A BUMP IN THE ROAD IN ELEMENTARY TOPOLOGY

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Abstract. We observe a subtle and apparently generally unnoticed difficulty with the definition of the relative topology on a subset of a topological space, and with the weak topology defined by a function.

“‘Obvious’ is the most dangerous word in mathematics.”

E. T. Bell

1. Relative Topology

One of the most elementary constructions in general topology is the definition of the relative or subspace topology on a subset of a topological space. But it turns out it is not quite as elementary to do this properly as has generally been thought.

If \((X, \mathcal{T})\) is a topological space and \(Y \subseteq X\), the relative topology, or subspace topology, on \(Y\) from \(\mathcal{T}\) is

\[\mathcal{T}_Y = \{ U \cap Y : U \in \mathcal{T} \}\]

i.e. the open sets in \(Y\) (called the relatively open sets) are the intersections with \(Y\) of the open sets in \(X\).

The main issue we discuss is whether \(\mathcal{T}_Y\) is really a topology on \(Y\). This is generally considered “obvious” or “trivial.” We write out the “obvious” argument:

**Proposition 1.1.** \(\mathcal{T}_Y\) is a topology on \(Y\).

**Proof.** We have \(\emptyset = \emptyset \cap Y\) and \(Y = X \cap Y\), so \(\emptyset, Y \in \mathcal{T}_Y\). If \(U_1 \cap Y, \ldots, U_n \cap Y \in \mathcal{T}_Y\), where \(U_1, \ldots, U_n \in \mathcal{T}\), then

\[(U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap \cdots \cap U_n) \cap Y \in \mathcal{T}_Y\]

since \(U_1 \cap \cdots \cap U_n \in \mathcal{T}\). If \(\{U_i \cap Y : i \in I\}\) is a collection of sets in \(\mathcal{T}_Y\), where each \(U_i \in \mathcal{T}\), then

\[\bigcup_{i \in I} (U_i \cap Y) = \left( \bigcup_{i \in I} U_i \right) \cap Y \in \mathcal{T}_Y\]

since \(\cup_{i \in I} U_i \in \mathcal{T}\). \(\square\)

Most standard topology references, e.g. [Bon98, Dug78, Eng89, HS55, HY88, Kas09, Kel75, Mun75, Wil04], either give this argument explicitly or state that the result is “trivial” or “easily verified,” presumably using this argument.

But actually there is a subtle problem with the last part of the argument: how do we know that every indexed collection of sets in \(\mathcal{T}_Y\) is of the form \(\{U_i \cap Y : i \in I\}\) for some \(U_i \in \mathcal{T}\)? In fact, the Axiom of Choice (AC) is needed to assert this,
since for a given $V \in \mathcal{T}_Y$ there are in general many $U \in \mathcal{T}$ for which $V = U \cap Y$, and one must somehow be chosen. (The same comment might apply to the finite intersection argument, but there only finitely many choices need to be made so the AC is not needed.)

When the AC was first formulated and its nature understood, it was observed that mathematicians had already been using it extensively without comment and generally without notice. The relative topology example shows that this is still happening.

But does it really require the AC? There is, in fact, a simple way to avoid it: there is a systematic way to choose the $U_i$ (I am indebted to S. Jabuka for this observation). If $V \in \mathcal{T}_Y$, there is a largest open set $U \in \mathcal{T}$ such that $V = U \cap Y$, namely the union of all $W \in \mathcal{T}$ for which $V = W \cap Y$. A correct phrasing of the proof would thus be:

**Proof.** We have $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$, so $\emptyset, Y \in \mathcal{T}_Y$. If $V_1, \ldots, V_n \in \mathcal{T}_Y$, then, for each $k$, $V_k = U_k \cap Y$ for some $U_k \in \mathcal{T}$; so

$$V_1 \cap \cdots \cap V_n = (U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap \cdots \cap U_n) \cap Y \in \mathcal{T}_Y$$

since $U_1 \cap \cdots \cap U_n \in \mathcal{T}$. If $\{V_i : i \in I\}$ is a collection of sets in $\mathcal{T}_Y$, for each $i \in I$ let $U_i$ be the union of all $W \in \mathcal{T}$ such that $V_i = W \cap Y$. Then $U_i \in \mathcal{T}$ and $V_i = U_i \cap Y$ for each $i$, so

$$\bigcup_{i \in I} V_i = \bigcup_{i \in I} (U_i \cap Y) = \left( \bigcup_{i \in I} U_i \right) \cap Y \in \mathcal{T}_Y$$

since $\cup_{i \in I} U_i \in \mathcal{T}$.

There is an alternate argument which avoids the AC in [Kur66] (which is the only topology book I have found with a complete correct proof of [1.1]). Recall that a Kuratowski closure operation on a set $Y$ is an assignment $A \mapsto \overline{A}$ for each subset $A$ of $Y$, with the properties $\emptyset = \emptyset$, $A \subseteq \overline{A} = \overline{\overline{A}}$ for all $A$, and $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all $A, B$. It is easy to show (the argument is in many standard references and can be found in [Bla], and does not use the AC) that any Kuratowski closure operation defines closure with respect to a unique topology for which the closed sets are precisely the sets $A$ for which $\overline{A} = A$.

To prove [1.1] for $A \subseteq Y$ define $\overline{A} = A \cap Y$, where $\overline{A}$ is the closure of $A$ in $X$. It is nearly trivial to check (without using the AC) that $A \mapsto \overline{A}$ is a Kuratowski closure operation on $Y$, and that the closed sets with respect to the corresponding topology are precisely the complements of the sets in $\mathcal{T}_Y$. It follows that the topology defined by this closure operation is $\mathcal{T}_Y$, and in particular $\mathcal{T}_Y$ is a topology.

2. The Weak Topology Defined by a Function

If $(X, \mathcal{T})$ is a topological space, $Y$ a set, and $f : Y \to X$ a function, there is a weakest topology on $Y$ making $f$ continuous. It should be

$$\mathcal{T}_Y = \{ f^{-1}(U) : U \in \mathcal{T} \}.$$

But is $\mathcal{T}_Y$ actually a topology? If $f$ is surjective, there is no difficulty verifying this (using that preimages respect unions and intersections). However, if $f$ is not surjective, we run into the same problem as in [1.1] (which is actually just the case where $f$ is injective), since many different open sets in $X$ can have the same preimage in $Y$, so the AC must apparently be used to show that $\mathcal{T}_Y$ is a topology.
To show that $\mathcal{T}_Y$ is a topology without using the AC, the union trick works, and the argument via Kuratowski closure operations works here too: for $A \subseteq Y$, set $A = f^{-1}(f(A))$. There is also an alternate argument. Let $Z = f(Y) \subseteq X$. By [1.1] we have that $\mathcal{T}_Z$ is a topology on $Z$. If $\mathcal{S}$ is a topology on $Y$, then $f$ is continuous as a function from $(Y, \mathcal{S})$ to $(X, \mathcal{T})$ if and only if it is continuous as a function from $(Y, \mathcal{S})$ to $(Z, \mathcal{T}_Z)$, and it is easily verified that $\mathcal{T}_Y = (\mathcal{T}_Z)_Y$. But $f : Y \to Z$ is surjective, so the AC is not needed to prove that $(\mathcal{T}_Z)_Y$ is a topology.

3. SHOULD WE WORRY ABOUT THE AC?

Most modern mathematicians have no serious qualms about using the AC, and largely share the opinion of Ralph Boas [Boa96, p. xi]:

"[A]fter Gödel’s results, the assumption of the axiom of choice can do no mathematical harm that has not already been done."

As an analyst, I have no qualms about it myself. But I do believe:

1. When the AC is used, it should be mentioned.
2. The AC should not be used if it is not needed.

There is a gray area with 2: the AC can drastically simplify proofs of some results which can be proved without it. But the relative topology case is one where use of the AC is of highly doubtful benefit.

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