Non-radial oscillations of the magnetized rotating stars with purely toroidal magnetic fields

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ABSTRACT
We calculate non-axisymmetric oscillations of uniformly rotating polytropes magnetized with a purely toroidal magnetic field, taking account of the effects of the deformation due to the magnetic field. As for rotation, we consider only the effects of Coriolis force on the oscillation modes, ignoring those of the centrifugal force, that is, of the rotational deformation of the star. Since separation of variables is not possible for the oscillation of rotating magnetized stars, we employ finite series expansions for the perturbations using spherical harmonic functions. We calculate magnetically modified normal modes such as $g$, $f$, $p$, $r$, and inertial modes. In the lowest order, the frequency shifts produced by the magnetic field scale with the square of the characteristic Alfvén frequency. As a measure of the effects of the magnetic field, we calculate the proportionality constant for the frequency shifts for various oscillation modes. We find that the effects of the deformation are significant for high frequency modes such as $f$- and $p$-modes but unimportant for low frequency modes such as $g$-, $r$-, and inertial modes.

Key words: – stars: magnetic fields – stars: neutron – stars: oscillations.

1 INTRODUCTION
Since the first discovery report of quasi-periodic oscillations (QPOs) in the tail of the giant X/$\gamma$-ray flare from the soft $\gamma$-ray repeater (SGR) 1806-20 (Israel et al. 2005), extensive theoretical studies have been carried out to identify physical mechanisms responsible for the QPOs. SGRs belong to what we call magnetars, neutron stars possessing an extremely strong magnetic field as strong as $10^{15}$ G at the surface. Giant flares have so far been observed only from three SGRs, that is, SGR 0526-66, 1900+44, and 1806-20, and just once for each of the SGRs, indicating that giant flares from magnetars are quite rare events. For magnetars, see reviews, for example, by Woods & Thompson (2006) and Mereghetti (2008). Analyzing archival data of another magnetar candidate SGR 1900+14, Strohmayer & Watts (2005) have succeeded in identifying QPOs in the X-ray giant flare that was observed in 1998. QPOs frequencies now identified in the giant flares from the two magnetar candidates are 18, 30, 92.5, 150, 626 Hz for SGR 1806-20 (Israel et al. 2005; Strohmayer & Watts 2006; Watts & Strohmayer 2006), and 28, 53.5, 84, 155 Hz for SGR 1900+14 (Strohmayer & Watts 2005). Employing Bayesian statistics, Hambaryan et al. (2011) have reanalyzed the data for SGR 1806-20 to identify QPO frequencies at 16.9, 21.4, 36.8, 59.0, 61.3, and 116.3 Hz. For the giant flare in 1979 from SGR 0526-66, Watts (2011) mentioned in her review paper a report of a QPO at $\sim 43$ Hz, but she also suggested the difficulty in the frequency analysis in the impulsive phase of the burst. Here, it is worth mentioning a promising recent attempt to find QPOs in short recurrent bursts in SGRs. Happenkothen et al. (2014) have succeeded in identifying candidate signals at 260, 93, and 127 Hz from J1550-5418, where they used Bayesian statistics for the analysis.

The QPOs are now regarded as a manifestation of global oscillations of the underlying neutron stars, and it is expected that they can be used for seismological studies of the magnetars. Seismological studies of magnetars may have started with a paper by Duncan (1998), who suggested that frequent starquakes in SGRs could excite crustal toroidal modes and the burst emission modulated at the mode frequencies would be detected. The detection of QPOs in the giant flare from SGR 1806-20 in 2004 (Israel et al. 2005) has given a huge trigger leading to subsequent intensive theoretical studies of QPOs in magnetars. In the early studies of QPOs in magnetars (e.g., Piro 2005; Lee 2007), the oscillations were assumed to be practically confined in the solid crust, as Duncan (1998) anticipated, and the effects of a magnetic field in the fluid core were ignored. Since magnetars are believed to possess an extremely strong magnetic field threading both the solid crust and the fluid core, to determine the oscillation frequency spectra of the stars, we need to correctly take account of the effects of the strong magnetic field in both regions on the oscillations. Applying a toy model, Glampedakis, Samuelsson, and Andersson (2006) discussed toroidal oscillations as global discrete modes residing in the fluid core and in the solid crust both threaded by a strong magnetic field,
and showed that the modes that are most likely to be excited in magnetars are such that the crust and the core oscillate in concert. Also using a toy model, Levin (2006, 2007) put forward a different idea that Alfvén modes in the fluid core may lead to the formation of continuum frequency spectra and that toroidal crust modes will be rapidly damped as a result of resonant absorption in the core, and even suggested that there exist no discrete normal modes for the strongly magnetized neutron stars. However, assuming a pure poloidal field threading both the solid crust and the fluid core, Lee (2008) and Asai & Lee (2014) carried out normal mode calculations of axisymmetric toroidal modes and found discrete toroidal modes. Later on, van Hoven & Levin (2011, 2012), employing spectral method of calculation, have succeeded in suggesting the existence of discrete modes in the gaps between continuum frequency bands. In general relativistic frame work, Sotani et al. (2007) computed normal modes of magnetized neutron stars in the weak magnetic field limit.

Oscillations in magnetized stars are governed by a set of linearized partial differential equations. Normal mode analysis of the oscillations of magnetized stars are not necessarily easy to conduct, partly because separation of variables between the radial and the angular coordinates is in general impossible to represent the perturbations, and partly because the possible existence of continuum bands in the frequency spectra make it difficult to properly calculate oscillation modes, particularly belonging to the continua (see, e.g., Goedbloed & Poedts 2004). In normal mode calculations, we usually employ series expansions of a finite length to represent the perturbations so that the set of linearized partial differential equations reduces to a set of linear ordinary differential equations for the expansion coefficients. This process can be very cumbersome when we try to carry out modal analyses for various configurations of magnetic fields. This may be one of the reasons why MHD simulations have been used to investigate the small amplitude oscillations of magnetized neutron stars by many authors including, e.g., Cotikos & Stergioulas (2008), Cerda-Durán et al. (2009), Colaiuda & Kokkotas (2011, 2012), Gabler et al. (2011, 2012, 2013a,b), Lander et al. (2010), Passamonti & Lander (2013, 2014). In the analyses with MHD simulations, QPOs are believed to be associated with continuum spectra and should be properly distinguished from discrete normal modes by closely watching motions and phases of various points in the interior.

Configurations of magnetic fields in magnetars are highly uncertain (e.g., Thompson & Duncan 1993, 1996; Thompson, Lyutikov & Kulkarni 2002). As shown by the core-collapse supernova MHD simulations (e.g., Kotake, Sato & Takahashi 2006), toroidal fields can be easily produced and amplified when winding of the initial seed poloidal fields is effective in the differentially rotating core even if there is no initial toroidal fields. For modal analyses, it is desirable to examine various magnetic field configurations. Most of the modal analyses so far carried out have been for a purely poloidal magnetic field (Lee 2007, 2008; Sotani et al 2007, 2008; Cerda-Durán et al. 2009; Colaiuda & Kokkotas 2011, 2012; Gabler et al. 2011, 2012; Passamonti & Lander 2013, 2014; Asai & Lee 2014). Recently, however, some authors, using MHD simulations, started investigating small amplitude oscillations for a purely toroidal magnetic field (Lander et al. 2010; Passamonti & Lander 2013), and even for mixed poloidal and toroidal field configurations (Gabler et al. 2013). Using MHD simulations, for example, Lander et al (2010), calculated rotational modes (r-modes and inertial modes) of magnetized stars and showed that r-modes at rapid rotation tend to magnetic modes in the slow rotation limit.

In this paper we carry out normal mode analysis of polytropic models with a purely toroidal magnetic field for various non-axisymmetry oscillation modes, including p-, f-, g-modes, and rotational modes such as r-modes and inertial modes, where we include the effects of equilibrium deformations due to the magnetic field on the oscillations. To calculate rotational modes, we only take account of Coriolis force and include no effects of the centrifugal force. The oscillation equations for magnetically deformed rotating stars are derived by following the formulation similar to that by Saio (1981) (see also Lee 1993; Yoshida & Lee 2000a). The numerical method to compute normal modes of magnetized rotating stars is the same as that in Lee (2005) (see also Lee 2007), who employed series expansions of a finite length in terms of spherical harmonic functions for the perturbations. This paper is organized as follows. §2 describes the method used to construct a magnetically deformed equilibrium stellar model, and perturbation equations for non-axisymmetric oscillation modes in magnetized rotating stars are derived in §3. Numerical results are summarized in §4 and we conclude in §5. The details of the oscillation equations solved in this paper and suitable boundary conditions imposed at the stellar center and surface are given in Appendix A.

## 2 EQUILIBRIUM MODEL

We consider the oscillations of uniformly rotating polytropes with purely toroidal magnetic fields. Equilibrium structures of stars having purely toroidal magnetic fields have so far been studied with non-perturbative approaches within the framework of Newtonian mechanics (Miketinac 1973) and of general relativity (Kiuchi & Yoshida 2008; Friezen & Rezzolla 2012). In this study, we employ a perturbative approach to construct stars deformed by a purely toroidal magnetic field. Following Miketinac (1973), a purely toroidal magnetic field imposed on the stars in equilibrium is given by

$$
B_r = 0, \quad B_\theta = 0, \quad B_\phi = k_\rho r \sin \theta,
$$

where $k = B_0/(\sqrt{2} \rho_0 R)$ is a constant, $B_0$ is the parameter used for the strength of the magnetic field in the interior, $\rho_0$ is the density at the stellar center, and $R$ is the radius of the star. Here, we use spherical polar coordinates ($r, \theta, \phi$). The magnitude of the magnetic field is given by $|\mathbf{B}| = |B_\phi| = (B_0/\sqrt{2}) \hat{\rho} x \sin \theta$, where $x = r/R$ and $\hat{\rho} = \rho/\rho_0$. The fluid velocity $\mathbf{v}$ in equilibrium is assumed to be given by

$$
v_r = 0, \quad v_\theta = 0, \quad v_\phi = r \sin \theta \Omega.
$$
where $\Omega$ denotes the angular velocity of the uniformly rotating star. In this study, the deformation of the star is assumed to be solely caused by the magnetic fields and the effects of the centrifugal force are ignored. By the assumptions (1) and (2), the induction and continuity equations are automatically satisfied and need not be considered any further. For the toroidal field (1), we can write the Lorentz force per unit mass as a potential force, that is,

$$\frac{1}{4\pi\rho}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left( \frac{B_0^2}{8\pi\rho_c} \hat{\rho} x \sin^2 \theta \right).$$

(3)

The structure of a star in equilibrium is then determined by the hydrostatic equation, the Poisson equation, and the equation of state:

$$\nabla p = -\rho \nabla \Phi,$$

(4)

$$\nabla^2 \Phi = 4\pi G \rho,$$

(5)

$$p = K_c \rho^{1+1/n},$$

(6)

where $n$ and $K_c$ are the polytropic index and the structure constant given by the mass and the radius of the star, $G$ is the gravitational constant, $\Phi$ is the gravitational potential, and $\Psi$ is the effective potential defined by

$$\Psi = \Phi + 1$$

where

$$\frac{1}{\nabla \omega}$$

on the right hand side of equation (7) to $\Phi$ may be given by $\bar{\omega}$ where

$$\omega \equiv \omega_2$$

and

$$r, \theta$$

from which we obtain the following linear ordinary differential equations for $\psi_0(x)$ and $\psi_2(x)$:

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d \psi_0}{dx} \right) = k(x) \psi_0 + f_0(x),$$

(12)

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d \psi_2}{dx} \right) = \left[ k(x) + \frac{6}{x^2} \right] \psi_2 + f_2(x),$$

(13)

where

$$f_0(x) = -\frac{1}{6} \left( r^2 \frac{d^2 \hat{\rho}_0}{dr^2} + 6r \frac{d \hat{\rho}_0}{dr} + 6\hat{\rho}_0 \right), \quad f_2(x) = \frac{1}{6} \left( r^2 \frac{d^2 \hat{\rho}_0}{dr^2} + 6r \frac{d \hat{\rho}_0}{dr} \right),$$

(14)

and

$$k(x) = 4\pi G R^2 \frac{d \hat{\rho}_0}{d \Psi_0}.$$
In order to numerically integrate the differential equations (12) and (13) from the stellar center, we need to impose the regularity condition at the center, which may be obtained by substituting the expansion around the center $x = 0$

$$\psi_j = x^a \sum_{n=0}^{\infty} (a_n^{(j)} x^n)^{(16)}$$

into (12) and (13) for $j = 0$ and $j = 2$, respectively. Since $k(x) \rightarrow k(0)$, $f_0(x) \rightarrow f_0(0)$, and $f_2(x) \rightarrow f_2 x^2$ as $x \rightarrow 0$, where $k(0)$, $f_0(0)$, and $f_2$ are constants, values of the exponent $s$ are given by $s = j$ for the regular solution at the center and the expansion coefficients $a_0^{(0)}$ and $a_0^{(2)}$ remain undetermined. Assuming that the density at the center $x = 0$ is independent of $\omega_A$, we have $a_0^{(0)} = 0$ for $\psi_0$. The expansion coefficient $a_0^{(2)}$ for $\psi_2$ must be specified by applying the surface boundary condition:

$$3\psi_{2}(1) + \frac{d\psi_2}{dx}(1) = \frac{1}{6} \frac{d^2}{dx^2}(1)$$

(17)

See Appendix B for the derivation of the boundary condition.

### 3 PERTURBATION EQUATIONS

For the modal analysis of magnetically deformed stars, we introduce the parameter $a$ that labels equi-potential surfaces of $\Psi(r, \theta)$. The parameter $a$ is defined such that $\Psi(r, \theta) = \Psi_0(a)$, that is,

$$\Psi_0(a) = \Psi_0(r) - 2R^2 \omega_A^2 [\psi_0(x) + \psi_2(x) P_2(\cos \theta)]$$

(18)

which may define the equipotential surface as given by a function $r(a, \theta)$. Assuming the deviation of the equipotential surface $r = r(a, \theta)$ from the spherical surface $r = a$ is small, we define the function $r(a, \theta)$ as

$$r = a [1 + \epsilon(a, \theta)]$$

(19)

and we assume that $\epsilon$ is the quantity of order of $R^2 \omega_A^2 / \Psi_0(R)$. By substituting equation (19) into (18), we obtain the explicit expression for the function $\epsilon(a, \theta)$ up to the order of $\omega_A^2$:

$$\epsilon(a, \theta) = \alpha(a) + \beta(a) P_2(\cos \theta)$$

(20)

where

$$\alpha(a) = \frac{2c_1 \omega_A^2}{x^2} \psi_0(x), \quad \beta(a) = \frac{2c_1 \omega_A^2}{x^2} \psi_2(x)$$

(21)

where $c_1 = (a/R)^3/[M(a)/M]$, and $M(a)$ denotes the mass inside the $a$-constant surface and $M = M(R)$.

Hereafter, we employ the parameter $a$ instead of the polar radial coordinate $r$ as the radial coordinate. In this coordinate system $(a, \theta, \phi)$, the line element is given by

$$ds^2 = (1 + 2\epsilon) da^2 + a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2 + 2a \frac{\partial \epsilon}{\partial a} da^2 + 2a \frac{\partial \epsilon}{\partial \theta} d\theta d\phi$$

(22)

Note that in this coordinate system, the pressure, the density and the effective potential of a magnetized star depend only on the radial coordinate $a$, although the orthogonality of the basis vectors is lost.

The governing equations of non-radial oscillations of a magnetized and uniformly rotating star are obtained by linearizing the basic equations. As for rotation effects on the oscillations, as mentioned in the previous section, we consider only the effects of the Coriolis force and ignore those of the centrifugal force, where we assume the rotation axis is parallel to the magnetic axis. Since the equilibrium state is assumed to be stationary and axisymmetric, the perturbation quantities are proportional to $\exp(\omega t + i m \phi)$, where $\omega$ is the frequency observed in an inertial frame and $m$ is the azimuthal wave number. Then, the linearized basic equations which govern the adiabatic, non-radial oscillations of a magnetized and uniformly rotating star are written in the coordinate system $(a, \theta, \phi)$, to second order in $\omega_A$, as (Saio 1981; Lee 1993; Yoshida & Lee 2000a)

$$-\sigma^2 \left[ (1 + 2\epsilon) A + a\xi \nabla_0 A + a(\xi \cdot \nabla_0 A) e_a \right] = -\nabla_0 \Phi' - \frac{1}{\rho'} \nabla_0 p' + \frac{\rho'}{\rho A} \left[ \frac{dp}{da} e_a - \frac{1}{4\pi} (\nabla_0 \times B) \times B \right] + i \sigma D$$

$$+ \frac{1}{4\pi \rho} \left[ (\nabla_0 \times B') \times B + (\nabla_0 \times B) \times B' \right]$$

(23)

$$\rho' + \nabla_0 \cdot (\rho \xi) + \rho \xi \cdot \nabla_0 \left( 3\epsilon + \frac{\partial \epsilon}{\partial a} a A \right) = 0$$

(24)

$$\frac{\dot{p}}{p} = \frac{\dot{p}}{\Gamma_1 p} - \frac{\xi a}{a} a A$$

(25)

$$(B')^i = \frac{1}{\sqrt{g}} \epsilon^{ijk} \frac{\partial}{\partial x^j} \left( \sqrt{g} \xi_{\ell m k} \xi^l B^m \right)$$

(26)
Stars having purely toroidal magnetic field

where $\sigma = \omega + m\Omega$ denotes the oscillation frequency observed in the corotating frame of the star, $\xi(a, \theta, \phi)$ is the displacement vector, the prime (') indicates the Eulerian perturbation, $\epsilon_{ijk}$ and $\epsilon^{ijk}$ are the Levi-Civita permutation symbols, $g$ is the determinant of the metric $g_{ij}$, and

$$\nabla_0 = \lim_{\epsilon \to 0} \left[ e_a \frac{\partial}{\partial a} + e_\theta \frac{1}{a} \frac{\partial}{\partial \theta} + e_\phi \frac{1}{a \sin \theta} \frac{\partial}{\partial \phi} \right],$$

(27)

and $e_a$, $e_\theta$, and $e_\phi$ are the basis vectors in the $a$, $\theta$, and $\phi$ directions, respectively. Here, the vector $D$ in equation (30) comes from the Coriolis force and is given by (see, e.g., Lee 1993; Yoshida & Lee 2000a)

$$D_a = 2\Omega \left( 1 + 2\epsilon + a \frac{\partial \epsilon}{\partial a} \right) \sin \theta \xi^a, \quad D_\theta = 2\Omega \left( 1 + 2\epsilon + \frac{\sin \theta}{\cos \theta} \frac{\partial \epsilon}{\partial \theta} \right) \cos \theta \xi^\phi,$$

(28)

and $aA$ in equation (25) denotes the Schwarzschild discriminant defined as

$$aA = \frac{d \ln \rho}{d \ln a} - 1 - \frac{d \ln p}{d \ln a} \Gamma_1 \Gamma_2 \frac{d \ln a}{d \ln \rho},$$

(29)

where $\Gamma_1 = (\partial \ln p/\partial \ln \rho)_{ad}$. In this paper, for simplicity, we employ the Cowling approximation, neglecting $\Phi'$.

Because of the Lorentz and Coriolis terms in the equation of motion (23), separation of variables for the perturbations is impossible between the radial coordinate ($a$) and the angular coordinates ($\theta$, $\phi$). We therefore expand the perturbations in terms of the spherical harmonic functions $Y_{lm}^m(\theta, \phi)$ with different $l$’s for a given azimuthal index $m$. The displacement vector $\xi$ is then given by (see e.g., Lee 2005, 2007)

$$\xi^a = \sum_{j=1}^{\text{max}} a S_j(a) Y_{lj}(\theta, \phi),$$

(30)

$$\xi^\theta = \sum_{j=1}^{\text{max}} \left[ a H_{lj}(a) \frac{\partial}{\partial \theta} Y_{lj}^m(\theta, \phi) - ia T_{lj}(a) \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_{lj}^m(\theta, \phi) \right],$$

(31)

$$\xi^\phi = \sum_{j=1}^{\text{max}} \left[ a H_{lj}(a) \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_{lj}^m(\theta, \phi) + ia T_{lj}(a) \frac{\partial}{\partial \theta} Y_{lj}^m(\theta, \phi) \right],$$

(32)

and the vector $B'$ is given by

$$\frac{(B^a')}{kp} = \sum_{j=1}^{\text{max}} \left[ i a h^S_{lj}(a) Y_{lj}^m(\theta, \phi) \right],$$

(33)

$$\frac{(B^\theta')}{kp} = \sum_{j=1}^{\text{max}} \left[ i a h^H_{lj}(a) \frac{\partial}{\partial \theta} Y_{lj}^m(\theta, \phi) - ah^T_{lj}(a) \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_{lj}^m(\theta, \phi) \right],$$

(34)

$$\frac{(B^\phi')}{kp} = \sum_{j=1}^{\text{max}} \left[ i a h^H_{lj}(a) \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_{lj}^m(\theta, \phi) + ah^T_{lj}(a) \frac{\partial}{\partial \theta} Y_{lj}^m(\theta, \phi) \right],$$

(35)

where $l_j = |m| + 2(j - 1)$ and $l'_j = l_j + 1$ for even modes, and $l_j = |m| + 2j - 1$ and $l'_j = l_j - 1$ for odd modes, respectively, and $j = 1, 2, 3, ..., \text{max}$.

The Euler perturbations of the pressure and density are given by

$$p' = \sum_{j=1}^{\text{max}} p_{lj}(a) Y_{lj}^m(\theta, \phi), \quad \rho' = \sum_{j=1}^{\text{max}} \rho_{lj}(a) Y_{lj}^m(\theta, \phi).$$

(36)

In this paper, we usually use $\text{max} = 12$ to obtain solutions with sufficiently high-angular resolution. Substituting the expansions (30)-(36) into the linearized basic equations (23)-(26), we obtain a finite set of coupled linear ordinary differential equations for the expansion coefficients $S_j(a)$ and $p_{lj}(a)$, which we call the oscillation equations to be solved in the interior of magnetized and uniformly rotating stars. The set of oscillation equations obtained for the magnetized rotating star is given in Appendix A.
**Figure 1.** Equi-magnetic field strength contours (solid lines) and equi-density contours (long-dashed lines) on the meridional cross sections are plotted, from left to right panels, for the polytropes of the index \(n = 1, 1.5, \text{ and } 3\), respectively. The outer-most solid circles show stellar surfaces. The solid contours correspond to \(B/B_{\text{max}} = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, \text{ and } 0.9\), and the long-dashed contours to \(\rho/\rho_c = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, \text{ and } 0.9\).

**Figure 2.** Functions \(\alpha' \equiv \alpha/\bar{\omega}_A^2\) and \(\beta' \equiv \beta/\bar{\omega}_A^3\) versus the fractional radius \(a/R\) for the \(n = 1\) polytrope.

### 4 NUMERICAL RESULTS
In this study, the three polytropes with indices \(n = 1, 1.5, \text{ and } 3\) are used for modal analyses of the magnetized stars. The polytropes with \(n = 1 \text{ and } 1.5\) and with \(n = 3\) are regarded as simplified models of the neutron star and the normal star, respectively. For these polytropes, distributions of the density and the imposed magnetic fields are shown in Fig. 1. In this figure, the equi-density and equi-magnetic field strength contours on the meridional cross sections are given. We see that the density and magnetic field distribution of the models with a larger polytropic index tend to be concentrated in the central region of the star. Typical values of the mass \(M\) and radius \(R\) for the neutron star and the normal star are assumed to be \((M, R) = (1.4M_\odot, 10^6\text{ cm})\) and \((M, R) = (M_\odot, R_\odot)\), respectively. Thus, we have \(\bar{\omega}_A = 4.42 \times 10^{-3}\) for the neutron star with the field strength \(B_0 = 10^{16}\text{ G}\), and \(\bar{\omega}_A = 7.30 \times 10^{-4}\) for the normal star with the field strength \(B_0 = 10^9\text{ G}\). For the polytrope with \(n = 1\), the functions \(\alpha(a)/\bar{\omega}_A^2\) and \(\beta(a)/\bar{\omega}_A^3\) for the magnetic deformation are plotted as functions of \(a/R\) in Fig. 2.

#### 4.1 \(g\), \(f\), and \(p\)-modes
We first calculate \(f\)-, low radial order \(g\)-, and \(p\)-modes of the polytropes taking account of the effects of the toroidal magnetic field. In these calculations, no effects of rotation are considered. To study oscillation modes for the neutron star and normal star models, the adiabatic indices for perturbations are assumed to be

\[
\frac{1}{\Gamma_1} = \frac{n}{n+1} + \gamma
\]

with \(\gamma\) being a constant, for which \(aA = \gamma(d\ln p/d\ln a)\). In this subsection, we use \(\gamma = -10^{-4}\) for the polytropes with the indices \(n = 1 \text{ and } 1.5\), but for the polytrope with \(n = 3\), we assume \(\Gamma_1 = 5/3\), and hence \(\gamma = -3/20\). Since all the magnetic terms in the oscillation equations are proportional to \(\bar{\omega}_A^2\), an oscillation frequency of the modes may be written by (see
where $\bar{\sigma}_0$ is the oscillation frequency of the non-magnetized star, and $\bar{E}_2$ is a proportionality coefficient and can be obtained by calculating the oscillation frequency of the mode for two different values of $\bar{\omega}_A$, that is, $\bar{\omega}_A = 0$ and $\bar{\omega}_A \approx 10^{-6}$, for example. Here, $\bar{\sigma}_0$ and $\bar{E}_2$ are the quantities normalized by the Kepler frequency of the star $\Omega$. This coefficient $\bar{E}_2$ for a mode may also be calculated by using the eigenfunctions of the non-magnetized star by treating $\bar{\omega}_A$ as a small parameter. We have used the symbol $\bar{E}_2^g$ to denote the coefficient computed by using the eigenfunctions for the non-magnetized star. The derivation and the explicit expression for $\bar{E}_2^g$ are given in Appendix C.

In Tables 1–3, we tabulate the coefficients $E_2, E_2'$ and $E_2^0$ for $f$-, $g$-, and $p$-modes of $l = m$ for the polytropic model with $n = 1$ and $\gamma = -10^{-4}$, along with the equations (38).

| mode | $\bar{\sigma}_0$ | $E_2$ | $E_2'$ | $E_2^0$ |
|------|-----------------|-------|--------|--------|
| $E_2$ | $-1.578(+0)$ | $-1.490(+0)$ | $-1.491(+0)$ |
| $E_2'$ | $-2.051(+0)$ | $-2.095(+0)$ | $-2.095(+0)$ |
| $E_2^0$ | $-3.063(+0)$ | $-3.064(+0)$ | $-3.064(+0)$ |
| $E_2$ | $3.26931$ | $6.162(-1)$ | $6.164(-1)$ | $3.553(-1)$ |
| $E_2'$ | $5.09325$ | $1.113(+0)$ | $1.114(+0)$ | $5.650(-1)$ |
| $E_2^0$ | $6.85013$ | $1.550(+0)$ | $1.554(+0)$ | $7.586(-1)$ |

Table 1. Coefficients $E_2, E_2'$, and $E_2^0$ for $g$-, $f$-, and $p$-modes of $l = m$ for the polytropic model with $n = 1$ and $\gamma = -10^{-4}$.

* We use the notation of $1.000 \times 10^N \equiv 1.000(N)$. Appendixes A & C and Unno et al. 1989)
Table 2. Coefficients $E_2$, $E'_2$, and $E''_2$ for $g$, $f$, and $p$-modes of $l = m$ for the polytropic model with $n = 1.5$ and $\gamma = -10^{-4}$.

| mode | $\sigma_0$ | $E_2$ | $E'_2$ | $E''_2$ |
|------|------------|-------|--------|---------|
| $m = 1$ | | | | |
| $g_1$ | 0.00788 | -8.899(-1) | -8.120(-1) | -8.123(-1) |
| $g_2$ | 0.01057 | -1.179(+0) | -1.193(+0) | -1.193(+0) |
| $g_3$ | 0.01626 | -1.797(+0) | -1.856(+0) | -1.856(+0) |
| $p_1$ | 3.08199 | 5.763(-1) | 5.761(-1) | 9.406(-2) |
| $p_2$ | 4.64233 | 9.672(-1) | 9.668(-1) | 1.600(-1) |
| $p_3$ | 6.15692 | 1.329(+0) | 1.328(+0) | 2.190(-1) |
| $m = 2$ | | | | |
| $g_1$ | 0.01214 | 2.316(+1) | 2.286(+1) | 2.286(+1) |
| $g_2$ | 0.01564 | 1.628(+1) | 1.607(+1) | 1.607(+1) |
| $g_3$ | 0.02217 | 8.198(+0) | 8.153(+0) | 8.152(+0) |
| $f$ | 1.84930 | 4.027(-1) | 4.027(-1) | 1.621(-1) |
| $p_1$ | 3.55537 | 7.093(-1) | 7.093(-1) | 1.265(-1) |
| $p_2$ | 5.1450 | 1.073(+0) | 1.073(+0) | 1.808(-1) |
| $p_3$ | 6.91114 | 1.429(+0) | 1.428(+0) | 2.454(-1) |
| $m = 3$ | | | | |
| $g_1$ | 0.01547 | 4.950(+1) | 4.872(+1) | 4.872(+1) |
| $g_2$ | 0.01935 | 3.546(+1) | 3.511(+1) | 3.511(+1) |
| $g_3$ | 0.02596 | 1.941(+1) | 1.908(+1) | 1.908(+1) |
| $f$ | 1.84930 | 4.027(-1) | 4.027(-1) | 1.621(-1) |
| $p_1$ | 3.55537 | 7.093(-1) | 7.093(-1) | 1.265(-1) |
| $p_2$ | 5.1450 | 1.073(+0) | 1.073(+0) | 1.808(-1) |
| $p_3$ | 6.91114 | 1.429(+0) | 1.428(+0) | 2.454(-1) |

* We use the notation of $1.000 \times 10^N \equiv 1.000(N)$.

Figure 3. Eigenfunctions of $m = 2$ even modes for the polytrope with $n = 1$ and $\gamma = -10^{-4}$ for $B_0 = 10^{16}$ G, where, from left to right panels, the eigenfunctions plotted are those of the $g_1$, $f$, and $p_1$ modes. The solid lines, the long dashed lines and the short dashed lines are for the functions $xS_1$, $xH_1$, and $xT_{1+1}$ with $x = a/R$, and the amplitude normalization is given by $S_1 = 1$ at the surface.

$p_1$ modes of $m = 1$ and $m = 2$ are 0.6164 and 0.7100. We therefore think that the coefficients $E_2$ in the tables have at least three significant digits.

In Figure 3, we plot the expansion coefficients $S_{1l}$, $H_{1l}$, and $T_{1l}$ of the $g_1$, $f$, and $p_1$-modes of $l = m = 2$ for the $n = 1$ polytrope with $B_0 = 10^{16}$ G. The first expansion coefficients associated with the harmonic degree $l_1$ and $l'_1$ are dominant over the coefficients with $l_j$ and $l'_j$ for $j > 1$, and the difference in the dominant expansion coefficients of the modes between the magnetized and non-magnetized models is almost indiscernible. Because of the surface boundary condition (A1) and an algebraic relation (A1) in Appendix A, when $\Omega = 0$ and $\omega_0^2 \ll 1$ we have $H_{1l} \simeq S_{1l} / \sigma^2$ at the surface and hence $H_{1l}$ at the surface can be very large for $g$-modes having very low frequencies $\sigma \ll 1$ for the normalization $S_{1l} = 1$. In Figure 4, we plot magnetic field perturbations $S_{bl}$, $H_{bl}$, and $T_{bl}$ of the $g_1$, $f$, and $p_1$-modes of $l = m = 2$ for the $n = 1$ polytrope with $B_0 = 10^{16}$ G, where $S_{bl} \equiv k \rho h^S_{bl} / B_0$, $H_{bl} \equiv k \rho h^H_{bl} / B_0$, and $T_{bl} \equiv k \rho h^T_{bl} / B_0$. 
concerned with purely magnetic effects on the rotational mode, in this subsection, we focus on non-stratified stars or isentropic.*

& Lee (2000b), the stratification of the stellar interior strongly affects modal properties of the rotational mode. Since we are

force is the restoring force and the oscillation frequency is proportional to the rotation frequency Ω. As shown by Yoshi
data

\[ C \equiv k\rho a h \]

\[ \bar{E}_{\Sigma} \equiv 3 \text{ and } \Gamma_{1} = 5/3 \]

\[ 0.88994 \quad -1.088(-3) \quad -7.875(-4) \quad -7.747(-3) \]
\[ 1.16154 \quad -1.270(-3) \quad -1.079(-3) \quad -9.976(-3) \]
\[ 1.68082 \quad 9.069(-4) \quad 7.333(-4) \quad -1.306(-2) \]
\[ 3.81006 \quad 2.906(-1) \quad 2.905(-1) \quad 2.689(-3) \]
\[ 5.01208 \quad 4.335(-1) \quad 4.331(-1) \quad 2.803(-3) \]
\[ 6.25522 \quad 5.541(-1) \quad 5.538(-1) \quad 3.836(-3) \]

\[ m = 2 \]
\[ 1.35792 \quad 1.948(-1) \quad 1.930(-1) \quad 1.802(-1) \]
\[ 1.70580 \quad 1.502(-1) \quad 1.494(-1) \quad 1.332(-1) \]
\[ 2.29614 \quad 1.103(-1) \quad 1.099(-1) \quad 8.008(-2) \]
\[ 3.06379 \quad 2.193(-1) \quad 2.194(-1) \quad 1.814(-2) \]
\[ 4.14666 \quad 3.521(-1) \quad 3.521(-1) \quad 1.064(-2) \]
\[ 5.39097 \quad 4.792(-1) \quad 4.788(-1) \quad 7.335(-3) \]
\[ 6.65382 \quad 5.990(-1) \quad 5.978(-1) \quad 6.608(-3) \]

\[ m = 3 \]
\[ 1.70370 \quad 4.161(-1) \quad 4.122(-1) \quad 3.942(-1) \]
\[ 2.07374 \quad 3.245(-1) \quad 3.227(-1) \quad 3.007(-1) \]
\[ 2.64527 \quad 2.338(-1) \quad 2.334(-1) \quad 1.920(-1) \]
\[ 3.12498 \quad 2.567(-1) \quad 2.566(-1) \quad 3.173(-2) \]
\[ 4.37567 \quad 3.947(-1) \quad 3.946(-1) \quad 1.118(-2) \]
\[ 5.68481 \quad 5.193(-1) \quad 5.189(-1) \quad 8.675(-3) \]
\[ 6.98164 \quad 6.392(-1) \quad 6.383(-1) \quad 8.076(-3) \]

\[ * \text{We use the notation of } 1.000 \times 10^N \equiv 1.000(N). \]

**Figure 4.** Same as Figure 3 but for the eigenfunctions \( S_{\ell m} \equiv k p a h_{r}^2 / B_0 \) (solid lines), \( H_{\ell m} \equiv k p a h_{r}^H / B_0 \) (long dashed lines), and \( T_{\ell m+1} \equiv k p a h_{l}^T / B_0 \) (short dashed lines).

For slowly rotating stars, we may write the inertial frame oscillation frequency \( \omega \) of a mode as

\[ \omega = \omega_0 + m(C_1 - 1)\Omega + E_2\omega_\Lambda^2 + \cdots , \]

where \( C_1 \) represents the response of the mode frequency to the slow rotation. Since \( \omega_\Lambda \approx 10^{-5} \sim 10^{-3} \) for \( B_0 = 10^{16} \sim 10^{17} \) G and \( E_2 \sim 10 \), the rotational effects may dominate the magnetic ones for \( \Omega \gtrsim 10^{-1} \).

### 4.2 Rotational modes

We consider the effects of the magnetic field on rotational modes such as inertial modes and \( r \)-modes, for which the Coriolis force is the restoring force and the oscillation frequency is proportional to the rotation frequency \( \Omega \). As shown by Yoshida & Lee (2000b), the stratification of the stellar interior strongly affects modal properties of the rotational mode. Since we are concerned with purely magnetic effects on the rotational mode, in this subsection, we focus on non-stratified stars or isentropic
the magnetic field perturbations $\xi$ of the "modes" in that frequency range have discontinuities as a function of $\xi$ continuum band of the frequency spectrum (see, section 7.4 of Goedbloed & Poedts 2004). This discontinuity of the functions rotational modes are quite small, which is similar to the case of low frequency $B_\sigma$ none. We obtained only solutions having pure imaginary $\xi_1$-modes. We looked for very low frequency modes to find magnetic modes having real frequencies for non-rotating stars, but we found 4.3 magnetic modes index and (A5) and used to eliminate the variables $\xi$ by using the eigenfunctions of non-magnetized slowly rotating stars. The coefficient $\kappa_1$ (2004) discussed the simplest case of a second order ordinary differential equation that possesses a frequency band in which stars, in which the adiabatic index for perturbations is given by $\Gamma_1 = 1 + n^{-1}$. To represent the effects of the magnetic field on the rotational modes, it is convenient to use the frequency ratio $\kappa \equiv \sigma / \Omega$, where $\sigma = \omega + m \Omega$ denotes the frequency observed in the co-rotating frame of the star, and for small values of $\omega_A^2 (\ll \Omega^2)$ we may write

$$\kappa = \kappa_0 (\Omega) \left[ 1 + \eta_2 \frac{\omega_A^2}{\Omega^2} \right] + \cdots ,$$

(40)

where the coefficient $\kappa_0$ may depend on the rotation rate $\Omega$ and the coefficient $\eta_2$ is a constant in the limit of $\omega_A^2 / \Omega^2 \to 0$ (see Appendix C). Inertial modes and r-modes of uniformly rotating isentropic polytropes were studied, for example, by Lockitch & Friedman (1999) and Yoshida & Lee (2000a). The r-modes of $m \neq 0$ and $l' > |m|$ are non-axisymmetric and retrograde modes and the frequency ratio $\kappa_0$ tends to $2m/(l' + 1)$ as $\Omega \to 0$. The ratio $\kappa_0$ for an inertial mode also tends to a definite value as $\Omega \to 0$, depending on $m$, $l$, and the polytropic index $n$ (see, e.g., Yoshida & Lee 2000a), and we may use $\kappa_0(0)$ as a labeling of the inertial modes for a given $m$.

Since stars with strong magnetic fields are frequently very slow rotators, we consider rotational modes in slowly rotating stars. In Table 4, the coefficients $\eta_2$ and $\eta_2'$ as well as $\kappa_0$ are tabulated for the $l' = |m|$ r-modes and inertial modes for $m = 2$ of isentropic (i.e., $\gamma = 0$) polytropes with three different indices $n$. We use the symbol $\eta_2'$ to denote the coefficient computed by using the eigenfunctions of non-magnetized slowly rotating stars. The coefficient $\eta_2$ of the fitting formula $y = \eta_2 x$, where $x \equiv 1/\Omega^2$ and $y \equiv E_2/\sigma_0$, can be calculated by a least-square method. In the table, $l_0 - m = 1$ means the r-modes, and $l_0 - m \geq 2$ correspond to inertial modes (see Yoshida & Lee 2000a), and the even and odd numbers of $l_0 - m$ indicate even and odd parity, respectively. Since using $\kappa_0$, the frequencies of the rotational modes can be written by $\kappa = \kappa_0 + \kappa_2 \Omega^2$, the intercept $\kappa_0$ of the fitting formula $y = \kappa_0 + \kappa_2 x$ can also be calculated by a least-square method. We find that the coefficients $\eta_2$ and $\eta_2'$ are in good agreement with each other. It is important to note that the effects of the magnetic deformation on the rotational modes are quite small, which is similar to the case of low frequency $g$-modes. We note that the frequency we obtain for $r$ mode is consistent with that by Lander et al. (2010).

In Figure 5, we show the eigenfunctions of the $m = 2$ rotational modes of the $n = 1$ polytrope for $B_0 = 10^{16}$ G, where we assume $\Omega = 0.05$ and the amplitude normalization is given by $T_{l_0} (R) = 1$. We find that the expansion coefficients for the $m = 2$ inertial mode of $\kappa_0(0) = 1$ are the same as those shown in figure 1 of Yoshida & Lee (2000a). In Figure 6, we plot the magnetic field perturbations $S_{\Omega_0}$, $H_{\Omega_0}$, and $T_{l_0}$ of the $m = 2$ rotational modes for $B_0 = 10^{16}$ G. In Figures 7 and 8, we plot the eigenfunctions and magnetic field perturbations of the $m = 2$ rotational modes of the isentropic polytrope with the index $n = 3$ for $B_0 = 10^6$ G.

4.3 magnetic modes

We looked for very low frequency modes to find magnetic modes having real frequencies for non-rotating stars, but we found none. We obtained only solutions having pure imaginary $\sigma$ for a given value of $j_{\text{max}}$, but we found that these solutions are dependent on $j_{\text{max}}$ and cannot be regarded as correct solutions we look for. It is to be noted that we could not obtain very low frequency $g$-modes either in the frequency range where magnetic modes having real frequencies might coexist. The functions $\xi$ of the “modes” in that frequency range have discontinuities as a function of $a$, which suggests that the “modes” are in a continuum band of the frequency spectrum (see, section 7.4 of Goedbloed & Poedts 2004). This discontinuity of the functions $\xi$, which occurs in a certain low frequency range, may be caused by the relation $A \Psi = B \Phi$ coming from equations (A4) and (A5) and used to eliminate the variables $\Phi$ in equations (A2) and (A3), where $\Psi = (H,T)^T$ and $\Phi = (y_1,y_2)^T$, and $A$ and $B$ are matrices, that is, there appears a point at which the determinant of the matrix $A$ vanishes. Goedbloed & Poedts (2004) discussed the simplest case of a second order ordinary differential equation that possesses a frequency band in which

Figure 5. Eigenfunctions of $m = 2$ rotational modes for the isentropic $n = 1$ polytrope for $B_0 = 10^{16}$ G: r mode of $\kappa_0 = 0.6667$ (left), and inertial modes of $\kappa_0 = 1.1000$ (center) and $\kappa_0 = 0.5173$ (right). The solid lines, the long dashed lines, and the short dashed lines are for the functions $xS_{l_0}$, $xH_{l_0}$, and $xT_{l_0}$ with $x = a/R$, respectively, and the amplitude normalization is given by $\max(xT_{l_0}) = 1$. 
Table 4. Coefficient $\eta_2$ of $m = 2$ rotational modes for isentropic polytropes with the indices $n = 1, 1.5, \text{and } 3^*$. 

| $l_0 - |m|$ | $\kappa_0$ | $\eta_2$ | $\eta_2^*$ |
|-----------|-----------|-----------|-------------|
| $n = 1$   |           |           |             |
| 1         | 0.66666   | 8.324(−1)| 8.326(−1)  |
| 2         | -0.55660  | 9.105(−1)| 9.171(−1)  |
|           | 1.10002   | 2.398(−1)| 2.401(−1)  |
| 3         | -1.02590  | 3.304(−1)| 3.316(−1)  |
|           | 0.51734   | 1.784(+0)| 1.785(+0)  |
|           | 1.35777   | 1.402(−1)| 1.404(−1)  |
| 4         | -1.27290  | 2.481(−1)| 2.511(−1)  |
|           | -0.27533  | 7.103(+0)| 7.100(+0)  |
|           | 0.86296   | 5.805(−1)| 5.820(−1)  |
|           | 1.51956   | 9.950(−2)| 9.957(−2)  |
| $n = 1.5$ |           |           |             |
| 1         | 0.66666   | 5.247(−1)| 5.244(−1)  |
| 2         | -0.69650  | 3.688(−1)| 3.696(−1)  |
|           | 1.06527   | 1.752(−1)| 1.753(−1)  |
| 3         | -1.12782  | 1.439(−1)| 1.443(−1)  |
|           | 0.53564   | 1.071(+0)| 1.069(+0)  |
|           | 1.31001   | 1.103(−1)| 1.104(−1)  |
| 4         | -1.34198  | 8.378(−2)| 8.382(−2)  |
|           | -0.36425  | 2.999(+0)| 2.993(+0)  |
|           | 0.85864   | 3.516(−1)| 3.520(−1)  |
|           | 1.47217   | 8.210(−2)| 8.237(−2)  |
| $n = 3$   |           |           |             |
| 1         | 0.66667   | 1.329(−1)| 1.328(−1)  |
| 2         | -1.07669  | 2.505(−2)| 2.520(−2)  |
|           | 0.99492   | 3.902(−2)| 3.892(−2)  |
| 3         | -1.37189  | 1.849(−2)| 1.872(−2)  |
|           | 0.57976   | 2.620(−1)| 2.622(−1)  |
|           | 1.20940   | 2.680(−2)| 2.665(−2)  |
| 4         | -1.51785  | 6.058(−3)| 6.101(−3)  |
|           | -0.66228  | 2.063(−1)| 2.064(−1)  |
|           | 0.85853   | 6.254(−2)| 6.253(−2)  |
|           | 1.36256   | 2.973(−2)| 3.023(−2)  |

* We use the notation of $1.000 \times 10^N \equiv 1.000(N)$. 

Figure 6. Same as Figure 5 but for the eigenfunctions $S_{b1} \equiv kpa h_f^S/B_0$ (solid lines), $H_{b1} \equiv kpa h_f^H/B_0$ (long dashed lines), and $T_{b1} \equiv k\rho a h_f^T/B_0$ (short dashed lines).
the differential equation becomes singular at a point, and the situation we have in our calculations for low frequency range is essentially the same as the second order differential equation.

This situation is largely different from that met by Lander et al. (2010), who found polar and axial Alfvén modes for magnetized rotating stars with a purely toroidal background magnetic field. They suggested that pure Alfvén modes of a non-rotating star or purely inertial modes of an unmagnetized rotating star may be replaced by hybrid magneto-inertial modes for magnetized and rotating stars (see also Mathis & Brye 2011, 2012), and that in the limit of \( M/T \to 0 \) or \( \bar{\Omega} A / \Omega^2 \to 0 \) the hybrid modes reduce to purely inertial modes, where \( M \) and \( T \) are the magnetic and kinetic energies of the equilibrium model, respectively. The method of calculation Lander et al (2010) use is a MHD simulation that follows the time development of linear oscillations around the equilibrium model and is different from the method we use in this paper. At this moment we do not understand why we find no magnetic modes for a purely toroidal background magnetic field.

5 CONCLUSION

In this paper, we have calculated non-axisymmetric oscillations of rotating stars magnetized with purely toroidal magnetic fields. Here we have used polytropic models of the indices \( n = 1, 1.5, \) and \( 3 \), and included the effects of the deformation caused by the toroidal magnetic fields. We have obtained discrete normal non-radial oscillation modes such as \( g, f, \) and \( p \) modes for non-rotating case. The frequency change due to the magnetic field for the modes scale with the square of typical stellar Alfvén frequency \( \omega_A \equiv \sqrt{B_0^2/(4\pi \rho_0 R^2)} \), and the proportional coefficients for the frequency changes are estimated by two different methods. For rotating stars, we have obtained rotational modes such as \( r \) and inertial modes. The frequency changes for the rotational modes also scale with the square of typical stellar Alfvén frequency \( \omega_A \). From Tables 1 to 4, we find that high frequency modes such as \( f \) and \( p \) modes are susceptible to the stellar deformation, while the lower frequency modes such as \( g, r, \) and inertial modes are almost insensitive to the deformation. It may be important to note that in the present analysis we could not find \( j_{\text{max}} \)-independent magnetic modes, the existence of which are suggested by Lander et al (2010). The reason for the difference between the two calculations is not yet well understood.

The present analysis is a part of our study of the oscillations of magnetized stars. Even for a purely poloidal magnetic
field, it is difficult to determine frequency spectra of non-axisymmetric modes and those of axisymmetric spheroidal modes. We note that the stability of a magnetic field configuration is another difficult problem. It is well known that a purely poloidal and purely toroidal magnetic fields are unstable and the energy of the field is dissipated quickly, that is, for several ten milliseconds (e.g., Goossens 1979; Kiuchi, Yoshida, & Shibata 2011; Laskey et al. 2011; Cioffi & Rezzolla 2012), although stellar rotation may weaken the instability of a purely poloidal or purely toroidal magnetic fields (e.g., Lander & Jones 2011a, 2011b). It is thus anticipated that a mixed poloidal and toroidal magnetic field configuration such as twisted-torus magnetic field (e.g., Braithwaite & Spruit 2004; Yoshida & Erguchi 2006; Yoshida, Yoshida & Erguchi 2006; Cioffi et al. 2009) can be stable, that is, such a magnetic field configuration can last stably for a long time. If this is the case, it will be important to investigate the oscillation modes of stars threded by both toroidal and poloidal magnetic fields. As suggested by Colaiuda & Kokkotas (2012), however, the presence of a toroidal field component can significantly change the properties of the oscillation modes of magnetized neutron stars. In the presence of both poloidal and toroidal field components, toroidal and spheroidal modes are coupled, which inevitably breaks equatorial symmetry and antisymmetry of perturbations.

As an important physical property inherent to cold neutron stars, we need to consider the effects of superfluidity and superconductivity of neutrons and protons on the oscillation modes (e.g., Andersson et al. 2009; Galmedakis et al. 2011). It is believed that neutrons become a superfluid both in the inner crust and in the fluid core while protons can be superconducting in the core. For example, if the fluid core is a type I superconductor, magnetic fields will be expelled from the core region, because of the Meissner effect, and hence confined to the solid crust (e.g., Colaiuda et al. 2008; Sotani et al. 2008). In this case, we only have to consider oscillations of a magnetized crust so long as toroidal modes are concerned, and we have toroidal crust modes modified by a magnetic field, while spheroidal oscillations can be propagative both in the magnetic crust and in the non-magnetic fluid core. However, a recent analysis of the spectrum of timing noise for SGR 1806-20 and SGR 1900+14 has suggested that the core region is a type II superconductor (Arras, Cumming & Thompson 2004). If this is the case, the fluid core can be threaded by a magnetic field and hence the frequency spectra of oscillation modes will be affected by the superconductivity in the core (e.g., Colaiuda et al. 2008; Sotani et al. 2008).

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APPENDIX A: PULSATION EQUATIONS FOR THE SLOWLY ROTATING STAR WITH PURELY TOROIDAL MAGNETIC FIELDS

To describe the master equations concisely, it is useful to introduce the following column vectors composed of the expansion coefficients for the perturbation quantities: the vectors $\mathbf{S}$, $\mathbf{H}$, $\mathbf{h}^S$, $\mathbf{h}^H$, $\mathbf{T}$, $\mathbf{h}^T$, and $\mathbf{y}_2$, defined by

$$
(S)_j = S_{j1}, \quad (H)_j = H_{j1}, \quad (T)_j = T_{j1}, \quad (h^S)_j = h^S_{j1}, \quad (h^H)_j = h^H_{j1}, \quad (h^T)_j = h^T_{j1}, \quad (y_2)_j = \frac{p_j}{\rho q_j},
$$

(A1)

where $(X)_j$ denotes the $j$-th component of the vector $X$ and $g = GM(a)/a^2$ is the gravitational acceleration. The perturbed continuity equation (A4), and the $a$, $\theta$- and $\phi$-components of the perturbed Euler equation (A5), respectively, reduce to

$$
a \frac{dS}{da} = \left\{ V_G - 3a \frac{d\theta(a)}{da} \right\} S - V_G y_2 + \left[ A_0 + 3\theta(\beta)B_0 \right] H + 3m\vartheta(\beta)Q_0 T,
$$

(A2)

$$
a \frac{dy_2}{da} = [c_1 \sigma^2 \left[ 1 + 2\eta(a) \right] + 2\pi(\beta)A_0 + a \Omega] S - \frac{aA}{3} \left( 2 + \frac{\ln \rho}{\ln a} \right) \frac{\rho c \varphi^2}{\sigma^2} (1 - A_0) S
$$

$$
+ (1 - aA - U) y_2 + V_G \left( 2 + \frac{\ln \rho}{\ln a} \right) \frac{\rho c \varphi^2}{\sigma^2} (1 - A_0) y_2
$$

$$
- \left\{ 2m c_1 \sigma \left[ 1 + \alpha + \eta(a) \right] 1 + 2m c_1 \sigma \Omega \left[ \beta + \eta(\beta) \right] A_0 + 3c_1 \sigma^2 \beta B_0 \right\} H
$$

$$
- \left\{ 2c_1 \sigma \Omega \left[ 1 + \alpha + \eta(a) \right] C_0 + 2c_1 \sigma \Omega \left[ \beta + \eta(\beta) \right] A_0 C_0 + 3m c_1 \sigma^2 \beta Q_0 \right\} T
$$

$$
- \left\{ m \left[ 1 + \alpha + \eta(a) \right] 1 + m \left[ \beta + \eta(\beta) \right] A_0 - 3\beta(2A_0 + B_0) \right\} S + \frac{a A p c_2^2}{\sigma^2} \left( 2 + \frac{\ln \rho}{\ln a} \right) \frac{\rho c \varphi^2}{\sigma^2} (1 - A_0) S
$$

$$
+ \frac{1}{c_1 \sigma^2} A_0 y_2 - \frac{\rho c^2}{\sigma^2} V_G \left( 2A_0 + B_0 \right) y_2
$$

$$
+ \left[ (1 + 2a) A_0 L_0 + 2B \left( A_0 A_0 + 3B_0 \right) - 2m \vartheta(1 + 6A_0) \right] H
$$

$$
+ \left\{ -\nu (1 + 2a - 2A_0 M_1 - 4r \beta(2A_0 A_0 + M_1 + 3Q_0 B_1) + 6m \beta T
$$

$$
+ \frac{1}{2m \rho c^2 \sigma^2} \left( 2 + \frac{\ln \rho}{\ln a} \right) h^S + m \rho c^2 \sigma^2 \bar{\rho}^2 h^2 - \frac{\rho c^2}{\sigma^2} A_0 R h^T \right) = 0,
$$

(A4)
\( \{ \nu [1 + \alpha + \eta(\alpha)] A_1 K + \nu [\beta + \eta(\beta)] (A_1 A_2 K + 3Q_1 Q_0 Q_1 - 3Q_1) - 3m\beta Q_1 \} S - m A_1 \rho \frac{\partial^2}{\partial \sigma^2} Q_1 S + m \rho \frac{\partial^2}{\partial \sigma^2} V_0 Q_1 y_2 \\
+ [-\nu (1 + 2\alpha - 2\beta) A_1 M_0 - 4\nu (A_1 A_2 M_0 + 3Q_1 B_0) + 6m\beta Q_1] H \\
+ [(1 + 2\alpha) A_1 L_1 + 2\beta (A_1 A_2 + 3B_1) - 2m\beta (I + 6A_1)] T \\
- \frac{1}{2} \rho \frac{\partial^2}{\partial \sigma^2} \left( 2 + \frac{\ln \rho}{\ln a} \right) A_1 K h^S + \rho \frac{\partial^2}{\partial \sigma^2} A_1 M_0 h^H + \frac{1}{2} m \rho \frac{\partial^2}{\partial \sigma^2} (A_I - 2I) h^T = 0. \) \quad (A5)

The \( a^- \), \( \theta^- \) and \( \phi^- \) components of the perturbed induction equation [20], respectively, lead

\( h^S = m S, \) \quad (A6)

\( h^H = m A_1 \Lambda^{-1}_0 S - m V_0 \Lambda_0^{-1} y_2 + m H, \) \quad (A7)

\( h^T = a A K S - V_0 K y_2 - m T. \) \quad (A8)

Here,

\( U = \frac{d \ln M(a)}{d \ln a}, \quad V_0 = -\frac{1}{\Gamma_1} \frac{d \ln p}{d \ln a}, \quad \theta(\alpha) = 3\alpha + a \frac{d \alpha}{d \ln a}, \quad \eta(\alpha) = \alpha + a \frac{d \alpha}{d \ln a}, \) \quad (A9)

and \( \sigma \equiv \pi/(GM/R^3)^{1/2} \) is the frequency in the unit of the Kepler frequency, and \( \nu \equiv 2\Omega/\sigma \). The quantities \( Q_0, Q_1, C_0, C_1, K, M_0, M_1, A_0, A_1, R, L_0, L_1, A_0, A_1, B_0, \) and \( B_1 \) denote the matrices defined as follows:

For even modes,

\( (Q_0)_{j,j} = J_{j,j+1}^{m}, \quad (Q_0)_{j,j+1,j} = J_{j,j+2}^{m}, \quad (Q_1)_{j,j} = J_{j+1,j}^{m}, \quad (Q_1)_{j,j+1,j} = J_{j+2,j}^{m}, \)

\( (C_0)_{j,j} = -(l_j + 2) J_{j,j}^{m}, \quad (C_0)_{j,j+1,j} = (l_j + 1) J_{j+1,j}^{m}, \quad (C_1)_{j,j} = l_j J_{j,j}^{m}, \quad (C_1)_{j,j+1,j} = -(l_j + 3) J_{j+1,j}^{m}, \)

\( (K)_{j,j} = \frac{J_{j+1,j}^{m} + l_j}{l_j + 1}, \quad (K)_{j+1,j} = -\frac{J_{j,j}^{m} + l_j}{l_j + 1}. \)

\( (M_0)_{j,j} = \frac{l_j}{l_j + 1} J_{j,j}^{m}, \quad (M_0)_{j,j+1,j} = \frac{l_j + 3}{l_j + 2} J_{j,j+2}^{m}, \quad (M_1)_{j,j} = \frac{l_j + 2}{l_j + 1} J_{j,j+1}^{m}, \quad (M_1)_{j+1,j} = \frac{l_j + 1}{l_j + 2} J_{j+1,j}^{m}, \)

\( (A_0)_{j,j} = l_j (l_j + 1), \quad (A_1)_{j,j} = (l_j + 1)(l_j + 2). \)

\( (R)_{j,j} = \frac{(l_j + 2)(l_j + 1)}{l_j + 1} J_{j,j}^{m}, \quad (R)_{j,j+1,j} = \frac{(l_j + 2)(l_j + 1)}{l_j + 1} J_{j,j+2}^{m}. \)

\( L_0 = 1 - m \nu A_0^{-1}, \quad L_1 = 1 - m \nu A_1^{-1}, \quad A_0 = \frac{1}{2} (3 Q_0 Q_1 I), \quad A_1 = \frac{1}{2} (3 Q_1 Q_0 I), \quad B_0 = Q_0 C_1, \quad B_1 = Q_1 C_0, \)

where \( l_j = |m| + 2 - 2 \) for \( j = 1, 2, 3, ..., j_{\text{max}}, \) and

\( J_{j,j}^{m} = \left[ \frac{(l_j + m)(l_j - m)}{(2l_j - 1)(2l_j + 1)} \right]^{1/2}. \) \quad (A11)

For odd modes,

\( (Q_0)_{j,j} = J_{j,j}^{m}, \quad (Q_0)_{j,j+1} = J_{j+1,j}^{m}, \quad (Q_1)_{j,j} = J_{j+1,j}^{m}, \quad (Q_1)_{j+1,j} = J_{j,j+2}^{m}, \)

\( (C_0)_{j,j} = l_j J_{j,j}^{m}, \quad (C_0)_{j,j+1,j} = -(l_j + 3) J_{j+1,j}^{m}, \quad (C_1)_{j,j} = -(l_j + 2) J_{j,j}^{m}, \quad (C_1)_{j+1,j} = (l_j + 1) J_{j+1,j}^{m}, \)

\( (K)_{j,j} = \frac{J_{j+1,j}^{m} + l_j}{l_j + 1}, \quad (K)_{j+1,j} = \frac{J_{j,j}^{m} + l_j}{l_j + 1}. \)

\( (M_0)_{j,j} = \frac{l_j + 1}{l_j + 1} J_{j,j}^{m}, \quad (M_0)_{j,j+1,j} = \frac{l_j + 1}{l_j + 2} J_{j,j+2}^{m}, \quad (M_1)_{j,j} = \frac{l_j}{l_j + 1} J_{j,j+1}^{m}, \quad (M_1)_{j+1,j} = \frac{l_j + 3}{l_j + 2} J_{j+1,j}^{m}, \)

\( (A_0)_{j,j} = (l_j + 1)(l_j + 2), \quad (A_1)_{j,j} = l_j (l_j + 1). \)

\( (R)_{j,j} = \frac{l_j (l_j + 3)}{l_j + 1} J_{j,j}^{m}, \quad (R)_{j,j+1,j} = \frac{l_j (l_j + 3)}{l_j + 1} J_{j,j+2}^{m}, \)

\( L_0 = 1 - m \nu A_0^{-1}, \quad L_1 = 1 - m \nu A_1^{-1}, \quad A_0 = \frac{1}{2} (3 Q_0 Q_1 I), \quad A_1 = \frac{1}{2} (3 Q_1 Q_0 I), \quad B_0 = Q_0 C_1, \quad B_1 = Q_1 C_0. \) \quad (A12)
where \( l_j = |m| + j - 1 \) for \( j = 1, 2, 3, \ldots, j_{\text{max}} \).

Eliminating the variables \( h^\chi, h^H, \) and \( h^T \) from equations (A3)-(A5) by using equations (A6)-(A8), and eliminating \( H \) and \( T \) by using equations (A4) and (A5), we may reduce equations (A2) and (A3) to a set of coupled first-order linear ordinary differential equations for the functions \( y_1 = S \) and \( y_2 \), which is formally written as:

\[
\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathcal{F} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]

The surface boundary conditions are

\[
-y_1 + y_2 + \rho_1 \bar{\omega}_A \left( 1 + \frac{1}{\rho} \frac{d \rho}{d \ln a} \right) (1 - A_0) S + \frac{1}{2} B_0 H + \frac{1}{2} m Q_0 T - \frac{1}{2} m h^H + \frac{1}{2} C_0 h^T = 0.
\]

which means \( \frac{1}{2\pi} \Delta \left( p + \frac{1}{8\pi} R^3 \right) = 0 \) at the stellar surface, where \( \Delta Q \) denotes the Lagrangian change of the quantity \( Q \). The boundary conditions at the stellar center are the regularity conditions for the eigenfunctions \( y_1 \) and \( y_2 \).

**APPENDIX B: SURFACE BOUNDARY CONDITION FOR THE FUNCTION \( \psi_2 \)**

To determine the function \( \psi_2 \) satisfying the differential equation (13), we need the surface boundary conditions. Assuming the deviation of the surface \( r = R(R, \theta) \) of the magnetized star from the surface \( r = R \) of the non-magnetized star is small, we may write

\[
R(R, \theta) = R(1 + \delta \zeta(\theta)).
\]

Since we have \( \rho(R_0, \theta) = 0 \) and \( \rho_0(R) = 0 \) at the stellar surface, from equation (10) we can obtain

\[
\delta \zeta = 2R^2 \omega_A^2 \frac{dx}{d\psi_0} \left[ \psi_0(1) + \psi_2(1) P_2(\cos \theta) \right].
\]

Using equation (9), the gravitational potential \( \Phi(r, \theta) \) and its derivative \( \partial \Phi(r, \theta) / \partial x \) inside the star are given by

\[
\Phi(r, \theta) = \Psi_0(r) + c_0 - 2R^2 \omega_A^2 [c_{1,0} + \psi_0(x) + \psi_2(x) P_2(\cos \theta)] - \frac{1}{3} \omega_A^2 x^2 \hat{\rho} [1 - P_2(\cos \theta)],
\]

\[
\frac{\partial \Phi(r, \theta)}{\partial x} = \frac{\partial \Psi_0(r)}{\partial x} - 2R^2 \omega_A^2 \left[ \frac{d \psi_0}{dx}(x) + \frac{d \psi_2}{dx}(P_2(\cos \theta)) \right] - \frac{1}{3} \omega_A^2 \left( x^2 \frac{d \hat{\rho}}{dx} + 2R \hat{\rho} \right) [1 - P_2(\cos \theta)],
\]

where we have set the constant \( C \) in equation (9) as \( C = c_0 - 2R^2 \omega_A^2 c_{1,0} \). Since \( \hat{\rho}(R_0, \theta) = 0 \) and \( \Psi_0(R) \approx \Psi_0(R) \) \((\partial \Psi_0(r) / \partial x) \approx 0 \) on \( R = GM/R^2 \approx (GM/R^2)(1 - 2\delta \zeta) \) at the deformed surface, the gravitational potential \( \Phi(R, \theta) \) and its derivative \( \partial \Phi(R, \theta) / \partial x \) reduce to

\[
\Phi(R, \theta) = \Psi_0(R) + c_0 - 2R^2 \omega_A^2 c_{1,0},
\]

\[
\frac{\partial \Phi(R, \theta)}{\partial x} = \frac{\partial \Psi_0(R)}{\partial x} - 2R^2 \omega_A^2 \left[ \frac{d \psi_0}{dx}(1) + \frac{d \psi_2}{dx}(P_2(\cos \theta)) \right] - \frac{1}{3} \omega_A^2 R^2 \frac{d \hat{\rho}}{dx} [1 - P_2(\cos \theta)] - 4R^2 \omega_A^2 \left[ \psi_0(1) + \psi_2(1) P_2(\cos \theta) \right],
\]

where we have used \( d \Psi_0 / dx = GM/R^2 \) at the surface. On the other hand, the gravitational potential outside the star can be written as

\[
\Phi = -\frac{\kappa_0}{x} - 2R^2 \omega_A^2 \left[ \frac{\kappa_{1,0}}{x} + \frac{\kappa_{1,2}}{x^3} P_2(\cos \theta) \right],
\]

\[
\frac{\partial \Phi}{\partial x} = \frac{\kappa_0}{x^2} + 2R^2 \omega_A^2 \left[ \frac{\kappa_{1,0}}{x^2} + \frac{3\kappa_{1,2}}{x^4} P_2(\cos \theta) \right],
\]

and at the stellar surface \( x = x_s \equiv 1 + \delta \zeta \) we have

\[
\Phi = -\kappa_0 - 2R^2 \omega_A^2 [\kappa_{1,0} + \kappa_{1,2} P_2(\cos \theta)] + \kappa_0 \delta \zeta,
\]

\[
\frac{\partial \Phi}{\partial x} = \kappa_0 + 2R^2 \omega_A^2 [\kappa_{1,0} + 3\kappa_{1,2} P_2(\cos \theta)] - 2\kappa_0 \delta \zeta,
\]

where \( \kappa_0, \kappa_{1,0}, \) and \( \kappa_{1,2} \) are arbitrary constants. If we require \( \Phi \) and \( \partial \Phi / \partial x \) inside and outside the star are continuous at the stellar surface, by comparing the zeroth-order terms, we find

\[
\Psi_0(R) + c_0 = -\kappa_0, \quad \kappa_0 = \frac{\partial \Psi_0(R)}{\partial x}.
\]
and hence we obtain
\[ \Psi_0(R) = -\kappa_0 = -\frac{\partial \Psi_0(R)}{\partial x}, \quad c_0 = 0. \]  
(B12)

By comparing the perturbed terms we can obtain following relations:
\[ -c_{1,0} = -\kappa_{1,0} + \psi_0(1), \quad \kappa_{1,2} = \psi_2(1), \]  
(B13)

\[ \kappa_{1,0} = -\frac{d\psi_0}{dx}(1) - \frac{1}{6} \frac{d\phi}{dx}(1), \quad 3\kappa_{1,2} = -\frac{d\psi_2}{dx}(1) + \frac{1}{6} \frac{d\phi}{dx}(1). \]  
(B14)

From the relations, we note that the unknown constants \( c_{1,0} \) and \( \kappa_{1,0} \) for the function \( \psi_0 \) are determined uniquely by integrating equation (12) to the surface:
\[ \kappa_{1,0} = -\frac{d\psi_0}{dx}(1) - \frac{1}{6} \frac{d\phi}{dx}(1), \quad c_{1,0} = -\frac{d\psi_0}{dx}(1) - \psi_0(1) - \frac{1}{6} \frac{d\phi}{dx}(1). \]  
(B15)

On the other hand, for the function \( \psi_2 \) we obtain by eliminating the constant \( \kappa_{1,2} \)
\[ 3\psi_2(1) + \frac{d\psi_2}{dx}(1) = \frac{1}{6} \frac{d\phi}{dx}(1), \]  
(B16)

which gives the outer boundary condition for \( \psi_2 \) at the surface.

**APPENDIX C: FREQUENCY CHANGES DUE TO THE MAGNETIC FIELDS**

Using the continuity equation (23) and the adiabatic relation (24), we may rewrite the Euler equation (26) as
\[ -\sigma^2[(1 + 2\epsilon) \cdot (\xi + a(\xi \cdot \nabla)\epsilon)\epsilon_a] = -\nabla_0 \cdot \epsilon + \epsilon_a \frac{\Gamma_{1\rho}}{\rho} \cdot A \left[ \nabla_0 \cdot \epsilon + (\xi \cdot \nabla_0) \left( 3\epsilon + a \frac{\partial \epsilon}{\partial a} \right) \right] + i\sigma \mathbf{D} + \frac{\epsilon \cdot \nabla_0 \ln P + \nabla_0 \cdot \epsilon}{4\pi \rho} \mathbf{B} \times \mathbf{B} + \frac{1}{4\pi \rho} \left[ \nabla_0 \times \mathbf{B} \right] \times \mathbf{B} \right), \]  
(C1)

where \( \chi \equiv \rho' / \rho \). We write the eigenfunctions and eigenfrequency as follows (for a similar treatment, see, e.g., Saio 1981):
\[ \xi = \xi_0 + \xi_2, \]  
(C2)

\[ \chi = \chi_0 + \chi_2, \]  
(C3)

\[ \sigma = \sigma_0 + \sigma_2, \]  
(C4)

where quantities with subscripts 0 and 2 denote quantities of order \( \omega_0^2 \) and \( \omega_2^2 \), respectively. The Coriolis term, \( \mathbf{D} \), is then written by
\[ \mathbf{D} = D^{(0)}[\xi_0] + D^{(0)}[\xi_2] + D^{(2)}[\xi_0], \]  
(C5)

where
\[ D^{(0)}_a[\xi] = 2\Omega \sin \theta \xi_a, \quad D^{(0)}_a[\xi] = 2\Omega \cos \theta \xi_a, \quad D^{(0)}_a[\xi] = -2\Omega \left( \sin \theta \xi_a + \cos \theta \xi_a \right), \]  
(C6)

\[ D^{(2)}_a[\xi] = 2\Omega \left( \frac{2\epsilon + a \frac{\partial \epsilon}{\partial a}}{\frac{\partial \epsilon}{\partial a}} \sin \theta \xi_a + \cos \theta \xi_a \right), \quad D^{(2)}_a[\xi] = 2\Omega \left( \frac{2\epsilon + \frac{\sin \theta \frac{\partial \xi}{\partial a}}{\sin \theta \frac{\partial \epsilon}{\partial a}}}{\frac{\partial \xi}{\partial a}} \sin \theta \xi_a + \cos \theta \xi_a \right). \]  
(C7)

Introducing equations (C2)-(C5) into equation (C1) and grouping quantities of the same order in \( \omega_A \), we obtain
\[ -\sigma_0^2 \xi_0 = -\nabla_0 \chi_0 + \epsilon_a \frac{\Gamma_{1\rho}}{\rho} A \nabla_0 \cdot \epsilon + i\sigma_0 D^{(0)}[\xi_0]. \]  
(C6)

for order \( \omega_0^2 \), and
\[ -\sigma_0^2 \xi_2 = -\sigma_0^2 \xi_0 - \sigma_0^2 \xi_0 \nabla_0 \epsilon - \sigma_0^2 (\xi_0 \cdot \nabla_0) \epsilon_a - 2\sigma_0 \sigma_2 \xi_0 \]  
(C7)

\[ = -\nabla_0 \chi_2 + \epsilon_a \frac{\Gamma_{1\rho}}{\rho} A \left[ \nabla_0 \cdot \epsilon + \chi_0 \cdot \nabla_0 \left( 3\epsilon + a \frac{\partial \epsilon}{\partial a} \right) \right] + i\sigma_0 D^{(0)}[\xi_2] + i\sigma_2 D^{(0)}[\xi_0] + i\sigma_0 D^{(2)}[\xi_0] \]  
(C7)
\[ + \frac{\epsilon_a \cdot \nabla_0 \ln P + \nabla_0 \cdot \epsilon}{4\pi \rho} \left( \nabla_0 \times \mathbf{B} \right) \times \mathbf{B} + \frac{1}{4\pi \rho} \left[ \nabla_0 \times \mathbf{B} \right] \times \mathbf{B}. \]  
(C7)
for order $\omega_2^A$. Note that Equation (C6) for order $\omega_2^A$ describes the oscillation of the unmagnetized slowly rotating star. The functions $\chi_0$ and $\chi_2$, and $B'$ are, in terms of $\xi_0$ and $\xi_2$, given by

$$
\chi_0 = -\frac{\rho_0}{\rho} (\nabla_0 \cdot \xi_0 + \xi_0 \cdot \nabla_0 \ln \rho - \xi_0 \cdot e_A),
$$
$$
\chi_2 = -\frac{\rho_0}{\rho} \left\{ \nabla_0 \cdot \xi_2 + \xi_2 \cdot \nabla_0 \left( 3 \epsilon + a \frac{\partial \epsilon}{\partial a} \right) + \xi_2 \cdot \nabla_0 \ln \rho - \xi_2 \cdot e_A \right\},
$$

\begin{equation}
(B')^I = \frac{1}{a^3 \sin \theta} \frac{i \partial}{\partial x^I} \left( a^2 \sin \theta \epsilon_{lmk} \xi_m^I B^m \right). 
\end{equation}

Multiplying Equation (C7) by the complex conjugate of the displacement vector $\xi_0^*$ and integrating over mass, we obtain the integral relation that includes no term related to $\xi_2$, given by

$$
-2 \sigma_0^3 \int_0^M \epsilon |\xi_0|^2 dM_a - 2 \sigma_2 \int_0^M (a \xi_0^* \nabla_0 \epsilon) \cdot \xi_0^* dM_a - 2 \sigma_0 \sigma_2 \int_0^M |\xi_0|^2 dM_a
$$
$$
= \int_0^M \chi_0 \cdot \xi_0 \left( 3 \epsilon + a \frac{\partial \epsilon}{\partial a} \right) dM_a + i \sigma_0 \int_0^M \left\{ \xi_0^* dM_a + 2 \sigma_0 \sigma_2 \right\} dM_a + \frac{1}{4 \pi} \int_0^M \frac{1}{\rho} (\chi_0 + e_a) \left[ (\nabla_0 \times B) \times B \right] \cdot \xi_0^* dM_a,
$$

where $rM_a = r(a)^2 \sin \phi d\phi d\phi$. Substituting the $\omega_2^A$-order eigenfunctions expanded into Equations (29)-(35) into Equation (C9) and taking $\sigma_2 = E_2^0 \omega_2^A$, we may obtain the integral expression for the coefficient $E_2^0$, given by

$$
E_2^0 = \left[ \frac{\Omega_K}{4 \pi_0} \int_0^R f_2(a) \rho_0^a da + \sigma_0 \int_0^R f_2(a) \rho_0^a da + \frac{\Omega_K}{2 \sigma_0} \int_0^R \frac{1}{c_1} f_2(a) \rho_0^a da + \Omega \int_0^R f_2(a) \rho_0^a da \right]/W_I,
$$

where

$$
W_I = \int_0^R \left\{ |S|^2 + H^T \Lambda_0 H + T^T \Lambda_1 T \right\} \rho_0^a da
$$
$$
- \frac{\Omega}{\sigma_0} \int_0^R \left\{ m \left( S^2 H^T + S H^T S + S |H|^2 + |T|^2 \right) + H^T \Lambda_0 T - T^T \Lambda_1 H S^2 \Lambda_0 M_1 T + T^T \Lambda_1 M_0 H \right\} \rho_0^a da.
$$

$$
f_1(a) = S^T \left\{ m \frac{dH}{da} - mH^S + \left( 2 + \frac{d \ln \rho}{da} \right) H^H - \Lambda_0 \left[ \frac{dH^T}{da} + 2 \left( 2 + \frac{d \ln \rho}{da} \right) H^T \right] \right\}
$$
$$
+ H^T \left[ \left( 2 + \frac{d \ln \rho}{da} \right) \Lambda_0 H^H + \Lambda_0 H^S + 2 \Lambda_0 M_0 h^H + m(1 - 2I) h^T \right]
$$
$$
- \frac{2aA}{3} \left( 2 + \frac{d \ln \rho}{da} \right) S^T (1 - \Lambda_0) S + \frac{2V_2}{3} \left( 2 + \frac{d \ln \rho}{da} \right) S^T (1 - \Lambda_0) y_2
$$
$$
+ 2aAH^T (2\Lambda_0 + B_0) S - 2V_2H^T (2\Lambda_0 + B_0) y_2 - 2aAH^T Q_1 S + 2mV_2 H^T Q_1 y_2.
$$

$$
f_2(a) = S^T \left\{ \left[ \bar{\alpha} + \eta(\bar{\alpha}) \right] A_0 \right\} S + \bar{\alpha} \left( H^T \Lambda_0 H + T^T \Lambda_1 T \right) + \bar{\beta} H^T (A_0 \Lambda_0 + 3B_0) H + \bar{\beta} T^T (A_1 \Lambda_1 + 3B_1) T
$$
$$
- \frac{3}{2} S^T B_0 H + \frac{3}{2} \bar{\beta} H^T (2\Lambda_0 + B_0) S + 3m\bar{\beta} \left( H^T Q_0 T + T^T Q_1 H \right) - \frac{3}{2} m \bar{\beta} \left( S^T Q_0 T + T^T Q_1 S \right),
$$

$$
f_3(a) = y_2^T \left[ \frac{d \bar{\alpha}(\bar{\alpha})}{da} \right] I + \frac{d \bar{\beta}(\bar{\beta})}{da} \right\} S - 3\bar{\beta} y_2^T B_0 H - 3m\bar{\beta} y_2^T Q_0 T,
$$

$$
f_4(a) = \left\{ \bar{\alpha} + \eta(\bar{\alpha}) \right\} I + \left[ \bar{\beta} + \eta(\bar{\beta}) \right] A_0 \right\} H - mH^T \left\{ \left[ \bar{\alpha} + \eta(\bar{\alpha}) \right] I + \left[ \bar{\beta} + \eta(\bar{\beta}) \right] A_0 \right\} S
$$
$$
- 2m\bar{\alpha} \left( H^T + |T|^2 \right) - 4 \bar{\beta} \left( A_0 \Lambda_0 + 3B_0 \right) H
$$
$$
- S^T \left\{ \left[ \bar{\alpha} + \eta(\bar{\alpha}) \right] C_0 + \left[ \bar{\beta} + \bar{\alpha}(\bar{\beta}) \right] A_0 C_0 \right\} H - 2(\bar{\alpha} - \bar{\beta}) T^T \Lambda_1 M_0 H - 2m \bar{\beta} H^T \left( 1 + 6\Lambda_0 \right) H - 4 \bar{\beta} H^T (A_0 \Lambda_0 + 3B_0) T
$$
$$
- 2m \bar{\beta} T^T (1 + 6\Lambda_0) T + \left[ \bar{\alpha} + \eta(\bar{\alpha}) \right] T^T \Lambda_1 K S + [\bar{\beta} + \bar{\alpha}(\bar{\beta})] T^T (A_0 \Lambda_0 K + 3Q_0 Q_1 - 3Q_1) S.
$$

Here, $\bar{\alpha} \equiv \alpha/\omega_2^A$, $\bar{\beta} \equiv \beta/\omega_2^A$, $\Omega_K \equiv (GM/R^4)^{1/2}$, and $X^T$ means the Hermitian conjugate of the complex matrix $X$. The magnetic perturbations $H^S$, $H^H$, and $H^T$ are given by

$$
H^S = m S,
$$

(C16)
\[ h^H = maA\Lambda_0^{-1}S - mV_G\Lambda_0^{-1}y_2 + mH, \]  
\[ h^T = aA\Lambda_S - V_G y_2 - mT. \]

In the expression for \( E_2' \) given above, all the eigenfunctions, \( S, H, T, \) and \( y_2 \) are for unmagnetized stars even though the subscript \( "0" \) is not attached.

When inertial modes are considered for the \( \omega_A^\omega \)-order eigensolution, for which \( \lim_{\Omega \to 0} \sigma_0 = \kappa_0 \) with \( \kappa_0 \) being a constant, from Equation (C10), we see that

\[ \frac{\sigma_2}{\sigma_0} \to \frac{\eta'_2\omega_A^2}{\Omega^2} \quad \text{as} \quad \Omega \to 0, \]  
where \( \eta'_2 \) is a constant depending on the mode considered, which is given by

\[ \eta'_2 = -\lim_{\Omega \to 0} \frac{1}{4\kappa_0^2\Omega^4} \left[ \int_0^R f_1(a)\rho a^4 da + 2\int_0^R \frac{1}{c_1} f_3(a)\rho a^4 da \right]. \]

Equation (C19) implies that for the inertial mode, our expression for \( \sigma_2 \) becomes inappropriate in the case of \( \Omega^2 \leq \omega_A^2 \). For the inertial mode, therefore, the condition \( \omega_A^2 \ll \Omega^2 \ll 1 \) is required for the expression for \( \sigma_2 \) to be applicable. Similar expressions to Eq. (C19) but for stars with general magnetic field distribution have been obtained by Morsink & Rezania (2002).

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