Criteria for analyticity of subordinate semigroups

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Abstract

Let \( \psi \) be a Bernstein function. A. Carasso and T. Kato obtained necessary and sufficient conditions for \( \psi \) to have a property that \( \psi(A) \) generates a quasibounded holomorphic semigroup for every generator \( A \) of a bounded \( C_0 \)-semigroup in a Banach space, in terms of some convolution semigroup of measures associated with \( \psi \). We give an alternative to Carasso-Kato’s criterium, and derive several sufficient conditions for \( \psi \) to have the above-mentioned property.

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1. Introduction

The well known theorem due to Yosida \([15]\) states that for every generator \( A \) of a bounded \( C_0 \)-semigroup on a Banach space \( X \) its fractional power \( -(−A)^\alpha, 0 < \alpha < 1 \) is a generator of a holomorphic semigroup on \( X \). The present paper is devoted to some generalizations and analogs of Yosida’s Theorem in terms of so-called Bochner-Phillips calculus \([1, 12]\) (see also \([5, \text{Chap. XIII}]; \, \, [8, 13, 9, 2]\)). Though the majority of works on Bochner-Phillips calculus use the class \( B \) of (positive) Bernstein functions, we prefer the class \( T \) of negative one. The corresponding reformulation of Bochner-Phillips calculus is trivial in view of the fact that \( \phi(x) \in B \) if and only if \( -\phi(-s) \in T \).

We say that the function \( \psi: (−∞, 0] \rightarrow (−∞, 0] \) belongs to the class \( T \) of negative Bernstein functions if \( \psi \in C^\infty((−∞, 0)) \cap C((−∞, 0]) \) and its derivative is absolutely monotonic, i.e. \( \psi^{(n)} \geq 0 \) for all \( n \in \mathbb{N} \). It is known that in this case \( \psi \) extends analytically to the left half-plane \( \Pi_− = \{\Re z < 0\} \), the extension is continuous on \( \{\Re z \leq 0\} \), and has the following integral representation

\[
\psi(z) = c_0 + \int_{\mathbb{R}_+} (e^{zu} - 1)u^{-1}d\rho(u), \quad Rez \leq 0 \tag{1}
\]

where \( c_0 = \psi(0) \), the positive measure \( \rho \) on \( \mathbb{R}_+ \) is uniquely determined by \( \psi \) and \( \int_{[0,1]}d\rho < \infty, \int_{[1,\infty]}u^{-1}d\rho(u) < \infty; \) the integrand in (1) is defined for \( u = 0 \) to be equal to \( z \).

Moreover, there is a convolution semigroup \((\nu_t)_{t \geq 0}\) of sub-probability measures on \( \mathbb{R}_+ \) with the Laplace transform

\[
g_t(z) := e^{t\psi(z)} = \int_{\mathbb{R}_+} e^{zu}d\nu_t(u), \quad Rez \leq 0 \tag{2}
\]

(see \([14, 5, \text{Chap. XIII}]\)).

The class \( T \) is a cone which is closed with respect to compositions and pointwise convergence on \( (−∞, 0] \), and contains a number of important functions, including (up to affine changes of variable) fractional powers, the logarithm, the inverse hyperbolic cosine, and polylogarithms \( Li_p \) of all orders \( p \in \mathbb{N} \) \([10]\).

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For a negative Bernstein function $\psi$ with integral representation (1) and a generator $A$ of a bounded $C_0$-semigroup $T$ on a complex Banach space $X$ the value of $\psi$ at $A$ for $x \in D(A)$, the domain of $A$, is defined by the Bochner integral

$$\psi(A)x = c_0x + \int_{\mathbb{R}^+} (T(u) - I)xu^{-1}d\rho(u).$$

The closure of this operator, which is also denoted by $\psi(A)$, is a generator of a bounded $C_0$-semigroup $g_t(A)$ on $X$ (the "subordinate semigroup"), too.

In the following, without loss of generality we shall assume that $c_0 = 0$. The corresponding subclass of $T$ will be denoted by $T_0$. We shall denote also by $M^{b}(\mathbb{R}^+, \mathbb{C})(\mathcal{M}(\mathbb{R}^+, \mathbb{R}^+))$ the space of all bounded complex valued (respectively positive) measures on $\mathbb{R}^+$, and by $C_0(\mathbb{R}^+)$ the space of all continuous complex valued functions on $\mathbb{R}^+$ which vanish at infinity; $X$ stands for a complex Banach space.

Another result by Yosida [16] asserts that if the bounded $C_0$-semigroup $T$ with generator $A$ on $X$ satisfies

$$T(t)X \subset D(A), t > 0, \text{ and } \limsup_{t \downarrow 0} (t\|AT(t)\|) < \infty,$$  

then for any $\beta > 0$, $\psi(A)$ can be extended to a bounded holomorphic semigroup on $X$.

We shall denote by $T_Y$ the set of all $\psi \in T$ such that $\psi(A)$ generates a bounded $C_0$-semigroup with property (Y) for every generator $A$ of a bounded $C_0$-semigroup in a Banach space. The class $T_Y$ is a cone [3, Theorem 6]. Moreover, it is clear that the composition $\psi_1 \circ \psi_2 \in T_Y$ if $\psi_1 \in T_Y$, $\psi_2 \in T$. But the class $T_Y$ is not closed with respect to pointwise convergence.

A. Carasso and T. Kato [3, Theorem 4] obtained necessary and sufficient conditions for a function $\psi$ to be in $T_Y$ in terms of the semigroup $(\nu_t)_{t \geq 0}$. They also gave two necessary conditions in terms of $\psi$ itself. Y. Fujita [6] obtained sufficient conditions for $\psi$ to be in $T_Y$ in terms of analytical continuation of $\psi$ and regular variation.

We proceed as follows. First we prove the multiplication rule which connects the Bochner-Phillips and Hille-Phillips calculi and then derive the alternative to [3] necessary and sufficient conditions for the inclusion $\psi \in T_Y$ (see Theorem 2 below; the variant of this theorem with $C_0(\mathbb{R}^+)$ instead of $E(\mathbb{R}^+)$ (for the definition of the last class see below) first appeared in [11]). Then we deduce two theorems from this criterion that give sufficient conditions for $\psi$ to be in $T_Y$ in terms of $\psi$. It should be noted that the assumptions of Theorem 4 below contain necessary conditions, obtained by Carasso and Kato (the idea to employ the Hausdorff-Young inequality in this context belongs to Carasso and Kato, too). Finally, we give one more condition, that is sufficient for the inclusion $\psi \in T_Y$. Several examples have been considered.

2. The multiplication rule for the Bochner-Phillips and Hille-Phillips calculi, and the criterion for $\psi$ to be in $T_Y$.

In [7, Chap.XV] the functional calculus (the Hille-Phillips calculus) of generators of $C_0$-semigroups have been constructed. In particular let $a \in M^{b}(\mathbb{R}^+, \mathbb{C})$ and

$$g(s) = La(s) := \int_{\mathbb{R}^+} e^{su}da(u) \quad (s \leq 0)$$

be the Laplace transform of $a$. Then for a generator $A$ of a bounded $C_0$-semigroup $T$ on a complex Banach space $X$ the value of $g$ at $A$ is the bounded operator on $X$ defined by the Bochner integral

$$g(A)x = \int_{\mathbb{R}^+} T(u)xda(u), \quad x \in X.$$
Our Theorem 1 connects the Bochner-Phillips and Hille-Phillips calculi. It is a generalization of Lemma 1 in [11]. But first we need the following approximation lemma. We shall denote by $E(\mathbb{R}_+)$ the complex space of exponential polynomials of the form

$$p(t) = \sum_{j=1}^{n} c_j e^{s_j t}, \quad c_j \in \mathbb{C}, s_j < 0,$$

endowed with sup-norm on $\mathbb{R}_+$.

**Lemma 1.** For every bounded function $q \in C^1(\mathbb{R}_+)$ with bounded derivative there exists a sequence $q_n \in E(\mathbb{R}_+)$ such that

1) $q_n \to q$, and $q_n' \to q'$ pointwise on $\mathbb{R}_+$;
2) $(q_n)$ and $(q_n')$ are uniformly bounded on $\mathbb{R}_+$.

**Proof.** Let us pick a sequence $q_n \in C^1(\mathbb{R}_+)$ such that $q_n(t) = q(t)$ for $t \in [n, n+1]$, and $(q_n)$ and $(q_n')$ are uniformly bounded, $|q_n| < C_1$, $|q_n'| < C_1$. Define $f_n(x) = q_n(\log x)$ for $x \in [0, 1]$ (for $n = 0$). Then $f_n \in C^1([0,1])$, $|f_n| < C_1$ for $x \in [0,1]$, and $|f_n'(x)| < C_1 x^{-1}$ for $x \in (0,1]$. It is well known (see, e. g., [11, Theorem 8.4.1]) that for every natural $n$ the algebraic polynomial $p_n$ exists such that

$$|f_n(x) - p_n(x)| < n^{-1}, \quad |f_n'(x) - p_n'(x)| < n^{-1}, \quad x \in [0,1].$$

Then $|p_n(0)| < n^{-1}$, $|p_n(x)| < C_1 + 1$, and $|p_n'(x)| < C_1 x^{-1} + 1$ for $x \in (0,1]$. Since $f_n(x) = q(\log x)$ for $x \in [0,1]$, and $n > -\log x$, we have

$$|q(\log x) - p_n(x)| < n^{-1}, \quad x \in (0,1], \quad n > -\log x.$$

Let $q_n(t) := p_n(e^{-t}) - p_n(0)$. Then $q_n \in E(\mathbb{R}_+)$, $q_n \to q$ on $\mathbb{R}_+$, and $(q_n)$ and $(q_n')$ are uniformly bounded on $\mathbb{R}_+$. Finally

$$|q'(\log x)(x^{-1}) - p_n'(x)| < n^{-1}, \quad x \in (0,1); n > -\log x.$$

Putting hear $x = e^{-t}$ we have for all natural $n > t$ ($t \in \mathbb{R}_+$) that $|q'(t) - q_n'(t)| < n^{-1}$. This completes the proof.

For measures $a \in M^b(\mathbb{R}_+, \mathbb{C})$, and $\rho \in M(\mathbb{R}_+, \mathbb{R}_+)$ let

$$K(a, \rho) = \sup_{\phi \in S} \left| \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \phi(r) d\rho(a(r - u) - a(r)) u^{-1} d\rho(u) \right|$$

(if the right hand side exists), where $S$ is the unit sphere of the space $E(\mathbb{R}_+)$ with respect to sup-norm on $\mathbb{R}_+$. Here we assume that $a = 0$ on $(-\infty; 0)$. See the proof of Theorem 5 for an estimate for $K(a, \rho)$ with bounded positive measure $a$, but $K(\delta, \delta) = \infty$.

**Theorem 1.** Let $g = La, \ a \in M^b(\mathbb{R}_+, \mathbb{C}),$ and $\psi \in T_0$ has integral representation (1).

If $K(a, \rho) < \infty$, then

1) the function $h := \psi g$ has the form $h = Lb$, where $b \in M^b(\mathbb{R}_+, \mathbb{C}), \ |b| = K(a, \rho)$;
2) $g(A)X \subset D(\psi(A)), \ h(A) = \psi(A)g(A),$ and $\|h(A)\| \leq MK(a, \rho)$ for every operator $A$ in a Banach space $X$, which generates a bounded $C_0$-semigroup $T$ with $\|T(t)\| \leq M$.

**Proof.** Let $a(r)$ denotes the distribution function for $a$, $a(r) = 0$ for $r \in (-\infty, 0]$. Then for $s < 0$

$$g(s) = \int_{\mathbb{R}_+} e^{sr} da(r) = (-s) \int_{\mathbb{R}_+} e^{sr} a(r) dr.$$
Thus for $u \geq 0$ and $s < 0$ we have

$$(e^{su} - 1)g(s) = (e^{su} - 1)(-s) \int e^{sr}a(r)dr$$

$$= (-s) \left( \int e^{s(r+u)}a(r)dr - \int e^{sr}a(r)dr \right) = (-s) \int e^{sr}(a(r-u) - a(r))dr = Lb^u(s),$$

where $b^u(r) = a(r-u) - a(r)$ has bounded variation and is concentrated on $\mathbb{R}_+$. Therefore for $\psi \in \mathcal{T}_0$ with integral representation (1) we get

$$h(s) = \int (e^{su} - 1)g(s)u^{-1}d\rho(u) = \int \int e^{sr}db^u(r)u^{-1}d\rho(u). \quad (3)$$

For $\phi \in E(\mathbb{R}_+)$ let

$$b(\phi) := \int b^u(\phi)u^{-1}d\rho(u) = \int \int \phi(r)d_r(a(r-u) - a(r))u^{-1}d\rho(u)$$

be the linear functional on $E(\mathbb{R}_+)$ (we use the notation $b^u(\phi)$ for $\int \phi db^u$). By the hypothesis of the theorem $\|b\| = K(a, \rho) < \infty$, and since $E(\mathbb{R}_+)$ is dense in $C_0(\mathbb{R}_+)$ by Stone-Weierstrass Theorem, $b$ extends to a measure $b \in \mathcal{M}^b(\mathbb{R}_+, \mathbb{C})$. Furthermore,

$$b = \int b^u u^{-1}d\rho(u)$$

(the weak integral; $\mathcal{M}^b(\mathbb{R}_+, \mathbb{C})$ is endowed with vague topology).

We claim that for every bounded function $q \in C^1(\mathbb{R}_+)$ with bounded derivative the following equality holds (we write $b(q)$ instead of $\int_{\mathbb{R}_+} qdb$ in the rest of the proof)

$$b(q) = \int b^u(q)u^{-1}d\rho(u). \quad (4)$$

In fact, let $(q_n)$ be as in Lemma 1, and $|q_n| < C$, $|q'_n| < C$ for some constant $C > 0$. Putting $p_n(u) := b^u(q_n)$ we have

$$p_n(u) = \int q_n(r)d_r(a(r-u) - a(r)) = \int (q_n(r + u) - q_n(r))da(r). \quad (5)$$

Now let $p(u) := b^n(q)$. Then $p_n(u) \to p(u)$ ($n \to \infty$) pointwise by Lebesgue Theorem. We have $|q_n(r + u) - q_n(r)| \leq Cu$, and $\leq 2C$. If we take $w(u) = \min\{u, 1\}$, then $w \in L^1(u^{-1}d\rho(u))$ and (5) implies that $|p_n(u)| \leq 2\|a\|w(u)$. Thus by the Lebesgue Theorem

$$\int p_n(u)u^{-1}d\rho(u) \to \int p(u)u^{-1}d\rho(u)(n \to \infty).$$

On the other hand,

$$\int p_n(u)u^{-1}d\rho(u) = \int b^u(q_n)u^{-1}d\rho(u) = b(q_n) \to b(q) \quad (n \to \infty).$$
The theorem is proved. In particular, for \( q(r) = e^{sr} \) \((s \leq 0)\) (4) and (3) imply the equality \( h = Lb \) which proves the first statement of the theorem.

To prove the second one, fix a bounded linear functional \( f \in X' \), vector \( x \in D(A) \), and let \( q(r) = f(T(r)x) \). Then \( q \in C^1(\mathbb{R}_+) \) and \( q \) is bounded together with the derivative \( q'(r) = f(T(r)Ax) \) \((r \geq 0)\). For such \( q \) equation (4) implies that

\[
f \left( \int_{\mathbb{R}_+} T x \, db \right) = \int_{\mathbb{R}_+} f(T(r)x) \, db(r) = \int_{\mathbb{R}_+} f \left( \int_{\mathbb{R}_+} T(r)x \, db(r) \right) u^{-1} \, dp(u).
\]

So by the definition of the weak integral

\[
\int_{\mathbb{R}_+} \left( \int_{\mathbb{R}_+} T(r)x \, db(r) \right) u^{-1} \, dp(u) = \int_{\mathbb{R}_+} T x \, db.
\]

In addition, the interior integral in the left hand side here exists in the sense of Bochner, and

\[
\int_{\mathbb{R}_+} T(r)x \, db(r) = \int_{[u, \infty)} T(r)x \, da(r) - \int_{\mathbb{R}_+} T(r)x \, da(r)
\]

\[
= \int_{\mathbb{R}_+} T(r+u)x \, da(r) - \int_{\mathbb{R}_+} T(r)x \, da(r) = (T(u) - I)g(A)x.
\]

Therefore for \( x \in D(A) \) we have

\[
h(A)x = \int_{\mathbb{R}_+} T(r)x \, db(r) = \int_{\mathbb{R}_+} (T(u) - I)g(A)x u^{-1} \, dp(u) = \psi(A)g(A)x.
\]

Since the operator \( h(A) \) is bounded, and, on the other hand, the operator \( \psi(A)g(A) \) is closed (as the product of a closed and a bounded operators), the last equality holds for all \( x \in X \). In particular, \( g(A)X \subset D(\psi(A)) \). Finally

\[
\|h(A)\| \leq \int_{\mathbb{R}_+} \|T(r)\| \, db(r) \leq M \|b\| = MK(\nu_t, \rho).
\]

The theorem is proved.

**Theorem 2.** Let \( \psi \in \mathcal{T}_0 \). Then \( \psi \in \mathcal{T}_\gamma \) if and only if

\[
K(\nu_t, \rho) = O(t^{-1}), \quad t \Downarrow 0
\]

holds (see formulas (1) and (2) for the definitions of \( \rho \) and \( \nu_t \)).

**Proof.** Let (6) holds. Putting \( a = \nu_t \) in Theorem 1 we get that for sufficiently small \( t > 0 \) the function \( h_t = \psi g_t \) has the form \( h_t = Lb_t \), where \( b_t \) is a bounded measure on \( \mathbb{R}_+ \), \( \|b_t\| = K(\nu_t, \rho) \). In addition, \( g_t(A)X \subset D(\psi(A)) \) for all \( t > 0 \) (\( \psi(A) = \) generator of the semigroup \( g_t(A) \)) and

\[
\|h_t(A)\| = \|\psi(A)g_t(A)\| \leq MK(\nu_t, \rho).
\]

Now (6) implies (Y) with \( g_t(A) \) instead of \( T(t) \).

To prove the converse, consider \( X = C_0(\mathbb{R}_+) \) with sup-norm, let \( \psi \in \mathcal{T}_\gamma \), and let \( T \) be the \( C_0 \)-semigroup of shifts on \( X \), \( (T(r)x)(v) = x(v + r) \) (in this concrete situation \( A \) is a derivation
with appropriate domain). Then, for each \( x \in C^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+), \quad t > 0 \) integration by parts gives

\[
y(v) := g_t(A)x(v) = \int_{\mathbb{R}_+} x(v + r)d\nu_t(r) = -\int_{\mathbb{R}_+} x'(v + r)\nu_t(r)dr.
\]

Therefore

\[
\psi(A)g_t(A)x(v) = \int_{\mathbb{R}_+} (y(v + u) - y(v))u^{-1}d\rho(u)
\]

\[
= \int_{\mathbb{R}_+} \left( -\int_{\mathbb{R}_+} x'(v + u + r)\nu_t(r)dr + \int_{\mathbb{R}_+} x'(v + r)\nu_t(r)dr \right) u^{-1}d\rho(u).
\]

Since \( \nu_t \) is concentrated on \( \mathbb{R}_+ \), we get

\[
\int_{\mathbb{R}_+} x'(v + u + r)\nu_t(r)dr = \int_{\mathbb{R}_+} x'(v + r)\nu_t(r - u)dr,
\]

and thus

\[
\psi(A)g_t(A)x(v) = \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}_+} (\nu_t(r) - \nu_t(r - u))x'(v + r)dr \right) u^{-1}d\rho(u).
\]

But integration by parts gives since \( \nu_t(0) = \nu_t(-u) = 0, \)

\[
\int_{\mathbb{R}_+} (\nu_t(r) - \nu_t(r - u))x'(v + r)dr = \int_{\mathbb{R}_+} x(v + r)d_r(\nu_t(r - u) - \nu_t(r)).
\]

Finally, for each \( x \in C^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+), \quad v \geq 0 \)

\[
\psi(A)g_t(A)x(v) = \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}_+} x(v + r)d_r(\nu_t(r - u) - \nu_t(r)) \right) u^{-1}d\rho(u).
\]

Taking into account that \( t|\psi(A)g_t(A)| \leq C \) for some \( C > 0 \) and all \( t \in (0, 1] \) we have for our \( x \) with \( \|x\| = 1 \) that \( |\psi(A)g_t(A)x(v)| \leq Ct^{-1}. \) So for each \( v \geq 0, \ t \in (0, 1] \)

\[
\left| \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}_+} x(v + r)d_r(\nu_t(r - u) - \nu_t(r)) \right) u^{-1}d\rho(u) \right| \leq Ct^{-1}.
\]

Since \( C^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+) \) is dense in \( C_0(\mathbb{R}_+) \), it follows for \( v = 0 \) that \( K(\nu_t, \rho) = O(t^{-1}), \quad t \downarrow 0, \) as desired.

3. **Sufficient conditions for \( \psi \) to be in \( \mathcal{T}_\gamma \) in terms of \( \psi \)**

In the following we shall denote by \( \mathcal{F} \) the Fourier transform on \( \mathbb{R}, \)

\[
\mathcal{F}f(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda t}f(t)dt,
\]
and by $\mathcal{F}^{-1}$ the inverse of $\mathcal{F}$. Let

$$F_t(\lambda) = e^{t\psi(i\lambda)} \psi(i\lambda) \quad (\text{Im} \lambda \geq 0, t > 0).$$

The restriction $F_t|\mathbb{R}$ will be also denoted by $F_t$.

**Theorem 3.** Let $\psi \in \mathcal{T}_0$. Assume that

(i) the derivative $\partial/\partial y F_t(y)$ exists for a.e. $y \in \mathbb{R}$ and each sufficiently small $t > 0$;

(ii) for some $p \in (1, 2]$ functions $F_t$ and $\partial/\partial y F_t$ both belong to $L^p(\mathbb{R})$ for each sufficiently small $t > 0$;

(iii) $\mathcal{F} F_t$ is concentrated on $\mathbb{R}_+$ for each sufficiently small $t > 0$;

(iv) $\|F_t\|_p^{1/q} \|\partial/\partial y F_t\|_p^{1/p} = O(t^{-1})$ as $t \downarrow 0$ ($p^{-1} + q^{-1} = 1$).

Then $\psi \in \mathcal{T}_Y$.

**Proof.** First we prove that $f_t := \mathcal{F} F_t \in L^1(\mathbb{R}_+)$, and $F_t = \mathcal{F}^{-1} f_t$. Indeed, $f_t \in L^q(\mathbb{R}_+)$, and $\mathcal{F}(\partial/\partial y F_t)(y) = iy f_t(y) \in L^q(\mathbb{R}_+).$ By Hölder’s inequality $f_t(y) = (iy f_t(y))(iy)^{-1} \in L^1(\{|y| > 1\})$, and so $f_t \in L^1(\mathbb{R}_+)$. Now by the Inverse Theorem for the Fourier transform, $F_t(y) = \mathcal{F}^{-1} f_t(y)$ a.e. $y \in \mathbb{R}$, and by the continuity the last equality holds for all $y \in \mathbb{R}$. Therefore we have for the Laplace transform

$$L f_t(z) = \int_{\mathbb{R}_+} e^{r z} f_t(r) dr = \sqrt{2\pi} e^{t \psi(z)} \psi(z), \quad \text{Re} z \leq 0,$$

because both sides here are analytic on the left half-plane $\Pi_-$, continuous on its closure, and coincide on its boundary $i\mathbb{R}$. In particular, $L f_t(s) = \sqrt{2\pi} e^{t \psi(s)} \psi(s)$ for all $s \leq 0$. It follows that for an arbitrary exponential polynomial $\phi \in E(\mathbb{R}_+)$, $\phi(r) = \sum_j c_j e^{s_j r}$ ($c_j \in \mathbb{C}, s_j < 0$) we have

$$\int_{\mathbb{R}_+} \phi(r) f_t(r) dr = \sqrt{2\pi} \sum_j c_j e^{t \psi(s_j)} \psi(s_j).$$

On the other hand,

$$\int_{\mathbb{R}_+} \phi(r) d_r (\nu_t(r - u) - \nu_t(r)) = \int_{[-u, \infty)} \phi(r + u) d\nu_t(r) - \int_{\mathbb{R}_+} \phi(r) d\nu_t(r)$$

$$= \int_{\mathbb{R}_+} (\phi(r + u) - \phi(r)) d\nu_t(r) = \sum_j c_j (e^{s_j u} - 1) e^{t \psi(s_j)},$$

and thus

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \phi(r) d_r (\nu_t(r - u) - \nu_t(r)) u^{-1} d\rho(u) = \sum_j c_j e^{t \psi(s_j)} \psi(s_j).$$

Now we conclude that $(E(\mathbb{R}_+)$ is dense in $C_0(\mathbb{R}_+))

$$K(\nu_t, \rho) = \frac{1}{\sqrt{2\pi}} \sup_{\phi \in E} \left| \int_{\mathbb{R}_+} \phi(r) f_t(r) dr \right| = \frac{1}{\sqrt{2\pi}} \|f_t\|_1.$$
Next, for any $v > 0$ Hölder’s inequality gives
\[
\int_{[0,v]} |f_t(u)| du \leq \|f_t\|_{q} v^{1/p},
\]
\[
\int_{[v,\infty)} (u|f_t(u)|) u^{-1} du \leq \|k_t\|_q (p-1)^{-1/p} v^{-1/q}.
\]

Then for any $v > 0$
\[
\|f_t\|_1 \leq \|f_t\|_{q} v^{1/p} + \|k_t\|_q (p-1)^{-1/p} v^{-1/q} \leq \|F_t\|_p v^{1/p} + \|\partial/\partial y F_t\|_p (p-1)^{-1/p} v^{-1/q}.
\]

Therefore, on choosing $v = (p-1)^{1/q}/\partial/\partial y F_t|_p/\|F_t\|_p$, it follows that
\[
K(\nu_t, p) = \frac{1}{\sqrt{2\pi}} \|f_t\|_1 \leq \text{const} \|F_t\|_p^{1/q} \|\partial/\partial y F_t\|_p^{1/p} = O(t^{-1}) \quad \text{as} \quad t \downarrow 0.
\]

Application of Theorem 2 completes the proof.

Before formulating the next theorem we note that by [2] Theorem 4 every $\psi \in T_Y \cap T_0$ maps $\Pi_-$ into a truncated sector
\[
S(\theta, \beta) := (\beta + \{| \arg(-z) | < \theta\}) \cap \Pi_-
\]
for some $\beta \geq 0, \theta \in (0, \pi/2)$, and there exist constants $k, k > 0$, and $\gamma, \gamma \in (0, 1)$, such that $|\psi(z)| \leq k|z|^\gamma$, $|z| \geq 1$, $\text{Re} z \leq 0$. The problem is what one can add to this conditions to obtain (necessary and) sufficient conditions for $\psi$ to be in $T_Y$. Now we shall deduce the partial answer to this question from Theorem 3.

**Theorem 4.** Let $\psi \in T_0$, and assume that the following conditions hold:

(i) $\psi : \Pi_- \to S(\theta, \beta)$ for some $\beta \geq 0, \theta \in (0, \pi/2)$;

(ii) $b|z|^\alpha \leq |\psi(z)| \leq k|z|^\gamma$ for $z \in \Pi_-, |z| \geq R$;

(iii) the function $y \mapsto \psi(iy)$ is differentiable for a.e. $y \in \mathbb{R}$ and
\[
|\psi'(iy)| \leq k|y|^{\delta}, \quad \text{a.e.} \quad y \in \mathbb{R}, \quad |y| \geq R,
\]

for some $\delta \in (\alpha - \gamma - 1, 2\alpha - \gamma - 1)$ if $\alpha < \gamma$, and $\delta = \gamma - 1$ if $\alpha = \gamma$;

(iv) $\psi'(iy) \in L^p([0, R])$ for some $p \in (1, 2)$ such that $p = \min\{2, (\alpha-\gamma-\delta)^{-1}, (\alpha-\delta-1)/(\gamma-\alpha)\}$ if $\alpha < \gamma$, and $p \leq \min\{2, (1-\gamma)\}$ if $\alpha = \gamma$.

Then $\psi \in T_Y$.

**Proof.** We shall verify all the conditions of Theorem 3 for $\psi$. Let $a_1 = \max\{|\psi(z)| \geq \Pi_-, |z| \geq R\}$, $m_1 = \min\{|\psi(z)| - b|z|^\alpha |z| \in \Pi_-, |z| \leq R\}$. Then $b|z|^\alpha + a_2 \leq |\psi(z)| \leq k|z|^{\gamma} + a_1$ for $z \in \Pi_-$, where $a_2 = \min\{0, m_1\}$. Since $\psi(iy) - \beta \in S(\theta, 0)$, we have $-\text{Re} \psi(iy) + \beta \geq \cos \theta(|\psi(iy)| - \beta)$, and $\text{Re} \psi(iy) \leq -c_1|y|^{\alpha} + c_2$, where $c_1 = b\cos \theta > 0, c_2 \in \mathbb{R}$. It follows that
\[
|F_t(y)| \leq e^{c_2 t} e^{-c_1 t}|y|^{\alpha} (k|y|^{\gamma} + a_1),
\]
and ($p \geq 1$)
\[
\|F_t\|_p \leq e^{c_2 t} 2^{1/p} \left( \int_{\mathbb{R}^+} e^{-c_1 p|y|^{\alpha}} (k|y|^{\gamma} + a_1)^p dy \right)^{1/p}.
\]

Putting $x = ty^{\alpha}$ we get for some constant $c_3 > 0$
\[ \| F_t \|_p \leq c_3 e^{c_2t} t^{-\gamma/\alpha - 1/\alpha p} \left( \int_{\mathbb{R}_+} e^{-c_1px} (kx^{\gamma/\alpha} + a_1 t^{\gamma/\alpha} p x^{1/\alpha - 1} dx) \right)^{1/p}. \]

The integral converges for all \( t \geq 0, p \geq 1 \), and by B. Levi’s Theorem

\[ \| F_t \|_p = O(1) t^{-\gamma/\alpha - 1/\alpha p} \quad \text{as} \quad t \downarrow 0. \] (7)

Let \( \alpha < \gamma, p = \min\{2, (\alpha - \gamma - \delta)^{-1}, (\alpha - \delta - 1)/(\gamma - \alpha)\} \), \( \delta \in (\alpha - \gamma - 1, 2\alpha - \gamma - 1) \). Since

\[ |\partial/\partial y F_t(y)| \leq e^{c_2t} e^{-c_1t|y|\alpha} (tk|y|\gamma + ta_1 + 1)|\psi'(iy)|, \]

we have

\[ \| \partial/\partial y F_t \|_p \leq e^{c_2t} 2^{1/p} \left( \int_{[0,R]} e^{-c_1pt|y|\alpha} (tk|y|\gamma + ta_1 + 1)p|\psi'(iy)|^p dy \
+ k^p \int_{[R,\infty)} e^{-c_1pt|y|\alpha} (tk|y|\gamma + ta_1 + 1)p|\psi'(iy)|^p dy \right)^{1/p}. \]

Putting \( x = ty^\alpha \) in the second integral we get

\[ \| \partial/\partial y F_t \|_p \leq e^{c_2t} 2^{1/p} t^{-\frac{1}{\alpha} + \frac{1}{\alpha p} - 1} \left( t^{\frac{1}{\alpha} + \frac{1}{\alpha p} - 1} \int_{[0,R]} e^{-c_1pt|y|\alpha} (tk|y|\gamma + ta_1 + 1)p|\psi'(iy)|^p dy \
+ k^p \int_{[R^\alpha,\infty)} e^{-c_1ptx} (kx^\gamma + ta_1 + 1)p x^{\frac{\delta p + 1}{\alpha} - 1} dx \right)^{1/p}. \] (8)

The second integral in (8) converges for all \( t \geq 0 \) because \((\gamma + \delta)p/\alpha + 1/\alpha - 1 > -1\) for our \( p \) and \( \delta \). Note that \((\gamma + \delta)/\alpha + 1 + 1/\alpha p - 1 \geq 0 \). Therefore (8) implies

\[ \| \partial/\partial y F_t \|_p = O(1) t^{-(\gamma + \delta)/\alpha - 1/\alpha p + 1} \quad \text{as} \quad t \downarrow 0. \] (9)

It follows from (7) and (9) that for our \( \delta \) we have

\[ \| F_t \|_{1/q} \| \partial/\partial y F_t \|_p^{1/p} = O(1) t^{-\gamma/\alpha - 1/\alpha p - (\delta/\alpha - 1)/p} = O(t^{-1}) \quad \text{as} \quad t \downarrow 0, \]

because \( \gamma/\alpha + 1/\alpha p + (\delta/\alpha - 1)/p < 1 \).

The case \( \gamma = \alpha, \delta = \gamma - 1, 1 < p < \min\{2, (1 - \gamma)^{-1}\} \) can be examined in the same manner.

Finally since \( \psi(i\lambda) - \beta \in S(\theta, 0) \) for \( \lambda \in \mathbb{C} \) with \( \text{Im}\lambda \geq 0 \), we have for such \( \lambda \) (as above)

\[ |F_t(\lambda)| \leq e^{c_2t} e^{-c_1t|\lambda|\alpha} (k|\lambda|\gamma + a_1). \]

Then for \( t > 0 \) (\( \lambda = s + iy, y > 0 \))

\[ \int_{\mathbb{R}_s} |F_t(s + iy)| ds \leq 2 e^{c_2t} \int_{\mathbb{R}_+} e^{-c_1t(s^2 + y^2)^{\alpha/2}} (k(s^2 + y^2)^{\gamma/2} + a_1) ds \]

\[ \int_{[y^2, \infty)} [s^2 + y^2 = v] e^{c_2t} \int_{\mathbb{R}_+} e^{-c_1tv^{\alpha/2}} (kv^{\gamma/2} + a_1)(v - y^2)^{-1/2} dv. \]
Thus
\[
\int_{|y^2+y^2+1|} e^{-ctv^{\alpha/2}}(kv^\gamma/2 + a_1)(v-y^2)^{-1/2}dv \leq \max_{v \geq 0} e^{-ctv^{\alpha/2}}(kv^\gamma/2 + a_1) \int_{[0,1]} u^{-1/2}du.
\]
Furthermore
\[
\int_{|y^2+1,\infty|} e^{-ctv^{\alpha/2}}(kv^\gamma/2 + a_1)(v-y^2)^{-1/2}dv \leq \int_{[1,\infty]} e^{-ctv^{\alpha/2}}(kv^\gamma/2 + a_1)dv.
\]
Thus $F_t$ belongs to the Hardy class $H^1(|\text{Im} \lambda > 0)$ for all $t > 0$ and therefore $FF_t$ is concentrated on $R_+$. This completes the proof.

**Corollary 1.** Let $\psi \in T_0$, and assume that the following conditions hold:

(i) $\psi: \Pi_- \to S(\theta, \beta)$ for some $\beta \geq 0, \theta \in (0, \pi/2)$;

(ii) $\psi(z) \asymp z^\gamma$ for some $\gamma \in (0,1)$ ($z \to \infty, \ z \in \Pi_-$);

(iii) the function $y \mapsto \psi(iy)$ is differentiable for a.e. $y \in R$ and
\[
|\psi'(iy)| \leq k|y|^{\gamma-1}, \ a.e. \ y \in R.
\]

Then $\psi \in T_Y$.

**Example 1** [15]. Let $\psi(z) = e^z - (1-z)^\alpha, \ \alpha \in (0,1), \ c \geq 0$. In this case, all the conditions of Corollary 1 (and hence of Theorems 3 and 4) are clear.

Now we shall give an example of a function $\psi \in T_0$ that satisfies all the conditions of Theorem 4, but conditions of the Theorem in [6] do not hold for $-\psi(-x)$.

**Example 2.** Let $0 < \alpha < \beta < 1$, and
\[
\psi(z) = -(z)^\alpha + (e^{-z})^\beta - 1.
\]

Since the summands map $\Pi_-$ into a sector and into a truncated sector respectively, the condition (i) of Theorem 4 holds. It is easy to verify that $\psi(z) \sim z^\alpha$ as $z \to \infty, z \in \Pi_-$, $\psi'(iy) \sim \alpha|y|^\alpha-1$ as $y \to \infty$. Finally (iv) holds for $p \in (1, \min\{2,(1-\alpha)^-1\})$. At the same time, $-\psi(-x)$ is not regularly varying.

**4. Further sufficient conditions for $\psi$ to be in $T_Y$**

In this section, we shall deduce further conditions from Theorem 2, that are sufficient for $\psi \in T_Y$.

**Theorem 5.** Let $\psi \in T_0$ and the function $r \mapsto \nu_t([r-u,r])$ is monotone decreasing on $[u, +\infty)$ ($u \geq 0$) for each sufficiently small $t > 0$. If
\[
\int_{\mathbb{R}_+} \nu_t([0,u])u^{-1}d\rho(u) = O(t^{-1}) \quad \text{as} \quad t \downarrow 0,
\]
then $\psi \in T_Y$.

**Proof.** Let $a \in M^b(\mathbb{R}_+, \mathbb{R}_+)$, and the function $r \mapsto a([r-u,r])$ is monotone decreasing on $[u, +\infty)$ ($u \geq 0$). Since $\lim_{r \to +\infty} a([r-u,r]) = 0$ for every $u > 0$, for all and $\phi \in E(\mathbb{R}_+)$ with $\sup |\phi| \leq 1$ we find $(a(r) = a([0,r]))$ for $r > 0$, and $a(r) = 0$ for $r \in (-\infty,0]$}
\[
| \int_{\mathbb{R}_+} \phi(r)d\nu_t(a(r-u) - a(r)) | \leq Var_{r \in \mathbb{R}_+} (a(r-u) - a(r)) =
\]

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\[ \text{Var}_{r \in [0,u]} a(r) + \text{Var}_{r \in [u,+\infty)} (a(r-u) - a(r)) = 2a([0,u]). \]

Thus
\[ K(a, \rho) \leq 2 \int_{\mathbb{R}_+} a([0,u]) u^{-1} d\rho(u). \]

It remains to put here \( a = \nu_t \) and to apply Theorem 2.

\textit{Example 3} (cf. \cite[Example 1]{3}). Let for \( b > 0 \)
\[ \psi(z) = \log b - \log(b - z). \]

It is well known that \( d\rho(u) = e^{-bu} du \) and
\[ e^{t\psi(s)} = b^t (b-s)^{-t} = b^t \Gamma(t)^{-1} \int_{\mathbb{R}_+} e^{sr} r^{t-1} e^{-br} dr. \]

So \( d\nu_t(r) = b^t \Gamma(t)^{-1} r^{t-1} e^{-br} dr \), and \( \nu_t \) has monotone decreasing density for \( t \in (0,1) \). Therefore
\[ \int_{\mathbb{R}_+} \nu_t([0,u]) u^{-1} d\rho(u) = \int_{\mathbb{R}_+} \left( \int_{[0,u]} b^t \Gamma(t)^{-1} r^{t-1} e^{-br} dr \right) u^{-1} e^{-bu} du \leq \]
\[ b^t \Gamma(t)^{-1} \int_{\mathbb{R}_+} \left( \int_{[0,u]} r^{t-1} dr \right) u^{-1} e^{-bu} du = \frac{1}{t}. \]

Thus \( \psi \in \mathcal{T}_Y \) by Theorem 5.

\textit{Example 4} (cf. \cite{11}). Let
\[ \psi(s) = \text{acosh} b - \text{acosh} (b - s) \quad (b \geq 1, s \leq 0). \]

Since \( \psi \in \mathcal{T}_Y \) implies \( -\psi(-c) + \psi(s - c) \in \mathcal{T}_Y \) for all \( c \geq 0 \), one can to restrict ourselves to the case \( b = 1 \). In this case, \( \psi \in \mathcal{T}_0 \) with \( d\rho(u) = e^{-u} I_0(u) du \) (the corresponding integral representation (1) can be verified by differentiation under the integral sign), and \( e^{t\psi(s)} = L f_t(s) \) with \( f_t(r) = tr^{-1} e^{-r} I_t(r), \quad r > 0 \) (\( I_t \) denotes the Bessel function of the first kind). Hence, \( d\nu_t(r) = f_t(r) dr \), and \( \nu_t \) has monotone decreasing density for \( t \in (0,1) \) (see \cite{11}). The calculations from Example 3 in \cite{11} show, that the conditions of Theorem 5 hold. So, \( \psi \in \mathcal{T}_Y \).

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