Lagrangian evolution of global strings

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We establish a method to trace the Lagrangian evolution of extended objects consisting of a multicomponent scalar field in terms of a numerical calculation of field equations in three dimensional Eulerian meshes. We apply our method to the cosmological evolution of global strings and evaluate the energy density, peculiar velocity, Lorentz factor, formation rate of loops, and emission rate of Nambu-Goldstone (NG) bosons. We confirm the scaling behavior with a number of long strings per horizon volume smaller than the case of local strings by a factor of ~ 10. The strategy and the method established here are applicable to a variety of fields in physics.

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Extended objects consisting of scalar fields such as topological defects or solitons play important roles in many fields of physical science ranging from condensed matter physics to cosmology. Much work has been done on the cosmological formation of topological defects by thermal or nonthermal phase transitions, their subsequent evolution, and cosmological implications of Q-balls. On the other hand, it has been pointed out that cosmological defect formation may be realizable in a laboratory. The phase transition of liquid helium can also be described in terms of Ginzburg-Landau theory as in the case of cosmological phase transitions. The order parameter, which is an expectation value of a scalar field in cosmological phase transition, is just replaced by a Bose condensate wave function. Following Zurek’s suggestion, the formation of topological defects has been studied in $^3$He and $^4$He. In fact, vortex formation is observed in $^3$He. Thus, it is very important to investigate the formation and subsequent evolution of extended objects, such as topological defects, produced both in cosmology and laboratories.

Among various types of topological defects predicted in high energy physics, strings hold a unique position, which, unlike magnetic monopoles and domain walls, do not overclose the universe because they settle down to the scaling solution. The key mechanism to achieve it is intercommutation of infinite strings to lose their energy by producing closed loops, which decay through radiating gravitational waves for local strings or Nambu-Goldstone (NG) bosons for global strings.

While many grand unified theories may contain local strings, there are no particle physics motivations for their existence. Nonetheless they were extensively studied because it was expected that they might be the origin of density fluctuations which seeded cosmic large-scale structures. In particular, time evolution of local cosmic strings have been investigated by a number of authors who used the Nambu-Goto action that applies for infinitely thin strings. These numerical analyses confirmed the scaling behavior and the scaling parameter, $\xi$, which is defined as $\xi = \rho_s l^2 / \mu$ with $\rho_s$ energy density of strings and $\mu$ string tension per unit length, converged to $\xi \approx 10$ in the radiation dominated universe. Recent observations of anisotropies of cosmic microwave background radiation, however, disfavor cosmic-string scenario of structure formation, and motivations to study local strings have somewhat diminished.

Global strings, on the contrary, are well motivated in particle physics, whose existence is predicted by the Peccei-Quinn solution to the strong CP problem. They are produced when the Peccei-Quinn U(1) symmetry is broken. They radiate axions as associated NG bosons which are one of the best candidates of cold dark matter. Furthermore, they are also realized in condensed matter physics. Vortices of $^3$He and $^4$He studied in a laboratory are analogous to global strings.

In contrast to local strings, energy density of global strings is dominated by the gradient energy of the NG scalar field rather than the potential energy of the core, because there are no gauge fields to cancel the former. As a result they have a long-range interaction and their dynamics cannot be analyzed with the Nambu-Goto action. It should be described with the Kalb-Ramond action which incorporates NG bosons and their coupling with the string core in addition to the Nambu-Goto action. In this action, however, the line density of the string is not well defined which has a logarithmic dependence on the radius. It is also very difficult to analyze it numerically. Hence the dynamics of strings governed by the Kalb-Ramond action has not been clarified in cosmological context yet even though evolution of axionic strings has very important implications in cosmology and

*The possibility has also been pointed out that infinite strings lose their energy by directly emitting massive particles for local strings.*
Indeed if with the horizon scale coefficients as \( \kappa \), long strings, global strings completely, we need to establish the in the same way on cosmological scales. both types of long strings should behave quantitatively bosons is much larger in simulation with small dynamic Shellard [26,27] who claim that we have obtained incor-

rect emission of NG bosons or axions, respectively, and incorrectely turned out to be much larger than \( \xi \), which is significantly smaller than the case of local counterparts as seen in the case of local [21] and global [22,23] monopoles.

In order to surmount such a situation, we have investigat-
ed evolution of global strings by solving equation of motion of the complex scalar field numerically in a three dimensional lattice in the expanding universe instead of dealing with the Kalb-Ramond action in collaboration with Kawasaki [24,25]. In these fully Eulerian simulations, it is difficult to identify strings and follow their dynamics. In the previous publications, whether a lattice contains a string core was judged by comparing po-
dynamics. In the previous publications, whether a lat-
tions, it is difficult to identify strings and follow their evolu-
tion in physics with some proper modifications.

We express the complex scalar field, which constitute global strings, in terms of two real scalar fields \( \phi_i(x) \) \( (i = 1, 2) \) and define their Lagrangian density as

\[
\mathcal{L}[\phi_i] = \frac{1}{2} g_{\mu\nu} (\partial^\mu \phi_i)(\partial^\nu \phi_i) - V[\phi_i, T],
\]

where \( g_{\mu\nu} \) is the flat Robertson-Walker metric. The finite-temperature effective potential \( V[\phi_i, T] \) is taken as

\[
V[\phi_i, T] = \frac{\lambda}{4} (\phi^2 - \sigma^2)^2 + \frac{\lambda}{6} T^2 \phi^2, \quad \phi^2 \equiv \phi_1^2 + \phi_2^2
\]

which exhibits a second-order phase transition with the critical temperature \( T_c = \sqrt{3} \sigma \) which produces global strings by breaking a global U(1) symmetry.

Equations of motion for the scalar fields are given by

\[
\ddot{\phi}_i(x) + 3H \dot{\phi}_i(x) - \frac{1}{R(t)^2} \nabla^2 \phi_i(x) = -\frac{\partial V}{\partial \phi_i},
\]

where a dot denotes time derivative and \( R(t) \) is the cosmic scale factor. In the radiation dominated universe, the Hubble parameter \( H = \dot{R}/R(t) \) and cosmic time \( t \) are given by

\[
H^2 = \frac{8\pi \pi^2}{3M_{pl}^2} \frac{g_* T^4}{30}, \quad t = \frac{1}{2H},
\]

respectively, where \( M_{pl} = 1.2 \times 10^{19} \text{GeV} \) is the Planck mass and \( g_* \) is the total number of degrees of freedom for the relativistic particles. For the ease of numerical calculations we take \( \sigma = 0.1(45/16 \pi^2 g_*)^{1/2} M_{pl} \) and \( \lambda = 0.08 \) but the results are insensitive to these choices. We start numerical simulation at the temperature \( T_i = 2T_c \) corresponding to \( t_i = t_{cr} \), adopting as an initial condition the thermal equilibrium state with a mass squared equal to the inverse curvature of the potential at that time. We have simulated the system from ten different thermal initial conditions under the periodic boundary condition. The number of lattices is 128\(^3\) with the lattice spacing \( \delta x = \sqrt{3 M_{pl}}/8 \) and the time step \( \Delta t = 0.01 t_i \). Thus, the box size is nearly equal to horizon volume \( (H^{-1})^3 \) and

\[
\frac{d\rho_\text{sec}}{dt} = -2H(1 + \langle v^2 \rangle)\rho_\text{sec} - \Gamma_\text{loop}\rho_\text{sec} - \Gamma_\text{NG}\rho_\text{sec},
\]

where the second and the third terms on the right-hand side represent energy loss due to loop formation and direct emission of NG bosons or axions, respectively, and \( \langle v^2 \rangle \) denotes average square velocity of string segments. In the scaling regime, the string network is characterized by a scale \( L \), which is defined as \( L \equiv \sqrt{\mu/\rho_\text{sec}} \) and grows with the horizon scale \( L \propto t \). Here \( \mu \) is the effective line energy density of a global string. We can introduce a loop production coefficient \( c \) and an emission coefficient \( \kappa \) by

\[
\Gamma_\text{loop}\rho_\text{sec} = c \frac{\rho_\text{sec}}{L}, \quad \Gamma_\text{NG}\rho_\text{sec} = \kappa \frac{\rho_\text{sec}}{L},
\]

Then, the scaling parameter \( \xi \) is characterized by these coefficients as

\[
\xi = \left( \frac{1 - \langle v^2 \rangle}{c + \kappa} \right)^2.
\]

Indeed if \( \kappa \) incorrectly turned out to be much larger than \( c \), we would find a smaller value of \( \xi \) than it should really be. Hence it is essential to evaluate contribution of each term in [1] with the help of numerical analysis. For this purpose we must know how each string segment moves and intercommutes with each other to trace formation of loops, which is impossible in conventional Eulerian simulations. Notice that this problem is ubiquitous because in many fields of physics we must solve equation of motion in Eulerian meshes, for example, in order to trace time evolution of solitons like Q-balls [8].
the lattice spacing to a core size of a string $\delta = 1/(\sqrt{2\pi s})$ at the final time $t_f = 200t_s$.

The above setup of calculation is the same as our previous fully Eulerian simulations \cite{24,25}. In order to reproduce Lagrangian evolution of string segments in the Eulerian meshes, we develop a different method to identify string cores from scalar field configurations, because the previous method \cite{24,25} was inadequate for configurations with large curvature such as small loops and it was impossible to find more correct position of string core in a box beyond the lattice spacing. Here we use a new two-step algorithm to identify the locus of a string more accurately. The first step is to find plaquettes which a string penetrates. This is done in terms of the Vachaspati-Vilenkin algorithm \cite{23} by monitoring phase rotation around each square. Then using the value of $\phi_a$ at each vertex of a plaquette penetrated by a string, we linearly interpolate $\phi_0(x)$ to calculate the position where both $\phi_1(x)$ and $\phi_2(x)$ vanish. We can thus find where a string penetrate in each plaquette and find more accurate trajectory of a string by connecting these points.

The next and the more important task to obtain Lagrangian evolution of string network, that has been missing in the previous Eulerian analyses, is to find velocity of each string segment. Since motion tangential to a string is a gauge mode, we should evaluate velocity normal to it. Suppose that a string core exists at a point $x_0$ at time $t_0$, namely, $\phi_1(x_0, t_0) = \phi_2(x_0, t_0) = 0$. In order to estimate where this string segment moves $\Delta t$ later, we expand scalar fields $\phi_a(x, t_0 + \Delta t)$ around $\phi_a(x_0, t_0)$ up to the first order,

$$\phi_a(x, t_0 + \Delta t) \approx \phi_a(x_0, t_0) + \nabla \phi_a(x_0, t_0) \cdot (x - x_0) + \frac{\partial \phi_a(x_0, t_0)}{\partial t} \Delta t \quad (a = 1, 2).$$

Again the loci of string core at $t = t_0 + \Delta t$ are given by the line where both scalar fields $\phi_2(x, t_0 + \Delta t)$ vanish. That is, under the first-order approximation \cite{8} it lies on the intersection of two planes given by

$$\nabla \phi_1(x_0, t_0) \cdot (x - x_0) + \frac{\partial \phi_1(x_0, t_0)}{\partial t} \Delta t = 0,$$

with $a = 1, 2$. Now suppose that the line normal to the string segment at $(x, t_0)$ intersect with the trajectory of string at $t = t_0 + \Delta t$ at $x = x_1(x_0, t_0, \Delta t)$. Then, making use of the fact that $x_1$ satisfies Eq. \cite{8}, we can easily calculate the velocity of this string segment, and its magnitude is given by

$$v(x_0, t_0) = \lim_{\Delta t \to 0} \frac{x_1(x_0, t_0, \Delta t) - x_0}{\Delta t} = \frac{\dot{\phi}_1 \nabla \phi_2 - \dot{\phi}_2 \nabla \phi_1}{|\nabla \phi_1 \times \nabla \phi_2|}.$$  \hspace{1cm} (10)

One should note that the formula obtained here applies to all soliton-like objects consisting of a field configuration with some proper modifications.

Now that we have developed a method to identify the location and velocity of strings at each time, the Lagrangian evolution of the string network can be traced just as in the case of numerical simulation of the evolution of local strings based on the Nambu-Goto action. In fact, our simulation, being based on scalar field equations, contains even more information, that is, we can find fate of intersecting string segments, namely, whether they reconnect or simply pass through each other, without assigning the probability of reconnection by hand unlike in the case of simulations based on the Nambu-Goto action. We can therefore calculate formation rate and spectrum of string loops without ambiguities.

Thus we can calculate all the terms of eq. \cite{1} directly from simulation data except for the last term, which can be evaluated by use of this equation itself. As pointed out in \cite{27}, we should use $\mu = \gamma \mu_s$ as the energy per length to calculate the energy density of long strings, $\rho_{\infty}$. Here $\gamma$ is the average Lorentz factor and $\mu_s$ is the line density of a static string given by $\mu_s \sim 2\pi^2 l_0^2 (m/\delta k^{1/2})$.

Figure 4 depicts time evolution of $\xi$ with three different identification methods of strings. Filled squares represent results of our new identification scheme, while blank circles and blank squares correspond to our previous method \cite{24,25} and Vachaspati-Vilenkin algorithm, which was adopted in \cite{27}, respectively. Both methods estimate total string length by simply counting the number of boxes penetrated by a string. As is easily seen, our previous method slightly overestimates $\xi$ and Vachaspati-Vilenkin approach overestimates $\xi$ by a factor of 1.4. We confirm that $\xi$ becomes constant after some relaxation period and find that the more correct value of the scaling parameter in the radiation dominated era is given by $\xi \simeq 0.80$.

Figures 5 depict time evolution of average velocity, average velocity squared, and average Lorentz factor. The first two quantities are also found to relax to constant values after some relaxation time, namely, $\langle v \rangle \simeq 0.65$ and $\langle v^2 \rangle \simeq 0.50 \gg \langle v \rangle^2$. The average Lorentz factor, however, has a large scatter in time, although long-time average is fairly constant with $\bar{\gamma} \simeq 1.8$. Since string segments moving with a speed close to light velocity have extremely large Lorentz factor and push up the average dramatically, fluctuation in the number of such string segments results in such large scatter. This also explains why the average Lorentz factor $\gamma = 1.6 - 2.0$ is larger than $1/\sqrt{1 - \langle v^2 \rangle}$ and $1/\sqrt{1 - \langle v \rangle^2}$. Thus the energy per unit length of a string is enhanced by a factor $\bar{\gamma} \simeq 1.8$.

Calculating the length of all infinite strings and that of loops at each time step and comparing with those at the preceding time step, the loop production parameter is found to be $c = 0.43 - 0.53$. Then taking account of the relation \cite{5}, the emission parameter $\kappa$ is calculated as $\kappa = 0.03 - 0.13$. We therefore conclude that the direct emission of NG bosons is a subdominant channel of energy loss of long strings even in the case cosmic horizon scale is not much larger than the string width due to
the limitation of the dynamic range of numerical simulations. Thus, even if $\kappa$ might become significantly smaller in the cosmological situation because it is proportional to $\ln(t/\delta)$ as pointed out in [24,27], the scaling parameter $\xi$ would not increase to coincide with that of local strings. The result of our simulation with $\ln(t/\delta) \sim 5$ implies that $\kappa \lesssim 0.01$ in the cosmological situations. Inserting this new value of $\kappa$ into the relation (3) together with $c$, $\kappa$, and $\langle v^2 \rangle$ yields $\xi = 0.9 - 1.4$ in the cosmological situations, which is still significantly smaller than the case of local strings. Hence, it is confirmed that although global strings relax to the scaling solution just as local strings, their number density is much smaller than the local ones. This is mainly because global strings intercommute more often and $\langle v^2 \rangle$ is larger due to long-range attractive forces between strings. Finally, we apply our results into the constraint of the breaking scale $f_a$ of the Peccei-Quinn U(1) symmetry. Then, in accordance with Ref. [24], $f_a$ is constrained as $f_a \lesssim (0.20 - 1.6) \times 10^{12}$ GeV for the normalized Hubble parameter $h = 0.7$ with $\xi = 0.9 - 1.4$ and $\gamma \simeq 1.6 - 2.0$.

In summary, we have developed a new method to trace Lagrangian evolution of topological defects solving scalar field equation on three dimensional Eulerian lattices and given all the quantities characterizing cosmological evolution of the global string network, namely, energy density, peculiar velocity, Lorentz factor, intercommutation rate, and emission rate of NG bosons. All these quantities can be obtained in Eulerian simulations only after establishing the method given in this Rapid Communication, which enables us to extract Lagrangian information from simulations done in Eulerian meshes. Our method is directly applicable to situations in a variety of fields in physics.

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FIG. 1. Time development of $\xi$ is depicted. Blank circles represent time development of $\xi$ for the identification method done in Refs. [24,25]. Filled squares represent the new identification method. Blank squares represent the identification method based on the Vachaspati-Vilenkin algorithm.

FIG. 2. Time development of average velocity, an average of velocity squared, and an average Lorentz factor of global strings is shown.