Context-free Grammars and Multivariate Stable Polynomials over Stirling Permutations

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Abstract

Recently, Haglund and Visontai established the stability of the multivariate Eulerian polynomials as the generating polynomials of the Stirling permutations, which serves as a unification of some results of Bóna, Brenti, Janson, Kuba, and Panholzer concerning Stirling permutations. Let $B_n(x)$ be the generating polynomials of the descent statistic over Legendre-Stirling permutations, and let $T_n(x) = 2^n C_n(x/2)$, where $C_n(x)$ are the second-order Eulerian polynomials. Haglund and Visontai proposed the problems of finding multivariate stable refinements of the polynomials $B_n(x)$ and $T_n(x)$. We obtain context-free grammars leading to multivariate stable refinements of the polynomials $B_n(x)$ and $T_n(x)$. Moreover, the grammars enable us to obtain combinatorial interpretations of the multivariate polynomials in terms of Legendre-Stirling permutations and marked Stirling permutations. Such stable multivariate polynomials provide solutions to two problems posed by Haglund and Visontai.

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1 Introduction

This paper presents an approach to the construction of stable combinatorial polynomials from the perspective of context-free grammars. The framework of using context-free grammars to generate combinatorial polynomials was proposed by Chen \cite{6}. More specifically, we introduce the structure of marked Stirling permutations, and we find context-free grammars that lead to multivariate stable polynomials over marked Stirling permutations and Legendre-Stirling permutations. These multivariate stable polynomials provide solutions to two problems posed by Haglund and Visontai \cite{13} in their study of multivariate stable refinements of the second-order Eulerian polynomials.

Let us first review some backgrounds on the second-order Eulerian polynomials. These polynomials were first introduced by Gessel and Stanley \cite{10}, which are defined as the generating functions of the descent statistic over Stirling permutations. Recall that a Stirling permutation of order $n$ is a permutation $\pi = \pi_1 \pi_2 \cdots \pi_{2n-1} \pi_{2n}$ of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$, denoted by $[n]_2$, which satisfies the following condition: if $\pi_i = \pi_j$ then $\pi_k > \pi_i$ whenever $i < k < j$. For $1 \leq i \leq 2n$, we say that $i$ is a descent of $\pi$ if $i = 2n$ or $\pi_i > \pi_{i+1}$. Analogously, $i$ is called an ascent of $\pi$ if $i = 1$ or $\pi_{i-1} < \pi_i$. Let $Q_n$ denote the set of Stirling permutations of order $n$. Let $C(n, k)$ be the number of Stirling
permutations of \([n]_2\) with \(k\) descents, and let
\[
C_n(x) = \sum_{k=1}^{n} C(n, k)x^k.
\]

Gessel and Stanley \cite{10} showed that
\[
\sum_{n=0}^{\infty} S(n+k, k)x^n = \frac{C_n(x)}{(1-x)^{2k+1}},
\]
where \(S(n+k, k)\), as usual, denotes the Stirling number of the second kind. The numbers \(C(n, k)\) are called the second-order Eulerian numbers by Graham, Knuth and Patashnik \cite{11}, and accordingly the polynomials \(C_n(x)\) are called the second-order Eulerian polynomials by Haglund and Visontai \cite{13}.

The Stirling permutations were further studied by Bóna \cite{1}, Brenti \cite{5}, Janson \cite{14} and Janson, Kuba and Panholzer \cite{15}. Bóna \cite{1} introduced a statistic, called plateau, on Stirling permutations, and proved that ascents, descents and plateaux have the same distribution over \(Q_n\). Given a Stirling permutation \(\pi = \pi_1 \pi_2 \ldots \pi_{2n} \in Q_n\), the index \(i\) is called a plateau of \(\pi\) if \(\pi_{i-1} = \pi_i\).

Analogous to that of the classical Eulerian polynomials, Bóna \cite{1} obtained the real-rootedness of the second-order Eulerian polynomials \(C_n(x)\).

**Theorem 1.1** For any positive integer \(n\), the roots of the polynomial \(C_n(x)\) are all real, distinct, and non-positive.

It should be noted that the real-rootedness of \(C_n(x)\) is essentially the real rootedness of the generating function of generalized Stirling permutations obtained by Brenti \cite{5}. A permutation \(\pi\) of the multiset \(\{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\}\) is called a generalized Stirling permutation of rank \(n\) if \(\pi\) satisfies the same condition as for a Stirling permutation. Let \(Q^*_n\) denote the set of generalized Stirling permutations of rank \(n\). In particular, if \(r_1 = r_2 = \cdots = r_n = r\) for some \(r\), then \(\pi\) is called an \(r\)-Stirling permutation of order \(n\). Let \(Q_n(r)\) denote the set of \(r\)-Stirling permutations of order \(n\). It is clear that 1-Stirling permutations are ordinary permutations and 2-Stirling permutations are the Stirling permutations. Brenti \cite{5} showed that the descent generating polynomials over \(Q^*_n\) have only real roots.

Janson \cite{14} defined the following trivariate generating function
\[
C_n(x, y, z) = \sum_{\pi \in Q_n} x^{\text{des}(\pi)}y^{\text{asc}(\pi)}z^{\text{plat}(\pi)},
\]
where \(\text{des}(\pi)\), \(\text{asc}(\pi)\), and \(\text{plat}(\pi)\) denote the numbers of descents, the number of ascents, and the number of plateaux of \(\pi\), respectively, and proved that \(C_n(x, y, z)\) is symmetric in \(x, y, z\). This implies the equidistribution of these three statistics derived by Bóna.

The symmetric property of \(C_n(x, y, z)\) was further extended to \(r\)-Stirling permutations by Janson, Kuba and Panholzer \cite{15}. For an \(r\)-Stirling permutation, they introduced the notion of a \(j\)-plateau. For an \(r\)-Stirling permutation \(\pi = \pi_1 \pi_2 \ldots \pi_{nr}\) and an integer \(1 \leq j \leq r - 1\), a number \(1 \leq i < nr\) is called a \(j\)-plateau of \(\pi\) if \(\pi_i = \pi_{i+1}\) and there are \(j-1\) indices \(l < i\) such that \(\pi_l = \pi_i\), i.e., the number \(\pi_i\) appears \(j\) times up to the \(i\)-th position of \(\pi\). Let \(\text{j-plat}(\pi)\) denote the number of \(j\)-plateaux of \(\pi\). Meanwhile, define a descent and an ascent of \(\pi\)
similar as ordinary permutations, and let des(\(\pi\)) and asc(\(\pi\)) denote the number of descents and ascents of \(\pi\). Janson, Kuba and Panholzer [13] showed that the distribution of \((\text{des}, 1\text{-plat}, 2\text{-plat}, \ldots, (r-1)\text{-plat}, \text{asc})\) is symmetric over the set of \(r\)-Stirling permutations.

Based on the theory of multivariate stable polynomials recently developed by Borcea and Brändén [24], Haglund and Visontai [13] presented a unified approach to the stability of the generating functions of Stirling permutations and \(r\)-Stirling permutations. A polynomial \(f(z) \in \mathbb{C}[z] = \mathbb{C}[z_1, z_2, \ldots, z_m]\) is said to be stable, if whenever the imaginary part \(\text{Im}(z_i) > 0\) for all \(i\) then \(f(z) \neq 0\). Clearly, a univariate polynomial \(f(z) \in \mathbb{R}[z]\) has only real roots if and only if it is stable.

For the case of univariate real polynomials, Pólya and Schur [14] characterized all diagonal operators preserving stability or real-rootedness. Recently, Borcea and Brändén [2–4] characterized all linear operators preserving stability of multivariate polynomials, see also the survey of Wagner [18]. This implies a characterization of linear operators preserving stability of univariate polynomials.

A multivariate polynomial is called multiaffine if the degree of each variable is at most 1. Borcea and Brändén showed that each of the operators preserving stability of multiaffine polynomials has a simple form. Using this property, Haglund and Visontai [13] obtained a stable multiaffine refinement of the second-order Eulerian polynomial \(C_n(x)\). Given a Stirling permutation \(\pi = \pi_1 \pi_2 \cdots \pi_{2n} \in Q_n\), let

\[
A(\pi) = \{i|\pi_{i-1} < \pi_i\},
\]
\[
D(\pi) = \{i|\pi_i > \pi_{i+1}\},
\]
\[
P(\pi) = \{i|\pi_{i-1} = \pi_i\}
\]

denote the set of ascents, the set of descents and the set plateaux of \(\pi\), respectively. Define

\[
C_n(x, y, z) = \sum_{\pi \in Q_n} \prod_{i \in D(\pi)} x_{\pi_i} \prod_{i \in A(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.
\]

Haglund and Visontai [13] proved the stability of \(C_n(x, y, z)\).

**Theorem 1.2** The polynomial \(C_n(x, y, z)\) is stable.

It is worth mentioning that, as observed by Haglund and Visontai, the recurrence relation between \(C_{n-1}(x, y, z)\) and \(C_n(x, y, z)\) can be used to derive the symmetry of \(C_n(x, y, z)\), which implies the symmetry of \(C_n(x, y, z)\) obtained by Janson, Kuba and Panholzer [13].

Moreover, Haglund and Visontai extended the stability of \(C_n(x, y, z)\) to generating polynomials of \(r\)-Stirling permutations by taking the \(j\)-plateau statistic into consideration. Let \(P_j(\pi)\) denote the set of \(j\)-plateaux of \(\pi\). Haglund and Visontai [13] obtained the following multivariate stable polynomial over \(r\)-Stirling permutations

\[
E_n(x, y, z_1, \ldots, z_{r-1}) = \sum_{\pi \in Q_n(r)} \left( \prod_{i \in D(\pi)} x_{\pi_i} \right) \left( \prod_{i \in A(\pi)} y_{\pi_i} \right) \prod_{j=1}^{r-1} \left( \prod_{i \in P_j(\pi)} z_{j, \pi_i} \right).
\]
They also obtained a similar multivariate stable polynomial for generalized Stirling permutations.

In view of the real-rootedness of \( C_n(x) \) and its multivariate stable refinement \( C_n(x, y, z) \), Haglund and Visontai posed the problem of finding multivariate stable polynomials as refinements of the generating polynomials of the descent statistic over Legendre-Stirling permutations. The Legendre-Stirling permutations were introduced by Egge \[17\] as a generalization of Stirling permutations in the study of Legendre-Stirling numbers of the second kind. For any \( n \geq 1 \), let \( M_n \) be the multiset \( \{1, 1, 2, 2, 2, \ldots, n, n, \bar{n}\} \). A permutation \( \pi = \pi_1 \pi_2 \ldots \pi_{3n} \) on \( M_n \) is called a Legendre-Stirling permutation if whenever \( i < j < k \) and \( \pi_i = \pi_k \) are both unbarred, then \( \pi_j > \pi_i \). For a Legendre-Stirling permutation \( \pi \) on \( M_n \), we say that \( i \) is a descent if either \( i = 3n \) or \( \pi_i > \pi_{i+1} \). Let \( B_{n,k} \) denote the number of Legendre-Stirling permutations of \( M_n \) with \( k \) descents. Define

\[
B_n(x) = \sum_{k=1}^{2n-1} B_{n,k} x^k.
\]

Egge proved the real-rootedness of \( B_n(x) \).

**Theorem 1.3** For \( n \geq 1 \), \( B_n(x) \) has distinct, real, non-positive roots.

In order to derive a multivariate stable refinement of \( B_n(x) \), we introduce an approach of generating stable polynomials by a sequence of grammars. Based on the Stirling grammar given by Chen and Fu \[7\], we find a sequence \( G_1, G_2, \ldots \) of context-free grammars to generate Legendre-Stirling permutations. We show that the formal derivative with respect to \( G_n \) preserves stability by applying Borcea and Brändén’s characterization of linear operators preserving stability. This leads to a multivariate stable refinement \( B_n(x, y, z, u, v) \) of \( B_n(x) \). On the other hand, according to the grammars, we obtain the following combinatorial interpretation

\[
B_n(x, y, z, u, v) = \sum_{\pi \in A(n)} \prod_{i \in D(\pi)} x_{\pi_i} \prod_{i \in A(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i} \prod_{i \in U(\pi)} u_{\pi_i} \prod_{i \in V(\pi)} v_{\pi_i}.
\]

The real-rootedness of \( B_n(x) \) is a consequence of the stability of \( B_n(x, y, z, u, v) \) by setting \( v_i = y_i = y \) and \( x_i = z_i = u_i = 1 \) for \( 0 \leq i \leq n \).

Haglund and Visontai also posed the problem of finding multivariate stable refinements of the polynomials \( T_n(x) \), which are given by

\[
T_n(x) = 2^n C_n \left( \frac{x}{2} \right) = \sum_k 2^{n-k} C(n, k) x^k,
\]

where \( C(n, k) \) and \( C_n(x) \), as before, denote the second-order Eulerian numbers and the second-order Eulerian polynomials respectively. The polynomials \( T_n(x) \) were introduced by Riordan \[14\].

In light of the relation \((1.1)\) between \( T_n(x) \) and \( C_n(x) \), we introduce the structure of marked Stirling permutations and the following multivariate polynomials

\[
T_n(x, y, z) = \sum_{\pi \in A(n)} \prod_{i \in D(\pi)} x_{\pi_i} \prod_{i \in A(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i},
\]

where \( \pi \) ranges over marked Stirling permutations of \([n]_2\). We shall show that the polynomials \( T_n(x, y, z) \) are stable. The polynomial \( T_n(x) \) becomes the specialization of \( T_n(x, y, z) \) by setting \( x_1 = z_1 = 1 \) and \( y_i = x \) for \( 0 \leq i \leq n \). This implies that \( T_n(x) \) is real-rooted.
This paper is organized as follows. In Section 2, we give an overview of differential operators associated with context-free grammars. We find context-free grammars to generate the polynomials $C_n(x, y, z)$. In Section 3, we obtain context-free grammars that lead to the multivariate generating polynomials $B_n(x, y, z, u, v)$. In Section 4, we introduce the structure of marked Stirling permutations, and we give context-free grammars to generate the multivariate polynomials $T_n(x, y, z)$. In Section 5, based on Borcea and Brändén’s characterization of stability preserving linear operators, we present an approach to proving the stability of polynomials generated by context-free grammars. In particular, we prove the stability of multivariate polynomials $B_n(x, y, z, u, v)$ and $T_n(x, y, z)$.

2 Context-free grammars

In this section, we give an overview of the idea of using context-free grammars $G$ to generate combinatorial polynomials and combinatorial structures as developed by Chen [6]. A context-free grammar $G$ over an alphabet $A$ is defined to be a set of production rules. Roughly speaking, a production rule means to substitute a letter in the alphabet $A$ by a polynomial in $A$ over a field. Given a context-free grammar, one may define a formal derivative $D$ as a linear operator on polynomials in $A$, where the action of $D$ on a letter is defined by the substitution rule of the grammar and the action of $D$ on a product of two polynomials $u$ and $v$ is defined by the Leibnitz rule, that is,

$$D(uv) = D(u)v + uD(v).$$

Many combinatorial polynomials can be generated by context-free grammars. Meanwhile, context-free grammars can be used to generate combinatorial structures. More precisely, one may use a word on an alphabet to label a combinatorial structure such that the context-free grammar serves as the procedure to recursively generate the combinatorial structures. Such a labeling of a combinatorial structure is called a grammatical labeling in [7].

For example, the grammar

$$G = \{ a \rightarrow ab, b \rightarrow b \}$$

is used in [6] to generate the set of partitions of $[n]$ and the Stirling polynomials,

$$S_n(x) = \sum_{i=0}^{n} S(n, k)x^{k},$$

where $S(n, k)$ denotes the Stirling number of the second kind. For a partition $P$, we label a block of $P$ by letter $b$ and label the partition itself by letter $a$, and we define the weight of a partition by the product of its labels. So a partition $P$ with $k$ blocks has the weight $w(P) = ab^k$. For example, the partition $\{\{1, 2\}, \{3\}\}$ is labeled as follows

$$\begin{array}{ccc}
{1,2} & {3} \\
\hline
b & b & a
\end{array}.$$  

In the above notation, we write a partition $P = \{P_1, P_2, \ldots, P_k\}$ of $[n]$ in such a way that the blocks are ordered in the increasing order of their minimum elements. Moreover, we put the letter $a$ at the end of the partition.
Using the above grammatical labeling of a partition, we deduce that

\[ D^n(a) = \sum_P w(P) = \sum_{k=1}^n S(n, k)ab^k. \]  

(2.1)

Many properties of the Stirling polynomials follow from the above expression in terms of the differential operator \( D \) with respect to the grammar \( G \).

Let us explain how the grammar works for the generation of partitions. For \( n = 1 \), there is one partition of \([1]\), that is, \( \{\{1\}\} \), whose label is \( ab \). Assume that we have generated all the partitions of \([n - 1]\) by applying the operator \( D^{n-2} \) to \( \{\{1\}\} \) with the initial grammatical labeling.

Let us give an example to demonstrate the action of the differential operator \( D \) with respect to the grammar \( G \) to a partition of \([n]\) with the aforementioned grammatical labeling. Consider the following partition of \( \{1, 2, 3, 4, 5, 6\} \)

\[ \{1, 3, 6\} \{2, 5\} \{4\} \]

\[ b \ b \ b \ a . \]

If we apply the substitution rule to the letter \( a \), then we get \( ab \) which we rewrite as \( ba \), where \( a \) still serves as the label of the new partition, and \( b \) stands for a new block \( \{7\} \). In this case, we get a partition

\[ \{1, 3, 6\} \{2, 5\} \{4\} \{7\} \]

\[ b \ b \ b \ b \ a . \]

If we apply the substitution rule to the second letter \( b \), then we get \( b \). In this case, we insert the element \( 7 \) in the second block, and we are led to the following partition with consistent grammatical labeling

\[ \{1, 3, 6\} \{2, 5, 7\} \{4\} \]

\[ b \ b \ b \ a . \]

Starting with the empty partition with label \( a \), we get

\[ D(a) = \{1\} \]

\[ b \ a, \]

\[ D^2(a) = \{1\} \{2\} \]

\[ b \ b \ a + b \ a, \]

\[ D^3(a) = \{1\} \{2\} \{3\} \]

\[ \{1\} \{2, 3\} \{1, 3\} \{2\} \{1, 2\} \{3\} \]

\[ b \ b \ b \ a + b \ b \ a + b \ b \ a + b \ a . \]

Without considering the combinatorial structures during the applications of the differential operator \( D \), we may directly compute \( D^n(x) \) to derive the Stirling polynomials \( S_n(x) \).

As the second example, we consider the context-free grammar

\[ G = \{x \to xy, y \to xy\} \]

introduced by Dumont [8] which is used to compute the Eulerian polynomials. For a permutation \( \pi = \pi_1\pi_2 \ldots \pi_n \) of \([n]\), let

\[ A(\pi) = \{i \mid \pi_{i-1} < \pi_i\}, \]

\[ D(\pi) = \{i \mid \pi_i > \pi_{i+1}\} \]
denote the set of ascents and the set of descents of \( \pi \), respectively. Here we set \( \pi_0 = \pi_{n+1} = 0 \). In other words, for any permutation \( \pi \) of \([n]\), 1 is always an ascent and \( n \) is always a descent. An element \( \pi_i \) is called a descent top of \( \pi \) if \( i \in D(\pi) \), and \( \pi_i \) is called an ascent top if \( i \in A(\pi) \), see Haglund and Visontai [13].

The grammatical labeling of a permutation \( \pi \) is defined as follows. If \( \pi_i \) is an ascent top of \( \pi \), then we label \( \pi_{i-1} \) with the letter \( x \). If \( \pi_i \) is a descent top, then we label \( \pi_i \) by the letter \( y \). For this labeling, the weight of \( \pi \) is given by

\[
w(\pi) = x^{|A(\pi)|} y^{|D(\pi)|}.
\]

Then for \( n \geq 1 \), we have

\[
D^n(x) = \sum_{\pi \in S_n} w(\pi) = \sum_{m=1}^{n} A(n, m) y^m x^{n+1-m},
\]

where \( A(n, m) \) is the Eulerian number, namely, the number of permutations of \([n]\) with \( m \) descents, see Dumont [8].

For \( n = 1 \), there is only one permutation of \([1]\), that is 1, whose label is \( xy \). Assume that we have generated all the permutations of \([n-1]\) by applying the operator \( D^{n-2} \) to 1.

Next we give an example to illustrate the action of \( D \) on a permutation of \([6]\). Take a permutation

\[
x y x y x y y.
\]

If we apply the substitution rule \( x \to xy \) to the third letter \( x \), we insert 7 after 5. As for the grammatical labeling, we keep all the labels and assign the element 7 a new label \( y \) as if it comes from the substitution rule \( x \to xy \). Indeed, it is easily checked that what we get is a permutation with a consistent grammatical labeling, namely,

\[
x y x 5 7 6 4 1 y y y.
\]

Similarly, if we apply the substitution rule \( y \to xy \) to the second letter \( y \), then we insert 7 after 6. In this case, we need to change the label of 6 from \( y \) into \( x \), and assign \( y \) to the new element 7. In other words, the label \( y \) becomes \( xy \) just like the substitution rule. So we get the following permutation with a grammatical labeling,

\[
x y x y x 7 6 4 1 y y y.
\]

Indeed, the above examples indicate that permutations of \([n]\) and the Eulerian polynomials \( A_n(x) \) can be generated by the operator \( D \) associated with the grammar \( G \).

In order to generate combinatorial structures with more parameters, we may use a sequence of grammars. Let us consider the the multivariate refinement of Eulerian polynomials \( A_n(x, y) \) introduced by Haglund and Visontai [13], which involve the sets of ascent tops and descent tops, not just the numbers of ascents and descents. More precisely,

\[
A_n(x, y) = \sum_{\pi \in S_n} \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i}.
\]

We shall introduce a sequence of grammars \( \{G_n\} \) to generate the multivariate polynomials \( A_n(x, y) \).
For \( n \geq 1 \), define
\[
G_n = \{ x_i \rightarrow x_iy_n, y_i \rightarrow x_iy_n, 0 \leq i < n \},
\]
and denote by \( D_n \) the formal differential operator with respect to \( G_n \). The multivariate polynomials \( A_n(x, y) \) can be generated by the sequence of grammars \( G_n \).

**Theorem 2.1** For \( n \geq 1 \), we have
\[
D_n D_{n-1} \cdots D_1(x_0) = A_n(x, y).
\]

**Proof.** We define the grammatical labeling of a permutation \( \pi \) as follows. For a permutation \( \pi \), if \( \pi_i \) is an ascent top, we label \( \pi_{i-1} \) by the letter \( x_{\pi_i} \); if \( \pi_i \) is a descent top, we label \( \pi_i \) by the letter \( y_{\pi_i} \). So the weight of \( \pi \) is given by
\[
w(\pi) = \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i}.
\]

We proceed to show by induction that \( D_n D_{n-1} \cdots D_1(x_0) \) equals the sum of the weights of permutations of \([n]\). For \( n = 1 \), the theorem is valid since the weight of the permutation 1 is \( x_1y_1 \). Assume that the theorem holds for \( n - 1 \), that is,
\[
D_{n-1} \cdots D_1(x_0) = \sum_{\pi \in S_{n-1}} w(\pi).
\]

We now use an example to illustrate the action of \( D_n \) on a permutation of \([n-1]\).
Let \( \pi = 325641 \). The grammatical labeling is as follows
\[
x_3 \ y_3 \ x_5 \ x_6 \ y_6 \ y_4 \ y_1.
\]

If we apply the substitution rule \( x_6 \rightarrow x_7y_7 \) to the letter \( x_6 \), we define the action as the insertion of 7 immediately after 5. The labels of 5 and 7 will be changed to \( x_7 \) and \( y_7 \) as given by the grammar. It is not hard to see that the permutation we obtain has a consistent grammatical labeling,
\[
x_3 \ y_3 \ x_5 \ x_6 \ y_7 \ y_6 \ y_4 \ y_1.
\]

Similarly, if we apply the substitution rule \( y_6 \rightarrow x_7y_7 \) to the letter \( y_6 \), we obtain a permutation with a consistent grammatical labeling
\[
x_3 \ y_3 \ x_5 \ x_6 \ y_7 \ y_7 \ y_4 \ y_1.
\]

It is clear that all permutations of \([n]\) can be obtained this way. So we conclude that
\[
D_n D_{n-1} \cdots D_1(x_0) = D_n \left( \sum_{\pi \in S_{n-1}} w(\pi) \right) = \sum_{\sigma \in S_n} w(\sigma).
\]

Hence the theorem holds for all positive numbers \( n \) by induction. □

For \( n = 0 \), the empty permutation is labeled by \( x_0 \). The values of \( A_n(x, y) \) for \( n = 1, 2, 3 \) are given below.
\[
D_1(x_0) = x_1^1 y_1,
\]
\[
D_2 D_1(x_0) = x_2^2 y_2^1 y_1 + x_1^1 x_2^1 y_2,
\]
\[
D_3 D_2 D_1(x_0) = x_3^3 y_3^2 y_1 y_1 + x_2^2 x_3^2 y_3 + x_2^2 y_2^1 x_3^3 y_3 + x_3^3 y_3^1 y_2 + x_1^1 x_3^3 y_2 + x_1 x_2^2 x_3^3 y_3.
\]

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Let us now consider the grammar to generate Stirling permutations. Chen and Fu [7] showed that the grammar
\[ G = \{ x \rightarrow x^2y, y \rightarrow x^2 \} \]
can be used to generate Stirling permutations. Let \( D \) denote the differential operator associated with the grammar \( G \). It has been shown in [7] that
\[ D^n(x) = x \sum_{m=1}^{n} C(n, m)x^{2n-m}y^m, \]
where \( C(n, m) \) denotes the second-order Eulerian number. Notice that
\[ D^n(x) \big|_{x=1} = C_n(y), \]
where \( C_n(y) \) is the second-order Eulerian polynomial.

The grammatical labeling of a Stirling permutation is defined as follows. For a Stirling permutation \( \pi \), if \( \pi_i \) is an ascent top, we label \( \pi_i - 1 \) by the letter \( x \); if \( \pi_i \) is a descent top, we label \( \pi_i \) by the letter \( y \); and if \( \pi_i \) is a plateau, we label \( \pi_i - 1 \) by the letter \( z \). For example, the Stirling permutation \( \pi = 233211 \) has the following grammatical labeling
\[ x^2 x^3 x^3 y^2 y^1 x^1 y. \]

Next we show that one can use a refinement of the grammar \( G \) to derive the multivariate polynomials \( C_n(x, y, z) \) of Haglund and Visontai [13]. Recall that
\[ C_n(x, y, z) = \sum_{\pi \in Q_n} \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}. \]

As a refinement of the grammar \( G \), we define
\[ G_n = \{ x_i \rightarrow x_n y_n z_n, y_i \rightarrow x_n y_n z_n, z_i \rightarrow x_n y_n z_n, 0 \leq i < n \}. \]

and we denote by \( D_n \) the differential operator associated with the grammar \( G_n \).

**Theorem 2.2** For \( n \geq 1 \), we have
\[ D_n D_{n-1} \cdots D_1(z_0) = C_n(x, y, z). \]

**Proof.** First, let us define the grammatical labeling of a Stirling permutation \( \pi \). For a Stirling permutation \( \pi \), if \( \pi_i \) is an ascent top, we label \( \pi_i - 1 \) by the letter \( x_{\pi_i} \); if \( \pi_i \) is a descent top, we label \( \pi_i \) by the letter \( y_{\pi_i} \); and if \( \pi_i \) is a plateau, we label \( \pi_i - 1 \) by the letter \( z_{\pi_i} \). For this labeling, the weight of \( \pi \) is given by
\[ w(\pi) = \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}. \]

We aim to show that \( D_n D_{n-1} \cdots D_1(z_0) \) equals the sum of weights of Stirling permutations of \([n]_2\). Let us use induction on \( n \). The theorem is obvious for \( n = 0 \) since the weight of the empty permutation is \( z_0 \). Assume that the theorem holds for \( n - 1 \), that is,
\[ D_{n-1} \cdots D_1(z_0) = \sum_{\pi \in Q_{n-1}} w(\pi). \]
Let us use an example to demonstrate the action of $D$ on a Stirling permutation of $[n-1]_2$. Let $\pi = 233211$. The grammatical labeling of $\pi$ is as follows

$$x_2 2 3 3 2 1 1 .$$

In general, if we apply a substitution rule of $G_4$ to any letter in $\pi$, we get $x_4y_4z_4$. Here we insert the two elements 44 after the element whose label is replaced by the substitution rule, and we use the labels $x_4$, $y_4$ and $z_4$ to relabel the three elements that are affected by the substitution. For example, if we apply the substitution rule $x_2 \rightarrow x_4y_4z_4$ to the above Stirling permutation, then we get a Stirling permutation with the following grammatical labeling

$$x_4 4 4 3 3 2 2 1 1 .$$

It is easily seen that the application of any substitution rule of $G_n$ to any Stirling permutation of $[n-1]_2$ leads to a Stirling permutation of $[n]_2$ with a consistent grammatical labeling. Hence we deduce that

$$D_nD_{n-1}\cdots D_1(z_0) = D_n \left( \sum_{\pi \in Q_{n-1}} w(\pi) \right) = \sum_{\sigma \in Q_n} w(\sigma).$$

Thus, the theorem holds for $n$. This completes the proof.

For $n = 0$, the empty permutation is labeled by $z_0$. The values of the polynomials $C_n(x, y, z)$ for $n = 1, 2$ are as follows,

$$D_1(z_0) = x_1 \frac{1}{1} y_1,$$

$$D_2D_1(z_0) = x_1 2 2 2 2 1 1 1 + x_2 \frac{1}{2} \frac{1}{2} y_2 y_1 + x_1 \frac{1}{1} y_1 \frac{1}{1} \frac{1}{1} 2 2 2 .$$

We shall give further refinements of the above two sequences of grammars as solutions to the problems of Haglund and Visontai [13]. On one hand, we use these refined grammars to construct multivariate polynomials for Legendre-Stirling permutations and marked Stirling permutations. On the other hand, we use the grammars to construct stability preserving operators leading to the stability of the multivariate polynomials.

## 3 Legendre-Stirling permutations

In this section, we introduce several statistics on Legendre-Stirling permutations of

$$M_n = \{ 1, 1, \bar{1}, 2, 2, \bar{2}, \cdots, n, n, \bar{n} \}.$$

In terms of these statistics, we obtain multivariate polynomials $B_n(x, y, z, u, v)$ as refinements of $B_n(x)$. In fact, the combinatorial construction of the multivariate polynomials $B_n(x, y, z, u, v)$ is obtained from further refinements of the grammars to generate permutations and Stirling permutations with respect to the numbers of descents. Using these grammars, we derive the combinatorial interpretation by giving a suitable grammatical labeling. In Section 4, we shall use grammars to prove the stability of $B_n(x, y, z, u, v)$. This leads to a solution to the problem of Haglund and Visontai.
Let $L_n$ denote the set of Legendre-Stirling permutations of $M_n$. For a Legendre-Stirling permutation $\pi \in L_n$, define

\[
X(\pi) = \{ i \mid \pi_{i-1} \leq \pi_i, \pi_i \text{ is unbarred and appears the first time} \},
\]

\[
Y(\pi) = \{ i \mid \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is unbarred} \},
\]

\[
Z(\pi) = \{ i \mid \pi_{i-1} \leq \pi_i, \pi_i \text{ is unbarred and appears the second time} \},
\]

\[
U(\pi) = \{ i \mid \pi_{i-1} \leq \pi_i \text{ and } \pi_i \text{ is barred} \},
\]

\[
V(\pi) = \{ i \mid \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is barred} \}.
\]

Here we set $\pi_0 = \pi_{3n+1} = 0$.

For example, let $\pi = 112233231$. Then we have $X(\pi) = \{2, 4, 5\}$, $Y(\pi) = \{6, 9\}$, $Z(\pi) = \{6\}$, $U(\pi) = \{1, 3, 8\}$ and $V(\pi) = \{8\}$.

Define

\[
B_n(x, y, z, u, v) = \sum_{\pi \in L_n} \prod_{i \in X(\pi)} x_{\pi_i} \prod_{i \in Y(\pi)} y_{\pi_i} \prod_{i \in Z(\pi)} z_{\pi_i} \prod_{i \in U(\pi)} u_{\pi_i} \prod_{i \in V(\pi)} v_{\pi_i}.
\]

We define the grammars $\{G_n\}$ as follows,

\[
G_{2n-1} = \{ x_i, y_i, z_i, u_i, v_i \rightarrow u_nv_n, 0 \leq i < n \},
\]

\[
G_{2n} = \{ x_i, y_i, z_i, u_i, v_i \rightarrow x_ny_nz_n, 0 \leq i < n; \}
\]

\[
u_n \rightarrow x_nz_nu_n, v_n \rightarrow x_ny_nz_n \}.
\]

Notice that $G_{2n-1}$ is a refinement of the grammar

\[
G = \{ x \rightarrow xy, y \rightarrow xy \}.
\]

and $G_{2n}$ is a refinement of the grammar

\[
G = \{ x \rightarrow x^2y, y \rightarrow x^2y \}
\]

The grammatical labeling of a Legendre-Stirling permutation is defined as follows. Let $\pi$ be a Legendre-Stirling permutation on $M_n$. For $i \in X(\pi)$, $i \in Z(\pi)$ or $i \in U(\pi)$, we label $\pi_{i-1}$ by the letter $x_{\pi_i}$, $z_{\pi_i}$ or $u_{\pi_i}$, respectively; for $i \in Y(\pi)$ or $i \in V(\pi)$, we label $\pi_i$ by the letter $y_{\pi_i}$ or $v_{\pi_i}$, respectively. For example, the above Legendre-Stirling permutation $\pi = 112233231$ has the following grammatical labeling

\[
\bar{u} \quad \bar{1} \quad \bar{1} \quad \bar{1} \quad \bar{2} \quad \bar{2} \quad \bar{2} \quad \bar{3} \quad \bar{3} \quad \bar{3} \quad \bar{2} \quad \bar{2} \quad 1 \quad 1.
\]

The following theorem shows that the polynomials $B_n(x, y, z, u, v)$ can be generated by the grammars $G_n$.

**Theorem 3.1** For $n \geq 1$, we have

\[
D_{2n}D_{2n-1} \cdots D_1(x_0) = B_n(x, y, z, u, v). \tag{3.1}
\]
Proof. We use induction on \( n \). The case for \( n = 0 \) is obvious since the empty permutation is labeled by \( x_0 \). Assume that the theorem holds for \( n - 1 \), that is,

\[
D_{2n-2} \cdots D_1(x_0) = \sum_{\pi \in L_{n-1}} w(\pi). \tag{3.2}
\]

Note that any Legendre-Stirling permutation of \( M_n \) can be obtained from a Legendre-Stirling permutation of \( M_{n-1} \) through two operations: (1) Insert a barred element \( \bar{n} \); (2) Insert two elements \( nn \). We use an example to show that the operators \( D_{2n-1} \) and \( D_{2n} \) correspond to these two operations.

Consider the Legendre-Stirling permutation \( \pi = 112233231 \), whose grammatical labeling is given by

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\bar{u} & x_1 & u_2 & \bar{u} & x_2 & x_3 & y_1 & u_3 & v_3 & v_1 & y_1 & .
\end{array}
\]

The first operation is just the procedure of generating permutations. In general, if we apply a substitution rule of \( G_7 \) to \( \pi \), we always get \( u_4 \bar{v}_4 \). Here we insert \( 4 \) after the element whose label is replaced by the substitution rule. At the same time, we relabel the two involved elements by the letters \( u_4 \) and \( v_4 \). For example, if we apply the substitution rule \( z_3 \to u_4 \bar{v}_4 \) to \( \pi \), then we obtain a Legendre-Stirling permutation with a consistent grammatical labeling

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\bar{u} & x_1 & u_2 & x_2 & x_3 & y_1 & u_3 & v_3 & v_1 & y_1 & .
\end{array}
\]

As for the second operation, consider the above Legendre-Stirling permutation \( \sigma = 1122343231 \). The two substitution rules \( u_4 \to x_4 \bar{z}_4 \) and \( v_4 \to x_4 \bar{y}_4 \bar{z}_4 \) of \( G_8 \) correspond to the operations of inserting two elements \( 44 \) before \( \bar{4} \) or after \( \bar{4} \), respectively. So we get two Legendre-Stirling permutations \( 112234443231 \) or \( 112234443231 \).

Next we consider the rest of substitution rules of \( G_8 \). If we apply any of the remaining substitution rules of \( G_8 \) to \( \sigma \), we always get \( x_4 \bar{y}_4 \bar{z}_4 \). Here we insert two elements \( 44 \) into \( \sigma \) between \( \pi_1 \) and \( \pi_{i+1} \), which are elements less than \( 4 \). For example, by applying the production rule \( u_2 \to x_4 \bar{y}_4 \bar{z}_4 \), we obtain the Legendre-Stirling permutation

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\bar{u} & x_1 & x_4 & \bar{z}_4 & \bar{y}_4 & x_2 & x_3 & u_4 & v_4 & y_3 & u_3 & v_3 & v_1 & y_1 & .
\end{array}
\]

It can be checked that any of applications of the substitution rules of \( G_8 \) to \( \sigma \) leads to consistent grammatical labelings. Moreover, it can be verified that the action of \( D_{2n}D_{2n-1} \) on the Legendre-Stirling permutations of \( M_{n-1} \) generates all the Legendre-Stirling permutations of \( M_n \). So we conclude that

\[
D_{2n}D_{2n-1} \cdots D_1(x_0) = \sum_{\pi \in L_n} w(\pi).
\]

Then the theorem follows by induction. \( \blacksquare \)

For \( n = 0 \), the empty permutation is labeled by \( x_0 \), and \( B_1(x, y, z, u, v) \) is calculated as follows,

\[
\begin{align*}
D_1(x_0) &= u_1 \bar{v}_1, \\
D_2D_1(x_0) &= x_1 \bar{z}_1 u_1 \bar{v}_1 + u_1 x_1 \bar{z}_1 \bar{y}_1.
\end{align*}
\]
4 Marked Stirling permutations

In this section, we introduce the structure of marked Stirling permutations, and we define several statistics in order to construct multivariate polynomials $T_n(x, y, z)$ as refinements of $T_n(x)$. We also give a sequence of grammars to generate $T_n(x, y, z)$ as well as marked Stirling permutations with suitable grammatical labelings. By using the grammars, the stability of $T_n(x, y, z)$ can be established in Section 5. This gives a solution to the problem of Haglund and Visontai concerning a stable refinement of $T_n(x)$.

A marked Stirling permutation is defined by the following marking rule. Given a Stirling permutation $\pi = \pi_1 \pi_2 \cdots \pi_{2n}$, if $\pi_i$ is an element of $\pi$ such that $\pi_i$ occurs the second time in $\pi$ and $\pi_i < \pi_{i+1}$, then we may mark the element $\pi_i$. We denote a marked element by $\bar{i}$. A marked Stirling permutation is a Stirling permutation with some elements marked according to the above rule.

For example, there is only one marked Stirling permutation of $[1]_2$: 11, whereas there are four marked Stirling permutations of $[2]_2$: 2211, 1221, 1122, 1\bar{1}22.

Let $Q_n$ denote the set of marked Stirling permutations of $[n]_2$. We use $A(\pi)$, $D(\pi)$, $P(\pi)$ to denote the set of descents, the set of ascents and the set of plateaux of $\pi$. More precisely, given a marked Stirling permutation $\pi = \pi_1 \pi_2 \cdots \pi_{2n} \in Q_n$, let
\[
A(\pi) = \{ i \mid \pi_{i-1} < \pi_i \}, \\
D(\pi) = \{ i \mid \pi_i > \pi_{i+1} \}, \\
P(\pi) = \{ i \mid \pi_{i-1} = \pi_i \}
\]
denote the set of ascents, the set of descents and the set of plateaux of $\pi$, respectively. Let $T(n, m)$ be the number of marked Stirling permutations of $[n]_2$ with $m$ descents. It follows from relation (1.1) that
\[
T_n(x) = \sum_{m=1}^{n} T(n, m) x^m.
\]

Note that Riordan [17] introduced the polynomials $T_n(x)$ and proved that $T_n(1)$ equals the Schröder number, namely, the number of series-reduced rooted trees with $n + 1$ labeled leaves.

We shall prove that the polynomials $T_n(x)$ can be generated by the grammar $G$ defined by
\[
G = \{ x \rightarrow x^2 y, y \rightarrow 2x^2 y \}.
\]
The grammatical labeling of a marked Stirling permutation can be described as follows. Let $\pi$ be a marked Stirling permutation of $[n]_2$. If $i \in D(\pi)$, we label $\pi_i$ by $y$. If $i \in A(\pi)$ or $i \in P(\pi)$, we label $\pi_{i-1}$ by $x$. The weight of a marked Stirling permutation $\pi$ of $[n]_2$ with $m$ descents is given by
\[
w(\pi) = x^{2n+1-m} y^m.
\]

**Theorem 4.1** For $n \geq 1$, we have
\[
D_n(x) = \sum_{m=1}^{n} T(n, m) x^{2n-m+1} y^m.
\]
Setting \( x = 1 \), we have
\[
D^n(x) |_{x=1} = T_n(y).
\]

**Proof.** We aim to show that \( D^n(x) \) equals the sum of the weights of marked Stirling permutations of \([n]_2\). We use induction on \( n \). The case for \( n = 0 \) follows from the fact that the weight of the empty permutation is \( x \). Assume that the theorem holds for \( n - 1 \), that is,
\[
D^{n-1}(x) = \sum_{\pi \in Q_{n-1}} w(\pi).
\]

We now use an example to demonstrate the action of \( D \) on a marked Stirling permutation of \([n-1]_2\). Let \( \pi = 12 \overline{2331} \) with the following grammatical labeling:
\[
1 \ 2 \ 2 \ 3 \ 3 \ 1.
\]

If we apply the substitution rule \( x \rightarrow x^2y \) to the fourth letter \( x \), then we insert the two elements 44 after 2. We keep all the labels except that we assign the labels \( x \) and \( y \) to the two new letters 44. It is not difficult to see that the generated marked Stirling permutation has a consistent grammatical labeling:
\[
1 \ 2 \ 2 \ 4 \ 4 \ 3 \ 3 \ 1.
\]

If we apply the substitution rule \( y \rightarrow 2x^2y \) to the first letter \( y \), then we insert 44 after the second element 3. We change the label of the second element 3 from \( y \) into \( x \) and assign \( x \) and \( y \) to the two new elements 44. According to the marking rule, the second element 3 may be marked or unmarked. These two choices correspond the coefficient 2 in the substitution rule \( y \rightarrow 2x^2y \). So we are led to the following two marked Stirling permutations with consistent grammatical labelings,
\[
1 \ 2 \ 2 \ 3 \ 3 \ 4 \ 4 \ 1,
\]
and
\[
1 \ 2 \ 2 \ 3 \ 3 \ 4 \ 4 \ 1.
\]

It can be verified that the above process generates all marked Stirling permutations of \([n]_2\). It follows that
\[
D^n(x) = D(D^{n-1}(x)) = D \left( \sum_{\pi \in Q_{n-1}} w(\pi) \right) = \sum_{\sigma \in Q_n} w(\sigma).
\]

Hence the proof is complete by induction.

As a multivariate refinement of \( T_n(x) \), we define the following generating polynomial of marked Stirling permutations of \([n]_2\),
\[
T_n(x, y, z) = \sum_{\pi \in Q_n} \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.
\]

Let
\[
G_n = \{x_i \rightarrow x_n y_n z_n, y_i \rightarrow 2x_n y_n z_n, z_i \rightarrow x_n y_n z_n, 0 \leq i < n\}.
\]

The grammatical labeling of a marked Stirling permutation can be described as follows. For a marked Stirling permutation \( \pi \) of \([n]_2\), if \( i \in A(\pi) \), we label \( \pi_{i-1} \)
by \(x_i\); if \(i \in D(\pi)\), we label \(\pi_i\) by \(y_i\); and if \(i \in P(\pi)\), we label \(\pi_i-1\) by \(z_i\). Then the weight of \(\pi\) equals

\[
w(\pi) = \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.
\]

The following theorem shows that the polynomials \(T_n(x, y, z)\) can be generated by the grammars \(G_n\).

**Theorem 4.2** For \(n \geq 1\), we have

\[
D_n D_{n-1} \cdots D_1(z_0) = T_n(x, y, z).
\]

The proof of the above theorem is analogous to that of Theorem 4.1. Hence the details are omitted. Here we use an example to illustrate the action of \(D_4\) to the above marked Stirling permutation \(\pi = 122331\) with the grammatical labeling

\[
x_1 \ x_2 \ x_3 \ x_4 \ y_3 \ y_1.
\]

If we apply the substitution rule \(x_3 \rightarrow x_4y_4z_4\) of \(G_4\) to the letter \(x_3\), then we insert the two elements 44 after 2 to get a marked Stirling permutation with the following consistent grammatical labeling

\[
x_1 \ x_2 \ y_2 \ x_3 \ x_4 \ y_4 \ z_3 \ y_3 \ y_1.
\]

If we apply the substitution rule \(y_3 \rightarrow 2x_4y_4z_4\) of \(G_4\) to the letter \(y_3\), then we insert 44 after the second element 3 to get the following two marked Stirling permutations with consistent grammatical labelings

\[
\begin{align*}
x_1 \ x_2 \ x_3 \ x_4 \ y_3 \ y_4 \ z_4 \ y_1, \\
& \quad \text{and} \quad x_1 \ x_2 \ x_3 \ x_2 \ x_4 \ y_4 \ z_4 \ y_1.
\end{align*}
\]

For \(n = 0\), the empty permutation is labeled by \(z_0\). For \(n = 1, 2\), \(T_n(x, y, z)\) are given below,

\[
\begin{align*}
D_1(z_0) &= x_1 \ z_1 \ y_1, \\
D_2D_1(z_0) &= x_2 \ y_2 \ z_1 \ y_1 \ z_1 \ y_1 + x_1 \ z_1 \ y_2 \ y_1 + x_1 \ x_2 \ y_2 \ y_1 \\
&\quad + x_1 \ z_1 \ x_2 \ y_2 \ y_1.
\end{align*}
\]

**5 Grammars preserving stability**

In this section, we prove the stability of the multivariate polynomials \(B_n(x, y, z, u, v)\) and \(T_n(x, y, z)\) based on context-free grammars and the characterization of stability preserving linear operators due to Borcea and Brändén [3].

Our idea of proving the stability of the polynomials by a sequence of context-free grammars \(\{G_n\}\) goes as follows. Since the initial polynomial \(x\) is stable, if \(D_1, D_2, \ldots, D_n, \ldots\) preserve stability, then \(D_n D_{n-1} \cdots D_1(x)\) is stable. If \(D_n\) is
not stability preserving, then we try to find a sequence of stability preserving operator \( \{ T_n \} \) such that

\[
T_n T_{n-1} \ldots T_1(x) = D_n D_{n-1} \ldots D_1(x).
\]

If such operators \( T_n \) exist, then we reach the conclusion that the multivariate polynomials \( D_n D_{n-1} \ldots D_1(x) \) are stable.

Note that \( B_n(x,y,z,u,v) \) and \( T_n(x,y,z) \) are all multiaffine polynomials in the sense that the degree in each variable is at most 1. In order to construct the stability preserving operators \( T_n \) based on the grammars \( G_n \), we consider some equivalent forms of production rules when we restrict our attention to multiaffine polynomials.

For example, let

\[
G_n = \{ a \to a b_n, b_i \to b_n, 0 \leq i < n \}.
\]

Observe that as far as the computation is concerned, the formal differential operator \( D_n \) with respect to \( G_n \) is in accordance with the following operator

\[
T_n = b_n(1 + \sum_{i=1}^{n} \partial/\partial b_i),
\]

when they are applied to certain polynomials. Thus we obtain

\[
T_n T_{n-1} \ldots T_1(a) = D_n D_{n-1} \ldots D_1(a).
\]

However, \( D_n \) and \( T_n \) are different operator in general, since

\[
D_n (a + b_1) \neq T_n (a + b_1).
\]

For multiaffine polynomials, the characterization of stability preserving operators is simpler than that for the general case. For the purpose of this paper, we only need the following sufficient condition to prove the stability of \( B_n(x,y,z,u,v) \) and \( T_n(x,y,z) \), see Borcea and Brändén [3].

**Lemma 5.1** Let \( f \in \mathbb{C}[z_1, z_2, \ldots, z_n] \) be a stable multiaffine polynomial and let \( T \) denote a linear operator acting on the polynomials in \( \mathbb{C}[z_1, z_2, \ldots, z_n] \). If

\[
T \left( \prod_{i=1}^{n} (z_i + w_i) \right) \in \mathbb{C}[z_1, z_2, \ldots, z_n, w_1, \ldots, w_n]
\]

is stable, then \( T(f) \) is either stable or identically 0.

Next we show how to prove the stability of polynomials generated by context-free grammars. Let us consider the multiaffine polynomials \( C_n(x,y,z) \) defined by Haglund and Visontai [3]. Let

\[
G_n = \{ x_i \to x_n y_n z_n, y_i \to x_n y_n z_n, z_i \to x_n y_n z_n, 0 \leq i < n \},
\]

and let \( G_n \) denote the differential operator associated with the grammar \( G_n \). Let \( f_n = D_n D_{n-1} \ldots D_1(z_0) \). From the grammatical labelings, it is clear that \( f_n \) is multiaffine. We wish to prove the stability of \( f_n \) by induction on \( n \). Since \( z_0 \) is stable, it suffices to prove that the operator \( D_{n+1} \) preserves stability of multiaffine polynomials.
Let
\[ F = \prod_{i=0}^{n}(x_i + w_i)(y_i + v_i)(z_i + u_i). \]
By Lemma 5.1 it suffices to check the stability of \( D_{n+1}(F) \), that is,
\[ D_{n+1}(F) = x_{n+1}y_{n+1}z_{n+1}F \sum_{i=0}^{n} \left( \frac{1}{x_i + w_i} + \frac{1}{y_i + v_i} + \frac{1}{z_i + u_i} \right) \]
is stable.

If \( x_i, y_i, z_i, w_i, v_i \) and \( u_i \) have positive imaginary parts for all \( 0 \leq i \leq n \), then
\[ \xi = \sum_{i=0}^{n} \left( \frac{1}{x_i + w_i} + \frac{1}{y_i + v_i} + \frac{1}{z_i + u_i} \right) \]
has negative imaginary part. Thus,
\[ D_{n+1}(F) = x_{n+1}y_{n+1}z_{n+1}F \xi \neq 0. \]
Hence \( D_{n+1}(F) \) is stable. By Lemma 5.1 we find that \( D_{n+1}(f_n) \) is a stable polynomial. So we conclude that
\[ f_{n+1} = D_{n+1}D_nD_{n-1}\cdots D_1(z_0) \]
is stable.

The stability of \( A_n(x, y) \) can be proved in the same way. Indeed, let
\[ G_n = \{ x_i \to x_iy_i, y_i \to x_iy_i, 0 \leq i < n \}, \]
and let \( D_n \) denote the differential operator with respect to \( G_n \). It turns out that the operator \( D_n \) preserves the stability of multiaffine polynomials.

It is worth mentioning that the formal differential operators used in the above two examples are essentially equivalent to the operators given by Haglund and Visontai in their proofs of the stability of \( C_n(x, y, z) \) and \( A_n(x, y) \).

Next we construct stable multivariate refinements of \( S_n(x) \), the Stirling polynomials. Recall that the grammar
\[ G = \{ a \to ab, b \to b \} \]
generates the polynomials \( S_n(x) \). Define
\[ G_n = \{ a \to ab_n, b_i \to b_n, 1 \leq i < n \}, \]
and let \( D_n \) denote the formal differential operator associated with \( G_n \). We define the grammatical labeling of a partition as follows. For a partition \( P = \{ P_1, P_2, \ldots, P_k \} \), we label the partition itself by the letter \( a \) and label a block \( P_i \) by the letter \( b_m \), where \( m \) is the maximum element in \( P_i \). Then the weight of \( P \) is given by
\[ w(P) = a \prod_{m=1}^{k} b_{m}, \]
where \( m_i \) is the maximum element in \( P_i \). Denote by \( S_n(a, b) \) the sum of weights of partitions of \([n]\). The next theorem shows that \( S_n(a, b) \) can be generated by \( G_n \). However, in this case, the differential operator \( D_n \) associated with \( G_n \) is not stability preserving even for multiaffine polynomials. Instead, we shall find an equivalent operator \( T_n \) that preserves stability for multiaffine polynomials.
Theorem 5.2 For \( n \geq 1 \), we have
\[
D_nD_{n-1} \ldots D_1(a) = S_n(a, b).
\]

The proof of the above theorem is analogous to that of \([24]\). Here we use the same example to demonstrate the action of \( D_7 \) on a partition of \( [6] \). Consider the partition of \( \{1, 2, 3, 4, 5, 6\} \) with the following grammatical labeling
\[
\{1, 3, 6\} \{2, 5\} \{4\} \\
b_6 \quad b_5 \quad b_4 \quad a.
\]

If we apply the substitution rule \( a \to ab \) of \( G_7 \) to the letter \( a \), then we get a partition with a consistent labeling
\[
\{1, 3, 6\} \{2, 5, 7\} \{4\} \\
b_6 \quad b_7 \quad b_4 \quad a.
\]

Theorem 5.3 For \( n \geq 1 \), the multivariate polynomial \( S_n(a, b) \) is stable.

Proof. From the grammatical labelings, we see that \( S_n(a, b) \) is multiaffine. Note that \( S_n(a, b) \) is multiaffine in \( a, b_1, b_2, \ldots, b_n \) with every term containing \( a \) as a factor. Since
\[
G_{n+1} = \{a \to ab_{n+1}, b_i \to b_{n+1}, 1 \leq i \leq n\},
\]
for each multiaffine monomial of \( S_n(a, b) \) which is of the form \( ah \), we have
\[
D_{n+1}(ah) = ab_{n+1}h + aD_{n+1}(h).
\]

It follows that
\[
S_{n+1}(a, b) = D_{n+1}(S_n(a, b)) \\
= b_{n+1}S_n(a, b) + b_{n+1} \sum_{i=1}^n \frac{\partial}{\partial b_i} (S_n(a, b)).
\]

Define
\[
T_{n+1} = b_{n+1} \left( 1 + \sum_{i=1}^n \frac{\partial}{\partial b_i} \right).
\]

Then we have \( S_{n+1}(a, b) = T_{n+1}(S_n(a, b)) \).

We proceed to prove the stability of \( S_n(a, b) \) by induction on \( n \). Since \( a \) is stable, we only need to show that the linear operator \( T_{n+1} \) preserves stability of multiaffine polynomials.

Let
\[
F = (a + w) \prod_{i=1}^n (b_i + v_i).
\]
Let $B_n$ vanish. It follows that $T_{n+1}$ is stable. This completes the proof.

Moreover, each factor of $F$ has positive imaginary part, and so does $b_{n+1}$. This yields that $F$ and $b_{n+1}$ do not vanish. It follows that

$$T_{n+1}(F) = b_{n+1}F\xi \neq 0.$$ 

Hence $T_{n+1}(F)$ is stable. In view of Lemma 5.1 we see that $T_{n+1}(S_n(a, b))$ is stable. This completes the proof.  

It is worth mentioning that we use the operator $T_{n+1}$ instead of $D_{n+1}$ in the above proof because the operator $D_{n+1}$ does not satisfy the condition in Lemma 5.1. Take $D_2$ as an example. It can be seen that $D_2((a + w)(b_1 + u))$ is not stable. Note that

$$D_2((a + w)(b_1 + u)) = b_2(a(b_1 + u + 1) + w).$$

Let $a = \frac{i-1}{2}, b_1 = \frac{1}{2} - 1, u = \frac{i}{2} - 1$ and $w = i$. But we have $D_2((a + w)(b_1 + u)) = 0$. This implies that $D_2((a + w)(b_1 + u))$ is not stable.

Next we prove the stability of $B_n(x, y, z, u, v)$.

**Theorem 5.4** For $n \geq 1$, the multivariate polynomial $B_n(x, y, z, u, v)$ is stable.

**Proof.** Let $f_n = D_nD_{n-1} \ldots D_1(x_0)$. From the grammatical labelings, it can be seen that $f_n$ is multiaffine. We proceed to prove the stability of $D_2nD_{2n-1} \ldots D_1(x_0)$ by induction on $n$. The stability of $x_0$ is evident.

We now assume that $f_{2n-2}$ is stable. Let us consider the actions of $D_{2n-1}$ and $D_{2n}$. By using the argument in the proof of the stability of $C_n(x, y, z)$, it can be shown that the operator $D_{2n-1}$ preserves stability of multiaffine polynomials. This leads to the stability of $f_{2n-1}$ since $f_{2n-1} = D_{2n-1}(f_{2n-2})$.

Recall that

$$G_{2n} = \{x_i, y_i, z_i, u_i, v_i \rightarrow x_ ny_n z_n, 0 \leq i < n;$$

$$u_n \rightarrow x_n z_n, v_n \rightarrow x_n y_n z_n\}.$$ 

Let $B$ denote the following alphabet

$$\{x_i, y_i, z_i, u_i, v_i, 0 \leq i < n\} \cup \{v_n\}.$$ 

Then we have

$$T_{n+1}(F) = b_{n+1}F + b_{n+1}\sum_{i=1}^{n} \frac{\partial}{\partial v_i}(F)$$

$$= b_{n+1}F + b_{n+1}F \frac{1}{b_1 + v_i}$$

$$= b_{n+1}F \left(1 + \sum_{i=1}^{n} \frac{1}{b_1 + v_i}\right).$$

To prove that $T_{n+1}(F)$ is stable, we assume that $a$, $w$, $b_i$ and $v_i$ have positive imaginary parts for all $1 \leq i \leq n$. Consequently,

$$\xi = 1 + \sum_{i=1}^{n} \frac{1}{b_1 + v_i}$$

is nonzero since it has negative imaginary part. Moreover, each factor of $F$ has positive imaginary part, and so does $b_{n+1}$. This yields that $F$ and $b_{n+1}$ do not vanish. It follows that

$$T_{n+1}(F) = b_{n+1}F\xi \neq 0.$$ 

Hence $T_{n+1}(F)$ is stable. In view of Lemma 5.1 we see that $T_{n+1}(S_n(a, b))$ is stable. This completes the proof.

It is worth mentioning that we use the operator $T_{n+1}$ instead of $D_{n+1}$ in the above proof because the operator $D_{n+1}$ does not satisfy the condition in Lemma 5.1. Take $D_2$ as an example. It can be seen that $D_2((a + w)(b_1 + u))$ is not stable. Note that

$$D_2((a + w)(b_1 + u)) = b_2(a(b_1 + u + 1) + w).$$

Let $a = \frac{i-1}{2}, b_1 = \frac{1}{2} - 1, u = \frac{i}{2} - 1$ and $w = i$. But we have $D_2((a + w)(b_1 + u)) = 0$. This implies that $D_2((a + w)(b_1 + u))$ is not stable.

Next we prove the stability of $B_n(x, y, z, u, v)$.

**Theorem 5.4** For $n \geq 1$, the multivariate polynomial $B_n(x, y, z, u, v)$ is stable.

**Proof.** Let $f_n = D_nD_{n-1} \ldots D_1(x_0)$. From the grammatical labelings, it can be seen that $f_n$ is multiaffine. We proceed to prove the stability of $D_2nD_{2n-1} \ldots D_1(x_0)$ by induction on $n$. The stability of $x_0$ is evident.

We now assume that $f_{2n-2}$ is stable. Let us consider the actions of $D_{2n-1}$ and $D_{2n}$. By using the argument in the proof of the stability of $C_n(x, y, z)$, it can be shown that the operator $D_{2n-1}$ preserves stability of multiaffine polynomials. This leads to the stability of $f_{2n-1}$ since $f_{2n-1} = D_{2n-1}(f_{2n-2})$.
Since \( f_{2n-1} \) is multiaffine and each term in \( f_{2n-1} \) contains \( u_n \), we may write a monomial of \( f_{2n-1} \) in the form \( u_n h \). Then we have
\[
D_{2n}(u_nh) = (x_n z_n u_n) h + x_n y_n z_n D_{2n}(h).
\]
Thus,
\[
f_{2n} = D_{2n}(f_{2n-1}) = x_n z_n f_{2n-1} + x_n y_n z_n \sum_{w \in B} \frac{\partial}{\partial w}(f_{2n-1}).
\]
Hence we may write \( f_{2n} \) as \( T(f_{2n-1}) \), where \( T \) is a linear operator as given by
\[
T = x_n z_n + x_n y_n z_n \sum_{w \in B} \frac{\partial}{\partial w}.
\]

It remains to show that \( T \) preserves the stability of multiaffine polynomials. Let
\[
F = (u_n + r_{u_n}) \prod_{w \in B} (w + r_w).
\]
By Lemma 5.1 it suffices to verify the stability of the following polynomial
\[
T(F) = x_n z_n F + x_n y_n z_n \sum_{w \in B} \frac{1}{w + r_w}.
\]

Suppose that all the variables \( x_i, y_i, z_i, u_i, v_i, r_x, r_y, r_z, r_u \), and \( r_v \) have positive imaginary parts for \( 0 \leq i \leq n \). Then
\[
\xi = \frac{1}{y_n} + \sum_{w \in B} \frac{1}{w + r_w}
\]
has negative imaginary part, and so it is nonzero. Meanwhile, every factor of \( F \) is nonzero since its imaginary part is positive. Note that under the above assumption, \( x_n, y_n \) and \( z_n \) have positive imaginary parts, and hence they are nonzero. Consequently, \( T(F) = x_n y_n z_n F \xi \) does not vanish. This leads to the stability of \( T(F) \).

In light of Lemma 5.1, we deduce that \( f_{2n} \) is stable. This completes the proof.

The proof of the stability of \( C_n(x, y, z) \) applies to the stability of \( T_n(x, y, z) \). The details are omitted.

**Theorem 5.5** For \( n \geq 1 \), the multivariate polynomial \( T_n(x, y, z) \) is stable.

Multivariate stable polynomials can be reduced to real-rooted univariate polynomials by diagonalization and specialization, see Wagner [18]. More precisely, if \( f \in \mathbb{R}[z_1, z_2, \ldots, z_n] \) is stable, then \( f(z_1, \ldots, z_n)|_{z_i = z} \) and \( f(z_1, \ldots, z_n)|_{z_i = a} \) are also stable, where \( 1 \leq i \neq j \leq n \) and \( a \in \mathbb{R} \). For example, setting \( a = 1 \) and \( b_1 = b_2 = \cdots = b_n = x \) in \( S_n(a, b) \) leads to the real-rootedness of \( S_n(x) \), see Harper [12].
For the multivariate stable polynomials \( B_n(x, y, z, u, v) \), applying the diagonalization \( y_i = v_i = x \) and the specialization \( x_i = u_i = z_i = 1 \) for all \( 0 \leq i \leq n \), we are led to Theorem \[1\].

Let \( M(n, k) \) denote the number of Legendre-Stirling permutations of \( M_n \) with \( k \) barred descents. By setting \( x_i = u_i = y_i = z_i = 1 \), and \( v_i = x \) for all \( 0 \leq i \leq n \), we obtain the real-rootedness of the generating function of \( M(n, k) \).

**Corollary 5.6** For \( n \geq 1 \), the polynomial

\[
M_n(x) = \sum_{k=1}^{n} M(n, k)x^k
\]

has only real roots.

For the multivariate stable polynomials \( T_n(x, y, z) \), by setting \( x_i = z_i = 1 \) and \( y_i = y \) for all \( 0 \leq i \leq n \), we are led to the real-rootedness of \( T_n(y) \), which is equivalent to the real-rootedness of \( C_n(x) \).

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