FOUR-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH COMMUTING HIGHER ORDER JACOBI OPERATORS

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Abstract. We consider four-dimensional Riemannian manifolds with commuting higher order Jacobi operators defined on two-dimensional orthogonal subspaces (polygons) and on their orthogonal subspaces.

More precisely, we discuss higher order Jacobi operator $J(X)$ and its commuting associated operator $J(X^\perp)$ induced by the orthogonal complement $X^\perp$ of the vector $X$, i.e. $J(X) \circ J(X^\perp) = J(X^\perp) \circ J(X)$.

At the end some new central theorems have been cited. The latter are due to P. Gilkey, E. Puffini and V. Videv, and have been recently obtained.

1. Preliminaries

Let $(M, g)$ be a $n$-dimensional Riemannian manifold with a metric tensor $g$. Tangent space at a point $p \in M$ we denote by $M_p$, and let $S_p(M)$ be the set of unit vectors in $M_p$, i.e. $S_p(M) := \{ z \in M_p \mid \|g(z, z)\| = 1 \}$. Let $\mathcal{F}(M)$ be the algebra of all smooth functions on $M$ and $\mathcal{A}(M)$ be the $\mathcal{F}(M)$-module of all smooth vector fields over $M$. Let also

$$R(X, Y, Z, U) := g(R(X, Y, Z)U)$$

be the $(1, 3)$ curvature tensor of the Levi-Civita connection $\nabla$. We define

$$R(X, Y, Z, U) := g(R(X, Y, Z)U)$$

to be the associated $(0, 4)$-curvature tensor which satisfied the following algebraic properties:

i) $R(X, Y, Z, U) = -R(Y, X, Z, U)$,

ii) $R(X, Y, Z, U) = -R(X, Y, U, Z)$,

iii) $R(X, Y, Z, U) + R(Y, Z, X, U) + R(Z, X, Y, U) = 0$ (first Bianchi identity),

iv) $R(X, Y, Z, U) = R(Z, U, X, Y)$.

In the Riemannian geometry the following differential equality is also true:

v) $(\nabla_X R)(Y, Z, W) + (\nabla_Y R)(Z, X, W) + (\nabla_Z R)(X, Y, W) = 0$ (second Bianchi identity),

where

$$(\nabla_X R)(Y, Z, W) := \nabla_X (R(Y, Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W,$$

and $\nabla_X R$ is the covariant derivative of the $(1, 3)$-curvature tensor $R$ with respect to $X$, $X, Y, Z \in \mathcal{A}(M)$.

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Let $\mathcal{J}: M_p \to M_p$ be the Jacobi operator defined by:

$$J(X)U = R(U, X, X).$$

(1.1)

One can easily see that $J(X)X = 0$ and $g(J(X)Y, Z) = g(Y, J(X)Z)$ which means that Jacobi operator is a symmetric linear operator.

Jacobi operator can be diagonalized in the Riemannian geometry. In this case we say that $g$ is Osserman metric if the eigenvalues of the Jacobi operator are constant over the tangent bundle $S(M) := \bigcup_{p \in M} S_p M$. If $(M, g)$ is a rank one locally symmetric space, i.e. $\nabla R = 0$, where $\nabla$ is the connection with all positive or all negative sectional curvatures [11] or $(M, g)$ is flat, i.e. $R = 0$, the group of local isometries acts transitively on $S(M)$ and each Jacobi operator has constant eigenvalues. Osserman [8] conjectured that the opposite is also true and this was confirmed by Chi when $n = 4$ and $n \equiv (2 \mod 4)$ [2] and by Nikolaevsky [7] when $n \neq 16$.

Gilkey, Stanilov and Videv [5] introduced a new operator which they called general Jacobi operator of order $k$ or $k$-order Jacobi operator. More precisely, if $\{Y_i\}_{i=1}^k$ is any orthonormal basis for an arbitrary $k$-plane $\pi \in M_p$, the higher $k$-order Jacobi operator is defined by:

$$J(\pi): Y \to \sum_{1 \leq i \leq k} R(Y, Y_i)Y_i = \sum_{1 \leq i \leq k} J(Y_i)Y.$$  

(1.2)

It can be easily verified that this operator does not depend on the basis of $\pi$.

2. Some commutativity conditions

Another variety of problems, connected to the higher order Jacobi operator, emerged thanks to Stanilov and Videv [11]. They are connected with some commutativity conditions forced on (1.2). Recently Brozos-Vázquez and Gilkey [1] were able to prove the following

**Theorem 2.1.** Let $(M, g)$ be a Riemannian manifold, dim $M \geq 3$. Then

(A) $(M, g)$ is flat iff $J(X)J(Y) = J(Y)J(X)$ for arbitrary vectors $X, Y \in M_p$;

(B) $(M, g)$ is a manifold with a constant sectional curvature iff $J(X)J(Y) = J(Y)J(X)$ for arbitrary vectors $X, Y \in M_p$ such that $X \perp Y$.

In this paper authors will characterize indecomposable four-dimensional Riemannian manifolds that satisfy the following two conditions:

For arbitrary unit vector $X \in M_p, p \in M$, we have:

(C1) $J(X) \circ J(X^\perp) = J(X^\perp) \circ J(X)$, where $X^\perp$ is the orthogonal complement of $X$ in $M_p$.

For arbitrary 2-plane $\alpha \subset M_p, p \in M$, we have

(C2) $J(\alpha) \circ J(\alpha^\perp) = J(\alpha^\perp) \circ J(\alpha)$, where $\alpha^\perp$ is the orthogonal complement of $\alpha$ in $M_p$.

Our main goal is to prove the following

**Theorem 2.2.** Let $(M, g)$ be a four-dimensional indecomposable Riemannian manifold. Then the following are equivalent:

(a) Equality (C1) holds for arbitrary unit vector $X \in M_p, p \in M$;

(b) Equality (C2) holds for arbitrary 2-plane $\alpha \subset M_p, p \in M$;

(c) $(M, g)$ is Einstein.
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Proof. (a) $\implies$ (c) Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for $M_p$, $p \in M$. Curvature operator matrices $J_{\{e_1, e_2, e_3\}}$ and $J_{\{e_4\}}$ then have the form:

\[
\begin{pmatrix}
K_{12} + K_{13} & R_{1332} & R_{1223} & \rho_{14} \\
R_{1332} & K_{12} + K_{13} & R_{2113} & \rho_{24} \\
R_{1223} & R_{2113} & K_{13} + K_{23} & \rho_{34} \\
\rho_{14} & \rho_{24} & \rho_{34} & K_{14} + K_{24} + K_{34}
\end{pmatrix},
\]

(2.1)

and

\[
\begin{pmatrix}
K_{14} & R_{1442} & R_{1443} & 0 \\
R_{1442} & K_{24} & R_{2443} & 0 \\
R_{1443} & R_{2443} & K_{34} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(2.2)

where $\rho_{ij} = \rho(e_i, e_j) := \sum_{k=1}^{4} g(R(e_k, e_i)e_j, e_k)$ are the components of the Ricci $(0, 2)$-tensor $\rho$ and $K_{ij} := g(R(e_i, e_j)e_j, e_i)$, $i, j = 1, \ldots, 4$.

We have the matrix equality

\[
J_{\{e_1, e_2, e_3\}} \circ J_{\{e_4\}} = J_{\{e_4\}} \circ J_{\{e_1, e_2, e_3\}},
\]

which leads us to the equations:

\[
\begin{align*}
(e_1) & \quad K_{14}(R_{1224} + R_{1334}) + R_{1442}(R_{2114} + R_{2334}) + R_{1443}(R_{3114} + R_{3224}) = 0, \\
(e_2) & \quad R_{1442}(R_{1224} + R_{1334}) + K_{24}(R_{2114} + R_{2334}) + R_{2443}(R_{3114} + R_{3224}) = 0, \\
(e_3) & \quad R_{1443}(R_{1224} + R_{1334}) + R_{2443}(R_{2114} + R_{2334}) + K_{34}(R_{3114} + R_{3224}) = 0.
\end{align*}
\]

We do a cyclic change of indecies $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ in $(e_1)$, $(e_2)$, $(e_3)$, and get the equations

\[
\begin{align*}
(e_1') & \quad K_{12}(R_{1332} + R_{1442}) + R_{2113}(R_{1223} + R_{1443}) + R_{2114}(R_{1224} + R_{1334}) = 0, \\
(e_2') & \quad R_{2113}(R_{1332} + R_{1442}) + K_{13}(R_{1223} + R_{1443}) + R_{3114}(R_{1224} + R_{1334}) = 0, \\
(e_3') & \quad R_{2114}(R_{1332} + R_{1442}) + R_{3114}(R_{1223} + R_{1443}) + K_{14}(R_{1224} + R_{1334}) = 0.
\end{align*}
\]

Doing a cyclic change of indecies $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ in $(e_1')$, $(e_2')$ and $(e_3')$, we get

\[
\begin{align*}
(e_1'') & \quad K_{23}(R_{2113} + R_{2443}) + R_{3224}(R_{2114} + R_{2334}) + R_{1223}(R_{1332} + R_{1442}) = 0, \\
(e_2'') & \quad R_{3224}(R_{2113} + R_{2443}) + K_{24}(R_{2114} + R_{2334}) + R_{1224}(R_{1332} + R_{1442}) = 0, \\
(e_3'') & \quad R_{1224}(R_{2113} + R_{2443}) + R_{1223}(R_{2114} + R_{2334}) + K_{12}(R_{1332} + R_{1442}) = 0.
\end{align*}
\]

Another cycling change $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ in $(e_1'')$, $(e_2'')$ and $(e_3'')$ will give us

\[
\begin{align*}
(e_1''') & \quad K_{34}(R_{3114} + R_{3224}) + R_{1334}(R_{1223} + R_{1443}) + R_{2334}(R_{2113} + R_{2443}) = 0, \\
(e_2''') & \quad R_{1334}(R_{3114} + R_{3224}) + K_{13}(R_{1223} + R_{1443}) + R_{1332}(R_{1223} + R_{1443}) = 0, \\
(e_3''') & \quad R_{1332}(R_{3114} + R_{3224}) + R_{1333}(R_{1223} + R_{1443}) + K_{23}(R_{2113} + R_{2443}) = 0.
\end{align*}
\]

Solving $(e_1)$, $(e_2)$, $(e_3)$ with respect to $R_{1224} + R_{1334}$, $R_{2114} + R_{2334}$, $R_{3114} + R_{3224}$, using Maple, we get:

\[
R_{1224} + R_{1334} = R_{2114} + R_{2334} = R_{3114} + R_{3224} = 0,
\]
since the above is in fact the trivial solution to the system of equations \((e_1), \ (e_2), \ (e_3)\) which is homogeneous:

\[
\begin{pmatrix}
R_{1442} & R_{1443} \\
R_{1443} & K_{24} \\
R_{2443} & K_{34}
\end{pmatrix}
\begin{pmatrix}
R_{1224} + R_{1334} \\
R_{2114} + R_{2334} \\
R_{3114} + R_{3224}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Analogously, solving the other two homogeneous systems \((e_1^1), \ (e_1^2), \ (e_1^3), \ (e_2^1), \ (e_2^2), \ (e_2^3), \ (e_3^1), \ (e_3^2), \ (e_3^3)\) we get that

\[
R_{1332} + R_{1442} = R_{1223} + R_{1443} = R_{1224} + R_{1334} = 0,
\]

and

\[
R_{1223} + R_{1443} = R_{2113} + R_{2443} = R_{3114} + R_{3224} = 0.
\]

From (2.3) we also get

\[
\begin{aligned}
(e_4) & \quad (K_{14} - K_{24})R_{1332} + R_{2113}R_{1443} - R_{2443}R_{1223} = 0, \\
(e_5) & \quad (K_{14} - K_{34})R_{1223} + R_{2113}R_{1442} - R_{2443}R_{1332} = 0, \\
(e_6) & \quad (K_{24} - K_{34})R_{2113} + (K_{13} - K_{12})R_{2443} + R_{1223}R_{1442} - R_{1332}R_{1443} = 0.
\end{aligned}
\]

By a cycling change of indices \(1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1\) in \((e_4), \ (e_5)\) and \((e_6),\) we get

\[
\begin{aligned}
(e_1^1) & \quad (K_{12} - K_{13})R_{2443} + R_{2114}R_{3224} - R_{3114}R_{2334} = 0, \\
(e_1^2) & \quad (K_{12} - K_{14})R_{2334} + R_{3224}R_{2113} - R_{3114}R_{2443} = 0, \\
(e_1^3) & \quad (K_{13} - K_{14})R_{3224} + (K_{24} - K_{23})R_{3114} + R_{2334}R_{2113} - R_{2443}R_{2114} = 0.
\end{aligned}
\]

Repeating the same procedure two more times, we get

\[
\begin{aligned}
(e_3^1) & \quad (K_{23} - K_{24})R_{3114} + R_{1334}R_{1223} - R_{1224}R_{1443} = 0, \\
(e_3^2) & \quad (K_{23} - K_{12})R_{1443} + R_{3224}R_{1334} - R_{1224}R_{3114} = 0, \\
(e_3^3) & \quad (K_{24} - K_{12})R_{1334} + (K_{13} - K_{34})R_{1224} + R_{1443}R_{3224} - R_{3114}R_{1223} = 0.
\end{aligned}
\]

and

\[
\begin{aligned}
(e_4^1) & \quad (K_{34} - K_{13})R_{1224} + R_{2334}R_{1442} - R_{1332}R_{2114} = 0, \\
(e_4^2) & \quad (K_{34} - K_{23})R_{2114} + R_{1334}R_{1442} - R_{1332}R_{1224} = 0, \\
(e_4^3) & \quad (K_{13} - K_{23})R_{1334} + (K_{24} - K_{14})R_{1332} + R_{1334}R_{2114} - R_{2334}R_{1224} = 0.
\end{aligned}
\]

Further, from \((e_4), \ (e_5), \ (e_6); \ (e_1^1), \ (e_1^2), \ (e_1^3); \ (e_2^1), \ (e_2^2), \ (e_2^3); \ (e_3^1), \ (e_3^2), \ (e_3^3); \ (e_4^1), \ (e_4^2), \ (e_4^3)\) and using that

\[
R_{2113} + R_{2443} = 0, \quad R_{1332} + R_{1442} = 0 \\
R_{1223} + R_{1443} = 0, \quad R_{1224} + R_{1334} = 0 \\
R_{2114} + R_{3224} = 0, \quad R_{3114} + R_{3224} = 0,
\]

we get the system of equations

\[
\begin{aligned}
K_{12} & = K_{34}, \\
K_{13} & = K_{24}, \\
K_{14} & = K_{23}
\end{aligned}
\]

with respect to the basis \(\{e_1, e_2, e_3, e_4\}.\) The latter is equivalent to \((M, g)\) being an Einstein\[10]. \square
Corollary 2.1. Let $p \in \text{FOUR-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH COMMUTING HIGHER ORDER JACOBI OPERATORS}$

(c) $\implies$ (a) Suppose $(M, g)$ is a four-dimensional Einstein manifold and let $X \in M_p$, $p \in M$ and $X^\perp$ is the orthogonal complement of $X$. Then $\rho = \lambda \text{Id}$, $\lambda = \text{const.}$, and hence

$$\mathcal{J}(X) \circ \mathcal{J}(X^\perp) - \mathcal{J}(X^\perp) \circ \mathcal{J}(X) = \mathcal{J}(X) \circ \mathcal{J}(X^\perp) + \mathcal{J}(X) - \mathcal{J}(X) \circ \mathcal{J}(X) - \mathcal{J}(X^\perp) \circ \mathcal{J}(X) =$$

$$\mathcal{J}(X) \circ \left[\mathcal{J}(X^\perp) + \mathcal{J}(X)\right] - \left[\mathcal{J}(X) + \mathcal{J}(X^\perp)\right] \circ \mathcal{J}(X) = \mathcal{J}(X) \circ \rho - \rho \circ \mathcal{J}(X) =$$

$$\lambda \left(\mathcal{J}(X) \circ \text{Id} - \text{Id} \circ \mathcal{J}(X)\right) = 0.$$

\qed

Analogously, one can prove, using \cite{9}, the following

**Corollary 2.1.** Let $(M, g)$ be a three-dimensional Riemannian manifold. Then the next two conditions are equivalent:

(i) $\mathcal{J}(X) \circ \mathcal{J}(X^\perp) = \mathcal{J}(X^\perp) \circ \mathcal{J}(X)$ for arbitrary $X$, $X^\perp \in M_p$, $p \in M$.

(ii) $(M, g)$ has a constant sectional curvature $\kappa$ such that $R(X, Y, Z) = \kappa (g(Y, Z)X - g(X, Z)Y)$, $X, Y, Z \in M_p$.

(b) $\implies$ (c) Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for $M_p$, $p \in M$. Then the curvature operator matrices $\mathcal{J}_{\{e_1, e_2\}}$ and $\mathcal{J}_{\{e_3, e_4\}}$ have the form:

\begin{equation}
\mathcal{J}_{\{e_1, e_2\}} = \begin{pmatrix}
K_{12} + K_{13} & 0 & R_{1223} & R_{1224} \\
0 & K_{12} & R_{2113} & R_{2114} \\
R_{1223} & R_{2113} & K_{13} + K_{23} & \rho_{34} \\
R_{1224} & R_{2114} & \rho_{34} & K_{14} + K_{24}
\end{pmatrix},
\end{equation}

and

\begin{equation}
\mathcal{J}_{\{e_3, e_4\}} = \begin{pmatrix}
K_{13} + K_{14} & \rho_{12} & R_{1443} & R_{1334} \\
\rho_{12} & K_{23} + K_{24} & R_{2443} & R_{2334} \\
R_{1443} & R_{2443} & K_{34} & 0 \\
R_{1334} & R_{2334} & 0 & K_{34}
\end{pmatrix}.
\end{equation}

Let also

\begin{equation}
(A) = \begin{pmatrix}
K_{12} + K_{13} & 0 & R_{1223} & R_{1224} \\
0 & K_{12} & R_{2113} & R_{2114} \\
R_{1223} & R_{2113} & K_{13} + K_{23} & \rho_{34} \\
R_{1224} & R_{2114} & \rho_{34} & K_{14} + K_{24}
\end{pmatrix},
\end{equation}

\begin{equation}
\begin{pmatrix}
K_{13} + K_{14} & \rho_{12} & R_{1443} & R_{1334} \\
\rho_{12} & K_{23} + K_{24} & R_{2443} & R_{2334} \\
R_{1443} & R_{2443} & K_{34} & 0 \\
R_{1334} & R_{2334} & 0 & K_{34}
\end{pmatrix}.
\end{equation}
Simple computations for the matrix \((A)\) give us:

\[
\begin{align*}
a_{11} &= K_{12}(K_{13} + K_{14}) + R_{1223}R_{1443} + R_{1334}R_{1224}, \\
a_{12} &= K_{12}\rho_{12} + R_{1223}R_{2443} + R_{2334}R_{1224}, \\
a_{13} &= K_{12}R_{1443} + K_{34}R_{1224}, \\
a_{14} &= K_{12}R_{1334} + K_{34}R_{1224}, \\
a_{21} &= K_{12}\rho_{12} + R_{1443}R_{2113} + R_{2114}R_{1334}, \\
a_{22} &= K_{12}(K_{23} + K_{24}) + R_{2443}R_{2113} + R_{2114}R_{2334}, \\
a_{23} &= K_{12}R_{2443} + K_{34}R_{2114}, \\
a_{24} &= K_{12}R_{2334} + K_{34}R_{2114}, \\
a_{31} &= (K_{13} + K_{14})R_{1223} + (K_{13} + K_{23})R_{1443} + \rho_{12}R_{2113} + \rho_{34}R_{1443}, \\
a_{32} &= (K_{23} + K_{24})R_{2113} + (K_{13} + K_{23})R_{2443} + \rho_{12}R_{1223} + \rho_{34}R_{2334}, \\
a_{33} &= (K_{13} + K_{23})K_{34} + R_{1223}R_{1443} + R_{2443}R_{2113}, \\
a_{34} &= \rho_{34}K_{34} + R_{1223}R_{1334} + R_{2334}R_{2113}, \\
a_{41} &= (K_{13} + K_{14})R_{1224} + (K_{14} + K_{24})R_{1334} + \rho_{12}R_{2114} + \rho_{34}R_{1443}, \\
a_{42} &= (K_{23} + K_{24})R_{2114} + (R_{14} + K_{24})R_{2334} + \rho_{12}R_{1224} + \rho_{34}R_{2443}, \\
a_{43} &= \rho_{34}K_{34} + R_{1443}R_{1224} + R_{2114}R_{2334}, \\
a_{44} &= (K_{14} + K_{24})K_{34} + R_{1334}R_{1224} + R_{2114}R_{2334}.
\end{align*}
\]

On the other hand let

\[
(B) = \begin{pmatrix} K_{13} + K_{14} & \rho_{12} & R_{1443} & R_{1334} \\ \rho_{12} & K_{23} + K_{24} & R_{2443} & R_{2334} \\ R_{1443} & R_{2443} & K_{34} & 0 \\ R_{1334} & R_{2334} & 0 & K_{34} \end{pmatrix} \begin{pmatrix} K_{12} & 0 & R_{1223} & R_{1224} \\ 0 & K_{12} & R_{2113} & R_{2114} \\ R_{1223} & R_{2113} & K_{13} + K_{23} & \rho_{34} \\ R_{1224} & R_{2114} & \rho_{34} & K_{14} + K_{24} \end{pmatrix}.
\]

For the matrix \((B)\) elements we derive:
From \([C2]\) we have \(a_{12} = b_{12}\) and according to (2.6) and (2.7), we get:

\[
\begin{align*}
(e_7) & \quad R_{1223}R_{2443} + R_{2334}R_{1224} - R_{1443}R_{2113} - R_{2114}R_{1334} = 0, \\
(e_8) & \quad R_{1223}R_{1334} + R_{2334}R_{2113} - R_{1443}R_{1224} - R_{2114}R_{2443} = 0.
\end{align*}
\]

By a cycling change of indices 1 \(\rightarrow\) 2 \(\rightarrow\) 3 \(\rightarrow\) 4 \(\rightarrow\) 1 in \((e_7)\) and \((e_8)\), we get

\[
\begin{align*}
(e^7_1) & \quad R_{2334}R_{3114} + R_{1443}R_{1332} - R_{2114}R_{3224} - R_{1223}R_{1442} = 0, \\
(e^8_1) & \quad R_{2334}R_{1442} + R_{1443}R_{3224} - R_{2114}R_{1332} - R_{1223}R_{3114} = 0.
\end{align*}
\]

We solve, using Maple, \((e_7),(e_8),(e^7_1)\) and \((e^8_1)\) together and arrive at the homogeneous system

\[
(2.8) \quad \begin{pmatrix} R_{2443} & R_{1224} & -R_{2113} & -R_{1334} \\ R_{1334} & R_{2113} & -R_{1224} & -R_{2443} \\ -R_{1442} & R_{3114} & R_{1332} & -R_{3224} \\ -R_{3114} & R_{1442} & R_{3224} & -R_{1332} \end{pmatrix} \begin{pmatrix} R_{1223} \\ R_{2334} \\ R_{1443} \\ R_{2114} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Solving (2.8) with respect to \(R_{1223}, R_{2334}, R_{1443}\) and \(R_{2114}\), we get at least the trivial solution

\[
(2.9) \quad R_{2114} = R_{1223} = R_{1443} = R_{2334} = 0.
\]

We also have \(J_{\{e_1,e_3\}} = J_{\{e_2,e_4\}}\) and \(J_{\{e_1,e_4\}} = J_{\{e_2,e_3\}}\), and using (2.9), it follows that

\[
R_{2114} = R_{1223} = R_{1443} = R_{2334} = 0.
\]
We solve (2.10) with respect to the tensor $R$ components $R_{1332}, R_{1442}, R_{3114}, R_{3224}, R_{1224}, R_{1334}, R_{2113}, R_{2114}$ using, for example Maple, and as a result we get

$$R_{1224}(-K_{34} + K_{13} + K_{14}) = R_{1334}(K_{12} - K_{14} - K_{24})$$

and

$$R_{2113}(-K_{34} + K_{23} + K_{24}) = R_{2443}(K_{12} - K_{13} - K_{23})$$

Further, by changing the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with the orthonormal basis $\{\frac{e_1 - e_2}{\sqrt{2}}, \frac{e_1 + e_2}{\sqrt{2}}, \frac{e_3 - e_4}{\sqrt{2}}, \frac{e_3 + e_4}{\sqrt{2}}\}$ and using (2.11) and (2.12) we receive a new system of equations with respect to the tensor curvature components which is equivalent to (2.10). From there we can conclude that

$$R_{1334} = R_{2443} = 0.$$

From (2.9), (2.11) and (2.12) it follows that all of the components $R_{i,j,k}$ are equal to zero for all $i, j, k = 1, 2, 3, 4$.

If we reformulate the second and third equation in (2.10) by changing the basis as shown above and transforming the curvature components using (2.9), (2.11) and (2.12), we get

$$(K_{13} - K_{23} + K_{14} - K_{24})(R_{1432} + R_{1342} + K_{12}) = 0$$

and

$$(K_{14} + K_{23} - K_{14} - K_{24})(R_{1432} + R_{1342} + K_{34}) = 0.$$

By a cycling change of indices $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ above and some extra tedious computations, we get the system:

$$\begin{align*}
(K_{13} + K_{14} - K_{23} - K_{24})(K_{12} + R_{1342} - R_{1432}) &= 0 \\
(K_{12} + K_{14} - K_{23} - K_{34})(K_{13} + R_{1234} - R_{1432}) &= 0 \\
(K_{12} + K_{13} - K_{24} - K_{34})(K_{14} + R_{1234} - R_{1342}) &= 0 \\
(K_{12} + K_{24} - K_{13} - K_{34})(K_{23} + R_{1234} - R_{1342}) &= 0 \\
(K_{12} + K_{23} - K_{14} - K_{34})(K_{24} + R_{1234} - R_{1432}) &= 0 \\
(K_{13} + K_{23} - K_{14} - K_{24})(K_{34} + R_{1342} - R_{1432}) &= 0
\end{align*}$$

Solving (2.14) and (2.15) with respect to the sectional curvature components $K_{12}, K_{13}$ and $K_{14}$, it follows that

$$K_{14} = K_{23}, \quad K_{13} = K_{24}, \quad K_{12} = K_{34},$$

and since the basis $\{e_1, e_2, e_3, e_4\}$ has been arbitrary chosen in $M_p$, it follows that $(M, g)$ is Einstein [10].
(b) $\implies$ (a) If $(M, g)$ is an Einstein manifold then $\rho = \lambda \text{Id}$, $\lambda = \text{const.}$, and if $\alpha$ is a 2-plane in $M$, $p \in M$, it follows that

$$\mathcal{J}(\alpha) \circ \mathcal{J}(\alpha^\perp) - \mathcal{J}(\alpha^\perp) \circ \mathcal{J}(\alpha) = \mathcal{J}(\alpha) \circ \mathcal{J}(\alpha^\perp) + \mathcal{J}(\alpha) \circ \mathcal{J}(\alpha) - \mathcal{J}(\alpha^\perp) \circ \mathcal{J}(\alpha) - \mathcal{J}(\alpha) \circ \mathcal{J}(\alpha) =$$

$$\rho \mathcal{J}(\alpha) \circ \left[ \mathcal{J}(\alpha^\perp) + \mathcal{J}(\alpha) \right] - \left[ \mathcal{J}(\alpha^\perp) + \mathcal{J}(\alpha) \right] \circ \mathcal{J}(\alpha) = \mathcal{J}(\alpha) \circ \rho - \rho \circ \mathcal{J}(\alpha) = \lambda (\mathcal{J}(\alpha) \circ \text{Id} - \text{Id} \circ \mathcal{J}(\alpha)) = 0.$$

That completes the proof. \hfill \Box

3. NEW APPROACHES AND RESULTS

Recently Gilkey, Puffini and Videv \cite{4} were able to generalize the results above. They define $\mathcal{M} := (V, \langle \cdot, \cdot \rangle, A)$ to be a 0-model if $\langle \cdot, \cdot \rangle$ is a non-degenerate inner product of signature $(p, q)$ on a finite dimensional vector space $V$ of dimension $m = p + q$ and if $A \in \otimes^4 V^*$ is an algebraic curvature tensor.

Let $\text{Gr}_{r,s}(V, \langle \cdot, \cdot \rangle)$ be the Grassmannian of all non-degenerate linear subspaces of $V$ which have signature $(r, s)$; the pair $(r, s)$ is said to be \textit{admissible} if and only if $\text{Gr}_{r,s}(V, \langle \cdot, \cdot \rangle)$ is non-empty and does not consist of a single point or, equivalently, if the inequalities $0 \leq r \leq p$, $0 \leq s \leq q$, and $1 \leq r + s \leq m - 1$ are satisfied. Let $[A,B] := AB - BA$ denote the commutator of two linear maps. Then they establish the following result:

\textbf{Theorem 3.1.} Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, A)$ be a 0-model. The following assertions are equivalent; if any is satisfied, then we shall say that $\mathcal{M}$ is a Puffini–Videv 0-model.

1. There exists $(r_0, s_0)$ admissible so that

$$\mathcal{J}(\pi) \circ \mathcal{J}(\pi^\perp) = \mathcal{J}(\pi^\perp) \circ \mathcal{J}(\pi) \text{ for all } \pi \in \text{Gr}_{r_0,s_0}(V, \langle \cdot, \cdot \rangle).$$

2. $\mathcal{J}(\pi) \circ \mathcal{J}(\pi^\perp) = \mathcal{J}(\pi^\perp) \circ \mathcal{J}(\pi) \text{ for every non-degenerate subspace } \pi.$

3. $[\mathcal{J}(\pi), \rho] = 0 \text{ for every non-degenerate subspace } \pi.$

We say that $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, A)$ is \textit{decomposable} if there exists a non-trivial orthogonal decomposition $V = V_1 \oplus V_2$ which decomposes $A = A_1 \oplus A_2$; in this setting, we shall write $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ where $\mathcal{M}_i := (V_i, \langle \cdot, \cdot \rangle|_{V_i}, A_i)$. One says that $\mathcal{M}$ is \textit{indecomposable} if $\mathcal{M}$ is not decomposable.

By Theorem 3.1, any Einstein 0-model is Puffini–Videv. More generally, the direct sum of Einstein Puffini–Videv models is again Puffini–Videv; the converse holds in the Riemannian setting:

\textbf{Theorem 3.2.} Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, A)$ be a Riemannian 0-model. Then $\mathcal{M}$ is Puffini–Videv if and only if $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$ where the $\mathcal{M}_i$ are Einstein.

In the pseudo-Riemannian setting, a somewhat weaker result can be established. One says that a 0-model is \textit{pseudo-Einstein} either if the Ricci operator $\rho$ has only one real eigenvalue $\lambda$ or if the Ricci operator $\rho$ has two complex eigenvalues $\lambda_1, \lambda_2$ with $\bar{\lambda}_1 = \lambda_2$. This does not imply that $\rho$ is diagonalizable in the higher signature setting and hence need not be Einstein.

\textbf{Theorem 3.3.} Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, A)$ be a 0-model of arbitrary signature. If $\mathcal{M}$ is Puffini–Videv, then we may decompose $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$ as the direct sum of pseudo-Einstein 0-models $\mathcal{M}_i$.\hfill \Box
References

[1] M. Brozos-Vázquez and P. Gilkey. The global geometry of Riemannian manifolds with commuting curvature operators. preprint.
[2] Q.-Sh. Chi. A curvature characterization of certain locally rank-one symmetric spaces. J. Diff. Geom., 28:187–202, 1988.
[3] E. García-Río, D. Kupeli, and R. Vázquez-Lorenzo. Osserman Manifolds in Semi-Riemannian Geometry. Number 1777 in Lecture Notes in Math. Springer-Verlag, Berlin, 2002.
[4] P. Gilkey, E. Puffini, and V. Videv. Puffini-Videv Models and Manifolds. J. of Geom. (to appear), 2006. arXiv: math.DG/0605464.
[5] P. Gilkey, G. Stanilov, and V. Videv. Pseudo-Riemannian manifolds whose generalized Jacobi operator has a constant characteristic polynomial. J. of Geom., 62:144–153, 1998.
[6] S. Kobayashi and K. Nomizu. Foundations of Differential Geometry, volume I. Interscience Publish., New York, 1963.
[7] Y. Nikolaevsky. Two theorems on Osserman manifolds. Diff. Geom. & Appl., 18:239–253, 2003.
[8] R. Osserman. Curvature in the eighties. Amer. Math. Monthly, 97:731–756, 1990.
[9] J.A. Schouten and D. J. Struik. On some properties of general manifolds relating to Einstein’s theory of gravitation. Amer. J. Math, 43:213–216, 1921.
[10] I. Singer and J. Thorpe. The curvature of 4-dimensional Einstein spaces. Global Analysis, 1969. Papers in Honor of K. Kodaira. University of Tokyo Press, Princeton University Press.
[11] G. Stanilov and V. Videv. On the commuting Stanilov’s curvature operators. Mathematics and Education in Mathematics (Proc. of the 33rd Spring Conference of the Union of Bulgarian Mathematicians, Borovets, April 1-4, 2004), pages 176–180, 2004.
[12] Gr. Stanilov. Differential geometry. Nauka i izkustvo, Sofia, 1988. (in Bulgarian).
[13] V. Videv and M. Ivanova. Four-dimensional Riemannian manifolds with commuting Stanilov curvature operators with respect to the orthogonal planes. Mathematics and Education in Mathematics (Proc. of the 33rd Spring Conference of the Union of Bulgarian Mathematicians, Borovets, April 1-4, 2004), pages 180–184, 2004.
[14] H. C. Wang. Two-point homogeneous spaces. Ann. of Math., 55:177–191, 1952.

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