THE SUBPOWER MEMBERSHIP PROBLEM FOR BANDS

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Abstract. Fix a finite semigroup $S$ and let $a_1, \ldots, a_k, b$ be tuples in a direct power $S^n$. The subpower membership problem (SMP) for $S$ asks whether $b$ can be generated by $a_1, \ldots, a_k$. For bands (idempotent semigroups), we provide a dichotomy result: if a band $S$ belongs to a certain quasivariety, then SMP($S$) is in P; otherwise it is NP-complete.

Furthermore we determine the greatest variety of bands all of whose finite members induce a tractable SMP. Finally we present the first example of two finite algebras that generate the same variety and have tractable and NP-complete SMPs, respectively.

1. Introduction

How hard is deciding membership in a subalgebra of a given algebraic structure? This problem occurs frequently in symbolic computation. For instance if $F$ is a fixed field and we are given vectors $a_1, \ldots, a_k, b$ in a vector space $F^n$, we often want to decide whether $b$ is in the linear span of $a_1, \ldots, a_k$. This question can be answered using Gaussian elimination in polynomial time in $n$ and $k$.

Depending on the formulation of the membership problem, the underlying algebra may be part of the input. For example if we are given transformations on $n$ elements, we may have to decide whether they generate a given transformation under composition. These functions belong to the full transformation semigroup $T_n$ on $n$ elements. In this case $n$ and the algebra $T_n$ are part of the input. Kozen proved that this problem is PSPACE-complete [9]. However, if we restrict the input to permutations on $n$ elements, then the problem is in P using Sims’ stabilizer chains [3].

In this paper we investigate the membership problem formulated by Willard in 2007 [14]. Fix a finite algebra $S$ with finitely many basic operations. We call a subalgebra of some direct power of $S$ a subpower of $S$. The subpower membership problem SMP($S$) is the following decision problem:

SMP($S$)
Input: $\{a_1, \ldots, a_k\} \subseteq S^n, b \in S^n$
Problem: Is $b$ in the subalgebra of $S^n$ generated by $\{a_1, \ldots, a_k\}$?

In this problem the algebra $S$ is not part of the input.

The SMP is of particular interest within the study of the constraint satisfaction problem (CSP) [8]. Recall that in a CSP instance the goal is to assign values of a given domain to a set of variables such that each constraint is satisfied. Constraints
are usually represented by constraint relations. In the algebraic approach to the CSP, each relation is regarded as a subpower of a certain finite algebra \( S \). Instead of storing all elements of a constraint relation, we can store a set of generators. Checking whether a given tuple belongs to a constraint relation represented by its generators is precisely the SMP for \( S \).

The input size of \( \text{SMP}(S) \) is essentially \((k + 1)n\). We can always decide the problem using a straightforward closure algorithm in time exponential in \( n \). For some algebras there is no faster algorithm. This follows from a result of Kozik [10], who actually constructed a finite algebra with \( \text{EXPTIME} \)-complete \( \text{SMP} \). However, there are structures whose \( \text{SMP} \) is considerably easier. For example, the \( \text{SMP} \) for a finite group is in \( \text{P} \) by an adaptation of Sims’ stabilizer chains [15]. Mayr [11] proved that the \( \text{SMP} \) for Mal’cev algebras is in \( \text{NP} \). He also showed that the \( \text{SMP} \) for every finite Mal’cev algebra which has prime power size and a nilpotent reduct is in \( \text{P} \).

In this paper we investigate the \( \text{SMP} \) for bands (idempotent semigroups). For semigroups in general the \( \text{SMP} \) is in \( \text{PSPACE} \) by a result of Bulatov, Mayr, and the present author [1]. There is no better upper bound since various semigroups were shown to have a \( \text{PSPACE} \)-complete \( \text{SMP} \), including the full transformation semigroup on 3 or more letters and the 6-element Brandt monoid [1, 13]. For commutative semigroups, however, the \( \text{SMP} \) is in \( \text{NP} \). In [1] a dichotomy result was provided: if a commutative semigroup \( S \) embeds into a direct product of a Clifford semigroup and a nilpotent semigroup, then \( \text{SMP}(S) \) is in \( \text{P} \); otherwise it is \( \text{NP} \)-complete [1]. These were also the first algebras known to have an \( \text{NP} \)-complete \( \text{SMP} \). Similar to the case of commutative semigroups, we will establish a \( \text{P}/\text{NP} \)-complete dichotomy for the \( \text{SMP} \) for bands in the present paper. Before that we introduce the notions of variety, identity, and quasiidentity.

Let \( v, w \) be words over variables \( x_1, \ldots, x_k \). For a semigroup \( S \) we define the function \( w^S : S^k \to S \) such that, when applied to \((s_1, \ldots, s_k) \in S^k \), it replaces every occurrence of \( x_i \) in \( w \) by \( s_i \) for all \( i \in \{1, \ldots, k\} \) and computes the resulting product. We refer to \( w^S \) as the \((k\text{-ary}) \) term function induced by \( w \). An expression of the form \( v \approx w \) is called an identity over \( x_1, \ldots, x_k \). A semigroup \( S \) satisfies the identity \( v \approx w \) (in symbols \( S \models v \approx w \)) if

\[
\forall s_1, \ldots, s_k \in S : v^S(s_1, \ldots, s_k) = w^S(s_1, \ldots, s_k).
\]

A class \( \mathcal{V} \) of semigroups is called a variety if there is a set \( \Sigma \) of identities such that \( \mathcal{V} \) contains precisely the semigroups that satisfy all identities in \( \Sigma \).

Given identities \( v_1 \approx w_1, \ldots, v_m \approx w_m \) and \( p \approx q \) over \( x_1, \ldots, x_k \), we call an expression \( \mu \) of the form

\[
v_1 \approx w_1 \& \ldots \& v_m \approx w_m \rightarrow p \approx q
\]

a quasiidentity over \( x_1, \ldots, x_k \). A semigroup \( S \) satisfies the quasiidentity \( \mu \) (in symbols \( S \models \mu \)) if for all \( \bar{s} = (s_1, \ldots, s_k) \) in \( S^k \),

\[
v_1^S(\bar{s}) = w_1^S(\bar{s}), \ldots, v_m^S(\bar{s}) = w_m^S(\bar{s}) \quad \text{implies} \quad p^S(\bar{s}) = q^S(\bar{s}).
\]

A class \( \mathcal{V} \) of semigroups is a quasivariety if there is a set \( \Sigma \) of quasiidentities such that \( \mathcal{V} \) contains precisely the semigroups that satisfy all identities in \( \Sigma \).

Green’s equivalences are denoted by \( \mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{D}, \mathcal{H} \) [7, p. 45]. For the preorders \( \preceq_{\mathcal{L}}, \preceq_{\mathcal{R}}, \preceq_{\mathcal{J}} \) see [7, p. 47].
The dichotomy result for the SMP for bands is based upon the following two quasiidentities.

\[
\begin{align*}
\lambda & \quad \begin{cases} 
    dyxe \approx de \\
    hx \approx x \\
    he \approx e \\
    d \leq \mathcal{J} e \leq \mathcal{J} x, y
\end{cases} \quad \rightarrow \quad dxfe \approx de. \\
\bar{\lambda} & \quad \begin{cases} 
    eyxd \approx ed \\
    xh \approx x \\
    eh \approx e \\
    d \leq \mathcal{J} e \leq \mathcal{J} x, y
\end{cases} \quad \rightarrow \quad edx \approx ed.
\end{align*}
\]

We will see that for every band the last condition on the left hand side of both \( \lambda \) and \( \bar{\lambda} \) is equivalent to

\[ ded \approx d \quad \& \quad exe \approx e \quad \& \quad eye \approx e. \]

This will follow from Lemma 2.3. Thus the atomic formulas of \( \lambda \) and \( \bar{\lambda} \) are really identities. Also note that \( \bar{\lambda} \) is the quasiidentity obtained when each word of \( \lambda \) is reversed. We obtain the following observation.

**Lemma 1.1.** A semigroup \( S \) satisfies \( \bar{\lambda} \) if and only if the dual semigroup \( \bar{S} \) satisfies \( \lambda \).

**Proof.** Straightforward. \( \square \)

We state the main result of this paper.

**Theorem 1.2.** If a finite band \( S \) satisfies \( \lambda \) and \( \bar{\lambda} \), then \( \text{SMP}(S) \) is in \( P \). Otherwise \( \text{SMP}(S) \) is \( \text{NP}\)-complete.

For the proof see Section 4. This means that under the assumption \( P \neq \text{NP} \), the finite bands with tractable SMP are precisely the finite members of the quasivariety determined by \( \lambda \) and \( \bar{\lambda} \). This quasivariety is not a variety by the following theorem.

**Theorem 1.3.** There are a 9-element band \( S_9 \) and a 10-element band \( S_{10} \) such that

(a) \( S_9 \) and \( S_{10} \) generate the same variety,

(b) \( S_9 \) is a homomorphic image of \( S_{10} \), and

(c) \( \text{SMP}(S_{10}) \) is in \( P \), whereas \( \text{SMP}(S_9) \) is \( \text{NP}\)-complete.

See Definition 5.1 for the multiplication tables of \( S_9 \) and \( S_{10} \). This is the first example of two finite algebras which generate the same variety and induce tractable and \( \text{NP}\)-complete SMPs, respectively. The band \( S_{10} \) is also the first finite algebra known to have a tractable SMP and a homomorphic image with \( \text{NP}\)-hard SMP. We will prove Theorem 1.3 in Section 5.

In the characterization of all varieties of bands [6], the sequences of words \( G_n \), \( H_n \), and \( I_n \) over \( x_1, \ldots, x_n \) for \( n \geq 2 \) play a fundamental role. We list the first four words of each sequence.

**Definition 1.4** (cf. [6, Notation 5.1]).

| \( n \) | \( G_n \) | \( H_n \) | \( I_n \) |
|---|---|---|---|
| 2  | \( x_2 x_1 \) | \( x_2 \) | \( x_2 x_1 x_2 \) |
| 3  | \( x_3 x_1 x_2 \) | \( x_3 x_1 x_2 x_3 x_2 \) | \( x_3 x_1 x_2 x_3 x_2 x_1 x_2 \) |
| 4  | \( x_4 x_2 x_1 x_3 \) | \( x_4 x_2 x_1 x_3 x_4 x_2 x_3 x_2 x_1 x_3 \) | \( x_4 x_2 x_1 x_3 x_4 x_2 x_1 x_2 x_3 x_2 x_1 x_3 \) |
For two words \( v \) and \( w \) we let \([v \approx w]\) denote the variety of bands that satisfy \( v \approx w \). We call a variety proper if it is smaller than the variety of all bands. By \( \bar{v} \) we denote the dual word of a word \( v \), which is the word obtained when the order of the variables of \( v \) is reversed.

**Theorem 1.5** ([6, Diagram 1]). Every variety of bands is of the form \([v \approx w]\) for some identity \( v \approx w \). The lattice of proper varieties of bands is given by Figure 1.

The following result on the SMP for bands will be proved in Section 5.

**Theorem 1.6.** Assume \( P \neq NP \). Then \( [\bar{G}_4 G_4 \approx \bar{H}_4 H_4] \) is the greatest variety of bands all of whose finite members induce a tractable SMP.

Bands that satisfy the identity \( \bar{G}_3 G_3 \approx \bar{I}_3 I_3 \) are called regular. From Theorem 1.6 and Figure 1 we obtain the following result.

**Corollary 1.7.** The SMP for every regular band is in \( P \).

2. **Varieties of bands**

The following lemma states the well-known fact that every band is a semilattice of rectangular bands. A band is called rectangular if it satisfies \( xyz \approx xz \).

**Lemma 2.1** (cf. [7, Theorem 4.4.1]). Let \( S \) be a band.

(a) \( J \) is a congruence on \( S \).

(b) \( S/J \) is a semilattice, and for all \( x, y \in S \) we have \( xy \ J x \) if and only if \( x \leq_J y \).

(c) Each \( J \)-class is a rectangular band.

**Lemma 2.2** ([6, Lemma 2.2]). Let \( x, y, z \) be elements of a band \( S \) such that \( x \ J z \). Then \( x \leq_J y \) if and only if \( xyz = xz \).

**Proof.** First assume \( x \leq_J y \). Lemma 2.1 (b) implies \( x \ J xy \). We have \( xyz = x(yz)z = xz \) by idempotence and Lemma 2.1 (c).

For the converse assume \( xyz = xz \). Since \( S/J \) is a semilattice, we have \( x \ J xz = xyz \). Thus \( x \ J x \), and Lemma 2.1 (b) yields \( x \leq_J y \). \( \square \)

We will use the following well-known rules throughout this paper.

**Lemma 2.3.** Let \( x, y \) be elements of a band \( S \). Then

(a) \( x \leq_J y \) if and only if \( xyx = x \),

(b) \( x \leq_L y \) if and only if \( xy = x \),

(c) \( x \leq_R y \) if and only if \( yx = x \).

**Proof.** (a) is immediate from Lemma 2.2.

(b) By the definition of \( \leq_L \) and by idempotence we have

\[
  x \leq_L y \quad \text{iff} \quad \exists u \in S^1: uy = x \quad \text{iff} \quad xy = x.
\]

(c) is proved similarly to (b). \( \square \)

From Lemma 2.3 follows that \( x \leq_J y \) holds in \( S \) if and only if it holds in some subsemigroup of \( S \) that contains \( x \) and \( y \). The same is true for \( \leq_L \) and \( \leq_R \).

We write \([n] := \{1, \ldots, n\}\) for \( n \in \mathbb{N} \). A tuple \( a \) in a direct power \( S^n \) is considered as a function \( a: [n] \to S \). Thus the \( i \)-th coordinate of this tuple is denoted by \( a(i) \) rather than \( a_i \). The subsemigroup generated by a set \( A = \{a_1, \ldots, a_k\} \) may be denoted by \( \langle A \rangle \) or \( \langle a_1, \ldots, a_k \rangle \).
Figure 1. The lattice of proper varieties of bands, taken from [6]. For two words $v$ and $w$ the expression $[v \approx w]$ denotes the variety of bands that satisfy the identity $v \approx w$. 
Lemma 2.4. Let \( S \) be a band and \( x, y \in S^n \) for some \( n \in \mathbb{N} \). Let \( \preceq \) be one of the preorders \( \preceq_J, \preceq_C, \preceq_R \). Then
\[
x \preceq \ y \quad \text{if and only if} \quad \forall i \in [n]: x(i) \preceq_R y(i).
\]

Proof. We prove the statement for \( \preceq_J \). By Lemma 2.3 (a) the following are equivalent:
\[
x \preceq_J y,
xyx = x,
xyx(i) = x(i) \quad \text{for all } i \in [n],
x(i) \preceq_J y(i) \quad \text{for all } i \in [n].
\]

For \( \preceq_C \) and \( \preceq_R \) the equivalence is proved similarly. \( \square \)

In the remainder of this section, we use some well-established results on the lattice of varieties of bands. For a full characterization the reader is referred to [6]. The following notation was introduced there.

Definition 2.5 (cf. [6, Notation 2.1]).
\( S \) the dual semigroup of \((S, \cdot)\) is the semigroup \((S, \ast)\) with \( x \ast y := y \cdot x \).
\( S^1 \) the semigroup obtained when an identity is adjoined to \( S \).
\( X \) the countably infinite set of variables \( \{x_1, x_2, \ldots\} \).
\( F(X) \) the free semigroup over \( X \).
\( \emptyset \) the empty word, i.e. the identity of \( F(X)^1 \).
\( \bar{w} \) the dual of a word \( w \in F(X)^1 \), i.e. the word obtained when reversing the order of the variables of \( w \).
\( c(w) \) the content of a word \( w \in F(X)^1 \), i.e. the set of variables occurring in \( w \).
\( s(w) \) the longest left cut of the word \( w \) that contains all but one of the variables of \( w \): For \( w \neq \emptyset \) let \( u, v \in F(X)^1 \) and \( x \in X \) such that \( w = u xv \) and \( c(u) \neq c(ux) = c(w) \). Then define \( s(w) := u \). Let \( s(\emptyset) := \emptyset \).
\( \sigma(w) \) for \( w \neq \emptyset \), the last variable in \( w \) under the order of the first occurrence, starting from the left. We define \( \sigma(\emptyset) := \emptyset \).
\( \bar{f} \) for a (partial) function \( f: F(X)^1 \to F(X)^1 \), we define \( \bar{f}(w) := \overline{f(w)} \).

Definition 2.6 (cf. [6, Notation 3.1]). For \( n \geq 2 \) we define \( h_n: F(X)^1 \to F(X)^1 \),
\[
h_n(\emptyset) := \emptyset \quad \text{for } n \geq 2,
h_2(w) := \text{the first variable of } w \text{ if } w \neq \emptyset,
h_n(w) := h_n s(w) \sigma(w) h_{n-1}(w) \quad \text{for } n \geq 3, \ w \neq \emptyset.
\]

From [6] we obtain the following upper bound on the length of \( h_n(w) \) for \( n \geq 2 \). This will allow us to prove that the SMP for every band is in NP.

Lemma 2.7. For every integer \( n \geq 2 \) there is a polynomial \( p_n \) such that for all \( k \in \mathbb{N} \) and all \( k \)-ary terms \( t \) the length of \( h_n(t) \) is at most \( p_n(k) \).

Proof. We use induction on \( n \). For the base case note that the length of \( h_2(t) \) is 1 for all terms \( t \). Now assume the assertion is true for some \( n \geq 2 \). Let
\[
p_{n+1}(k) := k(1 + p_n(k)) \quad \text{for } k \in \mathbb{N}.
\]
Let $k \in \mathbb{N}$, and $t$ be a term over $x_1, \ldots, x_k$. Let $\ell \leq k$ be the number of variables actually occurring in $t$. By repeated application of the recursion in Definition 2.6 we obtain

$$h_{n+1}(t) = h_{n+1}s(t) \cdot \sigma(t)\bar{h}_n(t)$$
$$= h_{n+1}s^2(t) \cdot \sigma s(t)\bar{h}_n s(t) \cdot \sigma(t)\bar{h}_n(t)$$
$$\vdots$$
$$= h_{n+1}s^{\ell}(t) \cdot \sigma s^{\ell-1}(t)\bar{h}_n s^{\ell-1}(t) \cdots \sigma s^1(t)\bar{h}_n s^1(t) \cdot \sigma(t)\bar{h}_n(t)$$
$$= \prod_{i=\ell}^0 \sigma s^i(t)\bar{h}_n s^i(t) \quad \text{since } s^i(t) = \emptyset.$$

The length of $\sigma s^i(t)$ is 1 for all $i$ by the definition of $\sigma$. Thus the length of each factor of the product is at most $1 + p_n(k)$. Therefore the length of $h_{n+1}(t)$ is bounded by $(1 + p_n(k))$, which is at most $p_{n+1}(k)$.

**Lemma 2.8.** Let $S$ be a finite band. Then there is a polynomial $p$ such that every $k$-ary term function on $S$ is induced by a term of length at most $p(k)$.

**Proof.** It is well-known that the variety of bands is not finitely generated. Thus $S$ belongs to a variety $[G_n \approx H_n]$ for some $n \geq 3$ by [6]. Let $p := p_n$ be the polynomial from Lemma 2.7. Now let $f$ be a $k$-ary term function on $S$ induced by some term $t$. By [5, Theorem 4.5] $S$ satisfies $t \approx h_n(t)$. Thus $h_n(t)$ also induces $f$. By Lemma 2.7 the length of $h_n(t)$ is at most $p_n(k)$. \hfill \Box

**Theorem 2.9.** The SMP for a finite band $S$ is in NP.

**Proof.** Fix an instance $\{a_1, \ldots, a_k\} \subseteq S^n$, $b \in S^n$ of SMP($S$). If $b \in \langle a_1, \ldots, a_k \rangle$, then there is a term function $f$ such that $b = f(a_1, \ldots, a_k)$. By Lemma 2.8 $f$ is induced by a term $t$ whose length is at most $p(k)$. We can verify $b = f(a_1, \ldots, a_k)$ in $O(np(k))$ time. Thus $t$ is a witness for $b \in \langle a_1, \ldots, a_k \rangle$. \hfill \Box

3. **Quasidentities**

Recall the quasidentities $\lambda$ and $\bar{\lambda}$ from Section 1. For every finite band $S$ we introduce two intermediate problems, INFIX($S$) and SUFFIX($S$). If $S$ satisfies $\lambda$, these problems can be solved in polynomial time. If $S$ also satisfies the dual quasidentity $\bar{\lambda}$, then SMP($S$) is in P. In Section 4 we will show that the SMP for the remaining finite bands is NP-complete.

For finite bands $S$, we define INFIX($S$) as follows.

**INFIX($S$)**

Input: $c, d, e \in S^n$ such that $c \text{ } J \text{ } d$ and $d \leq J \text{ } e$,

$A \subseteq S^n$ such that $\forall a \in A: e \leq J \text{ } a$.

Output: Some $y \in \langle A \rangle$ such that $dy = e$ if it exists; false otherwise.

We call an output $y \in \langle A \rangle$ a solution of INFIX($S$). The next result allows us to combine and reduce solutions of INFIX($S$) if $S$ satisfies the quasiderinty $\lambda$.

**Lemma 3.1.** Let $S$ be a finite band that satisfies $\lambda$. Let $c, d, e \in S^n$, $A \subseteq S^n$ be an instance of INFIX($S$) and $x, y, z \in \langle A \rangle$.

(a) If $x$ and $z$ are solutions of INFIX($S$) and there exists $h \in S^n$ with $hx = x$ and $hz = z$, then $xz$ is a solution of INFIX($S$).
(b) If \(xyz\) is a solution of \(\text{Infix}(S)\) with \(x \leq_y y\) and \(x \leq_z z\), then \(xz\) is also a solution of \(\text{Infix}(S)\).

Proof. (a) Let \(x, y, z, h\) be as as above. We claim that

\[
dx(ye)(ze) = d(ze) \\
hx = x \\
h(ze) = (ze) \\
d \leq_J (ze) \leq_J x, (ye).
\]

For proving (1) note that the condition on \(A\) and Lemma 2.1 imply \(e \leq_J g\) for all \(g \in \langle A \rangle\). So \(e \leq_J z\), and thus \(eze = e\) by Lemma 2.3 (a). Hence \(dxyeze = dxye = c = dze\), proving the first statement. The next two statements are clear from the assumptions. Lemma 2.1 and \(y, z \in \langle A \rangle\) imply \(e \leq_J z\) and \(d \leq_J e \leq_J x\) prove the last statement. By \(\lambda\) the identities (1) imply \(dx(ze) = d(ze)\). Thus \(dxze = c\). Item (a) is proved.

(b) Let \(x, y, z\) be as above. We prove that

\[
(dxy)(zx)y(ze) = (dxy)(ze) \\
z(xz) = (xz) \\
z(ze) = (ze) \\
d \leq_J (ze) \leq_J (xz), y.
\]

The first statement holds since \(x(yz)x = x\) by Lemma 2.3 (a). Statements two and three are clear. Lemma 2.1 implies \(e \leq_J ze\) and \(x \leq_J zx\). This and \(d \leq_J e \leq_J x, y\) prove the last statement. Now (2) implies \((dxy)(zx)(ze) = (dxy)(ze)\) by \(\lambda\). As \(xyzx = x\), we obtain \(dxze = dxyze\). This proves item (b).

\(\square\)

**Theorem 3.2.** Let \(S\) be a finite band that satisfies \(\lambda\). Then Algorithm 1 solves \(\text{Infix}(S)\) in polynomial time.

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**Algorithm 1**

Solves \(\text{Infix}(S)\) in polynomial time if the band \(S\) satisfies \(\lambda\).

**Input:** \(c, d, e \in S^n, A \subseteq S^n\) as in the definition of \(\text{Infix}(S)\).

**Output:** \(y \in \langle A \rangle\) such that \(dye = c\) if it exists; false otherwise.

```
1: for \(a_0 \in A\) such that \(dye = c\) if it exists; false otherwise. do
2: \(s := a_0s\)
3: \(y := a_0\)
4: until \(\exists a_1 \in A: a_1 \geq_J y, dya_1e = c\) do
5: if \(\exists a_2, a_3 \in A: a_2 \geq_J y, a_3 \not\geq_J y, dya_2a_3se = c\) then
6: \(y := ya_2a_3\)
7: else
8: continue for loop
9: end if
10: end until
11: return \(ya_1\) \(\triangleright a_1\) from line 4
12: end for
13: return false
```
Proof. Before we deal with correctness and complexity, we need some preparation. We fix an instance $c, d, e \in S^n$, $A \subseteq S^n$ of INFIX(S). We claim that

$$\forall i \in [n]: \text{ya}_2a_3(i) \leq_J y(i),$$

$$\exists i \in [n]: \text{ya}_2a_3(i) <_J y(i).$$

Hence $\text{ya}_2a_3$ is strictly smaller than $y$ in the preorder $\leq_J$. We can decrease $y$ at most $n(h-1)$ times. We proved (3). In particular the algorithm stops.

Correctness of Algorithm 1. First assume Algorithm 1 returns some $z \neq \text{false}$. This can only happen in line 11, and thus $z = \text{ya}_1$. From line 4 follows $\text{dy}a_1e = c$. We have $y \in \langle A \rangle$ since the only lines where $y$ is modified are 3 and 6. So $z = \text{ya}_1$ is a solution of INFIX(S).

Conversely assume that INFIX(S) has a solution $z \in \langle A \rangle$. Our goal is to show that Algorithm 1 returns some solution rather than false. Let $b_1, \ldots, b_m \in A$ such that $z = b_1 \cdots b_m$. For the remainder of the proof of correctness we may assume that

$$\text{the variable } a_0 \text{ of the for loop is set to } b_1.$$ 

It suffices to show that the algorithm returns some solution in line 11 for this case. The $s$ in line 1 exists since $s := z$ is one possibility. Fix the value assigned to $s$ in line 2. For each value assigned to $y$ in Algorithm 1, we claim that

$$y \in \langle A \rangle, \quad a_0y = y, \quad \text{and } dyse = c.$$

If $y$ obtained its value in line 3, then clearly (5) holds. If not, then $y$ obtained its value by one or more calls of the assignment $y := \text{ya}_2a_3$ in line 6. So the first two statements in (5) follow by induction, and line 5 implies $dyse = c$.

Next we claim that for each value assigned to $y$,

$$\text{if } y \leq_J z, \text{ then the condition in line 4 is fulfilled.}$$

Fix such $y \leq_J z$. By (5) and the assumptions we have

$$dyse = c, \quad dze = c, \quad a_0y = y, \quad a_0z = z.$$ 

Now apply Lemma 3.1 (a) and obtain $dyze = c$. Note that instead of $A$ we use $A' := \{a \in S^n \mid a \geq_J e\}$ for the hypothesis of Lemma 3.1. For $a_1 := b_m$ we have $z = za_1$ and thus

$$dyza_1e = c, \quad y \leq_J z \leq_J a_1.$$ 

Lemma 3.1 (b) yields $dyza_1e = c$. This proves (6).

Similar to (6) we claim that for each value assigned to $y$,

$$\text{if } y \not\leq_J z, \text{ then the condition in line 5 is fulfilled.}$$

Fix such $y \not\leq_J z$. Let $b_1$ be the first generator of $z$ with $b_1 \not\geq_J y$. By (4) and (5) we have $b_1 = a_0 \geq_J y$. Thus $b_1 \neq b_1$. Let $z_1 := b_1 \cdots b_{i-1}, a_2 := b_{i-1},$ and $a_3 := b_i$. 

Idempotence implies \( z_1a_2a_3z = z \). We have
\[
d(z_1a_2a_3)ze = c,\]
\[
dse = c \quad \text{by lines 1 and 2,}\]
\[
a_0z_1 = z_1 \quad \text{by (4),}\]
\[
a_0s = s \quad \text{by line 2.}\]

We apply Lemma 3.1 (a) and obtain \( d(z_1a_2a_3)se = c \). Now we have
\[
dyse = c \quad \text{by (5)},\]
\[
(d(z_1a_2a_3)s)e = c,\]
\[
a_0y = y \quad \text{by (5)},\]
\[
a_0z_1 = z_1.\]

Apply Lemma 3.1 (a) again and obtain \( dy(z_1a_2a_3)s)e = c \). For \( e' := a_3se \) we have \( e' \leq_J e \) and
\[
dyz_1a_2e' = c,\]
\[
e' \leq_J y \leq_J z_1 \leq_J a_2 \quad \text{by the definitions of } z_1 \text{ and } a_2.\]

From Lemma 3.1 (b) we obtain \( dya_2e' = dya_2(a_3se) = c \). This proves (7).

We are ready to complete the proof of correctness. By the assumption of (4) and by (6) and (7), Algorithm 1 does not enter the else branch in line 7. By (3) the until loop stops after \( O(n) \) iterations. After that \( ya_1 \) is returned in line 11. By (5) and line 4, \( ya_1 \) is a solution of \( \text{INFIX}(S) \).

**Complexity of Algorithm 1.** By (3) the until loop iterates at most \( O(n) \) times. Evaluating the condition in line 4 requires \( O(|A|) \) multiplications in \( S^n \), and the condition in line 5 requires \( O(|A|^2) \) multiplications. Thus \( O(n|A|^2) \) multiplications in \( S \) are performed in one iteration of the until loop. Therefore the effort for the until loop is \( O(n^2|A|^2) \).

The tuple \( s \) in line 1 can be found componentwise. In particular, for each \( i \in [n] \) we find \( s(i) \in S \) such that \( da_0se(i) = c(i) \). This process requires \( O(n) \) steps. Lines 2 and 3 require \( O(n) \) steps. Altogether one iteration of the for loop requires \( O(n^2|A|^2) \) time. Hence Algorithm 1 runs in \( O(n^2|A|^3) \) time. 

For finite bands \( S \) we define the intermediate problem \( \text{SUFFIX}(S) \). We will show that \( \text{SUFFIX}(S) \) is in P if \( S \) satisfies \( \lambda \).

**SUFFIX(S)**

**Input:** \( A \subseteq S^n, \ b \in S^n \).

**Output:** Some \( x \in \langle A \rangle \) such that \( x \leq b \) if it exists; false otherwise.

As usual we call an output \( x \in \langle A \rangle \) a **solution** of \( \text{SUFFIX}(S) \). Every solution \( x \) fulfills \( bx = b \), and thus \( x \) is a ‘suffix’ of \( b \). Hence the name of the problem.

**Theorem 3.3.** Let \( S \) be a finite band that satisfies \( \lambda \). Then Algorithm 2 solves \( \text{SUFFIX}(S) \) in polynomial time.

**Proof.** Before we prove correctness and complexity, we need some preparation. To show that the algorithm always stops, we claim that

\[
\text{the while loop iterates at most } O(n) \text{ times. (8)}
\]
Algorithm 2
Solves SUFFIX(S) in polynomial time if the band S satisfies λ.

Input: A ⊆ Sn, b ∈ Sn
Output: x ∈ ⟨A⟩ such that x \not≤ b if it exists; false otherwise.
1: find x ∈ A such that bx = b
2: return false if no such x exists
3: while x \not≤ b do
4: find a ∈ A, a \geq J b, a \not\geq J x and y ∈ ⟨A⟩, y \geq J x such that (ba)yx = b
▷ at most |A| instances of Infix(S)
5: return false if no such a, y exist
6: x := ayx
7: end while
8: return x

Let h be the height of the semilattice S/\mathcal{J}. In each iteration, either the algorithm terminates, or x is modified by x := ayx. By line 4 a \not\geq J x. Thus also ayx \not\geq J x. We have
\[ \forall i ∈ [n]: ayx(i) \leq \mathcal{J} x(i), \]
\[ \exists i ∈ [n]: ayx(i) < \mathcal{J} x(i). \]
Therefore the number of modifications of x is at most n(h − 1). We proved (8).
Later in the proof we use the following:
(9) x \not≤ b if and only if x \not\mathcal{J} b and bx = b.
This follows immediately since every \mathcal{J}-class is a rectangular band.
Correctness of Algorithm 2. First we claim that
(10) each value assigned to x fulfills x ∈ ⟨A⟩ and bx = b.
After initializing x in line 1, (10) clearly holds. The value of x is possibly modified by x := ayx in line 6. If x ∈ ⟨A⟩, then also ayx ∈ ⟨A⟩. From line 4 we know that bayx = b. This proves (10).
Now assume Algorithm 2 returns some x ≠ false. Then the while loop has finished. This implies x \not≤ b. By (10) x is a solution of SUFFIX(S).
Conversely assume SUFFIX(S) has a solution z ∈ ⟨A⟩. Our goal is to show that Algorithm 2 returns some solution. We fix c1, . . . , cm ∈ A such that z = c1 · · · cm. The x in line 1 exists. For instance for x = cm we have b = bz = bzx = bx.
Now fix a value assigned to x in line 1 or line 6. After this assignment, the while loop is called. If x \not≤ b, then the algorithm returns x, which is a solution by (10). Assume x \not\mathcal{J} b. By (9) and (10) we have x > J b. We claim that a and y as defined in line 4 exist, that is
(11) \exists a ∈ A, a \geq J b, a \not\geq J x \exists y ∈ ⟨A⟩, y \geq J x: bayx = b.
Since z \mathcal{J} b and b \not\geq J x, there is a ci with ci \not\geq J x. Assume i ∈ [m] is maximal with this property. If i = m, then let y := x. Otherwise let y := ci+1 · · · cm. In both cases y ∈ ⟨A⟩ and y \geq J x. Idempotence implies zx = zcyx. Hence
b = bzx = bzcxy = bcxy.
Note that \( c_i \geq_J b \) since \( S / J \) is a semilattice. This proves (11). Now we know that false is never returned in the while loop. By (8) the while loop finishes after finitely many iterations. After that \( x L b \) holds, and the solution \( x \) is returned.

**Complexity of Algorithm 2.** Line 1 requires at most \( O(n |A|) \) multiplications in \( S \). By (8) the while loop iterates at most \( O(n) \) times. In line 4 the algorithm iterates over \( a \) and \( y \). If we consider \( a \) as fixed, then the algorithm tries to find \( y \in \langle A \rangle \), \( y \geq_J x \) such that \( (ba)yx = b \). Finding such \( y \) is an instance of \( \text{Infix}(S) \). Thus in line 4 at most \( |A| \) instances of \( \text{Infix}(S) \) have to be solved. By the proof of Theorem 3.2, one instance runs in time \( O(n^2 |A|^3) \). Thus the while loop requires \( O(n^3 |A|^4) \) steps. Altogether Algorithm 2 runs in time \( O(n^3 |A|^4) \). \( \square \)

**Theorem 3.4.** Let \( S \) be a finite band that satisfies \( \lambda \) and \( \bar{\lambda} \). Then Algorithm 3 decides \( \text{SMP}(S) \) in polynomial time.

**Algorithm 3**

Decides \( \text{SMP}(S) \) in polynomial time if the band \( S \) satisfies \( \lambda \) and \( \bar{\lambda} \).

**Input:** \( A \subseteq S^n \), \( b \in S^n \)

**Output:** true if \( b \in \langle A \rangle \); false otherwise.

1: return \( \exists x, y \in \langle A \rangle : bx = b = yb \)

**Proof. Correctness of Algorithm 3.** If Algorithm 3 returns true, then \( bx = b = yb \). Thus \( b = ybx = yx \) by Lemma 2.3 (a) and hence \( b \in \langle A \rangle \). Conversely assume \( b \in \langle A \rangle \). Then \( x \) and \( y \) as defined in line 1 exist since we can set \( x = y = b \). Algorithm 3 returns true.

**Complexity of Algorithm 3.** In line 1 one instance of \( \text{Suffix}(S) \) and one of \( \text{Suffix}(\bar{S}) \) are solved. By the proof of Theorem 3.3 and by Lemma 1.1 both can be decided in \( O(n^3 |A|^4) \) time. \( \square \)

### 4. NP-hardness

In the previous section we showed that the SMP for a finite band that satisfies both \( \lambda \) and \( \bar{\lambda} \) is in P. In this section we will prove our dichotomy result Theorem 1.2 by showing that the SMP for the remaining finite bands is NP-complete.

**Definition 4.1.** Let \( S \) be a band. We say \( d, e, x, y, h \) witness \( S \not\models \lambda \) if they satisfy the premise of \( \lambda \), but not the implication.

Witnesses have the following properties.

**Lemma 4.2.** Let \( S \) be a finite band such that \( d, e, x, y, h \in S \) witness \( S \not\models \lambda \). Then

(a) \( d <_J e <_J x <_J h \),

(b) \( e, xe, y \) are distinct,

(c) \( d, dx, de, dxe \) are distinct.

**Proof.** (a) We have

\[
dxe = de, \quad hx = x, \quad he = e, \quad d \leq_J e \leq_J x, y \quad \text{and} \quad dxe \neq de.
\]

Thus \( d <_J e <_J x <_J h \). We show the strictness of each inequality by assuming the opposite and deriving the contradiction \( dxe = de \).
If \( d \not J e \), then \( dxe = de \) by Lemma 2.2.
If \( e \not J x \), then \( xe = xye \) by Lemma 2.2. Thus \( dxe = dxye = de \).
If \( x \not J h \), then \( hxe = h \) by Lemma 2.3 (a). Thus \( dxe = d(hx)(he) = dhe = de \).

(b) Assuming any equality, we derive the contradiction \( dxe = de \).
If \( e = xe \), then \( dxe = dxee = dxye = de \).
If \( xe = y \), then \( dxe = dxee = dxye = de \).
(c) Assuming any equality, we derive the contradiction \( dxe = de \).
If \( d = dx \), then \( dxe = de \).
If \( d = de \), then \( dxe = dxee = dxye = de \) since \( exe = e \).
If \( dx = de \), then \( dxe = dxee = de \).
If \( dx = dxe \), then \( dxe = dxee = dxye = de \).

□

If \( S \not| \lambda \), then there are witnesses with additional properties by the following lemma.

**Lemma 4.3.** Let \( S \) be a finite band that does not satisfy \( \lambda \). Then there are \( d, e, x, y, h \in S \) such that for \( T := \langle d, e, x, y, h \rangle \) the following holds:

(a) \( h \) is the identity of \( T \), and we have the following partial multiplication table whose entries are distinct.

|   | \( x \) | \( e \) | \( xe \) | \( y \) | \( d \) |
|---|---|---|---|---|---|
| \( x \) | \( x \) | \( xe \) | \( xe \) | \( y \) | \( xd \) |
| \( e \) | \( e \) | \( e \) | \( e \) | \( e \) | \( d \) |
| \( xe \) | \( xe \) | \( xe \) | \( xe \) | \( xe \) | \( xd \) |
| \( y \) | \( y \) | \( y \) | \( y \) | \( y \) | \( yd \) |
| \( d \) | \( dx \) | \( de \) | \( dxe \) | \( de \) | \( d \) |

(b) \( d, e, x, y, h \) witness \( T \not| \lambda \) and \( S \not| \lambda \).
(c) \( d/R \) is the smallest \( J \)-class of \( T \).
\( d/R = \{d, dx, de, dxe\} \), where the 4 elements are distinct, and
\( d/L = \{d, xd, yd\} \).
(d) Either of the following holds:
(1) \( d = xd = yd \) and \( |T| = 9 \).
(2) \( d = xd \neq yd \) and \( |T| = 13 \).
(3) \( d \neq xd = yd \) and \( |T| = 13 \).
(4) \( d, xd, yd \) are distinct and \( |T| = 17 \).

**Proof.** Let \( \bar{d}, \bar{e}, \bar{x}, \bar{y}, \bar{h} \in S \) witness \( S \not| \lambda \). Define
\[
d := \bar{x}\bar{h}\bar{d}\bar{h}, \quad e := \bar{e}\bar{h}, \quad x := \bar{x}\bar{h},
\]
\[
y := \bar{x}\bar{y}\bar{x}\bar{h}, \quad h := \bar{h}.
\]

(a) Since \( \bar{h} \) is a left identity for \( \bar{x} \) and \( \bar{e} \) and by idempotence, \( h = \bar{h} \) is an identity for \( d, e, x, y \). In the first row of the multiplication table, the only nontrivial entry is
\[
xy = (\bar{x}\bar{h})(\bar{x}\bar{y}\bar{x}\bar{h}) = \bar{x}\bar{y}\bar{x}\bar{h} = y.
\]
For the second row we use idempotence and Lemma 2.3 (a). We obtain

\[
\begin{align*}
ex &= (\bar{e} \bar{x} \bar{h})(\bar{x} \bar{h}) = \bar{e} \bar{x} \bar{h} = e, \\
ey &= (\bar{e} \bar{x} \bar{h})(\bar{x} \bar{y} \bar{e} \bar{x} \bar{h}) = \bar{e} \bar{x} \bar{h} = e.
\end{align*}
\]

The remaining entries follow from these. The third row is immediate from the second one. For the last two rows it suffices to show that

\[
\begin{align*}
yx &= (\bar{x} \bar{y} \bar{e} \bar{x} \bar{h})(\bar{x} \bar{h}) = \bar{x} \bar{y} \bar{e} \bar{x} \bar{h} = y, \\
ye &= (\bar{x} \bar{y} \bar{e} \bar{x} \bar{h})(\bar{e} \bar{x} \bar{h}) = \bar{x} \bar{y} \bar{e} \bar{x} \bar{h} = y, \\
dy &= (\bar{e} \bar{x} \bar{h} \bar{d} \bar{h})(\bar{x} \bar{y} \bar{e} \bar{x} \bar{h}) = (\bar{e} \bar{x} \bar{h} \bar{d} \bar{h})(\bar{e} \bar{x} \bar{h}) = de.
\end{align*}
\]

We will show that \(x, e, xe, y, d\) are distinct after proving (b).

(b) First we show that

\[
\begin{align*}
dxye &= de \neq dxe.
\end{align*}
\]

The equality follows from the multiplication table in (a). Now suppose \(dxe = de\). Then \(\ddxe \neq \ddde\). By Lemma 2.3 (a) we obtain

\[
\begin{align*}
\ddaxe &= \ddde, \\
&= \ddde.
\end{align*}
\]

Now \(\ddxe = \ddde\), which is impossible. Hence (12) holds.

From the multiplication table we see that

\[
\begin{align*}
d &\leq_J e \leq_J x, y.
\end{align*}
\]

Now (12), (13), \(hx = x\), and \(he = e\) yield item (b).

For item (a) it remains to show that \(x, e, xe, y, d\) are distinct. By item (b) and Lemma 4.2 (b) \(e, xe, y\) are distinct. From Lemma 4.2 (a) follows \(d <_J e <_J x <_J h\), and from the multiplication table \(e \in J\) \(xe \in J\) \(y\). Thus the elements are all distinct.

(c) By (13) and since \(T/J\) is a semilattice by Lemma 2.1, \(d/J\) is the smallest \(J\)-class of \(T\). Thus we have \(d/R = dT\) and \(d/L = Td\) in \(T\). The multiplication table yields

\[
dT = d\langle d, e, x, y, h\rangle = \{d, dx, de, dxe\}.
\]

The elements are distinct by Lemma 4.2 (c). For \(Td\) we obtain

\[
\begin{align*}
Td &= \langle d, e, x, y, h \rangle d \\
&= \langle e, x, y, h d \rangle \\
&= \langle x, y, h d \rangle \\
&= \{xd, yd, d\}
\end{align*}
\]

by the multiplication table.

Item (c) is proved.
(d) We count the elements of $T$. Note that $T = \{h, x, e, xe, y\} \cup d/J$ and

$$d/J = \{\ell dr \mid \ell \in \{h, x, y\}, r \in \{h, x, e, xe\}\}.$$ 

If $d = yd$, then $xd = xyd = yd = d$, and item (d)(1) holds. If $d \neq yd$, then one of the remaining cases applies. Using GAP [4] and the semigroups package [12], it is easy to check that the semigroups for the cases (d)(1) to (d)(4) actually exist. $\square$

**Corollary 4.4.** Let $S$ be a finite band. Then $S$ does not satisfy $\lambda$ if and only if one of the four non-isomorphic bands $T$ described in Lemma 4.3 embeds into $S$.

**Proof.** First we show that the two 13-element bands are not isomorphic. Let $T := \langle d, e, x, y, h \rangle$ and $\bar{T} := \langle \bar{d}, \bar{e}, \bar{x}, \bar{y}, \bar{h} \rangle$ be bands that fulfill properties (a) to (c) of Lemma 4.3. Assume $T$ fulfills (d)(2), and $\bar{T}$ fulfills (d)(3). Suppose there is an isomorphism $\alpha: T \rightarrow \bar{T}$. Isomorphisms preserve the relations $\leq_{T}$ and $J$. We have $T/J = \{\{h\}, \{x\}, \{e, xe, y\}, d/J\}$ and a similar partition for $\bar{T}$. Thus $\alpha$ maps $h$ to $\bar{h}$ and $x$ to $\bar{x}$. We apply $\alpha$ to the inequality $xe \neq e$ and obtain $\bar{x} \alpha(e) \neq \alpha(e)$. In order to fulfill the latter inequality and as $\alpha(e) \in \{\bar{e}, \bar{x}, \bar{y}\}$, we have $\alpha(e) = \bar{e}$. Therefore $\alpha(xe) = \bar{x}\bar{e}$, and thus $\alpha(y) = \bar{y}$. By items (c) and (d)(3) of Lemma 4.3 there is an $\ell \in \{\bar{h}, \bar{x}\}$ such that

$$\alpha(d) \in \{\ell \bar{d}, \ell \bar{d} \bar{x}, \ell \bar{d} \bar{e}, \ell \bar{d} \bar{x} \bar{e}\}.$$ 

Lemma 4.3 (c) implies $dx \neq d$, and thus $\alpha(d)x \neq \alpha(d)$. In order to fulfill this inequality and condition (14), $\alpha(d)$ must equal $\ell \bar{d}$. It remains to determine $\ell$. From $xd = d$ follows $x \alpha(d) = d$. Thus $\alpha(d) = \bar{x}d$. However $\alpha(yd) = \bar{y} \bar{x} \bar{d} = \bar{x} \bar{d} = \alpha(d)$. Thus $\alpha$ is not injective, which yields a contradiction. We proved that $T$ and $\bar{T}$ are not isomorphic.

Now the ($\Rightarrow$) direction of the corollary is immediate from Lemma 4.3. The ($\Leftarrow$) direction follows from the fact that none of the four bands described by Lemma 4.3 satisfies $\lambda$. $\square$

**Lemma 4.5.** Let $S$ be a finite band that does not satisfy $\lambda$ or $\bar{\lambda}$. Then $\text{SMP}(S)$ is NP-hard.

**Proof.** We may assume that $S$ does not satisfy $\lambda$. Let $d, e, x, y, h \in S$ witness $S \not\models \lambda$ such that properties (a) to (c) of Lemma 4.3 hold. Denote $h$ by 1 and let $T := \langle d, e, x, y, 1 \rangle$. We reduce the Boolean satisfiability problem SAT to $\text{SMP}(T)$. SAT is NP-complete [2] and defined as follows.

\begin{itemize}
  \item **SAT**
  \begin{itemize}
    \item **Input:** clauses $C_1, \ldots, C_n \subseteq \{x_1, \ldots, x_k, \neg x_1, \ldots, \neg x_k\}$
    \item **Problem:** Do truth values for $x_1, \ldots, x_k$ exist for which the Boolean formula $\phi(x_1, \ldots, x_k) := (\lor C_1) \land \ldots \land (\lor C_n)$ is true?
  \end{itemize}
\end{itemize}

Fix a SAT instance $C_1, \ldots, C_n$ on $k$ variables. For all $j \in [k]$ we may assume that $x_j$ or $\neg x_j$ occurs in some clause $C_i$. We define the corresponding $\text{SMP}(T)$ instance

$$A := \{u, v, a_{0}^{0}, \ldots, a_{k}^{0}, a_{1}^{0}, \ldots, a_{k}^{1}\} \subseteq T^{n+2k}, \quad b \in T^{n+2k}.$$
The first $n$ positions of the tuples correspond to the $n$ clauses. The remaining $2k$ positions control the order in which tuples can be multiplied. Let

$$
\begin{align*}
  b &:= ( \, de \cdots de \, de \cdots \cdots \cdots \cdots de \, ), \\
  u &:= ( \, d \cdots d \, d \cdots \cdots \cdots \cdots d \, ), \\
  v &:= ( \, xe \cdots xe \, y \cdots \cdots \cdots \cdots y \, ), \\
  a_j^0 &:= ( \, 1 \cdots 1 \, x \, e \cdots 1 \cdots 1 \, ) \quad \text{for } j \in [k], \\
  a_j^1 &:= ( \, 1 \cdots 1 \, e \, x \cdots 1 \cdots 1 \, ) \quad \text{for } j \in [k].
\end{align*}
$$

For $j \in [k]$ and $i \in [n]$ let

$$
(15) \quad a_j^0(i) := \begin{cases} 
  e & \text{if } \neg x_j \in C_i, \\
  1 & \text{otherwise,}
\end{cases} \quad a_j^1(i) := \begin{cases} 
  e & \text{if } x_j \in C_i, \\
  1 & \text{otherwise.}
\end{cases}
$$

Note that the size of the SAT instance is at least linear in the number of clauses $n$ and in the number of variables $k$. Hence we have a polynomial reduction from SAT to SMP($T$). In the remainder of the proof we show that

$$
(16) \quad \text{the Boolean formula } \phi \text{ is satisfiable if and only if } b \in \langle A \rangle.
$$

For the ($\Rightarrow$) direction let $z_1, \ldots, z_k \in \{0, 1\}$ such that $\phi(z_1, \ldots, z_k) = 1$. We claim that

$$
(17) \quad u a_1^{z_1} \cdots a_k^{z_k} v = b.
$$

For $i \in [n]$ the clause $\bigvee C_i$ is satisfied under the assignment $x_1 \mapsto z_1, \ldots, x_k \mapsto z_k$. Thus there is a $j \in [k]$ such that $x_j \in C_i$ and $z_j = 1$, or $\neg x_j \in C_i$ and $z_j = 0$. In both cases $a_j^0(i) = e$ by (15). Thus $a_1^{z_1} \cdots a_k^{z_k}(i) = e$, and hence

$$
ua_1^{z_1} \cdots a_k^{z_k} v(i) = dexe = de = b(i)
$$

by the multiplication table in Lemma 4.3. For $i \in [2k]$ we have $a_1^{z_1} \cdots a_k^{z_k}(n+i) \in \{x,e\}$. Thus

$$
ua_1^{z_1} \cdots a_k^{z_k} v(n+i) \in \{dxy,dey\} = \{de\}.
$$

We proved (17). Thus $b \in \langle A \rangle$.

For the ($\Leftarrow$) direction of (16) assume $b \in \langle A \rangle$. It is easy to see that $b = uv$. Thus there is a minimal $\ell \in \mathbb{N}_0$ such that $b = u g_1 \cdots g_\ell v$ for some $g_1, \ldots, g_\ell \in A$. We claim that

$$
(18) \quad u, v \notin \{g_1, \cdots, g_\ell\}.
$$

If $g_j = u$ for some $j \in [\ell]$, then $u g_1 \cdots g_j = u$ by Lemma 2.3 (a). Thus $b = u g_{j+1} \cdots g_\ell v$, contradicting the minimality of $\ell$. By a similar argument $v \notin \{g_1, \ldots, g_\ell\}$. We proved (18). Thus

$$
b = u a_{j_1}^{z_1} \cdots a_{j_\ell}^{z_\ell} v
$$

for some $j_1, \ldots, j_\ell \in [k]$ and $z_1, \ldots, z_\ell \in \{0, 1\}$. For $r, s \in [\ell]$ we claim:

$$
(19) \quad \text{If } j_r = j_s, \text{ then } z_r = z_s.
$$

Suppose there is a minimal index $r \in [\ell]$ such that $j_r = j_s$ and $z_r \neq z_s$ for some $s \in \{r+1, \ldots, \ell\}$. Then there is an $i \in \{2j_r - 1, 2j_r\}$ such that $a_{j_r}^{z_r}(n+i) = x$ and
$a^z_j(n+i) = e$. By the minimality of $r$ we have $a^z_j(n+i) = \ldots = a^{z_{r-1}}_j(n+i) = 1$. Thus $a^z_{j_1} \cdots a^z_{j_r}(n+i) = xe$, and hence $a^z_{j_1} \cdots a^z_{j_r}(n+i) = xe$. Therefore

$$ua^z_{j_1} \cdots a^z_{j_r}v(n+i) = dxe = dxe \neq b(n+i),$$

which contradicts our assumption. We proved (19).

Now we define an assignment

$$\theta: x_{j_1} \mapsto z_1, \ldots, x_{j_r} \mapsto z_r,$$

$$x_j \mapsto 0 \quad \text{for } j \in [k] \setminus \{j_1, \ldots, j_r\},$$

and show that

$$\theta$$ satisfies the formula $\phi$.

Let $i \in [n]$. We show that $\theta$ satisfies $\lor C_i$. Observe that $a^z_{j_1} \cdots a^z_{j_r}(i)$ is either 1 or $e$. In the first case $ua^z_{j_1} \cdots a^z_{j_r}v(i) = dxe \neq b(i)$, which is a contradiction. Thus $a^z_{j_1} \cdots a^z_{j_r}(i) = e$. Since not all factors in $a^z_{j_1} \cdots a^z_{j_r}(i)$ can be 1, we have $a^z_{j_r}(i) = e$ for some $r \in [l]$. By (15) either $z_r = 1$ and $x_{j_r} \in C_i$, or $z_r = 0$ and $\neg x_{j_r} \in C_i$. In both cases $\theta$ satisfies $\lor C_i$. We proved (20) and (16).

Finally we state an alternative version of our dichotomy result Theorem 1.2.

**Theorem 4.6.** Let $S$ be a finite band. Then $\text{SMP}(S)$ is in P if one of the following equivalent conditions holds:

(a) $S$ satisfies $\lambda$ and $\bar{\lambda}$.

(b) None of the four bands given in Lemma 4.3 embeds into $S$ or $\bar{S}$.

Otherwise $\text{SMP}(S)$ is NP-complete.

**Proof.** The conditions (a) and (b) are equivalent by Corollary 4.4. If they are fulfilled, then $\text{SMP}(S)$ is in P by Theorem 3.4. Otherwise $\text{SMP}(S)$ is NP-hard by Lemma 4.5 and in NP by Theorem 2.9. □

**Proof of Theorem 1.2.** Immediate from Theorem 4.6. □

5. PROOF OF THEOREMS 1.3 AND 1.6

In this section we prove the last two theorems of the introduction.

**Definition 5.1.** Let $S_9$ and $S_{10}$ be the bands with the following multiplication tables.

| $S_9$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|---|---|---|---|---|---|---|---|
| 1    | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2    | 2 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3    | 3 | 3 | 3 | 3 | 6 | 7 | 8 |
| 4    | 4 | 4 | 4 | 4 | 6 | 7 | 8 |
| 5    | 5 | 5 | 5 | 5 | 6 | 7 | 8 |
| 6    | 6 | 7 | 8 | 9 | 8 | 6 | 7 |
| 7    | 7 | 7 | 9 | 8 | 9 |
| 8    | 8 | 8 | 8 | 9 | 9 |
| 9    | 9 | 9 | 9 | 9 | 6 |

| $S_{10}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------|---|---|---|---|---|---|---|---|---|---|
| 1        | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2        | 2 | 2 | 4 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3        | 3 | 3 | 3 | 3 | 3 | 3 | 6 | 7 | 8 | 9 |
| 4        | 4 | 4 | 4 | 4 | 6 | 7 | 8 | 9 |
| 5        | 5 | 5 | 5 | 5 | 6 | 7 | 8 | 9 | 10 |
| 6        | 6 | 6 | 7 | 8 | 8 | 9 | 10 |
| 7        | 7 | 7 | 9 | 9 | 9 | 9 | 10 |
| 8        | 8 | 8 | 8 | 9 | 9 | 9 | 10 |
| 9        | 9 | 9 | 9 | 9 | 6 | 7 | 8 | 9 | 10 |

| 10       | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |

Note that $S_9$ is isomorphic to the 9-element band from Lemma 4.3 (d)(1) by renaming the elements as follows.

| $h$ | $x$ | $e$ | $xe$ | $y$ | $d$ | $dx$ | $de$ | $dxe$ |
|-----|-----|-----|------|-----|-----|------|------|------|
| 1   | 2   | 3   | 4    | 5   | 6   | 7    | 8    | 9    |
For the next result recall $G_n$, $H_n$, and $I_n$ from Definition 1.4.

**Lemma 5.2.** The bands $S_9$ and $S_{10}$ both generate the variety $[\bar{G}_3 \approx \bar{I}_3]$. Furthermore $S_9$ is the homomorphic image of $S_{10}$ under

$$\alpha: S_{10} \rightarrow S_9, \ x \mapsto \begin{cases} x & \text{if } x \leq 9, \\ 8 & \text{if } x = 10. \end{cases}$$

**Proof.** From the multiplication tables it is immediate that $\alpha$ is a homomorphism. Using the software GAP [4, 12] it is easy to show that both $S_9$ and $S_{10}$ satisfy the identity $\bar{G}_3 \approx \bar{I}_3$. It remains to show that $S_9$ and $S_{10}$ do not belong to a proper subvariety of $[\bar{G}_3 \approx \bar{I}_3]$. By Figure 1 every proper subvariety of $[\bar{G}_3 \approx \bar{I}_3]$ is contained in $[G_4 \approx H_4]$. For $v := (2, 1, 3, 6)$ and $S \in \{S_9, S_{10}\}$ we have

$$G_4^S(v) = 6123 = 9, \quad H_4^S(v) = 6123613123 = 8.$$ 

Thus neither $S_9$ nor $S_{10}$ satisfies $G_4 \approx H_4$. 

**Lemma 5.3.** The band $S_{10}$ satisfies $\lambda$ and $\bar{\lambda}$, whereas $S_9$ does not satisfy $\lambda$. 

**Proof.** By Corollary 4.4 $S_9$ does not satisfy $\lambda$. From the multiplication table for $S_{10}$ we see that every 9-element subsemigroup is of the form $S_{10} \setminus \{s\}$ for $s \in \{1, 2, 3, 5, 6\}$. None of these semigroups is isomorphic to $S_9$ or its dual $S_9$. Thus $S_{10}$ satisfies $\lambda$ and $\bar{\lambda}$ by Corollary 4.4. 

**Proof of Theorem 1.3.** Items (a) and (b) follow from Lemma 5.2, and item (c) from Theorem 4.6 and Lemma 5.3. 

**Lemma 5.4.** Let $S$ be a finite band that satisfies $G_4 \approx H_4$. Then $S$ satisfies $\lambda$. 

**Proof.** Let $d, e, x, y, h \in S$ such that

$$dx = x, \quad he = e, \quad d \leq_s e \leq_s x, y$$

Set $v := (y, x, e, d)$. From $S \models G_4 \approx H_4$ and the definitions of $G_4$ and $H_4$ follows

$$dxye = G_4^S(v) = H_4^S(v) = dxyedxeye.$$ 

Lemma 2.3 (a) implies $dxyed = d$ and $exye = e$. Therefore the right hand side equals $dxe$, and thus $de = dxye = dxe$. Hence $S$ satisfies $\lambda$. 

**Proof of Theorem 1.6.** From Figure 1 we see that

$$[\bar{G}_4 G_4 \approx \bar{H}_4 H_4] = [G_4 \approx H_4] \cap [\bar{G}_4 \approx \bar{H}_4].$$

Let $S$ be a finite band in this variety. Since $S$ and $\bar{S}$ belong to $[G_4 \approx H_4]$, both bands satisfy $\lambda$ by Theorem 5.4. Thus $S$ satisfies $\lambda$ and $\bar{\lambda}$. By Theorem 4.6 SMP($S$) is in $P$.

Now let $V$ be a variety of bands greater than $[\bar{G}_4 G_4 \approx \bar{H}_4 H_4]$. First assume $V \not\subseteq [G_4 \approx H_4]$. Then $[G_3 \approx I_3] \subseteq V$ by Figure 1. By Lemma 5.2 $S_9$ belongs to $V$. If $V \not\subseteq [G_4 \approx H_4]$, then the dual band $\bar{S}_9$ belongs to $V$ by a similar argument. Since the SMP for both $S_9$ and $\bar{S}_9$ is NP-complete by Lemma 4.3, the result follows. 

6. Conclusion and open problems

In [1] it was shown that the SMP for every finite semigroup is in PSPACE. Moreover, semigroups with NP-complete and PSPACE-complete SMPs were provided. In the present paper we presented the first examples of completely regular semigroups with NP-hard SMP. We established a P/NP-complete dichotomy for the case of bands. However, it is unknown if the SMP for every completely regular semigroup is in NP. Thus the following problem is open.

**Problem 6.1.** Is the SMP for every finite completely regular semigroup in NP? In particular, is there a P/NP-complete dichotomy for completely regular semigroups?

So far there is no algebra known for which the SMP is neither in P nor complete in NP, PSPACE, or EXPTIME. In the case of semigroups, the following open problem arises.

**Problem 6.2.** Is there a P/NP-complete/PSPACE-complete trichotomy for the SMP for semigroups?

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