Quantum Bit Commitment can be Unconditionally Secure

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Abstract

It is generally believed that unconditionally secure quantum bit commitment (QBC) is proven impossible by a “no-go theorem”. We point out that the theorem only establishes the existence of a cheating unitary transformation in any QBC scheme secure against the receiver, but this fact alone is not sufficient to rule out unconditionally secure QBC as a matter of principle, because there exists no proof that the cheating unitary transformation is known to the cheater in all possible cases. In this work, we show how to circumvent the “no-go theorem” and prove that unconditionally secure QBC is in fact possible.

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Quantum information and quantum computation is a field of intense activities in recent years. The idea of applying quantum mechanics in cryptography was first introduced in the late 1960’s \[1, 2\]. So far, the most well known and successful applications are found in the area of quantum key distribution \[3, 4\]. Other important quantum cryptographic protocols include quantum bit commitment (QBC) \[5, 6\], quantum oblivious transfer \[7, 8\], quantum coin tossing \[3, 5\], and so on. In particular, QBC is a basic protocol, or primitive, which can be used to construct other more sophisticated protocols. Moreover, it has the potential of being the building block of any secure two-party cryptographic protocols \[5, 10, 11\]. Hence the security of QBC is an issue of great importance in quantum information theory.

A QBC protocol involves a sender (Alice) and a receiver (Bob). To begin with, Alice secretly commits to a bit \( b \) (0 or 1) which is to be revealed to Bob at a later time. In order to assure Bob that she will not change her mind, Alice gives Bob a quantum mechanical wave function \( |\psi_B^{(b)}\rangle \) which can later be used to verify her honesty (in the simplest model). A QBC protocol is secure if (1) Alice cannot change her commitment without being discovered (binding), and (2) Bob can obtain no information about the commitment before Alice discloses it (concealing) – that means the density matrix \( \rho_B^{(b)} \) of \( |\psi_B^{(b)}\rangle \) must be independent of \( b \). An unconditionally secure protocol is one which is secure even if Alice and Bob were endowed with unlimited computational power.

The purpose of this letter is to propose a new QBC scheme and prove its unconditional security. Our result contradicts the widely held belief in a “no-go theorem” which claims that unconditionally secure QBC is impossible \[12, 13, 14, 15\]. The central issue of the problem can be explained as follows. Instead of honestly sending \( |\psi_B^{(b)}\rangle \) to Bob, Alice can always prepare another state \( |\psi_{AB}^{(b)}\rangle \), in which sectors \( A \) and \( B \) are entangled, and delivers only sector \( B \) to Bob. Clearly, as long as

\[
\rho_B^{(b)} = |\psi_B^{(b)}\rangle \langle \psi_B^{(b)}| = \text{Tr}_A |\psi_{AB}^{(b)}\rangle \langle \psi_{AB}^{(b)}|,
\]

Bob will not know the difference. Furthermore, from the Schmidt decomposition of \( |\psi_{AB}^{(b)}\rangle \) and the fact that \( \rho_B^{(0)} = \rho_B^{(1)} \) (concealing), we know there exists an unitary transformation \( U_A \) acting on sector-\( A \) only, such that \[12, 13, 14, 15, 16, 17\]

\[
|\psi_{AB}^{(1)}\rangle = U_A |\psi_{AB}^{(0)}\rangle.
\]

The fact that Alice can by herself transform \( |\psi_{AB}^{(0)}\rangle \) into \( |\psi_{AB}^{(1)}\rangle \) (and vice versa) implies that she can cheat with the following sure-win strategy (so-called EPR attack): Alice always commits to \( b = 0 \) in the beginning, and if she wants to change to \( b = 1 \) later on, she can simply apply the unitary transformation \( U_A \) to her particles before revealing her
bit. So in this simple model of QBC, if the scheme is concealing, it cannot be binding
at the same time. The “no-go theorem” claims that this result is universally valid, and
unconditionally secure QBC is ruled out as a matter of principle [12, 13, 14, 15].

Despite its widespread acceptance, we find that the “no-go theorem” is actually of
limited validity only. It is true that the theorem proves the existence of a cheating unitary
transformation $U_A$ in any QBC scheme which is secure against Bob, but this fact alone
does not rule out unconditionally secure QBC as a matter of principle, unless one could
also prove that $U_A$ must be known to Alice in all such schemes. The “no-go theorem”
simply asserts without proof that this is the case. Consequently we see that, as it is,
the theorem only rules out a restricted class of QBC schemes where, at the end of the
commitment phase, Alice has detailed knowledge of the wave function in Bob’s hand. It
says nothing about other possibilities.

In the rest of this letter, we show with a concrete example how to circumvent the ”no-
go theorem”. Before proceeding, let us define the notations to be used and establish two
preliminary results. First of all, we write the four Bell states in terms of the eigenstates
($|\uparrow_z\rangle$, $|\downarrow_z\rangle$) of the Pauli matrix $\sigma_z$:

$$
|0\pm\rangle \equiv \frac{1}{\sqrt{2}} \left( |\uparrow_z\downarrow_z\rangle \pm |\downarrow_z\uparrow_z\rangle \right),
$$

(3)

$$
|1\pm\rangle \equiv \frac{1}{\sqrt{2}} \left( |\uparrow_z\uparrow_z\rangle \pm |\downarrow_z\downarrow_z\rangle \right).
$$

(4)

Also, we shall use the notation,

$$
|\Psi\rangle = \{|\phi_1\rangle, |\phi_2\rangle; q_1, q_2\},
$$

(5)

where $q_i \geq 0$ and $q_1 + q_2 = 1$, to denote a mixed state with density matrix

$$
|\Psi\rangle\langle \Psi| = q_1 |\phi_1\rangle\langle \phi_1| + q_2 |\phi_2\rangle\langle \phi_2|.
$$

(6)

Let $S_n^{(b)}$ ($b = 0$ or 1) be an ordered sequence of $n$ pairs of particles with wave function

$$
|S_n^{(b)}\rangle = |\psi_1^{(b)}\rangle |\psi_2^{(b)}\rangle ... |\psi_n^{(b)}\rangle,
$$

(7)

where each $|\psi_i^{(b)}\rangle$ is a mixed state given by

$$
|\psi_i^{(b)}\rangle = \{|b+\rangle, |b-\rangle; 1/2, 1/2\}.
$$

(8)

As a short-hand notation, we write

$$
|S_n^{(b)}\rangle = \{|b+\rangle, |b-\rangle; 1/2, 1/2\}^n.
$$

(9)
We label the particles in $S_n^{(b)}$ as

$$S_n^{(b)} = \{(1_1 2_1), (1_2 2_2), \ldots, (1_n 2_n)\},$$

(10)

where particles with the same subscript belong to the same Bell state. Let

$$\Sigma_n^{(b)} \equiv \{1_1, 1_2, \ldots, 1_n\},$$

(11)

$$\Sigma_n^{(b)} \equiv \{2_1, 2_2, \ldots, 2_n\},$$

(12)

so that we can write

$$S_n^{(b)} = \Sigma_n^{(b)} + \Sigma_n^{(b)}.$$  

(13)

Clearly the density matrix $\rho^{(b)}$ of $|S_n^{(b)}\rangle$ is different for $b = 0$ and $b = 1$:

$$\rho^{(0)} \neq \rho^{(1)}.$$  

(14)

Therefore, given $|S_n^{(b)}\rangle$, Bob can readily find out the value of $b$. It is easy to see that $|S_n^{(0)}\rangle$ and $|S_n^{(1)}\rangle$ have the same density matrices as

$$|S_n^{(0)}\rangle = \{|\uparrow_z \downarrow_z\rangle, |\downarrow_z \uparrow_z\rangle; \ 1/2, 1/2\}^n,$$

(15)

and

$$|S_n^{(1)}\rangle = \{|\uparrow_z \uparrow_z\rangle, |\downarrow_z \downarrow_z\rangle; \ 1/2, 1/2\}^n,$$

(16)

respectively. Using these new representations of $\rho^{(b)}$, one can easily prove two preliminary results about $|S_n^{(b)}\rangle$:

(I) If the particle order in $S_n^{(b)}$ is randomized, then the only way to distinguish $S_n^{(0)}$ from $S_n^{(1)}$ is by measuring the total spin sum $S_z$. That is, $S_z$ vanishes for $b = 0$, but not necessarily so for $b = 1$.

(II) Suppose Bob is forced to measure each particle in $\Sigma_n^{(b)}$ along any arbitrary axis in the $xy$-plane, then the resultant sequence $\tilde{S}_n^{(b)}$ will appear to be identical for $b = 0$ and 1.

We are now ready to specify our new QBC scheme, which has two crucial features:

(1) In the commitment phase, Alice encodes the committed bit $b$ in a quantum sequence whose density matrix is different for $b = 0$ and $b = 1$. (2) Bob is forced to perform certain random measurements on the sequence so that the two cases become indistinguishable to him. At first sight, it would seem that our scheme is still covered by the “no-go theorem”, since at the end of the commitment phase, the density matrix of the particles in Bob hand is independent of $b$. However this density matrix depends on Bob’s random choices unknown to Alice, consequently she cannot cheat.

(1) Commitment Phase:
(1a) Let \( n \) and \( m \) be the security parameters, and \( N = n + m \). Alice decides the value of \( b \) (0 or 1) and accordingly prepares a sequence of \( S_N^{(b)} \), such that
\[
|S_N^{(b)}\rangle = \{|b+,|b-\rangle; 1/2,1/2\}^N.
\] (17)

Similar to Eq. (13), we decompose \( S_N^{(b)} \) into two subsequences,
\[
S_N^{(b)} = \Sigma^{(b)} + \Sigma^{(b)}_{N2}.
\] (18)

Alice sends \( \Sigma^{(b)}_{N2} \) to Bob.

(1b) Bob measures each particle in \( \Sigma^{(b)}_{N2} \) along an arbitrary axis \( \hat{e}_i \) in the \( xy \)-plane, and reports the outcomes (but not the \( \hat{e}_i \)’s) to Alice. He then returns the measured particles, \( \tilde{\Sigma}^{(b)}_{N2} \), to Alice.

(1c) Alice randomly chooses \( m \) particles in \( \tilde{\Sigma}^{(b)}_{N2} \) for testing, and asks Bob to disclose the axes along which he measured them. She can then check if Bob is honest by measuring these particles and their entangled partners in \( \Sigma^{(b)}_{N1} \). Alice terminates the protocol if Bob is found cheating. Otherwise she discards the \( 2m \) test particles, so that
\[
\Sigma^{(b)}_{N1} \rightarrow \Sigma^{(b)}_{n1},
\] (19)
\[
\tilde{\Sigma}^{(b)}_{N2} \rightarrow \tilde{\Sigma}^{(b)}_{n2},
\] (20)
and proceeds to the next step.

(1d) Let
\[
\tilde{S}_n^{(b)} = \Sigma^{(b)}_{n1} + \tilde{\Sigma}^{(b)}_{n2}.
\] (21)
Alice randomizes the particle order in \( \tilde{S}_n^{(b)} \), and sends the resultant sequence \( \tilde{S}_n^{(b)} \) to Bob.

(2) Unveiling Phase:

(2a) Alice reveals the committed bit \( b \). She also informs Bob how to recover \( \tilde{S}_n^{(b)} \) from \( \tilde{S}_n^{(b)} \), and specifies the individual spin states in the commitment sequence \( S_n^{(b)} \) (i.e., \( S_N^{(b)} \) less \( m \) pairs of test particles).

(2b) Bob verifies Alice’s honesty by checking the characteristic spin correlation in each state in \( S_n^{(b)} \). Incorrect spin correlation in any one of the states signals cheating by Alice.

Having specified the new QBC scheme, we proceed to prove that it is concealing. For simplicity, and without loss of generality, we shall use the representations of \( \rho^{(b)} \) as given in Eqs. (15, 16). Obviously, if Bob is honest in the commitment phase, then he cannot cheat afterward — this is just preliminary result (II) we established earlier. From preliminary result (I), the only way Bob can cheat is by finding out the spin sum \( S_z \) of the commitment sequence. Therefore a dishonest Bob would try to measure only the test
particles as prescribed by the scheme, and the rest along \( \hat{z} \). The problem is that he does not know which particles Alice will choose for testing.

Let us first consider Bob’s classical cheating strategies: (1) Clearly Bob can cheat if he could correctly guess which particles in \( \Sigma_{N_2}^{(b)} \) Alice would choose for testing, however his chance of success is only of order \( 2^{-2n} \) for \( m \approx n \). (2) If Bob measures all the particles in \( \Sigma_{N_2}^{(b)} \) along \( \hat{z} \) instead of \( \{\hat{e}_i\} \) in the \( xy \)-plane, then his chance of passing Alice’s check is \( 2^{-m} \). It is not hard to show that other similar strategies also do not work for large \( n \) and \( m \).

Next, we examine Bob’s quantum cheating strategy. In this case, upon receiving \( \Sigma_{N_2}^{(b)} \) from Alice, Bob determines the only spin projections (to be reported to Alice) but leaves the corresponding measuring axes \( \{\hat{e}_i\} \) undetermined at the quantum level; he will fix the axis information only when requested by Alice. Let \( |\varphi_i\rangle \) be the wave function of \( i \)-th particle in \( \Sigma_{N_2}^{(b)} \). Through unitary operations, Bob can entangle ancilla particles with \( |\varphi_i\rangle \) to form

\[
|\Phi_i(\varphi_i)\rangle = \sum_{k=1}^{K} \sum_{\alpha=\uparrow,\downarrow} |\alpha \hat{e}_i^k \rangle \langle \alpha \hat{e}_i^k | \varphi_i \rangle \ p_i^k |\chi^k \rangle |\xi^\alpha \rangle,
\]

where \( \{\hat{e}_i^k\} \) is a set of \( K \) randomly chosen unit vectors in the \( xy \)-plane (i.e., \( \hat{e}_i^k \cdot \hat{z} = 0 \)), \( |\alpha \hat{e}_i^k \rangle \) denotes an eigenstate of \( \hat{\sigma} \cdot \hat{e}_i^k \) (with spin projection \( \alpha \)), \( \{ |\chi^k \rangle \} \) and \( \{ |\xi^\alpha \rangle \} \) are orthonormal sets of ancilla states, and finally

\[
\sum_{k=1}^{K} |p_i^k|^2 = 1.
\]

With \( |\Phi_i(\varphi_i)\rangle \), Bob can measure the \( \xi \)-ancilla to obtain the spin projection \( \alpha_i \), and separately the \( \chi \)-ancilla to determine the corresponding axis.

Measuring the \( \xi \)-ancilla reduces \( |\Phi_i(\varphi_i)\rangle \) to

\[
|\tilde{\Phi}_i(\uparrow_z)\alpha_i\rangle = \sum_{k=1}^{K} |\alpha_i \hat{e}_i^k \rangle p_i^k |\chi^k \rangle
\]

for \( |\varphi_i\rangle = |\uparrow_z\rangle \), and

\[
|\tilde{\Phi}_i(\downarrow_z)\alpha_i\rangle = \sum_{k=1}^{K} e^{-i\theta_i^k} |\alpha_i \hat{e}_i^k \rangle p_i^k |\chi^k \rangle
\]

for \( |\varphi_i\rangle = |\downarrow_z\rangle \); where \( \alpha_i = \uparrow \) or \( \downarrow \) is the outcome of the measurement, and \( \cos(\theta_i^k) = \hat{e}_i \cdot \hat{x} \). Now if \( K = 2 \) and

\[
\{\hat{e}_i^1, \hat{e}_i^2\} = \{|\hat{e}_i, -\hat{e}_i\},
\]

where \( \hat{e}_i \) is any unit vector satisfying \( \hat{e}_i \cdot \hat{z} = 0 \), then Eq. (24) and Eq. (25) become respectively

\[
|\tilde{\Phi}_i(\uparrow_z)\alpha_i\rangle = p_i^1 |\alpha_i \hat{e}_i \rangle |\chi^1 \rangle + p_i^2 |\alpha_i -\hat{e}_i \rangle |\chi^2 \rangle,
\]

(27)
and

\[ |\tilde{\Phi}_i(\downarrow_z)\alpha_i\rangle = p_1^i |\alpha_i\hat{\epsilon}_i\rangle |\chi_1\rangle - p_2^i |\alpha_i - \hat{\epsilon}_i\rangle |\chi_2\rangle. \]  

(28)

Notice that these two expressions are in fact general because, no matter what \( K \) is, Eq. (24) and Eq. (25) can always be rewritten in these forms by Schmidt decomposition. In any case, since \( |\varphi_i\rangle = \{|\uparrow_z\rangle, |\downarrow_z\rangle; \frac{1}{2}, \frac{1}{2}\} \), therefore the combined wave function of the \( i \)-th particle and the \( \chi \)-ancilla is a \( b \)-independent mixed state given by

\[ |\tilde{\Phi}_i(\varphi_i)\alpha_i\rangle = \{ |\tilde{\Phi}_i(\uparrow_z)\alpha_i\rangle, |\tilde{\Phi}_i(\downarrow_z)\alpha_i\rangle; 1/2, 1/2\}. \]  

(29)

As far as its density matrix is concerned, we can equivalently write

\[ |\tilde{\Phi}_i(\varphi_i)\alpha_i\rangle = \{ |\alpha_i\hat{\epsilon}_i\rangle |\chi_1\rangle, |\alpha_i - \hat{\epsilon}_i\rangle |\chi_2\rangle; |p_1^i|^2, |p_2^i|^2\}. \]  

(30)

It is more transparent to refer to this representation in the following discussion, since it involves only product states.

Now, Bob keeps the \( \chi \)-ancillas, and sends all the other particles (i.e., \( \tilde{\Sigma}^{(b)}_{n_2} \)) to Alice for checking [step (1c)]. Since \( \hat{\epsilon}_i \cdot \hat{\epsilon}_i = 0 \), Bob is guaranteed to pass Alice’s check. Then, after discarding the \( 2m \) test particles, Alice sends the particles of \( \tilde{\Sigma}^{(b)}_{n_2} \) and \( \Sigma^{(b)}_{n_1} \) to Bob in random order [see step (1d)]. As explained below, this randomization of the particle order conceals the \( b \)-dependent correlations among the particles from Bob.

From Eqs. (15, 16) and Eq. (30), we get

\[ |\Sigma^{(b)}_{n_1}\rangle = \{|\uparrow_z\rangle, |\downarrow_z\rangle; 1/2, 1/2\|^n, \]  

(31)

\[ |\tilde{\Sigma}^{(b)}_{n_2}\rangle = \prod_{i=1}^n \{ |\alpha_i\hat{\epsilon}_i\rangle, |\alpha_i - \hat{\epsilon}_i\rangle; |p_1^i|^2, |p_2^i|^2\}, \]  

(32)

which are both independent of \( b \). Therefore \( |\tilde{\Sigma}^{(b)}_{n}\rangle \), being a random mixture of the \( |\Sigma^{(b)}_{n_1}\rangle \) and \( |\tilde{\Sigma}^{(b)}_{n_2}\rangle \), is independent of \( b \). From Eq. (30), we see that the collective wave function of the \( \chi \)-ancillas is also a \( b \)-independent mixed state:

\[ |X\rangle = \prod_{i=1}^n \{ |\chi_1\rangle, |\chi_2\rangle; |p_1^i|^2, |p_2^i|^2\}. \]  

(33)

It follows that the density matrix of all the particles in Bob’s hand does not depend on \( b \), therefore he cannot cheat. From a more intuitive point of view, after the particles of \( \Sigma^{(b)}_{n_1} \) and \( \tilde{\Sigma}^{(b)}_{n_2} \) are randomly mixed, it becomes impossible for Bob to extract the total spin sum \( S_z \) of \( \Sigma^{(b)}_{n_1} \) and \( \tilde{\Sigma}^{(b)}_{n_2} \). Then, according to preliminary result (I), he can no longer gain any information about the value of \( b \). We thus conclude that this scheme is concealing.

It remains to be proven that our new scheme is binding. First of all, we note that, in the unveiling phase, Alice has to provide Bob with two pieces of classical information
about the properties of the particles in his hand: (1) The permutation sequence that takes \( S^{*}\) back to \( S^{(b)} \), and (2) The quantum states in the original sequence \( S^{(b)} \). The “no-go theorem” \cite{12, 13, 14, 15} claims that, if Alice leaves these parameters undetermined at the quantum level until just before opening her commitment, then she would be able to cheat by “EPR attack”. We show below that this strategy does not work in this scheme.

As we have already shown, whether Bob honestly measured the particles as prescribed or adopted a quantum strategy instead, at the end of the commitment of phase, the density matrix of the particles in Bob’s hand is independent of \( b \). So there indeed exists an unitary transformation \( U_A \) which can map the \( b = 0 \) case into \( b = 1 \). However it is clear from from Eqs. \cite{27, 28} that \( U_A \) depends on Bob’s random choices, \( p^b_i \) and \( \{ \hat{e}_i \} \). Since Alice does not know these parameters, cheating is impossible. The point to note is that, as long as \( \hat{e}_i \cdot \hat{z} = 0 \), our scheme does not further specifies how Bob should choose \( \{ \hat{e}_i \} \); hence, at the end of the commitment phase, Alice does not have precise knowledge of the wave function in Bob’s hand. This concludes the proof that our new QBC scheme is unconditionally secure.

In summary, we have proposed a new QBC scheme which is not covered by the “no-go theorem” \cite{12, 13, 14, 15}. The crucial observation we made is that the so-called “no-go theorem” only establishes the existence of a cheating unitary transformation \( U_A \) in any QBC scheme which is concealing, however it has never been proven that \( U_A \) must be known to Alice in all possible cases. Our scheme provides a concrete example in which \( U_A \) depends on parameters not known to Alice, so that cheating by EPR attack is impossible. We conclude that QBC can be unconditionally secure after all.

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