Focal schemes to families of secant spaces to canonical curves

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Abstract For a general canonically embedded curve $C$ of genus $g \geq 5$, let $d \leq g - 1$ be an integer such that the Brill–Noether number $\rho(g,d,1) = g - 2(g - d + 1) \geq 1$. We study the family of $d$-secant $\mathbb{P}^{d-2}$'s to $C$ induced by the smooth locus of the Brill–Noether locus $W^d_d(C)$. Using the theory of foci and a structure theorem for the rank one locus of special 1-generic matrices by Eisenbud and Harris, we prove a Torelli-type theorem for general curves by reconstructing the curve from its Brill–Noether loci $W^d_d(C)$ of dimension at least 1.

Key words: Focal scheme, Brill-Noether locus, Torelli-type theorem

Subject Classifications: 14H51, 14M12, 14C34

1 Introduction and motivation

For a general canonically embedded curve $C$ of genus $g \geq 5$ over $\mathbb{C}$, we study the local structure of the Brill–Noether locus $W^d_d(C)$ for an integer $\left\lceil \frac{g+3}{2} \right\rceil \leq d \leq g - 1$. Our main object of interest is the focal scheme associated to the family of $d$-secant $\mathbb{P}^{d-2}$'s to $C$. The focal scheme arises in a natural way as the degeneracy locus of a map of locally free sheaves associated to a family of secant spaces to a curve. In other words, the focal scheme (or the scheme of first-order foci) consists of all points where a secant intersects its infinitesimal first-order deformation.

In [5] and [6], Ciliberto and Sernesi studied the geometry of the focal scheme associated to the family of $(g-1)$-secant $\mathbb{P}^{g-3}$'s induced by the singular locus $W^1_{g-1}(C)$ of the theta divisor, and they gave a conceptual new proof of Torelli’s theorem. Using higher-order focal schemes for general canonical curves of genus $g = 2m + 1$, they showed in [7] that the family of $(m+2)$-secants induced by
\textbf{2 The theory of foci}

We recall the definition as well as the construction of the family of \(d\)-secant \(\mathbf{P}^{d-2}\)'s induced by an open dense subset of \(C_d^{1}\). Afterwards we introduce the characteristic or focal map and define the scheme of first-order foci of rank \(k\) associated to the above family. We give a slightly generalised definition of the scheme of first- and second-order foci compared to [6]. In Section 2.2 we recall the basic properties of the scheme of first-order foci. Our approach follows [8].

\textbf{2.1 Definition of the scheme of first-order foci}

Let \(C\) be a Brill–Noether general canonically embedded curve of genus \(g \geq 5\), and let \(d \leq g - 1\) be an integer such that the Brill–Noether number \(\rho := \rho(g,d,1) = \frac{g + 3}{2}\), and \(g\) odd. The article [2] of Bajravani can be seen as a first extension of the previous results to another Brill–Noether locus \((g = 8\) and \(d = 6 = \lceil \frac{g + 3}{2} \rceil\)). Combining methods of [6], [7] and [8], we will give a unified proof which shows that the canonical curve is contained in the focal schemes parametrised by the smooth locus of any \(W_d^1(C)\) if \(d \leq g - 1\) and \(\rho(g,d,1) = g - 2(g - d + 1) \geq 1\). Moreover, we have the following Torelli-type theorem (see also Corollary 4).

\textbf{Theorem 1. A general canonically embedded curve of genus} \(g\) \textbf{can be reconstructed from its Brill–Noether locus} \(W_d^1(C)\) \textbf{if} \(\lceil \frac{g + 3}{2} \rceil \leq d \leq g - 1\).

In [12], G. Pirola and M. Teixidor i Bigas proved a generic Torelli-type theorem for \(W_d^r(C)\) if \(\rho(g,d,r) \geq 2\), or \(\rho(g,d,r) = 1\) and \(r = 1\). Whereas they used the global geometry of the Brill–Noether locus to recover the curve, our theorem is based on the local structure around a smooth point of \(W_d^1(C) \subset W_d(C)\). Only first-order deformations are needed.

Our proof follows [7]. We show that the first-order focal map is in general 1-generic and apply a result of D. Eisenbud and J. Harris [9] in order to describe the rank one locus of the focal matrix. Two cases are possible. The rank one locus of the focal matrix consists either of the support of a divisor \(D\) of degree \(d\) corresponding to a line bundle \(\mathcal{O}_C(D) \in W_d^1(C)\) or of a rational normal curve. Even if we are not able to decide which case should occur on a general curve (see Section 4 for a discussion), we finish our proof by studying focal schemes to a family of rational normal curves induced by the first-order focal map.
Let $C^1_d$ be the variety parametrising effective divisors of degree $d$ on $C$ moving in a linear system of dimension at least 1 (see [11, IV, §1]). Let $\Sigma \subset W_d^1(C)$ be the smooth locus of $W_d^1(C)$. Furthermore, let $\alpha_d : C^1_d \to W_d^1(C)$ be the Abel-Jacobi map (see [11, I, §3]) and let $S = \alpha_d^{-1}(\Sigma)$. Then $\alpha : S \to \Sigma$ is a $P^1$-bundle, and in particular $S$ is smooth of pure dimension $p + 1$. For every $s \in S$, we denote by $D_s$ the divisor of degree $d$ on $C$ defined by $s$ and $\Lambda_s = D_s \subset P^{s-1}$ its linear span, which is a $d$-secant $P^{d-2}$ to $C$. We get a $(p + 1)$-dimensional family of $d$-secant $P^{d-2}$s parametrised by $S$:

$$\Delta \subset S \times P^{s-1} \xrightarrow{q} P^{s-1}$$

We denote by $f : \Delta \to P^{s-1}$ the induced map.

**Construction 1 (of the family $\Delta$).** Let $D_d \subset C_d \times C$ be the universal divisor of degree $d$ and let $D_S \subset S \times C$ be its restriction to $S \times C$. We denote by $\pi : S \times C \to S$ the projection. We consider the short exact sequence

$$0 \to \mathcal{O}_{S \times C} \to \mathcal{O}_{S \times C}(D_S) \to \mathcal{O}_{D_s}(D_S) \to 0.$$  

By Grauert’s Theorem, the higher direct image $R^1\pi_* (\mathcal{O}_{D_s}(D_S)) = 0$ vanishes and we get a map of locally free sheaves on $S$

$$R^1\pi_* (\mathcal{O}_{S \times C}) \to R^1\pi_* (\mathcal{O}_{S \times C}(D_S)) \to 0$$

whose kernel is a locally free sheaf $\mathcal{F} \subset R^1\pi_* (\mathcal{O}_{S \times C}) \cong \mathcal{O}_S \otimes H^1(C, \mathcal{O}_C)$ of rank $d - 1 = g - (g - d + 1)$. The family $\Delta$ is the associated projective bundle

$$\Delta = P(\mathcal{F}) \subset S \times P^{s-1}.$$  

**Remark 1.** We can also construct the family $\Delta$ from the Brill-Noether locus $W_d(C)$ and its singular locus $W_d^1(C)$. At a singular point $L \in W_d^1(C) \setminus W_d^2(C)$, the projectivised tangent cone to $W_d^1(C)$ at $L$ in the canonical space $P^{s-1}$ coincides with the scroll

$$X_L = \bigcup_{D \in |L|} T_D$$

swept out by the pencil $g^1_d = |L|$. Hence, the ruling of $X_L$ is the one-dimensional family of secants induced by $|L|$. Varying the point $L$ yields the family $\Delta$. See also [5, Theorem 1.2]. We conclude that the family $\Delta$ is determined by $W_d(C)$ and its singular locus $W_d^1(C)$.

In order to define the first-order focal map of the family $\Delta$, we make a short digression. We consider a flat family $F$ of closed subschemes of a projective scheme $X$ over a base $B$, that is,
Let $T(\pi_1)|_F := \pi_1^*(T_B)|_F$ be the tangent sheaf along the fibers of $\pi_2$ restricted to the family $F$ and let $\mathcal{N}_{F/B \times X}$ be the normal sheaf of $F \subset B \times X$. There is a map

$$\psi : T(\pi_1)|_F \to \mathcal{N}_{F/B \times X}$$

called the global characteristic map of the family $F$ which is defined by the following exact and commutative diagram:

$$
\begin{array}{c}
0 \\
\downarrow \\
T(\pi_1)|_F \\
\downarrow \psi \\
\mathcal{N}_{F/B \times X} \\
\downarrow \\
T_F \\
\downarrow \\
T_{B \times X}|_F \\
\downarrow d(\pi_2|_F) \\
(\pi_2|_F)^*(T_X) \\
\downarrow \\
\pi_2^*(T_X)|_F \\
\end{array}
$$

For every $b \in B$ the homomorphism $\psi$ induces a homomorphism

$$\psi_b : T_{B,b} \otimes \mathcal{O}_{\pi_1^{-1}(b)} \to \mathcal{N}_{\pi_1^{-1}(b)/X}$$

called the (local) characteristic map of the family $F$ at a point $b$. Since $F$ is a flat family, we get a classifying morphism

$$\varphi : B \to \text{Hilb}_Y$$

by the universal property of the Hilbert scheme $\text{Hilb}_Y$. The linear map induced by the characteristic map

$$H^0(\psi_b) : T_{B,b} \to H^0(\mathcal{N}_{\pi_1^{-1}(b)/X})$$

is the differential $d\varphi_b$ at the point $b$ (see also [11, p. 198 f]). Assuming that $B, Y$ and the family $F$ are smooth, all sheaves in the above diagram are locally free and by diagram-chasing, it follows that

$$\ker(d(\pi_2|_F)) = \ker(\psi) \quad \text{and} \quad \dim(\pi_2(F)) = \dim(F) - \text{rk}(\ker(\psi)).$$
We come back to the smooth family $\Lambda$ and fix some notation for the rest of the article. Let $N := N_\Lambda/S \times \mathbb{P}^{s-1}$ be the normal bundle of $\Lambda$ in $S \times \mathbb{P}^{s-1}$ and let $T(p)|_\Lambda := p^*(T_S)|_\Lambda$ be the restriction of the tangent bundle along the fibers of $q$ to $\Lambda$. Let

$$\chi : T(p)|_\Lambda \to N$$

be the global characteristic map defined as above. For every $s \in S$ the homomorphism $\chi$ induces a homomorphism

$$\chi_s : T_{S,s} \otimes \mathcal{O}_{\Lambda_s} \to N_{\Lambda_s}/\mathbb{P}^{s-1}$$

also called the characteristic map or first-order focal map of the family $\Lambda$ at a point $s$.

**Remark 2.** Fix an $s \in S$. We have $\Lambda_s = \mathbb{P}(U)$, where $U \subset V = H^1(C, \mathcal{O}_C)$ is a vector subspace of dimension $d - 1$. The normal bundle of $\Lambda_s$ in $\mathbb{P}^{s-1}$ is given by

$$N_{\Lambda_s}/\mathbb{P}^{s-1} = V/U \otimes \mathcal{O}_{\Lambda_s}(1)$$

and

$$H^0(\Lambda_s, N_{\Lambda_s}/\mathbb{P}^{s-1}) = \text{Hom}(U, V/U).$$

The characteristic map is of the form

$$\chi_s : T_{S,s} \otimes \mathcal{O}_{\Lambda_s} \to V/U \otimes \mathcal{O}_{\Lambda_s}(1).$$

Hence, it is given by a matrix of linear form on $\Lambda_s$.

We define the first- and the second-order foci (of rank $k$) of a family $\Lambda$.

**Definition 1.**

(a) Let $V(\chi)_k$ be the closed subscheme of $\Lambda$ defined by

$$V(\chi)_k = \{ p \in \Lambda \mid \text{rk}(\chi(p)) \leq k \}.$$

Then, $V(\chi)_k$ is the scheme of first-order foci of rank $k$ and the fiber of $V(\chi)_k$ over a point $s \in S$

$$(V(\chi)_k)_s = V(\chi_s)_k \subset \Lambda_s$$

is the scheme of first-order foci of rank $k$ at $s$.

(b) Assume that $V(\chi)_k$ induces a family of rational normal curves $\Gamma$, that is, for a general $s \in S$ the fiber $\Gamma_s$ is a rational normal curve. Let $\psi$ be the global characteristic map of $\Gamma$. We call the first-order foci of rank $k$ of the family $\Gamma$, that is,

$$V(\psi)_k = \{ p \in \Gamma \mid \text{rk}(\psi(p)) \leq k \}$$

the second-order foci of rank $k$ of the family $\Lambda$.

**Remark 3.** Our definition of scheme of first-order foci is a slight generalisation of the definition given in [5, 6]. Note that if
we get the classical definition of first-order foci. Furthermore, our definition of the second-order foci of rank \(k\) is inspired by the definition of higher-order foci of [8].

**Remark 4.** (a) The equality \(V(\chi)_1 = V(\chi_s)_1\) is shown in [4, Proposition 14].
(b) If \(\chi\) has maximal rank, that is, if \(\chi\) is either injective or has torsion cokernel, then \(V(\chi)_k\) is a proper closed subscheme of \(\Lambda\) for \(k \leq \min\{\text{rk}(T(p)|_\Lambda), \text{rk}(\mathcal{M})\} - 1\).
(c) In Section 3 we study the scheme of first-order foci of rank 1 of the family \(\Lambda\).

### 2.2 Properties of the scheme of first-order foci

We assume in this section that \(C\) is a Brill–Noether general curve. The following proposition is proven in [6] which can be easily generalised to the case of divisors of degree \(d < g - 1\).

**Proposition 1.** For \(s \in S\), we have

\[ D_s \subset V(\chi_s)_1. \]

In particular, the canonical curve \(C\) is contained in the scheme of first-order foci.

**Proof.** Let \(p \in \text{Supp}(D_s)\). Then there exists a codimension 1 family of effective divisors and hence \(d\)-secants containing the point \(p\). Therefore, there is a codimension 1 subspace \(T \subset T_{S,s}\) such that the map \(\chi_s(p)|_T\) is zero. We conclude that the focal map \(\chi_s\) has rank at most 1 in points of \(\text{Supp}(D_s)\). \(\square\)

An important step in the proof of our main theorem is to show that the first-order focal map \(\chi_s\) is 1-generic. The general definition of 1-genericity can be found in [10]. In our case, a reformulation of the definition is the following.

**Proposition 2.** The matrix \(\chi_s\) is 1-generic if and only if for each nonzero element \(v \in T_{S,s}\), the homomorphism

\[ H^0(\chi_s)(v) \in \text{Hom}(U, V/U) \]

is surjective.

We recall what is known about the 1-genericity of the matrix \(\chi_s\).

**Proposition 3 ([6, Theorem 2.5], [7, Theorem 2], [2]).** Let \(s \in S\) be a general point.

(a) If \(D_s\) is a divisor of degree \(g - 1\) cut on \(C\) by \(\Lambda_s\), then the matrix \(\chi_s\) is 1-generic (equivalently, \(V(\chi_s)_1\) is a rational normal curve) if and only if the pencil \(|D_s|\) is base point free.
(b) If \(p = \rho(g, d, 1) = 1\), then the matrix \(\chi_s\) is 1-generic (equivalently, \(V(\chi_s)_1\) is a rational normal curve).
(c) If \( g = 8 \) and \( d = 6 \), then the matrix \( \chi_s \) is 1-generic.

**Remark 5 ([13, p. 253]).** Another fact related to the 1-genericty of \( \chi_s \) is the following: Let
\[
\Delta_s \subset \text{Spec}(C[\varepsilon]) \times \mathbb{P}^{g-1}
\]
be the first order deformation of \( \Lambda_s \) defined by \( H^0(\chi_s)(v) \) for a vector \( v \in T_{S,s} \). Then, \( H^0(\chi_s)(v) \) is surjective if and only if \( q(\Delta_s) \subseteq \mathbb{P}^{g-1} \) is not contained in a hyperplane. Furthermore, the definition of the first-order foci at a point \( s \in S \) depends only on the geometry of the family \( \Delta_s \) in a neighbourhood of \( s \). A point in \( V(\chi_s)_k \) is a point where the fiber \( \Lambda_s \) intersects a codimension \( k \) family of its infinitesimally near ones.

### 3 Proof of the main theorem

The strategy of the proof is the same as in [7]. We assume that the canonically embedded curve \( C \) is a Brill–Noether general curve. Recall that \( g \) and \( d \) are chosen such that the Brill–Noether number \( \rho := \rho(g, d, 1) \geq 1 \). We begin by showing some standard properties of a line bundle over a Brill–Noether general curve which we will use later on. Then we prove that the matrix \( \chi_s \) is 1-generic for general \( s \in S \) and study the rank one locus of \( \chi_s \) which will be the divisor \( D_s \) or a rational normal curve. In the second case, we study the second-order focal locus. In both cases we can recover the canonical curve.

**Lemma 1.** Let \( C \) be a Brill–Noether general curve and let \( L \in W^1_d(C) \) be a smooth point. Then \( |L| \) is base point free, \( H^1(C, L^2) = 0 \) and \( g^{\rho+2}_{2d} = |L^2| \) maps \( C \) birational to its image (it is not composed with an involution).

**Proof.** All of our claims follow directly from the generality assumption. We just mention that the map induced by \( |L^2| \) can not be composed with an irrational involution. Hence, if the map is not birational, it is composed with a \( g^{d'}_{d''} \) for \( d' \leq \frac{2d^2}{\rho^2} \) which is impossible for a Brill–Noether general curve. \( \square \)

**Corollary 1.** Let \( C \) be a Brill–Noether general curve and let \( L \in W^1_d(C) \) be a smooth point. For \( i \geq 1 \) and \( p_1, \ldots, p_i \in \text{Supp}(D) \) for \( D \in |L| \) general, we have
\[
H^0(C, L^2(-p_1 - \cdots - p_i)) = 2d - i + 1 - g.
\]

In particular, \( H^0(C, L^2(-p_1 - \cdots - p_{\rho+1})) = H^0(C, L) \) and \( H^1(C, L^2(-p_1 - \cdots - p_i)) = 0 \) for \( i = 1, \ldots, \rho + 1 \).

**Proof.** \( H^0(C, L^2(-p_1 - \cdots - p_i)) = H^0(C, L^2(-p_1 - \cdots - p_{\rho+1})) \) if the images under \( |L^2| \) of the two points \( p_i \) and \( p_{i+1} \) are the same point. Since \( |L^2| \) maps \( C \) birational to its image, this does not happen for a general choice. \( \square \)
Using Lemma\[\text{[1]}\] and Corollary\[\text{[1]}\] our proof of the following lemma is identical to \[\text{[7]}\] Theorem 2. We clarify and generalise the arguments given in \[\text{[7]}\] Theorem 2.

**Lemma 2.** With the assumptions of Lemma\[\text{[7]}\] the focal matrix $\chi_s : T_{S,s} \otimes \mathcal{O}_{A_s} \to \mathcal{N}_{A_s}/P^{-1}$ is 1-generic for a sufficiently general $s \in S$.

**Proof.** By Proposition\[\text{[2]}\] the matrix $\chi_s$ is 1-generic if and only if for each nonzero element $v \in T_{S,s}$, the homomorphism $H^0(\chi_s)(v) \in \text{Hom}(U, V/U)$ is surjective.

We consider the first order deformation $\mathcal{A}_s \subset \text{Spec}(k[\varepsilon]) \times P^{s-1}$ defined by $H^0(\chi_s)(\theta)$ for a nonzero vector $\theta \in T_{S,s}$. Note that $H^0(\chi_s)(\theta)$ is surjective if and only if the image $q(\mathcal{A}_s) \subset P^{s-1}$ is not contained in a hyperplane. Let $D_e \subset \text{Spec}(k[\varepsilon]) \times P^{s-1}$ be the first order deformation of the divisor $D_e$ defined by $\theta \in T_{S,s}$. Then

$$q(\mathcal{A}_s) \supset q(D_e)$$

and the curvilinear scheme $q(D_e)$ corresponding to a divisor on $C$ satisfies

$$D_s \leq q(D_e) \leq 2D_s.$$  

We show for all possible cases that $q(D_e)$ is not contained in a hyperplane.

**Case 1:** The vector $\theta$ is tangent to $\alpha_d^{-1}(L)$, equivalently the family $D_e$ deforms the divisor $D_s$ in the linear pencil $|L|$. Let $\phi_L$ be the morphism defined by the pencil. Then we get

$$q(D_e) = \phi_L(\theta),$$

where we identify $\theta$ with a curvilinear scheme of $P^1$ supported at the point $s \in P^1$. Since $|L|$ is base point free, we have $q(D_e) = 2D_s$. Therefore, the curvilinear scheme $q(D_e)$ is not contained in a hyperplane since $H^0(C, K_C - 2D_s) = H^1(C, 2D_s) = 0$. We are done in this case.

**Case 2:** We assume that $\theta \in T_{S,s} \setminus \{0\}$ is not tangent to $\alpha_d^{-1}(L)$ at $s$. Let

$$q(D_e) = p_1 + \cdots + p_k + 2(p_{k+1} + \cdots + p_d)$$

where $D_s = p_1 + \cdots + p_d$ and $k \geq 0$.

**Case 2 (a):** We assume $k \leq \rho$. We have

$$H^0(C, K_C - q(D_e))^* = H^1(C, p_1 + \cdots + p_k + 2(p_{k+1} + \cdots + p_d))$$

$$= H^1(C, 2D_s - p_1 - \cdots - p_k)$$

$$= H^1(C, L^2(-p_1 - \cdots - p_k)) = 0$$

by Corollary\[\text{[1]}\]. Hence, the curvilinear scheme $q(D_e)$ is not contained in a hyperplane and $H^0(\chi_s)(\theta)$ is surjective.

**Case 2 (b):** We assume $k \geq \rho + 1$. In the following, we will show that this case cannot occur. The vector $\theta$ is also tangent to $p_1 + \cdots + p_k + C_{d-k}$. We denote by $E_s$ the divisor $E_s = p_{k+1} + \cdots + p_d$. Then the tangent space to $p_1 + \cdots + p_k + C_{d-k}$ is
Given by $H^0(E_s, \mathcal{O}_{E_s}(D_s))$ which is a subspace of $H^0(D_s, \mathcal{O}_{D_s}(D_s))$. The short exact sequence

$$0 \to \mathcal{O}_C \to L \to \mathcal{O}_{D_s}(D_s) \to 0$$

induces a linear map

$$H^0(D_s, \mathcal{O}_{D_s}(D_s)) \xrightarrow{\delta} H^1(C, \mathcal{O}_C)$$

which we identify with the differential of $\alpha_d$ at $s$ (see [1] IV, §2, Lemma 2.3]). The image of $\theta \in H^0(E_s, \mathcal{O}_{E_s}(D_s))$ is therefore contained in the linear span of $E_s$. After projectivising, we get

$$[\delta(\theta)] \in \mathbb{P}_s = p_{k+1} + \cdots + p_d \subset \Lambda_v \subset \mathbb{P}^{g-1}.$$

Since $\theta$ is not tangent to $\alpha_d^{-1}(L)$, the vector $\theta$ is also tangent to $W_d^1(C)$ and therefore the image point $[\delta(\theta)]$ is contained in the vertex $V = T_L(W_d^1(C))$ of $X_L$, the scroll swept out by the linear pencil $L$. Hence, for every sufficiently general $D \in [L]$, there is an effective divisor $E$ of degree $d - \rho - 1$ such that $D = E + p_1 + \cdots + p_{\rho+1}$ and $V \cap E \neq \emptyset$. Hence, dim$(D_s + E) \leq d - 2 + d - \rho - 1$ and equivalently,

$$h^0(C, D_s + E) = \deg(D_s + E) + \dim(D_s + E) + 1 \geq 3.$$

But by Corollary [1] $H^0(C, L^2(-p_1 - \cdots - p_{\rho+1})) = H^0(C, L)$, a contradiction. \qed

Note that

$$\chi_s : T_{S, s} \otimes \mathcal{O}_{\Lambda_v} \to \mathcal{N}_{X_s/\mathbb{P}^{g-1}}$$

is a map between rank $\rho + 1$ and $n = h^1(C, L)$ vector bundles of linear forms in $\mathbb{P}^{d-2} = \Lambda_v$. Since $d = \rho + 1 + n$ and $\chi_s$ is 1-generic by Lemma 2 we may apply the following theorem due to Eisenbud and Harris.

**Theorem 2** ([9] Proposition 5.1). Let $M$ be an $(a + 1) \times (b + 1)$ 1-generic matrix of linear forms on $\mathbb{P}^{a+b}$. If $D_1(M) = \{ x \in \mathbb{P}^{a+b} \mid \rk(M(x)) \leq 1 \}$ contains a finite scheme $\Gamma$ of length $\geq a + b + 3$, then $D_1(M)$ is the unique rational normal curve through $\Gamma$ and $M$ is equivalent to the catalecticant matrix.

We get the following corollary.

**Corollary 2.** For $s \in S$ sufficiently general, the rank one locus $V(\chi_s)_1$ is either $D_s$ or a rational normal curve through $D_s$.

**Proof.** By Lemma 2 we may apply Theorem 2. Note that $D_s \subset V(\chi_s)_1$ (there exists a codimension 1 family in $S$ of $\Lambda_v$ containing a point of the support of $D_s$). \qed

**Remark 6.** (a) The scheme of first-order foci at $s \in S$ of the family $\Delta$ is a secant variety to $V(\chi_s)_1$.

(b) If $d = g - 1$ or $\rho = 1$, the focal matrix $\chi_s$ is a $2 \times (g - 3)$ or $n \times 2$-matrix, respectively. Hence, the rank one locus is the scheme of first-order foci, which is a rational normal curve in $\Lambda_v$. We recover the cases of [6] and [7].
Corollary 3. Let $C$ be a Brill–Noether general canonically embedded curve. If $V(\chi_s)_1 = D_s$ for sufficiently general $s \in S$, the family $\Delta$ determines the canonical curve $C$.

For the rest of this section, we assume that $\Gamma_s = V(\chi_s)_1$ is a rational normal curve for $s \in S$ sufficiently general.

Let $\Sigma$ be the smooth locus of $W_1^d(C)$ and $L \in \Sigma$. Let $U \subset \alpha_d^{-1}(L)$ be a Zariski open dense set such that $\Gamma_s = V(\chi_s)_1$ for all $s \in U$. We define the surface

$$\Gamma_L = \bigcup_{s \in U} \Gamma_s$$

and

$$\Gamma_{\mathbb{P}^{g-1}} = \bigcup_{L \in \Sigma} F_L.$$

Let

$$\bigcup \subset S' \times \mathbb{P}^{g-1} \xrightarrow{q} \mathbb{P}^{g-1}$$

be the family induced by all rational normal curves, that is, for $s \in S' \subset S$, $\Gamma_s = V(\chi_s)_1$ is a rational normal curve. The family $\bigcup$ is the rank one locus of the global characteristic map $\chi$ and the variety $\Gamma_{\mathbb{P}^{g-1}}$ is the image of the family $\bigcup$ under the second projection $q$.

Remark 7. In the cases $d = g - 1$ or $\rho = 1$ the rational surface $\Gamma_L$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ or a quadric cone in $\mathbb{P}^3$, respectively. This can be explained in terms of the curve $C$ and the line bundle $L$.

For $d = g - 1$ we consider the birational image $C'$ of $C \xrightarrow{|L| \times |\alpha \otimes L|^{-1}} \mathbb{P}^1 \times \mathbb{P}^1$ given by the line bundle $L$ and its Serre dual $\alpha \otimes L^{-1}$. Then the rational surface $\Gamma_L$ is the image of the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ along the singular points of $C'$ under the adjoint morphism.

For $\rho = 1$ we consider the birational image $C'$ of the curve $C$ in the quadric cone $Q$ in $\mathbb{P}^3$ induced by the line bundle $L^2$. Note that $H^0(C, L^2)$ is four-dimensional and the multiplication map $H^0(C, L) \otimes H^0(C, L) \rightarrow H^0(C, L^2)$ has a one-dimensional kernel. Then the rational surface $\Gamma_L$ is again the image of the blow up of $Q$ along the singular points of $C'$ under the adjoint morphism.

We have not found a similar geometrical meaning of the surface $\Gamma_L$ in the other cases (see also Question 1).

Lemma 3. The variety $\Gamma_{\mathbb{P}^{g-1}}$ has dimension at least 3.

Proof. Note that there is a map $\Gamma_L \rightarrow \mathbb{P}^1 = \alpha_d^{-1}(L)$ such that the general fiber is a rational curve. Hence, the surface $\Gamma_L$ is rational. Assume that $\Gamma_L = \Gamma_{L'}$ for all $L' \in \Sigma$. Since the scrolls $X_{L'}$ are algebraically equivalent to each other, the rulings on them
cut out a $(\rho + 1)$-dimensional family of algebraically equivalent rational curves on $\Gamma_L$, the focal curves. (We can also argue that all $d$-secant to $C$ are algebraically equivalent, thus the intersection with $\Gamma_L$ yields a $(\rho + 1)$-dimensional family of algebraically equivalent focal curves.) On the desingularization of $\Gamma_L$, all of them are linear equivalent since $\Gamma_L$ is regular ($H^1(\Gamma_L, \mathcal{O}_{\Gamma_L}) = 0$). This implies that all $g^1_d$'s on $C$ are linear equivalent, hence $C$ has a $g^1_d$. A contradiction to the generality assumption on $C$.

For the convenience of the reader, we recall the definition of the second-order foci of the family $\Lambda$ (see also Definition 1). We apply the theory of foci to the family $\Gamma_s \subset S' \times \mathbb{P}^{g-1}$ and get the characteristic map

$$\psi : T(p)_{|\Gamma_s} \rightarrow \mathcal{N}_{\Gamma_s/\mathbb{P}^{g-1}}$$

of vector bundles of rank $\rho + 1$ and $g - 2$, respectively. For $s \in S'$, we call the closed subscheme of $\Gamma_s$ defined by rank$(\psi_s) \leq k$ the scheme of second-order foci of rank $k$ at $s$ (of the family $\Lambda$).

We will show that the scheme of second-order foci of rank 1 at $s \in S'$ of the family $\Lambda$ is a finite scheme containing the divisor $D_s$ and compute its degree.

**Lemma 4.** Let $\psi_s : T_{S', s} \otimes \mathcal{O}_{\Gamma_s} \rightarrow \mathcal{N}_{\Gamma_s/\mathbb{P}^{g-1}}$ be the characteristic map for general $s \in S'$. Then the rank of $\psi_s$ at a general point of $\Gamma_s$ is at least 2.

**Proof.** We recall the connection of the rank and the dimension of $\Gamma_s$ as in [8, page 6]. Since dim$(\Gamma_{pe-1}) = \rho + 2 - \text{rank}(\ker(\psi))$, the rank of $\psi_s$ at the general point $p \in \Gamma_s$ is

$$\text{rank}(\psi_s(p)) = \text{dim}(T(p)_{|\Gamma_s}) - \text{rank}(\ker(\psi)) = \rho + 1 - \text{rank}(\ker(\psi)) = \text{dim}(\Gamma_{pe-1}) - 1.$$  

The lemma follows from Lemma 3. This fact is also shown in [4, page 98].

We now consider for a general $s \in S'$ the rank one locus of $\psi_s$ which is a proper subset of $\Gamma_s$ by Proposition 4.

**Lemma 5.** The degree of $V(\psi_s)_1 \subset \Gamma_s = V(\chi_s)_1$ is at most $d + \rho$.

**Proof.** We imitate the proof of [2, Theorem 3]. Let $s \in S' \subset C$ be a general point and let $\Gamma_s \subset \mathbb{P}^{d-2} = \Lambda_s$ be the rank 1 locus of the map

$$\chi_s : T_{S', s} \otimes \mathcal{O}_{\Lambda_s} \rightarrow \mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}}.$$  

Note that the normal bundle of $\Gamma_s$ splits

$$\mathcal{N}_{\Gamma_s/\mathbb{P}^{g-1}} = (\mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}} \otimes \mathcal{O}_{\Gamma_s}) \oplus \mathcal{N}_{\Gamma_s/\mathcal{A}_s} = \mathcal{O}_{\Gamma_s}(d - 2)^{\oplus n} \oplus \mathcal{O}_{\Gamma_s}(d)^{\oplus d - 3}.$$  

Hence, the map $\psi_s$ is given by a matrix.
\[ \psi_s = \begin{pmatrix} A \\ B \end{pmatrix} \]

where \( A \) is a \( n \times (\rho + 1) \)-matrix and \( B \) is a \( (d - 3) \times (\rho + 1) \)-matrix. The matrix \( A \) represents the map \( \chi_s : V_s \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{N}_{\mathbb{P}^1} \) and therefore has rank 1 and is equivalent to a catalecticant matrix. Let \( \{ s, t \} \) be a basis of \( H^0(\mathcal{O}_L(1)) \). In an appropriate basis, the matrix \( A \) is of the following form

\[
A = \begin{pmatrix}
t^{d-2} & t^{d-3}s & \cdots & t^{d-\rho} s^\rho \\
t^{d-3}s & \cdots & t^{d-\rho-1} s^\rho+1 \\
\vdots & \vdots & \ddots & \vdots \\
t^{d-2-n+1} s^1 & t^{d-2-n} s^n & \cdots & s^{n-2}
\end{pmatrix}
= \begin{pmatrix}
t^{n-1} & t^{n-1} s & \cdots & s^\rho \\
t^{n-1} s & \cdots & s^\rho+1 \\
\vdots & \vdots & \ddots & \vdots \\
s^{n-1} & t^{n-1} s & \cdots & s^\rho
\end{pmatrix}
\]

We see that the rank 1 locus of \( \psi_s \) is the rank 1 locus of the following matrix

\[
N = \begin{pmatrix}
t^\rho & t^{\rho-1} s & \cdots & s^\rho \\
B
\end{pmatrix}
\]

Since \( V(\psi_s)_1 \neq \Gamma_s \) by Lemma 4, we have

\[
\deg(V(\psi_s)_1) = \deg(D_1(N)) \leq \min \{ \text{degree of elements of } I_{2 \times 2}(N) \} \leq \rho + d.
\]

**Proposition 4.** Let \( s \in L \) be a sufficiently general point. Then, \( V(\psi_s)_1 \) is the union of \( D_s \) and \( \rho \) points which are the intersection of \( \Gamma_s = V(\chi_s)_1 \), and the vertex \( V \) of the scroll \( X_L \) swept out by the pencil \( |L| \).

**Proof.** As in the proof of Proposition 1, one can show that the points in the support of \( D_s \) are contained in \( V(\psi_s)_1 \).

Next, we show that the vertex in \( \Lambda_s \) is given by a column of the matrix \( \chi_s \). Again, we imitate the proof of [5, Proposition 4.2]. Each column of the \( n \times (\rho + 1) \)-matrix \( \chi_s \) is a section of the rank \( n \) vector bundle \( V/U \otimes \mathcal{O}_{\mathbb{P}^1}(1) \) (where \( U \subset V \) is the affine subspace representing \( \Lambda_s \)) corresponding to an infinitesimal deformation of \( \Lambda_s \). Each section vanishes in a \( \rho - 1 = (d - 2 - n) \)-subspace of \( \Lambda_s \) which is a \( \rho \)-secant of \( \Gamma_s \). Since \( \chi_s \) is 1-generic, we get a \( (\rho + 1) \)-dimensional family of infinitesimal deformations of \( \Lambda_s \) induced by all columns. Hence, one column corresponds to the deformation in the scroll \( X_L \). The corresponding section vanishes at the vertex. \( \square \)

As in the case \( V(\chi_s)_1 = D_s \), we get the following Torelli-type theorem using Remark 1.

**Corollary 4.** A Brill–Noether general canonically embedded curve \( C \) is uniquely determined by the family \( \Lambda \). More precise, the canonical curve \( C \) is a component of the scheme of first- or second-order foci of the family \( \Lambda \) induced by the Brill–Noether locus \( W_d(C) \) and (the smooth locus of) its singular locus \( W_d^1(C) \) of dimension at least one (equivalently \( \frac{2g-2}{d} \leq d \leq g - 1 \)).
4 The first-order focal map

For a general curve $C$ and a sufficiently general point $s \in S$, the rank one locus of the focal map $\chi_s$ at $s$ is either $d$ points or a rational normal curve. In the second case, the focal matrix at $s$ is catalecticant (see Corollary 2).

As mentioned above, the articles [6] and [7] of Ciliberto and Sernesi are the extremal cases ($d = g - 1$ and $\rho = 1$, respectively), where the rank one locus is always a rational normal curve. We propose the following question.

**Question 1.** When is the focal matrix $\chi_s$ catalecticant for a general curve $C$ and a sufficiently general point $s \in S$?

We conjecture that only in the extremal cases $d = g - 1$ and $\rho = 1$ the rank one locus of $\chi_s$ is a rational normal curve for a general curve $C$ and a general point $s \in S$. For the rest of this section we explain the reason for our conjecture.

Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus $g$ and let $L \in W_1^d(C)$ be a smooth point such that the rank one locus of the focal matrix $\chi_s: T_{S_s} \otimes \mathcal{O}_{L_s} \rightarrow \mathcal{O}_s \otimes \mathcal{O}_{L_s}$ is a rational normal curve $\Gamma_s$ in $\mathbb{P}^{d-2}$ for $s \in |L|$ sufficiently general. Let $X_L = \bigcup_{s \in |L|} \Gamma_s$ be the scroll swept out by the pencil $|L|$. We get a rational surface

$$\Gamma_L = \bigcup_{s \in |L| \, \text{gen}} \Gamma_s \subset X_L$$

defined as in the previous section. The rational normal curve $\Gamma_s$ intersects the vertex $V$ of $X_L$ in $\rho = \rho(g,d,1)$ points by Proposition 4. Note that the scroll $X_L$ is a cone over $\mathbb{P}^1 \times \mathbb{P}^{d_1(C,L)-1}$ with vertex $V$. Hence, projection from the vertex $V$ yields a rational surface in $\mathbb{P}^1 \times \mathbb{P}^{d_1(C,L)-1}$ whose general fiber in $\mathbb{P}^{d_1(C,L)-1}$ is again a rational normal curve. We have shown the following proposition.

**Proposition 5.** Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus $g$ and let $L \in W_1^d(C)$ be a smooth point such that the rank one locus of the focal matrix $\chi_s$ is a rational normal for $s \in |L|$ sufficiently general. Then, the image of $C$ in $\mathbb{P}^1 \times \mathbb{P}^{d_1(C,L)-1}$ given by $|L| \times |\omega_C \otimes L^{-1}|$ lies on a rational surface of bidegree $(d', h^1(C,L) - 1)$ for some $d'$.

**Proof.** The proposition follows from the preceding discussion. We only note that the map given by $|L| \times |\omega_C \otimes L^{-1}|$ is the same as the projection of $\mathbb{P}^{g-1}$ along the vertex $V$ of the canonically embedded $C$. \hfill \Box

**Example 1.** We explain the above circumstance for a curve $C$ of genus 8 with a line bundle $L \in W_6^1(C)$. The residual line bundle $\omega_C \otimes L^{-1}$ has degree 8 and $H^0(C, \omega_C \otimes L^{-1})$ is three-dimensional. Let $C'$ be the image of $C$ in $\mathbb{P}^1 \times \mathbb{P}^2$ given by $|L| \times |\omega_C \otimes L^{-1}|$. We think of $C' \rightarrow \mathbb{P}^1$ as a one-dimensional family of six points in the plane. If our assumption of Proposition 5 is true, the six points lie on a conic in every fiber over $\mathbb{P}^1$. Computing a curve of genus 8 with a $g_6^1$ in Macaulay2 shows that these conics do not exist. Hence, our assumption of Proposition 5 that is, the rank
one locus of the focal matrix $\chi_s$ is a rational normal curve for $s \in |L|$ sufficiently general, does not hold for a general curve.

If $\rho(g,d,1) = 2d - g - 2 \geq 2$ and $d < g - 1$, we do not expect the existence of such a rational surface for a curve of genus $g$ and a line bundle of degree $d$ as above. Indeed, $m$ general points in $\mathbb{P}^r$ do not lie on a rational normal curve if $m > r + 3$. But the inequality $\rho(g,d,1) = 2d - g - 2 \geq 2$ implies $d > (h^1(C,L) - 1) + 3$. Using our Macaulay2 package (see [3]), we could show in several examples $((g,d) = (8,6), (9,7), (10,8), (9,6))$ that the rational surface of bidegree $(d',h^1(C,L) - 1)$ of Proposition 5 does not exist. This confirms our conjectural behaviour of the first-order focal map.

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