ANALYTIC SOLUTIONS FOR THE APPROXIMATION OF 
\( p \)-LAPLACIAN PROBLEM

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ABSTRACT. This paper mainly investigates the analytic solutions for the approximation of \( p \)-Laplacian problem. Through an approximation mechanism, we convert the nonlinear partial differential equation with Dirichlet boundary into a sequence of minimization problems. And a sequence of analytic minimizers can be obtained by applying the canonical duality theory. Moreover, the nonlinear canonical transformation gives a sequence of perfect dual maximization(minimization) problems, and further discussion shows the global extrema for both primal and dual problems.

1. INTRODUCTION

Fractional order operators are very important mathematical models describing plenty of anomalous dynamic behaviors in applied sciences \([12, 14, 18]\). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a sufficiently smooth boundary \( \Gamma \). In this manuscript, we are interested in exploring the analytic solutions for the approximation of the following fractional order \( p \)-Laplacian problem(also called \( p \)-harmonic problem) with Dirichlet boundary in higher dimensions,

\[
\begin{cases}
\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u) + f = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma,
\end{cases}
\]

where \( f \in L^p(\Omega), 1/p+1/p^* = 1 \). This nonhomogeneous problem is intensively studied in many multidisciplinary fields, such as mean curvature analysis \((p = 1)\), compressible fluid in a homogeneous isotropic rigid porous medium \((p = 3/2)\), glaciology \((p \in (1, 4/3])\), nonlinear elasticity \((p > 2)\) and probabilistic games \((p = \infty)\), etc. Interested readers can refer to \([2, 4, 5, 14, 18]\) to become familiar with more useful applications in these respects.

Well-posedness and numerical simulations through finite element methods(FEMs) for \( p \in (1, \infty) \) are well established \([3, 8, 19, 22]\). General \( p \)-supersolution in the viscosity
sense for the \( p \)-Laplacian can be found in [20]. In particular, if \( \Omega \) is a bounded domain of class \( C^{1,\beta} \) for some \( \beta \in (0,1) \) and \( f \in L^\infty(\Omega) \), the unique weak solution \( u \) of (1) belongs to \( C^0(\overline{\Omega}) \) [1]. In effect, from the minimization problem corresponding to (1)

\[
(\mathcal{P}) : \min_{u \in W^{1,p}_0(\Omega)} \left\{ I[u] := 1/p \int_\Omega |\nabla u|^p dx - \int_\Omega fu dx \right\},
\]

it is evident that, for \( x \in \Omega \), if \( \text{Sgn}(f) = 1 \), then there exists a unique positive minimizer; while if \( \text{Sgn}(f) = -1 \), then there exists a unique negative minimizer [21]. For both Damascelli-Pacella’s weak comparison principle and Cuesta-Takač’s strong comparison principle of positive solutions, please refer to [1].

Here, we mainly address the nonlinear case \( p \in (1, 2) \cup (2, \infty) \). It is worth noticing that, at the critical points (\( \nabla u = 0 \)), the operator is degenerate elliptic for \( p > 2 \) and singular for \( p < 2 \). To tackle the singularity, one applies the perturbation method (or penalty function method frequently used in the image processing computation) proposed in [7, 20], namely,

\[
\Delta_{p,\epsilon} u_\epsilon := \text{div}(\nabla u_\epsilon^2 + \chi(p)\epsilon^2)^{(p-2)/2} \nabla u_\epsilon + f = 0 \quad \text{in} \quad \Omega,
\]

\[
\text{on} \quad \Gamma,
\]

where the cut-off function \( \chi \) is defined as

\[
\chi(p) := \begin{cases} 
1, & p \in (1, 2); \\
0, & p \in (2, +\infty).
\end{cases}
\]

The term \( (|\nabla u_\epsilon|^2 + \chi(p)\epsilon^2)^{(p-2)/2} \) is called the transport density. Clearly, (3) is a highly nonlinear PDE which is difficult to solve by the direct approach [10, 15]. As a matter of fact, given the distributed source term \( f \in L^p(\Omega) \), (3) is the Euler-Lagrange equation of the minimization problem

\[
(\mathcal{P}^{(\epsilon)}) : \min_{u_\epsilon \in \mathcal{V}} \left\{ I^{(\epsilon)}[u_\epsilon] := \int_\Omega L_{\epsilon,p}(\nabla u_\epsilon, u_\epsilon, x) dx = \int_\Omega H_{\epsilon,p}(\nabla u_\epsilon) dx - \int_\Omega fu_\epsilon dx \right\},
\]

where \( \mathcal{V} = W^{1,p}_0(\Omega) \), the function (so-called stored strain energy density) \( H_{\epsilon,p} : \mathbb{R}^n \to \mathbb{R} \) is given by

\[
H_{\epsilon,p}(\vec{\gamma}) := (|\vec{\gamma}|^2 + \chi(p)\epsilon^2)^{p/2}/p.
\]

Moreover, \( L_{\epsilon,p}(\vec{\gamma}, z, x) : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R} \) satisfies the following coercivity inequality and is convex in the variable \( \vec{\gamma} \),

\[
L_{\epsilon,p}(\vec{\gamma}, z, x) \geq p \epsilon |\vec{\gamma}|^p - q, \quad \vec{\gamma}, z \in \mathbb{R}^n, z \in \mathbb{R}, x \in \Omega,
\]

for certain constants \( p_\epsilon \) and \( q_\epsilon \). \( I^{(\epsilon)} \) is called the potential energy functional and is sequentially (weakly) lower semicontinuous with respect to the weak topology of \( W^{1,p}_0(\Omega) \). By Rellich-Kondrachov compactness theorem and Riesz’s mean convergence theorem, we have the following a priori estimates,

**Theorem 1.1.** Assume that \( f \in L^p(\Omega) \) does not change its sign on \( \overline{\Omega} \), that is to say, \( \text{Sgn}(f) = 1 \) or \( \text{Sgn}(f) = -1 \). Then, there exists at least one minimizer \( \bar{u}_\epsilon \in W^{1,p}_0(\Omega) \) of (4) satisfying

\[
\bar{u}_\epsilon \geq 0 \quad \text{a.e. in} \quad \Omega \quad \text{for} \quad \text{Sgn}(f) = 1,
\]
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Analytic solution for \( p \)-Laplacian problem

\[(7) \quad \bar{u}_\epsilon \leq 0 \text{ a.e. in } \Omega \text{ for } \text{Sgn}(f) = -1, \]
\[(8) \quad \bar{u}_\epsilon \to \bar{u} \text{ in } W^{1,p}_0(\Omega) \text{ as } \epsilon \to 0, \]
where \( \bar{u} \) is the unique positive solution to (2).

Next, we are going to explore the analytic solution of (3) rather than finite element approximation, which is of great use in the discussion of Mong-Kantorovich mass transfer problems \([7]\), etc. Here, let \( \Omega = B(O, R_2) \setminus B(O, R_1), R_2 > R_1 > 0, \) where both \( B(O, R_1) \) and \( B(O, R_2) \) denote open balls with center \( O \) and radii \( R_1 \) and \( R_2 \) in \( \mathbb{R}^n \), and \( \Gamma_1 \) and \( \Gamma_2 \) denote the corresponding boundary sphere, respectively. We are interested in the radially symmetric and continuous solutions for (3). Correspondingly, let the feasible function space \( \mathcal{N} \) in (4) be defined as
\[(9) \quad \mathcal{N} := \left\{ u \in W^{1,p}_0(\Omega) \cap C(\Omega) : u \text{ is radially symmetric} \right\}. \]

In the following, one is to construct the radially symmetric analytic solution for (3) through canonical duality method introduced by G. Strang et al. This theory was originally proposed to find minimizers for a non-convex strain energy functional with a double-well potential. During the past few years, considerable effort has been taken to illustrate these non-convex problems from the theoretical point of view. By applying this technique, the authors characterized the local and global energy extrema for both hard and soft devices and finally obtained the analytical solutions \([9]\).

At the moment, we would like to introduce the main theorem.

**Theorem 1.2.** Assume that the source term \( f \in C(\overline{\Omega}) \) is radially symmetric and satisfies
\[(10) \quad \text{Sgn}(f) = 1 \text{ or } \text{Sgn}(f) = -1 \text{ on } \overline{\Omega}. \]
Moreover, let
\[(11) \quad E_{c,p}(r) := r^{(2p-2)/(p-2)} - \chi(p) r^2. \]
It is evident that, when \( p \in (1, 2) \), \( E_{c,p} \) is strictly decreasing with respect to \( r \in (0, e^{p-2}] \); while when \( p \in (2, \infty) \), \( E_{c,p} \) is strictly increasing with respect to \( r \in [0, \infty) \). In either case, let \( E_{c,p}^{-1} \) stand for the inverse of \( E_{c,p} \). Then, there exists a sequence of global minimizers \( \{ \tilde{u}_\epsilon \}_\epsilon \) from \( \mathcal{N} \) in (9) for the approximation problems \( (P^{(r)}) \), which is at the same time a sequence of analytic solutions for the Euler-Lagrange equations (3) in the following explicit form \( \tilde{u}_\epsilon(r) \) (without any confusion with respect to \( \tilde{u}_\epsilon(x) \), as well as \( f(r) \)),
\[(12) \quad \tilde{u}_\epsilon(r) = \int_{R_1}^r F_\epsilon(\rho) \rho / E_{c,p}^{-1}(F_\epsilon^2(\rho) \rho^2) d\rho, \quad r \in [R_1, R_2], \]
where \( F_\epsilon \) is defined as
\[(13) \quad F_\epsilon(r) := R_1^n C_\epsilon / r^n + \int_r^{R_1} f(\rho) \rho^{n-1} / r^n d\rho, \quad r \in [R_1, R_2], \]
and \( \{ C_\epsilon \}_\epsilon \) is a positive number sequence given by (33).
Remark 1.3. For $p > 2$, the above theorem gives the exact radially symmetric solution of the $p$-Laplacian (1). While for $p < 2$, using the techniques in the proof of Theorem 1.1, we can choose from $\{\bar{u}_\epsilon\}_\epsilon$ a subsequence of radially symmetric solutions of (3) which converge to the exact radially symmetric solution of (1) in $W_0^{1,p}(\Omega)$ as $\epsilon \to 0$.

Remark 1.4. From the structure of the number sequence $\{C_\epsilon\}_\epsilon$ in the proof of Theorem 1.2, one finds out, if $\text{Sgn}(f) = 1$, then $\bar{u}_\epsilon \geq 0$ on $\overline{\Omega}$, while if $\text{Sgn}(f) = -1$, then $\bar{u}_\epsilon \leq 0$ on $\overline{\Omega}$, which is in accordance with Theorem 1.1. In addition, the special form of $\{C_\epsilon\}_\epsilon$ does not allow $R_1 = 0$.

Remark 1.5. When $p = \infty$, it turns out $E_{\epsilon,\infty} = r^2$, which is invertible on $[0, \infty)$. Then, one has $|du_\epsilon/dr| = 1$, and an exact solution is in the zigzag form. The uniqueness of the solution fails in this limit case [7]. On the other hand, for the case $p = 1$, (8) in Theorem 1.1 fails. These two important cases remain to be discussed [7, 13].

The rest of the paper is organized as follows. In Section 2, first, we introduce some useful notations which will simplify our proofs considerably. Then, we prove Theorem 1.1 by applying Rellich-Kondrachov compactness theorem and Riesz’s mean convergence theorem. Next, we apply the canonical dual transformation to deduce a sequence of perfect dual problems $(P_{\epsilon}^{(d)})$ corresponding to $(P_{\epsilon}^{(c)})$ and a pure complementary energy principle in order to prove Theorem 1.2.

2. PROOF OF THE MAIN RESULTS

2.1. Notations. Before proceeding, first and foremost, we introduce some useful notations.

- $\vec{\theta}_\epsilon$ is the corresponding Gâteaux derivative of $H$ with respect to $\nabla u_\epsilon$ given by
  $$\vec{\theta}_\epsilon(x) = (\theta_\epsilon^{(1)}(x), \ldots, \theta_\epsilon^{(n)}(x)) := ((\nabla u_\epsilon)^2 + \epsilon^2 \chi(p))^{(p-2)/2} \nabla u_\epsilon.$$

- $\Phi_\epsilon : \mathcal{N} \to L^{p/2}(\Omega)$ is a nonlinear geometric mapping given by
  $$\Phi_\epsilon(u_\epsilon) := |\nabla u_\epsilon|^2 + \epsilon^2 \chi(p).$$
  For convenience’s sake, denote $\xi_\epsilon := \Phi_\epsilon(u_\epsilon)$.

- $\Psi$ is the canonical energy defined as
  $$\Psi(\xi_\epsilon) := \xi_\epsilon^{p/2}/p.$$

- $\zeta_\epsilon$ is the corresponding Gâteaux derivative of $\Psi$ with respect to $\xi_\epsilon$ given by
  $$\zeta_\epsilon = \xi_\epsilon^{(p-2)/2},$$
  which is invertible with respect to $\xi_\epsilon$ and belongs to the following function space $\mathcal{K}_\epsilon$,
  $$\mathcal{K}_\epsilon := \begin{cases} \{ \phi : 0 < \phi \leq \epsilon^{p-2}/2 \} , & p \in (1, 2); \\ \{ \phi : \phi \in L^{p/(p-2)}(\Omega) \} , & p \in (2, \infty). \end{cases}$$

- $\Psi_*$ is Legendre transformation defined as
  $$\Psi_*(\zeta_\epsilon) := \xi_\epsilon \zeta_\epsilon - \Psi(\xi_\epsilon) = (p-2)(2\zeta_\epsilon)^{(p-2)/(2p)}.$$
2.2. **Proof of Theorem 1.1.** Without loss of generality, we prove the case for $Sgn(f) = 1$. Let $\bar{u}_\epsilon$ be a solution for (4), then, by the minimization property, $\bar{u}_\epsilon \geq 0$ a.e. in $\Omega$. Next, we prove the existence and convergence of a sequence of solutions. By applying Hölder’s inequality, one has

$$\int_\Omega |fu_\epsilon| dx \leq \left( \int_\Omega |f|^{p^*} dx \right)^{1/p^*} \left( \int_\Omega |u_\epsilon|^p dx \right)^{1/p}.$$ 

Since $\Omega$ is bounded, for any $u_\epsilon \geq 0$ a.e. from $W^{1,p}_0(\Omega)$, by applying Poincaré’s inequality, one obtains

$$\epsilon^p/p \text{ meas}(\Omega) \geq \int_\Omega H_\epsilon(\nabla u_\epsilon) dx - \int_\Omega fu_\epsilon dx \\
\geq 1/p \int_\Omega (|\nabla u_\epsilon|^2 + \epsilon^2)^{p/2} dx - \left( \int_\Omega |f|^{p^*} dx \right)^{1/p^*} \left( \int_\Omega |u_\epsilon|^p dx \right)^{1/p} \\
\geq 1/p \int_\Omega (|\nabla u_\epsilon|^2 + \epsilon^2)^{p/2} dx - C(p) \left( \int_\Omega |f|^{p^*} dx \right)^{1/p^*} \left( \int_\Omega |\nabla u_\epsilon|^p dx \right)^{1/p}.$$ 

Using the contradiction method, it is easy to check that, $u_\epsilon$ is uniformly bounded in $W^{1,p}_0(\Omega)$ for $p > 1$, and consequently, $I(\epsilon)$ is bounded from below.

We see that, there exists at least one solution $\bar{u}_\epsilon \in W^{1,p}_0(\Omega)$ to (4) due to the fact that $L_\epsilon(\vec{\gamma}, z, x)$ satisfies the coercivity inequality (5) and is convex in the variable $\vec{\gamma}$ [6]. On the one hand, according to the lower semicontinuity of $I(\epsilon)$ and the unique solvability of (2), it holds

(14) $\bar{u}_\epsilon \rightharpoonup \bar{u}$ weakly in $W^{1,p}_0(\Omega)$ as $\epsilon \to 0$,

where $\bar{u}$ is the unique solution to (2). Moreover, since the imbedding $W^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, then,

(15) $\bar{u}_\epsilon \to \bar{u}$ in $L^1(\Omega)$ as $\epsilon \to 0$.

On the other hand, from the minimization property $I[\bar{u}] \leq I[\bar{u}_\epsilon] \leq I(\epsilon)[\bar{u}_\epsilon] \leq I(\epsilon)[\bar{u}]$, we have

(16) $I[\bar{u}_\epsilon] \to I[\bar{u}]$ as $\epsilon \to 0$.

The above convergence properties (15) and (16) indicate that

(17) $\int_\Omega |\nabla \bar{u}_\epsilon|^p dx \to \int_\Omega |\nabla \bar{u}|^p dx$ as $\epsilon \to 0$.

Finally, combining (14) and (17), we reach the conclusion.

2.3. **Proof of Theorem 1.2.**

**Definition 2.1.** By Legendre transformation, one defines a total complementary energy functional $\Xi$,

$$\Xi(u_\epsilon, \zeta_\epsilon) := \int_\Omega \left\{ \Phi_\epsilon(u_\epsilon)\zeta_\epsilon - \Psi_\star(\zeta_\epsilon) - fu_\epsilon \right\} dx.$$
Next, we introduce an important criticality criterion for the total complementary energy functional.

**Definition 2.2.** $(\bar{u}_\epsilon, \zeta_\epsilon)$ is called a critical pair of $\Xi$ if and only if

\begin{align}
D_u \Xi(\bar{u}_\epsilon, \zeta_\epsilon) &= 0, \\
D_\zeta \Xi(\bar{u}_\epsilon, \zeta_\epsilon) &= 0,
\end{align}

where $D_u, D_\zeta$ denote the partial Gâteaux derivatives of $\Xi$, respectively.

Indeed, by variational calculus, one has the following observations from (18) and (19).

**Lemma 2.3.** On the one hand, for any fixed $\zeta_\epsilon \in \mathcal{V}_\epsilon$, (18) is equivalent to the equilibrium equation

$$\text{div}(2\zeta_\epsilon \nabla \bar{u}_\epsilon) + f = 0,$$

in $\Omega$.

On the other hand, for any fixed $u_\epsilon$ from $\mathcal{N}$, (19) is consistent with the constructive law

$$\Phi_\epsilon(u_\epsilon) = D_\zeta \Psi_*(\zeta_\epsilon).$$

Lemma 2.3 indicates that $\bar{u}_\epsilon$ from the critical pair $(\bar{u}_\epsilon, \zeta_\epsilon)$ solves the Euler-Lagrange equation (3).

**Definition 2.4.** From Definition 2.2, one defines the pure complementary energy $I_d^{(\epsilon)}$ in the following form

$$I_d^{(\epsilon)}[\zeta_\epsilon] := \Xi(\bar{u}_\epsilon, \zeta_\epsilon),$$

where $\bar{u}_\epsilon$ solves the Euler-Lagrange equation (3).

To further the discussion, one uses another representation of the pure energy $I_d^{(\epsilon)}$ given by the following lemma.

**Lemma 2.5.** The pure complementary energy functional $I_d^{(\epsilon)}$ can be rewritten as

$$I_d^{(\epsilon)}[\zeta_\epsilon] = - \frac{1}{2} \int_\Omega \left\{ \frac{\left| \nabla \bar{u}_\epsilon \right|^2}{2\zeta_\epsilon} - 2\chi(p)e^2\zeta_\epsilon + (p - 2)(2\zeta_\epsilon)^{p/(p-2)}/p \right\} dx,$$

where $\nabla \bar{u}_\epsilon$ satisfies

$$\text{div} \bar{\theta}_\epsilon + f = 0 \quad \text{in} \, \Omega,$$

equipped with a hidden boundary condition.

**Proof.** Through integrating by parts, one has

$$I_d^{(\epsilon)}[\zeta_\epsilon] = - \int_\Omega \left\{ \text{(I)} \div(2\zeta_\epsilon \nabla \bar{u}_\epsilon) + f \right\} \bar{u}_\epsilon dx$$

$$- \int_\Omega \left\{ \text{(II)} \zeta_\epsilon |\nabla \bar{u}_\epsilon|^2 - \chi(p)e^2\zeta_\epsilon + (p - 2)(2\zeta_\epsilon)^{p/(p-2)}/(2p) \right\} dx.$$
Since \( \bar{u}_e \) solves the Euler-Lagrange equation (3), then the first part (I) disappears. Keeping in mind the definitions of \( \vec{\theta}_\epsilon \) and \( \zeta_\epsilon \), one reaches the conclusion. \( \square \)

With the above discussion, in the following, we establish a sequence of dual variational problems \( (\mathcal{P}_d^{(e)}) \) corresponding to the approximation problems \( (\mathcal{P}^{(e)}) \), namely,

- \( p \in (1, 2) \)
  \[
  \min_{\zeta_\epsilon \in \mathcal{H}_\epsilon} \left\{ I_d^{(e)}[\zeta_\epsilon] = -1/2 \int_\Omega \left\{ |\vec{\theta}_\epsilon|^2/(2\zeta_\epsilon) - 2\chi(p)\epsilon^2\zeta_\epsilon + (p - 2)(2\zeta_\epsilon)^{p/(p-2)}/p \right\} dx \right\};
  \]
- \( p \in (2, \infty) \)
  \[
  \max_{\zeta_\epsilon \in \mathcal{H}_\epsilon} \left\{ I_d^{(e)}[\zeta_\epsilon] = -1/2 \int_\Omega \left\{ |\vec{\theta}_\epsilon|^2/(2\zeta_\epsilon) - 2\chi(p)\epsilon^2\zeta_\epsilon + (p - 2)(2\zeta_\epsilon)^{p/(p-2)}/p \right\} dx \right\}.
  \]

Indeed, by calculating the Gâteaux derivative of \( I_d^{(e)} \) with respect to \( \zeta_\epsilon \), one has

**Lemma 2.6.** The variation of \( I_d^{(e)} \) with respect to \( \zeta_\epsilon \) leads to the Dual Algebraic Equation (DAE), namely,

\[
|\vec{\theta}_\epsilon|^2 = (2\zeta_\epsilon)^{(2p-2)/(p-2)} - \chi(p)\epsilon^2(2\zeta_\epsilon)^2,
\]

where \( \zeta_\epsilon \) is from the critical pair \( (\bar{u}_e, \zeta_\epsilon) \).

Let \( \lambda_\epsilon := 2\zeta_\epsilon \), the identity (23) can be rewritten as

\[
|\vec{\theta}_\epsilon|^2 = E_{\epsilon\chi}(\lambda_\epsilon) = \lambda_\epsilon^{(2p-2)/(p-2)} - \chi(p)\epsilon^2\lambda_\epsilon^2.
\]

From the above discussion, one deduces that, once \( \vec{\theta}_\epsilon \) is given, then the analytic solution of the Euler-Lagrange equation (3) can be represented as

\[
\bar{u}_k(x) = \int_{x_0}^x \vec{\eta}_\epsilon \, dt,
\]

where \( x \in \bar{\Omega}, x_0 \in \Gamma, \vec{\eta}_\epsilon = (\eta^{(1)}_\epsilon, \eta^{(2)}_\epsilon, \ldots, \eta^{(n)}_\epsilon) := \vec{\theta}_\epsilon / \lambda_\epsilon \), which satisfies the condition for path independent integrals, namely,

\[
\partial_i \eta^{(j)}_\epsilon - \partial_j \eta^{(i)}_\epsilon = 0, \quad i, j = 1, \ldots, n.
\]

At the moment, we verify that \( \bar{u}_e \) is exactly a global minimizer over \( \mathcal{N} \) for \( (\mathcal{P}^{(e)}) \) and \( \zeta_\epsilon \) is a global extremum over \( \mathcal{H}_\epsilon \) for \( (\mathcal{P}_d^{(e)}) \).

**Lemma 2.7.** (Canonical duality theory) Assume that \( f \in C(\bar{\Omega}) \) is radially symmetric and satisfies \( \text{Sgn}(f) = 1 \) or \( \text{Sgn}(f) = -1 \), then, there exists a unique sequence of radially symmetric solutions \( \{\bar{u}_e\}_\epsilon \) for the Euler-Lagrange equations (3) with Dirichlet boundary in the form of (25), which is at the same time a sequence of global minimizers over \( \mathcal{N} \) for the approximation problems \( (\mathcal{P}^{(e)}) \). And the corresponding \( \{\zeta_\epsilon\}_\epsilon \) is a sequence of global minimizers for the dual problems (21), or a sequence of global maximizers for the dual problems (22). Moreover, the following duality identities hold,

\[
I^{(e)}[\bar{u}_e] = \min_{u_e \in \mathcal{N}} I^{(e)}[u_e] = \Xi(\bar{u}_e, \zeta_\epsilon) = \min_{\zeta_\epsilon \in \mathcal{H}_\epsilon} I_d^{(e)}[\zeta_\epsilon] = I_d^{(e)}[\zeta_\epsilon] \text{ for } p \in (1, 2);
\]
I^{(e)}[\bar{u}_\varepsilon] = \min_{u_\varepsilon \in \mathcal{N}} I^{(e)}[u_\varepsilon] = \Xi(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon) = \max_{\zeta_\varepsilon \in \mathcal{V}} I^{(e)}[\zeta_\varepsilon] = I^{(e)}[\bar{\zeta}_\varepsilon] \text{ for } p \in (2, \infty).

**Remark 2.8.** Lemma 2.7 shows that the minimization (maximization) of the pure complementary energy functional \( I^{(e)} \) is perfectly dual to the minimization of the potential energy functional \( I^{(e)} \) for \( p \in (1, 2) \). Indeed, both identities (26) and (27) indicate there is no duality gap between them.

**Proof.** The proof is divided into two parts. In the first part, we discuss the uniqueness of \( \theta^e \) for both \( \text{Sgn}(f) = 1 \) and \( \text{Sgn}(f) = -1 \). Global extrema will be studied in the second part.

**First Part:**

Let \( O = (a_1, a_2, \cdots, a_n) \). Actually, due to the radial symmetry of the solution \( \bar{u}_\varepsilon \) of (3), one has,

\[
\theta^e = F_e(r)(x_1 - a_1, \cdots, x_n - a_n) = F_e \left( \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} \right)(x_1 - a_1, \cdots, x_n - a_n),
\]

where \( F_e(r) \) in (13) is a general solution of the nonhomogeneous linear differential equation

\[
F_e'(r) + nF_e(r)/r = -f(r)/r, \quad r \in (R_1, R_2).
\]

Recall that \( \bar{u}_\varepsilon(R_1) = 0 \), as a result,

\[
\bar{u}_\varepsilon(r) = \int_{R_1}^{r} \left( R_1^n C_\varepsilon + \int_{\rho}^{R_1} f(\rho)r^{n-1}d\rho \right) \left( r^{n-1} \lambda_\varepsilon(\rho) \right) d\rho, \quad r \in (R_1, R_2).
\]

Recall that \( \bar{u}_\varepsilon(R_2) = 0 \), and one can determine the positive constant \( C_\varepsilon \) uniquely. Indeed, let

\[
\mu_e(r, y) := \left( \text{Sgn}(f)yR_1^n + \int_{r}^{R_1} f(\rho)r^{n-1}d\rho \right) / \left( r^{n-1} \lambda_e(\rho) \right),
\]

\[
M_e(y) := \int_{R_1}^{R_2} \mu_e(r, y)d\rho,
\]

where \( \lambda_e(r, y) \) is from (24). It is evident that \( \lambda_e \) depends on \( C_\varepsilon \). As a matter of fact, similar as the discussion in [23], one can check that, if \( \text{Sgn}(f) = 1 \), then \( M_e \) is strictly increasing with respect to \( y \in (0, \infty) \); while if \( \text{Sgn}(f) = -1 \), then \( M_e \) is strictly decreasing with respect to \( y \in (0, \infty) \). As a result, for both cases, one has

\[
C_\varepsilon = M_e^{-1}(0).
\]

The uniqueness of the solution \( \bar{u}_\varepsilon \) is concluded.

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**First Part:**

Let \( O = (a_1, a_2, \cdots, a_n) \). Actually, due to the radial symmetry of the solution \( \bar{u}_\varepsilon \) of (3), one has,

\[
\theta^e = F_e(r)(x_1 - a_1, \cdots, x_n - a_n) = F_e \left( \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} \right)(x_1 - a_1, \cdots, x_n - a_n),
\]

where \( F_e(r) \) in (13) is a general solution of the nonhomogeneous linear differential equation

\[
F_e'(r) + nF_e(r)/r = -f(r)/r, \quad r \in (R_1, R_2).
\]

Recall that \( \bar{u}_\varepsilon(R_1) = 0 \), as a result,

\[
\bar{u}_\varepsilon(r) = \int_{R_1}^{r} \left( R_1^n C_\varepsilon + \int_{\rho}^{R_1} f(\rho)r^{n-1}d\rho \right) \left( r^{n-1} \lambda_\varepsilon(\rho) \right) d\rho, \quad r \in (R_1, R_2).
\]

Recall that \( \bar{u}_\varepsilon(R_2) = 0 \), and one can determine the positive constant \( C_\varepsilon \) uniquely. Indeed, let

\[
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\]

\[
M_e(y) := \int_{R_1}^{R_2} \mu_e(r, y)d\rho,
\]

where \( \lambda_e(r, y) \) is from (24). It is evident that \( \lambda_e \) depends on \( C_\varepsilon \). As a matter of fact, similar as the discussion in [23], one can check that, if \( \text{Sgn}(f) = 1 \), then \( M_e \) is strictly increasing with respect to \( y \in (0, \infty) \); while if \( \text{Sgn}(f) = -1 \), then \( M_e \) is strictly decreasing with respect to \( y \in (0, \infty) \). As a result, for both cases, one has

\[
C_\varepsilon = M_e^{-1}(0).
\]

The uniqueness of the solution \( \bar{u}_\varepsilon \) is concluded.
Second Part:

On the one hand, for any test function $\phi \in W_0^{1,\infty}(\Omega)$, the second variational form $\delta^2 I^{(e)}$ is equal to

$$\int_{\Omega} \left\{ (p-2) \left( \left( |\nabla \bar{u}_e|^2 + \chi(p) \epsilon^2 \right)^{(p-4)/2} |\nabla \bar{u}_e \cdot \nabla \phi|^2 \right) + \left( |\nabla \bar{u}_e|^2 + \chi(p) \epsilon^2 \right)^{(p-2)/2} |\nabla \phi|^2 \right\} dx. \quad (34)$$

On the other hand, for any test function $\psi \in \mathcal{V}_\epsilon$, the second variational form $\delta^2 I^{(d)}_d$ is equal to

$$-1/2 \int_{\Omega} \left\{ |\tilde{\theta}_e|^2/\tilde{\zeta}_e^3 + 8 \cdot (2\tilde{\zeta}_e)^{(1-p)/(p-2)}/(p-2) \right\} \psi^2 dx. \quad (35)$$

From (34), by keeping in mind the fact

$$|\nabla \bar{u}_e \cdot \nabla \phi|^2 \leq |\nabla \bar{u}_e|^2 |\nabla \phi|^2,$$

one deduces immediately that, for $p \in (1, 2) \cup (2, +\infty)$,

$$\delta^2 I^{(e)}(\bar{u}_e) \geq 0. \quad (36)$$

From (35), taking the definition of $\tilde{\theta}_e$ into account, one has

$$\delta^2 I^{(d)}_d(\tilde{\zeta}_e) \geq 0, \quad p \in (1, 2) \quad (37)$$

and

$$\delta^2 I^{(d)}_d(\tilde{\zeta}_e) \leq 0, \quad p \in (2, +\infty). \quad (38)$$

Summarizing the above discussion, one proves Theorem 1.2.

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