D=(0|2) Dirac–Maxwell–Einstein Theory
as a Way for Describing Supersymmetric Quartions

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Abstract

Drawing an analogy with the Dirac theory of fermions interacting with electromagnetic
and gravitational field we write down supersymmetric equations of motion and construct
a superfield action for particles with spin $\frac{1}{4}$ and $\frac{3}{4}$ (quartions), where the role of quartion
momentum in effective (2+1)-dimensional space-time is played by an abelian gauge superfield propagating in a basic two-dimensional Grassmann-odd space with a cosmological
constant showing itself as the quartion mass. So, the (0|2) (0 even and 2 odd) dimensional
model of quartions interacting with the gauge and gravitational field manifests itself as
an effective (2+1)-dimensional supersymmetric theory of free quartions.

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1 Introduction

In modern theoretical physics there are two branches of research being closely related to fundamental problems of space-time, namely the twistor program \([1]\) and supersymmetry \([2]\). Both of them are based on the fundamental role played by the commuting and anticommuting spinors in establishing the relationship between quantum mechanical properties of physical objects and the geometrical structure of space-time.

In the present paper we suggest another argument for the idea that spinors are in the foundation of “everything” by proposing a field-theoretical model in \((0|2)\)-dimensional Grassmann-odd spinor space for describing particles with spin \(\frac{1}{4}\) and \(\frac{3}{4}\) called quartions \([3]\). The counterparts of the quartions with respect to quantum statistics are known under the name ‘semions’ \([4]\) (because of the middle position they take between bosons and fermions). Since the semions are mainly considered in the Chern–Simons approach to the anyons, which has not been proved to be completely equivalent to the group-theoretical approach, and since in the group-theoretical approach the Pauli spin-statistics principle for anyons has not been completely proved yet (though there is a strong evidence \([11]\) that it does take place) \([\dagger]\) we distinguish the quartions and the semions by their names, and just the quartions are the subject of the present paper.

The quartions and semions attract great deal of attention because of their possible relation to the problem of high-\(T_c\) superconductivity and other problems of strongly correlated quantum electron systems in two-dimensional space \([5,6,7]\), and due to their peculiar group-theoretical \([3,7]\) and interaction \([8]\) properties. Besides, quartions may provide a consistent way of describing solitons with the spin values \(s = \frac{1}{4}, \frac{3}{4}\) which were discovered in a \(D = 2 + 1\) chiral \(\sigma\)-model by Balachandran et. al. \([9]\), and one may presume some correspondence between this chiral model and a model of interacting quartions in analogy to the well known equivalence of the sine-Gordon and the Thirring model \([10]\).

The development of the group-theoretical approach to anyons is hindered by a lack of reliable symmetry and geometrical ground, which results in problems with constructing field equations of motion and actions for anyons. Drawing an analogy with the Dirac theory of fermions interacting with electromagnetic and gravitational field allows us to write down supersymmetric equations of motion and construct a superfield action for quartions, where the role of quartion momentum in effective \((2+1)\)-dimensional space-time is played by an abelian gauge superfield propagating in a basic two-dimensional Grassmann-odd space with a cosmological constant manifesting itself as the quartion momentum.

\(\dagger\)On general problems of group-theoretical approach to describing particles with fractional spin and statistics see \([11,12,13,14]\) and references therein.
mass. The quartion equations of the model under consideration coincide with that of an
alternative D=2+1 supersymmetric field model proposed in [7]. So, the (0|2)–dimensional
model of quartions interacting with the gauge and gravitational field manifests itself as
the effective (2+1)-dimensional supersymmetric theory.

2 Group-theoretical background

We start to draw the analogy between fermions and quartions by comparing their Lorentz
group representations. To describe the Lorentz transformations of spinors one introduces
the Dirac matrices \((\gamma^m)_{\alpha\beta}\) transforming as the D-dimensional vector representation of the
Lorentz group and forming the Clifford algebra characterized by the anticommutator
\[
\{\gamma^m, \gamma^n\} \equiv \gamma^m \gamma^n + \gamma^n \gamma^m = 2g^{mn},
\]
where \(g^{mn}\) is the Lorentz metric whose signature is chosen to be (+,-,...,-). Small latin
letters stand for vector indices and small greek letters stand for spinor indices.

The commutators of the Dirac matrices form the generators of the spinor representa-
tion of the Lorentz group
\[
S^{mn} = \frac{i}{4} [\gamma^m, \gamma^n].
\]

From the other hand, to describe quartions in D=2+1 one has to consider infinite-
dimensional representations of the Lorentz group \(SO(1,2) \sim SL(2,R)\) characterized by
the lowest spin weights \(\frac{1}{4}\) and \(\frac{3}{4} \) [15], which may be constructed by means of a Majorana
spinor operator \(L^\alpha (\alpha = 1, 2) \) [3, 7] forming the Heisenberg algebra with respect to the
commutator
\[
[L^\alpha, L^\beta] \equiv L^\alpha L^\beta - L^\beta L^\alpha = i\varepsilon^{\alpha\beta},
\]
where \(\varepsilon^{\alpha\beta} = \varepsilon_{\alpha\beta} = -\varepsilon^{\beta\alpha} \) \((\varepsilon^{12} = 1)\) is the metric in a spinor space. An infinite
dimensional representation of the Heisenberg–Weyl group generated by the algebra (3) is well-known
and describes the energy eigenstates of a one-dimensional quantum oscillator
\[
|n> = \frac{1}{\sqrt{n!}} (a^+)^n |0> \quad (n = 0, 1, ..., \infty),
\]
where \(a^+ = \frac{1}{\sqrt{2}}(L_1 - iL_2)\) and \(a = \frac{1}{\sqrt{2}}(L_1 + iL_2)\) are the raising and lowering operator,
respectively, and \(a|0> = 0\).

The anticommutators of the \(L^\alpha\) components
\[
S^{\alpha\beta} = \frac{1}{4} \{L^\alpha, L^\beta\}
\]
form the infinite-dimensional group representation we are looking for. Indeed, as one can
directly verify using (3), \(S^{\alpha\beta}\) satisfy the commutation relations for the Lorentz generators,
and the Casimir operator $\frac{1}{2}S_{\alpha\beta}S^{\alpha\beta}$ of the representation has the eigenvalue $s_0(s_0 - 1) = -\frac{3}{16}$. Thus $s_0$ is equal to either $\frac{1}{4}$ or $\frac{3}{4}$, and with respect to $SL(2, R)$ the Heisenberg-Weyl group representation $\{\}$ splits into the Lorentz group representations with the lowest weights $\frac{1}{4}$ and $\frac{3}{4}$, which correspond, respectively, to even and odd values of $n$ in Eq. (4), both representations being transformed into each other by $L_\alpha$.

An important assumption we make here is that since the difference between spin $\frac{3}{4}$ and $\frac{1}{4}$ is $\frac{1}{2}$, the relative statistics of corresponding quartionic states upon proper quantization should be fermionic, and, hence, $L_\alpha$ are to be odd operators in accordance with Eq. (5). So, the algebra $\{\}$, with $L_\alpha$ satisfying the additional conditions $\{\}$, generates an infinite dimensional $OSp(1, 2)$ group representation. Note that both the $OSp(1, 2)$ representation and the Heisenberg–Weyl group representation are realized on the same Hilbert space $\{\}$.

We see that $\gamma^m$ and $L_\alpha$ are the antipodes in the sense that where the commutator arises for the Dirac matrices the anticommutator arises for $L_\alpha$ and vice versa. Below we use this interchange in the commuting and anticommuting properties for constructing the quartion model in analogy with the Dirac–Maxwell–Einstein theory.

### 3 Equations of motion

Fermions are well-known to be described by spinor wave functions $\psi_\alpha(x)$ of bosonic space-time coordinates $x^m$. So, there is a correspondence between $x^m$ and $\gamma^m$ (they both form the vector representation of the Lorentz group), which is required for writing down the Dirac equation

$$ (i\gamma^m \partial_m + m)\psi(x) = 0, \quad (6) $$

for free fermions with a mass $m$.

Following the correspondence of the group-theoretical structure of space-time coordinates to that of the basic operators $\{\}$ and $\{\}$ we consider the quartion wave functions $\Phi(\theta)$ to be determined in a $(0|2)$–dimensional Grassmann-odd space parametrized by anticommuting Majorana spinor coordinates $\theta_\alpha$ and transformed as the irreducible representation of the Heisenberg-Weyl group $\{\}$ (or more strictly speaking $OSp(1, 2)$)

$$ \Phi(\theta) = \sum_{n=0}^{+\infty} \Phi^{(n)}(\theta)|n>. \quad (7) $$

Thus $\Phi^{(n)}(\theta)$ are polynomials of the form

$$ \Phi^{(n)}(\theta) = A^{(n)} + i\theta^\alpha \psi^{(n)}_\alpha + i\theta^\alpha \theta_\alpha C^{(n)}. \quad (8) $$

Note that $\Phi^{(n)}(\theta)$ are superfields with respect to the $SL(2, R)$ rotations and the shifts
\[ \theta_\alpha \rightarrow \theta_\alpha + \epsilon_\alpha \] in the Grassmann space and have relative fermionic statistics for even and odd \( n \) respectively.

By analogy with the Dirac equation (6), the “free” equation of motion for quaternions is written as

\[ (iL^\alpha \partial_{\theta^\alpha} + \kappa)\Phi(\theta) = 0, \quad (9) \]

where \( \kappa \) is a “mass” parameter whose physical meaning will be determined below.

Next step is to introduce the interaction of the objects in question with an abelian gauge field. In the Dirac–Maxwell theory this is achieved by lengthening the derivative in (6) with a bosonic vector field \( A_m(x) \) described by the Maxwell action

\[ S_{A_m} = -\frac{1}{4} \int d^D x F_{mn} F^{mn}, \quad (10) \]

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One can easily see that $a_\alpha$ and $p_{\beta\alpha}^{\alpha\beta}$ component of $A_\alpha(\theta)$ may be gauged away; hence, $p_{\alpha\beta}$ is a real symmetric matrix represented in the form

$$p_{\beta\alpha} = ip_m \gamma_{\alpha\beta}, \quad (17)$$

where $p_m$ is a constant vector to be interpreted below as the quarton momentum in the effective (2+1)-dimensional Minkowski space-time. Note also that the $c_\alpha$ component of $A_\alpha$ has the meaning of a quarton current (see next section). So the gauge field turns out to carry such dynamical characteristics of the quartions as the momentum and the current.

In the presence of the gauge field $A_\alpha(\theta)$ the quarton equation takes the form

$$(iL^\alpha D_\alpha + \kappa)\Phi(\theta) \equiv (iL^\alpha(\partial_\alpha + A_\alpha(\theta)) + \kappa)\Phi(\theta) = 0, \quad (18)$$

whose integrability condition

$$\left(\frac{i}{2} D_\alpha D^\alpha + \frac{1}{2} F_{\alpha\beta} L^\alpha L^\beta + \kappa^2\right)\Phi(x) = 0 \quad (19)$$

indicates that the quartons possess nonzero “magnetic” moment, which will be interpreted as the quarton helicity times a quarton mass in the effective (2+1)-dimensional space-time.

Note the presence of the imaginary unit $i$ in the transformation law (15) for $A_\alpha(\theta)$ and the absence of $i$ in the spinor derivative (18). This is because $\partial/\partial \theta$, in contrast to $\partial/\partial x^m$, is a Hermitian operator.

Let us consider now the simplest case corresponding to the propagation of a quarton wave function in an external gauge field $A_\alpha(\theta)$ satisfying the free equations of motion

$$\frac{\partial F_{\alpha\beta}}{\partial \theta^\alpha} = 0. \quad (20)$$

It is easy to verify that the single solution of Eq. (20) is $c_\alpha = 0$; hence, the only physically nontrivial component of the superfield (14) is the vector $p_{\alpha\beta}$, and (18) and (19) are reduced to

$$(iL^\alpha D_\alpha + \kappa)\Phi(\theta) = 0, \quad (21)$$

$$(iL^\alpha D_\alpha - \kappa)(iL^\alpha D_\alpha + \kappa)\Phi = \left(\frac{i}{2} D_\alpha D^\alpha \Phi(\theta) + p_{\alpha\beta} L^\alpha L^\beta - \kappa^2\right)\Phi(\theta) = 0. \quad (22)$$

where in $D_\alpha \equiv \partial/\partial \theta^\alpha + \theta^\beta p_{\beta\alpha}$ one may recognize the supercovariant derivative of the N=1, $D = 2 + 1$ SUSY theory [10] in the momentum representation, while, due to Eq. (3), the “magnetic moment” term $p_{\alpha\beta} L^\alpha L^\beta$ is nothing but the Pauli-Lyubanski scalar of the $D = 2 + 1$ super Poincarè group (see, for example, [6]). Indeed, since for $A_\alpha(\theta)$ with $c_\alpha = 0$ the following anticommutation relations take place:

$$\{D_\alpha, D_\beta\} = 2p_{\alpha\beta}, \quad \{Q_\alpha, Q_\beta\} = -2p_{\alpha\beta}, \quad \{Q_\alpha, D_\beta\} = 0, \quad (23)$$
(where \( Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - A_\alpha(\theta) \) is a supersymmetry generator), it follows that Eq. (21) is invariant under supersymmetry transformations

\[
\delta \Phi(\theta) = i \epsilon^\alpha Q_\alpha \Phi(\theta).
\]  

(24)

Note that when \( c_\alpha = 0 \) the gauge condition resulting in \( p_{\alpha \beta} = p_{\beta \alpha} \) looks as \( \partial_\alpha A^\alpha(\theta) = 0 \), which is analogous to the Lorentz condition \( \partial \lambda A^\lambda(x) = 0 \) for the electromagnetic field.

As in the case of the Majorana equation [17], Eq. (21) describes an infinite tower of spinning states with tachyon being present in the spectrum. To single out quartionic states with positive mass square and definite values of helicity one has to impose the constraint

\[
\frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} = 2m^2 = p_{\alpha \beta} p^{\alpha \beta},
\]

(25)

which plays the role of the mass shell condition for quartenions (\( m \) is a constant). To find the solutions of Eqs. (21) and (22), with (25) being taken in to account, let us write down the equations for \( \Phi(\theta) \) components which follow from (21) and (22). We get

\[
L^{\alpha} \psi_\alpha = \kappa A,
\]

\[
L^{\beta} p_{\beta \alpha} A - 2i L^{\alpha} C = \kappa \psi_\alpha,
\]

\[
\frac{i}{2} L^{\beta} p_{\beta \alpha} \psi^{\alpha} = \kappa C
\]

(26)

and

\[
2C = (p_{\alpha \beta} L^{\alpha} L^{\beta} - \kappa^2)A,
\]

\[
\frac{1}{4} p_{\alpha \beta} p^{\alpha \beta} A = (p_{\alpha \beta} L^{\alpha} L^{\beta} - \kappa^2)C;
\]

\[
p_{\alpha \beta} \psi^{\beta} = i (p_{\beta \gamma} L^{\beta} L^{\gamma} - \kappa^2) \psi_\alpha.
\]

(27)

For simplicity we shall perform further consideration of Eqs. (26) and (27) in the rest frame

\[
p_0^n = (m, 0, 0), \quad (p_0)_\alpha^\beta = i m \gamma^0_\alpha^\beta
\]

. From Eqs. (27) and (23), with taking into account

\[
p_{\alpha \beta} L^{\alpha} L^{\beta} \sum_{n=0}^{+\infty} A^{(n)} |n> = m \sum_{n=0}^{+\infty} (2n + 1) A^{(n)} |n >,
\]

(28)

it follows that

\[
((2n + 1)m - \kappa^2)^2 = m^2,
\]

(29)

and \( \kappa^2 \) and \( m \) are to be connected with each other by the equation

\[
\kappa^2 = 2mn,
\]

(30)
or

\[ \kappa^2 = 2m(n' + 1), \]  
\[ (31) \]

It is essential that for given values of \( \kappa^2 \) and \( m \) Eq. (22) (or (27)) singles out quartion states characterized by the number \( n \) or \( n' = n - 1 \). So the only nonzero components of \( A \) and \( C \) are

\[ mA^{(n)} = 2C^{(n)}, \]  
\[ (32) \]

for \( \kappa^2 \) determined by (30), and

\[ mA^{(n-1)} = 2C^{(n-1)}, \]  
\[ (33) \]

for \( \kappa^2 \) determined by (31) with \( n' = n - 1 \). In the first case (Eqs. (30), (32)) we have for \( \sum_{n=0}^{+\infty} \psi^{(n)}_\alpha |n \rangle \) the equation

\[ ip_\beta^{(n')}_\alpha \psi^{(n')}_\beta = (2(n' - n) + 1)m\psi^{(n')}_\alpha = 0, \]  
\[ (34) \]

from which it follows that for (34) to be consistent with (25) \( n' \) must be equal to either \( n \) or \( n - 1 \), the first value being excluded by Eqs. (26), which indicates that the nonzero values of \( n \) of quartionic states corresponding to the “scalar” and “spinor” components of \( \Phi(\theta) \) are differed by 1. Hence \( \psi^{(n-1)}_\alpha \) satisfy the Dirac equation

\[ ip_\beta^{(n-1)}_\alpha \psi^{(n-1)}_\beta + m\psi^{(n-1)}_\alpha = 0, \]  
\[ (35) \]

which leaves nonzero only \( \psi^{(n-1)}_- = \frac{1}{\sqrt{2}}(\psi_1^{(n-1)} + i\psi_2^{(n-1)}) \) component of \( \psi^{(n-1)}_\alpha \); \( \psi^{(n-1)}_- \) being connected with \( A^{(n)} \) due to Eq. (24):

\[ i(-1)^n\psi^{(n-1)}_- = \sqrt{2mA^{(n)}}. \]  
\[ (36) \]

Factor \((-1)^n\) appears in (36) because of the (anti)commuting properties of \( L_\alpha \) and the corresponding field components.

Starting with (31) and following the same reasoning as above, we find the nonzero components of \( \Phi(\theta) \) to be \( A^{(n-1)} \) and \( C^{(n-1)} \) which satisfy (32) and, \( \psi^{(n)}_+ = \frac{1}{\sqrt{2}}(\psi_1^{(n)} - i\psi_2^{(n)}) \) component of \( \psi_\alpha \) which satisfies the Dirac equation

\[ ip_\beta^{(n)}_\alpha \psi^{(n)}_\beta - m\psi^{(n)}_\alpha = 0, \]  
\[ (37) \]

and

\[ i(-1)^n\psi^{(n)}_- = -\sqrt{2mA^{(n-1)}}. \]  
\[ (38) \]

Thus, the general solution of (20) (in the rest frame) is

\[
\Phi(\theta) = A^{(n)}(|n> + \sqrt{2m}(-1)^n\theta_+|n - 1> + im\theta_+\theta_-|n>) \\
+ A^{(n-1)}(|n - 1> + \sqrt{2m}(-1)^n\theta_-|n> + im\theta_+\theta_-|n - 1>)
\]  
\[ (39) \]
(no summation over \( n \) is implied!). It should be noted once again that for \( \Phi(\theta) \) to possess definite statistics the components and the basic vectors, characterized by indices \( n \) differed by 1, must have the odd relative statistics.

Consider now the case \( \kappa = 0 \). At this Eq. (29) has the only one solution (30) with \( n = 0 \). The solution (31) should be skipped, since for our choice of the quaternion representations \( n \) is the non-negative integer. As a result all \( \Phi(\theta) \) components satisfying (26) correspond to the lowest state of the representation (4), and \( \Phi(\theta) \) obeys the equations

\[
L^\alpha D_\alpha \Phi(\theta) = 0, \tag{40}
\]

\[
p_{\alpha\beta} L^\alpha L^\beta \Phi(\theta) = m, \tag{41}
\]

\[
(D_\alpha D^\alpha - 2im) \Phi(\theta) = 0, \tag{42}
\]

where (42) is nothing but the equation of motion for the massive scalar superfield in \( N = 1, D = 2 + 1 \) superspace. Eq. (10) together with (12) reproduces equations of motion for a quartion superfield with the lowest helicity \( \frac{1}{4} \) obtained in [7]. Thus, the propagation of a two-dimensional quartionic field in the free gauge field \( A_\alpha(\theta) \), that is one described by Eqs. (20) and (25), is equivalent to the propagation of free superparticle quartionic states with the mass \( m \) and the superhelicities \( \frac{1}{4} + (n - 1) \) and \( \frac{1}{4} + n \) (for \( \kappa = 2mn \neq 0 \)), and a single free superparticle quartionic state with the superhelicity \( \frac{1}{4} \) (for \( \kappa = 0 \)).

4 Dirac–Maxwell–Einstein action for quartions

Let us firstly try to construct a quartion action, from which the quartion equation of motion (18) can be obtained, as the direct counterpart of the Dirac–Maxwell theory:

\[
S = \int d^2\theta (\Phi^\dagger(\theta) (iL^\alpha D_\alpha + \kappa) \Phi(\theta) - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}) \tag{43}
\]

where \( \Phi^\dagger(\theta) \) is Hermitian conjugate to \( \Phi(\theta) \).

Equations of motion for \( A_\alpha(\theta) \), which follow from the action (43), have the form

\[
\frac{\partial F^{\beta\alpha}}{\partial \theta^\beta} = i \Phi^\dagger(\theta) L^\alpha \Phi(\theta), \tag{44}
\]

or

\[
c_\alpha = \frac{1}{6} \Phi^\dagger(\theta) L_\alpha \Phi(\theta), \tag{45}
\]

where \( \Phi^\dagger(\theta) L^\alpha \Phi(\theta) \) is a conserved odd current analogous to the current \( \bar{\psi}\gamma_m \psi \) of Dirac fermions. Just as the Dirac current is nonzero when the fermion is charged (i.e. described by two irreducible representations of the Lorentz group) the semion current is nonzero if
the both types of the quartions (with spin $\frac{1}{4}$ and $\frac{3}{4}$) propagate in the space; at this $c_\alpha$ component of the gauge field is completely determined by the quartion current (Eq. (45)) and does not have independent degrees of freedom, analogous to the Chern-Simons gauge field which is completely determined by the current of charged matter fields \[18\]. Note that since $c_\alpha$ is a constant independent of $\theta$ the components of $\Phi^\dagger$ and $\Phi$ should satisfy the following relations

$$
\frac{\partial}{\partial \theta^\beta} \Phi^\dagger L_\alpha \Phi = A^\dagger L_\alpha \psi_\beta - \psi^\dagger_\beta L_\alpha A = 0, \\
(\frac{i}{2} \frac{\partial}{\partial \theta^\gamma} \varepsilon^{\gamma\delta} \frac{\partial}{\partial \theta^\delta}) \Phi^\dagger L_\alpha \Phi = A^\dagger L_\alpha C + C^\dagger L_\alpha A + \frac{i}{2} \psi^\dagger_\beta L_\alpha \psi_\beta = 0,
$$

Substituting Eq. (44) into (43) we get an action for quartions with “point-like” four-quartion interaction

$$
S = \int d^2\theta \left( \Phi^\dagger(\theta)(iL_\alpha D_\alpha + \kappa)\Phi(\theta) - \frac{1}{12} \delta(\theta)(\Phi^\dagger L_\alpha \Phi)(\Phi^\dagger L_\alpha \Phi) \right),
$$

where $\delta(\theta) = \theta_\alpha \theta^\alpha$.

This quartion interaction differs from a current-current interaction of anyons caused by an induced magnetic moment \[19, 8\] since the latter involves anyons with a single value of spin.

The model based on action (43) or (46) describes interacting quartions. In general, it contains a tachyon sector since $p_{\alpha\beta}$ may take space-like values. As we have already seen above, a possible way to select a physical sector is to restrict the space of quartion wave functions by the constraint (25) on the gauge field stress tensor. Eq. (25) thus imposed is an external constraint of the model which would be better to obtain from an action as an equation of motion.

The constraint (25) can be formally introduced into the action (43) with a corresponding Lagrange multiplier. It occurs possible to give this multiplier a geometrical meaning. To this end we generalize (43) to include gravity in the (0|2)–space \[20\].

Let us extend the global transformations $\theta_\alpha \rightarrow \theta_\alpha + \varepsilon_\alpha$ and the $SL(2, \mathbb{R})$ rotations of the $\theta_\alpha$ coordinates of the (0|2)–space to general coordinate transformations \[20\]

$$
\theta^{\alpha} = \theta^{\prime \alpha}(\theta).
$$

These transformations leave invariant the following length element in the Grassmann-odd space

$$
g'_{\alpha\beta}(\theta')d\theta'^\alpha d\theta'^\beta = g_{\alpha\beta}(\theta)d\theta^\alpha d\theta^\beta,
$$

where $g_{\alpha\beta}(\theta)$ is an arbitrary antisymmetric metric which can be represented in the form

$$
g_{\alpha\beta}(\theta) = \frac{1}{G(\theta)} \varepsilon_{\alpha\beta}
$$
so that $G(\theta)$ has the transformation properties of a scalar density

$$G'(\theta') = \text{det} \left( \frac{\partial \theta'^{\alpha}}{\partial \theta^{\beta}} \right) G(\theta)$$

(50)

and plays the role of the gravitation field. Note that one may fix the gauge, relative to transformations (47), in such a way that $G(\theta)$ power expansion in $\theta^{\alpha}$ is reduced to

$$G(\theta) = 1 + \frac{i}{2} \theta_{\alpha} \theta^{\alpha} G_0.$$

(51)

The Christoffel symbol and the spin connection satisfying the condition of the absence of torsion in the $(0|2)$–space are, respectively

$$\Gamma_{\alpha\beta}^{\gamma} = -\Gamma_{\beta\alpha}^{\gamma} = \varepsilon^{\gamma\delta} \partial_{\delta} (\ln G) \varepsilon_{\alpha\beta}$$

(52)

and

$$\omega_{\alpha a b} = \omega_{ab \alpha} = -e^\beta_b \left( \partial_{\alpha} e_{\beta a} - \Gamma^\gamma_{\alpha\beta} e^\gamma_a \right) = -e^\beta_b \left( \partial_{\alpha} e_{\beta a} + \frac{1}{G} e^\gamma_a e_{\alpha\beta} \partial_{\gamma} (\ln G) \right),$$

(53)

where $e^a_\alpha(\theta)$ are zweinbeins determined by the relations

$$Ge^a_\alpha e^b_\beta \varepsilon^{\alpha\beta} = \varepsilon^{ab}, \quad Ge^a_\alpha e^b_\beta \varepsilon_{a b} = \varepsilon_{\alpha\beta};$$

$a, \ b$ are indices of the tangent spinor space.

Taking into account (51) one may choose a local tangent space symmetry gauge in such a way that $e^a_\alpha(\theta) = \delta^a_\alpha (1 - \frac{i}{4} \theta_{\beta} \theta^{\beta} G_0)$.

The curvature tensor is determined as

$$R_{\alpha\beta\gamma}^\delta = i (\partial_{\alpha} \Gamma^\delta_{\beta\gamma} + \partial_{\beta} \Gamma^\delta_{\alpha\gamma} + \Gamma^\rho_{\alpha\gamma} \Gamma^\delta_{\rho\beta} + \Gamma^\rho_{\beta\gamma} \Gamma^\delta_{\rho\alpha}),$$

from which one gets the Ricci tensor and the scalar curvature in the form

$$R_{\alpha\beta} = \frac{3i}{2} \partial_{\alpha} \partial^{\gamma} G \varepsilon_{\alpha\beta}$$

and

$$R = -GR_{\alpha\beta} \varepsilon^{\beta\alpha} = 3i (\partial_{\gamma} \partial^{\gamma} G) = 6G_0.$$

(54)

Eqs. (54) indicate that the $(0|2)$–space is an Einstein space of a constant curvature [20].

In Eq. (54) and below the indices are raised and lowered by $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$, respectively.

Now, taking into account that the integration measure $d^2\theta$ transforms under (47) as

$$d^2\theta' = \frac{1}{\text{det} \left( \frac{\partial \theta'^{\alpha}}{\partial \theta^{\beta}} \right)} d^2\theta,$$

(55)
one may write down the generalization of (43) as follows

\[ S_G = \int d^2 \theta G \left( \Phi^\dagger(\theta) (iL^a e^\alpha_a \nabla_\alpha + \kappa) \Phi - R - \frac{1}{4} G^2 F_{\alpha\beta} F^{\alpha\beta} + 6m^2 \right), \tag{56} \]

where \( \nabla_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{4} \omega_{abc} L^b L^c + A_\alpha(\theta) \), and \( 6m^2 \) is a “cosmological” constant which should be positive for the model to be free of tachyons (see below). Note that this situation is, in some sense analogous to that in conventional supergravity theories, where a cosmological constant corresponding to the anti–De–Sitter space-time is consistent with supersymmetry while a De–Sitter cosmological constant is not (see, for example, \[22\] and references therein). Let us also remark that in contrast to the ordinary \( D = 2 \) gravity \( GR \) term in (56) is not a total derivative. This is because the odd directions imply the “negative” dimensionality of the space.

Varying Eq. (56) over \( e^\alpha_a(\theta) \) and converting the result with \( e^\alpha_a(\theta) \) we get the Einstein equation in the form

\[ 2R = 6i \partial_\gamma \partial^\gamma G = GT^\alpha_\alpha - 3\left(\frac{1}{4} G^2 F^{\alpha\beta} F_{\alpha\beta} - 2m^2 \right), \tag{57} \]

where

\[ T^\alpha_\beta = -T_{\beta\alpha} = \frac{i}{2} (\Phi^\dagger(\theta) L^a e_{a[\alpha} \nabla_{\beta]} \Phi - (L^a e_{a[\alpha} \nabla_{\beta]} \Phi)^\dagger \Phi) = -\frac{\kappa}{2G} \varepsilon_{\alpha\beta} \Phi^\dagger \Phi \tag{58} \]

is the quartion energy-momentum tensor.

In the case of \( \kappa \neq 0 \) the solution of Eq. (57) does not reproduce the solution (9) of (21) and (25) for quartions with higher helicities, since the matter energy-momentum tensor (58) contributes to the gravitation equations of motion causing the quartion dynamics in this case to be more complicated.

For \( \kappa = 0 \) we observe that \( T^\alpha_\beta \) is equal to zero on the mass shell (as a consequence of the quartion equations of motion). This indicates that quartions with superhelicity \( \frac{1}{4} \) get nontrivial dynamics only owing to interactions with the gauge and gravitational field. So \( T^\alpha_\beta \) drops from Eq. (57) and this allows one to get Eq. (25) from (57).

In components (with gauge conditions (17) and (51) imposed, and \( \kappa = 0 \) Eq. (57) looks as

\[ p^\alpha_\beta p^{\alpha\beta} = 2(m^2 - 2G_0), \]

\[ p^\alpha_\beta c^{\beta} = 0, \]

\[ 3c^\alpha c_\alpha = 2i(m^2 - 2G_0)G_0. \tag{59} \]

The system of Eqs. (58) has two solutions. The first one is \( G_0 = \frac{m^2}{2} \), \( p^\alpha_\beta p^{\alpha\beta} = 0 \) (the latter being explicitly solved by the Cartan relation \( p^\alpha_\beta = \mu^\alpha_\beta \), where \( \mu^\alpha_\beta \) is a commuting
Majorana spinor) and \( c_\alpha = \varrho \mu_\alpha \) (where \( \varrho \) is an anticommuting number). This solution is valid also for the case of free gravity, when in \((0|2)\)–space all matter fields are absent, that is it is caused by the gravitational degree of freedom. At present time it is not clear whether this solution leads to interesting consequences for anyon physics.

By dropping the curvature term out of the action we can single out the second solution of Eqs. (58) which is just one we are looking for: \( c_\alpha = G_0 = 0 \), which indicates that on the mass shell the geometry of \((0|2)\)–space is flat, \( A_\alpha(\theta) \) is a free gauge field with \( p_{\alpha\beta} \) being timelike, and the cosmological constant plays the role of the quartion mass. In the case considered one may directly verify that Eqs. (40)–(42) for \( \Phi^{(1/4)} \) and \( \Phi^{(3/4)} \) are received from (56) as equations of motion and integrability conditions thereof, their solutions being \( \Phi^{(3/4)} = 0 \) and the one-superparticle quartion state with the superhelicity \( 1/4 \) and mass \( m \) for \( \Phi^{(1/4)} \).

## 5 Conclusion

We have constructed the field-theoretical model in the \((0|2)\)–space for describing quartions, which turned out to be effectively equivalent to the momentum representation of the quartion field theory in \( N = 1, D = 2 + 1 \) superspace. We assume that the action of the corresponding \( D = 2 + 1 \) model may be obtained as an effective action by a functional integration of Eq. (43) or (56) over all possible independent configurations of \( A_\alpha(\theta) \).

We have shown that supersymmetry proved to be intrinsic to the quartions, since the helicities of the particles with \( s = \frac{1}{4} \) and \( s = \frac{3}{4} \) are differed by \( \frac{1}{2} (\text{mod } n) \) and their relative statistics is fermionic.

The model proposed is essentially based on analogy with the Dirac–Maxwell–Einstein theory, which allowed one to provide all fields of the model, including the Lagrange multipliers, with clear geometrical and physical meaning.

Note that this analogy can be drawn even further by starting with the comparison of the mechanics of a relativistic spinning particle [21] and that of quartions [7]. The spin part of the relativistic spin 1/2 particle action is described by the term

\[
S = \int d\tau i \chi \frac{d}{d\tau} \chi^m,
\]

where \( \tau \) is a world-line parameter and \( \chi^m \) are anticommuting classical counterparts of the Dirac matrices \( \gamma^m \). By generalizing Eq. (60) to be invariant under local supersymmetry transformations in \((\tau, \eta)\) world-line superspace \( (\eta^2 = 0) \) we restore a superfield action for the spin 1/2 particle [21]

\[
S = i \int d\tau d\eta E(\tau, \eta)(\partial_\eta + i\eta \partial_\tau)X_m \partial_\tau X^m,
\]
where the superfield $X^m = x^m(\tau) + i\eta \chi^m(\tau)$ contains $\chi^m$ and a particle space coordinate $x^m$ as the components. $E = e(\tau) + i\eta \psi(\tau)$ is the supereinbein which ensures the local superinvariance of (61).

By analogy with the spin 1/2 particle we can write down a spin part of a classical quartion action in the form

$$S = \int d\tau \alpha(\tau) \cdot d\tau \alpha(\tau),$$

(62)

where $\alpha(\tau)$ are commuting classical counterparts of $L_\alpha$, and get a complete action for a quartion particle moving in the $(0|2)$–space by generalizing (62) to the superfield action

$$S = i \int d\tau d\eta E(\tau, \eta)(\partial_\eta + i\eta \partial_\tau)\Theta \cdot \partial_\tau \Theta,$$

(63)

where $\Theta = \theta(\tau) + \eta \lambda(\tau)$ contains $\lambda$ and $\theta$ coordinates of the quartionic particle.

Adding to (63) a term $\int d\tau d\eta A_\alpha(\Theta)(\partial_\eta + i\eta \partial_\tau)\Theta$ we get classical mechanics counterpart of the model considered.

One may hope that this amusing fermion-quartion analogy, even if being a formal trick, may occur to be useful in making deeper insight into problems of the field-theoretical description of anyons such as anyon interactions, quantization and spin-statistics correspondence [11, 12, 13, 14]. Moreover, the above simple model demonstrates that the role of odd coordinates in describing the real space-time may be more fundamental then of even ones, the latter being a manifestation of some fibre bundle structure on a Grassmann base.

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