Approximation Methods for the Distributed Order Calculus Using the Convolution Quadrature

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Abstract. In this article we generalize the convolution quadrature (CQ) method, which aims at approximating the fractional calculus, to the case for the distributed order calculus. Our method is a natural expansion that the approximation formulas, convergence results and correction technique reduce to the cases for the CQ method if the weight function $\mu(\alpha)$ is defined by $\delta(\alpha - \alpha_0)$. Further, we explore a new structure of the solution of an ODE with the distributed order fractional derivative, which differs from those of the solutions of traditional fractional ODEs, and propose a new correction technique for this new structure to restore the optimal convergence rate. Numerical tests with smooth and nonsmooth solutions confirm our theoretical results and the efficiency of our correction technique.

1. Introduction. During the last few decades, equations with fractional calculus or distributed order calculus have been successfully applied to model the anomalous transport processes that do not obey the Gaussian statistics in the areas such as physics, biology, chemistry, geological sciences, etc., [12, 18, 19, 33]. Models with properly defined fractional calculus can faithfully reflect the process of the slow diffusion, which is featured in the mean square displacement of the diffusing particles of the power type $t^\alpha$, in contrast to the distributed order fractional calculus that can describe the ultra-slow diffusion which is just of logarithmic growth [3,4,32]. Hence, these models have attracted a growing interest both for its widespread applications and theoretical analysis.
For fractional calculus equations, one generally cannot get the solution in closed form, thus different numerical methods have been proposed to efficiently obtain the approximate solution, see [2, 5, 7, 8, 10, 20, 21, 24, 25, 31, 38, 41, 43, 49]. As is well known that the solution of fractional differential equations shows some singularity at the initial node, different techniques were employed to restore the optimal convergence rate, see [15, 23, 26, 27, 40, 42, 44, 45, 47]. For equations with distributed order calculus, things seem worse as the distributed order calculus is a natural generalization of the fractional calculus and hence is more complicated. In this paper, we aim at developing novel approximation formulas for the distributed order calculus, and for readers interested in the mathematical theory such as the uniqueness, existence and asymptotic properties of solutions, see [17, 22, 29] and references therein. Compared with the extensive studies on the numerical approximation for the fractional calculus, there are only a few studies on the numerical methods design or numerical analysis for the distributed order calculus, see, for example, [1, 6, 11, 13, 14, 16, 30, 34, 37, 39, 48, 50]. Most of the numerical methods transform the distributed order calculus into a multi-term fractional calculus by a quadrature formula and then approximate the resulted fractional calculus by the methods well studied. Exceptions are the work of Jin et al. [16] as well as Mashayekhi and Razzaghi [30]. Unlike the methods mentioned above, we develop our theory that is based directly on the work of Lubich [27], where the convolution quadrature (CQ) method was developed for fractional calculus. See also [27, 28] for more information.

We summarize the contributions of this paper as follows.

• Develop new approximation formulas based on the CQ method for the distributed order calculus $I^\mu f(x)$ which directly generalize the approximation methods for single-term fractional calculus in the sense that for $\mu(\alpha)$ taking the Dirac delta function $\delta(\alpha - \alpha_0)$, our formulas coincide with those of the CQ method used to approximate $I^{\alpha_0} f(x)$.

• Examine the solution structure of the equation $I^\mu f(x) = \lambda f(x)$ with the solution expansion near $x = 0$, and devise a novel correction technique to improve the error accuracy for the distributed order calculus. Although some researchers have explored the asymptotic properties (as $x \to 0$ or $\infty$) of the solution for the distributed order fractional equations, we for the first time point out the expansion of the solution at $x = 0$.

• Conduct exhaustive numerical tests with smooth and nonsmooth solutions for different methods we design. To the best of our knowledge, it is the first paper that deals with the weak regularity of solutions of distributed order fractional equations with higher-order numerical methods.

The rest of the paper is organized as follows:

In Sect. 2, we define the distributed order calculus $I^\mu$ as well as some notations for the Riemann-Liouville and Caputo fractional differential operators, make a brief overview for the convolution quadrature that our numerical methods based on and introduce some lemmas useful for the following theoretical analysis or the numerical experiment implementation.

In Sect. 3, we prove our main results of this paper. The approximation formula is stated in (20) and (21) which is constructed by the generating function of the CQ method. Some symbols $\Omega_h^{\mu}$ and $E_h^{\mu}$ for the distributed order calculus are designed for a better presentation of the analysis. The homogeneity property of $I^\mu$ and $\Omega_h^{\mu}$ is derived in Lemma 3.1. In Theorem 3.2, we obtain the error convergence rate for
our novel approximation formulas (20) and in Theorem 3.3 we consider the method by adding correction terms to recover a high error accuracy. Considering most papers assumed in numerical tests the structure of the solution being the power type, we further examine the solution structure (the Lemma 3.4) of the equation (60) and devise new correction strategies particularly for distributed order calculus.

In Sect. 4, we apply our approximation formulas to the distributed order fractional diffusion equation with smooth or nonsmooth solutions. Like most researchers did in their numerical experiments with the known solution structure of $u(t, \cdot) = t^\ell(\ell$ is positive and large enough), our methods can produce the desired convergence rate. For those $\ell$ that are not large enough, which means a weak regularity for the solution at the initial node, we can still obtain the desired convergence rate by adding correction terms. For the application of the discovery of the Lemma 3.4 that reveals the different structure of the solutions of the distributed order fractional equations, we examine our method for an ODE whose solution is known in advance of the form $f(x) = 1 + x \log x + \cdots$. The new and traditional correction technique has been applied to this experiment to show the efficiency of the methods.

In Sect. 5, we make some concluding remarks and indicate the potential difficulties in numerically analyzing the distributed fractional PDEs. Finally, we give the arguments for some lemmas in the Appendix.

2. Preliminaries. In this section, we give some definitions of the distributed order calculus and introduce the CQ method for the fractional calculus. Some lemmas are also stated to facilitate the analysis in the following sections.

2.1. Definition of the distributed order calculus. Define the distributed order calculus of $f$ with a weight function $\mu(\alpha)$ by

$$I^{[\alpha]} f(x) = \int_a^b \mu(\alpha) (I^\alpha f)(x) \, \, d\alpha,$$

where $I^\alpha$ denotes the Riemann-Liouville calculus

$$I^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) \, ds, & \text{if } \alpha > 0 \\ \frac{d^k}{dx^k} I^{\alpha+k} f(x), & \text{if } -k < \alpha < 1-k \\ f^{(k-1)}(x), & \text{if } \alpha = 1-k. \end{cases}$$

Here and in this section, we denote by $k$ a positive integer.

Suppose $f(x)$ has the required regularity such that the definition (1) does make sense. For the weight function $\mu(\alpha)$ in our article, we mainly focus on two cases:

(i) $\mu(\alpha) \in L^1([0,1])$, or

(ii) $\mu(\alpha) = \sum_{j=0}^Q \delta(\alpha - \alpha_j)$, where $\delta(\alpha)$ denotes the Dirac delta function.

Another closely related fractional derivative is proposed by Caputo, which is defined by for $f(x) \in C^k([0,X])$,

$$cD^\beta f(x) = \frac{1}{\Gamma(k-\beta)} \int_0^x \frac{f^{(k)}(s)\, ds}{(x-s)^{\beta+1-k}}, \text{ for } k-1 < \beta < k.\quad (3)$$

The well known relationship between the operators $I^{-\beta}$ and $cD^\beta$ is [35]

$$I^{-\beta} f(x) = \sum_{j=0}^{k-1} \frac{f^{(j)}(0)x^{j-\beta}}{\Gamma(1+j-\beta)} + cD^\beta f(x), \quad k-1 < \beta < k,\quad (4)$$
hence, if \( f^{(j)}(0) = 0, j = 0, \cdots, k - 1 \), we obtain \( I^{-\beta} f(x) = c D^{\beta} f(x) \). For simplicity, we define \( D^{\beta} = I^{-\beta} \).

2.2. Convolution quadrature for \( I^{\alpha} \). For arbitrary \( \alpha \) and arbitrary function \( f(x) \) of the form \( x^\ell g(x) \) with \( \ell \neq -1, -2, \cdots \), and sufficiently differentiable \( g(x) \), the convolution quadrature states that \( I^{\alpha} f(x) \) can be approximated by a convolution structure (see [27]) at node \( x = hn \),

\[
I^{\alpha} f(x) = \Omega_n^{\alpha} f(x) + h^\alpha \sum_{j=1}^{m} w_{n,j} f(jh) + O(h^p), \quad p \text{ is a positive integer},
\]

where \( h \) denotes the mesh size and \( h = X/N \) for a positive integer \( N \). The convolution part \( \Omega_n^{\alpha} f(x) \) is defined by

\[
\Omega_n^{\alpha} f(x) = h^\alpha \sum_{j=0}^{n} \varpi_{n-j}(\alpha) f(jh), \quad x = hn,
\]

where the weights \( \varpi_j \) independent of \( h \) are coefficients of a generating function of the form \( r_1(\xi)\alpha r_2(\xi) \) \((r_1(\xi), r_2(\xi) \) are rational functions), expressed by (see Lemma 3.2 and Lemma 3.3 in [27])

\[
\varpi(\alpha, \xi) = \sum_{j=0}^{\infty} \varpi_j(\alpha) \xi^j
\]

\[
= (1 - \xi)^{-\alpha} \left[ \gamma_0 + \gamma_1(1 - \xi) + \cdots + \gamma_{p-1}(1 - \xi)^{p-1} \right] + (1 - \xi)^p \varrho(\alpha, \xi),
\]

with the coefficients \( \varrho_n \) of \( \varrho(\alpha, \xi) \) having the asymptotic property \( \varrho_n = O(n^{\alpha-1}) \) and \( \gamma_i \)'s satisfying

\[
\sum_{i=0}^{\infty} \gamma_i(1 - \xi)^i = \left( \frac{\ln \xi}{\xi - 1} \right)^{-\alpha}.
\]

In our article, we always assume \( \varpi_j(\alpha) \) is continuous on \([a, b] \), which is indeed satisfied for all the \( \varpi(\alpha, \xi) \) mentioned below. The starting part \( h^\alpha \sum_{j=1}^{m} w_{n,j} f(jh) \) in (5) plays an essential role to improve the error accuracy of the formula for those \( f \) with initial singularity, and hence can be omitted provided \( f \) is smooth. For the calculation of the starting weights \( w_{n,j} \), we refer interested readers to [27, 44].

Some well studied CQ methods with \( \varpi(\alpha, \xi) \) satisfying (7) and (8) include (see [27, 46]),

- the generalized Newton-Gregory formulas, with \( \varrho(\alpha, \xi) \) in (7) simply replaced by 0,
- the fractional BDF-\( p \) which generalizes the traditional BDF-\( p \) to the fractional cases with generating function

\[
\varpi(\alpha, \xi) = \left[ \sum_{i=1}^{p} \frac{1}{i!} (1 - \xi)^i \right]^{-\alpha}, \quad 1 \leq p \leq 6,
\]

- the fractional trapezoidal rule (only for \( \alpha > 0 \))

\[
\varpi(\alpha, \xi) = \frac{(1 + \xi)^{\alpha}}{2^{\alpha}(1 - \xi)^{\alpha}},
\]

- the fractional BT-\( \theta \) method

\[
\varpi(\alpha, \xi) = \left[ \frac{1 - \theta + \theta \xi}{(3/2 - \theta) - (2 - 2\theta)\xi + (1/2 - \theta)\xi^2} \right]^{\alpha}, \quad \theta \in (-\infty, \frac{1}{2}),
\]
• the fractional BN-θ method
  \[ \varpi(\alpha, \xi) = \frac{1 - \alpha \theta + \alpha \theta \xi}{[(3/2 - \theta) - (2 - 2\theta)\xi + (1/2 - \theta)\xi^2]^{\alpha}}, \quad \theta \in (-\infty, 1], \quad \alpha \theta \leq \frac{1}{2}. \]  

We remark that the second-order generalized Newton-Gregory formula, the fractional BDF2 and the fractional trapezoidal rule are special cases of (11) and (12), see reference [46] for more information.

We end this subsection by introducing some key properties of the operator \( I^\alpha \) and \( \Omega_h^\alpha \) for the CQ method (see [27]), and extend them to the case of distributed order calculus in the next section.

(i) The homogeneity of \( I^\alpha \) and \( \Omega_h^\alpha \)

\[
\begin{align*}
I^\alpha f(x) &= x^\alpha (I^\alpha f(tx))(1), \\
\Omega_h^\alpha f(x) &= x^\alpha (\Omega_h^\alpha f(tx))(1).
\end{align*}
\]

(ii) The commutation with convolution

\[
\begin{align*}
I^\alpha (f * g) &= (I^\alpha f) * g, \\
\Omega_h^\alpha (f * g) &= (\Omega_h^\alpha f) * g.
\end{align*}
\]

(iii) Let \( E_h^\alpha = \Omega_h^\alpha - I^\alpha \). Then for \( \varpi(\alpha, \xi) \) defined by (7), we have

\[
(E_h^\alpha t^\ell)(1) = O(h^{\ell+1}) + O(h^p) \quad \text{for any } \ell \neq -1, -2, \cdots.
\]  

The convolution of \( f \) and \( g \) in (14) is denoted by

\[
(f * g)(x) = \int_0^x f(x-s)g(s)ds,
\]

and the commutation with convolution for the operator \( I^\alpha \) is satisfied for all \( \alpha \geq 0 \), and for those \( -k \leq \alpha < -k + 1 \) with \( f^{(j)}(0) = 0 \) (\( j = 0, \cdots, k - 1 \)).

2.3. Some lemmas. The Laplace transform for the Caputo fractional derivative is useful for the following analysis.

Lemma 2.1. (See [35]) Assume \( k - 1 < \beta \leq k \) for some positive integer \( k \), then the Laplace transform formula for \( C D^\beta f(x) \) is

\[
\mathcal{L}\{C D^\beta f(x)\}(s) := \int_0^\infty e^{-sx} C D^\beta f(x)dx = s^\beta \hat{f}(s) - \sum_{j=0}^{k-1} s^{\beta-j-1} f^{(j)}(0),
\]

where \( \hat{f}(s) \) denotes \( \mathcal{L}\{f(x)\}(s) \) for short.

Remark 1. Using Lemma 2.1, we can easily get the Laplace transform for the Caputo distributed order fractional derivative \( \int_0^1 \mu(\alpha)C D^\beta f(x) d\alpha \) by exchanging the Laplace operator \( \mathcal{L} \) with the integral operator.

In our numerical tests, we use two different numerical integration formulas to approximate the integration with respect to \( \alpha \) to make sure that the errors introduced will not affect the convergence rate of our scheme. Divide the interval \([a, b]\) by a uniform partition \( a = \alpha_0 < \alpha_1 < \cdots < \alpha_M = b \) and let \( \Delta \alpha = \frac{b-a}{M} \).
Lemma 2.2. (The composite trapezoidal formula, see [36])
If \( s(\alpha) \in C^2([a, b]) \), then
\[
\int_a^b s(\alpha) \, d\alpha = \Delta \alpha \left[ \frac{1}{2} s(\alpha_0) + s(\alpha_1) + \cdots + s(\alpha_{M-1}) + \frac{1}{2} s(\alpha_M) \right] - \frac{b-a}{12} \Delta \alpha^2 s''(\xi),
\]
where \( \xi \in (a, b) \) and \( \alpha_j = a + j \Delta \alpha \).

Lemma 2.3. (The composite Cavalieri-Simpson formula, see [36])
If \( s(\alpha) \in C^4([a, b]) \), then
\[
\int_a^b s(\alpha) \, d\alpha = \frac{\Delta \alpha}{6} \left[ s(\alpha_0) + 2 \sum_{j=1}^{M-1} s(\alpha_{2j}) + 4 \sum_{r=0}^{M-1} s(\alpha_{2r+1}) + s(\alpha_{2M}) \right] - \frac{b-a}{180} (\Delta \alpha/2)^4 s^{(4)}(\xi),
\]
where \( \xi \in (a, b) \) and \( \alpha_j = a + j \Delta \alpha/2 \).

3. Main results. In this article, we will approximate the operator \( I^{[\mu]} \) by a convolution part \( \Theta^{[\mu]}_h \) defined by
\[
\Theta^{[\mu]}_h f(x) = \sum_{j=0}^{n} \omega_j^{[\mu]}(h) f(jh), \quad x = nh,
\]
where the weights \( \omega_j^{[\mu]}(h) \) depending on the mesh size \( h := X/N \) are coefficients of the generating function
\[
\omega_j^{[\mu]}(\xi) = \sum_{j=0}^{\infty} \omega_j^{[\mu]}(h) \xi^j = \int_a^b \mu(\alpha) h^{\alpha} \omega(\alpha, \xi) \, d\alpha.
\]
Considering (7), we can obtain
\[
\omega_j^{[\mu]}(h) = \int_a^b \mu(\alpha) h^{\alpha} \omega_j(\alpha) \, d\alpha,
\]
and a direct property of \( \omega_j^{[\mu]}(h) \) is that
\[
\omega_j^{[\mu]}(hx) = \omega_j^{[\mu+1]}(h).
\]
We remark here for \( \mu(\alpha) \) taking \( \delta(\alpha - \alpha_0) \), the convolution part \( \Theta^{[\mu]}_h f(x) \) reduces to \( \Theta^\mu_n f(x) \) defined by (6). For the asymptotic property of \( \omega_j^{[\mu]}(h) \) with \( \mu(\alpha) \in L^1(a, b) \), we have the result
\[
\omega_j^{[\mu]}(h) = O(n^{-1}).
\]
Actually, for a \( \omega(\alpha, \xi) \) defined by (7) the coefficients \( \omega_n(\alpha) \) is of \( O(n^{\alpha-1}) \) (see the Definition 2.1 in [27]). Now with (22) we can easily get (24). Comparing with \( O(n^{\alpha-1}) \) of \( \omega_n(\alpha) \) that depends on \( \alpha \), the result (24) may be unsatisfactory for those applications where the boundedness of series \( \sum_{n=0}^{\infty} |\omega_n^{[\mu]}(h)| \) is required.

Define the error operator \( E^{[\mu]}_h \) such that
\[
E^{[\mu]}_h = \Theta^{[\mu]}_h - I^{[\mu]},
\]
then, as a generalization of the property (13) for \( I^{[\mu]} \), \( \Theta^{[\mu]}_h \) and \( E^{[\mu]}_h \), we have,
Lemma 3.1. The homogeneity of $I^\mu$ and $\Omega_h^\mu$ can be stated as,

$$I^\mu f(x) = (I^{\mu x^\alpha} f(tx))(1), \quad \Omega_h^\mu f(x) = (\Omega_{h/x}^{\mu x^\alpha} f(tx))(1), \quad x = hn.$$  \hspace{1cm} (26)

Hence, by (25) and for $f = t^\ell$ with $\ell \neq -1, -2, \ldots$, we have

$$\left( E_h^\mu t^\ell \right) (x) = x^\ell (E_{h/x}^{\mu x^\alpha} t^\ell)(1), \quad x = hn.$$  \hspace{1cm} (27)

**Proof.** Combining the definition of $(I^\mu f)(x)$ in (1) and the homogeneity of $I^\alpha$ in (13), we can easily get

$$I^\mu f(x) = \int_a^b \mu(\alpha) (I^\alpha f)(x) d\alpha = \int_a^b \mu(\alpha) x^\alpha (I^\alpha f(tx))(1) d\alpha$$

$$= (I^{\mu x^\alpha} f(x))(1),$$

and similarly, by (20), (22) and the homogeneity of $\Omega_h^\mu$ in (13), we have

$$\Omega_h^\mu f(x) = \int_a^b \mu(\alpha) h^\alpha \sum_{j=0}^n \alpha_j (x)(f(jh)) d\alpha = \int_a^b \mu(\alpha) (\Omega_h^\mu f(x)) d\alpha$$

$$= \int_a^b \mu(\alpha) x^\alpha (\Omega_{h/x}^{\mu x^\alpha} f(tx))(1) d\alpha = (\Omega_{h/x}^{\mu x^\alpha} f(tx))(1).$$

Now, (27) can be derived directly by (25) and (26) for $f = t^\ell$. The proof of the lemma is completed. \hfill \Box

Remark 2. The homogeneity property of the operators $I^\mu$, $\Omega_h^\mu$ and $E_h^\mu$ plays the role of simplifying the arguments in our article. With the relation (27), we can first analyse $(E_h^\mu t^\ell)(1)$, and then obtain the general case $(E_h^\mu t^\ell)(x)$ with a little extra work.

The main theorem of this paper can be stated as

**Theorem 3.2.** Let $\ell \neq -1, -2, \ldots$, and assume $\mu(\alpha) \in L^1(a,b)$. Then,

$$\left( E_h^\mu t^\ell \right) (x) = \left[ O(x^{-1} h^{\ell+1}) + O(x^{\ell-p} h^p) \right] \int_a^b \mu(\alpha) x^\alpha d\alpha, \quad x \in (0, X].$$  \hspace{1cm} (30)

If $\mu(\alpha) \in L^\infty(a,b)$ additionally, then

$$\left( E_h^\mu t^\ell \right) (x) = O \left( \frac{x^b-x^a}{x^{1-p}} h^{\ell+1} \right) + O \left( \frac{x^b-x^a}{x^{p-1} \log x} h^p \right), \quad x \in (0, X].$$  \hspace{1cm} (31)

**Proof.** By (15), (25) and the fact that $E_h^\mu = \Omega_h^\mu - I^\alpha$, we have

$$\left( E_h^\mu t^\ell \right) (1) = (\Omega_h^\mu t^\ell)(1) - (I^\mu t^\ell)(1) = \int_a^b \mu(\alpha) (E_h^\mu t^\ell)(1) d\alpha$$

$$= \int_a^b \mu(\alpha) (O(h^{\ell+1}) + O(h^p)) d\alpha.$$  \hspace{1cm} (32)

Replacing $\mu$ with $\mu x^\alpha$ and $h$ with $h/x$, then multiplying (32) by $x^\ell$, we can get by (27) that

$$\left( E_h^\mu t^\ell \right) (x) = x^\ell (E_{h/x}^{\mu x^\alpha} t^\ell)(1) = \int_a^b \mu(\alpha) x^\alpha (O(x^{-1} h^{\ell+1}) + O(x^{\ell-p} h^p)) d\alpha$$

$$= O(x^{-1} h^{\ell+1} + x^{\ell-p} h^p) \int_a^b \mu(\alpha) x^\alpha d\alpha.$$  \hspace{1cm} (33)
If we further assume $\mu(\alpha) \in L^\infty(a, b)$, then
\[
|\left( E_h^{[\mu]} t^\ell \right)(x) | \leq C(x^{-1}t^{\ell+1} + x^{\ell-p}h^p) \text{ess sup}_{\alpha \in (a,b)} \mu(\alpha) \int_a^b x^\alpha d\alpha \\
\leq C\left( \frac{x^b - x^a}{\log x} t^{\ell+1} + \frac{x^b - x^a}{x^{p-\ell} \log x} h^p \right).
\]  
(34)

The proof of the theorem is completed. \hfill \square

**Remark 3.** Now with the result (31) it is obvious for those $\ell - p + a < 0$, the optimal convergence rate $p$ may not be maintained uniformly, which is similar to the case for the fractional calculus. For the following distributed order time-fractional equation with suitable initial conditions
\[
\int_a^b \mu(\alpha) C D^\alpha u(t, x) d\alpha = \frac{\partial^2 u}{\partial x^2} + F(t, x), \quad (t, x) \in [0, T] \times \Omega,
\]  
(35)
or, equivalently,
\[
\int_{-a}^b \mu(-\alpha) I^\alpha u(t, x) d\alpha = \frac{\partial^2 u}{\partial x^2} + F(t, x),
\]  
(36)
researchers mainly studied the following cases: (i) the time-fractional diffusion equation with $a = 0, b = 1$, (ii) the time-fractional wave equation with $a = 1, b = 2$, (iii) the time-fractional diffusion-wave equation with $a = 0, b = 2$. Now by the error estimate (31) we obtain for the above three cases that (assume $\mu(\alpha) \in L^\infty(a, b)$)
\[
\text{Case (i)}: |E_h^{[\mu]} t^\ell| \leq C\epsilon_0 t^{-2} t^{\ell+1} + C\epsilon_0 t^{\ell-p-1} t^p,
\]  
(37)
\[
\text{Case (ii) and (iii)}: |E_h^{[\mu]} t^\ell| \leq C\epsilon_0 t^{-3} t^{\ell+1} + C\epsilon_0 t^{\ell-p-2} t^p,
\]  
(38)
where $\epsilon_0$ denotes $\min\{1/|\log t|, 1\}$ and $\tau$ is the time mesh size. $C$ is a constant free of $t$ and $\tau$. Few researchers have explored the case $a < b \leq 0$ for the equation (35). If we take $a = -1, b = 0$, the error estimate is
\[
|E_h^{[\mu]} t^\ell| \leq C\epsilon_0 t^{-1} t^{\ell+1} + C\epsilon_0 t^{\ell-p} t^p.
\]  
(39)
For all the estimates (37) and (38) with some small $\ell$ (i.e., the solution is of weak regularity), and taking $t = \tau$, the convergence order $p$ can not be maintained uniformly. See the first numerical experiment and Table 2 in section 4.

To restore high-order convergence rate, we consider in the following analysis adding some correction terms to $\Omega_h^{[\mu]}$ defined by (20). Denote by $I_h^{[\mu]} f(x)$ the distributed order convolution quadrature defined by
\[
I_h^{[\mu]} f(x) = \Omega_h^{[\mu]} f(x) + S_h^{[\mu]} f(x), \quad x = hn,
\]  
(39)
where the starting part $S_h^{[\mu]} f(x)$ is a linear combination of several terms of $f$,
\[
S_{h, m}^{[\mu]} f(x) = \sum_{j=1}^m w_{n,j}^{[\mu]}(h) f(jh), \quad x = hn,
\]  
(40)
and the starting weights $w_{n,j}^{[\mu]}(h)$ are derived by letting $I_h^{[\mu]}$ is exactly $I^{[\mu]}$ for the weak regular part of $f(x)$. For clarity, assume $f(x)$ can be expressed by
\[
f(x) = f_0(x) + f_1(x) + \cdots,
\]  
(41)
then we require that for $i = 0, 1, \cdots, m - 1$,

$$E_n^{[i]} f_i(x) + \sum_{j=1}^{m} w_{n,j}^{[i]}(h)f_i(jh) = 0, \quad x = hn,$$

(42)

which forms a linear system for $w_{n,j}^{[i]}(h), j = 1, 2, \cdots, m$. When $\mu(x) = \delta(\alpha - \alpha_0)$, if we define $w_{n,j}^{[i]}(h)$ satisfying

$$w_{n,j}^{[i]}(h) = h^\alpha w_{n,j},$$

(43)

then the condition (42) indeed reduces to the CQ case by which the weights $w_{n,j}$ in (5) are determined.

Considering most researchers assume the solution of an equation involving the distributed order fractional derivative is of the form (near the point $x = 0$),

$$f(x) = x^\ell g_1(x) + x^{\ell_2}g_2(x) + \cdots, \quad g_i(x) \text{ are sufficiently differentiable},$$

(44)

when conducting their numerical experiments (see [6, 11, 34]), we first explore the technique of adding starting part for solutions having the structure as (44), and for simplicity, we further assume

$$f(x) = x^\ell g(x), \quad \ell \neq -1, -2, \cdots, \text{ and } g(x) \text{ is sufficiently differentiable}.$$  

(45)

**Theorem 3.3.** Let $f(x)$ is defined by (45) and let the equation (42) holds for $f_i(x) = x^{\ell+i}$ ($i = 0, 1, \cdots, m - 1$). Then we can get

$$w_{n,j}^{[i]}(h) = O(n^{-1}) + O(n^{\ell+i-p}),$$

(46)

and the estimate

$$I_h^{[i]} f(x) - I_h^{[i]} f(x) = O(x^{\ell+m-p}h^p) \int_a^b \mu(\alpha)x^\alpha d\alpha.$$  

(47)

If further we assume $\mu(\alpha) \in L^\infty(a, b)$, then we have

$$I_h^{[i]} f(x) - I_h^{[i]} f(x) = O(x^{\ell+m-p}h^p) \frac{x^b - x^a}{\log x}.$$  

(48)

**Proof.** Replacing $f_i$ of (42) with $t^{\ell+i}$ and using the estimate (30), we have

$$\sum_{j=1}^{m} w_{n,j}^{[i]}(h)j^{\ell+i} = [O(n^{-1}) + O(n^{\ell+i-p})] \int_a^b \mu(\alpha)x^\alpha d\alpha.$$  

(49)

For $i = 0, 1, \cdots, m - 1$, (49) gives a Vandermonde type system of equations for the weights $w_{n,j}^{[i]}(h)$ ($j = 1, \cdots, m$). Noting that $\mu(\alpha) \in L^1(a, b)$ and $x^\alpha \in C([a, b])$, we have the asymptotic relation that

$$w_{n,j}^{[i]}(h) = O(n^{-1}) + O(n^{\ell+i-p}).$$  

(50)

Next, we expand $f$ as a fractional Taylor series with Bernoulli remainder term (see [27])

$$f(x) = \sum_{i=0}^{N} \frac{f^{(i+\ell)}(0)}{\Gamma(i + \ell + 1)} x^{\ell+i} + \frac{1}{\Gamma(N + \ell + 1)}(t^{N+\ell} + f(N+\ell+1))(x),$$  

(51)
where $f^{(s)}$ denotes $D^s f$ and assume $N > m$. Before showing the estimate (47), we prove the commutation property of $E_h^{[\alpha]}$ with convolution, i.e.,

$$E_h^{[\alpha]}(f * g) = (E_h^{[\alpha]} f) * g.$$  \hfill (52)

Actually, by (25), the relation (52) depends on the commutation of $I^{[\alpha]}$ and $\Omega_h^{[\alpha]}$ with convolution. By (14), we can get

$$I^{[\alpha]}(f * g)(x) = \int_a^b \mu(\alpha) I^{[\alpha]}(f * g)(x) d\alpha$$

$$\quad = \int_a^b \mu(\alpha) \int_0^x (I^\alpha f)(s) g(x - s) ds d\alpha$$

$$\quad = \int_0^x \left( \int_a^b \mu(\alpha) (I^\alpha f)(s) d\alpha \right) g(x - s) ds$$

$$\quad = \int_0^x (I^{[\alpha]} f)(s) g(x - s) ds = (I^{[\alpha]} f) * g(x).$$ \hfill (53)

Using the similar analysis, we can derive for $\Omega_h^{[\alpha]}$ that

$$\Omega_h^{[\alpha]}(f * g) = (\Omega_h^{[\alpha]} f) * g.$$ \hfill (54)

Now by (42), (30), (52) and the boundedness of $f^{(N+\ell+1)}$, we can get

$$I_h^{[\alpha]} f(x) - I^{[\alpha]} f(x)$$

$$= E_h^{[\alpha]} f(x) + \sum_{j=1}^m w_{n,j}^{[\alpha]}(h) f(jh)$$

$$= \sum_{i=m}^N \frac{f^{(\ell+i)}(0)}{\Gamma(i + \ell + 1)} E_h^{[\alpha]}(t^{\ell+i})(x) + \frac{1}{\Gamma(N + \ell + 1)} (E_h^{[\alpha]} f^{(N+\ell+1)})(x)$$

$$= \int_a^b \mu(\alpha) x^\alpha d\alpha \left( \sum_{i=m}^N O(x^{\ell+i-p}h^p) + O(x^{\ell+N-p+1}h^p) \right)$$

$$= O(x^{\ell+\alpha-p}h^p) \int_a^b \mu(\alpha) x^\alpha d\alpha.$$ \hfill (55)

If $\mu(\alpha) \in L^\infty(a,b)$, then we can get

$$I_h^{[\alpha]} f(x) - I^{[\alpha]} f(x) = O(x^{\ell+\alpha-p}h^p) \int_a^b x^\alpha d\alpha = O(\epsilon_0 x^{\alpha-p}h^p).$$ \hfill (56)

The proof of the theorem is completed. \hfill \Box

**Remark 4.** Theorem 3.3 shows that by adding starting part $S_{h,m}^{[\alpha]} f$ to $\Omega_h^{[\alpha]} f$, we can improve the error accuracy. There are two cases deserving our attention:

**Case I.** $p - \ell - 1 < m \leq p - \ell$ or $m = p - \ell - \theta$ for $\theta \in [0,1)$

We have $i \leq m - 1 \leq p - 1 - \ell$, then by (46), it holds that $w_{n,m}^{[\alpha]}(h) = O(n^{-1})$ which is the same as $\omega_{n,m}^{[\alpha]}(h)$, see (24). For the error (48), we have

$$I_h^{[\alpha]} f(x) - I^{[\alpha]} f(x) = O(\epsilon_0 x^{\alpha-p}h^p),$$ \hfill (57)

where $\epsilon_0 = \min\{1/|\log x|, 1\}$.

**Case II.** $p - \ell - a \leq m < p - \ell - a + 1$
In this case, we have $w_{n,j}^{[\mu]}(h) = O(n^{-1}) + O(n^{-a})$, and the error estimate is

$$I_h^{[\mu]} f(x) - I^{[\mu]} f(x) = O(\epsilon_0 h^p). \quad (58)$$

Similar to the fractional calculus (see the subsection 4.2 of [27]), we can only take case II to devise the starting part on short intervals $[0, X]$, as for $a < 1$ we have $w_{n,j}^{[\mu]}(h) = O(n^{-a})$ which dominates $\omega_n^{[\mu]}$ for large $n$.

From the above analysis, we see that the assumption of $f(x)$ defined by (44) is a key ingredient for our error estimate. Indeed, for the spacial case $\mu(\alpha) = \delta(\alpha - \alpha_0)$, the equation $I^{[\mu]} f(x) = \lambda f(x)$ can be analytically solved by using the Mittag-Leffler function with the following solution structure

$$f(x) = c_1 x^{\ell_1} + c_2 x^{\ell_2} + c_3 x^{\ell_3} + \cdots \quad (59)$$

where $\ell_i$ can be expressed by $\ell_i(\alpha_0)$ and is known in advance. In the rest of the section, we derive the asymptotic expression of the solution at node $x = 0$ for the simplest equation [17],

$$\int_0^1 \mu(\alpha) C \alpha f(x) \, d\alpha = \lambda f(x), \quad \lambda < 0, \ \mu(\alpha) \equiv 1, \quad (60)$$

with the initial condition $f_0 = f(0) = 1$, show the singularity of the solution at $x = 0$, and then propose a method to try to restore the high-order error accuracy. We remark that by (4) the equation (60) can be rewritten into the following form

$$\int_{-1}^0 I^\alpha (f - f_0)(x) \, d\alpha = \lambda f(x), \quad \lambda < 0. \quad (61)$$

As one can see, if $\mu(\alpha)$ in (60) takes $\delta(\alpha - 1)$, then we can easily get $f(x) = e^{\lambda x}$, and if $\mu(\alpha)$ takes $\delta(\alpha - \alpha_0)$, we can still express the solution by the Mittag-Leffler function as $f(x) = E_{\alpha_0}(\lambda x^{\alpha_0})$, where generally $E_{\alpha}(x)$ is defined by

$$E_{\alpha}(x) := \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(j \alpha + 1)}, \text{ for those } x \text{ that make the series converge.} \quad (62)$$

The next lemma shows the singularity of the solution for (60), which is quite different from the solutions of single- or multi-term fractional differential equations. We leave the proof and specific information about the constants involved in the lemma to the Appendix.

**Lemma 3.4.** The solution of (60) can be expressed by the following form in the neighbourhood of origin

$$f(x) = 1 + x(c_{10} \log x + c_{11}) + x^2(c_{20} \log^2 x + c_{21} \log x + c_{22})$$

$$+ x^3(c_{30} \log^3 x + c_{31} \log^2 x + c_{32} \log x + c_{33}) + o(x^3), \quad (63)$$

where constants $c_{kj}$ only depend on $\lambda$.

**Remark 5.** For the case $\mu(\alpha) = \delta(\alpha - \alpha_0)$, the solution $f(x)$ can be expressed by the series

$$f(x) = 1 + \frac{\lambda}{\Gamma(\alpha_0 + 1)} x^{\alpha_0} + \frac{\lambda^2}{\Gamma(2\alpha_0 + 1)} x^{2\alpha_0} + \cdots \quad (64)$$
Hence, to obtain a high-order convergence rate, Lubich [27] added starting part to remove the effect of the weak regular terms such as \( x^{\alpha_0}, x^{2\alpha_0}, \ldots \). For the solution (63), we find quite different weak regular terms compared with (64) that are

\[
x \log x, \ x^2 \log^2 x, \ x^2 \log x, \ x^3 \log^3 x, \ x^3 \log^2 x, \ x^3 \log x \cdots,
\]

which reminds us to design new starting parts with aspect to the above terms (see the second numerical tests and Table 3 in section 4). Specifically, we let \( I_h[\mu]g(x) = I^{[\nu]}_h g(x) \) for \( g(x) \) taking the weak regular terms (65). We can actually expand the solution \( f(x) \) by a combination of any finite many terms \( x^k \log^j x \) as the proof of the lemma shows, i.e., in the form

\[
f(x) \approx f_n(x) := 1 + \sum_{k=1}^{n} \sum_{j=0}^{k} c_{kj} x^k \log^{k-j} x,
\]

which is sufficient for a numerical method with a prescribed convergence rate to restore the desired error accuracy by correcting a few of the weak regular terms.

In Fig.1, we depict the expansions \( f_n(x) \) for \( n = 1, 2, 5 \) and the numerical solution obtained by the D-Euler method (see section 4) with sufficiently small mesh size \( h \). One can see that \( f_5(x) \) already coincides with the numerical solution very well in our selected interval \( \Omega = (0, 1) \).

![Figure 1. Figures of \( f_n(x) \) and the numerical solution](image)

4. Applications. In this section, we examine the efficiency of our numerical formula (20) by choosing different generating functions (21). To improve the error accuracy for those weak regular solutions we add correction terms as stated in the formula (39). For clarity, we call a method “D-METHOD” by constructing the generating function \( \omega^h_\mu(\xi) \) based on \( \varpi(\alpha, \xi) \) of the “METHOD” that is listed in the Preliminaries section. For example, “D-Euler” (which actually means “D-BDF1”) is the method with \( \varpi(\alpha, \xi) = (1 - \xi)^\alpha \) taken from the fractional BDF1, after replacing \( \alpha \) by \(-\alpha\), as both of the numerical tests in the following focus on differential equations.
4.1. Distributed order fractional diffusion equations. We apply the novel approximation formulas (20) to the equation (35) for the case(i) discussed in Remark 3. Let \( \Omega = (0, 1) \) and \( T = 1 \). The initial-boundary condition is \( u(t, x) = 0, (t, x) \in [0, T] \times \partial \Omega \) and \( u(0, x) = 0, x \in \Omega \). Take the exact solution as

\[
 u(t, x) = (t^\ell + t^4) \sin 2\pi x, \quad \ell > 0,
\]

then by letting \( \mu(\alpha) = \Gamma(5 - \alpha)\Gamma(\ell + 1 - \alpha) \), the source terms can be derived as

\[
 F(t, x) = \sin 2\pi x [\mathcal{M}(\ell, t) + 4\pi^2 (t^\ell + t^4)],
\]

where

\[
 \mathcal{M}(\ell, t) = \int_0^1 t^{\ell-\alpha}\Gamma(1 + \ell)\Gamma(5 - \alpha) + 24t^{\ell-\alpha}\Gamma(\ell - \alpha + 1)\,d\alpha.
\]

To derive the fully discrete scheme of (35), we introduce the following subspace of \( H_0^1(\Omega) \)

\[
 X^\ell_n = \{ \chi \in H_0^1(\Omega) : \chi|_e \in \mathbb{P}_r(z), e \in \mathcal{T}_h \},
\]

where \( \mathcal{T}_h = \{ e_i : e_i = [x_{i-1}, x_i], 1 \leq i \leq N_S \} \) is the uniform partition with the mesh size \( h = x_i - x_{i-1} = 1/N_S \) for a positive integer \( N_S \), and \( \mathbb{P}_r(z) \) is the set of polynomials of degree less than or equal to \( r \) \( (r \in \mathbb{Z}^+) \). Divide the temporal interval such that \( 0 = t_0 < t_1 < \cdots < t_{N_T} = 1 \) and \( \tau = t_i - t_{i-1} = 1/N_T \). Then by the relation (4), the fully discrete scheme is to find \( U^n : [0, T] \rightarrow X^\ell_n \) such that

\[
 \sum_{j=1}^{n} \omega^{[\mu]}_{n-j}(\tau)(U^j, \chi_h) + w^{[\mu]}_{n}(\tau)(U^1, \chi_h) + (\nabla U^n, \nabla \chi_h) = (F^n, \chi_h),
\]

holds for any \( \chi_h \in X^\ell_n \), where \( U^j \) denotes the numerical solution and \( F^n : = F(t_n, x) \). The symbol \( (\cdot, \cdot) \) is the inner product of the \( L^2(\Omega) \) space, and define \( \| \cdot \| = \sqrt{(\cdot, \cdot)} \).

According to the result (48) of Theorem 3.3 and the Case II of Remark 4, \( w^{[\mu]}_{n} \) are derived by letting

\[
 \sum_{j=1}^{n} \omega^{[\mu]}_{n-j}(\tau)t^j + w^{[\mu]}_{n}(\tau)t^1 = \int_0^1 \mu(\alpha)(D^\alpha t^\ell)(t_n)\,d\alpha, \quad n = 1, 2, \ldots ,
\]

provided \( \ell < p + 1 \), with \( p \) denoting the convergence rate for a given numerical scheme. If \( \ell \geq p + 1 \), which means the solution is sufficiently smooth for the scheme, we omit the starting parts or correction terms \( w^{[\mu]}_{n}(\tau)(U^1, \chi_h) \) from (71).

Considering the calculation of convolution weights (22) and starting weights (72) as well as the source term \( F(t, x) \) (68) may depend on the integration of the order \( \alpha \), we use in Table 1 two different numerical integration formulas that are known as the composite trapezoidal formula with the error accuracy of \( O(\Delta \alpha^2) \) and the composite Cavalieri-Simpson formula with the error accuracy of \( O(\Delta \alpha^4) \), see Lemma 2.2 and Lemma 2.3. The experiment shows that for schemes of convergence rate up to 3, both of the integration formulas can produce almost the same results. Hence, for Table 2 to Table 4, we will take the composite trapezoidal formula for simplicity.

In Table 1, we examine the convergence rates for sufficiently smooth solutions \( (\ell = 4) \) by using different numerical methods. We denote by \( E_T (\tau, h, \Delta \alpha) = \max_{1 \leq j \leq N_T} ||U^j - u(t_j, x)|| \) obtained when using the composite trapezoidal formula, and by \( E_S (\tau, h, \Delta \alpha) = \max_{1 \leq j \leq N_T} ||U^j - u(t_j, x)|| \) obtained when using the composite Cavalieri-Simpson formula. By taking the space mesh size \( h \) sufficiently small that \( h = 1/2000 \), we vary the time mesh size and let \( \Delta \alpha = \tau \).
The numerical results show that for smooth solutions we can get optimal temporal convergence rates. An interesting thing is that even for the D-BDF3, the composite trapezoidal formula still results in relatively accurate errors.

In Table 2, we choose \( \ell \) such that \( \ell < p + 1 \) where \( p \) is the convergence rate for corresponding methods. For these \( \ell \) the solution (67) is of weak regularity at the initial time. We define \( E_U(\tau, h, \Delta \alpha) = \max_{1 \leq j < N_T} \| U_j - u(t_j, x) \| \) as the error obtained without the correction terms, and define \( E_C(\tau, h, \Delta \alpha) = \max_{1 \leq j < N_T} \| U_j - u(t_j, x) \| \) as the error obtained by adding correction terms. As one can see, for a solution with a relatively strong singularity (\( \ell = 0.1 \)), the rate of the D-Euler method is even negative. However, with correction terms all the methods tested show the optimal convergence rates, which confirms the efficiency of the correction technique.

### Table 1. Temporal convergence rates for smooth solutions, \( h = \frac{1}{2000} \)

| Methods | \( \tau \) | \( E_T \) | Rate | CPU(s) | \( E_S \) | Rate | CPU(s) |
|---------|----------|-------|------|-------|-------|------|-------|
| D-Euler | 1/10     | 9.03E-02 | -    | 0.401 | 9.10E-02 | -    | 0.441 |
|         | 1/20     | 4.59E-02 | 0.98 | 0.513 | 4.60E-02 | 0.98 | 0.453 |
|         | 1/40     | 2.31E-02 | 0.99 | 0.554 | 2.31E-02 | 0.99 | 0.492 |
|         | 1/80     | 1.16E-02 | 1.00 | 0.688 | 1.16E-02 | 1.00 | 0.617 |
| D-BDF2  | 1/10     | 1.48E-02 | -    | 0.441 | 1.49E-02 | -    | 0.426 |
|         | 1/20     | 3.98E-03 | 1.89 | 0.453 | 3.98E-03 | 1.90 | 0.435 |
|         | 1/40     | 1.03E-03 | 1.95 | 0.474 | 1.03E-03 | 1.95 | 0.468 |
|         | 1/80     | 2.62E-04 | 1.98 | 0.579 | 2.62E-04 | 1.98 | 0.607 |
| D-BT-\( \theta \) (\( \theta = 0.45 \)) | 1/10     | 5.36E-03 | -    | 0.405 | 5.41E-03 | -    | 0.452 |
|         | 1/20     | 1.36E-03 | 1.97 | 0.454 | 1.37E-03 | 1.98 | 0.449 |
|         | 1/40     | 3.44E-04 | 1.99 | 0.516 | 3.44E-04 | 1.99 | 0.469 |
|         | 1/80     | 8.61E-05 | 2.00 | 0.604 | 8.61E-05 | 2.00 | 0.598 |
| D-BN-\( \theta \) (\( \theta = 1 \)) | 1/10     | 2.63E-02 | -    | 0.413 | 2.65E-02 | -    | 0.438 |
|         | 1/20     | 7.17E-03 | 1.88 | 0.453 | 7.18E-03 | 1.88 | 0.437 |
|         | 1/40     | 1.87E-03 | 1.94 | 0.490 | 1.87E-03 | 1.94 | 0.494 |
|         | 1/80     | 4.76E-04 | 1.97 | 0.623 | 4.76E-04 | 1.97 | 0.583 |
| D-BDF3  | 1/10     | 2.00E-03 | -    | 0.473 | 2.03E-03 | -    | 0.407 |
|         | 1/20     | 2.65E-04 | 2.92 | 0.475 | 2.66E-04 | 2.93 | 0.440 |
|         | 1/40     | 3.39E-05 | 2.97 | 0.496 | 3.39E-05 | 2.97 | 0.496 |
|         | 1/80     | 4.20E-06 | 3.01 | 0.611 | 4.20E-06 | 3.01 | 0.614 |

4.2. **Distributed order fractional ODEs.** For the distributed order fractional ODEs, we mainly explore the numerical methods for the equation

\[
\int_0^1 C^{D^n} f(x) \, d\alpha = -f(x) + F(x), \quad \Omega = (0, 1),
\]

with the exact solution taken as

\[
f(x) = 1 + x \log x - (1 - \gamma)x - \frac{1}{5} x^3, \quad \gamma \text{ is defined by (90)}. \]

The first three terms of \( f(x) \) are exactly taken from the expansion of the solution of (60) by Lemma 3.4. The term \(-\frac{1}{5} x^3\) is chosen such that \( f(x) \) is monotone decreasing.
Table 2. Comparison of rates for methods with and without correction terms, $h = \frac{1}{2000}$

| Methods  | $\tau$ | $E_U$       | Rate | CPU(s) | $E_C$       | Rate | CPU(s) |
|----------|--------|-------------|------|--------|-------------|------|--------|
| D-Euler  | 1/10   | 2.66E-01    | –    | 0.441  | 5.14E-02    | –    | 0.435  |
|          | 1/20   | 2.90E-01    | –    | 0.426  | 2.55E-02    | 1.01 | 0.461  |
|          | 1/40   | 3.05E-01    | –    | 0.480  | 1.28E-02    | 1.00 | 0.496  |
|          | 1/80   | 3.10E-01    | –    | 0.611  | 6.40E-03    | 1.00 | 0.611  |
| D-BDF2   | 1/20   | 5.54E-02    | –    | 0.442  | 4.20E-03    | –    | 0.435  |
|          | 1/40   | 4.97E-02    | 0.16 | 0.426  | 2.55E-02    | 1.01 | 0.461  |
|          | 1/80   | 4.23E-02    | 0.23 | 0.461  | 1.28E-02    | 1.00 | 0.496  |
| D-BT-$\theta$ | 1/20 | 1.55E-01 | –    | 0.450  | 1.87E-02    | –    | 0.442  |
|          | 1/40   | 1.50E-01    | 0.05 | 0.443  | 1.11E-02    | 1.92 | 0.481  |
|          | 1/80   | 1.38E-01    | 0.11 | 0.461  | 2.85E-04    | 1.96 | 0.481  |
| D-BN-$\theta$ | 1/20 | 1.27E-02 | –    | 0.448  | 6.40E-03    | –    | 0.435  |
|          | 1/40   | 9.47E-03    | 0.33 | 0.467  | 1.82E-03    | 2.00 | 0.461  |
|          | 1/80   | 6.69E-03    | 0.50 | 0.619  | 4.70E-04    | 2.02 | 0.461  |

on $[0, 1]$. Careful calculation shows that the source term is

$$F(x) = \int_{0}^{1} \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} \left[ \log x + \frac{d}{d\alpha} \log \Gamma(2-\alpha) \right] - \frac{6x^{3-\alpha}}{5\Gamma(4-\alpha)} d\alpha$$

$$+ 1 + x \log x - (1 - \gamma)x - \frac{1}{5} x^3.$$  \hspace{1cm} (75)

We divide the space by the partition $0 = x_0 < x_1 < \cdots < x_N = 1$ with the mesh size $h = 1/N$ for a positive integer $N$. To deal with the weak regularity of the term $x \log x$ and $x$ in the solution, we add new correction terms as discussed in Remark 5 that are different from those designed for solutions of the form (67).

In Table 3, we denote by $E_U(h, \Delta \alpha) = \max_{1 \leq j \leq N} |f^j - f(x_j)|$ the errors without correction terms where $f^j$ denotes the numerical solution at $x_j$, and by $E_C(h, \Delta \alpha) = \max_{1 \leq j \leq N} |f^j - f(x_j)|$ the errors obtained by using the new correction technique. Since the singularity of $x \log x$ and $x$ is not so strong, we can see that the convergence rates for the D-Euler method without correction terms are approximately 1. Nonetheless, all of the three second-order numerical methods without correction terms show a much lower error accuracy, whose convergence rates are only about 0.8. With the new correction technique, four numerical methods indeed recover the optimal convergence rate, and the computing time is almost the same as that without correction terms. In Fig. 2 we depict the errors at each node $x_j$ for the D-BDF2 with the new correction technique, and in Fig. 3 we also depict the errors without adding correction terms. One can see that for methods without correction terms, the error at the first level dominates the final error we record, which greatly affected the convergence rate that we can get.

In Table 4, we further explore the efficiency of traditional correction technique used in Table 2 for equation (73) with the solution (74). Specifically, we correct the
The traditional correction technique by letting $I^{|\mu|}\phi(x) = I^{|\mu|}_h\phi(x)$ for power functions $\phi(x)$ such as (76) is easy to implement and turns out to be efficient even if $\phi(x)$ is not part of the solution. For the case $\mu(\alpha) = \delta(\alpha - \alpha_0)$, by guessing some correction terms that may not be the parts of the solution, the error accuracy can be dramatically improved and this phenomenon has been proved by Zeng et al. [47]. As one can see, the term $x \log x$ is of weak singularity since its derivative blows up at $x = 0$. Instead of correcting the term itself, we correct the terms in (76). With $E_C$ and $E_U$ denote respectively the errors obtained with and without correction terms, we find that the error accuracy has been improved by the traditional correction technique without consuming too much extra computing time.

Table 3. Temporal convergence rates for methods with and without correction terms.

| Methods   | $h$    | $E_U$    | Rate | CPU(s) | $E_C$    | Rate | CPU(s) |
|-----------|--------|----------|------|--------|----------|------|--------|
| D-Euler   | 1/20   | 1.82E-02 | –    | 1.516  | 5.18E-03 | –    | 1.579  |
|           | 1/40   | 9.85E-03 | 0.89 | 3.526  | 2.64E-03 | 0.97 | 3.266  |
|           | 1/80   | 5.23E-03 | 0.91 | 7.992  | 1.33E-03 | 0.99 | 8.236  |
|           | 1/160  | 2.74E-03 | 0.93 | 33.652 | 6.65E-04 | 1.00 | 34.729 |
| D-BDF2    | 1/20   | 5.32E-02 | –    | 1.525  | 2.79E-04 | –    | 1.507  |
|           | 1/40   | 3.28E-02 | 0.70 | 3.285  | 7.09E-05 | 1.97 | 3.203  |
|           | 1/80   | 1.96E-02 | 0.75 | 7.897  | 1.79E-05 | 1.99 | 8.142  |
|           | 1/160  | 1.19E-02 | 0.72 | 34.870 | 4.49E-06 | 1.99 | 32.841 |
| D-BT-$\theta$ ($\theta=0.2$) | 1/20 | 5.94E-02 | –    | 1.468  | 1.95E-04 | –    | 1.523  |
|           | 1/40   | 3.68E-02 | 0.69 | 3.254  | 4.97E-05 | 1.97 | 3.308  |
|           | 1/80   | 2.20E-02 | 0.74 | 7.876  | 1.25E-05 | 1.99 | 8.160  |
|           | 1/160  | 1.28E-02 | 0.78 | 34.141 | 3.14E-06 | 1.99 | 34.439 |
| D-BN-$\theta$ ($\theta=0.5$) | 1/20 | 5.44E-02 | –    | 1.509  | 2.46E-04 | –    | 1.517  |
|           | 1/40   | 3.35E-02 | 0.70 | 3.281  | 6.24E-05 | 1.98 | 3.479  |
|           | 1/80   | 1.99E-02 | 0.75 | 8.090  | 1.57E-05 | 1.99 | 7.998  |
|           | 1/160  | 1.20E-02 | 0.73 | 35.025 | 3.95E-06 | 1.99 | 34.751 |
Table 4. Temporal convergence rates for methods with and without correction terms.

| Methods  | $h$   | $E_U$                         | CPU(s) | $E_C$                         | CPU(s) |
|----------|-------|-------------------------------|--------|-------------------------------|--------|
| D-Euler  | 1/20  | 1.8231582E-02                 | 1.010  | 1.40274270E-02               | 1.054  |
|          | 1/40  | 9.8461370E-03                 | 2.195  | 2.16534296E-03               | 2.278  |
|          | 1/80  | 5.2308982E-03                 | 5.646  | 1.18484214E-03               | 5.556  |
|          | 1/160 | 2.7366136E-03                 | 22.973 | 6.24937949E-04               | 23.143 |
| D-BDF2   | 1/20  | 5.3244469E-02                 | 0.982  | 1.40274270E-02               | 0.998  |
|          | 1/40  | 3.2848868E-02                 | 2.202  | 2.16534296E-03               | 2.233  |
|          | 1/80  | 2.1674708E-02                 | 5.452  | 2.69407811E-04               | 5.554  |
|          | 1/160 | 1.1896171E-02                 | 23.290 | 4.6232965E-06                | 23.297 |
| D-BT-$\theta$ ($\theta=0.2$) | 1/20 | 5.9461511E-02                 | 0.971  | 1.40274270E-02               | 0.998  |
|          | 1/40  | 3.6795394E-02                 | 2.226  | 2.16534295E-03               | 2.187  |
|          | 1/80  | 2.1969727E-02                 | 5.443  | 2.69407811E-04               | 5.579  |
|          | 1/160 | 1.3222071E-02                 | 23.077 | 4.62329657E-06               | 23.388 |
| D-BN-$\theta$ ($\theta=0.5$) | 1/20 | 5.4412922E-02                 | 0.959  | 1.40274270E-02               | 1.013  |
|          | 1/40  | 3.3572144E-02                 | 2.210  | 2.16534296E-03               | 2.199  |
|          | 1/80  | 2.1902404E-02                 | 5.520  | 2.69407811E-04               | 5.530  |
|          | 1/160 | 1.2015071E-02                 | 23.077 | 4.62329657E-06               | 23.571 |

5. Conclusion. In this paper, we generalize the convolution quadrature to approximate the distributed order calculus by constructing new generating functions based on those used in the CQ method. The convergence result and correction technique of the CQ method have now been extended such that we can recover them in our methods by simply assuming $\mu(\alpha) = \delta(\alpha - \alpha_0)$. However, the distributed order calculus shows some unique properties that are quite different from the single- or multi-term fractional calculus, as is proved in Lemma 3.4. See also [22]. Based on the new structure of solutions of distributed order fractional ODEs, we design a new correction technique and numerically show that the new technique can restore the optimal convergence rate. Further, we examine the efficiency of traditional correction technique when applied to new structure solutions. Numerical results confirm that the traditional correction technique is effective when solving distributed order fractional equations.

As we have pointed out below (24), the asymptotic properties of the weights $\omega_j^{[\mu]}(h)$, as well as other properties of our methods, differ from those of the CQ methods, raising new challenges for the numerical analysis for PDEs with distributed order calculus. To derive the stability and error estimates for distributed order fractional PDEs is our future work.

Appendix. Before proving the Lemma 3.4, we first state three lemmas that are needed in the following arguments. For simplicity, we use $f(x) \sim g(x)$ to denote $\lim_{x \to 0} \frac{f(x)}{g(x)} = 1$.

**Lemma 5.1.** (See [17]) If $\mu(\alpha) \in C^2([0, 1])$ and $\mu(1) \neq 0$, then the solution $f(x)$ of the equation (60) is continuous at the origin $x = 0$ and belongs to $C^\infty(0, X)$.

**Lemma 5.2.** (The Karamata’s Abel-Tauber theorem, see [9]) Let $U$ be a measure and define $U(s)$ such that

$$
\hat{U}(s) = \int_0^\infty e^{-sx}U(x)dx = s \int_0^\infty e^{-sx}U(x)dx,
$$

(77)
defined for $s > 0$. Assume $\tau t = 1$. Then each of the relations
\[
\frac{\hat{U}(ts)}{U(t)} \to \frac{1}{s^{\rho}}, \quad t \to \infty
\]  
(78)
and
\[
\frac{U(\tau x)}{U(\tau)} \to x^{\rho}, \quad \tau \to 0
\]  
(79)
implies the other as well as
\[
U(x) \sim \frac{\hat{U}(1/x)}{\Gamma(\rho + 1)}, \quad x \to 0.
\]  
(80)

Lemma 5.3. If $\lim_{x \to 0} \frac{h(x)}{g(x)} \neq 1$, then $h(x) - f(x) \sim g(x)$ means $f(x) \sim h(x) - g(x)$.

Proof. By $h(x) - f(x) \sim g(x)$, we have the relation that
\[
f(x) = h(x) - g(x) + o(g(x)).
\]  
(81)
Hence, with the condition $\lim_{x \to 0} \frac{h(x)}{g(x)} \neq 1$, we can obtain
\[
\frac{f(x)}{h(x) - g(x)} = 1 + \frac{o(g(x))}{h(x) - g(x)} = 1 + \frac{\frac{o(1)}{h(x) - g(x) - 1}}{\frac{h(x)}{g(x)} - 1} \to 1, \quad x \to 0.
\]  
(82)
The proof of the lemma is completed. \qed

Proof of Lemma 3.4. With the assumption $\mu(\alpha) \equiv 1$ in Lemma 3.4 and Lemma 5.1, we have the relation that
\[
f(x) = f(0) = 1, \quad x \to 0.
\]  
(83)
Taking the Laplace transform for the equation (60), we have
\[
\hat{f}(s) = \frac{\kappa(s)}{sk(s) - \lambda},
\]  
(84)
where $\kappa(s) = \int_0^1 s^{\alpha-1} \mu(\alpha) d\alpha = (1 - \frac{1}{\lambda})/\log s$. Define $f_1(x) := 1 - f(x)$. We assume that there exists a $\delta_1 > 0$ such that $f_1(x)$ is monotone in $(0, \delta_1)$. If $f_1(x)$ is non-increasing, we simply replace $f_1(x)$ by $f(x) - 1$. Then actually we have defined a measure $f_1(x)$. Now, replacing $U(x)$ of (77) with $f_1(x)$, we can get
\[
\frac{\hat{f}_1(s)}{s} = \int_0^\infty e^{-sx} (1 - f(x)) dx = \frac{1}{s} - \hat{f}(s).
\]  
(85)
Combining (84) and (85), we obtain
\[
\hat{f}_1(s) = \frac{\lambda - \frac{1}{\log s}}{\lambda - \frac{s-1}{\log s}}
\]  
(86)
Considering (78), we derive that
\[
\frac{\hat{f}_1(ts)}{f_1(t)} = \frac{\lambda - \frac{t-1}{\log t}}{\lambda - \frac{t-1}{\log ts}} \to \frac{1}{s}, \quad t \to \infty.
\]  
(87)
By (80) and (86) we can get the following relation that
\[
1 - f(x) = f_1(x) \sim \frac{\hat{f}_1(1/x)}{\Gamma(2)} \sim \lambda x \log x.
\]  
(88)
Define $f_2(x) := 1 - \lambda x \log x - f(x)$, one can check that $f_2(x) \to 0$ as $x \to 0$. Similarly, we assume $f_2(x)$ is monotone in $(0, \delta_2)$ for some $\delta_2 > 0$. By replacing $U(x)$ of (77) with $f_2(x)$, we can get

$$\frac{\hat{f}_2(s)}{s} = \frac{\hat{f}_1(s)}{s} - \frac{\lambda}{s^2} (1 - \gamma - \log s),$$

(89)

where $\gamma = 0.57721 \cdots$ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

(90)

Combining (89) with (86), we have

$$\hat{f}_2(s) = \frac{\lambda(\gamma - 1 + \log s)}{s} + \frac{\lambda \log s}{\lambda \log s - s + 1}.$$  

(91)

Considering (78), we can easily get

$$\frac{\hat{f}_2(ts)}{f_2(t)} = \frac{1}{s}, \quad t \to \infty.$$  

(92)

Now by (80) and (91), we have derived that

$$1 - \lambda x \log x - f(x) = f_2(x) \sim \frac{\hat{f}_2(1/x)}{\Gamma(2)} \sim \lambda(\gamma - 1)x.$$  

(93)

Define $f_3(x) := 1 - \lambda x \log x - \lambda(\gamma - 1)x - f(x)$, and repeat the procedures discussed above, we can get

$$1 - \lambda x \log x - \lambda(\gamma - 1)x - f(x) = f_3(x) \sim -\frac{\lambda^2}{2} x^2 \log^2 x.$$  

(94)

And, similarly, by careful calculations we can prove that

$$1 + x(c_{10} \log x + c_{11}) + x^2(c_{20} \log^2 x + c_{21} \log x + c_{22}) + x^3(c_{30} \log^3 x + c_{31} \log^2 x + c_{32} \log x + c_{33}) - f(x) \sim -\frac{\lambda^4}{24} x^4 \log^4 x.$$  

(95)

By Lemma 5.3, (95) gives the desired result of the lemma. The coefficients $c_i$’s are defined as follows

$$c_{10} = -\lambda,$$

$$c_{11} = -\lambda(\gamma - 1),$$

$$c_{20} = \frac{\lambda^2}{2},$$

$$c_{21} = \frac{\lambda}{2}(2\lambda \gamma - 3\lambda - 1),$$

$$c_{22} = -\frac{\lambda}{12} \left[ (-6\gamma^2 + \pi^2 + 18\gamma - 21)\lambda + 6\gamma - 9 \right],$$

$$c_{30} = -\frac{\lambda^3}{6},$$

$$c_{31} = \frac{\lambda^3}{6}(2\lambda \gamma - 3\lambda - 1),$$

$$c_{32} = -\frac{\lambda^3}{12} (6\gamma - 9),$$

$$c_{33} = -\frac{\lambda^3}{72} (6\gamma - 9).$$
\[ c_{31} = -\frac{\lambda^2}{12} (6\lambda \gamma - 11\lambda - 4), \]
\[ c_{32} = \frac{\lambda}{12} \left[ -2 + \left( \pi^2 - 6\gamma^2 + 22\gamma - \frac{85}{3} \right) \lambda^2 + \left( 8\gamma - \frac{44}{3} \right) \lambda \right], \]
\[ c_{33} = \frac{\lambda}{12} \left[ -4\zeta(3)\lambda^2 + \left[ -2\gamma^3 + 11\gamma^2 + \left( \pi^2 - \frac{85}{3} \right) \gamma + \frac{575}{18} - \frac{11}{6} \pi^2 \right] \lambda^2 \right. \]
\[ + \left. \left( -\frac{2}{3} \pi^2 + 4\gamma^2 - \frac{44}{3} \gamma + \frac{170}{9} \right) \lambda - 2\gamma + \frac{11}{3} \right] \lambda. \]  

(96)

\[ \zeta(3) \] in \( c_{33} \) is the value of the Riemann zeta function \( \zeta(s) \) defined by
\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \]
and \( \zeta(3) = 1.20205 \cdots \).

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