Entanglement entropy of quantum wire junctions

Pasquale Calabrese\textsuperscript{1,2}, Mihail Mintchev\textsuperscript{1,2} and Ettore Vicari\textsuperscript{1,2}

\textsuperscript{1} Dipartimento di Fisica dell’Università di Pisa, Largo Pontecorvo 3, 56127 Pisa, Italy
\textsuperscript{2} Istituto Nazionale di Fisica Nucleare, Largo Pontecorvo 3, 56127 Pisa, Italy

E-mail: mintchev@df.unipi.it

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Abstract

We consider a fermion gas on a star graph modeling a quantum wire junction and derive the entanglement entropy of one edge with respect to the rest of the junction. The gas is free in the bulk of the graph, the interaction being localized in its vertex and described by a non-trivial scattering matrix. We discuss all point-like interactions, which lead to a unitary time evolution of the system. We show that for a finite number of particles $N$, the Rényi entanglement entropies of one edge grow as $\ln N$ with a calculable prefactor, which depends not only on the central charge, but also on the total transmission probability from the considered edge to the rest of the graph. This result is extended to the case with a harmonic potential in the bulk.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum field theory on graphs has recently attracted much attention mainly in relation to the study [1–6] of the transport properties of quantum wire networks. Different frameworks [7–20] have been developed to investigate the phase diagram and the conductance of these structures. Despite the fact that the universal properties in the bulk are described by the well-known Luttinger liquid theory, the different boundary conditions at the junctions lead to exotic phase diagrams [10–12, 15–19] whose degree of universality is not completely understood and is still under investigation. The results, concerning the charge transport, confirm that the conductance properties of the quantum wire networks are strongly affected by the boundary conditions as well.

In this paper, we analyze another physical quantity—the entanglement entropy of one edge of the junction with respect to all the other edges. Lots of studies on the entanglement properties of many-body systems in the last decade have unveiled new (universal) features of these systems and somehow put their global understanding on a deeper level (see e.g. the reviews [21]). In particular, von Neumann and Rényi entanglement entropies of the reduced
density matrix $\rho_A$ of a subsystem $A$ turned out to be particularly useful for 1D systems. Rényi entanglement entropies are defined as

$$ S^{(\alpha)} = \frac{1}{1-\alpha} \ln \text{Tr} \rho_A^{\alpha}. $$  

(1.1)

For $\alpha \to 1$, this definition gives the most commonly used von Neumann entropy $S^{(1)} = -\text{Tr} \rho_A \ln \rho_A$, while for $\alpha \to \infty$ it gives the logarithm of the largest eigenvalue of $\rho_A$ also known as single copy entanglement [22]. Furthermore, knowledge of the $S^{(\alpha)}$ for different $\alpha$ characterizes the full spectrum of non-zero eigenvalues of $\rho_A$ [23].

One of the most remarkable results is the universal behavior displayed by the entanglement entropy at 1D conformal quantum critical points, determined by the central charge [24] of the underlying conformal field theory (CFT) [25–28]. For a partition of an infinite 1D system into a finite piece $A$ of length $\ell$ and the remainder, the Rényi entanglement entropies for $\ell$ much larger than the short-distance cutoff $a$ are

$$ S^{(\alpha)} = \frac{c}{6} \left( 1 + \frac{1}{\alpha} \right) \ln \frac{\ell}{a} + c_\alpha, $$  

(1.2)

where $c$ is the central charge and $c_\alpha$ is a non-universal constant.

Given the importance of this result (and also many others not mentioned here) for homogeneous systems, it is natural to wonder whether in the case of junctions the entanglement entropies can shed some light on the universality and relevance of the parameters defining the junction. Previous studies on the subject [29–37] have been limited to the case of only two edges (i.e. an infinite line with a defect) and most often performed for lattice models. These results provide strong evidence that the logarithmic behavior in equation (1.2) remains valid even in the presence of defects, but the prefactor does not depend only on the central charge of the bulk CFT when the defect is a marginal perturbation (in the renormalization group sense) as is known to happen for free fermions [1].

In order to tackle the problem of entanglement in a junction with an arbitrary number of wires, we use the recently developed systematic framework [38, 39] for calculating the bipartite entanglement entropy of spatial subsystems of one-dimensional quantum systems in continuous space. We consider only a free fermion gas in bulk in which the junction introduces a marginal perturbation. The junction boundary conditions define a specific scattering matrix $S$, encoding all possible point-like interactions in the vertex which give rise to unitary time evolution. Focussing on the scale invariant case, we show that for a finite number of particles $N$ and for edges of equal length $L$, the Rényi entanglement entropies of any of the edges grow as $\ln N$. In contrast to the case in the absence of the point-like interaction (i.e. equation (1.2)), the prefactor of this logarithm does not depend only on the central charge, but also on the total transmission probability $(1 - |S_{ii}|^2)$ from the considered edge $i$ to the rest of the graph. Some of the results presented here have been anticipated in the short communication [38]. We show also that the presence of an external harmonic potential in the bulk (acting identically on all edges) does not alter this result.

The paper is organized as follows. In the next section, we describe the basic features of the model and the scattering matrices generated by the point-like interactions at the junction. We discuss in detail the scale invariant case and derive the two-point correlation function. The entanglement entropy associated with this system is analytically computed in section 3. In section 4, we extend our considerations, adding a harmonic potential in the bulk. Section 5 is devoted to the conclusions and discussion of some further developments in the subject.
2. Schrödinger junction

2.1. The general setting

In this section, we consider a gas of $N$ spinless fermions on a quantum wire junction. We consider only the ground state of such a system and refer to it as the ground state of $N$ particles, having in mind that the particles are the original fermions and not the excitations above the ground state (which are usually referred to as particles in the field theory literature). A simple model, describing the junction, is represented by a star graph $\Gamma$ with $M$ edges of finite length $L$, as shown in figure 1. Each point $P$ in the bulk of $\Gamma$ is parametrized by $(x, i)$, where $0 \leq x \leq L$ is the distance of $P$ from the vertex $V$ of the graph and $i$ labels the edge. We assume that in the bulk ($x \neq 0$ and $x \neq L$), the gas is free and is described by the Schrödinger field $\psi_i(t, x)$, which satisfies

$$\left( i\partial_t + \frac{1}{2m} \partial_x^2 \right) \psi_i(t, x) = 0$$

and standard equal-time canonical anticommutation relations. The only non-trivial interactions are localized in the vertex $V$ of $\Gamma$ and are encoded in boundary conditions at $x = 0$. These conditions are fixed in turn by imposing that the bulk Hamiltonian $-\partial_x^2$ admits a self-adjoint extension on the whole graph. In such a way, all point-like interactions, leading to a unitary time evolution of the system, are covered. The most general boundary conditions, implementing this natural physical requirement, are \cite{41, 42} at the vertex

$$\sum_{j=1}^M [\lambda(I - U)_{ij} \psi_j(t, 0) - i(I + U)_{ij}(\partial_x \psi_j)(t, 0)] = 0,$$

where $U$ is an arbitrary $M \times M$ unitary matrix and $\lambda$ is a real parameter with the dimension of mass. To fully specify the problem, we also need to impose boundary conditions at the external ends of the edges. The most general ones are

$$\partial_x \psi_i(t, L) = \mu_i \psi_i(t, L),$$

where $\mu_i$ are again real parameters with the dimension of mass. Equation (2.3) is the familiar Robin (mixed) boundary condition. Note that equation (2.2) extends this condition to the vertex $V$ of the graph.

It has been established in \cite{41, 42} that the point-like interaction, induced by (2.2), generates the scattering matrix

$$S(k) = -\frac{[\lambda(I - U) - k(I + U)]}{[\lambda(I - U) + k(I + U)]}.$$
Besides the unitarity
\[ S(k)S^\dagger(k) = I, \]
\( S(k) \) satisfies the Hermitian analyticity
\[ S^\dagger(k) = S(-k) \]
as well. Note also that
\[ S(\lambda) = U, \quad S(-\lambda) = U^{-1}, \]
showing that the unitary matrix \( U \), entering the boundary conditions (2.2), is actually the scattering matrix at the scale \( \lambda \).

The main difficulty in solving the Schrödinger equation (2.1) on the graph \( \Gamma \) is the mixing between the different edges codified in the boundary conditions (2.2) and (2.3). In order to simplify this problem, we impose that the boundary conditions at the ends of each arm are all the same, in such a way to restore (at the level of the Hamiltonian) the permutation symmetry of the edges of the graph. It should be clear from the physical point of view that, being interested in the thermodynamic limit with \( L, N \to \infty \), the boundary conditions at \( L \) must not affect the final result. Thus, we assume from now on that
\[ \mu_1 = \mu_2 = \cdots = \mu_M \equiv \mu. \]  
Under this condition, equations (2.2) and (2.3) can be rewritten in equivalent forms without mixing. Indeed, let us introduce the unitary matrix \( U \) diagonalizing \( \mu \), namely
\[ U \cup U^\dagger = U_d = \text{diag}(e^{-2i\omega_1}, e^{-2i\omega_2}, \ldots, e^{-2i\omega_N}), \quad -\frac{\pi}{2} < \omega_i \leq \frac{\pi}{2}. \]  
Remarkably enough, \( U \) diagonalizes also \( S(k) \) for any \( k \):
\[ S_d(k) = U^\dagger S(k) U = \text{diag} \left( \frac{k + i\eta_1}{k - i\eta_1}, \ldots, \frac{k + i\eta_M}{k - i\eta_M} \right), \]
where
\[ \eta_i \equiv \lambda \tan(\omega_i). \]
It is quite natural at this point to introduce the fields
\[ \psi_i(t, x) = \sum_{j=1}^M U_{ij} \phi_j(t, x), \]  
which obviously satisfy equation (2.1). In terms of \( \psi_i \), the boundary conditions (2.2) and (2.3) decouple,
\[ (\partial_t \psi_i)(t, 0) = \eta_i \psi_i(t, 0), \]
\[ (\partial_t \psi_i)(t, L) = \mu \psi_i(t, L), \]
defining a simple spectral problem on the tensor product \( \mathcal{H} = \bigotimes_{i=1}^M L^2[0, L] \), which is analyzed below.

It is worth stressing that \( \psi_i(t, x) \) is a superposition of the values of the original field \( \psi_i(t, x) \) at the same distance \( x \) from the vertex, but on different edges of the junction. Being so delocalized, \( \psi_i(t, x) \) is unphysical and provides only a convenient basis for dealing with the boundary conditions. The physical observables and correlation functions will always be expressed in terms of the physical fields \( \psi_i(t, x) \).

The eigenfunctions of \( -\partial_x^2 \), obeying (2.13) and (2.14), are
\[ \phi_i(k, x) = c_i \left( e^{ikx} + \frac{k + i\eta_i}{k - i\eta_i} e^{-ikx} \right), \quad k \geq 0, \]
where \( c_i \) are some constants to be fixed below and \( k \) satisfy

\[
e^{2ikL} = \left( \frac{k + i\eta_i}{k - i\eta_i} \right) \left( \frac{k - i\mu}{k + i\mu} \right).
\]

In order to determine the spectrum of \( k \) explicitly, we simplify the problem further by requiring scale invariance.

2.2. The scale-invariant case

Scale invariance of the boundary conditions (2.13) and (2.14) implies the values

\[
\begin{align*}
\mu &= \begin{cases} 
0, & (\alpha_i = 0), \\
\infty, & (\alpha_i = \pi/2),
\end{cases} \quad \text{Neumann b.c.,} \\
\eta_i &= \begin{cases} 
0, & (\alpha_i = 0), \\
\infty, & (\alpha_i = \pi/2),
\end{cases} \quad \text{Dirichlet b.c.} 
\end{align*}
\]

In other words, the critical points are fixed by the \((M+1)\) vector \((\mu, \eta_1, \eta_2, \ldots, \eta_M)\) whose components take values (2.17). For an edge \( i \), one has the following possibilities:

(a) \( \mu = 0 \) (Neumann condition at \( x = L \)): equation (2.16) gives

\[
e^{2ikL} = \left( \frac{k + i\eta_i}{k - i\eta_i} \right),
\]

which gives

\[
\eta_i = 0 \implies \phi_i(n, x) = \frac{2}{L} \cos \left[ \left( n - 1 \right) \frac{\pi x}{L} \right], \quad n = 1, 2, \ldots
\]

(b) \( \mu = \infty \) (Dirichlet condition at \( x = L \)): equation (2.16) implies

\[
e^{2ikL} = - \left( \frac{k + i\eta_i}{k - i\eta_i} \right),
\]

which gives

\[
\eta_i = 0 \implies \phi_i(n, x) = \frac{2}{L} \cos \left[ \left( n - \frac{1}{2} \right) \frac{\pi x}{L} \right], \quad n = 1, 2, \ldots
\]

\[
\eta_i = \infty \implies \phi_i(n, x) = \frac{2}{L} \sin \left( n \frac{\pi x}{L} \right), \quad n = 1, 2, \ldots
\]

Note that any of the sets \((2.18), (2.19), (2.22)\) and \( (2.23) \) represent a complete orthonormal system in \( L^2[0, L] \).

2.3. Scale-invariant scattering matrices

Observing that the eigenvalue of \( S \) is 1 for \( \eta_i = 0 \) and \(-1\) for \( \eta_i = \infty \), one concludes that the most general scale-invariant scattering matrix, compatible with a unitary time evolution, is given by

\[
S = U S_d U^\dagger, \quad S_d = \operatorname{diag}(\pm 1, \pm 1, \ldots \pm 1),
\]

where \( U \) is a generic \( M \times M \) unitary matrix. From the group-theoretical point of view, any critical \( S \) matrix is a point in the orbit of some \( S_d \) under the adjoint action of the unitary group \( U(M) \). Obviously, one can enumerate the edges in such a way that the first \( p \) eigenvalues of \( S \) are \(+1\) and the remaining \( M - p \) are \(-1\). The cases \( p = M \) and \( p = 0 \) correspond to \( S = I \).
and \( S = -1 \) and are not interesting for the entanglement. In these two cases in fact, the single wires are decoupled (there is no transmission), which implies a vanishing entanglement.

It follows from (2.24) that besides being unitary, \( S \) is also Hermitian (in agreement with (2.6) at criticality). Therefore, all diagonal elements \( S_{ii} \) are real. Note however that in general \( S \) is not symmetric. If this is the case, time reversal invariance is broken [19].

In the nontrivial case \( p = 1 \), the most general \( 2 \times 2 \) scale-invariant scattering matrix depends on two parameters and can be written in the form

\[
S(\epsilon, \theta) = \frac{1}{1 + \epsilon^2} \begin{pmatrix} \epsilon^2 - 1 & 2\epsilon e^{i\theta} \\ 2\epsilon e^{-i\theta} & 1 - \epsilon^2 \end{pmatrix}, \quad \epsilon \in \mathbb{R}, \quad \theta \in [0, 2\pi).
\]  

Time reversal invariance is broken for \( \theta \neq 0, \pi \).

For \( M = 3 \), one has two families corresponding to \( p = 1 \) and \( p = 2 \). In order to avoid cumbersome formulae, we display only two representatives of these families, namely

\[
S_{p=1}(\epsilon_1, \epsilon_2) = \frac{1}{1 + \epsilon_1^2 + \epsilon_2^2} \begin{pmatrix} 2\epsilon_1 & 2\epsilon_2 & 1 - \epsilon_1^2 - \epsilon_2^2 \\ 2\epsilon_2 & -\epsilon_1^2 + \epsilon_2^2 - 1 & 2\epsilon_1 \epsilon_2 \\ \epsilon_1^2 - \epsilon_2^2 - 1 & 2\epsilon_1 \epsilon_2 & 2\epsilon_1 \end{pmatrix},
\]

\[
S_{p=2}(\epsilon_1, \epsilon_2) = -\frac{1}{1 + \epsilon_1^2 + \epsilon_2^2} \begin{pmatrix} \epsilon_1^2 - \epsilon_2^2 - 1 & 2\epsilon_1 \epsilon_2 & 2\epsilon_1 \\ 2\epsilon_1 \epsilon_2 & -\epsilon_1^2 + \epsilon_2^2 - 1 & 2\epsilon_2 \\ 2\epsilon_1 & 2\epsilon_2 & 1 - \epsilon_1^2 - \epsilon_2^2 \end{pmatrix},
\]

where \( \epsilon_1, \epsilon_2 \in \mathbb{R} \).

2.4. Two-point correlation function

Now we are in position to construct the physical field \( \psi_i(t, x) \) and the relative two-point function needed in the computation of the entanglement entropy. First of all, we write the unphysical field in terms of the eigenfunctions \( \phi_i(n, x) \),

\[
\psi_i(t, x) = \sum_{n=1}^{\infty} e^{-in\mu x} \phi_i(n, x)a_i(n),
\]

where the fermion annihilation and creation operators satisfy the standard anti-commutation relations

\[
[a_i(m), a_j^\dagger(n)]_+ = \delta_{ij} \delta_{mn}, \quad [a_i(m), a_j(n)]_+ = \left[a_i^\dagger(m), a_j^\dagger(n)\right]_+ = 0,
\]

and the energies are given by

\[
\omega_i(n) = \begin{cases} 
\frac{1}{2m} \left[ (n - 1) \frac{\pi}{L} \right]^2, & \text{if } 1 \leq i \leq p, \\
\frac{1}{2m} \left[ (2n - 1) \frac{\pi}{2L} \right]^2, & \text{if } p < i \leq M,
\end{cases}
\]

\[
\omega_i(n) = \begin{cases} 
\frac{1}{2m} \left[ (2n - 1) \frac{\pi}{2L} \right]^2, & \text{if } 1 \leq i \leq p, \\
\frac{1}{2m} \left[ \frac{\pi}{2L} \right]^2, & \text{if } p < i \leq M,
\end{cases}
\]

for \( \mu = 0 \) and \( \mu = \infty \), respectively. Note that different ‘unphysical’ edges may have different dispersion relations, which is not a problem because these edges are totally isolated from each other. By means of (2.12), one obtains the physical fields

\[
\psi_i(t, x) = \sum_{j=1}^{M} \mathcal{U}^{ij} \psi_j(t, x) = \sum_{j=1}^{M} \sum_{n=1}^{\infty} \mathcal{U}^{ij}_n e^{-in\mu x} \phi_j(n, x)a_j(n).
\]
One easily verifies that
\[ \left[ \psi(t, x), \psi_j(t, y) \right]_+ = \delta_{ij} \delta(x - y), \tag{2.33} \]
which fixes the normalization of the fields.

The equal-time two-point correlation function of the physical field \( \psi_1(t, x) \) on a given state \( |\Psi\rangle \) is
\[ C^\psi_{ij}(x, y) = \langle \psi_1(t, x) \psi_j(t, y) | \Psi \rangle \]
\[ = \sum_{k,l} \sum_{n,m=1}^\infty \mathcal{U}_{kl}^1 \mathcal{U}_{lk}^1 e^{i(\omega_j(m) - \omega_i(n)) t} \phi_k(n, x) \overline{\phi}_l(m, y) \langle \psi_1^k(n) \psi_1^l(m) | \Psi \rangle, \tag{2.34} \]
where the correlator \( \langle \psi_1^k(n) \psi_1^l(m) | \Psi \rangle \) can be deduced from the action of the algebra generated by \( \{ \phi_i(n), \phi^*_j(m) \} \) on the state \( |\Psi\rangle \). In particular, we are interested in the case when \( |\Psi\rangle \) is the ground state of the system formed by \( N \) fermions in the whole junction. It is then useful to rewrite \( N \) as
\[ N = MN', \tag{2.35} \]
where \( N' \) represents the average number of particles for each wire. The action of the annihilation and creation operators on the ground state is obvious since it is annihilated by all \( a_i(m) \) with \( m > N' \) and so \( \langle \psi_1^k(n) \psi_1^l(m) | \Psi \rangle = \delta_{kl} \delta_{mn} \delta(N' - n) \). Using this relation, equation (2.34), restricted to the same edge \((i = j)\) which is needed actually for computing the entanglement entropy, becomes
\[ C^\psi_{ii}(x, y) = \sum_{k=1}^M \sum_{n=1}^{N'} |\mathcal{U}_{k1}\rangle^2 \phi_k(n, x) \overline{\phi}_k(n, y), \tag{2.36} \]
where \( N' \) can also be interpreted as an ultraviolet cut-off for the series in (2.34).

It is convenient for what follows to rewrite (2.36) in more explicit terms. For this purpose, we consider any critical point characterized by the integer \( 1 < p < M \), i.e. a scale-invariant scattering matrix with \( p \) eigenvalues equal to +1. The two sums in (2.36) factorize and one obtains
\[ C^\psi_{ii}(x, y) = \sum_{k=1}^p |\mathcal{U}_{k1}\rangle^2 \sum_{n=1}^{N'} f_+(n, x) f_+(n, y) + \sum_{k=p+1}^M |\mathcal{U}_{k1}\rangle^2 \sum_{n=1}^{N'} f_-(n, x) f_-(n, y), \tag{2.37} \]
where
\[ f_+(n, x) = \begin{cases} \sqrt{\frac{2}{L}} \cos \left[ \left( n - \frac{1}{2} \right) \pi \frac{x}{L} \right], & \mu = 0, \\ \sqrt{\frac{2}{L}} \cos \left[ \frac{\pi}{2} \frac{x}{L} \right], & \mu = \infty, \end{cases} \tag{2.38} \]
\[ f_-(n, x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \left[ \frac{\pi}{2} \frac{x}{L} \right], & \mu = 0, \\ \sqrt{\frac{2}{L}} \sin \left[ \left( n - \frac{1}{2} \right) \pi \frac{x}{L} \right], & \mu = \infty. \end{cases} \tag{2.39} \]
Because of (2.24), the sums over \( k \) in (2.37) give
\[ \sum_{k=1}^p |\mathcal{U}_{k1}\rangle^2 = \frac{1}{2} (1 + S_{ii}), \quad \sum_{k=p+1}^M |\mathcal{U}_{k1}\rangle^2 = \frac{1}{2} (1 - S_{ii}). \tag{2.40} \]
Moreover, using that $S$ is both unitary and Hermitian, one has

$$S_{ii}^2 = 1 - \sum_{j=1}^{M} |S_{ij}|^2 = 1 - T_i^2,$$

(2.41)

where $T_i^2$ is the total transmission probability from the edge $i$ to the rest of the graph.

In conclusion, correlator (2.36) can be fully expressed in terms of the transmission probability $T_i^2$ and the one-particle wavefunctions as follows:

$$C^N_{ij}(x, y) = \frac{1}{2} \left( 1 + \sqrt{1 - T_i^2} \right) \sum_{n=1}^{N'} f_+(x, n) f_+(y, n) + \frac{1}{2} \left( 1 - \sqrt{1 - T_i^2} \right) \sum_{n=1}^{N'} f_-(x, n) f_-(y, n),$$

(2.42)

which is the basic input for deriving the entanglement entropy in the next section. We observe that $C^N_{ij}$ involves only the diagonal elements of $S$ and, consequently, does not depend on the behavior of $S$ under transposition. Therefore, contrary to the conductance [19], the entanglement entropy in our case is not sensitive to the breaking of time-reversal invariance.

3. Entanglement entropy

In order to compute the bipartite Rényi entanglement entropies defined as in equation (1.1) of a subsystem $A$ in the ground state our star graph, we use the method recently introduced in [38, 39]. The starting point to deal with a system made of a finite number of particles in continuous space is the Fredholm determinant

$$\mathbb{D}_A(\lambda) = \det \left[ \lambda \delta_A(x, y) - C_A(x, y) \right],$$

(3.1)

where $C_A(x, y)$ is the restriction of the correlation matrix $C(x, y)$ defined in equation (2.34) to $A$, i.e., $C_A = P_A C P_A$, where $P_A$ is the projector on $A$. The same definition holds for $\delta_A(x, y) = P_A \delta(x - y) P_A$. Following the ideas for the lattice model [43], $\mathbb{D}_A(\lambda)$ can be introduced in such a way that it is a polynomial in $\lambda$ having as zeros the eigenvalues of $C_A$.

Since we are dealing only with free fermions in the bulk, the reduced density matrix $\rho_A$ is Gaussian [44] and so one can easily derive [43, 38, 39]

$$S^{(\alpha)} = \frac{\ln \text{Tr} \rho_A^\alpha}{1 - \alpha} = \oint_{\gamma} \frac{d\lambda}{2\pi i} e_\alpha(\lambda) \frac{d \ln \mathbb{D}_A(\lambda)}{d\lambda},$$

(3.2)

where the integration contour encircles the segment $[0, 1]$, and

$$e_\alpha(\lambda) = \frac{1}{1 - \alpha} \ln[\lambda^\alpha + (1 - \lambda)^\alpha].$$

(3.3)

For $\alpha \to 1$, $e_1(\lambda) = -x \ln x - (1 - x) \ln(1 - x)$ and equation (3.2) gives the von Neumann entropy.

The Fredholm determinant is turned into a standard one by introducing the reduced overlap matrix $\hat{A}$ (also considered in [45]) with elements

$$\hat{A}_{nm} = \int_{\mathbb{R}} dz \bar{\phi}_n(z) \phi_m(z), \quad n, m = 1, \ldots, D,$$

(3.4)

where in general $\phi_n(x)$ represent the eigenfunctions corresponding to the $D$ lowest energy levels occupied in the ground state of the system. The matrix $\hat{A}$ satisfies $\text{Tr} C_A^k = \text{Tr} \hat{A}^k$ and so [39]

$$\ln \mathbb{D}_A(\lambda) = -\sum_{k=1}^{\infty} \frac{\text{Tr} C_A^k}{k \lambda^k} = -\sum_{k=1}^{\infty} \frac{\text{Tr} \hat{A}^k}{k \lambda^k} = \ln \det [\lambda I - \hat{A}] = \sum_{m=1}^{D} \ln(\lambda - a_m),$$

(3.5)

In conclusion, correlator (2.36) can be fully expressed in terms of the transmission probability $T_i^2$ and the one-particle wavefunctions as follows:
where $a_m$ are the eigenvalues of $\hat{A}$. Inserting (3.5) into integral (3.2), we obtain

$$S^{(\alpha)} = \oint \frac{d\lambda}{2\pi i} \sum_{m=1}^{D} \frac{e_{\alpha}(\lambda)}{\lambda - a_m} = \sum_{m=1}^{D} e_{\alpha}(a_m) = \frac{1}{1-\alpha} \text{Tr} \ln[\hat{A}^\alpha + (\mathbb{I} - \hat{A})^\alpha],$$  

(3.6)

as a consequence of the residue theorem.

In the following, we will be interested only in the entanglement entropy of any edge $i$ of the wire with respect to the rest of the junction in the global ground state of the star graph. As we have seen above in equation (2.42), the two-point correlation function for a finite number of particles $N$ in the full star graph can be written in the form (we omit the edge index $i$ hereafter)

$$C^N(x,y) = \sum_{n=1}^{2N} \mathcal{X}(x,n) \chi(y,n),$$  

(3.7)

where $\chi(x,n)$ are proportional to the one-particle eigenfunctions $f_{\pm}(x,n)$ in equation (2.42). Then the Rényi entanglement entropy of the subsystem represented by a single wire is given by equation (3.6) where the eigenvalues $a_m$ of $\hat{A}$ are numerically calculated from the overlap matrix built with the correlation function above.

### 3.1. The case $M = 2$

It is instructive to consider first the case with two edges only. In this case, one has actually a segment of length $2L$ with a point-like conformal defect placed in the middle. This situation has been investigated on the lattice [35, 36] by other methods and represents a useful check for our framework. We set $L = 1$ for simplicity and consider the case $\mu = \infty$ (Dirichlet condition at $x = L$).

Combining the explicit form (2.25) of the $S$-matrix with (2.41), one obtains the following transmission amplitudes:

$$T_1 = T_2 = \frac{2\varepsilon}{1 + \varepsilon^2} \equiv T,$$

(3.8)

which are the square root of the transmission probability. Accordingly,

$$C^N_{11}(x,y) = C^N_{22}(x,y) = \sum_{n=1}^{2N} \mathcal{X}(x,n) \chi(y,n),$$  

(3.9)

with

$$\chi(k,x) = \begin{cases} \frac{\varepsilon}{\sqrt{1 + \varepsilon^2}} \sqrt{2} \cos\left(\frac{k\pi x}{2}\right), & k = 1, 3, \ldots, 2N - 1, \\ \frac{1}{\sqrt{1 + \varepsilon^2}} \sqrt{2} \sin\left(\frac{k\pi x}{2}\right), & k = 2, 4, \ldots, 2N, \end{cases}$$  

(3.10)

where $N = 2N$ is the total particle number according to equation (2.35). The matrix $\hat{A}$, defined by (3.4), reads in our case

$$\hat{A}_{mn} = \begin{cases} \frac{\varepsilon^2}{(1 + \varepsilon^2)} \delta_{mn}, & m, n\text{—odd}, \\ \frac{1}{(1 + \varepsilon^2)} \delta_{mn}, & m, n\text{—even}, \\ \frac{2\varepsilon}{(1 + \varepsilon^2)} \frac{2n}{\pi (n^2 - m^2)}, & m\text{—odd}, n\text{—even}, \end{cases}$$  

(3.11, 3.12, 3.13)

3. The case $\mu = 0$ (Neumann boundary conditions at $x = L$) can be treated along the same lines.
\[ \hat{h}_{nm} = \frac{2\epsilon}{(1 + \epsilon^2)} \frac{2m}{\pi (m^2 - n^2)}, \quad m - \text{even}, \quad n - \text{odd}. \quad (3.14) \]

Since \( C_{11}^N = C_{22}^N \), the entanglement entropy of edge 1 equals that of edge 2 and is given by

\[ S^{(\alpha)}(T; N) = \frac{1}{1 - \alpha} \text{Tr} \ln [\hat{\rho}^{\alpha} + (\mathbb{I} - \hat{\rho})^{\alpha}] = \sum_{n=1}^{N} e_n(a_n), \quad (3.15) \]

with \( a_n \) being the eigenvalues of matrix (3.11)–(3.14). Using (3.15), we will show below that

\[ S^{(\alpha)}(T; N) = C^{(\alpha)}(T) \ln N + O(1), \quad (3.16) \]

with a prefactor \( C^{(\alpha)}(T) \) that depends on the transmission amplitude and not only on the central charge \( c = 1 \).

Before considering a generic value of \( \alpha \), it is instructive to discuss the Rényi entropy \( \alpha = 2 \). In this case, equation (3.15) takes the simple form

\[ S^{(2)}(T; N) = \text{Tr} \ln [\mathbb{I} - 2\mathbb{E}(T)] = \sum_{k=1}^{\infty} \frac{2^k}{k} \text{Tr} \mathbb{E}^k(T), \quad (3.17) \]

where the combination

\[ \mathbb{E}(T) \equiv \mathbb{A}(\mathbb{I} - \mathbb{A}) \quad (3.18) \]

represents the natural variable for performing the computation. The main idea at this point is to reduce the evaluation of (3.17) to the case of full transmission \( T = 1 \), when the defect is absent and one can use therefore the known [40] behavior of \( \text{Tr} \mathbb{E}^k \) for free fermion gas on the interval, namely

\[ \text{Tr} \mathbb{E}^k(T = 1) = \frac{1}{2\pi^2} \frac{[(k - 1)]^2}{(2k - 1)!} \ln N + O(1). \quad (3.19) \]

For this purpose, we first establish the fundamental relation

\[ \text{Tr} \mathbb{E}^k(T) = T^{2k} \text{Tr} \mathbb{E}^k(T = 1), \quad (3.20) \]

which captures the impact of the point-like interaction in the junction on the entanglement entropy (3.17). In order to prove (3.20), we exploit the property that a reordering of rows and columns of \( \mathbb{E} \) does not change the trace of \( \mathbb{E}^k \) we are interested in. We then write the overlap matrix in the following block form:

\[ \mathbb{A} = \begin{pmatrix} \frac{\epsilon^2}{1 + \epsilon^2} \mathbb{I} & T \mathbb{B}_1 \\ T \mathbb{B}_2 & \frac{1}{1 + \epsilon^2} \mathbb{I} \end{pmatrix}, \quad (3.21) \]

with quadratic blocks of size \( \mathcal{N} \). Here \( \mathbb{I} \) is the \( \mathcal{N} \times \mathcal{N} \) identity matrix and with respect to (3.11)–(3.14), we have reordered the lines and rows of \( \mathbb{A} \) in such a way that the upper block on the left and the lower one on the right coincide with (3.11) and (3.12), respectively (i.e. after this reordering, the first lines and rows are the odd indices \( n, m \), while the right-lower block is formed by even \( n, m \)). \( \mathbb{B}_1 \) and \( \mathbb{B}_2 \) are \( T \)-independent matrices, whose explicit form is not essential for the proof. Using representation (3.21) and relation (3.8), one obtains

\[ \mathbb{E} = \mathbb{A}(\mathbb{I} - \mathbb{A}) = \begin{pmatrix} \frac{\epsilon^2}{1 + \epsilon^2} \mathbb{I} & T \mathbb{B}_1 \\ T \mathbb{B}_2 & \frac{1}{1 + \epsilon^2} \mathbb{I} \end{pmatrix} \begin{pmatrix} \frac{1}{1 + \epsilon^2} \mathbb{I} & -T \mathbb{B}_1 \\ -T \mathbb{B}_2 & \frac{\epsilon^2}{1 + \epsilon^2} \mathbb{I} \end{pmatrix} \]

\[ = T^2 \begin{pmatrix} \mathbb{I} + \mathbb{B}_1 \mathbb{B}_2 & 0 \\ 0 & \mathbb{I} + \mathbb{B}_2 \mathbb{B}_1 \end{pmatrix}. \quad (3.22) \]
which proves (3.20). Finally, plugging (3.19) and (3.20) into (3.17), one obtains

\[
C^{(2)}(T) = \sum_{k=1}^{\infty} \frac{2^{k-1}[(k-1)!]^2}{\pi^2 k(2k-1)!} T^{2k} = \frac{2}{\pi^2} \arcsin^2 \left( \frac{T}{\sqrt{2}} \right),
\]

where we used

\[
\sum_{k=1}^{\infty} \frac{[(k-1)!]^2}{(2k)!} (2x)^{2k} = 2 \arcsin^2 x.
\]

We stress that each integer value of \( \alpha \geq 2 \) can be treated in an analogous way. For \( \alpha = 3, 4 \), one finds for instance

\[
C^{(3)}(T) = \sum_{k=1}^{\infty} \frac{3^k[(k-1)!]^2}{6\pi^2 k(2k-1)!} T^{2k} = \frac{2}{3\pi^2} \left[ \arcsin^2 \left( \frac{T}{\sqrt{2}} \sqrt{2 + \sqrt{2}} \right) + \arcsin^2 \left( \frac{T}{\sqrt{2}} \sqrt{2 - \sqrt{2}} \right) \right],
\]

and

\[
C^{(4)}(T) = \sum_{k=1}^{\infty} \frac{[(2 + \sqrt{2})^k + (2 - \sqrt{2})^k][(k-1)!]^2}{6\pi^2 k(2k-1)!} T^{2k} = \frac{2}{3\pi^2} \left[ \arcsin^2 \left( \frac{T}{\sqrt{2}} \sqrt{2 + \sqrt{2}} \right) + \arcsin^2 \left( \frac{T}{\sqrt{2}} \sqrt{2 - \sqrt{2}} \right) \right].
\]

### 3.1.1. Result for the generic integer \( \alpha \)

In order to obtain the result for the generic integer \( \alpha \), we first need to write the combination \( \tilde{A}^\alpha + (1 - \tilde{A})^\alpha \) in terms of the matrix \( E \). This can be achieved by formally inverting equation (3.18) as

\[
\tilde{A} = \frac{1}{4} (1 \pm \sqrt{1 - 4E}),
\]

with an ambiguity in the choice of the sign reflecting the degeneration of the spectrum of \( E \). However, in the needed combination

\[
\tilde{A}^\alpha + (1 - \tilde{A})^\alpha = 2^\alpha \left[ (1 \pm \sqrt{1 - 4E})^\alpha + (1 \mp \sqrt{1 - 4E})^\alpha \right],
\]

the choice of this sign is unimportant. Note that despite the apparent presence of a square root, for integer \( \alpha \) the above expression is a polynomial in \( E \) of degree the integer part \( \lceil \alpha/2 \rceil \) of \( \alpha/2 \). Using the binomial theorem \((1 + x)^\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{k} x^k\), we have

\[
\tilde{A}^\alpha + (1 - \tilde{A})^\alpha = 2^\alpha \sum_{k=0}^{\alpha} \binom{\alpha}{k} (1 - 4E)^{\alpha/2} (1 + (-1)^k)
\]

\[
= 2^{1-\alpha} \sum_{k=0}^{\lceil \alpha/2 \rceil} \binom{\alpha}{2k} (1 - 4E)^k,
\]

which makes the polynomial form explicit. Using again the binomial theorem for \((1 - 4E)^k\), we have

\[
\tilde{A}^\alpha + (1 - \tilde{A})^\alpha = 2^{1-\alpha} \sum_{k=0}^{\lceil \alpha/2 \rceil} \binom{\alpha}{2k} \sum_{p=0}^{k} \binom{k}{p} (-4E)^k = -\sum_{p=0}^{\lceil \alpha/2 \rceil} v_p (-4E)^p,
\]

where we defined

\[
v_p \equiv -2^{1-\alpha} \sum_{k=p}^{\lceil \alpha/2 \rceil} \binom{\alpha}{2k} \binom{k}{p} = -\binom{\alpha}{2p} \frac{\Gamma(\alpha - p) \Gamma(p + 1/2)}{\sqrt{\pi} \Gamma(\alpha)}.
\]
In order to calculate the Rényi entropies, we expand in series of $E$ the quantity $\ln[A^\alpha + (1-A)^\alpha]$, obtaining

$$\ln[A^\alpha + (1-A)^\alpha] = \ln\left[1 - \sum_{p=1}^{\lfloor \alpha/2 \rfloor} v_p (-4E)^p\right] = -\sum_{j=1}^{\infty} \frac{1}{j} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} v_p (-4E)^p \right]^j. \quad (3.32)$$

Using now the multinomial identity, we have

$$\ln[A^\alpha + (1-A)^\alpha] = -\sum_{j=1}^{\infty} \sum_{k_1,...,k_{\lfloor \alpha/2 \rfloor}} \frac{j!}{k_1! \cdots k_{\lfloor \alpha/2 \rfloor}!} \prod_{p=1}^{\lfloor \alpha/2 \rfloor} v_p (-4E)^p \right]^j, \quad (3.33)$$

where we introduced the symbol $\sum_{k_1,...,k_{\lfloor \alpha/2 \rfloor}}$ for the constrained sum $\sum_{k_1,...,k_{\lfloor \alpha/2 \rfloor}}$ with $\sum k_i = j$. We now introduce a sum over $K$ which will be equal to $\sum pk_p$ with the help of a Kronecker delta:

$$\ln[A^\alpha + (1-A)^\alpha] = -\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k_1,...,k_{\lfloor \alpha/2 \rfloor}} (j-1)! \prod_{p=1}^{\lfloor \alpha/2 \rfloor} v_p (-4E)^p \right]^j, \quad (3.34)$$

Using the contour integral representation of the Kronecker delta over the unitary circle $|z| = 1$,

$$\delta_{a,b} = \frac{1}{2\pi i} \oint dz z^{a-b-1}, \quad (3.35)$$

we have that the above expression equals

$$\frac{1}{2\pi i} \oint dz \sum_{K=1}^{\infty} (-4E)^K z^{K-1} \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k_1,...,k_{\lfloor \alpha/2 \rfloor}} j! \prod_{p=1}^{\lfloor \alpha/2 \rfloor} v_p (-4E)^p \right]^j$$

$$= \frac{1}{2\pi i} \sum_{K=1}^{\infty} (-4E)^K \oint dz z^{K-1} \ln \left[1 + \sum_{p=1}^{\lfloor \alpha/2 \rfloor} v_p (-4E)^p \right]^j$$

$$= \frac{1}{2\pi i} \sum_{K=1}^{\infty} (-4E)^K \oint dz z^{K-1} \ln \left[1 + \sqrt{1 + z^{-1}} \right] \right]^{\alpha} + (1 - \sqrt{1 + z^{-1}} \right]^{\alpha} \right]^{2\alpha} \quad (3.36)$$

where, in the last line, we recognized the expansion of the function from which we started from for a complex argument (no matrices).

The integral over $z$ can be performed with standard techniques of integrals on the complex plane. The various contributions come from the discontinuities at the cuts which are placed between the zeros of the argument of the logarithm, i.e. at the $z_p$ satisfying

$$(1 + \sqrt{1 + z_p^{-1}}) + (1 - \sqrt{1 + z_p^{-1}}) = 0. \quad (3.37)$$

All the solutions of this equation are simply found as

$$z_p = -\cos^2 \frac{\pi (2p - 1)}{2\alpha}, \quad \text{with} \quad p = 1, \ldots, \alpha/2. \quad (3.38)$$

Thus, the integral is given by

$$\frac{1}{2\pi i} \oint dz z^{K-1} \ln \left[1 + \sqrt{1 + z^{-1}} \right] \right]^{\alpha} + (1 - \sqrt{1 + z^{-1}} \right]^{\alpha} \right]^{2\alpha} \right] = \sum_{p=1}^{\lfloor \alpha/2 \rfloor} z_p^K \right]^{\alpha} \quad (3.39)$$

implying

$$\ln[A^\alpha + (1-A)^\alpha] = -\sum_{K=1}^{\infty} (-4E)^K \frac{(-1)^K}{K} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \cos 2K \left(\frac{2p - 1}{2\alpha}\right). \quad (3.40)$$
In this final form, it is straightforward to take the trace using equation (3.20) to obtain

$$\text{Tr} \ln [A^\alpha + (1 - A)^\alpha] = -\frac{1}{2\pi^2} \ln N \sum_{K=1}^\infty \frac{T^{2K}4^K}{(2K-1)!} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \cos^{2K} \left( \frac{2p - 1}{2\alpha} \right).$$

(3.41)

Inverting the order of the sums, the sum over \( K \) can now be performed using equation (3.24) and we have

$$\text{Tr} \ln [A^\alpha + (1 - A)^\alpha] = -\ln N \frac{2}{\pi^2} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \arcsin^2 \left( \frac{T \cos \left( \frac{2p - 1}{2\alpha} \right)}{2\alpha} \right),$$

(3.42)

which leads to the coefficient

$$C^{(\alpha)}(T) = \frac{1}{\alpha - 1} \frac{2}{\pi^2} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \arcsin^2 \left( \frac{T \cos \left( \frac{2p - 1}{2\alpha} \right)}{2\alpha} \right).$$

(3.43)

For \( \alpha = 2, 3, 4 \), it coincides with the result reported in the previous subsection.

We observe that \( C^{(\alpha)}(0) = 0 \) and \( C^{(\alpha)}(1) \) provide useful checks. The value \( T = 0 \) describes a totally reflecting defect; the two edges are isolated and indeed the entanglement vanishes. For \( T = 1 \), we have

$$C^{(\alpha)}(T = 1) = \frac{1}{\alpha - 1} \frac{2}{\pi^2} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \left[ \frac{\pi}{2} - \frac{(2p - 1)\pi}{2\alpha} \right]^2 = \frac{1}{12} \left( 1 + \frac{1}{\alpha} \right).$$

(3.44)

This corresponds to full transmission, i.e., the defect is absent and one considers the entanglement entropy of half of a system of length \( 2L \) with boundaries obtained in [38] and compatible with the standard CFT result [27].

### 3.1.2. The analytic continuation

In order to investigate the non-integer values of \( \alpha \), we exploit the results of [35, 36] for the Ising and XX spin chains. Using methods based on the corner transfer matrix and conformal mappings, Eisler and Peschel derived the entanglement entropy of the subsystem on (let us say) the right of the defect, for a chain of length \( 2L \) with a defect in the middle. For the XX chain, which after a Jordan–Wigner transformation corresponds to a lattice gas of free spinless fermions, the result can be written in the form [35]

$$S^{(\alpha)}(T; L) = C_L^{(\alpha)}(T) \ln L + O(1),$$

(3.45)

with \( C_L^{(\alpha)}(T) \) given by

$$C_L^{(\alpha)}(T) = \frac{2}{\pi^2(1 - \alpha)} \int_0^\infty dx \ln \left[ \frac{1 + e^{-2\omega(x, T)}}{(1 + e^{-2\omega(x, T)})^{\alpha}} \right],$$

(3.46)

where

$$\omega(x, T) = \text{acosh} \left[ \frac{\cosh(x) + T}{T} \right].$$

(3.47)

Actually only the result for \( \alpha = 1 \) has been reported in [35], but the derivation for general \( \alpha \) is straightforward from the results reported there. A result of \( \alpha = 2 \) has been reported in [36] and for the general integer \( \alpha \) in [46]. For a simple comparison between our work and [35], we mention that the transmission amplitude \( T \) in [35] is called \( s \).

For the integer \( \alpha \), it is straightforward to check numerically that \( C_L^{(\alpha)}(T) \) and \( C^{(\alpha)}(T) \) in equation (3.43) are equal, as it is possible to show analytically [46]. This coincidence is not unexpected. Indeed, since equation (3.45) is valid for the finite density \( N/2L = 1/2 \) on the lattice, it can be turned in the entanglement entropy as a function of \( N \), simply by replacing \( L \)
Figure 2. The von Neumann edge entanglement entropy in the two-wire junction for $T = 3/5$ and $T = 4/5$ where we considered only even values of $N$. Left: we plot $S^{(1)}(T; N) - C^{(1)}(T) \ln N$, where $C^{(1)}(T)$ is given by equation (3.46). The lines show fits of the data for $N \geq 400$, where the correction to the leading behavior $C^{(1)}(T) \ln N$ is a polynomial.

with $N$, but assuming (as we did in [38] on the basis of numerical data) that the dependence on $T$ is universal. Taking now the continuum limit, we straightforwardly deduce that

$$C^{(\alpha)}(T) = C^{(\alpha)}_L(T).$$

(3.48)

However, the above computation proves the universal $T$ dependence with no assumption. The computation of [46] also shows that the result in equation (3.46) is the analytic continuation of equation (3.43) to the non-integer values of $\alpha$. In particular, in the von Neumann case ($\alpha = 1$), the integral can be performed and one has [35]

$$C^{(1)}_L(T) = \frac{1}{\pi^2} \left\{ [(1 + T) \ln(1 + T) + (1 - T) \ln(1 - T)] \ln T 
+ (1 + T) \text{Li}_2(-T) + (1 - T) \text{Li}_2(T) \right\}.$$  \hspace{1cm} (3.49)

3.1.3. Comparison with numerical computation. We now turn to briefly present the numerical data for the entanglement entropies which have been fundamental for the conceptual understanding that led to the exact computation in the previous subsection. Furthermore, numerical computations give non-trivial insights into the corrections to the asymptotic behavior. In particular, we anticipate that in analogy to what was found for the half-space Rényi entanglement entropies in homogeneous systems [39, 47], the leading suppressed corrections turn out to be $O(N^{-1/\alpha})$.

The numerical estimates of the factor $C^{(1)}(T)$, obtained from our results for the entanglement entropy up to $N \approx 500$, perfectly match function (3.49), within a precision better than $O(10^{-6})$. Figure 2 shows the results for $T = 3/5$ and $T = 4/5$ for even $N$. Fits to the $T = 4/5$ data for $400 \lesssim N \lesssim 500$,

$$S^{(1)}(T; N) = a \ln N + b_0/N + b_1/N + b_2/N^2 + b_3/N^3,$$

(3.50)

give $a = 0.118841$, $b_0 = 0.342056$, $b_1 = -0.1404$, etc . . . , where $a$ should be compared with $C^{(1)}(4/5) = 0.11884065 . . .$. Analogous results are obtained for $T = 3/5$; we find $a = 0.076078$ to be compared with $C^{(1)}(3/5) = 0.07607750$. 

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Complete agreement is also found for the Rényi entropies. For $\alpha = 2$ for instance, the data fit the asymptotic behavior

$$S^{(2)}(T; N) = C^{(2)}(T) \ln N + b_0 + b_1/N^{1/2} + b_2/N + b_3/N^{3/2} + \cdots,$$

(3.51)

where the leading coefficient $C^{(2)}(T)$ is given by equation (3.23), as shown in figure 3.

In both equations (3.50) and (3.51), we use for the subleading corrections the form $O(N^{-1/\alpha})$. We mention that considering also odd values of the particle's number $N$, we observe in the graph entanglement entropies corrections to the scaling which depend on the parity of $N$, in analogy to what happens in the absence of defects both in the continuum [39] and on the lattice [47].

3.2. The case of $M > 2$ edges

The generalization to $M > 2$ edges is now straightforward. The novelty is that now the entanglement entropy on the edge $i$ with respect to the whole junction depends on $i$. As expected from (2.42), this dependence is encoded in the transmission amplitudes $T_i$, which for $M > 2$ do not coincide in general for different $i$. In order to investigate this aspect, it is convenient to introduce the variable

$$\Upsilon_i^2 = \frac{1}{2} \left( 1 + \sqrt{1 - T_i^2} \right).$$

(3.52)

The two-point function (2.42) now takes the form

$$C^N_{\mu}(x, y) = \Upsilon_i^2 \sum_{n=1}^{N} f_+(x, n)f_+(y, n) + \left( 1 - \Upsilon_i^2 \right) \sum_{n=1}^{N} f_-(x, n)f_-(y, n).$$

(3.53)

Equivalently, using (2.38) and (2.39), one finds in the case $\mu = \infty$

$$C^N_{\mu}(x, y) = \sum_{n=1}^{2N} \chi_i(x, n)\chi_i(y, n),$$

(3.54)
with
\[ \chi_i(k, x) = \begin{cases} \sqrt{2} \cos \left( k \pi x \frac{1}{2} \right), & k = 1, 3, \ldots, 2N - 1, \\ \sqrt{1 - \frac{1}{2} \sqrt{2} \sin \left( k \pi x \frac{1}{2} \right)}, & k = 2, 4, \ldots, 2N. \end{cases} \tag{3.55} \]

The reduced overlap matrix now also carries an edge index \( i = 1, \ldots, M \) and is given by
\[ K_{mn}^{(i)} = 2 \gamma_i \delta_{mn}, \quad m, n - \text{odd}, \tag{3.56} \]
\[ K_{mn}^{(i)} = 2(1 - \gamma_i^2) \delta_{mn}, \quad m, n - \text{even}, \tag{3.57} \]
\[ K_{mn}^{(i)} = 2 \gamma_i \sqrt{1 - \gamma_i^2} \frac{2n}{\pi (n^2 - m^2)}, \quad m - \text{odd}, \quad n - \text{even}, \tag{3.58} \]
\[ K_{mn}^{(i)} = 2 \gamma_i \sqrt{1 - \gamma_i^2} \frac{2m}{\pi (m^2 - n^2)}, \quad m - \text{even}, \quad n - \text{odd}. \tag{3.59} \]

Comparing matrices (3.11)–(3.14) and (3.56)–(3.59), one concludes that the entanglement entropy \( S_i^{(\alpha)} \) of the edge \( i \) with respect to the whole junction depends on the edge via the transmission amplitude
\[ T_i = 2 \gamma_i \sqrt{1 - \gamma_i^2} \equiv \sqrt{1 - \delta_i^2} \tag{3.60} \]
and can be expressed by
\[ S_i^{(\alpha)} (T_i; N) = S^{(\alpha)} (T_i; N), \tag{3.61} \]
where \( S^{(\alpha)} \) is entropy (3.15) in the case \( M = 2 \). Therefore, the asymptotic behavior for large \( N \) is given by
\[ S_i^{(\alpha)} (T_i; N) = C^{(\alpha)} (T_i) \ln N + O(1), \tag{3.62} \]
where \( C^{(\alpha)} \) is the universal function defined by (3.43), (3.46) and (3.47). At this point, all the analytical and numerical results in the previous subsections concerning \( C^{(\alpha)} \) apply.

4. Schrödinger junction with harmonic potential

In this section, we consider a star graph \( \Gamma \) with \( M \) infinite edges and a harmonic potential \( V(x) = \frac{1}{2} m \omega^2 x^2 \) trapping the gas in the bulk (harmonic trap). The Schrödinger field \( \psi_i(t, x) \) thus satisfies
\[ \left( i \partial_t + \frac{1}{2m} \partial_x^2 - \frac{1}{2} m \omega^2 x^2 \right) \psi_i(t, x) = 0, \quad x > 0. \tag{4.1} \]

Since \( V(x) \) defines a self-adjoint multiplication operator, the vertex boundary conditions controlling the self-adjointness of the total Hamiltonian \( -\partial_x^2 + V(x) \) are still given by (2.2). Accordingly, the critical \( S \) matrices are parametrized by (2.24). The field \( \psi_i(t, x) \), defined by (2.12), satisfies
\[ (\partial_i \psi_i)(t, 0) = 0, \quad 1 \leq i \leq p, \tag{4.2} \]
\[ \psi_i(t, 0) = 0, \quad p < i \leq M, \tag{4.3} \]
for all \( t \). The eigenfunctions of \( -\partial_x^2 + V(x) \), obeying these boundary conditions, are
\[ \phi_i(n, x) = \begin{cases} f_+(n, x), & 1 \leq i \leq p, \\ f_-(n, x), & p < i \leq M, \end{cases} \tag{4.4} \]
where \( f_\pm(n, x) \) are expressed in terms of the Hermite polynomials as follows\(^4\):

\[
\begin{align*}
  f_+(n, x) &= \frac{1}{\pi^{1/4} \sqrt{2^{2n-1}(2n)!}} H_{2n}(x) e^{-x^2/2}, \quad n = 0, 1, \ldots \quad (4.5) \\
  f_-(n, x) &= \frac{1}{\pi^{1/4} \sqrt{2^{2n}(2n+1)!}} H_{2n+1}(x) e^{-x^2/2}, \quad n = 0, 1, \ldots \quad (4.6)
\end{align*}
\]

Inserting (4.5) and (4.6) into (3.53), one obtains the two-point correlation function

\[
C^N_{ii}(x, y) = \sum_{n=0}^{2N} \chi_i(x, n) \chi_i(y, n),
\]

with

\[
\chi_i(k, x) = \begin{cases} 
  \frac{\Upsilon_i}{\pi^{1/4} \sqrt{2^{2k-1}k!}} H_k(x) e^{-x^2/2}, & k = 1, 3, \ldots, 2N - 1, \\
  \frac{1 - \Upsilon_i}{\pi^{1/4} \sqrt{2^{2k}k!}} H_k(x) e^{-x^2/2}, & k = 0, 2, \ldots, 2N.
\end{cases} \quad (4.8)
\]

At this point, one can derive the relative \( \Lambda \)-matrix and compute the entanglement entropy. The numerical data, displayed in figure 4, confirm that in the harmonic case, \( S^{(\alpha)}(T; N) \) has precisely the behavior described by equation (3.62). An educated guess for the large-\( N \) asymptotic behavior of the edge entanglement entropy is that it is the same as that of the hard-wall case, i.e.

\[
S^{(\alpha)}(T; N) = C^{(\alpha)}(T) \ln N + O(1),
\]

where the functions \( C^{(\alpha)} \) are the same as those in equation (3.46). Difference occurs at the level of the \( O(1) \) term, as in the case without defect [38, 48, 49].

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\(^4\) We set for simplicity \( \omega = 1 \) and \( m = 1 \).
In order to check it for nontrivial defects, we have computed the entanglement entropy up to $N \approx 50$ for the two-edge problem, and for two values of $T$, $T = 3/5$ and $T = 4/5$. The results for the von Neumann entropy are shown in figure 4. They are perfectly consistent with conjecture (4.9). For example, a fit of the data for $T = 4/5$ and even $N$ to

$$S^{(1)}(T; N) = a \ln N + b_0 + b_1/N + b_2/N^2 + b_3/N^3 \quad (4.10)$$

gives $a = 0.118842$, $b_0 = 0.424421$, etc... , where $a$ should be compared with $C^{(1)}(T = 4/5) = 0.11884065...$

5. Outlook and conclusions

We have studied the entanglement entropies of one edge $i$ with respect to the rest of a junction with $M$ edges. Our main result is that when working with a finite number of particles $N$ in edges of finite length $L$, the Rényi entanglement entropies can be derived analytically, obtaining

$$S^{(\alpha)} = C^{(\alpha)}(T) \ln N + O(N^0), \quad (5.1)$$

where $T$ is the total transmission probability from the edge $i$ to the rest of the graph and the prefactor is independent of the number of edges. The analytical computation of $C^{(\alpha)}$ is given by equation (3.43) for the integer $\alpha$ and its analytical continuation to the non-integer $\alpha$ is given by the prefactor of Eisler and Peschel [35] obtained for the spatial entanglement of a line with a defect reported in equation (3.46). Clearly the value of the total transmission does depend on the number of edges and of the kind of junction. The same asymptotic behavior in $N$ also describes the entanglement entropies of systems in which on each arm there is a confining parabolic potential.

We can turn the above asymptotic behavior in a more standard expression for the dependence of the entanglement entropies on the length of the subsystem (i.e. the edge in our case). Indeed, assuming a uniform density of particles, we have $N \propto L$ and so

$$S^{(\alpha)} = C^{(\alpha)}(T) \ln L + O(L^0), \quad (5.2)$$

which is expected to be valid also for lattice models and in particular for $M$ XX spin chains of length $L$ joined at a single common vertex.

Finally, we mention that the asymptotic result (5.2) has not yet been derived from CFT. Although it is clear that such derivation must be possible because the CFT encodes all the required ingredients, the practical calculation is very cumbersome, as the similar (but different) result of [33] shows.

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