Successful strategies for a queens placing game on an $n \times n$ chess board

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1 Abstract and introduction

In his list of open problems ([1]), Martin Erickson described a certain game:
“Two players alternately put queens on an $n \times n$ chess board so that each new queen is not in range of any queen already on the board (the color of the queens is unimportant). The last player who can move wins.”
Then he asked: “Who should win?”
Obviously, for $n$ up to 3, the first player wins, if he does not miss to start at the central position in the case $n = 3$.
In this article, we give very simple always winning strategies for the first player if $n$ is 4 or odd.
The additionally (in the source package) provided computer program QPGAME3 has been used to check that there are successful strategies for the first player if $n$ is 6 or 8, and for the second player if $n$ is 10, 12, 14, or 16.
As discovered during the submission process of the first version of this article, Hassan A Noon presented consistent results concerning values of $n$ which are odd or at most 10, in his B.A. thesis ([2]) and, together with Glen Van Brummelen, in a journal article ([3]).

2 The case $n = 4$

The first player places a queen at one of the four inner positions. When * indicates an unavailable cell and . indicates an available cell, we do have this situation (up to meaningless rotations of the complete board in steps of 90 degrees):

```
* * * .
* * * *
* * * .
. * . *
```
No matter which of the four available cells the second player takes, it remains exactly one available cell for the first player to place the last queen.

3 Notation and conflict-free positions

The rows and columns of the board are numbered from 0 to $n - 1$. A position is an (ordered) pair of a row number $r$ and a column number $c$, written as $(r,c)$.
If two queens are placed at $(r_1,c_1)$ and $(r_2,c_2)$, then the game rules require:
$r_1 \neq r_2$ (different rows),
c_1 \neq c_2$ (different columns),
\[ r_1 + c_1 \neq r_2 + c_2 \text{ (different falling diagonals), and} \]
\[ r_1 - c_1 \neq r_2 - c_2 \text{ (different rising diagonals)}. \]

4 The case of odd \( n \)

The first player puts the first queen to the cell at the central position \(( (n-1)/2, (n-1)/2) \).

From then on: When the second player has put a queen to the cell at a position \((r,c)\), the first player puts a queen to the cell at position \((R,C)\), where \(R = n - 1 - r\) and \(C = n - 1 - c\). This is a valid position, received from \((r,c)\) by mirroring the two coordinate values with respect to the middle lines, or, with the same effect, by rotating the board a half turn.

This strategy is always successful for the following reasons:

The central cell lies just at the middle of the straight line between \((r,c)\) and \((R,C)\). Thus, a conflict between queens at these positions exists if and only if there is a conflict between \((r,c)\) and the queen at the central position; but this is forbidden by the game rules. If \((R,C)\) was available before a queen is put to \((r,c)\), then it is still available after that move.

After the placing of the first queen, there is (half turn) rotational symmetry. Because of this symmetry, \((R,C)\) is available if and only if \((r,c)\) is available. After putting two queens to the cells at these positions, the (half turn) rotational symmetry exists again.

Eventually, no cell is available (for the second player) and the first player wins.

5 The provided computer program QPGAME3

As like as its two predecessors, the source code file QPGAME3.PAS has been developed for PASCAL compilers compatible with Turbo Pascal 4.0. Lines are at most 78 characters long. For inspections the use of an ASCII compatible monospaced font is strongly recommended. The intended indentation is by one character per structure level, using blanks (instead of tabs).

The program includes a good portion of comments (enclosed in curly braces). So it should be fairly understandable at least by readers knowing at least one imperative programming language.

Basically, the core procedure \texttt{wins} performs a general backtracking search. Of course, all selected positions have to be legal.

When a player in a certain situation finds a winning move position \(P_1\) as an answer to his opponent’s move to a position \(P_0\), it is clear that \(P_0\) would be a winning answer to \(P_1\) if the situation before both moves did not change. Therefore such a winning move position \(P_1\) is registered in the internal variable \texttt{forbidden} and skipped in further checks. That would give no advantage if \(P_1\) has been checked before \(P_0\). Therefore, excluding (“for historical reasons”) the second move of a game, the search of an answering move starts in a row that follows the row of the move of the previous move with respect to the search order for the previous move.

The newly added Boolean routine parameter \texttt{rotsym} indicates, whether all currently involved moves of player 2 had re-established the initial (half-turn) rotation symmetry by choosing the unique position just a half-turn away from the position of the previous move of player 1. If the value of \texttt{rotsym} is \texttt{True} when player 1 has to make a move, the symmetry is used by just considering the rows up to the center of the board. (Source code fragment: \texttt{“(not rotsym or (r<=nm1div2))”}).

In addition, there are these further conditions (in order to reduce the set of positions to be checked):

For the first move, the (then complete) symmetry is used.

(Source code fragment: \texttt{“(c<=nm1div2) and (r<=c)”}).

If \( n \) is odd, the above described strategy for the first player is used.

If \( n \) is even and not greater than 8, the first player is forced to start at a certain of the four inner positions. (Source code fragment: \texttt{“(r=nm1div2) and (c=nm1div2)”}).
Tests have shown, that this restriction does not prevent the first player from winning (in that case) while significantly reducing the sum of moves performed during the search in the case \( n = 8 \).

Finally, there is a special treatment of the case \( n = 16 \) by player 2, described in the following subsection. (Source code fragment:

```plaintext
" (n<>16) or 
    (n_moves>3) or 
    (n_moves=3) and n16_player_2_round_2(prev_r,prev_c,r,c) or 
    (n_moves=1) and n16_player_2_round_1(prev_r,prev_c,r,c)\n)".
```

5.1 Special treatment of the case \( n = 16 \)

In order to reduce the otherwise very long execution time, QPGAME3.PAS uses previously found and checked moves of player 2 to treat the case \( n = 16 \) in a special way: Two routines (\( n16_player_2_round_1 \) and \( n16_player_2_round_2 \)) select unique moves of player 2 when he has to answer to the moves of player 1 in the first two rounds.

In \( n16_player_2_round_1 \), the table \( T \) contains for each possible first move of player 1 a hexadecimally noted byte value, where \$FF indicates an invalid (unexpected) move of player 1 and each other value encodes an answering move of player 2: The upper/lower half-byte (first/second hexadecimal) gives the row/column number.

Because the first move of player 1 is restricted to positions satisfying “\((c<=n\text{m}1\text{div}2) \text{ and } (r<=c)\)” here “\((c<=7) \text{ and } (r<=c)\)”, just 36 move positions have to be considered. In order to be able to use the rotation symmetry, it has been tried to use \((15-r0,15-c0)\) as often as possible. Clearly, this was impossible for the 8 positions on the diagonal \((r0=c0)\). In these cases a position \((r1,c1)\) is given such that “\((c1<=7) \text{ and } (r1<=c1)\)” and therefore two positions can and will be checked by one actual check. So, 28 actual checks have to be executed.

By the way, for some non-diagonal positions \((r0,c0)\) turned out that player 2 could not win by choosing \((15-r0,15-c0)\), for instance \((1,2)\) and \((6,7)\).

In \( n16_player_2_round_2 \), the situation is more complicated, because the answering second move of player 2 depends not just on the second move of player 1 (third move of the game), but also on the first move of player 1 (first move of the game). Therefore, that first move has been stored before in the global variables \( r0 \) and \( c0 \) and is used to address an index value in the table \( A \). Here, the value 0 means, that the addressing position is invalid (by design); its use would cause a range check error and abort the execution. Otherwise, that value is the first index value to address an entry in the table \( B \). As the other two index values, the row and the column number of the second move of player 1 (routine parameters \( r2 \) and \( c2 \)) are used. The type, notation, meaning, and usage of the entries of \( B \) are as described above for the table \( T \).

Some of the (sub-)tables contain unused space or redundant information. In addition, the usage of the tables in a predicative (checking) instead of a functional (constructive) way is not optimal for minimal runtime. But the latter can be ignored because the selection of a second or fourth move of a game is an extremely seldom event under the move selections performed during a whole case check (search) if \( n \) is not very little. And because the program is still small, also the waste of space is no real problem here.

5.2 General properties of QPGAME3

The program does not use the heap or any pointer operation at all. If you don’t change the respective compiler directives, range checks and stack overflow checks will be generated. So the resulting executable will be extremely safe. It is also small and needs only a few kilobytes for the stack.
The program ignores any command line parameters or inputs other than pressing Ctrl-C to cancel the execution - where the speed of response depends on the used compiler (slow in the case of Turbo Pascal). It writes only to the standard output device and into the automatically created or rewritten, resp., file QPGAME.LST within the working directory. In the default case, the standard output device will be the monitor screen. But one could redirect that output (e.g. to a file). In most cases that will be unnecessary because the listing in QPGAME.LST is an essence of the data written to the standard output device. Just some execution state indicators are omitted. Even after the execution has been stopped by cancelling, the listing file should be readable and contain the data written before the end of the execution. During the execution, its content is not accessible (by other processes).

5.3 Progress indication and summaries

As for the older program versions, a + will be emitted (written to the standard output device) after each 1000000 moves. Those characters (and corresponding new-lines) will not be written into the listing file.

When the check of a certain case (value of \( n \)) begins or ends, an information line will be written (to the standard output device and into the listing file).

5.3.1 The sum of calls

QPGAME1, the program provided with the first version of this article, used a simple longint (signed 32 bit integer) variable to count the calls (of the core routine / of a player to try to find a winning move), then displayed as \textit{Sum of moves}, which is actually smaller (by one). Because the upper limit of that counter was \( 2^{31} - 1 = 2147483647 \), overflows (reductions by \( 2^{32} \)) could occur in realistic cases. Those overflows can not influence the search but mislead the user. So, the counter has been redesigned in QPGAME2.PAS. Since then, it will not overflow below \( (2^{31} - 1) \times 10^6 > 2.14 \times 10^{15} \).

5.3.2 Detailed sub-case information

In order to allow to give more detailed information about the state and subresults of the execution, especially in very extensive checks, now the generation of additional output can be advised:

If the Boolean variable \texttt{first\_move\_checking\_statistics} does have the value \texttt{True}, a subresult summary line (containing the position of the first move (by player 1), the resulting outcome, and the sum of calls during the check of that first move) will be written to the standard output device and into the listing file.

If the Boolean variable \texttt{indicate\_third\_moves\_checking} does have the value \texttt{True}, the start and the end of the check of the situation established by a third move will be indicated (but not written to the listing file). To be more concrete: On the start of that check, the positions of the first and third move of the game (both by player 1) will be written to the standard output device. On the end of that check, the result of the routine \texttt{wins} (the sub-case success of player 2) will be indicated by a digit (0 means loss, 1 means win). Typically, there are some + symbols between these two outputs.

An example (from the check of the case \( n = 16 \)):

\[
[1:\{0,0\}]\ 3:\{2,3\}+=++=+\rightarrow 1
\]

In the unchanged QPGAME3.PAS, “\( n \geq 16 \)” is the evaluating expression for both mentioned control variables.

5.4 Compiling and running, actually checked cases

In principle, the program is able to check the cases of \( n \) from 1 to 16 or 32 (depending on the compiler symbol \texttt{BIGN}, see below) without large changes. Because some mentioned cases were already solved mathematically
and checking the case \( n = 14 \) will take some time, in the published version only checks for even values of \( n \) from 6 to 12 are called from the main loop.

In order to avoid a compilation result depending on the settings you could use the command line versions of the compilers (TPC for Turbo Pascal, BPC for Borland Pascal 7, DCC32 for Borland Delphi (32 bit versions; do not miss to use the -CC option in order to generate a console executable), VPC for Virtual Pascal, FPC for Free Pascal) instead of the compilers integrated in the IDEs.

In order to make the execution faster than that of QPGAME1 (provided with the first version of this article), alternative structure and usage of the variables containing the information on already used rows, columns, and diagonals as described in [4] have been implemented with QPGAME2. The program became indeed faster, but because the new code will not work if \( n \) is greater than 16 and its function is not just that obvious as that of the old code, the old code is still in the source file and can be used instead of the new code by compiling with the symbol BIGN defined. For this purpose, the appropriate command line option would be -DBIGN for compilers from Borland (Turbo Pascal, Delphi), and -dBIGN for Free Pascal.

The program has been successfully compiled and executed on a 1 GHz Intel PIII PC running MS Windows 98 SE. These are the used compilers and the respective two execution times from compilations with defined/undefined compiler symbol BIGN:

| Compiler                  | Defined BIGN | Undefined BIGN |
|---------------------------|--------------|----------------|
| Turbo Pascal 5.5          | 1:44 min / 1:16 min | 1:16 min / 1:16 min |
| Turbo Pascal 7.01         | 1:39 min / 0:41 min | 0:41 min / 0:41 min |
| Borland Delphi 4.0 build 5.37 | 0:13 min / 0:11 min | 0:11 min / 0:11 min |
| Virtual Pascal 2.1 build 279 | 0:18 min / 0:14 min | 0:14 min / 0:14 min |
| Free Pascal 2.4.4 i386-Win32 | 0:15 min / 0:11 min | 0:11 min / 0:11 min |

Here is the content of QPGAME.LST after compiling the (unchanged) QPGAME3.PAS and running the generated QPGAME3.EXE:

```plaintext
=== Checking solutions for the queens placing game problem ===
=== Version 3 Copyright (c) 2014-04-17 Thomas Jenrich ===

Hints:
Output listing into file QPGAME.LST within the working directory.
After each 1000000 moves a + will be emitted.
To cancel the execution press Ctrl-C.

Starting search with n = 6
Search completed. Result of player 1: win. Sum of calls: 54

Starting search with n = 8
Search completed. Result of player 1: win. Sum of calls: 2266

Starting search with n = 10
Search completed. Result of player 1: loss. Sum of calls: 653007

Starting search with n = 12
Search completed. Result of player 1: loss. Sum of calls: 11334613

== Regular program stop ==

After changing the upper limit of the main loop from 13 to 14, Free Pascal has compiled QPGAME3.PAS again. The execution of the resulting QPGAME3.EXE took 18:51 min. This is the (relevant part of the) additional output:

Starting search with n = 14
Search completed. Result of player 1: loss. Sum of calls: 1161385667

After changing the lower limit and the upper limit of the main loop into 16, Free Pascal has compiled QPGAME3.PAS again. The execution of the resulting QPGAME3.EXE took 22:55:30 h. Parallel activities
on that computer may have caused some delay, but probably not more than one hour.

Here is the content of the generated QPGAME.LST:

```plaintext
=== Checking solutions for the queens placing game problem ===
=== Version 3   Copyright (c) 2014-04-17 Thomas Jenrich ===

Hints:
Output listing into file QPGAME.LST within the working directory.
After each 1000000 moves a + will be emitted.
To cancel the execution press Ctrl-C.

Starting search with n = 16
pl. 1: (0,0) -> pl. 2: win. Sum of calls: 4470810024
pl. 1: (0,1) -> pl. 2: win. Sum of calls: 3905839444
pl. 1: (0,2) -> pl. 2: win. Sum of calls: 3734401972
pl. 1: (0,3) -> pl. 2: win. Sum of calls: 3184149898
pl. 1: (0,4) -> pl. 2: win. Sum of calls: 4328139348
pl. 1: (0,5) -> pl. 2: win. Sum of calls: 4425220446
pl. 1: (0,6) -> pl. 2: win. Sum of calls: 2278166076
pl. 1: (0,7) -> pl. 2: win. Sum of calls: 2676717214
pl. 1: (1,2) -> pl. 2: win. Sum of calls: 3876758648
pl. 1: (1,3) -> pl. 2: win. Sum of calls: 2063718676
pl. 1: (1,4) -> pl. 2: win. Sum of calls: 2120404822
pl. 1: (1,5) -> pl. 2: win. Sum of calls: 2114679020
pl. 1: (1,7) -> pl. 2: win. Sum of calls: 2290066796
pl. 1: (2,2) -> pl. 2: win. Sum of calls: 3360372268
pl. 1: (2,3) -> pl. 2: win. Sum of calls: 1952040408
pl. 1: (2,4) -> pl. 2: win. Sum of calls: 1993953094
pl. 1: (2,5) -> pl. 2: win. Sum of calls: 1965208480
pl. 1: (2,6) -> pl. 2: win. Sum of calls: 2088818438
pl. 1: (2,7) -> pl. 2: win. Sum of calls: 2192529724
pl. 1: (3,4) -> pl. 2: win. Sum of calls: 1719300574
pl. 1: (3,5) -> pl. 2: win. Sum of calls: 1834911490
pl. 1: (3,6) -> pl. 2: win. Sum of calls: 1981765458
pl. 1: (3,7) -> pl. 2: win. Sum of calls: 1964816772
pl. 1: (4,5) -> pl. 2: win. Sum of calls: 1671384528
pl. 1: (4,6) -> pl. 2: win. Sum of calls: 1857679532
pl. 1: (4,7) -> pl. 2: win. Sum of calls: 2007652010
pl. 1: (5,6) -> pl. 2: win. Sum of calls: 1690434580
pl. 1: (5,7) -> pl. 2: win. Sum of calls: 1712035496

Search completed. Result of player 1: loss. Sum of calls: 71461975237

== Regular program stop ==
```

References

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[5] *Turbo Pascal* versions 1.0, 3.02, and 5.5 (binaries only)
http://edn.embarcadero.com/museum/antiquesoftware#
For downloading one has to register or sign-in.

[6] *Virtual Pascal* (Closed Source freeware)
One ZIP-file including binaries and documentation for Win32, OS/2, and Linux
Official forum:
http://vpascal.ning.com/
Forum entry Where can I download VP?
http://vpascal.ning.com/forum/topic/show?id=854411%3ATopic%3A9

[7] *Free Pascal* (Open Source freeware)
Sources, documentation, and binaries for several systems
http://www.freepascal.org

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