ON THE MODIFIED MOD $p$ LOCAL LANGLANDS CORRESPONDENCE FOR $\text{GL}_2(\mathbb{Q}_\ell)$

DAVID HELM

We give an explicit description of the modified mod $p$ local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_\ell)$ of [EH], Theorem 5.1.5, where $p$ is an odd prime different from $\ell$.

1. Introduction

In [EH], Matthew Emerton and the author introduce a “modified mod $p$ local Langlands correspondence,” a “mod $p$” version of the local Langlands correspondence that is well-behaved under specialization and has useful applications to the cohomology of modular curves and the “local Langlands correspondence in families” of [EH]. Section 5 of [EH] gives a general characterization of this mod $p$ correspondence in terms of its basic properties. If one restricts to the group $\text{GL}_2$, it is easy in most cases to go from this list of characterizing properties to an explicit description of this correspondence. These easy cases are discussed in detail in section 5.2 of [EH]. When $p$ is odd the cases discussed come close to a complete description of the correspondence, but omit certain more difficult special cases. The purpose of this note is to explicitly describe the correspondence in these more difficult cases and thus complete the description of the modified mod $p$ local Langlands correspondence for $\text{GL}_2$ and odd $p$.

Let $F$ be a finite extension of $\mathbb{Q}_\ell$ whose residue field has order $q$, let $p$ be an odd prime distinct from $\ell$, and let $k$ be a finite field of characteristic $p$. The modified mod $p$ local Langlands correspondence is an association $\rho \mapsto \pi(\rho)$, where $\rho : G_F \to \text{GL}_n(k)$ is a continuous $n$-dimensional representation of the absolute Galois group of $F$, and $\pi(\rho)$ is a finite length indecomposable smooth representation of $\text{GL}_n(F)$. Its interest arises from its nice behaviour under specialization, which we discuss below, and also from the fact that it arises “in nature” in the cohomology of the tower of modular curves. Indeed, in [Em], Emerton considers the following situation:

Let $\Sigma$ be a finite set of primes containing $p$, and let $H^1_{\Sigma}$ be the direct limit:

$$\lim_{\text{Supp } N \subseteq \Sigma} H^1_{\text{et}}(X(N)_{\overline{\mathbb{Q}}}, k).$$

(Here $\text{Supp } N$ denotes the set of primes dividing an integer $N$, so the limit is over $N$ divisible only by primes in $\Sigma$, ordered by divisibility.) The space $H^1_{\Sigma}$ acquires actions of $G_{\Sigma}$, of $\text{GL}_2(\mathbb{Q}_p)$, and of $\text{GL}_2(\mathbb{Q}_\ell)$ for $\ell \neq p$, as well as of the Hecke operators $T_r$ for $r \notin \Sigma$ and the diamond operators $(d)$ for $d$ not divisible by any prime of $\Sigma$. Let $\mathcal{T}_\Sigma$ be the subalgebra of $\text{End}_k(H^1_{\Sigma})$ generated by these Hecke operators and diamond operators.

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Let $\rho: G_\mathbb{Q} \to \text{GL}_2(k)$ be a modular Galois representation unramified outside $\Sigma$. Then there is a maximal ideal $m$ of $\mathbb{T}_\Sigma$ attached to $\rho$, and (under certain hypotheses on the local behavior of $\rho$ at $p$), Emerton has shown (Em, Theorem 6.2.13 and Proposition 6.1.20) that $H^1_\sigma[m]$ is a product of “local factors”:

$$H^1_\sigma[m] \cong \overline{\rho} \otimes \pi_p \otimes \bigotimes_{\ell \neq p, \ell \in \Sigma} \pi_\ell$$

where $\pi_p$ is attached to $\overline{\rho}|_{G_{\mathbb{Q}p}}$ by considerations from the $p$-adic Langlands program (see Em, section 3, for details) and each $\pi_\ell$ is the representation $\overline{\rho}|_{G_{\mathbb{Q}\ell}}$ attached to the restriction of $\rho$ to a decomposition group at $\ell$ via the modified mod $p$ local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_\ell)$. Thus an explicit description of the modified mod $p$ local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_\ell)$ gives an explicit description of the action of $\text{GL}_2(\mathbb{Q}_\ell)$ on the cohomology of the modular tower.

We now recall more precisely the definition of the modified mod $p$ local Langlands correspondence. The starting point is the characteristic zero “generic local Langlands correspondence” of Breuil-Schneider (BS). We refer the reader to sections 4.2 and 4.3 of EH for the basic properties of this correspondence. In particular, this correspondence associates to any $n$-dimensional Frobenius-semisimple Weil-Deligne representation $(\rho, N)$ over a field $K$ containing $\mathbb{Q}_p$ an indecomposable (but often reducible) admissible representation $\pi(\rho, N)$ of $\text{GL}_n(F)$.

The representations $\pi(\rho, N)$ have several nice properties. In particular, they are essentially AIG representations, a concept introduced in section 3.2 of EH. A smooth representation $\pi$ of $\text{GL}_n(F)$ over a field $K$ is called essentially AIG if

- the socle of $\pi$ is absolutely irreducible and generic,
- $\pi$ contains no generic irreducible subquotients other than its socle, and
- $\pi$ is the sum of its finite length submodules.

Such representations have several useful properties. In particular, their only endomorphisms are scalars (EH, Lemma 3.2.3), any submodule of an essentially AIG representation is essentially AIG, any nonzero map of essentially AIG representations is an embedding, and such an embedding, if it exists at all, is unique up to a scalar factor (EH, Lemma 3.2.2). Moreover, if $\pi$ is an absolutely irreducible generic representation of $\text{GL}_n(F)$, then there is an essentially AIG representation $\text{env}(\pi)$, known as the essentially AIG envelope of $\pi$, such that the socle of $\text{env}(\pi)$ is isomorphic to $\pi$ and any essentially AIG representation $\pi'$ with socle isomorphic to $\pi$ embeds in $\text{env}(\pi)$ (EH, Proposition 3.2.7). Moreover, all the subquotients of $\text{env}(\pi)$ (or, more generally, of any essentially AIG representation) have the same supercuspidal support (EH, Corollary 3.2.14).

A final useful property of essentially AIG representations is that they contain distinguished lattices (up to homothety). In particular, let $\mathcal{O}$ be a discrete valuation ring with residue field $k$ and field of fractions $K$, and let $\pi$ be an essentially AIG representation over $K$. Suppose further that $\pi$ is $\mathcal{O}$-integral; that is, contains a $\text{GL}_n(F)$-invariant $\mathcal{O}$-lattice. Then there is an $\mathcal{O}$-lattice $\pi^{\mathcal{O}}$ in $\pi$, unique up to homothety, such that $\pi^{\mathcal{O}} \otimes_\mathcal{O} k$ is essentially AIG (EH, Proposition 3.3.2).

This last property is crucial because it allows for a definition of the modified mod $p$ local Langlands correspondence via “compatibility with reduction mod $p$” from the characteristic zero correspondence of Breuil-Schneider. More precisely, one has:
Theorem 1.1 (EH, Theorem 5.1.5). Let $k$ be a finite field of characteristic $p$. There is a map $\overline{\rho} \mapsto \overline{\pi}(\overline{\rho})$ from the set of isomorphism classes of continuous representations $G_F \to \text{GL}_n(k)$ to the set of isomorphism classes of finite length admissible smooth $\text{GL}_n(F)$-representations over $k$, uniquely determined by the following three conditions:

1. For any $\overline{\rho}$, the associated $\text{GL}_n(F)$-representation $\overline{\pi}(\overline{\rho})$ is essentially AIG.
2. If $K$ is a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$ and residue field $k'$ containing $k$, $\rho : G_F \to \text{GL}_n(\mathcal{O})$ is a continuous representation lifting $\overline{\rho} \otimes_k k'$, and $\pi^o$ is the unique $\mathcal{O}$-lattice in $\pi(\rho)$ such that $\pi^o \otimes_{\mathcal{O}} k'$ is essentially AIG, then there is an embedding

$$\pi^o \otimes_{\mathcal{O}} k' \hookrightarrow \overline{\pi}(\overline{\rho}) \otimes_k k'.$$

3. The representation $\overline{\pi}(\overline{\rho})$ is minimal with respect to satisfying conditions (1) and (2); that is, for any $\overline{\rho}$, and any representation $\pi$ satisfying these conditions with respect to $\overline{\rho}$, there is an embedding of $\overline{\pi}(\overline{\rho})$ in $\pi$.

The construction of $\overline{\pi}(\overline{\rho})$ is straightforward. One shows that for a given $\overline{\rho}$, and any lift $\rho$ of $\overline{\rho}$ as in condition (2), the socle of $\pi^o \otimes_{\mathcal{O}} k'$ is the unique absolutely irreducible generic representation $\overline{\pi}^{\text{gen}}$ of $\text{GL}_n(F)$ whose supercuspidal support corresponds to $\overline{\rho}^{ss}$ under the mod $p$ semisimple local Langlands correspondence of Vigneras [V2]. Thus $\pi^o \otimes_{\mathcal{O}} k'$ embeds in the essentially AIG envelope $\text{env}(\overline{\pi}^{\text{gen}})$, so that $\text{env}(\overline{\pi}^{\text{gen}})$ satisfies conditions (1) and (2) of the theorem, but may be too large. One obtains $\overline{\pi}(\overline{\rho})$ by taking the sum, inside $\text{env}(\overline{\pi}^{\text{gen}}) \otimes_k \overline{k}$, of the subobjects $\pi^o \otimes_{\mathcal{O}} \overline{k}$ over all lifts $\rho$ as in (2), and descending from $\overline{k}$ to $k$.

When $n = 2$ and $p$ is odd, this perspective is all that one needs to explicitly describe the modified mod $p$ local Langlands correspondence. In particular, when $\overline{\rho}^{ss}$ is not a twist of the direct sum $1 \oplus \overline{\omega}$, where $\overline{\omega}$ is the mod $p$ cyclotomic character, then $\text{env}(\overline{\pi}^{\text{gen}})$ is irreducible, and thus the inclusions $\overline{\pi}^{\text{gen}} \subseteq \overline{\pi}(\overline{\rho}) \subset \text{env}(\overline{\pi}^{\text{gen}})$ are all equalities. When $\overline{\rho}^{ss}$ is a twist of $1 \oplus \overline{\omega}$, the situation is slightly more complicated, but still easy as long as $q$ is not congruent to $\pm 1$ modulo $p$. We refer the reader to section 5.2 of [EH] for details.

In section 2 we cover the case when $q$ is congruent to $-1$ modulo $p$, and $\overline{\rho}^{ss}$ is a twist of $1 \oplus \overline{\omega}$. This case was worked out independently by Emerton in unpublished work. It is similar to the case when $q$ is not congruent to $\pm 1$ modulo $p$, but is slightly more complicated because $\text{env}(\overline{\pi}^{\text{gen}})$ has length $3$ instead of $2$.

The remaining sections are devoted to the case when $q$ is congruent to $1$ modulo $p$, and $\overline{\rho}^{ss}$ is a twist of $1 \oplus \overline{\omega}$. (Note that $\overline{\omega}$ is trivial in this setting.) This case is the most difficult because in this setting there is a one-parameter family of representations $\overline{\rho}$ whose semisimplification is trivial. It turns out (Theorem 4.8) in this case that the modified mod $p$ local Langlands correspondence is sensitive enough to distinguish between these distinct extensions. This is in stark contrast to the situation in characteristic zero, where the Breuil-Schneider correspondence is insensitive to Frobenius-semisimplification. This “extra sensitivity” is quite striking and would be worth investigating in situations where $n$ is greater than two.

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In this section we write $G$ for $GL_2(F)$, for conciseness. Let $B$ be the standard Borel subgroup of $G$. Suppose that $q$ is congruent to $-1$ modulo $p$, and that $\overline{\rho}$ is a twist of $1 \oplus \omega$. Since the modified mod $p$ local Langlands correspondence is compatible with twisting, we may assume that $\overline{\rho}$ is equal to $1 \oplus \omega$. The semisimple mod $p$ local Langlands correspondence of Vigneras then shows that $\pi_{\text{gen}}$, and indeed every Jordan-Hölder constituent of $\text{env}(\overline{\pi}_{\text{gen}})$, has supercuspidal support given by the two characters $| \cdot |^{\pm \frac{1}{2}}$. Thus every Jordan-Hölder constituent of $\text{env}(\overline{\pi}_{\text{gen}})$ is also a Jordan-Holder constituent of the normalized parabolic induction $i^G_B | \cdot |^\frac{1}{2} \otimes | \cdot |^{-\frac{1}{2}}$. There are three such constituents in this setting (c.f. Example II.2.5 of [Vi1]): the trivial character of $G$, the character $| \cdot | \circ \det$ (which has values in $\pm 1$ because of our assumption on $q$), and a cuspidal subquotient, which is the unique generic subquotient and is thus isomorphic to $\overline{\pi}_{\text{gen}}$. More precisely, we have exact sequences:

$$0 \to W \to i^G_B | \cdot |^{-\frac{1}{2}} \otimes | \cdot |^\frac{1}{2} \to | \cdot |\circ \det \to 0$$

$$0 \to 1_G \to W \to \overline{\pi}_{\text{gen}} \to 0$$

for a suitable representation $W$. Both of these sequences are nonsplit, as $\overline{\pi}_{\text{gen}}$ is cuspidal and thus is neither a subobject nor a quotient of any parabolic induction.

**Lemma 2.1.** Any nonsplit extension of $\overline{\pi}_{\text{gen}}$ by the trivial character of $G$ is isomorphic to $W$.

**Proof.** Let $W'$ be such an extension. The parabolic restriction $r^P_{GL_2(F)} W'$ is isomorphic to $| \cdot |^{-\frac{1}{2}} \otimes | \cdot |^\frac{1}{2}$. As parabolic induction is a right adjoint to parabolic restriction, this isomorphism gives rise to a nonzero map:

$$W' \to i^G_B | \cdot |^{-\frac{1}{2}} \otimes | \cdot |^\frac{1}{2}$$

It is clear that such a map must be injective with image $W$. $\square$

Twisting by $| \cdot | \circ \det$ we find that $W \otimes (| \cdot | \circ \det)$ is the unique nonsplit extension of $\overline{\pi}_{\text{gen}}$ by $| \cdot | \circ \det$.

As $\overline{\pi}_{\text{gen}}$ is self-dual, it is clear that $W^\vee$ and $W^\vee \otimes (| \cdot | \circ \det)$ are essentially AIG representations with socle $\overline{\pi}_{\text{gen}}$. It follows that the representation $V$ obtained as the pushout of the diagram:

$$\begin{array}{ccc}
W^\vee & \to & V \\
\uparrow & & \uparrow \\
\overline{\pi}_{\text{gen}} & \to & W^\vee \otimes (| \cdot | \circ \det)
\end{array}$$

is also essentially AIG. Note that $V$ is an extension of $1_G \oplus (| \cdot | \circ \det)$ by $\overline{\pi}_{\text{gen}}$. 

**Proposition 2.2.** The representation $V$ is an essentially AIG envelope of $\overline{\pi}_{\text{gen}}$.

**Proof.** We must show that $V$ is not properly contained in an essentially AIG representation $V'$. Suppose there were such a $V'$. Then the socle of $V'/V$ is a contains no generic summand, and is thus isomorphic to a direct sum of characters, each of which is either trivial or isomorphic to $| \cdot | \circ \det$. Observing that $V$ is isomorphic to $V \otimes (| \cdot | \circ \det)$ we can ensure, twisting $V'$ if necessary, that $V'/V$ contains a one-dimensional subspace on which $G$ acts trivially. Let $V''$ be the preimage of this subspace under the surjection $V' \to V'/V$. 

Then $V''$ is an essentially AIG representation containing $V$, such that $V''/V$ is the character $1_G$, and it suffices to show that such a representation cannot exist.

Note that $V''$ is essentially AIG, and hence its only endomorphisms are scalars. In particular the center of $G$ acts on $V''$ by a character, and this character must be trivial since the center of $G$ acts trivially on $\pi^{\text{gen}}$. On the other hand, $V''/\pi^{\text{gen}}$ is an extension of $1_G$ by $1_G \oplus |\circ \det|$, and it is easy to see that such an extension must split if the center of $G$ acts trivially.

Thus $V''$ is an extension of $1_G \oplus 1_G \oplus (|\circ \det|)$ by $\pi^{\text{gen}}$. Applying duality to Lemma 2.1 shows that there is a unique nonsplit extension of $1_G$ by $\pi^{\text{gen}}$, and one deduces from this that the socle of $V''$ contains at least one copy of $1_G$, contradicting the hypothesis that $V''$ was essentially AIG. \hfill \Box

We now turn to understanding the modified mod $p$ local Langlands correspondence. If $\pi''$ is isomorphic to $1 \oplus \pi$, then $\pi'$ is either a nonsplit extension of $\pi$ by $1$, a nonsplit extension of $1$ by $\pi$, or the direct sum $1 \oplus \pi$.

By contrast, let us consider the representations $\rho : G_F \rightarrow \text{GL}_2(K)$ whose mod $p$ reduction has semisimplification $1 \oplus \pi$. There are several cases:

1. $\rho$ is irreducible, in which case its reduction modulo $p$ can be any of the three possibilities described above.

2. $\rho$ is a nonsplit extension of $\chi_1$ by $\chi_2$, where the mod $p$ reduction of $\chi_1$ is trivial and the mod $p$ reduction of $\chi_2$ is $\pi$. In this case the mod $p$ reduction of $\rho$ has a subrepresentation isomorphic to $\pi$, and thus cannot be a nonsplit extension of $\pi$ by $1$.

3. $\rho$ is a nonsplit extension of $\chi_2$ by $\chi_1$, where the mod $p$ reduction of $\chi_1$ is trivial and the mod $p$ reduction of $\chi_2$ is $\pi$. In this case the mod $p$ reduction of $\rho$ cannot be a nonsplit extension of $1$ by $\pi$.

4. $\rho$ is a direct sum of two characters. In this case the mod $p$ reduction of $\rho$ must be the direct sum $1 \oplus \pi$.

It is straightforward to describe $\pi(\rho)$, and the reduction $\pi(\rho)^{\circ} \otimes \omega k'$, in each of the above cases:

1. $\pi(\rho)$ is irreducible and cuspidal. In this case there is a unique homothety class of lattices in $\pi''$, and the reduction $\pi(\rho)^{\circ} \otimes \omega k'$ is also cuspidal, hence isomorphic to $\pi^{\text{gen}} \otimes_k k'$.

2. $\pi(\rho)$ is a twist of the Steinberg representation by a character that is trivial modulo $p$. The reduction mod $p$ of $\pi(\rho)$ then has two Jordan-Hölder constituents, isomorphic to $\pi^{\text{gen}}$ and $|\circ \det|$. In particular $\pi(\rho)^{\circ} \otimes \omega k'$ is a nonsplit extension of $(|\circ \det|) \otimes_k k'$ by $\pi^{\text{gen}} \otimes_k k'$, and is thus isomorphic to $W^\vee \otimes (|\circ \det|) \otimes k'$.

3. $\pi(\rho)$ is a twist of the Steinberg representation by a character that is congruent to $|\circ \det|$ modulo $p$. In this case $\pi(\rho)^{\circ} \otimes \omega k'$ is isomorphic to $W^\vee \otimes k'$.

4. $\pi(\rho)$ is a parabolic induction, that contains a lattice whose reduction modulo $p$ is $\pi^\text{gen} \otimes_k k'$, and thus cannot be a nonsplit extension of $\pi''$ by $\pi$.

It is now easy to establish the modified mod $p$ local Langlands correspondence for these $\pi$.

**Theorem 2.3.** Let $\pi'$ be a representation of $G_F$ such that $\pi'' \cong 1 \oplus \pi$.

1. If $\pi = 1 \oplus \pi'$, then $\pi(\pi')$ is isomorphic to $V$. 


(2) If \( \overline{\rho} \) is a nonsplit extension of 1 by \( \varpi \), then \( \pi(\overline{\rho}) \) is isomorphic to \( W^\vee \otimes (| \circ \det) \).

(3) If \( \overline{\rho} \) is a nonsplit extension of \( \varpi \) by 1, then \( \pi(\overline{\rho}) \) is isomorphic to \( W^\vee \).

Proof. In case (1), \( \overline{\rho} \) has a lift \( \rho \) of type (4), and then \( \pi(\rho)^\varpi \otimes_O k' \) is isomorphic to \( V \otimes_k k' \). On the other hand \( \pi(\overline{\rho}) \) is contained in \( V \) and \( \pi(\overline{\rho}) \otimes_k k' \) contains \( \pi(\rho)^\varpi \otimes_O k' \). We must thus have \( \pi(\overline{\rho}) = V \).

In case (2), \( \overline{\rho} \) has lifts of type (1) and (2), but not (3) or (4). Thus \( \pi(\rho)^\varpi \otimes_O k' \) is contained in \( W^\vee \otimes (| \circ \det) \otimes_O k' \), and the two are sometimes equal. We must thus have \( \pi(\overline{\rho}) = W^\vee \otimes (| \circ \det) \). Case (3) follow from case (2) by twisting by \( | \circ \det \). \( \square \)

3. Lattices in direct sums of characters

Before we turn to the case where \( q \equiv 1 \mod p \), we need a technical result. For this section, let \( G \) be an arbitrary locally profinite group. Let \( O \) be a discrete valuation ring with residue field \( k \), and field of fractions \( K \).

Let \( \chi_1 \) and \( \chi_2 \) be two distinct characters \( G \to O^\times \) that are trivial modulo \( \varpi \). We will attach a class \( \sigma(\chi_1, \chi_2) \) in \( H^1(G, k) \) (where \( G \) acts trivially on \( k \)) to this pair of characters. Let \( a \) be the largest integer such that \( \chi_1 \) and \( \chi_2 \) are congruent modulo \( a \). Then for \( g \in G \) we define \( \sigma(\chi_1, \chi_2)_g \) to be the reduction modulo \( \varpi \) of the element \( \frac{1}{g} \chi(\chi_1(g)) - \chi_2(g) \).

Note that \( H^1(G, k) \) is isomorphic to \( \text{Ext}^1_G(1_G, 1_G) \), where \( 1_G \) denotes the trivial character of \( G \) with values in \( k \). There is thus a bijection between lines in \( H^1(G, k) \) and nonsplit extensions of \( 1_G \) by \( 1_G \). This bijection can be made entirely explicit as follows: let \( E \) be such an extension, let \( e_1 \) span the invariant line in \( E \), and complete this to a \( k \)-basis \( \{e_1, e_2\} \) of \( E \). For any \( g, ge_2 - e_2 \) is equal to \( \sigma_g e_1 \) for some \( \sigma_g \) in \( k \); the cocycle \( g \mapsto \sigma_g \) represents a class in \( H^1(G, k) \) that is nontrivial because \( E \) is not split. A different choice of basis \( \{e_1, e_2\} \) changes \( \sigma \) by a nonzero scalar, and this gives the desired bijection of extensions \( E \) with lines in \( H^1(G, k) \).

Our goal in this section is to interpret the class \( \sigma(\chi_1, \chi_2) \) in terms of this isomorphism. Let \( L \) be a free \( O \)-module of rank two, with basis \( e_1 \) and \( e_2 \). Define an action of \( G \) on \( L \) by \( ge_1 = \chi_1(g)e_1 \) and \( ge_2 = \chi_2(g)e_2 \). Let \( L' \) be a \( G \)-stable \( O \)-lattice in \( L \otimes K \). Then \( L' / \varpi L' \) is an extension of \( 1_G \) by \( 1_G \), and we have:

**Proposition 3.1.** Suppose \( L' / \varpi L' \) is nonsplit. Then \( \sigma(\chi_1, \chi_2) \) generates the line in \( H^1(G, k) \) corresponding to the extension \( L' / \varpi L' \).

Proof. Since replacing \( L' \) with \( \varpi L' \) does not change the extension \( L' / \varpi L' \), we may assume without loss of generality that \( L \subset L' \) but \( L \not\subset \varpi L' \). Then the map

\[ L / \varpi L \to L' / \varpi L' \]

has one-dimensional image. Swapping \( e_1 \) and \( e_2 \) (and thus \( \chi_1 \) and \( \chi_2 \)) if necessary we may assume that \( e_1 \) generates the image of \( L / \varpi L \) in \( L' / \varpi L' \). (Note that this only changes \( \sigma(\chi_1, \chi_2) \) by a sign.

Since \( e_1 \) is nonzero in \( L' / \varpi L' \) we may complete it to a basis \( e_1, e_3 \) of \( L' \). Let \( b \) be the smallest integer greater than zero such that \( \varpi^b e_3 \) lies in \( L \), and write \( \varpi^b e_3 = \alpha e_1 + \beta e_2 \) for \( \alpha, \beta \in O \). We then have \( ge_3 = \chi_2(g)e_3 + \frac{1}{\varpi^b} \alpha(\chi_1(g) - \chi_2(g))e_1 \).

Note that by assumption the coefficient of \( e_1 \) lies in \( O \), as \( L' \) is \( G \)-stable.

Let \( \overline{e}_1, \overline{e}_3 \) be the images of \( e_1 \) and \( e_2 \) in \( L' / \varpi L' \). The action of \( G \) fixes \( \overline{e}_1 \), whereas \( g\overline{e}_3 = \overline{e}_3 + \sigma_g \overline{e}_1 \), where \( \sigma_g \) is the reduction modulo \( \varpi \) of \( \frac{1}{\varpi^b} \alpha(\chi_1(g) - \chi_2(g)) \). As
$L'/\varpi L'$ is nonsplit, $\sigma_g$ is nonzero for some $g$, and thus $\frac{1}{\varpi} \alpha$ must lie in $\frac{1}{\varpi} O^\times$. Thus $\sigma$ is a scalar multiple of $\sigma(\chi_1, \chi_2)$ as claimed. \hfill \square

4. $q \equiv 1 \mod p$

We now consider the case in which $q \equiv 1 \mod p$. In this case $\varpi$ is the trivial character, and, so, up to twist, the only case it remains to consider is when $\varpi^a$ is the two-dimensional trivial representation of $G = \text{GL}_2(F)$. As above, we begin by computing the appropriate essentially AIG envelope.

In this setting every subquotient of the essentially AIG envelope that contains $\pi(\varpi)$ has supercuspidal support given by two copies of the trivial character, and is thus isomorphic to a subquotient of the parabolic induction $i^G_B 1_T$, where $B \subset G$ is the standard Borel, $T$ is the standard torus, and $1_T$ is the trivial character of the torus over $k$. This induction has two Jordan-Hölder constituents: the trivial character $1_G$, and the Steinberg representation $\text{St}$ of $G$ over $k$. We thus have $\varpi_{\text{gen}} = \text{St}$.

**Lemma 4.1.** There is an isomorphism:

$$i^G_B 1_T \cong 1_G \oplus \text{St}.$$  

**Proof.** The restrictions $r^G_B 1_G$ and $r^G_B \text{St}$ are both isomorphic to the trivial character $1_T$, because the norm character $| \cdot |$ is trivial. We thus have:

$$\text{Hom}_G(1_G, i^G_B 1_T) = \text{Hom}_T(1_T, 1_T)$$

$$\text{Hom}_G(\text{St}, i^G_B 1_T) = \text{Hom}_T(1_T, 1_T)$$

and the claim follows. \hfill \square

**Lemma 4.2.** The space $\text{Ext}^1_G(1_G, \text{St})$ is two-dimensional.

**Proof.** Adjointness of parabolic induction and restriction gives an isomorphism:

$$\text{Ext}^1_G(1_G, i^G_B 1_T) \cong \text{Ext}^1_T(1_T, 1_T)$$

and the latter is four dimensional. On the other hand

$$\text{Ext}^1_G(1_G, i^G_B 1_T) \cong \text{Ext}^1_G(1_G, 1_G) \oplus \text{Ext}^1_G(1_G, \text{St}).$$

One easily sees that $\text{Ext}^1_G(1_G, 1_G)$ is two-dimensional, and the result follows. \hfill \square

Let $V$ be the “universal extension” of $1_G$ by $\text{St}$, in other words the unique extension of $1_G \oplus 1_G$ by $\text{St}$ that contains every isomorphism class of extension of $1_G$ by $\text{St}$ (more prosaically $V$ may be constructed as the pushout:

$$\begin{array}{ccc}
W & \rightarrow & V \\
\uparrow & & \uparrow \\
\text{St} & \rightarrow & W'
\end{array}$$

where $W$ and $W'$ are any two nonisomorphic extensions of $1_G$ by $\text{St}$.) We then have:

**Proposition 4.3.** The representation $V$ is an essentially AIG envelope of $\text{St}$. 

Proof. Suppose not. Then (just as in the $q \equiv -1 \mod p$ case), there is an essentially AIG representation $V'$ containing $V$ with $V'/V$ isomorphic to $1_G$. The quotient $V'/V$ is an extension of $1_G$ by $1_G \oplus 1_G$ on which the center of $G$ acts trivially; since $p$ is odd such an extension is split. Thus $V'$ is an extension of $1_G \oplus 1_G \oplus 1_G$ by $St$; since $\text{Ext}^1_G(1_G, St)$ is only two dimensional we must have that $1_G$ is a direct summand of $V'$, contradicting the fact that $V'$ is essentially AIG. \hfill \Box

It will be useful to be able to classify the nonsplit extensions of $1_G$ by $St$. Observe:

Lemma 4.4. Let $W$ be a nonsplit extension of $1_G$ by $St$. Then $r^B_GW$ is a nonsplit extension of $1_T$ by $T$. (Equivalently, the map

$$\text{Ext}^1_G(1_G, St) \to \text{Ext}^1_T(1_T, 1_T)$$

induced by $r^B_G$ is injective.)

Proof. Suppose $r^B_GW$ is split. Then $\text{Hom}_T(r^B_GW, 1_T)$ is two dimensional, so $\text{Hom}_G(W, i^G_B T)$ is also two dimensional. It follows that there is a surjection of $W$ onto $St$, implying that $W$ must also be split. \hfill \Box

It is not difficult to characterise the image of this map:

Lemma 4.5. Let $E$ be an extension of $1_G$ by $1_G$. Then there exists an extension $W$ of $1_G$ by $St$ with $r^B_GW = E$ if, and only if, the center $Z$ of $G$ acts trivially on $E$.

Proof. The representation $W$ is essentially AIG and so $Z$ acts on $W$ by scalars, and hence trivially. Thus the same is true of $r^B_GW$. Thus the image of the map

$$r^B_G : \text{Ext}^1_G(1_G, St) \to \text{Ext}^1_T(1_T, 1_T)$$

is contained in the subspace $\text{Ext}^1_{T/Z}(1_{T/Z}, 1_{T/Z})$ of $\text{Ext}^1_T(1_T, 1_T)$. This subspace is two-dimensional, as is the image of $r^B_G$, proving the claim. \hfill \Box

The sequence of isomorphisms: $W^b_F \cong F^\times \cong T/Z$ (where the last isomorphism sends $x \in F^\times$ to the class of the diagonal matrix with entries $a$ and $1$) induces a chain of isomorphisms:

$$\text{Ext}^1_{G_F}(1_{G_F}, 1_{G_F}) \cong \text{Ext}^1_{W_F}(1_{W_F}, 1_{W_F}) \cong \text{Ext}^1_{F^\times}(1_{F^\times}, 1_{F^\times}) \cong \text{Ext}^1_{T/Z}(1_{T/Z}, 1_{T/Z}).$$

Denote the composition of these morphisms by $\phi$. We observe:

Lemma 4.6. Let $K$ be a finite extension of $\mathbb{Q}_p$ with uniformizer $\varpi$, residue field $k'$ and ring of integers $\mathcal{O}$. Let $\hat{\chi}_1$ and $\hat{\chi}_2$ be distinct characters of $G_F$ with values in $\mathcal{O}$, whose reductions mod $\varpi$ are trivial, and let $\chi_1$ and $\chi_2$ be the corresponding characters of $F^\times$. Then the sequence of maps:

$$H^1(G_F, 1_G) \cong \text{Ext}^1_{G_F}(1_G, 1_G) \xrightarrow{\phi} \text{Ext}^1_{T/Z}(1_{T/Z}, 1_{T/Z}) \cong H^1(T/Z, 1_{T/Z})$$

takes $\sigma(\chi_1, \chi_2)$ to a nonzero multiple of the class $\sigma(\chi_1 \otimes \chi_2, \chi_2 \otimes \chi_1)$.

Proof. This is an easy computation. \hfill \Box

We are now in a position to describe $\mathbf{π}(\mathcal{P})$ for each $\mathcal{P}$. We first enumerate the possible lifts $\rho$ of $\mathcal{P}$. There are four cases:

1. $\rho$ is a two-dimensional representation of $G_F$ on which $G_F$ acts via a single character $\hat{\chi}$. In this case $\mathcal{P}$ must be trivial.
(2) $\rho$ is the direct sum of two distinct characters $\hat{\chi}_1$ and $\hat{\chi}_2$ whose reductions are trivial. In this case $\overline{\eta}$ is either trivial or the unique nonsplit extension of $1_{G_F}$ by $1_{G_F}$ of class $\sigma(\hat{\chi}_1, \hat{\chi}_2)$.

(3) $\rho$ is a nonsplit extension of a character $\hat{\chi}$ by the character $\omega \hat{\chi}$.

(4) $\rho$ is a twist of the unique unramified extension of the trivial representation of $G_F$ over $K$ by itself. In this case $\overline{\eta}$ is either trivial or the unique unramified extension of $1_{G_F}$ by $1_{G_F}$.

The next step is to describe $\pi(\rho) \otimes_O k'$. We first observe:

**Lemma 4.7.** Let $K$ be a finite extension of $Q_p$, and let $\pi$ be the irreducible parabolic induction $i_B^G 1_{T,K}$, where $1_{T,K}$ is the one-dimensional trivial representation of $T$ over $K$. Then $r_B^G \pi$ is the unique nonsplit extension of $1_{T,K}$ by $1_{T,K}$ on which the action of $T$ factors through the quotient: $T \to T/Z \cong F^\times \to Z$.

**Proof.** It is clear that the extension $r_B^G \pi$ is nonsplit, as we have isomorphisms:

$$K \cong \text{End}_G(\pi) \cong \text{Hom}_T(r_B^G \pi, 1_{T,K}).$$

It is also clear that the action of $T$ on $r_B^G \pi$ factors through $T/Z$. On the other hand the representation $\pi$ is an irreducible representation with an Iwahori fixed vector, and it is well-known that for such $\pi$, the subgroup $O^x \times O^x$ of $T$ acts trivially on $r_B^G \pi$. \hfill \Box

We can now describe $\pi(\rho) \otimes_O k'$ in each of the above cases.

(1) In this case $\pi(\rho)$ is a twist of $i_B^G 1_{T,K}$, and the lemma above then implies that $\pi(\rho) \otimes_O k'$ is the unique extension $W$ of $1_G$ by $\text{St}$ such that the action of $T$ on $r_B^G W$ factors through $T \to T/Z \cong F^\times \to Z$.

(2) In this case $\pi(\rho)$ is the parabolic induction $i_B^G \chi_1 \otimes \chi_2$, where $\chi_1$ and $\chi_2$ are the characters of $F^\times$ arising from $\hat{\chi}_1$ and $\hat{\chi}_2$ by local class field theory. It follows that $r_B^G \pi(\rho)$ is the direct sum of the characters $\chi_1 \otimes \chi_2$ and $\chi_2 \otimes \chi_1$, and $r_B^G(\pi(\rho) \otimes_O k')$ is then the nonsplit extension of $1_T$ by $1_T$ of class $\sigma(\chi_1 \otimes \chi_2, \chi_2 \otimes \chi_1)$.

(3) In this case $\pi(\rho)$ is a twist of the Steinberg representation, and $\pi(\rho) \otimes_O k'$ is isomorphic to $\text{St} \otimes_O k'$.

(4) In this case $\pi(\rho)$ is a twist of $i_B^G 1_{T,K}$, and the same discussion as in case (1) applies.

**Theorem 4.8.** If $\overline{\eta}$ is trivial, then $\overline{\pi(\eta)} = V$. On the other hand, if $\overline{\eta}$ is the nonsplit extension of $1_{G_F}$ by $1_{G_F}$ represented by $\sigma \in \text{Ext}^1_{G_F}(1_{G_F}, 1_{G_F})$, then $\overline{\pi(\eta)}$ is the unique nonsplit extension of $1_G$ by $\text{St}$ such that $r_B^G \pi(\eta)$ represents the class $\phi(\sigma)$ in $\text{Ext}^1_{T}(1_T, 1_T)$.

**Proof.** If $\overline{\eta}$ is trivial, then $\overline{\eta}$ has lifts of type (2) above for an arbitrary choice of $\hat{\chi}_1$ and $\hat{\chi}_2$. Thus $\pi(\rho) \otimes_O k'$ can be an arbitrary nonsplit extension of $1_G$ by $\text{St}$. As $\overline{\pi(\eta) \otimes_O k'}$ must contain all of these extensions, and is contained in $V$, we must have $\overline{\pi(\eta)} = V$.

If $\overline{\eta}$ is nontrivial and ramified, then $\overline{\eta}$ has lifts of type (2) and possibly (3), but not (1) or (4). If $\rho$ is a lift of type (3) then $\pi(\rho) \otimes_O k$ is isomorphic to $\text{St}$ and thus tells us nothing about $\overline{\pi(\eta)}$. On the other hand, if $\rho = \hat{\chi}_1 \oplus \hat{\chi}_2$ is a lift of type (2), we have $\sigma = \sigma(\hat{\chi}_1, \hat{\chi}_2)$. Then $\pi(\rho) \otimes_O k'$ is the extension of $1_G$ by $\text{St}$ corresponding to the class $\sigma(\chi_1 \otimes \chi_2, \chi_2 \otimes \chi_1)$ in $\text{Ext}^1_{T}(1_T, 1_T)$. This class is a nonzero multiple of $\phi(\sigma)$. It is thus clear that $\overline{\pi(\eta)}$ is the extension corresponding to $\phi(\sigma)$ as claimed.
If \( \mathfrak{p} \) is nontrivial but unramified, the discussion of the previous paragraph applies but one must also consider lifts of type (4). It suffices to check that these produce the same extension of \( 1_G \) by \( \text{St} \) as the lifts of type (2); that is, that when \( \sigma \) is the class attached to an unramified nonsplit extension of \( 1_{G_F} \) by \( 1_{G_F} \), then \( \phi(\sigma) \) corresponds to the extension of \( 1_T \) by \( 1_T \) on which the action of \( T \) factors through \( T \to T/Z \cong F^\times \to \mathbb{Z} \). This is a straightforward calculation. \( \square \)

References

[BS] Breuil C., Schneider P., First steps towards \( p \)-adic Langlands functoriality, J. Reine Agnew. Math. 610 (2007), 149–180.

[Em] Emerton, M. Local-global compatibility in the \( p \)-adic Langlands program for \( GL_2/\mathbb{Q} \), preprint, 2011.

[EH] Emerton, M., and Helm, D. The local Langlands correspondence for \( GL_n \) in families, submitted, [arXiv:1104.0321]

[Vi1] Vigneras, M.-F., Représentations \( \ell \)-modulaires d’un groupe réductif \( p \)-adique avec \( \ell \neq p \), Birkhauser (1996).

[Vi2] Vigneras, M.-F., Correspondance locale de Langlands semi-simple pour \( GL(n,F) \) modulo \( \ell \neq p \), Invent. Math. 144 (2001), 177–223.