THE SPHALERON BARRIER IN THE PRESENCE OF FERMIONS

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Abstract

We calculate the minimal energy path over the sphaleron barrier in the presence of fermions, assuming that the fermions of a doublet are degenerate in mass. This allows for spherically symmetric ansätze for the fields, when the mixing angle dependence is neglected. While light fermions have little influence on the barrier, the presence of heavy fermions ($M_F \sim \text{TeV}$) strongly deforms the barrier, giving rise to additional sphalerons for very heavy fermions ($M_F \sim 10 \text{ TeV}$). Heavy fermions form non-topological solitons in the vacuum sector.
1 Introduction

In 1976 't Hooft [1] observed that the standard model does not absolutely conserve baryon and lepton number due to the Adler-Bell-Jackiw anomaly. The process 't Hooft considered was spontaneous fermion number violation due to instanton induced transitions. Fermion number violating tunnelling transitions between topologically distinct vacua might indeed be observable at high energies at future accelerators [2,3].

Manton considered the possibility of fermion number violation in the standard model from another point of view [4]. Investigating the topological structure of the configuration space of the Weinberg-Salam theory, Manton showed that there are noncontractible loops in configuration space, and predicted the existence of a static, unstable solution of the field equations, a sphaleron [5], representing the top of the energy barrier between topologically distinct vacua.

At finite temperature this energy barrier between topologically distinct vacua can be overcome due to thermal fluctuations of the fields, and fermion number violating vacuum to vacuum transitions involving changes of baryon and lepton number can occur. The rate for such baryon number violating processes is largely determined by a Boltzmann factor, containing the height of the barrier at a given temperature and thus the energy of the sphaleron. Baryon number violation in the standard model due to such transitions over the barrier may be relevant for the generation of the baryon asymmetry of the universe [6-10].

While the barrier between topologically distinct vacua is traversed, the Chern-Simons number changes continuously from $N_{CS}$ in one vacuum sector to $N_{CS} + 1$ in the neighbouring vacuum sector. At the same time one occupied fermion level crosses from the positive continuum to the negative continuum, leading to the change in fermion number. When considered in the background field approximation this level crossing phenomenon predicts the existence of a fermion zero mode precisely at the top of the barrier, at the sphaleron. For massless fermions this zero mode is known analytically [11-13].

Considering the minimal energy path over the barrier [14,15], the fermionic level crossing was demonstrated recently in the background field approximation for the Weinberg-Salam theory [16], assuming that the fermions of a doublet are degenerate in mass. This assumption, violated in the standard model, allows for spherically symmetric ansätze for all of the fields, when the mixing angle dependence is neglected (which is an excellent approximation [17,18]).

With these assumptions we here calculate the minimal energy path over the barrier self-consistently, varying both the fermion mass and the Higgs mass. The presence of the fermions influences the energy and the shape of the barrier, which need no longer be symmetric with respect to the sphaleron. Neither must the fermion zero mode occur precisely at the top of the barrier.
While our main concern is the study of the barrier for light fermions up to about the top quark mass, we also consider heavy fermions with masses in the TeV region, because the possibility of rapid anomalous decay was discussed recently for heavy fermions, and the existence of a new type of soliton was conjectured [19], which is distinct from the known weak non-topological soliton [19-21] representing a weak chiral soliton. Even for heavy fermions we employ the valence fermion approximation, neglecting all boson loops and the effects of the Dirac sea (as in [19]).

We briefly review in section 2 the Weinberg-Salam lagrangian and the anomalous currents for vanishing mixing angle and for degenerate fermion doublets. In section 3 we present our ansatz, obtain the energy functional and the Chern-Simons number and derive the equations of motion. In section 4 we present our results. First we discuss the influence of the presence of fermions on the sphaleron. Then we exhibit the change in energy and shape of the barrier and present the fermion eigenvalue along the minimal energy path. We end this section with a discussion of the non-topological soliton, which, for heavy fermions, is reached at the end of the path instead of the vacuum. We present our conclusions in section 5.

2 Weinberg-Salam lagrangian

We consider the bosonic sector of the Weinberg-Salam theory in the limit of vanishing mixing angle. In this limit the U(1) field decouples and can consistently be set to zero

\[ \mathcal{L}_b = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} + (D^a_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda (\Phi^\dagger \Phi - \frac{1}{2} v^2)^2 \]  

with the SU(2) field strength tensor

\[ F^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + g \epsilon^{abc} V^b_\mu V^c_\nu, \]  

and the covariant derivative for the Higgs field

\[ D^a_\mu \Phi = (\partial_\mu - \frac{1}{2} i g \tau^a V^a_\mu) \Phi. \]  

The SU(2) gauge symmetry is spontaneously broken due to the non-vanishing vacuum expectation value \( v \) of the Higgs field

\[ \langle \Phi \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]  

leading to the boson masses

\[ M_W = M_Z = \frac{1}{2} g v, \quad M_H = v \sqrt{2} \lambda. \]
We employ the values \( M_W = 80 \text{ GeV} \), \( g = 0.65 \).

For vanishing mixing angle, considering only fermion doublets degenerate in mass, the fermion lagrangian reads

\[
\mathcal{L}_f = \bar{q}_L i \gamma^\mu D_\mu q_L + \bar{q}_R i \gamma^\mu \partial_\mu q_R - f(q) \bar{q}_L (\Phi u_R + \Phi d_R) - f(q) (\bar{d}_R \Phi^\dagger + \bar{u}_R \Phi^\dagger) q_L ,
\]

where \( q_L \) denotes the lefthanded doublet \((u_L, d_L)\), while \( q_R \) abbreviates the righthanded singlets \((u_R, d_R)\), with covariant derivative

\[
D_\mu q_L = (\partial_\mu - \frac{1}{2} ig \tau^a V^a_\mu) q_L ,
\]

and with \( \Phi = i \tau_2 \Phi^* \). The fermion mass is given by

\[
M_F = \frac{1}{\sqrt{2}} f(q) v .
\]

Due to the U(1) anomaly baryon number and lepton number are not conserved

\[
\partial^\mu j^B.L_\mu = - f_g \partial_\mu K_\mu ,
\]

where

\[
K_\mu = \frac{g^2}{16 \pi^2} \varepsilon_{\mu \nu \rho \sigma} \text{Tr}(\mathcal{F}^\nu_\rho \mathcal{V}^\sigma + \frac{2}{3} ig \mathcal{V}^\nu \mathcal{V}^\rho \mathcal{V}^\sigma)
\]

\((\mathcal{F}_{\nu \rho} = 1/2 \tau^i F^i_{\nu \rho}, \mathcal{V}_\sigma = 1/2 \tau^i V^i_\sigma)\) is the Chern-Simons current and \( f_g \) is the number of generations. In the unitary gauge the topological baryon number \( Q_B \), carried by a configuration, is determined by its Chern-Simons number \( N_{CS} \),

\[
N_{CS} = \int d^3 r K^0 .
\]

For the vacua the Chern-Simons number is identical to the integer winding number, while the sphaleron at the top of the barrier carries half integer Chern-Simons number [5].

### 3 Equations of Motion

The sphaleron barrier can be obtained by constructing a family of field configurations for the fermionic, gauge and Higgs fields, which interpolates smoothly from one vacuum sector to another, minimizing the energy along this path as a function of the Chern-Simons number [14]. Employing a spherically symmetric ansatz for the fields, we now derive the equations of motion, the energy functional and the Chern-Simons number.
3.1 Ansatz

In the limit of vanishing mixing angle the general static, spherically symmetric ansatz for the gauge and Higgs fields is given by [22]

\[ V_i^a = \frac{1 - f_A(r)}{gr} \hat{\varepsilon}_{a i j} \hat{j}_j + \frac{f_B(r)}{gr} (\delta_{i a} - \hat{\varepsilon}_{i a} \hat{r}_a) + \frac{f_C(r)}{gr} \hat{r}_i \hat{r}_a , \]  
\( V_0^a = 0 , \)

\[ \Phi = \frac{v}{\sqrt{2}} \begin{pmatrix} H(r) + i \vec{\tau} \cdot \hat{r} K(r) \\ 0 \\ 1 \end{pmatrix} , \]

(12) (13) (14)

and involves the five radial functions \( f_A(r) , f_B(r) , f_C(r) , H(r) \) and \( K(r) \).

To retain spherical symmetry we consider only fermion doublets degenerate in mass. The corresponding spherically symmetric ansatz for the fermion eigenstates is the hedgehog ansatz,

\[ q_L(\vec{r}, t) = e^{-i \omega t} M^3_W [G_L(r) + i \vec{\sigma} \cdot \hat{r} F_L(r)] \chi_h , \]

\[ q_R(\vec{r}, t) = e^{-i \omega t} M^3_W [G_R(r) - i \vec{\sigma} \cdot \hat{r} F_R(r)] \chi_h , \]

(15) (16)

where the normalized hedgehog spinor \( \chi_h \) satisfies the spin-isospin relation

\[ \vec{\sigma} \chi_h + \vec{r} \chi_h = 0 . \]

(17)

The ansatz is form-invariant under a spherically symmetric gauge transformation with the unitary matrix

\[ U(\vec{r}) = \exp(i \frac{\Theta(r)}{2} \vec{r} \cdot \hat{r}) . \]

(18)

Under such a U(1) transformation the functions transform as follows

\[ f_A + if_B \rightarrow \exp(i \Theta)(f_A + if_B) , \]

\[ f_C \rightarrow f_C + r \Theta' , \]

\[ H + iK \rightarrow \exp(i \frac{\Theta}{2})(H + iK) , \]

\[ F_L + iG_L \rightarrow \exp(i \frac{\Theta}{2})(F_L + iG_L) , \]

\[ F_R + iG_R \rightarrow F_R + iG_R . \]

(19)

3.2 Energy functional

The ansatz eqs.(12)-(17) leads to the spherically symmetric energy functional

\[ E = E_b + E_r , \]

(20)
with the bosonic part
\[ E_b = \frac{4\pi M_W}{g^2} \int_0^\infty dx \left[ \frac{1}{2x^2}(f_A^2 + f_B^2 - 1)^2 + \left( \frac{f'_A + \frac{f_B f_C}{x}}{x} \right)^2 + \left( \frac{f'_B - \frac{f_A f_C}{x}}{x} \right)^2 \right] + \left( K^2 + H^2 \right)(1 + f_A^2 + f_B^2 + \frac{f_C^2}{2}) + 2f_A(K^2 - H^2) - 4f_B H K \]
\[ - 2xf_C(K' H - KH') + 2x^2(H'^2 + K'^2) + \epsilon x^2(H^2 + K^2 - 1)^2 \],

where
\[ \epsilon = \frac{4\lambda}{g^2} = \frac{1}{2} \left( \frac{M_H}{M_W} \right)^2, \]

and the fermionic part
\[ E_f = \frac{4\pi M_W}{g^2} \int_0^\infty dx x^2 \left[ \left( G^2_R + F^2_R \right) - \left( G^2_L + F^2_L \right) \right] - 2 \frac{1-f_A}{x} G_L F_L + \frac{f_B}{x} (G^2_L - F^2_L) + \frac{f_C}{2x} (G^2_L + F^2_L) \]
\[ + 2\tilde{M}_F H (G_R G_L - F_R F_L) - 2\tilde{M}_F K (F_R G_L + F_L G_R) \].

We have introduced the dimensionless coordinate
\[ x = M_W r \]

and expressed the fermion mass in units of \( M_W \)
\[ \tilde{M}_F = M_F/M_W \]

The energy functional is invariant under the U(1) gauge transformation eq.(18).

The fermion functions entering the energy functional need to be normalized. When \( N \) fermions occupy the eigenstate eqs.(15)-(16) the normalization condition is
\[ 4\pi \int_0^\infty dx x^2(G^2_R + F^2_R + G^2_L + F^2_L) = N. \]

To construct the minimal energy path over the sphaleron barrier, we then consider the functional
\[ W = E - \omega N + \frac{8\pi^2 M_W}{g^2} N \xi N_{CS}, \]

where \( \omega \) is the fermion eigenvalue and \( \xi \) is a dimensionless lagrange multiplier [14]. The Chern-Simons number of a given configuration is
\[ N_{CS} = \frac{1}{2\pi} \int_0^\infty dx \left[ \left( f_A^2 + f_B^2 \right) \left( \frac{f_C}{x} - \Theta' \right) - \left( \frac{f_C}{x} - \Theta' \right) - \left( \sqrt{f_A^2 + f_B^2} \sin(\theta - \Theta) \right)' \right], \]

where we have defined the function \( \theta(x) = \arctan(f_B/f_A) \). The function \( \Theta(x) \) is an arbitrary radial function, associated with a general U(1) gauge transformation (18). From the expression (28) the Chern-Simons number is readily obtained in an arbitrary gauge, and in particular in the gauge, where the identification with the topological baryon number can be made.
3.3 Gauge choice

We perform the calculations in the radial gauge, where \( f_C = 0 \). In this gauge the spatial part of the Chern-Simons current contributes to the topological baryon number. We therefore rotate to the unitary gauge, where only the Chern-Simons number determines the topological baryon number. The corresponding gauge transformation involves the function \( \Theta(x) \), which satisfies \( \Theta(0) = 0 \) and \( \Theta(\infty) = \theta(\infty) \). We then find for the Chern-Simons number the expression

\[
N_{CS} = \frac{1}{2\pi} \int_0^\infty dx (f_B f'_A - f_A f'_B) + \frac{\theta(\infty)}{2\pi} . \tag{29}
\]

3.4 Equations

Variation of the functional eq.(27) leads to the following set of equations for the bosons

\[
f'^{''}_A = \frac{f_A}{x^2} (f_A^2 + f_B^2 - 1) + f_A (K^2 + H^2) + K^2 - H^2 + g^2 x F_L G_L - \xi f'_B , \tag{30}
\]

\[
f'^{''}_B = \frac{f_B}{x^2} (f_A^2 + f_B^2 - 1) + f_B (K^2 + H^2) - 2HK + \frac{1}{2} q^2 x (G_L^2 - F_L^2) + \xi f'_A , \tag{31}
\]

\[
H'' = -\frac{2}{x} H' + \frac{H}{2x^2} \left((1 - f_A)^2 + f_B^2\right) - \frac{K}{x^2} f_B + \epsilon (H^2 + K^2 - 1) H
+ \frac{g^2 \tilde{M}_F}{2} (G_R G_L - F_R F_L) , \tag{32}
\]

\[
K'' = -\frac{2}{x} K' + \frac{K}{2x^2} \left((1 + f_A)^2 + f_B^2\right) - \frac{H}{x^2} f_B + \epsilon (H^2 + K^2 - 1) K
- \frac{g^2 \tilde{M}_F}{2} (F_R G_L + F_L G_R) , \tag{33}
\]

and for the fermions

\[
\tilde{\omega} G_L - F'_L = -\frac{2}{x} F_L + \frac{1 - f_A}{x} F_L - \frac{f_B}{x} G_L + \tilde{M}_F (-HG_R + KF_R) = 0 , \tag{34}
\]

\[
\tilde{\omega} F_L + G'_L = \frac{1 - f_A}{x} G_L + \frac{f_B}{x} F_L + \tilde{M}_F (HF_R + KG_R) = 0 , \tag{35}
\]

\[
-\tilde{\omega} G_R + F'_R = -\frac{2}{x} G_R + \tilde{M}_F (HG_L - KF_L) = 0 , \tag{36}
\]

\[
\tilde{\omega} F_R + G'_R + \tilde{M}_F (HF_L + KG_L) = 0 . \tag{37}
\]
Here $\tilde{\omega}$ is the fermion eigenvalue $\omega$ in units of $M_W$

$$\tilde{\omega} = \frac{\omega}{M_W}.$$ (38)

For $\xi = 0$, finite energy solutions of these equations correspond to extrema of the Weinberg-Salam theory, such as the sphaleron or the non-topological soliton. The above set of equations can then also be obtained by substituting the ansatz eqs. (12)-(17) into the general equations of motion.

### 3.5 Boundary conditions

The boundary conditions are chosen such that the energy density and the energy both are finite.

At the origin the boson functions satisfy the boundary conditions

$$f_A(0) - 1 = f_B(0) = H'(0) = K(0) = 0,$$ (39)

while the fermion functions satisfy

$$F_R(0) = F_L(0) = 0,$$ (40)

and

$$G_R(0) = c_R, \quad G_L(0) = c_L,$$ (41)

where $c_R$, $c_L$ are unknown constants, subject to the normalization condition (26).

At infinity the gauge and Higgs field functions lie on a circle

$$f_A(\infty) + if_B(\infty) = \exp(i\theta(\infty)), \quad H(\infty) + iK(\infty) = \exp(i\frac{\theta(\infty)}{2}),$$ (42)

and $\theta(\infty)$ is an unknown function of $\xi$. Therefore we choose the boundary conditions

$$f'_A(\infty) = f'_B(\infty) = H'(\infty) = K'(\infty) = 0$$ (43)

for the boson functions. The fermion functions all vanish at infinity

$$F_R(\infty) = F_L(\infty) = G_R(\infty) = G_L(\infty) = 0.$$ (44)

### 4 Minimal Energy Path

We construct the minimal energy path from one vacuum sector with $N_{CS} = 0$ to the neighbouring vacuum sector with $N_{CS} = 1$ by solving the equations of motion (30)-(37), varying the Chern-Simons number along the path by means of the lagrange multiplier $\xi$. For convenience we start at the top of the barrier, at the sphaleron solution, where $\xi = 0$, and then vary the lagrange multiplier $\xi$ to positive and to negative values to follow the path down to the vacua on both sides of the barrier.
4.1 Sphaleron

The sphaleron represents the configuration at the top of the barrier between topologically distinct vacua. It is a solution of the classical equations of motion of the Weinberg-Salam theory with one unstable mode. Previously the sphaleron solution was obtained only in the bosonic sector of the Weinberg-Salam theory, neglecting the fermions. Here we investigate the influence which the presence of fermions has on the sphaleron solution.

Without fermions, the sphaleron (for $\theta_w = 0$) has spherical symmetry and parity reflection symmetry. (We do not consider here the bisphalerons [23-24], where parity reflection symmetry is broken, and which arise at large values of the Higgs mass.) Parity reflection symmetry simplifies the ansatz for the bosonic fields of the sphaleron eqs.(12)-(14). Two of the three gauge field functions, namely $f_B$ and $f_C$, and one of the Higgs field functions, $H$, vanish in the commonly used ansatz for the sphaleron. The sphaleron then has a Chern-Simons number of precisely $1/2$ [5].

Fermions have a zero mode in the background field of the sphaleron. For zero mass fermions this zero mode is known analytically in terms of the sphaleron functions [11-13]. Recently this zero mode has been constructed explicitly for the case, where the fermions of a weak doublet are degenerate in mass [16]. In this case two of the four fermion functions, namely $F_R$ and $F_L$, vanish.

In a self-consistent treatment the presence of fermions affects the bosonic fields through the coupled equations of motion. The equation for the parity violating gauge field function $f_B$ now contains a source term from the fermions, which does not vanish, even if $F_L$ were zero. Similarly the equation for the previously vanishing Higgs field function $H$ now contains a non-vanishing source term from the fermions (except for zero mass fermions, where $G_R = 0$). Thus the presence of fermions violates the parity reflection symmetry of the sphaleron. The previously vanishing parity violating functions now have source terms from the fermions and can therefore no longer vanish.

As a consequence the fermions couple to a less symmetric bosonic configuration and therefore also the previously vanishing fermion functions $F_R$ and $F_L$ now have non-vanishing source terms, even if the fermion eigenvalue would still be zero. The self-consistent sphaleron solution thus involves eight radial functions, four for the bosons and four for the fermions. With the parity reflection symmetry of the sphaleron lost, the barrier no longer needs to be symmetric with respect to the Chern-Simons number $N_{CS} = 1/2$. Therefore the Chern-Simons number of the sphaleron at the top of the barrier may deviate from $1/2$, and the eigenvalue of the fermions at the sphaleron solution may deviate from zero.

In Fig. 1 we show the boson functions for the sphaleron for the Higgs mass $M_H = M_W$ and for the fermion masses $M_F = 130$ GeV and $M_F = 1.3$ TeV for one bound fermion. We see, that even at a fermion mass of $M_F = 130$ GeV, corresponding to a mass on the order of the expected top quark mass, the functions $f_B$ and $H$ are
still almost vanishing, and the deviation of the functions $f_A$ and $K$ from those of the purely bosonic sphaleron is with only about 0.1% even an order of magnitude smaller. In contrast, for heavy fermions with masses in the TeV region $f_B$ and $H$ differ considerably from zero, while the deviation of the functions $f_A$ and $K$ from those of the purely bosonic sphaleron is still small.

In Fig. 2 we show the corresponding fermion functions. At the fermion mass of $M_F = 130 \text{ GeV}$ the functions $F_R$ and $F_L$ are still almost vanishing in the self-consistent calculation, they are smaller than $10^{-3}$, while the functions $G_R$ and $G_L$ are almost identical to those of the background field calculation. For heavy fermions with masses in the TeV region the functions $F_R$ and $F_L$ differ markedly from zero.

In Fig. 3 we show the energy of the sphaleron for three values of the Higgs mass, $M_H = 50, 80, 100 \text{ GeV}$, as a function of the fermion mass, for one and three bound fermions, respectively. For massless fermions the energy of the self-consistent sphaleron is almost unaltered compared to the purely bosonic sphaleron. For $M_H = M_W$ it increases by 0.0003 % (0.0034 %) for one (three) bound fermion(s). With increasing fermion mass the sphaleron energy first increases slightly until about $M_F = 120 \text{ GeV}$ and then decreases for larger values of the fermion mass. For a fermion mass on the order of the top quark mass, $M_F = 130 \text{ GeV}$, the sphaleron energy decreases by only 0.003 % (0.028) with respect to zero mass fermions for $M_H = M_W$. While for fermion masses approaching the TeV region, the sphaleron energy decreases more strongly, e. g. for a fermion mass of 1.3 TeV the energy changes by 1.8 % (12.5 %) for one (three) bound fermion(s) and $M_H = M_W$.

Fig. 4 shows the corresponding energy in the boson fields for the sphaleron as a function of the fermion mass. Here we see an increase in energy for large fermion masses. The decrease of the total energy for large values of the fermions mass must thus be due to the fermion eigenvalue, which must decrease considerably for large values of the fermion mass. The fermion eigenvalue is shown in Fig. 5, where we observe indeed a strong decrease for heavy fermions. Thus the fermion eigenvalue of the self-consistent sphaleron deviates strongly from zero for heavy fermions.

For light and intermediate fermions the eigenvalue is shown in Fig. 6. Here we observe small positive values of the eigenvalue. At a fermion mass of zero the eigenvalue starts at 0.07 GeV (0.2 GeV) for one (three) bound fermion(s), rises to a maximum of 0.6 GeV (1.8 GeV) for a fermion mass on the order of the top quark mass, and then declines, passing zero for a fermion mass of about 0.2 TeV in both cases for $M_H = M_W$.

In Fig. 7 the Chern-Simons number of the sphaleron solution is shown for the same set of Higgs masses and for one (three) bound fermion(s) as a function of the fermion mass. For zero mass fermions the Chern-Simons number approaches 0.499 (0.497) for one (three) bound fermion(s). For fermions with mass $M_F = 130 \text{ GeV}$ the corresponding Chern-Simons number is 0.495 (0.485), while for heavy fermions with mass $M_F = 1.3 \text{ TeV}$ the Chern-Simons number changes by 10 % (31 %) for $M_H = M_W$. 
Thus the Chern-Simons number differs appreciably from the value $N_{CS} = 1/2$ only for heavy fermions.

### 4.2 Barrier

When the barrier is calculated self-consistently in the presence of fermions the barrier is no longer symmetric with respect to $N_{CS} = 1/2$. The minimal energy path from the sphaleron down to the vacuum sector with $N_{CS} = 0$ is shorter and steeper than the path down to the other side to the vacuum sector with $N_{CS} = 1$.

The minimal energy path is shown in Fig. 8 for $M_H = M_W$ for one light fermion with $M_F = 1/10M_W$ and for one (three) heavy fermion(s) with $M_F = 10M_W$. For the light fermion the path agrees well with the purely bosonic path \[14\]. The deviation is very small at the top and increases slightly along the path towards both vacuum sectors. There the positive mass of the fermions is approached for $N_{CS} \to 0$ and the negative mass for $N_{CS} \to 1$, because there is now a free fermion present with positive or negative energy.

For large values of the fermion mass, the barrier is altered considerably. With increasing fermion mass the top of the barrier moves to smaller values of the Chern-Simons number, making the barrier increasingly asymmetric. Beyond a fermion mass of about 0.7 GeV for one bound fermion the minimal energy path heading towards the $N_{CS} = 0$ sector does not reach the bosonic vacuum configuration with one (three) free fermion(s) any more. Instead a new bound state is reached, a non-topological soliton \[19-21\], discussed below. The minimal energy path down to the other side of the sphaleron towards the $N_{CS} = 1$ sector does reach a bosonic vacuum configuration with one (three) free fermion(s) with negative energy.

In Fig. 9 we show the dependence of the Chern-Simons number on the lagrange multiplier $\xi$ along the minimal energy path for $M_H = M_W$, for one light fermion with $M_F = 1/10M_W$ and for one (three) heavy fermion(s) with $M_F = 10M_W$. For small fermion masses the relation is monotonic and almost linear. While $N_{CS}$ varies from zero to one, the lagrange multiplier $\xi$ changes from -2 to +2. In contrast for heavy fermions, where non-topological soliton solutions exist, $\xi$ reaches a minimum close to -2, and then rapidly increases again, approaching zero at the non-topological soliton solution as shown in Fig. 10. (Note, that the Chern-Simons number of the non-topological soliton is nonzero.)

In Fig. 11 we show the fermion eigenvalue $\omega$ along the minimal energy path in the self-consistent calculation for $M_H = M_W$ for one light fermion with $M_F = 1/10M_W$ for one intermediate fermion with $M_F = M_W$, and for one (three) heavy fermion(s) with $M_F = 10M_W$. Comparing the eigenvalues obtained in the self-consistent calculation with those of the background field calculation \[16\] we see little change for light and intermediate fermions. The eigenvalue still looks (almost) antisymmetric with respect
to $N_{CS} = 1/2$. In contrast, for heavy fermions the self-consistent minimal energy path starts from the non-topological soliton and thus with an eigenvalue $\omega < M_F$. The eigenvalue decreases monotonically with increasing Chern-Simons number and reaches zero long before the sphaleron is reached (e.g. for $M_F = 0.8$ TeV the fermion mode has eigenvalue zero at $N_{CS} = 0.4354$, while the sphaleron is reached at $N_{CS} = 0.4700$ for $M_H = M_W$). Then the eigenvalue gradually approaches the negative mass, $\omega \to -M_F$.

There exist critical values of the fermion mass where, at a given value of the Chern-Simons number $N_{CS}$, the bound state enters the positive or the negative continuum. For light and intermediate mass fermions the critical values of the self-consistent calculation almost coincide with those of the background field calculation [16]. For instance for $M_F = 80$ GeV the bound state enters the positive continuum at $N_{CS} = 0.0773$ and the negative continuum at $N_{CS} = 0.9216$ in the self-consistent calculation for one fermion, and at $N_{CS} = 0.0748$ and $N_{CS} = 0.9252$ in the background field calculation [16]. For heavy fermions the self-consistent solution reaches the non-topological soliton in the $N_{CS} = 0$ sector, leading to a very different behaviour for $N_{CS} \to 0$.

In Fig. 12 we show the value of the Chern-Simons number where, along the minimal energy path, the fermions have eigenvalue zero for $M_H = M_W$ and one (three) bound fermion(s). For massless fermions the eigenvalue is zero precisely at a Chern-Simons number of $1/2$. (The contributions to $N_{CS}$ from the integral and from the angle $\theta(\infty)$ in eq.(29) cancel to better than $10^{-5}$.) The Chern-Simons number of the zero mode then decreases with increasing fermion mass rapidly towards zero. The end configurations correspond to non-topological solitons with zero energy eigenstates, which are reached at $M_F = 2.416$ TeV (1.384 TeV) with $N_{CS} = 0.00038$ (0.00038) for one (three) bound fermion(s).

Let us now compare the self-consistent path with the path obtained in [19]. For very heavy fermions the self-consistent minimal energy path no longer decreases monotonically from the sphaleron towards the vacua. Instead a new critical behaviour appears for fermion masses around 4 TeV. The path down to the vacuum sector $N_{CS} = 0$ turns backward at a first critical point and forward again at a second critical point. This is illustrated in Fig. 13 for the fermion masses $M_F = 4$ TeV, $M_F = 6$ TeV, and $M_F = 10$ TeV for $M_H = M_W$. In Fig. 14 the corresponding dependence of the lagrange multiplier $\xi$ on the Chern-Simons number $N_{CS}$ is shown. While for $M_F = 4$ TeV and $M_F = 6$ TeV there is only one configuration with $\xi = 0$, there are three such configurations for $M_F = 10$ TeV. Since all configurations with $\xi = 0$ correspond to solutions of the classical field equations of the (simplified) Weinberg-Salam theory, there are additional unstable solutions for these high fermions masses, new sphalerons.

In [19] a restricted variational ansatz is used for the boson functions, which approach the vacuum configuration on both sides of the barrier, while the fermion functions are obtained perturbatively (in $f(q)/g$), and the fermion energy approaches the positive and negative free mass. Thus this variational calculation does not approach the non-
topological soliton. Not surprisingly this variational ansatz leads to a very different
dependence of the barrier on the fermion mass. When the fermion mass runs across
the TeV region critical values of the fermion mass are encountered, where a new local
energy minimum first appears and then disappears again along path [19]. The new
local minimum is associated with a new kind of soliton in [19], different from the non-topological soliton [19-21], and existing only for very heavy fermions with 9.0 TeV
\[ M_F \leq 12.4 \text{ TeV} \] for \[ M_H = M_W. \]

There are several reasons to doubt the argument for the existence of a new soliton,
given in [19]. The calculation of [19] is only variational and therefore gives only an upper
bound on the energy. Our calculation shows, that no such new smooth minimum ap-
pears along the self-consistent path, although a different critical behaviour does appear
for fermion masses around 4 TeV. Furthermore, for fermion masses in the critical range
9.0 TeV \[ M_F \leq 12.4 \text{ TeV} \] the non-topological soliton has a negative energy, indicating
the breakdown of the valence fermion approximation [20].

4.3 Non-topological soliton

We have seen, that for heavy fermions one encounters a non-topological soliton along
the minimal energy path when approaching the vacuum sector with \[ N_{\text{CS}} = 0. \] We now
take a closer look at this solution of the Weinberg-Salam model, which represents a
minimum and not a saddle point like the sphaleron.

Since the gauge field has little influence on the non-topological soliton [19], we first
consider a simpler model by setting the gauge field equal to zero. In this case the
equations for the left-handed and for the right-handed fermions are equivalent and we
are essentially left with a chiral linear \( \sigma \)-model, scaled up in energy since the vacuum
expectation value of the Higgs field is about 25 times bigger than \( f_\pi \).

In Fig. 15 we show the total energy of the \( \sigma \)-model non-topological soliton as a
function of the fermion mass for one bound fermion and for several values of the Higgs
mass ranging from 50 GeV to 5 TeV. We see, that for a given Higgs mass, there exists
a critical value of the fermion mass above which the fermion can form a stable bound
state, the non-topological soliton, which is lower in energy than the mass of one free
fermion. These results agree with those of [19], where a direct minimization technique
was used.

For a given fermion mass the energy of the non-topological soliton increases monotonically with the Higgs mass. Thus the energy is clearly bounded from above by the
energy obtained in the non-linear \( \sigma \)-model, where the Higgs field is confined to the
chiral circle [19-21]. The energy of the solitons becomes negative for fermion masses
in the range \( 3.6 \leq M_F \leq 3.8 \) TeV for the Higgs masses considered. This behaviour is
largely due to the valence fermion approximation, since taking into account the effects
of the Dirac sea leads to a qualitatively different behaviour [20-21].
We observe that the soliton branches approach the free fermion branch in two distinct ways, depending on the value of the Higgs mass. For low Higgs masses the energy along the soliton branch is always lower than the corresponding free energy, and the soliton branch simply bifurcates from the free fermion branch at a critical value of the fermion mass. Above a critical Higgs mass of about $M_H = 0.7$ TeV for one bound fermion, however, the behaviour is different. Here the soliton branch first crosses the free branch, then reaches a cusp at a critical value of the fermion mass, where a new (unstable) branch emerges. The new branch then approaches the free branch from above. This latter behaviour is well known for non-topological solitons [25].

The different critical behaviour of the soliton branches at various Higgs masses is also seen in Fig. 16. There we show the fermion eigenvalue as a function of the fermion mass for the same set of values of the Higgs mass. When for large values of the Higgs mass the energy branch approaches the cusp, the fermion eigenvalue exhibits an infinite slope.

In Fig. 17 we show the radial functions for the non-topological soliton for $M_H = M_W$ and $M_F = 2.6$ TeV for one bound fermion. We see, that the fermions are localized in a small region of space, while the Higgs field approaches its vacuum expectation value, where $H(x) \to 1$ and $K(x) \to 0$, much more slowly.

Finally we consider the effect of the presence of the gauge field on the non-topological solitons in the Weinberg-Salam model. In Fig. 18 we show the gauge field functions of the non-topological soliton for the physical gauge coupling $g = 0.65$ for $M_H = M_W$ and $M_F = 1.3$ TeV and $M_F = 2.6$ TeV for one bound fermion. They differ only little from zero.

The change in energy of the non-topological soliton due to the presence of the gauge field is smaller than 0.8 % for $M_F = 2.6$ TeV and for $M_H = M_W$. The deviation decreases for smaller fermion masses. The non-topological solitons possess a small, but finite Chern-Simons number, $N_{CS} \leq 0.0004$ for $M_F \leq 2.6$ TeV and $M_H = M_W$.

5 Conclusions

In the Weinberg-Salam theory topologically distinct vacuum sectors are separated by a barrier, whose height is determined by the sphaleron energy. Here we have constructed the minimal energy path over the barrier from one vacuum sector to the neighbouring one self-consistently in the valence fermion approximation. To retain spherical symmetry we have neglected the weak mixing angle and assumed that the fermions of a doublet are degenerate in mass. While the first approximation is very good [17,18], the latter is badly broken and needs to be improved.

The presence of the fermions affects the energy and the shape of the barrier. The sphaleron solution on top of the barrier loses its parity reflection symmetry due to the coupling to the fermions. It carries no longer exactly Chern-Simons number $N_{CS} = 1/2$, ...
and the fermion eigenvalue at the top of the barrier is no longer precisely zero. The deviations of the self-consistent sphaleron from the purely bosonic sphaleron depend on the mass and on the number of bound fermions. The presence of massless fermions hardly affects the sphaleron, while fermions with masses on the order of the top quark mass have a small but noticeable effect. The Chern-Simons number is decreased by about one percent from \( N_{\text{CS}} = 1/2 \), and the fermion eigenvalue differs slightly from zero, being on the order of one percent of the fermion mass. Though the height of the barrier is still insensitive in this mass range. Due to these small changes the barrier is very slightly tilted to the left, becoming asymmetric with respect to the top. The fermion eigenvalue along the self-consistent path differs little from the one of the background field calculation [16] for light and intermediate mass (top quark) fermions. We conclude that in this mass range the sphaleron and the barrier are well approximated by neglecting the fermions, and that the fermion eigenvalues are obtained with rather good accuracy in the background field calculation, as compared to the self-consistent valence fermion approximation. Clearly other approximations, such as neglecting the effect of the Dirac sea, still need to be investigated.

For heavy fermions with masses approaching the TeV region the barrier deforms stronger. The height of the barrier then decreases monotonically with increasing fermion mass, and decreases the stronger the more fermions occupy the valence level. Likewise the Chern-Simons number of the sphaleron and the fermion eigenvalue at the sphaleron decrease.

Heavy fermions with masses greater than about 0.7 TeV can form non-topological solitons in the vacuum sector. For such heavy fermions the self-consistent minimal energy path over the barrier does not end in the vacuum configuration but instead reaches the non-topological soliton as its end configuration. This is in contrast to the variational approximate path considered by Petriashvili [19], which for any value of the fermion mass ends in a vacuum with a free fermion. Due to this choice of end configuration Petriashvili finds a new minimum along the path for very heavy fermions, and therefore conjectures the existence of a new type of soliton [19]. Our self-consistent calculations also show a new critical behaviour for very heavy fermions. But we do not observe a new smooth minimum along the path. Instead we see the occurrence of two bifurcations for very heavy fermions. The minimal energy path then encounters two critical points. It winds backward at the first critical point, and forward again at the second critical point. This critical behaviour allows for the existence of additional unstable solutions, new sphalerons. Our self-consistent calculation does not support the conjecture, that a new type of soliton is present for very heavy fermions.

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6 Figure Captions

Figure 1: The gauge field functions $f_A$ and $f_B$ and Higgs field functions $H$ and $K$ of the sphaleron are shown with respect to the dimensionless variable $x$ for $M_H = M_W$ and for $M_F = 130$ GeV (solid) and $M_F = 1.3$ TeV (dotted).

Figure 2: The righthanded and lefthanded fermion functions $G$ and $F$ of the sphaleron are shown with respect to the dimensionless variable $x$ for $M_H = M_W$ and for $M_F = 130$ GeV (solid and dot-dashed) and $M_F = 1.3$ TeV (dotted and dashed).

Figure 3: The total energy (in TeV) of the sphaleron is shown as a function of the fermion mass (in TeV) for three values of the Higgs mass, $M_H = 50$ GeV, $M_H = 80$ GeV, and $M_H = 100$ GeV, for one fermion (solid) and three fermions (dotted).

Figure 4: The boson energy (in TeV) of the sphaleron is shown as a function of the fermion mass (in TeV) for three values of the Higgs mass, $M_H = 50$ GeV, $M_H = 80$ GeV, and $M_H = 100$ GeV, for one fermion (solid) and three fermions (dotted).

Figure 5: The fermion eigenvalue (in TeV) of the sphaleron is shown as a function of the fermion mass (in TeV) for three values of the Higgs mass, $M_H = 50$ GeV (solid), $M_H = 80$ GeV (dashed), and $M_H = 100$ GeV (dot-dashed), for one fermion and three fermions.

Figure 6: The fermion eigenvalue (in GeV) of the sphaleron is shown as a function of the fermion mass (in TeV) for light and intermediate mass fermions for three values of the Higgs mass, $M_H = 50$ GeV (solid), $M_H = 80$ GeV (dashed), and $M_H = 100$ GeV (dot-dashed), for one fermion and three fermions.

Figure 7: The Chern-Simons number of the sphaleron is shown as a function of the Chern-Simons number for $M_H = M_W$ for one fermion with $M_F = 1/10M_W$ and $M_F = 10M_W$ (solid) and for three fermions with $M_F = 10M_W$ (dotted).

Figure 9: The lagrange parameter $\xi$ along the minimal energy path is shown as a function of the Chern-Simons number for $M_H = M_W$ for one fermion with $M_F = 1/10M_W$ and $M_F = 10M_W$ (solid) and for three fermions with $M_F = 10M_W$ (dotted).

Figure 10: The lagrange parameter $\xi$ along the minimal energy path is shown in the vicinity of the non-topological soliton as a function of the Chern-Simons number for $M_H = M_W$ for one (solid) and three (dotted) fermions.

Figure 11: The eigenvalue (in units of $M_F$) along the minimal energy path is shown as a function of the Chern-Simons number for $M_H = M_W$ for one fermion with $M_F = 1/10M_W$, $M_F = M_W$ and $M_F = 10M_W$ (solid) and for three fermions with $M_F = 10M_W$ (dotted).
Figure 12: The Chern-Simons number of the zero mode along the minimal energy path is shown as a function of the fermion mass (in TeV) for $M_H = M_W$ for one fermion and three fermions.

Figure 13: The energy (in TeV) along the minimal energy path is shown as a function of the Chern-Simons number for $M_H = M_W$ for one heavy fermion with $M_F = 4$ TeV, $M_F = 6$ TeV, and $M_F = 10$ TeV. (The dotted parts are extrapolated.)

Figure 14: The lagrange multiplier $\xi$ along the minimal energy path is shown as a function of the Chern-Simons number for $M_H = M_W$ for one heavy fermion with $M_F = 4$ TeV $M_F = 6$ TeV, and $M_F = 10$ TeV. (The dotted parts contain non-negligible numerical errors.) The curves cross zero at sphaleron solutions.

Figure 15: The energy (in TeV) of the non-topological soliton for vanishing gauge coupling constant is shown as a function of the fermion mass (in TeV) for $M_H = 80$ GeV (dot-dashed), $M_H = 0.5$ TeV (dashed), and $M_H = 1$ TeV (dotted) and $M_H = 5$ TeV (solid) for one bound fermion.

Figure 16: The fermion eigenvalue (in TeV) of the non-topological soliton is shown as a function of the fermion mass (in TeV) for $M_H = 80$ GeV (dot-dashed), $M_H = 0.5$ TeV (dashed), and $M_H = 1$ TeV (dotted) and $M_H = 5$ TeV (solid) for one bound fermion for vanishing gauge coupling constant.

Figure 17: The fermion field functions $G$ (solid) and $F$ (dotted) and Higgs field functions $H$ (dashed) and $K$ (dot-dashed) of the non-topological soliton are shown with respect to the dimensionless variable $x$ for $M_H = M_W$ and for $M_F = 2.6$ TeV for vanishing gauge coupling constant.

Figure 18: The gauge field functions $1 - f_A$ and $f_B$ of the non-topological soliton are shown with respect to the dimensionless variable $x$ for $M_H = M_W$ and for $M_F = 2.6$ TeV (solid) and $M_F = 1.3$ TeV (dotted) for the physical value of the gauge coupling.