Uniform stability for a spatially discrete, subdiffusive Fokker–Planck equation

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Abstract
We prove stability estimates for the spatially discrete, Galerkin solution of a fractional Fokker–Planck equation, improving on previous results in several respects. Our main goal is to establish that the stability constants are bounded uniformly in the fractional diffusion exponent \( \alpha \in (0, 1) \). In addition, we account for the presence of an inhomogeneous term and show a stability estimate for the gradient of the Galerkin solution. As a by-product, the proofs of error bounds for a standard finite element approximation are simplified.

Keywords  Fractional calculus · Finite element method · Ritz projector · Discontinuous Galerkin method · Stability analysis

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1 Introduction

We consider the stability of semidiscrete Galerkin methods for the time-fractional Fokker–Planck equation [1, 6]

\[
\partial_t u - \nabla \cdot \left( \partial_t^{1-\alpha} \kappa \nabla u - F \partial_t^{1-\alpha} u \right) = g \\
\text{for } x \in \Omega \text{ and } 0 < t \leq T,
\]

\[
u = u_0(x) \quad \text{for } x \in \Omega \text{ when } t = 0,
\]

\[
u = 0 \quad \text{for } x \in \partial \Omega \text{ and } 0 < t \leq T.
\]

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Here, $\partial_t = \partial/\partial t$, the spatial domain $\Omega$ is bounded and Lipschitz in $\mathbb{R}^d$ ($d \geq 1$), the fractional exponent satisfies $0 < \alpha < 1$ and the fractional time derivative is understood in the Riemann–Liouville sense: $\partial_t^{1-\alpha} = \partial_t \mathcal{I}^\alpha$ where the fractional integration operator $\mathcal{I}^\alpha$ is defined as usual in (9) below. The diffusivity $\kappa \in L^\infty(\Omega)$ is assumed independent of time, positive and bounded below: $\kappa(x) \geq \kappa_{\text{min}} > 0$ for $x \in \Omega$. The forcing vector $F$ may depend both on $x$ and $t$, and we assume that $F$, $\partial_t F$, $\nabla \cdot F$ and $\nabla \cdot \partial_t F$ are bounded on $\Omega \times [0, T]$. Note that if $\alpha \to 1$ then $\partial_t^{1-\alpha} \phi \to \phi$ so the governing equation in (1) reduces to a classical Fokker–Planck equation.

If $F$ is independent of $t$, then by applying $\mathcal{I}^{1-\alpha}$ to both sides of the governing equation we find that (1) is equivalent to

$$C \partial_t^\alpha u - \nabla \cdot (\kappa \nabla u - Fu) = \mathcal{I}^{1-\alpha} g,$$

(2)

where $C \partial_t^\alpha u = \mathcal{I}^{1-\alpha} \partial_t u$ is the Caputo fractional derivative of order $\alpha$. In this form, numerous authors have studied the numerical solution of the problem, mostly for a 1D spatial domain $\Omega = (0, L)$ and with $g \equiv 0$. For instance, Deng [3] considered the method of lines, Jiang and Xu [8] proposed a finite volume method, Yang et al. [22] a spectral collocation method, and Duong and Jin [5] a Wasserstein gradient flow formulation.

For both continuous and discrete solutions to fractional PDEs, it is natural to expect stability constants to remain bounded as $\alpha \to 1$ if the limiting classical problem is stable. In applications, the value of $\alpha$ is typically estimated from measurements, and this process might be treated as an inverse problem. It would then be desirable that simulations of the forward problem are uniformly stable in $\alpha$, particularly if the diffusion turns out to be classical ($\alpha = 1$) or only slightly subdiffusive ($\alpha$ close to 1). Perhaps for these reasons, interest in the question of $\alpha$-uniform stability and convergence seems to be growing. Note that growth in the stability constant as $\alpha \to 0$ is of less concern, since very small values of $\alpha$ have not been observed in real physical systems.

In the special case $F \equiv 0$ of fractional diffusion, Chen and Stynes [2] showed that as $\alpha$ tends to 1 the solution of (2) tends to the solution of the classical diffusion problem, uniformly in $x$ and $t$. They also discussed several examples of numerical schemes for which the error analysis leads to constants that remain bounded as $\alpha \to 1$ (said to be $\alpha$-robust bounds), as well as several for which the constants blow up ($\alpha$-nonrobust bounds). Recent examples in the former category include Jin et al. [9, see Remark 4], Huang et al. [7, see Section 5] and Mustapha [19, Lemma 3.1, Theorem 3.5].

We work with the weak solution $u : (0, T] \to H^1_0(\Omega)$ of (1) characterized by

$$\langle u', v \rangle + \langle \partial_t^{1-\alpha} \kappa \nabla u, \nabla v \rangle - \langle F \partial_t^{1-\alpha} u, \nabla v \rangle = \langle g, v \rangle \quad \text{for } v \in H^1_0(\Omega)$$

(3)

and $0 < t \leq T$, with $u(0) = u_0$, where $u' = \partial_t u$, $\langle u, v \rangle = \int_\Omega uv$ and $\langle u, v \rangle = \int_\Omega u \cdot v$. Strictly speaking, to allow minimal assumptions on the regularity of the data $u_0$ and $g$, we define the solution $u$ by requiring that it satisfy the time-integrated equation

$$\langle u, v \rangle + \langle \mathcal{I}^\alpha \kappa \nabla u, \nabla v \rangle - \langle \mathcal{B}_1 u, \nabla v \rangle = \langle f, v \rangle,$$

(4)
where
\[(\mathcal{B}_1 \phi)(t) = \int_0^t \left( F \partial_t^{1-\alpha} \phi \right)(s) \, ds \quad \text{and} \quad f(t) = u_0 + \int_0^t g(s) \, ds. \quad (5)\]

In previous work, we have established that this problem is well-posed \([12, 16]\).

For a fixed, finite dimensional subspace \(X \subseteq H_0^1(\Omega)\), the semidiscrete Galerkin solution \(u_X : [0, T] \to X\) is given by
\[u_X(t) = v_0 + \int_0^t g(s) \, ds. \quad (6)\]
with \(f_X(t) = u_0 + \int_0^t g(s) \, ds\) and with \(u_X(0) = u_0 \in \mathbb{X}\) a suitable approximation to \(u_0\). Previously, we studied this problem in the particular case when \(X = S_h\) is a space of continuous, piecewise-linear finite element functions corresponding to a conforming triangulation of \(\Omega\) with maximum element size \(h > 0\). We showed that the Galerkin finite element solution \(u_h(t)\) is stable in the norm of \(L_2(\Omega)\) when \(g(t) \equiv 0\), satisfying the bound \([11, \text{Theorem 4.5}]\)
\[\|u_h(t)\| \leq C_\alpha \|u_0\| \quad \text{for} \quad 0 \leq t \leq T \quad \text{and} \quad 0 < \alpha < 1, \quad (7)\]
where \(u_h(0) = u_0 \in S_h\) approximates \(u_0\). The method of proof relied on estimates for fractional integrals \([11, \text{Lemmas 3.2–3.4}]\) involving powers of \((1-\alpha)^{-1}\), leading to a stability constant \(C_\alpha\) that blows up as \(\alpha \to 1\). However, in the limiting case \(\alpha = 1\) the semidiscrete finite element method is easily seen to be stable \([11, \text{Remark 4.7}]\), that is, (7) holds with \(C_1 < \infty\). In the absence of forcing, that is, in the simple case \(F \equiv 0\) of fractional diffusion, the stability constant equals one: \(\|u_h(t)\| \leq \|u_0\|\) for \(0 < \alpha < 1\).

Our primary aim in what follows is to improve the results of our earlier paper \([11]\) via a new analysis of (6). We obtain a stability constant that is uniformly bounded for \(0 < \alpha < 1\), as well as allow for the presence of a non-zero source term \(g\). Recently, Huang et al. \([7]\) have addressed the same question using a different analysis that requires \(1/2 < \alpha < 1\) and \(u_0 \in H^1(\Omega)\), with a stability constant that blows up as \(\alpha \to 1/2\).

Throughout the paper, \(C\) denotes a generic constant that may depend on \(T, \Omega, \kappa\) and \(F\). Dependence on any other parameters will be shown explicitly, and in particular, we write \(C_\alpha\) to show that the constant may also depend on \(\alpha\). After citing some technical lemmas in Section 2, we present the stability proof in Section 3, stating our main result as Theorem 1. Using similar arguments, Section 4 establishes an estimate for the gradient of \(u_X\) in Theorem 2. Combining these results gives, for \(0 \leq t \leq T\) and \(0 < \alpha < 1\),
\[\|u_X(t)\| + t^{\alpha/2} \|\nabla u_X(t)\| \leq C \left( \|u_0X\| + \int_0^t \|g(s)\| \, ds \right) + C \left( \frac{1}{t} \int_0^t \|g(s)\|^2 \, ds \right)^{1/2} \leq C \left( \|u_0X\| + t^{1/2} \int_0^t \|g(s)\|^2 \, ds \right). \quad (8)\]
(Here, the second bound follows from the first by Lemma 9 with \(\eta = 1\).) At the end of Section 4, in Remark 1, we note that the exact solution \(u\) has the same uniform stability property as the semidiscrete Galerkin approximation \(u_X\), that is, (8) holds with
u and u₀ replacing uₓ and u₀ₓ, respectively. Also, in Remark 3, we discuss briefly the implications of enforcing a zero-flux boundary condition instead of the homogeneous Dirichlet one in problem (1). Section 5 applies our new stability analysis to the piecewise linear Galerkin finite element solution uh, showing in Theorem 4 that \( \|u_h - u\| = O((r^{-\alpha(2-r)/2}h^2)) \) and \( \| \nabla u_h - \nabla u\| = O((r^{-\alpha(2-r)/2}h)) \). Unfortunately, these error bounds are not uniform in \( \alpha \) because of the way they depend on regularity estimates (Theorem 3) involving the parameter \( r \in [0, 2] \). Finally, Section 6 considers the time discretization of (1) in the special case \( F \equiv 0 \) of plain fractional diffusion using the discontinuous Galerkin method, proving a fully discrete stability result in Theorem 5.

2 Preliminaries

This brief section introduces notations and gathers together results from the literature that we will use in our subsequent analysis. Denote the fractional integral operator of order \( \mu > 0 \) by

\[
(\mathcal{I}^\mu \phi)(t) = \int_0^t \omega_\mu(t - s)\phi(s) \, ds \quad \text{for } t > 0, \quad \text{where } \omega_\mu(t) = \frac{t^{\mu-1}}{\Gamma(\mu)},
\]

with \( \mathcal{I}^0 \phi = \phi \), and observe that \( (\mathcal{I}^1 \phi)(t) = \int_0^t \phi(s) \, ds \). If we denote the Laplace transform of \( \phi \) by \( \hat{\phi}(z) = \int_0^\infty e^{-zt}\phi(t) \, dt \) then \( (\mathcal{I}^\mu \phi)(z) = \hat{\omega}_\mu(z)\hat{\phi}(z) \) and \( \hat{\omega}_\mu(z) = z^{-\mu} \). Hence, assuming \( \phi \) is real-valued with (say) compact support in \([0, \infty)\), we find that

\[
\int_0^\infty \langle \phi, \mathcal{I}^\mu \phi \rangle \, ds = \frac{\cos(\pi \mu/2)}{\pi} \int_0^\infty y^{-\mu} \|\hat{\phi}(iy)\|^2 \, dy \geq 0 \quad \text{if } 0 < \mu < 1. \tag{10}
\]

Our analysis relies on properties of \( \mathcal{I}^\mu \) stated in the next three lemmas.

**Lemma 1** If \( 0 \leq \mu \leq \nu \leq 1 \), then for \( t > 0 \) and \( \phi \in L^2((0, t), L^2(\Omega)) \),

\[
\int_0^t \| (\mathcal{I}^\nu \phi)(s) \|^2 \, ds \leq 2t^{2(\nu-\mu)} \int_0^t \| (\mathcal{I}^\mu \phi)(s) \|^2 \, ds.
\]

**Proof** See Le et al. [11, Lemma 3.1].

**Lemma 2** If \( 0 < \mu \leq 1 \), then for \( t > 0 \) and \( \phi \in L^2((0, t), L^2(\Omega)) \),

\[
\int_0^t \| (\mathcal{I}^\mu \phi)(s) \|^2 \, ds \leq 2 \int_0^t \omega_\mu(t - s) \int_0^s \| \phi(q), (\mathcal{I}^\mu \phi)(q) \| \, dq.
\]

**Proof** See McLean et al. [16, Lemma 2.2].

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Lemma 3 Let $0 \leq \mu < \nu \leq 1$. If $\phi : [0, T] \to L_2(\Omega)$ is continuous with $\phi(0) = 0$, and if its restriction to $(0, T]$ is differentiable with $\|\phi'(t)\| \leq Ct^{-\mu}$ for $0 < t \leq T$, then

$$\|\phi(t)\|^2 \leq 2\omega_{2-\nu}(t) \int_0^t \langle \phi'(s), (\mathcal{I}^\nu \phi')(s) \rangle ds.$$ 

\[\text{Proof} \quad \text{See McLean et al. [16, Lemma 2.3].} \]

We will also require the following fractional Gronwall inequality involving the Mittag-Leffler function $E_\mu(z) = \sum_{n=0}^{\infty} z^n / \Gamma(1 + n\mu)$.

Lemma 4 Let $\mu > 0$ and assume that $a$ and $b$ are non-negative and non-decreasing functions on the interval $[0, T]$. If $y : [0, T] \to \mathbb{R}$ is an integrable function satisfying

$$0 \leq y(t) \leq a(t) + b(t) \int_0^t \omega_\mu(t-s)y(s) ds \quad \text{for } 0 \leq t \leq T,$$

then

$$y(t) \leq a(t)E_\mu(b(t)t^\mu) \quad \text{for } 0 \leq t \leq T.$$

\[\text{Proof} \quad \text{See Dixon and McKee [4, Theorem 3.1] and Ye, Gao and Ding [23, Corollary 2].} \]

We recall the definition of the linear operator $\mathcal{B}_1$ in (5), and introduce two other linear operators $\mathcal{M}$ and $\mathcal{B}_2$, defined by

$$(\mathcal{M} \phi)(t) = t \phi(t) \quad \text{and} \quad \mathcal{B}_2 \phi = (\mathcal{M} \mathcal{B}_1 \phi)'.$$ 

With the help of the identity

$$\mathcal{M} \mathcal{I}^\alpha \mathcal{I}^\alpha \mathcal{M} = \alpha \mathcal{I}^{\alpha+1} \quad (11)$$

and using our assumption that $F$ and $F'$ are bounded, one can show the following technical estimates.

Lemma 5 Let $\mu > 0$. If $\phi : [0, T] \to L_2(\Omega)$ is continuous, and if its restriction to $(0, T]$ is differentiable with $\|\phi'(t)\| \leq C t^{\alpha-1}$ for $0 < t \leq T$, then

$$\int_0^t \| \mathcal{B}_1 \phi \|^2 ds \leq C \int_0^t \| \mathcal{I}^\alpha \phi \|^2 ds$$

and

$$\int_0^t \| \mathcal{B}_2 \phi \|^2 ds \leq C \int_0^t \left( \| \mathcal{I}^\alpha (\mathcal{M} \phi) \|^2 + \| \mathcal{I}^\alpha \phi \|^2 \right) ds.$$ 

\[\text{Proof} \quad \text{The estimate for } \mathcal{B}_1 \text{ was proved by Le et al. [11, Lemma 4.1], who also proved the bound for } \mathcal{B}_2 \text{ (denoted there by $B_3$) but with the extra term } C \int_0^t \| \mathcal{I}^\alpha \mathcal{M} \phi \|^2 ds. \text{ However, we may omit this extra term because it is bounded by } Ct^2 \int_0^t \| \mathcal{I}^\alpha \phi \|^2 ds, \text{ as one sees from Lemma 1 and (11).} \]
3 Stability analysis

To simplify the error analysis of Section 5, we will include an additional term on the right-hand side of (6) and study the stability of \( u_\alpha : [0, T] \to \mathbb{X} \) satisfying

\[
\langle u_\alpha, \chi \rangle + \langle \kappa \mathcal{J}^\alpha \nabla u_\alpha, \nabla \chi \rangle - \langle \mathcal{B}_1 u_\alpha, \nabla \chi \rangle = \langle f_1, \chi \rangle + \langle \mathcal{B}_2, \nabla \chi \rangle \quad \text{for } \chi \in \mathbb{X},
\]

(12)

with \( u_\alpha(0) = u_{0\alpha} \); the semidiscrete Galerkin solution is then given by the special case \( f_1 = f_\alpha \) and \( f_2 = 0 \). Our goal is to bound \( \|u_\alpha(t)\| \) pointwise in \( t \), and for this purpose our overall strategy is to apply Lemma 3 with \( \phi = \mathcal{M} u_\alpha \), which in turn requires an estimate for \( \int_0^t \langle \mathcal{J}^\alpha (\mathcal{M} u_\alpha)', (\mathcal{M} u_\alpha)' \rangle \, ds \). The technical details are worked out in Lemmas 6 and 7 below, and the stability estimate itself is then obtained in Lemma 8 for (12), and in Theorem 1 for the original semidiscrete Galerkin problem (6).

Our method of proof begins by multiplying both sides of (12) by \( t \) and using the identity (11) to obtain

\[
\langle \mathcal{M} u_\alpha, \chi \rangle + \langle \kappa \mathcal{J}^\alpha (\mathcal{M} \nabla u_\alpha), \nabla \chi \rangle + \alpha \kappa \mathcal{J}^{\alpha + 1} \nabla u_\alpha - \mathcal{B}_1 u_\alpha, \nabla \chi \rangle = \langle \mathcal{M} f_1, \chi \rangle + \langle \mathcal{M} f_2, \nabla \chi \rangle.
\]

Differentiating this equation with respect to time then yields

\[
\langle (\mathcal{M} u_\alpha)', \chi \rangle + \langle \kappa \mathcal{J}^{\alpha - 1} (\mathcal{M} \nabla u_\alpha) + \alpha \kappa \mathcal{J}^\alpha \nabla u_\alpha - \mathcal{B}_2 u_\alpha, \nabla \chi \rangle = \langle (\mathcal{M} f_1)', \chi \rangle + \langle (\mathcal{M} f_2)', \nabla \chi \rangle.
\]

(13)

**Lemma 6** For \( i \in \{0, 1\} \) and \( 0 < t \leq T \), the solution of (12) satisfies

\[
\int_0^t \left( \| \mathcal{J}^\alpha (\mathcal{M} i u_\alpha) \|^2 + t^\alpha \| \mathcal{J}^\alpha (\nabla \mathcal{M} i u_\alpha) \|^2 \right) \, ds \leq C t^{2(\alpha + i)} \int_0^t \left( \| f_1 \|^2 + t^{-\alpha} \| f_2 \|^2 \right) \, ds.
\]

**Proof** Choose \( \chi = (\mathcal{J}^\alpha u_\alpha)(t) \) in (12) so that

\[
\langle u_\alpha, \mathcal{J}^\alpha u_\alpha \rangle + \kappa \min \| \mathcal{J}^\alpha \nabla u_\alpha \|^2 \leq \langle f_2 + \mathcal{B}_1 u_\alpha, \mathcal{J}^\alpha \nabla u_\alpha \rangle + \langle f_1, \mathcal{J}^\alpha u_\alpha \rangle \leq \kappa^{-1}_\min \| f_2 \|^2 + \| \mathcal{B}_1 u_\alpha \|^2 + \frac{1}{2} \kappa \min \| \mathcal{J}^\alpha \nabla u_\alpha \|^2 + \left( t^{\alpha/2} \| f_1 \| (t^{-\alpha/2} \| \mathcal{J}^\alpha u_\alpha \|) \right).
\]

After cancelling \( \frac{1}{2} \kappa \min \| \mathcal{J}^\alpha \nabla u_\alpha \|^2 \), integrating in time and applying Lemma 5, we deduce that

\[
\int_0^t \left( \langle u_\alpha, \mathcal{J}^\alpha u_\alpha \rangle + \frac{1}{2} \kappa \min \| \mathcal{J}^\alpha \nabla u_\alpha \|^2 \right) \, ds \leq C_0 t^{-\alpha} \int_0^t \| \mathcal{J}^\alpha u_\alpha \|^2 \, ds + \int_0^t \left( t^\alpha \| f_1 \|^2 + \kappa^{-1}_\min \| f_2 \|^2 \right) \, ds,
\]

(14)

where \( C_0 \) is a fixed constant depending on \( T, \kappa \), and \( F \). Apply \( \mathcal{J}^\alpha \) to both sides of (12) and choose \( \chi = \mathcal{J}^\alpha u_\alpha(t) \) to obtain, for any \( \eta > 0 \),

\[
\| \mathcal{J}^\alpha u_\alpha \|^2 + \langle \kappa \mathcal{J}^\alpha (\mathcal{J}^\alpha \nabla u_\alpha), \mathcal{J}^\alpha \nabla u_\alpha \rangle = \langle \mathcal{J}^\alpha (f_2 + \mathcal{B}_1 u_\alpha), \mathcal{J}^\alpha \nabla u_\alpha \rangle + \langle \mathcal{J}^\alpha f_1, \mathcal{J}^\alpha u_\alpha \rangle \leq \eta \left( \| \mathcal{J}^\alpha f_2 \|^2 + \| \mathcal{J}^\alpha (\mathcal{B}_1 u_\alpha) \|^2 \right) + \frac{1}{2} \eta^{-1} \| \mathcal{J}^\alpha \nabla u_\alpha \|^2 + \frac{1}{2} \| \mathcal{J}^\alpha f_1 \|^2 + \frac{1}{2} \| \mathcal{J}^\alpha u_\alpha \|^2.
\]
Simplifying, integrating in time, and noting \( \int_0^t (\mathcal{I}^\alpha (\mathcal{I}^\alpha \nabla u_{\mathcal{I}}), \mathcal{I}^\alpha \nabla u_{\mathcal{I}}) \, ds \geq 0 \) by (10), we observe that

\[
\int_0^t \| \mathcal{I}^\alpha u_{\mathcal{I}} \|^2 \, ds \leq \int_0^t \| \mathcal{I}^\alpha f_1 \|^2 \, ds + 2\eta \int_0^t \left( \| \mathcal{I}^\alpha f_2 \|^2 + \| \mathcal{I}^\alpha (\mathcal{B}_1 u_{\mathcal{I}}) \|^2 \right) \, ds + \frac{1}{\eta} \int_0^t \| \mathcal{I}^\alpha \nabla u_{\mathcal{I}} \|^2 \, ds.
\]

By Lemmas 1 and 5,

\[
\int_0^t \| \mathcal{I}^\alpha (\mathcal{B}_1 u_{\mathcal{I}}) \|^2 \, ds \leq 2t^{2\alpha} \int_0^t \| \mathcal{B}_1 u_{\mathcal{I}} \|^2 \, ds \leq Ct^{2\alpha} \int_0^t \| \mathcal{I}^\alpha u_{\mathcal{I}} \|^2 \, ds,
\]

and so, again with the help of Lemma 1,

\[
\int_0^t \| \mathcal{I}^\alpha u_{\mathcal{I}} \|^2 \, ds \leq 2t^{2\alpha} \int_0^t (\| f_1 \|^2 + 2\eta \| f_2 \|^2) \, ds + C\eta t^{2\alpha} \int_0^t \| \mathcal{I}^\alpha u_{\mathcal{I}} \|^2 \, ds + \frac{1}{\eta} \int_0^t \| \mathcal{I}^\alpha \nabla u_{\mathcal{I}} \|^2 \, ds. \tag{15}
\]

However, by (14),

\[
\int_0^t \| \mathcal{I}^\alpha \nabla u_{\mathcal{I}} \|^2 \, ds \leq \frac{2C_0}{\kappa_{\min}} t^{-\alpha} \int_0^t \| \mathcal{I}^\alpha u_{\mathcal{I}} \|^2 \, ds + C \int_0^t (t^\alpha \| f_1 \|^2 + \| f_2 \|^2) \, ds,
\]

and by Lemma 2,

\[
\int_0^t \| \mathcal{I}^\alpha u_{\mathcal{I}} \|^2 \, ds \leq \frac{C}{\kappa_{\min}} t^{-\alpha} \int_0^t \| \mathcal{I}^\alpha u_{\mathcal{I}} \|^2 \, ds + C \int_0^t \omega_\alpha (t-s) \int_s^t \left( u_{\mathcal{I}} (q), (\mathcal{I}^\alpha u_{\mathcal{I}}) (q) \right) dq \, ds. \tag{16}
\]

Inserting these two estimates in the right-hand side of (15) and choosing \( \eta = 4C_0 t^{-\alpha} / \kappa_{\min} \) yields

\[
\int_0^t \| \mathcal{I}^\alpha u_{\mathcal{I}} \|^2 \, ds \leq Ct^\alpha \int_0^t (t^\alpha \| f_1 \|^2 + \| f_2 \|^2) \, ds + C t^\alpha \int_0^t \omega_\alpha (t-s) \int_0^s \left( u_{\mathcal{I}}, \mathcal{I}^\alpha u_{\mathcal{I}} \right) dq \, ds. \tag{17}
\]

Let \( y(t) = \int_0^t \left( \| u_{\mathcal{I}}, \mathcal{I}^\alpha u_{\mathcal{I}} \| + \| \mathcal{I}^\alpha \nabla u_{\mathcal{I}} \|^2 \right) \, ds \), and deduce from (14) and (17) that

\[
y(t) \leq C \int_0^t (t^\alpha \| f_1 \|^2 + \| f_2 \|^2) \, ds + C \int_0^t \omega_\alpha (t-s) y(s) \, ds. \tag{18}
\]

It follows by Lemma 4 that, for \( 0 \leq t \leq T \),

\[
y(t) \leq CE_\alpha (t^\alpha) \int_0^t (t^\alpha \| f_1 \|^2 + \| f_2 \|^2) \, ds \leq C \int_0^t (t^\alpha \| f_1 \|^2 + \| f_2 \|^2) \, ds. \tag{19}
\]

Together, (16) and (19) imply

\[
\int_0^t \left( \| \mathcal{I}^\alpha u_{\mathcal{I}} \|^2 + t^\alpha \| \mathcal{I}^\alpha \nabla u_{\mathcal{I}} \|^2 \right) \, ds \leq Ct^\alpha \int_0^t (t^\alpha \| f_1 \|^2 + \| f_2 \|^2) \, ds + C t^\alpha \int_0^t \omega_\alpha (t-s) y(s) \, ds,
\]
and using (19) a second time gives
\[
\int_0^t \omega(t - s)y(s)\,ds \leq C \left( \int_0^t \omega(t - s)\,ds \right) \int_0^t \left( t^\alpha \| f_1 \|^2 + \| f_2 \|^2 \right)\,ds
\]
\[
\leq Ct^\alpha \int_0^t \left( t^\alpha \| f_1 \|^2 + \| f_2 \|^2 \right)\,ds,
\]
completing the proof for the case \( i = 0 \).

The identity (11) implies that
\[
\| (\mathcal{I}^\alpha \mathcal{M} u \chi)(t) \|^2 \leq 2t^2 \| (\mathcal{I}^\alpha u \chi)(t) \|^2 + 2a^2 \| (\mathcal{I}^{a+1} u \chi)(t) \|^2
\]
so, using Lemma 1,
\[
\int_0^t \mathcal{I}^\alpha \mathcal{M} u \chi \| ds \leq 2t^2 \int_0^t \mathcal{I}^\alpha u \chi \| ds + 4a^2 t^2 \int_0^t \mathcal{I}^\alpha u \chi \| ds \leq 6t^2 \int_0^t \mathcal{I}^\alpha u \chi \| ds.
\]
Similarly,
\[
\int_0^t \mathcal{I}^\alpha \mathcal{M} \nabla u \chi \| ds \leq 6t^2 \int_0^t \mathcal{I}^\alpha \nabla u \chi \| ds,
\]
so the case \( i = 1 \) follows from the already proven case \( i = 0 \).

The next lemma makes use of the identity
\[
(\partial_t^{1-a} \phi)(t) = (\mathcal{I}^\alpha \phi)'(t) = \phi(0)\omega(t) + (\mathcal{I}^\alpha \phi')(t).
\] (20)

Lemma 7 For \( 0 < t \leq T \), the solution of (12) satisfies
\[
\int_0^t \left( \left( \mathcal{M} u \chi \right)' , \mathcal{I}^\alpha (\mathcal{M} u \chi)' + \mathcal{I}^\alpha (\mathcal{M} \nabla u \chi)' \right) \| ds \leq Ct^\alpha \int_0^t \left( \| f_1 \|^2 + \| (\mathcal{M} f_1)' \|^2 \right)\,ds + C \int_0^t \left( \| f_2 \|^2 + \| (\mathcal{M} f_2)' \|^2 \right)\,ds.
\]

Proof Rewriting (13) as
\[
\left\{ (\mathcal{M} u \chi)', \chi \right\} + \left\{ \kappa \partial_t^{1-a} (\mathcal{M} \nabla u \chi), \nabla \chi \right\}
\]
\[
= \left\{ (\mathcal{M} f_2)', \mathcal{B}_2 u \chi - \alpha \kappa \mathcal{I}^\alpha \nabla u \chi, \nabla \chi \right\} + \left\{ (\mathcal{M} f_1)', \chi \right\},
\] (21)
we note that the first term on the right is bounded by
\[
\frac{1}{2} \kappa \min \| \nabla \chi \|^2 + \frac{3}{2} \kappa \min \left( \| (\mathcal{M} f_1)' \|^2 + \| \mathcal{B}_2 u \chi \|^2 + \alpha^2 \| \kappa \mathcal{I}^\alpha \nabla u \chi \|^2 \right).
\]
Choose \( \chi = \partial_t^{1-a} (\mathcal{M} u \chi)(t) = (\mathcal{I}^\alpha (\mathcal{M} u \chi)'(t) \) and note that, since \( (\mathcal{M} u \chi)(0) = 0 \), the identity (20) implies \( \chi = \mathcal{I}^\alpha (\mathcal{M} u \chi)' \) so
\[
\left\{ (\mathcal{M} u \chi)', \mathcal{I}^\alpha (\mathcal{M} u \chi)' \right\} + \frac{1}{2} \kappa \min \| \mathcal{I}^\alpha (\mathcal{M} \nabla u \chi)' \|^2
\]
\[
\leq \| (\mathcal{M} f_1)' \| \| \mathcal{I}^\alpha (\mathcal{M} u \chi)' \| + C \| (\mathcal{M} f_2)' \| \| \mathcal{B}_2 u \chi \|^2 + C \| \mathcal{I}^\alpha \nabla u \chi \|^2.
\]
Thus, by Lemma 5,
\begin{align*}
y(t) & \equiv \int_0^t \left(\left\langle \left(\mathcal{M}u_X\right)', \mathcal{I}^\alpha(\mathcal{M}u_X)'\right\rangle + \|\mathcal{I}^\alpha(\mathcal{M}\nabla u_X)\|^2 \right) ds \\
& \leq C \int_0^t \left(t^\alpha \|\mathcal{M}f_1\|^2 + \|\mathcal{M}f_2\|^2\right) ds \\
& \quad + C \int_0^t \left(\|\mathcal{I}^\alpha(\nabla u_X)\|^2 + \|\mathcal{I}^\alpha u_X\|^2\right) ds + C_0 t^{-\alpha} \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}u_X)'\|^2 ds.
\end{align*}

The second integral on the right is bounded by \(C t^\alpha \int_0^t \left(\|f_1\|^2 + t^{-\alpha} \|f_2\|^2\right) ds\) via Lemma 6, giving
\begin{equation}
y(t) \leq C t^\alpha \int_0^t \left(\|f_1\|^2 + \|\mathcal{M}f_2\|^2\right) ds + \int_0^t \left(\|\mathcal{I}^\alpha u_X\|^2\right) ds + C_0 t^{-\alpha} \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}u_X)'\|^2 ds.
\end{equation}

Now apply \(\mathcal{I}^\alpha\) to both sides of (21), again with \(\chi = \partial_t^{1-\alpha}(\mathcal{M}u_X)(t) = \mathcal{I}^\alpha(\mathcal{M}u_X)'(t)\), to conclude that
\begin{align*}
\|\mathcal{I}^\alpha(\mathcal{M}u_X)\|^2 & + 2\kappa \mathcal{I}^\alpha(\mathcal{I}^\alpha(\mathcal{M}\nabla u_X)'), \mathcal{I}^\alpha(\mathcal{M}\nabla u_X)') \\
& \leq \left(\|\mathcal{I}^\alpha(\mathcal{M}f_2)'\|^2 + \|\mathcal{I}^\alpha(\mathcal{B}2 u_X)\|^2 + \|\mathcal{I}^{2\alpha} \nabla u_X\|^2\right) \mathcal{I}^\alpha(\mathcal{M}\nabla u_X)') \\
& \quad + \frac{1}{2} \|\mathcal{I}^\alpha(\mathcal{M}f_1)'\|^2 + \frac{1}{2} \|\mathcal{I}^\alpha(\mathcal{M}u_X)\|^2.
\end{align*}

After cancelling the last term on the right, we have, for any \(\eta > 0\),
\begin{align*}
\|\mathcal{I}^\alpha(\mathcal{M}u_X)\|^2 & + 2\kappa \mathcal{I}^\alpha(\mathcal{I}^\alpha(\mathcal{M}\nabla u_X)'), \mathcal{I}^\alpha(\mathcal{M}\nabla u_X)') \\
& \leq 3\eta \left(\|\mathcal{I}^\alpha(\mathcal{M}f_2)'\|^2 + \|\mathcal{I}^\alpha(\mathcal{B}2 u_X)\|^2 + \|\mathcal{I}^{2\alpha} \nabla u_X\|^2\right) \\
& \quad + \eta^{-1} \|\mathcal{I}^\alpha(\mathcal{M} \nabla u_X)\|^2 + \|\mathcal{I}^\alpha(\mathcal{M}f_1)'\|^2.
\end{align*}

Since the integral over \((0, t)\) of the second term on the left is non-negative, it follows using Lemma 1 that
\begin{align*}
\int_0^t \|\mathcal{I}^\alpha(\mathcal{M}u_X)'\|^2 ds & \leq 6\eta \int_0^t \left(\|\mathcal{M}f_2\|^2 + \|\mathcal{B}2 u_X\|^2 + \|\mathcal{I}^{\alpha} \nabla u_X\|^2\right) ds \\
& \quad + \eta^{-1} \int_0^t \|\mathcal{I}^\alpha(\mathcal{M} \nabla u_X)\|^2 ds + 2\eta^2 \int_0^t \|\mathcal{M}f_1\|^2 ds.
\end{align*}

By Lemmas 5 and 6,
\begin{align*}
\int_0^t \left(\|\mathcal{B}2 u_X\|^2 + \|\mathcal{I}^{\alpha} (\nabla u_X)\|^2\right) ds & \leq C t^\alpha \int_0^t \left(\|f_1\|^2 + t^{-\alpha} \|f_2\|^2\right) ds \\
& \quad + C \int_0^t \left(\|\mathcal{I}^\alpha(\mathcal{M}u_X)'\|^2 + \|\mathcal{I}^\alpha u_X\|^2\right) ds \\
& \leq C \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}u_X)'\|^2 ds + C \left(\alpha + \eta^2\right) \int_0^t \left(\|f_1\|^2 + t^{-\alpha} \|f_2\|^2\right) ds,
\end{align*}
and consequently,
\[
C_0 t^{-\alpha} \int_0^t \|\varphi(\mathcal{M} u_{\mathbb{X}})\|^2 ds \leq C t^\alpha \int_0^t (\eta t^\alpha \|f_1\|^2 + \|f_2\|^2 + \|\mathcal{M} f_1\|\|^2) ds
+ C \eta t^\alpha \int_0^t \|f_2\|^2 + \|\mathcal{M} f_2\|\|^2 ds + C \int_0^t \|f_2\|^2 + \|\mathcal{M} \nabla u_{\mathbb{X}}\|^2 ds
\]
\[
+ \frac{C_0 t^{-\alpha}}{\eta} \int_0^t \|\varphi(\mathcal{M} \nabla u_{\mathbb{X}})\|^2 ds.
\]
Choosing \(\eta = 2C_0 t^{-\alpha}\), we see from (22) that
\[
y(t) \leq C t^\alpha \int_0^t (\eta t^\alpha \|f_1\|^2 + \|f_2\|^2 + \|\mathcal{M} f_2\|\|^2) ds + C \int_0^t \|f_2\|^2 + \|\mathcal{M} \nabla u_{\mathbb{X}}\|^2 ds.
\]
The desired estimate follows after applying Lemma 2 to bound the last integral on the right in terms of \(y\), and then applying Lemma 4.

**Lemma 8** For \(0 < t \leq T\), the solution of (12) satisfies
\[
\|u_{\mathbb{X}}(t)\|^2 \leq \frac{C}{t} \int_0^t (\|f_1\|^2 + \|\mathcal{M} f_1\|\|^2) ds + \frac{C}{t^{1-\alpha}} \int_0^t (\|f_2\|^2 + \|\mathcal{M} f_2\|\|^2) ds.
\]

**Proof** The function \(\phi = \mathcal{M} u_{\mathbb{X}}\) satisfies \(\|\phi(t)\| \leq C_\alpha t\) and \(\|\phi'(t)\| \leq C_\alpha t^\alpha\) \([16,\text{Theorems 3.1 and 3.2}]\). We may thus apply Lemma 3, followed by Lemma 7, to conclude that
\[
t^2 \|u_{\mathbb{X}}(t)\|^2 = \|\mathcal{M} u_{\mathbb{X}}(t)\|^2 \leq \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^t \|\mathcal{M} u_{\mathbb{X}}\| \cdot (\mathcal{M} u_{\mathbb{X}}) ds
\]
\[
\leq C t \int_0^t (\|f_1\|^2 + \|\mathcal{M} f_1\|\|^2) ds + C t^{1-\alpha} \int_0^t (\|f_2\|^2 + \|\mathcal{M} f_2\|\|^2) ds,
\]
and then divide by \(t^2\).

**Theorem 1** The semidiscrete Galerkin solution, defined by (6), satisfies
\[
\|u_{\mathbb{X}}(t)\| \leq C \left( \|u_{0\mathbb{X}}\| + \int_0^t \|g(s)\| ds \right)^{1/2}
+ \left( \frac{1}{t} \int_0^t \|s g(s)\|^2 ds \right)^{1/2}
\]
for \(0 < t \leq T\), where the stability constant \(C\) depends on \(T\), \(\Omega\), \(\kappa\) and \(F\), but not on \(\alpha\) or the subspace \(\mathbb{X}\).

**Proof** Apply Lemma 8 with \(f_1 = f_{\mathbb{X}}\) and \(f_2 = 0\), noting that
\[
\frac{1}{t} \int_0^t \|f_{\mathbb{X}}\|^2 ds \leq \max_{0 \leq s \leq t} \|f_{\mathbb{X}}(s)\|^2 \leq \left( \|u_{0\mathbb{X}}\| + \int_0^t \|g(s)\| ds \right)^2
\]
and \((\mathcal{M} f_{\mathbb{X}})' = f_{\mathbb{X}} + \mathcal{M} g\).
The terms in $g$ from the above estimate can be bounded as follows. In particular, by choosing $\eta = 1$ we see that $\|u_X(t)\| \leq C\|u_0X\| + Ct^{1/2}\|g\|_{L^2(0,T);L^2(\Omega)}$.

**Lemma 9** For $0 < t \leq T$ and $0 < \eta \leq 1$,
\[
\left(\int_0^t \|g(s)\| \, ds\right)^2 + \frac{1}{t} \int_0^t \|sg(s)\|^2 \, ds \leq (1 + \eta^{-1})t^{\eta} \int_0^t s^{1-\eta}\|g(s)\|^2 \, ds.
\]

**Proof** Using the Cauchy–Schwarz inequality,
\[
\left(\int_0^t \|g(s)\| \, ds\right)^2 = \left(\int_0^t s^{-(1-\eta)/2}s^{(1-\eta)/2}\|g(s)\| \, ds\right)^2 \leq \int_0^t s^{\eta-1}ds \int_0^t s^{1-\eta}\|g(s)\|^2 \, ds = \frac{t^{\eta}}{\eta} \int_0^t s^{1-\eta}\|g(s)\|^2 \, ds,
\]
and furthermore,
\[
\frac{1}{t} \int_0^t \|sg(s)\|^2 \, ds \leq \int_0^t s\|g(s)\|^2 \, ds \leq t^{\eta} \int_0^t s^{1-\eta}\|g(s)\|^2 \, ds.
\]

\[\blacksquare\]

### 4 Gradient bounds

A similar strategy to the one used in Section 3 will allow us to bound $\|\nabla u_X(t)\|$ pointwise in $t$: we once again apply Lemma 3, this time with $\phi = \mathfrak{M}\nabla u_X$. The key result is stated as Lemma 12 for the generalized problem (12), and as Theorem 2 for the semidiscrete Galerkin equation (6). The proofs rely on the following estimates; cf. Lemma 5.

**Lemma 10** Let $\mu > 0$. If $\phi : [0, T] \to H^1(\Omega)$ is continuous, and if its restriction to $(0, T)$ is differentiable with $\|\phi'(t)\|_{H^1(\Omega)} \leq C t^{\mu-1}$ for $0 < t \leq T$, then
\[
\int_0^t \|\nabla \cdot (\mathcal{B}_1\phi)\|^2 \, ds \leq C \int_0^t \left(\|\mathcal{I}\alpha\phi\|^2 + \|\mathcal{I}\alpha\nabla\phi\|^2\right) \, ds
\]
and
\[
\int_0^t \|\nabla \cdot (\mathcal{B}_2\phi)\|^2 \, ds \leq C \int_0^t \left(\|\mathcal{I}\alpha\phi\|^2 + \|\mathcal{I}\alpha(\mathcal{M}\phi)'\|^2 + \|\mathcal{I}\alpha\nabla\phi\|^2 + \|\mathcal{I}\alpha(\mathcal{M}\nabla\phi)'\|^2\right) \, ds.
\]

**Proof** Integration by parts gives $(\mathcal{B}_1\phi)(t) = \mathcal{F}(\mathcal{I}\alpha\phi)(t) - \int_0^t \mathcal{F}'(\mathcal{I}\alpha\phi$)
\[
\nabla \cdot (\mathcal{B}_1\phi) = (\nabla \cdot \mathcal{F})(\mathcal{I}\alpha\phi) + \mathcal{F}'(\mathcal{I}\alpha\phi) - \int_0^t \left((\nabla \cdot \mathcal{F}'(\mathcal{I}\alpha\phi) + \mathcal{F}'(\mathcal{I}\alpha\nabla\phi)) \, ds,
\]
implicating the first estimate. Furthermore,
\[
\mathcal{B}_2\phi = \mathcal{F}'(\mathcal{I}\alpha\mathcal{M}\phi + \alpha\mathcal{I}\alpha+1\phi) + \mathcal{F}(\mathcal{I}\alpha\mathcal{M}\phi)' + \alpha\mathcal{I}\alpha\phi) - \mathcal{I}\alpha(\mathcal{F}'\mathcal{I}\alpha\phi) - \mathcal{M}\mathcal{F}'\mathcal{I}\alpha\phi,
\]
which implies the second estimate.
\[\blacksquare\]
The next result builds on the estimates of Lemmas 6 and 7.

**Lemma 11** The solution of (12) satisfies, for $0 < t \leq T$,
\[
\int_0^t \left( \| \mathcal{M} u_X \| \| \kappa \mathcal{A} \mathcal{M} \nabla u_X \|, (\mathcal{M} \nabla u_X) \right) ds \leq C \int_0^t \| u_X \|^2 ds \\
+ C \int_0^t \left( \| f_1 \|^2 + \| \mathcal{M} f_1 \|^2 + \| \mathcal{M} f_2 \|^2 + \| \mathcal{M} \cdot f_2 \|^2 + \| \mathcal{A} \mathcal{M} \cdot f_2 \|^2 \right) ds.
\]

**Proof** Using the first Green identity, we deduce from (21) that
\[
(\mathcal{M} u_X, \chi) + \kappa (\mathcal{A} \mathcal{M} \nabla u_X, \nabla \chi) = (\mathcal{M} f_1 - (\mathcal{M} \nabla \cdot f_2) - \nabla \cdot (\mathcal{B}_2 u_X), \chi)
- \alpha (\kappa \mathcal{A} \mathcal{M} \nabla u_X, \nabla \chi),
\]
and from (12) that
\[
(\mathcal{M} u_X, \chi) + \kappa (\mathcal{A} \mathcal{M} \nabla u_X, \nabla \chi) = (f_3 + \alpha \nabla \cdot (\mathcal{B}_1 u_X) - \nabla \cdot (\mathcal{B}_2 u_X) + \alpha u_X, \chi)
\leq \frac{1}{2} \| \chi \|^2 + 2 \| f_3 \|^2 + \| \nabla \cdot (\mathcal{B}_1 u_X) \|^2 + \| \nabla \cdot (\mathcal{B}_2 u_X) \|^2 + \| u_X \|^2,
\]
where $f_3 = (\mathcal{M} f_1') - \alpha f_1 - (\mathcal{M} \nabla \cdot f_2)' + \alpha \nabla \cdot f_2$. Choose $\chi = (\mathcal{M} u_X)'$, cancel the term $\frac{1}{2} \| \chi \|^2$, and integrate in time to obtain
\[
\int_0^t \left( \| \mathcal{M} u_X \|^2 + \kappa \mathcal{A} \mathcal{M} \nabla u_X, (\mathcal{M} \nabla u_X) \right) ds \leq C J(t) + C \int_0^t \| f_3 \|^2 ds
\leq C J(t) + C \int_0^t \| f_3 \|^2 ds
\leq C J(t) + C \int_0^t \| f_3 \|^2 ds
\]
where, by Lemma 10,
\[
J(t) = \int_0^t \left( \| \mathcal{M} u_X \|^2 + \| \mathcal{A} \mathcal{M} \nabla u_X \|^2 + \| \mathcal{A} \mathcal{M} \nabla u_X \|^2 + \| \mathcal{A} \mathcal{M} \nabla u_X \|^2 + \| u_X \|^2 \right) ds.
\]
If we let $y(t) = \int_0^t (\mathcal{M} u_X)'$, $\mathcal{A} \mathcal{M} (\mathcal{M} u_X)' ds$ then, by Lemma 2,
\[
\int_0^t \| \mathcal{A} \mathcal{M} (\mathcal{M} u_X)' \|^2 ds \leq 2 \int_0^t \omega \alpha (t - s) y(s) ds \leq 2 \omega \alpha_1 (t) \max_{0 \leq s \leq t} y(s),
\]
and so, using Lemmas 6 and 7,
\[
J(t) \leq C t \alpha \int_0^t \left( \| f_1 \|^2 + \| (\mathcal{M} f_1)' \|^2 \right) ds + C \int_0^t \left( \| f_2 \|^2 + \| (\mathcal{M} f_2)' \|^2 \right) ds.
\]
Since
\[
\int_0^t \| f_3 \|^2 ds \leq 4 \int_0^t \left( \| f_1 \|^2 + \| (\mathcal{M} f_1)' \|^2 + \| \nabla \cdot f_2 \|^2 + \| (\mathcal{M} \nabla \cdot f_2)' \|^2 \right) ds,
\]
the desired estimate now follows from (23). \qed
Lemma 12 For $0 < t \leq T$,
\[
t^{\alpha/2} \|\nabla u_X(t)\|^2 \leq \frac{C}{t} \int_0^t \left( \|f_1\|^2 + \|\mathcal{M} f_1\|^2 \right) ds \\
+ \frac{C}{t} \int_0^t \left( \|f_2\|^2 + \|\mathcal{M} f_2\|^2 + \|\nabla \cdot f_2\|^2 + \|\mathcal{M} \nabla \cdot f_2\|^2 \right) ds.
\]

Proof The function $\phi(t) = \mathcal{M} \nabla u_X(t)$ satisfies $\|\phi(t)\| \leq C\alpha t^{1-\alpha/2}$ and $\|\phi'(t)\| \leq C\alpha t^{-\alpha/2}$ [17, Theorem 1 and Corollary 10]. Since $t^\alpha \|\nabla u_X(t)\|^2 = t^{\alpha-2} \|\phi(t)\|^2$, Lemmas 3 and 11 imply that
\[
t^{\alpha/2} \|\nabla u_X(t)\|^2 \leq 2\omega^2 - t^{2 - \alpha} \int_0^t \left( \|\mathcal{M} \nabla u_X\|^2 \right) ds \\
+ C \int_0^t \left( \|f_1\|^2 + \|\mathcal{M} f_1\|^2 + \|f_2\|^2 + \|\mathcal{M} f_2\|^2 + \|\nabla \cdot f_2\|^2 + \|\mathcal{M} \nabla \cdot f_2\|^2 \right) ds,
\]
and it suffices to estimate $\int_0^t \|u_X\|^2 ds$. Choose $\chi = u_X(t)$ in (12) and use the first Green identity to deduce that
\[
\|u_X\|^2 + \kappa \mathcal{S}^{\alpha} u_X, \mathcal{S}^{\alpha} u_X = \left( f_1 - \nabla \cdot f_2 - \nabla \cdot \mathcal{B}_1 u_X, u_X \right) \\
\leq \frac{1}{2} \|u_X\|^2 + \frac{3}{2} (\|f_1\|^2 + \|\nabla \cdot f_2\|^2 + \|\nabla \cdot \mathcal{B}_1 u_X\|^2).
\]
After cancelling $\frac{1}{2} \|u_X\|^2$, integrating in time and using (10), we have
\[
\int_0^t \|u_X\|^2 ds \leq 3 \int_0^t (\|f_1\|^2 + \|\nabla \cdot f_2\|^2) ds + 3 \int_0^t \|\nabla \cdot (\mathcal{B}_1 u_X)\|^2 ds,
\]
and by Lemmas 10 and 6,
\[
\int_0^t \|\nabla \cdot (\mathcal{B}_1 u_X)\|^2 ds \leq C \int_0^t \left( \|\mathcal{S}^{\alpha} u_X\|^2 + \|\mathcal{S}^{\alpha} \nabla u_X\|^2 \right) ds \\
\leq Ct^{\alpha} \int_0^t (\|f_1\|^2 + t^{-\alpha} \|f_2\|^2) ds,
\]
which completes the proof.

The main result for this section now follows easily; once again, the terms in $g$ may be estimated using Lemma 9.

Theorem 2 The semidiscrete Galerkin solution, defined by (6), satisfies
\[
t^{\alpha/2} \|\nabla u_X(t)\| \leq C \left( \|u_{0X}\| + \int_0^t \|g(s)\| ds \right) + C \left( \frac{1}{t} \int_0^t \|sg(s)\|^2 ds \right)^{1/2}
\]
for $0 < t \leq T$, where the constant $C$ depends on $T$, $\Omega$, $\kappa$ and $F$, but not on $\alpha \in (0, 1]$ or the subspace $X$.

Proof Choose $f_1 = f_X$ and $f_2 = 0$ in Lemma 12, and estimate $f_X$ in terms of $u_{0X}$ and $g$ using the same steps as in the proof of Theorem 1.
Remark 1 The uniform stability estimate (8) for the semidiscrete Galerkin solution carries over to the weak solution $u$ of the continuous problem (1), that is,

$$
\|u(t)\| + t^{\alpha/2} \|\nabla u(t)\| \leq C \left( \|u_0\| + \int_0^t \|g(s)\| \, ds \right) + C \left( \frac{1}{t} \int_0^t \|s g(s)\|^2 \, ds \right)^{1/2}.
$$

Essentially, it suffices to repeat the steps in an earlier stability proof [16, Theorem 4.1] using (8) as a drop-in replacement for an estimate [16, Theorem 3.3] in which the stability constant was dependent on $\alpha$.

Remark 2 The exponent $\alpha/2$ from the term $t^{\alpha/2} \|\nabla u(t)\|$ in (24) is sharp. To see why, consider the simple case $F \equiv 0$ of no forcing with $g \equiv 0$, and suppose for a contradiction that there is an exponent $\mu < \alpha / 2$ such that

$$
t^{\alpha/2} \|\nabla u(t)\| \leq C \|u_0\| \quad \text{for } 0 < t \leq T \text{ and all } u_0 \in L^2(\Omega).
$$

Let $(\phi_n, \lambda_n)$ be the $n$th Dirichlet eigenpair of $-\kappa \nabla^2$ on $\Omega$ with the normalisation $\|\phi_n\| = 1$. Since $\langle \nabla \phi_n, \nabla v \rangle = \lambda_n \langle \phi_n, v \rangle$ for all $v \in H^1(\Omega)$, it follows that $\|\nabla \phi_n\| = \lambda_n^{1/2}$. If we choose $u_0 = \phi_n$, then $u(t) = E_\alpha(\lambda_n t^{\alpha}) \phi_n$ [13, Section 2] so

$$
t^{\alpha/2} \|\nabla u(t)\| = t^{\alpha/2} E_\alpha(-\lambda_n t^{\alpha}) \|\nabla \phi_n\| = (\lambda_n t^{\alpha})^{1/2} E_\alpha(-\lambda_n t^{\alpha}),
$$

where we used the fact that $E_\alpha(-\xi) > 0$ for all $\xi > 0$. Now choose $t = \lambda_n^{-1/\alpha}$ to deduce that

$$
E_\alpha(-1) = t^{\alpha/2} \|\nabla u(t)\| \leq C t^{\alpha/2-\mu} = C \lambda_n^{-\nu} \quad \text{for } \nu = \frac{1}{\alpha} \left( \frac{\alpha}{2} - \mu \right) > 0.
$$

Since $\lambda_n \to \infty$ as $n \to \infty$, we arrive at the contradiction $E_\alpha(-1) = 0$.

Remark 3 By introducing a flux vector $Q u = -\partial_t^{1-\alpha} \kappa \nabla u + F \partial_t^{1-\alpha} u$ we can write the fractional Fokker–Planck (1) as a conservation law: $\partial_t u + \nabla \cdot Q u = g$. It is then natural to consider a zero-flux boundary condition,

$$
n \cdot Q u = 0 \quad \text{for } \mathbf{x} \in \partial \Omega \text{ and } 0 < t \leq T,
$$

where $n$ denotes the outward unit normal to $\Omega$. (Notice that this boundary condition is non-local in time.) In this case, the weak solution $u : (0, T] \to H^1(\Omega)$ is again characterized by (3), and hence satisfies (4), but with the test functions $v$ now taken from the larger space $H^1(\Omega)$. We can then choose a finite dimensional subspace $X \subseteq H^1(\Omega)$ and again define the Galerkin solution $u_X : [0, T] \to X$ by (6). The analysis of Section 3 goes through with no change, and in particular $u_X$ is again stable in $L^2(\Omega)$. However, the first step in the proof of Lemma 11 fails because boundary terms are introduced if one integrates by parts in space, so our analysis no longer yields a bound for $t^{\alpha/2} \|\nabla u_X(t)\|$.

5 Error estimates

We now decompose the error in the semidiscrete Galerkin solution as

$$u_X - u = \theta_X - \rho_X \quad \text{where} \quad \theta_X = u_X - R_X u \quad \text{and} \quad \rho_X = u - R_X u,$$
and where $R_X$ denotes the Ritz projector for the (stationary) elliptic problem

$$-\nabla \cdot (\kappa \nabla v) + v = g \quad \text{in} \; \Omega, \quad \text{with} \; v = 0 \; \text{on} \; \partial \Omega. \quad (27)$$

Thus, $R_X : H^1_0(\Omega) \to \mathbb{X}$ satisfies

$$(\kappa \nabla R_X v, \nabla \chi) + \langle R_X v, \chi \rangle = \langle \kappa \nabla v, \nabla \chi \rangle + \langle v, \chi \rangle \quad \text{for} \; v \in H^1_0(\Omega) \; \text{and} \; \chi \in \mathbb{X}, \quad (28)$$

or in other words, $R_X : v \mapsto v_X$ where $v_X \in \mathbb{X}$ is the Galerkin solution of the elliptic problem (27). Note that, by including the lower-order term $v$, the Ritz projector $R_X : H^1(\Omega) \to \mathbb{X}$ would also be well-defined for the zero-flux boundary condition (26).

It follows from (4), (6) and (28) that $\theta_X : [0, T] \to \mathbb{X}$ satisfies

$$\langle \theta_X(t), \chi \rangle + \langle \mathcal{J}^\alpha (\kappa \nabla \theta_X) - \mathcal{B} \theta_X, \nabla \chi \rangle = \langle f_1, \chi \rangle + \langle f_2, \nabla \chi \rangle \quad \text{for} \; \chi \in \mathbb{X}, \quad (29)$$

where

$$f_1 = (u_{0X} - P_X u_0) + (\rho_X - \mathcal{J}^\alpha \rho_X), \quad f_2 = -\mathcal{B} \rho_X, \quad (30)$$

and $P_X : L^2(\Omega) \to \mathbb{X}$ is the orthoprojector given by $\langle P_X v, \chi \rangle = \langle v, \chi \rangle$ for $v \in L^2(\Omega)$ and $\chi \in \mathbb{X}$. If $u_0 \in H^1_0(\Omega)$ so that $R_X u_0$ exists, then

$$\tilde{f}_1 = (u_{0X} - R_X u_0) + (\rho_X - \rho_X(0)) - \mathcal{J}^\alpha \rho_X. \quad (31)$$

We estimate $\theta_X$ in terms of $\rho_X$ and the error in the discrete initial data $u_{0X}$, as follows; cf. Thomée [21, Lemma 3.3] for the limiting case $\alpha \to 1$ of a parabolic PDE.

**Lemma 13** For $0 < t \leq T$,

$$\|\theta_X(t)\|^2 \leq C \|u_{0X} - P_X u_0\|^2 + \frac{C}{t} \int_0^t \left( \|\rho_X\|^2 + s^2 \|\rho_X'\|^2 \right) ds.$$

**Proof** Noting that (29) has the same form as (12), with $\theta_X$ playing the role of $u_X$, we may apply Lemma 8 and conclude that

$$\|\theta_X(t)\|^2 \leq \frac{C}{t} \int_0^t \left( \|f_1\|^2 + s^2 \|f_1'\|^2 \right) ds + \frac{C}{t^{1+\alpha}} \int_0^t \left( \|f_2\|^2 + s^2 \|f_2'\|^2 \right) ds.$$

Since $f_1' = \rho_X' - \partial_1^{1-\alpha} \rho_X$, we find with the help of Lemma 1 that

$$\frac{C}{t} \int_0^t \left( \|f_1\|^2 + s^2 \|f_1'\|^2 \right) ds \leq C \|u_{0X} - P_X u_0\|^2$$

$$+ \frac{C}{t} \int_0^t \left( \|\rho_X\|^2 + s^2 \|\rho_X'\|^2 + s^2 \|\partial_1^{1-\alpha} \rho_X\|^2 \right) ds.$$

Using the identity (11) and noting that $(\mathcal{M} \rho_X)(0) = 0$,

$$s \partial_1^{1-\alpha} \rho_X = s \partial_1 \mathcal{J}^\alpha \rho_X = \partial_1 (\mathcal{M} \mathcal{J}^\alpha \rho_X) - \mathcal{J}^\alpha \rho_X = \partial_1 \left( \mathcal{J}^\alpha \mathcal{M} \rho_X + \alpha \mathcal{J}^{\alpha+1} \rho_X - \mathcal{J}^\alpha \rho_X \right)$$

$$= \mathcal{J}^\alpha (\mathcal{M} \partial_1 \rho_X)' + (\alpha - 1) \mathcal{J}^\alpha \rho_X = \mathcal{J}^\alpha \left( \mathcal{M} \rho_X + \alpha \rho_X \right),$$
so by Lemma 1,
\[
\int_0^t s^2 \| \partial_s^{1-\alpha} \rho_X \|^2 \, ds \leq 2t^{2\alpha} \int_0^t \| s \rho_X' + \alpha \rho_X \|^2 \, ds \leq 4t^{2\alpha} \int_0^t (\| \rho_X \|^2 + s^2 \| \rho_X' \|^2) \, ds.
\] (32)
and hence
\[
\frac{C}{t} \int_0^t (\| f_1 \|^2 + s^2 \| f'_1 \|^2) \, ds \leq C \| u_{0X} - P_X u_0 \|^2 + \frac{C}{t} \int_0^t (\| \rho_X \|^2 + s^2 \| \rho_X' \|^2) \, ds. \] (33)

Recalling (5), we have \( f_2(t) = - (F \partial_t^{1-\alpha} \rho_X)(t) \) and therefore by Lemma 5,
\[
\frac{C}{t^{1+\alpha}} \int_0^t (\| f_2 \|^2 + s^2 \| f'_2 \|^2) \, ds \leq \frac{C}{t} \int_0^t (\| \mathcal{G} \rho_X \|^2) \, ds + \frac{C}{t^{1+\alpha}} \int_0^t s^2 \| \partial_t^{1-\alpha} \rho_X \|^2 \, ds,
\]
which is bounded by the second term on the right-hand side of (33), as one sees by once again applying (32) and Lemma 1.

Two similar bounds hold for \( \nabla \theta_X \), but with additional terms involving \( \nabla \rho_X \) and \( \nabla \rho_X' \).

**Lemma 14** For \( 0 < t \leq T \),
\[
t^{\alpha} \| \nabla \theta_X(t) \|^2 \leq C \| u_{0X} - P_X u_0 \|^2 + \frac{C}{t} \int_0^t (\| \rho_X \|^2 + s^2 \| \rho_X' \|^2) \, ds
\]
\[
+ Ct^{2\alpha-1} \int_0^t (\| \nabla \rho_X \|^2 + s^2 \| \nabla \rho_X' \|^2) \, ds.
\]

If \( u_0 \in H^1_0(\Omega) \), then we also have the alternative bound
\[
t^{\alpha} \| \nabla \theta_X(t) \|^2 \leq C \| u_{0X} - R_X u_0 \|^2 + \frac{C}{t} \int_0^t (\| \rho_X - \rho_X(0) \|^2 + s^2 \| \rho_X' \|^2) \, ds
\]
\[
+ Ct^{2\alpha-1} \int_0^t (\| \rho_X \|^2 + \| \nabla \rho_X \|^2 + s^2 \| \nabla \rho_X' \|^2) \, ds.
\]

**Proof** With \( f_1 \) and \( f_2 \) given by (30), we apply Lemma 12 to (29) and bound \( t^{\alpha} \| \nabla \theta_X(t) \|^2 \) by
\[
\frac{C}{t} \int_0^t \left( \| f_1 \|^2 + \| (\mathcal{M} f_1)' \|^2 + \| f_2 \|^2 + \| (\mathcal{M} f_2)' \|^2 + \| \nabla \cdot f_2 \|^2 + \| (\mathcal{M} \nabla \cdot f_2)' \|^2 \right) \, ds.
\]
The terms in \( f_1 \) can be bounded as in (33), and since \( f_2 = - (B_1 \rho_X) \) and \( (\mathcal{M} f_2)' = - B_2 \rho_X \) we see from Lemma 5 followed by Lemma 10 and then Lemma 1 that
\[
\int_0^t \left( \| f_2 \|^2 + \| (\mathcal{M} f_2)' \|^2 + \| \nabla \cdot f_2 \|^2 + \| (\mathcal{M} \nabla \cdot f_2)' \|^2 \right) \, ds
\]
\[
\leq C \int_0^t \left( \| \mathcal{G} \rho_X \|^2 + \| \mathcal{G} (\mathcal{M} \rho_X)' \|^2 + \| \mathcal{G} \nabla \rho_X \|^2 + \| \mathcal{G} (\mathcal{M} \nabla \rho_X)' \|^2 \right) \, ds
\]
\[
\leq Ct^{2\alpha} \int_0^t \left( \| \rho_X \|^2 + s^2 \| \rho_X' \|^2 + \| \nabla \rho_X \|^2 + s^2 \| \nabla \rho_X' \|^2 \right) \, ds.
\]
which completes the proof of the first bound. If $u_0 \in H^1_0(\Omega)$, then we can replace $f_1$ with $\tilde{f}_1$ from (31), and since $\tilde{f}_1' = f_1'$ the second bound follows easily via the arguments leading to (33) (with $R_X$ replacing $P_X$).

To obtain more explicit error bounds we will use the regularity properties stated in the next theorem. The seminorm $|\cdot|_r$ and norm $\|\cdot\|_r$ in the (fractional-order) Sobolev space $\dot{H}^r(\Omega)$ is defined in the usual way [21] via the Dirichlet eigenfunctions of the Laplacian on $\Omega$, and this spatial domain is assumed convex to ensure $H^2$-regularity for the elliptic problem. The proof relies on results [17, Lemma 2,Theorems 11–13] involving constants that blow up as $\alpha \to 1$. Nevertheless, the estimates (34)–(36) hold in the limiting case $\alpha = 1$, when the problem reduces to the classical Fokker–Planck PDE; see Thomée [21, Lemmas 3.2 and 4.4] for a proof if $M = 0$.

**Theorem 3** Assume that $\Omega$ is convex, $0 < \alpha < 1$, $0 \leq r \leq 2$ and $\eta > 0$. If $u_0 \in \dot{H}^r(\Omega)$ and if $g : (0, T] \to L^2(\Omega)$ is continuously differentiable with $\|g(t)\| + t\|g'(t)\| \leq M t^\eta - 1$, then the weak solution of (1) satisfies, for $0 < t \leq T$,

$$
\|u(t)\|_1 \leq C_{\alpha, \eta}(\|u_0\|_r t^{-\alpha(1-r)/2} + M t^{\eta - \alpha/2}) \quad \text{if } r \leq 1,
$$

and

$$
t^{-\alpha/2}\|u(t) - u_0\|_1 \leq C_{\alpha, \eta}(\|u_0\|_r t^{-\alpha(2-r)/2} + M t^{\eta - \alpha}) \quad \text{if } r \geq 1,
$$

and

$$
t^{1-\alpha/2}\|u'(t)\|_1 + \|u(t)\|_2 + t\|u'(t)\|_2 \leq C_{\alpha, \eta}(\|u_0\|_r t^{-\alpha(2-r)/2} + M t^{\eta - \alpha}).
$$

**Proof** We showed [17, Theorem 11] that

$$
\|u(t)\|_1 \leq C_{\alpha, \eta}(\|u_0\| t^{-\alpha/2} + M t^{\eta - \alpha/2})
$$

and [17, Theorem 12] that

$$
\|u(t) - u_0\|_1 + t\|u'(t)\|_1 \leq C_{\alpha, \eta}(\|u_0\|_1 + M t^{\eta - \alpha/2}).
$$

Hence, $\|u(t)\|_1 \leq C(\|u_0\|_1 + M t^{\eta - \alpha/2})$ and (34) follows by interpolation. The estimates (35) and (36) were proved already [17, Theorems 12 and 13].

Now, consider the concrete example in which $X = S_h$ is the usual continuous piecewise-linear finite element space for a triangulation of $\Omega \subseteq \mathbb{R}^d$ with maximum element diameter $h$, and use a subscript $h$ instead of $X$, writing $u_h$, $\theta_h$, $\rho_h$ etc. The error in the Ritz projection satisfies

$$
\|\rho_h(t)\| + h\|\nabla \rho_h(t)\| \leq C h^r |u(t)|_r \quad \text{for } r \in \{1, 2\},
$$

allowing us to prove the following error bounds for $u_h$ and $\nabla u_h$. Notice that if $0 < \alpha < 1/2$, then the restriction $\alpha(2 - r) < 1$ is satisfied for all $r \in [0, 2]$, but if $1/2 \leq \alpha < 1$ (and hence $0 \leq 2 - \alpha^{-1} < 1$) then we are limited to $r \in (2 - \alpha^{-1}, 2]$. 

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Theorem 4 Let \( 0 \leq r \leq 2 \) with \( \alpha(2 - r) < 1 \), and assume that the assumptions of Theorem 3 are satisfied with \( \eta \geq \alpha r / 2 \). Then, for \( 0 < t \leq T \), the semidiscrete finite element solution \( u_h \) satisfies the error bound

\[
\| u_h(t) - u(t) \| \leq C\| u_0 - P_h u_0 \| + C_{\alpha} h^{2 - \alpha(2-r)/2} \frac{t}{\sqrt{1 - \alpha(2 - r)}} \| u_0 \|_r + M,
\]

and the gradient of \( u_h \) satisfies

\[
\| \nabla u_h(t) - \nabla u(t) \| \leq C t^{-\alpha/2} \| u_0 - Q_{r,h} u_0 \| + C_{\alpha} h^{2 - \alpha(2-r)/2} \frac{t}{\sqrt{1 - \alpha(2 - r)}} \| u_0 \|_r + M,
\]

where \( Q_{r,h} \) is either \( P_h \) if \( r \leq 1 \), or else \( R_h \) if \( r \geq 1 \).

Proof For brevity, let \( K_r = (\| u_0 \|_r + M)^2 \). Using (37) followed by (36), we have

\[
\| \rho_h(t) \|^2 + s^2 \| \rho_h'(t) \|^2 \leq C h^4 (\| u(s) \|_1^2 + s^2 \| u'(s) \|_1^2) \leq C_{\alpha} K_r h^4 s^{-\alpha(2-r)},
\]

so, because of the assumption \( \alpha(2 - r) < 1 \),

\[
\frac{1}{t} \int_0^t (\| \rho_h(s) \|^2 + s^2 \| \rho_h'(s) \|^2) \, ds \leq C_{\alpha} K_r h^4 \frac{t^{-\alpha(2-r)}}{1 - \alpha(2-r)}.
\]

Since \( \| u_h - u \| \leq \| \theta_h \| + \| \rho_h \| \) and \( \| \rho_h(t) \|^2 \leq C_{\alpha} K_r t^{-\alpha(2-r)} h^4 \), the error bound for \( u_h \) follows by Lemma 13.

To estimate the error in \( \nabla u_h \), we apply (37) and (34) to obtain

\[
\| \nabla \rho_h(s) \|^2 + s^2 \| \nabla \rho_h'(s) \|^2 \leq C h^2 (\| u(s) \|_1^2 + s^2 \| u'(s) \|_1^2) \leq C_{\alpha} K_r h^2 s^{-\alpha(1-r)}
\]

if \( r \leq 1 \), and (36) to obtain

\[
\| \nabla \rho_h(s) \|^2 + s^2 \| \nabla \rho_h'(s) \|^2 \leq C h^2 (\| u(s) \|_1^2 + s^2 \| u'(s) \|_1^2) \leq C_{\alpha} K_r h^2 s^{-\alpha(2-r)},
\]

so

\[
\frac{1}{t} \int_0^t (\| \rho_h(s) \|^2 + s^2 \| \rho_h'(s) \|^2) \, ds \leq t^\alpha C_{\alpha} K_r h^2 \frac{t^{-\alpha(2-r)}}{1 - \alpha(2-r)}
\]

if \( r \leq 1 \), and

\[
t^{\alpha-1} \int_0^t (\| \nabla \rho_h \|^2 + s^2 \| \nabla \rho_h'(s) \|^2) \, ds \leq \frac{C_{\alpha} K_r t^\alpha h^2}{1 - \alpha(2-r)} = t^\alpha C_{\alpha} K_r h^2 \frac{t^{-\alpha(2-r)}}{1 - \alpha(2-r)}.
\]

Since \( \| \nabla u_h(t) - \nabla u(t) \| \leq \| \nabla \theta_h(t) \| + \| \nabla \rho_h(t) \| \), the first estimate of Lemma 14 implies that the error bound for \( \nabla u_h \) holds for the case \( r \leq 1 \).

If \( r \geq 1 \), then we see using (34)–(37) that

\[
\| \rho_h(s) - \rho_h(0) \|^2 + s^2 \| \rho_h'(s) \|^2 \leq C h^2 (\| u(s) - u(0) \|_1^2 + s^2 \| u'(s) \|_1^2) \leq s^\alpha C_{\alpha} K_r h^2 s^{-\alpha(2-r)}
\]

and \( \| \rho_h(s) \|^2 \leq C h^2 \| u(s) \|_1^2 \leq C_{\alpha} K_r h^2 \leq C_{\alpha} K_r h^2 \), so

\[
\frac{1}{t} \int_0^t (\| \rho_h(s) \|^2 + s^2 \| \rho_h'(s) \|^2) \, ds + t^{\alpha-1} \int_0^t \| \rho_h \|^2 \, ds \leq t^\alpha C_{\alpha} K_r h^2 \frac{t^{-\alpha(2-r)}}{1 - \alpha(2-r)}.
\]
Hence, using the second estimate of Lemma 14 and (39), the error bound for $\nabla u_h$ follows also for the case $r \geq 1$.

**Remark 4** If $r = 2$ then by choosing $u_{0h} = R_h u_0$ we obtain an error bound that is uniform in time:

$$
\|u_h(t) - u(t)\| + h \|\nabla u_h(t) - \nabla u(t)\| \leq C_\alpha h^2 (\|u_0\|_2 + M) \quad \text{for } 0 < t \leq T.
$$

**Remark 5** As a consequence of Remark 3, if the zero-flux boundary condition (26) is imposed then the proof of the error bound for $u_h$ in Theorem 4 remains valid, but not that of the error bound for $\nabla u_h$.

**Remark 6** We observed already that if $1/2 \leq \alpha < 1$ and $0 \leq r \leq 2 - \alpha^{-1}$, then the assumption $\alpha(2-r) < 1$ of Theorem 4 is not satisfied. In this case, and when $g \equiv 0$, we may appeal to our previous analysis [11, Theorem 5.4] for the error estimate

$$
\|u_h(t) - u(t)\| \leq C \|u_{0h} - P_h u_0\| + C_\alpha t^{-\alpha(2-r)/2} h^2 \|u_0\|_r,
$$

valid for $0 < t \leq T$ and $0 \leq r \leq 2$. However, $C_\alpha$ now depends on $\alpha$ both via the (previous) stability bound and also via the regularity estimate. Notice also that we may use, instead of (38),

$$
\|\rho_h(s)\|^2 + s^2 \|\rho'_h(s)\|^2 \leq C h^2 \left( |u(s)|^2_1 + s^2 |u'(s)|^2_1 \right) \leq C_\alpha K_r h^2 s^{-\alpha(1-r)} \quad \text{for } 0 \leq r \leq 1,
$$

and hence, because $\alpha(1-r) < 1$,

$$
\frac{1}{t} \int_0^t \left( \|\rho_h(s)\|^2 + s^2 \|\rho'_h(s)\|^2 \right) ds \leq C_\alpha K_r h^2 \frac{t^{-\alpha(1-r)}}{1 - \alpha(1-r)}.
$$

Choosing $r = 0$ and, for simplicity, $u_{0h} = P_h u_0$ and $g \equiv 0$, yields

$$
\|u_h(t) - u(t)\| \leq C_\alpha h t^{-\alpha/2} \|u_0\|.
$$

In the limiting case $\alpha = 1$ and with $F \equiv 0$, Thomée [21, Equation (3.12)] shows in a similar fashion that $\|u_h(t) - u(t)\| \leq C h t^{-1/2} \|u_0\|$. However, to then obtain an optimal error bound $\|u_h(t) - u(t)\| \leq C h^{2-r-1} \|u_0\|$, he employs an iteration argument that relies on the semigroup property of the solution operator for the heat equation. Some new approach seems needed to obtain (40) with a constant $C$ independent of $\alpha$ for $0 < \alpha < 1$, $0 \leq r \leq 2$ and a non-zero forcing $F(x, t)$.

### 6 Discontinuous Galerkin time stepping when $F \equiv 0$

We briefly consider a fully discrete scheme for the fractional diffusion equation, that is, for the problem (1) in the case $F \equiv 0$. For time levels $0 = t_0 < t_1 < \cdots < t_N = T$ we denote the $n$th time interval by $I_n = (t_{n-1}, t_n)$ and the $n$th step size by $k_n = t_n - t_{n-1}$. We choose an integer $p_n \geq 0$ for each time interval $I_n$, and define the vector space $\mathcal{V}$ consisting of all functions $X : \bigcup_{n=1}^N I_n \to \mathbb{R}$ such that the restriction
is a polynomial in $t$ of degree at most $p_n$ with coefficients in $\mathbb{X}$. For any $X \in \mathcal{W}$, write

$$X^n_+ = \lim_{\varepsilon \downarrow 0} X(t_n + \varepsilon), \quad X^n_- = \lim_{\varepsilon \downarrow 0} X(t_n - \varepsilon), \quad [X]^n = X^n_+ - X^n_-,$$

then the discontinuous Galerkin (DG) solution $U \in \mathcal{W}$ is defined by requiring that

$$\left( [U]^n, X^n_- \right) + \int_{I_n} \langle \partial_t U, X \rangle \, dt + \int_{I_n} \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla X \rangle \, dt = \int_{I_n} \langle g, X \rangle \, dt$$

(41)

for $X \in \mathcal{W}$ and $1 \leq n \leq N$, with $U^0_- = u_{0X}$ (so that $[U]^0 = U^0_+ - u_{0X}$). To state a stability estimate for this scheme, let $C_\Omega$ denote the constant arising in the Poincaré inequality for $\Omega$,

$$\|v\|^2 \leq C_\Omega \|\nabla v\|^2 \quad \text{for } v \in H^1_0(\Omega),$$

(42)

and define $\Psi : (0, 1) \to \mathbb{R}$ by

$$\Psi(\alpha) = \frac{1}{\pi^{1-\alpha}} \frac{(2 - \alpha)^{2-\alpha}}{(1-\alpha)^{1-\alpha}} \frac{1}{\sin(\frac{1}{2} \pi \alpha)} \quad \text{for } 0 < \alpha < 1.$$

Notice that $\Psi(1) = \lim_{\alpha \to 1} \Psi(\alpha) = 1$ but $\Psi(\alpha) \sim 8 \pi^{2-\alpha-1}$ blows up as $\alpha \to 0$.

We will use the inequality [14, Theorem A.1]

$$\int_0^T \langle \partial_t^{1-\alpha} v, v \rangle \, dt \geq \frac{T^{1-\alpha}}{\Psi(\alpha)} \int_0^T \|v\|^2 \, dt$$

(43)

**Theorem 5** If $0 < \alpha < 1$, then the DG solution of the fractional diffusion problem satisfies

$$\|U^n\|^2 + \sum_{j=1}^{n-1} \|U^n_j\|^2 + \int_0^{t_n} \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla U \rangle \, dt \leq \|u_{0X}\|^2 + C_\Omega \Psi(\alpha) \int_0^{t_n} \|g(t)\|^2 \, dt$$

for $1 \leq n \leq N$.

**Proof** Let $B(U, X)$ denote the bilinear form

$$\langle U^n_+, X^n_+ \rangle + \sum_{n=1}^{N-1} \langle [U]^n, X^n_- \rangle + \sum_{n=1}^N \int_{I_n} \langle \partial_t U, X \rangle \, dt + \int_0^T \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla X \rangle \, dt,$$

and observe that the time-stepping equation (41) are equivalent to

$$B(U, X) = \langle u_{0X}, X^n_- \rangle + \int_0^T \langle g, X \rangle \, dt \quad \text{for } X \in \mathcal{W}.$$

Taking $X = U$, we find by arguing as in the proof of Mustapha [18, Theorem 1] that

$$B(U, U) = \frac{1}{2} \left( \|U^n_+\|^2 + \|U^n_-\|^2 + \sum_{n=1}^{N-1} \|[U]^n\|^2 \right) + \int_0^T \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla U \rangle \, dt.$$
and so
\[
\|U_+^0\|^2 + \|U_+^N\|^2 + \sum_{n=1}^{N-1} \|U^n\|^2 + 2 \int_0^T \{\partial_t^{1-\alpha} \kappa \nabla U, \nabla U\} dt
\]
\[
= 2\langle u_0X, U_+^0 \rangle + 2 \int_0^T \langle g, U \rangle dt.
\]

For any constant \( M > 0 \),
\[
2\langle u_0X, U_+^0 \rangle + 2 \int_0^T \langle g, U \rangle dt \leq \|U_+^0\|^2 + \|u_0X\|^2 + M \int_0^T \|g\|^2 dt + \frac{1}{M} \int_0^T \|U\|^2 dt,
\]
and using (42) and (43),
\[
\frac{1}{M} \int_0^T \|U\|^2 dt \leq \frac{C_O}{M} \int_0^T \|\nabla U\|^2 dt \leq \frac{C_O \Psi(\alpha)}{M \kappa_{\text{min}} T^{1-\alpha}} \int_0^T \{\partial_t^{1-\alpha} \kappa \nabla U, \nabla U\} dt.
\]

Choosing \( M = C_O \Psi(\alpha)/(\kappa_{\text{min}} T^{1-\alpha}) \) implies the estimate in the case \( n = N \), which completes the proof since \( T = t_N \).

Remark 7 If \( p_n = 0 \) then we have \( \|U(t)\| \leq \|U^n\| \) for \( t \in I_N \), and likewise if \( p_n = 1 \) then \( \|U(t)\| \leq \max(\|U^n\|, \|U^1\|) \) for \( t \in I_n \). Thus, for the piecewise constant [15] and piecewise linear [20] DG schemes, we can prove stability in \( L_\infty(L_2) \), uniformly for \( \alpha \) bounded away from zero.

Remark 8 For the solution \( u \) of the continuous fractional diffusion problem, we have the analogous stability property
\[
\|u(t)\|^2 + \int_0^t \{\partial_t^{1-\alpha} \kappa \nabla u, \nabla u\} dt \leq \|u_0\|^2 + \frac{C_O \Psi(\alpha)}{\kappa_{\text{min}} T^{1-\alpha}} \int_0^t \|g(s)\|^2 ds \quad \text{for} \quad 0 \leq t \leq T.
\]

The proof follows the same lines as above, except that now
\[
\int_0^T \{\partial_t u, u\} dt + \int_0^T \{\partial_t^{1-\alpha} \kappa \nabla u, \nabla u\} dt = \int_0^T \langle g, u \rangle dt
\]
and
\[
\int_0^T \{\partial_t u, u\} dt = \frac{1}{2}(\|u(T)\|^2 - \|u_0\|^2).
\]

Remark 9 Le et al. [10] proved stability and convergence of the DG scheme with general \( F \), but only for the lowest-order \( (p_n = 0) \) case and with no spatial discretization. Although the constants are bounded as \( \alpha \to 1 \), they blow up as \( \alpha \to 1/2 \) and thus the fractional exponent is restricted to the range \( 1/2 < \alpha < 1 \). Huang et al. [7] proved similar results for a slightly modified scheme.
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