ORBITAL STABILITY AND UNIQUENESS OF THE GROUND STATE FOR NLS IN DIMENSION ONE

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Abstract. We prove that standing-waves solutions to the non-linear Schrödinger equation in dimension one whose profiles can be obtained as minima of the energy over the mass, are orbitally stable and non-degenerate, provided the non-linear term $G$ satisfies a Euler differential inequality. When the non-linear term $G$ is a combined pure power-type, then there is only one positive, symmetric minimum of the energy constrained to the constant mass.

1. Introduction

The purpose of this paper is to prove the orbital stability of solitary-wave solutions to a non-linear Schrödinger equation

\[(\text{NLS}) \quad i\partial_t \phi(t, x) + \Delta_x \phi(t, x) - f(\phi(t, x)) = 0, \quad \phi: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C} \]

in dimension $n = 1$ for general class of nonlinear functions $f$ such that $f: \mathbb{C} \to \mathbb{C}$ is $C^1$ and

\[(1.1) \quad f(\overline{s}) = \overline{f(s)}, \quad f(zs) = zf(s), \quad \forall z \in \mathbb{C} \text{ such that } |z| = 1.\]

From (1.1), if $s$ is a real number, then $f(s)$ is a real number. We denote by $g: \mathbb{R} \to \mathbb{R}$ the restriction of $f$ to $\mathbb{R}$. From the second equality of (1.1), $g$ is an even function. Let $G$ be the primitive of $g$ such that $G(0) = 0$. We define for every complex number $s$

\[F(s) := G(|s|).\]

A solitary-wave is a solution to (NLS) of the form

\[(1.2) \quad \phi(t, x) = e^{i\omega t} R(x), \quad (t, x) \in [0, \infty) \times \mathbb{R} \]

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so that the gauge invariance condition (1.1) implies that
\[(SW) \quad \phi(t, x) = ze^{i(vx-tu^2)}e^{iwt} R(x - 2tv), \quad (t, x) \in [0, \infty) \times \mathbb{R}\]
is also a solution to (NLS) for any \(v\) in \(\mathbb{R}\) and any complex number \(z\) such that \(|z| = 1\). The profile \(R\) is a real-valued function. If \(\phi\) in (SW) is a solution to (NLS), then \(R\) satisfies the differential equation
\[
-R''(x) + f(R(x)) + \omega R(x) = 0.
\]
In this paper we address solutions to the equation above which can be obtained as minima of the energy functional
\[
E: H^1(\mathbb{R}; \mathbb{C}) \to \mathbb{R}, \quad E(u) := \frac{1}{2} \int_{-\infty}^{+\infty} |u'(x)|^2 dx + \int_{-\infty}^{+\infty} F(u(x)) dx
\]
under the constraint
\[
S(\lambda) := \{u \in H^1 \mid M(u) = \lambda\},
\]
where \(M(u) := \int_{-\infty}^{+\infty} |u(x)|^2 dx\). We will assume that the non-linearity satisfies conditions which guarantee the global well-posedness of the initial value problem of (NLS) in \(H^1\); that is, given an initial datum \(u_0 \in H^1(\mathbb{R}; \mathbb{C})\), there exists a unique solution \(\phi(t, x) \in C([0, +\infty); H^1(\mathbb{R}; \mathbb{C}))\) to the Schrödinger equation such that \(\phi(0, x) = u_0(x)\).

The global well-posedness determines a one-parameter family of operators \(U_t\) on \(H^1(\mathbb{R}; \mathbb{C})\). To a real number \(\lambda > 0\), we associate two subsets of \(H^1(\mathbb{R}; \mathbb{C})\):
\[
\mathcal{G}_\lambda := \{u \in H^1(\mathbb{R}; \mathbb{C}) \mid E(u) = \inf_{S(\lambda)} E\}
\]
\[
\mathcal{G}_\lambda(u) := \{zu(\cdot + y) \mid (z, y) \in S^1 \times \mathbb{R}\}.
\]
The first one is called ground state. The second set is a subset of the ground state; if \(u(x) = R(x)\), then \(\mathcal{G}_\lambda(R)\) contains the orbit of \(R\), that is, the point \(U_t(R)\) for every \(t \geq 0\).

On \(H^1(\mathbb{R}; \mathbb{C})\), we consider the distance given by the scalar product
\[
(u, w)_{H^1(\mathbb{R}; \mathbb{C})} := \text{Re} \int_{-\infty}^{+\infty} u \overline{w}(x) dx + \text{Re} \int_{-\infty}^{+\infty} \nabla u \cdot \nabla \overline{w}(x) dx.
\]
A set \(S \subseteq H^1(\mathbb{R}; \mathbb{C})\) is said stable if for every \(\varepsilon > 0\), there exists \(\delta > 0\) such that
\[
\text{dist}(u, S) < \delta \implies \text{dist}(U_t(u), S)
\]
for every \(t \geq 0\). One of the first result of stability is the work of T. Cazenave and P. L. Lions in 1982, [8], where \(f\) is a pure power function. Extensions to more general non-linearities have been obtained in [20] and [2, 16]. However, while in [8] the stability of both \(\mathcal{G}_\lambda\) and \(\mathcal{G}_\lambda(u)\) has been proved, in [2, 16] only the stability of the ground state is proved. The pure power case
\[
f(s) = -|s|^{p-2}s
\]
is very special as it exhibits the rescaling invariance \(f(ts) = t^{p-1}f(s)\) for every \(t \geq 0\). As a consequence, \(G_\lambda = G_\lambda(u)\) for every \(u\) in \(G_\lambda\). In fact, it is possible to give a precise description of an element \(u\) of the ground state:
\[
u(x) = \pm z \omega^{\frac{1}{p-1}} R_1(\omega^{1/2}(x + y)), \quad \omega^{\frac{5-p}{2p-1}} \|R_1\|_{L^2}^2 = \lambda
\]
where $R_1$ is the unique positive solution to (1.3) when $\omega = 1$, $y$ is in $\mathbb{R}$, and $z$ is complex with $|z| = 1$. Therefore, the set $\mathcal{G}_\lambda(u)$ is stable because the ground state is stable. When more general non-linearities are considered, the rescaling property fails, and it is not clear anymore whether the equality $\mathcal{G}_\lambda = \mathcal{G}_\lambda(u)$ holds. We list our assumptions:

(G1) there exists $s_0 \in \mathbb{R}$ such that $G(s_0) < 0$

(G2) there exist $C, p, q, p_*$ and $s_*$ such that

\[ |G'(s)| \leq C(|s|^{p-1} + |s|^{q-1}), \quad \forall s \in \mathbb{R} \]

\[ -C|s|^{p_*-1} \leq G''(s), \quad \forall s \geq s_* \]

where $2 < p_* < 6$ and $2 < p \leq q$.

**Theorem A** (Orbital stability of $\mathcal{G}_\lambda$). Suppose that (G1) and (G2) hold. Then, there exists $\lambda_* \geq 0$ such that for every $\lambda > \lambda_*$, the functional $E$ has a minimum, and $\mathcal{G}_\lambda$ is orbitally stable.

A proof of the stability of $\mathcal{G}_\lambda$ has been made in [2] in dimension $n \geq 3$ and in [16 3]. Here we present a few improvements with respect to the assumptions made in the quoted references. Firstly, we do not use (at this point) the growth condition (G2), required [2] F4, to obtain the splitting property (ii) of Proposition 2.2, which follows directly from [6]; secondly, (G2) weakens [16 F4], where $s_*$ is set to zero. Instead, we allow the non-linearity to be critical nearby the origin. We prove the orbital stability of $\mathcal{G}_\lambda$ with a version of the Concentration-Compactness Lemma of P. L. Lions, [14], introduced by V. Benci and D. Fortunato in [4] where the classic definitions of Concentration, Compactness and Vanishing are expressed in terms of weak convergence, instead of the Concentration Function used in [14].

Concentration. There exists a subsequence $(u_{n_k})$, a sequence $(y_k)$ and $u$ such that

(C) $u_{n_k} (\cdot + y_{n_k}) \rightharpoonup u$ in $H^1(\mathbb{R}; \mathbb{C})$.

Dichotomy. There exists a subsequence $(u_{n_k})$ and $(y_k)$ such that

(D) $u_{n_k} (\cdot + y_k) \rightharpoonup u$ in $H^1(\mathbb{R}; \mathbb{C})$

for some $u$ such that $0 < \|u\|_{H^1} < \liminf_{k \to +\infty} \|u_{n_k}\|_{H^1}$.

Vanishing

(V) $u_{n_k} (\cdot + y_k) \rightharpoonup u \implies u = 0$.

The functional $E$ is defined on $H^1(\mathbb{R}; \mathbb{C})$ instead of real-valued functions. This perspective of the minimization problem has the value of highlighting features of the minima which are essential in the proof of the stability of the other set, $\mathcal{G}_\lambda(u)$: if $u$ is a minimum, then $u = zR$, where $R$ is a real-valued minimum and $z$ a complex number with $|z| = 1$, [16] of Lemma 2.4. This fact relies on the Convex Inequality for the Gradient, [13 Lemma 7.8]. In the next assumption $G$ is $C^2$ and

(G3) $12G(R(x)) - 7R(x)G'(R(x)) + R(x)^2G''(R(x)) \geq 0$

for every solution $R$ of (1.3) and $x \in \mathbb{R}$.

(G4) $|G''(s)| \leq C(|s|^{p-2} + |s|^{q-2})$ for every $s \in \mathbb{R}$
and \( p, q \) as in \((G2)\). We denote with \( H^1_r(\mathbb{R}) \) the space of real-valued \( H^1 \) functions which are even on \( \mathbb{R} \). Let \( \lambda_* \) be as in Theorem A.

**Theorem B.** Suppose that \((G1), (G2), (G3) \) and \((G4)\) hold. Then, for \( \lambda > \lambda_* \), minima of \( E \) on \( S(\lambda) \cap H^1_r \) are non-degenerate.

Our work presents some changes with respect to the one of M. We instein, \([20]\) for the one-dimensional case. As we mentioned earlier in the introduction, in order to have stability the condition

\[
\int_{-\infty}^{+\infty} \left[ f(R^2) + 2R^2 f'(R^2)|R'(x)|^2 \right] \, dx \neq 0
\]

was required (in his notation \( f(s)s = G'(s) \)). In \((G3)\) we offer a different approach, as we prescribe a condition on the non-linearity rather than on a solution to \((1.3)\).

The non-degeneracy implies that the set \( \mathcal{G}_\lambda \cap H^1_r \) is finite. This should be compared with the pure-power case, where \( \mathcal{G}_\lambda \cap H^1_r \) consists of exactly two functions. Consequently, under the same assumptions as the theorem above and adding the assumption

**Theorem C.** Then the set \( \mathcal{G}_\lambda(u) \) is stable for every \( u \in \mathcal{G}_\lambda \).

Finally, we show that under an additional assumption, a uniqueness condition holds, just like the pure-power case. Then \( \mathcal{G}_\lambda = \mathcal{G}_\lambda(u) \), Corollary 5.1. For \( \omega > 0 \), we define

\[
V(s) := -\frac{2G(s)}{s^2}
\]

and

\[
R_\omega(s) := \inf\{ s > 0 \mid \omega = V(s) \}
\]

whenever the set on the right is non-empty.

\((G5)\) The set \( \mathcal{A} = \{ \omega \mid V'(R_\omega(\omega)) > 0 \} \) is an interval.

**Theorem D** (Uniqueness). If \((G1-5)\) hold, then \( \mathcal{G}_\lambda \cap H^1_r \) consists of exactly two functions, \( R_+ \) and \( R_- \). The first is positive while \( R_- = -R_+ \).

Both the proofs of the uniqueness and the stability of \( \mathcal{G}_\lambda \) rely on the function \( d(\omega) \) defined by W. Strauss in \([15]\) and \([12]\). Condition which guarantees the stability \( d''(\omega) \geq 0 \) follows from \((G3)\), and Lemma 5.1. Here, we deduce the stability of \( \mathcal{G}_\lambda(u) \) from the fact that the set \( H^1_r \cap \mathcal{G}_\lambda \) is finite, rather than checking the assumptions of \([12]\) directly.

We define

\[
L(s) = 12G(s) - 7sG'(s) + s^2G''(s).
\]

When \( L = 0 \) on \((0, +\infty)\), then \( F \) satisfies a Euler equation whose solutions are linear combinations of \( s^2 \) and \( s^6 \). If \( G \) is a pure-power \(-as^p \) with \( a > 0 \), then

\[
L(s) = a(p-2)(6-p)s^p
\]
which is strictly positive if $p$ is sub-critical. Therefore (G3) can be interpreted as a sub-critical assumption. However, this interpretation fails as we consider sub-critical pure-power combined non-linearities as

$$F(s) = -a|s|^p + b|s|^q, \quad p < q$$

where

$$L(s) = a(p - 2)(6 - p)s^p - b(q - 2)(6 - q)s^q$$

and can be negative on $(0, +\infty)$. However, (G3) prescribes the behaviour of $L$ only on the union of the images of the solutions of (1.3) (for arbitrary $\omega$). In fact, it turns out that (1.6) does satisfy (G3).

The paper is organized as follows. In §2 we show the Concentration-Compactness behaviour of minimizing sequences; in §3 we discuss the non-degeneracy of minima, in §4 the stability of the two sets $G_\lambda$ and $G_\lambda(u)$, in §5 the uniqueness of positive, even solutions in $G_\lambda$. In §6 we show that (1.6) satisfies all the assumptions mentioned above.

2. Properties of the functional $E$

In Lemma 6.1 we show that $G$ can be written as sum of two terms $G_1$ and $G_2$ which satisfy estimates (G2) and (G4) having only a single power on the right term. Since all the properties we will prove are well-behaved with respect to linear combination, we will assume that in (G2) and (G4) there is only the power $p$.

Some of the properties listed in the next proposition have been already thoroughly proved in [2] in dimension $n \geq 3$. We fill the details of the proof in the dimension $n = 1$. Throughout this section we will assume that (G2) holds.

**Proposition 2.1.** The functional $E$ satisfies the following properties:

(i) given $e, \lambda > 0$, there exists $C(e, \lambda)$ such that

$$E(u) \leq e \text{ and } M(u) \geq \lambda \implies \|u\|_{H^1} \leq C.$$  

Then minimizing sequences of $E$ over $S(\lambda)$ are bounded

(ii) if (G2) holds, given a weakly converging sequence $u_n \rightharpoonup u$

$$E(u_n) = E(u_n - u) + E(u) + o(1)$$

$$M(u_n) = M(u_n - u) + M(u) + o(1)$$

(iii) given a bounded sequence $(u_n)$ in $H^1$ and a sequence $(\lambda_n)$ such that $\lambda_n \to \lambda$, then

$$\lim_{n \to +\infty} (E(\lambda_n u_n) - E(\lambda u_n)) = 0$$

(iv) $E$ is bounded from below on $S(\lambda)$.

**Proof.** (i) and (iv). From the Sobolev-Gagliardo-Nirenberg inequality there exists a $S \in \mathbb{R}$ such that

$$\|u\|_{L^d}^d \leq S^d \|u\|_{L^2}^{d/2} \|u\|^{d-2}_{L^2}$$

(2.1)
for every $d \geq 2$. We set $A := \{|u| \geq s_*\}$. On $A$, $F(u(x))$ is bounded from below by $-C|u(x)|^p$ or $-Cs_*^{p-1}|u(x)|^{p}$, depending on whether $p \leq p_*$. On $A^c$, we use $-C|u(x)|^{p_*}$. In any case, from (2.2)
\[
\int_{-\infty}^{+\infty} F(u(x))dx \geq -C \max\{1, s_*^{p-p_*}\} \|u\|_{L^d}^d
\]
where $d < 6$. Then, from (2.1)
\[
(2.2) \quad 2e \geq 2E(u) \geq \|u'\|_{L^2}^2 - C' \lambda^{\frac{d+2}{d}} \|u'\|_{L^2}^{\frac{d+2}{d}}.
\]
Then $C(e, \lambda)$ exists because $(d-2)/2 < 2$. Then minimizing sequences are bounded \(\blacksquare\). We refer to the paper of H. Brezis and E. Lieb [6].

(iii). We write the proof only for the non-linear part $\int F(u)dx$, for which we use the same notation $E$. We define
\[
k_n(x) := F(\lambda_n u_n(x)) - F(\lambda u_n(x))
\]
\[
= \int_0^1 F'(t\lambda u_n(x) + (1-t)\lambda_n u_n(x))(\lambda_n - \lambda)u_n(x)dt.
\]
Then
\[
|k_n(x)| \leq C|\lambda_n - \lambda| \int_0^1 |t\lambda u_n(x) + (1-t)\lambda_n u_n(x)|^{p-1}|u_n(x)|dt
\]
\[
\leq 2^{p-2}C|\lambda_n - \lambda| \left( \int_0^1 |t\lambda u_n(x)|^{p-1} + |(1-t)\lambda_n u_n(x)|^{p-1} \right) |u_n(x)|dt
\]
\[
\leq 2^{p-2}C|\lambda_n - \lambda|(|\lambda|^{p-1} + |\lambda_n|^{p-1})|u_n(x)|^{p-1}.
\]
By integrating on $\mathbb{R}$, we obtain
\[
|E(\lambda_n u_n) - E(u_n)| \leq \int_{-\infty}^{+\infty} |k_n(x)|dx
\]
\[
\leq 2^{p-2}C|\lambda_n - \lambda|(|\lambda|^{p-1} + |\lambda_n|^{p-1}) \|u_n\|_{L^p}^p.
\]
Since $(u_n)$ is bounded in $H^1$, as we take the limit as $n \to +\infty$, we obtain the conclusion. \(\blacksquare\)

We are then allowed to define
\[
I(\lambda) := \inf_{\mathcal{S}(\lambda)} E.
\]

**Proposition 2.2.** The function $I$ satisfies the following properties:

(i) the function $I$ is non-positive 

(ii) for every $\theta \geq 1$ and $\lambda > 0$, there holds
\[
I(\theta \lambda) \leq \theta I(\lambda).
\]

If equality holds, then either $\theta = 1$ or $\theta > 1$ and $I(\lambda) = 0$

(iii) there exists $\lambda_* > 0$ such that
\[
I < 0 \text{ on } (\lambda_*, +\infty), \quad I = 0 \text{ on } (0, \lambda_*].
\]

If $\lambda \leq \lambda_*$, then $\mathcal{G}_\lambda$ is empty.
Proof. (i). The proof of this fact follows from [16, Lemma 2.3].

(ii). Such property of $I$ has been proved in [16, Lemma 3.2] and [2, Proposition 15] using rescalings. However, in both references it is assumed that $E$ achieves its infimum on $S(\lambda)$. Here, we just apply the same rescaling to a minimizing sequence $(u_n)$ over $S(\lambda)$

\[ u_{n,\vartheta}(x) = u_n(\vartheta^{-1}x). \]

Clearly, $u_n \in S(\vartheta \lambda)$. Then

\[ I(\vartheta \lambda) \leq E(u_{n,\vartheta}) = \vartheta \left( \frac{\vartheta^{-2}}{2} \int_{-\infty}^{+\infty} |u'_n|^2 dx + \int_{-\infty}^{+\infty} F(u_n) dx \right) \]

\[ = \vartheta \left( E(u_n) - \frac{1 - \vartheta^{-2}}{2} \|u'_n\|_{L^2}^2 \right) \]

\[ \leq \vartheta I(\lambda) - \frac{\vartheta(1 - \vartheta^{-2})}{2} \cdot \|u'_n\|_{L^2}^2 \leq \vartheta I(\lambda). \]

Clearly, if equality holds and $\vartheta > 1$, then the sequence of gradients converges to zero. From (6.4) and (2.1), we obtain

\[ \int_{-\infty}^{+\infty} F(u_n) dx \to 0. \]

Therefore, $I(\lambda) = 0$.

(iii). From [2, Lemma 5] it follows that there exists $\lambda_1$ such that $I(\lambda_1) < 0$.

Now, suppose that there exists $\lambda_0$ such that $I(\lambda_0) = 0$. If $\lambda' \leq \lambda_0$, then

\[ 0 = I(\lambda') = I(\lambda'/\lambda_0 \cdot \lambda_0) \leq (\lambda'/\lambda_0)I(\lambda_0) \leq 0. \]

The first inequality follows from (ii), while the last inequality follows from (i). Therefore, the set $\{ \lambda \mid I(\lambda) = 0 \}$ is bounded or is equal to $(0, +\infty)$. The second case is ruled about by $\lambda_1$. On the first case, we define

\[ \lambda_* := \sup \{ \lambda \mid I(\lambda) = 0 \}. \]

Since $I$ is continuous (from [16, Lemma 2.3]), $I(\lambda_*) = 0$. Now, we consider the case $\lambda \leq \lambda_1$. Let $u \in G_\lambda$ be a minimum. We apply to $u$ the same rescaling as in (2.3). For every $\vartheta$ the endpoints of the inequalities are zero, then $\|u''\|_{L^2}^2 = 0$ which gives $u = 0$, and obtain a contradiction with $\lambda > 0$. □

We define $J_k$ the open interval $(k, k + 1)$. The result of the next lemma is well-known from [14, Lemma I.1]: if a sequence $(u_n)$ vanishes, then all the $L^d$ norms converge to zero. In [14] they show that if

\[ \lim_{n \to +\infty} \sup_{k \in \mathbb{Z}} \|u_n\|_{L^2(J_k)} = 0, \]

then the sequence of $L^d$ norms of $(u_n)$ also converges to zero. Here we write a proof which provides an estimate of the $L^d$ norm by a product of the $H^1$ norm and (2.4). We need the Sobolev-Gagliardo-Nirenberg inequality for the interval $J_k$

\[ \|u\|_{L^d(J_k)} \leq s^d \|u\|_{L^2(J_k)}^{\frac{d-2}{2}} \|u\|_{H^1(J_k)}^{\frac{d-2}{2}}, \]
Lemma 2.1. For every $u \in H^1(\mathbb{R}; \mathbb{C})$ there holds

$$
\|u\|_{L^d}^d \leq s^d \left( \sup_{k \in \mathbb{Z}} \|u\|_{L^2(J_k)} \right)^{d-2} \|u\|_{H^1}^2
$$

if $d \geq 6$ and

$$
\|u\|_{L^d}^d \leq s^d \left( \sup_{k \in \mathbb{Z}} \|u\|_{L^2(J_k)} \right)^{d-2} \|u\|_{H^1}^2
$$

if $d \leq 6$.

Proof. First case: $d \geq 6$, that is $(d-2)/2 \geq 2$.

Then, in (2.6) we have the product of the bounded sequence

$$
a_k := \|u\|_{L^2(J_k)}^{d-2} \|u\|_{H^1(J_k)} \|u\|_{H^1(J_k)}, \quad \|a\|_{\infty} = \left( \sup_{k \in \mathbb{Z}} \|u\|_{L^2(J_k)} \right)^{d-2} \|u\|_{H^1}^{d-6}
$$

and the summable sequence

$$
b_k := \|u\|_{H^1(J_k)}, \quad \|b\|_1 = \|u\|_{H^1}^2.
$$

Therefore,

$$
\|u\|_{L^d}^d \leq s^d \left( \sup_{k \in \mathbb{Z}} \|u\|_{L^2(J_k)} \right)^{d-2} \|u\|_{H^1}^{d-6}.
$$

Second case: $d \leq 6$. Then $(d-2)/2 \leq 2$, and we have

$$
\frac{6-d}{2} = 2 - \frac{d-2}{2} < \frac{d+2}{2}.
$$

Then,

$$
\|u\|_{L^2(J_k)}^{d-2} \|u\|_{H^1(J_k)} \leq \|u\|_{L^2(J_k)}^{d-2} \|u\|_{H^1(J_k)}^{d-2} \|u\|_{H^1(J_k)}^{d-2} = \|u\|_{L^2(J_k)}^{d-2} \|u\|_{H^1(J_k)}^2.
$$

Again, taking the sum over $\mathbb{Z}$, we obtain

$$
\|u\|_{L^d}^d \leq s^d \left( \sup_{k \in \mathbb{Z}} \|u\|_{L^2(J_k)} \right)^{d-2} \|u\|_{H^1}^2.
$$

□

Lemma 2.2. Let $(w_n) \subseteq H^1(\mathbb{R}; \mathbb{C})$ be a sequence such that

$$
E(w_n) \to I(\lambda) \quad \text{and} \quad M(w_n) \to \lambda.
$$

Suppose that $(w_n)$ converges to $w$ in $L^2$. Then there exists a subsequence of $(w_n)$ which converges strongly in $H^1(\mathbb{R}; \mathbb{C})$.

Proof. From (2.2), $(w_n)$ is bounded in $H^1$ (in the inequality $\lambda = M(w_n)$). Then, there exists a subsequence $(w_{n_k})$ which converges weakly to $w$ in $H^1$, and pointwise a.e.

$$
o(1) + I(\lambda) = E(w_{n_k}) = \frac{1}{2} \int_{-\infty}^{+\infty} \left| u_{n_k}'(x) \right|^2 dx + \int_{-\infty}^{+\infty} F(w_{n_k}(x)) dx
$$

$$
\geq \frac{1}{2} \int_{-\infty}^{+\infty} \left| w'(x) \right|^2 dx + \int_{-\infty}^{+\infty} F(|w(x)|) dx \geq I(\lambda) + o(1).
$$
Since \((w_{nk})\) converges pointwise a.e. and \(L^2\), by (2.1), \(\int F(w_{nk})dx\) converges to \(\int F(w)dx\). From this fact and the weak lower-semicontinuity of \(\int |u'|^2dx\), we obtained the first inequality. The second inequality follows from the strong convergence in \(L^2\) which implies that \(w\) is in \(S(\lambda)\). Then, taking the limit, 
\[
\lim_{k \to +\infty} \int_{-\infty}^{+\infty} |w'_{nk}(x)|^2dx = \int_{-\infty}^{+\infty} |w'(x)|^2dx
\]

implying the convergence of \(w'_{nk}\) to \(w'\) in \(L^2\). In the next lemma, \(\lambda_\ast\) is as in Proposition 2.2. □

**Lemma 2.3.** Let \(\lambda\) be such that \(\lambda > \lambda_\ast\). A subsequence of a sequence \((u_n)\) such that 
\[
M(u_n) \to \lambda, \quad E(u_n) \to I(\lambda)
\]
satisfies the concentration behaviour \((\mathcal{C})\).

**Proof.** We show that \((u_n)\) does not vanish and does not have a dichotomy. If \((u_n)\) vanishes, from (V), up to extract a subsequence
\[
\lim_{n \to +\infty} \sup_{k \in \mathbb{Z}} \|u_n\|_{L^2(J_k)} = 0.
\]
Otherwise, there exists \(\varepsilon_0 > 0\) and a sequence \((k_n)\) such that 
\[
\|u_{k_n}(\cdot - k_n)\|_{L^2((0,1))} = \|u_{k_n}\|_{L^2(J_{k_n})} \geq \varepsilon_0
\]
However, a subsequence of \(u_{k_n}(\cdot + y_k)\) converges weakly to zero and, since \((0,1)\) is bounded, the \(L^2\)-norm converges to zero, giving a contradiction.

From (iii), \(I(\lambda) > 0\). From the inequalities (2.6) and (2.7), and (2.8), a subsequence of \((u_{k_n})\) converges to zero in \(L^p \cap L^q\). Therefore, \(I(\lambda) = 0\), and we have a contradiction.

Let \((u_{nk}), (y_k)\) and \(u\) be as in \((\mathcal{D})\). Firstly, we observe that the inequality 
\[
0 < \|u\|_{L^2} < \liminf_{k \to +\infty} \|u_{nk}\|_{L^2}
\]
holds too. Otherwise, we had strong convergence in \(L^2\) and thus, strong convergence in \(H^1\), by Lemma 2.2 and a contradiction with the dichotomy assumption. We define 
\[
\lambda^k_1 := \|u_{nk}(\cdot + y_k) - u\|^2_{L^2}.
\]
Up to extract a subsequence, we can suppose that \(\lambda^k_1\) converges. We use the notation \(\lambda_1\) for its limit and we have 
\[
0 < \lambda_1 < \lambda.
\]
We set 
\[
w_k := \frac{\lambda_1}{\lambda^k_1} \cdot (u_{nk}(\cdot + y_k) - u).
\]
By (i) of Proposition 2.1, the sequence \((w_k)\) is bounded. Then, we can apply (iii) and (iii) 
\[
(2.9) \quad E(u_{nk}) = E(u_{nk}(\cdot + y_k) - u) + E(u) + o(1) = E(w_k) + E(u) + o(1).
\]
Here, we use the argument of \([4, Lemma 20, p. 5]\). We define 
\[
\Lambda(u) := \frac{E(u)}{M(u)}.
\]
By (v) of Proposition 2.1 and (2.9) we have
\[ o(1) + \frac{I(\lambda)}{\lambda} = \frac{E(w_k) + E(u)}{M(w_k) + M(u)} + o(1) \geq \min\{\Lambda(w_k), \Lambda(u)\} + o(1) \]

Let us suppose that the term of \(w_k\) is the smaller (on the other case, the argument is the same). Then
\[ o(1) + \frac{I(\lambda)}{\lambda} \geq \frac{I(\lambda_1)}{\lambda_1} + o(1) \geq \frac{I(\lambda)}{\lambda} + o(1). \]

The last inequality is a consequence of (iii) of Proposition 2.1: the function \(I(\lambda)/\lambda\) is decreasing. Then, all the inequalities are equalities.

\[ I(\lambda) = \frac{I(\lambda_1)}{\lambda_1}. \]

From (iii) of Proposition 2.1, either \(\theta := \lambda/\lambda_1 = 1\), which is ruled out by the dichotomy assumption, or \(I(\lambda) = 0\), which contradicts the assumptions on \(\lambda\). Thus, the sequence is not dichotomy. \(\square\)

**Proposition 2.3.** \(G_\lambda \neq \emptyset\) for every \(\lambda > \lambda_\ast\), and the Lagrange multiplier is negative. If \(\lambda \leq \lambda_\ast\), then \(G_\lambda\) is empty.

**Proof.** The second part is just (iii) of Proposition 2.2. Let \((u_n)\) be a minimizing sequence. Since \(\lambda > \lambda_\ast\), the assumptions of Lemma 2.3 are satisfied and there exists \((y_n) \subseteq \mathbb{R}\) and \(u \in S(\lambda)\) such that
\[ u_n(\cdot + y_n) \rightarrow u. \]

From Proposition 6.2, \(E\) is continuous. Then, taking the limit as \(n \rightarrow +\infty\) in
\[ o(1) + I(\lambda) = E(u_n) = E(u_n(\cdot + y_n)) = o(1) + E(u) \]
we obtain \(E(u) \in S(\lambda)\). Now, we do not set any restriction on \(\lambda\) and just assume that \(u \in G_\lambda\). By (i) of Proposition 6.2, there exists \(\omega \in \mathbb{R}\) such that
\[ u''(x) - f(u(x)) - \omega u(x) = 0. \]

Taking the scalar product in \(\mathbb{C}\) with \(u'\) and obtain
\[ u'(x)^2 - 2F(u(x)) - \omega u(x)^2 = d \]
for some constant \(d\). On the left side we have a sum of \(L^1\) functions. Therefore, \(d = 0\). Integrating on \(\mathbb{R}\), we obtain
\[ \int_{-\infty}^{+\infty} |u'(x)|^2 dx - 2 \int_{-\infty}^{+\infty} F(u(x))dx - \omega \lambda = 0. \]

Since \(u\) is a minimum, the equality above becomes
\[ 2 \left( \int_{-\infty}^{+\infty} |u'(x)|^2 dx - I(\lambda) \right) = \omega \lambda. \]

From (iii) of Proposition 2.2, the left term is non-negative. Then \(\omega > 0\). \(\square\)

**Remark 2.1.** The critical case \(G(s) = \alpha s^6\) has been already ruled out by the assumption (G2). In this case, a minimum does not exist. On the contrary, the rescaling
\[ u_\eta = \eta^{1/2} u(\eta x) \]
gives \( E(u_\eta) = \eta^2 E(u) = \eta^2 I(\lambda) \). Therefore, \( I(\lambda) = 0 \) unless \( E \) is unbounded from below. By (iii) of Proposition 2.2, a minimum does not exist.

We conclude this section by showing general properties satisfied by minima of \( E \) over \( S(\lambda) \).

**Lemma 2.4.** Let \( u \) be a minimum of \( E \) over \( S(\lambda) \). Then \( R(x) := |u(x)| \) satisfies the following properties:

(i) \( \lim_{|x| \to +\infty} R(x) = 0 \)

(ii) \( R \) is symmetrically decreasing with respect to a point of \( \mathbb{R} \)

(iii) \( R \) is positive

(iv) there exists \( z \) such that \( |z| = 1 \) and \( u(x) = z R(x) \) for every \( x \in \mathbb{R} \).

**Proof.** Clearly, \( R \) is in \( S(\lambda) \). From the equality \( F(s) = F(|s|) \) and the Convex Inequality for the Gradient, \[13,\] Theorem 7.8], there holds \( E(u) \geq E(R) \). Since \( u \) is a minimum, necessarily

\[
(2.12) \quad E(u) = E(R).
\]

Thus, \( R \) is solution to \[(1.3)\] for some \( \omega \). Since \( R \) is \( H^1 \) it is also \( L^\infty \). From \[(2.10)\] and the continuity of \( F \), the function \( |R'| \) is bounded. Since \( R \in L^2 \), we obtain

\[
(2.13) \quad R(x) \to 0 \text{ as } |x| \to +\infty
\]

which is the condition \([5, 6.1]\) (following their notation \( G'(s) \) should be replaced with \( -G'(s) - \omega s \)). By \([5,\) (ii), Proof of Theorem 5], \( R \) is positive; by \([5,\) (i,iv), Proof of Theorem 5], \( R \) is a symmetrically decreasing function with respect to a point in \( y \) in \( \mathbb{R} \). Then, \( R' \) converges to zero as well. So, we proved (i,ii) and (iii).

(iv). From \[(2.12)\] and \( F(s) = F(|s|) \), there also holds \( \|u'\|_{L^2}^2 = \|u'\|_{L^2}^2 \). Since \( R > 0 \), by \([10,\) Lemma 5.1], there exists a complex number \( z \) such that \( |z| = 1 \) and \( u(x) = z |u(x)| \) for every \( x \). \(\square\)

### 3. Non-degeneracy of the minima on \( H^1_+(\mathbb{R}) \)

In this section we prove the non-degeneracy of the functional \( E \) when restricted to the sub-manifold \( S(\lambda) \cap H^1_+(\mathbb{R}) \) on minima. We need the notation

\[
Q(\omega, s) := \omega s^2 + 2G(s).
\]

We have

\[
(3.2) \quad R_\omega(\omega) = \inf \{ s > 0 \mid Q(\omega, s) = 0 \}
\]

where \( R_\omega(\omega) \) has been defined in \[(1.5)\].

**Remark 3.1.** If \((G2)\) holds, then \( R_\omega \) is a positive non-decreasing function defined on \((0, +\infty)\).

Let \( R_0 \) be an element of \( G_\lambda \cap H^1_+(\mathbb{R}) \). Then, there exists \( \omega_0 \) such that

\[
(3.3) \quad R''_0(x) = G'(R_0(x)) + \omega_0 R_0(x).
\]
By (11) of Lemma 2.4, \( R_0 \) converges to zero and then satisfies the condition [5, 6, 1]. Then, by (iii) of [5, Theorem 5],

\[
R_0(0) = R_*(\omega_0), \quad \partial_s Q(\omega_0, R_*(\omega_0)) < 0.
\]

By the Implicit Function Theorem, there exists \( \varepsilon_0 > 0 \) such that \( R_* \) is continuously differentiable on \((\omega_0 - \varepsilon_0, \omega_0 + \varepsilon_0)\) and

\[
\partial_s Q(\omega, R_*(\omega)) < 0.
\]

Also, since \( \omega_0 > 0 \), by Proposition 2.3 and 2.12, on this interval \( \omega > 0 \). We consider the solution of the initial value problem

\[
R_\omega(x) = G(\omega_\omega(x)) + \omega R_\omega(x), \quad R_\omega(0) = 0, \quad R_\omega(0) = R_*(\omega).
\]

From [5, Theorem 5], \( R_\omega \) converges to zero and, by [5, Remark 6.3] and the fact that \( \omega > 0 \), we obtain \( R_\omega \in H^1 \). Since \( Q(\omega, R_*(\omega)) = 0 \), differentiating with respect to \( \omega \), we obtain

\[
(2G'(R_*(\omega)) + 2\omega R_*(\omega))R_*'(\omega) + R_*^2(\omega) = 0.
\]

We define

\[
\lambda: (\omega_0 - \varepsilon_0, \omega_0 + \varepsilon_0) \to \mathbb{R}, \quad \lambda(\omega) := \|R_\omega\|^2_{L^2}.
\]

\[
\omega R_*(\omega)^2 + 2G(R_*(\omega)) = 0.
\]

**Lemma 3.1.** \( \lambda'(\omega) \geq 0 \) and \( \lambda'(\omega_0) > 0 \), provided \( (44) \) holds.

**Proof.** From (iv) of [5, Theorem 5], \( R_\omega \) is a strictly decreasing function on \(|x|\). Then, since \( R_\omega \) is real valued, from (2.10) we have

\[
R_\omega'(x)^2 = \omega R_\omega(x)^2 + 2G(R_\omega(x))
\]

and then

\[
R_\omega'(x) = -\sqrt{\omega R_\omega(x)^2 + 2G(R_\omega(x))}.
\]

We can write

\[
\lambda = 2 \int_0^\infty R_\omega(x)^2 dx = 2 \int_0^\infty \frac{R_\omega(x)^2}{-\sqrt{\omega R_\omega(x)^2 + 2G(R_\omega(x))}} R_\omega'(x) dx
\]

\[
= 2 \int_0^{R_*'(\omega)} \rho^2 d\rho \sqrt{\omega \rho^2 + 2G(\rho)} = 2 \int_0^1 \frac{\theta^2 d\theta}{\sqrt{\Psi(\theta, R_*(\omega), \omega)}}.
\]

The third and the fourth equalities follow from the substitutions \( \rho = R(x) \) and \( \rho = R_*(\omega) \), and

\[
\Psi(\theta, s, \omega) = \omega \theta^2 s^{-4} + 2s^{-6}G(s\theta).
\]

We prove that the function

\[
\omega \to \Psi(\theta, R_*(\omega), \omega)
\]

is non-increasing in \( \omega \). Then, we have to check that

\[
\partial_s \Psi(\theta, R_*(\omega), \omega) \leq 0.
\]

In turn, the derivative above is equal to

\[
(R_*(\omega))^{-4} \theta^2 + \left[-4\omega \theta^2 (R_*(\omega))^{-5} - 12(R_*(\omega))^{-7} G(R_*(\omega)) \theta + 2(R_*(\omega))^{-6} \theta G'(R_*(\omega))\right] R_*'(\omega).
\]
From Remark 3.1 the term

\[ R_*(\omega)^7 (R'_*(\omega))^{-1} \]

is positive. Then, dividing \( \partial_\omega \Psi \) by that term and using the relation (3.6), we obtain

\[
I(\omega, \theta) = - R_*(\omega) \theta^2 \left[ 2\omega R_*(\omega) - 2 G'(R_*(\omega)) \right] - 4 \omega \theta^2 (R_*(\omega))^2 \\
- 12 G(R_*(\omega) \theta) + 2 R_*(\omega) \theta G'(R_*(\omega) \theta)
\]

so using (3.7) we see that

\[
I(\omega, \theta) = 12 \theta^2 G(R_*(\omega)) - 2 \theta^2 R_*(\omega) G'(R_*(\omega)) \\
+ 12 G(R_*(\omega) \theta) - 2 R_*(\omega) \theta G'(R_*(\omega) \theta).
\]

Setting

\[ H(s) := -6 G(s) + s G'(s), \]

we obtain

\[
I(\omega, \theta) = 2H(R_*(\omega) \theta) - 2 \theta^2 H(R_*(\omega)) \\
= 2 \theta^2 R_*(\omega)^2 \left( \frac{H(R_*(\omega) \theta)}{R_*(\omega)^2 \theta^2} - \frac{H(R_*(\omega))}{R_*(\omega)^2} \right).
\]

Now we prove that the function \( H(s)/s^2 \) is monotonically non-decreasing on the interval \((0, R_*(\omega))\). Equivalently, we need to check that

\[ H's - 2H = 12G(s) - 7sG'(s) + s^2G''(s) \geq 0. \]

If we require (3.3), the inequality holds. Moreover, \( I(\omega, 1) = 0 \). Then, for every \( 0 \leq \theta \leq 1 \), we have \( I(\omega, \theta) \leq 0 \). In conclusion,

\[
\frac{d\lambda}{d\omega} = - \int_0^1 \frac{\theta^2 \partial_\omega (\Psi(\theta, R_*(\omega), \omega))d\theta}{(\Psi(\theta, R_*(\omega), \omega))^3/2} \geq 0.
\]

We are now able to prove that \( \lambda'(\omega_0) > 0 \). On the contrary,

\[
\partial_\omega (\Psi(\theta, R_*(\omega_0), \omega_0)) = 0
\]

for every \( 0 < \theta < 1 \), and the same applies to \( I \). Therefore,

\[ 12G(s) - 7sG'(s) + s^2G''(s) = 0 \text{ for every } s \in (0, R_*(\omega_0)). \]

Then \( G(s) = as^6 \) on \((0, R_*(\omega_0))\). By [5, Theorem 5], there is only one solution to (3.3) which is positive and converges to zero at infinity. Then \( R_{\omega_0}(0) = R_0(0) \), so the image of \( R_0 \) is contained in a set of \( \mathbb{R} \) where \( G \) is a pure-power critical non-linearity. From Remark 2.1 \( \mathcal{G}_\lambda \) is empty, giving a contradiction. \( \square \)

We wish to evaluate the Hessian operator of \( E \) at the critical point \( R_0 \), in a vector of the tangent space of \( R_0 \)

\[
(3.9) \quad TR_0(S_*(\lambda)) = \{ v \in H^1_r(\mathbb{R}) \mid (v, R)_{L^2} = 0 \}.
\]

We consider a curve in \( S_*(\lambda) \) as in

\[
(3.10) \quad u(t) = R + tv + \alpha(t) R.
\]

By the Implicit Function Theorem, there exists \( \delta > 0 \) and \( \alpha : (-\delta, \delta) \to \mathbb{R} \) such that

\[ M(R + tv + \alpha(t) R) = \lambda. \]
From the Taylor expansion of $M$ we get
\[ \alpha(t) = \alpha_0 t^2 + o(t^2), \quad \alpha_0 = -\frac{\|v\|^2_{L^2}}{2\lambda}, \]
and from the expansion of $E(u(t))$ we get
\[ 2E(u(t)) = \|R_0\|_{L^2}^2 + 2t(R_0, v')_{L^2} + t^2 \left( \|v'\|^2 + 2\alpha_0(R_0, v')_{L^2} \right) + o(t^2) \]
so using (3.5) and (3.9) we find
\[ 2E(u(t)) = 2E(R_0)v^2 \left( \|v'\|^2 + ((G''(R_0) + \omega R_0)v, v)_{L^2} \right) + o(t^2). \]
Therefore,
\[ D^2E(R_0)[v, v] = (E \circ u)''(0) \]
\[ = \int_{-\infty}^{+\infty} (|v'(x)|^2 + (G''(R_0(x)) + \omega_0 v(x))^2) dx =: \xi(v). \]
In order to show that $R_0$ is non-degenerate, we have to prove that the infimum of $\xi$ is positive $TR_0(S_r(\lambda)) \cap S(1)$. If $v$ is $H^2(\mathbb{R})$, then
\[ \xi(v) = (L_+ v, v)_{L^2}, \quad L_+(v) := v'' - G''(R_0)v - \omega_0 v. \]

**Proof of Theorem 3.2** Since $R_0$ is a minimum, $\xi(v) \geq 0$. The infimum of $\xi$ achieved. A proof of this can be found in [19, Proposition 2.10]. Suppose that the infimum is achieved and that $\xi(v) = 0$. Then, $v$ is $H^2$ and satisfies
\[ L_+(v) = \beta R_0 \]
for some $\beta \in \mathbb{R}$ with $\beta \neq 0$. Taking the derivative with respect to $\omega$ of (3.5), and evaluating at $\omega = \omega_0$, we obtain
\[ (3.11) \quad L_+(\partial_\omega R(\omega_0, \cdot)) = R_0. \]
Then $y := \beta \omega_0 + v$ solves the differential equation $L_+(y) = 0$. The kernel of the operator $L_+$ is generated by $R_0$, which is an odd function. Since $y$ is even, we obtain $y = 0$. Since $\beta \neq 0$,
\[ (L_+(\partial_\omega R(\omega_0, \cdot)), \partial_\omega R(\omega_0, \cdot))_{L^2} = 0 \]
However, from (3.11) and the definition $\lambda$ given in Lemma 3.1 we have
\[ 0 = (R_0, \partial_\omega R(\omega_0, \cdot))_{L^2} = \lambda'(\omega_0) \]
which gives a contradiction with the lemma. \hfill \Box

**Corollary 3.1.** The set $G_\lambda \cap H^1_0(\mathbb{R})$ is finite.

**Proof.** Let $(R_n) \subseteq G_\lambda \cap H^1_0(\mathbb{R}; \mathbb{R})$ be a sequence of minima. By Lemma 2.3 up to extract a subsequence, we can suppose that $R_n(\cdot) + y_n$ converges in $H^1$, for some sequence $(y_n) \subseteq \mathbb{R}$. By [3, Theorem 5], $R_n$ is symmetric and radially decreasing with respect to the origin. Therefore,
\[ \|R_n - R_m\|^2_{L^2} \leq \|R_n(\cdot) + y_n) - R_m(\cdot) + y_m)\|^2_{L^2}. \]
Then $(R_n)$ is a Cauchy sequence and there exists $R_0$ such that $R_n \to R_0$ in $L^2$. By Lemma 2.2, the convergence is strong in $H^1$, which contradicts the fact that $R_0$ is non-degenerate, thus isolated, minimum. \hfill \Box

**Proposition 3.1.** As $u$ varies in $G_\lambda$, there are finitely many $G_\lambda(u)$.
Proof. By (ii) and (iii) of Lemma 2.4 there exists \( y \in \mathbb{R} \) and a complex number \(|z| = 1 \) such that \( u(x) = zR(x + y) \), where \( R \in \mathcal{G}_\lambda \cap H^1_{r,+}(\mathbb{R}) \). Therefore, there are as many different \( \mathcal{G}_\lambda(u) \) as \# \( \mathcal{G}_\lambda \cap H^1_{r,+}(\mathbb{R}) \). \( \square \)

4. Stability of \( \mathcal{G}_\lambda \) and \( \mathcal{G}_\lambda(u) \)

According to [9, Theorem 3.5.1, p. 77] the equation (NLS) is locally well posed in \( H^1(\mathbb{R}; \mathbb{C}) \). That is, given \( u \in H^1(\mathbb{R}; \mathbb{C}) \), there exists a map

\[
U \in C^1([0, +\infty); L^2(\mathbb{R}; \mathbb{C})) \cap C([0, +\infty); H^1(\mathbb{R}; \mathbb{C}))
\]

such that \( U_0 = u \) and \( \phi(t, \cdot) := U_t(u) \) is a solution to (NLS). We briefly check that \( G' \) satisfies the the condition of [9, Example 3.2.4, p. 59]: since \( G'' \) is continuous,

\[
|G'(s_1) - G'(s_2)| \leq L(K)|s_1 - s_2|, \quad L(K) := \sup_{[0,K]} |G''|
\]

for every \( s_1, s_2 \in [0, K] \). And the function \( L \) is continuous because \( G'' \) is continuous. The global well-posedness follows from the apriori estimates that one can derive from (iii) of Proposition 2.1.

Proof of Theorem A. The proof of the stability is made with a contradiction argument: let \( (u_n), \varepsilon_0 > 0 \) and \( (t_n) \) be such that

\[
dist(u_n, \mathcal{G}_\lambda) \to 0, \quad \text{dist}(U_{t_n}(u_n), \mathcal{G}_\lambda) \geq \varepsilon_0.
\]

Since \( E \) and \( M \) are continuous functions, and constant on the orbits \( U_t(u_n) \),

\[
E(U_{t_n}(u_n)) \to I(\lambda), \quad M(u_n) \to \lambda.
\]

We set \( v_n := U_{t_n}(u_n) \). From Lemma 2.3, there exists a subsequence \( u_{n_k} \), a sequence \( (y_k) \) and \( u \in S(\lambda) \) such that

\[
\|u_{n_k}(\cdot + y_k) - u\|_{H^1(\mathbb{R}; \mathbb{C})} \to 0
\]

implying

\[
\|u_{n_k} - u(\cdot - y_k)\|_{H^1(\mathbb{R}; \mathbb{C})} \to 0
\]

and giving a contradiction with (4.2). \( \square \)

Proof of Theorem C. Stability of \( \mathcal{G}_\lambda(u) \). By Proposition 3.1

\[
\mathcal{G}_\lambda = \bigcup_{i=1}^n \mathcal{G}_\lambda(R_i).
\]

We prove that

\[
\text{dist}(\mathcal{G}_\lambda(R_h), \mathcal{G}_\lambda(R_k)) > 0 \quad \text{for} \quad h \neq k.
\]

In fact, the distance between two arbitrary points in the two sets is

\[
\text{dist}(z_1R_h(\cdot + y_1), z_2R_k(\cdot + y_2)) = \text{dist}(R_h, zR_k(\cdot + y)) \geq \text{dist}(R_h, R_k) > 0,
\]

where \( z = \overline{z}_1z_2 \) and \( y := y_2 - y_1 \). The first inequality follows the fact that both \( R_i \) are symmetrically decreasing with respect to the origin, from (ii) of Lemma 2.4. Then

\[
d := \inf_{h \neq k} \text{dist}(\mathcal{G}_\lambda(R_h), \mathcal{G}_\lambda(R_k)) > 0.
\]
Now we prove that $\mathcal{G}_\lambda(R_i)$ is stable. Let $\delta > 0$ be such that

\begin{equation}
B(R_i, \delta) \cap \mathcal{G}_\lambda \cap H^1_R = \{R_i\}, \quad \delta < \frac{d}{3}.
\end{equation}

We define

\[ E_\delta^i := \inf_{B(\mathcal{G}_\lambda(R_i), \delta)} E \]

where the metric restricted on $S(\lambda)$. We claim that

\begin{equation}
E_i^\delta > I(\lambda).
\end{equation}

Otherwise, we would have a sequence $(u_n)$ such that $E(u_n) \to I(\lambda)$, $M(u_n) = \lambda$.

By Lemma 2.3, there exists a subsequence $(u_{n_k})$, $u_{n_k}$ in $S(\lambda)$ and $(y_k)$ such that

\begin{equation}
(\mathcal{G}_\lambda(R_i)(\cdot + y_k) \to u \in \mathcal{G}_\lambda.
\end{equation}

From (4.4) and the choice of $\delta$, it follows that $u$ is in $\mathcal{G}_\lambda(R_i)$. However, since $\text{dist}(u_{n_k}(\cdot + y_k), \mathcal{G}_\lambda(R_i)) = \delta$ there also hold $\text{dist}(u, \mathcal{G}_\lambda(R_i)) = \delta$, giving a contradiction with (4.6). We are now able to prove that $\mathcal{G}_\lambda(R_i)$ is stable; again, we use a contradiction argument. Let $(u_n)$, $(t_n)$ and $\varepsilon_0 > 0$ be such that

\begin{equation}
\text{dist}(u_n, \mathcal{G}_\lambda(R_i)) \to 0, \quad \text{dist}(U_{t_n}(u_n), \mathcal{G}_\lambda(R_i)) \geq \varepsilon_0.
\end{equation}

We set $v_n := U_{t_n}(u_n)$. Since $\mathcal{G}_\lambda$ is stable, there exists $k$ such that

\begin{equation}
\text{dist}(v_n, \mathcal{G}_\lambda(R_k)) \to 0, \quad \mathcal{G}_\lambda(R_i) \neq \mathcal{G}_\lambda(R_k).
\end{equation}

Let $n_0$ be such that

\begin{equation}
\max\{\text{dist}(u_n, \mathcal{G}_\lambda(R_i)), \text{dist}(v_n, \mathcal{G}_\lambda(R_k))\} < \delta
\end{equation}

and

\begin{equation}
E(u_n) < \min\{E_\delta^i \mid 1 \leq i\}
\end{equation}

for every $n \geq n_0$. Along the curve

\[ \alpha: [0, t_{n_0}] \to H^1(\mathbb{R}; C), \quad \alpha(t) = U_t(u_{n_0}) \]

the quantities $E$ and $M$ are constant, while the function

\[ \beta: \mathbb{R} \to \mathbb{R}, \quad \beta(t) := \text{dist}(U_t(u_{n_0}), \mathcal{G}_\lambda(R_i)) \]

is continuous, from (4.7). From (4.7) and (4.8), we have

\[ \beta(0) < \delta, \quad \beta(t_{n_0}) \geq 2\delta. \]

Therefore, there exists $t_*$ such that

\[ \beta(t_*) = \delta. \]

Then,

\[ E(\alpha(t_*)) \geq E_\delta^i. \]

However from the conservation of $E$ and (4.5)

\[ E(\alpha(t_*)) = E(\alpha(0)) < E_\delta^i. \]

And from (4.5), we obtain a contradiction. □
5. Uniqueness

We assume that (G1-5) hold. We fix \( \lambda > 0 \).

*Proof of Theorem D.* Let \( R_0 \) and \( R_1 \) be two positive functions in \( G_\lambda \cap H^1_\lambda \). The set \( \mathcal{A} \) introduced in (5) is the set of \( \omega \) such that a solution to (1.3) exists. If \( \mathcal{A} \) is connected, then the function \( R_* \) defined on \( (\omega_0 - \varepsilon_0, \omega_0 + \varepsilon_0) \), in (3.4), can be extended to \( \mathcal{A} \), so the function \( \lambda \). Let \( \omega_1 \) be the Lagrange multiplier associated to \( R_1 \). Then \( \omega_1 \in A \). Since \( R_0 \) and \( R_1 \) belong to the same constraint,

\[
\lambda(\omega_0) = \lambda(\omega_1).
\]

By Lemma 3.1 \( \lambda' \geq 0 \) on \( [\omega_0, \omega_1] \). Then \( \lambda'(\omega) = 0 \) on the whole interval. Then \( \lambda'(\omega_0) = 0 \) giving a contradiction with Lemma 3.1. Hence \( \omega_0 = \omega_1 \) and \( R_0 \) and \( R_1 \) solve the same initial value problem (3.5). Then \( R_0 = R_1 =: R_+ \). The other solution is \( R_- := -R_+ \).

**Corollary 5.1.** If (G1-5) hold, then \( G_\lambda = G_\lambda(u) \) for every \( u \in G_\lambda \).

*Proof.* We prove that an arbitrary \( v \in G_\lambda \) belongs to \( G_\lambda \). In fact, by (16) of Lemma 2.4 there are two complex numbers \( z, w \in \mathbb{C} \) such that \( |z| = |w| = 1 \) and

\[
v(x) = zR_1(x), \quad u(x) = wR_2(x)
\]

where \( R_1, R_2 \in G_\lambda \cap H^1_\lambda \) and symmetric with respect to two points \( y_1 \) and \( y_2 \), respectively, by (11) of Lemma 2.4. Then \( R_1(\cdot - y_1) \) and \( R_2(\cdot - y_2) \) are two positive solutions in \( G_\lambda \cap H^1_\lambda \). By Theorem D

\[
R_1(\cdot - y_1) = R_2(\cdot - y_2).
\]

Then

\[
v(x) = zR_1(x) = zR_2(x - y_2 + y_1) = w^{-1}zwR_2(x - y_2 + y_1) = w^{-1}zu(x - y)
\]

where \( y := y_1 - y_2 \). Then \( v \in G_\lambda(u) \). \( \square \)

6. The combined power-type case

An example of non-linearity \( G \) satisfying all the assumptions (G1-G5) is

\[
G(s) := -a|s|^p + b|s|^q, \quad 2 < p < 6, \quad 2 < q
\]

with \( c_1, c_2 > 0 \). Regularity and power-type estimate assumptions contained in (G1), (G2) and (G3).

(G3) is satisfied. Let \( s_0 > 0 \) be the zero of the function \( Q_\omega \) such that \( Q_\omega(s_0) = 0 \) and \( Q'_\omega(s_0) < 0 \). First, we prove that \( L(s_0) \geq 0 \). In fact, the two conditions on \( s_0 \) give

\[
\omega - c_1s_0^{p-2} + c_2s_0^{q-2} = 0, \quad 2\omega - pc_1s_0^{p-2} + qc_2s_0^{q-2} < 0.
\]

A substitution yields

\[
c_1(p-2)s_0^{p-2} - c_2(q-2)s_0^{q-2} > 0.
\]
Then
\[ L(s_0) = c_1(p - 2)(6 - p)s_0^p - c_2(q - 2)(6 - q)s_0^q > c_2(q - 2)(6 - p)s_0^p - c_2(q - 2)(6 - q)s_0^q = c_2(q - 2)(q - p)s_0^{p-2} > 0. \]

We can show that \( L \) is non-negative on the interval \((0, R_*(\omega))\). Let \( s < R_*(\omega) \) be such that \( L(s) < 0 \). Then \( L(R_*(\omega)) < 0 \), because \( L \) has only one zero on \((0, +\infty)\). By definition of \( R_*(\omega) \), we have \( Q_*(R_*(\omega)) = 0 \) and \( Q'_*(R_*(\omega)) < 0 \), which implies \( L(R_*(\omega)) > 0 \) and gives a contradiction.

Then we choose \( R \) is satisfied. Let \( s_1 \) be the unique local maximum of \( G \). Then \( A = H((0, s_1)) \), thus connected. Therefore, from Theorem D when \( G \) is a combined power pure-power non-linearity, there exists only one positive function \( R \in G_0 \cap H^1 \).

**APPENDIX**

We show that a function satisfied a combined power-type estimate can be written as sum of two functions satisfying a power-type estimate. As a consequence, we can suppose that \( G \) satisfies
\[ |G'(s)| \leq c|s|^{p-1}, \quad G(0) = 0 \]
in place of (G2), and
\[ |G''(s)| \leq c|s|^{p-2} \]
in place of (G4).

**Proposition 6.1.** Let \( G \) be a function satisfying (G2). Then, there are two functions \( C^1 \) functions \( G_1 \) and \( G_2 \) and \( c \in \mathbb{R} \) such that
\[ |G_1'(s)| \leq c|s|^{p-1}, \quad |G_2'(s)| \leq c|s|^{q-1}, \quad G = G_1 + G_2. \]

If \( G \) satisfies (G4) as well, then \( G_1, G_2 \) and \( c \) can be chosen in such a way that the inequalities
\[ |G_1''(s)| \leq c|s|^{p-2}, \quad |G_2''(s)| \leq c|s|^{q-2}. \]
are also satisfied.

**Proof.** In both cases, the function can be obtained as follows: we consider a non-negative continuous function \( \sigma \) such that
\[ \sigma = \begin{cases} 1 & \text{on } [-1, 1] \\ 0 & \text{on } (-\infty, -2) \cup [2, +\infty) \end{cases} \]
and \( 0 \leq \sigma \leq 1, \ |\sigma'| \leq 2 \). Then we choose \( G_1 \) and \( G_2 \) as the unique functions such that \( G_1' = \sigma G'_1, \ G_2' = (1 - \sigma)G'_2 \) and \( c = 2^r - p + 1 \). If (G4) holds, \( G_1'' = \sigma G''_1 \) and \( G_2'' = (1 - \sigma)G''_2 \), while \( c \) does not change. \( \square \)

The next proposition is about the regularity of \( E \). The gradient part of \( E \) is smooth; the regularity of the non-linear part it is obtained with the same techniques used by A. Ambrosetti and G. Prodi in [1, Theorem 2.2]. We include the details of this proof in view of slight differences with the quoted
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where $\mathbb{R}$ is replaced by a bounded domain $\Omega$, and the class of regularity $C^2(H^1(\mathbb{R}; \mathbb{C}), \mathbb{R})$ is replaced by $C(L^p(\Omega), L^q(\Omega))$.

The regularity of $E$ depends on the regularity of $F$ and the power-type estimates which, in turn, are related to the estimates of $G$ (6.1) and (6.2).

We identify $\mathbb{C}$ with $\mathbb{R}^2$. If $G$ is derivable, then

$$F(s) = G(|s|)$$

$$F'(s) = G'(|s|)\frac{s}{|s|}, \quad F'(0) = 0$$

for every $s \in \mathbb{C} - \{0\}$. If $G$ is two-times derivable,

$$\partial^2_{ij} F(s) = G''(|s|)\frac{s_i s_j}{|s|^2} + G'(|s|)\frac{\delta_{ij}|s|^2 - s_i s_j}{|s|^3}, \quad \partial^2_{ij} F(0) = 0$$

Moreover,

$$(\partial^2_{ij} F)(s)^2 = G''(s)^2 \frac{s_i s_j}{|s|^2} + G'(s)^2 \left( \frac{\delta_{ij}|s|^2 - s_i s_j}{|s|^3} \right)^2$$

$$+ 2G''(s)G'(s) \frac{s_i s_j (\delta_{ij}|s|^2 - s_j s_j)}{|s|^5}.$$
We set
\[ h^*(x) := \int_0^1 |F'(u_0(x) + th(x)) - F'(u_0(x))|dt \]

From (6.3) and (6.5) there exists \( C_1 = C_1(C, p) \) such that
\[ |F'(u_0 + th) - F'(u_0)| \leq C_1(|u_0|^{p-1} + t^{p-1}|h|^{p-1}) \]

After the integration on the interval \([0, 1]\), we obtain
\[ h^*(x)^{p/(p-1)} \leq C_2 \left(|u_0(x)|^p + |h(x)|^p\right). \]

We prove that \( A = o(\|h\|) \) with a contradiction argument. If it is false, there exists \( \varepsilon > 0 \) and a sequence \((h_n)\) converging to zero in \( H^1 \) such that
\[ \tag{6.6} |A| \geq \varepsilon \|h_n\|. \]

Up to extract a subsequence, we can suppose that \( |h_n| \) converges to zero pointwise and it is dominated by an \( H^1 \) function \( h_0 \). Then
\[ h_n^*(x)^{p/(p-1)} \leq C_2(|u_0(x)|^p + h_0(x)^p). \]

We have a dominated sequence in \( L^1(\mathbb{R}) \) converging pointwise a.e. to zero. Therefore,
\[ \int_{-\infty}^{+\infty} |h_n^*(x)|^{p/(p-1)}dx \to 0. \]

From the Hölder inequality (with exponents \( p \) and \( p/(p-1) \) and functions \( h_n \) and \( h_n^* \)), and (2.4),
\[ |A| \leq C_2\|h_n\|_p \int_{-\infty}^{+\infty} |h_n^*(x)|^{p/(p-1)} \leq C_3\|h_n\|_H^1 \int_{-\infty}^{+\infty} |h_n^*(x)|^{p/(p-1)}. \]

Then \( |A| \leq \|h_n\|o(1) \), giving a contradiction with (6.6). Therefore, \( E \) is differentiable and
\[ \langle dE(u_0), k \rangle = \int_{-\infty}^{+\infty} F'(u_0(x)) \cdot k(x)dx. \]

(ii). If (6.1) and (6.2) hold, then \( E \) is two-times differentiable. We set
\[ B := dE(u_0 + h) - dE(u_0) - \int_{-\infty}^{+\infty} F''(u_0(x))h(x)dx \]

and prove that \( B = o(\|h\|_{H^1}) \). In fact,
\[ B = \int_{-\infty}^{+\infty} \int_0^1 \left(F''(u_0(x) + th(x)) - F''(u_0(x))\right)h(x)dt. \]

Let \( k \) be an arbitrary vector of \( H^1 \). Then
\[ |\langle B, k \rangle| \leq \int_{-\infty}^{+\infty} |h(x)||k(x)||h^*(x)|dx \]

where
\[ h^*(x) := \int_0^1 \|F''(u_0(x) + th(x)) - F''(u_0(x))\|_{L(\mathbb{R}^2)}dt. \]

From (6.3) and (6.5), there exists \( C_1' = C_1'(C, p) \) such that
\[ \|F''(u_0 + th) - F''(u_0)\|_{L(\mathbb{R}^2)} \leq C_1'(|u_0(x)|^{p-2} + t^{p-2}|h(x)|^{p-2}). \]
After the integration on $[0,1]$, from (6.5) we obtain
\[ h^{**}(x)^{p/(p-2)} \leq C'_2 (|u_0(x)|^p + |h(x)|^p). \]

Now, suppose that there exists $\varepsilon_0 > 0$ and a dominated sequence $(h_n)$ converging in $H^1$, pointwise a.e. to zero, such that
\[ \|B\|_{L(H^1,\mathbb{R})} \geq \varepsilon_0 \|h_n\|_{H^1}. \]

By the Hölder inequality and (2.1)
\[ |\langle B, k \rangle| \leq C'_2 \|h_n\|_p \|k\|_p \|h^{**}_n\|_{p/(p-2)} \leq C'_3 \|h_n\| \|k\| \|h^{**}_n\|_{p/(p-2)} \]
for every $k \in H^1$. Then,
\[ \|B\|_{L(H^1,\mathbb{R})} \leq C'_3 \|h_n\| \|h^{**}_n\|_{p/(p-2)} = \|h_n\| o(1) \]
which gives a contradiction with (6.7). \qed

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