Low dimensional models of the finite split Cayley hexagon

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Abstract. We provide a model of the split Cayley hexagon arising from the Hermitian surface $H(3, q^2)$, thereby yielding a geometric construction of the Dickson group $G_2(q)$ starting with the unitary group $SU_3(q)$.

1. Introduction

A generalised polygon $\Gamma$ is a point-line incidence structure such that the incidence graph is connected and bipartite with girth twice that of its diameter. If the valency of every vertex is at least 3, then we say that $\Gamma$ is thick, and it turns out that the incidence graph is then biregular\(^1\). By a famous result of Feit and Higman [8], a finite thick generalised polygon is a complete bipartite graph, projective plane, generalised quadrangle, generalised hexagon or generalised octagon. There are many known classes of finite projective planes and finite generalised quadrangles but presently there are only two known families, up to isomorphism and duality, of finite generalised hexagons; the split Cayley hexagons and the twisted triality hexagons.

The split Cayley hexagons $H(q)$ are the natural geometries for Dickson’s group $G_2(q)$, and they were introduced by Tits [21] as the set of points of the parabolic quadric $Q(6, q)$ and an orbit of lines of $Q(6, q)$ under $G_2(q)$. If $q$ is even, then the polar spaces $W(5, q)$ and $Q(6, q)$ are isomorphic geometries, and hence $H(q)$ can be embedded into a five-dimensional projective space. Thas and Van Maldeghem [19] proved that if $H$ is a finite thick generalised hexagon embedded into the projective space $PG(d, q)$, then $d \leq 7$ and this embedding is equivalent to one of the standard models of the known generalised hexagons. So in particular, it is impossible to embed the split Cayley hexagon $H(q)$ into a three-dimensional projective space. However, there is an elegant model of $H(q)$ which begins with geometric structures lying in $PG(3, q)$, and it is equivalent to the model provided by Cameron and Kantor [6, Appendix]:

**Theorem 1.1 (Cameron and Kantor (paraphrased) [6]).**

Let $(p, \sigma)$ be a point-plane anti-flag of $PG(3, q)$ and let $\Omega$ be a set of $q(q^2 - 1)(q^2 + q + 1)$ parabolic congruences\(^3\) each having axis not incident with $p$ or $\sigma$, but having a pencil of lines with one line incident with $p$ and another incident with $\sigma$. Suppose that for each pencil $L$ with vertex not in $\sigma$ and plane not incident with $p$, there are precisely $q + 1$ elements of $\Omega$ containing $L$, whose union are the lines of some linear complex (i.e., the lines of a symplectic geometry $W(3, q)$). Then the following incidence structure $\Gamma$ is isomorphic to the split Cayley hexagon $H(q)$.

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Moreover, \( \Gamma \) is isomorphic to the split Cayley hexagon \( H(q) \).

The proof that \( \Gamma \) is a generalised hexagon is presented in Section 2.1. Note that the lines of type (i) form a spread of \( H(q) \). There exists a natural candidate for \( \Omega \) which we explain in detail in Section 2.2, and it is essentially the only one (Theorem 2.6), and this implies the ultimate result that \( \Gamma \) is isomorphic to \( H(q) \).

By the deep results of Thas and Van Maldeghem \[18, 19\] and Cameron and Kantor \[6\], if a set of points \( P \) and lines \( L \) of \( PG(6, q) \) form a generalised hexagon, then it is isomorphic to the split Cayley hexagon \( H(q) \) if \( P \) spans \( PG(6, q) \) and for any point \( x \in P \), the points collinear to \( x \) span a plane. A similar result was proved recently by Thas and Van Maldeghem \[20\], by foregoing the assumption that \( P \) and \( L \) form a generalised hexagon, and instead instituting the following five axioms: (i) the size of \( L \) is \( (q^6 - 1)/(q - 1) \), (ii) every point of \( PG(6, q) \) is incident with either 0 or \( q + 1 \) elements of \( L \), (iii) every plane of \( PG(6, q) \) is incident with 0, 1 or \( q + 1 \) elements of \( L \), (iv) every solid of \( PG(6, q) \) contains 0, 1, \( q + 1 \) or 2\( q + 1 \) elements of \( L \), and (v) every hyperplane of \( PG(6, q) \) contains at most \( q^3 + 3q^2 + 3q \) elements of \( L \).

One could instead characterise the split Cayley hexagon viewed as points and lines of the parabolic quadric \( Q(6, q) \), and the best result we have to date follows from a result of Cuypers and Steinbach \[7\] (Theorem 1.1):

**Theorem 1.3** (Cuypers and Steinbach \[7\] (paraphrased)). Let \( L \) be a set of lines of \( Q(6, q) \) such that every point of \( Q(6, q) \) is incident with \( q + 1 \) lines of \( L \) spanning a plane, and such that the concurrency graph of \( L \) is connected. Then the points of \( Q(6, q) \) together with \( L \) define a generalised hexagon isomorphic to the split Cayley hexagon \( H(q) \).

In Section 3 we will give an elementary proof of Theorem 1.3 by using Theorem 2.6.

**Some remarks on notation:** In this paper, the relative norm and relative trace maps will be defined for the quadratic extension \( GF(q^2) \) over \( GF(q) \). The relative norm \( N \) is the multiplicative function which maps an element \( x \in GF(q^2) \) to the product of its conjugates of \( GF(q^2) \) over \( GF(q) \). That is, \( N(x) = x^{q+1} \). The relative trace is instead the sum of the conjugates, \( T(x) := x + x^q \).
2. The 3-dimensional Hermitian surface and its Baer substructures

The two (classical) generalised quadrangles of particular importance in this note are $H(3,q^2)$ and $Q^-(5,q)$. First there is the incidence structure of all points and lines of a non-singular Hermitian variety in $\text{PG}(3,q^2)$, which forms the generalised quadrangle $H(3,q^2)$ of order $(q^2,q)$. Its point-line dual is isomorphic to the geometry of points and lines of the elliptic quadric $Q^-(5,q)$ in $\text{PG}(5,q)$, which yields a generalised quadrangle of order $(q,q^2)$ (see [16] 3.2.3). To construct $H(3,q^2)$ given a prime power $q$, we take a non-degenerate Hermitian form such as

$$\langle X,Y \rangle = X_0Y_{0}^q + X_1Y_{1}^q + X_2Y_{2}^q + X_3Y_{3}^q$$

and the totally isotropic subspaces of the ambient projective space, with respect to this form. Most of the material contained in this section can be found in Barwick and Ebert’s book [2] and Hirschfeld’s book [11] Chapter 7.

Every line of $\text{PG}(3,q^2)$ is (i) a generator (i.e., totally isotropic line) of $H(3,q^2)$, (ii) meets $H(3,q^2)$ in one point (i.e., a tangent line), or (iii) meets $H(3,q^2)$ in a Baer subline (also called a hyperbolic line). A Baer subline of the projective line $\text{PG}(1,q^2)$ is a subset of $q+1$ points in $\text{PG}(1,q^2)$ which form a $\mathbb{GF}(q)$-linear subspace. We may also speak of Baer subplanes and Baer subgeometries of $\text{PG}(3,q^2)$ as sets of points giving rise to projective subgeometries isomorphic to $\text{PG}(2,q)$ and $\text{PG}(3,q)$ respectively. A Baer subgenerator of $H(3,q^2)$ is a Baer subline of a generator of $H(3,q^2)$. We will often use the fact that three collinear points determine a unique Baer subplane ([2] Theorem 2.6) and a planar quadrangle determines a unique Baer subplane ([2] Theorem 2.8). In particular, if $b$ and $b'$ are two Baer sublines of $\text{PG}(2,q^2)$ sharing a point, but not spanning the same line, then there is a unique Baer subplane containing both $b$ and $b'$. We say that it is the Baer subplane spanned by $b$ and $b'$.

One class of important objects for us in this paper will be the degenerate Hermitian curves of rank 2. Suppose we have a fixed hyperplane, $\pi : X_3 = 0$ say, meeting $H(3,q^2)$ in a Hermitian curve $\mathcal{O}$. Let $\ell$ be a generator of $H(3,q^2)$. Then the polar planes of the points on $\ell$ meet $\pi$ in the $q^2+1$ lines through $L := \ell \cap \mathcal{O}$. Now suppose we have a Baer subgenerator $b$ contained in $\ell$, and containing the point $L$. Then the polar planes of the points of $b$ meet $\pi$ in $q+1$ lines through the point $L$ giving a dual Baer subline of $\pi$ with vertex $L$. Moreover, the points lying on this dual Baer subline define a variety with Gram matrix $U$; a Hermitian matrix of rank 2. So they correspond to solutions of $XU(X^\pi)^T = 0$ where $U$ satisfies $U^q = U^T$. For example, if we consider a point $P$ in $\pi$, say $(1,\omega,0,0)$ where $N(\omega) = -1$, and two points $A : (a_0,a_0\omega,a_2,1), B : (b_0,b_0\omega,b_2,1)$ spanning a line with $P$, then $P, A, B$ determine a Baer subline. In fact, if we suppose $B = P + \alpha A$ for some $\alpha \in \mathbb{GF}(q^2)^*$, then this Baer subline is $\{A\} \cup \{(p+\alpha t \cdot a_\alpha) | t \in \mathbb{GF}(q)\}$ where $A = \langle a \rangle$ and $P = \langle p \rangle$.

Let $u$ be the polarity defining $H(3,q^2)$. Since $P$ is precisely the nullspace of $U$, and the tangent line $P^u \cap \pi$ is contained in the dual Baer subplane, it is not difficult to calculate that $U$ can be written explicitly as

$$U = \begin{pmatrix} -\delta \omega^q & \delta & -\gamma \omega \\ \delta & \delta^q & \delta^q \omega \\ -\gamma \omega \delta & \gamma^q & 0 \end{pmatrix}, \quad \delta \omega^q = \delta^q \omega.$$

If we also suppose that the points of $A^u \cap \pi$ and $B^u \cap \pi$ are contained in the dual Baer subplane defined by $U$, then we can solve for $\alpha$ and $\gamma$ (but the expressions might be ugly!). Here we explore a simple example where $A : (0,0,1,\omega)$. Now $A^u \cap \pi$ has points of the form $(r,s,0,0)$, $N(r) + N(s) = 0$. So if $(r,s,0,0)$ also satisfies $(r,s,0)U(r^q,s^q,0)^T = 0$, then

$$(r,s,0)U(r^q,s^q,0)^T = T(r,s \delta^q) + 2N(s)\delta^q \omega.$$ 

So $T(r,s \delta) + 2N(s)\delta^q \omega = 0$ for every $(r,s,0,0)$ satisfying $N(r) + N(s) = 0$. In particular, $\delta$ is forced to be zero. Therefore, we can write

$$U = \begin{pmatrix} 0 & 0 & -\gamma \omega \\ 0 & 0 & \gamma \\ -\gamma \omega \delta & \gamma^q & 0 \end{pmatrix}.$$
2.1. Proof of the first part of Theorem \[1,2\] Here we prove that the incidence structure \( \Gamma \) of Theorem \[1,2\] is a generalised hexagon. Our approach is to use a definition of a generalised hexagon which is equivalent to the one stated in the introduction: (i) it contains no ordinary \( k \)-gon for \( k \in \{2, 3, 4, 5\} \), (ii) any pair of elements is contained in an ordinary hexagon, and (iii) there exists an ordinary heptagon (see \[22, \S 1.3.1\]). A thick generalised polygon has order \((s, t)\) if every line has \( s + 1 \) points and every point is incident with \( t + 1 \) lines. A counting argument shows that if we know that the number of points and lines of a generalised hexagon are \((s + 1)(1 + st + s^2t^2)\) and \((t + 1)(1 + st + s^2t^2)\), then the conditions (ii) and (iii) automatically follow from the first condition.

Proof. First we show that \( \Omega \) induces a point-partition of each generator (minus its point in the Hermitian curve \( \mathcal{O} \)). Let \( \ell \) be a generator of \( H(3, q^2) \) and let \( P \) be a point of \( \ell \mathcal{O} \). For a point \( X \), we will let \( X^* \) be the \( q + 1 \) elements of \( \Omega \) which lie on \( X \). Consider the \((q + 1) \) elements \( P^* \) of \( \Omega \) on \( P \). Since \( P^* \) covers the points of a Baer subplane, it follows that there is a unique element of \( \Omega \) contained in \( \ell \) and containing \( P \). Therefore \( \Omega \) induces a point-partition of each generator minus its point in the Hermitian curve \( \mathcal{O} \). It follows immediately that \( \Gamma \) is a partial linear space (i.e., every two points lie on at most one line).

Since \( H(3, q^2) \) is a generalised quadrangle, \( \Gamma \) has no triangles. So suppose now that we have a quadrangle \( R, S, T, U \) of \( \Gamma \). Then at least three of these points are necessarily affine points. For example, if two of these points were of type \((a)\), two points of type \((b)\), and with one line of type \((i)\) and three of type \((ii)\) making up the quadrangle, the three lines of type \((ii)\) would yield a triangle of generators. So this case is clearly impossible. At least three points, \( S, T, U \) say, are necessarily affine points and the lines of the quadrangle are elements of \( \Omega \). Moreover, \( R \) is also an affine point, since if \( R \) were a generator then \( S \) and \( U \) would lie on \( R \) and \( ST, TU, SU \) would then be a triangle in \( H(3, q^2) \); a contradiction. So all the four points \( R, S, T, U \) of a quadrangle must be affine.

Recall that \( u \) is the polarity defining \( H(3, q^2) \). Note that \( R^u \cap T^u \) is equal to \( SU \cap H(3, q^2) \) and \( SU \cap H(3, q^2) \) is a Baer subline with a point on \( \mathcal{O} \). Indeed \( R^u \) spans a Baer subplane fully contained in \( H(3, q^2) \) and it meets \( \mathcal{O} \) in a Baer subline and since \( R^u \cap T^u \cap H(3, q^2) \) is a Baer subplane contained in \( R^u \) then \( SU \cap H(3, q^2) \) has a point in \( \mathcal{O} \). Likewise \( S^u \cap T^u \) equal to \( RT \) and \( RT \cap H(3, q^2) \) is a Baer subline with a point in \( \mathcal{O} \). So \( SU \) and \( RT \) are parallel to each other under \( u \), but then each point of \( H(3, q^2) \) on \( SU \) is collinear with each point of \( H(3, q^2) \) on \( RT \), while the points of \( \mathcal{O} \) are pairwise non-collinear, a contradiction. Hence \( \Gamma \) has no quadrangles.

Suppose we have a pentagon \( R, S, T, U, W \) of \( \Gamma \). Now points of type \((b)\), which are affine points, are collinear in \( \Gamma \) if they are incident with a common element of \( \Omega \). Since each element of \( \Omega \) spans a generator, points of type \((b)\) are also collinear in \( H(3, q^2) \). So since \( H(3, q^2) \) is a generalised quadrangle, we see immediately that each point of our pentagon is an affine point. Suppose, by way of contradiction, that our pentagon has a point of type \((a)\), that is, a generator \( \ell \) of \( H(3, q^2) \). Then we would have four generators of \( H(3, q^2) \) forming a quadrangle and we obtain a similar “forbidden” quadrangle of affine points (i.e., \( RSTU \)) from the above argument. So there are no pentagons in \( \Gamma \).

A trivial counting argument shows that \( \mathcal{L} \) has size \((q^6 - 1)/(q - 1)\), which is equal to the sum of the number of affine points and the number of generators of \( H(3, q^2) \), and so it follows that \( \Gamma \) is a generalised hexagon (of order \((q, q))\). \( \square \)

2.2. Exhibiting a suitable set of Baer subgenerators. In this section, we describe a natural candidate for a set \( \Omega \) of Baer subgenerators satisfying the hypotheses of Theorem \[1,2\] Consider the stabiliser \( G_\mathcal{O} \) in \( \text{PGU}_4(q) \) of the Hermitian curve \( \mathcal{O} = \pi \cap H(3, q^2) \), where \( \pi \) consists of the elements whose last coordinate is zero. Then the elements of \( G_\mathcal{O} \) can be thought of (projectively) as matrices \( M_A \) of the form

\[
M_A := \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad A \in \text{GU}_3(q).
\]

Lemma 2.1. The group \( G_\mathcal{O} \) acts transitively on the set of Baer subgenerators which have a point in \( \mathcal{O} \).

Proof. Inside the group \( \text{PGU}_4(q) \), the stabiliser \( J \) of a generator \( \ell \) induces a \( \text{PGL}_2(q^2) \) acting 3-transitively on the points of \( \ell \). So the stabiliser in \( J \) of a point \( P \) of \( \ell \) acts transitively on the Baer sublines within \( \ell \) which contain \( P \). Now \( J \) meets \( G_\mathcal{O} \) in the stabiliser of a point of \( \ell \), and so \( G_\mathcal{O}, \ell \)
acts transitively on Baer subgenerators contained in $\ell$. Since $G_\mathcal{O}$ acts transitively on $\mathcal{O}$, the result follows.

The key to this construction is the action of a particular subgroup of $G_\mathcal{O}$. We will see later that this group naturally corresponds to the stabiliser in $G_2(q)$ of a non-degenerate hyperplane $Q^-(5, q)$ of $Q(6, q)$.

**Definition 2.2 (SU$_3$).** Let SU$_3$ be the group of collineations of $H(3, q^2)$ obtained from the matrices $M_A$ where $A \in SU_3(q)$.

In short, the orbits of SU$_3$ on Baer subgenerators with a point in $\mathcal{O}$, each form a suitable candidate for $\Omega$, as we will see.

**Lemma 2.3.** Let $\mathcal{O} = \pi \cap H(3, q^2)$, where $\pi$ is the hyperplane $X_3 = 0$ of $PG(3, q^2)$, and let $G_\mathcal{O}$ be the stabiliser of $\mathcal{O}$ in PGU$_4(q)$. Let $b$ be a Baer subgenerator of $H(3, q^2)$ with a point in $\mathcal{O}$. Then the stabiliser of $b$ in $G_\mathcal{O}$ is contained in SU$_3$.

**Proof.** Recall from the beginning of Section 2 that given a Baer subgenerator $b$ of $H(3, q^2)$ with a point $B$ in $\mathcal{O}$, there is a dual Baer subline of $\pi$ with vertex $B$. So there is a set of $3 \times 3$ Hermitian matrices $U$ of rank 2, which are equivalent up to scalar multiplication in $\mathbb{GF}(q^2)^*$. Now $G_\mathcal{O}$ induces an action on the pairs $[U, \ell]$, where $U$ is a Hermitian matrix of rank 2 and $\ell$ is a generator containing the nullspace of $U$, which we can write out explicitly by

$$[U, \ell]^{M_A} = [A^{-1}UA, \ell^{M_A}].$$

Let $\omega$ be an element of $\mathbb{GF}(q^2)$ satisfying $N(\omega) = -1$, and let $U_0$ and $\ell_0$ be

$$U_0 := \begin{pmatrix} 0 & 0 & \omega \\ -\omega & 1 & 0 \end{pmatrix}, \quad \ell_0 := \langle (1,\omega,0,0), (0,0,1,\omega) \rangle.$$

Since $G_\mathcal{O}$ acts transitively on Baer subgenerators with a point in $\mathcal{O}$ (Lemma 2.1), we need only calculate the stabiliser of $[U_0, \ell_0]$. Now let $M_A$ be an element of $G_\mathcal{O}$ fixing $[U_0, \ell_0]$. Since $M_A$ fixes $\ell_0$, we can see by direct calculation that $A$ is of the form

$$A = \begin{pmatrix} a & b & -f\omega \\ d & e & f \\ g & \omega & 1 \end{pmatrix},$$

with $(a + d\omega)\omega = b + \omega\omega$.

Now we see what it means for $A$ to centralise $U_0$ up to a scalar $k$, that is, $U_0A = kAU_0$. Hence

$$\begin{pmatrix} -g\omega & -g\omega^2 & -\omega \\ g & g\omega & 1 \\ d - a\omega^q & e - b\omega^q & 0 \end{pmatrix} = k \begin{pmatrix} -f\omega & -f\omega^q & b - a\omega \\ -f & \omega & e - d\omega \\ -\omega^q & 1 & 0 \end{pmatrix},$$

and we obtain

$$A = \begin{pmatrix} k^{-1} - b\omega^q & b & -k^{-1}g\omega^2 \\ k^{-1} - k - b\omega^q & k + b\omega^q & k^{-1}g\omega \\ g & \omega & 1 \end{pmatrix}$$

where $k \in \mathbb{GF}(q)$, $N(g) = k^2 + T(b\omega) - 1$ and $T(g\omega) = 0$ (in order for this matrix to be unitary).

The determinant of $A$ is

$$1 - g^2\omega(N(\omega) + 1)(\omega k^{-2} + b(N(\omega) + 1)k^{-1} + \omega) = 1$$

and therefore, the stabiliser of $[U_0, \ell_0]$ in $G_\mathcal{O}$ is contained in SU$_3$. \hfill $\square$

The above lemma allows us to attach a value to a Baer subgenerator that is an invariant for the action of SU$_3$.

**Definition 2.4 (Norm of a Baer subgenerator).** Let $\mathcal{O}$ be the Hermitian curve $H(3, q^2) \cap \pi$, where $\pi$ is the hyperplane $X_3 = 0$ of $PG(3, q^2)$ and let $G_\mathcal{O}$ be the stabiliser of $\mathcal{O}$ in PGU$_4(q)$. Fix a Baer subgenerator $b_0$ of $H(3, q^2)$ with a point in $\mathcal{O}$. Let $b$ be a Baer subgenerator of $H(3, q^2)$ with a point in $\mathcal{O}$, and suppose $M_A$ is an element of $G_\mathcal{O}$ such that $b = b_0^{M_A}$. Then the norm of $b$ is

$$||b|| := \det(A).$$

Moreover (by Lemma 2.3), the map $b \mapsto ||b||$ induces a group homomorphism $\phi$ from $G_\mathcal{O}$ to the multiplicative subgroup of elements of $\mathbb{GF}(q^2)^*$ satisfying $N(x) = 1$. 
Note that the kernel of $\phi$ is $SU_3$. The homomorphism $\phi$ is surjective and hence there is a natural partition of Baer subgenerators with a point in $O$ into $q + 1$ classes. Each orbit of $SU_3$ consists of Baer subgenerators with a common value for their norm.

**Lemma 2.5.** Let $\mu$ be an element of $GF(q^2)$ such that $N(\mu) = 1$. Let $O$ be a Hermitian curve of $H(3, q^2)$ defined by $X_3 = 0$, and let $\Omega$ be a set of Baer subgenerators with a point in $O$ which have norm equal to $\mu$. Then:

(i) Every affine point is on $q + 1$ elements of $\Omega$ covering a Baer subplane.

(ii) For every point $X \in \Omega$ and for every affine point $Y$ in $X^u$, there is a unique element of $\Omega$ through $X$ and $Y$.

**Proof.** Recall that $\Omega$ is an orbit of $SU_3$ on Baer subgenerators and $SU_3$ acts transitively on the affine points $H(3, q^2) \setminus O$, and so clearly every affine point is on $q + 1$ elements of $\Omega$. Moreover, such a set of $q + 1$ elements of $\Omega$ will cover a Baer subplane, as we show now. Let $Y$ be an affine point, let $Y^*$ be the set of $q + 1$ elements of $\Omega$ through $Y$ and let $b_0$ be one particular element of $Y^*$. Then every other element of $Y^*$ is in the orbit of $b_0$ under the stabiliser of $Y$ in $SU_3$. Now for every $g \in (SU_3)_Y$, we know that $(b_0^n) = (b_0)^g \in Y^\perp$ and so every element of $Y^*$ lies in the plane $Y^\perp$. At infinity, $Y^\perp$ meets $O$ in a Baer subline and so we have a triangle of Baer sublines spanning a Baer subplane of $Y^\perp$, and it is covered completely by the elements of $Y^*$.

To complete the proof, we need only prove (ii). Since the stabiliser of a point in $O$ is transitive on the set of affine points in the perp of that point, we can assume that $X = (1, \omega, 0, 0)$ and $Y = (0, 0, 1, \omega)$ for some $\omega$ satisfying $N(\omega) = -10$. We have already seen, in the proof of Lemma 2.3, that $X$ and $Y$ lie on a Baer subgenerator, which we can assume without loss of generality, is in $\Omega$. This Baer subgenerator is uniquely defined by a $3 \times 3$ Hermitian matrix $U$ of rank 2 and the generator $\ell$ spanning $X$ and $Y$, and we assume (as before) that $U$ has the form

$$U := \begin{pmatrix} 0 & 0 & -\omega \\ 0 & 0 & 1 \\ -\omega^q & 1 & 0 \end{pmatrix}.$$ 

Then the two-point stabiliser of $X$ and $Y$ inside $SU_3$ consists of elements $M_A$ with $A$ of the form

$$A = \begin{pmatrix} a & b & 0 \\ d & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where $(a + d\omega)\omega = b + e\omega$ and $(a \ b \ 0 \ 0 \ 0 \ 1)$.

Let's consider one of these elements $M_A$. Then

$$(A^q)^TU = \begin{pmatrix} 0 & 0 & -a^q\omega + d^q \\ -a^q + d & -b^q + e & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and we see that this matrix is a scalar multiple of $U$ (the scalar being $(-b\omega^q + e)$). Therefore $M_A$ fixes the Baer subgenerator defined by $[U, \ell]$. Hence there is a unique element of $\Omega$ on $X$ and $Y$. □

2.3. Classifying the suitable sets of Baer subgenerators.

**Theorem 2.6.** Suppose $\Omega$ is a set of Baer subgenerators of $H(3, q^2)$ with a point in $O$, such that every affine point is on $q + 1$ elements of $\Omega$ spanning a Baer subplane. Then $\Omega$ is an orbit under $SU_3$.

**Proof.** Let $b$ be a Baer subgenerator of $H(3, q^2)$ with a point in $O$. If $b'$ is another Baer subgenerator of $H(3, q^2)$ with a point in $O$ such that $b$ and $b'$ meet in an affine point and span a fully contained Baer subplane, then we will show that there is some element of $SU_3$ which maps $b$ to $b'$. Without loss of generality, we can choose our favourite Baer subgenerator $b$ and our favourite affine point. Suppose we have a fixed Baer subgenerator $b$ giving the dual Baer subline defined by

$$U = \begin{pmatrix} 0 & 0 & -\omega \\ 0 & 0 & 1 \\ -\omega^q & 1 & 0 \end{pmatrix}$$

and on the generator $\ell = \{(1, \omega, 0, 0), (0, 0, 1, \omega)\}$ where $N(\omega) = -1$. Let $P$ be the affine point $(0, 0, 1, \omega)$ and consider an arbitrary generator $\ell'$ on $P$ where $\ell' := \{(0, 0, 1, \omega), (1, \nu, 0, 0)\}$ and $N(\nu) = -1$. Suppose we have a Baer subgenerator $b'$ on $P$, on the generator $\ell'$, defined by the matrix $U'$. Since every element of $P^u \cap O$ is in the dual Baer subline defined by $U'$, we have that $U'$ can be written as

$$\begin{pmatrix} a & 0 & b \\ 0 & a & \gamma \\ \beta & \gamma & \beta^q \end{pmatrix}.$$
where $a \in \mathbb{GF}(q)$ and $\beta, \gamma \in \mathbb{GF}(q^2)$. For $(1, \nu, 0, 0)$ to be in the nullspace of $U'$, we must have $a = 0$ and $\beta = -\gamma \nu$. That is, $U'$ is just

$$
\begin{pmatrix}
0 & 0 & -\gamma \nu \\
0 & 0 & \gamma \\
-\gamma \nu & \gamma & 0 \\
\end{pmatrix}.
$$

Now $b$ and $b'$ span a fully contained Baer subplane if and only if the dual Baer sublines defined by $U$ and $U'$ share only the points of $P^u \cap \mathcal{O}$, on $\mathcal{O}$. Indeed suppose, by way of contradiction, that there is a point $Z$ of $\mathcal{O}$ in common between the dual Baer sublines defined by $U$ and $U'$. Then $Z^u$ meets $b$ in a point $Q$, different from $L$ and $P$ and it meets $b'$ in a point $Q'$ different from $L' = (L' = \pi \cap b')$ and $P$. Thus $Z^u \cap P^u$ meets $\mathbb{H}(3, q^2)$ in a Baer subline $b''$ containing $Q$ and $Q'$. Now the Baer subplane spanned by $b$ and $b'$ is fully contained if and only if if $b''$ has a point $T$ in $\mathcal{O}$. This implies that $T$ and $Z$ are points of $\mathcal{O}$ collinear on $\mathbb{H}(3, q^2)$; a contradiction.

Note that $P^u \cap \mathcal{O}$ consists of the points of the form $(1, \delta, 0, 0)$ together with $(0, 1, 0, 0)$. Suppose $(1, \delta, \eta, 0)$ is an element of both dual Baer sublines. That is, $(1, \delta, \eta)U(1, \delta^q, \eta^q)^T = 0$ and $(1, \delta, \eta)U'(1, \delta^q, \eta^q)^T = 0$. Now

$$(1, \delta, \eta)U(1, \delta^q, \eta^q)^T = (1, \delta, \eta) \begin{pmatrix} 0 & 0 & -\omega \\ -\omega & 1 & 0 \end{pmatrix} (1, \delta^q, \eta^q)^T = -\eta \omega^q + \eta \delta^q + (-\omega + \delta) \eta^q = T(\eta(\delta - \omega)^q),$$

$$(1, \delta, \eta)U'(1, \delta^q, \eta^q)^T = (1, \delta, \eta) \begin{pmatrix} 0 & 0 & -\gamma \nu \\ -\gamma \nu & \gamma & 0 \end{pmatrix} (1, \delta^q, \eta^q)^T = -\eta \gamma^q \nu^q + \eta \gamma^q \delta^q + (-\gamma \nu + \delta \gamma + \gamma \nu) \eta^q = -(\eta \gamma \nu^q + \eta \gamma \nu + \eta \gamma \delta + c \eta \nu^t + 1) = T(\eta \gamma^q (\delta - \nu)^q) + cN(\eta).$$

Since $1 + N(\delta) + N(\eta) = 0$, we see that our equations become

\text{(*)} \quad T(\eta(\delta - \omega)^q) = 0$$

So since the dual Baer sublines defined by $U$ and $U'$ share only the points of $P^u \cap \mathcal{O}$, then whenever condition \textbf{(\dagger)} holds for a choice of $\delta, \eta$, we will have $\eta = 0$. Therefore, we must have a priori that $c = 0$ and $\gamma \notin \mathbb{GF}(q)$.

Let $\eta = (\gamma \nu - \gamma^2 \nu) \omega^q$ and

$$
\delta = \frac{-\eta \omega^q + \eta \gamma^q (\omega - \nu)^q}{\gamma^q - \gamma}.
$$

Then a straightforward calculation shows that $1 + N(\delta) + N(\eta) = 0$, $T(\eta(\delta - \omega)) = 0$ and $T(\eta \gamma(\delta - \nu)) = 0$, so condition \textbf{(\dagger)} holds, and hence $\eta = 0$. Therefore, $\nu = \omega \gamma^q - 1$ and

$$
U' = \begin{pmatrix} 0 & 0 & -\gamma \omega \\ -\gamma \omega & \gamma & 0 \end{pmatrix}.
$$

We want to show that $U'$ is conjugate to $U$ under some element of $\text{SU}_3(q)$. Now the group $\text{SU}_2(q)$ of invertible $2 \times 2$ matrices with unit determinant, and fixing the form $X_0 Y_0^q + X_1 Y_1^q = 0$ on $\mathbb{GF}(q^2)^2$, has $q + 1$ orbits on totally isotropic vectors of $\mathbb{GF}(q^2)^2$. Each orbit consists of vectors $(x, y)$ where $y/x^q$ attains a common value. Therefore, there exists some element $C_0$ of $\text{SU}_2(q)$ such that $C_0(-\gamma \nu, \gamma)^T = (-\omega, 1)$. Let

$$
C := \begin{pmatrix} C_0 & 0 & \gamma \\ 0 & 0 & \gamma \omega \\ \gamma & 0 & 1 \end{pmatrix}.
$$

Then one can check easily that $C$ has determinant $1$ and $CU(C^q)^T = U'$. Therefore, there is some element of $\text{SU}_3$, which maps $b$ to $b'$.

For every affine point $P$, let $P^*$ be the set of $q + 1$ elements of $\Omega$ incident with $P$. Then by the above, every element of $P^*$ is contained in a common orbit of $\text{SU}_3$. Note that $\text{SU}_3$ is transitive on generators of $\mathbb{H}(3, q^2)$, and the stabiliser of a point $X$ of $\mathcal{O}$ in $\text{SU}_3$ is transitive on the affine points of $X^u$. Suppose now that $b$ and $b'$ do not meet in an affine point. Let $P \in b$. Then $P^* \subset b\text{SU}_3$. Now there
exists $g \in \text{SU}_3$ such that $\langle b \rangle^g = \langle b' \rangle$ and $P^g \in b'$. Thus $b' \in (P^g)^* \subset (b')^\text{SU}_3$. Note also that $P^g \in b^g$, and hence $b'^g \in (b')^\text{SU}_3$. Therefore $b$ and $b'$ are in the same orbit under $\text{SU}_3$. □

In Section 4 we will use the above result to prove Theorem 1.3.

3. The connection with the 6-dimensional parabolic quadric

A non-degenerate hyperplane section of $Q(6, q)$ can be of one of two types (up to isometry): it could induce a hyperbolic quadric $Q^+(5, q)$ or it could induce an elliptic quadric $Q^-(5, q)$. The stabiliser of a hyperbolic quadric section in $G_2(q)$ is isomorphic to $\text{SL}_3(q) : 2$, whilst the stabiliser of an elliptic quadric section in $G_2(q)$ is isomorphic to $\text{SU}_3(q) : 2$ (see [12]). These two maximal subgroups bring forth the two low-dimensional models of the Split Cayley hexagon that appear in this paper, and a second way to explain the interplay between these ‘linear’ and ‘unitary’ models is via Curtis-Tits and Phan systems; see Section 5. We begin first with some observations about the situation where we fix a $Q^+(5, q)$ hyperplane section.

The stabiliser $\text{SL}_3(q) : 2$ of $Q^+(5, q)$ fixes two disjoint planes $p'$ and $\sigma'$ of $Q^+(5, q)$, and then the lines of $\mathcal{H}(q)$ contained in $Q^+(5, q)$ are just the lines of $Q^+(5, q)$ which meet both $p'$ and $\sigma'$ in a point.

It was noticed in [6] that we can reconstruct $\mathcal{H}(q)$ from these two fixed planes together with an orbit $\Omega$ of $\text{SL}_3(q)$ on affine lines (of size $(q^3 - q)(q^2 + q + 1)$). We can capture the affine points by noticing that the $q + 1$ hexagon-lines through an affine point span a totally isotropic plane (sometimes known as an $\mathcal{H}(q)$-plane) meeting $Q^+(5, q)$ in a line disjoint from both $p'$ and $\sigma'$. Similarly, we can take the polar image of an affine line and consider its intersection with $Q^+(5, q)$. This results in a 3-dimensional quadratic cone of $Q^+(5, q)$ meeting both $p'$ and $\sigma'$ in a point, but having vertex not in $p'$ nor $\sigma'$. We can then employ the Klein correspondence to map our projection of $\mathcal{H}(q)$ on $Q^+(5, q)$, to $PG(3, q)$ (see [10] §15.4 for more on the Klein correspondence). We summarise this correspondence below:

| $PG(3, q)$ | $Q(6, q)$ |
|------------|-----------|
| **Point-plane anti-flag** $(p, \sigma)$ | A latin $p'$ and greek plane $\sigma'$ defining a hyperbolic quadric $Q^+(5, q)$ |
| **Pencils with vertex not in $\sigma$ and plane not through $p$** | Affine points of $Q(6, q) \backslash Q^+(5, q)$ |
| **Lines** | Points of $Q^+(5, q)$ |
| **Pencils with vertex in $\sigma$ and plane through $p$** | Lines of $Q^+(5, q)$ meeting $p'$ and $\sigma'$ in a point |
| **Parabolic congruences** | Affine lines of $Q(6, q)$, quadratic cones of $Q^+(5, q)$ |
| **Parabolic congruences having axis not incident with $p$ or $\sigma$, but having a pencil of lines with one line incident with $p$ and another incident with $\sigma$** | Affine lines of $\mathcal{H}(q)$ |

**Table 1. The extended Klein representation.**

Now we describe how we can view $\mathcal{H}(q)$ as substructures of the 3-dimensional Hermitian surface $H(3, q^2)$. A $t$-spread of $PG(d, q)$ is a collection of $t$-dimensional subspaces which partition the points of $PG(d, q)$. So necessarily, $t + 1$ must divide $d + 1$ and the size of a $t$-spread of $PG(d, q)$ is $(q^{d+1} - 1)/(q^{t+1} - 1)$. If $t + 1$ is half of $d + 1$, we usually call a $t$-spread just a spread of $PG(d, q)$. Suppose we have a $t$-spread $S$ of $PG(d, q)$ and embed $PG(d, q)$ as a hyperplane in $PG(d + 1, q)$. If we define the blocks to be the $(t + 1)$-dimensional subspaces of $PG(d + 1, q)$ not contained in $PG(d, q)$ incident with an element of the $t$-spread, then together with the affine points $PG(d + 1, q) \backslash PG(d, q)$, we obtain a linear space; in fact, a $2 - (q^{d+1}, q^{t+1}, 1)$ design. This linear representation of a $t$-spread is a generalisation of the commonly called André/Bruck-Bose construction (where $t + 1 = (d + 1)/2$), and is fully explained by Barlotti and Cofman [1]. More generally, it is possible that this construction produces a Desarguesian affine space and we then say that the given $t$-spread is Desarguesian. It turns out that a $t$-spread $S$ is Desarguesian if and only if $S$ induces a spread in any subspace generated by two distinct elements of $S$ (see [13] and [17]).
Now consider $\text{PG}(3,q^2)$ and a hyperplane $\pi_\infty$ therein, and identify $\text{AG}(3,q^2)$ with the affine geometry $\text{PG}(3,q^2)\setminus\pi_\infty$. We will be considering the correspondence between objects in $\text{H}(3,q^2)$ and $\text{Q}(6,q)$, where $\mathcal{S}$ is a Hermitian spread of a non-degenerate hyperplane section $\mathcal{Q}^-\langle 5,q \rangle$ of $\text{Q}(6,q)$. One can also obtain this correspondence via field reduction from $\text{H}(3,q^2)$ to $\text{Q}^\perp\langle 7,q \rangle$, and then slicing with a non-degenerate hyperplane section (see [14]). We will call this correspondence the Barlotti-Cofman-Segre representation of $\text{H}(3,q^2)$. Below we summarise the various correspondences between objects in $\text{H}(3,q^2)$ and objects in $\text{Q}(6,q)$ obtained by the Barlotti-Cofman-Segre representation of $\text{H}(3,q^2)$.

Throughout, we fix a hyperplane $\Sigma_\infty$ at infinity intersecting $\text{Q}(6,q)$ in a $\mathcal{Q}^\perp\langle 5,q \rangle$, which corresponds to a fixed non-degenerate hyperplane $\pi_\infty$ of $\text{H}(3,q^2)$, and we let $\mathcal{S}$ denote a Hermitian spread of $\Sigma_\infty$.

| $\text{H}(3,q^2)$ | $\text{Q}(6,q)$ |
|-------------------|-----------------|
| Hermitian curve $\mathcal{O}$ of $\pi_\infty$ | Hermitian spread $\mathcal{S}$ of $\mathcal{Q}^\perp\langle 5,q \rangle$ |
| Affine points $\text{H}(3,q^2)\setminus\pi_\infty$ | Affine points of $\text{Q}(6,q)\setminus\mathcal{Q}^\perp\langle 5,q \rangle$ |
| Generators of $\text{H}(3,q^2)$ | Generators of $\text{Q}(6,q)$ incident with some element of $\mathcal{S}$ |
| Baer subplane contained in $\text{H}(3,q^2)$ meeting $\mathcal{O}$ in a Baer subline | Generators of $\text{Q}(6,q)$ not incident with any element of $\mathcal{S}$ |
| Baer subgenerators with a point in $\mathcal{O}$ | Affine lines of $\text{Q}(6,q)$ |

Table 2. The Barlotti-Cofman-Segre representation.

The table below shows how we can directly obtain the model for the split Cayley hexagon on the 3-dimensional Hermitian surface via the Barlotti-Cofman-Segre correspondence. We can recover the affine points of $\text{Q}(6,q)$ by noticing that a plane incident with a spread element will correspond to a hexagon-plane; a point of $\text{H}(q)$ together with its $q+1$ incident lines.

| In $\text{H}(3,q^2)$ | Barlotti-Cofman-Segre image in $\text{Q}(6,q)$ |
|-------------------|---------------------------------------------|
| POINTS (a) Lines of $\text{H}(3,q^2)$ | Planes of $\text{Q}(6,q)$ containing a spread element. |
| (b) Affine points of $\text{H}(3,q^2)\setminus\mathcal{O}$ | Affine points of $\text{Q}(6,q)\setminus\mathcal{Q}^\perp\langle 5,q \rangle$. |
| LINES (i) Points of $\mathcal{O}$ | Lines of the Hermitian spread. |
| (ii) Elements of $\Omega$ | Affine lines spanning a totally isotropic plane with a spread element. |

Table 3. The split Cayley hexagon in $\text{H}(3,q^2)$.

4. Characterising the split Cayley hexagon in the 6-dimensional parabolic quadric

By the Barlotti-Cofman-Segre correspondence, we can translate Theorem 1.2 to a statement about substructures of $\text{Q}(6,q)$. However, the information that can be transferred via this correspondence is not sufficient to characterise a set of lines $\text{Q}(6,q)$ as the lines of a generalised hexagon; there is an additional case. The natural model of the split Cayley hexagon was revised in the introduction, and here we briefly point out a characterisation of it as a set of lines of $\text{Q}(6,q)$. It is a special case of a result of Cuypers and Steinbach [7] Theorem 1.1, but we give a direct proof for completeness.

THEOREM 4.1. Let $\mathcal{L}$ be a set of lines of $\text{Q}(6,q)$ such that every point of $\text{Q}(6,q)$ is incident with $q+1$ lines of $\mathcal{L}$ spanning a plane. Then one of the following occurs:

(a) There is a spread $\mathcal{S}$ of $\text{Q}(6,q)$ such that $\mathcal{L}$ is equal to the union of the lines contained in each generator of $\mathcal{S}$.

(b) The points of $\text{Q}(6,q)$ together with $\mathcal{L}$ define the points and lines of a generalised hexagon, and a plane of $\text{Q}(6,q)$ contains 0 or $q+1$ elements of $\mathcal{L}$ in it.
PROOF. Let $\Gamma$ be the geometry having the points of $Q(6, q)$ as its points, and having $\mathcal{L}$ as its set of lines. Clearly $\Gamma$ is a partial linear space where there are $q + 1$ lines through every point, and $q + 1$ points through every line. We will write $P^*$ for the pencil of $q + 1$ lines of $\mathcal{L}$ incident with $P$.

Since every plane of $\text{PG}(6, q)$ meets $Q(6, q)$ in a full plane, a conic, a line, a pair of concurrent lines or a point, it follows that every plane intersects $\mathcal{L}$ in $q^2 + q + 1$, $q + 1$ or 0 lines. We will show now that the first possibility leads to case (a). Suppose there is a plane $\pi$ with $q^2 + q + 1$ lines of $\mathcal{L}$. Let $\ell$ be an element of $\mathcal{L}$ not contained in such a plane. Then the $q + 1$ planes on the tangent quadric containing $\ell$ (i.e., the points collinear to all the points on $\ell$) contain $q + 1$ elements of $\mathcal{L}$. Since there is always at least one point $p$ of $\pi$ collinear with all points of $\ell$, we see that the point $p$ is now incident with at least $q + 2$ elements of $\mathcal{L}$; a contradiction. Hence either every point is in a plane with $q^2 + q + 1$ elements of $\mathcal{L}$ (and we obtain the spread of $Q(6, q)$), or no point is.

Suppose now that $\mathcal{L}$ is not partitioned by a spread of $Q(6, q)$. So no plane of $\text{PG}(6, q)$ contains $q^2 + q + 1$ elements of $\mathcal{L}$, and therefore, every plane intersects $\mathcal{L}$ in 0, 1 or $q + 1$ lines. We continue now to prove that $\Gamma$ is a generalised hexagon. Clearly there is no triangle formed by lines of $\mathcal{L}$, so suppose we have a quadrangle $R, S, T, U$ in $\Gamma$. Note that these points do not lie in a common plane. The planes spanning $T^*, U^*$ and $R^*$ are three totally singular planes contained in a common 3-space, which implies that this 3-space is also totally singular; a contradiction. Suppose now we have a pentagon $R, S, T, U, W$ of $\Gamma$, (and the ordering of these points is important). So $RSTU$ spans a 3-space intersecting $Q(6, q)$ in two totally singular planes, namely $S^*$ and $T^*$. Now $W$ is collinear (in $\mathcal{L}$) with $R$ and $U$, and therefore the line $RU$ is totally singular; which implies that $RSTU$ is totally singular, a contradiction. So there are no $k$-gons in $\Gamma$ with $k < 6$. Since $\mathcal{L}$ has size equal to the number of points of $Q(6, q)$, it follows that $\Gamma$ is a generalised hexagon of order $q$.

Let $N_i$ be the number of planes of $Q(6, q)$ containing $i$ elements of $\mathcal{L}$. So $N_0 + N_1 + N_{q+1} = (q + 1)(q^2 + 1)(q^3 + 1)$. Now each point is on a unique plane containing $q + 1$ elements of $\mathcal{L}$, and so $N_{q+1} = (q^3 + 1)(q^2 + q + 1)$. Now for a given point $X$, all but one of the planes on $X$ would have no lines of $\mathcal{L}$ on it, which accounts for $N_0 = q^2(q^3 + 1)$ planes (n.b., there are $(q + 1)(q^2 + 1)$ planes on any point, and a plane contains $q^2 + q + 1$ points). So it follows that $N_1 = 0$.

LEMMA 4.2. Let $\mathcal{L}$ be a set of lines of $Q(6, q)$ such that every point $X$ of $Q(6, q)$ is incident with $q + 1$ lines of $\mathcal{L}$ spanning a plane $X^*$, and such that the concurrency graph of $\mathcal{L}$ is connected. Suppose $\Pi$ is a nondegenerate hyperplane meeting $Q(6, q)$ in a $Q^-(5, q)$-quadric. Then the set $S := \{X^* \cap \Pi : X \in Q^-(5, q)\}$ defines a Hermitian spread of $Q^-(5, q)$.

PROOF. Any pair of lines of $\mathcal{S}$ are disjoint since otherwise they would intersect in a point $P$ and the plane $P^*$ spanned by the $q + 1$ elements of $\mathcal{L}$ incident with $P$ would then be contained in $Q^-(5, q)$. Therefore, $\mathcal{S}$ forms a spread of $Q^-(5, q)$.

Now consider two elements $\ell$ and $m$ of $\mathcal{S}$. The solid $\langle \ell, m \rangle$ meets $Q^-(5, q)$ in a $Q^+ (3, q)$ section. The polar image of $\langle \ell, m \rangle$, within $Q(6, q)$, is then a plane meeting $Q(6, q)$ in a non-degenerate conic $\mathcal{C}$. Let $r$ be a line in the regulus determined by $\ell$ and $m$, and suppose for a proof by contradiction that $r$ is not an element of $\mathcal{S}$. Then each of the $q + 1$ points $Z_i$ on $r$ defines a different element $\ell_i := Z_i^* \cap \Pi$ of $\mathcal{S}$. Since the lines contained in $\langle \ell, m \rangle$ concurrent with $r$ form the opposite-regulus to that defined by $\ell$ and $m$, it follows that none of the $\ell_i$ are contained in $\langle \ell, m \rangle$.

Since $\ell$ is a line of $\mathcal{L}$ and, by Theorem 4.1, a plane of $Q(6, q)$ has 0 or $q + 1$ elements of $\mathcal{L}$ contained in it, each of the $q + 1$ planes $\langle \ell, X_\ell \rangle$ for $X_\ell \in \mathcal{C}$ is a plane $Y^*$ for some $Y \in \ell$. Similarly, each of the $q + 1$ planes $\langle m, X_m \rangle$ is a plane $Y^*$ for some $Y \in m$. Hence for each $X_\ell \in \mathcal{C}$ the plane $X_\ell^*$ meets $\langle \ell, m \rangle$ in a line of the opposite regulus. Therefore, there is a one-to-one correspondence between points $X_\ell$ of $\mathcal{C}$ and points $Z_i$ of $r$. That is, the line $X_\ell Z_i$ is a line of $\mathcal{L}$ for every $i$.

Recall that the concurrency graph of $\mathcal{L}$ is connected, and so by Theorem 4.1, $\mathcal{L}$ forms the lines of a generalised hexagon. Let $Z_1$ and $Z_2$ be two elements on $r$. Now $Z_1^*$ is a hyperplane and $Z_2^*$ is a plane, so we have two cases: (i) $Z_2^*$ is contained in $Z_1^+$, or (ii) $Z_2^*$ meets $Z_1^+$ in a line $n$. The first case cannot arise as a plane of $Q(6, q)$ contained in $Z_1^+$ must go through $Z_1$, and we have assumed that $r$ is not in $\mathcal{L}$. Suppose we have the second case. Since $Z_2$ lies in $Z_1^+$, the line $n$ lies on $Z_2$ and so $n$ is a line of $\mathcal{L}$ in $Z_2^*$. Note that $\langle Z_1, n \rangle$ is a plane of $Q(6, q)$ having at least one element of $\mathcal{L}$ contained in it. By Theorem 4.1, a plane of $Q(6, q)$ has 0 or $q + 1$ elements of $\mathcal{L}$ contained in it. Therefore, there is some point $V$ on $n$ such that $Z_1 \in V^*$. Hence we have a line of $\mathcal{L}$ going through $Z_1$ concurrent with
n, and $Z_1$ and $Z_2$ are at distance 4. This requirement then forces $r$ to lie in $V^*$. and hence each $Z_i^*$ goes through $V$.

Now $⟨C⟩$ is a non-degenerate plane through $Π^⊥$ and $Π^⊥ ∉ ⟨Z_i^* : Z_i ∈ r⟩$. Therefore, each $Z_i^*$ meets the conic only in the point $X_r$. The lines $X_1Z_1$ and $VZ_1$ are lines of $L$ and since $X_1, V; Z_1 ∈ r^⊥$, we have that $Z_i^*$ is contained in $r^⊥$; a contradiction. (Otherwise, $Z_i^*$ would be a plane through $Z_2$). Hence $r ∈ S$ and $S$ is closed under taking reguli. By [4], §3.1.2 and [15], such a spread of $Q^−(5, q)$ is necessarily a Hermitian spread of $Q^−(5, q)$. □

**Proof of Theorem 1.13.** First we will translate the hypothesis to the 3-dimensional Hermitian variety $H(3, q^2)$ via the Barlotti-Cofman-Segre correspondence. So let us fix a non-degenerate hyperplane section $Q^−(5, q)$ and consider the set of lines of $L$ that are contained in $Q^−(5, q)$. By Lemma 4.2, $S$ is a Hermitian spread of $Q^−(5, q)$ and so we have the ingredients for the Barlotti-Cofman-Segre correspondence, whereby the spread $S$ corresponds to a fixed Hermitian curve $O$ of $H(3, q^2)$. Recall that the elements of $L$ not contained in $Q^−(5, q)$ are mapped to a subset $Ω$ of the Baer subgenerators having a point in $O$. Also, the affine points of $Q(6, q)$ are mapped to the affine points of $H(3, q^2)\O$. We will show that $Ω$ satisfies the hypotheses of Theorem 1.12 that is, a generator spanned by $q + 1$ elements of $L$ corresponds to a Baer subplane of $H(3, q^2)$. Now by Theorem 1.11 we have either (a) $L$ is the union of lines of the planes of a spread $S$ of $Q(6, q)$, or (b) $L$ forms the lines of a generalised hexagon. Case (a) cannot occur as the concurrency graph of $L$ is connected. So $L$ is the lines of a generalised hexagon embedded into $Q(6, q)$. Let $P$ be an affine point of $Q(6, q)$ and let $P^*$ be the $q + 1$ elements of $L$ incident with $P$. By our hypothesis, $P^*$ spans a plane $π_P$. If this plane were to be incident with an element of $S$, then $π_P$ would contain more than $q + 1$ elements of $L$ thus implying that $π_P$ would have all of its lines in $L$; this would then imply that the concurrency graph of $L$ is disconnected (see the proof of Lemma 1.1). Therefore, $π_P$ is not incident with any element of $S$, and hence, $π_P$ meets $Q^−(5, q)$ in a transversal line to $q + 1$ elements of $S$. By the Barlotti-Cofman-Segre correspondence, $π_P$ corresponds to a Baer subplane of $H(3, q^2)$, as required.

By Theorem 2.6, $Ω$ is an orbit of $SU_3$. Moreover, this group $SU_3$ lies within the stabiliser in $PGU_4(q)$ of a non-degenerate hyperplane, and so corresponds to a subgroup $SU_3$ of the stabiliser of $S$. Now there are $q + 1$ split Cayley hexagons whose lines not lying in $Q^−(5, q)$ form an orbit under $SU_3$, so it remains to observe that $SU_3$ has only $q + 1$ orbits of size $q(q + 1)(q^3 + 1)$. Indeed, the orbits of $SU_3$ on lines of $Q(6, q)$ can be described completely geometrically from the corresponding orbits of objects in $H(3, q^2)$ (see Table 2). Therefore, $Ω$ is the set of lines of some split Cayley hexagon (having a set of lines containing $S$).

| Orbits in $H(3, q^2)$ | Orbits on lines of $Q(6, q)$ | Size |
|----------------------|-----------------------------|------|
| Hermitian curve $O$ of $π_∞$ | Hermitian spread $S$ of $Q^−(5, q)$ | $q^3 + 1$ |
| Affine points $H(3, q^2)\π_∞$ | Lines of $Q^−(5, q)$ not in $S$ | $q^2(q^3 + 1)$ |
| Baer subgenerators with no point in $O$ | Affine lines not meeting an element of $S$ in a totally singular plane | $q^2(q^2 - 1)(q^3 + 1)$ |
| Baer subgenerators with a point in $O$ | Affine lines meeting an element of $S$ in a totally singular plane | $(q + 1) × q(q + 1)(q^3 + 1)$ |

**Table 4. Orbits of $SU_3$ on lines of $Q(6, q)$.**

□

### 5. A connection with Phan theory

In the theory of linear algebraic groups, if a simply connected simple algebraic group $G$ of type $B_n$, $C_n$, $D_2n$, $E_7$, $E_8$, $F_4$ or $G_2$ has a Curtis-Tits system for its extended Dynkin diagram then there is a twisted version known as a Phan system for associated finite groups corresponding to fixed points of so-called Frobenius maps of $G$, where the $SL_2$-subgroups of the Curtis-Tits system are replaced with
certain SU$_2$-subgroups. This phenomenon has been known since the 1970’s to both group theorists and those working in the theory of twin buildings. In a Curtis-Tits system for a finite group $G$ (defined over $\text{GF}(q)$), if $K_\alpha$ and $K_\beta$ are two $\text{SL}_2$-subgroups for two fundamental roots $\alpha$ and $\beta$ joined by a single bond, then $(K_\alpha, K_\beta)$ is isomorphic to $(\text{P}SL_3(q))$. Whereas in the corresponding Phan system, a single bond represents an amalgam $(K_\alpha, K_\beta)$ isomorphic to $(\text{P}SU_3(q))$. (See [3], [5] and [9] for more on Phan systems). The geometric model of the split Cayley hexagon that we presented in this paper was inspired by a unitary analogue of the $\text{SL}_3$-model introduced by Cameron and Kantor [6].

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Curtis-Tits system} & \textbf{Phan system} \\
\text{$G_2(q)$} & \text{$SU_3(q)$} \\
\text{$\alpha$} & \text{$\alpha$} \\
\text{$\beta$} & \text{$\beta$} \\
\hline
\text{$-3\alpha - 2\beta$} & \text{$-3\alpha - 2\beta$} \\
\hline
\end{tabular}
\caption{A summary of the Curtis-Tits and Phan systems for the extended Dynkin diagram of $G_2(q)$.}
\end{table}

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