Quantum entanglement in soliton fractionalisation process

S. Arunagiri

The Institute of Mathematical Sciences, Chennai 600 113, India

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Quantum state, in relativistic quantum mechanics, itself turns out to be an entangled state due to its own degrees freedom such as spin and momentum. This peculiar entanglement leaves the transformed state mixed. We consider the fractional charge state that arises in a theory of fermion interacting with scalar background in this context. The apparent entanglement occurs between fermion ans scalar through Yukawa-type interaction. However, the spontaneous symmetry breaking causes appearance of the c-number zero energy solution of the Dirac equation as a pure state. Quantum entanglement in such relativistic system is proposed to have a microscopic view of the spontaneous symmetry breaking which has been realised in condensed matter system like polyacetylene.

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I. INTRODUCTION

Quantum state of fractional fermion number appears in a theory of fermion (electron) interacting with scalar field background instead of QED vacuum in quantum field theory [1]. In the above theory, the zero energy solution of the Dirac equation corresponds to soliton state with fermion number \( \frac{1}{2} \). Similar phenomenon occurs in polyacetylene resulting in separation of spin-charge of the fermion with formation of a charge neutral spin state and a spinless charge state [2]. The charge fractionalisation in both quantum field theory and condensed matter physics is consequence of spontaneous symmetry breaking. The concept of spontaneous symmetry breaking is widely studied in quantum field theory and interesting consequence in particle physics. It is realised in polymer system [3]. In the context of fractional quantum Hall effect in solids, the fractionally charged states given by the Laughlin’s wavefunction ansatz is crucial [4]. Recently, fractionalisation due to vertex is discussed in graphene [5]. This appealing quantum state of fractional charge will have another interesting aspect when the state is relativistic(-like), namely, quantum entanglement of a state due to its own degrees of freedom.

In quantum field theory, a quantum state is described by field operators in real space or creation and annihilation operators in energy momentum space with parameters of positive and negative frequency components or momentum, spin, etc respectively. These parameters are as important as the dynamical variables in relativistic dynamics. For example, a Dirac wavepacket of initial mean momentum and spin, \((+p,+j)\) at time \(t=0\), evolves in time into a state which is admixture of \((+p,+j)\) and \((-p,\pm j)\). This is never found to occur in nature because of the impossibility of localisation of the wavepacket into a size smaller than the corresponding wavelength. However, the dynamics of a quantum state in its parameters space is interesting for quantum information point of view.

Peres, Scudo and Treno [6] found that while the density matrix of spin state is initially a pure state, \(i.e.,\) the entropy is zero, the same state is looked mixed with an increase in entropy for an uniformly accelerated observer. When the spin state undergoes Wigner rotation under Lorentz boost, its direction and magnitude depend on the momentum. It is shown [6] that the entropy is positive. The spin entropy is, therefore, not Lorentz invariant. This is the consequence of Unruh effect [6]. Alternatively, the transformed state is a superposition of states involving momenta in all direction. Hence, such state is claimed to be quantum entangled in its own parameter space [6].

The phenomenon of soliton fractionalisation is proceeded by spontaneous breaking of symmetry with respect to the scalar field and appears as a c-number solution of the Dirac equation with the value of \(\phi(+x)\) or \(\phi(-x)\). As a result, the c-number solution is a real valued, nondegenerate, zero energy state. Before spontaneous symmetry breaking, the interaction induces a mixed state of \(\psi_{x}\) and \(\phi(\pm x)\). This mixed state involves entanglement and becomes a pure state given by the zero energy solution of Dirac equation as consequence of symmetry breaking which acts as quantum measurement.

It is the purpose of this note to examine the fractionalised soliton state from the quantum entanglement point of view. The fractional state is the superposition of states corresponding to soliton and antisoliton. The fractional state is found to be a pure state. In the next section \(\text{III}\) the phenomenon of soliton fractionalised is briefly discussed. The entanglement aspect is presented in section \(\text{III}\). Some remarks are made in section \(\text{IV}\).

II. SOLITON FRACTIONALISATION

In \((1+1)\) dimension, a fermion field, \(\psi\), interacting with a scalar field, \(\phi\), is described by [1]:

\[
\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_{\phi-\psi} \tag{1}
\]
where
\[ \mathcal{L}_\phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \]  \hspace{1cm} (2)
\[ \mathcal{L}_\psi = i \bar{\psi} \gamma^\mu \partial_\mu \psi \]  \hspace{1cm} (3)
\[ \mathcal{L}_{\phi - \psi} = \bar{\psi} \mathcal{V}(\phi) \psi \]  \hspace{1cm} (4)
where \( g \) is the coupling constant. The scalar part of \( \mathcal{L} \) can be solved for various choice of potential \( U(\phi) \). For a double well potential,
\[ U(\phi) = \frac{1}{2} \lambda^2 (1 - \phi^2) \]  \hspace{1cm} (5)
where \( \lambda \) is a parameter, we have the following solutions:
\[ \phi(x) = \begin{cases} \pm 1, \\ \pm \tanh(\lambda x) \end{cases} \]  \hspace{1cm} (6)
In the above, \( \phi = \pm 1 \) corresponds to the fermion in QED vacuum. The second of eq. (6) above are time-independent stationary, soliton, solutions with the signs \( \pm \) standing for soliton and anti-soliton. Then we have
\[ V(\phi) = \phi(x) \]  \hspace{1cm} (7)

For the fermion and its interaction part, we have the Dirac Hamiltonian
\[ H = \alpha \cdot p + \beta \mu \phi(x) \]  \hspace{1cm} (8)
where \( \alpha = \sigma_2 \), \( \beta = \sigma_1 \) and the dynamical fermion mass \( \mu = gm \) with \( m \) the fermion bare mass.

For \( \psi = (\psi_1, \psi_2)^T \) with \( \psi_1 \) and \( \psi_2 \) being conjugate states and symmetric to each other, we have the following set of Dirac equations
\[ \begin{align*}
\left( -\partial_x + \mu \phi(x) \right) \psi_2(x) &= \varepsilon \psi_1(x) \\
\left( \partial_x + \mu \phi(x) \right) \psi_1(x) &= \varepsilon \psi_2(x)
\end{align*} \]  \hspace{1cm} (9)
\[ \begin{align*}
\left( -\partial_x + \mu \phi(x) \right) \psi_2(x) &= \varepsilon \psi_1(x) \\
\left( \partial_x + \mu \phi(x) \right) \psi_1(x) &= \varepsilon \psi_2(x)
\end{align*} \]  \hspace{1cm} (10)
where the energy \( \varepsilon = \pm \sqrt{p^2 + \mu^2} \). For \( \varepsilon = 0 \), the conjugation symmetry is broken and there is only one state \( \psi_0 \). When \( m = 0 \) and \( \phi(x) = \pm 1 \) there exists zero energy solution as guaranteed by index theorem. With scalar field given by the soliton solutions, the second set of eq. (9) above are time-independent stationary, soliton, solutions with the signs \( \pm \) standing for soliton and anti-soliton. Then we have
\[ \psi_0 = \left( e^{-m \int_0^x \mathrm{d}y \phi(y)} \right) \\
\bar{\psi}_0 = \left( e^{-m \int_0^x \mathrm{d}y \phi(y)} \right) \]  \hspace{1cm} (11)
\[ \text{Now in the limit } L \to \infty \text{, the zero energy state is written as superposition of soliton and antisoliton states.} \]

### III. ENTANGLEMENT

The fermion-scalar interaction leads to a mixed state from the initial state which is given by
\[ |\psi\rangle \otimes |\phi\rangle \rightarrow \sum c_i |\psi_i\rangle |\phi_i\rangle \]  \hspace{1cm} (13)
For \( \psi = \psi_1 + \psi_2 \) and \( \phi = \phi_1 + \phi_2 \), the mixed state is given by the reduced matrix
\[ \tau = \sum |c_i|^2 |\psi_i\rangle \langle \psi_i| |\phi_i\rangle \langle \phi_i| \]  \hspace{1cm} (14)
It is the action of symmetry breaking that makes mixed state evolves into a pure state which corresponds to zero energy mode.

The zero energy solution of the above Dirac equations is given as sum of the two possible (linearly independent) states
\[ \psi_0 = f^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + f^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]  \hspace{1cm} (15)
where
\[ f^{\pm}(x) = \exp \left[ \pm \mu \int_0^x \phi(y) \mathrm{d}y \right] \]  \hspace{1cm} (16)
where \( \mu = gm \). Writing eq. (15) in a convenient form
\[ \Psi = \frac{1}{2} (\psi_0 + \bar{\psi}_0) \]  \hspace{1cm} (17)
where \( \psi_0 \) and \( \bar{\psi}_0 \) are self conjugate to each other.

The density matrix of the zero energy state is
\[ \rho = |\Psi \rangle \langle \Psi| \]  \hspace{1cm} (18)
\[ = \begin{pmatrix} f^+ f^{+*} & f^+ f^{-*} \\ f^{+*} f^- & f^{-*} f^{-*} \end{pmatrix} \]  \hspace{1cm} (19)
To express in terms of Bloch vector, let us have
\[ n_x = \int \left| f^+ \right|^2 - \left| f^- \right|^2 \mathrm{d}x \]  \hspace{1cm} (20)
\[ n_x + \mathrm{i} n_y = \int f^{+*} f^- \mathrm{d}x \]  \hspace{1cm} (21)
\[ n_x - \mathrm{i} n_y = \int f^+ f^{-*} \mathrm{d}x \]  \hspace{1cm} (22)
where \( n_z = 1 \) is the normalisation. Then the reduced matrix, \( \rho_R \) for the state of eigenvalue \( \frac{1}{2} \) is written in terms of the Bloch vector as
\[ \rho_R = \frac{1}{2} \begin{pmatrix} 1 + n_z & n_+ \\ n_+ & 1 - n_z \end{pmatrix} \]  \hspace{1cm} (23)
The eigen values of \( \rho_R \) are
\[ \lambda_i = (1 \pm |n|)/2 \]  \hspace{1cm} (24)
The linear entropy is given by

\[ S = 1 - Tr(\rho_R^2) \]  
\[ = 1 - \sum_i \lambda_i^2 = (1 - |n|^2)/2 \]  

(25)  
(26)

The linear entropy is measure of the mixedness of the zero energy state. The entropy of this state is independent of the fermion-scalar coupling constant \( g \) the coupling constant. As we see, the entropy is zero and hence the fractional charge state is a pure state.

**IV. CONCLUSION**

Because of the sharp value of the fermion number, the linearly superposed state of fractional charge turns out to be a pure state with zero entropy. This may be related to wavefunction collapse upon measurement. Such measurement process is given by the spontaneous symmetry breaking in the model theory of Jackiw and Rebbi [1]. Although there are fluctuations around the soliton position, they are shown to disappear when the volume of the system is taken to infinity [11, 12, 13].

Despite the inferred claim, in this note we have not explicitly shown the entanglement process and its entropy. But the curious point is envisaged. Alternately, the concept of the spontaneous symmetry breaking can be studied microscopically in terms dynamical variable plus the parameters using the idea of quantum entanglement. Such realisation would be helpful towards a picture of what the origin of the symmetry breaking is about.

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