open sets in \(\partial \mathcal{O}\), then in the class of functions \(z : \overline{\mathcal{O}} \to \mathbb{R}\) locally bounded, continuous on \(\partial \mathcal{O}\), and verifying the growth condition (45) [resp., (46)] there is at most one solution \(u\) to BVP and \(u\) turns out to be continuous in \(\overline{\mathcal{O}}\). The same conclusion holds if \(u\) solves BVP with the boundary condition “\(G(x, u, Du, D^2u) \geq 0\) on \(\Gamma_2\)” in place of “\(G(x, u, Du, D^2u) \leq 0\) on \(\Gamma_2\)” .

**Remark 8.** Since we deal with problems involving a generalized Dirichlet condition, solutions may happen to grow as \(x\) tends to infinity slower than the boundary datum \(\psi\). For instance, from the proof of Theorem 4.4 it immediately follows that if there exists a solution of the boundary value problem belonging to the class of functions verifying (45) such a solution turns out to be the unique solution in its class, independently from the growth of \(\psi\) (even if the stronger assumption (H2) does not hold). This is actually the case of the problem introduced in Example 1.

**Remark 9.** Parabolic problems. The previous results can be extended to the following parabolic boundary value problem, denoted by PBVP,

\[
\begin{align*}
 u_t + G(t, x, u, Du, D^2u) &= 0 \quad \text{in } ]0, T[ \times \mathcal{O}, \\
 u(t, x) &= \psi(t, x) \quad \text{or} \quad u_t + G(t, x, u, Du, D^2u) = 0 \quad \text{on } ]0, T[ \times \Gamma_1, \\
 u_t + G(t, x, u, Du, D^2u) &\leq 0 \quad \text{on } ]0, T[ \times \Gamma_2, \\
 u(t, x) &= \psi(t, x) \quad \text{on } ]0, T[ \times \Gamma_3, \\
 u(0, x) &= \psi(0, x) \quad \text{on } \{0\} \times \overline{\mathcal{O}},
\end{align*}
\]

where, with a small abuse of notation, the operator \(G\) is now of the form

\[
G(t, x, r, p, S) \doteq \inf_{a \in A, b \in B} \left\{-\frac{1}{2} \text{Tr} \{\Sigma(t, x, a, b) \Sigma^T(t, x, a, b) S\} - \langle d(t, x, a, b), p \rangle \right. \\
- \epsilon(t, x, a, b) + m(t, x, a, b) r \}
\]

for some nonempty sets \(A \subset \mathbb{R}^M\) and \(B \subset \mathbb{R}^r\). This extension is not an immediate application of the results of the previous sections even if we replace the variable \(x\) in BVP with \((t, x)\), since, in order to exploit the comparison theorems of Sections...
3 and 4 one has to identify new boundary subsets of $\partial([0,T] \times \mathcal{O})$, say $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$, and $\tilde{\Gamma}_3$ verifying (H0) and such that e.g. $\tilde{\Gamma}_2$ is relatively open, while in PBVP there are parts of the boundary where no conditions are assumed. Under the following (HP) hypothesis all the comparison and uniqueness results of Sections 3 and 4 can be extended to PBVP.

(HP) The functions that appear in the definition of $G$ depend continuously on the new variable $t$ and they verify (H1) and (H2) (or (H2)') uniformly with respect to $t \in [0,T]$. The strong monotonicity hypothesis (H3) may be relaxed by taking $\lambda_0 \in \mathbb{R}$ instead of $\lambda_0 > 0$.

In the special case where $G$ in PBVP is replaced with the following Bellman-Isaacs operator

$$H(t, x, r, p) = \inf_{a \in A_B} \sup_{b \in B} \{-(d(t, x, a, b), p) - c(t, x, a, b) + \lambda_0 r\},$$

when the data $c$ and $\psi$ are bounded from below we could also deduce a uniqueness result without growth conditions on solutions from above. Indeed, if $u$ and $v$ are both solutions such that $\inf u, \inf v > -C$ for some $C > 0$, first we may suppose that $\lambda_0 = 0$. If not, it is sufficient to replace the functions $u$ and $v$ with $\hat{u}(t, x) = e^{\lambda_0 t}u(t, x)$ and $\hat{v}(t, x) = e^{\lambda_0 t}v(t, x)$, respectively. Moreover, by straightforward calculations the functions

$$\hat{u}(t, x) = 1 - e^{-u(t,x)-C}, \quad \hat{v}(t, x) = 1 - e^{-v(t,x)-C}$$

are bounded solutions of a boundary value problem analogous to that solved by $u$ and $v$.

5. On the existence of solutions. We set

$$\mathcal{S} = \{w : w : \mathcal{O} \to \mathbb{R}, \quad G(x, w, Dw, D^2w) \geq 0 \text{ in } \mathcal{O}\}.$$

If $\Gamma_1$ is a relatively open subset of $\partial \mathcal{O}$, we also define

$$\mathcal{S} = \{w : w : \mathcal{O} \cup \Gamma_1 \to \mathbb{R}, \quad G(x, w, Dw, D^2w) \geq 0 \text{ in } \mathcal{O}, \quad w \geq \psi \text{ or } G(x, w, Dw, D^2w) \geq 0 \text{ on } \Gamma_1\}.$$

Perron’s method applied to mixed boundary value problems of the form of BVP or BVP1 leads to the following statement.

Proposition 4. Let the same assumptions of Theorem 4.1 [resp., of Theorem 4.5] on $\mathcal{O}$, $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\psi$ hold. Let $U$, $V : \overline{\mathcal{O}} \to \mathbb{R}$ be locally bounded functions, continuous on $\partial \mathcal{O}$, usc and lsc in $\overline{\mathcal{O}}$, respectively, and verifying the growth condition (42) [resp., (44) for the same $r$ for which $\psi$ verifies (43)].

i) If $\Gamma_2$ is a relatively open set in $\partial \mathcal{O}$, $U$ and $V$ verify (34), (35), respectively, and $U \geq \psi$ on $\Gamma_1$, then the function

$$\hat{W}(x) = \inf \left\{ w(x) : w \in \mathcal{S}, U \leq w \leq V \right\} \quad x \in \mathcal{O},$$

extended to $\overline{\mathcal{O}}$ by setting

$$\hat{W}(x) = \liminf_{y \to x} \left\{ W(y) : y \in \mathcal{O} \right\} \quad \forall x \in \partial \mathcal{O}$$

is a solution to BVP1 in the class of functions $z : \overline{\mathcal{O}} \to \mathbb{R}$ locally bounded and such that (45) [resp., (46)] holds true.
ii) If both $\Gamma_1$ and $\Gamma_2$ are relatively open sets in $\partial \mathcal{O}$ and $U$ and $V$ verify (34) and (39), then the function
\[
W(x) \doteq \inf \{ w(x) : w \in \mathcal{S}, U \leq w \leq V \} \quad \forall x \in \mathcal{O} \cup \Gamma_1,
\]
extended to $\overline{\mathcal{O}}$ by setting
\[
W(x) \doteq \liminf_{y \to x} \{ W(y) : y \in \mathcal{O} \cup \Gamma_1 \} \quad \forall x \in \overline{\mathcal{O}}
\]
is a solution to BVP in the class of functions $z : \overline{\mathcal{O}} \to \mathbb{R}$ locally bounded and such that (45) [resp., (46)] holds true.

**Remark 10.** In case i) similar conclusions hold for the problem obtained from BVP1 when the conditions “$u \leq \psi$ and $G(x, u, Du, D^2u) \geq 0$ if $u < \psi$ on $\Gamma_1$” and “$G(x, u, Du, D^2u) \geq 0$ on $\Gamma_2$” replace the conditions “$u \geq \psi$ and $G(x, u, Du, D^2u) \leq 0$ if $u > \psi$ on $\Gamma_1$” and “$G(x, u, Du, D^2u) \leq 0$ on $\Gamma_2$”, respectively. Analogously, in case ii) similar conclusions hold for the problem obtained from BVP when the condition “$G(x, u, Du, D^2u) \leq 0$ on $\Gamma_2$” is replaced by “$G(x, v, Dv, D^2v) \geq 0$ on $\Gamma_2$”. Of course, in these cases in the definitions of $\tilde{W}$ and $W$ instead of the infimum of supersolutions one takes the supremum over subsolutions.

**Proof of Proposition 4.** The proof of Proposition 4 consists in a careful adaptation of Perron’s method. We limit ourselves to discuss the case where the growth condition (45) is assumed, the other case being completely similar.

i) The set $\tilde{\mathcal{S}}$ is nonempty since $V \in \tilde{\mathcal{S}}$. Moreover, by the comparison Theorem 4.4, i), it follows that $U(x) \leq V(x) \quad \forall x \in \overline{\mathcal{O}}$. Therefore the set involved in the definition of the function $W$ is nonempty. $W$ trivially verifies the growth assumption (45) and the proof that $W_*$ is a supersolution of $G(x, u, Du, D^2u) = 0$ in $\mathcal{O}$ is standard (see e.g. [13]). At the points $x \in \Gamma_3$ it turns out that $W_*(x) = W(x) = \tilde{W}^*(x) = \psi(x)$ because of the continuity of $U$, $V$ and $\psi$ on $\partial \mathcal{O}$. If $x \in \Gamma_1$, one has $W_*(x) \geq U(x) \geq \psi(x)$ for any $x \in \Gamma_1$ because of the continuity of $U$ on $\partial \mathcal{O}$. Therefore $\tilde{W}_*$ is a supersolution of (35).

As a second step, one has to show that $\tilde{W}^*$ is a subsolution of (34). If $x \in \Gamma_3$, we have already proved that $\tilde{W}$ is continuous at $x$ and coincides with $\psi(x)$, while at $x \in \Gamma_1$ we have seen that $\tilde{W}^*(x) \geq \tilde{W}(x) \geq \psi(x)$. If either $x \in \mathcal{O} \cup \Gamma_2$ or $x \in \Gamma_1$ and $\tilde{W}^*(x) > \psi(x)$ suppose by contradiction that there is some $\varphi \in C^2(\mathbb{R}^N)$ such that $\tilde{W}^* - \varphi$ has a (global) maximum in $\overline{\mathcal{O}}$ at $x$, $\tilde{W}^*(x) = \varphi(x)$, and
\[
G(x, \varphi(x), D\varphi(x), D^2\varphi(x)) = 2
\]
for some $\nu$. If $\tilde{W}^*(x) = U(x)$, (in case $x \in \Gamma_1$, $U(x) > \psi(x)$ and) in a neighborhood of $x$ one has $U(y) - \varphi(y) \leq \tilde{W}^*(y) - \varphi(y) \leq 0 = U(x) - \varphi(x)$, so $U - \varphi$ has a local maximum in $\overline{\mathcal{O}}$ at $x$. Since $U$ verifies (34) we have the required contradiction to (47). The other possible case is $\tilde{W}^*(x) - \delta \geq U(x) + \delta$ for some $\delta > 0$. Arguing as in the proof of the so called Bump Lemma (see [7]), it is possible to choose two positive parameters $\varepsilon$ and $\delta$ such that the $C^2$ function $\tilde{\varphi}(y) \equiv \varphi(y) - \varepsilon + |y - x|^4$ is a classical strict supersolution, that is,
\[
G(y, \tilde{\varphi}(y), D\tilde{\varphi}(y), D^2\tilde{\varphi}(y)) \geq \nu \quad \forall y \in B(x, \delta) \cap \overline{\mathcal{O}}.
\]
Furthermore, setting
\[
w(y) \doteq \begin{cases} \inf \{ \tilde{\varphi}(y), \tilde{W}(y) \} & \text{in } B(x, \delta) \cap \overline{\mathcal{O}} \\ \tilde{W}(y) & \text{in } (\mathbb{R}^N \setminus B(x, \delta)) \cap \overline{\mathcal{O}} \end{cases}
\]
one obtains a function \( w \) such that
\[
w \leq \bar{W} \leq V \quad \text{in } \overline{\mathcal{O}}, \quad w < \bar{W} \quad \text{at some point of } \overline{\mathcal{O}}.
\]

We claim that \( w_* \) verifies (35). In fact, by the property of viscosity solutions this reduces to show that \( \bar{\varphi} \) verifies (35) in \( B(x, \delta) \cap \overline{\mathcal{O}} \). If \( x \in \mathcal{O} \) this means that \( \bar{\varphi} \) has to be a supersolution of \( G(x, u, Du, D^2u) = 0 \) at \( x \) and this result follows from (48). If \( x \in \Gamma_2 \), instead, it is well known that a classical supersolution may fail to be a supersolution in the viscosity sense. However, if \( x \in \Gamma_2 \), which is relatively open in \( \partial \mathcal{O} \), on the basis of (35) we have to prove that \( \bar{\varphi} \) is a supersolution just at the points of the open set \( B(x, \delta) \cap \mathcal{O} \), which is again a straightforward consequence of (48). Moreover, in case \( x \in \Gamma_3 \) we notice that by the assumption \( \bar{W}^*(x) > \psi(x) \) it follows that \( \bar{\varphi}(y) > \psi(y) \) in a neighborhood of \( x \) as soon as \( \varepsilon \) is small enough. Finally, since \( \Gamma_3 \) is relatively closed in \( \partial \mathcal{O} \), reducing \( \delta \) if necessary, we can always assume that \( w = \bar{W} \) on \( \Gamma_3 \). Therefore \( w_* \geq \psi \) on \( \Gamma_1 \cup \Gamma_3 \) and this concludes the proof that \( w_* \) verifies (35). At this point it is standard to show that being \( \bar{W}^*(x) - \delta \geq U(x) + \delta \) we can make \( w \geq U \) in \( \overline{\mathcal{O}} \) so that \( w \in \mathcal{S} \) and since \( w < \bar{W} \) at some point of \( \overline{\mathcal{O}} \) this yields a contradiction with the definition of \( W \). Thus (48) cannot hold and the proof of Proposition 4 in case i) is concluded.

ii) As in the previous case, by the comparison Theorem 4.4, ii), it follows that \( U(x) \leq V(x) \, \forall x \in \overline{\mathcal{O}} \) and the set involved in the definition of \( W \) is nonempty. \( W \) verifies the growth assumption (45), one can deduce that \( W_* \) is a supersolution of \( G(x, u, Du, D^2u) = 0 \) in \( \mathcal{O} \) arguing as in [13] and at the points \( x \in \Gamma_3 \), \( W \) is continuous and \( W(x) = \psi(x) \), as above. In order to prove that \( W \) verifies (39), it remains now to show that \( W_* \) is a supersolution of the generalized Dirichlet condition \( G(x, u, Du, D^2u) = 0 \) or \( u = \psi \) on \( \Gamma_1 \). Let \( x \in \Gamma_1 \) and let us assume that \( W(x) = W_*(x) < \psi(x) \). Let \( \varphi \in C^2(\mathbb{R}^N) \) be such that \( W_* - \varphi \) has a local minimum in \( \mathcal{O} \cup \Gamma_1 \) at \( x \), \( W(x) = \varphi(x) \) and \( W_*(y) - \varphi(y) > |y - x|^2 \, \forall y \) in some neighbourhood of \( x \). Let us point out that we can disregard \( \Gamma_2 \) and \( \Gamma_3 \) in this definition since \( \Gamma_1 \) is relatively open in \( \partial \mathcal{O} \). From now on we can argue as usual (see e.g. [7]) to deduce that there exist some sequences \( \{x_n\}_n \subset \mathcal{O} \cup \Gamma_1 \) and \( \{w_n\}_n \subset \mathcal{S} \) such that \( x_n \) is a local minimum point of \( (w_n)_* - \varphi \), \( x_n \to x \), and \( (w_n)_*(x_n) \to \varphi(x) = W(x) \). For \( n \) large enough, if \( x_n \in \Gamma_1 \) in view of the continuity of \( \psi \) at \( x \) one has \( (w_n)_*(x_n) < \psi(x_n) \). Thus, by the definition of \( \mathcal{S} \) it follows that \( G(x_n, (w_n)_*(x_n), D\varphi(x_n), D^2\varphi(x_n)) \geq 0 \) for all \( n \), which yields that \( W_* \) is a supersolution of \( G(x, u, Du, D^2u) = 0 \) at \( x \) by passing to the limit as \( n \to +\infty \).

As a second step, one has to show that \( W^* \) is a subsolution of (34). The proof proceeds as in i) except that in case \( x \in \Gamma_1 \), where one supposes now both that \( W^*(x) > \psi(x) \) and that there is some \( \varphi \in C^2(\mathbb{R}^N) \) such that \( W^* - \varphi \) has a (global) maximum in \( \overline{\mathcal{O}} \) at \( x \), \( W^*(x) = \varphi(x) \), and (47) holds true. One constructs the function \( w \) as above and in order to find a contradiction one has to show that \( w \) belongs to \( \mathcal{S} \) instead of \( \mathcal{S} \), as in i). Hence the difference concerns just the case \( x \in \Gamma_1 \), but arguing as above (and recalling that now \( \Gamma_1 \) is relatively open in \( \partial \mathcal{O} \)) by the assumption \( W^*(x) > \psi(x) \) it follows that \( \bar{\varphi}(y) > \psi(y) \) in a neighborhood of \( x \), so that it is a supersolution of the generalized Dirichlet condition \( G(x, u, Du, D^2u) = 0 \) or \( u = \psi \) in (35) on \( B(x, \delta) \cap \partial \mathcal{O} \) in a trivial way. Therefore \( w_* \) verifies (35) and the proof of Proposition 4 is concluded. \( \square \)
It is well known that sub- and supersolutions to BVP and BVP1 do not exist in general. In the next Proposition we apply Proposition 4 to prove the existence of a solution to BVP in a special case. To this aim, we introduce the following assumption.

(H4) There exists a neighborhood of $\Gamma_2$ where the distance function defined by
$$\rho(x) = \text{dist}(x, \mathbb{R}^N \setminus \mathcal{O}) \quad \forall x \in \mathbb{R}^N$$
is of class $C^2$. Moreover, for any $x \in \Gamma_2$ there is some $a(x) \in A$ such that
\begin{enumerate}[i)]
\item $\frac{1}{\rho^2} \text{Tr} \left( \Sigma(x, a(x), b)\Sigma^T(x, a(x), b)D^2\rho(x) \right) + \langle d(x, a(x), b), D\rho(x) \rangle \geq 0 \quad \forall b \in B,$
\item $(D\rho(x))^T \Sigma(x, a(x), b)\Sigma^T(x, a(x), b)D\rho(x) = 0 \quad \forall b \in B.$
\end{enumerate}

**Proposition 5.** Let the same assumptions of Theorem 4.4, ii), on $\mathcal{O}$, $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, and $\psi$ hold. Let (H4) be verified and let the cost function $c$ in the definition (11) of $G$ satisfy the growth assumption (45) uniformly with respect to the controls $a \in A$, $b \in B$. Then there exists a solution to BVP in the class of functions that verify (45).

**Proof.** Let $C_1$, $C_2$ be positive constants to be determined later and define the functions
$$U(x) \doteq -C_1 - C_2g(x), \quad V(x) \doteq C_1 + C_2g(x) \quad \forall x \in \overline{\mathcal{O}},$$
where $g(x) \doteq \log(1 + |x|^2)$. Owing to the sublogarithmic growth of $\psi$ at infinity, it is always possible to choose $C_1$ and $C_2$ such that $U$ and $V$ trivially verify the boundary conditions
$$U(x) \leq \psi(x), \quad V(x) \geq \psi(x) \quad \forall x \in \Gamma_1 \cup \Gamma_3.$$Let us introduce for all $(x, r, p, S, a, b)$ the map
$$G(x, r, p, S, a, b) \doteq -\frac{1}{2} \text{Tr} \left\{ \Sigma(x, a, b)\Sigma^T(x, a, b)S \right\} - \langle d(x, a, b), p \rangle - c(x, a, b) + m(x, a, b)r.$$

Straightforward calculations yield that $\forall x \in \overline{\mathcal{O}}$ one has
$$G(x, U(x), DU(x), D^2U(x), a, b) \leq -\lambda_0(C_1 + C_2g(x)) + 2C_2\tilde{C}^2 - \inf_{a \in A, b \in B} c(x, a, b),$$
and
$$G(x, V(x), DV(x), D^2V(x), a, b) \geq \lambda_0(C_1 + C_2g(x)) - 2C_2\tilde{C}^2 - \sup_{a \in A, b \in B} c(x, a, b)$$
for all $a \in A$ and $b \in B$, where the positive constant $\tilde{C}$ is the same as in the proof of Theorem 4.2. In view of the growth hypothesis on $c$ it is now clear that there exist $C_1$, $C_2$ large enough to get
$$\lambda_0(C_1 + C_2g(x)) - 2C_2\tilde{C}^2 - \sup_{a \in A, b \in B} c(x, a, b) \geq 0,$$
$$-\lambda_0(C_1 + C_2g(x)) + 2C_2\tilde{C}^2 - \inf_{a \in A, b \in B} c(x, a, b) \leq 0.$$With such a choice of $C_1$, $C_2$ at any point $x \in \overline{\mathcal{O}}$ we have
$$G(x, U(x), DU(x), D^2U(x), a, b) \leq 0, \quad G(x, V(x), DV(x), D^2V(x), a, b) \geq 0 \quad \text{for all } a \in A, b \in B.$$Therefore $U$ and $V$ are respectively a continuous subsolution and a continuous supersolution of the equation
$$G(x, u(x), Du(x), D^2u(x)) = 0 \quad \text{in } \mathcal{O},$$
and $U$ turns out to be a classical subsolution of this equation on $\Gamma_2$. By well known properties of viscosity solutions (see e.g. [7]), $U$ is also a viscosity subsolution on $\Gamma_2$ if $\forall x \in \Gamma_2$ one has

$$G(x, U(x), DU(x) + \lambda D\rho(x), D^2U(x) + \lambda D^2\rho(x) + \mu D\rho(x) \otimes D\rho(x)) \leq 0 \quad (50)$$

$\forall \lambda \geq 0$ and $\mu \in \mathbb{R}$, where $\rho$ is the distance function introduced in assumption (H4).

Let $x \in \Gamma_2$ and let $a(x) \in A$ be the same as in (H4). By $i), ii)$ in (H4) and by (49) it follows that

$$G(x, U(x), DU(x) + \lambda D\rho(x), D^2U(x) + \lambda D^2\rho(x) + \mu D\rho(x) \otimes D\rho(x), a(x), b) = G(x, U(x), DU(x), a(x), b)$$

$$-\lambda \left[ \frac{1}{2} Tr \left( \Sigma(x, a(x), b) \right) (x, b) \right] \leq 0$$

for all $b \in B$. This easily yields (50). Therefore the proof that $U$ and $V$ are respectively a sub- and a supersolution of BVP, both continuous and verifying the growth condition (42), is concluded. The statement of the proposition follows now from Proposition 4. \qed

6. Appendix.

Proof of Proposition 1. Step 1. We start by showing that $u^\varepsilon$ is a subsolution of $G_\varepsilon(x, u, Du, D^2u) = 0$ in $\mathcal{O}^\varepsilon$. Let $\varphi \in C^2(\mathcal{O}^\varepsilon)$, $x_0 \in \mathcal{O}^\varepsilon$ and $r > 0$ be such that $(u^\varepsilon - \varphi)(x_0) = \max_{B(x_0,r) \cap \mathcal{O}^\varepsilon} (u^\varepsilon - \varphi)$. Then by definition of $u^\varepsilon$ there exists $y_0 \in \mathcal{O}^\varepsilon \cap B(x_0, (C + C_0)\sqrt{\varepsilon})$ such that $u^\varepsilon(x_0) = u(y_0) - \frac{1}{2}|x_0 - y_0|^2$. If either $y_0 \in \Gamma_1$ and $u(y_0) \leq v(y_0)$ or $y_0 \in \Gamma_3$ we have $u^\varepsilon(x_0) = u(y_0) - \frac{1}{2}|x_0 - y_0|^2 \leq v(y_0)$. Then the definition of $G_\varepsilon$ implies in this case that

$$G_\varepsilon(x_0, u^\varepsilon(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq$$

$$\inf \{ G(y_0, u^\varepsilon(x_0), D\varphi(x_0), D^2\varphi(x_0)), u^\varepsilon(x_0) - \hat{v}(y_0) \} \leq u^\varepsilon(x_0) - v(y_0) \leq 0,$$

being $\hat{v}(y_0) = v(y_0)$, since $y_0 \notin \Gamma_2$.

If instead either $y_0 \in \Gamma_1$ and $u(y_0) > v(y_0)$ or $y_0 \in \Gamma_2 \cup \Omega$, let us notice that

$$u(y) - \frac{1}{2}|x - y|^2 - \varphi(x) \leq u(y_0) - \frac{1}{2}|x_0 - y_0|^2 - \varphi(x_0)$$

for all $x \in \mathcal{O}^\varepsilon \cap B(x_0, r)$ and all $y \in \mathcal{O}^\varepsilon \cap B(x, (C + C_0)\sqrt{\varepsilon})$. The choice of $x = y - y_0 + x_0$ implies that the map $y \to u(y) - \varphi(y - y_0 + x_0)$ has a maximum in $y = y_0$. Since $u$ is subsolution of $G = 0$ at $y_0$, $u^\varepsilon(x_0) \leq u(y_0)$ by definition of $y_0$, and $G$ is increasing in $u$,

$$G(y_0, u^\varepsilon(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq G(y_0, u(y_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

Since $|y_0 - x_0| \leq (C + C_0)\sqrt{\varepsilon}$, by the definition of $G_\varepsilon$ we conclude that

$$G_\varepsilon(x_0, u^\varepsilon(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq G(y_0, u^\varepsilon(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

Step 2. In order to show that $v_\varepsilon$ is a supersolution of $G_\varepsilon(x, u, Du, D^2u) = 0$ in $\mathcal{O}^\varepsilon$, let us consider $\varphi \in C^2(\mathcal{O}^\varepsilon)$, $x_0 \in \mathcal{O}^\varepsilon$ and $r > 0$ such that $(v_\varepsilon - \varphi)(x_0) = \min_{B(x_0,r) \cap \mathcal{O}^\varepsilon} (v_\varepsilon - \varphi)$. Then by definition of $v_\varepsilon$ there exists $y_0 \in \mathcal{O}^\varepsilon \cap B(x_0, (C + C_0)\sqrt{\varepsilon})$ such that $v_\varepsilon(x_0) = v(y_0) + \frac{1}{2}|x_0 - y_0|^2$. If $y_0 \in \Gamma_1 \cup \Gamma_2$, then the definition of $G_\varepsilon$ and $\hat{u}(y_0) = -\infty$ implies that

$$G_\varepsilon(x_0, v_\varepsilon(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq B_2(y_0, v_\varepsilon(x_0)) = +\infty;$$
if $y_0 \in \Gamma_0$ then $\hat{u}(y_0) = u(y_0)$, $v_z(x_0) \geq v(y_0) \geq u(y_0)$ so that

$$G^*(x_0, v_z(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq \sup(G(y_0, v_z(x_0), D\varphi(x_0), D^2\varphi(x_0)), v_z(x_0) - u(y_0)) \geq v_z(x_0) - u(y_0) \geq 0.$$  

If $y_0 \in O$ instead, similarly to what observed in Step 1, one has

$$v(y) + \frac{1}{\varepsilon}|x-y|^2 - \varphi(x) \geq v(y_0) + \frac{1}{\varepsilon}|x_0 - y_0|^2 - \varphi(x_0)$$

for all $x \in \overline{O} \cap B(x_0, r)$ and all $y \in \overline{O} \cap B(x, (C_0+\varepsilon)\sqrt{\varepsilon})$. The choice of $x = y-y_0+x_0$ implies that the map $y \to v(y) - \varphi(y-y_0+x_0)$ has a minimum in $y = y_0$. Since $v$ is supersolution of $G$ at $y_0$, $v_z(x_0) \geq v(y_0)$, and $G$ is increasing in its second variable,

$$G(y_0, v_z(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq G(y_0, v(y_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$$

which implies

$$G^*(x_0, v_z(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq G(y_0, v_z(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.$$  

**Step 3.** The fact that $G_z$ is lsc and $G^*$ is usc is a straightforward consequence of the quite trivial results listed in Lemma 6.1 below. 

**Lemma 6.1.** Given some closed set $\Lambda \subset \mathbb{R}^l$ for some $l$ and two functions $M$, $N : \Lambda \to \mathbb{R} \cup \{\pm \infty\}$, then

1. if $\inf \{M(y), N(y)\} \in \mathbb{R} \cup \{\pm \infty\}$ $\forall y \in \Lambda$, for any $C > 0$ one has
   $$I(x) := \inf \{\inf \{M(y), N(y)\} : y \in B(x, C) \cap \Lambda\} = \inf \{\inf \{M_*(y), N_*(y)\} : y \in B(x, C) \cap \Lambda\} \quad \forall x \in \mathbb{R}^l;$$

2. if $\sup \{M(y), N(y)\} \in \mathbb{R} \cup \{\pm \infty\}$ $\forall y \in \Lambda$, for any $C > 0$ one has
   $$S(x) := \sup \{\sup \{M(y), N(y)\} : y \in B(x, C) \cap \Lambda\} = \sup \{\sup \{M^*(y), N^*(y)\} : y \in B(x, C) \cap \Lambda\} \quad \forall x \in \mathbb{R}^l.$$

Therefore under the hypotheses above, $I$ is lsc and $S$ is usc in $\mathbb{R}^l$.

**Proof of Proposition 3.** The fact that $u^c$ [resp. $v_z$] is a subsolution [resp. supersolution] to $G_z(x,u,Du,D^2u) = 0$ [resp. $G^*(x,u,Du,D^2u) = 0$] in $O^c$ follows the same lines of Proposition 1, once we substitute $\hat{v}$ [resp. $\hat{u}$] with $\hat{\psi}_1$ [resp. $\hat{\psi}_2$].

The fact that $G_z$ is lsc and $G^*$ is usc follows from Lemma 6.1.

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