Quantum fluctuations of the vacuum stress-energy tensor are highly non-Gaussian, and can have unexpectedly large effects on the spacetime geometry. In this paper, we study a two-dimensional dilaton gravity model coupled to a conformal field theory, in which the distribution of vacuum fluctuations is well understood. By analyzing the geodesic deviation in this model, we show that a pencil of massive particles propagating on this fuzzy spacetime eventually converges and collapses. The collapse time depends on the velocity of the congruence of particles, but for ultra-relativistic particles the collapse probability as a function of time converges to an exponential distribution, consistent with our earlier analysis of null geodesics [Phys. Rev. Lett. 107, 021303 (2011)]. We thus find further evidence for the influence of vacuum fluctuations on the small scale causal structure of spacetime.

I. INTRODUCTION

Quantum fluctuations of vacuum energy are highly non-Gaussian, and can have unexpectedly large effects on spacetime geometry. In Ref. [1], we studied the impact of the vacuum fluctuations of a conformal field on the causal structure of spacetime. In order to do this, we analyzed the Raychaudhuri equation for a pencil of light in a two-dimensional dilaton gravity model for which the probability distribution for the fluctuations is exactly known; the dilaton field in this case played the role of a transverse area in the “missing” dimensions. We showed that the fluctuations of the stress-energy tensor lead to a sharp focusing of light cones near the Planck scale, breaking up the causal structure of spacetime at such small scales.

Additional evidence for this phenomenon, coming from perturbative algebraic quantum field theory, was obtained in Ref. [2]. The connection between vacuum fluctuations and spacetime geometry was further studied in Ref. [3]. However, since the exact probability distribution for the vacuum fluctuations is not known in four dimensions, in that case only the variance of the relative velocity and the mean squared distance fluctuation could be obtained through Riemann correlation functions. On the other hand, in two spacetime dimensions the exact probability distribution for fluctuations of the stress-energy tensor is known, at least for conformal fields [4, 5]. Finite results require that the stress-energy operator be smeared by a test function, but as shown in [5], the probability distribution for the fluctuations is given by a shifted Gamma distribution for a wide variety of smearings. This allows us to explore another aspect of vacuum fluctuations in dilaton gravity, which was neglected in [1]. Besides the direct effect on the dilaton, the vacuum fluctuations of the stress-energy tensor couple to the (two-dimensional) metric, inducing fluctuations of the spacetime itself. Pure Einstein gravity has no dynamics in two dimensions, since the Einstein-Hilbert action is a topological invariant. In dilaton gravity, though, the spacetime curvature is determined by the dilaton potential. Thus, once we find how the dilaton responds to vacuum fluctuations of a quantum field, we will be able to determine the fluctuations of the geometry and their effects on particle propagation. This is the main purpose of this work.

II. THE MODEL

Our two-dimensional dilaton gravity model can be obtained by dimensional reduction from higher-dimensional general relativity. Under such a reduction, the dilaton $\varphi$ is essentially the transverse area element. In a previous paper [1], we considered the direct effect of these fluctuations on $\varphi$, viewed as an area, by using a version of the Raychaudhuri equation in which the fluctuations acted as a stochastic noise term. Here we consider a slightly less direct process: fluctuations of the vacuum stress-energy tensor induce fluctuations of the curvature, which in turn affect the behavior of timelike geodesics. As we shall see, the two analyses lead to a consistent picture, in which vacuum fluctuations at the Planck scale lead to a rapid convergence of geodesics, the “collapse” of a pencil of geodesics, at that scale.

In a dilatonic theory, with the appropriate redefinitions, it is always possible to bring the action into the form [6, 7]

$$S = S_V + S_M,$$

with

$$S_V = \int d^2x \sqrt{-g} [\varphi R + V(\varphi)]$$

$$S_M = \int d^2x \sqrt{-g} [\text{matter lagrangian}]$$

arXiv:1809.08265v2 [gr-qc] 13 Oct 2018
being the geometrical action and
\[ S_M = \int d^2x \sqrt{-g} \mathcal{L}_M \] \hspace{1cm} (3)

being the action for the matter field. In what follows, we will take \( S_M \) to describe a conformal field with central charge \( c = 1 \). We will approximate the vacuum fluctuations of the matter stress-energy tensor by their flat spacetime values; as we will see later, the geometry fluctuates around the Minkowski spacetime \( R = 0 \), so this is a good approximation for the background geometry.

For a positive energy conformal field theory with central charge \( c = 1 \) in two-dimensional Minkowski spacetime, the right-moving/left-moving components of the smeared stress-energy tensor have probability distributions for individual measurements given by a shifted Gamma distribution [1].

\[ P(T_{R/L} = \omega) = \Theta(\omega + \omega_0) \frac{\beta^\alpha(\omega + \omega_0)^{\alpha - 1}}{\Gamma(\alpha)} e^{-\beta(\omega + \omega_0)}, \] \hspace{1cm} (4)

with
\[ \omega_0 = \frac{1}{24\pi\Delta^2}, \quad \alpha = \frac{1}{24}, \quad \beta = \pi\Delta^2, \] \hspace{1cm} (5)

where \( \Delta \) is the characteristic width of the smearing function.

The connection between geometry and vacuum fluctuations comes into being as follows. Varying the action [1] with respect to \( \varphi \), we arrive at
\[ R = -V'(\varphi). \] \hspace{1cm} (6)

This equation determines the curvature of the two-dimensional spacetime in terms of the dilaton \( \varphi \). The equation for the dilaton field is obtained through the variation of the action with respect to the metric \( g_{\mu\nu} \). It is given by
\[ \nabla_\mu \nabla_\nu \varphi = \frac{1}{2} g_{\mu\nu} V(\varphi) + g_{\mu\nu} T - T_{\mu\nu}, \] \hspace{1cm} (7)

where \( T_{\mu\nu} \) is the stress-energy tensor associated with the matter field, and \( T = 0 \) since we are dealing with a conformal field. We now make a few simplifications to the model. First, the potential \( V(\varphi) \) will be taken to be small near \( \varphi = 0 \); in this paper we consider the particular model
\[ V(\varphi) = \frac{1}{2} V_0 \varphi^2, \quad \text{with } V_0 \ll 1. \] \hspace{1cm} (8)

Next, since the vacuum fluctuations are highly centered around the average \( \langle T_{R/L} \rangle = 0 \), the curvature will also fluctuate around that of Minkowski spacetime. We will therefore approximate the covariant derivatives in (7) by partial derivatives since the spacetime is flat on average. With these simplifications, Eq. (7) yields
\[ \partial_\mu \partial_\nu \varphi = -T_{\mu\nu}. \] \hspace{1cm} (9)

In light-cone coordinates, \( \eta = t - x \) and \( \xi = t + x \), this equation becomes
\[ \partial_\eta \varphi = -T_R, \quad \partial_\xi \varphi = -T_L, \] \hspace{1cm} (10)

with a simple solution given by
\[ \varphi = -\left[ T_R \frac{\eta^2}{2} + T_L \frac{\xi^2}{2} \right]. \] \hspace{1cm} (11)

Now note that the average fluctuations of \( T_R \) and \( T_L \) are zero. If we set \( \Delta = 1 \)—that is, choosing the width of the smearing function to be the Planck length—a simple calculation shows that \( P(-0.1 < \omega < 0.1) \approx 96\% \). It is thus clear that typical values of \( \varphi^2 \) are highly suppressed relative to \( \varphi \). This provides a justification for our simplification \( V \sim 0 \) in Eq. (7), and for our use of the flat spacetime probability distribution [4].

By Eq. (6), the scalar curvature is now given by
\[ R = V_0 \left[ T_R \frac{\eta^2}{2} + T_L \frac{\xi^2}{2} \right], \] \hspace{1cm} (12)

and thus fluctuates with the components \( T_R \) and \( T_L \) of the stress-energy tensor. Now consider a pencil of massive particles with velocity \( v \), i.e., a congruence of geodesics parametrized by proper time \( \tau \) with worldlines \( u^\mu = \frac{1}{\sqrt{1 - v^2}}(\tau, v\tau) \). The curvature scalar in this case reads
\[ R = V_0 \left[ \frac{\tau^2}{1 - v^2} \left( T_R + T_L \right) \frac{(1 + v^2)}{2} + (T_L - T_R)v \right]. \] \hspace{1cm} (13)

Note that when \( v = 0 \), \( R \) fluctuates with the energy density \( T_{00} = T_R + T_L \) (see [3]). When \( 0 < v < 1 \), there is also a contribution from the momentum \( T_{01} = T_L - T_R \) associated with the conformal field.

III. COLLAPSE TIME

The Raychaudhuri equation for a congruence of timelike particles in two dimensions is given by
\[ \frac{d\theta(\tau)}{d\tau} = -\theta(\tau)^2 - R_{\mu\nu} u^\mu u^\nu. \] \hspace{1cm} (14)

Since in two dimensions the Ricci tensor \( R_{\mu\nu} \) has the simple form \( R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \), we have, using Eq. (13),
To work with congruences of geodesics at the Planck scale, we set \( \Delta = 1 \) in Eq. (3). Given \( T_R \) and \( T_L \), Eq. (15) can be solved exactly in terms of Bessel functions. We instead approached the problem numerically, using the software Mathematica [8]. We solved the equation in steps \( \Delta \tau = 1 \), with the initial condition at each step taken from the previous one and with \( \theta(0) = 0 \), that is, an initially parallel pencil of particles. At each step, a numerical value for \( T_{R/L} \) was randomly chosen using the probability distribution (4). It is easy to see that eventually the solution collapses, \( \theta(\tau_0) \rightarrow -\infty \), where \( \tau_0 \) is the “collapse time”. This collapse time depends on the particular values of \( T_R \) and \( T_L \), and changes from experiment to experiment. We performed \( 10^7 \) repetitions each for several values of the velocity \( v \), with \( 0 \leq v < 1 \). The calculated values for the averaged collapse time are shown in Fig. 1. To ensure \( V(\varphi) \sim 0 \), we set \( V_0 = 10^{-2} \).

![FIG. 1: Collapse time as a function of \( v/c \). The inset shows the behavior of this curve in a log scale for \( v \) approaching 1, with \( v = 1 - 10^{-n} \) and \( n = 0, 1, \ldots, 13 \).](image)

For an initial collection of stationary particles \((v = 0)\), the average collapse time is given by \( t_f^{(v=0)} = 36.59t_P \), where \( t_P \) is the Planck time. The histogram representing the relative frequency of events versus the collapse times is shown in Fig. 2(a), while a typical “trajectory” of \( \theta(\tau) \) is shown in Fig. 2(b). Notice that the collapse of the pencil of geodesics occurs abruptly as a result of a large enough random fluctuation.

When the velocity of the pencil of massive particles is increased to \( v = 0.8 \), the average collapse time drops to \( t_f^{(v=0.8)} = 28.16t_P \). The histogram and typical trajectories for this case (Figs. 2(c) and 2(d)) are similar to those of the stationary case, but the peak of the histogram is shifted to the left. In both cases, we could not find an exact expression for a distribution that fits each histogram.

Our most interesting result occurs for the case of ultra-relativistic particles, wherein \( v \rightarrow 1 \) (Figs. 2(e) and 2(f)). Here, the collapse time drops considerably to \( t_f^{(v=1)} = 8.54t_P \). The peak of the histogram now occurs at \( t_f = 0 \), and an exponential distribution of the form \( 1/t_f \exp(-\tau/t_f) \) fits quite well, as shown in Fig. 2(f). In [1] we found the same behavior for the collapse of light rays, but there we were dealing with a stochastic equation with a time-independent source. Despite this different set-up, the result here for ultra-relativistic particles is completely consistent with the analysis of Ref. [1], and reinforces the evidence for the influence of vacuum fluctuations on the causal structure of spacetime.

Finally, to show further evidence of the convergence of the collapse times as \( v \rightarrow 1 \), we performed numerical experiments for velocities \( v \) close to 1. The results are shown in the inset of Fig. 1 where the collapse time is plotted as a function of \( v(n) = 1 - 10^{-n} \), with \( n = 0, 1, \ldots, 13 \). We see that as \( n \) grows—that is, as the velocity approaches the speed of light—the collapse times approach the limiting value \( t_f^{(v=1)} = 8.54t_P \).

### IV. CONCLUSIONS

The results presented here agree with those of our earlier work [1]: vacuum fluctuations at the Planck scale lead to a rapid “collapse” of a pencil of geodesics. This behavior can be traced back to a phenomenon known in probability theory as “Gambler’s ruin,” sometimes stated as the maxim that if you are betting against an opponent with infinite resources, you will always lose in the end. In [1] the behavior was fairly simple. Gravity is attractive—positive energy fluctuations cause geodesics to converge, while negative energy fluctuations cause divergence. In the probability distribution (4), most fluctuations are negative. But these are bounded below, while there is an infinite tail of positive fluctuations. When a large enough positive fluctuation occurs, the system can no longer recover: the nonlinear term in the Raychaudhuri equation (14) becomes dominant, forcing the expansion \( \theta \) to \(-\infty\).

Here the analysis is more subtle due to the time and velocity dependence of the energy fluctuations in Eq. (15). Nevertheless, the qualitative behavior remains the same, and for velocities \( v \) near the speed of light, the quantitative behavior—the exponential distribution of collapse times shown in Fig. 2(e)—matches our earlier results. As discussed in Ref. [1], this behavior may have important implications for the small scale structure of spacetime: it is akin to “asymptotic silence” [9], a phenomenon in
which nearby points become causally disconnected from each other.

FIG. 2: Plots (a), (c) and (e) show the relative frequency of the collapse times for $v = 0$ (stationary particles), $v = 0.8$ and $v = 0.999999999$ (ultra-relativistic particles), respectively. Notice the different horizontal scale in (e). For each value of $v$, $10^7$ numerical experiments were performed. Plot (e) also shows that, as $v \to 1$, the histogram is well fitted by the probability distribution $1/t_f \exp (-\tau/t_f)$. In (b), (d) and (f) we show typical trajectories of $\theta(\tau)$ for $v = 0$, $v = 0.8$ and $v = 0.999999999$, respectively. In each case, an initial spreading caused by negative energy fluctuations ends abruptly when a rare large positive fluctuation induces a collapse ($\theta \to -\infty$).

**Acknowledgments**

S.C. received support from US Department of Energy grant DE-FG02-91ER40674. R.A.M. and J.P.M.P. acknowledge support from FAPESP grant 2013/09357-9. J.P.M.P. also acknowledges support from FAPESP Grant No. 2016/07057-6.

[1] S. Carlip, R.A. Mosna and J.P.M. Pitelli, *Vacuum fluctuations and the small scale structure of spacetime*, Phys. Rev. Lett. **107**, 021303 (2011).

[2] N. Drago and N. Pinamonte, *Influence of quantum matter fluctuations on geodesic deviation*, J. Phys. A **47**, 375202 (2014).

[3] H.S. Vieira, L.H. Ford and V.B. Bezerra, *Spacetime geometry fluctuations and geodesic deviation*, Phys. Rev. D **98**, 086001 (2018).

[4] C.J. Fewster, L.H. Ford and T.A. Roman, *Probability distributions of smeared quantum stress tensors*, Phys. Rev. D **81**, 121901 (2010).
[5] C.J. Fewster and S. Hollands, *Probability distributions for the stress tensor in conformal field theories*, arXiv:1805.04281 [math-ph].

[6] J. Gegenberg, G. Kunstatter and D. Louis-Martinez, *Observables far two-dimensional black holes*, Phys. Rev. D 51, 1781 (1995).

[7] M. Navarro, *Symmetries in two-dimensional dilaton gravity with matter*, Phys. Rev. D 56, 7792 (1997).

[8] Wolfram Research, Inc., Mathematica 11 (Champaign, IL, 2017).

[9] J.M. Heinzle, C. Uggla, and N. Rohr, *The cosmological billiard attractor*, Adv. Theor. Math. Phys. 13, 293 (2009).