Risk Aware Minimum Principle for Optimal Control of Stochastic Differential Equations

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Abstract—We present a probabilistic formulation of risk aware optimal control problems for stochastic differential equations. Risk awareness is in our framework captured by objective functions in which the risk neutral expectation is replaced by a risk function, a nonlinear functional of random variables that accounts for the controller’s risk preferences. We state and prove a risk aware minimum principle that gives necessary and sufficient conditions for optimality of generalized control processes taking values on probability measures defined on a given action space. We show that going from the risk neutral to the risk aware case, the minimum principle is modified by the introduction of one additional real-valued stochastic process that acts as a risk adjustment factor. This adjustment process is explicitly given as the expectation, conditional on the filtration at the given time, of an appropriately defined functional derivative of the risk function evaluated at the random total cost. The control model we employ differs from standard relaxed controls, and we elaborate on the differences, and benefits and drawbacks, of the control types; we further give conditions under which the generalized control can be realized using a strict control process. We present an application of the results for a portfolio allocation problem and show that the risk awareness of the objective function gives rise to a risk premium term that is characterized by the risk adjustment process described above. This suggests uses of our results in e.g. pricing of risk modeled by generic risk functions in financial applications.

Index Terms—Differential equations, optimal control, optimization, risk analysis, stochastic processes.

I. INTRODUCTION

We consider the problem of optimal control of stochastic differential equations of the form

\[ x_t = \xi + \int_0^t b(s, x_s, a_s) \, ds + \int_0^t \sigma(s, x_s, a_s) \, dw_s \]  

(1)

over a finite time horizon, \( t \in [0, T] =: [0, \infty) \), and where \( \xi \) is a random initial value, \( x = (x_t)_{t \in T} \) and \( a = (a_t)_{t \in T} \) are the state and control processes, respectively, taking values on spaces \( \mathbb{X} := \mathbb{R}^{d_x} \) and \( \mathbb{A} \subset \mathbb{R}^{d_a}, d_x, d_a \in \mathbb{N} := \{1, 2, \ldots\} \). The process \( w = (w_t)_{t \in T} \) is a standard \( d_w \)-dimensional Brownian motion, \( d_w \in \mathbb{N} \), and \( b \) and \( \sigma \) are deterministic functions \( b : \mathbb{T} \times \mathbb{X} \times \mathbb{A} \to \mathbb{R}^{d_x}, \sigma : \mathbb{T} \times \mathbb{X} \times \mathbb{A} \to \mathbb{R}^{d_x \times d_w} \).

Our focus here is on the problem of risk aware control of the diffusion process. The conventional optimal control theory of stochastic processes considers risk neutral problems, understood here as the minimization of expected costs accrued over the solution time interval

\[ \inf_{a = (a_t)_{t \in T}} \mathbb{E} \left[ \int_0^T c(t, x_t, a_t) \, dt + g(x_T) \right] \]

where \( c : \mathbb{T} \times \mathbb{X} \times \mathbb{A} \to \mathbb{R} \) is a cost rate function, and \( g : \mathbb{X} \to \mathbb{R} \) is a terminal cost function. In the risk aware control problems we consider here, the expectation in the objective is supplanted by a risk function \( \rho \) that describes the controller’s preferences that are not sufficiently modeled by the expected value. Formally, the risk aware problem is stated as

\[ \inf_{a = (a_t)_{t \in T}} \rho \left( \int_0^T c(t, x_t, a_t) \, dt + g(x_T) \right) \]

(2)

where we suppose that the risk function \( \rho \) is some generic mapping from random variables, representing total costs, to real values quantifying the magnitude of the risk associated with a given random variable. Convex or coherent risk measures form an important subset of the functions \( \rho \) that our results attempt to cover [6], [23], [24].

Literature review. The stochastic minimum principle has a long history. Its early derivations can be found in the works [10]–[12], [31], with the modern version often being attributed to [37]. The minimum principle is extensively covered in [47], with numerous additional references. For applications of the minimum principle, and backward stochastic differential equations, see [20] and [42]. Much of the recent work on the topic of control under uncertainty, broadly understood as random variability not accounted for by an expectation under full observations, has been done using dynamic risk measures [1] or nonlinear expectations such as Peng’s \( g \)-expectation [38], [39] and its generalization, the \( G \)-framework [40], [41]. Compared to static risk functions, these approaches impose additional structure, most notably time-consistency that allows for the use of, e.g., the dynamic programming principle. While it is well known that \( g \)-expectations give rise to convex risk functions, the converse is generally
true only for risk functions that are time-consistent [43]. In our approach, we consider objectives that are given in terms of static, law invariant risk measures, and in particular we do not impose time-consistency on the risk function. Moreover, since the risk function is not expressed as a $g$-expectation, we do not need to consider forward-backward stochastic differential equations as the starting point, as was done in, e.g., [35], where a minimum principle was derived for stochastic differential equations driven by Lévy processes.

Finally, we note that there is a connection between risk aware and mean field type control problems, also known as McKean–Vlasov problems. In the latter class of problems, the distribution of the controlled processes state enters the drift, diffusion, and cost functions, $b$, $\sigma$, $c$, and $g$. In contrast to mean field games, here the controller is aware that their decisions affect the state distribution. One can view risk aware problems as a type of mean field type control problem, since here, the distribution of costs enters the objective function. Mean field games and mean field type problems are discussed in detail in [14] and [15]. This connection has provided us with valuable tools used in the technical sections of this article.

**Contributions and Organization of the Article.** The contributions of this article can be summed up as follows: 1) We generalize the control problem informally stated in (1) and (2) to feature measure valued control processes. Albeit the control model and the notion of a solution we utilize has been considered by some authors under the name of relaxed controls, we opt for a new term of “vague controls.” We justify the nomenclature by demonstrating key differences between relaxed and vague controls, and further show why the latter notion of a solution can be particularly useful. 2) We introduce law invariant risk functions into a framework that allows a natural notion of functional differentiability that can subsequently be applied in deriving variational conditions for optimality of controls. 3) Using these results, we formulate and prove a risk aware version of the stochastic Pontryagin’s minimum principle for vague controls, and in doing so, we give a characterization of the optimal control of the risk aware problem (2). We find that in comparison to the risk neutral problem, the minimum principle is modified by a risk adjustment process that is related to the functional derivative of the risk function, evaluated at the terminal cost. Finally, 4), we demonstrate by means of solving a simple example that in financial applications, risk awareness creates nontrivial but intuitive risk pricing effects.

The primary focus of the article, and our main original contribution to the literature, is the introduction of a generic risk function into the objective, the optimality conditions in the form of the risk aware minimum principle, and the implications of risk awareness on the optimal control processes. Of secondary importance is the vague control model, however, we obtain new results, e.g., regarding the sufficiency of first-order adjoint processes in determining the optimal control.

In the next section, we will describe the notations used in the article, and state the control problem we consider. Section III outlines some necessary differentiability properties of the risk functions that will subsequently be needed for the probabilistic formulation of the problem that is given in the following Section IV. This section derives necessary and sufficient conditions for the optimality of a control process. We present an application of the theory in Section V, where we characterize the optimal controls of a simple portfolio allocation problem. Section VI concludes concludes this article. Technical proofs are deferred to the Appendix.

**II. Model**

Throughout the article we will use the following notations and definitions: For any probability space $(\Omega, \Sigma, P)$, a Banach space $(V, \langle \cdot, \cdot \rangle)$ and $p \geq 0$, we denote $L^p(\Omega, \Sigma, P; V)$, or $L^p(\Omega; V)$ for short, as the set of random variables $q \in \Omega \to V$, such that $\mathbb{E}_P[|q|^p] < \infty$, where $\mathbb{E}_P$ stands for the expectation with respect to the measure $P$. If $P$ is clear from the context, we simply use the symbol $\mathbb{E}$. In addition, $C^\infty(\Omega; V)$ denotes the space of $\mathbb{E}$-essentially bounded random variables. We shall use $\|\cdot\|_p$ to denote the norm on $L^p(\Omega; V)$, $p \in [1, \infty]$. For a real Banach space $V$, we use $V^*$ to denote its continuous dual, and $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ for the duality pairing.

Borel probability measures on a topological space $V$ are denoted by $\mathcal{B}(V)$, and the Borel $\sigma$-algebra on $V$ is denoted by $\mathcal{B}(V)$. The Dirac measure centered at $x \in V$ is denoted by $\delta_x$. When $V$ is a metric space, we use $P^0(V), p \in [1, \infty]$ to denote probability measures $\mu \in \mathcal{P}(V)$, such that $\int d(x, v_0) \mu(dx) < \infty$ for some $v_0 \in V$; $\mathcal{P}^\infty(\Omega; V)$ denotes probability measures with bounded support. The law or distribution of a random variable $V \in L^p(\Omega; V), p \in [1, \infty]$, is denoted by $\mathcal{L}_P(V)$, that is, $\mathcal{L}_P(V)(\Gamma) := P \circ V^{-1}(\Gamma)$ for all $\Gamma \in \mathcal{B}(\mathbb{R})$; if the probability measure is clear from the context, we use the symbol $\mathcal{L}$ instead. The extended reals will be denoted $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ and elements of $\mathbb{R}_\infty^n$, $n \in \mathbb{N}$, are by default interpreted as column vectors, i.e., $\mathbb{R}_\infty^n := \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$.

For a given filtered probability space $(\Omega, \Sigma, F, P)$, a Banach space $(V, \langle \cdot, \cdot \rangle)$, a $p \in [1, \infty]$, and for all $V$-valued $F$-predictable continuous processes on $[0, T]$, we set

$$\|x\|_{S^p_F} := \mathbb{E}\left[\sup_{t \in [0, T]}|x_t|^p\right]^{\frac{1}{p}}$$

and define $S^p_F(\Omega, \Sigma, P; V)$, or $S^p_F(\Omega; V)$ for short, as the subset of processes for which the above is finite.

In addition, for all $F$-predictable $V$-valued processes on $[0, T]$ we set

$$\|z\|_{H^p_F} := \mathbb{E}\left[\left(\int_0^T|z_t|^2 dt\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}$$

and similarly define $H^p_F(\Omega; V) = H^p_F(\Omega, \Sigma, P; V)$ as the subset on which $\|\cdot\|_{H^p_F}$ is finite. Finally, we set $S^p_F(\Omega; V) := \cap_{p \in [1, \infty]} S^p_F(\Omega; V)$ and $H^p_F(\Omega; V) := \cap_{p \in [1, \infty]} H^p_F(\Omega; V)$.

For every differentiable function $f : \mathbb{R}^n \to \mathbb{R}^k$, $n, k \in \mathbb{N}$, the Jacobian and Hessian of $f$ are denoted $\nabla f$ and $\mathcal{H} f$, so that $\nabla f : \mathbb{R}^n \to \mathbb{R}^{k \times n}$, $\mathcal{H} f : \mathbb{R}^n \to \mathbb{R}^{k \times n \times k}$, and $(\nabla f(x))_{ij} := \partial f_i(x)/\partial x_j$, $(\mathcal{H} f(x))_{ijl}$ for all $i, j, l \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n\}$; in particular, the gradient of a real-valued function is a row vector. For multivariate functions, we use $\nabla V_N$ to indicate that the derivative is taken with respect to the argument taking values in the space $U$. For convenience,
for all \( A \in \mathbb{R}^{n \times m} \) and \( B \in \mathbb{R}^{n \times \ell} \), \( n, m, \ell \in \mathbb{N} \), we denote \( A \cdot B := (A^\top B)^\top = B^\top A \in \mathbb{R}^{\ell \times m} \), where \((\cdot)^\top\) stands for the transpose. For matrices, \( | \cdot | \) shall denote the Frobenius norm.

We generalize (1) to feature measure valued controls and introduce the notion of vague controls (throughout \( X = \mathbb{R}^d \), and \( \mathbb{F} \subset \mathbb{R}^{d \times d} \) shall be our given state and action spaces, however, we will also consider solutions on extended state spaces, and hence the definitions below should be understood to hold for any analogously defined finite dimensional state and action spaces).

**Definition 1:** (Vague controlled solution) Let \( X := \mathbb{R}^d \), \( \mathcal{W} := \mathbb{R}^{d \times u} \), and \( \mathbb{A} \subset \mathbb{R}^{d \times d} \), and let \( b : T \times X \times A \to X \), \( \sigma : T \times X \times A \to \mathbb{R}^{d \times d} \), \( g : X \rightarrow \mathbb{R} \), and admissible control processes are such that there are constants \( L > 0, \bar{p}_1 \in [0, 1], \bar{p}_2 \in [0, \infty), \bar{p}_3 \in [0, \infty) \), \( \bar{p} \in [1, \infty), \bar{p}_1 \in [0, \infty), \bar{p}_2 \in [0, \infty), \bar{p}_3 \in [0, \infty) \) satisfying: 1) if \( \bar{p}_3 = \infty \), then \( \mathbb{A} \) is compact; 2) for all \( (t, x, a) \in T \times X \times A \)

\[
|b(t, x, a)| \leq L (1 + |x|^{\bar{p}_1} + |a|^{\bar{p}_2})
\]

(6a)

\[
|\sigma(t, x, a)| \leq L (1 + |x|^{\bar{p}_1} + |a|^{\bar{p}_2})
\]

(6b)

3) for all \( (t, x, a) \in T \times X \times A \), the functions \( x \to b(t, x, a) \) and \( x \to \sigma(t, x, a) \) are continuously differentiable, and the derivatives are bounded by \( L; 4) \) for all \( (t, x, a) \in T \times X \times A \)

\[
|c(t, x, a)| \leq L (1 + |x|^{p_1} + |a|^{p_2})
\]

(7a)

\[
|g(x)| \leq L (1 + |x|^{p_1})
\]

(7b)

5) for all \( (t, a) \in T \times A \), the functions \( x \to c(t, x, a) \) and \( x \to g(x) \) are continuously differentiable, and satisfy, for all \( (t, a) \in T \times A \)

\[
|\nabla_X c(t, x, a)| \leq L \left(1 + |x|^{p_1} + |a|^{p_2}\right)
\]

(8a)

\[
|\nabla_X g(x)| \leq L \left(1 + |x|^{p_1}\right)
\]

(8b)

6) the initial distribution \( \nu \in \mathcal{P}(\mathbb{X}); 7) \) all control processes are \( \bar{p}_3 \)-admissible, i.e., satisfy (4) for \( r = \bar{p}_3 \).

**Definition 4:** Let \( p \in [1, \infty) \). We say that a vague controlled solution \( \pi \in \mathfrak{U}(b, \sigma, \nu) \) is \( p \)-feasible and denote \( \pi \in \mathfrak{U}^p(b, \sigma, \nu) \), if there exists \( \tilde{p}, \bar{p}_i, i \in \{1, 2, 3\}, p_1, p_2 \) satisfying Assumption 3 and the following inequalities:

\[
p < \tilde{p} \leq \bar{p}_3
\]

(9a)

\[
\bar{p}_2 \leq \frac{\bar{p}_3}{\tilde{p}}
\]

(9b)

\[
p_1 \leq p_1, \quad p_2 \leq p_2
\]

(9c)

\[
p_1, p_2 < \frac{\bar{p}_3}{\tilde{p}} - 1.
\]

(9d)

To give an intuition on the meanings and uses of these constants (Proposition 5 below gives a more formal statement), \( p \) and \( \tilde{p} \) shall, respectively, represent the order up to which the costs \( C^{\pi} \) and the state space variables \( x^{\pi}_t, t \in T \), are integrable. We allow for unbounded cost rates and terminal costs, in fact even superlinear growth is admissible \( \{p_1, p_2 > 1 \} \) in (7)), but in order to guarantee that costs are in \( \mathcal{L}^p(\Omega; \mathbb{R}) \), bounds on the integrability of the state and action variables need to be imposed. The system (9) amounts to sufficient conditions for such integrability to hold.
Given a filtered probability space with a Brownian motion and a progressively measurable \( \mathcal{P}(\mathcal{A}) \)-valued control process, stochastic differential equations satisfying Assumption 3 and the inequalities of (9) for a given \( p \in [1, \infty) \) have strong solutions. Together, these comprise a \( p \)-feasible vague controlled solution, and moreover, if \( \pi \in \mathcal{P}(b, \sigma, \nu) \), then the costs \( C^\pi \in \mathcal{L}^p(\Omega; \mathbb{R}) \) hold. Let \( \pi = (\pi_t)_{t \in \mathbb{T}} \) be an \( \mathcal{F} \)-progressively measurable \( \mathcal{P}(\mathcal{A}) \)-valued stochastic process that satisfies (4) for \( \tilde{p}_3 \), then there exists a pathwise unique solution \( x^\pi = (x^\pi_t)_{t \in \mathbb{T}} \) to the stochastic differential equation (3), such that

\[
x^\pi \in S^b_\mathbb{P}(\Omega; \mathcal{H}), \quad C^\pi \in \mathcal{L}^p(\Omega; \mathbb{R})
\] (10)

so that \( (\Omega, \Sigma, \mathcal{F}, \mathbb{P}, \pi, x^\pi) \) is in \( \mathcal{P}(b, \sigma, \nu) \) or in other words is a \( p \)-feasible vague controlled solution.

In order to state the risk aware control problem, we need to first establish some basic properties of risk functions. We collate our discussions on their properties in the next section, where we first describe the subset of risk functions that can be used to evaluate the risk associated with \( C^\pi \) when \( \pi \in \mathcal{P}(b, \sigma, \nu) \). Before that, however, we digress to discuss the differences between vague and relaxed controlled solutions, and in particular, give conditions when vague controls can be realized or approximated using strict controls.

On Vague Versus Relaxed Controlled Solutions. Vague controlled solutions are in the current literature frequently referred to as relaxed controlled solutions, however, these two concepts are quite different in how the control affects the diffusion function. In fact, up to the knowledge of the authors, vague controlled solutions have never been called anything else but relaxed controls, and the differences between the definitions are not always explicitly noted. 1A relaxed controlled solution \( \pi \) corresponds to the drift and diffusion functions \( b, \sigma \) can be defined (see, e.g., [18, Definition 2.4]) as a filtered probability space together with a stochastic process \( (x^\pi_t, \pi_t)_{t \in \mathbb{T}}, x^\pi_0 \in \mathcal{X}, \pi_t \in \mathcal{P}(\mathcal{A}) \) for all \( t \in \mathbb{T} \), satisfying items 1–3) of Definition 1, but characterized by the condition that the processes \( (m^\pi_t)_{t \in \mathbb{T}} \)

\[
m^\pi_t := f(x^\pi_t) - f(x^\pi_0) - \int_0^t \left\{ \nabla f(x^\pi_s)b(s, x^\pi_s, a) \right\} ds + \frac{1}{2} \text{Tr} \left[ Hf(x^\pi_s)\sigma(s, x^\pi_s, a)\sigma(s, x^\pi_s, a)^\top \right] \] (11)

are martingales for all \( t \) under continuous differentiable \( f \) with compact support on \( \mathcal{X} \). Essential here is not how this defines a stochastic process, but rather how the \( \mathcal{A} \)-integral controls the term the diffusion \( \sigma \). In the relaxed controlled model, the control integral runs over square of the diffusion or the covariance function \( D := \sigma\sigma^\top \), while in the vague controlled integral is taken directly over the diffusion function \( \sigma \).

In some cases, squaring the diffusion function before averaging can make a very significant difference.

Example 6: Consider a one dimensional, \( \mathbb{X} = \mathbb{R} \) problem with \( \mathcal{A} = \{a_1, a_2\} \subseteq \mathbb{R} \) and \( \sigma(t, x, a) = a \) for all \( (t, x, a) \in \mathbb{T} \times \mathbb{X} \times \mathcal{A} \). The relaxed model allows for covariances \( D \) in the set \( D_1 := [a_1^2 \land a_2^2, a_1^2 \lor a_2^2] \), while in the vague case \( D \) may range in \( D_2 := \{(\alpha a_1 + (1 - \alpha)a_2)^2 \mid \alpha \in [0, 1]\} \). Clearly, \( D_1 \subset D_2 \) and the inclusion may be strict, and hence the vague control has more latitude in controlling the noise. If \( a_1a_2 < 0 \), then zero is in \( D_2 \) but not \( D_1 \), and the vague control may cancel the noise completely while the relaxed control may not.

Note that if there is no control in the diffusion function, or if the control is strict, then the two control models coincide in the sense that they yield state space processes \( (x^\pi_t)_{t \in \mathbb{T}} \) with the same finite dimensional distributions. The canonical reference on stochastic processes defined via the above martingale property is [45].

Examples of works where vague controls are used include [3], [5], [8], and [34]. We note, as [5], that vague controlled stochastic differential equations can be related to controlled stochastic processes driven by nonorthogonal martingale measures, whereas the more canonical relaxed controlled model can be identified with equations driven by orthogonal martingale measures [17], [46]. It also bears pointing out that the topology conventionally assigned to relaxed controls, see, e.g., [22], may be too coarse for vague controlled problems to guarantee the continuity of the mapping from controls to the stochastic trajectories, which has implications for, e.g., applying the shattering lemma [19, Th. 2.2] to vague controls. Indeed, as Example 7 below demonstrates, unlike in the relaxed controlled case [18], it may not always be possible to find strict controls and associated solutions of (3) that approximate a given vague controlled solution. A similar example has been featured earlier in [7].

Example 7: Consider \( \mathbb{X} = \mathbb{R} \), \( \mathcal{A} = \{-1, +1\} \) and \( b(t, x, a) = 0, \sigma(t, x, a) = a \) for all \( (t, x, a) \in \mathbb{T} \times \mathbb{X} \times \mathcal{A} \), and \( \nu = \delta_0 \). Then for all strict controls \( \pi \in \mathcal{B}(b, \sigma, \nu), \pi^\pi(\pi) \) is an \( \mathcal{F} \)-Brownian motion (the control is ±1, nonanticipating, and the increments from the Brownian noise are symmetric and zero mean; (11) also implies that all relaxed controls yield Brownian motions), but there exists a vague controlled solution \( \pi^\pi \in \mathcal{B}(b, \sigma, \nu), \) such that \( \pi^\pi = (\delta_{-1} + \delta_{+1})/2 \) and \( x^\pi_t = 0 \) for all \( t \in \mathbb{T} \). Consequently, considering, e.g., a control problem of \( \inf_{\pi} \mathbb{E}[\int (x^\pi_t)^2] \), it is clear that a vague controlled solution may attain a strictly lower optimal value than what can be found using strict or relaxed controls.

Our main reason for considering vague controls is that the optimality conditions obtained from a stochastic minimum principle are considerably simpler, and thus, easier to use in practice. In the classical risk neutral case, and when the control set \( \mathcal{A} \) nonconvex and the diffusion coefficient depends on the control, first- and second-order adjoint equations are needed to characterize the optimal control, see, e.g., the classic work by Peng [37] and more recent results for relaxed controls in [9].

1We note that, e.g., [8] incorrectly equates vague and relaxed control models by first stating a definition for relaxed controls but proceeding to use the vague control model instead. The reader is thus encouraged to use care when perusing the literature.
and [33]. The issues resulting from the need for second-order expansions are exacerbated in the risk aware setting, where the second-order expansions will also require us to compute second-order functional derivatives of the nonlinear risk functional that appears in the objective function.

For vague controls, first-order expansions turn out to be sufficient. We emphasize that there is no contradiction between the need for second-order adjoints in the relaxed control case, and the sufficiency of first-order adjoints in the vague control case. As noted in Examples 6 and 7, the vague control model treats the noise rather differently from the standard, relaxed control model. We demonstrate the effect of this difference below in Example 8, which is specifically intended as a counter to the classical examples of [37] and [47] for the necessity of second-order expansions in the relaxed controlled case.

Example 8: Consider the problem of [47, Example 4.1]. We set $X = \mathbb{R}$, $\mathcal{A} = \{0, 1\}$ and $b(t, x, a) = 0$, $\sigma(t, x, a) = a$ for all $(t, x, a) \in T \times X \times \mathcal{A}$, $\nu = \delta_0$, and consider minimizing $\mathbb{E}[x_T^2]$ over strict controls. Clearly, $(x_t = 0, \pi_t = \delta_0)_{t \in T}$ is optimal. In [47, Example 4.1] it is shown that a standard spike perturbation of size $\epsilon$ leads to response of order $\epsilon$ in the resulting stochastic trajectory $(x_t')_{t \in T}$ in the sense that $\sup_{t \in T} \mathbb{E}[|x_t' - x_t|^2] = \epsilon^2$. However, if we consider a vague control formed by a convex combination of the optimal control $\pi = (\pi_t = \delta_0)_{t \in T}$ and an arbitrary progressively measurable $q = (q_t)_{t \in T}$ so that $\pi_t' = (1 - \epsilon)\pi_t + \epsilon q_t$ for all $t \in T$, we find that

$$\sup_{t \in T} \mathbb{E}[|x_t' - x_t|^2] \leq 4T \epsilon^2.$$ 

Therefore, perturbations to vague controls result in an $O(\epsilon^2)$ response in the state space paths (in the above sense), whereas for strict controls, we only have $O(\epsilon)$. This suggests that computing the first-order response may indeed be sufficient for establishing necessary conditions for optimality of vague controlled solutions.

Importantly, other authors have obtained similar, first-order minimum principles, see, e.g., [3], [8], and [11]. On the other hand, we note that [5] considers vague controls and finds a second-order minimum principle; this work builds on the results of [34]. However, the findings of these papers are not applicable to our setup, as, e.g., [34, Th. 3.6] is in contradiction of Example 7.

The natural downside to considering vague controls is, as Example 7 demonstrated, that the optimal vague control may be something that cannot be approximated by strict controls. This may be an issue in practice, if a vague control cannot realistically be implemented. This is in contrast to the case of usual relaxed controls, for which it is typically possible to construct an $\epsilon$-optimal strict control from an optimal relaxed control. However, first, strict controls are naturally a subset of vague controls, and hence if a strict optimal control is obtained, it is also the optimal relaxed control. Second, we can use convexity conditions similar to those commonly used in the literature on relaxed controls to give conditions for which vague controls can be realized using strict controls. This is stated in the following theorem.

**Theorem 9:** Suppose $\Sigma$ is compact, $b(t, x, \cdot), c(t, x, \cdot)$, and $\sigma(t, x, \cdot)$ are continuous for all $(t, x) \in T \times X$, and that the set $\{b(t, x, a), c(t, x, a), \sigma(t, x, a) \mid a \in A\}$ is convex for all $(t, x) \in T \times X$. Then there exists a strict controlled solution $\pi'$ defined on the same filtered probability space $(\Omega, \Sigma, (\mathcal{F}_t)^T, \mathbb{P})$, such that $x_T^2(\omega) = x_t^2(\omega)$ and $C^2(\omega) = C^2(\omega)$ for all $(t, \omega) \in T \times \Omega$. Third and finally, if the diffusion function $\sigma$ is independent from the control, it can be readily shown that the vague and relaxed control models coincide.

**Proposition 10:** Suppose $\sigma$ is independent of the control, that is, $\sigma(t, x, a) = \sigma(t, x)$ for all $(t, x, a) \in T \times X \times A$ for some $\sigma : T \times X \to \mathbb{R}^{d_x \times d_x}$. Then a vague controlled solution $\pi \in \mathbb{P}(b, \sigma, \nu)$ is also a relaxed controlled solution.

The above proposition immediately implies that, when the diffusion function is independent of the control, standard results on relaxed controls can be applied to the vague controlled case. In particular, vague controlled solutions and the associated costs can then be shown to be limits of strict controlled solutions and their costs. The detailed statement and its proof are omitted.

### III. Risk Functions

**Risk Aware Objective Function.** We presume we are given a risk function $\rho : L^p(\Omega; \mathbb{R}) \to \mathbb{R}$, $p \in [1, \infty]$, defined on some unspecified probability space $(\Omega, \Sigma, \mathbb{P})$, mapping an $L^p(\Omega; \mathbb{R})$ random variable to a real-valued measure of risk that quantifies the risk of this random variable. Since in general the probability space for a given $\rho$ is fixed, we cannot use $\rho$ to evaluate the risk of $C^\pi$ when the probability space potentially varies with each $\pi \in \mathbb{P}(b, \sigma, \nu)$. To remedy this issue, we restrict ourselves to law invariant risk functions.

**Definition 11:** Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, $p \in [1, \infty]$, and let $V$ be a Banach space. 1) Suppose $\phi : L^p(\Omega, V) \to \mathbb{R}$. A function $\psi : \mathbb{P}(V) \to \mathbb{R}$ is a $P^\phi$-representation of $\phi$ if $\phi(U) = \psi(\mathcal{L}(U))$ for all $U \in L^p(\Omega, V)$. 2) Suppose $\psi : \mathbb{P}(V) \to \mathbb{R}$. If there is a probability space $(\Omega, \Sigma, \mathbb{P})$ and a function $\phi : L^p(\Omega, V) \to \mathbb{R}$, such that $\psi(\mathcal{L}(U)) = \phi(U)$ for all $U \in L^p(\Omega, V)$, then we say $\phi$ is an $L^p(\Omega)$-representation of $\psi$.

**Definition 12:** Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. A mapping $\phi : L^p(\Omega; \mathbb{R}) \to \mathbb{R}$, $p \in [1, \infty]$, is law invariant if it has a $P^\phi$-representation. Law invariant risk functions have been extensively studied in the literature, in particular they admit well-known and widely exploited representation theorems [25], [29], [32]. Here however, the law invariance property allows us to state the risk aware problem that is defined for any probability space corresponding to the given vague controlled solution. For any law invariant $\rho : L^p(\Omega; \mathbb{R}) \to \mathbb{R}$, $p \in [1, \infty]$, we define Problem $P_3$ as

$$P_3 : \inf_{\pi \in \mathbb{P}(b, \sigma, \nu)} \rho(C^\pi).$$

We adopt the view that a law invariant risk function can be equivalently seen as a mapping from $L^p(\Omega; \mathbb{R})$-random variables to reals, or as a function from $\mathbb{P}(\mathbb{R})$-measures to reals. This latter representation of risk functions has been used also in previous works, see, e.g., [26]–[28].
Remark 13: While law invariance allows one to consider a risk function \( \rho : L^p(\Omega; \mathbb{R}) \to \mathbb{R} \) equally well as a function \( \tilde{\rho} : P^p(\mathbb{R}) \to \mathbb{R} \), we note that many properties such as convexity and coherence are better defined for \( L^p(\Omega; \mathbb{R}) \)-functionals (see, e.g., Definition 14 below). Indeed, it is quite often true that if a risk function \( \rho : L^p(\Omega; \mathbb{R}) \to \mathbb{R} \) is convex, then its representation as a function \( \tilde{\rho} : P^p(\mathbb{R}) \to \mathbb{R} \), \( \rho(X) = \tilde{\rho}(X) \) for all \( X \in L^p(\Omega; \mathbb{R}) \) is concave [2]; this has obvious implications for minimization of such functionals. Second, we will in the following need some notion of differentiability of risk functions, and functional differentiability is more readily defined on the Banach spaces \( L^p(\Omega; \mathbb{R}) \).

In order to impose more structure on the set of risk functions we consider, some of the following properties, frequently considered in the literature [6] and [23], are assumed.

Definition 14: Let \((\Omega, \Sigma, \mathbb{P})\) be a probability space and denote \( L := L^p(\Omega; \mathbb{R}), \mathbb{P} \) a probability space on which it is defined. To accommodate varying probability spaces, a law invariant notion of a functional derivative is needed. Here, we use the approach of [13]–[15] to define a useful notion of a derivative of a law invariant risk function.

Definition 16: Let \( \phi : P^p(\mathbb{R}^n) \to \mathbb{R} \), \( n \in \mathbb{N}, p \in [1, \infty] \), and suppose there is a probability space \((\Omega, \Sigma, \mathbb{P})\) and an \( L^p(\Omega)\)-representation of \( \phi \), denoted \( \psi \). 1) We say the function \( \phi \) is \( L \)-differentiable at \( \mu_0 \in P^p(\mathbb{R}^n) \) if its \( L^p(\Omega)\)-representation \( \psi \) is Fréchet differentiable at any point \( U_0 \in L^p(\Omega; \mathbb{R}^n) \), such that \( L(U_0) = \mu_0 \).

2) The function \( \phi \) is continuously \( L \)-differentiable, if the Fréchet derivative of \( \psi \) as viewed as a function \( L^p(\Omega; \mathbb{R}^n) \ni X \to D\psi(X) \in L^q(\Omega; \mathbb{R}^n), q = p/(p-1) \), is continuous.

3) Given \( \mu \in P^p(\mathbb{R}^n) \), we say the function \( f : \mathbb{R}^n \to \mathbb{R}^{1 \times n} \) is an \( L \)-derivative of \( \psi \) at \( \mu \), if the Fréchet derivative of \( \psi \) at \( \mu \) is given by \( \{ f(X, Y) = E[f(X)Y] \} \) for all \( X, Y \in L^p(\Omega; \mathbb{R}) \), such that \( L(X) = \mu \). We will denote a representative \( L \)-derivative by \( D\phi(\mu) \).

Remark 17: A coherent risk function \( \rho : L^p(\Omega; \mathbb{R}) \to \mathbb{R} \) that is nonlinear cannot be everywhere Fréchet differentiable. Specifically, \( \rho \) cannot be Fréchet differentiable at \( X = 0 \), if positive homogeneity, Definition 14 3), holds. In this case the directional derivative \( D\rho(0) := d[\rho(\lambda Y)]/d\lambda|_{\lambda=0} = \rho(Y) \), which is not linear; this was earlier pointed out in [21, Proposition 3.1]. In Proposition 18 we demonstrate, by way of an example, that a risk function can be shown to be differentiable at the cost random variable \( C^\pi \).

We have the following result concerning the existence of \( L \)-derivatives. It demonstrates that \( L \)-derivatives exist under mild conditions, and are not limited to exceptional cases of risk functions.

Proposition 18: Suppose \( \phi : P^p(\mathbb{R}^n) \to \mathbb{R}, n \in \mathbb{N}, p \in [2, \infty) \), is continuously \( L \)-differentiable. Then an \( L \)-derivative exists, and is unique in the sense that if \( f_1 \) and \( f_2 \) are \( L \)-derivatives at \( \mu \in P^p(\mathbb{R}^n) \), then \( f_1(x) = f_2(x) \) for \( \mu \)-almost every \( x \in \mathbb{R}^n \).

Our main use of the \( L \)-derivative is in evaluating the first-order response of functions of probability measures under varying probability spaces. If \( \mu, \mu_0 \in P^p(\mathbb{R}^n) \), then for any random variables \( U, U_0 \in L^p(\Omega; \mathbb{R}) \), we denote \( \phi(\mu, \mu_0) := E[\tilde{\phi}(\mu_0)(U)] \).

Proposition 19: The risk function \( \rho \) is such that: 1) \( \rho : L^p(\Omega; \mathbb{R}) \to \mathbb{R} \), \( p \in [1, \infty) \), and \( \rho \) is law invariant. 2) The risk function \( \rho \) has an \( L \)-derivative on some open subset \( U \subset P^p(\mathbb{R}) \).

3) The law of the cost functional is in \( \mathcal{U} \), i.e., \( F(C^\pi) \subset \mathcal{U} \) for all \( \pi \in \Omega(\mathbb{R}^n) \).

These assumptions simply assert that \( \rho \) is differentiable over a sufficiently large set of random variables. Note that convexity is not yet assumed; it will be needed when we state sufficient conditions for optimality of controls.

For brevity of notations, for any probability space \((\Omega, \Sigma, \mathbb{P})\) with a law invariant \( \rho : L^p(\Omega; \mathbb{R}) \to \mathbb{R} \), and any possibly different probability space \((\Omega, \Sigma, \mathbb{P})\) and a random variable \( X \in L^p(\Omega; \mathbb{R}) \), we set \( D\rho(X) := D\rho(L^p(\mathbb{R})) \) whenever the \( P^p \)-representation of \( \rho \) has an \( L \)-derivative at \( L^p(\mathbb{R}) \).

IV. RISK AWARE MINIMUM PRINCIPLE

Main Results. We begin by stating our risk aware version of the stochastic Pontryagin’s minimum principle for Problem \( P_1 \).

We denote \( \mathcal{Y} := \mathbb{R}^{1 \times d}, \mathcal{Y}' := \mathbb{R}^d, \) and \( \mathcal{Z} := \mathbb{R}^{d_0 \times d} \), and define the Hamiltonian \( H \) as

\[
H(t, x, y, y', z, a) := yb(t, x, a) + y'c(t, x, a) + Tr[z\sigma(t, x, a)]
\]

\[
\nabla(t, x, y, y', z, a) \in T_x \times Y \times Y' \times Z \times A.
\]

Note that compared to the risk neutral case, the term involving the cost rate function has been modified to feature an additional
adjoint variable $y' \in \mathcal{Y}'$. We shall elaborate on this significant point later in more detail. We will give both necessary and sufficient conditions for the $\mathcal{P}_1$-optimality of a control $\pi \in \mathcal{P}^p(b, \sigma, \nu)$. For sufficiency, we need an additional convexity assumption.

**Assumption 20:** Suppose Assumptions 3 and 19 hold, and that additionally: 1) The function $x \to g(x)$ is convex, and

$$\mathcal{A}(x, \pi) \to \int_{\mathcal{A}} H(t, x, y, y', z, a) \pi(da)$$

is (jointly) convex for all $(t, y, y', z) \in T \times \mathcal{Y} \times \mathcal{Y}' \times \mathcal{Z}$, and 2) the risk function $\rho$ is convex as a mapping $\rho : L^p(\Omega; \mathbb{R}) \to \mathbb{R}$.

The risk aware minimum principle can then be stated as follows.

**Theorem 21 (Risk aware minimum principle):** 1) Suppose $\rho : \mathcal{P}^p(\mathbb{R}) \to \mathbb{R}$, $p \in [1, \infty)$, satisfies Assumption 19. If $\pi \in \mathcal{P}^p(b, \sigma, \nu)$ is $\mathcal{P}_1$-optimal, then there exist unique $\mathcal{F}$-adapted continuous processes $y^\pi \in S_{\mathcal{F}}^{p/(p-1)}(\Omega; Y)$ and $y'^\pi \in S_{\mathcal{F}}^{p/(p-1)}(\Omega; Y')$, and a unique $\mathcal{F}$-predictable $z^\pi \in \mathcal{H}_{\mathcal{F}}^{p/(p-1)}(\Omega; \mathcal{Z})$ that satisfy the backward stochastic differential equation

$$dy^\pi_t = -\nabla \pi H(t, x^\pi_t, y^\pi_t, y'^\pi_t, z^\pi_t, a_t) dt + z^\pi_t \cdot dw_t$$

$$y^\pi_T = y'^\pi_T \nabla \pi g(x^\pi_T)$$

and the representation

$$y^\pi_t = \mathbb{E} \left[ D\rho(C^{\pi}) \mid \mathcal{F}_t \right] \quad \forall t \in T. \quad (15)$$

Moreover, the Hamiltonian of (13) is optimized in the sense that

$$\mathcal{A}(x, \pi) = \inf_{\pi \in \mathcal{P}^p(\mathbb{R})} \int_{\mathcal{A}} H(t, x^\pi_t, y^\pi_t, y'^\pi_t, z^\pi_t, a) \pi_t(da)$$

is $\mathbb{P}$-almost surely, for Lebesgue almost all $t \in T$.

2) Suppose the stronger assumption, Assumption 20 holds. If $\pi \in \mathcal{P}^p(b, \sigma, \nu)$, and if there exist processes $y^\pi \in S_{\mathcal{F}}^{p/(p-1)}(\Omega; Y)$, $y'^\pi \in S_{\mathcal{F}}^{p/(p-1)}(\Omega; Y')$, $z^\pi \in \mathcal{H}_{\mathcal{F}}^{p/(p-1)}(\Omega; \mathcal{Z})$ satisfying (14)–(16), then $\pi$ is $\mathcal{P}_1$-optimal.

Importantly, the above stochastic minimum principle features only first-order adjoints. We stress, however, that this is not in conflict with existing results stating that in general, second-order adjoints are required to characterize the optimal control (see, e.g., the classic work [37]). This should not be a controversial claim, as it is known that first-order adjoints may be sufficient even in the case, where the diffusion function depends on the control [12]. In the event that Theorem 21 yields an optimal control that is strict, then this control is indeed also optimal within the set of strict controls, as these are a subset of vague controls. Yet it does not follow that first-order adjoints are always sufficient to obtain optimal controls within the space of strict controls. If the optimal vague control described by Theorem 21 lies outside of strict controls, then one should, in general, invoke a stochastic minimum principle that features second-order adjoints, in order to obtain a strict or relaxed optimal control.

Our minimum principle can therefore be used to test whether an optimal strict control can be found using only first-order adjoints. Based on the above discussion, the following corollary regarding strict optimal controls is an immediate consequence of Theorems 9 and 21.

**Corollary 22:** If additionally the conditions of Theorem 9 hold, then Theorem 21 gives necessary and sufficient conditions for a strict control to be optimal.

We further note that a minimum principle for strict controls cannot be obtained by naively constraining the control set to those in the form of Dirac measures. In general, one cannot add a constraint to an optimization problem and expect that the optimality conditions remain the same. Here specifically, the restriction to strict controls would invalidate the proofs of Theorem 21, since within them, vague controls are always used as perturbations testing the optimality of a given reference control, even in the strict case.

We can now see how risk awareness affects the form of the minimum principle: Going from the risk neutral to the risk aware case, an additional process $(y'^\pi_t)_{t \in T}$ is introduced which acts as a rescaling or adjustment factor for given cost rate and terminal cost functions $c$ and $g$. As per (15), the values $y'^\pi_t$, $t \in T$ of the process represent the controller’s time $t \in T$ expectation of the derivative of the risk function evaluated at the total cost $C^\pi$. Indeed, if $\rho$ is the expectation, a risk neutral minimum principle is recovered with the process $(y'^\pi_t)_{t \in T}$ disappearing in a natural way.

**Corollary 23:** Suppose that the assumptions of Theorem 21 hold, and additionally, $p = 1$ and $\rho$ is the expectation. Then the statement of the theorem holds, with the Hamiltonian $H$ replaced by

$$H_0(t, x, y, z, a) := c(t, x, a) + yb(t, x, a) + \text{Tr}[z\sigma(t, x, a)]$$

$$\forall(t, x, y, z, a) \in T \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \mathcal{A} \quad (17)$$

and where the process $(y'^\pi_t)_{t \in T}$ is constant, $y'^\pi_t = 1$ for all $t \in T$.

**Proof:** Follows from Theorem 21 due to the fact that if $\rho = \mathbb{E}$, then the $\mathcal{C}$-derivative $D\rho(\cdot)(\cdot)$ is identically one, and by (15), we have $y'^\pi_t = 1$ for all $t \in T$. Thus, we may also set $y' = 1$ in the definition of the Hamiltonian $H$, (13) to recover a risk neutral minimum principle.

**Remark 24:** The process $(y'^\pi_t)_{t \in T}$ in the statement of Theorem 21 also satisfies a backward stochastic differential equation that is obtained in an intermediate step when proving the minimum principle. Specifically, there is a unique $\mathcal{F}$-predictable process $z' \in \mathcal{H}_{\mathcal{F}}^{p/(p-1)}(\Omega; \mathcal{Z'})$, $\mathcal{Z'} := \mathbb{R}^{d_x \times 1}$, such that

$$dy'^\pi_t = z'^\pi_t \cdot dw_t, \quad y'^\pi_T = D\rho(C^\pi). \quad (18)$$

Therefore together, (3), (14), and (18) form a forward-backward system of stochastic differential equations with $d_x$ and $d_x + 1$ state and adjoint state variables, respectively.
Proofs of the Main Results. The rest of this section is dedicated to proving the risk aware minimum principle, Theorem 21. We present our intermediate steps in reaching the main result, but defer the details of their proofs to Appendix VI-C.

We adopt the following short-hand: For every Borel measurable function $f : A \to \mathbb{V}$ and every $\pi_1, \pi_2 \in \mathcal{P}(A)$ and $a_1, a_2 \in \mathbb{R}$, we denote
\[
f(a_1 \pi_1 + a_2 \pi_2) := a_1 \int_A f(a_1 \pi_1(\{a\}) + a_2 \int_A f(a_2 \pi_2(\{a\}).
\]
In addition to the original stochastic differential equation (26), describing a controlled process $(x^T_t)_{t \in T}$, we introduce the additional, coupled differential equation for an $\mathbb{R}$-valued, $\mathcal{F}$-adapted continuous process, the running costs, $x' = (x'_t)_{t \in T}$, defined as
\[
x'_t = \int_0^t \int_A c(s, x^s, a) \pi_s(\{a\}) \, da \, ds.
\] (19)

We can then rewrite the total cost as
\[C^\pi = x^\pi_T + g(x^\pi_T).
\]
We shall use $X' = \mathbb{R}$ to indicate the range of the process $x'$.

In order to establish optimality conditions for vague controlled solutions, we need some means of comparing pairs of solutions. Since our assumptions guarantee the existence of unique strong solutions, and the definition of a vague controlled solution allows the use of convenient probability spaces, we shall in the following suppose that pairs of vague controlled solutions are defined on the same probability space.

In addition, for every pair of vague controlled solutions $\pi, q \in \mathcal{P}_\mathbb{V}(b, \sigma, \nu)$, the convex combination of their control processes shall be denoted by the short-hand $\pi(\alpha, q)$, that is, for all $\pi, q$ and $\alpha \in [0, 1]$
\[
\pi(\alpha, q) := \pi_t + \alpha(q_t - \pi_t) \quad \forall t \in T.
\]

We will consider the control $q$ as a perturbation of the original, reference control $\pi$, and our goal is to derive optimality conditions from variational equations representing the response of the solution to $q$.

We begin with a few auxiliary results, variations of which have appeared in the literature. The following lemma states that solutions corresponding to perturbed controls are, uniformly in time, good approximations of the unperturbed solutions.

**Lemma 25:** For all $\pi, q \in \mathcal{P}_\mathbb{V}(b, \sigma, \nu)$ and $\alpha \in [0, 1]$
\[
\left\|x^\pi(\alpha, q) - x^\pi\right\|_{S^p} \in \mathcal{O}(\alpha)
\] (20a)
\[
\left\|x'^\pi(\alpha, q) - x'^\pi\right\|_{S^p} \in \mathcal{O}(\alpha)
\] (20b)

where $p$ is as in Assumption 3. In addition, we have for the terminal cost
\[
\left\|g(x^\pi(\alpha, q)) - g(x^\pi)\right\|_p \in \mathcal{O}(\alpha).
\] (21)

The following lemma provides the means for computing the first-order response of solutions to perturbations of the control process.

**Lemma 26:** Let $\pi, q \in \mathcal{P}_\mathbb{V}(b, \sigma, \nu)$ be arbitrary. Then there exists an $\mathbb{X}$-valued process $\delta^\pi q = (\delta^\pi q_t)_{t \in T}$ that is the unique strong solution of
\[
\begin{align*}
\delta^\pi q_t = & \int_0^t \nabla_X b(s, x^s, \pi_s) \delta^\pi q_s + b(s, x^s, q_s - \pi_s) \, ds \\
& + \int_0^t \nabla_X c(s, x^s, \pi_s) \delta^\pi q_s + c(s, x^s, q_s - \pi_s) \, dw_s.
\end{align*}
\] (22)

Moreover, defining $\delta'^\pi q := (\delta'^\pi q_t)_{t \in T}$ as
\[
\delta'^\pi q_t := \int_0^t \nabla_X c(s, x^s, \pi_s) \delta'^\pi q_s + c(s, x^s, q_s - \pi_s) \, ds
\] (23)
we have that there is $r > p$, such that
\[
\left\|\delta^\pi q\right\|_{S^r} < \infty, \quad \left\|\delta'^\pi q\right\|_{S^r} < \infty
\] (24a)
and, for all $\alpha \in [0, 1]$
\[
\left\|x^\pi(\alpha, q) - x^\pi - \alpha \delta^\pi q\right\|_{S^p} \in \mathcal{O}(\alpha)
\] (25a)
\[
\left\|x'^\pi(\alpha, q) - x'^\pi - \alpha \delta'^\pi q\right\|_{S^p} \in \mathcal{O}(\alpha).
\] (25b)

The next results connect the response of the dynamics to the perturbation, described by the process $(\delta^\pi q, \delta'^\pi q)$ and characterized by the above lemmas, to the risk aware objectives. For brevity, we set
\[
\theta(x, x') := g(x) + x' \quad \forall(x, x') \in \mathbb{X} \times \mathbb{X}'.
\] (26)

If $\pi \in \mathcal{P}_\mathbb{V}(b, \sigma, \nu)$ is $\mathcal{P}_1$-optimal, then by definition for any $q \in \mathcal{P}_\mathbb{V}(b, \sigma, \nu)$ and $\alpha \in [0, 1]$ we have that
\[
0 \leq \rho(C^{\pi(\alpha, q)}) - \rho(C^\pi).
\] (27)

We will use (27) as a starting point for deriving our optimality conditions.

**Lemma 27:** Suppose Assumption 19 holds. Let $\pi \in \mathcal{P}_\mathbb{V}(b, \sigma, \nu)$ be $\mathcal{P}_1$-optimal and $q \in \mathcal{P}_\mathbb{V}(b, \sigma, \nu)$ arbitrary, and let the process $(\delta^\pi q, \delta'^\pi q)_{t \in T}$ be as in the statement of Lemma 26. Then
\[
0 \leq \mathbb{E} \left[ \rho(C^\pi) \left( \nabla_X g(x^\pi_T) \delta^\pi q_T + \delta'^\pi q_T \right) \right].
\] (28)

We can now construct the adjoint processes $(y^\pi_T, y'^\pi_T, z^\pi_T, z'^\pi_T)_{t \in T}$ appearing in (14) and (18), and use them to restate the optimality condition of (28). The proof of the next lemma follows roughly the same ideas as used in the risk neutral case, see, e.g., [10]–[12], [37], and relies primarily on the martingale representation theorem. In the risk aware case, we need to additionally handle the nonlinearity of the risk aware objective, which gives rise to the risk adjustment process.

**Lemma 28:** Suppose Assumption 19 holds, and that
\[
(\Omega, \Sigma, \mathcal{F}) = (\mathcal{F}_t)_{t \in T}, \mathbb{P}, \mathbb{X} = (x^\pi_T)_{t \in T},
\]
\[
w = (w_t)_{t \in T}, \pi = (\pi_t)_{t \in T} \in \mathcal{P}_\mathbb{V}(b, \sigma, \nu)
\]
is $\mathcal{P}_1$-optimal. Then there are unique $\mathcal{F}$-adapted continuous processes $y^\pi_T \in \mathcal{S}_\mathcal{F}^{\bar{p}(-1)}(\Omega; \mathbb{Y})$ and $y'^\pi_T \in \mathcal{S}_\mathcal{F}^{\bar{p}(-1)}(\Omega; \mathbb{Y}')$, and unique $\mathcal{F}$-predictable processes $z^\pi_T \in \mathcal{Y}_\mathcal{F}^{\bar{p}(-1)}(\Omega; \mathbb{Z})$ and
\[ z^\tau \in \mathcal{H}_p^{(p-1)}(\Omega; Z') \text{ satisfying the backward stochastic differential equations} \]
\[ dy^\pi_t = -\nabla_x H(t,x^\pi_t, y^\pi_t, z^\pi_t, \pi_t) \, dt + z^\pi_t \cdot dw_t \]  
(29a)
\[ dy^\pi_t = z^\pi_t \cdot dw_t \]  
(29b)
\[ y^\pi_T = D\rho(C^\pi) \nabla_x g(x^\pi_T), \quad y^\pi_0 = D\rho(C^\pi) \]  
(29c)
where \( H \) is as defined in (13), and (28) implies that for all \( q \in \mathbb{Q}^p(b, \sigma, \nu) \)
\[ 0 \leq \mathbb{E} \left[ \int_0^T H(t,x^\pi_t, y^\pi_t, y^\pi_T, z^\pi_t, q_t) \, dt \right. \]
\[ \left. - \int_0^T H(t,x^\pi_t, y^\pi_t, y^\pi_T, z^\pi_t, \pi_t) \right] . \]  
(30)

Finally, we show that existence of the adjoints and minimization of the Hamiltonian is sufficient to establish optimality, provided that the additional convexity conditions of Assumption 20 hold.

**Lemma 29:** Suppose Assumption 20 holds, \( \pi \in \mathbb{Q}^p(b, \sigma, \nu) \), and there exists processes \( y^\pi \in \mathbb{S}_p^{(p-1)}(\Omega; \mathbb{Y}), \ y^\pi \in \mathbb{S}_p^{(p-1)}(\Omega; \mathbb{Y}') \), \( z^\pi \in \mathcal{H}_p^{(p-1)}(\Omega; Z) \) satisfying (14)–(16). Then
\[ \rho(C^\pi) - \rho(C^q) \leq 0 \]
for every \( q \in \mathbb{Q}^p(b, \sigma, \nu) \).

We can now collect the above together and give the proof of our main result, Theorem 21.

**Proof of Theorem 21:** The first part of the theorem now follows directly from Lemma 28 and (30), while the second is a direct consequence of Lemma 29. The representation of (18) follows directly from Lemma 28.

### V. EXAMPLES OF DIFFERENTIABLE RISK FUNCTIONS AND A PORTFOLIO ALLOCATION PROBLEM

The purpose of this section is to present an application of the results of previous sections, and hence the problem we consider is selected for simplicity while attempting to retain a reasonable degree of practical significance.

**Risk Functions.** As examples of law invariant risk functions, we use the mean-deviation, the (smoothed) mean-semideviation, and entropic risk functionals.

**Definition 30:** Let \( (\Omega, \Sigma, \mathbb{P}) \) be a probability space. 1) Mean-deviation risk function \( \rho^{MD} : \mathcal{L}^2(\Omega; \mathbb{R}) \to \mathbb{R} \) is defined as the mapping
\[ \rho^{MD}(X) := \mathbb{E}[X] + \beta \mathbb{E}\|X - \mathbb{E}[X]\|_2 \]  
(31)
for all \( X \in \mathcal{L}^2(\Omega; \mathbb{R}) \) and where \( \beta \in (0, 1) \). 2) Mean-semideviation risk function \( \rho^{MD+} : \mathcal{L}^1(\Omega; \mathbb{R}) \to \mathbb{R} \) and the \( \epsilon \)-smoothed mean-semideviation risk function \( \rho^{MD+}_\epsilon : \mathcal{L}^1(\Omega; \mathbb{R}) \to \mathbb{R}, \epsilon > 0, \) are defined as
\[ \rho^{MD+}(X) := \mathbb{E}[X] + \beta \mathbb{E}\|(X - \mathbb{E}[X])_+\| \]
\[ \rho^{MD+}_\epsilon(X) := \mathbb{E}[X] + \beta \mathbb{E}\|(X - \mathbb{E}[X])_{\epsilon+}\| \]
for all \( X \in \mathcal{L}^1(\Omega; \mathbb{R}) \) and where \( \beta \in (0, 1] \), and \( (\cdot)_{\epsilon+} : \mathbb{R} \to \mathbb{R}_{>0} \) and \( (\cdot)_{\epsilon+} : \mathbb{R} \to \mathbb{R}_{>0} \) are the positive part and \( \epsilon \)-smoothed positive part functions, \( (x)_{\epsilon+} := x \vee 0 \) and \( (x)_{\epsilon+} := x + \epsilon \ln(1 + e^{-x/\epsilon}) \) for all \( x \in \mathbb{R} \) and \( \epsilon > 0 \). 2) Entropic risk function is the risk measure \( \rho^{\text{Ent}} : \mathcal{L}^\infty(\Omega; \mathbb{R}) \to \mathbb{R} \) defined as
\[ \rho^{\text{Ent}}(X) := \frac{1}{\theta} \ln \mathbb{E}[e^{\theta X}] \quad \forall X \in \mathcal{L}^\infty(\Omega; \mathbb{R}) \]  
(32)
where \( \theta > 0 \).

We note that the mean-deviation risk function is convex, positively homogeneous, and translation invariant, that is, it satisfies Definition 14 items 2)–4). The \( \mathcal{L}^1(\Omega; \mathbb{R}) \) mean-semideivation risk measure \( \rho^{MD+} \) was considered in, e.g., [44], and it too is convex, positively homogeneous, and translation invariant, but is additionally monotonic, satisfying Definition 14 1).

As noted in Remark 17, the positive homogeneity of these functionals implies that they cannot be everywhere Fréchet differentiable. We demonstrate in the example problem below that this is not necessarily an issue for our purposes. Moreover, the \( \epsilon \)-smoothed mean-semideviation risk function \( \rho^{MD+}_\epsilon \) uniformly approximates \( \rho^{MD+} \), that is,
\[ 0 < \rho^{MD+}_\epsilon(X) - \rho^{MD+}(X) \leq \epsilon \beta \ln 2 \]
for all \( X \in \mathcal{L}^1(\Omega; \mathbb{R}) \) and \( \epsilon > 0 \), but its restriction to \( \mathcal{L}^2(\Omega; \mathbb{R}) \) is in fact everywhere Fréchet differentiable (this will be established in Lemma 31 below). The smoothed mean-semideivation is also convex and monotonic which, along with the above estimate, follows directly from the properties of the \( \epsilon \)-smoothed positive part function [16]. Our definition of \( \rho^{MD+}_\epsilon \) was inspired by the construction of a smoothed conditional value-at-risk risk functional in [30]. The entropic risk function \( \rho^{\text{Ent}} \) on the other hand satisfies monotonicity, convexity, and translation invariance properties, or items 1), 2), and 4) of Definition 14. It serves as an example of a commonly used risk function that is everywhere Fréchet differentiable.

**Lemma 31:** 1) The mean-deviation risk function is Fréchet differentiable at every \( X \in \mathcal{L}^2(\Omega; \mathbb{R}) \) that is not almost surely constant, with the derivative \( D\rho^{MD}(X) \in \mathcal{L}^2(\Omega; \mathbb{R}) \) being
\[ D\rho^{MD}(X) = 1 + \beta \frac{X - \mathbb{E}[X]}{\|X - \mathbb{E}[X]\|_2} \]  
(33)
Moreover, the derivative does not exist at \( X \in \mathcal{L}^2(\Omega; \mathbb{R}) \), such that \( X = \mathbb{E}[X] \). It additionally has the \( \mathcal{F} \)-derivative \( D\rho^{MD} : \mathcal{P}^2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R} \) that reads, for all \( \mu \in \mathcal{P}^2(\mathbb{R}) \) that are not a Dirac measures
\[ D\rho^{MD}(\mu)(x) = 1 + \beta \frac{x - \int x' \mu(dx')}{\left[ \int (x'' - x' \mu(dx'))^2 \mu(dx'') \right]^{\frac{1}{2}}} \]  
(34)
for all \( x \in \mathbb{R} \).

2) The \( \mathcal{L}^2(\Omega; \mathbb{R}) \)-restriction of the \( \epsilon \)-smoothed mean-semideivation risk function \( \rho^{MD+}_\epsilon \), \( \epsilon > 0 \), is Fréchet differentiable at every \( X \in \mathcal{L}^2(\Omega; \mathbb{R}) \), and has the Fréchet- and

---

3Specifically, from the inequality \( 0 < (x)_{\epsilon+} - (x)_{\epsilon+} \leq \epsilon \ln 2 \forall x \in \mathbb{R} \), and the monotonicity and convexity of \( (\cdot)_{\epsilon+} \).
\( \mathcal{L} \)-derivatives

\[
\begin{align*}
Dp^\text{MD} & (X) = 1 + \beta \left\{ U_e(X - \mathbb{E}[X]) - \mathbb{E} [U_e(X - \mathbb{E}[X])] \right\} \\
& \in \mathcal{L}^\infty(\Omega; \mathbb{R}) \quad \forall X \in \mathcal{L}^2(\Omega; \mathbb{R}) \\
Dp^\text{Ent} & (\mu)(x) = 1 + \beta \left\{ U_e \left( x - \int x' \mu(dx') \right) - \int U_e \left( x'' - \int x' \mu(dx') \right) \mu(dx'') \right\} \\
& \in (1 - \beta, 1 + \beta) \quad \forall \mu \in \mathcal{P}^2(\mathbb{R}), x \in \mathbb{R}.
\end{align*}
\]

respectively, and where \( U_e(x) := d(x)e^{+}dx = 1/(1 + e^{-x/\epsilon}) \) for all \( x \in \mathbb{R} \).

3) The entropic risk measure is Fréchet differentiable at every \( X \in \mathcal{L}^\infty(\Omega; \mathbb{R}) \), with the Fréchet- and \( \mathcal{L} \)-derivatives \( Dp^\text{Ent}(X) \in \mathcal{L}^1(\Omega; \mathbb{R}) \) and \( Dp^\text{Ent}(\mu)(x) \in \mathcal{P}(\mathbb{R}) \), \( \mu \in \mathcal{P}^\infty(\mathbb{R}) \)

\[
\begin{align*}
Dp^\text{Ent}(X) &= \frac{\mathbb{E}[\mathcal{E}[\sigma X]]}{\mathbb{E}[\mathcal{E}[\sigma X]]} \\
Dp^\text{Ent}(\mu)(x) &= \frac{\mathbb{E}[\mathcal{E}[\sigma X]]}{\mathbb{E}[\mathcal{E}[\sigma X]]} \quad \forall x \in \mathbb{R}.
\end{align*}
\]

**Portfolio Allocation Problem.** As a practical example, we consider a simplified portfolio allocation problem. An agent manages a portfolio consisting of a risk free bond, yielding a constant return rate \( r > 0 \), and a risky stock whose price \( Q_t \) evolves according to \( dQ_t = \mu Q_t dt + \sigma Q_t dw_t \), \( Q_0 = 1, \mu > 0, \sigma > 0 \). Let \( N_t = B_t + Q_t S_t \) be the net value of the agent’s portfolio, where \( B_t \) and \( S_t \) represent the agent’s bond and stock holdings at any \( t \in \mathbb{T} \), respectively. Let \( \phi_t := Q_t S_t/N_t \) be the proportion of the agent’s portfolio allocated to the risky asset, so that \( N_t \) follows the stochastic differential equation

\[
dN_t = \left[ r + (\mu - r)\phi_t \right] N_t dt + \sigma \phi_t N_t dw_t \tag{36}
\]

with a given initial condition \( N_0 \). Trading is costless and unconstrained so that \( \phi_t \) is a choice variable for each \( t \in \mathbb{T} \).

We suppose \( \phi_t \) is constrained to the interval \( \bar{\phi} = [\phi, \tilde{\phi}] \), where \( 0 < \phi < \tilde{\phi} < \infty \), and the agent optimizes the allocation so that the\( \bar{\phi} \) of the utility of \( N_T \) is minimized. Here, the agent values their profits or losses using a logarithmic utility, so that their utility of total cost evaluates to \(-\ln N_T \).

Rewriting (36) for the logarithm of \( N_t, x_t^T := \ln N_t \) for all \( t \in \mathbb{T} \), and generalizing to a vague controlled process, we have that

\[
dx_t = \left[ r + (\mu - r) \int H \phi_t \pi_t(d\phi) - \frac{1}{2} \sigma^2 \int H \phi_t \pi_t(d\phi) \right] dt \\
+ \sigma \int H \phi_t \pi_t(d\phi) dw_t \tag{37}
\]

where \( x_0^T = x_0 \in \mathbb{R} \) is given. The vague control has no particular interpretation here, we expect the optimal control to come out as a strict, Dirac-\( \delta \)-valued process. However, note that vague controlled solutions can be used to model actual portfolio choice, see [5].

Let \( b_\phi \) and \( \sigma_\phi \) be the drift and diffusion coefficients of (37), and let \( \nu_\phi = \delta_{x_0} \). Assumption 3 is now satisfied, with \( \bar{p}_1 = 0, \bar{p}_2 = \infty, p_1 = 1, p_2 = 0, \tilde{p}_1 = 0, \) and \( \tilde{p}_2 = 0 \). Since the initial condition is deterministic, \( \bar{p} \) may be selected to be arbitrarily large. It is easy to verify that (9) holds for any \( p \in [1, \infty) \), so that we may consider \( p \)-feasible solutions \( \pi \in \mathfrak{W}^p(b_\phi, \sigma_\phi, \nu_\phi) \).

The risk aware control problem, Problem \( \mathcal{P}_\phi \), becomes

\[
\mathcal{P}_\phi : \quad \inf_{\pi \in \mathfrak{W}^p(b_\phi, \sigma_\phi, \nu_\phi)} \rho(-x_T^T).
\]

We note that for instance the mean-deviation risk function of (31) is \( \mathcal{L} \)-differentiable at \(-x_T^T \).

**Proposition 32:** There is no \( \pi \in \mathfrak{W}^p(b_\phi, \sigma_\phi, \nu_\phi) \), such that \(-x_T^T \) is almost surely bounded.

**Proof:** Since for any \( \pi \in \mathfrak{W}^p(b_\phi, \sigma_\phi, \nu_\phi) \) the drift is bounded, and the diffusion bounded away from zero, \( x_T^T \) can take arbitrarily large values.

Since \(-x_T^T \) is not bounded, it cannot be constant, and therefore \( \rho^\text{MD} \) is \( \mathcal{L} \)-differentiable at the terminal cost. In addition, e.g., the mean-deviation risk function or the \( \mathcal{L} \)-restriction of the \( \epsilon \)-smoothed mean-semideviation risk function together with the cost \(-x_T^T \) satisfy Assumption 19.

We can now use the risk aware minimum principle to characterize an optimal allocation process. For simplicity, we assume that the \( \mathcal{L} \)-derivative of the risk function is positive (this is the case for, e.g., the \( \epsilon \)-smoothed mean-semideviation).

**Proposition 33:** Suppose \( \rho : \mathcal{L}^2(\Omega; \mathbb{R}) \to \mathbb{R}, p \in [1, \infty) \), is convex, satisfies Assumption 19, and has a positive \( \mathcal{L} \)-derivative, i.e., \( D_p^\rho(x) > 0 \) for all \( x \in \mathfrak{W}^p(\mathbb{R}) \). The optimal portfolio allocation for Problem \( \mathcal{P}_\phi \) is a strict control \( \pi \in \mathfrak{W}^p(b_\phi, \sigma_\phi, \nu_\phi) \), such that \( \pi_t = \delta_{\phi_t} \) for all \( t \in \mathbb{T} \), where

\[
\phi_t = \frac{\phi + \mu - r + t\epsilon}{\sigma^2} \left( 1 + \frac{1}{2} \sigma \int H \phi_t \pi_t(d\phi) \right),
\]

and where

\[
t_\epsilon := \frac{\sigma^2 \pi_t}{y_t}, \quad \forall t \in \mathbb{T}
\]

is a risk premium in which

\[
y_t = \mathbb{E} \left[ D\rho(-x_T^T) \mid \mathcal{F}_t \right]
\]

and

\[
\mathbb{E} \left[ D\rho(-x_T^T) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ D\rho(-x_T^T) \right] + \int_0^t z_t^\pi dw_t
\]

for all \( t \in \mathbb{T} \).

We note that interestingly, the risk aware version of the objective function has now given rise to the additional risk premium process \( (t_\epsilon)_{t \in \mathbb{T}} \) defined in (38). To wit, the risk premium vanishes if \( \rho \) is the expectation, since then as noted in Corollary 23, \( y_t^{\epsilon} = 1 \) for all \( t \in \mathbb{T} \) implying that \( z_t^\pi = 0 \) for all \( t \in \mathbb{T} \). Thus, the risk aware minimum principle may open new possibilities in, e.g., risk pricing theory.

**VI. Conclusion**

In Theorem 21 we have given a risk aware version of the stochastic minimum principle. A notable feature of the result is the way risk is captured via the risk adjustment process, essentially the marginal risk at a given time \( t \in \mathbb{T}, (15) \). In
the result we obtained, this risk accounting is represented by the \( F_t \)-conditional expectation of the \( \mathcal{L} \)-derivative of the risk function evaluated at the terminal cost.

The control model employed here is somewhat nonstandard (see Definition 1): We consider probability measure-valued generalized controls that are nonetheless distinct from the usual relaxed controls, and we opted for the term “vague controlled solutions,” to distinguish the two. The benefit of the vague control model is that within it, the minimum principle requires only first-order adjoint processes, which has the particular advantage in the risk aware case in that it avoids the need for second-order functional derivatives of the risk functions. Though the sufficiency of first-order adjoints may appear controversial, any vague control has an equivalent strict control, and vague controls become accumulation points of strict controls.

**APPENDIX**

**PROOFS OF THE RESULTS**

**A. Proofs for Section II**

**Proof of Proposition 5:** The existence of a unique strong solution \( x = (x_t)_{t \in T} \in \mathcal{S}_F^p(\Omega; X) \) can be shown using, e.g., [36, Th. 3.17]. The necessary growth and boundedness conditions are readily checked using Assumption 3 and (9). Costs can then be estimated using the bounds given in (7), and the finiteness of the \( \mathcal{L}^p \)-norms of the total running and terminal costs follows again using the conditions of (9).

**Proof of Example 8:** The main inequality of the example follows from a straight-forward application of the definition of the perturbed control and the Burkholder–Davis–Gundy (BDG) inequality, [36, Th. 1.76].

**Proof of Theorem 9:** The proof of [18, Th. 2.10] applies here as well, with the modification that instead of the squares of the diffusion function, \((\sigma \sigma')\), one needs to consider the functions \(\sigma\) instead. For completeness and clarity, we give a short, self-contained version of the proof.

Let \( \Lambda(t, x, a) := \langle b(t, x, a), c(t, x, a), \sigma(t, x, a) \rangle \) for all \((t, x, a) \in T \times X \times A\), and let \((\Omega, \Sigma, \mathcal{F}, \mathcal{P}, \omega, w, \pi) \in \mathcal{Q}(b, \sigma, \nu)\). We define \( G \) as the \(\sigma\)-algebra of progressively measurable functions

\[
G := \bigcap_{t \in T} \left\{ A \in \mathcal{B}(T) \otimes \mathcal{F}_T \mid A \cap ([0, t] \times \Omega) \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t \right\}.
\]

Let then \( F((t, \omega), a) := \int_{\Omega} \Lambda(t, x_\omega^T(\omega), a) \pi_\omega(\omega)(da) - \Lambda(t, x_\omega^T(\omega), a) \) for all \((t, \omega) \in T \times \Omega\) and \( a \in A\), and \( \Phi(t, \omega) := \{a \in A \mid F(t, \omega, a) = 0\} \) for all \((t, \omega) \in T \times \Omega\). Note that, for all \((t, \omega) \in T \times \Omega\), \( \Phi(t, \omega) \) is compact (as a closed subset of the compact set \(A\)), and by the convexity of \( \Lambda(t, x, A) \) also nonempty. By [4, Corollary 17.8(2), Lemma 17.4(3)], \( \Phi \) is an \(G\)-measurable correspondence. The Kuratowski–Ryll–Nardzewski measurable selection theorem [4, Th. 17.13] now yields us a \(G\)-measurable selection \((a_t)_{t \in T}\), such that \(a_t(\omega) \in \Phi(t, \omega)\) for all \((t, \omega) \in T \times \Omega\). It is then clear that \((\Omega, \Sigma, \mathcal{F}, \mathcal{P}, \omega, w, \pi') = (\delta_\omega)_{t \in T} \) is a strict controlled solution and that the costs are the same.

**Proof of Proposition 10:** Let \( \pi \in \mathcal{Q}(b, \sigma, \nu)\). It now suffices to consider an arbitrary continuous, compactly supported \( f : X \rightarrow \mathbb{R} \) with continuous and compactly supported first and second derivatives, and applying the Itô’s Lemma to see that

\[
f(x_t) - f(x_0) - \int_0^t G f(s, x_s, a) \pi(s) \, ds
\]

is an \(\mathcal{F}_\pi\)-martingale, and where \( G \) is the generator, \( Gf(t, x, a) := b(t, x, a) \nabla f(x) + (1/2) Tr \tilde{\sigma}(t, x) \tilde{\sigma}(t, x)^T H f(x) \).

**B. Proofs for Section 3**

**Proof of Proposition 18:** This result is proven in [14, Proposition 5.25] for the case of \( p = 2 \); here we are merely pointing out that the statement naturally holds also in the “smaller” spaces \( L^p(\Omega; \mathbb{R}^n) \), \( p \in (2, \infty) \). Let \( \psi \) be an \( L^p(\Omega)\)-representation of \( \phi \), whose Fréchet derivative is continuous. Since the embedding of \( L^p(\Omega; \mathbb{R}^n) \) into \( L^2(\Omega; \mathbb{R}^n) \) is continuous, the Fréchet derivative is continuous on \( L^2(\Omega; \mathbb{R}^n) \) as well. Therefore, there is an almost surely unique \( L \)-derivative \( f \) such that \( Y = \Omega \ni \omega \rightarrow f(X(\omega)) \in L^2(\Omega; \mathbb{R}^n) \). As an element of \( L^2(\Omega; \mathbb{R}^n) \), \( Y \) is also in \( L^p(\Omega; \mathbb{R}^n) \), \( q = p/(p - 1) \).

**C. Proofs for Section IV**

For the detailed proofs, we need to extend our notations somewhat. Throughout, \( C \) shall be a universal constant, that may depend only on \( T, L \), and the powers \( \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_1, p_2, p_3 \), and \( p_4 \) as given in Assumption 3. Let \( n, m \in \mathbb{N} \) and \( k_i \in \mathbb{N} \) for all \( i \in \{1, \ldots, m\} \). For all differentiable functions \( f : \mathbb{R}^n \rightarrow \mathbb{R}^{k_1 \times \cdots \times k_m} \), we define \( \nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^{k_1 \times \cdots \times k_m} \) so that \( (\nabla f(x))_{i_1, \ldots, i_m} := \partial f_{i_1, \ldots, i_m}(x)/\partial x_{j} \) for all \( x \in \mathbb{R}^n \), \( i = \{1, \ldots, k_i\}, \ell \in \{1, \ldots, m\} \).

Let \( N, M, n_1, m_1 \in \mathbb{N} \) for all \( i \in \{1, \ldots, N\}, j \in \{1, \ldots, M\} \), and let \( U \in \mathbb{R}^{n_1 \times \cdots \times N_\times N} \) and \( V \in \mathbb{R}^{m_1 \times \cdots \times M} \). The array \( UV \in \mathbb{R}^{n_1 \times \cdots \times n_1 \times \cdots \times N_\times N} \) is defined analogously to a standard matrix product by multiplying elements with matching first and last indices and taking the sum; the array \( UV \cdot V \in \mathbb{R}^{n_1 \times \cdots \times n_1 \times \cdots \times N_\times N} \) is defined the same way but multiplication and summation extends over last and first, and second to last, and second indices. In addition, for all \( X \in \mathbb{R}^{nN-1} \), we define \( U \cdot X \in \mathbb{R}^{n_1 \times \cdots \times N_\times N} \) as

\[
U_1 \ldots, i_{N-1}, i_N := \sum_{k=1}^{m_1} U_{i_1, \ldots, i_{N-2}, k, i_N} X_k
\]
for all \(i \in \{1, \ldots, n\}, \ell \in \{1, \ldots, N-2, N\}\).

We will also repeatedly use the following identity and estimates: For at least once continuously differentiable \(f: \mathbb{R}^n \to \mathbb{R}^m, n, m \in \mathbb{N}\)

\[
f(x) - f(y) = \int_0^1 \nabla f ((1 - \lambda)x + \lambda y) \,(x - y) \,d\lambda.
\]

\(\forall x, y \in \mathbb{R}^n\) \hspace{1cm} (41a)

\[
\int_0^1 \left| (1 - \lambda)x + \lambda y \right|^\gamma \,d\lambda \leq \ell_\gamma \left[ \left( |x|^\gamma + |y|^\gamma \right) \vee \left( |x + y|^\gamma \right) \right]
\]

\(\forall x, y \in \mathbb{R}^n, n \in \mathbb{N}, \gamma \in \mathbb{R}_{\geq 0}\)

\[
\ell_\gamma := 2^{-\gamma} \vee \frac{2^{\gamma-1}}{\gamma + 1} \quad \forall \gamma \in \mathbb{R}_{\geq 0}.
\] \hspace{1cm} (41b)

In addition, we shall frequently apply the BDG inequality (see, e.g., [36, Th. 1.76]).

**Proof of Lemma 25:** Let \(T_0 \in T\) be arbitrary. Using standard estimates, including the BDG, and Assumption 3, (9), and Proposition 5, we obtain

\[
\mathbb{E}\left[ \sup_{t \in [0, T_0]} \left| x_t^{\pi(a, q)} - x_t^{\bar{\rho}} \right|^{p}\right]
\]

\[
\leq C \left\{ \mathbb{E}\left[ \int_0^{T_0} \sup_{s \in [0, t]} \left| x_s^{\pi(a, q)} - x_s^{\bar{\rho}} \right|^{p} \, dt \right] + a^p S^{\pi, q}\right\}
\]

where \(S^{\pi, q} \in \mathbb{R}_{\geq 0}\). Using Grönwall’s inequality (see, e.g., [36, Corollary 6.60]), we find that

\[
\mathbb{E}\left[ \sup_{t \in [0, T_0]} \left| x_t^{\pi(a, q)} - x_t^{\bar{\rho}} \right|^{p}\right] \leq C S^{\pi, q} a^p \quad \forall T_0 \in T
\]

from where (20a) follows.

To prove (20b), consider the running cost processes, \((x_t^{\pi(a, q)})_{t \in T}\) and \((x_t^{\bar{\rho}})_{t \in T}\). We have that

\[
\mathbb{E}\left[ \sup_{t \in T} \left| x_t^{\pi(a, q)} - x_t^{\bar{\rho}} \right|^{p}\right] \leq 2^{p-1} \mathbb{E}\left[ \left( \int_0^T \left| c(s, x_s^{\pi(a, q)}, \pi_s(a, q)) - c(s, x_s^{\bar{\rho}}, \pi_s(a, q)) \right| \, ds \right)^p \right]
\]

\[
+ 2^{p-1} \mathbb{E}\left[ \left( \int_0^T \left| c(s, x_s^{\pi(a, q)}, \pi_s(a, q)) - c(s, x_s^{\bar{\rho}}, \pi_s(a, q)) \right| \, ds \right)^p \right].
\] \hspace{1cm} (42)

The latter term is \(O(a^p)\), which can be verified using similar estimates and assumptions as before. To see this for the first term as well, we apply Assumption 3 and (41a) and (41b) to get

\[
\left| c(t, x_2, a) - c(t, x_1, a) \right| \leq L \left| x_2 - x_1 \right|
\]

\[
\times \left( 1 + \ell_p |x_1 + x_2|^{p_1} \vee \left( |x_1|^{p_1} + |x_2|^{p_1} \right) + |a|^{p_2} \right)
\]

\[
\forall (t, x_1, x_2, a) \in T \times X \times X \times A.
\]

Using Hölder’s and Jensen’s inequalities and (20a), we get

\[
\mathbb{E}\left[ \left( \int_0^T \left| c(s, x_s^{\pi(a, q)}, \pi_s(a, q)) - c(s, x_s^{\bar{\rho}}, \pi_s(a, q)) \right| \, ds \right)^p \right]
\]

\[
\leq C \left\{ 1 + \sum_{l \in [1, 2]} \mathbb{E}\left[ Z_l \right] \mathbb{E}\left[ \left| x_l^{\pi(a, q)} - x_l^{\bar{\rho}} \right|^{p} \right] \right\}
\]

\[
\times \mathbb{E}\left[ \sup_{l \in T} \left| x_l^{\pi(a, q)} - x_l^{\bar{\rho}} \right|^{p} \right]^{\frac{p}{p}}
\] \hspace{1cm} (43)

and

\[
Z_1 := \sup_{l \in T} \left| x_l^{\pi(a, q)} - x_l^{\bar{\rho}} \right|^{p_1} \vee \left( \left| x_l^{\pi(a, q)} \right|^{p_1} + \left| x_l^{\bar{\rho}} \right|^{p_1} \right)
\]

\[
Z_2 := \sup_{l \in T} \int_A \left| a |x_l^{\pi(a, q)}|^{p_1} \pi_l(a, q) (da) \right| < \infty
\]

where the latter factor in (43) is \(O(a^p)\), again by (20a), and the former terms are easily verified finite using Assumption 3, (9), and elementary estimates. Therefore (20b) holds.

Finally, (21) is proven analogously.

**Proof of Lemma 26:** The drift and diffusion coefficients appearing in (22) are Lipschitz, since by Assumption 3 3) the gradients of \(b\) and \(\sigma\) are bounded. In addition, the terms in (22) that do not depend on \(\delta^{\pi, q}\) satisfy

\[
\left\| \int_0^1 b(s, x_s^{\pi}, q_s - \pi_s) \, ds \right\|_{S^p} < \infty
\]

\[
+ \int_0^1 \sigma(s, x_s^{\pi}, q_s - \pi_s) \, dw_s \right\|_{S^p} < \infty
\]

where we have used the growth conditions of Assumption 3 2), (9b), \(x^{\pi} \in S^p\), and the \(\tilde{b}\) and \(\tilde{\rho}\) admissibility of the control. A unique strong solution of (22) now exists by, e.g., [36, Th. 3.17], which also satisfies the first part of (24a).

To prove the second part of (24a), let \(r \in (1, \bar{p})\) be a solution of

\[
r = \frac{\bar{p}}{\gamma_r p_1} \wedge \frac{\bar{p}_3}{\gamma_r p_2} \wedge \frac{\bar{p}}{p_1} \wedge \frac{\bar{p}_3}{p_2}
\]

\[
\zeta_r := \frac{1}{1 - r/\bar{p}}
\]

The right hand side is continuous and decreasing in \(r\), and maps \((0, \bar{p})\) to \((0, a)\) for some \(a > 0\), and so a solution exists. We note that \(r > \bar{p}\), which follows from (9c) and (9d).

We use the growth conditions on \(c\) and \(\nabla_X c\), Assumptions 3 4), 5), and again standard methods to estimate \(\delta^{\pi, q}\)

\[
\|\delta^{\pi, q}\|_{S^p} \leq C \left\{ 1 + \left| x^{\pi} \right|^{r\bar{p}_1} + \int_A \left| a \right|^{r\bar{p}_2} \left( \pi (da) \right) \right\} \|x^{\pi}\|_{S^{\pi, q}}^r
\]

\[
+ \left| \delta^{\pi, q} \right| \|x^{\pi}\|_{S^{\pi, q}}^r
\]

\[
+ \left| \delta^{\pi, q} \right| \|x^{\pi}\|_{S^{\pi, q}}^r
\]

\[
\times \left( 1 + \left| x^{\pi} \right|^{r\bar{p}_1} + \frac{1}{2} \int_A \left| a \right|^{r\bar{p}_2} \left( \pi + q (da) \right) \right\} \|x^{\pi}\|_{S^{\pi, q}}^r.
\]
Proof of Lemma 26: We prove the property for \( p > 1 \), but comment at relevant places on changes needed to accommodate the \( p = 1 \) case. The statement of the lemma amounts to expressing (28) by using processes that are constructed to satisfy (29). For brevity, we set \( B^T_t := \nabla_X b(t, x^T_t, \pi_t) \in \mathbb{R}^{d_s \times d_s} \), \( F^T_t := \nabla_X c(t, x^T_t, \pi_t) \in \mathbb{R}^{1 \times d_s} \), \( S^T_t := \nabla_X \sigma(t, x^T_t, \pi_t) \in \mathbb{R}^{d_s \times d_s \times d_s} \) for all \( t \in \mathbb{T} \). We note that the proof is relatively standard, but requires special care to ensure that the appropriate norms of the involved processes and random variables are finite. In addition, the contribution of the risk function in the objective to the statement of the theorem is naturally not covered by standard proofs.

Let \( (U^T_t)_{t \in \mathbb{T}} \) be the fundamental solution of (22), i.e., \( U^T_t \in \mathbb{R}^{d_s \times d_s} \), \( t \in \mathbb{T} \), \( U^T_0 = I \), where \( I \) is the identity matrix, and
\[
dU^T_t = B^T_t U^T_t \, dt + (S^T_t U^T_t)^. \quad \text{(44)}
\]
and let \( V^T_t := (U^T_t)^{-1} \). It is readily verified that \( U^T, V^T \in \mathcal{F}_T^\mathbb{T}(\Omega; \mathbb{R}^{d_s \times d_s}) \), \( V^T \) satisfies
\[
dV^T_t = V^T_t (-B^T_t + S^T_t -. S^T_t) \, dt - (V^T_t S^T_t)^. \quad \text{dw}_t
\]
\( V^T_0 = I \), and that the solutions \( U^T \) and \( V^T \) are unique strong solutions of their respective differential equations.

We define \( (Q^T_t)_{t \in \mathbb{T}} \) and \( \Xi^T \) as
\[
Q^T_t := \int_0^t F^T_s U^T_s \, ds \quad \forall t \in \mathbb{T}
\]
\[
\Xi^T := D\rho(C^T) \nabla_X g(x^T_T) U^T_T + D\rho(C^T) Q^T_T. \quad \text{(45)}
\]
Let \( p^* := (\bar{p} / p_1') \land (\bar{p}_3 / p_2') \), \( \bar{p} := 1/(1/p - 1/p^*) \), and note that \( p < p^* \), \( p < \bar{p} < \bar{p}_3 \). Let \( \bar{p} \in (\bar{p}, \bar{p}_3) \) be arbitrary. We claim that
\[
\Xi^T \in L^{p^*}((\bar{p} - 1)(\Omega; \mathbb{R}^{1 \times d_s})
\]
which implies \( \Xi^T \in L^{p^*}((\bar{p} - 1)(\Omega; \mathbb{R}^{1 \times d_s}) \). To prove this, we now set \( \bar{q} := 1/(1/p - 1/p^*) \) and first show that \( \nabla_X g(x^T_T) U^T_T \in L^{\bar{q}}((\bar{p} - 1)(\Omega; \mathbb{R}^{1 \times d_s}) \) and \( Q^T_T \in L^{\bar{q}}((\bar{p} - 1)(\Omega; \mathbb{R}^{1 \times d_s}) \). We explicitly prove only the latter inclusion, the former is established in very much the same way. Basic estimates along with a judicious application of Hölder’s inequality shows that \( (p^* > \bar{q}) \)
\[
E \left[ \left| Q^T_T \right|^{\frac{p^*}{p^* - 1}} \right] \leq \mathcal{C} E \left[ \sup_{t \in \mathbb{T}} \left| U^T_t \right|^{p^*/(p^* - 1)} \right]^{1 - \frac{1}{p^*}} \times \mathbb{E} \left[ \sup_{t \in \mathbb{T}} \left( 1 + \left| x^T_t \right|^{p^*} \right)^p \right]^{\frac{1}{p^*}} \times \left( \mathbb{E} \left[ \left| \nabla_X g(x^T_T) U^T_T \right|^{p^*/(p^* - 1)} \right]^{1 - \frac{1}{p^*}} \right)
\]
\[
\times \left( \mathbb{E} \left[ |\nabla_X g(x^T_T) U^T_T|^{p^*/(p^* - 1)} \right]^{1 - \frac{1}{p^*}} \right). \quad \text{(47)}
\]
The right-hand side is finite, and hence \( Q^T_T \in L^{\bar{q}}((\bar{p} - 1)(\Omega; \mathbb{R}^{1 \times d_s}) \). Turning to proving (46), based on the above it is enough to show that \( D\rho(C^T) G \in L^{\bar{q}}((\bar{p} - 1)(\Omega; \mathbb{R}^{1 \times d_s}) \) for all \( G \in L^{\bar{q}}((\bar{p} - 1)(\Omega; \mathbb{R}^{1 \times d_s}) \). Let \( G \) be any such random variable. Then, again using Hölder’s inequality
\[
E \left[ \left| D\rho(C^T) G \right|^{\frac{\bar{q}}{\bar{q} - 1}} \right] \leq E \left[ \left| D\rho(C^T) \right|^{\frac{\bar{q}}{\bar{q} - 1}} \right] \times E \left[ \left| G \right|^{\frac{\bar{q}}{\bar{q} - 1}} \right]^{1 - \frac{1}{\bar{q}}} \times \left( \mathbb{E} \left[ \left| \nabla_X g(x^T_T) U^T_T \right|^{p^*/(p^* - 1)} \right]^{1 - \frac{1}{p^*}} \right)
\]
\[
\times \left( \mathbb{E} \left[ \left| \nabla_X g(x^T_T) U^T_T \right|^{p^*/(p^* - 1)} \right]^{1 - \frac{1}{p^*}} \right). \quad \text{(47)}
\]
and since $G \in \mathcal{L}^\bar{q}(\Omega; \mathbb{R}^{1 \times d_x})$, $\mathbb{D} \rho(C^n)G \in \mathcal{L}^\bar{q}(\rho^{-1})(\Omega; \mathbb{R})$. Equation (46) is now proven.

We can now apply the martingale representation theorem for $\mathcal{L}^r$-random variables, $r > 1$, given, e.g., in [36, Th. 2.42], to $\Xi^\pi$ and $\mathbb{D} \rho(C^n)$. This provides us with unique $\mathcal{F}$-predictable processes $\xi^n = (\xi^n_t)_{t \in T} \in \mathcal{H}^\rho(n)(\Omega; \mathbb{R}^{1 \times d_x})$ and $z^n = (z^n_t)_{t \in T} \in \mathcal{H}^\rho(n)(\Omega; \mathbb{R}^{d_x})$, taking, respectively, values in $\mathbb{R}^{d_x}$ and $\mathbb{R}$, such that

$$\Xi^\pi = \mathbb{E}[\Xi^\pi] + \int_0^T \xi^n_s \cdot \, dw_s \tag{48}$$

$$\mathbb{D} \rho(C^n) = \mathbb{E}[\mathbb{D} \rho(C^n)] + \int_0^T z^n_s \cdot \, dw_s. \tag{49}$$

Moreover, we define the processes $\Lambda^n_t = (\Lambda^n_t)_{t \in T} \in \mathcal{S}^\rho(n)(\Omega; \mathbb{R}^{1 \times d_x})$ and $y^n = (y^n_t)_{t \in T} \in \mathcal{S}^\rho(n)(\Omega; \mathbb{R})$ as the $\mathcal{F}_t$-conditional expectations of $\Xi^\pi$ and $\mathbb{D} \rho(C^n)$, which now by [36, Corollary 2.44] satisfy for all $t \in T$

$$\Lambda^n_t := \mathbb{E}[\Xi^\pi | \mathcal{F}_t] = \mathbb{E}[\Xi^\pi] + \int_0^t \xi^n_s \cdot \, dw_s. \tag{50}$$

$$y^n_t := \mathbb{E}[\mathbb{D} \rho(C^n) | \mathcal{F}_t] = \mathbb{E}[\mathbb{D} \rho(C^n)] + \int_0^t z^n_s \cdot \, dw_s. \tag{51}$$

We note that if $p = 1$, in the application of the martingale representation theorem we may instead pick an arbitrary $q \in (1, \infty)$ instead of $p/(p - 1)$.

We next define the processes $y^n = (y^n_t)_{t \in T}$, and $z^n = (z^n_t)_{t \in T}$ such that for all $t \in T$

$$y^n_t := (\Lambda^n_t - y^n_t \xi^n_t) V_T^n \tag{52}$$

$$z^n_t := (\xi^n_t - z^n_t \xi^n_t) V_T^n - y^n_t S_T^n. \tag{53}$$

The process $(y^n_t, z^n_t, x^n_t, y^n_t, z^n_t)_{t \in T}$ solves (29). This is already shown above for $(y^n_t, z^n_t, x^n_t, y^n_t, z^n_t)_{t \in T}$. To show that $(y^n_t, z^n_t)_{t \in T}$ satisfies its respective backward stochastic differential equation, we apply Itô’s lemma to $y^n$ to obtain

$$dy^n_t = -\nabla_x H(t, x^n_t, y^n_t, y^n_t, z^n_t) \, dt + z^n_t \cdot \, dw_t.$$

The terminal condition $y^n_T = \mathbb{D} \rho(C^n) \nabla_x g(x^n_T)$ is readily verified from the definitions of $y^n_T$ and $\Lambda^n_T$.

We can now establish that $y^n \in \mathcal{S}^\rho(n)(\Omega; \mathbb{Y})$ and $z^n \in \mathcal{H}^\rho(\rho^{-1})(\Omega; \mathbb{Z})$, or in fact, a slightly strengthened version thereof. Let $\tilde{p} = \bar{p}$, and set $\tilde{q} := 1/(1/\tilde{p} - 1/\bar{p})$. Consider first the process $y^n_T$, and let us estimate the terms in its definition (50) individually. For the first term we obtain

$$\mathbb{E}\left[\sup_{t \in T} |y^n_t V_T^n|^{\rho/(\rho - 1)}\right] < \infty$$

for all $t \in T$ using an application of Hölder’s inequality. The second term is treated similarly

$$\mathbb{E}\left[\sup_{t \in T} |y^n_t Q^n_t V_T^n|^{\rho/(\rho - 1)}\right] \leq CE\left[\sup_{t \in T} |y^n_t|^{\rho(\rho - 1)/\rho}ight]^{1/\rho} \tag{54}$$

where $\rho_1, \rho_2, \rho_3 \in (1, \infty)$ and $\rho_1^{-1} + \rho_2^{-1} + \rho_3^{-1} = 1$. We select $\rho = [p/(p - 1)]/[\rho/(\rho - 1)]$ and $\rho = \rho/\rho/(\rho - 1)],$ these are the largest choices still ensuring the finiteness of the first and second factors, respectively. Now $r_3$ may be arbitrarily large, and we then only need to verify that $\rho_1^{-1} + \rho_2^{-1} < 1,$ which is straight-forward. We now have that $y^n \in \mathcal{S}^\rho(n)(\Omega; \mathbb{Y})$ and $z^n \in \mathcal{H}^\rho(n)(\Omega; \mathbb{Z})$.

Finally, we prove (30). The solution of (22) can be written using the processes $U^n$ and $V^n$ as

$$\delta^n_t = U^n_t \int_0^t V^n_s \left[b(s, x^n_s, q_s - \pi_s) - S^n_s \cdot \sigma(s, x^n_s, q_s - \pi_s)\right] \, ds$$

$$+ U^n_t \int_0^t V^n_s \sigma(s, x^n_s, q_s - \pi_s) \, dw_s. \tag{55}$$

Consider next the processes $\gamma^n_t = (\gamma^n_t)_{t \in T} \in \mathcal{X}^\rho(\rho^{-1})(\Omega; \mathbb{Y})$ and $\gamma^n_t = (\gamma^n_t)_{t \in T} \in \mathcal{X}^\rho(\rho^{-1})(\Omega; \mathbb{Y})$. The right-hand side of the inequality of (28) equals the left-hand side of the inequality of (28)

$$\mathbb{E}[\Lambda^n_T \gamma^n_T + y^n_T \gamma^n_T]$$

$$\mathbb{E}\left[\mathbb{D} \rho(C^n) \nabla_x g(x^n_T) \delta^n_T + \mathbb{D} \rho(C^n) \delta^n_T \right]. \tag{56}$$

To compute the left-hand side of the above, we differentiate $\Lambda^n_T \gamma^n_T + y^n_T \gamma^n_T$, evaluate the result at $t = T$, and take the expectation, and use (53) to get

$$\mathbb{E}\left[\mathbb{D} \rho(C^n) \nabla_x g(x^n_T) \delta^n_T + \mathbb{D} \rho(C^n) \delta^n_T \right]$$

$$= \mathbb{E}\left[\int_0^T \left\{y^n_s \nabla_x \left[b(s, x^n_s, q_s - \pi_s) + y^n_s \sigma(s, x^n_s, q_s - \pi_s)\right]ight.\right.$$

$$+ \left.\text{Tr} \left[z^n_s \sigma(s, x^n_s, q_s - \pi_s)\right]\right] \, ds \right] + \mathbb{M}^{\rho, q} \tag{57}$$

where

$$\mathbb{M}^{\rho, q} := \mathbb{E}\left[\int_0^T \left(z^n_T + y^n_T S^n_T\right) \delta^n_t \right]$$

$$+ y^n_T \sigma(t, x^n_t, q_t - \pi_t) \, dw_t$$

$$+ \int_0^T \left(z^n_T \delta^n_t \right) \, dw_t.$$
a straight-forward application of Hölder’s inequality. If \( p = 1 \), the last term is somewhat special, since we cannot apply Hölder’s inequality in the same way. However, the estimate of norm of \( \delta^\pi \cdot \alpha \) in (24) is slightly stronger than that of \( \delta^\pi \cdot \alpha \) precisely to accommodate this edge case. Therefore, \( M^\pi \cdot \alpha = 0 \), and the proof is complete.

**Proof of Lemma 29:** Let \( q \in \mathcal{D}(b, a, \nu) \) be arbitrary. We have from the convexity of \( \rho \), convexity of \( g \), and the construction of the process \((y^n_t, y^n_{-1})_{t \in T}\), that

\[
\rho(C^n) - \rho(C^\alpha) \leq \mathbb{E} \left[ y^n_T - y^n_T + y^n_T(x^n_T - x^n_T) \right].
\]

We can evaluate the above expectation by first differentiating \( y^n_T(x^n_T - x^n_T) + y^n_T(x^n_T - x^n_T) \), using the \( y \) and \( x \) differential equations (3), (19), (29), evaluating the integrals at \( T \), and then taking expectations so that

\[
\mathbb{E} \left[ y^n_T(x^n_T - x^n_T) + y^n_T(x^n_T - x^n_T) \right] = \mathbb{E} \left[ \int_0^T \left( -\nabla \chi \phi (t, x^n, y^n_t, y^n_t, x^n_\pi, \pi_t) (x^n_T - x^n_T) + H(t, x^n_t, y^n_t, y^n_t, x^n_\pi, \pi_t) \right) dt \right].
\] (55)

The expectation of the integrals against the Brownian motion have vanished, since the integrands are in \( \mathcal{H}_2(\Omega; \mathbb{R}) \).

Let us denote

\[
h_t(x, q) := H(t, x, y^n_t, y^n_t, y^n_t, q, q) \]

\[
\nabla h_t(x, q) := \nabla \phi (t, x^n, y^n_t, y^n_t, x^n_\pi, x^n_\pi, q) \]

\[\forall (t, x, q) \in T \times \chi \pi \times \mathcal{D}(A) \times \mathcal{D}(A).\]

Using the joint convexity and differentiability of \( H \) we then get

\[
\nabla h_t(x_0, \pi_0)(x_0 - x_0) + h_t(x_0, \pi_0) - h_t(x_0, \pi_0) - h_t(x_0, \pi_0)
\]

\[\leq h_t(x_0, \pi_0) - h_t(x_0, \pi_0)\]

for all \((t, x_0, \chi_0, \nu_0) \in T \times \chi \pi \times \mathcal{D}(A) \times \mathcal{D}(A).\)

Using the above, along with the assumption that \( \pi \) minimizes \( \eta \rightarrow h_t(x^n_T, \pi) \) for \( \mathbb{P} \times dt \)-almost every \((\omega, t) \in \Omega \times T \), we have that

\[
h_t(x^n_T, q_t) - h_t(x^n_T, \pi_t) \geq \nabla h_t(x^n_T, \pi_t)(x^n_T - x^n_T) \]

\[\mathbb{P} \times dt \)-always. Applying this estimate in (55), we get

\[
\mathbb{E} \left[ y^n_T(x^n_T - x^n_T) + y^n_T(x^n_T - x^n_T) \right] \leq 0,
\]

implying that \( \rho(C^n) - \rho(C^\alpha) \leq 0 \), and the proof is complete.

\[\mathbb{D} \text{. Proofs for Section V}\]

**Proof of Lemma 31:** 1) Note that \( X = \mathbb{E}[X] \) if and only if \( X \) is almost surely constant. The Fréchet derivative of the first term in (31), the expectation, is clearly \( \mathbb{D} \mathbb{E}[X] = 1 \). Focusing then on the derivative of the second, norm term, we first note that the derivative of \( \mathcal{L}(\Omega; \mathbb{R}) \) \( X \rightarrow ||X||_2 \in \mathbb{R} \rightarrow ||X||_2^2 \in X \in \mathcal{L}(\Omega; \mathbb{R}) \), which suggests that for all \( Y \in \mathcal{L}(\Omega; \mathbb{R}) \)

\[
\mathbb{D} \mathbb{E}[X] \mathbb{E}[X] = \left( \frac{X - \mathbb{E}[X]}{||X - \mathbb{E}[X]||_2}, Y - \mathbb{E}[Y] \right).
\]

This can be verified through a direct calculation. Noting that \( \langle X - \mathbb{E}[X], 1 \rangle = 0 \), (33) follows. The nondifferentiability at almost surely constant random variables follows from the positive homogeneity and translation invariance of \( \rho \). (34) is easily found from the form of the Fréchet derivative of (33).

2) We first note that, for all \( x, h \in \mathbb{R} \)

\[
(x + h)_{x+} - (x)_{x+} = h \int_0^1 U(x + h) \xi \, d\xi
\]

\[
U(x + h) - U(x) = h \int_0^1 U'(x + h) \xi \, d\xi.
\] (56)

Since \( U'(x) = e^{-1} - 1/(1 + e^{x/\epsilon})^2 \in (0, 1/(4\epsilon)] \forall x \in \mathbb{R} \), from the second equality it follows that

\[
|U(x + h) - U(x)| \leq \frac{|h|}{4\epsilon} \forall x, h \in \mathbb{R}.
\] (57)

Now (35) can be verified by a direct calculation using (56) and (57) which give

\[
\left| \rho_{\text{MD}}(X + H) - \rho_{\text{MD}}(X) - \mathbb{E}[D\rho_{\text{MD}}(X)H] \right| \leq \frac{\beta}{8c} \|H - \mathbb{E}[H]\|_2^2
\]

and the given form of the Fréchet derivative is proven. The form of the \( \mathcal{L} \)-derivative is easily verified from \( \mathcal{D}\rho_{\text{MD}}(X) \).

3) It suffices to show that the limits in the directional derivatives, \( \lim_{t \rightarrow 0} \mathbb{E}^{\text{Ent}}(X + tY - \rho_{\text{Ent}}(X)) / t \), are attained uniformly over \( Y \in \mathcal{L}^\infty(\Omega; \mathbb{R}) \), such that \( \|Y\|_\infty = 1 \). We now have that \( \mathbb{E}[e^{\theta(x + tY)}] - \mathbb{E}[e^{\theta X}] = \mathbb{E}[e^{\theta X} + o(\epsilon)] \) by Taylor series expanding the exponential and using the fact \( Y(\omega) \leq \|Y\|_\infty \) almost everywhere. By the chain rule of differentiation, \( \mathcal{D}\rho_{\text{Ent}}(X) = e^{\theta X} / \mathbb{E}[e^{\theta X}] \) follows. The \( \mathcal{L} \)-derivative is similarly easily found.

**Proof of Proposition 33:** We first note that Assumption 3 are easily verified for the stochastic differential equations of Problem \( \mathcal{P}_\phi \). Suppose \( \pi \in \mathbb{Q}^\phi(b, a, \nu, \phi) \) is \( \mathcal{P}_\phi \)-optimal. By Theorem 21, we know that there exists a process \((y^n_t, y^n_0, z^n_0, \pi^n_0)_{t \in T}\), where by (14), (15), (18), \( y^n_t = \mathbb{E}[D\rho_{\text{MD}}(X + y^n_T)|F_t] \), \( dy^n_t = \pi^n_t dw_t \), and \( d\pi^n_t = \pi^n_t dw_t \), \( st = 0 \). This yields (39) and (40). By the uniqueness of solutions of Lipschitz backward differential equations, see, e.g., [36, Th. 5.17], we have that \( y^n_t = -y^n_0 \) and \( z^n_t = -z^n_0 \) for all \( t \in T \). \( \mathcal{P}_\phi \)-almost surely. From the assumption of positivity of \( \mathbb{D}\rho_{\text{MD}}(\cdot) \) we infer that \( y^n_0 > 0 \) and \( y^n_0 < 0 \) for all \( t \in T \). We can thus assume in the following that \( \bar{Y} = \mathbb{R} \lt 0 \) and \( \bar{Y} = \mathbb{R}_{>0} \).

Clearly, a minimizer of (16) can be found from Dirac \( \delta \)-measures (use either Jensen’s inequality, or Theorem 9), and the given form is readily verified. Since \( H \) and the terminal cost function are convex in \( (x, \phi) \) and \( x \), respectively, and \( \rho \) is convex by the assumption of the proposition, Assumption 20 holds and by Theorem 21 the above properties are also sufficient for \( \phi_t = \phi(y^n_t, z^n_t) \) to be \( \mathcal{P}_\phi \)-optimal.
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