EXISTENCE RESULTS TO A NONLINEAR $p(k)$-LAPLACIAN DIFFERENCE EQUATION

MOHSEN KHALEGHI MOGHADAM AND MUSTAFA AVCI

Abstract. In the present paper, by using variational method, the existence of non-trivial solutions to an anisotropic discrete non-linear problem involving $p(k)$-Laplacian operator with Dirichlet boundary condition is investigated. The main technical tools applied here are the two local minimum theorems for differentiable functionals given by Bonanno.

1. Introduction

The main goal of the present paper is to establish the existence of non-trivial solution for the following discrete anisotropic problem
\[
\begin{aligned}
-\Delta(w(k-1)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) + q(k)|u(k)|^{p(k)-2}u(k) &= \lambda f(k, u(k)), \\
u(0) &= u(T + 1) = 0,
\end{aligned}
\]
for any $k \in [1, T]$, where $T$ is a fixed positive integer, $[1, T]$ is the discrete interval \{1, ..., $T$\}, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda > 0$ is a parameter and $w : [0, T] \to [1, \infty)$ is a fix function such that
\[
w^+ := \max_{k \in [0, T]} w(k), \quad w^- := \min_{k \in [0, T]} w(k),
\]
and $\Delta u(k) = u(k + 1) - u(k)$ is the forward difference operator and the function $p : [0, T + 1] \to [2, \infty)$ is bounded, we denote for short
\[
p^+ := \max_{k \in [0, T + 1]} p(k) \quad \text{and} \quad p^- := \min_{k \in [0, T + 1]} p(k),
\]
and the function $q : [0, T + 1] \to [1, \infty)$ is bounded such that
\[
q^- := \min_{k \in [1, T + 1]} q(k) \geq 1, \quad q^+ := \max_{k \in [1, T + 1]} q(k).
\]

We want to remark that problem (1.1) is the discrete variant of the variable exponent anisotropic problem
\[
\begin{aligned}
-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( w_i(x) \frac{\partial u}{\partial x_i} \right) |\frac{\partial u}{\partial x_i}|^{p_i(x)-2} \frac{\partial u}{\partial x_i} + q(x)|u|^{p_i(x)-2}u &= \lambda f(x, u), \\
u = 0, & \quad x \in \partial \Omega,
\end{aligned}
\]
where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain with smooth boundary, $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is given function that satisfy certain properties and $p_i(x), w_i(x) \geq 1$ and $q(x) \geq 1$ are continuous functions on $\overline{\Omega}$ with $2 \leq p_i(x)$ for each $x \in \Omega$ and every $i \in \{1, \ldots, N\}$. 2000 Mathematics Subject Classification. 39F20, 34B15.

Key words and phrases. Discrete nonlinear boundary value problem; Non trivial solution; Variational methods; Critical point theory.
\{1, 2, \cdots, N\}, \lambda > 0 \text{ is a real number.}

The importance of difference equations arises from its applications to many different fields of research, such as mechanical engineering, control systems, economics, social sciences, computer science, physics, artificial or biological neural networks, cybernetics, ecology, to name a few. In this context, anisotropic discrete nonlinear problems involving \(p(k)\)-Laplacian operator seem to have attracted a great deal of attention due to its usefulness of modelling some more complicated phenomenon such as fluid dynamics and nonlinear elasticity. We refer the reader to [1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 24] and references therein, where they could find the detailed background as well as many different approaches and techniques applied in the related area.

In this paper, based on two local minimum theorems, i.e. Theorem 2.1 and Theorem 2.2 due to Bonanno [8], we obtain the exact intervals for the parameter \(\lambda\), in which the problem (1.1) admits non-trivial solutions. In section 2, we recall the main tools (Theorem 2.1 and Theorem 2.2) and give some basic knowledge. In Section 3, we state and prove our main results of the paper containing several theorems as well as corollaries. Finally, we prove a special case of the main result (Theorem 1.1) and illustrate the results by giving concrete examples as applications to (1.1).

At the very beginning, as an example, we give the following special case of our main results.

**Theorem 1.1.** Let \(T\) is a fixed positive integer. Assume that there exist two positive constants \(c\) and \(d\) such that

\[
d^3 < \left( \frac{3}{(T + 2)(T + 4)(2T + 2)^{T+3}} \right) \frac{3}{(T + 2)(T + 4)} < d^3\]

Let \(g : \mathbb{R} \to \mathbb{R}\) be a non-negative continuous function such that \(\int_0^t g(s)ds < c_0 (1 + t^2)\) for any \(t \in \mathbb{R}\) and some \(c_0 > 0\) and

\[
\frac{\int_0^t g(\xi)d\xi}{d^{T+4}} < \left( \frac{3}{(T + 4)(T + 2)(2T + 2)^{T+3}} \right) \frac{\int_0^d g(\xi)d\xi}{d^3}.
\]

Then, for each \(\lambda \in \Lambda_{c,d}\)

\[
\lambda \in \Lambda_{c,d} = \left[ \frac{d^3(T + 2)}{3T \int_0^d g(\xi)d\xi}, \frac{3c^{T+4} - d^3(T + 4)(T + 2)(2T + 2)^{T+3}}{3T(T + 4)(2T + 2)^{T+3} \int_0^d g(\xi)d\xi} \right],
\]

the problem

\[
\begin{align*}
- \Delta (|\Delta u(k-1)|^k \Delta u(k-1)) + |u(k)|^{k+1} u(k) = \lambda g(u(k)), & \quad k \in [1, T], \\
u(0) = u(T + 1) = 0,
\end{align*}
\]

admits at least one positive solution in the space \(\{u : [0, T + 1] \to \mathbb{R} : u(0) = u(T + 1) = 0\}\).
2. Preliminaries

First, we give the following definition. For given a set $X$ and two functionals $\Phi$, $\Psi : X \to \mathbb{R}$, we defined the following functions

$$
\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}([r_1, r_2])} \frac{\sup_{u \in \Phi^{-1}([r_1, r_2])} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},
$$

(2.1)

and

$$
\rho_1(r_1, r_2) = \sup_{v \in \Phi^{-1}([r_1, r_2])} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}([r_1, r_2])} \Psi(u)}{\Phi(v) - r_1},
$$

(2.2)

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$.

$$
\rho_2(r) = \sup_{v \in \Phi^{-1}([r, \infty])} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}([r, \infty])} \Psi(u)}{\Phi(v) - r},
$$

(2.3)

for all $r \in \mathbb{R}$.

Theorem 2.1. ([8] Theorem 5.1, Proposition 2.1) Let $X$ be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on $X^*$ and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda \Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that

$$
\beta(r_1, r_2) < \rho_1(r_1, r_2),
$$

where $\beta$ and $\rho_1$ are given by (2.1) and (2.2). Then, for each $\lambda \in \Lambda = \left[ \frac{1}{\rho_1(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)} \right]$ there is $u_{0, \lambda} \in \Phi^{-1}([r_1, r_2])$ such that $I_\lambda(u_{0, \lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}([r_1, r_2])$ and $I_\lambda'(u_{0, \lambda}) = 0$.

Theorem 2.2. ([8] Theorem 5.5) Let $X$ be a real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on $X^*$ and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Fix $\inf X \Phi < r < \sup X \Phi$ and assume that

$$
\rho_2(r) > 0,
$$

and for each $\lambda > \frac{1}{\rho_2(r)}$, the functional $I_\lambda := \Phi - \lambda \Psi$ is coercive. Then, for each $\lambda \in \left[ \frac{1}{\rho_2(r)}, +\infty \right]$ there is $u_{0, \lambda} \in \Phi^{-1}[r, +\infty]$ such that $I_\lambda(u_{0, \lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}[r, +\infty]$ and $I_\lambda'(u_{0, \lambda}) = 0$.

Theorems 2.1 is a consequence of a local minimum theorem [8] Theorem 3.1 which is a more general version of the Ricceri Variational Principle (see [24]) and Theorem 2.2 is a consequence of [8] Theorem 4.2 which is a relevant variant of [8] Theorem 3.1.

Let $T \geq 2$ be a fixed positive integer, $[1, T]$ denote a discrete interval $\{1, ..., T\}$. Define $T$-dimensional function space by

$$
W := \{ u : [0, T + 1] \to \mathbb{R} : u(0) = u(T + 1) = 0 \},
$$
which is a Hilbert space under the norm

$$\|u\| = \left\{ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} + q(k)|u(k)|^{p^+} \right\}^{1/p^-}.$$

Since $W$ is finite-dimensional, we can also define the following equivalent norm on $W$

$$\|u\|_+ = \left\{ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+} + q(k)|u(k)|^{p^-} \right\}^{1/p^+}.$$

It is clear that by weighted Hölder inequality, one can conclude

$$K_0 \|u\| \leq \|u\|_+ \leq 2^{\frac{p^+-p^-}{p^-}} K_0 \|u\|,$$

where,

$$K_0 = \left\{ (2T + 2) \max\{w^+, q^+\} \right\}^{\frac{p^-}{p^- - p^+}}.$$

Now, let $\varphi : W \to \mathbb{R}$ be given by the formula

$$\varphi(u) := \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p(k-1)} + q(k)|u(k)|^{p(k)}.$$

In the rest of the paper, the following lemma will be very useful.

**Lemma 2.3.** For any $u \in W$, there exists a positive constant $C_1$ such that

$$\|u\| < 1 \Rightarrow \|u\|^{p^+}_+ \leq \varphi(u) \leq \|u\|^{-1},$$

$$\|u\| \geq 1 \Rightarrow \|u\|^{p^-}_+ - C_1 \leq \varphi(u) \leq \|u\|^{p^+}_+ + C_1.$$

where $C_1 = (T + 1)(w^+ + q^+)$. 

**Proof.** Let $u \in W$ be fixed. By a similar approach argued in [23], we set

$$A^< := \{k \in [0, T + 1] : |\Delta u(k)| < 1\}, \quad B^< := \{k \in [0, T + 1] : |u(k)| < 1\},$$

$$A^\geq := \{k \in [0, T + 1] : |\Delta u(k)| \geq 1\}, \quad B^\geq := \{k \in [0, T + 1] : |u(k)| \geq 1\}.$$

If $\|u\| < 1$, then $A^\geq = B^\geq = \emptyset$ and $A^< = B^< = [0, T + 1]$. It follows that $|\Delta u(k-1)|, |u(k)| < 1$ for each $k \in [1, T + 1]$. Hence, we have

$$\varphi(u) \leq \sum_{k=1}^{T+1} \left[ w(k-1)|\Delta u(k-1)|^{p^-} + q(k)|u(k)|^{p^-} \right] = \|u\|^{p^-}.$$

$$\|u\|^{p^+}_+ = \sum_{k=1}^{T+1} \left[ w(k-1)|\Delta u(k-1)|^{p^+} + q(k)|u(k)|^{p^+} \right] \leq \varphi(u),$$

which means that (2.3) holds. If $\|u\| \geq 1$, we have

$$\sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} \right]$$

$$= \left[ \left( \sum_{k \in A^<} + \sum_{k \in A^\geq} \right) w(k-1)|\Delta u(k-1)|^{p(k-1)} \right]$$

$$\leq \left[ \sum_{k \in A^<} w(k-1)|\Delta u(k-1)|^{p^-} + \sum_{k \in A^\geq} w(k-1)|\Delta u(k-1)|^{p^+} \right]$$
Combining the above inequalities we obtain

\[
= \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+} \\
+ \sum_{k \in A^-} (w(k-1)(|\Delta u(k-1)|^{p^-} - |\Delta u(k-1)|^{p^+})) \\
\leq \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+} \\
+ w^+ \sum_{k \in A^-} (|\Delta u(k-1)|^{p^-} - |\Delta u(k-1)|^{p^+}) \\
\leq \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+} + (T+1)w^+.
\]

By a similar argument, it reads

\[
\sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \right] \\
= \left[ \left( \sum_{k \in A^-} + \sum_{k \in A^\geq} \right) w(k-1)|\Delta u(k-1)|^{p(k-1)} \right] \\
\geq \left[ \sum_{k \in A^-} w(k-1)|\Delta u(k-1)|^{p^+} + \sum_{k \in A^\geq} w(k-1)|\Delta u(k-1)|^{p^-} \right] \\
\geq \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} \\
- \sum_{k \in A^-} (w(k-1)(|\Delta u(k-1)|^{p^-} - |\Delta u(k-1)|^{p^+})) \\
\geq \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} \\
- w^+ \sum_{k \in A^-} (|\Delta u(k-1)|^{p^-} - |\Delta u(k-1)|^{p^+}) \\
\geq \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} - w^+(T+1).)
\]

Combining the above inequalities we obtain

\[
\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+} - w^+(T+1) \leq \sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \right] \\
\leq \sum_{k=1}^{T+1} \left[ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} + w^+(T+1) \right].
\]

In the same manner we get

\[
\sum_{k=1}^{T+1} q(k)|u(k)|^{p^-} - q^+(T+1) \leq \sum_{k=1}^{T+1} \left[ \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right]
\]
Combining the above double inequalities we obtain
\[ \|u\|_{p^-} - (T + 1)(w^+ + q^+) \leq \varphi(u) \leq \|u\|_{p^+} + (T + 1)(w^+ + q^+). \]

Let \( \Phi \) and \( \Psi \) be as in the following
\[
\Phi(u) := \sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)} \Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right], \quad (2.8)
\]
\[
\Psi(u) := \sum_{k=1}^{T} F(k, u(k)), \quad (2.9)
\]
where \( F(k, t) := \int_{t}^{k} f(k, \xi) d\xi \) for every \( (k, t) \in [1, T] \times \mathbb{R} \).

In the sequel, we will use the following inequality
\[
\|u\|_{\infty} := \max_{k \in [1, T]} |u(k)| \leq (2T + 2)^{\frac{1}{p-1}} \|u\|, \quad \forall u \in W, \quad (2.10)
\]

To study the problem (1.1), we consider the functional \( I_{\lambda, \mu} : W \rightarrow \mathbb{R} \) defined by
\[
I_{\lambda}(u) = \sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)} \Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right] - \lambda \sum_{k=1}^{T} F(k, u(k)). \quad (2.11)
\]

We want to remark that since problem (1.1) is settled in a finite-dimensional Hilbert space \( W \), it is not difficult to verify that the functional \( I_{\lambda} \) satisfies the regularity properties. Therefore \( I_{\lambda} \) is of class \( C^1 \) on \( W \) (see, e.g., [13]) with the derivative
\[
I_{\lambda}'(u)(v) = \sum_{k=1}^{T+1} \left[ w(k-1)|\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1) \right] + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2} u(k)v(k) - \sum_{k=1}^{T} [\lambda f(k, u(k))] v(k),
\]
for all \( u, v \in W \).

**Lemma 2.4.** The critical points of \( I_{\lambda} \) and the solutions of the problem (1.1) are exactly equal.

**Proof.** Let \( \pi \) be a critical point of \( I_{\lambda} \) in \( W \). Thus, for every \( v \in W \), taking \( v(0) = v(T + 1) = 0 \) into account and applying summation by parts, one has
\[
0 = I_{\lambda}'(\pi)(v) = -\sum_{k=1}^{T} \left[ \Delta (w(k-1)|\Delta \pi(k-1)|^{p(k-1)-2} \Delta \pi(k-1)) \right] - \sum_{k=1}^{T} \left[ \lambda f(k, \pi(k)) \right] v(k).
\]
we have 

\[ -\Delta (w(k-1)|\Delta \overline{u}(k-1)|^{p(k)-2}\Delta \overline{u}(k-1) + q(k)|\overline{u}(k)|^{p(k)-2}\overline{u}(k) = \lambda f(k, \overline{u}(k)), \]

for every \( k \in \{1, T\} \). Therefore, \( \overline{u} \) is a solution of \( (\text{1.1}) \). So by bearing in mind that \( \overline{u} \) is arbitrary, we conclude that every critical point of the functional \( I_{\lambda} \) in \( W \), is exactly a solution of the problem \( (\text{1.1}) \).

Vice versa, if \( \overline{u} \in W \) be a solution of problem \( (\text{1.1}) \), by multiplying the difference equation in problem \( (\text{1.1}) \) by \( v(k) \) as an arbitrary element of \( W \) and summing and using the fact that

\[
\sum_{k=1}^{T+1} w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1))\Delta v(k-1) = -\sum_{k=1}^{T} \Delta (w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1))) v(k),
\]

we have \( I'_{\lambda,\mu}(\overline{u})(v) = 0 \), hence \( \overline{u} \) is a critical point for \( I_{\lambda,\mu} \). Thus the vice versa holds and the proof is completed. \( \square \)

3. Main Results

First, put

\[ A = \left( w(0) + w(T) + \sum_{k=1}^{T} q(k) \right) \]

and

\[ K = (2T + 2)^{1-p^{-}} \left\{ \max\{w^+, q^+\} \right\} \frac{p^{-}-p^{+}}{p^{+}p^{-}}. \]

Moreover, for given two non-negative constants \( c \) and \( d \) with \( \frac{1}{p^{-}} (cK)^{p^{+}} \neq \frac{dp^{-}}{p^{-}A} \), define

\[ a_d(c) := \frac{\sum_{k=1}^{T} \max_{|\xi| \leq c} F(k, \xi) - \sum_{k=1}^{T} F(k, d)}{\frac{1}{p^{-}} (cK)^{p^{+}} - \frac{dp^{-}}{p^{-}A}}. \]

\( (F1) \) There exist a constant \( c_0 > 0 \) and a function \( \alpha : \mathbb{Z}[1, T] \to [2, +\infty) \), with \( \max_{k \in [1, T]} \alpha(k) := \alpha^+ < p^+ \), such that for all \( (k, t) \in \mathbb{Z}[1, T] \times \mathbb{R} \),

\[ F(k, t) \leq c_0 \left( 1 + |t|^\alpha(k) \right). \]

Now, we are ready to state our first main result as follows.

**Theorem 3.1.** Assume the condition \( (F1) \) holds and assume that there exist a non-negative constant \( c_1 \) and two positive constants \( c_2 \) and \( d \) such that

\[
\left\{ \frac{K}{A^{p^{-}}} \right\} c_1 < d < \left( \frac{p^{-}K^{p^{+}}}{p^{+}A} \right)^{\frac{p^{+}c_2}{c_2}} < \left( \frac{p^{-}}{p^{+}A} \right)^{\frac{1}{p^{+}}}, \tag{3.1}
\]

and

\( (F2) \) \( a_d(c_2) < a_d(c_1) \).

Then for any \( \lambda \in \left( \frac{1}{a_d(c_1)}, \frac{1}{a_d(c_2)} \right) \) the problem \( (\text{1.1}) \) has at least one non-trivial solution \( u_0 \in W \).
Proof. Our aim is to apply Theorem 2.1 to problem (1.1). To settle the variational framework of problem (1.1), take $X = W$, and put $\Phi, \Psi$ as defined in (2.8) and (2.9), respectively for every $u \in W$. Due to $(F2)$, the interval $\left[ \frac{1}{a_d(c_1)}, \frac{1}{a_d(c_2)} \right]$ is non-empty. Therefore, if we fix $\bar{\lambda}$ in this interval, we can write

$$ I_{\bar{\lambda}} = \Phi - \bar{\lambda}\Psi. $$

Again, because $W$ is finite dimensional, an easy computation ensures that $\Phi$ and $\Psi$ are of class $C^1$ on $W$ with the derivatives

$$ \Phi'(u)(v) = \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) $$

$$ + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2}u(k)v(k) $$

$$ = -\sum_{k=1}^{T} \Delta(w(k-1)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1))v(k) $$

$$ + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2}u(k)v(k), $$

and

$$ \Psi'(u)(v) = \sum_{k=1}^{T} f(k, u(k))v(k), $$

for all $u, v \in W$. Hence $\Phi$ is sequentially weakly semicontinuous functional. Also $\Phi$ is coercive. Indeed, let $u \in W$ be a fixed member with $\|u\| > 1$. From (2.7), we have

$$ \Phi(u) \geq \frac{1}{p^+}\varphi(u) \geq \frac{\|u\|^p}{p^+} - C_1. $$

Therefore $\Phi(u) \to \infty$ as $\|u\| \to \infty$, i.e. $\Phi$ is coercive. Also by similar argument in [2], $\Phi'$ has an inverse mapping $(\Phi')^{-1} : W^* \to W$ which is continuous. Additionally, functional $\Psi$ is a continuously Gâteaux differentiable functional and from $(F1)$ whose Gâteaux derivative is compact. The solutions of the equation $I'_{\bar{\lambda}} = \Phi' - \bar{\lambda}\Psi' = 0$ are exactly the solutions for problem (1.1), by Lemma 2.4. Hence, to prove our result, it is enough to apply Theorem 2.1.

Let us define the function $\varpi : Z[0, T+1] \to \mathbb{R}$ belonging to $W$ by

$$ \tilde{v}(t) = \begin{cases} 
  d, & k \in [1, T], \\
  0, & k = 0, T + 1,
\end{cases} \quad (3.2) $$

$$ r_1 = \frac{1}{p^+} (c_1 K)^{p^+}, $$

and

$$ r_2 = \frac{1}{p^+} (c_2 K)^{p^+}. $$

By (3.1), $r_1, r_2 < \frac{1}{p^+}$ and

$$ \Psi(\tilde{v}) = \sum_{k=1}^{T} F(k, \tilde{v}(k)) = \sum_{k=1}^{T} F(k, d), \quad (3.3) $$
On the other hand, one has

\[ r \text{ by again (3.1), } r < \Phi(\bar{v}) < r_2. \]

By (3.1), one can conclude that \( d \in (0, 1) \), therefore \( \frac{d^+}{p^+} - A < \Phi(\bar{v}) < \frac{d^-}{p^-} - A \), hence by again (3.1), \( r_1 < \Phi(\bar{v}) < r_2 \). Let be \( u \in \Phi^{-1}(\Phi(\bar{v})), i = 1, 2 \) for all \( u \in W \), then one has \( \max_{k \in [1, T]} |u(k)| \leq c_i, i = 1, 2 \). Indeed, by (2.5), \( \varphi(u) < p^+ \Phi(u) < r_i p^+ < 1 \), this means that \( \|u\| < 1 \), and therefore bearing in mind (2.6) and (2.4),

\[
K_0^+ \|u\|^T < \|u\|_+^T < \varphi(u) < r_i p^+;
\]

hence, \( \max_{k \in [1, T]} |u(k)| \leq (2T + 2) \frac{r_i}{p^+} \|u\| < (2T + 2) \frac{r_i p^+}{K_0} \|u\| < c_i \). Therefore,

\[
\sup_{u \in \Phi^{-1}(\Phi(\bar{v}))} \Psi(u) = \sup_{u \in \Phi^{-1}(\Phi(\bar{v}))} \sum_{k=1}^{T} F(k, u(k)) \leq \sum_{k=1}^{T} \max_{|\xi| \leq c_i} F(k, \xi), \quad i = 1, 2.
\]

Thus,

\[
0 \leq \beta(r_1, r_2) \leq \frac{\sup_{u \in \Phi^{-1}(\Phi(\bar{v}))} \Psi(u) - \Psi(\bar{v})}{p^+ (c_2 K)^{p^+} - \Phi(\bar{v})} \leq \frac{\sum_{k=1}^{T} \max_{|\xi| \leq c_i} F(k, \xi) - \sum_{k=1}^{T} F(k, d)}{p^+ (c_2 K)^{p^+} - \frac{d^+}{p^+} - A} = a_d(c_2).
\]

On the other hand, one has

\[
\rho_1(r_1, r_2) \geq \frac{\Psi(\bar{v}) - \sup_{u \in \Phi^{-1}(\Phi(\bar{v}))} \Psi(u)}{\Phi(\bar{v}) - r_1} \geq \frac{\sum_{k=1}^{T} F(k, d) - \sum_{k=1}^{T} \max_{|\xi| \leq c_i} F(k, \xi)}{\Phi(\bar{v}) - \frac{1}{p^+} (c_1 K)^{p^+}} \geq \frac{\sum_{k=1}^{T} F(k, d) - \sum_{k=1}^{T} \max_{|\xi| \leq c_i} F(k, \xi)}{\frac{d^-}{p^-} - \frac{1}{p^+} (c_1 K)^{p^+}} = a_d(c_1).
\]

Hence, from Assumption (F2), we get \( \beta(r_1, r_2) < \rho_1(r_1, r_2) \). Therefore, owing to Theorem 2.1 for each \( \lambda \in [\lambda_{u_0}, \lambda_{u_0}] \), the functional \( I_\lambda \) admits one critical point \( u_0 \in W \) such that \( r_1 < \Phi(u_0) < r_2 \). Hence, the proof is complete. \( \square \)

**Corollary 3.2.** If \( u_0 \) be the ensured solution in the conclusions of Theorems 3.1 then

\[
\left( \frac{p^-}{p^+} \right)^{\frac{1}{p^+}} (c_1 K)^{\frac{p^+}{p^-}} < \|u_0\| < c_2 (2T + 2) \frac{d^-}{p^-}.
\]
Therefore the conditions F1 and F2 hold. Then, by Theorem 3.1, for every \( \lambda \in [0.000000003, 111] \) the problem

\[
\begin{align*}
-\Delta(e^{k(10-k)} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1)) + 2k |u(k)|^{p(k)-2} u(k) &= \lambda \left( \frac{e^{k(10-k)} |\Delta u(k-1)|^{p(k-1)}}{u(k)^{2} + \| u(k) \|^{2} - 111} \right), \\
u(0) &= u(11) = 0,
\end{align*}
\]

for every \( k \in [1, 10] \), has at least one non-trivial solution \( u_0 \) that by Corollary 3.3.

\[
\frac{0.6}{10^{15} \times 22^{2} \times e^{-98}} < \| u_0 \| < \frac{10^{9}}{22^{7}}.
\]

Here we point out an immediate consequence of Theorem 3.1 as follows.
**Theorem 3.4.** Assume the condition $(F1)$ holds and assume that there exist two positive constants $c$ and $d$ with $Ap^+d^p < Kp^+p^-c^p < p^-$ such that

\[ (F3) \sum_{k=1}^{T} \max_{|\xi| \leq c} F(k, \xi) < \frac{p^-(cK)^p^+}{p^+d^p_A} \sum_{k=1}^{T} F(k, d). \]

Then, for each $\lambda \in \left[ \frac{d_-A}{\sum_{k=1}^{T} F(k, d)} \right.$
\[ \frac{1}{p^+} (cK)^p^+ - \frac{d_-A}{p^+} \left.] \right\rceil \]
the problem $(1.1)$ has at least one non-trivial solution $u_0 \in W$ such that $\|u_0\|_{\infty} < c$.

**Proof.** By applying Theorem 3.1 and picking $c_1 = c_2 = c$ the conclusion follows at once. Indeed, owing to our assumptions, one has

\[ a_d(c) = \frac{\sum_{k=1}^{T} \max_{|\xi| \leq c} F(k, \xi) - \sum_{k=1}^{T} F(k, c)}{\frac{1}{p^+} (cK)^p^+ - \frac{d_-A}{p^+}} \]
\[ < \frac{\sum_{k=1}^{T} F(k, d)}{\frac{d_-A}{p^+}} = a_d(0). \]

Hence, Theorem 3.1 along with Corollary 2.10 and Corollary 3.2 ensures the conclusion. \(\square\)

**Remark 3.5.** If $f$ is non-negative, then, by consideration [9] Theorem 2.2, the obtained solution $u_0$ in the conclusions of Theorems 3.1 and 3.6 is non-negative. If $f(k, 0) = 0$ for all $k \in [0, T]$, owing to [9] Theorem 2.3, the obtained solution $u_0$ is positive (see [9] Remark 2.1).

Next, we consider the following problem, as a special case of the problem $(1.1)$,

\[ \begin{cases} -\Delta(w(k-1))u(k-1)|p^{(k-1)-2} \Delta u(k-1)| + q(k) |u(k)|p^{(k)-2}u(k) = \lambda \beta(k) g(u(k)), \\ u(0) = u(T+1) = 0, \end{cases} \tag{3.6} \]

for any $k \in [1, T]$, where $\beta : [1, T] \rightarrow \mathbb{R}$ is a nonnegative function and $g \in C(\mathbb{R}, \mathbb{R})$ is a continuous function. Put $G(t) = \int_0^t g(\xi) d\xi$ for all $t \in \mathbb{R}$.

**Theorem 3.6.** Assume the condition $(F1)$ holds and assume that there exist two positive constants $c$ and $d$ with $Ap^+d^p < Kp^+p^-c^p < p^-$ such that

\[ (F4) \max_{|\xi| \leq c} G(\xi) = \frac{\max_{|\xi| \leq c} G(\xi)}{G(\xi)} < \frac{p^-(cK)^p^+}{p^+d^p_A} G(d). \]

Then, for each $\lambda \in \left[ \frac{d_-A}{G(d) \sum_{k=1}^{T} \beta(k)} \right.$
\[ \frac{1}{p^+} (cK)^p^+ - \frac{d_-A}{p^+} \left.] \right\rceil \]
the problem $(3.6)$ has at least one non-trivial solution $u_0 \in W$ such that $\|u_0\|_{\infty} < c$. 
Proof. Again, by applying Theorem 3.1 and picking $c_1 = 0$ and $c_2 = c$ we have the conclusion. Indeed, owing to our assumptions, one has
\[
a_d(c) = \sum_{k=1}^{T} \beta(k) \left[ \max_{|\xi| \leq c} G(\xi) - G(d) \right] \frac{1}{p^+} (cK)^{p^+} - \frac{2p}{p^+} A < \frac{G(d) \sum_{k=1}^{T} \beta(k)}{d^p A} = a_d(0).
\]
Thus, considering Theorem 3.1, (2.10) and Corollary 3.2, we obtain the desired conclusion. \(\square\)

We now proceed with the proof of Theorem 1.1.

**PROOF OF THEOREM 1.1**: This follows from Theorem 3.6 at once, by letting $p(k) = k + 3$, $\alpha(k) = 2$ and $w(k) = q(k) = \beta(k) = 1$ for every $k \in [1, T]$.

Here, we present the following example to illustrate the result of Theorem 1.1.

**Example 3.7.** Consider the problem
\[
\begin{cases}
-\Delta(|\Delta u(k-1)|^k \Delta u(k-1)) + |u(k)|^{k+1} u(k) = \frac{\lambda}{400} u(0) + \frac{\lambda}{400} u(11), & k \in [1, 10], \\
u(0) = u(11) = 0.
\end{cases}
\]

Put $T = 10$ and select $d = 0.1$, $c = 17.1$ and $c_0 = 0.0039$ that satisfying (1.3), growth condition and (1.4), that is $d^3 < \frac{1}{50 \times 122^{14}} < \frac{1}{56}$ and $\arctan(400c) < c_0(1 + t^2)$, for every $t \in \mathbb{R}$ and
\[
\frac{\arctan(400c)}{c^{14}} < \frac{1}{56 \times 122^{13}} \frac{\arctan(400d)}{d^3},
\]
respectively, then for each $\lambda \in [0.1035061724, 67.87674577]$ the problem (3.7) has at least one non-trivial solution $u_0 \in \{ u : [0, 10] \to \mathbb{R} : u(0) = u(11) = 0 \}$, that by Remark 3.3 is non-negative.

Now we state the second main result of the paper. We will apply Theorem 2.12.

To do so, we provide the following theorem.

**Theorem 3.8.** Assume the condition (F1) holds and
\[(F5) \text{ There exist constants } d, c_3 > 0 \text{ with } \frac{1}{p^{+}} > d > c_3 \frac{K}{p^{+}} \text{ such that}
\]
\[
\frac{p^{+}}{(c_3 K)^{p^{+}}} T c_0 (1 + \max\{c_3^{\alpha^+} + c_3^{-\alpha^-}\}) < \tilde{d}^{-1} \sum_{k=1}^{T} F(k, d),
\]
where $\tilde{d} = \frac{w(0) p(0)}{p(0)} + \frac{w(T) p(T)}{p(T)} + \sum_{k=1}^{T} \frac{q(k)}{p(k)} d^{p(k)}$. Then for each $\lambda \in \Lambda_d := \left[ \tilde{d}^{-1} \sum_{k=1}^{T} F(k, d), +\infty \right]$, the problem (1.1) admits at least one non-trivial weak solution.

**Proof.** As mentioned in the proof of Theorem 2.11 the regularities of $\Phi$ and $\Psi$ hold. Let us define the function $\mathcal{V} : \mathbb{Z}[0, T + 1] \to \mathbb{R}$ belonging to $W$ by the formula (3.2).
Then from (3.4) we deduce that
\[ \Phi(\tau) > \frac{A}{p^+} d^{p^+}. \]

Let fix \( r = (c_1 K)^{p^+} \). Since \( d^{p^+} > (c_1 K)^{p^+} \), we get \( \Phi(\tau) > r \). On the other hand, by Lemma 2.3 we have that \( \Phi \) is bounded on \( W \). Therefore, since \( \tau \in W \) and \( \inf_{u \in W} \Phi(u) = \Phi(0) = 0 \) we obtain
\[ \inf_{u \in W} \Phi(u) < r < \Phi(\tau) < \sup_{u \in W} \Phi(u). \]

For each \( u \in \Phi^{-1} ] - \infty, r [ \), by similar argument for obtaining (3.5), we have
\[ ||u|| < \left( \frac{rp^+}{K_0} \right)^{\frac{1}{p^+}}, \]
which leads us, by (2.10), to
\[ \max_{k \in [1, T]} |u(k)| \leq (2T + 2)^{\frac{p^+ - 1}{p^-}} ||u|| < (2T + 2)^{\frac{p^+ - 1}{p^-}} \left( \frac{rp^+}{K_0} \right)^{\frac{1}{p^+}} = : c_3. \]

Therefore, from the condition \((F1)\) and \((F5)\), it reads
\[ \sup_{u \in \Phi^{-1} (-\infty, r)} \Psi(u) \leq \sum_{k=1}^{T} \max_{|\xi| \leq c_3} F(k, \xi) \leq \sum_{k=1}^{T} \max_{|\xi| \leq c_3} c_0 (1 + |\xi|^{\alpha(k)}) < T c_0 (1 + \max \{ c_3^{\alpha^+}, c_3^{\alpha^-} \}) < r \frac{\Psi(\tau)}{\Phi(\tau)}. \]

Considering (3.8), we obtain
\[ \rho_2(r) = \frac{\sup_{u \in \Phi^{-1} (-\infty, r)} \Psi(u)}{\Phi(\tau) - r} \geq \frac{\Psi(\tau) - \sup_{u \in \Phi^{-1} (-\infty, r)} \Psi(u)}{\Phi(\tau) - r} = \frac{\Psi(\tau) - r}{\Phi(\tau) - r} > 0. \]

Let us proceed with the coercivity of \( I_\lambda \). To this end, let \( u \in W \) such that \( ||u|| \to +\infty \). Then, without loss of generality, we can assume that \( ||u|| > 1 \). Then from (2.9), (2.10) and condition \((F1)\), it reads
\[ I_\lambda(u) \geq \frac{||u||^{p^+}}{p^+} - \lambda \sum_{k=1}^{T} c_0 (1 + |u(k)|^{\alpha(k)}) \geq \frac{||u||^{p^+}}{p^+} - \lambda \sum_{k=1}^{T} c_0 (1 + \left( \max_{k \in [1, T]} |u(k)| \right)^{\alpha(k)}) \geq \frac{||u||^{p^+}}{p^+} - \lambda \sum_{k=1}^{T} c_0 (1 + (2T + 2)^{\frac{p^+ - 1}{p^-}} ||u||^{\alpha(k)}) \geq \frac{||u||^{p^+}}{p^+} - \lambda T c_0 (2T + 2)^{\frac{p^+ - 1}{p^-}} ||u||^{\alpha^+} - \lambda T c_0, \]
that due to $\alpha^+ < p^-$, it follows that $I_\lambda$ be coercive. Consequently, all assumptions of Theorem 2.2 are verified. Therefore, for each $\lambda \in \Lambda_d$, the problem (1.1) admits at least one nontrivial weak solution. 

In the following we give a corollary which is based on Theorem 2.4 of [10].

**Corollary 3.9.** Assume that (F1) and (F5) holds. Then for each \( \lambda \in \Lambda_r := \left\{ \frac{1}{d-1} \sum_{k=1}^{r} \frac{F(k,d)}{T \sigma(1 + \max\{c_3^+, c_3^-\})} \right\} \), the problem (1.1) admits at least three distinct critical points.

**Proof.** So far we have already obtained that $\Phi$ is a continuously Gâteaux differentiable, coercive and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on $W^*$, and $\Psi$ is continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, and $\inf_{x \in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Moreover, since $I_\lambda$ is coercive on $\Lambda_d$, it is coercive on $\Lambda_r$ as well because of the relation $\Lambda_r \subseteq \Lambda_d$. The rest of the proof is quite similar to that of Theorem 3.8. However, some remarks are in order. Since, apparently

\[
\Phi(I) = \hat{d} = \frac{u(0)d^p(0)}{p(0)} + \frac{u(T)d^p(T)}{p(T)} + \sum_{k=1}^{T} \frac{q(k)}{p(k)} \]

and

\[
\Psi(I) = \sum_{k=1}^{T} F(k,d),
\]

it reads, from (3.8) and (F5), that

\[
\sup_{u \in \Phi^{-1}(\infty, r)} \Psi(u) < T \sigma(1 + \max\{c_3^+, c_3^-\}) \leq \frac{\Psi(I)}{\Phi(I)}
\]

Overall, all assumptions of Theorem 2.4 of [10] are verified. Therefore, for each $\lambda \in \Lambda_r$, the functional $I_\lambda$ admits at least three distinct critical points that are weak solutions of Problem (1.1). \qed

Here, we present the following example to illustrate the result of Theorem 3.8.

**Example 3.10.** Let all assumptions in Example 3.3 hold, that is, $T = 10$, $p(k) = \frac{1}{d}k + 3$, $q(k) = 2^k$, $w(k) = e^{k(10-k)^2}$, $\alpha(k) = 2$ and $f(k,x) = e^{k+2}(k+1)^3(x+10)^{10}$. \( k = 1, 2, 3, ..., 10 \) and $x \in \mathbb{R}$. Hence $p^+ = 3$, $p^- = 5$, $\alpha^+ = 2$, $A = 2^{11}$ and $K = 22^\frac{1}{d}e^{-\frac{10}{d}}$ and $F(k,x) = \frac{10^{11}}{e^{k+2}(k+1)^3(x+10)^{10}}$. Put $c_0 = 0.000012$, $c_3 = 0.05$ and $d = 0.0000000005$ satisfying $\frac{1}{\sqrt{A}} > d > c_3^{-1}\frac{K}{\sqrt{A}}$. Simple calculations with Maple software show that

\[
F(k,t) \leq 0.000012(1 + |t|^2), \quad \forall (k,t) \in \mathbb{Z}[1, 10] \times \mathbb{R},
\]

\[
\left(\frac{p}{c_3K}\right)^p T \sigma(1 + \max\{c_3^+, c_3^-\}) \simeq 2.086867833 \times 10^{13},
\]

and

\[
\hat{d}^{-1} \sum_{k=1}^{T} F(k,d) \simeq 1.338709020 \times 10^{16},
\]
Therefore the conditions $F_1$ and $F_5$ hold. Then, by Theorem 3.8, for every $\lambda \in [7.469883186 \times 10^{-17}, \infty)$ the problem

$$
\begin{aligned}
-\Delta(e^{k(10-k)^2}\Delta u(k-1)) & |u(k-1)|^{p(k)-2} \Delta u(k-1) + 2^k |u(k)|^{p(k)-2} u(k) = \lambda \left( \frac{(u(k)^2 e^{(k+2)(k-1)})}{(u(k)^2 + 10^{-17})^2} \right), \\
u(0) = u(11) = 0,
\end{aligned}
$$

for every $k \in [1, 10]$, has at least one non-trivial solution and by Corollary 3.9, for every $\lambda \in [7.469883186 \times 10^{-17}, 4.791870305 \times 10^{-14}]$ the above problem has at least three solutions.

References

[1] R.P. Agarwal, K. Perera, and D. O’Regan, Multiple positive solutions of singular discrete $p$-Laplacian problems via variational methods, Adv. Diff. Equ. 2 (2005) 93-99.

[2] G.A. Afrouzi, A. Hadjian and S. Heidarkhani, Non-trivial solutions for a two-point boundary value problem, Ann. Poli.Math. 108.1 (2013) 75-84.

[3] M. Avci, Existence results for anisotropic discrete boundary value problems, EJDE, 148 (2016), 1-11.

[4] M. Avci and A. Pankov, Nontrivial solutions of discrete nonlinear equations with variable exponent, J.Math.Anal.Appl. 431 (2015), 22-33.

[5] R. Avery and J. Henderson, Existence of three positive pseudo-symmetric solutions for a one dimensional discrete $p$-Laplacian, J. Differ. Equ. Appl. 10 (2004) 529-539.

[6] D. Bai and Y. Xu, Nontrivial solutions of boundary value problems of second-order difference equations, J. Math. Anal. Appl. 326 (2007) 297-302.

[7] L.-H. Bian, H.-R. Sun and Q.-G. Zhang, Solutions for discrete $p$-Laplacian periodic boundary value problems via critical point theory, J. Differ. Equ. Appl. 18(3) (2012) 345-355.

[8] G. Bonanno, A Critical point theorem via the Ekeland variational principle, Nonl. Anal. TMA 75 (2012) 2992-3007.

[9] G. Bonanno and P. Candito, Infinitely many solutions for a class of discrete non-linear boundary value problems, Appl. Anal. 884 (2009) 605-616.

[10] G. Bonanno, A. Chinnì, Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent, J. Math. Anal. Appl. 418 (2014), 812–827.

[11] C. Bereanu, P. Jebelean, C. Serban,Periodic and Neumann problems for discrete $p$-Laplacian, J. Math. Anal. Appl. 399 (2013), 75-87.

[12] P. Candito and G. D’Aguì, Three solutions to a perturbed nonlinear discrete Dirichlet problem, J. Math. Anal. Appl. 375 (2011) 594-601.

[13] J. Chu and D. Jiang, Eigenvalues and discrete boundary value problems for one-dimensional $p$-Laplacian, J. Math. Anal. Appl. 305 (2005) 452-465.

[14] M. Galewska, G. Molica Bisci and R. Wieteska, Existence and multiplicity of solutions to discrete inclusions with the $p(k)$-Laplacian problem, J. Difference Equ. Appl. 21(10), (2015) 887-903.

[15] A. Guiro, B. Kone, and S. Ouaro Weak heteroclinic solutions and competition phenomena to anisotropic difference equations with variable exponents, Opuscula Math. 34, no. 4 (2014), 733-745.

[16] S. Heidarkhani and M. Khaleghi Moghadam, Existence of Three solutions for Perturbed nonlinear difference equations, Opuscula Math. 344 (2014), 747-761.

[17] J. Henderson and H.B. Thompson, Existence of multiple solutions for second order discrete boundary value problems, Comput. Math. Appl. 43 (2002), 1239-1248.

[18] L. Jiang and Z. Zhou, Three solutions to Dirichlet boundary value problems for $p$-Laplacian difference equations, Adv. Diff. Equ. 2008 (2008) 1-10.

[19] M. Khaleghi Moghadam, S. Heidarkhani and J. Henderson, Infinitely many solutions for perturbed difference equations, J. Difference Equ. Appl. 207 (2014), 1055-1068.

[20] M. Khaleghi Moghadam, S. Heidarkhani, Existence of a nontrivial solution for nonlinear difference equations, Differ. Equ. Appl. 64 (2014), 517-525.

[21] Y. Li and L. Lu, Existence of positive solutions of $p$-Laplacian difference equations, Appl. Math. Lett. 19 (2006) 1019-1023.
[22] Y. Liu and W. Ge, *Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator*, J. Math. Anal. Appl. 278 (2003) 551-561.
[23] M. Mihăilescu, V. Rădulescu and S. Tersian, *Eigenvalue problems for anisotropic discrete boundary value problems*, J. Difference Equ. Appl. 15 (2009) 557–567.
[24] B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math. 113 (2000) 401-410.

Mohsen Khaleghi Moghadam  
Department of Basic Science, Sari Agricultural Sciences and Natural Resources University, 578 Sari, Iran

E-mail address: mohsen.khaleghi@rocketmail.com and m.khaleghi@sanru.ac.ir

Mustafa Avci  
Faculty of Economics and Administrative Sciences, Batman University, Turkey

E-mail address: avcixmustafa@gmail.com