Contribution of Kaluza-Klein modes to vacuum energy in models with large extra dimensions & the Cosmological constant

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Abstract

In this paper, the generation of topological energy in models with large extra dimensions is investigated. The origin of this energy is attributed to a topological deformation of the standard Minkowski vacuum due to compactification of extra dimensions. This deformation is seen to give rise to an effective, finite energy density due to massive Kaluza-Klein modes of gravitation. It’s renormalized value is seen to depend on the size of the extra dimensions instead of the UV cut-off of the theory. It is shown that if this energy density is to contribute to the observed cosmological constant, there will be extremely stringent bounds on the number of extra dimensions and their size.

Introduction

The Einstein equations with the cosmological constant are given by

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = g_{\mu\nu} \Lambda - 8\pi G T_{\mu\nu} \]  

where \( \Lambda \) is the cosmological constant, or vacuum energy. Modulo the factor of \( 8\pi G \), this constant is identical to the stress tensor associated with vacuum. It can

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be seen on grounds of covariance that the vacuum expectation value of the stress tensor is of the form [1]

\[ \langle T_{\mu \nu} \rangle = \rho_{\text{vac}} g_{\mu \nu} \]  

(2)

No known symmetry forbids a cosmological term in Einstein equations. Based on observations, it is seen that in suitable units, \( \Lambda \) is at most of the order of \( 10^{-47} \) (GeV)\(^4\). At the same time, the vacuum energy evaluated with an ultraviolet cut-off at the Planck scale \( M_{\text{pl}} \sim 10^{19} \) GeV gives

\[ \langle E_{\text{vac}} \rangle \sim \frac{1}{2} \int_0^{M_{\text{pl}}} \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + m^2} \sim M_{\text{pl}}^4 \]  

(3)

which is about 120 orders of magnitude higher. Attempts have been made to account for this discrepancy [2, 3] and it has been realized that even if the cut-off is taken at the Supersymmetry (SUSY) scale, the discrepancy is still about 60 orders of magnitude. It is seen that an ultraviolet cut-off of the order of \( 10^{-3} \) eV is required to account for the observed value of the vacuum energy. Therefore, even very low energy quantum effects at sizes much larger than the size of atoms lead to untolerable contributions to the cosmological constant.

Several suggestions such as SUSY, scalar dilatons, 3-form fields and quintessence have been made, but no satisfactory resolution is seen to emerge (for a review, see [4]).

Recently, there have been developments in Kaluza-Klein theories that claim to remove the so-called hierarchy problem between the Electroweak and Planck scales [5, 6]. In these recent models, it is assumed that all matter is confined to a four dimensional hypersurface, a 3-brane, in a higher dimensional spacetime. Gravity, unlike matter, is assumed to propagate in all dimensions. There are essentially two models and their variants. The first model, the ADD model [5], assumes the existence of \( n \) large extra dimensions with size of the order of a millimeter and a bulk scale of gravitation of the order of a TeV, instead of \( M_{\text{pl}} \). The metric of spacetime is a direct product of the usual 4-dimensional spacetime and the compact extra dimensional space. Due to the large volume of the compact space, gravitation on the 3-brane is diluted down to the observed Planck scale, instead of a TeV scale, which is the gravitational scale of the bulk spacetime. The large size of the extra dimensions removes the hierarchy between the two scales and is not in conflict with micro-experiments, since matter cannot directly probe these extra dimensions, being stuck on the 3-brane. The most studied model from a cosmological point of view is the RS model [6] and it’s variants. In this model, there is only one extra dimension and one or more branes. This model differs from the ADD model in that spacetime is not a direct product of the four dimensional spacetime and the extra dimensional manifold (as is the case in the ADD scenario),
but has a warp factor multiplying the four dimensional metric. This warp factor is a function of the extra dimension. There are cosmological constant terms on the 3-branes and in addition, a bulk cosmological constant. The 5-dimensional equations are solved to obtain the 4-dimensional effective gravitational constant and the effective induced 4-dimensional cosmological constant. However, fine-tuning is required to get the effective cosmological constant to agree with the observational value. For a recent review on various approaches used to solve the cosmological constant problem, see [4].

In the ADD model, the vacuum solution is taken to be the standard bulk Minkowski metric, with the n extra dimensions compactified on an n-torus. A torus is obviously a zero curvature manifold. However, it is topologically very dissimilar to a situation in which the extra dimensions are non-compact. As a result, the vacuum of the former manifold is dissimilar to the vacuum of the latter, non-compact manifold. It is a standard result in quantum field theory that if a quantum field propagates in all the dimensions, the vacuum energy associated with it will be different in the two topologically dissimilar manifolds considered above. To get a finite, renormalized value for the vacuum energy associated with the compact manifold, the non-compact contribution is subtracted out. (For a detailed discussion, see [7]). This is equivalent to stating that the absolute zero of energy is taken to be a state in which all dimensions are non-compact, and the metric Minkowskian. In the ADD model, the gravitational field is the only field which propagates in all the dimensions, with the matter fields being forever stuck on the 3-brane. Even in the so-called vacuum state, since the extra dimensions are compact, there will be a vacuum stress due to compactification, giving rise to zero point vacuum fluctuations which after renormalization will be shown to be finite. It will be seen that this net vacuum energy is nothing but the renormalized zero-point energy of all the massive Kaluza-Klein modes associated with the gravitational field. In the following, this vacuum energy is explicitly calculated for two extra compact dimensions, and the result generalized to n extra dimensions. It is seen that if this vacuum energy is to contribute to the observed cosmological constant, this imposes serious constraints on the number of extra dimensions and their size.

A Toy Model

We begin by studying a toy model in 1+1 dimensions, in which the single spatial dimension is compact (See [7]) and spacetime is $R^1 \times S^1$. The spatial points $x$ and $x + R$ are identified, where $R$ is the periodicity length or 'circumference' of
the spatial section. Consider a massless scalar field propagating in this spacetime. Then, the effect of the compactification is to restrict the modes of propagation of the scalar field to the form

$$\phi_k = \frac{1}{\sqrt{2R\omega}} e^{i(kx-\omega t)}$$

(4)

where $k = (2m\pi)/R$ and $\omega = |k|$, $m$ being an integer. The field $\phi$ is expanded in terms of creation and annihilation operators and the modes $\phi_k$ as

$$\phi(t, x) = \sum_m [a_m\phi_m + a_m^\dagger\phi_m^*]$$

(5)

The vacuum expectation value of the energy density is calculated to be

$$\langle 0_R|T_{00}|0_R \rangle = \frac{2\pi}{R^2} \sum_{m=0}^{\infty} m$$

(6)

where $|0_R\rangle$ is the vacuum state for this compact manifold and is defined by $a_m|0_R\rangle = 0$. As expected, this is infinite. However, we renormalize it by demanding that the vacuum energy associated with the manifold $R^1 \times R^1$ should be zero, which is the standard Minkowski manifold in the limit $R \to \infty$. This renormalization implies that we subtract the vacuum contribution as $R \to \infty$ from the expression given in eqn.(6), i.e., the renormalized energy density of the compactified vacuum is given by

$$\langle 0_R|T_{00}|0_R \rangle_{\text{ren}} = \langle 0_R|T_{00}|0_R \rangle - \langle 0_R|T_{00}|0_R \rangle |_{R \to \infty}$$

(7)

This can be achieved by standard regularization techniques (see [7]) in which the vacuum expectation value is regularized by a cutoff and the limit $R \to \infty$ is subtracted from the result and finally, the regularization removed. The result is finite, giving

$$\langle 0_R|T_{00}|0_R \rangle_{\text{ren}} = -\frac{\pi}{6R^2}$$

(8)

One convenient regularization is the zeta function regularization (see [7]), in which the expression $\sum_{m=0}^{\infty} m$ is identified with the analytically continued value of the Reimann zeta function $\zeta(s)$ at the point $s = -1$. This is finite, and is given by $\zeta(-1) = -1/12$, giving the vacuum energy in eqn.(8) directly, without any explicit subtraction. This technique is very common in the study of quantum field theory in curved spacetime. This is the technique we shall employ in this paper.
Calculation Of Vacuum Energy In The ADD Model

The calculation of the vacuum energy due to compactification will now be carried out in the ADD scenario. We shall consider a spacetime which is a direct product of a non-compact 4-dimensional spacetime and two compact spatial extra dimensions compactified to a 2-torus. The matter fields forever reside on the 3-brane (the non-compact 4-dimensional spacetime) and gravity propagates in the bulk. The vacuum metric is Minkowskian. We consider the propagation of linearized gravity in this bulk spacetime. The Lagrangian for the system is the Fierz-Pauli Lagrangian [8] given by

\[ \mathcal{L} = \frac{1}{4} \left( \partial^\mu \hat{h}^{\nu\rho} \partial_\mu \hat{h}_{\nu\rho} - \partial^\mu \hat{h} \partial_\mu \hat{h} - 2 \hat{h} \hat{h}_{\mu\nu} + 2 \hat{h} \hat{h}_{\mu} \right) \] (9)

where \( \hat{h} = \hat{h}_\mu^\mu, \hat{h}_\nu = \partial^\mu \hat{h}_{\mu\nu} \) and the Greek indices run over the 6 bulk coordinates. Raising and lowering of indices has been carried out using the bulk Minkowski metric \( \eta^\mu_\nu = \text{diag}(1, -1, -1, -1, -1) \). The equations of motion in the de Donder gauge are the d’Alembert equations

\[ \Box_{(4+2)} \left( \hat{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \hat{h} \right) = 0 \] (10)

Now, a Kaluza-Klein reduction is carried out as (see [8])

\[ \hat{h}_{\mu\nu} = \frac{1}{R} \left( h_{\mu\nu} + \eta_{\mu\nu} \phi A_{\mu}, A_{\nuj} \right) \] (11)

where \( R \) is the size of each extra spatial dimension, \( \phi = \phi_i, \mu, \nu = 0, 1, 2, 3 \) and \( 1, j = 1, 2 \). The bulk linearized field can be expanded in terms of creation and annihilation operators. For notational simplicity, we shall suppress the tensor indices and the polarization states of the bulk graviton field and denote it by \( \Phi \). Then, in terms of the modes, it is given by

\[ \Phi(x, y) = \frac{1}{R} \sum_{n_1, n_2} \int \frac{d^3k}{(2\pi)^3 2\omega(k)_{n_1, n_2}} \left[ a(k)_{n_1, n_2} e^{-ik \cdot x} e^{i\frac{2\pi}{R} (n_1 y_1 + n_2 y_2)} + h.c. \right] \] (12)

where \( x \) denotes the four brane spacetime coordinates and \( y \) are the two compact coordinates. The commutation relations satisfied by the operators \( a(k)_{n_1, n_2} \) are

\[ [a(k)_{n_1, n_2}, a(k')_{m_1, m_2}] = \delta_{n_1 m_1} \delta_{n_2 m_2} 2(2\pi)^3 \omega(k)_{n_1, n_2} \delta(k - k') \] (13)

where \( \omega(k)_{n_1, n_2} = \sqrt{k^2 + 4\pi^2 (n_1^2 + n_2^2)/R^2} \).

The vacuum state is defined as \( a(k)_{n_1, n_2} |0_R \rangle = 0 \). The vacuum expectation
value of the stress-tensor associated with the field $\Phi$ is calculated and identifying the quantity $R\Phi$ as the effective four dimensional field (analogous to eqn.(11)), the four dimensional vacuum energy density is given by

$$\langle 0_R | T_{00} | 0_R \rangle = \langle E_{\text{vac}} \rangle = \sum_{n_1,n_2=1}^{\infty} \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + \frac{4\pi^2(n_1^2 + n_2^2)}{R^2}}$$

(14)

To regularize the $k$ integration, we write the integral as

$$I(m_n) = \mu^{2\epsilon} \int \frac{d^3k}{(2\pi)^3} (k^2 + m_n^2)^{\frac{1}{2} - \epsilon}$$

(15)

where $m_n^2 = \frac{4\pi^2(n_1^2 + n_2^2)}{R^2}$ and $\mu$ is a parameter with dimensions of mass. The parameter $\epsilon$ is introduced for regularization, and will finally be taken to be zero. In this form, eqn.(14) is seen to represent simply the zero point energy of the massive Kaluza-Klein tower on the brane. The $k$ integral is given in terms of the beta function as

$$\begin{align*}
I(m_n) &= \frac{m_n^4}{4\pi^2} \left( \frac{\mu^2}{m_n^2} \right)^\epsilon B(3/2, -2 + \epsilon) \\
&= -\frac{m_n^4}{32\pi^2} \left( \frac{1}{\epsilon} + 2 \log 2 - \frac{1}{2} - \log \left( \frac{m_n^2}{\mu^2} \right) \right)
\end{align*}$$

(16)

where the final expression holds for infinitesimal $\epsilon$. Then, the vacuum energy is given by

$$\langle E_{\text{vac}} \rangle = -\frac{1}{32\pi^2} \sum_{n_1,n_2=1}^{\infty} m_n^4 \left( \frac{1}{\epsilon} + 2 \log 2 - \frac{1}{2} - \log \left( \frac{m_n^2}{\mu^2} \right) \right)$$

(17)

The $k$ integral has been regularized, but the summation over modes is still to be regularized and the renormalized value obtained. This operation is carried out using the zeta-function regularization. We identify the summations over $n_1$ and $n_2$ as analytically continued values of the zeta-function and it’s derivatives. Then,

$$\sum_{n_1,n_2=1}^{\infty} m_n^4 = \left( \frac{4\pi^2}{R^2} \right)^2 \sum_{n_1,n_2=1}^{\infty} (n_1^4 + n_2^4 + 2n_1^2n_2^2)$$

$$= \left( \frac{4\pi^2}{R^2} \right)^2 2[\zeta(0)\zeta(-4) + \zeta^2(-2)]$$

$$= 0$$

(18)
since \( \zeta(-2m) = 0 \), if \( m \) is a natural number. Therefore, the resulting vacuum energy is given by

\[
\langle E_{\text{vac}} \rangle = \frac{1}{32\pi^2} \sum_{n_1,n_2=1}^{\infty} m_2^4 \log \left( \frac{m_2^2}{\mu^2} \right) (n_1^2 + n_2^2) \log(n_1^2 + n_2^2)
\]

\[
= \frac{\pi^2}{2R^4} \sum_{n_1,n_2=1}^{\infty} (n_1^2 + n_2^2) \log(n_1^2 + n_2^2)
\]

\[
= -\frac{\pi^2}{R^4} \zeta(0) \zeta'(-4)
\]  

(20)

where \( \zeta(0) = -1/2 \) and \( \zeta'(-4) = 0.008 \). Since we had suppressed the graviton polarization indices in making this calculation, we multiply the final result with the number of degrees of freedom for a graviton in 4 + 2 dimensions, which is 9. Therefore, the final vacuum energy is given by

\[
\langle E_{\text{vac}} \rangle_{4+2} = -\frac{9\pi^2}{R^4} \zeta(0) \zeta'(-4)
\]  

(21)

A straightforward calculation generalizes the result to 4 + \( n \) dimensions, and is given by

\[
\langle E_{\text{vac}} \rangle_{4+n} = -\frac{\pi^2}{R^4} \left[ \frac{(2+n)(3+n)}{2} \right] [\zeta(0)]^{n-1} \zeta'(-4)
\]  

(22)

where the factor in the brackets is just the number of graviton degrees of freedom in 4 + \( n \) dimensions.

**Constraints On The ADD Model**

If the vacuum energy associated with the massive Kaluza-Klein modes is to account for the origin of the cosmological constant, it would put severe constraints on the number of extra dimensions and the compactification radius of the extra dimensions. Firstly, it is to be noted that the sign of the vacuum energy depends on the number of the extra dimensions, being positive for even and negative for odd number of extra dimensions. We would like to put bounds on \( n \) and \( R \) in the above expression by demanding that

\[
\langle E_{\text{vac}} \rangle_{4+n} \leq 10^{-47}(GeV)^4
\]  

(23)

Then, it is seen that the bound on \( R \) is not very sensitive to the number of extra
dimensions, $n$. It is seen that for $n$ varying from 2 to 10, the above inequality approximately constrains $R$ to essentially the same lower value, $R \geq 0.1$ mm. However, there is a consistency condition relating the 4-dimensional Planck scale $M_{pl}$ and the bulk scale of gravitation, $M_s$. This relation goes as [8]

$$M_{pl}^2 \sim R^n M_s^{n+2}$$

(24)

For $R$ constrained to $R \geq 0.1$ mm., this consistency equation gives sensible values of $M_s$ for $n = 1, 2$ only. For $n = 2$, this gives $M_s \leq 1$ TeV, which is acceptable and interesting from the hierarchy problem point of view. For $n = 3$, $M_s$ is constrained to lie below a few GeV, which is phenomenologically ruled out. Higher values of $n$ become more and more unacceptable. The situation for $n = 1$ gives $M_s \leq 10^8$ GeV, the equality holding for $R = 0.1$ mm, and the inequality for $R > 0.1$ mm. Therefore, since $R > 0.1$ mm. will be in conflict with macroscopic gravitational experiments, this situation is acceptable only for $M_s \sim 10^8$ GeV, which would however not get rid of the hierarchy problem.

$n = 2$ therefore seems to be the only viable possibility which would solve the hierarchy problem and not be in conflict with high-energy experiments at the same time. However, since the lower constraint on $R$ is of the order of 0.1 mm., this simply means that macroscopic tests of gravitation ought to reveal deviations from Newton’s law of gravitation at distances comparable to this. Thus, if the ADD model is the correct description of spacetime and if this vacuum energy is to contribute to the observed cosmological constant, then deviations from Newtonian gravity at scales of the order of 0.1 mm. should show up. Any failure in observing such deviations at length scales $\geq 0.1$ mm. will imply that unless there exists some deeper symmetry/mechanism that removes this finite vacuum energy associated with compactification, the ADD model fails to give an accurate description of spacetime.

**Conclusions**

We have calculated the topological vacuum energy of linearized gravity in the ADD scenario and shown that if this is to account for the observational bounds on the cosmological constant, the size of the compact dimensions should be $\geq 0.1$ mm. The vacuum energy is shown to be just the zero point energy of the massive Kaluza-Klein modes. The calculation has been carried out for a toroidal topology of the compact manifold, but the essential features of the calculation are not expected to change if some other locally flat non-trivial topology is taken. The result after zeta-function regularization and appropriate renormalization condition
is seen to be finite, and not sensitive to the UV cut-off of the theory, as would have been dimensionally expected. The reason is the subtraction of infinite (or large but finite in case of a finite UV cut-off) vacuum energy associated with the trivial topology when the extra dimensions are not curled up or compact. The result has been generalized for the case of an n-torus. If this vacuum energy is to account for observational constraints on the cosmological constant, this would put severe constraints on the volume of the compact manifold and it’s dimensionality.

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