An analytical solution with a given accuracy for a nonlinear mathematical model of a console-type construction

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Abstract. On the basis of the results obtained for one of the variants of the mathematical model of the console-type construction with sudden application of the load described by a nonlinear differential equation, an analytical approximate solution with a given accuracy is constructed. The results of the numerical experiment are presented. The proposed mathematical approach allows on the basis of analytical solutions to calculate for any point the coordinates fracture during bending of the beam.

1. Introduction
Rod structures, as well as console-type structures, are widespread, including in construction. A large number of scientific researches are devoted to mathematical and numerical modeling of such constructions [1-11]. It should be noted that it is real objects that are used in the construction, aerospace or oil industry under certain conditions can be considered and calculated as a cantilever structure. Various approaches to solving such problems indicate the need to obtain an analytical solution with a given accuracy for a nonlinear mathematical model of a console-type construction [12]. In the presented work the variant of the solution of this actual problem is offered.

2. Problem statement
The basis of the mathematical model of the console-type construction in General must be represented by a nonlinear differential equation [13,14].
Consider the problem of calculating the console-type construction using the mathematical approach presented in [13,14,15,16]. The axis X is directed along the axis of the structure and the axis Y is perpendicular to the longitudinal axis. External load – a sudden loading of the free end of the console-type beam concentrated force, represented by two components: static and dynamic (inertial).

In [13], was presented the mathematical model of the structure cantilever type, obtained by a change of variables in the form:

$$Y' = \Psi(x)\sqrt{1+Y'^2}$$

$$\Psi(x) = \frac{M_x}{EJ}$$

Here, $M_x$ – internal bending moment caused by static action of perturbing load; $EJ$ – bending stiffness of the beam.
Consider a mathematical generalization of equation (1)
\[ Y' = \Psi(x)\sqrt{1 + Y'^2} + F_x \]  

(2)

Here, \( F_x \) is the additional internal bending moment from the dynamic component of the external load (some function).

Equations (1) and (2) are nonlinear with movable singular points of the same character. If equation (1) can be solved in quadratures in general, then equation (2) in general is not solvable in quadratures. The search area for the solution of these equations consists of the area of analyticity and the neighborhood of movable singular point. It should be noted that solvability in quadratures in general requires numerical calculations using existing numerical methods [17, 18].

Due to the specificity of the movable singular point, the existing numerical methods do not allow obtaining a reliable result, since they are based on derivatives of the desired function, and the latter are associated with finite differences of the first order. At the first stage, the area of analyticity is considered, for which the idea proposed in [19] and successfully tested for a number of classes of nonlinear differential equations was developed [15, 16, 20]. At the same time, a posteriori estimates of the computational error are used to optimize a priori estimates of the analytical approximate solution.

The movable singular points require a special approach related to the nature of the movable singular point and their physical interpretation for building structures as the coordinate of failure.

3. Methods and results of the study

In the area of analyticity, for equation (2) with the initial condition

\[ Y(x_0) = Y_0 \]  

(3)

the theorem of existence and uniqueness of the solution was proved [12], which allows to construct an analytical approximate solution.

\[ Y_N(x) = \sum_{0}^{N} C_n(x-x_0)^n \]  

(4)

The following theorem allows us to obtain a priori error estimate for the analytical approximate solution (4), thus guaranteeing the accuracy of the calculations.

3.1. Theorem

Let:

1) \( \Psi(x), F_x \in C^\infty \) in the field \( |x-x_0| < \rho_1 \), \( \rho_1 = \text{const} \neq 0 \);

2) exist are \( M_{1,n} \) and \( M_{2,n} \) such that

\[ \left| \frac{\Psi^{(n)}(x_0)}{n!} \right| \leq M_{1,n}, \quad \left| \frac{F_x^{(n)}(x_0)}{n!} \right| \leq M_{2,n} \quad \forall n = 0, 1, ... \]

Then for the approximate solution (4) of the problem (2)-(3) in the domain \( |x-x_0| < \rho_2 \) the error estimate is valid

\[ \Delta Y_N(x) \leq \frac{3^{N+1}M(M+1)N^{N+1}|x-x_0|^{N+1}}{1-3(M+1)^N|x-x_0|} \]

here:

\[ M = \max_n \{ Y_0, M_{1,n}, M_{2,n} \}, \quad n = 0, 1, 2, ... \]

\[ \rho_2 = \min \left\{ \rho_1, \frac{1}{3(M+1)^N} \right\} \]
3.2. Proof of theorem

The condition of the theorem allows to decompose functions $\Psi(x)$ and $F_x$ into series:

$$\Psi(x) = \sum_{n=0}^{\infty} D_n (x-x_0)^n, \quad F_x = \sum_{n=0}^{\infty} A_n (x-x_0)^n$$

Then, based on the results of [12] we have:

$$Y(x) = \sum_{n=0}^{\infty} C_n (x-x_0)^n$$

(5)

in the field $|x-x_0| < \rho_2$, herewith

$$\rho_2 = \min \left\{ \rho_1, \frac{1}{3(M+1)^3} \right\}, \quad M = \max \{ |Y_0|, M_{1,2}, M_{2,2} \}, \quad n = 0, 1, 2, ...$$

At the same time, for the coefficients $C_n$ of the series b (5), an estimate was obtained

$$|C_n| \leq 3^n M (M+1)^{3n}.$$  (6)

Therefore,

$$|\Psi(x) - Y_{\chi}(x)| = \Delta Y_{\chi}(x) = \left| \sum_{n=1}^{\infty} C_n (x-x_0)^n \right|$$

or, given the assessment (6),

$$\Delta Y_{\chi}(x) = \left| \sum_{n=1}^{\infty} C_n (x-x_0)^n \right| \leq \sum_{n=1}^{\infty} 3^n (M+1)^{3n} M (x-x_0)^n = \frac{3^{N+1} M (M+1)^{N+1}}{1-3(M+1)} |x-x_0|^{N+1}$$

The assessment is fair in the field of

$$|x-x_0| < \frac{1}{3(M+1)^3}$$

Given the conditions of the theorem, we finally obtain the domain $|x-x_0| < \rho_2$, where

$$\rho_2 = \min \left\{ \rho_1, \frac{1}{3(M+1)^3} \right\}$$

4. Numerical experiment

The fulfillment of the conditions of the above theorem will be shown by an example.

Consider the task Cauchy (2)-(3), where $\Psi(x) = x$, $F_x = x^2$.

For source data (starting conditions) $x_0 = 0.5$, $Y(x_0) = 0.1$ and $x_1 = 0.54$ performed calculations.

In this case

$$M = 1, \quad \rho_2 = \frac{1}{3(M+1)^3} = \frac{1}{24}$$

The model problem calculation results are presented in table 1., in which $x_i$ – current argument value; $Y_{\chi}(x_i)$ – analytical approximate solution (4); $\Delta_i Y$ – a priori error estimation (theorem); $\Delta_i$ – a posteriori error estimation.
Table 1. Characteristics of the analytical approximate solution

| $x_i$ | $Y_i(x_i)$ | $\Delta Y$ | $\Delta_\iota$ |
|-------|------------|------------|---------------|
| 0.54  | 0.13283718 | 0.81537270 | 0.0005        |

For $\Delta_\iota = 0.0005$ the structure of the approximate solution (4) requires $N = 265$. Given, as the calculations show, that

$$\sum_{n=0}^{265} C_n (x - x_0)^n < \Delta_\iota$$

we conclude that the error of the analytical approximate solution $Y_i(x_i)$ does not exceed the value 0.0005.

Calculations of coefficients $C_n$ were carried out on recurrent ratios

$$C_n = \frac{1}{n} \left( \sum_{i=0}^{n-1} D_i B_{n-i} + A_{n-1} \right)$$

$$B_n = \left( \sum_{i=0}^{n-1} \left( \sum_{j=0}^{n-i} C_{1,n-j} C_{1,j} \right) C_{1,n-i}^{**} - \sum_{i=1}^{n} B_{n-i} B_i \right) \frac{1}{2B_0}, \quad \forall n = 2, 3, ...$$

Herewith

$$C_0 = Y(x_0), \quad B_0 = \sqrt{(C_0^2 + 1)}^{1}, \quad C_1 = D_0 B_0 + A_0$$

$$B_1 = \frac{3(C_0^2 + 1)^2 C_0 C_1}{B_0}, \quad Y^2(x) = \sum_{n=0}^{\infty} C_n^*(x - x_0)^n, \quad 1 + Y^2(x) = \sum_{n=0}^{\infty} C_n^{**}$$

$$C_{1,n}^{**} = C_n^*, \quad n = 1, 2, ..., \quad C_{1,0}^{**} = C_0^* + 1, \quad (1 + Y^2(x))^2 = \sum_{n=0}^{\infty} C_n^{**} (x - x_0)^n$$

$$\left(1 + Y^2(x)\right)^2 = \sum_{n=0}^{\infty} C_n^{**}, \quad C_{3,n}^{**} = \sum_{i=0}^{n} C_{2,n-i} C_{1,i}^{**}, \quad C_{2,n}^{**} = \sum_{i=0}^{n} C_{1,n-i} C_{1,i}^{**}$$

5. Conclusion

In this paper, an analytical approximate solution for the calculation of dynamic relocations at any point in the analyticity region of a console-type structure in bending from the action of a concentrated force is constructed, which allows obtaining the results of the stress-strain state. A priori estimates of the error of the calculations can be significantly improved with the help of a posteriori estimates, which is presented in the calculations. It is also possible to control the accuracy of the calculations. The results obtained in the work, which guarantee the specified accuracy, are the basis of the algorithm when writing a program for computer modeling and calculation of the console structure.

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