Involution factorizations of Ewens random permutations

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Abstract

An involution is a bijection that is its own inverse. Given a permutation $\sigma$ of $[n]$, let $\text{invol} (\sigma)$ denote the number of ways $\sigma$ can be expressed as a composition of two involutions of $[n]$. We prove that the statistic $\text{invol}$ is asymptotically lognormal when the symmetric groups $\mathfrak{S}_n$ are each equipped with Ewens Sampling Formula probability measures of some fixed positive parameter $\theta$. This paper strengthens and generalizes previously determined results about the limiting distribution of $\log(\text{invol})$ for uniform random permutations, i.e. the specific case of $\theta = 1$. We also investigate the first two moments of $\text{invol}$ itself, detailing the phase transition in asymptotic behavior at $\theta = 1$, and provide a functional refinement and a demonstrably optimal convergence rate for the log-Gaussian limit law.

Keywords: asymptotic lognormality, Erdős-Turán law, Ewens Sampling Formula, Feller coupling, involutions, Mellin transform, saddle-point method.

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1 Introduction

An involution is a bijection that is its own inverse. The involutions in the symmetric group $\mathfrak{S}_n$ are precisely those permutations whose cycles are all of length 1 or 2. This paper is concerned with decomposing random permutations into two involutive factors and the probabilistic analysis thereof. Recent interest in involution factorizations in $\mathfrak{S}_n$ can be traced to Roberts and Vivaldi [30] who proposed that the dynamical behavior of reversible rational maps over a finite field can be modeled by compositions of two random involutions with a prescribed number of fixed points. Unrestricted products of two involutions, however, are of singular interest as well. For instance, Petersen and Tenner [28] outline hypothetical
links between enumerating involution factorizations, rim-hook tableaux, and the irreducible characters of $\mathcal{S}_n$ that are still in need of further clarification. Other applications have surfaced in comparative genomics [6], [11] and the analysis of algorithms [19], [35].

If $\sigma$ is a permutation of $[n]$, let $\text{invol}(\sigma)$ denote the number of function compositional factorizations of $\sigma$ into the product of two involutions of $[n]$, that is, the number of ordered pairs $(\tau_1, \tau_2)$ of involutions in $\mathcal{S}_n$ such that $\sigma = \tau_2 \circ \tau_1$. It is known that

$$\text{invol}(\sigma) = B(\sigma) \prod_{k=1}^{n} \sum_{j=0}^{\lfloor c_k/2 \rfloor} \frac{(c_k)_{2j}}{(2k)^{j}j!},$$

where $(x)_r$ is the falling factorial $x(x-1) \cdots (x-r+1)$, $c_k = c_k(\sigma)$ denotes the number of cycles of length $k$ (or, more succinctly, $k$-cycles) that $\sigma$ possesses, and $B(\sigma) = \prod k^{c_k}$ is the product of its cycle lengths. To the author’s knowledge, the earliest proofs of (1.1) are independently attributed to Lugo [24] and Petersen and Tenner [28]. Both of their derivations rely on identifying involutions with partial matchings and, consequently, the composition of two involutions with dichromatic edge-colored graphs in which every vertex is incident to exactly two edges, one edge per color. A derivation of (1.1) that merely exploits the canonical group action of $\mathcal{S}_n$ on $[n]$ appears in [10].

Even without (1.1), one notices that $\text{invol}(\sigma)$ is a class function by virtue of the conjugacy invariance of the set of involutions. It is therefore natural to wonder what the distribution of $\text{invol}$ is for permutations drawn from conjugation-invariant probability measures on $\mathcal{S}_n$ where cycles are multiplicatively weighted. The archetypal example of such a measure is the Ewens Sampling Formula with parameter $\theta > 0$, hereafter abbreviated ESF($\theta$). Read Crane’s survey article [12] for a lucid and concise synopsis of the literature on ESF($\theta$).

For each positive integer $n$, the Ewens probability measure $\mathbb{P}_{\theta,n}$ is a deformation of the uniform measure $\nu_n$ on $\mathcal{S}_n$ obtained by incorporating the following sampling bias:

$$\mathbb{P}_{\theta,n}(\sigma = \sigma_0) = \frac{n! \theta^K}{\theta(n)} \nu_n(\sigma = \sigma_0) = \frac{\theta^K}{\theta(n)},$$

where $K = \sum c_k$ is the total number of cycles that $\sigma$ has and $\theta(n)$ is the rising factorial $\theta(\theta+1) \cdots (\theta+n-1)$. (Note, in particular, that $\mathbb{P}_{1,n} = \nu_n$.) More specifically, the probability that a permutation of $[n]$ chosen according to ESF($\theta$) has cycle index $(a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ is

$$\mathbb{P}_{\theta,n}(c_k = a_k \text{ for } k = 1, \ldots, n) = 1 \left\{ \sum_{\ell=1}^{n} \ell a_\ell = n \right\} \frac{n!}{\theta(n)} \prod_{k=1}^{n} \left( \frac{\theta}{k} \right)^{a_k} \frac{1}{a_k!},$$

where $1\{\cdot\}$ denotes the indicator function. We shall also refer to random permutations chosen according to ESF($\theta$) as $\theta$-weighted permutations, just as Lugo does in [25].

A pivotal feature of ESF($\theta$) is that the joint distribution of the cycle counts $c_k$ converges to that of independent Poisson random variables of respective intensity $\theta/k$. Now consider a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of mutually independent random variables with $\alpha_k \sim \text{Poisson}(\theta/k)$,
for $k = 1, 2, \ldots$, and define
\[
\text{invol}_n(\alpha) = B_n(\alpha) \prod_{k=1}^n \sum_{j=0}^{\lfloor \alpha_k/2 \rfloor} \frac{(\alpha_k)_{2j}}{(2k)^j j!},
\]
where $B_n(\alpha) = \prod_{k=1}^n k^{\alpha_k}$. (1.4)

Note that
\[
\mathbb{E}|\log \text{invol}_n(\alpha) - \log B_n(\alpha)| = \sum_{k=1}^n \mathbb{E} \log \left( \sum_{j=0}^{\lfloor \alpha_k/2 \rfloor} \frac{(\alpha_k)_{2j}}{(2k)^j j!} \right)
\]
\[
\leq \sum_{k=1}^n \frac{1}{2k} \mathbb{E} \alpha_k^2 = O(1),
\]
whereas
\[
\text{Var}(\log B_n(\alpha)) = \sum_{k=1}^n \frac{\theta \log^2 k}{k} = \frac{\theta}{3} \log^3 n(1 + o(1)).
\]

Markov’s inequality thus dictates that
\[
\frac{|\log \text{invol}_n(\alpha) - \log B_n(\alpha)|}{\sqrt{\text{Var}(\log B_n(\alpha))}} \xrightarrow{p} 0.
\]

One would then suspect that something similar holds for $\log \text{invol}$ and $\log B$ in $\text{ESF}(\theta)$, assuming both are plausibly well-approximated by their Poissonized counterparts.

In this paper, we will establish both central limit and functional limit theorems for $\log(\text{invol})$ over Ewens random permutations. This will be accomplished by generalizing the key results of [10]. It was particularly shown in [10] that invol is asymptotically lognormal when each $\mathcal{S}_n$ is endowed with the uniform probability measure. The proof of this hinged on the observation that, for asymptotically almost all permutations $\sigma \in \mathcal{S}_n$, the statistic invol($\sigma$) is readily controlled by the simpler statistic $B(\sigma)$ together with the fact that $B$ is itself an asymptotically lognormal random variable over uniform random permutations. As we shall soon witness, these facts remain true in the setting of $\text{ESF}(\theta)$.

On the other hand, the distribution of invol itself, when left untouched by any rescaling or standardization, is heavily right-skewed. Practically all of its range is concentrated far below its mean. To demonstrate this, we will derive the leading-term asymptotics of the first and second moments of invol for general $\theta$-weighted permutations. The latter will highlight certain discrepancies in the variability of invol for $0 < \theta \leq 1$ versus $\theta > 1$.

Some classical results and techniques from asymptotic analysis are cited in a piecemeal fashion throughout this paper. As is common for random permutation and integer partition statistics, we will encounter generating functions that demand a careful examination by way of the saddle-point method and Mellin transforms. Technical preliminaries are briefly discussed in an appendix at the end.
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2 The Expected and Typical Values of invol

Let $\mathbb{P}_n$ denote the probability measure on $\mathfrak{S}_n$ corresponding to ESF($\theta$). For the sake of notational brevity, we will henceforth omit the $\theta$ in $\mathbb{P}_{\theta,n}$ and associated operators.

**Theorem 2.1.** The average number of involution factorizations for a permutation chosen according to the Ewens measure $\mathbb{P}_n$ is

$$
\mathbb{E}_{n,\text{invol}} = \frac{\theta^{\frac{1}{2}} \Gamma(\theta)}{2^{\frac{3}{2}} + e^{\frac{\theta}{2} \sqrt{\frac{\pi}{2}}}} e^{2\sqrt{\theta}} n^{\frac{\theta^2}{2} - \frac{1}{4}} \left(1 + O(n^{-\frac{1}{2}})\right).
$$

**Proof.** For each $\lambda \in \mathbb{Z}_{\geq 0}^n$, set $A_{\lambda} = \{\sigma \in \mathfrak{S}_n : \sigma \text{ has cycle type } 1^{\lambda_1} \cdots n^{\lambda_n}\}$. Consider the power series

$$
F(z) = 1 + \sum_{n=1}^{\infty} \left(\sum_{\lambda \vdash n} \mathbb{P}_n(\sigma \in A_{\lambda}) \mathbb{E}_n(\text{invol}(\sigma) \mid \sigma \in A_{\lambda})\right) \frac{\theta(n)}{n!} z^n.
$$

(2.1)

By the law of total expectation,

$$
F(z) = 1 + \sum_{n=1}^{\infty} \left(\sum_{\lambda \vdash n} \mathbb{P}_n(\sigma \in A_{\lambda}) \mathbb{E}_n(\text{invol}(\sigma) \mid \sigma \in A_{\lambda})\right) \frac{\theta(n)}{n!} z^n,
$$

(2.2)

where $\lambda \vdash n$ means that $\lambda \in \mathbb{Z}_{\geq 0}^n$ and $\sum k \lambda_k = n$. As pointed out in Theorem 2.5 of [10],

$$
\text{invol}(\sigma) = \prod_{k=1}^{n} k^{c_k(\sigma)} \frac{H_{c_k(\sigma)}(i \sqrt{k})}{(i \sqrt{k})^{c_k(\sigma)}},
$$

(2.3)

where $He_m$ is the probabilists’ Hermite polynomial $He_m(x) = m! \sum_{r=0}^{\lfloor m/2 \rfloor} (-1)^r \frac{x^{m-2r}}{r!(m-2r)!}$. Thus

$$
F(z) = 1 + \sum_{n=1}^{\infty} \left(\sum_{\lambda \vdash n} \left(\frac{n!}{\theta(n)} \prod_{k=1}^{n} \left(\frac{\theta}{k}\right)^{\lambda_k} \frac{1}{\lambda_k!}\right) \left(\prod_{k=1}^{n} k^{\lambda_k} \frac{H_{c_k(\sigma)}(i \sqrt{k})}{(i \sqrt{k})^{c_k(\sigma)}}\right)\right) \frac{\theta(n)}{n!} z^n
$$

(2.4)

$$
= \prod_{k=1}^{\infty} \left(\sum_{\lambda_k=0}^{\infty} \frac{H_{c_k(\sigma)}(i \sqrt{k})}{\lambda_k!} \frac{\theta(z^{k} \lambda_k)}{i \sqrt{k}}\right).
$$

(2.5)
Invoking the exponential generating function \( \exp(xt - t^2/2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \) (an assortment of derivations are available in [21], [22], [23]), we see that

\[
F(z) = \prod_{k=1}^{\infty} \exp\left(\theta z^k + \frac{\theta^2 z^{2k}}{2k}\right) = \frac{\exp(\theta z/(1 - z))}{(1 - z^2)^{\frac{\theta^2}{4}}}. \tag{2.6}
\]

Hence

\[
E_n\text{invol} = \frac{n!}{\theta(n)} [z^n] \exp(\theta z/(1 - z)) \frac{1}{(1 - z^2)^{\frac{\theta^2}{4}}}. \tag{2.7}
\]

Applying Corollary 6.2 with \( \beta = -\theta^2/2 \), \( \phi(z) = e^{-\theta(1 + z)} - \theta^2/2 \), and \( \alpha = \theta \), together with the Stirling’s approximations \( n! = \sqrt{2\pi n} e^{-n} n^n (1 + O(n^{-1})) \) and \( \theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} = \frac{\sqrt{2\pi e}}{\Gamma(\theta)} e^{-n^2/2} (1 + O(n^{-1})) \) completes the proof.

We can further adapt the function constructed in the proof of Theorem 2.1 to compute the average number of involution factorizations across permutations with a fixed number of cycles. Indeed, if we set

\[
F(u, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} E_{1,n}(1\{K(\sigma) = m\}\text{invol}(\sigma)) u^m z^n, \tag{2.8}
\]

then

\[
F(u, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{\lambda \vdash n} \left( \prod_{k=1}^{\lambda_k} \left( \frac{1}{k} \lambda_k \frac{1}{\lambda_k!} \right) \right) \left( \prod_{k=1}^{\lambda_k} \frac{He_{\lambda_k}(i\sqrt{k})}{(i\sqrt{k})^{\lambda_k}} \right) \left( \prod_{k=1}^{\lambda_k} u^{\lambda_k} \right) z^n \tag{2.9}
\]

\[
= 1 + \sum_{n=1}^{\infty} \sum_{\lambda \vdash n} \left( \prod_{k=1}^{\lambda_k} \left( \frac{u^k}{k} \lambda_k \frac{1}{\lambda_k!} \right) \right) \left( \prod_{k=1}^{\lambda_k} \frac{He_{\lambda_k}(i\sqrt{k})}{(i\sqrt{k})^{\lambda_k}} \right) z^n \tag{2.10}
\]

\[
= \frac{\exp(uz/(1 - z))}{(1 - z^2)^{\frac{\theta^2}{4}}} = \exp\left(\frac{uz}{1 - z} + \frac{1}{2}u^2 \log \frac{1}{1 - z^2}\right) \tag{2.11}
\]

is the bivariate ordinary generating function for the expected value of \( 1\{K(\sigma) = m\}\text{invol}(\sigma) \) over uniform random permutations, counted according to the size of the ground set (indicated by the variable \( z \)) and the total number of cycles (indicated by the variable \( u \)). The diagonal of \( F(u, z) \) is given by

\[
\lim_{z \to 0} F(x/z, z) = e^{x + \frac{x^2}{2}}, \tag{2.12}
\]

which is the exponential generating function for the class of involutions. This makes sense, as the only permutation of \([n]\) with \( n \) cycles is the identity permutation, which is precisely the square of every involution in \( \mathfrak{S}_n \). By expanding the exponential series in (2.11), we can
work towards retrieving the conditional expectation of \texttt{invol} for permutations with \( m \) cycles by using the “vertical” generating function

\[
[u^m]F(u, z) = \sum_{k=\lfloor m/2 \rfloor}^{m} \frac{(m-k)}{k!2^{m-k}} \left( \frac{z}{1-z} \right)^{2k-m} \log^{m-k} \left( \frac{1}{1-z^2} \right). \tag{2.13}
\]

For example, \([u]F(u, z) = z/(1-z)\), and so \([uz^n]F(u, z) = 1\) for \( n \geq 1 \). This too makes sense as the probability of picking an \( n \)-cycle is \( 1/n \), and there are exactly \( n \) ways to write an \( n \)-cycle as a product of two involutions.

**Theorem 2.2.** For each positive integer \( m \),

\[
\mathbb{E}_n(\text{invol}(\sigma) \mid K(\sigma) = m) = \frac{n^m}{m! \log^{m-1} n} \left( 1 + O \left( \frac{1}{\log n} \right) \right)
\]

as \( n \to \infty \).

**Proof.** Because \( K \) is a sufficient statistic for the underlying parameter \( \theta \), which means that the conditional distribution of \texttt{invol} does not depend on \( \theta \) (see section 2.2 of [12]), the conditional expectation is constant across the support of \( K \). Therefore

\[
\mathbb{E}_n(\text{invol}(\sigma) \mid K(\sigma) = m) = \frac{[u^m z^n]F(u, z)}{[n]_m/n!}, \tag{2.14}
\]

where \([n]_m\) are the unsigned Stirling numbers of the first kind. Based on the Flajolet-Odlyzko transfer theorem [17]

\[
[z^n](1-z)\alpha \log^k \left( \frac{1}{1-z} \right) = \frac{1}{\Gamma(-\alpha)} n^{-\alpha-1} \log^k n + O(n^{-\alpha-2} \log^k n) \quad \text{for } \alpha \notin \mathbb{Z}_{\geq 0}, \tag{2.15}
\]

we find that

\[
[u^m z^n]F(u, z) = \frac{1}{m!} [z^n] \left( \frac{z}{1-z} \right)^m + O(n^{m-2}) = \frac{n^{m-1}}{m!(m-1)!} + O(n^{m-2}) \tag{2.16}
\]

Evoke the following crude simplification of Wilf’s asymptotic [32]

\[
\frac{[n]_m}{n!} = \frac{\log^{m-1} n}{(m-1)! n} + O \left( \frac{\log^{m-2} n}{n} \right) \tag{2.17}
\]

to finish the calculation.

Most of the contribution to the mean comes from a small subset of permutations that admit an exceptionally large number of involution factorizations. Scrupulous glances at (1.1) should convince the reader that giving a permutation \( \sigma \) numerous occurrences of a distinct long cycle length or an exorbitant number of short cycles exaggerates the magnitude of
invol(σ). But it is rare for a permutation to have such a cycle profile. The typical permutation has much fewer involution factorizations. Direct combinatorial arguments can verify this for uniform random permutations, which have roughly \( e^{\sqrt{n}/\sqrt{8\pi n}} \) involution factorizations on average whereas the median is \( O(e^{\log n}) \) according to the following theorem and its corollary. (In fact, it is closer to \( e^{\frac{1}{2}\log^2 n} \), as shown in [10].)

**Theorem 2.3.** Let \( \xi_1 \) and \( \xi_2 \) be integers such that \( 20 \leq \xi_1, \xi_2 \leq n \) and \( \xi_1 \xi_2 < n \), where \( T_r \) is the \( r \)th triangular number. If \( H_n \) denotes the \( n \)th harmonic number and

\[
s_n^2 = 2 \sum_{j=1}^{n-1} \frac{H_j}{j+1} - H_n(H_n-1),
\]

then at least \( n!(2 - 1/\xi_1^2 - \exp(15/(\xi_1 + 1)! + 3/\xi_2)) \) permutations \( \sigma \in \mathcal{S}_n \) satisfy the inequality

\[
invol(\sigma) < (\xi_1!)^{\xi_1 n}[H_n + \xi_1 s_n] - \xi_2 e^{1/2 \xi_1 H_n}. \tag{2.20}
\]

*Proof.* Let \( \mathcal{P}_{\xi_1, \xi_2} \) be the set of all permutations \( \sigma \in \mathcal{S}_n \) for which \( c_k(\sigma) \leq \xi_1 \) for all \( k \leq \xi_2 \) and \( c_k(\sigma) \leq 1 \) for all \( k > \xi_2 \). Observe that the condition \( \xi_1 \xi_2 < n \) forces each permutation in \( \mathcal{P}_{\xi_1, \xi_2} \cap \mathcal{E}_1 \) to have at least one cycle longer than \( \xi_2 \). Further let \( \kappa(\sigma) \) denote the number of distinct cycle lengths present in the cycle decomposition of \( \sigma \), and let \( \mathcal{E}_{\xi_1} \) be the event that \( \kappa(\sigma) \leq H_n + \xi_1 s_n \).

It is folklore that \( H_n \) and \( s_n \) are the mean and standard deviation, respectively, of \( K \) over uniform random permutations of \([n]\). Erdős and Turán proved in Lemma III of [14] that

\[
\nu_{n}(\sigma \notin \mathcal{P}_{\xi_1, \xi_2} \cap \mathcal{E}_{\xi_1}) \leq e^{15 \xi_1^{-1} + 3/\xi_2} - 1. \tag{2.18}
\]

These together with the union bound and Chebyshev’s inequality tell us that

\[
\nu_{n}(\mathcal{P}_{\xi_1, \xi_2} \cap \mathcal{E}_{\xi_1}) \leq e^{15 \xi_1^{-1} + 3/\xi_2} - 1 + \nu_{n}(\kappa(\sigma) \geq H_n + \xi_1 s_n) \tag{2.19}
\]

\[
\leq e^{15 \xi_1^{-1} + 3/\xi_2} - 1 + \nu_{n}(|K(\sigma) - H_n| \geq \xi_1 s_n) \tag{2.20}
\]

\[
\leq e^{15 \xi_1^{-1} + 3/\xi_2} - 1 + \frac{1}{\xi_1^{\xi_1}}. \tag{2.21}
\]

Now let \( \sigma^{\text{max}} \) be a permutation for which invol attains its maximum value over \( \mathcal{P}_{\xi_1, \xi_2} \cap \mathcal{E}_{\xi_1} \). We claim that \( \sigma^{\text{max}} \) has the maximally obtainable amount of short cycles. For a cursory explanation as to why, note that if \( c_k(\sigma) \geq 1 \) and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) is a partition of \( k \) with \( \lambda_1 \neq 1 \), then any permutation obtained by splitting up each \( k \)-cycle of \( \sigma \) into \( \lambda_r \)-cycles for \( r = 1, 2, \ldots, k \) generally enlarges the number of involution factorizations. (Cf. the first two paragraphs in the proof of Theorem 16 in [31] which discuss how this strategy optimizes the product of the parts of a partition.)

So if we set \( \xi_3 = \lfloor H_n + \xi_1 s_n \rfloor \), then \( c_k(\sigma^{\text{max}}) = \xi_1 \) for \( 1 \leq k \leq \xi_2 \), and \( c_k(\sigma^{\text{max}}) = 1 \) for up to an additional \( \xi_3 - \xi_2 \) larger integers \( k \). Hence,

\[
invol(\sigma^{\text{max}}) = B(\sigma^{\text{max}}) \prod_{k=1}^{n} \sum_{j=0}^{\lfloor c_k(\sigma^{\text{max}})/2 \rfloor} \frac{(c_k(\sigma^{\text{max}}))_{2j}}{(2k)^{j} j!} \tag{2.22}
\]
\[
\leq \left( \prod_{k=1}^{\xi_2} k^{\xi_1} \right) \cdot \left( \prod_{k=\xi_2+1}^{\xi_3} n \right) \cdot \left( \prod_{k=1}^{\xi_2} \sum_{j=0}^{[\xi_2/2]} \frac{(\xi_1)_{2j}}{(2k)^j j!} \right) \tag{2.23}
\]
\[
\leq (\xi_2!)^{\xi_1} \cdot n^{\xi_3-\xi_2} \cdot \prod_{k=1}^{\xi_2} \exp \left( \frac{\xi_1^2}{2k} \right) = (\xi_2!)^{\xi_1} n^{\xi_3-\xi_2} e^{\frac{1}{2} \xi_1^2 H_{\xi_2}}. \tag{2.24}
\]

Setting \(\xi_1 = \xi_2\) in Theorem 2.3 and employing the Robbins-Stirling bound [29]
\[n! < \sqrt{2\pi n}^{e-\frac{1}{2}e^{-n}}\]
along with the elementary bounds \(H_n < \log n + 1\) and \(s_n < \sqrt{\log n}\) begets the following.

**Corollary 2.4.** If \(\xi = \xi(n) \to \infty\), but slowly enough so that \(\xi = o(\sqrt{\log n})\), then all but \(o(n!)\) permutations \(\sigma \in S_n\) satisfy the inequality
\[\text{invol}(\sigma) < e^{(1 + O(\xi \log^{-1/2} n))) \log^2 n}.\]

Goncharov’s limit law for \(K\) over ESF(\(\theta\)) (Theorem 3.5, [2]) and the upcoming Lemma 4.2 indicate that asymptotically almost all \(\theta\)-weighted permutations satisfy the above inequality, regardless of what \(\theta\) is. However, we can conclude something even stronger. Based on Theorem 4.4 if \(\sigma\) is a \(\theta\)-weighted permutation of \([n]\) and \(\epsilon > 0\), then w.h.p.,
\[e^{(\frac{\theta}{2} - \epsilon) \log n} < \text{invol}(\sigma) < e^{(\frac{\theta}{2} + \epsilon) \log^2 n}.\]

### 3 A Close Look at the Second Moment of invol

Ultimately, the asymptotic lognormality of invol for \(\theta\)-weighted permutations prevents, say, the method of moments from succeeding. (See, for instance, Example 30.2 on pg. 389 of Billingsley [7].) Ignoring that, we can still observe that the second moment of invol is quite obstreperous; its growth rate vastly outpaces the first moment, and the extent to which it does depends on whether \(\theta > 1\).

For a hint at this disparity, note that the Ewens Sampling Formula favors permutations with many cycles whenever \(\theta > 1\). This inflates the likelihood of drawing a permutation with a lot of short cycles, which is when invol(\(\sigma\)) is at its biggest. So \(E_n \text{invol}^2\) will grow faster than it does in the \(\theta < 1\) case and falls under an entirely different asymptotic regime.

The ensuing analysis is included here not only to contextualize the sheer futility of the method of moments in this situation but also to help cast light on the statistical spread of invol for \(\theta\)-weighted permutations and how \(\theta = 1\) acts as a “phase transition point” for the variance. Reprising the notation presented in the proof of Theorem 2.1 we consider the generating function
\[G(z) = 1 + \sum_{n=1}^{\infty} (E_n \text{invol}^2) \frac{\theta^{(n)}}{n!} z^n \tag{3.1}\]
\[
= 1 + \sum_{n=1}^{\infty} \left( \sum_{\lambda \vdash n} \mathbb{P}_{n}(\sigma \in \mathcal{A}_{\lambda}) \mathbb{E}_{n}(\text{invol}^{2} | \sigma \in \mathcal{A}_{\lambda}) \right) \frac{\theta(n)}{n!} z^n \tag{3.2}
\]

\[
= 1 + \sum_{n=1}^{\infty} \left( \sum_{\lambda \vdash n} \frac{n!}{\theta(n)} \prod_{k=1}^{\lambda} \left( \frac{\lambda}{k} \right) \frac{1}{\lambda_k!} \right) \left( \prod_{k=1}^{n} k^{2\lambda_k} \frac{H\epsilon_{\lambda_k}(i\sqrt{k})^2}{(i\sqrt{k})^{2\lambda_k}} \right) \frac{\theta(n)}{n!} z^n \tag{3.3}
\]

\[
= \prod_{k=1}^{\infty} \left( \sum_{\lambda_k=0}^{\infty} \frac{(H\epsilon_{\lambda_k}(i\sqrt{k})^2)^2}{\lambda_k!} \cdot (-\theta z^k)^{\lambda_k} \right) \tag{3.4}
\]

We now recall the probabilists’ version of the Mehler kernel (see Kibble [20])

\[
\sum_{n=0}^{\infty} H\epsilon_{n}(x) H\epsilon_{n}(y) \frac{t^n}{n!} = \frac{1}{\sqrt{1-t^2}} \exp \left( -\frac{t^2(x^2 + y^2) - 2txy}{2(1-t^2)} \right)
\]

to observe that

\[
G(z) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{1-\theta^2 z^2k^2}} \exp \left( \frac{\theta k z^k}{1-\theta z^k} \right). \tag{3.5}
\]

If \(\theta > 1\), then \(G\) has a dominant singularity at \(z = 1/\theta\) and the function \(\phi\) given by

\[
\phi(z) = \frac{\exp(-1)}{\sqrt{1+\theta z}} \prod_{k=2}^{\infty} \frac{1}{\sqrt{1-\theta^2 z^2k^2}} \exp \left( \frac{\theta k z^k}{1-\theta z^k} \right) \tag{3.6}
\]
is regular in the disc \(|z| < 1/\theta\). Corollary [6.2] can thus be applied to get

\[
[z^n]G(z) = [z^n](1-\theta z)^{-\frac{1}{2}} \phi(z) \exp \left( \frac{1}{1-\theta z} \right) = \frac{\theta^n}{n^{\frac{1}{2}}} K_{\theta} \exp \left( \frac{2\sqrt{n} - 1/2}{2\sqrt{2\pi}} \right) \left( 1 + O(n^{-\frac{1}{2}}) \right) \tag{3.7}
\]

where

\[
K_{\theta} = \prod_{k=2}^{\infty} \frac{1}{\sqrt{1-\theta^2 z^2k^2}} \exp \left( \frac{k}{\theta^{k-1} - 1} \right). \tag{3.8}
\]

Hence

\[
\mathbb{E}_n(\text{invol}^{2}) = \frac{n!}{\theta(n)} [z^n]G(z) = \frac{K_{\theta} \Gamma(\theta)}{2\sqrt{2\pi}} \theta^n e^{2\sqrt{n} - \frac{1}{2} n^{\frac{1}{2}} - \theta} \left( 1 + O(n^{-\frac{1}{2}}) \right), \tag{3.9}
\]

from which it follows that \(\mathbb{E}_n(\text{invol}^{2}) = \omega(\mathbb{E}_n(\text{invol})). \) However, if \(0 < \theta \leq 1\), then \(G\) has a radius of convergence of 1, but behaves too chaotically around the unit circle to be analytically continued beyond it. In this case, we appeal to the exp-log reorganization of (3.5):

\[
G(z) = \exp \left( \sum_{k=1}^{\infty} \left[ -\frac{1}{2} \log(1-\theta^2 z^2k^2) + \frac{\theta k z^k}{1-\theta z^k} \right] \right) \tag{3.10}
\]
We therefore consider the cases $0 < \theta < \infty$ to transition from (3.15) to (3.16). (Apply the arithmetic mean-harmonic mean inequality on the terms of the second summation).

From this, we can attain the following upper bound on $|\log G(z)|$ for when $0 < |z| < 1$:

$$|\log G(z)| \leq \frac{\theta |z|}{1 - |z|^2} + \sum_{\ell=1}^{\infty} \frac{\theta^{2\ell}}{2\ell} \frac{|z|^{2\ell}}{1 - |z|^{2\ell}} + \sum_{\ell=2}^{\infty} \frac{\theta^\ell |z|^\ell}{1 - |z|^\ell}$$

$$< \frac{\theta}{1 - |z|^2} + \sum_{\ell=1}^{\infty} \frac{\theta^{2\ell}}{(2\ell)^2 (1 - |z|^{2\ell})} + \sum_{\ell=2}^{\infty} \frac{\theta^\ell |z|^\ell}{(1 - |z|^\ell)^2}$$

$$< \frac{\theta}{1 - |z|^2} + \frac{1}{1 - |z|^2} \sum_{\ell=1}^{\infty} \frac{\theta^{2\ell}}{(2\ell)^2} (|z|^{-1} + |z|^{-2} + \cdots + |z|^{-2\ell}) + \frac{1}{1 - |z|^2} \sum_{\ell=2}^{\infty} \frac{\theta^\ell}{1 - |z|^\ell} (|z|^{-1} + |z|^{-2} + \cdots + |z|^{-(\ell-1)})$$

$$< \frac{\theta}{1 - |z|^2} + \frac{1}{1 - |z|^2} \sum_{\ell=1}^{\infty} \frac{\theta^{2\ell}}{4\ell^2} + \frac{1}{(1 - |z|^2)^2} \sum_{\ell=2}^{\infty} \frac{\theta^\ell}{\ell^2}$$

$$= \frac{\theta}{1 - |z|^2} + \frac{\text{Li}_2(\theta^2)}{4(1 - |z|^2)} + \frac{\text{Li}_2(\theta) - \theta}{(1 - |z|^2)^2}.$$  (3.16)  

(Apply the arithmetic mean-harmonic mean inequality on the terms of the second summation to transition from (3.15) to (3.16).)

Now in setting $z = e^{-u}$, the function

$$L(u) = \log G(e^{-u}) = \sum_{\ell=1}^{\infty} \left[ \frac{\theta^{2\ell} e^{-2\ell u}}{2\ell (1 - e^{-2\ell u})} + \frac{\theta^\ell e^{-\ell u}}{(1 - e^{-\ell u})^2} \right]$$

is a harmonic sum well-suited for a Mellin transform analysis. Consult section 6.2 of the appendix to deduce that the Mellin transform of $L(u)$ is

$$L^*(s) = (2^{-s-1}\text{Li}_{s+1}(\theta^2)\zeta(s) + \text{Li}_s(\theta)\zeta(s-1))\Gamma(s),$$

where $\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ is the polylogarithm function of order $s$.

Note that $\text{Li}_s(z)$, when treated as a function of $s$ and with $z$ held constant, is analytic for $z \in [0, 1)$, because $\text{Li}_s(z)$ converges absolutely for all $s \in \mathbb{C}$ when $|z| < 1$. If $z = 1$, then the polylogarithm is simply the Riemann zeta function which has a simple pole at $s = 1$.

We therefore consider the cases $0 < \theta < 1$ and $\theta = 1$ separately.
**Case 1:** \( \theta \in (0,1) \). The Mellin transform \( L^*(s) \) has simple poles at \( s = 2, 1, 0, -1, \ldots \). Using the standard transfer rule for the transform, we ascertain that

\[
L(t) = \lim_{t \to 0^+} \frac{\text{Li}_2(\theta)}{t^2} + \frac{\text{Li}_2(\theta^2)}{4t} + \frac{1}{4} \log(1 - \theta^2) - \frac{\theta}{12(1 - \theta)} + O(t) \tag{3.18}
\]

which, because of the Laurent series expansions of the powers of \( t = -\log(z) \) at \( z = 1 \) and the fact that (3.18) extends to \( \mathbb{C} - \mathbb{R}_{<0} \), corresponds to

\[
\log G(z) = \lim_{z \to 1} \frac{\text{Li}_2(\theta)}{(1 - z)^2} + \frac{\text{Li}_2(\theta^2) - 4\text{Li}_2(\theta)}{4(1 - z)} + \frac{1}{12} \text{Li}_2(\theta) - \frac{1}{8} \text{Li}_2(\theta^2) + \frac{1}{4} \log(1 - \theta^2) - \frac{\theta}{12(1 - \theta)} + O(1 - z) \tag{3.19}
\]

from within the unit disc. We can now conclude that

\[
G(z) = \tau(z)[1 + O(1 - z)], \tag{3.20}
\]

provided \(|z| < 1\) and

\[
\tau(z) = (1 - \theta^2)^{1/4} \exp \left( \frac{\text{Li}_2(\theta)}{(1 - z)^2} + \frac{\text{Li}_2(\theta^2) - 4\text{Li}_2(\theta)}{4(1 - z)} + \frac{1}{12} \text{Li}_2(\theta) - \frac{1}{8} \text{Li}_2(\theta^2) - \frac{\theta}{12(1 - \theta)} \right) \tag{3.21}
\]

We claim next that \([z^n]G(z) = [z^n]\tau(z)(1 + o(1))\). To see this, consider \( \gamma \), the positively oriented circle \(|z| = 1 - (2\text{Li}_2(\theta)/n)^{1/3}\), and split \( \gamma \) into the two contours

\[
\gamma_1 = \left\{ z \in \gamma : |1 - z| < \left( \frac{3\text{Li}_2(\theta)}{n} \right)^{\frac{1}{3}} \right\}
\]

and \( \gamma_2 = \gamma - \gamma_1 \). By Cauchy's integral formula,

\[
[z^n](G(z) - \tau(z)) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{G(z) - \tau(z)}{z^{n+1}} \, dz + \frac{1}{2\pi i} \int_{\gamma_2} \frac{G(z) - \tau(z)}{z^{n+1}} \, dz \tag{3.22}
\]

The length of \( \gamma_1 \) is \( O(n^{-1/3}) \). This together with (3.20) shows that the first integral is

\[
= O \left( \int_{\gamma_1} |z|^{-n} |1 - z| \exp \left( \frac{\text{Li}_2(\theta)}{(1 - |z|)^2} + \frac{\text{Li}_2(\theta^2) - 4\text{Li}_2(\theta)}{4(1 - |z|)} \right) \, |dz| \right) \tag{3.23}
\]

\[
= O \left( \exp \left( \frac{2\text{Li}_2(\theta)n^{\frac{2}{3}}}{(2\text{Li}_2(\theta))^{\frac{2}{3}}} \right) \cdot n^{-\frac{1}{3}} \cdot \exp \left( \frac{\text{Li}_2(\theta)n^{\frac{2}{3}}}{(2\text{Li}_2(\theta))^{\frac{2}{3}}} + \frac{(\text{Li}_2(\theta^2) - 4\text{Li}_2(\theta))n^{\frac{1}{3}}}{4(2\text{Li}_2(\theta))^{\frac{2}{3}}} \right) \right) \cdot O(n^{-\frac{1}{3}}) \tag{3.24}
\]

\[
= O \left( n^{-\frac{2}{3}} \exp \left( \frac{3\text{Li}_2(\theta)n^{\frac{2}{3}}}{(2\text{Li}_2(\theta))^{\frac{2}{3}}} + \frac{(\text{Li}_2(\theta^2) - 4\text{Li}_2(\theta))n^{\frac{1}{3}}}{4(2\text{Li}_2(\theta))^{\frac{2}{3}}} \right) \right) \tag{3.25}
\]

\[\text{(3.25)}\]
On the other hand, estimate (3.17) applies over $\gamma_2$, and so the second integral in (3.22) is

$$
= O\left(\int_{\gamma_2} |z|^{-n} \left[ \exp \left( \frac{\theta}{1 - |z|^2} + \frac{\text{Li}_2(\theta^2)}{4(1 - |z|)} + \text{Li}_2(\theta) - \theta \right) + \exp \left( \frac{\text{Li}_2(\theta)}{1 - |z|^2} + \frac{\text{Li}_2(\theta^2) - 4\text{Li}_2(\theta)}{4|1 - z|} \right) \right] |dz| \right)
$$

$$
= O\left( \exp \left( \frac{2\text{Li}_2(\theta)n^{2/3}}{(2\text{Li}_2(\theta))^{4/3}} \right) \cdot \exp \left( \frac{\theta n^{2/3}}{(3\text{Li}_2(\theta))^{4/3}} + \frac{\text{Li}_2(\theta^2)n^{4/3}}{4(2\text{Li}_2(\theta))^{7/3}} + \frac{(\text{Li}_2(\theta) - \theta)n^{4/3}}{4(2\text{Li}_2(\theta))^{7/3}} \right) \right)
$$

$$
+ \exp \left( \frac{2\text{Li}_2(\theta)n^{2/3}}{(2\text{Li}_2(\theta))^{4/3}} \right) \cdot \exp \left( \frac{\text{Li}_2(\theta)n^{2/3}}{(3\text{Li}_2(\theta))^{4/3}} + \frac{(\text{Li}_2(\theta) - 4\text{Li}_2(\theta))n^{4/3}}{4(3\text{Li}_2(\theta))^{7/3}} \right)
$$

$$
= O\left( \exp \left( An^{2/3} \right) \right),
$$

for some $A < 3\text{Li}_2(\theta)/(2\text{Li}_2(\theta))^{2/3}$. Therefore

$$
[z^n]G(z) = [z^n]\tau(z) + O\left( n^{-2/3} \exp \left( \frac{3\text{Li}_2(\theta)n^{2/3}}{(2\text{Li}_2(\theta))^{4/3}} + \frac{(\text{Li}_2(\theta^2) - 4\text{Li}_2(\theta))n^{4/3}}{4(2\text{Li}_2(\theta))^{7/3}} \right) \right)
$$

(3.29)

It is now a sophomore’s homework exercise to use Theorem 6.3 on $\tau(z)$, especially considering that the required singular expansion is already accounted for in (3.18). The error term in (3.29) is smaller than the leading term asymptotic of $[z^n]\tau(z)$ by a decaying factor of $O(\exp(-\text{Li}_2(\theta)n^{1/3}/(2\text{Li}_2(\theta))^{1/3}))$. In the end, we find that

$$
[z^n]G(z) \sim \frac{(1 - \theta^2)^{1/4}(2\text{Li}_2(\theta))^{1/2}}{\sqrt{6\pi n^{3/4}}} \exp \left( \frac{3\text{Li}_2(\theta)n^{2/3}}{(2\text{Li}_2(\theta))^{4/3}} + \frac{\text{Li}_2(\theta^2)n^{4/3}}{(2\text{Li}_2(\theta))^{7/3}} \right) \frac{\theta}{12(1 - \theta)}.
$$

(3.30)

and so

$$
E_n\text{invol}^2 \sim \frac{(1 - \theta^2)^{1/4}(2\text{Li}_2(\theta))^{1/2}\Gamma(\theta)n^{3/4 - \theta}}{\sqrt{6\pi}} \exp \left( \frac{3\text{Li}_2(\theta)n^{2/3}}{(2\text{Li}_2(\theta))^{4/3}} + \frac{\text{Li}_2(\theta^2)n^{4/3}}{(2\text{Li}_2(\theta))^{7/3}} - \frac{\theta}{12(1 - \theta)} \right).
$$

(3.31)

**Case 2:** $\theta = 1$. Then $G(z)$ is precisely the generating function of $E_n\text{invol}^2$, and the Mellin transform becomes

$$
L^*(s) = (2^{-s-1}\zeta(s + 1) + \zeta(s - 1))\zeta(s)\Gamma(s),
$$

(3.32)

which has simple poles at $s = 2, 1, 0, -1, \ldots$, and a double pole at $s = 0$. Hence

$$
L(t) = \pi^2 \frac{\zeta(2)}{6t^2} + \frac{\pi^2 - 12}{24t} + \frac{1}{4} \log t - \log \sqrt{2\pi} + \frac{1}{24} + O(t),
$$

(3.33)

which corresponds to

$$
\log G(z) \sim \frac{\pi^2}{6(1 - z)^2} - \frac{\pi^2 - 4}{8(1 - z)} + \frac{1}{4} \log(1 - z) + \frac{7}{24} - \frac{\pi^2}{144} - \log \sqrt{2\pi} + O(1 - z)
$$

(3.34)
from within the unit disc. Therefore
\[ G(z) = \tau(z)[1 + O(1 - z)], \tag{3.35} \]
provided \(|z| < 1\) and
\[ \tau(z) = (2\pi(1 - z))^{\frac{1}{4}} \exp\left(\frac{\pi^2}{6(1 - z)^2} - \frac{\pi^2 - 4}{8(1 - z)} + \frac{7}{24} - \frac{\pi^2}{144}\right). \tag{3.36} \]

At this point, the outstanding justifications are practically the same as they were for Case 1. It is tedious, but requires nothing new from earlier to figure out that
\[ [z^n]G(z) \sim \frac{1}{\sqrt{2} \sqrt[7]{\pi(3n)^{\frac{7}{12}}} \exp\left(\frac{1}{2}(3\pi n)^{\frac{1}{2}} - \frac{\pi^2 - 4}{8} \left(\frac{3n}{\pi^2}\right)^{\frac{1}{4}} + \frac{7}{24} - \frac{\pi^2}{144}\right)}. \tag{3.37} \]

4 A Central Limit Theorem for \(\log\text{invol}\)

In this section, we emulate the proof of Theorem 4.3 in [10], except now each \(\mathcal{G}_n\) is equipped with the ESF(\(\theta\)) probability measure \(\mathbb{P}_n\) with \(\theta > 0\) fixed. We proceed in three steps. First, \(\text{invol}(\sigma)\) is well-approximated by \(B(\sigma)\) for permutations \(\sigma\) in which only \(k\)-cycles with \(k\) under a certain threshold \(\xi\), to be stipulated \(a\ posteriori\), can occur more than once but still not more than \(\xi\) times. Call the set of all such permutations \(\mathcal{P}_\xi\). The second step is to show that \(\mathbb{P}_n(\sigma \in \mathcal{P}_\xi) \to 1\) as \(n \to \infty\). Lastly, the previous two steps are combined to compare the distribution of \(\log\text{invol}\) to the distribution of \(\log B\).

There are two notable improvements here over the presentation of [9] and [10]. Our present proof of the aforementioned second step is cleaner and considerably streamlined. Furthermore, a concrete error term for the approximate Gaussian law will be added in the next section. Perhaps unsurprisingly, the error term matches the convergence rate of the Erdős-Turán limit law for the order of a random permutation that was first determined in [5].

**Lemma 4.1.** (Burnette, Schmutz) Suppose \(\sigma \in \mathcal{P}_\xi\). Then there is a constant \(c > 0\), not dependent on \(\sigma\) nor \(\xi\), such that \(B(\sigma) \leq \text{invol}(\sigma) \leq B(\sigma) \cdot (c \xi)^{\xi}\).

**Lemma 4.1** is entirely deterministic and thus needs not be revisited. The next lemma generalizes Lemma III in [14].

**Lemma 4.2.** If \(\xi = \xi(n) \to \infty\), then \(\mathbb{P}_n(\sigma \not\in \mathcal{P}_\xi) = O(\xi^\xi)\).

**Proof.** Due to (1.3) and the exponential formula for labeled combinatorial structures,
\[ \mathbb{P}_n(\sigma \in \mathcal{P}_\xi) = \frac{n!}{\theta(n)[z^n]} \left[ \prod_{k=1}^{\lfloor \xi \rfloor} \sum_{j=1}^{\lfloor \xi \rfloor} \left( \frac{\theta z^k}{k} \right)^{\frac{j}{j!}} \right] \cdot \left( \prod_{k=\lfloor \xi \rfloor + 1}^{\infty} \left( 1 + \frac{\theta z^k}{k} \right) \right). \tag{4.1} \]
The product in (4.1) can be written as \((1 - z)^{-\theta} \Omega(z)\), where

\[
\Omega(z) = \left( \prod_{k=1}^{\lfloor \xi \rfloor} \left( 1 - e^{-\frac{\theta}{k}} \sum_{j=\lfloor \xi \rfloor + 1}^{\infty} \left( \frac{\theta}{k} \right)^j \frac{1}{j!} \right) \right) \cdot \left( \prod_{k=\lfloor \xi \rfloor + 1}^{\infty} \left( 1 - e^{-\frac{\theta}{k}} \sum_{j=2}^{\infty} \left( \frac{\theta}{k} \right)^j \frac{1}{j!} \right) \right).
\] (4.2)

Equating the corresponding coefficients reveals that

\[
\mathbb{P}_n(\sigma \in \mathcal{P}_\xi) - 1 = \frac{n!}{\theta(n)} \sum_{\ell=1}^{n} \left( \theta + n - \ell - 1 \right) [z^\ell] \Omega(z) \leq \sum_{\ell=1}^{n} [z^\ell] \Omega(z). \tag{4.3}
\]

If we replace every instance of the term \(-e^{-\frac{\theta}{k}}\) by \(e^{\frac{\theta}{k}}\) in each factor of \(\Omega(z)\), we obtain an entire function \(\Omega^*(z)\) whose coefficients are all positive and majorize the absolute values of the corresponding coefficients of \(\Omega(z)\). Hence

\[
\mathbb{P}_n(\sigma \notin \mathcal{P}_\xi) = 1 - \mathbb{P}_n(\sigma \in \mathcal{P}_\xi) \tag{4.4}
\]

\[
\leq \sum_{\ell=1}^{n} [z^\ell] \Omega^*(z) \tag{4.5}
\]

\[
< \Omega^*(1) - \Omega^*(0) \tag{4.6}
\]

\[
= \left( \prod_{k=1}^{\lfloor \xi \rfloor} \left( 1 + e^{\frac{\theta}{k}} \sum_{j=\lfloor \xi \rfloor + 1}^{\infty} \left( \frac{\theta}{k} \right)^j \frac{1}{j!} \right) \right) \cdot \left( \prod_{k=\lfloor \xi \rfloor + 1}^{\infty} \left( 1 + e^{\frac{\theta}{k}} \sum_{j=2}^{\infty} \left( \frac{\theta}{k} \right)^j \frac{1}{j!} \right) \right) - 1. \tag{4.7}
\]

For sufficiently large \(n\), we have

\[
\prod_{k=1}^{\lfloor \xi \rfloor} \left( 1 + e^{\frac{\theta}{k}} \sum_{j=\lfloor \xi \rfloor + 1}^{\infty} \left( \frac{\theta}{k} \right)^j \frac{1}{j!} \right) < \exp \left( \sum_{k=1}^{\lfloor \xi \rfloor} e^{\frac{\theta}{k}} \sum_{j=\lfloor \xi \rfloor + 1}^{\infty} \left( \frac{\theta}{k} \right)^j \frac{1}{j!} \right) \tag{4.8}
\]

\[
< \exp \left( e^{2\theta \lfloor \xi \rfloor + 1} \sum_{k=1}^{\lfloor \xi \rfloor + 1} \frac{\lfloor \xi \rfloor + 1}{k(\lfloor \xi \rfloor + 1)} \right) \tag{4.9}
\]

\[
< \exp \left( K_1 \frac{\theta \lfloor \xi \rfloor + 1}{(\lfloor \xi \rfloor + 1)!} \right) \tag{4.10}
\]

for some constant \(K_1\), and

\[
\prod_{k=\lfloor \xi \rfloor + 1}^{\infty} \left( 1 + e^{\frac{\theta}{k}} \sum_{j=2}^{\infty} \left( \frac{\theta}{k} \right)^j \frac{1}{j!} \right) < \exp \left( \sum_{k=\lfloor \xi \rfloor}^{\infty} e^{\frac{\theta}{k}} \sum_{j=2}^{\infty} \left( \frac{\theta}{k} \right)^j \frac{1}{j!} \right) \tag{4.11}
\]

\[
< \exp \left( \frac{e^{\theta^2} + 2}{2} \sum_{k=\lfloor \xi \rfloor}^{\infty} \frac{1}{k^2} \right) \tag{4.12}
\]
for some constant $K_2$. Therefore, as $n \to \infty$,

$$\mathbb{P}_n(\sigma \notin \mathcal{P}_\xi) < \exp\left(\frac{K_1\theta[\xi]+1}{(|\xi|+1)!} + \frac{K_2}{\xi}\right) - 1 = O\left(\frac{1}{\xi}\right).$$

(4.14)

We now cite the asymptotic lognormality of $B$ for $\theta$-weighted permutations.

**Lemma 4.3.** (Arratia, Barbour, Tavaré)

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}_n\left(\frac{\log B(\sigma) - \mu_n}{\sigma_n} \leq x\right) - \Phi(x) \right| = O\left(\frac{1}{\sqrt{\log n}}\right),$$

where $\mu_n := \sum_{k=1}^{n} \frac{\theta \log k}{k} \sim \frac{\theta}{2} \log^2 n$, $\sigma_n^2 := \sum_{k=1}^{n} \frac{\theta \log^2 k}{k} \sim \frac{\theta}{3} \log^3 n$, and $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$.

Erdős and Turán [15] were the first to confirm the asymptotic lognormality of $B$ for uniform random permutations. Progressively sharper versions of the Erdős-Turán limit law for $B$ have emerged on multiple occasions, e.g. [8], [13], [26], [27], [36]. Proofs of Lemma 4.3 itself appear in the aggregate works of Arratia, Barbour, and Tavář [2], [3], [4], [5].

**Theorem 4.4.** For all real $x$,

$$\lim_{n \to \infty} \mathbb{P}_n\left(\frac{\log \text{invol}(\sigma) - \mu_n}{\sigma_n} \leq x\right) = \Phi(x)$$

Proof. Because $\log \text{invol}(\sigma) \geq B(\sigma)$ for all $\sigma \in \mathfrak{S}_n$, an upper bound is immediate:

$$\mathbb{P}_n\left(\frac{\log \text{invol}(\sigma) - \mu_n}{\sigma_n} \leq x\right) \leq \mathbb{P}_n\left(\frac{\log B(\sigma) - \mu_n}{\sigma_n} \leq x\right) = \Phi(x) + o(1).$$

(4.15)

For a lower bound, we capitalize on the uniform continuity of $\Phi$. We also invoke the conditional bound from Lemma 4.1 where $\xi$ will now be chosen to our liking. Due to Lemma 4.2 this bound holds with probability $1 - O(1/\xi)$.

The remainder of the proof is virtually identical to that of Theorem 4.3 in [10], which is included here for the sake of completeness. Let $\epsilon > 0$ be a fixed but arbitrarily small positive number. We can choose $\delta > 0$ so that $|\Phi(x) - \Phi(a)| < \epsilon$ whenever $|x - a| < \delta$. If we choose $\xi = \sqrt{\log n}$, then we have $\log((c\xi^2)^{\frac{\epsilon}{2}}) = o(\sigma_n)$. Therefore we can choose $N_\epsilon$ so that, for all $n \geq N_\epsilon$, we have $\log((c\xi^2)^{\frac{\epsilon}{2}}) < \frac{\sigma_n}{2}$. It follows that, uniformly in $x$,

$$\mathbb{P}_n\left(\frac{\log \text{invol}(\sigma) - \mu_n}{\sigma_n} \leq x\right) \geq \mathbb{P}_n\left(\frac{\log B(\sigma) + \log((c\xi^2)^{\frac{\epsilon}{2}}) - \mu_n}{\sigma_n} \leq x\right) + o(1)$$

(4.16)
\[ \geq \mathbb{P}_n \left( \frac{\log B(\sigma) + \delta \sigma_n/2 - \mu_n}{\sigma_n} \leq x \right) + o(1) \quad (4.17) \]
\[ = \mathbb{P}_n \left( \frac{\log B(\sigma) - \mu_n}{\sigma_n} \leq x - \frac{\delta}{2} \right) + o(1) \quad (4.18) \]
\[ = \Phi \left( x - \frac{\delta}{2} \right) + o(1) > \Phi(x) - \epsilon + o(1). \quad (4.19) \]

Yet \( \epsilon > 0 \) was arbitrary, and so \( \mathbb{P}_n \left( \frac{\log \text{invol}(\sigma) - \mu_n}{\sigma_n} \leq x \right) \geq \Phi(x) + o(1). \]

5 A Functional Refinement and Convergence Rate

A stochastic process variant of Theorem 4.4 can also be formulated. Let \( C^{(n)} = (c_1^{(n)}, c_2^{(n)}, \ldots) \) be distributed according to \( \text{ESF}(\theta) \) and \( W_n \) be the random element of the Skorokhod space \( \mathcal{D}[0,1] \) of càdlàg functions on \([0,1]\) defined by

\[ W_n(t) = \frac{\log \text{invol}[n] \left( C^{(n)} \right) - \frac{\theta n^2}{2} \log^2 n}{\sqrt{\frac{2}{3} \log^3 n}}. \quad (5.1) \]

**Theorem 5.1.** It is possible to construct \( C^{(n)} \) and a standard Brownian motion \( W \) on the same probability space in such a way that

\[ \mathbb{E} \left\{ \sup_{0 \leq t \leq 1} \left| W_n(t) - W(t^3) \right| \right\} = O \left( \frac{\log \log n}{\sqrt{\log n}} \right). \]

The Feller coupling, which expresses the Ewens Sampling Formula in terms of the spacings between successes in a sequence of independent Bernoulli trials, offers the ideal terrain for this task. Arratia, Barbour, and Tavaré [1] recently wrote a brief expository note about the Feller coupling to act as a companion to [12]. To summarize, consider a sequence \( \beta = (\beta_1, \beta_2, \ldots) \) of independent Bernoulli random variables in which \( \Pr(\beta_j = 1) = \theta/(\theta + j - 1), j = 1,2,\ldots \)

Say that a \( k \)-spacing occurs in a sequence of zeros and ones, starting at position \( \ell - k \) and ending at position \( \ell \), if \( \beta_{\ell - k} = \beta_\ell = 1 \) and \( \beta_j = 0 \) for \( \ell - k + 1 \leq j \leq \ell - 1 \). If for all positive integers \( k \) and \( n \) we define

\[ c_k^{(n)} = \text{#}k\text{-spacings in } 1\beta_2\beta_3 \cdots \beta_n 1, \quad (5.2) \]

it can be shown that the distribution of \( C^{(n)} = (c_1^{(n)}, c_2^{(n)}, \ldots) \) is exactly the same as \( \text{ESF}(\theta) \). If we further define \( Z_m = (Z_{1,m}, Z_{2,m}, \ldots) \), where

\[ Z_{k,m} = \text{#}k\text{-spacings in } \beta_{m+1}\beta_{m+2} \cdots, \quad k, m \geq 1, \quad (5.3) \]

then \( Z_m \xrightarrow{a.s.} 0 \) as \( m \to \infty \) due to the Borel-Cantelli lemma, and so \( Z_0 \) is equivalent to the Poisson point process to which \( C^{(n)} \) itself converges.
We now assemble the pieces needed to prove Theorem 5.1. Our plan is to show that within the Feller coupling, \( \log \text{invol}_{n}(C^{(n)}) \) is “close enough” to \( \log \text{invol}_{n}(Z_0) \), which in turn is “close enough” to \( \log B_n(Z_0) \), which is known to conform to a Wiener process. To this end, we raise two lemmas. The first is a trivial consequence of the preceding definitions.

**Lemma 5.2.** For all positive integers \( k \) and \( n \),

\[
Z_{k,0} - Z_{k,n} - 1{\{L_n + R_n = k + 1\}} \leq c_{k}^{(n)} \leq Z_{k,0} + 1{\{L_n = k\}},
\]

where \( L_j = \min\{j \geq 1 : \beta_{n-j+1} = 1\} \) and \( R_j = \min\{j \geq 1 : \beta_{n+j} = 1\} \).

**Lemma 5.3.** If \( a \) and \( b \) are two vectors such that \( 0 \leq a_k \leq b_k \) for \( k \in [n] \), then

\[
\text{invol}_j(b) \leq \text{invol}_j(a) n^{\|b-a\|_1} \prod_{k=1}^{n} \exp\left(\frac{b_k^2 - a_k^2}{2k}\right),
\]

for all \( j \in [n] \), where \( \|b-a\|_1 \) denotes the usual 1-norm.

**Proof.** Observe that

\[
\text{invol}_j(b) = \text{invol}_j(a) \left(\prod_{k=1}^{j} b_k - a_k\right) \cdot \left(\prod_{k=1}^{j} \frac{V_{b_k}(k)}{V_{a_k}(k)}\right),
\]

where we have set

\[
V_m(k) = \sum_{\ell=0}^{\lfloor m/2 \rfloor} \frac{(m)_{2\ell}}{(2k)^\ell \ell!}.
\]

Note that \( 1 \leq V_m(k) \leq \exp\left(\frac{m^2}{2k}\right) \) for all positive integers \( m \) and \( k \).

**Lemma 5.4.**

\[
\mathbb{E}\left\{ \sup_{0 \leq t \leq 1} |\log \text{invol}_{n^t}(C^{(n)}) - \log B_{n^t}(Z_0)| \right\} = O(\log n).
\]

**Proof.** Set \( Y_n := \sum_{k=1}^{n} Z_{k,n}, D_n := \sum_{k=1}^{n} \frac{c_{k}^{(n)} + Z_{k,n+1}}{2k}, \) and \( D_n^* := \sum_{k=1}^{n} \frac{(Z_{k,0}+1)^2}{2k} \). It follows from Lemmas 5.2 and 5.3 that

\[
\log \text{invol}_j(Z_0) \leq \log \text{invol}_j(C^{(n)}) + Z_n + (1{\{L_n + R_n = 2}\}, \ldots, 1{\{L_n + R_n = n+1}\})) \leq \log \text{invol}_j(C^{(n)}) + (Y_n + 1) \log n + D_n
\]

and

\[
\log \text{invol}_j(C^{(n)}) \leq \log \text{invol}_j(Z_0 + (1{\{L_n = 1}\}, \ldots, 1{\{L_n = n}\})) \leq \log \text{invol}_j(Z_0) + \log n + D_n^*.
\]
for all \( j \in [n] \). By noticing that
\[
D_n \leq \sum_{k=1}^{n} \frac{(2Z_{k,0} + 2)^2}{2k} = 4D^*_n, \tag{5.9}
\]
we see that
\[
| \log \text{invol}_{n^t_1}(C^{(n)}) - \log \text{invol}_{n^t_1}(Z_0) | \leq (Y_n + 1) \log n + 4D^*_n \tag{5.10}
\]
for \( 0 \leq t \leq 1 \). Barbour and Tavaré proved in [5] that \( EY_n \leq \theta^2 \). Additionally,
\[
ED^*_n = \sum_{k=1}^{n} \frac{1}{2k} \left( \frac{\theta^2}{k^2} + \frac{3\theta}{k} + 1 \right) = O(\log n). \tag{5.11}
\]
Hence,
\[
E\left\{ \sup_{0 \leq t \leq 1} | \log \text{invol}_{n^t_1}(C^{(n)}) - \log \text{invol}_{n^t_1}(Z_0) | \right\} = O(\log n). \tag{5.12}
\]
Due to (1.5) and (1.6),
\[
E\left\{ \sup_{0 \leq t \leq 1} | \log \text{invol}_{n^t_1}(Z_0) - \log B_{n^t_1}(Z_0) | \right\} = O(1). \tag{5.13}
\]
Triangle inequality implies the desired result. \( \Box \)

The remainder of the proof of Theorem 5.1 is immediate in light of the following.

**Theorem 5.5.** (Barbour, Tavaré [5]) Suppose for all \( t \in [0, 1] \) we define
\[
B_n(t) = \frac{\log B_{n^t_1}(Z_0) - \frac{\theta t^2}{2} \log^2 n}{\sqrt{\frac{\theta}{3} \log^3 n}}.
\]
Then \( B_n \) converges weakly in \( \mathcal{D}[0, 1] \) to a standard Brownian motion \( B \), in the sense that
\[
E\left\{ \sup_{0 \leq t \leq 1} | B_n(t) - B(t^3) | \right\} = O\left( \frac{\log \log n}{\sqrt{\log n}} \right).
\]

We now articulate the proximity of the quantities \( \log \text{invol}_n(C^{(n)}) \) and \( \log B_n(C^{(n)}) \) with the goal of appending a convergence rate to Theorem 4.4. Let us introduce the two standardized random variables
\[
S_{1,n} = \frac{\log B_n(C^{(n)}) - \frac{\theta}{2} \log^2 n}{\sqrt{\frac{\theta}{3} \log^3 n}} \quad \text{and} \quad S_{2,n} = \frac{\log \text{invol}_n(C^{(n)}) - \frac{\theta}{2} \log^2 n}{\sqrt{\frac{\theta}{3} \log^3 n}}.
\]
The following generalization of Lemma 2.5 in [5] will help us show that the distributions of \( S_{1,n} \) and \( S_{2,n} \) are almost the same.
Lemma 5.6. Let $U$ and $X$ be random variables. Suppose that
\[
\sup_{x \in \mathbb{R}} |\mathbb{P}(U \leq x) - G(x)| \leq \eta,
\]
where $G(x)$ is a Lipschitz function over $\mathbb{R}$. Then, for any $\epsilon > 0$,
\[
\sup_{x \in \mathbb{R}} |\mathbb{P}(U + X \leq x) - G(x)| \ll \eta + \epsilon + \mathbb{P}(|X| > \epsilon)
\]
where the constant implicit in the symbol $\ll$ depends only on $\|G'\|_\infty$.

Theorem 5.7. If $C^{(n)}$ is distributed according to ESF($\theta$), then
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{\log \text{invol}_n(C^{(n)}) - \frac{\theta}{2} \log^2 n}{\sqrt{\frac{\theta}{3} \log^3 n}} \leq x \right) - \Phi(x) \right| = O\left( \frac{1}{\sqrt{\log n}} \right).
\]

Proof. We begin by applying Lemma 5.6 with $U = S_{1,n}$, and $X = S_{2,n} - S_{1,n}$. By Lemma 4.3 there exists a constant $c = c(\theta)$ such that $\sup_x |\mathbb{P}(S_{1,n} \leq x) - \Phi(x)| \leq c \log^{-1/2} n$. So if we take $\eta = c \log^{-1/2} n$ and $\epsilon = \log^{-1/2} n$, then applying a generic Chernoff bound yields
\[
\mathbb{P}(|S_{2,n} - S_{1,n}| > \epsilon) = \mathbb{P}\left( \log \text{invol}_n(C^{(n)}) - \log B_n(C^{(n)}) > \epsilon \sqrt{\frac{\theta}{3} \log^3 n} \right)
\]
\[
\leq \frac{e^\epsilon \sqrt{\frac{\theta}{4} \log^4 n}}{\epsilon \sqrt{\frac{\theta}{4} \log^4 n}} \left[ \prod_{k=1}^{n} \sum_{j=0}^{\lceil c_k/2 \rceil} \frac{(c_k^{(n)})_{2j}}{(2k)^j j!} \right]
\]
\[
\leq n^{-\sqrt{\frac{\theta}{4} \log^4 n}} \left[ \prod_{k=1}^{n} \sum_{j=0}^{\lceil c_k/2 \rceil} \frac{(Z_{k,0} + 1 \{ L_n = k \})_{2j}}{(2k)^j j!} \right].
\]

Leverage the fact that $1 \{ L_n = m \}$ for at most one integer $m$ to see that
\[
\mathbb{E} \left[ \prod_{k=1}^{n} \sum_{j=0}^{\lceil c_k/2 \rceil} \frac{(Z_{k,0} + 1 \{ L_n = k \})_{2j}}{(2k)^j j!} \right] \leq \mathbb{E} \left[ \prod_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{(Z_{m,0} + 1)_{2j}}{(2m)^j j!} \prod_{1 \leq k \leq n, k \neq m} \sum_{j=0}^{\infty} (Z_{k,0})_{2j} (2k)^j j! \right]
\]
\[
= \mathbb{E} \left[ \prod_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{Z_{m,0} + 1}{Z_{m,0} - 2j + 1} \frac{(Z_{m,0})_{2j}}{(2m)^j j!} \prod_{1 \leq k \leq n, k \neq m} \sum_{j=0}^{\infty} (Z_{k,0})_{2j} (2k)^j j! \right]
\]
\[
\text{(5.19)}
\]
Yet since the $Z_{k,0}$ are pairwise independent and $Z_{k,0} \sim \text{Poisson}(\theta/k)$, we can interchange the product and expectation operator and then compute the factorial moments to get

$$\begin{align*}
(5.20) &= \left( \sum_{j=0}^{\infty} \frac{(1 + 2j)E(Z_{m,0})_{2j}}{(2m)^{j}j!} \right) \prod_{1 \leq k \leq n, k \neq m} \sum_{j=0}^{\infty} \frac{E(Z_{k,0})_{2j}}{(2k)^{j}j!} \\
&= \left( \sum_{j=0}^{\infty} \frac{(1 + 2j)(\theta/m)^{2j}}{(2m)^{j}j!} \right) \prod_{1 \leq k \leq n, k \neq m} \sum_{j=0}^{\infty} \frac{(\theta/k)^{2j}}{(2k)^{j}j!} \\
&= \left( 1 + \frac{\theta^2}{m^3} \right) e^{\frac{\theta^2}{2m^2}} \prod_{1 \leq k \leq n, k \neq m} e^{\frac{\theta^2}{2k^2}} \leq (1 + \theta^2) e^{\frac{\theta^2}{2} \zeta(3)}. 
\end{align*}$$

Therefore $P(|S_{2,n} - S_{1,n}| > \epsilon) = O(n^{-\sqrt{\theta/3}})$, and so

$$\sup_{x \in \mathbb{R}} |P(S_{2,n} \leq x) - \Phi(x)| \ll \frac{c + 1}{\sqrt{\log n}} + O(n^{-\sqrt{\theta/3}}) = O\left(\frac{1}{\sqrt{\log n}}\right). \quad (5.24)$$

According to Zacharovas [30],

$$\sup_{x \in \mathbb{R}} \left| P\left( \frac{\log B_n(C^{(n)}) - \frac{1}{2} \log^2 n}{\sqrt{\frac{1}{3} \log^3 n}} \leq x \right) - \Phi(x) - \frac{3^{3/2}}{24 \sqrt{2\pi}} \frac{(1 - x^2)e^{-x^2/2}}{\sqrt{\log n}} \right| \ll \left( \frac{\log \log n}{\log n} \right)^{\frac{2}{3}}.$$ 

Putting, in (5.6),

$$U = \frac{\log B_n(C^{(n)}) - \frac{1}{2} \log^2 n}{\sqrt{\frac{1}{3} \log^3 n}}, \quad X = -\frac{\log B_n(C^{(n)}) - \log \text{invol}_n(C^{(n)})}{\sqrt{\frac{1}{3} \log^3 n}},$$

$$\epsilon = ((\log \log n)/\log n)^{2/3}, \quad \eta = \log^{-3/4} n,$n, and observing that

$$P(|S_{2,n} - S_{1,n}| \geq \epsilon) \leq (1 + \theta^2) e^{\frac{\theta^2}{2} \zeta(3) - \epsilon \sqrt{\frac{\theta}{2} \log n}} = O\left( e^{-(\log \log n)^{2/3} \log^{7/6} n} \right) \quad (5.25)$$

yields the following.
Theorem 5.8. If $C^{(n)}$ is distributed according to ESF(1), then
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\log \text{invol}_n(C^{(n)}) - \frac{1}{2} \log^2 n}{\sqrt{\frac{1}{3} \log^3 n}} \leq x \right) - \Phi(x) - \frac{3^{3/2}}{24 \sqrt{2 \pi}} \frac{(1 - x^2)e^{-x^2/2}}{\sqrt{\log n}} \right| \ll \left( \frac{\log \log n}{\log n} \right)^{3/2}.
\]

Zacharova’s result on the optimality of the convergence rate for the Erdős-Turán limit law is constructive and can be mimicked for $\theta$-weighted permutations in general. For this reason, the rate of convergence stated in Theorem 5.7 is the best possible. We refrain from verifying this here.

Although we will decline from doing so, we remark that it is straightforward to compute explicit bounds for the constants implied by the error terms in Theorems 5.1 and 5.7. The estimates needed to measure the distance between the distributions of $\log \mathcal{B}_n(C^{(n)})$ and $\log \mathcal{B}_n(Z)$ already appear in [5]. From there, a Berry-Essén inequality will disclose how quickly $\log \mathcal{B}_n(Z)$ tends to its Gaussian limit law. The constants implied by (5.11) and (5.12) require more delicate asymptotic evaluation, but can be safely ignored since the dominant error term for Theorem 5.1 resides in Theorem 5.5.

Concluding Remarks

The nonuniformity of invol makes it evident that the composition of two uniformly random involutions is not a uniformly random permutation. Furthermore, if we let $t_n$ denote the total number of involutions in $\mathfrak{S}_n$, then
\[
t_n^2 \sim n! \cdot \frac{e^{2\sqrt{n}}}{\sqrt{8\pi n}} \sim |\mathfrak{S}_n| \cdot \mathbb{E}_{1,n} \text{invol}. \quad (5.26)
\]
(Refer to pg. 583, Example VIII.9, [18].) Therefore, using pairs of random involutions to generate random permutations introduces a significant sampling bias.

Like ESF($\theta$) for $\theta > 1$, a composition of two random involutions is likelier to have a lot of cycles. However, the distribution of $c_k$ looks quite different in this model. The factors of the exponential generating function in (2.6) suggests that $c_k$ converges in law to the distribution of $X + 2Y_k$, where $X$ and $Y_k$ are independent Poisson random variables of mean 1 and $1/(2k)$, respectively. Lugo proved as much in an unpublished manuscript [24]. The author intends to expand on these and other observations in a followup paper.

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6 Appendix

This appendix compiles some technical prerequisites which are used sporadically throughout the manuscript.

6.1 Wright’s Expansions

The following consequence of the saddle-point method applied to functions with exponential singularities is often helpful in unraveling the asymptotic development of various random permutation statistics.

**Theorem 6.1.** (Wright [34]) Suppose

\[
f(z) = (1 - z)^{-\beta} \varphi(z) e^{P(z)},
\]

where \( \varphi \) is regular in the unit disc, \( \varphi(1) \neq 0 \),

\[
P(z) = \sum_{m=1}^{M} \frac{\alpha_m}{(1 - z)^{\rho_m}},
\]

\( \rho_1, \ldots, \rho_M \) are real, \( \beta, \alpha_1, \ldots, \alpha_M \) are complex, \( \rho_1 > 0 \), and \( \alpha_1 \neq 0 \). If in addition

\[
P(e^{-u}) = A u^{-\rho} \left(1 + \sum_{\ell=1}^{L} A_\ell u^{\sigma_\ell}\right) + O(u^K), \quad u \to 0^+,
\]

where \( A \in \mathbb{C} - \mathbb{R}_{\leq 0}, 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_L \leq \rho \), and \( L, A, \rho, the \sigma_\ell, and the A_\ell \) are all calculable in terms of the \( \rho_m \) and \( \alpha_m \), then

\[
[z^n]f(z) = \frac{N^{\beta-\frac{1}{2}} e^{\Omega(N)} \varphi(1)}{\sqrt{2\pi n(\rho + 1)}} (1 + O(n^{-K})), \quad n \to \infty,
\]

\[
= A u^{-\rho} \left(1 + \sum_{\ell=1}^{L} A_\ell u^{\sigma_\ell}\right) + O(u^K), \quad u \to 0^+,
\]

where \( A \in \mathbb{C} - \mathbb{R}_{\leq 0}, 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_L \leq \rho \), and \( L, A, \rho, the \sigma_\ell, and the A_\ell \) are all calculable in terms of the \( \rho_m \) and \( \alpha_m \), then

\[
[z^n]f(z) = \frac{N^{\beta-\frac{1}{2}} e^{\Omega(N)} \varphi(1)}{\sqrt{2\pi n(\rho + 1)}} (1 + O(n^{-K})), \quad n \to \infty,
\]
where

\[
N = \left( \frac{n}{A\rho} \right)^{\frac{1}{\rho+1}},
\]

\[
\Omega(N) = A(1 + \rho)N^\rho + A \sum_{\ell=1}^{L} A_{\ell} N^{\rho-\sigma_{\ell}} + \sum_{s=2}^{\infty} \frac{(-1)^{s-1}(A\rho)^{1-\frac{s}{\rho+1}}}{s!(\rho+1)} \left[ \frac{d^{s-2}}{dy^{s-2}}(y^{\rho+1-2}(\vartheta(y))^s) \right] \bigg|_{y=n}, \text{ and}
\]

\[
\vartheta(y) = \frac{1}{\rho} \sum_{\ell=1}^{L} (\rho - \sigma_{\ell}) A_{\ell} \left( \frac{y}{A\rho} \right)^{\frac{\sigma_{\ell}}{\rho+1}}.
\]

With one exception in section 3, we solely need this simpler instance of Theorem 6.1.

**Corollary 6.2.** (Wright [33]) The leading-term asymptotic for

\[
s_n = [z^n](1 - z)^\beta \phi(z) \exp \left( \frac{\alpha}{1 - z} \right),
\]

where \( \beta \) is a complex number, \( \phi \) is regular in the unit disc, \( \phi(1) \neq 0 \), and \( \alpha \) is a nonzero real number, is given by

\[
s_n = \frac{1}{n^{\frac{3\alpha}{2} + \frac{1}{2}}} \left[ \frac{\exp(2\sqrt{\alpha n})}{2\sqrt{\pi}} \phi(1) e^{\frac{\alpha}{2}} \right] \left( 1 + O(n^{-\frac{1}{2}}) \right).
\]

### 6.2 The Mellin Transform

To conclude this section, we review some pertinent facts about the Mellin transform as a means of extracting asymptotic estimates. Given a locally integrable function \( f \) defined over \( \mathbb{R}_{>0} \), the Mellin transform of \( f \) is the complex-variable function

\[
f^*(s) := \int_{0}^{\infty} f(x)x^{s-1} dx.
\]

The transform is also occasionally denoted by \( \mathcal{M}[f] \) or \( \mathcal{M}[f(x); s] \). If \( f(x) = O(x^{-a}) \) as \( x \to 0^+ \) and \( f(x) = O(x^{-b}) \) as \( x \to +\infty \), then \( f^* \) is an analytic function on the “fundamental strip” \( a < \Re(z) < b \) and is often continuable to a meromorphic function on the whole of \( \mathbb{C} \).

Notably, the Mellin transform is a linear operator that satisfies the rescaling rule

\[
\mathcal{M}[f(\mu x); s] = \mu^{-s} f^*(s),
\]

for any \( \mu > 0 \). This together with the linearity of \( \mathcal{M} \) yields the “harmonic sum rule”

\[
\mathcal{M}\left[ \sum_{k} \lambda_k f(\mu_k x); s \right] = \left( \sum_{k} \lambda_k \mu_k^{-s} \right) f^*(s),
\]

25
provided that the integral and summation are interchangeable. Sums of the form $\sum_k \lambda_k f(\mu_k x)$ are known as harmonic sums with base function $f$, amplitudes $\lambda_k$, and frequencies $\mu_k$. In other words, the Mellin transform of a harmonic sum factors into the product of the Mellin transform of the base function and a generalized Dirichlet series associated with the corresponding sequence of amplitudes and frequencies. For instance, we can obtain the useful Mellin pairs

$$\mathcal{M} \left[ \frac{e^{-x}}{1 - e^{-x}}; s \right] = \mathcal{M} \left[ \sum_{k=1}^{\infty} e^{-kx}; s \right] = \left( \sum_{k=1}^{\infty} k^{-s} \right) \mathcal{M}[e^{-x}; s] = \zeta(s) \Gamma(s) \quad (6.1)$$

and

$$\mathcal{M} \left[ \frac{e^{-x}}{(1 - e^{-x})^2}; s \right] = \mathcal{M} \left[ \sum_{k=1}^{\infty} ke^{-kx}; s \right] = \left( \sum_{k=1}^{\infty} k^{-s+1} \right) \mathcal{M}[e^{-x}; s] = \zeta(s - 1) \Gamma(s). \quad (6.2)$$

Akin to many other valuable integral transforms, the Mellin transform is essentially involutive. If $f$ is continuous in an interval containing $x$, then $f(x)$ can be recovered from $f^*(s)$ via the inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) t^{-s} ds,$$

where the abscissa $c$ must be chosen from the fundamental strip of $f$. If the continuation of $f^*(s)$ decays quickly enough as $\Im(c) \to +\infty$, then the inverse Mellin integral and Cauchy’s residue theorem can be combined to derive an asymptotic series for $f(x)$ as $x \to 0^+$. We have the following transfer rule in particular: a pole of order $k + 1$ at $s = s_0$ with

$$f^*(s) \underset{s \to s_0}{\sim} a_{s_0} \frac{(-1)^k k!}{(s - s_0)^{k+1}}$$

translates to the term $a_{s_0} x^{-s_0} (\log x)^k$ in the singular expansion of $f(x)$ near $x = 0$. (The specific requirements for this transfer rule are all fulfilled by the special functions appearing in this paper, as the gamma function decays rapidly along vertical lines in the complex plane whereas polylogarithms undergo only moderate growth.) Detailed explanations along with further uses of the Mellin transform towards solving asymptotic enumeration problems can be found in [16] and [18].