ON CABLED KNOTS, DEHN SURGERY, AND LEFT-ORDERABLE FUNDAMENTAL GROUPS

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Abstract. Previous work of the authors establishes a criterion on the fundamental group of a knot complement that determines when Dehn surgery on the knot will have a fundamental group that is not left-orderable [6]. We provide a refinement of this criterion by introducing the notion of a decayed knot; it is shown that Dehn surgery on decayed knots produces surgery manifolds that have non-left-orderable fundamental group for all sufficiently positive surgeries. As an application, we prove that sufficiently positive cables of decayed knots are always decayed knots. These results mirror properties of L-space surgeries in the context of Heegaard Floer homology.

Definition 1. A group $G$ is left-orderable if there exists a partition of the group elements $G = \mathcal{P} \sqcup \{1\} \sqcup \mathcal{P}^{-1}$ satisfying $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$ and $\mathcal{P} \neq \emptyset$. The subset $\mathcal{P}$ is called a positive cone.

This is equivalent to $G$ admitting a left-invariant strict total ordering. For background on left-orderable groups relevant to this paper see [2, 6]; a standard reference for the theory of left-orderable groups is [12]. As established by Boyer, Rolfsen and Wiest [2] (compare [11]), the fundamental group $\pi_1(K)$ of the complement of a knot $K$ in $S^3$ is always left-orderable. Indeed, this follows from the fact that any compact, connected, irreducible, orientable 3-manifold with positive first Betti number has left-orderable fundamental group [2, Theorem 1.1]. However, the question of left-orderability for fundamental groups of rational homology 3-spheres is considerably more subtle (see [2, 6]) and seems closely tied to certain codimension one structures on the 3-manifold (see [2, 3, 17]). Continuing along the lines of [6] this paper focuses on Dehn surgery, an operation on knots that produces rational homology 3-spheres. We recall this construction in order to fix notation and conventions.

For any knot $K$ in $S^3$ there is a preferred generating set for the peripheral subgroup $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(K)$ provided by the knot meridian $\mu$ and the Seifert longitude $\lambda$. The latter is uniquely determined (up to orientation) by the existence of a Seifert surface for $K$. We orient $\mu$ so that it links positively with $K$, and orient $\lambda$ so that $\mu \cdot \lambda = 1$. For any rational number $r$ with reduced form $\frac{p}{q}$ we denote the peripheral element $\mu^p \lambda^q$ by $\alpha_r$. At the level of the fundamental group, the result of Dehn surgery along $\alpha_r$ is summarized by the short exact sequence

$$1 \rightarrow \langle \langle \alpha_r \rangle \rangle \rightarrow \pi_1(K) \rightarrow \pi_1(S^3_r(K)) \rightarrow 1.$$
Here \(\langle \langle \alpha_r \rangle \rangle\) denotes the normal closure of \(\alpha_r\), and \(S^3_\nu(K)\) is the 3-manifold obtained by attaching a solid torus to the boundary of \(S^3 \setminus \nu(K)\), sending the meridian of the torus to a simple closed curve representing the class \([\alpha_r] \in H_1(\partial (S^3 \setminus \nu(K)); \mathbb{Z})/\{\pm 1\}\).

We will blur the distinction between \(\alpha_r\) as an element of the fundamental group or as a primitive class in the (projective) first homology of the boundary, and refer to these peripheral elements as slopes.

While many examples of rational homology 3-spheres have left-orderable fundamental group [2], there exist infinite families of knots for which sufficiently positive Dehn surgery (that is, along a slope parametrized by a suitable large rational number) yields a manifold with non-left-orderable fundamental group [6]. To make this precise, consider the set of slopes

\[ S_r = \{\alpha_{r'} | r' \geq r\} \]

for some fixed rational \(r\).

**Definition 2.** A nontrivial knot \(K\) in \(S^3\) is called \(r\)-decayed if, for any positive cone \(\mathcal{P}\) in \(\pi_1(K)\), either \(\mathcal{P} \cap S_r = S_r\) or \(\mathcal{P} \cap S_r = \emptyset\).

The existence of decayed knots is established in [6]. For example, the torus knot \(T_{p,q}\) is \((pq - 1)\)-decayed (for \(p,q > 0\)), and the \((-2,3,q)\)-pretzel knot is \((10 + q)\)-decayed for odd \(q \geq 5\) (see Theorem 11). Our interest in this property stems from the following:

**Theorem 3.** If \(K\) is \(r\)-decayed then \(\pi_1(S^3_{r'}(K))\) is not left-orderable for all \(r' \geq r\).

As a result, it is not restrictive to assume that \(r\) is a positive rational number since \(\pi_1(S^3_0(K))\) is always left-orderable [2]. Notice however that it is not immediately clear how Theorem 3 might be applied in practice, as there is no obvious method for checking when a knot is \(r\)-decayed. For this reason, in Section 1 we describe an equivalent formulation of \(r\)-decay whose statement is more technical, but easier to verify, than the definition. Together with the proof of Theorem 3, the results of Section 1 provide a useful refinement of the ideas from [6].

Results connecting left-orderability and Dehn surgery may be expected to mirror similar results relating to L-spaces, since there is no known example of an L-space with left-orderable fundamental group, while many L-spaces have fundamental group that is not left-orderable. (see [1, 5, 6, 16, 21]). Recall that an L-space is a rational homology sphere with Heegaard Floer homology that is as simple as possible, in the sense that \(\text{rk} \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|\) (see [15]). Theorem 3 mirrors a fundamental property of knots admitting L-space surgeries: if \(S^3_0(K)\) is an L-space, then \(S^3_r(K)\) is an L-space as well for any \(r \geq n\).

In the interest of further investigating left-orderability of fundamental groups of 3-manifolds along the lines of [6], we consider the behaviour of Dehn surgery on cables of \(r\)-decayed knots (for necessary background, see Section 2). Denoting the \((p,q)\)-cable of the knot \(K\) as \(C_{p,q}(K)\), the main theorem of this article is:
Theorem 4. If $K$ is $r$-decayed then $C_{p,q}(K)$ is $pq$-decayed whenever $\frac{q}{p} > r$.

The proof of Theorem 4 is contained in Section 2. Notice that combining Theorem 11 and Theorem 4 provides a rather large class of knots for which sufficiently positive surgery yields a non-left-orderable fundamental group.

Dehn surgery on cabled knots and non-left-orderability of the resulting fundamental groups may again be viewed in the context of Heegaard Floer homology. Referring to knots admitting L-space surgeries as L-space knots, Hedden proves:

Theorem 5. [9, Theorem 1.10] If $K$ is an L-space knot then $C_{p,q}(K)$ is an L-space knot whenever $\frac{q}{p} \geq 2g(K) - 1$.

Here, the quantity $g(K)$ is the Seifert genus of $K$. Note that the converse of this statement has been recently established by Hom [10].

In order to assess the strength of Theorem 4, it is natural to ask when Dehn surgery on a cable knot yields a manifold that has left-orderable fundamental group. It turns out that, in the case that $K$ is $r$-decayed, Theorem 4 is close to describing all possible non-left-orderable surgeries on a cable knot $C_{p,q}(K)$, in the following sense:

Theorem 6. Suppose that $C$ is the $(p, q)$-cable of some knot. If $r \in \mathbb{Q}$ satisfies $r < pq - p - q$, then $\pi_1(S^3_r(C))$ is left-orderable.

This result is a special case of a more general observation pertaining to satellite knots that is discussed in Section 3. Notice that Theorem 6 makes no reference to the original knot being $r$-decayed. However, restricted to $r$-decayed knots, Theorem 4 and Theorem 6 combine to produce an interval of surgery coefficients for which the left-orderability of the associated quotient is not determined. More precisely:

Question 7. If $K$ is $r$-decayed and $C$ is a $(p, q)$-cable of $K$ with $\frac{q}{p} > r$, can Theorem 4 and Theorem 6 be sharpened to determine when $\pi_1(S^3_{r'}(C))$ is left-orderable for $r'$ satisfying $pq - p - q < r' \leq pq$?

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1. A practical reformulation of Theorem 8

We begin with a reformulation of $r$-decay that will be essential in connecting this work with the results of [6]. This will require the following lemma:

Lemma 8. Let $G$ be a left-orderable group containing elements $g, h$. If $g \in \mathcal{P}$ implies $h \in \mathcal{P}$ for every positive cone $\mathcal{P}$, then $g \in \mathcal{P}$ if and only if $h \in \mathcal{P}$.
Proof. We need only show the converse, namely $h \in \mathcal{P}$ implies $g \in \mathcal{P}$ for every positive cone $\mathcal{P} \subset G$. For a contradiction, suppose this is not the case, so there exists a positive cone such that $h \in \mathcal{P}$ and $g \notin \mathcal{P}$. Consider the positive cone $\mathcal{Q} = \mathcal{P}^{-1}$, defining the reverse ordering of $G$. This gives $g \in \mathcal{Q}$ and $h \notin \mathcal{Q}$, contradicting our assumption. \hfill \Box

**Proposition 9.** A knot $K$ is $r$-decayed if and only if for every positive cone $\mathcal{P} \subset \pi_1(K)$ there exists a strictly increasing sequence of positive rational numbers $\{r_i\}$ with $r_i \to \infty$ satisfying

(1) $r = r_0$, and
(2) $\alpha_r \in \mathcal{P}$ implies $\alpha_{r_i} \in \mathcal{P}$ for all $i$.

Proof. Suppose that $K$ is $r$-decayed, and let $\mathcal{P}$ be any positive cone. Choose a strictly increasing sequence of rational numbers $\{r_i\}$ with $r_0 = r$ and $r_i \to \infty$. Whenever $\alpha_r = \alpha_{r_0} \in \mathcal{P}$ we have $\mathcal{S}_r \cap \mathcal{P} \neq \emptyset$, so that $\mathcal{S}_r \cap \mathcal{P} = \mathcal{S}_r$ since $K$ is $r$-decayed. It follows that $\alpha_{r_i} \in \mathcal{S}_r \subset \mathcal{P}$ for all $i$.

To prove the converse, let $\mathcal{P}$ be a positive cone for $\pi_1(K)$. Fix a strictly increasing sequence $\{r_i\}$ of rational numbers limiting to infinity and satisfying (1) and (2). Suppose that $\alpha_r \in \mathcal{P}$, then by assumption $\alpha_{r_i} \in \mathcal{P}$ for all $i > 0$.

Now suppose that $\mu^m \lambda^n$ is an element of $\mathcal{S}_r$. Choose $r_i, r_{i+1}$ with corresponding reduced forms $\frac{p_i}{q_i}, \frac{p_{i+1}}{q_{i+1}}$ such that $r_i < \frac{m}{n} < r_{i+1}$. By solving

$$q_i a + q_{i+1} b = cn$$
$$p_i a + p_{i+1} b = cm$$

we can find positive integers $a, b$ and $c$ such that $(\mu^{p_i} \lambda^{q_i})^a (\mu^{p_{i+1}} \lambda^{q_{i+1}})^b = (\mu^m \lambda^n)^c$. Explicitly, Cramer’s rule gives

$$a = \begin{vmatrix} n & q_{i+1} \\ m & p_{i+1} \end{vmatrix}, \quad b = \begin{vmatrix} q_i & n \\ p_i & m \end{vmatrix}, \quad c = \begin{vmatrix} q_i & q_{i+1} \\ p_i & p_{i+1} \end{vmatrix};$$

note that all these quantities are positive because of our restriction $r_i < \frac{m}{n} < r_{i+1}$ (compare \[\text{Lemma 17}\]) . This shows that $\mu^m \lambda^n$ is positive, since its $c$-th power is expressed as a product of positive elements. Hence $\mathcal{S}_r \cap \mathcal{P} = \mathcal{S}_r$.

This establishes the implication $\alpha_r \in \mathcal{P} \Rightarrow \mathcal{S}_r \subset \mathcal{P}$ for every positive cone $\mathcal{P}$. By Lemma \[\text{13}\], this is equivalent to $\mathcal{S}_r \cap \mathcal{P} = \mathcal{S}_r$ or $\mathcal{S}_r \cap \mathcal{P} = \emptyset$ for every positive cone $\mathcal{P}$, so that $K$ is $r$-decayed. \hfill \Box

**Remark 10.** In practice, it is often more natural to establish $\alpha_r \in \mathcal{P}$ implies $\alpha_{r^w_i} \in \mathcal{P}$ for all $i$, where $w_i \in \mathbb{N}$ (see in particular the proofs of Lemma \[\text{13}\] and Lemma \[\text{14}\]). This situation arises when one constructs (for a given positive cone $\mathcal{P}$) a sequence of unreduced rationals $\{r_i\} = \{\frac{p_i}{q_i}\}$ for which $\gcd(p_i, q_i) = w_i \geq 1$, and $\mu^{p_i} \lambda^{q_i} \in \mathcal{P}$ implies $\mu^{p_{i+1}} \lambda^{q_{i+1}} \in \mathcal{P}$ for all $i$. Notice that the implication $\alpha_r \in \mathcal{P}$ implies $\alpha_{r^w_i} \in \mathcal{P}$ still allows us to apply Proposition \[\text{14}\] since $\alpha_{r^w_i} \in \mathcal{P}$ if and only if $\alpha_{r_i} \in \mathcal{P}$ (this simple observation holds in any left-orderable group). Ultimately, this results in more flexibility in selecting the sequence $\{r_i\}$. 
The equivalence established in Proposition 9 shows that all examples considered in [6] are \( r \)-decayed for certain \( r \), as [6, Corollary 11] is a special case of Proposition 9.

**Theorem 11.** [6, Theorem 24, Theorem 28 and Theorem 30]

1. The \((p, q)\)-torus knot is \((pq - 1)\)-decayed for all positive, relatively prime pairs of integers \( p, q \).
2. The \((-2, 3, q)\)-pretzel knot is \((10 + q)\)-decayed for all odd \( q \geq 5 \).
3. The \((3, q)\)-torus knot with one positive full twist added along two strands is \((3q + 2)\)-decayed, for all positive \( q \) congruent to 2 modulo 3.

**Proof.** We consider the case of \( K_q \), the \((-2, 3, q)\)-pretzel knot with \( q \geq 5 \) odd, the other cases are similar. Set \( r = 10 + q \), and \( r_i = r + i \). It is shown in [6] that for every positive cone \( \mathcal{P} \) in \( \pi_1(K_q) \), the implication \( \alpha_r \in \mathcal{P} \Rightarrow \alpha_{r_i} \in \mathcal{P} \) holds for all \( i \geq 0 \). This means that for every left-ordering of \( \pi_1(K_q) \), the integer sequence \( \{r_i\} \) satisfies the properties required by Proposition 9 and we conclude that \( K_q \) is \( r \)-decayed.

Note that the above proof illustrates some particularly special behaviour, as the rational sequences \( \{r_i\} \) required by Proposition 9 (which a priori may be different for each left-ordering) are replaced by a single integer sequence sufficient for every left-ordering. Thus, Proposition 9 provides a more workable method (than used previously) for checking when a knot has surgeries that yield a non-left-orderable fundamental group. Combined with the material established in [6, Section 2], we provide a short proof of Theorem 3.

**Proof of Theorem 3.** For contradiction, assume that \( \pi_1(S^3_{r'}(K)) \) is left-orderable for some \( r' \geq r \), and consider the short exact sequence

\[
1 \rightarrow \langle \langle \alpha_r' \rangle \rangle \xrightarrow{i} \pi_1(K) \xrightarrow{f} \pi_1(S^3_{r'}(K)) \rightarrow 1,
\]

as defined in the introduction. Let \( \mu, \lambda \in \pi_1(K) \) denote the meridian and longitude. Since \( \pi_1(S^3_{r'}(K)) \) is left-orderable, \( \langle \langle \alpha_r' \rangle \rangle \cap \langle \mu, \lambda \rangle = \langle \alpha_r' \rangle \) (see proof of [6, Proposition 20]). In particular, if we fix an arbitrary rational number \( s_0 > r' \), then \( f(\alpha_{s_0}) \neq 1 \). Thus, we may choose a positive cone \( \mathcal{Q} \) in \( \pi_1(S^3_{r'}(K)) \) that contains \( f(\alpha_{s_0}) \). Next, choose a positive cone \( \mathcal{Q}' \subset \langle \langle \alpha_r' \rangle \rangle \) not containing \( \alpha_r' \), and define a positive cone \( \mathcal{P} \subset \pi_1(K) \) by

\[
\mathcal{P} = i(\mathcal{Q}') \cup f^{-1}(\mathcal{Q}).
\]

Note that \( \alpha_r' \notin \mathcal{P} \), and \( \alpha_{s_0} \in \mathcal{P} \).

This is a standard construction for creating a left-ordering of a group using a short exact sequence, here the result is a left-ordering of \( \pi_1(K) \) with positive cone \( \mathcal{P} \), relative to which the subgroup \( \langle \langle \alpha_r' \rangle \rangle \) is convex. Because \( \langle \langle \alpha_r' \rangle \rangle \) is convex, the intersection \( \langle \langle \alpha_r' \rangle \rangle \cap \langle \mu, \lambda \rangle = \langle \alpha_r' \rangle \) is convex in the restriction ordering of \( \langle \mu, \lambda \rangle \). Therefore, [6, Proposition 18] shows that all slopes \( \alpha_s \) with \( s > r' \) must have the same sign. In particular, since \( \alpha_{s_0} \) is positive it follows that all slopes \( \alpha_s \) with \( s > r' \) are positive, so that

\[
Q \cap S_r = \{ \alpha_s \mid s > r' \}.
\]

Therefore, \( K \) is not \( r \)-decayed. \( \square \)
We remark that there is a more geometric argument establishing Theorem 3, that relies upon an understanding of the topology of the space of left-orderings of \( \mathbb{Z} \oplus \mathbb{Z} \) (see [18, Section 3] and [4, Chapter 6]). Roughly, every left-ordering of the knot group \( \pi_1(K) \) restricts to a left-ordering of the peripheral subgroup that defines a line in \( \mathbb{Z} \oplus \mathbb{Z} \), with all positive elements of \( \mathbb{Z} \oplus \mathbb{Z} \) on one side of the line, and all the negative elements on the other side. As a result, given two rationals \( r_1 < r_2 \) corresponding to slopes \( \alpha_{r_1} \) and \( \alpha_{r_2} \) that have the same sign in every left-ordering, no left-ordering can restrict to an ordering of the peripheral subgroup with corresponding slope \( s \) between \( r_1 \) and \( r_2 \). The proof of Theorem 3 then follows from checking that whenever \( \pi_1(S_{r_1}^3(K)) \) is left-orderable, we can define a left-ordering of \( \pi_1(K) \) that restricts to yield a line of slope \( r' \) in the peripheral subgroup (compare [6, Proof of Theorem 9]).

2. The proof of Theorem 4

We recall the construction of a cabled knot in order to fix notation. Consider the \((p, q)\)-torus knot \( T_{p,q} \), where \( p, q > 0 \) are relatively prime. As the closure of a \( p \)-strand braid, this knot may be naturally viewed in a solid torus \( T \) by removing a tubular neighbourhood of the braid axis. The complement of \( T_{p,q} \) in \( T \) is referred to as a \((p, q)\)-cable space. Now given any knot \( K \) in \( S^3 \), the cable knot \( C_{p,q}(K) \) is obtained by identifying the boundary of \( T_{p,q} \) with \( \nu(K) \), identifying the longitude of \( T_{p,q} \) with the longitude \( \lambda \) of \( K \). We will denote this cable knot by \( C \) whenever this simplified notation does not cause confusion.

The knot group \( \pi_1(C) \) may be calculate via the Seifert-Van Kampen Theorem, by viewing the complement \( S^3 \setminus \nu(C) \) as the identification of the boundaries of \( S^3 \setminus \nu(K) \) and a solid torus \( D^2 \times S^1 \) along an essential annulus with core curve given by the slope \( \mu^q \lambda^p \). If \( \pi_1(D^2 \times S^1) = \langle t \rangle \) then this gives rise to a natural amalgamated product

\[
\pi_1(C) \cong \pi_1(K) \ast_{\mu^q \lambda^p = t^p} \mathbb{Z}.
\]

Consulting [6 Section 3], the meridian and longitude for \( C \) may be calculated as

\[
\mu_C = \mu^u \lambda^v t^{-v} \quad \text{and} \quad \lambda_C = \mu_C^{-pq} t^p
\]

where \( u \) and \( v \) are positive integers satisfying \( pu - qv = 1 \) (compare [20, Proof of Theorem 3.1]).

Suppose that the knot \( K \) is \( r \)-decayed, and choose cabling coefficients \( p \) and \( q \) so that \( q/p > r \). To begin, we choose a positive cone \( \mathcal{P} \subset \pi_1(C) \) and assume that \( \mu_C^{pq} \lambda = t^p \) is positive. This means that \( t^p = \mu^q \lambda^p \in \mathcal{P} \), so every element \( \mu^m \lambda^n \) is positive whenever \( m/n > r \), since \( K \) is \( r \)-decayed.

Our method of proof will be to check that the cable is \( pq \)-decayed by using the equivalence from Proposition 9. In particular, we will show that for the given positive cone \( \mathcal{P} \subset \pi_1(C) \) there exists an unbounded sequence of increasing rationals \( \{r_i\} \) with \( r_0 = pq \), such that our assumption \( \alpha_{pq} = \mu_C^{pq} \lambda_C \in \mathcal{P} \) implies \( \alpha_{r_i} \in \mathcal{P} \) for all \( i > 0 \).
First consider the case when $\mu_C$ is positive in the left-ordering defined by $P$. Here, $\mu_C^{pq+N}\lambda_C$ is positive for $N \geq 0$, as it is a product of positive elements. Therefore in this case it suffices to choose $r_i = pq + i$ for all $i \geq 0$.

For the remainder of the proof, we assume that $\mu_C$ is negative. For repeated use below, we also observe the crucial identity

$$((t-v)^p(\mu^u\lambda^v))^p = (t^p-v^p\mu^u\lambda^v) = (\mu^{-pq}\lambda^{-pv}\mu^u\lambda^v = \mu^{pv-qv} = \mu,$$

and recall that $t^p$ commutes with $\mu, \lambda, \mu_C$, and $\lambda_C$. Therefore, we also have

$$(\mu^u\lambda^v)^p(t^{-v})^p = (t^{-v})^p(\mu^u\lambda^v)^p = \mu.$$ 

Let $k$ be an arbitrary non-negative integer, and consider the element

$$\mu^{-k}(t^{-v}\mu^u\lambda^v)\mu^k.$$ 

If this element is positive for some $k$, then the required sequence is provided by Lemma 13 (proved below). Therefore, we may assume that

$$(1) \quad \mu^{-k}(t^{-v}\mu^u\lambda^v)\mu^k \notin P$$

for all $k$.

Similarly, for $k$ a non-negative integer, we consider

$$(\mu^{-k}t^{-v}\mu^k)^{p-1}(\mu^u\lambda^v)^{p-1}.$$ 

If this element is positive for some non-negative $k$, then we can create the required sequence using Lemma 14 (proved below). Therefore, we may assume that

$$(2) \quad (\mu^{-k}t^{-v}\mu^k)^{p-1}(\mu^u\lambda^v)^{p-1} \notin P$$

for all $k$.

Observe that

$$(\mu^{-k}t^{-v}\mu^k)^{p-1}(\mu^u\lambda^v)^{p-1} = (\mu^{-k}t^p\mu^k)(\mu^{-k}t^{-vp}\mu^k)(\mu^u\lambda^v)$$

which, recalling that $t^p$ commutes with the elements $\mu$ and $\lambda$, simplifies to give

$$(\mu^{-k}t^p\mu^k)(\mu^{-u}\lambda^{-v})t^{-vp}\mu^u\lambda^v = (\mu^{-k}t^p\mu^k)(\mu^{-u}\lambda^{-v})\mu = \mu^{-k}t^p\lambda^{-v}\mu^{-u}\mu^{k+1} \notin P$$

for all $k$. Taking inverses yields

$$\mu^{-k-1}\mu^u\lambda^v\mu^{-v}\mu^k = \mu^{-k-1}\mu_C\mu^k \in P.$$ 

For the following lemmas, let $>_{\text{left}}$ denote the left-ordering defined by the positive cone $P$, so that $h > g$ whenever $g^{-1}h \in P$. We can then calculate:

**Lemma 12.** If (1) and (2) hold for all $k \geq 0$, then $\mu^{N+q}\lambda^p$ must be positive for all $N \geq 0$. 
Proof. Since \( \mu^{-k-1} \mu_C \mu^k > 1 \), left-multiplying by \( \mu^{k+1} \) gives \( \mu_C \mu^k > \mu^{k+1} \) for all \( k \geq 0 \). Setting \( k = 0 \) we obtain \( \mu_C > \mu \), so that left-multiplying by \( \mu_C \) gives rise to

\[
\mu_C^2 > \mu C \mu.
\]

By setting \( k = 1 \), we get

\[
\mu_C \mu > \mu^2,
\]

which combines with the previous expression to give \( \mu_C^2 > \mu^2 \). Continuing in this manner, we obtain \( \mu_C^N > \mu^N \) for all \( N \geq 0 \). Left-multiplying by \( t^p \), it follows that

\[
\mu_N^{np} \lambda_C = \mu_C^N t^p > \mu^N t^p = \mu^{N+q} \lambda > 1,
\]

where the final inequality follows from the fact that \( (N + q)/p > q/p > r \) and \( K \) is \( r \)-decayed. \( \square \)

As in the first case, we may now choose the sequence of rationals \( r_i = pq + i \) for all \( i \geq 0 \), and the requirements of Proposition 9 are met.

To conclude the proof, we establish Lemma 13 and Lemma 14.

**Lemma 13.** If \( \mu^{-k}(t^{-v} \mu^u \lambda^v) \mu^k \in \mathcal{P} \) for some \( k \geq 0 \), then there exists a sequence of rationals \( \{r_i\} \) such that \( \alpha_{r_i} > 1 \) for all \( i \).

**Proof.** For \( N \geq 0 \), we rewrite \( \mu_C^N \) as

\[
\mu_C^N = \mu^u \lambda^v \mu^k (\mu^{-k}(t^{-v} \mu^u \lambda^v) \mu^k) \mu^{-u-k} \lambda^{-v}.
\]

Fix a positive integer \( s \) that is large enough so that \( (sq - u - k)/(sp - v) > r \), this is possible because \( q/p > r \). Next, the product \( \mu_C^{N+qs} \lambda_C^s \mu_C^{N+qs} \lambda^{sp} \) becomes \( \mu_C^N \mu^{sq} \lambda^{sp} \), which is equal to

\[
\mu^{u+k} \lambda^v (\mu^{-k}(t^{-v} \mu^u \lambda^v) \mu^k) (\mu^{qs-u-k} \lambda^{ps-v}).
\]

This is a product of positive elements, because:

1. \( \mu^{u+k} \lambda^v > 1 \) because \( (u + k)/v > q/p > r \), and
2. \( \mu^{qs-u-k} \lambda^{ps-v} > 1 \) because \( (sq - u - k)/(sp - v) > r \),

while the quantity \( \mu^{-k}(t^{-v} \mu^u \lambda^v) \mu^k \) is positive by assumption. Therefore, in this case we choose our sequence of rationals to be

\[
\alpha_{r_i} = \frac{pq + i}{s}
\]

for \( i \geq 0 \), this guarantees that the associated slopes \( \alpha_{r_i} \) are positive in the given left-ordering. \( \square \)

**Lemma 14.** If \( \mu^{-k}(t^{-v} \mu^u \lambda^v) \mu^k \notin \mathcal{P} \) for all \( k \geq 0 \), and \( (\mu^{-k}t^{-v} \mu^k)^{p-1}(\mu^u \lambda^v)^{p-1} \in \mathcal{P} \) for some \( k \geq 0 \), then there exists a sequence of rationals \( \{r_i\} \) such that \( \alpha_{r_i} > 1 \) for all \( i \).
Proof. Fix \( k \geq 0 \) such that \( \mu^{-k}(t^{-v}u^\lambda v^\mu)^k < 1 \) and \( (\mu^{-k}t^{-v}u^\lambda v^\mu)^{p-1}(\mu^u\lambda^v)^{p-1} > 1 \), and let \( n \) be the smallest positive integer such that
\[
(\mu^{-k}t^{-v}u^\lambda v^\mu)^n(\mu^u\lambda^v)^n > 1,
\]
and
\[
(\mu^{-k}t^{-v}u^\lambda v^\mu)^{n-1}(\mu^u\lambda^v)^{n-1} < 1,
\]
note that \( 1 < n \leq q - 1 \). Note that we may rearrange these two expressions, so that
\[
\mu^{-k}t^{-vn}(\mu^u\lambda^v)^n\mu^k > 1,
\]
and
\[
\mu^{-k}(\mu^u\lambda^v)^{1-n}t^{-v(1-n)}\mu^k > 1.
\]
Then we can rewrite \( \mu_C^N \) for \( N \geq 1 \) as follows:
\[
\mu_C^N = \mu^{u+k}\lambda^v(\mu^{-k}t^{-v(1-n)}\mu^k)(\mu^{-k}t^{-vn}(\mu^u\lambda^v)^n\mu^k)\mu^{-k}(\mu^u\lambda^v)^{1-n}t^{-v(1-n)}\mu^k)^{N-1}\mu^{-k}t^{-vn}.
\]
In the above expression, the quantity inside the square brackets is a product of positive elements. Denote this quantity by \( P \). Choose an integer \( s \) such that \( (qs - k)/ps > r \). Then considering the slope \( \mu_C^{N+pq(v+s)}\lambda_C^{v+s} = p_C^{N}\mu^{(v+s)} \), we find
\[
\mu_C^{N}\mu^{(v+s)} = \mu^{u+k}\lambda^v(\mu^{-k}t^{-v(1-n)}\mu^k)\mu^{N-1}\mu^{qs-k}\lambda^{ps}t^{pv-vn}.
\]
This is a product of positive elements, because:

1. \( \mu^{u+k}\lambda^v > 1 \), since \( (u + k)/v > q/p > r \).
2. \( \mu^{-k}t^{-v(1-n)}\mu^k > 1 \), because if we consider its \( p \)-th power, we can use the fact that \( t^p \) commutes with all peripheral elements so that
   \[
   (\mu^{-k}t^{-v(1-n)}\mu^k)^p = t^{-pv(1-n)} > 1.
   \]
   The final inequality follows from \( -pv(1-n) > 0 \).
3. \( \mu^{qs-k}\lambda^{ps} > 1 \), because \( s \) is chosen so that \( (qs - k)/ps > r \).
4. \( t^{pv-vn} > 1 \), because \( pv - vn > 0 \).

Therefore, in this case we may choose our sequence of rationals to be
\[
r_i = \frac{i + pq(v+s)}{v + s}
\]
for \( i \geq 0 \), as the corresponding elements \( \mu_C^{i+pq(v+s)}\lambda_C^{v+s} \) are positive in the left-ordering for \( i \geq 0 \).

\[
3. \text{Surgery on satellites}
\]

Let \( T \) denote the solid torus containing a knot \( K^P \), we require that \( K^P \) is not contained in any 3-ball inside \( T \). The knot \( K^P \) will be called the pattern knot. Let \( K^C \) denote a knot in \( S^3 \), \( K^C \) will be called the companion knot. We construct the satellite knot \( K \) with pattern \( K^P \) and companion \( K^C \) as follows.
Let \( h : \partial T \to \partial(S^3 \setminus \nu(K^C)) \) denote a diffeomorphism from the boundary of \( T \) to the boundary of the complement of \( \nu(K^C) \), which carries the longitude of \( \partial T \) onto the longitude of the knot \( K^C \). The knot \( K \) is then realized as the image of the knot \( K^P \) in the manifold \( S^3 \setminus \nu(K^C) \cup_h T = S^3 \).

**Lemma 15.** [19] Proposition 3.4 There exists a homomorphism \( \phi : \pi_1(K) \to \pi_1(K^P) \) that preserves peripheral structure.

**Proof.** We can compute the fundamental group \( \pi_1(K) \) by using the Seifert-Van Kampen theorem. Since

\[
S^3 \setminus \nu(K) = S^3 \setminus \nu(K^C) \cup_h T \setminus \nu(K^P),
\]

the group \( \pi_1(K) \) is the free product \( \pi_1(K^C) \ast \pi_1(T \setminus \nu(K^P)) \), with amalgamation as follows: The meridian of \( K^C \) is identified with the meridian of \( T \), and the longitude of \( K^C \) is identified with the longitude of \( T \).

Let \( N \) denote the normal closure in \( \pi_1(K) \) of the commutator subgroup of \( \pi_1(K^C) \). The quotient \( \pi_1(K)/N \) can be considered as the result of killing the longitude of \( T \). Topologically we can think of this quotient as gluing a second solid torus \( T' \) to the torus \( T \) containing \( K^P \), in such a way that the meridian of \( T' \) is glued to the longitude of \( T \). The result is that \( \pi_1(K^C) \) collapses to a single infinite cyclic subgroup, and the group \( \pi_1(K)/N \) is isomorphic to \( \pi_1(K^P) \). The desired homomorphism \( \phi \) is the quotient map \( \pi_1(K) \to \pi_1(K)/N \). \( \square \)

**Proposition 16.** Suppose that \( K \) is a satellite knot with pattern knot \( K^P \), and \( r \in \mathbb{Q} \) is any rational number. If \( \pi_1(S^3_r(K^P)) \) is left-orderable and \( S^3_r(K) \) is irreducible, then \( \pi_1(S^3_r(K)) \) is left-orderable.

**Proof.** By Lemma [19] there exists a homomorphism \( \phi : \pi_1(K) \to \pi_1(K^P) \) that preserves peripheral structure, so there exists an induced map

\[
\phi_r : \pi_1(S^3_r(K)) \to \pi_1(S^3_r(K^P))
\]

for every \( r \in \mathbb{Q} \). Whenever \( \pi_1(S^3_r(K^P)) \) is left-orderable the image of \( \phi_r \) is nontrivial and \( \pi_1(S^3_r(K)) \) is left-orderable [2] Theorem 1.1. \( \square \)

**Proof of Theorem 4** By [7], \( pq \)-surgery on a \((p,q)\)-cable knot yields a reducible manifold. Since the minimal geometric intersection number between reducible slopes is \( \pm 1 \) [8], \( r \)-surgery on a \((p,q)\)-cable yields an irreducible manifold whenever \( r < pq - p - q \). Moreover, a \((p,q)\)-cable knot can be described as a satellite knot with pattern knot \( T_{p,q} \), the \((p,q)\)-torus knot. Therefore, for \( r < pq - p - q \) we can apply Proposition [16] to conclude that \( \pi_1(S^3_r(K)) \) will be left-orderable whenever \( \pi_1(S^3_r(T_{p,q})) \) is left-orderable.

We may now combine known results for surgery on torus knots in this setting. On the one hand, \( \pi_1(S^3_r(T_{p,q})) \) is an L-space whenever \( r \geq 2g - 1 \) [14] Proposition 9.5 (see in particular [9] Lemma 2.13), where \( g = g(T_{p,q}) \) is the Seifert genus given by \( g(T_{p,q}) = \frac{1}{2}(p - 1)(q - 1) \). On the other, since \( S^3_r(T_{p,q}) \) is Seifert fibred or a connect sum of lens spaces for every \( r \) [13], \( S^3_r(T_{p,q}) \) is an L-space if and only if \( \pi_1(S^3_r(T_{p,q})) \) is not left-orderable [1] (see also [16] 21).
In particular, $\pi_1(S^3(T_{p,q}))$ is left-orderable whenever $r$ is less than $2g(T_{p,q}) - 1$ and the result follows.

\begin{flushright}
$\square$
\end{flushright}

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