Hyperbolic groups admit proper affine isometric actions on $l^p$-spaces

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1 Introduction

Let $X$ be a Banach space and $\Gamma$ be a countable discrete group. An affine and isometric action $\alpha$ of $\Gamma$ on $X$ is said to be proper if $\lim_{g \to \infty} \|\alpha(g)\xi\| = \infty$ for every $\xi \in X$. If $\Gamma$ admits a proper isometric affine action on Hilbert space, then $\Gamma$ is said to be of Haagerup property [9] or a-T-menable [12].

Bekka, Cherix and Valette proved that an amenable group admits a proper affine isometric action on Hilbert space [3]. This result has important applications to K-theory of group $C^*$-algebras [13] [14].

It is well known that an infinite Property (T) group doesn’t admit a proper affine isometric action on Hilbert space. The purpose of this paper is to prove the following result.

**Theorem 1.1.** If $\Gamma$ is a hyperbolic group, then there exists $2 \leq p < \infty$ such that $\Gamma$ admits a proper affine isometric action on an $l^p$-space.

We remark that the constant $p$ depends on the hyperbolic group $\Gamma$ (in the special case that $\Gamma$ is the fundamental group of a negatively curved compact manifold, $p$ depends on the dimension of the manifold), and $p$ is strictly greater than 2 if the hyperbolic $\Gamma$ is infinite and has Property (T). Recall that a theorem of A. Zuk states that hyperbolic groups are generically of Property (T) [22].

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In [1], Bader and Gelander studied Property (T) for $L^p$-spaces. Their work has extremely interesting applications in Fisher and Margulis’ theory of local rigidity [6]. Bader and Gelander raised the question if any affine isometric action of a Property (T) group on an $L^p$-space has a fixed point (Question 12 in [1]). Theorem 1.1 implies that the answer to this question is negative for infinite hyperbolic groups with Property (T).

The proof of Theorem 1.1 is based on a construction of Igor Mineyev [18] and is reminiscent of Alain Connes’ construction of Chern character of finitely summable Fredholm modules for rank one groups [5].

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2 Hyperbolic groups and bicombings.

In this section, we recall the concepts of hyperbolic groups and bicombings.

2.1 Hyperbolic groups.

Let $\Gamma$ be a finitely generated group. Let $S$ be a finite generating set for $\Gamma$. Recall that the Cayley graph of $\Gamma$ with respect to $S$ is the graph $G$ satisfying the following conditions:

1. the set of vertices in $G$, denoted by $G^{(0)}$, is $\Gamma$;
2. the set of edges is $\Gamma \times S$, where each edge $(g, s) \in \Gamma \times S$ spans the vertices $g$ and $gs$.

We endow $G$ with the path metric $d$ induced by assigning length 1 to each edge. Notice that $\Gamma$ acts freely, isometrically and cocompactly on $G$. A geodesic path in $G$ is a shortest edge path. The restriction of the path metric $d$ to $\Gamma$ is called the word metric.
A finitely generated group $\Gamma$ is called hyperbolic, if there exists a constant $\delta \geq 0$ such that all the geodesic triangles in $G$ are $\delta$-fine in the following sense: if $a$, $b$, and $c$ are vertices in $G$, $[a, b]$, $[b, c]$, and $[c, a]$ are geodesics from $a$ to $b$, from $b$ to $c$, and from $c$ to $a$, respectively, and points $\bar{a} \in [b, c]$, $v, \bar{c} \in [a, b]$, $w, \bar{b} \in [a, c]$ satisfy
\[
d(b, \bar{c}) = d(b, \bar{a}), \quad d(c, \bar{a}) = d(c, \bar{b}), \quad d(a, v) = d(a, w) \leq d(a, \bar{c}) = d(a, \bar{b}),
\]
then $d(v, w) \leq \delta$.

The above definition of hyperbolicity does not depend on the choice of the finite generating set $S$. See [8] for other equivalent definitions.

For vertices $a$, $b$, and $c$ in $G$, the Gromov product is defined by
\[
(b|c)_a := d(a, \bar{b}) = d(a, \bar{c}) = \frac{1}{2} \left[ d(a, b) + d(a, c) - d(b, c) \right].
\]
The Gromov product can be used to measure the degree of cancellation in the multiplication of group elements in $G$.

2.2 Bicombings.

Let $\Gamma$ be a finitely generated group. Let $G$ be its Cayley graph with respect to a finite generating set. A bicombing $q$ in $G$ is a function assigning to each ordered pair $(a, b)$ of vertices in $G$ an oriented edge-path $q[a, b]$ from $a$ to $b$. A bicombing $q$ is called geodesic, if each path $q[a, b]$ is geodesic, i.e. a shortest edge path. A bicombing $q$ is $\Gamma$-equivariant if $q[g \cdot a, g \cdot b] = g \cdot q[a, b]$ for each $a, b \in G^{(0)}$ and each $g \in \Gamma$.

3 A construction of Mineyev.

The purpose of this section is to recall Mineyev’s contraction for hyperbolic groups and its properties [18].

Let $\Gamma$ be a hyperbolic group and $G$ be a Cayley graph of $\Gamma$ with respect to a finite generating set. We endow $G$ with the path metric $d$, and identify $\Gamma$ with $G^{(0)}$, the set of vertices of $\Gamma$. Let $\delta \geq 1$ be a positive integer such that all the geodesic triangles in $G$ are $\delta$-fine.
The ball \( B(x, R) \) is the set of all vertices at distance at most \( R \) from the vertex \( x \). The sphere \( S(x, R) \) is the set of all vertices at distance \( R \) from the vertex \( x \). Pick an equivariant geodesic bicombing \( q \) in \( G \). By \( q[a, b](t) \) we denote the point on the geodesic path \( q[a, b] \) at distance \( t \) from \( a \). Recall that \( C_0(\Gamma, \mathbb{Q}) \) is the space of all finitely supported 0-chains (in \( \Gamma = G^{(0)} \)) with coefficients in \( \mathbb{Q} \), i.e. \( C_0(\Gamma, \mathbb{Q}) = \{ \sum_{\gamma \in \Gamma} c_{\gamma} \gamma : c_{\gamma} \in \mathbb{Q} \}, \) where \( \sum_{\gamma \in \Gamma} c_{\gamma} \gamma \) is finitely supported.

For each \( p \geq 1 \), endow \( C_0(\Gamma, \mathbb{Q}) \) with the \( l^p \)-norm \( || \cdot ||_p \). We identify \( \Gamma \) with the standard basis of \( C_0(\Gamma, \mathbb{Q}) \). Therefore the left action of \( \Gamma \) on itself induces a left action on \( C_0(G, \mathbb{Q}) \).

For \( v, w \in \Gamma \), the flower at \( w \) with respect to \( v \) is defined to be

\[
Fl(v, w) := S(v, d(v, w)) \cap B(w, \delta) \subseteq \Gamma.
\]

For each \( a \in \Gamma \), we define \( pr_a : \Gamma \to \Gamma \) by:

1. \( pr_a(a) := a \);
2. if \( b \neq a \), \( pr_a(b) := q[a, b](t) \), where \( t \) is the largest integral multiple of \( 10\delta \) which is strictly less than \( d(a, b) \).

Now for each pair \( a, b \in \Gamma \), we define a 0-chain \( f(a, b) \) in \( \Gamma \) inductively on the distance \( d(a, b) \) as follows:

1. if \( d(a, b) \leq 10\delta \), \( f(a, b) := b \);
2. if \( d(a, b) > 10\delta \) and \( d(a, b) \) is not an integral multiple of \( 10\delta \), let \( f(a, b) := f(a, pr_a(b)) \);
3. if \( d(a, b) > 10\delta \) and \( d(a, b) \) is an integral multiple of \( 10\delta \), let

\[ f(a, b) := \frac{1}{\# Fl(a, b)} \sum_{x \in Fl(a, b)} f(a, pr_a(x)). \]

The following result is due to Mineyev [18].

**Proposition 3.1.** The function \( f : \Gamma \times \Gamma \to C_0(\Gamma, \mathbb{Q}) \) defined above satisfies the following conditions.
(1) For each \( a, b \in \Gamma \), \( f(b, a) \) is a convex combination, i.e. its coefficients are non-negative and sum up to 1.

(2) If \( d(a, b) \geq 10\delta \), then \( \text{supp} f(b, a) \subseteq B(q[b, a](10\delta), \delta) \cap S(b, 10\delta) \).

(3) If \( d(a, b) \leq 10\delta \), then \( f(b, a) = a \).

(4) \( f \) is \( \Gamma \)-equivariant, i.e. \( f(g \cdot b, g \cdot a) = g \cdot f(b, a) \) for any \( g, a, b \in \Gamma \).

(5) There exist constants \( L \geq 0 \) and \( 0 \leq \lambda < 1 \) such that, for all \( a, a', b \in \Gamma \),
\[
\| f(b, a) - f(b, a') \|_1 \leq L \lambda^{(a|a')_b}.
\]

Let \( p \geq 2 \). For each pair \( b, a \in \Gamma \), define
\[
h(b, a) = \frac{1}{\| f(b, a) \|_p} f(b, a),
\]
where \( f \) is as in Proposition 3.1.

**Corollary 3.2.** The function \( h : \Gamma \times \Gamma \to C_0(\Gamma, \mathbb{Q}) \) defined above satisfies the following conditions.

(1) For each \( a, b \in \Gamma \), \( \| h(b, a) \|_p = 1 \).

(2) If \( d(a, b) \geq 10\delta \), then \( \text{supp} h(b, a) \subseteq B(q[b, a](10\delta), \delta) \cap S(b, 10\delta) \).

(3) If \( d(a, b) \leq 10\delta \), then \( h(b, a) = a \).

(4) \( h \) is \( \Gamma \)-equivariant, i.e. \( h(g \cdot b, g \cdot a) = g \cdot h(b, a) \) for any \( g, a, b \in \Gamma \).

(5) There exist constants \( C \geq 0 \) and \( 0 \leq \rho < 1 \) such that, for all \( a, a', b \in \Gamma \),
\[
\| h(b, a) - h(b, a') \|_p \leq C \rho^{(a|a')_b}.
\]

**Proof:** (1), (2), (3) and (4) of Corollary 3.2 follow from Proposition 3.1.

By (2) of Proposition 3.1, we have
\[
\# \text{supp} h(b, a) \leq \# S(b, 10\delta), \quad \# \text{supp} h(b, a') \leq \# S(b, 10\delta).
\]

It follows that
\[
\| h(b, a) - h(b, a') \|_p \leq 2(\# S(b, 10\delta))^{\frac{1}{p}} \| h(b, a) - h(b, a') \|_1.
\]

Now (5) of Corollary 3.2 follows from (5) of Proposition 3.1.
4 Proof of the main result.

In this section, we prove Theorem 1.1.

**Proof of Theorem 1.1:**

Let $v > 0$ such that $\#B(x, r) \leq v^r$ for all $x \in \Gamma$ and $r > 0$. Let $\rho$ be as in Corollary 3.2. Choose $p \geq 2$ such that $\rho^p v < \frac{1}{2}$.

Let $l_p(\Gamma)$ be the completion of $C_0(\Gamma, \mathbb{Q})$ with respect to the norm $\| \cdot \|_p$. Notice that the $\Gamma$ action on $C_0(\Gamma, \mathbb{Q})$ can be extended to an isometric action on $l_p(\Gamma)$.

Let $X = \{ \xi : \Gamma \to l_p(\Gamma) : \| \xi \|_p = \left( \sum_{\gamma \in \Gamma} \| \xi(\gamma) \|_p \right)^{\frac{1}{p}} < \infty \}$. Observe that $X$ is isometric to $l_p(\Gamma \times \Gamma)$.

Let $\pi$ be the isometric action of $\Gamma$ on $X$ defined by:

$$(\pi(g)\xi)(\gamma) = g(\xi(g^{-1}\gamma))$$

for all $\xi \in X$ and $g, \gamma \in \Gamma$.

Define $\eta \in X$ by:

$$\eta(\gamma) = h(\gamma, e)$$

for all $\gamma \in \Gamma$, where $e$ is the identity element in $\Gamma$.

For each $g \in \Gamma$, by Corollary 3.2 and the choice of $p$, we have:

$$\| \pi(g)\eta - \eta \|_p^p = \sum_{\gamma \in \Gamma} \| g(\xi(g^{-1}\gamma, e)) - h(\gamma, e) \|_p^p$$

$$\leq \sum_{\gamma \in \Gamma} \| h(\gamma, g) - h(\gamma, e) \|_p^p$$

$$\leq \sum_{\gamma \in \Gamma} C \rho^p (\rho(\delta(\gamma, e) - \delta(g, e)))$$

$$\leq \sum_{n=0}^{\infty} C \rho^p (\rho(n - \delta(g, e))) \cdot v^n$$

$$\leq 2C \rho^{-pd(g, e)}.$$

It follows that $\pi(g)\eta - \eta$ is an element in $X$ for each $g \in \Gamma$.

We now define an affine isometric action $\alpha$ on $X$ by $\Gamma$ by:
\[ \alpha(g)\xi = \pi(g)\xi + \pi(g)\eta - \eta \]
for all \( \xi \in X \) and \( g \in \Gamma \).

If \( \gamma \) is a vertex on the oriented geodesic \( q[g, e] \) satisfying \( d(\gamma, e) \geq 10\delta \) and \( d(\gamma, g) \geq 10\delta \), we have

\[
B(q[\gamma, e](10\delta), \delta) \cap B(q[\gamma, g](10\delta), \delta) = \emptyset.
\]

Otherwise, if there exists \( z \in B(q[\gamma, e](10\delta), \delta) \cap B(q[\gamma, g](10\delta), \delta) \), then

\[
d(g, e) \leq d(g, z) + d(z, e) \\
\leq (d(g, q[\gamma, g](10\delta)) + \delta) + (\delta + d(q[\gamma, e](10\delta), e)) \\
= ((d(g, \gamma) - 10\delta) + \delta) + (\delta + (d(\gamma, e) - 10\delta)) \\
= d(g, e) - 18\delta.
\]

This is a contradiction since \( \delta > 0 \).

By (2) of Corollary 3.2, we have

\[
supp h(\gamma, g) \cap supp h(\gamma, e) = \emptyset
\]
if \( \gamma \) is a vertex on the oriented geodesic \( q[g, e] \) satisfying \( d(\gamma, e) \geq 10\delta \) and \( d(\gamma, g) \geq 10\delta \).

It follows that there exist at least \( d(g, e) - 100\delta \) number of vertices \( \gamma \) on the oriented path \( q[g, e] \) such that

\[
\|g(h(g^{-1}\gamma, e)) - h(\gamma, e)\|_p = \|h(\gamma, g) - h(\gamma, e)\|_p \geq 1.
\]

Hence

\[
\|\pi(g)\eta - \eta\|_p^p \geq d(g, e) - 100\delta
\]
for all \( g \in \Gamma \).

As a consequence, for every \( \xi \in X \), we have

\[
\|\alpha(g)\xi - \pi(g)\xi\|_p \rightarrow \infty
\]
as \( g \rightarrow \infty \).
This, together with the fact that $\pi(g)$ is an isometry, implies that $\alpha$ is proper.

We should mention that it remains an open question if $SL(n,\mathbb{Z})$ admits a proper affine isometric action on some uniformly convex Banach space for $n \geq 3$. A positive answer to this question would have interesting applications to K-theory of group $C^*$-algebras [16].

References

[1] U. Bader and T. Gelander, Propert (T) and unitary representations on $L_p$. Preprint, 2004.

[2] P. Baum and A. Connes, K-theory for discrete groups, Operator Algebras and Applications, (D. Evans and M. Takesaki, editors), Cambridge University Press (1989), 1–20. Bekka, M. E.

[3] M. E. B. Bekka, P.-A. Cherix, and A. Valette, Proper affine isometric actions of amenable groups. Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), 1–4, London Math. Soc. Lecture Note Ser., 227, Cambridge Univ. Press, Cambridge, 1995.

[4] N. Brown and E. Guentner, Uniform embedding of bounded geometry spaces into reflexive Banach spaces. Preprint, 2003.

[5] A. Connes, Noncommutative Geometry, Academic Press, 1994.

[6] A. Connes and H. Moscovici, Cyclic cohomology, the Novikov conjecture and hyperbolic groups, Topology 29 (1990), 345–388.

[7] D. Fisher and G. A. Margulis, Almost isometric actions, Property (T), and local rigidity. Preprint, 2004.

[8] M. Gromov, Hyperbolic groups, MSRI Publ. 8, 75-263, Springer, 1987.
[9] M. Gromov, Asymptotic invariants for infinite groups, Geometric Group Theory, (G. A. Niblo and M. A. Roller, editors), Cambridge University Press, (1993), 1–295.

[10] M. Gromov, Problems (4) and (5), Novikov Conjectures, Index Theorems and Rigidity, Vol. 1, (S. Ferry, A. Ranicki and J. Rosenberg, editors), Cambridge University Press, (1995), 67.

[11] M. Gromov, Spaces and questions. GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part I, 118–161.

[12] U. Haagerup, An example of a nonnuclear $C^*$-algebra, which has the metric approximation property. Invent. Math. 50 (1978/79), no. 3, 279–293.

[13] N. Higson and G. G. Kasparov, Operator K-theory for groups which act properly and isometrically on Hilbert space, Electronic Research Announcements, AMS 3 (1997), 131–141.

[14] N. Higson and G. G. Kasparov, $E$-theory and $KK$-theory for groups which act properly and isometrically on Hilbert space. Invent. Math. 144 (2001), no. 1, 23–74.

[15] G. Kasparov and G. Yu, Uniform convexity and the coarse geometric Novikov conjecture. Preprint, 2004.

[16] G. Kasparov and G. Yu, Uniform convexity and K-theory of group $C^*$-algebras. In preparation.

[17] V. Lafforgue, $K$-thorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. Invent. Math. 149 (2002), no. 1, 1–95.

[18] I. Mineyev, Straightening and bounded cohomology of hyperbolic groups. Geom. Funct. Anal. 11 (2001), no. 4, 807–839.

[19] I. Mineyev and G. Yu, The Baum-Connes conjecture for hyperbolic groups. Invent. Math. 149 (2002), no. 1, 97–122.
[20] M. Puschnigg, The Kadison-Kaplansky conjecture for word-hyperbolic groups. Invent. Math. 149 (2002), no. 1, 153–194.

[21] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. Invent. Math. 139 (2000), no. 1, 201–240.

[22] A. Zuk, Property (T) and Kazhdan constants for discrete groups. Geom. Funct. Anal. 13 (2003), no. 3, 643–670.

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