Distributed Resource Allocation Over Multiple Interacting Coalitions: A Game-Theoretic Approach

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Abstract—Despite the fact that many distributed resource allocation (DRA) algorithms have been reported in the literature, it is still unknown how to allocate resources optimally over multiple interacting coalitions. One major challenge in solving such a problem is that the relevance of one coalition’s decision to other coalitions’ benefits may lead to conflicts of interest among these coalitions. Within this context, a new game model is formulated in the present note, termed as resource allocation game, where each coalition contains multiple agents cooperating to maximize the coalition-level benefit subject to an intracoalition resource constraint described by a coupled equality. Inspired by techniques such as variable replacement, gradient tracking, and leader-following consensus, a new DRA algorithm is developed. It is shown that the proposed algorithm converges linearly to the Nash equilibrium (NE) of the proposed game while satisfying the resource constraint during the whole NE-seeking process. Finally, the effectiveness of the proposed allocation algorithm is verified by numerical simulations.

Index Terms—Distributed optimization, distributed resource allocation, multicoalition game, distributed Nash equilibrium (NE) seeking, multiagent system.

I. INTRODUCTION

The past decade has witnessed a significant progress on distributed resource allocation (DRA) over multiagent networks (MANs), where interacting individual agents cooperate to make the best decision on allocating the group-level resources via information exchange among neighboring agents [1]. The task can be modeled as a distributed optimization problem regarding a group-level objective function subject to the resource constraint described by a coupled equality, and the problem has been extensively studied from various aspects with discrete-time [2], [3], [4], [5], [6], [7], [8] and continuous-time [9], [10], [11], [12], [13], [14] DRA algorithms developed.

Considering the complex interactions of real-world networked systems, the resource allocation problems of multiple coalitions may be coupled with each other. For example, in public finance management, in deciding the allocation of a provincial government’s revenue fund for economic development, the influence of other provinces’ economic developments would be taken into account, since cooperation and competition may exist across provinces. In such cases, the DRA problem of multiple coalitions cannot be decoupled into several independent single-coalition DRA problems and solved separately by employing existing DRA algorithms.

Inspired by the above observations, a new model is formulated in this article for the resource allocation problem of multiple interacting coalitions. In this model, the inputs of each individual agent’s objective function may include the states of agents not only within but also outside the coalition. The group-level objective function of each coalition is the sum of objective functions of all the individual agents therein. In each coalition, the individual agents cooperate to minimize the group-level objective function while subject to the resource constraint described by a coupled equality.

The proposed model can be viewed as a type of multicoalition game, as it shares the core feature of capturing the cooperation among agents belonging to the same coalition and the conflicts of interest among different coalitions. Existing studies on multicoalition games can be found in [15], [16], [17], [18], [19], [20], [21], and [22] and the literature cited therein. It is worth noting that in most existing studies, the states of individual agents in each coalition are required to reach an agreement [15], [16], [20], [21], [22], or without any coupled equality constraints [17], [18], [19]. To the best of the authors’ knowledge, research on the proposed multicoalition game with coupled equality constraint in the context of resource allocation has not been reported.

To fill this gap and partly motivated by the aforementioned studies, a new distributed game-based algorithm is developed to address the present DRA problem over multiple interacting coalitions, where the techniques of variable replacement, gradient descent, gradient-tracking and leader-following consensus are subtly integrated. The proposed algorithm is theoretically proven to converge linearly to the NE of the proposed game while meeting the intracoalition coupled equality constraint during the iterations. The main contribution of this article lies in the following aspects.

1) A new model is formulated for the resource allocation problem, which captures the cooperation of individual agents on resource allocation in each coalition as well as the conflicts of interest among different coalitions. The model includes the commonly studied distributed optimization model for DRA problem over MANs as a special case where all the agents are assumed to be cooperative.

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2) A new DRA algorithm is designed and utilized such that the decisions of the agents can converge linearly to the NE of the considered resource allocation game. The methodology developed in this article generalizes the existing ones on DRA over MANs, as the developed algorithms could deal with the DRA problem in the presence of conflicts of interest among different coalitions.

3) Another distinguished feature of the proposed algorithm for the considered game is that the resource constraints can be guaranteed at each iteration, which enables the algorithm to be executed in an online manner. Such a feature plays an important role in online solving various DRA problems or their variants such as the distributed economic dispatch problem of smart grid with multiple generating units subject to the constraint of supply–demand balance.

The rest of this article is organized as follows. In Section II, a game model is formulated for the problem of resource allocation over multiple interacting coalitions. In Section III, a DRA algorithm is developed and the convergence analysis is presented. A numerical example is presented in Section IV to verify the effectiveness of the proposed algorithm, and finally, Section V concludes this article.

### Notation

The sets of natural numbers, positive integers, and real numbers are represented by \( \mathbb{N} \) and \( \mathbb{N}^+ \) and \( \mathbb{R} \), respectively. The set of \( n \)-dimensional real column vectors is denoted by \( \mathbb{R}^n \). \( I_n \) is the \( n \)-dimensional identity matrix, and \( I_n(0_n) \) is the \( n \)-dimensional column vector with all the entries being \( 0 \). Symbol \( \otimes \) is the Kronecker product and \( \| \cdot \| \) denotes the Euclidian norm. \( \text{diag} \) denotes the diagonal block matrix with the matrix \( B \), and \( \mathcal{O} \) denotes the partial derivative of the function \( f(x) \) with respect to \( x_i \) at \( x \). A vector \( f_i(x) \) can receive information from agent \( i \) for any \( i \in I \), and is affected by only the collective decision of its members, i.e.,

\[
\sum_{ij \in \mathcal{E}} x_{ij} = R_i \quad (1)
\]

where \( R_i : \mathbb{R}^{n_{\text{num}}} \rightarrow \mathbb{R} \) and \( f_i : \mathbb{R}^{n_{\text{num}}} \rightarrow \mathbb{R} \) are the objective functions of agent \( i \) and coalition \( i \), respectively. Here, we consider the minimization setting without loss of generality, since the case of welfare maximization can be easily transformed into a minimization problem. The study of this model has potential applications in the fields of economics and engineering.

**Example 1 (Business Budget Allocation):** Consider that multiple firms, each of which has several product lines, manufacture related products in a competitive market. The revenue a product line generates will be influenced by the budgets assigned to the product line as well as other homogeneous product lines. Each firm wants to efficiently and effectively use its resource to maximize its total revenue. Such a problem can be modeled as the resource allocation game (1).

Next, the definition of NE will be introduced. Let \( x \triangleq (x_i, x_{-i}) \) for notational brevity. For each \( i \in I \), define the admissible set of coalition decision as \( \Omega \triangleq \{ x_i \in \mathbb{R}^{n_i} \mid x_i \in R_i \} \). The admissible set of collective decision for the game \( \Omega \) is then the Cartesian product of these sets, i.e., \( \Omega = \prod_{i \in \mathcal{V}} \Omega_i \).

**Definition 1:** An NE of the resource allocation game (1) is a vector \( x^\ast \triangleq (x^\ast_i, x^\ast_{-i}) \) in \( \Omega \) with the property that \( f_i(x_i, x^\ast_{-i}) \leq f_i(x_i, x^\ast_i) \), \( \forall x_i \in \Omega_i \), \( \forall i \in I \).

Define the pseudogradient function \( P : \mathbb{R}^{n_{\text{num}}} \rightarrow \mathbb{R}^{n_{\text{num}}} \)

\[
P(x) = [\nabla_{x_1} f_1(x), \nabla_{x_2} f_2(x), \ldots, \nabla_{x_N} f_N(x)]^T
\]

In this article, the objective functions of all the agents in game (1) are assumed to satisfy the following assumptions.

**Assumption 1:** For each agent \( i \in V \), the objective function \( f_i(x_i, x_{-i}) \) is convex and continuously differentiable in \( x_i \), for each \( x_{-i} \in \mathbb{R}^{n_{-i}} \). Moreover, \( \nabla_{x_i} f_i(x_i) \) is globally Lipschitz with the constant \( l_i \).

Under Assumption 1, it is not difficult to verify that \( \nabla_{x_i} f_i(x_i) \) is Lipschitz with a constant \( l_i = \sum_{j=1}^{m_i} \| a_{ij} \| \), where \( a_{ij} \in \mathbb{R}^{n_i} \), which is useful for the forthcoming algorithm design and the convergence analysis.

**Assumption 2 (Strongly Monotone Pseudogradient):** \( (a_1 - a_2)^T (P(a_1) - P(a_2)) \geq \| a_1 - a_2 \|^2 \), \( \forall a_1, a_2 \in \mathbb{R}^{n_{\text{num}}} \), where \( \mu \) is a positive constant.

**Remark 1:** By treating each coalition as a virtual player, the proposed \( N \)-coalition game with intracoalition coupled equality constraints can be viewed as an \( N \)-player noncooperative game with local convex constraint on each player. The NE of these two games is equivalent. Noticing this, together with Assumptions 1 and 2 commonly used in the studies of distributed NE computation [17], [18], [19], [21], [22], ensure the existence and uniqueness of NE in the proposed game (1).

### B. Network Topology and the Associated Matrices

The underlying network topology among the \( n_{\text{num}} \) game participants is depicted by an undirected graph \( G(V, \mathcal{E}) \) with \( V \) and \( \mathcal{E} \subseteq V \times V \) respectively denoting the node (agent) set and the edge (communication link) set. A pair \( (i, j) \in \mathcal{E} \) is an edge of \( G \) if agent \( j \) can receive information from agent \( i \). If \( (i, j) \in \mathcal{E} \), then agent \( i \) is called a neighbor of agent \( j \).

The graph \( G \) is assumed to be undirected, i.e., for any \( (i, j) \in \mathcal{E} \), \( (j, i) \in \mathcal{E} \). A path from agent \( i \) to agent \( j \) is a sequence of edges \((i_{m+1}, j), \ldots, (i_1, j) \) in \( \mathcal{E} \), \( m = 1, \ldots, l-1 \). The

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undirected graph $G$ is called connected if for any agent, there exist paths to all other agents. Define the induced subgraph $G_i(V_i, E_i)$ with the node set $V_i$ and the edge set $E_i = \{(ij, ik) | (ij, il) \in E\} \subseteq V_i \times V_i$. Obviously, $G_i$ characterizes the underlying topology among the agents in coalition $i$. For each agent $ij \in E$, define the neighbor set $N_i = \{il | (il, ij) \in E_i\}$, the degree $d_i = |N_i|$ and the intra coalition degree $d_{ij} = |N_{ij}|$.

The adjacency matrix of graph $G$ is defined as $A = \left[a_{ij}\right]_{n \times n}$, with $a^{ij} = 1$, and $a^{ij} = 0$ if $(pq, ij) \in E$ and 0, respectively. $a^{ij}$ denotes the element of $A$ on the $(i, j)$th row and $(j, i)$th column. Similarly, the adjacency matrix of the subgraph $G_i$ is defined as $A_i = \left[a^{ij}_i\right]_{n_i \times n_i}$, with $a^{ij}_i$ denoting the $(i, j)$th entry of $A_i$. Obviously, $A, \ldots, A_N$ are the diagonal blocks of $A$. The Laplacian matrix of $G$ is defined as $L_i = \left[l^{ij}_i\right]_{n_i \times n_i}$, with $l^{ij}_i = \sum_{k=1}^{n_i} a^{ij}_i$ and $l^{ij}_i = -a^{ij}_i, j \neq l$, where $l^{ij}_i$ denotes the $(i, j)$-entry of $L_i$.

Apart from the above matrices, a weighted adjacency matrix of the graph $G$ is defined as $W = \left[w^{ij}\right]_{n \times n}$, with $w^{ij} = 0, w^{ij} > 0$ if $(pq, ij) \in E$ and 0, otherwise, and $\forall v \in V, \sum_{p \in V_i} w^{ij}$ < 1 (similarly to the superscripts and subscripts in entries of the adjacency matrix $A$, $w^{ij}$ denotes the element of $W$ on the $(i, j)$th row and $(j, i)$th column). For example, the entries $w^{ij}, \forall v, pq \in V$ can be set as $w^{ij} = 0, h_{ij} = \sum_{k=1}^{n_i} a^{ij}_i$ and $h_{ij} = \max_{pq \in V} a^{ij}_i$. The entries of $C_i$ can be set as $c^{ij}_i = 1 - d^{ij}_i / n_i$ and $c^{ij}_i = a^{ij}_i / n_i$, $\forall v, \exists i \neq j$.

**Assumption 3:** The graph $G$ is undirected and connected, and the subgraphs $G_i, \forall i \in I$ are undirected and connected.

### C. Design Objective

In the previous sections, the resource allocation problem over multiple interacting coalitions has been formulated as a coalition game, and the communication topology among the individual agents in these coalitions has also been described. In this article, the problem is considered in the context of MANs. Specifically, individual objective function $f_{i}$ and decision variable $x_{ij}$ are private to agent $ij$, and the agents can merely share values of local variables with their neighbors through the communication network.

Next, distributed NE seeking algorithms will be developed for the proposed resource allocation game (1). The design objective is to make the collective agent decision $x$ converge to the NE $x^*$ of the proposed game, with the information utilization adapting to the network topology $G$.

### III. DISTRIBUTED NE COMPUTATION

In this section, we design a new algorithm for the proposed game and present the convergence analysis.

### A. Algorithm Design

To make the collective agent state converge to the NE in game (1), a distributed algorithm is designed for each agent $ij \in V \forall k \in \mathbb{N}$ as follows:

$$x_{ij}(k) = x_{ij}(0) - \sum_{i \in N_{ij}} (\eta_{ij}(k) - \eta_{im}(k))$$  \hspace{0.5cm} (2a)
interpreted as a pseudogradient descent law for the equivalent unconstrained problem, because \( \nabla_{x_{ij}} g_i(x_{ij}) = L_i \nabla_{x_i} f_i(x) \). Specifically, \( \psi_{ij}^{\mu} \) in the leader-following consensus law (2d) is for agent \( ij \) to estimate the value of \( x_{\mu ij} \). This is similar to existing studies on networked games [23], [24], [25], where each agent keeps an estimate of all other agents’ states based on the idea of consensus to cope with the lack of knowledge. With (2d) incorporated into (2c), \( \psi_{ij}^{\mu} \) computed by agent \( ij \) is to estimate the value of \( \nabla_{x_{ij}} f_i(x) \). The structure of (2c) is inspired by the gradient tracking technique in distributed optimization [27]. Then, the auxiliary variables \( \psi_{ij}^{\mu} \) \( \forall i \in V_i \), governed by (2c) are integrated into the update of \( \eta_{ij} \) in (2b).

Remark 3: For the special case where \( N = 1 \) and the argument of each individual objective function \( f_{ij} \) includes only the individual decision variable \( x_{ij} \), rather than the entire decision vector \( x \), the resulting model reduces to the basic form that has been previously studied in the literature on DRA [12], [13]. Algorithm (2) degenerates into the discrete-time version of the DRA algorithm in [13] for undirected graphs.

B. Analysis on Steady States

To proceed, we illustrate a property of the NE of game (1) in the following lemma, which is helpful for the NE seeking algorithm design.

Lemma 2: Suppose Assumptions 1 and 3 hold. A vector \( x \in \Omega \) is the NE of game (1) if and only if \( x \) satisfies \( L_i \nabla_{x_i} f_i(x) = 0 \), \( \forall i \in I \).

Proof: Define the Lagrangian function for each coalition as \( H_i(x, \lambda_i) = f_i(x) + \lambda_i (1_{n_i} \cdot x - R_i) \), where \( \lambda_i \in \mathbb{R} \) is the Lagrange multiplier. By the method of Lagrange multiplier [26], it can be verified that, for a vector \( x \in \Omega \), the following two statements are equivalent under Assumption 1:

i) \( x \) is the NE of game (1);

ii) there exist \( \lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R} \), such that \( \nabla_{x_i} H_i(x, \lambda_i) = \nabla_{x_i} f_i(x) + \lambda_i 1_{n_i} = 0 \), \( \forall i \in I \).

Note that \( 1_{n_i} \) is an eigenvector of the Laplacian matrix \( L_i \) associated with its zero eigenvalue. Moreover, zero is a simple eigenvalue of \( L_i \) if \( g_i \) is connected. Thus, it can be verified that, under Assumption 3, the statement ii) is equivalent to \( L_i \nabla_{x_i} f_i(x) = 0 \), \( \forall i \in I \). This completes the proof.

Remark 4: Unlike multicoalition games with intracoalition consistency constraints or those without any coupled equality constraints, the intracoalition coupled equality constraint in the context of resource allocation considered here results in the consensus of \( \nabla_{x_{11}} f_1(x), \ldots, \nabla_{x_{nn}} f_n(x) \) at the NE.

Define the following variables for notational brevity:

\[ \hat{\psi}_i = \frac{1}{n_i} (1_{n_i}^T \otimes I_{n_i}) \psi_i, \hat{Q}_i(\cdot) = \frac{1}{n_i} (1_{n_i}^T \otimes I_{n_i}) Q_i(\cdot) \]

Since the initial value of \( \psi_{ij}^{\mu} \) is set as \( \psi_{ij}^{\mu}(0) = \nabla_{x_{ij}} f_{ij}(\xi_{ij}(0)) \), one has \( \hat{\psi}_i(0) = \hat{Q}_i(\xi_i(0)) \). Note that \( 1_{n_i}^T C_i = 1_{n_i} \). Then, one can derive from (3c) that

\[ \hat{\psi}_i(k) = \hat{Q}_i(\xi_i(k)) \quad \forall k \in \mathbb{N}. \]  

One can also obtain by definition that

\[ \hat{Q}_i(1_{n_i} \otimes \mathbf{x}) = \frac{1}{n_i} \cdot \nabla_{x_i} f_i(x). \]  

The above two equations are quite critical for the forthcoming convergence analysis.

Next, we will present a steady-state analysis of the proposed algorithm, which can facilitate the error system construction and the convergence analysis. Suppose that the algorithm variables \( x_i(k) \), \( \psi_i(k) \), and \( \xi(k) \) will settle on some points \( x_i(\infty) \), \( \psi_i(\infty) \), and \( \xi(\infty) \), respectively. Then, from (3b), (3c), and (3d), the steady states satisfy

\[ \hat{L}_i \psi_i(\infty) = 0 \]  

\[ \psi_i(\infty) = (C_i \otimes I_{n_i}) \xi_i(\infty) \]  

\[ (I_{n_i}^T - M_i) (\xi_1(\infty) - 1_{n_i} \otimes \mathbf{x}(\infty)) = 0 \]

where \( M = W \otimes I_{n_m} + W \). One can obtain from (7) that \( \psi_i(\infty) = 1_{n_i} \otimes \tau_i \), where \( \tau_i \) is a constant vector to be determined later. Since \( (I_{n_i}^T - M_i) \) is nonsingular, one can get from (8) that \( \xi_i(\infty) = 1_{n_i} \otimes \mathbf{x}(\infty) \). Combining the above equations yields \( \psi_i(\infty) = 1_{n_i} \otimes (\frac{1}{n_i} \cdot \nabla_{x_i} f_i(\mathbf{x}(\infty))) \). Substituting this into (6) yields \( L_i \nabla_{x_i} f_i(\mathbf{x}(\infty)) = 0 \). Then one has \( \mathbf{x}(\infty) = \mathbf{x}^* \) from Lemmas 1 and 2.

C. Error System Construction and Convergence Analysis

From (3a) and (3b), one has

\[ x_i(k+1) - x_i(k) = -\alpha L_i \hat{L}_i \psi_i(k) \]

which can be rewritten in the following collective form:

\[ x(k+1) - x(k) = -\alpha \hat{L}_i \hat{L}_i \psi_i(k) \]

where \( \hat{L} = \text{diag} \{L_1, \ldots, L_N\} \) and \( \hat{L} = \text{diag} \{\hat{L}_1, \ldots, \hat{L}_N\} \). Define the estimation errors

\[ e_{\xi} = \xi - 1_{n_m} \otimes \mathbf{x}. \]

Combining (3d), (10), and (11), one can derive that

\[ e_{\xi}(k+1) = M \hat{L}_i \hat{L}_i \psi_i(k) - 1_{n_m} \otimes \mathbf{x}(k+1)
\]

\[ = (M + \hat{W})(1_{n_m} \otimes \mathbf{x}) - 1_{n_m} \otimes \mathbf{x}(k+1)
\]

\[ = M e_{\xi}(k) + 1_{n_m} \otimes (\alpha \hat{L}_i \hat{L}_i \psi_i(k)) \]

where the third equality is obtained by using the fact that \( (M + \hat{W})(1_{n_m} \otimes x) = 1_{n_m} \otimes \mathbf{x} \). Since the graph \( G \) is connected, it is easy to verify from Gershgorin’s circle theorem that \( M \) is a Schur matrix. Therefore, there exist a symmetric positive definite matrix \( W_{M} \) such that \( M^T W_{M} M - W_{M} = -I_{n_m} \).

Define the convergence errors

\[ e_{\psi_i}(k) = \psi_i(k) - 1_{n_i} \otimes \hat{\psi}_i(k) \]

and \( e_{\psi} = [e_{\psi_1}^T, e_{\psi_2}^T, \ldots, e_{\psi_N}^T]^T \). One can obtain from the iteration of \( \psi_i \) in (3c) that

\[ e_{\psi_i}(k+1) = (C_i \otimes I_{n_i}) \psi_i(k) - 1_{n_i} \otimes 1_{n_i} \hat{Q}_i(\xi_i(k+1))
\]

\[ - 1_{n_i} \otimes \left( \frac{1}{n_i} (1_{n_i}^T \otimes I_{n_i}) (\hat{Q}_i(\xi_i(k)) - \hat{Q}_i(\xi_i(k+1))) \right) \]

\[ = (C_i \otimes I_{n_i}) e_{\psi_i}(k) + (I_i \otimes I_{n_i}) (\hat{Q}_i(\xi_i(k)) - \hat{Q}_i(\xi_i(k+1))) \]

where \( C_i = C_i - \frac{1_{n_i}^T 1_{n_i}}{n_i} \), \( \hat{I}_i = I_{n_i} - \frac{1_{n_i}^T 1_{n_i}}{n_i} \), and the last equality is derived by using the fact that \( \frac{1_{n_i}^T 1_{n_i}}{n_i} \otimes I_{n_i} \). Then, it is obvious that \( C_i \) is a Schur...
matrix. Therefore, there exists a symmetric positive definite matrix \( W_{e_i} \) such that \( \nabla^T_i W_{e_i} \nabla_i = -I_{n_i} \).

**Theorem 1**: Suppose that Assumptions 1–3 hold. Under the proposed DRA algorithm (2), the collective agent state \( x \) will converge linearly to the NE of the resource allocation game (1), if \( \alpha \) satisfies

\[
\alpha \leq \min \left\{ \frac{\mu}{2 \left( \sum_{i=1}^{N} \left( \nabla^T_i L_i \| \nabla_i \|^2 / n_i \right) + \gamma_b \| \nabla_i \|^2 \right)} \right\}
\]

\[
\gamma_b \geq \frac{8}{\mu} \max_{i \in I} \left[ \gamma_i \left( \| L_i \| / n_i \right) \right] \}
\]

\[
\gamma_b \geq \frac{8}{\mu} \max_{i \in I} \left[ \gamma_i \left( \| L_i \| / n_i \right) \right] \}
\]

where

\[
\gamma_b = 4 \max_{i \in I} \left[ \| L_i \|^2 / n_i \right], \quad b = n_{\text{num}} \| \mathcal{M} \|^2 + \| W_M \| \}
\]

\[
\gamma_i = 4 \left( \max_{i \in I} \left( \frac{\| L_i \|^2 / n_i}{I_{ij}} \right) + 2 \gamma_i \max_{i,j \in I_i} \left\{ \left(2 \| \nabla^T_i L_i \| / n_i \right) \right\} \}
\]

\[
+ \frac{\gamma_i}{\max_{i \in I} \left( \| L_i \|^2 / n_i \right)} \| L_{ij} \| \right) \}
\]

Moreover, the equality constraint in (1) is always satisfied during the iterations.

Before proceeding, we present some useful lemmas, the proofs of which can be found in Appendix.

**Lemma 3 (Analysis of \( \xi \))**: Suppose that Assumption 3 holds. Under algorithm (2), for the function \( V_{e}(k) = e_{\xi}(k)^T W_{e_i} e_{\xi}(k) \) with \( W_{e} = \text{diag}(W_{e_1} \otimes I_{n_1}, \ldots, W_{e_N} \otimes I_{n_N}) \) and \( W_{e_1}, \ldots, W_{e_N} \) being the positive definite matrices defined below (12), the following inequality holds for all \( k \in \mathbb{N} \):

\[
V_{\xi}(k+1) - V_{\xi}(k) \leq -\frac{1}{2} \| e_{\xi}(k) \|^2 + \alpha^b \| \hat{L} \| \| \hat{L} e_{\xi}(k) \|^2 .
\]

**Lemma 4 (Analysis of \( e_{\psi} \))**: Suppose that Assumptions 1 and 3 hold. Under algorithm (2), for the function \( V_{e}(k) = e_{\psi}(k)^T W_{e_i} e_{\psi}(k) \) with \( W_{e} = \text{diag}(W_{e_1} \otimes I_{n_1}, \ldots, W_{e_N} \otimes I_{n_N}) \) and \( W_{e_1}, \ldots, W_{e_N} \) being the positive definite matrices defined below (13), the following inequality holds for all \( k \in \mathbb{N} \):

\[
V_{\psi}(k+1) - V_{\psi}(k) \leq -\frac{1}{2} \| e_{\psi}(k) \|^2 + 2 \max_{i \in I} \left( \left\{ \| \nabla^T_i L_i \| / n_i \right\} \right) \}
\]

\[
+ \frac{\max_{i \in I} \left( \| L_i \|^2 / n_i \right)}{\max_{i \in I} \left( \| L_i \|^2 / n_i \right)} \| e_{\psi}(k) \|^2 .
\]

**Lemma 5 (Analysis of \( L_i, \nabla_i, f_i(x) \))**: Suppose that Assumptions 1, 2, and 3 hold. Under algorithm (2), for the function \( V_{e}(k) = \sum_{i=1}^{N} \frac{1}{2n_i} \| L_i \|^2 + \| \nabla_i f_i(x) \| \|^2 \), the following inequality holds for all \( k \in \mathbb{N} \):

\[
V_{k}(k+1) - V_{k}(k) \leq -\frac{\mu}{\alpha} \left( \sum_{i=1}^{N} \frac{\| L_i \|^2 / n_i}{n_i} \right) \| e_{\psi}(k) \|^2 + \alpha^b \| \hat{L} \| \| \hat{L} e_{\psi}(k) \|^2 .
\]

Now we are in the position to demonstrate Theorem 1.

**Proof of Theorem 1**: Consider the following Lyapunov function \( V(k) = V_{e}(k) + V_{\psi}(k) + \gamma_{\psi} V_{\psi}(k) \), where \( V_{e}(k), V_{\psi}(k), \) and \( V_{\psi}(k) \) have been defined in Lemmas 3, 4, and 5, respectively, and \( \gamma_{\psi}, \gamma_b \) are positive constants defined in Theorem 1. From (14), one has \( \alpha \leq \mu / (2 \left( \sum_{i=1}^{N} \| L_i \|^2 / n_i \right) + \gamma_b \)\). Then, combining Lemmas 3, 4, and 5 yields

\[
V(k+1) - V(k) \leq -\frac{\mu}{\alpha} \left( \sum_{i=1}^{N} \frac{\| L_i \|^2 / n_i}{n_i} \right) \| e_{\psi}(k) \|^2 + \alpha^b \| \hat{L} \| \| \hat{L} e_{\psi}(k) \|^2 .
\]

Noting by definitions of \( L_i \) and \( e_{\psi} \), one has

\[
\hat{L} e_{\psi} = \hat{L} \| \psi_i - \nabla_i \psi_i \| = \hat{L} \| \psi_i - \hat{L} \psi_i \| .
\]

From (4) and (16), one can derive that

\[
-\| \hat{L} \| \| \hat{L} e_{\psi} \|^2 = -\sum_{i=1}^{N} \| L_i \| \| \hat{L} \psi_i \|^2 \}
\]

\[
= -\sum_{i=1}^{N} \| L_i \| \left( L_i e_{\psi} + L_i \hat{L} \psi_i - \frac{1}{n_i} L_i \nabla \xi_i f_i(x) \right) \}
\]

\[
+ \| L_i \| \| \nabla \xi_i f_i(x) \| \|^2 \}
\]

\[
\leq \sum_{i=1}^{N} \left( \sum_{i=1}^{N} \| L_i \| \| \hat{L} \psi_i \|^2 \right) + \| L_i \| \| \hat{L} \psi_i \|^2 \}
\]

\[
\leq \sum_{i=1}^{N} \left( \sum_{i=1}^{N} \| L_i \| \| \hat{L} \psi_i \|^2 \right) + \| L_i \| \| \hat{L} \psi_i \|^2 \}
\]

where the following inequality (18), obtained under Assumption 1, is applied in the final step.

\[
\| \nabla \xi_i f_i(x) \| \|^2 \}
\]

\[
\leq \sum_{i=1}^{N} \left( \sum_{i=1}^{N} \| L_i \| \| \hat{L} \psi_i \|^2 \right) + \| L_i \| \| \hat{L} \psi_i \|^2 \}
\]

Substituting (17) back into (15) yields

\[
V(k+1) - V(k) \leq -\frac{\mu}{\alpha} \left( \sum_{i=1}^{N} \frac{\| L_i \|^2 / n_i}{n_i} \right) \| e_{\psi}(k) \|^2 \}
\]

\[
+ \alpha^b \| \hat{L} \| \| \hat{L} e_{\psi}(k) \|^2 \}
\]

\[
-\frac{\gamma_{\psi}}{4} \| e_{\psi}(k) \|^2 \}
\]

\[
-\frac{\gamma_b}{4} \| e_{\psi}(k) \|^2 \}
\]

\[
\leq -\varepsilon V(k),
\]

where \( \varepsilon = \min \left( \sum_{i=1}^{N} \frac{\lambda_i}{n_i}, \frac{\lambda_i}{\max(1, \| W_M \|)} \right) \), and the second last inequality can be obtained since \( \alpha \) satisfies (14). The above inequality indicates that \( V(k) \) converges to zero with a linear rate \( O((1-\varepsilon)^k) \). This implies that \( e_{\psi}(k), e_{\psi}(k) \), and
[(L_1 \nabla_{x_1} f_1(x(k)))^T, \ldots, (L_N \nabla_{x_N} f_N(x(k)))^T]^T} also linearly converge to zero vectors. Recalling Lemmas 1 and 2, one can verify that \( x(k) \) converges to \( x^* \).

To further prove the linear convergence of \( x(k) \) to \( x^* \), one can take note of (4), (9), and (16), which yields

\[
\begin{align*}
|x_{i}(k+1) - x_{i}(k)| &= -\alpha L_i \hat{L}_i \epsilon_i(k) - \frac{\alpha L_i}{n_i} L_i \nabla_{x_i} f_i(x(k)) \\
&+ \frac{\alpha L_i^2}{n_i} \left( \nabla_{x_i} f_i(x(k)) - n_i \hat{Q}_i(\xi_i(k)) \right).
\end{align*}
\]

It follows from (18) that \( \|x_{i}(k+1) - x_{i}(k)\| \leq \alpha \|L_i \hat{L}_i\| \|\epsilon_i(k)\| + \frac{\alpha L_i^2}{n_i} \|L_i \nabla_{x_i} f_i(x(k))\| + \frac{\alpha L_i^2}{n_i} \sum_{j=1}^{n_i} \|L_j \nabla_{x_j} f_j(x(k))\| \] (1.4). Since \( \|x_{i}(k+1) - x_{i}(k)\| \leq m^k \rho^k \) for all \( k \in \mathbb{N} \). Moreover, since \( x(\infty) = x^* \), one has \( \|x(k) - x^*\| = \|x(k) - x(k+1) + x(k+1) - x(k+2) + \ldots - x(\infty)\| \leq m^k \rho^k \sum_{i=0}^{\infty} \rho^k = \frac{m^k \rho^k}{1 - \rho} \). Thus, one can conclude that the collective state \( x \) linearly converges to the NE \( x^* \).

Remark 5: Distinct from existing results on DRA over MANs with cooperative agents, the proposed algorithm (2) can deal with the DRA problem with conflicts of interest among the agents as well as the influence of some agents’ decisions on other agents’ individual benefits. Moreover, under proposed algorithms, the intracoalition coupled equality constraints can be satisfied at each iteration. Such a feature is favorable in online solving some practical problems such as economic dispatch of smart grid with multiple generating units, where balancing the power supply and demand while seeking the optimal solution is highly desired.

## IV. NUMERICAL SIMULATIONS

Numerical examples are provided in this section to test the effectiveness of the proposed algorithms. Consider three coalitions that contain four, five, and six agents, respectively, i.e., \( N = 3, T = \{1, 2, 3\}, n_1 = 4, n_2 = 5, n_3 = 6 \). The network topology is shown in Fig. 1.

The objective function of each agent \( ij \in V \) is \( f_{ij}(x) = 5(x_{ij} - d_{ij})^2 + \frac{1}{2} x_{ij}y_{ij} \), where \( y_{11} = x_{12} + x_{21} + x_{31} + x_{32} + x_{41} + x_{42} + x_{51} + x_{52} + x_{61} + x_{62} + x_{71} + x_{72}, y_{12} = x_{11} + x_{21} + x_{31} + x_{32}, y_{21} = x_{12} + x_{13} + x_{14} + x_{15} + x_{16}, y_{22} = x_{21} + x_{23} + x_{24} + x_{25} + x_{26}, y_{31} = x_{32} + x_{33} + x_{34} + x_{35} + x_{36}, y_{32} = x_{31} + x_{33} + x_{34}, y_{41} = x_{42} + x_{43} + x_{44}, y_{42} = x_{41} + x_{43} + x_{45} + x_{46}, y_{51} = x_{52} + x_{53} + x_{54} + x_{55} + x_{56}, y_{52} = x_{51} + x_{53} + x_{54} + x_{55} + x_{56}, y_{61} = x_{62} + x_{63} + x_{64} + x_{65} + x_{66}, y_{62} = x_{61} + x_{63} + x_{64} + x_{65} + x_{66}, y_{71} = x_{72} + x_{73} + x_{74} + x_{75} + x_{76}, y_{72} = x_{71} + x_{73} + x_{74} + x_{75} + x_{76}, y_{81} = x_{82} + x_{83} + x_{84} + x_{85} + x_{86}, y_{82} = x_{81} + x_{83} + x_{84} + x_{85} + x_{86}, y_{91} = x_{92} + x_{93} + x_{94}, y_{92} = x_{91} + x_{93} + x_{94}, y_{101} = x_{102} + x_{103} + x_{104}, y_{102} = x_{101} + x_{103} + x_{104}, y_{111} = x_{112} + x_{113} + x_{114}, y_{112} = x_{111} + x_{113} + x_{114}, y_{121} = x_{122} + x_{123} + x_{124}, y_{122} = x_{121} + x_{123} + x_{124}, y_{131} = x_{132} + x_{133}, y_{132} = x_{131} + x_{133}, y_{141} = x_{142} + x_{143}, y_{142} = x_{141} + x_{143}, y_{151} = x_{152}, y_{152} = x_{151}, y_{161} = x_{162}, y_{162} = x_{161} \). The quantities of the resources in the three coalitions are \( R_1 = 100, R_2 = 150, \) and \( R_3 = 120 \). By direct calculation, one can obtain the NE \( x^* = [9.08, 20.19, 29.27, 41.46, 48.78, 35.07, 23.96, 15.54, 26.65, 10.14, 21.25, 28.87, \) 28.87, 21.0, 9.89]^T, and the values of the coalition-level objective functions at the NE are \( f_1 = 6598, f_2 = 7295, f_3 = 9347 \), respectively. The proposed algorithm (2) is employed with the algorithm parameter set as \( \alpha = 0.01 \), and the simulation results are presented in Fig. 2, showing that the agent states achieve fast convergence to the NE. Moreover, the trajectory of the sum of agent states in each coalition demonstrates that the intracoalition coupled equality constraints are satisfied at every iteration.

## V. CONCLUSION

In this article, the problem of DRA over multiple interacting coalitions is investigated by developing game-theoretic approaches. To characterize the cooperation of individual agents on resource allocation in each coalition as well as the conflicts of interest among different coalitions, a multicoalition game model with intracoalition coupled equality constraints is formulated. Inspired by techniques such as variable replacement, gradient tracking and leader-following consensus, a new DRA algorithm is developed. One favorable feature of the designed DRA algorithm is that the resource constraints can be satisfied during the whole allocation process. Furthermore, linear convergence of the proposed DRA algorithm is successfully established. In the future, further research will consider fully distributed algorithm design for the proposed problem by introducing an adaptive step-size for each agent.

## APPENDIX

### A. Proof of Lemma 3

From (12), one has

\[
\begin{align*}
V_k(k+1) - V_k(k) &= e_k^T(t_k)(M^T W_M M - W_M) e_k(t_k) \\
&+ 2e_k^T(t_k)M^TW_M(1/n_{sum} \odot (\alpha \hat{L}^2 \hat{P}(\xi(k)))) \\
&+ (1/n_{sum} \odot (\alpha \hat{L}^2 \hat{P}(\xi(k))))^TW_M(1/n_{sum} \odot (\alpha \hat{L}^2 \hat{P}(\xi(k)))) \\
&\leq -\|e_k(k)\|^2 + 2\sqrt{n_{sum}}\|M^TW_M\|\|e_k(k)\|\|\alpha \hat{L}^2 \hat{P}(\xi(k))\|^2 \\
&+ n_{sum}\|W_M\|\|\alpha \hat{L}^2 \hat{P}(\xi(k))\|^2 \\
&\leq -\frac{1}{2}\|e_k(k)\|^2 + \alpha^2 b \|\hat{L}^2 \hat{P}(\xi(k))\|^2.
\end{align*}
\]
\section*{C. Proof of Lemma 5}

One can derive that
\begin{align*}
2 \left( L_i \nabla_x f_i (x(k + 1)) - L_i \nabla_x f_i (x(k)) \right)^T L_i \nabla_x f_i (x(k))
&= \frac{2n_i}{\alpha} \left( \nabla_x f_i (x(k + 1)) - \nabla_x f_i (x(k)) \right)^T (x(k + 1) - x(k)) \\
&\quad + 2 \left( L_i \nabla_x f_i (x(k + 1)) - L_i \nabla_x f_i (x(k)) \right)^T \\
&\quad \times \left( L_i \nabla_x f_i (x(k)) - n_i \bar{Q}_i (x_i (k)) \right) \\
&\leq \frac{2n_i}{\alpha} \left( \nabla_x f_i (x(k + 1)) - \nabla_x f_i (x(k)) \right)^T (x(k + 1) - x(k)) \\
&\quad + \| L_i \nabla_x f_i (x(k + 1)) - L_i \nabla_x f_i (x(k)) \|^2 \\
&\quad + 2 \left( L_i \nabla_x f_i (x(k + 1)) - L_i \nabla_x f_i (x(k)) \right)^T \\
&\quad \times \left( L_i \nabla_x f_i (x(k)) - n_i \bar{Q}_i (x_i (k)) \right) \\
&\quad + 2 \left\| n_i \bar{L}_i e_{\psi_i} (k) \right\|^2.
\end{align*}

where the first equality is obtained from (3a), (3b), (4), and (16).

B. \textbf{Proof of Lemma 4}

One can derive from the iteration of \( e_{\psi_i} \) in \((13)\) that
\begin{align*}
V_{\psi_i} (k+1) - V_{\psi_i} (k) \\
&= \sum_{i=1}^{N} \left( e_{\psi_i} \right)^T (C_i W_{e_i} \bar{C}_i - W_{e_i}) \odot I_{n_e} \left( e_{\psi_i} \right) \\
&\quad + 2 e_{\psi_i} \left( \bar{C}_i^T W_{e_i} \bar{L}_i \odot I_{n_e} \right) \left( Q_i (\xi_i (k + 1)) - Q_i (\xi_i (k)) \right) \\
&\quad \times \left( (Q_i (\xi_i (k + 1)) - Q_i (\xi_i (k)))^T (\bar{L}_i^T W_{e_i} \bar{L}_i \odot I_{n_e}) \right) \\
&\leq \sum_{i=1}^{N} \left( \| e_{\psi_i} (k) \|^2 + \frac{1}{2} \| e_{\psi_i} (k) \|^2 \\
&\quad + 2 \| \bar{C}_i^T W_{e_i} \bar{L}_i \| \| Q_i (\xi_i (k + 1)) - Q_i (\xi_i (k)) \|^2 \\
&\quad + \| \bar{L}_i^T W_{e_i} \bar{L}_i \| \| Q_i (\xi_i (k + 1)) - Q_i (\xi_i (k)) \|^2 \right). \tag{18}
\end{align*}

Under Assumption 1, one has
\begin{align*}
\| Q_i (\xi_i (k + 1)) - Q_i (\xi_i (k)) \|^2 &\leq \max_{i \in \mathcal{V}} \left\{ \| f_i \|^2 \right\} \| \xi_i (k + 1) - \xi_i (k) \|^2.
\end{align*}

Combining the above inequalities yields
\begin{align*}
V_{\psi_i} (k+1) - V_{\psi_i} (k) \\
&\leq - \frac{1}{2} \| e_{\psi_i} (k) \|^2 + \sum_{i=1}^{N} \left( 2 \| \bar{C}_i^T W_{e_i} \bar{L}_i \| + \| \bar{L}_i^T W_{e_i} \bar{L}_i \| \right) \\
&\times \max_{i \in \mathcal{V}} \left\{ \| \xi_i (k + 1) - \xi_i (k) \|^2 \right\} \\
&\leq - \frac{1}{2} \| e_{\psi_i} (k) \|^2 + 2 \max_{i \in \mathcal{V}} \left\{ \left( \| \bar{C}_i^T W_{e_i} \bar{L}_i \| + \| \bar{L}_i^T W_{e_i} \bar{L}_i \| \right) t_{ij} \right\} \\
&\times \| I_{n_e} - M \| \| e_{\psi_i} (k) \|^2
\end{align*}

where the last inequality is obtained by recalling \((10), (11), \) and \((12)\).
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