1 Introduction

Strongly compact cardinals were introduced by Tarski to generalize the fundamental compactness theorem of first-order logic to infinitary logic.

**Definition.** An uncountable cardinal $\kappa$ is *strongly compact* if for any $\mathcal{L}_{\kappa,\omega}$ theory $\Sigma$, if every $<\kappa$-sized subtheory of $\Sigma$ has a model, then $\Sigma$ itself has a model.

It was soon established that every strongly compact cardinal is measurable. An immediate question is whether the least strongly compact cardinal is strictly larger than the least measurable cardinal. To motivate our work we describe Solovay’s ultimately unsuccessful attack on this problem, Menas’s refutation of Solovay’s conjecture, and Magidor’s remarkable independence result.

In the late 1960s, Solovay initiated a program to answer this question positively. He began by reducing the notion of strong compactness to its pure set theoretic content.

**Definition.** Suppose $\kappa$ is a cardinal and $A$ is a set. Then $P_\kappa(A)$ denotes the collection of subsets of $A$ of cardinality less than $\kappa$.

An ultrafilter $\mathcal{U}$ on $P_\kappa(A)$ is *fine* if for all $a \in A$, for $\mathcal{U}$-almost all $\sigma \in P_\kappa(A)$, $a \in \sigma$.

**Theorem** (Solovay). A cardinal $\kappa$ is strongly compact if and only if $P_\kappa(A)$ carries a fine $\kappa$-complete ultrafilter for every set $A$.

Solovay’s approach to Tarski’s problem was motivated by the well-known fact that if $\kappa$ carries a $\kappa$-complete uniform ultrafilter (or equivalently if $P_\kappa(\kappa)$ carries a $\kappa$-complete fine ultrafilter), then $\kappa$ carries a normal ultrafilter.
Solovay observed that many of the notions of infinitary combinatorics generalize from their classical context on a regular uncountable cardinal $\kappa$ to a higher order analog on $P_\kappa(A)$ for any set $A$. Take for example the notion of a normal ultrafilter:

**Definition.** Suppose $\kappa$ is a cardinal and $A$ is a set. A fine ultrafilter $\mathcal{U}$ on $P_\kappa(A)$ is *normal* if any function $f : P_\kappa(A) \to A$ such that $f(\sigma) \in \sigma$ for $\mathcal{U}$-almost all $\sigma$ assumes a constant value for $\mathcal{U}$-almost all $\sigma$.

Given the analogy between $\kappa$ and $P_\kappa(A)$, it was natural to conjecture that an arbitrary fine $\kappa$-complete ultrafilter on $P_\kappa(A)$ could be massaged into a *normal* fine $\kappa$-complete ultrafilter on $P_\kappa(A)$. Solovay therefore introduced the notion of a supercompact cardinal.

**Definition.** A cardinal $\kappa$ is *supercompact* if $P_\kappa(A)$ carries a normal fine $\kappa$-complete ultrafilter for every set $A$.

Solovay conjectured that every strongly compact cardinal is supercompact. His conjecture would have implied a positive answer to Tarski’s question, since the set of measurable cardinals less than a supercompact cardinal $\kappa$ is stationary in $\kappa$. But Solovay’s conjecture is false assuming large cardinal hypotheses.

**Theorem (Menas).** Every measurable limit of strongly compact cardinals is strongly compact.

Using Menas’s theorem, it is easy to see that if $\kappa$ is the least measurable limit of strongly compact cardinals, the set of measurable cardinals below $\kappa$ is nonstationary, and therefore $\kappa$ is not supercompact, even though $\kappa$ is strongly compact, again by Menas’s theorem.

While this result implies that Solovay’s program cannot be carried out naively, the true subtlety of Tarski’s question was not understood until Magidor’s independence results.

**Theorem (Magidor).** Suppose $\kappa$ is strongly compact. Then there is a partial order $\mathbb{P} \subseteq V_{\kappa+1}$ that preserves the strong compactness of $\kappa$ while forcing that $\kappa$ is the least measurable cardinal.

Therefore one cannot prove that the least strongly compact cardinal is strictly larger than the least measurable cardinal. On the other hand, Magidor also showed that one cannot refute Solovay’s conjecture at the least strongly compact cardinal. An extension of this result that is relevant to us is due to Kimchi-Magidor:
Theorem (Kimchi-Magidor). There is a definable class partial order $P$ that preserves all supercompact cardinals while forcing that every strongly compact cardinal is either supercompact or a measurable limit of supercompact cardinals.

In other words, Menas’s theorem provides the only provable counterexamples to Solovay’s conjecture.

The approach to Tarski’s problem and Solovay’s conjecture taken here is inspired by Solovay’s, except that we are informed by Menas’s counterexamples, and we are forced by Magidor’s theorems to adopt a new principle.

The Ultrapower Axiom (UA) is a structural principle in the combinatorics of countably complete ultrafilters that holds in all known canonical inner models. If the inner model program reaches an inner model with a supercompact cardinal, then work of Woodin [1] suggests that UA must be consistent with all large cardinal axioms.

The axiom itself is simple enough, even though the reasons for believing it to be consistent with a strongly compact cardinal are subtle.

Ultrapower Axiom. For all countably complete ultrafilters $U$ and $W$, there exist $W_\ast \in M_U$ and $U_\ast \in M_W$, countably complete ultrafilters in their respective models, such that $M^{M_U}_{W_\ast} = M^{M_W}_{U_\ast}$ and $j^{M_U}_{W_\ast} \circ j_U = j^{M_W}_{U_\ast} \circ j_W$.

The Ultrapower Axiom is motivated by the Comparison Lemma of inner model theory. In fact the Comparison Lemma implies UA by a general argument. The Comparison Lemma is the central feature of modern inner model theory, so if one could rule out the Ultrapower Axiom from a supercompact cardinal, one would in fact rule out any sort of inner model theory for supercompact cardinals.

The theory of countably complete ultrafilters under ZFC alone is buried in independence results (for example, see [2], [3], [4], [5], [6]), but also contains hints of a deeper underlying structure ([7], [8]). Assuming UA, this structure comes to the surface, and there is quite a bit one can prove. We sketch some general facts in the theory of countably complete ultrafilters assuming UA in Section 3. The main ingredient is a wellfounded partial order on the class of countably complete uniform ultrafilters on ordinals generalizing the Mitchell order. This order is called the seed order and denoted $<_S$. The linearity of the seed order is equivalent to UA.

The focus of the rest of the paper is the theory of strong compactness and supercompactness under UA. Assuming UA, there is in fact a generalization of the argument producing a normal ultrafilter on $\kappa$ from a $\kappa$-complete ultrafilter on $\kappa$ that brings Solovay’s ideas described above to fruition. This is the subject of Section 4 whose main theorem is below:
Theorem (UA). Suppose $\delta$ is a regular cardinal and $\kappa \leq \delta$ is the least $\delta$-strongly compact cardinal. Then the $<_{S}$-least uniform countably complete ultrafilter on $\delta$ witnesses that $\kappa$ is $<_{\delta}$-supercompact.

If $\delta$ is not strongly inaccessible then in fact the $<_{S}$-least ultrafilter $U$ must be fully $\delta$-supercompact. An interesting question is whether this holds in general. We conjecture that the answer is no. This discussed in Section 5.5.

In particular, we have the following theorem:

Theorem (UA). The least strongly compact cardinal is supercompact.

What about the other strongly compact cardinals? This is the subject of Section 6 whose main result is that under the Ultrapower Axiom, the Kimchi-Magidor consistency result becomes a theorem.

Theorem (UA). Every strongly compact cardinal is either supercompact or a measurable limit of supercompact cardinals.

The full characterization of strongly compact cardinals above cannot be proved without first characterizing the least one by a completely different argument. One essentially propagates the supercompactness of the first strongly compact cardinal to the others. This dynamic hints at the special nature of the first supercompact cardinal in inner model theory, identified first by Woodin in [9] and [1].

2 Preliminaries

2.1 Notation and Conventions

We set up some conventions for discussing countably complete ultrafilters.

Definition 1. An ultrafilter $U$ on an ordinal $\delta$ is uniform if for all $\xi < \delta$, $\delta \setminus \xi \in U$. The set of uniform countably complete ultrafilters on $\delta$ is denoted by $\text{Un}_\delta$. The class of uniform countably complete ultrafilters is denoted by $\text{Un}$.

Definition 2. For $U \in \text{Un}$, we denote by $\text{sp}(U)$ the space of $U$, which is unique ordinal $\delta$ such that $U \in \text{Un}_\delta$.

Equivalently, $\text{sp}(U)$ is the unique $\delta$ such that $\delta \in U$.

In the case that $\delta$ is regular, the notion of uniformity defined above is the usual one. In the case that $\delta$ is a singular cardinal, there are two definitions of uniformity in use. We have chosen the weaker one, but we occasionally use the stronger one.
Definition 3. An ultrafilter $U$ on a set $X$ is strongly uniform if for all $A \in U$, $|A| = |X|$.

According to our official definition of uniformity, a principal ultrafilter can be uniform.

Definition 4. For any ordinal $\alpha$, $P_\alpha$ denotes the uniform principal ultrafilter concentrated at $\alpha$.

Thus $sp(P_\alpha) = \alpha + 1$.

We use standard notation for ultrapowers.

Definition 5. If $U$ is an ultrafilter, then $j_U : V \to M_U$ denotes the ultrapower of $V$ by $U$.

More generally, if $N$ is an inner model and $U$ is an $N$-ultrafilter, then $j_U^N : N \to M_U^N$ denotes the ultrapower of $N$ by $U$ using only functions in $N$.

We will use the notation $j_U^N$ and $M_U^N$ even when $U \notin N$.

Every ultrapower considered in this paper will be wellfounded and therefore identified with its transitive collapse.

We will be interested in limits of ultrafilters, occasionally in a slightly more general sense than the usual one.

Definition 6. Suppose $U$ is an ultrafilter and $W_\ast$ is a uniform $M_U$-ultrafilter on $X_\ast$. Suppose $X$ is such that $X_\ast \subseteq j_U(X)$. The $U$-limit of $W_\ast$ on $X$ is the ultrafilter

$$U^-(W_\ast, X) = \{ A \subseteq X : j_U(A) \cap X_\ast \in W_\ast \}$$

The main novelty of this definition is that we do not require $W_\ast \in M_U$.

This is useful in the proof of Theorem 110.

We single out two special cases of Definition 6 in which there is a canonical choice for the underlying set of the limit ultrafilter. These are the only cases we will really consider in this paper.

Definition 7. Suppose $U$ is an ultrafilter and $W_\ast$ is a uniform $M_U$-ultrafilter on $\delta_\ast$. Then the $U$-limit of $W_\ast$, denoted $U^-(W_\ast)$, is the $U$-limit of $W_\ast$ on $\delta$ where $\delta$ is least such that $\delta_\ast \leq j_U(\delta)$.

Note that $\delta$ is chosen so that $U^-(W_\ast)$ is uniform.

The second special case of Definition 8 generalizes the first one, but we will use it much less (only in the proof of Theorem 110).

Definition 8. Suppose $U$ is an ultrafilter and $W_\ast$ is a fine $M_U$-ultrafilter on $P_{\delta_\ast}(\delta_\ast)$. Then the $U$-limit of $W_\ast$, denoted $U^-(W_\ast)$, is the $U$-limit of $W_\ast$ on $P_{\delta}(\delta)$ where $\delta$ is least such that $\delta_\ast \leq j_U(\delta)$. 

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Note that $\delta$ is chosen so that $U^-(W_*)$ is a fine ultrafilter on $P_\delta(\delta)$.

It is usually easier to think about limits in terms of ultrapower embeddings, which is possible by the following lemma.

**Lemma 9.** Suppose $U, W \in \text{Un}$ and $U_* \in \text{Un}^{MW}$. Then $U = W^-(U_*)$ if and only if there is an elementary embedding $k : M_U \to M_{U_*}^{MW}$ such that $k \circ j_U = j_{U_*}^{MW} \circ j_W$ and $k([\text{id}]_U) = [\text{id}]_{U_*}$.

### 2.2 The Seed Order

The key to this work is a new order on the class of uniform countably complete ultrafilters.

**Definition 10.** The seed order is defined on $U, W \in \text{Un}$ by setting $U <_S W$ if there is some $U_* \in \text{Un}^{MW}$ with $\text{sp}(U_*) \leq [\text{id}]_W$ such that $U = W^-(U_*)$.

There is a useful characterization of the seed order in terms of elementary embeddings using Lemma 9.

**Corollary 11.** Suppose $U, W \in \text{Un}$. Then $U <_S W$ if and only if there is some $U_* \in \text{Un}^{MW}$ and an elementary embedding $k : M_U \to M_{U_*}^{MW}$ with $k \circ j_U = j_{U_*}^{MW} \circ j_W$ and $k([\text{id}]_U) < j_{U_*}^{MW}([\text{id}]_W)$.

When $U$ is an ultrafilter, the point $[\text{id}]_U$ is sometimes called the seed of $U$, which along with Corollary 11 should explain the name “seed order.”

A more detailed exposition of the seed order will appear elsewhere.

The following technical lemma allows the structure of the seed order in $V$ to be copied into its ultrapowers.

**Lemma 12.** Suppose $U, W, Z \in \text{Un}$ and $U <_S W$. Suppose $W_* \in \text{Un}^{MZ}$ is such that $Z^-(W_*) = W$. Then there is some $U_* <_S W_*$ with $Z^-(U_*) = U$.

We state without proof a very basic fact characterizing the simple relationship between the space of an ultrafilter and the seed order.

**Lemma 13.** Suppose $U, W \in \text{Un}$. If $\text{sp}(U) < \text{sp}(W)$ then $U <_S W$.

As an easy corollary of Lemma 12 and Lemma 13 one can prove the main structural fact about the seed order.

**Theorem 14.** The seed order is a set-like wellfounded strict partial order.
2.3 The Ultrapower Axiom

The Ultrapower Axiom is an abstract comparison principle motivated by the comparison process of inner model theory.

**Ultrapower Axiom.** For any countably complete ultrafilters $U$ and $W$, there exist countably complete ultrafilters $W_*$ and $U_*$ of $M_U$ and $M_W$ respectively such that the following hold:

\[
M_{W_*}^{M_U} = M_{U_*}^{M_W}, \\
J_{W_*}^{M_U} \circ j_U = j_{U_*}^{M_W} \circ j_W
\]

The Ultrapower Axiom holds in all known canonical inner models and is expected to hold in canonical inner models with supercompact cardinals if such models exist.

The key consequence of the Ultrapower Axiom is the linearity of the seed order, which is an immediate consequence of Corollary 11. Moreover the characterization of the seed order in terms of elementary embeddings becomes fully symmetric:

**Proposition 15 (UA).** The seed order is linear. In fact if $U,W \in \text{Un}$, then $U < S W$ if and only if there are countably complete ultrafilters $W_*$ and $U_*$ of $M_U$ and $M_W$ respectively such that the following hold:

\[
M_{W_*}^{M_U} = M_{U_*}^{M_W}, \\
J_{W_*}^{M_U} \circ j_U = J_{U_*}^{M_W} \circ j_W, \\
J_{W_*}^{M_U}(\text{id}_U) < j_{U_*}^{M_W}(\text{id}_W)
\]

By Corollary 34 below, one can further simplify the definition of the seed order under UA by removing the commutativity requirement.

3 Ultrafilter Theory under UA

3.1 Reciprocity

In this section we prove the converse of Proposition 15: the linearity of the seed order implies the Ultrapower Axiom. This shows the equivalence between comparison for ultrafilters on the one hand and a combinatorial generalization of the linearity of the Mitchell order on the other.

The proof introduces the very useful concept of a translation function.
**Definition 16.** Assume the seed order is linear. We associate to each countably complete ultrafilter $U$ a translation function $t_U : Un \to Un^{MU}$ as follows:

For any $W \in Un$, $t_U(W)$ denotes the $<_{MU}$-least $W_* \in Un^{MU}$ such that $U^-(W_*) = W$.

It is convenient to define an operation $\oplus$ with the property that for any $U \in Un$ and $W \in Un^{MW}$, $U \oplus W \in Un$ and $j_U \circ j_W = j_U \circ j_{U \oplus W}$. (The usual ultrafilter sum operation does not have range contained in Un.) There are various ways in which one could do this, and our choice is motivated mostly by the desire that this operation work smoothly with the seed order; see for example Lemma 21.

**Definition 17.** For $\alpha, \beta \in \text{Ord}$, $\alpha \oplus \beta$ denotes the *natural sum* of $\alpha$ and $\beta$, which is obtained as follows:

First write $\alpha$ and $\beta$ in Cantor normal form:

\[
\alpha = \sum_{\xi \in \text{Ord}} \omega^\xi \cdot m_\xi
\]

\[
\beta = \sum_{\xi \in \text{Ord}} \omega^\xi \cdot n_\xi
\]

where $m_\xi, n_\xi < \omega$ are equal to 0 for all but finitely many $\xi \in \text{Ord}$. Then

\[
\alpha \oplus \beta = \sum_{\xi \in \text{Ord}} \omega^\xi \cdot (m_\xi + n_\xi)
\]

In other words one adds the Cantor normal forms of $\alpha$ and $\beta$ as polynomials.

The fact that natural addition is commutative and associative follows easily from the corresponding facts for addition of natural numbers. We mostly need the following triviality:

**Lemma 18.** If $\alpha_0 < \alpha_1$ and $\beta$ are ordinals, then $\alpha_0 \oplus \beta < \alpha_1 \oplus \beta$.

**Definition 19.** If $U \in Un$ and $W_* \in Un^{MU}$ then the *natural sum of $U$ and $W_*$* is the uniform ultrafilter derived from $j_W^{MU} \circ j_U$ using $j_U \circ j_W^{MU} = [\text{id}]_{W_*} \oplus [\text{id}]_{W_*} ([\text{id}])$.

The next lemma says that the natural sum of ultrafilters is Rudin-Keisler equivalent to the usual sum of ultrafilters.

**Lemma 20.** For any $U \in Un$ and $W_* \in Un^{MU}$, $M_{U \oplus W_*} = M_{W_*}$ and $j_U \circ j_W^{MU} = j_{W_*}^{MU} \circ j_U$. 

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Natural sums also interact quite simply with the seed order:

**Lemma 21.** Suppose \( U \in \text{Un} \). Suppose \( W_0, W_1 \in \text{Un}^{\cdot M_U} \). Then \( W_0 <^S \cdot W_1 \) if and only if \( U \oplus W_0 <^S U \oplus W_1 \).

We will use the following theorem whose proof we defer to another paper.

**Theorem 22** (Minimality of Definable Embeddings). Suppose \( M \) and \( N \) are inner models and \( j, i : M \to N \) are elementary embeddings. If \( j \) is definable from parameters over \( M \), then \( j(\alpha) \leq i(\alpha) \) for all ordinals \( \alpha \).

The proof generalizes the Dodd-Jensen Lemma from inner model theory. It is inspired by a similar theorem in [1].

**Theorem 23** (Reciprocity Theorem). Assume the seed order is linear. Then for any uniform countably complete ultrafilters \( U \) and \( W \),

\[
U \oplus t_U(W) = W \oplus t_W(U)
\]

**Proof.** Assume towards a contradiction that \( U \oplus t_U(W) < S W \oplus t_W(U) \).

By Lemma 9 there is an inner model \( N \) admitting an elementary embedding \( k : M_{U \oplus t_U(W)} \to N \) and an ultrapower embedding \( i : M_{W \oplus t_W(U)} \to N \) such that \( k \circ j_{U \oplus t_U(W)} = i \circ j_{W \oplus t_W(U)} \) and

\[
k([id]_{U \oplus t_U(W)}) < i([id]_{W \oplus t_W(U)})
\]

In other words,

\[
k([id]_{U \oplus t_U(W)}^M) \oplus j_{U \oplus t_U(W)}([id]_U^M) < i([id]_{W \oplus t_W(U)}^M) \oplus j_{W \oplus t_W(U)}([id]_U^M)
\]

Claim 1.

\[
i([id]_{U \oplus t_U(W)}^M) \leq k\left(j_{U \oplus t_U(W)}([id]_U^M)\right)
\]

Claim 2.

\[
i\left(j_{W \oplus t_W(U)}([id]_U^M)\right) \leq k\left([id]_{U \oplus t_U(W)}^M\right)
\]

Using Lemma 18 these two claims contradict (1), so the assumption that \( U \oplus t_U(W) < S W \oplus t_W(U) \) was false.

**Proof of Claim** 1 Let \( U_* \) be the \( M_W \)-ultrafilter derived from \( i \circ j_{W \oplus t_W(U)}^M \) using \( k\left(j_{U \oplus t_U(W)}([id]_U^M)\right) \). Let \( h : M_{U_*} \to N \) be the factor embedding. Note that \( W^{-}(U_*) = U \): this is an easy calculation using Lemma 9, noting that there is an elementary embedding \( M_U \to M_{U_*}^{M_W} \) witnessing the hypotheses of Lemma 9, namely \( h^{-1} \circ k \circ j_{U \oplus t_U(W)}^{M_U} \).

If \( h([id]_{U_*}^M) < i([id]_{W \oplus t_W(U)}^M) \), then \( U_* <^S W \oplus t_W(U) \) by Corollary 11 contrary to the minimality of \( t_W(U) \). Thus \( i([id]_{W \oplus t_W(U)}^M) \leq h([id]_{U_*}^M) = k\left(j_{U \oplus t_U(W)}([id]_U^M)\right) \), as desired.

\( \square \)
Proof of Claim 2. Let $h : M_W \rightarrow M_{t_W(U)}^{M_U}$ be the elementary embedding given by Lemma 9. Then $k \circ h([id]_W) = k([id]_{t_W(U)}^{M_U})$. Since $i \circ j_{t_W(U)}^{M_W}$ and $k \circ h$ are elementary embeddings $M_W \rightarrow N$, and since $i \circ j_{t_W(U)}^{M_W}$ is definable from parameters over $M_W$,

$$i \circ j_{t_W(U)}^{M_W} \upharpoonright \text{Ord} \leq k \circ h \upharpoonright \text{Ord}$$

by Theorem 22. In particular, $i \circ j_{t_W(U)}^{M_W}([id]_W) \leq k \circ h([id]_W) = k([id]_{t_W(U)}^{M_U})$. □

Similarly we cannot have $W \oplus t_W(U) < S U \oplus t_U(W)$. By the linearity of the seed order, $U \oplus t_U(W) = W \oplus t_W(U)$, finishing the proof of the Reciprocity Theorem. □

An immediate corollary of Lemma 20 and Theorem 23 is the following:

**Corollary 24.** The following are equivalent:

1. The seed order is linear.
2. The Ultrapower Axiom holds.

Regarding the translation functions $t_U$, we also have:

**Proposition 25** (UA). For any countably complete ultrafilter $Z$, the function $t_Z : (\text{Un}, <_S) \rightarrow (\text{Un}^Z, <_{M_Z}^Z)$ is order preserving.

**Proof.** Suppose $U,W \in \text{Un}$ and $U <_S W$. By Lemma 12, $Z^{-1}(t_Z(W)) = W$, there is some $U_* <_{M_Z} t_Z(W)$ such that $Z^{-1}(U_*) = U$. By the minimality of $t_Z(U)$, we have $t_Z(U) \leq_{M_Z} U_*$. By the transitivity of the seed order, $t_Z(U) <_{M_Z} t_Z(W)$, as desired. □

### 3.2 Divisibility

**Definition 26.** Suppose $U$ and $W$ are countably complete ultrafilters. We say $U$ divides $W$ and write $U \leq_D W$ if there is some $W_* \in \text{Un}^{M_U}$ such that $M_W^{M_U} = M_W$ and $j_{W_*}^{M_U} \circ j_U = j_W$. 

The division order is sometimes called the Rudin-Frolik order, although that name is often reserved for a suborder of the division order. We continue to work with uniform ultrafilters on ordinals here, but notice that many of the facts we prove in this section are invariant under Rudin-Keisler equivalence.

The division order has a number of obvious combinatorial equivalents, one of which we put down below.

**Lemma 27.** Suppose $U, W \in \text{Un}$. Then $U$ divides $W$ if and only if there is some $W_* \in \text{Un}^{M_U}$ such that $U \oplus W_* \equiv W$.

The factor ultrafilter $W_*$ is unique up to equivalence:

**Lemma 28.** Suppose $U, W \in \text{Un}$. Then there is at most one internal embedding $i : M_U \to M_W$ such that $i \circ j_U = j_W$.

**Proof.** We may assume $U \in \text{Un}$. Such an embedding $i$ is determined by its values on $j_U[V] \cup \{[\text{id}]_U\}$. Any two such embeddings agree on $j_U[V]$ by the requirement $i \circ j_U = j_W$. Moreover they agree on $\{[\text{id}]_U\}$ by Theorem 22. 

Similarly, we have an absoluteness fact for division.

**Lemma 29.** Suppose $U \in \text{Un}$ and $D_*, W_* \in \text{Un}^{M_U}$. Then $D_* \leq_D W_*$ if and only if $U \oplus D_* \leq_D U \oplus W_*$. 

**Proof.** The only nontrivial part is proving that if $i : M_U \oplus M_{U \oplus W_*} \to M_{D_* \oplus W_*}$ is an internal ultrapower embedding with $i \circ j_{U \oplus W_*} = j_{U \oplus W_*}$ then $i \circ j_{D_*} = j_{W_*}$. It suffices to show that $i(j_{M_U}([\text{id}]_U)) = j_{M_U}([\text{id}]_U)$. This follows from Theorem 22. 

Under UA, the division order is closely related to the translation functions of the previous subsection.

**Definition 30 (UA).** For $U, W \in \text{Un}$, let $U \lor W$ denote $U \oplus t_U(W) = W \oplus t_W(U)$.

We chose this notation because $U \lor W$ is the least upper bound of $U$ and $W$ in the division order.

**Theorem 31 (UA).** Suppose $U, W, Z \in \text{Un}$. Then $U, W \leq_D Z$ if and only if $U \lor W \leq_D Z$. 

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Proof. For ease of notation let $D = U \lor Z$.

We claim that $t_Z(D)$ is principal in $M_Z$. To see this, it suffices to show that $[\text{id}]_{i_Z(D)}^{M_Z} \in \text{ran } j_{i_Z(D)}^{M_Z}$. Let $i_U : M_U \to M_Z$ and $i_W : M_W \to M_Z$ be internal ultrapower embeddings witnessing that $U, W$ divide $Z$. Note that

$$j_{t_Z(D)}^{M_Z} \circ i_U \upharpoonright \text{Ord} = j_{i_D(Z)}^{M_D} \circ j_{i_U(W)} \upharpoonright \text{Ord}$$

by Theorem 22. Hence

$$j_{i_D(Z)}^{M_D} (j_{i_U(W)} ([\text{id}]_U)) \in \text{ran } j_{t_Z(D)}^{M_Z}$$

Similarly,

$$j_{i_D(Z)}^{M_D} (j_{i_W(U)} ([\text{id}]_W)) \in \text{ran } j_{t_Z(D)}^{M_Z}$$

By Theorem 23,

$$[\text{id}]_{i_Z(D)}^{M_Z} = j_{i_D(Z)}^{M_D} ([\text{id}]_D) = j_{i_D(Z)}^{M_D} \left( j_{i_U(W)}^{M_U} ([\text{id}]_U) \oplus j_{i_W(U)}^{M_W} ([\text{id}]_W) \right)$$

Hence $[\text{id}]_{i_Z(D)}^{M_Z} \in \text{ran } j_{t_Z(D)}^{M_Z}$, so $t_Z(D)$ is principal in $M_Z$, as claimed.

It follows by Theorem 23 that

$$D \oplus t_D(Z) = Z \oplus t_D(D) \equiv Z$$

Hence $D \leq_D Z$ as desired. \qed

Similarly we can characterize the $M_{U \lor W}$-ultrafilters that belong to $M_{U \lor W}$:

**Corollary 32** (UA). Suppose $U, W \in \text{Un}$ and $F$ is a countably complete $M_{U \lor W}$-ultrafilter. Then $F \in M_{U \lor W}$ if and only if $F \in M_U \cap M_W$.

**Proof.** For ease of notation let $D = U \lor Z$. We may assume without loss of generality $F$ is $M_D$-uniform on some ordinal.

Clearly $F \in M_D$ implies $F \in M_U \cap M_W$.

Conversely assume $F \in M_U \cap M_W$. Then $U$ and $W$ divide $D \oplus F$ via $j_{M_D}^{F_D} \circ j_{i_U(W)}$ and $j_{M_D}^{F_D} \circ j_{i_W(U)}$. Hence $D$ divides $D \oplus F$.

Let

$$i : M_D \to M_{F_D}$$

be an internal ultrapower embedding witnessing that $D$ divides $D \oplus F$. We must verify that $i = j_{F_D}^{M_D}$. Since $i \circ j_D = j_{F_D}^{M_D} \circ j_D$ by the definition of division,
it suffices to show that \( i([\text{id}]_D) = j^M_F([\text{id}]_D) \). For this it is enough to show that

\[
i(j^M_{t_W(U)}([\text{id}]_U)) = j^M_{t_W(U)}([\text{id}]_U)
\]

\[
i(j^M_{t_W(U)}([\text{id}]_W)) = j^M_{t_W(U)}([\text{id}]_W)
\]

This follows immediately from the uniqueness of definable embeddings on the ordinals, Theorem 22. □

This has the following useful corollary.

**Theorem 33 (UA).** For \( U, W \in \text{Un} \), the following are equivalent:

1. \( U \) divides \( W \).
2. \( M_W \subseteq M_U \).
3. \( t_W(U) \in M_U \).
4. For some \( U_* \equiv^{M_W} t_W(U) \), \( U_* \in M_U \).
5. \( t_W(U) \) is principal.

**Proof.** This is arranged as a round-robin proof, and the only nontrivial implication is from (4) to (5).

This is a generalization of the proof that a nonprincipal ultrafilter does not belong to its target model. Suppose \( U_* \in M_U \) for some \( U_* \equiv^{M_W} t_W(U) \). Let \( \delta = \text{sp}(U_*) \). Then \( j^{M_W}_{t_W(U)} \upharpoonright \delta \in M_U \) since it can be computed in any inner model containing \( U_* \). Let \( N = M_{U \upharpoonright W} \). Consider the class \( C \) of sequences of ordinals \( s \) of the form \( f \circ (j^{M_W}_{t_W(U)} \upharpoonright \delta) \) for some \( f \in N \). Clearly \( C \) is a definable class of both \( M_U \) and \( M_W \).

We claim that \( C = \text{Ord}^s \cap M_W \). To see this, fix \( s \in \text{Ord}^s \cap M_W \), and we will show \( s \in C \). Working in \( M_W \), choose \( \langle g_\alpha : \alpha < \delta \rangle \) such that \( s_\alpha = [g_\alpha]_{t_W(U)} \) for all \( \alpha < \delta \). Then let \( \langle h_\alpha : \alpha < j^{M_W}_{t_W(U)}(\delta) \rangle = j^{M_W}_{t_W(U)}(\langle g_\alpha : \alpha < \delta \rangle) \), and let \( f = \langle h_\alpha([\text{id}]_{t_W(U)}) : \alpha < j^{M_W}_{t_W(U)}(\delta) \rangle \). Easily \( f \circ (j^{M_W}_{t_W(U)} \upharpoonright \delta) = s \).

Since \( C \) is a definable class of \( M_U \), one can form in \( M_U \) the ultrapower of \( \text{Ord} \) by \( U_* \) using only functions from \( C \). The ultrapower embedding is precisely \( j^{M_W}_{t_W(U)} \upharpoonright \text{Ord} \). But now repeating the argument of the previous paragraph, it follows that \( P(\text{Ord}) \cap M_W \) is a definable subclass of \( M_U \), and hence \( M_W = L(P(\text{Ord}) \cap M_W) \) is a definable subclass of \( M_U \).

In particular, \( j^{M_W}_{t_W(U)} : M_W \to N \) is definable over \( M_U \). Let \( i = j^{M_W}_{t_W(U)} \upharpoonright N \). Then \( i \) is definable both over \( M_U \) and over \( M_W \). Since \( i \) is induced by a
countably complete $\mathcal{N}$-ultrafilter (see Proposition 44), using Corollary 32, it follows that $i$ is an internal ultrapower embedding of $\mathcal{N}$.

Note however that $i$ and $j_{t_{W}(U)}^{M_{W}(U)}$ are definable elementary embeddings of $\mathcal{N}$ with the same target model. Hence they agree on the ordinals by Theorem 22. Assuming towards a contradiction that $t_{W}(U)$ is nonprincipal with completeness $\kappa$, we have

$$\text{crt}(i) = \kappa < j_{t_{W}(U)}^{M_{W}(U)}(\kappa) = \text{crt}(i_{t_{W}(U)}^{M_{W}(U)})$$

a contradiction. Therefore $t_{W}(U)$ is principal, as desired.

The following corollary has the psychological benefit that we never have to check that diagrams of internal ultrapowers commute.

**Corollary 34 (UA).** Suppose $U$ and $U'$ are countably complete ultrafilters such that $M_{U} = M_{U'}$. Then $j_{U} = j_{U'}$.

As a corollary of Theorem 31, we have another reciprocity result:

**Corollary 35 (UA).** Suppose $U, W, D \in \text{Un}$. Then $t_{U}(D)$ divides $t_{U}(W)$ in $M_{U}$ if and only if $t_{W}(D)$ divides $t_{W}(U)$ in $M_{W}$.

**Proof.** This is a calculation using Theorem 31:

$$t_{U}(D) \text{ divides } t_{U}(W) \text{ in } M_{U} \iff U \vee D \text{ divides } U \vee W \iff D \text{ divides } U \vee W \iff W \vee D \text{ divides } U \vee W \iff t_{W}(D) \text{ divides } t_{W}(U) \text{ in } M_{W}$$

In the second and third equivalences we use Theorem 31.

We can also prove that translation functions preserve division.

**Theorem 36 (UA).** Suppose $U$ is a countably complete ultrafilter and $D \leq_{D}$ $W$ are uniform ultrafilters. Then $t_{U}(D)$ divides $t_{U}(W)$ in $M_{U}$.

**Proof.** Note that $D \leq_{D} W \leq_{D} U \vee W$ so $D \leq_{D} U \vee W$. Therefore by Theorem 31 $U \vee D \leq_{D} U \vee W$. This implies $t_{U}(D) \leq_{M_{U}} t_{U}(W)$ by Lemma 29.

As an immediate corollary, Rudin-Keisler equivalent ultrafilters translate to Rudin-Keisler equivalent ultrafilters.

**Corollary 37 (UA).** If $U \equiv U'$ then $t_{W}(U) \equiv_{M_{W}} t_{W}(U')$.

Actually one does not need much machinery to prove the preceding corollary, and it is essentially provable without UA: one can show in ZFC that if $U_{*}$ is minimal such that $W^{-}(U_{*}) = U$ and $f : sp(U) \to \text{Ord}$ is one-to-one on a set in $U$, then setting $U' = f_{*}(U)$ and $U'_{*} = j_{W}(f)_{*}(U_{*})$, $U'_{*}$ is minimal such that $W^{-}(U'_{*}) = U'$.
3.3 The Internal Relation

In this subsection we define a version of the generalized Mitchell order called the internal relation that is compatible with the abstract techniques we have developed so far. The analysis of the internal relation (and similar notions) under UA is instrumental in the analysis of supercompactness.

Once the supercompactness analysis is carried out, however, we will be able to characterize the precise relationship between the internal relation and generalized Mitchell order assuming UA + GCH; they are essentially interdefinable. The details appear in Section 5.5. One can therefore view the internal relation as no more than a transitory definition aiding in the analysis of the Mitchell order. If one is interested in the ZFC theory, this view probably does not hold up.

**Definition 38.** The internal relation is defined on countably complete ultrafilters $U$ and $W$ by setting $U \sqsubseteq W$ if and only if $j_U \upharpoonright M_W$ is an amenable class of $M_W$.

To help the reader get his or her bearings, we include some immediate observations regarding the internal relation.

**Proposition 39.** If $W$ is $\delta$-supercompact and $U \in \text{Un}_{\leq \delta}$ then $U \sqsubseteq W$ if and only if $U <_M W$.

Unlike the generalized Mitchell order, however, assuming there are two measurable cardinals $\kappa_0 < \kappa_1$, the internal relation is neither strict or transitive on nonprincipal ultrafilters (which is why it is not called the internal order). To see this, note that while the internal relation is irreflexive on nonprincipal ultrafilters, there exist pairs of ultrafilters $U, W \in \text{Un}$ with $U \sqsubseteq W$ and $W \sqsubseteq U$: by the following theorem, any $\kappa_0$-complete ultrafilter on $\kappa_0$ and $\kappa_1$-complete ultrafilter on $\kappa_1$ furnish an example.

**Theorem 40 (Kunen).** Suppose $U, W \in \text{Un}$ satisfy $\text{sp}(U) < \text{crt}(W)$. Then $j_{j_U(W)} = j_W \upharpoonright M_U$ and $j_{j_W(U)} = j_U \upharpoonright M_W$. Therefore $U \sqsubseteq W$ and $W \sqsubseteq U$.

The conclusion of Theorem 40 is often abbreviated by saying “$U$ and $W$ commute,” since in particular it implies $j_U \circ j_W = j_W \circ j_U$. The results Theorem 47 and and more powerfully those of Section 5.5 argue that Theorem 40 is essentially the only way in which the internal relation fails to be strict.

On the other hand, restricted to $\text{Un}_\delta$ for a fixed $\delta$, the internal relation is strict and indeed wellfounded. In fact the seed order extends the internal relation on $\text{Un}_\delta$:
Proposition 41. For any ordinal $\delta$, the seed order extends the internal relation on $Un_\delta$.

We give a proof right after Corollary 45. It is not hard to prove Proposition 41 directly, but we are about to introduce notation that makes it transparent.

We introduce an ultrafilter $s_W(U)$ with the property that $U \subseteq W$ if and only if $s_W(U) \in M_W$. (The three functions $j_W, t_W, s_W$ are right inverse to the operation $W^-$.)

Definition 42. Suppose $U \in Un_\delta$ and $W$ is a countably complete ultrafilter. Then the pushforward of $U$ by $j_W$ restricted to $M_W$ is the $M_U$-ultrafilter $s_W(U)$ defined by

$$s_W(U) = \{ A \in P^{M_W}(\sup j_W[\delta]) : j_W^{-1}[A] \in U \}$$

One could easily define $s_W(U)$ for an arbitrary ultrafilter $U$, but we have no need for this here.

Lemma 43. If $U \in Un$ and $W \in Un$, then $s_W(U)$ is the $M_W$-uniform ultrafilter derived from $j_U \upharpoonright M_W$ using $j_M^{M_U}(\text{id}_U)$.

Proof. For $A \in P^{M_W}(j_W(X))$,

$$A \in s_W(U) \iff j_W^{-1}[A] \in U \iff \{ x \in X : j_W(x) \in A \} \in U \iff j_M^{M_U}(\text{id}_U) \in j_U(A)$$

with the last equivalence following from Los’s Theorem.

Proposition 44. For any $U, W \in Un$, $j_{s_W(U)}^{M_W} = j_U \upharpoonright M_W$.

Proof. By the previous theorem, there is an embedding $k : M_{s_W(U)}^{M_W} \to j_U(M_W)$ such that $k([\text{id}]_{s_W(U)}) = j_U(j_W)([\text{id}_U])$ and $k \circ j_{s_W(U)}^{M_W} = j_U \upharpoonright M_W$. It suffices to show that $k$ is surjective. Note that $k[M_{s_W(U)}]$ contains $j_U \circ j_W[V] = j_U(j_W) \circ j_U[V]$ as well as $j_U(j_W)(\text{id}_U)$. Hence

$$j_U(j_W)[M_U] \subseteq k[M_{s_W(U)}]$$

Moreover $k[M_{s_W(U)}]$ contains $k \circ j_{s_W(U)}^{M_W}([\text{id}_W]) = j_U([\text{id}_W]) = [\text{id}_U]_{j_U(W)}$, so

$$[\text{id}]_{j_U(W)} \in k[M_{s_W(U)}]$$

But $j_U(M_W) = M_{j_U(W)}$ is the definable hull of $j_U(j_W)(M_U) \cup \{ [\text{id}]_{j_U(W)} \}$, and it follows that $k$ is surjective.
Corollary 45. For any $U, W \in \text{Un}$, $U \sqsubseteq W$ if and only if $s_W(U) \in M_W$.

Proof. The forwards direction is clear from Lemma 43 and the reverse from Proposition 44.

Proof of Proposition 41. Suppose $U \sqsubseteq W$ belong to $\text{Un}_d$. Then $s_W(U) \in M_W$ and $W^-(s_W(U)) = U$. Moreover $s_W(U) \in M_W$ and $s_W(U) = [\text{id}]_W$. Thus by definition $U <_S W$.

Note that even under ZFC, $U \sqsubseteq W$ implies that $U <_S W$ in the stronger sense of Proposition 15; that is, both embeddings of the comparison are internal.

Proposition 46 (UA). Suppose $U, W \in \text{Un}$ and $U \sqsubseteq W$. Then $t_W(U) = s_W(U)$ and $t_U(W) = j_U(W)$.

Proof. Since $U$ and $W$ divide $U \oplus j_U(W) = W \oplus s_W(U)$, $U \vee W$ divides $U \oplus j_U(W)$.

Let $i : M_{U \vee W} \to M_{U \oplus j_U(W)}$ witness this. By Theorem 22 we have

$$i(j_W^M([\text{id}]_U)) = j_W^M([\text{id}]_U)$$
$$i(j_W^M([\text{id}]_W)) = j_W^M([\text{id}]_W)$$

Thus $[\text{id}]_{U \oplus j_U(W)} \in \text{ran}(i)$, so $i$ is the identity. Now the equations above imply $t_W(U) = s_W(U)$ and $t_U(W) = j_U(W)$.

Using this we can characterize how the internal relation fails to be strict.

Theorem 47 (UA). Suppose $U \sqsubseteq W$ and $W \sqsubseteq U$. Then $j_U(W) = s_U(W)$ and $j_W(U) = s_W(U)$. Consequently, $j_U^M = j_U \upharpoonright M_U$ and $j_W^M = j_W \upharpoonright M_W$.

Proof. By Proposition 46 since $U \sqsubseteq W$, $t_U(W) = j_U(W)$ and $t_W(U) = s_W(U)$. On the other hand since $W \sqsubseteq U$, $t_U(W) = s_U(W)$ and $t_W(U) = j_W(U)$. Equating like terms, $j_U(W) = s_U(W)$ and $j_W(U) = s_W(U)$. By Proposition 44 this implies the last statement of the theorem.

Theorem 47 can be a surprisingly powerful tool in proofs by contradiction. Good examples of this technique are Lemma 85, Lemma 87, and Theorem 94.

Is Theorem 47 provable in ZFC?

We can also prove the converse of Corollary 45.

Proposition 48 (UA). Suppose $U, W \in \text{Un}$ and $t_U(W) = j_U(W)$. Then $U \sqsubseteq W$.
Proof. We claim that $j^M_W : M_W \downarrow W$. Note that $j_U : M_W$ is the unique elementary embedding $i : M_W \rightarrow j_U(M_W)$ such that $i([\text{id}]_W) = j_U([\text{id}]_W)$ and $i \circ j_W = j_U \circ j_W$, since any elementary embedding of $M_W$ is determined by its target model and its values on $j_W[V] \cup \{[\text{id}]_W\}$. We claim that $j^M_W : M_W \downarrow W$ has these same properties, and hence the claim that $j^M_W : M_W \downarrow W$ follows.

Note first that $M^M_W = M^M_{j_U(W)} = j_U(M_W)$

Note second that $j^M_W([\text{id}]_W) = [\text{id}]^M_{j_U(W)} = [\text{id}]^M_{j_U(W)} = j_U([\text{id}]_W)$

Note finally that $j^M_W \circ j_W = j^M_{j_U(W)} \circ j_U = j_U \circ j_W$

This completes the proof. 

Finally we will need the following theorem relating fixed points to the internal relation.

**Theorem 49 (UA).** Suppose $\kappa$ and $\lambda$ are ordinals. Suppose $U$ is an ultrafilter fixing $\lambda$. Suppose $W$ is the $<S$-least ultrafilter such that $j_W(\kappa) \geq \lambda$. Then $U \subseteq W$.

**Proof.** Since $U(\Lambda_W) = W$, by the minimality of $t_U(W)$, $t_U(W) \leq M^W U^W$. We will show that $j_W(W) \leq M^W U^W$, so $t_U(W) = j_W(W)$ and hence $U \subseteq W$ by Proposition 48.

Note that $j^M_{t_U(W)}(j_U(\kappa)) = j^M_{t_U(W)}(j_U(\kappa)) \geq j^M_{j_U(W)}(\lambda) \geq \lambda = j_U(\lambda)$

In $M_U$, $j_U(W)$ is the $<S$-least ultrafilter $W_*$ such that $j_U(W) \geq j_U(\lambda)$. Thus $j_U(W) \leq M^W_U t_U(W)$, as desired.

An important distinction between the internal relation and the Mitchell order is that the internal relation propagates supercompactness.

**Lemma 50.** Suppose $U, W \in Un$ and $U \subseteq W$. Suppose $W$ is $<\kappa$-supercompact. Then

$$\text{Ord}^{<j_U(\kappa)} \cap M_U \subseteq M_W$$

**Proof.** Note that $j_U(M_W) \subseteq M_W$, so $\text{Ord}^{<j_U(\kappa)} \cap M_U = j_U(\text{Ord}^{<\kappa}) \subseteq M_W$. 

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Corollary 51. Suppose $U, W \in \text{Un}$ and $U \sqsubset W$. If $W$ is $<\kappa$-supercompact, $U$ is $\lambda$-supercompact, and $j_U(\kappa) > \lambda$, then $W$ is $\lambda$-supercompact.

We will mostly apply a souped up version of Corollary 51 whose proof uses the Kunen inconsistency theorem, even though we could often just appeal to the somewhat more natural Corollary 51.

Proposition 52. Suppose $U, W \in \text{Un}$ and $U \sqsubseteq W$. Let $\kappa = \text{crt}(U)$. Suppose $W$ is $<\kappa$-supercompact. If $U$ is $\lambda$-supercompact, then $W$ is $\lambda$-supercompact.

Proof. Let $\langle \kappa_n : n < \omega \rangle$ be the critical sequence of $U$. By Kunen’s inconsistency theorem, we can fix $n < \omega$ least such that $\lambda < \kappa_{n+1}$. We prove by induction that $W$ is $<\kappa_m$ supercompact for $m \leq n$: if $W$ is $<\kappa_m$-supercompact and $m < n$, then since $U$ is $<\kappa_{m+1}$-supercompact and $j_U(\kappa_m) = \kappa_{m+1}$, $W$ is $<\kappa_{m+1}$-supercompact by Corollary 51.

Therefore $W$ is $<\kappa_n$-supercompact. Since $j_U(\kappa_n) > \lambda$, one more application of Corollary 51 implies $W$ is $\lambda$-supercompact. \qed

Similarly, the internal relation propagates strong compactness.

Proposition 53. Suppose $U, W \in \text{Un}$ and $U \sqsubseteq W$. If $W$ is $<\kappa$-supercompact, $U$ has the $(\delta, \lambda)$-covering property, and $\lambda < j_U(\kappa)$, then $W$ has the $(\delta, \lambda)$-covering property.

Proof. Suppose $A \subseteq [\text{Ord}]^\delta$. Let $D \in M_U$ be such that $A \subseteq D$ and $|D|^{M_U} = \lambda$. Then $D \in M_W$ by Lemma 50, so it suffices to show that $|D|^{M_W} = \lambda$. But this follows easily from the fact that $\text{Ord}^\delta \cap M_U \subseteq M_W$, again by Lemma 50. \qed

4 The First Strongly Compact Cardinal

4.1 The Least Ultrafilter

Definition 54. A countably complete ultrafilter on a limit ordinal $\delta$ is 0-order if it is weakly normal and concentrates on the set of ordinals that do not carry countably complete uniform ultrafilters.

We begin with a trivial lemma that turns out to be useful.

Lemma 55. Suppose $U_*$ is a 0-order ultrafilter on a singular ordinal $\delta_*$ of cofinality $\delta$. Let $U$ be the weakly normal ultrafilter on $\delta$ derived from $U_*$. Then $U \equiv U_*$, and in fact for any continuous cofinal function $p : \delta \rightarrow \delta_*$, $p_*(U) = U_*$. 

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Proof. Let \( p : \delta \to \delta_* \) be continuous and cofinal. Then

\[
\begin{align*}
  j_{U_*}(p)(\sup j_{U_*}[\delta]) &= \sup j_{U_*}(p)[\sup j_{U_*}[\delta]] \\
  &= \sup j_{U_*}(p) \circ j_{U_*}[\delta] \\
  &= \sup j_{U_*} \circ p[\delta] \\
  &= \sup j_{U_*}[\delta_*]
\end{align*}
\]

Since \( U_* \) is weakly normal, \([id]_{U_*} = \sup j_{U_*}[\delta_*]\), so the calculation implies \( p_*(U) = U_* \), which proves the lemma.

Therefore the interesting 0-order ultrafilters lie on regular cardinals, and all other 0-order ultrafilters are reducible to them.

Generalizing an observation due to Solovay, Ketonen introduced the seed order on weakly normal ultrafilters and proved the following fact.

**Theorem 56** (Ketonen). A uniform countably complete ultrafilter on a limit ordinal \( \delta \) is 0-order if and only if it is an \(<_S\)-minimal element of \( U\delta \).

Proof. Suppose \( U \in U\delta, \alpha \) is an ordinal such that \( \sup j_U[\delta] \leq \alpha \leq [id]_U \), and \( W_* \in U\alpha M_U \). Then \( U^-(W_*) <_S U \) and \( U^-(W_*) \in U\delta \). Conversely if \( W <_S U \) and \( W \in U\delta \), then for some ordinal \( \alpha \) such that \( \sup j_U[\delta] \leq \alpha \leq [id]_U \) and \( W_* \in U\alpha M_U \), \( W = U^-(W_*) \).

Thus \( U \) is an \(<_S\)-minimal element of \( U\delta \) if and only if for all ordinals \( \alpha \) such that \( \sup j_U[\delta] \leq \alpha \leq [id]_U \),

\[
U\alpha M_U = \emptyset
\]

Recalling that \( U\alpha \) is nonempty whenever \( \alpha \) is a successor ordinal, this holds if and only if \( \sup j_U[\delta] = [id]_U \) and \([id]_U\) carries no uniform ultrafilters in \( M_U \), or equivalently if and only if \( U \) is 0-order.

In particular, if a limit ordinal \( \delta \) carries a countably complete uniform ultrafilter, it carries a 0-order ultrafilter. If the seed order is linear, then minimal elements of \( U\delta \) are minimum elements, which yields the following corollary.

**Corollary 57** (UA). A limit ordinal \( \delta \) carries at most one 0-order ultrafilter.

**Definition 58** (UA). If \( \delta \) is a limit ordinal that carries a countably complete uniform ultrafilter, we call the unique 0-order ultrafilter on \( \delta \) the least ultrafilter on \( \delta \).
In the context of UA, if $U$ is a least ultrafilter, then $U$ is irreducible in a very strong sense.

**Theorem 59 (UA).** Let $U$ be the least ultrafilter on a limit ordinal $\delta$. Fix $W \in \text{Un}_U$ and let $\delta_* = \sup j_W[\delta]$. Then one of the following holds:

1. $t_W(U)$ is the least ultrafilter on $\delta_*$ as computed in $M_W$.
2. $t_W(U) = P^M_{\delta_*}$.

**Proof.** Let $D$ be the ultrafilter derived from $W$ using $\delta_*$. Then $U \leq S_D$, so $t_W(U) \leq M_W S t_W(D) \leq M_W S P^M_W \delta_*$ with the first inequality following from Proposition 25 and the second from definition of translation functions and the fact that $D = W - (P^M_W \delta_*)$. On the other hand, $\text{sp}(t_W(U)) \geq \delta_*$ since otherwise $\text{sp}(U) = \text{sp}(W - (t_W(U))) < \delta$ contradicting that $\text{sp}(U) = \delta$.

It follows from Lemma 13 that either $t_W(U) \in \text{Un}^M_{\delta_*}$ or $t_W(U) = P^M_{\delta_*}$.

Suppose $t_W(U) \in \text{Un}^M_{\delta_*}$. We show $t_W(U)$ is $<^M_{\delta_*}$-minimal in $\text{Un}^M_{\delta_*}$, which proves the theorem. Fix $Z \in \text{Un}^M_{\delta_*}$ with $Z <^M_{\delta_*} t_W(U)$. Then $W^-(Z) <_S U$ since $t_W(W^-(Z)) \leq S Z < S t_W(U)$ and $t_W$ is order-preserving by Proposition 25. Hence $W^-(Z) \in \text{Un}^{<\delta}$, and therefore $Z \in \text{Un}^{<\delta_*}$, as desired.

This leads to a useful characterization of the internal ultrapower embeddings of a least ultrapower.

**Theorem 60 (UA).** Let $U$ be the least ultrafilter on a limit ordinal $\delta$. Suppose $W$ is a countably complete ultrafilter and $k : M_U \rightarrow M_W$ is an elementary embedding with $k \circ j_U = j_W$. Then $k$ is definable over $M_U$ if and only if $k$ is continuous at $\sup j_U[\delta]$.

**Proof.** One direction is obvious. Suppose conversely that $k$ is continuous at $\sup j_U[\delta]$. Then $k(\sup j_U[\delta]) = \sup j_W[\delta]$. Therefore $\sup j_W[\delta]$ carries no uniform ultrafilters in $M_W$. It follows that $t_W(U) = P^M_{\sup j_W[\delta]}$ by Theorem 59. Therefore by Theorem 23, $j_{U(W)}^M \circ j_U = j_W$ and $j_{U(W)}^M(\sup j_U[\delta]) = \sup j_W[\delta]$. It follows that $k = j^M_U \circ j_U(W)$, so $k$ is definable over $M_U$, as desired.
Restated, this is a characterization (reminiscent of Corollary 32) of the countably complete $M_U$-ultrafilters that belong to $M_U$:

**Theorem 61 (UA).** Let $U$ be the least ultrafilter on a limit ordinal $\delta$. Suppose $W_*$ is a countably complete $M_U$-ultrafilter. Then $W_* \in M_U$ if and only if $j_{W_*}$ is continuous at $\text{cf}^{M_U}(\sup j_U[\delta])$.

If $\delta$ is a regular cardinal and $U$ is a countably complete ultrafilter, a theorem of Ketonen [7] states that the value of $\text{cf}^{M_U}(\sup j_U[\delta])$ determines the covering property of $M_U$ with respect to $\delta$-sequences. This makes Theorem 61 extremely useful for the purpose of gauging the strong compactness and supercompactness of least ultrafilters under UA. We briefly discuss the relationship between cofinalities and covering properties of ultrapowers.

**Definition 62.** Suppose $M$ is an inner model, $\delta$ is a cardinal, and $\lambda$ is an $M$-cardinal. Then $M$ has the $(\delta, \lambda)$-covering property if for every set $A \subseteq \text{Ord}$ such that $|A| \leq \delta$, there exists a set $B \subseteq \text{Ord}$ belonging to $M$ such that $A \subseteq B$ and $|B|^M \leq \lambda$.

**Theorem 63 (Ketonen).** Suppose $\delta$ is a regular cardinal and $U$ is a countably complete ultrafilter. Let $\lambda = \text{cf}^{M_U}(\sup j_U[\delta])$. Then $M_U$ has the $(\delta, \lambda)$-covering property.

**Proof.** By a standard argument, it suffices to show that there is a set $B \in M_U$ with $j_U[\delta] \subseteq B$ and $|B|^M \leq \lambda$.

Let $C \in M_U$ be cofinal subset of $\sup j_U[\delta]$ of order type $\lambda$. Since $\delta$ is regular, we can partition $\delta$ into bounded intervals $\langle I_\alpha : \alpha < \delta \rangle$ with the property that for all $\alpha < \delta$, $j_U(I_\alpha) \cap C \neq \emptyset$. (This is achieved by setting

$$I_\alpha = \left[\sup \left\{ \eta + 1 : \eta \in \bigcup_{\beta < \alpha} I_\beta \right\}, \xi \right)$$

where $\xi < \delta$ is least ensuring $j_U(I_\alpha) \cap C \neq \emptyset$.

Let

$$\langle J_\alpha : \alpha < j_U(\delta) \rangle = j_U(\langle I_\alpha : \alpha < \delta \rangle)$$

and let

$$B = \{ \alpha < j_U(\delta) : J_\alpha \cap C \neq \emptyset \}$$

Clearly $B \in M_U$. Moreover since $|C|^M = \lambda$ and the $J_\alpha$ are disjoint, it follows that $|B|^M \leq \lambda$. Finally $j_U[\delta] \subseteq B$ since we arranged that $J_{j_U(\alpha)} \cap C = j_U(I_\alpha) \cap C \neq \emptyset$ for all $\alpha < \delta$. 

$\square$
A second proof proceeds by noting that it suffices to show that for some \(C\) of cardinality \(\delta\), \(j_U[C]\) is covered by a set \(D \in M_U\) with \(|D|^{M_U} = \lambda\). Then let \(D\) be any club of order type \(\lambda\) in \(\sup j_U[\delta]\) and let \(C = j_U^{-1}[D]\).

**Definition 64.** We say \(U\) has the tight covering property at \(\delta\) if \(M_U\) has the \((\delta, \delta)\)-covering property.

As a corollary of Theorem 61 and Theorem 63, we immediately obtain the following dichotomy:

**Corollary 65 (UA).** Let \(U\) be the least ultrafilter on a regular cardinal \(\delta\). Either \(U\) has the tight covering property at \(\delta\) or \(U \cap M_U \in M_U\).

**Proof.** Of course \(\delta \leq \text{cf}M_U(\sup j_U[\delta])\). If equality holds, then tight covering holds by Theorem 63. If instead \(\delta < \text{cf}M_U(\sup j_U[\delta])\), then \(j_U^{M_U}M\) is continuous at \(\text{cf}M_U(\sup j_U[\delta])\) so \(U \cap M_U \in M_U\) by Theorem 61.

It is not clear whether it is consistent with ZFC that there is a countably complete nonprincipal ultrafilter \(U\) on an ordinal such that \(U \cap M_U \in M_U\), but the usual proofs that \(U \notin M_U\) do not give much insight into this question. In any case we can rule out that \(U \cap M_U \in M_U\) in various contexts. (See for example Theorem 69.)

Theorem 61 yields the amenability of many ultrafilters to a least ultrapower. We will use the following classical theorem due to Hausdorff to transform this into the amenability of many sets of ordinals. This is the key to obtaining supercompactness from strong compactness.

**Definition 66.** Suppose \(\kappa \leq \delta\) are cardinals. A family \(F\) of subsets of \(\delta\) is \(\kappa\)-independent if for any subfamilies \(F_0, F_1 \subseteq F\) of cardinality less than \(\kappa\),

\[|\{\alpha < \delta : \forall X \in F_0 \alpha \in X \text{ and } \forall X \in F_1 \alpha \notin X\}| = \delta\]

Let \(F_\delta\) denote the filter of \(X \subseteq \delta\) such that \(|\delta \setminus X| < \delta\). A family \(F\) is \(\kappa\)-independent if and only if for any \(G \subseteq F\),

\[G \cup \{\delta \setminus X : X \notin G\} \cup F_\delta\]

generates a \(\kappa\)-complete filter. The official definition has the benefit that it is obviously absolute between \(V\) and any \(\kappa\)-closed inner model.

**Theorem 67 (Hausdorff).** Suppose \(\kappa \leq \delta\) are cardinals and \(\delta^{<\kappa} = \delta\). Then there is a \(\kappa\)-independent family of subsets of \(\delta\) of cardinality \(2^\kappa\).
We emphasize that the next lemma lies at the center of the relationship between strong compactness and supercompactness.

**Lemma 68 (UA).** Suppose $\kappa$ is a cardinal and $U$ is a $\kappa$-supercompact ultrafilter. Suppose $\gamma$ is an $M_U$-cardinal with $\text{cf}^{M_U}(\gamma) \geq \kappa$ such that for any $W \in \mathcal{U}_n, W \cap M_U \in M_U$. Suppose $\lambda \leq (2^{\gamma})^{M_U}$ and $\kappa$ is $\lambda$-strongly compact. Then $P(\lambda) \subseteq M_U$.

**Proof.** First note that $\kappa$ is $<\gamma$-strongly compact in $M_U$. To prove this, it suffices by a theorem of Ketonen [7] to show that every $M_U$-regular $\iota \in [\kappa, \gamma)$ carries a uniform $\kappa$-complete ultrafilter in $M_U$. Note that $\text{cf}(\iota) \in [\kappa, \gamma)$ since $M_U$ is closed under $\kappa$-sequences. Hence $\iota$ carries a uniform ultrafilter $W$ since $\kappa$ is $\gamma$-strongly compact in $V$. But by assumption $W \cap M_U \in M_U$, so $\iota$ carries a uniform $\kappa$-complete ultrafilter in $M_U$.

Since $\kappa$ is $<\gamma$-strongly compact in $M_U$ and $\text{cf}^{M_U}(\gamma) \geq \kappa$, $(\gamma^{<\kappa})^{M_U} = \gamma$. Applying Theorem 67 in $M_U$, let $\langle A_\alpha : \alpha < (2^{\gamma})^{M_U} \rangle$ be a $\kappa$-independent family of subsets of $\gamma$ relative to $M_U$. Since $M_U$ is closed under $\kappa$-sequences, $M_U$ is correct about $\kappa$-independence, so $\langle A_\alpha : \alpha < (2^{\gamma})^{M_U} \rangle$ is truly $\kappa$-independent.

Fix $X \subseteq \lambda$. We will show $X \in M_U$. The filter $F$ generated by

$$\{A_\alpha : \alpha \in X\} \cup \{\gamma \setminus A_\alpha : \alpha \in \lambda \setminus X\}$$

is $\kappa$-complete by $\kappa$-independence. Since $\kappa$ is $\lambda$-strongly compact, any $\kappa$-complete filter on $\gamma$ generated by $\lambda$ sets extends to a $\kappa$-complete ultrafilter. Therefore let $D$ be a $\kappa$-complete ultrafilter on $\gamma$ extending $F$. Since $D \cap M_U \in M_U$, $X \in M_U$:

$$X = \{\alpha < \lambda : A_\alpha \in D \cap M_U\}$$

This completes the proof. □

**Theorem 69 (UA).** Let $U$ be the least ultrafilter on a regular cardinal $\delta$. Let $\kappa$ be the completeness of $U$. Assume that $\kappa$ is $\delta$-strongly compact. Then $U$ is $<\delta$-supercompact and has the tight covering property at $\delta$. If $\delta$ is not strongly inaccessible then $U$ is $\delta$-supercompact.

**Proof.** Assume towards a contradiction that $U$ does not have the tight covering property at $\delta$. Then by Theorem 61, for any $W \in \mathcal{U}_n$, $W \cap M_U \in M_U$. But now by Lemma 68, $P(\delta) \subseteq M_U$. But then $U \cap M_U = U$, so $U \in M_U$, which is impossible. It follows that $U$ has the tight covering property at $\delta$.

By Theorem 61 for all $W \in \mathcal{U}_n$, $W \cap M_U \in M_U$. Therefore by Lemma 68, $\bigcup_{\gamma<\delta} P(\gamma) \subseteq M_U$. 

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Suppose $\gamma < \delta$ is a regular cardinal. Note that $\text{cf}^{M_U}(\sup j_U[\gamma]) \leq \delta$ by the tight covering property. Since $\delta$ is regular, it follows that $\text{cf}^{M_U}(\sup j_U[\gamma]) < \delta$. By Theorem 63, $j_U[\gamma]$ can be covered by a set of $M_U$-cardinality $\lambda < \delta$. Since $P(\lambda) \subseteq M_U$, $j_U[\gamma] \in M_U$. Hence $U$ is $\gamma$-supercompact.

Finally suppose $\delta$ is not a strong limit cardinal, and we will show that $U$ is $\delta$-supercompact. It suffices to show that $P(\delta) \subseteq M_U$: then by the tight covering property, $j_U[\delta]$ is covered by a set of $M_U$-cardinality $\delta$, and since $P(\delta) \subseteq M_U$, $j_U[\delta] \in M_U$.

We split into two cases.

Case 1. For some $\gamma < \delta$ with $\text{cf}(\gamma) \geq \kappa$, $2^\gamma \geq \delta$.

Note that $\delta \leq 2^\gamma \leq (2^\gamma)^{M_U}$, where the final inequality follows from the fact that $P(\gamma) \subseteq M_U$. We must therefore have $P(\delta) \subseteq M_U$ by Lemma 68 with $\lambda = \delta$.

Case 2. Otherwise.

Since $\delta$ is not a strong limit cardinal, there must be some $\lambda < \delta$ with $2^\lambda \geq \delta$. Since we are not in Case 1, $\lambda^\kappa \geq \delta$: otherwise $\gamma = \lambda^\kappa$ witnesses the hypotheses of Case 1. Note that $U$ is $\lambda$-supercompact since $\lambda$ is a singular limit cardinal and $U$ is $\gamma$-supercompact for all regular $\gamma < \lambda$. Since $j_U \upharpoonright \lambda \in M_U$ and $U$ is $\kappa$-complete, $j_U \upharpoonright P_\kappa(\lambda) \in M_U$. Thus $U$ is $\lambda^\kappa$-supercompact. So $U$ is $\delta$-supercompact.

We now show that in certain circumstances we can obtain the hypotheses of Theorem 69, leading to a proof from UA that the least strongly compact cardinal is supercompact.

This involves another theorem due to Ketonen, who used the combinatorics of 0-order ultrafilters to give a second proof of his characterization of strongly compact cardinals in terms of uniform countably complete ultrafilters. We include the version that is most relevant to us, since this is not exactly what Ketonen proved.

**Theorem 70 (Ketonen).** Suppose $U$ is a 0-order ultrafilter on a regular cardinal $\delta$. Then for any $\gamma \leq \delta$, $U$ is $(\gamma, \delta)$-regular if and only if every regular cardinal in the interval $[\gamma, \delta]$ carries a uniform countably complete ultrafilter.

**Proof.** If there is a $(\gamma, \delta)$-regular ultrafilter then easily every regular cardinal in the interval $[\gamma, \delta]$ carries a uniform countably complete ultrafilter.

Conversely assume every regular cardinal in the interval $[\gamma, \delta]$ carries a uniform countably complete ultrafilter. Since $U$ is 0-order, $\sup j_U[\delta]$ carries no uniform countably complete ultrafilter in $M_U$. Therefore $\text{cf}^{M_U}(\sup j_U[\delta])$
carries no uniform countably complete ultrafilter in $M_U$. It follows that $\text{cf}^{M_U}(\sup j_U[\delta]) \notin j_U([\gamma, \delta])$. Hence $\text{cf}^{M_U}(\sup j_U[\delta]) < j_U(\gamma)$, which implies $U$ is $(\gamma, \delta)$-regular.

**Lemma 71 (UA).** Let $U$ be the least ultrafilter on a regular cardinal $\delta$. Let $\kappa$ be its completeness, and let $\bar{\kappa} \leq \kappa$ be the least ordinal such that for some $W$, $j_W(\bar{\kappa}) > \kappa$. Suppose $W$ is a countably complete ultrafilter with $j_W(\bar{\kappa}) > \kappa$ and $U \sqsubseteq W$. Then $U$ and $W$ are $(\bar{\kappa}, \delta)$-regular. Hence $\bar{\kappa} = \kappa$.

For the proof we need a version of the Kunen inconsistency. For this we require a useful lemma that under favorable cardinal arithmetic conditions often allows us to replace arbitrary ultrafilters with ultrafilters on small sets.

**Lemma 72.** Suppose $U$ is an ultrafilter on a set $X$ and $\langle f_i : i \in I \rangle$ is a sequence of functions from $X$ to a set $Y$. Then there is a function $p : X \rightarrow Y^I$ such that letting $W = p_*(U)$ and $k : M_W \rightarrow M_U$ be the factor embedding, $[f_i]_U \in \text{ran}(k)$ for all $i \in I$.

*Proof.* Take $p : X \rightarrow Y^I$ such that $p(x)(i) = f_i(x)$. Let $W = p_*(U)$ and let $h = [\text{id}]_W \in j_W(Y^I)$. Then

$$k(h(j_W(i))) = j_U(p)([\text{id}]_U)(j_U(i)) = j_U(\langle f_i : i \in I \rangle)_{j_U}(i)[\text{id}]_U = j_U(f_i)([\text{id}]_U) = [f_i]_U$$

so $[f_i]_U \in \text{ran}(k)$. $\square$

**Corollary 73.** Suppose $U$ is a countably complete ultrafilter and $\lambda$ is a cardinal. There is a function $p : \text{sp}(U) \rightarrow 2^\lambda$ such that letting $W = p_*(U)$ and $k : M_W \rightarrow M_U$ be the factor embedding, $\text{crt}(k) > \lambda$.

The notation $\text{Un}_{< \kappa} \sqsubseteq U$ abbreviates the statement that $W \sqsubseteq U$ for all $W \in \text{Un}_{< \kappa}$.

**Lemma 74 (UA).** Suppose $U$ is a countably complete ultrafilter and $\lambda$ is the first fixed point of $j_\lambda$ above its critical point. Then for some $\nu < \lambda$, $\text{Un}_{< 2^\nu} \not\sqsubseteq U$.

*Proof.* By Kunen’s inconsistency theorem, there is some $\gamma < \lambda$ such that $U$ is not $\gamma$-supercompact. Let $\nu = \sup j_\nu[\gamma]$, so $\nu < \lambda$. By Lemma 72 there is some $W \in \text{Un}_{< 2^\nu}$ such that $j_W \upharpoonright \gamma = j_\nu \upharpoonright \gamma$. Note that $W \not\sqsubseteq U$ since otherwise $j_\nu \upharpoonright \gamma \in M_U$, contradicting that $U$ is not $\gamma$-supercompact. $\square$
Corollary 75 (UA). Suppose $U$ is a countably complete ultrafilter and $\lambda$ is the first fixed point of $j_U$ above its critical point. Suppose $\kappa$ is a strong limit cardinal and $\text{Un}_{<\kappa} \subseteq U$. Then $\kappa < \lambda$.

Corollary 76 (UA). Suppose $\kappa$ is a strong limit cardinal that is closed under ultrapowers. Suppose $U$ is a countably complete ultrafilter such that $\text{Un}_{<\kappa} \subseteq U$. Then $U$ is $\kappa$-complete.

Proof of Lemma 71. Let $\delta_* = \text{sup} j_W[\delta]$. The key point is that we have both that $t_W(U) = s_W(U)$ by Proposition 56 and also that $t_W(U)$ is the least ultrafilter on $\delta_*$ by Theorem 59. Hence

$$j_U \upharpoonright M_W = j_{U_*}^{M_W}$$

(2)

where $U_*$ is the least ultrafilter of $M_W$ on $\text{cf}^{M_W}(\delta_*)$. (Here we use Lemma 53 in $M_W$ to see that in $M_W$, $U_*$ is equivalent to the least ultrafilter on $\delta_*$.)

Suppose towards a contradiction that $W$ is not $(\bar{\kappa}, \delta)$-regular, and therefore $\text{cf}^{M_W}(\delta_*) \geq j_W(\bar{\kappa})$. Work in $M_W$. Note that $j_W(\bar{\kappa})$ is a strong limit cardinal that is closed under ultrapowers, and moreover $\text{Un}_{<j_W(\bar{\kappa})}^{M_W} \subseteq U_*$ by Theorem 61. Therefore $U_*$ is $j_W(\bar{\kappa})$-complete by Corollary 76. But $j_W(\bar{\kappa}) > \kappa$, contradicting that the critical point of $j_{U_*}^{M_W}$ is $\kappa$ by (2). Therefore our assumption was false, so $W$ is $(\bar{\kappa}, \delta)$-regular.

Finally we conclude that $\bar{\kappa} = \kappa$: first, $\bar{\kappa} \leq \kappa$ by definition, and second, by Theorem 70 $U$ is $(\bar{\kappa}, \delta)$-regular, so $\kappa \leq \bar{\kappa}$. □

Corollary 77 (UA). Suppose there are arbitrarily large regular cardinals carrying countably complete uniform ultrafilters. Suppose $\kappa$ is the least cardinal mapped arbitrarily high by countably complete ultrafilters. Then $\kappa$ is supercompact.

Proof. Note that $\kappa$ is closed under ultrapowers.

Let $\delta \geq \kappa$ be a regular cardinal carrying a countably complete uniform ultrafilter. We claim the least ultrafilter $U$ on $\delta$ witnesses $\kappa$ is $<\delta$-supercompact.

Let $\lambda$ be the least fixed point of $U$ above $\text{crt}(U)$. Let $W$ be the $<_\delta$-least ultrafilter such that $j_W(\kappa) \geq \lambda$. Then by Theorem 49 $U \subseteq W$. Clearly $j_W(\kappa) > \text{crt}(U)$. Since $\kappa$ is closed under ultrapowers, $\kappa$ is the least ordinal such that $j_Z(\kappa) > \text{crt}(U)$ for some $Z \in \text{Un}$.

Therefore by Lemma 71 $\text{crt}(U) = \kappa$ and $U$ is $(\kappa, \delta)$-regular. Now by Theorem 69 $U$ witnesses that $\kappa$ is $<\delta$-supercompact. □

Corollary 78 (UA). The least strongly compact cardinal is supercompact.
4.2 The Next Ultrafilter

We continue our investigation of 0-order ultrafilters, proving some local refinement of the results we have seen so far.

We first point out that this is much easier if one assumes UA + GCH. We will use the following theorem essentially due to Ketonen.

**Theorem 79** (Ketonen). Suppose $\gamma$ is regular and $U$ is a countably complete uniform ultrafilter on $\gamma^+$. Then $U$ is $\gamma$-decomposable.

**Proof.** Note that $\text{cf}(\sup j_U[\gamma^+]) \leq j_U(\gamma)$. If equality holds, then $\text{cf}(j_U(\gamma)) = \gamma^+$, so $j_U$ is discontinuous at $\gamma$. If strict inequality holds, then $U$ has the $(\gamma^+, \lambda)$-covering property for some $\lambda < j_U(\gamma)$, by Theorem 63 which implies again that $j_U$ is discontinuous at $\gamma$. □

**Proposition 80** (UA). Suppose $U$ is the least ultrafilter on a regular cardinal $\delta$ and for all $\gamma$ with $\gamma < \delta$, $2^\gamma = \gamma^+$. Then $\text{crt}(U)$ is $\delta$-strongly compact.

**Sketch.** We will use the fact that if $\gamma$ is regular and $\gamma^+$ carries a uniform $\bar{\kappa}$-complete ultrafilter, so does $\gamma$.

Let $\bar{\kappa}$ be least such that for some $W$, $j_W(\bar{\kappa}) > \delta$. One shows $\bar{\kappa}$ is a strong limit cardinal that is closed under ultrapowers. Therefore $\bar{\kappa} \leq \kappa$. We show $\bar{\kappa}$ is $\delta$-strongly compact. This will imply the theorem: since $\bar{\kappa}$ is $\delta$-strongly compact, $U$ is $\bar{\kappa}$-complete by Theorem 70 so $\kappa \leq \bar{\kappa}$, and hence $\kappa = \bar{\kappa}$ is $\delta$-strongly compact.

Let $\lambda$ be the largest limit cardinal with $\lambda \leq \delta$. It suffices by GCH and the first sentence of this proof to show that every regular $\gamma$ with $\bar{\kappa} \leq \gamma < \lambda$ carries a $\bar{\kappa}$-complete ultrafilter. Using Lemma 72 and GCH, one shows that the space of the least ultrafilter $W$ sending $\bar{\kappa}$ above $\gamma^+$ is at most $\gamma^{++}$. On the other hand $\text{sp}(W) \geq \gamma$ since $\bar{\kappa}^{<\gamma} = \gamma$. Moreover since every ultrafilter in $\text{Un}_{<\bar{\kappa}}$ fixes $\gamma^+$, $\text{Un}_{<\bar{\kappa}} \subseteq W$ by Theorem 49. Thus $W$ is $\bar{\kappa}$-complete by Corollary 76. Hence $\gamma$ carries a uniform $\bar{\kappa}$-complete ultrafilter by the first sentence. □

This result suffices for the analysis under GCH of higher strongly compact cardinals, so the reader who is not interested in the fine structure of least ultrafilters under UA without assuming GCH can skip ahead to Section 5.

Without GCH, we will show the following:

**Theorem 81** (UA). Suppose $\gamma < \delta$ are regular cardinals and $\text{Un}_{\delta}, \text{Un}_{\gamma} \neq \emptyset$. Then the least ultrafilter on $\delta$ is $\gamma$-supercompact.

Before proving Theorem 81 which will take several pages, we give some applications.
Corollary 82 (UA). Suppose $U$ is the least ultrafilter on a regular cardinal $\delta$. Let $\kappa$ be the completeness of $U$.

(1) If $\delta$ is the successor of a regular cardinal $\lambda$, then $U$ is $\delta$-supercompact.

(2) If $\delta$ is the successor of a singular limit $\lambda$ of regular $\gamma$ with $U\gamma \neq \emptyset$, then $U$ is $\delta$-supercompact.

(3) Therefore if $\delta$ is the successor of a singular cardinal $\lambda$ with $\text{cf}(\lambda) < \kappa$, then $U$ is $\delta$-supercompact.

(4) If $\delta$ is weakly inaccessible and $\text{sup} j_U[\delta]$ is not regular in $M_U$, then $U$ is $\text{<}\delta$-supercompact.

(5) Therefore if $\delta$ is weakly inaccessible but not weakly $<\delta^+\text{-Mahlo}$, then $U$ is $<\delta$-supercompact.

Proof of (1). Since $\lambda$ is regular and $U\lambda^+ \neq \emptyset$, $U\lambda \neq \emptyset$. Hence $U$ is $\lambda$-supercompact by Theorem 81. Since $\kappa$ is $\lambda$-supercompact and $\delta$ carries a $\kappa$-complete uniform ultrafilter, $\kappa$ is $\delta$-strongly compact by Ketonen’s characterization of strong compactness. Hence $U$ is $\delta$-supercompact by Theorem 69.

Proof of (2). Again $U$ is $<\lambda$-supercompact by Theorem 81. Since $\kappa$ is $\lambda$-supercompact and $\delta$ carries a $\kappa$-complete uniform ultrafilter, $\kappa$ is $\delta$-strongly compact by Ketonen’s characterization of strong compactness. Hence $U$ is $\delta$-supercompact by Theorem 69.

Proof of (3). By a result of Ketonen [7], the hypotheses of (2) follow from those of (3) using Theorem 63. The proof is similar to the proof of (4) below.

Proof of (4). If $\text{sup} j_U[\delta]$ is not regular in $M_U$, then for some $\iota < \delta$,

$$\text{cf}^{M_U}(\text{sup} j_U[\delta]) < j_U(\iota)$$

In other words $U$ is $(\iota, \delta)$-regular, and hence discontinuous at every regular $\gamma \in [\iota, \delta]$. Hence every regular $\gamma \in [\iota, \delta]$ carry countably complete uniform ultrafilters. So $U$ is $<\delta$-supercompact by Theorem 81.

Proof of (5). Suppose $U$ is not $<\delta$-supercompact. Then by (4), $\text{sup} j_U[\delta]$ is regular in $M_U$. Therefore $U$ concentrates on regular cardinals. Since $U$ is weakly normal, $U$ is closed under decreasing diagonal intersections and every set in $U$ is stationary.

Moreover $U$ is closed under the Mahlo operation by a well-known argument. To see this it suffices to show that for any $X \in U$, $j_U(X)$ reflects to $\text{sup} j_U[\delta]$. 29
We may assume that \( X \subseteq \text{Reg} \). Suppose \( C \subseteq \sup_j \U \[ \delta \] \) is club. Let \( \bar{C} = j_U^{-1}[C] \). Then since \( j_U \) is continuous at all sufficiently large regular cardinals, \( \lim(C) \cap \text{Reg} \subseteq \bar{C} \). Since \( X \) is stationary, there is some \( \gamma \in \lim(C) \cap X \). But \( \gamma \in \bar{C} \) since \( X \subseteq \text{Reg} \). Thus \( j_U(\gamma) \in C \cap j_U(X) \). Since \( C \) was arbitrary, it follows that \( j_U(X) \) reflects to \( \sup j_U[\delta] \), as desired.

It follows that \( \delta \) is \( <\delta^+ \)-Mahlo. \( \Box \)

We do not believe the previous corollary exhausts the supercompactness provable from UA alone. For example, if the least ultrafilter on an inaccessible \( \delta \) fails to be \( <\delta \)-supercompact, the consequences are truly bizarre:

**Proposition 83** (UA). Suppose \( \delta \) is a strongly inaccessible cardinal such that the least ultrafilter \( U \) on \( \delta \) is not \( <\delta \)-supercompact. Let \( \kappa = \text{crt}(U) \) and \( \delta_* = \sup j_U[\delta] \).

1. \( j_U(\kappa) > \delta \).
2. \( \delta \) is \( <\delta^+ \)-Mahlo.
3. \( U \) is the unique countably complete ultrafilter on \( \delta \) extending the club filter.
4. \( \delta \) is not measurable.
5. For all sufficiently large singular strong limits \( \gamma < \delta \) of cofinality less than \( \kappa \), \( 2^\gamma \geq \gamma^+ \).
6. For all sufficiently large regular \( \gamma < \delta \), \( \text{Refl}(S^\delta_{\gamma}) \).
7. For any countably complete \( M_U \)-ultrafilter \( W \) with \( \text{sp}(W) < \delta_* \), \( W \in M_U \).
8. In particular \( U \cap M_U \in M_U \).

If \( \delta \) is the least such cardinal, then \( U \cap M_U \) is a normal ultrafilter in \( M_U \).

We omit the proof, parts of which are similar to Proposition 80 and Corollary 82 (5).

**Conjecture 84.** It is provable from ZFC + UA that the least ultrafilter on a strongly inaccessible cardinal \( \delta \) is \( <\delta \)-supercompact.

In our first step toward Theorem 81, we prove a very simple weakening of Theorem 81 that serves as a local version of Theorem 70.

**Lemma 85** (UA). Suppose \( U \) is the least ultrafilter on a regular cardinal \( \delta \). Let \( \kappa \) be the completeness of \( U \). Suppose \( \tilde{\delta} \in [\kappa, \delta] \) is regular and \( U \cap \tilde{\delta} \) is nonempty. Then \( U \) is discontinuous at \( \tilde{\delta} \).
Proof. Assume towards a contradiction that $\delta$ is the least cardinal at which the lemma fails. Let $\tilde{\delta} \in [\kappa, \delta]$ witness this. Of course $\tilde{\delta} \in (\kappa, \delta)$.

Let $\bar{U}$ be the least ultrafilter on $\delta$. Let $\bar{\kappa}$ be the completeness of $\bar{U}$. Since $U$ is continuous at $\tilde{\delta}$, $U \supseteq \bar{U}$ by Theorem 61. Since $\tilde{\delta} < \delta$, $\bar{U} \supseteq U$ by Theorem 61. Thus by Theorem 47 $U$ and $\bar{U}$ commute. It follows in particular that $j_U(\kappa) = \kappa$ and $j_U(\bar{\kappa}) = \kappa$.

Since $\bar{\kappa} < \delta$ is a strong limit cardinal fixed by $U$, $\bar{\kappa} < \kappa$ by Corollary 75.

Now $\bar{U}$ is the least ultrafilter on the regular cardinal $\bar{\delta}$, but $\bar{U}$ fixes the measurable cardinal $\kappa \in [\bar{\kappa}, \bar{\delta}]$. This contradicts the minimality of $\delta$. □

The second step toward Theorem 81 is a result that looks like Lemma 71 but is really much more complicated.

**Theorem 86 (UA).** Let $U$ be the least ultrafilter on a regular cardinal $\delta$. Let $\kappa$ be its completeness. Suppose $W$ is a countably complete ultrafilter with $j_W(\kappa) > \kappa$ and $U \subseteq W$. Then $\kappa$ is closed under ultrapowers and $U$ and $W$ are $(\kappa, \delta)$-regular.

To simplify the notation below, we note that to prove Theorem 86, it suffices to prove the following lemma:

**Lemma 87 (UA).** Suppose $U$ is the least ultrafilter on a regular cardinal $\delta$. Let $\kappa$ be its completeness. Suppose there is a countably complete ultrafilter $W$ satisfying $j_W(\kappa) > \kappa$ and $U \subseteq W$. Then $\kappa$ is closed under ultrapowers.

**Proof of Theorem 86 given Lemma 87.** By Lemma 87 $\kappa$ is closed under ultrapowers. Therefore $\kappa$ itself is the least ordinal $\bar{\kappa}$ such that for some $Z \in U_n$ Now an application of Lemma 71 implies the theorem. □

**Proof of Lemma 87.** Assume towards a contradiction that $\bar{\kappa} < \kappa$ is the least ordinal such that for some $Z$, $j_Z(\bar{\kappa}) > \kappa$. Since $\kappa$ is measurable, $\bar{\kappa}$ is also the least ordinal mapped arbitrarily high below $\kappa$.

Denote by $W$ the $<_S$-least uniform countably complete ultrafilter such that $j_W(\kappa) > \kappa$ and $U \subseteq W$.

**Claim 1.** $U_{<\kappa} \subseteq W$.

**Proof of Claim 1.** Fix $Z \in U_{<\kappa}$. By Kunen’s Theorem 40 $j_Z(j_U) = j_U \upharpoonright M_Z$ and in particular $U \subseteq Z$. Since $U \subseteq Z$ and $U \subseteq W$, we have $U \subseteq Z \cup W$ by Corollary 32. This implies $s_Z(U) \subseteq M_Z \cap t_Z(W)$; that is,

$$j_Z(U) \subseteq M_Z \cap t_Z(W)$$

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Moreover \( j_{t_Z(W)}^{M_Z}(j_Z(\kappa)) = j_{t_W(Z)}^{M_W}(j_W(\kappa)) > \kappa = j_Z(\kappa) \), or more briefly:

\[
j_{t_Z(W)}^{M_Z}(j_Z(\kappa)) > j_Z(\kappa)
\]

Therefore in \( M_Z, t_Z(W) \) satisfies the conditions for which \( W \) was minimized with their parameters moved by \( j_Z \). Hence \( j_Z(W) \leq_M t_Z(W) \). By the definition of translation functions, \( t_Z(W) \leq_M j_Z(W) \), so \( t_Z(W) = j_Z(W) \). By Proposition 48, \( Z \subseteq W \).

\textbf{Claim 2.} \( j_W(\bar{\kappa}) < \kappa \).

\textit{Proof of Claim 2.} If \( j_W(\bar{\kappa}) > \kappa \) then \( \bar{\kappa} = \kappa \) by Lemma 71, a contradiction. Thus \( j_W(\bar{\kappa}) \leq \kappa \), and we must show the inequality is strict.

By Claim 1, \( U_{<\kappa} \subseteq W \). Using Lemma 72, it follows that \( \bar{\kappa} \) is mapped arbitrarily high below \( \kappa \) by ultrapowers in \( M_W \). If \( j_W(\bar{\kappa}) = \kappa \), then by elementarity, in \( V \) some ordinal \( \alpha < \bar{\kappa} \) is mapped arbitrarily high below \( \bar{\kappa} \). But \( \alpha \) is then mapped arbitrarily high below \( \kappa \), contradicting the minimality of \( \bar{\kappa} \).

We now break into two cases, based on whether or not \( \kappa \) is a limit of regular cardinals carrying uniform countably complete ultrafilters.

\textbf{Case 1.} \( \kappa \) is not a limit of regular cardinals that carry uniform countably complete ultrafilters.

\textit{Proof in Case 1.} We claim \( \kappa = \delta \). Otherwise the normal ultrafilter on \( \kappa \) derived from \( U \) belongs to \( M_U \) by Theorem 61 and so \( \kappa \) is easily a limit of measurable cardinals and much more.

We can also show in this case that \( W \) is discontinuous at \( \kappa \): otherwise since \( \delta = \kappa \), \( W \subseteq U \), so \( W \) and \( U \) commute by Theorem 47. This means \( j_W(j_U) = j_U \upharpoonright M_W \), which contradicts that \( j_W(\text{crt}(U)) \neq \text{crt}(U) \).

Since \( U \subseteq W \), \( t_W(U) = s_W(U) \) by Proposition 46. Therefore \( \text{sp}(t_W(U)) = \sup j_W[\kappa] \). Since \( W \) is discontinuous at \( \kappa \), we conclude \( \text{sp}(t_W(U)) < j_W(\kappa) \).

Since \( j_W(\kappa) \) is inaccessible, we can therefore find a fixed point \( \xi \) of \( t_W(U) \) above \( \kappa \) and below \( j_W(\kappa) \).

Let \( Z \) be \( \leq_M \) least such that \( j_Z^{M_W}(j_W(\bar{\kappa})) > \xi \), which exists since in \( M_W \), \( j_W(\bar{\kappa}) \) is mapped arbitrarily high below \( j_W(\kappa) \) by elementarity. By Theorem 49, we have \( t_W(U) \subseteq Z \).

Since \( j_W(\bar{\kappa}) \) is least mapped above \( j_W(\kappa) \) in \( M_W \), \( j_W(\bar{\kappa}) \) is closed under ultrapowers in \( M_W \), and hence is least mapped above \( \kappa \) in \( M_W \). Thus \( Z \) witnesses the hypotheses of Lemma 71 in \( M_W \). It follows that \( j_W(\bar{\kappa}) = \text{crt}(t_W(U)) \).

But \( \text{crt}(t_W(U)) = \text{crt}(U) = \kappa \). This contradicts Claim 2.
Case 2. \( \kappa \) is a limit of regular cardinals that carry uniform countably complete ultrafilters.

Proof in Case 2. Let \( \delta' = \text{cf}^{M_W}(\sup j_W[\delta]) \). Assume towards a contradiction that \( j_W(\bar{\kappa}) \) is \( \delta' \)-strongly compact in \( M_W \). Then by Theorem 59 and Theorem 70, \( t_W(U) \) has critical point less than or equal to \( j_W(\bar{\kappa}) \). But \( j_W(\bar{\kappa}) < \kappa \) while \( \text{crt}(t_W(U)) = \text{crt}(U) = \kappa \), a contradiction. Therefore our assumption was false, so \( j_W(\bar{\kappa}) \) is not \( \delta' \)-strongly compact in \( M_W \).

Let \( \iota_0 = \kappa \) and for each \( n < \omega \), let \( \iota_{n+1} = j_W(\iota_n) \).

Claim 3. For all \( n < \omega \):

1. \( U \) is \( < \iota_n \)-supercompact.
2. \( \bar{\kappa} \) is \( < \iota_n \)-supercompact.
3. \( W \) is \( < \iota_n \)-supercompact.
4. \( \iota_n \) is strongly inaccessible.

Proof of Claim 3. The proof is by induction.

We begin with the case \( n = 0 \). Since \( U \) is \( \kappa \)-complete, \( U \) is \( \kappa \)-supercompact, which yields (1). Since \( \kappa \) is an inaccessible limit of regular cardinals carrying uniform countably complete ultrafilters, Corollary 78 applied in \( V_\kappa \) implies that \( \bar{\kappa} \) is \( < \kappa \)-supercompact, which yields (2). Since \( U \) is \( < \kappa \)-supercompact and \( \kappa \) is measurable, \( \kappa \) is inaccessible, which yields (4).

Suppose the claim holds when \( n = k \), and we prove it is true for \( n = k + 1 \).

By elementarity, \( j_W(\bar{\kappa}) \) is \( < \iota_{k+1} \)-supercompact in \( M_W \). Since \( \bar{\kappa} \) is not \( \delta' \)-strongly compact in \( M_W \), \( \iota_{k+1} < \delta' \). For any \( M_W \)-regular \( \gamma < \iota_{k+1} \), fix \( Z \in \text{Un}^{M_W}_\gamma \) witnessing \( j_W(\bar{\kappa}) \) is \( \gamma \)-supercompact. Since \( \gamma < \delta' \), \( Z \subseteq t_W(U) \) by Theorem 61. Since \( \text{crt}(Z) = j_W(\bar{\kappa}) < \kappa = \text{crt}(t_W(U)) \), by Proposition 52 this implies \( t_W(U) \) is \( \gamma \)-supercompact in \( M_W \). Since \( \iota_{k+1} \) is strongly inaccessible in \( M_W \), it follows that \( t_W(U) \) is \( < \iota_{k+1} \)-supercompact in \( M_W \).

Thus for all \( \xi < \iota_{k+1} \), \( j_W^M \upharpoonright \xi \in M_{t_W(U)} \subseteq M_U \). Since \( j_W^M \upharpoonright \text{Ord} = j_U \upharpoonright \text{Ord} \), it follows that \( U \) is \( < \iota_{k+1} \)-supercompact. This shows (1).

Since \( \bar{\kappa} \) is \( < \kappa \)-supercompact, the fact that \( U \) is \( < \iota_{k+1} \)-supercompact implies that \( \bar{\kappa} \) is \( < \iota_{k+1} \)-supercompact. This shows (2).

Finally since \( W \) is \( < \kappa \)-supercompact and \( U \subseteq W \), Proposition 52 implies that \( W \) is \( < \iota_{k+1} \)-supercompact as well. This shows (3).
It follows that \( \iota_{k+1} \) is inaccessible: \( \iota_{k+1} = j_W(\iota_k) \) is inaccessible in \( M_W \) by elementarity and our inductive hypothesis. This is absolute to \( V \) since \( \bigcup_{\gamma < \iota_{k+1}} P(\gamma) \subseteq M_W \) by the supercompactness of \( W \). This shows (4).

Let \( \lambda = \sup_n \iota_n \). Then \( W \) is \( <\lambda \)-supercompact by Claim 3. Since \( M_W \) is closed under countable sequences, \( W \) is \( \lambda \)-supercompact. But \( j_W(\lambda) = \lambda \) and \( \text{crt}(W) < \lambda \). This contradicts Kunen’s inconsistency theorem.

We therefore reach contradictions in Case 1 and Case 2. It follows that our assumption was false, which completes the proof of Lemma 87.

In the next theorem we obtain the hypotheses of Lemma 87 from a large cardinal axiom that appears to be one ultrafilter away from optimal.

**Theorem 88 (UA).** Suppose \( \delta \) is a regular cardinal. Suppose there are distinct countably complete ultrafilters extending the club filter on \( \delta \). Let \( U_0 <_s U_1 \) be the \( <_s \)-least two.

Let \( \kappa \leq \delta \) be the least ordinal such that for some countably complete ultrafilter \( Z \), \( j_Z(\kappa) > \delta \). Then the following hold:

1. \( U_0 \) is \( \kappa \)-complete and \( (\kappa, \delta) \)-regular.

2. \( U_1 \) is the weakly normal ultrafilter of a normal fine \( \kappa \)-complete ultrafilter on \( P_\kappa(\delta) \), and \( U_0 <_M U_1 \).

(1) has strong consequences just short of supercompactness by Theorem 69. We will use this below to show that (1) implies (2).

For the proof of Theorem 88 we use a lemma that is probably part of the folklore, at least in the special case that \( F \) is a normal ultrafilter on \( \lambda \). Here we just use the case that \( F \) is the club filter on \( \lambda \), but we prove the lemma at a higher level of generality since we need the more general version in Section 5.

**Lemma 89.** Suppose \( F \) is a normal fine filter on \( P(\lambda) \). Suppose \( D \) is a countably complete ultrafilter on \( \lambda \). Let \( B = \{ \sigma \in P^{\text{MP}}(j_D(\lambda)) : [\text{id}]_D \in \sigma \} \). Then \( j_D[F] \cup \{ B \} \) generates \( j_D(F) \).

**Proof.** Suppose \( X \in j_D(F) \). Then \( X = j_D(\langle X_\alpha : \alpha < \lambda \rangle)([\text{id}]_D) \) with \( X_\alpha \in F \) for all \( \alpha < \lambda \). By normality \( \Delta_{\alpha<\lambda} X_\alpha \in F \). We claim \( j_D(\Delta_{\alpha<\lambda} X_\alpha) \cap B \subseteq X \).

Suppose \( \sigma \in j_D(\Delta_{\alpha<\lambda} X_\alpha) \cap B \). Since \( \sigma \in j_D(\Delta_{\alpha<\lambda} X_\alpha) \), we have \( \sigma \in \bigcap_{\alpha \in \sigma} X'_\alpha \) where \( \langle X'_\alpha : \alpha < j_D(\lambda) \rangle = j_D(\langle X_\alpha : \alpha < \lambda \rangle) \). Since \( \sigma \in B \), \( [\text{id}]_D \in \sigma \), so

\[
\sigma \in X'_{[\text{id}]_D} = j_D(\langle X_\alpha : \alpha < \lambda \rangle)([\text{id}]_D) = X
\]

as desired. \( \square \)
Proof of Theorem 88. We first show that for all \( W <_S U_1, W \sqsubset U_1 \). Suppose \( W <_S U_1 \). Then by Lemma 69 with \( \mathcal{F} \) the club filter on \( \delta \), \( t_W(U_1) \) extends the club filter on \( j_W(\delta) \). Of course \( t_W(U_1) \neq j_W(U_0) \), since \( W^-(j_W(U_0)) = U_0 \) while \( W^-(t_W(U_1)) = U_1 \). So \( j_W(U_0) <^M_W t_W(U_1) \) and therefore \( j_W(U_1) \leq^M_W t_W(U_1) \) since in \( M_W \), \( j_W(U_1) \) is the \( <^M_W \)-least ultrafilter extending the club filter on \( j_W(\delta) \) apart from \( j_W(U_0) \). By definition \( t_W(U_1) \leq^M_W j_W(U_1) \), so \( t_W(U_1) = j_W(U_1) \). This implies \( W \sqsubset U_1 \) by Proposition 48.

Since in particular \( U_0 \sqsubset U_1 \), it follows that \( j_W[\delta] \) carries a weakly normal ultrafilter in \( M_{U_1} \). So the weakly normal ultrafilter on \( \delta \) derived from \( U_1 \) is not equal to \( U_0 \). Since this derived weakly normal ultrafilter extends the club filter on \( \delta \), it is equal to \( U_1 \) by the minimality of \( U_1 \). Hence \( U_1 \) is weakly normal.

We next show that (1) implies (2). Note that if (1) holds, we may apply Theorem 69 to obtain that \( U_0 \) has the tight \( \delta \)-covering property. Note that \( U_1 \) is \( \kappa \)-complete since \( \kappa \) is closed under ultrapowers. Since \( U_0 \sqsubset U_1 \), Proposition 53 implies that \( U_1 \) has the tight covering property at \( \delta \). We repeat the argument of Lemma 68 to show that \( P(\delta) \sqsubseteq M_{U_1} \). Fix a \( \kappa \)-independent family of subsets of \( \delta, \langle X_\alpha : \alpha < \delta \rangle \in M_{U_1} \), which exists since \( M_{U_1} \) is closed under \( \kappa \)-sequences and \( (\delta^{<\kappa})^M_{U_1} = \delta \). For any \( A \subseteq \delta \), there is some \( W \leq_{RK} U_0 \) on \( \delta \) such that \( X_\alpha \in W \) if and only if \( \alpha \in A \). Since \( U_0 \sqsubset U_1 \), we have \( W \sqsubset U_1 \), and therefore \( A \in M_{U_1} \). It follows that \( P(\delta) \sqsubseteq M_{U_1} \), which combined with the tight covering property implies that \( U_1 \) is \( \delta \)-supercompact.

Note that \( j_{U_1}(\kappa) > \kappa \); otherwise \( \kappa \) is \( <_{j_{U_1}(\delta)} \)-supercompact in \( M_{U_1} \), and this implies that there are many weakly normal ultrafilters \( W \) on \( \delta \) in \( M_{U_1} \), and these are truly weakly normal and internal to \( U_1 \) since \( U_1 \) is \( \delta \)-supercompact, and this contradicts the \(<_S \delta \)-minimality of \( U_1 \) since the seed order extends the internal relation on \( U_{\delta} \). Now \( j_{U_1}(\kappa) > \delta \) since otherwise \( \kappa \) is huge, which contradicts that \( \kappa \) is closed under ultrapowers. Thus \( U_1 \) is the weakly normal ultrafilter of a normal fine ultrafilter on \( P(\kappa)(\delta) \). Finally \( U_0 <_M U_1 \) since \( U_0 \sqsubset U_1 \) and \( U_1 \) is \( \delta \)-supercompact.

Thus (1) implies (2).

We finally prove (1). Let \( \kappa_0 = \text{crt}(U_0) \). If \( j_{U_1}(\kappa_0) > \kappa_0 \), then (1) follows from Lemma 87. Assume instead \( j_{U_1}(\kappa_0) = \kappa_0 \). Then in \( M_{U_1} \), \( j_{U_1}(U_0) \) has critical point \( \kappa_0 \). Let \( \delta' = \text{cf}^M_{U_1}(\text{sup} j_{U_1}[\delta]) \), and let \( U_* \) denote the least ultrafilter on \( \delta' \) as computed in \( M_{U_1} \). By Lemma 55 \( U_* \) is equivalent in \( M_{U_1} \) to \( t_{U_1}(U_0) = s_{U_1}(U_0) \), and hence \( j_{U_*} = j_{U_0} \upharpoonright M_{U_1} \).

Since \( \delta' < j_{U_1}(\delta) \), Theorem 61 implies \( U_* \equiv^M_{U_1} j_{U_1}(U_0) \). Hence in \( M_{U_1} \), \( W = j_{U_1}(U_0) \) witnesses the hypothesis of Theorem 86 with respect to \( U \). It follows that in \( M_{U_1} \), \( \kappa_0 \) is closed under ultrapowers and \( U_* \) is \( (\kappa_0, \delta') \)-regular.
Since for all $W < S U_1$, $W \subseteq U_1$, this implies $\kappa_0$ is closed under ultrapowers in $V$. Moreover, since $U_*$ is $(\kappa_0, \delta')$-regular in $M_{U_1}$ and $j_{U_*}^{M_{U_1}} = j_{U_0} | M_{U_1}$, $U_0$ is discontinuous at every regular cardinal in the interval $[\kappa_0, \delta]$. Hence $U_0$ is $(\kappa_0, \delta)$-regular by Theorem 70. Therefore (1) holds in this case as well. \hfill \Box

As a corollary, if a least ultrafilter $U$ interacts nontrivially with an ultrafilter above it, then $U$ is well-behaved in the sense of Theorem 69:

**Corollary 90 (UA).** Let $U$ be the least ultrafilter on a regular cardinal $\delta$, and let $\kappa$ be its completeness. Suppose there is a countably complete ultrafilter that is neither divisible by $U$ nor internal to $U$. Then $\kappa$ is $\delta$-supercompact and closed under ultrapowers.

**Proof.** Let $W$ be such an ultrafilter. If $W$ is continuous at $\delta$, then $W \subseteq U$ by Theorem 61. Consider the weakly normal ultrafilter $D$ on $\delta$ derived from $W$. To finish, it suffices by Theorem 88 to show that $D \neq U$. But if $D = U$, then $U$ divides $W$, this time by Theorem 59. \hfill \Box

Using Lemma 85 and Corollary 90, we can prove Theorem 81.

**Proof of Theorem 81.** Let $W$ be the least ultrafilter on $\gamma$. Let $\kappa$ be its completeness. Clearly $W$ does not divide $U$. By Theorem 61, $W \subseteq U$. By Lemma 85, $U$ is discontinuous at $\gamma$, so since $W$ is 0-order with respect to $\gamma$, $U \nsubseteq W$.

Now by Corollary 90, $\kappa$ is $\gamma$-supercompact and closed under ultrapowers. By Corollary 76, $U$ is $\kappa$-complete. Since $U \nsubseteq U$ by Theorem 61, $U$ is $\gamma$-supercompact by Proposition 52. \hfill \Box

### 4.3 Some cardinal arithmetic

Part of the reason for proving these theorems with as few cardinal arithmetic assumptions as we can manage is that it allows us to improve the result that UA implies GCH.

We begin by mentioning a result that suffices to prove GCH above a supercompact.

**Definition 91.** We say $\gamma$ is $M$-commanded if every $A \subseteq P(\gamma)$ belongs to $M_W$ for some $W \in Un_{\leq \gamma}$.

Of course $M$-command follows from supercompactness by an argument due to Solovay:
Theorem 92 (Solovay). Suppose that $U$ is $\lambda$-supercompact, $\text{cf}(\lambda) \geq \text{CRT}(U)$, and the pre-normal ultrafilter $D$ on $\lambda$ derived from $U$ belongs to $M_U$. Then $\lambda$ is $M$-commanded in $M_U$.

Proof. Suppose not. Let $k : M_D \to M_U$ be the factor embedding. Since $k(\lambda) = \lambda$, $\lambda$ is not $M$-commanded in $M_D$. Take $A \subseteq P(\lambda)$ with $A$ in $M_D$ such that for no $Z \in \text{Un}_{\leq \lambda}^M$ does $A$ belong to $M_Z^{M_D}$. Then since $k(A) = A$, for no $Z \in \text{Un}_{\leq \lambda}^M$ does $A$ belong to $M_Z^{M_U}$. But $A \in M_D^{M_U}$: by Kunen’s inconsistency theorem there is some inaccessible $\kappa \leq \delta$ with $j_D(\kappa) > \delta$, and

$$A \in V_{j_D(\kappa)}^{M_D} = j_D(V_\kappa) = j_D(V_\kappa^{M_U}) = V_{j_D(\kappa)}^{M_U^{M_D}}$$

Since $D \in \text{Un}_{\leq \lambda}^M$, this is a contradiction. \hfill $\square$

The first proof of GCH above a supercompact from UA and large cardinals was based on the following fact.

Theorem 93 (UA). Suppose $\gamma$ is $M$-commanded. Suppose $\delta > \gamma$ is a regular cardinal that carries a uniform countably complete ultrafilter. Then $2^{\gamma} < 2^\delta$.

Sketch. Let $U$ be the least ultrafilter on $\delta$. Assume towards a contradiction that $2^{\gamma} = 2^\delta$. We can then code $U$ by $A \subseteq P(\gamma)$, so fix $W \in \text{Un}_{\leq \gamma}$ such that $U <_M W$. Note that $W \triangleleft U$ by Theorem [61] and similarly $U \triangleleft^M j_W(U)$ since $j_W(\delta) > \delta$ by Kunen’s inconsistency theorem. This implies $j_U^M \triangleleft \text{Ord}$ is amenable to $M_U$. In particular, $j_U \upharpoonright \gamma = j_U^M \upharpoonright \gamma \in M_U$ so $U$ is $\gamma$-supercompact. But then $W \triangleleft U$ implies $W <_M U$, contradicting the strictness of the Mitchell order. \hfill $\square$

Buried in the reductio was the first hint that UA might prove the supercompactness of least ultrafilters.

In fact, using the theory of the internal relation developed here, one can actually prove the following theorem:

Theorem 94 (UA). Suppose $\delta$ is an infinite cardinal. Suppose $W \in \text{Un}_\delta$ and $\gamma$ is such that $\text{Un}_{\leq \gamma} \triangleleft W$. Then $|\text{Un}_{\leq \gamma}| \leq 2^\delta$.

Proof. Assume by induction that the theorem is true for all $\alpha < \gamma$. Assume towards a contradiction that $|\text{Un}_{\leq \gamma}| \geq (2^\delta)^+$. For all $\alpha < \gamma$, $|\text{Un}_{\leq \xi}| \leq 2^\xi$, and so since we must have $\gamma < \delta$, $|\text{Un}_{\leq \gamma}| \leq \gamma \cdot 2^\delta = 2^\delta$. Hence $|\text{Un}_{\gamma}| \geq (2^\delta)^+$.

Note that for any $U \in \text{Un}_\gamma$, if $D <_S U$ then there is a sequence $\langle D_\alpha : \alpha < \gamma \rangle$ such that $\text{sp}(D_\alpha) \leq \alpha$ for all $\alpha < \gamma$ with the following property:

$$D = \{X \subseteq \text{sp}(D) : \{\alpha < \gamma : X \cap \text{sp}(D_\alpha) \in D_\alpha\} \in U\}$$
(Using Los’s theorem this just says $D = U^-(|D|_U)$. Thus $U$ has at most $\prod_{\alpha < \gamma} |Un_{\alpha}| \leq (2^\delta)^\gamma = 2^\delta$ predecessors in the seed order. It follows that the seed order on $Un_\gamma$ has order type exactly $(2^\delta)^+$: it is a wellorder of cardinality $(2^\delta)^+$ with initial segments of cardinality $2^\delta$.

For $U \in Un$, let $|U|_S$ denote the rank of $U$ in the seed order. For any $U, D \in Un$, we claim $|U|_S \leq |t_D(U)|^{MD}_S$. To see this, note that $t_D$ maps the $<_S$-predecessors of $U$ into the $<_S^{MD}$-predecessors of $t_D(U)$, preserving the seed order, by Proposition 25. In particular if $U$ is nonprincipal, then

$$|U|_S \leq |t_U(U)|^{MU}_S < |j_U(U)|^{MU}_S = j_U(|U|_S)$$

In other words, every nonprincipal ultrafilter moves its own seed rank. (More generally if $j_D(|U|_S) = |U|_S$ then $D \sqsubseteq U$. In many cases, for example for ultrafilters extending the club filter, we can also show the converse.)

Let $\eta = |U|_S$ where $U$ is least on $\gamma$. If $\eta \leq |D|_S < (2^\delta)^+$, then $D$ is a uniform ultrafilter on $\gamma$, and in particular $D$ is nonprincipal, so $D$ moves $|D|_S$. It follows that every $\alpha$ such that $\eta \leq \alpha < (2^\delta)^+$ is moved by some $D \in Un_\gamma$. (We remark that this hypothesis can be obtained in ZFC from $\delta$-compactness, and this is due to Kunen.)

Note that $j_W((2^\delta)^+) = (2^\delta)^+$ since $sp(W) = \delta$. Therefore $W$ has an $\omega$-club of fixed points below $(2^\delta)^+$. Moreover since $|Un_{<\gamma}| \leq 2^\delta$ and each $D \in Un_{<\gamma}$ fixes $\delta$, the set of common fixed points of elements of $Un_{<\gamma}$ is $\omega$-club in $(2^\delta)^+$. Let $\xi \in [\eta, (2^\delta)^+]$ be fixed by $W$ and by all ultrafilters in $Un_{<\gamma}$.

Let $\bar{W}$ be the least ultrafilter moving $\xi$. Then $\bar{W} \in Un_{\gamma}$. By Theorem 49, $W \sqsubseteq \bar{W}$ and $Un_{<\gamma} \sqsubseteq \bar{W}$. By assumption, $\bar{W} \sqsubseteq W$. So by Theorem 47, $W$ and $\bar{W}$ commute.

Let $\bar{\kappa} = \text{crt}(\bar{W})$ and $\kappa = \text{crt}(W)$. Commutativity implies $j_W(\bar{\kappa}) = \bar{\kappa}$ and $j_W(\kappa) = \kappa$. In particular $\bar{\kappa} \neq \kappa$.

Also $\kappa \neq \gamma$ since $j_W$ fixes $\kappa$ but not $\gamma$. We cannot have $\gamma < \kappa$ since $|Un_\gamma| > 2^\delta > \kappa$ while $\kappa$ is strongly inaccessible. It follows that $\kappa < \gamma$.

Note that $\kappa$ is a strong limit cardinal fixed by $j_W$ and $Un_{<\kappa} \sqsubseteq Un_{<\gamma} \sqsubseteq \bar{W}$. By Corollary 75, it follows that $\kappa \leq \bar{\kappa}$.

Similarly, $\bar{\kappa}$ is a strong limit cardinal fixed by $j_W$ and $Un_{<\bar{\kappa}} \sqsubseteq Un_{<\gamma} \sqsubseteq W$. By Corollary 75, it follows that $\bar{\kappa} \leq \kappa$.

Since $\bar{\kappa} \neq \kappa$, we have reached a contradiction.

The following generalization of Theorem 61 to singular cardinals is useful for obtaining the hypotheses of Theorem 94 (for example in Corollary 96 and Theorem 100) at singular cardinals. On the other hand, it seems quite possible that the hypotheses of Theorem 95 actually imply $\lambda$ is regular. In any case, the idea behind Theorem 95 might lead to a proof of this.
Theorem 95 (UA). Suppose $\lambda$ is a cardinal that carries a strongly uniform ultrafilter but $\lambda$ is not a singular limit of cardinals carrying strongly uniform ultrafilters. Suppose $W$ is the least strongly uniform ultrafilter on $\lambda$. Then $\text{Un}_{\lambda} \subseteq W$.

Proof. Suppose $D \in \text{Un}_{\lambda}$. Then $t_D(W)$ is equivalent to a strongly uniform ultrafilter on a cardinal $\lambda_* \geq \sup j_D[\lambda]$ since $W$ divides $D \oplus t_D(W)$. On the other hand $\lambda_* \leq j_D(\lambda)$ since $t_D(W) \leq S_D j_D(W)$.

Assume $\lambda_* < j_D(\lambda)$ towards a contradiction. Then $D$ is discontinuous at $\lambda$, so $\lambda$ is singular. Moreover by the usual reflection argument, $\lambda$ is a limit of cardinals carrying strongly uniform ultrafilters, contradicting the minimality of $\lambda$. Therefore our assumption was false, and $\lambda_* = j_D(\lambda)$.

Thus $t_D(W)$ is equivalent to a strongly uniform ultrafilter on $j_D(\lambda)$. Since $t_D(W) \leq S_D j_D(W)$ and $j_D(W)$ is the least strongly uniform ultrafilter on $j_D(\lambda)$, $t_D(W) = j_D(W)$. Thus $D \subseteq W$ by Proposition 18.

The following corollary generalizes a well-known fact regarding the Mitchell order on normal ultrafilters. A direct generalization of this that does not use UA will only show that $U \in \text{Un}_\delta$ has $2^{2^{\delta}}$ many predecessors in the seed order. This is for good reason: suppose $2^{2^{\delta}} = 2^{2^{\delta^+}}$, and $|\text{Un}_\delta| > 2^{\delta}$, and $\text{Un}_{\delta^+} \neq \emptyset$, a hypothesis that is easily proved consistent with ZFC from the existence of a cardinal $\delta$ that is $2^{\delta}$-supercompact. Then by Lemma 13, any $W \in \text{Un}_{\delta^+}$ lies above every $U \in \text{Un}_\delta$ in the seed order, and hence has more than $2^{\delta}$ predecessors. So Corollary 96 is not provable in ZFC.

Corollary 96 (UA). Suppose $\delta$ is a cardinal that carries a strongly uniform ultrafilter. Then any $U \in \text{Un}_\delta$ has at most $2^{\delta}$ predecessors in the seed order.

Proof. Assume by induction that the theorem holds at all cardinals below $\delta$.

If $\delta$ is not a singular limit of cardinals carrying strongly uniform ultrafilters, then using Theorem 95, the least strongly uniform $W$ on $\delta$ satisfies the hypotheses of Theorem 94 with respect to any $\alpha < \delta$, and therefore $|\text{Un}_\alpha| \leq 2^\alpha$ for all $\alpha < \delta$. Suppose instead that the set $A \subseteq \delta$ of cardinals carrying strongly uniform ultrafilters is unbounded in $\delta$. Let $B$ be the set of successor elements of $A$. Then every $\lambda \in B$ satisfies the hypotheses of Theorem 94 by Theorem 95. Now for any $\alpha \leq \delta$, take $\lambda \in B$ with $\alpha < \lambda$, and note that again by Theorem 94, $|\text{Un}_\alpha| \leq 2^\lambda \leq 2^\delta$.

In either case, therefore, $|\text{Un}_{<\delta}| \leq 2^\delta$. By the counting argument from the beginning of the proof of Theorem 94, it follows that any $U \in \text{Un}_\delta$ has at most $\prod_{\alpha < \delta} \text{Un}_{\leq \alpha} \leq 2^\delta$ predecessors.
An old observation of Solovay is that the linearity of the Mitchell order on normal ultrafilters on a cardinal $\kappa$ that carries $2^{2^\kappa}$ normal ultrafilters implies $2^{2^\kappa} = 2^{\kappa^+}$. By the previous theorem, one has the following generalization:

**Theorem 97 (UA).** Suppose $|\text{Un}_\delta| = 2^{2^\delta}$. Then $2^{2^\delta} = (2^\delta)^+$. 

**Sketch.** Let $\bar{\delta} \leq \delta$ be least such that $|\text{Un}_{\leq \bar{\delta}}| = 2^{2^\bar{\delta}}$. Easily $|\text{Un}_{< \bar{\delta}}| < 2^{2^\bar{\delta}}$. If $\bar{\delta}$ does not carry a strongly uniform ultrafilter then an easy counting argument implies $|\text{Un}_\delta| \leq [\delta]^{< \delta}, |\text{Un}_{< \delta}| < 2^{2^\delta}$, a contradiction. So $\bar{\delta}$ carries a strongly uniform ultrafilter. Therefore by Corollary 96, $\text{Un}_\delta$ is wellordered by the seed order with initial segments of cardinality $2^\delta$. It follows that $2^{2^\delta} = (2^\delta)^+$ as desired. □

In fact Theorem 97 implies much more interesting instances of GCH.

**Proposition 98 (UA).** Suppose $\delta$ is a cardinal, $|\text{Un}_\delta| = 2^{2^\delta}$, and $|\text{Un}_{\delta^+}| > 2^{\delta^+}$. Let $\lambda > \delta^+$ be least carrying a strongly uniform ultrafilter. Then $2^\delta < \lambda$.

**Proof.** Since $\text{Un}_{< \delta} \subseteq U$ where $U$ is least on $\delta^+$ by Theorem 61 we can apply Theorem 94 to obtain

$$2^{2^\delta} \leq |\text{Un}_{< \delta}| \leq 2^{\delta^+}$$

Since $\text{Un}_{< \lambda} \subseteq U$ where $U$ is the $<_S$-least strongly uniform ultrafilter on $\lambda$ by Theorem 95 we can apply Theorem 94 to obtain

$$2^{\delta^+} < |\text{Un}_{\delta^+}| \leq 2^\lambda$$

Combining these facts, $2^{2^\delta} < 2^\lambda$. Hence $2^\delta < \lambda$. □

In the case that $\lambda = \delta^{++}$, we have the following corollary:

**Theorem 99 (UA).** Suppose $\delta$ is a cardinal, $|\text{Un}_\delta| = 2^{2^\delta}$, and $|\text{Un}_{\delta^+}| > 2^{\delta^+}$, and $\text{Un}_{\delta^{++}} \neq \emptyset$. Then $2^\delta = \delta^+$. 

A somewhat subtler argument using the techniques of this paper improves the hypotheses above.

**Theorem 100 (UA).** Suppose $\delta$ is a cardinal and $\text{Un}_{(2^\delta)^+} \neq \emptyset$. Then $2^\delta = \delta^+$. 

**Proof.** Let $\delta$ be the least cardinal at which the theorem fails.

Let $U$ be the least ultrafilter on $(2^\delta)^+$. Let $\tau = \text{cf}(2^\delta)$, so $\tau > \delta$ by Konig’s theorem. Note that $U$ is $\tau$-supercompact: if $U$ is continuous at $2^\delta$ then $U$ is $(2^\delta)^+$-supercompact by the argument of Corollary 82 (3), and if $U$ is discontinuous at $2^\delta$ then $\text{Un}_\tau \neq \emptyset$ so we can appeal to Theorem 81 to conclude that $U$ is $\tau$-supercompact.
Assume first that $\delta$ is a singular cardinal. Then by the $\iota$-supercompactness of $U$, $\delta$ is a limit of regular cardinals carrying uniform ultrafilters. By Theorem 92 at all sufficiently large regular $\delta < \delta$, one has in $M_U$ the hypotheses of Theorem 100 for some $\lambda < \delta$. Hence $2^\delta < \lambda$. It follows that $\delta$ is a strong limit cardinal. Therefore by Solovay’s theorem [8] on SCH, $2^\delta = \delta^+$ since $\delta$ is singular.

We may therefore assume that $\delta$ is regular. By Theorem 92, $M_U \models |\text{Un}_\gamma| = 2^{2^\gamma}$ for $\gamma \in \{\delta, \delta^+\}$.

Since $\delta^+ < 2^\delta$, if $U$ is $\delta^{++}$-supercompact, then by Theorem 92 we have the hypotheses of Theorem 99 in $M_U$, so that $2^\delta = \delta^+$ in $M_U$, which is absolute to $V$ since $P(\delta) \subseteq M_U$. We may therefore assume $\iota < \delta^{++}$, so that $\iota = \delta^+$.

If $2^\delta$ is regular then the fact that $\iota = \delta^+$ implies the theorem. So we may assume $2^\delta$ is singular and in particular is a limit cardinal.

Assume first that $U$ has the tight covering property at $(2^\delta)^+$. Then since $j_U(2^\delta) > (2^\delta)^+$, $U$ is $(2^\delta, (2^\delta)^+)$-regular, and hence $U$ is discontinuous at cofinally many regular cardinals below $2^\delta$. It follows that $U$ is $(2^\delta)^+$-supercompact by Corollary 82 (2).

Therefore we may assume that $U$ does not have the tight covering property at $(2^\delta)^+$.

Assume first that $((2^\delta)^+)M_U = (2^\delta)^+$. Then since $j_U(2^\delta) > (2^\delta)^+$, $U$ is $(2^\delta, (2^\delta)^+)$-regular, and hence $U$ is discontinuous at cofinally many regular cardinals below $2^\delta$. It follows that $U$ is $(2^\delta)^+$-supercompact by Corollary 82 (2).

Therefore we may assume that $U$ does not have the tight covering property at $(2^\delta)^+$.

Assume first that $((2^\delta)^+)M_U < (2^\delta)^+$. Then the hypotheses of the theorem remain true in $M_U$, since $U \cap M_U \in M_U$ by Corollary 65. Note that $j_U(\delta) > \delta$ by Kunen’s inconsistency theorem, and so $\delta$ is below the least failure of the theorem in $M_U$. Therefore $2^\delta = \delta^+$ in $M_U$, and so since $P(\delta) \subseteq M_U$, $2^\delta = \delta^+$ in $V$, a contradiction.

Finally assume that $((2^\delta)^+)M_U < (2^\delta)^+$. Work in $M_U$. Since $|\text{Un}_\delta| = 2^{2^\delta}$, we have $2^{2^\delta} = (2^\delta)^+$. Hence $2^{\delta^+} \leq (2^\delta)^+$.

Returning to $V$, since $P(\delta^+) \subseteq M_U$, the fact that $M_U \models 2^{\delta^+} \leq (2^\delta)^+$ implies $2^{\delta^+} \leq |((2^\delta)^+)M_U| = 2^\delta$. Therefore $2^{\delta^+} = 2^\delta$. By König’s theorem, this contradicts the fact that $\text{cf}(2^\delta) = \iota = \delta^+$. \hfill $\Box$

Assume UA. If $\kappa$ is $\kappa$-supercompact, can $2^\kappa$ be weakly inaccessible? The previous theorem does not give much insight since $(2^\delta)^+$ is a bit of a moving target. Our next theorem rules this out.

**Theorem 101 (UA).** Suppose $\delta$ is a regular cardinal and $\delta^{++}$ carries two countably complete ultrafilters extending the club filter. Then $2^\delta = \delta^+$.

**Proof.** Let $W$ be the second such ultrafilter on $\delta^{++}$. By Theorem 88, $W$ is $\delta^{++}$-supercompact and equivalent to a normal fine $\kappa$-complete ultrafilter on $P_\kappa(\delta^{++})$ for some $\kappa < \delta^{++}$. 

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Suppose $\gamma = \delta$ or $\gamma = \delta^+$. (The argument that follows works in either case.) Let $U$ be the pre-normal ultrafilter on $\gamma$ derived from $W$; thus $U$ is equivalent to a normal fine $\kappa$-complete ultrafilter on $P_\kappa(\gamma)$. Note that $U <_M W$ by the proof of Theorem 88. By an argument due to Solovay, for any $A \subseteq P(\gamma)$, if $A \in M_W$, then for some normal fine normal fine $\kappa$-complete ultrafilter $\mathcal{D}$ on $P_\kappa(\gamma)$ with $\mathcal{D} <_M W$, $A \in M^M_{\mathcal{D}W}$. Such an ultrafilter $\mathcal{D}$ satisfies $|P(P(\gamma)) \cap M^M_{\mathcal{D}W}| = 2^\gamma$, and so a simple counting argument implies that $M_W$ thinks there are $2^{2^\gamma}$ such ultrafilters. Since each such ultrafilter is equivalent to a unique weakly normal ultrafilter on $\gamma$ by Solovay’s lemma, it follows that $M_W$ satisfies $|\text{Un}_\gamma| = 2^{2^\gamma}$.

But now $M_W$ satisfies the hypotheses of Theorem 100 at $\delta$. Since $\lambda = \delta^{++}$ carries a uniform ultrafilter in $M_W$ by Theorem 88 in $M_W$, $2^{\delta} < \delta^{++}$. Hence $M_W$ thinks $2^{\delta} = \delta^+$. Since $P(\delta) \subseteq M_W$, in fact $2^{\delta} = \delta^+$ in $V$. 

**Corollary 102 (UA).** If $\delta$ is a regular cardinal and $2^{\delta}$ carries a strongly uniform countably complete ultrafilter, then $2^{\delta} \leq \delta^{++}$.

**Corollary 103 (UA).** Suppose $\lambda$ is a limit of regular cardinals carrying uniform countably complete ultrafilters. Then GCH holds on a tail below $\lambda$.

As a corollary of this, if $\lambda$ is singular and $\lambda^+$ carries a two countably complete ultrafilters extending the club filter then $2^{\lambda} = \lambda^+$; by Theorem 88 some $\kappa < \lambda$ is $\lambda^+$-supercompact, so by Corollary 103 $\lambda$ is a strong limit cardinal, and hence by Solovay 10, $2^{\lambda} = \lambda^+$.

The following theorem appears in 11:

**Theorem 104 (UA).** Suppose $\lambda$ satisfies $2^{<\lambda} = \lambda$. Then the internal relation is linear on normal fine ultrafilters on $P(\lambda)$.

As we explained there, this is essentially the same as saying that the Mitchell order is linear. By the results here we can remove the GCH hypothesis in all but two cases. Actually that was the original impetus for this work, although its other applications turned out to be much more interesting. As a corollary of the local GCH results, in many cases the hypothesis $2^{<\lambda} = \lambda$ can be omitted since it simply follows from the existence of a normal fine ultrafilter on $P(\lambda)$.

**Theorem 105 (UA).** Suppose $\lambda$ is a limit cardinal, the successor of a singular cardinal, or the double successor of a cardinal of cofinality greater than or equal to the least $\lambda$-supercompact cardinal. Then the internal relation is linear on normal fine ultrafilters on $P(\lambda)$.

In other words, the only cases we cannot handle by current techniques are successors of inaccessible cardinals and double successors of singulars of small cofinality.
5 The Next Strongly Compact Cardinal

The point of this section is to extend the global results of the previous section beyond the least strongly compact cardinal and the local results beyond the least ultrafilters. Our target theorem is the following:

**Theorem 106 (UA).** Suppose \( \kappa \) is strongly compact. Then either \( \kappa \) is supercompact or \( \kappa \) is a measurable limit of supercompact cardinals.

In a sense this is best possible, since every measurable limit of supercompact cardinals is strongly compact by a construction of Menas. Better yet, UA + GCH yields a fairly satisfying local analysis of strong compactness that implies that the only way to obtain strong compactness in the absence of supercompactness is Menas's construction.

5.1 Factorization into irreducibles

**Definition 107.** An ultrafilter \( W \) is **irreducible** if for all \( U \leq_D W \), either \( U \) is principal or \( U \equiv W \).

A key structural consequence of UA, which relatively easy compared to the results of this paper, is an ultrafilter factorization theorem:

**Theorem 108 (UA).** Every countably complete ultrafilter factors as a finite iteration of irreducible ultrafilters.

**Proof.** Suppose towards a contradiction that \( U \in \text{Un} \) is \( <_S \)-least at which the theorem fails. Then \( U \) is reducible. Let \( D \) be a proper divisor of \( U \). Then without loss of generality \( D <_S U \). Moreover by Theorem 31 \( U \equiv D \oplus t_D(U) \). Since \( D \leq_D U \) and \( D \) is nonprincipal, \( D \nsubseteq U \) otherwise \( j_D \upharpoonright \text{Ord} \) is amenable to \( M_U \subseteq M_D \), a contradiction. Hence \( t_D(U) \neq j_D(U) \) by Proposition 48. It follows that \( t_D(U) <_{S_{M_D}} j_D(U) \) by the definition of translation functions and the linearity of the seed order.

By the minimality of \( U \), \( D \) factors as a finite iteration of irreducible ultrafilters. In \( M_D \), by the minimality of \( j_D(U) \), \( t_D(U) \) factors as a finite iteration of irreducible ultrafilters. Composing the two factorizations yields a factorization of \( U \) as a finite iteration of irreducible ultrafilters, a contradiction. \( \square \)

We acknowledge that the argument in Theorem 108 that \( t_D(U) \neq j_D(U) \) does not really require using the internal relation, let alone UA. We are ultimately reproving with our notation the standard fact that if \( U \) and \( D \) are
countably complete ultrafilters and $D$ is nonprincipal then $U \neq D \times U$. Obvi-
ous as it may appear, Theorem 108 itself is not provable in ZFC by a theorem
of Gitik [3].

The factorization theorem is only useful if one can analyze irreducible ultra-
filters. In the next section we will show that they are supercompact up to their
spaces. For this we need a different sort of factorization lemma, which states,
assuming enough GCH, that ultrafilters factor at their continuity points. Its
proof is the source of most of our cardinal arithmetic woes.

**Theorem 109 (UA).** Suppose $U$ is a $\kappa$-complete ultrafilter and $\delta$ is a regular
 cardinal such that $j_U(\delta) = \sup j_U[\delta]$. Let $\lambda = \text{ot}(\text{Un}_{<\delta}, <_S)$. Suppose $2^\lambda < \delta^{+\kappa}$. Then $U$ factors as $D \oplus Z$ where $\text{sp}(D) < \delta$ and $\text{crt}(Z) > j_D(\delta)$.

**Proof.** Take $p : \text{sp}(U) \to 2^\lambda$ such that letting $W = p_*(U)$ and $k : M_W \to M_U$
be the factor embedding, $\text{crt}(k) > \lambda$. Note that if $U$ projects to a uniform
ultrafilter on a cardinal between $\delta$ and $\delta^{+\kappa}$, then $U$ is discontinuous at $\delta$. Hence
we may replace $W$ with an equivalent uniform ultrafilter $D$ on some cardinal
$\gamma < \delta$.

Note that the map $Z \mapsto D \oplus Z$ is an order embedding from $(\text{Un}_{<\delta}^{M_D}, <_S^{M_D})$
to $(\text{Un}_{<\delta}, <_S)$ by Lemma 21. Therefore $j_D(\lambda) = \text{ot}(\text{Un}_{<j_D(\delta)}^{M_D}, <_S^{M_D}) \leq \lambda$, so
$j_D(\lambda) = \lambda$. Thus since $\text{crt}(k) > \lambda$, $\text{ot}(\text{Un}_{<j_D(\delta)}^{M_U}, <_S^{M_U}) = \lambda$. It follows that
$\text{Un}_{<j_D(\delta)}^{M_U} \subseteq \text{ran}(k)$. By Theorem 23, $t_U(D) \leq S P_{k[[\text{id}]](D)}^{M_U}$. But since $\text{sp}(D) = \gamma$,
we have $[\text{id}]_D < j_D(\gamma)$, so $k([\text{id}]) < j_D(\delta)$. So $t_U(D) \in \text{Un}_{<j_D(\delta)}^{M_U}$.

Hence $t_U(D) \in \text{ran}(k)$. Let $D_* = k^{-1}(t_U(D))$. It is easy to see that
$D^-(D_*) = D$. So by the definition of translation functions,

$$D_* \geq_S^{M_D} t_D(D) = P_{[\text{id}]_D}^{M_D}$$

On the other hand since $t_U(D) \leq_S^{M_U} P_{k[[\text{id}]]_D}^{M_U}$,

$$D_* \leq_S^{M_D} P_{[\text{id}]_D}^{M_D}$$

It follows that $D_* = P_{[\text{id}]_D}^{M_D}$. Hence $t_U(D) = P_{k[[\text{id}]]_D}^{M_U}$, so $D$ divides $U$ and
moreover $k = j_D(D)$ by Theorem 23.

Taking $Z = t_D(U)$ we therefore have $U \equiv D \oplus Z$ where $\text{sp}(D) < \delta$ and
$\text{crt}(Z) > \lambda = j_D(\lambda) > j_D(\delta)$. This proves the theorem.

**5.2 The structure of irreducible ultrafilters**

The main theorem of this subsection is a structure theorem for irreducible ultrafilters. This structure leads almost immediately to Theorem 106.


Theorem 110 (UA + GCH). Suppose $W$ is an irreducible strongly uniform ultrafilter on $\lambda$. Then for every successor cardinal $\delta \leq \lambda$, $W$ is $\delta$-supercompact.

Before proving Theorem 110, we prove a well-known fact that is a version of Solovay’s Lemma.

Lemma 111 (Solovay). There is a formula $\varphi(x)$ in the language of set theory with an extra predicate $\hat{S}$ with the following property:

Suppose $\delta$ is a regular cardinal and $S = \langle S_\alpha : \alpha < \delta \rangle$ is a partition of $S^4$ into stationary sets. Set $A = \{\sigma : (P_\delta(\delta), \in, \hat{S}) \models \varphi(\sigma)\}$. Then the sup function is one-to-one on $A$ any normal fine ultrafilter on $P_\delta(\delta)$ contains $A$.

Proof. We let $\varphi(x)$ be the following formula:

$$x = \{\alpha \in \text{Ord} : \hat{S}_\alpha \text{ meets every closed cofinal subset of } \text{sup}(x)\}$$

This works by the proof of Solovay’s Lemma. \qed

Definition 112. Suppose $\delta$ is a regular cardinal and $S$ is a partition of $S^4$ into stationary sets. The Solovay set defined from $S$ at $\delta$ is the set $A$ defined by the formula $\varphi$ of Lemma 111.

Proof of Theorem 110. We may assume without loss of generality that $\text{crt}(W) < \delta$.

We first prove that $W$ is discontinuous at $\delta$; assume towards a contradiction that it is not. Otherwise by Theorem 109, $W \equiv D \oplus Z$ where $D \in \text{Un}_{<\delta}$ and $Z \in \text{Un}^{M_D}$ is $\leq j_D(\delta)$-complete. Since $W$ is irreducible, either $D$ or $Z$ is principal. Since $W$ is strongly uniform on $\lambda \geq \delta$, $D$ is principal. Thus $\text{crt}(W) > \delta$, a contradiction.

Let $U$ be the least ultrafilter on $\delta$. Note that $W$ is neither divisible by $U$ nor internal to $U$ (since $W$ is discontinuous at $\delta$), so by Corollary 90, $\text{crt}(U)$ is $\delta$-supercompact. By Theorem 69, since $\delta$ is a successor, $U$ is $\delta$-supercompact. Let $W_* = t_U(W)$

We will show $\text{crt}(W_*) > \delta$. This implies the theorem: since $\text{Ord}^\delta \subseteq M_U$, $\text{crt}(W_*) > \delta$ implies $\text{Ord}^\delta \subseteq M_{W_*} \subseteq M_W$, which implies $W$ is $\delta$-supercompact, as desired.

Since $\delta$ carries no uniform ultrafilters in $M_U$, by Theorem 109, $W_*$ factors in $M_U$ as $D \oplus Z$ for $D \in \text{Un}_{<\delta}^{M_U}$ and $Z \in \text{Un}^{(M_D)^M_U}$ with $\text{crt}(Z) > j_D(\delta)$. We will prove $D$ is principal, and hence $\text{crt}(W_*) > \delta$. Assume towards a contradiction that $D$ is nonprincipal.
Note that $D$ is a (total) ultrafilter since $U$ is $\delta$-supercompact. Moreover $D \sqsubseteq U$, so $t_U(D) = s_U(D)$ divides $W_*$ in $M_U$. Let

$$U_* = t_W(U)$$

Since $t_U(D)$ divides $W_*$ in $M_U$, $t_W(D)$ divides $U_*$ in $M_W$ by Corollary 35.

Let

$$\gamma = \text{cf}^M_W(\sup j_W[\delta])$$

By Theorem 59, $U_*$ is either principal or is the least ultrafilter on $\sup j_W[\delta]$ in $M_W$, and therefore by Lemma 55 is equivalent to the least ultrafilter on $\gamma$, which is irreducible in $M_W$ by Theorem 59 applied in $M_W$. Since $D$ is nonprincipal by assumption, and since $W$ is irreducible, $t_W(D)$ is nonprincipal in $M_W$. Since $U_*$ is either principal or irreducible in $M_W$, and $t_W(D)$ divides $U_*$ in $M_W$, $t_W(D) \equiv^M_W U_*$. It follows in particular that $U_*$ is nonprincipal and hence equivalent to the least ultrafilter on $\gamma$. (This is easy to prove directly.)

We next show that $\gamma = \dot{j}_D(\delta)$. Since $U_*$ has the tight covering property at $\gamma$ in $M_W$ and $\text{cf}^M_W(\sup j_{U_*}^M \circ j_W[\delta]) = \text{cf}^M_W(\sup j_W[\delta]) = \gamma$,

$$\gamma = \text{cf}^{M_U}_{U_*}(\sup j_{U_*}^M \circ j_W[\delta])$$

Now we calculate:

$$\text{cf}^{M_U}_{U_*}(\sup j_{U_*}^M \circ j_W[\delta]) = \text{cf}^{M_U}_{U_*}(\sup j_{U_*}^M \circ j_U[\delta])$$

$$= \text{cf}^{M_U}_{U_*}(\sup j_{Z_D}^M \circ j_D^M \circ j_U[\delta])$$

$$= \text{cf}^{M_U}_{U_*}(j_{Z_D}^M \circ j_D^M(\sup j_U[\delta]))$$

$$= j_{Z_D}^M \circ j_D^M(\text{cf}^M_U(\sup j_U[\delta]))$$

$$= j_D^M(\text{cf}^M_U(\sup j_U[\delta]))$$

$$= j_D^M(\delta)$$

$$= j_D(\delta)$$

Let $U'$ be the least ultrafilter on $\gamma$ in $M_W$, so $U'$, $U_*$, and $t_W(D)$ are all equivalent in $M_W$.

**Claim 1.** $U' = \dot{j}_D(U)$.

The claim leads immediately to a contradiction: since $U' \equiv^M_W t_W(D)$, if $U' \in M_D$ then $U'$ is principal by Theorem 33 while clearly $j_D(U)$ is nonprincipal. Therefore to complete the proof of the theorem, we just need to prove the claim.
Note that $\text{Ord}^\gamma \cap M_D = \text{Ord}^\gamma \cap M_W$ since $t_D(W) \equiv Z$ and $t_W(D) \equiv U''$ are $\gamma$-supercompact in $M_D$ and $M_W$ respectively. Let $\mathcal{U}$ be the normal fine ultrafilter on $P(\delta)$ derived from $U$ and $\mathcal{U}'$ the $M_W$-normal fine ultrafilter on $P^M_W(\gamma)$ derived from $U'$. Since $\text{Ord}^\gamma \cap M_D = \text{Ord}^\gamma \cap M_W$, we have $P^M_W(\gamma) = P^M_D(\gamma)$ and $P^M_W(P^M_W(\gamma)) = P^M_D(P^M_D(\gamma))$. Thus $\mathcal{U}'$ is an $M_D$-normal fine ultrafilter on $P^M_D(\gamma)$.

We claim that $D^-(\mathcal{U}') = \mathcal{U}$. It then follows from Lemma \[9] that $\mathcal{U}' = j_D(\mathcal{U})$: we have $j_D[\mathcal{U}] \subseteq \mathcal{U}'$ since $D^-(\mathcal{U}') = \mathcal{U}$ and $B = \{ \sigma \in P^M_D(\gamma) : [\text{id}]_D \in \sigma \} \in \mathcal{U}'$ since $\mathcal{U}'$ is fine. But then the weakly normal ultrafilter on $\gamma$ derived from $\mathcal{U}'$ is equal to $j_D(\mathcal{U})$, or in other words $U'' = j_D(\mathcal{U})$, as claimed.

To show that $D^-(\mathcal{U}') = \mathcal{U}$, we first show that $D^-(\mathcal{U}') = U$. It suffices by Corollary \[7] to show that $D^-(\mathcal{U}')$ is weakly normal and concentrates on the set $A \subseteq \delta$ of ordinals that do not carry uniform countably complete ultrafilters. Note that $j_D(A)$ is the set of ordinals that do not carry uniform countably complete ultrafilters in $M_D$, which is the same as the set of ordinals that do not carry uniform countably complete ultrafilters in $M_W$. Thus $j_D(A) \in \mathcal{U}'$ since $U'$ is the least ultrafilter on $\gamma$ in $M_W$. To show $D^-(\mathcal{U}')$ is weakly normal, note that $[\text{id}]^{M_D}_{\mathcal{U}'} = [\text{id}]^{M_W}_{\mathcal{U}'} = \sup j^{M_W}_{\mathcal{U}'}[\gamma] = \sup j^{M_D}_{\mathcal{U}'} \circ j_D[\delta]$ and hence $[\text{id}]_{D^-(\mathcal{U}')} = \sup j_D^{-1}(\mathcal{U})[\delta]$.

We finally show $D^-(\mathcal{U}') = \mathcal{U}$. Fix a stationary partition $\vec{S}$ of $S^\delta_\omega$. Let $A$ be the Solovay set defined from $\vec{S}$ at $\delta$. Then in $M_D$, $j_D(A)$ is a the Solovay set defined from $j_D(\vec{S})$ at $\gamma$. Note that $j_D(\vec{S}) \in M_W$ and $j_D(\vec{S})$ is a stationary partition of $S^\omega_\omega$ in $M_W$, since $P^{M_D}(\gamma) = P^{M_W}(\gamma)$. Since $P^{M_W}(\gamma) = P^{M_D}(\gamma)$, $j_D(A)$ is a the Solovay set defined from $j_D(\vec{S})$ in $M_W$ at $\gamma$. It follows from Lemma \[11] applied in $M_W$ that $j_D(A) \in \mathcal{U}'$.

For any $X \subseteq P^\delta_\delta(\delta)$,

\[
X \in D^-(\mathcal{U}') \iff j_D(X) \in \mathcal{U}' \iff \{ \sup \sigma : \sigma \in j_D(X) \cap j_D(A) \} \in U' \iff j_D(\{ \sup \sigma : \sigma \in X \cap A \}) \in U' \iff \{ \sup \sigma : \sigma \in X \cap A \} \in D^-(U') \iff \{ \sup \sigma : \sigma \in X \cap A \} \in U \iff X \in \mathcal{U}
\]

The second and the last equivalences follow from Lemma \[11] \qed
5.3 More supercompact cardinals

In order to prove Theorem 106, we prove a special case of Theorem 110 that requires a more manageable form of GCH. The proof is a minor variant on the proof of Theorem 110. We prove only what is needed for Theorem 106 and leave it to the reader to figure out exactly the optimal local result one can get out of this variant argument.

**Proposition 113 (UA).** Suppose $W$ is an irreducible strongly uniform ultrafilter of completeness $\kappa$ on a cardinal $\lambda$. Suppose $\nu$ is a strong limit singular cardinal of countable cofinality and $2^\nu = \nu^+ < \lambda$. Then $W$ is $\nu^+$-supercompact.

We use the following version of Theorem 109.

**Lemma 114 (UA).** Suppose $W \in \Un$ is continuous at $\nu^+$ where $\nu$ is a strong limit singular cardinal of cofinality less than the critical point $\kappa$ of $W$. Assume $2^\nu = \nu^+$. Then $W$ factors as $D \oplus Z$ where $D \in \Un_{\nu^+}$ and $\text{crt}(Z) > j_D(\nu^+)$.  

**Proof.** By Lemma 72 there is a function $p : \SP(W) \to \nu^+$ such that the factor embedding $k : M_{p_*(W)} \to M_W$ has critical point greater than $\nu$. Since $W$ is continuous at $\nu$ and $\nu^+$, there is some $D \in \Un_{\nu^+}$ with $D \equiv p_*(W)$. Note that $j_D(\nu^+) = \nu^+$ so in fact $\text{crt}(k) > j_D(\nu^+)$. Moreover
\[
t_W(D) \in \Un_{\nu^+}^{M_U} = \Un_{\nu^+}^{M_U}
\]
Since $\nu$ is a strong limit and $\text{crt}(k) > \nu$, $t_W(D) \in M_D$. Hence $t_W(D)$ is principal by Theorem 33. This implies $D$ divides $W$, so fix $Z$ witnessing this. Then by Theorem 22 $j_Z \upharpoonright \text{Ord} \leq k \upharpoonright \text{Ord}$. It follows that $\text{crt}(Z) > j_D(\nu^+)$.  

**Sketch of Proposition 113.** The proof is very similar to that of Theorem 110, with $\delta = \nu^+$, so we only highlight the differences.

We may assume without loss of generality that $\kappa < \nu$.

The only difference in showing that $\nu^+$ carries a uniform ultrafilter and that the least $U \in \Un_{\nu^+}$ is $\nu^+$-supercompact lies in replacing Theorem 109 with Lemma 114.

As in Theorem 110 we factor $W_* = t_U(W)$ across $\nu^+$: since $U$ is 0-order, by Lemma 114 applied in $M_U$, $t_U(W)$ factors as $D \oplus Z$ where $D \in \Un_{\nu^+}^{M_U}$ and $Z \in \Un_{\nu^+}^{M_U}$ is $j_D^{M_U}(\nu^+)$-complete. To apply Lemma 114 here, we need that $2^\nu = \nu^+$ in $M_U$, but this follows from the fact that $2^\nu = \nu^+$ in $V$ combined with the $\nu^+$-supercompactness of $U$.

We now prove by contradiction that $D$ is principal, which implies $W_*$ has critical point above $\nu^+$, yielding the theorem. The only part of what remains
of Theorem 110 that uses GCH is the fact that $U_* = t_W(U)$ has the tight covering property at $\gamma = c^{M_W}(\sup_j j_W(\nu^+))$ in $M_W$, which we would like to use to prove that $\gamma = j_D(\nu^+)$. But actually the situation is a bit easier in the current context: since $\nu$ is a strong limit singular cardinal and $D \in \text{Un}_{<\nu}$, $j_D(\nu^+) = \nu^+$. The argument there establishes $\gamma \leq j_D(\nu^+)$, but obviously $\nu^+ \leq \gamma$, and hence $\gamma = \nu^+.$

The remainder of the proof is identical to that of Theorem 110. □

Proof of Theorem 106. Suppose $\kappa$ is strongly compact. By a theorem due to Solovay, SCH holds above $\kappa$, which is enough to justify all our uses of Proposition 113 below.

Let $\gamma > \kappa$ be a strong limit cardinal of uncountable cofinality and let $\lambda = \gamma^+$. Let $U$ be the $<_S$-least $\kappa$-complete ultrafilter on $\lambda$. By a variant of Theorem 70 due to Ketonen, $U$ is $(\kappa, \lambda)$-regular.

If $U$ is irreducible, then by Proposition 113, $U$ is $<_\gamma$-supercompact and hence $\kappa$ is $<_\gamma$-supercompact. Suppose $U$ is not irreducible. By Theorem 108, let $D$ be a divisor of $U$ such that $U_* = t_D(U)$ is irreducible. Of course $U \equiv D \oplus U_*$ by Theorem 31. Since $U$ is weakly normal, $D \in \text{Un}_{<\delta}$. Note that $U_* <_S j_D(U)$ since $U \not\subseteq D$.

By Lemma 89, $U_*$ extends the club filter on $j_D(\lambda)$, and hence is uniform on $j_D(\lambda)$. Let $\kappa_* = \text{crt}(U_*)$. Since $U \equiv D \oplus U_*$, $\kappa_* \geq \kappa$. On the other hand $\kappa_* < j_D(\kappa)$ since $j_D(U) >_S U_*$ is the $<_{\text{S} M_D}$-least $j_D(\kappa)$-complete uniform ultrafilter on $j_D(\lambda)$. By Proposition 113 applied in $M_D$, $U_*$ witnesses $\kappa_* \in [\kappa, j_D(\kappa))$ is $<_j(\gamma)$-supercompact in $M_D$.

Obviously $\text{crt}(D) \geq \kappa_*$, but since as we have seen $\kappa \leq \kappa_* < j_D(\kappa)$, in fact $j_D(\kappa) \neq \kappa$, so $\text{crt}(D) = \kappa$. Since in $M_D$ there is a $<_j(\gamma)$-supercompact cardinal $\kappa_* \in [\kappa, j_D(\kappa))$, the usual reflection argument implies $\kappa$ is a limit of $<_j(\gamma)$-supercompact cardinals.

The theorem is proved by taking $\gamma$ to absolute infinity and using a simple pigeonhole argument. □

5.4 Some applications

In this section we give a few applications of Theorem 110: a characterization of weakly normal ultrafilters and an application to huge cardinals.

Our first application is essentially a restatement of Theorem 110 that clarifies how it is related to Solovay’s program described in the introduction.

Definition 115. A countably complete strongly uniform ultrafilter $U$ on a cardinal $\lambda$ is pre-normal if it is Rudin-Keisler minimal among all strongly uniform ultrafilters on $\lambda$.
If $\lambda$ is regular, then $U$ is pre-normal if and only if $U$ is weakly normal. On the other hand, if $\lambda$ is singular, then no weakly normal ultrafilter on $\lambda$ is strongly uniform by Lemma 55.

**Definition 116.** Suppose $\lambda$ is a cardinal and $U$ is a $\lambda$-decomposable countably complete ultrafilter. The *pre-normal ultrafilter on $\lambda$ derived from $U$* is the ultrafilter derived from $U$ using the least generator $\theta$ of $U$ such that $\theta \geq \sup j_U[\lambda]$.

The name is inspired by the following theorem from [11]:

**Theorem 117.** Suppose $U$ is a $\lambda$-supercompact, $\lambda$-decomposable ultrafilter. Then the pre-normal ultrafilter on $\lambda$ derived from $U$ is equivalent to a normal fine ultrafilter on $P(\lambda)$.

This should be seen as a generalization of Solovay’s Lemma to singular cardinals.

Returning to the discussion in the introduction, suppose one wanted to generalize the proof that if $\kappa$ carries a $\kappa$-complete ultrafilter, then $\kappa$ carries a $\kappa$-complete normal ultrafilter. That proof really shows:

**Proposition 118.** If $U$ is a $\kappa$-complete, $\kappa$-decomposable ultrafilter then the pre-normal ultrafilter on $\kappa$ derived from $U$ is normal.

One might attempt to generalize this by starting with an arbitrary countably complete $\lambda$-decomposable ultrafilter, deriving its pre-normal ultrafilter $D$ on $\lambda$, and trying to prove that $D$ is equivalent to a normal fine ultrafilter on $P(\lambda)$.

This cannot work for several reasons. One is that Magidor’s independence result shows that such a statement cannot be provable from ZFC alone. Another is that there are pre-normal ultrafilters that are not equivalent to normal ultrafilters. All the known examples, for example those produced by Menas, are built by hitting a small ultrafilter to produce a failure of supercompactness and then hitting a large ultrafilter to produce strong uniformity, and finally showing that the resulting iterated ultrapower is equivalent to its derived pre-normal ultrafilter. There is no known provable example that is irreducible.

There is a good reason for this: as a corollary of our results, assuming UA + GCH, the naive attempt to generalize Proposition 118 succeeds under the simplest condition that rules out Menas’s counterexamples:

**Theorem 119 (UA + GCH).** Suppose $\lambda$ is an accessible cardinal. Suppose $U$ is a $\lambda$-decomposable ultrafilter that is not divisible by any $W \in \text{Un}_{<\lambda}$. Then the pre-normal ultrafilter on $\lambda$ derived from $U$ is equivalent to a normal fine ultrafilter on $P(\lambda)$. 
Proof. By Theorem 117, it suffices to show that $U$ is $\lambda$-supercompact.

The proof of Theorem 110 shows that $U$ is $\lambda$-supercompact if $\lambda$ is a successor and $<\lambda$-supercompact if $\lambda$ is a limit. In the latter case since $\lambda$ is accessible, $\lambda$ is singular, so in fact $U$ is $\lambda$-supercompact in this case as well. □

Our next application derives large cardinal strength in the realm of huge cardinals from a simpler ultrafilter theoretic statement.

**Lemma 120 (UA).** Suppose $\delta$ is a regular cardinal. Suppose $W$ is the $<_S$-least countably complete weakly normal ultrafilter on $\delta$ concentrating on $S^\delta_\kappa$ for some $\kappa < \delta$. Then $W$ is irreducible.

Proof. Suppose $D$ divides $W$ and $D <_S W$. Let $W_* = t_D(W)$. Since $W$ is pre-normal, $\text{sp}(D) < \delta$. It follows from Lemma 89 that $W_*$ lies on $j_D(\delta)$. Note that $W_*$ is weakly normal since $[\text{id}_W]_{W_*} = [\text{id}]_W = \sup j_W[\delta] = \sup j_{W_*}^{M_D} j_D(\delta)$. Moreover $j_W(S^\delta_\kappa) \in W_*$ since $D^-(W_*) = W$. Therefore $j_D(W) \leq_S W_*$. It follows that $W_* = j_D(W)$, so $D \sqsubseteq W$ by Proposition 48. Hence $D$ is principal. □

**Corollary 121 (UA + GCH).** Suppose $\delta$ is a regular cardinal carrying a countably complete ultrafilter concentrating on $S^\delta_\kappa$ for some $\kappa < \delta$. The $<_S$-least such ultrafilter $W$ is $<\delta$-supercompact. If $\delta$ is a successor, then $W$ is $\delta$-supercompact. In any case, $W$ has the tight covering property at $\delta$.

Proof. This is all immediate except for the tight covering property, which is only relevant when $\delta$ is inaccessible. In fact, in this case, it is not hard to show that every ultrafilter has the tight covering property at $\delta$:

**Lemma 122 (UA + GCH).** Suppose $\delta$ is inaccessible or a successor of a strong limit cardinal of countable cofinality. Then every countably complete ultrafilter has the tight covering property at $\delta$.

Proof. Suppose $\delta$ carries no countably complete ultrafilter. Then using Theorem 109 or Lemma 114, it is easy to see that every ultrapower fixes $\delta$ and hence has the tight covering property.

Otherwise let $U$ be the least ultrafilter on $\delta$. Suppose $W \in \text{Un}$. In $M_U$, $\delta$ carries no countably complete ultrafilter, so $t_U(W)$ fixes $\delta$. As in Theorem 110, one calculates that $\text{cf}^{M_U} (\sup j_W[\delta]) = j_{t_U(W)}^{M_U}(\delta)$, but $j_{t_U(W)}^{M_U}(\delta) = \delta$, so we are done by Theorem 63. □

This completes the proof of Corollary 121.

Similarly one can show the following fact for singular cardinals:
Corollary 123 (UA+GCH). Suppose there is a cardinal \( \delta \) as in Corollary 121. Then there is an almost huge cardinal.

Proof. Let \( \delta \) be such a cardinal and \( W \) the least countably complete weakly normal ultrafilter concentrating on \( S^\delta_\kappa \) for some \( \kappa < \delta \). We then have \( j_W(\kappa) = \text{cf}^{M_W}(\sup j_W[\delta]) = \delta \), where the first equality follows from Los’s theorem and the second follows from the tight covering property, which holds by Corollary 121.

Note that \( \text{crt}(W) \leq \kappa \). Since \( W \) is \(<\delta\)-supercompact and \( j_W(\text{crt}(W)) \leq j_W(\kappa) = \delta \), \( W \) witnesses that \( \kappa \) is almost huge. \( \square \)

If one assumes \( \delta \) is weakly inaccessible, Corollary 121 is provable without assuming GCH:

Theorem 124 (UA). Suppose \( \delta \) is a weakly inaccessible cardinal carrying a countably complete ultrafilter concentrating on \( S^\delta_\kappa \) for some \( \kappa < \delta \). The \(<S\)-least such ultrafilter \( W \) is \(<\delta\)-supercompact.

Proof. Let \( W \) be the \(<S\)-least such ultrafilter. As above it is easy to show that \( W \) is irreducible.

Note that \( \text{cf}^{M_U}(\sup j_U[\delta]) = j_U(\kappa) \leq \sup j_U[\delta] \). Therefore \( \delta \) is a limit of ultrafiltered regular cardinals, so by Corollary 52 the least ultrafilter \( U \) on \( \delta \) is \(<\delta\)-supercompact and has critical point less than or equal to \( \text{crt}(W) \). By Corollary 103 we have GCH on a tail below \( \delta \). (If one assumes \( \delta \) is strongly inaccessible, one can just use Solovay’s theorem on SCH here.) It follows that we can apply Proposition 113 to conclude that \( W \) is \(<\delta\)-supercompact. \( \square \)

As our last application, we show one can rule out the existence of cardinality-preserving embeddings of the universe into an inner model assuming UA; this sort of embedding is considered in [?]. This is somewhat interesting in that the hypothesis does not explicitly involve ultrafilters.

Caicedo observed that the following lemma is useful in this context:

Lemma 125. If \( j: V \to M \) is elementary, \( \tau \) is a successor cardinal, \( j(\tau) \) is a cardinal, and \( j(\tau) \neq \tau \), then \( \sup j[\tau] < j(\tau) \).

Proof. Since \( \tau \) is a successor cardinal, \( j(\tau) \) is a successor cardinal of \( M \), and hence since \( j(\tau) \) is a cardinal, it must be a successor cardinal in \( V \). Therefore \( j(\tau) \) is regular, so \( \sup j[\tau] < j(\tau) \). \( \square \)

We will use this several times. The following improves a lemma in Caicedo’s paper, using Theorem 79.

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Lemma 126. Suppose $j : V \rightarrow M$ is an elementary embedding and $\lambda$ is its first fixed point above its critical point $\kappa$. Suppose that for all $\gamma \in [\kappa, \lambda]$, $j(\gamma)$ is a cardinal. Then $j$ is discontinuous at every regular cardinal in $[\kappa, \lambda]$.

Proof. Note that $\lambda$ is a limit cardinal since the elements of the critical sequence of $j$ are cardinals. By Lemma 125, $j$ is continuous at every successor cardinal in $[\kappa, \lambda]$. Suppose $\delta \in [\kappa, \lambda]$ is regular. Let $U$ be the weakly normal ultrafilter derived from $j$ on $\delta^+$. Then $U$ is $\delta$-decomposable. So since $j$ factors through $j_U$, $j$ is discontinuous at $\delta$.

Lemma 127 (UA). Suppose $j : V \rightarrow M$ is an elementary embedding and $\lambda$ is its first fixed point above its critical point $\kappa$. Suppose $\lambda$ is a strong limit cardinal and $2^\lambda = \lambda^+$. Then $j$ is continuous at $\lambda^{+\kappa+1}$.

Proof. Suppose not, and let $U$ be the weakly normal ultrafilter on $\lambda^{+\kappa+1}$ derived from $j$. By Theorem 108, $U$ can be factored as $D \oplus Z$ where $D \in \text{Un}_{\lambda^{+\kappa+1}}$ and $Z \in j_D(\text{Un}_{\lambda^{+\kappa+1}})$ is irreducible in $M_Z$.

Note that $j_D(\lambda) = \lambda$ and $j_Z(\lambda) = \lambda$ since $j$ factors through $j_Z \circ j_D$ and fixes $\lambda$. By Proposition 113, $Z$ is therefore $\lambda^+$-supercompact in $M_D$. We have $\text{crt}(Z) < \lambda$ since in $M_D$ there are no measurable cardinals in the interval $[\lambda, j_D(\lambda^{+\kappa+1})]$. But the existence of such a $Z$ contradicts Kunen’s inconsistency theorem applied in $M_Z$.

Corollary 128 (UA). Suppose $j : V \rightarrow M$ is an elementary embedding and $\lambda$ is its first fixed point above its critical point $\kappa$. Then for some cardinal $\gamma \leq \lambda^{+\kappa+1}$, $j(\gamma)$ is not a cardinal.

Proof. Suppose not. By our assumptions, every element of the critical sequence of $j$ is a cardinal. So $\lambda$ is a limit cardinal. Since $\kappa$ is $\lambda$-strongly compact, it follows by Corollary 103 that $\lambda$ is a strong limit cardinal. Let $U$ be the ultrafilter on $\lambda^{+\kappa+1}$ derived from $j$. Again since $\kappa$ is $\lambda$-strongly compact, the least ultrafilter on $\lambda^{+\kappa+1}$ is $\lambda$-supercompact by Corollary 52. Since $\lambda$ has countable cofinality, this ultrafilter is $\lambda^+$-supercompact. Therefore $2^\lambda = \lambda^+$ by Solovay’s theorem.

Applying Lemma 127, $j$ is continuous at $\lambda^{+\kappa+1}$. Since $j(\lambda^{+\kappa+1})$ is a cardinal and $j(\lambda^{+\kappa+1}) > j(\lambda^+) \geq \lambda^{+\kappa+1}$, this contradicts Lemma 125.

5.5 Pathological Ultrafilters

We begin this subsection by showing quite easily that assuming UA + GCH, the internal relation and the Mitchell order essentially coincide.
Lemma 129. Suppose $\text{Un}_{<\lambda} \subseteq U$, $W \in \text{Un}_{\lambda}$, and $Z <_S W \subseteq U$. Then $Z \subseteq U$.

Sketch. Fix $Z_* \in \text{Un}_{<\lambda}^{MW}$ with $Z = W^-(Z_*).$ Since $\text{Un}_{<\lambda} \subseteq U$,

$$Z_* \in \text{Un}_{<\lambda}^{MW} = j_W(\text{Un}_{<\lambda}) \subseteq j_W(U)$$

Therefore $Z_* \subseteq j_W(U).$ Since $M_{j_W(U)}^{MW} \subseteq M_U$, it follows that $W \oplus Z_* \subseteq U$. Since $Z \subseteq \text{RK} W \oplus Z_*$, $Z \subseteq U$. \qed

Proposition 130 (UA). Suppose $\delta$ is a regular cardinal, $\text{Un}_{<\delta} \subseteq U$, and for some $W \in \text{Un}_{\delta}$, $W \subseteq U$. Then $U$ is $\delta$-supercompact.

Proof. If $\text{crt}(U) \geq \delta$, we are done, so assume not.

Let $Z$ be the least ultrafilter on $\delta$. By Lemma 129, $Z \subseteq U$. Assume towards a contradiction that $U \subseteq Z$. Then $U, Z$ commute by Theorem 40 and therefore fix each other’s critical points, contradicting Corollary 73. Therefore $U \nsubseteq Z$. By Corollary 90 it follows that $Z$ is $\kappa$-complete and $(\kappa, \delta)$-regular for some $\kappa \leq \delta$ closed under ultrapowers. Note that it follows from Corollary 73 that $\text{crt}(U) \geq \kappa$. Using this we can show by the argument of Lemma 68 that $P(\delta) \subseteq M_U$. Moreover by Proposition 53, $U$ has the tight covering property at $\delta$. Hence $U$ is $\delta$-supercompact. \qed

The following gives in many cases a converse to Kunen’s commuting ultrapowers lemma, Theorem 40.

Lemma 131. Suppose for $i = 0, 1$ that $U_i \in \text{Un}_{\delta_i}$ with completeness $\kappa_i$. Suppose $U_i$ fixes no measurable cardinals in $[\kappa_i, \delta_i]$. Assume $j_{U_0}(j_{U_1}) = j_{U_1} \upharpoonright M_{U_0}$ and $j_{U_1}(j_{U_0}) = j_{U_0} \upharpoonright M_{U_1}$. Then $\delta_0 < \kappa_1$ or $\delta_1 < \kappa_0$.

Proof. If $j_{U_0}(j_{U_1}) = j_{U_1} \upharpoonright M_{U_0}$ and $j_{U_1}(j_{U_0}) = j_{U_0} \upharpoonright M_{U_1}$ then in particular $j_{U_0}(\kappa_1) = \kappa_1$ and $j_{U_1}(\kappa_0) = \kappa_0$. It follows that $\kappa_1 \notin [\kappa_0, \delta_0]$ and $\kappa_0 \notin [\kappa_1, \delta_1]$. It follows that $\delta_0 < \kappa_1$ or $\delta_1 < \kappa_0$. \qed

Theorem 132 (UA + GCH). Suppose $U$ and $W$ are irreducible strongly uniform ultrafilters. Then $U \subseteq W$ if and only if $U <_M W$ or $\text{sp}(W) < \text{crt}(U)$.

Proof. Let $\lambda = \text{sp}(U)$.

For the reverse direction, assume $U <_M W$ or $\text{sp}(W) < \text{crt}(U)$. Obviously if the latter holds then $U \subseteq W$. We need to show that if $U <_M W$ then $W$ is $\lambda$-supercompact. Note that if $U <_M W$ then $P(\lambda) \subseteq M_W$. In particular by GCH, $\text{sp}(W) \geq \lambda$. If $\lambda$ is not inaccessible then $W$ is $\lambda$-supercompact by
Theorem 110. If $\lambda$ is inaccessible then $W$ has tight covering at $\lambda$, but since $U \ll M W$, $P(\lambda) \subseteq M W$, and therefore $W$ is $\lambda$-supercompact.

We now prove the forward direction. Assume $U \sqsubseteq W$ and we will show that $U < M W$ or $\text{sp}(W) < \text{crt}(U)$.

Suppose first that $U \sqsubseteq W$ and $W \not\subseteq U$. We claim $U < M W$. It suffices to show that $W$ is $\lambda$-supercompact. Note that since $W \not\subseteq U$, we must have $\text{sp}(W) \geq \lambda$. If $\lambda$ is not inaccessible then by Theorem 110, this implies $W$ is $\lambda$-supercompact. When $\lambda$ is inaccessible, we can only conclude that $W$ is $< \lambda$-supercompact and has the tight covering property at $\lambda$. In this case we can apply Proposition 130 to conclude that $W$ is $\lambda$-supercompact.

Suppose instead that $U \sqsubseteq W$ and $W \sqsubset U$. An irreducible ultrafilter fixes no regular cardinal between its critical point and its space by Theorem 110. Therefore by Lemma 131, either $\text{sp}(W) < \text{crt}(U)$ or $\text{sp}(U) < \text{crt}(W)$. If the former holds, we are done. If the latter holds then $U < M W$, so we are done again. This concludes the proof of the forwards direction. □

Since every ultrafilter factors into irreducible strongly uniform ultrafilters, this leads to a somewhat complicated characterization of the internal relation in terms of the Mitchell orders of ultrapowers. Even the statement is a bit tedious, so we omit it.

We now turn to the main subject of this section: pathological ultrafilters.

**Question 133 (UA + GCH).** Suppose $U$ is an irreducible ultrafilter on an inaccessible cardinal $\delta$. Must $U$ be $\delta$-supercompact?

We note that this is the same as the question of whether the least ultrafilter on $\delta$ is $\delta$-supercompact. This uses a very easy version of the argument of Theorem 110.

**Proposition 134 (UA + GCH).** Suppose $\text{Un}_{< \delta} \sqsubseteq W$ and $U$ is the least ultrafilter on $\delta$. Then $\text{crt}(t_U(W)) > \delta$.

**Proof.** Since $\text{Un}_{< \delta} \sqsubseteq U, W$, by Corollary 32, $\text{Un}_{< \delta} \sqsubseteq U \vee W$. It follows that $\text{Un}_{< \delta}^{M_U} \sqsubseteq t_U(W)$. But $t_U(W)$ is continuous at $\delta$ since $\delta$ carries no uniform ultrafilter in $M_U$. Therefore by Theorem 109 in $M_U$, $t_U(W)$ factors as $D \oplus Z$ where $D \in \text{Un}_{< \delta}$ and $\text{crt}(Z) > \delta$. But $\text{Un}_{< \delta} \sqsubseteq t_U(W)$ implies $D \sqsubseteq t_U(W)$, which implies $D$ is principal. □

**Proposition 135 (UA + GCH).** Suppose the least ultrafilter $U$ on $\delta$ is $\delta$-supercompact. Then every irreducible strongly uniform ultrafilter $W$ on $\lambda \geq \delta$ is $\delta$-supercompact.
**Sketch.** By Theorem 110, $W$ is $<\delta$-supercompact. It follows that $U_{\text{Un}} \subset W$. Therefore by Proposition 134, $\text{crt}(t_U(W)) > \delta$. Hence $\text{Ord}^\delta = \text{Ord}^\delta \cap M_U \subset M_W$. 

**Definition 136.** Suppose $\delta$ is a regular cardinal. An ultrafilter $U \in \text{Un}_\delta$ is called *pathological* if $U$ is $<\delta$-supercompact and has the tight covering property at $\delta$ but is not $\delta$-supercompact.

An immediate consequence of our main theorems place stringent constraints on the type of pathologies that can arise under UA + GCH.

**Proposition 137 (UA + GCH).** Suppose $\delta$ is a regular cardinal. An irreducible ultrafilter on $\delta$ is either $\delta$-supercompact or pathological. In the latter case $\delta$ must be strongly inaccessible.

The following variant of our main question is open in ZFC.

**Question 138.** Is it consistent (relative to large cardinals) that there is a pathological ultrafilter?

A very interesting special case of this question asks whether one can in fact prove in ZFC that a $\kappa$-complete uniform ultrafilter on $\kappa^+$ with the tight covering property is $\kappa^+$-supercompact. Of course this follows from UA by Corollary 82.

Under UA, to understand pathological ultrafilters it to some extent suffices to understand weakly normal pathological ultrafilters:

**Theorem 139 (UA+GCH).** Suppose that $U$ is a pathological ultrafilter on $\delta$ and $W$ is the weakly normal ultrafilter on $\delta$ derived from $U$. Then $W$ is a pathological ultrafilter and $\text{Ord}^\delta \cap M_W = \text{Ord}^\delta \cap M_U$.

**Proof.** Since $\delta$ carries a pathological ultrafilter. Let $k : M_W \to M_U$ be the factor embedding. Note that $k(\delta) = \delta$ since $k(\sup j_W[\delta]) = \sup j_U[\delta]$ and $\text{cf}^M_W(\sup j_W[\delta]) = \text{cf}^M_U(\sup j_U[\delta]) = \delta$ by tight covering.

We first show that $W$ is irreducible. By Theorem 108 there is some divisor $D \in \text{Un}_{<\delta}$ of $W$ such that $M_D \cap V_\delta = M_W \cap V_\delta$: one obtains $D$ by “iterating the least factor” until one reaches a strongly uniform ultrafilter on $\delta$. To see that $W$ is irreducible it suffices to show that $D$ is principal. Note that $k \circ j_D \restriction V_\delta$ is an elementary embedding from $V_\delta$ to itself. By Kunen’s inconsistency theorem, it is the identity. Hence $D$ is principal. So $W$ is irreducible, and therefore $W$ is pathological.

Since $W$ is irreducible, $V_\delta \cap M_W = V_\delta$. Therefore the same argument shows $k \restriction \delta$ is the identity. Since $W$ has the tight covering property at $\delta$,
\[ \delta^{+M_W} = \delta^+. \] It follows that \( \text{crt}(k) > \delta^+. \) Since we assume GCH, it follows that \( P(\delta) \cap M_W = P(\delta) \cap M_U. \) Combined with the tight covering property, it follows that \( \text{Ord}^g \cap M_U = \text{Ord}^g \cap M_W. \)

Must every irreducible pathological ultrafilter be weakly normal? One can show assuming UA + GCH that if \( U \) is \( \delta \)-supercompact, then either \( U \) is divisible by its derived pre-normal ultrafilter \( D \) on \( \delta \) or else \( D \subset U. \) Since pathological ultrafilters have no uniform ultrafilters on \( \delta \) internal to them, if one could generalize this fact to pathological ultrafilters, one would rule out pathology that is not essentially reducible to a weakly normal ultrafilter. But perhaps an alternate hierarchy of pathological ultrafilters emerges on inaccessible cardinals, distinct from the familiar hierarchy of supercompact ultrafilters under the Mitchell order.

By Theorem 58, if a regular cardinal \( \delta \) carries two weakly normal ultrafilters, the second is not pathological. Denote the least two weakly normal ultrafilters by \( U_0 <_S U_1. \) In \( M_{U_0}, \delta \) carries a unique weakly normal ultrafilter, while in \( M_{U_1}, \delta \) carries no ultrafilter whatsoever. By Theorem 51, any \( W \) with the property that \( \delta \) carries no weakly normal ultrafilter in \( M_W \) is divisible by \( U_0. \) Does this generalize to \( U_1: \) suppose \( \delta \) carries a unique weakly normal ultrafilter in \( M_W. \) Is \( W \) divisible by \( U_1? \)

If the answer is positive, then the least ultrafilter is the only source of pathology on \( \delta: \)

**Proposition 140 (UA + GCH).** Suppose \( U_0 <_S U_1 \) are the least weakly normal ultrafilters on \( \delta. \) Assume that every \( W \in \text{Un}_S \) such that \( \delta \) carries a unique weakly normal ultrafilter in \( M_W \) is divisible by the \( U_1. \) Then every pathological ultrafilter on \( \delta \) is divisible by \( U_0. \)

**Proof.** We first show that given such an ultrafilter \( W, \delta \) does not carry a unique weakly normal ultrafilter in \( M_W. \) Assume towards a contradiction that it does. Then \( U_1 \) divides \( W. \) Note that \( t_{U_1}(W) \) is \( \delta \)-strong and continuous at \( \delta \) (since otherwise \( U_0 \leq^{M_{U_1}} t_{U_1}(W) \) contradicting that \( \delta \) carries an ultrafilter in \( M_W. \)). Therefore \( \text{crt}(t_{U_1}(W)) > \delta. \) This contradicts that \( W \) is not \( \delta \)-supercompact.

We now finish the proof in general. Suppose \( \delta \) carries a second weakly normal ultrafilter \( U_1^* \) in \( M_W. \) By assumption, \( U_1 \) divides \( W \oplus U_1^*. \) Moreover since \( U_1^* \) is \( \delta \)-supercompact in \( M_W, W' = W \oplus U_1^* \) is pathological and \( \delta \) carries a unique weakly normal ultrafilter in \( M_{W'}. \) But the first paragraph rules out the existence of such a \( W'. \)
ultrafilters. Proposition [130] shows that we cannot use the Mitchell order (or the internal relation) to understand pathological ultrafilters: they are all incomparable. It is tempting to try to use a weaker Mitchell relation: if $W$ is a pathological ultrafilter, for which $U \cap M_W$ belong to $M_W$? Note that if some cardinal is strong compact past $\delta$, then every irreducible ultrafilter on $\delta$ in $M_W$ is of the form $U \cap M_W$ for some $U$ (and perhaps many).

As we have remarked, it is not even clear that this relation is irreflexive on nonprincipal ultrafilters in ZFC. But our main point here is that assuming UA + GCH, even under this weaker Mitchell relation, there is an enormous amount of incomparability.

**Lemma 141.** Suppose $U$ is a $\kappa$-complete ultrafilter with the tight covering property at $\delta$. Then there is a function $f : \delta \to \sup j_U[\delta]$ such that $f \in M_U$ and \{\alpha < \delta : f(\alpha) = j_U(\alpha)\} is $<\kappa$-club.

**Proof.** In fact if $f : \delta \to \sup j_U[\delta]$ is any increasing continuous cofinal function with $f \in M_U$ then since $j_U[\delta]$ is $<\kappa$-club, $f(\alpha) = j_U(\alpha)$ on a $<\kappa$-club of $\alpha$. 

**Corollary 142.** Suppose $U$ is a $\kappa$-complete ultrafilter with the tight covering property at $\delta$. Suppose $W$ is a countably complete ultrafilter extending the $<\kappa$-club filter on $\delta$. Then $j_{W \cap M_U} = j_W \upharpoonright M_U$.

**Proof.** Let $k : M_{W \cap M_U}^M_U \to j_W(M_U)$ be the factor embedding. By Proposition [14] it suffices to show that $j_W(j_{W \cap M_U}(f))(\id_{W \cap M_U}) \in \text{ran}(k)$. Take $f \in M_U$ such that \{\alpha < \delta : f(\alpha) = j_U(\alpha)\} is $<\kappa$-club. Then $k(j_{W \cap M_U}(f))(\id_{W \cap M_U}) = j_W(f)(\id_W)$ by the definition of the factor embedding, and $j_W(f)(\id_W) = j_W(j_U)(\id_W)$ since $f = j_U$ almost everywhere with respect to $W$, since $W$ extends the $<\kappa$-club filter. 

By Proposition [130] we have the following corollary.

**Corollary 143 (UA + GCH).** Suppose $U$ is a $\kappa$-complete pathological ultrafilter. Then for any countably complete ultrafilter $W$ extending the $<\kappa$-club filter on $\delta$, $W \cap M_U \notin M_U$. In particular, for any weakly normal ultrafilter $W$ on $\delta$ such that $j_W(\kappa) > \delta$, $W \cap M_U \notin M_U$. 

6 Questions

This paper probably represents only the beginning of the structure theory past strong compactness that can be established assuming the Ultrapower Axiom. There are many combinatorial questions we expect to be solvable above the least strongly compact.
**Question 144** (UA + GCH). Let $\kappa$ be the least strongly compact cardinal.

1. Do pathological ultrafilters exist? Or is the Mitchell order linear on irreducible ultrafilters? Is every irreducible ultrafilter Dodd solid?

2. Is there a nonreflecting stationary subset of $S^{\kappa^+}$?

3. The linearity of the Mitchell order yields $\diamondsuit(S^{\delta^+})$ whenever $\text{cf}(\delta) \geq \kappa$. Does $\diamondsuit^+(\kappa^+)$ hold? What about $\diamondsuit(S^{\kappa^+})$?

4. A theorem of Usuba \[1\] states that if there is a hyperhuge cardinal, the generic multiverse has a minimum element. Does this follow from the existence of a strongly compact cardinal assuming UA?

5. A theorem from \[2\] states that for any $X \subseteq \kappa$, $V = \text{HOD}_X$. Letting $\delta = \kappa^{++}$, Vopenka’s theorem implies that $V$ is a generic extension of $\text{HOD}$ by a partial order of cardinality $\delta$. Is $V$ a $\kappa$-cc extension of $\text{HOD}$? Does GCH hold above $\kappa$ in $\text{HOD}$?

6. Suppose $\gamma$ is the least tall cardinal above $\kappa$. Is $\gamma$ a strong cardinal?

7. Suppose $\lambda$ is least such that there is an elementary embedding $j : V \rightarrow M$ such that $j(\lambda) = \lambda$, $\text{crt}(j) < \lambda$, and $M$ computes cofinalities correctly below $\lambda$. Does $I_2$ hold at $\lambda$?

8. Suppose $U, W \in \text{Un}$. Must $\text{crt}(U \lor W) = \min\{\text{crt}(U), \text{crt}(W)\}$?

These are just the first questions that come to mind. There are many more raised implicitly in this paper. Of course there are the rather technical questions in the style of Section \[4\] of what can be proved from UA without assuming GCH, most interestingly whether there is a GCH free proof of Theorem \[10\] which seems extremely likely.

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