GENERALIZED INTERSECTION BODIES ARE NOT EQUIVALENT

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ABSTRACT. In [Kol00], A. Koldobsky asked whether two types of generalizations of the notion of an intersection-body, are in fact equivalent. The structures of these two types of generalized intersection-bodies have been studied in [Mil06b], providing substantial evidence for a positive answer to this question. The purpose of this note is to construct a counter-example, which provides a surprising negative answer to this question in a strong sense. This implies the existence of non-trivial non-negative functions in the range of the spherical Radon transform, and the existence of non-trivial spaces which embed in $L_p$ for certain negative values of $p$.

1. Introduction

Let $\text{Vol}(L)$ denote the Lebesgue measure of a set $L \subset \mathbb{R}^n$ in its affine hull, and let $G(n, k)$ denote the Grassmann manifold of $k$ dimensional subspaces of $\mathbb{R}^n$. Let $D_n$ denote the Euclidean unit ball, and $S^{n-1}$ the Euclidean sphere. All of the bodies considered in this note will be assumed to be centrally-symmetric star-bodies (even if the central-symmetry assumption is omitted). A centrally-symmetric star-body $K$ is a compact set with non-empty interior such that $K = -K$, $tK \subset K$ for all $t \in [0, 1]$, and such that its radial function $\rho_K(\theta) = \max\{r \geq 0 \mid r\theta \in K\}$ for $\theta \in S^{n-1}$ is an even continuous function on $S^{n-1}$.

This note concerns two generalizations of the notion of an intersection body, first introduced by E. Lutwak in [Lut75] (see also [Lut88]). A star-body $K$ is said to be an intersection body of a star-body $L$, if $\rho_K(\theta) = \text{Vol}(L \cap \theta^\perp)$ for every $\theta \in S^{n-1}$, where $\theta^\perp$ is the hyperplane perpendicular to $\theta$. $K$ is said to be an intersection body, if it is the limit in the radial metric $d_r$ of intersection bodies $\{K_i\}$ of star-bodies $\{L_i\}$, where $d_r(K_1, K_2) = \sup_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|$. This is equivalent (e.g. [Lut88], [Gar94a]) to $\rho_K = R^*(d\mu)$, where $\mu$ is a non-negative Borel measure on $S^{n-1}$, $R^*$ is the dual transform (as in (1.3)) to the Spherical Radon Transform $R : C(S^{n-1}) \to C(S^{n-1})$, which is defined for $f \in C(S^{n-1})$ as:

$$R(f)(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi)d\sigma_\theta(\xi),$$

where $\sigma_\theta$ the Haar probability measure on $S^{n-1} \cap \theta^\perp$.

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The notion of an intersection body has been shown to be fundamentally connected to the Busemann-Petty Problem (first posed in [BP56]), which asks whether two centrally-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ satisfying:

\[(1.2) \quad \text{Vol}(K \cap H) \leq \text{Vol}(L \cap H) \quad \forall H \in G(n, n-1)\]

necessarily satisfy $\text{Vol}(K) \leq \text{Vol}(L)$. It was shown in [Lut88], [Gar94a] that the answer is equivalent to whether all centrally-symmetric convex bodies in $\mathbb{R}^n$ are intersection bodies, and in a series of results ([LR75], [Bal88], [Bou91], [Gar94a], [Gia90], [Pap92], [Gar94b], [Kol98], [Zha99], [GKS99]) that this is true for $n \leq 4$, but false for $n \geq 5$.

In [Zha96], G. Zhang considered a generalization of the Busemann-Petty problem, in which $G(n, n-1)$ in (1.2) is replaced by $G(n, n-k)$, where $k$ is some integer between 1 and $n-1$. Zhang showed that the generalized $k$-codimensional Busemann-Petty problem is also naturally associated to a class of generalized intersection-bodies, which will be referred to as $k$-Busemann-Petty bodies (note that these bodies are referred to as $n-k$-intersection bodies in [Zha96] and generalized $k$-intersection bodies in [Kol00]), and that the generalized $k$-codimensional problem is equivalent to whether all centrally-symmetric convex bodies in $\mathbb{R}^n$ are $k$-Busemann-Petty bodies. It was shown in [BZ98] (see also [RZ04]), and later in [Kol00], that the answer is negative for $k < n-3$, but the cases $k = n-3$ and $k = n-2$ remain open (the case $k = n-1$ is obviously true). Several partial answers to these cases are known. It was shown in [Zha96] (see also [RZ04]) that when $K$ is a centrally-symmetric convex body of revolution then the answer is positive for the pair $K, L$ with $k = n-2, n-3$ and any star-body $L$. When $k = n-2$, it was shown in [BZ98] that the answer is positive if $L$ is a Euclidean ball and $K$ is convex and sufficiently close to $L$. This was extended in [Mil06a], where it was shown that this is again true for $k = n-2$ and $k = n-3$, when $L$ is an arbitrary star-body and $K$ is sufficiently close to a Euclidean ball (but to an extent depending on its curvature). Several other generalizations of the Busemann-Petty problem were treated in [RZ04], [Zva05], [Yas05], [Yas06].

Before defining the class of $k$-Busemann-Petty bodies we shall need to introduce the $m$-dimensional Spherical Radon Transform, acting on spaces of continuous functions as follows:

\[R_m : C(S^{n-1}) \longrightarrow C(G(n, m))\]

\[R_m(f)(E) = \int_{S^{n-1} \cap E} f(\theta) d\sigma_E(\theta),\]

where $\sigma_E$ is the Haar probability measure on $S^{n-1} \cap E$. It is well known (e.g. [Hel99]) that as an operator on even continuous functions, $R_m$ is injective.
The dual transform is defined on spaces of signed Borel measures $\mathcal{M}$ by:

$$
R^*_m : \mathcal{M}(G(n, m)) \rightarrow \mathcal{M}(S^{n-1})
$$

$$
\int_{S^{n-1}} f R^*_m(d\mu) = \int_{G(n, m)} R_m(f)d\mu \quad \forall f \in C(S^{n-1}),
$$

and for a measure $\mu$ with continuous density $g$, the transform may be explicitly written in terms of $g$ (see [Zha96]):

$$
R^*_m g(\theta) = \int_{\theta \in E \in G(n, m)} g(E)d\nu_{m,\theta}(E),
$$

where $\nu_{m,\theta}$ is the Haar probability measure on the homogeneous space \{E $\in G(n, m) | \theta \in E$\}.

We shall say that a body $K$ is a $k$-Busemann-Petty body if $\rho^k_K = R^*_{n-k}(d\mu)$ as measures in $\mathcal{M}(S^{n-1})$, where $\mu$ is a non-negative Borel measure on $G(n, n - k)$. We shall denote the class of such bodies by $\mathcal{BP}^n_k$. Choosing $k = 1$, for which $G(n, n - 1)$ is isometric to $S^{n-1}/\mathbb{Z}_2$ by mapping $H$ to $S^{n-1} \cap H^\perp$, and noticing that $R$ is equivalent to $R_{n-1}$ under this map, we see that $\mathcal{BP}^n_1$ is exactly the class of intersection bodies.

In [Kol00], a second generalization of the notion of an intersection body was introduced by A. Koldobsky, who studied a different analytic generalization of the Busemann-Petty problem. Following [Kol00], a centrally-symmetric star-body $K$ is said to be a $k$-intersection body of a star-body $L$, if $\text{Vol}(K \cap H^\perp) = \text{Vol}(L \cap H)$ for every $H \in G(n, n - k)$. $K$ is said to be a $k$-intersection body, if it is the limit in the radial metric of $k$-intersection bodies $\{K_i\}$ of star-bodies $\{L_i\}$. We shall denote the class of such bodies by $\mathcal{I}^n_k$. Again, choosing $k = 1$, we see that $\mathcal{I}^n_1$ is exactly the class of intersection bodies.

In [Kol00], Koldobsky considered the relationship between these two types of generalizations, $\mathcal{BP}^n_k$ and $\mathcal{I}^n_k$, and proved that $\mathcal{BP}^n_k \subset \mathcal{I}^n_k$ (see also [Mil06b]). Koldobsky also asked whether the opposite inclusion is equally true for all $k$ between 2 and $n - 2$ (for 1 and $n - 1$ this is true):

**Question (Kol00):** Is it true that $\mathcal{BP}^n_k = \mathcal{I}^n_k$ for $n \geq 4$ and $2 \leq k \leq n - 2$?

If this were true, as remarked by Koldobsky, a positive answer to the generalized $k$-codimensional Busemann-Petty problem for $k \geq n - 3$ would follow, since for those values of $k$ any centrally-symmetric convex body in $\mathbb{R}^n$ is known to be a $k$-intersection body ([Kol99a, Kol99b, Kol00]).

In [Mil06b], it was shown that these two classes $\mathcal{BP}^n_k$ and $\mathcal{I}^n_k$ share many identical structural properties, suggesting that it is indeed reasonable to believe that $\mathcal{BP}^n_k = \mathcal{I}^n_k$. Using techniques from Integral Geometry for the class $\mathcal{BP}^n_k$ and Fourier transform of distributions techniques for the class $\mathcal{I}^n_k$, the following structure Theorem was established (see [Mil06b] for an account of particular cases which were known before). We define the $k$-radial sum of two star-bodies $L_1, L_2$ as the star-body $L$ satisfying $\rho^k_L = \rho^k_{L_1} + \rho^k_{L_2}$.
Structure Theorem ([Mil06b]) Let $C = \mathcal{I}$ or $C = BP$ and $k, l = 1, \ldots, n - 1$. Then:

1. $C^n_k$ is closed under full-rank linear transformations, $k$-radial sums and taking limit in the radial metric.
2. $C^n_1$ is the class of intersection-bodies in $\mathbb{R}^n$, and $C^n_{n-1}$ is the class of all symmetric star-bodies in $\mathbb{R}^n$.
3. Let $K_1 \in C^n_{k_1}$, $K_2 \in C^n_{k_2}$ and $l = k_1 + k_2 \leq n - 1$. Then the star-body $L$ defined by $\rho_L = \frac{k_1}{k_2} \rho_{K_1}^{k_2}$ satisfies $L \in C^n_l$. As corollaries:
   - (a) $C^n_{k_1} \cap C^n_{k_2} \subset C^n_{k_1+k_2}$ if $k_1 + k_2 \leq n - 1$.
   - (b) $C^n_k \subset C^n_l$ if $k$ divides $l$.
   - (c) If $K \in C^n_k$ then the star-body $L$ defined by $\rho_L = \frac{k}{l}$ satisfies $L \in C^n_l$ for $l \geq k$.
4. If $K \in C^n_k$ then any $m$-dimensional central section $L$ of $K$ (for $m > k$) satisfies $L \in C^n_m$.

Despite this and other evidence from [Mil06b] for a positive answer to Koldobsky’s question, we give the following negative answer. Let $O(n)$ denote the orthogonal group on $\mathbb{R}^n$. Recall that a star-body $K$ is called a body of revolution if its radial function $\rho_K \in C(S^{n-1})$ is invariant under the natural action of $O(n-1)$ identified as some subgroup of $O(n)$.

**Theorem 1.1.** Let $n \geq 4$ and $2 \leq k \leq n - 2$. Then there exists an infinitely smooth centrally-symmetric body of revolution $K$ such that $K \in \mathcal{I}_k^n$ but $K \notin BP^n_k$.

Note that Theorem 1.1 does not imply a negative answer to the unresolved cases $k = n - 2, n - 3$ (for $n \geq 5$) of the generalized Busemann-Petty problem, which pertains to convex bodies. Indeed, the $K$ we construct cannot be a convex body in those ranges of $k$, since as already mentioned, convex bodies of revolution are known ([Zha96], see also [RZ04]) to belong to $BP^n_{n-2}$ and $BP^n_{n-3}$. Theorem 1.1 does however imply that if one wishes to prove a positive answer to these unresolved cases by means of comparing $k$-intersection bodies to $k$-Busemann-Petty bodies, it is essential to restrict one’s attention to convex bodies.

Let $I : C(G(n, k)) \to C(G(n, n - k))$ denote the operator defined by $I(f)(E) = f(E^\perp)$ for all $E \in G(n, n - k)$. Let $R_{n-k}(C(S^{n-1})) = \text{Im } R_{n-k}$ denote the range of $R_{n-k}$. As explained in Section 2, Theorem 1.1 can be equivalently reformulated as follows:

**Theorem 1.2.** Let $n \geq 4$ and $2 \leq k \leq n - 2$. Then there exists an infinitely smooth function $g \in C(G(n, n - k))$ such that $R^*_n g \geq 1$ and $(I \circ R_k)^* g \geq 1$ as functions in $C(S^{n-1})$, but $g$ is not non-negative as a functional on $R_{n-k}(C(S^{n-1}))$. In other words, there exists a non-negative $h \in R_{n-k}(C(S^{n-1}))$ such that $\int_{G(n, n - k)} g(E) h(E) d\eta_{n,n-k}(E) < 0$, where $\eta_{n,n-k}$ is the Haar probability measure on $G(n, n - k)$. Moreover, both $g$ and $h$ can be chosen to be invariant under the action of $O(n-1)$. 

In [Mil06b], several equivalent formulations to Koldobsky’s question were obtained using cone-duality and the Hahn-Banach Theorem for convex cones. Let \( C_+(S^{n-1}) \) denote the cone of non-negative continuous functions on the sphere, and let \( R_{n-k}(C(S^{n-1}))_+ \) denote the cone of non-negative functions in the image of \( R_{n-k} \). Let \( A \) denote the closure of a set \( A \) in the corresponding normed space. Note that by the results from [Mil06b], \( \text{Im} \, I \circ R_k = \text{Im} \, R_{n-k} \), and hence:

\[
R_{n-k}(C(S^{n-1}))_+ \supset R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1})).
\]

As formally verified in [Mil06b], the dual formulation to Theorem 1.2 then reads:

**Theorem 1.3.** Let \( n \geq 4 \) and \( 2 \leq k \leq n - 2 \). Then:

\[
R_{n-k}(C(S^{n-1}))_+ \setminus R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1})) \neq \emptyset.
\]

In other words, there exists an (infinitely smooth) function \( f \in R_{n-k}(C(S^{n-1}))_+ \) which can not be approximated (in \( C(G(n,n-k)) \)) by functions of the form \( R_{n-k}(g) + I \circ R_k(h) \) with \( g, h \in C_+(S^{n-1}) \).

Other equivalent formulations using the language of Fourier transforms of homogeneous distributions are given in section 5. We comment here that one such formulation pertains to embeddings in \( L_p \) for negative values of \( p \). The definition of embedding into such a space (for \(-n < p < 0\)) was given by Koldobsky in [Kol00] by means of analytic continuation of the usual definition for \( p > 0 \). It is known (see Section 5) that for \( p \geq -1 \) (\( p \neq 0 \)) and for \(-n < p \leq -n + 1\), any star-body \( K \) such that \( (\mathbb{R}^n, \|\cdot\|_K) \) embeds in \( L_p \) can be generated by starting with the Euclidean ball \( D_n \), applying full-rank linear transformations, \((-p)\)-radial sums and taking the limit in the radial metric. Our results imply that \( p = -1 \) and \( p = -n + 1 \) are critical values for this property, and that this is no longer true for \( p = -k, 2 \leq k \leq n - 2 \). In other words, there exist “non-trivial” \( n \)-dimensional spaces which embed in \( L_{-k} \) for \( 2 \leq k \leq n - 2 \).

The rest of this note is organized as follows. In Section 2 we provide some additional background which is required to see why Theorem 1.2 implies Theorem 1.1 and Theorem 1.3. In Section 3 we develop several formulas for the Spherical Radon Transform and its dual for functions of revolution, i.e. functions invariant under the action of \( O(n - 1) \). In Section 4 we use these formulas to prove Theorem 1.2, thereby constructing the desired counter-example to Koldobsky’s question. In Section 5 we give several additional equivalent formulations to Theorem 1.1 using the language of Fourier transforms of homogeneous distributions.

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In this section, we summarize the relevant results needed for this note. We also explain why Theorem 1.1 and 1.3 follow from Theorem 1.2. We refer to [Mil06b] for more details.

For a star-body $K$ (not necessarily convex), we define its Minkowski functional as $\|x\|_K = \min \{t \geq 0 \mid x \in tK\}$. When $K$ is a centrally-symmetric convex body, this of course coincides with the natural norm associated with it. Obviously $\rho_K(\theta) = \|\theta\|_K^{-1}$ for $\theta \in S^{n-1}$.

It was shown by Koldobsky in [Kol00] that for a star-body $K$ in $\mathbb{R}^n$, $K \in I^n_k$ iff $\|\cdot\|^{-k}_K$ is a positive definite distribution on $\mathbb{R}^n$, meaning that its Fourier transform (as a distribution) $(||\cdot||^{-k}_K)^\wedge$ is a non-negative Borel measure on $\mathbb{R}^n$. We refer the reader to Section 5 for more on Fourier transforms of homogeneous distributions, as this will not be of essence in the ensuing discussion. To translate this result to the language of Radon transforms, it was shown in [Mil06b, Corollary 4.2] that for an infinitely smooth star-body $K$ and a (signed) Borel measure $\mu \in \mathcal{M}(G(n, n - k))$:

\begin{equation}
\|\cdot\|^{-k}_K = R^*_{n-k}(d\mu) \iff (||\cdot||^{-k}_K)^\wedge = c(n, k) (I \circ R)^*_k (d\mu),
\end{equation}

where $c(n, k)$ is some positive constant and the equalities above are interpreted as equalities between measures on $S^{n-1}$. Hence, it follows ([Mil06b, Lemma 5.3]) that for an infinitely smooth star-body $K$ in $\mathbb{R}^n$, $K \in I^n_k$ iff there exists a (possibly signed) Borel measure $\mu \in \mathcal{M}(G(n, n - k))$, such that as measures $\rho^k_K = R^*_{n-k}(d\mu) \geq 0$ and $(I \circ R)^*_k (d\mu) \geq 0$.

This should be compared with the definition of $k$-Busemann-Petty bodies: $K \in \mathcal{BP}^n_k \iff \rho^k_K = R^*_{n-k}(d\mu)$ as measures on $S^{n-1}$ for a non-negative Borel measure $\mu \in \mathcal{M}(G(n, n - k))$. Since for such a measure, $(I \circ R)^*_k (d\mu) \geq 0$, it follows that every infinitely smooth $k$-Busemann-Petty body is also a $k$-intersection body, and this easily implies (see [Mil06b, Corollary 4.4]) that $\mathcal{BP}^n_k \subset I^n_k$ in general, as first showed by Koldobsky in [Kol00].

$R^*_{n-k}$ is known (e.g. [Hel99]) to be injective on the space of even functions in $C(S^{n-1})$, so by duality $R^*_{n-k}$ is onto a dense subset of even measures in $\mathcal{M}(S^{n-1})$, which is known to include even measures with infinitely smooth densities. However, it is important to note that for $2 \leq k \leq n - 2$, the image of $R^*_{n-k}$ is not dense in $C(G(n, n - k))$, and equivalently, $R^*_{n-k}$ has a non-trivial kernel. The above implies that for any infinitely smooth star-body $K$, we can find a measure $\mu$ such that $\rho^k_K = R^*_{n-k}(d\mu)$, but if $2 \leq k \leq n - 2$ this measure will not unique. Nevertheless, as a functional on $R^*_{n-k}(C(S^{n-1}))$, such a measure $\mu$ is determined uniquely. The conclusion is that if we need to determine whether $K \in \mathcal{BP}^n_k$ given a representation $\rho^k_K = R^*_{n-k}(d\mu)$ for some measure $\mu \in \mathcal{M}(G(n, n - k))$, a necessary and sufficient condition is that $\mu$ is a non-negative functional on $R^*_{n-k}(C(S^{n-1}))$, i.e. $\int_{G(n, n - k)} R_{n-k}(h)(E) d\mu(E) \geq 0$ for any $h \in C(S^{n-1})$ such that $R_{n-k}(h) \geq 0$. Indeed, any non-negative functional on $R^*_{n-k}(C(S^{n-1}))$ can be extended
to a non-negative functional on $C(G(n, n - k))$ by a version of the Hahn-Banach Theorem (see the remarks before [Mil06b Lemma 5.2] for more details).

The above discussion explains why Theorem 1.1 is an immediate consequence of Theorem 1.2. Given the infinitely smooth function $g$ provided by Theorem 1.2, we define the centrally-symmetric star-body $K$ given by $\rho^*_K = R^*_n - k (g)$. Note that this indeed defines a star-body since $R^*_n - k (g) \geq 0$. In fact, $K$ is an infinitely smooth star-body since it is known (e.g. [GGR84]) that $R^*_n - k (g)$ is an infinitely smooth function on $S^{n-1}$ if $g$ is infinitely smooth; and since $\rho^*_K = R^*_n - k (g) \geq 1$, it follows that $\rho_K$ itself is infinitely smooth. In addition $K \in T^+_k$ since $(I \circ R_k)^* (g) \geq 0$. But since $g$ is not a non-negative functional on $R^*_n - k (C(S^{n-1}))$, it follows that $K \notin \mathcal{B}^n_k$.

To explain why Theorem 1.1 is equivalent to Theorem 1.3, we recall another result from [Mil06b]. Denote $\mathcal{M} = M(G(n, n - k))$ for short, and let:

$\mathcal{M} (\mathcal{B}^n_k) = \{ \mu \in \mathcal{M}; \mu \text{ is a non-negative functional on } R^*_n - k (C(S^{n-1})) \}$

and:

$\mathcal{M} (T^+_k) = \{ \mu \in \mathcal{M}; R^*_n - k (d\mu) \geq 0 \text{ and } (I \circ R_k)^* (d\mu) \geq 0 \}$.

It should already be clear from the above discussion that the statement $\mathcal{B}^n_k = T^+_k$ is equivalent to the statement $\mathcal{M} (\mathcal{B}^n_k) = \mathcal{M} (T^+_k)$. By the Hahn-Banach Theorem for convex cones, it is not hard to see ([Mil06b Theorem 5.6]) that the latter statement is dual to:

(2.2) $\overline{R^*_n - k (C(S^{n-1}))} = \overline{R^*_n - k (C_+ (S^{n-1})) + I \circ R_k (C_+ (S^{n-1}))}$.

As follows from (2.1), $\text{Ker } R^*_n - k = \text{Ker } (I \circ R_k)^*$, and therefore $\text{Im } R^*_n - k = \text{Im } I \circ R_k$. This explains why the right-hand side of (2.2) is always a subset of the left. Theorem 1.1 shows that it is a proper subset, implying Theorem 1.3. Since this Theorem is attained using a convex separation argument, we have no constructive way of finding the function $f$ of the Theorem. Albeit, we can always find an infinitely smooth $f$, since the subspace of infinitely smooth functions in $R^*_n - k (C(S^{n-1}))$ is known to be dense in $R^*_n - k (C(S^{n-1}))$, and hence in $\overline{R^*_n - k (C(S^{n-1}))}$.

3. Radon Transform for Functions of Revolution

Fix $n \geq 3$ and $\xi_0 \in S^{n-1}$. We denote by $O_{\xi_0} (n - 1)$ the subgroup of $O(n)$ whose natural action on $S^{n-1}$ leaves $\xi_0$ invariant, and by $C_{\xi_0} (S^{n-1})$ the linear subspace of functions in $C_v (S^{n-1})$ invariant under $O_{\xi_0} (n - 1)$. Clearly $O_{\xi_0} (n - 1)$ is isometric to $O(n - 1)$. We refer to members of $C_{\xi_0} (S^{n-1})$ as spherical functions of revolution. For $\xi_1, \xi_2 \in S^{n-1}$, let $\angle (\xi_1, \xi_2)$ denote the angle in $[0, \pi / 2]$ between $\xi_1$ and $\xi_2$, i.e. $\cos \angle (\xi_1, \xi_2) = |\langle \xi_1, \xi_2 \rangle|$. We also denote $\angle (\xi_1, 0) = \pi / 2$. Clearly $F \in C_{\xi_0} (S^{n-1})$ iff $F (\xi) = f (\angle (\xi, \xi_0))$ for $f \in C([0, \pi / 2])$. In that case, we denote by $\tilde{f} \in C([0, 1])$ the function
Lemma 3.1. Let \( f \in C([0, \pi/2]) \) and \( 2 \leq k \leq n - 1 \). Then:

\[
R_k(f)(\phi) = c_k \int_0^{\pi/2} f(\cos^{-1}(\cos \phi \cos \theta)) \sin^{k-2} \theta d\theta,
\]

where the value of \( c_k \) is found by using \( f \equiv 1 \), in which case \( R_k(f) \equiv 1 \).

Remark 3.2. This lemma, together with the subsequent ones, extend to the case \( k = 1 \), if we properly interpret the (formally) diverging integral as integration with respect to an appropriate delta-measure. Note also that the value \( c_k \) is consistent with the one used in (3.1).

Proof. Let \( F \in C_{\xi_0}(S^{n-1}) \) be given by \( F(\xi) = f(\angle(\xi, \xi_0)) \). Let \( E \in G(n,k) \) be such that \( \angle(\xi_0, E) = \phi \). Hence, if \( \xi_1 = \text{Proj}_E \xi_0 \) then \( \angle(\xi_0, \xi_1) = \phi \). For \( \xi \in S^{n-1} \cap E \), since \( \xi \rightarrow \text{Proj}_E \xi \) and \( \xi_0 \rightarrow \text{Proj}_E \xi_0 \) are orthogonal, it follows that \( \text{Proj}_E \xi = \text{Proj}_E(\text{Proj}_E \xi) \). Hence \( \cos \angle(\xi_0, \xi_0) = \cos \angle(\xi_0, \xi_1) \cos \angle(\xi_1, \xi_0) = \cos \angle(\xi_0, \xi_1) \cos \phi \). Since the function \( F \) is even, a standard polar integration
formula then gives:

\[
R_k(f)(\phi) = R_k(F)(E) = \int_{S^{n-1} \cap E} F(\xi) d\mu_E(\xi) = \int_{S^{n-1} \cap E} f(\angle(\xi, \xi_0)) d\mu_E(\xi)
\]

\[
= \int_{S^{n-1} \cap E} f(\cos^{-1}(\cos(\angle(\xi, \xi_1) \cos \phi))) d\mu_E(\xi)
\]

\[
= c_{n,k} \int_0^{\pi/2} f(\cos^{-1}(\cos \phi \cos \theta)) \sin^{k-2} \theta d\theta.
\]

\[\blacksquare\]

Performing the change of variables \( t = \cos \theta, \ s = \cos \phi \) above, we immediately have:

**Corollary 3.3.** Let \( \tilde{f} \in C[0,1] \) and \( 2 \leq k \leq n-1 \). Then:

\[
\tilde{R}_k(\tilde{f})(s) = c_k \int_0^1 \tilde{f}(st)(1-t^2)^{\frac{k-3}{2}} dt,
\]

where the value of \( c_k \) is the same as in Lemma 3.1.

Next, we introduce \( C_{\xi_0}(G(n,k)) \), the linear subspace of all functions in \( C(G(n,k)) \) invariant under the action of \( O_{\xi_0}(n-1) \). We refer to members of \( C_{\xi_0}(G(n,k)) \) as functions of revolution on the Grassmannian. As before, it is clear that \( G \in C_{\xi_0}(G(n,k)) \) iff \( G(E) = g(\angle(\xi_0, E)) \) for \( g \in C([0, \pi/2]) \). We have the following:

**Lemma 3.4.** Let \( G \in C_{\xi_0}(G(n,k)) \) such that \( G(E) = g(\angle(\xi_0, E)) \), and let \( \tilde{g} = T(g) \). Then:

\[
\int_{G(n,k)} G(E) d\eta_{n,k}(E) = b_{n,k} \int_0^{\pi/2} \tilde{g}(s) (1-s^2)^{\frac{n-k-2}{2}} s^{k-1} ds,
\]

where \( \eta_{n,k} \) is the Haar probability measure on \( G(n,k) \), and the value of \( b_{n,k} \) may be deduced by using \( G \equiv g \equiv \tilde{g} \equiv 1 \).

**Proof.** Clearly:

\[
\int_{G(n,k)} G(E) d\eta_{n,k}(E) = \int_0^{\pi/2} g(\phi) d\eta_{n,k} \{ E \in G(n,k); \angle(\xi_0, E) \leq \phi \}.
\]

Since \( \sigma_n \) and \( \eta_{n,k} \) are rotation-invariant, it follows that \( \eta_{n,k} \{ E \in G(n,k); \angle(\xi_0, E) \leq \phi \} = \sigma_n \{ \xi \in S^{n-1}; \angle(\xi, E_0) \leq \phi \} \) for any \( E_0 \in G(n,k) \). Using bi-polar coordinates (e.g. [Yi68 Chapter IX]), it is easy to see that:

\[
d\sigma_n \{ \xi \in S^{n-1}; \angle(\xi, E_0) \leq \phi \} = b_{n,k} \sin^{n-k-1} \phi \cos^{k-1} \phi d\phi,
\]

for some \( b_{n,k} \). This concludes the proof of the first equality of the lemma, and the second one follows by the change of variables \( s = \cos(\phi) \). \[\blacksquare\]
Next, we find an expression for the dual spherical Radon-Transform of a function in $C_{\xi_0}(G(n,k))$. As before, it is clear that if $F \in C_{\xi_0}(S^{n-1})$ then $R_k(F) \in C_{\xi_0}(G(n,k))$, and that if $G \in C_{\xi_0}(G(n,k))$ then $\tilde{R}_k^*(G) \in C_{\xi_0}(S^{n-1})$. If $G \in C_{\xi_0}(G(n,k))$ is given by $G(E) = g(\angle(\xi, E))$, we denote by $R_k^*(g) \in C([0,\pi/2])$ the function given by $R_k^*(g)(\angle(\xi, \xi_0)) = R_k^*(G)(\xi)$. As usual, we define $\tilde{R}_k^*: C[0,1] \to C[0,1]$ by $\tilde{R}_k^* = T \circ R_k^* \circ T^{-1}$. The standard duality relation:

$$\int_{S^{n-1}} R_k^*(G)(\xi)F(\xi)d\sigma_n(\xi) = \int_{G(n,k)} G(E)R_k(F)(E)d\eta_{n,k}(E)$$

is immediately translated using (3.1) and Lemma 3.4 into the following duality relation between $\tilde{R}_k$ and $\tilde{R}_k^*$ on $C([0,1])$:

**Lemma 3.5.** Let $\tilde{f}, \tilde{g} \in C([0,1])$ and $1 \leq k \leq n - 1$. Then:

$$\int_0^1 \tilde{R}_k^*(\tilde{g})(t)\tilde{f}(t)(1 - t^2)^{-\frac{n-3}{2}} dt = d_{n,k} \int_0^1 \tilde{g}(s)\tilde{R}_k(\tilde{f})(s)(1 - s^2)^{-\frac{n-k-2}{2}} s^{k-1} ds,$$

where the value of $d_{n,k}$ is found by using $\tilde{f}, \tilde{g} \equiv 1$, in which case $\tilde{R}_k(\tilde{f}), \tilde{R}_k^*(\tilde{g}) \equiv 1$.

We can now deduce an expression for $\tilde{R}_k^*$:

**Lemma 3.6.** Let $\tilde{g} \in C([0,1])$ and $2 \leq k \leq n - 1$. Then:

$$\tilde{R}_k^*(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}(\sqrt{1 - s^2(1 - t^2)})(1 - s^2)^{\frac{k-3}{2}} s^{n-k-1} ds,$$

where the value of $e_{n,k}$ is found by using $\tilde{g} \equiv 1$, in which case $\tilde{R}_k^*(\tilde{g}) \equiv 1$.

**Proof.** We start with Lemma 3.5 and use the formula for $\tilde{R}_k$ given in Corollary 3.3:

$$\int_0^1 \tilde{R}_k^*(\tilde{g})(t)\tilde{f}(t)(1 - t^2)^{-\frac{n-3}{2}} dt = d_{n,k} \int_0^1 \tilde{g}(s)\tilde{R}_k(\tilde{f})(s)(1 - s^2)^{-\frac{n-k-2}{2}} s^{k-1} ds$$

$$= d_{n,k}c_k \int_0^1 \tilde{g}(s) \int_0^1 \tilde{f}(st)(1 - t^2)^{-\frac{k-3}{2}} dt(1 - s^2)^{-\frac{n-k-2}{2}} s^{k-1} ds$$

$$= d_{n,k}c_k \int_0^1 \tilde{f}(v) \int_v^1 \tilde{g}(s) \left(1 - \frac{s^2}{2}\right)^{\frac{k-3}{2}} (1 - s^2)^{-\frac{n-k-2}{2}} s^{k-2} ds dv.$$

Since this is true for any $\tilde{f} \in C([0,1])$, setting $e_{n,k} = d_{n,k}c_k$, we conclude that:

$$\tilde{R}_k^*(\tilde{g})(t) = e_{n,k}(1 - t^2)^{-\frac{n-3}{2}} \int_t^1 \tilde{g}(s) \left(1 - \frac{t^2}{s^2}\right)^{\frac{k-3}{2}} (1 - s^2)^{-\frac{n-k-2}{2}} s^{k-2} ds.$$

By the change of variable $s = \sqrt{1 - (s')^2(1 - t^2)}$, one easily checks that the assertion of the lemma is obtained. \hfill \Box
We now recall the definition of the “perp” operator $I$ from the Introduction, and extend it to the context of functions of revolution. For every $k = 1, \ldots, n - 1$, we define $I : C(G(n, k)) \to C(G(n, n-k))$ as $I(f)(E) = f(E^\perp)$ for all $E \in G(n, n-k)$, without specifying the index $k$. $I$ is obviously self-adjoint:

$$\int_{G(n,n-k)} I(F)(H)G(H)d\eta_{n-k}(H) = \int_{G(n,k)} F(E)I(G)(E)d\eta_k(E),$$

for all $F \in C(G(n,k))$ and $G \in C(G(n, n-k))$, where $\eta_m$ denotes the Haar probability measure on $G(n, m)$.

Since $\langle \xi_0, E \rangle = \pi/2 - \langle \xi_0, E^\perp \rangle$, it is clear that for $G \in C_{\xi_0}(G(n,k))$ such that $G(E) = g(\langle \xi_0, E \rangle)$ for every $E \in G(n,k)$, $I(G)(H) = g(\pi/2 - \langle \xi_0, H \rangle)$ for every $H \in G(n, n-k)$. We therefore define $I : C([0, \pi/2]) \to C([0, \pi/2])$ as $I(g)(\phi) = g(\pi/2 - \phi)$. Similarly, for $\tilde{g} \in C([0,1])$, we define $I(\tilde{g})(s) = \tilde{g}(\sqrt{1 - s^2})$. Clearly, if $G(E) = \tilde{g}(\langle \xi_0, E \rangle)$ then $I(G)(H) = I(\tilde{g})(\langle \xi_0, H \rangle)$. Hence in both cases $I$ must be self-adjoint, and this can be also verified directly. As an immediate corollary of 3.6, we have:

**Corollary 3.7.** Let $\tilde{g} \in C([0,1])$ and $2 \leq k \leq n - 1$. Then:

$$(I \circ \tilde{R}_k)^*(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}(s \sqrt{1 - t^2}) (1 - s^2)^{k-1} s^{n-k-1} ds,$$

where the value of $e_{n,k}$ is the same as in Lemma 3.6.

We are now ready to construct the counter-example to Koldobsky’s question, as described in the next section.

**4. The Construction**

The main step in the proof of Theorem 1.2 is the following:

**Proposition 4.1.** For any $n \geq 4$, $2 \leq k \leq n-2$ and $s_0 \in (0,1)$, there exists an infinitely smooth function $\tilde{g} \in C([0,1])$ such that:

1. For all $t \in [0,1]$:

$$\tilde{R}^*_{n-k}(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}(s \sqrt{1 - t^2}) (1 - s^2)^{k-1} s^{n-k-1} ds \geq 1.$$

2. For all $t \in [0,1]$:

$$(I \circ \tilde{R}_k)^*(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}(s \sqrt{1 - t^2}) (1 - s^2)^{k-1} s^{n-k-1} ds \geq 1.$$

3. $\tilde{g}(s_0) = -1$.

**Proof.** The proof is straightforward. We provide the details nevertheless. Let $\varepsilon > 0$ be such that $[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \subset (0,1)$. Let $T_1, T'_1 \in C([0,1])$ be defined by $T_1(s) = \sqrt{1 - s^2(1 - t^2)}$ and $T'_1(s) = s \sqrt{1 - t^2}$, and let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$. It is elementary to check that the maximum of
\[ \lambda \{ T_t^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \} \text{ over } t \in [0, 1] \text{ is attained at } t = s_0 - 2\varepsilon, \text{ in which case it is equal to:} \]

\[
\delta_1 := \max_{t \in [0, 1]} \lambda \{ T_t^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \} = 1 - \sqrt{\frac{1 - (s_0 + 2\varepsilon)^2}{1 - (s_0 - 2\varepsilon)^2}} < 1.
\]

An analogous computation shows that the maximum of \( \lambda \{ (T_t')^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \} \) over \( t \in [0, 1] \) is attained at \( t = \sqrt{1 - (s_0 + 2\varepsilon)^2} \), in which case it is equal to:

\[
\delta_2 := \max_{t \in [0, 1]} \lambda \{ (T_t')^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \} = \frac{4\varepsilon}{s_0 + 2\varepsilon} < 1.
\]

Set \( \delta := \max(\delta_1, \delta_2) < 1 \). Now denote by \( \mu_{n,m} \) the measure \( e_nm(1 - s^2)^{m-3}s^{n-m-1}ds \) on \([0,1]\), for \( 2 \leq m \leq n - 2 \). These are probability measures, as witnessed by using \( \tilde{g} \equiv 1 \) in Lemma 3.6, in which case \( R_k^*(\tilde{g}) \equiv 1 \).

Since their densities (with respect to \( \lambda \)) are absolutely continuous and do not vanish on \((0,1)\), a compactness argument shows that (fixing \( n \)):

\[
\gamma := \sup_{v \in [0,1], 2 \leq m \leq n-2} \mu_{n,m}(\{v, v + \delta\}) < 1.
\]

Set \( \gamma^* = \frac{1+\gamma}{1-\gamma} \). We conclude by constructing \( \tilde{g} \) as follows. Set \( \tilde{g}(s) = -1 \) for \( s \in [s_0 - \varepsilon, s_0 + \varepsilon] \), \( \tilde{g}(s) = \gamma^* \) for \( s \in [0,1] \setminus [s_0 - 2\varepsilon, s_0 + 2\varepsilon] \), and for \( s \in [s_0 - 2\varepsilon, s_0 + 2\varepsilon] \setminus [s_0 - \varepsilon, s_0 + \varepsilon] \) set \( \tilde{g}(s) \in [-1, \gamma^*] \) so that the resulting function \( \tilde{g} \in C[0,1] \) is in fact infinitely smooth (using standard methods).

Alternatively, we could simply define \( \tilde{g}(s) = (\gamma^* + 1)(\frac{s-s_0}{2\varepsilon})^2 - 1 \) on \([0,1]\).

Setting:

\[
\beta_1(t) := \mu_{n,n-k} \{ s \in [0,1]; T_t(s) \in [s_0 - 2\varepsilon, s_0 + 2\varepsilon] \},
\]

the definition of \( \gamma \) and \( \delta \) imply that \( \beta_1(t) \leq \gamma \) for all \( t \in [0,1] \), hence:

\[
\int_0^1 \tilde{g}(s) \sqrt{1 - s^2(1 - t^2)} \, d\mu_{n,n-k}(s) \geq \gamma^*(1 - \beta_1(t)) - \beta_1(t) \geq 1
\]

for all \( t \in [0,1] \). Similarly, setting:

\[
\beta_2(t) := \mu_{n,k} \{ s \in [0,1]; T_t'(s) \in [s_0 - 2\varepsilon, s_0 + 2\varepsilon] \},
\]

we have \( \beta_2(t) \leq \gamma \) for all \( t \in [0,1] \), and:

\[
\int_0^1 \tilde{g}(s) \sqrt{1 - t^2} \, d\mu_{n,k}(s) \geq \gamma^*(1 - \beta_2(t)) - \beta_2(t) \geq 1
\]

for all \( t \in [0,1] \). This concludes the proof. \( \square \)

**Remark 4.2.** Note that for \( k = 1 \) and \( k = n - 1 \) the above reasoning fails, as the measure \( \mu_{n,1} \) is a singular measure.

**Remark 4.3.** Note also that the function \( \tilde{g} \) we have constructed in fact satisfies the claims (1) and (2) for all values of \( k \) in the range \( 2 \leq k \leq n - 2 \).
We can now almost conclude the proof of Theorem 1.2. We still need one last observation, since a-priori, the fact that $\tilde{g}(s_0) < 0$ does not guarantee that the function $G \in C(G(n, n - k))$ defined as $G(E) = \tilde{g}(\cos(\angle(\xi_0, E)))$, is not a non-negative functional on $R_{n-k}(C(S^{n-1}))$. This is resolved by the following:

**Lemma 4.4.** The polynomials on $[0, 1]$ are in the range of $\tilde{R}_{n-k}(C([0, 1]))$.

**Proof.** This is immediate by Corollary 3.3 because if $\tilde{p}(t) = t^m$ ($m \geq 0$), then:

$$\tilde{R}_k(\tilde{p})(s) = c_k \int_0^1 \tilde{p}(st)(1 - t^2)^{k-3/2} dt = d_{k,m} s^m,$$

with $d_{k,m} > 0$. Hence polynomials are mapped to polynomials by $\tilde{R}_{n-k}$, and any polynomial in the range may be obtained. \hfill \Box

By the Weierstrass approximation theorem, if follows that:

**Corollary 4.5.** The range of $\tilde{R}_{n-k}$ is dense in $C([0, 1])$.

We can now turn to the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let $\tilde{g} \in C[0, 1]$ be the infinitely smooth function constructed in Proposition 4.1, with, say $s_0 = 1/2$. Fix some $\xi_0 \in S^{n-1}$, and let $G \in C_{\xi_0}(G(n, n - k))$ be defined by $G(E) = \tilde{g}(\cos(\angle(\xi_0, E)))$ for every $E \in G(n, n - k)$. Since the functions $\tilde{g}$, cos and $\angle(\xi, \cdot)$ are infinitely smooth on their corresponding domains, so is their composition, hence $G$ is infinitely smooth on $G(n, n - k)$. By the construction of $\tilde{g}$ and the compatibility of $R_{n-k}^*$ and $(I \circ R_k)^*$ with $\tilde{R}_{n-k}^*$ and $(I \circ \tilde{R}_k)^*$, respectively, it follows that $R_{n-k}^*(G) = \tilde{R}_{n-k}^*(\tilde{g}) \geq 1$ and $(I \circ R_k)^*(G) = (I \circ \tilde{R}_k)^*(\tilde{g}) \geq 1$. It remains to show that $G$ is not a non-negative functional on $R_{n-k}(C(S^{n-1}))$. Let $H \in C_{\xi_0}(S^{n-1})$ be such that $H(\xi) = \tilde{h}(\cos(\angle(\xi_0, \xi)))$ for some $\tilde{h} \in C([0, 1])$. Then by Lemma 3.3

$$\int_{G(n,n-k)} G(E) R_{n-k}(H)(E) d\eta_{n,k} = \int_0^1 \tilde{g}(s) \tilde{R}_{n-k}(\tilde{h})(s)(1 - s^2)^{n-k-2} s^{k-1} ds.$$

Since $\tilde{g}(s)(1 - s^2)^{n-k-2} s^{k-1}$ is a continuous function on $[0, 1]$ whose value at $s_0$ is negative, by Corollary 4.5 we can find a function $\tilde{h} \in C([0, 1])$ such that the integral in (1.1) is negative. This concludes the proof. \hfill \Box

5. Additional formulations

In this section, we give several additional equivalent formulations to the main result of this note, using the language of Fourier transforms of homogeneous distributions (we refer the reader to [Kol05] for more on this subject).

We denote by $S(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable test functions in $\mathbb{R}^n$, and by $S'(\mathbb{R}^n)$ the space of distributions over
\[ S(\mathbb{R}^n) \]. The Fourier transform \( \hat{f} \) of a distribution \( f \in S'(\mathbb{R}^n) \) is defined by 
\[ \langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle \] for every test function \( \phi \), where \( \hat{\phi}(y) = \int \phi(x) \exp(-i \langle x, y \rangle) dx \).

A distribution \( f \) is called homogeneous of degree \( p \in \mathbb{R} \) if \( \langle f, \phi(\cdot/t) \rangle = |t|^p \langle f, \phi \rangle \) for every \( t > 0 \), and it is called even if the same is true for \( t = -1 \). An even distribution \( f \) always satisfies \( (\hat{f})^* = (2\pi)^n f \). The Fourier transform of an even homogeneous distribution of degree \( p \) is an even homogeneous distribution of degree \( -n - p \). A distribution \( f \) is called positive if \( \langle f, \phi \rangle \geq 0 \) for every \( \phi \geq 0 \), implying that \( f \) is necessarily a non-negative Borel measure on \( \mathbb{R}^n \). We use Schwartz’s generalization of Bochner’s Theorem (\cite{GS64}) as a definition, and call a homogeneous distribution positive-definite if its Fourier transform is a positive distribution.

The following characterization was given by Koldobsky in \cite{Kol00}:

**Theorem 5.1 (Koldobsky).** The following are equivalent for a centrally-
symmetric star-body \( K \) in \( \mathbb{R}^n \):

1. \( K \) is a \( k \)-intersection body.
2. \( \| \cdot \|^{-k}_K \) is a positive definite distribution on \( \mathbb{R}^n \), meaning that its 
   Fourier-transform \( (\| \cdot \|^{-k}_K)^\wedge \) is a non-negative Borel measure on \( \mathbb{R}^n \).
3. The space \((\mathbb{R}^n, \| \cdot \|_K)\) embeds in \( L_{-k} \).

For completeness, we give the definition of embedding in \( L_{-k} \). For \( p > -1 \)
(and \( p \neq 0 \), the case \( p = 0 \) requires passing to the limit), it is well known
(e.g. \cite{Kol00}) that \((\mathbb{R}^n, \| \cdot \|)\) embeds in \( L_p \) iff:

\[ \int_{\mathbb{R}^n} \| x \|^{-p} \phi(x) dx = \int_{S^{n-1}} \int_0^\infty t^{-p-1} \hat{\phi}(t\theta) t dt d\mu(\theta), \]  

(5.1)

for some \( \mu \in \mathcal{M}_+(S^{n-1}) \), the cone of non-negative Borel measures on \( S^{n-1} \).

Unfortunately, this characterization breaks down at \( p = -1 \) since the above
integral no longer converges. However, Koldobsky showed that it is possible
to regularize this integral by using Fourier-transforms of distributions, and
gave the following definition: \((\mathbb{R}^n, \| \cdot \|)\) embeds in \( L_{-p} \) for \( 0 < p < n \) iff there
exists a measure \( \mu \in \mathcal{M}_+(S^{n-1}) \) such that for any even test-function \( \phi \):

\[ \int_{\mathbb{R}^n} \| x \|^{-p} \phi(x) dx = \int_{S^{n-1}} \int_0^\infty t^{-p-1} \hat{\phi}(t\theta) t dt d\mu(\theta). \]  

(5.2)

In addition to the characterization (3) in Theorem 5.1 of \( T_k^n \) as the class of
unit-balls of subspaces of scalar \( L_{-k} \) spaces, a functional analytic characterization
of \( \mathcal{B}P_k^n \) as the class of unit-balls of subspaces of certain vector-valued
\( L_{-k} \) spaces was given in \cite{Kol00}. To explain this better, we state the defini-
tion given by Koldobsky: \((\mathbb{R}^n, \| \cdot \|)\) embeds in \( L_{-p}(\mathbb{R}^k) \) for \( 0 < p < n \) iff
there exists a measure \( \mu \in \mathcal{M}_+(\mathbb{R}^{nk}) \) such that for any even test-function \( \phi \):

\[ \int_{\mathbb{R}^n} \| x \|^{-p} \phi(x) dx = \int_{\mathbb{R}^{nk}} \int_{\mathbb{R}^k} \| v \|^{2-p-k} \hat{\phi} \left( \sum_{i=1}^k v_i \xi_i \right) dv d\mu(\xi). \]  

(5.3)
For $k = 1$ it is easy to see that this coincides with the definition of embedding in $L_{-p}$. Using this definition, it was shown in [Ko00] that $K \in \mathcal{BP}_k^n$ iff $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_{-k}(\mathbb{R}^k)$. For $p > 0$, it is known that every separable vector valued $L_p$ space is isometric to a subspace of a scalar $L_p$ space and vice-versa. Translating Theorem 1.1 into the language of $L_p$ spaces, we see that this is no longer true when $p = -k, 2 \leq k \leq n - 2$:

**Corollary 5.2.** Let $n \geq 4$ and $2 \leq k \leq n - 2$. Then there exists an infinitely smooth centrally-symmetric body of revolution $K$ such that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_{-k}$ but does not embed in $L_{-k}(\mathbb{R}^k)$.

Next, we describe another property of $L_p$ spaces which breaks down when passing the critical value of $p = -1$. Let us denote the set of all star-bodies $K$ in $\mathbb{R}^n$ for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_p$ ($p \neq 0$) by $SL_p^n$. For $p \neq 0$, let the $p$-norm sum of two bodies $L_1, L_2$ be defined as the body $L$ satisfying $\|\cdot\|^p_L = \|\cdot\|^p_{L_1} + \|\cdot\|^p_{L_2}$. Obviously, the $p$-norm sum coincides with the $(-p)$-radial sum, defined in the introduction (before the Structure Theorem). We will denote by $D_p^n$ the class of bodies created from the Euclidean ball $D_n$ by applying full-rank linear-transformations, $p$-norm sums, and taking the limit in the radial metric. Using the characterization in (5.1), it is easy to show (e.g. [GZ99] Theorem 6.13) that for $p > -1$ ($p \neq 0$), $SL_p^n = D_p^n$. In order to understand what happens when $p \leq -1$, we turn to the following characterization of $\mathcal{BP}_k^n$, first proved by Goodey and Weil (slightly different) in [GW95] for intersection-bodies (the case $k = 1$), and extended to general $k$ by Grinberg and Zhang in [GZ99]:

**Theorem 5.3** (Grinberg and Zhang). A star-body $K$ in $\mathbb{R}^n$ is a $k$-Busemann-Petty body ($1 \leq k \leq n - 1$) iff it is the limit of $\{K_i\}$ in the radial metric, where each $K_i$ is a finite $k$-radial sum of ellipsoids $\{E_i\}$ in $\mathbb{R}^n$ having non-empty interior:

$$
\rho_{K_i}^k = \rho_{E_i}^k + \ldots + \rho_{E_{m_i}}^k.
$$

In other words, Theorem 5.3 states that $D^n_{k, k} = \mathcal{BP}_k^n$ for $k = 1, \ldots, n - 1$. Recall that $\mathcal{I}^n_1 = \mathcal{BP}^n_1$ is the class of all intersection-bodies in $\mathbb{R}^n$ and $\mathcal{I}^n_{n-1} = \mathcal{BP}^n_{n-1}$ is the class of all centrally-symmetric star-bodies in $\mathbb{R}^n$ (this is clear from the definitions, see also the Structure Theorem in the introduction). Since $\mathcal{I}^n_k = SL^n_{-k}$ by characterization (3) of Theorem 5.1, we see that $SL^n_{-k} = D^n_{-k}$ for $k = 1$ and $k = n - 1$. However, Theorem 1.1 implies that this is no longer true for $2 \leq k \leq n - 2$:

**Corollary 5.4.** Let $n \geq 4$ and $2 \leq k \leq n - 2$. Then $SL^n_{-k} \setminus D^n_{-k} \neq \emptyset$.

Note that since $\mathcal{BP}^n_k \subset \mathcal{I}^n_k$, it is always true that $D^n_{-k} \subset SL^n_{-k}$ (in fact, this is straightforward to check directly, implying that $\mathcal{BP}^n_k \subset \mathcal{I}^n_k$ by using Theorems 5.1 and 5.3). In some sense, the members of $D^n_{-k}$ are the “trivial” elements of $SL^n_{-k}$, since obviously $D_n \in SL^n_{-k}$, and $SL^n_{-k}$ is closed under
taking full-rank linear transformations, $(-k)$-norm sums and and limit in the radial-metric. Corollary 5.4 therefore says that there are also “non-trivial” elements in $SL^n_{-k}$, for $2 \leq k \leq n - 2$.

We conclude by translating Corollary 5.4 into the language of Fourier transforms of homogeneous distributions. Given an even $f \in C(S^{n-1})$, we denote by $E_p(f)$ its homogeneous extension of degree $p$ onto $\mathbb{R}^n$ (formally excluding $\{0\}$ if $p < 0$, i.e. $E_p(f)(t\theta) = t^pf(\theta)$ for $t > 0$ and $\theta \in S^{n-1}$. We denote by $E_p^\wedge(f)$ the Fourier transform of $E_p(f)$ as a distribution. Note that $E_p^\wedge(f)$ need not necessarily be a continuous function on $\mathbb{R}^n \setminus \{0\}$, nor even a measure on $\mathbb{R}^n$. In order to ensure that $E_p^\wedge(f)$ is a continuous function, we need to add some smoothness assumptions on $f$ (Kol05). We remark that for an infinitely smooth function $f \in C(S^{n-1})$, $E_p^\wedge(f)$ is infinitely smooth on $\mathbb{R}^n \setminus \{0\}$ for any $p \in (-n, 0)$. Whenever $E_p^\wedge(f)$ is continuous on $\mathbb{R}^n \setminus \{0\}$, it is uniquely determined by its value on $S^{n-1}$ (by homogeneity), so we identify (abusing notation) between $E_p^\wedge(f)$ and its restriction to $S^{n-1}$.

Clearly $E_{-k}(\rho^k_K) = ||\cdot||_{k}$ for a star-body $K$. Given a full-rank linear transformation $T$ in $\mathbb{R}^n$, we denote $T(E_p(f)) = E_p(f) \circ T^{-1}$, so $T(E_{-k}(\rho^k_K)) = E_{-k}(\rho^k_{T^k(K)})$ for a star-body $K$. Again, we identify (by homogeneity) between $T(E_p(f))$ and its restriction on $S^{n-1}$.

It is easy to check (e.g. Mil06b) that for any infinitely smooth $K \in D^n_{-k}$, we have $E_{-k}(\rho^k_K) \geq 0$ (and clearly $\rho^k_K \geq 0$). In fact, this immediately follows from the fact that this is true for $D_n \in D^n_{-k}$, the linearity of the Fourier transform, and its behavior under full-rank linear transformations. With Theorem 6.3 and characterization (2) of Theorem 5.1 in mind, asking whether $BP^n_k = \mathcal{T}^n_k$ is equivalent to asking whether the only infinitely smooth functions $f \in C(S^{n-1})$ such that $f \geq 0$ and $E_{-k}^\wedge(f) \geq 0$, are the ones such that $f = \rho^k_K$ for some $K \in D^n_{-k}$. In other words, whether every such $f$ can be approximated (in the maximum norm in $C(S^{n-1})$, which is clearly the same for $f$ and for $f^{1/k}$) by functions of the form $\sum_{i=1}^m T_i(E_{-k}(1))$, where $T_i$ are full-rank linear transformations. The following is thus an immediate consequence of Theorem 4.1

**Corollary 5.5.** Let $n \geq 4$ and $2 \leq k \leq n - 2$. Then there exists a “non-trivial” infinitely smooth function of revolution $f \in C(S^{n-1})$ such that $f \geq 0$ and $E_{-k}^\wedge(f) \geq 0$. By “non-trivial”, we mean that $f$ cannot be approximated in the maximum norm on $C(S^{n-1})$ by functions of the form $\sum_{i=1}^m T_i(E_{-k}(1))$, where $\{T_i\}$ are full-rank linear transformations in $\mathbb{R}^n$.

To conclude, we comment that although the original definitions of $BP^n_k$ and $\mathcal{T}^n_k$ make sense only for integer values of $k$ (between 1 and $n - 1$), some of the alternative characterizations of these classes stated in this section make sense for arbitrary real-valued $k$, for $0 < k < n$. In particular, characterizations (2) and (3) of Theorem 5.1 for the class $\mathcal{T}^n_k$ and Theorem 5.3 for the class $BP^n_k$ may be taken as definitions for these classes of star-bodies in this
extended range of $k$. It then makes sense to ask whether Theorem 1.1 also holds for any non-integer $1 < k < n - 1$. Although we do not proceed in this direction, the answer should be positive, since our construction of the function $\tilde{g}$ in Proposition 4.1 is purely analytic, and everything still works for arbitrary real-valued $k$, for $1 < k < n$.

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