Path integral and instantons for the process and phase transition rate of the RNAdS black hole

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We propose a new approach to study the dynamical phase transition of RNAdS black holes on the underlying free energy landscape. By formulating a path integral framework, we can quantify the kinetic paths representing the history from the initial state to the end state, which provides us a visualized yet quantified picture about how the phase transition proceeds. Based on these paths, we derive the analytical formulas for the time evolution of the transition probability and provide a physical interpretation of the contribution to the probability from one “pseudomolecule” (“anti-pseudomolecule”), which is actually the phase transition rate from the small(large) to the large(small) black hole state. These numerical results show a good consistency with the underlying free energy landscape topography.

I. INTRODUCTION

In general relativity, the classical black holes are emerged from the solutions of Einstein’s equation. The black holes have some fascinating features. They are perfect absorbers but emit nothing. As known, an object with non-zero temperature has the thermal radiation. This implies that the physical temperature of the classical black hole is zero and the black hole thermodynamics seems to be impossible. This has all been changed since the appearance of the theorem \cite{1}, stating that the event horizon area of the black hole can never decrease with the time. Bekenstein incisively noticed the similarities with the second law of thermodynamics, and proposed that every black hole should have its own entropy which is associated with the event horizon area by a directly proportional relationship \cite{2}. Thereafter, the four laws of black hole mechanics \cite{3} were formulated, analogous to the four laws of thermodynamics. However, since the temperature of the classical black hole is zero, it implies that these similarities are merely formal and do not have profound physical implications.

The whole picture has been significantly changed since the quantum effects were considered, leading to the famous Hawking radiation, which shows that the black holes emit radiation with a blackbody spectrum \cite{4}. The Hawking radiation and the four laws of black hole mechanics indicate the black holes are thermodynamic systems with temperature. Since then, the black hole thermodynamics has been widely used to study the black hole physics. A famous example is the Hawking-Page transition occurring in the asymptotically anti-de sitter space \cite{5}. A first-order phase transition has been found between the thermal radiation and the large stable schwarzschild anti-de sitter black hole. The subsequent researches show that the Hawking-Page transition can be interpreted as the confinement/deconfinement phase transition in QCD \cite{6}. By viewing the cosmological constant as thermodynamic pressure in the AdS space, the analogues of charged-AdS black holes and the Van der Waals fluids have been explored \cite{7,11}. The behaviors of the black hole thermodynamics at the triple point phase transition have also been investigated \cite{12}. All of these studies provide us a more profound understanding to black hole thermodynamics.

However, the dynamics of black hole phase transition has not been investigated adequately until very recently. The very recent studies of black hole thermal dynamical phase transition have been explored on the free energy landscape by solving the corresponding Fokker-Planck equation, giving rise to the mean first passage time and the fluctuations \cite{13,14}. The complete description of the dynamics should include two aspects. The rate shows how fast the black hole phase transition occurs and the path shows how the process proceeds in the phase transition. Thus, it is necessary for us to quantify the phase transition path to explore the underlying dynamical process.

Since the first appearance in \cite{15}, the path integral methods have been developed for a long time and used to study many physical and chemical problems successfully \cite{16,20}. The advantage of such method is that one can quantify the path with weight representing the history from the initial state to the end state. The path will provide us a quantitative and visual picture of the phase transition process. This certainly helps us to understand the dynamics of phase transition better.

The phase transition of RNAdS black holes takes place in the asymptotically AdS space with the negative cosmological constant. By interpreting the cosmological constant as thermodynamic pressure \cite{7,21,22}, one can formulate the extended phase space and study the van der waals type phase transition in RNAdS black holes. By choosing the black hole radius as the order parameter, the free energy landscape can be quantified along this order parameter. The phase transition can then be easily analyzed on the free energy landscape \cite{13,14}. There are three macroscopic emergent phases, the small, the intermediate and the large black hole states. The small

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and large black hole states are locally stable and the intermediate black hole state is unstable. Under the thermodynamic fluctuations, the phase transition is possible between the stable small black hole state and the stable large black hole state.

In this paper, we study the process of such a phase transition by using a path integral method [19, 20, 23–29]. The weights of different paths are from the exponentials through the path integral actions. This implies that the weight of the dominant path is significantly larger than that of the other paths. Then we can just consider the contribution from the dominant path. The dominant path should satisfy the Euler-Lagrangian equation. In the recent research, it was addressed the crucial kinetic path issue and provide a more profound understanding of the phase transition process of the RNAdS black holes.

The paper is organized as follows. In sec. II, we illustrate the thermodynamic properties of RNAdS black holes under the underlying free energy landscape. In sec. III, we introduce the path integral framework and apply it to the phase transition of RNAdS black holes. Then, the kinetic paths, phase transition rates, and the time evolutions of the probabilities are presented. In sec. IV, we present the conclusions.

II. THERMODYNAMICS OF RNADS BLACK HOLE AND THE FREE ENERGY LANDSCAPE

In this section, we will briefly review the thermodynamic properties of RNAdS black holes [7, 13, 14, 30, 31].

The metric of RNAdS black hole is given by \( G = 1 \) units

\[
ds^2 = -f(R)dt^2 + \frac{dR^2}{f(R)} + R^2 d\Omega^2,
\]

where \( f(R) \) is given by

\[
f(R) = 1 - \frac{2m}{R} + \frac{q^2}{R^2} + \frac{R^2}{L^2}.
\]

The parameter \( m \) represents the black hole mass, \( q \) is the black hole charge, and \( l \) is the AdS curvature radius which is associated with the negative cosmological constant \( \Lambda \) by \( l = \sqrt{-\frac{3}{\Lambda}} \).

In the AdS space, the cosmological constant is interpreted as the thermodynamic pressure [7, 32, 34]:

\[
P = \frac{3}{8\pi l^2}.
\]

The Hawking temperature is given by

\[
T_H = \frac{1}{4\pi r} (1 + 8\pi P r^2 - \frac{q^2}{r^2}).
\]

We should note that there is a critical pressure \( P_c = \frac{1}{96\pi r^2} \) [7, 14]. When \( P > P_c \), the Hawking temperature \( T_H \) is a monotonic increasing function of \( r \). When \( P < P_c \), \( T_H \) has a local minimum value \( T_{\text{min}} \) and a local maximum value \( T_{\text{max}} \), which are determined by \( \frac{\partial T_H}{\partial r} = 0 \). We will focus on the regime \( P < P_c \) and \( T_{\text{min}} < T < T_{\text{max}} \), where there are three on-shell solutions to the stationary Einstein field equation as the small, the intermediate and the large black holes.

On the free energy landscape, the free energy of the system is defined as a continuous function of the order parameter. It is necessary to introduce a series of off-shell states for the study of the black hole phase transition dynamics. In general, we can choose the radius of the AdS black hole as the order parameter and assume a canonical ensemble which is composed of various black hole spacetime states with different radii at the specific temperature [5, 13, 14]. This includes all the possible states appearing during the phase transition. These states are characterized by the different black hole radii. Expect for the small, the intermediate and the large black holes, all the other states are off-shell and do not obey the stationary Einstein field equation. In the recent research, it was found that there is a lower bound for the order parameter which corresponds to the extremal black hole [33]. We denote the lower bound of the order parameter as \( r_{\text{ex}} \).

Replacing the Hawking temperature \( T_H \) by the ensemble temperature \( T \) in the on-shell Gibbs free energy expression formulate \( G = m - T_H S \), we can generalize the on-shell Gibbs free energy to the off-shell free energy as [14, 30, 31]:

\[
G = m - TS = \frac{r}{2} (1 + \frac{8}{3} \pi P r^2 + \frac{q^2}{r^2}) - \pi T r^2,
\]

where the order parameter \( r \) can take the continuous values from \( r_{\text{ex}} \) to infinity.

We choose \( P = 0.4P_c \) and \( q = 1 \) in all the next calculation, \( r_{\text{ex}} \) can be calculated as 0.984, which is smaller than the radius of the small black hole at various temperatures. In Fig. [1], we have plotted the free energy as a function of black hole radius \( r \) at different temperatures.
can solve eq. (6) and obtain the values of $r$ corresponding to the free energy maximum is unstable while the intermediate black hole and large black hole states corresponding to the free energy minimum are stable. Furthermore, it can be seen from the Fig. 1, the small and large black hole states corresponding to the free energy minima are stable while the intermediate black hole state corresponding to the free energy maximum is unstable. Because of the thermal fluctuations, it is possible for the phase transitions between the stable small black hole state and the stable large black hole state. We will study the dynamics of such phase transitions by using path integral methods explored in the next section.

**III. PATH-INTEGRAL AND PHASE TRANSITION RATE OF RNAdS BLACK HOLE**

**A. The path integral of the RNAdS black hole phase transition**

The stochastic dynamics of the RNAdS black hole under the thermal fluctuations have been described by the probabilistic Fokker-Planck equation in [13, 14]. In order to formulate the path integral framework, we prefer to use the equivalent Langevin equation as:

$$\frac{dr}{dt} = -\frac{\partial G(r)}{\partial r} + \eta(r,t),$$  

(7)

where $\gamma$ is the friction coefficient; $-\frac{\partial G(r)}{\partial r}$ is the driving force; $\eta(r,t)$ is the fluctuating stochastic force. We assume $\eta(r,t)$ to be the Gaussian white noise, which satisfies the equations $\langle \eta(r,t) \rangle = 0$ and $\langle \eta(r,t)\eta(r,0) \rangle = 2D(r)\delta(t)$. The diffusion coefficient $D(r)$ is associated with the friction coefficient by the Einstein relationship

$$D(r)\gamma = k_BT.$$  

(8)

The dynamics can be described by the Onsager-Machlup functional path integral as [16, 23]:

$$P(r_t, t, r_0, 0) = \int Dr \exp[-\int L[r(t)]dt] = \int Dr \exp[-\int \left\{\frac{1}{4} \frac{dr}{dt} + \frac{D(r)\partial \beta G(r)}{\partial r}\right\}^2 D(r) \frac{1}{2} \frac{\partial (D(r)\partial \beta G(r))}{\partial r}] dt,$$

(9)

where $Dr$ represents the sums of all the paths connecting the initial state and the end state, $L[r(t)]$ is the stochastic Lagrangian (also called the Onsager-Machlup functional):

$$L = \frac{1}{4} \left\{\frac{dr}{dt} + \frac{D(r)\partial \beta G(r)}{\partial r}\right\}^2 - \frac{1}{2} \frac{\partial (D(r)\partial \beta G(r))}{\partial r}.$$  

(10)

We assume the diffusion coefficient is very small, then the last term of the Lagrangian in eq. (10) can be ignored as:

$$L = \frac{1}{4} \left\{\frac{dr}{dt} + \frac{D(r)\partial \beta G(r)}{\partial r}\right\}^2.$$  

(11)

From Eq. (9), we can see that the different paths contribute a different weight, which is on the exponential. This indicates that the dominant path has the largest weight, which can be significantly larger than the weights of the other paths. Thus, we can just consider the contributions of dominant path. The dominant path should obey the Euler-Lagrangian equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0.$$  

(12)

Substituting the Eq. (11) into the Eq. (12), we can obtain

$$\frac{d^2 r}{dt^2} - \frac{1}{2} \frac{\partial (D(r))}{\partial r} r^2 - 2D(r)\frac{\partial u}{\partial r} = 0,$$

(13)

where

$$u(r) = \frac{D(r)}{4} \left(\frac{\partial \beta G(r)}{\partial r}\right)^2.$$  

(14)

Integrating the Eq. (13), we can obtain:

$$\frac{(\frac{dr}{dt})^2}{4D(r)} - u(r) = E,$$

(15)
where $E$ is a constant.

The eq. (15) can be regarded as an energy conservation equation. $\frac{1}{2D(T)}(\frac{dr}{dt})^2$ is the kinetic energy term, $V(r) = -u(r)$ is the effective potential, and $E$ is the total energy. Thus, the dynamics of the phase transition can be regarded as the problem of one-dimensional particle with mass $\frac{1}{2D(T)}$ moving in the effective potential $V(r)$ [17] [20].

We assume $D$ is a constant and choose $D = k = 1$ without loss of generality. In Fig. 2, we have plotted the effective potential as the functions of the black hole radius at different temperature. It can be seen when $T < T_{\text{min}}$ or $T > T_{\text{max}}$, there is only one point whose effective potential is zero. When $T = T_{\text{min}}$ or $T = T_{\text{max}}$, there are two such points. And when $T_{\text{min}} < T < T_{\text{max}}$, there are three such points. Analyzing the eq. (10) and eq. (14), we can know all these radii of zero effective potential points are precisely the on-shell solutions to the stationary Einstein field equation.

When $T_{\text{min}} < T < T_{\text{max}}$, there are three local maximum points whose effective potentials are zero, representing the small, the intermediate and the large black hole states. Corresponding, we denote their radii as $r_s$, $r_m$ and $r_l$. In the long time limit, the phase transitions between the small black hole state and the large black hole state can take place many times because of the thermodynamic fluctuations. This implies that the equivalent particle can go back and forth many times between the point $r = r_s$ and the point $r = r_l$ in the effective potential $V(r)$. The dominant path is composed of a series of smallest units of such jumps named pseudomolecules. These pseudomolecules should start at the stable states and end also at the stable states. Actually, every pseudomolecule is composed of a pair of instantons (or named pseudoparticles) whose paths are between the point of $r_s$ or $r_l$ and the point of $r_m$. There are four kinds of pseudomolecules in total: $a$ pseudomolecule has the trajectory $r_s \rightarrow r_m \rightarrow r_s$ with an instanton $r_s \rightarrow r_m$ and an anti-instanton $r_m \rightarrow r_s$, whose contribution to the probability is named $m_1$; $b$ pseudomolecule has the trajectory $r_s \rightarrow r_m \rightarrow r_l$ with a pair of instantons $r_s \rightarrow r_m$ and $r_m \rightarrow r_l$, whose contribution to the probability is named $m_2$; $c$ pseudomolecule has the trajectory $r_l \rightarrow r_m \rightarrow r_l$ with an anti-instanton $r_l \rightarrow r_m$ and an instanton $r_m \rightarrow r_l$, whose contribution to the probability is named $m_3$; $d$ pseudomolecule has the trajectory $r_l \rightarrow r_m \rightarrow r_s$ with a pair of anti-instantons $r_l \rightarrow r_m$ and $r_m \rightarrow r_s$, whose contribution to the probability is named $m_4$. We assume that there are no interactions between the instantons, so that we can calculate the final contribution by summing in the dilute gas approximation [28] [35].

In order to calculate the probability for the phase transition, we need to obtain the pseudomolecule paths. Based on the eq. (13), we can plot the black hole radius as the function of time $t$ from small (large) black hole state to the intermediate black hole state at different temperatures in Fig. 3, they are actually the paths of $a$ and $c$ pseudomolecules. From the paths, it can be seen that the
phase transition between the small and large black holes will not have a residence time in the intermediate states. However, it will stay at the small and large black hole states with certain time interval. This indicates that the small and large black hole states are stable, while the off-shell states are unstable transient states. Furthermore, the paths mainly pass through the off-shell states, this is also the reason we introduce the off-shell states to show such a dynamical process when the phase transition occurs.

The weight of one pseudomolecule contribution to the probability is given by:

\[ M = \exp[-S] = \exp[-\int_{t_{\text{initial}}}^{t_{\text{end}}} L(r(t))dt], \quad (16) \]

where \( S \) represents the action of the path:

\[ S = \int_{t_{\text{initial}}}^{t_{\text{end}}} L(r(t))dt = \frac{1}{4} \int_{t_{\text{initial}}}^{t_{\text{end}}} \left( \frac{dr}{dt} \right)^2 + 2 \frac{dr}{dt} \frac{\partial \beta G(r)}{\partial r} + D \left( \frac{\partial \beta G(r)}{\partial r} \right)^2 dt. \quad (17) \]

For the weight contributions of the paths to the probability, we assume that the initial point is located at the stable small or stable large black hole state whose effective potential and kinetic energy are zero. The energy conservation equation \([13, 29, 36]\) becomes:

\[ \frac{dr}{dt} = \pm \sqrt{4Du(r)} = \pm D \left| \frac{\partial \beta G(r)}{\partial r} \right|. \quad (18) \]

Based on the free energy figure in Fig.1 we know that the \( \frac{\partial \beta G(r)}{\partial r} \) is greater than zero between \( r_s \) and \( r_m \), and less than zero between \( r_m \) and \( r_l \). The sign of \( \frac{dr}{dt} \) is determined by the process in which the system is proceeding. It is positive when the system translates from the small to large black hole state and negative from the large to small black hole state. Then, we can obtain:

From \( r_s \) to \( r_m \), \( \frac{dr}{dt} = D \frac{\partial \beta G(r)}{\partial r} \).  
From \( r_m \) to \( r_l \), \( \frac{dr}{dt} = -D \frac{\partial \beta G(r)}{\partial r} \).  
From \( r_l \) to \( r_m \), \( \frac{dr}{dt} = D \frac{\partial \beta G(r)}{\partial r} \).  
From \( r_m \) to \( r_s \), \( \frac{dr}{dt} = -D \frac{\partial \beta G(r)}{\partial r} \).

Substituting these equations back into eq.\([17]\), and being careful of the signs, we can see that the action \( S = 0 \) for the trajectories from \( r_m \) to \( r_s \) and from \( r_m \) to \( r_l \). This indicates that only the trajectory from the small (large) black hole state to the intermediate black hole state contributes to the probability for the phase transition from small (large) black hole state to large (small) black hole state, which also corresponds to the free energy figures in Fig.1. In the transition from the small to large black hole states, the free energy is uphill for the trajectory from the small black hole state to the intermediate black hole state and downhill from the intermediate black hole state to the large black hole state. The probability for the latter trajectory is 1, so we just need to consider the probability contribution for the former trajectory. The similar analysis can be made in the transition from the large black hole state to the small black hole state. Thus, in the phase transition from small (large) black hole state to large (small) black hole state, we do not need to obtain the whole path between the small black hole state and the large black hole state and just need to consider the path from small (large) black hole state to the intermediate black hole state.

Based on the above analysis, we know that the trajectories from \( r_m \) to \( r_l \) and from \( r_m \) to \( r_s \) do not have a contribution, so we have the equations:

\[ m_1 = -m_2 = -M_1; \quad (19) \]

\[ m_3 = -m_4 = -M_2, \quad (20) \]

where the minus sign appearing in the expressions is associated with the presence of a turning point on the trajectory \([29, 36]\). We call the \( M_1 \) as one pseudomolecule contribution to the probability and the \( M_2 \) as one anti-pseudomolecule contribution to the probability.

Then, the probability \( P(r_s, t; r_s, t_0) \) and \( P(r_l, t; r_l, t_0) \) can be given analytically. Because the intermediate black hole state of radium \( r_m \) is unstable, we can neglect the time interval staying at this state at long time limit.

At first, one can calculate the probability \( P(r_s, t; r_s, t_0) \). When there is zero pseudomolecule, the probability becomes \( e^{-u(r_s)(t-t_0)} \), where \( t-t_0 \) represents the time interval staying at the small black hole state;

When there is one pseudomolecule, only an \( a \) pseudomolecule contributes and the probability is

\[ \int_{t_0}^{t_1} dt_1 \int_{r_0}^{r_1} dt_2 (-M_1)e^{-u(r_s)(t_1-t_0)}e^{-u(r_s)(t-t_1)}, \quad (21) \]

where the \(-M_1\) is the one \( a \) pseudomolecule contribution to the probability, \( t_1 - t_0 \) and \( t - t_1 \) represent the time interval staying at the small black hole state;

When there are two pseudomolecules, two \( a \) pseudomolecules contribute, or \( b \to d \), the arrow represents the time sequence of the pseudomolecules. The probability is

\[ \int_{t_0}^{t_1} dt_1 \int_{r_0}^{r_1} dt_2 (-M_1)^2 e^{-u(r_s)(t_1-t_0)}e^{-u(r_s)(t_2-t_1)}e^{-u(r_s)(t-t_2)} \]

\[ + \int_{t_0}^{t_1} dt_1 \int_{r_0}^{r_1} dt_2 (M_1 M_2) e^{-u(r_s)(t_1-t_0)}e^{-u(r_s)(t_2-t_1)}e^{-u(r_s)(t-t_2)} \]

In the first term, \((-M_1)^2\) represents the two \( a \) pseudomolecule contribution to the probability, \( t_1 - t_0 \), \( t_2 - t_1 \) and
FIG. 3: The paths of $a$ and $c$ pseudomolecules and corresponding instantons and anti-instantons at different temperatures $T = 0.031$, $T = 0.033$ and $T = 0.035$.

$t - t_2$ are the time intervals staying at the small black hole state. In the second term, $M_1M_2$ represents the one $b$ and one $d$ pseudomolecule contribution to the probability, $t_1 - t_0$ and $t - t_2$ are the time intervals staying at the small black hole state, $t_2 - t_1$ is the time interval staying at the large black hole state.

When there are three pseudomolecules, three $a$ pseudomolecules contribute, $a \rightarrow b \rightarrow d$, $b \rightarrow d \rightarrow a$, or $b \rightarrow c \rightarrow d$, the probability is
\[
\int_{t_0}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \int_{t_2}^{\infty} dt_3 (-M_1^3 e^{-u(r_s)(t_1-t_0)} e^{-u(r_s)(t_2-t_1)} e^{-u(r_s)(t_3-t_2)} e^{-u(r_s)(t-t_3)} + 
\int_{t_0}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \int_{t_2}^{\infty} dt_3 (-M_1^2 M_2^2) e^{-u(r_s)(t_1-t_0)} e^{-u(r_s)(t_2-t_1)} e^{-u(r_s)(t_3-t_2)} e^{-u(r_s)(t-t_3)} + 
\int_{t_0}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \int_{t_2}^{\infty} dt_3 (-M_1^2 M_2^2) e^{-u(r_s)(t_1-t_0)} e^{-u(r_s)(t_2-t_1)} e^{-u(r_s)(t_3-t_2)} e^{-u(r_s)(t-t_3)}. \]

(23)

In the long time limit, the probability \( P(r_s, t; r_s, t_0) \) is the sums of the pseudomolecule number from 0 to \( \infty \); Thus, based on the condition \( u(r_s) = u(r_l) = u(r_m) = u \), the probability is simplified as:

\[
P(r_s, t; r_s, t_0) = e^{-u(t-t_0)} - M_1 \int_{t_0}^{\infty} dt_1 e^{-u(t_1-t_0)} e^{-u(t-t_1)} + M_1 (M_1 + M_2) \int_{t_0}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 e^{-u(t_1-t_0)} e^{-u(t_2-t_1)} e^{-u(t-t_2)} - M_1 (M_1 + M_2)^2 \int_{t_0}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \int_{t_2}^{\infty} dt_3 e^{-u(t_1-t_0)} e^{-u(t_2-t_1)} e^{-u(t_3-t_2)} e^{-u(t-t_3)} + \ldots
\]

\[
= e^{-u(t-t_0)} + M_1 \sum_{n=1}^{\infty} (-1)^n (M_1 + M_2)^{n-1} \int_{t_0}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \ldots \int_{t_{n-1}}^{\infty} dt_n e^{-u(t_1-t_0)} e^{-u(t_2-t_1)} \ldots e^{-u(t-t_n)},
\]

where \( t_0 = 0 \).

By using the Laplace transform, we can obtain

\[
P(s) = \frac{1}{s + u} - \frac{M_1}{M_1 + M_2} \left[ \frac{1}{s + u} - \frac{1}{s + u + M_1 + M_2} \right].
\]

(25)

Inverting the Laplace transform, we can simplify the Eq.(24) as:

\[
P(r_s, t; r_s, 0) = e^{-ut} \frac{M_2 + M_1 e^{-(M_1 - M_2)t}}{M_1 + M_2}. \]

(26)

In our problem, \( u(r_s) = u(r_m) = u(r_l) = u = 0 \), so

\[
P(r_s, t; r_s, 0) = \frac{1}{M_1 + M_2} \left[ M_2 + M_1 e^{-(M_1 - M_2)t} \right].
\]

(27)

An similar procedure can be applied to the calculation of \( P(r_l, t; r_l, 0) \), one can obtain:

\[
P(r_l, t; r_l, 0) = \frac{1}{M_1 + M_2} \left[ M_1 - M_1 e^{-(M_1 + M_2)t} \right].
\]

(28)

B. The physical significance of one pseudomolecule or one anti-pseudomolecule contribution to the probability and the kinetic rates

We can consider a model for a particle moving in a double well potential with the two stable states denoted as \( A \) and \( B \). We assume that the translation rate from state \( A \) to state \( B \) is \( k_A \) and the transition rate from state \( B \) to state \( A \) is \( k_B \), while the initial state of the particle is state \( A \). The \( P_A(\tau) \) represents the probability of the particle staying at the state \( A \) at time \( \tau \), and the \( P_B(\tau) \) represents the probability of the particle staying at the state \( B \) at time \( \tau \). Then, one can write the classical master equation as:

\[
\frac{dP_A(\tau)}{d\tau} = -k_A P_A(\tau) + k_B P_B(\tau).
\]

(29)

The total probability should be conserved, we can obtain

\[
P_A(\tau) + P_B(\tau) = 1.
\]

(30)

When substituting the Eq. (30) back to Eq. (29) and integrating the \( \tau \) from 0 to \( t \), we can obtain:

\[
P_A(t) = \frac{1}{k_A + k_B} [k_B + k_A e^{-(k_A + k_B)t}].
\]

(31)

The \( P_B(t) \) is given by

\[
P_B(t) = 1 - P_A(t) = \frac{1}{k_A + k_B} [k_A - k_A e^{-(k_A + k_B)t}].
\]

(32)

In the small-large black hole phase transition, the Gibbs free energy landscape has the double well shape as shown in Fig. 1. Eq. (31) and Eq. (32) can then be used to describe the time evolution of the transition probability during the phase transition, and we should obtain the same results as Eq. (27) and Eq. (28) after taking the
state \( A \) and \( B \) as the small and large black hole state respectively. When we compare the equation \( (27) \) and \( (28) \) to the equation \( (31) \) and \( (32) \), the physical significance of \( M_1 \) and \( M_2 \) can be easily seen: The \( M_1 \) represents the transition rate from the small black hole to large black hole, and the \( M_2 \) represents the transition rate from the large black hole to the small black hole.

Furthermore, based on the equation \( (27) \) and \( (28) \), the total kinetic rate is given by

\[
k = M_1 + M_2, \quad (33)
\]
which determines the rate or the time scale (inverse of the rate \( \frac{1}{\tau} \)) of the probability evolution for \( P(r_s, t; r_s, 0) \) and \( P(r_1, t; r_s, 0) \).

C. The second order effects

The fluctuation effects on the dominate path can be considered. Then the phase transition rate and the probability evolution will be modified.

We replace \( D\beta G(r) \) by \( U(r) \) in Eq. \( (9) \) for simplification, and the probability is given by:

\[
P(r_1, t_1; r_0, t_0) = \exp\left[-\frac{[U(r_1) - U(r_0)]}{2D}\right]
\]

\[
\int_{r_0}^{r_1} Dr \exp\left\{-\frac{1}{D} \int_{r_0}^{t_1} dt \left(\frac{d^2}{2} + \frac{(U'(r))^2}{4} - \frac{D}{2} U''(r)\right)\right\}
\]

\[
= \exp\left[-\frac{U(r_1) - U(r_0)}{2D}\right] \int_{r_0}^{r_1} Dr \exp\left[-\frac{S(r(t))}{D}\right]
\]

\[
(34)
\]

We denote \( \int_{r_0}^{r_1} Dr \exp\left[-\frac{S(r(t))}{D}\right] \) as \( K(r_1, t_1; r_0, t_0) \). The action can be expanded around the classical path to the second order variation in \( y(t) = r(t) - r_{cl}(t) \), one yields:

\[
K(r_1, t_1; r_0, t_0) = e^{-\frac{S[r_{cl}(t)]}{D}} \int D y(t)
\]

\[
\exp\left\{-\frac{1}{2D} \int_{r_0}^{t_1} dt \left(\frac{d^2}{2} + V''(r_{cl}(t))\right)y(t)\right\}
\]

\[
(35)
\]

where \( V(r) = \frac{(U'(r))^2}{4} - \frac{D}{2} U''(r) \).

We expand \( y(t) \) on an infinite orthogonal basis \( \{y_n(t)\} \) which is also the eigenfunction of \( \frac{-1}{2} \frac{d^2}{dt^2} + V''(r_{cl}(t)) \). These eigenfunctions satisfy the equation \( (A2), (A3), (A4) \) and \( (A5) \). By using the Gaussian integral, the equation \( (35) \) becomes \( (24) \) and \( (25) \):

\[
K(r_1, t_1; r_0, t_0) = \frac{N}{\sqrt{\prod_n \lambda_n}} \exp\left[-\frac{S[r_{cl}(t)]}{D}\right]
\]

\[
= \frac{N}{\sqrt{\prod_n \lambda_n}} \exp\left[-\frac{S(r_{cl}(t))}{D}\right],
\]

\[
(36)
\]

where \( \lambda_n \) is the eigenvalue of the operator \( \frac{1}{2} \frac{d^2}{dt^2} + V''(r_{cl}(t)) \). \( N \) is a constant. More description is given in the appendix.

This equation holds for monostable potentials, but will break down for the potential in our case which always has an eigenfunction \( r_{cl}(t) \) with zero eigenvalue. Thus, the Gaussian approximation of the corresponding fluctuation modes will break down. These modes are called zero modes and their physical origin is the time translational invariance of the system \( (21) \) and \( (22) \). Considering the case of one zero mode (or one instanton), the equation \( (35) \) will be replaced by \( (28) \) and \( (29) \). We denote \( \lambda_0 \) and \( \psi_{\lambda_0}(t) \) satisfy the equation \( (A8), (A9) \) and \( (A10) \), and \( \sqrt{\frac{S(r_{cl})}{4\pi D}} \) is the integration of the variables \( \tau_0 \).

The path integral problems of the second order in the symmetric double well have been explored by \( (24) \), \( (28) \), \( (37) \), \( (38) \), while the problems become quite difficult in the asymmetric double well. Because of the asymmetry, the instanton has the different asymptotic behaviours in the two sides of the time axis, and the calculations become tedious. This has been explored in \( (29), (39) \), which is based on the method of \( (28) \), \( (37) \), \( (38) \). The main procedures are shown in the appendix. Assume there are no interactions between these instantons, then the dilute gas approximation can be used to obtain the final probability by summing over the multi-instantons. After a heavy algebra, the final results show the probabilities driven by the classical paths have a correction as \( (29), (39) \):

\[
P(r_s, t; r_s, 0) = \sqrt{\frac{\beta G''(r_s)}{2\pi}} \frac{1}{M_1 + M_2} [M_2 + M_1 e^{-(M_1 - M_2)t}],
\]

\[
P(r_1, t; r_s, 0) = \sqrt{\frac{\beta G''(r_1)}{2\pi}} \frac{1}{M_1 + M_2} [M_1 - M_1 e^{-(M_1 + M_2)t}],
\]

\[
(38)
\]

\[
(39)
\]

where \( M_1 \) and \( M_2 \) have a correction as:

\[
M_1 \rightarrow \frac{\beta D \sqrt{[G''(r_n)]G''(r_s)}}{2\pi} \ast M_1,
\]

\[
M_2 \rightarrow \frac{\beta D \sqrt{[G''(r_n)]G''(r_1)}}{2\pi} \ast M_2.
\]

D. The numerical results

The phase transition rate is an important entity in the dynamics of phase transition process, which quantifies the time scale of the small (large) black hole state switching to the large (small) black hole state. Based on the equation \( (16) \), we use the classical pseudomolecule paths to obtain the temperature dependence of the phase transition rates in Fig. \( 4 \) (red lines). If we take into account of the second order effects, there is a correction to the phase transition rates as shown in \( (40) \) and \( (41) \). We also plot the temperature dependence of the phase transition rates including the second order effects in Fig. \( 4 \) (blue lines).
Note that the vertical coordinate is the logarithm of the phase transition rates.

When we analyze the phase transition rate without the second order effects (red lines). The results show, upon the increase of the temperature, the kinetic rate of phase transition will increase from the small to large black hole states transition and decrease from the large to small black hole states transition. As shown in Fig. 1, we know that the barrier height from the small black hole state to the large black hole state through the intermediate black hole state will decrease with the temperature. This indicates that the small black hole state should be easier to be switched to the large black hole state. The barrier height from the large to the small black hole states through the intermediate black hole state will increase with the temperature. Then the large black hole state should be more difficult to switch to the small black hole state. These are consistent with the picture of quantified rates in Fig. 4. Furthermore, when the Gibbs free energies of small and large black holes are equal ($T = 0.0298$), the phase transition rates of $M_1$ and $M_2$ should be equal. This also corresponds to our resulting rates well.

When we analyze the phase transition rate including the second order effects (blue lines), we can see that both these curves are near to the corresponding curves without the second order effects (red lines). Furthermore, we note that the curve in the left panel has an inflection point comparing with the curve without the second order effects. It is an interesting phenomenon which means that the second order effects can become significant compared to the zero order effects when the temperature is high. After taking into account of the second order effects, the phase transition rate should be determined by both the barrier height of the free energy landscape and the second derivatives of the free energy landscape at the basin and at the barrier (saddle). When the temperature is high, the barrier height of the free energy landscape from the small black hole to the large black hole through the intermediate black hole does not change significantly upon the increase of the temperature. However, the second order derivatives of the free energy landscape at both the small black hole basin and the intermediate black hole barrier or saddle decrease clearly. Thus, the second order effects become more important than the zero order effects, and the rate of phase transition decreases accordingly. In the right panel, we can not observe such an inflection point because of the obvious variation of the barrier height.

The temperature dependence of the total kinetic rates in both the cases of zero order (red lines) and of the one including the second order (blue lines) have been plotted in Fig. 5. It can be seen that the total kinetic rate decreases with the temperature first and then increases with the temperature in the zero order effects. The total kinetic rate indicates the time scale of the probability evolution, which is the combination of the rates from small to large and from large to small black hole phase transition. The two kinetic behaviors with the temperature reflect the temperature dependence of each individual transition rate. When the temperature is high, the rate from the small to large black hole state dominates in the total kinetic rate, and the inflection point of the total kinetic rate will appear after taking into account the second order effects for the same reason as the left panel in Fig. 4.

Furthermore, the time evolutions of $P(r_s, t; r_s, 0)$ and $P(r_l, t; r_s, 0)$ at different temperatures in both two cases are given in Fig. 6 and Fig. 7. As seen, the $P(r_s, t; r_s, 0)$ and $P(r_l, t; r_s, 0)$ become steady when $t$ is large, the time of the probability being steady is determined by the total kinetic rate in Fig. 5.

The stationary probability can reflect the thermodynamic stability, and it is determined by the value of the Gibbs free energy via the Boltzmann distribution. Whether we take into account of the second order effects or not, the stationary probability should be equal. As shown in Fig. 6 and Fig. 7, the red line (without the second order effects) and the blue line (with the second order effects) are steady to the same value at the same temperature. When we analyze the Fig. 6 and Fig. 7 separately, we can see that the steady state probability $P(r_s, t; r_s, 0)$ decreases and the steady state probability $P(r_l, t; r_s, 0)$ increases with the temperature increasing. As shown in Fig. 1, when the temperature increases, the free energy of the small black hole state decreases slower than that of the large black hole state.
The Boltzmann distribution tells us that the steady state probability $P(r_s, t; r_s, 0)$ decreases and the steady state probability $P(r_l, t; r_s, 0)$ increases with the temperature. When we compare the Fig. 6 with the Fig. 7, we can perform the following analyses. At $T = 0.0298$, the Gibbs free energies of the small black hole state and the large black hole state are equal, the steady state probability $P(r_s, t; r_s, 0)$ and $P(r_l, t; r_s, 0)$ should be equal. As shown in Fig. 7 and Fig. 8, they are both equal to 0.5. When $T = 0.029 < 0.0298$, the free energy of the small black hole state is smaller than the free energy of the large black hole state. This indicates that the small black hole state is more stable, thus the steady state probability $P(r_s, t; r_s, 0)$ is larger than the steady probability $P(r_l, t; r_s, 0)$. When $T = 0.03$ and 0.031, they are both larger than $T = 0.0298$. Then the large black hole state becomes more stable. Correspondingly, the steady state probability $P(r_l, t; r_s, t)$ is larger than the steady state probability $P(r_s, t; r_s, 0)$.

![Figure 6](image1)

**FIG. 6**: The time evolution of the probability $P(r_s, t; r_s, 0)$ at different temperatures $T = 0.029, 0.0298, 0.03$ and 0.031.

![Figure 7](image2)

**FIG. 7**: The time evolution of the probability $P(r_l, t; r_s, 0)$ at different temperatures $T = 0.029, 0.0298, 0.03$ and 0.031.

IV. CONCLUSIONS

In conclusion, we have formulated a path integral framework to investigate the dynamical phase transition of RNAdS black hole under the free energy landscape. There are three macroscopic emergent phases in the extended phase space. The small and large black hole states are stable and the intermediate black hole state is unstable. Under the thermal fluctuations, the phase transition is possible between the small and large black hole states. The corresponding dynamics can be described by the stochastic Langevin equation, where the thermodynamic driving force is provided by the underlying Gibbs free energy and the stochastic force comes from the thermal fluctuations. The contribution of the different paths to the weight or probability is on the exponential. Thus, the dominant path, which satisfies the Euler-Lagrangian equation, can be regarded as the main path in which the phase transition proceeds. Based on the dominant path, we derive the analytical formula for the time evolution of the transition probability. After comparing with a model for a particle moving in the double well potential, we find that the contribution to the probability from one “pseudomolecule” (“anti-pseudomolecule”) can be interpreted as the phase transition rate from the small (large) to large (small) black hole state. The numerical results show a good consistency with the underlying free energy landscape topography. This work provides a new framework to investigate the dynamics of black hole phase transition, which can address the equally important issues of both the kinetic path and the phase transition rate. This framework can also be used to investigate other kinds of black hole phase transition problems.

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Appendix A: The second order effects

The Eq. (35) is written again as:

$$K(r_l, t_1; r_0, t_0) = e^{-rac{d r_{cl}(t)}{d t} \int D y(t)} \exp \left\{ -\frac{1}{2D} \int_{t_0}^{t_1} dy(t) \left[ -\frac{1}{2} \frac{d^2}{dt^2} + V''(r_{cl}(t)) \right] y(t) \right\}$$ (A1)
One can expand \( y(t) \) on an infinite orthogonal basis \( \{y_n(t)\} \) which is also the eigenfunction of the second variational derivative of \( S \) [28]:

\[
y(t) = \sum_n c_n y_n(t), \tag{A2}
\]

\[
\left[-\frac{1}{2} \frac{d^2}{dt^2} + V''(r_{cl}(t))\right] y_n(t) = \lambda_n y_n(t), \tag{A3}
\]

\[
y_n(t_0) = y_n(t_1) = 0, \tag{A4}
\]

\[
\int_{t_0}^{t_1} y_n(t) y_m(t) dt = \delta_{mn}. \tag{A5}
\]

By using the Gauss integral, the equation (A1) becomes

\[
K(r_1, t_1; r_0, t_0) = \frac{N}{\det\left[-\frac{1}{2} \frac{d^2}{dt^2} + V''(r_{cl}(t))\right]} \exp\left[-\frac{S(r_{cl}(t))}{D}\right]
\]

\[
= \frac{N}{\det\left[-\frac{1}{2} \frac{d^2}{dt^2} + \frac{1}{2} w^2 \right]} \det\left[-\frac{1}{2} \frac{d^2}{dt^2} + V''(r_{cl}(t))\right] \exp\left[-\frac{S(r_{cl}(t))}{D}\right]
\]

\[
= \frac{N}{\sqrt{\prod_n \lambda_n^{(h)}}} \exp\left[-\frac{S(r_{cl}(t))}{D}\right]. \tag{A6}
\]

Here we have brought in the well-known harmonic solution \( \frac{N}{\sqrt{\prod_n \lambda_n^{(h)}}} \) to eliminate the constant \( N \), then the key issue becomes to calculate the factor \( \sqrt{\prod_n \lambda_n^{(h)}} \). [24] [28].

Based on the same zero points and the same pole points at the two sides of the following equation, it was proved that [28] [37] [38]

\[
\det\left[-\frac{1}{2} \frac{d^2}{dt^2} + V(1) - \lambda \right] = \psi_1^{(1)}(T/2) \psi_2^{(2)}(T/2), \tag{A7}
\]

where \( \psi_\lambda(t) \) is the corresponding solution satisfying:

\[
\left(-\frac{1}{2} \frac{d^2}{dt^2} + V(i) \right) \psi_\lambda^{(i)}(t) = \lambda \psi_\lambda^{(i)}(t), \tag{A8}
\]

\[
\psi_\lambda^{(i)}(t_0) = 0, \quad \partial_t \psi_\lambda^{(i)}(t_0) = 1, \tag{A9}
\]

where \( i = 1, 2 \).

The operator \(-\frac{1}{2} \frac{d^2}{dt^2} + V(i)\) has an eigenvalue \( \lambda_n \), only if

\[
\psi_\lambda^{(i)}(t_1) = 0. \tag{A10}
\]

Taking \( \lambda = 0 \) in Eq. (A7), the problem is changed to evaluate the ratio of the corresponding lowest eigenfunction. This condition holds for monostable potentials. However, there is always an eigenfunction \( \dot{r}_{cl}(t) \) with zero eigenvalue in our case, and the Gaussian approximation of the corresponding fluctuation modes will break down. These modes are called zero modes [28] [29] [37][39]. The eq. (A6) should factor out the zero modes and evaluate the determinant with zero eigenvalue omitted, when considering that there is only one instanton (or one zero mode), we can rewrite Eq. (A6) as:

\[
K(r_1, t_1; r_0, t_0) = \frac{N}{\sqrt{\prod_n \lambda_n^{(h)}}} \int_{t_0}^{t_1} \psi_\lambda^{(h)}(t_1) \psi_\lambda^{(h)}(t_0) \int dc_0 \exp\left[-\frac{S(r_{cl}(t))}{D}\right]. \tag{A11}
\]

Based on the time invariance of the instantons, one can obtain:

\[
\delta(r(t + \tau_0)) = \frac{dr(t)}{dt} \delta_{r_0} = y_0(t) \delta_{c_0} \tag{A12}
\]

\[
= \left( \frac{S_{cl}}{m} \right)^{\frac{1}{2}} \dot{r}_{cl}(t) \delta_{c_0}. \tag{A13}
\]

Then, one can replace the \( dc_0 \) integration by an integration over the position of the center of the instanton \( d\tau_0 \):

\[
dc_0 = \sqrt{\frac{S_{cl}}{m}} d\tau_0. \tag{A14}
\]

The \( K(r_1, t_1; r_0, t_0) \) becomes as:

\[
K(r_1, t_1; r_0, t_0) = \int_{t_0}^{t_1} d\tau_0 \sqrt{\frac{\lambda_0}{4\pi D \psi_\lambda^{(h)}(t_1) \psi_\lambda^{(h)}(t_0)}} \exp\left[-\frac{S(r_{cl}(t))}{D}\right]. \tag{A15}
\]

Taking the derivative with respect to the time, one can obtain:

\[
W = x_1(t) \dot{y}_1(t) - y_1(t) \dot{x}_1(t), \tag{A16}
\]

where \( W \) is actually the Wronskian determinant.

After using the classical equation of motion, one can obtain the asymptotic expression of the instanton solution \( \dot{r}_{cl}(t) \) when \( t \ll \tau_0 \) and \( t \gg \tau_0 \) (the detailed derivation can be seen in [29]). Then the asymptotic expression of \( x_1(t) \) is driven by the equation

\[
x_1(t) = \sqrt{\frac{m}{S_{cl}}} \dot{r}_{cl}(t), \tag{A17}
\]

where \( \sqrt{\frac{m}{S_{cl}}} \) is the normalized factor. The asymptotic expression of \( y_1(t) \) when \( t \ll \tau_0 \) and \( t \gg \tau_0 \) can be given based on the D’Alembert’s construction (15). Thus, the function \( \psi_{\lambda_0} \) satisfying the equation (A8), (A9) and...
can be given by the linear combination of $x_1(t)$ and $y_1(t)$. By transforming the equation (A8) into the integral equation and iterate once, one can obtain:

$$\psi_{\lambda_0}(t) = \psi(t) - \frac{2\lambda_0}{W} \int_{t_0}^{t} dt' [y_1(t)x_1(t') - x_1(t)y_1(t')] \psi_{\lambda_0}(t'), \tag{A18}$$

where $\psi_{\lambda_0}(t_0) = 0$.

One can take $t = t_1$ and use the equation (A10), the ratio of $\frac{\lambda_0}{\psi(t_1)}$ can be obtained:

$$\frac{\lambda_0}{\psi(t_1)} = \frac{W}{2} \left\{ \int_{t_0}^{t} dt' [y_1(t)x_1(t') - x_1(t)y_1(t')] \psi_{\lambda_0}(t') \right\}^{-1}. \tag{A19}$$

Based on the known asymptotic expression of $x_1(t)$ and $y_1(t)$, the $\frac{\lambda_0}{\psi(t_1)}$ can be calculated analytically. Taking into account all the above terms, one can calculate the corresponding $K(r_1, t_1; r_0, t_0)$ which only has one instanton. For the multi-instantons, the probability can be summed by the dilute gas approximation. After a heavy algebra, the results are finally given as the following:

$$P(r_1, t; r_s, 0) = \sqrt{\frac{\beta G''(r_s)}{2\pi}} \frac{1}{M_1 + M_2} [M_2 + M_1 e^{-(M_1 + M_2)t}], \tag{A20}$$

where $M_1$ and $M_2$ have a correction as:

$$M_1 \rightarrow \beta D \sqrt{(G''(r_m) G''(r_s))} + M_1, \tag{A22}$$

$$M_2 \rightarrow \beta D \sqrt{(G''(r_m) G''(r_1))} + M_2. \tag{A23}$$
World Scientific Press (2009).

[25] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press, Oxford (2021).

[26] J. Zinn-Justin, *Path Integrals in Quantum Mechanics*, Oxford University Press, Oxford (2005).

[27] A. Das, *Field Theory: A Path Integral Approach*, World Scientific Press (2006).

[28] S. Coleman, *Aspects of symmetry*, Cambridge University Press, Cambridge (1985).

[29] B. Caroli, C. Caroli and B. Roulet, *Diffusion in a Bistable Potential: The Functional Integral Approach*, J. Sat. Phys. 26 (1981) 83-111.

[30] J. W. York, *Black-hole thermodynamics and the Euclidean Einstein action*, Phys. Rev. D 33 (1986) 2092.

[31] R. André and J. P. S. Lemos, *Thermodynamics of five-dimensional Schwarzschild black holes in the canonical ensemble*, Phys. Rev. D 102 (2020) 024006 [hep-th/2006.10050].

[32] M. M. Caldarelli, G. Cognola and D. Klemm, *Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories*, Class. Quant. Grav. 17 (2000) 399-420 [hep-th/9908022].

[33] T. Padmanabhan, *Classical and quantum thermodynamics of horizons in spherically symmetric spacetimes*, Class. Quant. Grav. 19 (2002) 5387-5408 [gr-qc/0204019].

[34] D. Kastor, S. Ray and J. Traschen, *Enthalpy and the Mechanics of AdS Black Holes*, Class. Quant. Grav. 26 (2009) 195011 [hep-th/0904.2765].

[35] S. J. Yang, R. Zhou, S. W. Wei and Y. X. Liu, *Dynamics and kinetics of phase transition for Kerr AdS black hole on the free energy landscape*, arXiv: gr-qc/2105.00491.

[36] R. F. Dashen, B. Hasslacher and A. Neveu, *Nonperturbative methods and extended-hadron models in field theory. I. Semiclassical function methods*, Phys. Rev. D 10 (1974) 4114.

[37] S. Coleman, *Fate of the false vacuum: Semiclassical theory*, Phys. Rev. D 15 (1977) 2929-2936.

[38] C. G. Callan and S. Coleman, *Fate of the false vacuum. II. First quantum corrections*, Phys. Rev. D 16 (1977) 1762-1768.

[39] U. Weiss, *The uses of instantons for diffusion in bistable potentials*, Chaos and Order in Nature (1981).