THE GROWTH AND DISTORTION THEOREMS FOR SLICE REGULAR FUNCTIONS

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Abstract. The sharp growth and distortion theorems are established for the slice regular extension to the setting of quaternions or Clifford algebras for a univalent holomorphic function of one complex variable on the unit disc. It turns out that the geometric function theory for the slice regular extension to higher dimensions of a univalent holomorphic function holds without extra geometric assumptions, in contrast to the setting of several complex variables in which the growth and distortion theorems are failed in general and only hold for some subclasses with the starlike or convex assumption. However, the corresponding covering theorem is unsettled, which indicates that the extensions are indirect.

1. Introduction

In geometric function theory of holomorphic functions of one complex variable, the following well-known growth and distortion theorems (cf. see [8]) remark the beginning of the systematic theory of univalent functions.

Theorem 1.1 (Growth and Distortion Theorems). Let $F$ be a univalent function on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $F(0) = 0$ and $F'(0) = 1$, then for all $z \in \mathbb{D}$,

\begin{equation}
\frac{|z|}{(1 + |z|)^2} \leq |F(z)| \leq \frac{|z|}{(1 - |z|)^2};
\end{equation}

\begin{equation}
\frac{1 - |z|}{(1 + |z|)^3} \leq |F'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3};
\end{equation}

\begin{equation}
\frac{1 - |z|}{1 + |z|} \leq \left| \frac{z F'(z)}{F(z)} \right| \leq \frac{1 + |z|}{1 - |z|}.
\end{equation}

Moreover, equality holds for one of these six inequalities at some point $z \neq 0$ if and only if $F$ is a rotation of the Koebe function, i.e.

$$F(z) = \frac{z}{(1 - e^{i\theta} z)^2}, \quad \forall \ z \in \mathbb{C},$$

for some $\theta \in \mathbb{R}$.

The extension of geometric function theory to higher dimensions was suggested by H. Cartan [2] in 1933. But, the first meaningful result was only made in 1991 by Barnard, Fitzgerald and Gong [1]. Since then, the geometric function theory in several complex variables has been extensively studied, see [13, 14] and the

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references therein. In particular, the growth theorems hold for starlike mappings on starlike circular domains [16], and for convex mappings on convex circular domains [17].

However, nearly nothing have been done about the corresponding theory for other function classes, perhaps due to the failure of closeness under multiplication or composition.

The purpose of this article is to generalize the geometric function theory to the noncommutative setting for slice regular functions. The theory of slice regular functions is initiated recently by Gentili and Struppa [11, 12]. The detailed up-to-date theory appears in the monographs [10, 7]. In particular, the slice regular product was introduced in the setting of quaternions in [9] for slice regular power series and in [3] for slice regular functions on symmetric slice domains. In the setting of Clifford algebras, the slice monogenic product was introduced in [4] for slice monogenic power series, mimicking the standard multiplication of polynomials in a skew field (see, e.g., [15]) and in [6] for slice monogenic functions on symmetric slice domains.

We will establish the growth and distortion theorems for slice monogenic extensions of univalent holomorphic functions on the unit disc of the complex plane to the setting of quaternions or Clifford algebras. We remark that the corresponding covering theorem remains unknown, which indicates that the extensions are indirect.

Because an elementary function of a real variable can be extended as a holomorphic function, it is quite natural to study the further extensions to the setting of quaternions or Clifford algebras.

The classical approach can not be applied in our case since there lacks a fruitful theory of regular compositions for slice regular functions. We shall reduce the problem to the classical one via a new convex combination identity:

\[ |f(x + yJ)|^2 = \frac{1 + \langle I, J \rangle}{2} |f(x + yI)|^2 + \frac{1 - \langle I, J \rangle}{2} |f(x - yI)|^2 \]

for any regular function \( f \) on a symmetric slice domain \( U \) such that \( f(U_I) \subseteq \mathbb{C}_I \) for some \( I \in S \).

We remark that in contrast to the setting of several complex variables in which the growth and distortion theorems are failed in general [2] and can only be restricted to the starlike or convex subclasses, our results on the slice monogenic extensions of univalent holomorphic functions hold without extra geometric assumptions. We now state our main results for Clifford algebra \( \mathbb{R}_n \).

**Theorem 1.2.** Let \( F : \mathbb{D} \to \mathbb{C} \) be a normalized univalent holomorphic function on the unit disc \( \mathbb{D} \), and let \( f : \mathbb{B} \to \mathbb{R}_n \) be the slice monogenic extension of \( F \). Then for all \( x \in \mathbb{B} \),

\[
\frac{|x|}{(1 + |x|)^2} \leq |f(x)| \leq \frac{|x|}{(1 - |x|)^2}; \tag{1.4}
\]

\[
\frac{1 - |x|}{(1 + |x|)^3} \leq |f'(x)| \leq \frac{1 + |x|}{(1 - |x|)^3}; \tag{1.5}
\]

\[
\frac{1 - |x|}{1 + |x|} \leq |xf'(x) + f^{-\ast}(x)| \leq \frac{1 + |x|}{1 - |x|}. \tag{1.6}
\]
Moreover, equality holds for one of these six inequalities at some point \( x \neq 0 \) if and only if
\[
f(x) = x(1 - xe^{i\theta}e^2), \quad \forall \ x \in \mathbb{B}.
\]

The similar results also hold in the setting of quaternions. It deserves to point out that the results in fact hold in even larger function class including slice regular automorphism group of the open unit ball \( \mathbb{B} \) of \( \mathbb{H} \).

The outline of this paper is as follows. In Section 2, we set up basic notations and give some preliminary results. In Section 3, we establish the growth and distortion theorems for slice monogenic functions in the setting of Clifford algebras. The analogous results for slice regular functions in the quaternionic setting are established in Section 4. Finally, Section 5 come the concluding remarks.

2. Preliminaries

Let \( F : \mathbb{D} \rightarrow \mathbb{C} \) be a normalized univalent holomorphic function on the unit disc \( \mathbb{D} \) of the complex plane with Taylor expansion
\[
F(z) = z + \sum_{m=2}^{\infty} z^m a_m, \quad a_m \in \mathbb{C}.
\]

We consider the canonical imbedding \( \mathbb{C} \subset \mathbb{R}^{n+1} \) by expanding the basis \( \{1, i\} \) of \( \mathbb{C} \) to the basis \( \{1, e_1, \ldots, e_n\} \) of \( \mathbb{R}^{n+1} \) with \( e_1 = i \). Therefore we can construct a natural extension of \( F \) to the setting of Clifford algebras via
\[
f(x) = x + \sum_{m=2}^{\infty} x^m a_m, \quad x \in \mathbb{R}^{n+1}.
\]

we remark that the slice monogenic extension of holomorphic function on the unit disc \( \mathbb{D} \) of the complex plane can result in the theory of slice monogenic elementary functions. We refer to [7] for the corresponding functional calculus and applications.

Our aim is to establish the growth and distortion theorems for the slice monogenic extension \( f \) of univalent function \( F \). To this end, we shall apply the theory of slice monogenic functions. Now we recall some preliminary results on slice monogenic functions. To have a more complete insight on the theory, we refer the reader to [7].

The real Clifford algebra \( \mathbb{R}_n = Cl_{0,n} \) is the associative algebra over \( \mathbb{R} \) generated by \( n \) basis elements \( e_1, e_2, \ldots, e_n \), subject to the relations
\[
e_i e_j + e_j e_i = -2\delta_{ij} e_0, \quad i, j = 1, 2, \ldots, n.
\]

As a real vector space \( \mathbb{R}_n \) has dimension \( 2^n \). Each element in \( \mathbb{R}_n \) can be represented as
\[
b = \sum_{A} b_A e_A,
\]
where \( b_A \in \mathbb{R} \), \( e_A = e_{h_1} \cdots e_{h_r} \), and \( e_0 = e_0 = 1 \) with \( A = \{h_1, \ldots, h_r\} \subseteq \{1, \ldots, n\} \) such that \( 1 \leq h_1 < \cdots < h_r \leq n \). Moreover, the inner product on \( \mathbb{R}_n \cong \mathbb{R}^{2^n} \) is given by
\[
\langle a, b \rangle := \text{Sc}(ab) = \sum_{A} a_A b_A
\]
for any \( a = \sum_{A} a_A e_A, \ b = \sum_{A} b_A e_A \in \mathcal{A} \), then by definition
\[
\langle a, b \rangle = \langle b, a \rangle = \langle \bar{a}, \bar{b} \rangle = \langle \bar{b}, \bar{a} \rangle.
\]
In particular, for any \( x, y \in \mathbb{R}^n \),
\[
\langle x, y \rangle = Sc(x\bar{y}) = -\frac{1}{2}(xy + yx),
\]
and consequently,
\[
xy = -(x, y) + x \wedge y,
\]
where \( x \wedge y := \frac{1}{2}(xy - yx) \) is the wedge product of \( x \) and \( y \).

As usual we identify \( e_0 \) with the real number 1 so that \( \mathbb{R} \) can be regarded as a subspace of \( \mathbb{R}^{n+1} \). A vector \( x \) in \( \mathbb{R}^{n+1} \) can be taken as a Clifford number
\[
x = \sum_{i=0}^{n} x_i e_i,
\]
so that it has inverse
\[
x^{-1} = \frac{\bar{x}}{|x|^2},
\]
where \( \bar{x} \) is the conjugate of \( x \) given by \( \bar{x} = x_0 e_0 - \sum_{i=1}^{n} x_i e_i \) and the norm of \( x \) is induced by the inner product given above, i.e., \( |x| = \langle x, x \rangle^{\frac{1}{2}} \). For any \( x = x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}^{n+1} \) is composed by the scalar part \( Sc(x) = x_0 \) and the vector part \( \underline{x} = x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}^n \). Every \( x \in \mathbb{R}^{n+1} \) can be expressed as \( x = u + Iv \), where \( x, y \in \mathbb{R} \) and
\[
I = \frac{\underline{x}}{|\underline{x}|}
\]
if \( \underline{x} \neq 0 \), otherwise we take \( I \) arbitrarily such that \( I^2 = -1 \). Then \( I \) is an element of the unit \( (n-1) \)-sphere of unit 1-vectors in \( \mathbb{R}^n \),
\[
S = \{ \underline{x} = x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1 \}.
\]

For every \( I \in S \) we will denote by \( \mathbb{C}_I \) the plane \( \mathbb{R} \oplus I\mathbb{R} \), isomorphic to \( \mathbb{C} \), and, if \( U \subseteq \mathbb{R}^{n+1} \), by \( U_I \) the intersection \( U \cap \mathbb{C}_I \). Also, for \( R > 0 \), we will denote the open ball of \( \mathbb{R}^{n+1} \) centred at the origin with radius \( R \) by
\[
B(0, R) = \{ x \in \mathbb{R}^{n+1} : |x| < R \}.
\]

We can now recall the definition of slice monogenicity.

**Definition 2.1.** Let \( U \) be a domain in \( \mathbb{R}^{n+1} \). A function \( f : U \rightarrow \mathbb{R}_n \) is called slice monogenic if, for all \( I \in S \), its restriction \( f_I \) to \( U_I \) is holomorphic, i.e., it has continuous partial derivatives and satisfies
\[
\partial_I f(u + vI) := \frac{1}{2} \left( \frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) f_I(u + vI) = 0
\]
for all \( u + vI \in U_I \).

The natural domains of definition in the theory of slice monogenic functions are symmetric slice domains.

**Definition 2.2.** Let \( U \) be a domain in \( \mathbb{R}^{n+1} \). \( U \) is called a slice domain if \( U \) intersects the real axis and \( U_I \) is a domain of \( \mathbb{C}_I \) for any \( I \in S \).

Moreover, if \( u + vI \in U \) implies \( u + vS \subseteq U \) for any \( u, v \in \mathbb{R} \) and \( I \in S \), then \( U \) is called a symmetric slice domain.
From now on, we will focus mainly on slice monogenic functions on $B(0, R) = \{ x \in \mathbb{R}^{n+1} : |x| < R \}$. For slice monogenic functions the natural definition of derivative is given by the following.

**Definition 2.3.** Let $f : B(0, R) \to \mathbb{R}^n$ be a regular function. The slice derivative of $f$ at $x = u + vI$ is defined by

$$\partial f(u + vI) := \frac{1}{2} \left( \frac{\partial}{\partial u} - I \frac{\partial}{\partial v} \right) f(u + vI).$$

Notice that the operators $\partial$ and $\bar{\partial}$ commute, and $\partial f = \frac{\partial f}{\partial u}$ for slice monogenic functions. Therefore, the slice derivative of a slice monogenic function is still regular so that we can iterate the differentiation to obtain the $n$-th slice derivative

$$\partial^n f = \frac{\partial^n f}{\partial u^n}, \quad \forall \ n \in \mathbb{N}.$$

In what follows, for the sake of simplicity, we will directly denote the $n$-th slice derivative $\partial^n f$ by $f^{(n)}$ for every $n \in \mathbb{N}$.

As shown in [5], a paravector power series $\sum_{n=0}^{\infty} x^n a_n$ with $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_n$ defines a slice monogenic function in its domain of convergence, which proves to be an open ball $B(0, R)$ with $R$ equal to the radius of convergence of the power series. The converse result is also true.

**Theorem 2.4 (Taylor Expansion).** A function $f$ is slice monogenic on $B = B(0, R)$ if and only if $f$ has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} x^n a_n \quad \text{with} \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

A fundamental result in the theory of slice monogenic functions is described by the splitting lemma, which relates slice monogenicity to classical holomorphy (see [5]).

**Lemma 2.5 (Splitting Lemma).** Let $f$ be a slice monogenic function on $B = B(0, R)$. For any $I = I_1 \in \mathbb{S}$, let $I_2, \ldots, I_n$ be a completion to a basis of $\mathbb{R}_n$ satisfying the defining relations $I_i I_j + I_j I_i = -2\delta_{ij}$. Then there exist $2^{n-1}$ holomorphic functions $F_A : B_I \to \mathbb{C}_I$ such that for every $z = u + vI \in B_I$,

$$f_I(z) = \sum_{|A|=0}^{n-1} F_A(z) I_A,$$

where $I_A = I_{i_1} \cdots I_{i_r}$, and $I_0 = 1$ with $A = \{i_1, \cdots, i_r\} \subseteq \{1, \cdots, n\}$ such that $2 \leq i_1 < \cdots < i_r \leq n$.

The following version of the identity principle is one of the first consequences (see [5]).

**Theorem 2.6 (Identity Principle).** Let $f$ be a slice monogenic function on $B = B(0, R)$. Denote by $Z_f$ the zero set of $f$,

$$Z_f = \{ x \in B : f(x) = 0 \}.$$

If there exists an $I \in \mathbb{S}$ such that $B_I \cap Z_f$ has an accumulation point in $B_I$, then $f$ vanishes identically on $B$. 
Another useful result is the following (see [4]).

**Theorem 2.7 (Representation Formula).** Let \( f \) be a slice monogenic function on a symmetric slice domain \( U \subseteq \mathbb{R}^{n+1} \) and let \( I \in \mathbb{S} \). Then for all \( u + vJ \in U \) with \( J \in \mathbb{S} \), the following equality holds

\[
  f(u + vJ) = \frac{1}{2} \left( f(u + vI) + f(u - vI) \right) + \frac{1}{2} JI \left( f(u - vI) - f(u + vI) \right).
\]

In particular, for each sphere of the form \( u + vS \) contained in \( U \), there exist \( b, c \in \mathbb{R}^n \) such that \( f(u + vI) = b + Ic \) for all \( I \in \mathbb{S} \).

Thanks to this result, it is possible to recover the values of a function on symmetric slice domains, which are more general than open balls centred at the origin, from its values on a single slice. This yields an extension theorem that in the special case of functions that are regular on \( B(0, R) \) can be obtained by means of their power series expansion.

**Remark 2.8.** Let \( f_I \) be a holomorphic function on a disc \( B_I = B(0, R) \cap \mathbb{C}_I \) and let its power series expansion take the form

\[
  f_I(z) = \sum_{n=0}^{\infty} z^n a_n
\]

with \( \{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{H} \). Then the unique slice monogenic extension of \( f_I \) to the whole ball \( B(0, R) \) is the function defined by

\[
  f(x) := \text{ext}(f_I)(x) = \sum_{n=0}^{\infty} x^n a_n.
\]

The uniqueness is guaranteed by the Identity Principle 2.6.

In Section 3, we will establish the growth and distortion theorems for such a function class.

**Definition 2.9.** Let \( f, g : B = B(0, R) \to \mathbb{R}^{n+1} \) be two slice monogenic functions and let

\[
  f(x) = \sum_{n=0}^{\infty} x^n a_n, \quad g(x) = \sum_{n=0}^{\infty} x^n b_n
\]

be their series expansions. The slice monogenic product (\( \ast \)-product) of \( f \) and \( g \) is the function defined by

\[
  f \ast g(x) = \sum_{n=0}^{\infty} x^n \left( \sum_{k=0}^{n} a_k b_{n-k} \right)
\]

regular on \( B \).

We now recall more definitions from [7].

**Definition 2.10.** Let \( f(x) = \sum_{n=0}^{\infty} x^n a_n \) be a slice monogenic function on \( B = B(0, R) \). We define the slice monogenic conjugate of \( f \) as

\[
  f^c(x) = \sum_{n=0}^{\infty} x^n \bar{a}_n,
\]
and the *symmetrization* of $f$ as

$$f^s(x) = f \ast f^c(x) = f^c \ast f(x) = \sum_{n=0}^{\infty} x^n \left( \sum_{k=0}^{n} a_k \bar{a}_{n-k} \right).$$

Both $f^c$ and $f^s$ are slice monogenic functions on $B$.

Now the inverse element of a slice monogenic function $f$ with respect to the $\ast$-product can be defined. Let $Z_{f^s}$ denote the zero set of the symmetrization $f^s$ of $f$.

**Definition 2.11.** Let $f$ be a slice monogenic function on $B = B(0, R)$ such that, for some $I \in S$ its restriction $f_I$ to $B_I$ satisfies the condition that $(f_I \ast f^c_I)(B_I) \subseteq \mathbb{C}_I$. If $f$ does not vanish identically, its slice monogenic inverse is the function defined by

$$f^{-\ast}(x) := f^s(x)^{-1} f^c(x)$$

slice monogenic on $B \setminus Z_{f^s}$.

## 3. Growth and Distortion Theorems for Slice Monogenic Functions

In this section, we establish the growth and distortion theorems for slice monogenic functions in the setting of Clifford algebra $\mathbb{R}_n$. We begin with a main technical lemma.

To present our main technical lemma more generally, we will digress for a moment to slice monogenic functions on symmetric slice domains.

**Lemma 3.1.** Let $f$ be a slice monogenic function on a symmetric slice domain $U \subseteq \mathbb{R}^{n+1}$. Then, for any $x = u + vJ \in U$ and any $I \in S$, there holds the identity

$$|f(x)|^2 = \frac{1}{2} \langle I, J \rangle |f(y)|^2 + \frac{1}{2} \langle I, J \rangle |f(\bar{y})|^2 - \frac{1}{2} \left( f(y) \overline{f(y)} - f(\bar{y}) \overline{f(\bar{y})}, I \wedge J \right),$$

where $y = u + vI$ and $\bar{y} = u - vI$.

**Proof.** It follows from Theorem 2.7 that

$$f(x) = \frac{1}{2} \left( f(y) + f(\bar{y}) \right) - \frac{1}{2} JI (f(y) - f(\bar{y}))$$

for any $x = u + vJ \in U$. Here, for simplicity, we denote $y = u + vI$ and $\bar{y} = u - vI$ for the given $I \in S$.

Notice that, in vector notation,

$$\langle I, J \rangle = Sc(IJ) = -\frac{1}{2}(IJ +JI),$$

and

$$I \wedge J = \frac{1}{2}(IJ -JI).$$

We shall use the simple identity that

$$|a + b|^2 = |a|^2 + |b|^2 + 2\langle a, b \rangle$$

for any $a, b \in \mathcal{A} \simeq \mathbb{R}^{2n}$. 


Taking modulus on both sides of (3.2) and applying (3.3) to obtain
\[
|f(x)|^2 = \frac{1}{4} \left( |f(y) + f(\bar{y})|^2 + |f(y) - f(\bar{y})|^2 \right) - \\
\frac{1}{2} \left( f(y) + f(\bar{y}), JJ(f(y) - f(\bar{y})) \right) \\
= : A - \frac{1}{2} B.
\]

Again applying (3.5), it is evident that
\[
A = \frac{1}{2} |f(y)|^2 + |f(\bar{y})|^2.
\]

To calculate the term \( B \), it first follows from the very definition of inner product that
\[
B = \left\langle (f(y) + f(\bar{y}))(\overline{f(y)} - \overline{f(\bar{y})}), JJ \right\rangle = : B_1 + B_2,
\]
where \( B_1 = \left\langle f(y)\overline{f(y)} - \overline{f(\bar{y})}f(\bar{y}), JJ \right\rangle \), and \( B_2 = \left\langle f(\bar{y})\overline{f(\bar{y})} - f(y)\overline{f(y)}, JJ \right\rangle \).

We next claim that
\[
B_1 = -\langle I, J \rangle (|f(y)|^2 - |f(\bar{y})|^2),
\]
and
\[
B_2 = \left\langle f(y)\overline{f(y)} - f(\bar{y})\overline{f(\bar{y})}, I \wedge J \right\rangle.
\]
Indeed, applying the fact that \( \langle a, b \rangle = \langle \bar{a}, \bar{b} \rangle \) from (2.1) to \( B_1 \) yields that
\[
B_1 = \left\langle f(y)\overline{f(y)} - f(\bar{y})\overline{f(\bar{y})}, I \wedge J \right\rangle
\]
Combining this, (3.3) and the initial nation of \( B_1 \), we thus obtain
\[
B_1 = \frac{1}{2} \left\langle f(y)\overline{f(y)} - f(\bar{y})\overline{f(\bar{y})}, IJ + JJ \right\rangle \\
= -\left\langle f(y)\overline{f(y)} - f(\bar{y})\overline{f(\bar{y})}, I, J \right\rangle \\
= -\langle I, J \rangle \left\langle f(y)\overline{f(y)} - f(\bar{y})\overline{f(\bar{y})}, 1 \right\rangle \\
= -\langle I, J \rangle (|f(y)|^2 - |f(\bar{y})|^2).
\]
Similarly,
\[
B_2 = \left\langle f(\bar{y})\overline{f(y)} - f(y)\overline{f(\bar{y})}, IJ \right\rangle = \left\langle f(\bar{y})\overline{f(y)} - f(y)\overline{f(\bar{y})}, -IJ \right\rangle,
\]
from which and the initial nation of \( B_2 \) it follows that
\[
B_2 = \frac{1}{2} \left\langle f(\bar{y})\overline{f(y)} - f(y)\overline{f(\bar{y})}, JJ - IJ \right\rangle \\
= -\left\langle f(\bar{y})\overline{f(y)} - f(y)\overline{f(\bar{y})}, I \wedge J \right\rangle \\
= \left\langle f(\bar{y})\overline{f(y)} - f(y)\overline{f(\bar{y})}, I \wedge J \right\rangle.
\]
In the second equation we have used (3.4).
Now substituting (3.7) - (3.10) into (3.6) yields that
\[
|f(x)|^2 = \frac{1 + \langle I, J \rangle}{2} |f(y)|^2 + \frac{1 - \langle I, J \rangle}{2} |f(\bar{y})|^2 - \frac{1}{2} \left\langle f(y)\overline{f(y)} - f(\bar{y})\overline{f(\bar{y})}, I \wedge J \right\rangle,
\]
which completes the proof. □

The preceding theorem shows that when \( f \) preserves at least one slice, then the squared norm of \( f \) can thus be expressed as a convex combination of those in the preserved slice.

**Lemma 3.2.** Let \( f \) be a slice monogenic function on a symmetric slice domain \( U \subseteq \mathbb{R}^{n+1} \) such that \( f(U_I) \subseteq \mathbb{C}_I \) for some \( I \in S \). Then the convex combination identity

\[
(3.15) \quad |f(u + vJ)|^2 = \frac{1 + \langle I, J \rangle}{2} |f(u + vI)|^2 + \frac{1 - \langle I, J \rangle}{2} |f(u - vI)|^2
\]

holds for any \( u + vJ \in U \).

**Proof.** As mentioned before, the Lemma is a direct consequence of the preceding theorem. But here, we will give an alternative easier method to prove it under having no idea about Lemma 3.1.

Fact: For any \( I, J \in S \), the set \( \{1, I, I \wedge J, I(I \wedge J)\} \) is an orthogonal set of \( \mathbb{R}_n \simeq \mathbb{R}^2n \).

As in the preceding theorem, it follows from Theorem 2.7 that

\[
(3.16) \quad f(x) = \frac{1}{2} (f(y) + f(\bar{y})) - \frac{1}{2} JJ \left( f(y) - f(\bar{y}) \right)
\]

for any \( x = u + vJ \in U \) with \( y = u + vI \) and \( \bar{y} = u + vI \).

Notice that, in vector notation,

\[
JJ = -\langle I, J \rangle + J \wedge I.
\]

Hence, we can rewrite (3.16) as

\[
f(x) = \frac{1}{2} \left( (1 + \langle I, J \rangle) f(y) + (1 - \langle I, J \rangle) f(\bar{y}) \right) + \frac{1}{2} J \wedge I \left( f(\bar{y}) - f(y) \right)
\]

\[
= \frac{1}{2} A + \frac{1}{2} (J \wedge I) B
\]

By assumption \( f(U_I) \subseteq \mathbb{C}_I \), we thus have

\[
A \in \mathbb{C}_I, \quad B \in \mathbb{C}_I.
\]

From the above fact, taking modulus on both sides yields

\[
(3.17) \quad |f(x)|^2 = \frac{1}{4} |A|^2 + \frac{1}{4} |J \wedge I|^2 |B|^2.
\]

A simple calculation shows that

\[
(3.18) \quad |A|^2 = \left( (1 + \langle I, J \rangle)^2 |f(y)|^2 + (1 - \langle I, J \rangle)^2 |f(\bar{y})|^2 \right) + 2(1 - \langle I, J \rangle)^2 \langle f(y), f(\bar{y}) \rangle
\]

and

\[
(3.19) \quad |B|^2 = |f(y)|^2 + |f(\bar{y})|^2 - 2 \langle f(y) , f(\bar{y}) \rangle.
\]

Notice that

\[
(3.20) \quad |J \wedge I|^2 = 1 - \langle I, J \rangle^2.
\]

Now inserting (3.18), (3.19) and (3.20) into (3.17) yields

\[
|f(x)|^2 = \frac{1 + \langle I, J \rangle}{2} |f(y)|^2 + \frac{1 - \langle I, J \rangle}{2} |f(\bar{y})|^2,
\]
which completes the proof. □

Remark 3.3. The counterpart of the convex combination identity (3.15) in Lemma 3.2 also hold for slice regular functions defined on Octonions or more general real alternative algebras under the extra assumption that \( f \) preserves at least one slice. This can be verified similarly as in the proof of Lemma 3.1 and is left to the interested reader to verify.

As a direct consequence of Lemma 3.2, we conclude that the maximum as well as minimum modulus of \( f \) is actually attained on the preserved slice.

Corollary 3.4. Let \( f \) be a slice monogenic function on a symmetric slice domain \( U \subseteq \mathbb{R}^{n+1} \) such that \( f(U_I) \subseteq \mathbb{C}_I \) for some \( I \in \mathbb{S} \). Then for any \( u + vS \subseteq U \), the following equalities hold
\[
\max_{J \in \mathbb{S}} |f(u + vJ)| = \max \left( |f(u + vI)|, |f(u - vI)| \right),
\]
and
\[
\min_{J \in \mathbb{S}} |f(u + vJ)| = \min \left( |f(u + vI)|, |f(u - vI)| \right).
\]

We are now in a position to state the growth and distortion theorems for slice monogenic functions.

Theorem 3.5 (Growth and Distortion Theorems for Paravectors). Let \( f \) be a slice monogenic function on the open unit ball \( B \) such that its restriction \( f_I \) to \( B_I \) is injective and \( f(B_I) \subseteq \mathbb{C}_I \) for some \( I \in \mathbb{S} \). If \( f(0) = 0 \) and \( f'(0) = 1 \), then for all \( x \in B \), the following inequalities hold
\[
\frac{|x|}{(1 + |x|)^2} \leq |f(x)| \leq \frac{|x|}{(1 - |x|)^2}; \tag{3.21}
\]
\[
\frac{1 - |x|}{(1 + |x|)^3} \leq |f'(x)| \leq \frac{1 + |x|}{(1 - |x|)^3}; \tag{3.22}
\]
\[
\frac{1 - |x|}{1 + |x|} \leq |xf'(x) + f^*(-x)| \leq \frac{1 + |x|}{1 - |x|}; \tag{3.23}
\]
Moreover, equality holds for one of these six inequalities at some point \( x \neq 0 \) if and only if \( f \) is of the form
\[
f(x) = x(1 - xe^{I\theta})^{-2}, \quad \forall x \in B,
\]
for some \( \theta \in \mathbb{R} \).

Proof. Notice that \( f_I : B_I \to C_I \) is a univalent holomorphic function by our assumption. Theorem 1.1 with \( f_I \) in place of \( F \) can thus be invoked to deduce that
\[
\frac{|y|}{(1 + |y|)^2} \leq |f(y)| \leq \frac{|y|}{(1 - |y|)^2}; \tag{3.24}
\]
\[
\frac{1 - |y|}{(1 + |y|)^3} \leq |f'(y)| \leq \frac{1 + |y|}{(1 - |y|)^3}; \tag{3.25}
\]
\[
\frac{1 - |y|}{1 + |y|} \leq \frac{|yf'(y)|}{f(y)} \leq \frac{1 + |y|}{1 - |y|}; \tag{3.26}
\]
for all \( y = u + vI \in B_I \).
On the other hand,
\[ |f(x)|^2 = \frac{1 + \langle I, J \rangle}{2} |f(y)|^2 + \frac{1 - \langle I, J \rangle}{2} |f(\bar{y})|^2. \]
for any \( x = u + vJ \in \mathbb{B} \) due to Lemma 3.2.

Since (3.24) holds for all \( y = u + vI \) and \( \bar{y} = u - vI \in \mathbb{B} \), it also holds for any \( x = u + vJ \in \mathbb{B} \) due to the convex combination identity above. This proves inequalities in (3.21).

Since slice monogenic functions \( f'(x) \) and \( xf'(x) * f^{-*}(x) \) (by definition 2.11, \( f^{-*} \) is well-defined under our assumption) also satisfy the assumptions given in Lemma 3.2, we have for any \( x = u + vJ \in \mathbb{B} \),
\[ |f'(x)|^2 = \frac{1 + \langle I, J \rangle}{2} |f'(y)|^2 + \frac{1 - \langle I, J \rangle}{2} |f'(\bar{y})|^2, \]
and
\[ |xf' * f^{-*}(x)|^2 = \frac{1 + \langle I, J \rangle}{2} |yf'(y)|^2 + \frac{1 - \langle I, J \rangle}{2} |\bar{y}f'(\bar{y})|^2, \]
from which we can prove inequalities in (3.22) and (3.23) using the same arguments as above.

Furthermore, if equality holds for one of six inequalities in (3.21), (3.22) and (3.23) at some point \( x_0 = u_0 + v_0J \neq 0 \) with \( J \in \mathbb{S} \), then the corresponding equality also holds at \( y_0 = u_0 + v_0I \) and \( \bar{y}_0 = u_0 - v_0I \). By using Theorem 1.1 we obtain that
\[ f_I(y) = \frac{y}{(1 - e^{i\theta}y)^2}, \quad \forall y \in \mathbb{C}_I, \]
for some \( \theta \in \mathbb{R} \), which implies
\[ f(x) = x(1 - xe^{i\theta})^{-2}, \quad \forall x \in \mathbb{B}. \]
The converse part is obvious. Now the proof is completed. \( \square \)

Let \( F : \mathbb{D} \to \mathbb{C} \) be a normalized univalent holomorphic function on the unit disc \( \mathbb{D} \) of the complex plane with Taylor expansion
\[ F(z) = z + \sum_{m=2}^{\infty} z^m a_m, \quad a_m \in \mathbb{C}. \]
We consider the canonical imbedding \( \mathbb{C} \subset \mathbb{R}^{n+1} \) by expanding the basis \( \{1, i\} \) of \( \mathbb{C} \) to the basis \( \{1, e_1, \ldots, e_n\} \) of \( \mathbb{R}^{n+1} \) with \( e_1 = i \). Therefore we can construct a natural extension of \( F \) to the setting of Clifford algebras via
\[ f(x) = x + \sum_{m=2}^{\infty} x^m a_m, \quad x \in \mathbb{R}^{n+1}. \]
It is evident that \( f \) is a slice monogenic function on the open unit ball \( \mathbb{B} = B(0, 1) \) such that its restriction \( f|_D = F \) is injective and satisfying \( F(\mathbb{D}) \subset \mathbb{C} \). Clearly, \( f(0) = 0 \) and \( f'(0) = 1 \). Thus \( f \) satisfies all assumptions of Theorem 3.5 and this proves the Theorem 1.2.

The following proposition is of independent interest.

**Proposition 3.6.** Let \( f \) be a slice monogenic function on a symmetric slice domain \( U \subset \mathbb{R}^{n+1} \) such that its restriction \( f_I \) to \( U_I \) is injective and \( f(U_I) \subset \mathbb{C}_I \) for some \( I \in \mathbb{S} \). Then its restriction \( f_J : U_J \to A \) is also injective for any \( J \in \mathbb{S} \).
Proof. Suppose that there are \( x = \alpha + \beta J \) and \( y = \gamma + \delta J \) such that \( f(x) = f(y) \), it suffices to prove that \( x = y \). If \( J = \pm I \), the result follows from the assumption. Otherwise, from Theorem 2.7 one can deduce that
\[
\begin{align*}
\frac{1}{2}((f(z) + f(\bar{z})) - \frac{1}{2}JI(f(z) - f(\bar{z})))
\end{align*}
\]
and
\[
\begin{align*}
\frac{1}{2}((f(w) + f(\bar{w})) - \frac{1}{2}JI(f(w) - f(\bar{w}))).
\end{align*}
\]
Here \( z = \alpha + \beta I \) and \( \bar{w} = \gamma + \delta I \) for the given \( I \in S \). Therefore,
\[
\begin{align*}
\left( (f(z) + f(\bar{z})) - (f(w) + f(\bar{w})) \right) - JI\left( (f(z) - f(\bar{z})) - (f(w) - f(\bar{w})) \right) = 0.
\end{align*}
\]
Since \( f(U_I) \subseteq C_I \), 1 and \( J \) are linearly independent on \( C_I \) we obtain that
\[
\begin{align*}
f(z) + f(\bar{z}) = f(w) + f(\bar{w})
\end{align*}
\]
and
\[
\begin{align*}
f(z) - f(\bar{z}) = f(w) - f(\bar{w}),
\end{align*}
\]
which implies that \( f(z) = f(w) \). Thus it follows from the injectivity of \( f_I \) that \( z = w \) and consequently, \( x = y \). \( \square \)

Remark 3.7. Let \( f \) be as described in Proposition 4.8 then \( f_J : U_J \rightarrow A \) is also injective for any \( J \in S \) by the preceding proposition. Unfortunately, the authors do not know whether \( f : U \rightarrow A \) is injective.

4. Growth and Distortion Theorems for Slice Regular Functions

Let \( \mathbb{H} \) denote the non-commutative, associative, real algebra of quaternions with standard basis \( \{1, i, j, k\} \), subject to the multiplication rules
\[
i^2 = j^2 = k^2 = ijk = -1.
\]

Let \( \langle \cdot, \cdot \rangle \) denote the standard inner product on \( \mathbb{H} \cong \mathbb{R}^4 \), i.e.,
\[
\langle p, q \rangle = \text{Re}(p\bar{q}) = \sum_{n=0}^{3} x_n y_n
\]
for any \( p = x_0 + x_1 i + x_2 j + x_3 k \), \( q = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H} \).

In this section, we shall consider the slice regular functions defined on open sets \( \Omega \) of quaternions \( \mathbb{H} \) with values in \( \mathbb{H} \). Thus these functions do not coincide with slice monogenic functions that are obtained by setting \( n = 2 \).

To introduce the theory of slice regular functions, we will denote by \( S \) the unit 2-sphere of purely imaginary quaternions, i.e.,
\[
S = \{ q \in \mathbb{H} : q^2 = -1 \}.
\]

For every \( I \in S \) we will denote by \( C_I \) the plane \( \mathbb{R} \oplus I\mathbb{R} \), isomorphic to \( \mathbb{C} \), and, if \( \Omega \subseteq \mathbb{H} \), by \( \Omega_I \) the intersection \( \Omega \cap C_I \). Also, we will denote by \( B \) the open unit ball centred at the origin in \( \mathbb{H} \), i.e.,
\[
B = \{ q \in \mathbb{H} : |q| < 1 \}.
\]

We can now recall the definition of slice regularity.
Definition 4.1. Let $\Omega$ be a domain in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is called slice regular if, for all $I \in \mathbb{S}$, its restriction $f_I$ to $\Omega_I$ is holomorphic, i.e., it has continuous partial derivatives and satisfies

$$\partial_If(x+yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + J \frac{\partial}{\partial y} \right) f_I(x+yI) = 0$$

for all $x+yI \in \Omega_I$.

The notions of slice domain, of symmetric slice domain and of slice derivative are similar to the ones already given in Section 2. Moreover, the corresponding results still hold for the slice regular functions in the setting of quaternions, such as the splitting lemma, the representation formula, the power series expansion and so on.

Now we can establish the following results by obvious modifications in the quaternionic setting.

Lemma 4.2. Let $f$ be a slice regular function on a symmetric slice domain $\Omega \subseteq \mathbb{H}$ and let $I \in \mathbb{S}$. Then, for any $q = x + yJ \in \Omega$ and any $I \in \mathbb{S}$, there holds the identity

$$|f(q)|^2 = \frac{1}{2} |f(z)|^2 + \frac{1}{2} |f(\bar{z})|^2 - \langle \text{Im}(f(z)f(\bar{z})), I \wedge J \rangle,$$

where $z = x + yI$ and $\bar{z} = x - yI$.

Before presenting the key lemma to establish the growth and distortion theorems, we first make an equivalent characterization of the vanishment of the third term on the right-hand side of (4.2), thanks to the speciality of quaternions.

Theorem 4.3. Let $f$ be a slice regular function on a symmetric slice domain $\Omega \subseteq \mathbb{H}$ and let $I \in \mathbb{S}$. Then

$$\langle \text{Im}(f(z)f(\bar{z})), I \wedge J \rangle = 0$$

for any $J \in \mathbb{S}$ and any $z \in \Omega_I$ if and only if there exist $u \in \partial \mathbb{B}$ and a slice regular function $g$ on $\Omega$ that preserves the slice $\Omega_I$ such that

$$f(q) = g(q)u$$

on $\Omega$.

Proof. Let $f_I(z) = F(z) + G(z)K$ be the splitting of $f$ with $I, K \in \mathbb{S}$ and $I \perp K$. Take $L \in \mathbb{S}$ such that $\{1, I, K, L\}$ is an orthonormal basis of quaternions $\mathbb{H}$ and let $V$ denote the real vector space generated by the set $\{I \wedge J : J \in \mathbb{S}\}$, then it is clear that

$$V = K\mathbb{R} \oplus L\mathbb{R}. \quad (4.1)$$

Moreover, a simple calculation gives

$$f(z)f(\bar{z}) = \left( F(z)\overline{F(\bar{z})} + G(z)\overline{G(\bar{z})} \right) K,$$

from which and (4.1) it follows that

$$\langle \text{Im}(f(z)f(\bar{z})), I \wedge J \rangle = 0, \quad \forall J \in \mathbb{S},$$

if and only if

$$F(z)G(\bar{z}) = F(\bar{z})G(z), \quad \forall z \in \Omega_I. \quad (4.2)$$
If $G \equiv 0$ on $\Omega_I$, there is nothing to prove and the result follows. Otherwise, $G \not\equiv 0$, let $Z_G$ denote the zero set of $G$, then $Z_G$ has no accumulation point in $\Omega_I$ by the Identity Principle and so does $\overline{Z_G} := \{ \bar{z} \in \Omega_I : z \in Z_G \}$. Therefore by (4.2)

\[
\frac{F(z)}{G(z)} = \frac{F(\bar{z})}{G(\bar{z})}
\]

is both holomorphic and conjugate holomorphic on $\Omega_I \setminus (Z_G \cup \overline{Z_G})$, which is still a domain of $\mathcal{C}_I$, thus there exist a constant $\lambda \in \mathcal{C}_I$ such that

\[
\frac{F(z)}{G(z)} = \frac{F(\bar{z})}{G(\bar{z})} = \lambda,
\]

which implies that $F = \lambda G$ on $\Omega_I \setminus (Z_G \cup \overline{Z_G})$ and hence on $\Omega_I$ by the Identity Principle.

Now let $g$ be the slice regular extension of $G$, i.e., $g = \text{ext}(G)$ and set

\[
u = (1 + |\lambda|^2)^{-\frac{1}{2}} (\lambda + K) \in \partial \mathcal{B}_2,
\]

then $g$ is a regular function on $\Omega$ such that $g(\Omega_I) \subset \mathcal{C}_I$ and $f = gu$, which completes the proof.

\[\square\]

The precious theorem shows that when $f$ preserves one slice up to a suitable rotation from the group

\[
\mathbb{S} := \left\{ \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} : u \in \partial \mathcal{B}_2 \right\} \subset \mathcal{S}(1,1),
\]

e.g., $f(\Omega_I) \subset \mathcal{C}_I u$ for some $I \in \mathbb{S}$ and some $u \in \partial \mathcal{B}_2$, the squared norm of $f$ can thus be expressed as a convex combination of those in the preserved slice.

**Lemma 4.4.** Let $f$ be a slice regular function on a symmetric slice domain $\Omega \subset \mathbb{H}$ such that $f(\Omega_I) \subset \mathcal{C}_I u$ for some $I \in \mathbb{S}$ and some $u \in \partial \mathcal{B}_2$. Then

\[
|f(x + yI)|^2 = \frac{1 + (I, J)}{2} |f(x + yI)|^2 + \frac{1 - (I, J)}{2} |f(x - yI)|^2
\]

holds for any $x + yJ \in \Omega$.

In particular, each element $f$ from the slice regular automorphism group of the open unit ball $\mathcal{B}$ of $\mathbb{H}$

\[
\text{Aut}(\mathcal{B}) = \left\{ f(q) = (1 - q \bar{a})^{-*} (q - a) u : a \in \mathcal{B}, u \in \partial \mathcal{B}_2 \right\}
\]

satisfies the condition in Lemma 4.4.

As a direct consequence of Lemma 4.4 we conclude that the maximum as well as minimum modulus of $f$ is actually attained on the preserved slice.

**Corollary 4.5.** Let $f$ be a slice regular function on a symmetric slice domain $\Omega \subset \mathbb{H}$ such that $f(\Omega_I) \subset \mathcal{C}_I$ for some $I \in \mathbb{S}$. Then for any $x + y\mathbb{S} \subset \Omega$, the following equalities hold

\[
\max_{J \in \mathbb{S}} |f(x + yJ)| = \max \left( |f(x + yI)|, |f(x - yI)| \right),
\]

and

\[
\min_{J \in \mathbb{S}} |f(x + yJ)| = \min \left( |f(x + yI)|, |f(x - yI)| \right).
\]
We are now in a position to state the growth and distortion theorems for quaternions.

**Theorem 4.6** (Growth and Distortion Theorems for Quaternions). Let \( f \) be a slice regular function on the open unit ball \( B = B(0, 1) \) such that its restriction \( f_I \) to \( B_I \) is injective and \( f(B_I) \subseteq \mathbb{C}_I \) for some \( I \in \mathbb{S} \). If \( f(0) = 0 \) and \( f'(0) = 1 \), then for all \( q \in \mathbb{B} \), the following inequalities hold

\[
\frac{|q|}{(1 + |q|)^2} \leq |f(q)| \leq \frac{|q|}{(1 - |q|)^2};
\]
\[
\frac{1 - |q|}{(1 + |q|)^3} \leq |f'(q)| \leq \frac{1 + |q|}{(1 - |q|)^3};
\]
\[
\frac{1 - |q|}{1 + |q|} \leq |qf'(q) \ast f^{-*}(q)| \leq \frac{1 + |q|}{1 - |q|}.
\]

Moreover, equality holds for one of these six inequalities at some point \( q \neq 0 \) if and only if \( f \) is of the form

\[
f(q) = q(1 - qe^{i\theta})^{-*2}, \quad \forall q \in \mathbb{B},
\]

for some \( \theta \in \mathbb{R} \).

Let \( F : \mathbb{D} \to \mathbb{C} \) be a normalized univalent holomorphic function on the unit disc \( \mathbb{D} \) of the complex plane with Taylor expansion

\[
F(z) = z + \sum_{n=2}^{\infty} z^n a_n, \quad a_n \in \mathbb{C}.
\]

With the canonical imbedding \( \mathbb{C} \subseteq \mathbb{H} \), we can construct a natural regular extension of \( F \) to quaternions via

\[
f(q) = q + \sum_{n=2}^{\infty} q^n a_n, \quad q \in \mathbb{H}.
\]

It is evident that \( f \) is a slice regular function on the open unit ball \( \mathbb{B} = B(0, 1) \) such that its restriction \( f|_{\mathbb{D}} = F \) is injective and satisfying \( F(\mathbb{D}) \subseteq \mathbb{C} \). Clearly, \( f(0) = 0 \) and \( f'(0) = 1 \). Thus \( f \) satisfies all assumptions of Theorem 4.6 and this proves the following theorem.

**Theorem 4.7.** Let \( F : \mathbb{D} \to \mathbb{C} \) be a normalized univalent holomorphic function on the unit disc \( \mathbb{D} \), and let \( f : \mathbb{B} \to \mathbb{H} \) be the slice regular extension of \( F \). Then for all \( q \in \mathbb{B} \),

\[
\frac{|q|}{(1 + |q|)^2} \leq |f(q)| \leq \frac{|q|}{(1 - |q|)^2};
\]
\[
\frac{1 - |q|}{(1 + |q|)^3} \leq |f'(q)| \leq \frac{1 + |q|}{(1 - |q|)^3};
\]
\[
\frac{1 - |q|}{1 + |q|} \leq |qf'(q) \ast f^{-*}(q)| \leq \frac{1 + |q|}{1 - |q|}.
\]

Moreover, equality holds for one of these six inequalities at some point \( q \neq 0 \) if and only if

\[
f(q) = q(1 - qe^{i\theta})^{-*2}, \quad \forall q \in \mathbb{B}.
\]
Proposition 4.8. Let $f$ be a slice regular function on a symmetric slice domain $\Omega \subseteq \mathbb{H}$ such that its restriction $f_I$ to $\Omega_I$ is injective and $f(\Omega_I) \subseteq C_I$ for some $I \in S$. Then its restriction $f_J : \Omega_J \to \mathbb{H}$ is also injective for any $J \in S$.

5. Concluding remarks

As pointed out in Remark 3.3, the counterpart of the convex combination identity (3.15) in Lemma 3.2 also hold for slice regular functions defined on Octonions or more general real alternative algebras under the extra assumption that $f$ preserves at least one slice. Therefore the results analogous to those given in the preceding sections can be easily generalized by slight modification to the setting of Octonions and more general real alternative algebras. Finally, we conclude with some questions connected with the paper’s subject which we could not answer.

We define

$$S := \{ f \in R(\mathbb{B}) : f_I \text{ injective and } f_I(\mathbb{B}_I) \subseteq C_I \text{ for some } I \in S \}$$

Open question: A natural question arises of whether the class $S$ of slice regular functions is the largest subclass of $R(\mathbb{B})$ in which the corresponding growth and distortion theorems hold.

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