Transfer Matrices for the Partition Function of the Potts Model on Toroidal Lattice Strips

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We present a method for calculating transfer matrices for the $q$-state Potts model partition functions $Z(G, q, v)$, for arbitrary $q$ and temperature variable $v$, on strip graphs $G$ of the square (sq), triangular (tri), and honeycomb (hc) lattices of width $L_y$ vertices and of arbitrarily great length $L_x$ vertices, subject to toroidal and Klein bottle boundary conditions. For the toroidal case we express the partition function as

$$Z(\Lambda, L_y \times L_x, q, v) = \sum_{d=0}^{L_y} \sum_j b^{(d)}_j(\lambda_{Z,\Lambda,L_y,d,j})^m,$$

where $\Lambda$ denotes lattice type, $b^{(d)}_j$ are specified polynomials of degree $d$ in $q$, $\lambda_{Z,\Lambda,L_y,d,j}$ are eigenvalues of the transfer matrix $T_{Z,\Lambda,L_y,d}$ in the degree-$d$ subspace, and $m = L_x \ (L_x/2)$ for $\Lambda = sq, tri \ (hc)$, respectively. An analogous formula is given for Klein bottle strips. We exhibit a method for calculating $T_{Z,\Lambda,L_y,d}$ for arbitrary $L_y$. In particular, we find some very simple formulas for the determinant $\det(T_{Z,\Lambda,L_y,d})$, and trace $\text{Tr}(T_{Z,\Lambda,L_y})$. Corresponding results are given for the equivalent Tutte polynomials for these lattice strips and illustrative examples are included.

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I. INTRODUCTION

The $q$-state Potts model has served as a valuable model for the study of phase transitions and critical phenomena \[1, 2\]. On a lattice, or, more generally, on a (connected) graph $G$, at temperature $T$, this model is defined by the partition function

$$Z(G, q, v) = \sum_{\{\sigma_i\}} e^{-\beta H}$$

with the (zero-field) Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$$

where $\sigma_i = 1, ..., q$ are the spin variables on each vertex (site) $i \in G$; $\beta = (k_B T)^{-1}$; and $\langle ij \rangle$ denotes pairs of adjacent vertices. The graph $G = G(V, E)$ is defined by its vertex set $V$ and its edge set $E$; we denote the number of vertices of $G$ as $n = n(G) = |V|$ and the number of edges of $G$ as $e(G) = |E|$. We use the notation

$$K = \beta J, \quad v = e^K - 1$$

so that the physical ranges are $v \geq 0$ for the Potts ferromagnet, and $-1 \leq v \leq 0$ for Potts antiferromagnet, corresponding to $0 \leq T \leq \infty$. One defines the (reduced) free energy per site $f = -\beta F$, where $F$ is the actual free energy, via $f(\{G\}, q, v) = \lim_{n \to \infty} \ln[Z(G, q, v)^{1/n}]$, where we use the symbol $\{G\}$ to denote the formal limit $\lim_{n \to \infty} G$ for a given family of graphs. In the present context, this $n \to \infty$ limit corresponds to the limit of infinite length for a strip graph of fixed width and some prescribed boundary conditions.

In this paper we shall present transfer matrices for the $q$-state Potts model partition functions $Z(G, q, v)$, for arbitrary $q$ and temperature variable $v$, on toroidal and Klein bottle strip graphs $G$ of the square, triangular, and honeycomb lattices of width $L_y$ vertices and of arbitrarily great length $L_x$ vertices. We label the lattice type as $\Lambda$ and abbreviate the three respective types as $sq, tri$, and $hc$. Each strip involves a longitudinal repetition of $m$ copies of a particular subgraph. For the square-lattice strips, this is a column of squares. It is convenient to represent the strip of the triangular lattice as obtained from the corresponding strip of the square lattice with additional diagonal edges connecting, say, the upper-left to lower-right vertices in each square. In both these cases, the length is $L_x = m$ vertices. We represent the strip of the honeycomb lattice in the form of bricks oriented horizontally. In this case, since there are two vertices in 1-1 correspondence with each horizontal side of a brick, $L_x = 2m$ vertices. Summarizing for all of three lattices, the relation between the
The number of vertices and the number of repeated copies is

$$L_x = \begin{cases} 
  m & \text{if } \Lambda = \text{sq or tri} \\
  2m & \text{if } \Lambda = \text{hc}
\end{cases}$$  \hspace{1cm} (1.4)

For the toroidal case the partition function has the general form

$$Z(\Lambda, L_y \times L_x, \text{tor.}, q, v) = \sum_{d=0}^{L_y} \sum_j b_j^{(d)}(\lambda_{Z,\Lambda, L_y, d,j})^m \hspace{1cm} (1.5)$$

where $\lambda_{Z,\Lambda, L_y, d,j}$ are eigenvalues of the transfer matrices in the degree-$d$ subspace $T_{Z,\Lambda, L_y, d}$ and they are independent of the length of the strip length $L_x$. The size of the transfer matrices $T_{Z,\Lambda, L_y, d}$ is equal to $d! n_{Z,\text{tor}}(L_y, d)$ that will be discussed in the following section. Here because the dimensions of $T_{Z,\Lambda, L_y, d}$ are the same for all three lattices $\Lambda = \text{sq, tri, hc}$, we omit $\Lambda$ in the notation. In contrast to cyclic and Möbius strips, there can be more than one coefficient, denoted as $b_j^{(d)}$, for each degree $d$. The $b_j^{(d)}$'s are polynomials of degree $d$ in $q$ and play a role analogous to multiplicities of eigenvalues $\lambda_{Z,\Lambda, L_y, d,j}$, although this identification is formal, since $b_j^{(d)}$ may be zero or negative for the small physical values of $q$. We exhibit our method for calculating $T_{Z,\Lambda, L_y, d}$ for arbitrary $L_y$, and we shall construct an analogous formula for Klein bottle strips. Explicit results for arbitrary $L_y$ are given for (i) $T_{Z,\Lambda, L_y, d}$ with $d = L_y$, (ii) the determinant $\text{det}(T_{Z,\Lambda, L_y, d})$, and (iii) the trace $\text{Tr}(T_{Z,\Lambda, L_y})$. The results in (i) and (iii) apply for $\Lambda = \text{sq, tri, hc}$, while the results in (ii) apply for $\Lambda = \text{sq, hc}$. Corresponding results are given for the equivalent Tutte polynomials for these lattice strips and illustrative examples are included. We have calculated the transfer matrices up to widths $L_y = 4$ for the square and honeycomb lattices and up to $L_y = 3$ for the triangular lattice. Since the dimensions of these matrices increase rapidly with strip width (e.g., $\text{dim}(T_{Z,\Lambda, L_y, d}) = 14, 35, 56, 48$ for $L_y = 4$ and $0 \leq d \leq 3$), it is not feasible to present many of the explicit results here; instead, we concentrate on general methods and results that hold for arbitrary $L_y$.

Various special cases of the Potts model partition function $Z(G, q, v)$ are of interest. For example, if one considers the case of antiferromagnetic spin-spin coupling, $J < 0$ and takes the temperature to zero, so that $K = -\infty$ and $v = -1$, then

$$Z(G, q, -1) = P(G, q) \hspace{1cm} (1.6)$$

where $P(G, q)$ is the chromatic polynomial (in $q$) expressing the number of ways of coloring the vertices of the graph $G$ with $q$ colors such that no two adjacent vertices have the same color. The minimum number of colors necessary for such a coloring of $G$ is called the chromatic number, $\chi(G)$. The $q$-state Potts antiferromagnet (AF) exhibits nonzero ground state entropy, $S_0 > 0$ (without frustration) for sufficiently large $q$ on a given lattice $\Lambda$. 


or, more generally, on a graph $G = (V, E)$. This is equivalent to a ground state degeneracy per site $W > 1$, since $S_0 = k_B \ln W$. Thus

$$W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n} .$$  \hspace{1cm} (1.7)

For a strip graph $G_s$ of a lattice $\Lambda$ with given boundary conditions, following our earlier notation \cite{3} we denote the sum of coefficients (generalized multiplicities) $c_{Z,G,j}$ of the $\lambda_{Z,G,j}$ as

$$C_{Z,G} = \sum_{j=1}^{N_{Z,G,\lambda}} c_{Z,G,j}$$  \hspace{1cm} (1.8)

while for the chromatic polynomial as

$$C_{P,G} = \sum_{j=1}^{N_{P,G,\lambda}} c_{P,G,j} .$$  \hspace{1cm} (1.9)

These sums are independent of the length $m$ of the strip. General results for these sums will be given below for the strips of interest.

We recall some previous related work. The partition function $Z(G, q, v)$ for the Potts model was calculated for arbitrary $q$ and $v$ on strips of the lattice $\Lambda$ with toroidal or Klein bottle longitudinal boundary conditions was calculated for (i) $\Lambda = sq$, $L_y = 2, 3$ in \cite{8}, and (ii) for $L_y = 3$ on the square-lattice with next-nearest-neighbor spin-spin couplings in \cite{8}. Matrix methods for calculating chromatic polynomials were developed and used in \cite{10, 11, 12, 13} and more recently in \cite{14, 15, 16}. Ref. \cite{17, 18, 19} developed transfer matrix methods for both $Z(G, q, v)$ and the special case $v = -1$ of chromatic polynomials on strips of the square and triangular lattices with free longitudinal boundary conditions and used them to calculate the latter polynomials for a large variety of widths. These have been termed transfer matrices in the Fortuin-Kasteleyn representation (see eq. (1.10) below). These methods were applied to calculate the full Potts model partition function for strips of the square and triangular lattices with free boundary conditions and a number of widths in Refs. \cite{20, 21}, and then were generalized for strips of the square, triangular and honeycomb lattices with cyclic and Möbius strips \cite{22, 23}. Here we extend these transfer matrix methods for calculating Potts model partition functions on toroidal and Klein bottle lattice strips and present general results for these lattice strips of arbitrary width. Clearly, new exact results on the Potts model are of value in their own right.

Let $G' = (V, E')$ be a spanning subgraph of $G$, i.e. a subgraph having the same vertex set $V$ and an edge set $E' \subseteq E$. $Z(G, q, v)$ can be written as the sum \cite{28, 29}

$$Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} \ell^{E'}$$  \hspace{1cm} (1.10)
where \( k(G') \) denotes the number of connected components of \( G' \). Since we only consider connected graphs \( G \), we have \( k(G) = 1 \). The formula (1.10) enables one to generalize \( q \) from \( \mathbb{Z}_+ \) to \( \mathbb{R}_+ \) for physical ferromagnetic \( v \). More generally, eq. (1.10) allows one to generalize both \( q \) and \( v \) to complex values, as is necessary when studying zeros of the partition function in the complex \( q \) and \( v \) planes.

The Potts model partition function \( Z(G, q, v) \) is equivalent to an object of considerable current interest in mathematical graph theory, the Tutte polynomial, \( T(G, x, y) \), given by

\[
T(G, x, y) = \sum_{G' \subseteq G} (x-1)^{k(G')-k(G)} (y-1)^{c(G')}
\]

where \( c(G') = |E'| + k(G') - |V| \) is the number of independent circuits in \( G' \). Now let

\[
x = 1 + \frac{q}{v}, \quad y = v + 1
\]

so that

\[
q = (x - 1)(y - 1).
\]

Then the equivalence between the Potts model partition function and the Tutte polynomial for a graph \( G \) is

\[
Z(G, q, v) = (x - 1)^{k(G)} (y - 1)^{n(G)} T(G, x, y).
\]

Given this equivalence, we can express results either in Potts or Tutte form. We will use both, since each has its own particular advantages. The Potts model form, involving the variables \( q \) and \( v \) is convenient for physical applications, since \( q \) specifies the number of states and determines the universality class of the transition, and \( v \) is the temperature variable. The Tutte form has the advantage that many expressions are simpler when written in terms of the Tutte variables \( x \) and \( y \).

From (1.14), one can write the Tutte polynomial as

\[
T(\Lambda, L_y \times L_x, tor., x, y) = \frac{1}{x-1} \sum_{d=0}^{L_y} \sum_j b_j^{(d)} (\lambda T, \Lambda, L_y, d, j)^m
\]

where \( m \) is given in terms of \( L_x \) by eq. (1.4) and it is convenient to factor out a factor of \( 1/(x - 1) \). (This factor is always cancelled, since the Tutte polynomial is a polynomial in \( x \) as well as \( y \).)

From eq. (1.14) it follows that [3, 33]

\[
\lambda_{Z, \Lambda, L_y, d, j} = v^{p L_y} \lambda_{T, \Lambda, L_y, d, j}
\]
and

\[ T_{Z,\Lambda,L_y,d} = v^{pL_y}T_{T,\Lambda,L_y,d} \]

where

\[ p = \begin{cases} 1 & \text{if } \Lambda = \text{sq} \text{ or } \text{tri} \\ 2 & \text{if } \Lambda = \text{hc} \end{cases} \]

(1.18)

Note that the factor of \((x - 1)\) in eq. (1.14) cancels the factor \(1/(x - 1)\) in eq. (1.15).

II. TRANSFER MATRIX METHOD

The chromatic polynomials for strips with periodic boundary condition in the longitudinal direction were discussed in Refs. [14, 15, 16] in terms of a compatibility matrix, and the bases of the matrix are given by \([X|\xi]\), where \(X\) is a subset of \(L_y\) vertices and \(\xi\) is an injection from \(X\) to \(\{1, 2, ..., q\}\). We now use the same bases for the transfer matrix of the full Potts model partition function. The degree \(d = 0\) subspace of the transfer matrix (equivalently called the “level 0” subspace in [16]) corresponds to the empty set \(X = \emptyset\), and the bases of the transfer matrix are all of the possible non-crossing partitions of \(L_y\) vertices. (For the zero-temperature antiferromagnetic Potts model, a non-nearest-neighbor requirement is imposed; this will be discussed further in the next section.) The dimension of this matrix is \(n_{Z,\text{tor}}(L_y, 0) = C_{L_y}\), the Catalan number [17], as in eq. (2.21) below. The eigenvalues of the transfer matrix for a cylindrical strip are a subset of the eigenvalues of the transfer matrix \(T_{Z,\Lambda,L_y,d=0}\) in this degree \(d = 0\) subspace for the corresponding toroidal strip. The degree \(d = 1\) subspace of the transfer matrix is given by all of the possible non-crossing partitions with a color assignment to one vertex (with possible connections with other vertices); i.e., \(X\) contains one vertex, and the multiplicity is \(q - 1 \equiv b^{(1)}\). This follows because there are \(q\) possible ways of making this color assignment, but one of these has to be subtracted, since the effect of all the possible color assignments is equivalent to the choice of no specific color assignment, which has been taken into account in the level 0 subspace. In this derivation and subsequent ones we assume that \(q\) is a sufficiently large integer to begin with, so that the multiplicities are positive-definite; we then analytically continue them downward to apply in the region of small \(q\) where the coefficients can be zero or negative. For the next subspace we consider all of the non-crossing partitions with two-color assignments to two separated vertices (with possible connections with other vertices). Now the multiplicity can be understood by the sieve formula of [14; 15; 16]. Since the two assigned colors should be different, there are \(q(q - 1)\) ways of making these assignments. This includes the \(q\) possible color assignments for each of the two vertices that have been considered in level 1 and hence these must be
subtracted. In doing this, the no-color assignment was subtracted twice, and one of these
has to be added back. Therefore, the multiplicity is
\[ q(q - 1) - 2q + 1 = q^2 - 3q + 1 \equiv b^{(2)}. \] (2.1)

Recall that a slice of cyclic strips is a tree graph, but a slice of toroidal strips is a circuit
graph. For toroidal strips, these coefficients \( b^{(d)} \) with \( 0 \leq d \leq 2 \) are the same as those for
cyclic strips \( c^{(d)} \) in Ref. [35] (see also [34]) because when the number of vertices is 1 or 2 a
tree graph is equivalent to a circuit graph modulo multiple edges. For toroidal strips, the
multiplicity for three-color assignment is
\[ q(q - 1)(q - 2) - 3q(q - 1) + 3q - 1 = q^3 - 6q^2 + 8q - 1 \equiv b^{(3)}. \] (2.2)

By similar reasoning, the multiplicity for four-color assignment is
\[ q(q - 1)(q^2 - 3q + 3) - 4q(q - 1)^2 + 4q(q - 1) + 2q^2 - 4q + 1 \]
\[ = q^4 - 8q^3 + 20q^2 - 15q + 1 \equiv b^{(4)}. \] (2.3)

Further, we find
\[ b^{(5)} = q^5 - 10q^4 + 35q^3 - 50q^2 + 24q - 1 \]
\[ b^{(6)} = q^6 - 12q^5 + 54q^4 - 112q^3 + 105q^2 - 35q + 1 \]
\[ b^{(7)} = q^7 - 14q^6 + 77q^5 - 210q^4 + 294q^3 - 196q^2 + 48q - 1 \]
\[ b^{(8)} = (q^2 - 3q + 1)(q^6 - 13q^5 + 64q^4 - 147q^3 + 155q^2 - 60q + 1) \]
\[ b^{(9)} = q^9 - 18q^8 + 135q^7 - 546q^6 + 1287q^5 - 1782q^4 + 1386q^3 - 540q^2 + 80q - 1 \]
\[ b^{(10)} = q^{10} - 20q^9 + 170q^8 - 800q^7 + 2275q^6 - 4004q^5 + 4290q^4 - 2640q^3 + 825q^2 - 99q + 1 \] (2.4)

and so forth for higher values of \( d \). We show the calculation for these multiplicities pictorially
for \( 2 \leq d \leq 4 \) in Fig. 1. Let us define
\[ q = 2 + t + t^{-1} = 2 + 2 \cos \theta = 4 \cos^2(\theta/2). \] (2.5)
Then, in terms of these variables, we find that $b^{(d)}$ can be written as

$$b^{(0)} = 1$$

$$b^{(1)} = t + 1 + t^{-1} = 1 + 2 \cos \theta$$

and, for $d \geq 2$,

$$b^{(d)} = (-1)^d(t + 1 + t^{-1}) + t^d + t^{-d} = (-1)^d(1 + 2 \cos \theta) + 2 \cos(d \theta)$$

Therefore, we have the recursion relation

$$b^{(d+2)} + (-1)^{d+1}b^{(1)} = (t + \frac{1}{t})(b^{(d+1)} + (-1)^db^{(1)}) - (b^{(d)} + (-1)^{d-1}b^{(1)}) \quad \text{for } d \geq 2 , \quad (2.9)$$

that is

$$b^{(d+2)} = (q - 2)b^{(d+1)} - b^{(d)} + q(-1)^db^{(1)} \quad \text{for } d \geq 2 . \quad (2.10)$$

Now the coefficient $c^{(d)}$ is given by

$$c^{(d)} = U_{2d}(\frac{\sqrt{q}}{2}) = \sum_{j=0}^{d}(-1)^j \binom{2d-j}{j} q^{d-j} \quad (2.11)$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind. Equivalently, in terms of the angle $\theta$ in eq. (2.5),

$$c^{(d)} = \frac{\sin \left( \frac{(2d + 1)\theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} \quad (2.12)$$

In general, we find that the relation between $b^{(d)}$ and $c^{(d)}$ is as follows,

$$c^{(d)} = b^{(d)} \quad \text{for } 0 \leq d \leq 2$$

$$c^{(d)} = \sum_{d'=1}^{d} b^{(d')} \quad \text{for odd } d$$

$$c^{(d)} = \sum_{d'=2}^{d} b^{(d')} \quad \text{for even } d , \quad (2.13)$$

and

$$b^{(d)} = c^{(d)} - c^{(d-1)} + (-1)^d c^{(1)}$$

$$= U_{2d}(\frac{\sqrt{q}}{2}) - U_{2d-2}(\frac{\sqrt{q}}{2}) + (-1)^d(q - 1)$$
\begin{equation}
q^d + \sum_{j=1}^{d} (-1)^j \frac{2d}{2d-j} \binom{2d-j}{j} q^{d-j} + (-1)^d (q-1) \quad \text{for } d \geq 2 \quad (2.14)
\end{equation}

At the value \( q = 0 \), \( b^{(d)} \) has the property that
\begin{equation}
b^{(d)}(q = 0) = (-1)^d \quad . \quad (2.15)
\end{equation}

We also find the following properties of the \( b^{(d)} \) polynomials. For \( q = 1 \) and \( d \geq 2 \), \( b^{(d)} \) is equal to 2 if \( d \equiv 0 \) mod 3 and \(-1\) if \( d = 1 \) or \( d = 2 \) mod 3. For \( q = 2 \) and \( d \geq 2 \), \( b^{(d)} \) is equal to 3 if \( d \equiv 0 \) mod 4 and \(-1\) if \( d = 1, 2, 3 \) mod 4. One can derive similar relations for other values of \( q \). For \( q = 3 \), \( b^{(d)} \) takes on the pattern of values \(-4, 1, -1, 4, -1, 1\) for \( d = 3, 4, 5, 6, 7, 8 \), and then this pattern repeats for higher integral values of \( d \). For \( q = 4 \) and \( d \geq 2 \), \( b^{(d)} = 5 \) if \( d \) is even and \( b^{(d)} = -1 \) if \( d \) is odd. For larger values of \( q \), \( b^{(d)} \) does not cycle through a fixed pattern of values as \( d \) increases, but instead increases. The fixed patterns of values of \( b^{(d)} \) for integer \( q \) from 0 to 4 are related to special sum rules for Potts model partition functions on lattice strips with toroidal boundary conditions, similar to those that we exhibited in Refs. [9] and [8].

\begin{align*}
d = 2 & \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig1a.png}
\end{array} - 2 \circ + 1 \\
d = 3 & \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig1b.png}
\end{array} - 3 \circ + 3 \circ - 1 \\
d = 4 & \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig1c.png}
\end{array} - 4 \circ + (4 \circ + 2 \circ ) - 4 \circ + 1
\end{align*}

FIG. 1: Multiplicities \( b^{(d)} \) for \( 2 \leq d \leq 4 \).

The partitions for toroidal strips are more involved than those for cyclic strips because of the rotational symmetry of the slice of toroidal strips. We list graphically all the possible partitions for strips of widths \( L_y = 2 \) and \( L_y = 3 \) in Figs. 2 and 3, respectively, where white circles are the original \( L_y \) vertices and each black circle corresponds to a specific color assignment. In the following discussion, we will simply use the names white and black circles with these meanings understood implicitly. The connections between the black circles and white circles should obey the non-crossing restriction. The apparent crossings in some partitions in Figs. 2 and 3 do not violate this rule when white circles are rotated appropriately, owing to the rotational symmetry for toroidal strips. For example, the crossing in the last partition of \( d = 1 \) in Fig. 3 can be removed by moving the bottom white circle to the top so that it becomes equivalent to the fourth partition of \( d = 1 \), or moving the top white circle to the bottom so that it becomes equivalent to the eighth partition of \( d = 1 \) in the figure. Another example is the crossing in the eleventh partition of \( d = 2 \) in Fig. 3 that
can be removed by the rotational symmetry of the white circles. In addition, it is necessary to include all the arrangements of black circles when the transfer matrix is constructed. For example, the two partitions of \( d = 2 \) in Fig. 2 correspond to exchanging the black circles, i.e. the assignments of the colors. We arrange the partitions of \( d = 2 \) in Fig. 2 into pairs such that every two partitions are equivalent when the two black circles are exchanged. We show the six partitions of \( d = 3 \) in Fig. 3 which correspond to all the possible arrangements of black circles.

We denote the partitions \( \mathcal{P}_{L_y,d} \) for \( 2 \leq L_y \leq 4 \) as follows:

\[
P_{2,0} = \{I; 12\}, \quad P_{2,1} = \{\bar{2}; \hat{1}; \overline{12}\}, \quad P_{2,2} = \{\hat{1}, \hat{2}, \hat{1}, \hat{2}\} \quad \tag{2.16}
\]

\[
P_{3,0} = \{I; 12; 13; 23; 123\}, \quad P_{3,1} = \{\overline{3}; \bar{2}; \hat{1}; 12, 3; \overline{12}; \overline{13}; \overline{23}; 23, 1; \overline{123}; 13, \hat{2}\} ,
\]

\[
P_{3,2} = \{\overline{2}; \bar{3}; \hat{2}; 2; 3; 1; \overline{3}; 1; \hat{3}; 1; \hat{2}; 1; \overline{2}; \overline{12}; \overline{3}; \overline{12}; 3; 1; \overline{123}; 13; \hat{2}\} ,
\]

\[
P_{3,3} = \{\bar{1}; \hat{2}, \bar{3}; \hat{1}, \hat{2}, \hat{3}; 1; \hat{2}, \bar{3}; 1; \hat{2}, 3; 1; \hat{2}; \hat{3}\} \quad \tag{2.17}
\]
where partitions are separated by a colon, and overline, hat, check and tilde in each partition correspond to different color assignments. It follows that the size of the transfer matrix $T_{Z, A, L_d}$ always has the factor $d!$. Let us denote the reduced size of the transfer matrix as
\[ n_{Z,\text{tor}}(L_y, d), \text{i.e., we neglect the permutation of black circles. As the derivation of } b^{(d)} \text{ does not take into account permutations, i.e. a set of } d! \text{ eigenvalues should have their coefficients summed to be equal to } b^{(d)}, \text{ we find that the sum of all coefficients for a toroidal strip graph, which is equal to the dimension of the total transfer matrix, i.e., the number of ways to color } L_y \text{ vertices, is} \]

\[ C_{Z,\Lambda,L_y} = \dim(T_{Z,\Lambda,L_y}) = \sum_{d=0}^{L_y} n_{Z,\text{tor}}(L_y, d)b^{(d)} = q^{L_y} \quad \text{for } \Lambda = sq, tri, hc . \quad (2.19) \]

By substituting the expression for \( b^{(d)} \) in eq. (2.19), we determine the \( n_{Z,\text{tor}}(L_y, d) \), as follows:

\[ n_{Z,\text{tor}}(L_y, d) = 0 \quad \text{for } d > L_y \quad (2.20) \]
\[ n_{Z,\text{tor}}(L_y, 0) = C_{L_y} \quad (2.21) \]
\[ n_{Z,\text{tor}}(L_y, 1) = \left(\frac{2L_y - 1}{L_y - 1}\right) \quad (2.22) \]
\[ n_{Z,\text{tor}}(L_y, d) = \left(\frac{2L_y}{L_y - d}\right) \quad \text{for } 2 \leq d \leq L_y \quad (2.23) \]

where \( C_n \) is the Catalan number which occurs in combinatorics and is defined by

\[ C_n = \frac{1}{(n+1)}\left(\frac{2n}{n}\right) . \quad (2.24) \]

The first few Catalan numbers are \( C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, \) and \( C_5 = 42 \). Special cases of \( n_{Z,\text{tor}}(L_y, d) \) that are of interest here include

\[ n_{Z,\text{tor}}(L_y, L_y) = 1 \quad (2.25) \]
\[ n_{Z,\text{tor}}(L_y, L_y - 1) = 2L_y \quad \text{for } L_y \geq 3 \quad (2.26) \]

In Table III we list the first few numbers \( n_{Z,\text{tor}}(L_y, d) \). We notice the following recursion relations:

\[ n_{Z,\text{tor}}(L_y, 1) = n_{Z,\text{tor}}(L_y, 0) - n_{Z,\text{tor}}(L_y - 1, 0) + 2n_{Z,\text{tor}}(L_y - 1, 1) + n_{Z,\text{tor}}(L_y - 1, 2) \]

\[ \text{for } L_y \geq 2 \]

\[ n_{Z,\text{tor}}(L_y, 2) = -n_{Z,\text{tor}}(L_y - 1, 0) + 2n_{Z,\text{tor}}(L_y - 1, 1) + 2n_{Z,\text{tor}}(L_y - 1, 2) + n_{Z,\text{tor}}(L_y - 1, 3) \]

\[ \text{for } L_y \geq 2 \]
\[ n_{Z,\text{tor}}(L_y, d) = n_{Z,\text{tor}}(L_y - 1, d - 1) + 2n_{Z,\text{tor}}(L_y - 1, d) + n_{Z,\text{tor}}(L_y - 1, d + 1) \]

for \( L_y \geq 3, \ 3 \leq d \leq L_y \). \hspace{1cm} (2.27)

The last line here is the same as the recursion relation for \( n_{Z,\text{cyc}}(L_y, d) \) when \( 1 \leq d \leq L_y \) for cyclic strips (which was denoted as \( n_Z(L_y, d) \) in \[33\]). By substituting \( q = 0 \) into eq. (2.19) and using eq. (2.15), we have the relation

\[ \sum_{d=0}^{L_y} n_{Z,\text{tor}}(L_y, d)(-1)^d = 0 \hspace{0.5cm} \text{for} \hspace{0.5cm} \Lambda = sq, \text{tri}, \text{hc} \]. \hspace{1cm} (2.28)

**TABLE I:** Table of numbers \( n_{Z,\text{tor}}(L_y, d) \). See text for general formulas.

| \( L_y \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1  | 1 | 1 |   |   |   |   |   |   |   |   |   |
| 2  | 2 | 3 | 1 |   |   |   |   |   |   |   |   |
| 3  | 5 | 10| 6 | 1 |   |   |   |   |   |   |   |
| 4  | 14| 35| 28| 8 | 1 |   |   |   |   |   |   |
| 5  | 42| 126| 120| 45| 10| 1 |   |   |   |   |   |
| 6  | 132| 462| 495| 220| 66| 12| 1 |   |   |   |   |
| 7  | 429| 1716| 2002| 1001| 364| 91| 14| 1 |   |   |   |
| 8  | 1430| 6435| 8008| 4368| 1820| 560| 120| 16| 1 |   |   |
| 9  | 4862| 24310| 31824| 18564| 8568| 3060| 816| 153| 18| 1 |   |
| 10 | 16796| 92378| 125970| 77520| 38760| 15504| 4845| 1140| 190| 20| 1 |

Another way to realize the number \( n_{Z,\text{tor}}(L_y, d) \) is as follows. For \( d = 0 \), i.e. no color assignment, the partitions are identical to those for cyclic strips, and the number of these partitions is \( n_{Z,\text{tor}}(L_y, 0) = n_{Z,\text{cyc}}(L_y, 0) = C_{L_y} \) given by eq. (2.21). Recall that the number of partitions of size \( n \) with \( k \) components is given by the Narayana number \( N(n, k) \), which is sequence A001263 in \[36\] (see also Ref. \[37\] and the references therein), given by

\[ N(n, k) = \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1} = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}. \hspace{1cm} (2.29) \]
The first few rows of the triangle of Narayana numbers, or Catalan triangle, are shown in Fig. 4. It is clear that the sum of terms in each row in the triangle is equal to the corresponding Catalan number:

\[
N(L_y, k) = \frac{L_y}{k+1} \binom{L_y}{k} = C_{L_y} = n_{Z,tor}(L_y, 0). \tag{2.30}
\]

For \(d = 1\), we need to assign a color for each component of the level 0 partitions. Therefore, we have

\[
n_{Z,tor}(L_y, 1) = \sum_{k=1}^{L_y} k N(L_y, k) = \sum_{k=1}^{L_y} \binom{L_y - 1}{k - 1} \binom{L_y}{k - 1} = \binom{2L_y - 1}{L_y - 1}
\]

\[
= \frac{1}{2} \left( \frac{2L_y}{L_y} \right) = \frac{L_y + 1}{2} C_{L_y} = \frac{L_y + 1}{2} n_{Z,tor}(L_y, 0). \tag{2.31}
\]

Let us compare \(n_{Z,tor}(L_y, d)\) with the corresponding number \(n_{Z,cyc}(L_y, d)\) given in Table 3 of [35]. In general, \(n_{Z,tor}(L_y, d)\) is larger than \(n_{Z,cyc}(L_y, d)\), except for \(d = 0\) where they are the same. This is a consequence of the fact that certain forbidden partitions for cyclic strips for \(d \geq 1\) are allowed for toroidal strips with rotational translation. We list the first few \(n_{Z,tor}(L_y, d) - n_{Z,cyc}(L_y, d)\) in Table II. Comparing Table II with Table I, we infer the relation

\[
n_{Z,tor}(L_y, d) - n_{Z,cyc}(L_y, d) = n_{Z,tor}(L_y, d + 1) \quad \text{for } d \geq 2. \tag{2.32}
\]

This can be understood as follows. The extra partitions for toroidal strips, as contrast with cyclic strips, are those with removable crossings. That is, for any connection between two non-adjacent white circles, all the color assignments should be inside or outside the connection while these two white circles can be either color-assigned or not. Therefore, there are three possible correspondences from the partitions for \(n_{Z,tor}(L_y, d + 1)\) to the partitions for \(n_{Z,tor}(L_y, d) - n_{Z,cyc}(L_y, d)\) with \(d \geq 2\): (i) if the top color-assigned vertex and the bottom color-assigned vertex are not within a connection, then connect these two vertices (so that only one color is assigned to them); (ii) if all color-assigned vertices are within a connection and the connection is also assigned a color, then remove the assignment for the connection; (iii) if all color-assigned vertices are within a connection and the connection is not assigned a color, then connect the top and bottom color-assigned vertices (within that connection). These account for all of the extra partitions that toroidal strips have, relative to cyclic strips. As an example, consider the case of width \(L_y = 4\) and degree \(d = 2\) in eq. (2.18). For this case there are eight partitions that are allowed for toroidal strips but not for cyclic strips. These are \(\overline{13}, 2; \overline{14}, 3; \overline{14}, 2; 14, \overline{2}, 3; \overline{24}, 3; \overline{14}, 23; \overline{124}, 3; \overline{134}, 2\). Here we only use overline to indicate the color assignment, since permutations of black circles are
not considered for \( n_{Z,\text{tor}}(L_y, d) \). These partitions are in one-to-one correspondence with the partitions for \( L_y = 4 \) and \( d = 3 \) (without color permutation), namely \( \bar{1}, \bar{2}, 3; \bar{1}, 3, \bar{4}; \bar{1}, \bar{2}, \bar{4}; \bar{14}, \bar{2}, 3; \bar{2}, 3, \bar{4}; \bar{23}, \bar{1}, \bar{4}; \bar{12}, 3, \bar{4}; \bar{34}, \bar{1}, \bar{2}. \) Eq. (2.32) is equivalent to the relation

\[
n_{Z,\text{tor}}(L_y, d) = \sum_{d' = d}^{L_y} n_{Z,\text{cyc}}(L_y, d') \quad \text{for } d \geq 2 .
\]

FIG. 4: Triangle of Narayana numbers.

The construction of the transfer matrix for each level (= degree) \( d \) can be carried out by methods similar to those for cyclic strips [23]. Notice that for the honeycomb lattice, the number of vertices in the transverse direction, \( L_y \), should be an even number, and the smallest value without degeneracy is \( L_y = 4 \). Using the bases described above (e.g. eqs. (2.16)-(2.18) for \( 2 \leq L_y \leq 4 \)), we define \( J_{L_y,d,i,i+1} \) as the join operator between vertices \( i \) and \( i + 1 \) and \( D_{L_y,d,i} \) as the detach operator on vertex \( i \) for each subspace \( d \). As compared with the situation for cyclic strips, the corresponding toroidal strips with the same \( L_y \) and \( d \) have one more join operator, namely \( J_{L_y,d,L_y,1} \equiv J_{L_y,d,L_y,L_y+1} \). The transfer matrix \( T_{Z,\Lambda,L_y,d} \) for each \( d \) is the product of the transverse and longitudinal parts, \( H_{Z,\Lambda,L_y,d} \) and \( V_{Z,\Lambda,L_y,d} \), which can be expressed as

\[
H_{Z,\text{sq},L_y,d} = \prod_{i=1}^{L_y} (I + vJ_{L_y,d,i,i+1})
\]

\[
H_{Z,\text{tri},L_y,d} = J_{L_y+1,d,L_y+1,1} \prod_{i=1}^{L_y} (I + vJ_{L_y+1,d,i,i+1})
\]
TABLE II: Table of numbers \( n_{Z,\text{tor}}(L_y,d) - n_{Z,\text{cyc}}(L_y,d) \).

| \( L_y \) \( d \) | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------------|----|----|----|----|----|----|----|----|----|----|----|
| 1               | 0  | 0  |    |    |    |    |    |    |    |    |    |
| 2               | 0  | 0  | 0  |    |    |    |    |    |    |    |    |
| 3               | 0  | 1  | 1  | 0  |    |    |    |    |    |    |    |
| 4               | 0  | 7  | 8  | 1  | 0  |    |    |    |    |    |    |
| 5               | 0  | 36 | 45 | 10 | 1  | 0  |    |    |    |    |    |
| 6               | 0  | 165| 220| 66 | 12 | 1  | 0  |    |    |    |    |
| 7               | 0  | 715| 1001|364| 91 | 14 | 1  | 0  |    |    |    |
| 8               | 0  | 3003|4368|1820|560|120|16 | 1  | 0  |    |    |
| 9               | 0  | 12376|18564|8568|3060|816|143|18 | 1  | 0  |    |
| 10              | 0  | 50388|77520|38760|15504|4845|1140|190|20 | 1  |    |

\[
H_{Z,\text{hc},L_y,d,1} = \frac{L_y}{2} \prod_{i=1}^{L_y} (I + vJ_{L_y,d,2i-1,2i}) , \quad H_{Z,\text{hc},L_y,d,2} = \frac{L_y}{2} \prod_{i=1}^{L_y} (I + vJ_{L_y,d,2i,2i+1})
\]

\[
V_{Z,\text{sq},L_y,d} = V_{Z,\text{hc},L_y,d} = \prod_{i=1}^{L_y} (vI + D_{L_y,d,i})
\]

\[
V_{Z,\text{tri},L_y,d} = D_{L_y+1,d,1} \prod_{i=1}^{L_y} [(I + vJ_{L_y+1,d,i,i+1})(vI + D_{L_y+1,d,i+1})]
\]

(2.34)

where \([\nu]\) denotes the integral part of \( \nu \), and \( I \) is the identity matrix. Notice that for triangular strips with periodic transverse boundary conditions, one has to work with width- \( (L_y + 1) \) matrices and then identify the two end vertices, as shown in [17]. We have

\[
T_{Z,\text{sq},L_y,d} = V_{Z,\text{sq},L_y,d} H_{Z,\text{sq},L_y,d} , \quad T_{Z,\text{tri},L_y,d} = V_{Z,\text{tri},L_y,d} H_{Z,\text{tri},L_y,d}
\]

\[
T_{Z,\text{hc},L_y,d} = (V_{Z,\text{hc},L_y,d} H_{Z,\text{hc},L_y,d,2})(V_{Z,\text{hc},L_y,d} H_{Z,\text{hc},L_y,d,1}) \equiv T_{Z,\text{hc},L_y,d,2} T_{Z,\text{hc},L_y,d,1} . \quad (2.35)
\]

As discussed above, the definition of \( b^{(d)} \) does not include the possible permutations of black circles. In principle, there can be \( d! \) eigenvalues with their coefficients \( b_j^{(d)} \) added up to
In our explicit calculation for toroidal strips, $b^{(d)}$ decomposes into two $b^{(d)}_j$ (with possible multiplication by an even integer) for the square and honeycomb lattices when $1 < d < L_y$ up to $L_y = 4$ and for the triangular lattice when $1 < d \leq L_y$ up to $L_y = 3$. As discussed above, we know that there is only one coefficient for both $d = 0$ and $d = 1$, i.e., $b^{(0)} = 1$, $b^{(1)} = q - 1$. For $d = 2, 3$, the coefficients are

\[
\begin{align*}
  b^{(2)}_1 &= \frac{1}{2} q(q - 3) \\
  b^{(2)}_2 &= \frac{1}{2} (q - 1)(q - 2) \\
  b^{(3)}_1 &= \frac{1}{3} (q - 1)(q^2 - 5q + 3) \\
  &= \frac{1}{6} q(q - 1)(q - 5) + \frac{1}{6} (q - 1)(q - 2)(q - 3) \\
  b^{(3)}_2 &= \frac{1}{6} q(q - 2)(q - 4)
\end{align*}
\]

(2.36)

with possible multiplication. In contrast to cyclic and Möbius strips, certain eigenvalues for toroidal strips may appear in more than one level, so that their coefficients are the summation of the corresponding $b^{(d)}_j$'s.

For the Klein bottle strip of the square and triangular lattices or the honeycomb lattice with $L_y$ even, the sum of coefficients is the same as the sum for the corresponding strip with Möbius longitudinal boundary conditions,

\[
C_{Z,Ly,Kb} \equiv \sum_{j=1}^{N_{Z,Ly,Kb,\lambda}} c_{Z,Ly,Kb,j} = \begin{cases} 
  q^{L_y/2} & \text{for even } L_y \\
  q^{(L_y+1)/2} & \text{for odd } L_y
\end{cases}.
\]

(2.37)

Consider the coefficients for Klein bottle strips. For the square and honeycomb lattices, when the longitudinal boundary condition is changed from toroidal to Klein bottle, we observe the following changes of coefficients:

\[
\begin{align*}
  b^{(0)} &\rightarrow \pm b^{(0)} \\
  b^{(1)} &\rightarrow \pm b^{(1)} \\
  b^{(2)}_1 &\rightarrow \pm b^{(2)}_1 \\
  b^{(2)}_2 &\rightarrow \pm b^{(2)}_2
\end{align*}
\]
We find that certain coefficients become zero when the boundary condition is changed from toroidal to Klein bottle, and therefore the number of eigenvalues for Klein bottle strips always appears to be less than the number for the corresponding toroidal strips. This has been observed before in Ref. [38, 39]. The partition functions for Klein bottle strips have the same expressions as eqs. (1.5) and (1.15) with \( b_j^{(d)} \) replaced by appropriate coefficients, as in eq. (2.38). Illustrative calculations will be given below.

III. ZERO-TEMPERATURE POTTS ANTIFERROMAGNET

For the chromatic polynomials, it is necessary that adjacent vertices are not assigned the same color, i.e., in the partition diagrams, adjacent vertices cannot be connected by color-connection lines. For the square and triangular lattices, a transverse slice is a circuit graph; while for the honeycomb lattice, a transverse slice is a set of \( L_y/2 \) two-vertex tree graphs.

A. Square and triangular lattices

For the square and triangular lattices, the second partition of the degree \( d = 0 \) subspace and the third partition of the degree \( d = 1 \) subspace for width \( L_y = 2 \) in Fig. 2 are not allowed. For \( L_y = 3 \) in Fig. 3 we keep the first partition in the \( d = 0 \) subspace, the first three partitions of the \( d = 1 \) subspace, and the first six partitions of the \( d = 2 \) subspace. Let us denote the reduced size of the transfer matrix for the square and triangular lattices as \( n_{P,tor}(L_y, d) \), excluding permutations of black circles. The sum of all coefficients is the dimension of the total transfer matrix (independent of \( L_y \)), which is the chromatic polynomial of the circuit graph given by Theorem 10 of [35], so that for \( L_y \geq 2 \),

\[
C_{P,\Lambda,L_y} = \sum_{d=0}^{L_y} n_{P,tor}(L_y, d) b_j^{(d)} = (q - 1)^{L_y} + (q - 1)(-1)^{L_y} \quad \text{for} \quad \Lambda = sq, tri .
\]  

(3.1)

By substituting the expression for \( b_j^{(d)} \) in eq. (2.19), we determine the \( n_{P,tor}(L_y, d) \). We list the first few numbers \( n_{P,tor}(L_y, d) \) in Table III. The strip with \( L_y = 1 \) is obtained by identifying the free boundaries of the cyclic strip with \( L_y = 2 \), namely, each vertex is attached by a loop such that its chromatic polynomial is identically zero. The column \( n_{P,tor}(L_y, 0) \) in Table III is the number of non-crossing non-nearest-neighbor partitions of \( L_y \) vertices on a circle, and is given by the Riordan number \( R_{L_y} \) [40, 41]. If we compare \( n_{P,tor}(L_y, 0) \) with the
corresponding number \( n_{P,cyc}(L_y, 0) \) (which was denoted as \( n_P(L_y, d) \)) for cyclic strips, given in Table 1 of [35], we find the relation

\[
n_{P,cyc}(L_y, 0) - n_{P,tot}(L_y, 0) = n_{P,tot}(L_y - 1, 0) \quad \text{for } L_y \geq 2 .
\] (3.2)

This can be understood as follows. For toroidal strips we have additional bonds connecting the top and bottom vertices of the corresponding cyclic strips, so these vertices cannot have the same color. That is, the \( d = 0 \) partitions for cyclic strips with the top and bottom vertices connected (having the same color) is not allowed for the corresponding toroidal strips. The number of these partitions is given by \( n_{P,tot}(L_y - 1, 0) \). This is clear, since we know \( n_{P,cyc}(L_y, 0) = M_{L_y - 1} \) [35], where \( M_n \) is the Motzkin number, and the relation \( M_n = R_n + R_{n+1} \). We notice the following recursion relations:

\[
n_{P,tot}(L_y, 1) = n_{P,tot}(L_y, 0) + n_{P,tot}(L_y - 1, 1) + n_{P,tot}(L_y - 1, 2) + (-1)^{L_y} \quad \text{for } L_y \geq 2
\]

\[
n_{P,tot}(L_y, 2) = -n_{P,tot}(L_y - 1, 0) + 2n_{P,tot}(L_y - 1, 1) + n_{P,tot}(L_y - 1, 2) \\
+ n_{P,tot}(L_y - 1, 3) + (-1)^{L_y} \quad \text{for } L_y \geq 2
\]

\[
n_{P,tot}(L_y, d) = n_{P,tot}(L_y - 1, d - 1) + n_{P,tot}(L_y - 1, d) + n_{P,tot}(L_y - 1, d + 1)
\]

for \( L_y \geq 3, \ 3 \leq d \leq L_y \). \( \cdots \) (3.3)

The last line here is the same as the recursion relation for \( n_{P,cyc}(L_y, d) \) when \( 1 \leq d \leq L_y \) for cyclic strips [35]. By substituting \( q = 0 \) into eq. (3.1) and using eq. (2.15), we have

\[
\sum_{d=0}^{L_y} n_{P,tot}(L_y, d) (-1)^d = 0 \quad \text{for } \Lambda = sq, tri . \] (3.4)

Let us denote the number of diagonal dissections of a convex \( n \)-gon into \( k \) regions as \( D(n, k) \), which is sequence A033282 in [36]. (See also Ref. 37.)

\[
D(n, k) = \frac{1}{k} \binom{n-3}{k-1} \binom{n+k-2}{k-1} \quad \text{for } n \geq 3, 1 \leq k \leq n-2 .
\] (3.5)

The first few numbers are \( D(3, 1) = 1, D(4, 1) = 1, D(4, 2) = 2, D(5, 1) = 1, D(5, 2) = 5, D(5, 3) = 5 \). We also know that the Riordan number for \( n > 1 \) is given by the number of dissections of a convex polygon by nonintersecting diagonals with \( n + 1 \) edges. To relate \( D(n, k) \) with the Riordan number, define \( D'(n, k) \) as

\[
D'(n, k) = D(n + 2 - k, k) = \frac{1}{k} \binom{n-1-k}{k-1} \binom{n}{k-1} \quad \text{for } n \geq 2 .
\] (3.6)
TABLE III: Table of numbers \( n_{P,\text{tor}}(L_y, d) \) for the square and triangular lattices.

| \( L_y \backslash d \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---------------------|---|---|---|---|---|---|---|---|---|---|-----|
| 1                   | 0 | 0 |   |   |   |   |   |   |   |   |     |
| 2                   | 1 | 2 | 1 |   |   |   |   |   |   |   |     |
| 3                   | 1 | 3 | 3 | 1 |   |   |   |   |   |   |     |
| 4                   | 3 | 10| 10| 4 | 1 |   |   |   |   |   |     |
| 5                   | 6 | 25| 30| 15| 5 | 1 |   |   |   |   |     |
| 6                   | 15| 71 |90 |50 |21| 6 | 1 |   |   |   |     |
| 7                   | 36| 196|266|161| 77|28| 7 | 1 |   |   |     |
| 8                   | 91| 554|784|504|266|112|36| 8 | 1 |   |     |
| 9                   | 232|1569|2304|1554|882|414|156|45| 9 | 1 |     |
| 10                  | 603|4477|6765|4740|2850|1452|615|210|55| 10|     |

It is clear that the entries of \( D'(n, k) \) are zero for \( [n/2] + 1 \leq k \leq n \). Let us also define \( D'(1, 1) \equiv 0 = n_{P,\text{tor}}(1, 0) \), and list the first few rows of the triangle of \( D'(n, k) \) for \( n \geq 1, 1 \leq k \leq n \) in Fig. 5. The summation of each row in the triangle gives the Riordan number:

\[
\sum_{k=1}^{[L_y/2]} D'(L_y, k) = R_{L_y} = n_{P,\text{tor}}(L_y, 0) .
\] (3.7)

For \( d = 1 \), we need to assign a color for each component of level 0 partitions, so that

\[
n_{P,\text{tor}}(L_y, 1) = \sum_{k=1}^{[L_y/2]} (L_y + 1 - k)D'(L_y, k) = \sum_{k=1}^{[L_y/2]} \binom{L_y}{k} \left( \frac{L_y - 1 - k}{k - 1} \right) .
\] (3.8)

For entries \( n_{P,\text{tor}}(L_y, d) \) in Table III with \( d > 1 \), we find the following relation with known number sequences. Motivated by eq. (3.2), let us define \( n'_{P,\text{tor}}(L_y, d) \) such that \( n'_{P,\text{tor}}(1, d) = n_{P,\text{tor}}(1, d) \) and

\[
n'_{P,\text{tor}}(L_y, d) = n_{P,\text{tor}}(L_y, d) + n_{P,\text{tor}}(L_y - 1, d) \text{ for } L_y \geq 2 .
\] (3.9)

We list the first few numbers \( n'_{P,\text{tor}}(L_y, d) \) in Table IV. Comparing these with the triangular array given by sequence A025564 in [36] (which result from pairwise sums of entries in the
FIG. 5: Triangle of \( D'(n, k) \).

We list the first few rows of this triangular array and the number of compact rooted directed animals of size \( L_y \geq 3 \) having 3 source points. Combining the results in Table V and Table IV, we infer the relation

\[
n'_{P,\text{tor}}(L_y, d) - n_{P,\text{cyc}}(L_y, d) = n'_{P,\text{tor}}(L_y, d + 1) \quad \text{for } d \geq 2 ,
\]

or equivalently,

\[
n'_{P,\text{tor}}(L_y, d) = \sum_{d' = d}^{L_y} n_{P,\text{cyc}}(L_y, d') \quad \text{for } d \geq 2 .
\]

We recall that the sum of the coefficients in (1.9) for a Klein bottle strip of the square or triangular lattice of width \( L_y \) is (independent of \( L_x \)), given as Theorem 11 in [35], is

\[
C_{P,L_y,K}\lambda = \sum_{j=1}^{N_{P,L_y,K}\lambda} c_{P,L_y,K\lambda,j} = 0
\]
TABLE IV: Table of numbers $n'_{P,tor}(L_y, d)$.

| $L_y \backslash d$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|------------------|----|----|----|----|----|----|----|----|----|----|----|
| 1                | 0  | 0  |    |    |    |    |    |    |    |    |    |
| 2                | 1  | 2  | 1  |    |    |    |    |    |    |    |    |
| 3                | 2  | 5  | 4  | 1  |    |    |    |    |    |    |    |
| 4                | 4  | 13 | 13 | 5  | 1  |    |    |    |    |    |    |
| 5                | 9  | 35 | 40 | 19 | 6  | 1  |    |    |    |    |    |
| 6                | 21 | 96 | 120| 65 | 26 | 7  | 1  |    |    |    |    |
| 7                | 51 | 267| 356| 211| 98 | 34 | 8  | 1  |    |    |    |
| 8                | 127| 750|1050| 665| 343|140 |43  | 9  | 1  |    |    |
| 9                | 323| 2123|3088|2058|1148|526 |192 |53  |10  | 1  |    |
| 10               | 835| 6046|9069|6294|3732|1866|771 |255 |64  |11  | 1  |

B. Honeycomb lattice

For a strip of the honeycomb lattice with toroidal boundary conditions, the width $L_y$ must be even. In each transverse slice of such a strip, the $L_y$ vertices are connected in a pairwise manner, just as for the strips with cyclic boundary conditions. Let us denote the reduced size of the transfer matrix as $n_{P,tor,hc}(L_y, d)$, without considering permutations of black circles in the partition diagram. The sum of all coefficients is the dimension of the total transfer matrix (independent of $L_x$), which is the same as eq. (6.19) of [33] for cyclic strips, namely

$$C_{P,hc,L_y} = \sum_{d=0}^{L_y} n_{P,tor,hc}(L_y, d)b^{(d)} = (q(q - 1))^{L_y/2} .$$

(3.14)

By substituting the expression for $b^{(d)}$ in eq. (2.19), we determine the $n_{P,tor,hc}(L_y, d)$. We list the first few numbers $n_{P,tor,hc}(L_y, d)$ in Table VI. We infer the following recursion relations for $L_y \geq 4$:

$$n_{P,tor,hc}(L_y, 0) = -n_{P,tor,hc}(L_y - 2, 0) + 4n_{P,tor,hc}(L_y - 2, 1) - n_{P,tor,hc}(L_y - 2, 2) - n_{P,tor,hc}(L_y - 2, 3)$$
FIG. 6: Triangle of \( T(n, k) \).

TABLE V: Table of numbers \( n'_{P,tor}(L_y, d) - n_{P,cyc}(L_y, d) \).

| \( L_y \) \( d \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|---|---|
| 2              | 0 | 0 | 0 |   |   |   |   |   |   |   |   |
| 3              | 0 | 1 | 1 | 0 |   |   |   |   |   |   |   |
| 4              | 0 | 4 | 5 | 1 | 0 |   |   |   |   |   |   |
| 5              | 0 | 14| 19| 6 | 1 | 0 |   |   |   |   |   |
| 6              | 0 | 45| 65| 26| 7 | 1 | 0 |   |   |   |   |
| 7              | 0 | 140| 211| 98 | 34 | 8 | 1 | 0 |   |   |   |
| 8              | 0 | 427| 665| 343 | 140 | 43 | 9 | 1 | 0 |   |   |
| 9              | 0 | 1288| 2058| 1148 | 526 | 192 | 53 | 10 | 1 | 0 |   |
| 10             | 0 | 3858| 6294| 3732 | 1866 | 771 | 255 | 64 | 11 | 1 | 0 |

\[
n_{P,tor,hc}(L_y, 1) = -3n_{P,tor,hc}(L_y - 2, 0) + 10n_{P,tor,hc}(L_y - 2, 1) + n_{P,tor,hc}(L_y - 2, 2)
\]

\[
n_{P,tor,hc}(L_y, 2) = -3n_{P,tor,hc}(L_y - 2, 0) + 8n_{P,tor,hc}(L_y - 2, 1) + 4n_{P,tor,hc}(L_y - 2, 2)
\]

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\[ +3n_{P,\text{tor,hc}}(L_y - 2, 3) + n_{P,\text{tor,hc}}(L_y - 2, 4) \]

\[ n_{P,\text{tor,hc}}(L_y, 3) = -n_{P,\text{tor,hc}}(L_y - 2, 0) + 2n_{P,\text{tor,hc}}(L_y - 2, 1) + 3n_{P,\text{tor,hc}}(L_y - 2, 2) \]

\[ +4n_{P,\text{tor,hc}}(L_y - 2, 3) + 3n_{P,\text{tor,hc}}(L_y - 2, 4) + n_{P,\text{tor,hc}}(L_y - 2, 5) \]

\[ n_{P,\text{tor,hc}}(L_y, d) = n_{P,\text{tor,hc}}(L_y - 2, d - 2) + 3n_{P,\text{tor,hc}}(L_y - 2, d - 1) + 4n_{P,\text{tor,hc}}(L_y - 2, d) \]

\[ +3n_{P,\text{tor,hc}}(L_y - 2, d + 1) + n_{P,\text{tor,hc}}(L_y - 2, d + 2) \quad \text{for } 4 \leq d \leq L_y \] (3.15)

By substituting \( q = 0 \) into eq. (3.14) and using eq. (2.15), we have

\[ \sum_{d=0}^{L_y} n_{P,\text{tor,hc}}(L_y, d)(-1)^d = 0 . \] (3.16)

**TABLE VI:** Table of numbers \( n_{P,\text{tor,hc}}(L_y, d) \) for the honeycomb lattice.

| \( L_y \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------|---|---|---|---|---|---|---|---|---|---|----|
| 2         | 1 | 2 | 1 |   |   |   |   |   |   |   |    |
| 4         | 6 | 18| 17| 6 | 1 |   |   |   |   |   |    |
| 6         | 43| 179| 213| 108| 39| 9 | 1 |   |   |   |    |
| 8         | 352| 1874| 2518| 1512| 721| 264| 70| 12| 1 |   |    |
| 10        | 3114| 20202| 29265| 19425| 10800| 4953| 1830| 525| 110| 15| 1  |

It is clear that the number of degree \( d = 0 \) partitions \( n_{P,\text{tor,hc}}(L_y, 0) \) is the same as the corresponding number \( n_{P,\text{cyc,hc}}(L_y, 0) \) for the corresponding cyclic strips (which was denoted \( n_P(hc, L_y, d) \) in [33]). Combining these results with those in Table 2 of [33], we observe that

\[ n_{P,\text{tor,hc}}(L_y, 1) = \frac{N_{P,\text{cyc,hc},L_y,\lambda}}{2} \] (3.17)

where \( N_{P,\text{cyc,hc},L_y,\lambda} \) is the total number of \( \lambda \) for cyclic strips. We list the first few \( n_{P,\text{tor,hc}}(L_y, d) - n_{P,\text{cyc,hc}}(L_y, d) \) in Table VII. Compare Table VII with Table VI, it is easy to see the relation

\[ n_{P,\text{tor,hc}}(L_y, d) - n_{P,\text{cyc,hc}}(L_y, d) = n_{P,\text{tor,hc}}(L_y, d + 1) \quad \text{for } d \geq 2 , \] (3.18)
or equivalently,

\[ n_{P,\text{tor},hc}(L_y, d) = \sum_{d' = d}^{L_y} n_{P,\text{cyc},hc}(L_y, d') \quad \text{for } d \geq 2 \ . \tag{3.19} \]

We also have

\[ n_{P,\text{tor},hc}(L_y, 1) - n_{P,\text{cyc},hc}(L_y, 1) = \sum_{d' = 3}^{L_y} (-1)^{d' + 1} n_{P,\text{tor},hc}(L_y, d') = \sum_{\text{odd } d' = 3}^{L_y - 1} n_{P,\text{cyc},hc}(L_y, d') . \tag{3.20} \]

TABLE VII: Table of numbers \( n_{P,\text{tor},hc}(L_y, d) - n_{P,\text{cyc},hc}(L_y, d) \) for the honeycomb lattice.

| \( L_y \) \( d \) | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 2               | 0   | 0   | 0   |     |     |     |     |     |     |     |     |
| 4               | 0   | 5   | 6   | 1   | 0   |     |     |     |     |     |     |
| 6               | 0   | 77  | 108 | 39  | 9   | 1   | 0   |     |     |     |     |
| 8               | 0   | 996 | 1512| 721 | 264 | 70  | 12  | 1   | 0   |     |     |
| 10              | 0   | 12177| 19425| 10800| 4953| 1830| 525 | 110 | 15  | 1   | 0   |

IV. PROPERTIES OF TRANSFER MATRICES AT SPECIAL VALUES OF PARAMETERS

In this section we derive some properties of the transfer matrices \( T_{Z,A,L_y,d} \) at special values of \( q \) and \( v \), and, correspondingly, properties of \( T_{T,A,L_y,d} \) at special values of \( x \) and \( y \).

A. \( v = 0 \)

From (1.1) or (1.10) it follows that for any graph \( G \), the Potts model partition function \( Z(G, q, v) \) satisfies

\[ Z(G, q, 0) = q^{n(G)} . \tag{4.1} \]

Since this holds for arbitrary values of \( q \), in the context of the lattice strips considered here, it implies that

\[ (T_{Z,A,L_y,d})_{v=0} = 0 \quad \text{for } 1 \leq d \leq L_y , \tag{4.2} \]
i.e. these are zero matrices. Secondly, restricting to toroidal strips for simplicity, and using the basic results \( n = L_y L_x = L_y m \) for \( \Lambda = sq, tri \) and \( n = 2 L_y m \) for \( \Lambda = hc \), eq. (4.1) implies that

\[
Tr[(T_{Z,\Lambda,L_y})^m]_{v=0} = \begin{cases} 
q^{L_y m} & \text{for } \Lambda = sq, tri \\
q^{2L_y m} & \text{for } \Lambda = hc 
\end{cases}.
\]

(4.3)

With our explicit calculations, we have

\[
(T_{Z,\Lambda,L_y,0})_{jk} = 0 \quad \text{for} \quad v = 0, \quad \text{and} \quad j \geq 2,
\]

(4.4)
i.e., all rows of these matrices except the first vanish, and

\[
(T_{Z,\Lambda,L_y,0})_{11} = q^{p L_y} \quad \text{for} \quad v = 0,
\]

(4.5)
where \( p \) was given in eq. (4.8). For this \( v = 0 \) case, since all of the rows except the first are zero, the elements \((T_{Z,\Lambda,L_y,0})_{1k}\) for \( k \geq 2 \) do not enter into \( Tr[(T_{Z,\Lambda,L_y,0})^m] \), which just reduces to the \( m \)'th power of the (1,1) element:

\[
Tr[(T_{Z,\Lambda,L_y,0})^m] = [(T_{Z,\Lambda,L_y,0})_{11}]^m.
\]

(4.6)
Corresponding to this, all of the eigenvalues \( \lambda_{Z,\Lambda,L_y,d,j} \) vanish except for one, which is equal to \((T_{Z,\Lambda,L_y,0})_{11}\) in eq. (4.5). As will be seen, this is reflected in the property that \( \text{det}(T_{Z,\Lambda,L_y,d}) \) has a nonzero power of \( v \) as a factor for all of the lattice-\( \Lambda \) strips considered here. Note that the condition \( v = 0 \) is equivalent to the Tutte variable condition \( y = 1 \).

**B. \( q = 0 \)**

Another fundamental relation that follows, e.g., by setting \( q = 0 \) in eq. (1.10), is

\[
Z(G, 0, v) = 0.
\]

(4.7)
Now the coefficients \( b^{(d)} \) evaluated at \( q = 0 \) satisfy eq. (2.15). Although \( b^{(d)} \) decomposes into two \( b^{(d)}_j \) for \( d > 1 \), one of them is equal to zero and the other is \((-1)^d\) at \( q = 0 \) for the cases we have, namely \( b^{(2)}_1 = 0, b^{(2)}_2 = 1, b^{(3)}_1 = -1, b^{(3)}_2 = 0 \). Hence, in terms of transfer matrices, we derive the sum rule

\[
\sum_{0 \leq \text{even } d \leq L_y} \sum_j (\lambda_{Z,\Lambda,L_y,d,j}(0, v))^m - \sum_{1 \leq \text{odd } d \leq L_y} \sum_j (\lambda_{Z,\Lambda,L_y,d,j}(0, v))^m = 0.
\]

(4.8)
This is similar to a sum rule that we obtained in Ref. [42]. Since it applies for arbitrary \( m \), it implies that there must be a pairwise cancellation between various eigenvalues in different
degree-\(d\) subspaces, which, in turn, implies that at \(q = 0\) there are equalities between these eigenvalues. For example, for the \(L_y = 2\) strip of the square lattice and \(q = 0\), two of the eigenvalues of \(T_{Z,sq,2,1}\) become equal to the eigenvalues of \(T_{Z,sq,2,0}\), while the third eigenvalue of \(T_{Z,sq,2,1}\) becomes equal to \(\lambda_{Z,sq,2,2} = v^2\). Note that setting \(q = 0\) does not, in general, lead to any additional vanishing eigenvalues for the \(T_{Z,\Lambda,L_y,d}\) of the \(\Lambda = sq, hc\) lattices, and hence our formulas below for \(T_{Z,\Lambda,L_y,d}\) do not contain overall factors of \(q\). However, for the triangular lattice, certain eigenvalues do vanish at \(q = 0\).

In terms of the variables \(x\) and \(y\) in the Tutte polynomial, the value \(q = 0\) is equivalent to the value \(x = 1\) (unless \(v = 0\)). In contrast to the vanishing of \(Z(G, q, v)\) at \(q = 0\), the Tutte polynomial \(T(G, x, y)\) is nonzero for general \(y\). This different behavior can be traced to the feature that in eq. (1.14), \(Z(G, q, v)\) is proportional to \(T(G, x, y)\) multiplied by the factor \((x - 1)^k(G) = (x - 1)\), so at \(x = 1\), \(Z(G, 0, v) = 0\) even if \(T(G, 1, y) \neq 0\).

C. \(v = -1\), i.e., \(y = 0\)

As discussed above, the special value \(v = -1\), i.e., \(y = 0\), corresponds to the zero-temperature Potts antiferromagnet, and in this case the Potts model partition function reduces to the chromatic polynomial, as indicated in eq. (1.6). In the previous two sections we have determined how the dimensions \(n_{Z,tor}(L_y, d)\) for the square, triangular, and honeycomb lattice strips reduce to the dimensions \(n_{P,tor}(L_y, d)\). In all cases, in each degree-\(d\) subspace for \(0 \leq d \leq L_y - 1\) for \(\Lambda = sq, tri, hc\), some eigenvalues vanish, so that \(n_{P,tor,\Lambda}(L_y, d) < n_{Z,tor,\Lambda}(L_y, d)\) and hence \(\det(T_{Z,\Lambda,L_y,d}) = 0\) at \(v = -1\). This property is reflected in the powers of \((v + 1)\) and \(y\) that appear, respectively, in our formulas below for \(\det(T_{Z,\Lambda,L_y,d})\) and \(\det(T_{T,\Lambda,L_y,d})\).

D. \(v = -q\), i.e., \(x = 0\)

For the graph \(G = G(V, E)\), setting \(x = 0\) and \(y = 1 - q\) in the Tutte polynomial \(T(G, x, y)\) yields the flow polynomial \(F(G, q)\), which counts the number of nowhere-0 \(q\)-flows (without sinks or sources) that there are on \(G\) \(\text{[13]}\):

\[
F(G, q) = (-1)^{e(G) - n(G) + 1}T(G, 0, 1 - q) .
\]  

Therefore, the flow polynomial for toroidal lattice strips has the form

\[
F(\Lambda, L_y \times L_x, tor., q) = \sum_{d=0}^{L_y} \sum_{j} b_j^{(d)} (\lambda_{F,\Lambda,L_y,d,j})^m .
\] (4.10)
We found that for all $L_y \geq 1$ and $0 \leq d \leq L_y - 1$ for $\Lambda = sq, hc$, when one sets $x = 0$ in the Tutte polynomial, or equivalently, $q = -v$ in the Potts model partition function, some of the eigenvalues in each degree-$d$ subspace vanish, and hence $\det(T_{Z, \Lambda, L_y, d})$ and $\det(T_{T, \Lambda, L_y, d})$ vanish. This is reflected in the powers of $(q + v)$ and $x$ that appear, respectively, in our formulas below for $\det(T_{Z, \Lambda, L_y, d})$ and $\det(T_{T, \Lambda, L_y, d})$. The expressions for the determinants for strips of the triangular lattice are more complicated because of the fact alluded to above that one has to work with width-$(L_y + 1)$ matrices and then identify the two end vertices.

V. GENERAL RESULTS FOR TOROIDAL STRIPS

In this section we present general results that we have obtained for transfer matrices of toroidal strips and their properties. These are valid for arbitrarily large strip widths $L_y \geq 2$ (as well as arbitrarily great lengths).

A. Determinants

For the square lattice, we find

$$\det(T_{T, sq, L_y, d}) = (xy)^d \frac{L_y n_{Z, tor}(L_y - 1, d)}{L_y} \quad (5.1)$$

where $n_{Z, tor}(L_y, d)$ was given by eq. (2.21)-(2.23). This result applies for all $d$, i.e., $0 \leq d \leq L_y$ since $n_{Z, tor}(L_y - 1, d) = 0$ for $d > L_y - 1$ (c.f. eq. (2.20)). This is equivalent to the somewhat more complicated expression for $\det(T_{Z, sq, L_y, d})$:

$$\det(T_{Z, sq, L_y, d}) = v^d \frac{L_y n_{Z, tor}(L_y, d)}{L_y} \left[ \left( 1 + \frac{q}{v} \right) (1 + v) \right]^d \frac{L_y n_{Z, tor}(L_y - 1, d)}{L_y}. \quad (5.2)$$

These determinant formulas can be explained as follows. By the arrangement of the partitions, as shown in Figs. 2 and 3, the matrix $J_{L_y, d, i, i+1}$ has the lower triangular form and the matrix $D_{L_y, d, i}$ has the upper triangular form. From the definition of the transfer matrix in eq. (2.35), the determinant of $T_{Z, sq, L_y, d}$ is the product of the diagonal elements of $H_{Z, sq, L_y, d}$ and $V_{Z, sq, L_y, d}$. The diagonal elements of $H_{Z, sq, L_y, d}$ have the form $(1 + v)^r$, where $r$ is the number of edges in the corresponding partition, and the diagonal elements of $V_{Z, sq, L_y, d}$ have the form $v^{L_y} (1 + q/v)^s$, where $s$ is the number of vertices which do not connect to any other vertex in the corresponding partition. Therefore, the power of $(1 + v)$ in eq. (5.2) is the sum of the number of nearest-neighbor edges of all the $(L_y, d)$-partitions. Let us compare the $(L_y, d)$-partitions and $(L_y - 1, d)$ partitions. Since each edge of a $(L_y, d)$-partition
corresponds to adding a new vertex in a \((L_y - 1, d)\)-partition (at \(L_y\) possible places) and connecting it with the vertex above it, the power of \((1 + v)\) is \(d! L_y n_{Z,tor}(L_y - 1, d)\), where the factor \(d!\) comes from the permutation of black circles. It is clear that the power of \(v\) is \(d! L_y n_{Z,tor}(L_y, d)\). The power of \((1 + q/v)\) in eq. 5.2 is the sum of the number of unconnected vertices of all the \((L_y, d)\)-partitions. Now consider the \((L_y, d)\)-partitions and \((L_y - 1, d)\) partitions again. Since each unconnected vertex of a \((L_y, d)\)-partition corresponds to adding a unconnected vertex to a \((L_y - 1, d)\)-partition in \(L_y\) possible ways, the power of \((1 + q/v)\) is again \(d! L_y n_{Z,tor}(L_y, d)\).

Next, taking into account that the generalized multiplicity is \(b^{(d)}\), we have, for the total determinant

\[
\det(T_{Z, sq, L_y}) = \prod_{d=0}^{L_y} (y - 1)^{L_y n_{Z,tor}(L_y, d)b^{(d)}} (xy)^{L_y n_{Z,tor}(L_y - 1, d)b^{(d)}}
\]

\[
= (y - 1)^{L_y \sum_{d=0}^{L_y} n_{Z,tor}(L_y, d)b^{(d)}} (xy)^{L_y \sum_{d=0}^{L_y} n_{Z,tor}(L_y - 1, d)b^{(d)}}. \tag{5.3}
\]

Using eq. 2.19 together with eq. 2.20 so that \(\sum_{d=0}^{L_y} n_{Z,tor}(L_y - 1, d)b^{(d)} = \sum_{d=0}^{L_y-1} n_{Z,tor}(L_y - 1, d)b^{(d)}\), we have, finally,

\[
\det(T_{Z, sq, L_y}) = (y - 1)^{L_y q^{L_y}} (xy)^{L_y q^{L_y - 1}}. \tag{5.4}
\]

This agrees with the conjecture given as eq. (3.70) of our earlier Ref. 8 for the determinant of the transfer matrix of the toroidal strip of the square lattice with arbitrary width \(L_y\).

For the honeycomb lattice, we find

\[
\det(T_{T, hc, L_y, d}) = (x^2 y)^{d! L_y n_{Z,tor}(L_y - 1, d)}. \tag{5.5}
\]

Equivalently,

\[
\det(T_{Z, hc, L_y, d}) = (v^2)^{d! L_y n_{Z,tor}(L_y, d)} \left[\left(1 + \frac{q}{v}\right)^2 (1 + v)\right]^{d! L_y n_{Z,tor}(L_y - 1, d)}. \tag{5.6}
\]

This can be understood as follows: by an argument similar to that given before, the power of \((1 + v)\) is the same as for the square lattice case. Comparing \(T_{Z, sq, L_y, d}\) and \(T_{Z, hc, L_y, d}\) in eq. 2.35, one sees that \(V_{Z, hc, L_y, d} = V_{Z, sq, L_y, d}\) has been multiplied twice for the honeycomb lattice, so that the powers of \(v\) and \((1 + q/v)\) become twice of the corresponding powers for the square lattice.

Taking into account that the generalized multiplicity is \(b^{(d)}\), the total determinant for the \(hc\) lattice is given by

\[
\det(T_{T, hc, L_y}) = (x^2 y)^{L_y q^{L_y - 1}}. \tag{5.7}
\]
Equivalently, \( \det(T_{Z, hc, L_y}) = (v^2)_{L_y} q^{L_y} \left[ \left( 1 + \frac{q}{v} \right)^2 (1 + v) \right] \left[ q^{L_y-1} \right] \). (5.8)

It is clear that \( \det(T_{T, hc, L_y, d}) \) is related to \( \det(T_{T, sq, L_y, d}) \) by the replacement \( x \to x^2 \) (holding \( y \) fixed). This, together with the fact that \( n_{Z, tor}(L_y, d) \) is the same for these lattices means that the total determinants \( \det(T_{T, hc, L_y}) \) is related to \( \det(T_{T, sq, L_y}) \) by the same respective replacements. Correspondingly, \( \det(T_{Z, hc, L_y, d}) \) is related to \( \det(T_{Z, sq, L_y, d}) \) by the replacements of the respective factors \( v \) by \( v^2 \) (cf. eq. (1.18)) and \( (1 + q/v) \) by \( (1 + q/v)^2 \).

**B. Traces**

The trace of the total transfer matrix is the \( m = 1 \) case in eq. (1.8) or (1.19). For strips of the square lattice, this corresponds to a \( L_y \)-vertex circuit with a loop attached to each vertex, as illustrated in Fig. 7. We have \( Tr(T_{T, sq, L_y}) \) given by the corresponding Tutte polynomial,

\[
Tr(T_{T, sq, L_y}) = y^{L_y} \left( y - 1 + \frac{x^{L_y} - 1}{x - 1} \right).
\] (5.9)

Equivalently, we find

\[
Tr(T_{Z, sq, L_y}) = (1 + v)^{L_y} [(v + q)^{L_y} + (q - 1)v^{L_y}] \right].
\] (5.10)

In principle, we can also consider the \( m = 1 \) case for the Klein bottle strips, but the result is not as simple as that listed here.

For the triangular lattice, the total trace corresponds to a \( L_y \)-vertex circuit with each edge doubled and with a loop attached to each vertex as illustrated in Fig. 8. Therefore, the trace is given by the Tutte polynomial of this graph

\[
Tr(T_{T, tri, L_y}) = y^{L_y} \left( (y - 1)(y + 1)^{L_y} + \sum_{j=0}^{L_y-1} (y + 1)^j (x + y)^{L_y-1-j} \right).
\] (5.11)
Equivalently,

\[
Tr(T_{Z,\text{tri},L_y}) = q(v + 1)^{L_y} \left( v^2 + 2v \right)^{L_y} + \sum_{j=0}^{L_y-1} (v^2 + 2v)^j (v^2 + 2v + q)^{L_y-1-j} \right), \quad (5.12)
\]

For the honeycomb lattice, the total trace corresponds to a \(2L_y\)-vertex circuit with every other edge doubled as shown in Fig. 9. Hence, the trace is given by the Tutte polynomial of this graph

\[
Tr(T_{T,\text{hc},L_y}) = (x+y)^{L_y} \left( \frac{x^{L_y} - 1}{x - 1} \right) + (y-1)(y+1)^{L_y} + \sum_{j=0}^{L_y-1} (y+1)^j (x+y)^{L_y-1-j} \right). \quad (5.13)
\]

Equivalently,

\[
Tr(T_{Z,\text{hc},L_y}) = (v^2 + 2v + q)^{L_y} \left( (v + q)^{L_y} - v^{L_y} \right)
+ qv^{L_y} \left( v^2 + 2v \right)^{L_y} + \sum_{j=0}^{L_y-1} (v^2 + 2v)^j (v^2 + 2v + q)^{L_y-1-j} \right). \quad (5.14)
\]
C. Eigenvalues for $d = L_y$ and $d = L_y - 1$ for $\Lambda = sq, hc$

It was shown earlier \[38, 39\] that the $\lambda$’s for a strip with Klein bottle boundary conditions are a subset of the $\lambda$’s for the same strip with torus boundary conditions. From eq. \(2.23\) one knows that there is only one $\lambda$, denoted as $\lambda_{Z,\Lambda, L_y, L_y}$, for degree $d = L_y$ for strips of the square and honeycomb lattices because all the permutation of black circles are equivalent. That is, for this value of $d$, the transfer matrix reduces to $1 \times 1$, i.e. is a scalar. We found that for a toroidal or Klein bottle strip of the square, or honeycomb lattice with width $L_y$,

$$\lambda_{T,\Lambda, L_y, L_y} = 1.$$ (5.15)

Equivalently, in terms of Potts model variables,

$$\lambda_{Z, sq, L_y, L_y} = v^{L_y}$$ (5.16)

$$\lambda_{Z, hc, L_y, L_y} = v^{2L_y}.$$ (5.17)

For the width $L_y$ triangular strips, because one has to start by constructing the width-$(L_y + 1)$ transfer matrices, there can be more than one $\lambda$ even for degree $d = L_y$.

Concerning the $\lambda$’s for degree $d = L_y - 1$, we find that one of these is the same, independent of $L_y$, for the toroidal square strips we have calculated, namely

$$\lambda_{T, sq, L_y, L_y - 1, 1} = x.$$ (5.18)

This also appears for degree $d = L_y - 1$ in the case of the cyclic square-lattice strips, and we conjecture that all the toroidal square-lattice strips have this eigenvalue. For the strips of the square lattice with $L_y = 3, 4$, there is another common $\lambda$ for degree $d = L_y - 1$, namely

$$\lambda_{T, sq, L_y, L_y - 1, 2} = y.$$ (5.19)

For the toroidal honeycomb strip with $L_y = 4$, one of the eigenvalues for degree $d = L_y - 1$, say that for $j = 1$, is given by

$$\lambda_{T, hc, L_y, d=L_y-1, j=1} = x^2$$ (5.20)

as for the corresponding cyclic strip. We conjecture that all of the toroidal strips of the honeycomb lattice have this eigenvalue. In terms of Potts model quantities these results are

$$\lambda_{Z, sq, L_y, L_y - 1, 1} = v^{L_y - 1}(v + q)$$

$$\lambda_{Z, sq, L_y, L_y - 1, 2} = v^{L_y}(v + 1) \quad \text{except for } L_y = 2$$

$$\lambda_{Z, hc, L_y, L_y - 1, 1} = v^{2(L_y-1)}(v + q)^2.$$ (5.21)

The expressions for the other eigenvalues are, in general, more complicated.
VI. SOME ILLUSTRATIVE CALCULATIONS

A. Square-Lattice Strip, \( L_y = 2 \)

The toroidal strip of the square lattice with width \( L_y = 2 \) is equivalent to the ladder graph with all the transverse edges doubled. The Potts model partition function \( Z(sq, L_y \times m, q, v) \) and Tutte polynomial \( T(sq, L_y \times m, x, y) \) were calculated for the toroidal and Klein bottle strips of the square lattice with width \( L_y = 2 \) in Ref. [8]. We express the results here in terms of transfer matrices \( T_{Z,sq,d} \). For \( d = 0 \), we have

\[
T_{Z,sq,0} = \begin{pmatrix}
q^2 + 4qv + qv^2 + 5v^2 + 2v^3 (q + 2v)(1 + v)^2 \\
v^3(2 + v) \\
v^2(1 + v)^2
\end{pmatrix}.
\]

The eigenvalues of \( T_{Z,sq,0} \) are the same as \( \lambda_{Z,s2t,(5,6)} \) given as eqs. (3.16) and (3.17) of [8]. For \( d = 1 \), we have

\[
T_{Z,sq,1} = v \begin{pmatrix}
q + 3v + v^2 & v(2 + v) & (1 + v)^2 \\
v(2 + v) & q + 3v + v^2 & (1 + v)^2 \\
v^2(2 + v) & v^2(2 + v) & v(1 + v)^2
\end{pmatrix}.
\]

The eigenvalues of \( T_{Z,sq,1} \) are the same as \( \lambda_{Z,s2t,(2,3,4)} \) given as eqs. (3.13) to (3.15) of [8]. The matrix \( T_{T,sq,2,2} = v^2 \) has been given above.

B. Square-Lattice Strip, \( L_y = 3 \)

For the \( L_y = 3 \) toroidal and Klein bottle strips of the square lattice, the Potts model partition function \( Z(sq, L_y \times m, q, v) \) and Tutte polynomial \( T(sq, L_y \times m, x, y) \) were calculated in Ref. [8]. We express the results here in terms of transfer matrices \( T_{Z,sq,3,d} \). For \( d = 0 \), we have

\[
T_{Z,sq,0} = \begin{pmatrix}
s_1 & v_1s_2 & v_1s_2 & v_1s_2 & v_1^3s_5 \\
v_3^3s_3 & v^2v_1s_4 & v^3v_1v_2 & v^3v_1v_2 & v^2v_1^3 \\
v_3^3s_3 & v^3v_1v_2 & v^3v_1v_2 & v^3v_1v_2 & v^2v_1^3 \\
v_3^3s_3 & v^3v_1v_2 & v^3v_1v_2 & v^3v_1s_4 & v^2v_1^3 \\
v_5v_3 & v^4v_1v_2 & v^4v_1v_2 & v^4v_1v_2 & v^3v_1^3
\end{pmatrix}.
\]

where we use the notations \( v_1 = 1 + v \), \( v_2 = 2 + v \), \( v_3 = 3 + v \) and

\[
s_1 = q^3 + 6q^2v + 15qv^2 + qv^3 + 16v^3 + 3v^4
\]
The eigenvalues of $T_{\lambda}$ consist of a linear term $v^2(v+q)(v+1)$ and roots of a cubic equation. The linear term, denoted as $\lambda_{Z,sq,10}$ in [8], has multiplicity two, so the corresponding coefficient is $2b^{(0)} = 2$. The cubic equation is the same as eqs. (3.48) to (3.51) of [8]. For $d = 1$, we have

$$s_2 = q^2 + 5qv + qv^2 + 8v^2 + 3v^3$$
$$s_3 = q + 4v + v^2$$
$$s_4 = q + 3v + v^2$$
$$s_5 = q + 3v .$$

(6.4)

The eigenvalues of $T_{Z,sq,3,0}$ consist of the roots of a cubic equation, which enter with multiplicity two, and the roots of a quartic equation. The cubic equation is given by eqs. (2.23) to (2.26) in [8], and since it has multiplicity two, its roots have coefficients $2b^{(1)} = 2(q-1)$. The quartic equation is the same as that given in eqs. (3.43) to (3.47) of [8]. For $d = 2$, we

where

$$s_6 = q^2 + 5qv + 8v^2 + v^3 .$$

(6.6)
The eigenvalues of $T_{Z,sq,3,2}$ consist of two linear terms and roots of three quadratic equations; two of the quadratic equations occur with multiplicity two. Both of the linear terms $v^2(v+q)$ and $v^3(v+1)$ have the coefficient $b^{(2)}_2 = (q-1)(q-2)/2$. One of the quadratic equations is given by eq. (3.39) of [8] with coefficient $q(q-3)/2$. The other two quadratic equations have multiplicity two, as noted; one of them is given by eq. (2.20) of [8] and its roots have coefficient $2b^{(2)}_2 = (q-1)(q-2)$. The other is given by eq. (2.21) of [8] and its roots have coefficient $2b^{(2)}_1 = q(q-3)$. The matrix $T_{T,sq,3,3} = v^3$ has been given above and has coefficient $b^{(3)}$.

C. Square-Lattice Strip, $L_y = 4$

For the $L_y = 4$ toroidal and Klein bottle strips of the square lattice, the sizes of matrices are $14 \times 14$, $35 \times 35$, $56 \times 56$, and $48 \times 48$ for levels $0 \leq d \leq 3$, as indicated in Table I. The eigenvalues of $T_{Z,sq,4,0}$ are roots of three quadratic equations and a sixth degree equation. One of the quadratic equations has multiplicity two. The eigenvalues of $T_{Z,sq,4,1}$ consist of a linear term $v^3(1+v)(q+v)$, roots of a cubic equation, a sixth-degree equation, an eighth-degree equation and a ninth-degree equation. The eighth-degree equation has multiplicity two. The eigenvalues of $T_{Z,sq,4,2}$ consist of a linear term, roots of two cubic equations, three quartic equations, two sixth-degree equations and an eighth-degree equation. The linear term is again $v^3(1+v)(q+v)$ which has multiplicity four with coefficient $2(q^2 - 3q + 1)$. The eighth-
degree equation and one of the sixth-degree equations have multiplicity two. The eigenvalues of $T_{Z, sq, 4, 3}$ consist of two linear terms $v^4(1 + v)$ and $v^3(v + q)$, and roots of four quadratic equations and a quartic equation. The linear terms and one of the quadratic equations have multiplicity two, and the other equations have multiplicity four. The matrix $T_{Z, sq, 4, 4} = v^4$ has been given above with coefficient $b^{(4)}$. Some of these equations are too lengthy to list here; they are available from the authors. At the special value $v = -1$, the Potts model partition function reduces to the chromatic polynomial. The non-zero eigenvalues are the same as $\lambda_{st4, j}$ for $1 \leq j \leq 33$ from eqs. (2.3) to (2.21) in [39].

D. Triangular-Lattice Strip, $L_y = 2$

The toroidal strip of the triangular lattice with $L_y = 2$ is equivalent to the ladder graph with next-nearest-neighbor coupling and all transverse edges doubled. The corresponding Klein bottle strip is the same as the toroidal strip. For $d = 0$, we have

$$T_{Z, tri, 2, 0} = \begin{pmatrix}
q^2 + 6qv + qv^2 + 12v^2 + 8v^3 + 2v^4 & (q + 4v + 2v^2)(1 + v)^2 \\
v^2(2q + 12v + 13v^2 + 6v^3 + v^4) & v^2(2 + v)^2(1 + v)^2
\end{pmatrix}.$$ (6.8)

The eigenvalues of $T_{Z, tri, 2, 0}$, denoted as $\lambda_{tt2, (1,2)}$, are roots of the following equation:

$$\xi^2 - (v^6 + q^2 + 6v^5 + 20v^3 + 15v^4 + 16v^2 + 2qv^2 + 6vq)v^2(1 + v)^2(2v^4 + 4v^3 + 4v^2q + 11vq^2 + q^2 + 4q^2v + 2q^2) = 0.$$ (6.9)

For $d = 1$, we have

$$T_{Z, tri, 2, 1} = v \begin{pmatrix}
q + 6v + 4v^2 + v^3 & q + 6v + 4v^2 + v^3 & (2 + v)(1 + v)^2 \\
q + 6v + 4v^2 + v^3 & q + 6v + 4v^2 + v^3 & (2 + v)(1 + v)^2 \\
v(q + 12v + 13v^2 + 6v^3 + v^4) & v(q + 12v + 13v^2 + 6v^3 + v^4) & v(2 + v)^2(1 + v)^2
\end{pmatrix}.$$ (6.10)

The eigenvalues of $T_{Z, tri, 2, 1}$ consist of a zero, $\lambda_{tt2, 0} = 0$, and roots of the following quadratic equation:

$$\xi^2 - v(v^5 + 15v^3 + 20v^2 + 16v + 2q)v^2(2 + v)(1 + v)^2(v^2 + vq + q) = 0.$$ (6.11)

with roots $\lambda_{tt2, (3,4)}$. Their coefficients are equal to $b^{(4)} = q - 1$. For $d = 2$, we have

$$T_{Z, tri, 2, 2} = v^2 \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}.$$ (6.12)
$T_{Z,\text{tri},2,2}$ has two eigenvalues. One of them is $\lambda_{t2,5} = 2v^2$ with coefficient $b_1^{(2)} = q(q - 3)/2$; while the other eigenvalue is zero with coefficient $b_2^{(2)} = (q - 1)(q - 2)/2$. Therefore, the Potts model partition function for the triangular lattice strip with $L_y = 2$ and toroidal boundary condition is given by

$$Z(tri, 2 \times L_x, \text{tor.}, q, v) = b^{(0)} \sum_{j=1}^{2} (\lambda_{t2,j})^{L_x} + b^{(1)} \sum_{j=3}^{4} (\lambda_{t2,j})^{L_x} + b_1^{(2)} (\lambda_{t2,5})^{L_x} . \quad (6.13)$$

**E. Triangular-Lattice Strip, $L_y = 3$**

We illustrate our results for the $L_y = 3$ toroidal strip of the triangular lattice. For $d = 0$, we have

$$T_{Z,\text{tri},3,0} = \begin{pmatrix}
t_1 & v_1t_2 & v_1t_2 & v_1t_2 & v_1t_5 \\
v^2t_3 & v^2v_1v_2t_4 & v^2v_1v_2t_4 & v^2v_1t_6 & v^2v_3^2v_2^2 \\
v^2t_3 & v^2v_1t_6 & v^2v_1v_2t_4 & v^2v_1v_2t_4 & v^2v_3^2v_2^2 \\
v^2t_3 & v^2v_1v_2t_4 & v^2v_1t_6 & v^2v_1v_2t_4 & v^2v_3^2v_2^2 \\
v^4v_3t_7 & v^3v_1v_2t_8 & v^3v_1v_2t_8 & v^3v_1v_2t_8 & v^3v_3^3v_2^2 \\
\end{pmatrix} . \quad (6.14)$$

where we define

$$
\begin{align*}
t_1 &= q^3 + 9q^2v + 33qv^2 + 4qv^3 + 50v^3 + 21v^4 + 3v^5 \\
t_2 &= q^2 + 8qv + 2qv^2 + 20v^2 + 14v^3 + 3v^4 \\
t_3 &= q^2 + 10qv + 2qv^2 + 30v^2 + 22v^3 + 7v^4 + v^5 \\
t_4 &= q + 6v + 4v^2 + v^3 \\
t_5 &= q + 6v + 3v^2 \\
t_6 &= 2q + 14v + 14v^2 + 6v^3 + v^4 \\
t_7 &= 3q + 18v + 15v^2 + 6v^3 + v^4 \\
t_8 &= q + 12v + 13v^2 + 6v^3 + v^4 .
\end{align*}
$$

(6.15)

The characteristic equation of $T_{Z,\text{tri},3,0}$ yields the following cubic and quadratic equations:

$$\xi^3 + f_{31}\xi^2 + f_{32}\xi + f_{33} = 0 \quad (6.16)$$

where

$$f_{31} = -(q^3 + v^9 + 39v^2q + 9v^8 + 36v^7 + 84v^6 + 129v^5 + 137v^4 + 96v^3 + 12v^3q + 2v^4q + 9q^2v)$$

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\[ f_{32} = v^2(1 + v)(46q^3v + 76q^3v^2 + 246q^2v^5 + 56v^3q^3 + 1190v^6q + 72q^2v^6 + q^3v^6 + 24v^9q \\
+ 2v^{10}q + 3q^4 + 35v^{10} + 3v^{11} + 9v^7q^2 + 534v^7q + 147v^8q + 483v^3q^2 + 213v^2q^2 \\
+ 8q^3v^5 + 28q^3v^4 + 1272v^4q + 2q^4v + 414v^3q + 794v^5 + 192v^4 + 1618v^5q + 1091v^7 \\
+ 1243v^6 + 188v^9 + 580v^8 + 455v^4q^2) \]

\[ f_{33} = -v^5(1 + v)^4(22q^3v + 72q^3v^2 + 105q^2v^5 + 105v^3q^3 + 72v^6q + 24q^2v^6 + 6q^4 + 19v^7q \\
+ 90v^3q^2 + 30v^2q^2 + 2v^4q^4 + 12q^3v^3 + 24q^4v^2 + 12q^5v^5 + 62q^3v^4 + 60v^4q + 19q^4v \\
+ 24v^3q + 24v^5 + 8v^4 + 90v^5q + 22v^7 + 30v^6 + 6v^8 + 148v^4q^2) \] (6.17)

with roots \( \lambda_{3,j} \) for \( 1 \leq j \leq 3 \) and

\[ \xi^2 - v^3(q - 2)(1 + v)\xi + v^6(q - 2)^2(1 + v)^2 = 0 \] (6.18)

with roots \( \lambda_{3,j} \) for \( j = 4, 5 \). For \( d = 1 \), we have

\[ T_{Z,tri,3,1} = v \times \]

\[
\begin{pmatrix}
  t_9 & t_9 & vt_{10} & v_1t_{11} & v_1t_4 & v_1t_4 & v_1v_2s_3 & v_1v_2v_2^2 & v_1^3v_2 & v_1t_{11} \\
vt_{10} & t_9 & t_9 & v_1v_2v_2^2 & v_1v_2s_3 & v_1t_4 & v_1t_4 & v_1t_{11} & v_1^3v_2 & v_1t_{11} \\
  t_9 & vt_{10} & t_9 & v_1t_{11} & v_1t_4 & v_1v_2s_3 & v_1t_4 & v_1t_{11} & v_1^3v_2 & v_1v_2v_2^2 \\
v^2t_{12} & v^2t_{12} & 0 & v^2v_1v_2 & 0 & 0 & v^2v_1v_2 & 0 & 0 & v^2v_1v_2 \\
v^2t_{13} & v^2t_{13} & vt_{14} & v^2v_1v_2t_{15} & v_1v_2t_4 & v_1v_2t_4 & v_1t_8 & v_1v_1t_{16} & v_1^3v_2v_2 & v_1^2v_1v_2t_{15} \\
v^2t_{14} & v^2t_{13} & v^2t_{13} & v_1v_1t_{16} & v_1t_8 & v_1v_1t_4 & v_1v_2t_4 & v^2v_1v_2t_{15} & v_1^3v_2v_2 & v_1^2v_1v_2t_{15} \\
v^2t_{13} & vt_{14} & v^2t_{13} & v^2v_1v_2t_{15} & v_1v_2t_4 & v_1v_2t_4 & v_1t_8 & v_1v_1t_4 & v^2v_1v_2t_{15} & v_1^3v_2v_2 \\
v^2t_{12} & 0 & v^2t_{12} & v^2v_1v_2 & 0 & v^2v_1v_2 & 0 & v^2v_1v_2 & 0 & 0 \\
v^3v_3t_{17} & v^3v_3t_{17} & v^3v_1v_2v_3t_{18} & v^2v_1v_2t_8 & v^2v_1v_2t_8 & v^2v_1v_2t_8 & v^2v_1v_2t_8 & v^3v_1v_2v_3t_{18} & v^2v_1^3v_2^2 & v^3v_1v_2v_3t_{18} \\
  0 & v^2t_{12} & v^2t_{12} & 0 & v^2v_1v_2 & 0 & 0 & v^2v_1v_2 & 0 & v^2v_1v_2 \\
\end{pmatrix}
\]

(6.19)

where

\[ t_9 = q^2 + 8qv + qv^2 + 20v^2 + 8v^3 + v^4 \]
\[ t_{10} = 2q + 10v + 5v^2 + v^3 \]
\[ t_{11} = q + 8v + 5v^2 + v^3 \]
\[ t_{12} = q + 6v + 2v^2 \]
\[ t_{13} = 4q + qv + 24v + 20v^2 + 7v^3 + v^4 \]
\[ t_{14} = q^2 + 9qv + 2qv^2 + 30v^2 + 22v^3 + 7v^4 + v^5 \]
\[ t_{15} = 5 + 4v + v^2 \]
\[ t_{16} = q + 14v + 14v^2 + 6v^3 + v^4 \]
\[ t_{17} = 2q + 18v + 15v^2 + 6v^3 + v^4 \]
\[ t_{18} = 4 + 3v + v^2 . \quad (6.20) \]

The characteristic equation of \( T_{Z,tri,3,1} \) yields the following quartic equation and sixth-degree equation:

\[ \xi^4 + f_{41}\xi^3 + f_{42}\xi^2 + f_{43}\xi + f_{44} = 0 \quad (6.21) \]

where

\[ f_{41} = -v(84v^5 + 36v^6 + 9v^7 + 2v^3q + 9v^2q + 2q^2 + v^8 + 140v^3 + 130v^4 + 98v^2 + 23vq) \]
\[ f_{42} = v^3(1 + v)(4v^{10} + 46v^9 + 2v^9q + 242v^8 + 22v^8q + 118v^7q + 736v^7 + 383v^6q + 2v^2q^6 + 1388v^5 + 797v^5q + 16q^2v^5 + 1638v^5 + 1071v^4q + 1164v^4 + 56v^4q^2 + 920v^3q + 384v^3 + 109v^3q^2 + 396v^2q + 141v^2q^2 + 94q^2v + 3q^3v + 6q^3) \]
\[ f_{43} = -v^6(1 + v)^2(33q^3v + 48q^3v^2 + 111q^2v^5 + 42v^3q^3 + 834v^6q + 19q^2v^6 + 4v^9q + 3v^{10} + 262v^7q + 46v^6q + 348v^3q^2 + 277v^2q^2 + 3q^3v^5 + 18q^3v^4 + 100q^2v + 360v^2q + 1836v^4q + 1248v^3q + 1277v^5 + 812v^4 + 12q^3 + 1585v^5q + 196v^3 + 602v^7 + 1127v^9 + 35v^9 + 194v^8 + 268v^4q^2) \]
\[ f_{44} = v^9(v + 2)^2(1 + v)^5(6v^6 + 18v^5q + 10v^5 + 34v^4q + 18v^4q^2 + 10v^4 + 43v^3q^2 + 4v^3 + 6v^3q^2 + 25v^3q + 10v^2q + 18q^3v^2 + 34v^2q^2 + 18q^3v + 10q^2v + 6q^3) \quad (6.22) \]

with roots \( \lambda_{3,j} \) for \( 6 \leq j \leq 9 \) and

\[ \xi^6 + f_{61}\xi^5 + f_{62}\xi^4 + f_{63}\xi^3 + f_{64}\xi^2 + f_{65}\xi + f_{66} = 0 \quad (6.23) \]

where

\[ f_{61} = -v(7v^3 + 2v^4 + 12v^2 + q^2 + v^3q + 7vq + 3v^2q) \]
\[ f_{62} = v^2(4v^8 + 168v^4 + 208v^5 + 117v^6 + 32v^7 + 3v^7q + q^4 + q^2v^6 + 14q^3v + 2v^3q^3 + 6q^3v^2 + 73v^2q^2 + 21v^6q + 26v^4q^2 + 168v^4q + 172v^3q + 55v^3q^2 + 6q^2v^5 + 84v^5q) \]
\[ f_{63} = -v^6(1 + v)(2v^7q - 8v^7 + 11v^6q + q^2v^6 - 60v^6 + 24v^5q + 7q^2v^5 - 200v^5 - 10v^4q - 336v^4 + 27v^4q^2 + 2v^3q^3 + 47v^3q^2 - 116v^3q - 272v^3 - 200v^2q + 7q^3v^2 + 32v^2q^2) \]
\[+13q^3v - 48q^2v + q^4 - 4q^3\]

\[f_{64} = v^8(1 + v)^2(22q^3v + 9q^3v^2 + 38q^2v^5 + 17v^3q^3 + v^8q^2 - 212v^6q + 28q^2v^6 + 2q^4 + 8v^7q^2 - 64v^7q^2 - 8v^8q - 14v^3q^2 + 104v^2q^4 + 4v^4q^2 + 2q^3v^5 + 9q^3v^4 - 4v^4q + q^4v + 344v^3q + 872v^5 + 672v^4 - 288v^5q + 136v^7 + 492v^6 + 16v^8 + 3v^4q^2)\]

\[f_{65} = v^{12}(q - 4)(v + 2)(1 + v)^3(v^3 + 4v^2 + 6v + q)(q + 2v)^2\]

\[f_{66} = v^{14}(v + 2)^2(1 + v)^4(q + 2v)^4\]

with roots \(\lambda_{r3,j}\) for \(10 \leq j \leq 15\). Their coefficients are equal to \(b^{(1)} = q - 1\). For \(d = 2\), we have

\[
T_{Z,tri,3,2} = v^2 \\
\begin{pmatrix}
  t_{19} & v & t_{12} & 0 & t_{19} & v & v_{1}v_{2} & 0 & v_{1} & v_{1} & v_{1}v_{2} & 0 \\
  v & t_{19} & 0 & t_{12} & v & t_{19} & 0 & v_{1}v_{2} & v_{1} & v_{1} & 0 & v_{1}v_{2} \\
  v & t_{19} & t_{19} & v & t_{12} & 0 & v_{1} & v_{1} & 0 & v_{1}v_{2} & v_{1}v_{2} & 0 \\
  t_{19} & v & v & t_{19} & 0 & t_{12} & v_{1} & v_{1} & v_{1}v_{2} & 0 & 0 & v_{1}v_{2} \\
  0 & t_{12} & v & t_{19} & t_{19} & v & 0 & v_{1}v_{2} & 0 & v_{1}v_{2} & v_{1}v_{2} & 0 \\
  t_{12} & 0 & t_{19} & v & v & t_{19} & v_{1}v_{2} & 0 & v_{1}v_{2} & 0 & v_{1} & v_{1} \\
  v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & 0 \\
  v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & 0 \\
  v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & 0 \\
  0 & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & 0 \\
  0 & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & 0 \\
  0 & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & 0 \\
  v_{1}t_{12} & 0 & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & 0 \\
  v_{1}t_{12} & 0 & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & 0 \\
  v_{1}t_{12} & 0 & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & 0 \\
  v_{1}t_{12} & 0 & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & v_{2}v_{3} & 0 \\
  0 & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & 0 \\
  0 & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & v_{1}t_{12} & 0 \end{pmatrix}
\]

(6.25)

where

\[t_{19} = q + 5v + v^2.\]

(6.26)

The characteristic equation \(T_{Z,tri,3,2}\) yields two linear equations, a quadratic equation and two quartic equations. The eigenvalues which are the solutions of the two linear equations, \(\lambda_{r3,16} = v^3(v + 1)(v + 2)\) and \(\lambda_{r3,17} = v^2(2v + q)\), have the coefficient \(b^{(2)} = (q - 1)(q - 2)/2\).

One of the quartic equations is given by

\[
\xi^4 - v^2(9v^2 + 2v^3 + 14v + 2q)\xi^3 + v^4(36v^5 + 135v^4 + 246v^3 + 192v^2 + 4v^6 + 7v^3q + 33v^2q + 54vq + 4q^2)\xi^2
\]

40
\[ -v^7(v + 2)(1 + v)(9v^2 + 2v + 14v + 2q)(q + 2v)\xi + v^{10}(v + 2)(1 + v)^2(q + 2v)^2 = 0 \]
\[ (6.27) \]

with roots \( \lambda_{tt,3,j} \) for \( 18 \leq j \leq 21 \) that also have \( b_2^{(2)} \) as their coefficient. The roots of the following two equations have coefficient \( b_1^{(2)} = q(q - 3)/2 \):
\[ \xi^2 - v^2(3q + 24v + 13v^2 + 3v^3)\xi + v^5(1 + v)(6v^2 + 5vq + 6q) = 0 \]
\[ (6.28) \]

with roots \( \lambda_{tt,22,23} \) and
\[ \xi^4 + v^4\xi^3 + v^6(q + vq + v^2)\xi^2 - v^{10}q(1 + v)\xi + v^{12}q^2(1 + v)^2 = 0 \]
\[ (6.29) \]

with roots \( \lambda_{tt,3,j} \) for \( 24 \leq j \leq 27 \). For \( d = 2 \), we have
\[ T_{Z,tri,3,3} = v^3 \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \]
\[ (6.30) \]

All the eigenvalues of \( T_{Z,tri,3,3} \) have multiplicity two. These eigenvalues are \( \lambda_{tt,28} = 2v^3 \) with coefficient \( b_1^{(3)} = (q - 1)(q^2 - 5q + 3)/3 \) and the roots of \( \xi^2 - v^3\xi + v^6 = 0 \), denoted as \( \lambda_{tt,29,30} \), with coefficient \( 2b_2^{(3)} = q(q - 2)(q - 4)/3 \). Therefore, the Potts model partition function for the triangular lattice strip with \( L_y = 3 \) and toroidal boundary condition is given by
\[ Z(tri, 3 \times L_x, tor., q, v) = b_0^{(0)} \sum_{j=1}^{5} (\lambda_{tt,3,j})^{L_x} + b_1^{(1)} \sum_{j=6}^{15} (\lambda_{tt,3,j})^{L_x} + b_2^{(2)} \sum_{j=16}^{21} (\lambda_{tt,3,j})^{L_x} + b_1^{(3)} \sum_{j=22}^{27} (\lambda_{tt,3,j})^{L_x} + b_1^{(3)} (\lambda_{tt,28})^{L_x} + b_2^{(3)} \sum_{j=29}^{30} (\lambda_{tt,3,j})^{L_x}. \]
\[ (6.31) \]

For the corresponding Klein bottle strip, the coefficients for \( \lambda_{tt,3,j} \) with \( 10 \leq j \leq 15, 18 \leq j \leq 21, 24 \leq j \leq 27 \) and \( j = 4, 5, 29, 30 \) become zero. The coefficient for \( \lambda_{tt,17} \) changes sign and the coefficient for \( \lambda_{tt,28} \) becomes \( -b_1^{(1)} = 1 - q \). Consequently, the Potts model partition function for the triangular lattice strip with \( L_y = 3 \) and Klein bottle boundary condition is given by
\[ Z(tri, 3 \times L_x, Kb., q, v) = b_0^{(0)} \sum_{j=1}^{3} (\lambda_{tt,3,j})^{L_x} + b_1^{(1)} \left( \sum_{j=6}^{9} (\lambda_{tt,3,j})^{L_x} - (\lambda_{tt,28})^{L_x} \right) \]
At the special value \( v = -1 \), the Potts model partition function reduces to the chromatic polynomial. The non-zero eigenvalues are the same as \( \lambda_{tt,j} \) for \( 1 \leq j \leq 11 \) from eqs. (4.2) to (4.9) in [44].

**F. Honeycomb-Lattice Strip, \( L_y = 4 \)**

For the \( L_y = 4 \) toroidal and Klein bottle strips of the honeycomb lattice, the linear sizes of the matrices are 14, 35, 56, 48 for levels \( 0 \leq d \leq 3 \), as indicated in Table I. The eigenvalues of \( T_{Z, hc, 4,0} \) are roots of two quadratic equations and an eighth-degree equation. One of the quadratic equations has multiplicity two. The eigenvalues of \( T_{Z, hc, 4,1} \) are roots of a seventh-degree equation, an eighth-degree equation and a twelfth-degree equation. The eighth-degree equation has multiplicity two. The eigenvalues of \( T_{Z, hc, 4,2} \) are roots of a quartic equation, a fifth-degree equation, a sixth-degree equation, two eighth-degree equations and an eleventh-degree equation. The sixth-degree equation and one of the eighth-degree equations have multiplicity two. The eigenvalues of \( T_{Z, hc, 4,3} \) consist of a linear term \( v^6(v + q)^2 \), roots of a quadratic equation, a cubic equation and two quartic equations. The linear terms and the cubic equation have multiplicity two, and the other equations have multiplicity four. The matrix \( T_{Z, hc, 4,4} = v^8 \) has been given above with coefficient \( b^{(4)} \). Some of these equations are too lengthy to list here. They are available from the authors.

At the special value \( v = -1 \), the Potts model partition function reduces to the chromatic polynomial. The linear sizes of the matrices reduce to 6, 18, 34, 36 for levels \( 0 \leq d \leq 3 \), as indicated in Table VI. The eigenvalues of \( T_{P, hc, 4,0} \) consist of two linear terms and roots of a quartic equation. They are given by

\[
\lambda_{ht4,0} = 0 \tag{6.33}
\]
\[
\lambda_{ht4,1} = (q - 1)^2(q - 3)^2 \tag{6.34}
\]

\[
\xi^4 - (q^8 - 12q^7 + 66q^6 - 220q^5 + 496q^4 - 796q^3 + 922q^2 - 734q + 314)\xi^3 \\
+ (q^{12} - 20q^{11} + 188q^{10} - 1092q^9 + 4344q^8 - 12428q^7 + 26192q^6 - 41022q^5 + 47597q^4 \\
- 40318q^3 + 24247q^2 - 9782q + 2169)\xi^2 \\
-(4q^{12} - 80q^{11} + 736q^{10} - 4124q^9 + 15705q^8 - 42922q^7 + 86535q^6 - 129964q^5 \\
+ 144575q^4 - 116374q^3 + 64525q^2 - 22280q + 3680)\xi + 4(q - 1)^4(q - 2)^4 = 0 \tag{6.35}
\]
with roots $\lambda_{ht4,j}$ for $2 \leq j \leq 5$. $\lambda_{ht4,0} = 0$ appears as one of the eigenvalues of $T_{P,hc,A1}$ with multiplicity two. The remaining eigenvalues of $T_{P,hc,A1}$ are roots of a cubic equation, a quartic equation and a sixth-degree equation. The cubic equation has multiplicity two. These equations are given by

$$
\xi^3 - (q^6 - 10q^5 + 44q^4 - 110q^3 + 167q^2 - 150q + 68)\xi^2 \\
+ (q^8 - 14q^7 + 90q^6 - 342q^5 + 832q^4 - 1322q^3 + 1345q^2 - 812q + 230)\xi \\
- (q - 1)^2(2q^6 - 22q^5 + 102q^4 - 256q^3 + 373q^2 - 308q + 115) = 0
$$

(6.36)

with roots $\lambda_{ht4,j}$ for $j = 6, 7, 8$.

$$
\xi^4 - (q^6 - 8q^5 + 29q^4 - 64q^3 + 95q^2 - 86q + 37)\xi^3 \\
+ (q - 1)^2(q^8 - 14q^7 + 86q^6 - 302q^5 + 672q^4 - 1000q^3 + 1030q^2 - 724q + 280)\xi^2 \\
- (q - 1)^4(q^8 - 16q^7 + 112q^6 - 444q^5 + 1084q^4 - 1664q^3 + 1581q^2 - 886q + 253)\xi \\
+ (q - 3)^2(q - 1)^6 = 0
$$

(6.37)

with roots $\lambda_{ht4,j}$ for $9 \leq j \leq 12$.

$$
\xi^6 - (q^6 - 12q^5 + 65q^4 - 204q^3 + 407q^2 - 510q + 321)\xi^5 + (q^{10} - 20q^9 + 187q^8 \\
- 1064q^7 + 4046q^6 - 10670q^5 + 19694q^4 - 25280q^3 + 22278q^2 - 13120q + 4368)\xi^4 \\
- (q^{12} - 24q^{11} + 274q^{10} - 1948q^9 + 9541q^8 - 33732q^7 + 87865q^6 - 169306q^5 + 239311q^4 \\
- 243300q^3 + 172063q^2 - 79202q + 19177)\xi^3 + (4q^{12} - 92q^{11} + 992q^{10} - 6588q^9 \\
+ 29885q^8 - 97284q^7 + 232764q^6 - 412788q^5 + 540290q^4 - 512128q^3 + 336720q^2 \\
- 139512q + 27993)\xi^2 - 4(q - 1)^2(q^{10} - 20q^9 + 182q^8 - 984q^7 + 3483q^6 - 8402q^5 \\
+ 13956q^4 - 15756q^3 + 11602q^2 - 5078q + 1032)\xi + 16(q - 2)^2(q - 1)^6 = 0
$$

(6.38)

with roots $\lambda_{ht4,j}$ for $13 \leq j \leq 18$. $\lambda_{ht4,0} = 0$ is again one of the eigenvalues of $T_{P,hc,A2}$. The remaining eigenvalues of $T_{P,hc,A2}$ are roots of a quadratic equation, two cubic equations, two fifth-degree equations and a seventh-degree equation. They are given by

$$
\lambda_{ht4,(19,20)} = \frac{1}{2} [q^2 - 4q + 7 \pm (q - 3)(q^2 - 2q + 5)^{1/2}]
$$

(6.39)

$$
\xi^5 - (2q^4 - 14q^3 + 41q^2 - 60q + 41)\xi^4 + (q^8 - 14q^7 + 91q^6 - 358q^5 + 932q^4 - 1636q^3 \\
+ 1875q^2 - 1270q + 396)\xi^3 - (q^{10} - 18q^9 + 148q^8 - 726q^7 + 2339q^6 - 5148q^5 + 7824q^4 \\
- 8136q^3 + 5615q^2 - 2372q + 477)\xi^2 + (q - 1)^2(q^8 - 14q^7 + 85q^6 - 292q^5 + 626q^4 \\
- 874q^3 + 800q^2 - 448q + 125)\xi - 4(q - 1)^4 = 0
$$

(6.40)

with roots $\lambda_{ht4,j}$ for $21 \leq j \leq 25$. 

$$
\xi^5 - (2q^4 - 15q^3 + 51q^2 - 90q + 73)\xi^4 + (q^8 - 15q^7 + 104q^6 - 429q^5 + 1152q^4 - 2064q^3 \\
$$


+2415q^2 - 1694q + 570)\xi^3 - (q^{10} - 19q^9 + 167q^8 - 881q^7 + 3070q^6 - 7372q^5 + 12396q^4 \\
-14520q^3 + 11459q^2 - 5550q + 1265)\xi^2 + (q - 1)^2(q^8 - 15q^7 + 104q^6 - 428q^5 + 1144q^4 \\
-2039q^3 + 2392q^2 - 1706q + 575)\xi - 2q(q - 1)^6 = 0 

(6.41)

with roots \(\lambda_{ht4,j}\) for \(26 \leq j \leq 30\).

\[
\xi^3 - (q^4 - 7q^3 + 21q^2 - 32q + 24)\xi^2 + (q^6 - 11q^5 + 53q^4 - 139q^3 + 206q^2 - 164q + 58)\xi \\
-(q - 1)^2(q^4 - 7q^3 + 18q^2 - 24q + 15) = 0
\]

(6.42)

with roots \(\lambda_{ht4,j}\) for \(j = 31, 32, 33\).

\[
\xi^3 - (q^4 - 8q^3 + 27q^2 - 40q + 24)\xi^2 + (q - 1)^2(q^4 - 10q^3 + 44q^2 - 90q + 72)\xi \\
-(q - 3)^2(q - 1)^4 = 0
\]

(6.43)

with roots \(\lambda_{ht4,j}\) for \(j = 34, 35, 36\).

\[
\xi^7 - (3q^4 - 22q^3 + 75q^2 - 140q + 126)\xi^6 + (3q^8 - 44q^7 + 304q^6 - 1282q^5 + 3601q^4 \\
-6890q^3 + 8830q^2 - 7038q + 2781)\xi^5 - (q^{12} - 22q^{11} + 230q^{10} - 1504q^9 + 6839q^8 \\
-22756q^7 + 56785q^6 - 107160q^5 + 152314q^4 - 160280q^3 + 120302q^2 - 59250q + 15045)\xi^4 \\
+(q^{14} - 26q^{13} + 318q^{12} - 2414q^{11} + 12685q^{10} - 48806q^9 + 141948q^8 - 317702q^7 \\
+551665q^6 - 743122q^5 + 769969q^4 - 601782q^3 + 341290q^2 - 128886q + 25246)\xi^3 \\
-(q^{14} - 24q^{13} + 280q^{12} - 2088q^{11} + 11028q^{10} - 43286q^9 + 129310q^8 - 297078q^7 \\
+525432q^6 - 710236q^5 + 721717q^4 - 535598q^3 + 276095q^2 - 89634q + 14145)\xi^2 \\
+4(q - 1)^2(q^9 - 6q^8 - 14q^7 + 249q^6 - 1080q^5 + 2522q^4 - 3564q^3 + 3069q^2 - 1499q \\
+326)\xi - 4(q - 2)^2(q - 1)^6 = 0
\]

(6.44)

with roots \(\lambda_{ht4,j}\) for \(37 \leq j \leq 43\). The second fifth-degree equation and the first cubic equation given above have multiplicity two. \(\lambda_{ht4,0} = 0\) is again one of the eigenvalues of \(T_{P,hc,4,3}\) with multiplicity two. The remaining eigenvalues of \(T_{P,hc,4,3}\) are two linear terms, roots of a quadratic equation and two cubic equations. They are given by

\[
\lambda_{ht4,44} = q^2 - 4q + 5
\]

(6.45)

\[
\lambda_{ht4,45} = (q - 1)^2
\]

(6.46)

\[
\lambda_{ht4,(46,47)} = \frac{1}{2} [q^2 - 6q + 13 \pm (q - 3)(q^2 - 6q + 17)^{1/2}]
\]

(6.47)

\[
\xi^3 - (2q^2 - 8q + 11)\xi^2 + (q^4 - 8q^3 + 26q^2 - 38q + 23)\xi - (q - 1)^2 = 0
\]

(6.48)

with roots \(\lambda_{ht4,j}\) for \(j = 48, 49, 50\).

\[
\xi^3 - (2q^2 - 8q + 13)\xi^2 + (q^4 - 8q^3 + 26q^2 - 34q + 19)\xi - 3(q - 1)^2 = 0
\]

(6.49)
with roots $\lambda_{ht,j}$ for $j = 51, 52, 53$. $\lambda_{ht,j}$ for $j = 45, 46, 47$ have multiplicity two, and the other eigenvalues of $T_{P,hc,4,3}$ have multiplicity four. Finally, we have $\lambda_{ht,54} = 1$ as the eigenvalue of $T_{P,hc,4,4}$. The corresponding coefficients are

$$b_{ht,j} = 1 \quad \text{for} \quad 1 \leq j \leq 5$$

$$b_{ht,j} = 2(q - 1) \quad \text{for} \quad 6 \leq j \leq 8$$

$$b_{ht,j} = q - 1 \quad \text{for} \quad 9 \leq j \leq 18$$

$$b_{ht,j} = \frac{1}{2}(q - 1)(q - 2) \quad \text{for} \quad 19 \leq j \leq 25$$

$$b_{ht,j} = (q - 1)(q - 2) \quad \text{for} \quad 26 \leq j \leq 30$$

$$b_{ht,j} = q(q - 3) \quad \text{for} \quad 31 \leq j \leq 33$$

$$b_{ht,j} = \frac{1}{2}q(q - 3) \quad \text{for} \quad 34 \leq j \leq 43$$

$$b_{ht,44} = \frac{2}{3}(q - 1)(q^2 - 5q + 3)$$

$$b_{ht,j} = \frac{1}{3}(q - 1)(q^2 - 5q + 3) \quad \text{for} \quad 45 \leq j \leq 47$$

$$b_{ht,j} = \frac{2}{3}q(q - 2)(q - 4) \quad \text{for} \quad 48 \leq j \leq 53$$

$$b_{ht,54} = q^4 - 8q^3 + 20q^2 - 15q + 1. \quad (6.50)$$

Although $\lambda_{ht,0} = 0$ does not contribute to the chromatic polynomial, we can still calculate its coefficient for the toroidal strip, and we get $(q - 1)(2q^2 - 7q + 12)/6$. The sum of all the coefficients is equal to $q^2(q - 1)^2$, as dictated by the $L_y = 4$ special case of the general result (3.14). Hence, the chromatic polynomial for the honeycomb lattice strip with $L_y = 4$ and toroidal boundary condition is given by

$$P(hc, 4 \times L_x, tor., q) = b^{(0)} \sum_{j=1}^{5} (\lambda_{ht,j})^m + b^{(1)} \left( 2 \sum_{j=6}^{8} (\lambda_{ht,j})^m + \sum_{j=9}^{18} (\lambda_{ht,j})^m \right)$$

$$+ b^{(2)} \left( \sum_{j=19}^{25} (\lambda_{ht,j})^m + 2 \sum_{j=26}^{30} (\lambda_{ht,j})^m \right)$$

45
where \( m = L_x/2 \) as given in eq. (1.4). For the corresponding Klein bottle strip, \( \lambda_{ht4,j} \) for \( 6 \leq j \leq 8, 26 \leq j \leq 33, 48 \leq j \leq 53 \) and \( j = 44 \) do not contribute. The coefficient for \( \lambda_{ht4,45} \) becomes \( b^{(1)} = 1, \) the coefficients for \( \lambda_{ht4,46,47} \) become \(-b^{(1)} = 1 - q\) and the coefficients for \( \lambda_{ht4,54} \) become one. Therefore, the chromatic polynomial for the honeycomb lattice strip with \( L_y = 4 \) and Klein bottle boundary condition is given by

\[
P(hc, 4 \times L_x, Kb., q) = b^{(0)} \left[ -(\lambda_{ht4,1})^m + \sum_{j=2}^{5} (\lambda_{ht4,j})^m + (\lambda_{ht4,54})^m \right] \\
+ b^{(1)} \left[ -\sum_{j=9}^{12} (\lambda_{ht4,j})^m + \sum_{j=13}^{18} (\lambda_{ht4,j})^m + (\lambda_{ht4,45})^m - \sum_{j=46}^{47} (\lambda_{ht4,j})^m \right] \\
+ b^{(2)}_2 \left[ \sum_{j=19}^{20} (\lambda_{ht4,j})^m - \sum_{j=21}^{25} (\lambda_{ht4,j})^m \right] \\
+ b^{(2)}_1 \left[ -\sum_{j=34}^{36} (\lambda_{ht4,j})^m + \sum_{j=37}^{43} (\lambda_{ht4,j})^m \right].
\]  

(6.51)

The singular locus \( B \) for the \( L_x \to \infty \) limit of the strip of the honeycomb lattice with \( L_y = 4 \) and toroidal boundary conditions is shown in Fig. 10. For comparison, chromatic zeros are calculated and shown for length \( L_x = 13 \) (i.e., \( n = 104 \) vertices). The locus \( B \) crosses the real axis at the points \( q = 0, q = 2, \) and at the maximal point \( q = q_c, \) where

\[
q_c \simeq 2.480749 \quad \text{for} \quad \{ G \} = (hc, 4 \times \infty, PBC_y, PBC_x). 
\]  

(6.53)

We note that this is rather close to the value for the infinite 2D honeycomb lattice, namely \( q_c(hc) = (3 + \sqrt{5})/2 \simeq 2.61803. \) As is evident from Fig. 10 the locus \( B \) separates the \( q \) plane into different regions including the following: (i) region \( R_1, \) containing the semi-infinite intervals \( q > q_c \) and \( q < 0 \) on the real axis and extending outward to infinite \( |q|; \) (ii) \( R_2 \) containing the interval \( 2 < q < q_c; \) (iii) \( R_3 \) containing the real interval \( 0 < q < 2; \) and (iv) the complex-conjugate pair \( R_4, R_4^* \) centered approximately at \( q = 2.55 \pm 0.6i. \) The (nonzero)
FIG. 10: Singular locus $B$ for the $L_x \to \infty$ limit of the strip of the honeycomb lattice with $L_y = 4$ and toroidal boundary conditions. For comparison, chromatic zeros are shown for $L_x = 13$ (i.e., $n = 104$).

densities of chromatic zeros have the smallest values on the curve separating regions $R_1$ and $R_3$ in the vicinity of the point $q = 0$ and on the curve separating regions $R_2$ and $R_3$ in the vicinity of the point $q = 2$.

It is instructive to compare this locus $B$ in Fig. 10 with the corresponding loci that we obtained previously for strips of the honeycomb lattice with periodic longitudinal boundary conditions and free or periodic transverse boundary conditions. We recall that for a given infinite-length lattice strip with specified transverse boundary conditions, the locus $B$ is the same for periodic longitudinal and twisted periodic longitudinal boundary conditions (i.e., cyclic versus Möbius if the transverse B.C. are free, and toroidal versus Klein bottle if the transverse B.C. are periodic). For the cyclic strip of the honeycomb lattice, $B$ was given as (i) Fig. 1 of Ref. [25] for width $L_y = 2$, and (ii) Figs. 7, 8, and 9 of [22] for $L_y = 3, 4, 5$ (see also the zeros in Fig. 17 of Ref. [33] for $L_y = 3$). We recall that $L_y$ must be even for strips of the honeycomb lattice with toroidal boundary conditions. Comparing our results for $B$ for the width $L_y = 4$ toroidal strip of the honeycomb lattice in Fig. 10 with our earlier results for the $L_y = 4$ cyclic strip of this lattice in Fig. 8 of [22], we see that the locus $B$ is significantly simpler for toroidal, versus cyclic, boundary conditions, with
fewer regions and, in particular, no evidence of the tiny sliver phases that are present in the latter case. Furthermore, the value of \(q_c \approx 2.481\) that we obtain (eq. (6.53)) for the width \(L_y = 4\) infinite-length strip of the honeycomb lattice with toroidal boundary conditions is closer to the value of \(q_c(hc) \approx 2.618\) for the infinite 2D than the value \(q_c \approx 2.155\) for the corresponding cyclic \(L_y = 4\) strip. This is in agreement with one’s general expectations, since toroidal B.C. minimize finite-size effects to a greater extent than cyclic B.C. and hence expedite the approach to the \(L_y \to \infty\) limit.

In region \(R_1\), the dominant eigenvalue is the root of eq. (6.35) with largest magnitude. Denoting it as \(\lambda_{ht4,2}\), we have

\[
W = (\lambda_{ht4,2})^{1/8}, \quad q \in R_1.
\]

This is the same as \(W\) for the corresponding \(L_x \to \infty\) limit of the strip of the honeycomb lattice with the same width \(L_y = 4\) and cylindrical \((PBC_y, FBC_x)\) boundary conditions, calculated in [33]. This equality of the \(W\) functions for the \(L_x \to \infty\) limit of two strips of a given lattice with the same transverse boundary conditions and different longitudinal boundary conditions in the more restrictive region \(R_1\) defined by the two boundary conditions is a general result [4, 45]. In [33] we have listed values of \(W\) for a range of values of \(q\) for the \(L_x \to \infty\) limit of various strips of the honeycomb lattice, including \((hc, 4 \times \infty, PBC_y, FBC_x)\). Since \(W\) is independent of \(BC_x\) for \(q\) in the more restrictive region \(R_1\) defined by \(FBC_x\) and \((T)PBC_x\) (which is the \(R_1\) defined by \(PBC_x\) here), it follows, in particular, that

\[
W(4 \times \infty, PBC_y, (T)PBC_x, q) = W(4 \times \infty, PBC_y, FBC_x, q) \quad \text{for} \quad q \geq 2.48\ldots
\]

In region \(R_2\), the largest root of the seventh degree equation (6.44) is dominant; we label this as \(\lambda_{ht4,37}\), so that

\[
|W| = |\lambda_{ht4,37}|^{1/8}, \quad q \in R_2
\]

(in regions other than \(R_1\), only \(|W|\) can be determined unambiguously [4]). Thus, \(q_c\) is the relevant solution of the equation of degeneracy in magnitude \(|\lambda_{ht4,2}| = |\lambda_{ht4,37}|\). In region \(R_3\),

\[
|W| = |\lambda_{ht4,13}|^{1/8}, \quad q \in R_3
\]

where \(\lambda_{ht4,13}\) is the largest root of the sixth degree equation (6.38). In regions \(R_4, R_4^*\),

\[
|W| = |\lambda_{ht4,9}|^{1/8}, \quad q \in R_4, R_4^*
\]

where \(\lambda_{ht4,9}\) is the largest root of the quartic equation (6.37). We calculate the following specific values: \(|W(q = 1)| = 1.67290\ldots, |W(q = 2)| = 1.35817\ldots,\) and \(|W(q = q_c)| = 1.26949\ldots\)

These values may be compared with those listed in Ref. [33]. As is evident from Fig. 10, the curve \(B\) has support for \(Re(q) < 0\). For strips with finite lengths \(m \geq 3\), certain chromatic zeros have support for \(Re(q) < 0\).
VII. SUMMARY

In summary, in this paper we have presented a method for calculating transfer matrices for the $q$-state Potts model partition functions $Z(G, q, v)$, for arbitrary $q$ and temperature variable $v$, on strip graphs $G$ of the square, triangular, and honeycomb lattices with width $L_y$ and arbitrarily great length $L_x$, having toroidal and Klein bottle boundary conditions. Using this method, we have derived a number of general properties of these transfer matrices. In particular, we have found some very simple formulas for the determinant $\det(T_{Z,\Lambda,L_y,d})$, and trace $\text{Tr}(T_{Z,\Lambda,L_y})$. A number of explicit exact calculations were given.

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