Congruences involving alternating sums related to harmonic numbers and binomial coefficients

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Abstract: In 2017, Bing He investigated arithmetic properties to obtain various basic congruences modulo a prime for several alternating sums involving harmonic numbers and binomial coefficients. In this paper we study how we can obtain more congruences modulo a power of a prime number \( p \) (super congruences) in the ring of \( p \)-integer \( \mathbb{Z}_p \) involving binomial coefficients and generalized harmonic numbers.

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1 Introduction

In this paper we study how we can obtain more interesting congruences modulo a power of a prime number \( p \) (super congruences) in the ring of \( p \)-integer \( \mathbb{Z}_p \), involving binomial coefficients, harmonic numbers and generalized harmonic numbers. Many great mathematicians have been
interested to study congruences of sums concerning harmonic numbers and binomial coefficients such the work of Gould [4], Dilsher [2], Prodinger [10], Choi and Srivastava [1], Sun [13], Meštrović and Andjić [8], Wang [15] and other, in our work we are interested to find closed expressions for the sums of the form

$$\sum_{k=1}^{q} (-1)^{k+1} k^{l} \binom{n}{k} H_{k,m},$$

and congruences of the form

$$\sum_{k=1}^{p-1} (-1)^{k+1} k^{l} \binom{p-1}{k} H_{k} \pmod{p^3}, \quad \sum_{k=1}^{\left\lfloor \frac{q}{p} \right\rfloor} (-1)^{k+1} k^{l} \binom{p-1}{k} H_{k} \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} (-1)^{k+1} k^{l} \binom{np-1}{k} H_{k,m} \pmod{p^r}, \quad r \in \{2, 3, 4\},$$

for some values of $l, m$, where $H_{n,m}$ to be the $n$-th generalized harmonic number defined by

$$H_0 = 0, \quad H_{n,m} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}.$$

We denote

$$S(n, q, m) = \sum_{k=1}^{q} \frac{(-1)^k}{k^m} \binom{n}{k-1}.$$

The sequence $(S(n, q, m))$ satisfies

$$S(n, q, m - 1) = (n+1) S(n, q, m) - nS(n-1, q, m), \quad (1)$$

$$S(n, q, 0) = (-1)^q \frac{q}{n} \binom{n}{q}, \quad (2)$$

$$S(n, q, 1) = \frac{1}{n+1} \left( (-1)^q \binom{n}{q} - 1 \right), \quad (3)$$

$$S(n, q, m) = \frac{1}{n+1} \left( \sum_{j=1}^{n} S(j, q, m-1) - 1 \right). \quad (4)$$

In 2017, He [5] established the following congruences:

**Theorem 1.1** ([5]). Let $p > 3$ be a prime number. We have

$$\sum_{k=0}^{p-1} (-1)^{k+1} \binom{p-1}{k} H_{k,3} \equiv -\frac{1}{3} pB_{p-3} \pmod{p^2},$$

$$\sum_{k=0}^{p-1} (-1)^{k+1} \binom{p-1}{k} H_{k,2} \equiv \frac{1}{3} p^2 B_{p-3} \pmod{p^3},$$

where $B_n$ is the $n$-th Bernoulli number defined by

$$B_0 = 1, \quad B_n = \sum_{k=0}^{n-1} \binom{n}{k} B_k, \quad n \geq 1.$$
Motivated by the above work of He [5], we establish in this paper the following results.

**Theorem 1.2.** Let \( p \geq 3 \) be a prime, \( n \) be a positive integer and \( m, l \) be nonegative integers such \( m - l \in \{2, 3, \cdots, p - 2\} \). Then

\[
\sum_{k=0}^{p-1} (-1)^{k+1} \left( \binom{np - 1}{k} (k + 1)^l \right) H_{k,m} \equiv \left( n - 1 + \frac{1}{m - l} \right) pB_{p-m+l} \pmod{p^2},
\]

In particular, for \( l = 0, p > 3 \) and \( m \in \{2, \ldots, p - 2\} \) we get

\[
\sum_{k=1}^{p-1} (-1)^{k+1} \frac{(np - 1)}{k} H_{k,m} \equiv - \left( n - 1 + \frac{1}{m} \right) pB_{p-m} \pmod{p^2},
\]

and for \( l = 1 \) we get

\[
\sum_{k=0}^{p-1} (-1)^{k+1} \frac{(np - 1)}{k} H_{k,m} \equiv \left\{ \begin{array}{ll}
\frac{1}{2} \left( n - 1 + \frac{1}{m} \right) pB_{p-m} \pmod{p^2} & \text{if } m \text{ odd}, \\
-\frac{1}{2} \left( n - 1 + \frac{1}{m-1} \right) pB_{p+1-m} \pmod{p^2} & \text{if } m \text{ even}.
\end{array} \right.
\]

**Theorem 1.3.** Let \( p \geq 3 \) be a prime and \( m \) be a integer with \( m \in \{1, \ldots, \frac{p-3}{2}\} \). We have

\[
\sum_{k=0}^{p-1} (-1)^{k+1} \frac{(np - 1)}{k} H_{k,2m} \equiv \frac{A(m,n)}{2(2m+1)} p^2 p_{p-1-2m} \pmod{p^3},
\]

where

\[
A(m,n) = (m+1)(2m+1)n^2 + (m+1)(2m-1)n - 2m \cdot (2m+1).
\]

**Remark 1.** For \( m = 3 \) and \( n = 1 \) in (6) we obtain the congruence (1.2) given in [5, Theorem 1.2], and, for \( m = 1 \) and \( n = 1 \) in (8) we obtain the congruence (1.1) given in [5].

**Theorem 1.4.** Let \( p \) be a prime number and \( n \) be a positive integer. Then

1) if \( p \equiv 5 \), we have

\[
\sum_{k=1}^{p-1} (-1)^{k+1} \frac{(p - 1)}{k} H_{k} \equiv \left( q_2 - \frac{1}{2} \right) + \left( \frac{3}{2} q_2^2 + q_2 - \frac{1}{2} \right) p \\
+ \left( \frac{1}{3} q_2^3 + \frac{1}{2} q_2^2 + q_2 - \frac{1}{2} + \frac{7}{24} B_{p-3} \right) p^2 \pmod{p^3},
\]

2) if \( p \equiv 1 \pmod{3} \), we have

\[
\sum_{k=1}^{p-1} (-1)^{k+1} \frac{(np - 1)}{k} H_{k} \equiv \left( q_3 - \frac{1}{3} \right) + \left( \frac{3n - 1}{2} q_3^2 + \frac{n + 1}{2} q_3 - \frac{2n - 1}{3} + \frac{1}{9} B_{p-2} \left( \frac{1}{3} \right) \right) p \pmod{p^2}.
\]
and if \( p \equiv 2 \pmod{3} \), we have

\[
\sum_{k=1}^{p} (-1)^{k+1} \binom{np - 1}{k} H_k 
\equiv \frac{1}{2} q_3 - \frac{2}{3} + \left( \frac{3n - 1}{4} q_3 - \frac{n - 1}{2} q_3 - \frac{4n - 1}{3} - \frac{1}{18} B_{p-2} \left( \frac{1}{3} \right) \right) p \pmod{p^2},
\]

where \( q_a \) is the quotient of Fermat base \( a \) defined for a given prime number \( p \) by

\[
q_a = q_p (a) := \frac{a^{p-1} - 1}{p}, \quad a \in \mathbb{Z} - p\mathbb{Z},
\]

and \( \{ B_n(x) \} \) is the sequence of Bernoulli polynomials defined by

\[
B_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} B_{n-k} x^k, \quad n \geq 1.
\]

The next theorem is a generalization of Theorem 1.1 given [5].

**Theorem 1.5.** Let \( p > 5 \) be a prime number. We have

\[
\sum_{k=0}^{p} (-1)^{k+1} \binom{p - 1}{k} H_{k,2} \equiv \left( \frac{B_{2p-4}}{2p-4} - \frac{2}{p-3} \right) p^2 + \left( \frac{B_{2p-4}}{2p-4} - \frac{2}{p-3} \right)^3 \pmod{p^3}.
\]

**Corollary 1.5.1.** Let \( p \geq 5 \) be a prime number and \( n \) be a positive integer with \( n \not\equiv 0 \pmod{p} \). We have

\[
\sum_{k=0}^{p} (-1)^{k+1} \binom{n}{k} H_k \equiv \frac{1}{n} \left( 1 - \binom{n-1}{p-1} \right) \pmod{p^3}.
\]

## 2 Basic tools to prove the main theorems

In order to prove the main Theorems, we need some auxiliary results given by the following Lemmas.

**Lemma 2.1 ([11, Th. 5.1, Cor. 5.1]).** Let \( p > 3 \) be a prime. Then, for \( m \in \{1, 3, \ldots, p - 4\} \) being an odd number, we have

\[
\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv \frac{m(m+1)}{2} B_{p-2-m} p^2 \pmod{p^3}, \quad (9)
\]

and for \( m \in \{1, 2, \ldots, p - 2\} \), we have

\[
\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv \frac{mp}{m+1} B_{p-1-m} \pmod{p^2}. \quad (10)
\]

In particular, for \( m \in \{1, 2, \ldots, p - 2\} \), we have

\[
\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv 0 \pmod{p}. \quad (11)
\]
Lemma 2.2 ([13, Lem. 3.1]). Let \( n \) and \( l \) be positive integers with \( l \leq n - 1 \), and let \( p > n \) be a prime number. Then

\[
\sum_{k=1}^{p-1} \frac{H_k}{k^n} \equiv \begin{cases} 
\frac{(-1)^{l-1}}{n} \binom{n}{l} B_{p-n} \pmod{p}, & \text{if } n \text{ odd}, \\
\frac{pB_{p-n-1}}{2(n+1)} \left(n + (-1)^l \frac{n-2l}{l+1} \binom{n+1}{l}\right) \pmod{p^2}, & \text{if } n \text{ even.}
\end{cases}
\]  

Lemma 2.3 ([11, Th. 5.2]). Let \( p > 3 \) be a prime number. Then

\[
\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \equiv -2q_2 + pq_2^2 - \frac{2}{3}q_2^3 - \frac{7}{12}p^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3}.
\]  

Lemma 2.4. Let \( p \) be an odd prime number. Then, for \( n \in \{1, 2, \ldots, p-2\} \) we have

\[
\sum_{k=1}^{p-1} \frac{H_k}{k^n} \equiv B_{p-1-n} \pmod{p}
\]  

and for \( n \in \{1, 2, \ldots, \frac{p-3}{2}\} \) we have

\[
\sum_{k=1}^{p-1} \frac{H_k^2}{k^{2n-1}} \equiv B_{p-1-2n} \pmod{p}.
\]

Proof. If \( n \) is even, we set \( n = 2m \) and \( l = 1 \) in relation (12), and if \( n \) is odd, we use the fact that \( H_{p-k} = H_{p-1} - H_{k-1} \equiv -H_{k-1} \pmod{p} \) to get

\[
\sum_{k=1}^{p-1} \frac{H_k}{k^n} = \sum_{k=1}^{p-1} \frac{H_{p-k}}{(p-k)^n} = -\sum_{k=1}^{p-1} \frac{H_{k-1}}{k^n} = -\sum_{k=1}^{p-1} \frac{H_k}{k^n} + \sum_{k=1}^{p-1} \frac{1}{k^{n+1}},
\]

then

\[
2 \sum_{k=1}^{p-1} \frac{H_k}{k^n} \equiv \sum_{k=1}^{p-1} \frac{1}{k^{n+1}} \equiv 0 = B_{p-1-n} \pmod{p},
\]

because \( B_{p-1-n} = 0 \) when \( p - 1 - n \) is odd. Similarly, we have

\[
\sum_{k=1}^{p-1} \frac{H_k^2}{k^{2n-1}} = \sum_{k=1}^{p-1} \frac{H_{p-k}^2}{(p-k)^{2n-1}}
\]

\[
= -\sum_{k=1}^{p-1} \frac{1}{k^{2n-1}} \left(H_k - \frac{1}{k}\right)^2
\]

\[
= -\sum_{k=1}^{p-1} \frac{H_k^2}{k^{2n-1}} + 2 \sum_{k=1}^{p-1} \frac{H_k}{k^{2n}} - H_{p-1,2n} \pmod{p},
\]

so, we get

\[
2 \sum_{k=1}^{p-1} \frac{H_k^2}{k^{2n-1}} \equiv 2 \sum_{k=1}^{p-1} \frac{H_k}{k^{2n}} - H_{p-1,2n} \equiv 2B_{p-1-2n} \pmod{p},
\]

which completes the proof. \( \Box \)
Lemma 2.5. Let \( p \) be a prime number and let \( k \in \{1, 2, \ldots, p\} \). Then, for any integer \( n \geq 1 \), we have

\[
\binom{np - 1}{k - 1} \equiv (-1)^{k-1} \left(1 - npH_{k-1} + \frac{n^2p^2}{2} \left(H_{k-1}^2 - H_{k-1,2}\right) \right) \pmod{p^3}. \tag{16}
\]

Reducing the modulus in this congruence we get

\[
\binom{np - 1}{k - 1} \equiv (-1)^{k-1} (1 - npH_{k-1}) \pmod{p^2}. \tag{17}
\]

Proof. As of the proofs given [8, Lem. 2.3], we have

\[
\binom{np - 1}{k - 1} = \frac{(np - 1)(np - 2)\cdots(np - j)\cdots(np - (k - 1))}{1\cdot2\cdot j\cdots(k - 1)} = \frac{(np - 1)(np - 2)\cdots(np - j)\cdots(np - (k - 1))}{1\cdot2\cdot j\cdots(k - 1)}
\]

\[
= (-1)^{k-1} \left(1 - \frac{np}{1}\right) \left(1 - \frac{np}{2}\right) \cdots \left(1 - \frac{np}{j}\right) \cdots \left(1 - \frac{np}{k - 1}\right)
\]

\[
= (-1)^{k-1} \left(1 - np \sum_{i=1}^{k-1} \frac{1}{i} + \frac{n^2p^2}{2} \sum_{1 \leq i < j \leq k-1} \frac{1}{ij}\right)
\]

\[
= (-1)^{k-1} \left(1 - npH_{k-1} + \frac{n^2p^2}{2} \left(H_{k-1}^2 - H_{k-1,2}\right) \right) \pmod{p^3}. \]

\[
\]

Lemma 2.6. Let

\[
S_{n,p,m} := S(n^p - 1, p - 1, m) = \sum_{k=1}^{p-1} (-1)^{k} \frac{np - 1}{k^m}. \tag{18}
\]

Then, if \( m \in \{1, 2, \ldots, p - 3\} \), we obtain

\[
S_{n,p,m} \equiv 0 \pmod{p} \quad \text{and} \quad S_{n,p,m} \equiv \left(n - \frac{m}{m + 1}\right) pB_{p-1-m} \pmod{p^2},
\]

and, if \( m \in \{1, 2, \ldots, \frac{p-3}{2}\} \), we obtain

\[
S_{n,p,2m-1} \equiv -\frac{T(m,n)}{2(m+1)} p^2B_{p-2m} \pmod{p^3}, \tag{19}
\]

where \( T(m,n) = (2m + 1) (m - 1) n^2 + (2m - 1) (m + 1) n - 2m (2m - 1) \).

Proof. In view of the congruence (17) we have

\[
S_{n,p,m} = \sum_{k=1}^{p-1} (-1)^{k} \frac{1}{k^m} \left(np - 1\right).
\]

\[
= -\sum_{k=1}^{p-1} \frac{1 - npH_{k-1}}{k^m}
\]

\[
= -\sum_{k=1}^{p-1} \frac{1}{k^m} + np \sum_{k=1}^{p-1} \frac{H_{k} - \frac{1}{k}}{k^m}
\]

\[
= -\sum_{k=1}^{p-1} \frac{1}{k^m} + np \left(\sum_{k=1}^{p-1} \frac{H_{k}}{k^m} - \sum_{k=1}^{p-1} \frac{1}{k^{m+1}}\right) \pmod{p^2},
\]

using the relations of congruences (10), (11) and (14) the proof of (18) is complete.
The congruence (9) gives
\[
\sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} \equiv \frac{(2m-1)m}{p-2m-1} p^2 B_{p-2m-1} \equiv -m \frac{2m-1}{2m+1} p^2 B_{p-2m-1} \pmod{p^3} \tag{20}
\]
and by using the above congruences and the congruences (10), (11), (14), (15) and (16) we get
\[
S_{n,p,2m-1} = \sum_{k=1}^{p-1} (-1)^k \frac{1}{k^{2m-1}} \left( np - 1 \right)
\]
\[
= -\sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} \left( 1 - npH_{k-1} + \frac{n^2p^2}{2} \left( H_{k-1,2}^2 - H_{k-1,2} \right) \right)
\]
\[
= -\sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} + np \sum_{k=1}^{p-1} H_{k-1} \frac{1}{k^{2m-1}} - \frac{n^2p^2}{2} \left( \sum_{k=1}^{p-1} H_{k-1}^2 \frac{1}{k^{2m-1}} - \sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} \right)
\]
\[
= -\sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} + np \left( \sum_{k=1}^{p-1} \frac{H_k}{k^{2m-1}} - \sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} \right)
\]
\[
- \frac{n^2p^2}{2} \left( \sum_{k=1}^{p-1} H_k^2 \frac{1}{k^{2m-1}} - 2 \sum_{k=1}^{p-1} \frac{H_k}{k^{2m-1}} + 2 \sum_{k=1}^{p-1} \frac{1}{k^{2m+1}} \right)
\]
\[
= \frac{2m-1}{2m+1} p^2 B_{p-2m-1} + np \left( \frac{1}{3} \sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} \right)
\]
\[
- \frac{2m-1}{2m+1} p^2 B_{p-2m-1} + \frac{n^2p^2}{2} \left( B_{p-2m-1} - mB_{p-1-2m} \right) \pmod{p^3},
\]
and this is the congruence (19). \qedhere

**Lemma 2.7.** Let \( p > 3 \) be a prime number. Then
\[
\sum_{k=1}^{[\frac{p}{3}]} \frac{1}{k} \equiv -\frac{p}{6} L_{\frac{p}{3}} B_{p-2} \left( \frac{1}{3} \right) - \frac{3}{2} q_3 + \frac{3}{4} p q_3^2 \pmod{p^3},
\]
where \( L_{\frac{p}{3}} \) is Legendre symbol defined by
\[
L_{\frac{p}{3}} = \begin{cases} 
1 & \text{if } p \equiv 1 \pmod{3}, \\
-1 & \text{if } p \equiv 2 \pmod{3}.
\end{cases}
\]

**Proof.** We have
\[
\sum_{k=1}^{[\frac{p}{3}]} \frac{1}{k} = \sum_{k=1}^{[\frac{p}{3}]} \frac{p + 3k}{(p-3k)(p+3k)} = \sum_{k=1}^{[\frac{p}{3}]} \frac{p + 3k}{p^2 - 9k^2} \equiv -\frac{1}{9} \sum_{k=1}^{[\frac{p}{3}]} \frac{p + 3k}{k^2},
\]
i.e.
\[
\sum_{k=1}^{[\frac{p}{3}]} \frac{1}{k} \equiv -\frac{p}{9} \sum_{k=1}^{[\frac{p}{3}]} \frac{1}{k^2} - \frac{1}{3} \sum_{k=1}^{[\frac{p}{3}]} \frac{1}{k} \pmod{p^2},
\]
\[
\sum_{k=1}^{[\frac{p}{3}]} \frac{1}{k} \equiv -\frac{p}{9} \sum_{k=1}^{[\frac{p}{3}]} \frac{1}{k^2} - \frac{1}{3} \sum_{k=1}^{[\frac{p}{3}]} \frac{1}{k} \pmod{p^2},
\]
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and by the relation 42 given in [7] and Theorem 3.3 given in [12] we obtain

\[ \sum_{k=1}^{[\frac{q}{p}]} \frac{1}{p-3k} \equiv \frac{1}{2} q_3 - \frac{1}{4} pq_3^2 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{[\frac{q}{p}]} \frac{1}{k^2} \equiv \frac{1}{2} L_{\frac{q}{p}} B_{p^{-2}} \left( \frac{1}{3} \right) \pmod{p}. \]

So, we may state

\[ -\frac{1}{3} \sum_{k=1}^{[\frac{q}{p}]} \frac{1}{k} \equiv \frac{p}{18} L_{\frac{q}{p}} B_{p^{-2}} \left( \frac{1}{3} \right) + \frac{1}{2} q_3 - \frac{1}{4} pq_3^2 \pmod{p^2}. \]

**Theorem 2.8.** For \( m, l, n \geq 1, q \geq 1 \) be integers we have

\[
\sum_{k=0}^{q} (-1)^{k+1} \binom{n-k}{k} H_{k,m} = (-1)^{q+1} (n-q) \binom{n}{q} H_{q,m} - nS (n-1, q, m-l). \tag{22}
\]

In particular, for \( l = 0, 1 \) we get

\[
\sum_{k=0}^{q} (-1)^{k+1} n \binom{n-k}{k} H_{k,m} = (-1)^{q+1} (n-q) \binom{n}{q} H_{q,m} - nS (n-1, q, m), \tag{23}
\]

for \( q = n \) we get

\[
\sum_{k=0}^{n} (-1)^{k+1} \binom{n-k}{k} H_{k,m} = -nS (n-1, n, m-l). \]

and for \( m = 1 \) and \( l = 0 \) or \( 1 \) we get

\[
n \sum_{k=0}^{q} (-1)^{k+1} \binom{n}{k} H_k = (-1)^{q+1} (n-q) \binom{n}{q} \left( H_q + \frac{1}{n} \right) + 1 \quad \text{and} \tag{24}
\]

\[
\sum_{k=0}^{q} (-1)^{k+1} k \binom{n}{k} H_k = (-1)^{q+1} \frac{n-q}{n-1} \binom{n}{q} \left( H_q + \frac{q}{n-1} \frac{1}{n} \right) - \frac{1}{n-1}.
\]

**Proof.** We proceed as follows

\[
(-1)^q q^{l+1} H_{q,m} \binom{n}{q}
\]

\[
= \sum_{k=0}^{q} (-1)^k k^{l+1} \binom{n}{k} H_{k,m} - \sum_{k=0}^{q-1} (-1)^k k^{l+1} \binom{n}{k} H_{k,m}
\]

\[
= \sum_{k=0}^{q-1} (-1)^k (k+1)^{l+1} \binom{n}{k+1} H_{k+1,m} + \sum_{k=0}^{q-1} (-1)^k k^{l+1} \binom{n}{k} H_{k,m}
\]

\[ \quad \text{for } q = n \]

\[ \quad \text{and for } m = 1 \text{ and } l = 0 \text{ or } 1 \]
Also, for \( q = m \) and for \( m = 1 \) in Theorem 2.8 we have
\[
\sum_{k=0}^{q-1} (-1)^{k+1} \binom{n}{k} H_{k,m} = \sum_{k=0}^{q-1} (-1)^{k+1} \binom{n}{k} = 0.
\]
and since \( H_{p-1,m} \equiv 0 \) (mod \( p \)) we obtain
\[
= (-1)^p (n - 1)^{p' + 1} \binom{np - 1}{p - 1} H_{p-1,m} - (np - 1) S (np - 2, p - 1, m - l),
\]

\section{3 Proofs of the main theorems}

\textit{Proof of Theorem 1.2.} For \( q = p - 1 \) and \( n := np - 1 \) in the identity \((22)\) we get
\[
\sum_{k=0}^{p-1} (-1)^{k+1} \binom{np - 1 - k}{k} \binom{np - 1}{k} H_{k,m} = (-1)^p (n - 1)^{p' + 1} \binom{np - 1}{p - 1} H_{p-1,m} - (np - 1) S (np - 2, p - 1, m - l),
\]

\textbf{Remark 2.} In Theorem 2.8, if \( q = n \), we get the known identity (see, for example \([1, 2, 4]\)). For \( m = 1 \) we have
\[
\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} H_k = \frac{1}{n}
\]
and for \( m = 2 \) we have
\[
\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} H_{k,2} = -\frac{H_n}{n}.
\]
\[
\sum_{k=0}^{p-1} (-1)^{k+1} \left( (np - 1 - k) (k + 1) + k^{l+1} \right) \binom{np - 1}{k} H_{k,m}
\equiv - (np - 1) S (np - 2, p - 1, m - l) \pmod{p^2}.
\]

So, for \( q = p - 1 \) and \( n := np - 1 \) in the recurrence relation (1) and by using the congruence (18), we get

\[
(np - 1) S (np - 2, p - 1, m - l)
= npS (np - 1, p - 1, m - l) - S (np - 1, p - 1, m - l - 1)
= npS_{n,p,m-l} - S_{n,p,m-l-1}
\equiv -S_{n,p,m-l-1}
\equiv - \left( n - 1 + \frac{1}{m - l} \right) pB_{p-m+l} \pmod{p^2}.
\]

Hence

\[
\sum_{k=0}^{p-1} (-1)^{k+1} \left( (np - 1 - k) (k + 1) + k^{l+1} \right) \binom{np - 1}{k} H_{k,m}
\equiv \left( n - 1 + \frac{1}{m - l} \right) pB_{p-m+l} \pmod{p^2},
\]

which is the congruence (5).

For \( l = 0 \) in the congruence (5) we get

\[
(np - 1) \sum_{k=1}^{p-1} (-1)^{k+1} \binom{np - 1}{k} H_{k,m} \equiv \left( n - 1 + \frac{1}{m} \right) pB_{p-m} \pmod{p^2}, \tag{27}
\]

and since \( \frac{1}{np-1} \equiv -1 - np \pmod{p^2} \), the congruence (6) follows.

For \( l = 1 \) in the congruence (5) we get

\[
(np - 2) \sum_{k=1}^{p-1} (-1)^{k+1} k \binom{np - 1}{k} H_{k,m} + (np - 1) \sum_{k=1}^{p-1} (-1)^{k+1} \binom{np - 1}{k} H_{k,m}
\equiv \left( n - 1 + \frac{1}{m - l} \right) pB_{p-m+1} \pmod{p^2},
\]

and by the congruence (27) and the congruence

\[
\frac{1}{np - 2} \equiv - \frac{1}{2} - \frac{1}{4} np \pmod{p^2},
\]

the congruence (7) follows. □
Proof of Theorem 1.3. From the relations (23), (10), (18) and (19) we get

\[
(np - 1) \sum_{k=0}^{p-1} (-1)^{k+1} \binom{np - 1}{k} H_{k,2m} \\
= - (n - 1) p \binom{np - 1}{p - 1} H_{p-1,2m} - (np - 1) S (np - 2, p - 1, 2m) \\
≡ - (n - 1) pH_{p-1,2m} - (np - 1) S (np - 2, p - 1, 2m) \\
= - (n - 1) pH_{p-1,2m} - npS_{n,p,2m} + S_{n,p,2m-1} \\
= - (n - 1) \frac{2mp^2}{2m+1} B_{p-1,2m} - np^2 \left(n - \frac{2m}{2m+1}\right) B_{p-1,2m} - T (m,n) \frac{p^2 B_{p-1,2m}}{2(2m+1)} \\
≡ \left(-n^2 + \frac{-4m (n - 1) + 4nm - T (m,n)}{2 (2m+1)}\right) p^2 B_{p-1,2m} \pmod{p^3},
\]

and since

\[
\frac{1}{np - 1} \equiv -1 - np - n^2p^2 \pmod{p^3},
\]

the proof of theorem is complete. \(\square\)

Proof of Theorem 1.4. 1) If we let \(q = \frac{p}{2}\) and \(n = p - 1\) in relation (24), we have then

\[
\sum_{k=1}^{p-1} (-1)^{k+1} \binom{p - 1}{k} H_k = \frac{1}{2} (-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \left( H_{\frac{p}{2}} + \frac{1}{p - 1} \right) + \frac{1}{p - 1},
\]

using the known congruence obtained by Morley [9]

\[
\binom{p - 1}{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} 4^{p-1} = (-1)^{\frac{p-1}{2}} (1 + pq_2)^2 \pmod{p^3},
\]

and the congruence

\[
\frac{1}{p - 1} \equiv -1 - p - p^2 \pmod{p^3},
\]

and the relations (13), (24) we have

\[
\sum_{k=1}^{p-1} (-1)^{k+1} \binom{p - 1}{k} H_k \\
≡ -\frac{1}{2} (1 + pq_2)^2 \left(-2q_2 + p^2 q_2^2 - \frac{2}{3} p^2 q_2^3 - \frac{7}{12} p^2 B_{p-2} \left(\frac{1}{3}\right) + \frac{1}{p - 1}\right) + \frac{1}{p - 1} \pmod{p^3},
\]

the proof is complete.

2) a) For \(k - 1 = \left[\frac{p}{3}\right]\) the relation (17) becomes

\[
\binom{np - 1}{\left[\frac{p}{3}\right]} \equiv (-1)^\left[\frac{p}{3}\right] \left(1 - npH_{\left[\frac{p}{3}\right]}\right) \pmod{p^3},
\]

and since \(H_{\left[\frac{p}{3}\right]} \equiv -\frac{3}{2} q_3 \pmod{p}\) (see [3]) we obtain

\[
\binom{np - 1}{\left[\frac{p}{3}\right]} \equiv (-1)^\left[\frac{p}{3}\right] \left(1 + \frac{3}{2} npq_3\right) \pmod{p^2}.
\]
For \( p = 1 \pmod{3} \) we use the congruence
\[
\frac{1}{np - 1} \equiv -1 - np \pmod{p^2},
\]
and the relations (21), (24) to obtain
\[
\frac{p - 1}{3} \sum_{k=1}^{p-1} (-1)^{k+1} \binom{np - 1}{k} H_k = \left(1 + \frac{3}{2} npq_3\right) \frac{p - 1}{np - 1} \left(\frac{-p}{6} B_{p-2} + \frac{1}{3} - \frac{3}{2} q_3 + \frac{3}{4} pq_3^2 + \frac{1}{np - 1}\right) + \frac{1}{np - 1}
\]
\[
\equiv \left(q_3 - \frac{1}{3}\right) \{\left(\frac{3}{2} n - \frac{1}{2}\right) q_3^2 + \frac{n + 1}{2} q_3 - \frac{2}{3} n + \frac{1}{3} + \frac{1}{9} B_{p-2} + \frac{1}{3}\} \pmod{p^2}.
\]
b) The other congruences can be proved similarly as above. 

**Proof of Theorem 1.5.** By the relation (26) we get
\[
\sum_{k=1}^{p-1} (-1)^{k+1} \binom{p - 1}{k} H_{k,2} = H_{p-1} \pmod{p-1},
\]
using the congruence of Remark 5.1 given in [11]
\[
H_{p-1} \equiv - \left(\frac{B_{2p-4}}{2p - 4} - \frac{B_{p-3}}{p - 3}\right) p^2 \pmod{p^4},
\]
and the congruence
\[
\frac{1}{p - 1} = -1 - p - p^2 - p^3 \pmod{p^4}, \tag{28}
\]
the proof is complete. 

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