Testing isotropy of cosmic microwave background radiation

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ABSTRACT
We introduce new symmetry-based methods to test for isotropy in cosmic microwave background (CMB) radiation. Each angular multipole is factored into unique products of power eigenvectors, related multipoles and singular values that provide two new rotationally invariant measures mode by mode. The power entropy and directional entropy are new tests of randomness that are independent of the usual CMB power. Simulated Galactic plane contamination is readily identified. The ILC–WMAP data maps show seven axes well aligned with one another and the direction Virgo. Parameter free statistics find 12 independent cases of extraordinary axial alignment, low power entropy, or both having 5 per cent probability or lower in an isotropic distribution. Isotropy of the ILC maps is ruled out to confidence levels of better than 99.9 per cent, whether or not coincidences with other puzzles coming from the Virgo axis are included. Our work shows that anisotropy is not confined to the low l region, but extends over a much larger l range.

Key words: methods: data analysis – methods: statistical – cosmic microwave background.

1 INTRODUCTION
Studies of the cosmic microwave background (CMB) were long posed in terms of the temperature power spectrum. The power spectrum is invariant under rotations, and by itself cannot in principle test the basic postulate that the radiation field should be statistically isotropic. The availability of high-quality data from Wilkinson Microwave Anisotropy probe (WMAP) (Hinshaw et al. 2003, 2007) has allowed isotropy to be examined critically, and with surprising outcomes.

A long-standing question exists in the unexpected size of low-l multipoles. Interpretation of this is stalemated by inherent uncertainties of fluctuations called cosmic variance. Directional effects are much more decisive, because they confront a symmetry. de Oliveira-Costa et al. (2004) constructed an axial statistic for which they found modest statistical significance in multipoles for \( l = 2, 3 \) being rather well aligned. The fact that the dipole \( (l = 1) \) also aligns very closely with the multipoles \( l = 2, 3 \) was later highlighted by Ralston & Jain (2004) and Schwarz et al. (2004). When the dipole is interpreted to be due to our proper motion, it has sometimes been excluded as having ‘no cosmological significance’. However, there are many physical mechanisms which can correlate CMB observations with Galactic motion. There are good reasons not to exclude it, and the correlation of all three multipoles significantly aligned with the constellation Virgo is quite inconsistent with chance.

Subsequently there have been a large number of studies (Bielewicz, Gorski & Banday 2004; Eriksen et al. 2004; Hansen, Banday & Gorski 2004; Katz & Weeks 2004; Bielewicz et al. 2005; Prunet et al. 2005; Bernui et al. 2006; Copi et al. 2006; de Oliveira-Costa & Tegmark 2006; Wiaux et al. 2006; Freeman et al. 2006; Bernui et al. 2007; Copi et al. 2007; Helling, Schupp & Tesileanu 2007; Land & Magueijo 2007; Magueijo & Sorkin 2007) which claim the CMB is not consistent with isotropy. It is unlikely that the effect may arise due to Galactic foregrounds (Gaztanaga et al. 2003). Several physical explanations for the observed anisotropy have been put forward (Clune, Crotty & Lesgourgues 2003; Contaldi et al. 2003; Kesden, Kamionkowski & Cooray 2003; Armendariz-Picon 2004; Berera, Buniy & Kephart 2004; Gordon et al. 2005; Moffat 2005; Vale 2005;Abramo, Sodre & Wuenesch 2006; Land & Magueijo 2006; Rakic, Rasanel & Schwarz 2006; Gumrukcuoglu, Contaldi & Peloso 2007a; Campanelli, Cea & Tedesco 2007; Inoue & Silk 2006; Koivisto & Mota 2007; Naselsky, Verkhodanov & Nielsen 2007; Rodrigues 2007). It has also been suggested that the anisotropy may be due to foreground contamination (Slosar & Seljak 2004). Land & Magueijo (2005) find evidence that the detected anisotropy has positive mirror parity. Meanwhile some studies find no inconsistency (Efstathiou 2003; Hajian, Souradeep & Cornish 2004; Donoghue & Donoghue 2005; Hajian & Souradeep 2006). There have also been several studies of the primordial perturbations (Armendariz-Picon 2006; Battye & Moss 2006; Koivisto & Mota 2006; Gumrukcuoglu, Contaldi & Peloso 2007b; Pereira, Pitrou & Uzan 2007) and inflation (Hunt & Sarkar 2004; Buniy, Berera & Kephart 2006; Donoghue, Dutta & Ross 2007) in an anisotropic universe.

It is clearly necessary to explore new observables to determine whether anisotropy may be signs of physics beyond the standard paradigm. Here we develop new methods to test CMB data for...
isotropy. The new tests are possible because there exist many more invariant and vector-valued quantities than commonly examined. As conventional, the temperature distribution $\Delta T(\hat{n})$ is expanded in terms of the spherical harmonics, defining $\Delta T(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$, where $\hat{n}$ is a unit vector on the sky. The assumption of statistical isotropy is the statement that the ensemble average is given by

$$\langle a_{lm} a_{lm}^* \rangle = C_l \delta_{ij} \delta_{mn}.$$

(1)

As a consequence all tensors formed from products of $a_{lm}$ must be isotropic. We concentrate on the second rank tensors that can be formed, which include the isotropic power $C_l \delta_{ij}$ and the power tensor $A_{ij}(l)$. They are defined by

$$A(l) \delta_{ij} = \frac{1}{2l + 1} \sum_{nm'} a_{lm}^* \delta_{ij} \delta_{nm} a_{lm'};$$

$$A_{ij}(l) = \frac{1}{l(l+1)} \sum_{nm'} a_{lm}^* (J_i J_j)_{nm} a_{lm'}.$$

(2)

Here $J_i$ are the angular momentum operators in representation $l$ for Cartesian indices $i = 1, \ldots, 3$. From equation (1) statistical isotropy predicts the ensemble averages

$$\langle A \rangle = C_l;$$

$$\langle A_{ij}(l) \rangle = C_l \delta_{ij}.$$

(3)

Giving attention to the power tensor represents a modest step towards examining the data more thoroughly. Consider the region of $0 < l < 300$, which includes the acclaimed peak in plots of $l(l+1)C_l$. There are 299 data points from $2 < l < 300$ but the information makes a smooth curve, one that can be fit with three or four parameters. The existence of an acoustic peak is useful but only the beginning of tests. Each $a_{lm}$ has $2l + 1$ components, and the remaining 90,298 numbers, measured at great effort and considerable public expense, are never examined with the power spectrum. One reason comes from early tradition when data were poor. There also appears to be a widespread but false belief that the power spectrum is the only rotationally invariant quantity.

The approach of Copi et al. (2007) consists of factoring the spherical harmonics expansion into products of vector-valued terms. All spherical harmonics can be developed from traceless symmetric polynomials of vector products. There are $2l + 1$ real freedoms in multipoles of given $l$. Each multipole can be factored into $l$ unit vectors, each with two freedoms, making $2l$ terms (Weeks 2004, Katz and Weeks 2004). Finally there is one overall power coefficient. The analysis of 50 vectors for $l = 50$ (say) is inherently complicated, and complicated further by discovery that Maxwell-multipole vectors of isotropic data are kinematically correlated to ‘repel’ one another. Dennis (2005) discusses this and even solves the two-point correlation analytically.

Factoring into products of vectors is natural for tensors of rank 2. When we consider rank 2 tensors, we realize they have more than one invariant eigenvalue, so that the CMB power cannot even describe the quadrupole component. Tensors of rank higher than 2 have many types of invariants. One complication in finding them is the absence of a unique canonical form for high-rank tensors (da Silva & Lim 2006). Another technical obstacle is to obtain the invariants using quadratic functions of the data, which are statistically robust.

The technical problems are solved here with a procedure exploiting an invariant canonical form where unique and preferred vectors are factored from each multipole. The fact this can be done is both obvious from symmetry and systematic via Clebsch–Gordan series. It is a particular linear transformation of the $a_{lm}$ that leads to the power tensor $A_{ij}$ as quadratic form. When our procedure is applied to WMAP data there are numerous new invariants and several new signals of anisotropy.

We had two distinct motivations in beginning our study.

(i) Isotropy is a well-defined statistical data question that does not need any physical hypothesis to be motivated. Physical hypotheses are best characterized and tested through their symmetries. No free parameters are used in expressing isotropy, which can be confirmed or ruled out independent of the parameterization of model predictions. All of the new tests in this paper are also parameter-free. Isotropy of the CMB is also a separate issue from isotropy of other observables in the universe. This is because the CMB contains both information on formation in interaction with matter, and information on the propagation of light since the era of decoupling. By separating tests of isotropy of the CMB we maintain an orderly process of testing larger issues in cosmology.

(ii) Anisotropy of the CMB may be a signal of physics beyond electrodynamics and gravity. The interaction of light with an anisotropic axion-like condensate, or its conversion into weakly interacting particles, may give anisotropic signatures without disturbing the large scale distribution of matter.

Interestingly, there is a long history of ‘Virgo alignment’ of electromagnetic propagation effects which cannot be explained by any known media of conventional physics. Propagation of radiation in the sub-GHz region of radio telescopes, the 20–90 GHz region of CMB studies, and the optical region all reveal anisotropic effects. Many years ago Birch (1982) observed that the offsets of sub-GHz linear polarizations relative to radio galaxy axes were not distributed isotropically. The statistical significance of Birch’s signal was confirmed by Kendall & Young (1984) and a statistic used by Bietenholz & Kronberg (1984), but generally rejected nonetheless. The redshift-dependent signal found by Nodland & Ralston (1997) was also generally dismissed. Rejection was based on not finding correlation in statistics believed to test for the same signals. Yet one must be sure the same tests were really made. Paradoxes were finally resolved by classification under parity. Statistics finding no signal test for even parity distribution with no sense of twist, while those statistics of odd parity consistently show signals of high statistical significance. Subsequent analysis (Jain & Ralston 1999; Jain & Sarala 2006) has found a robust axial relationship of odd parity character and aligned with Virgo.

Until recently these explorations of anisotropy were conducted in the radio regime, which might indicate a conventional explanation in plasma electrodynamics. The discovery of optical frequency polarization correlations (Hutsemékers 1998; Hutsemékers & Lamy 2001; Jain, Narain & Sarala 2004) from cosmologically distant QSO’s is extraordinary. The optical correlations again generate an axis aligned with Virgo (Ralston & Jain 2004). The coincidence of 5 closely aligned axes coming from three distinct frequency domains makes an exceptional mystery that merits further study.

In this paper we make no attempt at physical explanation of anisotropy. Our new methods provide two independent new roads for data analysis. Order by order for each $l \geq 2$ there exist three rotationally invariant eigenvalues of $A_{ij}(l)$. The sum of three eigenvalues reproduce the usual power $C_l$. Meanwhile two independent combinations are new invariants. For the region of $2 \leq l \leq 50$, say, we have 98 new invariants never examined before. The isotropic CMB model predicts that all three eigenvalues should be degenerate and equal to $C_l/3$, up to fluctuations. We test this prediction by introducing the power entropy of $A_{ij}$. Inspired by quark statistical mechanics, the
power entropy is a stable and bounded measure of the randomness of eigenvalues of matrix $A_{ij}$. The entropy of the data turns out not to be consistent with the conventional CMB prediction. Next we examine the eigenvectors of $A_{ij}$. The eigenvectors are independent covariant quantities. Under the isotropic prediction they should be randomly oriented. For the range of $2 \leq l \leq 50$ there are 98 independent eigenvectors, and 49 ‘principal’ eigenvectors associated with the largest eigenvalues. The largest eigenvalue makes the largest contribution to the total power and gives an objective reason to be singled out. Statistical measures of the alignment of eigenvectors do not happen to support the random isotropic expectations. Instead 6 axes obtained from the range $2 \leq l \leq 50$ are found to be well aligned with the dipole or Virgo axis, each having independent probability of less than 5 per cent.

1.1 Outline of the paper

Transformation properties and signal processing of $a_{lm}$ are encoded by linear operations. To take advantage we first introduce $\psi_{m}^{l}$, which is the unique linear transformation of $a_{lm}$ that divides each $l$-multiplet into direct products of irreducible representations with vectors. We relate $\psi_{m}^{l}$ to the power tensor, and study the distribution of invariants in the Internal Linear Combination (ILC) map.

In the following section we describe the new methods. Applications to data follow. Not surprisingly, the power tensor method is generally more sensitive to anisotropy than power spectrum methods. Yet power spectrum methods find similar signals of anisotropy. Generally more sensitive to anisotropy than power spectrum methods. The standard power spectrum invariant is $\int_{\omega} \varphi_{\omega}^{l}(\omega) d\omega$, which produces a constraint in the usual basis convention

$$\int_{\omega} \varphi_{\omega}^{l}(\omega) d\omega \rightarrow |a(l)|^{2} = |\hat{a}(l)|^{2} + |\delta a(l)|^{2} - i\varphi_{\omega}^{l}(\omega)\langle a(l) | J^{l} J^{l} | a(l) \rangle \approx A^{l}.$$

Under what axes does the rotation make the most difference? Compute the Hessian of the change,

$$\frac{\partial^{2}}{\partial \theta_{1} \partial \theta_{2}}|\delta a(l)| = |\delta a(l)| = \langle a(l) | J^{l} J^{l} | a(l) \rangle = A^{l}.$$

By Rayleigh–Ritz variation, the maximum rotation is developed along the eigenvectors of $A^{l}$, which are natural principal axes of the $a_{lm}$ as objects order by order in $l$.

It is convenient to define a linear map or ‘wavefunction’ $\psi_{m}^{l}(l)$ (Ralston & Jain 2004) we call ‘vector factorization’:

$$\psi_{m}^{l}(l) = \frac{1}{\sqrt{2(l + 1)}} (l, m | J^{l} | a(l)).$$

The purpose of this transformation is to extract (algebraically divide out) a vector (spin $-1$) quantity from each representation of spin $-1$, so each can be covariantly compared across different $l$. Inserting a complete set of states gives

$$\psi_{m}^{l}(l) = \frac{1}{\sqrt{2(l + 1)}} \sum_{m'=-l}^{l} (l, m' | J^{l} | a(l)).$$

Under rotations each index of $\psi_{m}^{l}$ rotates by its representation $R(\theta)$, namely

$$\psi_{m}^{l} \rightarrow \psi_{m}^{l'} = R_{\mu\nu}(l) \psi_{m}^{l}.$$  \( \text{(7)} \)

The transformation $a \rightarrow \psi$ is invertible. The inverse relation can be written as

$$a_{lm} = \sum_{k' m'} \Gamma_{l m k}^{m'} \psi_{m'}^{k}.$$  \( \text{(5)} \)

where

$$\Gamma_{l m k}^{m'} = \frac{1}{\sqrt{2(l + 1)}} (l, m', m | J^{l} | a(l)).$$

Upon rotating $\psi$ by the rule of equation (7), then

$$a_{lm} \rightarrow a'_{lm} = \sum_{m' k} \Gamma_{l m k}^{m'} \psi_{m'}^{k} = R_{\mu\nu}(l) a_{lm}.$$  \( \text{(6)} \)

The standard power spectrum invariant is one of the invariants produced from $\psi_{l m}$:

$$\int_{\omega} \varphi_{\omega}^{l}(\omega) d\omega = \sum_{m k} \psi_{m}^{l} \psi_{m}^{k} = \sum_{m} \sum_{k} a_{lm} a_{km}^{*}.$$  \( \text{(8)} \)

$$\theta_{l},$$  \text{under which}  \( |a(l)| \rightarrow |a(l)|' = |a(l)| + |\delta a(l)|;$$

$$|\delta a(l)| = -i \theta_{l} J^{l} |a(l)|.$$
Here and in subsequent equations we suppress label \( l \) when it is obvious. Appendix A discusses factorization in general terms of a Clebsch–Gordan series.

The singular values \( \Lambda^a \) are invariants under rotations on the indices \( k \) and \( m \). (Indeed they are invariants under the even higher symmetric of independent \( SO(3) \times SO(3) \) rotations.) The various factors are constructed by diagonalizing the \( 3 \times 3 \) Hermitian matrices

\[
(\psi^l \psi^l)^{kk'}(l) = \sum_m \psi^\dagger_m(l) \psi^m_{km}(l),
\]

\[
= \sum_a e^{a^\dagger}_k (\Lambda^a)^2 e^{a^*}_k;
\]

\[
(\psi^l \psi^l)_{mm'}(l) = \sum_k \psi^{m*}_k(l) \psi^{km'}(l),
\]

\[
= \sum_a u^{a*}_{km}(\Lambda^a)^2 u^{a^*}_{m'k}.
\]

Since they are the eigenvectors of Hermitian matrices the \( e^a \) and \( u^a \) are generally orthogonal, except for exceptional degeneracies, and normalized to

\[
\sum_k e^{a^\dagger}_k e^{a}_k = \delta_{a\alpha};
\]

\[
\sum_m u^{a^*}_m u^{a}_m = \delta_{a\alpha}.
\]

Since \( (\Lambda^a)^2 \) are the eigenvalues of Hermitian matrices they are real and positive, with \( \Lambda^a > 0 \) defining the sign convention for \( e^a \). Since \( a_{\alpha\mu} \) are real, we also have \( e^a \) real valued.

The set of three orthogonal \( e^a \) defines the ‘frame vectors’, which make a preferred frame for the vector components of \( \psi \). In that preferred frame and the frame of three \( u^a \), the matrix \( \psi^\dagger_m \) is diagonal with three invariant singular values \( \Lambda^a \). We will call \( \Lambda^a \delta_{a\alpha} \) the ‘SV matrix’, SV standing for singular values.

2.1 Isotropy

The relation of \( \psi^\dagger_m \) to the power tensor \( A_{ij} \) is simple algebra. We have

\[
A^{ij}(l) = \frac{1}{l(l+1)} \text{tr}(J^i | a(l) (a(l)) J^j),
\]

where tr is the trace. Then

\[
A^{ij}(l) = \sum_m \psi^{i*}_m(l) \psi^{jm}(l),
\]

\[
= \sum_a e^{a^\dagger}_j (\Lambda^a)^2 e^{a^*}_m(l),
\]

which is just equation (9). Imposing the isotropic assumption of equation (1) yields

\[
(\psi^l)^{ij}(l) = \frac{C_j}{l(l+1)} \text{tr}(J^i J^j) = \frac{C_j}{3} g^{ij}.
\]

The last step comes from the orthogonality of \( J^i \) as group generators. Isotropy thus requires that the eigenvectors \( e^a \) be distributed isotropically on the sky, and that the eigenvalues \( \Lambda^a \) must be random variables concentrated at \( \sqrt{C_j/3} \).

Equation (10) can be contrasted with the correlations of Maxwell-multipole vectors, produced by dividing multipoles into sums of products of vectors. The Maxwell-multipole vectors have not been organized into irreps nor classified in invariant terms of importance.

Their distribution is not uncorrelated (Dennis 2005), but instead contains complicated kinematic ‘repulsions’ among their directions. In Appendix A we discuss correlations of \( \psi^\dagger_m \), which are projective transformations of the unit matrix. An advantage of studying isotropy using \( e^a(l) \) is that isotropy transforms to isotropy without annoying Jacobian factors.

Here is how to interpret the factors \( e^a(l) \). The highest possible degree of anisotropy produces one singular value equal to the total power, and two others that vanish. The corresponding \( a_{\alpha\mu} \) components can be written as the product of one vector \( e^{(1)} \) and one multiplet \( u^{(1)} \). We will call this special situation a ‘pure state’. This has a very simple realization for the quadrupole case \( l = 2 \). Any quadrupole is equivalent to a symmetric \( 3 \times 3 \) matrix made from sums of products of 3-vectors. A generic symmetric matrix has three real and unequal eigenvalues. If two eigenvalues vanish then the matrix is the outer product of a unique vector, a pure state. Conversely, if all eigenvalues are degenerate, the matrix is a (trace subtracted) multiple of the unit matrix, equivalent to the sum of outer products of three frame vectors by completeness, \( 1 = \sum_a |e^a(l) (e^a)| \), which is the isotropic prediction.

2.2 Tensor power entropy

The isotropy hypothesis that the SV matrix should be \( \sqrt{C_j/3} \) times the unit matrix can be tested in several different invariant ways.

The information entropy or simply power entropy of the power tensor comes by recognizing that \( \rho^{ij} = (\psi^l)^{ij}(l) \) is proportional to a density matrix on 3-space. The proportionality constant is the overall power. Removing it produces a normalized form

\[
\rho^{ij} = \frac{(\psi^l)^{ij}(l)}{\sum_l (\psi^l)^{ij}(l)}.
\]

Von Neumann (1932) found the entropy \( S \) of normalized Hermitian matrix \( \hat{\rho} \) to be

\[
S = -\text{tr}(\hat{\rho} \log(\hat{\rho})),
\]

\[
= \sum_u (\Lambda^u)^2 \log((\Lambda^u)^2),
\]

\( 0 \leq S \leq \log(3). \)

Here \( (\Lambda^u)^2 \) are normalized to sum to one, again removing the overall power from discussion:

\[
(\tilde{\Lambda}^u)^2 = \frac{(\Lambda^u)^2}{\sum_u (\Lambda^u)^2}.
\]

This normalizes \( \text{tr}(\tilde{\rho}) = 1 \).

The entropy is unique in being invariant, extensive, positive, additive for independent subsystems, and zero for pure states. A density matrix with no information is a multiple of the unit matrix, and has entropy equaling the log of its dimension. Very simply \( S_{\text{un}} \rightarrow \log(3) \) is the isotropic CMB prediction. Low entropy compared to isotropy is a measure of concentration of power along one or another eigenvector of \( \rho \). In the present context pure states have all the power in one singular value, define one single directional eigenvector, and have power entropy \( S_{\text{par}} \rightarrow 0 \).

2.3 Power alignments across \( l \)-classes

Our methods allow us to explore the alignment of power tensors between different \( l \)-classes.

One of the most interesting tests concerns the ‘principal axis \( \vec{e}(l) \)’, which means the eigenvector with the largest eigenvalue for each
From equation (10) the set of all principal frame vectors should be a symmetrical ball in the isotropic prediction. If the data are anisotropic a bundle of vectors may lie along an axis or in a preferred plane. It is important to remember that the sign of eigenvectors is meaningless, and determined by algorithms assigning signs in computer code. Thus the ‘average’ eigenvector is not a good statistic. Tensor products are the natural probe of a collection. For statistical studies we construct a matrix $X$ defined by

$$X_{ij}(l_{\max}) = \sum_{l=2}^{l_{\max}} \hat{e}_i \hat{e}_j,$$  \hfill (14)

The eigenvalues of $X$ are a probe of the shape of the bundle collected from $2 \leq l \leq l_{\max}$. We probe the isotropy of the collection with the directional entropy $S_x$ computed using equation (12) and $X \rightarrow \rho_\mathbf{X}$. The directional entropy does not use the singular values of the CMB data, and is independent of the power entropy. Confirmation of isotropy comes if $S_x \sim \log(3)$ up to random fluctuations. A signal of anisotropy would be an unusually low value of $S_x$ compared to $\log(3)$.

2.3.1 Traceless power tensor

We also compare alignments using a statistic that includes the weighting by singular values. For this purpose we define the traceless power tensor $B^{ij}$,

$$B^{ij}(l) = A^{ij}(l) - \frac{1}{3} \text{tr}(A(l)) \delta^{ij}.$$  

The eigenvectors of $B$ are the same as $A$, but the value of the total power has been removed. To compare two angular momentum classes we examine the correlation

$$Y(l, l') = \frac{\text{tr}(B(l)B(l'))}{\sqrt{\text{tr}(B(l)B(l'))} \sqrt{\text{tr}(B(l')B(l'))}}.$$  \hfill (15)

In the isotropic uncorrelated model $Y(l, l')$ should be proportional to $\delta_{ll'}$.

3 APPLICATIONS TO WMAP DATA

The WMAP–ILC team developed a foreground cleaned temperature anisotropy map using a method described as ILC. The process combines data from five bands of frequencies 23, 33, 41, 61 and 91 GHz. While there also exist several other cleaning procedures that are interesting to compare (Tegmark, de Oliveira-Costa & Hamilton 2003; Saha, Jain & Souradeep 2006; Eriksen et al. 2007), our main focus lies on the ILC map.

According to Hinshaw et al. (2007) the ILC map is known to become dominated by statistical instrumental noise for $l \gtrsim 400$. Errors on the $a_m$ come from several sources. Systematic errors may occur from inappropriate foreground subtractions. Ambiguities in combining frequency bands also contribute. The range $2 \leq l \leq 50$ is considered to be reliable under statements that statistical and systematic errors lie within ranges typical of cosmic variance. We restrict our studies to this range.

Fig. 1 shows the normalized eigenvalues $\Lambda^\alpha$, with a dashed line for the isotropic prediction $\Lambda^\alpha = \Lambda_i / \sqrt{3}$. The normalization is $\sum_{\alpha} (\Lambda^\alpha)^2 = 1$. The power entropy $S_{\text{ilc}}(l)$ mode by mode in $l$ for the ILC map is shown in Fig. 2. By Monte Carlo simulations we find that the points $l = 6, 16, 17, 30, 34, 40$ appear to be statistically unlikely.

We generated random $a_m$ and derived the power entropy using 10000 realizations of CMB maps. In our procedure real $a_m$ values were drawn from Gaussian distributions with zero mean and unit norm. Complex $a_m$ for $m = 1, \ldots, l$ were created by adding real and $i$ times real numbers from the same distribution and dividing by $\sqrt{2}$. As a consistency check, $P$-values were computed independently by different members of the group, both including the $C_l$ values, and omitting them, to verify the method and that the usual power statistic indeed drops out. Finally we computed $P$-values representing the frequency that power entropy in a random sample is less than that seen in the map. In Fig. 3 we show $P$-values of power entropy realized in the ILC maps versus mode number $l$. There are six unusually low power entropies of 5 per cent or smaller found at $l = 6, 16, 17, 30, 34$ and 40, with $P$-values of 0.040, 0.032, 0.041, 0.018, 0.045, 0.024, respectively.

It is interesting that the power entropy does not single out $l = 2$ or 3 as particularly significant. The power entropy itself does not have any information about the directional alignment of eigenvectors, which is independent and will be studied separately.
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Figure 3. \( \log_{10}(P) \)-values of the full-sky ILC map power entropy shown in Fig. 2. P-values are estimated from 10 000 realizations of random isotropic CMB maps. The dashed horizontal line shows \( P = 5 \) per cent.

Figure 4. \( \log_{10}(P) \)-values of largest power eigenvectors to align with the quadrupole versus \( l \). The dashed line shows the 5 per cent level of significance. There are six points at the level of \( P \lesssim 5 \) per cent or less.

Virgo suggests we compare the principal axes of the power tensors \( l \)-by-\( l \) with the quadrupole axis. The axes are compared by computing the quantity \( (1 - \hat{h} \cdot \hat{h}') \), where the unit vectors \( \hat{h} \) and \( \hat{h}' \) point in the directions of the principal axes corresponding to the quadrupole and the multipole \( l \), respectively.

Fig. 4 shows \( \log_{10}(P) \)-values in an isotropic distribution for power principal eigenvectors to align with the quadrupole. There are six axes aligned with a significance close to 5 per cent level or less; they are multipoles with \( l = 3 \), 9, 16, 21, 40, 43. The binomial probability of seeing 6 such events is 2.3 per cent. Hence the alignment with the quadrupole is seen over a relatively large range of \( l \) values. If coincidences are thrown out one by one, the estimated probability to find \((6, 5, 4, \ldots)\) small \(P\)-values seen in the data is \(2.3 \times 10^{-2}, 1.7 \times 10^{-2}, 1.4 \times 10^{-2}, \ldots\). The distribution of the principal axes for many different \( l \) are not consistent with the isotropic proposal. The Cartesian components of the well-aligned principal axes are listed in Table 1.

It may appear arbitrary to compare the axes to the \( l = 2 \) case. It is motivated by the literature and our previous history of work, discussed in the Introduction. For completeness \( P\)-values among all the independent pairs are listed in Table 2. It shows that choosing the \( l = 2 \) case causes no special bias, because once the axes are well aligned any one of them could be used for comparison. Correlation with the dipole, which is not strictly part of the ILC map, is also included for reference in Table 1. The statistical significance against isotropy is high whether or not one includes the dipole.

Table 1. Galactic Cartesian coordinates of principle axes ('large eigenvectors') \( \hat{h} \) that are extraordinarily aligned. Criteria for selection are \( P\)-values \( \lesssim 5 \) per cent relative to the \( l = 2 \) principal axis. The \( l = 1 \) or dipole axis is included for completeness. The axis \( \hat{h}_X \) is the principal eigenvector of the directional entropy of the set \( 2 \leq l \leq 50 \).

| \( l \) | Axis \( \hat{h}(x, y, z) \) |
|---|---|
| 2 | \((-0.209, -0.302, 0.930)\) |
| 3 | \((-0.251, -0.385, 0.888)\) |
| 9 | \((-0.224, -0.0974, 0.970)\) |
| 16 | \((-0.157, -0.175, 0.972)\) |
| 21 | \((-0.139, -0.536, 0.832)\) |
| 40 | \((-0.0928, 0.211, 0.973)\) |
| 43 | \((-0.217, -0.132, 0.967)\) |
| 1 | \((-0.0616, -0.664, 0.745)\) |
| \( \hat{h}_X \) | \((-0.435, 0.134, 0.890)\) |

1 In discrete data analysis each particular possibility is enumerated, as in throwing dice, so we put no emphasis on the cumulative distribution. The cumulative binomial probability to see six or more cases is slightly larger, \( f(k \geq 6 \text{ events} | n = 48, p = 0.045) \approx 0.020 \).
Table 2. P-values of coincidence between independent pairs of principal axes labelled by $l, l'$ shown in Table 1. Correlation with the $l' = 1$ axis, which is not strictly part of the ILC map, is included for completeness.

| $l' = 2$ | 3 | 9 | 16 | 21 | 40 | 43 |
|----------|---|---|----|----|----|----|
| 3        | 0.005 |     |    |    |    |    |
| 9        | 0.022 | 0.045 |    |    |    |    |
| 16       | 0.010 | 0.030 | 0.005 |    |    |    |
| 21       | 0.035 | 0.019 | 0.109 | 0.075 |    |    |
| 40       | 0.051 | 0.078 | 0.057 | 0.032 | 0.090 |    |
| 43       | 0.015 | 0.036 | 0.0006 | 0.003 | 0.094 | 0.051 |
| 1        | 0.094 | 0.067 | 0.199 | 0.150 | 0.015 | 0.141 | 0.178 |

3.1.1 Alignment entropy

In performing this research we first tested for clustering using the directional entropy $S_X$ of matrix $X$. Comparison with a Monte Carlo simulation (Fig. 5) using the same range $2 \leq l \leq 50$ does not show anything extraordinary, yielding a $P$-value of 69.0 per cent. The directional entropy $S_X$ did not signal significant clustering. Yet the directional entropy is rather insensitive statistic which misses the pattern in Table 2.

Curiously, however, the principal axis of $X$ is

$$\hat{n}_X = (-0.435, 0.134, 0.890).$$

The angular Galactic coordinates ($l = 162.88, b = 62.91$) are clearly not in the Galactic plane, but well aligned with Virgo. Recall that the principal vectors of the quadrupole and octupole in the ILC map have Galactic Cartesian components

$$\hat{n}(l = 2) = (-0.209, -0.302, 0.930),$$
$$\hat{n}(l = 3) = (-0.251, -0.386, 0.888),$$

respectively. These two axes are very closely aligned with one another, as so much noted. It is remarkable that they are aligned with the principal axis of $X$ coming from the whole ensemble.

Our first studies also used a Mollweide projection to examine visually for clustering of axes (Fig. 6). Since we are plotting an axis, and not a vector, we plot points only on one-half of the sphere. The ILC map does not visually show any striking clustering among different eigenvectors in Mollweide projection.

3.1.2 The region $1 \leq l \leq 11$

The method of Copi et al. (2007) finds highly significant signals for the region $1 \leq l \leq 11$. It is interesting to evaluate our results for this range.

There are four well-aligned principal vectors at $l = 1, 2, 3, 9$. Removing one as trivial, the probability to find three $P$-values of less than 5 per cent probability in a random sample in 11 trials is
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Figure 8. The P-values (in per cent) of the full-sky ILC map of the directional entropy $S_X$ in the range $2 \leq l \leq l_{\text{upper}}$.

about 1 per cent. One unusually low power entropy value exists in the range at $l = 6$.

What about the directional entropy? When $S_X$ is evaluated for the range $2 \leq l \leq 11$, random samples show $S_X$ equal or lower than the ILC map only 440 times out of 10 000, a $P$-value of 4.4 per cent. This is another indication of anisotropy. The $P$-values for different upper limits $l_{\text{upper}}$ are shown in Fig. 8. We see that many of the $P$-values in this range are smaller than 5 per cent. Finally the principal axis of $X$ is $\hat{n}_X(2 \leq l \leq 11) = (-0.531, 0.117, 0.839)$. It is well aligned with Virgo. The signals from $1 \leq l \leq 11$ evidently come from the same source as the signals from the entire range $2 \leq l \leq 50$.

3.1.3 The $Y$ matrix

The $Y$ matrix gives another method to examine visually the angular correlations. Fig. 9 shows contour maps (colour online) of the $Y(l, l')$ correlation. The correlation in ideal uncorrelated data should show a diagonal line up to fluctuations. Features of Fig. 9 confirm the other studies of unusual correlations in nominally uncorrelated random data. This study complements the study with matrix $X$, which uses principal axes and does not use the information contained in the singular values of $\psi$.

Figure 9. Contour maps of the correlation $Y(l, l')$ of traceless tensor power between different angular momentum indices $l$. Left-hand panel: a simple density plot. Right-hand panel: contours interpolated with trends more readily visible. A scale (colour online) shows the colour code of correlation values. Features confirm the other studies of unusual correlations in supposedly uncorrelated random CMB data.

3.2 Foreground contaminated maps

Here we study effects of simulated foreground contamination on the power tensor and entropy statistics.

The input of the simulation is shown in Fig. 10, top panel. The map is made by adding a random CMB map with a simulated synchrotron foreground map (Giardino et al. 2002) generated at 23 GHz frequency and normalized to 2 per cent of the actual foreground strength. Contamination is dominantly in the Galactic plane. Our procedure readily senses the contamination. Fig. 10, bottom panel, shows the spatial distribution of largest eigenvectors $\hat{e}_l$ for the range of multipole moments $2 \leq l \leq 50$. The vectors lie dominantly in the Galactic plane, providing a clear signal of alignment caused by Galactic contamination.

Calculation of the entropy of the largest eigenvectors $S_X(l_{\text{max}} = 50) = 0.995$ also shows significant anisotropy. The $P$-value for this to occur in a random isotropic sky is $P = 0.01$ per cent. The principal axis, or largest eigenvector of $X$ is $\hat{n}_X = (-0.233, -0.963, -0.132)$, which corresponds to $(l = 76.4, b = 7.56)$, lying close to the Galactic plane.

These studies show that our procedure detects Galactic contamination in a simple and statistically reliable way. They also indicate that the Virgo alignment observed in the preferred directions of the ILC data cannot be attributed to in-plane Galactic contamination.

4 SUMMARY AND CONCLUSIONS

We have developed several new symmetry-based methods to test CMB data for isotropy.

The wavefunction $\psi^i_{a_{\text{lin}}}(l)$ is a linear transformation of the amplitudes $a_{\text{lin}}$ which expresses them as products of vectors and
representations \( l \). The power tensor \( A_l \) is quadratic in the wavefunctions and \( a_{lm} \). Invariants of the power tensor are its eigenvalues, the singular values of \( \psi \). The statistical distribution of these invariants is a test of randomness and isotropy of the power tensor \( l \) by \( l \). The power entropy serves as a robust statistic. The power entropy is independent of the usual power, and also independent of the eigenvectors of \( A_l \), which contain information on the orientation of multipoles. To assess multipole orientations, we collected the principal axis of \( A_l \), which is the eigenvector with largest eigenvalue.

Both the power entropy and the axial vector distributions show features that are not consistent with isotropy. Fig. 11 summarizes some results of the ILC full sky study. The figure repeats Fig. 1, highlighting the exceptional cases. Cases of very low power entropy are shown by a short thin line. Cases of very high alignment of the principal axis with the quadrupole axis are shown by a longer thick grey line. There are 12 independent cases having 5\% or lower in an isotropic distribution, with two coincidences showing both exceptional power entropy and alignment. Ignoring the double coincidences, there are 98 independent trials in estimating \( 2 < l < 49 \) twice. The total probability to see the data values in uncorrelated isotropic CMB maps is of order \( 10^{-3} \).

Our study substantially extends previous studies that find low \( l \) multipoles of current CMB maps are highly unlikely on statistical grounds. Unlike methods that produce a large number of vector factors for each \( l \), our wavefunction method identifies a single invariant axis for each \( l \), and three independent invariants for each \( l \). The axial quantities are remarkably aligned with the direction of Virgo in rather systematic fashion for many \( l \). CMB data show seven total axes aligned over the range \( 1 < l < 50 \), with each case having independent probability at order 5\% or smaller. The values of \( l = 6, 16, 17, 30, 34, 40 \) have anomalously non-random power eigenvalue distributions giving exceptionally lower power entropy. Multipoles with \( l = 3, 9, 16, 21, 40, 43 \) have anomalous alignment with the quadrupole. It is no longer possible to restrict anomalous CMB statistics to the relative orientation or power seen in the dipole, quadrupole and octupole cases. Our study shows, for the first time, that the alignment of the multipoles with an axis pointing in the direction of Virgo is much more pervasive and not just confined to low \( l \) values.

As consistency checks, the anisotropies of CMB plus simulated synchrotron emission contamination from the plane of the Galaxy are detected by our methods in just the spatial regions where they are simulated. The observed anisotropies cannot readily be explained away by appealing to Galactic foreground contamination.

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**APPENDIX A: VECTOR DIVISION AND CLEBSCH SERIES**

Here we relate the factorization wavefunction $\psi_m^k$ to covariant canonical forms of the multipole $a_{\ell m}$. The implementation of angular momentum one conventionally expresses products of representations of dimensions $j_1, j_2$ into sums of states $\langle JM | \Psi \rangle$, where $| j_1 - j_2 | \leq j_1 \leq j_1 + j_2$. By that process ‘small’ spaces are normally used to build up larger ones. It is also possible to divide a representation $\langle JM | \Psi \rangle$ into products of certain representations. Let $\Psi'_M = \langle JM | \Psi \rangle$ be the matrix elements of a state $| \Psi \rangle$ transforming on representation $J$. We write

$$\Psi'_M \rightarrow \gamma_{m_1 m_2}^{j_1 j_2} = \Gamma_{J M}^{j_1 j_2} \Psi'_M,$$


(A1)

where the Clebsch–Gordan coefficients are

$$\Gamma_{J M}^{j_1 j_2} = \langle j_1 m_1 j_2 m_2 | J M \rangle.$$

Our analysis uses ‘vector division’, by which a given representation $J$ is expressed as products of 3-vectors ($j_1 = 1$) and representations of the original dimension ($j_2 = J$). The Clebsh’s for this decomposition are proportional to the angular momentum generators $J$. Recall that $J$ transforms like $m = 0$, and $J^\dagger \pm iJ^2$ transform like $m = \pm 1$. It follows that the Cartesian $(x, y, z)$ basis components of vector division are obtained by using the Cartesian components $J^\dagger$, which produces our rule $\psi_m^k = J^l_{\text{num}} a_{ml} / \sqrt{l(l+1)}$.

The process extends to arbitrary order. For example, traceless symmetric products of the generators $J^\dagger J$ transform like spin $-2$, and will factor a given representation into products of spin $-2$ and its own dimension. Any division consistent with the rules of addition of angular momentum can be explored. This explains the remark that no preferred canonical form exists for high-rank tensors.

The relation of $\psi_m^k$ to the canonical form of Copi et al. can be viewed as coming from two steps. First divide $|a\rangle \rightarrow (j_1 = 1) \otimes (j_2 = 1) \otimes (j_1 = 1)$, continuing down a chain of sums of products, to develop a large set of Maxwell multipoles:

$$|a\rangle = \text{constant} \sum_p \langle v_{1p}|v_{2p}| \cdots |v_{pp}\rangle.$$

Then represent the whole array as a single preferred 3-vector frame, times a preferred tensor of rank $k$:

$$\text{constant} \sum_p \langle v_{1p}|v_{2p}| \cdots |v_{pp}\rangle \rightarrow \psi_m^k = \sum_a a_{\text{num}} \Gamma_{\text{num}}^{|\psi_m^k|}.$$

The Clebsh $\Gamma_{\text{num}}$ makes traceless symmetric products, as necessary for an irreducible representation. This seems to be rather involved, and $\psi_m^k$ can simply be written down (Ralston & Jain 2004) skipping the Maxwell multipoles on the basis of the preferred 3-vector available from symmetry.

There appears to be a paradox in the number of degrees of freedom. Each list $a_{\text{num}}$ has $2j + 1$ real degrees of freedom. An arbitrary complex list $\psi_m^k$ would have $2 \times (2j + 1) \times 3$ degrees of freedom. There seem to be more freedoms in $\psi_m^k$. Yet the transformation from $a \rightarrow \psi \rightarrow a$ is completely invertible. The resolution lies in the transformation of the $SO(3)$ metric $121 + 121$. The inner product transforms

$$\sum_m a_{\text{num}} a_{\text{num}}^* \rightarrow \frac{1}{l(l+1)} \sum_m \psi_m^k \psi_m^k \psi_{m^*}^k \psi_{m^*}^k.$$

Thus the metric transforms

$$\delta_{\text{num}} \rightarrow \delta_{\text{num}}^k = \frac{1}{l(l+1)} \sum_m f_m^k f_{m^*}^k.$$

The transformed metric is a projection operator on the joint product space:

$$\sum_{m, m^*} \delta_{\text{num}}^k \gamma_{m^* m} = \delta_{\text{num}}^k.$$

In the last line the centre-dot indicates contracting indices by matrix product.

Since $g$ is a projector, it acts like the unit matrix on its ‘image’ subspace, and like zero on the orthogonal complement or ‘kernel’. The dimension of the image is just the original dimension. Our data analysis mapping $a \rightarrow \psi$ generates the constraint

$$g = \psi \cdot g.$$

This is a very special relation, which invariantly expresses that $\psi$ of our study are restricted to the image space, and then have just the same number of degrees of freedom as $a$. The norm reduces to $a^* \cdot a \rightarrow \psi^* \cdot \psi$, as required by construction.
These geometrical considerations are general facts of linear maps. For CMB purposes attention shifts to quadratic correlations. The determinant of the transformation from $a$ to $\psi$ (Jacobian) is unity, because the Clebsch transformation is unitary, when restricted to the invertible subspace $\psi = g \cdot \psi$. The map predicts that the 'unit correlation' $< a_m a_n^* > \sim \delta_{mn}$ transforms to a relation $< \psi_m \psi_n^\ast > \sim \delta_{mn}^{kk'}$. We did not need this relation in the paper, because correlations of $e_a^k$ are already contracted on indices $m$. Under isotropy $\sum_{m} g_{mm'}^{kk'} \rightarrow \text{constant} \delta_{kk'}$ goes to the obvious prediction of symmetry. These correlations would be relevant if one were to simulate $\psi^{kk'}$, or explore correlations of the $e_a^k$ and $u_a$ defined by equation (8).

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