Finite-element quantum electrodynamics. II. Lattice propagators, current commutators, and axial-vector anomalies

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Abstract

We apply the finite-element lattice equations of motion for quantum electrodynamics given in the first paper in this series to examine anomalies in the current operators. By taking explicit lattice divergences of the vector and axial-vector currents we compute the vector and axial-vector anomalies in two and four dimensions. We examine anomalous commutators of the currents to compute divergent and finite Schwinger terms. And, using free lattice propagators, we compute the vacuum polarization in two dimensions and hence the anomaly in the Schwinger model. A discussion of our choice of gauge-invariant current is provided.

11.15.Ha, 11.15.Tk, 12.20.Ds, 13.40.Fn

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I. INTRODUCTION

This paper summarizes the status of the finite-element approach to gauge theories, in which the Heisenberg operator equations of motion are converted to operator difference equations consistent with unitarity. For a review of the entire program, from quantum mechanics to quantum field theory, see Ref. \[1\]. The gauging of an Abelian theory on a finite-element lattice was first carried out in Ref. \[2\], while the generalization to a non-Abelian theory was given in Ref. \[3\]. The beginnings of constructing a canonical quantum electrodynamics along these lines were given in Ref. \[4\]. In particular, we showed there that, at least for background fields, the Dirac equation is unitary, in the sense that canonical anticommutation relations hold at each lattice time.

In this paper, we will first show, in Sec. \[II\], how to construct free lattice propagators for electrons and photons. We remind the reader how interactions are introduced, in Sec. \[III\], and then, using the gauge-invariant current, compute directly vector and axial-vector anomalies. Another anomalous aspect of the current operators is examined in Sec. \[IV\], where equal-time current commutators are computed, from which the divergent and finite Schwinger terms are computed. In Sec. \[V\] a lattice loop calculation is described. We compute the vacuum polarization in two dimensions using the electron propagator found in Sec. \[II\], and obtain results consistent with the known continuum Schwinger model. The continuing direction of our program is sketched in the concluding Sec. \[VI\]. In the Appendix we discuss a possible alternative definition of the current possible in Euclidean space, and show why such a current is unacceptable.
II. FREE DIRAC EQUATION. LATTICE PROPAGATORS

We begin by reminding the reader of the form of the free finite-element lattice Dirac equation

\[ \frac{i\gamma^0}{\hbar}(\psi_{m,n+1} - \psi_{m,n}) + \frac{i\gamma^j}{\Delta}(\psi_{m,j+1,m_{\perp},n} - \psi_{m,j,m_{\perp},n}) - \mu \psi_{m,n} = 0. \]  \hspace{1cm} (2.1)

Here \( \mu \) is the electron mass, \( \hbar \) is the temporal lattice spacing, \( \Delta \) is the spatial lattice spacing, \( m \) represents a spatial lattice coordinate, \( n \) a temporal coordinate, and overbars signify forward averaging:

\[ x_{\overline{m}} = \frac{1}{2}(x_{m+1} + x_m). \]  \hspace{1cm} (2.2)

By transforming this into momentum space we find that the momentum expansion of the canonical Dirac field is

\[ \psi_{m,n} = \sum_{s,p} \sqrt{\frac{\mu}{\omega}} (b_{p,s} u_{p,s} \lambda^{-n} e^{ip \cdot m_{\perp}/M} + d_{p,s}^\dagger v_{p,s} \lambda^n e^{-ip \cdot m_{\perp}/M}), \]  \hspace{1cm} (2.3)

where the spinors are normalized according to

\[ \sum_s \tilde{u}_\pm \tilde{u}_\pm^\dagger \gamma^0 = \pm \frac{\mu \pm \gamma \cdot \tilde{p}}{2\mu} \equiv \pm \Lambda_\pm, \]  \hspace{1cm} (2.4)

where \( u = \tilde{u}_- \) and \( v = \tilde{u}_+ \), and where

\[ \tilde{p} = \frac{2t}{\Delta}, \quad \omega = \tilde{p}^0 = \sqrt{\tilde{p}^2 + \mu^2}, \quad (t_p)_i = \tan p_i \pi / M. \]  \hspace{1cm} (2.5)

In (2.3) \( \lambda \) is the eigenvalue of the Dirac transfer matrix

\[ \lambda = \frac{1 + i\hbar \omega / 2}{1 - i\hbar \omega / 2} \equiv e^{i\Omega(h)\hbar}. \]  \hspace{1cm} (2.6)

Notice that we may solve (2.6) for \( \omega \):

\[ \begin{align*}
\text{In (2.3) } \lambda \text{ is the eigenvalue of the Dirac transfer matrix } \\
&\lambda = \frac{1 + i\hbar \omega / 2}{1 - i\hbar \omega / 2} \equiv e^{i\Omega(h)\hbar}. \\
&\text{Notice that we may solve (2.6) for } \omega: \\
&\text{1Note that we use a different sign of } \mu \text{ than in Ref. [4], and correspondingly, a different definition of spinors in (2.4) and a change of sign in the unitary time evolution operator } U, (3.2). \end{align*} \]
\[ \omega = \frac{2}{\hbar} \tan \frac{\hbar \Omega}{2}. \quad (2.7) \]

We take \( M \), the number of lattice points in a given spatial direction, to be odd, so that \( \psi \) is periodic on the spatial lattice. It follows from (2.3) that the canonical lattice anticommutation relations for the Dirac fields

\[ \{ \psi_{\mathbf{m},n}, \psi^\dagger_{\mathbf{m}',n} \} = \frac{1}{\Delta^3} \delta_{\mathbf{m},\mathbf{m}'} \]  

are satisfied if \( (L = M \Delta, \text{the length of the spatial lattice}) \)

\[ \{ b_{\mathbf{p},s}, b^\dagger_{\mathbf{p}',s'} \} = \frac{1}{L^3} \delta_{\mathbf{p},\mathbf{p}'} \delta_{s,s'}, \quad (2.9a) \]
\[ \{ d_{\mathbf{p},s}, d^\dagger_{\mathbf{p}',s'} \} = \frac{1}{L^3} \delta_{\mathbf{p},\mathbf{p}'} \delta_{s,s'}, \quad (2.9b) \]

and all other anticommutators of these operators vanish.

At this point we can construct a free lattice Dirac Green’s function according to

\[ iG_{\mathbf{m},n;\mathbf{m}',n'} = \langle T(\psi_{\mathbf{m},n} \psi^\dagger_{\mathbf{m}',n'}) \rangle = \langle \psi_{\mathbf{m},n} \psi^\dagger_{\mathbf{m}',n'} \rangle \eta(n - n') - \langle \psi^\dagger_{\mathbf{m}',n'} \psi_{\mathbf{m},n} \rangle \eta(n' - n), \quad (2.10) \]

where the step function is defined by

\[ \eta(x) = \begin{cases} 
1, & x > 0, \\
1/2, & x = 0, \\
0, & x < 0. 
\end{cases} \quad (2.11) \]

Using the fact that \( b \) and \( d \) are annihilation operators, we easily see from the Fourier expansion (2.3) that the free Green’s function has the following momentum expansion:

\[ \delta_{n,n'} \frac{h}{2L^3} \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot (\mathbf{m} - \mathbf{m}')} 2\pi/M (\mu - \gamma \cdot \mathbf{p}) \tan \hbar \Omega/2. \]  

\(^2\text{Inclusion of the } x = 0 \text{ value of the step function eliminates the following nonlocal term from (2.14):} \]

\[ \delta_{n,n'} \frac{h}{2L^3} \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot (\mathbf{m} - \mathbf{m}')} 2\pi/M (\mu - \gamma \cdot \mathbf{p}) \tan \hbar \Omega/2. \]
To recast (2.12) in four-dimensional momentum space, we can use the identity for integer $p$

$$\oint_{\gamma} \frac{dz}{2\pi i} \frac{z^{p-1}}{z-1} = \begin{cases} 1, & p \geq 1, \\ 0, & p \leq 0, \end{cases}$$

(2.13)

where $\gamma$ is a contour encircling 0 and 1 in a positive sense. It is then a straightforward exercise to show that, apart from a contact term\[3\],

$$G_{m,n',n'} = h \frac{1}{L^3} \sum_{\mathbf{p}} e^{i \mathbf{p} \cdot (\mathbf{m} - \mathbf{m}')} e^{i \Omega (n - n')} \frac{1}{L^3} \sum_{\mathbf{p}} e^{i \mathbf{p} \cdot (\mathbf{m} - \mathbf{m}') 2\pi/M}$$

This expression has the expected continuum limit: as $h \to 0$ and $nh \to t$, (2.14) becomes

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\hat{\Omega} e^{-i\hat{\Omega}(t-t')} \frac{1}{L^3} \sum_{\mathbf{p}} e^{i \mathbf{p} \cdot (\mathbf{m} - \mathbf{m}') 2\pi/M} \frac{\gamma \cdot \hat{p} - \mu}{\hat{p}^2 + \mu^2 - i\epsilon},$$

(2.15)

where $\hat{p} = (\hat{\Omega}, \tilde{\mathbf{p}})$.

In a similar way we can work out the free lattice Green’s function for the photon. We first note, from the lattice versions of the Maxwell equations in the temporal gauge, $A_0 = 0$, in part,

$$\mathbf{E} = \hat{\mathbf{A}}, \quad \dot{\mathbf{E}} + \nabla \times \mathbf{B} = 0,$$

(2.16)

namely,

$$\mathbf{E}_{m,n} = \frac{1}{h} (\mathbf{A}_{m,n+1} - \mathbf{A}_{m,n}),$$

(2.17)

\[3\]This contact term is

$$\frac{h}{4L^3} \delta_{n,n'} \sum_{\mathbf{p}} e^{i \mathbf{p} \cdot (\mathbf{m} - \mathbf{m}) 2\pi/M} (\mu - \gamma \cdot \tilde{\mathbf{p}}),$$

and serves to convert, in the numerator, $\cos^2 h\Omega/2$ to $\cos^2 h\hat{\Omega}/2$. 

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and
\[
\frac{1}{\hbar}[(E_i)_{\mathbf{m},n+1} - (E_i)_{\mathbf{m},n}] + \frac{1}{\Delta} \epsilon_{ijk}[(B_k)_{m_j+1,\mathbf{m},\pi} - (B_k)_{m_j,\mathbf{m},\mathbf{m}}] = 0, \tag{2.18}
\]
that the momentum-space transfer matrix for the photon is
\[
T = e^{i\Omega h} \tag{2.19}
\]
where
\[
\omega = \frac{2t}{\Delta} = \frac{2}{h} \tan \frac{h\Omega}{2}, \tag{2.20}
\]
just as in (2.7). [Incidentally, notice that this form of the free photon transfer matrix establishes unitarity of the free lattice Maxwell equations.]

Now we compute the photon propagator defined by
\[
iD_{m,n;m',n'}^{ij} = \langle T(A_{\mathbf{m},n}^i A_{\mathbf{m}',n'}^j) \rangle = \langle A_{\mathbf{m},n}^i A_{\mathbf{m}',n'}^j \rangle \eta(n - n') + \langle A_{\mathbf{m}',n'}^j A_{\mathbf{m},n}^i \rangle \eta(n' - n), \tag{2.21}\]
which, on use of the canonical commutation relations for the photon creation and annihilation operators, defined through the momentum expansion of the photon fields, (2.17) of Ref. [4],
\[
[a_k^i, a_{k'}^j] = 0, \quad [a_k^i, a_{k'}^j] = \delta_{k,k'} f^{ij}(k), \tag{2.22}
\]
where the transverse projection operator is
\[
f^{ij}(k) = \delta_{ij} - \frac{(t_k)_i(t_k)_j}{(t_k)^2}, \tag{2.23}
\]
leads to
\[
iD_{m,n;m',n'}^{ij} = \eta(n - n') \frac{1}{L^3} \sum_k \frac{1}{2\omega_k} e^{-i\Omega h(n-n')} e^{i\mathbf{k} \cdot (\mathbf{m} - \mathbf{m}')} 2\pi/M f^{ij}(k)
+ \eta(n' - n) \frac{1}{L^3} \sum_k \frac{1}{2\omega_k} e^{i\Omega h(n-n')} e^{-i\mathbf{k} \cdot (\mathbf{m} - \mathbf{m}')} 2\pi/M f^{ij}(k). \tag{2.24}
\]
Use of the identity (2.13) leads to the four-dimensional form, apart from a contact term,
\[
D^{ij}_{m,n;m',n'} = \frac{\hbar^2}{4\pi} \int_{-\pi/\hbar}^{\pi/\hbar} d\hat{\Omega} e^{-i\hbar\hat{\Omega}(n-n')} \frac{1}{L^3} \sum_k e^{i\mathbf{k} \cdot (m-m')} e^{2\pi/M f^{ij}(k)} \times \frac{\cos^2 \hbar\hat{\Omega}/2}{\cos h(\hat{\Omega} - i\epsilon) - \cos h\hat{\Omega}},
\]

completely analogous to the electron Green’s function (2.14).

### III. INTERACTIONS. AXIAL-VECTOR ANOMALY

Interactions of an electron with a background electromagnetic field is given in terms of a transfer matrix \( T \):

\[
\psi_{n+1} = T_n \psi_n,
\]

which is to be understood as a matrix equation in \( \mathbf{m} \). Explicitly, in the gauge \( A^0 = 0 \),

\[
T = 2U^{-1} - 1, \quad U = 1 + \frac{i\hbar\gamma^0}{2} - \frac{\hbar}{\Delta} \gamma^0 \gamma \cdot \mathbf{D},
\]

where

\[
\mathcal{D}^j_{m,m'} = -(-1)^{m_j + m'_j} \left[ \epsilon_{m_j, m'_j} \cos \hat{\zeta}_{m_j, m'_j} - i \sin \hat{\zeta}_{m_j, m'_j} \right] \sec \hat{\zeta}^{(j)} \delta_{m_\perp, m'_\perp}.
\]

Here

\[
\epsilon_{m,m'} = \begin{cases} 
1, & m > m', \\
0, & m = m', \\
-1, & m < m',
\end{cases}
\]

and (the following are local and unaveraged in \( \mathbf{m}_\perp, n \))

\[
\zeta_{m_j} = \frac{e\Delta}{2} A_{m_j-1}^j, \quad \zeta^{(j)} = \sum_{m_j=1}^M \zeta_{m_j},
\]

and

\[
\hat{\zeta}_{m_j, m'_j} = \sum_{m''_j=1}^M \text{sgn} (m''_j - m_j) \text{sgn} (m''_j - m'_j) \zeta_{m''_j},
\]
with
\[ \text{sgn} \left( m - m' \right) = \epsilon_{m,m'} - \delta_{m,m'} . \] (3.6)

Because \( D \) is anti-Hermitian, it follows that \( T \) is unitary, that is, that \( \phi_{m,n} = \psi_{m,n} \) is the canonical field variable satisfying the canonical anticommutation relations (2.8).

It is instructive, and very simple, to consider the Schwinger model, that is the case with dimension \( d = 2 \) and mass \( \mu = 0 \). Because the light-cone aligns with the lattice in that case, we set \( h = \Delta \). Then we see that the transfer matrix for positive or negative chirality, that is, eigenvalue of \( i\gamma_5 = \gamma^0\gamma^1 \) equal to \( \pm 1 \), is
\[ T_{\pm} = \frac{1 \pm D}{1 \mp D} . \] (3.7)

From (3.3) we see that the numerator of \( T \) is
\[ (1 \pm D)_{m,m'} = [\delta_{m,m'} e^{\pm i\zeta} \mp (-1)^{m+m'} \epsilon_{m,m'} e^{-i\epsilon_{m,m'} \zeta_{m,m'}}] \sec \zeta , \] (3.8)
while it is a simple calculation to verify that the inverse of this operator is
\[ (1 + D)^{-1} \] (3.9a)
\[ (1 - D)^{-1} \] (3.9b)

It is therefore immediate to find
\[ (T_+)_{m,m'} = \delta_{m,m'+1} e^{2i\zeta_m} , \] (3.10a)
\[ (T_-)_{m,m'} = \delta_{m+1,m'} e^{-2i\zeta_{m'}} , \] (3.10b)

which simply says that the \( + \) (\( - \)) chirality fermions move on the light-cone to the right (left), acquiring a phase proportional to the vector potential. Solution (3.10) directly implies the chiral anomaly in the Schwinger model, as shown in [2].

To compute the vector and axial-vector anomalies in \( (3 + 1) \) dimensions, we will find it useful to expand \( T \) in powers of the temporal lattice spacing \( h \) (but not in the spatial lattice spacing \( \Delta \)). This leads to a very simple form for the transfer matrix:
\[ T = 1 + h \left[ -i\mu\gamma^0 + \frac{2}{\Delta}\gamma^0\gamma \cdot \mathbf{D} \right] + \hbar^2 \left[ -\frac{\mu^2}{2} + \frac{2}{\Delta^2}\mathbf{D}^2 + \frac{2i}{\Delta^2}\sigma \cdot (\mathbf{D} \times \mathbf{D}) \right] + O(h^3). \quad (3.11) \]

On the lattice the gauge-invariant current is written in terms of the all-averaged field \( \Psi \),

\[ \Psi_{m,n} = \psi_{m,n} = \frac{1}{2}(\phi_{m,n+1} + \phi_{m,n}), \quad (3.12) \]

rather than the canonical field \( \phi \). That is, the current is

\[ J_{\mu} = e\Psi^\dagger \gamma^0\gamma^\mu \Psi = \frac{e}{4} [\phi^\dagger (1 + T_n^\dagger)]_m \gamma^0\gamma^\mu [1 + T_n]_n \phi |m|. \quad (3.13) \]

(For a discussion of why this choice of current is used, see the Appendix.) In the absence of interactions it is easy to show that

\[ \langle \partial_\mu J^\mu \rangle = 0, \quad (3.14) \]

where the quotation marks signify a finite-element lattice divergence. At the initial time \( n = 0 \) we introduce the momentum expansion (2.3), and compute the expectation value in the corresponding Fock-space vacuum. If we use the expression for the current in (3.13) and keep only terms of \( O(h) \), we find

\[ \langle J_{\mu,0} \rangle = \frac{1}{3}\sum_{m'} \frac{1}{\Delta} \langle \phi^\dagger_{m',0} \gamma^0\gamma \cdot \mathbf{D}(0) + \mathbf{D}(1) \rangle_{m',m} \gamma^0\gamma^\mu \phi_{m,0} + h.c. \quad (3.15) \]

Thus, to leading order in \( h \), the vacuum expectation value of the divergence of the current

is (averaging in \( m \) is implicit)

\[ \langle \partial_\mu J^\mu \rangle = \frac{\langle J_{\mu,1} \rangle - \langle J_{\mu,0} \rangle}{h} = -\frac{2e}{\Delta L^3} \sum_{m'} \sum_p \frac{\tilde{p}_j}{\omega} \sin[p_j(m'_{j} - m_j)2\pi/M] \tilde{\alpha}_{m_j,m'_{j}}, \quad (3.16) \]

where \( \tilde{\alpha} = -\text{Im} D \). Not only is this not zero, it is not gauge invariant, for under a gauge transformation,

\[ \delta \mathbf{D}_{m_j,m'_{j}} = i e(\delta\Omega_{m_j} - \delta\Omega_{m'_{j}}) \mathbf{D}_{m_j,m'_{j}}. \quad (3.17) \]

This seems to be a conundrum, because the current was explicitly constructed to be gauge invariant. The resolution lies in the recognition that since the canonical field \( \phi \) is covariant under time-independent gauge transformations,
\( \delta \phi_{m,n} = ie\delta \Omega_m \phi_{m,n}, \) \hspace{1cm} (3.18)

the states created by that operator must transform likewise. That is, we require

\( \delta \langle \phi^\dagger_{m_j} \phi_{m'_j} \rangle = ie(\delta \Omega_{m'_j} - \delta \Omega_{m_j})\langle \phi^\dagger_{m_j} \phi_{m'_j} \rangle. \) \hspace{1cm} (3.19)

This suggests we supply the following factor in computing the vacuum expectation values:

\( \langle \phi^\dagger_{m_j} \phi_{m'_j} \rangle \rightarrow \exp(i\hat{\zeta}_{m_j,m'_j}\epsilon_{m_j,m'_j})\langle \phi^\dagger_{m_j} \phi_{m'_j} \rangle, \) \hspace{1cm} (3.20)

because

\( \delta \hat{\zeta}_{m_j,m'_j} = e(\delta \Omega_{m_j} - \delta \Omega_{m'_j})\epsilon_{m_j,m'_j}. \) \hspace{1cm} (3.21)

Then, the derivative operator that occurs in (3.16) becomes

\( -\text{Im} e^{i\hat{\zeta}_{m_j,m'_j}\epsilon_{m_j,m'_j}} D^j_{m_j,m'_j} = 0, \) \hspace{1cm} (3.22)

since the term with \( m_j = m'_j \) does not contribute.

In the next order in \( h \) it appears that a nontrivial contribution could emerge. It is the following gauge-invariant structure (here only the leading term in \( \Delta A \) is written down):

\[ \langle \partial^\mu J_{\mu} \rangle \propto -\frac{2eh}{\Delta^5} M \sum_i \zeta^{(i)} \omega. \] \hspace{1cm} (3.23)

This is a nonlocal term without a continuum analogue, because its gauge invariance depends on the periodic boundary conditions on the spatial lattice. However, to this order, since \( A_{m,1} = A_{m,0} + O(h) \), a simple inspection shows that the coefficient of this term is zero.

The axial current is defined just as in (3.13) with the additional factor of \( i\gamma_5 \). We compute the divergence analogously to (3.16), except here the first apparently nonzero term is \( O(h^2) \):

\[ \langle \partial^\mu J_{\mu}^5 \rangle = \langle J_{5,\bar{m},1}^0 \rangle - \langle J_{5,\bar{m},0}^0 \rangle \hspace{1cm} (3.24) \]

Because of the diagonal property of \( D \) in the perpendicular coordinates [see (3.3)], (3.24) is zero. So, to \( O(h) \) both the vector and axial-vector currents are nonanomalous. This may be
a bit surprising since when \( h = \Delta \) an axial-vector anomaly emerges in \((1+1)\) dimensions. We are presently trying to understand this state of affairs by carrying out an exact calculation with \( h = \Delta \) in \((3+1)\) dimensions.

IV. ANOMALOUS CURRENT COMMUTATORS

It is extremely interesting to compute commutators of the current defined by (3.13), and compare with the anomalous commutators in the continuum \([5,6]\):

\[
\langle [J^0(0, \mathbf{x}), J(0)] \rangle = iS \nabla \delta(\mathbf{x}) + id \nabla^2 \delta(\mathbf{x}),
\]

(4.1)

where \( S \) is the quadratically divergent Schwinger term, and \( d = 1/12\pi^2 \) for the Bjorken-Johnson-Low regularization. Straightforward evaluation of the current-current commutators on the lattice requires use of the transfer matrix \( T \) for the free Dirac field:

\[
T = \left( \frac{i\gamma^0}{h} + \frac{\mathbf{\gamma} \cdot \mathbf{t}}{\Delta} - \frac{\mu}{2} \right)^{-1} \left( \frac{i\gamma^0}{h} - \frac{\mathbf{\gamma} \cdot \mathbf{t}}{\Delta} + \frac{\mu}{2} \right)
= \left( 1 + \frac{\mu^2 h^2}{4} + \frac{h^2}{\Delta^2 t^2} \right)^{-1} \left( 1 - \frac{\mu^2 h^2}{4} - \frac{h^2}{\Delta^2 t^2} + \frac{2h}{\Delta} i\gamma^0 \mathbf{\gamma} \cdot \mathbf{t} - \mu hi\gamma^0 \right),
\]

(4.2)

where \( \mathbf{t} = t_p \) is given by (2.5). The result is the following expression for a hypercubic lattice, \( h = \Delta \):

\[
\langle [J^0_{m,n}, J^0_{m',n}] \rangle = -\frac{4e^2}{(M\Delta)^6} \sum_{k,k'} \frac{\tan(\pi k'_j/M)}{(1 + (\Delta\omega_{k-k'}/2)^2)(1 + (\Delta\omega_{k'}/2)^2)(\Delta\omega_{k'}/2)} e^{2\pi i k \cdot (m - m')/M}.
\]

(4.3)

This matrix (in \( \mathbf{m} \) and \( \mathbf{m}' \)) is shown for lattice size \( M = 9 \) and mass \( \mu = 0 \) in Fig. 4. The abscissas of the plot correspond to one-dimensional representations of the three components of \( \mathbf{m} \) and \( \mathbf{m}' \), respectively. (That is, the base-\( M \) number \((m_1, m_2, m_3)\) is converted to a base-10 value of the abscissa.) It is apparent that the leading-order behavior of this commutator is a first derivative of the delta function, as expected.

To make this comparison quantitative, we fit this result to the functional form expected in the continuum. The lattice analog of the Dirac delta function is
\[ \delta(x) \approx \frac{1}{\Delta^3} \delta_{m,0} = \frac{1}{(M \Delta)^3} \sum_k e^{2\pi i k \cdot m/M}, \quad (4.4) \]

so we take as a trial function

\[ -e^2 \frac{1}{(M \Delta)^3} \frac{2\pi i}{M \Delta} \sum_{r=0}^R a_{2r+1} P_r(\Delta) (-1)^r \left( \frac{2\pi}{M \Delta} \right)^{2r} \sum_k k_j k_j e^{2\pi i k \cdot (m - m')/M}. \quad (4.5) \]

When \( \mu = 0 \), the coefficients \( a \) do not depend on the lattice spacing \( \Delta \); they will presumably remain finite in the continuum limit, \( \Delta \to 0 \). The functions \( P_r(\Delta) \) are inserted to force each term in the series to be of the same order in \( \Delta \) as the commutator \((4.3)\), namely \( 1/\Delta^6 \):

\[ P_r(\Delta) = \Delta^{2r-2}. \quad (4.6) \]

Therefore, in the continuum limit, the first term in the series \((4.5)\) will be quadratically divergent, the second will be finite, and the rest will vanish, as expected.

To perform the fit, the problem is first converted from a three-dimensional to a two-dimensional representation. (This is necessary because the generation of three-dimensional plots is impractical beyond \( M = 9 \) and three-dimensional fits are prone to difficulties.) Specifically, a plot such as Fig. 1 is projected onto a plane orthogonal to the \( m, m' \) plane and to the diagonal of the matrix. The resulting curve is then fit to a similar projection of \((4.5)\). In Fig. 2 we show the results for the coefficients of a fit including only the first three odd derivatives of the lattice delta function (that is, for \( R = 2 \) in \((4.3)\)) for various lattice sizes. The coefficient of the first derivative is roughly consistent with the Schwinger result \[5\]. Similarly, the result for the coefficient of the third derivative term in the commutator is in approximate agreement with the agreement with the BJL result \[6\]; but seems to be closer to that of Chanowitz \[7\]. It is not surprising that the results are regularization dependent, given the divergent nature of the commutator. A more detailed discussion of this calculation will appear elsewhere \[8\].

**V. LOOP CALCULATION. SCHWINGER MODEL**

It is illuminating to calculate physical quantities such as the magnetic moment or the axial-vector anomaly by doing perturbation theory on the lattice. That is, we calculate
quantities such as vacuum polarization by using weak-coupling perturbation theory together with the lattice propagators given in (2.14) and (2.23). We illustrate this idea in the simple context of the Schwinger model, two-dimensional QED. (An analogous calculation in four dimensions of both the axial-vector anomaly and the anomalous magnetic moment is in progress.)

We begin by discussing the Schwinger model \( (d = 2) \) electrodynamics in the continuum. To calculate the anomaly it is essential to work with massive \( \text{QED}_2 \) and then take the massless limit to avoid infrared singularities. The vacuum polarization is then

\[
\Pi^{\mu\nu} = ie^2 \int \frac{d^2 p}{(2\pi)^2} \frac{\text{tr} \left( \gamma^\mu (p - k - \mu) \gamma^\nu (p - \mu) \right)}{((p - k)^2 + \mu^2)(p^2 + \mu^2)}.
\]

(5.1)

We require that \( \Pi^{\mu\nu} \) be transverse:

\[
\Pi^{\mu\nu} = \Pi(k^2) (g^{\mu\nu} - \frac{k_\mu k_\nu}{k^2}).
\]

(5.2)

Contracting on the indices gives

\[
\Pi(k^2) = -ie^2 \int \frac{d^2 p}{(2\pi)^2} \frac{4\mu^2}{((p - k)^2 + \mu^2)(p^2 + \mu^2)}.
\]

(5.3)

which is easily expressed using Feynman parameters as

\[
\Pi(k^2) = -\frac{e^2}{\pi} \int_0^1 dx \frac{\mu^2}{x(x - 1)k^2 + \mu^2}.
\]

(5.4)

For the \( \mu \neq 0 \) theory, we require that \( \Pi(0) = 0 \). This implies the presence of a subtraction constant \( e^2/\pi \), which for the \( \mu = 0 \) theory is the square of the boson mass, or the anomaly. Rather than introduce a Feynman parameter we could obtain this result by setting \( k = 0 \) in (5.3) and performing the momentum integration explicitly. We have, then, for the unsubtracted vacuum polarization

\[
\Pi(k = 0) = -i\frac{e^2}{(2\pi)^2} \int dp_0 \int dp_1 \frac{4\mu^2}{(p_0^2 - p_1^2 - \mu^2 + i\epsilon)^2} = -\frac{e^2\mu^2}{\pi} \int_0^\infty dp_1 \frac{1}{(p_1^2 + \mu^2)^2},
\]

(5.5)

which again gives the usual result, \( \Pi(0) = -e^2/\pi \).

We now describe the analogous calculation in the finite-element lattice framework. The lowest-order vacuum polarization is given by the time-ordered products of currents,
\[ \Pi_{\mu, \nu}^{\mu, \nu} = i \langle J_{\mu, \nu}^{\mu} J_{\mu, \nu}^{\nu} \rangle = i e^2 \langle \Psi_{m, n} \gamma^{\mu} \Psi_{m, n} \Psi_{m', n'} \gamma^{\nu} \Psi_{m', n'} \rangle, \]  

(5.6)

where the gauge invariant electromagnetic current is given by (3.13). Using the lattice Green’s function (2.14) and the transfer matrix (4.2), we find

\[ \Pi_{\mu, \nu}^{\mu, \nu} = i e^2 \left( \frac{\hbar}{L^4 \pi} \right)^2 \sum_{p_1} \sum_{p_2} \int_0^{2\pi/\hbar} \int_0^{2\pi/\hbar} d\hat{\Omega}_1 d\hat{\Omega}_2 e^{i h \hat{\Omega}_1 (n'n') \hbar \hat{\Omega}_2 (n'n')} \times e^{-i p_1 (m+m')2\pi/\hbar} e^{i p_2 (m-m')2\pi/\hbar} \text{tr}[\gamma^{\mu} \mathcal{P}_1 \left( \gamma^{0} \sin(h \hat{\Omega}_1) + (\mu - \gamma \hat{p}_1) h \cos^2(h \hat{\Omega}_1/2) \right) \times \mathcal{P}_2] \times \frac{1}{(\cos h(\Omega_1 - i \epsilon) - \cos h \hat{\Omega}_1)(\cos h(\Omega_2 - i \epsilon) - \cos h \hat{\Omega}_2)}, \]  

(5.7)

where \( \mathcal{P} = (1 + T)/2 \) and \( \mathcal{P} = \gamma^{0} \mathcal{P}^\dagger \gamma^{0} \). We now perform a lattice Fourier transform:

\[ \Pi^{\mu, \nu}(q, \omega) = \Delta \sum_{m=1}^{M} \sum_{n=-\infty}^{\infty} 2 \sin \left( \frac{\omega h}{2} \right) e^{-i h \omega (n'n')} e^{i q (m-m')2\pi/\hbar} \Pi_{m, n; m', n'}^{\mu, \nu}. \]  

(5.8)

Note that

\[ \sum_{m=1}^{M} e^{i q m 2\pi/\hbar} = M \delta_{q, 0} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} e^{-i h \omega n} = 2\pi \delta(h \omega), \quad -\pi < h \omega \leq \pi. \]  

(5.9)

Upon contracting with \( g^{\mu, \nu} \) we find for the vacuum polarization

\[ \Pi(q, \omega) = -i e^2 \left( \frac{\hbar}{L^4 \pi} \right)^2 2\pi \Delta M \left( \frac{2}{\hbar \omega} \sin \left( \frac{\omega h}{2} \right) \right) \times \sum_{p_2} \int_0^{2\pi/\hbar} \frac{d\hat{\Omega}_2}{(\cos h(\Omega_1 - i \epsilon) - \cos h \hat{\Omega}_1)(\cos h(\Omega_2 - i \epsilon) - \cos h \hat{\Omega}_2)} \cos^2(h \hat{\Omega}_1/2) \cos^2(h \hat{\Omega}_2/2) \times \frac{1}{(1 + r^2 \nu^2 + r^2 t_2^2)(1 + r^2 \nu^2 + r^2 t_2^2)}, \]  

(5.10)

where \( \nu = \frac{1}{2} \mu \Delta \) and \( r = h/\Delta \) and where, in the trace, we recall that \( \gamma^\lambda \gamma^n \gamma_\lambda = 0 \) in two dimensions. Here we understand that \( \hat{\Omega}_1 = \omega + \hat{\Omega}_2 \) and \( t_1 = t_{p_1} = t_{q+p_2} \). The integral over \( \hat{\Omega}_2 \) can now be done to yield, when \( t_q = \omega = 0, \)

\[ \Pi(0, 0) = \frac{-e^2}{\pi} \sum_{p_2} 4\pi r^3 \frac{\nu^2 \cos^4 h \Omega_2/2}{M (1 + r^2 \nu^2 + r^2 t_2^2)^2} \cos h \Omega_2 \left( 1 - \frac{4 \sin^2 h \Omega_2/2}{\cos h \Omega_2} \right). \]  

(5.11)

For zero external momentum, \( \Omega_1 = \Omega_2 \), where

\[ h \Omega_2 = \tan^{-1}(r(t_2^2 + \nu^2)^{1/2}). \]  

(5.12)
We perform the momentum sum in (5.11) for various values of the lattice size $M$, mass parameter $\nu$, and ratio of temporal and spatial lattice spacing $r$. The parameter $\nu$ is constrained by $M^{-1} \ll \nu \ll 1$. Figure 3 is a graph of the sum for $M = 2533$ versus $\nu$. Plotted are curves with $r = 0.1, 0.75, 0.99$. The first two cases stay close to 1 for a significant range of $\nu$, and only for $r = 0.99$ does the value drop significantly below 1. Note that for very small $\nu$ the sum diverges, as one expects for $\nu \sim M^{-1}$. This feature is shown more explicitly in Fig. 4, that is a plot for $r = 0.1$ for lattice sizes $1105 \leq M \leq 3145$, with increasing endpoint for decreasing lattice size. For smaller lattices, the curve turns up for larger $\nu$, as the limit of the domain of validity of the lattice calculation is reached.

More detailed analysis shows that, as a function of $r$, $\mp(0,0)/(e^2/\pi)$ is on the average unity, but is very noisy, with wild fluctuations around 1. For small $r$, fluctuations are greatly diminished, but for $r \sim 1$ and for various values of the input parameters, $\hbar \Omega_2$ can get very close to $\pi$ and make a few terms in the sum very large. (One might think that the desired value of $r$ is $r \sim 1$, the case of a hypercubic lattice; as noted in Sec. II the anomaly in the Schwinger model has previously been calculated with the finite element lattice technique with $r = 1$.) This question of stability is also under intense scrutiny.

VI. CONCLUSIONS

This paper is a report of recent progress in our finite-element program as applied to electrodynamics. More complete treatments of the analyses of the last two sections will appear elsewhere [8,9]. Our immediate goal in QED remains the extraction of a nonperturbative value for the anomalous moment of the electron.

Of course, our real interest lies in non-Abelian theories. Our first task there is to complete the formulation in four dimensions [3,10]. This should be accomplished in the next few months. Then we can apply this approach to theories such as QCD.
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APPENDIX: CURRENT CONSTRUCTED FROM EUCLIDEAN LAGRANGIAN

In the text we simply assumed a form of the current (3.13) which was manifestly gauge invariant. We are at liberty to do so, because the Minkowski finite-element equations of motion are not derivable from a Lagrangian. The current cannot be derived from the Dirac equation, but is an independent source for the Maxwell equations, see Ref. [3]. However, if one were to work in Euclidean space-time (periodic in all four directions), it is possible to construct an action from which the equations of motion are derivable, and which therefore supplies a lattice current. The fermion part of that action is (a factor of $i$ is absorbed in going to Euclidean space)

$$W_f = \Delta^4 \sum_{m,m'} \bar{\psi}_{m'} \left( \frac{2}{\Delta} \Gamma \cdot \mathcal{D} + \mu \right)_{m,m'} \psi_m,$$

$$\Gamma = \left( \gamma^0 \gamma^k, \gamma^0 \right).$$

(A1)

If we vary (A1) with respect to $\psi^\dagger$ we obtain the Euclidean Dirac equation

$$\left( \frac{2}{\Delta} \Gamma \cdot \mathcal{D} + \mu \right) \psi = 0.$$  

(A2)

where, as in (A1), a four-dimensional scalar product is implied.

Given an action, we can construct a conserved vector current by making a local gauge transformation

$$\delta \psi_{m'} = ie\delta \Omega_m \psi_{m'}.$$  

(A3)

Because the Dirac equation, and hence the action, is invariant under the global version of (A3), $\delta \Omega_m = \delta \Omega = \text{constant}$, we must have by the action principle
\[\delta W_f = 0 = - \sum_m J^i_m \frac{1}{\lambda} (\delta \Omega_{m_i, m_{\perp}} - \delta \Omega_{m_i-1, m_{\perp}}),\]  
(A4)

from which we read off the conserved current

\[J^i_m = -e \sum_{m_i', m_i''} \psi^\dagger_{m_i', m_{\perp}} \Gamma^i \psi_{m_i'', m_{\perp}} \text{sgn}(m_i - m_i') \text{sgn}(m_i - m_i'') \times (-1)^{m_i' + m_i''} \sec \zeta^{(i)} \exp(-i \epsilon_{m_i', m_i''} \zeta_{m_i', m_i''}).\]  
(A5)

(The same result, of course, can be obtained by varying \(D\) with respect to \(A^{i}_{m_i-1, m_{\perp}}\).) The expression for this Euclidean current has been simplified by deleting constant terms. It is easy to verify explicitly that this current is both conserved and gauge invariant. Similarly, by making a chiral transformation,

\[\delta \psi_{\mathbf{m}} = \gamma^0 \gamma_5 \delta \Omega_{\mathbf{m}} \psi_{\mathbf{m}},\]  
(A6)

we can construct the axial-vector current \(J^i_{5m}\), which has the form of \(\text{(A5)}\) with the replacement

\[\epsilon \Gamma^i \rightarrow \gamma^0 i \gamma_5 \Gamma^i \equiv \Gamma^i_5, \quad \Gamma^i_5 = (-i \gamma^5 \gamma^i, -i \gamma_5).\]  
(A7)

By construction, these currents possess no anomalies. However, they appear to be completely unacceptable, because they are horribly nonlocal. In particular, they possess no Minkowski analogues, in the sense that it is not possible to analytically continue back to real unbounded times. Crucial to our formulation is the propagation of the operators from past times to the present time, so that we can solve for the field operators by time-stepping through the lattice. The Euclidean current \(\text{(A5)}\) involves fermion field operators at all Euclidean times, which would make it impossible to solve for the operators at time \(n\) in terms of operators at earlier times. Therefore, for the considerations of the text we use the gauge-invariant current \(\text{(3.13)}\) and its axial analogue, currents which do possess anomalies.
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FIGURES

FIG. 1. Three-dimensional plot of (4.3) for $M = 9$ and $\mu = 0$.

FIG. 2. Coefficients of three spectral fit for different lattice sizes ($\mu = 0$). For the first derivative term, the coefficient shown is $S\Delta^2$.

FIG. 3. Plot of the sum (5.11) for $M = 2533$ as a function of $\nu = \mu\Delta/2$. Shown are curves with $r = h/\Delta = 0.1, 0.75, 0.99$.

FIG. 4. Plot of (5.11) for $1105 \leq M \leq 3145$ and $r = 0.1$. 

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Lattice size $M$

Value of coefficients

- First derivative
- Third derivative
- Fifth derivative
- $2/3\pi^2$ (Schwinger)
- $1/12\pi^2$ (Boulware, et al)
- $1/15\pi^2$ (Chanowitz)
