Research Article

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Solutions for nonhomogeneous fractional $(p, q)$-Laplacian systems with critical nonlinearities

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Abstract: In this article, we aimed to study a class of nonhomogeneous fractional $(p, q)$-Laplacian systems with critical nonlinearities as well as critical Hardy nonlinearities in $\mathbb{R}^N$. By appealing to a fixed point result and fractional Hardy-Sobolev inequality, the existence of nontrivial nonnegative solutions is obtained. In particular, we also consider Choquard-type nonlinearities in the second part of this article. More precisely, with the help of Hardy-Littlewood-Sobolev inequality, we obtain the existence of nontrivial solutions for the related systems based on the same approach. Finally, we obtain the corresponding existence results for the fractional $(p, q)$-Laplacian systems in the case of $N = sp = lq$. It is worth pointing out that using fixed point argument to seek solutions for a class of nonhomogeneous fractional $(p, q)$-Laplacian systems is the main novelty of this article.

Keywords: fractional $(p, q)$-Laplacian system, critical nonlinearities, Choquard nonlinearities, fixed point theorem

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1 Introduction and main results

In the first part of our article, we are dedicated to the study of the following fractional $(p, q)$-Laplacian system in $\mathbb{R}^N$:

\[
\begin{cases}
(-\Delta)^s_p u - \frac{|u|^{p^*_s - 2} u}{|x|^s} = \frac{\alpha}{p^*_s} |u|^{p^*_s - 2} u|v|^{\beta} + \lambda_1 f(x) & \text{in } \mathbb{R}^N, \\
(-\Delta)^q_l v - \frac{|v|^{q^*_l - 2} v}{|x|^l} = \frac{\beta}{q^*_l} |v|^{q^*_l - 2} v|u|^{\alpha} + \lambda_2 g(x) & \text{in } \mathbb{R}^N,
\end{cases}
\]

(1.1)

where $0 < s, l < 1$, $2 \leq p < q$, $0 \leq \gamma \leq sp < N$, $0 \leq \tau \leq lq < N$, $p - \alpha < p^*_s$, $q - \beta < q^*_l$, $p^*_s = Np/(N - sp)$ and $q^*_l = Nq/(N - lq)$ are fractional Sobolev critical exponents, $\lambda_1$ and $\lambda_2$ are positive parameters, the functions $f(x)$ and $g(x)$ are considered to be perturbation terms and $f(x), g(x) \neq 0$. Moreover, $p^*_y = (N - y)p/(N - sp)$ and $q^*_y = (N - y)q/(N - lq)$ are fractional Hardy-Sobolev critical exponents. And, $(-\Delta)^m_m$ is the fractional $m$-Laplace operator defined by
where \( B_\varepsilon(x) \) denotes the ball of \( \mathbb{R}^N \) centered at \( x \in \mathbb{R}^N \) with radius \( \varepsilon > 0 \). More details about the fractional Sobolev space and the fractional \( p \)-Laplacian can be found in [17,38] and references therein.

In recent years, fractional and nonlocal operators are extremely widely used in many different contexts, such as physics, optimization, soft thin films, population dynamics, geophysical fluid dynamics, finance, phase transitions, stratified materials, water waves, game theory, anomalous diffusion, flame propagation, conservation laws, crystal dislocation, semipermeable membranes and so on, and we can see more details in [1,6,31]. For more useful results and applications of fractional and nonlocal operators, we refer to [2,7,36,37,43,49].

In the last 20 years, the investigation of the following fractional Schrödinger equations:

\[
(-\Delta)^s u + V(x)u = f(x, u) \quad \text{in} \quad \mathbb{R}^N,
\]

which was proposed by Laskin in the framework of quantum mechanics (see [31]), has attracted extensive attention. For some results of the above equation and its variants, we may refer the reader to [10,38,42]. Recently, it is worth noting that Autuori and Pucci [2] have considered a class of fractional Laplacian equations in \( \mathbb{R}^N \) and obtained the multiplicity and existence of weak solutions by using the mountain pass theorem and the direct method in variational methods. To be more precise, they considered the following problem:

\[
(-\Delta)^s u + V(x)u = Aw(x)|u|^{q-2}u - h(x)|u|^{r-2}u, \tag{1.4}
\]

where \( \lambda \in \mathbb{R}, s \in (0,1), N > 2s, 2 < q < \min\{r, \mathbb{Z}^*_+\} \) with \( \mathbb{Z}^*_+ = 2N/(N - 2s) \). As a result, the authors obtained the existence of three critical values of parameter \( \lambda \), which makes equation (1.4) have different numbers of solutions when \( \lambda \) belongs to different intervals. In [49], Xiang and Zhang established the existence of weak solutions for the following fractional \( p \)-Laplacian equations in \( \mathbb{R}^N \) by using the critical point theory:

\[
(-\Delta)^s u + a(x)|u|^{p-2}u = \lambda f(x)|u|^{r-2}u - f_2(x)|u|^{q-2}u, \tag{1.5}
\]

where \( s \in (0,1), 1 < p < r < \min\{q, p^*_r\} \) and \( \lambda \) is a real parameter. Under the same conditions for the nonlinearities, the following Kirchhoff-type equations was studied in [43],

\[
M(|u|^{p^*_s})(-\Delta)^s u + V(x)|u|^{p-2}u = \lambda a(x)|u|^{r-2}u - h(x)|u|^{r-2}u. \tag{1.6}
\]

The authors established multiplicity results for problem (1.1) by using the topological degree theory and variational methods, which depend on a real parameter \( \lambda \). Furthermore, with the help of the genus theory, the existence of infinitely many pairs of entire solutions was investigated for problem (1.1).

In particular, in this article we consider an interesting case, that is, a modified fractional \((p, q)\)-Laplacian system by adding Hardy terms, and we refer the interested reader to [25] for the motivation of the study for Hardy-type problems. Now, we briefly introduce some results in this respect, which are very enlightening to this article. Faraci and Livrea investigated a class of \( p \)-Laplacian equations with zero boundary value and Hardy term in [18] and obtained the existence of solutions for the following equation as \( \mu \) is close to 0:

\[
\begin{aligned}
&-\Delta_p u = \frac{\mu}{|x|^p}|u|^{p-2}u + \lambda f(x, u) \quad \text{a.e. in} \quad \Omega, \\
&u = 0, \quad \text{on} \quad \partial \Omega,
\end{aligned}
\tag{1.7}
\]

where \( 1 < p < N \ (N \geq 2), \lambda \) is a positive parameter. The following perturbed Kirchhoff-type fractional problems with singular exponential nonlinearity has been considered in [35]:

\[
\begin{aligned}
&M(|u|^{p^*_s})L_{\mu}^p u = f(x, u)|x|^{p^*_r} + Ah(x) \quad \text{in} \quad \Omega, \\
&u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,
\end{aligned}
\tag{1.8}
\]
where

\[ \|u\|^{N/s} = \iint_{\mathbb{R}^N} |u(x) - u(y)|^{N/s} K(x - y) \, dx \, dy. \]

When the nonlinearity satisfies the critical or subcritical exponential growth condition, the existence of solutions for equation (1.8) was obtained by using the mountain pass theorem and Ekeland’s variational principle. Besides, the authors also dealt with the existence of ground state solutions to equation (1.8) in the absence of perturbation and (AR) condition. Filippucci et al. in [20] studied a class of the double critical equations of Emden-Fowler type and obtained the existence of a positive weak solution by using the mountain pass theorem. Caponio and Pucci [8] investigated various properties of entire solutions for a kind of stationary Kirchhoff-type equations in \( \mathbb{R}^N \) involving the nonlocal fractional Laplacian and a Hardy term. In [21], Fiscella and Pucci discussed a series of Kirchhoff-type fractional \( p \)-Laplacian equations involving critical Hardy-Sobolev nonlinearities and nonnegative potentials in \( \mathbb{R}^N \). And the existence of the solutions is obtained by using different variational methods. For more related results in this trend, we refer to [4,22,27,29].

Recently, the elliptic systems involving the fractional Laplacian have been paid more and more attention. These elliptic systems arise from the following two-component reaction–diffusion problem:

\[
\begin{aligned}
    u_t &= \text{div}(D(u) \nabla u) + f(x, u, v), \\
v_t &= \text{div}(D(v) \nabla v) + g(x, u, v),
\end{aligned}
\]

and have been widely used in biophysics, plasma physics, chemical reaction design and so on. More physical background on this problem can be found in [14]. In fact, two-component systems are applicable to a wider range of possible phenomena than single-component systems. In [30], He et al. established the following elliptic system with the critical Sobolev exponent and concave-convex nonlinearities:

\[
\begin{aligned}
    (-\Delta)^s u &= \frac{2\alpha}{\alpha + \beta} |u|^{q-2} u |v|^\beta + \lambda |u|^{q-2} u \quad \text{in } \Omega, \\
    (-\Delta)^s v &= \frac{2\beta}{\alpha + \beta} |u|^\alpha |v|^{\beta-2} v + \mu |v|^{q-2} v \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.9)

where \( 2s < N \) with \( s \in (0, 1) \), \( \alpha > 1 \), \( \beta > 1 \) satisfy \( \alpha + \beta = \frac{2N}{N-2s} \), \( \lambda, \mu \) are regarded as positive parameters. As a result, the existence of at least two positive solutions for equation (1.9) is obtained by the Nehari manifold method. Furthermore, Chen and Deng in [12] extended equation (1.9) to the fractional \( p \)-Laplacian system and obtained the existence of at least two nontrivial solutions. In the circumstances of the Kirchhoff, the following critical Schrödinger-Kirchhoff-type system driven by nonlocal integro-differential operator was proposed in [51]

\[
\begin{aligned}
    M(||(u, v)||^p)(|\mathcal{L}^s_p u + \mathcal{V}(x)|u|^{p-2} u) &= \lambda \mathcal{H}_s(x, u, v) + \frac{\alpha}{p_s^*} |v|^{\beta} |u|^{q-2} u \quad \text{in } \mathbb{R}^N, \\
    M(||(u, v)||^p)(|\mathcal{L}^s_p v + \mathcal{V}(x)|v|^{p-2} v) &= \lambda \mathcal{H}_s(x, u, v) + \frac{\beta}{p_s^*} |u|^{\alpha} |v|^{\beta-2} v \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]

where \( \lambda > 0 \) is a real parameter, \( \alpha, \beta > 1 \) with \( \alpha + \beta = p_s^* \). Consequently, the existence of solutions for this kind of system is obtained for the first time by using the mountain pass theorem and Ekeland’s variational principle.

However, few work has been carried out on the fractional \((p, q)\)-Laplacian problems in the whole space. Xiang et al. [50] dealt with the following quasilinear fractional \((p, q)\)-Laplacian system with nonlinearities satisfying the (AR) condition:

\[
\begin{aligned}
    (-\Delta)^s_p u + a(x)|u|^{p-2} u &= \mathcal{H}_s(x, u, v) \quad \text{in } \mathbb{R}^N, \\
    (-\Delta)^s_p v + b(x)|v|^{q-2} v &= \mathcal{H}_s(x, u, v) \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]

(1.10)
where \( sp < N \) with \( 1 < q < p, a(x), b(x) \) are two continuous positive functions, and \( \mathcal{H}_u, \mathcal{H}_v \) represent the partial derivatives of \( \mathcal{H} \) with respect to variables \( u \) and \( v \), respectively. The authors obtained the existence of solutions for system (1.10) by the mountain pass theorem. The interaction between fractional \((p, q)\)-Laplacian operators and critical Sobolev-Hardy nonlinear terms in the whole \( \mathbb{R}^N \) is studied for the first time in [41]. In this existence of solutions for a class of fractional \((p, q)\)-Laplacian systems in \( \mathbb{R}^N \) is established. More precisely, they considered the system as follows:

\[
\begin{align*}
(-\Delta)^s_p u + \frac{(-\Delta)^t q u + |u|^{p-2} u + |u|^{q-2} u - \sigma |u|^q u}{|x|^q} &= \lambda H_u(x, u, v) + \frac{a}{q'} |v|^q |u|^{a-2} u, \\
(-\Delta)^s_p v + \frac{(-\Delta)^t q v + |v|^{p-2} v + |v|^{q-2} v - \sigma |v|^q v}{|x|^q} &= \lambda H_v(x, u, v) + \frac{b}{q'} |u|^p |v|^{b-2} v,
\end{align*}
\]

where \( 0 < s \leq t < 1 < p < q, N > tq \) with \( N \geq 2, 1 < \alpha, \beta \) satisfy \( \alpha + \beta = q', \alpha = Nq/(N - tq) \). Baldelli et al. in [3] considered an elliptic problem of \((p, q)\)-Laplacian type involving a critical term, nonnegative weights and a positive parameter \( \lambda \) in \( \mathbb{R}^N \). By using variational method and the concentration compactness principle, under suitable conditions, the authors proved the existence of infinite many weak solutions with negative energy as the parameter \( \lambda \) belongs to a certain interval. For more interesting results on fractional \((p, q)\)-Laplacian systems, we refer to [23–25,28] and references therein.

A main motivation of our article is based on Souza’s work concerning the following nonhomogeneous fractional \( p \)-Laplacian equations in [16]:

\[
(-\Delta)^s_p u + V(x)|u|^{p-2} u = f(x, u) + \lambda h \quad \text{in } \mathbb{R}^N.
\]

The existence of nontrivial weak solutions of equation (1.11) is proved by the fixed point result in [9]. A natural question is whether it is possible to study the fractional \( p \)-Laplacian systems with the method in [9]. Another question is what happens if we extend the problem to the fractional \((p, q)\)-Laplacian systems in this case. This article will give a positive answer.

Motivated by the above works, we consider to use a fixed point result to study system (1.1). More precisely, we should apply a fixed point theorem in [9] and the Hardy-Sobolev inequality in [11] to solve system (1.1). In particular, motivated by [41], we consider the double critical case in the system, and the two critical Sobolev-Hardy nonlinearities which interact. Obviously, the combination of two critical indices will undoubtedly bring more difficulties. In particular, to conquer the difficulty of exponential inconsistency caused by \( p \neq q \) in system (1.1), we have to develop some subtle techniques. To the best of our knowledge, none of the result has been obtained so far for the fractional \((p, q)\)-Laplacian systems via fixed point theorems. Particularly, we also consider a Choquard-type nonlinearity in the second part of this article, in this sense, our results are new even in the case \( p = q = 2 \). There is no doubt that our new approach employed in this article could be applied to study other elliptic equations and systems involving the critical exponents.

We first introduce some useful notations to facilitate the proof of our article. Let \( \mathcal{D}_{s,p}(\mathbb{R}^N) \) denote the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the Gagliardo seminorm

\[
[u]_{s,p} = \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \right)^{1/p}.
\]

By Theorems 1–2 of [34], we have

\[
\begin{align*}
\|u\|_{s,p}^p &\leq C_{N,p} \frac{s(1 - s)}{(N - ps)^{p-1}} [u]_{s,p}^p, \quad \forall u \in \mathcal{D}_{s,p}(\mathbb{R}^N), \\
\int_{\mathbb{R}^N} |\nabla v(x)|^p &\leq C_{N,p} \frac{s(1 - s)}{(N - ps)^p} [v]_{s,p}^p, \quad \forall v \in \mathcal{D}_{s,p}(\mathbb{R}^N),
\end{align*}
\]

and \( C_{N,p} \) is a positive constant depending only on \( N \) and \( p \).
endowed with the norm

$$|u|_{L^p(R^N, |x|^{-\gamma})} = \left( \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^\gamma} \, dx \right)^{1/p},$$

and $L^p(R^N)$ denotes a Lebesgue function space, endowed with the classical norm $\| \cdot \|_p$. Then it is standard to show that the embedding $\mathcal{D}_{s,p}(R^N) \hookrightarrow L^p(R^N, |x|^{-\gamma})$ and $\mathcal{D}_{s,p}(R^N) \hookrightarrow L^p_0(R^N)$ are continuous. We endow the space $\mathcal{D}_{s,p}$ with the norm $\| \cdot \|_{s,p} = \| \cdot \|_{\mathcal{D}_{s,p}}$, which is a reflexive Banach space endowed with the norm

$$\|(u, v)\| = \|u\|_{s,p} + \|v\|_{l,q}.$$

In order to obtain the weak solutions of system (1.1), we consider the subspace of $\mathcal{D}(R^N)$

$$Q(R^N) = \{(u, v) \in \mathcal{D}(R^N) : u \equiv 0, v \equiv 0 \ a.e. \ in \ R^N\}.$$

It is easy to see that $Q(R^N)$ is a close linear subspace of $\mathcal{D}(R^N)$, hence we conclude that $Q(R^N)$ is a Banach space. By using the Clarkson’s first inequality [5], it is standard to deduce that $Q(R^N, \| \cdot \|)$ is also a reflexive Banach space. Meanwhile, we consider the more general case of the perturbation functions, that is, $f, g$ belong to $Q'(R^N)$, which is the dual space of space $Q(R^N)$. Next, we denote that $\langle \cdot, \cdot \rangle$ is the duality between $Q(R^N)$ and $Q'(R^N)$. Now, we can give the following definition:

**Definition 1.1.** We say that $(u, v) \in Q(R^N)$ is a weak solution of system (1.1), if

$$\begin{align*}
(u, v)_{s,p} + (v, l,q) &= \int_{\mathbb{R}^N} \frac{|u|^{\alpha-2}u\omega}{|x|^{\gamma}} \, dx - \int_{\mathbb{R}^N} \frac{|v|^{\beta-2}v\nu}{|x|^{\gamma}} \, dx \\
&= \frac{\alpha}{p^*_s} \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} \omega \, dx + \frac{\beta}{l^*_q} \int_{\mathbb{R}^N} |v|^{\beta} |u|^{\alpha} \nu \, dx + \lambda_1 \langle f, (u, v) \rangle + \lambda_2 \langle g, (u, v) \rangle \\
&= \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{\alpha-2}(u(x) - u(y))(\omega(x) - \omega(y))}{|x - y|^{\gamma + mn}} \, dx dy,
\end{align*}$$

for all $(u, v) \in Q(R^N)$.

The existence result in the first part of our article is stated as follows:

**Theorem 1.1.** If \( \frac{\alpha}{\gamma} + \frac{\beta}{\gamma} = 1 \), there exists $\lambda_0 > 0$ such that for all $0 < \lambda_1$, \( \lambda_2 \leq \lambda_0 \), system (1.1) possesses a nontrivial nonnegative weak solution in $Q(R^N)$.

**Remark 1.1.** Theorem 1.1 only ensures the existence of a nontrivial nonnegative weak solution for the nonhomogeneous system (1.1).

The second part of our article is devoted to the research of the following Choquard-type fractional $(p, q)$-Laplacian system:

$$\begin{align*}
(-\Delta)^s_p u - \frac{|u|^{\alpha-2}u}{|x|^{\gamma}} &= (I_{\mathcal{E}_s \ast |u|^{\alpha}\omega})|u|^{\alpha} \omega - \frac{\alpha}{p^*_s} |u|^{\alpha-2}u \omega + \lambda_1 f(x) \quad \text{in} \ \mathbb{R}^N, \\
(-\Delta)^s_q v - \frac{|v|^{\beta-2}v}{|x|^{\gamma}} &= (I_{\mathcal{E}_s \ast |v|^{\beta}\nu})|v|^{\beta} \nu - \frac{\beta}{l^*_q} |v|^{\beta-2}v \nu + \lambda_2 g(x) \quad \text{in} \ \mathbb{R}^N,
\end{align*}$$

(1.13)
where $I_{\xi}(x) = |x|^{-\xi}$ is the Riesz potential of order $\xi \in (0, N)$, $i = 1, 2$ and $p_{\xi,i} = p(N - \xi_i/2)(N - ps)$ and $q_{\xi,i} = q(N - \xi_i/2)(N - q)$ are the upper critical exponents in the sense of the Hardy-Littlewood-Sobolev inequality.

As far as we know, in the last few decades, the Choquard-type equations have attracted much attention. Considering the modeling of quantum polaron, Fröhlich [26] and Pekar [40] first proposed the following Choquard equation:

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right)u, \quad \text{in } \mathbb{R}^3. \quad (1.14)$$

This model is equivalent to studying how the free electrons in an ionic lattice interact with phonons, which are related to the deformation of the lattice or the polarization it produces in the medium (the interaction of electrons with their pores). As time goes on, the following fractional Choquard equation has been widely studied:

$$(-\Delta)^a u + a(x)u = (I_{\xi} * F(u))f(u), \quad \text{in } \mathbb{R}^N. \quad (1.15)$$

When $f$ fulfills some suitable assumptions, the various properties of solutions for equation (1.15) have been investigated extensively, see for example [15, 45], and also see [39] for a survey of the study of the Choquard-type equations. It is worth mentioning that in the Kirchhoff setting, the following Kirchhoff-type Schrödinger-Chouard equation in $\mathbb{R}^N$ was first investigated in [44]:

$$(a + |x|^{(p-1)}u(-\Delta)^p u + P(x)|u|^{p-2}u = \lambda g(x, u) + (I_{\xi} * |u|^q)v)|u|^{q-2}u, \quad \text{in } \mathbb{R}^N. \quad (1.16)$$

where $p_{\xi,i} = p(N - \xi_i/2)(N - ps), a, b > 0$ satisfy $a + b > 0$. The existence of the solution of the problem (1.16) in the case of non-degenerate or degenerate is obtained by using the mountain pass theorem as the nonlinearities conform the sublinear or superlinear growth conditions. Very recently, Chen et al. in [13] studied the following problems:

$$\begin{cases}
(a + b|u|^{p_{\xi,p}^{(q-1)}}(-\Delta)^p u = \lambda(|x|^\xi + |u|^q)|u|^{q-2}u + \frac{|u|^{p_{\xi,p}^q-2}_u}{|x|^a}, \quad \text{in } \Omega, \\
u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \quad (1.17)$$

where $0 < a < ps < N, \xi \in (0, \min(N, 2sp)), a, b > 0$ with $a + b > 0, q \in (1, Np/(N - ps))$ satisfies some additional assumptions. By means of the mountain pass theorem and concentration-compactness principle, the existence of solutions for equation (1.17) is obtained as $q$ satisfies suitable ranges. Furthermore, in the study of fractional Laplacian systems, Wang et al. [48] studied the fractional Laplacian system involving critical Hardy-Sobolev-type nonlinearities and critical Sobolev-type nonlinearities in $\mathbb{R}^N$:

$$(\Delta)^a u = \mu \frac{u}{|x|^2} = (I_{\xi} * |u|^{\beta/(\gamma - 2)})|u|^{\gamma - 2}u + \frac{|u|^{p_{\xi,p}^{\Sigma(\gamma-2)}}}{|x|^\gamma} + \frac{\eta a}{\alpha + \beta} \frac{|u|^{p_{\xi,p}^{\Sigma(\gamma-2)}}}{|x|^\gamma},$$

$$(-\Delta)^b v = \nu \frac{v}{|x|^2} = (I_{\xi} * |v|^{\beta/(\gamma - 2)})|v|^{\gamma - 2}v + \frac{|v|^{p_{\xi,p}^{\Sigma(\gamma-2)}}}{|x|^\gamma} + \frac{\eta \beta}{\alpha + \beta} \frac{|v|^{p_{\xi,p}^{\Sigma(\gamma-2)}}}{|x|^\gamma}, \quad (1.18)$$

where $0 < \xi, \gamma < 2s < N, \alpha > 1, \beta > 1$ satisfy $a + \beta = 2\Sigma_{\alpha \beta}, \Sigma_{\alpha \beta} = 2(N - \gamma)/(N - 2s)$ and $\Sigma_{\alpha \beta} = (N + \xi)/(N - 2s)$ are fractional critical exponents. The existence of solutions and the extreme problem of the corresponding optimal fractional Hardy-Sobolev constant are studied by using the variational approaches. Filippucci and Ghergu [19] investigated the following inequality:

$$\text{div}(|x|^{-a}|\nabla u|^{m-2}\nabla u) \geq (I_{\xi} + u^p)u^q \quad \text{in } B_1 \setminus \{0\} \subset \mathbb{R}^N \quad (1.19)$$

and with the double inequality

$$a(I_{\xi} + u^p)u^q \geq \text{div}(|x|^{-a}|\nabla u|^{m-2}\nabla u) \geq b(I_{\xi} + u^p)u^q \quad \text{in } B_1 \setminus \{0\}, \quad (1.20)$$

where $\xi \in (0, N), a > 0, p, q > m - 1 > 0$ and $a > b > 0$. The authors first obtained the sharp condition for the existence of positive singular solutions of inequality (1.19) and further established the asymptotic
distribution of singular solutions of inequality (1.20), see [19] for more details. The Choquard-type equations are also of great significance in the study of magnetic fields. Such problems have been studied by many authors when the interactions between particles are taken into account, see [32] and references therein.

Inspired by the above work, we are interested in using the method in Theorem 1.1 to deal with the Choquard-type fractional $(p, q)$-Laplacian systems. In fact, under the assumption of Theorem 1.1, through the fixed point results in [9] and the Hardy–Littlewood–Sobolev inequality in [33], we obtain the existence of nontrivial weak solutions of system (1.13). To our best knowledge, this result is new even in the Laplacian case.

Similar to Definition 1.1, we can define the weak solution of system (1.13) as follows:

**Definition 1.2.** We say that $(u, v) \in Q(\mathbb{R}^N)$ is a weak solution of system (1.13), if

$$(u, \omega)_{s,p} + (v, \nu)_{l,q} = \int_{\mathbb{R}^N} |u|^{p_l^{s-2}}u\omega \, dx - \int_{\mathbb{R}^N} |v|^{q_l^{s-2}}v\nu \, dx$$

$$= \iint_{\mathbb{R}^{2N}} \frac{|u(y)|^{p_t^{s_1}}}{|x - y|^{s_1}} |u(x)|^{p_t^{s_2}} u(x)\omega(x) \, dx \, dy + \iint_{\mathbb{R}^{2N}} \frac{|v(y)|^{q_t^{s_1}}}{|x - y|^{s_1}} |v(x)|^{q_t^{s_2}} v(x)\nu(x) \, dx \, dy$$

$$+ \frac{\alpha}{p_t^s} \int_{\mathbb{R}^N} |u|^{\alpha} u\beta \, dx + \frac{\beta}{q_t^s} \int_{\mathbb{R}^N} |v|^{\beta} v\alpha \, dx + \lambda \mathcal{T}(f, (u, v)) + \lambda \mathcal{T}(g, (u, v)),$$

for all $(\omega, \nu) \in Q(\mathbb{R}^N)$.

The existence result in the second part of our article reads as follows:

**Theorem 1.2.** Under the same hypotheses of Theorem 1.1, there exists $\lambda_*>0$ such that for all $0<\lambda_1, \lambda_2 \leq \lambda_*$, system (1.13) admits a nontrivial nonnegative weak solution in $Q(\mathbb{R}^N)$.

Note, in the first two parts of our article, we conduct the existence research under the assumption that $N > sp$ and $N > lq$. For the case $N = sp = lq$, it is an appealing problem when considering the fractional $(p, q)$-Laplacian systems. Therefore, in the last part of this article, we will explore this situation.

The framework of this article is as follows. In Section 2, we give some preliminary results and properties of the $Q(\mathbb{R}^N)$. In Section 3, we show some useful results around the proof of the main theorems. In Section 4, we prove Theorems 1.1 and 1.2. In Section 5, we investigate the case of $N = sp = lq$.

Unless otherwise specified, we point out that $S, S_0, S_1, S_2, S_m$ ... are positive constants.

## 2 Preliminaries

We introduce some basic results, which are necessary to prove the results of this article.

Let $L_{p^*_t, q^*_t}(\mathbb{R}^N) = L^{p^*_t}(\mathbb{R}^N) \times L^{q^*_t}(\mathbb{R}^N)$, endowed with the following norm:

$$|| (u, v) ||_{p^*_t, q^*_t} = || u ||_{p^*_t} + || v ||_{q^*_t}, \quad \forall (u, v) \in L_{p^*_t, q^*_t}(\mathbb{R}^N).$$

We denote that $L_{r^*, r}(\mathbb{R}^N) = L^{r^*}(\mathbb{R}^N, |x|^r) \times L^{r^*}(\mathbb{R}^N, |x|^{-r})$, endowed with the following norm:

$$|| (u, v) ||_{r^*, r} = || u ||_{L^{r^*}(\mathbb{R}^N, |x|^{r^*})} + || v ||_{L^{r^*}(\mathbb{R}^N, |x|^{r^*})},$$

for all $(u, v) \in L_{r^*, r}(\mathbb{R}^N)$. Then we have

**Lemma 2.1.** The embedding $Q(\mathbb{R}^N) \hookrightarrow L_{p^*_t, q^*_t}(\mathbb{R}^N)$ is continuous, where $p^*_t$, $q^*_t$ are fractional Sobolev critical exponents with $p^*_t = Np/(N - sp)$, $q^*_t = Nq/(N - lq)$.
Proof. By Theorem 1 of [34], the fractional Sobolev embedding $\mathcal{D}_{s,p}(\mathbb{R}^N) \hookrightarrow L^p_s(\mathbb{R}^N)$ is continuous. There exists $S_0$ such that
\[
\|u\|_{\mathcal{D}_{s,p}(\mathbb{R}^N)} = \|u\|_{s,p} + \|v\|_{s,q} \leq S_0\|u\|_{s,p} + \|v\|_{s,q} = S_0\|(u, v)\|, \quad \forall (u, v) \in Q(\mathbb{R}^N).
\]
This completes the proof. \qed

Lemma 2.2. (See [22, Lemma 2.1]) Suppose that $N > sp \geq \gamma \geq 0$. Therefore, there exists $S_0$ such that
\[
\|u\|_{L^p_{\gamma}([0,]\mathbb{R}^N)} \leq S_0\|u\|_{s,p}, \quad \forall u \in \mathcal{D}_{s,p}(\mathbb{R}^N),
\]
where $S_0$ may only depend on $\gamma$, $s$, $p$, $N$.

Lemma 2.3. The embedding $Q(\mathbb{R}^N) \hookrightarrow L_{p,\gamma}(\mathbb{R}^N)$ is continuous, where $0 \leq p \leq N$ and $0 \leq \tau \leq lq < N$.

Proof. Obviously, by Lemma 2.2, the fractional Hardy-Sobolev embedding $\mathcal{D}_{s,p}(\mathbb{R}^N) \hookrightarrow L^p_s(\mathbb{R}^N, |x|^{-\gamma})$ is continuous. Then, similar to the proof of Lemma 2.1, it is easy to obtain the desired result. \qed

Theorem 2.1. (See [5, Theorem 5.16]) If $Q$ is a Banach space which is reflexive and $M : Q \rightarrow Q'$ is a continuous nonlinear map satisfying
\[
\{M(u_p, v_p) - M(u_t, v_t), (u_p, v_p) - (u_t, v_t)\} > 0,
\]
for all $(u_p, v_p), (u_t, v_t) \in Q$ with $(u_p, v_p) \neq (u_t, v_t)$, and
\[
\lim_{\|u,v\| \rightarrow \infty} \frac{\{M(u, v), (u, v)\}}{\|(u, v)\|} = \infty,
\]
then for every $(f_1, f_2) \in Q'$, the equation $M(u, v) = (f_1, f_2)$ has a unique solution $(u, v) \in Q$.

To facilitate the introduction of the following lemma, it is necessary to review some useful notations and concepts. Consider that $Q$ is a real Banach space and $Q_+ \neq 0$ is a nonempty subset of $Q$. We say that $Q_+$ is an order cone if the space $Q$ meets the following three conditions:

(a) $Q_+$ is closed and convex.
(b) If $\lambda \geq 0$ and $(u, v) \in Q_+$, then $\lambda(u, v) \in Q_+$.
(c) If $-(u, v) \in Q_+$ and $(u, v) \in Q_+$, then $(u, v) = (0, 0)$.

- We say $(Q, \simeq)$ is an ordered Banach space if
\[
(u_p, v_p) \simeq (u_t, v_t) \quad \text{if and only if} \quad (u_t - u_p, v_t - v_p) \in Q_+,
\]
where $Q_+$ is an order cone.
- We say $(Q, \| \cdot \|)$ is a lattice if $\inf\{(u_p, v_p), (u_t, v_t)\}$ and $\sup\{(u_p, v_p), (u_t, v_t)\}$ exist for all $(u_p, v_p), (u_t, v_t) \in Q$ with respect to $\leq$.
- We say $(Q, \| \cdot \|)$ is a Banach semilattice if
\[
\|(u, v)^+\| \leq \|(u, v)\| \quad \text{for all} \quad (u, v) \in Q,
\]
where $(u, v)^+ = (\sup\{0, u\}, \sup\{0, v\})$ and $(u, v)^- = (\inf\{0, u\}, \inf\{0, v\})$.

Lemma 2.4. (See [9, Corollary 2.2]) Let $Q$ be a reflexive Banach semilattice. Therefore, any closed ball of $Q$ is considered to have the fixed point property.

Remark 2.1. Consider that $(Q, \simeq)$ and $(Q', \simeq')$ are ordered Banach Spaces. If and only if there is $M(u_p, v_p) \simeq M(u_t, v_t)$ for all $(u_p, v_p), (u_t, v_t) \in Q$, $(u_p, v_p) \simeq (u_t, v_t)$, the operator $M : Q \rightarrow Q'$ is increasing. Assume that $Q_0$ is a subset of $Q$. If any increasing operator $M : Q_0 \rightarrow Q_0$ has a fixed point, $Q_0$ has fixed point property.
Lemma 2.5. (See [33, Theorem 4.3]) (Hardy-Littlewood-Sobolev inequality) Suppose that $1 < m, n < \infty$, $0 < \xi < N$ and

$$\frac{1}{m} + \frac{1}{n} + \frac{\xi}{N} = 2.$$ 

Thus, there exists $S(m, n, \xi, N)$ such that

$$\int_{\mathbb{R}^N} \frac{|I(x)||I(y)|}{|x - y|^\xi} \, dx \, dy \leq S(m, n, \xi, N) \|I\|_m \|I\|_n, \quad \forall I_1 \in L^m(\mathbb{R}^N), I_2 \in L^n(\mathbb{R}^N).$$

Remark 2.2. We point out that the Hardy-Littlewood-Sobolev inequality will be used to prove Theorem 1.2, which plays an important role in getting a crucial estimate.

3 Some auxiliary results

We define a functional $\mathcal{R} : Q(\mathbb{R}^N) \to Q'(\mathbb{R}^N)$ by

$$\langle \mathcal{R}(u, v), (\omega, v) \rangle = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\omega(x) - \omega(y))}{|x - y|^{N+sp}} \, dx \, dy + \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{q-2}(v(x) - v(y))(v(x) - v(y))}{|x - y|^{N+ql}} \, dx \, dy, \quad (u, v), (\omega, v) \in Q(\mathbb{R}^N).$$

It is obvious that $\mathcal{R}$ is a linear map. The following inequality can be derived by using the Hölder inequality,

$$|\langle \mathcal{R}(u, v), (\omega, v) \rangle| \leq (\|u\|_p^{p-1} + \|v\|_q^{q-1})\|\omega\|_r,$$

which means that $\mathcal{R}(u, v) \in Q'(\mathbb{R}^N)$ and therefore $\mathcal{R}$ is well defined.

Lemma 3.1. The operator $\mathcal{R} : Q(\mathbb{R}^N) \to Q'(\mathbb{R}^N)$ is continuous and invertible.

Proof. Set $(u_\varepsilon, v_\varepsilon) \in Q(\mathbb{R}^N)$ such that $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $Q(\mathbb{R}^N)$. Using Hölder’s inequality, for $(u, v) \in Q(\mathbb{R}^N)$ with $|\omega, v| \leq 1$ we have

$$|\langle \mathcal{R}(u, v) - \mathcal{R}(u_\varepsilon, v_\varepsilon), (\omega, v) \rangle|$$

$$\leq \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) - |u_\varepsilon(x) - u_\varepsilon(y)|^{p-2}(u_\varepsilon(x) - u_\varepsilon(y))}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{p}{p+1}}$$

$$+ \left( \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{q-2}(v(x) - v(y)) - |v_\varepsilon(x) - v_\varepsilon(y)|^{q-2}(v_\varepsilon(x) - v_\varepsilon(y))}{|x - y|^{N+ql}} \, dx \, dy \right)^{\frac{q}{q+1}} = P_1 + Q_2.$$

Now we recall a fundamental inequality:

$$|t_1|^{m-2} - |t_2|^{m-2} \leq S_m|t_1 - t_2|(|t_1| + |t_2|)^{m-2}, \quad \forall m \geq 2, t_1, t_2 \in \mathbb{R}. \quad (3.1)$$

By using (3.1) and Hölder’s inequality we can obtain that

$$P_1^{\frac{1}{p+1}} \leq S_p \int_{\mathbb{R}^N} \frac{|u(x) - u(y)| - |u_\varepsilon(x) - u_\varepsilon(y)|}{|x - y|^{N+sp}} \, dx \, dy \leq S_p \|u - u_\varepsilon\|_{L^p}^{\frac{p}{p+1}} (\|u\|_{L^p} + \|u_\varepsilon\|_{L^p})^{\frac{p-1}{p}}. \quad (3.2)$$
Analogously,
\[
\begin{align*}
Q_{xy}^{\frac{q}{p}} & \leq S_q \iint_{\mathbb{R}^N} |v(x) - v(y)| - (v(x) - v(y)) \frac{q}{p} \left( |v(x) - v(y)| + |v(x) - v(y)| \right) \frac{q-2}{p-1} |x - y|^{N+q} \ dx \ dy \\
& \leq S_q \|v\|_q^{\frac{q}{p}} + \|v\|_q^{\frac{q}{p}} \frac{q-2}{p-1}.
\end{align*}
\]
(3.3)

Since \((u_v, v_v) \rightarrow (u, v)\) in \(Q(\mathbb{R}^N)\), by (3.2) and (3.3), we have that
\[
\|(R(u, v) - R(u_v, v_v), (\omega_v, v_v))\|_q(\mathbb{R}^N) = \sup_{(u_v, v_v) \in Q(\mathbb{R}^N), (u_v, v_v) \in 1} |(R(u, v) - R(u_v, v_v), (\omega_v, v_v))| \leq P_1 + Q_2 \rightarrow 0,
\]
as \(\varepsilon \rightarrow 0\). Then the operator \(R\) is continuous.

Consider \(q > p \geq 2\) and
\[
(R(u, v), (u, v)) = \|u\|_{p}^{p} + \|v\|_{p}^{p}, \quad \forall (u, v) \in Q(\mathbb{R}^{N}).
\]

Thus, we have
\[
\lim_{(u_v, v_v) \rightarrow \infty} \frac{(R(u, v), (u, v))}{\|(u, v)\|} = \lim_{(u_v, v_v) \rightarrow \infty} \frac{\|u\|_{p}^{p} + \|v\|_{p}^{p}}{\|(u, v)\|} = \infty.
\]
(3.4)

Furthermore, by the following well-known inequality:
\[
|t_t^{\varepsilon-\varepsilon} - s_1^{\varepsilon-\varepsilon} - s_2^{\varepsilon-\varepsilon}|(t_1 - t_2) \geq S_1 |t_1 - t_2|^{\varepsilon}, \quad \forall n \geq 2, t_1, t_2 \in \mathbb{R},
\]
we can obtain
\[
(R(u_p, v_p) - R(u_v, v_v), (u_p, v_p) - (u_v, v_v)) > 0 \quad \forall (u_p, v_p), (u_v, v_v) \in Q(\mathbb{R}^{N}), (u_p, v_p) \neq (u_v, v_v).
\]

Therefore, according to Theorem 2.1, the operator \(R\) is said to be reversible. \(\square\)

In order to demonstrate the monotonicity of the invertible operator \(R^{-1}\), we endow the following partial order in \(Q(\mathbb{R}^{N})\):
\[
(u_p, v_p), (u_v, v_v) \in Q(\mathbb{R}^{N}), (u_p, v_p) \preceq (u_v, v_v) \Leftrightarrow u_p \leq u_v, v_p \leq v_v \quad \text{a.e. in } \mathbb{R}^{N}.
\]
(3.6)

Obviously, \((Q(\mathbb{R}^{N}), \preceq)\) is an ordered Banach space and for any \((u_p, v_p), (u_v, v_v) \in Q(\mathbb{R}^{N})\), there are \(\sup((u_p, v_p), (u_v, v_v))\) and \(\inf((u_p, v_p), (u_v, v_v))\) corresponding to partial order \(\preceq\). Because \(\delta(x) - \delta(x)\) \& \(\delta(x) - \delta(x)\) are almost everywhere in \(\mathbb{R}^{N}\), it is clear to obtain \(\|(u, v)\| \leq \|(u, v)\|\). Therefore, \((Q(\mathbb{R}^{N}), \preceq)\) is a reflexive Banach semilattice. We also note that the dual space \(Q'(\mathbb{R}^{N})\) of \(Q(\mathbb{R}^{N})\) is endowed with the following partial order:
\[
(\omega_p, v_p), (\omega_v, v_v) \in Q'(\mathbb{R}^{N}), (\omega_p, v_p) \preceq (\omega_v, v_v) \Leftrightarrow \langle (\omega_p, v_p), (\psi, \chi) \rangle \leq \langle (\omega_v, v_v), (\psi, \chi) \rangle,
\]
for all \((\psi, \chi) \in Q(\mathbb{R}^{N})\).

**Lemma 3.2.** The operator \(R^{-1} : (Q'(\mathbb{R}^{N}), \preceq \rightarrow (Q(\mathbb{R}^{N}), \preceq)\) is increasing.

**Proof.** Choosing \((\omega_p, v_p), (\omega_v, v_v) \in Q'(\mathbb{R}^{N})\) such that \((\omega_v, v_v) \preceq (\omega_v, v_v)\) and setting \((u_p, v_p) = R^{-1}(\omega_p, v_p), (u_v, v_v) = R^{-1}(\omega_v, v_v)\). Let \(\rho_1(x, y) = u_1(x) - u_2(y), \quad \sigma_1(x, y) = u_2(x) - u_2(y), \quad \rho_2(x, y) = v_1(x) - v_2(y), \quad \sigma_2(x, y) = v_2(x) - v_2(y)\). Then, for any \((u, v) \in Q(\mathbb{R}^{N})\) we have
\[
0 \leq \langle (\omega_v, v_v) - (\omega_p, v_p), (u, v) \rangle = \langle R(u_v, v_v) - R(u_p, v_p), (u, v) \rangle
\]
\[
= \iint_{\mathbb{R}^{N}} \frac{|\rho_1(x, y)|^{p-2}\rho_1(x, y) - |\sigma_1(x, y)|^{p-2}\sigma_1(x, y)(u(x) - u(y))}{|x - y|^{N+sp}} \ dx \ dy
\]
\[
+ \iint_{\mathbb{R}^{N}} \frac{|\rho_2(x, y)|^{q-2}\rho_2(x, y) - |\sigma_2(x, y)|^{q-2}\sigma_2(x, y)(v(x) - v(y))}{|x - y|^{N+sq}} \ dx \ dy.
\]
Taking \((u, v) = (u_t - u_p, v_t - v_p)^{-}\), \((u_t - u_p)^{-}, (v_t - v_p)^{-}\) \(\in Q(\mathbb{R}^N)\), we conclude that
\[
0 \leq \langle (\omega_t, v_t) - (\omega_p, v_p), (u_t - u_p, v_t - v_p)^{-}\rangle \leq -S_p \| (u_t - u_p)^{-}\|_{p, q}^p - S_q \| (v_t - v_p)^{-}\|_{q, p}^q \leq 0. \tag{3.7}
\]

In fact, set
\[
\begin{align*}
\Omega_a &= \{x \in \mathbb{R}^N : u_t(x) \leq u_p(x)\}, \\
\Omega_b &= \{x \in \mathbb{R}^N : v_t(x) \leq v_p(x)\}, \\
\Omega_{a^c} &= \{x \in \mathbb{R}^N : u_t(x) > u_p(x)\}, \\
\Omega_{b^c} &= \{x \in \mathbb{R}^N : v_t(x) > v_p(x)\}.
\end{align*}
\]

For the sake of calculation, we assume that
\[
\begin{align*}
\mathcal{T}_1 &= \int_{\mathbb{R}^{2N}} \frac{|\sigma_t(x, y)|^{p-2}\sigma_t(x, y)((u_t(x) - u_p(x)) - (u_t(y) - u_p(y)))}{|x - y|^{N+sp}} \, dx \, dy, \\
\mathcal{T}_2 &= \int_{\mathbb{R}^{2N}} \frac{|\rho_t(x, y)|^{p-2}\rho_t(x, y)((u_t(x) - u_p(x)) - (u_t(y) - u_p(y)))}{|x - y|^{N+sp}} \, dx \, dy, \\
\mathcal{T}_3 &= \int_{\mathbb{R}^{2N}} \frac{|\sigma_t(x, y)|^{p-2}\sigma_t(x, y)((v_t(x) - v_p(x)) - (v_t(y) - v_p(y)))}{|x - y|^{N+sp}} \, dx \, dy, \\
\mathcal{T}_4 &= \int_{\mathbb{R}^{2N}} \frac{|\rho_t(x, y)|^{p-2}\rho_t(x, y)((v_t(x) - v_p(x)) - (v_t(y) - v_p(y)))}{|x - y|^{N+sp}} \, dx \, dy.
\end{align*}
\]

Now we focus on \(\mathcal{T}_2 - \mathcal{T}_1\). Note that
\[
\begin{align*}
\mathcal{T}_1 &= \int_{\Omega_a} \int_{\Omega_a^c} \frac{|\sigma_t(x, y)|^{p-2}\sigma_t(x, y)((u_t(x) - u_p(x)) + (u_t(y) - u_p(y)))}{|x - y|^{N+sp}} \, dx \, dy \\
&\quad + \int_{\Omega_a} \int_{\Omega_a^c} \frac{|\sigma_t(x, y)|^{p-2}\sigma_t(x, y)((u_t(x) - u_p(x)))}{|x - y|^{N+sp}} \, dx \, dy \\
&\quad + \int_{\Omega_a} \int_{\Omega_a^c} \frac{|\sigma_t(x, y)|^{p-2}\sigma_t(x, y)((u_t(y) - u_p(y)))}{|x - y|^{N+sp}} \, dx \, dy
\end{align*}
\]
and
\[
\begin{align*}
\mathcal{T}_2 &= \int_{\Omega_a} \int_{\Omega_a^c} \frac{|\rho_t(x, y)|^{p-2}\rho_t(x, y)((u_t(x) - u_p(x)) + (u_t(y) - u_p(y)))}{|x - y|^{N+sp}} \, dx \, dy \\
&\quad + \int_{\Omega_a} \int_{\Omega_a^c} \frac{|\rho_t(x, y)|^{p-2}\rho_t(x, y)((u_t(x) - u_p(x)))}{|x - y|^{N+sp}} \, dx \, dy \\
&\quad + \int_{\Omega_a} \int_{\Omega_a^c} \frac{|\rho_t(x, y)|^{p-2}\rho_t(x, y)((u_t(y) - u_p(y)))}{|x - y|^{N+sp}} \, dx \, dy.
\end{align*}
\]

Then, by (3.5) we can obtain
\[
\begin{align*}
\int_{\Omega_a} \int_{\Omega_a^c} \frac{|\rho_t(x, y)|^{p-2}\rho_t(x, y) - |\sigma_t(x, y)|^{p-2}\sigma_t(x, y)((u_t(x) - u_p(x)) - (u_t(y) - u_p(y)))}{|x - y|^{N+sp}} \, dx \, dy \\
&\leq -S_p \int_{\Omega_a} \int_{\Omega_a^c} |\rho_t(x, y) - \sigma_t(x, y)|^p \frac{dx \, dy}{|x - y|^{N+sp}}.
\end{align*}
\]
Since the function $k(u) = |u|^{-2}u$ is increasing, it follows that
\[
\int_{\Omega} |\rho(x, y)|^{p-2} \rho(x, y)(-u(x) - u_p(x)) \, dx \leq \int_{\Omega} |\sigma(x, y)|^{p-2} \sigma(x, y)((u(x) - u_p(x)) \, dx
\]
and
\[
\int_{\Omega} |\rho(x, y)|^{q-2} \rho(x, y)((u(x) - u_p(x)) \, dx \leq \int_{\Omega} |\sigma(x, y)|^{q-2} \sigma(x, y)((u(x) - u_p(x)) \, dx.
\]
Therefore, we have
\[
\mathcal{T}_2 - \mathcal{T}_1 \leq -S_p \int_{\Omega} |\rho(x, y)| - \sigma(x, y)|^{p} \, dx \frac{dy}{|x-y|^{N+sp}}, \tag{3.8}
\]
Similarly, we can obtain
\[
\mathcal{T}_4 - \mathcal{T}_3 \leq -S_q \int_{\Omega} |\rho(x, y)| - \sigma(x, y)|^{q} \, dx \frac{dy}{|x-y|^{N+sq}}, \tag{3.9}
\]
Combining (3.8) and (3.9), we conclude that (3.7) is true. Now we can obtain $(u_t - u_p)^+ = 0$ and $(v_t - v_p)^+ = 0$, which yields that $u_p \leq u_t$ and $v_p \leq v_t$ almost everywhere in $\mathbb{R}^N$, then $\mathcal{R}^{-1}(\omega_p, v_p) \in \mathcal{R}^{-1}(\omega_t, v_t)$ and hence the proof is finished.

In the sequel, we consider the operator $\mathcal{K} : Q(\mathbb{R}^N) \to Q(\mathbb{R}^N)$ given by
\[
\langle \mathcal{K}(u, v), (\omega, v) \rangle = \frac{\alpha}{p^*_s} \int_{\mathbb{R}^N} |u|^{p^*_s-2}u|v|^{\beta^*_s} \omega \, dx + \frac{\beta}{q^*_s} \int_{\mathbb{R}^N} |v|^{q^*_s-2}v|u|^{\alpha^*_s} \nu \, dx
\]
\[+ \int_{\mathbb{R}^N} \frac{|u|^{p^*_s-2}u \omega}{|x|^{\gamma}} \, dx + \int_{\mathbb{R}^N} \frac{|v|^{q^*_s-2}v \nu}{|x|^{\gamma}} \, dx + \lambda_1(f,(\omega,v)) + \lambda_2(g,(\omega,v)),
\]
for all $(u, v), (\omega, v) \in Q(\mathbb{R}^N)$.

Lemma 3.3. Under the hypotheses of Theorem 1.1, there exist $S_\eta$, $S_\kappa$ and $S_\mu$ such that
\[
|\langle \mathcal{K}(u, v), (\omega, v) \rangle| \leq (S_\eta \|(u, v)\|^{p^*_s-1} + S_\kappa \|(u, v)\|^{q^*_s-1} + S_\mu \|(u, v)\|^{\alpha^*_s-1} + \lambda_1\|f\|_{L^2(\mathbb{R}^N)} + \lambda_1\|g\|_{L^2(\mathbb{R}^N)})(\omega, v),
\]
for any $(u, v) \in Q(\mathbb{R}^N)$.

Proof. By Lemma 2.3 and the Hölder inequality, there exists $S_\eta$ such that
\[
\int_{\mathbb{R}^N} \frac{|u|^{p^*_s-2}u \omega}{|x|^{\gamma}} \, dx \leq \left( \int_{\mathbb{R}^N} \frac{|u|^{p^*_s-1}}{|x|^{p^*_s-1}} \right)^{\frac{1}{p^*_s}} \left( \int_{\mathbb{R}^N} \frac{|u|^{p^*_s-1}}{|x|^{p^*_s-1}} \right)^{\frac{1}{p^*_s}} \leq S_\eta \|(u, v)\|^{p^*_s-1}((\omega, v)). \ (3.10)
\]
Analogously, there exists $S_\kappa$ such that
\[
\int_{\mathbb{R}^N} \frac{|v|^{q^*_s-2}v \nu}{|x|^{\gamma}} \, dx \leq S_\kappa \|(u, v)\|^{q^*_s-1}((\omega, v)). \ (3.11)
\]
If $\frac{\alpha}{p^*_s} + \frac{\beta}{q^*_s} = 1$, in view of Lemma 2.1 and the Hölder inequality, there exists $S_\mu$ such that
\[ \frac{a}{p_s} \int_{\mathbb{R}^N} |u|^{p_s-2} u|v|^p \omega dx \leq \frac{a}{p_s} \left( \int_{\mathbb{R}^N} |u|^{\alpha-1} |v|^2 \omega dx \right) \left( \int_{\mathbb{R}^N} |v|^q \omega dx \right) \left( \int_{\mathbb{R}^N} |\omega|^{\frac{q}{q'}} dx \right) \left( \int_{\mathbb{R}^N} |\omega|^\frac{q'}{\frac{q}{q'}} dx \right) \leq S_{\mu_1} \frac{a}{p_s^\alpha} \| (u, v) \|^{\alpha \beta - 1} \| (\omega, v) \|. \quad (3.12) \]

And by the same process, there exists \( S_{\mu_2} \) such that
\[ \frac{b}{q_l} \int_{\mathbb{R}^N} |v|^{q_l-2} v|u|^q \omega dx \leq S_{\mu_2} \frac{b}{q_l} \| (u, v) \|^{\alpha \beta - 1} \| (\omega, v) \|. \quad (3.13) \]

Since the perturbation functions \( f, g \) belong to \( Q'(\mathbb{R}^N) \), we have
\[ \langle f, (\omega, v) \rangle \leq \| f \|_{Q'(\mathbb{R}^N)} \| (\omega, v) \|, \quad \langle g, (\omega, v) \rangle \leq \| g \|_{Q'(\mathbb{R}^N)} \| (\omega, v) \|. \quad (3.14) \]

Put \( S_{\mu} = S_{\mu_1} \frac{a}{p_s^\alpha} + S_{\mu_2} \frac{b}{q_l} \), combining (3.10)–(3.14), we complete the proof. \( \square \)

Next, we define the operator \( M = \mathcal{R}^{-1} \circ \mathcal{K} \). To satisfy the conditions of Lemma 2.4, we also need the following key results:

**Lemma 3.4.** Assume that the hypotheses of Theorem 1.1 hold, then for any \( 0 < \lambda_0 < \lambda_0 < \lambda_0 \), there exists \( S > 0 \), such that
\[ M(O[0, S] \times O[0, S]) \subset O[0, S] \times O[0, S], \quad (3.15) \]
where \( O[0, S] \times O[0, S] = \{(u, v) \in Q(\mathbb{R}^N) : \| (u, v) \| \leq S \} \).

**Proof.** Let \((u, v) \in Q(\mathbb{R}^N), (\omega, v) = M(u, v) = (\mathcal{R}^{-1} \circ \mathcal{K})(u, v), \) since \( \langle \mathcal{R}(u, v), (u, v) \rangle = \| u \|^p_{L_p} + \| v \|^q_{L_q}, \) we have
\[
\begin{aligned}
2^{p-1} \langle \mathcal{R}(u, v), (u, v) \rangle &\geq 2^{p-1} \left( \| u \|^p_{L_p} + \| v \|^q_{L_q} \right) \geq \| u \|^p_{L_p} + \| v \|^q_{L_q} = \| (u, v) \|^p, \| u \|^p_{L_p} \geq 1, \| v \|^q_{L_q} \geq 1, \\
2^{q_l-1} \langle \mathcal{R}(u, v), (u, v) \rangle &\geq 2^{q_l-1} \left( \| u \|^p_{L_p} + \| v \|^q_{L_q} \right) \geq \| u \|^p_{L_p} + \| v \|^q_{L_q} = \| (u, v) \|^q, \| u \|^p_{L_p} \leq 1, \| v \|^q_{L_q} \leq 1, \\
2^{q-1} \langle \mathcal{R}(u, v), (u, v) \rangle &\geq 2^{q-1} \left( \| u \|^p_{L_p} + \| v \|^q_{L_q} \right) \geq \| u \|^p_{L_p} + \| v \|^q_{L_q} = \| (u, v) \|^q, \| u \|^p_{L_p} \leq 1, \| v \|^q_{L_q} \leq 1.
\end{aligned}
\]

Since \( M(u, v) = (\omega, v), \) from the above inequalities, we shall prove the lemma in four cases.

**Case 1:** If \( \| \omega \|_{L_p} \geq 1, \| v \|_{L_q} \geq 1 \), we have
\[ \| M(u, v) \|^p \leq 2^{p-1} \| \mathcal{K}(u, v) \|_{Q'(\mathbb{R}^N)} \| M(u, v) \|. \]

We consider that \( \| (u, v) \| \leq S, \) by Lemma 3.3, we can obtain
\[ \| M(u, v) \|^p \leq 2^{p-1} \langle \mathcal{K}(u, v), M(u, v) \rangle \leq 2^{p-1} \| \mathcal{K}(u, v) \|_{Q'(\mathbb{R}^N)} \| M(u, v) \|, \]

which implies that
\[ \frac{\| M(u, v) \|^p}{S^{p-1}} \leq 2^{p-1} \left( S_{\mu_1} \| S^{p_{\mu_1}} - p \| + S_{\mu_2} \| S^{q_{\mu_2}} - q_l \| + \frac{\lambda_1 \| f \|^p_{Q'(\mathbb{R}^N)} + \lambda_2 \| g \|^q_{Q'(\mathbb{R}^N)}}{S^{p-1}} \right). \quad (3.17) \]

Next, let \( S > 0 \) be sufficiently small such that
\[ 2^{p-1} \left( S_{\mu_1} \| S^{p_{\mu_1}} - p \| + S_{\mu_2} \| S^{q_{\mu_2}} - q_l \| + \frac{\lambda_1 \| f \|^p_{Q'(\mathbb{R}^N)} + \lambda_2 \| g \|^q_{Q'(\mathbb{R}^N)}}{S^{p-1}} \right) \leq \frac{1}{2}. \]
Put
\[\zeta_1 = \frac{S^{p-1}}{2^p(\|f\|_{Q(R^N)} + \|g\|_{Q(R^N)})^\gamma}.\]
Then for all \(0 < \lambda_1, \lambda_2 \leq \zeta_1\), we can derive from (3.17) that
\[\frac{\|M(u, v)\|^{p-1}}{S^{p-1}} \leq 1.\]

**Case 2:** If \(\|\omega\|_{k,p} \leq 1, \|\nu\|_{l,q} \leq 1\), we have
\[\|M(u, v)\|^{p} \leq 2^{p-1} \|\mathcal{K}(u, v), M(u, v)\| \leq 2^{p-1} \|\mathcal{K}(u, v)\|Q(R^N) \|M(u, v)\|.\]

Following steps similar to Case 1, let
\[\zeta_2 = \frac{S^{q-1}}{2^q(\|f\|_{Q(R^N)} + \|g\|_{Q(R^N)})^\gamma}.\]
Then for all \(0 < \lambda_1, \lambda_2 \leq \zeta_2\), we have
\[\frac{\|M(u, v)\|^{q-1}}{S^{q-1}} \leq 1.\]

**Case 3:** If \(\|\omega\|_{k,p} \leq 1, \|\nu\|_{l,q} \geq 1\), we obtain
\[\|M(u, v)\|^{p} \leq 2^{p-1} \|\mathcal{K}(u, v), M(u, v)\| \leq 2^{p-1} \|\mathcal{K}(u, v)\|Q(R^N) \|M(u, v)\|.\]

Similarly, let
\[\zeta_3 = \frac{S^{p-1}}{2^p(\|f\|_{Q(R^N)} + \|g\|_{Q(R^N)})^\gamma}.\]
Then for all \(0 < \lambda_1, \lambda_2 \leq \zeta_3\), we obtain
\[\frac{\|M(u, v)\|^{p-1}}{S^{p-1}} \leq 1.\]

**Case 4:** If \(\|\omega\|_{k,p} \geq 1, \|\nu\|_{l,q} \leq 1\), we have
\[\|M(u, v)\|^{p} \leq 2^{p} \|\mathcal{K}(u, v), M(u, v)\| \leq 2^{p} \|\mathcal{K}(u, v)\|Q(R^N) \|M(u, v)\|.\]
Let
\[\zeta_4 = \frac{S^{p-1}}{2^p(\|f\|_{Q(R^N)} + \|g\|_{Q(R^N)})^\gamma}.\]
Then for all \(0 < \lambda_1, \lambda_2 \leq \zeta_4\), we obtain
\[\frac{\|M(u, v)\|^{p-1}}{S^{p-1}} \leq 1.\]

Put \(\lambda_0 = \min(\zeta_1, \zeta_2, \zeta_3, \zeta_4)\), combining with the above four cases, the proof is finished. \(\square\)

Now, we focus on problem (1.13). For this, we demonstrate a new operator \(\mathcal{P} : Q(R^N) \rightarrow Q'(R^N)\) given by:
\[
\langle \mathcal{P}(u, v), (\omega, \nu) \rangle = \int_{R^N} \frac{|u|^{p-2} u \omega}{|x|^\gamma} \, dx + \int_{R^N} \frac{|v|^{q-2} v \nu}{|x|^\beta} \, dx + \int_{R^N} \frac{a |u|^{p-2} u |\nu|^\beta \omega}{|x|^\gamma} \, dx + \int_{R^N} \frac{b |v|^{q-2} v |\nu|^\beta \omega}{|x|^\beta} \, dx
+ \int_{R^N} \frac{|u(x)|^{\nu_0} \omega(x)}{|x - y|} \, dx + \int_{R^N} \frac{|v(x)|^{\omega_0} \nu(x)}{|x - y|} \, dx
+ \lambda_1 \langle f, (\omega, \nu) \rangle + \lambda_2 \langle g, (\omega, \nu) \rangle,
\]
for all \((u, v), (\omega, \nu) \in Q(R^N)\). Then, we can obtain an estimate similar to Lemma 3.3.
Lemma 3.5. Under the assumptions of Theorem 1.2, there exist $S_n$, $S_x$, $S_\mu$, $S_{\xi_1}$ and $S_{\xi_2}$ such that

$$\| (\mathcal{P}(u, v), (\omega, v)) \| \in (S_n \|(u, v)\|^{p^*_\alpha} + S_x \|(u, v)\|^{p^*_\beta} + S_\mu \|(u, v)\|^{\alpha + \beta - 1} + \lambda \|g\|_{Q(R^n)}$$

$$+ \lambda_1 \|f\|_{Q(R^n)} + S_{\xi_1} \|(u, v)\|^{2\beta - 1} + S_{\xi_2} \|(u, v)\|^{2\alpha - 1}),$$

for any $(\omega, v) \in Q(R^n)$.

Proof. By Hölder’s inequality and Lemma 2.5, there exist $S(m_i, n_i, \xi_i, N), S(m_2, n_2, \xi_2, N)$ such that

$$\int_0^1 \int_0^1 \frac{|u(x)|^{p^*_\alpha}}{|x - y|^{\xi_1}} |u(x)|^{p^*_\beta - 2} u(x) a(x) dx dy + \int_0^1 \int_0^1 \frac{|v(x)|^{p^*_\alpha}}{|x - y|^{\xi_1}} |v(x)|^{p^*_\beta - 2} v(x) a(x) dx dy$$

$$\leq S(m_1, n_1, \xi_1, N) u^{p^*_\alpha} |\xi_1|^{p^*_\alpha} + S(m_2, n_2, \xi_2, N) v^{p^*_\alpha} |\xi_2|^{p^*_\alpha}$$

$$\leq S(m_1, n_1, \xi_1, N) u^{p^*_\alpha} |\xi_1|^{p^*_\alpha} + S(m_2, n_2, \xi_2, N) v^{p^*_\alpha} |\xi_2|^{p^*_\alpha}$$

$$\leq (S_{\xi_1} \|(u, v)\|^{2\beta - 1} + S_{\xi_2} \|(u, v)\|^{2\alpha - 1}))(\omega, v).$$

Then, according to Lemma 3.3, the proof is completed.

Remark 3.1. We point out that $S_n$, $S_x$ and $S_\mu$ in Lemma 3.5 are consistent with those in Lemma 3.3.

We also define the operator $M_\alpha = R^{-1} \circ \mathcal{P}$. Following Lemma 3.4, we can obtain a similar result.

Lemma 3.6. Under the assumptions of Theorem 1.2, for any $0 < \lambda_1 < \lambda_2$, there exists $\mathcal{T} > 0$ such that

$$M_\alpha(O[0, \mathcal{T}] \times O[0, \mathcal{T}]) \subset O[0, \mathcal{T}] \times O[0, \mathcal{T}],$$

where $O[0, \mathcal{T}] \times O[0, \mathcal{T}] = \{(u, v) \in Q(R^n) : \|(u, v)\| \leq \mathcal{T}\}$.

Proof. Let $(\bar{u}, \bar{v}) \in Q(R^n), (\omega, u) = (\mathcal{P} \circ \mathcal{R})(\bar{u}, \bar{v}) = M_\alpha(\bar{u}, \bar{v})$. We still use the method in Lemma 3.4 to prove this lemma by four cases. Since $(\mathcal{R}(\bar{u}, \bar{v}), (\bar{u}, \bar{v})) = \|\bar{u}\|_p^p + \|\bar{v}\|_q^q$, we have

Case 1: If $\|\omega\|_p \geq 1, \|u\|_q \geq 1$, we have

$$\|M_\alpha(\bar{u}, \bar{v})\|^p \leq 2^p - 1 \|\mathcal{P}(\bar{u}, \bar{v}) \| Q(R^n) \|M_\alpha(\bar{u}, \bar{v})\|.$$

Suppose $\|\bar{u}\| \leq \mathcal{T}$, and by Lemma 3.5, we can deduce the following inequality:

$$\|M_\alpha(\bar{u}, \bar{v})\|^p \leq 2^p \|\mathcal{P}(\bar{u}, \bar{v}) \| Q(R^n) \|M_\alpha(\bar{u}, \bar{v})\|.$$

which implies that
\[
\frac{\|M((\bar{u}, \bar{v}))\|^p}{T^{p-1}} \leq 2^{p-1}\left( S_{\eta} T p^{-\frac{\alpha}{p}} + S_{\xi} T q^{-\frac{\beta}{q}} + S_{\mu} T \alpha \gamma_{\xi} - p + S_{\eta} T 2^p \xi_{\xi} - p \right).
\]

Let \( T > 0 \) be small enough such that
\[
2^{p-1}\left( S_{\eta} T p^{-\frac{\alpha}{p}} + S_{\xi} T q^{-\frac{\beta}{q}} + S_{\mu} T \alpha \gamma_{\xi} - p + S_{\eta} T 2^p \xi_{\xi} - p \right) \leq \frac{1}{2}.
\]

Let \( \eta_1 = T^{-p-1} \),
\[
\eta_2 = 2^{p-1}(\|f\|_{0^p} + \|g\|_{Q^p}),
\]
and
\[
\eta_3 = 2^{p-1}(\|f\|_{0^p} + \|g\|_{Q^p}).
\]

Then for all \( 0 < \lambda_1, \lambda_2 \leq \eta_1 \), we can derive from (3.22) that
\[
\frac{\|M((\bar{u}, \bar{v}))\|^p}{T^{p-1}} \leq 1.
\]

Proceeding as in the four cases in Lemma 3.4, we set that
\[
\eta_2 = 2^{p-1}(\|f\|_{0^p} + \|g\|_{Q^p}),
\]
and
\[
\eta_3 = 2^{p-1}(\|f\|_{0^p} + \|g\|_{Q^p}).
\]

Put \( \lambda = \min(\eta_1, \eta_2, \eta_3, \eta_4) \), for \( 0 < \lambda_1, \lambda_2 \leq \lambda \), we have
\[
M((O[0, T] \times O[0, T]) \subset O[0, T] \times O[0, T].
\]

This completes the proof.

4 Proofs of Theorems 1.1–1.2

4.1 Proof of Theorem 1.1

Now, all we have to do is to prove that \( M : (Q(R^N), \leq) \rightarrow (Q(R^N), \leq) \) is an increasing operator, and by Lemma 2.4, we can obtain the existence of the weak solutions. We have shown in Lemma 3.2 that \( R^{-1} : (Q', \leq) \rightarrow (Q, \leq) \) is increasing. Next, let us show that the operator \( K : (Q(R^N), \leq) \rightarrow (Q(R^N), \leq) \) is increasing. Choosing \((u_1, v_1), (u_2, v_2) \in Q(R^N)\) such that \( u_1 \leq u_2, v_1 \leq v_2 \) almost everywhere in \( R^N \). Since the function \( k(u) = |u|^{r-2}u \) and \( k_s(u, v) = |u|^{r-2}u|v|^s \) are increasing for \( u, v \in Q(R^N) \) as \( r \geq 2 \), we have
\[
\langle K(u_1, v_1), (\omega, v) \rangle \leq \langle K(u_2, v_2), (\omega, v) \rangle, \forall (\omega, v) \in Q(R^N).
\]

Consequently, the operator \( M : (Q(R^N), \leq) \rightarrow (Q(R^N), \leq) \) is increasing. By Lemmas 3.4 and 2.4, the operator \( M \) has a fixed point. Hence, there exists \((u_0, v_0) \in O[0, S] \times O[0, S]\) such that \( M(u_0, v_0) = (u_0, v_0) \). Since \( M = R^{-1} \circ K \) we have
\[
\langle R(u_0, v_0), (\omega, v) \rangle = \langle K(u_0, v_0), (\omega, v) \rangle, \forall (\omega, v) \in Q(R^N).
\]

Therefore, by the definitions of \( R \) and \( K, f, g \neq 0 \) imply that \((u_0, v_0)\) is a nontrivial nonnegative weak solution of problem (1.1) and this completes the proof of Theorem 1.1.
4.2 Proof of Theorem 1.2

Set \((\tilde{u}_1, \tilde{v}_1), (\tilde{u}_2, \tilde{v}_2) \in Q(\mathbb{R}^N)\) such that \((\tilde{u}_1, \tilde{v}_1) \leq (\tilde{u}_2, \tilde{v}_2)\) almost everywhere in \(\mathbb{R}^N\). Because the function\(k(u) = |u|^p\) is increasing for all \(u \geq 0\) as \(r \geq 2\), we deduce
\[
\langle \mathcal{P}(\tilde{u}_1, \tilde{v}_1), (\omega, v) \rangle \leq \langle \mathcal{P}(\tilde{u}_2, \tilde{v}_2), (\omega, v) \rangle .
\]
for any \((\omega, v) \in Q(\mathbb{R}^N)\). Then, by (4.2) and Lemma 3.2, \(\mathcal{M}_* : (Q(\mathbb{R}^N), \leq) \to (Q(\mathbb{R}^N), \leq)\) is increasing. By Lemmas 3.6 and 2.4, the operator \(\mathcal{M}_*\) has a fixed point. Therefore, there exists \((u_*, v_*) \in \mathcal{O}[0, T] \times \mathcal{O}[0, T]\) such that \(\mathcal{M}_*(u_*, v_*) = (u_*, v_*)\). Consequently, \(f, g \not= 0\) yield that \((u_*, v_*)\) is a nontrivial nonnegative weak solution of problem (1.13) and this ends the proof of Theorem 1.2.

5 The case of \(N = sp = lq\)

In this section, we consider the case of \(N = sp = lq\). For this purpose, we begin with a variant of system (1.1) as follows:

\[
\begin{aligned}
(-\Delta)_p u + V(x)|u|^{p-2} u - \frac{|u|^{p-2} u}{|x|^r} &= \frac{\alpha}{p} |u|^{a-2} u|v|^\beta + \lambda_1 f(x) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)_q v + V(x)|v|^{q-2} v - \frac{|v|^{q-2} v}{|x|^r} &= \frac{\beta}{q} |v|^{p-2} v|u|^\alpha + \lambda_2 g(x) \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]

where \(0 < s, l < 1, 2 \leq p < q, 0 \leq \tau, \tau < N, p < \alpha < +\infty, q < \beta < \infty\). Similarly, the variant of system (1.13) reads as follows:

\[
\begin{aligned}
(-\Delta)^s_{\xi_1} u + V(x)|u|^{p-2} u - \frac{|u|^{p-2} u}{|x|^r} &= (I_{\xi_1} \ast |u|^p)|u|^{p-2} u + \frac{\alpha}{p} |u|^{a-2} u|v|^\beta + \lambda_1 f(x) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^q_{\xi_2} v + V(x)|v|^{q-2} v - \frac{|v|^{q-2} v}{|x|^r} &= (I_{\xi_2} \ast |v|^q)|v|^{q-2} v + \frac{\beta}{q} |v|^{p-2} v|u|^\alpha + \lambda_2 g(x) \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]

where \(I_{\xi_i}(x) = |x|^{-\xi_i}\) is the Riesz potential of order \(\xi_i \in (0, N), i = 1, 2\). The perturbation terms \(f(x), g(x)\) in systems (5.1) and (5.2) satisfy \(f(x), g(x) \not= 0\). In addition, our hypothesis for the potential function \(V\) is given as follows:

(V) \(V \in C(\mathbb{R}^N)\) satisfies there exists a constant \(V_0 > 0\) such that \(\inf_{\mathbb{R}^N} V(x) \geq V_0\).

In order to investigate the existence of solutions to systems 5.1 and 5.2, we first recall some knowledge of fractional Sobolev space. The fractional Sobolev space \(W^{s,p}(\mathbb{R}^N)\) is defined by

\[W^{s,p}(\mathbb{R}^N) = \{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},\]

and it is endowed with the following norm:

\[\|u\|_{W^{s,p}(\mathbb{R}^N)} = (\|u\|_p^p + [u]_{s,p}^p)^{1/p}.\]

By Theorem 6.9 of [17], the fractional Sobolev embedding \(W^{s,p}(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N)\) is continuous for all \(p \leq m < \infty\) as \(sp = N\). Let \(\mathcal{W} = W^{s,p}(\mathbb{R}^N) \times W^{l,q}(\mathbb{R}^N)\) be endowed with the norm \(\|(u, v)\|_{\mathcal{W}} = \|u\|_{W^{s,p}(\mathbb{R}^N)} + \|v\|_{W^{l,q}(\mathbb{R}^N)}\).

As we all know, the Banach space \(W^{s,p}(\mathbb{R}^N)\) is uniformly convex, then \(\mathcal{W}\) is a reflexive Banach space. We introduce a subspace of \(W^{s,p}(\mathbb{R}^N)\), which is defined as follows

\[X_{s,p}(\mathbb{R}^N) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p \, dx < \infty\right\}.\]
with respect to the norm

\[ |u|_{X_{s,p}(\mathbb{R}^N)} = (|u|^p_{E,p} + |u|^p_{p,V})^{1/p}, \quad \|u\|^p_{p,V} = \int_{\mathbb{R}^N} V(x)|u(x)|^p \, dx. \]

We denote that \( \mathcal{X}(\mathbb{R}^N) = X_{s,p}(\mathbb{R}^N) \times X_{s,q}(\mathbb{R}^N) \), endowed with the norm

\[ |(u, v)|_2 = \|k\|_{X_{s,p}(\mathbb{R}^N)} + \|v\|_{X_{s,q}(\mathbb{R}^N)}, \]

By employing the Clarkson’s first inequality (see [5]), \( \mathcal{X}(\mathbb{R}^N) \) is a reflexive Banach space. Thanks to [42] and [47, Lemma 6], the embeddings \( X_{s,p}(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N) \) and \( X_{s,q}(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N), |x|^{-\gamma} \) are continuous for each \( m \in [p, \infty) \) and \( y \in [0, N) \).

In order to investigate the weak solutions of equations (5.1) and (5.2), we consider the subspace of \( \mathcal{X}(\mathbb{R}^N) \)

\[ \mathcal{Z}(\mathbb{R}^N) = \{(u, v) \in \mathcal{X}(\mathbb{R}^N) : u \geq 0, v \geq 0 \quad a.e. \quad in \quad \mathbb{R}^N\}. \]

Therefore, the space \( \mathcal{Z}(\mathbb{R}^N, \|\cdot\|_2) \) is a reflexive Banach space. Similar to the proof of Lemmas 2.1 and 2.3 in our article, we can obtain that the embedding \( \mathcal{Z}(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N) \times L^m(\mathbb{R}^N) \) is continuous for each \( m \in [p, \infty), m_2 \in [q, \infty) \) and \( \mathcal{Z}(\mathbb{R}^N) \) is continuously embedded in \( L^m(\mathbb{R}^N, |x|^{-\gamma}) \times L^m(\mathbb{R}^N, |x|^{-\gamma}) \) for \( 0 \leq y, \tau < N \). Next, we denote that \( \mathcal{Z}'(\mathbb{R}^N) \) is the dual space of \( \mathcal{Z}(\mathbb{R}^N) \).

In this setting, by setting and processing some operators as in our article, we proceed the proofs as in [46] with minor changes, then we can obtain the following results:

**Theorem 5.1.** Suppose that \( V \) satisfies (V) and \( f, g \) belong to \( \mathcal{Z}'(\mathbb{R}^N) \). If \( \frac{\alpha}{p} + \frac{\beta}{q} = 1 \), then there exists \( \lambda^* > 0 \) such that for all \( 0 < \lambda_1, \lambda_2 \leq \lambda^* \), system (5.1)) has a nontrivial nonnegative weak solution in \( \mathcal{Z}(\mathbb{R}^N) \).

**Theorem 5.2.** Suppose that \( V \) satisfies (V) and \( f, g \) belong to \( \mathcal{Z}'(\mathbb{R}^N) \). If \( \frac{\alpha}{p} + \frac{\beta}{q} = 1 \), then there exists \( \lambda_3 > 0 \) such that for all \( 0 < \lambda_1, \lambda_2 \leq \lambda_3 \), system (5.2) has a nontrivial nonnegative weak solution in \( \mathcal{Z}(\mathbb{R}^N) \).

**Remark 5.1.** We would like to mention that the main results in this section are similar to those in [46], but in the context of fractional \((p, q)\)-Laplacian systems, our results are still new.

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