Analytical features of extended Airy beams

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Abstract. The presence of widely used features, in particular, the autofocusing feature, of classical Airy beams leads to a thought of a benefit of their modification. There are several generalizations of Airy function both on base of the differential equation modification and variations in the integral notion. In the paper one more type of extended Airy functions is investigated theoretically and numerically. The type is constructed by generalization of the integral notion in a wide range of non-integer values of power. A principal attention in the paper is devoted to obtaining of approximate analytical expressions for extended Airy functions under an arbitrary exponent. On base of introduced functions the new type of autofocusing beams is formed. Extended Airy beams are formed if a diffractive optical element called the generalized lens is placed in an entrance plane of the optical scheme. The numerical examination of beams features is implemented by usage of the fractional Fourier transformation.

1. Introduction

The Airy wave packets, considered in the late seventies of the last century in the framework of quantum mechanics [1] as wave objects propagating along a curved trajectory, have attracted recently increased interest of optical scientists. Interest arose in connection with the successful physical generation of "accelerating" laser beams [2], consistent with the Airy functions. Special properties of Airy beams were used in many applications [3], including optical manipulation [4, 5].

Airy's functions theoretically have infinite energy, so their physical realization requires truncation. In [2], the Airy beams with finite energy representing the product of the classical Airy mode and exponential function were considered. Although multiplying by a Gaussian or exponential function allows such beams to be formed quite simply by means of a spatial light modulator, in both cases the beams formed are actually no longer diffraction-free, although they approximately retain their appearance up to a certain distance.

In [6], a different method of truncation of the infinite Airy mode was considered – using a rectangular aperture truncating the function in the positive part of the argument at its decay to almost zero, and in the negative part – to nth zero. In [6], the divergence degrees of three types of truncated Airy beams were compared: exponential, Gaussian and diaphragm-bounded. Numerically it was shown that in the latter case the oscillating beam structure and the clearly distinguished intensity maximum are preserved much longer than in the first two cases.

Among other non-diffraction modes, Airy beams are distinguished by a special property of "acceleration", which (when beams are propagated in free space) manifests itself as a deviation from the straight path according to the parabolic law [2, 7, 8].
In [9], a generalized family of two-dimensional "accelerating" beams based on Airy and parabolic beams was considered. In this case, the two-dimensional beam is represented as the product of a one-dimensional Airy beam (or parabolic beam) and any other one-dimensional distribution.

An interesting type of the beam, which is the product of three Airy functions, was proposed in the paper [10]. Such beams have triangular symmetry in the cross section, and the beam structure changes during propagation.

New so-called “mirror” Airy beams were proposed in [11]. They are the sum of two Airy functions that are truncated in the $n$th zero (or extremum), one of the functions being mirrored. The structure of such Airy beams resembles Hermite–Gaussian modes. The spectrum of mirror Airy beams is described by the cosine function of a nonlinear argument (with cubic and linear dependence) [11]. During propagation, the mirror Airy beams demonstrate a symmetrical “acceleration” in opposite directions (divergence proportional to the square of the distance traveled). The astigmatic transformation of two-dimensional mirror (or specular) Airy beams with indices $(n, m)$ makes it possible to obtain quasi-ring structures possessing an orbital angular momentum [12]. The vortex Airy beams are also considered in Ref. [12], the radial part of which is expressed in terms of the truncated Airy functions. During propagation, such beams retain a well-defined multi-ring structure, similar to the Laguerre–Gaussian modes, although the ratio of the radii of the rings varies.

Note that the type of vortex Airy functions, considered in the work [12], differs substantially from the circular Airy functions proposed in [13-15], for which sharp autofocusing is characteristic. This property can be useful for multiphoton polymerization [16, 17].

Although Airy beams are the solution of the paraxial wave equation, their propagation is increasingly considered under conditions of wide divergence angles and even in the area of evanescent waves [18-20].

Thus, a review of the properties and applications of Airy beams shows their versatility and high demand. This fact stimulates the search for new modifications and generalizations of the classical Airy functions.

There are several generalizations of the Airy functions both on the basis of modification of the differential equation [21, 22] and variations in the integral representation [23-28].

In this paper, another extension of Airy beams is proposed. Of the known generalizations, the closest to the extension we propose are the variants from [23, 24, 27], but we consider a wider range of the power dependence of the argument, including non-integer power values.

2. Generalized Airy functions

Before proceeding to the description and analysis of the new extension of Airy beams, we list briefly known generalizations.

One way of generalization is to modify the differential equation of the classical Airy function

$$\frac{d^2u}{dz^2} - zu = 0.$$  

In particular, in [21] instead of it the differential equation of the form

$$\frac{d^2u}{dz^2} - z^n u = 0$$

is considered, in which $n$ is a positive integer. The article [22] gives another generalization based on a similar change:

$$\frac{d^2u}{dz^2} - z^\alpha u = 0,$$

where $\alpha > -2$ is not necessarily an integer, but a limit $z > 0$ is introduced.

Another type of generalization is considered by F. Olver in [23]. The Olver function is defined by the following integral expression

$$O_m(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(a(i\lambda)^{m+1} + i\lambda x\right)d\lambda,$$

in which $m$ is a non-negative integer and $|a| = (m + 3)^{1/2}$. The classical Airy function is a special case when $m=0$. In the paper [24], a new class of non-diffraction beams is introduced on the basis of the Olver functions.

There are other generalizations of the integral representation of the Airy function. Functions, presented as contour integrals, were introduced in [25, 26]. For particular parameter values, they are equal to the Airy function, or expressed through it. The tables of several functions, including
$A_{i,q}(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \left( \frac{iy^p}{p} - \frac{qy^q}{q} + ixy \right) dy$, are given in [27]. Here $\alpha \geq 0$, $x$ is a real number, $p$ and $q$ are natural numbers such that $q$ is even and $p$ is odd and $p \geq 3$. In special cases, there is a connection with the classical Airy function: $A_{i,q}(0,x) = Ai(x)$, $A_{i,2}(\alpha,x) = \exp(\alpha x / 2) Ai(x + \alpha^2 / 4)$. In the work [28] the study of the function, which with a precision of notations coincides with one of the functions introduced in [27] is implemented.

A more detailed description of the known generalizations is given in the article [29], here more attention is paid to the analytical properties of the functions under consideration.

3. New type of extended Airy functions

We consider a new type of extension of the Airy function based on the integral representation:

$$\text{Fr}_{\gamma} (\omega) = \frac{D_{\gamma/2}}{D_{\gamma/2}} \exp[-i(k\alpha x)^{\gamma} \cdot \text{symm}(x^\gamma) - i(\omega x)] dx$$

(1)

corresponding to the one-dimensional Fourier transform in finite limits from the transmission function of a generalized lens or fractional axicon [30-32]. symm$(x^\gamma)$ is the symmetry operator applied to the expression $x^\gamma$, so that it makes sense when $x$ is negative and $\gamma$ is non-integer. For even symmetrization at $x<0$ we use $|p|$, and for odd $-|p|$. Next, we use the odd symmetrization, since it coincides with the original $x^\gamma$ for the odd $\gamma$. The latter is important when compared with the classical Airy function.

The expression (1) is close to that considered in [27], but a wider range of powers $\gamma$ is used here, including non-integer values. If $x$ is negative, it is assumed to use even or odd symmetry.

Consider the case where the pupil size is large enough and the limits of integration with sufficient accuracy can be replaced by infinite. This is valid for $\gamma>1$ and means that the influence of the pupil can be neglected. After applying the odd symmetrization, we obtain the following formula:

$$\text{Fr}_{\gamma,odd} (\omega) = \frac{D_{\gamma/2}}{D_{\gamma/2}} \exp[-i(k\alpha x)^{\gamma} - i(\omega x)] dx + \frac{D_{\gamma/2}}{D_{\gamma/2}} \exp[i(k\alpha x)^{\gamma} + i(\omega x)] dx = 2\pi \cos((k\alpha x)^{\gamma} + k\omega x) dx$$

(2)

Exponents in both integrals differ only by a sign, which will simplify the calculations. (With even symmetrization, this property is absent, and the result is more complex.)

In the particular case when $\gamma=3$ have: $\text{Fr}_{\gamma,odd} (\omega) = \frac{D_{\gamma/2}}{D_{\gamma/2}} \exp[-i(k\alpha x)^{\gamma} - i(\omega x)] dx$. This integral can be calculated using tables [33], and it is equal to

$$\text{Fr}_{\gamma=3,odd} (\omega) = \frac{2\pi}{k (3\alpha_0)^{3/2}} \left\{ \sqrt{\sin} \left( I_{1/1} \left( \frac{\omega}{3\alpha_0} \right) \right)^{3/2} - I_{1/1} \left( \frac{\omega}{3\alpha_0} \right) \right\}, \quad \omega > 0,$$

(3)

$$\text{Fr}_{\gamma=3,odd} (\omega) = \frac{2\pi}{k (3\alpha_0)^{3/2}} \left\{ \sqrt{-\cos} \left( J_{1/1} \left( \frac{\omega}{3\alpha_0} \right) \right)^{3/2} + J_{1/1} \left( \frac{\omega}{3\alpha_0} \right) \right\}, \quad \omega \leq 0.$$

This result can be written via the classical Airy function:

$$\text{Fr}_{\gamma=3,odd} (\omega) = \frac{2\pi}{\sqrt{3k\alpha_0}} \text{Ai} \left( \frac{\omega}{\sqrt{3\alpha_0}} \right)$$

(4)

For the general form (2), the exact solution cannot be found, but it is possible to obtain an asymptotic representation for large argument values.

If we use the stationary phase method [34] to calculate each of the two integrals in (2), we obtain:

$$2\pi \cos g(x) dx = \oint g(x) \cos \left( g(x) + \pi / 4 \right)$$

(5)
where $x_0$ is stationary point. We use for this function $g(x)$ the general formula (5). Derivatives are equal to $g(x) = (k\alpha_g)^\gamma x^{\gamma - 1} + k \omega$, $g'(x) = (k\alpha_g)^\gamma x^{\gamma - 1}$. Let $\omega$ be negative (positive are discussed below). Then we have the following expressions ($\alpha_g = -\omega$):

$$x_0 = \frac{1}{k} \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{1}{\gamma - 1}}; g(x_0) = \omega \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{1}{\gamma - 1}}; g'(x_0) = k^2 \omega (\gamma - 1) \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{1}{\gamma - 1}}.$$

(6)

Substituting (6) into (5), we get that when $\omega$ is negative and large in modulus there is an approximation

$$\text{Fr}_{\gamma}^{\text{odd}} (\omega < 0) \approx \frac{2}{k} \sqrt{\frac{2\pi}{\gamma - 1}} \cdot \frac{2\gamma}{\gamma \alpha_g^2} \times \cos \left[ \frac{\pi}{\gamma} \cdot \left( \gamma - 1 \right) - \frac{\pi}{4} \right].$$

(7)

When $\gamma = 3$, the expression (7) corresponds to the asymptotic representation of the Bessel function $J_{\gamma/2}(x)$. It follows from the expression (7) that the extended Airy functions oscillate at negative values of the argument, and the oscillation frequency increases with the increase in the absolute value of the argument. With the growth of the parameter $\gamma$, the thickening rate decreases, and the frequency tends to a constant value. The envelope decreases as a fractional-rational function. The integral in (2) converges at $\gamma > 1$, but at $1 < \gamma \leq 2$ the envelope dependence becomes non-physical. Therefore, the approximate formula (7) is applicable for the exponents $\gamma > 2$ (at $\gamma$, close to 2, the error will be large). It is possible that for $1 < \gamma \leq 2$, some transformation of expression (2) is required before applying the stationary phase method, a similar situation is described in [35, 36].

Now let $\omega$ be a large positive value. Then the stationary point is:

$$x_0 = \frac{1}{k} \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{2}{\gamma - 1}} \cdot e^{\frac{1}{\gamma}}.$$  

(8)

It is not real, so further calculations are to some extent formal. We took the root with the smallest imaginary part, as other roots will not contribute to the main summand of the final expression. For the same reason, we did not consider complex roots for $\omega < 0$. The values at a stationary point are:

$$g(x_0) = \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{1}{\gamma - 1}} \cdot (\gamma - 1) e^{-\frac{1}{\gamma}}; g'(x_0) = (\gamma - 1) k^2 e^{-\frac{1}{\gamma}} \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{1}{\gamma - 1}} \alpha_g^{-\frac{1}{\gamma - 1}} \alpha_g^{-\frac{1}{\gamma - 1}}.$$

(9)

Then the second integral in (2) in the substitution of values (9) is equal to

$$\int_0^\infty e^{i\omega x} dx \approx \sqrt{\frac{\pi}{g(x_0)}} \cdot e^{g(x_0)} \cdot 2e^{i\pi/4} = \sqrt{\frac{2\pi}{\gamma - 1}} \cdot e^{-\frac{1}{\gamma} \alpha_g^{-\frac{1}{\gamma - 1}}} \cdot \frac{1}{k} \cdot \frac{2\gamma}{\gamma \alpha_g^2} \cdot \exp \left[ \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{1}{\gamma - 1}} (\gamma - 1) \left( \cos \frac{\pi}{\gamma - 1} + i \sin \frac{\pi}{\gamma - 1} \right) \right].$$

(10)

And when calculating the first one, two changes are made – a symmetrical stationary point is taken (in contrast to (9), a multiplier of $\exp(-i\pi/(\gamma - 1))$ is used), and a minus sign is added when rooting from the second derivative. Indirectly, this can be justified by the fact that only in this case the answer is real and without exponential growth. As a result, the first integral is equal to

$$\int_0^\infty e^{i(-\omega) x} dx \approx \sqrt{\frac{\pi}{g(x_0)}} \cdot e^{-g(x_0)} \cdot 2e^{i\pi/4} = \sqrt{\frac{2\pi}{\gamma - 1}} \cdot e^{-\frac{1}{\gamma} \alpha_g^{-\frac{1}{\gamma - 1}}} \cdot \frac{1}{k} \cdot \frac{2\gamma}{\gamma \alpha_g^2} \cdot \exp \left[ -i \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{1}{\gamma - 1}} (\gamma - 1) \left( \cos \frac{\pi}{\gamma - 1} - i \sin \frac{\pi}{\gamma - 1} \right) \right].$$

(11)

Adding these expressions, we find an approximate formula

$$\text{Fr}_{\gamma}^{\text{odd}} (\omega > 0) \approx \frac{2}{k} \sqrt{\frac{2\pi}{\gamma - 1}} \cdot \frac{2\gamma}{\gamma \alpha_g^2} \times \exp \left[ -i \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{1}{\gamma - 1}} (\gamma - 1) \left( \cos \frac{\pi}{\gamma - 1} + \frac{\pi}{\gamma - 1} \right) \right] \cos \left( \frac{\omega}{\gamma \alpha_g} \right)^{\frac{1}{\gamma - 1}} (\gamma - 1) \left( \cos \frac{\pi}{\gamma - 1} + \frac{\pi}{\gamma - 1} \right).$$

(12)
which is true for $\gamma>2$. If in (3) a square bracket is expressed through a function $K_{\nu,\beta}(\cdot)$ and its form with a large argument value is used, the result is twice less than when $\gamma=3$ is substituted in (12). Therefore, we have to say that the formula (12) is correct to the accuracy of the numerical coefficient.

The formula (12) shows that the value $\gamma=3$ corresponding to the Airy function is exceptional, since only in this case $\cos -\frac{\pi}{\gamma-1}=0$, which means the conversion of the oscillating factor into a constant. That is, only the classical Airy function with a positive argument decreases exponentially to zero, remaining positive.

For positive values of the argument, the extended Airy function also oscillates and the envelope decreases exponentially. The type of frequency dependence on $\omega$ is the same as for the negative argument.

Note that in zero we can get the exact expression for (2) [33]:

$$F_{\gamma}(\omega=0) = \frac{2\Gamma(1/\gamma)}{\gamma k \alpha_0} \cos \left( \frac{\pi}{2\gamma} \right)$$

which is true for any $\gamma>1$.

### 4. Numerical simulation

To obtain the field distribution in any paraxial region, (in the Fresnel and Fraunhofer diffraction zones) the fractional Fourier transform (FrFT) can be applied [37-39]:

$$E(u,z) = \sqrt{\frac{i k}{2\pi \sin(\tau)}} \exp \left( \frac{i k u^2 \cos(\tau)}{2 f \sin(\tau)} \right) \int_{-\infty}^{\infty} g(x) \exp \left[ \frac{i k x^2 \cos(\tau)}{2 f \sin(\tau)} \right. - \frac{i k x u}{f \sin(\tau)} \left. \right] dx$$

where $k = 2\pi/\lambda$, $\lambda$ is radiation wavelength, $f$ is lens focal length, $\tau = \pi/(2f)$, $z$ is distance from the input plane, $D$ is field size in the input plane.

As an input function, we consider a phase optical element with a complex transmission function:

$$g(x) = \exp \left[ -i (k \alpha_0)^\gamma \cdot \text{symm}(x') \right]$$

where $R=D/2$, $\alpha_0$ is specified numerical aperture of the optical element.

Such elements were considered earlier in [30-32, 35, 40] and were called a generalized lens (for $\gamma>1$) and a fractional axicon (for $0<\gamma<1$). Note that element (15) has a certain focusing power not only at $\gamma>1$, but also at $0<\gamma<1$. In the latter case, a compact focal spot is formed in the near diffraction zone [40].

By analyzing the integrand in (14), it can be stated that when using the input function of the form (15), which has a convergent wave front, the "focus" will occur earlier than at $z=f$. In particular, when $\gamma=2$, the position of the new focus can be easily computed:

$$z_f = f \frac{\arctg}{\pi} \left( \frac{1}{2k\alpha_0 f} \right)$$

From (16) it is obvious that $z_f < f$.

The properties of the extended Airy beams (1) will depend not only on the parameter $\gamma$, but also on the parameter $\alpha_0$. To perform a comparative simulation of the propagation of Airy beams at different values of $\gamma$, we agree the $\alpha_0$ parameter on the numerical aperture of the optical element (15) as follows:

$$\alpha_0 = \left( \frac{NA}{\gamma (k R)^{-\gamma}} \right)^{1/\gamma}$$

where $R=D/2$, $NA$ is specified numerical aperture of the optical element.

Table 1 shows the formation and propagation of the extended Airy beams (1) at a distance of two focuses ($f=100$ mm) from the input plane under illumination of optical elements (15) with a size $D=2$ mm by radiation with a wavelength of 633 nm.
As can be seen from the results in Table 1, the energy distribution between the first and the rest of the beam petals depends on the parameter $\gamma$. The first petal becomes prevalent and the beam becomes similar to the classical Airy beam at $\gamma > 2.5$. When $\gamma >> 1$ extended Airy beam reminiscent of the Airy beam with exponential apodization [2]. In addition, the pattern of ray propagation depends on the parameter $\gamma$ (the second column of Table 1).

When $\gamma < 2$ initially converging wavefront after the focal plane $z_f$ will be divergent due to the quadratic phase function in operator (14). Therefore, in the focal plane of the lens, $z = f$, the picture will be less bright. Note that in this case the trajectory of the maximum intensity to the focal plane $z_f$ has a bulge up (the first line of Table 1). Z axis is directed from left to right, ($x$ axis – from top to bottom) on pictures of the field amplitude in Tables 1 and 2.

| Beam parameters \[ \gamma, \alpha_0 \] | Field amplitude in $xz$ plane $z \in [10 \text{ mm}, 190 \text{ mm}], x \in [-2 \text{ mm}, 1 \text{ mm}]$ | Intensity distribution in the focal plane $z = 100 \text{ mm}$ | $x$, mm |
|---|---|---|---|
| $\gamma = 1.5$, $\alpha_0 = 0.002$ | ![Field amplitude](image1.png) | ![Intensity distribution](image2.png) | |
| $\gamma = 2$, $\alpha_0 = 0.001$ | ![Field amplitude](image3.png) | ![Intensity distribution](image4.png) | |
| $\gamma = 2.5$, $\alpha_0 = 0.0006$ | ![Field amplitude](image5.png) | ![Intensity distribution](image6.png) | |
| $\gamma = 3$, $\alpha_0 = 0.0004$ | ![Field amplitude](image7.png) | ![Intensity distribution](image8.png) | |
| $\gamma = 6$, $\alpha_0 = 0.00018$ | ![Field amplitude](image9.png) | ![Intensity distribution](image10.png) | |

At $\gamma = 2$, the most pronounced intensity maximum is observed in the focus plane $z_f$. The trajectories to the focal plane $z_f$ look approximately straight.

For $\gamma > 2$, the trajectory of the propagation of the extended Airy beams has the form of a pronounced accelerating beam, similar to the classical Airy beams. At $\gamma > 2$, the divergence due to the influence of the quadratic phase will be less in the formed beam. In this case, the trajectory of maximum intensity has a bulge down to the focal plane of the lens (the third, fourth and fifth rows of Table 1). Note that for large $\gamma$, the central part of the input field is close to the planar beam (the phase is approximately...
constant), which will have the maximum value in the focal plane of the lens. The peripheral part of the input beam with the largest phase variations focuses very close to the input plane, so this plane is not visible in the illustration in Table 1 (second column, fifth row).

The simulation results at $0<\gamma<1$ are not given here (they are available in [29]), since the presence of the input pupil cannot be ignored for this range.

Obviously, using in (15) instead of the exponential function sine or cosine, it is possible to form so-called extended mirror Airy beams. Table 2 shows the formation and propagation of mirror beams obtained using the input function of the following form:

$$g(x) = \sin[(k\alpha_x)^{\gamma} \cdot \text{symm}(x')]$$

(18)

| Beam parameters | Field amplitude in $xz$ plane $z \in [10 \text{ mm, } 190 \text{ mm}], x \in [-1 \text{ mm, } 1 \text{ mm}]$ | Intensity distribution in the focal plane $(z=100 \text{ mm})$ |
|-----------------|-----------------------------------------------------------------|-------------------------------------------------------------|
| $\gamma=0.5$, $\alpha_x=1$ | ![Image](image1.png) | ![Image](image2.png) |
| $\gamma=1$, $\alpha_x=0.005$ | ![Image](image3.png) | ![Image](image4.png) |
| $\gamma=1.5$, $\alpha_x=0.001$ | ![Image](image5.png) | ![Image](image6.png) |
| $\gamma=2$, $\alpha_x=0.0005$ | ![Image](image7.png) | ![Image](image8.png) |
| $\gamma=3$, $\alpha_x=0.00025$ | ![Image](image9.png) | ![Image](image10.png) |
| $\gamma=10$, $\alpha_x=0.00012$ | ![Image](image11.png) | ![Image](image12.png) |
As can be seen from the results shown in Table 2, the propagation patterns become symmetric with respect to the optical axis, which is due to the propagation of symmetric beams: \( \sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i} \). However, there is not just a symmetrization of the amplitude of the above distributions, but also interference effects become apparent. In particular, for \( \gamma = 1 \) in the focal plane of the lens not just two symmetrical peaks, and two split peaks are formed. This effect for the binary axicon (the radial analogue of the considered element) was analyzed in [41, 42]. For small values of the numerical aperture NA and large values \( \gamma \) (the fifth and sixth rows in Table 2), two symmetric orders merge and form a distribution in the focal plane that is no longer similar to the sum of the two displaced fractional Airy beams discussed above. Thus, not only a new type of Airy functions, but also their superpositions allow obtaining beams with new properties.

5. Conclusion
The paper contains theoretical and numerical investigation of a new type of extended Airy functions, based on a generalization of the integral representation of a wide range of non-integer power values. The analytical asymptotic expression of extended Airy functions, valid for any \( \gamma > 2 \) is obtained.

The numerical study of the properties of beams generated by such functions is carried out using the fractional Fourier transform, which describes the beams transformations by paraxial optical systems. The view of the extended Airy functions was obtained numerically in the focal plane of the lens. It is shown that the distribution of intensity essentially depends on the parameter \( \gamma \): for \( \gamma > 1 \), at negative values of the argument, noticeable oscillations are observed, the frequency of which increases both with the increase in the absolute value of the argument and with the growth of the parameter \( \gamma \); for \( \gamma < 1 \), the reverse situation is observed – the oscillation frequency decreases with the increase of the absolute value of the argument. Also, the ratio of energy between the first and the rest of the petals of the beam depends on \( \gamma \). The first petal becomes prevalent and the beam becomes similar to the classical Airy beam at \( \gamma > 2.5 \). When \( \gamma > 1 \) extended Airy beam reminiscent of the Airy beam with exponential apodization.

In addition, the trajectory of the maximum intensity depends on the parameter \( \gamma \). For \( \gamma > 2 \), the trajectory of the propagation of the extended Airy beams has the form of a pronounced accelerating beam similar to the classical Airy beams. For the other values of the parameter \( \gamma \), the propagation path is more complex.

The symmetrized extended Airy beams demonstrate not only the mirror doubling of the initial beams, but also the interference effects, as a result of which it is possible to obtain beams with new properties. We hope that the new beams will be useful for the methods of optical micromanipulation and lens-free template construction. Diffraction optical elements are an effective means of forming extended Airy beams (including symmetrized ones). Questions related to the technology of their production are described in [43-46].

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