STRUCTURE OF THE UNITARY VALUATION ALGEBRA

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ABSTRACT. S. Alesker has shown that if $G$ is a compact subgroup of $O(n)$ acting transitively on the unit sphere $S^{n-1}$ then the vector space $\text{Val}^G$ of continuous, translation-invariant, $G$-invariant convex valuations on $\mathbb{R}^n$ has the structure of a finite dimensional graded algebra over $\mathbb{R}$ satisfying Poincaré duality. We show that the kinematic formulas for $G$ are determined by the product pairing. Using this result we then show that the algebra $\text{Val}^U$ is isomorphic to $\mathbb{R}[s,t]/(f_n+1, f_{n+2})$, where $s,t$ have degrees 2 and 1 respectively, and the polynomial $f_i$ is the degree $i$ term of the power series $\log(1 + s + t)$.

1. Introduction

In [6], Hadwiger showed that the vector space $\text{Val}^{SO(n)}$ of continuous convex valuations on $\mathbb{R}^n$ invariant under the group $SO(n)$ of orientation-preserving isometries has dimension $n+1$, with a basis consisting of the Minkowski “Quermassintegrals”, or intrinsic volumes in the terminology of [11]. An immediate consequence is the following form of the Principal Kinematic Formula of Blaschke: if $\Phi_0, \ldots, \Phi_n$ are the intrinsic volumes, indexed by degree, then there exist constants $c_{ij}^k$ such that

$$\int_{SO(n)} \Phi_k(A \cap \bar{g}B) \, d\bar{g} = \sum_{i+j=n+k} c_{ij}^k \Phi_i(A) \Phi_j(B)$$

for all compact convex bodies $A, B \subset \mathbb{R}^n$ (cf. also [12]). By applying the formula to appropriate lists of bodies $A, B$, one may then determine the constants $c_{ij}^k$ by explicit calculations of the integral. Though not essentially difficult, this procedure can be a bit troublesome, with many opportunities for computational errors; however, Nijenhuis [9] showed that if the basis $\{\Phi_i\}_{i=0}^n$ and the Haar measure $d\bar{g}$ are normalized appropriately then all of the $c_{ij}^k$ are equal to unity. He speculated that there might exist some underlying algebraic structure that would explain this fact.

More recently, in a series of fundamental papers [1], [2], [3] S. Alesker has shown that if $G$ is a compact subgroup of the orthogonal group $O(n)$ acting transitively on the unit sphere $S^{n-1}$, then the vector space $\text{Val}^G$ of $G$-invariant translation-invariant continuous valuations carries the structure of a finite-dimensional commutative graded algebra over $\mathbb{R}$. Furthermore the resulting algebra satisfies Poincaré duality: the top degree piece of $\text{Val}^G$ is one-dimensional and occurs in degree $n$, and the pairing $\langle a, b \rangle :=$ the degree $n$ component of $ab$ is perfect. One of the results of the present article is to show that this algebra structure satisfies Nijenhuis’s speculation, reducing the results of [9], which originally appeared to be

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a kind of miracle obtained by laborious calculations, to an obvious triviality based on the simple structure of the algebra $\text{Val}^{SO(n)}$.

Thus the case $G = SO(n)$ should be viewed as the ground case of a more subtle general theory that remains to be worked out in detail. The first serious case is the case $G = U(n)$. The main result of this paper is the explicit determination of the structure of $\text{Val}^{U(n)}$ (Thm. 3.1 below):

**Main Theorem.** The graded $\mathbb{R}$-algebra $\text{Val}^{U(n)}$ is isomorphic to $\mathbb{R}[s,t]/(f_{n+1}, f_{n+2})$, where the generators $s,t$ have degrees 2 and 1 respectively, and $f_j$ is the degree $j$ component of the power series $\log(1 + s + t)$.

Because of Thm. 2.6 below, this result determines in principle the kinematic formulas for all of the $U(n)$-invariant valuations. Nevertheless the problem of writing them down explicitly remains open. In fact this is only one of several open problems arising from a comparison between the $U(n)$ and the $SO(n)$ theories, which we discuss these in the closing section.

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## 2. General results

Throughout this section we let $V$ be a vector space over $\mathbb{R}$ of dimension $n < \infty$, endowed with a euclidean structure. Let $O(V)$ denote the corresponding orthogonal group, and fix a compact subgroup $G \subset O(V)$ that acts transitively on the unit sphere of $V$. Put $K(V)$ to be the space of all compact convex subsets of $V$, endowed with the Hausdorff metric. If $r \in \mathbb{R}$, $x \in V$ and $K \in K(V)$ then we put

$$x + K := \{x + p : p \in K\},$$

$$rK := \{rp : p \in K\}.$$

Denote by $\text{Val}^G(V)$ or simply $(\text{Val}^G)$ the vector space of continuous functions $\phi : K(V) \to \mathbb{R}$ enjoying the properties

- finite additivity: if $K, L, K \cup L \in K(V)$ then
  $$\phi(K \cup L) = \phi(K) + \phi(L) - \phi(K \cap L);$$
- translation-invariance: if $x \in V$ and $K \in K(V)$ then $\phi(x + K) = \phi(K);$  
- $G$-invariance: if $g \in G$ and $K \in K(V)$ then
  $$g\phi(K) := \phi(g^{-1}K) = \phi(K).$$
We put

\[ \omega_k := \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k+2}{2}\right)} \]

for the volume of the unit ball in \( \mathbb{R}^k \).

2.1. \textbf{Val}^G as an algebra. We begin by listing some basic properties of \textbf{Val}^G. Put \( \overline{G} := G \ltimes V \), equipped with a bi-invariant Haar measure. We leave the choice of normalizing constant undetermined for the moment.

Recall that a valuation \( \phi \) is said to have degree \( i \) if \( \phi(rK) = r^i \phi(K) \) for all \( r \geq 0 \). Put \( \text{Val}^G_i \) for the subspace of degree \( i \) elements in \( \text{Val}^G \). The first fact is a consequence of a theorem of P. McMullen [8].

\textbf{Theorem 2.1.}

\[ \text{Val}^G(V) = \bigoplus_{i=0}^{n} \text{Val}^G_i(V). \]

Furthermore \( \text{Val}^G_0(V) \) is one-dimensional, and is spanned by the volume.

The results listed next are simple consequences of results of Alesker ([1],[2],[3]). In fact we give them only in a restricted form sufficient for the purposes of the present paper.

\textbf{Theorem 2.2.}

\begin{itemize}
  \item \( \dim_{\mathbb{R}} \text{Val}^G < \infty. \)
  \item Given \( K \in \mathcal{K}(V) \), define \( \mu^G_K \in \text{Val}^G \) by
    \begin{align*}
      \mu^G_K(L) &:= \int_G \text{volume} \left( L - gK \right) dg \\
      &= \int_G \chi(L \cap \bar{g}K) d\bar{g}.
    \end{align*}
    Then there are \( K_1, \ldots, K_N \in \mathcal{K}(V) \) such that \( \text{Val}^G \) is spanned by \( \mu^G_{K_1}, \ldots, \mu^G_{K_N} \).
  \item There is a natural continuous multiplication on the space of all continuous translation-invariant valuations that restricts to a multiplication on \( \text{Val}^G \), given as follows: if \( \phi \in \text{Val}^G \) and \( K, L \in \mathcal{K}(V) \), then
    \[ (\phi \cdot \mu^G_K)(L) := \int_G \phi(L \cap \bar{g}K) d\bar{g}. \]
    Extending by linearity, the resulting product endows \( \text{Val}^G \) with the structure of a commutative graded algebra over \( \mathbb{R} \), with unit element given by the Euler characteristic \( \chi \).
  \item Let \( W \subset V \) be a linear subspace, and let \( H \subset G \) be the stabilizer of \( W \). Suppose that \( H \) acts transitively on the unit sphere of \( W \). Then the natural restriction map \( \text{Val}^G(V) \to \text{Val}^H(W) \) is a homomorphism of \( \mathbb{R} \)-algebras.
  \item The pairing \( \text{Val}^G \otimes \text{Val}^G \to \text{Val}^G_n \simeq \mathbb{R} \) given by
    \[ \text{PD} : (a, b) \mapsto (ab)_n \]
    \((\text{degree } n \text{ piece of } ab) \) is perfect.
\end{itemize}
Thus the pairing (6) may be thought of as a self-adjoint map $\text{PD} : \text{Val}^G \to (\text{Val}^G)^*$. For the moment we leave unspecified the choice of linear isomorphism $\text{Val}^G_n \sim \mathbb{R}$, and $\text{PD}$ inherits this imprecision. It is trivial to see that $\text{PD}$ is a homomorphism of $\text{Val}^G$-modules, where the $\text{Val}^G$-module structure on the dual space $(\text{Val}^G)^*$ is

\begin{equation}
(a\alpha)(b) := \alpha(ab),
\end{equation}

$a, b \in \text{Val}^G, \alpha \in (\text{Val}^G)^*$. Since the pairing is perfect, in fact $\text{PD}$ is an isomorphism, and it is clearly graded in the sense that $\text{PD} : \text{Val}^G_i \rightarrow (\text{Val}^G_{n-i})^*$. In other words $\text{Val}^G$ carries the structure of a graded Frobenius algebra ([5]).

It is well known and trivial to prove:

**Lemma 2.3.** If $A$ is a finite-dimensional graded algebra over a field $k$, then any two graded $A$-module isomorphisms $A \rightarrow A^*$ differ by multiplication by a unit of $A$ of pure degree 0.

2.2. **Kinematic formulas.** Let $\mu_1, \ldots, \mu_N$ be a basis for $\text{Val}^G$. It is straightforward to deduce that given $\phi \in \text{Val}^G$ there are constants $c_{ij}^\phi \in \mathbb{R}$ such that for all $K, L \in \mathcal{K}(V)$

\begin{equation}
\int_G \phi(K \cap gL) \, dg = \sum_{i,j=1}^N c_{ij}^\phi \mu_i(K) \mu_j(L).
\end{equation}

This situation may be abbreviated by defining the map

\[ k_G : \text{Val}^G \rightarrow \text{Val}^G \otimes \text{Val}^G \simeq \text{Hom}_\mathbb{R}((\text{Val}^G)^*, \text{Val}^G) \]

by

\[ k_G(\phi) := \sum_{i,j} c_{ij}^\phi \mu_i \otimes \mu_j. \]

Note that the precise definition of $k_G$ depends on the choice of normalization for the Haar measure $dg$. For the time being we prefer to leave this unspecified. It is straightforward to check that $k_G$ is a coassociative, cocommutative coproduct. Noticing the similarity with the definition of the product, coassociativity is equivalent to

**Lemma 2.4.** If $\phi, \psi \in \text{Val}^G$ then

\[ k_G(\psi \cdot \phi) = \sum_{i,j} c_{ij}^\phi (\psi \cdot \mu_i) \otimes \mu_j = \sum_{i,j} c_{ij}^\phi \mu_i \otimes (\psi \cdot \mu_j). \]

\[ \square \]

**Proposition 2.5.** For every $\varphi \in \text{Val}^G$, $k_G(\varphi)$ is a homomorphism of $\text{Val}^G$-modules when thought of as a map $(\text{Val}^G)^* \rightarrow \text{Val}^G$. Furthermore $k_G(1)$ is an isomorphism.

**Proof.** That $k_G(\varphi)$ is a homomorphism of $\text{Val}^G$-modules follows immediately from Lemma 2.4.
To see that $k_G(1)$ is an isomorphism it is enough to prove surjectivity. By Thm. 2.2 above, any given valuation $\phi \in \text{Val}^G$ may be written as

\begin{align}
\phi &= \sum_{i=1}^{N} a_i \mu^G_{K_i} \\
&= \sum_{i=1}^{N} a_i \int_G \chi(\cdot \cap \bar{g}K_i) \, dg \\
&= \sum_{i=1}^{N} a_i \, k_G(1)(\cdot, K_i).
\end{align}

for some $a_1, \ldots, a_N \in \mathbb{R}$. But this last expression is precisely the image under $k_G(1)$ of the element $\psi \mapsto \sum a_i \psi(K_i)$ of $(\text{Val}^G)^*$. □

In view of Lemma 2.3 this gives

**Theorem 2.6.** With appropriate choices of scaling factors,

$$k_G(1) = \mathbf{PD}^{-1}.$$

2.3. The classical case: $G = SO(n)$. The case of $G = SO(n)$ or $O(n)$ was substantially settled by Hadwiger in the 1950s. Nonetheless the perspective introduced by Alesker remains illuminating.

Hadwiger proved that there is exactly one $SO(n)$-invariant valuation on $\mathbb{R}^n$ in each degree between 0 and $n$, given by the coefficients of the polynomial giving the volume of a tubular neighborhood of variable radius $r$ (Steiner’s formula). Thus the bodies $K_i$ above may be taken to be balls of $n+1$ distinct radii $r_0 < r_1 < \cdots < r_n$.

An alternative approach is to take $K_i$ to be a disk of dimension $i$. Letting the radius of such a disk to tend to $\infty$ and normalizing appropriately, one arrives at the classical expression for the Hadwiger valuations in terms of intersections with affine subspaces: if $\mathcal{G}(n, k)$ denotes the affine Grassmannian of $k$-planes in $\mathbb{R}^n$ then the valuation $\mu_i$ of degree $i$ may be expressed

$$
\mu_i(K) = \int_{\mathcal{G}(n,n-i)} \chi(K \cap \bar{P}) \, d\bar{P}.
$$

Let $t := \mu_1$. Then

$$
t(K) = \lim_{r \to \infty} r^{1-n} \int_{SO(n)} \chi(K \cap \bar{g}D^{n-1}_r) \, dg,
$$

where $D^k_r$ is the open ball of radius $r$ in $\mathbb{R}^k$.\]
where $D^n_{r}$ is the disk of dimension $n - 1$ and radius $r$. Therefore the definition of the product gives

$$t^2(K) = \lim_{r \to \infty} r^{2-2n} \int_{SO(n)} \int_{SO(n)} \chi(K \cap \tilde{g}D^n_{r} \cap \tilde{h}D^n_{r}) \, d\tilde{g} \, d\tilde{h}$$

(13)

$$= \int_{G(n,n-2)} \chi(K \cap \tilde{Q}) \, d\tilde{Q}$$

(14)

$$= \mu_2(K).$$

(15)

Continuing in this way we arrive at the following result of Alesker:

**Theorem 2.7.** $\text{Val}^{SO(n)} \simeq \mathbb{R}[t]/(t^{n+1})$.

The Poincare duality pairing is obviously $\langle t^i, t^j \rangle = \delta_{i,j}$. As a map $\text{Val}^{SO(n)} \to (\text{Val}^{SO(n)})^*$ it takes $t^{n-i}$ to $(t^i)^*$, where $1^*, t^*, \ldots, (t^n)^*$ is the dual basis to the $t^i$. The principal kinematic formula $k_{SO(n)}(1) = \text{PD}^{-1}$ thus takes $(t^i)^*$ to $t^{n-i}$, or in different terms

$$k_{SO(n)}(1) = \sum_{i=0}^{n} t^i \otimes t^{n-i}.$$ 

More generally, Lemma 2.4 yields

$$k_{SO(n)}(t^k) = \sum_{i+j=n+k} t^i \otimes t^j.$$ 

Thus we recover a classical result of Nijenhuis [9]:

**Theorem 2.8.** There exists a graded basis $\mu_0, \ldots, \mu_n$ for $\text{Val}^{SO(n)}$ such that for an appropriate normalization of the Haar measure

$$\int_{SO(n)} \mu_k(K \cap \tilde{g}L) \, d\tilde{g} = \sum_{i+j=n+k} \mu_i(K) \mu_j(L),$$

i.e. the coefficients in the kinematic formulas for the $\mu_i$ in terms of the $\mu_i$ are all equal to unity.

2.4. The orthogonal complement of $\text{Val}^{SO(n)}$ in $\text{Val}^G$. Returning to the case of a general group $G$ (transitive on the sphere of course), we take $V = \mathbb{R}^n$. Then $\text{Val}^{SO(n)} \subset \text{Val}^G$. Put

$$A_i^G := \{ \mu \in \text{Val}^G : t^{n-i} \cdot \mu = 0 \}$$

and

$$A^G := \bigoplus_{i=1}^{n-1} A_i^G.$$ 

Thus $A_i^G$ is a subspace of codimension 1 of $\text{Val}^G$. Clearly $A_i^G = 0$ unless $1 \leq i \leq n - 1$, and

$$t \cdot A_i^G \subset A_{i+1}^G.$$ 

(16)

(Actually Alesker has shown that $\text{Val}^G$ is one-dimensional for $i = 1, n - 1$, hence $A_i^G = A_{n-1}^G = 0$ as well.)
Proposition 2.9. With appropriate scalings,

\[ k_G(t^k) = \sum_{i+j=n+k} t^i \otimes t^j \mod A^G \otimes A^G, \quad k = 0, \ldots, n. \]

Proof. The subspace \( A^G_i \) is the orthogonal complement of \( t^{n-i} \) under the product pairing. Theorem 2.6 may be interpreted as saying that \( k_G(1) \) is the associated pairing on the dual space, which implies (17) for \( k = 0 \). Now the general case follows from Lemma 2.4 and the definition of \( A^G \). \( \square \)

Corollary 2.10. \( A^G = \{ \mu \in \text{Val}^G : \mu(D^n_r) = 0 \text{ for all } r > 0 \} \).

Proof. Let \( \{ \alpha_i \} \) be a basis for \( A^G \). Then by Prop. 2.9 given \( K \in K(V) \)

\[ \sum_{i+j=n} t^i(K)t^j(D^n_r) + \sum_{k,l} d_{kl} \alpha_k(K)\alpha_l(D^n_r) = \int_G \chi(K \cap gD^n_r) \bar{d}g \]

\[ = \int_{SO(n)} \chi(K \cap gD^n_r) \bar{d}g \]

\[ = \sum_{i+j=n} t^i(K)t^j(D^n_r). \]

Since the pairing \( k_G(1) \) is symmetric and nonsingular, so is its restriction to \( (\text{Val}^{SO(n)})^\perp = A^G \), from which it follows that all \( \alpha_l(D^n_r) = 0 \). \( \square \)

Remark. In fact the group \( G \) plays no role here, as Alesker (4) has proved the following:

Let \( \text{Val}_k(\mathbb{R}^n) \) denote the space of all degree \( k \) translation-invariant degree \( k \) valuations on \( \mathbb{R}^n \). Put

\[ \mathcal{A}_k := \{ \phi \in \text{Val}_k(\mathbb{R}^n) : t^{n-k} \cdot \phi = 0 \} \]

and

\[ \mathcal{B}_k := \{ \phi \in \text{Val}_k(\mathbb{R}^n) : \phi(D^n(r)) = 0 \text{ for all } r > 0 \}. \]

Then \( \mathcal{A}_k = \mathcal{B}_k \).

3. The unitary case

3.1. Statement of the main theorem. It is natural to consider next the case \( V = \mathbb{C}^n, \ G = U(n) \). Let \( G_n(n, k) \) denote the affine Grassmannian of complex \( k \)-planes in \( \mathbb{C}^n \). We will assume that

\[ \mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \ldots \]

in the natural way, and will take the corresponding unitary groups and Grassmannians to be included in each other accordingly. Denote the unit ball in \( \mathbb{C}^k \) by \( D^n_{\mathbb{C}} \).
In order to state a precise result in this case we need to take more care in the choices of
constants. We normalize the measures $d\bar{g}$ on $\mathcal{U}(n)$ and $dh$ on $SO(2n)$ so that, given any
measurable set $E \subset \mathbb{C}^n$,
\begin{equation}
(21) \quad d\bar{g} \left( \{ g \in \mathcal{U}(n) : \bar{g}(0) \in E \} \right) = dh \left( \{ \bar{h} \in SO(2n) : \bar{h}(0) \in E \} \right) = |E|,
\end{equation}
where $|E|$ is the Lebesgue measure of $E$. We take the measures on the real affine Grass-
mannians $G(2n, k)$ and the complex affine Grassmannians $\overline{G}_C(n, l)$ so that
\begin{align}
(22) & \quad d\bar{P} \left( \{ \bar{P} \in \overline{G}(2n, k) : \bar{P} \cap D^{n}_C \neq \emptyset \} \right) = \omega_{2n-k}, \\
(23) & \quad d\bar{Q} \left( \{ \bar{Q} \in \overline{G}_C(n, l) : \bar{Q} \cap D^{n}_C \neq \emptyset \} \right) = \omega_{2n-2l} = \frac{\pi^{n-l}}{(n-l)!}.
\end{align}
These measures are compatible with the measures on $SO(2n)$ and $\mathcal{U}(n)$ in the following
sense. We define $V^2n_i, \ldots, V^2n_2_n \in \mathbf{Val}^{SO(2n)}$ by
\begin{equation}
(24) \quad V^2n_i(K) := \int_{\overline{G}(2n, 2n-i)} \chi(K \cap \bar{P}) d\bar{P} = d\bar{P} \left( \{ \bar{P} : \bar{P} \cap K \neq \emptyset \} \right)
\end{equation}
and $W^n_0, \ldots, W^n_n \in \mathbf{Val}^{U(n)}$ by
\begin{equation}
(25) \quad W^n_j(K) := \int_{\overline{G}_C(n, n-j)} \chi(K \cap \bar{Q}) d\bar{Q} = d\bar{Q} \left( \{ \bar{Q} : \bar{Q} \cap K \neq \emptyset \} \right),
\end{equation}
$K \in \mathcal{K}(\mathbb{C}^n)$. Then
\begin{align}
(26) & \quad W^n_j(K) = \lim_{R \to \infty} (\omega_{2n-2j}R^{2n-2j})^{-1} d\bar{g}(\{ g \in \mathcal{U}(n) : gD^{2n-2j}_C(R) \cap K \neq \emptyset \}), \\
(27) & \quad V^2n_i(K) = \lim_{R \to \infty} (\omega_{2n-i}R^{2n-i})^{-1} dh(\{ \bar{h} \in SO(2n) : \bar{h}D^{2n-i}_R(R) \cap K \neq \emptyset \}).
\end{align}
Clearly
\begin{equation}
\deg V^2n_i = i, \quad \deg W^n_j = 2j.
\end{equation}
Our main result is
\begin{itemize}
\item $\mathbf{Val}^{U(n)}$ is generated as an $\mathbb{R}$-algebra by $V^2n_i$ and $W^n_i$.
\item Consider the graded polynomial algebra $\mathbb{R}[s, t]$, where $\deg t = 1$ and $\deg s = 2$. Put $f_i$ for the component of total degree $i$ in the power series expansion of $\log(1 + s + t)$. Then the map $\varphi_n : \mathbb{R}[s, t] \to \mathbf{Val}^{U(n)}$ of graded $\mathbb{R}$-algebras determined by
\begin{align}
(28) & \quad \varphi_n(t) \geq 0, \\
(29) & \quad \varphi_n(t^2) = \frac{2(2n-1)}{\pi} V^2n_i, \\
(30) & \quad \varphi_n(s) = \frac{\pi}{2} W^n_i
\end{align}
covers an isomorphism
\begin{equation}
(31) \quad \mathbb{C}[s, t]/(f_{n+1}, f_{n+2}) \simeq \mathbf{Val}^{U(n)}.
\end{equation}
\end{itemize}
The polynomials $f_i$ satisfy the relation

\[(32)\quad nsf_n + (n + 1)tf_{n+1} + (n + 2)f_{n+2} = 0, \quad n \geq 1\]

and the diagram

\[
\begin{array}{cccc}
\vdots & \vdots & & r \\
C[s, t]/(f_{n+1}, f_{n+2}) & \xrightarrow{\varphi_n} & Val^U(n) & r \\
\downarrow & & \downarrow & \\
C[s, t]/(f_n, f_{n+1}) & \xrightarrow{\varphi_{n-1}} & Val^U(n-1) & \\
\vdots & \vdots & & r \\
C[s, t]/(f_2, f_3) & \xrightarrow{\varphi_1} & Val^U(1) &
\end{array}
\]

commutes, where the vertical maps on the right are given by restriction.

**Remark.** Since

\[
\log(1 + s + t) = (s + t) - \frac{1}{2}(s + t)^2 + \frac{1}{3}(s + t)^3 + \ldots
\]

it is easy to write down in closed form as many $f_i$ as desired. For example,

\[
\begin{align*}
    f_1 &= t, \\
    f_2 &= s - \frac{1}{2}t^2, \\
    f_3 &= -st + \frac{1}{3}t^3, \\
    f_4 &= -\frac{1}{2}s^2 + st^2 - \frac{1}{4}t^4,
\end{align*}
\]

etc.

The rest of this paper is devoted to the proof of Theorem 3.1. The reason for the coefficients in (29) and (30) is the following.

**Lemma 3.2.** For $1 \leq k \leq n$, put $r^n_k : Val^U(n) \to Val^U(k)$ to be the restriction map. Then

\[
\begin{align*}
    r^n_k \left( \frac{2(2n-1)}{\pi} V_2^{2n} \right) &= \frac{2(2k-1)}{\pi} V_2^{2k}, \\
    r^n_k \left( \frac{n}{\pi} W_1^n \right) &= \frac{k}{\pi} W_1^k.
\end{align*}
\]
Proof. It is clear that in each relation above the left and right sides are constant multiples of each other, so it will be enough to show that

\[ r_n^1 \left( \frac{2(2n-1)}{\pi} V_2^{2n} \right) = \frac{2}{\pi} V_2^2, \]
\[ r_n^1 \left( \frac{n}{\pi} W_1^n \right) = \frac{1}{\pi} W_1^1, \]

i.e. that

\[ V_2^{2n}(D_C^1) = \frac{\pi}{2n - 1}, \]
\[ W_1^n(D_C^1) = \frac{\pi}{n}. \]

To prove (38) we note that the Hadwiger valuation \( V_2^{2n} \) of a smooth body \( K \subset \mathbb{C}^n \cong \mathbb{R}^{2n} \) may be expressed as the integral of the \((2n - 3)\)rd symmetric function of the principal curvatures of the boundary of \( K \), multiplied by a certain constant \( c \). Therefore

\[ \pi = V_2^{2n}(D_C^0) = c \left( \frac{2n-1}{2n-3} \right) \alpha_{2n-1} \]
\[ = c \left( \frac{2n-1}{2n-3} \right) (2n) \frac{\pi^n}{n!}, \]

so

\[ c = \frac{(n-2)!}{(4n-2)\pi^{n-1}}. \]

On the other hand, the disk \( D_C^1 \) is the Hausdorff limit of its tubular neighborhoods of radius \( r \) as \( r \downarrow 0 \), which may be thought of as the union of \( D_C^1 \times D_C^{n-1}(r) \) together with a bundle of half-balls of real dimension \( 2n \) over the boundary circle. Computing the curvature integrals and passing to \( r = 0 \), the boundary term tends to 0 and we obtain

\[ V_2^{2n}(D_C^1) = c \alpha_{2n-3} \pi = \frac{(n-2)!}{(4n-2)\pi^{n-1}(2n-2)} \frac{\pi^{n-1}}{(n-1)!} \pi = \frac{\pi}{2n - 1}. \]

as claimed.

To prove (39) we apply Howard’s transfer principle for Poincaré-Crofton formulas \([7]\) to compare the integral geometry of \( \mathbb{C}^n \) under the holomorphic isometry group \( \overline{U(n)} \) with that of \( \mathbb{P}^n \) under its full isometry group \( U(n+1)/U(1) \). To remain consistent with the measure on \( \overline{U(n)} \) given in \([2] \), we select the Haar measure on \( U(n+1)/U(1) \) so that its total mass is equal to the volume \( \frac{\pi^n}{n!} \) of \( \mathbb{P}^n \). The transfer principle then implies that

\[ \frac{1}{n} = \int_{U(n+1)/U(1)} \#(\mathbb{P}^1 \cap h\mathbb{P}^{n-1}) \, dh = \int_{\overline{U(n)}} \#(D_C^1 \cap \tilde{g}D_C^{n-1}(R)) \, d\tilde{g} \]
\[ \frac{1}{\text{area}(D_C^1) \text{vol}(D_C^{n-1}(R))} = \int_{\overline{U(n)}} \#(D_C^1 \cap \tilde{g}D_C^{n-1}(R)) \, d\tilde{g} \]
On the other hand (26) may be written

\[
W^n_1(K) \sim \frac{\int_{U(n)} \chi(K \cap \bar{g}D^n_{\mathbb{C}}(R)) \, d\bar{g}}{\text{vol}(D^n_{\mathbb{C}}(R))}
\]

as \( R \to \infty \). Applying this formula to \( K = D^n_{\mathbb{C}} \), (44) implies that \( W^n_1(D^n_{\mathbb{C}}) = \pi^n \). \( \square \)

**Definition 3.3.** In view of this fact, for the sake of simplicity we will abuse notation by writing \( s \) for \( n \pi W^n_1 \), and \( t \) for the positive square root of \( \frac{4n-2}{\pi} V^n_2 \). In computations with these elements the dimension in which we work should be clear from the context.

**Corollary 3.4.** In \( \text{Val}^{U(n)} \),

\[
t^{2n} = \frac{2 \cdot (2n-1)!}{\pi^n (n-1)!} \cdot \text{volume}^{2n}.
\]

**Proof.** We again use Howard’s transfer principle, this time for the associated pairs \( (\mathbb{R}^n, \text{SO}(n)) \) and \( (S^n, \text{SO}(n+1)) \).

Put

\[
\Psi_i = \Psi^n_i := \alpha_{n-1}^{-1} \alpha_{n-i-1}^{-1} \Phi^n_i.
\]

Thus the restriction of \( \Psi_i \) to subsets of \( \mathbb{R}^n \) of dimension \( i \) is a multiple of the \( i \)-dimensional Hausdorff measure that transfers to \( S^n \) to give

\[
\Psi_i(S^i) = 1.
\]

Now the kinematic formula for \( \mathbb{R}^n \) may be expressed

\[
c_n(\Psi_k) = \sum_{i+j=n+k} a_i \Psi_i \otimes \Psi_j.
\]

By the transfer principle the same formula applies to subsets of \( S^n \)— this is true because the kinematic formula specializes to the Poincaré–Crofton formulas

\[
\int \text{volume}^k(M^i \cap \bar{g}N^j) \, d\bar{g} = \int \Psi_k(M^i \cap \bar{g}N^j) \, d\bar{g} = c \Psi_i(M^i) \Psi_j(N^j)
\]

for submanifolds \( M^i, N^j \) with \( i + j = n + k \). Taking the total measure of \( \text{SO}(n+1) \) to be \( \alpha_n \), and applying the formula to \( S^i, S^j \subset S^n \), we find that all of the constants are equal to \( \alpha_n \):

\[
c_n(\Psi_k) = \alpha_n \sum_{i+j=n+k} \Psi_i \otimes \Psi_j.
\]

By Lemma 2.4 since \( \Psi_0 = \frac{1}{2} \chi = \frac{1}{2} \),

\[
c_n(\Psi_k) = 2c_n(\Psi_k \cdot \Psi_0) = 2\alpha_n \sum_{i+j=n} (\Psi_k \cdot \Psi_i) \otimes \Psi_j.
\]

Therefore \( \Psi_i \cdot \Psi_k = \frac{1}{2} \Psi_{i+k} \), so \( (4\Psi_1)^2 = 8\Psi_2 \), and since

\[
8\Psi_2(S^2) = 8,
\]

\[
t^2(S^2) = \frac{2}{\pi} \text{area}(S^2) = 8,
\]

therefore \( 8\Psi_2(S^2) = 8 \).
it follows that
\[(48)\]
\[t = 4\Psi_1\]
and
\[(49)\]
\[t^n = 2^{n+1}\Psi_n = \frac{2^{n+1}}{\alpha_n}\text{volume}^n.\]
In particular
\[(50)\]
\[t^{2n} = 2^{2n+1}\left(\frac{(2n-1)(2n-3)\ldots3\cdot1}{\pi^n 2^{n+1}}\right)\text{volume}^{2n},\]
as claimed. \[\square\]

We note for future reference that
\[(51)\]
\[f_k = (-1)^{k+1} \sum_{i=0}^{[\frac{k}{2}]} (-1)^i \binom{k-i}{i}s^i t^{k-2i}\]
\[= (-1)^{k+1} \sum_{i=0}^{[\frac{k}{2}]} (-1)^i \binom{k-i-1}{i}s^i t^{k-2i}.\]
In particular
\[(52)\]
\[f_{n+1} = (-1)^n \sum_{i=0}^{[\frac{n+1}{2}]} (-1)^i \frac{(n-i)!}{i!(n-2i+1)!} s^i t^{n-2i+1}.\]

3.2. First deductions. Our starting point is the following result of Alesker [2]:

**Theorem 3.5.** The valuations
\[U_{k,p}(K) := \int_{G_C(n,n-p)} t^{k-2p}(K \cap \bar{P}) \, d\bar{P},\]
\[0 \leq p \leq \frac{1}{2} \min\{k, 2n-k\},\]
constitute a basis for \(\text{Val}^{U(n)}\). In particular, the Poincaré series of \(\text{Val}^{U(n)}\) is
\[(53)\]
\[P_{\text{Val}^{U(n)}}(x) = \frac{(1 - x^{n+1})(1 - x^{n+2})}{(1 - x)(1 - x^2)}.\]

**Proof.** The first assertion is due to Alesker in [2]. The assertion about the Poincaré series then follows from a simple comparison between the coefficients of the given polynomial and Alesker’s computation of the dimensions of the \(\text{Val}_k^{U(n)}\). \[\square\]

As in the discussion preceding the statement of Theorem 2.7, these valuations are monomials in \(s\) and \(t\):

**Proposition 3.6.**
\[U_{k,p}^n = s^p t^{k-2p}.\]
This establishes the first assertion of Theorem 3.1.

**Remark.** Thus the basis described in Theorem 3.5 may be understood as follows. Up through degree \(i = n\), there are no relations between \(s\) and \(t\). From degree \(i = n + 1\) through the highest degree \(i = 2n\), the basis elements are in one-to-one correspondence with those in degree \(2n - i\): in fact they are simply the products of the latter with \(t^{2i - 2n}\).

From general considerations we also find

**Lemma 3.7.** There are polynomials \(p_{n+1}, p_{n+2}\), of degrees \(n + 1, n + 2\) respectively, such that

\[
\ker \varphi_n = (p_{n+1}, p_{n+2}).
\]

**Proof.** By Alesker’s basis Theorem 3.5 there are relations \(p_{n+1}, p_{n+2} \in \mathbb{C}[s, t]\) in the given degrees such that \(p_{n+2} \neq t \cdot p_{n+1}\). We first show that these polynomials are relatively prime. Otherwise, let \(w \in \mathbb{C}[s, t]\) be an element of degree \(0 < k < n + 1\) dividing both, with \(p_j = w d_j, j = n + 1, n + 2\), where the \(d_j\) are relatively prime. It is clear that all of these elements are homogeneous. Therefore \(W := \mathbb{C}[s, t]/(d_{n+1}, d_{n+2})\) is a graded algebra with Poincaré series

\[
P_W(x) = \frac{(1 - x^{n+1-k})(1 - x^{n+2-k})}{(1 - x)(1 - x^2)},
\]

which has degree \(2n - 2k\). Thus the image of the linear map \(h : W \to \text{Val}^{U(n)}\) covered by multiplication by \(w\) in \(\mathbb{C}[s, t]\) meets the socle \(\text{Val}^{U(n)}_{2n}\) only at \(0\). However, this contradicts the fact that multiplication induces a perfect pairing on \(\text{Val}^{U(n)}\): since the image of \(w\) in \(\text{Val}^{U(n)}\) is nonzero, there exists \(g \in \mathbb{C}[s, t], \deg g = 2n - k\), such that \(w \cdot g \neq 0\) in \(\text{Val}^{U(n)}\). Thus the image of \(g\) in \(W\) under \(h\) is not zero. This is a contradiction.

It follows that the Poincaré series of \(\mathbb{C}[s, t]/(p_{n+1}, p_{n+2})\) is given by (55). Since \(\text{Val}^{U(n)}\) is isomorphic to a quotient of this algebra we obtain the desired conclusion. \(\square\)

From this and Lemma 3.2 we find that the last conclusion of Theorem 3.1 is valid with the \(f_i\) replaced by the polynomials \(p_i\), to be determined. Thus only the second assertion remains. The combinatorial part (52) follows at once by writing

\[
\log(1 + tx + sx^2) = \sum_{i=1}^{\infty} f_i(s, t)x^i,
\]

\[
f_1 + (tf_1 + 2f_2)x + \sum_{i=1}^{\infty} [isf_i + (i+1)tf_{i+1} + (i+2)f_{i+2}];x^{i+1} = (1 + tx + sx^2) \sum_{i=1}^{\infty} if_ix^{i-1}
\]

\[
= (1 + tx + sx^2) \frac{d}{dx} \log(1 + tx + sx^2)
\]

\[
= t + 2sx.
\]
3.3. Identifying \( A^n \). For the rest of the paper we will abbreviate \( A^n := A^{U(n)} \). Using Corollary 2.10 we can give an explicit basis for this subspace.

**Lemma 3.8.** For \( 2 \leq j \leq 2n - 2 \), the elements
\[
(n - i)s^{i}t^{j-2i} - (4n - 4i - 2)s^{i+1}t^{j-2i-2}, \quad 0 \leq i \leq \min\{j, 2n - j\} - 1
\]
constitute a basis for \( A^n_j \).

**Proof.** By Theorem 3.5, together with \( t^j \) the elements (60) constitute a basis for \( \text{Val}_j^{U(n)} \). Hence it is enough to show that they belong to \( A^n \).

Put \( r_k := kt^2 - (4k - 2)s \). By Lemma 3.2 \( r_k \in A^k \). By the definition of the generating valuation \( s \), it follows that \( s^{n-k}r_k \in A^n \) for \( k \leq n \). Now the relation (16) implies that \( t^i s^{n-k}r_k \in A^n \) for all \( i \geq 0 \). These include the elements (60). \( \square \)

Denote the ordered monomial basis \( (t^j, st^{j-2}, \ldots) \) for \( \text{Val}_j^{U(n)} \) given in Theorem 3.5 and Prop. 3.6 by \( b_j \), and the ordered basis consisting of \( t^j \) together with the degree \( j \) elements of (60) by \( c_j \). Thus if we define the \((k + 1) \times (k + 1)\) matrix
\[
A^k_j := \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-2(2n - 1) & 0 & 0 & \cdots & 0 & 0 \\
0 & (n - 1) & -2(2n - 3) & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & n - k + 1 & -2(2n - 2k + 1)
\end{bmatrix}
\]
then for \( 2k + 1 \leq n \) we have
\[
c_j = A^k_j b_j,
\]
\( j = 2k, 2k + 1, 2n - 2k - 1, 2n - 2k \).

3.4. Framework for induction. Recalling the remark following Prop. 3.6, if \( 2k + 1 \leq n \) then the \( \mathbb{R} \) vector spaces
\[
\text{Val}_{2k}^{U(n)}, \text{Val}_{2k+1}^{U(n)}, \text{Val}_{2n-2k-1}^{U(n)}, \text{Val}_{2n-2k}^{U(n)}
\]
all have dimension \( k + 1 \). In fact the maps
\[
a \mapsto t^j \cdot a,
\]
\( j = 1, 2n - 4k - 1, 2n - 4k \), are isomorphisms \( \text{Val}_{2k}^{U(n)} \to \text{Val}_{2k+1}^{U(n)}, \text{Val}_{2n-2k-1}^{U(n)}, \text{Val}_{2n-2k}^{U(n)} \) respectively. Since the Poincaré duality pairing is given by multiplication it is trivial to see:

**Proposition 3.9.** The pairings on \( \text{Val}_{2k}^{U(n)} \) given by
\[
\langle a, b \rangle := \text{PD}(a, t^{2n-4k}b)
\]
and
\[
\langle\langle a, b \rangle\rangle := \text{PD}(ta, t^{2n-4k-1}b)
\]
are identical to one another, and are symmetric and nondegenerate. \( \square \)
We denote by $P^n_k$ the $(k+1) \times (k+1)$ matrix giving this pairing with respect to the ordered monomial basis $t^{2k}, st^{2k-2}, \ldots, s^k$, and put $Q^n_k := (P^n_k)^{-1}$. Thus $Q^n_k$ is the matrix giving the associated pairing on the dual spaces. By Corollary 2.6 $Q^n_k$ is also the matrix of coefficients for the kinematic formula $k_n(1)$.

**Convention.** From this point on we normalize the $k_n$ so that equation (17) in Prop. 2.9 is literally true.

With this convention, Corollary 3.4 gives

$$\int_{U(n)} \chi(K \cap \bar{g} L) d\bar{g} = \chi(K) \text{volume}(L) + \ldots$$

$$= \frac{\pi^n(n-1)!}{2 \cdot (2n-1)!} (1 \otimes t^{2n} + \ldots)(K, L)$$

$$= \frac{\pi^n(n-1)!}{2 \cdot (2n-1)!} k_n(1)(K, L)$$

(66)

$$= \frac{\pi^n(n-1)!}{2 \cdot (2n-1)!} k_n(1)(K, L)$$

(67)

where

$$k_n(1) = \sum_{i=0}^{2n} (t^i, st^{i-2}, \ldots, s^k t^{i-2k}) \otimes Q^n_{\min\{i, \frac{2n-i}{2}\}} \left( \begin{array}{c} t^{2n-i} \\ st^{2n-i-2} \\ \vdots \\ s^k t^{2n-i-2k} \end{array} \right).$$

(68)

Here $Q^n_0 = 1$. Note that in this last sum there are four terms involving each of the matrices $Q^n_k$, $k \leq \frac{n}{2} - 1$; on the other hand there are three such terms when $n$ is odd and $k = \frac{n-1}{2}$, and one such term when $n$ is even and $k = \frac{n}{2}$.

Now by Prop. 2.9 for $k \geq 1$ there exists a nonsingular symmetric $k \times k$ matrix $\widetilde{Q}^n_k$ such that

$$Q^n_k = (A^n_k)^t \left[ \begin{array}{cc} 1 & 0 \\ 0 & \widetilde{Q}^n_k \end{array} \right] A^n_k.$$

(69)

We will determine the relations in $\text{Val}^{U(n)}$ between the generators $s$ and $t$ using an induction on $n$ and $k$.

**Definition 3.10.** For each $\bar{P} \in \bar{G}_C(n+1, n)$, choose a holomorphic isometry $\gamma_{\bar{P}} : \bar{P} \to \mathbb{C}^n$, and define the map $\iota : \text{Val}^{U(n)} \to \text{Val}^{U(n+1)}$ by

$$\iota(\mu)(K) := \int_{\bar{G}_C(n+1, n)} \mu(\gamma_{\bar{P}}(K \cap \bar{P})) d\bar{P}.$$

Clearly this is independent of the choices of the $\gamma_{\bar{P}}$.

**Proposition 3.11.** Under the maps

$$\text{Val}^{U(n+1)} \otimes \text{Val}^{U(n+1)} \xrightarrow{id \otimes \iota} \text{Val}^{U(n+1)} \otimes \text{Val}^{U(n)} \xleftarrow{\iota \otimes id} \text{Val}^{U(n)} \otimes \text{Val}^{U(n)}.$$
the elements $k_{n+1}(1)$ and $k_n(1)$ satisfy
\[(71) \quad \frac{\pi}{2(2n+1)}(id \otimes r)(k_{n+1}(1)) = (\iota \otimes id)(k_n(1)).\]

In terms of the notational abuse of Definition 3.3, this relation may be written
\[(72) \quad (n+1)k_{n+1}(1) = 2(2n+1)(s \otimes 1) \cdot k_n(1)
\]
as elements of $\text{Val}^{U(n+1)} \otimes \text{Val}^{U(n)}$.

Proof. Clearly the images of $s$ and $t$ under the restriction map satisfy
\[(73) \quad r(s_{n+1}) = s_n, \quad r(t_{n+1}) = t_n.\]

Furthermore, in view of our normalizing conventions for the various Haar measures,
\[(74) \quad \iota(s_it_jn) = \pi n + 1 s_{i+1}n + 1 t_{j+1}.\]

Returning to our usual abuse of notation we write more succinctly
\[(75) \quad r(s) = s, \quad r(t) = t, \quad \iota(s_it_j) = \pi n + 1 s_{i+1}t_{j+1}.\]

To prove the relation (71), for $\bar{P} \in \bar{G}_{\mathbb{C}}(n+1, n)$ we put $\bar{U}_{\mathbb{C}}(n, \bar{P})$ for the left coset of $\bar{U}_{\mathbb{C}}(n)$ in $\bar{U}_{\mathbb{C}}(n+1)$ consisting of the elements that map $\mathbb{C}^n$ to $\bar{P}$. Now if $K \in \mathcal{C}(\mathbb{C}^{n+1})$ and $L \in \mathcal{K}(\mathbb{C}^n) \subset \mathcal{K}(\mathbb{C}^{n+1})$ then (denoting by $i: \mathbb{C}^n \to \mathbb{C}^{n+1}$ the inclusion)
\[
\frac{\pi^{n+1} n!}{2 \cdot (2n+1)!} (id \otimes r)(k_{n+1}(1))(K, L) = \frac{\pi^{n+1} n!}{2 \cdot (2n+1)!} k_{n+1}(1)(K, i(L))
\]
\[
= \int_{\bar{U}(n+1)} \chi(K \cap \bar{g}L) \, d\bar{g}
\]
\[
= \int_{\bar{G}_{\mathbb{C}}(n+1, n)} \int_{\bar{U}_{\mathbb{C}}(n, \bar{P})} \chi(K \cap \bar{g}L) \, d\bar{g} \, d\bar{P}
\]
\[
= \int_{\bar{G}_{\mathbb{C}}(n+1, n)} \int_{\bar{U}_{\mathbb{C}}(n, \bar{P})} \chi((K \cap \bar{P}) \cap \bar{h}L) \, d\bar{h} \, d\bar{P}
\]
\[
= \frac{\pi^{n} (n-1)!}{2 \cdot (2n-1)!} \int_{\bar{G}_{\mathbb{C}}(n+1, n)} k_{n}(1)(\gamma_P(K \cap \bar{P}), L) \, d\bar{P}
\]
\[
= \frac{\pi^{n} (n-1)!}{2 \cdot (2n-1)!} (\iota \otimes id)(k_n(1))(K, L),
\]
which simplifies to (71). The relation (72) follows from this and (76). \qed
3.5. Proof of theorem.

Lemma 3.12. If $2k \leq n - 1$ then

\begin{align}
\frac{n}{2(2n-1)} Q_k^n P_k^{n-1} = \begin{pmatrix}
0 & 0 & 0 & \ldots & a_{n,k}^n \\
1 & 0 & 0 & \ldots & a_{n,k}^1 \\
0 & 1 & 0 & \ldots & a_{n,k}^2 \\
0 & 0 & 1 & \ldots & a_{n,k}^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & a_{n,k}^n
\end{pmatrix},
\end{align}

where, putting

\begin{align}
a_{k+1,k}^n &= 1, \\
\varphi_{n,k}^n &= \sum_{i=0}^{k+1} a_{n,k}^i \kappa_{2n-2i-1},
\end{align}

we have

\begin{align}
\varphi_{n,k}^n &= 0 \text{ in } \text{Val}_U^{(n)}, 0 \leq k \leq \frac{n}{2} - 1;
\end{align}

and if $n$ is odd then

\begin{align}
t\varphi_{n,k}^{n+1} &= 0 \text{ in } \text{Val}_U^{(n)}.
\end{align}

Observe that in the latter case $t\varphi_{n,k}^{n+1}$ is a polynomial even though $\varphi_{n,k}^{n+1}$ is only rational.

Proof. By (56), the terms of bidegree $(2n - 2k, 2k)$ in $k(n(1)$ may be written

\begin{align}
\left( t^{2n-2k}, \ldots, s^k t^{2n-4k} \right) \otimes Q_k^n \left( \ldots \right) \in \text{Val}_U^{(n)} \otimes \text{Val}_U^{(n)},
\end{align}

and similarly the terms of bidegree $(2n - 2k - 2, 2k)$ in $k(n(1)$ are given by

\begin{align}
\left( t^{2n-2k-2}, \ldots, s^k t^{2n-4k-2} \right) \otimes Q_k^{n-1} \left( \ldots \right) \in \text{Val}_U^{(n-1)} \otimes \text{Val}_U^{(n-1)}.
\end{align}

Therefore the relation (44) gives

\begin{align}
\frac{n}{2(2n-1)} \left( t^{2n-2k}, \ldots, s^k t^{2n-4k} \right) \otimes Q_k^n \left( t^{2k} \ldots \right) = \left( s^{2n-2k-2}, \ldots, s^{k+1} t^{2n-4k-2} \right) \otimes Q_k^{n-1} \left( \ldots \right)
\end{align}

in $\text{Val}_U^{(n)} \otimes \text{Val}_U^{(n-1)}$. Since $t^{2k}, \ldots, s^k$ constitute a basis for $\text{Val}_2^{k(n-1)}$ it follows that

\begin{align}
\frac{n}{2(2n-1)} \left( t^{2n-2k}, \ldots, s^k t^{2n-4k} \right) Q_k^n = \left( s^{2n-2k-2}, \ldots, s^{k+1} t^{2n-4k-2} \right) Q_k^{n-1},
\end{align}

which shows that the left hand side of (77) has the given form for some constants $a_{n,k}^i$, where the resulting polynomial $\varphi_{n,k}^n$ satisfies $t\varphi_{n,k}^{n+1} = 0$ in $\text{Val}_U^{(n)}$. 

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If \( k \leq \frac{n}{2} - 1 \) then the relations corresponding to (81) and (82) hold for the terms of bidegree \((2n - 2k - 1, 2k + 1)\) and \((2n - 2k - 3, 2k + 1)\). Arguing as above we then arrive at the relation (79). □

To prove theorem we will show that if \( n \) is odd (resp. even) then \( t\phi^{n, \frac{n-1}{2}} \) (resp. \( \phi^{n, \frac{n-1}{2}} \)) is a nonzero constant multiple of \( f_{n+1} \). Since the natural map \( \text{Val}^U(n+1) \to \text{Val}^U(n) \) is well-defined and \( f_{n+2} = f_{(n+1)+1} = 0 \) in \( \text{Val}^U(n+1) \), it follows that \( f_{n+2} = 0 \) in \( \text{Val}^U(n) \) as well.

In fact we prove more generally:

**Proposition 3.13.** For \( 2k \leq n - 1 \) and \( 0 \leq i \leq k \),

\[
a_i^{n,k} = (-2)^{i-1} \binom{k+1}{i} \frac{(n-i)(n-i-1)\ldots(n-k)}{(2n-2k-2i-1)(2n-2k-2i-3)\ldots(2n-4k-1)}.
\]

**Corollary 3.14.** If \( n \) is even then

\[
\phi^{n, \frac{n-1}{2}} = (-1)^{\frac{n-1}{2}} f_{n+1}.
\]

If \( n \) is odd then

\[
t\phi^{n, \frac{n-1}{2}} = (-1)^{\frac{n-1}{2}} \left( \frac{n+1}{2} \right) f_{n+1},
\]

The rest of this section is devoted to the proof of Prop. 3.13 using induction on \( k \), the starting point being the relation

\[
a_i^{n,0} = \frac{-n}{2(2n-1)}, \quad n \geq 2,
\]

which is clearly valid from the defining relation (77) since \( Q_0^n = 1 \) for all \( n \).

**Lemma 3.15.** Let

\[
R_k^n := \frac{n}{2(2n-1)}
\]

\[
\begin{bmatrix}
\frac{2(2n-1)}{n} & 1 & \frac{2(2n-5)}{n-2} & \frac{2(2n-7)}{n-3} & \ldots & \frac{2(2n-2k-1)}{n-k} \\
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
\end{bmatrix}
\]

Then

\[
R_k^n Q_k^n = \begin{bmatrix} 1 & 0 \\ 0 & Q_{k-1}^{n-1} \end{bmatrix}.
\]

**Remark.** Note that the first row of \( R_k^n \) may also be written as

\[
\begin{bmatrix} 2n - 1 \end{bmatrix}^{-1} \begin{bmatrix} 2n - 1 \\ n - 1 \end{bmatrix} \begin{bmatrix} 2n - 3 \\ n - 1 \end{bmatrix} \ldots \begin{bmatrix} 2n - 2k - 1 \end{bmatrix} \]
Proof. Equating the terms of bidegree \((2n-2k, 2k)\) in Prop. 3.11 we obtain for \(0 \leq 2k \leq n\):

\[
\frac{n}{2(2n-1)} (t^{2n-2k}, st^{2n-2k-2}, \ldots, s^k t^{2n-4k}) \otimes Q_k^n = M(94)
\]

\[
= (t^{2n-2k}, st^{2n-2k-2}, \ldots, s^{k-1} t^{2n-4k+2}) \otimes Q_{k-1}^{n-1}
\]

in \(\text{Val}^{U(n-1)} \otimes \text{Val}^{U(n)}\). Applying Lemma 3.12 it follows that

\[
(t^{2n-2k}, st^{2n-2k-2}, \ldots, s^{k-1} t^{2n-4k+2}) = (t^{2n-2k}, st^{2n-2k-2}, \ldots, s^{k-1} t^{2n-4k+2}) \left[ I_k - a^{n-1,k-1} \right]
\]

in \(\text{Val}^{U(n)}\). Since \(t^{2n-2k}, st^{2n-2k-2}, \ldots, s^{k-1} t^{2n-4k+2}\) constitute a basis for \(\text{Val}^{U(n-1)}\),

and \(t^{2k}, \ldots, s^k\) are a basis for \(\text{Val}^{U(n)}\), recalling (69) we find that

\[
\frac{n}{2(2n-1)} \left[ I_k - a^{n-1,k-1} \right] \left( A_k^n \right)^t \begin{bmatrix} 1 & 0 \\ 0 & Q_k^n \end{bmatrix} A_k^n = \frac{n}{2(2n-1)} \left[ I_k - a^{n-1,k-1} \right] Q_k^n = \begin{bmatrix} e_1 & 0 \\ 0 & Q_{k-1}^{n-1} \end{bmatrix}
\]

(93)

In view of the definition (61) of \(A_k^n\), the first row of \(\begin{bmatrix} 1 & 0 \\ 0 & Q_k^n \end{bmatrix} A_k^n\) is \(e_1 := (1, 0, \ldots, 0)\). Now if we denote by \(M\) the \((k + 1) \times (k + 1)\) matrix whose first row is \(e_1\) and whose bottom \(k\) rows are identical to \(\frac{n}{2(2n-1)} \left[ I_k - a^{n-1,k-1} \right] \left( A_k^n \right)^t\) then (93) becomes

\[
M \begin{bmatrix} 1 & 0 \\ 0 & Q_k^n \end{bmatrix} A_k^n = \begin{bmatrix} 1 & 0 \\ 0 & Q_{k-1}^{n-1} \end{bmatrix}.
\]

(94)

Using the identity

\[
(n - i) \binom{2n - 2i - 1}{n - i} - 2(2n - 2i - 1) \binom{2n - 2i - 3}{n - i - 1} = 0,
\]

a straightforward calculation shows that

\[
M = R_k^n \left( A_k^n \right)^t
\]

so the desired relation follows from (69) and (94). \(\Box\)

To complete the proof of theorem we use Lemma 3.15 to write

\[
Q_k^n P_k^{n-1} = (R_k^n)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & Q_{k-1}^{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P_{k-1}^{n-2} \end{bmatrix} R_k^{n-1} P_{k-1}^{n-2} = (R_k^n)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & Q_{k-1}^{n-1} P_{k-1}^{n-2} \end{bmatrix} R_k^{n-1} P_{k-1}^{n-2},
\]
yielding
\[
\frac{n-1}{2(2n-3)} P^n_k \left( \frac{n}{2(2n-1)} Q^n_k P^{n-1}_k \right) = \frac{n}{2(2n-1)} \left[ \frac{n-1}{2(2n-3)} \tilde{Q}^{n-1}_k P^{n-2}_k \right] R^{n-1}_k.
\]

Recalling Lemma 3.12 and the definition of $R^n_k$, if we equate the last columns on the right and the left we obtain the following relations among the constants $a_{i}^{n,k}, a_{i}^{n-1,k-1}, a_{i}^{n-2,k-1}$:
\[
\begin{align*}
-\sum_{i=0}^{k} \binom{2n-2i-1}{n-i} a_{i}^{n,k} &= \binom{2n-2k-3}{n-k-1}, \\
-a_{i}^{n,k} + a_{i}^{n-1,k-1} a_{k}^{n,k} &= -a_{i-1}^{n-2,k-1} + a_{i}^{n-1,k-1} a_{k-1}^{n-2,k-1}, i = 0, \ldots, k - 1,
\end{align*}
\]
where we set $a_{-1}^{n-2,k-1} := 0$. Equivalently,
\[
\begin{align*}
\sum_{i=0}^{k+1} \binom{2n-2i-1}{n-i} a_{i}^{n,k} &= 0, \\
a_{i-1}^{n-2,k-1} + a_{i}^{n-1,k-1} (a_{k}^{n,k} - a_{k-1}^{n-2,k-1}) &= a_{i}^{n,k}, i = 0, \ldots, k - 1.
\end{align*}
\]

Since all of the matrices above are invertible these relations determine the $a_{i}^{n,k}$ uniquely in terms of the $a_{i}^{n-1,k-1}, a_{i}^{n-2,k-1}$, i.e. the system above is nonsingular in $a_{0}^{n,k}, \ldots, a_{k}^{n,k}$. Therefore, to complete the proof of Prop. 3.13 by induction on $k$ it is enough to show that (96) and (97) are valid for the stated values (85).

Starting with the observation that, with these values,
\[
a_{k}^{n,k} - a_{k-1}^{n-2,k-1} = -\frac{n}{2(2n-4k-1)},
\]
the verification of (97) is a straightforward calculation. Meanwhile, fixing $n$ and $k$, the relation (96) reduces to
\[
\sum_{i=0}^{k+1} (-1)^{i} \binom{k+1}{i} \frac{(2n-2i-1) \cdots (2n-2k-1)}{(2n-2k-2i-1) \cdots (2n-4k-1)} = 0,
\]
where the $(k+1)$st term is understood to be $(-1)^{k+1}$— in fact, after substituting the values (85) into (96) we find that the ratio of the $i$th terms of respective left-hand sides of (96) and (98) is
\[
\frac{(2n-2k-3)(2n-2k-5) \cdots 3 \cdot 1(-1)^{k+1} 2^{n-k-2}}{(n-k-1)!},
\]
independent of $i$. Substituting $z := \frac{2n}{2n-1}$ and multiplying the sum by the function $(z - k - 1)(z - k - 2) \cdots (z - 2k)$, the relation (98) becomes
\[
\sum_{i=0}^{k+1} (-1)^{i} \binom{k+1}{i} (z - i)(z - i - 1) \cdots (z - i - k + 1) = 0.
\]
If $\Delta$ is the difference operator $\Delta(f(z)) := f(z) - f(z - 1)$ then the left-hand side is
\[
\Delta^{k+1} (z(z - 1) \cdots (z - k + 1))
\]
which vanishes identically since the subject polynomial has degree $k$. □
4. Open questions

1. Obviously $\text{Val}^{U(n)}$ is much more complicated than $\text{Val}^{SO(n)}$, and many questions that are trivial in the latter case are not in the former. As we have seen, the Poincaré duality pairing for $\text{Val}^{SO(n)}$ is essentially as simple as possible, so the deduction of the kinematic formulas via Thm. 2.6 is very easy. For $\text{Val}^{U(n)}$ the question of determining the pairing matrices $P^k_n$ and their inverses $Q^k_n$, which determine the kinematic formulas, is open. Using the MAGMA computer algebra package, Graham Matthews has calculated the $Q^k_n$ for $k \leq 11$. The entries are rational functions of $n$, with numerators and denominators having irreducible factors of even degree apparently growing without bound as $k$ increases. We have not been able to discern the patterns in the coefficients.

A seemingly simpler question is: are the $Q^k_n$ positive definite? They are for small values of $n$.

2. Hadwiger’s basis theorem for $\text{Val}^{SO(n)}$ implies that various methods for constructing $SO(n)$-invariant valuations lead to the same results. For example, if $K$ is a compact convex body then $t^i(K)$ is equal to the average value of $t^i(\pi_P(K))$ as $P$ ranges over the Grassmanian $G(n,j)$, $j \geq i$; or, if the body has smooth boundary, as the integral over the boundary of $K$ of the $(n-i-1)$st elementary symmetric functions of the principal curvatures. In the unitary case, we have seen that the monomials $s^it^j$ correspond to Alesker’s $U_{k,p}$ basis, given by integrating the Hadwiger valuations of the intersections of $K$ with affine complex planes. On the other hand Alesker also defined an alternative basis, denoted $C_{k,p}$, given by averaging the Hadwiger valuations of the projections of $K$ to the elements of the various complex Grassmannians. Furthermore, H. Park [10] has classified the $U(n)$-invariant differential forms on the sphere bundle of $\mathbb{C}^n$, whose pairings with $N(K)$ give valuations in $\text{Val}^{U(n)}$ as before. Determining the linear relations among these bases is a fundamental open problem.

3. Say that a valuation $\varphi$ is positive if $\varphi(K) \geq 0$ for all convex bodies $K$, and monotone if $\varphi(K) \geq \varphi(L)$ whenever $K \supset L$. It is easy to see that the cones of positive and monotone valuations in $\text{Val}^{SO(n)}$ coincide, and consist of all nonnegative linear combinations of the Hadwiger valuations. What are the positive and monotone cones in $\text{Val}^{U(n)}$?

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