Schwarz Boundary Value Problems for Polyanalytic Equation in a Sector Ring

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Abstract
In this article, we first give a modified Schwarz–Pompeiu formula in a general sector ring with angle \( \vartheta = \frac{\pi}{\alpha} \), \( \alpha \geq 1/2 \) by proper conformal mappings, and obtain the solution of the Schwarz problem for the Cauchy–Riemann equation in explicit forms. Furthermore, we construct some integral operators and discuss their properties in detail. Finally, by virtue of these operators, Schwarz problems for an inhomogeneous polyanalytic equation and for a generalized polyanalytic equation are investigated, respectively.

Keywords Schwarz–Pompeiu representation · Singular integral operators · Schwarz problem · Sector ring

1 Introduction

The theory of the classical boundary value problems for analytic functions has been applied directly or indirectly in many different fields [1–3], such as signal analysis, crack and elasticity, time-frequency and so on. This makes a great interest in the investigation of boundary value problems for complex partial differential equations in different domains [4–18]. Especially, in [14, 15], four basic boundary value problems for the Cauchy–Riemann equation and the Neumann problem for Bitsadze equation...
were studied in a ring domain by constructing kernel functions and integral expression formulas, respectively. In [16], some harmonic boundary value problems for the Poisson equation were investigated in upper half ring domain with the help of building a modified Cauchy–Pompeiu formula, harmonic Green and Neumann functions. Dirichlet and Neumann problems for the Poisson equation in a quarter sector ring and the general sector ring were respectively solved in [17, 18] through establishing proper Poisson kernels on basis of reflection principle.

Motivated by these, our purpose is to establish the theory of boundary value problems for polyanalytic equations and generalized polyanalytic equations in general sector rings through developing proper kernel functions, which generalizes the boundary value problem not only to the case of high order complex partial differential equation, but to more general sector ring domain. In this present paper, we first give a Schwarz–Pompeiu formula in the general sector ring with angle \( \vartheta = \frac{\pi}{\alpha}, \alpha \geq 1/2 \) by proper conformal mappings, and obtain the solution of Schwarz problem for the Cauchy–Riemann equation. It is just the case in [16] for \( \alpha = 1 \) and the case in [18] for \( \alpha = 2 \). In Sect. 3, a poly-Schwarz and a poly-Pompeiu operator are introduced, and then a Schwarz problem for the inhomogeneous polyanalytic equation in the general sector ring is investigated explicitly. In the end, the poly-Pompeiu operators are discussed in more detail for the upper half ring. On account of these poly-Pompeiu operators, we consider the Schwarz problem for the generalized inhomogeneous polyanalytic equations by transforming it into a singular integral equation.

Throughout this article, \( \Omega^* \) is a sector ring with \( \vartheta = \frac{\pi}{\alpha} \) (\( \alpha \geq 1/2 \)), that is, \( \Omega^* = \{ 0 < r < |z| < 1, \vartheta < \arg z < \vartheta \} \), whose boundary \( \partial \Omega^* = [r, 1] \cup \Gamma_1 \cup [\sigma, \omega] \cup \Gamma_2 \) is given counter-clockwisely, where four corner points are \( r, 1, \sigma = e^{i\vartheta}, \omega = re^{i\vartheta} \), respectively, and

\[
\Gamma_1 : \tau \mapsto e^{i\tau}, \quad \tau \in [0, \vartheta],
\]

\[
\Gamma_2 : \tau \mapsto re^{i\tau}, \quad \tau \in [\vartheta, 0].
\]

2 Schwarz Problem for the Cauchy–Riemann Equation

In order to discuss the Schwarz problem for the Cauchy–Riemann equation in \( \Omega^* \), we define kernel functions \( H_1(z, \zeta), H_2(z, \zeta) \) as follows.

\[
H_1(z, \zeta) = \frac{\zeta + z}{\zeta - z} - \frac{\bar{\zeta} + z}{\bar{\zeta} - z} + 2 \sum_{n=1}^{\infty} r^{2an} \left( \frac{\zeta}{r^{2an} \zeta - z} - \frac{z}{r^{2an} \zeta - z} - \frac{\bar{\zeta}}{r^{2an} \bar{\zeta} - z} + \frac{z}{r^{2an} \bar{\zeta} - z} \right), (1)
\]

\[
H_2(z, \zeta) = \frac{1}{\zeta - z} - \frac{1}{1 - z\zeta} + \sum_{n=1}^{\infty} r^{2an} \left( \frac{1}{r^{2an} \zeta - z} - \frac{z}{r^{2an} \zeta - z} - \frac{1}{\zeta(r^{2an} \zeta - \bar{\xi})} + \frac{z}{r^{2an} \zeta - \bar{\xi}} \right). (2)
\]
From the result in [16], we have the following expression by using the kernels $H_1$ and $H_2$.

**Lemma 2.1** ([16]) Any $w \in C^1(R^+; \mathbb{C}) \cap C(\overline{R^+}; \mathbb{C})$ can be expressed as

$$w(z) = \frac{1}{2\pi i} \int_{L_1 \cup L_2} \text{Re} w(\zeta) H_1(z, \zeta) \frac{d\zeta}{\zeta} + \frac{1}{\pi} \int_{L_1} \text{Im} w(\zeta) \frac{d\zeta}{\zeta} + \frac{1}{\pi i} \int_{[-1,-r] \cup [r,1]} \text{Re} w(\zeta) H_2(z, \zeta) d\zeta - \frac{1}{\pi} \int_{R^+} \left[ w^{-1}(\zeta) H_2(z, \zeta) - \overline{w^{-1}(\zeta)} H_2(z, \overline{\zeta}) \right] d\xi d\eta,$$

where $H_1(z, \zeta), H_2(z, \zeta)$ are given by (1) and (2) with $\alpha = 1$, $R^+ = \{ 0 < r < |z| < 1, \text{Im} z > 0 \}$, and its boundary $\partial R^+ = [-1, -r] \cup [r, 1] \cup L_1 \cup L_2$ with $L_1 = \{ |\tau| = 1, \text{Im} \tau > 0 \}$ and $L_2 = \{ |\tau| = r, \text{Im} \tau > 0 \}$ being oriented counterclockwise and clockwise, respectively.

By Lemma 2.1 and proper conformal mappings, we obtain the Schwarz–Pompeiu formula for the sector ring with angle $\pi/\alpha$ ($\alpha \geq 1/2$) as follows.

**Theorem 2.1** For any $w \in C^1(\Omega^*; \mathbb{C}) \cap C(\overline{\Omega^*}; \mathbb{C})$, it has the following expression formula

$$w(z) = \frac{\alpha}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} \text{Re} w(\tau) H_1(z^\alpha, \tau^\alpha) \frac{d\tau}{\tau} + \frac{\alpha}{\pi i} \int_{[\sigma, \omega] \cup [r, 1]} \text{Re} w(\tau) H_2(z^\alpha, \tau^\alpha) \tau^{\alpha-1} d\tau$$

$$+ \frac{\alpha}{\pi} \int_{\Gamma_1} \text{Im} w(\tau) \frac{d\tau}{\tau} - \frac{\alpha}{\pi} \int_{\Omega^*} \left[ \tau^{\alpha-1} H_2(z^\alpha, \tau^\alpha) w_\tau(\tau) - \tau^{\alpha-1} H_2(z^\alpha, \overline{\tau^\alpha}) \overline{w_\tau(\tau)} \right] d\tau_1 d\tau_2,$$

where $\tau = \tau_1 + i \tau_2$, $\tau_1, \tau_2 \in \mathbb{R}$, $H_1, H_2$ are given by (1) and (2), respectively.

**Proof** Suppose $\tilde{R}^+ = \{ 0 < r^\alpha < |z| < 1, \text{Im} z > 0 \}$, and $\tilde{L}_2 = \{ |\tau| = r^\alpha, \text{Im} \tau > 0 \}$ is oriented clockwise. Making a conformal mapping [19],

$$\Delta_1 : \Omega^* \rightarrow \tilde{R}^+$$

$$z \mapsto z^\alpha$$

whose branch is along $(-\infty, r)$, which transforms $\Gamma_1$ onto $L_1$, $\Gamma_2$ onto $\tilde{L}_2$, $[r, 1]$ onto $[r^\alpha, 1]$, and $[\sigma, \omega]$ is mapped onto $[-1, -r^\alpha]$, respectively. In the same time, the mapping

$$\Delta_2 : \tilde{R}^+ \rightarrow \Omega^*$$

$$z \mapsto z^{1/\alpha}$$
maps the boundary of $\tilde{R}^+$ onto the relative boundary of $\Omega^*$, respectively.

Assume that $f(z) \triangleq \partial_\tau w(z)$ and $W(z) \triangleq w(z^{1/\alpha})$ for $z \in \tilde{R}^+$, then, $W(z) \in C^1(\tilde{R}^+; \mathbb{C}) \cap C(\tilde{R}^+; \mathbb{C})$. According to Lemma 2.1, for $z \in \Omega^*$,

$$W(z^\alpha) = \frac{1}{2\pi i} \int_{L_1 \cup L_2} \Re W(\zeta) H_1(z^\alpha, \zeta) \frac{d\zeta}{\zeta} + \frac{1}{\pi i} \int_{[-1, -r^\alpha] \cup [r^\alpha, 1]} \Re W(\zeta) H_2(z^\alpha, \zeta) d\zeta$$

$$+ \frac{1}{\pi} \int_{L_1} \Im W(\zeta) d\zeta - \frac{1}{\pi} \int_{R^+ \setminus \mathbb{R}} \left[ W_\zeta(\zeta) H_2(z^\alpha, \zeta) - \frac{W(z^\alpha)}{\zeta} H_2(z^\alpha, \tilde{\zeta}) \right] d\zeta d\eta,$$

where $W_\zeta(\zeta) = \frac{1}{\alpha} \zeta^{\frac{1}{\alpha} - 1} f(\zeta^{1/\alpha})$. Taking $\zeta = \tau^\alpha$ with $\tau = \tau_1 + i \tau_2$, then $d\zeta d\eta = \alpha^2 |\tau|^{2(\alpha - 1)} d\tau_1 d\tau_2$. Thus, when $z \in \Omega^*$,

$$w(z) = W(z^\alpha) = \frac{\alpha}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} \Re W(\tau^\alpha) H_1(z^\alpha, \tau^\alpha) \frac{d\tau}{\tau} + \frac{\alpha}{\pi} \int_{\Gamma_1} \Im W(\tau^\alpha) \frac{d\tau}{\tau}$$

$$+ \frac{\alpha}{\pi i} \int_{[\sigma, \omega] \cup [r, 1]} \Re W(\tau^\alpha) H_2(z^\alpha, \tau^\alpha) \tau^{\alpha - 1} d\tau$$

$$- \frac{\alpha}{\pi} \int_{\Omega^*} \left[ \tau^{\alpha - 1} H_2(z^\alpha, \tau^\alpha) f(\tau) - \overline{\tau^{\alpha - 1}} H_2(z^\alpha, \overline{\tau^\alpha}) f(\overline{\tau}) \right] d\tau_1 d\tau_2.$$

The proof is completed. \qed

**Lemma 2.2** When $\gamma(\zeta) \in C(\partial \Omega^*; \mathbb{R})$ and $t \in \partial \Omega^*$,

$$\lim_{z \in \Omega^*, \ z \to t} \frac{\alpha}{2\pi i} \int_{\partial \Omega^*} \gamma(\zeta) [H_2(z^\alpha, \zeta^\alpha) - H_2(\overline{z^\alpha}, \zeta^\alpha)] \zeta^{\alpha - 1} d\zeta = \gamma(t).$$

**Proof** By simply computation,

$$H_2(z^\alpha, \zeta^\alpha) - H_2(\overline{z^\alpha}, \zeta^\alpha) = H(z, \zeta),$$

where

$$H(z, \zeta) = \frac{1}{\zeta^\alpha - z^\alpha} - \frac{1}{\zeta^\alpha - \overline{z^\alpha}} \frac{\overline{z^\alpha}}{1 - \overline{z^\alpha} \zeta^\alpha} - \frac{z^\alpha}{1 - z^\alpha \overline{\zeta}^\alpha}$$

$$+ \sum_{n=1}^{\infty} \left[ \frac{1}{r^{2an} \zeta^\alpha - z^\alpha} - \frac{1}{r^{2an} \zeta^\alpha - \overline{z^\alpha}} - \frac{\overline{z^\alpha}}{r^{2an} - z^\alpha \zeta^\alpha} + \frac{z^\alpha}{r^{2an} - \overline{z^\alpha} \overline{\zeta}^\alpha} \right]$$

$$- \sum_{n=1}^{\infty} \left[ \frac{1}{r^{2an} \zeta^\alpha - z^\alpha} - \frac{1}{r^{2an} \zeta^\alpha - \overline{z^\alpha}} - \frac{\overline{z^\alpha}}{r^{2an} - z^\alpha \zeta^\alpha} + \frac{z^\alpha}{r^{2an} - \overline{z^\alpha} \overline{\zeta}^\alpha} \right]$$

which is the same as in [17]. From Lemmas 2.1–2.8 in [17], we obtain

$$\lim_{z \in \Omega^*, \ z \to t} \frac{\alpha}{2\pi i} \int_{\partial \Omega^*} \gamma(\zeta) H(z, \zeta) \zeta^{\alpha - 1} d\zeta = \gamma(t).$$
Then the proof is completed. □

Combining Theorem 2.1 with Lemma 2.2, we can solve the following Schwarz problem.

**Theorem 2.2** The Schwarz problem for the Cauchy–Riemann equation in $\Omega^*$

\[
\begin{cases}
    w_\gamma = f \text{ in } \Omega^*, \\
    \text{Re } w = \gamma \text{ on } \partial \Omega^*, \\
    \frac{\alpha}{\pi i} \int_{\Gamma_1} \frac{\text{Im } w(\zeta)}{\zeta} \, d\zeta = c, \quad c \in \mathbb{R}
\end{cases}
\]

with $f \in L_p(\Omega^*; \mathbb{C})$, $p > 2$ and $\gamma \in C(\partial \Omega^*; \mathbb{R})$, is solvable by

\[
w(z) = \frac{\alpha}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} \gamma(\zeta) H_1(z^\alpha, \zeta^\alpha) \frac{d\zeta}{\zeta} + \frac{\alpha}{\pi i} \int_{[r, \sigma]} \gamma(\zeta) H_2(z^\alpha, \zeta^\alpha) \zeta^{\alpha-1} \, d\zeta
\]

\[+ic - \frac{\alpha}{\pi} \int_{\Omega^*} \left[ \frac{1}{\zeta^\alpha} H_2(z^\alpha, \zeta^\alpha) f(\zeta) - \frac{1}{\zeta^{\alpha-1}} H_2(z^\alpha, \overline{\zeta^\alpha}) \overline{f(\zeta)} \right] d\xi d\eta, \quad z \in \Omega^*, \quad (4)
\]

where $\alpha \geq \frac{1}{2}$, and $H_1$, $H_2$ are defined by (1), (2), respectively.

**Proof** By Theorem 2.1, it is sufficient to prove that (4) is the solution of the Schwarz problem. Obviously, the boundary integral and constant in (4) are analytic. Moreover, the kernel in area integral can be rewritten

\[
\frac{\alpha \zeta^{\alpha-1}}{\zeta^\alpha - z^\alpha} + \text{others} = \frac{1}{\zeta - z} + \rho(z, \zeta), \quad z, \zeta \in \Omega^*.
\]

Here, $\rho(z, \zeta)$ is analytic about $z$. Thus, it is easy to obtain $\partial_z w(z) = f(z)$.

Next, when $\zeta \in \Gamma_1$,

\[
\frac{\alpha}{2\pi i} \int_{\Gamma_1} H_1(z^\alpha, \zeta^\alpha) \frac{dz}{z} = \frac{1}{2\pi i} \int_{|z|=1} \left( \frac{\zeta^\alpha + z}{\zeta^\alpha - z} + 2 \sum_{m=1}^{\infty} \left[ \frac{r^{2m\alpha} \zeta^\alpha}{r^{2m\alpha} \zeta^\alpha - z} - \frac{r^{2m\alpha} \zeta^\alpha}{r^{2m\alpha} \zeta^\alpha - z} \right] \right) \frac{dz}{z} = 0.
\]

Further, for $\zeta \in \Gamma_2$, we have

\[
\frac{\alpha}{2\pi i} \int_{\Gamma_1} H_1(z^\alpha, \zeta^\alpha) \frac{dz}{z} = \frac{1}{\pi i} \int_{|z|=1} \left( \frac{\zeta^\alpha}{\zeta^\alpha - z} + \sum_{m=1}^{\infty} \left[ \frac{r^{2m\alpha} \zeta^\alpha}{r^{2m\alpha} \zeta^\alpha - z} - \frac{r^{2m\alpha} \zeta^\alpha}{r^{2m\alpha} - z} \right] \right) \frac{dz}{z} = 0.
\]
Also, for \( \zeta \in (r, 1) \cup (\varpi, \omega) \cup \Omega^* \),
\[
\frac{\alpha}{2\pi i} \int_{\Gamma_1} \frac{H_2(z^\alpha, \zeta^\alpha)}{z} \, dz
= \frac{1}{2\pi i} \int_{|z|=1} \left( \frac{1}{\zeta^\alpha - z} + \sum_{m=1}^{\infty} \left[ \frac{r^{2ma}}{\zeta^\alpha - z} - \frac{z}{\zeta^\alpha (z - r^{-2ma} \zeta^\alpha)} \right] \right) \frac{dz}{z} = 0.
\]
Similarly, when \( \zeta \in \Omega^* \),
\[
\frac{\alpha}{2\pi i} \int_{\Gamma_1} \frac{H_2(z^\alpha, \zeta^\alpha)}{z} \, dz = 0.
\]
Therefore, by interchange of integral order, we obtain
\[
\alpha \pi \int_{\Gamma_1} \text{Im} \frac{w(\zeta)}{\zeta} \, d\zeta = c.
\]
Suppose \( \tilde{w} \) be the area integral of (4), by simple calculation,
\[
\text{Re} \tilde{w}(z) = -\frac{\alpha}{2\pi} \iint_{\partial \Omega^*} \left[ f(\zeta) G^*(z, \zeta) + \overline{f(\zeta)} \overline{G^*(z, \zeta)} \right] \, d\zeta \, d\eta
\]
with \( G^*(z, \zeta) = \zeta^{\alpha-1} [H_2(z^{\alpha}, \zeta^{\alpha}) - H_2(z^{\overline{\alpha}}, \zeta^{\alpha})] \). Since for \( (z, \zeta) \in \partial \Omega^* \times \Omega^* \), \( G^*(z, \zeta) = 0 \), which means that \( \text{Re} \tilde{w}(z) = 0 \) is true for \( z \in \partial \Omega^* \). Then, we express
\[
\text{Re} w(z) = \frac{\alpha}{2\pi i} \int_{\partial \Omega^*} \gamma(\zeta) \left[ H_2(z^{\alpha}, \zeta^{\alpha}) - H_2(z^{\overline{\alpha}}, \zeta^{\alpha}) \right] \zeta^{\alpha-1} \, d\zeta + \text{Re} \tilde{w}(z). \tag{5}
\]
By Lemma 2.2 and (5), we obtain, for all \( t \in \partial \Omega^* \),
\[
\lim_{z \to t} \text{Re} w(z) = \gamma(t).
\]
The proof is completed. \( \square \)

**Remark 2.1** When \( \alpha = 1 \), Theorem 2.2 is just the one result in [16].

### 3 Schwarz Problem for Inhomogeneous Polyanalytic equations

To solve Schwarz problem for the inhomogeneous polyanalytic equation, we define a poly-Schwarz operator and a Pompeiu operator, respectively. For \( f \in L_p(\Omega^*; C) \), \( p > 2 \), and \( \rho_k \in C(\partial \Omega^*, \mathbb{R}) \), \( k = 0, 1 \ldots, n-1 \),
\[
S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](z)
\]
\[
\triangleq \sum_{k=0}^{n-1} \left( \frac{-1}{k!} \right) \left\{ \frac{\alpha}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} \rho_k(\zeta) \left( \zeta - z + \overline{\zeta - z} \right)^k H_1(z^\alpha, \zeta^\alpha) \, d\zeta \right\}
+ \frac{\alpha}{\pi i} \int_{[\varpi, \omega] \cup [r, 1]} \rho_k(\zeta) \left( \zeta - z + \overline{\zeta - z} \right)^k H_2(z^\alpha, \zeta^\alpha) \zeta^{\alpha-1} \, d\zeta, \quad z \in \Omega^*, \tag{6}
\]
and

\[ T_n[f](z) \triangleq \frac{(-1)^n \alpha}{\pi (n-1)!} \int_{\Omega} \left( \zeta - z + \bar{\zeta} - \bar{z} \right)^{n-1} \left[ \zeta^{\alpha-1} H_2(z^\alpha, \zeta^\alpha) f(\zeta) - \bar{\zeta}^{\alpha-1} H_2(\bar{z}^\alpha, \bar{\zeta}^\alpha) \bar{f}(\bar{\zeta}) \right] d\zeta d\bar{\zeta}, \quad z \in \Omega^* . \]  

(7)

The operators \(T_n\) and \(S_n\) have the following differentiability and boundary properties.

**Lemma 3.1** When \(\rho_0, \rho_1, \ldots, \rho_{n-1} \in C(\partial \Omega^*, \mathbb{R})\) and \(t \in \partial \Omega^*\), then

\[
\begin{cases}
\frac{\partial^n S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](z)}{\partial \overline{z}^n} = 0, & z \in \Omega^* , \\
\Re \frac{\partial^k S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}]}{\partial \overline{z}^k}(t) = \rho_k(t), & k = 0, 1, 2, \ldots, n-1
\end{cases}
\]

with \(\frac{\partial^k S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}]}{\partial \overline{z}^k}(z) = S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](z)\) for \(z \in \Omega^*\) and \(k = 0\).

**Proof** Obviously, according to (6), the first equation in this lemma is true. Furthermore, for \(k = 0, 1, 2, \ldots, n-1\),

\[
\frac{\partial^k S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](z)}{\partial \overline{z}^k} = \sum_{l=k}^{n-1} \sum_{j=0}^{l-k} \frac{(-1)^{l-k}}{j!(l-k-j)!} (-z - \bar{z})^{l-k-j} \left\{ \frac{\alpha}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} \rho_l(\xi) (\xi + \bar{\xi})^j H_1(z^\alpha, \xi^\alpha) \frac{d\xi}{\xi} \right. \\
+ \frac{\alpha}{\pi i} \int_{[\sigma, \omega] \cup [r, 1]} \rho_l(\xi) (\xi + \bar{\xi})^j H_2(z^\alpha, \xi^\alpha) \xi^{\alpha-1} d\xi \biggr\}.
\]

By simple calculation, for \(\xi \in \Gamma_1 \cup \Gamma_2\),

\[
\frac{1}{2\xi} \left[ H_1(z^\alpha, \xi^\alpha) + H_1(\bar{z}^\alpha, \bar{\xi}^\alpha) \right] = \left[ H_2(z^\alpha, \xi^\alpha) - H_2(\bar{z}^\alpha, \bar{\xi}^\alpha) \right] \xi^{\alpha-1},
\]

which implies

\[
\Re \left\{ \frac{\partial^k S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](t)}{\partial \overline{z}^k} \right\} = \sum_{l=k}^{n-1} \sum_{j=0}^{l-k} \frac{(-1)^{l-k}}{j!(l-k-j)!} \frac{\alpha}{2\pi i} \int_{\partial \Omega^*} \rho_l(\xi) (\xi + \bar{\xi})^j G(z, \xi) \xi^{\alpha-1} d\xi,
\]

where

\[
G(z, \xi) = H_2(z^\alpha, \xi^\alpha) - H_2(\bar{z}^\alpha, \bar{\xi}^\alpha), \quad z, \xi \in \Omega^*. \quad (8)
\]
By Lemma 2.2,
\[
\lim_{z \to t, t \in \partial \Omega^*} \frac{\alpha}{2\pi i} \int_{\partial \Omega^*} \rho_i(\xi) (\xi + \overline{\xi})^j G(z, \xi) \xi^{\alpha-1} d\xi = \rho_i(t) (t + \overline{t})^j,
\]
hence,
\[
\left\{ \text{Re} \frac{\partial^k S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}]}{\partial z^k} \right\}^+(t) = \sum_{l=k}^{n-1} \sum_{j=0}^{l-k} \frac{(-1)^j (t + \overline{t})^{l-k} \rho_l(\xi)}{j!(l-k-j)!} = \rho_k(t).
\]
Therefore, the proof is finished. \(\Box\)

**Lemma 3.2** When \( f \in L_p(\Omega^*; \mathbb{C}), \ p > 2, \ z \in \Omega^* \) and \( t \in \partial \Omega^* \),
\[
\begin{cases}
\frac{\partial T_k[f](z)}{\partial z} = T_{k-1}[f](z), \ k \geq 2, \ k \in \mathbb{N}, \\
\frac{\partial^n T_n[f](z)}{\partial z^n} = f(z), \ n \geq 1, \ n \in \mathbb{N}, \\
\{\text{Re} \frac{\partial^k T_n[f]}{\partial z^k}\}^+(t) = 0, \ k = 0, 1, \ldots, n - 1.
\end{cases}
\]

**Proof** When \( k \geq 2 \), we know
\[
T_k[f](z) = \frac{(-1)^{k-1}}{(k-1)!} \sum_{j=0}^{k-1} \binom{k-1}{j} (-z - \overline{z})^{k-j-1} T_1[(\xi + \overline{\xi}^j) f](z) \tag{9}
\]
with
\[
T_1[f](z) = -\frac{\alpha}{\pi} \int_{\Omega^*} \int_{\Omega^*} \left[ \xi^{\alpha-1} H_2(\xi^\alpha, \overline{\xi}^\alpha) f(\xi) - \overline{\xi}^{\alpha-1} H_2(\xi^\alpha, \overline{\xi}^\alpha) f(\overline{\xi}) \right] d\xi d\eta. \tag{10}
\]
From the proof in Theorem 2.2, \( \frac{\partial T_1[f](z)}{\partial \overline{z}} = f(z) \). Then
\[
\frac{\partial T_k[f](z)}{\partial \overline{z}} = \frac{(-1)^{k-2}}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} (-z - \overline{z})^{k-j-2} T_1[(\xi + \overline{\xi}^j) f](z)
\]
\[
= T_{k-1}[f](z),
\]
which implies \( \frac{\partial^n T_n[f](z)}{\partial \overline{z}^n} = \frac{\partial T_1[f](z)}{\partial \overline{z}} = f(z) \) for \( n \geq 1 \). Furthermore, by (9), we obtain for \( n \geq 1 \),
\[
\text{Re}[T_n[f](z)] = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} (-z - \overline{z})^{n-k-1} \text{Re} \left\{ T_1[(\xi + \overline{\xi})^k f](z) \right\}, \tag{11}
\]

where
\[
\text{Re} \left\{ T_1[(\zeta + \bar{\zeta})^k f](z) \right\} = -\frac{\alpha}{2\pi} \int_{\Omega} (\zeta + \bar{\zeta})^k \left[ f(\xi)G^*(z, \xi) + \bar{f}(\xi) \bar{G}^*(z, \xi) \right] d\xi d\eta
\]
with \(G^*(z, \xi)\) given in the proof of Theorem 2.2, and for \((z, \xi) \in \partial \Omega^* \times \Omega^*,\) \(G^*(z, \xi) = 0.\) Thus, \(\text{Re} \left\{ T_1[(\zeta + \bar{\zeta})^k f](z) \right\}^+ \) for \(t \in \partial \Omega^*\), which implies \(\text{Re} T_n[f]^+(t) = 0\) for \(n \geq 1.\) In addition, from the first equation of this lemma, \(\partial^k T_n[f](z) = T_{n-k}[f](z)\) for \(k = 0, 1, \ldots n - 1,\) then \(\text{Re} \partial^k T_n[f]^+(t) = 0\) for \(t \in \partial \Omega^*.\) Therefore, the proof is completed. \(\square\)

By Lemmas 3.1–3.2, we can solve the following Schwarz problem for polyanalytic equation.

**Theorem 3.1** For the Schwarz problem
\[
\begin{aligned}
\frac{\partial^n w(z)}{\partial \bar{z}^n} & = f(z), \quad z \in \Omega^*, \quad f \in L_p(\Omega^*, \mathbb{C}), \quad p > 2, \\
\{\text{Re} \partial^k \bar{\zeta} w\}^+ & (t) = \rho_k(t), \quad t \in \partial \Omega^*, \quad \rho_k \in C(\partial \Omega^*, \mathbb{R}), \quad k = 0, 1, \ldots n - 1, \\
\text{Im} \partial^k \bar{\zeta} w(z_0) & = c_k, \quad z_0 \in \Omega^*, \quad k = 0, 1, \ldots n - 1,
\end{aligned}
\]
(12)

it is solvable by
\[
w(z) = S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](z) + T_n[f](z) + i \sum_{k=0}^{n-1} (z + \bar{z})^k a_k.
\]

Here \(S_n, T_n\) are defined by (6) and (7), respectively. \(a_k \in \mathbb{R}\) is given by
\[
P = B(z_0 + \bar{z}_0) Q, \quad P = (a_0, a_1, \ldots a_{n-1})^T,
\]
(13)

where \(Q = (q_0, q_1, \ldots q_{n-1})^T,\)
\[
q_j = \begin{cases} 
  c_j - \text{Im} \partial^j \bar{\zeta} S_n(z_0) - \text{Im} \partial^j \bar{\zeta} T_n(z_0), & j = 1, 2, \ldots n - 1, \\
  c_0 - \text{Im} S_n(z_0) - \text{Im} T_n(z_0), & j = 0,
\end{cases}
\]

and
\[
B(z) = (d_{r,m})_{n \times n} \text{ with } d_{r,m} = \begin{cases} 
  (-1)^{m-r} \frac{z^{m-r}}{(r-1)! (m-r)!}, & r \leq m, \\
  0, & r > m.
\end{cases}
\]
Proof From Lemmas 3.1 and 3.2, it is easy to know that $S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](z) + T_n[f](z)$ satisfies the first two conditions of (12). So we write

$$w(z) = S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](z) + T_n[f](z) + V(z),$$

which means

$$\begin{cases}
\partial^a_z V(z) = 0, & z \in \Omega^*; \\
\{\text{Re}(\partial^k_z V)\}^+(t) = 0, & t \in \partial\Omega^*, \ k = 0, 1, \ldots, n-1.
\end{cases} \quad (14)$$

According to the result in [20], $V(z)$ of order $n$ has the expression as follows

$$V(z) = \sum_{k=0}^{n-1} (z + \bar{z})^k f_k(z), \quad z \in \Omega^* \quad (15)$$

with $f_k$ being analytic functions. Furthermore, from the second equation of (14), $\{\text{Re} f_k\}^+(t) = 0, \ t \in \partial\Omega^*$. Therefore, by the Theorem 2.1, we obtain $f_k(z) = ia_k$, where $a_k$ are free real numbers. Then,

$$w(z) = S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](z) + T_n[f](z) + i \sum_{k=0}^{n-1} (z + \bar{z})^k a_k. \quad (16)$$

Taking (16) into the third condition of (12), we obtain for $k = 0, 1, \ldots, n-1$,

$$c_k - \text{Im}(\partial^k_z S_n)(z_0) - \text{Im}(\partial^k_z T_n)(z_0) = \sum_{l=k}^{n-1} \frac{l!}{(l-k)!} (z + \bar{z})^{l-k} a_l,$$

which can be expressed in matrix form as $A(z_0 + \bar{z}_0) P = Q$, where

$$A(z) = (a_{r,m})_{n \times n} \quad \text{with} \quad a_{r,m} = \begin{cases}
\frac{(m-1)!}{(m-r)!} z^{m-r}, & r \leq m, \\
0, & r > m.
\end{cases}$$

Since $A(z)$ is an invertible upper triangular matrix, thus, the constants $a_k$ are defined by $P = A^{-1}(z_0 + \bar{z}_0) Q = B(z_0 + \bar{z}_0) Q$. The proof is completed. $\square$

4 Schwarz Problem for a Generalized Polyanalytic Equation

In this section, we consider Schwarz problem of a generalized polyanalytic equation for the domain $R^+ = \{0 < r < |z| < 1, \ \text{Im} z > 0\}$. We firstly discuss some properties in more detail for the operator $T_n$, which will be used in the sequel. Let
\[ \tilde{T}_k[f](z) = \frac{(-1)^k}{\pi(k-1)!} \int_{R^+} (\xi - z + \bar{\xi} - z)^{k-1} \left[ H(z, \xi) f(\xi) - H^*(z, \xi) f(\xi) - H^*(z, \bar{\xi}) f(\xi) \right] d\xi d\eta, \quad z \in R^+, \tag{17} \]

which is just (7) for \( \alpha = 1 \). Here

\[ H^*(z, \xi) = \frac{1}{\xi - z} - \frac{z}{1 - z \xi} \]

\[ + \sum_{n=1}^{\infty} \frac{r^{2n}}{r^{2n} \xi - z} - \frac{z}{\xi (r^{2n} \xi - \xi)} - \frac{1}{\xi (r^{2n} - \xi)} + \frac{z}{r^{2n} \xi - 1}. \tag{18} \]

Then from Leibniz rule, we obtain for \( 0 \leq l \leq k - 1 \)

\[ \frac{\partial^l \tilde{T}_k[f](z)}{\partial z^l} = \frac{1}{\pi} \int_{R^+} \sum_{j=0}^{l} \left( \begin{array}{c} l \\ j \end{array} \right) \frac{(-1)^{k-j} (l-j)!}{(k-j-1)!} (\xi - z + \bar{\xi} - z)^{k-j-1} \left[ f(\xi) g(z, \xi) - f(\xi) g(z, \bar{\xi}) \right] d\xi d\eta, \tag{19} \]

where

\[ g(z, \xi) = \frac{1}{(\xi - z)^{l-j+1}} - \frac{\xi^{l-j-1}}{(1 - z \xi)^{l-j+1}} + \sum_{n=1}^{\infty} \frac{r^{2n}}{(r^{2n} \xi - z)^{l-j+1}} \left( \begin{array}{c} r^{2n} \\ -1 \end{array} \right) \frac{(l-j)^{l-j-1}}{(r^{2n} - \xi)^{l-j+1}}. \tag{20} \]

Lemma 4.1 For \( f \in L_p(R^+, \mathbb{C}) \), \( p > 2 \) and \( k \in \mathbb{N} \), then

\[ \left| \frac{\partial^l \tilde{T}_k[f](z)}{\partial z^l} \right| \leq C(l, p) \| f \|_{L_p(R^+)}, \quad z \in R^+, \quad l = 0, 1, \ldots, k - 1, \]

where \( C(l, p) \) is a constant depending on \( l \) and \( p \).

Proof We rewrite \( g \) in (20) as

\[ g(z, \xi) = \frac{1}{(\xi - z)^{l-j+1}} - \frac{\xi^{l-j-1}}{(1 - z \xi)^{l-j+1}} - \frac{r^{2l} \xi^{l-j-1}}{(r^2 - z \xi)^{l-j+1}} + g^*(z, \xi), \tag{21} \]
with

\[ g^*(z, \zeta) = \sum_{n=1}^{\infty} \left[ \frac{r^{2n}}{(r^{2n}z - \zeta)^{l+j+1}} - \frac{(-1)^{l-j}r^{2n(l-j)}}{(r^{2n}z - \zeta)^{l+j+1}} - \frac{r^{2(n+1)}\zeta^{l-j-1}}{(r^{2n+2}z - \zeta)^{l+j+1}} \right. \]

\[ \left. + \frac{(-1)^{l-j}r^{2n(l-j)}\zeta^{l-j-1}}{(r^{2n}z \zeta - 1)^{l+j+1}} \right]. \tag{22} \]

By simple calculation, for \( z, \zeta \in R^+ \), there exists some \( M > 0 \) such that \( |r^{2n}z - \zeta| > M, |r^{2n}z \zeta - 1| > M \) for \( n \geq 1 \), and \( |r^{2n} - z\zeta| > M \) for \( n \geq 2 \), which means that \( g^*(z, \zeta) \) is bounded for \( z, \zeta \in R^+ \). Similarly, \( g^*(z, \bar{\zeta}) \) is also bounded for \( z, \zeta \in R^+ \). In addition, from (19), we obtain

\[ \frac{\partial^l \mathcal{F}_k[f](z)}{\partial z^l} = \sum_{j=0}^{l} \binom{l}{j} (-1)^{k-j} (l-j)! \left[ A_{l,j}(z) + B_{l,j}(z) \right], \tag{23} \]

where

\[ A_{l,j}(z) = \frac{1}{\pi} \int_{R^+} \left[ \frac{1}{(\zeta - z)^{l+j+1}} - \frac{\zeta^{l-j-1}}{(1 - z\zeta)^{l+j+1}} - \frac{r^{2n}\zeta^{l-j-1}}{(r^{2n}z - \zeta)^{l+j+1}} \right] \frac{1}{(\zeta - z)^{l+j+1}} - \frac{\zeta^{l-j-1}}{(1 - z\zeta)^{l+j+1}} - \frac{r^{2n}\zeta^{l-j-1}}{(r^{2n}z - \zeta)^{l+j+1}} \right] d\xi d\eta, \tag{24} \]

and

\[ B_{l,j}(z) = \frac{1}{\pi} \int_{R^+} \left[ f(\xi) - f(\bar{\zeta}) \right] g^*(z, \zeta) \left[ f(\xi) g^*(z, \zeta) - f(\bar{\zeta}) g^*(z, \bar{\zeta}) \right] d\xi d\eta, \tag{25} \]

with \( g^* \) given by (22). Since for \( z, \zeta \in R^+ \), we have the fact that

\[ |\zeta - z| < |\bar{\zeta} - z|, \quad |\zeta - z| < |1 - z\zeta| < |1 - z\bar{\zeta}|, \tag{26} \]

and

\[ r|\zeta - z| < |r^2 - z\zeta| < |r^2 - z\bar{\zeta}|, \tag{27} \]
thus, making use of Hölder inequality, for \( \frac{1}{p} + \frac{1}{q} = 1 \),

\[
|A_{l,j}(z)| \leq C(l, j) \left( \int_{\mathbb{R}^+} |\zeta - z|^{q(k-l-2)} \, d\xi \, d\eta \right)^{1/q} \| f \|_{L^p(\mathbb{R}^+)} , \quad j = 0, 1, \ldots l.
\]

Therefore,

\[
|A_{l,j}(z)| \leq C(l, p, j) \| f \|_{L^p(\mathbb{R}^+)}
\]

is true since \( k - l - 2 \geq -1 \) and \( p > 2 \), where \( C(l, p, j) \) is a constant depending on \( l \) and \( p \). Also,

\[
|B_{l,j}(z)| \leq C(l, j) \left( \int_{\mathbb{R}^+} |\zeta - z|^{q(k-j-1)} \, d\xi \, d\eta \right)^{1/q} \| f \|_{L^p(\mathbb{R}^+)} , \quad j = 0, 1, \ldots l,
\]

and \( k - j - 1 \geq 0 \), which implies \( |B_{l,j}(z)| \leq C(l, p, j) \| f \|_{L^p(\mathbb{R}^+)} \). Thus, the proof is finished. \( \square \)

**Lemma 4.2** Suppose \( f \in L^p(\mathbb{R}^+) \), \( p > 2 \), then for \( z_1, z_2 \in \mathbb{R}^+ \) and \( k \in \mathbb{N} \),

\[
\left| \frac{\partial^l \tilde{T}_k[f](z_1)}{\partial z^l} - \frac{\partial^l \tilde{T}_k[f](z_2)}{\partial z^l} \right| \leq C(l, p) \| f \|_{L^p(\mathbb{R}^+)} \begin{cases} |z_1 - z_2|, & 0 \leq l \leq k - 2, \\ |z_1 - z_2|^{(p-2)/p}, & l = k - 1, \end{cases}
\]

where \( C(l, p) \) is a constant depending on \( l \) and \( p \).

**Proof** When \( l = 0, 1, \ldots k - 2 \), from Lemma 4.1, we know

\[
\left| \frac{\partial^{l+1} \tilde{T}_k[f](z)}{\partial z^{l+1}} \right| \leq C(l, p) \| f \|_{L^p(\mathbb{R}^+)},
\]

then by the mean value theorem,

\[
\left| \frac{\partial^l \tilde{T}_k[f](z_1)}{\partial z^l} - \frac{\partial^l \tilde{T}_k[f](z_2)}{\partial z^l} \right| \leq C(l, p) \| f \|_{L^p(\mathbb{R}^+)} |z_1 - z_2|.
\] (28)

For \( l = k - 1 \), from (23),

\[
\left| \frac{\partial^{k-1} \tilde{T}_k[f](z_1)}{\partial z^{k-1}} - \frac{\partial^{k-1} \tilde{T}_k[f](z_2)}{\partial z^{k-1}} \right| \leq \sum_{j=0}^{k-1} \binom{k-1}{j} \left[ |A_{k-1,j}(z_1) - A_{k-1,j}(z_2)| + |B_{k-1,j}(z_1) - B_{k-1,j}(z_2)| \right]
\] (29)
with $A_{k-1,j} \cdot B_{k-1,j}$ given by (24) and (25) for $l = k - 1$, respectively.

Invoking the identity

$$b^{i+1}c^i - a^{i+1}d^i = (b - a) \sum_{j=0}^{i} b^{i-j} a^j c^{i-j} d^j + (c - d) \sum_{j=0}^{i-1} b^{i-j} a^{j+1} c^{i-1-j} d^j,$$

and $|\zeta - z| \leq |1 - z| \zeta$ for $z, \zeta \in R^+$, we obtain for $j = 0, 1, \ldots, k - 1$,

$$| (1 - z_2\zeta)^{k-j} (\zeta - z_1 + \zeta - z_1)^{k-j-1} - (1 - z_1\zeta)^{k-j} (\zeta - z_2 + \zeta - z_2)^{k-j-1} | \leq C(k)|z_1 - z_2||1 - z_1\zeta|^{k-j-1}|1 - z_2\zeta|^{k-j-1},$$

where $C(k)$ is a constant depending on $l$, which gives

$$\frac{|(\zeta - z_1 + \zeta - z_1)^{k-j-1} - (\zeta - z_2 + \zeta - z_2)^{k-j-1}|}{|1 - z_1\zeta|^{k-j} - |1 - z_2\zeta|^{k-j}} \leq C(k)|z_1 - z_2| \frac{|1 - z_1\zeta|^{k-j-1}|1 - z_2\zeta|^{k-j-1}}{|\zeta - z_1| |\zeta - z_2|}.$$

Similarly, for $z, \zeta \in R^+$, in terms of (26), (27) and (30), we obtain

$$| (\zeta - z_1 + \zeta - z_1)^{k-j-1} - (\zeta - z_2 + \zeta - z_2)^{k-j-1} | \leq C(k)|z_1 - z_2| \frac{|1 - z_1\zeta|^{k-j-1}|1 - z_2\zeta|^{k-j-1}}{|\zeta - z_1| |\zeta - z_2|},$$

thus,

$$|A_{k-1,j}(z_1) - A_{k-1,j}(z_2)| \leq C(k)|z_1 - z_2| \int \int_{R^+}$$
Then,
\[ |A_{k-1,j}(z_1) - A_{k-1,j}(z_2)| \leq C(k, p) |z_1 - z_2|^{(p-2)/p} \| f \|_{L^p(R^+)} . \] (31)

By (30), for \( z, \zeta \in R^+ \)
\[ \left| (r^{2n} \zeta - z_2)^{k-j} (\zeta - z_1 + \zeta - z_1)^{k-j-1} - (r^{2n} \zeta - z_1)^{k-j} (\zeta - z_2 + \zeta - z_2)^{k-j-1} \right| \leq C(k) |z_1 - z_2| . \]

Also, for \( z, \zeta \in R^+ \), there exists some \( M > 0 \) such that \( |r^{2n} \zeta - z| > M \), therefore,
\[ \left| \frac{(\zeta - z_1 + \zeta - z_1)^{k-j-1}}{(r^{2n} \zeta - z_1)^{k-j}} - \frac{(\zeta - z_2 + \zeta - z_2)^{k-j-1}}{(r^{2n} \zeta - z_2)^{k-j}} \right| \leq C(k) |z_1 - z_2| , \]
which implies
\[ \sum_{n=1}^{\infty} r^{2n} \left[ \frac{(\zeta - z_1 + \zeta - z_1)^{k-j-1}}{(r^{2n} \zeta - z_1)^{k-j}} - \frac{(\zeta - z_2 + \zeta - z_2)^{k-j-1}}{(r^{2n} \zeta - z_2)^{k-j}} \right] \leq C(k) |z_1 - z_2| . \]

Similarly,
\[ \sum_{n=1}^{\infty} (-r^{2n})^{k-j-1} \left[ \frac{(\zeta - z_1 + \zeta - z_1)^{k-j-1}}{(r^{2n} \zeta - z_1)^{k-j}} - \frac{(\zeta - z_2 + \zeta - z_2)^{k-j-1}}{(r^{2n} \zeta - z_2)^{k-j}} \right] \leq C(k) |z_1 - z_2| , \]
\[ \sum_{n=1}^{2(n+1)} \left[ \frac{(\zeta - z_1 + \zeta - z_1)^{k-j-1}}{(r^{2n+2} \zeta - z_2)^{k-j}} - \frac{(\zeta - z_2 + \zeta - z_2)^{k-j-1}}{(r^{2n+2} \zeta - z_2)^{k-j}} \right] \leq C(k) |z_1 - z_2| , \]
\[ \sum_{n=1}^{\infty} (-r^{2n})^{k-j-1} \left[ \frac{(\zeta - z_1 + \zeta - z_1)^{k-j-1}}{(r^{2n} z_2 \zeta - 1)^{k-j}} - \frac{(\zeta - z_2 + \zeta - z_2)^{k-j-1}}{(r^{2n} z_2 \zeta - 1)^{k-j}} \right] \leq C(k) |z_1 - z_2| , \]
and
\[ \sum_{n=1}^{\infty} r^{2n} \left[ \frac{(\zeta - z_1 + \zeta - z_1)^{k-j-1}}{(r^{2n} \zeta - z_1)^{k-j}} - \frac{(\zeta - z_2 + \zeta - z_2)^{k-j-1}}{(r^{2n} \zeta - z_2)^{k-j}} \right] \leq C(k) |z_1 - z_2| , \]
\[ \sum_{n=1}^{\infty} (-r^{2n})^{k-j-1} \left[ \frac{(\zeta - z_1 + \zeta - z_1)^{k-j-1}}{(r^{2n} \zeta - z_1)^{k-j}} - \frac{(\zeta - z_2 + \zeta - z_2)^{k-j-1}}{(r^{2n} \zeta - z_2)^{k-j}} \right] \leq C(k) |z_1 - z_2| , \]
\[ \sum_{n=1}^{2(n+1)} \left[ \frac{(\zeta - z_1 + \zeta - z_1)^{k-j-1}}{(r^{2n+2} z_2 \zeta - 1)^{k-j}} - \frac{(\zeta - z_2 + \zeta - z_2)^{k-j-1}}{(r^{2n+2} z_2 \zeta - 1)^{k-j}} \right] \leq C(k) |z_1 - z_2| , \]
\[ \sum_{n=1}^{\infty} (-r^{2n})^{k-j-1} \left[ \frac{(\zeta - z_1 + \zeta - z_1)^{k-j-1}}{(r^{2n} z_1 \zeta - 1)^{k-j}} - \frac{(\zeta - z_2 + \zeta - z_2)^{k-j-1}}{(r^{2n} z_2 \zeta - 1)^{k-j}} \right] \leq C(k) |z_1 - z_2| . \]
Thus, from the expression of $g^*(z, \zeta)$ in (22) for $l = k - 1$, we obtain
\[
\left| (\zeta - z_1 + \bar{\zeta} - z_1)^{k-j-1} g^*(z_1, \zeta) - (\zeta - z_2 + \bar{\zeta} - z_2)^{k-j-1} g^*(z_2, \zeta) \right| \\
\leq C(k)|z_1 - z_2|,
\]
and
\[
\left| (\zeta - z_1 + \bar{\zeta} - z_1)^{k-j-1} g^*(z_1, \bar{\zeta}) - (\zeta - z_2 + \bar{\zeta} - z_2)^{k-j-1} g^*(z_2, \bar{\zeta}) \right| \\
\leq C(k)|z_1 - z_2|,
\]
then from (25),
\[
|B_{k-1,j}(z_1) - B_{k-1,j}(z_2)| \leq C(k, p)|z_1 - z_2| \parallel f \parallel_{L^p(R^+)}.
\tag{32}
\]
Therefore, from the (29), (31) and (32),
\[
\left| \frac{\partial^{k-1} \tilde{T}_k[f](z_1)}{\partial z^{k-1}} - \frac{\partial^{k-1} \tilde{T}_k[f](z_2)}{\partial z^{k-1}} \right| \leq C(k, p) \parallel f \parallel_{L^p(R^+)} |z_1 - z_2|^{(p-2)/p}.
\]
That is, the proof is completed. \hfill \Box

For the sake of investigating the boundedness of \( \frac{\partial^k \tilde{T}_k f}{\partial z^k} \), the following operators are introduced.
\[
\Xi_1 f(z) = \frac{(-1)^k k}{\pi} \int_{R^+} \left[ (\bar{\zeta} - \zeta)^{k-1} f(\zeta) - (\zeta - \bar{\zeta})^{k-1} \frac{f(\zeta)}{\bar{\zeta} - \zeta} \right] d\xi d\eta,
\tag{33}
\]
\[
\Xi_2 f(z) = \frac{(-1)^{k+1} k}{\pi} \int_{R^+} \left\{ \left[ |\zeta|^2 - 1 + \xi (\zeta - \bar{\zeta}) \right]^{k-1} f(\zeta) - \left[ |\zeta|^2 - 1 + \xi (\zeta - \bar{\zeta}) \right]^{k-1} \frac{f(\zeta)}{\bar{\zeta} - \zeta} \right\} d\xi d\eta,
\tag{34}
\]
\[
\Xi_3 f(z) = \frac{(-1)^{k+1} kr^2}{\pi} \int_{R^+} \left\{ \left[ |\zeta|^2 - r^2 + \xi (\zeta - \bar{\zeta}) \right]^{k-1} f(\zeta) - \left[ |\zeta|^2 - r^2 + \xi (\zeta - \bar{\zeta}) \right]^{k-1} \frac{f(\zeta)}{r^2 - \zeta} \right\} d\xi d\eta.
\tag{35}
\]

**Lemma 4.3** For \( f \in L^p(R^+) \) and \( p > 2 \),
\[
\parallel \Xi_1 f \parallel_{L^p(R^+)} \leq C(p) \parallel f \parallel_{L^p(R^+)},
\tag{36}
\]
where \( \Xi_1 f \) is defined by (33) and \( C(p) \) is a constant depending on \( p \).
Proof Let $\Xi_1(z) = T f(z) - \tilde{T} f(z)$, where

$$T f(z) = \frac{(-1)^k k}{\pi} \int_{R^+} \left( \frac{\xi - z}{\xi - \zeta} \right)^{k-1} \frac{f(\zeta)}{(\xi - z)^2} d\xi d\eta$$

and

$$\tilde{T} f(z) = \frac{(-1)^k k}{\pi} \int_{R^+} \Omega_k(\zeta - z)\hat{f}(\zeta) d\xi d\eta.$$ 

Here

$$\Omega_k(z) = \left( \frac{\zeta}{z} \right)^{k-1} \left( 1 - \frac{z}{\zeta} \right) \Lambda(z),$$

with $\Lambda(z) = \left( \frac{z}{\zeta} \right)^k$.

From the result in [21], the singular integral operator $T$ is bounded on $L_p(R^+)$ for $p > 2$. Furthermore, similarly in Lemma 2.6 of [6], $\Lambda(z)$ is homogeneous of degree zero and $\int_{|z|=1} \Lambda(z)d\sigma(z) = 0$ with $d\sigma(z)$ being the arc length differential. Thus, the operator $\tilde{T}$ is bounded on $L_p(R^+)$ for $p > 2$, which means that $\Xi_1 f$ is bounded on $L_p(R^+)$ for $p > 2$. \(\square\)

Lemma 4.4 For $f \in L_p(R^+)$ and $p > 2$,

$$\| \Xi_2 f \|_{L_p(R^+)} \leq C(p) \| f \|_{L_p(R^+)},$$

(37)

where $\Xi_2 f$ is defined by (34) and $C(p)$ is a constant depending on $p$.

Proof According to (34), putting $\Xi_2(z) = P_1 f(z) + P_2 f(z)$ with

$$P_1 f(z) = \frac{(-1)^k k}{\pi} \int_{R^+} \left\{ \frac{(|\xi|^2 - 1 + \overline{\zeta}(\xi - z))^{k-1}}{(1 - z\overline{\zeta})^{k+1}} \right\} \hat{f}(\zeta) d\xi d\eta$$

and

$$P_2 f(z) = \frac{(-1)^{k+1} k}{\pi} \int_{R^+} \left\{ \frac{(|\xi|^2 - 1 + \zeta(\xi - z))^{k-1}}{(1 - z\zeta)^{k+1}} \right\} f(\zeta) d\xi d\eta.$$ 

Then, $P_1 f$ can be rewritten

$$P_1 f(z) = \frac{(-1)^k k}{\pi} \int_{D} \left\{ \frac{(|\xi|^2 - 1 + \overline{\zeta}(\xi - z))^{k-1}}{(1 - z\overline{\zeta})^{k+1}} \right\} \hat{f}(\zeta) d\xi d\eta.$$
with \( \mathbb{D} \) being the unit disc and

\[
\widehat{f}(z) = \begin{cases} 
0, & f \in \mathbb{D} \setminus R^+, \\
 f(z), & f \in R^+. 
\end{cases}
\]

By [22] and the Theorem 3.1 in [23], we obtain

\[
\| P_1 f \|_{L_p(\mathbb{D})} \leq C(p) \| \widehat{f} \|_{L_p(\mathbb{D})} = C(p) \| f \|_{L_p(R^+)},
\]

therefore,

\[
\| P_1 f \|_{L_p(R^+)} \leq \| P_1 f \|_{L_p(\mathbb{D})} \leq C(p) \| f \|_{L_p(R^+)},
\]

which implies that \( P_1 f \) is bounded on \( L_p(R^+) \) for \( p > 2 \).

Next, since \( |1 - z| > |\zeta - z| \) and \( |1 - z \zeta| > |1 - z \bar{\zeta}| \) for \( z, \zeta \in R^+ \), we obtain

\[
|P_2 f(z)| \leq C \sum_{m=0}^{k-1} \binom{k-1}{m} \int_{R^+} \int_{R^+} \frac{(1 - |\zeta|^2)^m |\zeta - \bar{\zeta}|^{k-1-m}}{|1 - z \zeta|^k} |f(\zeta)| d\xi d\eta
\]

\[
\leq C \sum_{m=0}^{k-1} \binom{k-1}{m} \int_{R^+} \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^m}{|1 - z \zeta|^m} |\zeta|^{|1 - \zeta|^2} |f(\zeta)| d\xi d\eta
\]

\[
= C \sum_{m=0}^{k-1} \binom{k-1}{m} \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^m}{|1 - z \zeta|^m} |\zeta|^{|1 - \zeta|^2} |f(\zeta)| d\xi d\eta
\]

with \( C \) being a constant. According to the proof of Theorem 3.1 in [23], \( P_2 \) is bounded on \( L_p(\mathbb{D}) \) for \( p > 2 \), thus

\[
\| P_2 f \|_{L_p(R^+)} \leq \| P_2 f \|_{L_p(\mathbb{D})} \leq C(p) \| \widehat{f} \|_{L_p(\mathbb{D})} = C(p) \| f \|_{L_p(R^+)}. 
\]

So, the proof is completed.

\( \square \)

**Lemma 4.5** For \( f \in L_p(R^+) \) and \( p > 2 \),

\[
\| \Xi_3 f \|_{L_p(R^+)} \leq C(p) \| f \|_{L_p(R^+)}, \tag{38}
\]

where \( \Xi_3 f \) is defined by (35) and \( C(p) \) is a constant depending on \( p \).

**Proof** By (35), suppose \( \Xi_3 f(z) = \Upsilon_1 f(z) + \Upsilon_2 f(z) \) with

\[
\Upsilon_1 f(z) = \frac{(-1)^k kr^2}{\pi} \int_{R^+} \int_{R^+} \frac{[|\xi|^2 - r^2 + \zeta(\xi - \bar{\zeta})]^{k-1}}{(r^2 - z \zeta)^k} f(\zeta) d\xi d\eta
\]

and

\[
\Upsilon_2 f(z) = \frac{(-1)^{k+1} kr^2}{\pi} \int_{R^+} \int_{R^+} \frac{[|\xi|^2 - r^2 + \zeta(\xi - \bar{\zeta})]^{k-1}}{(r^2 - z \zeta)^k} f(\zeta) d\xi d\eta.
\]
From the inequality \(|\zeta - z| < \frac{1}{r}|r^2 - \overline{z}\zeta|\), then

\[
|\mathcal{Y}_1 f(z)| \leq C_1 \sum_{m=0}^{k-1} \binom{k-1}{m} \int_{R^+} \int_{x|z|^2} \frac{(|\zeta|^2 - r^2)^m |\zeta - \overline{z}|^{k-1-m}}{|r^2 - \overline{z}\zeta|^{k+1}} |f(\zeta)| \, d\xi \, d\eta
\]

\[
\leq C_2 \sum_{m=0}^{k-1} \binom{k-1}{m} \int_{R^+} \int_{x|z|^2} \frac{(|\zeta|^2 - r^2)^m}{|r^2 - \overline{z}\zeta|^{k+1}} |f(\zeta)| \, d\xi \, d\eta.
\]

Let \(\tau = \frac{r}{\zeta}\) and \(z = \frac{r}{\mu}\), then \(\tau, \mu \in R^+\) for \(z, \zeta \in R^+\). Also suppose \(\tau = \tau_1 + i\tau_2\) with \(\tau_1, \tau_2 \in \mathbb{R}\), thus \(d\xi \, d\eta = \frac{r^2}{|\tau|^4} \, d\tau_1 \, d\tau_2\), and

\[
|\mathcal{Y}_1 f(z)| \leq C_2 |\mu|^{m+2} \frac{r^2}{|\tau|^4} \sum_{m=0}^{k-1} \binom{k-1}{m} \int_{R^+} \int_{x|z|^2} \frac{(1 - |\tau|^2)^m}{|\tau|^{m+2} |1 - \mu\tau|^{m+2}} |f\left(\frac{r}{\tau}\right)| \, d\tau_1 \, d\tau_2
\]

\[
\leq C_3 \sum_{m=0}^{k-1} \binom{k-1}{m} \int_{D} \frac{(1 - |\tau|^2)^m}{|1 - \mu\tau|^{m+2}} |\tilde{f}(\tau)| \, d\tau_1 \, d\tau_2
\]

with

\[
\tilde{f}(\tau) = \begin{cases} f\left(\frac{r}{\tau}\right), & \tau \in R^+, \\
0, & \tau \in \mathbb{D} \setminus R^+.
\end{cases}
\]

Since \(f \in L_p(R^+) \iff \tilde{f} \in L_p(D)\), and from the above Lemma 4.4,

\[
\| \mathcal{Y}_1 f \|_{L_p(R^+)} \leq \| \mathcal{Y}_1 f \|_{L_p(D)} \leq C(p) \| \tilde{f} \|_{L_p(D)} = C(p) \| \tilde{f} \|_{L_p(R^+)}
\]

\[
\leq C(p) \| f \|_{L_p(R^+)}.
\]

From \(|r^2 - z\zeta| > |r|\overline{z} - z|\) and \(|r^2 - z\zeta| > |r^2 - \overline{z}\zeta|\), we obtain

\[
|\mathcal{Y}_2 f(z)| \leq C_4 \sum_{m=0}^{k-1} \binom{k-1}{m} \int_{R^+} \int_{x|z|^2} \frac{(|\zeta|^2 - r^2)^m |\zeta - \overline{z}|^{k-1-m}}{|r^2 - \overline{z}\zeta|^{k+1}} |f(\zeta)| \, d\xi \, d\eta
\]

\[
\leq C_5 \sum_{m=0}^{k-1} \binom{k-1}{m} \int_{R^+} \int_{x|z|^2} \frac{(|\zeta|^2 - r^2)^m}{|r^2 - \overline{z}\zeta|^{k+1}} |f(\zeta)| \, d\xi \, d\eta,
\]

where \(C_i\) for \(i = 1, \ldots, 5\) are constants. According to the above result, \(\mathcal{Y}_2\) is bounded on \(L_p(R^+)\) for \(p > 2\), therefore,

\[
\| \mathcal{Y}_3 f \|_{L_p(R^+)} \leq C(p) \| f \|_{L_p(R^+)}.
\]

\[\square\]

**Lemma 4.6** Let \(f \in L_p(R^+, \mathbb{C})\), \(p > 2\), \(k \in \mathbb{N}\), and

\[
\tilde{\mathfrak{N}}_k f \triangleq \frac{\partial_k \tilde{f}}{\partial \overline{z}^k}.
\]

(39)
then
\[ \| \widehat{\Pi}_k f \|_{L^p(R^+)} \leq C(p) \| f \|_{L^p(R^+)}, \]

where \( C(p) \) is a constant depending on \( p \).

**Proof** From (19), (21) and (22) for \( l = k - 1 \), and differentiating with respect to \( z \), we obtain
\[ \frac{\partial^k \tilde{T}_k[f](z)}{\partial z^k} = \Xi_1 f(z) + \Xi_2 f(z) + \Xi_3 f(z) + \Xi_4 f(z) + \Xi_5 f(z), \tag{40} \]
where \( \Xi_1, \Xi_2, \Xi_3 \) are given by (33),(34) and (35), respectively, and
\[ \Xi_4 f(z) = \frac{1}{\pi} \int \int_{R^+} \Delta(z, \zeta) f(\zeta) d\xi d\eta, \]
\[ \Xi_5 f(z) = \frac{1}{\pi} \int \int_{R^+} \overline{\Delta(z, \zeta)} f(\zeta) d\xi d\eta \]
with
\[ \Delta(z, \zeta) = \sum_{j=0}^{k-2} \frac{(-1)^{k-j-1}(k-1)!}{j!(k-j-2)!} (\zeta - z + \overline{\zeta - z})^{k-j-2} h^*(z, \zeta) \]
\[ + \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^{k-j} (\zeta - z + \overline{\zeta - z})^{k-j-1} \partial_z h^*(z, \zeta), \]
where \( h^*(z, \zeta) = g^*(z, \zeta) \) for \( l = k - 1 \) given by (22). Here we know \( h^*(z, \zeta) \) and \( \partial_z h^*(z, \zeta) \) are bounded for \( z, \zeta \in R^+ \), which means that \( \Delta(z, \zeta) \) is bounded for \( z, \zeta \in R^+ \). Thus,
\[ \| \Xi_4 f \|_{L^p(R^+)} \leq C(p) \| f \|_{L^p(R^+)} . \tag{41} \]

Similarly,
\[ \| \Xi_5 f \|_{L^p(R^+)} \leq C(p) \| f \|_{L^p(R^+)} . \tag{42} \]
In terms of Lemmas 4.3–4.5, (41) and (42), \( \widehat{\Pi}_k \) is bounded. Then the proof is completed. \( \square \)

With the above preliminaries, we can solve the following Schwarz problem for a generalized inhomogeneous polyanalytic equation on basis of the properties of the operators \( \tilde{T}_k \).
Schwarz Problem For \( w \in W^{p,n}(R^+) \), we consider a generalized polyanalytic equation as follows

\[
\frac{\partial^n w}{\partial z^n} + \sum_{j=1}^{n} q_{1j}(z) \frac{\partial^n w}{\partial z^{n-j} \partial z^j} + \sum_{j=1}^{n} q_{2j}(z) \frac{\partial^n w}{\partial z^{n-j} \partial \bar{z}^j} \\
+ \sum_{l=0}^{n-1} \sum_{m=0}^{l} \left[ a_{ml}(z) \frac{\partial^l w}{\partial z^{l-m} \partial z^m} + b_{ml}(z) \frac{\partial^l w}{\partial z^{l-m} \partial \bar{z}^m} \right] = f(z), \quad z \in R^+,
\]

which satisfying for \( l = 0, 1, \ldots n - 1 \),

\[
\text{Re} \frac{\partial^l w}{\partial \bar{z}^l} = \rho_l, \quad z \in \partial R^+, \quad \text{Im} \frac{\partial^l w}{\partial \bar{z}^l} (z_0) = c_l, \quad z_0 \in R^+.
\]

Here, \( a_{ml}, b_{ml}, f \in L_p(R^+, \mathbb{C}), p > 2 \) and \( q_{1j}, q_{2j} \) are measurable bounded with

\[
\sum_{j=1}^{n} |q_{1j}(z)| + |q_{2j}(z)| < q_0 < 1.
\]

Similarly in [6], from (7), (17), Lemma 3.2 and Theorem 3.1, we have the following result.

Theorem 4.1 The Schwarz problem (43)–(45) can be transformed into the following equation

\[
(I + \tilde{\Xi} + \tilde{K}) g = \chi(z),
\]

with \( w = \tilde{T}_n g + \Phi(z) \),

\[
\tilde{\Xi} g = \sum_{j=1}^{n} \left( q_{1j} \tilde{\Pi}_j g + q_{2j} \tilde{\Pi}_j g \right),
\]

\[
\tilde{K} g = \sum_{l=0}^{n-1} \sum_{m=0}^{l} \left[ a_{ml}(z) \frac{\partial^m \tilde{T}_{n+m-l} g}{\partial z^m} + b_{ml}(z) \frac{\partial^m \tilde{T}_{n+m-l} g}{\partial \bar{z}^m} \right],
\]

\[
\Phi(z) = i \sum_{k=0}^{n-1} (z + \bar{z})^k a_k + S_n[\rho_0, \rho_1, \ldots, \rho_{n-1}](z),
\]
and
\[
\chi(z) = f - \sum_{j=1}^{n} \left( q_{1j}(z) \frac{\partial^n \Phi}{\partial z^{n-j} \partial z^j} + q_{2j}(z) \frac{\partial^k \Phi}{\partial z^{n-j} \partial z^j} \right) + \sum_{l=0}^{n-1} \sum_{m=0}^{l} \left[ a_{ml}(z) \frac{\partial^l \Phi}{\partial z^{l-m} \partial z^m} - b_{ml}(z) \frac{\partial^l \Phi}{\partial z^{l-m} \partial z^m} \right].
\]

Here, \( \tilde{T}_k \), \( \tilde{\Pi}_k \) are defined by (17) and (39), respectively. \( S_k \) is given by (6) for \( \alpha = 1 \). Moreover, \( a_k \) can be determined by the system of equations \( \text{Im} \frac{\partial^l \tilde{T}_n g}{\partial z^l} (z_0) + \text{Im} \frac{\partial^l \Phi}{\partial z^l} (z_0) = c_l \) for \( l = 0, 1, n - 1 \).

**Remark 4.1** Since the expression of \( \chi(z) \) in Theorem 4.1 involving the constants \( a_k \) through the function \( \Phi \), then \( g(z) \) given by (46) also includes \( a_k \). Thus from the boundary condition, we know \( a_k \) satisfies \( \text{Im} \frac{\partial^l \tilde{T}_n g}{\partial z^l} (z_0) + \text{Im} \frac{\partial^l \Phi}{\partial z^l} (z_0) = c_l \) for \( l = 0, 1, \ldots, n - 1 \), which is a system of equations for \( a_k \).

**Theorem 4.2** Suppose the condition
\[
q_0 \max_{1 \leq j \leq n} \| \tilde{\Pi}_j \|_{L_p(R^+)} < 1,
\]
then the solution for the above Schwarz problem is expressed by \( w = \tilde{T}_n g + \Phi(z) \), where \( g \in L_p(R^+) \), \( p > 2 \) satisfies the singular integral equation (46).

**Proof** By Lemmas 4.1–4.2 and Arzela Ascoli theorem, it is easy to know that \( \tilde{K} \) is compact. Moreover, Lemma 4.6 and (47) imply that \( I + \tilde{K} \) is an invertible operator on \( L^p(R^+) \). Then \( I + \tilde{K} \) is a Fredholm operator of index zero, which means that the singular integral equation (46) is solvable. Thus, by Theorem 4.1, the Schwarz problem (43)–(44) is solvable and its solution is \( w = \tilde{T}_n g + \Phi(z) \), where \( g \) is the solution of (46) given by \( (I + \tilde{\Xi} + \tilde{K})^{-1} \chi(z) \). The proof is completed. \( \square \)

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