On powers of Stieltjes moment sequences, II

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Abstract

We consider the set of Stieltjes moment sequences, for which every positive power is again a Stieltjes moment sequence, we and prove an integral representation of the logarithm of the moment sequence in analogy to the Lévy-Khinchin representation. We use the result to construct product convolution semigroups with moments of all orders and to calculate their Mellin transforms. As an application we construct a positive generating function for the orthonormal Hermite polynomials.

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1 Introduction and main results

In his fundamental memoir [15] Stieltjes characterized sequences of the form

\[ s_n = \int_0^\infty x^n \, d\mu(x), \quad n = 0, 1, \ldots, \]

where \( \mu \) is a non-negative measure on \([0, \infty]\), by certain quadratic forms being non-negative. These sequences are now called Stieltjes moment sequences. They are called normalized if \( s_0 = 1 \). A Stieltjes moment sequence is called S-determinate, if there is only one measure \( \mu \) on \([0, \infty]\) such that (1) holds; otherwise it is called S-indeterminate. It is to be noticed that in the S-indeterminate case there are also solutions \( \mu \) to (1), which are not supported by \([0, \infty]\), i.e. solutions to the corresponding Hamburger moment problem.

A Stieltjes moment sequence is either non-vanishing (i.e. \( s_n > 0 \) for all \( n \)) or of the form \( s_n = c \delta_{0n} \) with \( c \geq 0 \), where \( (\delta_{0n}) \) is the sequence \((1, 0, 0, \ldots)\). The latter corresponds to the Dirac measure \( \delta_0 \) with mass 1 concentrated at 0.

In this paper we shall characterize the set \( \mathcal{I} \) of normalized Stieltjes moment sequences \( (s_n) \) with the property that \( (cs_n^c) \) is a Stieltjes moment sequence for each \( c > 0 \). The result is given in Theorem 1.4, from which we extract the following:
A Stieltjes moment sequence \((s_n)\) belongs to \(\mathcal{I}\) if and only if there exist \(\varepsilon \in [0,1]\) and an infinitely divisible probability \(\omega\) on \(\mathbb{R}\) such that

\[
s_n = (1-\varepsilon)\delta_0 + \varepsilon \int_{-\infty}^{\infty} e^{-ny} d\omega(y).
\]  

(2)

We stress however that Theorem 1.4 also contains a kind of Lévy-Khintchine representation of \(\log s_n\) in the case \(\varepsilon \neq 0\), and this result is very useful for deciding if a given sequence belongs to \(\mathcal{I}\).

During the preparation of this paper our attention was drawn to the Ph.d.-thesis [10] of Shu-gwei Tyan, which contains a chapter on infinitely divisible moment sequences. The set \(\mathcal{I}\) is the set of infinitely divisible Stieltjes moment sequences in the sense of Tyan. Theorem 4.2 in [10] is a representation of \(\log s_n\) similar to condition (ii) in Theorem 1.4. As far as we know these results of [10] have not been published elsewhere, so we discuss his results in Section 3.

The motivation for the present paper can be found in the paper [4] by Durán and the author and which provides a unification of recent work of Bertoin, Carmona, Petit and Yor, see [7], [8], [9]. We found a procedure to generate sequences \((s_n) \in \mathcal{I}\). To formulate the motivating result of [4] we need the concept of a Bernstein function.

Let \((\eta_t)_{t>0}\) be a convolution semigroup of sub-probabilities on \([0,\infty]\) with Laplace exponent or Bernstein function \(f\) given by

\[
\int_0^{\infty} e^{-sx} d\eta_t(x) = e^{-tf(s)}, \quad s > 0,
\]

cf. [5], [6]. We recall that \(f\) has the integral representation

\[
f(s) = a + bs + \int_0^{\infty} (1-e^{-sx}) d\nu(x),
\]

(3)

where \(a, b \geq 0\) and the Lévy measure \(\nu\) on \([0,\infty]\) satisfies the integrability condition \(\int x/(1+x) d\nu(x) < \infty\). Note that \(\eta_t([0,\infty[) = \exp(-at)\), so that \((\eta_t)_{t>0}\) consists of probabilities if and only if \(a = 0\).

In the following we shall exclude the Bernstein function identically equal to zero, which corresponds to the convolution semigroup \(\eta_t = \delta_0, t > 0\).

Let \(\mathcal{B}\) denote the set of Bernstein functions which are not identically zero. For \(f \in \mathcal{B}\) we note that \(f'/f\) is completely monotonic as product of the completely monotonic functions \(f'\) and \(1/f\). Therefore there exists a non-negative measure \(\kappa\) on \([0,\infty[\) such that

\[
\frac{f'(s)}{f(s)} = \int_0^{\infty} e^{-sx} d\kappa(x).
\]

(4)

It is easy to see that \(\kappa(\{0\}) = 0\) using [3] and \(f'(s) \geq \kappa(\{0\}) f(s)\).
Theorem 1.1 (Berg-Durán [4], Berg [2]) Let \( \alpha \geq 0, \beta > 0 \) and let \( f \in B \) be such that \( f(\alpha) > 0 \). Then the sequence \((s_n)\) defined by

\[
s_0 = 1, s_n = f(\alpha)f(\alpha + \beta) \cdots f(\alpha + (n-1)\beta), \quad n \geq 1
\]

belongs to \( I \). Furthermore \((s_n^c)\) is S-determinate for \( c \leq 2 \).

In most applications of the theorem we put \( \alpha = \beta = 1 \) or \( \alpha = 0, \beta = 1 \), the latter provided \( f(0) > 0 \). The moment sequence \((s^c_n)\) of Theorem 1.1 can be S-indeterminate for \( c > 2 \). This is shown in [2] for the moment sequences

\[
s_n^c = (n!)^c \quad \text{and} \quad s_n^c = (n+1)^{c(n+1)}
\]

derived from the Bernstein functions \( f(s) = s \) and \( f(s) = s(1 + 1/s)^{c+1} \). For the Bernstein function \( f(s) = s/(s+1) \) the moment sequence \( s_n^c = (n+1)^{-c} \) is a Hausdorff moment sequence since

\[
\frac{1}{(n+1)^c} = \frac{1}{\Gamma(c)} \int_0^1 x^n(\log(1/x))^{c-1} \, dx,
\]

and in particular it is S-determinate for all \( c > 0 \). In Section 2 we give a new proof of Theorem 1.1.

In Section 4 we use the Stieltjes moment sequence \((\sqrt{n!})\) to prove non-negativity of a generating function for the orthonormal Hermite polynomials.

The sequence \((a)_n := a(a+1) \cdots (a+n-1), a > 0\) belongs to \( I \) and is a one parameter extension of \( n! \). For \( 0 < a < b \) also \((a)_n/(b)_n\) belongs to \( I \). These examples are studied in Section 5. Finally, in Section 6 we study a \( q \)-extension \((a;q)_n/(b;q)_n \in I \) for \( 0 < q < 1, 0 \leq b < a < 1 \). In Section 7 we give some complementary examples.

Any normalized Stieltjes moment sequence \((s_n)\) has the form \( s_n = (1-\varepsilon)\delta_{0n} + \varepsilon t_n \), where \( \varepsilon \in [0,1] \) and \((t_n)\) is a normalized Stieltjes moment sequence satisfying \( t_n > 0 \).

Although the moment sequence \((s^c_n)\) of Theorem 1.1 can be S-indeterminate for \( c > 2 \), there is a “canonical” solution \( \rho_c \) to the moment problem defined by “infinite divisibility”.

The notion of an infinitely divisible probability measure has been studied for arbitrary locally compact groups, cf. [12].

We need the product convolution \( \mu \odot \nu \) of two measures \( \mu \) and \( \nu \) on \([0, \infty[\): It is defined as the image of the product measure \( \mu \otimes \nu \) under the product mapping \((s,t) \mapsto st\). For measures concentrated on \([0, \infty[\) it is the convolution with respect to the multiplicative group structure on the interval. It is clear that the \( n \)'th moment of the product convolution is the product of the \( n \)'th moments of the factors.

In accordance with the general definition we say that a probability \( \rho \) on \([0, \infty[\) is infinitely divisible on the multiplicative group of positive real numbers, if it
has \( p \)'th product convolution roots for any natural number \( p \), i.e. if there exists a probability \( \tau(p) \) on \([0, \infty[\) such that \((\tau(p))^p = \rho\). This condition implies the existence of a unique family \((\rho_c)_{c > 0}\) of probabilities on \([0, \infty[\) such that \(\rho_c \circ \rho_d = \rho_{c+d}\), \(\rho_1 = \rho\) and \(c \mapsto \rho_c\) is weakly continuous. (These conditions imply that \(\lim_{c \to 0} \rho_c = \delta_1\) weakly.) We call such a family a product convolution semigroup. It is a (continuous) convolution semigroup in the abstract sense of \([5]\) or \([12]\). A \( p \)'th root \( \tau(p) \) is unique and one defines \( \rho_1^p = \tau(p), \rho_m^p = (\tau(p))^m, m = 1, 2, \ldots \).

Finally \( \rho_c \) is defined by continuity when \( c > 0 \) is irrational.

The family of image measures \((\log(\rho_c))\) under the log-function is a convolution semigroup of infinitely divisible measures in the ordinary sense on the real line considered as an additive group.

The following result generalizes Theorem 1.8 in \([2]\), which treats the special case \( \alpha = \beta = 1 \). In addition we express the Mellin transform of the product convolution semigroup \((\rho_c)\) in terms of the measure \( \kappa \) from \([4]\).

**Theorem 1.2** Let \( \alpha \geq 0, \beta > 0 \) and let \( f \in B \) be such that \( f(\alpha) > 0 \). The uniquely determined probability measure \( \rho \) with moments

\[
s_n = f(\alpha)f(\alpha + \beta) \cdots f(\alpha + (n-1)\beta), \quad n \geq 1
\]

is concentrated on \([0, \infty[\) and is infinitely divisible with respect to the product convolution. The unique product convolution semigroup \((\rho_c)_{c > 0}\) with \(\rho_1 = \rho\) has the moments

\[
\int_0^{\infty} x^n d\rho_c(x) = (f(\alpha)f(\alpha + \beta) \cdots f(\alpha + (n-1)\beta))^c, \quad c > 0, n = 1, 2, \ldots
\]

(6)

The Mellin transform of \( \rho_c \) is given by

\[
\int_0^{\infty} t^z d\rho_c(t) = e^{-c\psi(z)}, \quad \Re z \geq 0,
\]

(7)

where

\[
\psi(z) = -z \log f(\alpha) + \int_0^{\infty} \left((1-e^{-z\beta x}) - z(1-e^{-\beta x})\right) e^{-\alpha x} \frac{e^{-ax}}{x(1-e^{-ax})} d\kappa(x),
\]

(8)

and \( \kappa \) is given by \([4]\).

Proof of the theorem is given in Section 2.

In connection with questions of determinacy the following result is useful.

**Lemma 1.3** Assume that a Stieltjes moment sequence \((u_n)\) is the product \( u_n = s_n t_n \) of two Stieltjes moment sequences \((s_n), (t_n)\). If \( t_n > 0 \) for all \( n \) and \((s_n)\) is \( S \)-indeterminate, then also \((u_n)\) is \( S \)-indeterminate.
This is proved in Lemma 2.2 and Remark 2.3 in [4]. It follows that if \((s_n) \in \mathcal{I}\) and \((s_n^c)\) is S-indeterminate for \(c = c_0\), then it is S-indeterminate for any \(c > c_0\). Therefore one of the following three cases occur

- \((s_n^c)\) is S-determinate for all \(c > 0\).
- There exists \(c_0, 0 < c_0 < \infty\) such that \((s_n^c)\) is S-determinate for \(0 < c < c_0\) and S-indeterminate for \(c > c_0\).
- \((s_n^c)\) is S-indeterminate for all \(c > 0\).

We have already mentioned examples of the first two cases, and the third case occurs in Remark 1.7.

The question of characterizing the set of normalized Stieltjes moment sequences \((s_n)\) with the property that \((s_n^c)\) is a Stieltjes moment sequence for each \(c > 0\) is essentially answered in the monograph [3]. (This was written without knowledge about [10].) In fact, \(\delta_0\) has clearly this property, so let us restrict the attention to the class of non-vanishing normalized Stieltjes moment sequences \((s_n)\) for which we can apply the general theory of infinitely divisible positive definite kernels, see [3, Proposition 3.2.7]. Combining this result with Theorem 6.2.6 in the same monograph we can formulate the solution in the following way, where (iii) and (iv) are new:

**Theorem 1.4** For a sequence \(s_n > 0\) the following conditions are equivalent:

(i) \(s_n^c\) is a normalized Stieltjes moment sequence for each \(c > 0\), i.e. \((s_n) \in \mathcal{I}\).

(ii) There exist \(a \in \mathbb{R}, b \geq 0\) and a positive Radon measure \(\sigma\) on \([0, \infty)\) \({\{1}\})\) satisfying

\[
\int_0^\infty (1 - x)^2 \, d\sigma(x) < \infty, \quad \int_2^\infty x^n \, d\sigma(x) < \infty, \quad n \geq 3
\]

such that

\[
\log s_n = an + bn^2 + \int_0^\infty (x^n - 1 - n(x - 1)) \, d\sigma(x), \quad n = 0, 1, \ldots. \tag{9}
\]

(iii) There exist \(0 < \varepsilon \leq 1\) and an infinitely divisible probability \(\omega\) on \(\mathbb{R}\) such that

\[
s_n = (1 - \varepsilon)\delta_0 + \varepsilon \int_{-\infty}^\infty e^{-ny} \, d\omega(y). \tag{10}
\]

(iv) There exist \(0 < \varepsilon \leq 1\) and a product convolution semigroup \((\rho_c)_{c > 0}\) of probabilities on \([0, \infty]\) such that

\[
s_n^c = (1 - \varepsilon^c)\delta_0 + \varepsilon^c \int_0^\infty x^n \, d\rho_c(x), \quad n \geq 0, \quad c > 0. \tag{11}
\]
Assume \((s_n) \in \mathcal{I}\). If \((s_n^c)\) is \(S\)-determinate for some \(c = c_0 > 0\), then the quantities \(a, b, \sigma, \varepsilon, \omega, (\rho_c)_{c>0}\) from (ii)-(iv) are uniquely determined. Furthermore \(a = \log s_1, b = 0\) and the finite measure \((1 - x)^2 d\sigma(x)\) is \(S\)-determinate.

**Remark 1.5** The measure \(\sigma\) in condition (ii) can have infinite mass close to 1. There is nothing special about the lower limit 2 of the second integral. It can be any number \(> 1\). The conditions on \(\sigma\) can be formulated that \((1 - x)^2 d\sigma(x)\) has moments of any order.

**Remark 1.6** Concerning condition (iv) notice that the measures \(\tilde{\rho}_c = (1 - \varepsilon^c)\delta_0 + \varepsilon^c \rho_c, \ c > 0\) (12)

satisfy the convolution equation

\[ \tilde{\rho}_c \circ \tilde{\rho}_d = \tilde{\rho}_{c+d} \] (13)

and (11) can be written

\[ s_n^c = \int_0^\infty x^n d\tilde{\rho}_c(x), \ c > 0. \] (14)

On the other hand, if we start with a family \((\tilde{\rho}_c)_{c>0}\) of probabilities on \([0, \infty[\) satisfying (11), and if we define \(h(c) = 1 - \tilde{\rho}_c(\{0\}) = \tilde{\rho}_c(0, \infty[)\), then \(h(c + d) = h(c)h(d)\) and therefore \(h(c) = \varepsilon^c\) with \(\varepsilon = h(1) \in [0, 1]\). If \(\varepsilon = 0\) then \(\tilde{\rho}_c = \delta_0\) for all \(c > 0\), and if \(\varepsilon > 0\) then \(\rho_c := \varepsilon^{-c}(\tilde{\rho}_c|0, \infty[)\) is a probability on \([0, \infty[\) satisfying \(\rho_c \circ \rho_d = \rho_{c+d}\).

**Remark 1.7** In [4] was introduced a transformation \(\mathcal{T}\) from normalized non-vanishing Hausdorff moment sequences \((a_n)\) to normalized Stieltjes moment sequences \((s_n)\) by the formula

\[ \mathcal{T}[(a_n)] = (s_n); \quad s_n = \frac{1}{a_1 \cdots a_n}, \ n \geq 1. \] (15)

We note the following result:

If \((a_n)\) is a normalized Hausdorff moment sequence in \(\mathcal{I}\), then \(\mathcal{T}[(a_n)] \in \mathcal{I}\).

As an example consider the Hausdorff moment sequence \(a_n = q^n\), where \(0 < q < 1\) is fixed. Clearly \((q^n) \in \mathcal{I}\) and the corresponding product convolution semigroup is \((\delta_{q^n})_{c>0}\). The transformed sequence \((s_n) = \mathcal{T}[(q^n)]\) is given by

\[ s_n = q^{-\left(\frac{n+1}{2}\right)}, \]

which again belongs to \(\mathcal{I}\). The sequence \((s_n^c)\) is \(S\)-indeterminate for all \(c > 0\). The family of densities

\[ v_c(x) = \frac{q^{c/8}}{\sqrt{2\pi \log(1/q^c)}} \sqrt{x} \exp \left[ -\frac{(\log x)^2}{2\log(1/q^c)} \right], \ x > 0 \]
form a product convolution semigroup because

\[ \int_0^\infty x^z v_c(x) \, dx = q^{-cz(z+1)/2}, \quad z \in \mathbb{C}. \]

In particular

\[ \int_0^\infty x^nv_c(x) \, dx = q^{-c(n+1)/2}. \]

Each of the measures \( v_c(x) \, dx \) is infinitely divisible for the additive structure as well as for the multiplicative structure. The additive infinite divisibility is deeper than the multiplicative and was first proved by Thorin, cf. \[18\].

2 Proofs

We start by proving Theorem 1.4 and will deduce Theorem 1.1 and 1.2 from this result.

**Proof of Theorem 1.4**: The proof of “(i)⇒(ii)” is a modification of the proof of Theorem 6.2.6 in \[3\]: For each \( c > 0 \) we choose a probability measure \( \tilde{\rho}_c \) on \([0, \infty[\) such that for \( n \geq 0 \)

\[ s_n^c = \int_0^\infty x^n \, d\tilde{\rho}_c(x), \]

hence

\[ \int_0^\infty (x^n - 1 - n(x - 1)) \, d\tilde{\rho}_c(x) = s_n^c - 1 - n(s_1^c - 1). \]

(Because of the possibility of S-indeterminacy we cannot claim the convolution equation \( \tilde{\rho}_c \circ \tilde{\rho}_d = \tilde{\rho}_{c+d} \). If \( \mu \) denotes a vague accumulation point for \((1/c)(x - 1)^2 \, d\tilde{\rho}_c(x)\) as \( c \to 0\), we obtain the representation

\[ \log s_n - n \log s_1 = \int_0^\infty \frac{x^n - 1 - n(x - 1)}{(1 - x)^2} \, d\mu(x), \]

which gives \([9]\) by taking out the mass of \( \mu \) at \( x = 1 \) and defining \( \sigma = (x - 1)^{-2} \, d\mu(x) \) on \([0, \infty[\) \( \setminus \{1\}\). For details see \[3\].

“(ii)⇒(iii)” Define \( m = \sigma(\{0\}) \geq 0 \) and let \( \lambda \) be the image measure on \( \mathbb{R} \setminus \{0\} \) of \( \sigma - m\delta_0 \) under \( -\log x \). We get

\[ \int_{[-1,1]\setminus\{0\}} y^2 \, d\lambda(y) = \int_{[1/e,1]\setminus\{1\}} (1 - x)^2 \left( \frac{-\log x}{1 - x} \right)^2 \, d\sigma(x) < \infty, \]

and for \( n \geq 0 \)

\[ \int_{\mathbb{R}\setminus[-1,1]} e^{-ny} \, d\lambda(y) = \int_{[0,\infty[\setminus[1/e,1]} x^n \, d\sigma(x) < \infty. \] (16)
This shows that \( \lambda \) is a Lévy measure, which allows us to define a negative definite function

\[
\psi(x) = i\tilde{a}x + bx^2 + \int_{\mathbb{R}\setminus\{0\}} \left( 1 - e^{-ixy} - \frac{ixy}{1 + y^2} \right) \, d\lambda(y),
\]

where

\[
\tilde{a} := \int_{\mathbb{R}\setminus\{0\}} \left( \frac{y}{1 + y^2} + e^{-y} - 1 \right) \, d\lambda(y) - a.
\]

Let \( (\tau_c)_{c>0} \) be the convolution semigroup on \( \mathbb{R} \) with

\[
\int_{-\infty}^{\infty} e^{-ixy} \, d\tau_c(y) = e^{-c\psi(x)}, x \in \mathbb{R}.
\]

Because of (16) we see that \( \psi \) and then also \( e^{-c\psi} \) has a holomorphic extension to the lower half-plane. By a classical result (going back to Landau for Dirichlet series), see [20, p.58], this implies

\[
\int_{-\infty}^{\infty} e^{-ny} \, d\tau_c(y) < \infty, n = 0, 1, \ldots.
\]

For \( z = x + is, s \leq 0 \) the holomorphic extension of \( \psi \) is given by

\[
\psi(z) = i\tilde{a}z + bz^2 + \int_{\mathbb{R}\setminus\{0\}} \left( 1 - e^{-izy} - \frac{izy}{1 + y^2} \right) \, d\lambda(y),
\]

and we have

\[
\int_{-\infty}^{\infty} e^{-izy} \, d\tau_c(y) = e^{-c\psi(z)}.
\]

In particular we get

\[
-\psi(-in) = -\tilde{a}n + bn^2 + \int_{\mathbb{R}\setminus\{0\}} \left( e^{-ny} - 1 + \frac{ny}{1 + y^2} \right) \, d\lambda(y)
\]

\[
= -\tilde{a}n + bn^2 + \int_{\mathbb{R}\setminus\{0\}} \left( e^{-ny} - 1 - n(e^{-y} - 1) \right) \, d\lambda(y)
\]

\[
+ n \int_{\mathbb{R}\setminus\{0\}} \left( \frac{y}{1 + y^2} + e^{-y} - 1 \right) \, d\lambda(y)
\]

\[
= an + bn^2 + \int_{|x|,|y|\leq 1} (x^n - 1 - n(x - 1)) \, d\sigma(x),
\]

and therefore

\[
\log s_n = (n - 1)m - \psi(-in) \text{ for } n \geq 1,
\]

while \( \log s_0 = \psi(0) = 0 \).

The measure \( \omega = \delta_{-m} \ast \tau_1 \) is infinitely divisible on \( \mathbb{R} \) and we find for \( n \geq 1 \)

\[
s_n = e^{-m} e^{nm-\psi(-in)} = e^{-m} \int_{-\infty}^{\infty} e^{-ny} \, d\omega(y),
\]
so (10) holds with $\varepsilon = e^{-m}$.

“(iii)$\Rightarrow$(iv)” Suppose (10) holds and let $(\omega_c)_{c>0}$ be the unique convolution semigroup on $\mathbb{R}$ such that $\omega_1 = \omega$. Let $(\rho_c)_{c>0}$ be the product convolution semigroup on $[0, \infty]$ such that $\rho_1$ is the image of $\omega_c$ under $e^{-y}$. Then (11) holds for $c = 1, n \geq 0$ and for $c > 0$ when $n = 0$. For $n \geq 1$ we shall prove that

$$s_c^n = e^c \int_0^\infty x^n d\rho_c(x), \quad c > 0,$$

but this follows from (10) first for $c$ rational and then for all $c > 0$ by continuity.

“(iv)$\Rightarrow$(i)” is clear since $(s^n_c)$ is the Stieltjes moment sequence of $\tilde{\rho}_c$ given by (12).

Assume now $(s_n) \in \mathcal{I}$. We get $\log s_1 = a + b$. If $b > 0$ then $(s^n_c)$ is S-indeterminate for all $c > 0$ by Lemma 1.3 because the moment sequence $(\exp(cn^2))$ is S-indeterminate for all $c > 0$ by Remark 1.7.

If $(1 - x)^2 d\sigma(x)$ is S-indeterminate there exist infinitely many measures $\tau$ on $[0, \infty[ \backslash \{1\}$ with $\tau(\{1\}) = 0$ and such that

$$\int_0^\infty x^n (1 - x)^2 d\sigma(x) = \int_0^\infty x^n d\tau(x), \quad n \geq 0.$$

For any of these measures $\tau$ we have

$$\log s_n = an + bn^2 + \int_0^\infty \frac{x^n - 1 - n(x - 1)}{(1 - x)^2} d\tau(x),$$

because the integrand is a polynomial. Therefore $(s^n_c)$ has the S-indeterminate factor

$$\exp \left( c \int_0^\infty \frac{x^n - 1 - n(x - 1)}{(1 - x)^2} d\tau(x) \right)$$

and is itself S-indeterminate for all $c > 0$.

We conclude that if $(s^n_c)$ is S-determinate for $0 < c < c_0$, then $b = 0$ and $(1 - x)^2 d\sigma(x)$ is S-determinate. Then $a = \log s_1$ and $\sigma$ is uniquely determined on $[0, \infty[ \backslash \{1\}$. Furthermore, if $\varepsilon, (\rho_c)_{c>0}$ satisfy (11) then

$$s^n_c = \int_0^\infty x^n d\rho_c(x), \quad c > 0$$

with the notation of Remark 1.6 and we get that $\tilde{\rho}_c$ is uniquely determined for $0 < c < c_0$. This determines $\varepsilon$ and $\rho_c$ for $0 < c < c_0$, but then $\rho_c$ is unique for any $c > 0$ by the convolution equation.

We see that $\varepsilon, \omega$ are uniquely determined by (10) since (iii) implies (iv). □

Proof of Theorem 1.1 and 1.2
To verify directly that the sequence
\[ s_n = f(\alpha)f(\alpha + \beta) \cdots f(\alpha + (n-1)\beta) \]
of the form considered in Theorem 1.1 satisfies (9), we integrate formula (4) from \( \alpha \) to \( s \) and get
\[
\log f(s) = \log f(\alpha) + \int_{0}^{\infty} \frac{(e^{-ax} - e^{-sx})}{x} d\kappa(x).
\]
Applying this formula we find
\[
\log s_n = \sum_{k=0}^{n-1} \log f(\alpha + k\beta) = n \log f(\alpha) + \int_{0}^{\infty} \frac{(e^{-ax} - e^{-n\beta x})}{x} \frac{d\kappa(x)}{1 - e^{-\beta x}}.
\]
where \( \sigma \) is the image measure of \( e^{-\alpha x} d\kappa(x) \) under \( e^{-\beta x} \). Note that \( \sigma \) is concentrated on \([0,1]\). This shows that \((s_n) \in I \). It follows from the proof of Theorem 1.4 that the constant \( \varepsilon \) of (iii) is \( \varepsilon = 1 \), so (11) reduces to (6). The sequence \((s_n^c)\) is S-determinate for \( c \leq 2 \) by Carleman’s criterion stating that if
\[
\sum_{n=0}^{\infty} \frac{1}{n! s_n^c} = \infty,
\]
then \((s_n^c)\) is S-determinate, cf. [1], [14]. To see that this condition is satisfied we note that \( f(s) \leq (f(\beta)/\beta)s \) for \( s \geq \beta \), and hence
\[
s_n = f(\alpha)f(\alpha + \beta) \cdots f(\alpha + (n-1)\beta) \leq f(\alpha) \left( \frac{f(\beta)}{\beta} \right)^{n-1} \prod_{k=1}^{n-1} (\alpha + k\beta) = f(\alpha) f(\beta)^{n-1} (1 + \frac{\alpha}{\beta})^{n-1}.
\]
It follows from Stirling’s formula that (19) holds for \( c \leq 2 \).

We claim that
\[
\int_{1}^{\infty} \frac{e^{-ax}}{x} d\kappa(x) < \infty. \tag{20}
\]
This is clear if \( \alpha > 0 \), but if \( \alpha = 0 \) we shall prove
\[
\int_{1}^{\infty} \frac{d\kappa(x)}{x} < \infty.
\]
For $\alpha = 0$ we assume that $f(0) = a > 0$ and therefore the potential kernel
\[ p = \int_0^\infty \eta_t \, dt \]
has finite total mass $1/a$. Furthermore we have $\kappa = p \ast (b\delta_0 + x \, d\nu(x))$ since
\[ f'(s) = b + \int_0^\infty e^{-sx} \, d\nu(x), \]
so we can write $\kappa = \kappa_1 + \kappa_2$ with
\[ \kappa_1 = p \ast (b\delta_0 + x1_{]0,1]}(x) \, d\nu(x)), \quad \kappa_2 = p \ast (x1_{]1,\infty[}(x) \, d\nu(x)), \]
and $\kappa_1$ is a finite measure. Finally
\[ \int_1^\infty \frac{d\kappa_2(x)}{x} = \int_1^\infty \left( \int_0^\infty \frac{y}{x+y} \, dp(x) \right) \, d\nu(y) \leq \frac{\nu([1,\infty[)}{a} < \infty. \]

The function $\psi$ given by (8) is continuous in the closed half-plane $\Re z \geq 0$ and holomorphic in $\Re z > 0$ because of (20). Note that $\psi(n) = -\log s_n$ by (13). We also notice that $\psi(iy)$ is a continuous negative definite function on the additive group $(\mathbb{R}, +)$, cf. [5], because
\[ 1 - e^{-iyx} - iy(1 - e^{-x}) \]
is a continuous negative definite function of $y$ for each $x \geq 0$. Therefore there exists a unique product convolution semigroup $(\tau_c)_{c>0}$ of probabilities on $]0, \infty[$ such that
\[ \int_0^\infty t^y \, d\tau_c(t) = e^{-c\psi(iy)}, \quad c > 0, \, y \in \mathbb{R}. \quad (21) \]

By a classical result, see [20, p. 58]), the holomorphy of $\psi$ in the right half-plane implies that $t^z$ is $\tau_c$-integrable for $\Re z \geq 0$ and
\[ \int_0^\infty t^z \, d\tau_c(t) = e^{-c\psi(z)}, \quad c > 0, \, \Re z \geq 0. \quad (22) \]
In particular the $n$’th moment is given by
\[ \int_0^\infty t^n \, d\tau_c(t) = e^{-c\psi(n)} = e^{c\log s_n} = s_n^c, \]
so by S-determinacy of $(s_n^c)$ for $c \leq 2$ we get $\rho_c = \tau_c$ for $c \leq 2$. This is however enough to ensure that $\rho_c = \tau_c$ for all $c > 0$ since $(\rho_c)$ and $(\tau_c)$ are product convolution semigroups. \qed
3 Tyan’s thesis revisited

In [19] Tyan defines a normalized Hamburger moment sequence

\[ s_n = \int_{-\infty}^{\infty} x^n \mu(x), \quad n \geq 0 \]

to be infinitely divisible if

(i) \( s_n \geq 0 \) for all \( n \geq 0 \)

(ii) \( (s_n^c) \) is a Hamburger moment sequence for all \( c > 0 \).

Since the set of Hamburger moment sequences is closed under limits and products, we can replace (ii) by the weaker

(ii') \( \sqrt[k]{s_n} \) is a Hamburger moment sequence for all \( k = 0, 1, \ldots \).

Lemma 3.1 (Tyan) Let \( (s_n) \) be an infinitely divisible Hamburger moment sequence. Then one of the following cases hold:

• \( s_n > 0 \) for all \( n \).
• \( s_{2n} > 0, s_{2n+1} = 0 \) for all \( n \).
• \( s_n = 0 \) for \( n \geq 1 \).

Proof: The sequence \( (u_n) \) defined by

\[ u_n = \lim_{k \to \infty} \sqrt[k]{s_n} = \begin{cases} 1 \text{ if } s_n > 0 \\ 0 \text{ if } s_n = 0 \end{cases} \]

is a Hamburger moment sequence, and since it is bounded by 1 we have

\[ u_n = \int_{-1}^{1} x^n \, d\mu(x) \]

for some probability \( \mu \) on \([-1, 1]\).

Either \( u_2 = 1 \) and then \( \mu = \alpha \delta_1 + (1 - \alpha) \delta_{-1} \) for some \( \alpha \in [0, 1] \), or \( u_2 = 0 \) and then \( \mu = \delta_0 \), which gives the third case of the Lemma.

In the case \( u_2 = 1 \) we have \( u_1 = 2\alpha - 1 \), which is either 1 or 0 corresponding to either \( \alpha = 1 \) or \( \alpha = \frac{1}{2} \), which gives the two first cases of the Lemma. □

The symmetric case \( s_{2n} > 0, s_{2n+1} = 0 \) is equivalent to studying infinitely divisible Stieltjes moment sequences, while the third case is trivial.

Theorem 4.2 of [19] can be formulated:
Theorem 3.2  A Hamburger moment sequence \((s_n)\) such that \(s_n > 0\) for all \(n\) is infinitely divisible if and only if the following representation holds

\[
\log s_n = an + bn^2 + \int_{-\infty}^{\infty} (x^n - 1 - n(x-1)) d\sigma(x), \quad n \geq 0,
\]

where \(a \in \mathbb{R}, b \geq 0\) and \(\sigma\) is a positive measure on \(\mathbb{R} \setminus \{1\}\) such that \((1-x^2) d\sigma(x)\) is a measure with moments of any order. Furthermore \((s_n)\) is a Stieltjes moment sequence if and only if \(\sigma\) can be chosen supported by \([0, \infty[\).

The proof is analogous to the proof of Theorem 1.4.

Tyan also discusses infinitely divisible multidimensional moment sequences and obtains analogous results.

4 An application to Hermite polynomials

It follows from equation (23) that

\[
\sqrt{n!} = \int_0^{\infty} u^n d\sigma(u)
\]

for the unique probability \(\sigma\) on the half-line satisfying \(\sigma \circ \sigma = \exp(-t)1_{[0,\infty)}(t) dt\). Even though \(\sigma\) is not explicitly known, it can be used to prove that a certain generating function for the Hermite polynomials is non-negative.

Let \(H_n, n = 0, 1, \ldots\) denote the sequence of Hermite polynomials satisfying the orthogonality relation

\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2} dx = 2^n n! \delta_{nm}.
\]

The following generating function is well known:

\[
\sum_{k=0}^{\infty} \frac{H_k(x)}{k!} z^k = e^{x z - z^2}, \quad x, z \in \mathbb{C}.
\]

The corresponding orthonormal polynomials are given by

\[
h_n(x) = \frac{H_n(x)}{\sqrt{2^n n!}},
\]

and they satisfy the following inequality of Szasz, cf. [17]

\[
|h_n(x)| \leq e^{x^2/2}, x \in \mathbb{R}, n = 0, 1, \ldots
\]

Let \(\mathbb{D}\) denote the open unit disc in the complex plane.
Theorem 4.1 The generating function

$$G(t, x) = \sum_{k=0}^{\infty} h_k(x) t^k$$

is continuous for $(t, x) \in \mathbb{D} \times \mathbb{R}$ and satisfies $G(t, x) > 0$ for $-1 < t < 1, x \in \mathbb{R}$.

Proof: The series for the generating function (26) converges uniformly on compact subsets of $\mathbb{D} \times \mathbb{R}$ by the inequality of Szasz (25), so it is continuous.

By (23) we find

$$\sum_{k=0}^{n} h_k(x) t^k = \int_{0}^{\infty} \left( \sum_{k=0}^{n} \frac{H_k(x)}{k!} \left(\frac{tu}{\sqrt{2}}\right)^k \right) d\sigma(u),$$

which by (24) converges to

$$\int_{0}^{\infty} \exp(\sqrt{2}tux - t^2u^2/2) d\sigma(u) > 0 \text{ for } -1 < t < 1, x \in \mathbb{R},$$

provided we have dominated convergence. This follows however from (25) because

$$\int_{0}^{\infty} \left| \sum_{k=0}^{n} \frac{H_k(x)}{k!} \left(\frac{tu}{\sqrt{2}}\right)^k \right| d\sigma(u) \leq e^{x^2/2} \int_{0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{|u|^k}{\sqrt{k!}} \right) d\sigma(u)$$

$$= e^{x^2/2}(1 - |t|)^{-1} < \infty.$$

\[\Box\]

5 The moment sequences $(a)^c_n$ and $((a)_n/(b)_n)^c$

For each $a > 0$ the sequence $(a)_n := a(a+1) \cdot \ldots \cdot (a+n-1)$ is the Stieltjes moment sequence of the $\Gamma$-distribution $\gamma_a$:

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = \int x^n d\gamma_a(x) = \frac{1}{\Gamma(a)} \int_{0}^{\infty} x^{a+n-1} e^{-x} dx.$$

For $a = 1$ we get the moment sequence $n!$, so the following result generalizes Theorem 2.5 of [2].

Theorem 5.1 The sequence $(a)_n$ belongs to $\mathcal{I}$ for each $a > 0$. There exists a unique product convolution semigroup $(\gamma_{a,c})_{c>0}$ such that $\gamma_{a,1} = \gamma_a$. The moments are given as

$$\int_{0}^{\infty} x^n d\gamma_{a,c}(x) = (a)_n^c, \quad c > 0.$$
\[
\int_0^\infty x^z \, d\gamma_{a,c}(x) = \left( \frac{\Gamma(a + z)}{\Gamma(a)} \right)^c, \quad \text{Re} \, z > -a.
\]

The moment sequence \( ((a)_n^c) \) is S-determinate for \( c \leq 2 \) and S-indeterminate for \( c > 2 \).

**Proof:** We apply Theorem 1.1 and 1.2 to the Bernstein function \( f(s) = a + s \) and put \( \alpha = 0, \beta = 1 \). The formula for the Mellin transform follows from a classical formula about \( \log \Gamma \), cf. [11, 8.3417].

We shall prove that \( (a)_n^c \) is S-indeterminate for \( c > 2 \). In [2] it was proved that \( (n!)^c \) is S-indeterminate for \( c > 2 \), and so are all the shifted sequences \( ((n + k - 1)!)^c, k \in \mathbb{N} \). This implies that

\[
(k)_n^c = \left( \frac{(n + k - 1)!}{(k - 1)!} \right)^c
\]
is S-indeterminate for \( k \in \mathbb{N}, c > 2 \). To see that also \( (a)_n^c \) is S-indeterminate for \( a \notin \mathbb{N} \), we choose an integer \( k > a \) and factorize

\[
(a)_n^c = \left( \frac{(a)_n}{(k)_n} \right)^c (k)_n^c.
\]

By the following theorem the first factor is a non-vanishing Stieltjes moment sequence, and by Lemma 1.3 the product is S-indeterminate. \( \square \)

For \( 0 < a < b \) we have

\[
\frac{(a)_n}{(b)_n} = \frac{1}{B(a, b - a)} \int_0^1 x^{n+1-a} \left( 1 - x \right)^{-a-1} \, dx, \quad (27)
\]

where \( B \) denotes the Beta-function.

**Theorem 5.2** Let \( 0 < a < b \). Then \( ((a)_n/(b)_n) \) belongs to \( \mathcal{I} \) and all powers of the moment sequence are Hausdorff moment sequences. There exists a unique product convolution semigroup \((\beta(a, b)_c)_{c>0}\) on \([0, 1]\) such that

\[
\int_0^1 x^z \, d\beta(a, b)_c(x) = \left( \frac{\Gamma(a + z)}{\Gamma(a)} \frac{\Gamma(b + z)}{\Gamma(b)} \right)^c, \quad \text{Re} \, z > -a.
\]

**Proof:** We apply Theorem 1.1 and 1.2 to the Bernstein function \( f(s) = (a + s)/(b + s) \) and put \( \alpha = 0, \beta = 1 \).

The Stieltjes moment sequences \( ((a)_n/(b)_n)^c) \) are all bounded and hence Hausdorff moment sequences. The measures \( \gamma_{b,c} \circ \beta(a, b)_c \) and \( \gamma_{a,c} \) have the same moments and are therefore equal for \( c \leq 2 \) and hence for any \( c > 0 \) by the convolution equations. The Mellin transform of \( \beta(a, b)_c \) follows from Theorem 6.1. \( \square \)
6 The $q$-extension $((a; q)_n / (b; q)_n)^c$

In this section we fix $0 < q < 1$ and consider the $q$-shifted factorials

$$(z; q)_n = \prod_{k=0}^{n-1} (1 - zq^k), \quad z \in \mathbb{C}, \quad n = 1, 2, \ldots, \infty$$

and $(z; q)_0 = 1$. We refer the reader to [10] for further details about $q$-extensions of various functions.

For $0 \leq b < a < 1$ the sequence $s_n = (a; q)_n / (b; q)_n$ is a Hausdorff moment sequence for the measure

$$\mu(a, b; q) = \frac{(a; q)_\infty}{(b; q)_\infty} \sum_{k=0}^\infty \frac{(b/a; q)_k}{(q; q)_k} a^k \delta_{q^k},$$

which is a probability on $[0, 1]$ by the $q$-binomial Theorem, cf. [10]. The calculation of the $n$’th moment follows also from this theorem since

$$s_n(\mu(a, b; q)) = \frac{(a; q)_\infty}{(b; q)_\infty} \sum_{k=0}^\infty \frac{(b/a; q)_k}{(q; q)_k} a^k q^k = \frac{(a; q)_\infty}{(b; q)_\infty} \frac{(bq/a; q)_\infty}{(aq^n; q)_\infty} = \frac{(a; q)_n}{(b; q)_n}.$$

Replacing $a$ by $q^a$ and $b$ by $q^b$ and letting $q \to 1$ we get the moment sequences $(a)_n / (b)_n$, so the present example can be thought of as a $q$-extension of the former. The distribution $\mu(q^a, q^b; q)$ is called the $q$-Beta law in Pakes [13] because of its relation to the $q$-Beta function.

**Theorem 6.1** For $0 \leq b < a < 1$ the sequence $s_n = (a; q)_n / (b; q)_n$ belongs to $\mathcal{I}$. The measure $\mu(a, b; q)$ is infinitely divisible with respect to the product convolution and the corresponding product convolution semigroup $(\mu(a, b; q)_c)_{c>0}$ satisfies

$$\int t^z \, d\mu(a, b; q)_c(t) = \left( \frac{(bq^z; q)_\infty}{(b; q)_\infty} / \frac{(aq^z; q)_\infty}{(a; q)_\infty} \right)^c, \quad \text{Re } z > -\frac{\log q}{\log q}. \quad (29)$$

In particular

$$s_n^c = ((a; q)_n / (b; q)_n)^c \quad (30)$$

is the moment sequence of $\mu(a, b; q)_c$, which is concentrated on $\{q^k | k = 0, 1, \ldots\}$ for each $c > 0$.

**Proof:** It is easy to prove that $(a; q)_n / (b; q)_n$ belongs to $\mathcal{I}$ using Theorem [1.1] and [1.2] applied to the Bernstein function

$$f(s) = \frac{1 - aq^s}{1 - bq^s} = 1 - (a - b) \sum_{k=0}^\infty b^k q^{(k+1)s},$$
but it will also be a consequence of the following considerations, which gives information about the support of $\mu(a, b; q)$.

For a probability $\mu$ on $[0, 1]$ let $\tau = -\log(\mu)$ be the image measure of $\mu$ under $-\log$. It is concentrated on $[0, \infty]$ and

$$\int_0^1 t^{ix} d\mu(t) = \int_0^\infty e^{-itx} d\tau(t).$$

This shows that $\mu$ is infinitely divisible with respect to the product convolution if and only if $\tau$ is infinitely divisible in the ordinary sense, and in the affirmative case the negative definite function $\psi$ associated to $\mu$ is related to the Bernstein function $f$ associated to $\tau$ by $\psi(x) = f(ix), x \in \mathbb{R}$, cf. [5, p.69].

We now prove that $\mu(a, b; q)$ is infinitely divisible for the product convolution. As noticed this is equivalent to proving that the measure

$$\tau(a, b; q) := \frac{(a; q)_\infty}{(b; q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a; q)_k}{(q; q)_k} \frac{a^k}{q^k} \delta_k \log(1/q),$$

is infinitely divisible in the ordinary sense. To see this we calculate the Laplace transform of $\tau(a, b; q)$ and get by the $q$-binomial Theorem

$$\int_0^\infty e^{-st} d\tau(a, b; q)(t) = \frac{(aq^s; q)_\infty}{(bq^s; q)_\infty} / \frac{(aq^s; q)_\infty}{(a; q)_\infty}, \quad s \geq 0. \quad (31)$$

Putting

$$f_a(s) = \log \frac{(aq^s; q)_\infty}{(a; q)_\infty},$$

we see that $f_a$ is a bounded Bernstein function of the form

$$f_a(s) = -\log(a; q)_\infty - \varphi_a(s),$$

where

$$\varphi_a(s) = -\log(aq^s; q)_\infty = \sum_{k=1}^{\infty} \frac{a^k}{k(1 - q^k)} q^{ks}$$

is completely monotonic as Laplace transform of the finite measure

$$\nu_a = \sum_{k=1}^{\infty} \frac{a^k}{k(1 - q^k)} \delta_k \log(1/q).$$

From (31) we get

$$\int_0^\infty e^{-st} d\tau(a, b; q)(t) = \frac{(a; q)_\infty}{(b; q)_\infty} e^{\varphi_a(s) - \varphi_b(s)}.$$
and it follows that $\tau(a, b; q)$ is infinitely divisible and the corresponding convolution semigroup is given by the infinite series

$$
\tau(a, b; q)_c = \left( \frac{(a; q)_\infty}{(b; q)_\infty} \right)^c \sum_{k=0}^{\infty} \frac{c^k (\nu_a - \nu_b)^*}{k!}, \quad c > 0.
$$

Note that each of these measures are concentrated on $\{k \log(1/q) \mid k = 0, 1, \ldots\}$. The associated Lévy measure is the finite measure $\nu_a - \nu_b$ concentrated on $\{k \log(1/q) \mid k = 1, 2, \ldots\}$. This shows that the image measures

$$
\mu(a, b; q)_c = \exp(-\tau(a, b; q)_c), \quad c > 0
$$

form a product convolution semigroup concentrated on $\{q^k \mid k = 0, 1, \ldots\}$.

The product convolution semigroup $(\mu(a, b; q)_c)_{c>0}$ has the negative definite function $f(ix)$, where $f(s) = f_a(s) - f_b(s)$ for $\Re s \geq 0$, hence

$$
\int t^{ix} d\mu(a, b; q)_c(t) = \left( \frac{(bq^{ix}; q)_{\infty}}{(b; q)_{\infty}} \right)^c (a^{ix}; q)_{\infty}, \quad x \in \mathbb{R},
$$

and the equation (29) follows by holomorphic continuation. Putting $z = n$ gives (30).

\[\square\]

7 Complements

Example 7.1 Let $0 < a < b$ and consider the Hausdorff moment sequence

$$
a_n = (a)_n/(b)_n \in \mathcal{I}.
$$

By Remark 1.7 the moment sequence $(s_n) = \mathcal{T}[(a_n)]$ belongs to $\mathcal{I}$. We find

$$
s_n = \prod_{k=1}^{n} \frac{(b)_k}{(a)_k} = \prod_{k=0}^{n-1} \left( \frac{b + k}{a + k} \right)^{n-k}.
$$

Example 7.2 Applying $\mathcal{T}$ to the Hausdorff moment sequence $((a; q)_n/(b; q)_n)$ gives the Stieltjes moment sequence

$$
s_n = \prod_{k=1}^{n} \frac{(b; q)_k}{(a; q)_k} = \prod_{k=0}^{n-1} \left( \frac{1 - bq^k}{1 - aq^k} \right)^{n-k}.
$$

We shall now give the measure with moments (32).

For $0 \leq p < 1, 0 < q < 1$ we consider the function of $z$

$$
h_p(z; q) = \prod_{k=0}^{\infty} \left( \frac{1 - pq^k}{1 - zq^k} \right)^k,
$$
which is holomorphic in the unit disk with a power series expansion

\[ h_p(z; q) = \sum_{k=0}^{\infty} c_k z^k \]  

having non-negative coefficients \( c_k = c_k(p, q) \). To see this, notice that

\[ \frac{1 - pz}{1 - z} = 1 + \sum_{k=1}^{\infty} (1 - p) z^k. \]

For \( 0 \leq b < a < 1 \) and \( \gamma > 0 \) we consider the probability measure with support in \([0, \gamma]\)

\[ \sigma_{a,b,\gamma} = \frac{1}{h_{b/a}(a; q)} \sum_{k=0}^{\infty} c_k a^k \delta_{\gamma q^k}, \]

where the numbers \( c_k \) are the (non-negative) coefficients of the power series for \( h_{b/a}(z; q) \).

The \( n \)’th moment of \( \sigma_{a,b,\gamma} \) is given by

\[ s_n(\sigma_{a,b,\gamma}) = \gamma^n \frac{h_{b/a}(aq^n; q)}{h_{b/a}(a; q)}. \]

For \( \gamma = (b; q)_\infty/(a; q)_\infty \) it is easy to see that

\[ s_n(\sigma_{a,b,\gamma}) = \prod_{k=0}^{n-1} \left( \frac{1 - b q^k}{1 - a q^k} \right)^{n-k}, \]

which are the moments \([22]\).

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