Closed geodesics and bounded gaps

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Abstract: The bounded gaps property of the prime numbers, as proven by Yitang Zhang, is considered for sequences of lengths of closed geodesics, which by the theory of Selberg zeta functions are the geometric analogue of the prime numbers. It turns out that the property holds for congruence subgroups and it is conjectured to be false for a dense set in Teichmüller space.

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Introduction

In the paper [Zha13], Yitang Zhang proved that there are infinitely many primes $p, q$ such that $|p - q| < C$, where $C > 0$ is an explicit constant.

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Closed geodesics and bounded gaps

Viewing Ruelle zeta functions as a geometric analogue of the Riemann zeta functions, one is led to ask whether a similar result holds for the latter.

For a sequence \((x_n)\) of real numbers \(x_n > 0\) with \(x_{n+1} > x_n\) we define the limit gap of the sequence as

\[
\text{LG}((x_n)) = \lim \inf \frac{|x_{n+1} - x_n|}{n},
\]

Zhang’s result says that for the sequence \(\pi\) of prime numbers one has \(\text{LG}(\pi) < 7 \times 10^7\), where before it wasn’t even known to be finite. The number \(\text{LG}(\pi)\) is an integer \(\geq 2\) and the twin prime conjecture says that \(\text{LG}(\pi)\) should be equal to 2.

We say that an increasing sequence \((x_n)\) has bounded gaps, if \(0 < \text{LG}(\gamma) < \infty\). Zhang has proved that the ascending enumeration of the prime numbers has bounded gaps. Geometric zeta functions of Ruelle, Selberg and Ihara are defined by Euler products which don’t run over the primes but over lengths of closed geodesics in locally symmetric spaces or finite graphs respectively. The sequences of lengths (with their multiplicities) are also called the length spectra of the underlying spaces. So the lengths spectra may be viewed as a geometric analogue of the primes. This paper is concerned with the question which length spectra have bounded gaps. It is shown that congruence subgroups of \(\text{SL}_2(\mathbb{Z})\) define spaces with the bounded gaps property. On the other hand, there is reason to believe that there is a dense set in Teichmüller space where the bounded gaps property does not hold. We formulate the latter assertion as a conjecture. It remains open whether general arithmetic groups do have bounded gaps. They would, if the property was hereditary to subgroups of finite index, which is not known.

As an analogue of the Teichmüller space we define a moduli space of metric graphs and we find the behavior predicted by the conjecture in a simple case, where explicit calculations can be done.

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1 Riemann surfaces

Let $X$ denote a connected Riemann surface equipped with a hyperbolic metric such that $X$ has finite volume. Then the universal covering of $X$ is the upper half plane $\mathbb{H}$ in $\mathbb{C}$ and so $X$ is the quotient $\Gamma \backslash \mathbb{H}$, where $\Gamma$ is the fundamental group of $X$, acting as biholomorphic maps on $\mathbb{H}$, hence $\Gamma$ may be viewed as a discrete subgroup of $\text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/\pm 1$, which through the action by linear fractionals, \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is identified with the group of biholomorphic maps on $\mathbb{H}$.

The Ruelle zeta function \cite{Rue76,BO95} is defined as
\[ R(s) = \prod_c (1 - N(c)^{-s}), \]
where the product runs over all primitive closed geodesics on $X$ and $N(c) = e^{l(c)}$, where $l(c)$ is the length of the closed geodesic $c$. Here a closed geodesic is called primitive, if it is not a power of a shorter one. Using the identification between closed geodesics and hyperbolic conjugacy classes in $\Gamma$, one extends this theory to include non torsion-free lattices $\Gamma$ like $\text{PSL}_2(\mathbb{Z})$. It is known, \cite{Rue76,BO95,DE09}, that $R(s)$ extends to a meromorphic function on $\mathbb{C}$ and that in the compact case all its zeros and poles lie in the union of $\mathbb{R}$ with the two vertical lines $\{ \text{Re}(s) = \pm \frac{1}{2} \}$. If one compares the Ruelle function with the Riemann zeta function, the role of the primes is taken over by the numbers $N(c)$. So let $N(\Gamma)$ denote the sequence of values $N(c)$ in ascending order. Note that multiplicities occur, i.e., there may be different closed geodesics $c, c'$ which are not equal and not inverse to each other, such that $l(c) = l(c')$. We plainly ignore multiplicities, as they don’t influence our question. We shall write $\text{LG}(\Gamma)$ for $\text{LG}(N(\Gamma))$. We call $\Gamma$ a bounded gaps group if the sequence $N(\Gamma)$ has bounded gaps, i.e., if $0 < \text{LG}(\Gamma) < \infty$ holds.

If $\Sigma$ is a subgroup of finite index of $\Gamma$, we get
\[ \text{LG}(\Sigma) \geq \text{LG}(\Gamma). \]
Therefore, if $\Gamma' \subset \Gamma \subset \Gamma''$ and $\Gamma'$ and $\Gamma''$ both have bounded gaps, then so has $\Gamma$.

Question 1.1. If $\Sigma$ is a finite index subgroup of $\Gamma$, is it true that
\[ \text{\Sigma has bounded gaps } \iff \text{\Gamma has bounded gaps?} \]
In the next section we shall prove that congruence subgroups of $\text{PSL}_2(\mathbb{Z})$ and unit groups of quaternion algebras have bounded gaps. Since any arithmetic group is commensurable to one of these, a positive answer to the above question would imply that every arithmetic groups has bounded gaps.

**Conjecture 1.2.** One has $LG(\Gamma) = 0$ for $\Gamma$ in a dense subset of the Teichmüller space.

To support the conjecture, we let $\Gamma$ be a lattice in $G$ and for $x > 0$ let $v(x)$ be the number of closed geodesics $c$ with $N(c) \leq x$. By the Prime Geodesic Theorem [DE09] one has

$$v(x) \sim \frac{x^2}{\log(x^2)}$$

as $x \to \infty$. The point is, that there can be multiplicities, i.e., different closed geodesics can share the same length. But as $\Gamma$ moves around in Teichmüller space, all lengths of closed geodesics change continuously, so it seems likely that there is a dense set of $\Gamma$ for which there are no multiplicities. If $\Gamma$ has no multiplicities, the Prime Geodesic Theorem implies that $LG(\Gamma) = 0$.

## 2 Arithmetic groups

For $N \in \mathbb{N}$, $N \geq 2$, we consider the principal congruence subgroup $\Gamma(N)$, which is the kernel of the group homomorphism $\text{PSL}_2(\mathbb{Z}) \to \text{PSL}_2(\mathbb{Z}/N\mathbb{Z})$.

**Theorem 2.1.** (a) Let $\Gamma = \text{PSL}_2(\mathbb{Z})$. Then $LG(\Gamma) = 1$, so $\Gamma$ has bounded gaps.

(b) Let $N$ be an integer $\geq 2$. Then the congruence subgroup $\Gamma(N)$ has bounded gaps.

**Proof.** Let $\bar{\gamma} \in \Gamma$ be a hyperbolic element and let $\gamma \in \text{SL}_2(\mathbb{Z})$ be a preimage of $\gamma$. Let $M_2(\mathbb{Q})$ be the algebra of $2 \times 2$ matrices and let $M_2(\mathbb{Q})_{\gamma}$ be the centralizer of $\gamma$. This then is a real quadratic field and $\mathcal{O}_{\gamma} = M_2(\mathbb{Q})_{\gamma} \cap M_2(\mathbb{Z})$ is an order in this field, so it is isomorphic to some

$$\mathcal{O}_D = \left\{ \frac{a + b\sqrt{D}}{2} : a \equiv bD \mod (2) \right\},$$
where $D \in \mathbb{N}$, $D \equiv 0, 1 \mod (4)$ not a square. In this way one gets a map from the set of hyperbolic conjugacy classes in $\Gamma$ to the set of isomorphism classes of orders in real quadratic fields, which are parametrized by $D$ as above, under which a given order $\mathcal{O}_D$ has exactly $h(D)$ preimages, where $h(D)$ is the class number (see [Dei02]).

The unit group $\mathcal{O}_D^\times$ consist of all $\alpha \in \mathcal{O}_D$ of norm $N(\alpha) = a\overline{a} = \pm 1$, where we have written $a + b\sqrt{D} = a - \sqrt{D}$ for the non-trivial Galois homomorphism. The subgroup $\mathcal{O}_D^1$ of all $\alpha$ with $N(\alpha) = 1$ has index one or two. This group is isomorphic to $\{\pm 1\} \times \varepsilon_D \mathbb{Z}$ for a unique fundamental unit $\varepsilon_D > 1$.

If $\gamma$ is primitive, it gets mapped to $\pm \varepsilon_D$ and we choose $\gamma$ in a way that it gets mapped to $\varepsilon_D$. We then find that actually, $\varepsilon_D = N(\gamma) = e^{l(\gamma)}$.

We say that $D \equiv 0, 1 \mod (4)$ is a fundamental discriminant, if $D$ is square-free or $D/4$ is square-free respectively. A variation of the arguments in [Spr73] shows that the number of $\varepsilon_D \leq x$ for $D$ being a fundamental unit, is $x + O(x^{1/2})$. Hence the number of $\varepsilon_D \leq x$ grows at least like $x$. In the expression $\varepsilon_D = \frac{a + b\sqrt{D}}{2}$ we can replace $b$ by $1$ and $D$ by $D^2$ without changing the unit $\varepsilon_D$. Because of $a^2 - b\sqrt{D} = 4$ we get $\varepsilon_D = \frac{a + \sqrt{a^2 - 4}}{2}$. In particular, $\varepsilon_D$ is determined by $a$ and it grows like $a$ as $a \to \infty$. Together this means that the number of all $\varepsilon_D \leq x$ is asymptotically equivalent to $x$, which implies the claim (a).

Now for (b). We have already seen that $\varepsilon_D = \frac{a + b\sqrt{D}}{2}$ is determined by $a$. As the number of $\varepsilon_D \leq x$ grows like $x$, it follows that the set of all $a \in \mathbb{N}$, such that $\frac{a + \sqrt{a^2 - 4}}{2}$ is unit $\varepsilon_D$, has density one. Therefore the set of all such $a$ satisfying $a \equiv 2 \mod (N^4)$ has positive density $\frac{1}{N^4}$. Therefore, if we show that for every such $a$, there is an element of $\Gamma(N)$, which as element of $\Gamma(1)$ corresponds to $\varepsilon_D = \frac{a + \sqrt{a^2 - 4}}{2}$, we have finished the proof.

So let $a$ satisfy $a \equiv 2 \mod (N^4)$. Write $a^2 - 4 = b^2D$, where $D$ is a fundamental discriminant, so $b^2$ contains all square factors away from 2. By the choice of $a$, the number $N$ divides $b$. We show that the element

$$\gamma = \left( \frac{a + bD}{b}, \frac{bD(D-1)}{a-bD} \right)$$

does the job. Since $a \equiv bD \mod 2$, the diagonal entries are integers. Since
either $D$ or $D-1$ is divisible by 4, the off-diagonal entries are integers as well. The trace is $a$ and the determinant is one, so $\gamma$ corresponds to $\varepsilon_D$ indeed. It remains to show that $\gamma$ lies in $\Gamma(N)$. As $N$ divides $b$, the off-diagonal entries are divisible by $N$, so $\gamma$ is congruent to the diagonal matrix $\text{diag}(a + bD^2, a - bD^2)$ modulo $N$. Write $a = 2 + cN^4 = 2 + lN$, so $l = cN^4$ and write $b = kN$, then

$$\frac{a + bD}{2} = \frac{2 + (l + kD)N}{2} = 1 + \frac{l + kD}{2}N.$$ 

If $N$ is odd, $l + kD$ is even and $\frac{a + bD}{2} \equiv 1 \equiv \frac{a - bD}{2}$ mod $(N)$. If $N$ is even, so is $l$ and one has $k^2N^2D = b^2D = a^2 - 4 = 4cN^4 + c^2N^8$, which implies that $k$ is even and so $\gamma \in \Gamma(N)$.

Let $M$ be a non-split quaternion algebra over $\mathbb{Q}$ which splits over $\mathbb{R}$. This means that there is an isomorphism $\phi : M(\mathbb{R}) \to M_2(\mathbb{R})$ with $N_{\text{red}}(\alpha) = \det(\phi(\alpha))$ for every $\alpha \in M(\mathbb{R})$, where $N_{\text{red}}$ is the reduced norm. For any ring $R$, let $M^1(R)$ denote the group of all $\alpha \in M(R)$ with $N_{\text{red}}(\alpha) = 1$. Then $\phi$ induces an isomorphism $M^1(R) \cong \text{SL}_2(\mathbb{R})$.

**Theorem 2.2.** The arithmetic group $\Gamma = \phi(M^1(\mathbb{Z}))/\pm 1 \subset \text{PSL}_2(\mathbb{R})$ has bounded gaps.

**Proof.** Let $S$ be the finite the set of primes $p$ at which $M$ does not split and let $R = \prod_{p \in S} p$ be its radical. Note that $S$ has at least two elements. For a given $c \in \mathbb{N}$ which is coprime to $2R$ we set $a = 2 + 4cR$ and $D = a^2 - 4 = 16cR(1 + cR)$. Write $D = m^2\Delta$, where $\Delta$ is square-free. Then the discriminant $\text{disc}(K)$ of $K = \mathbb{Q}(\sqrt{D})$ is equal to $\Delta$ if $\Delta \equiv 1 \mod 4$ and $4\Delta$ otherwise. We now argue that every $p \in S$ is ramified in $K$, i.e. $p|\text{disc}(K)$.

If $p$ is odd, then $p$ divides $D$, but $p^2$ does not, hence $p|\Delta$, so $p$ divides the discriminant. If $2 \in S$, then from $D = 16cR(1 + cR)$ we see that $2|\Delta$, but $4 \nmid \Delta$, so $\Delta \equiv 2 \mod 4$, which means that $\text{disc}(K) = 4\Delta$ is even. This implies that all primes in $S$ are ramified in $K$. Next we need to show that $\mathcal{O}_D$ is maximal at each $p \in S$. This, however, follows from $p|\text{disc}(K)$ and $p^2 \nmid \text{disc}(K)$ which we have shown for each $p \in S$. We have shown that each $p \in S$ is non-decomposed in $K$ and that $\mathcal{O}_D$ is maximal at $p$. This implies (see [Dei02]), that there exists a primitive $\gamma \in \Gamma$ such that $\mathcal{N}(\gamma) = \varepsilon_D$. But the set of all $a = 2 + 4cR$ contains infinite arithmetic progressions, which implies $\text{LG}(\Gamma) < \infty$. Since, on the other hand, the sequence $\mathcal{N}(\Gamma)$ is a subsequence of $\mathcal{N}(\text{PSL}_2(\mathbb{Z}))$, we also have $\text{LG}(\Gamma) > 0$. 

\[\square\]
3 Metric graphs

By a graph we mean a pair \((V, \varphi)\) where \(V\) is a countable set, called the vertex set and \(\varphi : E \to \mathcal{P}_{1,2}(V)\) is a map, where \(E\) is a set, the edge set, and \(\mathcal{P}_{1,2}(V)\) is the set of subsets of \(V\) which contain one or two elements. So loops and multiple edges are generally allowed. A geometric realization of a graph \((V, E)\) is obtained as follows. For each edge \(k\) fix a copy \(I_k\) of the unit interval and choose a marking of the endpoints with \(a, b\) if \(k = \{a, b\}\). In the topological space \(\bigsqcup_{k \in E} I_k\), which is the disjoint union of all these intervals, we identify all endpoints marked with the same element of \(V\). The resulting space is a geometric realization of the graph. A metric graph is obtained from this by attaching to each edge \(e\) a length \(l(e) \in (0, \infty)\). One then derives an internal metric on the geometric realization, if the graph is connected. One defines the distance of points which lie on different connected components to be \(+\infty\) and thus gets a generalized metric on the entire graph. The moduli space \(\mathcal{M}(V, \varphi)\) is defined to be the set of all metrics in this sense, i.e., the set of all maps \(l : E \to (0, \infty)\). It carries a natural compact-open topology generated by all sets of the form

\[L(K, U) = \{l \in \mathcal{M}(V, \varphi) : l(K) \subset U\},\]

where \(U \subset (0, \infty)\) is an open subset and \(K \subset E\) is finite.

If \(E\) is finite itself, the moduli space can be naturally compactified, the boundary consisting of moduli spaces of degenerations of \((V, E)\).

Example 3.1. As a simple example we consider the graph \(X_a\) for \(a > 0\),

where the \(a\) indicates that this edge gets length \(a\) and all other edges get the length 1. If we vary \(a\), we get two degenerations of the graph. One is obtained by letting \(a\) tend to infinity and the other by letting it tend to zero:
For a finite metric graph $X$ we define the *Ihara zeta function* \cite{Iha66,Ser03,Sun86,Has89,Bas92} to be

$$Z_X(s) = \prod_c (1 - N(c)^{-s}),$$

where the product runs over all closed geodesics, $N(c) = e^{l(c)}$ and $l(c)$ is the length of the geodesic $c$.

**Theorem 3.2.** For a finite graph $X$, the product defining the Ihara zeta function converges for $\text{Re}(s)$ large enough and defines an entire function which indeed is a polynomial in the entries $e^{-sl(k)}$, where $k$ runs through the finite set of edges.

If a sequence $(X_j)$ of finite metric graphs converges in the compactified moduli space $\mathcal{M}(V, \varphi)$ to a graph $X$, then the corresponding Ihara zeta functions $Z_{X_j}(s)$ converge locally uniformly on $\mathbb{C}$ to $Z_X$.

**Proof.** Elementary estimates using the finiteness of the graph show that the sum $\sum_c e^{-s l(c)}$ converges for $\text{Re}(s)$ large enough. This settles convergence. For the second assertion let $\tilde{E}$ be the set of all oriented edges, i.e., $\tilde{E}$ is the pullback of $E \to \mathcal{P}(V)$ under the natural map $V \times V \to \mathcal{P}(V); (a, b) \mapsto \{a, b\}$. More precisely, the elements of $\tilde{E}$ are all pairs $(k, (a, b))$ in $E \times (V \times V)$ with the property that $\varphi(k) = \{a, b\}$. For a given oriented edge $(k, (a, b))$ the opposite edge is defined by $(k, (a, b)) = (k, (b, a))$.

For $(k, (a, b)) \in \tilde{E}$ we say that $a$ is the *source* and $b$ the *target* of $(k, (a, b))$. Let $F(\tilde{E})$ be the free $\mathbb{C}$-vector space spanned by $\tilde{E}$ and for $s \in \mathbb{C}$ define the linear operator $T_s : F(\tilde{E}) \to F(\tilde{E})$ by

$$T_s(k) = \sum_{k' \neq \tilde{k}, s(k') = t(k)} e^{-s l(k')} k',$$

where $k \in \tilde{E}$ now and $\tilde{k}$ is the edge opposite to $k$. We equip $F(\tilde{E})$ with the inner product making the given basis $(k)_{k \in \tilde{E}}$ an orthonormal basis. We denote by $L$ the length function which assigns the length 1 to every edge.
Then for any \( n \in \mathbb{N} \), the trace of \( T^n \) is
\[
\text{tr}(T^n) = \sum_k \langle T^n k, k \rangle
= \sum_{c : L(c) = n} e^{-sl(c)} L(c_0),
\]
where the sum runs over all closed geodesics and \( c_0 \) is the unique primitive geodesic underlying a given closed geodesic \( c \). The function \( s \mapsto \det(1 - T_s) \) is entire and is a polynomial in \( e^{-sk} \), where \( k \) ranges in \( E \). We will show that this function equals \( Z_X(s) \) from which all assertions of the proposition follow. We keep the notation marking primitive geodesics by \( c_0 \) and compute
\[
Z(s) = \prod_{c_0} (1 - e^{-sl(c_0)}) = \exp \left( \sum_{c_0} \log(1 - e^{-sl(c_0)}) \right).
\]
\[
= \exp \left( - \sum_{c_0} \sum_{n=1}^{\infty} \frac{e^{-snl(c_0)}}{n} \right)
= \exp \left( - \sum_{c} \frac{e^{-sl(c)}}{L(c)} L(c_0) \right) = \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} \text{tr}(T^n s) \right)
= \exp (\text{tr} \log (1 - T_s)) = \det(1 - T_s)
\]
We call a metric graph \( X \) a rational graph, if there exists \( \theta > 0 \) such that all lengths of edges lie in \( \theta \mathbb{Q} \).

For a metric graph \( X \), we write \( N(X) \) for the sequence of the values \( N(c) \) in ascending order and without repetition and we write \( \text{LG}(X) = \text{LG}(N(X)) \).

**Proposition 3.3.** For a finite rational graph \( X \), one has \( \text{LG}(X) = \infty \).

**Proof.** As \( X \) is finite, we can assume that there is \( \theta > 0 \) such that all lengths of edges are in \( \theta \mathbb{N} \). We write the length of a geodesic \( c \) as \( l(c) = \theta L(c) \), where \( L(c) \in \mathbb{Z} \). If \( l(c) < l(c') \), then by the mean value theorem there exists \( \eta \in [L(c), L(c')] \) such that
\[
|N(c) - N(c')| = |e^{\theta L(c)} - e^{\theta L(c')} |
\]
\[
= \theta e^{\theta \eta} |L(c) - L(c')|
\]
\[
\geq \theta e^{\theta \eta}
\]
\[
\geq \theta e^{\theta L(c)} = \theta e^{l(c)}.
\]
The convergence of the Euler product implies that there are only finitely many closed geodesics \( c \) of length below a given bound, which implies that for any given \( C > 0 \), there are only finitely many pairs \( (c, c') \) such that \( |N(c) - N(c')| \leq C \).

To learn more about the general situation, we consider the very simple example the graph \( Y_a \) defined as follows,

\[
\begin{array}{c}
  a \\
  \infty
\end{array}
\]

where the \( a \) indicated that one loop has irrational length \( a \in (0, 1) \), whereas the other has length 1.

**Proposition 3.4.** The set of all \( a \in (0, 1) \) such that \( \text{LG}(Y_a) = 0 \) is dense in \([0, 1]\).

**Proof.** Any closed geodesic in \( Y_a \) has to run the left loop for a certain number \( m \) of times and the right loop for another number \( n \) of times. The length then is

\[
l(m, n) = am + n,
\]

and since \( a \) is irrational, the length determines \( m \) and \( n \) uniquely. So for another pair \( (k, l) \) we have

\[
e^{l(m,n)} - e^{l(k,l)} = e^{am+n} - e^{ak+l}
\]

\[
= (e^{a(m-k)+(n-l)} - 1) e^{ak+l}
\]

For this to remain bounded for infinitely many geodesics, the bracket expression must be small, so the exponent \( a(m-k) + (n-l) \) must be close to zero. But close to zero, \( e^x - 1 \) behaves like \( x \), so that we consider

\[
|a(m-k) + (n-l)| e^{ak+l}.
\]

For \( 0 < a < 1 \), this expression is less than a given \( C > 0 \) if

\[
\left| a - \frac{(n-l)}{m-k} \right| \leq \frac{C}{e^{ak+l}(m-k)}.
\]
It is now easy to construct \( a \), such that this estimate is satisfied for infinitely many geodesics where \( C > 0 \) is arbitrary. You start with \( a_1 \) being any decimal number in \((0, 1)\) with finitely many nonzero decimals. Then you add a string of zeros, followed by a single 1, again a string of zeros and so on. By letting the length of the strings of zeros increase fast enough, one gets some \( a \) that satisfies the claim. Furthermore, these \( a \) are dense in \([0, 1]\), so that we have proven the proposition.

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