Smooth Value Function for a Consumption-Wealth Preference and Leverage Constraint

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Abstract

This paper considers an optimal consumption-investment problem for an investor whose instantaneous utility depends on consumption and wealth (as luxury goods or social status). The investor faces a general leverage constraint that the investment amount in the risky asset does not exceed an exogenous function of the wealth. We prove that the value function is second-order smooth, and the optimal consumption-investment policy are provided in a feedback form. Moreover, when the risky investment amount is bounded by a fixed constant, we show that under certain conditions, the leverage constraint is binding if and only if an endogenous threshold bounds the portfolio wealth. Our results encompass many well-developed portfolio choice models and imply new applications.

Keywords: Consumption-investment, smoothness, leverage constraint, constrained viscosity solution, value function.

JEL Classification Codes: G11, G12, G13, D52, and D90
1 Introduction

This paper considers an optimal consumption and investment problem with two remarkable features. First, the investor’s instantaneous utility depends on both consumption and wealth (as luxury goods or social status). Second, there is a dynamic leverage constraint on the risky investment, which is given by a general concave, increasing function of the portfolio wealth. We study the problem in a financial market over an infinite trading horizon with a risk-free asset and a risky asset. The risky asset price is driven by the process of Brownian Motion. In addition, shorting is allowed, but the wealth must stay nonnegative, i.e. bankruptcy is prohibited.

There have been numerous studies on the optimal portfolio choice problem under a leverage constraint for a standard time-separable preference. For instance, Grossman and Vila (1992), Vila and Zariphopoulou (1997) consider a leverage constraint on the investment rate, which linearly depends on the wealth process. Zariphopoulou (1994) considers the constraint on the risky investment by a generally concave, increasing function of the wealth process. Studies on the constraint over the consumption or the wealth process include Black and Perold (1992), Bardhan (1994), Dybvig (1995), El Karoui and Jeanblanc-Picque (1998), Elie and Touzi (2008), Dybvig and Liu (2010), Chen and Tian (2016), Xu and Yi (2016), Ahn, Choi and Lim (2019), among others. These previous studies often assume specifications of the preference to obtain the value function of the optimal portfolio choice problem, and the smoothness property of the value function follows directly.

However, the smoothness of the value function is a technical challenging for a general time-separable preference under constraint. Zariphopoulou (1994) shows the smoothness of the value function if the HJB equation is uniformly elliptic. Strulovici and Szydlowski (2015) consider a more general model and prove that the smoothness of the value function of an optimal stopping problem is continuously differentiable under some conditions. Briefly speaking, Zariphopoupou (1994) and Strulovici and Szydlowski (2015) show the smoothness of the value function by verifying that the HJB equation is uniformly elliptic. Nevertheless, the uniform elliptic condition, in a general situation, is not clear whether to be satisfied or not.

On the other hand, when wealth is viewed as luxury goods or social status, some authors have solved the optimal portfolio choice problems and derived asset pricing implications.
See, for instance, Carroll (2000, 2002), Bakshi and Chen (1996), Roussanov (2010), Smith (2001), assuming particular specification of the preference. The instantaneous utility might also depend only on the wealth in different contexts of optimal portfolio choice problems, in such as Liu and Loewenstein (2002), Tian and Zhu (2022). It is unknown about the smoothness property of the value function for a general non-standard preference depending on wealth only, let alone the preference depending on both consumption and wealth, when the investor faces a leverage constraint.

Our first main result is the smoothness of the value function of the optimal consumption problem with general utility function of both the consumption and wealth, under a general leverage constraint of risky investment. Specifically, we combine the uniformly elliptic approach in Zariphopoulou (1994) and Strulovici and Szydlowski (2015) and a dual method by Xu and Yi (2016), which use the method of dual transformation to transfer the original HJB equation into another auxiliary HJB equation. In the dual transformation approach, we can demonstrate the smoothness for the auxiliary HJB equation, which imply the smooth of the value function. We first characterize the value function as the unique viscosity solution of the HJB equation. See, for instance, Crandall, Ishii and Lions (1992) or Fleming and Soner (2006). Then, we split the domain of nonnegative wealth into two parts: the unconstrained domain and the constrained domain. In the constrained domain, under some conditions, we prove the smoothness of the value function by the uniform elliptic condition. In the unconstrained domain, by using the Legendre-Fenchel transformation, we show that the auxiliary HJB equation is non-degenerate, quasilinear ODE, which implies the smoothness of the auxiliary HJB equation (therefore, the original one). Finally, we prove the smoothness of the value function on the connection points of the constrained domain and unconstrained domain. The optimal control are then provided as the feedback form.

In the second main result of this paper, we characterize the constrained region explicitly. Given the smoothness property of the value function, we characterize the HJB equation into correspondingly two ODEs, in the unconstrained and constrained domains. We study the case that the risky investment is bounded from above by a positive constant. By the use of the comparison principle, we derive a sufficient condition for the existence of threshold \( x^* \) such that the constrained domain is exactly \( (x^*, \infty) \), and we name it as a two-regions property. We illustrate our results with several examples of utility functions.

The rest of the paper is organized as follows. In Section 2, we formulate a continuous-time optimal consumption problem under a general consumption-wealth preference with a
leverage constraint. In Section 3, we show that the value function is second-order smooth, and the optimal consumption-investment policy are derived in a feedback form. In Section 4, we show that under certain conditions the constrained domain is \((x^*, \infty)\). Finally, we present some examples to illustrate our major results in Section 5. We conclude the paper in Section 6. Technical proofs are given in Appendix.

2 Model Setup

There are two assets in a continuous-time economy with time \(t \in [0, \infty)\). Let \((\Omega, (\mathcal{F}_t), P)\) be a filtered probability space in which the information flow in the economy is generated by a standard Brownian motion \((B_t)\). The risk-free asset ("bond") grows at a continuously compounded, constant \(r\). The other asset ("the stock index") is a risky asset, and its price process \(S_t\) follows

\[
dS_t = S_t(\mu dt + \sigma dB_t),
\]

where \(\mu\) and \(\sigma\) are the expected return and the volatility of the stock return. We assume \(\mu > r\).

We consider the set of feasible strategy \((\pi_t, c_t)\) such that (1) \(c_t \geq 0\), \(\int_0^t c_s ds < \infty, a.s., \forall t > 0\), and \((c_t)\) is \(\mathcal{F}_t\) adapted; (2) \((\pi_t)\) is \(\mathcal{F}_t\) adapted, and \(\int_0^t c_s ds + \int_0^t \pi_s^2 ds < \infty, a.s., \forall t > 0\), and \(0 \leq \pi_t W_t \leq g(W_t), \forall t \geq 0\), where \(g(\cdot)\) is an increasing, concave, Lipschitz continuous and twice differentiable function on \([0, \infty)\) to \([0, \infty)\), moreover, there exists \(L > 0\) such that

\[
g(x) \geq L, \forall x > 0;
\]

and (3) there is a strong solution, \(W_t = W_t^{(\pi, c)}\), of the stochastic differential equation

\[
dW_t = \pi_t W_t((\mu - r) dt + \sigma dB_t) + r W_t dt - C_t dt, W_0 \geq 0.
\]

and \(W_t \geq 0, \forall t\). We denote \(\mathcal{A}(x)\) as the set of consumption-investment strategies \((c, \pi)\) such that \(W_0 = x\).

The investor’s expected utility is given by

\[
E \left[ \int_0^\infty e^{-\delta t} f(c_t, W_t) dt \right].
\]
We make the following conditions on \( f(c, w) \):

**Assumption 1**

1. \( f(c, w) \) is a \( C^2 \) smooth (\( f_{11}, f_{22}, f_{12} \) exist and are all continuous) function \( [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \) with \( f(0, 0) = 0 \).

2. For \( c, w > 0 \), \( f_1(c, w) > 0 \), \( f_2(c, w) > 0 \). That is, \( f(c, w) \) increases with respect to each component. We also assume \( f \) satisfies the Inada’s condition:

\[
\lim_{c \to 0, w \to 0} f_1(c, w) = \lim_{c \to 0, w \to 0} f_2(c, w) = \infty
\]

3. The Hessian matrix of the function \( f(c, w) \) is negative definite. That is, \( f_{11}(c, w) < 0 \) and \( f_{11}(c, w)f_{22}(c, w) - f_{12}(c, w)^2 > 0 \). It is known that the function \( f(c, w) \) is a concave function jointly with \( c \) and \( w \).

4. \( f(c, w) \leq M(1 + c^\gamma + w^\gamma) \) for some positive number \( M \) and \( 0 < \gamma < 1 \).

The optimal portfolio choice problem is to find the optimal trading strategy \((\pi_t)\) and the consumption rule \((c_t)\) in

\[
V(x) := \sup_{(c_t, \pi_t) \in A(x)} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} f(c_t, W_t) dt \right]. 
\] (4)

To guarantee the value function \( V(x) \) is well defined, throughout this paper, we assume that

\[
\delta > r\gamma + \frac{\gamma(\mu - r)^2}{2\sigma^2(1 - \gamma)}.
\]

For later use, the following notation will be used. For \( x, \zeta > 0 \), we define the Legendre-Fenchel transformation of the function \( f(c, x) \):

\[
p(x, \zeta) = \max_{c \geq 0} \{ f(c, x) - c\zeta \}. 
\] (5)

By the property of the function \( f(\cdot, \cdot) \), there exists a positive function \( I(x, \zeta) \) such that \( f_1(I(x, \zeta), x) = \zeta \).
3 Smoothness property of the value function

The following theorem is the first main result of this paper.

**Theorem 1** Under Assumption 1, the value function $V(x)$ is the unique $C^2((0, \infty)) \cup C([0, \infty))$ solution of:

$$
\delta V(x) = \max_{0 \leq \pi x \leq g(x)} \left[ (\mu - r)\pi x V'(x) + \frac{1}{2} \sigma^2 \pi^2 x^2 V''(x) \right] + \max_{c \geq 0} \{ f(c, x) - cV'(x) \} + rxV'(x), \quad (x > 0)
$$

in the class of concave functions with $V(0) = 0$. The optimal feedback control are

$$
c^*(x) = I(x, V'(x)), \quad \pi^*(x) = \min\left( \frac{(\mu - r)V'(x)}{\sigma^2 x V''(x)}, g(x) \right), \quad x > 0.
$$

We briefly explain its idea, and defer the proof of this result in the Appendix. The idea is to study the HJB equation (6) in two different regions. In the unconstrained domain $U$, assuming $V(x)$ is $C^2$ smooth,

$$
\frac{\mu - r}{\sigma^2} \left( -\frac{V'(x)}{V''(x)} \right) < g(x),
$$

the equation (6) reduces to

$$
\delta V(x) = -\frac{(\mu - r)^2 (V')^2(x)}{2\sigma^2 V''(x)} + p(x, V'(x)) + rxV'(x).
$$

(7)

Here, $p(x, \zeta)$ is the Legendre-Fenchel transformation of the preference function $f(c, x)$. Our definition of the unconstrained region is slightly different from standard literature in such as Vila and Zariaphpoilou (1997) in which $\frac{\mu - r}{\sigma^2} \left( -\frac{V'(x)}{V''(x)} \right) \leq g(x)$, and we will explain its reason shortly. In the constrained region $B$, that is,

$$
\frac{\mu - r}{\sigma^2} \left( -\frac{V'(x)}{V''(x)} \right) > g(x),
$$

the HJB equation reduces to

$$
\delta V(x) = (\mu - r)g(x)V'(x) + \frac{1}{2} \sigma^2 g^2(x)V''(x) + p(x, V'(x)) + rxV'(x).
$$

(8)
We first note that $V(x)$ is the unique viscosity solution of the HJB equation \( (6) \). This is essentially a deep theorem of Zariphopoulou (1994). There are several steps to show that $V(x)$ is smooth. In the constrained region, there exists unique smooth solution of the equation \( (8) \). A crucial step is to show that $V(x)$ is $C^1$ smooth in the unconstrained region. In this step, we make use of the viscosity solution characterization. We next show that $V(x)$ is $C^2$ smooth in the unconstrained region by a dual approach. The another crucial issue is to show that the value function is $C^1$ smooth at the connection point in $cl(U) \cap cl(B)$. This $C^1$ property at these points follow the $C^2$ smooth properties in each (open) region, unconstrained and constrained region, and the viscosity solution characterization of the value function. Finally, we show the $C^2$ smooth property at $cl(U) \cap cl(B)$. A standard verification argument then conclude the proof.

Theorem 1 encompass several known results in literature. A standard leverage constraint is first suggested in Grossman and Vila (1994) to study the optimal portfolio growth rate for $g(x) = k(x + L), k > 0$ in a continuous-time setting. Zariphopoulou (1994) demonstrates the $C^2$ smooth property of the value function for $f(c, x) = u(c)$ and a general leverage constraint. Vila and Zariphopoulou (1997) further characterize the value function explicitly for CRRA utility $u(c)$.

A general $C^2$ smooth property of the value function for $f(c, \pi, x)$ is proved in Strulovic and Szydlowski (2015), when $(c, \pi, x)$ belongs to a compact set and other technical conditions. The general theorem of Strulovic and Szydlowski (2015) can be applied to the case that $\pi x \in [a, b]$, for $0 < a < b$. However, it is yet open about the smoothness of the value function for a constraint that $g(x) = L$. In Tian and Zhu (2022), for $f(c, x) = \frac{c^{1-R}}{1-R}$, the $C^2$ smooth property of the value function is reduced to a unique solution of a non-linear equation of one-argument. In a general preference $f(c, x)$, the wealth $x$ is interpreted as luxury goods, and the optimal portfolio choice problem has been studied in Carroll (2000, 2002), Ait-Sahalia, Parker, and Yogo (2004), Watcher and Yogo (2010). Moreover, this kind of preference is often used to model the social status of the wealthy in Bakshi and Chen (1996), Roussanov (2010), Smith (2001). But the smoothness property of the value function under leverage constraint is not studied yet in previous studies. Theorem 1 states the smoothness property of the value function for those consumption-wealth preference and a general leverage constraint.
We next move to the optimal investment strategy. In the unconstrained region, the optimal proportion of wealth invested in the risky asset is

$$\pi^*(x) = -\frac{\mu - r}{\sigma^2} \frac{V'(x)}{V''(x)}.$$  

There is a standard way to introduce translate the non-linear PDE (7) into a standard linear PDE as follows. Introduce a new variable $y = V'(x)$. We write $x = H(y)$ and $V(x) = J(y)$. Then Equation (7) reduces to

$$\delta J(y) = -\frac{(\mu - r)^2}{2\sigma^2} y^2 H'(y) + p(H(y), y) + ryH(y), J'(y) = yH'(y).$$

Given a solution of $H(\cdot)$ (if plausible in many special situations), $\pi^* = -\frac{(\mu - r)yH'(y)}{\sigma^2 H(y)}$ in terms of the variable $y$. Moreover, the optimal consumption rate $c^*$, as a function of the variable $y$, is given by $I(H(y), y)$.

On the other hand, in the unconstrained region, the optimal proportion of wealth invested in the risky asset is

$$\pi^*(x) = \frac{g(x)}{x}.$$  

Under assumption on $g(x)$, $\lim_{x\to0} \frac{g(x)}{x} = +\infty$. The next lemma shows that the constrained region does not contains $(0, a)$ for a small number $a$. Therefore, the general leverage constraint is meaningful.

**Lemma 1** Under Assumption 7, there exists a positive number $W^*$ such that $(0, W^*) \subseteq U$.

The next result shows a general decreasing property of the proportion of financial wealth in the risky asset.

**Lemma 2** In the constrained region, the proportion $\pi^*(x)$ decreases with respect to the wealth.

It remains to characterize the constrained and unconstrained region geometrically, which is studied in the next section.
4 Characterization of the Constrained Region

In this section, we characterize the constrained region under the leverage constraint. When \( g(x) \equiv L \), the next theorem shows the two-regions property under certain conditions.

**Theorem 2** Let \( g(x) \equiv L \) and

\[
m(x) = -(\mu - r)p_1(x, V'(x)) - \sigma^2 L[p_{11}(x, V'(x)) + 2p_{12}(x, V'(x))] - \frac{1}{\sigma^2 L}p_{22}(x, V'(x))(\mu - r)^2(V'(x))^2 + r(\mu - r)V'(x). \tag{9}
\]

Under Assumption 2 and if the function \( m(\cdot) \) satisfies one of the following condition:

1. \( m(x) \) does not change sign on \((0, \infty)\);
2. \( m(x) \) changes the sign once and only once from positive to negative on \((0, \infty)\).

then there exists a real number \( x^* > 0 \) such that \( U = (0, x^*) \) and \( B = (x^*, \infty) \).

The proofs for Lemma 1, Lemma 2, and Theorem 2 are in the Appendix. By Theorem 2, an explicit solution of the optimal portfolio choice problem is reduced to find the endogenous number \( x^* \), which is charactering by the smooth-fit condition in \( U \) and \( B \).

5 Examples and Implications

In this section we illustrate the main results with several examples. To simplify notations, we assume \( r = 0 \).

We start with a time-separable preference \( f(c, x) = u(c) \).

**Corollary 5.1** Assume \( g(x) \equiv L \), \( f(c, x) = u(c) \) for any increasing and concave function \( u(\cdot) \). Then, the value function is second-order smooth and the two-regions property holds.

**Proof:** We check that this utility function satisfies the condition in Theorem 2. In this case

\[
p(x, \zeta) = u(K(\zeta)) - K(\zeta)\zeta
\]
where \( K(\zeta) = (u')^{-1}(\zeta) \). Clearly, \( p(x, \zeta) \) is independent of \( x \), hence \( p_1(x, \zeta) = p_{11}(x, \zeta) = p_{12}(x, \zeta) = 0 \). Moreover, by Lemma 6.3, \( p_{22}(x, \zeta) = -K'(\zeta) > 0 \). Therefore,

\[
m(x) = -\frac{1}{\sigma^2 L}p_{22}(x,V'(x))(\mu - r)^2(V'(x))^2 < 0
\]

satisfies the first condition of \( m(x) \) in Theorem 2. Therefore, the two-regions property holds. \( \square \)

The first part of Corollary 5.1 is known by Zariphopoulou (1994). For the second part, and \( u(c) = \frac{c^{1-R}}{1-R}, R > 0, R \neq 1 \) and \( g(x) = kx + L, k \geq 0, L > 0 \), Vila and Zariphopoulou (1997) show the two-regions property for \( k > 0 \), and under the assumption that

\[
\delta + \frac{(\mu - r)^2}{2\sigma^2} > r + \frac{k(\mu - r)}{2}.
\]

Tian and Zhu (2022) solves the situation for \( k = 0 \). Corollary 5.1 implies the two-regions property for a general utility function \( u(\cdot) \).

**Corollary 5.2** \( g(x) \equiv L, f(c, x) = u(x) \) for any increasing and concave function \( u \) such that either (1) \((\mu - r)u'(x) + \sigma^2 Lu''(x)\) is always positive on \((0, \infty)\), or (2) \((\mu - r)u'(x) + \sigma^2 Lu''(x)\) changes the sign once and only once from negative to positive on \((0, \infty)\), then the value function is second-order smooth and the two-regions property holds.

**Proof:** We check that this utility function satisfies the condition in Theorem 2. In this case \( p(x, \zeta) = u(x) \). It is clear that \( p_2(x, \zeta) = p_{12}(x, \zeta) = p_{22}(x, \zeta) = 0 \). Therefore,

\[
m(x) = - (\mu - r)u'(x) - \sigma^2 Lu''(x)
\]

By the condition added on \( u(x) \). It is clear that it satisfies the condition in Theorem 2. \( \square \)

Corollary 5.2 characterizes the constrained region explicitly by a positive number \( x^* \) for a general utility function \( u(\cdot) \). As an illustrative example, \( u(x) = \frac{x^{1-R}}{1-R}, R > 0, R \neq 1 \), it is easy to see that

\[
m(x) = -x^{-R-1}[(\mu - r)x - \sigma^2 RL]
\]

which exactly changes the sign once from positive to negative on \((0, \infty)\). Therefore, the two-regions property holds, which is proved under certain conditions in Tian and Zhu (2022).
Corollary 5.2 demonstrates this two-regions property \textit{unconditionally}. Moreover, by combining with Tian and Zhu (2022), we obtain an explicit expression of the consumption and investment strategy.

We next consider some specifications of $f(c, x)$ with both the consumption and wealth are involved.

\textbf{Corollary 5.3} Assume $f(c, x) = \alpha u(c) + \beta v(x)$ where $u(\cdot)$ and $v(\cdot)$ are increasing and concave function. Let $g(x) \equiv L$, and

$$m(x) = -\beta[(\mu - r)v'(x) + \sigma^2Lv''(x)] + \frac{1}{\alpha\sigma^2 L}K'(\frac{V'(x)}{\alpha})(\mu - r)^2(V')^2.$$ 

If $m(x)$ satisfies the conditions in Theorem 2, then the two-regions property holds.

In this additive specification, the two-region property is reduced to study the function $m(x)$. For example, let $f(c, x) = \frac{\alpha c^{1-R} \beta x^{1-R}}{1-R}$, i.e. $u(x) = v(x) = \frac{x^{1-R}}{1-R}$. Then, the value function is smooth for such a consumption-wealth preference and a constant leverage constraint. Moreover, by computation,

$$m(x) = -x^{-R-1}[(\mu - r)x - \sigma^2 RL] - \alpha \frac{1}{\sigma^2 LR} (\mu - r)^2 (V')^{1-\frac{1}{R}}.$$ 

If $m(x)$ satisfies the condition in Theorem 2, then the two-regions property holds. This example is studied in Tian and Zhu (2022).

Similarly, we can consider a multiplicative specification of the preference studied in Bakshi and Chen (1996). Assume $f(c, x) = \frac{c^a x^b}{1-R}$, where $a > 0, b > 0$ and $a + b < 1$. An explicit expression of the value function is given in Bakshi and Chen (1996). We have shown that the value function for such a preference and a constant leverage constraint. Moreover, by computation,

$$p(x, \zeta) = a^{\frac{1}{a(1-R)}} \left[ \frac{1}{a(1-R)} - 1 \right] \zeta^{\frac{a(1-R)}{a(1-R)-1}} x^{\frac{b(1-R)}{a(1-R)-1}},$$

and

$$m(x) = a^{\frac{1}{a(1-R)}} \left[ \frac{1}{a(1-R)} - 1 \right] (V'(x))^{\frac{a(1-R)}{a(1-R)-1}} x^{\frac{b(1-R)}{a(1-R)-1}} n(x).$$
where
\[
    n(x) : = - (\mu - r) \frac{b(1 - R)}{1 - a(1 - R)} x - \frac{(\mu - r)^2}{\sigma^2 L} a(1 - R) \frac{a(1 - R)}{a(1 - R) - 1} (\frac{a(1 - R)}{a(1 - R) - 1} - 1) x^2 \\
    - \sigma^2 L \left[ \frac{b(1 - R)}{1 - a(1 - R)} \left( \frac{b(1 - R)}{1 - a(1 - R)} - 1 \right) + 2 \frac{b(1 - R)}{1 - a(1 - R)} \frac{a(1 - R)}{a(1 - R) - 1} (V'(x))^{-1} x \right]
\]

Clearly, \( m(x) \) has the same sign with \( n(x) \) by the assumption. If \( n(x) \) satisfies the condition in Theorem 2, then the two-regions property holds.

6 Conclusion

This paper demonstrates a general smoothness property of the value function of an optimal consumption problem in which the investor has an instantaneous consumption-wealth utility and faces a general dynamic leverage constraint. We prove this deep result by combining uniformly elliptic conditions and the dual approach. Furthermore, we show that under some conditions, the constrained domain is \((x^*, \infty)\) for a threshold \(x^*\). The general form of \( f(c, w) \) include several well-studied preferences in the literature, including Bakshi and Chen (1994), Liu and Loewenstein (2002), Tian and Zhu (2022), and standard time-separable preference on the consumption rate. Moreover, the dynamic constraint is given by a general increasing and concave function of the wealth function on the risky investment.
Appendix: Proofs

We start with viscosity solution characterization of the value function. Briefly speaking, $V(x)$ is a viscosity subsolution of an elliptic second-order equation $F(x,u,u_x,u_{xx}) = 0$ if for any smooth function $\psi$ and a maximum point $x_0$ of $V - \psi$, the inequality

$$F(x_0,V(x_0),\psi_x(x_0),\psi_{xx}(x_0)) \leq 0$$

holds. Similarly, $V$ is a viscosity supersolution if for any smooth function $\psi$ and a minimum point $x_0$ of $V - \psi$, the inequality

$$F(x_0,V(x_0),\psi_x(x_0),\psi_{xx}(x_0)) \geq 0$$

holds. A viscosity solution is both a viscosity subsolution and supersolution. We refer to Fleming and Soner (2006) for the theory of viscosity solution.

**Proposition 1** The value function $V(x)$ is continuous, strictly increasing and strictly concave. Moreover, $V(x)$ is the unique viscosity solution of

$$\delta V(x) = \max_{0 \leq \pi x \leq g(x)} [(\mu - r)\pi V'(x) + \frac{1}{2}\sigma^2 \pi^2 V''(x)] + \max_{c \geq 0} \{f(c,x) - cV'(x)\} + rxV'(x), (x > 0)$$

(A-1)

in the class of concave functions with $V(0) = 0$.

**Proof:** The concave property of the value function follows from the concavity property of $f(c,x)$ and concave function $g(x)$. The part $V(0) = 0$ is standard. For the viscosity solution characterization of the value function $V(x)$, this is a deep theorem of Zariphopoulou (1994), in which we replace $u(c)$ by $f(c,x)$ throughout the entire proof. \(\square\)

The challenge of proving Theorem \(\square\) is to show that the viscosity concave solution is $C^2$. We need several preliminary results to demonstrate the smooth properties.

The first lemma is related to the derivative of a concave function. Since we cannot find the proof of this known result in available reference, we present its proof here.

**Lemma 6.1** Assume $f(x)$ is a concave function on $[a, b]$ for some real numbers $a$ and $b$. If $f(x)$ is differentiable on $(a, b)$, then $f(x)$ is $C^1$ on $(a, b)$. 
Proof: Since \( f(x) \) is differentiable and concave on \((a, b)\), \( f'(x) \) is a decreasing function on \((a, b)\). Fix \( x_0 \in (a, b) \), for all \( \epsilon > 0 \), by Darboux theorem, there exists \( x_1 \in (x_0, \frac{x_0 + b}{2}) \) such that

\[
f'(x_1) = \max(f'(x_0) - \epsilon, f'\left(\frac{x_0 + b}{2}\right))
\]

therefore, by \( f'(x) \) is decreasing, we have for all \( x \in (x_0, x_1) \)

\[
0 \leq f'(x) - f'(x_0) \leq \epsilon \quad (A-2)
\]

Similarly, for same \( \epsilon \), we could find \( x_2 \in \left(\frac{x_0 + x}{2}, x_0\right) \) such that when \( x \in (x_2, x_0) \), we have

\[
0 \leq f'(x) - f'(x_0) \leq \epsilon \quad (A-3)
\]

Combining \((A-2)\) and \((A-3)\), we have \( f \) is \( C^1 \) at \( x_0 \). Since \( x_0 \) is arbitrary, we have \( f \) is \( C^1 \) on \((a, b)\). \( \square \)

The next lemma is given in Tian and Zhu (2022). The proof below belongs to Prof. Jianfeng Zhang.

Lemma 6.2 Let \( F : (0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a continuous and elliptic operator, that is, \( F(x, r, p, X) < F(x, r, p, Y), \forall X \geq Y \). Assume \( V(x) \) is a continuous viscosity solution of a second-order (HJB) equation \( F(x, u, u_x, u_{xx}) = 0 \) and the region of \( x \) is \( D = (0, \infty) \). For \( x^* \in (0, \infty) \) if there exist two numbers \( 0 < x_1 < x^*, x^* < x_2 < \infty \), such that \( V(x) \) is smooth in \((x_1, x^*)\) and \((x^*, x_2)\), then \( V(x) \) must satisfies the smooth-fit condition at \( x^* \), that is, \( V'(x^-) = V'(x^+) \).

Proof: Since \( V(x) \) is smooth in both \((x_1, x^*)\) and \((x^*, x_2)\), \( V'(x^-) \) and \( V'(x^+) \) exist. Without lost of generality, we assume that \( V'(x^-) < 0 < V'(x^+) \) and derive a contradiction. Since there is no available test function, the subsolution holds automatically. We next check the supersolution. Let the test function in the form of

\[
\psi(x) \equiv V(x^*) + \frac{1}{2} \left[V'(x^-) + V'(x^+)\right] (x - x^*) + \alpha(x - x^*)^2
\]

We claim that \( \alpha \) can take any real value: To make \( \psi(x) \) the valid test function, we need to guarantee that \( \psi(x) \leq V(x) \) when \( x \) is in a small neighborhood of \( x^* \). However, when \( x \to x^* \), the linear term \( \frac{1}{2} \left[V'(x^-) + V'(x^+)\right] (x - x^*) \) will dominate the quadratic term.
\(\alpha(x - x^*)^2\). Therefore, when \(x\) and \(x^*\) are close enough, we could choose sufficiently large \(\alpha\) such that \(\psi(x) \leq V(x)\). It is now clear that \(\alpha\) can take any value.

Now, apply the viscosity property at \(x^*\), we have

\[
F \left( x^*, V(x^*), \frac{1}{2} \left[ V''(x^*) + V'(x^*)^2 \right], 2\alpha \right) \geq 0,
\]

which is impossible by the free choice of the parameter \(\alpha\).

Recall the Legendre-Fenchel transformation of the function \(f(c, x)\) is defined as

\[
p(x, \zeta) = \max_{c \geq 0} \{ f(c, x) - c\zeta \}.
\]

**Lemma 6.3** The function \(p(x, \zeta)\) is a convex function of the argument \(\zeta\).

**Proof:** It is clear that \(p(x, \zeta)\) is continuous on \(x\). Recall the definition of \(I(x, \zeta)\), we have \(f_1(I(x, \zeta)) = \zeta\). By \(f_{11}(c, x) < 0\), we have \(I_2(x, \zeta) < 0\). Rewrite \(p(x, \zeta) = f(I(x, \zeta), x) - I(x, \zeta)\zeta\). Then,

\[
p_2(x, \zeta) = f_1(I(x, \zeta), x)I_2(x, \zeta) - I_2(x, \zeta)\zeta - I(x, \zeta)
\]

and \(p_{22}(x, \zeta) = -I_2(x, \zeta) > 0\). \(\square\)

By Proposition 1 and the structure of equation (A-1), we define the unconstrained domain \(U\) be the set of \(x\) such that \(V(x)\) is the viscosity solution of

\[
\delta V(x) = \frac{(\mu - r)^2}{2\sigma^2} \frac{(V')^2(x)}{V''(x)} + p(x, V'(x)) + rxV'(x).
\] (A-4)

Also, define the constrained domain \(B\) be set of \(x\) such that \(V(x)\) is the viscosity solution of

\[
\delta V(x) = (\mu - r)g(x)V'(x) + \frac{1}{2}\sigma^2 g^2(x)V''(x) + p(x, V'(x)) + rxV'(x).
\] (A-5)

Note that \((0, \infty)\) will be divided into three parts: (1) The unconstrained domain \(U\); (2) the constrained domain \(B\); and (3) the connection points in \(cl(U) \cap cl(B)\).

**Proof of Theorem**
We divide the proof into several steps.

**Step 1:** By Proposition 1, the value function $V(x)$ is strictly increasing, strictly concave function. Moreover, $V(x)$ is the unique viscosity solution of (A-1).

**Step 2:** We show $V(x)$ is $C^2$ at $x \in \mathcal{B}$.

For any $x \in \mathcal{B}$, $V(x)$ is the viscosity solution of

$$
\delta V(x) = (\mu - r)g(x)V'(x) + \frac{1}{2}\sigma^2 g^2(x)V''(x) + p(x, V'(x)) + rxV'(x).
$$

Note that the coefficients of $V''(x)$ is $\frac{1}{2}\sigma^2 g^2(x)$. However, by the definition of $\mathcal{A}(x)$, we have

$$
\frac{1}{2}\sigma^2 g^2(x) \geq \frac{1}{2}\sigma^2 L^2 > 0, \quad \forall x \in \mathcal{B}
$$

which is uniformly positive. Hence, when $x \in \mathcal{B}$, $V(x)$ is $C^2$ due to the the equation is clearly non-degenerate. See Krylov (1987).

**Step 3:** We show that the value function $V(x)$ is $C^1$ at $x \in \mathcal{U}$.

Since $V$ is increasing and concave, we define its right and left derivative:

$$
V'(x+) = \lim_{h \to 0^+} \frac{V(x + h) - V(x)}{h} \geq 0 \quad \text{(A-6)}
$$

for all $x > 0$. Note that $0 \leq V'_x(x+) \leq V'_x(x-) < \infty$ for all $x > 0$. By Lemma 6.1, in order to show that $V(x)$ is $C^1$, it suffices to show that $V(x)$ is differentiable, i.e. $V'(x-) = V'(x+)$.

Now we prove the result by contradiction. Assume $V'(x_0+) < V'(x_0-)$ for some $x_0 \in \mathcal{U}$. Set $\eta$ satisfies $V'(x_0+) < \eta < V'(x_0-)$. Define

$$
\phi(x) = V(x_0) + \eta(x - x_0) - m(x - x_0)^2 \quad \text{(A-7)}
$$

for $m > 0$. It is clear that $\phi(x_0) = V(x_0), \phi'(x_0) = \eta$ and $\phi''(x_0) = -2m$.

By concavity of $V(x)$, when $0 < x_0 - x < \frac{1}{m}(V'(x_0-) - \eta)$, we have

$$
V(x) \leq V(x_0) + V'(x_0-)(x - x_0)
$$
Plugging (A-7) into it, we get

\[ V(x) \leq \phi(x) + (V'(x_0-) - \eta)(x - x_0) + m(x - x_0)^2 < \phi(x) \] (A-8)

Similarly, when \( 0 < x - x_0 < \frac{1}{m}(\eta - V'(x_0+)) \), we have

\[ V(x) \leq V(x_0) + V'(x_0+)(x - x_0) = \phi(x) + (V'(x_0+) - \eta)(x - x_0) + m(x - x_0)^2 < \phi(x) \] (A-9)

Basically, (A-8) and (A-9) imply that \( V(x) < \phi(x) \) in a small neighborhood of \( x_0 \), therefore we may use \( \phi(x) \) as the desired text function at \( x = x_0 \). Since when \( x \in \mathcal{U} \), \( V(\cdot) \) is a viscosity solution of (A-4), using the definition of viscosity subsolution at \( x_0 \), we have

\[ 0 \geq \delta \phi(x_0) + \frac{(\mu - r)^2 (\phi'(x_0))^2}{2\sigma^2} - p(x_0, \phi'(x_0)) - rx_0\phi'(x_0) + \frac{\mu^2\eta^2}{4\sigma^2m} - p(x_0, \eta) - rx_0\eta. \]

Sending \( m \to \infty \), we have

\[ 0 \geq \delta V(x_0) - p(x_0, \eta) - rx_0\eta. \] (A-10)

for \( \forall \eta \in (V'(x_0+), V'(x_0-)) \)

On the other hand, since \( V(\cdot) \) is concave, it is second differentiable almost everywhere. Then there exists \( \{x_n\}_{n \geq 0} \) increases to \( x_0 \) such that \( V(\cdot) \) is \( C^2 \) at all \( x_n \). Then by (A-4), we have

\[ 0 = \delta V(x_n) + \frac{(\mu - r)^2 V'(x_n)^2}{2\sigma^2} - p(x_n, V'(x_n)) - rx_nV'(x_n) \leq \delta V(x_n) - p(x_n, V'(x_n)) - rx_nV'(x_n) \] (A-11)

Sending \( x_n \uparrow x_0 \), we get

\[ 0 \leq \delta V(x_0) - p(x_0, V'(x_0-)) - rx_0V'(x_0-) \] (A-12)
Similarly, by choosing $x_n \downarrow x_0$, we get

$$0 \leq \delta V(x_0) - p(x_0, V'(x_0^+)) - r x_0 V'(x_0^+).$$  \hfill (A-13)

Define $q(x_0, \eta) = p(x_0, \eta) + r x_0 \eta$. It is clear that $q_{22}(x_0, \eta) = p_{22}(x_0, \eta) > 0$ by Lemma 6.3

Then $q(x_0, \eta)$ is a convex function of $\eta$. By (A-10), we have

$$q(x_0, \eta) \geq \delta V(x_0), \quad \forall \eta \in (V'(x_0^+), V'(x_0^-))$$  \hfill (A-14)

On the other hand, by convexity of $q(x_0, \eta)$ on $\eta$, we have

$$q(x_0, \eta) \leq \min(q(x_0, V'(x_0^-)), q(x_0, V'(x_0^+))), \quad \forall \eta \in (V'(x_0^+), V'(x_0^-))$$  \hfill (A-15)

Use (A-12) and (A-13), we have

$$q(x_0, \eta) \leq \delta V(x_0), \quad \forall \eta \in (V'(x_0^+), V'(x_0^-)).$$  \hfill (A-15)

Now, combining (A-14), (A-15), we conclude that $h(\eta)$ is a constant on $\eta \in (V'(x_0^+), V'(x_0^-))$. However, since

$$q_{22}(x_0, \eta) > 0,$$  \hfill (A-16)

Hence, $q(x_0, \eta)$ can not be a constant function on $\eta \in (V'(x_0^+), V'(x_0^-))$. We therefore conclude that the value function $V$ is $C^1$ when $x \in U$.

**Step 4:** We show the value function $V$ is $C^2$ at $x \in U$.

For this purpose, we define the dual transformation:

$$v(y) := \max_{x > 0} (V(x) - xy), \quad V'(\infty) < y < V'(0)$$  \hfill (A-17)

Then $v(\cdot)$ is a decreasing convex function on $(V'(\infty), V'(0))$. Since $V'(\cdot)$ is strictly decreasing, we denote the inverse function of $V'(x) = y$ by $I(y) = x$. Then $I(\cdot)$ is decreasing and mapping $(V'(\infty), V'(0))$ to $(0, \infty)$. Also, from (A-17), we get

$$v(y) = [V(x) - x V'(x)]|_{x=I(y)} = V(I(y)) - y I(y)$$  \hfill (A-18)
Differentiate (A-18) once and twice, we get
\[ v'(y) = V'(I(y))I'(y) - yI'(y) - I(y) = -I(y) \]  
(A-19)
and
\[ v''(y) = -I'(y) = -\frac{1}{V''(I(y))} \]  
(A-20)
Combining (A-18) and (A-19), we get
\[ V(I(y)) = v(y) - yv'(y) \]  
(A-21)
Now, plugging (A-19), (A-20), (A-21) into (A-4), we get
\[ \delta(v(y) - yv'(y)) = \left(\frac{\mu - r}{2\sigma^2}\right) y^2 v''(y) + p(-v'(y), y) - rv'(y)y, \quad V'(\infty) < y < V'(0) \]  
(A-22)
Equation (A-22) is a quasilinear ODE, which only degenerates at \( y = 0 \). Since \( V'(x) > 0 \) for all \( x > 0 \), we have \( 0 \notin (V'(\infty), V'(0)) \). Then, the coefficient of \( v''(y) \) in the above equation is \( \frac{(\mu - r)^2}{2\sigma^2} y^2 \), which is nonzero. It follows that
\[ v \in C^2(V'(\infty), V'(0)). \]  
(A-23)
By (A-20), we get \( V(\cdot) \) is \( C^2 \) when \( x \in \mathcal{U} \).

Step 5: In Step 2 and Step 4, we have proved that the value function \( V \) is \( C^2 \) when \( x \in \mathcal{U} \) and \( x \in \mathcal{B} \). In this step, we show that \( V \) is \( C^2 \) when \( x \) is the connection point in \( cl(\mathcal{U}) \cap cl(\mathcal{B}) \).
Assume \( x^* \) is the connection point in \( cl(\mathcal{U}) \cap cl(\mathcal{B}) \). Our goal is to show that \( V \) is \( C^2 \) at \( x^* \).
Without loss of generality, we assume that the left neighborhood of \( x^* \) is \( \mathcal{U} \) and right neighborhood of \( x^* \) is \( \mathcal{B} \). That is, we have
\[ \delta V(x^*-) = -\frac{(\mu - r)^2}{2\sigma^2} \frac{V'(x^*-)^2}{V''(x^*-)} + p(x^*-, V'(x^*-)) + rx^*V'(x^*) \]  
(A-24)
\[ \delta V(x^+) = (\mu - r)g(x^*)V'(x^+) + \frac{1}{2}\sigma^2(g(x^*))^2V''(x^+) + p(x^+, V'(x^+)) + rx^*V''(x^+). \] (A-25)

Since \( V \) is continuous on \((0, \infty)\), we have
\[ V(x^+) = V(x^-). \] (A-26)

Moreover, since \( V \) is the viscosity solution of \((A-1)\) and \( V \) is \( C^2 \) when \( x \in \mathcal{U} \) and \( x \in \mathcal{B} \), we obtain
\[ V'(x^+) = V'(x^-). \] (A-27)

Next, we use \((A-26)\) \((A-27)\) and \( u \) is a continuous function, compare the terms in \((A-24)\) and \((A-25)\), we have
\[ -\frac{(\mu - r)^2}{2\sigma^2} V'(x^-)^2 = (\mu - r)g(x^*)V'(x^+) + \frac{1}{2}\sigma^2(g(x^*))^2V''(x^+) \] (A-28)

Moreover, when \( x \in \mathcal{U} \), for the optimal \( \pi^* \), we have
\[ x^*\pi^*(x^-) = -\frac{(\mu - r)V'(x^-)}{\sigma^2V''(x^-)} = g(x^*) \] (A-29)

Plugging \((A-29)\) into \((A-28)\), we get
\[ \frac{\mu - r}{2}g(x^*)V'(x^-) = (\mu - r)g(x^*)V'(x^+) + \frac{1}{2}\sigma^2(g(x^*))^2V''(x^+) \]

Use \((A-27)\), we get
\[ -\frac{\mu - r}{2}g(x^*)V'(x^-) = \frac{1}{2}\sigma^2(g(x^*))^2V''(x^+) \] (A-30)

Finally, combine \((A-29)\) and \((A-30)\), we get
\[ V''(x^+) = V''(x^-) \] (A-31)

Since \( x^* \) is arbitrary connection point in \( cl(\mathcal{U}) \cap cl(\mathcal{B}) \), we have shown that \( V(x) \) is \( C^2 \) on all the connection points in \( cl(\mathcal{U}) \cap cl(\mathcal{B}) \).
**Step 6:** (Verification) In this step, we show that if $V(x)$ is the concave $C^2$ solution of (6). Then we must have $V(x) = V(x)$. Fix $T > 0$, for arbitrary $(c_t, \pi_t) \in A(x)$, define

$$
\tau_n = (T - \frac{1}{n})^+ \land \inf \{ s \in [0, T]; W_s \geq n \text{ or } W_s \leq \frac{1}{n} \lor \int_0^s \pi_u^2 W_u^2 du = n \} \n$$

apply Ito’s formula to $e^{-\delta t}V(W_t)$ on $[0, T \land \tau_n]$, we have:

$$
Ee^{-\delta(T \land \tau_n)}V(W_{T \land \tau_n}) = \nabla V(x) + \int_0^{T \land \tau_n} e^{-\delta t} [V'(W_t)(\pi_t W_t(\mu - r) + r W_t - C_t) + \frac{1}{2} \nabla'' V(x) \sigma^2 \pi_t^2 W_t^2 - \delta V(W_t)] dt \n$$

Since $V$ is the $C^2$ solution of (6), we have

$$
Ee^{-\delta(T \land \tau_n)}V(W_{T \land \tau_n}) \leq V(x) - \int_0^{T \land \tau_n} e^{-\delta t} f(c_t, W_t) dt \n$$

First send $n \to \infty$, then send $T \to \infty$, since $V$ is concave, it is easy to see the transversality condition $\lim_{T \to \infty} Ee^{-\delta T}V(W_T) = 0$, therefore, we have

$$
E \int_0^\infty e^{-\delta t} f(c_t, W_t) dt \leq V(x) \n$$

since $(c_t, \pi_t)$ is arbitrary, we have $V(x) \geq V(x)$.

On the other hand, if we choose $(c_t^*, \pi_t^*) \in A(x)$ such that

$$
c_t^* = h(W_t^*, \nabla V(W_t^*)), \pi_t^* = \min(g(W_t^*), \frac{-(\mu - r)W_t V(W_t^*)}{\sigma^2 W_t^2 \nabla V(W_t^*)}) \n$$

where $h(x, \zeta)$ satisfies $f_1(h(x, \zeta), x) = \zeta$. Then repeat the above process, we get

$$
V(x) = E \int_0^\infty e^{-\delta t} f(c_t^*, W_t^*) dt \leq V(x) \n$$

We therefore complete the proof. \(\Box\)

We first show the following result in order to prove Lemma 1:

**Lemma 6.4** $\lim_{x \to 0} V'(x) = \infty$
Proof: It is clear that for $x > 0$, $(c_t, \pi_t) = (rW_t, 0) \in A(x)$. Therefore

$$V(x) - V(0) \geq E \int_0^\infty e^{-\delta t}[f(rW_t, W_t) - f(0, 0)]dt$$

$$= E \int_0^\infty e^{-\delta t}[f(rx, x) - f(0, 0)]dt$$

Since in Theorem 1 it is already shown that $V(x)$ is $C^2$ smooth. Then by Fatou’s lemma and the Inada’s condition on $f$, we get

$$\lim_{x \to 0} V'(x) = \lim_{x \to 0} \frac{V(x) - V(0)}{x} \geq E \int_0^\infty e^{-\delta t} \lim_{x \to 0} \left[ \frac{f(rx, x) - f(0, 0)}{x} \right] dt$$

$$= \infty$$

□

Proof of Lemma 1. Assume not, then there exists a sequence $x_n \to 0$ such that $x_n \pi^*(x_n) = g(x_n)$ satisfies

$$\delta V(x_n) = (\mu - r)g(x_n)V'(x_n) + \frac{1}{2}\sigma^2 g^2(x_n)V''(x_n) + p(x_n, V'(x_n)) + rx_nV'(x_n). \quad (A-32)$$

By the definition of $B$, we have $\frac{\mu - \sigma^2}{x^2} \left( -\frac{V'(x_n)}{V''(x_n)} \right) > g(x_n)$, therefore (A-32), the nonnegativity of $p(x, \zeta)$ and $V'(x)$ implies that

$$\delta V(x_n) \geq \frac{1}{2}(\mu - r)g(x_n)V'(x_n) + p(x_n, V'(x_n)) + rx_nV'(x_n)$$

$$\geq \frac{1}{2}(\mu - r)LV'(x_n)$$

Sending $n \to \infty$, By the continuity of $V$, $\delta V(x_n)$ converges to $V(0) = 0$. However, by Lemma 6.4 $\frac{1}{2}(\mu - r)LV'(x_n)$ tends to infinity, contradiction. □

Proof of Lemma 2. It suffices to show that $\frac{g(x)}{x}$ is a decreasing function when $x > 0$. Note that

$$\left( \frac{g(x)}{x} \right)' = \frac{xg'(x) - g(x)}{x^2}$$
We will show that $h(x) := xg'(x) - g(x) \leq 0$ and therefore complete the proof. Note that $\lim_{x \to 0} h(x) = \lim_{x \to 0} xg'(x) - g(0)$. Since $g(x)$ is a Lipschitz function, therefore, $g'(x) \leq K, \forall x > 0$ for some positive constant $K$. Therefore, $\lim_{x \to 0} xg'(x) = 0$. We then have $\lim_{x \to 0} h(x) = -g(0) \leq 0$. Moreover, since $g(x)$ is a concave function

$$h'(x) = xg''(x) \leq 0$$

Therefore, $h(x)$ is a decreasing function. We conclude $h(x) \leq \lim_{x \to 0} h(x) \leq 0, \forall x > 0$. □

**Proof of Theorem 2**

When $g(x) \equiv L$, the ordinary differential equation for $V(x)$ in the unconstrained and constrained region are

$$\delta V(x) = \theta \frac{(V'(x))^2}{-V''(x)} + p(x, V'(x)) + rxV'(x), \theta \equiv \frac{(\mu - r)^2}{2\sigma^2}. \quad (A-33)$$

and

$$\delta V(x) = (\mu - r)LV''(x) + \frac{1}{2}\sigma^2 L^2 V''(x) + p(x, V'(x)) + rxV'(x). \quad (A-34)$$

We define a function

$$Y(x) = (\mu - r)V'(x) + \sigma^2 LV''(x), x > 0. \quad (A-35)$$

Then, $Y(x) > 0, \forall x \in \mathcal{U}$, and $Y(x) < 0$ for any $x \in \mathcal{B}$.

**Step 1:** In the unconstrained region, the value function $V(\cdot)$ satisfies the ODE (A-33) differentiate this ODE once and twice, we get

$$\delta V' = -2\theta V' + \frac{\theta(V')^2V''}{(V'')^2} + p_1(x, V') + p_2(x, V')V'' + rV' + rV''$$

and

$$\delta V'' = -2\theta V'' + \frac{\theta(V')^2V'''}{(V'')^2} + \frac{2\theta V'V'''}{(V'')^3} [(V'')^2 - V'V'''] + p_{11}(x, V') + p_{12}(x, V')V'' + [p_{21}(x, V') + p_{22}(x, V')V'']V'' + p_2(x, V')V''' + 2rV'' + rxV'''$$
By the definition of $Y(x)$, the last two equations imply that

\[
\delta Y = -2\theta Y + \frac{\theta (V')^2}{(V'')^2} Y'' + \frac{2\theta V'V'''}{(V'')^3} \left[ V'' Y' - \frac{V'}{\sigma^2 L} Y \right] + [p_2(x, V') + r x] Y' + r Y + (\mu - r)p_1(x, V') \\
+ \sigma^2 L[p_{11}(x, V') + 2p_{12}(x, V') + p_{22}(x, V')(V'')^2 + r V'']
\]

\[
= -2\theta Y + \frac{\theta (V')^2}{(V'')^2} Y'' + \frac{2\theta V'V'''}{(V'')^3} \left[ V'' Y' - \frac{V'}{\sigma^2 L} Y \right] + [p_2(x, V') + r x] Y' + r Y + (\mu - r)p_1(x, V') \\
+ \sigma^2 L[p_{11}(x, V') + 2p_{12}(x, V') + p_{22}(x, V')(\frac{1}{\sigma^2 L}(Y - (\mu - r)V'))^2 + \frac{r}{\sigma^2 L}(Y - (\mu - r)V')]
\]

\[
= -2\theta Y + \frac{\theta (V')^2}{(V'')^2} Y'' + \frac{2\theta V'V'''}{(V'')^3} \left[ V'' Y' - \frac{V'}{\sigma^2 L} Y \right] + [p_2(x, V') + r x] Y' + 2 r Y \\
+ \frac{1}{\sigma^2 L} p_{22}(x, V')(Y^2 - 2(\mu - r)V'Y) + (\mu - r)p_1(x, V') + \sigma^2 L[p_{11}(x, V') + 2p_{12}(x, V')]
\]

We then define an elliptic operator on the unconstrained region by

\[
\mathcal{L}^U[y] \equiv -\frac{\theta (V')^2}{(V'')^2} y'' - \frac{2\theta V'V'''}{(V'')^3} \left[ V'' y' - \frac{V'}{\sigma^2 L} y' \right] - [p_2(x, V') + r x] y' + (2\theta + \delta - 2 r)y \\
- \frac{1}{\sigma^2 L} p_{22}(x, V')(y^2 - 2(\mu - r)V'y) - (\mu - r)p_1(x, V') - \sigma^2 L[p_{11}(x, V') + 2p_{12}(x, V')]
\]

\[
- \frac{1}{\sigma^2 L} p_{22}(x, V') (\mu - r)^2(V')^2 + r(\mu - r)V'
\]

Therefore, $\mathcal{L}^U[Y] = 0$ in $U$.

**Step 2.** In the constrained region $\mathcal{B}$, by differentiate the ODE [A-34] once and twice, we get

\[
\delta V' = (\mu - r)LV'' + \frac{1}{2} \sigma^2 L^2 V''' + p_1(x, V') + p_2(x, V') V'' + r V' + r x V''
\]

and

\[
\delta V'' = (\mu - r)LV''' + \frac{1}{2} \sigma^2 L^2 V'''' + p_{11}(x, V') + p_{12}(x, V') V'' + [p_{21}(x, V') + p_{22}(x, V') V''']V'' \\
+ p_2(x, V') V''' + 2 r V'' + r x V'''
\]
Again, by the definition of $Y(x)$, we have

$$
\delta Y = (\mu - r)LY' + \frac{1}{2\sigma^2L^2}Y'' + [p_2(x, V') + rx]Y' + 2rY + \frac{1}{\sigma^2 L}p_{22}(x, V')(Y^2 - 2(\mu - r)V'Y)
$$

$$
+ (\mu - r)p_1(x, V') + \sigma^2 L[p_{11}(x, V') + 2p_{12}(x, V')] + \frac{1}{\sigma^2 L}p_{22}(x, V')(\mu - r)^2(V')^2 - r(\mu - r)V'
$$

Similarly, we define an elliptic operator

$$
L^B[y] = -\frac{1}{2\sigma^2L^2}y'' - (\mu - r)Ly' - [p_2(x, V') + rx]y' - 2ry - \frac{1}{\sigma^2 L}p_{22}(x, V')(y^2 - 2(\mu - r)V'y)
$$

$$
- (\mu - r)p_1(x, V') - \sigma^2 L[p_{11}(x, V') + 2p_{12}(x, V')] - \frac{1}{\sigma^2 L}p_{22}(x, V')(\mu - r)^2(V')^2 + r(\mu - r)V'
$$

Then $L^B[Y] = 0$ in $B$.

**Step 3.** By computation, we get

$$
L^B[0] = L^U[0] = -(\mu - r)p_1(x, V') - \sigma^2 L[p_{11}(x, V') + 2p_{12}(x, V')] - \frac{1}{\sigma^2 L}p_{22}(x, V')(\mu - r)^2(V')^2
$$

$$
+ r(\mu - r)V'
$$

Note that this function is exactly $m(x)$ defined in the theorem.

**Step 4.** By Lemma 1, there exists a real number $x_1 > 0$ such that $(0, x_1) \subseteq U$ and $Y(x_1) = 0$. Since $U \neq (0, \infty)$, we have $x_1 < \infty$. We show that $(x_1, \infty) \subseteq B$ by a contradiction argument.

Assume that, there exists $x_2 > x_1$ such that $(x_1, x_2) \subseteq B$ and $Y(x_2) = 0$. Moreover, there exists $x_3 > x_2$ such that $(x_2, x_3) \subseteq U$. We show this is impossible and thus finish the proof.

We first show that the constant function $y = 0$ is not the supersolution for $L^B[y] = 0$ in the region $(x_1, x_2)$. The reason is as follows. Otherwise, since $L^B[Y] = 0$ in the region $(x_1, x_2) \subseteq B$ and $Y(x_1) = Y(x_2) = 0$, then by the comparison principle, $Y(x) \leq y = 0$ for $x \in (x_1, x_2)$. However, by its definition of $B$, $Y(x) > 0$ for all $x \in (x_1, x_2)$. This contradiction show that the constant function $y = 0$ is not a supersolution of $L^B[y] = 0$ in the region $(W_1, W_2)$. Therefore, there exists some $x_0 \in (x_1, x_2)$ such that, at $x = x_0$,

$$
L^B[0] = m(x_0) < 0.
$$
We divide the proof into two situations because of the single crossing condition

**Case 1.** The function $m(x)$ does not change sign at all in $(0, \infty)$. In this case, $g(x) < 0$ for all $(0, \infty)$. We now consider the region $(x_2, x_3) \in U$ and the operator $L^U$. Since $L^U[0] \leq 0$ in this small region, the constant function $y = 0$ is the subsolution for $L^U[0] = 0$. Since $Y(x_2) = Y(x_3) = 0$, by the comparison principle, we obtain $Y(x) \geq 0, \forall x \in (x_2, x_3)$, which is impossible since $Y(x)$ is strictly negative over the region $(x_2, x_3) \subseteq U$, the unconstrained region.

**Case 2.** The function $m(x)$ change the sign once and only once from positive to negative on $(0, \infty)$.

By $m(x_0) < 0$ and the condition on $m(x)$, the function $m(x)$ must be negative for all $x > x_0$. In particular, $m(x) < 0, \forall x \in (x_2, x_3)$. Following the same proof as in Case 1, the constant $y = 0$ is the subsolution for $L^U[0] = 0$. It implies that $Y(x) \geq 0, \forall x \in (x_2, x_3)$. This leads a contradiction again by the definition of $U$.

By the above proof, we have shown that $(x_1, \infty) = B$ by a contradiction argument. □
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