Edge solitons in the QHE*

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Abstract

The spacelike reduction of the Chern-Simons Lagrangian yields a modified Nonlinear Schrödinger Equation (jNLS) where in the non-linearity the particle density is replaced by the current. When the phase is linear in the position, this latter is an ordinary NLS with time-dependent coefficients which admits interesting solutions, whose arisal is explained by the conformal properties of non-relativistic spacetime. Only the usual travelling soliton is consistent with the jNLS but adding a six-order potential converts it into an integrable equation.

1 Reduction of Chern-Simons

The Landau-Ginzburg theory of the Quantum Hall Effect [1] uses the Chern-Simons Lagrangian in (2 + 1) dimensions,

\[ L = \frac{1}{4\kappa} \epsilon_{\mu
u\rho} F_{\mu\nu} A_\rho + i\phi^* D_t \phi - \frac{1}{2} |\vec{D}\phi|^2 - V(\phi), \]

where the scalar field \( \phi \) is the order parameter and \( A_\mu \) is the statistical gauge field; \( D_\mu = \partial_\mu - i A_\mu \) is the covariant derivative. The constant \( \kappa \) is interpreted as the Hall conductivity. The second-order field equations are not integrable [2]; they admit integrable reductions, though. The simplest of these is when time-dependence is eliminated; then, for a judicious choice of the self-interaction potential \( V(\phi) \), the system admits finite-energy vortex solutions [3]. Here we focus our attention to another, space-like reduction [4]. Assuming independence from one spacelike coordinate and adding a suitable kinetic term yields in fact, after elimination of the gauge field using its equation of motion,

\[ L = i\phi^* \partial_t \phi - \frac{1}{2} |(\partial_x - i\kappa^2 \rho)\phi|^2 - V, \]

where \( \rho = |\phi|^2 \) is the particle density. This is the model proposed in Ref. [5] to describe the edge states in the QHE.

The field equations associated to (2) read

\[ i\partial_t \phi = -\frac{1}{2} (\partial_x - i\kappa^2 \rho)^2 \phi - \kappa^2 j \phi + \frac{\partial V}{\partial \phi^*}, \]

\[ j = \frac{1}{2\kappa^2} [\phi^* (\partial_x - i\kappa^2 \rho) \phi - \phi (\partial_x + i\kappa^2 \rho) \phi^*]. \]
Then the particle density and the current satisfy the continuity equation $\partial_t \rho + \partial_x j = 0$. Let us first assume that $V = 0$. Now the non-local transformation

$$\psi = \left( \exp[-i\kappa^2 \int_x^y \rho(y)dy] \right) \phi$$

(4)

takes into the modified non-linear Schrödinger equation in which the density in the non-linearity has been replaced by the current,

$$i\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi - 2\kappa^2 j \psi,$$

$$j = \frac{1}{2i}[\psi^* \partial_x \psi - \psi (\partial_x \psi)^*].$$

(5)

\section{The variable-coefficient NLS}

Decomposing $\psi$ into module and phase, $\psi = \sqrt{\rho} e^{i\theta}$, yields (formally) the ordinary cubic NLS with variable coefficient,

$$i\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi - F(t,x)|\psi|^2 \psi,$$

(6)

with $F(t,x) = 2\kappa^2 \partial_x \theta$. Then Aglietti et al. \cite{4} observe that, for $\theta = vx - \omega t$, Eq. (6) reduces to the usual non-linear Schrödinger equation with constant coefficient $F = 2\kappa^2 v$ which admits, for example, the travelling soliton solution

$$\psi_s = \pm e^{i(vx - \omega t)} \sqrt{\frac{1}{2\kappa^2 v \cosh \alpha(x - vt)}}\cosh \left[ -\frac{x}{vt} - x_0 \right].$$

(7)

The non-linearity in (6) has to be attractive, $F > 0$; the solution (7) is therefore chiral, $v > 0$.

It is natural to ask whether the travelling soliton (7) can be generalized. Let us first study the variable-coefficient NLS (6) on its own. It has been shown \cite{8} that this equation only passes the Painlevé test of Weiss, Tabor and Carnevale \cite{9}, when the coefficient of the non-linearity is

$$F = (a + bt)^{-1},$$

(8)

where $a$ and $b$ constants. For $b = 0$, $F(t,x)$ is a constant and we recover the constant-coefficient NLS. For $b \neq 0$, the equation becomes explicitly time-dependent. Assuming, for simplicity, that $a = 0$ and $b = 1$, it reads

$$i\partial_t \psi + \frac{1}{2}\partial_x^2 \psi + \frac{1}{t}|\psi|^2 \psi = 0.$$

(9)

This equation can also be solved. Generalizing the usual travelling soliton, we find, for example, the 1-soliton

$$\psi_0(t,x) = \frac{1}{\sqrt{t \cosh \left[ -x/t - x_0 \right]}} e^{i(x^2/4t - 1/2t)}.$$

(10)

It is worth pointing out that the steps followed in constructing (10) are essentially the same as those for the travelling soliton of the ordinary NLS — and this is not a pure coincidence. A short calculation shows in fact that

$$\psi(t,x) = \frac{1}{\sqrt{t}} \exp \left[ \frac{i x^2}{4t} \right] \Psi(-1/t, -x/t)$$

(11)

satisfies the time-dependent equation (9) if and only if $\Psi(t,x)$ solves Eqn. (9) with $F = 1$. Our soliton (10) comes in particular from the “standing soliton” $\Psi_s = \exp(it/2)(\cosh[x - x_0])^{-1}$ solution of the NLS.
3 Non-relativistic conformal transformations

Where does the formula (10) come from? The non-linear space-time transformation

\[
D : \begin{pmatrix} t \\ x \end{pmatrix} \rightarrow \begin{pmatrix} -1/t \\ -x/t \end{pmatrix}
\]  

(12)

has already been met in a rather different context, namely in describing planetary motion when the gravitational “constant” changes inversely with time, as suggested by Dirac [9]. One shows in fact that \( \vec{r}(t) = t \vec{r}^* \left(-\frac{1}{t}\right) \) describes planetary motion with Newton’s “constant” varying as \( G(t) = \frac{G_0}{t} \), whenever \( \vec{r}^*(t) \) describes ordinary planetary motion, i.e. the one with a constant gravitational constant \( G_0 \) [10], [11]. The strange-looking transformation (12) is indeed related to the conformal structure of non-relativistic space-time [12]. It has been noticed in fact almost thirty years ago, that the “conformal” space-time transformations

\[
\begin{pmatrix} t \\ x \end{pmatrix} \rightarrow \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} \delta^2 t \\ \delta x \end{pmatrix}, \quad 0 \neq \delta \in \mathbb{R} \quad \text{dilatations}
\]

\[
\begin{pmatrix} t \\ x \end{pmatrix} \rightarrow \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} t \\ \frac{1-\kappa t}{x} \\ \frac{-\kappa t}{x} \end{pmatrix}, \quad \kappa \in \mathbb{R} \quad \text{expansions}
\]

(13)

\[
\begin{pmatrix} t \\ x \end{pmatrix} \rightarrow \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} t + \epsilon \\ x \end{pmatrix}, \quad \epsilon \in \mathbb{R} \quad \text{time translations}
\]

implemented on wave functions according to

\[
\Psi(T, X) = \begin{cases} 
\delta^{1/2} u(t, x) \\
(1 - \kappa t)^{1/2} \exp \left[i \frac{\kappa x^2}{4(1 - \kappa t)}\right] \psi(t, x) \\
\psi(t, x)
\end{cases}
\]

(14)

permute the solutions of the free Schrödinger equation. In other words, they are symmetries of the free Schrödinger equation. The generators in (13) span in fact an \( SL(2, \mathbb{R}) \) group. (A Dirac monopole, an Aharonov-Bohm vector potential and an inverse-square potential can also be included). The transformation \( D \) in Eqn. (12) belongs to this symmetry group: it is in fact (i) a time translation with \( \epsilon = 1 \), (ii) followed by an expansion with \( \kappa = 1 \), (iii) followed by a second time-translation with \( \epsilon = 1 \). It is hence a symmetry for the free (linear) Schrödinger equation.

The cubic NLS with constant non-linearity is not more \( SL(2, \mathbb{R}) \) invariant: the transformation \( D \) in (12) implemented as in Eq. (14) carries the cubic term into the time-dependent term \((1/t)|u|^2 u\), just like Newton’s gravitational potential \( G_0/r \) with \( G_0 = \text{const.} \) is carried into the time-dependent Dirac expression \( t^{-1} G_0/r \). More generally, the the non-linear Schrödinger equation with time–dependent coefficients,

\[
i \partial_t \psi + p(t) \partial_x^2 \psi + F(t) |u|^2 \psi = 0,
\]

(15)

can be transformed into the constant–coefficient form, whenever

\[
p(t) = F(t) \left(a + b \int p(s) ds\right)
\]

(16)

This same condition was found later as the one needed for the Painlevé test [7].
Similar arguments explain the integrability of other NLS type equations. For example, electromagnetic waves in a non-uniform medium propagate according to

\[ i\partial_t \psi + \partial_x^2 \psi + (-2\alpha x + 2|\psi|^2)\psi = 0, \]

which can again be solved by inverse scattering \cite{13}. This is explained by observing that the potential term here can be eliminated by switching to a uniformly accelerated frame:

\[ \psi(t, x) = \exp \left[ -i(2\alpha xt + \frac{s}{3}\alpha^2 t^3) \right] \Psi(T, X), \]

\[ T = t, \quad X = x + 2\alpha t^2. \]

Then \( u(t, x) \) solves \((17)\) whenever \( \Psi(T, X) \) solves the free equation.

The transformation \((18)\) is again related to the structure of non-relativistic space-time. It can be shown in fact \cite{11} that the (linear) Schrödinger equation

\[ i\partial_t \psi + \partial_x^2 \psi - V(t, x) \psi = 0 \]

(19)

can be brought into the free form by a space-time transformation if and only if the potential is

\[ V(t, x) = \alpha(t)x \pm \omega^2(t)x^2. \]

For the uniform force field \((\omega = 0)\) the required transformation is precisely \((18)\).

For the oscillator potential \((\alpha = 0)\), one can use rather Niederer’s transformation \cite{14}

\[ \psi(t, x) = (\cos \omega t)^{-1/2} \exp \left[ -i\frac{\omega^2}{4} x^2 \tan \omega t \right] \Psi(T, X), \]

\[ T = \frac{\tan \omega t}{\omega}, \quad X = \frac{x}{\cos \omega t}. \]

Then

\[ i\partial_t \psi + \frac{1}{2}\partial_x^2 \psi - \frac{\omega^2 x^2}{4} \psi = (\cos \omega t)^{-5/2} \exp \left[ -i\frac{\omega}{4} \tan \omega t \right] \left( i\partial_T \Psi + \frac{1}{2}\partial_x^2 \Psi \right) \]

so that \( \psi \) satisfies the oscillator-equations iff \( \Psi \) solves the free equation.

The Niederer transformation \((20)\) leaves the inverse square potential invariant; this explains why the Calogero model in a harmonic background can be brought into the pure Calogero form \cite{15}. Restoring the nonlinear term allows us to infer also that

\[ i\partial_t \psi + \partial_x^2 \psi + \left( \frac{-\omega^2 x^2}{4} + \frac{1}{\cos \omega t} |\psi|^2 \right) \psi = 0 \]

(21)

is integrable, and its solutions are obtained from those of the “free” NLS by the transformation \((20)\). Let us mention that the covariance w. r. t. chronoprojective transformations was used before \cite{16} for solving the NLS in oscillator and uniform-field backgrounds.

Now the constant-coefficient, damped, driven NLS,

\[ i\partial_t \psi + \partial_x^2 \psi + F|\psi|^2 \psi = a(t, x)\psi + b(t, x), \]

passes the Painlevé test if

\[ a(t, x) = (\frac{1}{2}\partial_t \beta - \beta^2)x^2 + i\beta(t) + \alpha_1(t)x + \alpha_0(t), \]

\[ b(t, x) = 0 \]

\cite{17}, i. e., precisely when the potential can be transformed away by our “non-relativistic conformal transformations”.

4 An integrable extension

Unfortunately, the time-dependent travelling soliton (10) is inconsistent with the original equation (5), since its phase is quadratic in $x$ rather than linear, as required by consistency. The clue for finding integrable extensions is to observe that Eqn. (5) is in fact a Derivative Non-Linear Schrödinger equation (DNLS) [6]. Now the results of Clarkon and Cosgrove [18] say that the constant-coefficient equation

$$i\partial_t \psi + \frac{1}{2} \partial_x^2 \psi + ia \psi \psi^* \partial_x \psi + ib \psi^2 \partial_x \psi^* + c\psi^3 \psi^*^2 = 0$$

(24)

is integrable iff

$$c = \frac{1}{2}b(2b - a).$$

(25)

In our case $a = -b = -\kappa^2$ and $c = 0$; Eq. (5) is therefore not integrable. However, adding a 6th–order potential to the Lagrange density i. e. considering rather

$$i\partial_t \psi + \frac{1}{2} \partial_x^2 \psi + 2\kappa^2 \psi \partial_x \psi + \frac{3}{2}\kappa^4 |\psi|^4 \psi = 0$$

(26)

converts (5) into an integrable equation. Eq. (26) admits, e. g., the travelling wave solution $\psi = \sqrt{\rho} e^{i\theta}$, where

$$\rho = \frac{|v|}{2\kappa^2} \frac{1}{\sqrt{2} \cosh \left[v(x - \frac{v^2}{4} t)\right] + \text{sign} v}, \quad \theta = \frac{v}{2} x.$$  

(27)

This can be checked by observing that for the Ansatz $\psi = f(x,t)e^{ivx/2}$ the modified NLS (26) again reduces to a constant-coefficient equation. Then the imaginary part of (26) requires that $f(x,t) = f(x - (v/2)t)$, while the real part can be integrated by the usual trick of multiplication by $f'$. The asymptotic conditions fix the integration constant to vanish, yielding a six-order non-linear equation, only containing even powers of $f$. Then, introducing $\rho = f^2$, we end up with the equation

$$\left(\rho'\right)^2 - v^2 \rho^2 + 4\kappa^2 v \rho^3 + 4\kappa^4 \rho^4 = 0,$$

(28)

whose integration provides us with (27).

Another way of understanding how the integrability comes is to apply the non-local transformation (4) backwards, which carries (26) into a Derivative Non-linear Schrödinger equation of type II (DNLSII),

$$i\partial_t \phi + \frac{1}{2} \partial_x^2 \phi + 2i\kappa^2 \rho \partial_x \phi = 0,$$

(29)

which, consistently with Eq. (25), is integrable [19].

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