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Transition electromagnetic form factor in the Minkowski space Bethe-Salpeter approach

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Abstract Using the solutions of the Bethe-Salpeter equation in Minkowski space for bound and scattering states found in previous works, we calculate the transition electromagnetic form factor describing the electro-disintegration of a bound system.

Keywords Bethe-Salpeter equation · Minkowski space · Transition electromagnetic form factor

1 Introduction

During the last few years there has been a renaissance of the Bethe-Salpeter (BS) approach [1] in its original Minkowski space formulation. It is caused by the progress in finding new methods which allow to overcome the difficulties resulting from the singularities of the BS equation and to obtain solution in the Minkowski space both on- and off-mass shell. The on-shell solution, found few decades ago [2, 3], allows to compute the nucleon-nucleon (NN) scattering phase shifts and parametrize the NN-interaction. The off-shell solution for the bound [4] and scattering states [5, 6, 7, 8] is much more recent and allows to properly calculate the electromagnetic elastic [9, 10] and transition form factors describing the electro-disintegration of a bound system (e.g. the deuteron). Note that the Euclidean solution cannot be used for that purpose, since the Wick rotation cannot be done in the BS equation itself, but not in the integral expression giving the form factors [10].

The full BS program has been achieved for a NN system interacting via separable kernel [11] since in this case, the singular integrals – both in the equation and in the form factors – are performed analytically. However it is not yet fully realized for a field-theoretically inspired description. For one-boson exchange kernel the analytical treatment of singularities is also possible due to the fact that the bound state BS solution is found in the form of the Nakanishi integral representation. In this case one can also calculate the integrals analytically and express the form factor [9] through the non-singular Nakanishi weight function [12]. For the scattering states the development of this method (based on the Nakanishi representation [12]) is in progress [13] but did not yet enter in the stage of numerical calculations. On the other hand, we have recently developed a different approach [5, 6, 7, 8] based on the direct treatment of the singularities and which allows to find both the bound and scattering state solutions.

The direct treatment of singularities is required also when calculating the transition form factors. In the present work we briefly explain our approach to this problem and present the first numerical

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results. In absence, for the time being, of scattering state solutions in the Nakanishi form, our method is the only one that allows to calculate this quantity. We consider in this contribution only the case of spinless particles. The non-zero spin case complicates the algebra but do not change the singularities of the propagators.

2 Calculating form factor

![Feynman diagram for the EM transition form factor.](image)

Let us start with the expression for the transition current, represented graphically in figure 1

\[ \tilde{J}_\mu = i \int \frac{d^4k}{(2\pi)^4} \frac{(p_\mu + \tilde{p}_\mu - 2k_\mu)\Gamma_i (\frac{q}{2}p - k, p)\Gamma_f (\frac{q}{2}p' - k, p')}{(k^2 - m^2 + i\varepsilon)[(p - k)^2 - m^2 + i\varepsilon][(p' - k)^2 - m^2 + i\varepsilon]} \]  

(1)

The plane wave contribution will not be considered in this short presentation.

The quantities \( \Gamma_i \) and \( \Gamma_f \) are respectively the initial (bound state) and final (scattering) vertex functions. \( \Gamma_f \) is the half-off-shell S-wave scattering BS amplitude found in [6]. Like any scattering amplitude, \( \Gamma_f \) depends on the initial and final four-momenta satisfying the conservation law. Therefore, the expression (1) depends on the four-momenta \( p, p', q \) (excluding \( q \) from the conservation law) or alternatively on \( p, p' = p_1 + p_2 \) and \( p_s = (p_1 - p_2)/2 \). For a given partial wave, say S-wave, we average it over the angles of the vector \( s \) in the c.m. frame of the final state where \( p' = 0 \). After that the four-vector \( p_s \) does not enter in the decomposition of \( \tilde{J}_\mu \), though the result depends on the module \( |p_s| \) which we will still denote as \( p_s \). It determines the final state mass \( m_f = 2\sqrt{m^2 + p_s^2} \). Therefore we have in our disposal the four-vectors \( p, p' \) only.

Notice also that in the approximation given by fig. [1] restricted to the one-body electromagnetic current for the interaction of photons with constituents, the current \( \tilde{J}_\mu \) is not conserved, that is \( (p' - p)_\mu \tilde{J}_\mu \neq 0 \). It can be decomposed as:

\[ \tilde{J}_\mu = (p_\mu + \tilde{p}_\mu)F(Q^2) + (p'_\mu - p_\mu)F'(Q^2) \]

(2)

In this situation, one can chose one of the two following strategies: (i) find both \( F \) and \( F' \) and use the current (2), in spite of its non-conservation, to calculate the electro-disintegration cross section; (ii) start with (2) and construct the conserved current

\[ J_\mu = \tilde{J}_\mu - \frac{q_\mu}{q^2}(q\cdot \tilde{J}) = \left[(p_\mu + p'_\mu) + \frac{(M_f^2 - M_i^2)}{Q^2}(p'_\mu - p_\mu)\right] F(Q^2) \]

(3)

which satisfies to \( q\cdot J = 0 \) (\( q = p' - p, q^2 = (p' - p)^2, Q^2 = -q^2 \)). Due to the constraint \( q\cdot J = 0 \), the current \( J_\mu \) is determined only by the form factor \( F(Q^2) \), and this is the quantity that will be calculated below. The form factor \( F'(Q^2) \), if at all necessary, can be obtained analogously.

It is convenient to carry out the calculations in the system of reference where \( p'_0 = p_0 \) (i.e. \( q_0 = 0 \)) and \( p \) and \( p' \) are collinear (i.e., they are either parallel or anti-parallel to each other, depending on the \( Q^2 \) value). In the elastic case, this system coincides with the Breit frame \( p + p' = 0 \), where \( |p| = |p'| \) and \( p'_0 = p_0 \). In the inelastic case, since one still has \( p'_0 = p_0 \), the three-momenta are different \( |p| \neq |p'| \). This frame can be obtained by performing a Lorentz transform from the Breit frame.

Using this reference frame, the form factor \( F(Q^2) \) can be computed by taking the zero component of the current:

\[ J_0 = \tilde{J}_0 = 2p_0F(Q^2) \]
That is:

\[ F(Q^2) = \frac{i}{p_0} \int \frac{d^4k}{(2\pi)^4} \frac{(p_0 - k_0)\Gamma_i \left( \frac{i}{2}p - k, p \right) \Gamma_f \left( \frac{i}{2}p' - k, p' \right)}{(k_0^2 - E_k^2 + i\epsilon)(p_0^2 - E_p^2 + i\epsilon)(p_0^2 - E_{p'}^2 + i\epsilon)}, \tag{4} \]

where \( E_k = \sqrt{m^2 + k^2} \) is the on-shell energy (and similarly for other energies).

This 4D integral is in practice a 3D one, since the azimuthal integration gives simply a factor 2. To calculate it numerically, we treat the singularities in a similar way to what was done for solving the scattering states BS equation [6]. Namely, we represent each of three propagators as the sum of the three propagators as the sum of logarithmic or principal value (PV) integrals and the \( \delta \)-function contributions, that is, symbolically:

\[ F(Q^2) = \int (PV + i\pi\delta)(PV + i\pi\delta)(PV + i\pi\delta) \ldots \tag{5} \]

The non-zero contributions result from the terms containing two, one and zero \( \delta \)-functions. Performing explicit integrations of the \( \delta \)-functions, we get, correspondingly, the sum of 1D, 2D and 3D PV integrals. To treat the pole singularities in the \( k_0 \) variable in the PV integrals, we introduce a function \( g(k_0) \) which has the same pole singularities – and only them – and the same residues as the integrand of (4), once (5) being inserted. We perform usual subtraction technique in the integrand with \( g(k_0) \) in such a way that the difference is smooth function and can be safely integrated numerically. Since the function \( g(k_0) \) is the sum of a few pole terms only, its integral is calculated analytically. All other singularities (the vertex functions \( \Gamma_i \), or in \( k \) variable, after integration over \( k_0 \)) are logarithmic or weaker and therefore can be also integrated numerically, though with some care. In this way we can properly treat the singularities in the integral (4) and calculate the transition form factor. The details of this calculation will be given in a subsequent publication [14].

As a test of our calculation we have considered the case \( \Gamma_i = \Gamma_f = 1 \). We used the two-parameters Feynman parametrization to calculate analytically the 4D integral (4). The resulting integral is calculated analytically over one Feynman parameter and numerically over the second one. We took small value of \( \epsilon \) in the denominator of (4) and obtained a stable result relative to variation of \( \epsilon \). In this way, we reproduced both real and imaginary parts of the transition form factor vs. \( Q^2 \) found by the method described above. Taking \( M_f = M_i \), we reproduced the (real) elastic form factor.

![Fig. 2](image1.png) Real (dashed) and imaginary (dot-dashed) parts of the transition form factor \( F(Q^2) \) for \( p_s = 0.1 \).

![Fig. 3](image2.png) The same as in fig. 2 for \( p_s = 0.5 \).

3 Numerical results

We present the results corresponding to the one-boson exchange model with the constituent mass \( m = 1 \), exchange boson mass \( \mu = 0.5 \) and the coupling constant \( \alpha = 1.437 \), providing a bound state with the mass \( M_i = 1.99 \). The initial bound state BS amplitude \( \Gamma_i \) and the final scattering state...
amplitudes $T_f$ have been obtained by the method described in [8]. Using these solutions, the transition form factor $F(Q^2)$ was calculated by the methods presented above. Its real and imaginary parts vs. $Q^2$ for the parameters $M_i = 1.99$ and $M_f = 2.01$ ($p_s = 0.1$) are shown in fig. 2. For these kinematical parameters the form factor is almost real (like the elastic form factor) since the final mass is very close to the initial one.

For $p_s = 0.5$ ($M_f = 2\sqrt{p_s^2 + m^2} = 2.236$) and for the same values of other parameters the transition form factor is shown in fig. 3. Now, for considerably larger inelasticity (i.e., for larger effective final state mass), the imaginary part of form factor is comparable with its real part. In our calculations, we have no restrictions for the values of momentum transfer $Q^2$ and the final state mass $M_f$. Several aspects remain to be clarified, like e.g. the relative importance of the plane wave graph and the dependence on the binding energy of the initial state.

4 Conclusion

The solutions of Bethe-Salpeter equation in the natural Minkowski metric are difficult by the singularities presented in the integral equation and in the solution itself. The euclidean solutions are much more easy to get but they are not suitable for obtaining the scattering amplitudes as well as the electromagnetic and/or transition form factors due to the impossibility to perform the Wick rotation in the corresponding integral expressions.

A series of works have been undertaken in the last years to overcome this difficulty. They had the aim of safely obtaining the Minkowski solutions for bound and scattering states [3, 6; 7, 8; 13] and compute the elastic electromagnetic form factor [4; 10]. This contribution is inserted in this effort and presents the first results on the transition electromagnetic form factor.

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