Classification of matrix product ground states corresponding to one dimensional chains of two state sites of nearest neighbor interactions

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Abstract
A complete classification is given for one dimensional chains with nearest neighbor interactions having two states in each site, for which a matrix product ground state exists. The Hamiltonians and their corresponding matrix product ground states are explicitly obtained.

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1 Introduction

Recently many body problems have absorbed much interest. Among these problems are quantum spin chain systems. It is desirable to find the eigenvalues, eigenvectors, and correlation functions of such models. But in general it is a very difficult, if not impossible, task. In fact, there are few models for which the complete spectra can be obtained in a closed form. The main problem lies in the fact that the dimension of the Hilbert space for a general many-body system grows exponentially with the system size, and hence exponentially many variables are needed to specify the wavefunction of such a system. Moreover, in some cases although the ground state is known, its structure turns out to be quite complicated, making the calculation of correlation functions very difficult. When the interaction is local, the matrix product state representation turns out to be one of the techniques which may give us the opportunity to find the ground state of a many body system. Recently, extensive studies have been done on the matrix product state formalism [1–9]. In this method, first the ground state is constructed. Then, a Hamiltonian is written in such a way that the above mentioned state be its ground state. A matrix product state is a generalization of an uncorrelated state (the tensor product of one site states).

This method has been used in exactly solvable spin chain models, spin ladders, and spin systems on two dimensional lattices [3,10,13]. It can also be applied to many types of stochastic systems of interacting particles in one dimensional chains [14,16]. In [17], the matrix product formalism was used to find the exact ground states of two new spin-1 quantum chains with nearest neighbor interactions. In [18], the matrix product states having only spin-flip and parity symmetries, were classified. It was seen there that there are three distinct classes of such states. In [19], the matrix product states have been used to classify quantum phases.

In this article, a classification is presented for the ground states and their corresponding Hamiltonians, which are obtained by the matrix product states. This classification applies to models with nearest neighbor interactions on a one dimensional lattice, for which the number of states in each site is two. The Heisenberg spin 1/2 chain is an example of such models. We classify the models, for which a matrix product ground state can be obtained, and obtain the corresponding Hamiltonians and their matrix product ground states.

The scheme of the paper is the following. In section 2, some general techniques are introduced, mainly to fix the notation. In section 3 a partial classification is given, based on possible different Hamiltonians, and without taking into account that not for all of these Hamiltonians there exists a (nonzero) matrix product state as the ground state. In section 4 this classification is completed by finding the corresponding ground states, and the Hamiltonians and the ground states are explicitly presented. Section 5 is devoted to the concluding remarks.
2 General formulation

To fix the notation, let’s recall some basic facts about the matrix product state. The models addressed here are one dimensional quantum chains with $N$ sites. The number of possible states of each site is $d$, and the set $\{e_1, \ldots, e_d\}$ is an orthonormal basis for the Hilbert space corresponding to the states of each site. Consider a set of $D$ dimensional matrices $\{A^1, \ldots, A^d\}$. Then $\psi$, the (normalized) matrix product state (corresponding to these matrices) is defined as

$$
\psi := \frac{1}{\sqrt{Z}} \tr(A^{a_1} \cdots A^{a_N}) e_{a_1} \otimes \cdots \otimes e_{a_N}.
$$

The summation convention is assumed, so that for an index appeared once and only once as a subscript and once and only once as a superscript, a summation on that index is assumed. $Z$ is the normalization constant:

$$
Z = \tr(A^N),
$$

and

$$
A := \delta_{\alpha \beta} \overline{A}^\alpha \otimes A^\beta,
$$

where $\overline{X}$ is the complex conjugate of $X$.

Consider the following equation for the tensor $C$, which is of rank $k$.

$$
C_{\alpha_1 \cdots \alpha_k} A^{\alpha_1} \cdots A^{\alpha_k} = 0.
$$

The set of all tensors $C$ of rank $k$ satisfying (4), is obviously a vector space. Let $\{E^1, \ldots\}$ be a basis for that vector space:

$$
C_{\alpha_1 \cdots \alpha_k} = C_a E^a_{\alpha_1 \cdots \alpha_k}.
$$

Consider further a family of local Hamiltonians $h$ acting on the Hilbert space corresponding to $k$ consecutive sites:

$$
h := \Lambda_{a b} E^a \dagger E^b,
$$

where $\Lambda$ is Hermitian and positive semi-definite. These guarantee that $h$ is hermitian and positive semi-definite, respectively. In terms of the matrix elements, (6) can be written as

$$
h^{\alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_k} = \Lambda_{a b} (E^a \dagger)^{\alpha_1 \cdots \alpha_k} E_{\beta_1 \cdots \beta_k}^b,
$$

where

$$
h \left( e_{\beta_1} \otimes \cdots \otimes e_{\beta_k} \right) := h^{\alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_k} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k},
$$

and

$$
E_a := E_a^{\alpha_1 \cdots \alpha_k} e^{\alpha_1} \otimes \cdots \otimes e^{\alpha_k},
$$

and $\{e^1, \ldots, e^k\}$ is the basis dual to $\{e_1, \ldots, e_k\}$. Now define the full Hamiltonian $H$ (acting on the Hilbert space corresponding to the whole lattice) through

$$
H := \sum_{i=1}^{N-k+1} h_{i,i+k-1},
$$

where $N$ is the number of sites in the lattice.
where
\[ h_{i,i+k-1} := \underbrace{1 \otimes \cdots \otimes 1}_{N-k-i+1} \otimes h \otimes \underbrace{1 \otimes \cdots \otimes 1}_{i-1}, \]
(11)
and 1 is the identity matrix. Using (11), or equivalently
\[ E_{\alpha_1 \cdots \alpha_k} A^{\alpha_1} \cdots A^{\alpha_k} = 0, \]
(12)
it is seen that \( \psi \), defined through (1) is an eigenvector of \( H \) corresponding to the eigenvalue zero. Also, the fact that \( h \) is positive semi-definite ensures that \( H \) is positive semi-definite as well. So zero is the smallest eigenvalue of \( H \), hence \( \psi \) is a ground state corresponding to \( H \).

It is seen that the Hamiltonian \( H \) constructed through (10), describes an interaction in blocks consisting of \( k \) consecutive sites.

3 Hamiltonians, partial classification

We want to classify models with nearest neighbor interactions \((k = 2)\), so (4) changes to
\[ C_{\alpha \beta} A^{\alpha} A^{\beta} = 0. \]
(13)
We also consider cases where the Hilbert space corresponding to each site is two dimensional \((d = 2)\). So the set of all the matrices \( C \) for which \( A^{\alpha} \)'s are to satisfy (13), is a vector subspace \( V \) of the span(\( S \)), where
\[ S := \{ \tau_0, \tau_1, \tau_2, \sigma \}, \]
(14)
and
\[ \tau_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]
\[ \tau_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
\[ \tau_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
\[ \sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]
(15)
Denote the permutation operator by \( P \):
\[ (PC)_{\alpha \beta} := C_{\beta \alpha}, \]
(16)
and the projections to the eigenspaces of \( P \) corresponding to the eigenvalues \( \pm 1 \) by \( \Pi^\pm \):
\[ \Pi^\pm := \frac{1}{2} \left[ (1^* \otimes 1^*) \pm P \right], \]
(17)
where \( \mathbf{1}^* \) is the pullback of identity (1). Defining the symmetric and antisymmetric projected spaces \( (\mathcal{V}^+ \text{ and } \mathcal{V}^-) \) respectively as
\[
\mathcal{V}^\pm := \Pi^\pm \mathcal{V},
\]
it is seen that \( \mathcal{V}^\pm \) is a vector subspace of \( \text{span}(\mathcal{S}^\pm) \), where
\[
\mathcal{S}^+: = \{\tau_0, \tau_1, \tau_2\}, \\
\mathcal{S}^- := \{\sigma\}.
\]

The action of an invertible matrix \( \Gamma \) on a member of \( \text{span}(\mathcal{S}) \) is denoted by \( \text{O}_\Gamma \):
\[
(\text{O}_\Gamma C)_{\alpha \beta} := \Gamma^\gamma\alpha \Gamma^\delta\beta C_{\gamma \delta},
\]
which can be written more compactly as
\[
\text{O}_\Gamma C := (\Gamma^* \otimes \Gamma^*) C, \tag{21}
\]
or
\[
\text{O}_\Gamma C := \Gamma^* C \Gamma, \tag{22}
\]
where \( \Gamma^* \) is the pullback of \( \Gamma \):
\[
(\Gamma^*)^\beta\alpha := \Gamma^\alpha\beta. \tag{23}
\]

Two spaces \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are said to be equivalent to each other, if there is an invertible matrix \( \Gamma \) such that
\[
\mathcal{V}_2 = \text{O}_\Gamma \mathcal{V}_1. \tag{24}
\]

It is easy to see that the action of \( \Gamma \) and the action of any nonzero multiple of it on a subspace of \( \text{span}(\mathcal{S}) \) are the same. So in order to find all spaces equivalent to \( \mathcal{V}_1 \), it is sufficient to consider only those \( \Gamma \)’s which have a unit determinant, that is, only the matrices belonging to \( \text{SL}_2(\mathbb{C}) \). From now on it is assumed that the matrices \( \Gamma \) acting on subspaces of \( \text{span}(\mathcal{C}) \) are members of \( \text{SL}_2(\mathbb{C}) \).

Any member of \( \text{span}(\mathcal{S}) \) is characterized by the ordered quartet \( (\nu^0, \nu^1, \nu^2, u) \):
\[
C =: u^i \tau_i + u \sigma. \tag{25}
\]

Using
\[
[\Gamma^* \otimes \Gamma^*, P] = 0, \tag{26}
\]
it is seen that \( \text{span}(\mathcal{S}^+) \) and \( \text{span}(\mathcal{S}^-) \) are invariant subspaces of \( \Gamma^* \otimes \Gamma^* \). In fact, as
\[
(\Gamma^* \otimes \Gamma^*) \sigma = \det(\Gamma) \sigma, \tag{27}
\]
\( \text{span}(\mathcal{S}^-) \) is an eigenspace of \( (\Gamma^* \otimes \Gamma^*) \) with eigenvalue one. This means that under the action of \( \Gamma \), \( (\nu^0, \nu^1, \nu^2, u) \) is transformed to \( (\nu'^0, \nu'^1, \nu'^2, u) \), that is \( u \) remains invariant.
Using 
\[ \Gamma^{-1} \sigma^{-1} = \sigma^{-1} \Gamma^*, \tag{28} \]

it is seen that 
\[ \sigma^{-1} (O \Gamma C) \sigma^{-1} = \Gamma^{-1} (\sigma^{-1} C \sigma^{-1}) (\Gamma^*)^{-1}, \tag{29} \]

so that under the action of \( \Gamma \), \( (C_1 \sigma^{-1} C_2 \sigma^{-1}) \) is similarity transformed, which means that \( \text{tr}(C_1 \sigma^{-1} C_2 \sigma^{-1}) \) remains invariant. It is seen that 
\[ \text{tr}(C_1 \sigma^{-1} C_2 \sigma^{-1}) = 2 [-u_1 u_2 - v_0^1 v_0^2 + v_1^1 v_2^1 + v_2^2 v_1^2]. \tag{30} \]

As the action of \( \Gamma \) on \( C \) leaves \( u \) invariant, it is seen that this action leaves the product \( (v_1 \cdot v_2) \) invariant as well, where 
\[ v_1 \cdot v_2 := -v_1^0 v_2^0 + v_1^1 v_2^1 + v_2^2 v_1^2. \tag{31} \]

Now consider the action of an \( \text{SL}_2(\mathbb{C}) \) matrix on \( V^+ \). \( V^+ \) could be zero-, one-, two-, or three-dimensional. In the first and last cases, it is invariant under such an action. If \( V^+ \) is one dimensional, then it either is null (with respect to the product \( \langle 31 \rangle \)), or is not null. If it is null, there is a \( \Gamma \) such that 
\[ O \Gamma \mathcal{V}^+ = \text{span}\{\tau_0 + \tau_1\}. \tag{32} \]
If \( \mathcal{V}^+ \) is not null, then there is a \( \Gamma \) such that 
\[ O \Gamma \mathcal{V}^+ = \text{span}\{\tau_2\}. \tag{33} \]

If \( \mathcal{V}^+ \) is two dimensional, then the subspace of all vectors in \( \text{span}(\mathcal{S}) \) which are normal to it (again with respect to the product \( \langle 31 \rangle \)), is one dimensional. This one dimensional space \( (\mathcal{W}) \), either is null, or is not null. If it is null, then there is a \( \Gamma \) such that 
\[ O \Gamma \mathcal{W} = \text{span}\{\tau_0 + \tau_1\}, \tag{34} \]
which means that 
\[ O \Gamma \mathcal{V}^+ = \text{span}\{\tau_0 + \tau_1, \tau_2\}. \tag{35} \]
If \( \mathcal{W} \) is not null, then there is a \( \Gamma \) such that 
\[ O \Gamma \mathcal{W} = \text{span}\{\tau_1\}, \tag{36} \]
which means that 
\[ O \Gamma \mathcal{V}^+ = \text{span}\{\tau_0, \tau_2\}. \tag{37} \]

To summarize, it is seen that using suitable transformations one can bring \( \mathcal{V}^+ \) to one of these forms (the final form is denoted by \( \mathcal{V}^+ \) rather than \( O \Gamma \mathcal{V}^+ \)): 
\[ \mathcal{V}^+ = \{0\}, \tag{38} \]
\[ \mathcal{V}^+ = \text{span}\{\tau_2\}, \tag{39} \]
\[ \mathcal{V}^+ = \text{span}\{\tau_0 + \tau_1\}, \tag{40} \]
\[ \mathcal{V}^+ = \text{span}\{\tau_0, \tau_2\}, \tag{41} \]
\[ \mathcal{V}^+ = \text{span}\{\tau_0 + \tau_1, \tau_2\}, \tag{42} \]
\[ \mathcal{V}^+ = \text{span}\{\tau_0, \tau_1, \tau_2\}. \tag{43} \]
In each case, $V$ is either $(V^+ \oplus \text{span}\{\sigma}\})$, or a subspace of $(V^+ \oplus \text{span}\{\sigma}\})$ the dimension of which is one less than the dimension of $(V^+ \oplus \text{span}\{\sigma}\})$. One can say that in (23), $u$ is either independent of $v$ or a linear function of $v$. In the latter case, there exists a $w$ such that

$$u = w \cdot v. \quad (44)$$

Each of the cases (38) to (43) then can be further analyzed to obtain nonequivalent possible forms of $V$: Corresponding to (38), $u$ is either zero or arbitrary. For the cases (39) and (40), $u$ is either arbitrary or of the form (44). In the latter case,

$$u = \mu v^2, \quad (45)$$

for (39), and

$$u = \mu v^0, \quad (46)$$

for (40). In this case, one can further make $\mu$ equal to either zero or one. The reason is that one can take $w$ to be in $\text{span}(\tau_0 - \tau_1)$, and if it is nonzero there exists an $\text{SL}_2(\mathbb{C})$ induced transformation which leaves $V^+$ invariant and makes $w$ equal to $(\tau_1 - \tau_0)/2$.

For (41), if $u$ is not arbitrary then $w$ can be taken in $V^+$. If $w$ is not null, it can be transformed to a multiple of $\tau_2$. If $w$ is null, it can be transformed to $(\tau_1 - \tau_0)$.

For (42), if $u$ is not arbitrary then $w$ can be taken in the $\text{span}\{\tau_1 - \tau_0, \tau_2\}$. The $\text{SL}_2(\mathbb{C})$ transformations which leave $V^+$ invariant are

$$(\tau_0 + \tau_1) \rightarrow \frac{1}{\rho} (\tau_0 + \tau_1),$$

$$(\tau_1 - \tau_0) \rightarrow \rho (\tau_1 - \tau_0) + \lambda \tau_2 - \frac{\lambda^2}{2 \rho} (\tau_0 + \tau_1),$$

$$\tau_2 \rightarrow \pm \left[ \tau_2 - \frac{\lambda}{2 \rho} (\tau_0 + \tau_1) \right]. \quad (47)$$

It shows that $w$ can be transformed to either a constant multiple of $\tau_2$, or $(\tau_1 - \tau_0)/2$.

Finally, for (43) and if $u$ is not arbitrary, $w$ can be transformed to either a constant multiple of $\tau_2$, or $(\tau_1 - \tau_0)$. 

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One then arrives at the following nonequivalent forms for \( V \).

\[
\begin{align*}
V &= \{0\}, \quad (48) \\
V &= \text{span}\{\sigma\}, \quad (49) \\
V &= \text{span}\{\tau_2 + \mu \sigma\}, \quad (50) \\
V &= \text{span}\{\tau_2, \sigma\}, \quad (51) \\
V &= \text{span}\{\tau_0 + \tau_1\}, \quad (52) \\
V &= \text{span}\{\tau_0 + \tau_1 + \sigma\}, \quad (53) \\
V &= \text{span}\{\tau_0 + \tau_1, \sigma\}, \quad (54) \\
V &= \text{span}\{\tau_0, \tau_2 + \mu \sigma\}, \quad (55) \\
V &= \text{span}\{\tau_0 + \tau_2 + \sigma\}, \quad (56) \\
V &= \text{span}\{\tau_0 + \tau_1, \tau_2 + \mu \sigma\}, \quad (57) \\
V &= \text{span}\{\tau_0 + \tau_1 + \sigma, \tau_2\}, \quad (58) \\
V &= \text{span}\{\tau_0 + \tau_1, \tau_2, \sigma\}, \quad (59) \\
V &= \text{span}\{\tau_0, \tau_1, \tau_2 + \mu \sigma\}, \quad (60) \\
V &= \text{span}\{\tau_0 + \sigma, \tau_1 + \sigma, \tau_2\}, \quad (61) \\
V &= \text{span}\{\tau_0, \tau_1, \tau_2, \sigma\}. \quad (62)
\end{align*}
\]

In each case, one constructs a local Hamiltonian through (7), and a ground state for the total Hamiltonian would be (1). But this is true only if the matrix product state (1) does not vanish. In order to investigate this, and actually obtain that matrix product state, one has to classify the solutions for \( A^\alpha \)'s in (13), corresponding to each of the above cases.

Finally, the above classification for the matrices \( C \), is a classification of non-equivalent cases. In each case, it is possible to obtain new \( C \) matrices through the action of an \( \text{SL}_2(\mathbb{C}) \) matrix \( \Gamma \). That is, if \( \psi \) is a ground state of the total Hamiltonian the corresponding local Hamiltonian of which is \( h \), then \( \psi' \) is a ground state of a total Hamiltonian the corresponding local Hamiltonian of which is \( h' \), where

\[
\begin{align*}
\psi' := & (\underbrace{\Gamma^{-1} \otimes \cdots \otimes \Gamma^{-1}}_{\text{\# of states}}) \psi, \quad (63) \\
h' := & (\underbrace{\Gamma \otimes \cdots \otimes \Gamma}_{\text{\# of states}}) \, (\underbrace{\Gamma \otimes \cdots \otimes \Gamma}_{\text{\# of states}}) \, \psi, \quad (64)
\end{align*}
\]

Note that this (64) is not a similarity transformation on \( h \). So not all the eigenvectors of \( h' \) and \( h \) are related to each other through something like (63). But the eigenvectors corresponding to the eigenvalue zero are related to each other this way.
4 Ground states, full classification

In this section we want to find the matrices $A^\alpha$ satisfying (13), corresponding to each of the nonequivalent spaces of the previous section. Then, corresponding to each nontrivial pair $(C, A)$ one has a Hamiltonian and a corresponding ground state. $C$ is nontrivial if it is nonzero. $A$ is nontrivial if the corresponding matrix product state is nonzero.

In general, if one of the matrices $A^\alpha$ is zero, say $A^1$ is zero, then

$$\psi_0 := (e_0)^\otimes N$$

is a ground state, where $\{e_0, e_1\}$ is an orthonormal basis for the Hilbert space corresponding to one site. So in all the cases studied below, it is assumed that none of $A^0$ and $A^1$ vanish, unless otherwise stated.

For the case (48), $V = \{0\}$, so $C$ is trivial.

Cases (49) and (50) can be written as

$$\nu' A^0 A^1 = \nu A^1 A^0,$$

where among $\nu$ and $\nu'$ at least one is nonzero. If the ratio of $\nu'$ and $\nu$ is not a root of one, then the trace of any product of $A^0$'s and $A^1$'s containing both $A^0$ and $A^1$ is zero. Hence $(c_0 \psi_0 + c_1 \psi_1)$ is a ground state, where

$$\psi_1 := (e_1)^\otimes N.$$ (67)

This means that both $\psi_0$ and $\psi_1$ are ground states.

If the ratio of $\nu'$ and $\nu$ is a root of one, $M$ is the smallest positive integer where

$$\left(\frac{\nu'}{\nu}\right)^M = 1,$$ (68)

and $(N/M)$ is an integer, then another family of ground states is there as well. In this case, $\psi_k$ is a ground state where

$$\psi_k := \sum_{\mathcal{P}_{N,k,M}} \zeta(\mathcal{P}_{N,k,M}) [e_{\mathcal{P}_{N,k,M}(1)} \otimes \cdots \otimes e_{\mathcal{P}_{N,k,M}(N)}],$$ (69)

and $\mathcal{P}_{N,N'}$ is a function with the domain $\{1, \ldots, N\}$, so that $N'$ of its values are 0 and $(N - N')$ of its values are 1. $\zeta(\mathcal{P}_{N,N'})$ is defined as the following. $\mathcal{P}_{N,N'}$ corresponds to a sequence of 0's and 1's, where the position of the $\ell$'th 0 is $i_\ell$. Then

$$\zeta(\mathcal{P}_{N,N'}) := \left(\frac{\nu'}{\nu}\right)^\sum_{\ell} (i_\ell - \ell).$$ (70)

The corresponding local Hamiltonian is constructed using

$$E = \nu' e^0 \otimes e^1 - \nu e^1 \otimes e^0.$$ (71)

One has

$$h = g E^\dagger E,$$ (72)
where \( g \) is a positive constant. Using the Pauli matrices
\[
\sigma_1 := e_0 e^1 + e_1 e^0,
\sigma_3 := e_0 e^0 - e_1 e^1,
\sigma_+ := e_0 e^1,
\sigma_- := e_1 e^0,
\]
(73)
this can be rewritten as
\[
h = g \left[ \sqrt{|\nu|^2 + |\nu'|^2} \left( 1 \otimes 1 - \sigma_3 \otimes \sigma_3 \right) + \frac{|\nu|^2 - |\nu'|^2}{4} \left( \sigma_3 \otimes 1 - 1 \otimes \sigma_3 \right) \right.
\]
\[
- \sqrt{|\nu|^2 + |\nu'|^2} \left( \sigma_3 \otimes \sigma_+ - \sigma_- \otimes \sigma_+ \right].
\]
(74)

The full Hamiltonian is the Hamiltonian of a generalized version of the XYZ Heisenberg quantum chain, with magnetic field at the boundaries.

In the case (52), one has
\[
(A^0)^2 = 0.
\]
(75)
So the trace of any product of \( A^0 \)'s and \( A^1 \)'s vanish if that product contains two adjacent \( A^0 \)'s. One may guess that any tensor product of \( e_0 \)'s and \( e_1 \)'s is a ground state, provided that product does not contain two adjacent \( e_0 \)'s. This is in fact true, and it is easy to see it directly from the Hamiltonian. The local Hamiltonian corresponding to (52) is
\[
h = g (1 + \sigma_3) \otimes (1 + \sigma_3),
\]
(76)
where \( g \) is a positive constant. Obviously the Hamiltonian of the lattice contains only the operators \( (\sigma_3)_i \), which commute with each other. So any simultaneous eigenvector of these operators is an eigenvector of the Hamiltonian. Noting
\[
\sigma_3 e_\alpha = (1 - 2 \alpha) e_\alpha,
\]
(77)
it is seen that \( \lambda_{\alpha_1 \cdots \alpha_N} \), the eigenvalue of the Hamiltonian corresponding to the eigenvector \( e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_N} \), satisfies
\[
\lambda_{\alpha_1 \cdots \alpha_N} = (2 - 2 \alpha_1) (2 - 2 \alpha_2) + \cdots + (2 - 2 \alpha_N),
\]
(78)
which is zero if no two adjacent \( \alpha_i \)'s are zero, and positive otherwise.

In the case (53), one has
\[
2 (A^0)^2 + [A^0, A^1] = 0.
\]
(79)
Let \( \lambda \) be an eigenvalue of \( A^0 \) and \( \Pi_\lambda \) be a projector so that if \( v_\lambda \) is any generalized eigenvector of \( A^0 \) corresponding to \( \lambda \),
\[
\Pi_\lambda v_\lambda = \delta_{\lambda \lambda'} v_\lambda'.
\]
(80)
\( \Pi_\lambda \) commutes with \( A^0 \) and \((79)\) gives
\[
2 \Pi_\lambda (A^0)^2 \Pi_\lambda + [A^0, \Pi_\lambda A^1 \Pi_\lambda] = 0, \tag{81}
\]
so
\[
\text{tr}[\Pi_\lambda (A^0)^2 \Pi_\lambda] = 0, \tag{82}
\]
which shows that \( \lambda \) should be zero. So \( A^0 \) is nilpotent. \((79)\) shows that if \( A \) is a linear combination of \( A^0 \) and \( A^1 \),
\[
A^0 A = A' A^0, \tag{83}
\]
where \( A' \) is another linear combination of \( A^0 \) and \( A^1 \). This shows that if \( B \) is any matrix in the algebra constructed by \( A^0 \) and \( A^1 \) and \( 1 \), and if \( v \in \ker([A^0]^j] \), then \( (B v) \in \ker([A^0]^j] \). Now consider a basis in which \( A^0 \) is Jordanian. The vectors in this basis can be grouped into groups \( B^j \), where \( \text{span}(B^j) \) is \( V^j \), so that \( V^j \) is a subspace of \( \ker([A^0]^j] \) but has no nonzero vector in \( \ker([A^0]^j-1] \). We want to prove that
\[
\text{tr}(A^0 B) = 0, \tag{84}
\]
where \( B \) is an arbitrary matrix in the algebra constructed by \( A^0 \) and \( A^1 \) and \( 1 \). To do so, take and arbitrary vector \( v \) in \( V^j \). It is seen that \( B v \) is in the direct sum of \( V^1 \) to \( V^j \). So \( (A^0 B v) \) is in the direct sum of \( V^1 \) to \( V^j-1 \), which proves \((83)\). So \( \psi_1 \) is a ground state. In this case the local Hamiltonian is like \((72)\), but with
\[
E = 2 e^0 \otimes e^0 + e^0 \otimes e^1 - e^1 \otimes e^0, \tag{85}
\]
so,
\[
h = g \left[ \frac{3}{2} 1 \otimes 1 + 1 \otimes \sigma_3 + \sigma_3 \otimes 1 + \frac{1}{2} \sigma_3 \otimes \sigma_3 \\
+ (1 + \sigma_3) \otimes \sigma_1 - \sigma_1 \otimes (1 + \sigma_3) - \sigma_- \otimes \sigma_+ - \sigma_+ \otimes \sigma_- \right]. \tag{86}
\]

For \((71)\), the trace of any product of \( A^0 \)'s and \( A^1 \)'s containing both \( A^0 \) and \( A^1 \) vanishes. \( \psi_0 \) and \( \psi_1 \) are ground states. In this case, \( V \) is two dimensional and a basis for which is \( \{E^1, E^2\} \), where
\[
E^1 = e^0 \otimes e^1, \\
E^2 = e^1 \otimes e^0. \tag{87}
\]
The local Hamiltonian is then
\[
h = \frac{g_1 + g_2}{4} (1 \otimes 1 - \sigma_3 \otimes \sigma_3) + \frac{g_1 - g_2}{4} (\sigma_3 \otimes 1 - 1 \otimes \sigma_3) \\
+ g_3 \sigma_+ \otimes \sigma_- + \frac{g_3}{2} \sigma_- \otimes \sigma_+, \tag{88}
\]
where \( g_1 \) and \( g_2 \) are real and nonnegative, and
\[
g_1 g_2 \geq |g_3|^2. \tag{89}
\]

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The full Hamiltonian is a generalized version of the XYZ Heisenberg quantum chain, with magnetic field at the boundaries.

In the case \( (54) \), \((A_0)^2 \) vanishes and \( A_0 \) and \( A_1 \) commute with each other. So \( (90) \) holds and the ground state is \( \psi_1 \). In this case, \( \mathcal{V} \) is two dimensional and a basis for which is \( \{ E_1, E_2 \} \), where
\[
E_1 = e^0 \otimes e^0, \\
E_2 = e^0 \otimes e^1 - e^1 \otimes e^0.
\] (90)

The local Hamiltonian is
\[
h = \frac{g_1 + 2g_2}{4} (\sigma_3 \otimes 1 + 1 \otimes \sigma_3) + \frac{g_1 - 2g_2}{4} \sigma_3 \otimes \sigma_3 - g_2 (\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) + \frac{1 + \sigma_3}{2} \otimes (g_3 \sigma_+ + \frac{g_3}{\sqrt{3}} \sigma_-) - (g_3 \sigma_+ + \frac{g_3}{\sqrt{3}} \sigma_-) \otimes \frac{1 + \sigma_3}{2},
\] (91)

where \( g_1 \) and \( g_2 \) are real and nonnegative, and \( (89) \) holds. The full Hamiltonian corresponds to that of an XXZ quantum spin chain with constant magnetic field, and an additional boundary interaction.

In the case \( (55) \), one has \( (66) \), with the additional constraint
\[
(A_0)^2 + (A_1)^2 = 0.
\] (92)

This additional constraint means that ground states other than \( \psi_0 \) and \( \psi_1 \) are possible only if
\[
\left( \frac{\nu}{\nu'} \right)^2 = 1.
\] (93)

For \( A_0 \) and \( A_1 \) anticommuting, one obtains for even \( N \) a ground state
\[
\psi' := \sum_{k=0}^{N/2} (-1)^k \sum_{P_{N,2k}} \zeta(P_{N,2k}) [e_{P_{N,2k}(1)} \otimes \cdots \otimes e_{P_{N,2k}(N)}].
\] (94)

For \( A_0 \) and \( A_1 \) commuting, there are two ground states (apart from \( \psi_0 \) and \( \psi_1 \)), which are
\[
\psi'_0 := \sum_{k=1}^{[(N-1)/2]} (-1)^k \sum_{P_{N,2k+1}} [e_{P_{N,2k+1}(1)} \otimes \cdots \otimes e_{P_{N,2k+1}(N)}],
\] (95)

and
\[
\psi'_e := \sum_{k=0}^{[N/2]} (-1)^k \sum_{P_{N,2k}} [e_{P_{N,2k}(1)} \otimes \cdots \otimes e_{P_{N,2k}(N)}].
\] (96)

In this case, \( \mathcal{V} \) is two dimensional and a basis for which is \( \{ E^1, E^2 \} \), where
\[
E^1 = e^0 \otimes e^0 + e^1 \otimes e^1, \\
E^2 = \nu' e^0 \otimes e^1 - \nu e^1 \otimes e^0.
\] (97)
The local Hamiltonian is
\[ h = \frac{2g_1 + g_2 (|\nu'|^2 + |\nu|^2)}{4} 1 \otimes 1 + \frac{2g_1 - g_2 (|\nu'|^2 + |\nu|^2)}{4} \sigma_3 \otimes \sigma_3 \]
\[ + (g_1 - g_2 \nu' \nu) \sigma_+ \otimes \sigma_- + (g_1 - g_2 \nu' \nu) \sigma_- \otimes \sigma_+ \]
\[ + g_2 \frac{|\nu'|^2 - |\nu|^2}{4} (\sigma_3 \otimes 1 - 1 \otimes \sigma_3) \]
\[ + \frac{g_3}{2} [1 \otimes (\nu' \sigma_+ - \nu \sigma_-) + (\nu' \sigma_- - \nu \sigma_+) \otimes 1 \]
\[ + \sigma_3 \otimes (\nu' \sigma_+ + \nu \sigma_-) - (\nu' \sigma_- + \nu \sigma_+) \otimes \sigma_3] \]
\[ + \frac{g_3}{2} [1 \otimes (\nu' \sigma_- - \nu \sigma_+) + (\nu' \sigma_+ - \nu \sigma_-) \otimes 1 \]
\[ + \sigma_3 \otimes (\nu' \sigma_- + \nu \sigma_+) - (\nu' \sigma_+ + \nu \sigma_-) \otimes \sigma_3], \tag{98} \]
where \( g_1 \) and \( g_2 \) are real and nonnegative, and \( \mathbf{(89)} \) holds.

In the case \( \mathbf{(56)} \), \( (A^0 A^1) \) vanishes. So the trace of any product of \( A^0 \) and \( A^1 \) containing both \( A^0 \) and \( A^1 \) vanishes. But using
\[ (A^0)^2 + (A^1)^2 + A^0 A^1 - A^1 A^0 = 0, \tag{99} \]
it is seen that \( (A^0)^N \) and \( (A^1)^N \) themselves can both be written as products containing both \( A^0 \) and \( A^1 \), with vanishing traces. So no ground state is obtained this way.

In the case \( \mathbf{(57)} \), the trace of any product of \( A^0 \)'s and \( A^1 \)'s containing more than one \( A^0 \) vanishes. But as \( \mathbf{(56)} \) holds with \( \nu' \neq \nu \), the trace of any product of \( A^1 \)'s and one \( A^0 \) vanishes as well. So \( \psi_1 \) is a ground state. In this case, \( \mathcal{V} \) is two dimensional and a basis for which is \( \{E^1, E^2\} \), where
\[ E^1 = e^0 \otimes e^0, \]
\[ E^2 = \nu' e^0 \otimes e^1 - \nu e^1 \otimes e^0. \tag{100} \]

The local Hamiltonian is
\[ h = \frac{2g_1 + g_2 (|\nu'|^2 + |\nu|^2)}{4} 1 \otimes 1 + \frac{2g_1 - g_2 (|\nu'|^2 + |\nu|^2)}{4} \sigma_3 \otimes \sigma_3 \]
\[ - g_2 (\nu' \nu \sigma_+ \otimes \sigma_- + \nu' \nu \sigma_- \otimes \sigma_+) + g_2 \frac{|\nu'|^2 - |\nu|^2}{4} (\sigma_3 \otimes 1 - 1 \otimes \sigma_3) \]
\[ + \frac{g_3}{2} (\nu' 1 \otimes \sigma_+ - \nu \sigma_+ \otimes 1 + \nu' \sigma_3 \otimes \sigma_+ - \nu \sigma_+ \otimes \sigma_3) \]
\[ + \frac{g_3}{2} (\nu' 1 \otimes \sigma_- - \nu \sigma_- \otimes 1 + \nu' \sigma_3 \otimes \sigma_- - \nu \sigma_- \otimes \sigma_3), \tag{101} \]
where \( g_1 \) and \( g_2 \) are real and nonnegative, and \( \mathbf{(89)} \) holds.

For the case \( \mathbf{(58)} \), \( \mathbf{(79)} \) holds meaning that the trace of any product of \( A^0 \)'s and \( A^1 \)'s other than \( (A^1)^N \) vanishes. So \( \psi_1 \) is a ground state. In this case, \( \mathcal{V} \) is two dimensional and a basis for which is \( \{E^1, E^2\} \), where
\[ E^1 = 2 e^0 \otimes e^0 + e^0 \otimes e^1 + e^1 \otimes e^0, \]
\[ E^2 = e^0 \otimes e^1 - e^1 \otimes e^0. \tag{102} \]
The local Hamiltonian is
\[ h = \frac{3g_1 + g_2}{2} \sigma_3 \otimes \sigma_3 + \frac{g_1 - g_2}{2} \sigma_3 \otimes \sigma_3 + (g_1 - g_2) (\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_) \]
\[ + g_1 (\sigma_3 \otimes \sigma_1 + \sigma_1 \otimes \sigma_3) + g_1 [(\sigma_3 + \sigma_1) \otimes \mathbf{1} + \mathbf{1} \otimes (\sigma_3 + \sigma_1)] \]
\[ + \sigma_3 \otimes (g_3 \sigma_- + \frac{g_3}{\sqrt{3}} \sigma_+) - (g_3 \sigma_- + \frac{g_3}{\sqrt{3}} \sigma_+) \otimes \sigma_3 \]
\[ + (\frac{g_3}{\sqrt{3}} - g_3)(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_) \]
\[ + \mathbf{1} \otimes (g_3 \sigma_- + \frac{g_3}{\sqrt{3}} \sigma_+) - (g_3 \sigma_- + \frac{g_3}{\sqrt{3}} \sigma_+) \otimes \mathbf{1} \]
\[ + \frac{g_3}{\sqrt{3}} + g_3 (\sigma_3 \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_3), \]
where \( g_1 \) and \( g_2 \) are real and nonnegative, and (89) holds.

In the case (59), of the quadratic products of \( A_0 \) and \( A_1 \) only \( (A_0)^2 \) is nonvanishing, leading to \( \psi_1 \) as a ground state. In this case, \( V \) is two dimensional and a basis for which is \( \{ E_1, E_2 \} \), where
\[ E_1 = e^0 \otimes e^0, \]
\[ E_2 = e^0 \otimes e^1, \]
\[ E_3 = e^1 \otimes e^0. \]

The local Hamiltonian is any hermitian positive semi-definite matrix so that its kernel includes \( e_1 \otimes e_1 \). This is a nine (real) parameter family.

In the case (60), \( (A_0)^2 \) and \( (A_1)^2 \) vanish and (66) holds. So the trace of any product of \( A_0 \)'s and \( A_1 \)'s vanish and no ground state is obtained this way.

The case (61) is similar to the case (56) with an additional constraint. But (61) leads to vanishing of the trace of any product of \( A_0 \)'s and \( A_1 \)'s. So for (61) too, no ground state is obtained this way.

Finally, for (62) all of the quadratic products of \( A_0 \) and \( A_1 \) vanish, so that no ground state is obtained this way.

## 5 Concluding remarks

To summarize, the local Hamiltonians for which the full Hamiltonian has a matrix product ground state, and the corresponding matrix product ground state, are the following.
\[ h = g \left[ \left| \nu \right|^2 + \left| \nu' \right|^2 \right] \sigma_3 \otimes \sigma_3 + \left[ \left| \nu' \right|^2 - \left| \nu \right|^2 \right] \sigma_3 \otimes \sigma_3 \]
\[ - \nu' \sigma_+ \otimes \sigma_- - \nu' \sigma_- \otimes \sigma_+ \],
\[ \psi_0, \psi_1 \]

with \( \psi_0 \) and \( \psi_1 \) being ground states (of the full Hamiltonian). If \( (\nu'/\nu) \) is a root of 1, then \( \psi_k \)'s are also ground states, where
\[ \psi_k := \sum_{P_{N,k,M}} \zeta(P_{N,k,M}) [e_{P_{N,k,M}(1)} \otimes \cdots \otimes e_{P_{N,k,M}(N)}]. \]
and $M$ is the smallest positive integer so that $(\nu'/\nu)^M$ is 1.

$$h = g (1 + \sigma_3) \otimes (1 + \sigma_3), \quad (107)$$

with any tensor product of $e_0$’s and $e_1$’s, not containing two adjacent $e_0$’s being ground states.

$$h = g \left[ \frac{3}{2} 1 \otimes 1 + 1 \otimes \sigma_3 + \sigma_3 \otimes 1 + \frac{1}{2} \sigma_3 \otimes \sigma_3 
+ (1 + \sigma_3) \otimes \sigma_1 - \sigma_1 \otimes (1 + \sigma_3) - \sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+ \right], \quad (108)$$

with $\psi_1$ being a ground state.

$$h = \frac{g_1 + g_2}{4} (1 \otimes 1 - \sigma_3 \otimes \sigma_3) + \frac{g_1 - g_2}{4} (\sigma_3 \otimes 1 - 1 \otimes \sigma_3) 
+ g_3 \sigma_+ \otimes \sigma_- + \bar{g}_3 \sigma_- \otimes \sigma_+, \quad (109)$$

where $g_1$ and $g_2$ are real and nonnegative, and

$$g_1 g_2 \geq |g_3|^2. \quad (110)$$

Here $\psi_0$ and $\psi_1$ are ground states.

$$h = \frac{g_1 + 2g_2}{4} (1 \otimes 1 + \frac{g_1}{4} (\sigma_3 \otimes 1 + 1 \otimes \sigma_3) 
+ \frac{g_1 - 2g_2}{4} \sigma_3 \otimes \sigma_3 - g_2 (\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+)) 
+ \frac{1 + \sigma_3}{2} \otimes (g_3 \sigma_+ + \bar{g}_3 \sigma_-) - (g_3 \sigma_+ + \bar{g}_3 \sigma_-) \otimes \frac{1 + \sigma_3}{2}, \quad (111)$$

where $g_1$ and $g_2$ are real and nonnegative, and (110) holds. $\psi_1$ is a ground state.

$$h = \frac{2g_1 + g_2 (|\nu'|^2 + |\nu|^2)}{4} (1 \otimes 1 + \frac{2g_1 - g_2 (|\nu'|^2 + |\nu|^2)}{4} \sigma_3 \otimes \sigma_3 
+ (g_1 - g_2 \nu' \nu) \sigma_+ \otimes \sigma_- + (g_1 - g_2 \nu' \nu) \sigma_- \otimes \sigma_+) 
+ g_2 \frac{|\nu'|^2 - |\nu|^2}{4} (\sigma_3 \otimes 1 - 1 \otimes \sigma_3) 
+ \frac{g_3}{2} \left[ 1 \otimes (\nu' \sigma_+ - \nu \sigma_-) + (\nu' \sigma_- - \nu \sigma_+) \otimes 1 
+ \sigma_3 \otimes (\nu' \sigma_+ + \nu \sigma_-) - (\nu' \sigma_- + \nu \sigma_+) \otimes \sigma_3 \right] 
+ \frac{g_3}{2} \left[ 1 \otimes (\nu' \sigma_- - \nu \sigma_+) + (\nu' \sigma_+ - \nu \sigma_-) \otimes 1 
+ \sigma_3 \otimes (\nu' \sigma_- + \nu \sigma_+) - (\nu' \sigma_+ + \nu \sigma_-) \otimes \sigma_3 \right], \quad (112)$$

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where $g_1$ and $g_2$ are real and nonnegative, and (110) holds. $\psi_0$ and $\psi_1$ are ground states. If $\nu' = -\nu$, then $\psi'$ is a ground state as well, where

$$\psi' := \sum_{k=0}^{N/2} (-1)^k \sum_{\mathcal{P}_{N,2k}} \zeta(\mathcal{P}_{N,2k}) [e_{\mathcal{P}_{N,2k}(1)} \otimes \cdots \otimes e_{\mathcal{P}_{N,2k}(N)}]. \quad (113)$$

If $\nu' = \nu$, then $\psi'_0$ and $\psi'_1$ are ground states as well (apart from $\psi_0$ and $\psi_1$), where

$$\psi'_0 := \sum_{k=1}^{[(N-1)/2]} (-1)^k \sum_{\mathcal{P}_{N,2k+1}} [e_{\mathcal{P}_{N,2k+1}(1)} \otimes \cdots \otimes e_{\mathcal{P}_{N,2k+1}(N)}], \quad (114)$$

and

$$\psi'_1 := \sum_{k=0}^{[N/2]} (-1)^k \sum_{\mathcal{P}_{N,2k}} [e_{\mathcal{P}_{N,2k}(1)} \otimes \cdots \otimes e_{\mathcal{P}_{N,2k}(N)}]. \quad (115)$$

$$h = \frac{2g_1 + g_2 (|\nu|^2 + |\nu'|^2)}{4} 1 \otimes 1 + \frac{2g_1 - g_2 (|\nu'|^2 + |\nu|^2)}{4} \sigma_3 \otimes \sigma_3$$

$$- g_2 (\nu' \nu' \sigma_+ \otimes \sigma_- + \nu' \nu \sigma_- \otimes \sigma_+) + g_2 \frac{|\nu'|^2 - |\nu|^2}{4} (\sigma_3 \otimes 1 - 1 \otimes \sigma_3)$$

$$+ \frac{g_3}{2} (\nu' 1 \otimes \sigma_+ - \nu \sigma_+ \otimes 1 + \nu' \sigma_3 \otimes \sigma_+ - \nu \sigma_+ \otimes \sigma_3)$$

$$+ \frac{g_3}{2} (\nu' \sigma_- - \nu \sigma_- \otimes 1 + \nu' \sigma_3 \otimes \sigma_- - \nu \sigma_- \otimes \sigma_3), \quad (116)$$

where $g_1$ and $g_2$ are real and nonnegative, and (110) holds. $\psi_1$ is a ground state.

$$h = \frac{3g_1 + g_2}{2} 1 \otimes 1 + \frac{g_1 - g_2}{2} \sigma_3 \otimes \sigma_3 + (g_1 - g_2) (\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+)$$

$$+ g_1 (\sigma_3 \otimes \sigma_1 + \sigma_1 \otimes \sigma_3) + g_1 [(\sigma_3 + \sigma_1) \otimes 1 + 1 \otimes (\sigma_3 + \sigma_1)]$$

$$+ \sigma_3 \otimes (g_3 \sigma_- + \frac{g_3}{2} \sigma_+) - (g_3 \sigma_- + \frac{g_3}{2} \sigma_+) \otimes \sigma_3$$

$$+ (\frac{g_3}{2} - g_3) (\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+)$$

$$+ 1 \otimes (g_3 \sigma_- + \frac{g_3}{2} \sigma_+) - (g_3 \sigma_- + \frac{g_3}{2} \sigma_+) \otimes 1$$

$$+ \frac{g_3}{2} + g_3 (\sigma_3 \otimes 1 + 1 \otimes \sigma_3), \quad (117)$$

where $g_1$ and $g_2$ are real and nonnegative, and (110) holds. $\psi_1$ is a ground state.

Finally, for any local Hamiltonian the kernel of which includes $e_0 \otimes e_0$, a ground state of the full Hamiltonian is $\psi_1$.

All Hamiltonians related to the above through (64), have corresponding ground states constructed through (63).

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