Differential Message Importance Measure: A New Approach to the Required Sampling Number in Big Data Structure Characterization

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Abstract—Data collection is a fundamental problem in the scenario of big data, where the size of sampling sets plays a very important role, especially in the characterization of data structure. This paper considers the information collection process by taking message importance into account, and gives a distribution-free criterion to determine how many samples are required in big data structure characterization. Similar to differential entropy, we define differential message importance measure (DMIM) as a measure of message importance for continuous random variable. The DMIM for many common densities is discussed, and high-precision approximate values for normal distribution are given. Moreover, it is proved that the change of DMIM can describe the gap between the distribution of a set of sample values and a theoretical distribution. In fact, the deviation of DMIM is equivalent to Kolmogorov-Smirnov statistic, but it offers a new way to characterize the distribution goodness-of-fit. Numerical results show some basic properties of DMIM and the accuracy of the proposed approximate values.

Index Terms—Differential Message importance measure, Big Data, Kolmogorov-Smirnov test, Goodness of fit, distribution-free.

I. INTRODUCTION

The actual system of big data needs to process lots of data within a limited time generally, so many researches are on sample data to improve their efficiency [1], [2]. In fact, sampling technology is intensely effective for solving the challenges in big data, such as intrusion detection [3] and privacy-preserving approximate search [4]. One basic problem that can occur with sampling is that how many samples are required to have a good characterization of the big data structure, e.g. fitting the real distribution. Too many samples means wasting of resources, while too little samples is along with great bias. Distribution goodness-of-fit is generally used to describe this problem, which focuses on the error magnitude between the distribution of a set of sample values and the real distribution, and it plays a fundamental role in signal processing and information theory. This paper desires to solve this problem based on information theory.

Shannon entropy [5] is possibly the most important quantity in information theory, which describes the fundamental laws of data compression and communication [6]. Due to its success, numerous entropies have been provided in order to extend information theory. Among them, the most successful expansion is Rényi entropy [7]. There are many applications based on Rényi entropy, such as hypothesis testing [8], [9].

Actually, entropy is a quantity with respect to probability distribution, which satisfies the intuitive notion of what a measure of information should be [10]. Generally, the events are naturally endowed with importance label and the process of fitting is equivalent to the process of information collection. Therefore, in this paper, we propose differential message importance measure (DMIM) as a measure of information for continuous random variable to characterize the process of information collection. DMIM is expanded from discrete message importance measure (MIM) [11] which is such an information quantity coming from the intuitive notion of information importance for small probability event. Much of research in the last two decades has examined the application of small probability event in big data [12]–[14]. Recent studies also show that MIM has many applications in big data, such as information divergence measures [15] and compressed data storage [16].

Much of the research in the goodness of fit in the past several decades focused on the Kolmogorov-Smirnov test [17], [18]. Based on it, [19] gave an error estimation of empirical distribution. [21] presented a general method for distribution-free goodness-of-fit tests based on Kullback-Leibler discrimination information. The problem of testing goodness-of-fit in a discrete setting was discussed in [20]. All these result can describe the goodness of fit very well and guide us to choose the sampling numbers. However, they all consider this problem based on the divergence of two distributions, so the previous results can not describe the message carried by each sample and the information change with the increase of the sampling size, which means that they can not visually display the process of information collection. In fact, DMIM is the proper measure to help us consider the problem of goodness-of-fit in the view of the information collection of continuous random variables. Moreover, Compared with Kolmogorov-Smirnov statistic, DMIM also shows the relationship between the variance of a random variable and the error estimation of empirical distribution.
The rest of this paper is organized as follows. Section II introduces the definition and the relationship between MIM and DMIM. In Section III, the properties of DMIM are introduced. Then, the DMIM of some basic continuous distributions are discussed in Section IV, in which we give the asymptotic analysis of normal distribution. In Section V, the goodness of fit with DMIM is presented in order to analyze the process of information collection. The validity of proposed theoretical results is verified by the simulation results in Section VI. Finally, we finish the paper with conclusions in Section VII.

II. THE DEFINITION OF DMIM

A. Differential Message Important Measure

**Definition 1.** The DMIM \( l(X) \) of a continuous random variable \( X \) with density \( f(x) \) is defined as

\[
l(X) = \int_{-\infty}^{+\infty} f(x)e^{-f(x)}dx,
\]

where \( S \) is the support set of the random variable.

For most continuous random variables, the DMIM has no simple expression and the integral form is inconvenient for numerical calculation, so we will give another form of it.

**Theorem 1.** The DMIM of a continuous random variable \( X \) with density \( f(x) \) can be written as

\[
l(X) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} (f(x))^{n+1}dx.
\]

**Proof.** In fact, we obtain

\[
l(X) = \int_{-\infty}^{+\infty} f(x)e^{-f(x)}dx = \int_{-\infty}^{+\infty} f(x) \sum_{n=0}^{\infty} \frac{(-f(x))^n}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} (f(x))^{n+1}dx
\]

\[
= \int_{-\infty}^{+\infty} f(x)dx + \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} (-1)^n \frac{(f(x))^{n+1}}{n!} dx
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} (f(x))^{n+1}dx.
\]

B. Relation of DMIM to MIM

For a random variable \( X \) with density \( f(x) \), we divide the range of \( X \) into bins of length \( \Delta \). We also suppose that \( f(x) \) is continuous within the bins. According to the mean value theorem, there exists a value \( x_i \) within each bin such that \( f(x_i) \Delta = \int_{x_i}^{x_{i+1}} f(x)dx \). Then, we define a quantized random variable \( X^\Delta \), which is given by

\[
X^\Delta = x_i, \quad \text{if } i\Delta \leq X < (i+1)\Delta.
\]

Therefore, \( p_i = Pr\{X^\Delta = x_i\} = \int_{x_i}^{x_{i+1}} f(x)dx = f(x_i)\Delta \).

The MIM of \( X^\Delta \) is given by [11]

\[
L(X^\Delta) = \log \sum_{i=1}^{n} p_i e^{\omega(1-p_i)}
\]

\[
= \log \sum_{i=1}^{n} \Delta f(x_i) e^{\omega(1-\Delta f(x_i))}
\]

\[
= \log e^{\sum_{i=1}^{n} \Delta f(x_i) e^{-\omega\Delta f(x_i)}}
\]

\[
= \omega + \log \sum_{i=1}^{n} \Delta f(x_i) e^{-\omega\Delta f(x_i)},
\]

since \( \sum_{i=1}^{n} f(x_i)\Delta = 1 \). Substituting \( \omega = 1/\Delta \) in (5c), we obtain

\[
L = \frac{1}{\Delta} + \log \sum_{i=1}^{n} f(x_i)e^{-f(x_i)}\Delta.
\]

It is observed that the first term in (6) approaches infinity when \( \Delta \to 0 \). Therefore, the MIM of continuous random variables approaches infinity, which makes no sense. However, the second term in (6) can help us characterize the relative importance of continuous random variables. The logarithm operator does not change the monotonicity of a function, which is only to reduce the magnitude of the numerical results, so \( \sum_{i=1}^{n} f(x_i)e^{-f(x_i)}\Delta \) is adopted to measure the relative importance. If \( f(x)e^{-f(x)} \) is Riemann integrable, \( \sum_{i=1}^{n} f(x_i)e^{-f(x_i)}\Delta \) approaches the integral of \( f(x)e^{-f(x)} \) as \( \Delta \to 0 \) by definition of Riemann integrability.

III. THE PROPERTIES OF DMIM

In this section, the properties of DMIM are discussed in details.

A. Upper and Lower Bound

For any continuous random variable \( X \) with density \( f(x) \), it is noted that

\[
\int_{S} f(x)e^{-f(x)}dx \leq \int_{S} f(x)dx = 1.
\]

(7) is obtained for the fact that \( 0 \leq f(x) \leq 1 \), which leads to \( e^{-f(x)} \leq 1 \). As a result, \( f(x)e^{-f(x)} \leq f(x) \). Obviously, we also find \( l(x) \geq 0 \) because \( f(x) \geq 0 \). Hence, we obtain

\[
0 \leq l(X) \leq 1.
\]

B. Translation with Constant

Let \( Y = X + c \), where \( c \) is a real constant. Then \( f_Y(y) = f_X(y-c) \), and

\[
l(X + c) = \int_{-\infty}^{+\infty} f_X(x-c)e^{-f_X(x-c)}dx = l(X).
\]

As a result, the translation with a constant does not change the DMIM.
C. Stretching

Let \( Y = aX \), where \( a \) is a non-zero real number. Then \( f_Y(y) = \frac{1}{|a|} f_X \left( \frac{y}{a} \right) \), and

\[
\begin{align*}
  l(aX) &= \int f_Y(y) e^{-f_Y(y)} dy \\
        &= \int \frac{1}{|a|} f_X \left( \frac{y}{a} \right) e^{-\frac{1}{|a|} f_X \left( \frac{y}{a} \right)} dy \\
        &= \int f_X(x) e^{-\frac{1}{a} f_X(x)} dx.
\end{align*}
\]

Consider the extreme case, we get

\[
\lim_{a \to \infty} l(aX) = \lim_{a \to \infty} \int f(x) e^{-\frac{1}{a} f(x)} dx = 1,
\]

and

\[
\lim_{a \to 0} l(aX) = \lim_{a \to 0} \int f(x) e^{-\frac{1}{a} f(x)} dx = 0.
\]

Asymptotically, too small stretch factor will lead to lessen the relative importance of random variables. Nevertheless, when the stretch factor approaches infinity, DMIM reaches the maximum.

D. Relation of DMIM to Rényi Entropy

The differential Rényi entropy of a continuous random variable \( X \) with density \( f(x) \) is given by [9]

\[
h_\alpha(X) = \frac{1}{1-\alpha} \ln \int (f(x))^{\alpha} dx,
\]

where \( \alpha > 0 \) and \( \alpha \neq 1 \). As \( \alpha \) tends to 1, the Rényi entropy tends to the Shannon entropy.

Therefore, we obtain

\[
\int (f(x))^{\alpha} dx = e^{(1-\alpha)h_\alpha(X)}.
\]

Hence, we find

\[
l(X) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} (f(x))^{n+1} dx
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} e^{-nh_{n+1}(X)}.
\]

Obviously, the DMIM is an infinite series of Rényi Entropy.

E. Truncation Error

In this part, the remainder term of (2) will be discussed. In fact, the remainder term is limited in many cases, which is summarized as the following theorem.

Theorem 2. If \( \int (f(x))^{n+1} dx \leq \varepsilon \) for every \( n \geq m \), then

\[
\left| l(X) - \left( 1 + \sum_{n=1}^{m-1} \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} (f(x))^{n+1} dx \right) \right| \leq \varepsilon.
\]

Proof. Substituting (2) in the left of (16), we obtain

\[
\begin{align*}
  l(X) &= \left| \left( 1 + \sum_{n=1}^{m-1} \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} (f(x))^{n+1} dx \right) \right| \\
  &= \sum_{n=m}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} (f(x))^{n+1} dx
\end{align*}
\]

(17c) follows from \( \int (f(x))^{n+1} dx \leq \varepsilon \), when \( n \geq m \).

That is to say, if the integral of the density to the \((n+1)\)-th power is limited, the remainder term will be restricted.

Corollary 1. If \( \int (f(x))^{n+1} dx \leq \varepsilon \) for every \( n \geq m \), then

\[
l(X) - \left( 1 + \sum_{n=1}^{m-1} \frac{(-1)^n}{n!} e^{-nh_{n+1}(X)} \right) \leq \varepsilon e.
\]

Proof. Clearly we have

\[
\begin{align*}
  l(X) - (1 + \sum_{n=1}^{m-1} \frac{(-1)^n}{n!} e^{-nh_{n+1}(X)})
  &= \left| l(X) - \left( 1 + \sum_{n=1}^{m-1} \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} (f(x))^{n+1} dx \right) \right|
  \leq \varepsilon e,
\end{align*}
\]

(18a) follows from Theorem 2.

Remark 1. Letting \( m = 2 \) in (17c), after manipulations, we obtain

\[
\left| l(X) - (1 - e^{-h_{2}(X)}) \right| \leq (e - 2)e,
\]

\[
l(X) + e^{-h_{2}(X)} \leq 1 + (e - 2)e.
\]

Especially, if \( \varepsilon \) is too small, we have \( l(x) + e^{-h_{2}(X)} \approx 1 \). That means \( l(X) \) is approximately the dual part of Rényi entropy with order 2.

IV. THE DMIM OF SOME DISTRIBUTIONS

A. Uniform Distribution

For a random variable whose density is \( \frac{1}{b-a} \) for \( a \leq x \leq b \) and 0 elsewhere, we have

\[
l(X) = \int_{a}^{b} \frac{1}{b-a} e^{-\frac{1}{b-a} x} dx = e^{-\frac{1}{b-a}}.
\]

Note that

\[
\lim_{b-a \to 0} e^{-\frac{1}{b-a}} = 0,
\]

\[
\lim_{b-a \to \infty} e^{-\frac{1}{b-a}} = 1.
\]
B. Normal Distribution

Let \( X \sim \phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \) with \( \sigma \neq 0 \), then

\[
\int_{-\infty}^{+\infty} (\phi(x))^{n+1} dx = \int_{-\infty}^{+\infty} \left( \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right)^{n+1} dx
\]

\[
= \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^{n+1} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx
\]

\[
= \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^{n+1} \sqrt{\frac{2\pi\sigma^2}{n+1}}
\]

\[
= \frac{1}{\sqrt{n+1}} \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^n.
\]

Substituting (22c) in (2), we obtain

\[
l(X) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^n.
\]

1) When \( \sigma \) is large: If \( \sigma > 1/\sqrt{2\pi} \), \( \int_{-\infty}^{+\infty} (\phi(x))^{n+1} dx \)

will be less than or equal to \( 1/(2\sqrt{3\pi\sigma^2}) \) for every \( n \geq 2 \)

because \( \frac{1}{\sqrt{n+1}} \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \) monotonically decreases in this case.

According to Remark 1, we obtain

\[
\left| l(X) - (1 - e^{-b_2(X)}) \right| \leq \frac{(\epsilon - 2)}{2\sqrt{3\pi\sigma^2}}.
\]

If \( \sigma \) is big enough, \( \frac{(\epsilon - 2)}{2\sqrt{3\pi\sigma^2}} \approx 0 \). Moreover, the intensity of approximation error decreases as the inverse square of \( \sigma \). In this case, substituting \( b_2(X) = \ln 2 + 0.5\ln \pi + \ln \sigma \) in \( 1 - e^{-b_2(X)} \), we find

\[
1 - e^{-b_2(X)} = 1 - \frac{1}{2\sqrt{\pi\sigma}} \approx e^{-\frac{x}{2\sqrt{\pi\sigma}}},
\]

We define

\[
\hat{l}_1(X) = 1 - \frac{1}{2\sqrt{\pi\sigma}}, \quad \hat{l}_2(X) = e^{-\frac{x}{2\sqrt{\pi\sigma}}}.
\]

According to (24), \( \hat{l}_1(X) \) and \( \hat{l}_2(X) \) is very good approximate values for DMIM of normal distribution when \( \sigma \) is not too small, which will be shown by the numerical results in section VI.

2) When \( \sigma \) is small: However, the DMIM of normal distribution will be hard to calculate when \( \sigma \) is small. By Stirling formula, \( l(X) \) can also be written as

\[
l(X) \approx 1 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi\sigma\sigma(n+1)}} \left( -\frac{e}{\sqrt{2\pi\sigma}} \right)^n.
\]

If \( n > \frac{\epsilon}{\sqrt{2\pi\sigma}} \approx \frac{0.841}{\sigma} \), we will obtain \( \frac{e}{\sqrt{2\pi\sigma}} \approx \frac{1}{2\sqrt{3\pi\sigma^2}} \) \( \leq 1 \). Let \( n_0 = \left[ \frac{\epsilon}{\sqrt{2\pi\sigma}} \right] \) where \( \left[ x \right] \) is the largest integer smaller than or equal to \( x \). In this case, we define

\[
\hat{l}(X) = 1 + \sum_{n=1}^{n_0} \frac{(-1)^n}{n!} \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^n,
\]

as the approximate value when \( \sigma \) is small. The following theorem shows the validity of \( \hat{l}(X) \).

**Theorem 3**. \( X \) is a normal random variable with mean \( \mu \) and variance \( \sigma^2 \), \( \hat{l}(X) \) is the first \( n_0 \) terms of \( l(X) \), given by (28), where \( n_0 = \left[ \frac{\epsilon}{\sqrt{2\pi\sigma}} \right] \). If \( \sigma \) is relatively small, Then we have

\[
\left| l(X) - \hat{l}(X) \right| < \frac{3\sigma}{e},
\]

**Proof**. Refer to the Appendix A.

It is easy to see that the upper bound of error approaches 0 if \( \sigma \) approaches 0.

C. Exponential Distribution

Letting

\[
X \sim f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases},
\]

where \( \lambda > 0 \), we obtain

\[
\int_{-\infty}^{+\infty} (f(x))^{n+1} dx = \int_{0}^{+\infty} (\lambda e^{-\lambda x})^{n+1} dx
\]

\[
= \int_{0}^{+\infty} \lambda^{n+1} e^{-\lambda(n+1)x} dx
\]

\[
= \frac{\lambda^{n+1}}{n+1}.
\]

Substituting (31c) in (2), we obtain

\[
l(X) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \frac{\lambda^n}{n+1!} = \frac{1}{\lambda} (1 - e^{-\lambda}).
\]

It is noted that

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} (1 - e^{-\lambda}) = 1,
\]

\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} (1 - e^{-\lambda}) = 0.
\]

D. Gamma Distribution

In many cases, the Gamma distribution can be used to describe the distribution of the amount of time one has to wait until a total of \( n \) events has occurred in practice [22]. For a random variable obeying Gamma distribution, its density is

\[
X \sim f(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, & x \geq 0 \\ 0, & x < 0 \end{cases},
\]

where \( \lambda, \alpha > 0 \), we obtain

\[
\int_{-\infty}^{+\infty} (f(x))^{n+1} dx
\]

\[
= \int_{0}^{+\infty} \left( \frac{\lambda e^{-\lambda x}}{\Gamma(\alpha)} \right)^{n+1} dx
\]

\[
= \frac{\lambda^n}{(n+1)!} \Gamma(\alpha) \int_{0}^{+\infty} e^{-t(\alpha-1)(n+1)} dt
\]

\[
= \frac{\lambda^n}{(n+1)!} \Gamma(\alpha) \Gamma(\alpha-1)(n+1)
\]

(35a)

(35b)
Substituting (35b) in (2), we obtain

\[ l(X) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\lambda^n (an - n + \alpha)}{(n+1)^{\alpha n - n + \alpha} \Gamma^{n+1}(\alpha)}. \]  

(36)

E. Beta Distribution

The \( \beta \) distribution often arises to depict a random variable whose set of possible values is some finite interval, such as \([0, 1]\) [22]. For a random variable follows \( \beta \) distribution whose density is

\[ X \sim f(x) = \begin{cases} \frac{1}{B(a, b)} x^{a-1} (1 - x)^{b-1}, & 0 < x < 1, \\ 0, & \text{else} \end{cases} \]  

(37)

where \( B(a, b) = \int_0^1 x^{a-1} (1 - x)^{b-1} dx \) and \( a, b > 0 \). According to [22], we have

\[ B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)}. \]  

(38)

In fact, we find

\[
\int_{-\infty}^{\infty} (f(x))^{n+1} dx \\
= \int_0^1 \left( \frac{1}{B(a, b)} x^{a-1} (1 - x)^{b-1} \right)^{n+1} dx \\
= \frac{1}{B^{n+1}(a,b)} \int_0^1 x^{(a-1)(n+1)} (1 - x)^{(b-1)(n+1)} dx \\
= B(an - n + a, bn - n + b) \\
\frac{1}{B^{n+1}(a,b)} \cdot \int_0^1 x^{(a-1)(n+1)} (1 - x)^{(b-1)(n+1)} dx \\
= B(an - n + a, bn - n + b) \\
B^{n+1}(a,b),
\]

(39)

Hence, we obtain

\[ l(X) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{B(an - n + a, bn - n + b)}{B^{n+1}(a,b)}. \]  

(40)

F. Laplace Distribution

A random variable, whose density function is

\[ f(x) = \frac{\lambda}{2} e^{\lambda|x-\theta|}, \]  

(41)

has a Laplace distribution where \( \theta \) is a location parameter and \( \lambda > 0 \). In fact, we find

\[
\int_{-\infty}^{+\infty} \left( \frac{\lambda}{2} e^{\lambda|x-\theta|} \right)^{n+1} dx = \frac{1}{n+1} \left( \frac{\lambda}{2} \right)^n.
\]  

(42)

Substituting (42) in (2), we obtain

\[ l(X) = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \left( -\frac{\lambda}{2} \right)^n = \frac{2}{\lambda} \left( 1 - e^{-\frac{\lambda}{2}} \right). \]  

(43)

For simplicity to follow, the DMIM for these common densities are summarized in Table I.

V. Goodness of Fit with DMIM

In this section, we will consider the problem of distribution goodness-of-fit in a continuous setting. Let \( X_1, X_2, \ldots X_n \) be a sequence of independent and identically distributed random variables, each having mean \( \mu \) and variance \( \sigma^2 \). In practice, the real distribution is generally unknown and we usually use empirical distribution to substitute real distribution. Generally, the empirical distribution function is given by

\[ \hat{F}_n(x) = \frac{1}{n} \sum_{k=1}^{n} I(X_k \leq x), \]  

(44)

and the real distribution is \( F(x) \).

One practical problem that can occur with this strategy is that how many samples is required for fitting the real distribution with an acceptable bias in some degree. Many literatures studied this problem by Kolmogorov-Smirnov statistic [17]–[19]. When \( n \) is big enough, the confidence limits for a cumulative distribution are given by [19],

\[ P\{ D_n > d \} \approx 2 \sum_{k=1}^{\infty} (-1)^{k-1} e^{-2nk^2d^2}, \]  

(45)

where \( D_n \) is error bound between empirical distribution and real distribution, called Kolmogorov-Smirnov statistic, which is defined as

\[ D_n = \sup_x \left| \hat{F}_n(x) - F(x) \right|. \]  

(46)

Though this result can describe the goodness of fit very well and guide us to choose the sampling numbers, we need to give two artificial criterions, the deviation value \( d \) and the probability \( P\{ D_n > d \} \), in order to determine \( n \). In addition, this method do not take the message importance of samples into account, which makes the process of information collection not intuitionistic.

In this paper, we consider this problem from the perspective of DMIM. Firstly, we define

\[ \gamma(n) = l \left( \sum_{i=1}^{n} X_i \right) / l(X). \]  

(47)

as relative importance of these \( n \) sample points. According to central-limit theorem [22], when \( n \) is big enough, \( \sum_{i=1}^{n} X_i \) approximately obeys normal distribution \( N(n\mu, n\sigma^2) \). In fact, when \( \sqrt{n}\sigma \) is not too small (such a condition is satisfied because \( n \) is big enough), \( l(\sum_{i=1}^{n} X_i) \approx e^{-\frac{\pi}{2\lambda}} \) according to (25). Hence

\[ \gamma(n) \approx e^{-\frac{\pi}{2\lambda}}. \]  

(48)

We find \( \gamma(n) \) increases rapidly firstly, and then increases slowly by analyzing its monotonicity. Moreover, we obtain

\[ \gamma(\infty) = \lim_{n \to \infty} \gamma(n) = \lim_{n \to \infty} e^{-\frac{\pi}{2\lambda}} \frac{l(X)}{l(X)} = \frac{1}{l(X)}, \]  

(49)

which means \( \gamma(n) \) reaches limit as \( n \to \infty \). In fact, these two points are consistent with the characteristic of data fitting. Both \( \gamma(n) \) and data fitting have the law of diminishing of marginal utility. Furthermore, the goodness of fit can not increase
unboundedly and it reaches the upper bound when the number of sampling points approaches infinity. DMIM is bounded, while Shannon entropy and Rényi entropy do not possess these characteristic. In conclusion, we adopt $|\gamma(\infty) - \gamma(n)|$ to describe the goodness of fit.

**Theorem 4.** $X_1, X_2, X_3, \ldots, X_n$ are the $n$ sampling of a continuous random variable $X$, whose density is $f(x)$. If $|\gamma(\infty) - \gamma(n)| \leq \varepsilon$, we will obtain

$$P\left\{D_n > \sqrt{2\pi \sigma^2 \ln \frac{19}{9\beta}} \ln \frac{1}{1 - \varepsilon} \right\} \leq \beta. \tag{50}$$

**Proof.** Refer to the Appendix B. \hfill \Box

**Remark 2.** According to (65) in Appendix B, we obtain

$$\varepsilon = 1 - e^{-d(2\pi \sigma^2 \ln \frac{19}{9\beta})^{-1/2}}, \tag{51}$$

$$\beta = \frac{19}{9} e^{\frac{d^2}{2\pi \sigma^2 \ln^2(1 - \varepsilon)}}. \tag{51a}$$

Therefore, there is a ternary relation among $d$, $\beta$, and $\varepsilon$. If two of them are known, the third one can be obtained, easily.

**Remark 3.** For arbitrary positive number $d$ and $\beta \leq 1$, one can always find a $\varepsilon_0$, which can be obtained by (51), when $\varepsilon \leq \varepsilon_0$, $P\{D_n > d\} < \beta$ holds.

**Remark 4.** When $\varepsilon$ tends zero, which means $n \rightarrow \infty$, at this time, $P\{D_n > 0\} = 0$. Therefore, the real distribution is equal to empirical distribution with probability $1$ as $\varepsilon \rightarrow 0$. That is,

$$\hat{F}_n(x) \rightarrow F(x) \text{ as } \varepsilon \rightarrow 0. \tag{52}$$

Actually, the DMIM deviation characterizes the process of collection information in terms of data structure. With the growth of sampling number, the information gathers, and the empirical distribution approaches real distribution at the same time. In particular, when $n \rightarrow \infty$, all the information about the real distribution will be obtained. In this case, the empirical distribution is equal to real distribution, naturally.

**Remark 5.** For arbitrary continuous random variable with variance $\sigma^2$, if the maximal allowed DMIM deviation is $\varepsilon$, the sampling number should be bigger than $1/(4\pi \sigma^2 \ln^2(1 - \varepsilon))$ according to (64).

The sampling number only depends on one artificial criterion, the DMIM deviation, while the variance are the own attributes of the observed variable $X$. Furthermore, the sampling number in the new developed method has nothing to do with the distribution form, which means the new method is distribution-free.

**VI. Numerical Results**

In this section, we present some numerical results to validate the above results in this paper.

**A. The DMIM of Normal Distribution**

First of all, we analyze the DMIM in normal distribution by simulation. Its standard deviation $\sigma$ is varying from 0.01 to 10.

Fig. 1 depicts the DMIM versus standard deviation $\sigma$ in normal distribution. We observe that there are some constraints on DMIM in this case. That is, the DMIM grows with the increasing of $\sigma$. Furthermore, it increases rapidly when $\sigma$ is small ($\sigma < 1$), while it increases slowly when $\sigma$ is big ($\sigma > 4$). Besides, DMIM is non-negative and it is very close to zero when $\sigma$ approaches zero. In order to avoid complex calculations, we give two approximate value of DMIM in Gauss distribution, which are $l_1(X) = 1 - 1/(2\sqrt{\pi}\sigma)$ and $l_2(X) = e^{-1/(2\sqrt{\pi}\sigma)}$. Obviously, the gap between true value $l(X)$ and the approximate value $l_1(X)$ will be very small if $\sigma$ is big enough. However, $l_2(X)$ is smaller than $l(X)$ when $\sigma$ is small. For $l_2(x)$, the gap between it and $l(X)$ will be very small if $\sigma$ is not too small. In fact, there is only a slight deviation between them when $\sigma$ is small.

Moreover, Fig. 2 shows the absolute and relative error when we adopt approximations. Some observations are obtained. Both absolute and relative error are relatively small when $\sigma$ is big ($|l(X) - l_1(X)| / l(X) < 1\%$ when $\sigma > 2.5$, and $|l(X) - l_2(X)| / l(X) < 1\%$ when $\sigma > 1.25$), and they both decreases with increasing of $\sigma$ for two approximate values in most of time. In fact, when $\sigma$ is not too small ($\sigma > 0.3$), the relative error of $l_2(X)$ is smaller than $10\%$. When $\sigma < 6.25$, the relative error of $l_2(X)$ is smaller than that of $l_1(X)$ and the opposite is true when $\sigma > 6.25$. In summary, $l_2(X)$ is a good approximation for all the $\sigma$ and $l_1(X)$ is an excellent approximation when $\sigma$ is big enough.

Fig. 3 shows the truncation error $|l(X) - \bar{l}(X)|$ versus the number of series $N$ when $\sigma$ is small. Without loss of
shows the relationship between the probability of increasing of \( \alpha \) for the same variance. It also can be seen from the figure, that the gap between the DMIM of uniform distribution and that of normal distribution is negligibly small when variance is big enough. This is because that, for the same variance \( \sigma^2 \), these two DMIM respectively are \( e^{-1/(2\sqrt{3}\sigma)} \) and \( e^{-1/(2\sqrt{3}\sigma)} \) (approximate value when \( \sigma \) is large according to (25)), which are very close.

C. Goodness-of-fit with DMIM

Next we focus on conducting Monte Carlo simulation by computer to validate our results about goodness of fit. The samples are drawn by independent identically distributed Gaussian, each having mean zero. Their standard deviation is 1 or 2. The DMIM deviation \( \varepsilon \) is varying from 0.001 to 0.1. The confidence limit \( \beta \) is 0.001. For each value of \( \varepsilon \), the simulation is repeated 10000 times.

Fig. 5 shows the relationship between the probability of error bound \( P\{D > d\} \) and DMIM deviation \( \varepsilon \). Some observations can be obtained. The probability of error bound decreases with the decreasing of DMIM deviation. In fact, this process can be divided into three phases. In phase one, in which \( \varepsilon \) is very small \( (\varepsilon < 10^{-2.8}) \) when \( d = 0.01 \) and \( \sigma = 1 \), \( P\{D > d\} \) is close to zero. In phase two, \( \varepsilon \) is neither too
small nor too large ($10^{-2.8} < \varepsilon < 10^{-2}$ when $d = 0.01$ and $\sigma = 1$). In this case, $P\{D > d\}$ increases rapidly from zero to one. In the phase three, in which $\varepsilon$ is large ($\varepsilon > 10^{-2}$ when $d = 0.01$ and $\sigma = 1$), $P\{D > d\}$ approaches one. For the same standard deviation, $P\{D > d\}$ decreases with increasing of $d$ when $P\{D > d\} < 1$. Furthermore, for the same $d$, the probability of error bound increases with increasing of the standard deviation.

The simulation results of $P\{D > 0.01\}$ are listed in Table II, where the sampling number $n$ is given by (64) and the upper bound for the error probability $\beta$ is given by (51a). In this table, we take $d = 0.01$ as the criterion to evaluate the error between the empirical distribution and real distribution. To better validate our results, normal distribution, exponential distribution, uniform distribution and Laplace distribution are listed here. The standard deviation of these four distribution is $\sigma$. As a result, $\lambda = 1/\sigma$ in exponential distribution, and the density of uniform distribution is $1/(2\sqrt{\sigma})$. The $\lambda = \sqrt{2}/\sigma$ in Laplace distribution. The remaining parameter values are same with that in Fig. 5. We obtain that $\beta$ is indeed the upper bound of $P\{D > 0.01\}$ because every $P\{D > 0.01\}$ is smaller than $\beta$. For each distribution, it is noted that the sampling number increases with the decreasing of DMIM deviation. In addition, $P\{D > 0.01\}$ decreases with decreasing of the DMIM deviation. $P\{D > 0.01\}$ can even be zero when $\varepsilon = 0.001$ and $\sigma = 1$. For the same DMIM deviation, $\beta$ and $P\{D > 0.01\}$ increase with increasing of $\sigma$. Therefore, if one wants to have the same precision in different variance, it needs to select smaller $\varepsilon$ when $\sigma$ is larger, such as $\varepsilon = 0.002$ when $\sigma = 1$ and $\varepsilon = 0.001$ when $\sigma = 2$. Furthermore, when $n$ is not too small, for the same $\varepsilon$ and $\sigma$, $P\{D > 0.01\}$ of these four distribution is very close to each other, which means this method is distribution-free.

To demonstrate the effectiveness of our theoretical results, we illustrate our proposed sampling number to fit a common and complex distribution, the Nakagami distribution. Nakagami-$m$ distribution provides good fitting to empirical multipath fading channel [23]. The parameter $m$ in this part is 2 and $\Omega = 10$. Fig. 6 shows the cumulative distribution function (CDF) of empirical distribution and real distribution.

The simulated DMIM deviation $\varepsilon$ is 0.1, 0.05, 0.01 and 0.001. It is noted that the gap between the CDF of empirical distribution and that of real distribution is constrained by the DMIM deviation. Obviously, the gap decreases with the decreasing of the DMIM deviation. Particularly, the gap almost disappears when $\varepsilon = 0.001$. In general, there is a tradeoff between the sampling number and the accuracy for empirical distribution, but DMIM can provide a new viewpoint on this by taking message importance into account.

VII. CONCLUSION

This paper focused on the problem, that how many samples is required in big data collection, with taking DMIM into account. Firstly, we defined DMIM as an measure of message importance for continuous random variable to help us describe the information flows during sampling. It is an extension of MIM and similar to differential entropy. Then, the DMIM for some common distributions, such as normal and uniform distribution, were discussed. Moreover, we made the asymptotic analysis of Gaussian distribution. As a result, high-precision approximate values for DMIM of normal distribution were respectively given when variance is extremely big or relatively small.

Then we proved that the divergence between the empirical distribution and real distribution is controlled by the DMIM deviation, which shows the deviation of DMIM is equivalent to Kolmogorov-Smirnov statistic. In fact, compared with Kolmogorov-Smirnov test, the new method based on DMIM gives us another viewpoint of information collection because it visually shows the information flow with the increasing of sampling points, which helps us to design sampling strategy for the actual system of big data. Moreover, similar to Kolmogorov-Smirnov test, the sampling number in our method is distribution-free, which only depends on the DMIM deviation when the random variable is given.

Proposing the joint differential message importance measure and using it to design high-efficiency big data analytic system are of our future interests.
TABLE II

Table of probability of error bound $P(D > 0.01)$. The sampling number $n$ is given by (64) and the upper bound for the error probability $\beta$ is given by (51a).

| Distribution | DMIM deviation $\varepsilon$ | $\sigma = 1$ | $\sigma = 2$ |
|-------------|-------------------------------|--------------|--------------|
|             | $n$ | $\beta$ | $P(D > 0.01)$ | $n$ | $\beta$ | $P(D > 0.01)$ |
| Normal      | 0.01 | 787 | 1.8034 | 0.9994 | 196 | 2.0296 | 1 |
|             | 0.003 | 8815 | 0.3621 | 0.2286 | 2203 | 0.13586 | 0.9056 |
|             | 0.002 | 19854 | 0.0398 | 0.0207 | 4963 | 0.7823 | 0.5390 |
|             | 0.001 | 79497 | 2.63e-7 | 0 | 19874 | 0.0396 | 0.0221 |
| Exponent    | 0.01 | 787 | 1.8034 | 0.9994 | 196 | 2.0296 | 1 |
|             | 0.003 | 8815 | 0.3621 | 0.2011 | 2203 | 1.3586 | 0.8609 |
|             | 0.002 | 19854 | 0.0398 | 0.0164 | 4963 | 0.7823 | 0.4821 |
|             | 0.001 | 79497 | 2.63e-7 | 0 | 19874 | 0.0396 | 0.0161 |
| Uniform     | 0.01 | 787 | 1.8034 | 0.9996 | 196 | 2.0296 | 1 |
|             | 0.003 | 8815 | 0.3621 | 0.2791 | 2203 | 1.3586 | 0.9447 |
|             | 0.002 | 19854 | 0.0398 | 0.03 | 4963 | 0.7823 | 0.6043 |
|             | 0.001 | 79497 | 2.63e-7 | 0 | 19874 | 0.0396 | 0.0275 |
| Laplace     | 0.01 | 787 | 1.8034 | 0.9952 | 196 | 2.0296 | 1 |
|             | 0.003 | 8815 | 0.3621 | 0.1835 | 2203 | 1.3586 | 0.8466 |
|             | 0.002 | 19854 | 0.0398 | 0.0125 | 4963 | 0.7823 | 0.4509 |
|             | 0.001 | 79497 | 2.63e-7 | 0 | 19874 | 0.0396 | 0.0152 |

APPENDIX A

PROOF OF THEOREM 3

Proof. For convenience, we might as well take

$$T(N) = \frac{1}{\sqrt{2\pi n(n+1)}} \left( \frac{e}{\sqrt{2\pi n}} \right)^n,$$  \hspace{1cm} (53)

and let $n'_0 = \left[ \frac{e}{\sqrt{2\pi}} \right] = c/\sigma + c' = n_0 + 1$ where $[x]$ is the smallest integer larger than or equal to $x$ and $c = e/\sqrt{2\pi}$. Obviously, $0 \leq c' \leq 1$.

Hence

$$|l(X) - \hat{l}(X)| = \sum_{n=n_0'}^{\infty} \frac{(-1)^n}{\sqrt{2\pi n(n+1)}} \left( \frac{e}{\sqrt{2\pi n}} \right)^n.$$  \hspace{1cm} (54)

where (54a) follows from \(\sum_{n=n_0'}^{\infty} |T(n)| < \frac{3\sigma}{e} \) holds.

Then we find

$$T(n'_0) = \frac{1}{\sqrt{2\pi n'_0(n'_0+1)}} \left( \frac{e}{\sqrt{2\pi n'_0}} \right)^{n'_0} = 1$$ \hspace{1cm} (55a)

$$\frac{1}{\sqrt{2\pi (c/\sigma + c') (c/\sigma + c' + 1)}} \left( \frac{c}{\sigma (c/\sigma + c')} \right)^{c/\sigma + c'} \cdot \left( 1 + \frac{c'}{c} \right)^{-c'} \left( 1 + \frac{c'}{c} \right)^{-c'}.$$ \hspace{1cm} (55b)

When $N > n'_0$, we obtain

$$T(N) = \frac{1}{\sqrt{2\pi N(N+1)}} \left( \frac{e}{\sqrt{2\pi \sigma N}} \right)^N < \frac{1}{\sqrt{2\pi n'_0(n'_0+1)}} \left( \frac{e}{\sqrt{2\pi \sigma n'_0}} \right)^{N-n'_0}$$ \hspace{1cm} (56a)

$$= \frac{1}{\sqrt{2\pi n'_0(n'_0+1)}} \left( \frac{e}{\sqrt{2\pi \sigma n'_0}} \right)^{n'_0} \left( \frac{e}{\sqrt{2\pi \sigma}} \right)^{N-n'_0}$$ \hspace{1cm} (56b)

$$= \frac{1}{\sqrt{2\pi n'_0(n'_0+1)}} \left( \frac{e}{\sqrt{2\pi \sigma n'_0}} \right)^{n'_0} \left( \frac{e}{\sqrt{2\pi \sigma}} \right)^{N-n'_0}$$ \hspace{1cm} (56c)

where (56a) follows from \(\frac{1}{\sqrt{2\pi N(N+1)}} < \frac{1}{\sqrt{2\pi n'_0(n'_0+1)}}\) because $N > n'_0$. (56c) is obtained by removing \(\frac{1}{\sqrt{2\pi N(N+1)}}\) \(-n'_0\). It requires that $0 < \left(1 + \frac{N-n'_0}{n'_0}\right)^{-n'_0} < 1$. Such a condition is satisfied because $N > n'_0 > 0$. It is obtained that $\frac{1}{N-n'_{N-n'_0}} \leq \frac{1}{-N(n'_0-1)}$ when $N-n'_0 \geq 2$. Therefore (56d) holds.

Substituting (56d) in (54b), we have (57)-(58) (See the next page). Based on the discussions above, when $\sigma$ is relatively small, we have

$$\sum_{n=n_0'}^{\infty} |T(n)| < \frac{3\sigma}{e} \cdot e^{-c'}. \hspace{1cm} (59)$$
\[ \sum_{n=n_0'}^{\infty} |T(n)| < T(n_0') \left( 1 + \frac{e}{\sqrt{2\pi\sigma}} \frac{1}{n_0' + 1} + \frac{e}{\sqrt{2\pi\sigma}} \frac{1}{(n_0' + 1)(n_0' + 2)} + \frac{e}{\sqrt{2\pi\sigma}} \frac{1}{(n_0' + 2)(n_0' + 3)} + \ldots \right) \] (57)

\[ = T(n_0') \left( 1 + \frac{e}{\sqrt{2\pi\sigma}} \frac{1}{n_0' + 1} + \frac{e}{\sqrt{2\pi\sigma}} \frac{1}{n_0' + 1} - \frac{e}{\sqrt{2\pi\sigma}} \frac{1}{n_0' + 2} + \frac{e}{\sqrt{2\pi\sigma}} \frac{1}{n_0' + 2} + \ldots \right) \] (57a)

\[ = T(n_0') \left( 1 + \frac{2c}{\sigma} \frac{1}{n_0' + 1} \right) \] (57b)

\[ = \frac{1}{\sqrt{2\pi (c/\sigma + c') (c/\sigma + c' + 1)}} \left( \left( 1 + \frac{c'\sigma}{c} \right)^{\frac{1}{\sigma}} \right)^{-e'} \left( 1 + \frac{c'\sigma}{c} \right)^{-e'} \left( 1 + \frac{2c}{c + c'\sigma + \sigma} \right). \] (57c)

(57c) is obtained by substituting (55a) in (57b). In fact, we find

\[ \lim_{\sigma \to 0} \frac{1}{3\sigma e^{-c'} - 1} \left( \left( 1 + \frac{c'\sigma}{c} \right)^{\frac{1}{\sigma}} \right)^{-e'} \left( 1 + \frac{c'\sigma}{c} \right)^{-e'} \left( 1 + \frac{2c}{c + c'\sigma + \sigma} \right) = 1. \] (58)

Therefore, when \( \sigma \) is relatively small, \( 3\sigma e^{-c'} - 1 \) is a good approximate value for (57c).

In fact \( 0 \leq e' \leq 1 \), so we obtain

\[ \sum_{n=n_0'}^{\infty} |T(n)| < \frac{3\sigma}{e}. \] (60)

Hence,

\[ |l(x) - l(\hat{x})| < \frac{3\sigma}{e}. \] (61)

The proof is completed.

**APPENDIX B**

**PROOF OF THEOREM 4**

**Proof.** In fact, a upper bound of \( P\{D_n > d\} \) is given by

\[ P\{D_n > d\} \approx 2 \sum_{k=1}^{\infty} (-1)^{k-1} e^{-2nk^2d^2} \] (62)

\[ = 2 \sum_{m=1}^{\infty} e^{-2n(2m-1)^2d^2} - 2n(2m-1+1)^2d^2) \] (62a)

\[ = 2 \sum_{m=1}^{\infty} e^{-2n(2m-1)^2d^2} - 2n(4m-1)^2d^2) \] (62b)

\[ \leq 2 \sum_{m=1}^{\infty} e^{-2n(2m-1)^2d^2} \] (62c)

\[ \leq 2 \sum_{m=1}^{\infty} e^{-4nd^2(2m-1)+2nd^2} \] (62d)

\[ = 2 \sum_{m=1}^{\infty} e^{-8nd^2m+6nd^2} \] (62e)

\[ = \frac{2e^{-2nd^2}}{1-e^{-8nd^2}}. \] (62f)

(62c) is obtained for the fact that \( 1 - e^{2n(4m-1)d^2} \leq 1 \). (62d)

\[ 2n(2m-1)^2d^2 \leq -4nd^2(2m-1)+2nd^2. \]

Such a condition is satisfied because \( -2nd^2(2m-1-1)^2 \leq 0 \).

This means, we only need to check \( \frac{2e^{-2nd^2}}{1-e^{-8nd^2}} \leq \beta \) holds.

Substituting (48) and (49) in \( |\gamma(\infty) - \gamma(n)| \leq \varepsilon \), we get

\[ \left| \frac{1}{l(X)} - e^{-\frac{\beta^2}{4\pi^2\sigma^2 l(X)}} \right| \leq \varepsilon \Rightarrow n \geq \frac{1}{4\pi^2\sigma^2 l(X)(1-\varepsilon)}. \] (63)

Because \( 0 \leq l(X) \leq 1 \), we obtain

\[ n \geq \frac{1}{4\pi^2\sigma^2 l(X)(1-\varepsilon)} \geq \frac{1}{4\pi^2\sigma^2 l(X)} \] (64)

Letting

\[ d = \sqrt{2\pi^2 \ln \frac{19}{9\beta} \ln \frac{1}{1-\varepsilon}}, \] (65)

we have

\[ 2nd^2 \geq 2 \frac{2\pi^2 \ln \frac{19}{9\beta} \ln \frac{1}{1-\varepsilon}}{4\pi^2\sigma^2 l(X)(1-\varepsilon)} \Rightarrow e^{-2nd^2} \leq \frac{9\beta}{19}. \] (66)

It is easy to check

\[ \beta \left( e^{-2nd^2} \right)^4 + 2e^{-2nd^2} - \beta \leq 0, \] (67)

when \( \beta \leq \frac{19}{4} \sqrt{\frac{19}{19}} \approx 1.0112 \). In fact, \( \beta \) is a threshold value of the probability, so we usually take \( \beta \leq 1 \). Therefore, (67) holds all the time.

Hence,

\[ \frac{2e^{-2nd^2}}{1-e^{-8nd^2}} \leq \beta. \] (68)

Based on the discussions above, we get

\[ P\{D_n > \sqrt{2\pi^2 \ln \frac{19}{9\beta} \ln \frac{1}{1-\varepsilon}} \} < \beta. \] (69)

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