THE FULL AUTOMORPHISM GROUP OF $\overline{T}$

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Abstract. Let $G$ be the wonderful compactification of a simple affine algebraic group $G$ of adjoint type defined over $\mathbb{C}$. Let $T \subset G$ be the closure of a maximal torus $T \subset G$. We prove that the group of all automorphisms of the variety $\overline{T}$ is the semi-direct product $N_G(T) \rtimes D$, where $N_G(T)$ is the normalizer of $T$ in $G$ and $D$ is the group of all automorphisms of the Dynkin diagram, if $G \neq \text{PSL}(2, \mathbb{C})$. Note that if $G = \text{PSL}(2, \mathbb{C})$, then $\overline{T} = \mathbb{CP}^1$ and so in this case $\text{Aut}(\overline{T}) = \text{PSL}(2, \mathbb{C})$.

1. Introduction

Let $G$ be a simple affine algebraic group of adjoint type defined over the field of complex numbers. De Concini and Procesi constructed a very important compactification of $G$ [DP, p. 14, 3.1, THEOREM]; it is known as the wonderful compactification. The wonderful compactification of $G$ will be denoted by $\overline{G}$. Fix a maximal torus $T$ of $G$, and denote by $\overline{T}$ the closure of the variety $T$ in the wonderful compactification $\overline{G}$ [BJ, § 1]. Let $\text{Aut}(\overline{T})$ denote the group of all holomorphic automorphisms of $\overline{T}$. For $G \neq \text{PSL}(2, \mathbb{C})$, the connected component of $\text{Aut}(\overline{T})$ containing the identity element coincides with $T$ acting on $\overline{T}$ by translations [BKN, Theorem 3.1]. Our aim here is to compute the full automorphism group $\text{Aut}(\overline{T})$.

It may be noted that $\overline{T}$ is stable under the conjugation of the normalizer $N_G(T)$ of $T$ in $G$. This indicates that $\text{Aut}(\overline{T})$ need not be connected.

For $G$ different from $\text{PSL}(2, \mathbb{C})$, we prove that $\text{Aut}(\overline{T})$ is the semi-direct product $N_G(T) \rtimes D$, where $N_G(T)$ is the normalizer of $T$ in $G$, and $D$ is the group of all automorphisms of the Dynkin diagram (see Theorem 3.1).
2. Lie algebra and algebraic groups

We recall the set-up of [BKN]. Throughout $G$ will denote an affine algebraic group over $\mathbb{C}$ such that $G$ is simple and of adjoint type (equivalently, the center of the simple group is trivial). We will always assume that $G \neq \text{PSL}(2, \mathbb{C})$.

Fix a maximal torus $T$ of $G$. The group of all characters of $T$ will be denoted by $X(T)$. The Weyl group of $G$ with respect to $T$ is defined to be $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of $T$ in $G$. Let

$$R \subset X(T)$$

be the root system of $G$ with respect to $T$. For a Borel subgroup $B$ of $G$ containing the maximal torus $T$, let $R^+(B)$ denote the set of positive roots determined by $T$ and $B$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots in $R^+(B)$, where $n$ is the rank of $G$. Let $B^-$ denote the opposite Borel subgroup of $G$ determined by $B$ and $T$. So in particular $B \cap B^- = T$. For any $\alpha \in R^+(B)$, let $s_\alpha \in W$ be the reflection corresponding to $\alpha$.

The Lie algebras of $G$, $T$ and $B$ will be denoted by $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{b}$ respectively. The dual of the real form $\mathfrak{t}_\mathbb{R}$ of $\mathfrak{t}$ is $X(T) \otimes \mathbb{R} = \text{Hom}_\mathbb{R}(\mathfrak{t}_\mathbb{R}, \mathbb{R})$.

Now, let $\sigma$ be the involution of $G \times G$ defined by $\sigma(x, y) = (y, x)$. We note that the diagonal subgroup $\Delta(G)$ of $G \times G$ is the subgroup of fixed points of $\sigma$. The subgroup $T \times T \subset G \times G$ is a $\sigma$-stable maximal torus of $G \times G$, while $B \times B^-$ is a Borel subgroup of $G \times G$; this Borel subgroup $B \times B^-$ has the property that $\sigma(\alpha) \in -R^+(B \times B^-)$ for every $\alpha \in R^+(B \times B^-)$.

The group $G$ is identified with the symmetric space $(G \times G)/\Delta(G)$. Let $\overline{G}$ denote the corresponding wonderful compactification of $G$ (see [DP, p. 14, 3.1, THEOREM]). In particular $G \times G$ acts on $\overline{G}$. Let $\overline{T}$ be the closure of $T$ in $\overline{G}$. The action of the subgroup $N_G(T) \subset G = \Delta(G)$ on $\overline{G}$ preserves $\overline{T}$.

3. The automorphism group of $\overline{T}$

Let $\text{Aut}(\overline{T})$ denote the group of all holomorphic automorphisms of $\overline{T}$; any holomorphic automorphism is algebraic. Let $\text{Aut}^0(\overline{T}) \subset \text{Aut}(\overline{T})$ be the connected component containing the identity element. The translation action of $T$ on itself produces an isomorphism

$$\rho : T \longrightarrow \text{Aut}^0(\overline{T})$$

[BKN, p. 786, Theorem 3.1].

**Theorem 3.1.** The automorphism group $\text{Aut}(\overline{T})$ is the semi-direct product $N_G(T) \rtimes D$, where $N_G(T)$ is the normalizer of $T$ in $G$, and $D$ is the group of all automorphisms of the Dynkin diagram of $G$.

**Proof.** For notational convenience denote

$$A = \text{Aut}(\overline{T}).$$
Note that $\overline{T}$ is stable under the conjugation action of $N_G(T)$ on $T$. Let
\[ \tilde{\Delta} \subset t_\mathbb{R} \]  \hspace{1cm} (3)
be the fan of the toric variety $\overline{T}$. This $\tilde{\Delta}$ consists of cones associated to the Weyl chambers (see [BK] p. 187, 6.1.6, Lemma]. Note that any automorphism $\sigma$ of the Dynkin Diagram associated to set $S \subset R$ of simple roots with respect to $(T, B)$ preserves the fan $\tilde{\Delta}$. Therefore, we have [Co, p. 47]

\[ N_G(T) \rtimes D \subset A. \]

Next we will show that $N_G(T) \rtimes D = A$.

Since $\rho$ in (2) is an isomorphism, it follows immediately that $T$ is a normal subgroup of $A$. Therefore, the intersection $T \cap g(T)$ is a $T$ stable open dense subset of $\overline{T}$ for every element $g \in A$. Consequently, the open subset $T \subset \overline{T}$ is preserved by the natural action of $A$ on $\overline{T}$. Consequently, every automorphism $g \in A$ can be expressed as
\[ g = l_{t_0}h, \]  \hspace{1cm} (4)
where $l_{t_0}$ is the left translation by some $t_0 \in T$, and $h \in A$ satisfies the condition that $h(1) = 1$, with 1 being the identity element of $T$.

By a result of Rosenlicht, the action of the $h$ (in (4)) on $T$ is by group automorphism (see [MR] p. 986, Theorem 3]. Therefore, $h$ gives an automorphism of $X(T)$, and hence $h$ gives an automorphism of $t_\mathbb{R}$. Since $T$ is left invariant under the action of $h$ the toric variety data of $\overline{T}$ is preserved by $h$. Hence we see that the automorphism of $t_\mathbb{R}$ given by $h$ preserves the fan $\tilde{\Delta}$ in (3). Since $\tilde{\Delta}$ is given by the Weyl chambers and its faces, we see that the induced action of $h$ on $X(T)$ leaves the root system $R$ of $G$ in (1) invariant. Consequently, $h$ produces an automorphism of the root system $R$.

On the other hand, the automorphism group $\text{Aut}(R)$ of the root system $R$ is precisely
\[ N_G(T)/T \rtimes D = W \rtimes D \]
(see [HJ] p. 231, (A.8)]).

Corollary 3.2. The quotient group $\text{Aut}(\overline{T})/\text{Aut}^0(\overline{T})$ is isomorphic to $\text{Aut}(R) = W \rtimes D$.

Remark 3.3. The automorphism group $D$ is trivial except for the types $A_\ell$ with $\ell \geq 2$, $D_\ell$ and $E_6$ (see [HJ] p. 231, (A.8)]).

Remark 3.4. We note that the structure of the automorphism group of a complete simplicial toric variety is described by D. A. Cox (see [Co] p. 48, Corallary 4.7])

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