On birth of discrete Lorenz attractors under bifurcations of 3D maps with nontransversal heteroclinic cycles.

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Abstract. In this paper new global bifurcations of three-dimensional diffeomorphisms leading to the birth of discrete Lorenz attractors are studied. We consider the case of a heteroclinic cycle having one non-transversal heteroclinic orbit (quadratic tangency). Additional conditions are imposed onto the system, namely, the Jacobians in the saddles are taken such that the phase volumes near one point are expanded and contracted near another point. Thus here we are dealing with the contracting-expanding, or mixed case. Also it is assumed that one of the heteroclinic orbits (either the transverse one, or the one corresponding to a quadratic tangency) is non-simple. These conditions prevent from existence of a two-dimensional global center manifold and thus keeps the dynamics effectively three-dimensional, thus giving a possibility for Lorenz attractors to exist. The analogous case of a non-simple homoclinic tangency was studied in [21], but the birth of Lorenz attractor in the bifurcations of heteroclinic intersections with a non-simple geometry was not studied before.

Keywords: Homoclinic tangency, rescaling, 3D Hénon map, bifurcation, Lorenz-like attractor.

Mathematics Subject Classification: 37C05, 37G25, 37G35

Introduction

The dynamical chaos is a topic that attracts a high interest of researchers as dynamical models with complex behaviour can be widely met in applications. This relates, in particular, to climate models, see [1] [2] for example, where the influence of a chaotic evolutionary process on “index” cycles of large-scale atmospheric circulation was studied. Usually, the presence of chaos in a dynamical system is connected to the existence of strange attractors. The latter divide into two main types – genuine strange attractors and quasiattractors, this classification was introduced by Afraimovich and Shilnikov [3]. Many of well-known types of chaotic attractors, such as Hénon-like attractors, spiral attractors (Rössler attractors, attractors observed in the Chua circuits) etc., are quasiattractors as their arbitrary small perturbations lead to the appearance of stable periodic orbits, i.e.
systems with quasiattractors lie in the closure of so-called stability windows in the space of dynamical systems. This is not the case for genuine strange attractors. They exist in open domains in the space of dynamical systems and even in case they are structurally unstable (e.g. the Lorenz attractor), stable periodic orbits are not born under these bifurcations.

Among the known genuine strange attractors there are hyperbolic attractors, Lorenz attractors and wild hyperbolic attractors. The latter are remarkable by the fact that they, unlike the previous ones, allow homoclinic tangencies and therefore they belong to Newhouse domains [1, 5]. However, stable periodic orbits and other stable invariant subsets are not born in them under perturbations. An example of a wild spiral attractor was presented in [6] by Turaev and Shilnikov. In addition, such attractors are stable, closed and chain-transitive invariant sets (chain transitivity means that any point of attractor \( \Lambda \) is admissible by \( \varepsilon \)-orbits from any other point of \( \Lambda \); stability means the existence of an open adsorbing domain containing the attractor such that any orbit entering the domain tends to \( \Lambda \) exponentially fast). Thus, this definition coincides, in fact, with the definition of an attractor given by Ruelle and Conley [7, 8].

An important example of a wild hyperbolic attractor is the discrete Lorenz attractor which appears, in particular, in the Poincare maps for periodically perturbed flows with Lorenz attractors [9]. It is well-known that the Lorenz attractors do not allow homoclinic tangencies [10, 11]. However the latter can appear under small non-autonomous periodic perturbations. The reason why stable periodic orbits do not arise from such bifurcations is that Lorenz-like attractors possess a pseudo-hyperbolic structure and this property is preserved under small perturbations. One of the peculiarities of discrete Lorenz attractors is that such attractors can be born at local bifurcations of periodic orbits having three or more multipliers lying on the unit circle. Thus the corresponding attractors can be found in particular models which have a sufficient number of parameters to provide the mentioned degeneracy. The following 3D Hénon map

\[
\begin{align*}
\bar{x} &= y, \\ \bar{y} &= z, \\ \bar{z} &= M_1 + Bx + M_2 y - z^2
\end{align*}
\] (0.1)

controlled by three independent parameters \( M_1, M_2 \) and \( B \) is an example of such a model. In papers [12, 13, 14] it was shown that map (0.1) possesses a discrete Lorenz-like attractor in some open parameter domain near point \((M_1 = 1/4, B = 1, M_2 = 1)\), where the map has a fixed point with the triplet \((-1, -1, 1)\) of multipliers.

This result immediately implies the birth of discrete Lorenz attractors in global (homoclinic and heteroclinic) bifurcations in which map (0.1) appears. The first such example was considered in [13]. It was shown that the 3D Henon map is the asymptotic normal

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1This means that the differential \( Df \) of the corresponding map \( f \) in the restriction onto the adsorbing domain \( D \) of \( \Lambda \) admits an invariant splitting of form \( E^s_x \oplus E^{uc}_x \), for any point \( x \in D \), such that \( Df \) is strongly contracting along directions \( E^s \) and expands volumes in transversal to \( E^s \) sections \( E^{uc} \) (see [8, 9] for details).
form of the first return map in case of a quadratic homoclinic tangency to a saddle-focus fixed point with a unit Jacobian. Later analogous results were obtained for heteroclinic cycles containing saddle-foci \[\text{[15, 16, 17]}\]. Note that the presence of saddle-foci in these cases is a very important condition for the existence of Lorenz-like attractors as it prevents from the existence of lower-dimensional center manifolds and makes the dynamics to be effectively three-dimensional (see \[\text{[15]}\]). Another important condition for this is the restriction on the Jacobians in the fixed points. It is based on the fact that the orbits under consideration may spend unboundedly large time in the neighbourhoods of the saddle fixed point. In the homoclinic case this means that if the Jacobian differs from one, the phase volumes near such orbits will be either unboundedly expanded or unboundedly contracted, and the dynamics will have effective dimension less than three. In the same way, for the heteroclinic cases it is necessary to demand for all the Jacobians not to be simultaneously contracting \((< 1)\) or expanding \((> 1)\). Thus, in order to get Lorenz attractors in heteroclinic cycles, we need to consider “contracting-expanding” or “mixed” cases.

In the present paper we study the birth of discrete Lorenz attractors in the case of a heteroclinic cycle consisting of fixed points of saddle type, i.e. when all multipliers of the fixed points are real, such that one of the heteroclinic orbits is structurally unstable (it belongs to a quadratic tangency of invariant manifolds). In addition, the Jacobian in one of the points is greater than one and less than one in another point. Then, as it is known from \[\text{[19, 20]}\], in order to have the effective dimension of the corresponding problem to be equal to 3, we need to impose an additional degeneracy assumptions on the heteroclinic orbits. Namely, first we demand the quadratic tangency to be non-simple (or generalized by the terminology of \[\text{[20]}\]). Both possible cases of a non-simple quadratic homoclinic tangency were studied in \[\text{[21]}\], however in the heteroclinic cycle another degeneracy of this kind is possible — when the quadratic tangency is simple, but the second, transversal heteroclinic orbit is non-simple. The corresponding definitions are given below in section \[\text{[1]}\].

Remark 1 The notion of a simple quadratic homoclinic tangency (that is analogous to the notion of a quasitransversal homoclinic intersection \[\text{[4]}\]) was introduced in \[\text{[19]}\]. For three-dimensional maps with a homoclinic tangency to a saddle point \(O\) with multipliers \(\nu_i, i = 1, 2, 3\) such that \(|\nu_1| < |\nu_2| < |\nu_3|\), it implies the existence of a global two-dimensional invariant manifold \(\mathcal{M}\) for all nearby maps. This manifold contains all orbits entirely lying in a small fixed neighbourhood of the homoclinic orbit. In general, it is \(C^1\) only and particularly hyperbolic. If point \(O\) has type \((2, 1)\), i.e. \(|\nu_{1,2}| < 1 < |\nu_3|\), the manifold is center-stable; if point \(O\) has type \((1, 2)\), i.e. \(|\nu_1| < 1 < |\nu_{2,3}|\), the manifold is center-

\[\text{[2]}\] Also another case is possible when a saddle fixed point is resonant i.e. when its multipliers \(\lambda_1, \lambda_2, \gamma\), where \(|\lambda_i| < 1 < |\gamma|\), are such that \(|\lambda_1| = |\lambda_2|\). The case of a homoclinic tangency to a resonant saddle with \(\lambda_1 = -\lambda_2\) was considered in \[\text{[22]}\].

\[\text{3}\]
unstable. It implies that neither periodic nor strange attractors can be born at homoclinic bifurcations if $|\nu_2\nu_3| > 1$. However, as it was shown in [20], periodic attractors can appear even in these cases if the homoclinic tangency is non-simple, see also paper [23] in which the case of the point of type $(1, 2)$ was considered in more details. These results are very important for the theory of dynamical chaos since they show that the appearance of non-simple homoclinic tangencies can destroy the “strangeness” of attractors.

The main result of the paper is formulated in Theorem 1 which states that in the space of dynamical systems the system under consideration is a limit of a countable number of open subsets of systems possessing discrete Lorenz attractors. The idea of the proof is based on a construction of the first return map along the heteroclinic cycle, which is then brought to the standard form (asymptotically close to three-dimensional Henon map (0.1)) by means of variables scaling (see Lemma 1 for details).

The paper is organised as follows. In section 1 we give the statement of the problem, main definitions, identify three principally different cases for study and formulate the main theorem. In section 2 the first return map is constructed and the rescaling lemma for all three cases is formulated. Section 3 contains the proof of the rescaling lemma 1 from which the statement of the main theorem follows immediately.
1 Statement of the problem and main definitions

Consider a three-dimensional orientable $C^r$-diffeomorphism $f_0$, $r \geq 3$, satisfying the following conditions (see figure 1):

A) $f_0$ has two fixed points of saddle type: $O_1$ with multipliers $(\lambda_1, \lambda_2, \gamma_1)$ where $0 < \lambda_2 < \lambda_1 < 1 < \gamma_1$, and $O_2$ with multipliers $(\nu_1, \nu_2, \gamma_2)$ such that $0 < \nu_2 < \nu_1 < 1 < \gamma_2$;

B) $J_1 = J(O_1) \equiv \lambda_1 \lambda_2 \gamma_1 < 1$, $J_2 = J(O_2) \equiv \nu_1 \nu_2 \gamma_2 > 1$,

C) Invariant manifolds $W^u(O_1)$ and $W^s(O_2)$ intersect transversely at the points of a heteroclinic orbit $\Gamma_{12}$, the invariant manifolds $W^u(O_2)$ and $W^s(O_1)$ have a quadratic tangency at the points of a heteroclinic orbit $\Gamma_{21}$.

We also assume that the heteroclinic cycle has an additional degeneracy, namely $f_0$ satisfies to one of the following conditions:

D1) The transversal intersection of $W^u(O_1)$ and $W^s(O_2)$ is simple and the quadratic tangency of $W^u(O_2)$ and $W^s(O_1)$ is non-simple.

D2) The transversal intersection of $W^u(O_1)$ and $W^s(O_2)$ is non-simple and the quadratic tangency of $W^u(O_2)$ and $W^s(O_1)$ is simple.

We will study bifurcations of single-round periodic orbits in generic unfoldings of $f_0$. For this purpose we need to determine the necessary number of parameters to take. Diffeomorphisms close to $f_0$ and satisfying either conditions A–D1 or A–D2 compose locally connected bifurcation surfaces of codimension two in the space of $C^r$-diffeomorphisms, thus the number of parameters must be at least two. It is natural to choose the splitting parameter of invariant manifolds $W^u(O_2)$ and $W^s(O_1)$ with respect to some point of $\Gamma_{21}$ as the first parameter $\mu_1$. We also take the second parameter $\mu_2$ to control conditions D1 or D2 in such a way that for $\mu_1 = 0$ and $\mu_2 \neq 0$ the corresponding degeneracy disappears i.e. in the case D1 the tangency of $W^u(O_2)$ and $W^s(O_1)$ becomes simple and in the case D2 the intersection of $W^u(O_1)$ and $W^s(O_2)$ becomes simple. Also note that due to condition B we have the contracting-expanding (or mixed) case, which requires one more parameter $\mu_3$ that will control the values of the Jacobians $J_1$ and $J_2$. It is well known [15, 16, 17] that the following value effectively plays this role:

$$\mu_3 = S(f_\mu) - S(f_0),$$

(1.1)

where $S(f)$ is a functional defined as $S(f) = -\ln J_1 / \ln J_2$.

In order to define simple and non-simple heteroclinic orbits, we recall some facts from the normal hyperbolicity theory. Let $O$ be a saddle fixed points of type $(2,1)$ and $U_0$ be some small neighbourhood of it. It is known [24, 25, 26, 27] that diffeomorphism $f_\mu |_{U_0}$ for

The detailed definition of simple and non-simple heteroclinic orbits will be given below in this section.
each small $\mu$ can be represented in some $C^r$-smooth local coordinates $(x_1, x_2, y)$ as follows (the so-called main normal form):

\[
\begin{align*}
\dot{x}_1 &= \lambda_1(\mu)x_1 + \tilde{H}_2(y, \mu)x_2 + O(\|x\|^2|y|) \\
\dot{x}_2 &= \lambda_2(\mu)x_2 + \tilde{R}_2(x, \mu) + \tilde{H}_4(y, \mu)x_2 + O(\|x\|^2|y|) \\
\dot{y} &= \gamma(\mu)y + O(\|x\|^2|y|^2),
\end{align*}
\]

where $\tilde{H}_2, 4(0, \mu) = 0$, $\tilde{R}_2(x, \mu) = O(\|x\|^2)$. In coordinates \(1.2\) the invariant manifolds of saddle fixed point $O$ are locally straightened: stable $W^s_{\text{loc}}(O)$ : \(y = 0\), unstable $W^u_{\text{loc}}(O) : \{x_1 = 0, x_2 = 0\}$ and strong stable $W^s_{\text{loc}}(O) : \{x_1 = 0, y = 0\}$.

According to \[27, 28\], an important role in dynamics is played by an extended unstable manifold $W^u_{\text{loc}}(O)$, see figure \[2\]. By definition, it is a two-dimensional invariant manifold, tangent to the leading stable direction (corresponding to $\lambda_1$) at the saddle point and containing unstable manifold $W^u(O)$. Unlike the previous ones, the extended unstable manifold is not uniquely defined and its smoothness is, generally speaking, only $C^{1+\varepsilon}$. Locally, $W^s_{\text{loc}}(O) = W^s_{\text{loc}}(O) \cap U_0$, and the equation of $W^s_{\text{loc}}(O)$ has the form $x_2 = \varphi(x_1, y)$, where $\varphi(0, y) \equiv 0$ and $\varphi'_{x_1}(0, 0) = 0$. Note that despite the fact that $W^u_{\text{loc}}(O)$ is non-unique, all of them have the same tangent plane at each point of $W^u(O)$.

Another essential fact is the existence of the strong stable invariant foliation, see Figure \[2\]. In $W^s(O)$ there exists a one-dimensional strong stable invariant submanifold $W^{ss}(O)$, which is $C^r$-smooth and touches at $O$ the eigenvector corresponding to the strong stable (non-leading) multiplier $\lambda_2$. Moreover, manifold $W^s(O)$ is foliated near $O$ by the leaves of invariant foliation $F^{ss}$ which is $C^r$-smooth, unique and contains $W^{ss}(O)$ as a leaf.

Now consider diffeomorphism $f_0$. It has fixed points $O_1$ and $O_2$ and heteroclinic orbit $\Gamma_{21}$ in the points of which manifolds $W^u(O_2)$ and $W^s(O_1)$ have a quadratic tangency. Everything mentioned above on the existence of extended unstable manifolds and strong stable foliations can be applied to each of the saddles. Let $U_{01} \ni O_1$ and $U_{02} \ni O_2$ be some small neighbourhoods of the fixed points, $M^+_1 \in W^s_{\text{loc}}(O_1) \subset U_{01}$ and $M^-_2 \in W^u_{\text{loc}}(O_2) \subset U_{02}$ be two points of $\Gamma_{21}$ and $\Pi^+_1 \subset U_{01}$ and $\Pi^-_2 \subset U_{02}$ their respective neighborhoods. Note that there exists some integer $q_1$ such that $M^+_1 = f_0^{q_1}(M^-_2)$. Define the global map along $\Gamma_{21}$ for all small $\mu$ as $T_{21, \mu} : \Pi^-_2 \to \Pi^+_1 = f^{q_1}_{\mu}|_{\Pi^-_2}$ (for simpler notation, further we will omit the subscript $\mu$ for global and local maps, implicitly always assuming them to be the corresponding parametric families). The heteroclinic tangency of $W^u(O_2)$ and $W^s(O_1)$ is called simple if image $T_{21}(P^u(M^-_2))$ of tangent plane $P^u(M^-_2)$ to $W^u(O_2)$ at point $M^-_2$, intersects transversely the leaf $F^{ss}_1(M^+_1)$ of invariant foliation $F^{ss}_1$, containing point $M^+_1$. If this condition is not fulfilled we call such a quadratic tangency non-simple. Following \[20\], there may be only two generic cases of non-simple heteroclinic tangencies defined by condition D1:
Figure 2: Invariant structures near a saddle fixed point. A part of the strong stable foliation $F^{ss}$ containing the strong stable manifold $W^{ss}$ and a piece of one of the extended unstable manifolds $W^{ue}$ containing $W^u$ and being transversal to $W^{ss}$ at $O$.

Case I. The surface $T_{21}(P^{ue}(M^-_1))$ is transversal to the plane $W^{loc}_1(O_1)$ but is tangent to the line $F^{ss}_1(M^+_1)$ at point $M^+_1$, fig 3 (a).

Case II. The surfaces $T_{21}(P^{ue}(M^-_1))$ and $W^{loc}_1(O_1)$ have a quadratic tangency at $M^+_1$ and the curves $T_{21}(W^{u}_{loc}(O_2) \cap \Pi^+_2)$ and $F^{ss}_1(M^+_1)$ have a general intersection, fig 3 (b).

Thus, in Case I tangent vectors $l_u$ to $T_{21}(W^{u}_{loc}(O_2))$ and $l_{ss}$ to $F^{ss}_1(M^+_1)$ are collinear, while in Case II these vectors have different directions, the latter guarantees the absence of additional degeneracies.

Note that if saddle points $O_1$ and $O_2$ coincide, we formally obtain the known definition of a non-simple homoclinic tangency [20, 21, 23]. However, in distinct from the homoclinic case, the heteroclinic cycle under consideration allows one more degeneracy, related to the second heteroclinic orbit $\Gamma_{12}$. This is the case when the transversal heteroclinic intersection of manifolds $W^u(O_1)$ and $W^s(O_2)$ is non-simple. Consider two heteroclinic points $M^-_1 \in U_{01}$ and $M^+_2 \in U_{02}$ and their small respective neighbourhoods $\Pi^-_1 \subset U_{01}$ and $\Pi^+_2 \subset U_{02}$. Again, there exists some integer $q_2$ such that $M^+_2 = f^{q_2}_0(M^-_1)$ so that we define the global map from $U_{01}$ to $U_{02}$ as $T_{12} : \Pi^-_1 \rightarrow \Pi^+_2 = f^{q_2}_0|_{\Pi^-_1}$. Let $P^{ue}(M^-_1)$ be the tangent plane to $W^{ue}(O_1)$ at $M^-_1$ and $F^{ss}_1(M^+_2)$ be the leaf of invariant foliation $F^{ss}_2$ on $W^s(O_2)$ passing through $M^+_2$. The heteroclinic intersection of $W^u(O_1)$ and $W^s(O_2)$ is called simple if image $T_{12}(P^{ue}(M^-_1))$ and leaf $F^{ss}_2(M^+_2)$ intersect transversely. If this condition is not fulfilled the heteroclinic intersection is non-simple, see fig. 4. Thus, under
condition D2, we have

Case III. The surface $T_{12}(P_{ue}(M^-))$ is transversal to the plane $W_{loc}^s(O_2)$ but is tangent to the line $F_{2}^{ss}(M^+_2)$ at $M^+_2$.

In the present paper we study the birth of discrete Lorenz attractors in cases I – III. The main result is given by the following

**Theorem 1** Let $f_\mu$ be the three-parametric family under consideration ($f_0$ satisfies A – D and $f_\mu$ is a general unfolding of conditions B, C and D, where D is either D1 or D2). Then, in any neighbourhood of the origin $\mu = 0$ in the parameter space there exist infinitely many domains $\delta_{ij}$, where $\delta_{ij} \to (0,0,0)$ as $i,j \to \infty$, such that the diffeomorphism $f_\mu$ has at $\mu \in \delta_{ij}$ a discrete Lorenz attractor.

### 2 The first return map and the rescaling lemma

Let $U$ be a sufficiently small and fixed neighborhood of heteroclinic cycle $\{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$. It is composed as a union of small neighborhoods $U_{01}$ and $U_{02}$ of points $O_1$ and $O_2$ respectively, with a finite number of small neighborhoods $U_i$ of those points of heteroclinic orbits $\Gamma_{12}$ and $\Gamma_{21}$ which do not belong to $U_{01} \cup U_{02}$. Each single-round periodic orbit lying entirely in a small neighborhood of the heteroclinic cycle should have exactly one intersection point with each of $U_i$ and the rest points lying in $U_{01} \cup U_{02}$.
Figure 4: A non-simple heteroclinic intersection of $W^u(O_1)$ and $W^s(O_2)$.

Consider heteroclinic points $M_{1,2}^\pm$ and their respective small neighborhoods $\Pi_{1,2}^\pm$ described in the previous section. Define the first return map as a composition of two local and two global maps. Local maps $T_{01}$ and $T_{02}$ are the restrictions of $f_\mu$ onto $U_{01}$ and $U_{02}$ respectively and the global maps were defined in the following way: $T_{12} = f_{\mu_1} : \Pi_1^- \rightarrow \Pi_2^+$, $T_{21} = f_{\mu_2} : \Pi_2^- \rightarrow \Pi_1^+$.

Begin iterating $\Pi_1^+$ under the action of $T_{01}$. Starting from some number $i_0$ images $T_{01}^i\Pi_1^+$ will have a nonempty intersection with $\Pi_1^-$. The same applies for iterations of $\Pi_2^+$, there exists some $j_0$ such that for any $j \geq j_0$ image $T_{02}^j\Pi_2^+$ has a nonempty intersection with $\Pi_2^-$. The first return map $T_{ij} \equiv T_{21}T_{02}T_{12}T_{01}^i$ is thus defined on an infinite set of regions $V_{ij}$ that lie in $\Pi_1^+$ and shrink to $M_{1,2}^+$ as $i,j \rightarrow \infty$. Their images $f_{\mu_1}^i V_{ij}$ lie in $\Pi_1^-$, regions $f_{\mu_2}^{i+q_1} V_{ij}$ lie in $\Pi_2^+$, and regions $f_{\mu_2}^{i+q_1+j} V_{ij}$ lie in $\Pi_2^-$, so $f_{\mu_2}^{i+q_1+j+q_2} V_{ij}$ lie in $\Pi_1^+$ again.

To construct the first return map we need first to write both local and global maps in the most suitable form. For both local maps $T_{01}$ and $T_{02}$ it is the main normal form (1.2). One its important property is that the iterations $T_{0m}^k : U_{0m} \rightarrow U_{0m}$, $m = 1,2$, for any $k$ can be calculated in a simple way, namely, in a form close to linear (see, for example, [27], [26]). Namely, for small $\mu$ iterations $T_{01}^k(\mu) : (x_0, y_0) \rightarrow (x_k, y_k)$ can be represented as:

$$
\begin{align*}
x_{1k} &= \lambda_{1k} x_{10} + \hat{\lambda}_{1k}(x_{0}, y_{k}, \mu), \\
x_{2k} &= \lambda_{2k} x_{2k} + \hat{\lambda}_{2k}(x_{0}, y_{k}, \mu), \\
y_0 &= \gamma_{1}^{-k} y_{k} + \hat{\gamma}_{1}^{-k}(x_{0}, y_{k}, \mu),
\end{align*}
$$

(2.1)
and iterations $T^k_{02}(\mu) : (u_0, v_0) \rightarrow (u_k, v_k)$ as

\begin{align*}
    u_{1k} &= \mu^k u_{10} + \dot{\mu}^k \xi_{4k}(u_0, v_0, \mu), \\
    u_{2k} &= \dot{\mu}^k \xi_{5k}(u_0, v_0, \mu), \\
    v_0 &= \gamma_2^k v_k + \gamma_2^k \xi_{6k}(u_0, v_0, \mu). \\
\end{align*}

(2.2)

Here $0 < \hat{\lambda} < \lambda_1$, $0 < \dot{\nu} < \nu_1$, $\dot{\gamma}_{1,2} > \gamma_{1,2}$, functions $\xi_{mk}$ and their derivatives up to the order $(r - 2)$ are uniformly bounded and their higher order derivatives tend to zero.

In normal coordinates, local stable and unstable manifolds of $O_1$ in $U_1$ are $W^s_{loc} = \{ y = 0 \}$ and $W^u_{loc} = \{ x = 0 \}$, the local stable and unstable manifolds of $O_2$ in $U_2$ are $W^s_{loc} = \{ v = 0 \}$ and $W^u_{loc} = \{ u = 0 \}$. Assume that for $\mu = 0$, we have $M^-_1 = (0, 0, y^-) \in W^u_{loc}(O_1)$, $M^+_2 = (u^+_1, u^+_2, 0) \in W^s_{loc}(O_2)$, and $M^-_2 = (0, 0, v^-) \in W^u_{loc}(O_2)$, $M^+_1 = (x^+_1, x^+_2, 0) \in W^s_{loc}(O_1)$. Then global maps for all small $\mu$ are written as Taylor expansions near points $M^-_1$ and $M^-_2$:

\begin{align*}
    T_{12} : u - u^+ &= A^{(1)} x + b^{(1)}(y - y^-) + O(||x||^2 + ||x|| \cdot |y - y^-| + (y - y^-)^2), \\
    v &= (c^{(1)}_1)^\top x + d^{(1)}(y - y^-) + O(||x||^2 + ||x|| \cdot |y - y^-| + (y - y^-)^2), \\
\end{align*}

(2.3)

\begin{align*}
    T_{21} : \bar{x} - x^+ &= A^{(2)} u + b^{(2)}(v - v^-) + O(||u||^2 + ||u|| \cdot |v - v^-| + (v - v^-)^2), \\
    \bar{y} &= y^+ + (c^{(2)}_1)^\top u + d^{(2)}(v - v^-)^2 + O(||u||^2 + ||u|| \cdot |v - v^-| + (v - v^-)^3), \\
\end{align*}

(2.4)

where $d^{(1)} \neq 0$ and $d^{(2)} \neq 0$, since $W^u(O_1)$ and $W^s(O_2)$ intersect transversely and the tangency between $W^u(O_2)$ and $W^s(O_1)$ is quadratic for $\mu = 0$. Moreover, both maps $T_{12}$ and $T_{21}$ are diffeomorphisms, so that we have

\begin{align*}
    J_{12} &= \det \begin{pmatrix}
        a^{(1)}_{11} & a^{(1)}_{12} & b^{(1)}_1 \\
        a^{(1)}_{21} & a^{(1)}_{22} & b^{(1)}_2 \\
        c^{(1)}_1 & c^{(1)}_2 & d^{(1)}
    \end{pmatrix} \neq 0, \\
    J_{21} &= \det \begin{pmatrix}
        a^{(2)}_{11} & a^{(2)}_{12} & b^{(2)}_1 \\
        a^{(2)}_{21} & a^{(2)}_{22} & b^{(2)}_2 \\
        c^{(2)}_1 & c^{(2)}_2 & 0
    \end{pmatrix} \neq 0. \\
\end{align*}

(2.5)

In particular, this means that $\sqrt{b^{(2)}_1^2 + b^{(2)}_2^2} \neq 0$ and $\sqrt{c^{(1)}_1^2 + c^{(1)}_2^2} \neq 0$ for $\mu = 0$.

Now we consider conditions D1 and D2 separately.

Case I. Tangent plane $P^u_{ue}(M^-_2)$ to $W^u_{loc}(O_2)$ at $M^-_2$ has equation $v_2 = 0$. The equation of $T_{21}(P^u_{ue}(M^-_2))$ at $\mu = 0$ is obtained by putting $u_2 = 0$ into (2.4). Then the transversality of $T_{21}(P^u_{ue}(M^-_2))$ and $W^s_{loc}(O_1)$ which has the equation $\bar{y} = 0$, yields $c^{(2)}_1(0) \neq 0$. The tangent vector to the line $T_{21}(P^u_{ue}(M^-_2)) \cap W^s_{loc}(O_1)$ at point $M^+_1$ is $(b^{(2)}_1(0), b^{(2)}_2(0), 0)$. The equation of leaf $F^{ss}(M^+_1)$ is $\{ x_1 = x^+_1, y = 0 \}$. Therefore, the tangency of $T_{21}(P^u_{ue}(M^-_2))$ and $F^{ss}(M^+_1)$ implies $b^{(2)}_1(0) = 0$. In this case $b^{(2)}_2(0) \neq 0$ and $a^{(2)}_{11} + a^{(2)}_{12} \neq 0$ because of (2.5).
Case II. The equation of $T_1(P^{ue}(M^-_2))$ at $\mu = 0$ is the same as in Case I. Then the tangency of surfaces $T_{21}(P^{ue}(M^-_2))$ and $W_{loc}^{ss}(O_1)$ at $\mu = 0$ implies $c_1^{(2)}(0) = 0$. Also, the tangent vectors to the lines $T_{21}(W_{loc}^{ss}(O_2) \cap \Pi^-_2)$ and $F_{ss}(M^+_1)$ at point $M^+_1$ are non-parallel if $b_1^{(2)}(0) \neq 0$.

Case III. The equation of tangent plane $P^{ue}(M^-_1)$ to $W_{loc}^{ue}(O_1)$ at $M^-_1$ is $x_2 = 0$. Putting this to (2.3) gives the equation of its image under $T_{12}$. The equation of leaf $F_{ss}(M^+_2)$ is \{ $u_1 = u_1^+, v = 0$ \}. Thus this leaf will be tangent to $T_{12}(P^{ue}(M^-_1))$ at $\mu = 0$ if:

$$A_{11}^{(1)}(0) = a_{11}^{(1)}(0) - b_1^{(1)}(0)c_1^{(1)}(0)/d_1(0) = 0.$$  

We are now able to write the non-simple heteroclinic orbit conditions in the explicit form for all three cases:

Case I: $b_1^{(2)}(0) = 0, b_2^{(2)}(0) \neq 0, c_1^{(2)}(0) \neq 0, A_{11}^{(1)}(0) \neq 0$.
Case II: $b_1^{(2)}(0) \neq 0, c_1^{(2)}(0) = 0, c_2^{(2)}(0) \neq 0, A_{11}^{(1)}(0) \neq 0$.
Case III: $b_1^{(2)}(0) \neq 0, c_1^{(2)}(0) \neq 0, A_{11}^{(1)}(0) = 0$. (2.6)

We will construct a three-parameters family $f_\mu$, where $\mu = (\mu_1, \mu_2, \mu_3)$. As the first governing parameter we take the splitting parameter $\mu_1$ for the quadratic heteroclinic tangency so that

$$\mu_1 \equiv y^+(\mu).$$ (2.7)

The second parameter $\mu_2$ is responsible for the degeneracy related to, respectively, conditions (D1):

$$\mu_2 = b_1^{(2)} \quad \text{in Case I,}$$ (2.8)
$$\mu_2 = c_1^{(2)} \quad \text{in Case II}$$ (2.9)

or (D2):

$$\mu_2 = A_{11}^{(1)} = a_{11}^{(1)} - \frac{b_1^{(1)}c_1^{(1)}}{d_1} \quad \text{in Case III.}$$ (2.10)

The third parameter has been already given by formula (1.1).

Lemma 1 Let $f_{\mu_1,\mu_2,\mu_3}$ be the family under consideration. Then, in the space $(\mu_1, \mu_2, \mu_3)$ there exist infinitely many regions $\Delta_{ij}$ accumulating to the origin as $i, j \to \infty$, such that the first return map $T_{ij}$ in appropriate rescaled coordinates and parameters is asymptotically $C^{r-1}$-close to one of the following limit maps.

1) In Case I, the limit map is

$$\bar{X}_1 = - J_{ij}X_2 + M_2Y, \quad \bar{X}_2 = Y, \quad \bar{Y} = M_1 - X_1 - Y^2,$$ (2.11)
where
\[
M_1 = -d^{(1)^2}d^{(2)}_1\gamma_1^{2i}\gamma_2^{2j}(\mu_1 + \nu_1^j c_1^{(2)} u_1^+ - \gamma_1^{-i} y^- + \nu_{ij}^1),
\]
\[
M_2 = (\mu_2 + \rho_{ij}^1) c_1^{(2)} A_1^{(1)} \lambda_1^j \nu_1^j \gamma_2^j,
\]
\[
J_{ij} = J_{12} J_{21} (\lambda_1 \lambda_2 \gamma_1^i)(\nu_1 \nu_2 \gamma_2^j),
\]
and \( \nu_{ij}^1 = O(\hat{\gamma}_1^{-i} + \hat{\nu}^j + \gamma_1^{-i} \gamma_2^{-j}) \), \( \rho_{ij}^1 = O(\nu_1^j) \).

2) In Case II, the limit map is
\[
\bar{X}_1 = Y, \quad \bar{X}_2 = X_1, \quad \bar{Y} = M_1 + M_2 X_1 + BX_2 - Y^2,
\]
where
\[
M_1 = -d^{(1)^2}d^{(2)}_1\gamma_1^{2i}\gamma_2^{2j}(\mu_1 + \nu_1^j \mu_2 u_1^+ - \gamma_1^{-i} y^- + \nu_{ij}^2),
\]
\[
M_2 = (\mu_2 + \rho_{ij}^2) b_1^{(2)} A_1^{(1)} \lambda_1^j \nu_1^j \gamma_2^j,
\]
\[
B = J_{12} J_{21} (\lambda_1 \lambda_2 \gamma_1^i)(\nu_1 \nu_2 \gamma_2^j)
\]
and \( \nu_{ij}^2 = O(\hat{\gamma}_1^{-i} + \hat{\nu}^j + \gamma_1^{-i} \gamma_2^{-j} + \nu_1^j \left(\hat{\lambda}/\lambda_1\right)^j) \), \( \rho_{ij}^2 = O(\nu_1^j) \).

3) In Case III, the limit map is
\[
X_1 = Y, \quad X_2 = X_1, \quad Y = M_1 + M_2 X_1 + BX_2 - Y^2,
\]
where
\[
M_1 = -d^{(1)^2}d^{(2)}_1\gamma_1^{2i}\gamma_2^{2j}(\mu_1 + \nu_1^j c_1^{(2)} u_1^+ - \gamma_1^{-i} y^- + \nu_{ij}^3),
\]
\[
M_2 = (\mu_2 + \rho_{ij}^3) b_1^{(2)} c_1^{(2)} \lambda_1^j \nu_1^j \gamma_1^j \gamma_2^j,
\]
\[
B = J_{12} J_{21} (\lambda_1 \lambda_2 \gamma_1^i)(\nu_1 \nu_2 \gamma_2^j)
\]
and \( \nu_{ij}^3 = O(\hat{\gamma}_1^{-i} + \hat{\nu}^j + \gamma_1^{-i} \gamma_2^{-j}) \), \( \rho_{ij}^3 = O(\nu_1^j) \).

Thus, the rescaled first return map in Cases II and III is exactly the 3D Henon map \((1.1)\). In Case I for system \((2.11)\) we make an additional change of coordinates \(X_{1\text{new}} = X_1 - M_2 X_2\) and scale \(X_1\) by \((-J_{ij})\), bringing it again to the form \((1.1)\). Next, the statement of Theorem\([\text{II}]\) follows immediately – as shown in \([12,13]\), this three-dimensional Henon map possesses the discrete Lorenz attractor for an open set of parameters \((M_1, M_2, B)\). Hence for each sufficiently large \(i\) and \(j\), for which the Jacobian \(J_{ij}\) stays finite, the corresponding domain \(\delta_{ij}\) in the original parameters \((\mu_1, \mu_2, \mu_3)\) is determined from formulas \((2.12), (2.14)\) or \((2.16)\) respectively in Cases I–III. These domains accumulate to the origin when \(i\) and \(j\) unboundedly grow.
3 Proof of Lemma 1.

Note that the first return map $T_{ij}$ is rescaled differently in cases I – III, however, there is a preparation part of the proof that is conducted in the same way for all the cases.

Using formulas (2.1), (2.2), (2.3) and (2.4), we obtain the following expression for the first return map $T_{ij} \equiv T_{21}T_{02}T_{12}T_{01} : \Pi^+_i \to \Pi^+_i$

$$u_1 - u_1^+ = a_{11}^{(1)}(\lambda^1_i x_1 + \hat{\lambda}^1_i \xi_{1i}(x, y, \mu)) + a_{12}^{(1)} \hat{\lambda}^1_i \xi_{2i}(x, y, \mu) + b_1^{(1)}(y - y^-) + O(\lambda_i^2 x_1 \|x\|^2 + \lambda_i^1 x_1 \|y\| \cdot |y - y^-| + (y - y^-)^2),$$

$$u_2 - u_2^+ = a_{21}^{(1)}(\lambda^1_i x_1 + \hat{\lambda}^1_i \xi_{1i}(x, y, \mu)) + a_{22}^{(1)} \hat{\lambda}^1_i \xi_{2i}(x, y, \mu) + b_2^{(1)}(y - y^-) + O(\lambda_i^2 x_1 \|x\|^2 + \lambda_i^1 x_1 \|y\| \cdot |y - y^-| + (y - y^-)^2),$$

$$\gamma^{-j}_2(v + \frac{\gamma^-_2 \gamma^-_2^{-j}}{\gamma^-_1} \xi_{0j}(u, v, \mu))) = c_{11}^{(1)}(\lambda^1_i x_1 + \hat{\lambda}^1_i \xi_{1i}(x, y, \mu)) + c_{21}^{(1)} \lambda^1_i \xi_{2i}(x, y, \mu) +$$

$$+ d^{(1)}(y - y^-) + O(\lambda_i^2 x_1 \|x\|^2 + \lambda_i^1 x_1 \|y\| \cdot |y - y^-| + (y - y^-)^2),$$

$$\gamma^{-j}_1 \tilde{y} + (\gamma^{-j}_1 \gamma^{-j}_1) \xi_{3i}(\tilde{x}, \tilde{y}, \mu))) = \mu_1 + c_{12}^{(1)}(\nu_{i}^1 u_1 + \hat{\nu}^1 \xi_{4i}(u, v, \mu)) +$$

$$+ c_{22}^{(1)} \hat{\nu}^1 \xi_{5i}(u, v, \mu) + d^{(2)}(v - v^-)^2 + O(\nu_{i}^1 \|u\|^2 + \nu_{i}^1 \|u\| \cdot |v - v^-|^2 +$$

$$+(v - v^-)^3).$$

Make a coordinate shift $u_{new} = u - u^+ + \varphi_{ij}^1$, $v_{new} = v - v^+ + \varphi_{ij}^2$, $x_{new} = x - x^+ + \psi_{ij}^1$, $y_{new} = y - y^- + \psi_{ij}^2$, where $\varphi_{ij}^1, \psi_{ij}^2 = O(\gamma^{-j}_2 \lambda_i^1)$ and $\varphi_{ij}^2, \psi_{ij}^2 = O(\nu_{i}^1)$. With that, the nonlinearity functions in the left parts of the third equations of (3.1) and (3.2) can be expressed as Taylor expansions $\xi_{0j}(u + u^+, \varphi_{ij}^1, v + v^- + \varphi_{ij}^2, \mu)) = \xi_{0j}^0 + \xi_{0j}^1 \nu_{i}^1 v + \xi_{0j}^2 \xi_{3j}(u, v) + \xi_{0j}^3(v)$, $\xi_{3j}(x + x^+ + \psi_{ij}^1 \tilde{y} + y^- + \psi_{ij}^2, \mu)) = \xi_{3j}^0 + \xi_{3j}^1 \tilde{y} + \xi_{3j}^2 \tilde{y} + \xi_{3j}^3(\tilde{y})$, respectively, where coefficients $\xi_{0j}^0, \xi_{0j}^1, \xi_{0j}^2, \xi_{0j}^3, \xi_{3j}^0, \xi_{3j}^1, \xi_{3j}^2, \xi_{3j}^3$ are uniformly bounded in $i$ and $j$ for all small $\mu$ and $\xi_{0j}^2(u, v) = O(u)$, $\xi_{3j}^2(\tilde{x}, \tilde{y}) = O(\tilde{x})$, $\xi_{3j}^2(v) = O(\tilde{y})$, $\xi_{3j}^2(\tilde{y}) = O(\tilde{y})$. We select constants $\varphi_{ij}^1, \varphi_{ij}^2, \psi_{ij}^1, \psi_{ij}^2$ in such a way that all constant terms in equations (3.1), the constant terms in the first two equations and the linear in $v_{new}$ term in the last equation of (3.2) vanish. In addition, we plug the expressions for $u$ coordinates from the first two equations of (3.1) into the third one, this will cause additions of order $\gamma^{-j}_2$ to all the coefficients.

The system is rewritten as:

$$u_1 = a_{11}^{(1)}(\lambda^1_i x_1 + O(\lambda^1_i \gamma^{-j}_2))O(|x|) + b_1^{(1)}(y) + \lambda^1_i O(|x| \cdot |y|) + O(y^2),$$

$$u_2 = a_{21}^{(1)}(\lambda^1_i x_1 + O(\lambda^1_i \gamma^{-j}_2))O(|x|) + b_2^{(1)}(y) + \lambda^1_i O(|x| \cdot |y|) + O(y^2),$$

$$\gamma^{-j}_2(1 + q_{ij}^{(2)}(v) + \gamma^{-j}_2 O(v^2)) = c_{11}^{(1)}(\lambda^1_i x_1 + O(\lambda^1_i \gamma^{-j}_2))O(|x|) + d^{(1)}(y) +$$

$$+ \lambda^1_i O(|x| \cdot |y|) + O(y^2),$$

(3.3)
where \( q_{ij}^{(1)} = O \left( \left( \frac{\gamma_1}{\gamma_1'} \right)^{-i} \right) \), \( q_{ij}^{(2)} = O \left( \left( \frac{\gamma_2}{\gamma_2'} \right)^{-j} \right) \), coefficients marked with “tilde” are uniformly bounded for small \( \mu \) and the following expression is valid for \( M^1 \):

\[
M^1 = \mu_1 + \nu_1^j c_1^{(2)} u_1^i - \gamma_1^{-i} y^j + O(\gamma_1^{-i} + \nu_1^j + \gamma_1^{-i} \gamma_2^{-j}).
\]  

Next, we take the right-hand side of the third equation of (3.3) divided by factor \( \gamma_2^{-j}(1 + q_{ij}^{(2)}) \) from the left-hand side as the new variable \( y \) — the equation becomes the following \( v + \left( \left( \frac{\tilde{\gamma}_2}{\gamma_2} \right)^{-j} \right) O(v^2) = y \). We substitute this formula instead of the \( y \) variable to all equations. Defining \( u_{new} = u - (b^{(1)}/d^{(1)}) \gamma_2^{-j} v + O(\gamma_2^{-j} v^2) \) we eliminate all terms in the equation for \( u \) which depend on \( v \) alone. In addition, we substitute the expressions for \( \bar{x} \) to the last equation of (3.3). These actions cause the linear in \( v \) term of order \( O(\gamma_1^{-i} + \nu_1^j \gamma_2^{-j}) \) to appear in the equation for \( \bar{v} \). We will make it zero again later. Thus, we obtain

\[
\bar{x}_1 = a_{11}^{(2)} v_1^i u_1 + a_{12}^{(2)} \tilde{\nu}_1^j u_2 + b_1^{(2)} v + O(\tilde{\nu}_1^j u_1^i + \nu_1^j u_1^i \cdot |v| + v^2),
\]

\[
\bar{x}_2 = a_{21}^{(2)} v_1^i u_1 + a_{22}^{(2)} \tilde{\nu}_1^j u_2 + b_2^{(2)} v + O(\tilde{\nu}_1^j u_1^i + \nu_1^j u_1^i \cdot |v| + v^2),
\]

\[
\gamma_1^{-i} \gamma_2^{-j} \frac{d^{(1)}}{d^{(1)}} \bar{v}(1 + q_{ij}^{(3)}) + \gamma_1^{-i} \gamma_2^{-j} O(v^2) = M^1 + c_1^{(2)} \nu_1^j u_1 + \tilde{c}_2^{(2)} \tilde{\nu}_1^j u_2 +
\]

\[
+ O(\gamma_1^{-i} + \nu_1^j \gamma_2^{-j} v^2 + O(\nu_1^j u_1^i) \cdot |v| + |v|^3),
\]

where \( q_{ij}^{(3)} = O \left( \left( \frac{\gamma_1}{\gamma_1'} \right)^{-i} + \left( \frac{\gamma_2}{\gamma_2'} \right)^{-j} \right) \) and

\[
A_{11}^{(1)} = a_{11} - b_1^{(1)} c_1^{(1)}/d^{(1)}, \quad A_{21}^{(1)} = a_{21} - b_2^{(1)} c_1^{(1)}/d^{(1)}.
\]  

Next, we substitute \( u \) as a function of \( x \) and \( v \) from the first two equations to the last three ones. After this, in addition, we make a shift of \( (x, v) \) coordinates by a constant of order \( O(\gamma_1^{-i} + \nu_1^j \gamma_2^{-j}) \) to nullify the linear in \( v \) term in the last equation. This gives us
the following formula for the map $T_{ij} : (x, v) \mapsto (\bar{x}, \bar{v})$:

$$
\bar{x}_1 = A_{11}^{(1)} a_{11}^{(2)} \lambda_1^j \nu_1^j x_1 + \bar{a}_{12} s_{ij}^{(1)} x_2 + b_1^{(2)} v + O(\lambda_1^j \nu_1^j + \lambda_1^j \nu_1^j \gamma_2^{-j}) O(|x|^2) + \\
+ \lambda_1^j \nu_1^j O(|x| \cdot |v|) + O(v^2),
$$

$$
\bar{x}_2 = A_{11}^{(1)} a_{21}^{(2)} \lambda_1^j \nu_1^j x_1 + \bar{a}_{22} s_{ij}^{(2)} x_2 + b_2^{(2)} v + O(\lambda_1^j \nu_1^j + \lambda_1^j \nu_1^j \gamma_2^{-j}) O(|x|^2) + \\
+ \lambda_1^j \nu_1^j O(|x| \cdot |v|) + O(v^2),
$$

$$
\gamma_1^{-i} \gamma_2^{-j} \bar{v}(1 + q_{ij}^{(3)}) + \gamma_1^{-i} \gamma_2^{-j} O(\bar{v}^2) = M^1 + A_{11}^{(1)} c_1^{(2)} \lambda_1^j \nu_1^j x_1 + \bar{c}_{2} s_{ij}^{(3)} x_2 + d^{(2)} v^2 + \\
+ O(\lambda_1^j \nu_1^j + \lambda_1^j \nu_1^j \gamma_2^{-j}) O(|x|^2) + \lambda_1^j \nu_1^j O(|x| \cdot |v|) + O(v^3),
$$

where $s_{ij}^{(k)} = O(\lambda_1^j \nu_1^j + \lambda_1^j \nu_1^j \gamma_2^{-j})$.

**Case I.** The second parameter in the first case is introduced as $\mu_2 = b_1^{(2)}(\mu)$ and we also recall that $c_1^{(2)}$, $b_1^{(2)}$ and $A_{11}^{(1)}$ are bounded from zero due to (2.6). We make the following change of coordinates:

$$
x_{1\text{new}} = x_1 + O\left(\left(\frac{\dot{\lambda}}{\lambda_1}\right)^i + \gamma_2^{-j}\right) O(|x||), \quad x_{2\text{new}} = x_2, \quad v_{\text{new}} = v
$$

such that all the terms which depend only on $x$–coordinates are now put into $x_{1\text{new}}$ in the third equation. Then (3.8) is rewritten in the form:

$$
\bar{x}_1 = A_{11}^{(1)} a_{11}^{(2)} \lambda_1^j \nu_1^j x_1 + \bar{a}_{12} s_{ij}^{(1)} x_2 + (\mu_2 + \rho_{ij}^{(1)}) v + O(\lambda_1^j \nu_1^j + \lambda_1^j \nu_1^j \gamma_2^{-j}) O(|x|^2) + \\
+ \lambda_1^j \nu_1^j O(|x| \cdot |v|) + O(v^2),
$$

$$
\bar{x}_2 = A_{11}^{(1)} a_{21}^{(2)} \lambda_1^j \nu_1^j x_1 + \bar{a}_{22} s_{ij}^{(2)} x_2 + b_2^{(2)} v + O(\lambda_1^j \nu_1^j + \lambda_1^j \nu_1^j \gamma_2^{-j}) O(|x|^2) + \\
+ \lambda_1^j \nu_1^j O(|x| \cdot |v|) + O(v^2),
$$

$$
\gamma_1^{-i} \gamma_2^{-j} \bar{v}(1 + q_{ij}^{(3)}) + \gamma_1^{-i} \gamma_2^{-j} O(\bar{v}^2) = M^1 + A_{11}^{(1)} c_1^{(2)} \lambda_1^j \nu_1^j x_1 + d^{(2)} v^2 + \\
+ \lambda_1^j \nu_1^j O(|x| \cdot |v|) + O(v^3),
$$

where $\rho_{ij}^{(1)} = O(\nu_1^j)$. Now we rescale the coordinates as follows

$$
v = -\frac{\gamma_1^{-i} \gamma_2^{-j}}{d^{(1)}(d^{(2)})} (1 + q_{ij}^{(3)}) Y, \quad x_1 = \frac{\lambda_1^{-i} \gamma_2^{-j}}{c_1^{(2)} A_{11}^{(1)}(d^{(1)})^2(d^{(2)})} X_1, \quad x_2 = -\frac{b_2^{(2)} \gamma_1^{-i} \gamma_2^{-j}}{d^{(1)}(d^{(2)})} X_2.
$$

Then system (3.9) is rewritten in the new coordinates:

$$
\begin{align*}
X_1 &= -J_{ij} X_2 + M_2 Y + O(\lambda_1^j \nu_1^j), \\
X_2 &= Y + O(\lambda_1^j \nu_1^j + \lambda_1^j \nu_1^j \gamma_2^{-j} + \gamma_1^{-i} \gamma_2^{-j}), \\
Y &= M_1 - X_1 - Y^2 + O(\gamma_1^{-i} \gamma_2^{-j}),
\end{align*}
$$

(3.10)
where formulas (2.12) are valid for $M_1$ and $M_2$. Note, that the coefficient $J_{ij}$ is the Jacobian of map (3.10) up to asymptotically small in $i$, $j$ terms, and, hence, $J_{ij}$ coincides in the main order with the Jacobian of map $T_{12}T_{02}T_{12}T_{01}$, i.e. it is given by formula (2.12).

**Case II.** Now we have $\mu_2 \equiv c_1^{(2)}(\mu)$ and coefficients $b_1^{(2)}$, $c_2^{(2)}$ and $A_1^{(1)}$ are not zeros. Introduce the new coordinates as $x_{1\text{new}} = x_1$, $x_{2\text{new}} = x_2 - (b_2^{(2)}/b_1^{(2)})x_1$, $v_{\text{new}} = v$. Then (3.8) recasts as

\[
\begin{align*}
    \bar{x}_1 &= A_1^{(1)}a_{11}^{(2)}\lambda_1^{i}\nu_j^{j}x_1 + \bar{a}_{12}s_i^{(1)}x_2 + b_1^{(2)}v + O(\bar{\lambda}^{i}\nu_j^{j} + \lambda_1^{i}\nu_1^{j}\gamma_2^{-j})O(\|x\|^2) + \\
    &\quad + \lambda_1^{i}\nu_j^{j}O(\|x\| \cdot |v|) + O(v^2), \\
    \bar{x}_2 &= A_1^{(1)}a_{21}^{(2)}\lambda_1^{i}\nu_j^{j}x_1 + \bar{a}_{22}s_i^{(2)}x_2 + O(\bar{\lambda}^{i}\nu_j^{j} + \lambda_1^{i}\nu_1^{j}\gamma_2^{-j})O(\|x\|^2) + \\
    &\quad + \lambda_1^{i}\nu_j^{j}O(\|x\| \cdot |v|) + O(v^2), \\
    \gamma_1^{-i}\gamma_2^{-j}d^{(1)}\bar{v}(1 + q_{ij}^{(3)}) + \gamma_1^{-i}\gamma_2^{-j}O(\bar{v}^2) = M_1 + (\mu_2 + \rho_2^{(2)})A_1^{(1)}\lambda_1^{i}\nu_j^{j}x_1 + \tilde{c}_{2}s_i^{(3)}x_2 + \\
    &\quad + d^{(2)}v^2 + O(\lambda_1^{i}\nu_j^{j} + \lambda_1^{i}\nu_1^{j}\gamma_2^{-j})O(\|x\|^2) + \lambda_1^{i}\nu_j^{j}O(\|x\| \cdot |v|) + O(v^3),
\end{align*}
\]

where $\rho_2^{(2)} = O\left(\left(\lambda/\lambda_1\right)^i + (\bar{\nu}/\nu_1)^j\right)$, $A_2^{(2)} = a_{21}^{(2)} - (b_2^{(2)}/b_1^{(2)})a_{11}^{(2)} \neq 0$ due to (2.5) and (2.6).

Now we rescale the coordinates as follows

\[
v = -\gamma_1^{-i}\gamma_2^{-j}d^{(1)}d^{(2)}(1 + q_{ij}^{(3)})Y, \quad x_1 = -\frac{b_1^{(2)}\gamma_1^{-i}\gamma_2^{-j}}{d^{(1)}d^{(2)}}X_1, \quad x_2 = -\frac{b_1^{(2)}A_1^{(1)}A_2^{(2)}\lambda_1^{i}\gamma_1^{-i}\lambda_2^{j}\gamma_2^{-j}}{d^{(1)}d^{(2)}}X_2.
\]

After this, we can rewrite (3.11) in the following form

\[
\begin{align*}
    \bar{X}_1 &= Y + O(\lambda_1^{i}\nu_j^{j} + \gamma_1^{-i}\gamma_2^{-j}) , \\
    \bar{X}_2 &= X_1 + O(\lambda_1^{i}\nu_j^{j} + \gamma_1^{-i}\gamma_2^{-j}) , \\
    \bar{Y} &= M_1 + M_2X_1 + J_{ij}X_2 - Y^2 + O(\lambda_1^{i}\nu_j^{j} + \gamma_1^{-i}\gamma_2^{-j}).
\end{align*}
\]

and formulas (2.12) are valid for $M_1$, $M_2$ and $J_{ij}$.

**Case III.** Here we have $\mu_2 \equiv A_1^{(1)} = a_{11} - b_1^{(1)}c_1^{(1)}/d^{(1)}$, and coefficients $b_1^{(2)}$, $c_2^{(2)}$ are not zeros. Introduce the new coordinates as in the previous case: $x_{1\text{new}} = x_1$, $x_{2\text{new}} = x_2 - (b_2^{(2)}/b_1^{(2)})x_1$, $v_{\text{new}} = v$. The system (3.8) is then rewritten as:

\[
\begin{align*}
    \bar{x}_1 &= a_{11}^{(2)}\mu_2 + \rho_2^{(2)}\lambda_1^{i}\nu_j^{j}x_1 + \bar{a}_{12}s_i^{(1)}x_2 + b_1^{(2)}v + O(\bar{\lambda}^{i}\nu_j^{j} + \lambda_1^{i}\nu_1^{j}\gamma_2^{-j})O(\|x\|^2) + \\
    &\quad + \lambda_1^{i}\nu_j^{j}O(\|x\| \cdot |v|) + O(v^2), \\
    \bar{x}_2 &= a_{21}^{(2)}\mu_2 + \rho_2^{(2)}\lambda_1^{i}\nu_j^{j}x_1 + \bar{a}_{22}s_i^{(2)}x_2 + O(\bar{\lambda}^{i}\nu_j^{j} + \lambda_1^{i}\nu_1^{j}\gamma_2^{-j})O(\|x\|^2) + \\
    &\quad + \lambda_1^{i}\nu_j^{j}O(\|x\| \cdot |v|) + O(v^2), \\
    \gamma_1^{-i}\gamma_2^{-j}d^{(1)}\bar{v}(1 + q_{ij}^{(3)}) + \gamma_1^{-i}\gamma_2^{-j}O(\bar{v}^2) = M_1 + c_1^{(2)}(\mu_2 + \rho_2^{(2)})\lambda_1^{i}\nu_j^{j}x_1 + \tilde{c}_{2}s_i^{(3)}x_2 + \\
    &\quad + d^{(2)}v^2 + O(\lambda_1^{i}\nu_j^{j} + \lambda_1^{i}\nu_1^{j}\gamma_2^{-j})O(\|x\|^2) + \lambda_1^{i}\nu_j^{j}O(\|x\| \cdot |v|) + O(v^3),
\end{align*}
\]
where $A_{21}^{(2)} = a_{21}^{(2)} - (b_{2}^{(2)}/b_{1}^{(2)})a_{11}^{(2)}$ and $\rho_{ij}^{3,4,5} = O \left( \left( \hat{\lambda}/\lambda_{1} \right)^{i} + (\hat{\nu}/\nu_{1})^{j} \right)$. Now we will select $\mu_{2} = O \left( \left( \hat{\lambda}/\lambda_{1} \right)^{i} + (\hat{\nu}/\nu_{1})^{j} \right)$ in the way to make the value of $\delta_{ij} = \mu_{2} + \rho_{ij}^{2}$ asymptotically small as $i, j \rightarrow \infty$. Then we have $A_{21}^{(2)}(\mu_{2} + \rho_{ij}^{2}) = A_{21}^{(2)}\rho_{ij}^{6} = O \left( \left( \hat{\lambda}/\lambda_{1} \right)^{i} + (\hat{\nu}/\nu_{1})^{j} \right) \neq 0$ as otherwise the Jacobian of map $T_{21}T_{02}T_{12}T_{01}$ would be vanishing when $\delta_{ij}$ goes to zero.

Finally we rescale the coordinates as follows

$$v = -\frac{\gamma_{1}^{-i}\gamma_{2}^{-j}}{d^{(1)}_{1}d^{(2)}_{2}}(1 + q_{ij}^{(3)}) Y, \quad x_{1} = -\frac{b_{1}^{(2)}\gamma_{1}^{-i}\gamma_{2}^{-j}}{d^{(1)}_{1}d^{(2)}_{2}} X_{1}, \quad x_{2} = -\frac{b_{1}^{(2)}A_{21}^{(2)}\rho_{j}^{6}\lambda_{1}^{-i}\lambda_{2}^{-j}}{d^{(1)}_{1}d^{(2)}_{2}} X_{2}. $$

After this, we can rewrite (3.11) in the following form

$$\begin{align*}
\dot{X}_{1} &= Y + O(\lambda_{1}^{i}\nu_{1}^{j} + \gamma_{1}^{-i}\gamma_{2}^{-j}) , \\
\dot{X}_{2} &= X_{1} + O(\lambda_{1}^{i}\nu_{1}^{j} + \gamma_{1}^{-i}\gamma_{2}^{-j}) , \\
\dot{Y} &= M_{1} + M_{2}X_{1} + J_{ij}X_{2} - Y^{2} + O(\lambda_{1}^{i}\nu_{1}^{j} + \gamma_{1}^{-i}\gamma_{2}^{-j}) .
\end{align*} \tag{3.14}$$

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