ROBUST ERROR ESTIMATES FOR STABILIZED FINITE ELEMENT APPROXIMATIONS OF THE TWO DIMENSIONAL NAVIER-STOKES EQUATIONS WITH APPLICATION TO IMPLICIT LARGE EDDY SIMULATION

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Abstract. We consider error estimates in weak parametrised norms for stabilized finite element approximations of the two-dimensional Navier-Stokes’ equations. These weak norms can be related to the norms of certain filtered quantities, where the parameter of the norm, relates to the filter width. Under the assumption of the existence of a certain decomposition of the solution, into large eddies and fine scale fluctuations, the constants of the estimates are proven to be independent of both the Reynolds number and the Sobolev norm of the exact solution. Instead they exhibit exponential growth with a coefficient proportional to the maximum gradient of the large eddies. The error estimates are on a posteriori form, but using Sobolev injections valid on finite element spaces and the properties of the stabilization operators the residuals may be upper bounded uniformly, leading to robust a priori error estimates.

Key words. Navier-Stokes’ equations, stability, error estimates, large eddy simulation, finite element methods, stabilization

AMS subject classifications.

1. Introduction. In this paper we will be interested in stabilized finite element methods in the context of so called implicit large eddy simulation (ILES), see [2]. This is a numerical approach to the computation of turbulent flow where no modelling of the Reynolds stresses is performed on the continuous level. Instead the Navier-Stokes’ equations are approximated numerically using a method that dissipates sufficient energy on the scale of the mesh size. This eliminates the buildup of energy that creates spurious oscillations in any energy conservative approximation method. It has been argued that the truncation error of such methods by itself may act as a subgrid model [1, 18] and there exists numerical evidence that ILES methods work for the simulation of two dimensional turbulence, provided back scatter effects are not strong [16]. There is also numerical evidence of the potential for adaptive LES/DNS driven by adaptive, stabilized finite element simulations, see [12, 13] and [21].

Our objective in this paper is to provide a numerical analysis for stabilized finite element methods under minimal regularity assumptions and to provide sufficient conditions on the exact solution for the derivation of rigorous error estimates that are independent of both the Reynolds number and Sobolev norms of the exact solution. It is well known that provided the exact solution is sufficiently smooth the approximate solution \( u_h \) of the Navier-Stokes’ equations on velocity-pressure form can be proved to satisfy estimates of the type

\[
\| u - u_h \|_{L^2(\Omega)} \lesssim e \| \nabla u \|_{H^1(\Omega)} h^2 | u |_{L^2(\Omega; H^2(\Omega))},
\]

(1.1)

if a consistent stabilized finite element method with piecewise affine approximation is used. Here \( u \) denotes the flow velocity and \( h \) the mesh size. See [11, 2] for examples of analyses of Navier-Stokes’ equations on velocity-pressure form and [15, 14] for analyses on velocity-vorticity form. We also give a proof of (1.1) for one of the methods proposed herein in appendix. Here we use the notation \( a \lesssim b \) for \( a \leq C b \) with \( C \) a constant

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independent of the physical parameters of the problem, unless they can be expected to have $O(1)$ contribution, it can also include some dependence on initial data, that may be assumed to be $O(1)$. We will also use $a \sim b$ for $a \lesssim b$ and $b \lesssim a$. Note that there is no explicit dependence on the viscosity in the estimate \[E.\] For this estimate to be useful the included Sobolev norms must be small, which rarely is the case in the high Reynolds number regime and hence the dependence of the viscosity enters in an implicit manner. The purpose of the present paper is to propose an alternative approach, where the estimate is indeed independent of the Reynolds number, both in the sense that the estimate is free from inverse power of the viscosity in the upper bound, but also that the dependence on unknown Sobolev norms of the exact solution is strongly reduced. It does not seem possible to eliminate this dependence completely, due to the possible presence of backscatter. Observe that the presence of \[\|\nabla u\|_\infty\] in the exponential of \[E.\] reflects the effect of diverging characteristics in the transport equation and is present already in the linear convection–diffusion equation at high Péclet numbers. We may define a timescale for the flow separation, \[\tau := \|\nabla u\|_\infty^{-1}.\] The reason this time scale becomes so small is that it will be the smallest timescale of the flow and hence equal to the micro time, because of fine scale fluctuations of the velocity. From the physical point of view it is argued that LES will be successful for flows where both the quantities of interest and the rate-controlling processes are determined by the resolved large scales, see Pope [20] for a discussion. We will use this idea as a starting point for our assumptions on the flow.

To derive error estimates for a numerical method we need the following:

– continuous dependence on data, independent of the exact solution;

– some smooth quantity that we can apply approximation estimates to.

At a first glance both these prerequisites appear to fail for the two-dimensional Navier–Stokes’ equation. The first fails because of the presence of the exponential factor and the second fails because Sobolev norms of the exact solution can be huge for small viscosities. The following three points allow us to break this deadlock:

1. the use of a parametrized weak norm, corresponding to measuring the error in filtered quantities of the solution;

2. introduction of an assumption on the structure of the exact solution that is sufficient for an implicit large eddy simulation to be robust;

3. a stabilized finite element method, giving enhanced a priori control of residual quantities in the high Reynolds regime.

The idea of measuring error in filtered quantities was considered in [9, 10], but the estimates were not robust in the Reynolds number and the constant included high order Sobolev norms of the exact solution. In [4] weak norm estimates were used in order to derive robust estimates for the Burgers’ equation, where the constant in the right hand side only depended on initial data. The second point, which was not necessary in the case of the Burgers’ equation, reflects the difficulty to characterise the solution structure in higher dimension and the ensuing need of some structural assumptions in order to rule out strong backscatter effects. The third point allows us to control residual quantities independent of the viscosity.

As a first approximation it is reasonable to assume that for a solution to be amenable to large eddy simulation, there are relatively smooth eddies, with large associated Reynolds number, containing the bulk of the energy and small scale fluctuations that may vary rapidly in space, but carries a negligible part of the energy. To make this precise, we assume that there exists a decomposition of the exact solution in the spatially slowly varying large scales and an arbitrarily rough fine scale, with
small energy.

\[ u = \bar{u} + u', \quad \bar{u}, u' \in W^{1,\infty}(\Omega). \]

We then assume that the Reynolds number associated to the large scales, \( Re \), may be large, but \( \| \bar{u} \|_{W^{1,\infty}(Q)} \sim 1 \), whereas for the fine scales \( \| u' \|_{W^{1,\infty}(Q)} \) may be large, but the energy small. To give a precise meaning to small here, we introduce a global time scale for the flow, defined using the large scales

\[ \tau_F := \| \bar{u} \|_{W^{1,\infty}(Q)}^{-1} \sim 1. \]

This is in agreement with the statement that rate controlling processes are determined by the large scale. Using the viscosity coefficient we may make the following assumption on the energy content of the small scales

\[ \| u' \|_\infty^2 \sim \nu / \tau_F. \]

The length scale based on \(| u' |\) and \( \tau_F \) writes

\[ l' := | u' | \tau_F \sim | u' | \frac{\nu}{| u' |^2} \]

and it follows that the small scale Reynolds number is

\[ Re' := \frac{u' l'}{\nu} \sim \frac{| u' |^{\nu}}{| u' |^2} \sim 1. \quad (1.2) \]

Alternatively one may assume that the fine scale Reynolds number is one and that the large scale characteristic time, is the globally relevant time scale and then derive the bound on the energy. We will refer to the above as the large eddy assumption. Under this assumption we prove the following bound on the approximate velocities

\[ \sup_{t \in (0,T)} \| u - u_h \|_{L^2(\Omega)} \lesssim h^{\frac{1}{2}}. \quad (1.3) \]

The hidden constant in the above estimate only depends on initial data (maximum initial vorticity) and the mesh geometry, but is independent of the Reynolds number and Sobolev norms of the exact solution. The discussion is limited to two space dimensions and hence we do not properly speaking address the question of turbulent flows.

Let us end this introductory discussion by emphasising that what we compute is an approximation to the solution of the Navier-Stokes’ equations. For this approximate solution we can prove that provided \( \tau_F \) is not too small, corresponding to slowly varying large scale velocity field, the filtered part of the vorticity is stable under perturbations resulting in robust error estimates in weak norms for vorticity. Using these estimates we may then control the \( L^2 \)-norm of the velocity error as shown above. Herein our main concern will be the high mesh Reynolds number case

\[ Re_h := \frac{U_0 h}{\nu} > 1, \]

where \( U_0 := \| u_h (\cdot, 0) \|_{L^\infty(\Omega)} \sim 1 \) denotes a characteristic velocity of the flow, but many results are independent of the mesh Reynolds number. It will always be explicitly stated when a result only holds in the high Reynolds regime. If the local
Reynolds number is low, other approaches than those presented herein might be more appropriate. Let us also point out that another feature of our estimates is that they provide the first error estimates with an order in \( h \) for nonlinear stabilization schemes, satisfying a discrete maximum principle, in two space dimensions.

We will consider the Navier-Stokes’ equations written on vorticity-velocity form. Let \( \Omega \) be the unit square and assume that the boundary conditions are periodic in both cartesian directions. The \( L^2 \)-scalar product over some space-time domain will be denoted \( (\cdot, \cdot)_X \) with associated norm \( \| \cdot \|_X \) where the subscript may be dropped for \( X = \Omega \). Define the time interval \( I := (0, T) \) and the space-time domain \( Q := \Omega \times I \). The equations then writes,

\[
\begin{align*}
\partial_t \omega + \nabla \cdot (u \omega) - \nu \Delta \omega &= 0, \quad \text{in } Q, \\
-\Delta \Psi &= \omega, \quad \text{in } Q, \\
u = \text{rot } \Psi, \quad \omega(x, 0) &= \omega_0,
\end{align*}
\]

with \( \omega_0 \in L^\infty(\Omega) \). The associated weak formulation takes the form, for \( t > 0 \), find \( (\omega, \Psi) \in H^1(\Omega) \times H^1(\Omega) \cap L^\infty(\Omega) \) such that

\[
\begin{align*}
(\partial_t \omega, v)_\Omega + a(u; \omega, v) &= 0, \quad \text{(1.5)} \\
(\nabla \Psi, \nabla \Phi)_\Omega &= (\omega, \Phi)_\Omega, \quad \text{(1.6)} \\
u = \text{rot } \Psi,
\end{align*}
\]

for all \( (v, \Phi) \in H^1(\Omega) \times H^1(\Omega) \cap L^\infty(\Omega) \), where the semi-linear form \( a(\cdot; \cdot, \cdot) \) is defined by

\[
a(u; \omega, v) := (\nabla \cdot (u \omega), v)_\Omega + (\nu \nabla \omega, \nabla v)_\Omega.
\]

This problem is known to be well-posed, but a priori error estimates on the solution are in general strongly dependent on the viscosity coefficient reflecting the possible poor stability of the equations in the high Reynolds number regime.

2. Finite element discretization. Let \( \{T_h\}_{h>0} \) be a family of affine, simplicial Delaunay meshes of \( \Omega \). We assume that the meshes are kept fixed in time and that the family \( \{T_h\}_{h>0} \) is quasi-uniform. Mesh faces are collected in the set \( F \). For a smooth enough function \( v \) that is possibly double-valued at \( F \in F \) with \( F = \partial T^- \cap \partial T^+ \), we define its jump at \( F \) as \([v] := v|_{T^-} - v|_{T^+}\), and we fix the unit normal vector to \( F \), denoted by \( n_F \), as pointing from \( T^- \) to \( T^+ \). The arbitrariness in the sign of \([v]\) is irrelevant in what follows. Define \( V_h \) to be the standard space of piecewise polynomial, continuous periodic functions,

\[
V^k_h := \{v_h \in H^1(\Omega): v_h|_K \in P_k(K) ; \forall K \in T_h; \ v_h \ \text{periodic in } x \ \text{and } y \}.
\]

The set of gradients of functions in \( V^k_h \) will be denoted by

\[
W^{k-1}_h := \{w_h = \nabla v_h; \ v_h \in V^k_h\}.
\]

Let \( L := L^2(\Omega) \) and set \( L_* := \{q \in L; \int q = 0\} \). Let \( V_* := V^k_h \cap L_* \). We let \( \pi_L \) denote the \( L^2 \)-projection on \( V^k_h \) and \( \pi_V \) the \( H^1 \)-projection

\[
(\nabla \pi_V u, \nabla v)_\Omega = (\nabla u, \nabla v)_\Omega \quad \forall v_h \in V_h \ \text{and} \ \int (\pi_V u - u) \ dx = 0.
\]
We recall that the following approximation estimates hold for $\pi_L$ and $\pi_V$,
\[
\|\pi_L u - u_h\| + h\|\nabla (\pi_L u - u)\| \leq c_0 h^s |u|_s, \text{ with } 1 \leq s \leq k + 1
\]  
(2.1)
and
\[
\|\pi_V u - u_h\| + h\|\nabla (\pi_V u - u)\| \leq c_1 h^s |u|_s, \text{ with } 1 \leq s \leq k + 1.
\]  
(2.2)

We consider continuous finite elements with $k = 1$ to discretize the vorticity $\omega$ in space and $k = 1, 2$ for the stream function $\Psi$. The discrete velocity is given elementwise by $u_h|_K := \text{rot } \Psi_h := (\partial_y \Psi_h, -\partial_x \Psi_h)$. Note that using this definition $\nabla \cdot u_h = 0$ in $\Omega$, i.e. the discrete velocity is globally divergence free. We discretize in space using a stabilized finite element method. For $t > 0$ find $(\omega_h, \Psi_h) \in V_h^3 \times V_h^l$, with $l = 1, 2$, such that
\[
(\partial_t \omega_h, v_h)_M + a(u_h; \omega_h, v_h) + s(u_h; \omega_h, v_h) = 0, \tag{2.3}
\]
\[
(\nabla \Psi_h, \nabla \Phi_h)_\Omega - (\omega_h, \Phi_h)_\Omega = 0, \tag{2.4}
\]
\[
u h \omega_h = 0,
\]
for all $(v_h, \Phi_h) \in V_h \times V_h$ and with initial data $w_0 := \pi_L \omega(\cdot, 0)$. Here $s(\cdot : \cdot : \cdot)$ denotes a stabilization operator that is linear in its last argument and $(\partial_t \omega_h, v_h)_M$ denotes the bilinear form defining the mass matrix, this operator either coincides with $(\cdot , \cdot)_\Omega$ or is defined as $(\cdot , \cdot)_\Omega$ approximated using nodal quadrature, i.e. so called mass lumping. We will assume the stabilization term satisfies
\[
\inf_{v_h \in V_h} \|h^{\frac{1}{2}} (u_h \cdot \nabla \omega_h - v_h)\| \lesssim s(u_h, \omega_h; \omega_h) \lesssim h^{\frac{1}{2}} (U_0 + \|u_h\|_{L^\infty(\Omega)}) \int |\nabla \omega_h|| \quad (2.5)
\]
and
\[
s(u_h; \omega_h; v_h) \lesssim h^{\frac{1}{2}} (U_0 + \|u_h\|_{L^\infty(\Omega)}) s(u_h, \omega_h; \omega_h) \|\nabla v_h\|. \quad (2.6)
\]

The formulation (2.3)-(2.4) satisfies the following stability estimates.

**Lemma 2.1.**

\[
\sup_{t \in I} \|\omega_h(\cdot, t)\|_{M}^2 + 2\|\nu^{\frac{1}{2}} \omega_h\|_{Q}^2 + 2 \int_I s(u_h; \omega_h, \omega_h) \ dt \leq \|\omega_h(\cdot, 0)\|_{M}^2, \tag{2.7}
\]

and if exact integration is used for $(\cdot , \cdot)_M$,

\[
\|u_h(\cdot, T)\|_{M}^2 + 2\|\nu^{\frac{1}{2}} \omega_h\|_{Q}^2 = \|u_h(\cdot, 0)\|_{M}^2 - 2 \int_0^T s(u_h; \omega_h, \Psi_h) \ dt, \tag{2.8}
\]

\[
\|u_h(\cdot, T)\|_{L^q(\Omega)} \leq c_q \|\omega_h(\cdot, T)\|_{L^q(\Omega)} , \quad q > 2 \tag{2.9}
\]

and for $l = 1$, \n\[
\int_0^T \|\nabla \partial_t \omega_h\| \ dt \lesssim \int_0^T (h^{-\frac{1}{4}} (U_0 + \|u_h\|_{L^\infty(\Omega)}) s(u_h, \omega_h, \omega_h) + \nu h^{-\frac{1}{2}} \|\nabla \omega_h\|) \ dt. \tag{2.10}
\]

**Proof.** Inequality (2.7) is immediate by taking $v_h = \omega_h$ in (2.3). Inequality (2.8) is obtained by taking $v_h = \Psi_h$ in the equation (2.3) and deriving the equation (2.4)
in time and taking \( \Phi_h = \omega_h \). For the inequality (2.11), consider the auxiliary problem,
\( -\Delta \tilde{\Psi} = \omega_h \) in \( \Omega \) and note that by [22] there holds
\[
\| u_h(\cdot, t) \|_{L^\infty(\Omega)} \lesssim \| \tilde{\Psi}(\cdot, t) \|_{W^{1,\infty}(\Omega)}
\]
and adapting the analysis of [19] we have for the (simpler) case or periodic boundary conditions,
\[
\| \tilde{\Psi}(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq c_q \| \omega_h(\cdot, t) \|_{L^q(\Omega)}, \quad q > 2.
\]
To prove (2.10) finally we introduce a function \( \xi_h \in V_h \) such that
\[
(\xi_h, v_h)_M = (\nabla \partial_t \omega_h, \nabla v_h)_\Omega, \quad \forall v_h \in V_h,
\]
it follows by taking \( v_h = \xi_h \) and using the Cauchy-Schwarz inequality followed by an inverse inequality that
\[
(\xi_h, \xi_h)_M \sim \| \xi_h \| \lesssim h^{-1} \| \partial_t \nabla \omega_h \|.
\quad \quad (2.11)
\]
Observe that by norm equivalence on discrete spaces the \( L^2 \)-norm defined using nodal quadrature is equivalent to the consistent \( L^2 \)-norm. Taking \( v_h = \xi_h \) in (2.8) yields
\[
\| \partial_t \nabla \omega_h \|^2 = -(u_h \cdot \nabla \omega_h, \xi_h)_\Omega - (\nu \nabla \omega_h, \nabla \xi_h)_\Omega - s(u_h, \omega_h, \xi_h).
\]
We may then apply the Cauchy-Schwarz inequality in the second term of the right hand side and (2.10) in the last term, followed by inverse inequalities on \( \| \nabla \xi_h \| \) and the estimate (2.11). For the first term we write, using the properties of \( \nabla \xi_h \) and the bound
\[
|(v_h, \xi_h)_M - (v_h, \xi_h)_\Omega| \lesssim (h^2|\nabla v_h|, |\nabla \xi_h|)_\Omega,
\]
\[
|(u_h \cdot \nabla \omega_h, \xi_h)_\Omega| \lesssim (u_h \cdot \nabla \omega_h - v_h, \xi_h)_\Omega + (h^2|\nabla v_h|, |\nabla \xi_h|)_\Omega + (\partial_t \nabla \omega_h, \nabla v_h)_\Omega + (\partial_t \nabla \omega_h, \partial_t v_h)_\Omega
\]
Since both \( u_h \) and \( \nabla \omega_h \) are constant per element \( \nabla v_h|_K = \nabla (v_h - u_h \cdot \nabla \omega_h)|_K \). Using inverse inequalities and the bound (2.11) on \( \xi_h \) we have
\[
|(u_h \cdot \nabla \omega_h - v_h, \xi_h)| + (\partial_t \nabla \omega_h, \nabla v_h)|_\Omega + (h^2|\nabla v_h|, |\nabla \xi_h|)_\Omega \lesssim h^{-1} \| \partial_t \nabla \omega_h \| \| u_h \cdot \nabla \omega_h - v_h \|.
\]
The claim follows by the inequality (2.8), (2.11) and finally by integrating in time. It follows from (2.8) that the method is energy consistent if \( s(u_h; \omega_h, \Psi_h) = 0 \). Taking the difference of the formulations (1.5) - (1.6) (with \( v = v_h \)) and (2.3) - (2.4) and setting \( e_\omega = \omega - \omega_h \) and \( e_\Psi = \Psi - \Psi_h \), the following consistency relation holds
\[
(\partial_t e_\omega + u \cdot \nabla e_\omega + \nu e_\omega, v_h)_\Omega + (\nu \nabla e_\omega, \nabla v_h)_\Omega = (\partial_t e_\omega, v_h)_M - (\partial_t e_\omega, v_h)_\Omega + s(u_h, \omega_h; v_h) \text{ in } Q. \quad (2.12)
\]
\[
(\nabla e_\Psi, \nabla \Phi_h)_\Omega - (e_\omega, \Phi_h)_\Omega = 0 \text{ in } Q.
\]
As mentioned in the introduction, if the solution \( (u, \omega) \) is smooth one may prove an error estimate that is robust with respect to \( \nu \) using standard linear theory and perturbation arguments. For the methods we consider herein, this result is an extension of the works in [17] and [7] and we state it here only with the dominant terms present.
For the reader’s convenience, we briefly outline the proof using one stabilization operator (defined in equation (5.5)) in the appendix.

Proposition 2.2. Let \((u, \omega)\) be a smooth solution of \((1.3)-(1.6)\) and \((u_h, \omega_h)\) be the solution of \((2.3)-(2.4)\), where the stabilization operator satisfies the additional weak consistency property

\[ s(u_h; \pi_L\omega, \pi_L\omega) \leq c(u, \omega)h^{3/2} \]

then for \(l = 1, 2\)

\[ \||u - u_h||(\cdot, T) + \||\omega - \omega_h||(\cdot, T)\| \leq c_\omega(h^{3/2}||\omega||_{L^2(I; H^2(\Omega))} + h^l||\Psi||_{L^\infty(I; H^{l+1}(\Omega))}) \]

where \(c_\omega := e||\nabla \omega||_{L^\infty(\Omega)}\). In addition, there holds for the stabilization operator

\[ s(u_h; \omega_h, \omega_h)^{3/2} \leq c_\omega(h^{3/2}||\omega||_{L^2(I; H^2(\Omega))} + h^l||\Psi||_{L^\infty(I; H^{l+1}(\Omega))}). \]

Observe that the exponential factor here depends on \(||\nabla \omega||_{L^\infty(\Omega)}\), compared to \(||\nabla u||_{L^\infty(\Omega)}\) in \((1.1)\). This is the price we pay for estimating the \(L^2\)-error of the vorticity. As we shall see below, the use of weaker norms for the estimation of \(\omega_h\) allows us to revert back to the exponential factor of \((1.1)\) and under the large eddy assumption, the exponential growth is moderate.

3. Dual problem. From the consistency relation \((2.12)\) we deduce the following (homogeneous) perturbation formulation for the evolution of \((e_\omega, e_\phi)\)

\[
\begin{align*}
(\partial_t e_\omega + u \cdot \nabla e_\omega + \text{rot } e_\phi \cdot \nabla \omega_h, \varphi_1)_\Omega + (\nu \nabla e_\omega, \nabla \varphi_1)_\Omega &= 0 \text{ in } Q, \\
(\nabla e_\phi, \nabla \varphi_2)_\Omega - (e_\omega, \varphi_2)_\Omega &= 0 \text{ in } Q,
\end{align*}
\]

where \(\varphi_1, \varphi_2\) are the solutions to a dual adjoint perturbation equation related to the continuous equation \((1.3)-(1.6)\) and the discretization \((2.3)-(2.4)\). Since the jump of the tangential derivative of \(\omega_h\) is zero, we may integrate by parts in \((3.1)\), to arrive at the dual adjoint problem

\[
\begin{align*}
-\partial_t \varphi_1 - u \cdot \nabla \varphi_1 - \varphi_2 - \nu \Delta \varphi_1 &= 0 \text{ in } Q, \\
-\Delta \varphi_2 - \nabla \omega_h \cdot \text{rot } \varphi_1 &= 0 \text{ in } Q, \\
\varphi_1(x, T) &= \xi_0(x) \text{ in } \Omega,
\end{align*}
\]

where \(\xi_0(x)\) is some initial data to be fixed later, the choice of \(\xi_0\) determines the quantity of interest.

A key result for the present analysis is the following stability estimate for the dual adjoint solution

Proposition 3.1. The following stability estimate holds for the solution \((\varphi_1, \varphi_2)\) of \((3.3)-(3.5)\),

\[
\begin{align*}
\sup_{t \in I} ||\nabla \varphi_1(\cdot, t)|| + ||\nu^{3/2} D^2 \varphi_1||_Q &\leq e^{\tau_F} ||\nabla \xi_0|| \\
\int_I ||\nabla \varphi_2(\cdot, t)|| \ dt &\leq e^{\tau_F} \int_I ||\omega_h||_{L^\infty(\Omega)} \ dt \ ||\nabla \xi_0||
\end{align*}
\]

where \(\tau_F\) is defined in the proof. If the large eddy assumption holds \(\tau_F \sim 1\).
Proof. First multiply (3.3) by $-\Delta \varphi_1$ and (3.4) by $\varphi_1$ and integrate over $Q^*: = \Omega \times (t*, T)$, where $t*$ is a time to be chosen. By summing the two relations we obtain

$$(\partial_t \varphi_1, \Delta \varphi_1)_{Q^*} + (u \nabla \varphi_1, \Delta \varphi_1)_{Q^*} + (\nabla \omega_h \cdot \text{rot} \varphi_1, \varphi_1)_{Q^*} + \|\nu \Delta \varphi_1\|_{Q^*}^2 = 0.$$ 

We will now treat the terms $I_1 - I_4$ term by term. First note that by integration by parts in space and then integration in time we have

$$I_1 = -\frac{1}{2} \int_{t*}^T \frac{d}{dt} \|\nabla \varphi_1(\cdot, t)\|^2 dt = \frac{1}{2} \|\nabla \varphi_1(\cdot, t*)\|^2 - \frac{1}{2} \|\nabla \xi_0\|^2.$$ 

The second term is handled using the decomposition of $u$ in the large scale and fine scale component and then an integration by parts only in the large scale part. Here $\nabla_S u$ denotes the symmetric part of the gradient of the vector $u$.

$$I_2 = -((\nabla_S \bar{u} - \frac{1}{2}(\nabla \bar{\omega}) I_{2 \times 2}) \nabla \varphi_1, \nabla \varphi_1)_{Q^*} - (u' \nabla \varphi_1, \Delta \varphi_1)_{Q^*} \leq \int_Q (\Lambda(\bar{u}, u', \nu) \nabla \varphi_1)^T \nabla \varphi_1 \, dx \, dt + \frac{1}{2} \|\nu \Delta \varphi_1\|_{Q^*}^2,$$

where $\Lambda(\bar{u}, u', \nu)$ is a two by two, symmetric matrix defined by,

$$\Lambda(\bar{u}, u', \nu) = -\nabla_S \bar{u} + \frac{1}{2} \nabla \dot{\bar{\omega}} I_{2 \times 2} + \frac{1}{2\nu} u' u'.$$

We now define the global timescale $\tau_F^*$ of the flow by

$$(\tau_F^*)^{-1} := \inf_{\bar{u} \in L^\infty(\Omega)} \inf_{u' \in L^\infty(\Omega)} \|\sigma_p^+ (\Lambda(\bar{u}, u', \nu))\|_{L^\infty(Q)}.$$

Here $\sigma_p^+$ denotes the largest positive eigenvalue of the matrix. This results in a nontrivial minimization problem in $L^\infty$. We leave the precise study of this problem for further work and here simply observe that by computing the eigenvalues of the symmetric part of the gradient tensor we may write

$$(\tau_F^*)^{-1} \leq \inf_{\bar{u}} J(\bar{u}, u')$$

where

$$J(\bar{u}, u') := \sup_{t \in I} \left( \| (\partial_x \bar{u}_1 - \partial_x \bar{u}_2)^2 + (\partial_x \bar{u}_1 + \partial_x \bar{u}_2)^2 \|_{L^\infty(\Omega)} + \nu^{-1} \|u'\|_{L^\infty(\Omega)}^2 \right).$$

We observe that the global stability does not depend on the divergence component or the rotational of $\bar{u}$, only on the other two components of the velocity gradient matrix. Since $u' = u - \bar{u}$, it follows that we can minimize over all large scale vector fields $\bar{u} \in [W^{1, \infty}(\Omega)]$ and the infimum value obtained is the optimal timescale of the flow. Under the assumptions made in the introduction, that $\|\bar{u}\|_{W^{1, \infty}(\Omega)} \sim 1$ and $\nu^{-1} \|u'\|_{L^\infty(\Omega)}^2 \sim 1$, for all $t$, we immediately deduce that $\tau_F^* \sim 1$.

By an integration by parts and by using the relations $\nabla \cdot \text{rot} \varphi = 0$ and $\nabla \varphi \cdot \text{rot} \varphi = 0$ we have

$$I_3 = - (\omega_h \nabla \cdot \text{rot} \varphi_1, \varphi_1)_{Q^*} - (\omega_h \cdot \text{rot} \varphi_1, \nabla \varphi_1)_{Q^*} = 0.$$
Collecting the results for $I_1 - I_3$ we have
\[
\|
abla \varphi_1(t^*)\|^2 + \|\nu \frac{1}{T} \Delta \varphi_1\|^2_{L^2} \leq \tau_{E^1} \|\nabla \varphi_1\|^2_{L^2} + \|\nabla \xi_0\|^2_{L^2}.
\]

The inequality for $\varphi_1$ follows after a Gronwall’s inequality and by taking the supremum over $t^*$, resulting in
\[
\sup_{t \in I} \|
abla \varphi_1(t)\|^2 + \|\nu \frac{1}{T} D^2 \varphi_1\|^2_{L^2} \lesssim e^\frac{x}{T} \|
abla \xi\|^2.
\]

Elliptic regularity has been used for the second term.

For the bound on $\varphi_2$ multiply equation \((3.4)\) by $\varphi_2$ and integrate over $\Omega$,
\[
\|\nabla \varphi_2(t)\|^2 = -(\omega_h \text{rot} \, \varphi_1, \nabla \varphi_2)_{\Omega} \leq \|\omega_h(t)\|_{L^\infty(\Omega)} \|\nabla \varphi_1(t)\| \|\nabla \varphi_2(t)\|.
\]

Then divide by $\|\nabla \varphi_2(t)\|$, integrate in time and use that
\[
\int_0^T \|\omega_h(t)\|_{L^\infty(\Omega)} \|\nabla \varphi_1(t)\| \, dt \leq \int_0^T \|\omega_h(t)\|_{L^\infty(\Omega)} \, dt \sup_{t \in I} \|\nabla \varphi_1(t)\|.
\]

Finally use equation \((3.6)\) to bound the term in $\|\nabla \varphi_1(t)\|$. [□]

Note the dependence of $\omega_h$ in the bound \((3.7)\). This appearance of a finite element function in the stability estimate shows that the global stability depends on the monotonicity of the approximation scheme. However as we shall see, strict monotonicity is not necessary, only $L^\infty$-control of the vorticity.

4. **A posteriori and a priori error estimates for the abstract method.**

Let $e_\omega = \omega - \omega_h$ and let the filtered error $\tilde{e}_\omega$ be defined as the solution to the problem
\[
- \delta^2 \Delta \tilde{e}_\omega + \tilde{e}_\omega = e_\omega.
\]

We introduce a norm on $\tilde{e}_\omega$ such that $\|\tilde{e}_\omega\|^2 := \|\delta \nabla \tilde{e}_\omega\|^2 + \|\tilde{e}_\omega\| = (e_\omega, \tilde{e}_\omega)_{\Omega}$. This norm coincides with the $L^2$-norm for $\delta = 0$ and is related to the $H^{-1}$-norm for $\delta = 1$. By choosing $\delta = \delta(h)$, i.e. by reducing the filter width with the mesh size, we obtain a family of norms that become stronger as the mesh size is reduced.

Using the above norm and the relations \((2.12), (3.3)- (3.4)\) as well as the stability result of Proposition \[\[4.1\]\] we may derive a posteriori estimates for the filtered quantity $\tilde{e}_\omega$. We here derive the result for the abstract finite element element method \((2.3)- (2.4)\) and then show how these estimates can be transformed into a priori error estimates, depending on the properties of the stabilization operator $s(u_h, \omega_h; \nu_h)$. The use of weak norms and stabilized finite element methods in the following estimates draws on ideas from \[\[4.1\]\] and \[\[4.4\]\].

**Theorem 4.1.** (A posteriori error estimates)

\[
\|\tilde{\omega} - \tilde{\omega}_h\|_{\delta} \lesssim e^\frac{x}{T} \left( \frac{1}{h^2} \right)^{\frac{1}{2}} \sum_{i=0}^5 \mathcal{R}_i,
\]

with

\[
\mathcal{R}_0 := \|\omega - \omega_h(\cdot, 0)\|,
\]

\[
\mathcal{R}_1 := \int_0^T \inf_{v_h \in V_h} \|h^\frac{1}{2} (u_h \cdot \nabla \omega_h - v_h)\| \, dt,
\]

\[
\mathcal{R}_2 := \int_0^T \|\nabla \omega_h\| \|\nabla \omega_h\| \, dt,
\]

\[
\mathcal{R}_3 := \int_0^T \|\omega_h\| \|\nabla \omega_h\| \, dt,
\]

\[
\mathcal{R}_4 := \int_0^T \|\nabla \omega_h\|^2 \, dt,
\]

\[
\mathcal{R}_5 := \int_0^T \|\omega_h\| \|\nabla \omega_h\| \, dt.
\]
\[ \mathcal{R}_2 := \min(h, \nu^2 T^2)\|n \cdot \nabla \omega_h\|_{F \times I}, \]

\[ \mathcal{R}_3 := \int_0^T \|\omega_h(\cdot, t)\|_{L^2(\Omega)} \ dt \min(c_0 \sup_{t \in I} \|\Psi_h(\cdot, t)\|_{\Delta, \beta}, c_1 h^{2} \sup_{t \in I} \|\omega_h(\cdot, t)\|), \]

where

\[ \|\Psi_h(\cdot, t)\|_{\Delta, \beta} := \|h^2 \nabla \Psi_h(\cdot, t)\|_F + \inf_{v_h \in V_h} \left( \sum_{K \in T_h} \|h^{2} \Delta \Psi_h(\cdot, t) - v_h\|_K^2 \right)^{1/2}, \]

\[ \mathcal{R}_4 := h^2 \int_0^T \|\partial_t \omega_h\| \ dt \]

and

\[ \mathcal{R}_5 := (U_0 + \|u_h\|_{L^2(Q)}) \int_0^T \|s(u_h; \omega_h, \omega_h)\|^2 \ dt. \]

The term \( \mathcal{R}_4 \) is omitted if the consistent mass matrix is used. For the velocities we have the estimate, for all \( t \in I \),

\[ \|(u - u_h)(\cdot, t)\| \lesssim \left( \|\Psi_h(\cdot, t)\|_{\Delta, \beta} + \|\omega - \omega_h(\cdot, t)\|_1 \right) \tag{4.3} \]

where the second term in the right hand side may be a posteriori bounded by taking \( \delta = 1 \) in \([1.2]\).

Proof. By the definition of \( \tilde{e}_\omega \), we have, taking \( \xi_0 = \tilde{e}_\omega \) in \([1.5]\),

\[
\|\tilde{e}_\omega\|^2 = (e_\omega(T), \varphi_1(T))_\Omega + (e_\omega, -\partial_t \varphi_1 - u \cdot \nabla \varphi_1 - \nu \Delta \varphi_1)_Q

+ (e_\Psi, -\Delta \varphi_2 - \nabla \omega_h \cdot \text{rot } \varphi_1)_Q

= (e_\omega(0), \varphi_1(0))_\Omega + (\partial_t e_\omega + u \cdot \nabla e_\omega + \text{rot } e_\Psi \cdot \nabla \omega_h, \varphi_1)_\Omega + (\nu \nabla e_\omega, \nabla \varphi_1)_\Omega

+ (\nabla e_\Psi, \nabla \varphi_2)_\Omega - (e_\omega, \varphi_2)_Q.
\]

Using now the consistency relation \([2.2]\) we obtain

\[
\|\tilde{e}_\omega\|^2 = (e_\omega(0), (\varphi_1 - \pi_L \varphi_1)(\cdot, 0))_\Omega + (\partial_t e + u \cdot \nabla e + \text{rot } e_\Psi \cdot \nabla \omega_h, \varphi_1 - \pi_L \varphi_1)_Q

+ (\nu \nabla e, \nabla (\varphi_1 - \pi_L \varphi_1))_Q + (\nabla e_\Psi, \nabla (\varphi_2 - \pi_L \varphi_2))_Q - (e, \varphi_2 - \pi_L \varphi_2)_Q

- (\partial_t \omega_h, \pi_L \varphi_1)_M \cdot Q + (\partial_t \omega_h, \pi_L \varphi_1)_Q - s(u_h; \omega_h; \pi_L \varphi_1),
\]

where \( \Pi : H^1(\Omega) \to V_h^1 \) will be taken as either \( \pi_L \) or \( \pi_V \). Using the equations \([1.5]-[1.0]\) and the definitions of the projections \( \pi_L \) and \( \pi_V \) we deduce for \( \Pi := \pi_V \),

\[
\|\tilde{e}_\omega\|^2 = (e_\omega(0), (\varphi_1 - \pi_L \varphi_1)(\cdot, 0))_\Omega - (u_h \cdot \nabla \omega_h - v_h, \varphi_1 - \pi_L \varphi_1)_Q

- (\nu \nabla \omega_h, \nabla (\varphi_1 - \pi_L \varphi_1))_Q + (\omega_h, \varphi_2 - \pi_V \varphi_2)_Q

- (\partial_t \omega_h, \pi_L \varphi_1)_M \cdot Q + (\partial_t \omega_h, \pi_L \varphi_1)_Q - \int_0^T s(u_h, \omega_h; \pi_L \varphi_1) \ dt,
\]

and similarly for \( \Pi := \pi_L \),

\[
\|\tilde{e}_\omega\|^2 = (e_\omega(\cdot, 0), (\varphi_1 - \pi_L \varphi_1)(\cdot, 0))_\Omega - (u_h \cdot \nabla \omega_h - v_h), \varphi_1 - \pi_L \varphi_1)_Q

- (\nu \nabla \omega_h, \nabla (\varphi_1 - \pi_L \varphi_1))_Q - (\nabla \Psi_h, \nabla (\varphi_2 - \pi_L \varphi_2))_Q

- (\partial_t \omega_h, \pi_L \varphi_1)_M \cdot Q + (\partial_t \omega_h, \pi_L \varphi_1)_Q - \int_0^T s(u_h, \omega_h; \pi_L \varphi_1) \ dt.
After some standard manipulation including integrations by parts, Cauchy-Schwarz inequalities, trace inequalities the approximation results (2.1) and (2.2) we may conclude, for \( \Pi := \pi_V \),

\[
\| \hat{e}_\omega \|_\delta^2 \lesssim \left( \frac{h}{\delta} \right)^{\frac{1}{2}} \| e_\omega (\cdot, 0) \| + \int_0^T \inf_{v_h \in V_h} h^{\frac{1}{2}} (u_h \cdot \nabla \omega_h - v_h) \| dt \\
+ \min(h, \nu \frac{h^2}{T}) \| n_F \cdot \nabla \omega_h \|_{L^2(I)} + c_1 h^{\frac{1}{2}} \sup_{t \in I} \| \omega_h (\cdot, t) \| \int_0^T \| \omega_h (\cdot, t) \|_{L^\infty(Q)} \| dt \\
+ h^{\frac{3}{2}} \int_0^T \| \partial_t \nabla \omega_h \| dt + (U_0 + \| u_h \|_{L^\infty(Q)}) \int_0^T s(u_h; \omega_h, \omega_h) \| dt \\
\times \left( \sup_{t \in I} \| \delta \nabla \varphi_1 (\cdot, t) \| + \| \delta \nabla^2 \varphi_1 \|_{Q} \right).
\]

If \( \Pi := \pi_L \) the fourth term on the right hand side is replaced using

\[
(\nabla \Psi_h, \nabla (\varphi_2 - \pi_L \varphi_2))_Q \lesssim \left( \frac{h}{\delta} \right)^{\frac{1}{2}} c_0 \sup_{t \in I} \| \Psi_h (t) \|_{\Delta, 0} \int_0^T \| \delta \nabla \varphi_2 (\cdot, t) \| dt,
\]

followed by the bound (3.7) on \( \varphi_2 \). The estimate (4.2) now follows by taking the minimum of the two expressions and noting that by (3.6)

\[
\sup_{t \in I} \| \delta \nabla \varphi_1 (\cdot, t) \| + \| \delta \nabla^2 \varphi_1 \|_{Q} \lesssim e^{\frac{2}{\gamma}} \| \hat{e}_\omega \|_\delta.
\]

The velocity estimate (4.3) is obtained by noting that, with \( e_\Psi := \Psi - \Psi_h \),

\[
\| u - u_h \|_2^2 := \| \nabla e_\Psi \|_2^2 = (\nabla e_\Psi, \nabla (e_\Psi - \pi_L e_\Psi)) + (e_\omega, \pi_L e_\Psi).
\]

Using the equation (1.6) we have

\[
\| \nabla e_\Psi \|_2^2 = (\nabla \Psi_h, \nabla (e_\Psi - \pi_L e_\Psi)) + (\omega, e_\Psi) - (\omega_h, \pi_L e_\Psi) \\
= (\nabla \Psi_h, \nabla (e_\Psi - \pi_L e_\Psi)) + (\omega - \omega_h, e_\Psi)
\]

Let \( \hat{e} \) be the solution of (4.1) with \( \delta = 1 \). Then

\[
\| u - u_h \|_2^2 = (\nabla \Psi_h, \nabla (e_\Psi - \pi_L e_\Psi))_\Omega + (\nabla \hat{e}_\omega, \nabla e_\Psi)_\Omega + (\hat{e}_\omega, e_\Psi)_\Omega.
\]

By an integration by parts in the first term, followed by a Cauchy-Schwarz inequality and the Poincaré-Friedrichs inequality in the last term we may write

\[
\| u - u_h \|_2^2 \lesssim \| h^{\frac{1}{2}} \| \nabla \Psi_h \|_F \| h^{-\frac{1}{2}} (e_\Psi - \pi_L e_\Psi) \|_F \\
+ \left( \sum_{K \in T_h} \| h (\Delta \Psi_h - v_h) \|_K^2 \right)^{\frac{1}{2}} \| h^{-1} (e_\Psi - \pi_L e_\Psi) \|_1 \\
+ \| (\hat{\omega} - \hat{\omega}_h) \|_1 \| (u - u_h) \|.
\]

By elementwise trace inequalities and the approximation property (2.1) we have

\[
\| h^{-\frac{1}{2}} (e_\Psi - \pi_L e_\Psi) \|_F + \| h^{-1} (e_\Psi - \pi_L e_\Psi) \| \lesssim \| u - u_h \|.
\]
by which we conclude. □

If the stability properties of the stabilized method are sufficient, these a posteriori error estimates translate into a priori error estimates. We propose two strategies for this. One using stability concepts based on Sobolev injections for discrete spaces and one based on monotonicity, applicable to monotone stabilized finite element methods and monotone implicit large eddy methods. The advantage of the former is that it allows the derivation of a priori error estimates for quasi linear terms \( s(u_h; \omega_h, v_h) \) and the use of the consistent mass matrix. The latter technique on the other hand allows for the derivation of a priori error estimates with precise control of the constants in the estimates. We will use the notion of the discrete maximum principle (DMP) and the associated, DMP-property of the forms defining a finite element method introduced in [6].

**Proposition 4.2.** Assume that the mass \((\cdot, \cdot)_M\) is evaluated exactly and that in addition to (2.5) and (2.7) the following stability estimate holds for all \( t > 0 \),

\[
\| \omega_h \|_{L^\infty(\Omega)} \lesssim c(h) (\| \omega_h \| + s(u_h, \omega_h, \omega_h)^{1/2}). \tag{4.4}
\]

Then there holds for all \( \epsilon > 0 \),

\[
\| (\tilde{\omega} - \tilde{\omega}_h)(T) \|_{S} \lesssim \epsilon^{1/2} \left( \frac{h}{\delta^2} \right) \left( c_r h^{-\epsilon} + c(h) h^{1/2} \right)
\]

and

\[
\| (u - u_h)(\cdot, T) \| \lesssim \inf_{v_h \in \V_h} \| (u - v_h)(\cdot, T) \| + \epsilon^{1/2} \left( \frac{h}{\delta^2} \right) \left( c_r h^{-\epsilon} + c(h) h^{1/2} \right).
\]

**Proof.** First we recall that \( \| \omega_h(\cdot, 0) \| \leq \| \omega(\cdot, 0) \| \). Then by (2.9) and (2.7)

\[
\int_{0}^{T} \inf_{v_h \in \V_h} \| h^{1/2} (u_h \nabla \omega_h - v_h) \| \, dt \lesssim T^{1/2} \left( \int_{0}^{T} s(u_h; \omega_h, \omega_h) \, dt \right)^{1/2} \lesssim T^{1/2} |\omega(\cdot, 0)|.
\]

Using an elementwise trace inequality and (2.7) we also have

\[
\min(h, \nu^{1/2} T^{-1/2}) \| \nabla \omega_h \|_{\mathcal{F} \times I} \leq h^{1/2} \| \nabla \omega_h \|_{\mathcal{Q}} \lesssim h^{1/2} |\omega(\cdot, 0)|.
\]

For \( \mathcal{R}_3 \) we use the discrete Sobolev injection (4.3) to deduce

\[
h^{1/2} \sup_{t \in I} \| \omega_h(\cdot, t) \| \int_{0}^{T} \| \omega_h(\cdot, t) \|_{L^\infty(\Omega)} \, dt \lesssim h^{1/2} \| \omega_h(\cdot, 0) \| c(h) \int_{0}^{T} \left( \| \omega_h \| + s(u_h, \omega_h, \omega_h)^{1/2} \right) \, dt \lesssim h^{1/2} c(h) \| \omega_h(\cdot, 0) \|^2.
\]

The only remaining term is the stabilization term, which is not innocent since we do not have an a priori bound on the factor \( \| u_h \|_{L^\infty(\Omega)} \). Here we use (2.9) to deduce, for all \( t > 0 \) and \( q > 2 \),

\[
\| u_h \|_{L^\infty(\Omega)} \int_{0}^{T} s(u_h; \omega_h, \omega_h)^{1/2} \, dt \leq c_q \| \omega_h \|_{L^\infty(\Omega)} T^{1/2} \left( \int_{0}^{T} s(u_h; \omega_h, \omega_h) \, dt \right)^{1/2}
\]
and by a global inverse inequality and the bound (2.7) we may conclude
\[ \| u_h \|_{L^2(Q)} \int_0^T s(u_h; \omega_h, \omega_h) \frac{1}{2} \frac{1}{T} dt \lesssim c_q h^{\frac{2(q-2)}{q}} T^\frac{1}{q} \sup_{t \in I} \| \omega_h(\cdot, t) \| \| \omega_h(\cdot, 0) \| \]
and the estimate follows taking \( \epsilon = (q - 2)/q \). Note that the constant \( c_q \) explodes as \( q \to 2 \).

The bound in the \( L^2 \)-norm for the velocities follows as before from the vorticity estimate using, with \( C_h \) denoting the Clément interpolant,
\[ \| u - u_h \|^2 = \| \nabla \varepsilon \|^2 = (\nabla \varepsilon, \nabla (\Psi - C_h \Psi)) - (\varepsilon, \Psi_h - C_h \Psi) \leq \| u - u_h \| \| \nabla (\Psi - C_h \Psi) \| - (\Delta \varepsilon, \varepsilon, (C_h \Psi - \Psi_h)) \leq \| u - u_h \| \| \nabla (\Psi - C_h \Psi) \| + \| \varepsilon - \varepsilon_h \| \| C_h \Psi - \Psi_h \|_{H^1(Q)} . \]

We conclude by using the \( H^1 \)-stability of the Clément interpolant, a Poincaré inequality and finally by dividing both sides with \( \| u - u_h \| \). \( \Box \)

**Proposition 4.3.** (A priori error estimate using monotonicity) Assume that \( \text{Re}_h > 1 \), that the mass \( (\cdot, \cdot)_M \) is evaluated using nodal quadrature, that the form \( a(u_h; \omega_h, v_h) + s(u_h; \omega_h, v_h) \) has the DMP property as defined in [6] and that (2.5) - (2.0) are satisfied as well as the assumptions of Lemma 2.1 Then there holds
\[ \| (\hat{\omega} - \hat{\omega}_h)(T) \| \lesssim e^{\frac{1}{2}} \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} \]
and
\[ \| (u - u_h)(\cdot, T) \| \lesssim \inf_{v_h \in \mathcal{V}_h} \| (u - v_h)(\cdot, T) \| + e^{\frac{1}{2}} \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} \]

**Proof.** The terms \( R_0 - R_2 \) are bounded as in the proof of Proposition 4.2. Since by assumption the spatial discretization of (2.8) has the DMP property and the mass-matrix is evaluated using nodal quadrature, we know from [5, 6] that
\[ \| \omega_h \|_{L^\infty(Q)} = \| \omega_h(\cdot, 0) \|_{L^\infty(\Omega)} . \]
Hence by (2.0) \( \| u_h \|_{L^\infty(Q)} \leq c_{\infty} \| \omega_h(\cdot, 0) \|_{L^\infty(\Omega)} \). We may then use these \( L^\infty \)-bounds together with the stabilities of Lemma 2.1 to upper bound the remaining residual quantities of (2.12). Using (2.9) and (2.7) we immediately have
\[ h^{\frac{3}{2}} \sup_{t \in I} \| \omega_h(\cdot, t) \| \int_0^T \| \omega_h(\cdot, t) \|_{L^\infty(\Omega)} dt \lesssim h^{\frac{1}{2}} T \| \omega_h(\cdot, 0) \| \| \omega_h(\cdot, 0) \|_{L^\infty(\Omega)} \]
For the residual term resulting from the mass lumping we have using the stability (2.10)
\[ h^{\frac{3}{2}} \int_0^T \| \partial_t \nabla u_h \| dt \lesssim T^\frac{1}{2}(U_0 + \| u_h \|_{L^\infty(Q)}) \left( \int_0^T (s(u_h; \omega_h, \omega_h) + \| \nu \nabla \omega_h \|^2) dt \right)^\frac{1}{2} \leq T^\frac{1}{2} U_0^\frac{1}{2} \| \omega(\cdot, 0) \|. \]
The remaining contribution from the stabilization is bounded as before using the maximum principle and (2.7). The proof of the $L^2$-norm estimate on the velocities is identical to that of Proposition 4.2.

Note that only the proof of Proposition 4.3 uses the assumption $Re_h > 1$ and only to control the non-consistent mass term. This constraint is likely to vanish if the method is analysed using techniques appropriate for parabolic problems, since mass lumping is known to be stable for dominant diffusion (see for instance [24]).

5. Stabilized finite element methods. The estimates of Theorem 4.1 holds for any finite element method on the form (2.3)-(2.4). Indeed by taking $s(\cdot, \cdot, \cdot) \equiv 0$ the standard Galerkin method is included. This means that in general the effect of stabilization can be observed only a posteriori, by observing smaller residuals for the stabilized formulations. In Propositions 4.3 and 4.2 we propose a priori estimates derived from the a posteriori error estimates under special assumptions on the properties of the stabilizing terms. These can be proven to hold only for stabilized finite element methods, since the standard Galerkin method does not allow for a control of the second term of the right hand side of (4.2) independently of the viscosity, nor can (2.5) and (4.4) be proven to hold. In this section we will suggest some stabilization operators that satisfy the assumptions necessary for the results of the abstract analysis to hold. We will consider the following cases:

1. linear artificial viscosity, in which the numerical viscosity is increased so that the mesh Reynolds number always is one. Using a lumped mass matrix together with anisotropic viscosity we may design the scheme to satisfy a discrete maximum principle, giving a priori control of $\|\omega_h\|_{L^\infty(Q)}$. When the consistent mass matrix is used one may proved that (4.4) holds giving once again a priori estimates, at the price of a logarithmic factor.

2. high order stabilization, we propose to stabilize the jump of the streamline derivative. This scheme does not yield a maximum principle, so the residuals can not be completely a priori bounded. The scheme has some interesting conservation properties for two-dimensional Navier-Stokes’ computations that we will point out. If a nonlinear stabilization term is added and mass-lumping is used the solution may be made monotone and the a priori error estimate of Proposition 4.3 holds, this time with the possibility of higher order convergence in the smooth portion of the flow. Finally if the consistent mass matrix is used and stabilization is added also in the crosswind direction, an estimate of the type (4.4) can be shown to hold leading to a priori error bounds using Proposition 4.2.

5.1. Methods using consistent mass matrix. We consider first the stabilization method obtained by penalizing the jumps of the streamline derivative over element faces. We use the exact mass matrix in (2.3) and the stabilizing operator

$$s_{sd}(u_h, \omega_h, v_h) := \gamma \sum_{F \in \mathcal{F}} U_0^{-1}(h_F^2 \langle [u_h \cdot \nabla \omega_h], [u_h \cdot \nabla v_h] \rangle_F). \quad (5.1)$$

For this formulation the following stability estimates hold

**Lemma 5.1.**

$$\sup_{t \in I} \|\omega_h(\cdot, t)\|^2 + 2 \|\nabla \omega_h\|^2_Q + 2\gamma U_0^{-1} \|h_F [u_h \cdot \nabla \omega_h]\|_F^2 \leq \|\omega_h(\cdot, 0)\|^2 \quad (5.2)$$

and if the consistent mass matrix is used,

$$\|u_h(\cdot, T)\|^2 + 2 \|\nabla \omega_h\|^2_Q = \|u_h(\cdot, 0)\|^2 \quad (5.3)$$
Robust error estimates for the two dimensional Navier-Stokes equations

Proof. the proof of (5.2) is an immediate consequence of (2.7) and the definition (5.1). The inequality (5.3) follows by observing that
\[ s_{sd}(u_h, \omega_h, \Psi_h) = \gamma \sum_{F \in F} (U_0^{-1} h_F^2 \|u_h \times \nabla \omega_h\|, [\text{rot } \Psi_h \cdot \nabla \Psi_h]) = 0. \]

Observe that the method dissipates enstrophy but conserves energy exactly as the physics of the problem suggests. Using known results on interpolation between discrete spaces it is also straightforward to show (see [8]),
\[ \inf_{v_h \in V_h} \| h^{-\frac{1}{2}}(u_h \cdot \nabla \omega_h - v_h) \|^2 \lesssim s_{sd}(u_h, \omega_h, \omega_h). \]

Unfortunately this stabilization operator can not be shown to satisfy (4.4). For this we need the stabilization to act also in the crosswind direction. We therefore propose the following two stabilization operators, the first is the standard artificial viscosity method
\[ s_{av}(u_h; \omega_h, \omega_h) := (\gamma h(U_0 + |u_h|)^2 U_0^{-1} \nabla \omega_h, \nabla v_h) \] (5.4)
and the second is a modification of (5.1) where also the crosswind gradient is penalized defined by
\[ s_{cd}(u_h, \omega_h, v_h) := s_{sd}(u_h, \omega_h, v_h) + \gamma_1 \sum_{K \in T_h} U_0 h_K^2 \int_{\partial K} [n_F \cdot \nabla \omega_h] [n_F \cdot \nabla v_h] \, ds \] (5.5)

Observe that the first part of \( s_{cd} \) ensures the satisfaction of (2.5) and as we shall see the second part is necessary for (4.4) to hold.

Proposition 5.2. Both stabilization operators (5.4) and (5.5) satisfy (2.5) and (2.6). The stabilization operator \( s_{av}(\cdot;\cdot;\cdot) \) satisfy (4.4) with \( c(h) \sim h^{-\frac{1}{2}} (1 + |\log(h)|) \) and \( s_{cd}(\cdot;\cdot;\cdot) \) satisfy (4.4) with \( c(h) \sim h^{-\frac{1}{2} + \mu} (1 + |\log(h)|), \mu > 0 \).

Proof. The proofs of (2.9) - (2.10) are consequences of the Cauchy-Schwarz inequality and in the case of \( s_{cd} \) trace inequalities. To prove (4.4) we note that in two space dimensions there holds (see [23]),
\[ \| \omega_h \|_{L^\infty(\Omega)} \lesssim (1 + |\log(h)|) \| \omega_h \|_{H^1(\Omega)}. \]

This allows us to conclude for \( s_{av} \). For \( s_{cd} \) we use that
\[ \| \nabla \omega_h \| \leq \left( \sum_{F \in F} \int_F \| \nabla \omega_h \cdot n_F \| |\omega_h| \, ds \right)^{\frac{1}{2}}. \]

A Cauchy-Schwarz inequality followed by a trace inequality in the right hand side leads to
\[ \| \nabla \omega_h \| \lesssim \left( \sum_{K \in T_h} h^{-\frac{1}{2} + \mu} \| \omega_h \|_K [n_F \cdot \nabla \omega_h] \|_{\partial K} \right)^{\frac{1}{2}} \lesssim h^{-\frac{1}{2} + \mu} (\| \omega_h \| + s_{cd}(u_h; \omega_h, \omega_h)^{\frac{1}{2}}). \]

Since the assumptions of Proposition 4.2 are satisfied, we may conclude that the
method \cite{[23, 24]} using the stabilization \cite{[5, 4]} satisfy the a priori error bounds for \( \epsilon > 0 \)
\[
\| (\tilde{\omega} - \tilde{\omega}_h)(T) \|_\delta \lesssim e^{\frac{\delta}{\gamma}} \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} (c_h h^{-\epsilon} + 1 + |\log(h)|)
\]
and
\[
\| (u - u_h)(\cdot, T) \| \lesssim \inf_{v_h \in W_h^{-1}} \| (u - v_h)(\cdot, T) \| + e^{\frac{\delta}{\gamma}} \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} (c_h h^{-\epsilon} + 1 + |\log(h)|).
\]
Similarly we have the following estimates if the stabilization \cite{[5, 5]} is used.
\[
\| (\tilde{\omega} - \tilde{\omega}_h)(T) \|_\delta \lesssim e^{\frac{\delta}{\gamma}} \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} (c_h h^{-\epsilon} + (1 + |\log(h)|)h^{-\frac{1}{2}})
\]
and
\[
\| (u - u_h)(\cdot, T) \| \lesssim \inf_{v_h \in W_h^{-1}} \| (u - v_h)(\cdot, T) \| + e^{\frac{\delta}{\gamma}} h^{\frac{1}{2}} (c_h h^{-\epsilon} + (1 + |\log(h)|)h^{-\frac{1}{2}}).
\]
We see that if we take \( \mu = 1 \) in \cite{[5, 5]} we get the same order for the two methods, however if we want the method to have optimal convergence for smooth solutions we choose \( \mu = 2 \) and \( l = 2 \), resulting in an a priori convergence order of \( O(h^{\frac{4}{2}}) \) in the non-smooth case.

5.2. Monotone methods. Since the consistent mass matrix is non-monotone we herein only consider methods using lumped mass. Monotone methods can also be designed using a nonlinear switch that changes the local quadrature as a function of the operator results in an M-matrix on Delaunay meshes. Since by the maximum principle, any additional acute condition on the mesh, since the discretization of the Laplace operator results in an M-matrix on Delaunay meshes. Since by the maximum principle, the dispersion error known to haunt mass-lumping schemes, such methods are beyond the scope of the present paper.

5.2.1. Linear artificial viscosity. A monotone method using linear artificial viscosity is obtained by taking (see \cite{[5]})
\[
s(u_h, \omega_h, v_h) := \gamma \sum_K (\max(U_0, \| u_h \|_{\infty(K)})h_K^2 \sum_{F \in \partial K} (\nabla \omega_h \times n_F, \nabla v_h \times n_F)_F. \quad (5.6)
\]
Then the estimates \cite{[2, 7]} - \cite{[2, 8]} hold and we observe that there exists positive constants \( c_1, c_2 \) such that
\[
c_1 \| u_h \|^{\frac{1}{2}} h^{\frac{1}{2}} \nabla \omega_h \|_Q^2 \leq s(u_h, \omega_h, \omega_h) \leq c_2 \| u_h \|^{\frac{1}{2}} h^{\frac{1}{2}} \nabla \omega_h \|_Q^2.
\]
Let the mass matrix be evaluated using nodal quadrature so that the matrix corresponding to \( (\cdot, \cdot)_M \) is diagonal. We may use the theory of \cite{[5, 4]} to prove that the operator \( a(\omega, v_h) + s(u_h, \omega_h, v_h) \) has the DMP-property and hence the following discrete maximum principle holds
\[
\| \omega_h \|_{\infty(Q)} = \| \omega_h (\cdot, 0) \|_{\infty(\Omega)}.
\]
This requires the parameter \( \gamma \) to be chosen large enough, however it does not require any additional acute condition on the mesh, since the discretization of the Laplace operator results in an M-matrix on Delaunay meshes. Since by the maximum principle, \( \| u_h \|_{\infty(Q)} \lesssim 1 \| \omega_h (\cdot, 0) \|_{\infty(\Omega)} \) we have
\[
\| u_h \|^{\frac{1}{2}} h^{\frac{1}{2}} \nabla \omega_h \|_Q^2 \lesssim \| u_h \|^{\frac{1}{2}} h^{\frac{1}{2}} \nabla \omega_h \|_M \| s(u_h, \omega_h, \omega_h) \|_M dt \lesssim \| \omega_h (\cdot, 0) \|_M^2 \quad (5.7)
\]
which proves (2.5) with \( v_h = 0 \). It is straightforward to prove also (2.6). Comparing with Proposition 4.3 we conclude that the assumptions are satisfied and hence that the Proposition holds for (2.3)-(2.4) with stabilization given by (5.6) and the mass matrix evaluated using nodal quadrature.

5.2.2. Nonlinear artificial viscosity. Here we assume that \( l = 1 \) so that both \( \omega_h \) and \( \Psi_h \) are discretized using piecewise affine elements. We propose a stabilization term consisting of one linear part and one nonlinear part. The role of the nonlinear part is to ensure that the form \( a(\cdot; \cdot; \cdot) + s(\cdot; \cdot; \cdot) \) has the DMP property. The linear part is necessary to ensure that the inequality (2.5) holds. We define

\[
s(u_h; \omega_h, v_h) := s_{sd}(u_h; \omega_h, v_h) \tag{5.8}
\]

\[
+ \gamma_2 \sum_K h^2 \sum_{F \in \partial K} R_F(u_h, \omega_h)(\text{sign}(\nabla \omega_h \times n_F), \nabla v_h \times n_F)_F \tag{5.9}
\]

where

\[
R_F(u_h, \omega_h) := \| u_h \|_{L^\infty(\Delta F)} (1 + U_0^{-1}\| u_h \|_{L^\infty(\Delta F)}) m_F(\| n_F \cdot \nabla \omega_h \|)
\]

with \( \Delta_F := \cup_{K \in T_h; K \cap F \neq \emptyset} K \) and

\[
m_F(\| n_F \cdot \nabla \omega_h \|) = \max_{F' \in \partial F; K' \cap F = F} \| n_{F'} \cdot \nabla \omega_h \|.\]

It is shown in [6] that with this definition \( a(u_h; \omega_h, v_h) + s(u_h; \omega_h, v_h) \) has the DMP property for \( \gamma_2 \) large enough. Since the bounds (2.5)-(2.10) also hold, the assumptions of Proposition 4.3 are satisfied and its estimates hold. We conclude that for the methods defined by mass lumping and the stabilization operators (5.6) or (5.8) the following estimates hold

\[
\| (\tilde{\omega} - \tilde{\omega}_h)(T) \|_\delta \lesssim \varepsilon \tau F \left( \frac{h}{\delta^2} \right)^{1/2}
\]

and

\[
\| u - u_h \|_\delta \lesssim \inf_{v_h \in W^{1,1}_h} \| u - v_h \|_\delta + \varepsilon \tau F \left( \frac{h}{\delta^2} \right)^{1/2}.
\]

6. Conclusion. We have shown that under a certain structural assumption on the solution of the two dimensional Navier-Stokes’ equation one may derive robust error estimates with an order in \( h \), independent of both the Reynolds number and high order Sobolev norms of the exact solution. Robustness is obtained for a class of stabilized finite element methods. The estimates are both on a posteriori form, and on a priori form, providing an upper bound on the error. Due to the strong assumptions on the mesh the present a posteriori error estimates are not immediately suitable for use in adaptive algorithms, but a more detailed analysis may allow the mesh assumptions to be relaxed. If the solution is smooth we also prove that optimal convergence may be obtained, provided the stabilization operator is weakly consistent to the right order.

Observe that it is natural that the LES estimate has much poorer convergence order, since we may assume no smoothness of the exact solution. Even the large scales are assumed to have moderate gradients only.
We show how several stabilized methods enter the framework, both first and second order accurate ones. The interest of the first order artificial viscosity method is primarily its close relationship to the vertex centered finite volume method. Note also that the estimates with an order proposed herein for nonlinear monotone schemes to the best of our knowledge are the first of their kind in the literature.

It appears that for implicit large eddy simulations both the estimate (1.1) for smooth solutions and the estimate (1.3) for rough solutions derived herein are desirable properties for the theoretical justification of a method.

Future work will focus on numerical investigations both in two and three space dimensions. Of particular interest is to study the stability of the incompressible Euler equations to see if the limit estimate with no allowed small scales is sharp.

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Appendix A. Proof of Proposition 2.2. We introduce the discrete errors, with $I_h$ denoting the Lagrange interpolant $e_{h, \psi} := \psi_h - I_h \psi$ and $e_{h, \omega} := \omega_h - \pi_L \omega$.

First consider the second equation (2.4) and use Galerkin orthogonality

$$\|\nabla e_{h, \psi}\|^2 = (\nabla (\psi - I_h \psi), \nabla e_{h, \psi}) - (\omega - \omega_h, e_{h, \psi}).$$

Applying Poincaré’s inequality followed by Cauchy Schwarz inequality we obtain the following bound for $\psi$ in terms of the error in the vorticity

$$\|\nabla e_{h, \psi}\| \lesssim \|\nabla (\psi - I_h \psi)\| + \|\omega - \pi_L \omega_h\| + \|e_{h, \omega}\|.$$

Consider now the equation (2.3) taking $v_h = e_{h, \omega}$ and observing that there holds

$$\frac{1}{2} \frac{d}{dt} \|e_{h, \omega}\|^2 + s(u_h; e_{h, \omega}, e_{h, \omega})$$

$$= (\partial_t (\omega - \pi_L \omega), e_{h, \omega}) + (\omega, u \cdot \nabla e_{h, \omega})$$

$$- (\pi_L \omega, u_h \cdot \nabla e_{h, \omega}) - s(u_h; \pi_L \omega, e_{h, \omega}).$$

By integration by parts in time we see that the first term on the right hand side is zero, by the orthogonality of the $L^2$-projection. We then add and subtract $u_h$ in the second term on the right hand side to obtain

$$\frac{1}{2} \frac{d}{dt} \|e_{h, \omega}\|^2 + s(u_h; e_{h, \omega}, e_{h, \omega}) = (\omega, (u - u_h) \cdot \nabla e_{h, \omega})$$

$$+ (\omega - \pi_L \omega, u_h \cdot \nabla e_{h, \omega}) - s(u_h; \pi_L \omega, e_{h, \omega}) = I + II + III.$$

In the first term on the right hand side we now reintegrate by parts and use Cauchy-Schwarz inequality,

$$I \leq \|\omega\|_{W^{1, \infty}} \|u - u_h\| \|e_{h, \omega}\|$$

$$\lesssim \|\omega\|_{W^{1, \infty}} (\|\nabla (\psi - I_h \psi)\| + \|\omega - \omega_h\|) \|e_{h, \omega}\|$$

$$\lesssim \|\omega\|_{W^{1, \infty}} (\|\nabla (\psi - I_h \psi)\|^2 + \|\omega - \pi_L \omega\|^2 + \|e_{h, \omega}\|^2).$$
In the second term we use the orthogonality of the $L^2$-projection to retract some function $v_h$ and then apply (2.35),

$$II = (\omega - \pi_L \omega, u_h \cdot \nabla e_h, \omega - v_h) \leq c \|h^{-\frac{1}{2}}(\omega - \pi_L \omega)\|_{L^2} s(u_h; e_h, e_h) \frac{1}{T}$$

$$\leq c h^{2s-1} \|\omega\|_{H^s}^2 + \frac{1}{4} s(u_h; e_h, e_h).$$

For the stabilization term finally we apply the Cauchy-Schwarz inequality and an arithmetic-geometric inequality to obtain

$$III = s(u_h; \pi_L \omega, e_h, \omega) \leq s(u_h; \pi_L \omega, \pi_L \omega) + \frac{1}{4} s(u_h; e_h, e_h).$$

(A.1)

Then we observe that by adding and subtracting $I_h \rot \Psi$ we may write

$$s(u_h; \pi_L \omega, \pi_L \omega) \leq s(u_h - I_h \rot \Psi; \pi_L \omega, \pi_L \omega) + s(I_h \rot \Psi; \pi_L \omega, \pi_L \omega)$$

and using the definition (A.1) and the stability of the $L^2$-projection on quasi uniform meshes, we have,

$$s(u_h - I_h \rot \Psi; \pi_L \omega, \pi_L \omega) \leq \sum_{F \in \mathcal{F}} \int_F h^2 |u_h - I_h \rot \Psi| |\nabla \pi_L \omega h|^2 ds$$

$$\leq \|\nabla \omega\|_{L^2(\Omega)}^2 \|h u_h - I_h \rot \Psi\|^2$$

$$\leq \|\nabla \omega\|_{L^2(\Omega)}^2 h(\|\nabla \Psi - I_h \nabla \Psi\|^2 + \|\nabla (\Psi - I_h \Psi)\|^2 + \|\omega - \pi_L \omega\|^2 + \|e_h, \omega\|^2)$$

and then

$$s(I_h \rot \Psi; \pi_L \omega, \pi_L \omega)$$

$$\leq \|I_h \rot \Psi\|_{L^2(Q)}^2 \sum_K h_K (\|\nabla (\omega - \pi_L \omega)\|_K^2 + h_K^2 \|\nabla (\omega - \pi_L \omega)\|^2)$$

$$\leq \|u\|_{L^2(Q)}^2 C h_K^{2s-1} \|\omega\|_{L^2(\Omega)}^2.$$
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