GALOIS-TYPE EXTENSIONS AND EQUIVARIANT PROJECTIVITY

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Abstract

The theory of general Galois-type extensions is presented, including the interrelations between coalgebra extensions and algebra (co)extensions, properties of corresponding (co)translation maps, and rudiments of entwinings and factorisations. To achieve broad perspective, this theory is placed in the context of far reaching generalisations of the Galois condition to the setting of corings. At the same time, to bring together $K$-theory and general Galois theory, the equivariant projectivity of extensions is assumed resulting in the centrepiece concept of a principal extension. Motivated by noncommutative geometry, we employ such extensions as replacements of principal bundles. This brings about the notion of a strong connection and yields finitely generated projective associated modules, which play the role of noncommutative vector bundles. Subsequently, the theory of strong connections is developed. It is purported as a basic ingredient in the construction of the Chern character for Galois-type extensions (called the Chern-Galois character).

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1 Introduction

Taking advantage of Peter-Weyl theory, principal comodule algebras (faithfully flat Hopf-Galois extensions with bijective antipodes) have been shown [2] to generalise compact principal bundles in the sense of Henri Cartan (no local triviality assumed). On the other hand, there are examples of quantum spaces which, classically, correspond to principal bundles, yet do not fit the Hopf-Galois framework. More specifically, a natural source of examples of principal bundles is provided by homogeneous spaces. These can always be defined as quotients of a group by its subgroup. In the case of Hopf algebras understood as quantum groups, however, there is a rather limited number of quantum subgroups (given by surjections of Hopf algebras). As a result, not every quantum homogeneous space is a quotient of a quantum group by its quantum subgroup. For example, only one member of the family of quantum 2-spheres defined in [71] can be obtained as a quotient of $SU_q(2)$ by $U(1)$. The theory of Hopf-Galois extensions can only describe quantum homogeneous spaces that are quotients of quantum groups by quantum subgroups.

Thus it appears necessary to consider a wider class of extensions that, on one hand, would be close enough to principal comodule algebras, yet general enough to include examples coming from quantum homogeneous spaces. The basic idea is to replace a Hopf algebra in a Hopf-Galois extension by a coalgebra. This point of view for the first time was taken seriously in [21], where the studies of coalgebra principal bundles were initiated. Over the recent years and in a significant number of papers, the theory of coalgebra principal bundles or coalgebra-Galois extensions [17] has been developed and refined both in purely algebraic and differential geometric directions. On the algebraic side it has led to revival of the coring theory and provided new points of view on areas such as noncommutative descent theory [25]. On the differential geometric side, it has culminated in the introduction of principal extensions as noncommutative objects most closely describing principal bundles, and in the development of Chern-Weil theory for such extensions [18]. Most importantly, the abstract theory of principal extensions generalising principal comodule algebras was supported by new interesting examples such as noncommutative or quantum instanton bundles going beyond Hopf-Galois theory.

It seems that the theory of coalgebra-Galois and principal extensions has achieved a level of maturity at which it could be profitable to review recent progress and present it in a unified manner. This is the aim of the current article. The article consists of two parts. In the first part we analyse the algebraic side of coalgebra-Galois extensions. We give basic definitions and properties, we look at dual ways of defining Galois-type extensions (by algebras or by coalgebras), we also put Galois-type extensions in a wider framework of corings and quantum groupoids.

The second part is devoted to geometry motivated aspects of the coalgebra-Galois theory. In particular, we define modules associated to Galois-type extensions via corepresentations of their structure comodule coalgebras. They can be understood as modules of sections of associated noncommutative vector bundles. We describe basic elements of the theory of connections and strong connections, and derive consequences of the definition of a principal extension. The key idea here is that the concept of equivariant projectivity replaces that of faithful flatness used in Hopf-Galois theory. These two concepts are equivalent in the Hopf-Galois setting (bijective antipode assumed) but only the implication “equivariant projectivity” $\Rightarrow$ “faithful flatness” is known in general. Therefore, we build our theory on equivariant projectivity which guarantees that the aforementioned associated modules are finitely generated projective for any finite-dimensional corepresentation of the structure coalgebra. This way we arrive at the $K$-theory
of the coaction-invariant subalgebra. Now, we can apply the noncommutative Chern character mapping the $K_0$-group to the even cyclic homology.

Furthermore, strong connections give explicit formulae for idempotents. Although these formulae depend on the choice of strong connections, corresponding elements of the $K_0$-group are connection independent. Thus we obtain an explicit map from the Grothendieck group of isomorphism classes of finite-dimensional corepresentations of the structure coalgebra to the even cyclic homology of the coaction-invariant subalgebra. We call it the Chern-Galois character, and view as noncommutative Chern-Weil theory.

1.1 General conventions and standing assumptions

All (co)algebras are (co)unital and over a field $k$. We use the standard Heynemann-Sweedler notation (with the summation symbol suppressed) for coproducts and coactions, and $\ast$ for the convolution product of maps from a coalgebra to an algebra. The coproduct, counit, multiplication, and antipode are denoted by $\Delta$, $\varepsilon$, $m$, and $S$, respectively. The kernel of the multiplication map $A \otimes A \to A$ is written as $\Omega^1_A$, and called the space of universal differential 1-forms. The formula $da := 1 \otimes a - a \otimes 1$ defines the universal differential $A \to \Omega^1_A$.

Our typical notation for a left and a right coaction on a vector space $V$ is $V \Delta$ and $\Delta V$, or $V \varrho$ and $\varrho^V$, respectively. For actions on $V$, we use symbols like $\mu_V$ or $m_V$. For an algebra $B$ and a coalgebra $C$, the symbol $B \mathcal{M}^C$ stands for the category of left $B$-modules that are also right $C$-comodules with $B$-linear coactions. Morphisms in $B \mathcal{M}^C$ are left $B$-linear right $C$-colinear maps. The space of all colinear homomorphisms is denoted by $\text{Hom}^C$. Analogous symbols denote other categories of left (co)modules right (co)modules with the left and right structures being compatible and other homomorphism spaces.

1.2 Equivariant projectivity

The notion of equivariant projectivity of a (left) $B$-module $P$ occurs whenever $P$ has additional algebraic structure, compatible with the $B$-module structure. In this case we might like to require the properties of projectivity (such as the splitting of the product map) to respect this additional structure. A typical situation of key importance to the theory of principal extensions can be described as follows.

As in [18], an object $P \in B \mathcal{M}^C$ is called a $C$-equivariantly projective left $B$-module if for any two objects $M, N$ and morphisms $\pi : M \to N$, $f : P \to N$ in $B \mathcal{M}^C$, together with a right $C$-colinear splitting $i : N \to M$ of $\pi$ there exists a morphism $g : P \to M$ in $B \mathcal{M}^C$ such that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\pi} & N \\
\downarrow{i} & & \downarrow{f} \\
\exists g & \downarrow{g} & \rightarrow P
\end{array}
$$

Similarly to projective modules, the $C$-equivariant projectivity can be fully characterised by the splitting property of the multiplication map.

**Lemma 1.1.** An object $P \in B \mathcal{M}^C$ is a $C$-equivariantly projective left $B$-module if and only if there exists a left $B$-module right $C$-comodule section $s$ of the product map $B \otimes P \to P$. Here $B \otimes P$ is a right $C$-comodule with the tensor product coaction $\text{id}_B \otimes \Delta_P$. 


Proof. Given a section \( s \) of the multiplication map \( m_P : B \otimes P \rightarrow P \), and \( M, N, f, i \) and \( \pi \) as in the diagram above, one defines the map \( g : P \rightarrow M \) by \( g = m_M \circ (\text{id}_B \otimes (i \circ f)) \circ s \), where \( m_M : B \otimes M \rightarrow M \) is the \( B \)-multiplication map for \( M \). Conversely, in the defining diagram of a \( C \)-equivariantly projective \( B \)-module \( P \) take \( M = B \otimes P \), \( N = P \), \( \pi = m_P \), \( i : P \rightarrow B \otimes P \), \( p \mapsto 1_B \otimes p \) and \( f \) the identity map. Then \( g \) constructed through such diagram is the required splitting of the multiplication map.

In an analogous way, one calls a \((B, A)\)-bimodule \( P \) an \( A \)-equivariantly projective left \( B \)-module if for any two \((B, A)\)-bimodules \( M, N \) and \((B, A)\)-bilinear maps \( \pi : M \rightarrow N \), \( f : P \rightarrow N \) in \( _B \mathcal{M}_A \), together with a right \( A \)-linear splitting \( i : N \rightarrow M \) of \( \pi \) there exists a \((B, A)\)-bilinear map \( g : P \rightarrow M \) such that \( \pi \circ g = f \). This is equivalent to the existence of a \((B, A)\)-bilinear splitting of the multiplication map \( B \otimes P \rightarrow P \).

Since any right \( C \)-comodule is a left module of the convolution algebra \( C^* \), any object \( P \in _B \mathcal{M}_C \) is a \((B, A)\)-bimodule, where \( A = C^{\text{op}} \). In this case, \( P \) is a \( C \)-equivariantly projective left \( B \)-module if and only if it is an \( A \)-equivariantly projective left \( B \)-module (since there is a bijective correspondence between \( C \)-colinear and \( A \)-linear maps).

The notion of equivariant projectivity should be contrasted with that of relative projectivity. Given an algebra map \( \iota : A \rightarrow B \), any left \( B \)-module is also a left \( A \)-module via \( \iota \) and the multiplication in \( B \). In this situation, one often says that \( B \) is an \( A \)-ring or an algebra over \( A \) and that \( P \) is a module over an \( A \)-ring. The product map \( B \otimes P \rightarrow P \) descends to the map \( m_{P|A} : B \otimes_A P \rightarrow P \). \( P \) is called an \( A \) relatively projective left \( B \)-module provided the map \( m_{P|A} \) has a left \( B \)-linear section.

An equivariantly projective left \( B \)-module (be it \( A \)-equivariantly or \( C \)-equivariantly) is always a projective left \( B \)-module (a \((B, A)\)-linear splitting of the multiplication map is, in particular, left \( B \)-linear). Not every \((B, A)\)-bimodule \( P \) that is projective as a left \( B \)-module is an equivariantly projective module. For an \( A \)-ring \( B \), a projective left \( B \)-module is always an \( A \)-relatively projective left \( B \)-module, but the relative projectivity of \( P \) does not imply the projectivity of \( P \) (however, when \( A \) is a separable algebra the \( A \)-relative projectivity is equivalent to the projectivity of \( P \)).

2 Galois-type extensions and coextensions

This section is devoted to the definition and description of basic algebraic properties of general Galois-type extensions. We start in Section 2.1 by introducing the notion of equivariant projectivity, then give the definition of coalgebra-Galois extensions and two other types of algebra-Galois (co)extensions. Every such extension is determined by the existence of a (co)translation map, the properties of which are studied in Section 2.2. Furthermore, any coalgebra-Galois extension or an algebra-Galois coextension gives rise to an algebraic structure, which encodes the symmetries of extension and is known as an entwining structure. This is closely related (by semi-dualisation) to factorisation of algebras. Both are described in Section 2.3. Section 2.4 is devoted to the definition of a principal extension [18] which generalises the concept of a faithfully flat Hopf-Galois extension with bijective antipode and forms a cornerstone of the theory of noncommutative principal bundles. Representations of entwining structures are given in terms of entwined modules. These unify many categories of modules studied previously in Hopf algebra theory. Rudimentary properties of entwined modules are described in Section 2.5. In this section it is also shown, how the properties of such modules and Galois-type extensions can
be derived from the properties of corings and their comodules. The latter provide a conceptual and algebraic framework for Galois-type extensions.

2.1 Definitions and basic properties

2.1.1 Coalgebra-Galois extensions

Let $C$ be a coalgebra over a field $k$ and $P$ a $k$-algebra and right $C$-comodule with a comodule structure map $\Delta_P : P \to P \otimes C$. In attempting to define a coalgebra-Galois extension one first has to address the problem of defining the coaction invariants.

Recall that for Hopf-Galois extensions coinvariant elements are defined as $p \in P$ such that $\Delta_P(p) = p \otimes 1$, using the fact that the unit of a Hopf algebra is group-like. Since there might not necessarily exist such a group-like element in the coalgebra $C$, we can no longer obtain coaction invariants of a $C$-comodule $P$ in this way. Instead, we define the coaction invariants of $P$ by\footnote{We owe this definition to M. Takeuchi.}

$$ P^{coC} := \{ b \in P \mid \forall p \in P : \Delta_P(bp) = b\Delta_P(p) \} \quad (2.2) $$

First observe that $P^{coC}$ is a subalgebra of $P$. Indeed, for all $b,b' \in P^{coC}$ and $p \in P$,

$$ \Delta_P(bb'p) = b\Delta_P(b'p) = bb'\Delta_P(p). \quad (2.3) $$

Thus $bb' \in P^{coC}$, and since $1 \in P^{coC}$, we conclude that $P^{coC}$ is a subalgebra of $P$.

Another, and perhaps more intuitive, definition of coaction invariants is possible, if there exists a group-like element $e$ in the coalgebra $C$ such that $\Delta_P(1) = 1 \otimes e$. (We call coactions enjoying this property $e$-coaugmented.) Then one can define the set of $e$-coaction invariants as

$$ P^{coC}_e := \{ p \in P \mid \Delta_P(p) = p \otimes e \} \quad (2.4) $$

Note, however, that it is not always true that $P^{coC}_e$ is a subalgebra of $P$, although it is a subset of $P$ which contains 1. These two types of coaction invariants are related in the following way.

**Lemma 2.1.** Let $C$ be a coalgebra with a group-like element $e$, and let $P$ be an algebra and a right $C$-comodule such that $\Delta_P(1) = 1 \otimes e$. Then $P^{coC} \subseteq P^{coC}_e$.

**Proof.** If $b \in P^{coC}$, then $\Delta_P(b) = \Delta_P(b \cdot 1) = b\Delta_P(1) = b \cdot (1 \otimes e) = b \otimes e$, i.e. $b \in P^{coC}_e$. □

Although this is not immediately apparent, both definitions of coaction invariants are related to a group-like element. This is, however, not a group-like element in $C$ but a group-like element in $P \otimes C$, understood as a coalgebra over $P$ or a coring. More information about corings is given below, and the role of group-like elements is explained in Remark 2.49 (cf. Proposition 2.23).

We call an extension of algebras $B \subseteq P$ a $C$-extension if $B = P^{coC}$. The definition of $P^{coC}$ immediately implies that the coaction of a right $C$-comodule $P$ is a left $P^{coC}$-linear map. This observation allows us to define when a coaction of a coalgebra on an algebra is Galois, and thus to generalise the notion of a Hopf-Galois extension.
Definition 2.2 ([17]). Let $C$ be a coalgebra and $B \subseteq P$ a $C$-extension of algebras. We call the left $P$-module and right $C$-comodule homomorphism
\[
\text{can} : P \otimes B P \longrightarrow P \otimes C, \quad p \otimes p' \mapsto p \Delta_P(p'),
\]
the canonical map of the $C$-extension $B \subseteq P$. We say that this extension is a coalgebra-Galois extension if the canonical map is bijective. Furthermore, if there exists a group-like element $e$ such that $\Delta_P(1) = 1 \otimes e$, we call $B \subseteq P$ an $e$-coaugmented coalgebra-Galois $C$-extension.

A straightforward generalisation of [49] provides us with an alternative definition of a coalgebra-Galois extension.

Proposition 2.3. Let $C$ be a coalgebra and $B \subseteq P$ a $C$-extension of algebras. The extension is a coalgebra-Galois extension if and only if the following sequence is exact:
\[
0 \longrightarrow P(\Omega^1 B)P \longrightarrow \Omega^1 P \overset{\text{can}|_{\Omega^1 P}}{\longrightarrow} P \otimes C^+ \longrightarrow 0. \tag{2.6}
\]
Here $\text{can} : P \otimes P \rightarrow P \otimes_B P \xrightarrow{\text{can}} P \otimes C$ is the natural lifting of the canonical map, and $C^+ := \text{Ker} \varepsilon$ is the augmentation ideal of $C$.

Proof. Consider first the following commutative diagram (of left $P$-modules) with exact rows and columns:
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ker } \text{can}|_{\Omega^1 P} & \longrightarrow & \text{Ker } \text{can} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1 P & \longrightarrow & P \otimes P \xrightarrow{m} P & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P \otimes C^+ & \longrightarrow & P \otimes C \xrightarrow{id \otimes \varepsilon} P & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Coker } \text{can}|_{\Omega^1 P} & \longrightarrow & \text{Coker } \text{can} & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}
\] (2.7)

Applying the Snake Lemma to the above diagram we obtain the exact sequence
\[
0 \longrightarrow \text{Ker } \text{can}|_{\Omega^1 P} \longrightarrow \text{Ker } \text{can} \longrightarrow 0 \longrightarrow \text{Coker } \text{can}|_{\Omega^1 P} \longrightarrow \text{Coker } \text{can} \longrightarrow 0. \tag{2.8}
\]
It follows from the exactness of this sequence that
\[
\text{Ker } \text{can}|_{\Omega^1 P} = \text{Ker } \text{can}, \quad \text{Coker } \text{can}|_{\Omega^1 P} = \text{Coker } \text{can}. \tag{2.9}
\]
On the other hand, the Snake Lemma applied to
\[
\begin{array}{ccccccc}
0 & \longrightarrow & P(\Omega^1 B)P & \longrightarrow & \text{Ker } \text{can} & \longrightarrow & \text{Ker } \text{can} \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P(\Omega^1 B)P & \longrightarrow & P \otimes P & \longrightarrow & P \otimes_B P \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P \otimes C & \xrightarrow{\text{can}} & P \otimes C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Coker } \text{can} & \longrightarrow & \text{Coker } \text{can} & \longrightarrow & 0
\end{array}
\] (2.10)
yields the following exact sequence:

\[ 0 \rightarrow P(\Omega^1B)P \rightarrow \text{Ker } \tilde{\text{can}} \rightarrow \text{Ker } \text{can} \rightarrow 0 \rightarrow \text{Coker } \tilde{\text{can}} \rightarrow \text{Coker } \text{can} \rightarrow 0. \]  \hspace{1cm} (2.11)

Assume now that \( B \subseteq P \) is a coalgebra-Galois \( C \)-extension. Then \( \text{Ker } \text{can} = 0 = \text{Coker } \text{can} \), and, from the exactness of (2.11), we can infer that \( \text{Coker } \tilde{\text{can}} = 0 \) and \( \text{Ker } \tilde{\text{can}} = P(\Omega^1B)P \). Combining this with (2.9), we conclude that (2.6) is exact.

Conversely, assume that the sequence (2.6) is exact. Then \( \text{Ker } \tilde{\text{can}} = 0 = \text{Coker } \text{can} \), and, from the exactness of (2.11), we have that \( \text{Ker } \text{can} = 0 = \text{Coker } \text{can} \), i.e. \( B \subseteq P \) is a coalgebra-Galois extension.

Let \( X \) and \( X' \) be total spaces of principal bundles with the same base and structure group. Recall that any map \( X \rightarrow X' \) inducing identity on the base and commuting with the group action has to be bijective. We end this section with a coalgebra-Galois incarnation of this fact. It is a straightforward generalisation of [81].

**Lemma 2.4.** Let \( P \) and \( P' \) be coalgebra-Galois \( C \)-extensions of \( B \), and let \( P' \) be right faithfully flat over \( B \). Then any left \( B \)-linear right \( C \)-colinear map \( F : P \rightarrow P' \) is an isomorphism.

**Proof.** Consider \( P' \) as a right module over \( P \) via \( F \). The composition

\[ P' \otimes_P P \otimes_B P \longrightarrow P' \otimes_B P \stackrel{\text{id} \otimes F}{\longrightarrow} P' \otimes_B P' \longrightarrow P' \otimes C \longrightarrow P' \otimes_P P \otimes C \]  \hspace{1cm} (2.12)

coincides with

\[ \text{id} \otimes_P \text{can} : P' \otimes_P P \otimes_B P \longrightarrow P' \otimes_P P \otimes C, \]  \hspace{1cm} (2.13)

where \( \text{can} \) and \( \text{can}' \) are the respective canonical maps. Hence \( \text{id} \otimes_B F : P' \otimes_B P \rightarrow P' \otimes_B P' \) is an isomorphism. Therefore, so is \( F \) by the right faithful flatness of \( P' \) over \( B \).

**Remark 2.5.** As a special case of coalgebra-Galois extensions, obtained by replacing Hopf algebras in Hopf-Galois extensions by braided groups one can consider braided Hopf-Galois extensions. These provide an intermediate step in between the \( H \)- and \( C \)-Galois, and allows one to develop a braided group gauge theory [65].

### 2.1.2 Quotient-coalgebra and homogeneous Galois extensions

Though it is demonstrated in the previous and following sections that one can get away with the lack of a group-like element in defining and developing some general aspects of coalgebra-Galois theory, throughout this section all extensions will be coaugmented by some group-like element \( e \). The reason is that we are not aware of interesting examples of non-coaugmented extensions, and co-augmentation seems indispensable to prove some of the desired technical results.

On the other hand, a very interesting class of examples comes from the theory of Hopf-algebra quotients that is elaborated in [79], and that already in 1990 gave birth to coalgebra-Galois theory [81]. The setting is as follows. Let \( H \) be a Hopf algebra, \( P \) be a right \( H \)-comodule algebra and \( I \) a right ideal coideal of \( H \). Then the composite map

\[ P \xrightarrow{\Delta_P} P \otimes H \longrightarrow P \otimes (H/I) \]  \hspace{1cm} (2.14)
defines a right coaction of the quotient coalgebra $H/I$ on $P$. Demanding this coaction to be Galois defines a $1$-coaugmented coalgebra-Galois $H/I$-extension [81]. (Here $1$ is the class of $1$ in $H/I$.) Thus the coaugmentation of such extensions comes automatically from the Hopf-algebra symmetry that is fundamental in this definition. We call such extensions quotient-coalgebra Galois extensions.

The above construction is parallel to what happens in differential geometry. Let us explain it on the example of the principal instanton bundle $S^7 \to S^4$. The sphere $S^7$ is a homogeneous space of $SU(4)$. Viewing $SU(2)$ as a block-diagonal subgroup of $SU(4)$ gives an action of $SU(2)$ on $S^7$ that defines the principal instanton fibration: $S^7/SU(2) \cong S^4$. The most sophisticated example of a quotient-coalgebra Galois extension that we know of is a noncommutative deformation of the instanton bundle [11]. Here one starts with the Soibelman-Vaksman quantum sphere $S_q^7$ [92], which is a homogeneous space of $SU_q(4)$, and then, following the insight given by Poisson geometry [12], one constructs a coideal right ideal $I$ of the Hopf algebra $O(SU_q(4))$ such that the canonical surjection $O(SU_q(4)) \to O(SU_q(4))/I$ corresponds to the block-diagonal inclusion of $SU(2)$ in $SU(4)$ and the induced coaction is Galois yielding $O(S_q^4)$ as the coaction invariant subalgebra [11].

One of the reasons why this example is interesting is that it uses the full generality of quotient-coalgebra Galois extensions, i.e., we have $P \neq H$ and $I \neq 0$. Observe that for $I = 0$ we recover as a special case Hopf-Galois theory, whereas for $P = H$ we obtain what is called homogeneous coalgebra-Galois extensions. We devote the remainder of this section to the latter case. This is the case which deals with quantum homogeneous spaces or left coideal subalgebras (thus justifying the name “homogeneous coalgebra-Galois extension”). The aim is to try and reproduce in the general noncommutative setting a classical construction in which a homogeneous space $M$ of a group $G$ is viewed as a base for a principal bundle with the total space $G$.

Let $P$ be a Hopf algebra and $I$ a coideal right ideal of $P$, so that $P/I$ is a coalgebra and a right $P$-module. View $P$ as a right $P/I$-comodule via the induced coaction

$$\Delta_P := (\text{id} \otimes \pi_I) \circ \Delta, \quad P \xrightarrow{\pi_I} P/I. \quad (2.15)$$

The corresponding $P/I$-extension $B \subseteq P$ is called a homogeneous $P/I$-extension. The importance of extensions of this type stems from the fact that $B$ is a quantum homogeneous space or a left coideal subalgebra of $B$. Let us discuss this in more detail.

Since $1$ is a group-like element in a Hopf algebra $P$, its coalgebra projection $\pi_I(1)$ is a group-like element in $P/I$. Furthermore, $\Delta_P(1) = 1 \otimes \pi_I(1)$. Observe then that for a homogeneous $P/I$-extension, the coaction-invariant subalgebra $B$ is equal to the subalgebra of $\pi_I(1)$-coaction invariants

$$P^{P/I}_{\pi_I(1)} = \{ b \in P \mid \Delta_P(b) = b \otimes \pi_I(1) \}. \quad \text{Indeed, } B \subseteq P^{P/I}_{\pi_I(1)} \text{ by Lemma 2.1.}$$

Conversely, if $\Delta_P(b) = b \otimes \pi_I(1)$, then, for all $p \in P$,

$$\Delta_P(bp) = b_{(1)}p_{(1)} \otimes \pi_I(b_{(2)}p_{(2)}) = b_{(1)}p_{(1)} \otimes \pi_I(b_{(2)})p_{(2)} = b_{(1)}p_{(1)} \otimes \pi_I(1)p_{(2)} = bp_{(1)} \otimes \pi_I(p_{(2)}) = b\Delta_P(p),$$

where we have used the fact that $\pi_I$ is a right $P$-module map. Thus, $b \in B$ as required. Next, using the coassociativity of the coaction $\Delta_P$ and the description of $B$ as $\pi_I(1)$-coaction invariants, apply $\Delta \otimes \text{id}$ to equation $b_{(1)} \otimes \pi_I(b_{(2)}) = b \otimes \pi_I(1)$ to deduce that for all $b \in B$,

$$(\text{id} \otimes \Delta_P) \circ \Delta)(b) = \Delta(b) \otimes \pi_I(1).$$

This implies that

$$\forall b \in B, \quad b_{(1)} \otimes b_{(2)} \in P \otimes B, \quad (2.16)$$

Thus, $B$ is a right $P$-module.
i.e., $\Delta(B) \subseteq P \otimes B$, so that $B$ is a left coideal subalgebra of $P$ or a quantum homogeneous space of $P$.

Thus homogeneous $P/I$-extensions provide one with a suitable set-up for principal bundles over quantum homogeneous spaces. To exploit this fully, however, we need to address a question when a homogeneous $P/I$-extension is a coalgebra-Galois extension. The answer turns out to determine the structure of $I$ completely (cf. Lemma 5.2 in [20]).

**Theorem 2.6.** Let $B \subseteq P$ be a homogeneous $P/I$-extension. Then this extension is Galois if and only if $I = B^+ P$, where $B^+ := B \cap \text{Ker } \varepsilon$.

**Proof.** Assume first that $I = B^+ P$. Taking advantage of (2.16), for any $b \in B^+$, $p \in P$, we compute:

$$S(b(1)p(1)) \otimes_B b(2)p(2) = S(b(1)p(1))b(2) \otimes_B p(2) = S(p(1))\varepsilon(b) \otimes_B p(2) = 0. \quad (2.17)$$

Hence there is a well-defined map

$$T : P \otimes (P/I) \longrightarrow P \otimes_B P, \quad T(p \otimes [p]_I) := pS(p(1)) \otimes_B p'(2). \quad (2.18)$$

It is straightforward to verify that $T$ is the inverse of the canonical map $can$. Consequently, $P$ is a coalgebra-Galois $P/I$-extension.

To show the converse, let us first prove the following:

**Lemma 2.7.** Let $P$, $I$ and $B$ be as above. Then $B \subseteq P$ is a coalgebra-Galois $P/I$-extension if and only if $\left(\pi_B \circ (S \otimes \text{id}) \circ \Delta\right)(I) = 0$, where $\pi_B : P \otimes P \rightarrow P \otimes_B P$ is the canonical surjection.

**Proof.** By Proposition 2.3, $P$ is a coalgebra-Galois $P/I$-extension of $B$ if and only if the following sequence

$$0 \longrightarrow P(\Omega^1_B)P \longrightarrow P \otimes P \xrightarrow{\text{can}} P \otimes P/I \longrightarrow 0 \quad (2.19)$$

is exact. One can check that $\left\langle \text{can} \circ (S \otimes \text{id}) \circ \Delta\right\rangle(I) = 0$. Hence, it follows from the exactness of (2.19) that $\left\langle (S \otimes \text{id}) \circ \Delta\right\rangle(I) \subseteq P(\Omega^1_B)P$. Consequently, $\left\langle \pi_B \circ (S \otimes \text{id}) \circ \Delta\right\rangle(I) = 0$ due to the exactness of the sequence

$$0 \longrightarrow P(\Omega^1_B)P \longrightarrow P \otimes P \xrightarrow{\pi_B} P \otimes_B P \longrightarrow 0. \quad (2.20)$$

To prove the converse, one can proceed as in the considerations preceding this lemma. \hfill \Box

**Corollary 2.8.** Let $B \subseteq P$ be a coalgebra-Galois $P/I$-extension as above. Then the translation map $\tau := \text{can}^{-1}(1 \otimes \cdot)$ is given by the formula: $\tau([p]_I) := S(p(1)) \otimes_B p(2)$.

Assume now that $P$ is a coalgebra-Galois $P/I$-extension of $B$. It follows from the above corollary and (2.17) that $\tau([B^+ P]_I) = 0$. Hence, by the injectivity of $\tau$, we have $B^+ P \subseteq I$. Furthermore, there is a well-defined map

$$\text{can}' : P \otimes_B P \longrightarrow P \otimes (P/B^+ P), \quad p \otimes_B p' \longmapsto pp'(1) \otimes [p'(2)]_{B^+ P}. \quad (2.21)$$

Indeed, taking again advantage of (2.16), we obtain

$$p \otimes bp' \longmapsto pb(1)p'(1) \otimes [(b(2) - \varepsilon(b(2)))p'(2)]_{B^+ P} + \varepsilon(b(2))p'(2)]_{B^+ P}$$
$$= pb(1)p'(1) \otimes \varepsilon(b(2))[p'(2)]_{B^+ P}$$
$$= pbp'(1) \otimes [p'(2)]_{B^+ P}. \quad (2.22)$$
On the other hand, \( pb \otimes p' \mapsto pbp'_{(1)} \otimes [p'_{(2)}]_{B+P} \). Reasoning as in the first part of the proof, we can conclude that \( \text{can}' \) is bijective. Next, consider the following commutative diagram:

\[
P \otimes_B P \xrightarrow{\text{can}'} P \otimes (P/B^+P) \\
\downarrow \text{id} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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A step beyond classical spaces is to consider cocommutative coalgebras that do not admit a basis whose all elements are group-like. As an example, take the coalgebra dual to the algebra $A := \mathbb{C}[\theta]/(\theta^2)$ of dual numbers. Denote by $\{1^*, \theta^*\}$ the dual basis of $A^*$. Then the coalgebra structure on $A^*$ is given by the formulae:

$$
\Delta 1^* = 1^* \otimes 1^*, \quad \Delta \theta^* = 1^* \otimes \theta^* + \theta^* \otimes 1^*, \quad \varepsilon(1^*) = 1, \quad \varepsilon(\theta^*) = 0.
$$

(2.25)

The aforementioned example belongs to the realm of super geometry: it is neither classical nor noncommutative.

For a noncommutative example, let us consider the algebra of matrices

$$
M_n(\mathbb{C}) := \mathbb{C}\langle x, y \rangle/\langle x^n - 1, y^n - 1, xy - e^{\frac{2\pi i}{n}} \rangle.
$$

(2.26)

We can take as generating matrices

$$
x = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}, \quad y = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & q & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & q^{n-1}
\end{pmatrix}, \quad q = e^{\frac{2\pi i}{n}}.
$$

(2.27)

Then $\{x^ky^l \mid k, l \in \{0, \ldots, n-1\}\}$ is a linear basis of $M_n(\mathbb{C})$. Let $C_n$ denote the dual coalgebra $M_n(\mathbb{C})^*$. As before, we use the notation $(x^ky^l)^*$ for the dual basis elements. A direct calculation yields the coalgebra structure:

$$
\Delta((x^ky^l)^*) = \sum_{p,r,s,t=0}^{n-1} q^{rs}\delta_{k,p+s \text{mod } n}\delta_{l,r+t \text{mod } n} x^py^r \otimes x^sy^t, \quad \varepsilon((x^ky^l)^*) = \delta_{k,0}\delta_{l,0}.
$$

(2.28)

To make this example more tangible, put $n = 2$. Then we have explicitly

$$
\Delta(1^*) = 1^* \otimes 1^* + x^* \otimes x^* + y^* \otimes y^* - (xy)^* \otimes (xy)^*,
$$

(2.29)

$$
\Delta(x^*) = x^* \otimes 1^* + 1^* \otimes x^* + (xy)^* \otimes y^* - y^* \otimes (xy)^*,
$$

(2.30)

$$
\Delta(y^*) = y^* \otimes 1^* + 1^* \otimes y^* - (xy)^* \otimes x^* + x^* \otimes (xy)^*,
$$

(2.31)

$$
\Delta((xy)^*) = (xy)^* \otimes 1^* + x^* \otimes y^* - y^* \otimes x^* + 1^* \otimes (xy)^*.
$$

(2.32)

We can think of a coproduct as a set-theoretical map from the set of basis elements $X$ to its Cartesian square times the ground field: $X \times X \times k$. For the classical spaces, this map embeds $X$ in $X \times X \times k$ as the diagonal in $X \times X$ times $\{1\}$. For the “commutative spaces”, the value of this map on any element of $X$ is symmetric with respect to the plane $\{(x, x, \alpha) \in X \times X \times k \mid x \in X, \alpha \in k\}$. The noncommutativity of the space is measured by the lack of the aforementioned symmetry. Thus we can visualise the geometry of noncommutativity. The finite spaces fit perfectly this coalgebraic picture (the dual of any finite dimensional algebra is a coalgebra), but a significant adjustment seems to be required to accommodate the infinite case, especially if we want to go beyond the discrete topology. However, this point of view is, hopefully, tenable through some topological version of the concept of a coalgebra.

Since the coalgebra-Galois extensions are dual to the classical principal bundles, they are also dual (for each object involved) to the algebra-Galois coextensions. Therefore, in this section we dualise coalgebra-Galois extensions and derive results analogous to the results discussed in the previous section. As this dualisation in its full generality is somewhat involved, it is helpful
to consider first the definition from [84, p. 3346] that dualises the concept of a Hopf-Galois $H$-extension. Let $H$ be a Hopf algebra, $C$ a right $H$-module coalgebra with the action $\mu_C : C \otimes H \to C$. Then, since the action $\mu_C$ is a coalgebra map, i.e., $\Delta \circ \mu_C = (\mu_C \otimes \mu_C) \circ (\Delta \otimes \Delta)$, we have

$$\Delta(\mu_C(c, h) - \varepsilon(h)c) = \mu_C(c(1), h(1)) \otimes \mu_C(c(2), h(2)) - c(1) \otimes \varepsilon(h)c(2)$$

$$- \varepsilon(h(1))c(1) \otimes \mu_C(c(2), h(2)) + \varepsilon(h(1))c(1) \otimes \mu_C(c(2), h(2))$$

$$+ c(1) \otimes (\mu_C(c(2), h) - \varepsilon(h)c(2)), \quad (2.33)$$

so that $I := \{\mu_C(c, h) - \varepsilon(h)c \mid c \in C, h \in H\}$ is a coideal in $C$. Hence $B := C/I$ is a coalgebra. Using again the assumption that $\mu_C$ is a coalgebra map, it can be directly checked that $((C \otimes \mu_C) \circ (\Delta \otimes H))(C \otimes H) \subseteq C \square B C$. This way we arrive at:

**Definition 2.9.** We say that $C \to B$ is a (right) Hopf-Galois $H$-coextension if the canonical left $C$-comodule right $H$-module map $\text{cocalg} := (C \otimes \mu_C) \circ (\Delta \otimes H) : C \otimes H \to C \square B C$ is a bijection.

Now, to obtain a dualisation of the general coalgebra-Galois $C$-extension, we replace $H$ by an algebra $A$ and remove the condition that the action $\mu_C$ is a coalgebra map. At this level of generality, to formulate the definition of a coextension, we first need:

**Lemma 2.10 ([17]).** Let $A$ be an algebra and $C$ a coalgebra and right $A$-module with an action $\mu_C : C \otimes A \to C$. Denote by $I$ the vector space

$$\text{span}\{\mu_C(c, a)(1)\alpha(\mu_C(c, a)(2)) - c(1)\alpha(\mu_C(c(2), a)) \mid a \in A, c \in C, \alpha \in \text{Hom}(C, k)\}, \quad (2.34)$$

by $\pi : C \to C/I$ the canonical surjection, and put $B := C/I$. Then $I$ is a coideal of $C$, the action $\mu_C$ is left $B$-colinear, i.e., $((\pi \otimes C) \circ \Delta \circ \mu_C = (B \otimes \mu_C) \circ ((\pi \otimes C) \circ \Delta \otimes A)$, and $((C \otimes \mu_C) \circ (\Delta \otimes A))(C \otimes A) \subseteq C \square B C$.

Now we can conclude that we have a well-defined map

$$\text{cocalg} := (C \otimes \mu_C) \circ (\Delta \otimes A) : C \otimes A \to C \square B C, \quad (2.35)$$

and can consider:

**Definition 2.11 ([17]).** Let $A$ be an algebra, $C$ a coalgebra and right $A$-module, and $B = C/I$, where $I$ is the coideal of Lemma 2.10. We say that $C$ is a (right) algebra-Galois $A$-coextension of $B$ if the canonical left $C$-comodule right $A$-module map $\text{cocalg} := (C \otimes \mu_C) \circ (\Delta \otimes A) : C \otimes A \to C \square B C$ is bijective. An algebra-Galois coextension is said to be $\kappa$-augmented if there exists an algebra map $\kappa : A \to k$ such that $\varepsilon_C \circ \mu_C = \varepsilon_C \otimes \kappa$.

To see more clearly that Definition 2.11 dualises the notion of a Galois $C$-extension, one can notice that both $C \otimes A$ and $C \square B C$ are objects in $\mathcal{CM}_A$, which is dual to $\mathcal{M}^C$. The structure maps are $\Delta \otimes C, C \otimes m$ and $\Delta \square B C, C \square B \mu_C$, respectively. The canonical map $\text{cocalg}$ is a morphism in $\mathcal{CM}_A$. The right $A$-coextension $C \to B$ is algebra-Galois if $C \otimes A \cong C \square B C$ as objects in $\mathcal{CM}_A$ by the canonical map $\text{cocalg}$. (In what follows, we consider only right coextensions and omit “right” for brevity.)
Finally, note that for the group actions on spaces translated into group-ring actions on classical-space coalgebras, cocan can be reduced to

$$F_X^G : X \times G \longrightarrow X \times_{X/G} X, \quad F_X^G((x, g)) = (x, xg).$$

(2.36)

Here $X \times_{X/G} X$ is by definition the image of $F_X^G$. The bijectivity of cocan is then equivalent to the bijectivity of $F_X^G$. This guarantees that the action of $G$ on $X$ is free. However, to arrive at principal actions, one needs to introduce topology and go beyond the map $F_X^G$ to guarantee the properness of the action (see the preamble of the (co)translation map section).

### 2.1.4 Algebra-Galois extensions

The right action of a group $G$ on a space $X$ induces the right action of the group ring $(kG)^{op}$ on a suitable algebra of functions on $X$: $(f \circ g)(x) := f(xg)$. This model is a prototype of our considerations here. Let $P$ be an algebra and a right $A$-module via the action $P \otimes A \xrightarrow{\Delta} P$. Then one can define the invariant subalgebra

$$P^A = \{ b \in P \mid (bp) \triangleleft a = b(p \triangleleft a), \forall p \in P, a \in A \}.$$  

(2.37)

If $P$ is a comodule via the map $\Delta_P : P \to P \otimes C$ and the action of $(C^*)^{op}$ is given by the formula $p \triangleleft a = p(0)a(p(1))$, then $P^{(C^*)^{op}} = P^{coC}$ (cf. [68]). Now, if $A$ is a finite dimensional algebra, then the pullback of the multiplication and the unit map turns $A^* := \text{Hom}(A, k)$ into a coalgebra. Similarly, any action $M \otimes A \xrightarrow{\rho} M$ gives rise to a coaction

$$\rho : M \to M \otimes (A^{op})^*,$$

$$\rho(m) = \sum_{i=1}^{\dim A} m \triangleleft e_i \otimes e^i,$$

(2.38)

where $\{e_i\}_{i \in \{1, \ldots, \dim A\}}$ is a basis of $A$ and $\{e^i\}_{i \in \{1, \ldots, \dim A\}}$ the dual basis. In this situation, algebras and coalgebras as well as modules and comodules are equivalent concepts. In particular, one directly translates the Galois condition for coactions into an equivalent Galois condition for actions. This has been carried out in [23]. The aim of this section is to study the case $\dim A = \infty$, so that for the details concerning $\dim A < \infty$ we refer to [23]. Here let us only observe the following:

**Remark 2.12.** In this remark we follow the convention and notation of [23]. An algebra action $A \otimes P \to P$ cannot be Galois in the sense of [23, Proposition 2.2] when $\dim A = \infty$. Indeed, suppose that $\dim A = \infty$ and action is Galois. Choose a linear basis $\{e_i\}$ of $A$, and write $\chi^\#(1) = \sum_{i=1}^{n} t_i \otimes e_i$. Then

$$1 \otimes e_{n+1} = ((\chi \otimes \text{id}) \circ (\text{id} \otimes \chi^\#))(e_{n+1} \otimes 1)$$

$$= \sum_{i=1}^{n} (\chi \otimes \text{id})(e_{n+1} \otimes t_i \otimes e_i)$$

$$= \sum_{i=1}^{n} (e_{n+1} \triangleright t_i) \otimes e_i.$$

This contradicts the linear independence of $\{e_i\}$. (In the finite dimensional case one can take $n = \dim A$, and then there does not exist a linearly independent $e_{n+1}$.)

---

[2] This section is based on joint work with P. Schauenburg and H.-J. Schneider.
In the infinitely dimensional case, the point is that an action cannot always be turned to a coaction [68, p.11]. Therefore, we need a definition of the Galois property which avoids starting from a coaction.

**Definition 2.13.** Let $P$ be an algebra and a right $A$-module. Let $V \subseteq A^*$ and $j : P \otimes V \to \text{Hom}(A, P)$ be a linear map defined by $j(p \otimes v)(a) := pv(a)$. We say that the action of $A$ on $P$ is Galois provided

(i) the map $\text{Can} : P \otimes_{PA} P \to \text{Hom}(A, P)$, $\text{Can}(p \otimes_{PA} p')(a) := p(p' \triangleleft a)$ is injective,

(ii) there exists a subspace $V \subseteq A^*$ such that

(a) $\text{Can}(P \otimes_{PA} P) = j(P \otimes V)$ ($V$ is sufficiently small),

(b) $V(a) = 0 \Rightarrow a = 0$ ($V$ is sufficiently big).

A Galois action is said to be $\kappa$-augmented if there exists a character of $A$, $\kappa \in V$, such that $1_p \kappa(a) = 1_p \triangleleft a$ for all $a \in A$.

We say that an extension of algebras $B \subseteq P$ is an $A$-extension if $B = P^A$, and we call it algebra-Galois if the action of $A$ on $P$ is Galois. Finally an algebra-Galois extension with a $\kappa$-augmented action is called a $\kappa$-augmented algebra-Galois extension. Let us begin by extracting immediate properties of algebra-Galois extensions.

**Lemma 2.14.** Assume that the action $P \otimes A \to P$ satisfies condition (ii)(a) for some $V \subseteq A^*$. Let $\gamma : P^* \otimes P \to A^*$ be defined by the formula $\gamma(f \otimes p)(a) = f(p \triangleleft a)$. Then $V = \text{Im} \gamma$.

**Proof.** Condition (ii)(a) entails that

$$\forall v \in V \exists \sum p_i \otimes p_i' \in P \otimes P \forall a \in A : v(a) = \sum_i p_i(p_i' \triangleleft a). \quad (2.39)$$

Choose $f \in P^*$ such that $f(1) = 1$ and define $f_i$ by $f_i(p) = f(p_i p)$. Then

$$v(a) = f(v(a)) = f(\sum_i p_i(p_i' \triangleleft a)) = \sum_i f_i(p_i' \triangleleft a) = \gamma(\sum_i f_i \otimes p_i')(a), \quad (2.40)$$

as needed. \hfill \Box

Thus condition (ii)(a) uniquely determines $V$. The following lemma shows that it also forces the action of $A$ to be locally finite. On the other hand, condition (ii)(b) is responsible for the faithfulness of the action of $A$.

**Lemma 2.15.** The Galois action of $A$ on $P$ is always faithful ($P \triangleleft a = 0 \Rightarrow a = 0$) and locally finite ($\dim(p \triangleleft A) < \infty$ for any $p \in P$).

**Proof.** Let us prove first that $\dim(p \triangleleft A) < \infty$. It follows from condition (ii)(a) that

$$\forall \ p \in P \exists \sum_i e_i \otimes v_i \in P \otimes V \forall a \in A : p \triangleleft a = \sum_i e_i v_i(a). \quad (2.41)$$

Hence $p \triangleleft A \subseteq \text{span}\{e_i\}_{i \in \text{finite set}}$, so that $\dim(p \triangleleft A) < \infty$. Next, the faithfulness assertion follows immediately from condition (ii)(b) and Lemma 2.14. Indeed,

$$P \triangleleft a = 0 \Rightarrow V(a) = 0 \Rightarrow a = 0.$$

\hfill \Box
Furthermore, the axioms for $V$ turn out to be sufficiently strong to make it a coalgebra and $P$ a $V$-comodule reflecting the $A$-module structure.

**Lemma 2.16.** Assume that the action $P \otimes A \to P$ satisfies conditions (i) and (ii)(a). Then there exists a coalgebra structure on $V$ and a coaction $\Delta_P : P \to P \otimes V$ such that $(j \circ \Delta_P)(p)(a) = p \triangleleft a$ for any $p \in P$, $a \in A$. Moreover, $P$ is a coalgebra-Galois $V$-extension of $P^A$.

**Proof.** Since $\text{Can}(P \otimes_{P^A} P) = j(P \otimes V)$ and $j : P \otimes V \to \text{Hom}(A, P)$ is injective, there is a homomorphism

$$\Delta_P : P \longrightarrow P \otimes V, \quad \Delta_P(p) := j^{-1}(\text{Can}(1 \otimes_{P^A} p)). \tag{2.42}$$

To define the desired coproduct on $V$, let us consider the relationship between $\Delta_P$ and $\gamma$. Taking advantage of the natural embedding of $V \otimes V$ in $(A \otimes A)^*$, we obtain:

$$((\gamma \otimes \text{id}) \circ (\text{id} \otimes \Delta_P))(\varphi \otimes p)(a \otimes a') = \gamma(\varphi \otimes p \triangleleft a')(a)$$

$$= \varphi((p \triangleleft a') \triangleleft a)$$

$$= \gamma(\varphi \otimes p)(a'a). \tag{2.43}$$

Consequently, if $t \in \text{Ker} \gamma$, then $(\text{id} \otimes \Delta_P)(t) \in \text{Ker} (\gamma \otimes \text{id})$. Therefore, we have a commutative diagram defining the coproduct:

$$\begin{array}{ccc}
0 & \longrightarrow & \text{Ker} \gamma \\
\downarrow & & \downarrow \text{id} \otimes \Delta_P \\
0 & \longrightarrow & \text{Ker} \gamma \otimes V
\end{array}$$

$$\begin{array}{ccc}
P^* \otimes P & \longrightarrow & V \\
\gamma & \downarrow \Delta & \downarrow \gamma \otimes \text{id} \\
V \otimes V & \longrightarrow & 0
\end{array}$$

In the spirit of (2.43), we can verify that

$$(\Delta v)(a \otimes a') = \sum_i ((\gamma \otimes \text{id}) \circ (\text{id} \otimes \Delta_P))(\varphi_i \otimes p_i)(a \otimes a') = v(a'a). \tag{2.45}$$

With the help of the natural embedding $V \otimes V \subseteq (A \otimes A \otimes A)^*$, the above formula entails the coassociativity of $\Delta$. The counit $\varepsilon$ is given by the evaluation at 1. The map $\Delta_P$ is by construction compatible with the counit, i.e., $(\text{id} \otimes \varepsilon) \circ \Delta_P = \text{id}$, and the coassociativity of $\Delta_P$ can be proven the same way as the coassociativity of $\Delta$. Thus $V$ is a coalgebra and $P$ is a comodule such that $p_{(0)}p_{(1)}(a) = p \triangleleft a$ (see (2.42)). Since (by the injectivity of $j$)

$$(bp)_{(0)} \otimes (bp)_{(1)} = bp_{(0)} \otimes p_{(1)} \iff (bp) \triangleleft a = b(p \triangleleft a), \tag{2.46}$$

we have $P^{coV} = P^A$. Finally, the coaction $\Delta_P$ is clearly Galois as $\text{Can}$ is surjective by condition (ii)(a) and is injective due to the injectivity of $\text{Can}$. \hfill \Box

We have just shown that, if an $A$-extension $B \subseteq P$ satisfies conditions (i) and (ii)(a), then there exists a coalgebra $C \subseteq A^*$ such that $B \subseteq P$ is a coalgebra-Galois $C$-extension and $\text{Can} = j \circ \text{can}$. Hence, behind any algebra-Galois extension, there is a coalgebra-Galois extension. On the other hand, behind any coalgebra-Galois extension there is an algebra-Galois extension — just take $A = (C^*)^{op}$ and $V = i(C)$, where $i : C \hookrightarrow C^{**} = A^*$ is the canonical embedding. The injectivity of $\text{Can}$ and condition (ii)(a) follow immediately from the definition of the action of $(C^*)^{op}$ ($p \triangleleft a = p_{(0)}a(p_{(1)})$), and condition (ii)(b) is automatic. We now want to employ condition (ii)(b) to show that the aforementioned procedure of extracting the coalgebra-Galois structure from an algebra-Galois extension retrieves the original coalgebra-Galois $C$-extension we started from.
Lemma 2.17. Let $P$ be an algebra and a right $C$-comodule. The $C$-extension $B \subseteq P$ is coalgebra-Galois if and only if the induced action $(A := (C^*)^{op}, p \triangleleft a := p_{(0)}a(p_{(1)})$) is Galois.

Proof. If the $C$-coaction is Galois, we take $V = i(C)$ and verify that all works. In the converse direction, assume that the action of $A$ on $P$ is Galois. Since it is given by the formula $p \triangleleft a = p_{(0)}a(p_{(1)})$, we have that $P^{\text{co}C} \subseteq P^A$ and

$$Can = j \circ (\text{id} \otimes i) \circ \text{can}.$$ 

Hence, by condition (ii)(a),

$$P \otimes V = \text{Im}((\text{id} \otimes i) \circ \text{can}) \subseteq P \otimes i(C),$$

so that $V \subseteq i(C)$. We also have that $\text{can}$ is injective due to the injectivity of $Can$. Furthermore, now it follows from (ii)(a) that $i(C) \subseteq V$. Indeed, otherwise there exists $u \in i(C), u \notin V$. Let $\{c_i\}$ be a basis of $i^{-1}(V)$. Then $i^{-1}(u)$ is linearly independent of $\{c_i\}$, and we can complete the set $\{c_i\} \cup \{i^{-1}(u)\}$ to a linear basis of $C$. Using such a basis, define $a_0$ to be 1 on $i^{-1}(u)$ and 0 on all other basis elements. Then $a_0 \neq 0$ and $V(a_0) = a_0(i^{-1}(V)) = 0$, which contradicts (ii)(a). Therefore $V = i(C)$, and (2.48) implies that $P \otimes C$ is the image of $\text{can}$. Combining this with the injectivity of $\text{can}$ we conclude that $\text{can}$ is a bijection from $P \otimes_{P^{\text{co}C}} P$ to $P \otimes C$, i.e., the extension $P^{\text{co}C} \subseteq P$ is $C$-Galois.

We have shown that a coaction is Galois if and only if the induced action is Galois. To complete the picture, let us note that the coalgebra structure on $V$ constructed in Lemma 2.14 coincides with the coalgebra structure on $C$. First, it is straightforward to observe that their coactions on $P$ coincide:

$$\begin{array}{ccc}
P & \xrightarrow{\Delta} & P \otimes V \\
& \downarrow{\text{id} \otimes i} & \\
P \otimes C. & \end{array}$$

Indeed, applying the injection $j$ to the coaction of $V$ yields by (2.42) $Can(1 \otimes_{P^A} p) \in \text{Hom}(A, P)$, and applying it to the composed map gives $j(p_{(0)} \otimes p_{(1)}) \in \text{Hom}(A, P)$. Evaluating these maps on an arbitrary $a \in A$, one obtains

$$Can(1 \otimes_{P^A} p)(a) = p \triangleleft a = p_{(0)}a(p_{(1)}) = j(p_{(0)} \otimes i(p_{(1)}))(a).$$

Thus the diagram (2.49) is commutative as claimed. Next, let us choose $f \in P^*$ such that $f(1) = 1$. Putting together the constructions from Lemma 2.16 and embedding $i(C) \otimes i(C)$ in $(A \otimes A)^*$, we compute:

$$\begin{align}
(\Delta \circ i)(c)(a \otimes a') &= (\gamma(f(c^{[1]} \otimes_{P^A} c^{[2]}_{(0)}) \otimes i(c^{[2]}_{(1})))(a \otimes a') \\
&= f((c^{[1]}(c^{[2]}_{(0)} \triangleleft a))a'(c^{[2]}_{(1)}) \\
&= f((c^{[1]} \triangleleft a' \triangleleft a)) \\
&= f(Can(c^{[1]} \otimes_{P^A} c^{[2]}))(a'a)) \\
&= f((j \circ (\text{id} \otimes i) \circ \text{can})(c^{[1]} \otimes_{P^A} c^{[2]})(a'a)) \\
&= f(j(1 \otimes i(c))(a'a)) \\
&= f((a'a)(c)) \\
&= (a'a)(c). \end{align}$$

(2.51)
Here we abused the notation and denoted by $\gamma$ the map $P^* \otimes_A P \to A^*$. On the other hand,

$$
(i(c(1)) \otimes i(c(2)))(a \otimes a') = a(c(1))a'(c(2)) = (a'a)(c).
$$

(2.52)

The counitality of $i$ is also clear:

$$(\varepsilon_V \circ i)(c) = i(c)(1) = \varepsilon_C(c).$$

(2.53)

Hence $V$ and $C$ are isomorphic as coalgebras. This way we have shown that indeed the procedure from Lemma 2.16 applied to the algebra-Galois ($C^*$)op-extension recovers the original coalgebra-Galois $C$-extension.

**Lemma 2.18.** Let $P$ be an algebra and a right $C$-comodule. If the coaction is Galois, then it is isomorphic to the coaction of $V$ corresponding via Lemma 2.16 to the induced algebra-Galois ($C^*$)op-extension.

Now one might ask what happens if we start from an algebra-Galois $A$-extension, go to the coalgebra-Galois $V$-extension, and then to the algebra-Galois ($V^*$)op-extension. It turns out that $A$ is a subalgebra of ($V^*$)op via $A \hookrightarrow A^*$, and its action on $P$ factors via the action of ($V^*$)op. First, note that $V \subseteq A^*$ and, by condition (ii)(b), the pullback of this inclusion composed with $A \hookrightarrow A^*$, i.e., $i_A : A \hookrightarrow A^* \to V^*$, is injective. In fact, the injectivity of this map is equivalent to condition (ii)(b). Let us check now that $i_A$ is an algebra homomorphism from $A$ to ($V^*$)op. To this end, we choose any $v \in V$, and compute:

$$(i_A(a)i_A(a'))(v) = v(a')(a) = (\Delta v)(a' \otimes a) = v(aa') = (i_A(aa'))(v).$$

(2.54)

Here the penultimate identity follows from (2.45). The unitality of $i_A$ is also clear:

$$i_A(1_A)(v) = v(1_A) = \varepsilon(v) = 1_{(V^*)^*}(v).$$

(2.55)

Finally, the diagram

$$
\begin{array}{ccc}
P \otimes A & \xrightarrow{id \otimes i_A} & P \\
\downarrow{\text{id} \otimes i_A} & & \downarrow{\text{id} \otimes i_A} \\
P \otimes (V^*)^* & \xrightarrow{f} & P
\end{array}
$$

(2.56)

is commutative because, by virtue of (2.42), we have

$$
p \circ i_A(a) = p_0(i_A(a)(p_1)) = p_0f(p_1)(a) = j(p_0 \otimes p_1)(a) = Can(1 \otimes_A p)(a) = p \circ a.
$$

(2.57)

We can summarise much of the above in the following:

**Theorem 2.19.** Let $P$ be an algebra and a right $A$-module. The action of $A$ on $P$ is Galois if and only if there exists a coalgebra $V := C \subseteq A^*$ coacting on $P$ on the right and such that

(i) the induced action of ($C^*$)op on $P$ ($p \circ f = p_0f(p_1)$) is Galois,

(ii) $A$ is a subalgebra of ($C^*$)op via the composition of the canonical embedding with the pullback of the above inclusion: $i_A : A \hookrightarrow A^* \to (C^*)^*$,
(iii) the action of $A$ factors through $i_A$ and the action of $(C^*_{op})$ ($p \triangleleft a = p \triangleleft i_A(a)$).

**Proof.** If the action of $A$ on $P$ is Galois, then, by Lemma 2.16, there exists a right Galois coaction on $P$ by a coalgebra $C \subseteq A^*$. On the other hand, by Lemma 2.17, the induced action of $(C^*_{op})$ on $P$ is Galois. The remaining properties follow from the discussion preceding the theorem.

Assume now that conditions (i)–(iii) are satisfied. It follows from condition (i) and Lemma 2.17 that the coaction of $C$ is Galois. Next, condition (iii) entails that

$$p \triangleleft a = p \triangleleft i_A(a) = p(0)i_A(a)(p(1)) = p(0)p(1)(a) = j(p(0) \otimes p(1))(a). \quad (2.58)$$

Since $C \subseteq A^*$, the map $j : P \otimes C \to \text{Hom}(A, P)$ is injective. Therefore, (2.58) implies that

$$\forall \ p \in P, \ a \in A : (bp) \triangleleft a = b(p \triangleleft a) \iff \forall \ p \in P : (bp)(0) \otimes (bp)(1) = bp(0) \otimes p(1). \quad (2.59)$$

Consequently, $P^A = P^{coC}$. Now, using (2.58), one can immediately check that $Can = j \circ can$. Hence, the injectivity of $j$ and $can$ imply the injectivity of $Can$, and the surjectivity of $can$ implies condition (ii)(a) in Definition 2.13. Finally, condition (ii)(b) in Definition 2.13 follows from condition (ii). \qed

To illustrate the foregoing theory, let us consider an example with trivial invariants, so that we can focus on the subtleties particular to the Galois actions of infinite dimensional algebras.

**Example 2.20.** Let $P$ be the Hopf algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ acted upon by the group algebra $A = \mathbb{C}U(1)$ via the formula $z^\mu \triangleleft e^{i\theta} = e^{i\theta}z^\mu$. We want to show that this action is Galois. First, note that the invariant subalgebra $P^A$ is trivial:

$$\left( \forall \ \mu \in \mathbb{Z}, \ \theta \in [0, 2\pi) : \left( \sum_{k=-m}^{n} a_k z^k z^\mu \triangleleft e^{i\theta} = \sum_{k=-m}^{n} a_k z^k (z^\mu \triangleleft e^{i\theta}) \right) \Rightarrow a_k = \delta_{k0}a_0. \right)$$

One can guess that the Galois coaction standing behind this action is simply the coproduct on $P = \mathbb{C}[z, z^{-1}]$. (The canonical map has the form $p \otimes p' \to pp'(1) \otimes p'(2)$.) Therefore, in view of Lemma 2.16, we take $V \cong \mathbb{C}[z, z^{-1}]$ and view it as a subset of $(\mathbb{C}U(1))^*$ via the evaluation map: $i(z^\mu)(e^{i\theta}) := e^{i\mu\theta}$. The injectivity of the mapping $i$ from $\mathbb{C}[z, z^{-1}]$ to $(\mathbb{C}U(1))^*$ is clear, because a rational function that is 0 at infinitely many points is the 0 function. Now, it is straightforward to verify that $j \circ (\text{id} \otimes i) \circ can = Can$, so that condition (ii)(a) holds due to the surjectivity of $can$. Condition 1 (injectivity of $Can$) also holds, because $i$, $j$ and $can$ are injective. Finally, we need to check that

$$\left( \forall \ \mu \in \mathbb{Z}, \ \text{finite subset } I \ of \ U(1) : i(z^\mu)(\sum_{g \in I} \lambda_g g) = 0 \right) \Rightarrow (\forall \ g \in I : \lambda_g = 0).$$

To this end, note that the left-hand side can be thought of as a system of infinitely many linear equations, where the lambdas are variables and elements of $U(1)$ are complex coefficients. Let $n$ be the number of elements in $I$. Since the equation $\sum_{j=1}^{n} \lambda_j g_j = 0 \in \mathbb{C}$ has to be satisfied for all $\mu \in \mathbb{Z}$, it has to be satisfied for $\mu \in \{0, ..., n-1\}$. This way we obtain a system of $n$ linear equations with the coefficient matrix

$$\begin{pmatrix}
1 & 1 & \ldots & 1 \\
g_1 & g_2 & \ldots & g_n \\
\vdots & & & \\
g_1^{n-1} & g_2^{n-1} & \ldots & g_n^{n-1}
\end{pmatrix} \quad (2.60)$$

\[3\] We owe this argument to R. Matthes.
The freeness of action means that all the $g_j$s are pairwise different. This proves that the linear system has only the zero solution, whence $\sum_{g \in I} \lambda_g g = 0 \in \mathbb{C}U(1)$. Thus we can conclude that the action of $\mathbb{C}U(1)$ on $\mathbb{C}[z, z^{-1}]$ is Galois. Therefore, according to Theorem 2.19, the group algebra $\mathbb{C}U(1)$ can be viewed as a subalgebra of the convolution algebra $(\mathbb{C}[z, z^{-1}])^{op} \cong \text{Map}(\mathbb{Z}, \mathbb{C})$. The injective homomorphism is given by the formula

$$\mathbb{C}U(1) \ni \sum_{g \in I \subseteq U(1)} \lambda_g g \rightarrow f \in \text{Map}(\mathbb{Z}, \mathbb{C}), \quad f(\mu) = \sum_{g \in I \subseteq U(1)} \lambda_g g^\mu.$$  \hspace{1cm} (2.61)

### 2.2 The (co)translation map

Apart from its value as a technical tool, the translation map has a very nice geometric interpretation. In classical geometry a principal bundle can also be defined in the following way (cf. [54, Section 4.2]). Consider a topological space $X$ with a free action of a topological group $G$. The freeness of action means that $x \cdot g = x$ for $x \in X$, $g \in G$ implies that $g = 1$. It guarantees that there is a function $\hat{\tau} : X \times_{X/G} X \rightarrow G$ determined by the relation $x \cdot \hat{\tau}(x, x') = x'$. The function $\hat{\tau}$ is called a \textit{translation function}. One then says that $X$ is a principal $G$-bundle over $X/G$ provided the translation function is continuous and $X \times_{X/G} X$ is closed in $X \times X$. These two conditions are equivalent to the action being proper. Dualising the notions of a translation function $\hat{\tau}$ and of a free action one arrives at the translation map in Definition 2.21.

#### 2.2.1 Coalgebra extensions

In the studies of coalgebra-Galois extensions an important role is played by the notion of a translation map.

**Definition 2.21.** For a coalgebra-Galois $C$-extension $B \subseteq P$, the map

$$\tau : C \rightarrow P \otimes_B P \ , \ c \mapsto \text{can}^{-1}(1 \otimes c),$$

is called a \textit{translation map}. For each $c \in C$, the image $\tau(c)$ is denoted by $\tau(c) := c^{[1]} \otimes_B c^{[2]}$ (summation understood).

**Lemma 2.22 (Translation Map Lemma).** Let $C$-extension $B \subseteq P$ be a coalgebra-Galois extension. For all $c \in C$ and $p \in P$, the translation map $\tau$ has the following properties:

(i) $c^{[1]} \otimes_B c^{[2]}(0) \otimes c^{[2]}(1) = 1 \otimes c$;

(ii) $c^{[1]} \otimes c^{[2]} = \varepsilon(c)1$;

(iii) $p(0)p(1)^{[1]} \otimes_B p(1)^{[2]} = 1 \otimes_B p$;

(iv) $c^{[1]} \otimes c^{[2]}(0) \otimes c^{[2]}(1) = c^{[1]}c^{[2]}(1)$;

(v) $c^{[1]} \otimes 1 \otimes c^{[2]} = c^{[1]}c^{[2]}c^{[1]}c^{[2]}$;

(vi) Gauge invariance: for any algebra and right $C$-comodule map $P \xrightarrow{F} P$, $(F \otimes_B F) \circ \tau = \tau$;

(vii) In the case of an $e$-coaugmented coalgebra-Galois extension $B \subseteq P$: $e^{[1]} \otimes_B e^{[2]} = 1 \otimes_B 1$. 

\[20\]
Proof. (i) From the definition of the translation map it follows that

\[ c^{[1]}e^{[2]}(0) \otimes c^{[2]}(1) = (m \otimes \text{id}) \circ (\text{id} \otimes \Delta_P) \circ \tau(c) = \text{can} \circ \tau(c) = 1 \otimes c. \]

(ii) Apply id \otimes \varepsilon to property (i).

(iii) Applying \text{can} to both sides of the equation, on one hand we obtain

\[ \text{can}(p_{(0)}p_{(1)}^{[1]} \otimes_B p_{(1)}^{[2]}) = p_{(0)}p_{(1)}^{[1]}p_{(1)}^{[2]}(0) \otimes p_{(1)}^{[2]}(1) = p_{(0)} \otimes p_{(1)} \]

by property (i). On the other hand, \text{can}(1 \otimes p) = p_{(0)} \otimes p_{(1)}. Since \text{can} is a bijection, both arguments are equal.

(iv) We apply the isomorphism \text{can} \otimes \text{id}_C to both sides and use the fact that the canonical map \text{can} is right \( C \)-colinear to obtain

\[ \text{can}(c^{[1]} \otimes_B c^{[2]}(0)) \otimes c^{[2]}(1) = \text{can}(c^{[1]} \otimes_B c^{[2]}(0)) \otimes \text{can}(c^{[1]} \otimes_B c^{[2]}(1)) = 1 \otimes c(1) \otimes c(2) = \text{can}(c^{[1]} \otimes_B c^{[2]}(1)) \otimes c(2). \]

(v) Apply id \otimes_B \text{can}^{-1} to equality (iv) and then use property (iii).

(vi) For any \( c \in C \) compute

\[ \text{can}(F(c^{[1]} \otimes_B F(c^{[2]})) = F(c^{[1]}F(c^{[2]}(0)) \otimes F(c^{[2]}(1)) = F(c^{[1]}c^{[2]}(0)) \otimes c^{[2]}(1) = 1 \otimes c. \]

Now apply \text{can}^{-1} to deduce the assertion.

(vii) Since for an \( e \)-coaugmented coalgebra-Galois extension \( \Delta_P(e) = 1 \otimes e \), we can use property (iii) to compute

\[ e^{[1]} \otimes_B e^{[2]} = 1(0)1^{[1]}(1) \otimes_B 1^{[2]}(1) = 1 \otimes_B 1, \]

as required. \( \square \)

The properties listed in the Translation Map Lemma are simply dualisations of the properties of the classical translation function. For example, Lemma 2.22(ii) corresponds to the classical property \( \hat{\tau}(x, x) = 1 \), while Lemma 2.22(v) corresponds to the classical transitivity property of the translation function, \( \hat{\tau}(x, x')\hat{\tau}(x', x'') = \tau(x, x'') \), for all \( x, x', x'' \in X \). The properties of the translation map in the case of a Hopf-Galois extension were first studied in [82].

As the first application of the Translation Map Lemma we show that two notions of coaction invariants introduced earlier coincide in the case of an \( e \)-coaugmented coalgebra-Galois extension.

**Proposition 2.23.** \( P^\text{coC} = P^\text{coC} \) for an \( e \)-coaugmented coalgebra-Galois \( C \)-extension \( B \subseteq P \).

**Proof.** \( P^\text{coC} \subseteq P^\text{coC}_e \) is proven in Lemma 2.1. For the opposite inclusion, let \( b \in P^\text{coC}_e \), i.e. \( b_{(0)} \otimes b_{(1)} = b \otimes e \). Then using properties (iii) and (v) from Lemma 2.22 we obtain, for all \( p \in P \),

\[ \Delta_P(bp) = b_{(0)}b_{(1)}^{[1]} \Delta_P(b_{(1)}^{[2]}p) = be^{[1]} \Delta_P(e^{[2]}p) = b \Delta_P(p). \]

This shows that \( b \in P^\text{coC}_e \). \( \square \)
2.2.2 Algebra coextensions

Assume that $C \rightarrow B$ is an algebra-Galois $A$-coextension. Then $cocan$ is a bijection and there exists the cotranslation map $\hat{\tau} : C \square_B C \rightarrow A$, $\hat{\tau} := (\varepsilon \otimes id_C) \circ cocan^{-1}$. By dualising properties of the translation map (or directly from the definition of $\hat{\tau}$), one can establish the corresponding properties of the cotranslation map.

Lemma 2.24 (COTRANSITION MAP LEMMA). Let $C \rightarrow B$ be an algebra-Galois $A$-coextension. Then the cotranslation map $\hat{\tau}$ has the following properties, for all $c \otimes c' \in C \square_B C$ (summation implicit) and $a \in A$:

(i) $\hat{\tau} \circ cocan = \varepsilon \otimes id_A$;

(ii) $\hat{\tau} \circ \Delta = \varepsilon$;

(iii) $\mu_C \circ (id_C \otimes \hat{\tau}) \circ (\Delta_B \otimes id_C) = \varepsilon_B \otimes id_C$ or explicitly $\mu_C(c_{(1)}, \hat{\tau}(c_{(2)}, c')) = \varepsilon(c)c'$;

(iv) $\hat{\tau}(c, \mu_C(c', a)) = \hat{\tau}(c, c')a$;

(v) $\hat{\tau} \circ (id_C \square_B \varepsilon \otimes id_C) = m_A \circ (\hat{\tau} \otimes \hat{\tau}) \circ (id_C \square_B \Delta \otimes id_C)$;

(vi) Gauge invariance: for any right $A$-linear coalgebra map $F : C \rightarrow C$, $\hat{\tau} \circ (F \square_B F) = \hat{\tau}$;

(vii) In the case of a $\kappa$-augmented algebra-Galois $A$-coextension $C \rightarrow B$, $\kappa \circ \hat{\tau} = \varepsilon \square_B \varepsilon$.

2.2.3 Algebra extensions

Let $B \subseteq P$ be an algebra-Galois extension by $A$ (relative to $V \subseteq A^*$) as in Definition 2.13. Then for any $v \in V$ one can find a unique $v^{[1]} \otimes_B v^{[2]} \in P \otimes_B P$ (summation assumed) such that, for any $p \in P$, $Can(v^{[1]} \otimes_B v^{[2]}) = j(p \otimes v)$. The assignment $v \mapsto v^{[1]} \otimes_B v^{[2]}$ defines a map $\tau : V \rightarrow P \otimes_B P$ which is called a translation map for an algebra-Galois $A$-extension $B \subseteq P$. In view of Theorem 2.19, the map $\tau$ can be seen to coincide with the translation map of the corresponding coalgebra-Galois extension, so that the Translation Map Lemma can be used to derive the following properties of $\tau$.

Lemma 2.25. Let $B \subseteq P$ be an algebra-Galois extension by $A$ (relative to $V \subseteq A^*$). Then the translation map $\tau : V \rightarrow P \otimes_B P$ has the following properties for all $a \in A$, $v \in V$ and $p \in P$

(i) $v^{[1]}(v^{[2]} \triangleleft a) = 1_P v(a)$.

(ii) $v^{[1]} v^{[2]} = 1_P v(1_A)$.

(iii) Let $\sum_i p_i \otimes v_i = j^{-1}(Can(1_P \otimes_B p))$. Then $\sum_i p_i v_i^{[1]} \otimes_B v_i^{[2]} = 1 \otimes_B p$.

(iv) Consider $V$ as a right $A$-module by $(v \cdot a)(a') = v(aa')$. Then $v^{[1]} \otimes_B v^{[2]} \triangleleft a = (v \cdot a)^{[1]} \otimes_B (v \cdot a)^{[2]}$, i.e., $\tau$ is a right $A$-module map.

(v) Gauge invariance: for any right $A$-linear algebra map $F : P \rightarrow P$, $(F \otimes_B F) \circ \tau = \tau$.

(vi) In the case of a $\kappa$-augmented algebra-Galois extension, $\kappa^{[1]} \otimes_B \kappa^{[2]} = 1 \otimes_B 1$.

Proof. The properties listed in the lemma are simply properties of the translation map in Lemma 2.22 translated to the algebra-Galois case with the help of Theorem 2.19. We leave direct proofs to the reader as an exercise. Note only that (i) is simply the definition of a translation map.
2.3 Entwining and factorisation

2.3.1 Entwining

In the definition of a Hopf-Galois extension one requires that \( P \) is a comodule algebra of a Hopf algebra \( H \). Obviously, no such assumption is possible in the case of a coalgebra-Galois extension, since a priori \( C \) does not have any algebra structure. This might appear to be a problem, in particular when the differential geometry of coalgebra-Galois extensions is considered, and some replacement for an algebra structure on \( C \) compatible with an algebra structure on \( P \) is needed. Thus the original point of view taken in [21] was to require a compatibility between \( P \) and \( C \) in terms of an entwining, and then define the canonical map and require its bijectivity within this setting. It has been realised in [17] that given a coalgebra-Galois extension as defined in Definition 2.2, there actually exists a relation between the coalgebra structure of \( C \) and the algebra structure of \( P \) provided by an entwining.

**Definition 2.26.** An entwining structure consists of a triple \((A, C, \psi)\), where \( A \) is an algebra, \( C \) a coalgebra and \( \psi : C \otimes A \to A \otimes C \) a linear map such that the following “bow-tie”-diagram commutes:

\[
\begin{array}{ccc}
C \otimes A \otimes A & \xrightarrow{\psi \otimes \text{id}_A} & A \otimes C \otimes A \\
\Downarrow \text{id}_C \otimes m & & \Downarrow \Delta \otimes \text{id}_A \\
A \otimes C \otimes A & \xrightarrow{\text{id}_A \otimes \psi} & C \otimes A \\
\Downarrow \text{id}_A \otimes \eta & & \Downarrow \varepsilon \otimes \text{id}_A \\
A \otimes A \otimes C & \xrightarrow{m \otimes \text{id}_C} & A \otimes C \otimes C \\
\end{array}
\]

where \( m \) is the multiplication and \( \eta \) the unit of \( A \), and \( \Delta \) is comultiplication and \( \varepsilon \) the counit of \( C \).

The notion of an entwining structure was introduced in [21]. The bow-tie diagram expresses the most natural compatibility conditions between algebra and coalgebra structures. The right part of the diagram states that \( \psi \) respects the coproduct (right pentagon) and the counit (right triangle). The left part of the diagram expresses the compatibility of \( \psi \) with the product (left pentagon) and the unit (left triangle). Any algebra and a coalgebra can be provided with an entwining structure with \( \psi \) being the usual flip of tensor factors (for obvious reasons this can be called a trivial entwining structure). There are several extremely nice properties of an entwining structure. Most importantly, the notion of an entwining structure is self-dual in the following sense. The structure described by the bow-tie diagram is invariant under the operation that involves interchanging \( A \) with \( C \), \( \Delta \) with \( m \), \( \varepsilon \) with \( \eta \), and reversing all the arrows. Another property is related to tensor algebras and, consequently to universal differential algebras. Consider the tensor algebra \( A^\otimes = \oplus_n A^{\otimes n} \) with the product given by the concatenation, i.e.,

\[
(a_1 \otimes \ldots \otimes a^m)(a^{m+1} \otimes \ldots a^{m+n}) = a_1 \otimes \ldots \otimes a^m a^{m+1} \otimes \ldots a^{m+n}.
\]
Then \((A,C,\psi)\) induces an entwining structure \((A^\otimes,C,\psi^\otimes)\) with
\[
\psi^\otimes \mid_{C^\otimes A^\otimes n} = (\text{id}_A^\otimes \otimes \psi) \circ (\text{id}_A^{\otimes n-2} \otimes \psi \otimes \text{id}_A) \circ \ldots \circ (\psi \otimes \text{id}_A^{\otimes n-1}).
\]
As an exercise in the self-duality the reader can derive the corresponding entwining structure for a tensor coalgebra \(C^\otimes\).

To describe the action of \(\psi\) we use the following \(\alpha\)-notation: \(\psi(c \otimes a) = a\alpha \otimes c^\alpha\) (summation over a Greek index understood), for all \(a \in A, c \in C\), which proves very useful in concrete computations involving \(\psi\). The reader is advised to check that the bow-tie diagram is equivalent to the following four explicit relations:

- **left pentagon:** \((aa')_\alpha \otimes c^\alpha = a\alpha a'\beta \otimes c^{\alpha\beta},\)
- **left triangle:** \(1_\alpha \otimes c^\alpha = 1 \otimes c,\)
- **right pentagon:** \(a\alpha \otimes c(1)^\alpha \otimes c(2)^\beta = a \beta_\alpha \otimes c(1)^\alpha \otimes c(2)^\beta,\)
- **right triangle:** \(a\alpha \varepsilon(c(\alpha)) = a\varepsilon(c),\)

for all \(a, a' \in A, c \in C\).

One may (or perhaps even should) think of an entwining map \(\psi\) as a twist in the convolution algebra \(\text{Hom}(C,A)\). Namely, given an entwining structure, one can define the map \(*_\psi : \text{Hom}(C,A) \otimes \text{Hom}(C,A) \rightarrow \text{Hom}(C,A)\) via \((f *_\psi g)(c) = f(c(2))_\alpha g(c(1)^\alpha)\), for all \(f, g \in \text{Hom}(C,A)\) and \(c \in C\). One can easily check that \((\text{Hom}(C,A), *_\psi)\) is an associative algebra with unit \(\eta \circ \varepsilon\). This algebra is known as a \(\psi\)-twisted convolution algebra.

Directly from the definition of an entwining structure one obtains the following

**Lemma 2.27.** Let \((A,C,\psi)\) be an entwining structure. If \(e \in C\) is a group-like element, then \(A\) is a right \(C\)-comodule with the coaction
\[
\Delta_A : A \rightarrow A \otimes C, \quad a \mapsto \psi(e \otimes a).
\]

Dually, if \(\kappa : A \rightarrow k\) is a character (i.e., an algebra map), then \(C\) is a right \(A\)-module with the action \(c \cdot a = a\alpha \kappa(c^\alpha)\).

**Proof.** The fact that \(\Delta_A\) is a right \(C\)-coaction follows from the right part of the bow-tie diagram in Definition 2.26. In particular, from the right pentagon and using the fact that \(e\) is a group-like element, we have
\[
(\Delta_A \otimes \text{id}) \circ \Delta_A(a) = \Delta_A(a\alpha) \otimes e^\alpha = a\alpha_\beta \otimes e^\beta \otimes e^\alpha = a\alpha_\beta \otimes e(1)^\beta \otimes e(2)^\alpha = a\alpha \otimes e(1)^\alpha \otimes e(2)^\alpha = (\text{id} \otimes \Delta) \circ \Delta_A(a),
\]
while the right triangle implies that \(a\alpha \varepsilon(e(\alpha)) = a\varepsilon(e) = a\).

The second statement follows from the self-duality of the notion of an entwining structure.

We now give two generic examples of entwining structures.

**Example 2.28.** If \(H\) is a bialgebra, then \(\psi : H \otimes H \rightarrow H \otimes H,\) \(h \otimes g \mapsto g(1) \otimes hg(2)\) defines an entwining structure with \(C = A = H\). Conversely, given an algebra and coalgebra \(H\) such that the map \(\psi\) above is an entwining map, \(H\) is a bialgebra. The first statement can be checked by direct computations. Conversely, suppose that \(H\) is an algebra and a coalgebra and that
(H, H, ψ) is an entwining structure. We need to show that the coproduct Δ and the counit ε of H are algebra maps. Evaluating the left pentagon identity at 1 ⊗ h ⊗ h' for all h, h' ∈ H, we immediately conclude that Δ is a multiplicative map. Also, evaluating the left triangle at 1 ⊗ 1 we obtain Δ(1) = 1 ⊗ 1. Thus Δ is an algebra map as required. Furthermore the right triangle reads in this case h(1)ε(h′h(2)) = ε(h′)h. Thus applying ε we immediately deduce that the counit is a multiplicative map, as required.

The entwining constructed in Example 2.28 is known as a bialgebra entwining. It justifies the statement that an entwining structure is a generalisation of a bialgebra. A bialgebra entwining is a special case of the following more general example.

**Example 2.29.** Given a bialgebra H, let C be a right H-module coalgebra and A a right H-comodule algebra. Recall that this means that C is a right H-module and the module structure map µC : C ⊗ H → C is a coalgebra map, and that A is a right H-comodule with the coaction ΔA : A → A ⊗ H that is an algebra map. Consider a k-linear map

ψ : C ⊗ A → A ⊗ C, \quad c ⊗ a ↦ a_{(0)} ⊗ c · a_{(1)}.

Then (A, C, ψ) is an entwining structure. This is shown by direct calculations (cf. [25, 33.4]).

A triple (H, C, A) satisfying conditions of Example 2.29 is known as a (right-right) Doi-Koppinen datum or a Doi-Koppinen structure, the corresponding entwining structure is known as an entwining structure associated to a Doi-Koppinen datum. The studies of Doi-Koppinen structures where initiated independently by Doi in [39] and Koppinen in [59]. Doi-Koppinen data as a separate entity first appeared in [28], and then they were given a separate name in [26]. (Incidentally, they were called Doi-Hopf data.) Various properties and applications of Doi-Koppinen structures are studied in a monograph [27]. Finally, the entwining structure associated to a Doi-Koppinen datum was first introduced in [14].

Clearly, a bialgebra H itself forms a Doi-Koppinen datum \((H, H, H)\) in which H is viewed as an H-comodule algebra via the coproduct and H is an H-module coalgebra via the multiplication. Therefore the bialgebra entwining in Example 2.28 is a special case of the Doi-Koppinen entwining. Several other special cases of the Doi-Koppinen entwining are of particular interest. Most notably, the relative Hopf entwining in which C = H, and the dual-relative Hopf entwining in which A = H. Another important example of an entwining, which again is a special case of the Doi-Koppinen entwining, but is best verified directly, is the Yetter-Drinfeld entwining. In this case H is a Hopf algebra, A = C = H, and the entwining map ψ : H ⊗ H → H ⊗ H is given by ψ : h′ ⊗ h ↦ h(2) ⊗ (Sh(1))h′h(3). We encourage the reader to verify that ψ satisfies the bow-tie diagram, and to verify that this entwining comes from the Doi-Koppinen datum \((H^\text{op} ⊗ H, H, H)\) (remember that the first entry is the bialgebra), with the following module and comodule structures. The right multiplication by \(H^\text{op} ⊗ H\) is given by \(h · (h′ ⊗ h'') = h'h''\), and the right coaction of \(H^\text{op} ⊗ H\) on H is \(g^H(h) = \sum h(2) ⊗ Sh(1) ⊗ h(3)\) (cf. [26]).

The ψ-twisted convolution algebras corresponding to all those special cases of entwining structures were also studied. For example the ψ-twisted convolution algebra corresponding to a Doi-Koppinen datum was introduced in [59, Definition 2.2] and is also known as Koppinen’s smash product. The ψ-twisted convolution algebra corresponding to the relative-Hopf entwining was studied in [38] and [58].

A dual version of Example 2.29, in which A is a left H-module algebra and C is a left H-comodule coalgebra has been constructed by Schauenburg in [76], and is termed an alternative Doi-Koppinen datum. Although Doi-Koppinen and alternative Doi-Koppinen data provide a
rich source of entwining structures, and one can show that, if either $A$ or $C$ is finite dimensional, any $(A, C, \psi)$ is of an (alternative) Doi-Koppinen type, they do not exhaust all possibilities. An example of an entwining structure that does not come from Doi-Koppinen data is constructed in [76, Example 3.4]. This construction is based on earlier work of Tambara [91] on factorisations of algebras.

Given an entwining structure $(A, C, \psi)$, a right $A$-module and right $C$-comodule $M$ with coaction $\varrho^M : M \to M \otimes C$ is called an entwined module if for all $m \in M, a \in A$

$$\varrho^M(m \cdot a) = m_{(0)} \cdot \psi(m_{(1)} \otimes a) = m_{(0)} \cdot a \alpha \otimes m_{(1)} \alpha.$$ 

The category of entwined modules together with right $A$-linear and right $C$-colinear morphisms is denoted by $\mathcal{M}^C_A(\psi)$. The following example shows that entwined modules unify and generalise various categories of Hopf modules studied for the last 40 years.

**Example 2.30.** If the entwining structure $(A, C, \psi)$ comes from a Doi-Koppinen datum as in example 2.29, then $\mathcal{M}^C_A(\psi) = \mathcal{M}^C_A(H)$, the category of Doi-Koppinen modules introduced in [39], [59]. In particular, Sweedler’s Hopf modules [85] correspond to a bialgebra entwining in Example 2.28. In a similar way, relative Hopf modules introduced in [88], [37] are simply entwined modules associated to the relative Hopf entwining, dual-relative Hopf modules or $[C, A]$-Hopf modules of [37] correspond to the dual-relative Hopf entwining. Finally, Yetter-Drinfeld or crossed modules (cf. [97], [72]), which play an important role in the representation theory of quantum groups, are simply entwined modules of a Yetter-Drinfeld entwining.

For either finite dimensional $A$ or $C$, any category of entwined modules is equal to a category of (alternative) Doi-Koppinen modules.

Various properties of entwined modules, in particular the modules described in Example 2.30, in various conventions are studied in a monograph [27]. The theory of entwined modules is extremely rich, but of course, in these notes we are not able to cover it in any detail. From the point of view of coalgebra-Galois extensions, and their interpretation as generalised principal bundles of noncommutative geometry, the following theorem is of the highest importance.

**Theorem 2.31.** Let $C$-extension $B \subseteq P$ be a coalgebra-Galois extension. Then there exists a unique entwining structure $(P, C, \psi)$, such that $P$ becomes an entwined module in $\mathcal{M}^C_P(\psi)$ with the $P$-module structure given by the multiplication.

**Proof.** For a detailed proof of this fact we refer to [17]. We only remark that the entwining map in this case is defined by

$$\psi : C \otimes P \to P \otimes C, \quad c \otimes p \mapsto \text{can}(\tau(c) \cdot p),$$

where $\tau$ is the translation map of $C$-extension $B \subseteq P$. The bow-tie conditions from Definition 2.26 can now be verified using the Translation Map Lemma 2.22, while the uniqueness follows from the following simple argument. Suppose that there is an entwining map $\tilde{\psi}$ such that $P \in \mathcal{M}^C_P(\tilde{\psi})$ with structure maps $m_P$ and $\Delta_P$. Then, for all $p \in P, c \in C$,

$$\psi(c \otimes p) = c^1(c^2 p)(0) \otimes (c^2 p)(1) = c^1 c^2 (0) \tilde{\psi}(c^2 (1) \otimes p) = \tilde{\psi}(c \otimes p),$$

where we used the definition of the translation map to obtain the last equality.
Although in principle there is no relation between the coalgebra structure of \( C \) and the algebra structure of \( P \), the definition of the coalgebra-Galois extension is rigid enough to produce such a relationship in terms of an entwining. The entwining associated to the \( C \)-extension \( B \subseteq P \) in Theorem 2.31 is called the canonical entwining. Its existence allows one to discuss symmetry properties of coalgebra-Galois extensions, and to extend such symmetries to (universal) differential structures on the \( C \)-extension \( B \subseteq P \). This is crucial for the definition of a connection on \( B \subseteq P \).

To get a better feeling for canonical entwining structures it is instructive to consider the following

**Example 2.32.** Let \( B \subseteq P \) be a quotient-coalgebra-Galois \( H/I \)-extension. In this case, using the Translation Map Lemma and the fact that \( H/I \) is a right \( H \)-module, we can explicitly compute the formula (2.63) for the canonical entwining:

\[
\psi(h \otimes p) = \text{can} \left( h^{[1]} \otimes_B h^{[2]} p \right) = \left( h^{[1]} \otimes h^{[2]}(0) p(0) \right) \otimes \left( h^{[2]}(1) p(1) \right)
\]

(2.64)

Here \( p(0) \otimes p(1) \) is meant as the result of \( H \)-coaction on \( p \). Observe that whenever the antipode \( S \) of \( H \) is bijective, so is the above computed canonical entwining \( \psi \). Indeed, it is straightforward to verify that the formula \( \psi^{-1}(p \otimes h) = h^{-1} \otimes \left( p^{(1)} \right) \otimes p^{(0)} \) defines the inverse of \( \psi \). Note also that in particular, for \( I = 0 \), we get the canonical entwining of a Hopf-Galois extension, which is a relative Hopf entwining.

Very much as in the previous section, it turns out that every algebra-Galois \( A \)-coextension is equipped with an entwining structure. More precisely, we have the following dual version of Theorem 2.31:

**Theorem 2.33.** Let \( C \) be an algebra-Galois \( A \)-coextension of \( B \). Then there exists a unique map \( \psi : C \otimes A \to A \otimes C \) entwining \( C \) with \( A \) and such that \( C \in \mathcal{M}_C^A(\psi) \) with the structure maps \( \Delta \) and \( \mu_C \). (The map \( \psi \) is called the canonical entwining map associated to the algebra-Galois \( A \)-coextension \( C \to B \).)

Using the cotranslation map one defines a map \( \psi : C \otimes A \to A \otimes C \) by

\[
\psi = (\tilde{\tau} \otimes C) \circ (C \otimes \Delta) \circ \text{cocan},
\]

\[
\psi(c \otimes a) = \tilde{\tau}(c^{(1)}, \mu_C(c^{(2)}, a)_1) \otimes \mu_C(c^{(2)}, a)_2 .
\]

(2.65)

### 2.3.2 Factorisation

Given two algebras \( A, P \), one can study all possible algebra structures on the tensor product \( A \otimes P \) with unit \( 1 \otimes 1 \) and with the property that the multiplication becomes an \( (A, P) \)-bimodule map. It turns out (cf. [91], [64, pp. 299-300], [29]) that all such algebra structures are in one-to-one correspondence with maps \( \Psi : P \otimes A \to A \otimes P \) such that

\[
\Psi \circ (\mu_P \otimes \text{id}_A) = (\text{id}_A \otimes \mu_P) \circ (\Psi \otimes \text{id}_P) \circ (\text{id}_P \otimes \Psi), \quad \Psi(1 \otimes a) = a \otimes 1, \quad \forall a \in A \quad (2.66)
\]

\[
\Psi \circ (\text{id}_P \otimes \mu_A) = (\mu_A \otimes \text{id}_P) \circ (\text{id}_A \otimes \Psi) \circ (\Psi \otimes \text{id}_A), \quad \Psi(p \otimes 1) = 1 \otimes p, \quad \forall p \in P, \quad (2.67)
\]
where $\mu_A$ is the product in $A$ and $\mu_P$ is the product in $P$. A triple $(P, A, \Psi)$ is known as a factorisation structure. For example, every braided tensor product of algebras gives rise to a factorisation (given by braiding). Furthermore, crossed product algebras including bicrossproducts [63] correspond to factorisations.

There is a close relationship between factorisations and entwining structures. Let $(P, C, \psi)$ be an entwining structure with a finite-dimensional coalgebra $C$, and let $A = (C^*)^\text{op}$, i.e., the dual algebra with multiplication given by $(aa')(c) = a(c(2))a'(c(1))$, for $a, a' \in A$, $c \in C$. Define a map $\Psi : P \otimes A \to A \otimes P$, $p \otimes a \mapsto \sum_i a_i \otimes p_i$, where $\sum_i a_i(c)p_i = p_\alpha a(c_\alpha)$ for all $c \in C$. Then $\Psi$ is a factorisation, that is $A \otimes P$ has an algebra structure given by $(a \otimes p)(a' \otimes p') = \sum_i a a'_i \otimes p p'_i$.

The opposite statement holds for a factorisation $\Psi$ together with a finite dimensional algebra $A$. Then one can construct an entwining structure for the algebra $P$ and the coalgebra $C := A^\vee$.

This relationship between entwining structures and factorisations as well as the existence of a canonical entwining structure associated to a coalgebra-Galois extension allows one to develop the coalgebra-Galois theory on a purely algebraic (not coalgebraic) level, in terms of suitable factorisations. This point of view is taken and such a theory is developed in [23] (see Section 2.1.4).

Finally, we would like to remark that before the factorisations and entwining structures appeared in the current setup, similar structures were studied in category theory. In category theory in place of algebras one uses monads and in place of coalgebras one uses comonads. A structure corresponding to factorisation involves two monads and is known as distributive law [3] [1], while the structure corresponding to an entwining involves a monad and a comonad and is known as a mixed distributive law [93].

2.4 Principal extensions

The concept of a faithfully flat Hopf-Galois extension with a bijective antipode is a cornerstone of Hopf-Galois theory. The following notion of a principal extension generalises this key concept in a way that it encompasses interesting examples escaping Hopf-Galois theory, yet still enjoys a number of crucial properties of the aforementioned class of Hopf-Galois extensions. It is an elaboration of the Galois-type extension [17, Definition 2.3] which evolved from [81, p.182], [21] and other papers.

**Definition 2.34.** An $e$-coaugmented coalgebra-Galois $C$-extension $B \subseteq P$ is said to be principal if

(i) the canonical entwining map is bijective,

(ii) $P$ is $C$-equivariantly projective as a left $B$-module.

**Lemma 2.35.** Let $(A, C, \psi)$ be an entwining structure such that $\psi$ is bijective. Assume also that there exists a group-like element $e \in C$ such that $A$ is a right $C$-comodule via $\psi \circ (e \otimes \text{id})$ and a left $C$-comodule via $\psi^{-1} \circ (\text{id} \otimes e)$. Then $A$ is coflat as a right (resp. left) $C$-comodule if and only if there exists $j_R \in \text{Hom}^C(C, A)$ (resp. $j_L \in C\text{Hom}(C, A)$) such that $j_R(e) = 1$ (resp. $j_L(e) = 1$). (Here $C$ is a $C$-comodule via the coproduct.)

The comodule maps $j_R$ and $j_L$ are generalisations of total integrals of Doi. The latter are
used in [81] to prove coflatness results in the context of Hopf-Galois extensions. Analogous results can be proven also in our more general setting. The axioms of a principal extension guarantee that \((P,C,\psi)\) is an entwining structure satisfying the assumptions of the above lemma [17, Theorem 2.7]. Moreover, it can be shown that maps \(j_L\) and \(j_R\) as in Lemma 2.35 can be constructed for any principal \(C\)-extension, and one can prove the following [18]:

**Theorem 2.36.** Let \(B \subseteq P\) be a principal \(C\)-extension. Then:

1. \(P\) is a projective left and right \(B\)-module.
2. \(B\) is a direct summand of \(P\) as a left and right \(B\)-module.
3. \(P\) is a faithfully flat left and right \(B\)-module.
4. \(P\) is a coflat left and right \(C\)-comodule.

In (4), the left \(C\)-comodule structure of \(P\) is given by \(P \Delta(p) = \psi^{-1}(p \otimes e)\). Note that (3) follows from (1) and (2) by standard module-theoretic arguments.

### 2.4.1 Extensions by coseparable coalgebras

It turns out that in many cases of interest, for example, in those in which \(C\) corresponds to a coalgebra structure of a matrix quantum group, the injectivity of the canonical map implies its bijectivity. Recall that a coalgebra \(C\) is said to be *coseparable* provided the coproduct has a retraction in the category of \(C\)-bicomodules. Equivalently, \(C\) is a coseparable coalgebra if there exists a *cointegral*, i.e. a \(k\)-linear map \(\delta : C \otimes C \to k\) such that \(\delta \circ \Delta = \varepsilon\) and, for all \(c,c' \in C\),

\[
c_{(1)} \delta(c_{(2)} \otimes c') = \delta(c \otimes c'_{(1)})c'_{(2)}. \tag{2.68}
\]

Any cosemisimple coalgebra over an algebraically closed field is coseparable (cf. [79, Proposition 2.5.3]).

**Theorem 2.37** ([16], Theorem 4.6). Let \((P,C,\psi)\) an entwining structure such that the map \(\psi\) is bijective. Suppose that \(e \in C\) is a group-like element and view \(P\) as a right \(C\)-comodule with the coaction \(\Delta_P : P \to P \otimes C\), \(p \mapsto \psi(e \otimes p)\). If \(C\) is a coseparable coalgebra and the lifted canonical map

\[
\tilde{\text{can}} : P \otimes P \to P \otimes C, \quad p \otimes q \mapsto p \Delta_P(q),
\]

is surjective, then \(P\) is a principal \(C\)-extension of the coaction-invariant subalgebra \(B = P^{coC}\).

Theorem 2.37 is proven in [16] as a special case of a general structure theorem for principal comodules for a coring (see Theorem 2.48 below). It is also proven (over a commutative ring) in [80, Theorem 5.9]. The discussion of this proof goes beyond the scope of these notes. On the other hand, a direct proof of Theorem 2.37 has been recently presented in [4]. This uses the explicit form of a strong connection which we describe in more detail in Section 3.2.3.

The example of a noncommutative instanton bundle [11] described in Section 2.1.2 is principal. Since this is the situation of a quotient coalgebra extension, the group-like element in the coalgebra \(C = \mathcal{O}(SU_q(4))/I\) is the image of 1 under the canonical projection, and the invertibility of the canonical entwining map follows from the invertibility of the antipode of the Hopf algebra \(\mathcal{O}(SU_q(4))\) (see Example 2.32). The Galois property has been proven in [11] by brute force. The hard part of this proof was to verify the injectivity of the canonical map \(\text{can}\). Since the coalgebra \(C = \mathcal{O}(SU_q(4))/I\) is cosemisimple, this part is a direct consequence of Theorem 2.37. The same theorem also gives equivariant projectivity.
2.4.2 Hopf fibrations over the Podleś quantum 2-spheres

In our presentation we follow [23], where the reader can find detailed proofs of all the facts quoted. The total space of the bundle is given by the Hopf algebra of functions on the quantum group \( SL_q(2) \), \( P = \mathcal{O}(SL_q(2)) \). Recall that \( \mathcal{O}(SL_q(2)) \) is defined as a polynomial algebra with unit, generated by \( \alpha, \beta, \gamma \) and \( \delta \), with relations \( \alpha \beta = q \beta \alpha, \alpha \gamma = q \gamma \alpha, \beta \gamma = q \beta \gamma, \alpha \delta = \delta \alpha - (q - q^{-1}) \beta \gamma, \gamma \delta = q \delta \gamma, \beta \delta = q \delta \beta, \alpha \delta - q \beta \gamma = 1 \). Here \( q \) is any number which is not a root of unity. In the case of \( k = \mathbb{C} \), the algebra \( \mathcal{O}(SL_q(2)) \) can be made into a \( C^* \)-algebra \( C(SU_q(2)) \), provided \( q \in (0, 1) \).

Coideal subalgebras of \( P \) or (embeddable) homogeneous spaces of the quantum group \( \mathcal{O}(SL_q(2)) \) are known as quantum or Podleś spheres and were introduced in [71]. They are defined as subalgebras \( B = S_{q,s} \), where \( s \in k \) is a parameter, which are embedded in \( P \) and generated by

\[
\begin{align*}
\xi &= s(\alpha^2 - q^{-1} \beta^2) + (s^2 - 1) q^{-1} \alpha \beta, \\
\eta &= s(q \gamma^2 - \delta^2) + (s^2 - 1) \gamma \delta, \\
\zeta &= s(q \alpha \gamma - \beta \delta) + (s^2 - 1) q \beta \gamma.
\end{align*}
\]

(2.69)

In the case of the \( C^* \)-algebra \( C(SU_q(2)) \), \( S_{q,s} \) can be made into \( C^* \)-subalgebras of \( C(SU_q(2)) \), provided \( s \in [0, 1] \). The coideal \( I = B^+ P \) can be computed as \( I = (\langle \xi - s, \eta + s, \zeta \rangle P \). The corresponding quotient coalgebra \( C = P/I \) is spanned by group-like elements

\[
\begin{align*}
g_0 &= \pi_I(1), & g_n &= \pi_I(\prod_{k=0}^{n-1}(\alpha + q^k s \beta)), & g_{-n} &= \pi_I(\prod_{k=0}^{n-1}(\delta + q^{-k} s \gamma)),
\end{align*}
\]

(2.70)

where the multiplication increases from left to right (e.g., \( g_2 = \pi((\alpha + s \beta)(\alpha + q s \beta)) \), etc.). The coalgebra \( C \) can be equipped with numerous Hopf-algebra structures. For example, \( C \) can be an algebra generated by two elements \( Z \) and \( Z^{-1} \) such that \( ZZ^{-1} = 1 = Z^{-1} Z \) by setting \( Z^n = g_n \) and \( Z^{-n} = g_{-n} \). In this way, \( C \) can be viewed as an algebra of functions on the circle, i.e. \( C = \mathcal{O}(S^1) \) (or \( C = C(U(1)) \) in the \( C^* \)-algebra case). The constructed coalgebra-Galois extensions is therefore known as a quantum Hopf fibration, since the classical Hopf fibration is a principal bundle over the two-sphere with the circle as a fibre and the three-sphere \( SU(2) \) as the total space. Note that \( \pi_I \) is a Hopf-algebra map only if \( s = 0 \). This corresponds to the standard Podleś sphere.

2.5 The Galois condition in the setting of corings

In this section we outline some general properties of entwined modules, view them as a special case of comodules of corings, and then describe a generalisation of Hopf algebras in which the ground field is replaced by a noncommutative algebra. For details we refer to [25]. This section has a purely algebraic flavour and can be skipped by a more geometrically oriented reader (mind, however, that Example 2.53 is certainly of geometric origin and interest).

2.5.1 The structure theorems for entwined modules

Suppose that we have an entwining structure \((P,C,\psi)\) such that \( P \) itself is a \((P,C,\psi)\)-entwined module. This means in particular that \( P \) is a right \( C \)-comodule with the structure map \( \Delta_P : \)
$P \to P \otimes C$, and one can consider the subalgebra of coaction invariants of $P$, $B = P^{\co C}$. In this case the coaction invariants have an alternative description.

**Lemma 2.38.** Let $(P, C, \psi)$ be an entwining structure such that $P$ is a $(P, C, \psi)$-entwined module. Then

$$P^{\co C} = \{ b \in P \mid \Delta_P(b) = b\Delta_P(1) = b1(0) \otimes 1(1) \}.$$  

**Proof.** Clearly, if $b \in P^{\co C}$, then $\Delta_P(b) = b\Delta_P(1)$. Conversely, if $\Delta_P(b) = b\Delta_P(1)$, then

$$\Delta_P(bp) = b(0)p_a \otimes b(1) = b1(0)p_a \otimes 1(a) = b\Delta_P(1)p = b\Delta_P(p)$$

for all $p \in P$, as required. \hfill \Box

Note that in an $e$-coaugmented coalgebra-Galois extension, $\Delta_P(1) = 1 \otimes e$. Thus Lemma 2.38 combined with Theorem 2.31 imply Proposition 2.23, i.e., that in the case of an $e$-coaugmented coalgebra-Galois $C$-extension both definitions of coaction invariants coincide.

Given an entwining structure $(P, C, \psi)$ such that $P \in M^C_P(\psi)$, one can consider two functors between the categories of right $B = P^{\co C}$-modules and the entwined modules. First, there is the induction functor $- \otimes_B P : M_B \to M^C_P(\psi)$, which to each right $B$-module $N$ assigns a $(P, C, \psi)$-entwined module $N \otimes_B P$, and to a right $B$-module morphism $f$, a morphism of entwined modules $f \otimes_B id_N$. Here $N \otimes_B P$ is a right $P$-module by multiplication in $P$, i.e., $(n \otimes_B p) \cdot p' = n \otimes_B pp'$, and a right $C$-comodule with the coaction $\varrho^{N \otimes_B P} = id_N \otimes_B \Delta_P$. In the opposite direction, there is a coaction-invariants functor $(\cdot)^{\co C} : M^C_P(\psi) \to M_B$ that to each $M \in M^C_P(\psi)$ assigns the right $B$-module

$$M^{\co C} := \{ m \in M \mid \varrho^M(m) = m\Delta_P(1) \}.$$  

Note that $M^{\co C}$ is a right $B$-module by the definition of $B$ as subalgebra of coaction invariants of $P$. One can easily show that the coaction-invariants functor is the right adjoint of the induction functor, i.e., for any $N \in M_B$ and $M \in M^C_P(\psi)$, there is an isomorphism of vector spaces $\text{Hom}_B^C(N \otimes_B P, M) \cong \text{Hom}_B(N, M^{\co C})$, natural in $M$ and $N$. Here $\text{Hom}_B^C(\cdot, \cdot)$ denotes all right $P$-module right $C$-comodule maps. Perhaps the easiest way of seeing that this is so is to realise that the coaction invariants functor can be identified with the Hom-functor, $M^{\co C} = \text{Hom}_B^C(M, P)$, and then use the standard Hom-tensor relations. Explicitly, the unit of the adjunction reads, for all $N \in M_B$,

$$\eta_N : N \to (N \otimes_B P)^{\co C}, \quad n \mapsto n \otimes 1,$$

while, for all $M \in M^C_P(\psi)$, the counit of the adjunction reads

$$\sigma_M : M^{\co C} \otimes_B P \to M, \quad m \otimes p \mapsto m \cdot p.$$  

The structure theorems for entwined modules deal with the properties of the above adjoint pair of functors. The first theorem determines when the coaction-invariants functor is fully faithful, while the second theorem determines when the above adjunction is an equivalence of categories, i.e., when $\eta_N$ and $\sigma_M$ are isomorphisms (natural in $M$ and $N$). In this way one obtains a generalised version of Schneider’s theorem [81, 3.7 Theorem].

**Theorem 2.39.** Let $(P, C, \psi)$ be an entwining structure such that $P \in M^C_P(\psi)$, and define $B := P^{\co C}$. Then the following are equivalent:

---

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(a) $B \subseteq P$ is a coalgebra-Galois $C$-extension, $\psi$ is the canonical entwining map associated to $B \subseteq P$, and $B \otimes P$ is flat;

(b) the functor $(-)^{coC} : M^P_C(\psi) \to M_B$ is fully faithful, i.e., for all $M \in M^P_C(\psi)$, the counit of adjunction $\sigma^M : M^{coC} \otimes_B P \to M$ is an isomorphism.

**Proof.** Assume that $B \subseteq P$ is a coalgebra-Galois $C$-extension, and that $P$ is a flat left $B$-module. For all $M \in M^C_P(\psi)$, one can view $M \otimes C$ as a right $P$-module (as a $(P,C,\psi)$-entwined module in fact) with the structure map $(m \otimes c) \cdot p = m \cdot p_a \otimes c^a$. In particular, $P \otimes C$ is a right $P$-module, and the reader can check that the canonical map $can : P \otimes_B P \to P \otimes C$ is a right $P$-module map (i.e., it is a $(P,P)$-bimodule bijection). Now consider the following commuting diagram of right $C$-comodule right $P$-module maps

\[
\begin{array}{cccc}
0 & \longrightarrow & M^{coC} \otimes_B P & \longrightarrow & M \otimes_B P & \longrightarrow & (M \otimes C) \otimes_B P \\
& & \sigma^M \downarrow & & \text{id}_M \otimes_P can \downarrow & & (\text{id}_M \otimes \text{id}_C) \otimes_P \text{can} \\
0 & \longrightarrow & M & \longrightarrow & M \otimes C & \longrightarrow & \ell_{MC}M \otimes C \otimes C.
\end{array}
\]  

(2.71)

The maps in the top row are the obvious inclusion and

\[
m \otimes p \mapsto \varphi^M(m) \otimes_B p - m \cdot \Delta_P(1) \otimes_B p,
\]

\[
(2.72)
\]

while $\ell_{MC} = \varphi^M \otimes \text{id}_C - \text{id}_M \otimes \Delta$ is the coaction equalising map. The top row is exact since it is a defining sequence of $M^{coC}$ tensored with $P$ over $B$, and the functor $- \otimes_B P$ is exact. The bottom row is exact too. Since the canonical map $can$ is a $(P,P)$-bimodule and right $C$-comodule, also the maps $\text{id}_M \otimes_P can$ and $(\text{id}_M \otimes \text{id}_C) \otimes_P \text{can}$ are right $C$-comodule right $P$-module bijections. Therefore, $\sigma^M$ is an isomorphism in $M^P_C(\psi)$, i.e., $(-)^{coC}$ is fully faithful, as required.

Conversely, assume that $(-)^{coC}$ is fully faithful. Note that $P \otimes C$ is an entwined module with the action $(p \otimes c) \cdot p' = pp'_a \otimes c$, and the coaction $\varphi^{P \otimes C} = \text{id}_P \otimes \Delta$. Therefore, there is a corresponding counit of adjunction $\sigma_{P \otimes C} : (P \otimes C)^{coC} \otimes_B P \to P \otimes C$, and it is bijective. Next consider the map $\varphi : P \to (P \otimes C)^{coC}$, given by $p \mapsto p\Delta_P(1)$. This map is well defined since

\[
(p1_{(0')} \otimes 1_{(1')}) \cdot 1_{(0)} \otimes 1_{(1)} = p1_{(0')}1_{(0)\alpha} \otimes 1_{(1)\alpha} \otimes 1_{(1)} = p1_{(0)} \otimes 1_{(1)} \otimes 1_{(2)} = \varphi^{P \otimes C}(p1_{(0)} \otimes 1_{(1)}).
\]

Here $1_{(0')} \otimes 1_{(1')}$ denotes another copy of $\Delta_P(1)$, and we used the definition of the $P$-action on $P \otimes C$ and the fact that $P$ is an entwined module. Clearly, $\varphi$ is a left $P$-module map. It is also right $B$-linear since, for all $b \in B$ and $p \in P$, we have

\[
\varphi(p) \cdot b = (p1_{(0)} \otimes 1_{(1)}) \cdot b = p1_{(0)} b_\alpha \otimes 1_{(1)\alpha} = pb_{(0)} \otimes b_{(1)} = pb1_{(0)} \otimes 1_{(1)} = \varphi(pb).
\]

Here we used that $P$ is an entwined module and the definition of coaction invariants in $P$. Finally, $\varphi$ is an isomorphism of $(P,B)$-bimodules with the inverse $\varphi^{-1} = \text{id}_P \otimes \varepsilon$. Now take any $p,p' \in P$ and compute

\[
\sigma_{P \otimes C} \circ (\varphi \otimes_B \text{id}_P)(p \otimes_B p') = \sigma_{P \otimes C}(p1_{(0)} \otimes 1_{(1)} \otimes_B p') = (p1_{(0)} \otimes 1_{(1)}) \cdot p' = p1_{(0)}p'_{(0)} \otimes 1_{(1)\alpha} = pp'_{(0)} \otimes p'_{(1)},
\]

i.e., $can = \sigma_{P \otimes C} \circ (\varphi \otimes_B \text{id}_P)$. Therefore, the canonical map is a composition of isomorphisms and hence an isomorphism as required. Thus the extension $B \subseteq P$ is Galois.
To prove that \( BP \) is flat one uses the following argument from the category theory. Note that both kernels and cokernels of any morphism in \( \mathcal{M}_P^C(\psi) \) are \((P, C, \psi)\)-entwined modules, i.e., \( \mathcal{M}_P^C(\psi) \) is an Abelian category. In fact, one can prove that \( \mathcal{M}_P^C(\psi) \) is a Grothendieck category. Therefore, any sequence of \((P, C, \psi)\)-entwined module maps is exact if and only if it is exact as a sequence of additive maps. Thus to prove that \( BP \) is flat suffices it to show that the functor \(- \otimes_B P : \mathcal{M}_B \to \mathcal{M}_P^C(\psi)\) is exact. Note, however, that \(- \otimes_B P\) is a left adjoint of a fully faithful functor, and since \( \mathcal{M}_P^C(\psi) \) is a Grothendieck category, the functor \(- \otimes_B P\) is exact by the Gabriel-Popescu theorem (cf. [44, Theorem 15.26]).

**Theorem 2.40.** Let \((P, C, \psi)\) be an entwining structure such that \( P \in \mathcal{M}_P^C(\psi) \), and define \( B := P^{\text{co}C} \). Then the following are equivalent:

(a) \( B \subseteq P \) is a coalgebra-Galois \( C \)-extension, \( \psi \) is the canonical entwining map associated to \( C \)-extension \( B \subseteq P \) and \( BP \) is faithfully flat;

(b) the functor \(- \otimes_B P : \mathcal{M}_B \to \mathcal{M}_P^C(\psi)\) is a category equivalence.

**Proof.** Assume that \( B \subseteq P \) is a coalgebra-Galois \( C \)-extension, and that \( P \) is a faithfully flat left \( B \)-module. For all \( N \in \mathcal{M}_B \), consider the following commutative diagram of right \( B \)-module maps

\[
\begin{array}{c}
0 \rightarrow N \xrightarrow{\eta_N} N \otimes_B P \xrightarrow{\text{id}_N \otimes \text{can}} N \otimes_B P \otimes_B P \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow (N \otimes_B P)^{\text{co}C} \xrightarrow{\sigma_M} N \otimes_B P \xrightarrow{\text{id}_N \otimes \text{can}} N \otimes_B P \otimes B C.
\end{array}
\]

The maps in the top row are: \( n \mapsto n \otimes_B 1 \) and \( n \otimes_B P \mapsto n \otimes_B P \otimes_B 1 - n \otimes_B 1 \otimes_B P \), and the top row is exact by the faithfully flat descent. The bottom row is the defining sequence of \((N \otimes_B P)^{\text{co}C}\) and hence is exact. This implies that the unit of adjunction \( \eta_N \) is an isomorphism in \( \mathcal{M}_B \). Since \( BP \) is flat, also \( \sigma_M \) is an isomorphism for all \( M \in \mathcal{M}_P^C(\psi) \) by Theorem 2.39. Therefore \((-)^{\text{co}C}\) and \(- \otimes_B P\) are inverse equivalences.

Conversely, assume that \((-)^{\text{co}C}\) and \(- \otimes_B P\) are inverse equivalences. Then by Theorem 2.39 the extension \( B \subseteq P \) is Galois, and \( BP \) is flat. Since \(- \otimes_B P\) is an equivalence, it also reflects exact sequences. Therefore \( P \) is a faithfully flat left \( B \)-module.

### 2.5.2 Corings and Galois comodules

This part of the text is devoted to the introduction of an algebraic structure which helps to understand better properties of entwining structures and entwined modules. This structure will play no further role in studies of geometric aspects of coalgebra-Galois extensions presented below, but it is currently a subject of intensive studies. Interested readers are referred to [15] and to the monograph [25].

**Definition 2.41.** Let \( A \) be an algebra. An \((A, A)\)-bimodule \( \mathcal{C} \) is called an \( A \)-coring if there exist \((A, A)\)-bimodule maps

\[
\Delta_c : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C} \quad \text{and} \quad \varepsilon_c : \mathcal{C} \rightarrow A
\]

such that

\[
(\Delta_c \otimes_A \text{id}_c) \circ \Delta_c = (\text{id}_c \otimes_A \Delta_c) \circ \Delta_c, \quad (\varepsilon_c \otimes_A \text{id}_c) \circ \Delta_c = \text{id}_c = (\text{id}_c \otimes_A \varepsilon_c) \circ \Delta_c,
\]

i.e., \( \Delta_c \) and \( \varepsilon_c \) satisfy the axioms for a coproduct and counit.
The term coring was coined by Sweedler in the context of a semi-dual version of the Jacobson-Bourbaki theorem [86]. In late seventies, corings resurfaced under the name of bimodules over a category with a coalgebra structure, or BOCS’s for short, in the work of Rojter [73] and Kleiner [56] on algorithms for matrix problems. Corings play particularly important role in the theory of ring extensions, and the canonical example comes from such an extension.

**Example 2.42.** Consider an algebra extension $B \hookrightarrow A$. Let $C := A \otimes_B A$ with the obvious $(A, A)$-bimodule structure. Then $C$ is an $A$-coring with the coproduct

$$
\Delta_C : C \to C \otimes_A C \cong A \otimes_B A \otimes_B A, \quad a \otimes_B a' \mapsto a \otimes_B 1 \otimes_B a',
$$

and the counit

$$
\varepsilon_C : C \to A, \quad a \otimes_B a' \mapsto aa'.
$$

This coring is called the *canonical coring* associated to an extension of algebras $B \hookrightarrow A$, or simply the *Sweedler coring*.

For a long time, essentially only two types of examples of corings truly generalising coalgebras were known — one associated to a ring extension as in Example 2.42, the other associated to a matrix problem. The latter example was also studied in the context of differential graded algebras. This lack of examples obviously hampered the progress in general coring theory. At the end of the nineties, however, M. Takeuchi made a remarkable observation that connects entwining structures with corings, and thus provides one with a reach, new source of examples of corings. More precisely we have (cf. [15, Proposition 2.2])

**Theorem 2.43** (Takeuchi). Let $(A, C, \psi)$ be an entwining structure. Then $C := A \otimes C$ becomes an $A$-coring with the following structure:

1. $a \cdot (a' \otimes c) \cdot a'' = aa'\psi(c \otimes a'')$,
2. $\Delta_C : C \to C \otimes_A C$, $a \otimes c \mapsto a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)}$,
3. $\varepsilon_C : C \to A$, $a \otimes c \mapsto \varepsilon(c)a$.

Conversely, if $A$ is an algebra, $C$ a coalgebra, and $C = A \otimes C$ has the structure of an $A$-coring, then an entwining map $\psi$ is given by $\psi : C \otimes A \to A \otimes C$, $c \otimes a \mapsto (1 \otimes c) \cdot a$.

**Proof.** It is obvious that $A \otimes C$ is a left $A$-module with the specified action. The following simple calculations, performed for any $a, a', a'' \in A$ and $c \in C$,

$$(a \otimes c) \cdot (a' a'') = a (a' a'')_a \otimes c^a = aa' a''_\beta \otimes c^{a\beta} = (aa'_\alpha \otimes c^\alpha) a'' = ((a \otimes c) \cdot a') \cdot a''$$

and

$$(a \otimes c) \cdot 1 = a1_\alpha \otimes c^\alpha = a \otimes c$$

prove that $A \otimes C$ is a right $A$-module. Note how the left pentagon was used to derive the first result and the left triangle to obtain the second one. Thus $C$ is an $(A, A)$-bimodule.

Next one has to check that $\varepsilon_C$ and $\Delta_C$ are $(A, A)$-bimodule maps. Clearly, they are left $A$-linear. Take any $a, a' \in A$, $c \in C$, and compute

$$
\varepsilon_C((a \otimes c) \cdot a') = \varepsilon_C(aa'_\alpha \otimes c^\alpha) = aa'_\alpha \varepsilon(c^\alpha) = aa'\varepsilon(c) = \varepsilon_C(a \otimes c)a'.
$$
Here the right triangle was used to establish the penultimate equality. Furthermore,
\[
\Delta_C((a \otimes c) \cdot a') = a a'_\alpha \otimes c^\alpha(1) \otimes c^\alpha(2) = a a'_\alpha \otimes c(1) \otimes c(2)^\alpha = (a \otimes c(1)) \cdot a = (a \otimes c(1)) \otimes_A (1 \otimes c(2)) \cdot a' = \Delta_C(a \otimes c) \cdot a'.
\]

Here the second equality follows from the right pentagon. Thus \(\varepsilon_C\) and \(\Delta_C\) are \((A, A)\)-bimodule morphisms, as required. Now, the coassociativity of \(\Delta\) is a counit of \(\varepsilon_C\) and \(\Delta_C\) is an entwining structure, as claimed.

As an application of Theorem 2.43 one obtains a quick proof of the existence of the canonical entwining structure in Theorem 2.31. Indeed, for a coalgebra-Galois \(C\)-extension \(B \subseteq P\), Sweedler’s canonical coring \(C = P \otimes_B P\) induces through the bijectivity of \(\text{can} : P \otimes_B P \to P \otimes C\) a coring structure on \(P \otimes C\). Then Theorem 2.43 implies that there is an entwining structure \((P, C, \psi)\). This is precisely the canonical entwining structure \((P, C, \psi)\) from Theorem 2.31.

Similarly as for coalgebras, one can study the corepresentation theory of corings. Given an \(A\)-coring \(C\), a right \(C\)-comodule is a right \(A\)-module \(M\), together with a right \(A\)-module map \(\varrho^M : M \to M \otimes_A C\) satisfying the following axioms for a coaction:
\[
\begin{align*}
(\text{id}_M \otimes_A \Delta_C) \circ \varrho^M &= (\varrho^M \otimes_A \text{id}_C) \circ \varrho^M, \\
(\text{id}_M \otimes \varepsilon_C) \circ \varrho^M &= \text{id}_M .
\end{align*}
\]

A morphism of right \(C\)-comodules is a right \(A\)-module map \(f : M \to N\) such that \(\varrho^N \circ f = (f \otimes_A \text{id}_C) \circ \varrho^M\). The category of \(C\)-comodules is denoted by \(\text{MC}^C\).

The category of comodules of Sweedler’s coring is familiar from the (noncommutative) descent theory (cf. [13]).

Example 2.44. ([25, 25.3]) The category of comodules over Sweedler’s canonical coring \(C = A \otimes_B A\) is isomorphic to the category of (right) descent data \(\text{Desc}(A/B)\). The objects in category \(\text{Desc}(A/B)\), known as descent data, are pairs \((M, f)\), where \(M\) is a right \(A\)-module, and \(f : M \to M \otimes_B A\) is a right \(A\)-module morphism satisfying the following conditions. Let, for any \(m \in M\), \(f(m) = \sum_i m_i \otimes_B a_i\). Then
Descent theory provides answers to the following types of questions.

(i) \( \sum_i f(m_i) \otimes_B a_i = \sum_i m_i \otimes_B 1 \otimes_B a_i; \)

(ii) \( \sum_i m_i \cdot a_i = m. \)

A morphism \((M, f) \rightarrow (M', f')\) in \(\text{Desc}(A/B)\) is a right \(A\)-module map \(\varphi : M \rightarrow M'\) such that \(f' \circ \varphi = (\varphi \otimes_B \text{id}_A) \circ f.\)

The category \(\text{Desc}(A/B)\) is a noncommutative generalisation [30] of the category of descent data associated to an extension of commutative rings introduced by Knus and Ojanguren in [57], and forms a backbone of the noncommutative extension of the classical descent theory [47] [48]. Descent theory provides answers to the following types of questions.

- **Descent of modules**: given an algebra extension \(B \rightarrow A\) and a right \(A\)-module \(M\), is there a right \(B\)-module \(N\) such that \(M \cong N \otimes_B A\) as right \(A\)-modules?

- **Classification of \(A\)-forms**: given a right \(B\)-module \(N\), classify all right \(B\)-modules \(M\) such that \(N \otimes_B A \cong M \otimes_B A\) as right \(A\)-modules.

For recent developments in the descent theory we refer to [70]. Thus corings shed a new light and give new tools for studies of the descent theory.

From the point of view of coalgebra-Galois extensions, more important is the following

**Proposition 2.45.** For an entwining structure \((A, C, \psi)\) and its associated coring \(C = A \otimes C\), the category of \(C\)-comodules \(M^C\) is isomorphic to the category of entwined modules \(M_\psi^A\). Hence, the theory of entwined modules can be viewed as a special case of corepresentation theory of corings.

**Proof.** The key observation here is that, if \(M\) is a right \(A\)-module, then \(M \otimes C\) is a right \(A\)-module with the multiplication \((m \otimes c) \cdot a = m \cdot a \otimes c^a\). The statement \(M\) is an \((A, C, \psi)\)-entwined module is equivalent to the statement that \(g^M\) is a right \(A\)-module map. Using the canonical identification \(M \otimes C \cong M \otimes_A A \otimes C = M \otimes_A C\), one can view a right \(C\)-coaction as a right \(A\)-module map \(M \rightarrow M \otimes_A C\), i.e., as a right \(C\)-coaction. Conversely a right \(C\)-coaction can be viewed as a right \(A\)-module map \(g^M : M \rightarrow M \otimes C\) thus providing a right \(C\)-comodule with the structure of an \((A, C, \psi)\)-entwined module. \(\square\)

In view of Example 2.45, several properties of entwined modules can be derived from corresponding properties of more general comodules of a coring. For instance, Theorem 2.39 and Theorem 2.40 are special cases of the structure theorems for corings which are isomorphic to the Sweedler coring and known as Galois corings (cf. [15, Theorem 5.6] and [25, 28.19]). These, in turn, are special cases of a more general construction which extends both the canonical Sweedler coring and Galois-type extensions, and leads to structure theorems from which structure theorems for entwined modules, such as Theorem 2.39 and Theorem 2.40, can be derived as corollaries.

**Definition 2.46 ([40]).** Let \(C\) be an \(A\)-coring, \(M\) be a right \(C\)-comodule and let \(B = \text{End}^C(M)\) be the endomorphism ring of \(M\). View \(C\) as a right \(C\)-comodule with the regular coaction \(\Delta_C\). The comodule \(M\) is called a Galois (right) comodule if \(M\) is a finitely generated and projective right \(A\)-module, and the evaluation map

\[ \varphi_C : \text{Hom}^C(M, C) \otimes_B M \rightarrow C, \quad f \otimes_B m \mapsto f(m), \]

is an isomorphism of right \(C\)-comodules.
An equivalent definition of Galois comodules is obtained by first noting that \( M \) is a \((B, A)\)-bimodule and \( \text{Hom}^C(M, C) \simeq M^* = \text{Hom}_A(M, A) \) as \((A, B)\)-bimodules. If \( M_A \) is finitely generated projective, then \( M^* \otimes_B M \) is an \( A \)-coring with the coproduct \( \Delta_{M^* \otimes_B M}(\xi \otimes_B m) = \sum_i \xi \otimes_B e^i \otimes_A \xi^i \otimes_B m \), where \( \{ e^i \in M, \xi^i \in M^* \} \) is a dual basis of \( M_A \), and with the counit \( \varepsilon_{M^* \otimes_B M}(\xi \otimes_B m) = \xi(m) \) (cf. [40]). The coring \( M^* \otimes_B M \) is known as a comatrix coring. The map \( \varphi_C \) reduces to the canonical \( A \)-coring morphism

\[
\text{can}_M : M^* \otimes_B M \longrightarrow C, \quad \xi \otimes_B m \longmapsto \sum \xi(m_{(0)})m_{(1)}.
\]

A finitely generated projective right \( A \)-module \( M \) is a Galois comodule if and only if the canonical map \( \text{can}_M \) is an isomorphism of corings.

To see the relationship of Definition 2.46 with the coalgebra-Galois theory, take \( C = A \otimes C \), the coring corresponding to an entwining structure \((A, C, \psi)\). Assume \( A \) is an entwined module, hence a \( C \)-comodule, and take \( M = A \). This is obviously a finitely generated projective right \( A \)-module. The endomorphism ring \( B = \text{End}^C(A) \) can be easily identified with the coaction-invariant subalgebra, i.e., \( B \simeq A^{coC} \). Since \( A^* \simeq A \), the corresponding comatrix coring can be identified with the Sweedler coring \( A \otimes_B A \) of the extension \( B \hookrightarrow A \), and the canonical map \( \text{can}_A \) with the canonical map of the coalgebra-Galois theory. Thus \( A \) is a Galois comodule if and only if the \( C \)-extension \( B \hookrightarrow A \) is Galois. In view of this, Theorem 2.39 and Theorem 2.40 are corollaries of the following Galois Comodule Structure Theorem which, in part, was first formulated in [40, Theorem 3.2].

**Theorem 2.47 (The Galois Comodule Structure Theorem).** Let \( C \) be an \( A \)-coring and \( M \) be a right \( C \)-comodule that is finitely generated and projective as a right \( A \)-module. Set \( B = \text{End}^C(M) \).

1. The following are equivalent:
   a. \( M \) is a Galois comodule that is flat as a left \( B \)-module.
   b. \( C \) is a flat left \( A \)-module and \( M \) is a generator in \( M^C \).
   c. \( C \) is a flat left \( A \)-module and, for any \( N \in M^C \), the evaluation map \( \varphi_N : \text{Hom}^C(M, N) \otimes_B M \rightarrow N, f \otimes_B m \mapsto f(m) \), is an isomorphism of right \( C \)-comodules.
   d. \( C \) is a flat left \( A \)-module and the functor \( \text{Hom}^C(M, -) : M^C \rightarrow M_B \) is fully faithful.

2. The following are equivalent:
   a. \( M \) is a Galois comodule that is faithfully flat as a left \( B \)-module.
   b. \( C \) is a flat left \( A \)-module and \( M \) is a projective generator in \( M^C \).
   c. \( C \) is a flat left \( A \)-module and \( \text{Hom}^C(M, -) : M^C \rightarrow M_B \) is an equivalence with the inverse \( - \otimes_B M : M_B \rightarrow M^C \).

For the proof of this theorem we refer to [25, 18.27]. The assertion (d) in Theorem 2.47 (1) is simply a rephrasing of the assertion (c), since the natural map \( \varphi \) is the counit of the adjunction \((- \otimes_B M \dashv \text{Hom}^C(M, -)) \).

Thus Galois comodules provide one with a very general and conceptually clear point of view on Galois-type extensions. Going further in this direction one introduces the notion of a principal comodule as a Galois \( C \)-comodule \( M \) which is projective as a (left) module over its coaction-invariant algebra \( B = \text{End}^C(M) \). Principal extensions are examples of such comodules. Principal comodules are characterised by the following theorem proven in [16]:
Theorem 2.48. Let $C$ be an $A$-coring and $M$ a right $C$-comodule that is finitely generated and projective as a right $A$-module. Set $B = \text{End}^C(M)$.

(1) If $M$ is a principal comodule, then it is faithfully flat as a left $B$-module.

(2) View $M^* \otimes M$ as a left $C$-comodule via $M^*_g \otimes \text{id}_M$, where $M^*_g : M^* \to C \otimes_A M^*$ is the left coaction induced from the right $C$-coaction $\varrho^M : M \to M \otimes_A C$. Then the following statements are equivalent:

(a) The map $\widehat{\text{can}}_M : M^* \otimes M \to C$, $\xi \otimes m \mapsto \sum \xi(m_{(0)})m_{(1)}$, is a split epimorphism of left $C$-comodules.

(b) $M$ is a principal right $C$-comodule.

Several generalisations of Galois comodules discussed above have been introduced recently. Comatrix corings built out of modules which are not required to be finitely generated projective and the corresponding Galois comodules are discussed in [41]. This led in a natural way to considering corings and Galois comodules over firm rings without a unit in [46]. Going in a slightly different direction, the definition of a Galois comodule was proposed in [94]. It is required there that a certain morphism of functors, $\text{Hom}_A(M, -) \otimes_B M \to - \otimes_A C$, where $C$ is an $A$-coring, $M$ is a right $C$-comodule, and $B$ is its endomorphism algebra, is an isomorphism.

Yet another approach to the Galois condition for corings, deeply rooted in noncommutative (affine) algebraic geometry was recently developed by T. Maszczyk [66]. The idea is to depart from considering comodules and to start with a morphism of two $A$-corings $\widetilde{\gamma} : D \to C$. With such a morphism one can view $D$ as a $C$-$C$ bicomodule in a natural way and then define a ring $B = \mathcal{H}om_C(D, C)$. $D$ is a $B$-$B$ bimodule and one can consider the quotient $A$-coring $D/[D, B]$. The map $\gamma$ descends to this quotient, and one says that a Galois condition is satisfied if the resulting $A$-$A$ bimodule map $\gamma : D/[D, B] \to C$ is bijective (an isomorphism of $A$-corings). In particular, if one starts with a right $C$-comodule $M$ that is finitely generated and projective as a right $A$-module and sets $D := M^* \otimes M$ and $\widetilde{\gamma} := \widehat{\text{can}}_M$, then $B$ is isomorphic to $\text{End}^C(M)$, and $D/[D, B]$ can be identified with $M^* \otimes_B M$. The resulting $\gamma$ is the canonical map $\text{can}_M$.

All these definitions of the Galois condition for corings can be understood as instances of a Galois condition for comonads in general categories. The latter is discussed in detail in [45].

Remark 2.49. We have already discussed the problem that one cannot use group-like elements in a coalgebra $C$ when defining invariants in the context of a coalgebra Galois extension $C$-extension $B \subseteq P$. On the other hand, studying the corresponding coring $C = P \otimes C$, as given in Theorem 2.43, yields the advantage that there actually exists a group-like element $g := \Delta_P(1) \in P \otimes C$. A group-like element in a general $P$-coring $C$ is defined in an analogous way as in a coalgebra, i.e., $g \in C$ is group-like, if $\Delta_C(g) = g \otimes_P g$ and $\varepsilon_C(g) = 1$. One easily checks that in the case $g = 1_{(0)} \otimes 1_{(1)}$

$$g \otimes_P g = (1_{(0)}) \otimes (1_{(1)}) \otimes_P (1_{(0)}) \otimes 1_{(1)} = (1_{(0)}) \otimes (1_{(1)}) \cdot (1_{(0)}) \otimes 1_{(1)}$$

$$= 1_{(0)} 1_{(0)} \alpha \otimes 1_{(1)} \alpha \otimes 1_{(1)} = 1_{(0)} \otimes 1_{(1)} \otimes 1_{(2)}$$

$$= \Delta_C(1_{(0)} \otimes 1_{(1)})$$

where we used that $P$ is an entwined module, and, obviously, $\varepsilon_C(1_{(0)} \otimes 1_{(1)}) = 1_{(0)} \varepsilon(1_{(1)}) = 1$. Therefore $\Delta_P(1)$ is a group-like element in a $P$-coring $P \otimes C$. In view of Lemma 2.38, the coaction invariants of $P$ are defined as $\{b \in P | \Delta_P(b) = b \otimes_P g\}$.

Although the knowledge of corings is not essential for the studies of (noncommutative) geometry of coalgebra-Galois extensions, and in what follows we will make some simplifying
assumptions which will make working with coalgebra-Galois extensions and entwining structures more pleasant, corings are a very useful device, which allows one to view complicated algebraic structure through much simpler and more familiar objects (our intuition trained on coalgebras will also work for corings in the majority of cases). In the context of noncommutative geometry, it seems to be worthwhile to mention an interesting relationship between corings with a group-like element and differential graded algebras (see [73] or [25, Section 29] for more details). Given an \( A \)-coring \( C \) with a group-like element \( g \), one can introduce the structure of a graded differential algebra \( \Omega^\bullet \) with \( \Omega^0 = A \) and \( \Omega^1 = \text{Ker} \varepsilon_C \). Given a right \( A \)-module \( M \) one can study connections on \( M \) with values in this graded differential algebra (see below for the definition of a connection). It turns out that \( M \) is a right \( C \)-comodule if and only if it admits a flat connection. Thus the representation-theoretic notion of a comodule of a coring has a deep and somewhat unexpected (noncommutative) differential-geometric meaning.

2.5.3 Quantum groupoids

In noncommutative geometry, Hopf algebras are understood as deformed algebras of functions on groups. In algebra and geometry (in particular in Poisson geometry and the geometry of foliations), in addition to groups, one also considers groupoids (cf. [62], [67]), which are roughly defined as groups for which not every two points can be multiplied together. Precisely, a groupoid is a small category in which any morphism is an isomorphism. Thus underlying a groupoid are two sets: a set of points (objects) and a set of invertible arrows (maps). An arrow can be composed with another arrow only if the head of one of them coincides with the tail of the other. Thus not any pair of arrows can be composed together. On the other hand, if the groupoid has a single point (object), then the head of any arrow must be the same as its tail, hence any two maps can be composed with each other, and, since they are assumed to be isomorphisms, the set of arrows forms a group. A typical example of a groupoid is a fundamental groupoid of a manifold: the points are points of the manifold, the arrows are homotopy classes of paths, and the product is induced from the concatenation of paths.

As can be seen from the above example, groupoids have a strong geometric flavour. In fact, one can associate a groupoid to any principal bundle (this is known as a gauge or Ehresmann groupoid). An algebra of functions on a groupoid, however, is no longer a Hopf algebra. First, a groupoid is built on two sets and each of them gives rise to an algebra of functions. Second, recall that a product of elements of a group gives rise to a coproduct in the algebra of functions on it. But in the case of a groupoid, not every two arrows can be composed. As a result, the product does not provide the algebra of functions on a groupoid with a coalgebra structure, but with the structure of a coring over the algebra of functions on points. Thus an algebra of functions on a groupoid is also a coring with a suitable compatibility conditions between the product and coproduct. Noncommutative generalisation of this structure leads to the notions of a bialgebroid and a Hopf algebroid or a quantum groupoid.

What makes the definition of a bialgebroid non-trivial is the fact that if \( A \) is an algebra and \( \mathcal{H} \) is an \( A \)-bimodule and an algebra, the tensor product \( \mathcal{H} \otimes_A \mathcal{H} \) is not necessarily an algebra. Thus one cannot require that the coproduct \( \Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H} \) is an algebra map. A different compatibility condition must be used. Over the years, different authors have found different solutions to this problem. First Sweedler [87] (in the case of a commutative base) and Takeuchi [89] (in the case of a general base algebra) proposed to restrict the image of \( \Delta_{\mathcal{H}} \) to a subbimodule of \( \mathcal{H} \otimes_A \mathcal{H} \) on which the algebra structure is well-defined, and then to require it to be an algebra map. Later and independently, Lu [61] proposed to replace the algebra condition for \( \Delta_{\mathcal{H}} \) with
a weaker algebraic condition. Most recently, Xu [98] suggested a different definition supported with an additional map (an anchor). Amazingly, all these seemingly different solutions lead to the same algebraic structure (see [24]). The conceptual understanding of the definition of a bialgebroid $\mathcal{H}$ in terms of a monoidal structure of the category of its (left) modules has been provided by Schauenburg [75]. Our presentation here is based on [25, Section 31], where more details can be found.

Given a $k$-algebra $A$, an $A$-ring or an algebra over $A$ is a pair $(U, i)$, where $U$ is a $k$-algebra and $i : A \to U$ is an algebra map. If $(U, i)$ is an $A$-ring, then $U$ is an $(A, A)$-bimodule with the structure provided by the map $i$, $aua' := i(a)ui(a')$. A map of $A$-rings $f : (U, i) \to (V, j)$ is a $k$-algebra map $f : U \to V$ such that $f \circ i = j$.

Let $\bar{A} = A^{op}$ be the opposite algebra of $A$. For $a \in A$, $\bar{a} \in \bar{A}$ is the same $a$ but now viewed as an element in $\bar{A}$, that is, $a \mapsto \bar{a}$ is an (obvious) anti-isomorphism of algebras. Let $A^e = A \otimes \bar{A}$ be the enveloping algebra of $A$. Then a pair $(H, i)$ is an $A^e$-ring if and only if there exist an algebra map $s : A \to H$ and an anti-algebra map $t : A \to H$, such that $s(a)t(b) = t(b)s(a)$, for all $a, b \in A$. Explicitly, $s(a) = i(a \otimes 1)$ and $t(a) = i(1 \otimes \bar{a})$, and, conversely, $i(a \otimes \bar{b}) = s(a)t(b)$.

In the case of an $A^e$-ring $H$, $A$ is called a base algebra, $H$ a total algebra, $s$ a source map and $t$ a target map. To indicate explicitly the source and target maps we write $(H, s, t)$.

Take a pair of $A^e$-rings $(U, s_U, t_U)$ and $(V, s_V, t_V)$, view $U$ as a right $A$-module via the left multiplication by the target map $t_U$, and view $V$ as a left $A$-module via the left multiplication by the source map $s_V$. A Takeuchi $\times_A$-product is then defined as

$$U \times_A V := \{ \sum_i u_i \otimes_A v_i \in M \otimes_A N \mid \forall b \in A, \sum_i u_i t_U(b) \otimes v_i = \sum_i u_i \otimes v_i s_V(b) \}.$$ 

The importance of the notion of the Takeuchi $\times_A$-product is a direct consequence of the following observation ([86, Proposition 3.1], [89, Proposition 3.1]).

**Lemma 2.50.** For any pair of $A^e$-rings $(U, i)$ and $(V, j)$, the $(A^e, A^e)$-bimodule $U \times_A V$ is an $A^e$-ring with the algebra map $A \otimes \bar{A} \to U \times_A V$, $a \otimes \bar{b} \mapsto i(a) \otimes j(\bar{b})$, the associative product

$$(\sum_i u^i \otimes v^i)(\sum_j \bar{u}^j \otimes \bar{v}^j) = \sum_{i,j} u^i \bar{u}^j \otimes v^i \bar{v}^j,$$

and the unit $1_U \otimes 1_V$.

We are now ready to define bialgebroids.

**Definition 2.51.** Let $(\mathcal{H}, s, t)$ be an $A^e$-ring. View $\mathcal{H}$ as an $(A, A)$-bimodule, with the left $A$-action given by the source map $s$, and the right $A$-action that descends from the left $A$-action given by the target map $t$, that is,

$$ah = s(a)h, \quad ha = t(a)h, \quad \text{for all } a \in A, h \in \mathcal{H}.$$ 

We say that $(\mathcal{H}, s, t, \Delta_\mathcal{H}, \varepsilon_\mathcal{H})$ is an $A$-bialgebroid if

1. $(\mathcal{H}, \Delta_\mathcal{H}, \varepsilon_\mathcal{H})$ is an $A$-coring;
2. $\text{Im}(\Delta_\mathcal{H}) \subseteq \mathcal{H} \times_A \mathcal{H}$ and the corestriction of $\Delta_\mathcal{H}$ to $\Delta_\mathcal{H} : \mathcal{H} \to \mathcal{H} \times_A \mathcal{H}$ is an algebra map;
3. $\varepsilon_\mathcal{H}(1_\mathcal{H}) = 1_A$, and, for all $g, h \in \mathcal{H},$

$$\varepsilon_\mathcal{H}(gh) = \varepsilon_\mathcal{H}(gs(\varepsilon_\mathcal{H}(h))) = \varepsilon_\mathcal{H}(gt(\varepsilon_\mathcal{H}(h))).$$
There are many examples of bialgebroids. For instance, if $H$ is a bialgebra and $A$ is an algebra, then $\mathcal{H} = A \otimes H \otimes \bar{A}$ is a bialgebroid over $A$ with the natural tensor algebra structure and:

(i) the source map $s : a \mapsto a \otimes 1_H \otimes 1_A$;

(ii) the target map $t : a \mapsto 1_A \otimes 1_H \otimes \bar{a}$;

(iii) the coproduct $\Delta : a \otimes b \otimes \bar{a}' \mapsto \sum a \otimes b(1) \otimes 1_A \otimes b(2) \otimes \bar{a}'$;

(iv) the counit $\varepsilon : a \otimes b \otimes \bar{a}' \mapsto \varepsilon(a)\bar{a}'$;

The definition of a bialgebroid we present here is by now generally accepted. On the other hand, there seems to be no consensus as to how a Hopf algebroid should be defined. In [61], an anti-algebra map $\kappa : \mathcal{H} \to \mathcal{H}$ such that

(i) $\kappa \circ t = s$;

(ii) $\mu_H \circ (\kappa \otimes \text{id}_H) \circ \Delta_H = t \circ \varepsilon_H \circ \kappa$;

(iii) there exists a section $\gamma : \mathcal{H} \otimes_A \mathcal{H} \to \mathcal{H} \otimes_k \mathcal{H}$ of the natural projection $\mathcal{H} \otimes_k \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}$ such that $\mu_H \circ (\text{id}_H \otimes \kappa) \circ \gamma \circ \Delta_H = s \circ \varepsilon_H$,

is called an antipode. A bialgebroid with an antipode in this sense is often referred to as a Lu-Hopf algebroid. Another definition of a Hopf algebroid has been proposed by Schauenburg in [77]. For any $A$-bialgebroid $\mathcal{H}$, the category of its left modules is a monoidal category such that the forgetful functor to the category of $A$-modules preserves also the closed structure. In algebraic terms, one requires that the ‘canonical map’ $\mathcal{H} \otimes_A \mathcal{H} \to \mathcal{H} \otimes_k \mathcal{H}$, $g \otimes_A h \mapsto g(1) \otimes_A g(2)\bar{h}$ be invertible. One refers to such bialgebroids as $\times_A$-Hopf algebras. This definition, however, does not lead to an antipode as a map $\mathcal{H} \to \mathcal{H}$.

The most symmetric and closest to the Hopf algebra case definition of a Hopf algebroid was proposed by Böhm and Szlachányi in [10], [6]. This definition starts with the observation that, in order to define an antipode which would have similar properties to those of an antipode in a Hopf algebra, one needs to symmetrise the definition of a bialgebroid. Note that the Definition 2.51 is not symmetric in the sense that it considers $\mathcal{H}$ as an $A$-bimodule with the actions obtained by the left multiplication with the source and target maps. One therefore refers to the notion defined in Definition 2.51 more precisely as a left $A$-bialgebroid. Obviously, it is possible to define a similar object by using the right multiplication by the source and target maps. Such an object is then called a right $A$-bialgebroid. The notion of a Hopf algebroid proposed in [10], [6] requires existence of both left and right bialgebroid structures on the same algebra.

**Definition 2.52.** A Hopf algebroid consists of a left $L$-bialgebroid $\mathcal{H}$ with structure maps $s_L$, $t_L$, $\Delta_L$, $\varepsilon_L$, a right $R$-bialgebroid $\mathcal{H}$ with structure maps $s_R$, $t_R$, $\Delta_R$, $\varepsilon_R$, and a map $S : \mathcal{H} \to \mathcal{H}$ satisfying the following conditions:

(a) source-target compatibility,

$$s_L \circ \varepsilon_L \circ t_R = t_R, \quad t_L \circ \varepsilon_L \circ s_R = s_R, \quad s_R \circ \varepsilon_R \circ t_L = t_L, \quad t_R \circ \varepsilon_R \circ s_L = s_L;$$

(b) colinearity of coproducts,

$$((\Delta_L \otimes_R \text{id}_H) \circ \Delta_R = (\text{id}_H \otimes_L \Delta_R) \circ \Delta_L, \quad (\Delta_R \otimes_L \text{id}_H) \circ \Delta_L = (\text{id}_H \otimes_R \Delta_L) \circ \Delta_R.$$
(c) the $R \otimes L$-bilinearity of the antipode,
\[
S(t_L(l)ht_R(r)) = s_R(r)S(h)s_L(l), \quad S(t_R(r)ht_L(l)) = s_L(l)S(h)s_R(r),
\]
for all $r \in R$, $l \in L$ and $h \in \mathcal{H}$;
(d) antipode axioms,
\[
\mu_{\mathcal{H}} \circ (S \otimes L \text{id}_{\mathcal{H}}) \circ \Delta_L = s_R \circ \varepsilon_R, \quad \mu_{\mathcal{H}} \circ (\text{id}_{\mathcal{H}} \otimes R S) \circ \Delta_R = s_L \circ \varepsilon_L.
\]

Note that the axiom (b) in Definition 2.52 can be stated since condition (a) implies that $\Delta_L$ is $R$-bilinear and $\Delta_R$ is $L$-bilinear. Furthermore, Definition 2.52 implies that $R$ is isomorphic to the opposite algebra of $L$, so that $L = A$ and $R = \bar{A}$ with no loss of generality. It is also proven in [6, Proposition 2.3] that the antipode in a Hopf algebroid is an antimultiplicative and an anticomultiplicative map (in the latter case both coproducts must be used). Böhm and Szlachányi show that their definition is not equivalent to that of Lu, by constructing an explicit example of a Hopf algebroid which is not a Lu-Hopf algebroid (cf. [10, Example 4.9]). Many constructions familiar in Hopf algebra can be extended to Hopf algebroids (see [7] for a review). From the geometric point of view, the discussion of strong connections in Hopf algebroid extensions and the construction of the relative Chern-Galois character in [8] might be of particular interest.

Yet another definition of a Hopf algebroid has been proposed by Day and Street in [35] in the framework of monoidal bicategories. When specialised to the monoidal bicategory of $k$-algebras, bimodules and bimodule maps, this definition is shown in [10, Theorem 4.7] to be very close to (but slightly more restrictive than) the Böhm-Szlachányi definition.

There are many examples of Hopf algebroids in the sense of Definition 2.52. It is shown in [42] that weak Hopf algebras (with a bijective antipode) of Böhm, Nill and Szlachányi [9] are examples of bialgebroids. This result is then refined in [78], where it is shown that weak bialgebras are bialgebroids, while weak Hopf algebras are Hopf $\times A$-algebras in the sense of Schauenburg. In [10, Example 4.8], weak Hopf algebras are shown to be Hopf algebroids. Furthermore, it is shown in [55] that one can associate a bialgebroid to any depth-2 algebra extension. In case a depth-2 extension is Frobenius, the corresponding bialgebroid is a Hopf algebroid [10]. A class of Lu-Hopf algebroids associated to braided commutative algebras is constructed in [24]. These are shown to be Hopf algebroids in [10, Example 4.14] (provided the antipodes are bijective). Other examples of Hopf algebroids in the sense of Definition 2.52 considered in [10] include quantum torus and the Connes-Moscovici twisted Hopf algebras [32], [33] (cf. [5]). On the mathematical physics side, quantum groupoids arise from solutions to quantum dynamical Yang-Baxter equations (in the guise of dynamical quantum groups cf. [43]) as well as symmetries of certain models in statistical physics (in the guise of face algebras cf. [53]). From the point of view of Galois-type extensions, the following example appears to be most significant.

**Example 2.53.** Given a coalgebra-Galois $C$-extension $B \subseteq P$ with the translation map $\tau$, consider a $B$-bimodule
\[
\mathcal{C} = \{ \sum_i p^i \otimes \tilde{p}^i \in P \otimes P \mid \sum_i p^i_{(0)} \otimes \tau(p^i_{(1)}) \tilde{p}^i = \sum_i p^i \otimes \tilde{p}^i \otimes_B 1 \}.
\]
If $P$ is faithfully flat as a left $B$-module, then $\mathcal{C}$ is a $B$-coring with the coproduct and counit
\[
\Delta_c(\sum_i p^i \otimes \tilde{p}^i) = \sum_i p^i_{(0)} \otimes \tau(p^i_{(1)}) \otimes \tilde{p}^i, \quad \varepsilon_c(\sum_i p^i \otimes \tilde{p}^i) = \sum_i p^i \tilde{p}^i.
\]
The $B$-coring $C$ is called the Ehresmann or gauge coring associated to a (left faithfully flat) coalgebra-Galois $C$-extension $B \subseteq P$.

The Ehresmann coring associated to a (left faithfully flat) Hopf-Galois $H$-extension $B \subseteq P$ is a $B$-bialgebroid with the algebra structure of a subalgebra of $P_e$, the source map $s : p \mapsto p \otimes 1$ and the target map $t : p \mapsto 1 \otimes \bar{p}$.

For details of the proof we refer to [25, Section 34]. The Ehresmann bialgebroid corresponding to a Hopf-Galois extension was constructed in [75]. Both the Ehresmann coring and bialgebroid can be seen as a noncommutative version of the Ehresmann or gauge groupoid that can be associated to any principal bundle (cf. [67, Example 5.1(8)]).

3 Connections and associated modules

Recall that, from a geometric point of view, coalgebra-Galois extensions can be viewed as noncommutative principal bundles. Motivated by this relationship, one can develop geometric-type objects such as connections, sections of associated vector bundles, etc. These have physical meaning of gauge potentials (connections) or matter fields (sections). The aim of this section is to present differential-geometric aspects of coalgebra-Galois and principal extensions.

We begin in Section 3.1.1 by introducing connections in coalgebra-Galois extensions. We give various descriptions of such connections. We then proceed in Section 3.1.2 to define modules associated to coalgebra-Galois extensions, which can be understood as modules of sections of noncommutative associated vector bundles. In Section 3.1.3 gauge transformations of coalgebra-Galois extensions are discussed. Finally, in Section 3.2 strong connections on principal extensions are studied. These are objects that induce connections on associated modules, and thus guarantee the projectivity of the latter. This makes the desired link with $K$-theory.

3.1 General coalgebra-Galois extensions

3.1.1 Connections

If an algebra $P$ is a comodule of a coalgebra $C$, we would like to establish covariance properties of $\Omega^1 P$, i.e., we wish to define a $C$-coaction on $\Omega^1 P$. Although in general the coaction $\Delta_P : P \to P \otimes C$ does not extend to $\Omega^1 P$, it turns out that such an extension is possible for an entwining structure $(P, C, \psi)$ with $P \in \mathcal{M}_P^C(\psi)$.

**Proposition 3.1.** ([23, Proposition 2.2]) Consider an entwining structure $(P, C, \psi)$ with $P \in \mathcal{M}_P^C(\psi)$, and define a map

$$
\Delta_{P \otimes P} : P \otimes P \longrightarrow P \otimes P \otimes C , \ p \otimes p' \longmapsto p(0) \otimes \psi(p(1) \otimes p').
$$

(3.75)

The homomorphism $\Delta_{P \otimes P}$ is a $C$-coaction on $P \otimes P$, and it restricts to a coaction $\Delta_{\Omega^1 P} : \Omega^1 P \to \Omega^1 P \otimes C$.

**Proof.** The fact that $\Delta_{P \otimes P} : P \otimes P \to P \otimes P \otimes C$ defines a $C$-coaction can be checked using the right hand side of the bow-tie diagram in Definition 2.26. The fact that $\Delta_{P \otimes P}$ restricts to
the coaction $\Delta_{\Omega P} : \Omega^1 P \rightarrow \Omega^1 P \otimes C$ follows from the left hand side of the bow-tie diagram. Explicitly, since for all $\sum_i p_i \otimes p'_i \in \Omega^1 P$, $\sum_i p_i p'_i = 0$, we obtain
\[ \sum_i p_{i(0)} \psi(p_{i(1)} \otimes p'_i) = \Delta_P(\sum_i p_i p'_i) = \Delta_P(0) = 0. \] (3.76)

Hence $\Delta_{P \otimes P}(\Omega^1 P) \subseteq \Omega^1 P \otimes C$, as required.

Thus, in the case of a canonical entwining structure associated to a coalgebra-Galois $C$-extension $B \subseteq P$, both $P \otimes P$ and $\Omega^1 P$ become right $C$-comodules. This observation is crucial for the following:

**Definition 3.2.** Let $B \subseteq P$ be a coalgebra-Galois $C$-extension.

(1) A connection is a left $P$-module projection $\Pi : \Omega^1 P \rightarrow \Omega^1 P$ such that
   (i) $\text{Ker}(\Pi) = P(\Omega^1 B)P$ (horizontality),
   (ii) $\Pi \circ d : P \rightarrow \Omega^1 P$ is a right $C$-comodule map (covariance property).

(2) A connection form is a homomorphism $\omega : C \rightarrow \Omega^1 P$ such that
   (i) $1_{(0)} \omega(1_{(1)}) = 0$,
   (ii) for all $c \in C$, $(\tilde{\text{can}} \circ \omega)(c) = 1 \otimes c - 1_{(0)} \otimes 1_{(1)} \epsilon(c)$,
   (iii) $(\text{id} \otimes \psi) \circ (\psi \otimes \text{id}) \circ (\text{id} \otimes \omega) \circ \Delta = (\omega \otimes \text{id}) \circ \Delta$.

In view of the fact that the definition of a coalgebra-Galois extension is equivalent the exactness of the sequence (2.6), we can interpret connections as left-linear splittings of the sequence (2.6) with the covariance property Definition 3.2(1)(ii). This leads to the following (cf. [23, Proposition 3.3]):

**Theorem 3.3.** Let $B \subseteq P$ be coalgebra-Galois $C$-extension. Write $\tau(c) = c^{[1]} \otimes_B c^{[2]}$ for the action of the translation map on any $c \in C$. The following formulae
\[ \Pi \mapsto \omega_\Pi, \quad \omega_\Pi(c) = c^{[1]} \Pi(\text{de}^{[2]}), \] (3.77)
\[ \omega \mapsto \Pi^\omega, \quad \Pi^\omega(p \Delta d(p')) = pp'(0)\omega(p'(1)), \] (3.78)
define mutually inverse maps between the space of connections $\Pi$ and the space of connection forms $\omega$.

**Proof.** Let $\omega$ be a connection form. The map $\Pi^\omega$ is well defined because, for all $p \in P$, $\Pi^\omega(p \Delta d(1)) = p1_{(0)}\omega(1_{(1)}) = 0$, by Definition 3.2(2)(i). For any $p, p' \in P$, $b \in B$,
\[ \Pi^\omega(p \Delta d(b)p') = \Pi^\omega(p \Delta d(bp')) - \Pi^\omega(p \Delta d(p')) = p(bp')(0)\omega((bp')(1)) - pbp'(0)\omega(p'(1)) = 0, \]
as $\Delta_P$ is left $B$-linear. On the other hand, if $\sum_i p^i d(q^i) \in \text{Ker } \Pi^\omega$, then using Definition 3.2(2)(ii) we can compute
\[ 0 = \sum_i \tilde{\text{can}}(p^i q^i_{(0)} \omega(q^i_{(1)})) = \sum_i (p^i q^i_{(0)} \otimes q^i_{(1)} - p^i q^i 1_{(0)} \otimes 1_{(1)}) = \sum_i \tilde{\text{can}}(p^i d(q^i)). \]
Since $\ker \widehat{can} = P(\Omega^1 B)P$, we deduce that $\ker \Pi^\omega \subseteq P(\Omega^1 B)P$, i.e., $\ker \Pi^\omega = P(\Omega^1 B)P$. Finally, write $\psi^2$ for $(\text{id} \otimes \psi) \circ (\psi \otimes \text{id})$, and compute, for all $p \in P$,

$$\Delta_{\Omega^1 P}(\Pi^\omega(d(p))) = p(0)\psi^2(p(1) \otimes \omega(p(2))) = p(0)\omega(p(1)) \otimes p(2),$$

(by Definition 3.2(2)(iii))

so that $\Pi^\omega \circ d$ is a right $C$-comodule map, as required.

Conversely, let $\Pi$ be a connection in a coalgebra-Galois $C$-extension $B \subseteq P$. In particular, $\Pi$ is left $P$-linear and $\ker \Pi = P(\Omega^1 B)P$, so that, for all $b \in B$ and $p \in P$,

$$\Pi(d(bp)) = \Pi((db)p) + b\Pi(dp) = b\Pi(dp).$$

Therefore, the map $\omega_\Pi$ in equation (3.77) is well defined (despite the fact that the differential $d$ is not a left $B$-module map). Using the Translation Map Lemma Lemma 2.22(iii), one immediately finds

$$1(0)\omega_\Pi(1(1)) = 1(0)1(1)[1]\Pi(d1(1)[2]) = \Pi(d1) = 0.$$

Hence $\omega_\Pi$ satisfies condition Definition 3.2(2)(i).

Next, define a left $P$-linear map $\sigma_\Pi : P \otimes C^+ \to \Omega^1 P$, $p \otimes c \mapsto p\omega_\Pi(c)$. Again employ Lemma 2.22(iii) to note that, for all $p, p' \in P$, $\Pi(pd_p') = pp'(0)\omega_\Pi(p'(1))$. A short calculation reveals that $\Pi(pd_p') = \sigma_\Pi(\widehat{can}(pd_p'))$, i.e.,

$$\Pi = \sigma_\Pi \circ \widehat{can}.$$

Since $\widehat{can}$ is an epimorphism and, by the assumption on $\Pi$ and the exactness of sequence (2.6), $\ker \Pi = P(\Omega^1 B)P = \ker \widehat{can}$, this implies that the map $\sigma_\Pi$ is a monomorphism. Indeed, suppose $\sigma_\Pi(x) = 0$ for some $x \in P \otimes C^+$. Then there exists $y \in \Omega^1 P$ such that $x = \widehat{can}(y)$. Therefore,

$$\Pi(y) = \sigma_\Pi(\widehat{can}(y)) = \sigma_\Pi(x) = 0,$$

i.e., $y \in \ker \Pi = \ker \widehat{can}$, so that $x = \widehat{can}(y) = 0$. Now, as $\Pi$ is assumed to be a projection, we have

$$\sigma_\Pi \circ \widehat{can} = \Pi = \Pi \circ \Pi = \sigma_\Pi \circ \widehat{can} \circ \sigma_\Pi \circ \widehat{can}.$$

This means that $\widehat{can} \circ \sigma_\Pi = \text{id}_{P \otimes C^+}$, for $\sigma_\Pi$ is a monomorphism and $\widehat{can}$ is an epimorphism. Therefore, in view of the fact that $\omega_\Pi$ satisfies Definition 3.2(2)(i), we obtain, for all $c \in C$,

$$\widehat{can}(\omega_\Pi(c)) = \widehat{can}(1(0)\omega_\Pi(c \varepsilon(1(1)) - \varepsilon(c)1(1))) = \widehat{can}(\sigma_\Pi(1(0) \otimes (c \varepsilon(1(1)) - \varepsilon(c)1(1)))) = 1(0) \otimes c \varepsilon(1(1)) - 1(0) \otimes \varepsilon(c)1(1) = 1 \otimes c - \varepsilon(c)1(0) \otimes 1(1).$$

Thus $\omega_\Pi$ satisfies condition Definition 3.2(2)(ii) as well.

To prove that $\omega_\Pi$ satisfies condition Definition 3.2(2)(iii), first apply $\text{id} \otimes_B \text{id} \otimes \tau$ to Lemma 2.22(iv), multiply the second and third factors, and use Lemma 2.22(iii) to deduce that, for all $c \in C$,

$$c^{[1]} \otimes_B 1 \otimes_B c^{[2]} = c(1)^{[1]} \otimes_B c(1)^{[2]} c(2)^{[1]} \otimes_B c(2)^{[2]}.$$

(3.79)
This then facilitates the following calculation (in which \( \psi^2 \) is the shorthand for \((\text{id} \otimes \psi) \circ (\psi \otimes \text{id})\), as before):

\[
\psi^2(c(1) \otimes \omega_{\Pi}(c(2))) = \psi^2(c(1) \otimes c(2)[1] \Pi(dc(2)[2])) = c(2)[1] \alpha \psi^2(c(1) \alpha \otimes \Pi(dc(2)[2])) \quad \text{(left pentagon for } \psi) \\
= c(1)[1](c(1)[2]c(2)[1])_0 \psi^2((c(1)[2]c(2)[1])_1 \otimes \Pi(dc(2)[2])) = c[1]1_0 \psi^2(1(1) \otimes \Pi(dc[2])) \quad \text{(equation (3.79))} \\
= c[1] \Delta_{\Omega P}(\Pi/dc[2])) \quad \text{(def. of coaction } \Delta_{\Omega P}) \\
= c[1](\Pi(dc[2](0))) \otimes c[2](1) \quad \text{(colinearity of } \Pi \circ d) \\
= c(1)[1](\Pi(dc[2](1))) \otimes c(2) \quad \text{(Lemma 2.22(iv))} \\
= \omega_{\Pi}(c(1)) \otimes c(2).
\]

The third equality follows from the definition of the canonical entwining map in equation (2.63). Thus \( \omega_{\Pi} \) satisfies condition Definition 3.2(2)(iii), and we conclude that \( \omega_{\Pi} \) is a connection form.

Finally, a simple calculation that uses the Translation Map Lemma 2.22 reveals that the maps defined by equations (3.77) and (3.78) are inverses of each other. This completes the proof. \( \square \)

The connection and connection form are constructed in analogy with their classical counterparts that are adapted to de Rham differential forms. In our setting of the universal calculus, it appears more convenient to use the following formulation of the concept of connection.

**Definition 3.4.** Let \( B \subseteq P \) be a coalgebra-Galois \( C \)-extension. A connection lifting is a homomorphism \( \ell : C \rightarrow P \otimes P \) such that

\begin{align*}
(i) \quad 1_{(0)}\ell(1_{(1)}) &= 1 \otimes 1, \\
(ii) \quad \overline{\alpha n} \circ \ell &= 1 \otimes \text{id}, \\
(iii) \quad (\text{id} \otimes \psi) \circ (\psi \otimes \text{id}) \circ (\text{id} \otimes \ell) \circ \Delta &= (\ell \otimes \text{id}) \circ \Delta.
\end{align*}

It is straightforward to verify that the connection lifting and connection form are related by the following simple formula:

\[
\ell = \omega + 1 \otimes 1\varepsilon. \quad (3.80)
\]

This constant shift of a connection form allows one to view a connection as a lifting of the translation map \( \tau \). Indeed, one can equivalently write the condition Definition 3.4(ii) as \( \pi_B \circ \ell = \tau \), where \( \pi_B : P \otimes P \rightarrow P \otimes B \) is the canonical surjection.

**Remark 3.5.** In an \( e \)-coaugmented coalgebra-Galois \( C \)-extension \( B \subseteq P \), the external differential commutes with the coaction, \((d \otimes \text{id}_C) \circ \Delta_P = \Delta_{\Omega P} \circ d\). Indeed, for all \( p \in P \), we have

\[
d(p)(0) \otimes d(p)(1) = 1 \otimes p_\alpha \otimes e^\alpha - p(0) \otimes 1_\alpha \otimes p(1)^\alpha \\
= 1 \otimes p(0) \otimes p(1) - p(0) \otimes 1 \otimes p(1) = d(p(0)) \otimes p(1).
\]

Here we used the left triangle in the bow-tie diagram and the fact that \( P \) is an entwined module, i.e.,

\[
p_\alpha \otimes e^\alpha = 1_{(0)}p_\alpha \otimes 1_{(1)}^\alpha = (p1)(0) \otimes p(1) = \Delta_p(p). \quad (3.81)
\]

Thus, in this case, the universal differential calculus \( \Omega^1 P \) is a **right-covariant differential calculus** on \( P \) (cf. [96] for the definition of a right-covariant calculus). Moreover, we can now define a **covariant differential** \( D = d - \Pi \circ d \) which is right \( C \)-colinear.
The definition of a connection Definition 3.2 tries to follow the classical definition of a connection in principal bundles, albeit in a dual language. Obviously, not all the classical properties of connections can be recovered in this general algebraic setup. For example, if \( P \) is an algebra of functions on a total space \( X \) of a classical principal bundle, \( B \) is an algebra of functions on a base manifold \( M \), and \( \Omega^1(B) \) is the classical space of 1-forms on \( M \), then horizontal forms have several equivalent descriptions

\[
\Omega^1_{\text{hor}}(P) = P(\Omega^1(B))P = P(\Omega^1(B)) = (\Omega^1(B))P.
\]

Obviously, no such relationships exist in a general noncommutative setting, even more so in the case of the universal differential calculus. In a non-universal calculus case, and in particularly nice examples (e.g., in the case of the quantum Hopf fibering with the 3D-calculus discussed in [20]), the above equalities can be obtained. In a general situation with the universal differential calculus, in order to come closer to the classical geometric intuition, one is forced to consider also a stronger version of the notion of a connection (cf. [49]).

### 3.1.2 Associated modules

In noncommutative differential geometry [31] vector bundles are identified with finitely generated projective modules, via the extrapolation of the classical Serre-Swan theorem which states that the category of finite dimensional vector bundles over a compact Hausdorff space \( M \) is equivalent to the category of finitely generated projective modules over the algebra of functions \( C(M) \). The equivalence is given by assigning to each vector bundle its the module of continuous sections.

In classical geometry, all vector bundles (and hence projective modules of sections) arise as bundles associated to principal bundles. More precisely, consider a principal bundle with a total space \( X \), the structure group \( G \), and the base space \( M = X/G \). Take any finite-dimensional representation of \( G \), i.e., a finite-dimensional vector space \( V \) with a right \( G \)-action. Then sections of an associated vector bundle with a standard fibre \( V \) are identified with vector-valued maps \( \sigma : X \to V \) such that, for all \( x \in X \) and \( g \in G \), \( \sigma(x \cdot g) = \sigma(x) \cdot g \). Given a vector bundle \( E \) with a standard fibre \( k^n \), one can construct a \( GL_n(k) \)-principal bundle (the frame bundle) such that \( E \) is isomorphic to a bundle associated to this principal bundle. Since coalgebra-Galois extensions are to be interpreted as noncommutative principal bundles, it makes sense to study associated modules, which are to be interpreted as vector bundles. This construction can be performed for any coalgebra-Galois extension, but we restrict our attention to those extensions which lead to bona fide noncommutative vector bundles, i.e., finitely generated projective modules.

Recall that elements of \( \text{Hom}^C(V, P) \) are linear maps \( f : V \to P \) such that \( \Delta_P \circ f = (f \otimes \text{id}) \circ \Delta_V \), where \( \Delta_V : V \to V \otimes C \) is a coaction of \( C \) an \( V \).

**Definition 3.6.** For a coalgebra-Galois \( C \)-extension \( B \subseteq P \) and a right \( C \)-comodule \( V \), the module of sections of a bundle associated to the \( C \)-extension \( B \subseteq P \) with standard fibre \( V \) is defined as the space of right \( C \)-colinear maps \( E := \text{Hom}^C(V, P) \).

Since \( \Delta_P \) is by definition a left \( B \)-linear map we immediately obtain \( E \) is a left \( B \)-module with the action \( (b \cdot f)(v) = bf(v) \). Recall that for a finite-dimensional comodule \( V \), the associated module \( E = \text{Hom}^C(V, P) \) is isomorphic to \( P \square_C V^* \) in \( B \text{-Mod} \). Note that, if \( P \) is equivariantly projective, then \( E \) is projective. Indeed, \( \text{id} : P \square_C V^* \to B \otimes P \square_C V^* \) defines a splitting of the multiplication map. Furthermore, one can prove:
Lemma 3.7. Let $B \subseteq P$ be a symmetric (bijectivity of the canonical entwining assumed) Galois $C$-extension with $P$ faithfully flat in $M_B$ and $V$ a left $C$-comodule with $\dim_k V < \infty$. Then the associated module $E = \text{Hom}^C(V, P)$ is finitely generated as a left $B$-module.

3.1.3 Gauge transformations

In classical geometry, gauge transformations arise from automorphisms of principal bundles. In parallel to this and following [14], we consider:

Definition 3.8. Given a coalgebra-Galois $C$-extension $B \subseteq P$, a left $B$-linear, right $C$-colinear automorphism $F : P \to P$ such that $F(1) = 1$ is called a gauge automorphism of $B \subseteq P$. Gauge automorphisms form a group with respect to the opposite composition (i.e., $FG = G \circ F$) which is denoted by $GA^C(B \subseteq P)$.

The following theorem (cf. [14, Theorem 2.4]) provides one with an equivalent description of gauge automorphisms in terms of certain maps $C \to P$, which play the role of gauge transformations in the coalgebra-Galois setting.

Theorem 3.9. Let $B \subseteq P$ be a coalgebra-Galois $C$-extension, and let $\psi$ denote the canonical entwining. Then

1. Convolution invertible maps $f : C \to P$ such that
   
   (a) $1(0)f(1(1)) = 1$,
   
   (b) $\psi \circ (\text{id} \otimes f) \circ \Delta = (f \otimes \text{id}) \circ \Delta$,

   form a group with respect to the convolution product. This group is denoted by $GT^C(B \subseteq P)$ and called the group of gauge transformations of $B \subseteq P$.

2. The assignments

   $f \mapsto F_f$, \quad $F_f(p) = p(0)f(p(1))$ \quad (3.82)

   $F \mapsto f_F$, \quad $f_F(c) = c^{[1]}F(c^{[2]})$ \quad (3.83)

   define the mutually inverse isomorphisms of groups of gauge transformations and gauge automorphisms.

Proof. (1) It is clear that if $f, g : C \to P$ satisfy (a), then so does their convolution product $f \ast g$. Condition (b) reads explicitly, for any $c \in C$,

   $\psi(c_{(1)} \otimes f(c_{(2)})) = f(c_{(1)}) \otimes c_{(2)}$. \quad (3.84)

An easy calculation that uses the left pentagon in the bow-tie diagram reveals that if $f$ and $g$ satisfy (3.84), then so does the convolution product $f \ast g$. Thus $GT^C(B \subseteq P)$ is a semigroup.

It is clear that the map $\eta : c \mapsto \varepsilon(c)1_P$ satisfies condition (a). The left triangle in the bow-tie diagram ensures that this map also satisfies condition (b). Thus $\eta \in GT^C(B \subseteq P)$, and therefore $GT^C(B \subseteq P)$ is a monoid.

Now take any $f \in GT^C(B \subseteq P)$. We need to show that its convolution inverse $f^{-1}$ satisfies conditions (a) and (b). Since $P$ is an entwined module and $f$ satisfies conditions (a) and (b), we can compute

\[ 1(0) \otimes 1(1) = \Delta_P(1) = \Delta_P(1(0)f(1(1))) = 1(0)\psi(1(1) \otimes f(1(2))) = 1(0)f(1(1)) \otimes 1(2). \]
Apply $\text{id} \otimes f^{-1}$ and multiply to conclude that $1_{(0)} f^{-1}(1_{(1)}) = 1$, as required. Finally, the facts that $f$ satisfies (b) and the commutativity of the left pentagon in the bow-tie diagram, imply that, for all $c \in C$,
\[
 c_{(1)} \otimes 1 \otimes c_{(2)} = c_{(1)} \otimes \psi(c_{(2)} \otimes f(c_{(3)}) f^{-1}(c_{(4)})) = c_{(1)} \otimes f(c_{(2)}) \psi(c_{(3)} \otimes f^{-1}(c_{(4)})).
\]

Applying $f^{-1} \otimes \text{id} \otimes \text{id}$ to the above equality and multiplying the first two factors one finds that $f^{-1}$ satisfies (b). Thus $\text{GT}^C(B \subseteq P)$ is a group as claimed.

(2) The canonical isomorphism $\text{can} : P \otimes_B P \cong P \otimes C$ induces an isomorphism of spaces of left $P$-linear maps, $\mathcal{P}\text{Hom}(P \otimes_B P, P) \cong \mathcal{P}\text{Hom}(P \otimes C, P)$. This, in turn, reduces to an isomorphism
\[
 \mathcal{B}\text{Hom}(P, P) \cong \text{Hom}(C, P).
\]

One easily checks that the explicit form of this isomorphism is given by the maps in equations (3.82) and (3.83), and that the opposite composition of automorphisms corresponds to the convolution product. The $C$-colinearity of an automorphism of $P$ induces condition (1)(b) for the corresponding map on the right hand side. Similarly, normalisation leads to condition (1)(a).

Gauge transformations induce transformations of connections in coalgebra-Galois extensions. More precisely, the gauge group acts on the space of connections. This is of particular importance in the case of strong connections in principal extensions.

### 3.2 Strong connections on principal extensions

The original motivation for defining strong connections was to have a concept of connection that for a cleft Hopf-Galois $H$-extension would be expressible only in terms of the Hopf algebra $H$ and the coaction-invariant subalgebra $[49]$. Roughly speaking, in differential geometry this corresponds to the fact that horizontal subspaces are always isomorphic to tangent spaces of the base manifold, so that one can assemble a connection form on the total space out of a collection of locally defined forms on the base space. This turned out to be a notion that allowed one to construct a covariant derivative on the associated modules $[51]$, and thus link the Hopf-Galois theory of quantum principal bundles with connections on projective modules.

Here we first study strong connections in the general setting of coalgebra-Galois extensions. The definition of a general connection given before called for a replacement of a diagonal coaction, and only finding it allowed a definition analogous to its earlier Hopf-Galois version. Now, having a general connection at hand, one can phrase the strongness condition precisely as in Hopf-Galois theory $[49]$.

**Definition 3.10** ([23]). Let $\Pi$ be a connection on a coalgebra-Galois $C$-extension $B \subseteq P$. We call it **strong** if $(\text{id} - \Pi \circ \text{d})(P) \subseteq (\Omega^1 B)P$.

One can alternatively define a strong connection in the following way:

**Definition 3.11** ([36]). Let $B \subseteq P$ be a coalgebra-Galois $C$-extension. We call a unital left $B$-linear right $C$-colinear (with respect to $\text{id} \otimes \Delta_P$) splitting of the multiplication map $B \otimes P \rightarrow P$ a strong-connection splitting.
Lemma 3.12 ([36]). Let $B \subseteq P$ be a coalgebra-Galois $C$-extension. Then the equation
\[ s = \Pi \circ d + \text{id} \otimes 1 \] (3.85)
defines a one-to-one correspondence between strong connections and strong-connection splittings.

Proof. Let $s$ be such a splitting, and $\Pi^*(r dp) := r(s(p) - p \otimes 1)$. One can verify that this formula gives a well-defined left $P$-linear endomorphism of $\Omega^1 P$. Furthermore, by the left $B$-linearity of $s$, for any $\sum_i d b_i p_i \in (\Omega^1 B) P$, we have:
\[ \Pi^*(\sum_i d b_i p_i) = \sum_i \Pi^*(d b_i p_i) = \sum_i s(b_i p_i) - b_i d p_i \otimes 1 = \sum_i s(p_i) - p_i \otimes 1 = 0. \] (3.86)
Hence $P(\Omega^1 B) P \subseteq \text{Ker } \Pi^*$ by the left $P$-linearity of $\Pi^*$. On the other hand, if $m \circ s = \text{id}$ and $s(P) \subseteq B \otimes P$, we have $\pi_B(s(p)) = 1 \otimes_B p$, where $\pi_B : P \otimes P \to P \otimes_B P$ is the canonical surjection. Consequently,
\[ \pi_B(\Pi^*(p' dp)) = \pi_B(p'(s(p) - p \otimes 1)) = p' \pi_B(s(p)) - rp \otimes_B 1 = p' \otimes_B p - rp \otimes_B 1 = \pi_B(p' dp). \] (3.87)
Therefore, since $P(\Omega^1 B) P = \text{Ker } \pi_B$, we obtain $\text{Ker } \Pi^* \subseteq P(\Omega^1 B) P$. Thus $\ker \Pi^* = P(\Omega^1 B) P$.

Next, take any $p \in P$. It follows from $s(P) \subseteq B \otimes P$ that
\[ dp - \Pi^*(dp) = 1 \otimes p - p \otimes 1 - s(p) + p \otimes 1 = 1 \otimes p - s(p) \in B \otimes P. \] (3.88)
Since also $m(1 \otimes p - s(p)) = 0$, we have $dp - \Pi^*(dp) \in (\Omega^1 B) P \subseteq \text{Ker } \Pi^*$. By the left $P$-linearity of $\Pi^*$, we can conclude now that $\Pi^* \circ (\text{id} - \Pi^*) = 0$, i.e., $\Pi^* = (\Pi^*)^2 = \Pi^*$. It remains to show that $\Delta_{P \otimes P} \circ \Pi^* \circ d = \Pi^* \circ d$.

\[ \Delta_{P \otimes P}(p \otimes 1) = p(0) \otimes \psi(p(1) \otimes 1) = p(0) \otimes 1 \otimes p(1). \] (3.89)

Therefore,
\[ \Delta_{P \otimes P}(\Pi^*(dp)) = \Delta_{P \otimes P}(s(p)) - \Delta_{P \otimes P}(p \otimes 1) = s(p(0)) \otimes p(1) - p(0) \otimes 1 \otimes p(1) = (\Pi^* \circ d)(\Delta_P(p)), \] (3.90)
by the colinearity of $s$. Consequently, $\Pi^*$ is a connection, as claimed.

\[ \Box \]

Remark 3.13. Within the framework of Hopf-Galois theory the right coaction $\text{id} \otimes \Delta_P : B \otimes P \to B \otimes P \otimes H$ and the restriction $\Delta_{B \otimes P}$ of the diagonal coaction $\Delta_{P \otimes P}$ (3.75) coincide. Therefore, one can use either of them to define the colinearity of a splitting $s$ of the multiplication map $B \otimes P \to P$. In the general setting of coalgebra-Galois extensions
\[ \Delta_{P \otimes P}(b \otimes p) = (\text{id} \otimes \psi)(\Delta_P(b_1 \otimes p) = (\text{id} \otimes \psi)(b_1(0) \otimes 1(1) \otimes p) = b_1(0) \otimes \psi(1(1) \otimes p). \] (3.91)

On the other hand, if $B \subseteq P$ is $C$-Galois and $\psi$ is its canonical entwining structure [17, (2.5)], then, by [17, Theorem 2.7], $P$ is a $(P, C, \psi)$-module [14], so that we have
\[ \Delta_P(p' p) = p'(0) \psi(p' (1) \otimes p). \]
In particular, $\Delta_P(p) = 1(0) \psi(1(1) \otimes p)$. Hence
\[ (\text{id} \otimes \Delta_P)(b \otimes p) = b \otimes 1(0) \psi(1(1) \otimes p). \] (3.92)

Hence we need to distinguish between $\Delta_{B \otimes P}$ and $\text{id} \otimes \Delta_P$ in the $C$-Galois case.
The just discussed problem of the diagonal coaction in the general coalgebra-Galois setting was already encountered in Remark 3.5, where it obstructs the definition of a covariant differential associated to a general connection. For strong connections this problem disappears. First, we can define:

**Definition 3.14.** Let $B \subseteq P$ be a coalgebra-Galois $C$-extension. A strong covariant differential is a homomorphism $D : P \to (\Omega^1 B)P$ satisfying

1. $D(bp) = bD(p) + (db)p$, $\forall b \in B$, $p \in P$ (the Leibniz rule),
2. $(\text{id} \otimes \Delta_P) \circ D = (D \otimes \text{id}) \circ \Delta_P$ (covariance).

Taking advantage of Lemma 3.12 and reasoning as in [34, 36], now we can prove:

**Lemma 3.15.** Let $B \subseteq P$ be a coalgebra-Galois $C$-extension. Then the equation

$$D = (\text{id} - \Pi) \circ \Omega \tag{3.93}$$

defines a one-to-one correspondence between strong connections and strong covariant differentials.

*Proof.* Given $s$ as in (2), define the corresponding $D_s : P \to (\Omega^1 B)P$ via $p \mapsto 1 \otimes p - s(p)$. Conversely, given $D$ as in (3), define $s_D : P \to B \otimes P$, $p \mapsto 1 \otimes p - D(p)$. This establishes the equivalence between descriptions (2) and (3). The equivalence $(3) \iff (1)$ is established as follows. Given $D$, define $\Pi_D : pd(q) \mapsto pd(q) - pD(q)$, while given $\Pi$ define $D_\Pi = d - \Pi \circ d : P \to (\Omega^1 B)P$. \hfill \Box

**Remark 3.16.** In the light of Lemma 3.14, the same arguments as in [36, pp. 314–315] establish correspondence between strong connections on coalgebra-Galois extensions and Cuntz-Quillen connections on bimodules [34, p.283].

Let us now pass to strong-connection forms. One can define them simply as connection forms corresponding to strong connections via Theorem 3.3. However, it turns out that one needs to assume coaugmentation to define the strongness condition for connection forms in a more intrinsic way.

**Lemma 3.17.** Let $B \subseteq P$ be an e-coaugmented coalgebra-Galois $C$-extension such that the canonical entwining map is injective. For a connection 1-form $\omega$, the following are equivalent:

1. $\omega$ is a strong connection one-form;
2. $(\text{id} \otimes \Delta_P) \circ \omega(c) = 1 \otimes 1 \otimes c - \epsilon(c)1 \otimes 1 \otimes e + \omega(c_{(1)}) \otimes c_{(2)}$.

*Proof.* The injectivity of $\psi$ implies that $x \in B \otimes P$ if and only if $\Delta_{P \otimes P}(x) = (\text{id} \otimes \Delta_P)(x)$. Indeed, it is clear that if $x \in B \otimes P$, then $\Delta_{P \otimes P}(x) = (\text{id} \otimes \Delta_P)(x)$. Conversely, write $x = \sum_i r^i \otimes p^i$. In view of the definition of the coaction $\Delta_{P \otimes P}$ and the fact that $\Delta_P(p) = \psi(e \otimes p)$, the condition $\Delta_{P \otimes P}(x) = (\text{id} \otimes \Delta_P)(x)$ explicitly reads

$$(\text{id} \otimes \psi)(\sum_i r^i \otimes p^i) = (\text{id} \otimes \psi)(\sum_i r^i \otimes e \otimes p^i).$$

Since $\psi$ is injective, this implies that $\sum_i \Delta_P(r^i) \otimes p^i = \sum_i r^i \otimes e \otimes p^i$, i.e., for each of the $r^i$s, we have $\Delta_P(r^i) = r^i \otimes e$, so that all of them are elements of $B$, as claimed.
Let $\omega$ be a connection form and let $D$ be the corresponding covariant differential. By definition, $\omega$ is strong if and only if, for all $p \in P$, $D(p) \in (\Omega^1 B) P$. In view of the discussion above, this is equivalent to the condition
\[
(id \otimes \Delta_P) \circ D(p) = \Delta_{\Omega^1 P} \circ D(u), \quad \forall p \in P.
\] (3.94)
Using the explicit definition of $d$ and $D$, Theorem 3.3(iii), as well as the fact that $\Omega^1 P \in M_{\tilde{H}}(\psi \otimes \tilde{\psi})$, one finds that (3.94) implies that
\[
(id \otimes \Delta_P)(p(0)\omega(p(1))) = p(0) \otimes 1 \otimes p(1) - p \otimes 1 \otimes e + p(0) \omega(p(1)) \otimes p(2).
\] (3.95)
Next, for all $c$, let $\tau(c) = c^{[1]} \otimes_B c^{[2]}$ be the translation map. Using the Translation Map Lemma 2.22, we compute
\[
(id \otimes \Delta_P) \circ \omega(c) = (id \otimes \Delta_P)(c^{[1]} \omega(c^{[2]})) = c^{[1]}(id \otimes \Delta_P)(c^{[2]} \omega(c^{[2]})) = c^{[1]}c^{[2]}(0) \otimes 1 \otimes c^{[2]}(1) - c^{[1]}c^{[2]} \otimes 1 \otimes e + c^{[1]}c^{[2]}(0) \omega(c^{[2]}(1)) \otimes c^{[2]}(2)
\] 
\[= 1 \otimes 1 \otimes c - \varepsilon(c) 1 \otimes 1 \otimes e + \omega(c^{[1]}) \otimes c^{[2]}.
\]
Thus if $\omega$ is a strong-connection form, then the assertion (b) holds. Conversely, an easy calculation reveals that the assertion (b) implies equation (3.94), i.e., the connection is strong, as required. \hfill \Box

The structure of coalgebra-Galois extensions is even richer when the canonical entwining map is bijective. In the case of a Hopf-Galois $H$-extension $B \subseteq P$, the canonical entwining map has the form $\psi: h \otimes p \mapsto p(0) \otimes hp(1)$. Hence it is bijective, provided the antipode is bijective. The inverse of $\psi$ then reads, $\psi^{-1} : p \otimes h \mapsto hS^{-1}(p(1)) \otimes p(0)$ (cf. Example 2.32). Thus, whenever the bijectivity of $\psi$ is assumed one should keep in mind that this corresponds to the bijectivity of the antipode in the case of a Hopf-Galois extension. This heuristic understanding can be extended even further, once one realises that also in more general case of a Doi-Koppinen datum $(H, C, A)$ in Example 2.29, the corresponding entwining map $\psi$ is bijective provided $H$ is a Hopf algebra with a bijective antipode (in fact, suffices it assume that $H$ is a bialgebra with a twisted antipode, i.e., $H^{op}$ is a Hopf algebra).

Much as for strong connection forms, we can simply define strong-connection liftings as connection liftings corresponding to strong connections. This time, to obtain a more intrinsic characterisation of the strongness condition, we need additionally to assume that the canonical entwining map $\psi$ of our $e$-coaugmented coalgebra-Galois $C$-extension is bijective. In this case, we can first define a left coaction
\[
p\Delta : P \longrightarrow C \otimes P, \quad p\Delta(p) := \psi^{-1}(p \otimes e).
\] (3.96)

Remark 3.18. That $p\Delta$ is a coaction can be verified directly, but it can also be seen as follows. In Definition 2.26 we defined a right-right entwining structure, in the sense that the structures defined in Lemma 2.27 are a right comodule and a right module structures. One can easily define a left-left entwining structure, by flipping all tensor products in the bow-tie diagram (a left-left entwining would then be a map $A \otimes C \rightarrow C \otimes A$). One then immediately has the left-handed version of Lemma 2.27. Now, if $\psi$ is a right-right entwining map, then its inverse is a left-left entwining map. Thus the left-handed version of Lemma 2.27 implies that $p\Delta$ given by equation (3.96) is a left coaction.
In addition to Lemma 3.17, we obtain the following characterisation of strong connection forms.

**Lemma 3.19.** Let \( B \subseteq P \) be an \( e \)-coaugmented coalgebra-Galois \( C \)-extension whose canonical entwining map \( \psi \) is bijective. For a connection form \( \omega \) the following are equivalent:

(a) \( \omega \) is a strong connection one-form;

(b) \[ (P \Delta \otimes \operatorname{id}) \circ \omega(c) = c \otimes 1 \otimes 1 - e \otimes 1 \otimes \varepsilon(c)1 + c_{(1)} \otimes \omega(c_{(2)}). \]

**Proof.** Obviously, since \( \psi \) is bijective, Lemma 3.17 holds. Hence it suffices to apply the bijection \( \psi^{-1} = (\psi^{-1} \otimes \operatorname{id}) \circ (\operatorname{id} \otimes \psi^{-1}) \) to assertion (b) in Lemma 3.17.

Now a strong-connection lifting can be characterised as follows:

**Lemma 3.20 ([19]).** Let \( B \subseteq P \) be an \( e \)-coaugmented coalgebra-Galois \( C \)-extension whose canonical entwining map \( \psi \) is bijective, and let \( \ell \) be a homomorphism from \( C \) to \( P \otimes P \). It is a strong-connection lifting if and only if it satisfies the following conditions:

1. \( \ell(e) = 1 \otimes 1 \) (unitality),
2. \( \pi_B \circ \ell = \tau, \pi_B : P \otimes P \to P \otimes_B P \), (lifting property)
3. \( (P \Delta \otimes \operatorname{id}) \circ \ell = (\operatorname{id} \otimes \ell) \circ \Delta \) and \( (\operatorname{id} \otimes \Delta_P) \circ \ell = (\ell \otimes \operatorname{id}) \circ \Delta \) (bicolinearity).

Cogaugmented coalgebra-Galois extensions whose canonical entwining map is bijective are very symmetric. Following the usual convention of differential geometry, where one considers right rather than left bundles, we formulated the right-sided version of extensions. However, much as in differential geometry, any \( e \)-coaugmented coalgebra-Galois \( C \)-extension \( B \subseteq P \) enjoying the existence of \( \psi^{-1} \) can be equivalently formulated as a left coalgebra-Galois \( C \)-extension. To begin with, there already exists a left \( C \)-coaction on \( P \) given by the formula (3.96).

Next, we need to check that the left and right coaction invariants coincide:

**Lemma 3.21 ([23]).** Let \( B \subseteq P \) be an \( e \)-coaugmented coalgebra-Galois \( C \)-extension whose canonical entwining map \( \psi \) is bijective. Then \( \left< c_{\text{co}} P \right> = B \).

**Proof.** Since \( b \in B \) if and only if \( \psi(e \otimes b) = \Delta_P(b) = b \otimes e \) (cf. Proposition 2.23), applying \( \psi^{-1} \), we immediately conclude that \( b \in B \) if and if

\[ \left< c_{\text{co}} P \right>_e = \{ p \in P \mid \Delta_P(p) = p \otimes e \}. \] (3.97)

The left handed version of Lemma 2.1 implies that \( \left< c_{\text{co}} P \right> \subseteq \left< c_{\text{co}} P \right>_e \). Now, since \( P \) is a right-right \((P, C, \psi)\)-entwined module by Theorem 2.31, one easily checks that \( P \) is also a left-left \((P, C, \psi^{-1})\)-entwined module with coaction \( P \Delta \), i.e., \( P \in \mathcal{C}_P(\psi^{-1}) \) (cf. Remark 3.18). Thus, if \( b \in \left< c_{\text{co}} P \right>_e \), we can compute for any \( p \in P \),

\[ P \Delta(p b) = \psi^{-1}(p \otimes b_{(-1)})) \otimes b_{(0)} = \psi^{-1}(p \otimes e)b = \Delta_P(p) b. \]

Therefore, \( b \in \left< c_{\text{co}} P \right> \), so that \( \left< c_{\text{co}} P \right> = \left< c_{\text{co}} P \right>_e \). This completes the proof. \( \square \)
The right coaction is left $B$-linear, whereas the left coaction is right $B$-linear. (In the Hopf-Galois case, they are both $B$-bimodule homomorphisms.) One should also bear in mind that, in general, even under the assumption of commutativity, $P$ is not a bicomodule with respect to $p\Delta$ and $\Delta P$.

Now, the left canonical map can be defined as follows:

$$can_L : P \otimes_B P \longrightarrow C \otimes P, \quad can_L(x \otimes y) := \Delta_P(x)y.$$ (3.98)

It is straightforward to verify that the left and right canonical maps are related by the commutative diagram

$$\begin{array}{c}
P \otimes_B P \\
can_P \\
\downarrow \downarrow \\
\uparrow \uparrow \\
C \otimes P \\
\psi \\
P \otimes C.
\end{array}$$ (3.99)

This implies $\tau(c) = can_P^{-1}(1 \otimes c) = can_L^{-1}(c \otimes 1) = c^{[1]} \otimes_B c^{[2]}$. (Here we used a more symmetric convention $can_P = can$.) Therefore, even though there are left and right canonical maps, the left and right translation maps coincide. Consequently, the lifting formulation of a strong connection is also independent of the choice of left or right-sided formulation.

So far we restricted our attention to formulating the concept of a strong connections without asking when such a connection exists. It turns out that the exisitance of a strong connection both forces and is guaranteed by the equivariant projectivity. More precisely, we have:

**Lemma 3.22 ([19]).** An $e$-coaugmented coalgebra-Galois $C$-extension $B \subseteq P$ admits a strong connection if and only if $P$ is $C$-equivariantly projective as a left $B$-module.

Therefore, to have available all 5 formulations of a strong connection and to ensure their existence, we need to demand that our coalgebra-Galois extension is equivariantly projective, coaugmented and with bijective canonical entwining. Thus we arrive at the concept of a principal extension from postulating rich strong-connection theory. The following theorem summarises and completes this section on strong-connection theory. It is a generalisation of Theorem 2.3 and Theorem 4.1 in [36].

**Theorem 3.23. (Strong-Connection Theorem)** Let $B \subseteq P$ be a principal $C$-extension. For any $c \in C$, write the translation map $\tau(c) = c^{[1]} \otimes_B c^{[2]}$. The following maps

$$s \mapsto D_s, \quad D_s(p) = 1 \otimes p - s(p), \quad \forall p \in P,$$ (3.100)

$$D \mapsto \Pi, \quad \Pi(pdp') = pdp' - pD(p'), \quad \forall p, p' \in P,$$ (3.101)

$$\Pi \mapsto \omega, \quad \omega(c) = c^{[1]}\Pi(1c^{[2]}), \quad \forall c \in C,$$ (3.102)

$$\omega \mapsto \ell, \quad \ell(c) = \omega(c) + \varepsilon(c)1 \otimes 1, \quad \forall c \in C,$$ (3.103)

$$\ell \mapsto s, \quad s(p) = p_{(0)}\ell(p_{(1)}), \quad \forall p \in P,$$ (3.104)

give bijective correspondences between sets of strong-connection splittings $s$, strong covariant differentials $D$, strong connections $\Pi$, strong-connection forms $\omega$, and strong-connection liftings $\ell$. The explicit form of inverses of these maps is obtained by cyclic composition, e.g., the inverse of map (3.101) is obtained by composing maps (3.102) with (3.103), with (3.104) with (3.100), etc.
Proof. The equivalent descriptions of strong connections given by maps (3.100) and (3.101) are contained in Lemma 3.12 and Lemma 3.15, while the map (3.102) is described in Theorem 3.3. The map (3.103) originates from equation (3.80). That the connection splitting corresponding to a strong connection is a strong-connection splitting follows from Lemma 3.20. Thus we obtain maps between claimed spaces as required. One easily checks that cyclic compositions provide inverses, as described.

Since a gauge automorphism $F$ is unital, i.e., $F(1) = 1$, its left $B$-linearity implies that $F(b) = b$ for all $b \in B$. Furthermore, $F$ is $C$-colinear, so that given a strong-connection splitting $s : P \to B \otimes P$, the map $s^F = (\text{id} \otimes F^{-1}) \circ s \circ F : P \to B \otimes P$ is again a strong-connection splitting. It is clear from the form of $s^F$ that the assignment $F \mapsto s^F$ defines a left action of the group of gauge automorphisms on strong-connection splittings (remember that we use conventions in which the product of gauge automorphisms is given by the opposite composition). In view of the description of gauge automorphisms in terms of gauge transformations in Theorem 3.9 as well as various descriptions of strong connections in Theorem 3.23, we are led to the following.

**Theorem 3.24.** The group of gauge transformations of a principal $C$-extension $B \subseteq P$ acts on the spaces of strong connection splittings, covariant differentials, connections, connection forms and connection liftings in the following ways, for all $f \in GT^C(B \subseteq P)$, $p, r \in P$ and $c \in C$:

1. **Strong-connection splittings** $s : P \to B \otimes P$:
   $$(f \triangleright s)(p) := s(p(0)f(p(1)))f^{-1}(p(2));$$

2. **Strong covariant differentials** $D : P \to \Omega^1 BP$:
   $$(f \triangleright D)(p) := D(p(0)f(p(1)))f^{-1}(p(2));$$

3. **Strong connections** $\Pi : \Omega^1 P \to \Omega^1 P$:
   $$(f \triangleright \Pi)(rdp) := r\Pi(d(p(0)f(p(1))))f^{-1}(p(2)) + rp(0)f(p(1))df^{-1}(p(2));$$

4. **Strong connection forms** $\omega : C \to \Omega^1 P$:
   $$(f \triangleright \omega)(c) := f(c(1))\omega(c(2))f^{-1}(c(3)) + f(c(1))df^{-1}(c(2));$$

5. **Strong connection liftings** $\ell : C \to P \otimes P$:
   $$(f \triangleright \ell)(c) := f(c(1))\ell(c(2))f^{-1}(c(3)).$$

All these actions are compatible with the isomorphisms described in Theorem 3.23, i.e., the maps given by (3.100)–(3.104) are left $GT^C(B \subseteq P)$-module maps.

Proof. In view of Theorem 3.9 and the discussion preceding the theorem, to show that item (1) describes a left action of $GT^C(B \subseteq P)$ on the space of strong-connection splittings, it suffices to show that $f \triangleright s = s^{F_f} := (\text{id} \otimes F_f^{-1}) \circ s \circ F_f$, where $F_f$ is given by equation (3.82). Since $s$ is right $C$-colinear, and, for all $p \in P$, $F_f(p) = p(0)f(p(1))$ and $F_f^{-1}(p) = p(0)f^{-1}(p(1))$, one immediately finds that $((\text{id} \otimes F_f^{-1}) \circ s)(p) = s(p(0))f^{-1}(p(1))$. Furthermore, $P$ is an entwined module, so that

\[
\begin{align*}
  s^{F_f}(p) &= s((p(0)f(p(1)))(0))f^{-1}((p(0)f(p(1)))(1)) \\
  &= s(p(0)f(p(2))a)f^{-1}(p(1)a) \\
  &= s(p(0)f(p(1)))f^{-1}(p(2)).
\end{align*}
\]
Note that the final equality follows from the fact that $f$ is a gauge transformation, i.e., it satisfies equation (3.84). The formulae for the action in the case of other descriptions of a strong connection (items (2)–(5)) are obtained by applying maps described in Theorem 3.23. In particular, this implies that descriptions of actions of the gauge group are compatible with these maps. The reader can directly check these formulae, noting that the element $1 \otimes 1 \in P \otimes P$ is fixed under the gauge transformations. \hfill $\square$

### 3.2.1 Covariant derivatives on associated modules

**Theorem 3.25.** *Modules associated to equivariantly projective symmetric (bijectivity of the canonical entwining assumed) coalgebra-Galois $C$-extensions via finite-dimensional corepresentations are finitely generated projective.*

Once vector bundles are identified with projective modules one can study connections in such modules. In general [31], for an algebra $B$ and a left $B$-module $E$, a *connection* is defined as a linear map $\nabla : E \rightarrow \Omega^1 B \otimes_B E$, which satisfies the Leibniz rule in the form

$$\nabla(b \cdot f) = d(b) \otimes_B f + b\nabla(f),$$

for all $b \in B$ and $f \in E$. The theory of connections is of particular interest, and indeed, meaningful, in the case of projective modules, since a module admits a connection if and only if it is a projective module [34]. The connection constructed directly from an idempotent is known as the *Grassmann* connection. Note that there is no need for a projective module to be finitely generated in order to have a connection.

Given a strong connection 1-form $\omega$ in a symmetric coalgebra-Galois $e$-coaugmented $C$-extension $B \subseteq P$, the corresponding covariant differential induces a map on the associated module of sections $\text{Hom}^C(V, P)$:

$$\nabla : \text{Hom}^C(V, P) \rightarrow \text{Hom}^C(V, (\Omega^1 B) P), \ f \longmapsto \nabla f,$$

where $\nabla f(v) := df(v) - f(v_{(0)})\omega(v_{(1)})$.

**Proposition 3.26.** *If $V$ is finite dimensional then $\nabla$ is a connection on $\text{Hom}^C(V, P)$.***

**Proof.** The first crucial observation here is that if $V$ is finite dimensional, then $\text{Hom}^C(V, (\Omega^1 B) P) \cong \Omega^1 B \otimes_B \text{Hom}^C(V, P)$. Indeed, if a right $C$-comodule $V$ is finite dimensional, then the dual space $V^*$ is a left $C$-comodule with the coaction given explicitly, for all $v^* \in V^*$, $V^* \varrho(v^*) = \sum_{i=1}^n v^*(e_i) e_i \otimes e^i$, where $\{e_i \in V\}_{i=1,...,n}$ and $\{e^i \in V^*\}_{i=1,...,n}$ are dual to each other bases of $V$ and $V^*$, respectively. For any right $C$-comodule $W$, we have the canonical identification

$$\text{Hom}^C(V, W) \cong W \square_C V^*.$$ 

Here

$$W \square_C V^* := \left\{ \sum_i w_i \otimes v^*_i \in W \otimes V^* \mid \sum_i \Delta_W(w_i) \otimes v^*_i = \sum_i w_i \otimes V^* \varrho(v^*_i) \right\}$$

is the cotensor product of a right and a left $C$-comodule. We then have the following chain of identifications

$$\text{Hom}^C(V, (\Omega^1 B) P) = \text{Hom}^C((V, \Omega^1 B \otimes_B P) \cong (\Omega^1 B \otimes_B P) \square_C V^*$$

$$= \Omega^1 B \otimes_B (P \square_C V^*) \cong \Omega^1 B \otimes_B \text{Hom}^C(V, P).$$
The redistribution of brackets in the penultimate equality is possible because \( \Omega^1 B \) is a flat right \( B \)-module. Thus the map \( \nabla \) can be viewed as a map \( \nabla : \text{Hom}^C(V, P) \to \Omega^1 B \otimes_B \text{Hom}^C(V, P) \), and has a right range for a connection. Hence only the Leibniz rule needs to be verified. For all \( b \in B \) and \( f \in \text{Hom}^C(V, P) \), we have

\[
\nabla(b \cdot f)(v) = d(bf(v)) - bf(v_0)\omega(v_1) = d(b)f(v) + b(df(v) - f(v_0))\omega(v_1)) = d(b)f(v) + b\nabla(f)(v),
\]

as required.

**Remark 3.27.** Due to the right \( C \)-colinearity of the covariant differential \( D \), we can re-write point (2) of the above theorem in terms of the gauge automorphisms \( F \) to obtain the formula \( (F \triangleright D)(p) = F^{-1}(DF(p)) \). This formula coincides with the usual formula for the action of gauge transformations on projective-module connections, cf. [31, p.554]. Note, however, that since we use the opposite composition as a group operation in \( GA^C(B \subseteq P) \), we have a left rather than right action here.

### 3.2.2 Strong connections on pullback constructions

Let \( _C A_C \) be the category of unital algebras equipped with left and right (not necessarily commuting) coactions \( \Delta_A \) and \( \Delta_A \) of an \( e \)-coaugmented coalgebra \( C \) such that \( \Delta_A(1) = e \otimes 1 \) and \( \Delta_A(1) = 1 \otimes e \). Morphisms in this category are bicolinear algebra homomorphisms. Since we work over a field, this category is evidently closed under any pullback

\[
(3.105)
\]

The aim of this section is to show that the subcategory of principal extensions is closed under one-surjective pullbacks. Here the right coaction is the coaction defining a principal extension and the left coaction is the one defined by the inverse of the canonical entwining. The following theorem generalises the two-surjective pullback Hopf-Galois result of [50]:

**Theorem 3.28** ([52]). Let \( C \) be an \( e \)-coaugmented coalgebra and let \( P \) be the pullback of \( \pi_1 : P_1 \to P_{12} \) and \( \pi_2 : P_2 \to P_{12} \) in the category \( _C A_C \). If \( \pi_1 \) or \( \pi_2 \) is surjective and both \( P_1 \) and \( P_2 \) are principal \( C \)-extensions, then \( P \) is a principal \( C \)-extension.

**Proof.** Without the loss of generality, let us assume that \( \pi_2 \) is surjective. First step is to prove that any surjective morphism in \( _C A_C \) whose domain is a principal extension can be split by a left colinear map and by a right colinear map (not necessarily by a bicolinear map). This can be proved much the same way as in the Hopf-Galois case [50].

Let \( \alpha^2_L \) and \( \alpha^2_R \) be a left colinear splitting and a right colinear splitting of \( \pi_2 \), respectively.
Also, let $\alpha_1^R$ be a right colinear splitting of $\pi_1$ viewed as a map onto $\pi_1(P_1)$.

$$\begin{align*}
\alpha_1^R & \colon P_1 \to P_1, \\
\pi_1 & \colon P \to P_1, \\
\pi_2 & \colon P \to P_2.
\end{align*}$$

Since $P_1$ and $P_2$ are principal, they admit strong-connection liftings $\ell_1$ and $\ell_2$, respectively. For brevity, let us introduce the notation

$$\begin{align*}
\alpha_{12}^L & := \alpha_2^L \circ \pi_1, \\
\alpha_{12}^R & := \alpha_2^R \circ \pi_1, \\
\alpha_{21}^R & := \alpha_1^R \circ \pi_2 |_{\pi_1^{-1}(\pi_1(P_1))}, \\
L & := m_{P_2} \circ (\alpha_{12}^L \otimes \alpha_{12}^R) \circ \ell_1,
\end{align*}$$

where $m_{P_2}$ is the multiplication of $P_2$. In the light of Lemma 3.22, the proof boils down to verifying that the following formula defines a strong-connection lifting on $P$:

$$\ell := ((\text{id} + \alpha_{12}^L) \otimes (\text{id} + \alpha_{12}^R)) \circ \ell_1$$

$$+ (\text{pr}_2 \circ \varepsilon - L) \ast \left((\text{id} \otimes (\text{id} + \alpha_{21}^R)) \circ (\ell_2 - \ell_2 \ast L + (\alpha_{12}^L \otimes \alpha_{12}^R) \circ \ell_1)\right).$$

Here $\text{pr}_2$ is the canonical pullback map on the second component and $\ast$, as usual, stands for the convolution product.

### 3.2.3 Strong connections on extensions by coseparable coalgebras

In view of Theorem 2.37, an $e$-coaugmented bijectively entwined extension $B \subseteq P$ by a coseparable coalgebra $C$ is principal, provided the lifted canonical map is surjective. Following [4], we construct now an explicit form of a connection lifting in this case.

Assume that $C$ is a coseparable coalgebra with a cointegral $\delta$. Take an entwining structure $(P, C, \psi)$ such that the map $\psi$ is bijective. Suppose that $e \in C$ is a group-like element and view $P$ as a right $C$-comodule with the coaction $\Delta_P : P \to P \otimes C$, $p \mapsto \psi(e \otimes p)$, and as a left $C$-comodule with coaction $p \Delta : P \to C \otimes P$, $p \mapsto \psi^{-1}(p \otimes e)$. Let $\tilde{\sigma}$ be a $k$-linear section of the lifted canonical map

$$\tilde{\text{can}} : P \otimes P \to P \otimes C, \quad p \otimes q \mapsto p \Delta_P(q).$$

Since $\tilde{\text{can}}(1 \otimes 1) = 1 \otimes e$, the linear map $\sigma := \tilde{\sigma}(1 \otimes \cdot)$ can always be normalised (so that $\sigma(e) = 1 \otimes 1$) by making the linear change

$$\sigma \mapsto \sigma + 1 \otimes 1 \varepsilon - \sigma(e) \varepsilon.$$

We thus choose $\sigma$ that already is normalised in this way. By Theorem 2.37, $B \subseteq P$ is a principal $C$-extension. Define

$$\gamma = (\delta \otimes \text{id}_P) \circ (\text{id}_C \otimes p \Delta), \quad \alpha = (\text{id}_P \otimes \delta) \circ (\Delta_P \otimes \text{id}_C),$$

and

$$\ell = (\gamma \otimes \alpha) \circ (\text{id}_C \otimes \sigma \otimes \text{id}_C) \circ (\Delta \otimes \text{id}_C) \circ \Delta.$$
Theorem 3.29 ([4]). The map $\ell$ given by (3.110) is a strong-connection lifting.

**Proof.** Using (2.68) one easily checks that the map $\gamma$ is left $C$-colinear, where $C \otimes P$ as understood as a left $C$-comodule via $\Delta \otimes \text{id}$, and $\alpha$ is right $C$-colinear, where $P \otimes C$ is a right $C$-comodule via $\text{id} \otimes \Delta$. By the colinearity of $\gamma$ and $\alpha$, the map $\ell$ is $C$-bicolinear.

To prove that $\ell$ has a lifting property, we start with the following simple calculation, for all $p, q \in P$,

$$\psi^{-1}(p\Delta_P(q)) = \psi^{-1}(p\psi(e \otimes q)) = \psi^{-1}(p \otimes e)q = p\Delta(p)q.$$  

Here the first and last equalities follow from the definitions of the right and left $C$-coactions on $P$, and the second equality follows by the fact that $\psi^{-1}$ is the inverse of the entwining map $\psi$. Thus we obtain the equality

$$\psi^{-1}(pq(0) \otimes q(1)) \otimes q(2) = p\Delta(p)\Delta_P(q).$$  

For any $c \in C$, write explicitly $c^{(1)} \otimes c^{(2)} := \sigma(c)$, so that $c^{(1)}c^{(2)}(0) \otimes c^{(2)}(1) = 1 \otimes c$. This leads to the equality

$$c^{(1)} \otimes c^{(2)}(0) \otimes c^{(2)}(1) \otimes c^{(3)} = c^{(1)} \otimes 1 \otimes c^{(2)} \otimes c^{(3)}.$$  

Apply $(\text{id}_C \otimes \psi^{-1} \otimes \text{id}_C \otimes \Delta) \circ (\text{id}_C \otimes \text{id}_P \otimes \Delta \otimes \text{id}_C)$ and then use (3.111) on the left hand side, and the unitality of the entwining map (the left triangle in the bow-tie diagram) on the right hand side, to obtain

$$c^{(1)} \otimes p\Delta(c^{(2)}(1))\Delta_P(c^{(2)}(2)) \otimes c^{(3)} \otimes c^{(4)} = c^{(1)} \otimes c^{(2)} \otimes 1 \otimes c^{(3)} \otimes c^{(4)} \otimes c^{(5)}.$$  

Now apply $\delta \otimes \text{id}_P \otimes \delta \otimes \text{id}_C$ and use the definitions of maps $\gamma$ and $\alpha$ in terms of $\delta$ on the left hand side, and the properties of the cointegral (2.68) on the right, to conclude that

$$\gamma(c^{(1)} \otimes c^{(2)}(1))\alpha(c^{(2)}(2) \otimes c^{(3)}) \otimes c^{(4)} = 1 \otimes c.$$  

By the right $C$-colinearity of $\alpha$, this implies that $\overline{\text{can}} \circ \ell = 1 \otimes \text{id}_C$. Hence, as $\text{can}^{-1} \circ \overline{\text{can}} = \pi_B$ is the standard projection $P \otimes P \to P \otimes_B P$, we conclude that $\pi_B \circ \ell = \tau$, as required.

Finally, the definitions of left and right $C$-coactions on $P$ an (2.68) imply that $\alpha(1 \otimes e) = 1$ and $\gamma(e \otimes 1) = 1$. These equalities together with the chosen normalisation for $\sigma$ yield $\ell(e) = 1 \otimes 1$.

Therefore, the map $\ell$ defined in (3.110) satisfies all the properties of Lemma 3.20, i.e. it is a strong connection lifting, as stated. \qed

### 3.2.4 Strong connections on homogeneous Galois extensions

As a further illustration of the theory of strong connections in symmetric (bijectivity of the canonical entwining assumed) coalgebra-Galois extensions, we consider such connections in a coalgebra-Galois extension of a quantum homogeneous space. This is a preparation for an explicit example in the next section.

Let $P$ be a Hopf algebra and $B \subseteq P$ a left coideal subalgebra. Consider the homogeneous coalgebra-Galois $P/B^+P$-extension as in Section 2.1.2. Write $C := P/B^+P$ and $\pi : P \to C$ for the canonical epimorphism. If the antipode $S$ of $P$ is bijective, then the canonical entwining map $\psi$ given by $\psi(c \otimes p) = p(1) \otimes \pi(p'p(2)) = p(1) \otimes c \cdot p(2)$ for $p' \in \pi^{-1}(c)$ is bijective with inverse $\psi^{-1}(p \otimes e) = c \cdot S^{-1}(p(2)) \otimes p(1)$, so that $B \subseteq P$ is a symmetric coalgebra-Galois extension.
Now, if we consider the Hopf algebra $P$ as a $P$-bicomodule by the comultiplication $\Delta$ (regular coactions), then the universal differential calculus $\Omega^1 P$ is bicovariant (cf. [96]). More precisely, the diagonal $P$-coactions on $P \otimes P$ can be restricted to a right $\Omega^1 P \Delta$ and a left $\Omega^1 P \Delta$ coaction on $\Omega^1 P$. The coactions $\Delta_{\Omega^1 P}$ and $\Omega^1 P \Delta$ make $\Omega^1 P$ into a $P$-bicomodule. Furthermore, one easily checks that the universal differential $d : P \to \Omega^1 P$ is a $P$-bicomodule map, i.e.,

$$\Omega^1 P \Delta \circ d = (\text{id} \otimes d) \circ \Delta, \quad \Delta_{\Omega^1 P} \circ d = (d \otimes \text{id}) \circ \Delta.$$ 

Since $\Omega^1 P$ is a bicovariant calculus, one can, in particular, consider left-invariant forms, i.e., elements $\omega \in \Omega^1 P$ such that $\Omega^1 P \Delta(\omega) = 1 \otimes \omega$. Any bicovariant calculus on $P$, which necessarily is obtained as a quotient of the universal calculus $\Omega^1 P$, is generated by left-invariant forms (as a left or right $P$-module).

In the case of a coalgebra-Galois extension $B \subseteq P$, one can also study strong connections whose connection forms $\omega$ are left-invariant, i.e., such that, for all $c \in C$, $\Omega^1 P \Delta(\omega(c)) = 1 \otimes \omega(c)$. The following theorem classifies all left-invariant strong-connection forms in the symmetric case [23, Proposition 4.4].

**Theorem 3.30.** Consider a homogeneous coalgebra-Galois $C$-extension $B \subseteq P$. Assume that the antipode $S$ is bijective, i.e., $B \subseteq P$ is a symmetric coalgebra-Galois extension. View the Hopf algebra $P$ as a right $C$-comodule as $\Delta_P = (\text{id}_P \otimes \pi) \circ \Delta$ and as a left $C$-comodule via $P \Delta = (\pi \otimes \text{id}_P) \circ \Delta$. (Note that this is not the induced left coaction (3.96).) Then there is a one-to-one correspondence between left-invariant strong-connection forms and $C$-bicomodule maps $i : C \to P$ such that $\pi \circ i = \text{id}_C$, $i(\pi(1)) = 1$, and $\varepsilon_P \circ i = \varepsilon_C$. A connection 1-form is given by $\omega(c) = S(i(c(1)))d(i(c(2)))$.

**Proof.** Given such a splitting $i : C \to P$ of $\pi$, consider $\omega(c) = Si(c(1))d(i(c(2)))$, as stated. The normalisation conditions imply that $\omega(\pi(1)) = 0$ and $\tilde{\text{can}} \circ \omega(c) = 1 \otimes c - \varepsilon(c)1 \otimes \pi(1)$. Use the short-hand notation $\psi^2 := (\text{id}_P \otimes \psi) \circ (\psi \otimes \text{id}_P)$ and compute

$$\psi^2(c(1) \otimes \omega(c(2))) = Si(c(2)(2)\text{di}(c(2)(3)) \otimes \pi(i(c(1)))Si(c(2)(1)\text{di}(c(2)(4))) = Si(c(1)(3)\text{di}(c(1)(4)) \otimes \pi(i(c(1)))Si(i(c(2))\text{di}(c(2)(5))) \quad (i \text{ is left-colinear})$$

$$= Si(c(1)(1)\text{di}(c(1)(2)) \otimes \pi(i(c(1)))) \quad (i \text{ is right-colinear})$$

$$= \omega(c(1)) \otimes c(2) \quad (\pi \text{ is split by } i).$$

Theorem 3.3 implies that $\omega$ is a connection one-form. Finally, compute

$$(\text{id} \otimes \Delta_P)(\omega(c)) = Si(c(1)(1) \otimes i(c(1)(2)) \otimes \pi(i(c(1)(3)) - \varepsilon(c)1 \otimes 1 \otimes \pi(1)) = Si(c(1)(1) \otimes i(c(1)(2)) \otimes c(2) - \varepsilon_C(c)1 \otimes 1 \otimes \pi(1)) = \omega(c(1)) \otimes c(2) + 1 \otimes 1 \otimes c - \varepsilon_C(c)1 \otimes 1 \otimes \pi(1),$$

where the use of the fact that $i$ is a right colinear splitting was made in the derivation of the second equality. Lemma 3.17 now implies that the connection corresponding to $\omega$ is strong.

Conversely, assume that there is a strong connection with the left-invariant connection form $\omega$. Then the left-invariance of $\omega$ implies that there exists a splitting $i : C \to P$ of $\pi$ such that $\varepsilon_P \circ i = \varepsilon_C$ and $\omega(c) = Si(c(1))\text{di}(c(2))$ (cf. [22, Proposition 3.5]). The fact that $\omega(\pi(1)) = 0$ implies that $i(\pi(1)) = 1$. Applying $(\text{id} \otimes \Delta_P)$ to this $\omega$ and using Lemma 3.17, one deduces that $i$ is right-colinear. Since we are dealing with a symmetric coalgebra-Galois extension the
entwining map is bijective. The left coaction (3.96) induced by $\psi^{-1}$ is $p\Delta(p) = \pi(S^{-1}p(2)) \otimes p(1)$. By Lemma 3.19,

$$(p\Delta \otimes \text{id}_P)\omega(c) = \pi(i(c)(1)) \otimes Si(c)(2) \otimes i(c)(3) - \varepsilon_C(c)\pi(1) \otimes 1 \otimes 1 \quad (3.112)$$

must be equal to

$$c(1) \otimes S(i(c(2))(1)) \otimes i(c(2))(2) - \varepsilon_C(c)\pi(1) \otimes 1 \otimes 1. \quad (3.113)$$

Applying $\text{id} \otimes S^{-1} \otimes \varepsilon_C$ to this equality, one deduces that $i$ must be left-colinear (with respect to the coaction $p\Delta$). This completes the proof.  

Theorem 3.30 shows that strong connections in a symmetric coalgebra-Galois extensions over a coideal subalgebra can be obtained from purely (co)algebraic data. This observation allows one to construct concrete examples of strong connections.

### 3.2.5 Dirac monopoles over the Podleś 2-spheres

Consider the quantum Hopf fibration described in Section 2.4.2. In this case, the coalgebra $C$ is spanned by group-like elements $g_\mu$, $\mu \in \mathbb{Z}$, given by equations (2.70), and the bicolinear splitting $i$ of the projection $\pi$ can be relatively easily computed. Explicitly, for all positive integers $n$, it comes out as

$$i(g_n) = \prod_{k=0}^{n-1} \frac{\alpha + q^k s(\beta + \gamma) + q^{2k}s^2\delta}{1 + q^{2k}s^2},$$

$$i(g_{-n}) = \prod_{k=0}^{n-1} \frac{\delta - q^{-k}s(\beta + \gamma) + q^{-2k}s^2\alpha}{1 + q^{-2k}s^2}, \quad (3.114)$$

where the multiplication increases from left to right. Thus, in view of Theorem 3.30, we have constructed a strong left-invariant connection in the quantum Hopf fibration with connection lifting $\ell = (S \otimes \text{id}) \circ \Delta \circ i$. Such a connection in the classical Hopf fibration is known as the Dirac magnetic monopole, as it has a physical interpretation of a point particle which is a source of a magnetic field. (See [60] for very nice description of classical monopoles from the point of view of noncommutative geometry.) Motivated by this correspondence, the strong connection constructed from $i$ via Theorem 3.30 is called the Dirac $q$-monopole.

Furthermore, one can study the module of sections of a line bundle associated to the quantum Hopf fibration. As a right $C$-comodule we take the one-dimensional space $V = k$ with the coaction $\Delta_V(1) = 1 \otimes g_1$. Then the module of sections turns out to be $E = \text{Hom}^C(V, H) = \{x(\alpha + s\beta) + y(\gamma + s\delta) \mid x, y \in \mathcal{O}(S_{q,s}^2)\}$. Explicitly, $E$ is given by the following idempotent matrix:

$$E \cong (S_{q,s}^2)^2 \mathbf{p}, \quad \mathbf{p} = \frac{1}{1 + s^2} \begin{pmatrix} 1 - \zeta & \xi \\ -\eta & s^2 + q^{-2}\zeta \end{pmatrix}, \quad (3.115)$$

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