THE DENNIS’ SUPERTRACE AND THE HOCHSCHILD HOMOLOGY
OF SUPERMATRICES

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Dedicated to Professor Nicolae Telean, on his 65th birthday anniversary

Abstract. We construct, in this paper, a generalization of the Dennis trace (for matrices) to the case of the supermatrices over an arbitrary (not necessarily commutative) superalgebra with unit. By analogy with the ungraded case, we show how it is possible to use this map to construct an isomorphism from the Hochschild homology of the superalgebra to the Hochschild homology of the supermatrix algebra.

1. The supertrace and the supercommutator

We remind first a couple of things related to the \( \mathbb{Z}_2 \)-grading of the supermatrix algebra \( M_{p,q}(R) \). For the general material regarding the superalgebra the reader should consult the classical books of Bartocci, Bruzzo and Hernandez-Ruiperez ([3]) and Manin ([5]). A matrix \( A \in M_{p,q}(R) \) is considered to be homogeneous if it can be decomposed into blocks

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

where \( A_{11}, A_{12}, A_{21}, A_{22} \) are matrices of type \((p,p),(p,q),(q,p),(q,q)\), respectively and all the components of one of the matrix are homogeneous of the same parity. Moreover, it is assumed that the elements of \( A_{11} \) and \( A_{22} \) have the same parity and the same is true for the other pair of matrices. Now, a matrix \( A \) satisfying these conditions is

- **even** if the elements of the diagonal blocks are even, while the elements from the blocks off the diagonal are odd;
- **odd** if the elements of the diagonal blocks are odd, while the elements from the blocks off the diagonal are even.

Now, the supertrace of matrices is the \( R \)-module morphism \( \text{str} : M_{p,q}(R) \to R \), defined on homogeneous matrices \( A \) by

\[
\text{str}(A) = \text{tr} A_{11} + (-1)^{1+|A|} \text{tr} A_{22}
\]

where \( \text{tr} \) is the ordinary trace of a matrix, while with \( | \cdot | \) it is denoted the parity of an element. We shall use the same notation to denote the parity of an element of the algebra \( R \) and a matrix, because it will be always clear from the context what kind of object we are dealing with.

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It should be noted that, contrary to what one would expect, the supertrace coincides with the ordinary trace of a matrix \( A \) when the matrix \( A \) is odd and not even. On the other hand, for matrices from \( M_{p,0}(R) \) the supertrace is identical to the trace, no matter what parity the matrices might have. Finally, we remark that \( \text{str} \) is, indeed, an \( R \)-module morphism, in the sense that not only it is linear, but also preserves the parity.

Let, now, \( A, B \in M_{p,q}(R) \) be two supermatrices. Their supercommutator will be

\[
\{A, B\} \doteq A \cdot B - (-1)^{|A||B|} B \cdot A.
\]

Thus, if at least one of the two supermatrices is even, the supercommutator reduces to the ordinary commutator of two matrices. There is a difference in sign only in the case when both supermatrices are odd.

A very important property of the supercommutator, which will be useful also in the following is that it is related rather “nicely” to the supertrace.

Since the operations involved are either linear or bilinear, it is, clearly, enough to make the computations on a system of (homogeneous) generators of the algebra \( M_{p,q}(R) \). A very convenient such basis is constructed from matrices of the form \( E^{ij}(a) \), where \( i, j \in \{1, \ldots, p+q\} \), while \( a \) is a homogeneous element of the algebra \( A \). Here

\[
E^{ij}(a) = \delta^{ij}_{kl} \cdot a = \begin{cases} a & \text{if } i = k, j = l, \\ 0 & \text{if } i \neq k \text{ or } j \neq l. \end{cases}
\]

The parity of a matrix of the form \( E^{ij}(a) \) is related to the parity of the element \( a \) in a very simple manner: if \( a \) is in the diagonal block components, than the two objects have the same parity, otherwise their parity is opposed. More precisely,

\[
|E^{ij}(a)| = \begin{cases} |a| & \text{if } i \leq p, j \leq p \text{ or } i > p, j > p, \\ 1 + |a| & \text{if } i \leq p, j > p \text{ or } i > p, j \leq p. \end{cases}
\]

It is not difficult to see that the family of supermatrices

\[
\{E^{ij}(a) \mid 1 \leq i, j \leq p+q, a \in R_0 \text{ or } a \in R_1\}
\]

is an ideal in \( M_{p,q}(R) \):

\[
E^{ij}(a) \cdot E^{kl}(b) = \begin{cases} E^{il}(a \cdot b), & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases}
\]

Now,

\[
\{E^{ij}(a), E^{kl}(b)\} = E^{ij}(a) \cdot E^{kl}(b) - (-1)^{|E^{ij}(a)||E^{kl}(b)|} E^{kl}(b) \cdot E^{ij}(a) =
\]

\[
\begin{cases} 0, & j \neq k, i \neq l, \\ E^{il}(ab) & j = k, i \neq l, \\ -(-1)^{|E^{ij}(a)|+|E^{kl}(b)|} E^{kj}(ab), & j \neq k, i = l, \\ E^{li}(ab) - (-1)^{|E^{ij}(a)||E^{kl}(b)|} E^{kj}(ba), & j = k, i = l. \end{cases}
\]
We note first that the matrices $E^{ij}(a)$ with $i \neq j$ are commutators. Let us compute now the supertrace of the supercommutator of two generating matrices, separately for each combination of indices.

We get, obviously, 0 if $j \neq k$ and $i \neq l$. If $j = k, i \neq l$, we obtain

$$\text{str} \left\{ E^{ij}(a), E^{ji}(b) \right\} = \text{str} E^{il}(ab) = 0. \tag{8}$$

The same is true for the case $j \neq k, i = l$. The only interesting case is the last one. Now we have

$$\begin{align*}
\text{str} \left\{ E^{ij}(a), E^{ji}(b) \right\} &= \text{str} \left( E^{ii}(ab) - (-1)^{|E^{ij}(a)|} |E^{ij}(b)| E^{ji}(ba) \right) \\
&= \text{str} E^{ii}(ab) - (-1)^{|E^{ij}(a)|} |E^{ij}(b)| E^{ji}(ba).
\end{align*} \tag{9}$$

We have several subcases to consider here:

(i) Suppose we have $i = j$. In this case, we have $|E^{ii}(a)| = |a|$, $|E^{ii}(b)| = |b|$, thus,

$$\begin{align*}
\{E^{ii}(a), E^{ii}(b)\} &= E^{ii}(ab) - (-1)^{|a|-|b|} E^{ii}(ba) = E^{ii}(ab - (-1)^{|a|-|b|} ba) \\
&= E^{ii}(\{a, b\}),
\end{align*}$$

therefore

$$\begin{align*}
\text{str} \left\{ E^{ii}(a), E^{ii}(b) \right\} &= \text{str} E^{ii}(\{a, b\}) = \begin{cases} 
\{a, b\}, & i \leq p \\
(-1)^{1+|a|+|b|}\{a, b\}, & i > p.
\end{cases}
\end{align*}$$

(ii) $i < j \leq p$. In this case we have $|E^{ij}(a)| = |a|$, $|E^{ji}(b)| = |b|$, therefore

$$\begin{align*}
\{E^{ij}(a), E^{ji}(b)\} &= E^{ii}(ab) - (-1)^{|a|-|b|} E^{jj}(ba),
\end{align*}$$

hence

$$\text{str} \left\{ E^{ij}(a), E^{ji}(b) \right\} = \text{str} E^{ii}(ab) - (-1)^{|a|-|b|} \text{str} E^{jj}(ba) =$$

$$ab - (-1)^{|a|-|b|} ba = \{a, b\}.$$ 

(iii) $i \leq p < j$. Now $|E^{ij}(a)| = 1 + |a|$, $|E^{ji}(b)| = 1 + |b|$, and then

$$\begin{align*}
\{E^{ij}(a), E^{ji}(b)\} &= E^{ii}(ab) - (-1)^{1+|a|+|b|+|a|-|b|} E^{jj}(ba)
\end{align*}$$

and

$$\text{str} \left\{ E^{ij}(a), E^{ji}(b) \right\} = \text{str} E^{ii}(ab) - (-1)^{1+|a|+|b|+|a|-|b|} \text{str} E^{jj}(ba) =$$

$$ab - (-1)^{1+|a|+|b|+|a|-|b|} \cdot (-1)^{1+|a|+|b|} ba =$$

$$ab - (-1)^{|a|-|b|} ba = \{a, b\}.$$ 

(iv) $p < i < j$. In this situation, $|E^{ij}(a)| = |a|$, $|E^{ji}(b)| = |b|$, therefore

$$\begin{align*}
\{E^{ij}(a), E^{ji}(b)\} &= E^{ii}(ab) - (-1)^{|a|-|b|} E^{jj}(ba),
\end{align*}$$

but

$$\text{str} \left\{ E^{ij}(a), E^{ji}(b) \right\} = \text{str} E^{ii}(ab) - (-1)^{|a|-|b|} \text{str} E^{jj}(ba) =$$

$$(-1)^{1+|a|+|b|} \left(ab - (-1)^{|a|-|b|} ba\right) = (-1)^{1+|a|+|b|}\{a, b\}.$$
(v) $j < i \leq p$. This case is identical to the case (ii).
(vi) $j \leq p < i$. We have now, as in the case (iii), $|E^{ij}(a)| = 1 + |a|$, $|E^{ji}(b)| = 1 + |b|$, therefore
\[ \{E^{ij}(a), E^{ji}(b)\} = E^{ii}(ab) - (-1)^{1+|a|+|b|} E^{ij}(ba), \]
but
\[ \text{str} \{E^{ij}(a), E^{ji}(b)\} = \text{str} E^{ii}(ab) - (-1)^{1+|a|+|b|} \text{str} E^{ij}(ba) = \]
\[ = (-1)^{1+|a|+|b|} ab - (-1)^{1+|a|+|b|} ba = \]
\[ = (-1)^{1+|a|+|b|} \{a, b\}. \]
(vii) $p < j < i$. This case is identical to the case (iv).

2. The Hochschild homology of superalgebras

The Hochschild complex for superalgebras (Kassel, 1986), is very similar to the analogous complex for ungraded case. Namely, the chain groups are, as in the classical case, $C_m(R) = R^\otimes m+1$, where, of course, the tensor product should be understood in the graded sense, while the face maps and degeneracies are given by
\[
\delta^m_i (a_0 \otimes \cdots \otimes a_m) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_m, \quad \text{if } 0 \leq i < m, \tag{10}
\]
\[
\delta^m_m (a_0 \otimes \cdots \otimes a_m) = (-1)^{|a_m|(|a_0|+\cdots+|a_{m-1}|)} a_m a + a_0 \otimes a_1 \otimes \cdots \otimes a_{m-1}, \tag{11}
\]
\[
s^m_i (a_0 \otimes \cdots \otimes a_m) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_m, \quad 0 \leq i \leq m. \tag{12}
\]
Now the differential is defined in the usual way, meaning $d^m : C_m(R) \to C_{m-1}(R)$,
\[
d^m = \sum_{i=0}^{m} (-1)^i \delta^m_i. \tag{13}
\]
and the Hochschild homology of the superalgebra is just the homology of the complex $(C(R), d)$. In particular, it is easy to see that for any superalgebra $R$ we have
\[
H_0(R) = R/\{R, R\}, \tag{14}
\]
where $\{R, R\}$ is the subspace generated by the supercommutators.

3. The Dennis supertrace and its properties

If $A^1$ and $A^2$ are two square matrices over an arbitrary algebra $R$, then their product is
\[
(A^1 \cdot A^2)_{ij} = \sum_{k=1}^{n} A^1_{ik} A^2_{kj},
\]
therefore, the trace of the product is
\[
\text{tr} (A^1 \cdot A^2) = \sum_{i=1}^{n} \sum_{k=1}^{n} A^1_{ik} A^2_{ki}.
\]
Completely analogously, the trace of the product of $m + 1 \geq 2$ matrices is

$$\text{tr} \left( A^0 \cdot A^1 \ldots A^m \right) = \sum_{i_0=1}^{n} \sum_{i_1, \ldots, i_m} A^0_{i_0 i_1} \cdot A^1_{i_1 i_2} \ldots A^{m-1}_{i_{m-1} i_m} A^m_{i_m i_0},$$

where the second sum is taken after all the possible values of the indices $i_1, \ldots, i_m \in \{1, \ldots, n\}$.

Now, the very natural idea of Dennis was to define a generalized trace map (which is now often called \textit{Dennis trace}),

$$\text{Tr}^m : M_n(R)^{\otimes m+1} \to R^{\otimes m+1},$$

putting

$$\text{Tr}^m \left( A^0 \otimes A^1 \otimes \cdots \otimes A^m \right) = \sum_{i_0=1}^{n} \sum_{i_1, \ldots, i_m} A^0_{i_0 i_1} \otimes A^1_{i_1 i_2} \otimes \cdots \otimes A^{m-1}_{i_{m-1} i_m} A^m_{i_m i_0},$$

with the second summation sign having the same significance as above. Dennis used the generalized trace to construct an isomorphism between the Hochschild homology of the matrix algebra over an algebra $R$ and that of the algebra itself.

The Dennis’ construction can be carried out also in the case of superalgebras if we replace the trace with the supertrace and we pay attention to the signs.

Namely, it is easy to see that the supertrace of a product of two homogeneous supermatrices of type $(p, q)$ over a superalgebra $R$

$$\text{str} \left( A^1 \cdot A^2 \right) = \sum_{i=1}^{p} \sum_{k=1}^{q} A^1_{ik} \cdot A^2_{ki} + (-1)^{1+|A^1|+|A^2|} \sum_{i=p+1}^{p+q} \sum_{k=1}^{q} A^1_{ik} \cdot A^2_{ki},$$

while for $m + 1$ supermatrices we have

$$\text{str} \left( A^0 \cdot A^1 \ldots A^m \right) = \sum_{i_0=1}^{p} \sum_{i_1, \ldots, i_m} A^0_{i_0 i_1} \cdot A^1_{i_1 i_2} \ldots A^{m-1}_{i_{m-1} i_m} A^m_{i_m i_0} +$$

$$+ (-1)^{1+|A^0|+\cdots+|A^m|} \sum_{i_0=p+1}^{p+q} \sum_{i_1, \ldots, i_m} A^0_{i_0 i_1} \cdot A^1_{i_1 i_2} \ldots A^{m-1}_{i_{m-1} i_m} A^m_{i_m i_0},$$

where, as above, the second sum in each term is taken after all the values of the indices $i_1, \ldots, i_m \in \{1, \ldots, p+q\}$.

Now, it is clear that to have a consistent generalization of the Dennis trace for the $\mathbb{Z}_2$-graded case we should put (for homogeneous supermatrices)

$$\text{Str}^m \left( A^0 \otimes A^1 \otimes \cdots \otimes A^m \right) = \sum_{i_0=1}^{p} \sum_{i_1, \ldots, i_m} A^0_{i_0 i_1} \otimes A^1_{i_1 i_2} \otimes \cdots \otimes A^{m-1}_{i_{m-1} i_m} A^m_{i_m i_0} +$$

$$+ (-1)^{1+|A^0|+\cdots+|A^m|} \sum_{i_0=p+1}^{p+q} \sum_{i_1, \ldots, i_m} A^0_{i_0 i_1} \otimes A^1_{i_1 i_2} \otimes \cdots \otimes A^{m-1}_{i_{m-1} i_m} A^m_{i_m i_0}. $$
We shall call this generalized supertrace the Dennis supertrace map.

It is convenient to work, as before, with the (homogeneous) generators $E^{ij}(a)$ of the supermatrix algebra $M_{p,q}(R)$. The nice thing about them is that the Dennis supertrace can be written down very easily, for these generators, because, as one can see immediately,

$$\text{Str}^m (E^{i_0j_0}(a_0) \otimes \cdots \otimes E^{i_mj_m}(a_m)) \neq 0$$

if and only if we have

$$j_0 = i_1, j_1 = i_2, \ldots, j_{m-1} = i_m, j_m = i_0,$$

therefore we shall suppose all the time that these conditions are fulfilled. Now, it is easy to see that

**Proposition 1.** The Dennis supertrace can be written on the homogeneous generators as

$$(17) \quad \text{Str}^m (E^{i_0i_1}(a_0) \otimes \cdots \otimes E^{i_mi_0}(a_m)) = \begin{cases} a_0 \otimes \cdots \otimes a_m, & i_0 \leq p, \\ (-1)^{1+|a_0|+\cdots+|a_m|} a_0 \otimes \cdots \otimes a_m, & i_0 > p. \end{cases}$$

**Proof.** Clearly, the only thing that calls for a justification is the fact that

$$|E^{i_0i_1}(a_0)| + \cdots + |E^{i_mi_0}(a_m)| = |a_0| + \cdots + |a_m|.$$

But this follows immediately if we notice that

$$E^{i_0i_1}(a_0) \cdot E^{i_1i_2}(a_1) \cdots E^{i_mi_0}(a_m) = E^{i_0i_0}(a_0 \cdot a_1 \cdots a_m),$$

therefore, on the one hand

$$|E^{i_0i_1}(a_0) \cdot E^{i_1i_2}(a_1) \cdots E^{i_mi_0}(a_m)| = |E^{i_0i_1}(a_0)| + \cdots + |E^{i_mi_0}(a_m)|$$

and, on the other hand,

$$|E^{i_0i_1}(a_0) \cdot E^{i_1i_2}(a_1) \cdots E^{i_mi_0}(a_m)| = |E^{i_0i_0}(a_0 \cdot a_1 \cdots a_m)| = |a_0 \cdot a_1 \cdots a_m| = |a_0| + \cdots + |a_m|,$$

where, when we wrote the second equality, we took into account the fact that the only non-vanishing element of the matrix $E^{i_0i_0}(a_0 \cdot a_1 \cdots a_m)$ is on the diagonal, therefore the parity of the matrix is equal to the parity of that element.

**Theorem 1.** The family of mappings $\{\text{Str}^m : M_{p,q}(R)^{\otimes m+1} \rightarrow R^{\otimes m+1}\}$ defines a chain morphism between the Hochschild complex of the algebra $M_{p,q}(R)$ and the Hochschild complex of the ground algebra $R$.

**Proof.** Clearly, each Dennis supertrace is a linear map. All we have to do is to show that the Dennis traces commute with the face maps of the two Hochschild complexes, i.e. with the operators $\delta^m_k$, $k = 0, \ldots, m$. Again, it is enough to verify for elements of the form

$$E^{i_0i_1} \otimes \cdots \otimes E^{i_mi_0}.$$

We shall discuss first the case $k < m$. As we saw above, the Dennis supetrace, calculated on such an element is

$$\text{Str}^m (E^{i_0i_1}(a_0) \otimes \cdots \otimes E^{i_mi_0}(a_m)) = \begin{cases} a_0 \otimes \cdots \otimes a_m, & i_0 \leq p, \\ (-1)^{1+|a_0|+\cdots+|a_m|} a_0 \otimes \cdots \otimes a_m, & i_0 > p. \end{cases}$$
Thus, we have
\[
\delta_k^m \Str^m (E^{ioi_1}(a_0) \otimes \cdots \otimes E^{imio}(a_m)) = \\
= \begin{cases} 
  a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_m, & i_0 \leq p, \\
  (-1)^{1+|a_0|+\cdots+|a_m|} a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_m, & i_0 > p.
\end{cases}
\]

On the other hand,
\[
\delta_k^m (E^{ioi_1}(a_0) \otimes \cdots \otimes E^{imio}(a_m)) = \\
E^{ioi_1}(a_0) \otimes \cdots \otimes E^{ik_{k+1}}(a_k) E^{ik_{k+1}+2}(a_{k+1}) \otimes \cdots \otimes E^{imio}(a_m) = \\
E^{ioi_1}(a_0) \otimes \cdots \otimes E^{ik_{k+1}+2}(a_k a_{k+1}) \otimes \cdots \otimes E^{imio}(a_m),
\]
therefore
\[
\Str^{m-1} \delta_k^m (E^{ioi_1}(a_0) \otimes \cdots \otimes E^{imio}(a_m)) = \\
= \begin{cases} 
  a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_m, & i_0 \leq p, \\
  (-1)^{1+|a_0|+\cdots+|a_m|} a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_m, & i_0 > p.
\end{cases}
\]
\[
= \delta_k^m \Str^m (E^{ioi_1}(a_0) \otimes \cdots \otimes E^{imio}(a_m)),
\]
where we have use the fact that \(|a_k a_{k+1}| = |a_k| + |a_{k+1}|\). The only case that needs extra work is the case \(k = m\). In this case we have, on the one hand,
\[
\delta_m^m \Str^m (E^{ioi_1}(a_0) \otimes \cdots \otimes E^{imio}(a_m)) = \\
\begin{cases} 
  (-1)^{|a_m|} \sum_{i=0}^{m-1} |a_i| a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_m, & i_0 \leq p, \\
  (-1)^{1+|a_0|+\cdots+|a_m|} \sum_{i=0}^{m-1} |a_i| a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_m, & i_0 > p.
\end{cases}
\]
On the other hand,
\[
\Str^{m-1} \delta_m^m (E^{ioi_1}(a_0) \otimes \cdots \otimes E^{imio}(a_m)) = \\
= \Str^{m-1} \begin{cases} 
  (-1)^{|E^{imio}(a_m)|} \sum_{i=0}^{m-1} |E^{i,i+1}(a_i)| E^{im} a_0 \otimes \cdots \otimes E^{im-1} (a_{m-1}), & i_0 \leq p, \\
  (-1)^{|E^{im} a_0(a_m)|} \sum_{i=0}^{m-1} |E^{i,i+1}(a_i)| a_m a_0 \otimes \cdots \otimes a_{m-1}, & i_m \leq p, \\
  (-1)^{1+|E^{im} a_0(a_m)|} \sum_{i=0}^{m-1} |E^{i,i+1}(a_i)| + \sum_{k=0}^{m} |a_k| a_m a_0 \otimes \cdots \otimes a_{m-1}, & i_m > p.
\end{cases}
\]
Now, to prove that \(\Str^{m-1} \delta_m^m = \delta_m^m \Str^m\), we have to consider several cases.
(i) \( i_0 \leq p, i_m \leq p \). In this case we have to prove that

\[
|a_m \sum_{k=0}^{m-1} |a_k| = \left| E^{i_m i_0} (a_m) \right| \sum_{l=0}^{m-1} |E^{i_l i_{l+1}} (a_l)|.
\]

But, since we have \( i_0, i_m \leq p \), it follows that \( |E^{i_m i_0} (a_m)| = |a_m| \) and we have already seen that

\[
\sum_{k=0}^{m-1} |a_k| = \sum_{l=0}^{m-1} |E^{i_l i_{l+1}} (a_l)|.
\]

(ii) \( i_0 \leq p, i_m > p \). Now the identity we have to prove is

\[
|a_m \sum_{k=0}^{m-1} |a_k| = 1 + \sum_{k=0}^{m} |a_k| + \left| E^{i_m i_0} (a_m) \right| \sum_{l=0}^{m-1} |E^{i_l i_{l+1}} (a_l)|.
\]

In this case, \( \left| E^{i_m i_0} (a_m) \right| = 1 + |a_m| \). Moreover, we have

\[
\sum_{l=0}^{m-1} |E^{i_l i_{l+1}} (a_l)| = \sum_{l=0}^{m} |E^{i_l i_{l+1}} (a_l)| - \left| E^{i_m i_0} (a_m) \right| = \sum_{k=0}^{m} |a_k| - 1 - |a_m| = 1 + \sum_{k=0}^{m-1} |a_k|.
\]

It follows, therefore, that

\[
RHS = 1 + \sum_{k=0}^{m} |a_k| + (1 + |a_m|) \left(1 + \sum_{k=0}^{m-1} |a_k| \right) = 1 + \sum_{k=0}^{m} |a_k| + 1 + \sum_{k=0}^{m} |a_k| + |a_m| \sum_{k=0}^{m-1} |a_k| = a_m \sum_{k=0}^{m-1} |a_k|,
\]

so the identity is proven.

(iii) \( i_0 > p, i_m \leq p \). We have to show that

\[
1 + \sum_{k=0}^{m} |a_k| + |a_m| \sum_{k=0}^{m-1} |a_k| = \left| E^{i_m i_0} (a_m) \right| \sum_{l=0}^{m-1} |E^{i_l i_{l+1}} (a_l)|.
\]

The same reasoning we did before ensure us that we have \( |E^{i_m i_0} (a_m)| = 1 + |a_m| \) and

\[
\sum_{l=0}^{m-1} |E^{i_l i_{l+1}} (a_l)| = 1 + \sum_{k=0}^{m-1} |a_k|,
\]
hence
\[ RHS = (1 + |a_m|) \left( 1 + \sum_{k=0}^{m-1} |a_k| \right) = 1 + \sum_{k=0}^{m-1} |a_k| + |a_m| \sum_{k=0}^{m-1} |a_k| + |a_m| = 1 + \sum_{k=0}^{m} |a_k| + |a_m| \sum_{k=0}^{m-1} |a_k|, \]
so we are done.

(iv) \( i_0 > p, i_m > p \). The required identity reads
\[ 1 + \sum_{k=0}^{m} |a_k| + |a_m| \sum_{k=0}^{m-1} |a_k| = 1 + \sum_{k=0}^{m-1} |a_k| + |E^{i_0}_{i_0}(a_m)| \sum_{l=0}^{m-1} |E^{i_l i_{l+1}}(a_l)| \]
or, which is the same,
\[ |a_m| \sum_{k=0}^{m-1} |a_k| = |E^{i_0}_{i_0}(a_m)| \sum_{l=0}^{m-1} |E^{i_l i_{l+1}}(a_l)|, \]
which is obvious, since in this case \( |E^{i_0}_{i_0}(a_m)| = |a_m| \).

\[ \square \]

It is pretty clear that the Dennis trace maps are onto. In fact, we can consider the map
\[ \text{inc} : R \to M_{p,q}(R) \]
given by \( \text{inc}(a) = E^{11}(a) \). This map, which is, obviously, a linear morphism (\( |E^{11}(a)| = |a| \)), can be extended, for each natural \( m \), to a morphism
\[ \text{Inc}^m : R^\otimes m+1 \to M_{p,q}(R)^\otimes m+1, \]
\[ \text{Inc}^m(a_0 \otimes \cdots \otimes a_m) = E^{11}(a_0) \otimes \cdots \otimes E^{11}(a_m). \]
It can be shown immediately that

**Proposition 2.** The family of maps \( \{ \text{Inc}^m : R^\otimes m+1 \to M_{p,q}(R)^\otimes m+1, \ m \in \mathbb{N} \} \), is a chain map from the Hochschild complex of \( R \) to the Hochschild complex of \( M_{p,q}(R) \), which is a splitting of the Dennis supertrace.

### 4. The Hochschild homology of \( M_{p,q}(R) \)

Inc is a right inverse of the Dennis supertrace, but, obviously, it is not, also, a right inverse, so the Hochschild complexes of \( M_{p,q}(R) \) and \( R \) are not isomorphical. We shall prove that, however, the supertrace induces an isomorphism in homology. To prove this, it is enough to verify that Inc is a left quasi-inverse of the supertrace, i.e.

**Theorem 2.** There is a chain homotopy \( h : C(M_{p,q}(R)) \to C(M_{p,q}(R)) \) such that
\[ d \circ h + h \circ d = \text{Id} - \text{Inc} \circ \text{Str}. \]
Proof. We shall define the homotopy exactly as in the classical (non-graded) case and we shall check that it does the job equally in the supercase. Thus, let us consider
\[ h = \sum_{l=0}^{m} (-1)^l h_l : M_{p,q}(R)^{\otimes m+1} \to M_{p,q}(R)^{\otimes m+2}, \]
with
\[ h_l (E^{i_0 i_1}(a_0) \otimes \cdots \otimes E^{i_m i_0}(a_m)) = \]
\[ = E^{i_0 1}(a_0) \otimes E^{11}(a_1) \otimes \cdots \otimes E^{11}(a_l) \otimes E^{i_1 i_1+1}(1) \otimes E^{i_1+1 i_2+1}(a_{l+1}) \otimes \cdots \otimes E^{i_m i_0}(a_m) \]
Let us, verify, first, that it works for the particular case of \( m = 0 \). We have
\[ h_0 (E^{i_0 i_0}(a_0)) = E^{i_0 1}(a_0) \otimes E^{1 j_0}(1). \]
\[ d^1 \circ h_0 (E^{i_0 i_0}(a_0)) = \delta^1_0 \circ h_0 (E^{i_0 i_0}(a_0)) - \delta^1_0 \circ h_0 (E^{i_0 i_0}(a_0)) = \]
\[ = \delta^1_0 (E^{i_0 1}(a_0) \otimes E^{1 j_0}(1)) - \delta^1_0 (E^{i_0 1}(a_0) \otimes E^{1 j_0}(1)) = \]
\[ = \begin{cases} E^{i_0 i_0}(a_0) - (-1)|E^{i_0 1}(a_0)| |E^{1 j_0}(1)| E^{11}(a_0) & i_0 \neq j_0 \\ E^{i_0 i_0}(a_0) & i_0 = j_0 \leq p, = (\text{Id} - \text{Inc} \circ \text{Str}) (E^{i_0 i_0}(a_0)). & i_0 = j_0 > p \end{cases} \]
Thus, the claim is true at the lowest level. Take now an arbitrary \( m \in \mathbb{N} \). Let us compute first \( \delta^{m+1}_{m+1} \circ h_m \). We have
\[ \delta^{m+1}_{m+1} \circ h_m (E^{i_0 i_1}(a_0) \otimes \cdots \otimes E^{i_m i_0}(a_m)) = \]
\[ = \delta^{m+1}_{m+1} (E^{i_0 1}(a_0) \otimes E^{11}(a_1) \otimes \cdots \otimes E^{11}(a_m) \otimes E^{1 j_0}(1)) = \]
\[ = (-1)^{|E^{i_0 1}(a_0)| + \sum_{k=1}^{m} |a_k|} E^{11}(a_0) \otimes E^{11}(a_1) \otimes \cdots \otimes E^{11}(a_m) = \]
\[ = \begin{cases} E^{11}(a_0) \otimes E^{11}(a_1) \otimes \cdots \otimes E^{11}(a_m), & i_0 \leq p, \\ (-1)^{1+\sum_{k=0}^{m} |a_k|} E^{11}(a_0) \otimes E^{11}(a_1) \otimes \cdots \otimes E^{11}(a_m), & i_0 > p. \end{cases} \]
On the other hand,
\[ \text{Str}^m (E^{i_0 i_1}(a_0) \otimes \cdots \otimes E^{i_m i_0}(a_m)) = \begin{cases} a_0 \otimes \cdots \otimes a_m, & i_0 \leq p, \\ (-1)^{1+\sum_{k=0}^{m} |a_k|} a_0 \otimes \cdots \otimes a_m, & i_0 > p. \end{cases} \]
It follows then, immediately, that
\[ \delta^{m+1}_{m+1} \circ h_m = \text{Inc}^m \circ \text{Str}^m. \]
Moreover, we have

\[ \delta_{m+1} \circ h_0 (E_{i_0} a_0 \otimes \cdots \otimes E_{i_m} a_m) = \]
\[ = \delta_{m+1} (E_{i_0} a_0 \otimes E_{i_1} (1) \otimes E_{i_2} a_1 \otimes \cdots \otimes E_{i_m} a_m) = \]
\[ = \text{Id} (E_{i_0} a_0 \otimes \cdots \otimes E_{i_m} a_m). \]

Now, exactly as in the classical case, one verifies immediately that if \( 1 \leq l \leq m \) then

\[ \delta^{m+1}_l \circ h_l = \delta^{m+1}_l \circ h_{l-1}, \]

while, if \( k < l \leq m \), then

\[ \delta^{m+1}_k \circ h_l = h_{l-1} \circ \delta^{m}_k \]

and, also, if \( k \geq l \), then

\[ \delta^{m+1}_k \circ h_l = h_l \circ \delta^{m}_k. \]

To summarize, we have the following set of relations:

\begin{align*}
\text{(18)} & \quad \delta^{m+1}_m \circ h_m = \text{Inc}^m \circ \text{Str}^m; \\
\text{(19)} & \quad \delta^{m+1}_0 \circ h_0 = \text{Id}; \\
\text{(20)} & \quad \delta^{m+1}_l \circ h_l = \delta^{m+1}_l \circ h_{l-1}, \quad \text{if } 1 \leq l \leq m; \\
\text{(21)} & \quad \delta^{m+1}_k \circ h_l = h_{l-1} \circ \delta^{m}_k, \quad \text{if } k < l \leq m; \\
\text{(22)} & \quad \delta^{m+1}_k \circ h_l = h_l \circ \delta^{m}_k, \quad \text{if } k \geq l + 2. 
\end{align*}
We have everything we need to prove our assertion:

\[
d^{m+1} \circ h^m + h^m \circ d^m = \sum_{k=0}^{m+1} \sum_{l=0}^{m+1} (-1)^{k+l} \delta_k^{m+1} \circ h_l^m + \sum_{k=0}^{m} \sum_{l=0}^{m-1} (-1)^{k+l} h_l^m \circ \delta_k^m = \\
= \delta_0^{m+1} \circ h_0 - \delta^{m+1}_{m+1} \circ h_m + \sum_{l=1}^{m+1} \delta_l^{m+1} \circ h_l - \sum_{l=1}^{m} \delta_l^{m+1} \circ h_{l-1} + \\
+ \sum_{k=0}^{m} \sum_{l=0}^{m-1} (-1)^{k+l} h_l^m \circ \delta_k^m = \operatorname{Id} - \operatorname{Inc}^m \circ \operatorname{Str}^m + \\
+ \sum_{k=0}^{m} \sum_{l=1}^{m-1} (-1)^{k+l} h_l^m \circ \delta_k^m = \operatorname{Id} - \operatorname{Inc}^m \circ \operatorname{Str}^m - \\
\sum_{k=0}^{m} \sum_{l=0}^{m-1} (-1)^{k+l} h_l^m \circ \delta_k^m = \operatorname{Id} - \operatorname{Inc}^m \circ \operatorname{Str}^m + \\
= \operatorname{Id} - \operatorname{Inc}^m \circ \operatorname{Str}^m,
\]

where we used the relations (18) – (22). Thus, we have a quasi-isomorphisms between the two chain complexes, which means that the two Hochschild homologies are isomorphic. \(\square\)

5. Final remarks

The basic ideas of these proof are “super”-versions of the classical, ungraded proof (see [8]). They amount to an unpublished result of R.K. Dennis (whence the name). We notice, however, to avoid confusions, that the term “Dennis trace” is also used for another map (related to the generalized trace, also introduced by Dennis), establishing a connection between algebraic K-theory and Hochschild homology (see [8]).

The Dennis supertraces can be used, as well, to provide a proof of the Morita invariance of the cyclic homology of the superalgebras (see [2]). We also managed to prove, recently, the general Morita invariance of Hochschild homology of superalgebras, not only for the case of supermatrices (see [1]). We used there a spectral sequence argument. Probably the
more “economical” tools used by McCarthy ([7], see also the book of Loday [5]) can be adapted, as well, to the super-case.

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