Rotating magnetic solution in three dimensional Einstein gravity

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ABSTRACT: We obtain the magnetic counterpart of the BTZ solution, i.e., the rotating spacetime of a point source generating a magnetic field in three dimensional Einstein gravity with a negative cosmological constant. The static (non-rotating) magnetic solution was found by Clément, by Hirschmann and Welch and by Cataldo and Salgado. This paper is an extension of their work in order to include (i) angular momentum, (ii) the definition of conserved quantities (this is possible since spacetime is asymptotically anti-de Sitter), (iii) upper bounds for the conserved quantities themselves, and (iv) a new interpretation for the magnetic field source. We show that both the static and rotating magnetic solutions have negative mass and that there is an upper bound for the intensity of the magnetic field source and for the value of the angular momentum. The magnetic field source can be interpreted not as a vortex but as being composed by a system of two symmetric and superposed electric charges, one of the electric charges is at rest and the other is spinning. The rotating magnetic solution reduces to the rotating uncharged BTZ solution when the magnetic field source vanishes.

KEYWORDS: Classical Theories of Gravity, Black Holes
1. Introduction

The three dimensional BTZ black hole solution of Bañados, Teitelboim and Zanelli [1, 2] has been the subject of many studies [3]. The original article [1] considers the static and rotating uncharged BTZ black hole and the static electrically charged BTZ black hole. The extension to consider the rotating electrically charged BTZ black hole (mass, angular momentum and electric charge different from zero) has been done by Clément [4] and by Martínez, Teitelboim and Zanelli [5]. An extension to include a Brans-Dicke term has been made by Dias and Lemos [6].

The static magnetic counterpart of the BTZ solution has also been considered. Indeed Clément [4], Hirschmann and Welch [7] and Cataldo and Salgado [8], using different procedures, have found the spacetime generated by a static point source of magnetic field in three dimensional Einstein gravity with negative cosmological constant, that reduces to the BTZ solution when the magnetic field source vanishes. This solution is horizonless and has a conical singularity at the origin.

In this paper, we extend [7, 8] in order to include (i) angular momentum, (ii) the definition of conserved quantities (mass, angular momentum and electric charge), (iii) upper bounds for the conserved quantities and (iv) a new interpretation for the magnetic field source. This rotating magnetic solution reduces to the rotating uncharged BTZ solution when the magnetic field source vanishes.
The plan of this article is the following. In section 2 we study the static magnetic solution found in \cite{7,8} and its properties. Section 3 is devoted to the rotating magnetic solution. The angular momentum is added in section 3.1 through a rotational Lorentz boost. In section 3.2 we calculate the mass, angular momentum, and electric charge of both the static and rotating solutions and in section 3.3 we set relations between the conserved charges. The rotating magnetic solution is written as a function of its hairs in section 3.4 and we show that it reduces to the rotating BTZ solution when the magnetic source vanishes. In section 4 we give a physical interpretation for the origin of the magnetic field source. Finally, in section 5 we present the concluding remarks.

2. Static solution. Analysis of its general structure

2.1 Static solution

Einstein gravity with a negative cosmological constant and a source of magnetic field (or Einstein-Maxwell-anti de Sitter gravity) in three dimensions can be characterized by the action

\[ S = \frac{1}{4} \int d^3 x \sqrt{-g} \left( R - 2\Lambda - F^{\mu \nu} F_{\mu \nu} \right), \]

(2.1)

where \( g \) is the determinant of the metric \( g_{\mu \nu} \), \( R \) the Ricci scalar, \( F_{\mu \nu} \) the electromagnetic tensor given in terms of the vector potential \( A_\mu \) by \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( \Lambda \) is the cosmological constant. In this subsection we study the static point source solution of action (2.1) found by Clément \cite{4}, by Hirschmann and Welch \cite{7} and by Cataldo and Salgado \cite{8}. The point source generates a gravitational field and a magnetic field. Written in the gauge presented in \cite{7}, the static solution is given by the following metric and vector potential 1-form

\[ ds^2 = -(r^2/l^2 - m)dt^2 + [r^2 + \chi_m^2 \ln |r^2/l^2 - m|] d\varphi^2 + r^2(r^2/l^2 - m)^{-1} \left[ r^2 + \chi_m^2 \ln |r^2/l^2 - m| \right]^{-1} dr^2, \]

(2.2)

\[ A = \frac{1}{2} \chi_m \ln |r^2/l^2 - m| d\varphi, \]

(2.3)

where \( t, r \) and \( \varphi \) are the time, the radial and the angular coordinates, respectively, \( \chi_m \) is an integration constant which measures the intensity of the magnetic field source and \( l \equiv -1/\sqrt{\Lambda} \) is the cosmological length. This spacetime reduces to the three dimensional BTZ black hole solution of Bañados, Teitelboim and Zanelli \cite{1,2} when the magnetic source vanishes. The parameter \( m \) is the mass of this uncharged solution.

The \( g_{rr} \) function is negative for \( r < r_+ \) and positive for \( r > r_+ \), where \( r_+ \) is such that

\[ r_+^2 + \chi_m^2 \ln |r_+^2/l^2 - m| = 0, \]

(2.4)

and the condition \( r_+^2 > ml^2 \) is obeyed. One might then be tempted to say that the solution has an horizon at \( r = r_+ \) and consequently that one is in the presence of a magnetically charged black hole. However, this is not the case. In fact, one first notices that the metric components \( g_{rr} \) and \( g_{\varphi \varphi} \) are related by \( g_{\varphi \varphi} = \left[ g_{rr}(r^2 - ml^2)/(l^2 r^2) \right]^{-1} \). Then, when \( g_{rr} \) becomes negative (which occurs for \( r < r_+ \)) so does \( g_{\varphi \varphi} \) and this leads to an apparent
change of signature from +2 to −2. This strongly indicates that an incorrect extension is being used and that one should choose a different continuation to describe the region \( r < r_+ \). By introducing a new radial coordinate, \( \rho^2 = r^2 - r_+^2 \), one obtains a spacetime that is both null and timelike geodesically complete for \( r \geq r_+ \),

\[
ds^2 = -\frac{1}{l^2} (\rho^2 + r_+^2 - ml^2) dt^2 + \left[ \rho^2 + \chi_m^2 \ln \left( 1 + \frac{\rho^2}{r_+^2 - ml^2} \right) \right] d\varphi^2 + \]

\[+ l^2 \rho^2 (\rho^2 + r_+^2 - ml^2)^{-1} \left[ \rho^2 + \chi_m^2 \ln \left( 1 + \frac{\rho^2}{r_+^2 - ml^2} \right) \right]^{-1} d\rho^2,
\]

where \( 0 \leq \rho < \infty \). This static spacetime has no curvature singularity, but it presents a conical geometry and, in particular, it has a conical singularity at \( \rho = 0 \) which can be removed if one identifies \( \varphi \) with the period \( T_\varphi = 2\pi \nu \) where

\[
\nu = \frac{\exp (\beta/2)}{1 + \chi_m^2 \exp (\beta)/l^2},
\]

and \( \beta = r_+^2/\chi_m^2 \). Near the origin, metric (2.5) describes a spacetime which is locally flat but has a conical singularity at \( \rho = 0 \) with an angle deficit \( \delta \varphi = 2\pi (1 - \nu) \).

### 2.2 Geodesic structure

We want to study the geodesic motion and, in particular, to confirm that the spacetime described by (2.5) is both null and timelike geodesically complete, i.e., that every null or timelike geodesic starting from an arbitrary point either can be extended to infinite values of the affine parameter along the geodesic or ends on a singularity. The equations governing the geodesics can be derived from the lagrangian

\[
\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\frac{\delta}{2},
\]

where \( \tau \) is an affine parameter along the geodesic which, for a timelike geodesic, can be identified with the proper time of the particle along the geodesic. For a null geodesic one has \( \delta = 0 \) and for a timelike geodesic \( \delta = +1 \). From the Euler-Lagrange equations one gets that the generalized momenta associated with the time coordinate and angular coordinate are constants: \( p_t = E \) and \( p_\varphi = L \). The constant \( E \) is related to the timelike Killing vector \( (\partial/\partial t)^\mu \) which reflects the time translation invariance of the metric, while the constant \( L \) is associated to the spacelike Killing vector \( (\partial/\partial \varphi)^\mu \) which reflects the invariance of the metric under rotation. Note that since the spacetime is not asymptotically flat, the constants \( E \) and \( L \) cannot be interpreted as the energy and angular momentum at infinity.

From the metric we can derive the radial geodesic,

\[
\rho^2 = -\frac{1}{g_{\rho\rho}} \frac{E^2}{g_{\varphi\varphi}} + \frac{L^2}{g_{tt}} \frac{\delta}{g_{\rho\rho}}.
\]

Using the two useful relations \( g_{tt} g_{\varphi\varphi} = -\rho^2 / g_{\rho\rho} \) and \( g_{\varphi\varphi} = \left[ g_{\rho\rho} (\rho^2 + r_+^2 - ml^2) / (l^2 \rho^2) \right]^{-1} \), we can write eq. (2.8) as

\[
\rho^2 \frac{d^2 \rho^2}{d\rho^2} = \left[ \frac{l^2 E^2}{\rho^2 + r_+^2 - ml^2} - \delta \right] \frac{\rho^2}{g_{\rho\rho}} + L^2 g_{tt}.
\]
(i) Null geodesics ($\delta = 0$) – Noticing that $1/g_{\rho\rho}$ is always positive for $\rho > 0$ and zero for $\rho = 0$, and that $g_{tt}$ is always negative we conclude the following about the null geodesic motion. The first term in (2.9) is positive (except at $\rho = 0$ where it vanishes), while the second term is always negative. We can then conclude that spiraling ($L \neq 0$) null particles coming in from an arbitrary point are scattered at the turning point $\rho_{tp} > 0$ and spiral back to infinity. If the angular momentum $L$ of the null particle is zero it hits the origin (where there is a conical singularity) with vanishing velocity. (ii) Timelike geodesics ($\delta = +1$) – Timelike geodesic motion is possible only if the energy of the particle satisfies $E > (r_+^2 - ml^2)^{1/2}/l$. In this case, spiraling timelike particles are bounded between two turning points that satisfy $\rho_{a}^{tp} > 0$ and $\rho_{b}^{tp} < \sqrt{l^2(E^2 + m) - r_+^2}$, with $\rho_{b}^{tp} \geq \rho_{a}^{tp}$. When the timelike particle has no angular momentum ($L = 0$) there is a turning point located at $\rho_{b}^{tp} = \sqrt{l^2(E^2 + m) - r_+^2}$ and it hits the conical singularity at the origin $\rho = 0$. Hence, we confirm that the spacetime described by eq. (2.5) is both timelike and null geodesically complete.

3. Rotating magnetic solution

3.1 Addition of angular momentum

Now, we want to endow the spacetime solution (2.5) with a global rotation, i.e., we want to add angular momentum to the spacetime. In order to do so we perform the following rotation boost in the $t$-$\varphi$ plane (see e.g. [5, 6, 10])

\[ t \mapsto \gamma t - l\omega \varphi, \]
\[ \varphi \mapsto \gamma \varphi - \frac{\omega}{l} t, \]  
(3.1)

where $\gamma$ and $\omega$ are constant parameters. Substituting (3.1) into (2.5) and (2.3) we obtain the stationary spacetime generated by a magnetic source

\[ ds^2 = -\frac{1}{l^2} \left[ (\gamma^2 - \omega^2)\rho^2 + \gamma^2(r_+^2 - ml^2) - \omega^2 \chi_m^2 \ln \left( 1 + \frac{\rho^2}{r_+^2 - ml^2} \right) \right] dt^2 - \frac{\gamma \omega}{l} \left[ - (r_+^2 - ml^2) + \chi_m^2 \ln \left( 1 + \frac{\rho^2}{r_+^2 - ml^2} \right) \right] 2dtd\varphi + l^2 \rho^2 (\rho^2 + r_+^2 - ml^2)^{-1} \left[ \rho^2 + \chi_m^2 \ln \left( 1 + \frac{\rho^2}{r_+^2 - ml^2} \right) \right]^{-1} d\rho^2 + \left[ (\gamma^2 - \omega^2)\rho^2 - \omega^2 (r_+^2 - ml^2) + \gamma^2 \chi_m^2 \ln \left( 1 + \frac{\rho^2}{r_+^2 - ml^2} \right) \right] d\varphi^2, \]

(3.2)

\[ A = -\frac{\omega}{l} A(\rho)dt + \gamma A(\rho) d\varphi, \]
(3.3)

with $A(\rho) = \chi_m \ln [(\rho^2 + r_+^2)/l^2 - ml]/2$. We set $\gamma^2 - \omega^2 = 1$ because in this way when the angular momentum vanishes ($\omega = 0$) we have $\gamma = 1$ and so we recover the static solution.

Solution (3.2) represents a magnetically charged stationary spacetime and also solves the three dimensional Einstein-Maxwell-anti de Sitter gravity action (2.1). Transformations
generate a new metric because they are not permitted global coordinate transformations. Transformations (3.1) can be done locally, but not globally. Therefore, the metrics (2.5) and (3.2) can be locally mapped into each other but not globally, and as such they are distinct.

Clément [4], using a procedure, and Chen [12], through the application of T-duality to [7] have written a rotating metric. However, the properties of the spacetime were not studied.

3.2 Mass, angular momentum and electric charge of the solutions

Both the static and rotating solutions are asymptotically anti-de Sitter. This fact allows us to calculate the mass, angular momentum and the electric charge of the static and rotating solutions. To obtain these quantities we apply the formalism of Regge and Teitelboim [13] (see also [5, 6]). We first write (3.2) in the canonical form involving the lapse function $N^0(\rho)$ and the shift function $N_\phi(\rho)$

$$ds^2 = -(N^0)^2 dt^2 + \frac{d\rho^2}{f^2} + H^2 (d\phi + N_\phi dt)^2,$$

where $f^{-2} = g_{\rho\rho}$, $H^2 = g_{\phi\phi}$, $H^2 N^\phi = g_{t\phi}$ and $(N^0)^2 - H^2 (N^\phi)^2 = g_{tt}$. Then, the action can be written in the hamiltonian form as a function of the energy constraint $H$, momentum constraint $H_\phi$ and Gauss constraint $G$

$$S = -\int dt d^2 x [N^0 H + N^\phi H_\phi + A_t G] + B$$

$$= -\Delta t \int d\rho N \nu \left[ \frac{2\pi^2}{H^3} + 2f(f H_{,\rho})_{,\rho} + \frac{H}{f^2} + \frac{2H}{f} (E^2 + B^2) \right] +$$

$$+ \Delta t \int d\rho N^\phi \nu \left[ (2\pi)_{,\rho} + \frac{4H}{f} E^\rho B \right] + \Delta t \int d\rho A_t \nu \left[ - \frac{4H}{f} \partial_\rho E^\rho \right] + B,$$

where $N = \frac{N^0}{t}$, $\pi \equiv \pi_\rho = -\frac{fH^3 (N^\phi)}{2N^0}$ (with $\pi_\rho$ being the momentum conjugate to $g_{\rho\phi}$), $E^\rho$ and $B$ are the electric and magnetic fields and $B$ is a boundary term. The factor $\nu$ comes from the fact that, due to the angle deficit, the integration over $\phi$ is between 0 and $2\pi\nu$. Upon varying the action with respect to $f(\rho)$, $H(\rho)$, $\pi(\rho)$ and $E^\rho(\rho)$ one picks up additional surface terms. Indeed,

$$\delta S = -\Delta t N \nu \left[ H_{,\rho} \delta f^2 - (f^2)_{,\rho} \delta H + 2f^2 \delta (H_{,\rho}) \right] +$$

$$+ \Delta t N^\phi \nu [2\nu \delta \pi] + \Delta t A_t \left[ - \nu \frac{4H}{f} \delta E^\rho \right] + \delta B +$$

$$(\text{terms vanishing when the equations of motion hold}).$$

In order that the Hamilton’s equations are satisfied, the boundary term $B$ has to be adjusted so that it cancels the above additional surface terms. More specifically one has

$$\delta B = -\Delta t N \delta M + \Delta t N^\phi \delta J + \Delta t A_t \delta Q_e,$$

where one identifies $M$ as the mass, $J$ as the angular momentum and $Q_e$ as the electric charge since they are the terms conjugate to the asymptotic values of $N$, $N^\phi$ and $A_t$, respectively.
To determine the mass, the angular momentum and the electric charge of the solutions one must take the spacetime that we have obtained and subtract the background reference spacetime contribution, i.e., we choose the energy zero point in such a way that the mass, angular momentum and charge vanish when the matter is not present.

Now, note that (3.2) has an asymptotic metric given by

\[
\frac{-\gamma^2 - \omega^2}{l^2} dt^2 + \frac{l^2}{\rho^2} d\rho^2 + (\gamma^2 - \omega^2) \rho^2 d\varphi^2,
\]

where \(\gamma^2 - \omega^2 = 1\) so, it is asymptotically an anti-de Sitter spacetime. The anti-de Sitter spacetime is also the background reference spacetime, since the metric (3.2) reduces to (3.8) if the matter is not present \((m = 0, \chi_m = 0)\).

Taking the subtraction of the background reference spacetime into account we have that the mass, angular momentum and electric charge are given by

\[
M = \nu [ - H_{,\rho} (f^2 - f^2_{\text{ref}}) + (f^2)_{,\rho} (H - H_{\text{ref}}) - 2f^2 (H_{,\rho} - H_{\text{ref},\rho}) ] , \\
J = - 2\nu (\pi - \pi_{\text{ref}}) , \\
Q_e = 4\frac{H}{f} \nu (E^\rho - E^\rho_{\text{ref}}) .
\]

After taking the asymptotic limit, \(\rho \to +\infty\), we finally have that the mass and angular momentum are

\[
M = \nu [ (\gamma^2 + \omega^2) (m - r_+^2/l^2) - 2\chi_m^2/l^2 ] + \text{Div}_M(\chi_m, \rho) , \tag{3.10}
\]

\[
J = 2\nu \gamma \omega (m l^2 - r_+^2 - \chi_m^2)/l + \text{Div}_J(\chi_m, \rho) , \tag{3.11}
\]

where \(\text{Div}_M(\chi_m, \rho)\) and \(\text{Div}_J(\chi_m, \rho)\) are logarithmic terms proportional to the magnetic source \(\chi_m\) that diverge as \(\rho \to +\infty\) (see also [14]). The presence of these kind of divergences in the mass and angular momentum is a usual feature present in charged solutions. They can be found for example in the electrically charged point source solution [15], in the electrically charged BTZ black hole [16] and in the electrically charged black holes of three dimensional Brans-Dicke gravity [17]. Following [3, 8, 15] the divergences on the mass can be treated as follows. One considers a boundary of large radius \(\rho_0\) involving the system. Then, one sums and subtracts \(\text{Div}_M(\chi_m, \rho_0)\) to (3.10) so that the mass (3.10) is now written as

\[
M = M(\rho_0) + [\text{Div}_M(\chi_m, \rho) - \text{Div}_M(\chi_m, \rho_0)] , \tag{3.12}
\]

where \(M(\rho_0) = M_0 + \text{Div}_M(\chi_m, \rho_0)\), i.e.,

\[
M_0 = M(\rho_0) - \text{Div}_M(\chi_m, \rho_0) . \tag{3.13}
\]

The term between brackets in (3.12) vanishes when \(\rho \to \rho_0\). Then \(M(\rho_0)\) is the energy within the radius \(\rho_0\). The difference between \(M(\rho_0)\) and \(M_0\) is \(-\text{Div}_M(\chi_m, \rho_0)\) which is interpreted as the electromagnetic energy outside \(\rho_0\) apart from an infinite constant which is absorbed in \(M(\rho_0)\). The sum (3.13) is then independent of \(\rho_0\), finite and equal to the total mass. In practice the treatment of the mass divergence amounts to forgetting about \(\rho_0\) and take as zero the asymptotic limit: \(\lim \text{Div}_M(\chi_m, \rho) = 0\).
To handle the angular momentum divergence, one first notices that the asymptotic limit of the angular momentum per unit mass \((J/M)\) is either zero or one, so the angular momentum diverges at a rate slower or equal to the rate of the mass divergence. The divergence on the angular momentum can then be treated in a similar way as the mass divergence. So, one can again consider a boundary of large radius \(\rho_0\) involving the system. Following the procedure applied for the mass divergence one concludes that the divergent term \(-\text{Div}_J(\chi_m, \rho_0)\) can be interpreted as the electromagnetic angular momentum outside \(\rho_0\) up to an infinite constant that is absorbed in \(J(\rho_0)\).

Hence, in practice the treatment of both the mass and angular divergences amounts to forgetting about \(\rho_0\) and take as zero the asymptotic limits: \(\lim \text{Div}_M(\chi_m, \rho) = 0\) and \(\lim \text{Div}_J(\chi_m, \rho) = 0\) in (3.10) and (3.11).

Now, we calculate the electric charge of the solutions. To determine the electric field we must consider the projections of the Maxwell field on spatial hypersurfaces. The normal to such hypersurfaces is \(n^\nu = (1/N^0, 0, -N^\phi/N^0)\) and the electric field is given by \(E^\mu = g^{\mu\sigma}F_{\sigma\nu}n^\nu\). Then, from (3.9), the electric charge is

\[
Q_e = -4HfN_0^\nu(\partial_\rho A_t - N^\phi\partial_\rho A_\phi) = 2\nu^2 m \chi_m. \tag{3.14}
\]

Note that the electric charge is proportional to \(\omega \chi_m\). Since in three dimensions the magnetic field is a scalar (rather than a vector) one cannot use Gauss’s law to define a conserved magnetic charge. In the next section we will propose a physical interpretation for the origin of the magnetic field source and discuss the result obtained in (3.14).

The mass, angular momentum and electric charge of the static solutions can be obtained by putting \(\gamma = 1\) and \(\omega = 0\) on the above expressions [see (3.1)].

3.3 Relations between the conserved charges

Now, we want to cast the metric (3.2) in terms of \(M, J, Q_e\) and \(\chi_m\). We can use (3.10) and (3.11) to solve a quadratic equation for \(\gamma^2\) and \(\omega^2\). It gives two distinct sets of solutions

\[
\gamma^2 = \frac{Ml^2 + 2\chi_m^2}{2(ml^2 - r_+^2)} \frac{(2 - \Omega)}{\nu}, \quad \omega^2 = \frac{Ml^2 + 2\chi_m^2}{2(ml^2 - r_+^2)} \frac{\Omega}{\nu}, \tag{3.15}
\]

\[
\gamma^2 = \frac{Ml^2 + 2\chi_m^2}{2(ml^2 - r_+^2)} \frac{\Omega}{\nu}, \quad \omega^2 = \frac{Ml^2 + 2\chi_m^2}{2(ml^2 - r_+^2)} \frac{(2 - \Omega)}{\nu}, \tag{3.16}
\]

where we have defined a rotating parameter \(\Omega\), which ranges between \(0 \leq \Omega < 1\), as

\[
\Omega \equiv 1 - \sqrt{1 - \frac{(ml^2 - r_+^2)^2}{(Ml^2 + 2\chi_m^2)^2}} \frac{l^2J^2}{(ml^2 - r_+^2 - \chi_m^2)^2}. \tag{3.17}
\]

When we take \(J = 0\) (which implies \(\Omega = 0\)), (3.15) gives \(\gamma \neq 0\) and \(\omega = 0\) while (3.16) gives the nonphysical solution \(\gamma = 0\) and \(\omega \neq 0\) which does not reduce to the static original metric. Therefore we will study the solutions found from (3.15). The condition that \(\Omega\) remains real imposes a restriction on the allowed values of the angular momentum

\[
l^2J^2 \leq \frac{(ml^2 - r_+^2 - \chi_m^2)^2}{(ml^2 - r_+^2)^2} (Ml^2 + 2\chi_m^2)^2. \tag{3.18}
\]
The condition $\gamma^2 - \omega^2 = 1$ allows us to write $r_+^2 - ml^2$ as a function of $M$, $\Omega$ and $\chi_m$, 
\[ r_+^2 - ml^2 = (Ml^2 + 2\chi_m^2)(\Omega - 1)/\nu . \] (3.19)
This relation allows us to achieve interesting conclusions about the values that the parameters $M$, $\chi_m$ and $J$ can have. Indeed, if we replace (3.19) into (3.15) we get
\[ \gamma^2 = (2 - \Omega)/(1 - \Omega), \quad \omega^2 = \Omega/(1 - \Omega). \] (3.20)
Since $\Omega$ ranges between $0 \leq \Omega < 1$, we have $\gamma^2 > 0$ and $\omega^2 > 0$. Besides, one has that $r_+^2 > ml^2$ and $\nu > 0$ so from (3.12) we conclude that both the static and rotating solutions have negative mass. Therefore, from now one, whenever we refer to the mass of the solution we will set
\[ M = -|M|, \] (3.21)
unless otherwise stated.
Looking again to (3.19) we can also conclude that one must have
\[ \chi_m^2 < |M|^2/2, \] (3.22)
i.e., there is an upper bound for the intensity of the magnetic field strength.
From (3.11) we also see that the angular momentum is always negative indicating that the angular momentum and the angular velocity, $\omega$, have opposite directions. This is the expected result since $J$ is the inertial momentum times the angular velocity and the inertial momentum is proportional to the mass which is negative. Introducing (3.19) into (3.18) we find an upper bound for the angular momentum
\[ |J| \leq |M|^2 - 2\chi_m^2 + \nu\chi_m^2/(1 - \Omega). \] (3.23)
Note that from (3.17) we can get the precise value of $J$ as a function of $M$, $\Omega$ and $\chi_m$.
Finally, we remark that the auxiliary equations (2.4), (2.6) and (3.20) allow us to define the auxiliary parameters $r_+$, $\nu$ and $m$ as a function of the hairs $M$, $\Omega$ and $\chi_m$.

3.4 The rotating magnetic solution
We are now in position to write the stationary spacetime (3.3) generated by a source of magnetic field in three dimensional Einstein-Maxwell-anti de Sitter gravity as a function of its hairs,
\[
\begin{align*}
  ds^2 &= -\frac{1}{t^2} \left[ \rho^2 + \frac{1}{2\nu}(|M|^2 - 2\chi_m^2)(2 - \Omega) - \frac{Q_e^2}{4\nu} \ln \left( 1 + \frac{\nu\rho^2}{(|M|^2 - 2\chi_m^2)(\Omega - 1)} \right) \right] dt^2 + \\
  &\quad + \frac{j(|M|^2 - 2\chi_m^2)(\Omega - 1) + \nu\chi_m^2 \ln \left( 1 + \frac{\nu\rho^2}{(|M|^2 - 2\chi_m^2)(\Omega - 1)} \right)}{\nu(|M|^2 - 2\chi_m^2)(1 - \Omega) + \nu\chi_m^2} dtd\varphi + \\
  &\quad + l^2 \rho^2 \left[ \rho^2 + \chi_m^2 \ln \left( 1 + \frac{\nu\rho^2}{(|M|^2 - 2\chi_m^2)(1 - \Omega)} \right) \right]^{-1} d\rho^2 + \\
  &\quad + \left[ \rho^2 - (|M|^2 - 2\chi_m^2)\frac{\Omega}{2\nu} + \frac{2 - \Omega}{1 - \Omega} \frac{\chi_m^2}{2} \ln \left( 1 + \frac{\nu\rho^2}{(|M|^2 - 2\chi_m^2)(\Omega - 1)} \right) \right] d\varphi^2, \quad (3.24)
\end{align*}
\]
as well as the vector potential 1-form (3.3)

\[ A = \frac{2}{\sqrt{1 - \Omega}} \left[ -\frac{\sqrt{\Omega}}{l} A(\rho)dt + \sqrt{2 - \Omega} A(\rho) d\varphi \right], \]  

with \( A(\rho) = (\chi_m/2) \ln \left| \frac{\rho^2}{l^2} + (|M| - 2\chi_m^2/l^2)(1 - \Omega)/\nu \right|. \)

If we set \( \Omega = 0 \) (and thus \( J = 0 \) and \( Q_e = 0 \)) we recover the static solution (2.3) [see (3.1)]. Finally if we set \( \chi_m = 0 \) (and so \( \nu = 1 \)) one gets

\[ ds^2 = -\frac{1}{l^2} \left[ \rho^2 - ML^2 \frac{2 - \Omega}{2} \right] dt^2 - J dt d\varphi + \frac{l^2}{\rho^2 - ML^2(1 - \Omega)} d\rho^2 + \left[ \rho^2 + ML^2 \frac{\Omega}{2} \right] d\varphi^2, \]  

where we have dropped the absolute value of \( M \) since now the mass can be positive. This is the rotating uncharged BTZ solution written, however, in an unusual gauge. To write it in the usual gauge we apply to (3.26) the radial coordinate transformation

\[ \rho^2 = R^2 - ML^2 \frac{\Omega}{2} \Rightarrow d\rho^2 = \frac{R^2}{R^2 - ML^2 \frac{\Omega}{2}} dR^2 \]  

and use the relation \( J^2 = \Omega(2 - \Omega) ML^2 \) [see (3.17)] to obtain

\[ ds^2 = -\left( \frac{R^2}{l^2} - M \right) dt^2 - J dt d\varphi + \left( \frac{R^2}{l^2} - M + \frac{J^2}{4R^2} \right)^{-1} dR^2 + R^2 d\varphi^2. \]  

So, as expected, (3.24) reduces to the rotating uncharged BTZ solution \([1, 2]\) when the magnetic field source vanishes.

### 3.5 Geodesic structure

The geodesic structure of the rotating spacetime is similar to the static spacetime (see section II.2), although there are now direct (corotating with \( L > 0 \)) and retrograde (counter-rotating with \( L < 0 \)) orbits. The most important result that spacetime is geodesically complete still holds for the stationary spacetime.

### 4. Physical interpretation of the magnetic source

When we look back to the electric charge given in (3.14), we see that it is zero when \( \omega = 0 \), i.e., when the angular momentum \( J \) of the spacetime vanishes. This is expected since in the static solution we have imposed that the electric field is zero (\( F_{12} \) is the only non-null component of the Maxwell tensor).

Still missing is a physical interpretation for the origin of the magnetic field source. The magnetic field source is not a Nielson-Oleson vortex solution since we are working with the Maxwell theory and not with an abelian-Higgs model. We might then think that the magnetic field is produced by a Dirac point-like monopole. However, this is not also the case since a Dirac monopole with strength \( g_m \) appears when one breaks the Bianchi identity [16], yielding \( \partial_\mu (\sqrt{-g} F^\mu) = g_m \delta^2(\vec{x}) \) (where \( \tilde{F}^\mu = \epsilon^{\mu\nu\gamma} F_{\nu\gamma}/2 \) is the dual of the Maxwell field strength), whereas in this work we have that \( \partial_\mu (\sqrt{-g} \tilde{F}^\mu) = 0 \). Indeed, we are
clearly dealing with the Maxwell theory which satisfies Maxwell equations and the Bianchi identity

\[
\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{\mu \nu}) = \frac{\pi}{2} \frac{1}{\sqrt{-g}} j^\mu, \tag{4.1}
\]

\[
\partial_\mu (\sqrt{-g} \tilde{F}^\mu) = 0, \tag{4.2}
\]

respectively. In (4.1) we have made use of the fact that the general relativistic current density is $1/\sqrt{-g}$ times the special relativistic current density $j^\mu = \sum q \delta^2(\vec{x} - \vec{x}_0) \dot{x}^\mu$.

We then propose that the magnetic field source can be interpreted as composed by a system of two symmetric and superposed electric charges (each with strength $q$). One of the electric charges is at rest with positive charge (say), and the other is spinning with an angular velocity $\dot{\varphi}_0$ and negative electric charge. Clearly, this system produces no electric field since the total electric charge is zero and the magnetic field is produced by the angular electric current. To confirm our interpretation, we go back to eq. (4.1). In our solution, the only non-vanishing component of the Maxwell field is $F^{\varphi \rho}$ which implies that only $j^\varphi$ is not zero. According to our interpretation one has $j^\varphi = q \delta^2(\vec{x} - \vec{x}_0) \dot{\varphi}$, which one inserts in eq. (4.1). Finally, integrating over $\rho$ and $\varphi$ we have

\[
\chi_m \propto q \dot{\varphi}_0. \tag{4.3}
\]

So, the magnetic source strength, $\chi_m$, can be interpreted as an electric charge $q$ times its spinning velocity.

Looking again to the electric charge given in (3.14), one sees that after applying the rotation boost in the $t-\varphi$ plane to endow the initial static spacetime with angular momentum, there appears a net electric charge. This result was already expected since now, besides the scalar magnetic field ($F_{\rho \varphi} \neq 0$), there is also an electric field ($F_{\rho \rho} \neq 0$) [see (3.25)]. A physical interpretation for the appearance of the net electric charge is now needed. To do so, we return to the static spacetime. In this static spacetime there is a static positive charge and a spinning negative charge of equal strength at the center. The net charge is then zero. Therefore, an observer at rest ($S$) sees a density of positive charges at rest which is equal to the density of negative charges that are spinning. Now, we perform a local rotational boost $t' = \gamma t - l \omega \varphi$ and $\varphi' = \gamma \varphi - \frac{\omega}{\gamma} t$ to an observer ($S'$) in the static spacetime, so that $S'$ is moving relatively to $S$. This means that $S'$ sees a different charge density since a density is a charge over an area and this area suffers a Lorentz contraction in the direction of the boost. Hence, the two sets of charge distributions that had symmetric charge densities in the frame $S$ will not have charge densities with equal magnitude in the frame $S'$. Consequently, the charge densities will not cancel each other in the frame $S'$ and a net electric charge appears. This was done locally. When we turn into the global rotational Lorentz boost of eqs. (3.1) this interpretation still holds. The local analysis above is similar to the one that occurs when one has a copper wire with an electric current and we apply a translation Lorentz boost to the wire: first, there is only a magnetic field but, after the Lorentz boost, one also has an electric field. The difference is that in the present situation the Lorentz boost is a rotational one and not a translational one.
5. Conclusions

Clément [4], Hirschmann and Welch [7] and Cataldo and Salgado [8] have found the static magnetic counterpart of the static electric BTZ black hole [1, 2]. This static magnetic solution is horizonless and has a conical singularity at the origin. In this paper we have extended their work in order to include angular momentum, the definition of conserved quantities and a new interpretation for the magnetic field source. We have shown that both the static and rotating magnetic solutions have negative mass and that there is an upper bound for the intensity of the magnetic field strength and for the value of the angular momentum. Our rotating magnetic solution is the counterpart of the rotating electric BTZ black hole [3] and, as expected, our solution reduces to the rotating uncharged BTZ solution when the magnetic field source vanishes.

Hirschmann and Welch [7] interpreted the static magnetic source as a kind of magnetic monopole reminiscent of a Nielson-Olesen vortex solution. We prefer to interpret the static magnetic field source as being composed by a system of two symmetric and superposed electric charges. One of the electric charges is at rest and the other is spinning. This system produces no electric field since the total electric charge is zero and the scalar magnetic field is produced by the angular electric current. When we apply a rotational Lorentz boost to add angular momentum to the spacetime, there appears an electric charge and an electric field.

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References

[1] M. Bañados, C. Teitelboim and J. Zanelli, The black hole in three-dimensional space-time, Phys. Rev. Lett. 69 (1992) 1849 [hep-th/9204099].
[2] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Geometry of the (2+1) black hole, Phys. Rev. D 48 (1993) 1506 [gr-qc/9302012].
[3] S. Carlip, The (2+1)-dimensional black hole, Class. and Quant. Grav. 12 (1995) 2853 [gr-qc/9506079]; Quantum gravity in 2+1 dimensions, Cambridge University Press, Cambridge 1998.
[4] G. Clément, Classical solutions in three-dimensional Einstein-Maxwell cosmological gravity, Class. and Quant. Grav. 10 (1993) L45.
[5] C. Martínez, C. Teitelboim and J. Zanelli, Charged rotating black hole in three spacetime dimensions, Phys. Rev. D 61 (2000) 104013 [hep-th/9912259].
[6] O.J.C. Dias and J.P.S. Lemos, Static and rotating electrically charged black holes in three-dimensional brans-dicke gravity theories, Phys. Rev. D 64 (2001) 064001 [hep-th/0105183].
[7] E.W. Hirschmann and D.L. Welch, Magnetic solutions to 2+1 gravity, *Phys. Rev. D* **53** (1996) 5579 [hep-th/9510181].

[8] M. Cataldo and P. Salgado, Static Einstein-Maxwell solutions in (2+1)-dimensions, *Phys. Rev. D* **54** (1996) 2971.

[9] J.H. Horne and G.T. Horowitz, Exact black string solutions in three-dimensions, *Nucl. Phys. B* **368** (1992) 444 [hep-th/9108001].

[10] G.T. Horowitz and D.L. Welch, Exact three-dimensional black holes in string theory, *Phys. Rev. Lett.* **71** (1993) 328 [hep-th/9302126].

[11] J. Stachel, Globally stationary but locally static space-times: a gravitational analog of the Aharonov-Bohm effect, *Phys. Rev. D* **26** (1982) 1281.

[12] C.-M. Chen, T-duality and spinning solutions in 2+1 gravity, *Nucl. Phys. B* **544** (1999) 777 [hep-th/9810245].

[13] T. Regge and C. Teitelboim, Role of surface integrals in the hamiltonian formulation of general relativity, *Ann. Phys. (NY)* **88** (1974) 286.

[14] K.C.K. Chan, Comment on the calculation of the angular momentum for the (anti-) self dual charged spinning BTZ black hole, *Phys. Lett. B* **373** (1996) 296 [gr-qc/9509032].

[15] S. Deser, P.O. Mazur, Static solutions in D = 3 Einstein-Maxwell theory, *Class. and Quant. Grav.* **2** (1985) L51.

[16] W.A. Moura-Melo and J.A. Helayel-Neto, Remarks on Dirac-like monopoles, Maxwell and Maxwell-Chern-Simons electrodynamics in d = (2 + 1)-dimensions, *Phys. Rev. D* **63** (2001) 065013.