Entanglement Sudden Death as an Indicator of Fidelity in a Four-Qubit Cluster State

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I explore the entanglement evolution of a four qubit cluster state in a dephasing environment concentrating on the phenomenon of entanglement sudden death (ESD). Specifically, I ask whether the onset of ESD has an effect on the utilization of this cluster state as a means of implementing a single qubit rotation in the measurement based cluster state model of quantum computation. To do this I compare the evolution of the entanglement to the fidelity, a measure of how accurately the desired state (after the measurement based operations) is achieved. I find that ESD does not cause a change of behavior or discontinuity in the fidelity but may indicate when the fidelity of certain states goes to .5.

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I. INTRODUCTION

Entanglement is a uniquely quantum mechanical phenomenon in which quantum systems exhibit correlations above and beyond what is possible for classical systems. Entangled systems are thus an important resource for many quantum information processing protocols including quantum computation, quantum metrology, and quantum communication. Much work has been done in the area of research is to understand the possible degredation of entanglement under decoherence. Decoherence, unwanted interactions between the system and environment, is the major challenge confronting experimental implementations of quantum computation, metrology, and communication. Decoherence may be especially detrimental to highly non-classical, and hence the most potentially useful, entangled states. A manifestation of the detrimental affects of decoherence on entangled states is entanglement sudden death (ESD) in which entanglement is completely lost in a finite time despite the fact that the loss of system coherence is asymptotic. This aspect of entanglement has been well explored in the case of bi-partite systems and there are a number of studies looking at ESD in multi-partite systems. In addition, there have been several initial experimental ESD studies.

The ESD phenomenon is interesting on a fundamental level and important for the general study of entanglement. However, it is not yet clear what the affect of ESD is on quantum information protocols. Are different quantum protocols helped, hurt, or indifferent to ESD? Previous studies along these lines have been in the area of quantum error correction (QEC). An explicit study of the three-qubit phase flip code concludes that this specific code is indifferent to ESD. In this paper I take a first step in studying the affect of ESD on cluster state quantum computational gates. Specifically, I study a four qubit cluster state to see how ESD affects its utility as a means of implementing a general single qubit rotation for measurement based (cluster state) quantum computation. My approach will be to use an entanglement witness, the negativity and bi-partite concurrence as entanglement metrics and compare the behavior of these metrics under the influence of decoherence to the fidelity of the final state after the attempted single qubit rotation. In addition, I will study the entanglement that remains in the cluster state after two measurements and compare it to the fidelity of the state of the two unmeasured qubits.

The cluster state is a specific type of entangled state that can be used as an initial resource for a measurement based approach to quantum computation. A cluster state can be created by first rotating all qubits into the state \( \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \). Desired pairs of qubits are entangled by applying control phase (CZ) gates between them. In a graphical picture of a cluster state, qubits are represented by circles and pairs of qubits that have been entangled via a CZ gate are connected by a line. A cluster state with qubits arranged in a two-dimensional lattice, such that each qubit has been entangled with four nearest neighbors, suffices for universal QC.

After constructing the cluster state, any quantum computational algorithm can be implemented using only single-qubit measurements along axes in the \( x-y \) plane. These processing measurements are performed by column, from left to right, until only the last column is left unmeasured. The last column contains the output state of the quantum algorithm which can be extracted by a final readout measurement. One can view each row of the cluster-state lattice as the evolution of a single logical qubit in time. Two (logical) qubit gates are performed via a connection between two rows of the cluster state. CZ gates in particular are ‘built-in’ to the cluster state and simple measurement automatically implements the gate. Single qubit rotations can be performed when there is no connection between the measured qubit(s) and qubits in another row. In such a case the logical gate implemented by measurement along an angle \( \phi \) in the \( x-y \) plane is \( X(\pi m) H Z(\phi) \), where \( H \) is the Hadamard gate and \( Z(\alpha) (X(\alpha)) \) is a \( z \)- (\( x \)-) rotation by an angle \( \alpha \). The dependence of the logical operation on the
outcome of the measurement is manifest in \( m = 0, 1 \) for measurement outcome \(-1, +1\). An arbitrary single qubit rotation can be implemented via three logical single-qubit rotations of the above sort yielding

\[
HZ(\alpha + \pi m_\alpha)X(\beta + \pi m_\beta)Z(\gamma + \pi m_\gamma),
\]

where \((\alpha, \beta, \gamma)\) are the Euler angles of the rotation. For example, by drawing the Euler angles according to the Haar measure, a random single-qubit rotation can be implemented.

As with all quantum computing paradigms, cluster state quantum computation, both during the construction of the cluster state and during subsequent measurement, are subject to decoherence. We study a four qubit cluster chain, with no interaction between the qubits (beyond the initial conditional phase gates used to construct the cluster state) placed in a dephasing environment fully described by the Kraus operators

\[
K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}; \quad K_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}
\]

where we have defined the dephasing parameter \( p \). When all four qubits undergo dephasing we have 16 Kraus operators each of the form \( A_l = (K_j \otimes K_j \otimes K_k \otimes K_\ell) \) where \( l = 1, 2, ..., 16 \) and \( i, j, k, \ell = 1, 2 \). Though all of the above calculations are done with respect to \( p \), I implicitly assume that \( p \) increases with time, \( \tau \), at a rate \( \kappa \), such that \( p = 1 - e^{-\kappa \tau} \) and \( p \rightarrow 1 \) only at infinite times. For now I also assume equal dephasing for all four qubits.

In optical cluster state construction small (few qubit) cluster states are fused together to form larger cluster states [11]. The smaller states must be stored until they are needed and may be subject to decoherence (especially dephasing). In other cluster state implementations, where complete two-dimensional cluster states can be constructed in just a few steps [12], any four qubit chain may be attached to at least one other qubit. In this case our results may not be exact.

While entanglement is invariant to single qubit operations, decoherence is not and local operations may play a significant role in the entanglement dynamics of the state. Thus, if a cluster state must be stored in a decohering environment one would ideally like to choose a cluster state representation (within single qubit operations) that has the greatest immunity to the decoherence so as retain as much entanglement as possible. With this in mind a secondary aim of this paper is to study two representations of the four qubit chain cluster state and compare the affects of dephasing on these representations. The first representation of the four qubit cluster state is

\[
|C_4\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle).
\]

This representation minimizes the number of computational basis states having non-zero contribution. The second representation is:

\[
|C_{4H}\rangle = H_1 H_4 |C_4\rangle,
\]

where \( H_j \) is the single qubit Hadamard gate on qubit \( j \). This is the state one would get by initially rotating each qubit into the state \( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) and applying controlled phase gates \( CZ_{12}, CZ_{23}, \) and \( CZ_{34} \). We note that ‘connections’ between qubits may be added or removed by single qubit rotations (though the entanglement stays constant) through changing the operation performed via measurement [13].

The four qubit cluster has pure four qubit entanglement. Thus, for example, there is no bi-partite concurrence between any of the two qubits. As an entanglement metric we use the negativity, \( N \), for which we will simply use the most negative eigenvalue of the partial transpose of the density matrix [19]. There are a number of inequivalent forms of the negativity for the four qubit cluster state: the partial transpose may be taken with respect to any single qubit, \( N_1 \), or the partial transpose may be taken with respect to two qubits: \( N_{12} \), \( N_{13} \), or \( N_{14} \).

A further method of monitoring entanglement evolution is via the expectation value of the state with respect to an appropriate entanglement witness [20]. Entanglement witnesses are observables with positive or zero expectation value for all states in a specified class and a negative expectation value for at least one state of the specified class. Entanglement witnesses may allow for an efficient means of determining whether entanglement is present in a state (as opposed to inefficient state tomography). This is especially important for experimental implementations as it may be the only practical means of deciding whether or not sufficient entanglement is present in the system. The entanglement witnesses I use are designed to detect cluster states and will be either \( WC_{14} = \frac{1}{2} - |C_4\rangle\langle C_4| \) or \( WC_{4H} = \frac{1}{2} - |C_{4H}\rangle\langle C_{4H}| \) depending on the representation [21].

II. ESD IN A FOUR QUBIT CLUSTER STATE

Our first step is to determine at what dephasing strength, \( p \), (if any) the four qubit cluster state exhibits ESD. The final state of the four qubit system after dephasing is given by \( \rho_r(p) = \sum_l (A_l|C_4\rangle\langle C_4|A_l) \) where \( r = 4, 4H \). Figure 1 shows the evolution of our chosen entanglement metrics for initial cluster states as a function of \( p \). For the initial state \( |C_4\rangle \) the expectation value of the final state after dephasing with respect to the entanglement witness, \( WC_{44} \), is given by \(-\frac{1}{2}(p^2 - 4p + 2)\). Thus, cluster state entanglement can be detected by the entanglement witness for \( p < 2 - \sqrt{2} \approx 0.586 \). Interestingly, the expectation value with respect to the entanglement witness is equal to \( N_{12} \), the most negative eigenvalue of the partial transpose of the final state with respect to qubits 1 and 2, which thus exhibits ESD at the same value. \( N_{1}, N_{13}, \) and \( N_{14} \) do not undergo ESD. \( N_{1} \), the lowest eigenvalue of the partial transpose of the final state with respect to one qubit, is given by \(-\frac{1}{4}(p^2 - 3p + 2)\). The most negative eigenvalues of the partial transpose of the
and ESD occurs at \( p_N \) (entanglement witness, \( W \approx p_N \)) exhibited at \( p_N \) (chained line), dashed line) for initial states \(|C_4\rangle\) (left) and \(|C_{4H}\rangle\) (right) as a function of dephasing strength \( p \) on all four qubits. For initial state \(|C_4\rangle\) there is no ESD for \( N_1 = N_{12} = 16 \). ESD is exhibited at \( N_1 \approx .586 \). This is the same value for which \( N_1 \) exhibits ESD. ESD for \( N_3 \) is exhibited at \( p = .938 \). The entanglement witness, \( W_{C_{4H}} \) fails to detect entanglement for \( p \gtrsim .535 \).

For the initial state \(|C_4\rangle\) under dephasing bound entanglement is present in the state for \( p \gtrsim .586 \). For the initial state \(|C_{4H}\rangle\) \( N_1 = N_{12} \) with the most negative eigenvalue of the partial transpose of the final state given by \( \frac{1}{16} (-24 - 4\tilde{p}^2 + 6p - p^2) \), where \( \tilde{p} = \sqrt{1 - p} \). Both exhibit ESD at \( p = -2 + 2\sqrt{2} \approx .828 \). For \( N_{13} \) the most negative eigenvalue is given by \( \frac{1}{16} (-4\tilde{p} + 2p - p^2) \) and is the last negativity to exhibit ESD, which occurs when \( p \approx .938 \). For \( N_{14} \) the lowest eigenvalue is doubly degenerate and given by \( \frac{1}{16} (-4 + 4p + p^2) \). ESD is exhibited at \( p = -2 + 2\sqrt{2} \approx .828 \) which is the same dephasing value at which \( N_1 \) exhibits ESD. Again note the presence of bound entanglement for \( .828 \leq p \leq .938 \). The expectation value of the final state with respect to the entanglement witness, \( W_{C_{4H}} \) is given by \( \frac{1}{16} (-2\tilde{p} + p(8 + 4\tilde{p} - p)) \). Thus, the witness fails to detect entanglement for \( p > 2(-\sqrt{2} + 2^{3/4}) \approx .535 \). The evolution of the above entanglement metrics as a function of \( p \) are shown in Fig. 1.

### III. FINAL STATE FIDELITY

Having observed that some sort of ESD occurs for both of our chosen representations of the four qubit cluster state, we now seek to determine whether ESD affects the utilization of the cluster state as a means of implementing a general single qubit rotation in the measurement based cluster model of quantum computation. To implement such a rotation measurements at an angle \( \theta_l \) with respect to the positive \( x \) axis in the \( x-y \) plane are performed on the first three qubits, \( t = 1, 2, 3 \), giving a one qubit final state as a function of the measurement angles and the dephasing strength, \( \rho_f(p, \theta_1, \theta_2, \theta_3) \). We look at the fidelity of the state of the unmeasured qubit as compared to the same state without dephasing:

\[
F_r(p, \theta_1, \theta_2, \theta_3) = \text{Tr}[\rho_f(p, \theta_1, \theta_2, \theta_3)\rho_f(0, \theta_1, \theta_2, \theta_3)].
\]

(4)

For convenience we have assumed that the outcome of each measurement is \( -1 \) in the chosen measurement basis, such that \( m = 0 \) and no extra \( X \) rotations are necessary. A measurement of \( +1 \) would simply add the necessity for an \( X \) rotation. We note that the fidelity calculation was done only for initial states \(|C_4\rangle\) and \(|C_{4H}\rangle\) while full process tomography is needed to completely determine the dynamics of the single qubit rotation.

For initial state \(|C_4\rangle\) the fidelity can be determined analytically,

\[
F_{C4}(p, \theta_1, \theta_2, \theta_3) = \frac{1}{4}(4 + p(p - 3) + p(1 - p)\cos(2(\theta_1 + \theta_2))).
\]

(5)

Notice that for this representation, \( \theta_3 \) cancels and the other measurement angles contribute only as \( \theta_1 + \theta_2 \). The fidelity is plotted in Fig. 2 and shows an oscillating plane steadily and smoothly decreasing toward, but never reaching, \( F_{C4} = .5 \). The amplitude of the oscillations decrease at high and low values of \( p \) and reach a maximum at \( p \approx .5 \). We do not see any sort of sharp transition or discontinuity in the behavior of \( F_{C4} \) at \( p \approx .586 \) as one might expect due to the sudden disappearance of \( N_{12} \) for the complete four qubit cluster.

As mentioned above, the initial state \(|C_4\rangle\) undergoes ESD only with respect to \( N_{12} \). One may suggest that the reason ESD is not manifest in the fidelity degradation of the unmeasured qubit for this initial state is because there is still some entanglement, \( N_1 \), which does not exhibit ESD, present in the state. To explore this we now

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**FIG. 1:** (Color online) Entanglement evolution as measured by \( -\text{Tr}[W_4\rho_f(p)] \) (solid line), \( N_1 \) (large dashed line), \( N_{12} \) (chained line), \( N_{13} \) (medium dashed line), and \( N_{14} \) (small dashed line) for initial states \(|C_4\rangle\) (left) and \(|C_{4H}\rangle\) (right) as a function of dephasing strength \( p \) on all four qubits.

**FIG. 2:** (Color online) Fidelity of the state of the single unmeasured qubit from the four qubit cluster state \(|C_4\rangle\) as a function of the dephasing strength \( p \) and the sum of the first two measurement angles \( \theta_1 + \theta_2 \). The third measurement angle \( \theta_3 \) does not affect the fidelity. The unmeasured qubit is the final state of the cluster computational logical qubit after performance of an arbitrary single qubit rotation via measurement. There is no sign of any sort of discontinuity that might have been expected due to ESD at \( p \approx .586 \).
look at the initial state $|C_{4H}\rangle$ which, under dephasing, exhibits ESD for all negativity measures. Following the above, we find the fidelity of the final single qubit state as a function of $p$ and measurement angles $\theta_i, t = 1, 2, 3$ for the initial state $|C_{4H}\rangle$ to be:

$$
F_{C_{4H}}(p, \theta_1, \theta_2, \theta_3) = \frac{1}{64} (2p^3 \cos(2(\theta_1 - \theta_2)) + 4p^3 \cos(2(\theta_2 + \theta_3)) + 2p^3 \cos(2(\theta_1 + \theta_3)) + 2p^3 \cos(2(\theta_1 - \theta_3)) + 4p^3 \cos(2(\theta_1 - \theta_2 - \theta_3)) + 2p^3 \cos(2(\theta_2 - \theta_3)) + 2p^3 \cos(2(\theta_1 + \theta_2 - \theta_3)) + 4(p - 1) \cos(2\theta_1) (p - 2(p + 1) \cos \theta_3^2) + 12p' \cos(2\theta_3) + 4(11 + 5p^3 + 3 \cos(2\theta_3)) + 2p' \cos(2(\theta_1 + \theta_3)) + 2p' \cos(2(\theta_1 - \theta_2 + \theta_3)) + 2p' \cos(2(\theta_2 + \theta_3)) + 6p^3 \cos(2(\theta_1 + \theta_2 + \theta_3)) + 16 \cos(2\theta_2) \cos \theta_3^2 \sin \theta_1^2 + 8p^3 \cos(2\theta_2) \sin \theta_1^2 (1 + 2 \cos(2\theta_2) \sin \theta_1^2) - \cos \theta_2 \sin(2\theta_1) \sin(2\theta_3)) + 8p(-4 \cos \theta_3^2 (1 + \cos(2\theta_2) \sin \theta_1^2) + \cos \theta_2 \sin(2\theta_1) \sin(2\theta_3))).
$$

(6)

**FIG. 3:** (Color online) Left: using initial state $|C_{4H}\rangle$, fidelity of the state of the single unmeasured qubit such that an arbitrary rotation has been performed via the cluster state as a function of two of the measurement angles $\theta_1$ and $\theta_2$. The curves are $\theta_3 = \pi/16$ (gray) and $\pi/3$ (light) for $p = .3$. The black curve is the fidelity of the state of the single unmeasured qubit with dephasing for the initial state $|C_{4H}\rangle$. This is plotted so as to compare the range of fidelities of the two initial states given the same evolution. Right: fidelity as a function of dephasing strength and $\theta_1$ with $\theta_2 = \pi/4$ and the two curves again equal to $\theta_3 = \pi/16$ (gray) and $\pi/3$ (light). As a function of $p$ we see the overall fidelity decreases steadily toward .5 without any discontinuity.

**FIG. 4:** (Color online) The concurrence between unmeasured qubits 3 and 4 after measurement on the first two qubits having started with the state $|C_{4}\rangle$. The concurrence is plotted as a function of dephasing strength and measurement axes (which contribute only as $\theta_1 + \theta_2$). There is no ESD exhibited for this concurrence function.

**IV. TWO QUBIT FIDELITIES AND CONCURRENCE**

So far our exploration of fidelity decay and entanglement as functions of dephasing indicate that ESD does not affect the utility of a cluster state as a means of implementing an arbitrary logical single qubit rotation. However, the pictures changes when we explore fidelities and sudden bi-partite entanglement death of two qubits after having measured the other two qubits. To quantify the bi-partite entanglement between the two unmeasured qubits I use the concurrence $C_{jk}$. The concurrence between two qubits $j$ and $k$ with density matrix $\rho_{jk}$ is usually defined as the maximum of zero and $\Lambda$, where $\Lambda = \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}$ and the $\lambda_i$ are the eigenvalues of $\rho_{jk}(\sigma_{y}^j \otimes \sigma_{y}^k)\rho_{jk}^+(\sigma_{y}^j \otimes \sigma_{y}^k)$ in decreasing order. $\sigma_{y}^j$ is the $y$ Pauli matrix of qubit $i$. For the purposes of clearly seeing at what point ESD occurs we will use $\Lambda$ as the concurrence noting that ESD occurs when $\Lambda = 0$ in finite time (i. e. before $p \to 1$).

We start with the initial state $|C_{4}\rangle$, with measurements
FIG. 5: (Color online) Fidelity (dashed line) of the state of qubits 2 and 4 after measurements on qubits 1 and 3 of the initial state \( |C_4 \rangle \) compared to the concurrence (solid line) between these same qubits. Note that the fidelity crosses .5 (horizontal light line at \( p \approx 0.586 \)) and is given by

\[
f_{34}(\theta_1, \theta_2) = \frac{1}{16} (8 + \frac{1}{2} p + p(-5 - 4\tilde{p} + p) - (p-1)\cos(2\theta_1) - 2\cos(2\theta_2)\sin\theta_1^2)).
\]

The dephasing values at which ESD is exhibited approach \( p \approx 0.704 \) as \( \theta_1 \) goes to zero. Top right: concurrence as a function of \( \theta_1 \) and \( \theta_2 \) for \( \theta_2 = 0 \) (dotted line), \( \theta_2 = \pi/4 \) (solid line), and \( \theta_2 = \pi/2 \) (dashed line). Bottom right: contours of concurrence equal to zero showing where ESD occurs (values of \( \theta_2 \) as in previous contour plot). The dephasing values at which ESD is exhibited approach .704, the exact value for which the fidelity goes to .5.

Measurement on qubits 1 and 4 or qubits 2 and 3 give the exact same results as the measurements on 1 and 3.

We see similar correlations between fidelity and entanglement metrics when measuring certain pairs of qubits of the initial state \( |C_{4H} \rangle \). The fidelity of the state of qubits 3 and 4 after measuring qubits 1 and 2 is given by:

\[
F_{34}(\theta_1, \theta_2) = \frac{1}{16} (8(1 + \tilde{p}) + p(-5 - 4\tilde{p} + p) - (p-1)\cos(2\theta_1) - 2\cos(2\theta_2)\sin\theta_1^2)).
\]

As shown in Fig. 6 when \( p \approx 0.704 \) the fidelity goes to .5 as \( \theta_1 \) approaches 0 or \( \pi \) or when \( \theta_2 \) approaches \( \pi/2 \). This is also the maximum dephasing value for which we find ESD of the concurrence between unmeasured qubits 3 and 4 as shown in the figure (we do not have an analytical solution for the concurrence). Thus, while once again we do not have a change of fidelity behavior due to ESD, the sudden death of concurrence does indicate the lowering of fidelity to the critical value of .5.

The fidelity of the state of qubits 2 and 4 upon mea-
suring qubits 1 and 3 is given by:

\[
F_{24}(p, \theta_1, \theta_3) = \frac{1}{16}(10 + 6\tilde{\rho} - p(7 + 2\tilde{\rho} - p) + \cos(2\theta_4) \\
\times (-2 + 2\tilde{\rho} + 3p - p^2) + 2\cos(2\theta_1) \\
\times (\tilde{\rho} - \tilde{\rho}^3 + 2\cos(2\theta_4) - (p - 2)(p - 1)\sin\theta_1^2))
\]

and is plotted in Fig. 7 along with the concurrence between unmeasured qubits 2 and 4. This does not appear to be a correlation between the fidelity and concurrence with respect to these two unmeasured qubits. However, the maximum p at which the fidelity crosses .5, when \(\theta_1 = \theta_3 = 0\), is \(2\sqrt{2} - 2 \simeq .828\), the exact value where the four qubit state \(|C_{4|H}\rangle\) exhibits ESD for a number of entanglement measures. The minimum p at which the fidelity crosses \(p\) = 0.618, which is also equal to the maximum p at which ESD of concurrence is exhibited. Though the initial state in this example was \(|C_{4|H}\rangle\) this is the value at which ESD occurs for the initial state \(|C_4\rangle\). Such cross-correlation between the different cluster state representations can come from the measurements: measuring some of the qubits at certain angles transforms the state from one representation to the other.

The fidelity of the state of qubits 2 and 3 upon measuring qubits 1 and 4 is given by:

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F_{34}(p, \theta_2, \theta_4) = \frac{1}{16}(10 + 6\tilde{\rho} - p(7 + 6\tilde{\rho} - p) + \cos(2\theta_4) \\
\times (-2 + 2\tilde{\rho} + 3p - p^2) + 2\cos(2\theta_2) \\
\times (\tilde{\rho} - \tilde{\rho}^3 + 2\cos(2\theta_4) - (p - 2)(p - 1)\sin\theta_2^2)),
\]

and plotted in Fig. 8 along with the concurrence between unmeasured qubits 2 and 3. As in the previous case ESD may be an indicator of fidelity. The maximum p at which the fidelity crosses .5, which occurs for \(\theta_1 = \theta_3 = 0\), is \(2\sqrt{2} - 2 \simeq .828\), the exact value where the four qubit state \(|C_{4|H}\rangle\) exhibits ESD for a number of entanglement measures. The minimum p at which the fidelity crosses \(p\) = 0.618, which is also equal to the maximum p at which ESD of concurrence is exhibited. Though the initial state in this example was \(|C_{4|H}\rangle\) this is the value at which ESD occurs for the initial state \(|C_4\rangle\). Such cross-correlation between the different cluster state representations can come from the measurements: measuring some of the qubits at certain angles transforms the state from one representation to the other.

The fidelity of the state of qubits 2 and 3 upon measuring qubits 1 and 4 is given by:

\[
F_{14}(p, \theta_2, \theta_3) = \frac{1}{16}(10 + 6\tilde{\rho} - p(7 + 6\tilde{\rho} - p) \\
\times (-2 + 2\tilde{\rho} + 3p - p^2) + 2\cos(2\theta_2) \\
\times (\tilde{\rho} - \tilde{\rho}^3 + 2\cos(2\theta_4) - (p - 2)(p - 1)\sin\theta_2^2)),
\]

and plotted in Fig. 8 along with the concurrence between unmeasured qubits 2 and 3. As in the previous case ESD may be an indicator of fidelity. The maximum p at which the fidelity crosses .5, which occurs for \(\theta_1 = \theta_3 = 0\), is \(2\sqrt{2} - 2 \simeq .828\), the exact value where the four qubit state \(|C_{4|H}\rangle\) exhibits ESD for a number of entanglement measures. The minimum p at which the fidelity crosses \(p\) = 0.618, which is also equal to the maximum p at which ESD of concurrence is exhibited. Though the initial state in this example was \(|C_{4|H}\rangle\) this is the value at which ESD occurs for the initial state \(|C_4\rangle\). Such cross-correlation between the different cluster state representations can come from the measurements: measuring some of the qubits at certain angles transforms the state from one representation to the other.

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\times (-2 + 2\tilde{\rho} + 3p - p^2) + 2\cos(2\theta_1) \\
\times (\tilde{\rho} - \tilde{\rho}^3 + 2\cos(2\theta_4) - (p - 2)(p - 1)\sin\theta_1^2)),
\]

and plotted in Fig. 7 along with the concurrence between unmeasured qubits 2 and 4. This does not appear to be a correlation between the fidelity and concurrence with respect to these two unmeasured qubits. However, the maximum p at which the fidelity crosses .5, when \(\theta_1 = \theta_3 = 0\), is \(2\sqrt{2} - 2 \simeq .828\), the exact value where the four qubit state \(|C_{4|H}\rangle\) exhibits ESD for a number of entanglement measures. The minimum p at which the fidelity crosses \(p\) = 0.618, which is also equal to the maximum p at which ESD of concurrence is exhibited. Though the initial state in this example was \(|C_{4|H}\rangle\) this is the value at which ESD occurs for the initial state \(|C_4\rangle\). Such cross-correlation between the different cluster state representations can come from the measurements: measuring some of the qubits at certain angles transforms the state from one representation to the other.

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\times (\tilde{\rho} - \tilde{\rho}^3 + 2\cos(2\theta_4) - (p - 2)(p - 1)\sin\theta_2^2)),
\]

and plotted in Fig. 8 along with the concurrence between unmeasured qubits 2 and 3. As in the previous case ESD may be an indicator of fidelity. The maximum p at which the fidelity crosses .5, which occurs for \(\theta_1 = \theta_3 = 0\), is \(2\sqrt{2} - 2 \simeq .828\), the exact value where the four qubit state \(|C_{4|H}\rangle\) exhibits ESD for a number of entanglement measures. The minimum p at which the fidelity crosses \(p\) = 0.618, which is also equal to the maximum p at which ESD of concurrence is exhibited. Though the initial state in this example was \(|C_{4|H}\rangle\) this is the value at which ESD occurs for the initial state \(|C_4\rangle\). Such cross-correlation between the different cluster state representations can come from the measurements: measuring some of the qubits at certain angles transforms the state from one representation to the other.
in the behavior of the entanglement. Another assumption is that the dephasing strength is equal on all four qubits. This is unrealistic for a number of reasons but especially so if not all of the measurements are performed at the same time (non-simultaneous measurements are necessary when trying to implement a given logical rotation because the measurement axes for a given qubit depends on the outcome of the measurement on the previous qubit). A way to relax this assumption without significantly increasing the number of variables in the problem may be to add a $k\Delta \rho$ term to the dephasing strength where $\Delta \rho$ represents the dephasing the occurs during the time between subsequent measurements and $k$ is an integer.

V. CONCLUSIONS

In conclusion, I have studied the entanglement evolution of a four qubit (chain) cluster state in a dephasing environment. Specifically, I have looked at two representations of the state differing by single qubit rotations. Both of these representations exhibit entanglement sudden death under sufficient dephasing. The difference in the dephasing strength at which this occurs may be important when deciding in what representation to store a cluster state. The issue of storage is especially relevant during the construction of optical cluster states but may have relevance to other implementations as well.

I asked whether ESD affects the utility of the cluster state in implementing a general single qubit rotation in the cluster state measurement based quantum computation paradigm. Judging from the fidelity decay of the single unmeasured qubit as a function of dephasing strength and the measurement axes angles of the three measurements the answer would seem to be no. I see no indication in the fidelity behavior that ESD has taken place. Instead the fidelity decreases smoothly with increased dephasing with no discontinuities or dramatic changes in behavior. However, there are clear correlations (sometimes total and sometimes at certain limits) between the fidelity of the state of two qubits remaining from the four qubit cluster state after measurement on the other two qubits, and ESD of the negativity for the entire cluster state. ESD of the negativity for the entire cluster state may be an indicator of how badly a certain cluster state operation was carried out. However, this is not the same as saying that ESD itself negatively affects quantum information protocols.

The question of whether ESD affects quantum information protocols requires further study and may be related to the more general issue of the role of entanglement in quantum computation.

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TABLE I: The three parts of the table show 1) the values of \( p \) at which ESD occurs in four qubit cluster states subject to dephasing as measured by the expectation value of the proper entanglement witness (\( W_r \), \( r = 4, AH \)), the negativity with partial transpose taken with respect to one (\( N_1 \)) and two (\( N_{ik} , k = 2, 3, 4 \) qubits, 2) the value of \( p \) for which the concurrence, \( C_{jk} \), between two unmeasured qubits \( j \) and \( k \) of the four qubit cluster state goes to zero after measurements on the other two qubits, 3) the value of \( p \) for which the fidelity of the dephased states goes to .5 for two qubit states \( F_{jk} \), after measurement on the other two qubits, or the fidelity of the one qubit states \( F_r \), after measurement on three qubits.

| \( W_r \) | \( N_1 \) | \( N_{12} \) | \( N_{13} \) | \( N_{14} \) |
|---|---|---|---|---|
| \( C_{44} \) | .586 | none | .586 | none | none |
| \( C_{44}^{\text{tr}} \) | .535 | .828 | .828 | .938 | .828 |
| \( C_{44} \) | none | \( C_{24} \) | \( C_{23} \) | \( C_{14} \) | none |
| \( C_{24} \) | \( < .704 \) | \( < .618 \) | \( < .586 \) | \( .568 \leq p < .586 \) | none |
| \( F_{34} \) | none | \( F_{24} \) | \( F_{23} \) | \( F_{14} \) | \( F_r \) |
| \( C_{44} \) | none | .586 | .586 | none | none |
| \( F_{44} \) | .618 \( \leq p \leq .704 \) | .568 \( \leq p \leq .828 \) | .586 \( \leq p \leq .828 \) | .568 \( \leq p \leq .586 \) | none |

[1] M Nielsen, I. Chuang, *Quantum information and Computation* (Cambridge University Press, Cambridge, 2000).
[2] For a recent review see R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, arXiv:quant-ph/0702225.
[3] C. Simon and J. Kempe, Phys. Rev. A 65, 052327 (2002); W. Dur and H.-J. Briegel, Phys. Rev. Lett. 92, 180403 (2004); M. Hein, W. Dur, and H.-J. Briegel, Phys. Rev. A 71, 032350 (2005); S. Bandyopadhyay and D.A. Lidar, Phys. Rev. A 72, 042339 (2005); O. Guhne, F. Bodosky, and M. Blaauboer, Phys. Rev. A 78, 060301 (2008).
[4] L. Diosi, in *Irreversible Quantum Dynamics*, edited by F. Benatti and R. Floreanini, Lect. Notes Phys. 622, (Springer-Verlag, Berlin) 157 (2003); P.J. Dodd and J.J. Halliwell, Phys. Rev. A 69, 052105 (2004).
[5] T. Yu and J.H. Eberly, Phys. Rev. Lett. 93, 140404 (2004); ibid. 97, 140403 (2006).
[6] I. Sainz and G. Bjork, Phys. Rev. A 76, 042313 (2007).
[7] L. Aolita, R. Chaves, D. Cavalcanti, A. Acin, and L. Davidovich, Phys. Rev. Lett. 100, 080501 (2008).
[8] C.E. Lopez, G. Romero, F. Lastra, E. Solano, and J.C. Retamal, Phys. Rev. Lett. 101, 080503 (2008).
[9] M. Yonac, T. Yu, J.H. Eberly, J. Phys. B 39, 5621 (2006); ibid. 40, 545 (2007).
[10] I. Sainz and G. Bjork, Phys. Rev A 77, 052307 (2008).
[11] Y.S. Weinstein, Phys. Rev A 79, 012318 (2009).
[12] M.P. Almeida, et al., Science 316, 579 (2007); J. Laurat, K.S. Choi, H. Deng, C.W. Chou, and H.J. Kimble, Phys. Rev. Lett. 99, 180504 (2007); A. Salles, F. de Melo, M.P. Almeida, M. Hor-Meyll, S.P. Walborn, P.H. Souto Ribeiro, and L. Davidovich, Phys. Rev. A 78, 022322 (2008).
[13] H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. 86, 910 (2001).
[14] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).
[15] R. Raussendorf, D. E. Browne, and H. J. Briegel, Phys. Rev. A 68, 022312 (2003).
[16] D.E. Browne and T. Rudolph, Phys. Rev. Lett., 95, 010501, (2005).
[17] Y.S. Weinstein, C.S. Hellberg, and J. Levy, Phys. Rev. A 72, 020304 (2005); Y.S. Weinstein and C.S. Hellberg, Phys. Rev. Lett. 98, 110501 (2007); J.Q. You, X. Wang, T. Tanamoto, and F. Nori, Phys. Rev. A 75, 052319 (2007); L. Jiang, A.M. Rey, O. Romero-Isert, J.J. Garcia-Ripoll, A. Sanpera, and M.D. Lukin, arXiv:0811.3049.
[18] P. Walther, K.J. Resch, T. Rudolph, E. Schenk, H. Weinfurter, V. Vedral, M. Aspelmeyer, and A. Zeilinger, Nature (London) 434, 169 (2005); G. Gilbert, M. Hamrick, and Y.S. Weinstein, Phys. Rev. A 73, 064303 (2006).
[19] G. Vidal and R.F. Werner, Phys. Rev. A 65, 032314 (2002).
[20] B.M. Terhal, Phys. Lett. A 271, 319 (2000); M. Lewenstein, B. Kraus, J.I. Cirac, and P. Horodecki, Phys. Rev. A 62, 052310 (2000).
[21] G. Toth and O. Guhne, Phys. Rev. Lett. 94, 060501 (2005).
[22] S. Hill and W.K. Wootters, Phys. Rev. Lett 78, 5022 (1997).