MORSE THEORY ON LIE GROUPOIDS

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Abstract. In this paper we introduce Morse Lie groupoid morphisms and we study their main properties. We show that Morse Lie groupoid morphisms are Morita invariant, giving rise to a good notion of Morse function on a differentiable stack. We show a groupoid version of the Morse-Bott lemma, which allows us to speak about the index of a non-degenerate critical subgroupoid and hence to describe critical sub-levels by an attaching procedure in the category of topological groupoids. Motivated by the Morse-Bott complex defined by Austin and Braam, we introduce the groupoid Morse double complex, showing that its total cohomology is isomorphic to the Bott-Shulman-Stasheff cohomology of the underlying Lie groupoid. Then we study Morse Lie groupoid morphisms which are invariant under the action of a Lie 2-group yielding an equivariant version of the groupoid Morse double complex. We show that in this case, the associated cohomology is isomorphic to the corresponding equivariant cohomology of the Lie 2-group action. As an application, we show that the equivariant cohomology of toric symplectic stacks can be computed by means of groupoid Morse theory.

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1. Introduction

This is the first in a series of two papers devoted to the study of Morse theory on Lie groupoids and their differentiable stacks. Classical Morse theory is a powerful tool which allows us to extract both geometric and topological information of a manifold equipped with a Morse function, that is, a smooth real valued function on the manifold all of whose critical points are non-degenerate. Morse functions admit a local model around any critical point, this is the content of the Morse Lemma, which establishes that on a neighborhood of any critical point $x \in M$ a Morse function $f : M \to \mathbb{R}$ looks like a quadratic form $Q_f(x) = -(x_1^2 + \cdots + x_3^2) + x_{\lambda+1}^2 + \cdots + x_n^2$. The integer $\lambda$ is called the index of the critical point $x$ of $f : M \to \mathbb{R}$. As a consequence of the Morse Lemma one concludes that non-degenerate critical points are isolated. Some of the main results of Morse theory are: the fundamental theorem of Morse theory which describes how the topology of critical sub-levels changes after crossing a critical level; the Morse inequalities give a relation between the Betti numbers of the manifold and the alternate sum of the number of critical points grouped by their index; the Morse complex which computes the homology of the manifold. Morse theory has led to several interesting geometric results, including: the existence of closed geodesics on a Riemannian manifold as well as an infinite dimensional version of the Morse complex which led to Floer homology.

When dealing with manifolds equipped with a Lie group action, classical Morse theory is no longer the right setting to extract topological information out of an invariant Morse function. This is due to the fact that in equivariant Morse theory, critical points come in orbits hence they are no longer non-degenerate. The right framework to study an invariant Morse function on a manifold is given by Morse-Bott theory. Here, critical points are arranged in families of submanifolds which are non-degenerate in the sense that the normal Hessian is non-degenerate. Among the several applications of Morse-Bott theory one finds: the computation of the cohomology of complex Grassmannians, Bott periodicity theorem or the Atiyah-Guillemin-Sternberg theorem on the convexity of the image of moment maps for torus actions on symplectic manifolds.

In many situations a manifold comes with a Morse-Bott function whose critical submanifolds are not necessarily given by the orbits of a Lie group action but they still come from certain symmetries of the manifold. In this work we are interested in the case of Morse-Bott functions on a manifold whose critical submanifolds are given by the orbits of a Lie groupoid.

Morse theory on singular spaces given by the orbit space of a Lie groupoid has been studied by several authors. In this regard, in [27] Lerman and Tolman study torus actions on symplectic orbifolds, for which they proved some results on Morse-Bott theory on orbifolds. Similarly, in [24] Hepworth studies Morse theory on differentiable Deligne-Mumford stacks, showing for instance the Morse inequalities for orbifolds. Also, Cho and Hong introduce the Morse-Smale-Witten complex for orbifolds [12].

Recently, Lie groupoids equipped with geometric structures suitably compatible with Morita equivalence have been object of intense research. This is due to the fact that such structures descend to the quotient stack of a Lie groupoid. For instance, in [18] the notion of Morita equivalence of VB-groupoids plays a role to define vector bundles over differentiable stacks; Riemannian metrics on Lie groupoids were introduced in [16] showing that they behave well with respect to Morita equivalence, hence inducing a Riemannian metric on the associated quotient stack [17]; Lie algebroids over stacks are modeled by LA-groupoids [42], the space of multiplicative sections of an LA-groupoid [41] has the structure of a Lie 2-algebra which is Morita invariant and, in particular, the space of vector fields on a differentiable stack has a
natural structure of Lie 2-algebra [7, 41] as conjectured in [24]. Also, invariants of differentiable stacks can be defined in terms of their groupoid counterparts. As an example, the equivariant cohomology of a Lie group acting on a differentiable stack was introduced in [5] by looking at the Bott-Shulman cohomology of a certain action groupoid encoded by an equivariant atlas of a $G$-stack. Another invariant of a differentiable stack is introduced in [19] by mean of the cohomology of a Lie groupoid with coefficients in a representation up to homotopy of Lie groupoids.

This paper is concerned with Morse theory on differentiable stacks by looking at its groupoid counterpart. For that, we introduce Morse theory for Lie groupoid morphisms $F: G \to \mathbb{R}$ with values in the unital Lie groupoid $\mathbb{R} \rightrightarrows \mathbb{R}$. Such a morphism is completely determined by a basic function $f \in C^\infty(M)^G$, that is, a function satisfying $s^*f = t^*f$ and hence constant along the groupoid orbits. Note that $C^\infty(M)^G$ is clearly Morita invariant since it coincides with the $0^{th}$-degree groupoid cohomology $H^0_{diff}(G)$. Hence one thinks of either a Lie groupoid morphism $G \to \mathbb{R}$ or its corresponding basic function, as a function on the quotient stack $[M/G]$.

The paper is organized as follows. In Section 2 we review the basics on Morse-Bott theory, Lie groupoids and Riemannian metrics on Lie groupoids as introduced in [16]. In Section 3 we introduce the main objects of study of the paper, namely Morse Lie groupoid morphisms. We study their main properties and we show Proposition 3.10 which establishes that the property of being a Morse Lie groupoid morphism is Morita invariant. In Section 4 we give several examples of Morse Lie groupoid morphisms and we show an existence result for proper Lie groupoids whose base is compact, this is the content of Theorem 4.6. Then we recall the notion of Hamiltonian actions of Lie 2-groups on 0-symplectic groupoids [26] and we show that in the case of a Lie 2-group whose base is a torus the components of the moment map are Morse Lie groupoid morphisms. In Section 5 by means of tubular neighborhoods given by Euler-like vector fields [8], we show Theorem 5.4 which is our version of the Morse Lemma in the Lie groupoid setting. In particular, we introduce a notion of index of a non-degenerate critical subgroupoid as well as their positive and negative normal bundles. In Section 6 we study gradient vector fields of real valued Lie groupoid morphisms with respect to a 2-metric in the sense of [16]. We show Proposition 6.3 which says the such gradient vector fields are multiplicative. We also prove Proposition 6.2 which gives an attaching construction of Lie groupoids whose outcome is a topological groupoid. As a consequence, we show Proposition 6.5 and Theorem 6.8 which describe critical sub-levels of Morse Lie groupoid morphisms in terms of attaching groupoids.

In Section 7 we extend the classical results of Morse-Smale dynamics to the framework of Lie groupoids and we show Proposition 7.6 which says that the moduli space of gradient flow lines has a natural structure of Lie groupoid. In Section 8 we construct a double complex which is the Lie groupoid analogue of the Morse-Bott complex defined by Austin and Braam in [1]. The main result of this section is Theorem 8.10 which shows that the total cohomology of the groupoid double Morse complex is isomorphic to the Bott-Shulman-Stasheff cohomology of the underlying Lie groupoid. In particular, the cohomology of the double groupoid Morse complex is Morita invariant. We finish this section by introducing an equivariant groupoid Morse double complex with respect to an action of a Lie 2-group. We prove Proposition 8.15 which establishes that the equivariant cohomology of a Lie 2-group action as defined in [6] can be computed by means of the cohomology of the equivariant groupoid Morse double complex. We use this result to describe the equivariant cohomology of toric symplectic stacks by means of groupoid Morse theoretical tools.
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2. Preliminaries

In this section we briefly introduce the basic notions and results on both Morse–Bott theory and Lie groupoids which will be used throughout this paper. Much of the classical results about Morse theory may be found for instance in [1, 9, 21, 36, 40]. For the general notions regarding Lie groupoids, Morita equivalences and Riemannian groupoids we follow [15, 16, 37] closely.

2.1. Morse–Bott functions. Let \( f : M \to \mathbb{R} \) be a smooth function such that \( \text{Crit}(f) := \{ x \in M : df(x) = 0 \} \) contains a submanifold \( C \) of positive dimension. The choice of a Riemannian metric on \( M \) yields a decomposition
\[
TM|_C = TC \oplus \nu(C),
\]
where \( TC \) and \( \nu(C) \) are the tangent bundle and the normal bundle of \( C \), respectively. Let \( \mathcal{H}_x(f) \) be the Hessian of \( f \) at \( x \in C \), then \( T_xC \subseteq \text{ker}(\mathcal{H}_x(f)) \). Indeed, if \( v, w \in T_xC \) and \( \tilde{w} \in \mathcal{X}(M) \) is any extension of \( w \), then
\[
\mathcal{H}_x(f)(v, w) = v(\tilde{w} \cdot f) = 0,
\]
since \( df(\tilde{w})|_C = 0 \) because \( C \subseteq \text{Crit}(f) \). Therefore, the Hessian \( \mathcal{H}_x(f) \) induces a well defined symmetric bilinear form on \( \nu_x(C) \) referred to as the normal Hessian of \( f \) at \( x \in C \) and which we shall denote again as \( \mathcal{H}_x(f) \) only if there is no risk of confusion. A critical submanifold \( C \subseteq M \) of \( f \) is called non-degenerate if the normal Hessian at every \( x \in C \) is non-degenerate. This is equivalent to \( \text{ker}(\mathcal{H}_x(f)) = T_xC \) for every \( x \in C \).

Definition 2.1. A smooth function \( f : M \to \mathbb{R} \) is said to be Morse–Bott if \( \text{Crit}(f) \) is a disjoint union of connected submanifolds which are non-degenerate.

Examples of Morse–Bott functions include: usual Morse functions [36, 40], invariant smooth functions by the action of a compact Lie group which have non-degenerate critical orbits [44], and component functions of moment maps associated to Hamiltonian torus actions on symplectic manifolds [40, s. 3.5], among others.

If \( C \) is a connected non-degenerate critical submanifold for \( f \) then we may define a function \( Q_f : \nu(C) \to \mathbb{R} \) which is quadratic along the fibers. Namely,
\[
Q_f(v) = \frac{1}{2} \mathcal{H}_{\pi(v)}(f)(v, v), \quad v \in \nu(C),
\]
where \( \pi : \nu(C) \to C \) is the bundle projection. The Morse-Bott lemma establishes that there exist neighborhoods \( U \subseteq M \) of \( C \) and \( V \subseteq \nu(C) \) of the zero section \( C \), together with a diffeomorphism \( \phi : V \to U \) with
\[
\phi|_C = \text{id} \text{ and } \phi^* f = Q_f.
\]
Remark 2.2. One of the advantages of working with Morse–Bott functions is that they allow us to get similar results to those obtained with usual Morse functions but without assuming that their critical point set is isolated. For instance: we have for them a similar local linear representation provided by the so-called Morse–Bott lemma [9], they describe very well the topological behavior of a manifold around a non-degenerate critical submanifold [9], and using Morse–Bott–Smale dynamics it is possible to recover the de Rham cohomology of a compact oriented manifold by means of a cochain complex which is defined in terms of the de Rham complex of the critical point sets and gradient flow line spaces [1]. The latter fact yields a way to obtain the Morse–Bott inequalities.

2.2. Lie groupoids. A Lie groupoid $G \rightrightarrows M$ consists of a manifold $M$ of objects and a manifold $G$ of arrows, two surjective submersions $s, t : G \to M$ respectively indicating the source and the target of the arrows, and a smooth associative composition $m : G(2) \to G$ over the set of composable arrows $G(2) = G \times_M G$, admitting unit $u : M \to G$ and inverse $i : G \to G$, subject to the usual groupoid axioms. The collection of maps mentioned above are called structural maps of the Lie groupoid.

Special instances of Lie groupoids are given by manifolds, Lie groups, Lie group actions, surjective submersions, foliations, pseudogroups, principal bundles, vector bundles, among others. For specific details the reader is recommended to visit the references [15, 31, 37].

Let us now describe some features concerning the structure of a Lie groupoid. Let $G \rightrightarrows M$ be a Lie groupoid. For each $x \in M$, its isotropy group $G_x := s^{-1}(x) \cap t^{-1}(x)$ is a Lie group and an embedded submanifold in $G$. There is an equivalence relation on $M$ defined by $x \sim y$ if there exists $g \in G$ with $s(g) = x$ and $t(g) = y$. The corresponding equivalence class of $x \in M$ is denoted by $O_x \subseteq M$ and called the orbit of $x$. The previous equivalence relation defines a quotient space $M/G$ called the orbit space of $G \rightrightarrows M$. This space equipped with the quotient topology is in general a singular space, that is, it does not carry a differentiable structure making the quotient projection $M \to M/G$ a surjective submersion.

Given a Lie groupoid $G \rightrightarrows M$, its tangent groupoid is the Lie groupoid $TG \rightrightarrows TM$ obtained by applying the tangent functor to each of its structural maps. If $S \subseteq M$ is a saturated submanifold, i.e. it is given by the union of orbits, then we can restrict the groupoid structure to $G_S = s^{-1}(S) \cap t^{-1}(S)$, thus obtaining a Lie subgroupoid $G_S \rightrightarrows S$ of $G \rightrightarrows M$. Furthermore, the Lie groupoid structure of $TG \rightrightarrows TM$ induces a Lie groupoid $\nu(G_S) \rightrightarrows \nu(S)$ on the normal bundles, having the property that all of its structural maps are fiberwise isomorphisms. In particular, if $S = O$ is any orbit, the source map of the normal Lie groupoid $\nu(G_O) \rightrightarrows \nu(O)$ yields a vector bundle isomorphism $\nu(O)$.

The choice of a Riemannian metric on $M$ gives a decomposition $TM = TC \oplus \nu(C)$, yielding two subbundles $\nu^+(C), \nu^-(C) \subseteq \nu(C)$ spanned by the eigenvectors of $H_f$ with positive and negative eigenvalues, respectively. The index of the critical submanifold $C \subseteq M$ is defined as

$$\lambda(C, f) := \text{rk}(\nu^-(C)),$$

the rank of $\nu^-(C)$.

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$$\frac{dt}{ds} : \nu(O) \to t^*\nu(O),$$

defines a representation of $G_O \rightrightarrows O$ on the normal bundle $\nu(O)$. As a consequence, for every $x \in M$ the isotropy group $G_x$ has a canonical representation on the normal fiber $\nu_x := \nu_x(O_x)$, called the normal representation of $G_x$ on the normal direction.
**Definition 2.3.** A Lie groupoid morphism between $G ightrightarrows M$ and $G' ightrightarrows M'$ is a pair $\phi := (\phi^1, \phi^0)$ where $\phi^1 : G \to G'$ and $\phi^0 : M \to M'$ are smooth functions commuting with both source and target maps and preserving the composition maps.

A Morita map is a groupoid morphism $\phi : (G \rightrightarrows M) \to (G' \rightrightarrows M')$ which is fully faithful and essentially surjective, in the sense that the source/target maps define a fibred product of manifolds $G \cong (M \times M) \times_{(M' \times M')} G'$ and that the map $G' \times_{M'} M \to M$ sending $(\phi^0(x) \to y) \mapsto y$ is a surjective submersion, see [15, 37]. An important fact shown in [15] is that a Lie groupoid morphism is a Morita map if and only if it yields an isomorphism between transversal data. That is, the morphism must induce: a homeomorphism between the orbit spaces, a Lie group isomorphism between the normal representations $G_x \cong G_{\phi^0(x)}$ between the isotropies and isomorphisms between the product of manifolds $G \cong (M \times M) \times_{(M' \times M')} G'$.

**Definition 2.4.** Let $G ightrightarrows M$ and $G' \rightrightarrows M'$ be Lie groupoids. We say that $G$ and $G'$ are Morita equivalent if there exists a third Lie groupoid $H \rightrightarrows N$ with Morita maps $H \to G$ and $H \to G'$.

It is well known that a Morita equivalence can be always realized by Morita fibrations, that is, Morita maps covering a surjective submersion on objects. For more details see [15, 37].

### 2.3. Riemannian groupoids

Here we briefly recall the notion of Riemannian metric on a Lie groupoid introduced in [16]. Such a notion of Riemannian metric is compatible with the groupoid composition so that it plays an important role in our work. We start by recalling that a submersion $\pi : (E, \eta^E) \to B$ with $(E, \eta^E)$ a Riemannian manifold is said to be Riemannian if the fibers of it are equidistant (transverse condition). In this case the base $B$ inherits a metric $\eta^B := \pi_* \eta^E$ for which the linear map $d\pi(e) = (\ker(d\pi(e)))^\perp \to T_{\pi(e)} B$ is an isometry for all $e \in E$.

It is well known that given a Lie groupoid $G \rightrightarrows M$ every pair of composable arrows in $G^{(2)}$ may be identified with an element in the space of commutative triangles so that it admits an action of the symmetric group $S_3$ determined by permuting the vertices of such triangles. In these terms, a Riemannian groupoid is a pair $(G \rightrightarrows M, \eta)$ where $G \rightrightarrows M$ is a Lie groupoid and $\eta = \eta^{(2)}$ is a Riemannian metric on $G^{(2)}$ that is invariant by the $S_3$-action and transverse to the composition map $m : G^{(2)} \to G$. The metric $\eta$ induces metrics $\eta^{(1)} = \pi_2^* \eta^{(2)} = m_* \eta^{(2)} = \pi_1^* \eta^{(2)}$ on $G$ and $\eta^{(0)} = s_* \eta^{(1)} = t_* \eta^{(1)}$ on $M$ such that $\pi_2, m, \pi_1 : G^{(2)} \to G$ and $s, t : G \to M$ are Riemannian submersions. This is because the $S_3$-action permutes these face maps.

The metric $\eta^{(j)}$, for $j = 2, 1, 0$, is called a $j$-metric. It is important to mention that every proper groupoid can be endowed with a $2$-metric (more generally, an $n$-metric as defined below) and if a Lie groupoid admits a $2$-metric then it is weakly linearizable. For more details visit [16, 14].

We finish this section by giving a quick observation that will be very useful when working with the nerve of a Lie groupoid at the end of the paper.

**Remark 2.5.** The notion of $n$-metric on Lie groupoids for $n \geq 3$ was introduced in [16]. This is just a Riemannian metric on the set of $n$-composable arrows $G^{(n)}$ that is invariant by the canonical $S_{n+1}$-action on $G^{(n)}$ and transverse to one (hence to all) face map $G^{(n)} \to G^{(n-1)}$. We can push this $n$-metric forward with the different face maps $G^{(n)} \to G^{(n-1)}$ to define an $(n-1)$-metric on $G^{(n-1)}$ in such a way these face maps become Riemannian submersions. One can use this process to obtain $r$-metrics $\eta^{(r)}$ on $G^{(r)}$ for all $0 \leq r \leq n - 1$ so that we get Riemannian submersions $(G^{(r)}, \eta^{(r)}) \to (G^{(r-1)}, \eta^{(r-1)})$. 


3. Morse Lie groupoid morphisms

Let $G \rightrightarrows M$ be a Lie groupoid. The space of basic functions on $M$ is defined by

$$C^\infty(M)^G := \{ f \in C^\infty(M) : s^*f = t^*f \}.$$  

In other words, a basic function $f : M \to \mathbb{R}$ is just a function which is constant along the groupoid orbits. On the one hand, the space of basic functions is Morita invariant since

$$H^0_{\text{diff}}(G) = C^\infty(M)^G,$$

where $H^0_{\text{diff}}(G)$ denotes $0^{th}$-degree groupoid cohomology, which is well-known to be Morita invariant $[13]$. Hence, one can think of $H^0_{\text{diff}}(G) = C^\infty(M)^G$ as the space of smooth functions on the quotient stack $[M/G]$. On the other hand, any basic function $f \in C^\infty(M)^G$ induces a Lie groupoid morphism $F : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ given either by $F = s^*f$ or $F = t^*f$. It is clear that every Lie groupoid morphism $F : G \to \mathbb{R}$ has this form. Hence one can identify the space of real valued Lie groupoid morphisms with that of basic functions.

Let us see some elementary examples of real valued Lie groupoid morphisms.

**Example 3.1.** If $M$ is a smooth manifold and $M \rightrightarrows M$ is its underlying unit Lie groupoid, then having a Lie groupoid morphism $F : (M \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is the same as having a smooth function $f : M \to \mathbb{R}$.

**Example 3.2.** If $G$ is a Lie group acting on a smooth manifold $M$ and $G \ltimes M \rightrightarrows M$ is the corresponding action groupoid, then a Lie groupoid morphism $F : (G \ltimes M \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is completely determined by a $G$-invariant smooth function $f : M \to \mathbb{R}$. Such a function is constant on the orbits of the action so that it provides us with a well defined function on the quotient space $M/G$.

**Example 3.3.** Suppose that $\pi : M \to N$ is a surjective submersion and that $M \times_N M \rightrightarrows M$ is its corresponding submersion groupoid. To have a Lie groupoid morphism $F : (M \times_N M \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is the same as having a smooth map $f : M \to \mathbb{R}$ which is constant on the fibers of $\pi$. So, we get a well defined function on $M/N$.

**Example 3.4.** Let $G \rightrightarrows M$ be a proper Lie groupoid with proper Haar measure system $\{\mu^x\}$. For any smooth function $f : M \to \mathbb{R}$ we define

$$f^\mu(x) := \int_{g \in s^{-1}(x)} (f \circ t)(g) \mu^x(g), \quad x \in M.$$  

First, note that the properness of the Haar system $\{\mu^x\}$ ensures that the integral defining $f^\mu$ is finite and the smoothness tells us that $f^\mu$ is also smooth. Also, the right-invariance and the identity $t \circ m = t \circ \pi_1$ imply that for all $h \in G$

$$f^\mu(t(h)) = \int_{g \in s^{-1}(t(h))} (f \circ t)(g) \mu^x(g) = \int_{g \in s^{-1}(s(h))} (f \circ t)(kg) \mu^x(g) = \int_{g \in s^{-1}(s(h))} (f \circ t)(g) \mu^x(g) = f^\mu(s(h)).$$

So, $f^\mu$ is basic and hence defines a Lie groupoid morphism $F^\mu : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$.

Now we introduce the notion of critical arrow of a real valued Lie groupoid morphism.

**Definition 3.5.** A critical arrow of a Lie groupoid morphism $F : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is defined as a critical point of either $s^*f$ or $t^*f$. 
Let \( \text{Crit}(F) \subseteq G \) and \( \text{Crit}(f) \subseteq M \) denote the sets of critical points of \( F \) and \( f \), respectively. Note that if \( x \) is a critical point of \( f \) then its orbit \( \mathcal{O}_x \) is a critical submanifold of \( f \). Hence the subset \( \text{Crit}(f) \subseteq M \) is saturated.

**Lemma 3.6.** There exists a natural topological groupoid structure \( \text{Crit}(F) \rightrightarrows \text{Crit}(f) \).

**Proof.** Let us suppose that \( g \in G \) is a critical arrow of \( F \). It is simple to see that both \( s(g) \) and \( t(g) \) are critical points of \( f \) since both \( s \) and \( t \) are surjective submersions. This in turn implies that \( g^{-1} \) is critical arrow of \( F \) as well. Also, if \( x \in \text{Crit}(f) \) one easily sees that \( 1_x \in \text{Crit}(F) \).

Finally, using the identities \( s \circ m = s \circ \pi_2 \) and \( t \circ m = t \circ \pi_1 \), one concludes that if \( (g, h) \in G^{(2)} \) with either \( g \) or \( h \) a critical arrow of \( F \), then the composition \( gh \) is also a critical arrow of \( F \).

It follows from the previous lemma that \( s^{-1} \text{Crit}(f) = t^{-1} \text{Crit}(f) = \text{Crit}(F) \), since \( \text{Crit}(f) \) is saturated in \( M \). In particular, if \( \mathcal{O} \subseteq M \) is a critical orbit of \( f \), then \( G_{\mathcal{O}} = s^{-1}(\mathcal{O}) = t^{-1}(\mathcal{O}) \) is a critical submanifold of \( F \). Hence, the restricted Lie groupoid \( G_{\mathcal{O}} \rightrightarrows \mathcal{O} \) is a critical Lie subgroupoid of \( G \rightrightarrows M \).

**Proposition 3.7.** Let \( F : G \to \mathbb{R} \) be a Lie groupoid morphism covering \( f : M \to \mathbb{R} \). Let \( G_{\mathcal{O}} \rightrightarrows \mathcal{O} \) be a critical Lie subgroupoid defined by a critical orbit \( \mathcal{O} \subseteq M \). The following are equivalent:

i) \( \mathcal{O} \subseteq M \) is a non-degenerate critical orbit, and

ii) \( G_{\mathcal{O}} \subseteq G \) is a non-degenerate critical submanifold of \( F \).

**Remark 3.8.** Before proving Proposition 3.7, we observe that if \( \pi : Q \to S \) is a surjective submersion and \( f \in C^\infty(S) \), then after fixing a suitable system of local coordinates, the following identity holds:

\[
\mathcal{H}_q(f \circ \pi) = d\pi(q)^T \cdot \mathcal{H}_{\pi(q)}(f) \cdot d\pi(q) + (df(\pi(q)) \otimes I_n) \cdot \mathcal{H}_q(\pi),
\]

where \( n = \dim S \).

Now we are ready to prove Proposition 3.7.

**Proof.** (Proposition 3.7) Applying formula (5) to the surjective submersion \( s : G \to M \) given by the source map, one gets

\[
\mathcal{H}_g(f \circ s) = ds(g)^T \cdot \mathcal{H}_{s(g)}(f) \cdot ds(g) + (df(s(g)) \otimes I_n) \cdot \mathcal{H}_g(s),
\]

where \( n = \dim M \). Let us show that i) implies ii). For that, suppose that \( \mathcal{O} \) is a non-degenerate critical orbit of \( f \). It follows from identity (6) that if \( g \in G_{\mathcal{O}} \) then

\[
\mathcal{H}_g(f \circ s) = ds(g)^T \cdot \mathcal{H}_{s(g)}(f) \cdot ds(g),
\]

since \( s(g) \in \mathcal{O} \) is a critical point of \( f \).

Since \( ds : \nu(G_{\mathcal{O}}) \to \nu(\mathcal{O}) \) is a fiberwise isomorphism and \( \mathcal{H}_{s(g)}(f) \) is non-degenerate when restricted to \( \nu_{s(g)}(\mathcal{O}) \), it follows from (7) that \( \mathcal{H}_g(f \circ s) \) is non-degenerate when restricted to \( \nu_g(G_{\mathcal{O}}) \). The converse statement, i.e. ii) implies i), follows similarly.

Due to the previous result, the following definition is natural.

**Definition 3.9.** Let \( F : G \to \mathbb{R} \) be a Lie groupoid morphism covering \( f : M \to \mathbb{R} \). We say that \( F \) is a **Morse Lie groupoid morphism** if every critical orbit \( \mathcal{O} \subseteq M \) of \( f \) is non-degenerate.
It is worth noticing that \( F : G \to \mathbb{R} \) is a Morse Lie groupoid morphism if and only if every critical subgroupoid \( G_O \rightrightarrows \mathcal{O} \) is non-degenerate in the sense that \( \mathcal{O} \subset M \) is a non-degenerate critical orbit, hence \( G_O \subseteq G \) is a non-degenerate critical submanifold of \( F \).

We show now that the notion of Morse Lie groupoid morphism is Morita invariant. Recall that a Morita equivalence between \( G \rightrightarrows M \) and \( G' \rightrightarrows M' \) yields an isomorphism between zero degree groupoid cohomology, that is, an isomorphism between the corresponding spaces of basic functions. More precisely, if \( K \rightrightarrows N \) is a Lie groupoid together with Morita fibrations \( \phi : K \to G \) and \( \psi : K \to G' \), then \( \psi^*f' \in C^\infty(N)^K \) for every \( f' \in C^\infty(M')^G \). Also, there exists a unique \( f \in C^\infty(M)^G \) with \( \phi^*f = \psi^*f' \). This defines an isomorphism

\[
C^\infty(M')^G \to C^\infty(M)^G; f' \mapsto f.
\]  

(8)

In particular, the isomorphism (8) yields an isomorphism

\[
\text{Hom}_{Gpds}(G', \mathbb{R}) \to \text{Hom}_{Gpds}(G, \mathbb{R}); F' = s^*f' \mapsto F = s^*f,
\]

(9)

where \( f' \in C^\infty(M')^G \) and \( f \in C^\infty(M)^G \) are related by (8). Our main goal now is to show that (8) gives rise to an isomorphism between basic Morse functions, hence between Morse Lie groupoid morphisms.

**Proposition 3.10.** Let \( G \rightrightarrows K \to G' \) be a Morita equivalence covering surjective submersions at the level of objects. The isomorphism (9) preserves Morse Lie groupoid morphisms.

**Proof.** Suppose that \( f \in C^\infty(M)^G \) allows us to define a Morse Lie groupoid morphism. To prove that \( f' \in C^\infty(M')^G \) induces another Morse Lie groupoid morphism it suffices to show that \( \phi^*F \) is a Morse Lie groupoid morphism since \( \phi^*f = \psi^*f' \). Indeed, the Lie groupoid morphism \( \phi^*F : (K \rightrightarrows N) \to (\mathbb{R} \rightrightarrows \mathbb{R}) \) is given by the pair \( (F \circ \phi^1, f \circ \phi^0) \). Note that if \( x \in M \) is a critical point of \( f \circ \phi^0 \), then \( \phi^0(x) \) is a critical point of \( f \) since \( \phi^0 \) is a surjective submersion. Thus, from Identity (6) we get at \( x \) that

\[
\mathcal{H}_x(f \circ \phi^0) = d\phi^0(x)^T \cdot \mathcal{H}_{\phi^0(x)}(f) \cdot d\phi^0(x).
\]

From [15] we know that if \( \phi \) is a Morita map then \( d\phi^0 : \nu(O_x) \to \nu(O'_{\phi(x)}) \) is a fiberwise isomorphism. Thus, as \( \mathcal{H}_{\phi^0(x)}(f) \) is non-degenerate when restricted to \( \nu_{\phi^0(x)}(O'_{\phi^0(x)}) \) we conclude that \( \mathcal{H}_x(f \circ \phi^0) \) is non-degenerate when restricted to \( \nu_x(O_x) \), as desired. \( \square \)

4. **Examples and existence of Morse Lie groupoid morphisms**

In this short section we mention some examples of Morse Lie groupoid morphisms. Elementary examples are the following:

**Example 4.1.** If \( M \) is a smooth manifold then every Morse function \( f : M \to \mathbb{R} \) induces a Lie groupoid Morphism on the unit groupoid \( M \rightrightarrows M \).

**Example 4.2.** A Lie group bundle is a Lie groupoid \( G \rightrightarrows M \) such that \( s = t \). Therefore, any Morse function \( f : M \to \mathbb{R} \) induces a Lie groupoid morphism on \( G \rightrightarrows M \) since its orbits are points. In particular, on a Lie group \( G \rightrightarrows \{\ast\} \) every Lie groupoid morphism \( G \to \mathbb{R} \) is necessarily constant. Hence, there are no interesting examples of Morse Lie groupoid morphisms on Lie groups. Combining this with Proposition 3.10 we conclude the same for any transitive Lie groupoid \( G \rightrightarrows M \) since transitive Lie groupoids are always Morita equivalent to Lie groups.
Example 4.3. Let $G$ be a compact Lie group acting on a smooth manifold $M$ and consider the action groupoid $G \times M \rightrightarrows M$. Wasserman showed in [43] that the set of $G$-equivariant Morse functions on $M$ is dense in the set of $G$-equivariant functions. Hence, there always exist Morse Lie groupoid morphisms on $G \times M \rightrightarrows M$.

Example 4.4. Let $(M, \mathcal{F})$ be a complete transverse parallel foliated connected manifold (see for instance [39, s. 4.5] or [37, s. 4.1.2]). Let $p : \tilde{M} \to M$ denote the universal covering map and consider the induced foliation $\tilde{\mathcal{F}}$ on $M$. This foliation is simple so that we have that $X = \tilde{M}/\tilde{\mathcal{F}}$ is a Hausdorff manifold and the canonical projection $\pi_{\text{bas}} : \tilde{M} \to X$ is a surjective submersion. It turns out that with this data it is possible to obtain a natural structure of principal groupoid bi-bundle $(\tilde{M}, p, \pi_{\text{bas}}) : \text{Hol}(M, \mathcal{F}) \to \pi_1(M) \times X$ between the holonomy groupoid $\text{Hol}(M, \mathcal{F}) \rightrightarrows M$ and the action groupoid $\pi_1(M) \times X \rightrightarrows X$, that is, a Morita equivalence. Therefore, if $M$ has finite fundamental group then as a consequence of Wasserman’s result and Proposition 3.10 we get that there exist Morse Lie groupoid morphisms on $\text{Hol}(M, \mathcal{F}) \rightrightarrows M$.

Example 4.5. Suppose that $G \rightrightarrows M$ is a Lie groupoid for which the orbit space $M/G$ admits a structure of smooth manifold such that the canonical projection $\pi : M \to M/G$ is a surjective submersion. Any Morse function $\tilde{f} : M/G \to \mathbb{R}$ induces a basic function $f := \pi^*\tilde{f} : M \to \mathbb{R}$. The fact that every orbit $O \subseteq M$ is non-degenerate follows by applying (5) to the surjective submersion $\pi : M \to M/G$. In particular, $G$ has Morse groupoid morphisms.

4.1. Proper groupoids with compact base. Recall that isomorphism classes of orbifolds are in one-to-one correspondence with Morita equivalence classes of Lie groupoids which are proper and étale [38]. Thus, motivated by the detailed study of Morse theory in the context of orbifolds done by Hepworth in [24], we explain how to prove that Morse Lie groupoid morphisms on proper étale groupoids $G \rightrightarrows M$ are dense. More importantly, we adapt some of Hepworth’s ideas to prove that:

Theorem 4.6. If $G \rightrightarrows M$ is a proper Lie groupoid with $M$ compact then Morse Lie groupoid morphisms on $G \rightrightarrows M$ are dense in the space of all Lie groupoid morphisms $G \rightrightarrows M$.

Proof. Let us consider a proper Haar measure system $\{\mu^x\}$ for $G \rightrightarrows M$. By taking average with respect to $\{\mu^x\}$ we can define a surjective linear continuous operator $P^u : C^\infty(M) \to C^\infty(M)^G$ by sending $f$ to $f^u$ as defined in Example 3.4. This is consequence of having that $C^\infty(M)$ is a Fréchet space and the average operator $P^u$ is a projection, i.e. $(P^u)^2 = P^u$. In particular, by the Banach-Schauder Theorem it follows that $P^u$ is open. Mather proved in [33, Prop. 3.1] that $C^\infty(M)$ is a Baire space so that the previous facts imply that the space of basic functions $C^\infty(M)^G$ is also a Baire space in $C^\infty(M)$ with respect to the Whitney $C^\infty$-topology.

Crainic and Struchiner showed in [14] that any $x \in M$ has an open neighborhood $U_\alpha$ in $M$ such that $G_{U_\alpha} \rightrightarrows U_\alpha$ is Morita equivalent to the action groupoid $G_\times \nu_x(O_x) \rightrightarrows \nu_x(O_x)$. Therefore, by Proposition 3.10 combined with Wasserman’s result we have that there exist Morse Lie groupoid morphisms $F_\alpha = s^*f_\alpha = t^*f_\alpha$ on $G_{U_\alpha} \rightrightarrows U_\alpha$ and they are actually dense. We may assume that the open cover we obtain above is countable and it satisfies that for every $\alpha$ there exists a compact subset $K_\alpha \subseteq U_\alpha$ such that $\text{Crit}(f_\alpha) \subseteq K_\alpha$ since $M$ is compact.

Let $\mathcal{M}_\alpha(G)$ denote the subset formed by functions $f \in C^\infty(M)^G$ for which $F|_{G_{U_\alpha}}$ is a Morse Lie groupoid morphisms on $G_{U_\alpha} \rightrightarrows U_\alpha$ such that $\text{Crit}(f|_{U_\alpha}) \subseteq K_\alpha$. Note that the set $\mathcal{M}(G)$ of all Morse Lie groupoid morphisms on $G \rightrightarrows M$ agrees with $\bigcap_\alpha \mathcal{M}_\alpha(G)$. If we show that $\mathcal{M}_\alpha(G)$ is open and dense for all $\alpha$ then $\mathcal{M}(G)$ will be dense since $C^\infty(M)^G$ is a Baire space. On the one hand, the fact that $\mathcal{M}_\alpha(G)$ is open follows from [4, Lem. 5.32] since $\text{Crit}(f|_{U_\alpha}) \subseteq K_\alpha$. On
the other hand, pick \( f \in C_G(M) \) and let \( \mathcal{N} \) be an open neighborhood of \( f \). If we prove that \( \mathcal{N} \cap \mathcal{M}_\alpha(G) \) is nonempty then we will have that \( \mathcal{M}_\alpha(G) \) is dense. Consider the restriction \( F|_{G_{U_\alpha}} \) of \( F \) on \( G_{U_\alpha} \Rightarrow U_\alpha \). By either taking average with respect to \( \{ \mu^x \} \) or mimicking the proof of [24, Lem. 3.12] it follows that there exists a basic smooth function \( \phi_\alpha : U_\alpha \rightarrow \mathbb{R} \) such that \( \phi_\alpha \equiv 1 \) in a neighborhood of \( K_\alpha \) and its support is compact. This implies that the map \( C^\infty(U_\alpha)^{G_{U_\alpha}} \rightarrow C^\infty(M)^G \) given by \( g \mapsto \tilde{\phi_\alpha} g \), where the symbol \( \sim \) denotes smooth extension by zero (which in this case is well defined since \( \phi_\alpha \) has compact support), is continuous (compare also with [24, Lem. 6.12]). Therefore, there exists an open neighborhood \( \mathcal{N}' \) in \( C^\infty(U_\alpha)^{G_{U_\alpha}} \) small enough such that \( \tilde{f}(1 - \tilde{\phi_\alpha}) + \tilde{\phi_\alpha} g \in \mathcal{N} \) for all \( g \in \mathcal{N}' \). We already know that Morse Lie groupoid morphisms on \( G_{U_\alpha} \Rightarrow U_\alpha \) are dense so that we may assume that there exists \( g \in \mathcal{N}' \) defining a Morse Lie groupoid morphisms such that \( \tilde{f}(1 - \tilde{\phi_\alpha}) + \tilde{\phi_\alpha} g \in \mathcal{N} \). Recall that by construction \( \phi_\alpha \equiv 1 \) in a neighborhood of \( K_\alpha \subset U_\alpha \) for which \( \tilde{f}(1 - \tilde{\phi_\alpha}) + \tilde{\phi_\alpha} g \) restricts to \( g \) over such a neighborhood. So, the result follows.

\[ \square \]

**Remark 4.7.** It is worth mentioning that if we assume that \( G \Rightarrow M \) is proper and étale then we may prove that Morse Lie groupoid morphisms on \( G \Rightarrow M \) are dense **without requiring that** \( M \) **is compact.** We only have to use the local description of such a Lie groupoid around any point provided by [37, Prop. 5.30 and Cor. 5.31] together with Proposition 3.10 and Hepworth’s ideas in the context of orbifolds as we did in Theorem 4.6. In this case, the existence of the compact sets \( K_\alpha \) containing Crit(\( f|_{U_\alpha} \)) is for free. See [24] for specific details.

### 4.2. Moment maps on 0-symplectic groupoids

Our goal now is to show that moment maps for Hamiltonian 2-actions on 0-symplectic groupoids in the sense of [26] induce Morse Lie groupoid morphisms. This will be consequence of the results proved in [30] for Hamiltonian actions on presymplectic manifolds. We start by briefly introducing some necessary terminology which can be found in [26].

A **foliation groupoid** is a Lie groupoid \( G \Rightarrow M \) whose space of objects \( M \) is Hausdorff and whose isotropy groups \( G_x \) are discrete for all \( x \in M \). For instance, every étale Lie groupoid with Hausdorff objects manifold is a foliation groupoid. The converse is not true, however every foliation groupoid is Morita equivalent to an étale groupoid. As shown in [13], being a foliation groupoid is equivalent to the associated Lie algebroid anchor map \( \rho : A \rightarrow TM \) being injective. As a consequence, the manifold \( M \) comes with a regular foliation \( F \) tangent to the leaves of \( \text{im}(\rho) \subseteq TM \). Note that if \( G \Rightarrow M \) is source-connected the leaves of \( \text{im}(\rho) \subseteq TM \) coincide with the groupoid orbits.

A **basic** 2-form on a foliation groupoid \( G \Rightarrow M \) is given by a pair of 2-forms \( \omega = (\omega_1, \omega_0) \) with \( \omega_1 \in \Omega^2(G) \), \( \omega_0 \in \Omega^2(M) \) satisfying \( s^*\omega_0 = \omega_1 = t^*\omega_0 \). We say that \( \omega \) is **non-degenerate** if \( \ker(\omega_0) = \text{im}(\rho) \subseteq TM \). A basic 2-form \( \omega = (\omega_1, \omega_0) \) is **closed** if \( \omega_0 \) is closed.

**Definition 4.8.** [26] A **0-symplectic groupoid** is a foliation groupoid equipped with a closed and non-degenerate basic 2-form.

It is important to point out that this notion of symplectic groupoid differs from that of Weinstein introduced in [44], since \( \omega_1 \in \Omega^2(G) \) is not necessarily non-degenerate nor multiplicative.

It follows immediately from Definition 4.8 that \( (M, \omega_0) \) is a pre-symplectic manifold with \( \ker(\omega_0) = TF \) in the sense of [30]. Additionally, there is a left action of the product groupoid \( G \times G \Rightarrow M \times M \) on \( G \) along \((s, s)\) given by \((g, h)f = gh^{-1}f\). The components of the orbits of this action define a regular foliation \( F_1 \) of \( G \) satisfying \( TF_1 = \ker(ds) + \ker(dt) \). In particular, \( \omega \)
in Definition 4.8 is non-degenerate if and only if $\ker(\omega_1) = \ker(ds) + \ker(dt)$. As a consequence, $(G, \omega_1)$ is also a pre-symplectic manifold with $\ker(\omega_1) = TF_1$.

To define Hamiltonian 2-actions in this context we need to introduce the notion of Lie 2-group. A Lie 2-group is a Lie groupoid $K_1 \rightrightarrows K_0$ where both $K_1$ and $K_0$ are Lie groups and all the structural maps are Lie group morphisms. A 2-action of $K_1 \rightrightarrows K_0$ on $G \rightrightarrows M$ is a Lie groupoid morphism from the product groupoid $K_1 \times G \rightrightarrows K_0 \times M$ to $G \rightrightarrows M$ whose component maps are Lie group actions in the usual sense. If we apply the Lie functor to $K_1 \rightrightarrows K_0$ then we obtain a Lie 2-algebra $\mathfrak{t}_1 \rightrightarrows \mathfrak{t}_0$, i.e. a Lie groupoid where $\mathfrak{t}_1$ and $\mathfrak{t}_0$ are Lie algebras and all the structural maps are Lie algebra morphisms.

It is well known that there exists a bijective correspondence between Lie 2-groups and crossed modules of Lie groups; see [11]. By a crossed module of Lie groups we mean a quadruple $(K, H, \partial, \alpha)$ where $K$ and $H$ are Lie groups, $\partial : H \to K$ is a Lie group morphism, and $\alpha : K \to \text{Aut}(H)$ is an action of $K$ on $H$ subject to the requirements $\partial(\alpha(g)(h)) = g\partial(h)g^{-1}$ and $\alpha(\partial(h))(h') = hh'h^{-1}$ for all $g \in K$ and $h, h' \in H$. In particular, if $K_1 \rightrightarrows K_0$ is a Lie 2-group then its associated crossed module is determined by the data $K = G_0$, $H = \ker(s)$, $\partial = \mathfrak{t}_1|_M$, and $\alpha$ is the conjugation action of $K_1$ on $H$ composed with the identity bisection map $u : K_0 \to K_1$.

Let us denote by $\pi : \mathfrak{t} \to \mathfrak{t}/\mathfrak{h}$ the quotient map.

Lemma 4.9. [26] If $K_1 \rightrightarrows K_0$ is a foliation Lie 2-group with associated crossed module $(K, H, \partial, \alpha)$ then the pair $(\pi \circ \text{Lie}(t), \pi) : (\mathfrak{t}_1 \rightrightarrows \mathfrak{t}_0) \to (\mathfrak{t}/\mathfrak{h} \rightrightarrows \mathfrak{t}/\mathfrak{h})$ is a Morita morphism of Lie 2-algebras.

As a consequence of the previous result, the Lie groupoid morphism $\text{Ad} : (K_1 \times \mathfrak{t}_1 \rightrightarrows K_0 \times \mathfrak{t}_0) \to (\mathfrak{t}_1 \rightrightarrows \mathfrak{t}_0)$, which is formed by the adjoint actions $\text{Ad}_j$ of $K_j$ on $\mathfrak{t}_j$ (for $j = 0, 1$), descends to a well defined 2-action of $K_1 \rightrightarrows K_0$ on $\mathfrak{t}/\mathfrak{h} \rightrightarrows \mathfrak{t}/\mathfrak{h}$. By abuse of language, we will call this induced 2-action as the adjoint action and denote it by $\text{Ad}$ as well. Accordingly, this notion of adjoint action allows us to speak about the coadjoint action $\text{Ad}^* : (K_1 \times (\mathfrak{t}/\mathfrak{h})^* \rightrightarrows K_0 \times (\mathfrak{t}/\mathfrak{h})^*) \to (\mathfrak{t}/\mathfrak{h})^* \rightrightarrows (\mathfrak{t}/\mathfrak{h})^*$ which is nothing but the 2-action whose component maps are the coadjoint actions $\text{Ad}_j^*$ of $K_j$ on $\mathfrak{t}/\mathfrak{h}$ induced by the identification we mentioned above.

The final ingredient necessary to define Hamiltonian 2-actions is given by the notion of fundamental vector field associated to a 2-action. Namely, consider a 2-action of $K_1 \rightrightarrows K_0$ on $G \rightrightarrows M$. It is simple to check that for every $\xi \in \mathfrak{t} = \mathfrak{t}_0$ the pair $(\text{Lie}(u)(\xi), \xi_M)$, formed by the fundamental vector fields of the respective Lie group actions, determines a multiplicative vector field on $G \rightrightarrows M$. Therefore, if $K_1 \rightrightarrows K_0$ is a foliation Lie 2-group and $G \rightrightarrows M$ is a foliation groupoid then the fundamental vector field associated to the 2-action above is by definition the basic vector field on $G \rightrightarrows M$ determined by the pair $(\text{Lie}(u)(\xi), \xi_M)$. By applying this procedure it is possible to show that there exists a Lie algebra anti-morphism from $\mathfrak{t}/\mathfrak{h}$ to the Lie algebra of basic vector fields on $G \rightrightarrows M$; see [26, Pro. 6.9.2] for further details.

Definition 4.10. [26] Let $(G \rightrightarrows M, \omega)$ be a 0-symplectic groupoid and let $K_1 \rightrightarrows K_0$ be a foliation Lie 2-group with associated crossed module $(K, H, \partial, \alpha)$. A 2-action of $K_1 \rightrightarrows K_0$ on $(G \rightrightarrows M, \omega)$ is said to be Hamiltonian if the following conditions hold:

1. the action of $K_0$ on $M$ is presymplectic, and
2. there is a morphism of Lie groupoids called moment map

$$\mu = (\mu_1, \mu_0) : (G \rightrightarrows M) \to ((\mathfrak{t}/\mathfrak{h})^* \rightrightarrows (\mathfrak{t}/\mathfrak{h})^*),$$

verifying

i) for all $\xi \in \mathfrak{t}/\mathfrak{h}$ it satisfies $d\mu_0^\xi = \iota_{\xi_M}^* \omega_0$, and
ii) $\mu$ is equivariant with respect to the 2-action of $K_1 \rightrightarrows K_0$ on $(G \rightrightarrows M, \omega)$ and the coadjoint action of $K_1 \rightrightarrows K_0$ on $(\mathfrak{k}/\mathfrak{h})^* \rightrightarrows (\mathfrak{k}/\mathfrak{h})^*$.

If all these conditions are satisfied then we say that $(G \rightrightarrows M, \omega)$ is a Hamiltonian $(K_1 \rightrightarrows K_0)$-groupoid with moment map $\mu$.

Some observations about the previous definition come in order. First, as we are working with a 2-action and $\omega$ is basic then we immediately get that the action of $K_1$ on $G$ is also presymplectic. Additionally, $d\mu_1^\xi = \iota_{\text{Lie}(\mu)(\xi)}\omega_1$. This follows from the fact that $(\text{Lie}(\mu)(\xi), \xi_M)$ is a multiplicative vector field, $\omega$ is basic, and either $\mu_0 \circ s = \mu_1$ or $\mu_0 \circ t = \mu_1$. One also observes that
\[
s^*(\mu_0^\xi)(x) = \mu_0(s(x))(\xi) = \mu_1(x)(\xi) = \mu_0(t(x))(\xi) = t^*(\mu_0^\xi)(x).
\]

Therefore, for each $\xi \in \mathfrak{k}/\mathfrak{h}$ we have a well defined Lie groupoid morphisms $\mu^\xi : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ given either by $s^*(\mu_0^\xi)$ or $t^*(\mu_0^\xi)$. The required condition which will allow us to ensure that $\mu^\xi$ is a Morse Lie groupoid morphisms is determined in terms of the notion of “cleanness” introduced in [30].

Consider a left action of a connected Lie group $K$ on a presymplectic manifold $(M, \omega)$ with foliation $\mathcal{F}$ and set $n(\mathcal{F}) = \{\xi \in \mathfrak{k} : (\xi_M)(x) \in T_x\mathcal{F} \text{ for all } x \in M\}$. This space is an ideal in $\mathfrak{k}$. Let $N(\mathcal{F})$ be the connected immersed Lie subgroup in $K$ with Lie algebra $n(\mathcal{F})$.

**Definition 4.11.** [30] The action of $K$ on $M$ is clean if
\[
T_x(O_{N(\mathcal{F})}(x)) = T_x(O_K(x)) \cap T_x\mathcal{F},
\]
for all $x \in M$.

Summing up, we are in conditions to state:

**Proposition 4.12.** Let $(G \rightrightarrows M, \omega)$ be a Hamiltonian $(K_1 \rightrightarrows K_0)$-groupoid with moment map $\mu : G \to (\mathfrak{k}/\mathfrak{h})^*$. Suppose that $K_0$ is a torus and the action of $K_0$ on $M$ is clean. Then, for every $\xi \in \mathfrak{k}/\mathfrak{h}$ the map $\mu^\xi : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is a Morse Lie groupoid morphism with even index at every non-degenerate critical arrow.

**Proof.** Let $\xi \in \mathfrak{k}/\mathfrak{h}$ be fixed. The fact that $\mu^\xi$ is a Lie groupoid morphism implies that the critical point set of $\mu_0^\xi$ is saturated in $M$. Thus, when applying [30, Thm. 3.4.5] to our situation we get that every critical orbit in $\text{Crit}(\mu_0^\xi)$ is a non-degenerate submanifold of even index since the action of $K_0$ on $M$ is clean. Therefore, $\mu^\xi$ is a Morse Lie groupoid morphism with the required property.

\[\square\]

5. The Morse lemma

The goal of this section is to state a version of the Morse lemma in the Lie groupoid setting. Our approach relies on the nice geometric proof of existence of groupoid tubular neighborhoods around an orbit due to Meinrenken in [35] combined with his ideas used to give a proof of the Morse-Bott Lemma. A different approach that may also be used to achieve a groupoid version of the Morse lemma is the classical one provided in [21, App.B]. Recall that given a Morse-Bott function $f : M \to \mathbb{R}$ and a non-degenerate critical submanifold $C \subseteq M$, the Morse-Bott Lemma gives a local normal form for $f$ around $C$. Namely, on a suitable neighborhood of $C$ the function $f$ looks like the quadratic form $Q_f : \nu(C) \to \mathbb{R}$ defined in (1) up to a constant, see for instance [2].

We start by considering a Lie groupoid morphism $F : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ associated to a basic function $f \in C^\infty(M)^G$. Suppose that $O \subseteq M$ is a non-degenerate critical orbit of $f$ and
consider the restricted Lie groupoid $G_\mathcal{O} \rightrightarrows \mathcal{O}$. It follows from Proposition 3.7 that $G_\mathcal{O} \subseteq G$ is a non-degenerate critical submanifold of $F$. In particular, the construction of (1) applies to both $f$ and $F$ yielding quadratic forms $Q_F : \nu(G_\mathcal{O}) \rightarrow \mathbb{R}$ and $Q_f : \nu(\mathcal{O}) \rightarrow \mathbb{R}$.

**Lemma 5.1.** The pair $(Q_F, Q_f) : (\nu(G_\mathcal{O}) \ni \nu(\mathcal{O})) \rightarrow (\mathbb{R} \ni \mathbb{R})$ is a Lie groupoid morphism.

**Proof.** Let $g \in G_\mathcal{O}$ a critical arrow. Then, from identity (6) we get that

$$\mathcal{H}_g(f \circ s) = d_s(g)^T \cdot \mathcal{H}_s(f) \cdot d_s(g).$$

Given that $F = s^* f = t^* f$ we actually have $d_s(g)^T \cdot \mathcal{H}_s(f) \cdot d_s(g) = dt(g)^T \cdot \mathcal{H}_x(f) \cdot dt(g)$. It follows from the definition of $Q_F$ and $Q_f$ that the previous two identities immediately imply that $Q_{f os} = Q_f \circ d_s$ and $Q_{f ot} = Q_f \circ dt$ with

$$Q_F = Q_f \circ d_s = Q_f \circ dt.$$ 

So, we have a well defined Lie groupoid morphism $Q_F : (\nu(G_\mathcal{O}_x) \ni \nu(\mathcal{O}_x)) \rightarrow (\mathbb{R} \ni \mathbb{R})$, as claimed.

**Remark 5.2.** A similar argument shows that the Hessian $H(F) : \nu(G_\mathcal{O}) \oplus \nu(G_\mathcal{O}) \rightarrow \mathbb{R}$ is a Lie groupoid morphism covering $H(f) : \nu(\mathcal{O}) \oplus \nu(\mathcal{O}) \rightarrow \mathbb{R}$. In other words,

$$H(F) = (d_s \oplus d_s)^* H(f) = (dt \oplus dt)^* H(f).$$

In particular, $H(f) : \nu(\mathcal{O}) \oplus \nu(\mathcal{O}) \rightarrow \mathbb{R}$ is invariant under the normal representation (4).

### 5.1. Normal form around an orbit.

Let $G \rightrightarrows M$ be a Lie groupoid and $\mathcal{O} \subseteq M$ an orbit. Denote by $G_\mathcal{O} \rightrightarrows \mathcal{O}$ the restricted Lie groupoid. A **groupoid neighborhood** of $G_\mathcal{O} \rightrightarrows \mathcal{O}$ is given by an open Lie subgroupoid $(\tilde{U} \rightrightarrows \tilde{U}) \subseteq (G \rightrightarrows M)$ with $G_\mathcal{O} \subseteq \tilde{U}$ and $\mathcal{O} \subseteq \tilde{U}$. Let $\nu(G_\mathcal{O}) \ni \nu(\mathcal{O})$ be the normal groupoid associated to an orbit. We say that $G \rightrightarrows M$ is **weakly linearizable** around $\mathcal{O}$ if there are groupoid neighborhoods $(\tilde{U} \rightrightarrows \tilde{U}) \subseteq (G \rightrightarrows M)$ of $G_\mathcal{O} \rightrightarrows \mathcal{O}$ and $(\tilde{V} \rightrightarrows \tilde{V}) \subseteq (\nu(G_\mathcal{O}) \ni \nu(\mathcal{O}))$ of $G_\mathcal{O} \rightrightarrows \mathcal{O}$ seen as the zero section, and a Lie groupoid isomorphism

$$\phi : (\tilde{V} \rightrightarrows \tilde{V}) \rightarrow (\tilde{U} \rightrightarrows \tilde{U}),$$

which is the identity on $G_\mathcal{O} \rightrightarrows \mathcal{O}$. We refer to $\tilde{U} \rightrightarrows \tilde{U}$ as a **groupoid tubular neighborhood** of $G_\mathcal{O} \rightrightarrows \mathcal{O}$.

Let us now consider a non-degenerate critical subgroupoid $G_\mathcal{O} \rightrightarrows \mathcal{O}$ of a Lie groupoid morphism $F : (G \rightrightarrows M) \rightarrow (\mathbb{R} \ni \mathbb{R})$. Assume that $G \rightrightarrows M$ is weakly linearizable around $\mathcal{O}$ (e.g. if $G$ carries a 2-metric or an Euler-like multiplicative vector field as defined below) and consider the groupoid isomorphism $\phi : (\tilde{V} \rightrightarrows \tilde{V}) \rightarrow (\tilde{U} \rightrightarrows \tilde{U})$ given by a groupoid tubular neighborhood of $G_\mathcal{O} \rightrightarrows \mathcal{O}$. The **local model** of $F$ around $G_\mathcal{O}$ is defined as the Lie groupoid morphism

$$\tilde{F} := \phi^* F : (\tilde{V} \rightrightarrows \tilde{V}) \subseteq (\nu(G_\mathcal{O}) \ni \nu(\mathcal{O})) \rightarrow (\mathbb{R} \ni \mathbb{R}).$$ 

Note the zero section is a non-degenerate critical subgroupoid of $\tilde{F}$.

Mehrkenen’s proof of the existence of groupoid tubular neighborhoods for proper Lie groupoids is based on the existence of Euler-like multiplicative vector fields [35]. Namely, a vector field on a manifold $M$ is called **Euler-like** with respect to a submanifold $C$ if it vanishes along the submanifold and its linear approximation is the Euler vector field $\mathcal{E}$ on the normal bundle $\nu(C)$, i.e. $\mathcal{E}$ is the vector field having scalar multiplication by $e^{-t}$ as its flow at the normal direction. It is important to mention that an Euler-like vector field $X$ for $(M, C)$ determines a unique
maximal tubular neighborhood embedding \( \phi : \nu(C) \to M \) such that \( \phi^* X = \mathcal{E} \); compare with [8].

We recall the notion of multiplicative vector field introduced by Mackenzie and Xu in [32]. A **multiplicative vector field** on a Lie groupoid \( G \rightrightarrows M \) is a Lie groupoid morphism \( X : G \to TG \) satisfying \( p_G \circ X = \text{id} \) where \( p_G : TG \to G \) is the canonical projection. Hence, a multiplicative vector field encompasses vector fields \( X^1 \in \mathfrak{X}(G) \) and \( X^0 \in \mathfrak{X}(M) \) such that \( X^1 : G \to TG \) is a Lie groupoid morphism covering \( X^0 : M \to TM \).

In these terms we have:

**Theorem 5.3.** [35] If \( X : G \to TG \) is an Euler-like multiplicative vector field for \((G,G_\mathcal{O})\) then the tubular neighborhood embedding \( \phi : \nu(G_\mathcal{O}) \to G \) defined by \( X \) determines a Lie groupoid tubular neighborhood of \( G_\mathcal{O} \rightrightarrows \mathcal{O} \). In particular, if \( G \rightrightarrows M \) is proper then there always exists such an Euler-like multiplicative vector field for \((G,G_\mathcal{O})\).

A nice application mentioned in [35] says that if \( f : M \to \mathbb{R} \) is a Morse–Bott function with non-degenerate critical submanifold \( C \subseteq M \) then there exists an Euler-like vector field \( X \) for \((M,C)\) such that \( \mathcal{L}_X f = 2f \) near \( C \). Therefore, in the resulting tubular neighborhood embedding \( \phi \) associated to \( X \) we have \( \phi^* X = \mathcal{E} \), thus obtaining that \( \mathcal{L}_X \phi^* f = 2\phi^* f \). This means that \( \phi^* f \) is homogeneous of degree 2 and hence coincides with its quadratic approximation \( c + Q_f \) (here \( c \) is the common value of \( f \) on \( C \)). As a consequence, we have obtained that \( \phi^* f = c + Q_f \) which is just the Morse–Bott lemma.

Now we are ready to set up the first part of our version of the Morse lemma in the Lie groupoid context.

**Theorem 5.4** (Morse lemma). Let \( F : G \to \mathbb{R} \) be a Morse Lie groupoid morphism covering \( f : M \to \mathbb{R} \). If \( G \) is proper, then around a non-degenerate critical subgroupoid \( G_\mathcal{O} \rightrightarrows \mathcal{O} \) there is a Lie groupoid tubular neighborhood \( \phi : (\widetilde{V} \rightrightarrows V) \xrightarrow{\cong} (\widetilde{U} \rightrightarrows U) \) such that

\[
\phi^* F = c + Q_F.
\]

More precisely, \( (\phi_1^* F,\phi_0^* F) = (c + Q_F,c + Q_f) \) where \( c = f(\mathcal{O}_x) = F(G_{\mathcal{O}_x}) \) is the common value of \( f \) and \( F \) on \( G_{\mathcal{O}} \) and \( \mathcal{O} \), respectively.

**Proof.** We shall follow closely Meinrenken’s ideas to construct an Euler-like multiplicative vector field for \((G,G_\mathcal{O})\) combined with his application regarding the proof of the Morse-Bott lemma mentioned right above. As the groupoid orbit \( \mathcal{O} \) is non-degenerate for \( f \) there exists an Euler-like vector field \( Y \) for \((M,\mathcal{O})\) such that \( \mathcal{L}_Y f = 2f \) near \( \mathcal{O} \). Since \( s \) is a surjective submersion there is a vector field \( X \) on \( G \) such that \( X \) is \( s \)-related with \( Y \). The fact that \( \overline{ds} : \nu(G_{\mathcal{O}}) \to \nu(\mathcal{O}) \) is a fibrewise isomorphism implies that \( X \) is Euler-like for \((G,G_\mathcal{O})\). Furthermore, \( \mathcal{L}_X F = 2F \) near \( G_{\mathcal{O}} \) because \( F = s^* f \) and \( X \) is \( s \)-related with \( Y \).

Let us now consider a proper Haar measure system \( \{\mu^x\} \) for \( G \rightrightarrows M \). By following results due to Crainic and Stuchiner in [14] we can construct a multiplicative vector field \( \overline{X} : G \to TG \) by taking the average with respect to \( \{\mu^x\} \):

\[
\overline{X}(g) = \int_{a \in \mathcal{L}^{-1}(\nu(g))}^{\mu_{(g),a^{-1}}}(X(a g), d\alpha(X_a))\mu(a).
\]

This vector field is \( s \)-related with the vector field \( \overline{Y} \) on \( M \) defined as

\[
\overline{Y}(x) = \int_{a \in \mathcal{L}^{-1}(x)}^{\mu_{a}(X_a)} dt_a(X_a)\mu(a).
\]
Meinrenken showed that both \( \overline{X} \) and \( \overline{Y} \) are Euler-like vector fields, thus proving that \( G \rightrightarrows M \) is weakly linearizable around \( G_O \rightrightarrows O \) (see [35, Lem. 4.2 and Thm. 4.5]). Therefore, our result will follow once we prove that \( \mathcal{L}_{\overline{X}}f = 2f \) and \( \mathcal{L}_{\overline{Y}}F = 2F \) near \( O \) and \( G_O \), respectively. Indeed, on the one hand it follows that

\[
\mathcal{L}_{\overline{X}}f(x) = \int_{a \in t^{-1}(x)} d(f \circ t)_a(X_a)\mu(a) = \int_{a \in t^{-1}(x)} df_{s(a)}(Y(s(a))\mu(a) = 2 \int_{a \in t^{-1}(x)} (f \circ t)(a)\mu(a) = 2f(x),
\]

since \( X \) is \( s \)-related with \( Y \) and \( f \) is basic. On the other hand, as \( \overline{X} \) and \( \overline{Y} \) are \( s \)-related then from the previous computation we trivially obtain that \( \mathcal{L}_{\overline{Y}}F = 2F \) near \( G_O \), as desired. \( \square \)

**Remark 5.5.** It was shown in [16] that if \( (G \rightrightarrows M, \eta) \) is a Riemannian groupoid (see Section 2.3) and \( O \subseteq M \) is any orbit, then the exponential map defines a weak linearization of \( G \) around \( O \). Also, a result of [16] establishes that any proper Lie groupoid admits a 2-metric, hence if \( G \rightrightarrows M \) is proper groupoid, then it is weakly linearizable around \( O \). The proof of the Morse lemma we exhibited in Theorem 5.4 reveals the geometric richness derived from the existence of an Euler-like multiplicative vector field. Nevertheless, a less simple proof of such a result may be given by assuming the weakly linearization provided by a 2-metric combined with the classical ideas for the proof of the Morse-Bott lemma that can be found for instance in [21, App.B]. The latter approach has the advantage of not requiring our Lie groupoid to be proper.

From now on we assume that \( G \rightrightarrows M \) can be equipped with a Riemannian 2-metric \( \eta(2) \) on \( G^{(2)} \) which in turn induces a 1-metric \( \eta(1) \) on \( G \) and a 0-metric \( \eta(0) \) on \( M \). As it happens in classical Morse theory, the following natural number will be relevant to us in the sequel.

**Proposition 5.6.** Let \( F : G \to \mathbb{R} \) be a Morse Lie groupoid morphism covering \( f : M \to \mathbb{R} \). Let \( O \subseteq M \) be a non-degenerate critical orbit and \( G_O \rightrightarrows O \) the corresponding non-degenerate critical subgroupoid of \( F \). Then the index \( \lambda(G_O, F) \) coincides with the index \( \lambda(O, f) \).

**Proof.** Let \( g \in G_O \) be an arrow with \( s(g) = x \in O \). If we restrict the 1-metric \( \eta(1) \) to \( \nu(G_O) \) and the 0-metric \( \eta(0) \) to \( \nu(O) \) and use them to identify \( \nu_g(G_O) \cong T_g(G_O)^{\perp} \) and \( \nu_x(O) \cong T_x(O)^{\perp} \), then the fact that \( s \) is a Riemannian submersion implies that \( ds(g)^T = ds(g)^{-1} \). Since \( F = s^*f \), it follows from Formula (6) that \( \mathcal{H}_g(f \circ s) = \overline{ds(g)^{-1}} \cdot \mathcal{H}_x(f) \cdot \overline{ds(g)} \). Hence, as the Hessian forms \( \mathcal{H}_g(f \circ s) \) and \( \mathcal{H}_x(f) \) are non-degenerate when restricted to the normal bundles \( \nu(G_O) \) and \( \nu(O) \), respectively, we obtain that they have the same eigenvalues, thus obtaining that the indexes \( \lambda(G_O, F) \) and \( \lambda(O, f) \) agree. \( \square \)

**Remark 5.7.** One observes that the index of a non-degenerate critical orbit \( O \subseteq M \) is well-defined even if \( O \) is not connected. This follows immediately from the invariance of the Hessian by the normal representation, see Remark (5.2).

Throughout this paper we will use the following terminology.

**Definition 5.8.** Let \( F : G \to \mathbb{R} \) be a Morse Lie groupoid morphism covering \( f : M \to \mathbb{R} \). If \( O \subseteq M \) is a non-degenerate critical orbit with associated non-degenerate critical subgroupoid of \( F \), then any arrow \( g \in G_O \) will be called a **non-degenerate critical arrow** of \( F \) and its **index** is defined as \( \lambda(g, F) := \lambda(O, f) \).
With the aim of getting other consequences from the classical Morse–Bott lemma that emerge in this setting we fix from here a non-degenerate critical arrow \( y \xmapsto{\varphi} x \). It is well known that we have the splittings \( \nu(G_{O_x}) = \nu_{\perp}(G_{O_x}) \oplus \nu_{\parallel}(G_{O_x}) \) and \( \nu(O_x) = \nu_{\perp}(O_x) \oplus \nu_{\parallel}(O_x) \) which are defined by the eigenvectors corresponding to the positive/negative eigenvalues of \( \mathcal{H}_g(F) \) and \( \mathcal{H}_x(f) \), respectively.

**Lemma 5.9.** The Lie groupoid structure of \( \nu(G_{O_x}) \Rightarrow \nu(O_x) \) can be restricted to define two new Lie subgroupoids \( \nu_{\perp}(G_{O_x}) \Rightarrow \nu_{\perp}(O_x) \) and \( \nu_{\parallel}(G_{O_x}) \Rightarrow \nu_{\parallel}(O_x) \).

**Proof.** We will show why it is possible to restrict the groupoid structure in the first case since the second one is completely analogous. We already know that \( \mathcal{H}_g(f \circ s) = \overline{ds}(g)^{-1} \mathcal{H}_x(f) \overline{ds}(g) \).

If \( v \) is an eigenvector of \( \mathcal{H}_g(f \circ s) \) with negative eigenvalue \( c \), then
\[
\mathcal{H}_x(f)(\overline{ds}(g)(v)) = \overline{ds}(g)(\mathcal{H}_x(f \circ s)(v)) = \overline{ds}(g)(c \cdot v) = c \cdot \overline{ds}(g)(v).
\]

That is, \( \overline{ds}(g)(v) \) is an eigenvector of \( \mathcal{H}_x(f) \) with eigenvalue \( c \). Same conclusion may be obtained by arguing with the Riemannian submersion \( t \). Let \( v \) and \( u \) be eigenvectors of \( \mathcal{H}_g(f \circ s) \) and \( \mathcal{H}_h(f \circ s) \) with respective negative eigenvalues \( c_1 \) and \( c_2 \) such that \( \overline{dm}_{(g,h)}(v, u) \) is well defined.

Let us say \( z \xmapsto{\varphi} x \xmapsto{h} y \). Thus, by using the formula \( s \circ m = s \circ \pi_2 \) we get that
\[
\mathcal{H}_{gh}(f \circ s)(\overline{dm}_{(g,h)}(v, u)) = \overline{ds}(gh)^{-1}(\mathcal{H}_g(f)(\overline{ds}(gh)(\overline{dm}_{(g,h)}(v, u))))
\]
\[
= \overline{ds}(gh)^{-1}(\mathcal{H}_g(f)(\overline{ds}(h)(\overline{dm}_{(g,h)}(v, u)))) = \overline{ds}(gh)^{-1}(\mathcal{H}_g(f)(\overline{ds}(h)(u)))
\]
\[
= \overline{ds}(gh)^{-1}(\mathcal{H}_h(f \circ s)(u))) = c_2 \cdot \overline{ds}(gh)^{-1}(\overline{ds}(h)(u))
\]
\[
= c_2 \cdot \overline{ds}(gh)^{-1}(\overline{ds}_{\pi_2(g,h)}((\overline{dm}_{(g,h)}(v, u))))
\]
\[
= c_2 \cdot \overline{ds}(gh)^{-1}(\overline{ds}_{\pi_2(g,h)}((\overline{dm}_{(g,h)}(v, u))))
\]
\[
= c_2 \cdot \overline{ds}(gh)^{-1}(\overline{ds}(s \circ m)(g,h)(v, u)) = c_2 \cdot \overline{dm}_{(g,h)}(v, u).
\]

If we assume that \( F = t^*f \) then by using the identity \( t \circ m = s \circ \pi_1 \) we conclude that \( \mathcal{H}_{gh}(f \circ t)(\overline{dm}_{(g,h)}(v, u)) = c_1 \cdot \overline{dm}_{(g,h)}(v, u) \). This computation implies that the composition \( \overline{dm}_{(\nu_{\perp}(G_{O_x}))}^{(2)} \to \nu_{\perp}(G_{O_x}) \) is well defined when considering \( \overline{ds} \) and \( \overline{dt} \) restricted to \( \nu_{\perp}(G_{O_x}) \). The restriction of the unit map as \( \overline{du} : \nu_{\perp}(O_x) \to \nu_{\perp}(G_{O_x}) \) is also well defined since \( s \circ u = id \) holds true. Indeed, if \( v \) is an eigenvector of \( \mathcal{H}_x(f) \) with negative eigenvalue \( c \) we have that
\[
\mathcal{H}_{1c}(f \circ s)(\overline{du}(v)(x)(v)) = \overline{ds}(1c)^{-1}(\mathcal{H}_x(f)(\overline{ds}(1c)(\overline{du}(x)(v)(v)))) = \overline{ds}(1c)^{-1}(\mathcal{H}_x(f)(\overline{ds}(s \circ u)(x)(v)))
\]
\[
= c \cdot \overline{ds}(1c)^{-1}(v) = c \cdot \overline{ds}(1c)^{-1}(\overline{ds}(s \circ u)(v)(x)) = c \cdot \overline{du}(v)(x).
\]

So, \( \overline{du}(v)(x) \) is an eigenvector of \( \mathcal{H}_{1c}(f \circ s) \) with eigenvalue \( c \). Finally, with similar computations, using the identities \( t = soi, \) \( s = toi, \) and \( \mathcal{H}(f \circ s) = \mathcal{H}(f \circ t) \), we obtain that if \( v \) is an eigenvector of \( \mathcal{H}_c(f \circ s) \) with negative eigenvalue \( c \) then \( \mathcal{H}_{g \circ \varphi}(f \circ s)(\overline{du}(g)(v)) = c \cdot \overline{du}(g)(v) \). Thus, the restriction of the inverse map as \( \overline{dt} : \nu_{\perp}(G_{O_x}) \to \nu_{\perp}(G_{O_x}) \) is also well defined. The properties required to be satisfied by the composition, the inverse, and the unit map follow from those of \( \nu(G_{O_x}) \Rightarrow \nu(O_x) \).

Let us consider the negative unit disk bundle \( D_{\perp}(G_{O_x}) \) defined by
\[
D_{\perp}(G_{O_x}) = \{ v \in \nu_{\perp}(G_{O_x}) : \| v \|_1 \leq 1 \},
\]
where \( \| \cdot \|_1 \) is the norm on \( \nu_{\perp}(G_{O_x}) \) induced by \( \eta^{(1)} \). The positive unit disk bundle \( D_{\parallel}(G_{O_x}) \) is defined accordingly. Also, one has unit disk bundles at the level of objects \( D_{\perp}(O_x) \) and \( D_{\parallel}(O_x) \) defined by the induced metric \( \eta^{(0)} \) on \( M \).
As both the ranks of \( \nu_-(G_{\mathcal{O}_z}) \) and \( \nu_-(\mathcal{O}_z) \) agree and they are actually the index \( \lambda \) of the non-degenerate critical submanifolds, the fibers of the negative unit disk bundles are \( \lambda \)-dimensional disks. Moreover, the unit disk bundles define Lie groupoids of the normal groupoid.

**Lemma 5.10.** The Lie groupoid structure of \( \nu_-(G_{\mathcal{O}_z}) \Rightarrow \nu_-(\mathcal{O}_z) \) restricts to the unit disk bundle yielding a Lie subgroupoid \( D_-(G_{\mathcal{O}_z}) \Rightarrow D_-(\mathcal{O}_z) \).

**Proof.** Recall that \( s,t : G \to M \) as well as \( \pi_1,m,\pi_2 : G^{(2)} \to G \) are Riemannian submersions and that the inversion map \( i : G \to G \) is an isometry. Therefore, if we consider the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_0 \) with respect to the metrics \( \eta^{(1)} \) and \( \eta^{(0)} \) restricted to the normal bundles \( \nu(G_{\mathcal{O}_z}) \) and \( \nu(\mathcal{O}_z) \), respectively, then we get the identities

\[
\|v\|_1 = \|d\nu(x)\|_0, \quad \|v\|_1 = \|d\nu(x)\|_0, \quad \|(v,w)\|_2 = \|d\nu(x,w)\|_1 = \|d\nu(x)\|_0 = \|v\|_1,
\]

\[
\|(v,w)\|_2 = \|d\nu(x,w)\|_1 = \|d\nu(x)\|_0 = \|v\|_1, \quad \|d\nu(x)\|_1 = \|v\|_1, \quad \|d\nu(w)\|_1 = \|v\|_0.
\]

To deduce these formulas it is important to have in mind the identities \( s \circ m = s \circ \pi_2, t \circ m = t \circ \pi_1, t = s \circ i, s = t \circ i \), and \( s \circ u = id \). Hence, by mimicking the steps followed in Lemma 5.9 it is simple to see that \( D_-(G_{\mathcal{O}_z}) \Rightarrow D_-(\mathcal{O}_z) \) is a Lie subgroupoid of \( \nu_-(G_{\mathcal{O}_z}) \Rightarrow \nu_-(\mathcal{O}_z) \).

The Lie groupoid introduced in Lemma 5.10 will be called the **unit disk groupoid** of \( \nu_-(G_{\mathcal{O}_z}) \Rightarrow \nu_-(\mathcal{O}_z) \) with respect the 2-metric \( \eta^{(2)} \). Note that this is actually a Lie groupoid with boundary. Indeed, the boundary of \( D_-(G_{\mathcal{O}_z}) \) is the unit sphere bundle

\[
\partial D_-(G_{\mathcal{O}_z}) = \{ v \in \nu(G_{\mathcal{O}_z}) : \|v\|_1 = 1 \},
\]

with the natural projection onto \( G_{\mathcal{O}_z} \). The unit sphere bundle \( \partial D_-(\mathcal{O}_z) \) is similarly defined by using the norm \( \| \cdot \|_0 \) induced by \( \eta^{(0)} \). Observe that the fibers of these sphere bundles are indeed \( (\lambda - 1) \)-dimensional spheres. It is simple to check that there is a well defined Lie groupoid \( \partial D_-(G_{\mathcal{O}_z}) \Rightarrow \partial D_-(\mathcal{O}_z) \) whose structural maps are the induced ones. This Lie groupoid will be called the **unit sphere groupoid** of \( \nu_-(G_{\mathcal{O}_z}) \Rightarrow \nu_-(\mathcal{O}_z) \) with respect the 2-metric \( \eta^{(2)} \) and we shall usually refer to it as the boundary of \( D_-(G_{\mathcal{O}_z}) \Rightarrow D_-(\mathcal{O}_z) \).

Summing up, we have obtained the second part of our version of the Morse lemma in the Lie groupoid setting.

**Proposition 5.11.** There exists a natural Lie groupoid \( \nu_-(G_{\mathcal{O}_z}) \Rightarrow \nu_-(\mathcal{O}_z) \) defining a unit disk groupoid \( D_-(G_{\mathcal{O}_z}) \Rightarrow D_-(\mathcal{O}_z) \) with boundary the unit sphere groupoid \( \partial D_-(G_{\mathcal{O}_z}) \Rightarrow \partial D_-(\mathcal{O}_z) \).

As we mentioned above, the same conclusion can be obtained about the Lie groupoids defined in terms of the positive eigenvalues. The natural projection \( \partial D_-(G_{\mathcal{O}_z}) \to G_{\mathcal{O}_z} \) is a Lie groupoid morphism covering the projection \( \partial D_-(\mathcal{O}_z) \to \mathcal{O}_z \). This Lie groupoid morphism is just the restriction of the canonical projection \( TG \Rightarrow TM \to (G \Rightarrow M) \).

6. **Gradient vector field and Lie subgroupoid levels**

After having described the local features of a Morse Lie groupoid morphism around a non-degenerate critical arrow, in this section we deal with one of the most important results of Morse theory that, in turn, addresses the topological behavior of a Lie groupoid around a non-degenerate critical Lie subgroupoid. We start by introducing a notion of attaching groupoid and then we study multiplicative gradient vector fields.
6.1. Attaching groupoid. In this subsection we show how to construct topological groupoids by an attaching procedure between Lie groupoids.

**Definition 6.1.** Let \( G \rightrightarrows M \) be a Lie groupoid. A **groupoid attaching data** on \( G \) consists of:

i) A Lie groupoid \( G' \rightrightarrows M' \),

ii) closed submanifolds \( \partial G' \subset G' \) and \( \partial M' \subset M' \) such that \( \partial G' \rightrightarrows \partial M' \) is a Lie subgroupoid of \( G' \rightrightarrows M' \), and

iii) a Lie groupoid morphism \( (B, b) : (\partial G' \rightrightarrows \partial M') \to (G \rightrightarrows M) \).

A groupoid attaching data defines two topological spaces \( G \sqcup_B G' \) and \( M \sqcup_B M' \) defined by the usual attaching construction. Namely, the quotient space \( G \sqcup_B G' \) is defined by taking the disjoint union \( G \sqcup G' \) and then identifying \( g' \sim B(g') \) for all \( g' \in \partial G' \). The attaching space \( M \sqcup_B M' \) is defined in the same way.

One immediately observes that there is a natural topological groupoid \( G \sqcup G' \rightrightarrows M \sqcup M' \) whose structural maps are defined as the disjoint union of the corresponding structural maps of \( G \) and \( G' \). Our main goal in what follows is to show that this groupoid structure descends to the attaching spaces giving rise to a topological groupoid \( G \sqcup_B G' \rightrightarrows M \sqcup_B M' \). Since \( (B, b) : (\partial G' \rightrightarrows \partial M') \to (G \rightrightarrows M) \) is a Lie groupoid morphism, it follows that the source and target maps \( s \sqcup s', t \sqcup t : G \sqcup G' \to M \sqcup M' \) pass to the quotient, yielding surjective open maps

\[
\begin{align*}
\tau, \tau : G \sqcup_B G' & \to M \sqcup_B M', \\
\tau & : G \sqcup_B G' \\
\end{align*}
\]

As usual, we define the set of composable arrows \((G \sqcup_B G')^{(2)}\) as the fibered product induced by \( \tau \) and \( \tau \). Therefore, to define the composition map \( \tau \tau : (G \sqcup_B G')^{(2)} \to G \sqcup_B G' \) we have to consider the following four cases. Recall that \((1, x') \sim (b(x'), 2)\) are the only kinds of elements related in \( M \sqcup M' \) so that we set

\[
\tau \tau([(g, 2)]_B, [(h, 2)]_B) := [(gh, 2)]_B, \quad \tau \tau([(1, g')]_B, [(1, h')]_B) := [(1, g'h')]_B,
\]

\[
\tau \tau([(1, g')]_B, [(g, 2)]_B) := [(B(g')g, 2)]_B, \quad \tau \tau([(g, 2)]_B, [(1, g')]_B) := [(gB(g'), 2)]_B.
\]

It is simple to check that the well definition of this composition follows from the fact that \( \tau \) and \( \tau \) are also well defined and \((B, b)\) is a Lie groupoid morphism. The associative property of \( \tau \tau \) is satisfied because we have that \( m \) and \( m' \) are associative in \( G \) and \( G' \), respectively and \( B : \partial G' \to G \) satisfies \( B \circ m' = m \circ (B \times B) \). Indeed, we only have to be careful when verifying the following case:

\[
[(1, g')]_B \cdot [(1, h')]_B \cdot [(g, 2)]_B = [(1, g')]_B \cdot [(B(h')g, 2)]_B = [(B(g')(B(h')g), 2)]_B = \tau \tau([(g, 2)]_B, [(1, g')]_B) \cdot \tau \tau([(1, h')]_B, [(1, h')]_B) \cdot [(g, 2)]_B.
\]

The case \([(g, 2)]_B \cdot [(1, g')]_B \cdot [(1, h')]_B\) may be verified in a similar fashion and the other ones follow more directly.

As expected, the unit map \( \tau \tau : M \sqcup_B M' \to G \sqcup_B G' \) and the inverse \( \iota : G \sqcup_B G' \to G \sqcup_B G' \) are defined by passing to the quotient \( \tau \tau \) and \( \iota \) respectively. It follows easily that both \( \tau \tau \) and \( \iota \) satisfy the required conditions of the groupoid axioms.

Summing up, we have obtained:
Proposition 6.2. There exists a natural topological groupoid $G \sqcup_B G' \rightrightarrows M \sqcup_B M'$ whose structural maps are given by the ones $(\pi, T, \overline{\pi}, \overline{T})$ defined as above.

This groupoid will be called the **attaching groupoid** of $G \rightrightarrows M$ with respect to the Lie groupoid morphism $(B, b)$. A very special case is obtained when taking both $G' = M' = D\lambda$ a closed $\lambda$-disk and $\partial G' = \partial M' = \partial D\lambda$ its corresponding $(\lambda - 1)$-sphere with their underlying structure of unit groupoids. In this case we attach cells of the same dimension at both arrows and objects extending the groupoid structure. More important to us, it follows that as consequence of Proposition 5.11 we get that the Lie groupoid morphism $(B, b) : (\partial D_-(G_{\mathcal{O}_s}) \rightrightarrows \partial D_-(\mathcal{O}_s)) \to (G \rightrightarrows M)$, which is defined through the natural projections, gives rise to the attaching groupoid $G \sqcup_B D_-(G_{\mathcal{O}_s}) \rightrightarrows M \sqcup_B D_-(\mathcal{O}_s)$.

### 6.2. Lie subgroupoid levels

In this subsection we start by defining the gradient vector field of a Lie groupoid morphism $F : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ with respect to a 2-metric. Let $G \rightrightarrows M$ be a Lie groupoid equipped with a 2-metric $\eta^{(2)}$ with induced 1-metric $\eta^{(1)}$ on $G$ and 0-metric $\eta^{(0)}$ on $M$. Consider a Lie groupoid morphism $F : G \to \mathbb{R}$ induced by a smooth basic function $f : M \to \mathbb{R}$, i.e. $F = s^*f = t^*f$. The **gradient** of $F$ is defined as the pair $\nabla F = (\nabla(s^*f), \nabla f)$, where $\nabla(s^*f)$ and $\nabla f$ are the gradient vector fields of $s^*f$ and $f$ with respect to the 1-metric $\eta^{(1)}$ on $G$ and the 0-metric $\eta^{(0)}$ on $M$, respectively. We have the following result.

**Proposition 6.3.** The gradient $\nabla F$ is a multiplicative vector field on $G$.

**Proof.** Recall that $m$, $s$ and $t$ are Riemannian submersions. Thus, the vector fields $\nabla((f \circ s)$ and $\nabla f$ are $s$-related, $\nabla(f \circ t) = \nabla(f \circ s)$ and $\nabla f$ are $t$-related, and $\nabla((f \circ s) \circ m)$ and $\nabla(f \circ s)$ are $m$-related. Here $\nabla((f \circ s) \circ m)$ denotes the gradient vector field of $(f \circ s) \circ m$ with respect to $\eta^{(2)}$ on $G^{(2)}$ and we have that

$$dm \circ \nabla((f \circ s) \circ m) = \nabla(f \circ s) \circ m.$$

The crucial point is to prove that for $(g, h) \in G^{(2)}$ the identity

$$\nabla((f \circ s) \circ m)(g, h) = (\nabla(f \circ s)(g), \nabla(f \circ t)(h)),$$

holds true where $ds(g)(\nabla(f \circ s)(g)) = \nabla f(s(g)) = \nabla f(t(h)) = dt(h)(\nabla(f \circ t)(h))$. Take $(v, w) \in TG^{(2)}$ so that $ds(g)(v) = dt(h)(w)$. Thus, by using the identity mentioned in Remark 2.5 from [16] that relates $\eta^{(2)}$, $\eta^{(1)}$ and $\eta^{(0)}$ we have that

$$\eta^{(2)}_{(g, h)}(\nabla((f \circ s) \circ m)(g, h) - (\nabla(f \circ s)(g), \nabla(f \circ t)(h)), (v, w))$$

$$= d((f \circ s) \circ m)(g, h, v, w) - \eta^{(1)}_{g}(\nabla(f \circ s)(g), v) - \eta^{(1)}_{h}(\nabla(f \circ t)(h), w)$$

$$+ \eta^{(0)}_{s(g)}(ds(g)(\nabla(f \circ s)(g)), ds(g)(v))$$

$$= df(s(g))(ds(m)(g, h, v, w)) - df(s(g))(g)(v) - df(t(h))(w) + df(s(g))(ds(g)(v))$$

$$= df(s(h))(ds(h)(w)) - df(t(h))(w)$$

$$= df(s(h))(w) - df(t(h))(w) = 0,$$

since $s \circ m = s \circ \pi_2$ and $s^*f = t^*f$. Therefore, as $\eta^{(2)}$ is non-degenerate we get what we required and because of this $dm \circ (\nabla(f \circ s) \times \nabla(f \circ s)) = \nabla(f \circ s) \circ m$. \qed

It is clear that we also may define the gradient vector field as the pair $\nabla F = (\nabla(t^*f), \nabla f)$. 

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Remark 6.4. To have that $\nabla F$ is a multiplicative vector field it is very important to us since this is equivalent to require that the pair of flows $(\Phi_x, \Phi_y)$ induces (local) automorphisms on the Lie groupoid; see [32]. Namely, the following identities hold

$$s \circ \Phi_{x}(s') = \Phi_{x} \circ s, \quad t \circ \Phi_{y}(t') = \Phi_{y} \circ t, \quad \Phi_{x}(f s) \circ m = m \circ (\Phi_{x}(f s) \times \Phi_{x}(f s)).$$

Let us now define the Lie subgroupoid levels of the kind of Lie groupoid morphisms we are working with. For simplicity we assume that $G \rightrightarrows M$ is a proper Lie groupoid, that is, the anchor map $\rho : M \rightarrow G \times G$, which is defined by $\rho(g) = (s(g), t(g))$, is proper. As consequence we get that the isotropy groups $G_x$ are compact Lie groups and its orbits $O_x$ are closed embedded submanifolds. Let $F : (G \rightrightarrows M) \rightarrow (\mathbb{R} \rightrightarrows \mathbb{R})$ be a Lie groupoid morphism induced by a smooth basic function $f : M \rightarrow \mathbb{R}$. Take $a \in \mathbb{R}$ and consider the level submanifold $M^a = \{ x \in M : f(x) \leq a \}$. It is simple to check that since $s^*f = t^* f$ we have that $s^{-1}(M^a) = t^{-1}(M^a)$ which is equivalent to saying that $M^a$ is a saturated submanifold. This implies that $G^a = G_{M^a}$, so that we get a well defined Lie subgroupoid $G^a \rightrightarrows M^a$ of $G \rightrightarrows M$ that we call the **Lie subgroupoid level** of $F$ below $a$. It is important to notice that $G^a \rightrightarrows M^a$ is actually Lie groupoid with boundary in the sense that if $\partial M^a = \{ x \in M : f(x) = a \}$ then $\partial G^a = s^{-1}(\partial M^a) = t^{-1}(\partial M^a)$, thus obtaining that $\partial M^a \rightrightarrows \partial G^a$ is again a Lie subgroupoid of $G \rightrightarrows M$.

It is well known that the gradient vector field $X = \nabla f$ of a smooth function $f$ on a Riemannian manifold $(M, \eta)$ is actually a gradient-like vector field. That is, it satisfies both $\text{Zeros}(X) = \text{Crit}(f)$ and $df(X) > 0$ on $M \setminus \text{Crit}(f)$.

Thus, as consequence of Proposition 6.3 we obtain:

**Proposition 6.5.** Suppose that $[a, b]$ is a closed interval which does not contain critical values of $f$ and $f^{-1}[a, b]$ is compact. Then $G^a \rightrightarrows M^a$ and $G^b \rightrightarrows M^b$ are isomorphic Lie groupoids. Furthermore, $G^a \rightrightarrows M^a$ is a deformation retraction of $G^b \rightrightarrows M^b$.

**Proof.** Let us consider the smooth map $f \times f : M \times M \rightarrow \mathbb{R}^2$ defined as $(f \times f)(x, y) = (f(x), f(y))$. As $f^{-1}[a, b]$ is compact we get that $(f \times f)^{-1}[a, b] \times [a, b] = f^{-1}[a, b] \times f^{-1}[a, b]$ is compact in $M \times M$ and because of this $\rho^{-1}(f^{-1}[a, b] \times f^{-1}[a, b])$ is compact in $G$ since $\rho$ is a proper map. Another straightforward computation allows us to conclude that $\rho^{-1}(f^{-1}[a, b] \times f^{-1}[a, b]) = (s^*f)^{-1}[a, b]$ since $s^*f = t^* f$ and therefore $(s^*f)^{-1}[a, b]$ is compact in $G$.

Let us now follow some ideas stated in [36, 40] about the proof of this result in the classical case. It is clear that if $f^{-1}[a, b] \cap \text{Crit}(f) = \emptyset$, then $(s^*f)^{-1}[a, b] \cap \text{Crit}(F) = \emptyset$. We know that $\nabla X = \nabla f \circ s$ and $X = \nabla f$ are gradient-like vector fields on $G$ and $M$, respectively. Thus, consider the smooth function $\mu : M \rightarrow [0, \infty)$ which is defined by $\|Xf\|^{-1}$ in $f^{-1}[a, b]$ and that vanishes outside a compact neighborhood of this set. We can similarly construct a smooth function $\tilde{\mu} : G \rightarrow [0, \infty)$ by using instead $\nabla X$ and $(s^*f)^{-1}[a, b]$. On the one hand, observe that since $\nabla F = (\nabla X, X)$ is a multiplicative vector field we get that

$$\mu(s(g)) = \|(Xf)(s(g))\|^{-1} = \|df(s(g))(X(s(g)))\|^{-1} = \|df(s(g))(ds(g)(\nabla X)(g))\|^{-1} = \|d(f \circ s)(g)(\nabla X)(g))\|^{-1} = \|d(\tilde{\mu} \circ s)(g)(\nabla X)(g))\|^{-1} = \tilde{\mu}(g),$$

what means that we have $\tilde{\mu} = \mu \circ s$. On the other hand, let us consider the vectors fields $-\tilde{\mu} \nabla X$ and $-\mu X$. From [36, Lemma 2.4] we obtain that they are complete vector fields. Furthermore,
the pair \((\tilde{\mu}X, -\mu X)\) defines again a multiplicative vector field. Indeed, as consequence of Proposition 6.3 we get
\[
(ds \circ \tilde{\mu}X)(g) = \tilde{\mu}(g)ds(g)(\tilde{X}(g)) = \tilde{\mu}(g)X(s(g)) = \mu(s(g))X(s(g)) = (\mu X \circ s)(g).
\]
Given that \(\nabla(f \circ t) = \nabla(f \circ o)\) we actually have that \(\tilde{\mu} = \mu \circ s = \mu \circ t\) and thus we can analogously obtain that \(dt \circ \tilde{\mu}X = \mu X \circ t\). To prove the identity regarding the composition map observe that the formulas \(s \circ m = s \circ \pi_2\) and \(t \circ m = t \circ \pi_1\) imply that \(\tilde{\mu}(m(g, h)) = \tilde{\mu}(g) = \mu(h)\) for all \((g, h) \in G^{(2)}\). Therefore,
\[
(dm \circ (\tilde{\mu}X \times \tilde{\mu}X))(g, h) = dm(\tilde{\mu}(g)X, \tilde{\mu}(h)X) = \mu(m(g, h))dm(\tilde{\mu}(g)X, \tilde{\mu}(h)X) = \tilde{\mu}(m(g, h))X(m(g, h)) = (\tilde{\mu}X \circ m)(g, h).
\]
For all \(\tau \in \mathbb{R}\), let \(\tilde{\Phi}_\tau : G \to G\) and \(\Phi_\tau : M \to M\) denote the respective flows generated by the pair \((\tilde{\mu}X, -\mu X)\). From [32] we know that \((\tilde{\Phi}_\tau, \Phi_\tau)\) is a Lie groupoid isomorphism for all \(\tau \in \mathbb{R}\). As in the classical case (see [36, 40]) we get diffeomorphisms
\[
\Phi_{b-a}(M^b) = M^a, \quad \Phi_{a-b}(M^a) = M^b \quad \text{and} \quad \tilde{\Phi}_{b-a}(G^b) = G^a, \quad \tilde{\Phi}_{a-b}(G^a) = G^b
\]
so that \((\tilde{\Phi}_{b-a}, \Phi_{a-b}) : (G^b \rightrightarrows M^b) \to (G^a \rightrightarrows M^a)\) defines a Lie groupoid isomorphism with obvious inverse \((\Phi_{a-b}, \tilde{\Phi}_{b-a})\). Finally, we can define two deformation retractions \(H : [0,1] \times M^b \to M^b\) and \(\tilde{H} : [0,1] \times G^b \to G^b\) respectively as
\[
H(\tau, x) = \Phi_{\tau \cdot (f(x) \cdot a)}(x) \quad \text{and} \quad \tilde{H}(\tau, g) = \tilde{\Phi}_{\tau \cdot ((s \cdot f)(g) \cdot a)}(g).
\]
Here \(r^+ := \max\{r, 0\}\) for every real number \(r\). Hence, we obtain again a Lie groupoid morphism which allows us to conclude that \(G^a \rightrightarrows M^a\) is a deformation retraction of \(G^b \rightrightarrows M^b\).

\begin{remark}
Observe that to proof the previous result we may relax the properness assumption of our Lie groupoid by requiring only s-properness and the existence of a 2-metric. We are requiring the Lie groupoid to be proper because this property is a Morita invariant while being s-proper is not.
\end{remark}

Let us now analyze the case in which \(f^{-1}[a, b] \cap \text{Crit}(f) \neq \emptyset\). The main reference for the basic ideas we will be following in this case may be for instance [21, App. B]. Let \(y \nless x\) be a non-degenerate critical arrow. We will denote by \(\xi_0 : \nu(O_x) \to \nu_-(O_x)\) and \(\eta_0 : \nu(O_x) \to \nu_+(O_x)\) the two mutually complementary projections. They induce bundle morphisms between the bundle projections \(\pi : \nu(O_x) \to O_x\) and \(\pi_\pm : \nu_\pm(O_x) \to O_x\), respectively. Moreover, for every \(v \in \nu(O_x)\) we have that \(v = \xi_0(v) + \eta_0(v)\) and the expression
\[
\|v\|^2 = -Q_f(\xi_0(v)) + Q_f(\eta_0(v))
\]
defines a positive definite quadratic form (i.e. a norm) on \(\nu(O_x)\). As expected, we can define respective mutually complementary projections \(\xi_1 : \nu(G_{O_x}) \to \nu_-(G_{O_x})\) and \(\eta_1 : \nu(G_{O_x}) \to \nu_+(G_{O_x})\) which enjoy of similar properties as above. Namely, every \(\tilde{v} \in \nu(G_{O_x})\) can be rewritten as \(\tilde{v} = \xi_1(\tilde{v}) + \eta_1(\tilde{v})\) and we have the norm \(\|\tilde{v}\|_1^2 = -Q_F(\xi_1(\tilde{v})) + Q_F(\eta_1(\tilde{v}))\) on \(\nu(G_{O_x})\). Furthermore:
Lemma 6.7. Both \((\xi_1, \xi_0) : (\nu(G_{O_x}) \ni \nu(O_x)) \rightarrow (\nu_-(G_{O_x}) \ni \nu_-(O_x))\) and \((\eta_1, \eta_0) : (\nu(G_{O_x}) \ni \nu(O_x)) \rightarrow (\nu_+(G_{O_x}) \ni \nu_+(O_x))\) are Lie groupoid morphisms.

Proof. If \(\tilde{v} \in \nu(G_{O_x})\) then we get
\[
(\xi_0 \circ d\xi)(\tilde{v}) = (\xi_0 \circ d\xi)(\xi_1(\tilde{v}) + \xi_1(\tilde{v})) = \xi_0(d\xi(\xi_1(\tilde{v})) + d\xi(\eta_1(\tilde{v}))) = d\xi(\xi_1(\tilde{v})).
\]

We can analogously prove that \(\xi_0 \circ d\xi = d\xi \circ \xi_1\). Now, if \(\tilde{v}, \tilde{u} \in \nu(G_{O_x})\) are such that \(d\xi(\tilde{v}, \tilde{u})\) is defined then
\[
(\xi_1 \circ d\xi)(\tilde{v}, \tilde{u}) = (\xi_1 \circ d\xi)(\xi_1(\tilde{v}) + \eta_1(\tilde{v}), \xi_1(\tilde{u}) + \eta_1(\tilde{u})) = \xi_1(\xi_1(\xi_1(\tilde{v}), \eta_1(\tilde{u})) + d\xi(\eta_1(\tilde{v}), \eta_1(\tilde{u}))) = d\xi(\xi_1(\tilde{v}), \xi_1(\tilde{u})),
\]
so that \(\xi_1 \circ d\xi = d\xi \circ \xi_1\).

The fact that \((\eta_1, \eta_0)\) is a Lie groupoid morphism may be showed in a similar way. \(\square\)

With this in mind we have:

Theorem 6.8. Suppose that \([a, b]\) is a closed interval such that \(f^{-1}[a, b]\) is compact and the only non-degenerate critical orbit inside \(f^{-1}(a, b)\) is \(O_x\). Then \(G^n \cup_B D_-(G_{O_x}) \ni M^n \cup_B D_-(O_x)\) is a deformation retraction of \(G^n \ni M^n\).

Proof. Given that \(O_x \subset f^{-1}(a, b) \subset M\) is closed and embedded we get that \(O_x\) is also compact. Furthermore, as we previously saw we have that \((s^*f)^{-1}[a, b]\) is also compact and therefore \(G_{O_x}\) is the only compact non-degenerate critical submanifold inside \((s^*f)^{-1}(a, b)\).

Consider the mutually complementary projections \(\xi_1 : \nu(G_{O_x}) \rightarrow \nu_-(G_{O_x}), \eta_1 : \nu(G_{O_x}) \rightarrow \nu_+(G_{O_x})\) and \(\xi_0 : \nu(O_x) \rightarrow \nu_-(O_x), \eta_0 : \nu(O_x) \rightarrow \nu_+(O_x)\) as well as the norms
\[
\|\tilde{v}\|_1^n = -Q_F(\xi_1(\tilde{v})) + Q_F(\eta_1(\tilde{v})) \quad \text{and} \quad \|v\|_0^n = -Q_f(\eta_0(v)) + Q_f(\eta_0(v)),
\]
for all \(\tilde{v} \in \nu(G_{O_x})\) and \(v \in \nu(O_x)\). Because of Theorem 5.4 there is a Lie groupoid tubular neighborhood \(\phi : (\tilde{V} \ni V) \xrightarrow{\approx} (\tilde{U} \ni U)\) in the sense of weakly linearization such that
\[
(\phi_1^*F, \phi_1^*f) = (c + Q_F, c + Q_f),
\]
where \(c = f(O_x) = F(G_{O_x})\) is the common value of \(f\) and \(F\) on \(G_{O_x}\) and \(O_x\), respectively. By identifying \(\tilde{V} \cong \tilde{U}\) and \(V \cong U\) we may think of \(F\) on \(\tilde{U}\) and \(f\) on \(U\) as respectively given by
\[
F(\tilde{v}) = c + Q_F(\tilde{v}) = c - \|\xi_1(\tilde{v})\|_1^2 + \|\eta_1(\tilde{v})\|_1^2, \quad f(v) = c + Q_f(v) = c - \|\eta_0(v)\|_0^2 + \|\eta_0(v)\|_0^2.
\]
Let us now follow some ideas stated in \([21, 36, 40]\) about the proof of this result in the classical case. We choose \(\epsilon > 0\) small enough so that the interval \((c - \epsilon, c + \epsilon)\) is contained in \([a, b]\) and all points \(\tilde{v} \in \nu(G_{O_x})\) with \(\|\tilde{v}\|_1^2 \leq 2\epsilon\) belong to the neighborhood \(\tilde{U}\) and \(v \in \nu(O_x)\) with \(\|v\|_0^2 \leq 2\epsilon\) belong to the neighborhood \(U\). Similarly to how it was done in \([36, p. 16-19]\) we construct two smooth functions which are a modification of the pair \((F, f)\). Namely, let \(\mu : \mathbb{R} \rightarrow \mathbb{R}\) be the smooth function verifying the conditions
\[
\mu(0) > 0, \\
\mu(r) = 0 \quad \text{for all} \quad r \geq 2\epsilon, \\
-1 < \mu'(r) \leq 0 \quad \text{for all} \quad r.
\]
Consider the functions \(F_1 : G \rightarrow \mathbb{R}\) and \(f_1 : M \rightarrow \mathbb{R}\) which respectively coincide with \(F\) and \(f\) outside of \(\tilde{U}\) and \(U\) but within those neighborhoods they are given as
\[
F_1(\tilde{v}) = F(\tilde{v}) - \mu(\|\xi_1(\tilde{v})\|_1^2 + 2\|\eta_1(\tilde{v})\|_1^2) = c - \|\xi_1(\tilde{v})\|_1^2 + \|\eta_1(\tilde{v})\|_1^2 - \mu(\|\xi_1(\tilde{v})\|_1^2 + 2\|\eta_1(\tilde{v})\|_1^2),
\]
and \( f_1(v) = f(v) - \mu(\|\xi_0(v)\|^2_0 + 2\|\eta_0(v)\|^2_0) = c - \|\xi_0(v)\|^2_0 + \|\eta_0(v)\|^2_0 - \mu(\|\xi_0(v)\|^2_0 + 2\|\eta_0(v)\|^2_0) \).

These are well defined smooth functions. Let us prove that \( (F_1, f_1) : (G \rightleftharpoons M) \rightarrow (\mathbb{R} \rightleftharpoons \mathbb{R}) \) is also a Lie groupoid morphism. Since \( \phi \) is a Lie groupoid isomorphism and \( (F_1, f_1) \) agrees with \((F, f)\) outside \((\tilde{U}, U)\) we only have to see what happens inside \((\tilde{U}, U)\). As consequence of the classical Morse–Bott lemma we may identify the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_0 \) with those norms defined by the restrictions of \( \eta^{(1)} \) and \( \eta^{(0)} \) on \( \nu(G_{\mathcal{O}_x}) \) and \( \nu(\mathcal{O}_x) \), respectively. As \( s, t : G \rightarrow M \) are Riemannian submersions we obtain that \( \|\tilde{v}\|_1 = \|ds(\tilde{v})\|_0 \) and \( \|\tilde{v}\|_2 = \|dt(\tilde{v})\|_0 \). It is worth noticing that if we do not identify the norms as we did above we can also get the previous statement by using the fact that \((Q_F, Q_I), (\xi_1, \xi_0), \) and \((\eta_1, \eta_0)\) are Lie groupoid morphisms.

Therefore, by using again the latter fact we obtain that
\[
(ds f_1)(\tilde{v}) = f_1(ds(\tilde{v})) = c + Q_I(ds(\tilde{v})) - \mu(\|\xi_0(ds(\tilde{v}))\|^2_0 + 2\|\eta_0(ds(\tilde{v}))\|^2_0)
= c + Q_I(ds(\tilde{v})) - \mu(\|\xi_1(\tilde{v})\|^2_0 + 2\|\eta_1(\tilde{v})\|^2_0).
\]

We can analogously prove that \( df_1 = F_1 \) which implies what we desired.

Just as it was proved in [36, p. 16-19] we have the following assertions.

- The regions \( F_{-1}^{-1}(\infty, c+\epsilon) \) and \( f_{-1}^{-1}(\infty, c+\epsilon) \) coincide with the regions \( G_{c+\epsilon} \) and \( M_{c+\epsilon} \), respectively.
- The functions \( F \) and \( F_1 \) have the same critical arrows. Similarly, the functions \( f \) and \( f_1 \) have the same critical points.
- The sets \( F_{-1}^{-1}(c-\epsilon, c+\epsilon) \) and \( f_{-1}^{-1}(c-\epsilon, c+\epsilon) \) contain no critical points of \( F_1 \) and \( f_1 \), respectively.

Thus, from Proposition 6.5 we conclude that the Lie subgroupoid level \( F_{-1}^{-1}(\infty, c-\epsilon) \Rightarrow f_{-1}^{-1}(\infty, c-\epsilon) \) is a deformation retraction of \( G_{c+\epsilon} \Rightarrow M_{c+\epsilon} \).

Let us now consider the negative disk bundle \( D_-(\mathcal{O}_x) = \{v \in U : \|\xi_0(v)\|^2_0 \leq \epsilon, \eta_0(v) = 0\} \). As the value of \( f_1 \) on any point of \( D_-(\mathcal{O}_x) \) is less than \( c-\epsilon \) we have that \( M_{c-\epsilon} \cup D_-(\mathcal{O}_x) \subset f_{-1}^{-1}(\infty, c-\epsilon) \). Moreover, the intersection \( D_-(\mathcal{O}_x) \cap M_{c-\epsilon} \) coincides with the negative sphere bundle \( \partial D_-(\mathcal{O}_x) = \{v \in U : \|\xi_0(v)\|^2_0 = \epsilon, \eta_0(v) = 0\} \). Therefore, we can consider the attaching space \( M_{c-\epsilon} \cup bD_-(\mathcal{O}_x) \) with respect to the bundle projection \( b : \partial D_-(\mathcal{O}_x) \rightarrow M_{c-\epsilon} \) which actually we may be seen as the union \( M_{c-\epsilon} \cup D_-(\mathcal{O}_x) \) which is a deformation retract of \( f_{-1}^{-1}(\infty, c-\epsilon) \). The deformation retraction \( r_t : f_{-1}^{-1}(\infty, c-\epsilon) \rightarrow f_{-1}^{-1}(\infty, c-\epsilon) \) is identical outside \( U \) but inside \( U \) it acts as follows.

- In the domain \( \|\xi_0(v)\|^2_0 \leq \epsilon \) (i.e. in \( D_-(\mathcal{O}_x) \)) the deformation \( r_t \) is given by the formula
  \[
  r_t(v) = \xi_0(v) + t\eta_0(v).
  \]

- In the domain \( \epsilon \leq \|\xi_0(v)\|^2_0 \leq \|\eta_0(v)\|^2_0 + \epsilon \) we define \( r_t \) by
  \[
  r_t(v) = \xi_0(v) + s_t(v)\eta_0(v),
  \]
  where the number \( s_t(v) \in [0,1] \) is defined by \( s_t(v) = t + (1-t)\sqrt{\|\xi_0(v)\|^2_0 - \epsilon \|\eta_0(v)\|^2_0} \). The map \( r_0 \) takes values in \( f_{-1}^{-1}(c-\epsilon) \).
Within the domain \( \|\eta_0(v)\|_0^2 + \epsilon \leq \|\xi_0(v)\|_0^2 \) (i.e. in \( M^{c-\epsilon} \)) we set \( r_t \) to be the identity map, \( t \in [0, 1] \).

The expressions above agree on the intersection of the three domains and thus they define a continuous map \( r_t \) so that \( r_t \) is the identity map and \( r_0 \) is a retraction of \( f_1^{-1}(-\infty, c - \epsilon] \) onto \( M^{c-\epsilon} \cap D_-(O_x) \). As expected, we can similarly define the attaching space \( G^{c-\epsilon} \cup B D_-(G_{O_x}) \) by using the bundle projection \( B : \partial D_-(G_{O_x}) \to G^{c-\epsilon} \) as well as construct a deformation retraction \( \tilde{r}_t : F_1^{-1}(-\infty, c - \epsilon] \to F_1^{-1}(-\infty, c - \epsilon] \) by using the mutually complementary projections \( \xi_1, \eta_1 \) and the norm \( \| \cdot \|_1 \) to produce same formulas as we did above.

The key point now is to prove that \((\tilde{r}_t, r_t)\) defines a Lie groupoid morphism. As \( \tilde{r}_t \) and \( r_t \) are the identity outside \( \tilde{U} \) and \( U \), respectively, we only have to check this on the three special cases mentioned above. We will use again the fact that \((\xi_1, \xi_0)\) and \((\eta_1, \eta_0)\) are Lie groupoid morphisms. In the first domain we have that

\[
(r_t \circ \overline{ds})(\tilde{v}) = \xi_0(\overline{ds}(\tilde{v})) + t\eta_0(\overline{ds}(\tilde{v})) = \overline{ds}(\xi_1(\tilde{v})) + t\overline{ds}(\eta_1(\tilde{v})) = (\overline{ds} \circ \tilde{r}_t)(\tilde{v}).
\]

We can analogously prove that \( r_t \circ \overline{dt} = \overline{dt} \circ \tilde{r}_t \). Now, if \( \tilde{v} \) and \( \tilde{u} \) are such that \( dm(\tilde{v}, \tilde{u}) \) is defined then

\[
\tilde{r}_t(dm(\tilde{v}, \tilde{u})) = \xi_1(dm(\tilde{v}, \tilde{u})) + t\eta_1(dm(\tilde{v}, \tilde{u})) = \overline{dm}(\xi_1(\tilde{v}), \xi_1(\tilde{u})) + t\overline{dm}(\eta_1(\tilde{v}), \eta_1(\tilde{u}))
\]

what means that \( \tilde{r}_t \circ \overline{dm} = \overline{dm} \circ (\tilde{r}_t \times \tilde{r}_t) \). To verify the assertion in the second domain we only have to check that \( s_1(\overline{ds}(\tilde{v})) = \tilde{s}_1(\tilde{v}), s_1(\overline{dt}(\tilde{v})) = \tilde{s}_1(\tilde{v}) \) and \( \tilde{s}_1(\tilde{v}) = \tilde{s}_1(dm(\tilde{v}, \tilde{u})) = \tilde{s}_1(\tilde{u}) \). These formulas follows from the fact that \((\xi_1, \xi_0)\) and \((\eta_1, \eta_0)\) are Lie groupoid morphisms plus the identities

\[
\| \overline{dm}(\tilde{v}, \tilde{u}) \|_1 = \| \overline{ds}(\tilde{v}) \|_0 = \| \tilde{u} \|_1 \quad \text{and} \quad \| \overline{dm}(\tilde{v}, \tilde{u}) \|_1 = \| \overline{dt}(\tilde{v}) \|_0 = \| \tilde{v} \|_1
\]

which can be obtained by either the fact that \( s, t, m, \pi_1 \) and \( \pi_2 \) are Riemannian submersions or \((Q_F, Q_f), (\xi_1, \xi_0), (\eta_1, \eta_0)\) are Lie groupoid morphisms. The remaining computations are similar to those done when looking at the first domain. Finally, in the third domain the assertion directly follows since \( \tilde{r}_t \) and \( r_t \) are the identity maps over there.

In conclusion, the Lie groupoid \( G^{c-\epsilon} \cup B D_-(G_{O_x}) \to M^{c-\epsilon} \cupb D_-(O_x) \) is a deformation retraction of the Lie groupoid level \( F_1^{-1}(-\infty, c - \epsilon] \) and \( \eta_1 \) in turn, as consequence of Proposition 6.5, is a deformation retraction of the Lie groupoid level \( G^{c+\epsilon} \to M^{c+\epsilon} \). Hence, \( G^{c-\epsilon} \cup B D_-(G_{O_x}) \to M^{c-\epsilon} \cupb D_-(O_x) \) is a deformation retraction of \( G^{c+\epsilon} \to M^{c+\epsilon} \).

\section{Morse–Smale dynamics}

Let us now start by adapting some notions of the Morse–Smale dynamics to the Lie groupoid setting. Our goal here is to define the stable and unstable Lie groupoids of a Morse Lie groupoid morphism as well as to study some of their elementary properties. For more details see [1, 3, 40] and references therein. Let \( F : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R}) \) be a Morse Lie groupoid morphism induced by a smooth basic function \( f : M \to \mathbb{R} \). Throughout this section, apart from assuming that \( G \rightrightarrows M \) is proper, we will also assume that one of either \( G \) or \( M \) is compact. This automatically implies that the other one is compact as well. Under these assumptions it is clear that the multiplicative gradient vector field \( \nabla F \) is given by a pair of complete vector fields. Let \( \Phi_r : G \to G \) and \( \Phi_r : M \to M \) denote the flow of the vector fields \(-\nabla(s^*f)\) and
The stable and unstable submanifolds of a critical arrow $g \in \text{Crit}(F)$ are similarly defined by using the descending flow $\Phi_g$. Let $S_\lambda \subseteq \text{Crit}(f)$ denote the set formed by the orbits in $M$ with same index $\lambda$. We may assume that $S_\lambda$ consists of orbits with the same dimension, otherwise we split $S_\lambda$ into components consisting of orbits with the same dimension. Hence, we may assume that $S_\lambda$ is a manifold which, being saturated, yields a well-defined Lie groupoid $G_{S_\lambda} \cong S_\lambda$ defined by the restriction of $G \cong M$ to $S_\lambda$. It is important to observe that $S_\lambda$ is a non-degenerate critical submanifold for $f$ of index $\lambda$ and as consequence of what we did before we may conclude that $G_{S_\lambda}$ is also a non-degenerate critical submanifold for $F$ with same index $\lambda$. Thus, as we have that every point in $S_\lambda$ is a critical point for $f$ we define the stable and unstable submanifolds of $S_\lambda$ as the respective disjoint unions

$$W^s(S_\lambda) = \bigcup_{x \in S_\lambda} W^s(x) \quad \text{and} \quad W^u(S_\lambda) = \bigcup_{x \in S_\lambda} W^u(x).$$

The stable and unstable submanifolds $W^s(G_{S_\lambda})$ are $W^u(G_{S_\lambda})$ are defined in the same way since every arrow in $G_{S_\lambda}$ is a critical arrow for $F$. As a consequence of the multiplicativity of $-\nabla F = (-\nabla(s^*f), -\nabla f)$ we obtain the following expected result.

**Lemma 7.1.** The Lie groupoid structure of $G \cong M$ can be naturally restricted to define two Lie groupoids $W^s(G_{S_\lambda}) \cong W^s(S_\lambda)$ and $W^u(G_{S_\lambda}) \cong W^u(S_\lambda)$.

**Proof.** We will only show why the Lie groupoid structure may be well restricted for defining $W^s(G_{S_\lambda}) \cong W^s(S_\lambda)$ since the other case follows analogously. This is carried out in the following steps. First, if $h \in W^s(G_{S_\lambda})$, then $h \in W^s(g)$ for some critical arrow $g \in G_{S_\lambda}$. Thus, the identity $s \circ \Phi_g = \Phi_g \circ s$ implies that

$$\lim_{t \to -\infty} \Phi_g(s(h)) = \lim_{t \to -\infty} s(\Phi_g(h)) = s \left( \lim_{t \to -\infty} \Phi_g(h) \right) = s(g).$$

This means that $s(h) \in W^s(s(g)) \subseteq W^s(S_\lambda)$ and therefore $s : W^s(G_{S_\lambda}) \to W^s(S_\lambda)$ is well restricted. As for each critical arrow $g \in G_{S_\lambda}$ we have that $t(g) \in S_\lambda$ is a critical point, then by arguing in the exactly same way with the identity $t \circ \Phi_g = \Phi_g \circ t$ we get that $t : W^s(G_{S_\lambda}) \to W^s(S_\lambda)$ is well restricted. Now, let us consider the fibered product space $(W^s(G_{S_\lambda}))^{(2)}$ defined through $s$ and $t$. If $(h_1, h_2) \in (W^s(G_{S_\lambda}))^{(2)}$ then there exists a pair of critical arrows $(g_1, g_2) \in G_{S_\lambda} \times G_{S_\lambda}$ such that $h_1 \in W^s(g_1)$ and $h_2 \in W^s(g_2)$. On the one hand, observe that $(g_1, g_2) \in G_{S_\lambda}^{(2)}$. Indeed, since $s(h_1) = t(h_2)$ and $(\Phi_g, \Phi_f)$ is a Lie groupoid morphism we obtain

$$s(g_1) = s \left( \lim_{t \to -\infty} \Phi_f(h_1) \right) = \lim_{t \to -\infty} \Phi_f(s(h_1)) = \lim_{t \to -\infty} \Phi_f(t(h_2)) = t \left( \lim_{t \to -\infty} \Phi_f(h_2) \right) = t(g_2).$$

On the other hand, from the identity $\Phi_g \circ m = m \circ (\Phi_g \times \Phi_f)$ we get that

$$\lim_{t \to -\infty} \Phi_f(h_1 h_2) = \lim_{t \to -\infty} m(\Phi_f(h_1), \Phi_f(h_2)) = m \left( \lim_{t \to -\infty} \Phi_f(h_1), \lim_{t \to -\infty} \Phi_f(h_2) \right) = g_1 g_2.$$

Given that we know that the composition of two composable critical arrows produces again a critical arrow we have that $h_1 h_2 \in W^s(g_1 g_2) \subseteq W^s(G_{S_\lambda})$. Hence, the composition map $m : (W^s(G_{S_\lambda}))^{(2)} \to W^s(G_{S_\lambda})$ is also well restricted. Finally, let us now consider the inversion
map \( i : G \to G \) and the unit map \( u : M \to G \) of the Lie groupoid \( G \rightrightarrows M \). As \( (\Phi_r, \Phi_\tau) \) is a Lie groupoid morphism we have that \( \Phi_\tau \circ i = i \circ \Phi_\tau \) and \( \Phi_r \circ u = u \circ \Phi_r \). Recall that if \( g \) is a critical arrow, then \( g^{-1} \) is also a critical arrow and if \( x \) is a critical point, then \( 1_x \) is a critical arrow. So, by arguing as in the previous items we obtain that \( i : W^s(G_{S_\lambda}) \to W^s(G_{S_{\lambda'}}) \) and \( u : W^u(S_{\lambda}) \to W^u(G_{S_{\lambda'}}) \) are well restricted as well.

The Lie groupoids introduced in Lemma 7.1 will be respectively called **stable** and **unstable Lie groupoids** associated to the critical submanifold \( S_\lambda \). Consider now the smooth endpoint maps \( l_0 : W^s(S_{\lambda}) \to S_{\lambda} \) and \( u_0 : W^u(S_{\lambda}) \to S_{\lambda} \) which are respectively defined by

\[
l_0(x) = \lim_{\tau \to -\infty} \Phi_\tau(x) \quad \text{and} \quad u_0(x) = \lim_{\tau \to -\infty} \Phi_\tau(x).
\]

It was shown in [1, Prop. 3.2] that the endpoint maps (12) are smooth locally trivial fibrations. In order to state the groupoid analogue of this property, we need the following definition.

**Definition 7.2.** [31] A Lie groupoid fibration is a Lie groupoid morphism \( \phi^l : G \to G' \) covering a surjective submersion \( \phi^b : M \to N \) with the property that

\[
\hat{\phi} : G \to G' \times N ; \quad g \mapsto (\phi^l(g), s(g)),
\]

is a surjective submersion.

At the level of arrows, we can similarly define smooth endpoint maps \( l_1 : W^s(G_{S_{\lambda'}}) \to G_{S_{\lambda'}} \) and \( u_1 : W^u(G_{S_{\lambda'}}) \to G_{S_{\lambda'}} \) by using the descending flow \( \Phi_\tau \). Recall that we have the equality of indexes \( \lambda(G_{S_{\lambda'}}, F) = \lambda(S_{\lambda'}, f) = \lambda \). Thus, as consequence of [1, Prop. 3.2] we obtain that \( u_1 \) and \( l_1 \) are smooth fiber bundles with fibers diffeomorphic to the disks \( D_\lambda^k \) and \( D_\lambda^k \), respectively. Here \( k = \operatorname{codim}(G_{S_{\lambda'}}) = \operatorname{codim}(S_{\lambda'}) \) seen as submanifolds of \( G \) and \( M \), respectively. So, motivated by this and Lemma 7.1 one has the following result.

**Proposition 7.3.** The endpoint maps \( l = (l_1, l_0) : (W^s(G_{S_{\lambda'}}) \rightrightarrows W^u(S_{\lambda})) \to (G_{S_{\lambda'}} \rightrightarrows S_{\lambda}) \) and \( u = (u_1, u_0) : (W^u(G_{S_{\lambda'}}) \rightrightarrows W^u(S_{\lambda})) \to (G_{S_{\lambda'}} \rightrightarrows S_{\lambda}) \) are Lie groupoid fibrations.

**Proof.** On the one hand, the fact that both \( l \) and \( u \) define Lie groupoid morphisms follows by arguing exactly as we did in Lemma 7.1. Indeed, as the pair \( (\Phi_r, \Phi_\tau) \) determines a Lie groupoid morphism we have that

\[
(s \circ u_1)(h) = s \left( \lim_{\tau \to -\infty} \Phi_\tau(h) \right) = \lim_{\tau \to -\infty} s(\Phi_\tau(h)) = \lim_{\tau \to -\infty} \Phi_\tau(s(h)) = (u_0 \circ s)(h).
\]

We can analogously get that \( t \circ u_1 = u_0 \circ t \). If \( (h_1, h_2) \in (W^u(G_{S_{\lambda'}}))^{(2)} \) then

\[
(m \circ (u_1 \times u_1))(h_1, h_2) = m \left( \lim_{\tau \to -\infty} \Phi_\tau(h_1), \lim_{\tau \to -\infty} \Phi_\tau(h_2) \right) = \lim_{\tau \to -\infty} (m \circ (\Phi_\tau \times \Phi_\tau))(h_1, h_2)
\]

\[
= \lim_{\tau \to -\infty} (\Phi_\tau(m(h_1, h_2))) = (u_1 \circ m)(h_1, h_2).
\]

On the other hand, let us consider the fibred product given by the diagram below

\[
\begin{array}{ccc}
G_{S_{\lambda'}} \times_{S_{\lambda'}} W^u(S_{\lambda'}) & \xrightarrow{\pi_2} & W^u(S_{\lambda'}) \\
\pi_1 \downarrow & & \downarrow u_0 \\
G_{S_{\lambda'}} & \xrightarrow{s} & S_{\lambda}.
\end{array}
\]
Now we show that \( u \) has the fibration property (13), that is, the map \( \hat{u} : W^u(G_{S_\lambda}) \rightarrow G_{S_\lambda} \times_{S_\lambda} W^u(S_\lambda), h \mapsto (u_1(h), s(h)) \), is a surjective submersion. Indeed, since \( \pi_1 \) and \( \pi_2 \) are surjective submersions, \( \pi_1 \circ \hat{u} = u_1 \) and \( \pi_2 \circ \hat{u} = s \), then \( u \) is a surjective submersion because \( u_1 \) and \( s \) are so. Hence, as \( u_0 : W^u(S_\lambda) \rightarrow S_\lambda \) is also a surjective submersion we conclude that \( u \) is in fact a Lie groupoid fibration.

Let us now consider the notion of moduli space of gradient flow lines in the Lie groupoid setting. Some of the ideas stated below will be used to construct a double complex which in turn will allow us to recover the total cohomology of the Bott–Shulman–Stasheff double complex of the Lie groupoid we are working with. Note that there exists a natural free action of \( \mathbb{R} \) on \( W^u(S_\lambda) \) (resp. on \( W^s(S_\lambda) \)) defined by \( r \cdot y = \Phi_r(y) \) for all \( r \in \mathbb{R} \). This is well defined since if \( y \in W^u(S_\lambda) \) we have that

\[
\lim_{r \to -\infty} \Phi_r(r \cdot y) = \lim_{r \to -\infty} \Phi_r(\Phi_r(y)) = \lim_{r \to -\infty} \Phi_{r+r}(y) \in W^u(S_\lambda). \tag{14}
\]

In particular, it is simple to see that these actions induce a well defined action of \( \mathbb{R} \) on any intersection \( W^u(S_\lambda) \cap W^s(S_{\lambda_j}) \). We will refer to this action as action by flow translation. The moduli space of gradient flow lines in \( M \) associated to the non-degenerate critical saturated submanifolds \( S_\lambda \) and \( S_{\lambda_j} \) is defined as the quotient space obtained from the action by flow translation:

\[
\mathcal{M}(S_\lambda, S_{\lambda_j}) := (W^u(S_\lambda) \cap W^s(S_{\lambda_j}))/\mathbb{R}.
\]

The moduli space of gradient flow lines \( \mathcal{M}(G_{S_\lambda}, G_{S_{\lambda_j}}) \) is equally defined by using the action induced by the descending flow \( \Phi_r \). Thus, motivated by the Morse–Bott transversality condition (see [1, 3]) we set up the following definition.

**Definition 7.4.** A Morse Lie groupoid morphism \( F : (G \rightrightarrows M) \rightarrow (\mathbb{R} \rightrightarrows \mathbb{R}) \) is said to satisfy the **Morse–Smale transversality** condition with respect to a 2-metric on \( G \rightrightarrows M \) if for any two critical non-degenerate saturated submanifolds \( S_\lambda \) and \( S_{\lambda_j} \) with respect to \( f \) we have that \( W^u(y) \cap W^s(S_{\lambda_j}) \) for all \( y \in S_\lambda \).

It is important to notice that, unlike the classical Morse case, in the Morse–Bott case it is not always possible to perturb a Riemannian metric to make a given Morse-Bott function satisfy the Morse-Bott-Smale transversality condition. See [28, Rmk. 2.4] for an interesting counterexample. More importantly, such a condition is not always satisfied for \( G \)-invariant Morse functions; compare [28, s. 5]. As a consequence, not every Morse Lie groupoid morphism satisfies the Morse-Smale transversality condition in general. As it is argued in [1], when assuming the Morse–Smale transversality condition from Definition 7.4, we get that the spaces \( W^u(S_\lambda) \cap W^s(S_{\lambda_j}) \) and \( \mathcal{M}(S_\lambda, S_{\lambda_j}) \) are smooth manifolds since the action by flow translation is free and proper. The endpoint maps (12) descend to the moduli spaces yielding new endpoint maps \( (l_0)_j^j : \mathcal{M}(S_\lambda, S_{\lambda_j}) \rightarrow S_{\lambda_j} \) and \( (u_0)_j^j : \mathcal{M}(S_\lambda, S_{\lambda_j}) \rightarrow S_{\lambda_j} \) given respectively by

\[
(l_0)_j^j([x]) = l_0(x) \quad \text{and} \quad (u_0)_j^j([x]) = u_0(x). \tag{15}
\]

One can see that these maps are well defined in a similar manner as in (14). Also, the endpoint maps on the moduli spaces are locally trivial fibrations, see [1].

**Lemma 7.5.** Let \( F : G \rightrightarrows \mathbb{R} \) be a Morse Lie groupoid morphism having the Morse-Smale transversality property. Then the non-degenerate critical submanifolds \( G_{S_{\lambda_i}} \) and \( G_{S_{\lambda_j}} \) satisfy the Morse-Smale transversality condition with respect to \( F \).
Proof. Let \( y \leftarrow x \) be an arrow in \( W^u(G_{S_{\lambda_j}}) \) such that \( g \in W^u(h) \cap W^s(G_{S_{\lambda_j}}) \). Given that \( s : G \to M \) is a submersion we have that it is transverse to any submanifold in \( M \) so that, in particular, we obtain that \( s \cap W^u(s(h)) \) and \( s \cap W^s(S_{\lambda_j}) \). Since \( x \in W^u(s(h)) \cap W^s(S_{\lambda_j}) \) the Morse-Smale transversality condition with respect to \( f \) gives us

\[
T_g W^u(h) + T_g W^s(G_{S_{\lambda_j}}) = T_g(s^{-1}(W^u(s(h)))) + T_g(s^{-1}(W^s(S_{\lambda_j})))
\]

\[
= d s_g^{-1}(T_x W^u(s(h))) + d s_g^{-1}(T_x W^s(S_{\lambda_j}))
\]

\[
= d s_g^{-1}(T_x W^u(s(h))) + T_g W^s(S_{\lambda_j})
\]

\[
= d s_g^{-1}(T_x M) = T_g G.
\]

So, the result follows. \( \Box \)

As consequence of this observation we can analogously define smooth manifolds \( W^u(G_{S_{\lambda_j}}) \cap W^s(G_{S_{\lambda_j}}) \) and \( \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \) and smooth maps \( (l_i)_j : \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \to G_{S_{\lambda_j}} \) and \( (u_1)_j : \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \to G_{S_{\lambda_j}} \) so that \( (u_1)_j \) has the structure of a locally trivial bundle.

It is simple to check that by the nature of the structure of our stable and unstable Lie groupoids we may naturally define a Lie groupoid \( W^u(G_{S_{\lambda_j}}) \cap W^s(G_{S_{\lambda_j}}) \to W^u(S_{\lambda_j}) \cap W^s(S_{\lambda_j}) \). More importantly:

**Proposition 7.6** (Groupoid of gradient flow lines). There exists a unique structure of Lie groupoid \( \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \Rightarrow \mathcal{M}(S_{\lambda_j}, S_{\lambda_j}) \) on the moduli spaces of gradient lines, making the canonical projection \( W^u(G_{S_{\lambda_j}}) \cap W^s(G_{S_{\lambda_j}}) \to \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \) a Lie groupoid fibration.

**Proof.** The action of \( \mathbb{R} \) on \( W^u(G_{S_{\lambda_j}}) \cap W^s(G_{S_{\lambda_j}}) \Rightarrow W^u(S_{\lambda_j}) \cap W^s(S_{\lambda_j}) \) is given by Lie groupoid automorphisms. Since this action is free and proper, then the quotient inherits a Lie groupoid structure with the desired properties. \( \Box \)

As an application of Proposition 7.3 we easily get:

**Corollary 7.7.** The pair of endpoint maps \( u^j = ((u_1)_j, (u_0)_j) : (\mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \Rightarrow \mathcal{M}(S_{\lambda_j}, S_{\lambda_j})) \to (G_{S_{\lambda_j}} \Rightarrow S_{\lambda_j}) \) defines a Lie groupoid fibration.

Note that because of the well definition of index we have that \( W^u(S_{\lambda_j}) \cap W^s(S_{\lambda_j}) \) does not contain critical points of \( f \) since \( S_{\lambda_j} \cap S_{\lambda_j} = \emptyset \). This in particular implies that the action by flow translation on this space is free. Namely, the latter statement follows from the fact that a point is a singularity of a vector field if and only if its flow fixes such a point. In particular, the gradient vector field of \( f \) is nonzero at the regular points of \( f \) which implies that its flow not fixes such points. Same conclusion can be obtained at the arrow level. This key observation allows us to define the composition map of the Lie groupoid from Proposition 7.6 explicitly. Indeed, the source and target maps are respectively defined by setting \( \overline{\pi} : \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \to \mathcal{M}(S_{\lambda_j}, S_{\lambda_j}) \) as \( \overline{\pi}(g) = [s(g)] \) and \( \overline{T} : \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \to \mathcal{M}(S_{\lambda_j}, S_{\lambda_j}) \) as \( \overline{T}(g) = [t(g)] \), respectively. They are well-defined since our gradient vector field is multiplicative. Furthermore, if we take \([g] \) and \([h] \) in \( \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \) such that \( \overline{\pi}(g) = \overline{T}(h) \) then there exists \( r \in \mathbb{R} \) such that \( \Phi_r(s(g)) = t(h) \) which is equivalent to have \( s(\Phi_r(g)) = t(h) \) so that we can multiply \( m(\Phi_r(g), h) = \Phi_r(g)h \). Thus, we define \( \overline{m} : \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}})(2) \to \mathcal{M}(G_{S_{\lambda_j}}, G_{S_{\lambda_j}}) \) as \( \overline{m}([g], [h]) := [\Phi_r(g)h] \).

Having this in mind, an interesting consequence of Proposition 7.6 is the following.
Remark 7.8. As it can be viewed for instance in [1], the Morse–Smale transversality assumption allows to show that there exists a way for constructing a compactification of these moduli spaces of gradient flow lines. Without going into too much details, it can be done by applying a series of fibered products with respect to the endpoint maps $u$ and $l$. Therefore, as consequence of Proposition 7.6, [1, Lem. 3.3] and [15, Prop. 4.4.1] we may extend the Lie groupoid structure previously described to the respective compactifications. Here it is important to have in mind that $u$ and $l$ define Lie groupoid morphisms where $u$ is composed by locally trivial fibrations.

8. The double complex of a Morse Lie groupoid morphism

Given a Morse function $f : M \rightarrow \mathbb{R}$, the Morse-Smale-Witten complex of the pair $(M, f)$ is a complex whose homology computes de singular homology of the manifold $M$. In the setting of Morse-Bott theory, there are several versions of the Morse-Smale-Witten complex which computes either the homology or the de Rham cohomology of the manifold. We end the paper by introducing the groupoid version of the Morse complex. Given a Morse Lie groupoid morphism $F : G \rightarrow \mathbb{R}$ we construct a double cochain complex whose total cohomology recovers the total cohomology of the Bott–Shulman–Stasheff double complex associated to $G$. Our construction relies on Austin and Braam version of the Morse–Bott complex [1].

8.1. The Austin-Braam complex. Let $f : M \rightarrow \mathbb{R}$ be a Morse-Bott function and pick a Riemannian metric on $M$. Any non-degenerate critical submanifold $S \subseteq M$ determines endpoint maps $l_S : W^s(S) \rightarrow S$ and $u_S : W^u(S) \rightarrow S$ defined as in (12). Hence, for non-degenerate critical submanifolds $S, S'$ for $f$, one considers the endpoint maps $l_{S'} : \mathcal{M}(S, S') \rightarrow S'$ and $u_S : \mathcal{M}(S, S') \rightarrow S$ defined on the moduli spaces of gradient flow lines (12). It was shown in [1] that the endpoint maps are locally trivial fibrations.

Denote by $S_i$ the set of critical points of index $i$ and by $C^{i,j}_i := \Omega^j(S_i)$ the set of $j$-forms on $S_i$. The cochain complex defined in [1] is built out of certain maps $\partial_r : C^{i,j}_i \rightarrow C^{i+r,j-r+1}_i$ which are the composition of a pullback of forms followed by integration over the fibers of the endpoint maps on the moduli spaces. This requires the following:

**Austin-Braam Assumption**

1. $f$ is weakly self indexing in the sense that $\mathcal{M}(S_i, S_j) = \emptyset$.
2. For all $i, j$ and $x \in S_i$ we have that $W_i^u(x) := u_i^{-1}(x)$ intersects $W^s(S_j)$ transversally.
3. the critical submanifolds $S_i$ and the negative normal bundles $\nu_-(S_i)$ are orientable for all $i$.
4. $M$ is orientable.

With this, Austin and Braam defined the maps $\partial_r : C^{i,j}_i \rightarrow C^{i+r,j-r+1}_i$ as

$$\partial_r(\omega) := \begin{cases} d\omega, & r = 0 \\ (-1)^j(u_i^{i+r})_*((l_i^{i+r})^*)^*(\omega), & \text{otherwise} \end{cases}$$

where $(u_i^{i+r})_*$ denotes the integration of forms over the fibers of the bundle (15). One of the main results of [1] establishes that for each $j$

$$\sum_{m=0}^j \partial_{j-m} \circ \partial_m = 0,$$

hence the maps $\partial_r$ fit together into a cochain complex.
whose cohomology is isomorphic to the de Rham cohomology of $M$. In the next subsection we will see the Lie groupoid version of this complex, yielding a double complex whose total cohomology is isomorphic to the Bott-Shulman-Stasheff cohomology.

8.2. The double complex. Throughout this section a Lie groupoid is denoted as $G^{(1)} ightrightarrows G^{(0)}$. Let $F_1 : (G^{(1)} \rightrightarrows G^{(0)}) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ be a Morse Lie groupoid morphism which is induced by a smooth basic function $F_0 : G^{(0)} \to \mathbb{R}$. We assume that $G^{(1)} \rightrightarrows G^{(0)}$ is proper and either $G^{(1)}$ or $G^{(0)}$ is compact.

Let us consider the nerve $G(\bullet)$ of $G^{(1)} \rightrightarrows G^{(0)}$ that is depicted as

$$G(\bullet) : \cdots \Rightarrow G^{(n)} \Rightarrow \cdots \Rightarrow G^{(2)} \Rightarrow G^{(1)} \Rightarrow G^{(0)}.$$

The manifolds $G^{(n)}$ of $n$-composable arrows are given by

$$G^{(n)} = G^{(1)} \times_{G^{(0)}} \cdots \times_{G^{(0)}} G^{(1)} = \{(g_n, \cdots, g_1) : s(g_j) = t(g_{j-1}); j = 2, \cdots, n\},$$

and the left arrows represent the face maps $d^k_n : G^{(n)} \to G^{(n-1)}$ for $k = 0, \cdots, n$. These are defined as $d^0_n = t, d^1_n = s$ and

$$d^k_n(g_n, \cdots, g_1) = \begin{cases} (g_n, g_2) & \text{if } k = 0 \\ (g_n, \cdots, g_{k+1}, g_{k+2}, g_{k+3}, \cdots, g_1) & \text{if } k = 1, \cdots, n-1 \\ (g_{n+1}, \cdots, g_2) & \text{if } k = n. \end{cases}$$

The face maps are surjective submersions and they satisfy the so-called simplicial identities

$$d^{n-1}_k \circ d^n_{k'} = d^{n-1}_{k'} \circ d^n_k, \quad k < k'. \tag{18}$$

By our initial assumptions it follows that $G^{(n)}$ is compact since it is closed and is contained inside $G^{(1)} \times \cdots \times G^{(1)}$. From [16] we know that the Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$ admits an $n$-metric $\eta^{(n)}$ on $G^{(n)}$. It is important to remember that we can push $\eta^{(n)}$ forward with the different face maps $d^k_n : G^{(n)} \to G^{(n-1)}$ to define an $(n-1)$-metric $\eta^{(n-1)}$ on $G^{(n-1)}$ where $\eta^{(n-1)} = (d^k_n)_* \eta^{(n)} = (d^k_n)_* \eta^{(n)}$ for all $k, k'$ and every $d^k_n : (G^{(n)}; \eta^{(n)}) \to (G^{(n-1)}, \eta^{(n-1)})$ becomes a Riemannian submersion. One can use this process to obtain $r$-metrics $\eta^{(r)}$ on $G^{(r)}$ in such a way that $d^r_k : (G^{(r)}, \eta^{(r)}) \to (G^{(r-1)}, \eta^{(r-1)})$ is a Riemannian submersion for every $0 \leq r \leq n-1$.

Applying the nerve functor to $F_1 : G^{(1)} \rightrightarrows \mathbb{R}$ yields a simplicial function $F_\bullet = (F_n)_{n \in \mathbb{N}}$ between the nerves:

$$F_\bullet : \cdots \Rightarrow G^{(n)} \Rightarrow \cdots \Rightarrow G^{(2)} \Rightarrow G^{(1)} \Rightarrow G^{(0)}$$

$$\cdots \Rightarrow \mathbb{R} \Rightarrow \cdots \Rightarrow \mathbb{R} \Rightarrow \mathbb{R} \Rightarrow \mathbb{R}.$$

which, for $n \geq 2$, is inductively defined as $F_n = (d^m_n)^* F_{n-1} = (d^m_n)^* F_{n-1}$ for all $k, k'$. These smooth functions are well defined since $F_1$ is a Lie groupoid morphism and the simplicial identities (18) hold true. Recall that, in our case, the set of critical points of $F_0$ is given by a disjoint union of finite compact and connected Lie groupoid orbits $\text{Crit}(F_0) = \bigcup \mathcal{O}$ since $G_0$ is compact and $G^{(1)} \rightrightarrows G^{(0)}$ is proper.
Lemma 8.1. There exists a sub-nerve $G_i^{(\bullet)}$ of $G^{(\bullet)}$ formed by non-degenerate critical submanifolds of index $i$ for $F_\bullet$ of the form:

$$G_i^{(1)} : \ldots \xrightarrow{\cdots} G_i^{(n)} \xrightarrow{\cdots} \cdots \xrightarrow{G_i^{(2)}} \xrightarrow{G_i^{(1)}} G_i^{(0)},$$

where $G_i^{(n)} = \bigcup \{ G_O^{(n)} \subset \text{Crit}(F_n) : \lambda(F_n, G_O^{(n)}) = i \}$ and $G_O^{(n)}$ denotes the manifold of $n$-composable arrows of $G_O \Rightarrow O$.

Proof. This follows by applying simple arguments from the previous sections since $F_1$ is a Morse Lie groupoid morphism. Namely, $G_i^{(0)}$ is a saturated submanifold with $G_i^{(1)} = s^{-1}(G_i^{(0)}) = t^{-1}(G_i^{(0)})$ and all the face maps of $G^{(\bullet)}$ are Riemannian submersions verifying the simplicial identities (18).

Let us now consider the collection of vector fields $-\nabla F_\bullet = (-\nabla F_n)_{n \in \mathbb{N}}$ where $-\nabla F_n$ denotes the negative gradient vector field of $F_n$ on $G^{(n)}$ with respect to $\eta^{(n)}$. As the face maps $d_k^n : G^{(n)} \rightarrow G^{(n-1)}$ are Riemannian submersions we actually have that $-\nabla F_{\bullet}$ defines a simplicial vector field on the nerve $G^{(\bullet)}$ since $\eta^{(n-1)} = (d_k^n) \cdot \eta^{(n)}$ so that $d_k^n \circ \nabla F_n = \nabla F_{n-1} \circ d_k^n$. It turns out that the collection of descending flows $\Phi_\tau^n : G^{(n)} \rightarrow G^{(n)}$, which are defined for all $\tau \in \mathbb{R}$ since $G^{(n)}$ is compact, verifies the relations

$$d_k^n \circ \Phi_\tau^n = \Phi_{\tau-1} \circ d_k^n, \quad k = 0, \ldots, n \quad \text{and} \quad \forall \tau \in \mathbb{R},$$

thus obtaining a simplicial automorphism $\Phi^{\bullet} : G^{(\bullet)} \rightarrow G^{(\bullet)}$. The collection of smooth endpoint maps associated to the collection of stable and unstable submanifolds $W^s(G_i^{(n)})$ and $W^u(G_i^{(n)})$ will be denoted by $u_i(n) : W^u(G_i^{(n)}) \rightarrow G_i^{(n)}$ and $l_i(n) : W^s(G_i^{(n)}) \rightarrow G_i^{(n)}$. Recall that these maps are locally trivial fiber bundles respectively defined by sending $A \mapsto \lim_{\tau \rightarrow -\infty} \Phi_\tau^n(A)$ and $A \mapsto \lim_{\tau \rightarrow +\infty} \Phi_\tau^n(A)$. It follows directly from Identities (19) that

$$d_k^n \circ u_i(n) = u_i(n-1) \circ d_k^n \quad \text{and} \quad d_k^n \circ l_i(n) = l_i(n-1) \circ d_k^n,$$

for all $k = 0, \ldots, n$. For indexes $i$ and $j$ we can consider again the moduli spaces of gradient flow lines

$$M^n(G_i^{(n)}, G_j^{(n)}) = W^u(G_i^{(n)}) \cap W^s(G_j^{(n)}) / \mathbb{R},$$

which are the quotient spaces defined through the action by flow translation.

Lemma 8.2. The simplicial structure of the nerve $G^{(\bullet)}$ may be restricted to define the stable and unstable sub-nerves $W^s(G_i^{(\bullet)})$ and $W^u(G_i^{(\bullet)})$ of the sub-nerve $G_i^{(\bullet)}$ and a topological sub-nerve structure $\mathcal{M}^{\bullet}(G_i^{(\bullet)}, G_j^{(\bullet)})$ between the moduli spaces of gradient flow lines.

Proof. Note that the first statement follows by flowing to $\infty$ and $-\infty$ at both sides of Equations (19) so that $d_k^n : W^s(G_i^{(n)}) \rightarrow W^s(G_i^{(n-1)})$ and $d_k^n : W^u(G_i^{(n)}) \rightarrow W^u(G_i^{(n-1)})$ are well restricted. Furthermore, the maps $d_k^n : M^n(G_i^{(n)}, G_j^{(n)}) \rightarrow M^{n-1}(G_i^{(n-1)}, G_j^{(n-1)})$ defined by sending $[A]_n \mapsto [d_k^n(A)]_{n-1}$ are well defined, open and surjective since $d_k^n$ are so.

We are interested in inducing well behaved structures of smooth manifold and orientability over our moduli spaces of gradient flow lines. To obtain this we require the following analogue of the Austin-Braam assumption (see subsection 8.1).

Assumption 8.3. (a) $\mathcal{M}^0(G_i^{(0)}, G_j^{(0)}) = \emptyset$ if $i \leq j$ ($F_0$ is weakly self-indexing).
(b) For all \( i, j \) and \( x \in G_i^{(0)} \) we have that \( W_i^u(x) := u_i(0)^{-1}(x) \) intersects \( W^s(G_j^{(0)}) \) transversally:
\[
W_i^u(x) \cap W^s(G_j^{(0)}).
\]

(c) Both the critical submanifolds \( G_i^{(0)} \) and the negative normal bundles \( \nu_-(G_i^{(0)}) \) are orientable for all \( i \).

(d) \( G^{(n)} \) is orientable for all \( n \in \mathbb{N} \).

\textbf{Remark 8.4.} Although condition (d) from previous assumption may seem somewhat restrictive there are many cases where such a requirement can be achieved. For instance, if \( G^{(0)} \) is orientable then every manifold conforming the nerve of: the unit groupoid, the pair groupoid, the action groupoid defined through a smooth action of a Lie group on \( G^{(0)} \), and étale Lie groupoids over \( G^{(0)} \), are orientable.

\textbf{Lemma 8.5.} Conditions (a)-(b)-(c) from Assumption 8.3 are satisfied at every level of the nerve configuration.

\textbf{Proof.} First, by Lemma 8.2 and proceeding by induction over \( n \) it is simple to check that \( \mathcal{M}^{n-1}(G_i^{(n-1)}, G_j^{(n-1)}) = \emptyset \) if \( i \leq j \) implies that \( \mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) = \emptyset \) if \( i \leq j \). The step \( n = 1 \) is consequence of (a) and the well definition of \( d_k^u \).

Proceeding again by induction over \( n \), we may prove that if the Morse–Smale transversality condition (b) holds true at the level \( n - 1 \) then it also holds true at the level \( n \). The case \( n = 1 \) was argued in Lemma 7.5. The inductive cases follow by arguing in the exactly same way but with the submersions \( d_k^u \).

Finally, let us now look at the orientability requirements (c). On the one hand, by the Transverse Submanifold Theorem, it is well known that because of conditions (c) and (d) together the simplicial identities (18), and the relation between the Hessian forms of \( F_i \) and \( F_0 \). In consequence, by using Lemma 8.1 we may easily extend some of these arguments to construct a sub-nerve \( \nu_-(G_i^{(*)}) \) of \( \nu_-(G^{(*)}) \) where the face maps at every level are the collection of fiberwise isomorphisms \( d_i \). Therefore, proceeding again by induction over \( n \) we will have that if \( \nu_-(G_i^{n-1}) \) is orientable, then so is \( \nu_-(G_i^{n}) \). Indeed, by following [23, c. VII] and [29, c. 8], if \( t_x : \nu_-(G_i^{(0)})_x \curvearrowright \nu_-(G_i^{(0)}) \) denotes the canonical inclusion of each fiber and \( \omega \) is an \( i \)-form on \( \nu_-(G_i^{(0)}) \) inducing orientations \( t_x^* \omega_x \) at every fiber \( \nu_-(G_i^{(0)})_x \) then when pulling \( t_x^* \omega_x \) back by the isomorphism \( \overline{ds_g} : \nu_-(G_1^{(1)})_g \to \nu_-(G_i^{(1)})_x \) we get orientations \( (t_x \circ \overline{ds_g})^* \omega_x = (\overline{ds} \circ t_g)^* \omega_x \) for each fiber \( t_g : \nu_-(G_1^{(1)})_g \to \nu_-(G_i^{(1)})_x \) so that \( \overline{ds} \omega \) is an \( i \)-form inducing an orientation on \( \nu_-(G_i^{(1)}) \). Hence, the other orientations are inductively obtained by using any of the fiberwise isomorphisms \( d_i \).

The first important consequence of the previous result is that because of (b), for every \( n \in \mathbb{N} \) the moduli of gradient flow lines \( \mathcal{M}^n(G_i^{[n]}, G_j^{[n]}) \) is a smooth manifold where the new endpoint maps
\[
u_i(n) : \mathcal{M}^n(G_i^{[n]}, G_j^{[n]}) \to G_i^{[n]} \quad \text{and} \quad \nu_j(n) : \mathcal{M}^n(G_i^{[n]}, G_j^{[n]}) \to G_j^{[n]}
\]
defined by \( u_j^n([A]_n) := u_i(n)(A) \) and \( l_j^n([A]_n) := l_j(n)(A) \), are well defined smooth maps such that \( u_j^n(n) \) has the structure of locally trivial bundle; see [1]. Therefore, as the action by flow translations is free and proper we get that the maps \( d_k^n \) are smooth and from Identities (20) we obtain that the following diagrams are commutative

\[
\begin{array}{c}
\mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) \xrightarrow{d_k^n} \mathcal{M}^{n-1}(G_i^{(n-1)}, G_j^{(n-1)}) \\
\downarrow u_i^n(n) \quad \quad \quad \downarrow u_i^n(n-1) \\
G_i^{(n)} \xrightarrow{d_k^n} G_i^{(n-1)} \\
\end{array} \quad \text{and} \quad \begin{array}{c}
\mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) \xrightarrow{d_k^n} \mathcal{M}^{n-1}(G_i^{(n-1)}, G_j^{(n-1)}) \\
\downarrow l_j^n(n) \quad \quad \downarrow l_j^n(n-1) \\
G_j^{(n)} \xrightarrow{d_k^n} G_j^{(n-1)} \\
\end{array}
\]

so that we have the commutative identities

\[
d_k^n \circ u_j^n(n) = u_j^n(n - 1) \circ d_k^n \quad \text{and} \quad d_k^n \circ l_j^n(n) = l_j^n(n - 1) \circ d_k^n. \tag{22}
\]

In other words, we have obtained simplicial smooth maps

\[
u_j^n(\bullet) : \mathcal{M}^n(G_i^{(*)}, G_j^{(*)}) \to G_i^{(*)} \quad \text{and} \quad l_j^n(\bullet) : \mathcal{M}^n(G_i^{(*)}, G_j^{(*)}) \to G_j^{(*)}
\]

with \( u_j^n(\bullet) \) a locally trivial simplicial fibration.

Let us define our double cochain complex. For that, we will define maps which are the composition of the pullback operation followed by the integration along the fibers via (23).

For each \( n \in \mathbb{N} \) we set \( C^{i,j}(G^{(n)}) := \Omega^j(G_i^{(n)}) \) and define the operator \( \partial^n_r : C^{i,j}(G^{(n)}) \to C^{i+r,j-r+1}(G^{(n)}) \) as

\[
\partial^n_r(\omega) = \begin{cases} 
\frac{d\omega}{r = 0} \\
(-1)^j (u_i^{i+r}(n))_r (l_j^{i+r}(n))^{*}(\omega) \quad \text{otherwise,}
\end{cases}
\]

where \( (u_i^{i+r}(n))_r \) is integration along the fiber of the bundle (21). Let us now set

\[
C^n(G^{(n)}) := \bigoplus_{i+j=p} C^{i,j}(G^{(n)}) = \bigoplus_{i+j=p} \Omega^j(G_i^{(n)}) \quad \text{with} \quad \partial^n = \sum \partial^n_r. \tag{24}
\]

As it was shown for the classical case in [1], the operator \( \partial^n \) is a boundary operator, i.e., \((\partial^n)^2 = 0\). Let us now consider the simplicial differentials of the sub-nerve \( G_i^{(*)} \) from Lemma 8.1:

\[
\delta_i^n = \sum (-1)^k (d_k^n)^{*} : C^{i,j}(G_i^{(n-1)}) \to C^{i,j}(G_i^{(n)}).
\]

**Lemma 8.6.** The following diagram commutes for all \( i \) and \( r \)

\[
\begin{array}{c}
\Omega^j(G_i^{(n-1)}) \xrightarrow{\partial^n_r} \Omega^{j-r+1}(G_i^{(n-1)}) \\
\downarrow \delta_i^n \quad \quad \quad \downarrow \delta_i^{n+r} \\
\Omega^j(G_i^{(n)}) \xrightarrow{\partial^n_r} \Omega^{j-r+1}(G_i^{(n)}) \\
\end{array}
\]

**Proof.** Observe that if \( r = 0 \) then

\[
\delta_i^{n+r} \circ \partial_i^n = \partial_i^n \circ \delta_i^n \iff \delta_i^n \circ d = d \circ \delta_i^n,
\]

since the simplicial differential already commute with the de Rham differentials. Otherwise,

\[
(\partial^n_r \circ \delta^n_i)(\omega) = \partial^n_r \left( \sum (-1)^k (d_k^n)^{*}(\omega) \right) = \sum (-1)^k \partial^n_r ((d_k^n)^{*}(\omega)),
\]

so that we have the commutative identities

\[
d_k^n \circ u_j^n(n) = u_j^n(n - 1) \circ d_k^n \quad \text{and} \quad d_k^n \circ l_j^n(n) = l_j^n(n - 1) \circ d_k^n. \tag{22}
\]
where, as consequence of the base-change formula of the integration along the fiber operation together with Identities (22), we obtain

$$\partial_r^p((d_k^p)^*(\omega)) = (-1)^j(u_i^{i+r}(n))_*((l_i^{i+r}(n)))((d_k^p)^*(\omega))$$

Thus, we have that

$$\delta^n_r((d_k^p)^*(\omega)) = (-1)^j(u_i^{i+r}(n))_*(d_k^p \circ l_i^{i+r}(n))^*(\omega)$$

Summing up, we have obtained that:

$$\delta^n_r = (d_k^p)^*(\partial_r^p(\omega))$$

Thus, we have that

$$(\partial_r^p \circ \delta^n_r)\omega = \sum_k (-1)^k \partial_r^p((d_k^p)^*(\omega)) = \sum_k (-1)^k (d_k^p)^*(\partial_r^{p-1} \omega) = (\delta^n_{r+p} \circ \partial_r^{p-1} \omega).$$

Having the previous fact in mind we define $\overline{\delta^n} : C^p(G^{(n-1)}) \to C^p(G^{(n)})$ as

$$\overline{\delta^n}(\omega) = \begin{cases} \delta^n_r(\omega) & \text{if } w \in \Omega^p(G_i^{(n-1)}) \\ 0 & \text{otherwise.} \end{cases}$$

This operator may be thought of as $\overline{\delta^n} = \sum \delta^n_r$ verifying $\delta^n_r \circ \delta^n_r = \delta^n_r \circ \delta^n_r = 0$ since if $i \neq i'$ then $\delta^n_r \circ \delta^n_r = 0$ by definition and if $i = i'$ then $(\delta^n_r)^2 = 0$ also holds because $\delta^n_r$ is a simplicial differential of the sub-nerve $G_i^{(\bullet)}$. In particular, we have that $(\overline{\delta^n})^2 = 0$.

Note that we have defined two boundary operators $\partial$ and $\delta$ verifying $\partial^2 = 0$, $\delta^2 = 0$ and, moreover, from the commutativity property stated in Lemma 8.6 we get that they also satisfy $\partial \circ \delta = \delta \circ \partial$.

Summing up, we have obtained that:

**Proposition 8.7.** The triple $(C^\bullet(G^{(\bullet)}), \partial, \delta)$ determines a double cochain complex which may be depicted as

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots \\
\partial & \partial & \partial \\
C^2(G^{(0)}) & \overline{\delta} & C^2(G^{(1)}) & \overline{\delta} & C^2(G^{(2)}) & \overline{\delta} & \cdots \\
\partial & \partial & \partial \\
C^1(G^{(0)}) & \overline{\delta} & C^1(G^{(1)}) & \overline{\delta} & C^1(G^{(2)}) & \overline{\delta} & \cdots \\
\partial & \partial & \partial \\
C^0(G^{(0)}) & \overline{\delta} & C^0(G^{(1)}) & \overline{\delta} & C^0(G^{(2)}) & \overline{\delta} & \cdots
\end{array}
\]

We can make a total complex out of this double cochain complex by setting $C_T^p(G) = \bigoplus_{p+q=n} C^p(G^{(q)})$ and defining the total differential $\partial_T : C_T^p(G) \to C_T^{p+1}(G)$ as
\[ \partial_T(\omega) = (\tilde{\delta} + (-1)^n \partial)(\omega), \quad \omega \in C^p(G^{(q)}). \]

The sing change is introduced in order that \( \partial_T^2 = 0. \)

**Definition 8.8.** The **groupoid Morse cohomology** associated to the Morse Lie groupoid \( F_1 : (G^{(1)} \rightrightarrows G^{(0)}) \to (\mathbb{R} \rightrightarrows \mathbb{R}) \) and the \( n \)-metric \( \eta^{(n)} \) on \( G^{(n)} \) is defined to be the cohomology of \( (C^\bullet_T(G), \partial_T). \)

Finally, let us exhibit a morphism of double complexes between the Bott–Shulman-Stasheff double complex of \( G^{(1)} \rightrightarrows G^{(0)} \) and the double complex constructed above. The **Bott-Shulman-Stasheff double complex** \( (\Omega^\bullet(G^{(\bullet)}), d, \delta) \) may be depicted as

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\delta & \delta & \delta & \\
\Omega^2(G^{(0)}) & \rightarrow & \Omega^2(G^{(1)}) & \rightarrow & \Omega^2(G^{(2)}) & \rightarrow & \cdots \\
\delta & \delta & \delta & \\
\Omega^1(G^{(0)}) & \rightarrow & \Omega^1(G^{(1)}) & \rightarrow & \Omega^1(G^{(2)}) & \rightarrow & \cdots \\
\delta & \delta & \delta & \\
\Omega^0(G^{(0)}) & \rightarrow & \Omega^0(G^{(1)}) & \rightarrow & \Omega^0(G^{(2)}) & \rightarrow & \cdots \\
\end{array}
\]

where the vertical differential is the de Rham differential and the horizontal differential is the simplicial differential associated to the nerve \( G^{(\bullet)} \) of \( G^{(1)} \rightrightarrows G^{(0)}. \) We can similarly define a total cohomology for this double complex which is usually denoted by \( H^\bullet_{dR}(\Omega^\bullet(G^{(1)}), d). \)

Let us consider the collection of maps \( \Psi^\bullet := \{\Psi^n\}_{n \in \mathbb{N}} \) constructed as follows. Recall that for each \( n \in \mathbb{N} \) the map \( u_i(n) : W^n(G^{(n)}_{i}) \rightarrow G^{(n)}_{i} \) has the structure of locally trivial bundle. Thus, by using again integration along the fiber, we define \( \Psi^n_i : \Omega^p(G^{(n)}_{i}) \rightarrow C^{n,p-i}(G^{(n)}) \) as

\[ \Psi^n_i(\omega) := (u_i(n))_{\ast} \left( \omega|_{W^n(G^{(n)}_{i})} \right), \quad \omega \in \Omega^p(G^{(n)}_{i}), \]

and \( \Psi^n = \bigoplus \Psi^n_i : \Omega^p(G^{(n)}) \rightarrow C^{n,p}(G^{(n)}). \) As consequence of the results due to Austin and Braam in [1] we have that \( \Psi^\bullet \circ d = \partial \circ \Psi^\bullet \) and, more importantly, this collection of maps induces isomorphisms between the cohomology groups

\[ H^\bullet(C^\bullet(G^{(n)}), \partial) \cong H^\bullet_{dR}(\Omega^\bullet(G^{(n)}), d). \]

To see that the collection \( \Psi^\bullet \) defines a morphism of double complexes it remains to check that \( \Psi^\bullet \circ \delta = \tilde{\delta} \circ \Psi^\bullet. \) This will be consequence of showing the following identity.

**Lemma 8.9.** For all \( i \) and \( n, \) the following holds

\[ \Psi^n_i \circ \delta = \tilde{\delta} \circ \Psi^{n-1}_i. \]

**Proof.** Because of Identities (20), which in turn allow us to conclude that every face map \( d^n_k \) is well restricted to the stable/unstable manifolds, and by using the base-change property of the
integration along the fiber operation, we obtain that
\[
(\Psi_i^n \circ \delta)(\omega) = \Psi_i^n \left( \sum (-1)^k (d^n_k)^* (\omega) \right) = \sum (-1)^k (d^n_k)^* (\omega)
\]
\[
= \sum (-1)^k (u_i(n))_* \left( \left( (d^n_k)^* (\omega) \right)_{|W^n(G^{(n)})} \right)
\]
\[
= \sum (-1)^k (d^n_k)^* (u_i(n-1))_* \left( \omega_{|W^n(G^{(n-1)})} \right) = (\delta \circ \Psi_i^{n-1})(\omega),
\]
for all \( \omega \in \Omega^p(G^{(n-1)}) \) as desired. \( \square \)

So, \( \Psi^* \) defines a morphism of double complexes between \((\Omega^*(G^{(*)}), d, \delta)\) and \((C^*(G^{(*)}), \partial, \overline{\partial})\) inducing isomorphisms between the vertical cohomologies of the complexes. Therefore, as consequence of all the facts stated above, by using a usual argument of spectral sequences (see for instance [10, p. 108]) we conclude:

**Theorem 8.10.** The total cohomology of the double complex \((C^*(G^{(*)}), \partial, \overline{\partial})\) is isomorphic to the total cohomology of the Bott–Shulman-Stasheff double complex of \(G^{(1)} \rightrightarrows G^{(0)}\):

\[
H^*_T(G, \partial_T) \cong H^*_{dR}(G).
\]

In particular, the total cohomology on the left hand side is Morita invariant. This should be expected since the notion of Morse Lie groupoid morphism we are working with is also Morita invariant.

**Remark 8.11.** Fukaya in [22] gave a construction which is similar to Austin–Braam’s model, but using singular chains instead of differential forms. One of the main features of his construction is that it can be directly generalized to the infinite-dimensional case so that it allowed him to address problems in Floer homology. We plan to apply Fukaya’s approach to our setting with the hope of starting the study of Floer homology in the context of Lie groupoids and differentiable stacks.

### 8.3. Equivariant cohomology

Under certain some additional assumptions and motivated by the Austin–Braam’s equivariant Morse complex (see [1, s. 5.1]), we apply the constructions from the previous section to recover the equivariant cohomology associated to a 2-action on a Lie groupoid defined in [6]. Let us consider a Lie groupoid \(G^{(1)} \rightrightarrows G^{(0)}\) and a Morse Lie groupoid morphism \(F_1 : (G^{(1)} \rightrightarrows G^{(0)}) \to (\mathbb{R} \rightrightarrows \mathbb{R})\), induced by a basic function \(F_0 : G^{(0)} \to \mathbb{R}\), as those we considered in the previous section. Suppose that \(K^{(1)} \rightrightarrows K^{(0)}\) is a Lie 2-group, with \(K_1\) compact, determining a 2-action on \(G^{(1)} \rightrightarrows G^{(0)}\) such that \(F_0\) is \(K^{(0)}\)-invariant. The set of \(n\)-composable arrows \(K^{(n)}\) inherits a canonical Lie group structure from the direct product \((K^{(1)})^n\) and the 2-action above allows us to define a smooth left action of \(K^{(n)}\) on \(G^{(n)}\). In other words, we have a well defined simplicial left action of the nerve \(K^{(*)}\) on the nerve \(G^{(*)}\). It is simple to check that the face maps \((d^n_k)_K : K^{(n)} \to K^{(n-1)}\) and \((d^n_k)_{G} : G^{(n)} \to G^{(n-1)}\) satisfy the equivariant relations

\[
(d^n_k)_K (k \cdot g) = (d^n_k)_K (k) \cdot (d^n_k)_G (g), \quad (25)
\]

for all \( k \in K^{(n)} \) and \( g \in G^{(n)}\). This is consequence of the simplicial identities on \(G^{(*)}\) and the fact that we are working with a 2-action. Therefore, Formula (25) implies that the simplicial function \(F^*\) is \(K^{(*)}\)-invariant in the sense that \(F^*_n\) is \(K^{(n)}\)-invariant for all \(n\) since \(F_0\) is \(K^{(0)}\)-invariant. Let us now consider an \(n\)-metric \(\eta^{(n)}\) on \(G^{(n)}\).
Definition 8.12. [20] The 2-action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$ is said to be \textbf{isometric} if the action of $K^{(n)}$ on $(G^{(n)}, \eta^{(n)})$ is isometric.

A detailed study of several geometric aspects concerning isometric 2-actions as well as the structure of Riemannian groupoids (for instance, existence, Morita invariance, equivariant groupoid linearization, isometries, Lie 2-algebra of Killing vector fields, among other things), will be provided in [20]. As it was proved therein, the fact that the horizontals are given by action groupoids and the verticals are Lie groupoid products.

where the horizontals are given by action groupoids and the verticals are Lie groupoid products.

This will be denoted by $H^{(1)}$. The 2-action naturally allows us to define a double Lie groupoid $K^{(1)} \times G^{(1)} \rightrightarrows G^{(1)}$, where the horizontals are given by action groupoids and the verticals are Lie groupoid products. If we consider the nerve configuration associated to this double Lie groupoid then we obtain a bisimplicial smooth manifold so that we may work with the triple complex $C^{\bullet\bullet\bullet}$ where

\[ C^{n,p,q} = \Omega^{p}((K^{(n)})^{q} \times G^{(n)}), \]

with differentials given by the de Rham differential, the simplicial differential associated to the actions groupoids, and the simplicial differential associated to the product groupoids.

Definition 8.14. [6] The \textbf{equivariant cohomology} of the 2-action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$ is defined to be the total cohomology determined by the triple complex mentioned above. This will be denoted by $H^\bullet_K(G)$.

Two important features of this cohomology is that it is Morita invariant and can be recovered by the \textbf{Cartan model} as follows. Let us consider the nerve configuration $\mathfrak{k}^{(\bullet)}$ associated to the Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ of $K^{(1)} \rightrightarrows K^{(0)}$. The notion of simplicial equivariant forms introduced in [34] comes from a double complex $(C^{\bullet\bullet\bullet}_M, d_K, \delta_K)$ given by
\[ C_{n,k}^{CM} = \Omega_{K(n)}^k(G^{(n)}) = \bigoplus_{k=2p+q} (S^p((\mathfrak{t}^{(n)})^*) \otimes \Omega^q(G^{(n)}))^{K(n)}, \]

(26)

where \((\mathfrak{t}^{(n)})^*\) denotes the dual vector space of the Lie algebra \(\mathfrak{t}^{(n)}\), \(d_K\) is the Cartan differential, and \(\delta_K : \Omega_{K(n)}^k(G^{(n)}) \to \Omega_{K(n+1)}^k(G^{(n+1)})\) is defined by \(\delta_K = \delta_t \otimes \delta_G\) with \(\delta_t\) the simplicial differential of \(\mathfrak{t}^{(\bullet)}\) and \(\delta_G\) the simplicial differential of \(G^{(\bullet)}\). As it was proven in [6] the total cohomology of this double complex is isomorphic to the equivariant cohomology \(H^*_K(G)\) since \(K^{(1)}\) is assumed to be compact.

We claim that it is possible to recover the equivariant cohomology defined above by following the ideas from [1] together with what we did in the previous section. For that, we consider a Morse Lie groupoid morphism \(F^1 : G \to \mathbb{R}\) covering a basic function \(F^0 : M \to \mathbb{R}\) which is \(K_0\)-invariant. Using the Cartan model we define the equivariant version of the double complex (24):

\[ C^p(G^{(n)}) = \bigoplus_{i+r=p} \Omega_{K(n)}^i(G^{(n)}) = \bigoplus_{i+j+2q=p} (\Omega^j(\mathcal{G}^{(n)}_i) \otimes S^q((\mathfrak{t}^{(n)})^*))^{K(n)}, \]

(27)

with differential operators \(\partial_K^n\) and \(\overline{\delta}_K^n\) defined as follows. On the one hand, as before we split \(\partial_K^n : C^p(G^{(n)}) \to C^{p+1}(G^{(n)})\) as the sum \(\partial_K^n = \sum_v (\partial_K^n)_v\) where, for \(\omega \otimes \phi \in (\Omega^j(G^{(n)}_i) \otimes S^q((\mathfrak{t}^{(n)})^*))^{K(n)}\), we have that \((\partial_K^n)_v(\omega \otimes \phi) = d_K(\omega \otimes \phi)\) is the Cartan differential and for \(v > 0\) we set \((\partial_K^n)_v(\omega \otimes \phi) = \partial^n_v \omega \otimes \phi\). On the other hand, we define \(\overline{\delta}_K^n : C^p(G^{(n)}) \to C^p(G^{(n)+1})\) as \(\overline{\delta}_K^n(\omega \otimes \phi) = \delta^n \omega \otimes \delta^n \phi\). It is simple to check that these two operators \(\partial_K\) and \(\overline{\delta}_K\) commute and that \(\overline{\delta}_K^0 = 0\). Moreover, from [1] we also get that \(\overline{\delta}_K^2 = 0\). Therefore, we have actually obtained a double cochain complex \((C^{\bullet}(G^{(\bullet)}), \partial_K, \overline{\delta}_K)\), as claimed above.

Finally, let us now exhibit a morphism of double complexes between the double complex \((C_{CM}^{\bullet}, d_K, \delta_K)\) which is obtained by using the Cartan model (26) and \((C^{\bullet}(G^{(\bullet)}), \partial_K, \overline{\delta}_K)\) defined in (27). Let \(\Theta^{\bullet} : C^{\bullet}(G^{(\bullet)}) \to C_{CM}^{\bullet}\) be defined as a collection of maps \(\{\Theta^n\}_{n \in \mathbb{N}}\) defined by \(\Theta^n(\omega \otimes \phi) = \Psi^n(\omega) \otimes \phi\). From a straightforward computation it follows that \(\Theta^n \circ \partial_K = \overline{\delta}_K \circ \Theta^n\).

Also, as a consequence of what was proven in [1] we have that \(\Theta^n \circ d_K = \partial_K \circ \Theta^n\), and, more importantly, this collection of maps induces isomorphisms between the cohomology groups

\[ H^*(C^{\bullet}(G^{(n)}), \partial_K) \cong H^*(\Omega_{K(n)}^*(G^{(n)}), d_K) = H_{K^{(n)}}^*(G^{(n)}), \quad n \in \mathbb{N}. \]

Here \(H_{K^{(n)}}^*(G^{(n)})\) denotes the equivariant cohomology obtained through the Cartan model associated to the action of \(K^{(n)}\) on \(G^{(n)}\). So, \(\Theta^{\bullet}\) defines a morphism of double complexes between \((C_{CM}^{\bullet}, d_K, \delta_K)\) and \((C^{\bullet}(G^{(\bullet)}), \partial_K, \overline{\delta}_K)\) inducing isomorphisms between the vertical cohomologies of the complexes. Just as in the non equivariant case, by mean of a spectral sequence argument we conclude:

**Proposition 8.15.** The total cohomology \(H_T^*(G, \partial_{TK})\) of the double complex \((C^{\bullet}(G^{(\bullet)}), \partial_K, \overline{\delta}_K)\) is isomorphic to the total cohomology of the double complex determined by the Cartan model associated to the 2-action of \(K^{(1)} \Rightarrow K^{(0)}\) on \(G^{(1)} \Rightarrow G^{(0)}\). That is,

\[ H_T^*(G, \partial_{TK}) \cong H^*_K(G). \]

Toric symplectic stacks [25] are presented by 0-symplectic groupoids with a Hamiltonian action of a 2-torus [26]. Hence, as an application of Proposition 8.15 one can compute the equivariant cohomology of a toric symplectic stack by mean of groupoid Morse theory.
Example 8.16 (Equivariant cohomology of toric symplectic stacks). Let $G^{(1)} \rightrightarrows G^{(0)}$ be a 0-symplectic groupoid equipped with a Hamiltonian $(K^{(1)} \rightrightarrows K^{(0)})$-groupoid 2-action with moment map $\mu$ verifying Proposition 4.12. For every $\xi \in \mathfrak{k}/\mathfrak{h}$ we have a Morse Lie groupoid morphism $\mu^{\xi} : (G^{(1)} \rightrightarrows G^{(0)}) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ for which $\mu^{\xi}_{0}$ is $K^{(0)}$-invariant since $K^{(0)}$ is abelian. Therefore, if there exists $\xi \in \mathfrak{k}/\mathfrak{h}$ such that $\mu^{\xi}$ verifies Assumption 8.3, then Proposition 8.15 allows to compute the equivariant cohomology associated to the 2-action of the foliation Lie 2-group $(K^{(1)} \rightrightarrows K^{(0)})$ on the 0-symplectic groupoid $G^{(1)} \rightrightarrows G^{(0)}$ by using the equivariant version of the groupoid Morse cohomology.

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