Graph skein modules and symmetry of spatial graphs

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Abstract

In this paper, we compute the graph skein algebra of the punctured disk with two holes. Then, we apply the graph skein techniques developed here to establish necessary conditions for a spatial graph to have a symmetry of order $p$, where $p$ is a prime. The obstruction criteria introduced here extend some results obtained earlier for symmetric spatial graphs.

1 Introduction

A spatial graph in the three-sphere $S^3$ is the image of an embedding of a graph $G$ in $S^3$. It is worth mentioning that the graphs considered here may have multi-edges and loops. Moreover, we assume that the valency of each vertex is greater than or equal to 3. Let $p \geq 2$ be an integer, a spatial graph is said to be $\mathbb{Z}_p$-symmetric if it can be isotoped into a spatial graph which is invariant by a rotation of order $p$. We consider two kinds of $\mathbb{Z}_p$-symmetry. The first one is when the axis of the rotation does not intersect the spatial graph. The second case is when the axis of the rotation intercepts the graph only at one vertex $v$. In this paper, we address the following problem: let $G$ be a graph whose automorphism group has an element of order $p$ and let $\tilde{G}$ be a spatial embedding of $G$. Does $\tilde{G}$ has a $\mathbb{Z}_p$-symmetry?
Przytycki [14] introduced the notion of skein modules of three-manifolds. Przytycki’s first motivation was to extend the definition of the polynomial invariants of links in $S^3$ to links in other three-manifolds. Let $M$ be an oriented three-manifold. Let $\mathcal{L}$ be the set of isotopy classes of framed links in $M$. The Kauffman Bracket skein module of $M$ is defined to be the quotient of the free $\mathbb{Z}[A^{\pm 1}]$-module generated by $\mathcal{L}$, by the Kauffman bracket skein relations, see Section 3.

Inspired by the discovery of the quantum invariants of knots and links, Yamada [19] introduced a polynomial invariant of spatial graphs. This topological invariant can be defined recursively by a family of local relations of skein type on planar diagrams of spatial graphs. In spirit of the algebraic topology based on knots, we define a version of skein modules using embedded graphs instead of links. Here is an outline of our construction. Let $M$ be an oriented three-manifold. A ribbon graph in $M$ is an oriented surface in $M$ that retracts by deformation on a graph embedded in $M$. Let $\mathcal{G}$ be the set of all isotopy classes of ribbon graphs embedded in $M$. Let $\mathcal{R} = \mathbb{Z}[A^{\pm 1}, d^{-1}]$, where $d = -A^2 - A^{-2}$, we define the graph skein module of $M$, $\mathcal{Y}(M)$ to be the quotient of $\mathcal{R}(\mathcal{G})$ by the Yamada relations, see section 3. If $M = F_{g,n} \times I$ where $F_{g,n}$ is an oriented surface of genus $g$ and having $n$ boundary components, then $\mathcal{Y}(M)$ has an algebra structure defined in the same way as in the case of the Kauffman bracket skein module [2]. The unit of the multiplicative structure is the empty graph and the product of two ribbon graphs $G$ and $G'$ is obtained by taking a disjoint union of $G$ and $G'$ where $G$ is pushed isotopically into $F_{g,n} \times [1/2, 1]$ and $G'$ is pushed isotopically into $F_{g,n} \times [0, 1/2]$. Since the multiplication depends on the product structure on $M$, then we will denote the skein algebra of $F_{g,n} \times I$ by $\mathcal{Y}(F_{g,n})$. In [6], we proved that the skein algebra $\mathcal{Y}(F_{0,2})$ is isomorphic to the polynomial algebra $\mathcal{R}[b]$, where $b$ is pictured in Figure 1.
In this paper we prove the following theorem

**Theorem 1.1.** The graph skein algebra \( \mathcal{Y}(F_{0,3}) \) is isomorphic to the quotient of the polynomial algebra \( R[x, y, z, t] \) by the ideal generated by

\[
t^2 - 1 + d^{-2} - 2d^{-1} + (1 - 2d^{-1})x + (1 - 2d^{-1})y + z - 2d^{-1}t \\
+ (1 - 2d^{-2})xy + xz + yz - 2d^{-1}tx - 2d^{-1}ty - d^{-2}x^2 - d^{-2}y^2 + xyz,
\]

where \( x, y, z \) and \( t \) are as in the following picture.

Marui [12], studied the behavior of the Yamada polynomial of spatial graphs with \( \mathbb{Z}_p \)-symmetry in some special cases. In [6], we used the graph skein algebra of the annulus and the criteria of link periodicity introduced by Murasugi [13], Przytyki [15] and Traczyk [17] to obtain an extension of Marui’s result. In the present paper, we prove some congruence relationships satisfied by the Yamada polynomial of spatial graphs with \( \mathbb{Z}_p \)-symmetry, where \( p \) is a prime. These relationships are, a priori, more precise than what we obtained in [6]. This is due to
the fact that the congruence relations obtained here involve smaller ideals than the ones in [6]. Furthermore, what we obtain here apply to vertex-fixing \( \mathbb{Z}_p \)-symmetries as well. This case is not covered by the results obtained earlier in [6] and [12].

**Theorem 1.2.** Let \( p \) be a prime and \( \tilde{G} \) a ribbon spatial graph. Then we have the following congruences in the ring \( \mathbb{Z}[A^\pm 1, d^{-1}] \)

(a) If \( \tilde{G} \) has a \( \mathbb{Z}_p \)-symmetry with no fixed points then: 
\[
Y(\tilde{G})(A) \equiv (Y(\tilde{G})(A))^p \mod p, d^p - d^2.
\]

(b) If \( \tilde{G} \) has a vertex-fixing \( \mathbb{Z}_p \)-symmetry then: 
\[
Y(\tilde{G})(A) \equiv (Y(\tilde{G})(A))^p \mod p, d^p - 1.
\]

(c) If \( \tilde{G} \) has a \( \mathbb{Z}_p \)-symmetry then: 
\[
Y(\tilde{G})(A) \equiv Y(\tilde{G})(A^{-1}) \mod p, A^{8p} - 1.
\]

Where \( \tilde{G} \) is the the quotient spatial graph.

**Example.** Let us illustrate the last statement in Theorem 1.2 by an example. Consider the following embedding \( \tilde{P} \) of the Petersen graph.

It is known that the automorphism group of the Petersen graph is the symmetric group \( S_5 \). So it contains elements of order 3 and 5. We want to check whether the necessary condition (c) in Theorem 1.2 can be used to rule out the possibility of symmetries of order 3 or 5 for \( \tilde{P} \). The Yamada polynomial of \( \tilde{P} \) is given by

\[
Y(\tilde{P})(A) = -A^{-34} - 6A^{-30} - 15A^{-26} - 35A^{-22} - 65A^{-18} - 66A^{-14} - 36A^{-10} - 15A^{-6} - 5A^{-2} + 10A^6 + 35A^{10} + 61A^{14} + 66A^{18} + 40A^{22} + 15A^{26} + 10A^{30} + 6A^{34} + A^{38}.
\]

Easy computation shows that \( Y(\tilde{P})(A) \) is not congruent to \( Y(\tilde{P})(A^{-1}) \) modulo 5 and \( A^{40} - 1 \). Thus, \( \tilde{P} \) doesn’t have a \( \mathbb{Z}_5 \)-symmetry or a vertex fixing \( \mathbb{Z}_5 \)-symmetry. However, \( Y(\tilde{P})(A) \) is congruent to \( Y(\tilde{P})(A^{-1}) \) modulo \( A^{24} - 1 \) and 3. Which means that our criterion fails to decide whether \( \tilde{P} \) has \( \mathbb{Z}_3 \)-symmetry or not.

4
2 The Yamada polynomial

The Yamada polynomial \[19\] is an invariant \( R \) of regular isotopy of spatial graphs in the three-sphere. In our terms, \( R \) is a topological invariant of ribbon spatial graphs. It takes its values in the ring \( \mathbb{Z}[A^\pm 1] \) and may be defined recursively on planar diagrams of spatial graphs.

Yamada also introduced a similar invariant of trivalent graphs, with good weight associated with the set of edges \[20\]. This construction was extended by Yokota \[21\] using the linear skein theory introduced by Lickorich \[11\]. For our reasons, we find it more convenient to slightly change the recursive formulas introduced by Yamada, see Figure 3. Let \( d = -A^2 - A^{-2} \) and \( R = \mathbb{Z}[A^\pm 1, d^{-1}] \). We define the invariant \( Y \) recursively, by the four relations in the following figure. It is worth mentioning that the following identities hold for diagrams which are identical except in a small disk where they look as pictured below.

\[
Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array}) = A^4 Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array}) + A^{-4} Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array}) - d Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array})
\]

\[
Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array}) = Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array}) - d^{-1} Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array})
\]

\[
Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array}) = (d - d^{-1}) Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array})
\]

\[
Y(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\end{array}) = (d^2 - 1) Y(D), \text{ for any graph diagram } D.
\]

Figure 3

The invariant \( Y \) we obtain takes values in \( R = \mathbb{Z}[A^\pm 1, d^{-1}] \). It is related to Yamada’s invariant \( R \) by the formula:

\[
Y(\tilde{G})(A) = (-d)^{\alpha(G)} R(\tilde{G})(A^4),
\]

where \( \alpha(G) \) is equal to the number of edges of \( G \) minus the number of vertices of \( G \).
3 Graph skein modules of three-manifolds

3.1 The Kauffman Bracket skein module

Let $M$ be an oriented three-manifold and $\mathcal{L}$ the set of all isotopy classes of framed links in $M$. Let $\mathcal{R}=\mathbb{Z}[A^\pm, d^{-1}]$ and $K(M)$ be the free $\mathcal{R}$-module generated by all elements of $\mathcal{L}$. The Kauffman bracket skein module of $M$ with coefficients in $\mathcal{R}$ which we denote here by $\mathcal{K}(M)$ is defined as the quotient of $K(M)$ by the smallest submodule containing all expressions of the form:

$$\bigcirc \cup L - dL$$

$$L - AL_0 - A^{-1}L_\infty,$$

where $\bigcirc$ is the trivial circle and $L$, $L_0$ and $L_\infty$ are three links which are identical except in a small three-ball where they look like in the following picture.

![Diagram of links](image)

**Figure 4**

The Kauffman bracket skein module has been subject to an extensive literature. The existence and the uniqueness of the Kauffman bracket polynomial [9] is equivalent to the fact that the Kauffman bracket skein module of $S^3$ is isomorphic to $\mathcal{R}$ with the empty link $\emptyset$ as a generator. The case $M = F_{g,n} \times I$, where $F_{g,n}$ is the oriented surface of genus $g$ with $n$ boundary components and $I$ is the unit interval $[0, 1]$ is particularly interesting. Indeed, one may project on the surface and consider diagrams of links on the surface modulo the skein relations. Przytycki [14] proved that the Kauffman bracket skein module of $M$ is generated by all links on $F_{g,n}$ without trivial components but including the empty link. For instance, the skein module of the solid torus
$S^1 \times I \times I$ is generated by \( \{b^n; n \geq 0\} \) where \( b^n \) is the link in the annulus made up of \( n \) parallel copies of the boundary component \( b = S^1 \times \{0\} \times \{0\} \), see Figure 1.

Furthermore, there is an algebra structure on the skein module of \( M = F_{g,n} \times I \). The unit of the algebra is the empty link. The product of two elements \( L \) and \( L' \) is defined as follows: \( L \cdot L' \) is the disjoint union of \( L \) and \( L' \) where \( L \) is pushed isotopically into \( F_{g,n} \times [1/2, 1] \) and \( L' \) is pushed isotopically into \( F_{g,n} \times [0, 1/2] \). Since the multiplication depends on the product structure on \( M \), then we will denote the skein algebra of \( F_{g,n} \times I \) by \( \mathcal{K}(F_{g,n}) \). Here are two examples.

**Theorem 3.1.1 [3]**

1) The algebra \( \mathcal{K}(F_{0,2}) \) is isomorphic to the polynomial algebra \( \mathcal{R}[b] \).

2) The algebra \( \mathcal{K}(F_{0,3}) \) is isomorphic to the polynomial algebra \( \mathcal{R}[x, y, z] \), where \( x, y \) and \( z \) are the three boundary components.

### 3.2 The graph skein module

This subsection is devoted to introduce the theory of graph skein modules of three manifolds. This construction is inspired by Przytycki’s algebraic topology based on knots. Namely, we extend Przytycki’s construction to a similar theory using ribbon embedded graphs instead of framed links and Yamada relations, Figure 5, instead of the Kauffman bracket skein relations. Here is an outline of the theory and some basic results.

Let \( M \) be an oriented three-manifold and let \( \mathcal{G} \) be the set of all embeddings of ribbon graphs in \( M \) considered up to isotopy. Let \( \mathcal{RG} \) be the free \( \mathcal{R} \)-module generated by \( \mathcal{G} \). The graph skein module of \( M \) with coefficients in \( \mathcal{R} \), \( \mathcal{Y}(M, \mathcal{R}, A) \) is defined to be the quotient of \( \mathcal{RG} \) by the smallest submodule containing all expressions of the form:

\[
\begin{align*}
-A^4 & - A^{-4} + d & \\
\text{or} & -d^{-1} & \\
\end{align*}
\]
Throughout the rest of the paper and when there is no need to specify the value of $A$, we write $\mathcal{Y}(M)$ instead of $\mathcal{Y}(M, \mathcal{R}, A)$. The fact that the Yamada polynomial is uniquely determined by the Yamada skein relations [19] translates in the language of graph skein modules as follows:

**Theorem 3.2.1.** The graph skein module $\mathcal{Y}(S^3)$ is isomorphic to $\mathcal{R}$ with the empty graph $\emptyset$ as a generator.

In [6], we have studied the skein algebra of the annulus and proved the following:

**Theorem 3.2.2.** The skein algebra $\mathcal{Y}(F_{0,2})$ is isomorphic to the polynomial algebra $\mathcal{R}[b]$, where $b$ is pictured in Figure 1.

Before we prove Theorem 1.1, we will explore the relationship between our graph-algebra and the Kauffman bracket skein algebra. We first introduce the Jones-Wenzel idempotents [10]. Let $n$ be an integer, an $n$-tangle $T$ is a one-dimensional sub-manifold of $\mathbb{R}^2 \times I$, such that the boundary of $T$ is made up of $2n$ points $\{(0, i, 1), (0, i, 0) \mid 0 \leq i \leq n-1\}$. Let $\mathcal{T}_n$ be the free $\mathcal{R}$-module generated by the set of all $n$-tangles. We define $\tau_n$ to be the quotient of $\mathcal{T}_n$ by the Kauffman bracket skein relations. It is well known that $\tau_n$ is isomorphic to the Temperley-Lieb algebra. A set of generators $(U_i)_{0 \leq i \leq n-1}$ of $\tau_n$ is given as follows.
Let \((f_i)_{0 \leq i \leq n-1}\) denote the family of Jones-Wenzl projectors in \(\tau_n\). This family is defined by the following recursive formulas:

\[
f_0 = 1_n,
\]

\[
f_{k+1} = f_k - \mu_{k+1} f_k U_{k-1} f_k,
\]

where \(\mu_1 = d^{-1}\) and \(\mu_{k+1} = (d - \mu_k)^{-1}\).

In particular, we have \(f_1 = 1_n - d^{-1} U_1\). The elements \(f_k\) enjoy the following properties: \(f_k^2 = f_k\) and \(f_i U_j = U_j f_i = 0\) for \(j \leq i\). See [10] for more details.

Let \(G\) be a graph diagram lying in a given oriented surface. Let \(f_1\) be the Jones-Wenzl projector in \(\tau_2\)

\[
f_1 = \bigg| \big| -d^{-1} \big| \bigg|
\]

We define \(G'\) to be the linear combination of graph diagrams obtained from \(G\) by replacing each edge of \(G\) by two planar strands with a projector \(f_1\) in the cable, and by replacing each vertex of \(G\) by a diagram as follows

![Diagram](image)

Here, writing an integer \(n\) beneath an edge \(e\) means that this edge has to be replaced by \(n\) parallel ones.

Now, let \(\varphi\) be the map from \(\mathcal{R}(G)\) to \(\mathcal{K}(F)\) defined on the generators by \(\varphi(G) = G'\) and extended by linearity to \(\mathcal{R}(G)\), see also [20] and [21].
It was proved in [6] that $\varphi$ defines a map $\Phi$ from the graph skein module $\mathcal{Y}(F \times I)$ to the Kauffman bracket skein module $\mathcal{K}(F \times I)$, where $F$ is any oriented surface. Obviously, $\Phi$ is a homomorphism of algebras. In the case of the annulus, we have seen that both $\mathcal{K}(F_{0,2})$ and $\mathcal{Y}(F_{0,2})$ are isomorphic to the polynomial algebra $\mathcal{R}[b]$. We can easily see that $\Phi(b) = b^2 - 1$. Thus, $\Phi$ is injective. Actually, $\Phi$ defines an isomorphism between the graph algebra $\mathcal{Y}(F_{0,2})$ and the even part of the Kauffman bracket skein algebra $\mathcal{K}(F_{0,2})$.

3.3 Proof of Theorem 1.1

By the reduction arguments used in the case of the annulus [6], we can prove that any graph diagram in $F_{0,3}$ can be written as a linear combination of finite disjoint unions of bouquets each of which has no contractible cycles. Arguments similar to the ones used to prove Lemma 3.3.2 in [6] would enable us to prove that the graph skein module of $F_{0,3} \times I$ is generated by elements of type $x^i y^j z^k t^\epsilon$, where $i, j$ and $k$ are nonnegative integers, and $\epsilon \in \{0, 1\}$.

To prove that the algebra is exactly as depicted in Theorem 1.1, we use the fact that the homomorphism $\Phi$ is injective. The injectivity of $\Phi$ is due to the following identities:

$\Phi(x) = x^2 - 1$, $\Phi(y) = y^2 - 1$, $\Phi(z) = z^2 - 1$ and

$\Phi(t) = xyz - d^{-1}x^2 - d^{-1}y^2 + d^{-1}$. Let $\mathcal{E} = \langle x^2, y^2, z^2, xyz \rangle$ be the even part of the algebra $\mathcal{K}(F_{0,3})$. Then, $\Phi$ defines an isomorphism between $\mathcal{Y}(F_{0,3})$ and $\mathcal{E}$. The inverse isomorphism $\Psi$ is defined as follows: $\Psi(x^2) = x + 1$, $\Psi(y^2) = y + 1$, $\Psi(z^2) = z + 1$ and $\Psi(xyz) = t + d^{-1}x + d^{-1}y + d^{-1}$.

Using the relations above we can see that:

$$xyz = \Phi(t + d^{-1}x + d^{-1}y + d^{-1}).$$

Hence:

$$\Phi(t^2) = \Phi(t)^2 = \Phi((x + 1)(y + 1)(z + 1) + d^{-2}(x + 1)^2 + d^{-2}(y + 1)^2 + d^{-2} - 2d^{-1}(t + d^{-1}x + d^{-1}y + 1)(x + y + 1) + 2d^{-1}(x + 1)(y + 1) - 2d^{-2}(x + y + 2)),$$

10
Since $\Phi$ is injective, then

\[
t^2 = 1 + d^{-2} - 2d^{-1} + (1 - 2d^{-1})x + (1 - 2d^{-1})y + z - 2d^{-1}t \\
  + (1 - 2d^{-2})xy + xz + zy - 2d^{-1}tx - 2d^{-1}ty - d^{-2}x^2 - d^{-2}y^2 + xyz.
\]

This ends the proof of Theorem 1.1.

In a joint work, T. Fleming and the author [7] studied the graph skein algebras of the torus $F_{1,0}$ and the punctured torus $F_{1,1}$. They determined a set of generators for each of these algebras. It was proved that the graph skein algebra $\mathcal{Y}(F_{1,0})$ is generated by the three torus curves $(1,0),(0,1),(1,1)$ and the wedge $(1,0) \vee (0,1)$. A similar statement was proved for the punctured torus.

4 Proof of Theorem 1.2

Statement (a) in Theorem 1.2 is concerned with the case of $\mathbb{Z}_p$-symmetries in which the spatial graph does not intersect the axis of the rotation. Such a spatial graph is called $p$-periodic. Marui [12], used the Yamada polynomial to study the periodicity of spatial graphs with wrapping number 1 or 2. In [6], we used the criteria of link periodicity introduced by Murasugi [13], Przytyki [15] and Traczyk [17] to obtain a generalization of Marui’s result. We proved the following [6]:

**Theorem 4.1.** Let $p$ be a prime and $\hat{G}$ a ribbon spatial graph. If $\hat{G}$ is $p$-periodic, then

(a) $Y(\hat{G})(A) \equiv (Y(\hat{G})(A))^p$ modulo $p, d^p - d$.

(b) $Y(\hat{G})(A) \equiv Y(\hat{G})(A^{-1})$ modulo $p, A^{2p} - 1$.

Where the congruences hold in the ring $\mathbb{Z}[A^\pm 1, d^{-1}]$.

Now, we shall start the proof of the first congruence relation in Theorem 1.2. We shall then explain how to extend the result to vertex-fixing $\mathbb{Z}_p$-symmetry as in statement (b). The idea of the proof is to change the coefficients in the Yamada skein relations in order to define a kind of equivariant graph skein module. We already know that this idea works well for the
study of symmetries of links \([4, 5]\). Let \(\mathcal{R}_p = \mathbb{Z}/p\mathbb{Z}[A^\pm, d^{-1}]\) and let \(\mathcal{S}_p\) be the free \(\mathcal{R}_p\)-module generated by all isotopy classes of ribbon graphs embedded in the solid torus. Now, let \(\mathcal{Q}_p\) be the submodule of \(\mathcal{S}_p\) generated by all elements of the form:

\[
\begin{align*}
\includegraphics{figure8}
\end{align*}
\]

\(D \sqcup \circ - (d^2 - 1)^p D, \text{ for any graph diagram } D\)

**Figure 8**

We define \(\mathcal{Y}_p\) to be the quotient of \(\mathcal{S}_p\) by the submodule \(\mathcal{Q}_p\). By universal coefficient property of the skein modules, we can prove that the module \(\mathcal{Y}_p\) is isomorphic to \(\mathcal{R}_p[b]\). Let \(\pi : S^1 \times D^2 \longrightarrow S^1 \times D^2\) denote the \(p\)-fold cyclic cover defined by the action of the rotation \(h\) on the solid torus. Let \(F\) (resp. \(F'\)) be the map from \(\mathcal{S}_p\) to \(\mathcal{R}_p\) defined on the set of generators of \(\mathcal{S}_p\) by \(F(g) = Y(\pi^{-1}(g))\) (resp. \(F'(g) = (Y(g))^p\)) and extended to \(\mathcal{S}_p\) by linearity. Using the fact that \(p\) is a prime and that the finite cyclic group of order \(p\) acts semi-freely on the set of states of the Yamada resolution of the diagram of the periodic spatial graph \(\tilde{G} = \pi^{-1}(g)\), we should be able to easily prove the following lemma (see also Lemma 3.4 in [15]).

**Lemma 4.2.** \(F(\mathcal{Q}_p) = F'(\mathcal{Q}_p) = 0\).

According to this lemma, \(F\) (resp. \(F'\)) defines a map from the skein module \(\mathcal{Y}_p\) to \(\mathcal{R}_p\). We will denote this map by \(\bar{F}\) (resp. \(\bar{F}'\)).

The module \(\mathcal{Y}_p\) is generated by \(\{b^k, k \geq 0\}\). Let \(I\) be the submodule generated by \(\{\bar{F}(b^k) - \bar{F}'(b^k), k \geq 0\}\). Simple computations show that \(I\) is equal to the ideal generated by \((d^2 - 1)^p - \ldots\)
\((d^2 - 1)\). Consequently:

\[
\bar{F}(g) \equiv \bar{F}'(g) \mod (d^2 - 1)^p - (d^2 - 1).
\]

This ends the proof of statement (a).

Now we shall consider the case of vertex fixing \(\mathbb{Z}_p\)-symmetry. Assume that \(\tilde{G}\) is a spatial graph which is invariant by a rotation of order \(p\) such that the axis of the rotation intercepts the graph only at one vertex \(v\). Take a diagram of \(\tilde{G}\) which is invariant by a planar rotation centered at \(v\). If we use Yamada relations to reduce the graph diagram of \(\tilde{G}\), then we only need to consider equivariant states, since the contribution of non-equivariant states sums to zero modulo \(p\). Each resolution is actually a diagram which is made up of bouquets. We distinguish two cases:

- If the diagram of the state does not contain any bouquet centered at the vertex \(v\). In this case, we have a periodic diagram and we use the statement (a) to conclude that the contribution of the state is congruent to the contribution of the quotient state modulo \(p\), \(d^{2p} - d^2\).

- If the diagram of the state contains a bouquet \(B\) of \(kp\)-leaves centered at \(v\), then the quotient state should contain a bouquet \(\bar{B}\) with \(k\)-leaves centered at \(v\). Since \(Y(B) = (d - d^{-1})^{kp-1}(d^2 - 1)\) and \(Y(\bar{B}) = (d - d^{-1})^{k-1}(d^2 - 1)\), then the contribution of \(B\) to \(Y(\tilde{G})\) and the contribution of \(\bar{B}\) to \(Y(\tilde{G})^p\) are congruent modulo \(p\) and \(d^{p-1} - 1\). This ends the proof of statement (b).

It remains to prove the third statement in Theorem 1.2. Let \(\Delta\) be an unknotted circle in \(S^3\) and let \(\Gamma\) be a spatial graph in \(S^3\) which either does not intersect \(\Delta\) or intersects \(\Delta\) exactly at one vertex \(v\). Now, let \(\tilde{\Gamma}\) be the pre-image of \(\Gamma\) in the cyclic \(p\)-fold cover branched along \(\Delta\). We consider the two maps \(\alpha\) and \(\beta\) defined on the graph skein module of \(S^3\) as follows: \(\alpha(\Gamma) = Y(\tilde{\Gamma})(A)\) and \(\beta(\Gamma) = Y(\tilde{\Gamma}!)(A)\), where \(\tilde{\Gamma}!\) is the mirror image of \(\tilde{\Gamma}\).

Since \(Y(\tilde{\Gamma})(A) = Y(\tilde{\Gamma})(A^{-1})\) \([19]\), then arguments similar to the ones used in the previous paragraph should enable us to prove that both of the polynomials \(\alpha\) and \(\beta\) satisfy the following relations modulo \(p\),

\[
\alpha(\Gamma_+)(A) \equiv A^{4p} \alpha(\Gamma_0)(A) + A^{-4p} \alpha(\Gamma_\infty)(A) - d^p \alpha(\Gamma_\times),
\]
and

$$\beta(\Gamma_+)(A) \equiv A^{-4p}\beta(\Gamma_0)(A) + A^{4p}\beta(\Gamma_\infty)(A) - d^p\beta(\Gamma_\times)(A),$$

where $\Gamma_+$, $\Gamma_0$, $\Gamma_\infty$ and $\Gamma_\times$ are respectively the four graph diagrams which appear in the first Yamada skein relation, see figure 5.

It is obvious that if $A^{4p} = A^{-4p}$, then $\alpha$ and $\beta$ are defined using the same skein relations. Hence, $\alpha \equiv \beta$ modulo $p, A^{8p} - 1$. Since any spatial graph with $\mathbb{Z}_p$-symmetry or a vertex-fixing $\mathbb{Z}_p$-symmetry can be constructed as $\tilde{\Gamma}$ for some spatial graph $\Gamma$, then we conclude that the congruence in statement (c) holds. This ends the proof of Theorem 1.2.

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