Finite irreflexive homomorphism-homogeneous binary relational systems✩

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Abstract

A structure is called homogeneous if every isomorphism between finite substructures of the structure extends to an automorphism of the structure. Recently, P. J. Cameron and J. Nešetřil introduced a relaxed version of homogeneity: we say that a structure is homomorphism-homogeneous if every homomorphism between finite substructures of the structure extends to an endomorphism of the structure. In this paper we characterize all finite homomorphism-homogeneous relational systems with one irreflexive binary relation.

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1. Introduction

A structure is homogeneous if every isomorphism between finite substructures of the structure extends to an automorphism of the structure. For example, finite and countably infinite homogeneous directed graphs were described in [2]. In their recent paper [1] the authors discuss a generalization of homogeneity to various types of morphisms between structures, and in particular introduce the notion of homomorphism-homogeneous structures:

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Definition 1.1 (Cameron, Nešetřil [1]) A structure is called homomorphism-homogeneous if every homomorphism between finite substructures of the structure extends to an endomorphism of the structure.

In this short note we characterize all finite homomorphism-homogeneous relational systems with one irreflexive binary relation.

2. Preliminaries

A binary relational system is an ordered pair \((V, E)\) where \(E \subseteq V^2\) is a binary relation on \(V\). A binary relational system \((V, E)\) is reflexive if \((x, x) \in E\) for all \(x \in V\), irreflexive if \((x, x) \notin E\) for all \(x \in V\), symmetric if \((x, y) \in E\) implies \((y, x) \in E\) for all \(x, y \in V\) and antisymmetric if \((x, y) \in E\) implies \((y, x) \notin E\) for all distinct \(x, y \in V\).

Binary relational systems can be thought of in terms of digraphs (hence the notation \((V, E)\)). Then \(V\) is the set of vertices and \(E\) is the set of edges of the binary relational system/digraph \((V, E)\). Edges of the form \((x, x)\) are called loops. If \((x, x) \in E\) we also say that \(x\) has a loop. Instead of \((x, y) \in E\) we often write \(x \to y\) and say that \(x\) dominates \(y\), or that \(y\) is dominated by \(x\). By \(x \sim y\) we denote that \(x \to y\) or \(y \to x\), while \(x \ddash y\) denotes that \(x \to y\) and \(y \not\to x\). If \(x \ddash y\), we say that \(x\) and \(y\) form a double edge. We shall also say that a vertex \(x\) is incident with a double edge if there is a vertex \(y \neq x\) such that \(x \ddash y\).

Digraphs \((V, E)\) where \(E\) is a symmetric binary relation on \(V\) are usually referred to as graphs. Proper digraphs are digraphs \((V, E)\) where \(E\) is an antisymmetric binary relation. In this paper, digraphs \((V, E)\) where \(E\) is neither antisymmetric nor symmetric will be referred to as improper digraphs. In an improper digraph there exists a pair of distinct vertices \(x\) and \(y\) such that \(x \ddash y\) and another pair of distinct vertices \(u\) and \(v\) such that \(u \to v\) and \(v \not\to u\).

A digraph \(D' = (V', E')\) is a subdigraph of a digraph \(D = (V, E)\) if \(V' \subseteq V\) and \(E' \subseteq E\). We write \(D' \leq D\) to denote that \(D'\) is isomorphic to a subdigraph of \(D\). For \(\emptyset \neq W \subseteq V\) by \(D[W]\) we denote the digraph \((W, E \cap W^2)\) which we refer to as the subdigraph of \(D\) induced by \(W\).

Vertices \(x\) and \(y\) are connected in \(D\) if there exists a sequence of vertices \(z_1, \ldots, z_k \in V\) such that \(x = z_1 \sim \ldots \sim z_k = y\). A digraph \(D\) is weakly connected if each pair of distinct vertices of \(D\) is connected in \(D\). A digraph \(D\) is disconnected if it is not weakly connected. A connected component of
$D$ is a maximal set $S \subseteq V$ such that $D[S]$ is weakly connected. The number of connected components of $D$ will be denoted by $\omega(D)$.

Vertices $x$ and $y$ are **doubly connected in $D$** if there exists a sequence of vertices $z_1, \ldots, z_k \in V$ such that $x = z_1 \rightarrow \ldots \rightarrow z_k = y$. Define a binary relation $\theta(D)$ on $V(D)$ as follows: $(x, y) \in \theta(D)$ if and only if $x = y$ or $x$ and $y$ are doubly connected. Clearly, $\theta(D)$ is an equivalence relation on $V(D)$ and $\omega(D) \leq |V(D)/\theta(D)|$. We say that a digraph $D$ is $\theta$-**connected** if $\omega(D) = |V(D)/\theta(D)|$, and that it is $\theta$-**disconnected** if $\omega(D) < |V(D)/\theta(D)|$.

Note that a $\theta$-connected digraph need not be connected, and that a $\theta$-disconnected digraph need not be disconnected; a digraph $D$ is $\theta$-connected if every connected component of $D$ contains precisely one $\theta(D)$-class, while it is $\theta$-disconnected if there exists a connected component of $D$ which consists of at least two $\theta(D)$-classes. In particular, every proper digraph with at least two vertices is $\theta$-disconnected, and every graph is $\theta$-connected.

Let $K_n$ denote the complete irreflexive graph on $n$ vertices. Let 1 denote the trivial digraph with only one vertex and no edges, and let $1^\circ$ denote the digraph with only one vertex with a loop. An **oriented cycle with $n$ vertices** is a digraph $C_n$ whose vertices are 1, 2, \ldots, $n$, $n \geq 3$, and whose edges are $1 \rightarrow 2 \rightarrow \ldots \rightarrow n \rightarrow 1$.

For digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$, by $D_1 + D_2$ we denote the **disjoint union** of $D_1$ and $D_2$. We assume that $D + O = O + D = D$, where $O = (\emptyset, \emptyset)$ denotes the **empty digraph**. The disjoint union $\overbrace{D + \ldots + D}^k$ consisting of $k \geq 1$ copies of $D$ will be abbreviated to $k \cdot D$. Moreover, we let $0 \cdot D = O$.

Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be digraphs. We say that $f : V_1 \rightarrow V_2$ is a **homomorphism** between $D_1$ and $D_2$ and write $f : D_1 \rightarrow D_2$ if

$$x \rightarrow y \text{ implies } f(x) \rightarrow f(y), \text{ for all } x, y \in V_1.$$ 

An **endomorphism** is a homomorphism from $D$ into itself. A mapping $f : V_1 \rightarrow V_2$ is an **isomorphism** between $D_1$ and $D_2$ if $f$ is bijective and

$$x \rightarrow y \text{ if and only if } f(x) \rightarrow f(y), \text{ for all } x, y \in V_1.$$ 

Digraphs $D_1$ and $D_2$ are **isomorphic** if there is an isomorphism between them. We write $D_1 \cong D_2$. An **automorphism** is an isomorphism from $D$ onto itself.

A digraph $D$ is **homomorphism-homogeneous** if every homomorphism $f : W_1 \rightarrow W_2$ between finitely induced subdigraphs of $D$ extends to an endomorphism of $D$ (see Definition 1.1).
3. Finite irreflexive binary relational systems

Cameron and Nešetřil have shown in [1] that a finite irreflexive graph is homomorphism-homogeneous if and only if it is isomorphic to $k \cdot K_n$ for some $k, n \geq 1$. It was shown in [3, Theorem 3.10] that a finite irreflexive proper digraph is homomorphism-homogeneous if and only if it is isomorphic to $k \cdot 1$ for some $k \geq 1$ or $k \cdot C_3$ for some $k \geq 1$. In this section we show that these are the only finite homomorphism-homogeneous irreflexive binary relational systems by showing that no finite irreflexive improper digraph is homomorphism-homogeneous.

**Lemma 3.1** Let $D$ be a finite homomorphism-homogeneous irreflexive improper digraph. Then every vertex of $D$ is incident with a double edge.

**Proof.** Let $x \iff y$ be a double edge in $D$ and let $v$ be an arbitrary vertex of $D$. The mapping

$$f : \begin{pmatrix} x \\ v \end{pmatrix}$$

is a homomorphism between finitely induced subdigraphs of $D$, so it extends to an endomorphism $f^*$ of $D$ by the homogeneity requirement. Then $x \iff y$ implies $v = f^*(x) \iff f^*(y)$. □

**Lemma 3.2** Let $D$ be a finite homomorphism-homogeneous irreflexive improper digraph and let $S \in V(D)/\theta(D)$ be an arbitrary equivalence class of $\theta(D)$. Then $D[S] \cong K_n$ for some $n \geq 2$.

**Proof.** Lemma 3.1 implies that $|S| \geq 2$ for every $S \in V(D)/\theta(D)$.

Suppose that there is an $S \in V(D)/\theta(D)$ such that $D[S]$ is not a complete graph. Then there exist $u, v \in S$ such that $u \not\rightarrow v$ or $v \not\rightarrow u$. Let $z_1, z_2, \ldots, z_k \in V(D)$ be the shortest sequence of vertices of $D$ such that

$$u = z_1 \iff z_2 \iff \ldots \iff z_k = v.$$ 

Then $k \geq 3$ since $u \not\equiv v$, and the fact that $z_1, z_2, \ldots, z_k$ is the shortest such sequence implies that $z_1 \not\equiv z_3$. The mapping

$$f_1 : \begin{pmatrix} z_1 & z_3 \\ z_2 & z_3 \end{pmatrix}$$
is a homomorphism between finitely induced subdigraphs of $D$, so it extends to an endomorphism $f_1^*$ of $D$ by the homogeneity requirement. Let $x_1 = f_1^*(z_2)$. It is easy to see that $x_1 \notin \{z_1, z_2, z_3\}$ and $x_1 \rightleftharpoons y$ for all $y \in \{z_2, z_3\}$. Consider now the mapping
\[
f_2 : \left( \begin{array}{ccc}
z_1 & z_3 & x_1 \\
z_2 & z_3 & x_1 \\
\end{array} \right)
\]
which is clearly a homomorphism between finitely induced subdigraphs of $D$. It extends to an endomorphism $f_2^*$ of $D$. Let $x_2 = f_2^*(z_2)$. Again, it is easy to see that $x_2 \notin \{z_1, z_2, z_3, x_1\}$ and $x_2 \rightleftharpoons y$ for all $y \in \{z_2, z_3, x_1\}$. Analogously, the mapping
\[
f_3 : \left( \begin{array}{ccc}
z_1 & z_3 & x_1 & x_2 \\
z_2 & z_3 & x_1 & x_2 \\
\end{array} \right)
\]
is a homomorphism between finitely induced subdigraphs of $D$, so it extends to an endomorphism $f_3^*$ of $D$. Let $x_3 = f_3^*(z_2)$. Again, $x_3 \notin \{z_1, z_2, z_3, x_1, x_2\}$ and $x_3 \rightleftharpoons y$ for all $y \in \{z_2, z_3, x_1, x_2\}$. And so on. We can continue with this procedure as many times as we like, which contradicts the fact that $D$ is a finite digraph.

**Proposition 3.3** There does not exist a finite homomorphism-homogeneous irreflexive improper digraph.

**Proof.** Suppose that $D$ is a finite homomorphism-homogeneous irreflexive improper digraph. Then there exist vertices $x, y \in V(D)$ such that $x \rightarrow y$ and $y \not\rightarrow x$. Let $S = x/\theta(D)$ and $T = y/\theta(D)$. Clearly, $S \cap T = \emptyset$. Let $T = \{y, t_1, \ldots, t_k\}$. Since $D[T]$ is a complete graph (Lemma 3.2), the mapping
\[
f : \left( \begin{array}{cccc}
x & t_1 & \ldots & t_k \\
y & t_1 & \ldots & t_k \\
\end{array} \right)
\]
is a homomorphism between finitely induced subdigraphs of $D$, so it extends to an endomorphism $f^*$ of $D$ by the homogeneity requirement. Let us compute $f^*(y)$. From $f^*(t_1) \in T$ it follows that $f^*(T) \subseteq T$. Moreover, $f^*|_T$ is injective since there are no loops in $D$. Therefore, $f^*|_T : T \rightarrow T$ is a bijection. But $f^*(t_i) = t_i$ for all $i \in \{1, \ldots, k\}$, so it follows that $f^*(y) = y$. Now, $x \rightarrow y$ implies $f^*(x) \rightarrow f^*(y)$, that is, $y \rightarrow y$, which is impossible since there are no loops in $D$. \qed

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Corollary 3.4 Let $D$ be a finite irreflexive binary relational system. Then $D$ is homomorphism-homogeneous if and only if it is isomorphic to one of the following:

1. $k \cdot K_n$ for some $k, n \geq 1$;
2. $k \cdot C_3$ for some $k \geq 1$.

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