SMART PROOFS VIA SMART CONTRACTS:
SUCCINCT AND INFORMATIVE MATHEMATICAL DERIVATIONS
VIA DECENTRALIZED MARKETS

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Abstract. Modern mathematics is built on the idea that a proof should be translatable into a formal proof, whose validity is an objective question, decidable by a computer. In practice, however, proofs are informal, succinct, and omit numerous uninteresting details: their goal is to share insight among a community of agents. An agent considers a proof valid if they trust that it could (in principle) be expanded into a machine-verifiable proof. A proof’s validity can thus become a subjective matter, possibly leading to a debate; if agents' incentives are not aligned, it may be hard to reach a consensus. Hence, while the concept of valid proof is well-defined in principle, the process to establish a proof’s validity is itself a complex multi-agent problem.

In this paper, we introduce the SPRIG (Smart Proofs via Recursive Information Gathering) protocol, which allows agents to propose and verify succinct and informative proofs in a decentralized fashion; the trust is established by agents being able to request more details at steps where they feel there could be problems; debates, if they arise, need to isolate specific details of proofs; if they persist, they must go down to machine-level details, where they can be settled automatically. A structure of fees, bounties, and stakes is set to incentivize the agents to act in good faith, i.e. to not publish problematic proofs and to not ask for trivial details.

We propose a game-theoretic discussion of SPRIG, illustrating how agents with different types of information interact, leading to a verification tree with an appropriate level of detail, and to the invalidation of problematic proofs, and we discuss resilience against various attacks. We then provide an in-depth treatment of a simplified model, characterize its equilibria and analytically compute the agents' level of trust.

The SPRIG protocol is designed so that it can run fully autonomously as a smart contract on a decentralized blockchain platform, without a need for a central trusted institution. This allows agents to participate anonymously in the verification debate, being incentivized to contribute with their information. The smart contract mediates all the interactions between the agents, and settles debates on the validity of proofs, and guarantees that bounties and stakes are paid as specified by the protocol.

SPRIG also allows for a number of other applications, in particular the issuance of bounties for solving open problems, and the creation of derivatives markets, enabling agents to inject more information pertaining to mathematical proofs.

In addition to presenting the key novelties of the SPRIG protocol, this whitepaper is also designed as a brief survey. In order to reach a broader audience, we recall a variety of historical and technical details.

1. Introduction

1.1. Mathematical Proofs. Mathematical derivation, also sometimes called logical reasoning, rigorous derivation, formal rational reasoning, or mathematical proof is a process that allows one to derive mathematical statements from other mathematical statements. By relying on a collection of statements accepted to be fundamentally true, called axioms, this mechanism allows one to derive new mathematical truths, called proven statements. Depending on the context, such proven statements are also called propositions, theorems (when they are deemed interesting), or lemmas (when they are ancillary in the derivation of theorems); the derivation leading to a statement (starting from another statement, assumed or already established to be true) is called its proof.

This form of reasoning is at the heart of rational thinking (in mathematics, all the sciences, and way beyond), crucially leading to:

- One’s trust in the truth of statements derived.
- One’s insight into the reasons why such statements hold true.

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These two aspects of the question are discussed in Sections 1.1.1 and 1.1.2 below.

1.1.1. Proofs as a Means of Trust. In many ways, mathematically proven statements achieve the highest possible level of certainty one can have. The validity of an established theorem (say, for instance, Euclid’s theorem on the existence of an infinite number of primes) does not fluctuate with the evolution of knowledge. Conversely, for statements for which no known proof exists, the trust in their validity only grows progressively with empirical or heuristic evidence support, and it never quite reaches that of mathematically proven statements.

Once the trust in a statement is established via a proof, the statement can be used as a basis for establishing trust in new statements: the proofs of all the statements obtained this way can (if needed) be ‘unrolled’ down to the axioms. Hence, by propagation, various agents are able to build together a set of trusted statements relying upon each other (a ‘tower of knowledge’) without necessarily knowing all the details of all the proofs. This is the way modern mathematics is built.

More recently, due to the development of computer technologies, proofs have become fundamental not only for the construction of mathematics and science, but they have also become objects manipulated by e.g. cryptography or program verification systems. Most digital interactions are now mediated by cryptographic primitives, which aim in particular at establishing trust in confidentiality and authenticity: for instance, a party can prove their identity by providing a digital signature (more precisely, the mathematically proven statement is: either the party knows a secret encryption key, or they are extremely lucky, or a certain widely believed algorithmic hardness assumption is in fact wrong). Similarly, for critical systems, there is often a need for a proof that a piece of code meets some specification, such as termination or type safety.

Of course, in any case, the trust in a mathematically proven statement relies on the trust that the underlying proof is indeed correct, i.e. that:

- The derivation rules are clearly defined, computable and consistent.
- They are applied uniformly throughout the proof of the statement, as verified by computers or other mathematicians.
- The premises and axioms underlying the reasoning can be trusted.

These issues and the underlying challenges are briefly discussed in Section 1.2 below.

1.1.2. Proofs as a Means of Explanation and Insight. Arguably more important to most mathematicians’ minds than the idea that a mathematical derivation answers the question ‘how do we know this statement is true?’, is the resulting insight that it provides into the nature of the underlying problem (see e.g. [Thur94]).

For instance, there are famous statements such as Goldbach’s conjecture that are already widely believed to be true (as supported by various heuristics and numerical verifications), despite being unproven. Finding a proof of Goldbach’s conjecture would be considered a breakthrough not so much because it would tell us that it is true (which would hardly surprise anyone), but because it would tell us why, and because it may give some new deep insight into the nature of prime numbers.

The idea that proofs give insight is arguably a central driving force behind mathematical teaching and exposition: it motivates the great effort that goes into the presentation of theorems’ proofs. Similarly, the work towards finding new, simpler, more intuitive, or just different proofs of existing theorems is greatly valued: multiple proofs of the same statement can bring insights on various issues, a complementary value [AiZi10], or enhance one’s intuition of a result.

Conversely, certain proofs appear to bring little insight because of their complexity (for instance, computer-generated proofs such as the one of the four-color theorem [ApHa77, Thur94]). While the trust in such proofs is very high, they are considered somewhat unsatisfactory by many mathematicians, as they cannot be comprehended as well as more ‘elegant’ proofs.

Often, insightful proofs appear to be centered around a limited number of new ideas. In fact, this seems to be how educated agents should convince each other: convey credence about statements through the transmission of a limited amount of relevant information.

A balance between this idea of proof and that of Section 1.1.1, i.e. between ‘a short collection of insightful statements’ and ‘the list of all the statements needed to establish perfect trust’, is in principle possible, though somewhat delicate, as discussed in Section 1.2 below.

The SPRIG (Smart Proofs via Recursive Information Gathering) protocol allows, for a system of agents with various levels of interest and information, to reach this balance.
1.2. **Nature of Mathematical Derivations.** In this subsection, we describe the modern view of the idea of mathematical derivation, as it emerged at the beginning of the 20th century, that now serves as the basis of all contemporary mathematics. This view also lies at the heart of all computer-based proof systems (Section 1.3) and has, in recent years, faced a number of practical challenges (Section 1.4).

1.2.1. **Hilbert’s Program and Logicism.** The clarification of the foundations of mathematics, as advocated by Hilbert’s program, progressed greatly in the early 20th century. One key step was the idea of logicism [Korn60]: that mathematical statements should be written in a formal language (or unambiguously translatable into one), and that a mathematical derivation ought to consist of a sequence of well-defined manipulations of such statements. This is naturally connected to the then-emerging field of computer science: mathematical derivations, written in the appropriate language, ought to be verifiable by a computer program.

These new foundations led to two key developments:

- On the one hand, the emergence of the study of valid formal mathematical derivations as central objects, as a field in itself (namely proof theory), primarily associated with mathematical logic and theoretical computer science (see Section 1.2.2 below);
- On the other hand, the emergence of modern mathematics, with more rigorous and standardized definitions, theorems, and proofs, implicitly relying on the new foundations (see Section 1.2.3 below).

Interestingly, despite immense progress in computer technology, these two developments have only seen little interaction. Arguably, this is due to practical (rather than fundamental) reasons, and recent challenges and developments suggest that this has led to a somewhat unfortunate situation (see Section 1.3 below). The objective of the present paper is indeed to present a new way to make this interaction practical and fruitful.

1.2.2. **Logic View of Mathematical Derivations.**

"The development of mathematics toward greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules". – K. Gödel.

The controversies of the late 19th century led to convergence on the foundational theory of mathematics, by which the disagreements could be resolved. Ideas of Frege, Hilbert, Gödel, Turing, and others, led to a definition of formal proof, connected with the notion of computer (itself defined in terms of Turing machines; see e.g. [Wigd19] for a modern account).

**Definition 1.** A formal proof or formal derivation, or machine-level proof, or proof-object is a finite sequence of sentences in a formal language, each of which is an axiom, an assumption, or follows from the preceding sentences in the sequence, by a rule of inference. The validity of the application of the rules of inference can be checked by a computer.

In practice, most of modern mathematics, by convention, relies on a standard set of axioms (based on the Zermelo-Fraenkel set theory, with some version of the Axiom of Choice), and on higher-order logic. A number of computer proof systems implement such a framework (see Section 1.3.1 below).

Besides the important clarifications that they bring, the strength of formal proofs is that the verification of their validity is completely mechanical. As a result, they can be checked reliably by computers. In principle, computers can also then be used to try to produce proofs, as was already suggested by Gödel in his Lost Letter to Von Neumann [Lipt10]. The use of a formal language also allows in principle various areas of mathematics to communicate with each other, by allowing them to inter-operate unambiguously.

Unfortunately, formal proofs are extremely long in practice [Wied03] and hard to produce, even with modern computer proof assistants (see Section 1.3.1 below); and most importantly, they are often quite different from the way mathematicians think of proofs (a formal proof may bring little insight to a mathematician). Still, the stronger rigor of such proofs has influenced the shaping of modern mathematical derivations as mathematicians use them today, as described in Section 1.2.3 below: they must stay ‘in the back of mathematicians’ minds’, as they write and communicate their proofs, something that the SPRIG protocol allows us to formalize.

1.2.3. **Modern Mathematical Derivations in Practice.** The core of mathematical activity has seen little change over the last century, and the focus of mathematics has shifted away from formal verification and foundational issues, in favor of introducing new objects, discovering new ideas, and solving interesting problems. At the same time, mathematics still emphasizes strict rigor: for instance, heuristic arguments or numerical
simulations, however convincing, are not accepted as being parts of mathematical derivations and proofs, while, for instance in theoretical physics derivations, they are often deemed sufficient.

The following defines what it means for a mathematician to know the proof of a statement:

**Claim 2.** A mathematician knows how to prove a statement rigorously, if they have the confidence in the following: given access to a corpus of references, they would be able, if pressed and given enough time, to give details at an arbitrarily high level in the proof of each statement, down to a computer-checkable formal level if needed.

The working definition of ‘a proof’ that a modern mathematical text uses can be phrased as follows:

**Claim 3.** A written proof consists of

- A Proof Sketch \( P = D, S_1, \ldots, S_k \), consisting of a collection of definitions and references \( D \) and a list of statements (lemmas, propositions, theorems, remarks) \( S_1, \ldots, S_k \) using symbols in \( D \), where each \( S_j \) is allowed to assume that \( S_i \) holds true for \( i < j \).
- A text in free format \( F \) (including proof arguments, drawings, informal explanations, etc.), such that it is claimed that the proof is complete and valid in the eyes of mathematicians (of the given audience), in the following sense:

**Claim 4.** A written proof \((P, F)\) of a statement \( S \), with \( P = D, S_1, \ldots, S_k \) is considered complete and valid in the eye of a mathematician if, by using the text from \( F \) and standard mathematical knowledge if needed, they know how to prove:

- for each \( j = 1, \ldots, k \) the statement \( S_j \) assuming (if needed) \( S_1, \ldots, S_{j-1} \);
- the statement \( S \) from the statements \( S_j \), where \( j = 1, \ldots, k \).

**Remark 5.** Another way to phrase the structure of the proof sketch \( P \) is to say that the statements \( S_1, \ldots, S_k, S \) form a directed acyclic graph of dependence, with root \( S \) (where a statement points to the statements it assumes). The order in which the parts \( P \) are presented in a paper may not follow the order here, but the vertices of any directed acyclic graph can be ordered so the vertex \( i \to j \) implies \( i > j \).

**Remark 6.** In mathematical papers, definitions-statements may appear (for instance, defining Riemann’s \( \zeta \) function may require a proof of convergence); see Section 2.2.1 for a discussion of how such definitions-statements can be recast in the format of proof sketches above.

**Remark 7.** Proofs by contradictions can be written in the proof sketch format as above (see Section 2.2.1).

The free part \( F \) of a proof is what is sometimes called a Social Proof [Buss98], and the proof sketch part \( P \) should be directly translatable into a collection of formal statements, sometimes called a Formal Proof Sketch [Wied03]. In this article, we will consider that the proof sketches are always formal. It should be noted that in mathematical articles, these two parts are sometimes not clearly separated; also, in some cases, the entire proof is a social proof.

**1.2.4. Amount of Detail in Proofs.** As explained in Section 1.2.3 above, ‘being convinced by a proof given as a mathematical text’ means: knowing enough to be confident that one would be able to produce all the details, if needed, while at the same time knowing that this most likely will not be needed. Indeed, it would be so energy-consuming and uninformative that there would be no point in doing it. In the language of structured proofs above, every mathematician must build their own structured proof at a level of detail that they deem satisfying.

Being a mathematician hence crucially requires a great deal of self-discipline, in order not to delude oneself into thinking that one knows how to prove a statement. How to be confident in one’s ability to perform a task (providing machine language proof of new results), that one will (in all likelihood) never perform?

Moreover, what constitutes a complete proof (in the sense of Section 1.2.3) becomes as a result somewhat subjective, depending on the reader’s standards: the amount of detail required from a student at an exam will typically be very different from the level of detail in a research paper.

For research papers, it will depend on the subfield and the journal and editorial standards: a debate between the authors and referees can arise, in which the editor is the arbiter.

In any case, determining the relevant amount of detail to provide in a mathematical paper is a difficult task, which requires a delicate balance between the need to guarantee the validity of the reasoning, to limit
the pre-requisites and the work on the reader’s side, to stay within page limits, to avoid unnecessary clutter, and to keep only the essential arguments.

Within these constraints, there is a lot of room for subjective choices and writing modern mathematical proofs is largely an art, as discussed in e.g. [Lamp95, Lamp12]. The increasing complexity of proofs involved in contemporary mathematics has led to a number of challenges, as discussed in Section 1.4 below. At the same time, the development of computer-based proof systems offers great promises to help tackle such challenges.

The goal of SPRIG protocol is to unify these two visions of formal proofs, as verified by computers, and of informal proofs as done by mathematicians, to leverage the advantages of both (trust and insight, respectively): SPRIG will allow the agents to inject various levels of information to reveal a subtree of the proof tree as in Section 2.2.1 below.

1.2.5. Structured Proofs. An interactive way of viewing proofs as in Claim 3 is the following: instead of asking to be able to write down a proof of the formal statements in the machine-level language directly, one can think of being able to request for formal details for each of the statements (assuming perhaps an audience that is more and more curious into the details); and in our answer, if any requests for more details arises, one should again be able to provide them. This way, one should eventually be able to reach (if needed) the machine level after a reasonable number of steps, using a reasonable amount of space, keeping an informative structure (and abstracting away the free part $F$ of the proof).

This leads to the following definition of structured proof (written in a formal language), upon which the proof format of our protocol is based (see Section 2.2.1). A structured proof of a statement $S_*$ of level $L \geq 1$ consists of a tree with the following structure:

- The root (‘top-level’) is the statement $S_* : A_* \Rightarrow C_*$ itself, where $A_*$ includes axioms and accepted statements used to derive the conclusion $C_*$. 
- For each non-leaf (‘high-level’) statement $S : A \Rightarrow C$, its children $(S_j)_{j=1,\ldots,k}$ are statements $A_j \Rightarrow C_j$, where $A_j$ is of the form
  \[ A_j = A \cup \{C_i \text{ for } i \in I_j\} \]
  for some $I_j \subset \{0, 1, \ldots, j - 1\}$, and where $C_k = C$.
- For each leaf (‘low-level’) statement $S : A \Rightarrow C$, a machine-verifiable proof is provided.
- The tree height (distance between the root and leaves) is at most $L \geq 1$.

Remark 8. The idea of structured proof in our paper is very close to structured proofs suggested by Lamport [Lamp95, Lamp12]; the difference is that we make the level more explicit. Ideally, a proof tree should be well balanced (not too deep, and at the same time with a moderately large degree, with fairly short statements), and highest levels should be the most interesting to experienced mathematicians.

The SPRIG protocol is based on the assumption that valid known mathematical derivations can be structured with well-balanced trees (in principle, as the complete tree is as large as a machine-level proof), and that the existence (or non-existence) of a complete structured proof tree can be determined with high confidence by a mathematician only knowing a small subset of the tree (which may depend on the level of information of the mathematician).

The verification of a proof in a debate (between a teacher and a student, or a reviewer and an author 1.4 below) works largely with the idea of well-balanced tree: a teacher may ask a student to produce the highest levels of the tree, to reach the confidence that the student would know how to provide the rest of the tree (if given enough time).

As discussed in Section 1.2.4 above, determining the relevant amount of details to provide in a published proof is delicate and somewhat subjective. This paper aims at explicitly taking this subjectivity into account, by considering a system of agents with various levels of information and confidence in their ability to fill in the details of a proof, and proposing a protocol by which such agents can exchange information.

1.3. Recent Developments in Computer-Based Proofs. The second half of the 20th century saw the explosion, both theoretical and practical, of computer science. As discussed in Section 1.2.2 the modern notion of formal proof leads to the idea that such proofs are machine-verifiable, and as a result, we use formal proofs and machine-level proofs as synonyms. This has led to a desire to see a formalization of mathematics
in computer-checkable terms (see e.g. the QED Manifesto [Anon94]). At this point, this wish remains largely unfulfilled, and most modern mathematics has not benefited from the progress in computer-based proofs.

In this subsection, we discuss key features of such systems and the associated challenges.

1.3.1. Computer-Assisted Proofs. The idea that computers could (and perhaps should) verify the validity of proofs goes back at least to Gödel’s Lost Letter to Von Neumann [Lipt10]. It is extremely natural: in principle, any modern mathematical derivation can be translated into a sequence of formulae, which are progressively derived by applying specific rules (which we will call logic system); specifying the formulae and the applied derivation rules thus constitutes a computer-checkable proof, sometimes called proof object (see Section 1.3.2 below).

Computer-assisted derivations have yielded a number of successes in mathematics, and have been instrumental for the proofs of celebrated conjectures, which involve dealing with a large number of cases separately:

- Famously, computer-assisted proofs were instrumental to the first proof of the four-color theorem [ApHa77, AHK77]; a fully machine-verified proof was given later [Gont08].
- The Kepler conjecture about sphere packings was established by a machine-checked proof [Hal+17].

In addition to enabling proofs that are too hard for humans to write and check, computer-assisted proofs are also important in the field of software verification, where they allow one to guarantee that functions of a program will behave according to specification.

As a result of their appeal (as trusted elements of knowledge, as objects that computers can sometimes produce better than humans), a desire to formalize mathematics has grown over the years. SPRIG is based on the idea that to convey succinct, informative, and trustable proofs, entire proofs are not necessary: only a subset that is relevant to the agents exchanging information is needed. As discussed in Section 1.3.2, many proof systems exist; the rest of this paper is based, for concreteness, on one of them, which is particularly readable by mathematicians.

1.3.2. Computer Proof Systems. A number of computer-based systems such as Mizar, TLA+, Isabelle/Isar, Coq, Metamath, HOL, or Lean enable and facilitate the writing of formal proofs. These systems rely on some basic low-level language: proofs written in this language are called proof objects, and they are what the computer ultimately checks the validity of. At the same time, the users of such systems usually work with a higher-level language, called user-level proof, which yields proofs that are usually much shorter (yet still very long compared to proofs used by mathematicians [Wied12]).

This article is largely agnostic on the specific choice of computer-system but could be implemented easily in so-called declarative systems such as Mizar, Isar or Lean.

A user-level proof consists of a sequence of statements (which include equivalent of high-level mathematical proofs, with definitions, proof steps, justifications, sub-statements, cases, etc.) written in a formal language. The proof is called complete when the system is able to validate each of the justifications for the steps. While somewhat tedious to write and to read, formal proof sketches, which are valid proofs in which a number of steps have been removed, are particularly easy to understand by mathematicians, as pointed out in [Wied03] (see also Section 1.2.3 above).

In languages such as Isar, reading and writing simple statements or definitions (as opposed to writing complete justifications) is relatively easy. As a result, a mathematician can be expected to be able to determine if a simple statement deemed to have a short proof does indeed have one, without having to try to write it; this idea is at the heart of SPRIG.

1.3.3. Interactive Proofs. We conclude this subsection by discussing the development of a completely different take on proofs induced by the algorithmic reduction of proofs, motivated in particular by cryptographic applications: that of interactive proof protocols. An interactive proof protocol allows a prover to demonstrate their knowledge of a proof to a verifier, through their ability to answer the verifier’s questions.

The cornerstone of interactive proof checking relies on computational complexity theory and NP-completeness (see e.g. [Wigd19] for a modern account): for any statement \( S \) in a formal logic system \( \Lambda \), the statement ‘\( S \) has a proof of length \( \leq L \) in \( \Lambda \)’ can be reduced to the statement ‘\( \text{G is 3-colorable} \)’ for a certain graph \( \text{G} \) computable in polynomial time in terms of \( S, L, \text{ and } \Lambda ; \) proofs of length \( \leq L \) of \( S \) in \( \Lambda \) then are in one-to-one correspondence with 3-colorings of \( \text{G} \) (i.e. assignments of colors to the vertices of \( \text{G} \) such that any pair of adjacent vertices has different colors). Note that we focus on 3-colorings for concreteness, but any NP-complete problem would do the job as well. Through the prism of computational complexity, a prover
can demonstrate her knowledge of a proof of length $\leq L$ of a statement $S$ by demonstrating her ability to color the corresponding graph $G$.

This view of proofs allows in particular for a number of interesting applications (see e.g. \cite{Wig19}):

- Probabilistically checkable proofs: the prover may be able to map her 3-coloring problem to a 3-coloring problem with an amplified gap, and to use it to convince a skeptic of her knowledge of a proof (of length $\leq L$) via a limited number of interactions (independent on the proof length); in essence, if the prover bluff in her answers to questions, she will be caught with high probability.
- Zero-knowledge interactive proofs: by relying on cryptographic primitives, the prover may be able to demonstrate her knowledge of a proof (of length $\leq L$) without divulging anything about the proof itself.

The view of proofs underlying the field of interactive proofs is at odds with that of theorem proving in mathematics: the proofs in that world yield little, if any, insight to mathematicians (as discussed in Section 1.1.2 and 1.2.2 above) into why proven theorems are true (and for instance, in the case of zero-knowledge proofs, the goal is to give zero information about the proof).

Still, the idea of establishing trust via a few interactions is very appealing for proof verification in mathematics. In light of this question, SPRIG aims at allowing agents to communicate both trust and insight to each other, via a limited number of interactions.

1.4. Challenges in Modern Mathematical Derivations. In mathematical practice, the amount of detail needed to assess a proof’s validity is usually decided by a peer-review refereeing process, whose goal is also to determine how interesting the results and insights are. Usually, this happens within the context of a publication by a journal, in which a small number of independent experts assesses the validity of a proof (i.e. whether enough details are provided to transmit the confidence that the proof is correct, in the sense of Claim 2 above). In the case of higher-profile results, validation also comes from the larger community, where all experts of the field may discuss the results, identify weaknesses, and exchange comments with the authors.

In this context of ‘high-level’ proofs, the last decades have seen a number of developments which created additional challenges, in particular:

- The inflation in the complexity of (published) mathematical proofs makes their validation more difficult (see Section 1.4.1 below).
- The boundary conditions of the process (see Section 1.4.2).
- The alignment of various external incentives with the ones of the validation process (see Section 1.4.3).

In principle, as suggested in Section 1.2 above, all of these challenges are of a purely practical nature: given enough competent, reliable and properly incentivized experts, these challenges would not exist; or, alternatively, if all proofs were easy to write down in a format verifiable by a computer (as discussed in Section 1.3), there would be no need for expert verification. As discussed in the following subsections, however, these challenges are very real today; it is the aim of SPRIG protocol allowing to help tackle them.

1.4.1. Complexity of Proof Validation. Checking the validity of mathematical derivations is a time-intensive task, which naturally depends on the length of the result as published, and on the time it takes a referee to check the details (i.e. explicitly or implicitly filling in the blanks to convince oneself of its validity). Over the last 100 years, the complexity of published mathematical proofs has grown significantly:

- The average length of mathematical proofs has grown: for instance, the average length of a paper in Annals of Mathematics in 1950 was less than 17 pages, but in 2020 it was more than 58.
- The papers, in turn, usually rely on larger and larger bodies of work, and proofs are now rarely self-contained.
- Some proofs are split into many mathematical articles: for instance, the classification of the finite simple groups consists of tens of thousands of pages in several hundred journal articles, published over a 50-year period.

As a result of this inflation in complexity, the process of validating a proof has increased in difficulty. Some documented, high-profile, examples are:
- The Jacobian Conjecture: it has seen a large number of claims of proof in the 20th century, which have survived for a number of years, before being invalidated, and standing as an open problem [Wiki21].

- Poincaré’s Conjecture: after a number of incorrect claimed proofs were proposed throughout the 20th century, a collection of papers was published by Perelman in 2002–2003, which led to a validation by the Clay Institute in 2006 following many debates, including the publication of a number of papers, some of which filled in details, of which some claimed to be the first complete proof of the conjecture [NaGr06, Szpiro08].

- Hilbert’s 16-th Problem: currently an open problem, for which many attempted solutions have been proposed, some of which took decades to be invalidated [Ilya02].

- The ABC Conjecture: a solution has been proposed, which is considered wrong or incomplete by a significant number of experts, but at the same time considered correct by a significant number of experts [Cast20].

- The classification of finite simple groups was announced as completed in 1983. Yet a number of gaps have been found over the years, which were filled over the following decades [Solo01].

In a number of high-profile cases, the underlying debates have taken years to settle. In principle, such debates should not last: it should be enough to provide a computer-verifiable proof in practice, the sheer length of the relevant proofs in machine-level language makes such a task daunting. As discussed in Section 1.4.2, this poses a number of problems in terms of the boundary conditions of the process.

SPRIG aims at enabling various agents (including computers) with diverse areas of expertise to collaborate in the reviewing process and in the writing of the proofs’ details.

1.4.2. Boundary Conditions of the Reviewing Process. The reviewing process, as performed by a journal or a community as a whole, involves a number of delicate boundary conditions, which are usually not formalized:

- The reviewing process involves matching authors and expert reviewers. Picking experts may be a difficult task for a journal editor; in the case of public debates, for the community to decide whom to listen to requires the build-up of a consensus.

- The interaction protocol between authors and reviewers is not formalized. Should the authors or reviewers not act in good faith or have drastically views of what a complete proof means, the process will stall:
  - In principle, there is nothing that prevents reviewers from nitpicking or claiming they don’t understand some parts, and hence of deeming a correct proof incomplete: in some sense, a proof is indeed incomplete until it is completely written down in a machine-verifiable format.
  - Dually, an author whose proof is too vague or incomplete (or possibly void) may keep adding irrelevant details that do not address the heart of the issue, or claim there is nothing important to add and that the reviewers are nitpicking on trivialities.
  - Since going down to the machine level is not a feasible option, for a journal, the editor ends up being the ultimate arbiter; when the whole community discusses the issue, a consensus forms (or doesn’t form).

- The dual role of reviewers: they are expected to emit at the same time, a judgement on the validity and the interest of the result.

- Ultimately, there is no specification as to which of the ‘wrong until proven correct’ (the machine-level proof standard) or ‘correct until proven wrong’ principles prevail in case of disagreement, and in which timeframe, i.e. where the burden of proof lies.

While the above are theoretical weaknesses of the protocol of the reviewing process, they are not necessarily problematic in practice if the various agents work constructively towards aligned goals (such as uncovering mathematical truths, acting in good faith, etc.). But this is no longer the case as soon as conflicts of interest exist: see Section 1.4.3.

In light of the boundary conditions problem, SPRIG allows one to rely on computer-based proof verification algorithms (as in Section 1.3) as the ultimate arbiters, and to enforce explicit and transparent time constraints.

1.4.3. Alignment of Incentives. The functioning of the reviewing process involves a number of agents, whose identities may or may not be known; for high-profile proofs, the whole community may end up being involved. As discussed in Section 1.4.2, the boundary conditions can in principle be the source of problems; this can
in particular be the case if there is misalignment between the goal of thorough and quick validation and the agents’ objectives.

Arguably, a large part of the incentives underlying the reviewing process is implicit, rather than explicit (namely: desire to discover the truth, to be intellectually honest, to participate in the good functioning of the community, to be respected as an expert, etc.). However, the agents’ strategies may be also influenced by the presence of various external incentives whose alignment with the reviewing goals is unclear (e.g. funding, jobs, prizes, recognition).

In terms of explicit incentives, a number of problems may arise:

- The reviewing process is rarely explicitly incentivized (in the case of journals, reviewers are usually anonymous and not compensated), in particular related to their ability to spot e.g. mistakes; and there is an asymmetry of incentives: there are usually only negative consequences for not finding errors and no significant downside to rejecting a valid proof.
- For high-profile problems, there is a problematic asymmetry: numerous amateurs may see a lot of upside in submitting (mostly incorrect) proofs of famous conjectures, while at the same time fewer experts are available to spend their precious time on finding errors in these proofs (with no upside).
- Authors may be incentivized to publish vague, incomplete proofs to claim precedence over other authors.
- Independent experts are hard to find in very specialized fields, and the incentives to disclose conflicts of interest are limited. If the reviewers are competing with the authors (or conversely are interested in seeing them succeed) they may stall the reviewing process (or conversely be too lenient), as discussed in Section 1.4.2 above.
- In the case where proofs involve security issues (such as in cryptographic contexts), there may be additional problems, with experts having possibly incentives to keep discovered mistakes to themselves, as exploiting them may be worth money.

The above problems are illustrated by a number of high-profile examples [NaGr06, Cast20], and even when aware of the existence of alignment problems, it is hard for external agents to identify in which instances of the above problems a situation falls [Cast20, NaGr06].

In light of the alignment problem, the goal of SPRIG is to mediate multi-agent interactions with explicit incentives, designed in such a way as to align the agents’ objectives with the ones of the reviewing process.

1.5. Markets, Information, and Games. As discussed in Section 1.4, the validation process of proofs as performed by mathematicians involves a number of agents, with different levels of information, interacting in a variety of manners. These include publishing proofs, detecting issues in papers, asking for more details, and providing them.

Similarly, SPRIG invites repeated interactions between members of the mathematical community with potentially variable levels of information and degree of involvement.

In both the current validation process and the SPRIG protocol, understanding the set of involved agents and their interactions as an economic system is of paramount importance. Indeed, the raw data of the protocol outcome (say: number of refereeing rounds, final status: accepted/rejected) is in general insufficient to determine precisely what credence the community should have in the validity/invalidity of a proof.

Understanding the motives, incentives, and beliefs of the relevant agents allows one to assess what information we can actually extract from a protocol outcome. The simplest example is perhaps the case of an accepted paper, the author of which sits in the editorial board of the publishing review. All other things being equal, and because of an obvious incentive problem, it seems rational to (at least slightly) decrease the credence in the validity of the published results.

Section 1.4.3 lists a number of other incentive issues. For completeness, we now briefly review the core concepts of the modern microeconomics toolbox and the main economic theories pertaining to the discussion above. A deeper game-theoretic treatment of the validation process is provided in Section 4.

1.5.1. Agents, Bayesian Views, and Markets. The raison d’être of incentives problems is that each agent follows their own agenda, being driven by specific motives or preferences. These preferences can be represented by a utility function, a notion that traces back at least to Bentham and J. S. Mill and has become commonplace since the emergence of neoclassical economics. Von Neumann and Morgenstern [VNMo44] established conditions on agents’ preferences such that these can be ordered by an expected utility calculation, providing
the economics profession with a key tool for dealing with decision-making under uncertainty. However, other concepts were needed in order to think and make predictions about the way economic agents (inter)act.

In particular, a proper microeconomic treatment of incentive problems and their consequences was virtually impossible before the advent of two intellectual revolutions.

- The first one, largely initiated by Nash [Nash50], is game theory: it provided economics with a much-needed tool to model situations where uncertainty arises from one’s imperfect knowledge, not about the state of Nature, but rather from other agents’ actions.
- The second one, initiated by Muth [Muth61] and later supported, consolidated, and popularized by Lucas, is rational expectations: rational agents not only maximize their own utility but have the same knowledge as the economic modeler and are able to correctly compute the model’s outcome. This requires making assumptions about other agents’ behavior, but at the same time, allows them to predict this behavior (given that agents will take utility-maximizing decisions). Rationality requires that the predictions coincide with the assumptions. Key to the rational anticipation process is the ability to correctly compute expectations; in particular, rational agents are Bayesian updaters.

Several contexts, including the issues we investigate in the current paper, called for an extension of these tools to the case of asymmetric information: situations in which agents operate under different information sets and can extract some information from others’ actions. Akerlof [Aker70], Spence [Spen3], and the various works of Stiglitz and his co-authors in the seventies closed this gap. Contributions such as [ChKr87] and [FuTi91] provided refinements of the equilibrium concept for strategic interactions under asymmetric information. In this literature, agents have a ‘type’, i.e. a characteristic that is not directly observable but partially or fully inferred given a history of actions. As we shall see, in SPRIG, this type is the subjective probability that a proposed proof can be unrolled up to machine language level (itself a function of variables such as personal skill, amount of work, and carefulness which are not fully observable by outsiders).

Townsend’s model [Town79] with costly state verification initiated a large literature on optimal contracting under asymmetric information. In SPRIG, the goal is, indeed, to estimate the status (i.e. valid/invalid) of a proof. But the very design of our validation process implies that verification (while potentially costly in terms of time and intellectual energy) might in fact be beneficial to the verifiers, who can collect bounties.

The system formed by SPRIG and its users can be seen as a market for proofs, although not quite in the sense of a stock market. However, just as the (semi-strong) efficient market hypothesis [Fama70] stipulates that the utility-maximizing behavior of agents will lead stock prices to reflect any available public information, we expect that with properly chosen parameters, SPRIG will aggregate information and eventually disclose the actual status of a proof.

The dark markets reviewed by Duffie [Duff12], while different from our market in a variety of regards, also share similarities with SPRIG. These markets are dealer networks in which connected agents conduct bilateral negotiations to trade financial assets (there is no ‘market price’). Each transaction reveals part of the private information that dealers have, and therefore one can expect information to ‘percolate’ through the network.

While scoring rules can be used to aggregate the credences of various agents on statements (see e.g. [Hans03a, Hans03b]), SPRIG features two additional key characteristics. First, it provides a built-in termination date at which the validity of the proof/question is decided and thus “bets” can be settled. Second, the dynamic verification process generates explicit information about the strengths and weaknesses of a claim. In fact, SPRIG can be viewed as a multi-round security game (see e.g. [BCDPS13]), in which agents ‘debate’ the validity of claims ([ICA18]), with automatically set boundary conditions.

1.5.2. Economic Markets and Mathematical Truth. The idea of agents with a Bayesian view on the truth of mathematical questions dates back at least to the works of Solomonoff [Solo64]. Recent works on systems of such agents interacting through a market [GBCST16] have shed light on how such systems may be viewed as (decentralized) algorithms that estimate and refine probabilities of truths for mathematical statements. [GBCST16]’s algorithm, a logical inductor, dynamically assigns probabilities to mathematical statements and the belief system thus produced is shown to be consistent asymptotically. This consistency as well as other desirable properties of their algorithm derives from a logical induction criterion, which is essentially a no-arbitrage condition on a market defined as follows. Each mathematical statement ϕ is associated with a derivative that pays $1 if is true and zero otherwise; the market price of this derivative can then be seen as the current belief about the truth of ϕ. A trader can observe the history of prices up to time \( t - 1 \), make some
computations of their own, and post market orders at time \( t \), adjusting their portfolio of derivatives written on statements \( \phi_1, ..., \phi_n(t) \). Their trading strategy is adapted, i.e. a function of past prices. Importantly, this “past” includes time \( t \): naturally, the demand function depends on the price that will eventually be set. A market maker (a subroutine of the logical inductor) then sets prices in such a way that the demand of the trader is (approximately) zero for all derivatives. By construction then, a “fair price” obtains, which captures the probability of the statements underlying the derivatives.

While appealing in many aspects, these ideas remain unfortunately extremely theoretical. In the words of \[ \text{GBCST16}, \] ‘logical inductors are not intended for practical use.’

- The required computation times and spaces are unreasonably large.
- There is an important distinction between being true and being provable: the latter is the focus of mathematics research, and accumulating evidence for the veracity of result may not result in any progress towards proving it (for instance, verifying Goldbach’s conjecture up to a large \( N \) may increase one’s trust in the truth of the statement, but not bring any insight into how to prove it).
- There is no focus on the amount of insight associated with the agents’ discoveries.
- There is no obvious way to incorporate agents seeking information about specific statements, i.e. to shift the attention of the market towards a set of problems currently of interest to these agents.

SPRIG leverages on the view that the combination of Bayesian updating and individual profit-seeking behaviour leads the market to reveal information about fundamentals (in our context: the validity of mathematical claims). However, rather than aiming at constructing an “exhaustive encyclopedia” of mathematical propositions, our framework incentivizes agents to inject (or induce injection of) information about specific statements, which are relevant for the community either because they are mathematically interesting or because they correspond to critical points in a proof. Furthermore, our focus is on the effective provability of statements rather than on the credence that they are, in an abstract way, true. That is, SPRIG not only invites its users to focus on important mathematical statements, but also induces them to discuss/prove/refute those in a way that provides intuition and insights about why they are true or false.

1.6. **Blockchain and Related Technologies.** The last decade saw the emergence of fully decentralized computing systems, in particular in the context of public databases made of public, immutable records, called blockchains. These systems, running on a decentralized network of computers (typically connected to the internet), have grown out of the desire to build transparent, trustless consensus systems, relying on no central authority, or specific machine, following a secure, time-stamped, and easily auditable behavior.

While initially restricted to the context of the storage of digital assets into accounts (such as for Bitcoin), blockchains have grown in terms of applications and features. The introduction of smart contracts, allowing the blockchain to perform complex operations and transactions conditioned on taking various inputs, has opened new possibilities. A particularly prominent development is the creation of inexpensive, open, efficient, and trusted general-purpose markets.

SPRIG provides a way to construct such markets aimed at decentralized, public proof verification: its very design makes it perfectly suitable for running on the blockchain.

In this subsection, we review a number of key principles and mechanisms of blockchain technologies, upon which SPRIG relies.

1.6.1. **Distributed Systems.** A blockchain is a certain type of distributed system. A distributed system instance consists of execution instances of programs (often called clients) running on a network of computers (often called nodes), which communicate via a specified protocol. The protocol prescribes the communications that the nodes should emit and receive. Early examples of distributed systems across the internet include peer-to-peer file-sharing networks, in which a node has a number of files available for other nodes to request. As a whole, a distributed system can be viewed as an execution instance of a program: a peer-to-peer file network can, for instance, be viewed as a single database, emerging from the various nodes.

Unlike regular program instances, distributed systems cannot be viewed as running on any particular node, while emerging from the nodes; this makes distributed systems more tolerant to localized failures in the system.

A distributed system is called (fully) decentralized when there is no principal node coordinating the network. A fully decentralized system instance is hence a form of consensus, emerging from the execution of the clients on the nodes: the choice of the nodes to adhere to the protocol defining it, by running client programs that respect the protocol, is what gives life to the instance.
Remark 9. In some regards, the mathematical activity (on planet Earth) can be viewed as a decentralized system: mathematicians are agents who choose to adhere to a protocol of logic rules, and whose work should be accepted by other agents as long as it follows the protocol, and there is no central authority entitled to decide what is correct mathematics or who should be able to publish mathematics. Of course, an important difference is that the protocol’s rules are not completely specified in practice and that most nodes are not computers, but humans.

Distributed systems have grown in importance over the last decades, due in large part to the development of the internet. As discussed in Section 1.6.2 below, a special of distributed systems have risen to prominence in the last decade: blockchains. SPRIG is designed to run on such systems.

1.6.2. Blockchain and Cryptocurrency Basics. A blockchain is a specific type of distributed system, where the underlying database is made of a chain of immutable pieces of data called blocks, which grows over time (new blocks are appended as the instance evolves). This feature of blockchain allows for a consensus about time-stamped data to develop.

For instance, Bitcoin is a blockchain instance consisting of blocks describing transactions between accounts (represented by cryptographic public keys), where each node possesses an entire copy of the blockchain; about every 10 minutes a new block is added, which contains the transactions that have been validated in that time interval. The Bitcoin software is designed in such a way that the Bitcoin blockchain can be viewed as a public ledger of amounts of currency units (called bitcoins) owned by each account, and where validated transactions move bitcoins between accounts. More generally, blockchains can be used to implement crypto-currencies, and allow users to trade virtual assets, called tokens. In such systems, accounts are also represented by a cryptographic public key, and transactions from an account submitted to the network are accepted if they are signed by the private key associated with the account and the funds are still available.

A key feature of blockchains such as Bitcoin is that they are based on a publicly available protocol (typically with an open-source reference client implementation); as a result, the laws governing the system are transparent (the ‘code is law’ motto is sometimes used to describe such systems), and it informs the adherence of various nodes and stakeholders to the system.

The implementation of most blockchain systems usually relies on the internet’s infrastructure, and on cryptographic primitives to ensure both the integrity of the data and the authenticity of the transactions. The guarantee of data integrity provided by blockchains is at the heart of such protocols and of the interactions of the agents, who can thus use blockchains as a medium for information exchange and a trusted hub. As a result, a number of blockchains (such as Bitcoin, Ethereum, etc.) have emerged as focal points (or Schelling points) for a growing population of users [Brei17]. Blockchains thus play the role of trusted platforms for agents interested in exchanging information and digital assets.

The fact that the functions of blockchains are executed by code running on nodes has led to many extensions beyond the original application of token ledgers. In particular, the advent of smart contracts, discussed in Section 1.6.3 has opened many new possibilities, including the system proposed in this paper.

1.6.3. Smart Contracts. Smart contracts emerged naturally from the desire to leverage the power of blockchains as decentralized computing platforms to implement programs to perform automated transactions: for e.g. a cryptocurrency, one would like to be able to run a program that automatically moves some asset from an account to another, at a time when certain conditions are met. Such programs run on the blockchain (i.e. are executed by the nodes of the blockchain) and update its state (by contributing new blocks); once running on the blockchain, such a program can be viewed as a contract, guaranteeing the execution of certain transactions if pre-specified conditions are met, hence the name smart contract.

The behavior of a smart contract is specified by its code, together with inputs from the blockchain: for instance, a smart contract may be the recipient of a transaction from another agent on the blockchain, or it may act according to a signed information source (for instance, a trusted information feed from the physical world, known as an ‘oracle’). As a simple example, one can imagine a smart contract implementing a chess competition with automated rewards distribution: players submit their (cryptographically signed) moves to the blockchain, and the outcome is either determined by one party resigning, both parties agreeing to a draw, or by reaching a position computed as terminal by the smart contract.

A number of blockchains have come up with developed infrastructures for smart contracts, including Ethereum, Tezos, Algorand, Avalanche, ... Each of these platforms allows for the writing of smart contracts in fairly rich (sometimes Turing-complete, as for Ethereum) high-level languages, and their execution on the
blockchain against a fee (depending on the complexity of the operations, and payable in the blockchain’s cryptocurrency token).

The smart contract infrastructure has enabled the construction of numerous decentralized platforms, in particular in decentralized finance (as discussed in Section 1.6.4 below): exchanges, betting markets (relying on information feeds), automated market makers, stable coins, etc. can now be run as smart contracts. Such platforms allow for applications that previously needed to rely on the good behavior of expensive (and corruptible) trusted third-parties to enforce the execution of the contracts. Their transparency allows for a detailed risk analysis (see e.g. [AnCh20, AKCNC20]).

Smart contracts platforms can be used to build trusted interactions and consensus, to establish transparent, reliable, and efficient institutions. SPRIG is designed to run on smart contracts (without relying on external oracles), allowing it to aim for such goals, and to be a building block for further decentralized applications (see Section 5.5 below).

1.6.4. Decentralized Markets. We now briefly discuss advances in the applications of smart contracts to decentralized markets, which have experienced a surge of interest in recent years; SPRIG can be viewed as a form of decentralized market for mathematical derivations.

One of the first interesting applications of smart contracts is that of decentralized betting markets (e.g. Augur). In the simplest form of decentralized betting smart contracts, two parties decide to bet at given odds on the outcomes of some future event. To do so, they create a smart contract to which they send their bets (i.e. the contract acts as an escrow) and that can look up a pre-specified, commonly trusted data feed, aka the ‘oracle’; when the event has happened, the smart contract determines the outcome from the data feed and sends the wagered funds to the winner.

For certain betting markets, no external oracle is even needed since the relevant event occurs directly on the blockchain. The power of the blockchain to move assets based on the results of computations has attracted some attention in the mathematical community. Indeed, in principle, checking a proof (in a machine-verifiable format) can be performed by a smart contract (provided that the platform’s language is expressive enough, and enough computing resources are available): in particular, the projects Qeditas [Whit16] and Mathcoin [Su18] are based on such ideas. Both aim at constructing a ledger of agreed-upon mathematical statements where the prospect of financial rewards induces agents to contribute their knowledge.

Mathcoin: In the Mathcoin project proposed by [Su18], the ledger is constructed using a bottom-up approach: agents successively append statements that are logical consequences of the previous ones, starting from the Zermelo-Fraenkel axioms. These statements are appended if the agents provide a valid proof at the machine level. Connected to this growing ledger, a market allows the agents to bet on yet unproven propositions. A user in possession of a result potentially relevant to the proof of such a proposition can buy a derivative paying conditional on the validity of the proposition, then post their result on the ledger. They should subsequently benefit from an appreciation of the price of the derivative. Hence agents are incentivized to contribute their knowledge. However, the Mathcoin protocol and SPRIG differ in several important respects. First, the former produces a ledger of machine-level claims only, which is likely to be impractical for the scientific community as a whole; the machine-language requirement also presumably implies that growing the ledger will be a slow and cumbersome task. By contrast, we use machine-language expansion only as a boundary condition and expect SPRIG to produce concise, human-tailored proofs. Second, its pricing function exposes Mathcoin to an attack where agents are incentivized to post trivial claims. The associated token is initially priced at 0.5. The claimer can then immediately post a proof and collect 1, as their claim was proven. This attack pollutes the blockchain and more importantly drains the public fund, which is intended to reward agents who successfully bet on substantive propositions, effectively rendering the system unusable. While [Su18] mentions this attack, no satisfactory fix is provided.

Qeditas: The Qeditas system of [Whit16] also suggests the construction of a ledger of propositions. There, agents are incentivized to append a result (written in machine language) either to collect bounties from a foundation or an individual interested in the result or because they expect other agents to need it to prove something else in the future and hence to buy its ‘rights’. As for Mathcoin, the bottom-up approach combined with the requirement for complete machine-level proofs implies usability and practicability issues.
To sum up, while projects such as Qeditas or Mathcoin are promising endeavours, their functioning seems at odds with the way mathematicians work, as discussed in Sections 1.2.2 and 1.2.3. SPRIG creates a decentralized market for mathematical derivations, which allows one to avoid evaluating unneeded regions of the proof, while still relying on the ability of smart contracts to arbitrate, in case of disagreement, mathematical truths.

1.7. Outline. As discussed in the previous subsections, mathematical proofs aim at eliciting trust into the validity of statements and transmitting insight into their justifications. While trust relies on the confidence that a machine-verifiable proof could be produced if needed, machine-verifiable proofs are difficult to produce and convey little insight on a per-line basis. As a result, proofs are made at a high, informal level in practice, and are much more concise than their machine-level counterparts, omitting many details. The production and verification of such high-level proofs thus represent a challenge: various agents with various levels of information may disagree on what constitutes a complete and valid proof. As a result, while a proof’s validity is an objective question in principle, in practice it becomes a complex multi-agent problem, where incentive alignment problems may arise.

In this paper, we introduce the SPRIG protocol, which aims at enabling proof submission and verification in a decentralized manner, allowing agents to participate with their various degrees of information, and to be incentivized to behave honestly. It is designed to run on a blockchain, allowing a smart contract to handle the distribution of stakes and bounties, and to serve as an arbiter of debates without relying on any trusted institution.

More precisely, the structure of the following sections is as follows.

- In Section 2, the SPRIG protocol is presented.
  - In Section 2.1, the ideas leading to SPRIG are introduced, starting with a simple game between a claimer and a skeptic (Section 2.1.1), and introducing variants one by one (Sections 2.1.2, 2.1.3, and 2.1.4).
  - In Section 2.2.1, the basic version of SPRIG is described in detail: it is based on a hierarchical proof format, called Claim of Proof Format (Section 2.2.1), and a recursive structure of nested claims and questions (Sections 2.2.2 and 2.2.3).
  - In Section 2.3, the SPRIG protocol is illustrated, through interactions mediated by it, in a number of cases.
  - In Section 2.4, a number of variants of the basic version of SPRIG are presented, and their merits are discussed.
  - In Section 2.5, various aspects pertaining to the blockchain implementation of SPRIG are discussed.
- In Section 3, a game-theoretic perspective on SPRIG is introduced, presenting informally the effects of the incentives structure on the agents’ interactions, and the protocol’s resilience to attacks.
  - In Section 3.1, the strategic interaction between claimers and skeptics, and the results on the proofs constructed in this interaction are discussed.
  - In Section 3.2, the robustness properties of SPRIG against various types of attackers are discussed.
- In Section 4, an in-depth quantitative analysis of a simplified model of SPRIG is presented.
  - In Section 4.1, the simplified model is introduced, which consists of a two-player game of depth 2.
  - In Section 4.2, the solution of the model is presented, with a description of the minima.
  - In Section 4.3, key questions about the reliability of SPRIG are answered in terms of the model’s solution.
  - In Section 4.4, the dynamics and robustness of SPRIG are discussed in light of the analysis of the simplified model.
- In Section 5, a number of applications of SPRIG and outlook for future research are discussed:
  - In Sections 5.1, 5.2, 5.3, a number of direct applications of SPRIG to concrete verification situations are outlined: for theorem proof verification, for the creation of mathematical challenges with bounties, and for the elicitations of decentralized security audits.
In Sections 5.4 and 5.5, possible uses of SPRIG as a platform for new applications are discussed, in particular for the development of automated theorem proving and derivatives markets, allowing agents to inject various types of information. In Section 5.6, a number of other applications, relying on external oracles, are proposed.

2. The SPRIG Protocol

In this section, we describe the protocol at the heart of the present paper, which allows one to construct and incentivize a debate between claimers (provers) and skeptics to determine the validity of a high-level, declarative mathematical derivation: SPRIG, short for Smart Proofs via Recursive Information Gathering.

- In Section 2.1, the key ideas of the protocol are progressively introduced.
- In Section 2.2.1, the claim of proof format upon which SPRIG is based is introduced.
- In Sections 2.2.2, 2.2.3 and 2.3, SPRIG is presented in detail: first, via an informal top-down view, then via a formal bottom-up definition, and finally via illustrative examples.
- In Section 2.4, a number of natural variants and extensions of the basic SPRIG protocol are proposed.
- In Section 2.5, a number of aspects of the blockchain implementation of SPRIG are discussed.

2.1. Prologue. In this prologue, we proceed step by step to introduce SPRIG: we start by introducing a proof-checking protocol with two agents, a claimer and a skeptic, debating the provability of a statement, and using machine-level verification as the ultimate arbiter.

2.1.1. Claimer and Skeptic Debate. We first introduce the Claimer-Skeptic debate as a simple process between two agents, called Claimer (pronoun: she) and Skeptic (pronoun: he):

- Claimer claims to have proven a theorem in the sense of modern mathematical proofs: she has a high-level proof sketch of the theorem (a collection of statements claimed to break down the difficulty of the theorem into smaller pieces), and feels confident she could fill in the details, down to machine level if needed (i.e., she can provide a sequence of statements which follow from each other by the application of elementary rules of logic); at the same time, she cannot or does not want to provide all the details down to the machine level, because the proof would be impractically long.
- Skeptic sees what Claimer shows him. For any statement shown by Claimer, Skeptic has beliefs about the probability that Claimer could indeed, if pressed, provide the details.

For the game to allow for a verification of the proof, Claimer and Skeptic use the following protocol:

- Skeptic may ask for more detail on any proof statement shown by Claimer that is of higher level than machine-level detail.
- Skeptic may invalidate the proof by revealing a mistake in the machine-level proof details.
- Claimer cannot indefinitely propose high-level proofs: if pressed to give details down a certain number of levels (say 9), she must reach machine-level details, or her proof is considered invalid.
- Claimer has explicit bounds on the size of the proof sketches and of the machine-level details.
- After Claimer has published a proof sketch, Skeptic has a limited (fixed in advance) amount of time to request details, and after Skeptic has asked for details, Claimer has a limited (fixed in advance) amount of time to provide details on a proof statement.

Claimer and Skeptic play a role that is somewhat akin to that of an author and a reviewer; in practice, Skeptic can just be Claimer’s critical thinking, which probes for possible weaknesses of Claimer’s proof, to assess Claimer’s confidence in her proof being indeed complete.

The assessment made by this protocol is whether Claimer can provide details in the proofs quickly enough: there is indeed a limit in terms of how much into the details she can go (to avoid a case where Claimer would just state tautologies, instead of giving proofs).

A central weakness of the above protocol arises if there is no alignment between Claimer and Skeptic: Skeptic may start bombarding Claimer with useless questions, or conversely not ask any question at all. In order to prevent this, incentives can be put in place, as explained in Section 2.1.2 below.

2.1.2. Incentives: Claimer’s Stakes and Skeptic’s Bounties. In order to avoid the alignment problem in the debate between Claimer and Skeptic introduced in Section 2.1.1, incentives can be added:

- Claimer must put a stake with the claim: this stake encourages Skeptic to ask questions.
• Skeptic must pay a bounty to ask a question: this prevents Skeptic from bombing Claimer with useless questions.
• In case Claimer can answer a question from Skeptic, she gets Skeptic’s bounty: this compensates her for the work required to answer.
• In case Claimer cannot correctly answer a question from Skeptic, Skeptic gets Claimer’s stake.

Setting the parameters correctly will incentivize Claimer and Skeptic to do their work: Skeptic will only ask questions about the places where he feels there is a reasonably good chance Claimer cannot fill in the details; conversely, he will not ask about obvious points in the proof. This selective revealing of the proof may be useful to an external observer as well: the details that are revealed are only the interesting, non-trivial ones. At each stage, one side may disagree with the other; the first one to stop debating loses, unless we reach the machine level, where it is Claimer’s burden to prove her statement in machine-level language (otherwise she loses).

In this sense, the machine level serves as the ultimate arbiter of who is right; interestingly, if the incentives are set correctly, a debate between a rational claimer and a rational skeptic would probably end before reaching the machine level (as in a game between chess masters where a checkmate position is almost never reached: the losing side will resign beforehand).

2.1.3. Many Claimers and Skeptics. In Sections 2.1.1 and 2.1.2 we had a debate between only two agents (Claimer and Skeptic). With good incentives, the roles of Claimer and Skeptic can in fact be completely decentralized: we will have a market of agents, claimers and skeptics (where an agent can play both roles at various levels). The skeptics can ask details about any published proof sketch (by paying an upfront bounty, being the first to ask, and doing so within time limits); conversely, the claimers can propose proof sketches to any unanswered question (by paying an upfront stake and doing so within time limits). The claimer’s stakes are actually split in two: an ‘up’ stake and ‘down’ stake: if the claim of proof ends up being invalidated, the ‘up’ stake goes to the question that the claim of proof was trying to answer, and the ‘down’ stake to the question that first invalidated the claim.

This structure allows various agents to perform in various capacities: for instance, agents with a good high-level vision can propose high-level proof sketches, while agents who are more comfortable with low-level details will provide proofs of sub-sub-claims, etc. Again, if the bounties and stakes and times are set well, each agent will inject their own information into the system by either proposing claim of proofs and questions reflecting their beliefs, in a fully decentralized way.

2.1.4. Question as Root of the Process. A small variant of the protocol can be introduced in the case where ‘we start with Skeptic’: Skeptic may start by putting a bounty (for instance, because he is interested in sponsoring research about a question he cares about), to which claimers may propose proof sketches (paying an ‘up’ and ‘down’ stake upfront).

With this scheme, claimers should be able to submit several proofs for a statement, while compensating skeptics who may debunk wrong proofs. The rest of the process is completely symmetrical.

This concludes the prelude part of the SPRIG protocol description. In Section 2.2 a precise formalization of the SPRIG is detailed. In Section 2.4 a number of variants and extensions are presented. In Section 2.5 questions associated with the blockchain implementation are discussed.

2.2. SPRIG Protocol Description. Building upon the insights of the previous sections, we now formalize the Smart Proof by Recursive Information Gathering (SPRIG) protocol. At the root of SPRIG is either a question or a claim of proof; the protocol then builds a tree starting from the root, with questions following claims of proof and vice versa. All the questions and claims of proof consist of statements written in a formal mathematical language, leaving no room for ambiguity (see Section 2.2.1).

2.2.1. Claim of Proof Format. SPRIG is based on the communication of unambiguous mathematical statements, written in a formal proof language. The description we give here is agnostic of the specific formal system; our format description can be implemented using a declarative proof language such as Mizar, Isar, or Lean.

The format we describe is based on hierarchical proofs. A complete machine-verifiable proof of a statement is a proof that can be structured as a tree with nodes corresponding to statements, and with leaves corresponding to machine-verifiable statements. SPRIG allows agents to query and provide a subtree of the
complete proof tree that is large enough to reach a consensus about whether or not the tree can be completed into a complete tree (with given size and time constraints), as discussed in Sections 2.2.2 and 2.2.3 below.

Recalling the definition of Section 1.2.5 and setting aside the question of definitions for a moment (this will be discussed in Definition below 17 below), the structured proof format of level $L \geq 1$ is that of a tree with the following structure:

- The root (‘top-level’) is the statement $S_*: A_* \implies C_*$ itself, where $A_*$ includes axioms and accepted statements used to derive the conclusion $C_*$.
- For each non-leaf (‘high-level’) statement $S: A \implies C$, its children $(S_j)_{j=1,\ldots,k}$ are statements $A_j \implies C_j$, where $A_j$ is of the form
  \[
  A_j = A \cup \{C_i \mid i \in I_j\}
  \]
  for some $I_j \subset \{1, \ldots, j - 1\}$, where $C_k = C$.
- For each leaf (‘low-level’) statement $S: A \implies C$, a machine-verifiable proof is provided.
- The tree height (distance between the root and leaves) is at most $L \geq 1$.

Remark 10. In our framework, an assumption $A$ may include the introduction of notation (e.g. ‘let $x$ be such that ...’); some mechanism for the resolution of overloaded symbols is naturally needed (but not discussed here, being an implementation detail).

Remark 11. Theorems are often explicitly of the form $T: \alpha \implies \gamma$ (e.g. we could have $\alpha$ corresponding to ‘$f$ is a holomorphic function on $\mathbb{C}$’ and $\gamma$ corresponding to ‘$f$ has a convergent power series expansion on $\mathbb{C}$’). In such a case, we could write the statement with $\gamma = C_*$, and $A_*$ would include $\alpha$, as well as the list of axioms and assumed results used to derive $\gamma$.

Remark 12. Various formats of proof fit in this framework, including proofs by contradictions, etc. See Section 7 in the Appendix for examples.

To make the writing of statements effective, definitions are needed, which allow one to reserve notation to refer to properties of objects.

Definition 13. A collection of definitions $D$ introduces symbols specifying properties of objects, written in formal language, and specifies references from which other definitions can be imported.

Remark 14. For instance, a definition could be ‘is-group($G$, op)’ which would imply that $G$ is indeed a set, op is indeed a function $G \times G \to G$ and that the various properties of the op operation are satisfied.

Remark 15. In the format as we specify it, definitions need to be syntactically correct, but not necessarily consistent; they are to be thought of as mere shortcuts enabling more concise and clearer statement formulations.

Remark 16. In mathematics, definitions of objects such as Riemann’s zeta function $\zeta$ by a series such as $\sum_{n=1}^{\infty} n^{-s}$ on $\mathbb{H}_1 := \{s \in \mathbb{C} : \Re(s) > 1\}$ involve a lemma (saying the series converges on $\mathbb{H}_1$); in our framework, we would instead define a property ‘is-zeta-on-$\mathbb{H}_1$’ for a function $f: \mathbb{H}_1 \to \mathbb{C}$ which would mean that the series $\sum_{n=1}^{\infty} n^{-s}$ converges for any $s \in \mathbb{H}_1$ and its value is $f(s)$; a statement (needed to e.g. prove the prime number theorem) would then assert that there exists a unique function $\mathbb{H}_1 \to \mathbb{C}$ that satisfies the ‘is-zeta-on-$\mathbb{H}_1$’ property; a lemma such as $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ would then read ’fix a function $\zeta: \mathbb{H}_1 \to \mathbb{C}$; assume that $\zeta$ satisfies the ‘is-zeta-on-$\mathbb{H}_1$’ property; then for any $s \in \mathbb{H}_1$, we have $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$. A specific language may include shorthands to make the alleviate the notation, of course.

Adding definitions to the proof construction, we obtain the following format for statements:

Definition 17. The Claim of Proof Format (see Figure 2.1) consists of statements, high-level claims of proof, and machine-level claims of proof, associated with a fixed logic system $\Lambda$ (as in Section 1.3.2).

- A statement $S$ consists of a context $\Gamma$ of definitions and an implication $A \implies C$.
- A claim of proof $P_L$ of level $L \geq 1$ of a statement $S$ with context $\Gamma$ and implication $A \implies C$ consists of a chain of reasoning $D, S_1, \ldots, S_k$, where
  - $D$ is a collection of definitions (as in Definition 13).
  - $(S_j)_{j=1,\ldots,k}$ are statements with contexts $\Gamma \cup D$ and where $S_j$ is the implication $A_j \implies C_j$, such that...
\[ A \implies C \]

\[ A_1 \implies C_1 \quad A_2 \implies C_2 \quad \cdots \quad A_k \implies C_k \]

Figure 2.1. Claim of Proof Format. The boxes’ left sides correspond to assumptions, while the boxes’ right sides correspond to conclusions. The light dashed lines represent possible assumption dependencies.

\[ A_j \text{ is of the form } A \cup \{ C_i \text{ for } i \in I_j \} \text{ for some } I_j \subset \{1, \ldots, j-1\} \]

\[ C_k = C. \]

Remark 18. A high-level claim of proof comes with the following implicit claim: for each \( j = 1, \ldots, k \), deriving \( S_j \) is significantly easier than deriving \( S_* \).

The length of a claim of proof is measured by its aggregated length measure \( \mu \) that is an increasing function of the length of its chain of reasoning (measured in number of symbols in the language in which it is expressed, possibly with a weight associated with different symbols); for instance, a natural choice for \( \mu \) is simply the total length of the statements measured in number of symbols.

Remark 19.

2.2.2. Top-Down Informal View of SPRIG. Generalizing the examples, we now give an informal description of SPRIG, in the order in which the interactions between agents take place.

Given a context \( \Gamma \) and an aggregated length measure \( \mu \) the validation mechanism goes as follows:

1. Given a parameter \( L \geq 1 \), the root of the process consists of
   
   (a) either a claim \( C_L \) of level \( L \) (a statement \( S \) with context \( \Gamma \), together with a claim of proof \( P_L \) of level \( L \), as in Definition 2.2.1) with a pre-specified stake \( \sigma_L = \sigma^+_L \);
   
   (b) or a question \( Q_L \) (i.e. a statement \( S \) with context \( \Gamma \)) with status ‘unanswered’ and a pre-specified bounty \( \beta_L \).

   The root specifies maximum proof lengths \( \lambda_{L-1}, \ldots, \lambda_0 \), stakes \( \sigma^+_L, \sigma^+_L, \ldots, \sigma^+_1, \sigma^+_1, \sigma^+_0, \sigma_0 \) and bounties \( \beta_{L-1}, \ldots, \beta_0 \) to be used at each of the lower levels.

2. For \( \ell \geq 0 \), a claimer might attempt to answer a question \( Q_\ell = (S) \) by producing a claim \( C = (S, P) \) of level \( \ell' \in \{\ell, 0\} \), i.e. by providing:
   
   - A claim of proof \( P \) of level \( \ell' \) of the statement \( S \).
   
   - If \( \ell' = \ell \):
     
     - The claim of proof \( P \) must be of total length at most \( \mu(P) \leq \lambda_\ell \).
     
     - The claimer must lock a stake \( \sigma^+_\ell, \sigma^+_\ell \) (with \( \sigma^+_\ell = 0 \) if \( \ell = L \)).
   
   - If \( \ell' = 0 \):
     
     - The claim of proof \( P \) must be of length at most \( \lambda_0 \).
     
     - The claimer must lock a stake \( \sigma^+_0 \) and pay a computation cost \( c_0 \).
     
     - In this case, if the claim of proof is (automatically) validated, it gets the status ‘validated’, otherwise, it gets the status ‘invalidated’.
In all cases:
- All claims of proof addressing \( Q_\ell \) must be proposed within a response time \( \tau_\ell \) of \( Q_\ell \)'s publication.
- If a claim of proof addressing \( Q_\ell \) gets the status ‘validated’, then this claim is said to be answering \( Q_\ell \) and \( Q_\ell \) gets the status ‘answered’; if no such claim exists, then \( Q_\ell \) gets the status ‘unanswered’.

(3) For \( \ell \geq 1 \), a skeptic might dispute a level-\( \ell \) claim \( C_\ell = (S, P_\ell) \) by asking a question \( Q_{\ell-1} = (S) \) where \( S \) is one of the statements appearing in the claim of proof \( P_\ell \).

(a) The skeptic must lock a bounty \( \beta_\ell \) associated with the question.
(b) All questions about \( C_\ell \) must be asked within the verification time \( \theta_\ell \) of \( C_\ell \)'s publication.
(c) If a question originating from the claim gets the status ‘unanswered’, then this question is said to be a defeating question, and the claim gets the status ‘invalidated’; if no such question exists, the claim gets the status ‘validated’.

The incentivization mechanism for the proposal is based on bounties \( (\beta_\ell) \) and stakes \( (\sigma_\ell^+, \sigma_\ell^-) \) as follows:

(1) If a claim \( C_\ell \) addressing a question \( Q_\ell \) is the first one to get the status ‘validated’, then \( C_\ell \) receives the bounty \( \beta_\ell \) from \( Q_\ell \).
(2) If a claim \( C_\ell \) addressing a question \( Q_\ell \) gets the status ‘invalidated’, then \( Q_\ell \) receives the stake \( \sigma_\ell^+ \) from \( C_\ell \).
(3) If a question \( Q_\ell \) disputing a claim \( C_{\ell+1} \) is the first one to get the status ‘unanswered’, then \( Q_\ell \) receives the stake \( \sigma_{\ell+1}^- \) from \( C_\ell \).

In a nutshell, there are debates between claimers (agents providing claims of proof for statements) and skeptics (agents asking questions about proofs), where each side debates while having some ‘skin in the game’: claimers and skeptics must pay upfront to play, and they will be paid back if their point is valid (i.e. the claim is validated, or the question remains unanswered) and possibly further compensated (for a question, if it is the first to defeat the claim it originates from; for a claim, if it the first to answer the question it originates from). Winning occurs when one of the sides concedes, and in case no side wants to concede, after at most \( D \) steps, one reaches the point where only machine-level proofs are accepted; hence the ultimate judge is an algorithm that runs the checking of the machine-level proof.

Claimers and skeptics have dual roles. Let us simply point out the following differences:

- The skeptics only have a limited number of possible moves (limited by the number of steps in the claims of proof that have been published), while the provers have a virtually infinite number of possible moves (they can provide any purported claim of proof).
- While invalidated claims must share their stake to the question they address (if it exists) and of the first defeating question, the answered questions must only pay their bounties to the first validated claim of proof closing them.

Remark 20. The protocol interaction does not necessarily stop immediately after the root status has been set. For instance, a claim may be invalidated by a first unanswered question, but the status of questions that were raised after that first question may still be undecided; the protocol interaction must run until all questions and claims get a status.

2.2.3. Formal Description. We now give the formal description of the SPRIG protocol introduced in Section 2.2.2.

A context \( \Gamma \) is fixed by the root, as well as a high-level proof aggregated length measure \( \mu \).

We work with claims \( C_\ell \) and questions \( Q_\ell \) of levels \( \ell = 0, 1, \ldots \). We denote by \( C_\ell, Q_\ell \) the corresponding types (and we write e.g. \( C_\ell \in C_\ell \) to indicate that \( C_\ell \) is a claim of level \( C_\ell \)).

For simplicity, the protocol assumes that questions and claims are submitted in continuous time and cannot appear simultaneously. Similarly, we assume that the claims and questions are published at the moment of their creation. See Section 2.3 for a discussion of this issue in the context of smart contracts.

Type \( C_\ell \) for \( \ell \geq 1 \).
- Data:
  - An origin question \( Q_\ell \in Q_\ell \cup \{\text{none}\} \) (we say that the claim originates from \( Q_\ell \)).
  - A mathematical statement \( S \):
    * The statement \( S \) of the origin \( Q_\ell \) if \( Q_\ell \neq \text{none} \).
    * An independent mathematical statement if \( Q_\ell = \text{none} \).
A claim of proof $P_\ell = D, S_1, \ldots, S_k$ of level $\ell$ of $S$ of aggregated length $\mu(P_\ell) \leq \lambda_\ell$,
- Parameters $\pi_{C_\ell}$: max-length $\lambda_\ell$, stake pair $\left(\sigma^+_\ell, \sigma^-\ell\right)$ (with $\sigma^-\ell = 0$ if $Q_\ell = \text{none}$), verification time $\theta_\ell$, $Q_{\ell-1}$ parameters $\pi_{C_{\ell-1}}$ if $\ell \geq 1$ (bounty $\beta_{\ell-1}$, response time $\tau_{d-1}$, $\pi_{\ell-1}$ parameters).

- Necessary creation of initial funds: $\sigma^+_\ell + \sigma^-\ell$.

- Status outcome:
  - The claim gets the status ‘invalidated’ if there exists a defeating question, i.e. a question $Q_{\ell-1} \in Q_{\ell-1}$ that
    (1) originates from the claim and disputes from one of statements $S_1, \ldots, S_k$ of its claim of proof;
    (2) respects the parameters $\pi_{Q_{\ell-1}}$ (i.e. whose parameters are $\pi_{Q_{\ell-1}}$);
    (3) has the appropriate creation funds;
    (4) appears less than $\theta_\ell$ units of time after the publication of the claim;
    (5) gets the status ‘unanswered’.
  - Otherwise (if no defeating question exists): the claim has status ‘validated’.

- Stakes/bounties outcomes:
  - If the claim gets the status ‘validated’:
    * the stakes $\sigma^+_\ell, \sigma^-\ell$ are reimbursed to the claim owner;
    * if $Q_\ell \neq \text{none}$, and it is the first descendent of $Q_\ell$ to get the ‘validated’ status, the bounty $\beta_\ell$ of $Q_\ell$ is paid to the claim.
  - If the claim gets the status ‘invalidated’:
    * the stake $\sigma^+_\ell$ is paid to the $Q_\ell$ origin, if there is one;
    * the stake $\sigma^-\ell$ is paid to the first defeating question owner.

Type $Q_\ell$ for $\ell \geq 0$.

- Parameters $\pi_{Q_\ell}$: bounty $\beta_\ell$, response time $\tau_\ell$, $C_\ell$ parameters $\pi_{C_\ell}$ (max-length $\lambda_\ell$, stake pair $\left(\sigma^+_\ell, \sigma^-\ell\right)$), verification time $\theta_\ell$, $Q_{\ell-1}$ parameters $\pi_{Q_{\ell-1}}$ if $\ell \geq 1$.

- Necessary creation of initial funds: $\beta_\ell$.

- Data:
  - An origin claim $C_{\ell+1} \in C_{\ell+1} \cup \{\text{none}\}$, we say that the question originates from $C_{\ell+1}$.
  - A mathematical statement $S$ with context $\Gamma$:
    * One of the statements $S$ in the claim of proof $P_{\ell+1} = D, S_1, \ldots, S_k$ of $C_{\ell+1}$ if $C_{\ell+1} \neq \text{none}$; in such a case, we say that the question disputes $S$.
    * An independent mathematical statement if $C_{\ell+1} = \text{none}$.

- Outcome:
  - The stake $\sigma^+_\ell$ is paid to the question owner by any claim $C_\ell \in C_\ell$ that
    (1) originates from the question;
    (2) respects the $\pi_{C_\ell}$ parameters (i.e. whose parameters are $\pi_{C_\ell}$);
    (3) has the appropriate creation funds;
    (4) appears less than $\tau_\ell$ unit of time after the publication of the question;
    (5) gets the status ‘invalidated’.
  - If there is a claim $C \in C_\ell \cup C_0$ that
    (1) originates from the question;
    (2) respects the $\pi_{C_\ell}$ parameters;
    (3) has the appropriate creation funds;
    (4) appears less than $\tau_\ell$ unit of time after the publication of the question
    (5) gets the status ‘validated’
    then the question gets the status ‘answered’.

Otherwise (i.e. no such claim has appeared) the question is marked as ‘unanswered’.

- Stakes/bounties outcomes:
  (1) If the question gets the status ‘answered’:
    - The owner of the first validated claim $C \in C_\ell \cup C_0$ originating from the question gets the bounty $\beta_\ell$;
    - The next such claims get nothing (but lose nothing).
Figure 2.2. A basic validated claim of proof (top horizontal segment): one question was raised (vertical segment on the left). In answer to this question, a first claim of proof was proposed (middle horizontal segment) and invalidated by two unanswered questions (two short vertical segments), but then a second claim of proof was proposed, which was validated as no question was raised about it.

(2) If the question gets the status ‘unanswered’:
- The bounty $\beta d$ is reimbursed to the question owner.
- If the question is the first question originating from $C_{\ell+1}$ to get the ‘unanswered’ status, the stake $\sigma_{\ell+1}^\downarrow$ is paid by the claim to the question owner;
- The next such questions get nothing (but lose nothing).

Type $C_0$.
- Data:
  - An origin question $Q_\ell \in Q_\ell$ (we say that the claim originates from $Q_\ell$) for $\ell \geq 0$.
  - The statement $S$ of the origin $Q_\ell$.
  - A machine-verifiable claim of proof $P_0$ of length $\leq \lambda_0$.
  - Parameters: max-length $\lambda_0$, stake $\sigma_0^\uparrow$, computation cost $c_0$.
- Necessary creation initial funds: $\sigma_0^\uparrow + c_0$.
- Status outcome:
  - If $P_0$ is validated by the computer verification system, the claim gets the status ‘validated’.
  - Otherwise, the claim gets the status ‘invalidated’.
- Stakes/bounties outcomes:
  - If the claim gets the status ‘validated’: the stake $\sigma_0^\uparrow$ is reimbursed to the claim owner.
  - If the claim gets the status ‘invalidated’: the stake $\sigma_0^\uparrow$ is paid to the origin.
  - The computation cost $c_0$ is burnt.

Remark 21. As each claim and question contains the parameters that the claim and questions originating from it must respect, the parameters of the entire interaction defined by the SPRIG protocol are specified by the parameters of the root.

2.3. Illustrations Of SPRIG. In this subsection, we present illustrations of SPRIG-based interactions.
- Claims of proof are depicted as horizontal segments, with dashed segments representing machine-level proofs.
- Questions on parts of the proofs are represented as vertical segments.
- Small teal segments on the claims of proof/questions segments represent the end of the allotted response time.
- The validation status of a claim of proof at the end of the interaction is represented at the right end of the line (validated: ✓, invalidated: ×). Claims of proof written in machine-level language are marked with a green diamond.
- Questions are represented as vertical lines (emanating from a statement in a claim of proof), and their eventual status is represented at the bottom end of the line (answered: ✓, unanswered: ×).

Four basic examples are provided (Figures 2.2–2.5), which are subparts of two examples of SPRIG runs, one with a claim of proof as root (Figure 2.6) and another one with a question as root (Figure 2.7).
2.3. A basic invalidated claim of proof (top horizontal segment): one question was raised (vertical segment on the left); as an answer to this question, a claim of proof was proposed (second horizontal segment from the top); the claim of proof was itself questioned, and an additional claim of proof was proposed as an answer (second horizontal segment from the bottom); two questions were raised about that additional claim of proof (the two bottom-most vertical segments), the first of which was unanswered, and the second of which was answered by a validated claim of proof (bottom-most horizontal segment). As a result, the claim was invalidated.

2.4. A basic answered question (left-most vertical segment): a first claim of proof was proposed (top-most horizontal segment), which was then invalidated by an unanswered question, but then a second claim of proof was proposed which was validated after a question was raised (bottom-most vertical segment), and that question was answered by a claim (bottom-most horizontal segment) which itself was not questioned.

2.5. A basic unanswered question (left-most vertical segment). A claim of proof was proposed (top-most horizontal segment), which resisted a first question (second left-most vertical segment), as that question was answered by an unquestioned claim of proof, but which did not resist the second question, as the only claim of proof answering it (right-most horizontal segment) was invalidated by an unanswered question.

2.4. Variants and Extensions. In this section, we present a number of variants and extensions of the SPRIG protocol, which can be enabled to optimize for various goals under certain environments. Many more variants and extensions can in principle be considered, but we focus here on the ones that appear to be the most naturally motivated and that live directly on the protocol itself. A number of further interesting
In this case, the claim was eventually validated. Time goes horizontally in the lifetime of claims and vertically in the lifetime of questions, but it is not represented at scale. In this figure, four questions were asked and successfully answered; for the first question, a first claim of proof was proposed, which was then invalidated, followed by one which was later validated; the same is true for the fourth question that was asked. The second and third questions were answered by claims of proof which were later validated. The second claim of proof proposed for the first question was itself only validated after the first question about it saw two claims of proof (a first one, which was invalidated after going down 3 more levels), and a second one, which was validated after questions were asked and answered at the machine level.

Extensions can be then built upon protocol instances, in particular, decentralized markets for derivatives can rely on using protocol instances as oracles, as discussed in Section 5.5. While these variants and extensions appear to be promising, their detailed analysis is significantly more complex and goes beyond the scope of this article.

2.4.1. Time-Varying Stakes and Bounties. Intuitively (and as discussed in Sections 3 and 4 below), the trust in the fact that a claim of proof is correct depends on how favorable the incentives are to those asking questions: if a skeptic has little to gain, and too much to risk in asking questions (in terms of explicit incentives), he may not ask a question about a claim, unless he is very confident that the question cannot be answered (and he may not want invest time and energy to find questions about claims).

An agent publishing a claim of proof as a means to validate it may wish to establish a high level of trust in it (by offering a high stake, to be paid to a skeptic successfully challenging her claim), but may herself not be very confident in its ultimate validity: there may be a fairly obvious mistake (for instance of notation), and she would not want to pay a high price for an obvious mistake. Similarly, an organization may want to incentivize the solution to a given open problem, but may not want to pay too much for it, if the question turns out to be obvious.

A solution to this is to rely on time-dependent stakes and bounties, in a way similar to Dutch auctions: start with conditions that are very favorable to the defending side (the side at the root of the interaction), and make them more and more favorable for the challenging side. If there is an obvious challenge (i.e. a question of whether the root is a claim of proof, if the root is a question), challenging agents will still be incentivized to pose it as soon as possible (rather than to wait to increase their reward), as they are in competition with other challenging agents.
Figure 2.7. Illustration of the SPRIG protocol when the root is a question. The same notation convention is used as in Figure 2.6. In this case, the question was answered by the third claim of proof (and the interaction was ended as a result); a fourth claim of proof was submitted, but its status was not yet decided at the time the interaction has ended; all question marks indicate statuses that are not yet assigned.

At the same time, the owner of a claim may want to get back some of the liquidity that she locked into the smart contract after a while, while still incentivizing the search for mistakes in the proof; in this case, having the stakes decrease over time could prove useful.

In any case, the bounty and stake parameters at all levels of SPRIG can be replaced by time-evolving functions, which must be specified when the protocol instance is created.
2.4.2. **Generalized Bounties and Stakes.** In the basic version of SPRIG, a bounty $\beta$ must be locked to ask a question, which will be paid to the first validated claim of proof answering it; dually, a stake pair $(\sigma^\uparrow, \sigma^\downarrow)$ must be locked to propose a claim of proof, where, in case of invalidation, the part $\sigma^\uparrow$ is paid to the question the claim of proof was trying to answer, and the part $\sigma^\downarrow$ is paid to the first question that invalidated the claim of proof. A number of variants can be introduced in terms of distributions for stakes and bounties.

It may seem natural to propose an ‘upwards’ bounty $\beta^\uparrow$ to be paid to the claim of proof a question derives from; however, as discussed in Section*, such upwards bounties may however open the possibilities for certain attacks (called the ‘Plagiarist’s attack’ below).

Another possible extension to incentivize agents to disclose more information into challenging a claim consists in splitting the stake $\sigma^\downarrow$ into $p$ shares $\sigma^\downarrow_1, \ldots, \sigma^\downarrow_p$ for the first $p$ unanswered questions about the claim (this may require forbidding to ask the same question more than once). While a single unanswered question suffices to invalidate a claim, a stake may incentivize agents to look more closely at a claim of proof even after a first question has been raised. On the other hand, rewarding multiple claims of proof that successfully answer a question appears to be much more delicate: this opens the possibility to Plagiarist’s attacks that are difficult to counter.

2.4.3. **Claim-of-Proof-Dependent Parameters.** In the basic version of SPRIG, a fixed stake is associated with a claim of proof, together with a fixed amount of time to raise questions, and a fixed limit on the claim of proof length. While this setup incentivizes the publishing of correct claims of proof and incentivizes agents who spot a problem in a claim of proof to question it, it does account for the fact that claims of proof may be more or less hard to read. A way to account for this in the protocol is to allow for the stakes/bounties and verification times to depend on the complexity of the claims of proof submitted: in particular, longer claims of proof (as measured in the lengths of their claims, or the number of them) may warrant longer verification times (or the verification could be encouraged by requiring higher stakes); for claims of proof consisting of many statements, one may want to reduce the bounty to ask a question.

For such variants, the parameters set by the root owners should be replaced by functions (also set by the root owner) of the proof complexity and number of claims.

2.4.4. **Synchronous SPRIG.** The original version of SPRIG is intrinsically asynchronous in nature: each question and each claim of proof runs on an independent clock, and new questions or claims of proof may come at an arbitrary time, starting their own clocks. While this process incentivizes agents to disclose information as soon as they have it, and is more efficient at closing obvious cases sooner rather than later, a number of situations may suggest using a synchronous variant of the protocol: after a question is posed (say), a fixed response time is given to post claims of proof. Claims of proof are hidden until the response time is elapsed, at which point they are revealed. Then, a round of questioning starts, giving a fixed amount of time to ask questions about the various claims of proof; the questions are also hidden and revealed when the questioning round ends. Then a third round starts, in which claims of proof are proposed in response to the questions can be posted and revealed at the end of the third round.

This variant of the protocol may be useful for revealing information at specific times (e.g. yearly contest) or to make the best uses of a community’s resources (perhaps the best critics of a theorem’s claim of proof are other agents trying to submit at the same time their own claim of proof, and they can focus on criticizing others’ proofs some of the time, while focussing on answering questions the rest of the time).

2.4.5. **Exclusive Disclosure.** In the basic version of SPRIG, agents ask questions when they doubt the validity of a step in a claim of proof. However, an intrinsic motivation may come from a question-raiser: that they are curious, for independent reasons, about the answer to the question. In such a setup, it may be useful to guarantee to the first question-raiser an exclusive access to the answers to a question for a brief amount of time, before the answer is made public (the poster of answers would be incentivized not to disclose their answers to other parties in the exclusive time period, as it limits their attack surface).

2.4.6. **Expedited Validation or Invalidation.** In a number of cases, it may be desirable to expedite a validation process: a variant can be introduced that allows a claimer to reduce the validation time in exchange for higher stakes and/or lower response times for the lower levels. This feature may prove desirable in certain situations, but it must be dealt with carefully in order to not introduce flaws (leading to the validation of a proof that should not have been validated).
2.4.7. Open Questions and Multi-Question Bounties. One may want to put at the root of a SPRIG instance both a question and its negation: for instance, the Clay Institute offers a prize for the first proof of $P = NP$ or of $P \neq NP$. In this case, a single bounty (1M USD) is put at the root of the two questions, and it should go to the first validated claim of proof for either question that gets validated (as a result there is no bounty left for the other question; this should not be a problem as a statement and its negation should not have both validated proofs!).

More generally, an institute may want to put a single bounty for the first agent answering one of a list of questions; as soon as one of the questions is answered, there will be no bounty left for answering other questions will be cancelled. Even more generally, a limited number $K$ of bounties could be made available for the first $K$ questions answered, after which there will be no bounties left for answering the other questions.

2.4.8. Stake-Sharing and Bounty-Sharing. A possible downside of the system is the barrier of entry for a participating agent, who may not have enough funds, while at the same time possessing useful information.

For questions, such an agent could put a partial bounty, and wait until this bounty is completed by other agents: at that moment, the question is formally asked, and should the question be the first to invalidate the claim of proof, the stake will be shared by the agents who put the bounty, at the pro-rata of their bounty share.

For claims of proof, such an agent could put an encrypted claim of proof with a partial stake, try to find other agents who also believe in it to complete the stake (for instance, by proving her identity and using her reputation), and decrypt it when the stake is full (thus avoiding a plagiarist to copy her claim of proof); again, in this case, if there is a bounty to be won, it will be shared among the stake holders at the pro-rata of their stake (or according to some other pre-determined rule).

2.5. Blockchain Implementation. The SPRIG protocol presented in Section 2.2 and its variants and extensions presented in Section 2.4 are designed so it can be implemented on a blockchain, in a fully decentralized manner, without reliance on an external oracle. In this subsection, we discuss a number of design questions related to the implementation of the protocol on a blockchain infrastructure.

2.5.1. Automated Proof Settlement. As emphasized in the top-down view of SPRIG (Section 2.2.2), what settles the boundary conditions of the protocol (and hence ensures its good functioning) is the presence of an ultimate arbiter, in the form of a computer-based system to verify machine-level claims. Implementing SPRIG on a blockchain thus requires the ability to perform the necessary computations on the blockchain to ensure transparency of the result of the computation (or to offload the computation to another blockchain, or to find a verifiable way to ensure the relevant computations were done off-chain).

While a large number of powerful proof verification programs are available (see Section 1.3.2), their emphasis is usually on helping users to write proofs. The most desirable features for a blockchain-based proof verification system are somewhat different.

- **Low** memory usage: ultimately, a smart contract needs to be able to verify any step of the computation. The sequence of computations may not need to be performed entirely on-chain, as long as it is auditable (see e.g. [EbHe18]).
- **Syntax** making the writing of definitions and statements (as specified in Section 2.2.1) should be transparent to the agents. This may be helped by the development of open off-chain statement translators, assisting the users in the formalization of definitions and statements.

On the other end, the system living on the blockchain can be very primitive in its ability to assist users to write down proofs; should debates ever go down to the machine level, proof assistants can in principle be used off-chain to propose machine-level claims of proof. Still, if incentives are set well and agents are rational, the presence of a well-functioning proof system will only serve as a deterrent: close enough to the machine level, rational skeptics and agents should already agree on the existence of a machine-level proof and the side that is wrong is incentivized to concede early.

As a result of the above design goals and considerations, the development of a proof verification system tailored for them seems desirable.

2.5.2. Timing and Concurrency Issues. The block-based structure of blockchains serves crucially as a timestamp mechanism to validate the transactions: the consensus on the order of the blocks serves as the measure of the passage of time (and crucially at determining anteriority of modifications submitted to the blockchain:
this is in particular what prevents double-spending with Bitcoin). As a result, the natural time unit of a blockchain is the number of blocks emitted so far.

In the description of SPRIG, time is treated as a continuous resource, and time is asynchronous (except in the synchronous variant discussed in Section 2.4.4). For blockchains with a sufficiently short validation time, and for non-trivial enough problems discussed with the protocol, it is unlikely that two questions are asked simultaneously (i.e. in the same block); however, in such a case, a rule should be specified. Still, it is important to keep this granularity in mind to avoid attacks by e.g. a quick plagiarist who could copy a claim of proof and try to push it into the same block; the solution in such a case is simply to make claimers first commit a signed and encrypted version of their proof at least one block before disclosing its content.

2.5.3. **Stakes and Bounties Lock**. For the protocol implementation, the agents need to lock their bounties and stakes in the smart contract for a long time. In case they need liquidity, it is possible for them to resell (i.e. transfer ownership of) their stake in the contract to a third party.

For the variant with time-varying stakes and bounties 2.4.1, a number of challenges also arise: either the staker should put the maximum amount of capital upfront or they could be mandated to inject additional capital as time passes (at the risk of losing their stake if they don’t do so).

3. **Informal Game Theoretic Discussion**

3.1. **Strategic Interactions and Protocol Outcome**. As mentioned in Section 1.5, understanding how agents interact through the SPRIG protocol, as well as interpreting the validation process outcome requires taking an economic perspective. Indeed, while SPRIG is a set of rules that, given the decisions of various users, deterministically defines a tree, allocates rewards, and eventually settles the status of the claims and questions, these very decisions are in essence strategic.

A complete characterization of the strategic interaction between users is out of reach as they depend on a variety of elusive elements. First, we do not have access to the real world’s information structure (the information set of each user and their beliefs about others’ information sets), which at any rate would be highly intricate. Second, this information structure is endogenous since the incentive scheme can lead agents to work and gather additional information in a way that is hard to capture (it depends e.g. on the mathematical background of the agent, the difficulty of the problem). Third, the incentive scheme itself is not fully characterized by the protocol bounties and stakes as e.g. (i) a claimer presumably enjoys an (unobservable) intrinsic reward when their proof is accepted and (ii) there might be external incentives, too, as rewards related to SPRIG’s outcome might conceivably also be collectable in a secondary/derivatives market. However, we can identify for each category of agents a number of high-level features independent of the details discussed above:

**Provers**: The decision to enter (i.e. start interacting) in a SPRIG instance depends on one’s confidence about the validity of one’s claim of proof, the explicit incentives (stakes and bounties), the private incentives (intrinsic reward of having one’s claim of proof accepted), beliefs about the skeptics’ ability to identify a flaw in the claim and beliefs about their incentives for attempting to do so. Given that the skeptics’ incentives are also partly shaped by expectations about the incentives of subsequent claimers, the validation game is dynamic; the final, machine-level step provides the boundary condition. One important aspect of this dynamic process is that the initial claimer need not be the one to address all (or indeed any) subsequent questions from the skeptics. If the blockchain’s users get a sufficiently good grasp of the claimer’s argument, competition fostered by the incentive scheme makes it likely that ungrounded skeptics’ challenges are answered by third parties. This should deter ‘spamming’ by the skeptics. Hence, the initial claim must be sufficiently clear for the baton to be passed; but being too explicit and detailed does not seem optimal for the claimers either. Indeed, in that case, they perform upfront a task that would have only been needed in case of a question, so that, if the proof is correct, there is no need to do it immediately, and if it is incorrect, the excessive level of detail makes it easier to detect.

**Skeptics**: The decision-making process of skeptics is similar in the sense that it responds to the same incentive scheme and relies on the formation of beliefs over the same objects. First, and obviously, Skeptic’s incentives to challenge increase with their subjective probability of the claim of proof being wrong. Second, they decrease with the probability that any part of the initial claim can be converted into machine language in due time if necessary. A skeptic can deem this unlikely if they observe that
the claim of proof or parts of it are somewhat obscure, or even if they have a sense that the claim should be correct but the proof is too convoluted to be transformed into machine language before the deadline. But there are other incentives for a skeptic to challenge: they might want to obtain information that is also relevant to another ongoing validation process; or purely out of scientific interest.

(Claim of) proof shapes are endogenous in SPRIG, emerging from the interaction between claimers and skeptics. Indeed, it appears from the discussion above that with properly designed incentives, claimers would benefit from writing (claims of) proofs that are concise and elegant (and easier to convert into machine language if necessary), without being excessively terse (e.g. because no third party would have enough information to defend the claim if needed). Hence, beyond providing a decentralized way to produce a consensus about mathematical claims, SPRIG also naturally delivers balanced, ‘agent-tailored’ proofs: sufficiently detailed to be convincing but sufficiently concise to give intuition and be remembered. In particular, we expect that one would rarely, if ever, need to reach the final, machine-language step. As soon as the convertibility to machine level is credible, no skeptic would have an incentive to push the process to that step. (The classic analogy is with a government guaranteeing to intervene in case of a banking panic; if such a guarantee is credible, then the panic would not occur, and the intervention would never be needed).

The economic approach is also key for dealing with a crucial point: how to interpret the fact that a claim of proof has been accepted by the protocol? Understanding incentives is necessary for answering this question. The simplest example is one where a claim has been accepted without any challenge: is it because all users were fully convinced or because the incentive scheme makes it prohibitively costly (in expectation) to ask questions? In principle, given the correct economic model (data of all incentives and information structure), any Bayesian observer can use the protocol’s outcome to compute the probability that the proof is known by the market participants.

In Section 4, we investigate some of the game-theoretic aspects presented above in a stylized setting. In particular, we explain how to do a Bayesian estimate of the probability that a claim of proof is correct given that it has been accepted in a (highly) simplified version of the protocol.

3.2. Robustness Properties. As in many blockchain systems, strategies in SPRIG can, broadly speaking, be divided into ‘honest strategies’ and ‘attacks’. The former refers to actions whose motivations are aligned with the purpose of the protocol. The latter refers to attempts to game the system, i.e. take advantage of the incentive scheme without contributing to the end goal of the blockchain. We expect SPRIG to be robust. First, its trust model shares similarities with that of optimistic rollups [OptRollUp] [Arbitrum18], and of the TrueBit protocol [TrueBit19]. Second, it features a large array of parameters which we expect to be sufficiently rich to shape incentives that deter attacks. To guide the specific choice of the parameters, we now list several potential attacks together with which parameters are to be tuned to thwart them.

3.2.1. The Carpet-Bomber. A skeptic may decide to question all parts of a claim in the hope of stalling the process. The idea would be to induce the claimer to concede by lack of resources and because there are not enough third-party claimers available to help them defend. This is similar to a DDoS attack on the protocol. The skeptic’s goal is to collect the stake of the claim.

This attack can be thwarted by appropriately choosing the question bounties and the time allotted for the subsequent claimers’ replies. The former should not be too small relative to the stake and the latter should be sufficiently long.

3.2.2. The Nitpicker. A skeptic may decide to ask for more and more details about a claim of proof and refuse to concede until the machine level is reached. Such an attack is not only based on the hope that a flaw will be identified at some point but more importantly on the skeptic’s desire to delay the acceptance of the claim as much as possible. One reason could be that the skeptics are themselves a claimer of an identical or similar result, which they want to be accepted first.

This attack induces the claimer to present their proof with lemmas of similar complexity. This mitigates incentives to nitpick as it reduces the depth needed in order to expand the proof up to machine level. Well-balanced bounties (i.e. not too low) and deadlines that take into account the possibility of nitpicking (i.e. the maximal time allotted for expansion up to machine level might indeed be reached) contribute further to thwarting nitpicking attacks.
3.2.3. The Evasive Prover. A claimer may decide to be evasive, i.e. to stuff his claim with a combination of irrelevant lemmas (purposely looking intricate but for which they actually hold a machine level-proof) and one lemma of complexity similar to the initial theorem, such that it is not clear to outsiders which lemma to question. The goal is to deflect questions towards the irrelevant lemmas and hence get the Claim accepted and collect the questions’ bounties.

This attack can be thwarted by choosing the following parameters appropriately: the stakes, the time allotted for the subsequent skeptics’ questions, the maximal level of a proof, and the maximal length of a claim. The first two should be sufficiently large to incentivize skeptics to work and identify the weak link. The last two should be sufficiently small in order to cap the number of deflection targets and force the claimer to ‘show their hand’ quickly enough.

3.2.4. The Sandbagger. This attack mirrors the Carpet-Bombing one: a claimer may decide to answer a Question with a multitude of claims in the hope of stalling the process. The idea would be to induce the skeptic to concede by lack of resources and because there are not enough third-party skeptics available to help them continue challenging. The claimer’s goal is to collect the bounty of the Question.

This attack can be thwarted by appropriately choosing the claim stakes and the time allotted for the subsequent skeptics’ questions. The former should not be too small relative to the bounty and the latter should be sufficiently long.

3.2.5. The Misleader. A claimer may decide to stuff his claim with dubious lemmas and pursue one of the following two strategies:

• they attack the dubious lemmas and provide answers themselves in order to “intimidate” the skeptics (improving their general credence in the initial claim);
• they attack the dubious lemmas and postpone answers to the very last moment in order to mislead the skeptics into believing that other users are already challenging (so there is no point in joining the fray, as the stakes no longer seem earnable).

This attack can be handled similarly to the Sandbagger attack.

3.2.6. The Plagiarist. A mathematically illiterate agent can have a firm belief that a claimer is able to answer a given Question correctly. This may occur, for instance, on occasion, when an eavesdropper obtains information that a researcher has a proof for a Question submitted by an institution, or, more frequently, when the Question concerns the claim of another claimer. The agent may then attempt to appropriate the proof of the claimer in order to collect the Question’s bounty as illustrated below.

Consider a mathematician Alice who found a correct proof of a theorem. She posts a corresponding claim on the blockchain. Then, Bob asks a question about the claim, targeting statement $S$. At this stage, there could be an incentive for Charlie, the mathematically illiterate agent, to immediately reply to Bob’s question with a tautological answer: ‘the proof of $S$ is $S$’; and to stick to this strategy when questions are asked/repeated until Alice decides to provide an answer herself to ensure that her Claim is not rejected. From that point onwards, Charlie replicates any of Alice’s replies and challenges her using the questions skeptics ask him.

In doing so, people may prefer to ask the questions straight to Alice to know if she can provide a satisfying answer; they only challenge Charlie with the same question if she does not. This may lead to a faster approval of Charlie’s claim.

Fortunately, Alice can defend herself: as soon as she is challenged, she answers the question, then asks Charlie the same question if it was not already done by another skeptic and, instantaneously, provides the same answer. This thwarts the Plagiarist’s attack since it provides a zero-cost defense mechanism to ensure that Charlie’s Claim cannot be validated before her own Claim.

4. A Simplified Equilibrium Analysis

In this section, we analyze a tractable sequential game that captures several key features of the strategic interaction between claimers and skeptics through SPRIG. The adequate equilibrium concept for such dynamic games with incomplete information is that of Perfect Bayesian Equilibrium (PBE) [FuTi91]. Our setup consists of a game involving two players: Claimer (pronoun: she) and Skeptic (pronoun: he). In our setup, Skeptic does not observe the initial confidence of Claimer (a correctly estimated probability that her
proof is validatable, i.e. it is possible to unroll it down to machine level) and must therefore form beliefs about it to proceed. A PBE is a collection of actions and beliefs such that:

(1) Given beliefs, the action taken by any agent at any node of the game tree maximizes her or his expected utility.

(2) Beliefs at each node of the game tree are consistent with the history of actions, i.e. computed from Bayes’ rule.

Hence, by constructing PBEs, one recognizes that the mere fact of initiating a process in the protocol (or, in general, of pushing it further) has informational content: intuitively, if a claimer posts a proof, this should reflect the fact that she is relatively confident about her proof and it should lead to an upwards update of the outsiders’ beliefs. For simplicity, we focus on the signaling content of the entry decision, not of the parameters (deadlines, stakes, and bounties) chosen at initiation. One could assume there is a set of ‘default’ parameters suggested by the protocol and then using them conveys limited (although non-empty) signaling content. There is no fundamental obstacle in extending the solution of our game to the case where the choice of parameters is endogenous; but this would lead to a dramatic increase in complexity without altering the key messages that we want to convey in this section.

Section 4.5 discusses the strengths and limitations of our simplified protocol model and highlights directions in which it can be enriched.

4.1. Model Setup. We consider a highly stylized version of SPRIG. The maximal level is two and there are only two agents: Claimer and Skeptic. Both are risk-neutral and do not discount the future. Claimer is endowed with a claim of proof C (of some statement).

We say that the claim is validatable if it is possible to unroll it down to machine level and that it is accepted if either Skeptic renounces challenging or the claim is indeed unrolled down to machine level.

Claimer initially receives a (random) signal P ∈ [0, 1], uniform on [0, 1], such that

\[ E[X|P] = P \]

where \( X = 1_{\{C \text{ is validatable}\}} \) (where \( 1_A(x) = 1 \) if \( x \in A \) and \( 1_A(x) = 0 \) if \( x \notin A \)). We use the economic term signal to refer to a random variable whose realization can be informative about the variable X of interest. Here, we could define U to be an independent copy of P and assume that C is validatable exactly on the event \( \{U \leq P\} \). In words, Claimer has more information than Skeptic, as she knows an updated probability, the realization of P, that her claim of proof is validatable. By contrast, Skeptic initially only has the knowledge that P is uniformly distributed over [0, 1].

On top of the potential collection of bounties, Claimer derives private benefits from having her claim of proof accepted. We denote by \( B_2, B_1, B_0 \geq 0 \) the benefits of being accepted at level 2, 1, 0 respectively.

If Claimer decides not to post C, the game ends immediately, and both players receive a payoff of 0. If she posts C, the protocol specifies a stake \( \sigma_2 \) to be collected by Skeptic in case of a successful challenge. If C remains unchallenged (no questions are asked about it within time \( \theta_2 \) after publication), then Claimer gets \( B_2 \), and Skeptic gets 0. If Skeptic challenges the claim of proof within time \( \theta_2 \), staking a bounty \( \beta_1 \), we make the assumption that Claimer gets to know the realization of X and will be able to provide a machine-level proof of her claim if valid at level 0 of the protocol. Given this realization, she decides whether or not to post a claim at level 1 within time \( \tau_1 \) after the publication of Skeptic’s question. If she does not, her final payoff if \(-\sigma_2^1\) and Skeptic’s is \( \sigma_2^2 \). If she does, the protocol specifies two stakes \( \sigma_1^1, \sigma_1^+ \). Skeptic has a last chance to challenge the claim within time \( \theta_1 \) after its publication: if he does not, Claimer’s payoff is \( B_1 + \beta_1 \) and Skeptic’s is \(-\beta_1 \). If he does, he stakes a bounty \( \beta_0 \) and then Claimer posts the machine language proof if available within time \( \tau_0 \) after publication of Skeptic’s question. Claimer’s final payoff is \(-\sigma_2^1 - \sigma_1^1 - \sigma_1^2\) if she can provide a machine-level proof, and \( B_0 + \beta_0 + \beta_1\) if she can. The corresponding Skeptic’s payoffs are \( \sigma_2^1 + \sigma_1^1 + \sigma_1^2\) and \(-\beta_0 - \beta_1 \). Since we consider a single skeptic, the recipient of the up and down stakes \( \sigma_1^1, \sigma_1^+ \) is the same and hence we can aggregate those in \( \sigma_1 = \sigma_1^1 + \sigma_1^+ \). From now on, we also denote \( \sigma_2 = \sigma_2^2 \).

The game is represented in Figure 4.1.

The time lengths \( \theta_1, \theta_2, \tau_0, \tau_1 \) are key to make sure that the status (‘challenged or not’) of a claim/question is eventually settled but their value does not play a role in our stylized model. Hence, our game is fully characterized by the parameter set

\[ \Theta = \{B_0, B_1, B_2, \sigma_1, \sigma_2, \beta_0, \beta_1\} \]
4.2. Model Solution.

**Proposition 22.** The simplified protocol game possesses a unique Perfect Bayesian Equilibrium. Depending on the parameter set $\Theta$, this PBE takes one of the three types detailed below. In all of them, there is a threshold $\pi^* := \pi^*(\Theta) \in [0, 1)$ such that Claimer posts if and only if $P \geq \pi^*$ and:

- **Type 1:** Skeptic always challenges, Claimer replies if $X = 1$ and replies with probability $p := p(\Theta)$ if $X = 0$. Conditional on reply, Skeptic challenges w.p. $q_1 := q_1(\Theta)$ (and Claimer successfully terminates the process if and only if $X = 1$).
- **Type 2:** Skeptic challenges the initial claim with probability $q_2 := q_2(\Theta) \in (0, 1)$. Then actions unfold as in Type 1.
- **Type 3:** $\pi^* = 0$ and Skeptic never challenges.

The parameters $q_1, q_2, p$ and $\pi^*$ are known explicitly and their values are provided in Appendix 8.1. The proof of the proposition can be found in the same Appendix. For illustrations on how the nature of the equilibrium (type 1, 2, or 3) depends on the parameters, see Section 4.4.

4.3. **Extracting Relevant Information from the Protocol’s Outcome.** One key appeal of our model is that it allows us to compute various measures of protocol reliability, in particular the likelihood that type I or type II errors (in the statistics sense) occur. This is crucial because the outcome of the protocol’s validation process for a claim (accepted/validated or rejected/invalidated) does not say, in isolation, what credence the agents’ community should have in the claim. Section 4.4 discusses further these issues.

4.3.1. **Notation.** We shall need the following notation:

- $\mathbb{A}$ (resp. $\mathbb{A}^c$) is the event ‘The claim is accepted’ (resp. rejected, i.e. not accepted).
- $\mathbb{A}_2$ (resp. $\mathbb{A}_1$) is the event ‘The claim is accepted at level 2’, i.e. no question was asked (resp. level 1, i.e. one question was asked).
- $\mathbb{Q}_0$ is the event ‘Skeptic challenges Claimer after the reply of Claimer’, i.e. Skeptic posts a question at level 0.
- $\mathbb{R}$ is the event ‘Claimer replies to first challenge’ (i.e. posts a claim at level 1).

4.3.2. **Results.**

**Proposition 23.** In a Type 1 equilibrium:

- The probabilities that a claim of proof is accepted (resp. accepted and valid) are

\[
P(\mathbb{A}) = \frac{1}{2} (1 - \pi^*) (1 + \pi^* + (1 - \pi^*) p(1 - q_1))
\]
\[
P(\mathbb{A}, X = 1) = \frac{1}{2} (1 + \pi^*) (1 - \pi^*).
\]
The probabilities that a claim of proof is accepted given that it is valid (resp. false) are

\begin{align}
P(A | X = 1) &= (1 + \pi^*) (1 - \pi^*) \\
P(A | X = 0) &= (1 - \pi^*)^2 p(1 - q_1).
\end{align}

The probabilities that a claim of proof is valid given that it is accepted (resp. rejected) are

\begin{align}
P(X = 1 | A) &= \frac{1 + \pi^*}{1 + \pi^* + (1 - \pi^*) p(1 - q_1)} \\
P(X = 1 | A^c) &= \frac{\pi^*}{\pi^* + 1 - (1 - \pi^*)^2 p(1 - q_1)}.
\end{align}

The probabilities that a claim of proof is accepted at level 2 (resp. 1) given that it is accepted and valid are:

\begin{align}
P(A_2 | A, X = 1) &= 0 \\
P(A_1 | A, X = 1) &= 1 - q_1.
\end{align}

Such expressions can also be derived in the case of Type 2 and Type 3 equilibria: see the proof.

4.4. Results. We now use our model solution to explore various trade-offs faced by protocol designers. Obviously, the fact that we consider both a stylized model and a simplified information structure does not allow us to produce general and quantitative positive or normative statements about SPRIG parameters. However, our sequential game with imperfect information is rich enough to illustrate several important forces that must be taken into account by designers, and that can be illustrated through examples. As baseline parameters, we consider $B_2 = 10, B_1 = B_0 = 40$ and $\beta_1 = \sigma_1 = \beta_0 = 5$, and let the stake $\sigma_2$ vary. Of course, one could evidence similar trade-offs by varying another parameter, as well as the transitions between the different equilibrium types. We focus on varying $\sigma_2$ merely for the sake of brevity.

4.4.1. Stakes and bounties, entry and reliability ratio. The two following properties are desirable for the protocol:

- (i) Have as many correct claims of proof as possible passing through the protocol.
- (ii) Have a (very) high probability that an accepted claim of proof indeed corresponds to a proof (i.e. is correct). The left panel of Figure 4.2 evidences that the two objectives are, in general, in conflict with each other. In this plot, the solid line depicts the probability that a claim is produced and accepted by the protocol, while the dashed line depicts the probability that a correct claim is produced and accepted by the protocol. Define the reliability ratio $RR$ as the ratio of the latter by the former (‘dashed/solid’).

$RR$ is close to the desirable 100% as long as the equilibrium is of Type 1, and quickly deteriorates as we enter the Type 2 equilibrium region. While there is no ideal conciliation of Objectives (i) and (ii) above, the left panel of Figure 4.2 suggests that a good way to resolve the trade-off is to select a stake $\sigma_2$ just barely sufficient to incentivize Skeptic to systematically challenge. This maximizes the probability of having a claim going successfully through the protocol among Type 1 equilibria. Of course, by reducing $\sigma_2$ further, one could increase this probability further, but the ‘price’ to pay (the quick drop of $RR$) is likely to be prohibitive.

The right panel of Figure 4.2 indicates that $\pi^*$, the equilibrium entry threshold, increases with the bounty $\sigma_2$. This is consistent with intuition: if she must pay a large amount in case of a successful challenge, Claimer will only enter when she is very confident about her claim of proof. Hence, increasing $\sigma_2$ reduces entry; but it also increases the likelihood that a claim of proof is true conditional on entry. Thus, the impact of $\sigma_2$ on $P(A)$ was a priori non-trivial. The left panel of Figure 4.2 indicates that there is a monotone decreasing relationship between the two variables. In fact, this is always true, as one can easily deduce from the formulas of Proposition 23.
4.4.2. Statistical Type I and Type II Errors. In this section, we focus on the four quantities $P(A^c|X = 1)$, $P(A|X = 0)$, $P(X = 0|A)$ and $P(X = 1|A^c)$. All are measures of the likelihood that the protocol produces an undesirable outcome (at least from a scientific standpoint, as a claimer would presumably have no problem with having an incorrect claim accepted). The last two correspond to the standard statistical notions of Type I and Type II errors, respectively. The reliability ratio $RR$ introduced above is simply the complement to the probability of a Type I error.

Being able to compute such measures is of paramount importance for protocol users and for the mathematical community at large. Without them, there is no clear link between the outcome of the validation process and the credence that humans should give to a claim (or its negation). In particular, humans may want to lower their confidence in claims accepted in some particular equilibrium type and state. As an illustration, consider the right panel of Figure 4.3. If the equilibrium is of Type 3, all proofs are accepted, but this is irrelevant from a scientific standpoint, as the probability of Type I error is $\frac{1}{2}$. If stakes and bounties are designed in such a way that challenging is prohibitively expensive, one should not give too much credit to a claim simply because it has passed through the protocol. Such a design would be severely flawed. More generally, given a model that allows one to predict the equilibrium type, one can and should observe the blockchain in order to refine the statement ‘the claim has been accepted’ into ‘the claim has been accepted at level $d$ after history $h$’ and update the correctness probability accordingly.

While the right panel of Figure 4.3 illustrates that there are some entirely flawed protocol designs (if it generates a Type 3 equilibrium) it also highlights that there is no perfect design: one cannot simultaneously decrease the likelihood of Type I and Type II errors. Again, the juncture point between Type 1 and Type 2 equilibria seems to be a good candidate: for instance, any choice of a larger $\sigma_2$ would only very marginally decrease the probability that an accepted claim is incorrect, but significantly increase the probability that a rejected claim is correct.

Arguably, the aggregate costs of accepting invalid claims are much larger than the costs of rejecting correct ones. Indeed, once accepted, an invalid claim could be used repeatedly in subsequent research or applications, so that the mistake propagates and its consequences grow. Moreover, there might not be enough incentives or reasons to challenge the claim again in the future. By contrast, a wrongly rejected claimer could always rewrite her claim of proof, improve communication and post again at a later stage, getting another chance to be accepted.

The left panel of Figure 4.3 tells a similar story, with a perspective closer to the point of view of the claimer. Reducing the risk of rejecting a correct claim of proof increases the risk of accepting a wrong claim of proof. Once again, thinking about the aggregate costs of both types of error should allow designers to select their preferred parameters.

As can be seen, producing graphs such as those of Figure 4.3 gives a lot of information about which parameter values are the most effective. This will be precious for future (more realistic and quantitatively accurate) model descriptions of the protocol.
4.4.3. Terminations at Intermediate Level. Fixing the likelihood of statistical errors discussed above, a short acceptance process (termination after a few steps) has advantages and drawbacks. On the one hand, accepting correct claims of proof quickly saves significant time and intellectual energy that can be invested in tackling other problems. On the other hand, longer acceptance processes can have positive externalities, as they involve clarifying steps and new lemmas that could be useful in other contexts. In the current discussion, we wish to focus on the former point and consider that accepting a correct proof quickly is desirable—to the extent of course that it does not harm its credibility too much, see Section 4.4.2. But the latter requirement is key: in our model, claims accepted immediately after posting have little scientific relevance. Indeed, we saw that the reliability ratio quickly decreases away from 100% as the likelihood to terminate immediately increases away from 0. Hence, a good proxy for ‘having proofs that terminate before final level without harming reliability’ is the probability of termination after exactly 1 step, \( P(A_1 | A, X = 1) \). This quantity is depicted in Figure 4.3.

Again, the behaviour of this quantity as a function of \( \sigma_2 \) is non-trivial. Indeed a larger stake (i) increases the incentives to challenge (direct ‘greed’ effect) but also (ii) decreases them as the average quality of a proof is higher (this is evidenced by the fact that \( \pi^* \) is larger). As before, a good choice seems to take the lowest \( \sigma_2 \) that implements a Type 1 equilibrium.

4.5. Discussion of the Model’s Analysis. Our stylized model captures several important features. First, we take into account the likely information asymmetry that exists between claimers and skeptics (at least at the time a claim is posted). Then, the theory of signaling games allows us to predict how the mere fact of
posting a claim impacts the ‘market’s beliefs’. Second, our simplified game is dynamic. In particular, we can understand the impact of stakes and bounties at further levels on current decisions, and get estimates of the probability that the protocol stops before the final, machine-language step. Third, our model is rich enough to inform a Bayesian agent about the correctness of a claim, using its status in the protocol (accepted vs rejected).

The model can be enriched in several directions. First, Skeptic could be able to perform some work on his own: by paying some cost, he could be able to access a private signal about the correctness of Claimer’s proof. We have studied this possibility in a one-period version of the model. New insights appear, but the enlargement of Skeptic’s strategy set also implies the emergence of multiple equilibria. Those are potentially interesting but render predictions difficult. Second, the information structure could be enriched: there could be several skeptics (and several subsequent claimers replying to these skeptics), each potentially endowed with their own information. A challenging and very interesting question is to understand how information dynamically incorporates in such a context. Third, our stylized model is not able to answer questions such as whether we should expect SPRIG instances to generally terminate at the top level or the machine level, or instead in between, or regarding the structure of the tree that an initial claim or question generates.

5. Applications and Outlook

In this section, we discuss how the SPRIG protocol provides a solution to the challenges of mathematical derivation raised in Section 1, which are centered around the communication of trustable, succinct, and informative proofs in a system with agents with various levels of information.

5.1. Theorem Verification. The validation of a theorem’s proof by authors can be done through a protocol: they can place a stake (which may increase over time, as discussed in 2.4.1) and set up a SPRIG instance for a given amount of time, incentivizing anyone to find a gap in their proof. Compared to the classical publishing model, many more agents are incentivized to be skeptical of the proof (and no one is pressured to participate either), and their questions can be assumed to be made in good faith (since there is nothing to gain by asking trivial questions); also, the anonymity of the reviewers is guaranteed (unlike the reviewing process, which consists in the redaction of a report, and which relies on an editorial board, both of which may leak information). The results of the validations can thus be made transparent and convey information about the validation of theorems. At the same time, as they provide their claim of proof, the authors can also publish a paper written in an informal way, which may help the community participate in the process more rationally.

5.2. Bounty for Open Problem. The research on open questions in mathematics can be incentivized by bounties, such as the celebrated Millennium Problems posted by the Clay Institute. In cases such as that of the Millennium Problems, an open two-sided question is at the root of the problem, as discussed in Section 2.4.7. SPRIG allows one to outsource the validation of claims of proof (which in principle relies on a committee), to disincentivize bogus claims of proof (a stake must be put to propose a claim of proof), and to limit conflicts of interest.

Incentives for shorter answers can also be added, by creating extra challenges, with tighter limits on proof length, or by using claim-of-proof-dependent parameters (as in Section 2.4.3).

5.3. Security Proof Certification. An organization may want to elicit trust in its system. For instance, it may want to publish their source and incentivize the public to find security flaws in any of $N$ subsystems. It may have a limited number $K$ of bounties available for finding problems in any of the $N$ subsystems. This may be done using the multi-question bounty variant discussed in Section 2.4.7 and elicit a trust in the system (e.g. that there is no flaw in any of the subsystems) that is as strong as if there were $N$ bounties, while at the same time locking up and risking only $K$ bounties worth of capital.

5.4. Automated Theorem Proving. A great deal of effort has been put in recent years into constructing intelligent automated provers, relying on e.g. reinforcement learning techniques or text prediction mechanisms, with encouraging successes [UrJa20]. SPRIG can serve as a playground for the development of such agents, allowing them to participate using some level of information (in particular by first developing an ability to write low-level proofs or to validate them), and learning by playing.
5.5. **Derivatives Markets.** A promising feature of SPRIG is that its outcomes can then be used as oracles for other smart contracts. In particular, other prediction markets can run on such outcomes.

For instance, agents can inject information by betting that a certain question will or will not be answered before a certain time. Or they could bet that conditionally on there being an unanswered question, this question will challenge a specific step $S$ of the proof, thereby indicating that $S$ might be the weak link.

Securities markets relying on SPRIG may also prove to be useful for incentivizing different types of contributions for agents. For instance, an agent able to provide good formalizable heuristics but not knowing how to formalize them may participate in a market betting that a certain question will be answered before a certain time, buy (for relatively cheap) a security betting that it will be answered, and then publish her heuristics; if it looks like the heuristics can be formalized by some agent in time, the odds for betting that the question will be answered will change, and she can net a profit by re-selling her security or letting it mature. Thus, she can inject interesting information into the market, i.e. information that changes the feasibility landscape of proof construction by the community.

5.6. **Beyond Mathematical Reasoning.** Beyond mathematics, many fields rely on rigorous formal reasoning intertwined with external elements of reasoning. Adding support for external sources for SPRIG appears promising for a number of applications:

- Support for importing empirical knowledge in the protocol: this would allow it to submit and verify arguments pertaining to experimental sciences.
- Support for numerically-justified heuristic arguments or recognized heuristics: this would allow for useful derivations in e.g. theoretical physics.
- Support for validated time-stamped predictions: this could help rational discourse in disciplines based on forecasting. An economic model could be presented and challenged similarly to claims of proof in SPRIG. In lieu of the machine-level terminal condition, the final validation step would be given by the publication of official numbers. A ‘claim’ (i.e. a model) would then be validated if it has correctly predicted a n-tuple of economic variables (e.g. interest rate set by the Fed, GDP) up to a prespecified error margin.
- Support for oracles with zero-knowledge proofs: this would allow for auditable arguments in public debates in which certain sources must be protected.

6. **Conclusion**

In this paper, we introduced the Smart Proofs by Recursive Information Gathering (SPRIG) protocol, which allows agents to propose and verify succinct and informative proofs in a decentralized fashion. Claimers and skeptics ‘debate’ about statements and their proofs: consensus arises from the skeptics being able to request details on steps that they feel could be problematic and from the claimers being able to provide details answering the skeptics’ requests. Importantly, to participate in the process, claimers and skeptics must attach a bounty/stake to their moves: this gives the proper incentive for subsequent users to verify those. As a result, agents with various types of information can participate and inject their knowledge into the proof construction and verification process; this allows one to strike a balance between the ‘short collection of insightful statements’ vs ‘list of all the statements needed to establish perfect trust’ tradeoff in mathematics writing.

In our claim of proof format, mathematical proofs can be viewed as trees, in which claimers and skeptics expand branches containing the relevant level of detail for the agents in the community: branches only grow in places where there is uncertainty, until either that uncertainty is cleared or a specific problem is isolated. This resulting subtree thus serves as a proof that is useful to the community, as it makes the consensus-building process transparent and can help agents build their own credence in the validity of the proof.

Our analysis of SPRIG and its robustness is based on game-theoretic considerations that take into account the various incentives of the agents, address possible attacks, and leading up to a detailed equilibrium analysis of a simplified protocol. While the complete SPRIG protocol is very complex to study analytically, our results give a clear insight into a number of qualitative aspects of its strategic features.

We also present a number of variants and applications of SPRIG, allowing it to be useful in numerous contexts, and demonstrating its versatility.
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7. Appendix: Claim of Proof Examples

In this appendix, we give a number of examples of proofs written in the claim of proof structure described in Section 1.2.5 (we naturally only give subtrees of the entire trees). We denote by $A_0$ the standard background assumptions (including basic axioms and whatever statement is taken for granted).

7.1. A simple proof. A simple proof concerns the existence of an infinite number of primes.

In this case, we have $\gamma = \{\text{there exists an infinite number of primes}\}$ and the root of the claim of proof is

- $S_* : A_* \implies C_*$, with $A_* = A_0$ and $C_* = \gamma$.

The nodes at distance 1 from the root are:

- $S_1 : A_1 \implies C_1$, where $A_1 = A_0$ and $C_1$ corresponds to ‘For any $N \geq 2$, $N! + 1$ is not divisible by any $k \in \mathbb{N}$ with $2 \leq k \leq N$’;
- $S_2 : A_2 \implies C_2$, where $A_2 = A_0 \cup \{C_1\}$ and $C_2$ corresponds to ‘For any $N \geq 2$, there exists a prime number $p > N$’.
- $S_3 : A_3 \implies C_3$, where $A_3 = A_0 \cup \{C_2\}$ and $C_3 = C_*$.

If we expand the proof of $S_2$ (at distance 2 from the root), we find:

- $S_{2,1} : A_{2,1} \implies C_{2,1}$, where $A_{2,1} = A_2$ and $C_{2,1}$ corresponds to ‘For any $N \geq 2$, any prime factor of $N! + 1$ is larger than $N$’.
- $S_{2,2} : A_{2,2} \implies C_{2,2}$, where $A_{2,2} = A_2 \cup \{C_{2,1}\}$ and $C_{2,2} = C_2$.

7.2. A proof by contradiction. Proofs by contradiction can be formulated naturally in our framework. A classical proof by contradiction is that of the fundamental theorem of algebra $\alpha \implies \gamma$, where

- $\alpha$ corresponds to ‘$P$ is a complex polynomial of degree $\geq 1$’.
- $\gamma$ corresponds to ‘there exists $z \in \mathbb{C}$ such that $P(z) = 0$’.

In this case, the root of the claim of proof is

- $S_* : A_* \implies C_*$, where $A_* = A_0 \cup \{\alpha\}$, and $C_* = \gamma$.

The nodes at distance 1 from the root are:

- $S_1 : A_1 \implies C_1$, where $A_1 = A_*$ and $C_1$ corresponds to ‘There exists $M, R > 0$ such that $|P(z)| \geq M$ for all $|z| \geq R$’;
- $S_2 : A_2 \implies C_2$, where $A_2 = A_*$ and $C_2$ corresponds to ‘$|P|$ cannot have a nonzero minimum on $\mathbb{C}$’;
- $S_3 : A_3 \implies C_3$, where $A_3 = A_* \cup \{C_1, C_2\}$ and $C_3 = C_*$.

If we go further into the details of the proof of $S_2 : A_2 \implies C_2$ (the heart of the proof by contradiction), we have (at distance 2 from the root):

- $S_{2,1} : A_{2,1} \implies C_{2,1}$, where $A_{2,1} = A_2$ and $C_{2,1}$ corresponds to ‘If $P(z) \neq 0$, then there exists $z' \in \mathbb{C}$ such that $|P(z')| < |P(z)|$’;
- $S_{2,2} : A_{2,2} \implies C_{2,2}$, where $A_{2,2} = A_{2,1} \cup \{C_{2,1}\}$ and $C_{2,2}$ corresponds to ‘If $|P|$ has a nonzero minimum on $\mathbb{C}$, then we have a contradiction’;
- $S_{2,3} : A_{2,3} \implies C_{2,3}$, where $A_{2,3} = A_2 \cup \{C_{2,2}\}$ and $C_{2,3} = C_2$.

Ultimately, in the above format, proofs by contradictions must explicitly carry their ‘wrong assumption’ (i.e. the negation of the conclusion) in the conclusion part of the statements: if we wish to assume the negation $\neg C_*$ of the conclusion $C_*$ to arrive at a contradiction, this will involve substatements $S : A \implies C$, where $C$ will be of the form ‘if $\neg C_*$ then $\ldots$’. While this makes the proofs by contradiction heavier in notation, this makes individual statements easier to verify: if the proof is correct, the conclusion of each statement is correct (and not contingent on an assumption which itself is wrong).

7.3. Inverse Function Theorem Proof. A richer example of proof is that of the inverse function theorem.

In this case, we have:

- $\alpha$: Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be a function that is $C^1$ with derivative $x \mapsto Df|_x$. Let $x_* \in U$ be a point such that $Df|_{x_*}$ is an invertible matrix.
- $\gamma$: There exists an open neighborhood $V \subset U$ of $x_*$ and an open neighborhood $W$ of $f(x_*)$ such that $f|_{V} : V \to W$ is a bijection from $V$ to $W$, and an inverse $(f|_{V})^{-1} : W \to U$ that is differentiable at $f(x_*)$ with derivative $(Df|_{x_*})^{-1}$.
In this case, the root of the claim of proof is:

- **S_0**: \( A_s \rightarrow C_s \), where \( A_s = A_0 \cup \{ \alpha \} \) and \( C_s = \gamma \).

Informally, we first argue that without loss of generality, we may assume that \( x_s = 0, f(x_s) = 0, Df|_{x_s} = Id_n \). In this case, the nodes at distance 1 from the root are:

- **S_1**: \( A_1 \rightarrow C_1 \), where \( A_1 = A_s \) and \( C_1 = (\alpha_1 \Rightarrow \gamma) \) with
  \[ \alpha_1 = \{ x_s = 0, f(x_s) = 0, Df|_{x_s} = Id_n \} \].

- **S_2**: \( A_2 \rightarrow C_2 \), where \( A_2 = A_1 \cup \{ C_1 \} \) and \( C_2 = C_s \).

If we go further into the details of why **S_1** holds true, we find (at distance 2 from the root):

- **S_{1.1}**: \( A_{1,1} \rightarrow C_{1,1} \), where \( A_{1,1} = A_1 \) and, denoting by \( \| \cdot \|_{M_n} \) the operator norm on \( n \times n \) matrices, \( C_{1,1} = (\alpha_1 \Rightarrow \gamma_{1,1}) \), with \( \gamma_{1,1} \)
  \[ \gamma_{1,1} = \left\{ \exists r > 0 \text{ with } \|Df|_{x_s} - Id_n\|_{M_n} \leq \frac{1}{2} \forall x \in B(0,r) \right\} \].

- **S_{1.2}**: \( A_{1,2} \rightarrow C_{1,2} \), where \( A_{1,2} = A_1 \cup \{ C_{1,1} \} \) and \( C_{1,2} = (\alpha_1 \Rightarrow \gamma_{1,2}) \), with \( \gamma_{1,2} \)
  \[ \gamma_{1,2} = \left\{ \exists r > 0 \text{ such that } \forall y \in \mathbb{R}^n, \text{ the function } x \mapsto x + y - f(x) \text{ is } \frac{1}{2} \text{ - Lipschitz on } B(0,r) \right\} \],

- **S_{1.3}**: \( A_{1,3} \rightarrow C_{1,3} \), where \( A_{1,3} = A_1 \cup \{ C_{1,2} \} \) and \( C_{1,3} = (\alpha_1 \Rightarrow \gamma_{1,3}) \), with \( \gamma_{1,3} \)
  \[ \gamma_{1,3} = \left\{ \exists r > 0 : \forall y \in B(0,r/2) : \exists x \in B(0,r) \text{ such that } f(x) = y \right\} \],

- **S_{1.4}**: \( A_{1,4} \rightarrow C_{1,4} \), where \( A_{1,4} = A_1 \cup \{ C_{1,3} \} \) and \( C_{1,4} = (\alpha_1 \Rightarrow \gamma_{1,4}) \), with \( \gamma_{1,4} \)
  \[ \gamma_{1,4} = \{ \exists U, V \text{ open neighborhood of } 0 \text{ such that } f \text{ is a bijection } U \rightarrow V \} \],

- **S_{1.5}**: \( A_{1,5} \rightarrow C_{1,5} \), where \( A_{1,5} = A_1 \cup \{ C_{1,4} \} \) and \( C_{1,5} = (\alpha_1 \Rightarrow \gamma_{1,5}) \), with \( \gamma_{1,5} \)
  \[ \gamma_{1,5} = \gamma_{1,4} \cap \{ \exists f^{-1} : V \rightarrow U, f^{-1} \text{ is the inverse of } f \text{ and } f^{-1} \text{ is differentiable at } 0 \text{ with differential } Id_{n \times n} \} \].

- **S_{1.6}**: \( A_{1,6} \rightarrow C_{1,6} \), where \( A_{1,6} = A_1 \cup \{ C_{1,5} \} \) and \( C_{1,6} = C_1 \).

If we go further into the details of why **S_{1.5}** holds true, we find (at distance 3 from the root):
Remark 24. As the above example reveals, the context of a proof needs to be explicitly carried from statement to statement; a good concrete implementation of the claim of proof format should facilitate this operation in the writing of proofs.
8. Appendix: Game-Theoretic Analysis

In this appendix, we give the proofs of the statements of Section 4.

8.1. Proof of Proposition 22 First, note that the expected payoff of Claimer is increasing in \( P \). Hence, for any given anticipated actions of Skeptic, if initially posting at some \( P \) is optimal, then it is also the case for any \( P' > P \). Hence, the entry decision of the claimer must be of the threshold form given in the Proposition.

Let \( h_e \) and \( h_1 \) be the histories (Post, Challenge) and (Post, Challenge, Reply) respectively, and \( \pi_e = \mathbb{P}_{h_e}(X = 1), \pi_1 = \mathbb{P}_{h_1}(X = 1) \) Skeptic’s beliefs for these histories. Note that \( \pi_e = \frac{1}{2} (1 + \pi^*) \).

8.1.1. Subgame equilibria at \( h_e \). Let us check when (Reply, No Challenge) is an equilibrium of the subgame at \( h_e \). In this scenario, Claimer always replies so observing Reply has no informational content: \( \pi_e = \pi_1 \). For No Challenge to be the best response, we need:

\[
- \beta_1 \geq \pi_1 (-\beta_1 - \beta_0) + (1-\pi_1)(\sigma_2 + \sigma_1)
\]

(8.1)

(i.e. \( \pi_1 \geq \pi_1^* = \frac{\sigma_2 + \sigma_1 + \beta_1}{\sigma_2 + \sigma_1 + \beta_1 + \beta_0} \)).

Because Reply is trivially the best response to No Challenge, we have constructed an equilibrium of the subgame at \( h_e \) as soon as \( \pi_1 = \pi_e = \frac{1}{2} (1 + \pi^*) \) satisfies (8.1).

Now check when (Reply if \( X = 1 \), Reply if \( X = 0 \) w.p \( p \), Challenge w.p. \( q_1 \)) is an equilibrium of the subgame at \( h_e \). In this scenario, Bayes’ rule indicates that

\[
\pi_1 = \frac{\pi_e}{\pi_e + (1-\pi_e)p}.
\]

For Skeptic to be indifferent between challenging or not, we must have equality in (8.1). Using (8.2), we see that this implies:

\[
p = p(\pi_e) = \frac{\pi_e(1-\pi_1^*)}{\pi_1^*(1-\pi_e)}.
\]

(8.3)

Since \( p < 1 \), \( \pi_e < \pi_1^* \). When \( X = 1 \), it is trivially optimal for Claimer to reply. When \( X = 0 \), she must be indifferent. Not replying gives payoff \(-\sigma_2\), while replying gives the expected payoff \((1-q_1)(B_1 + \beta_1) - q_1(\sigma_2 + \sigma_1)\). This pins down the equilibrium value of \( q_1 \):

\[
q_1 = \frac{B_1 + \sigma_2 + \beta_1}{B_1 + \sigma_2 + \beta_1 + \sigma_1}.
\]

(8.4)

This completes the description of the equilibria of the subgame at \( h_e \). Indeed, Skeptic cannot be expected to challenge with certainty, for the best response of Claimer would be to never reply when \( X = 0 \), which in turn would make systematic challenge suboptimal.

Hence, we have characterized equilibrium expected profits at \( h_e \):

- if \( \pi_e \geq \pi_1^* \): \( (B_1 + \beta_1, -\beta_1) \)
- if \( \pi_e < \pi_1^* \):
  - \( X = 1 \): \( (q_1(B_0 + \beta_1 + \beta_0) + (1-q_1)(B_1 + \beta_1), -q_1(\beta_1 + \beta_0) - (1-q_1)\beta_1) \)
  - \( X = 0 \): \( (-\sigma_2, (1-p)\sigma_2 + p(-\beta_1)) \).

8.1.2. Type 1 Equilibria. For such an equilibrium to exist, we must have

\[
\pi_e = \frac{1}{2} (1 + \pi^*) < \pi_1^* \quad \text{(8.5)}
\]

\[
0 < \phi(p_e) \equiv \pi_e(-q_1(\beta_1 + \beta_0) - (1-q_1)\beta_1) + (1-\pi_e)((1-p(\pi_e))\sigma_2 - p(\pi_e)\beta_1). \quad \text{(8.6)}
\]

These conditions, obtained from the results of Section 4.1, ensure that Skeptic has a positive continuation value after Claimer posts \( C \). Indeed, his expected payoff at node \( h_e \) is positive. They are sufficient to guarantee an equilibrium of the Type 1 exists, as soon as Claimer is indeed willing to post if and only if \( P \geq \pi^* \). That is, we have an indifference condition at \( P = \pi^* \), where the expected profit of posting, \( \pi^*(q_1(B_0 + \beta_1 + \beta_0) + (1-q_1)(B_1 + \beta_1)) - (1-\pi^*)\sigma_2 \), must equate 0, the profit of not posting. Hence

\[
\pi^* = \frac{\sigma_2}{q_1(B_0 + \beta_1 + \beta_0) + (1-q_1)(B_1 + \beta_1) + \sigma_2}. 
\]
The “$1 - \pi_e$” term in the denominator of $p(\pi_e)$ cancels out with the “$1 - \pi_e$” term in (8.6), so that the function $\phi$ is in fact linear. Moreover $\phi(\pi^*_2) < 0$ as the only positive term of $\phi$, $(1 - p(\pi_e))\sigma_2$, vanishes. In particular, if $\phi(0) < 0$, a Type 1 equilibrium cannot exist. The properties of $\phi$ will also be important to characterize Type 2 equilibria.

A Type 1 equilibrium exists if and only if conditions (8.5), (8.6) and (8.7) are simultaneously satisfied.

8.1.3. Type 2 Equilibria. For such an equilibrium to exist, Skeptic must be indifferent between challenging the initial claim or not. Hence, we must have

\begin{align}
\pi_e & = \frac{1}{2} (1 + \pi^*) < \pi^*_1 \\
0 & = \phi(\pi_e).
\end{align}

If (8.8) is not satisfied, Skeptic makes a negative profit by continuing because he cannot challenge back upon reply of Claimer. Hence, he cannot be indifferent between challenging the initial claim or not. Equation (8.9) writes down explicitly the payoff of replying when (8.8) holds. (8.9) has a valid root if and only if

\begin{equation}
\phi(0) \geq 0.
\end{equation}

Claimer should also be indifferent between posting $C$ and not posting when $P = \pi^*$. That is, her expected payoff of posting, $(1 - q_2)B_2 + q_2(\pi^*(q_1(B_0 + \beta_1 + \beta_0) + (1 - q_1)(B_1 + \beta_1)) - (1 - \pi^*)\sigma_2)$, should be 0. This gives:

\begin{equation}
q_2 = \frac{B_2}{B_2 - \pi^*(q_1(B_0 + \beta_1 + \beta_0) + (1 - q_1)(B_1 + \beta_1)) + (1 - \pi^*)\sigma_2}.
\end{equation}

A Type 2 equilibrium exists if and only if conditions (8.8), (8.10) and (8.11) are simultaneously satisfied, with $0 < q_2 < 1$.

8.1.4. Type 3 Equilibria. From the previous analysis, it is now clear that if $\frac{1}{2} = \pi_e(\pi^* = 0) \geq \pi^*_1$ or $\frac{1}{2} < \pi^*_1$ but $\phi(0) < 0$ then we have a type 3 equilibrium: Claimer always enters the game and Skeptic never challenges.

8.1.5. Existence and Uniqueness. If $\frac{1}{2} \geq \pi^*_1$, a Type 3 equilibrium exists and no equilibrium of Type 1 or Type 2 can exist. From now on, assume $\frac{1}{2} < \pi^*_1$, and successively (i) $\phi(0) < 0$ and (ii) $\phi(0) \geq 0$.

Case (i): we know that a Type 3 equilibrium exists and we have seen that no equilibrium of Type 1 or Type 2 can exist. (That $\phi(0) < 0$ indicates that Skeptic does not want to challenge even under the worst possible belief about the correctness of $C$. Hence, for any belief $\pi^*$ about the posting threshold, Skeptic would also find it optimal not to challenge.)

Case (ii): we know that no equilibrium of Type 3 exists. Assume a Type 2 equilibrium exists, characterized by, say, $\pi^*_T$ and $q_2, T_2$. Recall that we have

\begin{equation}
0 = (1 - q_2, T_2) \frac{B_2}{B_2 + q_2, T_2} \frac{\pi^*_T(q_1(B_0 + \beta_1 + \beta_0) + (1 - q_1)(B_1 + \beta_1)) - (1 - \pi^*_T)\sigma_2}{\pi^*_T} > 0
\end{equation}

with $0 < q_2, T_2 < 1$. This implies that Claimer expects a negative profit conditional on being challenged at $\pi^*_T$. In particular, if a Type 1 equilibrium were to exist, it would need to feature $\pi^* = \pi^*_T > \pi^*_T$. But $\phi$ decreases (the incentives of Skeptic to challenge decrease with the probability that Claimer is right), so $\phi(\pi^*_T) < \phi(\pi^*_T) = 0$ and Skeptic has no incentive to challenge, so that one cannot construct a Type 1 equilibrium.

At this stage, we have seen that the different types of equilibria are mutually exclusive. To show that there is always one, remark that if Type 2 and Type 3 equilibria do not exist, then $\frac{1}{2} < \pi^*_1$, $\phi(0) \geq 0$ but there is no value of $q_2 \in (0, 1)$ such that (8.11) holds. This means that at the unique root $\pi$ of $\phi$ over $[0, \pi^*_1]$, for all $q_2 \in [0, 1]$, the expected payoff of posting is non-negative. In particular, this holds at $q_2 = 1$: at $\pi$ (meaning: when $P = \pi$ and under the belief that Claimer posts if and only if $P \geq \pi$), Claimer is willing to post even conditional on Skeptic always challenging, and Skeptic is indifferent between challenging or not. As the candidate $\pi^*$ decreases away from $\pi$, the incentives to Challenge increase, and the expected payoff of Claimer decrease at $\pi^*$. Hence, if we define $\pi^*$ as the infimum of the $\pi$ such that Claimer is willing to post even conditional on Skeptic always challenging (a bounded, non-empty set from what we saw above), we have at $\pi^*$ that Claimer is indifferent between posting or not, and that Skeptic will always challenge: we have constructed a Type 1 equilibrium.
8.2. Proof of Proposition 23. We will need the following:

Lemma 25. The probabilities that Claimer enters the game conditional on the claim of proof being correct (resp. incorrect) are

\[ \Pr(P \geq \pi^* | X = 1) = (1 + \pi^*) (1 - \pi^*) \]
\[ \Pr(P \geq \pi^* | X = 0) = (1 - \pi^*)^2. \]

Proof. From Bayes’ formula,

\[ \Pr(P \geq \pi^*) = \frac{\mathbb{E}[X | P \geq \pi^*] \Pr(P \geq \pi^*)}{\Pr(X = 1)}. \]

Since \( \mathbb{E}[X | P] = P \) and \( P \) is uniformly distributed,

\[ \Pr(P \geq \pi^*) = \mathbb{E}[\mathbb{E}[X | P] | P \geq \pi^*] = \mathbb{E}[P | P \geq \pi^*] = \frac{1 + \pi^*}{2}, \]
and \( \Pr(P \geq \pi^*) = 1 - \pi^*, \Pr(X = 1) = \frac{1}{2} \), which yields the first equality. The second one is obtained using similar arguments.

We are now in a position to prove the results relative to Type 1 equilibria. We first compute the probabilities that a proof is accepted/rejected given that it is correct/incorrect. Since a correct claim of proof is accepted if and only if Claimer posts,

\[ \Pr(A, X = 1) = \frac{1}{2} \Pr(A | X = 1) \]
\[ \Pr(A) = \frac{1}{2} (\Pr(A | X = 1) + \Pr(A | X = 0)). \]

This yields the result for the probability that a claim of proof is accepted and true as well as accepted.

The probabilities that a proof is true given that it is accepted/rejected are obtained by applying

\[ \Pr(X = 1 | A) = \frac{\Pr(A, X = 1)}{\Pr(A)} \]
\[ \Pr(X = 1 | A^c) = \frac{(1 - \Pr(A | X = 1)) \Pr(X = 1)}{1 - \Pr(A)}. \]

Finally, in a Type 1 equilibrium, Skeptic always challenges at the first step, so that \( \Pr(A_2 | A, X = 1) = 0 \). Moreover, if \( X = 1 \), the proof finishes after the Reply of Claimer if and only if Skeptic renounces to challenge. This occurs with probability \( 1 - q_1 \), hence \( \Pr(A_1 | A, X = 1) = 1 - q_1 \).

Using similar computations, one can obtain these event probabilities in the case of a Type 2 equilibrium. Specifically, in a Type 2 equilibrium:

- The probabilities that a claim of proof is accepted (resp. accepted and true) are

\[ \Pr(A) = \frac{1}{2} (1 + \pi^*) (1 - \pi^*) + \frac{1}{2} (1 - \pi^*)^2 (1 - q_2 + q_2 p (1 - q_1)) \]
\[ \Pr(A, X = 1) = \frac{1}{2} (1 + \pi^*) (1 - \pi^*). \]

- The probabilities that a claim of proof is accepted given that it is true (resp. false) are

\[ \Pr(A | X = 1) = (1 + \pi^*) (1 - \pi^*) \]
\[ \Pr(A | X = 0) = (1 - \pi^*)^2 (1 - q_2 + q_2 p (1 - q_1)). \]
• The probabilities that a claim of proof is true given that it is accepted (resp. rejected) are

\[
P(X = 1|A) = \frac{1 + \pi^*}{1 + \pi^* + (1 - \pi^*)(1 - q_2 + q_2p(1 - q_1))}
\]

(8.25)

\[
P(X = 1|A^c) = \frac{\pi^{*2}}{\pi^{*2} + 1 - (1 - \pi^*)^2(1 - q_2 + q_2p(1 - q_1))}.
\]

(8.26)

• The probabilities that a claim of proof is accepted at level 0 (resp. 1) given that it is accepted and true are:

\[
P(A_2|A, X = 1) = 1 - q_2
\]

(8.27)

\[
P(A_1|A, X = 1) = q_2(1 - q_1).
\]

(8.28)

The computations of the probabilities for an equilibrium of Type 3 are trivial and omitted.