Bayesian Fixed-domain Asymptotics: Bernstein-von Mises Theorem for Covariance Parameters in a Gaussian Process Model

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Abstract

Gaussian process models typically contain finite dimensional parameters in the covariance functions that need to be estimated from the data. We study the Bayesian fixed-domain asymptotic properties of the covariance parameters in a Gaussian process with an isotropic Matérn covariance function, which has many applications in spatial statistics. Under fixed-domain asymptotics, it is well known that when the dimension of data is less than or equal to three, the microergodic parameter can be consistently estimated with asymptotic normality while the variance parameter and the range (or length-scale) parameter cannot. Motivated by the frequentist theory, we prove a Bernstein-von Mises theorem for the covariance parameters in isotropic Matérn covariance functions. We show that under fixed-domain asymptotics, the joint posterior distribution of the microergodic parameter and the range parameter can be factored independently into the product of their marginal posteriors as the sample size goes to infinity. The posterior of the microergodic parameter converges in total variation norm to a normal distribution with shrinking variance, while the posterior distribution of the range parameter does not necessarily converge to any degenerate distribution in general. Our theory allows an unbounded prior support for the range parameter on the whole positive real line. Furthermore, we propose a new property called the posterior asymptotic efficiency in linear prediction, and show that the Bayesian kriging predictor at a new spatial location with covariance parameters randomly drawn from their posterior distribution has the same prediction mean squared error as if the true parameters were known. In the special case of one-dimensional Ornstein-Uhlenbeck process, we derive an explicit form for the limiting posterior distribution of the range parameter and an explicit posterior convergence rate for the posterior asymptotic efficiency in linear prediction. We verify these asymptotic results in numerical examples.

Keywords: Fixed-domain asymptotics, Bernstein-von Mises theorem, Matérn covariance function, Posterior asymptotic efficiency in linear prediction

1 Introduction

Gaussian process has wide applications in spatial statistics, computer experiments, machine learning, and many other fields, as it can be used as a flexible prior distribution over the space of functions. The properties of a Gaussian process are exclusively determined by its mean function and covariance function given the Gaussianity assumption. In this paper, we consider a mean-zero isotropic Gaussian stochastic process $X = \{ X(s), s \in S \subseteq [0, T]^d \}$ on a fixed domain $S \subseteq [0, T]^d$, where $0 < T < \infty$ is a constant and the dimension $d \in \{1, 2, 3\}$. Such a dimension $d$ is of primary interest in spatial statistics. We assume that the covariance function of $X$ is the
isotropic Matérn covariance function given by
\[
\text{Cov}(X(s), X(t)) = \sigma^2 \mathcal{K}_{\alpha, \nu}(s - t) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} (\alpha \|s - t\|)^\nu \mathcal{K}_\nu (\alpha \|s - t\|),
\]
(1)
for any \(s, t \in S\), where \(\nu > 0\) is the smoothness parameter, \(\sigma^2 > 0\) is the variance parameter, and \(\alpha > 0\) is the inverse range (or length-scale) parameter, \(\mathcal{K}_{\nu}(\cdot)\) is the modified Bessel function of the second kind (Kreh [41]), and \(\| \cdot \|\) is the Euclidean norm. The Matérn covariance function is popular in applications of spatial statistics and computer experiments because the smoothness parameter \(\nu\) provides flexibility in controlling the smoothness of sample paths (Stein [63]). We assume that the observed data are \(X_n = (X(s_1), \ldots, X(s_n))^T\), the realization of \(X\) on the sampling points \(S_n\), where for a positive integer \(n\), \(S_n = \{s_1, \ldots, s_n\}\) is a sequence of distinct sampling points in the fixed domain \(S\). Parameter estimation and prediction of the Gaussian process \(X\) (known as kriging) is based on the observed data \(X_n\). For simplicity, we call \(\alpha\) the range parameter in the rest of the paper.

In Bayesian inference on Gaussian process models, it is common practice to assign prior distributions on the parameters \((\sigma^2, \alpha)\) in the covariance function, and the prediction of \(X(s^*)\) at a new location \(s^*\) is based on the posterior distribution of \((\sigma^2, \alpha)\) (Handcock and Stein [31], De Oliveira et al. [20]). There is abundant literature in Bayesian spatial statistics on speeding up the costly Gaussian process posterior computation for spatial datasets with a large sample size \(n\) (Banerjee et al. [6], Sang and Huang [54], Datta et al. [19], Guhaniyogi et al. [28], Heaton et al. [32], etc.) However, there is a clear lack of theoretical understanding of the asymptotic properties of the Bayesian posterior distributions of covariance parameters \((\sigma^2, \alpha)\). This theory is important because in practice, the covariance parameters \((\sigma^2, \alpha)\) are usually drawn from their posterior using sampling algorithms such as Markov chain Monte Carlo. The asymptotic properties of the posterior of \((\sigma^2, \alpha)\) are crucial for justifying the performance of Bayesian prediction based on such randomly drawn covariance parameters \((\sigma^2, \alpha)\).

We study the limiting behavior of the Bayesian posterior distributions of the covariance parameters \((\sigma^2, \alpha)\) in the Matérn covariance function in (1), under the fixed-domain asymptotics framework (Stein [57], Stein [63], Zhang [32]). To the best of our knowledge, this paper is the first theoretical work on the fixed-domain asymptotic properties of the Bayesian posterior distributions of the finite dimensional parameters in Gaussian process covariance functions. In the following, we explain why Bayesian fixed-domain asymptotic theory is important, the main technical difficulties, and our main contributions.

1.1 Why fixed-domain asymptotics?

In the fixed-domain asymptotics regime, the domain \(S\) remains fixed and bounded regardless of the increasing sampling size \(n\). This implies that as \(n\) goes to infinity, the sampling points \(S_n\) become increasingly dense in the domain \(S\), leading to increasingly stronger dependence between adjacent observations in \(X_n\). This theory setup matches up with the reality in many spatial applications; for example, the advances in sensor technology make it possible to collected spatial data in larger volume and higher resolution in a given region (Sun et al. [65]). Besides the fixed-domain asymptotics regime, there are also increasing-domain asymptotics (Mardia and Marshall [50]) and mixed-domain asymptotics (Chang et al. [13]), in which the domain is assumed to increase as \(n\) goes to infinity. Therefore, the minimum distance between two adjacent sampling points is either not decreasing or decreasing slowly with \(n\). Compared to fixed-domain asymptotics, the fixed-domain asymptotics regime is more suitable for interpolation of the spatial process; see Section 3.3 of Stein [63] for a cogent argument. Furthermore, Zhang and Zimmerman [83] has shown that compared to the increasing-domain asymptotics regime, the fixed-domain asymptotics has equally good estimation performance for the microergodic parameter and better estimation performance for the non-microergodic parameters.
1.2 What are the main difficulties in Bayesian fixed-domain asymptotics?

Theoretically, the increasingly stronger spatial dependence among the observed data $X_n$ in fixed-domain asymptotics leads to a lack of consistent estimation for the covariance parameters $(\sigma^2, \alpha)$ (Zhang [82]) and therefore poses significant challenges to theory development. When the dimension of sampling points $d = 1, 2, 3$, a well known fixed-domain asymptotics result Zhang [82] says that it is only possible to consistently estimate the microergodic parameter $\theta = \sigma^2 \alpha^{-2\nu}$ in an isotropic Matérn covariance function, but not the individual variance parameter $\sigma^2$ and the range parameter $\alpha$; see page 163 of Stein [63] for a general definition of microergodic parameter. On the other hand, both the variance and range parameters $(\sigma^2, \alpha)$ can be consistently estimated if $d \geq 5$, with the case of $d = 4$ still open (Anderes [1]). However, the cases with $d = 1, 2, 3$ are of primary interest in spatial and spatiotemporal applications.

The standard Bayesian asymptotic theory consists of results such as posterior consistency, posterior convergence rates, and the Bernstein-von Mises theorem (Ghosal and van der Vaart [24]). For parametric models, the Berstein-von Mises (BvM) theorem typically relies on the local asymptotic normality (LAN) condition and the existence of uniformly consistent tests; see for example, Chapter 10 in van der Vaart [69]. Since no consistent frequentist estimator exists for $(\sigma^2, \alpha)$ under fixed-domain asymptotics, one cannot expect to establish posterior consistency for their posterior distribution. Instead, we will consider the microergodic parameter $\theta = \sigma^2 \alpha^{-2\nu}$ which can be consistently estimated, and reparametrize the covariance function (1) by $(\theta, \alpha)$. Crowder [18] is an early work on the asymptotic normality of MLE in the presence of dependent observations and nuisance parameters. The techniques in this paper will establish the LAN condition for the microergodic parameter $\theta$ by strengthening the frequentist fixed-domain asymptotic results for the maximum likelihood estimator (MLE) of $\theta$ (Du et al. [21], Wang and Loh [75], Kaufman and Shaby [38]). The LAN for $\theta$ in this paper relies on the spectral analysis of the Matérn covariance function and holds uniformly over the “nuisance” range parameter $\alpha$, a parameter that cannot be consistently estimated. Such a uniform LAN condition based on data with increasingly stronger dependence is new in the literature and differs significantly from the LAN in classic parametric models with i.i.d. or weakly dependent data. The BvM theorem for the microergodic parameter $\theta$ is crucial and guarantees the asymptotic efficiency in Bayesian prediction of the Gaussian process $X$ at a new location.

For Bayesian inference on the parameters in covariance functions of Gaussian processes, the only theoretical work we are aware of is Shaby and Ruppert [55], who have worked under the increasing-domain asymptotics regime and have established that the joint posterior of all parameters in the tapered covariance functions converges to a limiting normal distribution. The result in Shaby and Ruppert [55] is close to the classic BvM theorem since the dependence among $X_n$ does not get stronger under the increasing-domain asymptotics. A key assumption in Shaby and Ruppert [55] that the covariance matrix of the observed data $X_n$ have lower and upper bounded eigenvalues, which can only hold under increasing-domain asymptotics but no longer holds under fixed-domain asymptotics. In fact, things become dramatically different under fixed-domain asymptotics as both $\sigma^2$ and $\alpha$ have no consistent estimators and hence, no posterior consistency. Our paper is the first to consider the fixed-domain asymptotic theory from the Bayesian perspective.

1.3 Our Contributions

The Bayesian fixed-domain asymptotic theory in this paper addresses the difficult case with $d = 1, 2, 3$ in the covariance function (1), which is of primary interest in spatial applications. The main technical challenge arises from the strong dependence in the data and the absence of consistent estimators of the variance and range parameters in Matérn covariance functions. For these reasons, our results are substantial and serve at the first step towards a full Bayesian fixed-domain asymptotic theory for general Gaussian process models.
This paper makes two major contributions. First, using observations from a mean zero Gaussian process with a Matérn covariance function, we establish a novel Bernstein-von Mises theorem for the microergodic parameter \( \theta = \sigma^2 \alpha^2 \nu \) and the range parameter \( \alpha \) jointly under fixed-domain asymptotics. We show that the joint posterior distribution of the microergodic parameter \( \theta \) and the range parameter \( \alpha \) can be factored independently into the product of their marginal posteriors asymptotically, under a general set of sufficient conditions (Theorem 1). The marginal posterior distribution of the microergodic parameter converges in total variation norm to a normal distribution at the parametric rate (Theorem 2). This limiting normal distribution is the same as that of the frequentist maximum likelihood estimator (MLE). In contrast, the marginal posterior of the range parameter does not necessarily converge to any degenerate distribution in general. This phenomenon has been observed in many previous spatial statistics literature and our theory formally gives an expression of this posterior for the range parameter.

Our theory allows the range parameter \( \alpha \) to take an unbounded prior support from zero to infinity. The unbounded support for the range parameter \( \alpha \) is generally not allowed in most of the frequentist fixed-domain asymptotic results for the MLE. Furthermore, our general BvM theory works for arbitrary design of the sampling points \( S_n \), no matter it being a fixed design or a random design. For the special case of 1-dimensional Matérn covariance with smoothness parameter \( 1/2 \) (an Ornstein-Uhlenbeck process) observed on an equispaced grid, we derive an explicit parametric form of the limiting marginal posterior of the range parameter \( \alpha \), which is close to a polynomially tilted normal distribution (Bochkina and Green [10]) and does not converge to any degenerate distribution in the asymptotics (Theorem 3).

Our second contribution is to show a Bayesian version of the asymptotic efficiency in linear prediction. In a series of works Stein [57], Stein [58], Stein [59], Stein [61], Stein [62], and Stein [64], Stein has systematically studied the theoretical conditions under which the linear predictions using a misspecified covariance function are asymptotically efficient, in the sense that the prediction mean squared errors (MSEs) calculated from a misspecified covariance function are asymptotically equal to those from the true covariance function. The consistent estimation of the microergodic parameter plays a fundamental role in showing this asymptotic efficiency in linear prediction. Based on our BvM theorems, we show that the MSE based on the covariance parameters randomly drawn from their posterior distribution is asymptotically equal to the MSE based on the true covariance parameters (Theorems 4 and 5), in the sense that their ratio converges to 1 in the posterior as the sample size tends to infinity. We further quantify the posterior convergence rate of this asymptotic efficiency similar to Stein [59] and Stein [64], with an explicit posterior convergence rate of asymptotic efficiency for the 1-dimensional Ornstein-Uhlenbeck process (Theorem 6).

Although we mainly focus on the Matérn covariance function, our techniques can be potentially extended to other covariance functions such as the tapered Matérn covariance function (Kaufman et al. [37]), the generalized Wendland covariance function (Bevilacqua et al. [8]), and those parametric covariance functions whose spectral densities have similar tail behaviors to that of the Matérn covariance function. To focus on the main idea, we present the theory only for the model with the microergodic parameter and the range parameter. The inclusion of a mean parameter and a nugget parameter for the variance of measurement error will significantly change the asymptotic distribution and will be investigated in the future. We postpone the detailed discussion on these extensions to Section 5.

Our work has important relations to several research topics. First, our theory can be viewed as the Bayesian counterpart of the frequentist fixed-domain asymptotic theory on the maximum likelihood estimator in Ying [80], Ying [81], Zhang [82], Chen et al. [14], Loh [46], Du et al. [21], Wang and Loh [75], Kaufman and Shaby [38], Chang et al. [12], Velandia et al. [73], Bachoc et al. [3], Bachoc and Lagnoux [4], etc. Second, our posterior asymptotic efficiency result is a counterpart of Stein’s work in the Bayesian setup and guarantees the optimal estimation of prediction MSE. Third, our BvM theorems have some relation to the semiparametric BvM
results with nuisance parameters in Bickel and Kleijn [9] and Shen [30], though our proof techniques are different. Fourth, our model can be viewed as a partially identified model and be related to the previous Bayesian asymptotic results in Moon and Schorfheide [51] and Jiang [34]. We will elaborate on these relations in later sections.

The remainder of the paper is organized as follows. In Section 2 we introduce the basic model setup and present the main BvM theorems for covariance parameters under fixed-domain asymptotics. Section 3 presents the theory on posterior asymptotic efficiency in linear prediction. Section 4 shows the empirical results from simulation examples to verify the main theorems. Section 5 discusses potential extensions of our theory to more general model setups. Appendix A includes the proofs of main results. All the other technical proofs are in the supplementary materials.

2 Bernstein-von Mises Theorem for Covariance Parameters

We consider the Bayesian estimation of \( (\sigma^2, \alpha) \) in the covariance function \( \mathcal{L}_n \) based on the observed data \( X_n \). Throughout the paper, we assume that the smoothness parameter \( \nu > 0 \) is fixed and known. Estimation of the smoothness parameter \( \nu \) is an important research topic with some recent developments in frequentist literature (Loh [47], Loh et al. [48]), but is beyond the scope of the current paper. We let the true parameter values in the Matérn covariance function \( \sigma^2 \) and \( \alpha \) be related to the previous Bayesian asymptotic results in Moon and Schorfheide [51] and Jiang [34].

The asymptotic normality for the MLE of \( \theta = \sigma^2 \alpha^\nu \) can still be consistently estimated, for example the \( \tilde{\sigma}^2 \), \( \tilde{\alpha}^\nu \) as \( \tilde{\sigma}^2 = \frac{1}{n} \mathbf{X}^\top \mathbf{R}^{-1} \mathbf{X} \), \( \tilde{\alpha}^\nu = \frac{1}{n} \mathbf{X}^\top \mathbf{R}^{-1} \mathbf{X} \). We use the notation \( \mathbf{X} \) be the implied \( n \times n \) Matérn correlation matrix on \( \mathcal{S} \) indexed by \( \alpha \), whose \( (i, j) \)th entry is \( \mathbf{K}_{n, \nu} (s_i - s_j) \), for \( i, j \in \{1, \ldots, n\} \). We omit the dependence of \( R_n \) on \( \nu \). The covariance matrix of \( X_n \) is then \( \sigma^2 \mathbf{R}_n \). The log-likelihood function based on \( X_n \) is

\[
\mathcal{L}_n (\sigma^2, \alpha) = -\frac{n}{2} \log \sigma^2 - \frac{1}{2} \log |R_n| - \frac{1}{2 \sigma^2 X_n^\top R^{-1} X_n},
\]

where \( |A| \) is the determinant of a generic matrix \( A \).

In the spatial statistics literature, it is well known (Zhang [82]) that the parameters \( (\sigma^2, \alpha) \) cannot be consistently estimated under fixed-domain asymptotics. The main reason is that for two Gaussian measures \( \mathcal{G}(0, \sigma^2 K_{\alpha, \nu}) \) on the space of sample paths on the domain \( \mathcal{S} \subset \mathbb{R}^d \), they are equivalent (or mutually absolutely continuous) as long as \( \sigma^2 \alpha^\nu = \sigma^2 \alpha_2^{\nu'} \), and they are orthogonal otherwise. As a result, one cannot tell from a finite sample \( X_n \) which parameter values \( (\sigma^2, \alpha) \) are correct. Empirically, this phenomenon has been also observed (Fuglstad et al. [23]). Despite the lack of consistent estimator for \( (\sigma^2, \alpha) \), the microergodic parameter \( \theta = \sigma^2 \alpha^\nu \) can still be consistently estimated, for example the MLE (Zhang [82]). For a fixed \( \alpha > 0 \), the MLE of \( \theta \) can be obtained from maximizing \( \mathcal{L}_n (\sigma^2, \alpha) \) over \( \sigma^2 \),

\[
\tilde{\sigma}^2 = \frac{1}{n} \mathbf{X}^\top \mathbf{R}^{-1} \mathbf{X},
\]

such that the maximizer for the microergodic parameter \( \theta \) given a fixed \( \alpha \) is

\[
\tilde{\alpha} = \tilde{\sigma}^2 \alpha^\nu = \frac{\alpha^\nu}{n} \mathbf{X}^\top \mathbf{R}^{-1} \mathbf{X}.
\]

We can plug in \( \tilde{\alpha} \) in \( \mathcal{L}_n \) to obtain the profile log-likelihood function (up to an additive constant), which plays an important role in our BvM theorems:

\[
\tilde{\mathcal{L}}_n (\alpha) \equiv \mathcal{L}_n (\alpha, 2 \tilde{\alpha}) = -\frac{n}{2} \log \frac{\mathbf{X}^\top \mathbf{R}^{-1} \mathbf{X}}{n} - \frac{1}{2} \log |R_n|.
\]
and Loh \cite{75}, and Kaufman and Shaby \cite{38}. In particular, if $\alpha \in [\alpha_1, \alpha_2]$ for some constant lower and upper bounds $0 < \alpha_1 < \alpha_2 < \infty$, then the MLE of $\theta$, denoted by $\hat{\theta}$, which is also $\tilde{\theta}_n$ at the maximizer of $\alpha \in [\alpha_1, \alpha_2]$, satisfies that $\sqrt{n}(\hat{\theta} - \theta_0) \overset{D}{\rightarrow} \mathcal{N}(0, 2\theta_0^2)$ as $n \rightarrow \infty$ under fixed-domain asymptotics, where $\theta_0 = \sigma_0^2 \alpha_0^{2\nu}$ is the true value, $\overset{D}{\rightarrow}$ is the convergence in distribution, and $\mathcal{N}(u, v)$ is the normal distribution with mean $u$ and variance $v$. Without confusion, later on we also use notation such as $\mathcal{N}(d\theta|u, v)$ to denote the same normal distribution for $\theta$, in order to highlight the argument of the normal distribution.

We study the Bayesian posterior distribution based on the log-likelihood \cite{2}. We reparametrize the model using $(\theta, \alpha)$, with $\theta = \sigma^2 \alpha^{2\nu}$ being the microergodic parameter. This reparametrization has been suggested in Stein \cite{63} (p.175) and used in some previous work on Gaussian random field models Fuglstad et al. \cite{23}. For the clarity of notation, we will still maintain the parametrization of $(\sigma^2, \alpha)$ for the log-likelihood functions and quantities related to the probability distributions, such as $P_{(\sigma^2, \alpha)}$ for the probability distribution of $\text{GP}(0, \sigma^2 K_{\alpha, \nu})$, $\mathbb{E}_{(\sigma^2, \alpha)}(\cdot)$ and $\text{Var}_{(\sigma^2, \alpha)}(\cdot)$ for the mean and variance under $P_{(\sigma^2, \alpha)}$. The change of variable from $\sigma^2$ to $\theta = \sigma^2 \alpha^{2\nu}$ is often clear from the context. We assign prior distributions on $(\theta, \alpha)$ and write the joint prior density as $\pi(\theta, \alpha) = \pi(\theta|\alpha)\pi(\alpha)$. The joint posterior density of $(\theta, \alpha)$ is given by

$$
\pi(\theta, \alpha|X_n) = \frac{\exp\left\{\mathcal{L}_n(\theta/\alpha^{2\nu}, \alpha)\right\}}{\int_0^\infty \int_0^\infty \exp\left\{\mathcal{L}_n(\theta'/\alpha'^{2\nu}, \alpha')\right\}} \pi(\theta'|\alpha')\pi(\alpha')d\alpha'd\theta'.
$$

We will use $\Pi(d\theta, d\alpha|X_n)$ to denote the posterior probability measure with the density in \cite{4}.

We define some additional notation. Let $\mathbb{R}_+ = (0, +\infty)$. For two positive sequences $a_n$ and $b_n$, we use $a_n < b_n$ and $b_n > a_n$ to denote the relation $\lim_{n\rightarrow\infty} a_n/b_n = 0$, $a_n \leq b_n$, and $b_n \geq a_n$ to denote the relation $\lim \sup_{n\rightarrow\infty} a_n/b_n < +\infty$, and $a_n \asymp b_n$ to denote the relation $a_n \leq b_n$ and $a_n \geq b_n$.

### 2.1 Main Results

We first present a BvM theorem for $\theta$ conditional on a fixed $\alpha > 0$. For two probability measures $P_1, P_2$, let $\|P_1(\cdot) - P_2(\cdot)\|_{\text{TV}} = \sup_{A} |P_1(A) - P_2(A)|$, where the supremum is taken over all measurable set $A$. We need the following assumption on the prior, which is mild and satisfied in most applications.

(A.1) The conditional prior density of $\theta$ given $\alpha$, $\pi(\theta|\alpha)$, is a proper prior density that is continuously differentiable in $\theta$, continuous in $\alpha$, and finite everywhere for all $\theta \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+$. $\pi(\theta|\alpha)$ does not depend on $n$. $\pi(n|\theta_0|\alpha) > 0$ for all $\alpha > 0$.

**Theorem 1** (BvM Theorem for Conditional Posterior). Suppose that $\alpha > 0$ is fixed and does not depend on $n$. Under Assumption \cite{[A.1]}, the conditional posterior distribution of $\theta$ given $\alpha > 0$ satisfies that

$$
\left\|\Pi(d\theta|X_n, \alpha) - \mathcal{N}\left( d\theta|\tilde{\theta}_n, 2\theta_0^2/n\right) \right\|_{\text{TV}} \rightarrow 0,
$$

as $n \rightarrow \infty$ almost surely $P_{(\sigma_0^2, \alpha_0)}$, where $\tilde{\theta}_n$ is given in \cite{4}, and $\Pi(\cdot|X_n, \alpha)$ is the conditional posterior probability measure of $\theta$ given a fixed $\alpha > 0$ with the density

$$
\pi(\theta|X_n, \alpha) = \frac{p(X_n|\theta, \alpha)\pi(\theta|\alpha)}{\int_0^\infty p(X_n|\theta', \alpha)\pi(\theta'|\alpha)d\theta'}.
$$

Theorem \cite{1} shows that under fixed-domain asymptotics, the conditional posterior $\pi(\theta|X_n, \alpha)$ converges in total variation norm to the normal distribution $\mathcal{N}(\tilde{\theta}_n, 2\theta_0^2/n)$. For a given range parameter $\alpha > 0$, $\tilde{\theta}_n$ is the MLE of $\theta$, and $2\theta_0^2$ is the asymptotic variance of the MLE $\hat{\theta}_n$. Therefore, this result appears similar to the classic BvM theorem in regular parametric models.
for independent data, such as Theorem 8.2 in Lehmann and Casella [44] and Theorem 10.1 in van der Vaart [69], where the limiting normal distribution is centered at the MLE with variance equal to the asymptotic variance of MLE. The classic BvM theorem usually relies on the local asymptotic normality (LAN) condition and the existence of uniformly consistent tests (Theorem 10.1 in van der Vaart [69]). The main technical challenge in the proof of Theorem 1 is to establish the LAN condition for data with increasingly stronger dependence under fixed-domain asymptotics, which is fundamentally different from the BvM theorem for independent data. We need the asymptotic normality of $\tilde{\theta}_\alpha$ at a given range parameter $\alpha > 0$ which can be different from the truth $\theta_0$. Such asymptotic normality of $\tilde{\theta}_\alpha$ has been derived previously in Du et al. [21], Wang and Loh [75], and Kaufman and Shaby [38], which heavily depends on the spectral analysis of Matérn covariance functions. Our proof relies on a refined version of the analysis in Wang and Loh [75]; see Section S1 for the details.

In most spatial applications, the range parameter $\alpha$ is unknown and assigned a prior $\pi(\alpha)$. Next, we present the BvM theorem for the joint posterior of $(\theta, \alpha) \in \mathbb{R}^d_+$. The consistency of the MLE of $\theta$ and the nonexistence of consistent frequentist estimator for $\alpha$ indicates that the posterior of $\theta$ should converge to a normal limit, while the posterior of $\alpha$ does not necessarily converge to any fixed value under fixed-domain asymptotics. We prove this idea rigorously. We define two small positive constants $\kappa$ and $\overline{\kappa}$ that depend on the smoothness $\nu > 0$ and the dimension $d$ ($d \in \{1, 2, 3\}$), together with two deterministic sequences $\underline{\alpha}_n$ and $\overline{\alpha}_n$:

$$\kappa = \frac{1}{2} \min \left\{ \frac{3.9 - d}{(d + 3.94)(8\nu + 4d - 3.9)}, \frac{1}{4(3\nu + d)}, 0.1 \right\},$$

$$\overline{\kappa} = \frac{1}{2} \min \left\{ \frac{3.9 - d}{(d + 3.94)(8\nu + 4d + 3.9)}, \frac{1}{2(2\nu + d + 2)}, 0.04 \right\},$$

$$\underline{\alpha}_n = n^{-\kappa}, \quad \overline{\alpha}_n = n^{\overline{\kappa}}. \tag{9}$$

The choices of $\pi$ and $\kappa$ in (9) are not unique and can be replaced by other sufficiently small positive numbers. This will be made clear from our proofs in the supplementary materials. By definition, $\underline{\alpha}_n \to 0$ and $\overline{\alpha}_n \to +\infty$ as $n \to \infty$, and both are in slow polynomial rates. A key result is that uniformly for all $\alpha$ in the slowly expanding interval $[\underline{\alpha}_n, \overline{\alpha}_n]$, the difference between $\tilde{\theta}_\alpha$ and $\tilde{\theta}_{\alpha_0}$ converges to zero at a faster rate than $n^{-1/2}$.

**Lemma 1.** There exist a large integer $N_1$ and a positive constant $\tau \in (0, 1/2)$ that only depend on $\nu, d, T, \alpha_0$, such that for all $n > N_1$,

$$\Pr \left( \sup_{\alpha \in [\underline{\alpha}_n, \overline{\alpha}_n]} \sqrt{n} \left| \tilde{\theta}_\alpha - \tilde{\theta}_{\alpha_0} \right| \leq \theta_0 n^{-\tau} \right) \geq 1 - 2 \exp(-\log^2 n), \tag{10}$$

where $\underline{\alpha}_n$ and $\overline{\alpha}_n$ are defined in (9), and hereafter $\Pr(\cdot)$ denotes the probability under true probability measure $P_{(\theta_0^2, \alpha_0)}$.

The proof of Lemma 1 involves careful spectral analysis of the isotropic Matérn covariance functions. We leverage the powerful spectral analysis tools in Wang and Loh [75] and strengthen them with concentration inequalities to upper bound the difference between $\tilde{\theta}_\alpha$ and $\tilde{\theta}_{\alpha_0}$. To obtain the supremum convergence on the interval $[\underline{\alpha}_n, \overline{\alpha}_n]$, we use the important finding in Kaufman and Shaby [38] that $\tilde{\theta}_\alpha$ is a monotonically increasing function in $\alpha$, such that our proof for the supremum can completely circumvent any empirical process argument. We will present these detailed analysis in Section S2.

Based on the uniform convergence in Lemma 1, a heuristic argument to extend the conditional BvM result in Theorem 1 to the joint posterior of $(\theta, \alpha)$ is as follows: For each $\alpha \in [\underline{\alpha}_n, \overline{\alpha}_n]$, the conditional posterior $\pi(\theta | X_n, \alpha)$ can be approximated by the normal distribution $N(\tilde{\theta}_\alpha, 2\theta_0^2/n)$. Since the center $\tilde{\theta}_\alpha$ only differs from $\tilde{\theta}_{\alpha_0}$ by a higher order term $O(n^{-1/2-\tau})$,
this normal distribution can be further approximated by \(\mathcal{N}(\tilde{\theta}_{\alpha_0}, 2\theta_0^2/n)\), whose mean parameter only depends on the data \(X_n\) but not \(\alpha\). Hence, the limiting distribution of \(\theta\) is approximately independent of \(\alpha\).

To solidify this idea, we need additional prior conditions such that the posterior probabilities outside the interval \([\alpha_n, \pi_n]\) can be made small, such that the convergence to the normal limit inside \([\alpha_n, \pi_n]\) is dominant in driving the asymptotics of the joint posterior distribution of \((\theta, \alpha)\).

We specify the following general assumptions on the prior densities \(\pi(\theta|\alpha)\) and \(\pi(\alpha)\).

(A.2) There exist positive constants \(C_{\pi,1}, C_{\pi,2}\), and \(C_{\pi,3}\) that can depend on \(\nu, d, T, \alpha_0, \theta_0\), such that \(0 < C_{\pi,1} \leq 1/2, 0 < C_{\pi,3} < 1\), and for \(\alpha_n\) and \(\pi_n\) defined in (9), for all sufficiently large \(n\),

\[
\sup_{\alpha \in [\alpha_n, \pi_n]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \left| \frac{\partial \log \pi(\theta|\alpha)}{\partial \theta} \right| \leq n^{C_{\pi,1}}, \quad (11)
\]

\[
\sup_{\alpha \in [\alpha_n, \pi_n]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \frac{\pi(\theta|\alpha)}{\pi(\theta_0|\alpha)} \leq n^{C_{\pi,2}}, \quad (12)
\]

\[
\inf_{\alpha \in [\alpha_n, \pi_n]} \log \pi(\theta_0|\alpha) \geq -n^{C_{\pi,3}}. \quad (13)
\]

(A.3) The marginal prior \(\pi(\alpha)\) is a proper and continuous density function on \(\mathbb{R}_+\). \(\pi(\alpha)\) does not depend on \(n\). \(\pi(\alpha_0) > 0\). \(\int_0^{\infty} \pi(\theta_0|\alpha)\pi(\alpha) \, d\alpha < \infty\). There exist positive constants \(c_\pi < (\nu + d/2)\kappa\) and \(\tau_\pi < (\nu + d/2)\pi\) for \(\kappa\) and \(\pi\) defined in (9), such that for \(\alpha_n\) and \(\pi_n\) defined in (9), and for all sufficiently large \(n\),

\[
\int_0^{\alpha_n} \alpha^{-n(\nu + d/2)} \pi(\alpha) \, d\alpha \leq \exp \left( c_\pi n \log n \right), \quad (14)
\]

\[
\int_0^{\alpha_n} \alpha^{-n(\nu + d/2)} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha \leq \exp \left( c_\pi n \log n \right), \quad (15)
\]

We will discuss these two assumptions in greater detail after presenting our main BvM theorem for the joint posterior of \((\theta, \alpha)\).

**Theorem 2** (BvM Theorem for Joint and Marginal Posteriors). Suppose that both \((\theta, \alpha)\) are assigned priors. Under Assumptions [A.1], [A.2], and [A.3], the posterior distributions of \(\theta\) and \(\alpha\) are asymptotically independent, in the sense that the joint posterior distribution of \((\theta, \alpha)\) satisfies

\[
\left\| \Pi(d\theta, d\alpha|X_n) - \mathcal{N} \left( d\theta|\tilde{\theta}_{\alpha_0}, 2\theta_0^2/n \right) \times \tilde{\Pi}(d\alpha|X_n) \right\|_{TV} \to 0,
\]

as \(n \to \infty\) almost surely \(P_{(\pi^2, \alpha_0)}\), where \(\tilde{\Pi}(d\alpha|X_n)\) is the profile posterior distribution with density \(\tilde{\pi}(\alpha|X_n)\) given by

\[
\tilde{\pi}(\alpha|X_n) = \frac{\exp \left\{ \tilde{\mathcal{L}}_n(\alpha) \right\} \pi(\alpha|\theta_0)}{\int_0^{\infty} \exp \left\{ \tilde{\mathcal{L}}_n(\alpha') \right\} \pi(\alpha'|\theta_0) \, d\alpha'}, \quad (17)
\]

with the profile log-likelihood \(\tilde{\mathcal{L}}_n(\alpha)\) given in (5) and \(\pi(\alpha|\theta_0)\) being the conditional prior density of \(\alpha\) given \(\theta = \theta_0\). Furthermore, this profile posterior density \(\tilde{\pi}(\alpha|X_n)\) is always well defined for any data \(X_n\). As a result, the total variation distance between \(\Pi(d\theta|X_n)\) and \(\mathcal{N} \left( d\theta|\tilde{\theta}_{\alpha_0}, 2\theta_0^2/n \right)\) converges to zero, and the total variation distance between \(\Pi(d\alpha|X_n)\) and \(\tilde{\Pi}(d\alpha|X_n)\) converges to zero, as \(n \to \infty\) almost surely \(P_{(\pi^2, \alpha_0)}\).
Theorem\textsuperscript{2} provides a clear description of the limiting behavior of the joint posterior of \((\theta, \alpha)\). Under fixed-domain asymptotics, the microergodic parameter \(\theta\) and the range parameter \(\alpha\) have asymptotically independent posterior distributions. The posterior of \(\theta\) is centered at \(\hat{\theta}_n\) and the variance is the same \(2\theta_0^2/n\) as the limiting variance of MLE \(\hat{\theta}\). In fact, according to Lemma\textsuperscript{1} the center \(\hat{\theta}_n\) can be replaced by \(\hat{\theta}_1\) for any fixed \(\alpha_1 > 0\), since \(\alpha_1\) will be eventually covered by the slowly expanding interval \([\alpha_n, \pi_n]\), and the difference between \(\hat{\theta}_n\) and \(\hat{\theta}_1\) is negligible compared to the limiting normal standard deviation \(\sqrt{2\theta_0^2/n}\).

The posterior convergence of the microergodic parameter \(\theta\) shows that we can consistently estimate the equivalent class of Gaussian measures using the Bayesian procedure. An important consequence is that based on a random draw of parameters \((\theta, \alpha)\) from the posterior, the prediction variance at a new location is asymptotically equal to the prediction variance based on the true parameters \((\theta_0, \alpha_0)\). We will elaborate this in Section 3.

Theorem\textsuperscript{2} also shows that the marginal posterior density of \(\alpha\) can be approximated by the more abstract profile posterior with density \(\pi(\alpha|X_n)\), which is based on the profile likelihood of \(\alpha\). Using the result in Gu et al.\textsuperscript{27}, we can show that this profile posterior is always well defined if \(X_n\) is generated from a Matérn covariance function, with no requirement on the sampling design or distribution of \(S_n\). On the other hand, without further assumptions on \(S_n\), it is not likely that the form of the profile posterior density \(\pi(\alpha|X_n)\) can be simplified. In general, this profile posterior of \(\alpha\) does not necessarily converge to any degenerate distribution. In Theorem\textsuperscript{3} below, in the case of 1-dimensional Ornstein-Uhlenbeck process (Matérn with \(\nu = 1/2\)) observed on an equispaced grid, we approximate \(\pi(\alpha|X_n)\) using an explicit density of \(\alpha\) that asymptotically does not contract to any fixed value with high probability. This non-converging property of \(\pi(\alpha|X_n)\) matches with empirical observations in many spatial applications. We also verify this phenomenon using simulation examples in Section 3.

The difficulty in the estimation of the range parameter \(\alpha\) is a well-known problem in the Gaussian process literature (Kennedy and O’Hagan\textsuperscript{39}). Different values of \(\alpha\) in the Matérn covariance function\textsuperscript{1} can still generate Gaussian processes with similar sample paths (Fuglstad et al.\textsuperscript{23}) as long as they have the same microergodic parameter \(\theta\), thus making it difficult to decide an appropriate value for \(\alpha\) from the data. For finding the frequentist MLE of \(\alpha\), Zhang\textsuperscript{82} and many others have observed that for a fixed value of \(\theta > 0\), \(L_n(\theta/\alpha^{2\nu}, \alpha)\) has a long right tail in \(1/\alpha\) that creates problem for maximization over \(\alpha\). The sampling distribution of the MLE of \(\alpha\) does not show signs of convergence even as \(n \to \infty\). For Bayesian inference, Gu et al.\textsuperscript{27} identifies prior conditions using the objective priors in Berger et al.\textsuperscript{7} for robust estimation of \(1/\alpha\) in finite samples, in the sense that one can avoid the situation where the posterior mode of \(\alpha\) with a finite sample \(X_n\) is attained at either zero or infinity. Though our goal is not to address the estimation problem for \(\alpha\), our technical proofs have explored some properties of the profile log-likelihood function \(\hat{L}_n(\alpha)\) and the profile posterior \(\pi(\alpha|X_n)\), which could be of independent interest for Matérn covariance functions; see the supplementary materials for details.

2.2 On the Prior Assumptions

We discuss the two technical prior assumptions\textsuperscript{[A.2]} and \textsuperscript{[A.3]}. Both assumptions are used to ensure that the posterior converges to the normal limit for \(\theta\).\textsuperscript{11} and \textsuperscript{12} in \textsuperscript{[A.2] require that the conditional prior \(\pi(\theta|\alpha)\) does not vary too dramatically in a neighborhood of \(\theta_0\) and in the slowly expanding interval \([\alpha_n, \pi_n]\). The interval \((\theta_0/2, 2\theta_0)\) just used as a neighborhood of the truth \(\theta_0\), and in principle can be replaced by \((\theta_0 - \delta_0, \theta_0 + \delta_0)\) for any \(0 < \delta_0 < \theta_0\).\textsuperscript{13} in \textsuperscript{[A.2]} requires that the prior assigns a minimum of \(\exp(-nC_{\nu, 3})\) prior mass on the true value \(\theta_0\) over all \(\alpha \in [\alpha_n, \pi_n]\). Such minimal prior mass assumption is often necessary for achieving the basic posterior consistency in Bayesian models (Ghosal and van der Vaart\textsuperscript{24}). In particular, we can verify Assumption\textsuperscript{[A.2]} for the following examples of priors \(\pi(\theta|\alpha)\) which are commonly used in spatial applications.
Proposition 1. Suppose that the prior $\pi(\theta|\alpha)$ does not depend on the sample size $n$. Then Assumption (A.2) holds in either one of the following cases:

(i) $\pi(\theta|\alpha) = \pi(\theta)$ is independent of $\alpha$. $\pi(\theta)$ has continuous first derivative on $\mathbb{R}_+$ and $\pi(\theta) > 0$ for all $\theta \in \mathbb{R}_+$.

(ii) $\pi(\alpha)$ is supported on a compact interval $[\alpha_1, \alpha_2]$, with constants lower and upper bounds $0 < \alpha_1 < \alpha_2 < \infty$. $\pi(\theta|\alpha)$ is positive for all $(\theta, \alpha) \in \mathbb{R}_+^2$, is continuous in $\alpha \in \mathbb{R}_+$, and has continuous first derivative with respect to $\theta$ on $\mathbb{R}_+$ for all $\alpha \in \mathbb{R}_+$.

(iii) The prior of $\sigma^2$ is independent of $\alpha$ and belongs to the broad distribution family of the generalized beta of the second kind (or the Feller-Pareto family, Arnold [2]), with the density

$$
\pi(\sigma^2) = \frac{\Gamma(\gamma_1 + \gamma_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \frac{(\sigma^2/b)^{(\gamma_2/\gamma_1 - 1)}}{b^{\gamma_1}(\sigma^{2b} + \gamma_1 + \gamma_2)}
$$

with parameters $b > 0, \gamma > 0, \gamma_1 > 0, \gamma_2 > 0$.

Proposition 1 shows that Assumption (A.2) about $\pi(\theta|\alpha)$ is satisfied by a wide range of prior distributions on $\theta$ with continuously differentiable densities. Case (i) says that (A.2) holds as long as the priors of $\theta$ and $\alpha$ are independent. Case (ii) says that (A.2) holds as long as the support of the prior of $\alpha$ is bounded away from zero and infinity. Compactly supported priors for the range parameter $\alpha$ have been widely used in Bayesian spatial statistics literature, though the Gaussian process models are usually more complicated than ours; see for example, Banerjee et al. [6], Finley et al. [22], Banerjee et al. [5], Sang and Huang [54], Datta et al. [19], Guhaniyogi et al. [28], etc. Case (iii) provides the example in which an independent prior is assigned on the variance parameter $\sigma^2$ instead of on $\theta$. The generalized beta of the second kind (or Feller-Pareto family, Brazauskas [11], Arnold [2]) has polynomially decaying tails at both $\sigma^2 \to 0+$ and $\sigma^2 \to +\infty$. This family covers a wide range of continuous distributions on $(0, +\infty)$ including the half-Student’s $t$ distributions, the $F$ distributions, the log-logistic distributions, the Burr distributions, and many others (Arnold [2]). Case (iii) mainly illustrates that if $\pi(\alpha)$ has a full support on $[0, +\infty)$, then $\pi(\theta|\alpha)$ cannot decay too fast in the two tails. For example, if $\pi(\theta|\alpha)$ has exponentially decaying tails at either $\theta \to 0+$ and $\theta \to +\infty$, then (A.2) is not satisfied when $\pi(\alpha)$ has a full support on $[0, +\infty)$. Fortunately, most spatial applications use a compacted supported prior for $\alpha$, and (A.2) is satisfied as in Case (ii).

Next, we discuss Assumption (A.3) which imposes some technical conditions on the tail behavior of $\pi(\alpha)$ as $\alpha \to 0+$ and $\alpha \to +\infty$. The following proposition gives some concrete cases, where $p(\alpha)$ can be replaced by either $\pi(\alpha)$ or $\pi(\theta_0|\alpha)\pi(\alpha)$. Since \( \int_0^\infty \pi(\theta_0|\alpha)\pi(\alpha)d\alpha < \infty \) as in (A.3), the tail conditions on $\pi(\theta_0|\alpha)\pi(\alpha)$ are the same as the tail conditions on $\pi(\alpha|\theta_0)$.

Proposition 2. For any function $p(\alpha)$ defined for $\alpha > 0$, and $\kappa, \pi, \nu, \alpha_n, \pi_n$, defined in (9), there exists a constant $\overline{\alpha}_n$ such that $0 < \overline{\alpha}_n < (\nu + d/2)\overline{\kappa}$ and for all sufficiently large $n$,

$$
\int_{\pi_n}^\infty \alpha^{(\nu+d/2)}p(\alpha)d\alpha \leq \exp(\overline{\alpha}_n n \log n),
$$

if either one of the following conditions holds true:

(i) $p(\alpha) \leq \exp(-\alpha^{\delta_1})$ for all $\alpha > \overline{\alpha}_n$, for some constant $\delta_1 > 1/\pi$ and for all sufficiently large $n$;

(ii) $p(\alpha) \leq n^{\delta_2} \exp(-n^{\delta_2}\alpha)$ for all $\alpha > \overline{\alpha}_n$, for some constant $1 - \pi < \delta_2 \leq \delta_3 < \infty$ and all sufficiently large $n$.

Similarly, there exists a constant $\overline{c}_n$ such that $0 < \overline{c}_n < (\nu + d/2)\overline{\kappa}$ and for all sufficiently large $n$,

$$
\int_0^{\overline{c}_n} \alpha^{-n(\nu+d/2)}p(\alpha)d\alpha \leq \exp(\overline{c}_n n \log n),
$$

if either one of the following conditions holds true:
(i) \( p(\alpha) \leq \exp(-\alpha^{-\delta_1}) \) for all \( 0 < \alpha < \bar{\alpha}_n \), for some constant \( \delta_1 > 1/\kappa \) and for all sufficiently large \( n \);

(ii) \( p(\alpha) \leq n^{\delta_3} \exp(-n^{\delta_2}/\alpha) \) for all \( 0 < \alpha < \bar{\alpha}_n \), for some constant \( 1 - \kappa < \delta_3 < \infty \) and all sufficiently large \( n \).

\( p(\alpha) \) in Proposition 2 is to be replaced by \( \pi(\alpha) \) and \( \pi(\theta_0/\alpha) \pi(\alpha) \) in (14) and (15) in Assumption (A.3). Two types of tail decaying conditions are given in Proposition 2. In the first case, the tail of \( \pi(\alpha) \) or \( \pi(\theta_0) \) decays at the exponential power rate \( \exp(-\alpha^{\delta_1}) \) in the right tail (or \( \exp(-\alpha^{-\delta_1}) \) in the left tail), with some lower conditions on \( \delta_1 \) depending on the values of \( \bar{\alpha} \) (or \( \bar{\alpha} \)). This condition requires that \( \pi(\alpha) \) and \( \pi(\alpha/\theta_0) \) decay very fast in the right (or left) tail. One example of \( \pi(\alpha) \) is that \( \alpha^{1/\min(e,\pi)} \) follows the inverse Gaussian distribution, since the inverse Gaussian distribution has exponentially decaying tails at zero and infinity. In the second case of Proposition 2, we allow the tails of \( \pi(\alpha) \) and \( \pi(\alpha/\theta_0) \) to be upper bounded by some exponential rate in \( \alpha \) that depends on \( n \). These tail decaying conditions in Proposition 2 and Assumption (A.3) can ensure that the BvM type convergence to a normal limit will be dominant in the joint posterior of \((\theta, \alpha)\).

Theorem 2 allows an \( \mathbb{R}_+ \) support for \( \alpha \), which is stronger than a BvM theorem assuming a compactly supported prior for \( \alpha \). Nevertheless, the tail conditions in (A.3) are often stronger than necessary in practice. This is partly because our BvM result in Theorem 2 has no assumption on the design or distribution of the sampling points \( S_n \). Even when \( S_n \) is highly unevenly distributed in \( S \) or is not dense in the full space of \( S \), Theorem 2 still holds true under Assumption (A.3) which allows the prior \( \pi(\alpha) \) to have a full support in \([0, +\infty)\). If one is willing to impose more assumptions on \( S_n \), for example, the maximum distance between two adjacent points decreases at a certain rate to zero, then it is possible to relax the tail conditions in (A.3). For a general smoothness \( \nu \), this inevitably requires more sophisticated matrix theory for the properties of \( X_n R^{-1}_\alpha X_n \) and \( |R_\alpha| \) as \( \alpha \to 0+ \) and \( \alpha \to +\infty \), since these two terms determine the properties of the profile log-likelihood function [5]. We will see in Theorem 3 below that in a special case when the sampling points are from an equispaced grid, the tail conditions in (A.3) can be significantly weakened and the BvM holds for a broader class of priors on \( \alpha \).

### 2.3 BvM for 1-Dimensional Ornstein-Uhlenbeck Process

For a concrete example of Theorem 2, we consider the special case of \( d = 1 \), \( S = [0, 1] \), and \( \nu = 1/2 \) in the Matérn covariance function. The covariance function becomes \( \text{Cov}(X(s), X(t)) = \sigma^2 \exp(-\alpha|s - t|) \) for \( s, t \in [0, 1] \), which is also known as the exponential covariance function. The resulted stochastic process \( X \) is the 1-dimensional Ornstein-Uhlenbeck process (Rasmussen and Williams [53]). The frequentist MLE of this model under fixed-domain asymptotics has been studied in Ying [80], Ying [81], Chen et al. [14], Du et al. [21], etc. We assume that the sampling points in \( S_n \) are on the equispaced grid with \( s_i = i/n \) for \( i = 1, \ldots, n \). The inverse matrix \( R^{-1}_\alpha \) is given by

\[
(R^{-1}_\alpha)_{ii} = \begin{cases} 
1 - e^{-2\alpha/n}, & i = 1, n \\
1 + e^{-2\alpha/n} / (1 - e^{-2\alpha/n}), & i = 2, \ldots, n - 1 
\end{cases}
\]

\[
(R^{-1}_\alpha)_{i,i+1} = (R^{-1}_\alpha)_{i+1,i} = -e^{-\alpha/n} (1 - e^{-2\alpha/n})^{-1}, \quad i = 1, \ldots, n - 1,
\]

and all other entries of \( R_\alpha \) are zero. Furthermore, the determinant of \( R_\alpha \) is \( |R_\alpha| = (1 - e^{-2\alpha/n})^{n-1} \). The profile log-likelihood in (5) then has the explicit form

\[
\tilde{\mathcal{L}}_n(\alpha) = -\frac{n}{2} \log \left( A_1 e^{-2\alpha/n} - 2 A_2 e^{-\alpha/n} + A_3 \right) + \frac{1}{2} \log(1 - e^{-2\alpha/n}),
\]

where

\[
A_1 = \sum_{i=2}^{n-1} X(s_i)^2, \quad A_2 = \sum_{i=1}^{n-1} X(s_i) X(s_{i+1}), \quad A_3 = \sum_{i=1}^{n} X(s_i)^2.
\]
For the prior of $\alpha$, instead of Assumption (A.3), we use a weaker alternative assumption.

(A.3’) The marginal prior $\pi(\alpha)$ is a proper and continuous density on $\mathbb{R}_+$. $\pi(\alpha)$ does not depend on $n$. $\pi(\alpha_0) > 0$. $\int_0^\infty \pi(\theta_0|\alpha)\pi(\alpha) d\alpha < \infty$. $\int_0^\infty \sqrt{\alpha} \pi(\theta_0|\alpha)\pi(\alpha) d\alpha < \infty$. $\int_0^\infty \sqrt{\alpha} \pi(\alpha) d\alpha < \infty$. Furthermore, for $\alpha_n$ and $\pi_n$ defined in (9), the following relations hold as $n \to \infty$:

$$\sqrt{n} \int_0^{2\alpha_n} \sqrt{\alpha} \pi(\alpha) d\alpha \to 0, \quad \sqrt{n} \int_{\pi_n}^\infty \sqrt{\alpha} \pi(\alpha) d\alpha \to 0. \quad (22)$$

(A.3’) is considerably weaker than (A.3). (A.3’) only requires that $\pi(\alpha)$ and $\pi(\theta_0|\alpha)\pi(\alpha)$ (or equivalently, $\pi(\alpha|\theta_0)$) to have polynomially decaying tails at zero and infinity, compared to the exponential power tails as in Proposition 2. With appropriate choice of hyperparameters, $\pi(\alpha)$ in (A.3) can be taken as gamma, inverse gamma, inverse Gaussian, or the family of generalized beta of the second kind defined in Proposition 1.

**Theorem 3.** Consider the model with $d = 1$, $S = [0, 1]$, $\nu = 1/2$, and observations $X_n$ on the equispaced grid $s_i = i/n$ for $i = 1, \ldots, n$. Suppose that Assumptions (A.1), (A.2) and (A.3’) hold. Then

$$\left\| \Pi(d\theta, d\alpha|X_n) - \mathcal{N}\left(d\theta|\bar{\theta}_{\alpha_0}, 2\theta_0^2/n\right) \times \Pi(d\alpha|X_n) \right\|_{TV} \to 0, \quad (23)$$

$$\left\| \Pi(d\theta, d\alpha|X_n) - \mathcal{N}\left(d\theta|\bar{\theta}_{\alpha_0}, 2\theta_0^2/n\right) \times \Pi_*(d\alpha|X_n) \right\|_{TV} \to 0, \quad (24)$$

as $n \to \infty$ in $P(\sigma^2_0, \alpha_0)$-probability, where the density of the profile posterior distribution $\Pi(d\alpha|X_n)$ is given in (17) of Theorem 2 and the distribution $\Pi_*(d\alpha|X_n)$ has the density

$$\pi_*(\alpha|X_n) \propto \sqrt{\alpha} \exp\left\{ -\frac{(\alpha - u_*^2)}{2v_*} \right\} \pi(\alpha|\theta_0), \quad \text{for all } \alpha \in \mathbb{R}_+, \quad \text{where} \quad u_* = \frac{n(A_1 - A_2)}{A_1}, \quad v_* = \frac{n(A_1 - 2A_2 + A_3)}{A_1},$$

and $A_1, A_2, A_3$ are defined in (21). $v_* > 0$ and $v_* \asymp 1$ as $n \to \infty$ almost surely $P(\sigma^2_0, \alpha_0)$. Therefore, $\pi(\alpha|X_n)$ does not converge to any degenerate distribution as $n \to \infty$ in $P(\sigma^2_0, \alpha_0)$-probability.

Theorem 3 establishes the BvM type theorem for the joint posterior of $(\theta, \alpha)$ in the 1-dimensional Ornstein-Uhlenbeck process under fixed-domain asymptotics. Compared to Theorem 2, Theorem 3 shows the same limiting distribution under the weaker (A.3’). Furthermore, Theorem 3 simplifies the profile posterior density $\tilde{\pi}(\alpha)$ to a more explicit form $\pi_*(\alpha|X_n)$, which is a polynomially tilted normal density (Bochkina and Green [19]) times the conditional prior density $\pi(\alpha|\theta_0)$. The “normal” part of $\pi_*(\alpha|X_n)$ is centered at $u_*$ with scale $v_*$. The scale $v_*$ is of constant order almost surely $P(\sigma^2_0, \alpha_0)$. Moreover, (A.1) and (A.3’) ensure that $\pi(\alpha|\theta_0)$ is positive for all $\alpha \in \mathbb{R}_+$. Therefore, the limiting distribution $\pi_*(\alpha|X_n)$ has a continuous and positive density with a non-shrinking variance on $\mathbb{R}_+$. If $\pi(\alpha|\theta_0)$ does not depend on $n$, then as a result of the convergence in total variation distance in (24), the marginal posterior $\pi(\alpha|X_n)$ also cannot converge to any degenerate distribution as $n \to \infty$. Therefore, the posterior of $\alpha$ does not converge to the true parameter $\alpha_0$. This Bayesian asymptotic result matches with the frequentist theory in Zhang [82] that there exists no consistent estimator for $\alpha$ under fixed-domain asymptotics.

**2.4 Discussion on the BvM Results**

We discuss the relations of our results to the previous BvM results and to the Bayesian literature of partially identified models.
Relation to previous BvM results. In the presence of nuisance parameters, Shen \cite{Shen2011} and Bickel and Kleijn \cite{Bickel2011} have developed general machinery for proving BvM results in the presence of possibly nonparametric nuisance parameters. Suppose that \( \theta \) is the parameter of primary interest and \( \alpha \) is the nuisance parameter in the model. The method in Bickel and Kleijn \cite{Bickel2011} first establishes a LAN result for each value of \( \theta \) inside a neighborhood of the “least-favorable submodel”, which is a contracting neighborhood of \( \alpha \) around the minimizer of the Kullback-Leibler divergence from the probability measure with parameters \((\theta, \alpha)\) to the true measure with \((\theta_0, \alpha_0)\). Then the method in Bickel and Kleijn \cite{Bickel2011} obtains the integral LAN property with integration over \( \alpha \) (their Theorem 4.2). The existence of uniformly consistent test is achieved by a condition on the Hellinger distance (their Lemma 3.2). Bickel and Kleijn \cite{Bickel2011} further proposes a rate free BvM theorem in their Corollary 5.2 that does not require the posterior of \( \alpha \) to contract to the least favorable submodel as \( n \to \infty \), which is more general and can be related to our Gaussian process model, since the posterior of the range parameter \( \alpha \) in our model also does not converge to a point pass in fixed-domain asymptotics.

Despite this similarity, we adopt a more direct proof technique for our BvM Theorems 1, 2 and 3. Checking the condition on Hellinger distance in Bickel and Kleijn \cite{Bickel2011} adds complexity to the proof. There are also several other main challenges. First, the likelihood function in our Gaussian process model cannot be written in an i.i.d. product form. The design of the sampling points \( S_n \) is arbitrary, making \( R^{-1}_\alpha \) and \(|R_\alpha|\) intractable in general. These two features determine that the LAN condition in our model is fundamentally different from previous results for independent or weakly dependent data. We adopt the proof techniques in Wang and Loh \cite{Wang2015} to establish the LAN condition. We integrate out \( \theta \) for each fixed \( \alpha \) and obtain the profile posterior distribution of \( \alpha \) as in (17). The second challenge is the unbounded prior support for \( \alpha \), as the LAN condition for \( \theta \) does not hold for those \( \alpha \) outside \([\alpha_n, \bar{\alpha}_n]\). We derive sufficient conditions on the prior distribution such that the posterior probability mass outside \([\alpha_n, \bar{\alpha}_n]\) will vanish as \( n \to \infty \). This involves detailed analysis of the properties of the profile posterior distribution in (17); see Section S3 for more details.

As a consequence of fixed-domain asymptotics, the limiting posterior distributions of \((\theta, \alpha)\) in our Theorems 2 and 3 are not the standard multivariate normal distribution as in the classical parametric BvM theorems. We note that under the different regime of increasing-domain asymptotics, Shaby and Ruppert \cite{Shaby2015} has shown that for the tapered covariance functions, a standard BvM theorem for \((\sigma^2, \alpha)\) holds with \( n^{-1/2} \) convergence towards a nonsingular bivariate normal distribution. The contrast of our result to theirs is caused by the fundamental difference between the two asymptotics regimes. It has been shown in several examples in Zhang and Zimmerman \cite{Zhang2016} that the Fisher information for \((\sigma^2, \alpha)\) increases linearly with the sample size \( n \) under increasing-domain asymptotics, but remains fixed under fixed-domain asymptotics, thus causing the lack of consistent estimator for \((\sigma^2, \alpha)\) under fixed-domain asymptotics. Our BvM results show that this phenomenon also translates to the asymptotic behavior of the Bayesian posterior distribution for \( \alpha \).

In the broader sense, our work contributes a new example to the literature of BvM theorems for nonregular models; see for example, Chernozhukov and Hong \cite{Chernozhukov2005}, Kleijn and Knapik \cite{Kleijn2016}, Bochkina and Green \cite{Bochkina2015}, Jun et al. \cite{Jun2015}, Chen et al. \cite{Chen2016}, etc.

Relation to partially identified models. Our BvM theorems for the covariance parameters can also be related to the Bayesian literature of partially identified models. Such models have been studied extensively in statistics and econometrics literature (Manski \cite{Manski2008}, Tamer \cite{Tamer2013}, Gustafsson \cite{Gustafsson2015}). In a partially identified model, the probability distribution of the data are compatible with a set of different parameter values. This parameter set is referred to as the identification region. As a result, consistent point estimator for the true parameter value does not exist, though one can still consistently estimate the identification region. If the Matérn covariance function is parameterized by \((\sigma^2, \alpha)\), then under fixed-domain asymptotics, the distribution of \( X_n \) are asymptotically compatible with any parameters on the curve.
\[ \Gamma_{\theta_0} = \{(\sigma^2, \alpha) \in \mathbb{R}^2_+ : \sigma^2 \alpha^{2\nu} = \theta_0\}, \] which is the identification region in our problem. Therefore, in the asymptotic sense, our model can be viewed as a partially identified model.

In the Bayesian setup, the asymptotic property of posterior distributions in partially identified models have been studied in Moon and Schorfheide [51], Gustafson [29], Jiang [34], Chen et al. [15], Jiang and Li [33], etc. In particular, Moon and Schorfheide [51] proves that the posterior distribution of those parameters that can be identified from the data (called “reduced-form parameters”, \(\theta\) in our model) has the standard \(n^{-1/2}\)-convergence to a normal limit, and the posterior of the full partially identified parameter vector (called “structural parameters”, \((\theta, \alpha)\) in our model) converges to the conditional prior given the MLE of the reduced-form parameter. For our model, an analogous result to Theorem 1 in Moon and Schorfheide [51] would be that the joint posterior of \((\theta, \alpha)\) is asymptotically close to \(\mathcal{N}(d\theta; \hat{\theta}_0, 2\sigma^2_0/n) \times \Pi(d\alpha|\theta_0)\), where \(\Pi(d\alpha|\theta_0)\) is the prior of \(\alpha\) given \(\theta_0\). However, this is different from the limiting distribution in Theorem 2 which is proportional to \(\mathcal{N}(d\theta; \hat{\theta}_0, 2\sigma^2_0/n) \times \Pi(d\alpha|X_n)\), and the profile posterior density \(\hat{\pi}(d\alpha|X_n)\) is proportional to the profile likelihood function times the conditional prior \(\pi(d\alpha|\theta_0)\). The reason for this essential difference is that Moon and Schorfheide [51] has made the strong “marginal uninformativeness” assumption that the distribution of observed data only depends on the reduced-form parameter \(\theta\) but not \(\alpha\), which does not hold in our model since the distribution of \(X_n\) depends on both \(\theta\) and \(\alpha\). Unlike Theorem 1 in Moon and Schorfheide [51], the profile likelihood in our model also contributes nonnegligible information to the posterior of \(\alpha\). Jiang [34] has relaxed the marginal uninformativeness assumption of Moon and Schorfheide [51] to their Condition 1, such that the information about \(\alpha\) can be summarized in a nonstochastic function \(\tau(\cdot)\) in Jiang [34], which is the integral of the (quasi-)likelihood function over \(\theta\). This nonstochastic function eventually enters the limiting posterior distribution as an additional multiplicative factor; see Theorem 1 in Jiang [34]. However, given the intractability of the Matérn covariance matrix \(R_\alpha\), it is unclear if there exists such a nonstochastic function \(\tau(\cdot)\) that can satisfy their Condition 1. The limiting distribution in Theorem 1 of Jiang [34] also has slight difference from ours in Theorem 2 which is essentially caused by the difference between the integrated likelihood used in Jiang [34] and the profile likelihood used in our theory.

3 Posterior Asymptotic Efficiency in Linear Prediction

The BvM theorem in Section 2 shows that the posterior of the microergodic parameter \(\theta\) in the Matérn covariance function satisfies the same \(n^{-1/2}\)-convergence to a normal limit as in a regular parametric model. This result has an important implication for the kriging prediction with covariance parameters randomly drawn from the posterior distribution at a new location \(s^* \in S \setminus S_n\), i.e. \(s^*\) is an arbitrary point in \(S\) but different from the sampling points \(S_n\). We need the following dense assumption for some of our theory later.

(A.4) Suppose that the sequence of \(S_n = \{s_1, \ldots, s_n\}\) is getting dense in \(S\) as \(n \to \infty\), in the sense that \(\sup_{s^* \in S} \min_{1 \leq i \leq n} \|s^* - s_i\| \to 0\) as \(n \to \infty\).

The sequence of sets \(S_1, S_2, \ldots\) is increasingly dense in the fixed domain \(S\), so that we can predict at any new location accurately. But we do not require the sequence \(S_1, S_2, \ldots\) to be nested.

Consider the linear prediction (or kriging) of \(X(s^*)\) using the data \(X_n\). Let \(r_\alpha(s^*) = (K_{\alpha,\nu}(s_1 - s^*), \ldots, K_{\alpha,\nu}(s_n - s^*))^\top\) be the correlation vector between \(s^*\) and \(\{s_1, \ldots, s_n\}\). The best linear unbiased predictor (BLUP) for \(X(s^*)\) using an incorrect model GP \((0, \sigma^2 K_{\alpha,\nu})\) (Stein [63]) is

\[ \hat{X}_n(s^*; \alpha) = r_\alpha(s^*)^\top R_\alpha^{-1} X_n. \]

This kriging predictor only depends on \(\alpha\) but not \(\sigma^2\). Let \(e_\alpha(s^*; \alpha) = \hat{X}_n(s^*; \alpha) - X(s^*)\) be the prediction error. Then under the incorrect model GP \((0, \sigma^2 K_{\alpha,\nu})\), the prediction mean square
error (MSE), which is also the prediction variance, is given by
\[ E_{\sigma^2, \alpha} \left\{ \epsilon_n(s^*; \alpha)^2 \right\} = \sigma^2 \left\{ 1 - r_\alpha(s^*)^\top R_\alpha^{-1} r_\alpha(s^*) \right\}. \]  
(26)

But under the true model \( \text{GP}(0, \sigma_0^2 K_{\alpha, \nu}) \), the prediction mean square error is
\[ E_{\sigma_0^2, \alpha_0} \left\{ \epsilon_n(s^*; \alpha)^2 \right\} = \sigma_0^2 \left\{ 1 - 2r_\alpha(s^*)^\top R_\alpha^{-1} r_\alpha(s^*) + r_\alpha(s^*)^\top R_\alpha^{-1} R_\alpha R_\alpha^{-1} r_\alpha(s^*) \right\}. \]  
(27)

It is therefore of theoretical interest to study whether \( E_{\sigma^2, \alpha} \left\{ \epsilon_n(s^*; \alpha)^2 \right\} \) in (26) is close to the prediction MSE under the true measure \( E_{\sigma_0^2, \alpha_0} \left\{ \epsilon_n(s^*; \alpha)^2 \right\} \) in (27), as well as the optimal “oracle” prediction MSE \( E_{\sigma_0^2, \alpha_0} \left\{ \epsilon_n(s^*; \alpha_0)^2 \right\} \). The series of works Stein [57], Stein [58], Stein [59], and Stein [61] have shown that if an incorrect Gaussian process model is used for prediction, the prediction variance at \( s^* \) is asymptotically equal to the prediction variance at \( s^* \) using the incorrect model but evaluated under the true Gaussian process model, as long as the two Gaussian measures are compatible (or mutually absolutely continuous). For our Gaussian process model with mean-zero and Matérn covariance function, the compatibility of the incorrect model \( \text{GP}(0, \sigma^2 K_{\alpha, \nu}) \) and the true model \( \text{GP}(0, \sigma_0^2 K_{\alpha_0, \nu}) \) simplifies to the equivalence condition \( \sigma^2 \alpha^{2\nu} = \theta_0 = \sigma_0^2 \alpha_0^{2\nu} \), i.e., they have the same microergodic parameter \( \theta_0 \). If the equivalence condition holds, then Stein [57], Stein [58], and Stein [61] have shown that as \( n \to \infty \),

\[ \sup_{s^* \in S \setminus S_n} \left| \frac{\epsilon_{\sigma^2, \alpha} \left\{ \epsilon_n(s^*; \alpha)^2 \right\}}{\epsilon_{\sigma_0^2, \alpha_0} \left\{ \epsilon_n(s^*; \alpha)^2 \right\}} - 1 \right| \to 0, \]
\[ \sup_{s^* \in S \setminus S_n} \left| \frac{\epsilon_{\sigma_0^2, \alpha_0} \left\{ \epsilon_n(s^*; \alpha_0)^2 \right\}}{\epsilon_{\sigma_0^2, \alpha_0} \left\{ \epsilon_n(s^*; \alpha_0)^2 \right\}} - 1 \right| \to 0, \]  
(28)

which is called asymptotic efficiency in linear prediction. The first convergence shows that for the BLUP (25), the prediction MSEs are almost the same under either the incorrect Gaussian measure \( P_{\sigma^2, \alpha} \) or the true Gaussian measure \( P_{\sigma_0^2, \alpha_0} \). The second convergence shows that the prediction MSEs obtained from the incorrect model \( \text{GP}(0, \sigma^2 K_{\alpha, \nu}) \) is asymptotically equal to the optimal prediction MSE from the true model \( \text{GP}(0, \sigma_0^2 K_{\alpha_0, \nu}) \).

Using the weakened conditions in Stein [61], Theorem 4 of Kaufman and Shaby [38] shows that a consistent estimator of the microergodic parameter \( \theta \) can also achieve asymptotic efficiency. For a given \( \alpha > 0 \), the prediction based on the MLE of \( \sigma^2 \) for a fixed \( \alpha > 0 \) satisfies

\[ \sup_{s^* \in S \setminus S_n} \left| \frac{\epsilon_{\sigma_0^2, \alpha} \left\{ \epsilon_n(s^*; \alpha)^2 \right\}}{\epsilon_{\sigma_0^2, \alpha_0} \left\{ \epsilon_n(s^*; \alpha)^2 \right\}} - 1 \right| \to 0, \]  

as \( n \to \infty \) almost surely \( P_{\sigma_0^2, \alpha_0} \), where \( \hat{\sigma}^2_\alpha \) is the MLE of \( \sigma^2 \) in (3) for a given \( \alpha > 0 \).

Motivated by these works, we establish the Bayesian version of (28), called posterior asymptotic efficiency in linear prediction. In Bayesian inference, we randomly draw \( (\theta, \alpha) \) from the joint posterior distribution, and compute the prediction MSE at a new location \( s^* \in S \setminus S_n \) using the Gaussian measure \( P_{\sigma^2, \alpha}(\theta, \alpha) \) with \( \sigma^2 = \theta/\alpha^{2\nu} \). This prediction MSE is random due to the randomness in the posterior draw \( (\theta, \alpha) \). However, using the BvM results in Section 2 we are able to show that such a random prediction MSE is closed to the optimal prediction MSE based on the true model.

For a given \( \alpha > 0 \), we define the following sequence \( \varsigma_n(\alpha) \) which will be useful
\[ \varsigma_n(\alpha) = \max \left\{ \sup_{s^* \in S \setminus S_n} \left| \frac{\epsilon_{\theta_0/\alpha^{2\nu}, \alpha} \left\{ \epsilon_n(s^*; \alpha)^2 \right\}}{\epsilon_{\sigma_0^2, \alpha_0} \left\{ \epsilon_n(s^*; \alpha)^2 \right\}} - 1 \right|, \sup_{s^* \in S \setminus S_n} \left| \frac{\epsilon_{\theta_0/\alpha^{2\nu}, \alpha} \left\{ \epsilon_n(s^*; \alpha_0)^2 \right\}}{\epsilon_{\sigma_0^2, \alpha_0} \left\{ \epsilon_n(s^*; \alpha_0)^2 \right\}} - 1 \right| \right\}. \]  
(29)

Lemma 2. (Stein [58] and Stein [61]) If Assumption \([A.4]\) holds, then for a given \( \alpha > 0 \), \( \varsigma_n(\alpha) \to 0 \) as \( n \to \infty \).
The proof of Lemma 2 follows from Stein’s results since the two Gaussian measures \( P(\theta_0/\alpha^{2\nu}, \alpha) \) and \( P(\sigma_0^2, \alpha_0) \) are equivalent. In particular, the first rate in \( \varsigma_n(\alpha) \) in (29) converges to zero by Theorem 3.1 of Stein [58], and the second rate in (29) converges to zero by Theorem 2 of Stein [61].

Similar to our BvM theorems, we proceed in two steps by first considering the case when \( \alpha > 0 \) is fixed, and then letting both \( \theta \) and \( \alpha \) be random and assigned priors.

**Theorem 4. (Posterior asymptotic efficiency of linear predictions for fixed \( \alpha \))** Suppose that \( \alpha > 0 \) is fixed and the prior \( \pi(\theta|\alpha) \) satisfies Assumption [A.1].

(i) As \( n \to \infty \), almost surely \( P(\sigma_0^2, \alpha_0) \):

\[
\Pi \left[ \sup_{s^* \in \mathbb{S} \setminus \mathbb{S}_n} \frac{E_{\sigma_0^2, \alpha_0} \{e_n(s^*; \alpha_0)^2\}}{E_{\theta_0/\alpha^{2\nu}, \alpha} \{e_n(s^*; \alpha)^2\}} - 1 \right] > 7n^{-1/2} \log n \rightarrow 0;
\]

(ii) If Assumption [A.4] holds, then as \( n \to \infty \), almost surely \( P(\sigma_0^2, \alpha_0) \):

\[
\Pi \left[ \sup_{s^* \in \mathbb{S} \setminus \mathbb{S}_n} \frac{E_{\sigma_0^2, \alpha_0} \{e_n(s^*; \alpha_0)^2\}}{E_{\sigma_0^2, \alpha_0} \{e_n(s^*; \alpha_0)^2\}} - 1 \right] > \max \left\{ 16n^{-1/2} \log n, 2\varsigma_n(\alpha) \right\} \rightarrow 0;
\]

\[
\Pi \left[ \sup_{s^* \in \mathbb{S} \setminus \mathbb{S}_n} \frac{E_{\sigma_0^2, \alpha_0} \{e_n(s^*; \alpha_0)^2\}}{E_{\sigma_0^2, \alpha_0} \{e_n(s^*; \alpha_0)^2\}} - 1 \right] > \max \left\{ 16n^{-1/2} \log n, 2\varsigma_n(\alpha) \right\} \rightarrow 0.
\]

Part (i) of Theorem 4 shows that the prediction MSE at an arbitrary new location \( s^* \) evaluated under the measure \( P(\sigma_0^2, \alpha) \) is asymptotically equal to the prediction MSE evaluated under the measure \( P(\theta_0/\alpha^{2\nu}, \alpha) \), as if the true parameter \( \theta_0 \) were known. Based on posterior draws of \((\theta, \alpha)\), we can compute the prediction MSE under \( P(\sigma_0^2, \alpha) \) but not under \( P(\theta_0/\alpha^{2\nu}, \alpha) \), since the latter depends on the unknown true parameter \( \theta_0 \). Therefore, Part (i) compares the prediction performance from the posterior of a Gaussian process model to a “half-oracle” model, which has the true microergodic parameter \( \theta_0 \) but not the true range parameter \( \alpha_0 \). In fact, Part (i) is a direct consequence of the BvM result in Theorem 1. It only depends on the posterior convergence of \( \theta \) to \( \theta_0 \) and does not require the dense assumption [A.4]. We also give the explicit convergence rate \( n^{-1/2}\log n \), which has a \( \log n \) factor compared to the BvM result in Theorem 1 to ensure that the posterior convergence happens almost surely \( P(\sigma_0^2, \alpha_0) \).

Part (ii) of Theorem 4 establishes two posterior convergence results. The first convergence is about the ratio of the prediction MSEs using an incorrect range parameter \( \alpha \) evaluated under the measure \( P(\sigma_0^2, \alpha) \) and the true measure \( P(\sigma_0^2, \alpha_0) \), which implies that these two prediction MSEs are asymptotically equal. The convergence rate depends on two parts: one is the posterior convergence rate of \( \theta \) to \( \theta_0 \), which is as fast as \( n^{-1/2}\log n \); the other is the convergence rate from the convergence of the ratio \( E_{\sigma_0^2, \alpha_0} \{e_n(s^*; \alpha_0)^2\} / E_{\theta_0/\alpha^{2\nu}, \alpha} \{e_n(s^*; \alpha)^2\} \) towards 1. This is part of \( \varsigma_n(\alpha) \) in (29) and the convergence is guaranteed by Lemma 2.

The second convergence in Part (ii) of Theorem 4 is about the ratio of the prediction MSEs using the incorrect model \( P(\sigma_0^2, \alpha) \) and the “oracle” optimal prediction MSE using the true model \( P(\sigma_0^2, \alpha_0) \). This implies that the prediction MSE computed with random parameters \((\theta, \alpha)\) from the Bayesian posterior can exactly recover the oracle optimal prediction MSE in the asymptotics. The posterior convergence rate of asymptotic efficiency depends on the \( n^{-1/2} \) posterior convergence rate of \( \theta \) to \( \theta_0 \) and the convergence rate of the ratio \( E_{\theta_0/\alpha^{2\nu}, \alpha} \{e_n(s^*; \alpha)^2\} / E_{\sigma_0^2, \alpha_0} \{e_n(s^*; \alpha_0)^2\} \) towards 1. This latter convergence is also guaranteed by Lemma 2.

Our next goal is to let \( \alpha \) be random and assigned a prior. We specify the following uniform convergence assumption.
(A.5) As \( n \to \infty \), there exists a positive deterministic sequence \( \varsigma_n \to 0 \) as \( n \to \infty \), such that \( \sup_{\alpha \in [\underline{\alpha}, \overline{\alpha}]_n} \varsigma_n(\alpha) \leq \varsigma_n \) for the sequence \( \varsigma_n(\alpha) \) defined in (29).

Compared to Theorem 1, the additional requirement in (A.5) is that the convergence of these the two ratios in (29) towards 1 holds uniformly over the interval \([\underline{\alpha}_n, \overline{\alpha}_n]\). Although (A.5) is not easy to verify for Matérn covariance functions with a general smoothness parameter \( \nu > 0 \), the existence of such a uniform rate \( \varsigma_n \) is somehow expected given that the interval \([\underline{\alpha}_n, \overline{\alpha}_n]\) expands very slowly with \( n \). We will give an explicit form of the uniform convergence rate \( \varsigma_n \) in Theorem 6 for the 1-dimensional Ornstein-Uhlenbeck process with \( \nu = 1/2 \) in Section 2.3.

Before that, we state a general posterior asymptotic efficiency theorem for \((\theta, \alpha)\) drawn from their joint posterior distribution.

**Theorem 5. (Posterior asymptotic efficiency of linear predictions for random \( \alpha \))** Suppose that Assumptions (A.1)–(A.3) hold.

(i) As \( n \to \infty \), almost surely \( P(\sigma_0^2, \alpha_0) \),

\[
\begin{align*}
\Pi \left[ \sup_{s^* \in S \setminus S_n} \frac{E_{\sigma^2, \alpha} \{e_n(s^*; \alpha)^2\}}{E_{\theta_0, \alpha^2} \{e_n(s^*; \alpha)^2\}} - 1 > 6n^{-1/2} \log n \right] X_n \to 0;
\end{align*}
\]

(ii) If in addition, Assumption (A.4) and (A.5) hold, then as \( n \to \infty \), almost surely \( P(\sigma_0^2, \alpha_0) \),

\[
\begin{align*}
\Pi \left[ \sup_{s^* \in S \setminus S_n} \frac{E_{\sigma^2, \alpha} \{e_n(s^*; \alpha)^2\}}{E_{\sigma_0^2, \alpha^2} \{e_n(s^*; \alpha)^2\}} - 1 > \max \left( 16n^{-1/2} \log n, 2\varsigma_n \right) \right] X_n \to 0,
\end{align*}
\]

\[
\begin{align*}
\Pi \left[ \sup_{s^* \in S \setminus S_n} \frac{E_{\sigma^2, \alpha} \{e_n(s^*; \alpha)^2\}}{E_{\sigma_0^2, \alpha^2} \{e_n(s^*; \alpha^2)^2\}} - 1 > \max \left( 16n^{-1/2} \log n, 2\varsigma_n \right) \right] X_n \to 0 \quad (30)
\end{align*}
\]

Part (i) of Theorem 5 shows that the prediction MSE at an arbitrary new location \( s^* \) evaluated under the measure \( P(\sigma_0^2, \alpha_0) \) is asymptotically equal to the prediction MSE evaluated under the measure \( P(\theta_0, \alpha_0) \), as if the true parameter \( \theta_0 \) were known. This is in the same form as Part (i) of Theorem 1 with the same convergence rate, but now with \( \alpha \) being random and \((\theta, \alpha)\) assigned a joint prior that satisfies Assumptions (A.1)–(A.3). Part (i) is a direct consequence of our BvM result in Theorem 2 for the joint posterior of \((\theta, \alpha)\). It shows that the prediction performance from a random draw of \((\theta, \alpha)\) from the posterior is as good as the half-oracle model with the true microergodic parameter \( \theta_0 \) but not the true range parameter \( \alpha_0 \). Part (i) does not require the dense assumption (A.4) and the uniform assumption (A.5).

Part (ii) of Theorem 5 is similar to Part (ii) of 4 with the same interpretation of asymptotic efficiency, except that \( \alpha \) is now random and \((\theta, \alpha)\) is drawn from their joint posterior. Furthermore, Assumption (A.5) is involved to guarantee the uniform convergence over the majority of \( \alpha \) values in the interval \([\underline{\alpha}_n, \overline{\alpha}_n]\). Theorem 5 shows that the prediction MSE computed from random sampled parameters from the posterior is asymptotically equal to the oracle optimal prediction MSE with the true parameters. Theorems 4 and 5 together provide strong theoretical guarantees for the Bayesian posterior prediction performance with random samples of parameters, which has been widely adopted in the practice of real data analysis in spatial statistics.

To clarify the rate \( \varsigma_n \) in Assumption (A.5), we revisit the 1-dimensional Ornstein-Uhlenbeck process in Section 2.3 and derive an explicit form for \( \varsigma_n \) in this example.

**Theorem 6.** For the case of \( d = 1, \nu = 1/2, S = [0, 1] \), and equispaced grid \( s_i = i/n \), for \( i = 1, \ldots, n \), Assumption (A.5) is satisfied with \( \varsigma_n = 7n^{2\nu+\kappa-1} \), where \( \kappa \) and \( \nu \) are defined in (9). As a result, under Assumptions (A.1)–(A.4) as \( n \to \infty \), almost surely \( P(\sigma_0^2, \alpha_0) \),

\[
\begin{align*}
\Pi \left[ \sup_{s^* \in S \setminus S_n} \frac{E_{\sigma^2, \alpha} \{e_n(s^*; \alpha)^2\}}{E_{\sigma_0^2, \alpha^2} \{e_n(s^*; \alpha)^2\}} - 1 > 16n^{-1/2} \log n \right] X_n \to 0,
\end{align*}
\]

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\[
\Pi \left[ \sup_{s^* \in S_n \setminus S \cap |X_n|} \frac{E_{\sigma^2,\alpha} \left\{ e_n(s^*; \alpha) \right\}^2}{E_{\sigma_0^2,\alpha_0} \left\{ e_n(s^*; \alpha_0) \right\}^2} - 1 \right] > 16n^{-1/2} \log n \rightarrow 0.
\]

To prove Theorem 6, we use the result in Stein [59] and relate the rate \( c_n \) in Assumption (A.5) to the convergence rate of the finite sample version of the symmetrized Kullback-Leibler divergence between two equivalent Gaussian measures \( P_{\sigma^2, \alpha} \) and \( P_{\sigma_0^2, \alpha_0} \) towards its limit, where \( \sigma^2 \alpha^{2\nu} = \theta_0 = \sigma_0^2 \alpha_0^{2\nu} \); see the proof in Section 5 of the supplementary materials. Since \( \pi \) and \( \kappa \) are both small positive numbers as given in (9), it is clear that \( c_n \) in Theorem 6 converges to zero faster than \( n^{-1/2} \). As a result, the two posterior convergence rates for asymptotic efficiency in Theorem 6 are both as fast as \( n^{-1/2} \log n \).

The convergence rates of the oracle prediction MSE in Gaussian process models have been extensively studied in the literature (Yakowitz and Szidarovszky [77], Stein [58], Wang et al. [76], Tuo and Wang [68]). The posterior asymptotic efficiency developed in this section implies that in the Gaussian process model \( X \sim GP(0, \sigma^2 I_{m,\nu}) \), under fixed-domain asymptotics, the prediction MSE with a random draw of \((\theta, \alpha)\) from the Bayesian posterior distribution converges to zero at the same rate as the oracle prediction MSE based on the true parameter \((\theta_0, \alpha_0)\). Our theorems in this section are stronger than a convergence rate result, since our result implies that the two MSEs not only have the same convergence rate, but are asymptotically equal to each other. On the other hand, our results are not directly comparable with the previous Bayesian work on Gaussian process regression, such as van der Vaart and van Zanten [70], van der Vaart and van Zanten [71], van der Vaart and van Zanten [72], Yang et al. [78], Yang and Tokdar [79], etc., since our model assumes a random sample path \( X(\cdot) \) instead of a deterministic function and does not contain the additional measurement error term as in these literature. We provide more discussion on the Gaussian process model with a nugget parameter in Section 5.

4 Simulation Study

We verify our BvM theorems and posterior asymptotic efficiency using numerical examples. We consider the 1 and 2-dimensional Ornstein-Uhlenbeck process with \( \nu = 1/2 \) in the Matérn covariance function. The true parameters are \( \sigma^2_0 = 1, \alpha_0 = 0.5, \) and \( \theta_0 = 0.5 \). We assign independent gamma priors to \( \theta \) and \( \alpha \), with the same shape parameter 1.1 and rate parameter 0.1. This prior satisfies Assumptions (A.1) and (A.2) and the right tail condition (the second relation of (22)) in (A.3) but does not satisfy the left tail condition (the first relation of (22)) in (A.3) see Proposition 1. We will see that empirically this prior still yields convergent results. We consider two cases with dimensions \( d = 1 \) and \( d = 2 \). For the \( d = 1 \) case, we set \( S = [0,1] \) and the sampling points of \( S_n \) to be the grid \( s_i = \frac{2i-1}{2n} \) (\( i = 1, \ldots, n \)) perturbed by Uniform\([-0.0002, 0.0002]\) noise, for \( n = 25, 50, 100, 200, 400 \). When the \( d = 2 \) case, we set \( S = [0,1]^2 \) and the sampling points of \( S_n \) to be the regular grid \( \left( \frac{2i-1}{2m}, \frac{2j-1}{2m} \right) \) (\( i, j = 1, \ldots, m \)) perturbed by Uniform\([-0.001, 0.001] \times \text{Uniform}[0.001, 0.001]\) noise, for \( m = 10, 20, 30 \) and \( n = m^2 \). Then we draw \( X_n \) from the mean zero Gaussian process with the \( \nu = 1/2 \) Matérn covariance function observed on \( S_n \). We use the random walk Metropolis algorithm (RWM) to draw 5000 samples after 1000 burnins from the joint posterior \( \Pi(d\theta, d\alpha|X_n) \) and the limiting posteriors \( \mathcal{N}(\hat{\theta}_0, 2\sigma_0^2/n) \times \Pi(\alpha|X_n) \) in Theorem 2 respectively. For the \( d = 1 \) case, we further use RWM to draw 5000 samples from the limiting posterior \( \mathcal{N}(\hat{\theta}_0, 2\sigma_0^2/n) \times \Pi_{\nu}(\alpha|X_n) \) in Theorem 3.

We compare the true posterior distribution with the BvM limiting distributions using two criteria: (a) the closeness of our limiting distributions in Theorems 2 and 3 to the true posterior, and (b) the convergence of the two asymptotic efficiency measures in (28) with \((\theta, \alpha)\) drawn from the joint posterior. For (a), since it is generally difficult to evaluate the total variation distance between two 2-dimensional posterior distributions based on finite posterior samples,
we instead compute the Wasserstein-2 distance between the marginal posteriors for θ and α, respectively. The Wasserstein-2 ($W_2$) distance between two 1-dimensional distributions $F_1$ and $F_2$ has the simple expression $W_2(F_1, F_2)^2 = \int_0^1 |F_1^{-1}(u) - F_2^{-1}(u)|^2 du$, where $F_1^{-1}$ and $F_2^{-1}$ are the corresponding quantile functions. With finite samples from $F_1$ and $F_2$, $W_2(F_1, F_2)$ can be accurately estimated by replacing $F_1^{-1}$ and $F_2^{-1}$ with the empirical quantile functions (Li et al. [45]). In our simulation study, we replace $F_1$ and $F_2$ with $\Pi(d\theta|X_n)$ and $\mathcal{N}(d\theta|\hat{\theta}_0, 2\sigma_0^2/n)$ for $\theta$, and $\Pi(d\alpha|X_n)$ and $\Pi(d\alpha|X_n)$ for $\alpha$, respectively. For the $d = 1$ case, we also compute the $W_2$ distance between $\Pi(d\alpha|X_n)$ and $\Pi_1(d\alpha|X_n)$. The convergence in $W_2$ distance is equivalent to the weak convergence plus the convergence in the second moment (Villani [74]). Though slightly different from the convergence in total variation distance, it can provide some useful empirical evidence for convergence in the posterior means and variances of $\theta$ and $\alpha$.

For the $d = 1$ case, the results of the estimated posterior means under the true posterior $\Pi(\cdot|X_n)$, the limiting posterior $\Pi(\cdot|X_n)$ in Theorem 2, the limiting posterior $\Pi_\alpha(\cdot|X_n)$ in Theorem 3, and the $W_2$ distances between the marginal posteriors are reported in Table 1. The posterior mean estimates of the microergodic $\theta$ are accurate for the truth $\theta_0 = 0.5$ and the standard error decreases as $n$ increases. As expected, the posterior mean estimates of $\alpha$ are not consistent for the truth $\alpha_0 = 0.5$ in general, and show no sign of convergence for all three distributions. This verifies Theorem 3 that the marginal posterior of $\alpha$ does not converge as $n \to \infty$. For the BvM approximation accuracy, we can see that the $W_2$ distance between the true marginal posterior of $\theta$ and the normal limit in our BvM theorem decreases quickly to zero as $n$ increases. Furthermore, the $W_2$ distances between the true marginal posterior of $\alpha$ and the two approximations, the profile posterior $\Pi(d\alpha|X_n)$ and the polynomially tilted normal distribution $\Pi_1(d\alpha|X_n)$ in Theorem 3 also show clear decreasing trends towards zero as $n$ increases. These empirical observations have verified our BvM result in Theorem 3 for the 1-dimensional Ornstein-Uhlenbeck process.

Table 1: Parameter estimation and Wasserstein-2 distances between the true posterior and the limiting posteriors in Theorems 2 and 3 for the $d = 1$ case. $E(\cdot|X_n)$, $\hat{E}(\cdot|X_n)$, and $E_\alpha(\cdot|X_n)$ are the posterior means under the true posterior, the limiting posterior in Theorem 2 and the limiting posterior in Theorem 3. The true parameter values are $\theta_0 = 0.5$ and $\alpha_0 = 0.5$. All numbers are averaged over 100 macro replications. The standard errors are in the parentheses.

| $d = 1$ | $n = 25$ | $n = 50$ | $n = 100$ | $n = 200$ | $n = 400$ |
|---------|---------|---------|---------|---------|---------|
| $E(\theta|X_n)$ | 0.6511 | 0.5739 | 0.5390 | 0.5197 | 0.4993 |
| (0.1951) | (0.1335) | (0.0814) | (0.0535) | (0.0392) |
| $\hat{E}(\theta|X_n)$ | 0.4832 | 0.5004 | 0.5057 | 0.5033 | 0.4920 |
| (0.1368) | (0.1148) | (0.0764) | (0.0522) | (0.0385) |
| $E(\alpha|X_n)$ | 2.4145 | 2.4500 | 2.0546 | 2.2128 | 1.8724 |
| (2.4499) | (2.4175) | (1.7782) | (2.6410) | (1.9098) |
| $\hat{E}(\alpha|X_n)$ | 2.1848 | 2.3584 | 2.0008 | 2.1911 | 1.8614 |
| (2.1757) | (2.3164) | (1.7299) | (2.6215) | (1.9025) |
| $E_\alpha(\alpha|X_n)$ | 1.8594 | 2.1846 | 1.9263 | 2.1380 | 1.8445 |
| (1.7675) | (2.0579) | (1.6357) | (2.5243) | (1.8729) |

For the $d = 2$ case, the results are summarized in Table 2, showing similar trends to those from the $d = 1$ case. The posterior mean estimates of $\theta$ are accurate with standard errors decreasing with $n$, while the posterior mean estimates of $\alpha$ show no sign of convergence. The $W_2$ distance between the true marginal posteriors and the limiting posteriors in Theorem 2 converges to zero as $n$ increases. This has verified the BvM approximation in Theorem 2 for the 2-dimensional process.
Table 2: Parameter estimation and Wasserstein-2 distances between the true posterior and the limiting posteriors in Theorem 2 for the $d = 2$ case. $E(\cdot | X_n)$ and $\tilde{E}(\cdot | X_n)$ are the posterior means under the true posterior and the limiting posterior in Theorem 2. The true parameter values are $\theta_0 = 0.5$ and $\alpha_0 = 0.5$. All numbers are averaged over 100 macro replications. The standard errors are in the parentheses.

| $d = 2$          | $n = 10^2$ | $n = 20^2$ | $n = 30^2$ |
|------------------|------------|------------|------------|
| $E(\theta| X_n)$ | 0.5271     | 0.5088     | 0.5082     |
|                  | (0.0789)   | (0.0343)   | (0.0230)   |
| $\tilde{E}(\theta| X_n)$ | 0.5030 | 0.5028 | 0.5056 |
|                  | (0.0752)   | (0.0336)   | (0.0229)   |
| $E(\alpha| X_n)$ | 0.9278     | 0.8646     | 0.8410     |
|                  | (0.5281)   | (0.4700)   | (0.5290)   |
| $\tilde{E}(\alpha| X_n)$ | 0.9178 | 0.8641 | 0.8409 |
|                  | (0.5071)   | (0.4722)   | (0.5311)   |

| $W_2\left(\Pi(d\theta| X_n), \mathcal{N}\left(d\theta| d\tilde{\theta}_0, \frac{2\sigma^2}{n}\right)\right)$ | 0.0291 | 0.0072 | 0.0033 |
| $W_2(\Pi(d\alpha| X_n), \Pi(d\alpha| X_n))$ | 0.0479 | 0.0393 | 0.0388 |

Figure 1 illustrates the convergence of posterior densities for the $d = 1$ case. With $n = 50$, there exists noticeable difference between the true posterior and the limiting posteriors. But their difference gradually disappears as $n$ increases. Furthermore, as $n$ increases, the posterior shrinks along the direction of $\theta$, but remains spread out in the $\alpha$ direction. The ridge of the joint likelihood function $\tilde{\theta}_n$ increases with $\alpha$ as shown in Kaufman and Shaby [35], but becomes flatter as $n$ increases, indicating the convergence from $\tilde{\theta}_n$ to $\theta_0 = 0.5$ over each fixed value of $\alpha$.

For the posterior asymptotic efficiency in (b), we compute the two asymptotic efficiency measures in (28) and Theorems 5 and 6 empirically, using the posterior samples of $(\theta, \alpha)$. To approximate the suprema, we take the maximum of the ratios that depend on the random $(\sigma^2, \alpha)$ drawn from the posterior:

$$r_{1n}(s^*) = \frac{E_{\sigma^2, \alpha}\left(e_n(s^*|\alpha)^2\right)}{E_{\sigma^2, \alpha}\left(e_n(s^*|\alpha)^2\right)} - 1$$

over a large number of testing points $s^*$ from the Latin hypercube design. We use 1000 testing points in $S = [0, 1]$ for the $d = 1$ case, and 2500 testing points in $S = [0, 1]^2$ for the $d = 2$ case. Let the testing set be $S^*$. We report the estimated posterior mean $E[\max_{s^* \in S^*} r_{1n}(s^*) | X_n]$ and $E[\max_{s^* \in S^*} r_{2n}(s^*) | X_n]$. The results are summarized in Table 3. The simulation results show that the posterior means of the two ratios in (31) decrease as $n$ increases, and their standard errors also decrease. This is observed for both 1 and 2-dimensional domains.

Table 3: The posterior means of the two ratios of prediction MSEs defined in (31) maximized over 2500 testing points $s^*$, averaged over 100 macro replications. The standard errors are in the parentheses.

| $d = 1$          | $n = 25$   | $n = 50$   | $n = 100$  | $n = 200$  | $n = 400$  |
|------------------|------------|------------|------------|------------|------------|
| $E[\max_{s^* \in S^*} r_{1n}(s^*) | X_n]$ | 0.5006     | 0.3195     | 0.1956     | 0.1337     | 0.0868     |
|                  | (0.0431)   | (0.0220)   | (0.0131)   | (0.0086)   | (0.0054)   |
| $E[\max_{s^* \in S^*} r_{2n}(s^*) | X_n]$ | 0.4797     | 0.3008     | 0.1661     | 0.1241     | 0.0841     |
|                  | (0.0443)   | (0.0224)   | (0.0131)   | (0.0084)   | (0.0054)   |

| $d = 2$          | $n = 10^2$ | $n = 20^2$ | $n = 30^2$ |
|------------------|------------|------------|------------|
| $E[\max_{s^* \in S^*} r_{1n}(s^*) | X_n]$ | 0.1757     | 0.0799     | 0.0666     |
|                  | (0.0124)   | (0.0056)   | (0.0047)   |
| $E[\max_{s^* \in S^*} r_{2n}(s^*) | X_n]$ | 0.1662     | 0.0744     | 0.0572     |
|                  | (0.0123)   | (0.0052)   | (0.0036)   |
Figure 1: Contour plots of the true joint posterior density $\pi(\theta, \alpha | X_n)$ (in red), the limiting posterior density $\mathcal{N}(\theta | \theta_0, 2\theta_0^2/n) \times \pi(\alpha | X_n)$ in Theorem 3 Eq. (23) (in blue), and the limiting posterior density $\mathcal{N}(\theta | \theta_0, 2\theta_0^2/n) \times \pi_*(\alpha | X_n)$ in Theorem 3 Eq. (24) (in grey), for the 1-d Ornstein-Uhlenbeck process with sample size $n = 50, 100, 200, 400$. The dashed line is the “ridge” $\tilde{\theta}_n$ (given in (1)), the value of $\theta$ that maximizes the joint likelihood for each given $\alpha$. The true parameter values are $\theta_0 = 0.5$ and $\alpha_0 = 0.5$.

5 Discussion

The Gaussian process model $X \sim \text{GP}(0, \sigma^2 K_{\alpha, \nu})$ considered in this paper is simple and can be extended in several aspects. For simple kriging, one may include an unknown mean parameter $\mu \in \mathbb{R}$ and assume $X \sim \text{GP}(\mu, \sigma^2 K_{\alpha, \nu})$. However, it is known that the MLE of the mean parameter $\mu$ is inconsistent under fixed-domain asymptotics, even in the simple case of 1-dimensional Ornstein-Uhlenbeck process; see for example, Lemma 5 of Gu and Anderson [20].

The reason is that one cannot expect to estimate $\mu$ consistently with observations from one single sample path. In the Bayesian setting, it should be expected that the marginal posterior distribution for $\mu$ does not converge to the underlying truth under fixed-domain asymptotics.

The inclusion of the mean parameter $\mu$ will also slightly impact the limiting posterior distributions of $(\theta, \alpha)$, but we expect that the marginal posterior of $\theta$ still converges to the same normal limit with a shrinking variance. Accordingly, it is likely that we can establish the same posterior asymptotic efficiency in linear prediction as in Section 3, similar to the frequentist results in Putter and Young [52] where the mean parameter is also estimated from the data.

As in most applications in spatial statistics, one may also consider adding a measurement error term to the model, i.e. we observe the noisy $Y(s_i) = X(s_i) + \varepsilon(s_i)$ for $i = 1, \ldots, n$ and an independent noise process $\{\varepsilon(s) : s \in S\}$ that is independent of $X$. Often it is assumed that $\varepsilon(s) \sim \mathcal{N}(0, \tau^2)$ for all $s \in S$. The parameter $\tau^2$ is the nugget parameter (Cressie [17]). It is well known (Stein [60]) that the presence of nugget parameter significantly changes the convergence rate of the microergodic parameter $\theta$, due to the convolution with Gaussian noise. For example, as shown in the frequentist MLE result Chen et al. [14] for the 1-dimensional Ornstein-Uhlenbeck process ($\nu = 1/2$), the convergence rate of $\theta$ deteriorates from $n^{-1/2}$ to $n^{-1/4}$ under fixed-domain asymptotics, though both the microergodic parameter $\theta$ and the nugget parameter $\tau^2$ can still be consistently estimated; see also the recent development in Tang et al. [67]. Therefore, in the Bayesian fixed-domain asymptotics setting, we expect that the BvM theorem for $(\theta, \tau^2, \alpha)$ is quite different from Theorems 2 and 3.

Our technical proofs make extensive use of the properties of the spectral density of the Matérn covariance function. The explicit form of its spectral density allows us to derive explicit
probabilistic bounds for controlling the BvM approximation error. Our derivations can be possibly to extended to the tapered Matérn covariance functions, since the spectral analysis of the tapered covariance functions shares a lot of similarities to that of Matérn covariance functions (Du et al. [21], Wang and Loh [75]). Our derivations may also be extended to the generalized Wendland (GW) covariance functions (Gneiting [25]). The tails of the spectral densities of GW covariance functions also decay at polynomial rates (Bevilacqua et al. [8]). Furthermore, as shown in Lemma 1 of Bevilacqua et al. [8], the MLE of the microergodic parameter also has the monotonicity property with respect to the range parameter \( \beta \) for the GW covariance functions under some conditions on the parameters, similar to the case of Matérn covariance functions. Therefore, we conjecture that similar BvM results can be established for the tapered Matérn covariance functions and the GW covariance functions. In fact, since the covariance functions share a lot of similarities to that of Matérn covariance functions, possibly to extended to the tapered Matérn covariance functions, since the spectral analysis of the tapered covariance functions shares a lot of similarities to that of Matérn covariance functions. We will explore these directions in future research.

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A Proof of Theorems 1 and 2

In this section, we prove Theorems 1 and 2. The proofs of Theorems 3, 4, 5, and all technical lemmas are in the supplementary material.

We first establish a LAN condition for the microergodic parameter \( \theta \) for each given \( \alpha \). For a given \( \alpha > 0 \), let \( t = \sqrt{n}(\theta - \tilde{\theta}_\alpha) \) be the local parameter. We define the following function:

\[
\varrho_n(t; \alpha) = \exp \left\{ \mathcal{L}_n(\alpha^{-2\nu}(\tilde{\theta}_\alpha + \frac{t}{\sqrt{n}}), \alpha) - \mathcal{L}_n(\alpha^{-2\nu}\tilde{\theta}_\alpha, \alpha) \right\} \cdot \frac{\pi \left( \tilde{\theta}_\alpha + \frac{t}{\sqrt{n}} \middle| \alpha \right)}{\pi(\theta_0 | \alpha)} - e^{-\frac{t^2}{4\theta_0^2}}. \tag{32}
\]

Lemma 3. Suppose that Assumption (A.1) holds. Then for any fixed \( \alpha > 0 \), for any positive sequences \( \epsilon_{1n} \downarrow 0 \) and \( 1 \leq s_n < \min \left( n^{1/6}, \epsilon_{1n}^{-1/2} \right) \) that do not depend on \( \alpha \), for all sufficiently large \( n \), the \( \varrho_n \) function in (32) satisfies the following upper bound on the event \( \mathcal{E}_1(\epsilon_{1n}, \alpha) = \{ |\theta_0 - \theta_0| < \epsilon_{1n} \} \):

\[
\int_{\mathbb{R}} |\varrho_n(t; \alpha)| dt \leq B_n(\alpha), \tag{33}
\]

where

\[
B_n(\alpha) \equiv 4\theta_0 e^{-n/64} + \frac{\sqrt{n}}{\pi(\theta_0 | \alpha)} e^{-0.007n} \\
+ 10\theta_0 \exp \left( -\frac{4s_n^2}{125\theta_0^2} \right) \cdot \sup_{\theta \in \left( \frac{1}{2}\theta_0, \frac{1}{2}\theta_0 \right)} \frac{\pi(\theta | \alpha)}{\pi(\theta_0 | \alpha)} + 4\theta_0 \exp \left( -\frac{s_n^2}{4\theta_0^2} \right) \\
+ \frac{8}{\theta_0} \left( s_n^2 \epsilon_{1n} + \frac{12s_n^2}{\sqrt{n}} \right) \cdot \sup_{\theta \in \left( \frac{1}{2}\theta_0, \frac{1}{2}\theta_0 \right)} \frac{\pi(\theta | \alpha)}{\pi(\theta_0 | \alpha)} \\
+ 4\theta_0 \sup_{\theta \in \left( \frac{1}{2}\theta_0, \frac{1}{2}\theta_0 \right)} \frac{\partial \log \pi(\theta | \alpha)}{\partial \theta} \left| \sup_{\theta \in \left( \frac{1}{2}\theta_0, \frac{1}{2}\theta_0 \right)} \frac{\pi(\theta | \alpha)}{\pi(\theta_0 | \alpha)} \cdot \left( \epsilon_{1n} + \frac{s_n}{\sqrt{n}} \right) \right|. \tag{34}
\]

The proof of Lemma 3 is given in the supplementary material.
Proof of Theorem 1. From (8), the posterior distribution of $t$ can be written as

$$\pi(\theta|X_n, \alpha) = \frac{e^{L_n(\alpha - 2\nu_0, \alpha)} \pi(\theta|\alpha)}{\int_0^\infty e^{L_n(\alpha - 2\nu_0, \alpha)} \pi(\theta|\alpha) d\theta} = \frac{e^{L_n(\alpha - 2\nu_0, \alpha) - L_n(\alpha - 2\nu_0, \alpha)} \pi(\theta|\alpha)}{\int_0^\infty e^{L_n(\alpha - 2\nu_0, \alpha) - L_n(\alpha - 2\nu_0, \alpha)} \pi(\theta|\alpha) d\theta}. \quad (35)$$

We can rewrite (33) in Lemma 3 in terms of $\theta = \tilde{\theta}_n + n^{-1/2}t$:

$$\int_{\mathbb{R}} e^{L_n(\alpha - 2\nu_0, \alpha) - L_n(\alpha - 2\nu_0, \alpha)} \pi(\theta|\alpha) \pi(\theta|\alpha) - e^{-\frac{n(\theta - \tilde{\theta}_n)^2}{4\theta_0^2}} d\theta \leq B_n(\alpha) \frac{n}{\sqrt{n}}. \quad (36)$$

For the fixed $\alpha > 0$, define the events $E'_1(\epsilon, \alpha) = \{|\tilde{\theta}_n - \tilde{\theta}_0| < \epsilon\}$ and $E''_1(\epsilon) = \{|\tilde{\theta}_0 - \theta_0| < \epsilon\}$ for any $\epsilon > 0$. From Lemma 1, $\Pr\{E'_1(\theta_0 n^{-1/2-\gamma}, \alpha)\} \geq 1 - 2 \exp(-\log^2 n)$ for all sufficiently large $n$. From Lemma S.8, $\Pr\{E''_1(4\theta_0 n^{-1/2} \log n)\} \geq 1 - 2 \exp(-\log^2 n)$ for all sufficiently large $n$. Since when $n$ is sufficiently large,

$$E'_1(\theta_0 n^{-1/2-\gamma}, \alpha) \cap E''_1(4\theta_0 n^{-1/2} \log n, \alpha) \subseteq E_1(5\theta_0 n^{-1/2} \log n, \alpha),$$

we have that $\Pr\{E_1(5\theta_0 n^{-1/2} \log n, \alpha)\} \geq 1 - 4 \exp(-\log^2 n)$. In the expression of $B_n(\alpha)$ in (34), we set $\epsilon_n = 5\theta_0 n^{-1/2} \log n$ and $s_n = \log n$ which satisfies the conditions in Lemma 3. By Assumption (A.1) for a fixed $\alpha > 0$, there exists some finite constant $C_1 > 0$ that depends on $\alpha$, such that

$$\sup_{\theta \in (\frac{1}{2}\tilde{\theta}_0, \frac{3}{2}\tilde{\theta}_0)} \frac{\pi(\theta|\alpha)}{\pi(\theta|\alpha)} \leq C_1, \quad \sup_{\theta \in (\frac{1}{2}\tilde{\theta}_0, \frac{3}{2}\tilde{\theta}_0)} \frac{\partial \log \pi(\theta|\alpha)}{\partial \theta} \leq C_1.$$

Hence, on the event $E_1(5\theta_0 n^{-1/2} \log n, \alpha)$, the order of $B_n(\alpha)$ can be quantified from (34) in Lemma 3:

$$B_n(\alpha) \leq 4\theta_0 e^{-n/64} \frac{n}{\tilde{\theta}_0} e^{-0.007n} + 10C_1 \theta_0 \exp\left(- \frac{4 \log^2 n}{125 \theta_0^2}\right) + 4\theta_0 \exp\left(- \frac{\log^2 n}{4\theta_0^2}\right)$$

$$+ \frac{8C_1}{\theta_0^2} \left(5\theta_0 n^{-1/2} \log^3 n + 2 n^{-1/2} \log^3 n + 4C_1^2 \theta_0 (5\theta_0 + 1) n^{-1/2} \log n\right) \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty. \quad (37)$$

This together with (36) implies that on the event $E_1(5\theta_0 n^{-1/2} \log n, \alpha)$, the denominator of (35) converges to $\int_{\mathbb{R}} e^{-\frac{n(\theta - \tilde{\theta}_0)^2}{4\theta_0^2}} d\theta = 2\theta_0 \sqrt{\pi/n}$. Now in Lemma S.16, we set $f$ to be the numerator of (35) and $g$ to be $e^{-\frac{n(\theta - \tilde{\theta}_0)^2}{4\theta_0^2}}$. Using (37), we obtain that on the event $E_1(5\theta_0 n^{-1/2} \log n, \alpha)$, as $n \rightarrow \infty$,

$$\int_{\mathbb{R}} \left| \pi(\theta|X_n, \alpha) - \frac{1}{2\sqrt{\pi} \theta_0} e^{-\frac{n(\theta - \tilde{\theta}_0)^2}{4\theta_0^2}} \right| d\theta$$

$$\leq \frac{2B_n(\alpha)/\sqrt{n}}{2\theta_0 \sqrt{\pi/n}} = \frac{B_n(\alpha)}{2\theta_0 \sqrt{\pi}} \rightarrow 0.$$

Since $\Pr\{E_1(5\theta_0 n^{-1/2} \log n, \alpha)\} \leq 4 \exp(-\log^2 n)$ and $\sum_{n=1}^{\infty} 4 \exp(-\log^2 n) < \infty$, by the Borel-Cantelli lemma, we have shown that (7) holds as $n \rightarrow \infty$ almost surely $P_{(\sigma^2_0, \alpha_0)}$. \qed
Proof of Theorem \[3\]. It will be proved in Lemma \[S.15\] that the profile posterior density \[17\] is well defined for every \( n \) and \( X_n \). The convergence in total variation norm for the marginal posterior distributions of \( \theta \) and \( \alpha \) follows trivially once the convergence for the joint posterior is proved. The convergence in total variation norm for the joint posterior \[16\] is implied by adding the following inequalities using a triangle inequality:

\[
\int_0^\infty \int_\mathbb{R} \left| \pi(\theta, \alpha|X_n) - \frac{\sqrt{n}}{2\sqrt{\pi \theta_0}} e^{-\frac{(\theta-\tilde{\theta}_n)^2}{4\theta_0^2}} \cdot \tilde{\pi}(\alpha|X_n) \right| \ d\theta d\alpha \to 0, \tag{38}
\]

\[
\int_0^\infty \int_\mathbb{R} \left| \frac{\sqrt{n}}{2\sqrt{\pi \theta_0}} e^{-\frac{(\theta-\tilde{\theta}_n)^2}{4\theta_0^2}} - \frac{\sqrt{n}}{2\sqrt{\pi \theta_0}} e^{-\frac{(\theta-\tilde{\theta}_0)^2}{4\theta_0^2}} \right| \cdot \tilde{\pi}(\alpha|X_n) \ d\theta d\alpha \to 0, \tag{39}
\]

as \( n \to \infty \) almost surely \( P(\sigma^2_0, \alpha_0) \). We prove \[38\] and \[39\] respectively.

Proof of \[38\]:

In Lemma \[S.16\] we take

\[
f = e^{\mathcal{L}_n(\alpha^{-2\nu}, \alpha) - \mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}_n, \alpha) \pi(\theta|\alpha)} \cdot e^{\mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}_n, \alpha) \pi(\alpha)},
\]

\[
g = e^{-\frac{n(\theta-\tilde{\theta}_n)^2}{4\theta_0^2}} \pi(\theta_0|\alpha) \cdot e^{\mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}_n, \alpha) \pi(\alpha)},
\]

such that by applying Lemma \[S.16\] we can obtain that

\[
\int_0^\infty \int_\mathbb{R} \left| \pi(\theta, \alpha|X_n) - \frac{\sqrt{n}}{2\sqrt{\pi \theta_0}} e^{-\frac{(\theta-\tilde{\theta}_n)^2}{4\theta_0^2}} \cdot \tilde{\pi}(\alpha|X_n) \right| \ d\theta d\alpha
\]

\[
= \int_0^\infty \int_\mathbb{R} \left| \frac{\sqrt{n}}{2\sqrt{\pi \theta_0}} e^{-\frac{(\theta-\tilde{\theta}_n)^2}{4\theta_0^2}} - \frac{\sqrt{n}}{2\sqrt{\pi \theta_0}} e^{-\frac{(\theta-\tilde{\theta}_0)^2}{4\theta_0^2}} \right| \cdot \tilde{\pi}(\alpha|X_n) \ d\theta d\alpha
\]

\[
\leq \frac{N}{D}, \tag{40}
\]

where (with \( \Delta(t; \alpha) \) defined in \[32\])

\[
N = 2 \int_0^\infty \int_\mathbb{R} \left| e^{\mathcal{L}_n(\alpha^{-2\nu}, \alpha) - \mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}_n, \alpha) \pi(\theta|\alpha)} \cdot \frac{\pi(\theta|\alpha)}{\pi(\theta_0|\alpha)} - e^{-\frac{n(\theta-\tilde{\theta}_n)^2}{4\theta_0^2}} \right| \ d\theta d\alpha
\]

\[
= 2 \int_0^\infty \int_\mathbb{R} \left| \Delta(\sqrt{n}(\theta - \tilde{\theta}_n); \alpha) \right| e^{\mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}_n, \alpha) \pi(\theta|\alpha)} \ d\theta d\alpha, \tag{41}
\]

\[
D = \frac{2\theta_0\sqrt{n}}{\sqrt{n}} \int_0^\infty e^{-\frac{n(\theta^2-\tilde{\theta}_n^2)}{4\theta_0^2}} \cdot \mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}_n, \alpha) \pi(\theta|\alpha) \ d\theta d\alpha
\]

\[
= \frac{2\theta_0\sqrt{n}}{\sqrt{n}} \int_0^\infty e^{\mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}_n, \alpha) \pi(\theta|\alpha)} \ d\theta d\alpha, \tag{42}
\]

We decompose the numerator in \[41\] into three terms:

\[
N = N_1 + N_2 + N_3,
\]

\[
N_1 = 2 \int_\alpha \left| \Delta(\sqrt{n}(\theta - \tilde{\theta}_n); \alpha) \right| e^{\mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}_n, \alpha) \pi(\theta_0|\alpha)} \ d\theta d\alpha,
\]

\[
N_2 = \int_0^\infty \int_\mathbb{R} \left| \frac{\sqrt{n}}{2\sqrt{\pi \theta_0}} e^{-\frac{(\theta-\tilde{\theta}_n)^2}{4\theta_0^2}} - \frac{\sqrt{n}}{2\sqrt{\pi \theta_0}} e^{-\frac{(\theta-\tilde{\theta}_0)^2}{4\theta_0^2}} \right| \cdot \tilde{\pi}(\alpha|X_n) \ d\theta d\alpha,
\]

\[
N_3 = \int_0^\infty \int_\mathbb{R} \left| \frac{\sqrt{n}}{2\sqrt{\pi \theta_0}} e^{-\frac{(\theta-\tilde{\theta}_n)^2}{4\theta_0^2}} \cdot \tilde{\pi}(\alpha|X_n) \right| \ d\theta d\alpha.
\]
\[ N_2 = 2 \int_0^{\alpha_n} \int_{\mathbb{R}} |\Delta(\sqrt{n}(\theta - \tilde{\theta}_n); \alpha)| e^{\mathcal{L}_n(\alpha^{-2\nu}\tilde{\theta}_n, \alpha)} \pi(\theta_0) \pi(\alpha) d\theta d\alpha, \]
\[ N_3 = 2 \int_{\alpha_n}^{\infty} \int_{\mathbb{R}} |\Delta(\sqrt{n}(\theta - \tilde{\theta}_n); \alpha)| e^{\mathcal{L}_n(\alpha^{-2\nu}\tilde{\theta}_n, \alpha)} \pi(\theta_0) \pi(\alpha) d\theta d\alpha, \quad (43) \]

To show (38), from (40) and (43), it suffices to show that \( N_j / D \to 0 \) for \( j = 1, 2, 3 \) as \( n \to \infty \) almost surely \( P_{(\sigma_n^2, \alpha_0)} \).

**Proof of \( N_1 / D \to 0 \):**

We consider all \( \alpha \in [\alpha_n, \alpha_n] \). For any \( \epsilon > 0 \), define three events

\[ \mathcal{E}_2(\epsilon) = \left\{ \sup_{\alpha \in [\alpha_n, \alpha_n]} |\tilde{\theta}_n - \theta_0| < \epsilon \right\}, \quad \mathcal{E}_3(\epsilon) = \left\{ \sup_{\alpha \in [\alpha_n, \alpha_n]} |\tilde{\theta}_n - \tilde{\theta}_0| < \epsilon \right\}, \]
\[ \mathcal{E}_4(\epsilon) = \left\{ |\tilde{\theta}_0 - \theta_0| < \epsilon \right\}. \]

For sufficiently large \( n \), Lemma S.7 shows that \( \Pr(\mathcal{E}_3(\theta_0 n^{-1/2-\tau})) \geq 1 - 2 \exp(-\log^2 n) \) for some constant \( \tau \in (0, 1/2) \). Lemma S.8 shows that \( \Pr(\mathcal{E}_4(4\theta_0 n^{-1/2} \log n)) \geq 1 - 2 \exp(-\log^2 n) \).

By the triangle inequality, for sufficiently large \( n \),
\[ \mathcal{E}_2(5\theta_0 n^{-1/2} \log n) \supseteq \mathcal{E}_3(\theta_0 n^{-1/2-\tau}) \cap \mathcal{E}_4(4\theta_0 n^{-1/2} \log n), \]

it follows that \( \Pr(\mathcal{E}_2(5\theta_0 n^{-1/2} \log n)) \geq 1 - 4 \exp(-\log^2 n) \).

We again invoke the inequality (36) from the conclusion of Lemma 3 with \( B_n(\alpha) \) defined in (34) with \( \epsilon_1 n = 5\theta_0 n^{-1/2} \log n \) and \( s_n = \log n \). Since \( \mathcal{E}_1(5\theta_0 n^{-1/2} \log n, \alpha) \supseteq \mathcal{E}_2(5\theta_0 n^{-1/2} \log n) \) for every \( \alpha \in [\alpha_n, \alpha_n] \), Lemma 3 can be applied to all \( \alpha \in [\alpha_n, \alpha_n] \) with \( \epsilon_1 n = 5\theta_0 n^{-1/2} \log n \) and \( s_n = \log n \). Therefore, (36) holds uniformly for all \( \alpha \in [\alpha_n, \alpha_n] \) on the event \( \mathcal{E}_2(\theta_0 n^{-1/2} \log n) \), such that \( \Pr(\mathcal{E}_2(5\theta_0 n^{-1/2} \log n)) \geq 1 - 4 \exp(-\log^2 n) \).

Integrating (36) over the interval \([\alpha_n, \alpha_n]\) gives that
\[ \int_{\alpha_n}^{\alpha_n} \int_{\mathbb{R}} e^{\mathcal{L}_n(\alpha^{-2\nu}\theta, \alpha)} e^{\mathcal{L}_n(\alpha^{-2\nu}\tilde{\theta}_n, \alpha)} \pi(\theta_0) \pi(\alpha) d\theta d\alpha \]
\[ \times e^{\mathcal{L}_n(\alpha^{-2\nu}\tilde{\theta}_n, \alpha)} \pi(\theta_0) \pi(\alpha) d\theta d\alpha \]
\[ \leq \int_{\alpha_n}^{\alpha_n} \frac{B_n(\alpha)}{\sqrt{n}} e^{\mathcal{L}_n(\alpha^{-2\nu}\tilde{\theta}_n, \alpha)} \pi(\theta_0) \pi(\alpha) d\alpha \]
\[ \leq \sup_{\alpha \in [\alpha_n, \alpha_n]} \frac{B_n(\alpha)}{\sqrt{n}} \int_{\alpha_n}^{\alpha_n} e^{\mathcal{L}_n(\alpha^{-2\nu}\tilde{\theta}_n, \alpha)} \pi(\theta_0) \pi(\alpha) d\alpha. \quad (44) \]

According to Assumption (A.2), with \( \epsilon_1 n = 5\theta_0 n^{-1/2} \log n \) and \( s_n = \log n \), \( B_n(\alpha) \) as defined in (34) satisfies that for all sufficiently large \( n \),
\[
\sup_{\alpha \in [\alpha_n, \alpha_n]} B_n(\alpha) \\
\leq 4\theta_0 e^{-n/64} + \inf_{\alpha \in [\alpha_n, \alpha_n]} \frac{\sqrt{n}}{\pi(\theta_0) \pi(\alpha)} e^{-0.007n} \\
+ \sup_{\alpha \in [\alpha_n, \alpha_n]} \sup_{\theta \in (\pm \theta_0)} \frac{\pi(\theta_0) \pi(\alpha)}{\pi(\theta_0) \pi(\alpha)} \cdot 10\theta_0 \exp \left( -\frac{4\log^2 n}{125\theta_0^2} \right) + 4\theta_0 \exp \left( -\frac{\log^2 n}{4\theta_0^2} \right) \\
+ \frac{8}{\theta_0^2} \left( 5\theta_0 \log^3 n \sqrt{n} + 2 \log^3 n \right) \sup_{\alpha \in [\alpha_n, \alpha_n]} \sup_{\theta \in (\pm \theta_0)} \frac{\pi(\theta_0) \pi(\alpha)}{\pi(\theta_0) \pi(\alpha)} \
\]
Proof of Assumption (A.2).

Using the Borel-Cantelli lemma, we have shown that $N_n \to \infty$ as $n \to \infty$.

On the other hand, define the event $\mathcal{E}_2 = \{ \alpha : \sup_n \log n \sqrt{n} \leq 0 \}$.

By Assumption (A.2), we have $\sup_n \log n \sqrt{n} \leq 0$.

Then on the event $\mathcal{E}_2$, we have

$$
\Pr\{ \alpha \in \{ \alpha : \sup_n \log n \sqrt{n} \leq 0 \} \} \leq \frac{4 \log^2 n}{125 \theta_0^2} + 4 \log^2 n \times \frac{\log n}{\sqrt{n}} \to 0, \quad \text{as } n \to \infty,
$$

(45)

where in the last step, we have used the fact that $C_{\pi,3} < 1$ and $C_{\pi,1} + C_{\pi,2} < 1/2$ according to Assumption (A.2).

Therefore, (44), (45), (43), and (42) together imply that on the event $\mathcal{E}_2(5 \theta_0 n^{-1/2} \log n)$,

$$
\frac{N_n}{D} = \frac{2}{\theta_0 \sqrt{\pi}} \int_0^\infty \left( \Delta(\sqrt{n}(\theta - \tilde{\theta}_0)); \alpha \right) e^{\mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha)} \pi(\theta) \pi(\alpha) d\theta d\alpha
\leq \frac{\sup_{\alpha} e^{\mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha)} \pi(\theta) \pi(\alpha) d\alpha}{\theta_0 \sqrt{\pi}} \to 0,
$$

(46)

as $n \to \infty$. Since $\Pr\{ \mathcal{E}_2(5 \theta_0 n^{-1/2} \log n) \} \leq 4 \exp(-\log^2 n)$ and $\sum_{n=1}^\infty 4 \exp(-\log^2 n) < \infty$, by the Borel-Cantelli lemma, we have shown that $N_n / D \to 0$ as $n \to \infty$ almost surely $P_0(\sigma^2_0, \alpha_0)$.

Proof of $N_2 / D \to 0$:

We start with an upper bound for $N_2$:

$$
N_2 = 2 \int_0^{\alpha} \int_0^{\alpha} \left( e^{\mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha)} - \mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha) \right) \pi(\theta) \pi(\alpha) d\theta d\alpha
\leq 2 \int_0^{\alpha} \int_0^{\alpha} \left( e^{\mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha)} - \mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha) \right) \pi(\theta) \pi(\alpha) d\theta d\alpha
\leq 2 \int_0^{\alpha} \int_0^{\alpha} \left( e^{\mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha)} - \mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha) \right) \pi(\theta) \pi(\alpha) d\theta d\alpha
\leq 2 \int_0^{\alpha} e^{\mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha)} \pi(\alpha) d\alpha + \frac{4 \theta_0 \sqrt{\pi}}{\sqrt{n}} \int_0^{\alpha} e^{\mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha)} \pi(\theta) \pi(\alpha) d\alpha,
$$

(47)

where (i) follows from the fact that $\mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha) \leq \mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha)$ as $\tilde{\theta}_0$ is the maximizer of $\mathcal{L}_n(\alpha - 2\tilde{\theta}_0, \alpha)$ given $\alpha$.

On the other hand, define the event $\mathcal{E}_5$ to be the event that $\{5.57\}$ in Lemma 5.9 happens, such that $\Pr(\mathcal{E}_5) \geq 1 - 2 \exp(-\log^2 n)$. Then on the event $\mathcal{E}_5$, the denominator (42) can be
lower bounded by

\[ D \geq \frac{2\theta_0 \sqrt{\pi}}{\sqrt{n}} e^{\bar{L}_n(\omega_0)} \int_{\omega_0}^{e^\alpha \bar{L}_n(\omega_0) \pi(\theta_0) \pi(\alpha) \, d\alpha} \]

\[ \geq \frac{2\theta_0 \sqrt{\pi}}{\sqrt{n}} \exp\left\{ \bar{L}_n(\omega_0) - n^{1/2-\tau} \right\} \int_{\omega_0}^{e^\alpha \bar{L}_n(\omega_0) \pi(\theta_0) \pi(\alpha) \, d\alpha} \]

\[ = \frac{2\theta_0 \sqrt{\pi} c_{\pi,0}}{\sqrt{n}} \exp\left\{ \bar{L}_n(\omega_0) - n^{1/2-\tau} \right\}, \quad (48) \]

where \( c_{\pi,0} = \int_{\omega_0}^{e^\alpha \bar{L}_n(\omega_0) \pi(\theta_0) \pi(\alpha) \, d\alpha}. \) By Assumptions \([A.1]\) and \([A.3]\) since \( \pi(\theta_0) > 0 \) for all \( \alpha > 0 \) and \( \pi(\alpha) > 0 \) around a neighborhood of \( \omega_0, \pi(\theta_0) \pi(\alpha) \) is strictly positive on the interval \([\omega_0, 2^{1/(2\nu+d)} \omega_0]\) and hence \( c_{\pi,0} \) is a positive constant.

We combine \([47]\) and \([48]\) to obtain that

\[ \frac{N_2}{D} \leq \frac{\sqrt{n}}{\theta_0 \sqrt{\pi} c_{\pi,0}} \exp\left\{ n^{1/2-\tau} \int_0^{2\pi} e^{\bar{L}_n(\omega) - \bar{L}_n(\omega_0)} \pi(\theta_0) \pi(\alpha) \, d\alpha \right\} + \frac{2}{c_{\pi,0}} \exp\left\{ n^{1/2-\tau} \right\} \int_0^{2\pi} e^{\bar{L}_n(\omega) - \bar{L}_n(\omega_0)} \pi(\theta_0) \pi(\alpha) \, d\alpha. \quad (49) \]

To upper bound the two terms in \([49]\), we first derive a simple relation for the part \( \exp\{\bar{L}_n(\omega) - \bar{L}_n(\omega_0)\}. \) Let \( \mathcal{E}_6 \) be the event on which \([S.65]\) in Lemma \([S.10]\) happens. So \( \Pr(\mathcal{E}_6) \geq 1 - 2 \exp(-\log^2 n) \) for sufficiently large \( n. \) On the event \( \mathcal{E}_6, \) the monotonicity bound from Lemma \([S.14]\) and the upper bound from Lemma \([S.10]\) imply that for any \( \alpha \in (0, \omega_n), \)

\[ \exp\left\{ \bar{L}_n(\omega) - \bar{L}_n(\omega_0) \right\} = \exp\left\{ \bar{L}_n(\alpha) - \bar{L}_n(\omega_0) \right\} \cdot \exp\left\{ \bar{L}_n(\omega_0) - \bar{L}_n(\omega_0) \right\} < \left( \frac{\alpha_n}{\alpha} \right)^{n(\nu+d/2)} \exp\left\{ 2n^{1/2-\tau} \right\} = \alpha^{-n(\nu+d/2)} \exp\left\{ -(\nu + d/2) \kappa n \log n + 2n^{1/2-\tau} \right\}, \quad (50) \]

where \( \tau \in (0,1/2) \) and \( \kappa \in (0,1/2) \) are defined in Lemma \([S.7]. \) We now plug \([50]\) in \([49]\) and invoke Assumption \([A.3]\) to obtain that on the event \( \mathcal{E}_5 \cap \mathcal{E}_6, \)

\[ \frac{N_2}{D} \leq \frac{\sqrt{n}}{\theta_0 \sqrt{\pi} c_{\pi,0}} \exp\left\{ -(\nu + d/2) \kappa n \log n + 3n^{1/2-\tau} \right\} \int_0^{2\pi} \alpha^{-n(\nu+d/2)} \pi(\theta_0) \pi(\alpha) \, d\alpha \]

\[ + \frac{2}{c_{\pi,0}} \exp\left\{ -(\nu + d/2) \kappa n \log n + 3n^{1/2-\tau} \right\} \int_0^{2\pi} \alpha^{-n(\nu+d/2)} \pi(\theta_0) \pi(\alpha) \, d\alpha \]

\[ \leq \frac{\sqrt{n}}{\theta_0 \sqrt{\pi} c_{\pi,0}} \exp\left\{ -(\nu + d/2) \kappa n \log n + 3n^{1/2-\tau} \right\} \exp\left( c_\pi n \log n \right) \]

\[ + \frac{2}{c_{\pi,0}} \exp\left\{ -(\nu + d/2) \kappa n \log n + 3n^{1/2-\tau} \right\} \exp\left( c_\pi n \log n \right) \]

\[ \to 0, \text{ as } n \to \infty, \quad (51) \]

where the last step follows because \( c_\pi < (\nu+d/2) \kappa \) by Assumption \([A.3]. \) Since \( \Pr\{ (\mathcal{E}_5 \cap \mathcal{E}_6)^c \} < 4 \exp(-\log^2 n) \) and \( \sum_{n=1}^{\infty} 4 \exp(-\log^2 n) < \infty, \) by the Borel-Cantelli lemma, we have shown that \( N_2 / D \to 0 \text{ as } n \to \infty \text{ almost surely } P(\sigma^{\omega_0,\omega_0}). \)

Proof of \( N_3 / D \to 0: \)
We have the following upper bound for $N_3$, similar to the derivation of (47):

\[
N_3 = 2 \int_{\pi_n}^{\infty} \left| e^{L_n(\alpha-2\nu\theta_n,\alpha)} - e^{L_n(\alpha-\theta_n,\alpha)} \right| \pi(\alpha) d\alpha \\
\leq 2 \int_{\pi_n}^{\infty} \left( e^{L_n(\alpha-2\nu\theta_n,\alpha)} - e^{L_n(\alpha-\theta_n,\alpha)} \right) \pi(\alpha) d\alpha \\
\leq 2 \int_{\pi_n}^{\infty} \left( \int_{0}^{\infty} \pi(\theta) d\theta \right) e^{L_n(\alpha-\theta_n,\alpha)} \pi(\alpha) d\alpha \\
+ 2 \int_{\pi_n}^{\infty} \left( \int_{R} e^{-\frac{n(\theta-\theta_n)^2}{4\theta_0^2}} d\theta \right) e^{L_n(\alpha-\theta_n,\alpha)} \pi(\theta_0) \pi(\alpha) d\alpha \\
\leq 2 \int_{\pi_n}^{\infty} e^{L_n(\alpha-\theta_n,\alpha)} \pi(\alpha) d\alpha + \frac{4\theta_0}{\sqrt{\pi}} \int_{\pi_n}^{\infty} e^{L_n(\alpha-\theta_n,\alpha)} \pi(\theta_0) \pi(\alpha) d\alpha. \tag{52}
\]

(52) and (48) imply that on the event $\mathcal{E}_5$,

\[
\frac{N_3}{D} \leq \frac{\sqrt{n}}{\theta_0 \sqrt{\pi \epsilon_n \theta_n}} \exp \left( \int_{\pi_n}^{\infty} e^{L_n(\alpha-\theta_n,\alpha)} \pi(\alpha) d\alpha \right) \\
+ \frac{2}{\epsilon_n \theta_n} \exp \left( \int_{\pi_n}^{\infty} e^{L_n(\alpha-\theta_n,\alpha)} \pi(\theta_0) \pi(\alpha) d\alpha \right). \tag{53}
\]

Similar to the proof of $N_2/D \to 0$, we use Lemma S.14 and Lemma S.12 to obtain that for any $\alpha \in (\pi_n, +\infty)$,

\[
\exp \left\{ \tilde{L}_n(\alpha) - \tilde{L}_n(\alpha_0) \right\} \\
= \exp \left\{ \tilde{L}_n(\alpha) - \tilde{L}_n(\pi_n) \right\} \cdot \exp \left\{ \tilde{L}_n(\pi_n) - \tilde{L}_n(\alpha_0) \right\} \\
< \left( \frac{\alpha}{\pi_n} \right)^{n(\nu+d/2)} \exp (C_3 n^{\kappa_1} \log n) \\
= \alpha^{n(\nu+d/2)} \exp \left\{ -(\nu + d/2) \pi n \log n + C_3 n^{\kappa_1} \log n \right\}, \tag{54}
\]

where $C_3 > 0$ and $\kappa_1 \in (0, 1/2)$ are given in Lemma S.12 and $\pi \in (0, 1/2)$ is given in Lemma S.7. We now plug (54) in (53) and invoke Assumption (A.3) to obtain that on the event $\mathcal{E}_5$,

\[
\frac{N_3}{D} \leq \frac{\sqrt{n}}{\theta_0 \sqrt{\pi \epsilon_n \theta_n}} \exp \left\{ -(\nu + d/2) \pi n \log n + C_3 n^{\kappa_1} \log n + n^{1/2-\tau} \right\} \\
\times \int_{\pi_n}^{\infty} \alpha^{n(\nu+d/2)} \pi(\alpha) d\alpha \\
+ \frac{2}{\epsilon_n \theta_n} \exp \left\{ -(\nu + d/2) \pi n \log n + C_3 n^{\kappa_1} \log n + n^{1/2-\tau} \right\} \exp (\pi n \log n) \\
\times \int_{\pi_n}^{\infty} \alpha^{n(\nu+d/2)} \pi(\theta_0) \pi(\alpha) d\alpha \\
\leq \frac{\sqrt{n}}{\theta_0 \sqrt{\pi \epsilon_n \theta_n}} \exp \left\{ -(\nu + d/2) \pi n \log n + C_3 n^{\kappa_1} \log n + n^{1/2-\tau} \right\} \exp (\pi n \log n) \\
+ \frac{2}{\epsilon_n \theta_n} \exp \left\{ -(\nu + d/2) \pi n \log n + C_3 n^{\kappa_1} \log n + n^{1/2-\tau} \right\} \exp (\pi n \log n) \\
\to 0, \text{ as } n \to \infty, \tag{55}
\]
where the last step follows because $\tau_\pi < (\nu + d/2)\pi$ by Assumption \((\text{A.3})\). Since $\Pr(\mathcal{E}_5^c) < 2\exp(-\log^2 n)$ and $\sum_{n=1}^{\infty} 2\exp(-\log^2 n) < \infty$, by the Borel-Cantelli lemma, we have shown that $N_3/D \to 0$ as $n \to \infty$ almost surely $P(\sigma_{\tilde{\alpha},\bar{\alpha}})$.

Proof of \((\text{59})\):

We use Lemma \((\text{S.17})\) and obtain that

\[
\int_0^\infty \int_{\mathbb{R}} \left| \frac{\sqrt{n}}{2\sqrt{\pi}\theta_0} e^{-\frac{(\theta_0 - \bar{\theta}_\alpha)^2}{4\theta_0^2}} - \frac{\sqrt{n}}{2\sqrt{\pi}\theta_0} e^{-\frac{(\theta_0 - \bar{\theta}_{\alpha_0})^2}{4\theta_0^2}} \right| \tilde{\pi}(\alpha|X_n) d\theta d\alpha

= \int_0^\infty d_{TV} \left\{ N(\bar{\theta}_\alpha, 2\theta_0^2/n), N(\bar{\theta}_{\alpha_0}, 2\theta_0^2/n) \right\} \tilde{\pi}(\alpha|X_n) d\alpha

\leq \int_0^\infty 2 \left\{ 2\Phi \left( \frac{n^{1/2}|\bar{\theta}_\alpha - \bar{\theta}_{\alpha_0}|}{2\sqrt{\theta_0}} - 1 \right) \right\} \tilde{\pi}(\alpha|X_n) d\alpha

\leq \int_{\Delta_n} 2\sqrt{2/\pi} n^{1/2}|\bar{\theta}_\alpha - \bar{\theta}_{\alpha_0}| \tilde{\pi}(\alpha|X_n) d\alpha

+ 2 \int_{\Delta_n} \tilde{\pi}(\alpha|X_n) d\alpha + 2 \int_{\pi_n} ^\infty \tilde{\pi}(\alpha|X_n) d\alpha,

\tag{56}
\]

where in (i), we use the relation $\Phi(x) - 0.5 = \Phi(x) - \Phi(0) \leq \phi(0)x = x/\sqrt{2\pi}$ for all $x \geq 0$ (where $\phi(x)$ is the standard normal density), as well as the direct bound $|2\Phi(x) - 1| \leq 1$ for all $x \in \mathbb{R}$.

On the event $\mathcal{E}_3(n^{-1/2-\tau})$, we have that $n^{1/2}|\bar{\theta}_\alpha - \bar{\theta}_{\alpha_0}| \leq n^{-\tau}$ uniformly for all $\alpha \in \left[\alpha_n, \pi_n\right]$. Together with the fact that $\tilde{\pi}(\alpha|X_n)$ is a proper probability density from Lemma \((\text{S.15})\) we can derive from \((\text{56})\) that on the event $\mathcal{E}_3(\theta_0 n^{-1/2-\tau})$,

\[
\int_{\Delta_n} 2\sqrt{2/\pi} n^{1/2}|\bar{\theta}_\alpha - \bar{\theta}_{\alpha_0}| \tilde{\pi}(\alpha|X_n) d\alpha

\leq 2\sqrt{2/\pi} n^{-\tau} \int_{\pi_n} ^\infty \tilde{\pi}(\alpha|X_n) d\alpha \leq 2\sqrt{2/\pi} n^{-\tau} \to 0,

\tag{57}
\]

as $n \to \infty$. Since $\Pr \left\{ \mathcal{E}_3(\theta_0 n^{-1/2-\tau})^c \right\} \leq 2\exp(-\log^2 n)$ and $\sum_{n=1}^{\infty} 2\exp(-\log^2 n) < \infty$, by the Borel-Cantelli lemma, we have shown that \((\text{57})\) holds as $n \to \infty$ almost surely $P(\sigma_{\tilde{\alpha},\bar{\alpha}})$.

For the second term on the right-hand side of \((\text{56})\), we have that by the definition \((\text{17})\),

\[
2 \int_{\Delta_n} \tilde{\pi}(\alpha|X_n) d\alpha \leq 2 \int_{\Delta_n} e^{\tilde{Z}_n(\alpha)-\tilde{Z}_n(\alpha_0)} \pi(\theta_0|\alpha) \pi(\alpha) d\alpha.

\]

The denominator is lower bounded by $\exp(-n^{1/2-\tau})c_{\pi,0}$ on the event $\mathcal{E}_5$, similar to the proof of \((\text{48})\). The numerator can be upper bounded on the event $\mathcal{E}_6$, using the same derivation as in \((\text{50})\) and \((\text{49})\). As a result, on the event $\mathcal{E}_5 \cap \mathcal{E}_6$, using Assumption \((\text{A.3})\) we have that

\[
2 \int_{\Delta_n} \tilde{\pi}(\alpha|X_n) d\alpha \leq \frac{2 \exp \left\{ -(\nu + d/2)\kappa n \log n + n^{1/2-\tau} \right\} \int_{\Delta_n} \alpha^{-n(\nu+d/2)}\pi(\alpha) d\alpha}{\exp(-n^{1/2-\tau})c_{\pi,0}}

\leq \frac{2 c_{\pi,0}^{\nu+d/2} n \log n + 2n^{1/2-\tau} + c_{\pi}^{\nu+d/2} n \log n}{\exp(-n^{1/2-\tau})c_{\pi,0}}

\to 0, \text{ as } n \to \infty,

\tag{58}
\]

given that $c_{\pi,0} < (\nu + d/2)\kappa$ in Assumption \((\text{A.3})\). \((\text{58})\) holds as $n \to \infty$ almost surely $P(\sigma_{\tilde{\alpha},\bar{\alpha}})$ since $\Pr((\mathcal{E}_5 \cap \mathcal{E}_6)^c) < 4\exp(-\log^2 n)$ and $\sum_{n=1}^{\infty} 4\exp(-\log^2 n) < \infty$. 

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Similarly, for the third term on the right-hand side of (56), we have that by the definition (17),
\[
2 \int_{\pi_n}^{\infty} \tilde{p}(\alpha|X_n) d\alpha \leq 2 \frac{\int_{\pi_n}^{\infty} e^{\tilde{e}_n(\alpha) - \tilde{e}_n(\alpha_0)} \pi(\theta_0|\alpha) \pi(\alpha) d\alpha}{\int_{\alpha_0}^{\infty} e^{\tilde{e}_n(\alpha) - \tilde{e}_n(\alpha_0)} \pi(\theta_0|\alpha) \pi(\alpha) d\alpha}.
\]

The denominator is lower bounded by \(\exp(-n^{1/2-\tau})c_{\pi,0}\) on the event \(E_5\), similar to the proof of (48). The numerator can be upper bounded using the same derivation as in (54) and (55).

As a result, on the event \(E_5\), using Assumption (A.3), we have that\(\tilde{p}(\alpha|X_n)\) converges to zero as \(n \to \infty\),

\[
\tilde{p}(\alpha|X_n) \to 0, \quad \text{as} \quad n \to \infty, \quad (59)
\]
given that \(\bar{\pi}_n < (\nu + d/2)\bar{\pi}\) in Assumption (A.3) holds as \(n \to \infty\) almost surely \(P_{(\sigma^2_0,\alpha_0)}\) since \(\Pr\{E_5^c\} < 2 \exp(-\log^2 n)\) and \(\sum_{n=1}^\infty 2 \exp(-\log^2 n) < \infty\).

Finally, (57), (58), and (59) together imply that the right-hand side of (56) converges to zero as \(n \to \infty\) almost surely \(P_{(\sigma^2_0,\alpha_0)}\). This has proved (39), and hence has completed the proof of Theorem 2.

\[\square\]

Supplementary Materials

S1 Spectral Analysis of Matérn Covariance Functions

We first present a series of properties based on the spectral analysis of the Matérn covariance function. These results are strengthened versions of the spectral analysis in Wang and Loh [75]. We follow the same techniques in Section 4 of Wang and Loh [75], but make the upper bounds in Sections 4, 5, 7 of Wang and Loh [75] explicitly dependent on \(\alpha\).

For a generic \(\alpha > 0\), let \(U_{\alpha}\) be an \(n \times n\) invertible matrix that simultaneously diagonalizes \(R_{\alpha_0}\) and \(R_{\alpha}\):

\[
\sigma_0^2 U_{\alpha}^\top R_{\alpha_0} U_{\alpha} = I_n, \quad \sigma^2 U_{\alpha}^\top R_{\alpha} U_{\alpha} = \text{diag} \{ \lambda_{k,n}(\alpha) : k = 1, \ldots, n \},
\]
(S.1)

where \(\{ \lambda_{k,n}(\alpha), k = 1, \ldots, n \}\) are the positive diagonal entries. Let \(\nu = \sqrt{-1}\). For \(\omega \in \mathbb{R}^d\), let

\[
f_{\sigma,\alpha}(\omega) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\omega^\top x} \sigma^2 K_{\alpha,\nu}(x) dx = \frac{\Gamma(\nu + d/2)}{\Gamma(\nu)} \cdot \frac{\sigma^2 \alpha^{2\nu}}{\pi^{d/2} (\alpha^2 + ||\omega||^2)^{\nu+d/2}},
\]
(S.2)

be the isotropic spectral density of the Gaussian process with Matérn covariance function defined in [1]. For any given pair \((\sigma, \alpha)\), let \(\|\psi\|_{f_{\sigma,\alpha}}^2 = (\psi, \psi)_{f_{\sigma,\alpha}} = \int_{\mathbb{R}^d} |\psi(\omega)|^2 f_{\sigma,\alpha}(\omega) d\omega\) be the norm of a generic function \(\psi\) in the Hilbert space \(L_2(f_{\sigma,\alpha})\), with inner product \((\psi_1, \psi_2)_{f_{\sigma,\alpha}} = \int_{\mathbb{R}^d} \psi_1(\omega)\psi_2(\omega) f_{\sigma,\alpha}(\omega) d\omega\) for any \(\psi_1, \psi_2 \in L_2(f_{\sigma,\alpha})\).
According to the spectral analysis in Section 4 of Wang and Loh [75], using the same notation as theirs, for any given pair \((\sigma, \alpha)\) that satisfies \(\sigma^2 \alpha^{2\nu} = \theta_0 = \sigma_0^2 \alpha_0^{2\nu}\), there exist orthonormal basis functions \(\psi_1, \ldots, \psi_n \in L_2(f_{\sigma_0, \alpha_0})\) such that for any \(j, k \in \{1, \ldots, n\}\),
\[
\langle \psi_j, \psi_k \rangle_{f_{\sigma_0, \alpha_0}} = I(j = k), \quad \langle \psi_j, \psi_k \rangle_{f_{\sigma, \alpha}} = \lambda_{j,n}(\alpha) I(j = k). \tag{S.3}
\]

We have the following results for the spectral density \(f_{\sigma, \alpha}\) and the sequence \(\{\lambda_{k,n}(\alpha), k = 1, \ldots, n\}\).  

**Lemma S.1.** Let \(\nu > 0\) and \(d \in \{1, 2, 3\}\). For any pair \((\sigma, \alpha) \in \mathbb{R}_+^2\) that satisfies \(\sigma^2 \alpha^{2\nu} = \theta_0 = \sigma_0^2 \alpha_0^{2\nu}\), and for all \(\omega \in \mathbb{R}^d\), the following relations hold:

\[
\min \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d+1}, 1 \right\} \leq \frac{f_{\sigma,\alpha}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} \leq \max \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d+1}, 1 \right\}, \tag{S.4}
\]

\[
\left| \frac{f_{\sigma,\alpha}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} - 1 \right| \leq \frac{(2\nu + d) \max(\alpha_0^2, \alpha^2) \max(\sigma_0^{2\nu+d-2}, \alpha^{2\nu+d-2})}{\alpha^{2\nu+d-2}(\alpha^2 + \|\omega\|^2)}, \tag{S.5}
\]

\[
\lambda_{k,n}(\alpha) \leq \max \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d+1}, 1 \right\}, \tag{S.6}
\]

\[
\lambda_{k,n}(\alpha) \geq \min \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d+1}, 1 \right\}. \tag{S.7}
\]

**Proof of Lemma S.1.** For (S.4), when \(\sigma^2 \alpha^{2\nu} = \theta_0\), we have that
\[
\frac{f_{\sigma,\alpha}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} = \left( \frac{\alpha_0^2 + \|\omega\|^2 }{\alpha^2 + \|\omega\|^2} \right)^{\nu+d/2}. \]

If \(\alpha \geq \alpha_0\), then \(f_{\sigma,\alpha}(\omega)/f_{\sigma_0,\alpha_0}(\omega)\) is an increasing function in \(\|\omega\|\), which implies that \(f_{\sigma,\alpha}(\omega)/f_{\sigma_0,\alpha_0}(\omega) \leq 1\) (taken as \(\|\omega\| \to +\infty\)), and \(f_{\sigma,\alpha}(\omega)/f_{\sigma_0,\alpha_0}(\omega) \geq (\alpha_0/\alpha)^{2\nu+d}\) (taken as \(\|\omega\| \to 0\)). The case of \(\alpha < \alpha_0\) follows similarly. (S.4) summarizes the two cases.

For (S.5), if \(\nu + d/2 \geq 1\), then using a first order Taylor expansion, we have that
\[
\left| \frac{f_{\sigma,\alpha}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} - 1 \right| \leq \frac{(\nu + d/2)(\alpha_0^2 + \|\omega\|^2)^{\nu+d/2} - 1}{(\alpha^2 + \|\omega\|^2)^{\nu+d/2}} \]
\[
\leq (\nu + d/2)(\alpha_0^2 + \|\omega\|^2)^{\nu+d/2-1} \cdot 2\alpha_1 \cdot |\alpha - \alpha_0| \]
\[
\leq (\nu + d/2) \max(\alpha_0^2, \alpha^2) \left( \max(\alpha_0, \alpha)^2 + \|\omega\|^2 \right)^{\nu+d/2-1} \cdot \frac{1}{\alpha^2 + \|\omega\|^2} \]
\[
\leq (\nu + d/2) \max(\alpha_0^2, \alpha^2) \max(\alpha_0^{2\nu+d-2}, \alpha^{2\nu+d-2}) \frac{1}{\alpha^{2\nu+d-2}(\alpha^2 + \|\omega\|^2)}, \tag{S.8}
\]
where \(\alpha_1\) is a value between \(\alpha_0\) and \(\alpha\).

If \(\nu + d/2 < 1\), then we have that
\[
\left| \frac{f_{\sigma,\alpha}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} - 1 \right| \leq \frac{(\nu + d/2)(\alpha_0^2 + \|\omega\|^2)^{\nu+d/2} - 1}{(\alpha^2 + \|\omega\|^2)^{\nu+d/2}} \]
\[
\leq (\nu + d/2)(\alpha_0^2 + \|\omega\|^2)^{\nu+d/2-1} \cdot 2\alpha_1 \cdot |\alpha - \alpha_0| \]
\[
\leq (\nu + d/2) \max(\alpha_0^2, \alpha^2) \left( \alpha_0^{2\nu+d-2} + \|\omega\|^2 \right)^{1-(\nu+d/2)} \cdot \frac{1}{\alpha^2 + \|\omega\|^2}. \tag{S.9}
\]
In (S.5), if $\alpha \geq \alpha_1 \geq \alpha_0$, then the function \( \left( \frac{\alpha^2 + \|\omega\|^2}{\alpha_1^2 + \|\omega\|^2} \right)^{1-(\nu+d/2)} \) is decreasing in $\|\omega\|^2$, so
\[
\left( \frac{\alpha^2 + \|\omega\|^2}{\alpha_1^2 + \|\omega\|^2} \right)^{1-(\nu+d/2)} \leq \left( \frac{\alpha}{\alpha_1} \right)^{2-(2\nu+d)} = \left( \frac{\alpha_1}{\alpha} \right)^{2\nu-2} \leq \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d-2}.
\]
If $\alpha \leq \alpha_1 \leq \alpha_0$, then the function $\left( \frac{\alpha^2 + \|\omega\|^2}{\alpha_1^2 + \|\omega\|^2} \right)^{1-(\nu+d/2)}$ is increasing in $\|\omega\|^2$, so
\[
\left( \frac{\alpha^2 + \|\omega\|^2}{\alpha_1^2 + \|\omega\|^2} \right)^{1-(\nu+d/2)} \leq 1.
\]
Considering both cases, then from (S.5), we can derive that
\[
\begin{align*}
\left| \frac{f_{\sigma,\alpha}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} - 1 \right| &\leq (2\nu + d) \max (\alpha_0^2, \alpha^2) \left( \frac{\alpha^2 + \|\omega\|^2}{\alpha_1^2 + \|\omega\|^2} \right)^{1-(\nu+d/2)} \cdot \frac{1}{\alpha_1^2 + \|\omega\|^2} \\
&\leq (2\nu + d) \max (\alpha_0^2, \alpha^2) \max \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d-2}, 1 \right\} \\
&\leq \frac{(2\nu + d) \max (\alpha_0^2, \alpha^2) \max \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d-2}, \alpha^2 \right\}}{\alpha_1^{2\nu+d-2}(\alpha^2 + \|\omega\|^2)}.
\end{align*}
\]
For $\nu + d/2 \geq 1$ and (S.10) for $\nu + d/2 < 1$ lead to (S.5).

For (S.6) and (S.7), we use the relation $\lambda_{k,n}(\alpha) = \int_{\mathbb{R}^d} |\psi_k(\omega)|^2 f_{\sigma_0,\alpha_0}(\omega) \cdot f_{\sigma,\alpha}(\omega) d\omega$ for $k = 1, \ldots, n$ and the bounds in (S.4) to obtain that
\[
\begin{align*}
\lambda_{k,n}(\alpha) &\leq \sup_{\omega \in \mathbb{R}^d} \frac{f_{\sigma,\alpha}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} \cdot \int_{\mathbb{R}^d} |\psi_k(\omega)|^2 f_{\sigma_0,\alpha_0}(\omega) d\omega \leq \max \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d}, 1 \right\}, \\
\lambda_{k,n}(\alpha) &\geq \inf_{\omega \in \mathbb{R}^d} \frac{f_{\sigma,\alpha}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} \cdot \int_{\mathbb{R}^d} |\psi_k(\omega)|^2 f_{\sigma_0,\alpha_0}(\omega) d\omega \geq \min \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d}, 1 \right\}.
\end{align*}
\]

For any $a > 0$, define $m_a = \lfloor a + d/2 \rfloor + 1$. For $\omega \in \mathbb{R}^d$, let
\[
\begin{align*}
c_0(x) &= \|x\|^{\frac{\nu+d/2}{2m_a}} d I(\|x\| \leq 1), \\
\xi_0(\omega) &= \int_{\mathbb{R}^d} e^{-ix^T\omega} c_0(x) dx,
\end{align*}
\]
and $\xi_1(\omega) = \xi_0(\omega)^{2m_a}$ for all $\omega \in \mathbb{R}^d$. If $c_1 = c_0 * \ldots * c_0$ is the $2m_a$-fold convolution of the function $c_0$ with itself, then $\xi_1(\omega)$ is the Fourier transform of $c_1(x)$. Then Lemma 6 in Wang and Loh [25] has proved that for $d = 1, 2, 3$, $\xi_0(\omega) \asymp \|\omega\|^{-\frac{\nu+d/2}{2m_a}}$ as $\|\omega\| \to \infty$, which means that $\xi_1(\omega) \asymp \|\omega\|^{-(\nu+d/2)}$. This implies that if $a^2 \alpha^{2\nu} = \theta_0$, then $f_{\sigma,\alpha}(\omega)/\xi_1(\omega) \sim 1$ as $\|\omega\| \to \infty$. In fact, using Lemma 6 in Wang and Loh [25], we can prove the following lower and upper bound for his ratio.

Lemma S.2. For any pair $(\sigma, \alpha) \in \mathbb{R}_+^d$, the following holds for all $\omega \in \mathbb{R}^d$:
\[
\xi_0^2 \sigma^2 \alpha^{2\nu} \min \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d}, 1 \right\} \leq f_{\sigma,\alpha}(\omega) \leq \xi_0^2 \sigma^2 \alpha^{2\nu} \max \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d}, 1 \right\},
\]
where $c_0$ and $\xi_0$ are two positive constants that only depend on $d$ and $\nu$. 

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Proof of Lemma S.2. Lemma 6 in Wang and Loh [75] has proved that for $d = 1, 2, 3$, $\xi_0(\omega) \asymp \|\omega\|^{-\frac{\nu+d/2}{2m\nu}}$ as $\|\omega\| \to \infty$. This implies that there exists two positive absolute constants $c_0$ and $c_\xi$ that only depend on $d$ and $\nu$, such that

$$c_0 \leq (\alpha_0^2 + \|\omega\|^2)^{-\frac{\nu+d/2}{4m\nu}} \xi_0(\omega) \leq c_\xi,$$

for all $\omega \in \mathbb{R}^d$. According to the definition of $\xi_1(\omega)$, this implies that

$$c_0^{2m\nu} \leq (\alpha_0^2 + \|\omega\|^2)^{-\frac{\nu+d/2}{2m\nu}} \xi_0(\omega) \leq c_\xi^{2m\nu},$$

for all $\omega \in \mathbb{R}^d$. Now, from the definition of $f_{\sigma,\alpha}$ in (S.2), we have that

$$f_{\sigma,\alpha}(\omega) = \frac{\sigma^2 \alpha^{2\nu} (\alpha_0^2 + \|\omega\|^2)^{-\nu+d/2}}{\pi^{d/2} (\alpha^2 + \|\omega\|^2)^{\nu+d/2}} \cdot \frac{1}{(\alpha_0^2 + \|\omega\|^2)^{\nu+d/2} \xi_0(\omega)^2}. \quad (S.15)$$

Since

$$\min \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d}, 1 \right\} \leq \left( \frac{\alpha_0^2 + \|\omega\|^2}{\alpha^2 + \|\omega\|^2} \right)^{\nu+d/2} \leq \max \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d}, 1 \right\},$$

we have from (S.15) that

$$f_{\sigma,\alpha}(\omega) \leq \frac{\sigma^2 \alpha^{2\nu} (\alpha_0^2 + \|\omega\|^2)^{-\nu+d/2}}{\pi^{d/2} c_0^{4m\nu}} \cdot \min \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d}, 1 \right\},$$

$$f_{\sigma,\alpha}(\omega) \geq \frac{\sigma^2 \alpha^{2\nu} (\alpha_0^2 + \|\omega\|^2)^{-\nu+d/2}}{\pi^{d/2} c_\xi^{4m\nu}} \cdot \max \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d}, 1 \right\}.$$

Finally, we let $\epsilon_\rho = 1/(\pi^{d/2} c_0^{4m\nu})$ and $\epsilon_\xi = 1/(\pi^{d/2} c_\xi^{4m\nu})$ and the conclusion follows. \hfill $\square$

Now to proceed, we define the function

$$\eta(\omega) = \frac{f_{\sigma,\alpha}(\omega) - f_{\sigma_0,\alpha_0}(\omega)}{\xi_1(\omega)^2}, \quad \forall \omega \in \mathbb{R}^d. \quad (S.16)$$

Note that $\eta$ depends on $(\sigma, \alpha)$, but we suppress the dependence for the ease of notation.

For any given pair $(\sigma, \alpha) \in \mathbb{R}_+^2$, from (S.5) in Lemma S.1 and (S.14) in Lemma S.2 we have that

$$\int_{\mathbb{R}^d} \eta_\rho(\omega)^2 d\omega = \int_{\mathbb{R}^d} \left\{ \frac{f_{\sigma,\alpha}(\omega) - f_{\sigma_0,\alpha_0}(\omega)}{\xi_1(\omega)^2} \right\}^2 d\omega$$

$$= \int_{\mathbb{R}^d} \left\{ \frac{f_{\sigma,\alpha}(\omega) - f_{\sigma_0,\alpha_0}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} \right\}^2 \left( \frac{f_{\sigma_0,\alpha_0}(\omega)}{\xi_1(\omega)^2} \right)^2 d\omega$$

$$\leq \sup_{\omega \in \mathbb{R}^d} \left( \frac{f_{\sigma_0,\alpha_0}(\omega)}{\xi_1(\omega)^2} \right)^2 \cdot \int_{\mathbb{R}^d} \left| \frac{f_{\sigma,\alpha}(\omega)}{f_{\sigma_0,\alpha_0}(\omega)} - 1 \right|^2 d\omega$$

$$\leq \frac{\epsilon_\rho^2 \theta_0^2}{c_0^4} \int_{\mathbb{R}^d} \left\{ \frac{(2\nu+d) \max(\alpha_0^2, \alpha^2) \max(\alpha_0^{2(\nu+d-2)}, \alpha^{2(\nu+d-2)})}{\alpha^{2\nu+d-2}(\alpha^2 + \|\omega\|^2)} \right\}^2 d\omega$$

$$= \frac{\epsilon_\rho^2 \theta_0^2 (2\nu+d)^2 \max(\alpha_0^4, \alpha^4) \max(\alpha_0^{2(2\nu+d-2)}, \alpha^{2(2\nu+d-2)})}{\alpha^{2(2\nu+d-2)}},$$

$$\times \int_0^\infty \frac{r^{d-1}}{(\alpha^2 + r^2)^2} dr < \infty.$$
where the last integral is finite because $\alpha > 0$ and $4 - (d - 1) \geq 2$ for $d = 1, 2, 3$. Therefore, we have shown that $\eta(\omega)$ is a square-integrable function of $w$. From the theory of Fourier transforms of $L_2(\mathbb{R}^d)$, there exists a square-integrable function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$
\int_{\mathbb{R}^d} \{\eta(\omega) - \hat{g}_k(\omega)\}^2 \, d\omega, \text{ as } k \to \infty,
$$

where

$$
\hat{g}_k(\omega) = \int_{\mathbb{R}^d} e^{-i\omega^\top x} g(x) \mathcal{I}(|x|_\infty \leq k) \, dx. \tag{S.17}
$$

Furthermore, for any sequence $\varepsilon_n = o(1)$ as $n \to \infty$ and any positive constant $\alpha > 0$, we define the following functions similar to Equations (35) and (36) in Wang and Loh [75]. Let

$$
\tilde{c}_0(x) = \|x\|^{a + d/2} \mathcal{I}(|x| \leq 1), \quad \forall x \in \mathbb{R}^d,
$$

and $\tilde{c}_1(x) = c_0 \ast \ldots \ast c_0(x)$ be the $2m_n$-fold convolution of $c_0$ with itself. Let $C_q = \int_{\mathbb{R}^d} \tilde{c}_1(x) \, dx$. Define the following functions

$$
\tilde{\xi}_0(\omega) = \int_{\mathbb{R}^d} e^{-i\omega^\top w} \tilde{c}_0(x) \, dx, \quad \forall \omega \in \mathbb{R}^d,
$$

$$
\tilde{\xi}_1(\omega) = \int_{\mathbb{R}^d} e^{-i\omega^\top w} \tilde{c}_1(x) \, dx = \tilde{\xi}_0(\omega)^{2m_n}, \quad \forall \omega \in \mathbb{R}^d,
$$

$$
q_n(x) = \frac{1}{C_q^\varepsilon_n} \tilde{c}_1 \left( \frac{x}{\varepsilon_n} \right), \quad \forall x \in \mathbb{R}^d,
$$

$$
\tilde{q}_n(\omega) = \int_{\mathbb{R}^d} e^{-i\omega^\top x} q_n(x) \, dx = \frac{1}{C_q} \int_{\mathbb{R}^d} e^{-ix^\top \omega^\top x} \tilde{c}_1(x) \, dx = \frac{\tilde{\xi}_1(\varepsilon_n w)}{C_q}, \quad \forall \omega \in \mathbb{R}^d. \tag{S.18}
$$

Then using Lemma 6 of Wang and Loh [75], there exists a constant $C_q$ that only depends on $d, \nu, a$, such that

$$
|\tilde{q}_n(\omega)| \leq \frac{C_q}{(1 + \varepsilon_n \|\omega\|)^{a + d/2}}, \quad \forall \omega \in \mathbb{R}^d. \tag{S.19}
$$

**Lemma S.3.** Let $a > 0$ and $\beta \in (0, 4 - d)$ be fixed constants. For the $g$ function in (S.17) and the $q_n$ function in (S.18), there exists a positive constant $C_{g,q}$ that depends only on $\nu, d, a, \beta$, such that

$$
\left\{ \int_{\mathbb{R}^d} |q_n \ast g(x) - g(x)|^2 \, dx \right\}^{1/2} \leq C_{g,q} \max(\alpha_0^4, \alpha^4) \max\left\{ \alpha_0^{2(4\nu + d - 2)}, \alpha^{2(4\nu + d - 2)} \right\} \varepsilon_n^{\beta/2},
$$

where $q_n \ast g(x) = \int_{\mathbb{R}^d} q_n(y) g(x - y) \, dy$ for any $x \in \mathbb{R}^d$.

**Proof of Lemma S.3.** We have the following derivation:

$$
\left\{ \int_{\mathbb{R}^d} |q_n \ast g(x) - g(x)|^2 \, dx \right\}^{1/2} = \left\{ \int_{\mathbb{R}^d} \int_{\|y\| \leq 2m_n \varepsilon_n} \{g(x - y) - g(x)\} q_n(y) \, dy \right\}^{1/2} \leq \int_{\|y\| \leq 2m_n \varepsilon_n} \left( \int_{\mathbb{R}^d} |g(x - y) - g(x)|^2 \, dx \right)^{1/2} q_n(y) \, dy.
$$
\[
(i) \quad \int_{\|y\| \leq 2m_\varepsilon n} \left[ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| (e^{-ix^T y} - 1)\eta(x) \right|^2 \, dx \right]^{1/2} q_n(y) \, dy \\
(ii) \quad \int_{\|y\| \leq 2m_\varepsilon n} \left[ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| (e^{-ix^T y} - 1) \cdot \frac{f_{\sigma,\alpha}(\omega) - f_{\sigma_0,0}(\omega)}{f_{\sigma_0,0}(\omega)} \cdot \frac{f_{\sigma_0,0}(\omega)}{\xi_1(\omega)^2} \right|^2 \, d\omega \right]^{1/2} q_n(y) \, dy \\
(iii) \quad \leq \frac{1}{(2\pi)^{d/2}} \sup_{\omega \in \mathbb{R}^d} \frac{f_{\sigma_0,0}(\omega)}{\xi_1(\omega)^2}.
\]

\[
(iv) \quad \int_{\|y\| \leq 2m_\varepsilon n} \left[ \int_{\mathbb{R}^d} \left| (e^{-ix^T y} - 1) \cdot \left\{ \frac{f_{\sigma,\alpha}(\omega) - f_{\sigma_0,0}(\omega)}{f_{\sigma_0,0}(\omega)} \right\} \right|^2 \, d\omega \right]^{1/2} \|y\|^\beta q_n(y) \, dy \\
\leq \frac{2^{1-\beta/2}}{(2\pi)^{d/2}} \sup_{\omega \in \mathbb{R}^d} \frac{f_{\sigma_0,0}(\omega)}{\xi_1(\omega)^2} \\
\cdot \left[ \int_{\mathbb{R}^d} \left\{ (2\nu + d) \max(\alpha_0^2, \alpha^2) \max \left( \alpha_0^{2\nu+d-2}, \alpha^{2\nu+d-2} \right) \right\} \frac{\|y\|^{\beta/2} \, d\omega}{(\alpha^{2\nu+d-2} + \|y\|^{\beta/2})^{\beta/2}} \right]^{1/2} q_n(y) \, dy \\
\leq \frac{2^{1-\beta/2} \theta_0}{(2\pi)^{d/2}} \cdot \frac{2\sigma^2 \alpha^2}{\alpha^{2\nu+d}} \max \left\{ \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d}, 1 \right\} \\
\cdot \left( \int_{\mathbb{R}^d} \left( 2\nu + d \right) \max(\alpha_0^2, \alpha^2) \max \left( \alpha_0^{2\nu+d-2}, \alpha^{2\nu+d-2} \right) \right)^{1/2} \|y\|^\beta q_n(y) \, dy \\
\leq \left[ \int_{\mathbb{R}^d} \left( 2\nu + d \right) \max(\alpha_0^2, \alpha^2) \max \left( \alpha_0^{2\nu+d-2}, \alpha^{2\nu+d-2} \right) \right]^{1/2} \|y\|^\beta q_n(y) \, dy \\
\leq \left[ \int_0^\infty \frac{r^{\beta+d-1}}{(1 + r^2)^{\nu+d/2}} \, dr \right]^{1/2} \cdot (2m_\varepsilon n)^{\beta/2} \\
\cdot \frac{2\sigma \theta_0 (2\nu + d) m_\varepsilon \max(\alpha_0^4, \alpha^4) \max \left( \alpha_0^{2(2\nu+d-2)}, \alpha^{2(2\nu+d-2)} \right)}{(2\pi)^{d/2} \alpha^{4\nu+3d/2-\beta/2}} \cdot \varepsilon_n^{\beta/2}. \quad (S.20)
\]

In the derivations above: (i) follows from the Minkowski’s integral inequality. (ii) follows from the Plancherel’s theorem. (iii) is based on the definition of \(\eta(\omega)\) in (S.10). (iv) uses the fact that \(|e^{\alpha a} - 1|^2 = 4\sin^2 (a/2) \leq 2^{2-\beta} |a|^{\beta}\) for any \(a \in \mathbb{R}\) and all \(\beta \in (0, 2]\). (v) follows from (S.5) in Lemma S.1. (vi) follows from (S.14) in Lemma S.2. Since \(\beta < 4-d\), the integral in the last display exists and hence the conclusion follows.

\[\square\]

**Lemma S.4.** Let \((\sigma, \alpha) \in \mathbb{R}^2_+\) satisfy \(\sigma^2 \alpha^{2\nu} = \theta_0 = \sigma_0^2 \alpha_0^2\). Let \(a > 0\) and \(\beta \in (0, 4-d)\) be fixed constants. For the \(\lambda_{k,n}(\alpha)\) in (S.3), there exist positive constants \(C_1^+, C_1^-, C_2^+, C_2^-\) that depend only on \(\nu, d, T, \alpha_0, a, \beta\), such that

\[
\sum_{k=1}^n |\lambda_{k,n}(\alpha) - 1|
\]
\[
C_1 \max(\alpha_0^6, \alpha^6) \max \left\{ \frac{3(2\nu+d-2)}{\alpha^{4\nu+3d/2-\beta/2}}, \frac{\alpha^{3(2\nu+d-2)}}{\sqrt{n_c \beta/2}} \right\} + C_1 \max(\alpha_0^6, \alpha^6) \max \left\{ \frac{3(2\nu+d-2)}{\alpha^{3(2\nu+d)}} \right\}.
\] (S.21)

**Proof of Lemma [75]** For any \( x, y \in S \), let \( b(x, y) = E_{\sigma, a}{X(x)X(y)} - E_{\sigma, a}{X(x)X(y)} \). Then using the definition of \( c_0(x) \) in (S.12) and \( c_1(x) \) with the support of \( c_1 \) in \([-2m_\nu, 2m_\nu]^d \), Wang and Loh [75] Equation (21) has shown that for \( s, t \in S \),

\[
b(x, y) = (2\pi)^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s-t)c_1(x-s)c_1(y-t) \, ds \, dt
\]

\[
= (2\pi)^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q_n * g(s-t)c_1(x-s)c_1(y-t) \, ds \, dt
\]

\[
+ (2\pi)^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_n^*(s, t)c_1(x-s)c_1(y-t) \, ds \, dt,
\] (S.22)

where \( h_n^*(s, t) = [g(s-t) - q_n * g(s-t)] I(|s + t|_{\infty} \leq 4m_\nu + 2T) \), for any \( s, t \in \mathbb{R}^d \). Let \( \eta_n^* : \mathbb{R}^d \to \mathbb{C} \) be the Fourier transform of \( g - q_n * g \). This implies that

\[
\int_{\mathbb{R}^d} |\eta_n^*(\omega) - \hat{g}_{n,k}(\omega)|^2 \, d\omega \to 0, \text{ as } k \to \infty,
\]

where \( \hat{g}_{n,k}(\omega) = \int_{\mathbb{R}^d} e^{-i\omega^T x} \{g(x) - q_n * g(x)\} I(|x|_{\infty} \leq k) \, dx \) for all \( \omega \in \mathbb{R}^d \). As in Equation (38) of Wang and Loh [75],

\[
(2\pi)^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_n^*(s, t)c_1(x-s)c_1(y-t) \, ds \, dt
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(\omega^T x - v^T y)} \eta_n^*(\frac{w + v}{2}) \, \vartheta \left( \frac{w - v}{2} \right) \xi_1(\omega)\xi_1(v) \, d\omega \, dv,
\] (S.23)

where \( \vartheta(\omega) \) is defined in the same way as Equation (23) of Wang and Loh [75]:

\[
\vartheta(\omega) = \frac{1}{2d} \int_{\mathbb{R}^d} e^{-u^T w} I(|t|_{\infty} \leq 4m_\nu + 2T) \, dt,
\] (S.24)

Lemma 3 of Wang and Loh [75] has proved that \( \int_{\mathbb{R}^d} \vartheta(\omega)^2 \, d\omega < \infty \) and its value only depends on \( d, \nu, T \).

Now, for the first term in (S.22), we define the following function \( h_{n,K}^*(s, t) \):

\[
h_{n,K}^*(s, t) = \int_{\mathbb{R}^d} q_n(s-u)g(u-t) I(|u|_{\infty} \leq 2m_\nu + 2m_\omega + T) \, du,
\] (S.25)

Then \( h_{n,K}^* : \mathbb{R}^{2d} \to \mathbb{C} \) is square-integrable and following the same derivation as on page 257 of Wang and Loh [75], we have that

\[
(2\pi)^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q_n * g(s-t)c_1(x-s)c_1(y-t) \, ds \, dt
\]

\[
= (2\pi)^d \int_{\mathbb{R}^{2d}} h_{n,K}^*(s, t)c_1(x-s)c_1(y-t) \, ds \, dt
\]

\[
= (2\pi)^d \int_{\mathbb{R}^{2d}} e^{i(\omega^T x - v^T y)} \xi_1(\omega)\xi_1(v) \left\{ \int_{|u|_{\infty} \leq 2m_\nu + 2m_\omega + T} e^{-t(\omega^T u - v^T u)} \right\} \eta_n(\omega) \vartheta(\omega) \, d\omega \, dv.
\] (S.26)
where in (S.3), we have that for \( k \)
Hence, for any pair \( (x, y) \)
Therefore, we can plug in (S.26) and (S.23) into (S.22) and obtain that
Note that by the definition of covariance function,
We follow the derivations on page 258-259 of Wang and Loh [75]. By the Bessel’s inequality,
Hence, for any pair \( (\sigma, \alpha) \) that satisfies \( \sigma^2\alpha^{2\nu} = \theta_0 = \sigma_0^2\alpha_0^{2\nu} \), for the \( \{\psi_k : k = 1, \ldots, n\} \) functions in (S.3), we have that for \( k = 1, \ldots, n \),
where
We follow the derivations on page 258-259 of Wang and Loh [75]. By the Bessel’s inequality, we have that
\[
\sum_{k=1}^{n} \left| \zeta_{k,n}^+ \right|^2 \leq \left( \frac{1}{2\pi^d} \right)^2 \int_{\mathbb{R}^d} \mathcal{L}_n \left( \frac{w + v}{2} \right) \vartheta \left( \frac{w - v}{2} \right) \xi_1(\omega) \xi_1(v) d\omega dv
\]
\[
\leq \left( \frac{1}{2\pi^d} \right)^2 \sup_{\omega \in \mathbb{R}^d} \xi_1(\omega)^2 \int_{\mathbb{R}^d} |\mathcal{L}_n(\omega)|^2 d\omega \int_{\mathbb{R}^d} |\vartheta(v)|^2 dv
\]
\[
\leq \left( \frac{1}{2\pi^d} \right)^2 \left( \frac{\max \left\{ (\alpha/\alpha_0)\nu + d, 1 \right\}}{\xi_0} \right)^2 \cdot \int_{\mathbb{R}^d} |\vartheta(v)|^2 dv
\]
\[
\leq \left( \frac{1}{2\pi^d} \right)^2 \left[ \max (\alpha_0^4, \alpha_1^4) \max \left\{ \alpha_0^{2(\nu-2d+2)}, \alpha_0^{2(2\nu-2d-2)} \right\} \right]^2 \cdot \varepsilon_n
\]
\[
\leq (C_1)^2 \max (\alpha_0^{12}, \alpha_1^{12}) \max \left\{ \alpha_0^{6(2\nu-2d-2)}, \alpha_0^{6(2\nu+2d-2)} \right\} \varepsilon_n, \quad (S.31)
\]
where we have applied Lemma S.2 and Lemma S.3 in the step (i), and $C_1^\dagger$ is a positive constant that depends only on $\nu, d, T, \alpha_0, \alpha, \beta$.

For $\zeta_{k,n}^\dagger$, we apply the Bessel’s inequality to obtain that

\[
\sum_{k=1}^{n} |\zeta_{k,n}^\dagger| \leq \frac{1}{(2\pi)^d} \sum_{k=1}^{n} \int_{|\omega| \leq 2m_\nu + 2m_\alpha + T} \left| \int_{\mathbb{R}^d} e^{-i\omega^\top u} \psi_k(\omega) \xi_1(\omega) \hat{q}_n(\omega) d\omega \right| \times \left| \int_{\mathbb{R}^d} e^{i\nu^\top u} \psi_k(\nu) \xi_1(\nu) \eta(\nu) dv \right| du
\]

\[
\leq \frac{1}{(2\pi)^d} \int_{|\omega| \leq 2m_\nu + 2m_\alpha + T} \left\{ \left( \int_{\mathbb{R}^d} \frac{\xi_1(\omega)}{f_{\sigma,\alpha}(\omega)} \left| \hat{q}_n(\omega) \right|^2 d\omega \right)^2 + \left( \int_{\mathbb{R}^d} \frac{\xi_1(\nu)}{f_{\sigma,\alpha}(\nu)} \left| \eta(\nu) \right| d\nu \right)^2 \right\} du
\]

\[
\leq \frac{1}{2(2\pi)^d} \cdot (4m_\nu + 4m_\alpha + 2T)^d \cdot \left\{ \max \left\{ (\alpha/\alpha_0)^{2\nu+d}, 1 \right\} \right\}
\]

\[
\times \int_{\mathbb{R}^d} \left( 1 + \varepsilon_n \|\omega\| \right)^{2d+2} d\omega
\]

\[
+ \frac{1}{2(2\pi)^d} \cdot (4m_\nu + 4m_\alpha + 2T)^d \cdot \tau_\xi \theta_0 \max \left\{ \frac{\alpha_0}{\alpha} \right\} \max \left\{ (\alpha/\alpha_0)^{2\nu+d}, 1 \right\}
\]

\[
\times \int_{\mathbb{R}^d} \left\{ \frac{(2\nu + d) \max(\alpha_0^6, \alpha^2) \max(\alpha_0^{2\nu+d-2}, \alpha^{2\nu+d-2})}{\alpha^{2\nu+d-2}} \right\}^2 \frac{1}{(\alpha^2 + \|\nu\|^2)^2} dv
\]

\[
\leq \frac{(4m_\nu + 4m_\alpha + 2T)^d \cdot C_2^\dagger \max(\alpha_0, \alpha)^{2\nu+d}}{2(2\pi)^d} \tau_\xi \theta_0 \left\{ \int_0^\infty \frac{r^{d-1}}{(1 + r)^{2d+2}} dr \right\}
\]

\[
+ \frac{(4m_\nu + 4m_\alpha + 2T)d\tau_\xi \theta_0}{2(2\pi)^d} \cdot (2\nu + d)^2 \max(\alpha_0^6, \alpha^6) \max \left\{ \alpha_0^{3(2\nu+d-2)}, \alpha^{3(2\nu+d-2)} \right\}
\]

\[
\times \alpha^{d-4} \left\{ \int_0^\infty \frac{r^{d-1}}{(1 + r)^{3d+2}} dr \right\}
\]

\[
\leq C_1^\dagger \max(\alpha_0, \alpha)^{2\nu+d} \max(\alpha_0^6, \alpha^6) \max \left\{ \alpha_0^{3(2\nu+d-2)}, \alpha^{3(2\nu+d-2)} \right\},
\]

where we have used Lemma S.1 and Lemma S.2 in the step (i), and $C_1^\dagger, C_2^\dagger$ are positive constants that depend only on $\nu, d, T, \alpha_0, \alpha, \beta$.

Finally, we combine (S.31) and (S.32) to conclude that for any pair $(\sigma, \alpha)$ that satisfies
\[\sigma^2 \alpha^{2\nu} = \theta_0 = \sigma_0^2 \alpha_0^{2\nu},\]

\[\frac{\lambda_{k,n}(\alpha) - 1}{n} \leq \lambda_{k,n}(\alpha) - 1 \leq \left( \frac{\lambda_{k,n}(\alpha) - 1}{n} \right) \leq \left( n \sum_{k=1}^{n} |\lambda_{k,n}(\alpha)|^2 \right)^{1/2} + \sum_{k=1}^{n} |\lambda_{k,n}(\alpha)| \leq C_1 \max(\alpha_0^6, \alpha^6) \max \left\{ \alpha_0^{3(2\nu+d-2)}, \alpha^{3(2\nu+d-2)} \right\} \text{ for any vector } w\in \mathbb{R}^n \]

\[\leq C_1 \max(\alpha_0^6, \alpha^6) \frac{\alpha^{4\nu+3d/2-\beta/2}}{\alpha^{2(3\nu+d)}} + C_2 \max(\alpha_0^6, \alpha^6) \max \left\{ \alpha_0^{3(2\nu+d-2)}, \alpha^{3(2\nu+d-2)} \right\}.\]

S2 Proof of Lemma \[S.7\]

To prove Lemma \[S.7\] we cite an important result from Kaufman and Shaby [38].

**Lemma S.5.** (Kaufman and Shaby [38] Lemma 1) Let \( \tilde{\theta}_\alpha \) be defined as in \[S.1\]. Then for any \( 0 < \alpha_1 < \alpha_2 < \infty, \tilde{\theta}_{\alpha_1} \leq \tilde{\theta}_{\alpha_2} \) for any vector \( X_n \).

**Lemma S.6.** (Laurent and Massart [43] Lemma 1) Let \( Y_1, \ldots, Y_n \) be \( \mathcal{N}(0,1) \) random variables. Let \( \{w_i : i = 1, \ldots, n\} \) be nonnegative constants. Let \( \|w\|_\infty = \max_{1 \leq i \leq n} w_i, \|w\|_1 = \sum_{i=1}^{n} w_i, \text{ and } \|w\|^2 = \sum_{i=1}^{n} w_i^2 \). Then for any positive \( z > 0 \),

\[\Pr \left\{ \sum_{i=1}^{n} w_i Y_i^2 \geq \|w\|_1 + 2\|w\|_\infty z \right\} \leq e^{-z},\]

\[\Pr \left\{ \sum_{i=1}^{n} w_i Y_i^2 \leq \|w\|_1 - 2\|w\|_\infty z \right\} \leq e^{-z}.\]

We strengthen the conclusion of Lemma \[S.1\] to the following lemma. The proof of Lemma \[S.7\] follows from its proof.

**Lemma S.7.** There exist a large integer \( N_1 \) and a positive constant \( \tau \in (0,1/2) \) that only depend on \( \nu, d, T, \alpha_0 \), such that for all \( n > N_1 \),

\[\Pr \left( \sup_{\alpha \in [\alpha_n, \alpha_\infty]} \sqrt{n} \left| \tilde{\theta}_\alpha - \tilde{\theta}_{\alpha_0} \right| \leq \theta_0 n^{-\tau} \right) \geq 1 - \exp(-\log^2 n), \quad (S.33)\]

\[\Pr \left( \sup_{\alpha \in [\alpha_0, \pi_n]} \sqrt{n} \left| \tilde{\theta}_\alpha - \tilde{\theta}_{\alpha_0} \right| \leq \theta_0 n^{-\tau} \right) \geq 1 - \exp(-\log^2 n), \quad (S.34)\]

\[\Pr \left( \sup_{\alpha \in [\pi_n, \pi_\infty]} \sqrt{n} \left| \tilde{\theta}_\alpha - \tilde{\theta}_{\alpha_0} \right| \leq \theta_0 n^{-\tau} \right) \geq 1 - 2 \exp(-\log^2 n), \quad (S.35)\]

where \( \alpha_n \) and \( \pi_n \) are defined in \[S.9\].

**Proof of Lemma S.7.** First of all, using the simultaneous diagonalization in \[S.1\] and \[S.3\], for any pair \((\sigma, \alpha) \in \mathbb{R}_+^2\) that satisfies \( \sigma^2 \alpha^{2\nu} = \theta_0 = \sigma_0^2 \alpha_0^{2\nu} \), the difference \( \tilde{\theta}_\alpha - \tilde{\theta}_{\alpha_0} \) can be rewritten as

\[\sqrt{n} (\tilde{\theta}_\alpha - \tilde{\theta}_{\alpha_0}) = \frac{1}{\sqrt{n}} \left\{ \sigma^{-2} \left( \sigma^2 \alpha^{2\nu} \right) X_n \begin{bmatrix} R_n & R_n^{-1} \end{bmatrix} X_n - \sigma_0^{-2} \left( \sigma_0^2 \alpha_0^{2\nu} \right) X_n \begin{bmatrix} R_n & R_n^{-1} \end{bmatrix} X_n \right\} \]

\[= \frac{\theta_0}{\sqrt{n}} \left\{ \sigma^{-2} X_n \begin{bmatrix} R_n & R_n^{-1} \end{bmatrix} X_n - \sigma_0^{-2} X_n \begin{bmatrix} R_n & R_n^{-1} \end{bmatrix} X_n \right\} \]
\[ \frac{\lambda_i, n(\alpha)^{-1}}{\sqrt{n}} \sum_{i=1}^{n} \{ \lambda_i, n(\alpha)^{-1} - 1 \} Y_{i, n}(\alpha)^2, \]  

(S.36)

where the random variables \( \{ Y_{i, n}(\alpha) : i = 1, \ldots, n \} \) satisfy \( Y_\alpha = (Y_{1, n}(\alpha), \ldots, Y_{n, n}(\alpha))^\top = U_\alpha^\top X_n \sim \mathcal{N}(0, I_n) \), with \( U_\alpha \) given in (S.1).

Now we apply Lemma S.6 with \( w_i(\alpha) = |\lambda_i, n(\alpha)^{-1} - 1| / \sqrt{n} \), \( i = 1, \ldots, n \). Let \( w(\alpha) = (w_1(\alpha), \ldots, w_n(\alpha))^\top \). We set \( z = \log^2 n \) in Lemma S.6 such that \( e^{-z} \) is summable over \( n \). From Lemma S.1 and Lemma S.4, we can obtain that

\[ \|w(\alpha)\|_1 = \sum_{i=1}^{n} w_i(\alpha) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |\lambda_i, n(\alpha)^{-1} - 1| \leq \frac{1}{\sqrt{n}} \min_{1 \leq i \leq n} |\lambda_i, n(\alpha)^{-1} - 1| \leq \frac{\{ \max(\alpha_0, \alpha) \}^{2\nu + d}}{\sqrt{n} \alpha_0^{2\nu + d}} \times \left[ C_1^4 \frac{\max(\alpha_0^6, \alpha^6) \max \left\{ \alpha_0^{3(2\nu + d - 2)}, \alpha^{3(2\nu + d - 2)} \right\}}{\alpha^{4\nu + 3d/2 - \beta/2}} \right. \]

\[ + C_2^4 \frac{\max(\alpha_0^6, \alpha^6) \max \left\{ \alpha_0^{3(2\nu + d - 2)}, \alpha^{3(2\nu + d - 2)} \right\}}{\alpha^{2(3\nu + d)}} \right] \],

(S.37)

\[ \|w(\alpha)\|_2^2 = \sum_{i=1}^{n} w_i^2(\alpha) = \frac{1}{n} \sum_{i=1}^{n} |\lambda_i, n(\alpha)^{-1} - 1| \leq \frac{1}{n \{ \min_{1 \leq i \leq n} |\lambda_i, n(\alpha)\}^2} \sum_{i=1}^{n} |\lambda_i, n(\alpha) - 1|^2 \leq \frac{1}{n \{ \min_{1 \leq i \leq n} |\lambda_i, n(\alpha)\}^2} \left( \sum_{i=1}^{n} |\lambda_i, n(\alpha) - 1| \right)^2 \leq \frac{\{ \max(\alpha_0, \alpha) \}^{2(2\nu + d)}}{n \alpha_0^{2(2\nu + d)}} \times \left[ C_1^4 \frac{\max(\alpha_0^6, \alpha^6) \max \left\{ \alpha_0^{3(2\nu + d - 2)}, \alpha^{3(2\nu + d - 2)} \right\}}{\alpha^{4\nu + 3d/2 - \beta/2}} \right. \]

\[ + C_2^4 \frac{\max(\alpha_0^6, \alpha^6) \max \left\{ \alpha_0^{3(2\nu + d - 2)}, \alpha^{3(2\nu + d - 2)} \right\}}{\alpha^{2(3\nu + d)}} \left. \right] \right)^2 \],

(S.38)

We can see the upper bound in (S.38) is exactly the square of the upper bound in (S.37).

\[ \|w(\alpha)\|_\infty = \max_{1 \leq i \leq n} w_i = \frac{1}{\sqrt{n}} \max_{1 \leq i \leq n} |\lambda_i, n(\alpha)^{-1} - 1| \leq \frac{1}{\sqrt{n} \min_{1 \leq i \leq n} |\lambda_i, n(\alpha) - 1|} \leq \frac{\max_{1 \leq i \leq n} |\lambda_i, n(\alpha) + 1|}{\sqrt{n} \min_{1 \leq i \leq n} |\lambda_i, n(\alpha)|} \leq \frac{\max \left\{ \{ \alpha(0) \}^{2\nu + d}, 1 \right\} + 1}{\sqrt{n} \min \left\{ \{ \alpha(0) \}^{2\nu + d}, 1 \right\}} \leq \frac{\{ \max(\alpha_0, \alpha) \}^{2\nu + d} + \alpha^{2\nu + d} \{ \max(\alpha_0, \alpha) \}^{2\nu + d}}{\sqrt{n} \alpha_0^{2\nu + d} \alpha^{2\nu + d}} \leq \frac{2 \{ \max(\alpha_0, \alpha) \}^{2(2\nu + d)}}{\sqrt{n} \alpha_0^{2\nu + d} \alpha^{2\nu + d}}.

(S.39)
In the following, we choose \( \epsilon_n = n^{-1/(4a + 2d + \beta)} \), such that \( \sqrt{n}\epsilon_n^{3/2} = 1/\epsilon_n^{2a + d} = n(2a + d)/(4a + 2d + \beta) \). We also fix \( a = 0.01 \) and \( \beta = 3.9 - d \), which satisfies \( \beta \in (0, 4 - d) \). Now we analyze the upper bounds for this \( \epsilon_n \) and also the choice of \( z = \log^2 n \), \( \alpha_n \) and \( \pi_n \) as defined in (9). We consider two situations according to the value of \( \alpha \), each of which has two further sub-cases according to the sign of \( 2\nu + d - 2 \). We show that for the choice of \( \alpha_n \) and \( \pi_n \) given in (9), \( \|w(\alpha)\|_1 + 2\|w(\alpha)\|\sqrt{z} + 2\|w(\alpha)\|_{\infty}z = o(1) \) for any \( \alpha \in [\alpha_n, \pi_n] \) as \( n \to \infty \), and then we can apply Lemmas S.3 and S.4 to derive the desired error bounds.

(1) When \( \alpha \in [\alpha_0, \pi_n] \) and possibly \( \alpha \to +\infty \) as \( n \to \infty \):

In this case, in the upper bounds of (S.37) and (S.38), since \( \alpha \geq \alpha_0 \), we have that \( \max\{\alpha_0^{3(2\nu + d - 2)}, \alpha^{3(2\nu + d - 2)}\} \leq \alpha^{3(2\nu + d - 2)} \) if \( 2\nu + d - 2 \geq 0 \), and that \( \max\{\alpha_0^{3(2\nu + d - 2)}, \alpha^{3(2\nu + d - 2)}\} \leq 1 \) if \( -1 < 2\nu + d - 2 < 0 \). We discuss the two sub-cases respectively:

(1)-(i) When \( 2\nu + d - 2 \geq 0 \), we have \( \max\{\alpha_0^{3(2\nu + d - 2)}, \alpha^{3(2\nu + d - 2)}\} \leq \alpha^{3(2\nu + d - 2)} \). Using (S.37), (S.38), and (S.39), we can see that (neglecting all multiplicative constants by using the order relation \( \leq \)):

\[
\begin{align*}
\|w(\alpha)\|_1 + 2\|w(\alpha)\|\sqrt{z} + 2\|w(\alpha)\|_{\infty}z & \leq \frac{\alpha^{2\nu + d} \log n}{\sqrt{n}} \left( \alpha^{2\nu + 3d/2 + \beta/2} \sqrt{n}\epsilon_n^{3/2} + \frac{\alpha^{2\nu + d}}{\epsilon_n^{2a + d}} + \alpha^{2\nu + d} \log^2 n \right) \\
& \leq \frac{\alpha_n^{2\nu - 2d} \log n}{\sqrt{n}} \cdot \frac{\pi_n^{2\nu + d/2 + \beta/2} n^{(2a + d)/(4a + 2d + \beta)}}{\epsilon_n^{2a + d}} + \frac{\pi_n^{2\nu + d} \log^2 n}{\sqrt{n}} \\
& = \frac{\alpha_n^{2\nu - 2d/2 + \beta/2} \log n}{n^{\beta/(8a + 4d + 2\beta)}} + \frac{\alpha_n^{2\nu + d} \log^2 n}{\sqrt{n}}.
\end{align*}
\]

(S.40)

In order to make the last upper bound \( o(1) \), given that \( \pi_n > 1 \), we further need

\[
\pi_n < n^{(\frac{\beta}{8a + 4d + 2\beta})(\log n)^{-2}} \quad \text{or} \quad \pi_n < n^{\frac{1}{2(2\nu + d)}} \quad \text{for all} \quad \alpha \in [\alpha_0, \pi_n]
\]

(S.41)

With the choice \( \pi_n = n^\pi \) as given in (9) and \( a = 0.01 \), \( \beta = 3.9 - d \), we have that

\[
\pi \leq \frac{1}{2} \cdot \frac{3.9 - d}{(d + 3.94)(8\nu + 4d + 3.9)} < \frac{1}{2(2\nu + d)} < \frac{1}{2(2\nu + d)}.
\]

(S.42)

Therefore, (S.41) is satisfied. We have that uniformly for all \( \alpha \in [\alpha_0, \pi_n] \), \( \|w(\alpha)\|_1 + 2\|w(\alpha)\|\sqrt{z} + 2\|w(\alpha)\|_{\infty}z = o(1) \).

(1)-(ii) When \( -1 < 2\nu + d - 2 < 0 \), we have \( \max\{\alpha_0^{3(2\nu + d - 2)}, \alpha^{3(2\nu + d - 2)}\} \leq 1 \). Note that this special case can only happen when \( d = 1 \) and \( \nu \in (0, 1/2) \). Using (S.37), (S.38), and (S.39), we can see that (neglecting all multiplicative constants by using the order relation \( \leq \)):

\[
\begin{align*}
\|w(\alpha)\|_1 + 2\|w(\alpha)\|\sqrt{z} + 2\|w(\alpha)\|_{\infty}z & \leq \frac{\alpha^{2\nu + d} \log n}{\sqrt{n}} \left( \alpha^{6 - 4\nu - 3d + \beta/2} n^{(2a + d)/(4a + 2d + \beta)} + \alpha^{2\nu + d} n^{(2a + d)/(4a + 2d + \beta)} \right) \\
& + \alpha^{6 - 6\nu - 2d} \frac{\alpha^{2\nu + d} \log^2 n}{\sqrt{n}}.
\end{align*}
\]

(S.43)
Therefore, (S.43) further implies that

\[
\|w(\alpha)\|_1 + 2\|w(\alpha)\|\sqrt{2} + 2\|w(\alpha)\|_{\infty}\ 
\leq \frac{2d}{n} \log n \left\{ \frac{\|e^{-4\nu-3d+2+\beta/2}n^{(2a+d)/(4a+2d+\beta)} + \|e^{-6\nu-2d}\|}{\sqrt{n}} \right\} + \frac{2d}{n} \log^2 n \sqrt{n}. \tag{S.44}
\]

In order to make the last upper bound \(o(1)\), given that \(\pi_n > 1\) and \(d = 1\), we further need

\[
\lambda_n < n^{(4a+2+\beta)(11-4\nu+\beta)} (\log n)^{-1/(11-4\nu+\beta)},
\]

\[
\lambda_n < n^{(4a+2+\beta)} (\log n)^{-1/4\nu},
\]

\[
\lambda_n < n^{1/(2\nu+1)} (\log n)^{-2/2\nu+1}. \tag{S.45}
\]

Since \(d = 1\) and \(\nu \in (0, 1/2)\) in this case, the choice of \(\pi\) in (3), and the choice \(a = 0.01, \beta = 3.9 - d = 2.9\) imply that

\[
\pi \leq 0.02 < \frac{1}{2} \cdot \frac{3.9 - 1}{(5.94 - 1)(14.9 - 1)} = \frac{1}{2} \cdot \frac{\beta}{(4a + 2 + \beta)(11 + \beta)} < \frac{1}{2} \cdot \frac{\beta}{(4a + 2 + \beta)(11 - 4\nu + \beta)},
\]

\[
\pi \leq 0.02 < \frac{1}{10} < \frac{1}{2(5 - 4\nu)},
\]

\[
\pi \leq \frac{1}{2} \cdot \frac{1}{2(2\nu + 1)} < \frac{1}{2(2\nu + 1)}.
\]

Therefore, \(\pi_n = n\pi\) satisfies the requirement in (S.45). We have that uniformly for all \(\alpha \in [\alpha_0, \alpha_n], \|w(\alpha)\|_1 + 2\|w(\alpha)\|\sqrt{2} + 2\|w(\alpha)\|_{\infty}\ 
\leq o(1)\).

(2) When \(\alpha \in [\alpha_n, \alpha_0]\) and possibly \(\alpha \to 0^+\) as \(n \to \infty\):

In this case, in the upper bounds of (S.37) and (S.38), since \(\alpha \leq \alpha_0\), we have that

\[
\max\{\alpha_0^{3(2\nu+d-2)}, \alpha_0^{3(2\nu+d-2)}\} \leq 1 \text{ if } 2\nu + d - 2 \geq 0, \text{ and that }
\]

\[
\max\{\alpha_0^{3(2\nu+d-2)}, \alpha_0^{3(2\nu+d-2)}\} \leq \alpha_0^{3(2\nu+d-2)} \text{ if } -1 < 2\nu + d - 2 < 0. \text{ We discuss the two sub-cases respectively:}
\]

(2)-i) When \(2\nu + d - 2 \geq 0\), we have \(\max\{\alpha_0^{3(2\nu+d-2)}, \alpha_0^{3(2\nu+d-2)}\} \leq 1\) and \(\max(\alpha_0, \alpha) \leq 1\),

Using (S.37), (S.38), and (S.39), we can see that in this case (neglecting all multiplicative constants by using the order relation \(\leq\)):

\[
\|w(\alpha)\|_1 + 2\|w(\alpha)\|\sqrt{2} + 2\|w(\alpha)\|_{\infty}\ 
\leq \log n \left( \frac{\sqrt{n}e^{\beta/4\nu+3d/2-\beta/2}}{\alpha_n^{4\nu+3d/2-\beta/2}} + \frac{1}{\alpha_n^{2a+d}} + \frac{1}{\alpha_0^{2(3\nu+d)}} \right) + \frac{\log^2 n}{\sqrt{n}\alpha_0^{2\nu+d}}.
\]

\[
\leq \frac{\log n}{\sqrt{n}\alpha_n^{4\nu+3d/2-\beta/2}} + \frac{\log n}{\sqrt{n}\alpha_0^{2(3\nu+d)}} + \frac{\log^2 n}{\sqrt{n}\alpha_0^{2\nu+d}}. \tag{S.46}
\]
In order to make the last upper bound $o(1)$, given that $\lambda_n \prec 1$ and $d = 1$, we further need that
\[
\begin{align*}
\Omega_n &\succ n^{-\frac{\beta(4a+2d+\beta)(8\nu+3d-\beta)}{8\nu+3d-\beta}} (\log n)^{\frac{2}{8\nu+3d-\beta}}, \\
\chi_n &\succ n^{-\frac{1}{4(3\nu-d)}} (\log n)^{\frac{1}{2(3\nu-d)}}, \\
\omega_n &\succ n^{-\frac{1}{\pi(2\nu+d)}} (\log n)^{\frac{2}{2\nu+d}}.
\end{align*}
\] (S.47)

With the choice $\alpha_n = n^\kappa$ as given in (9) and $a = 0.01$, $\beta = 3.9 - d$, we have that
\[
\begin{align*}
\kappa &\leq \frac{1}{2} \frac{3.9 - d}{(d+3.94)(8\nu+4d-3.9)} \\
&= \frac{1}{2} \frac{\beta}{(4a+2d+\beta)(8\nu+3d-\beta)} < \frac{\beta}{(4a+2d+\beta)(8\nu+3d-\beta)}, \\
\kappa &\leq \frac{1}{2} \frac{1}{4(3\nu+d)} < \frac{1}{4(3\nu+d)}, \\
\kappa &\leq \frac{1}{2} \frac{1}{4(3\nu+d)} < \frac{1}{2(2\nu+d)}.
\end{align*}
\]
Therefore, $\Omega_n = n^\kappa$ satisfies the requirement in (S.47). We have that that uniformly for all $\alpha \in [\alpha_n, \alpha_0]$, $\|w(\alpha)\|_1 + 2\|w(\alpha)\|_{\sqrt{z}} + 2\|w(\alpha)\|_{\infty z} = o(1)$.

(\textbf{2)-(ii}) When $-1 < 2\nu + d - 2 < 0$, we have $\max\{\alpha_0^{3(2\nu+d-2)}, \alpha^3(2\nu+d-2)\} \lesssim \alpha^{3(2\nu+d-2)}$ and $\max(\alpha_0, \alpha) \lesssim \alpha^{3(2\nu+d-2)}$. Note that this special case can only happen when $d = 1$ and $\nu \in (0,1/2)$. Using (S.37), (S.38), and (S.39), we can see that in this case (neglecting all multiplicative constants by using the order relation $\prec$):
\[
\begin{align*}
\|w(\alpha)\|_1 + 2\|w(\alpha)\|_{\sqrt{z}} + 2\|w(\alpha)\|_{\infty z} \\ \leq \frac{\log n}{\sqrt{n}} \left( \frac{\sqrt{n}^{\frac{\beta}{2}}} {\alpha^{6-2\nu-3d/2-\beta/2}} + \frac{1}{2\alpha+d} + \frac{1}{\alpha^{6-d}} \right) + \frac{\log^2 n}{\sqrt{n}\alpha^{2\nu+d}} \\
\leq \frac{\log n}{\alpha_n^{6-2\nu-3d/2-\beta/2}} + \frac{\log n}{\sqrt{n}\alpha_n^{6-d}} + \frac{\log^2 n}{\sqrt{n}\alpha_n^{2\nu+d}}.
\end{align*}
\] (S.48)

In order to make the last upper bound $o(1)$, given that $\lambda_n \prec 1$ and $d = 1$, we further need that
\[
\begin{align*}
\Omega_n &\succ n^{-\frac{\beta(4a+2d+\beta)(9-4\nu-\beta)}{9-4\nu-\beta}} (\log n)^{\frac{2}{9-4\nu-\beta}} \text{ if } 9-4\nu-\beta > 0, \\
\chi_n &\succ n^{-\frac{1}{\pi}} (\log n)^{\frac{1}{2}}, \\
\omega_n &\succ n^{-\frac{1}{\pi(2\nu+1)}} (\log n)^{\frac{2}{2\nu+1}}.
\end{align*}
\] (S.49)

Since $d = 1$ and $\nu \in (0,1/2)$ in this case, the choice $a = 0.01$, $\beta = 3.9 - d = 2.9$ imply that $9-4\nu-\beta = 6.1 - 4\nu > 6.1 - 4 \cdot (1/2) = 4.1 > 0$. Together with the choice of $\kappa$ in (9), we can derive that
\[
\begin{align*}
\kappa &\leq \frac{1}{2} \cdot 0.1 < \frac{1}{2} \cdot \frac{3.9 - 1}{4.94(6.1 - 4 \cdot 0.5)} \leq \frac{1}{2} \cdot \frac{3.9 - d}{(4a+4.9)(6.1 - 4\nu)} \\
&= \frac{1}{2} \frac{\beta}{(4a+2+\beta)(9-4\nu-\beta)} < \frac{\beta}{(4a+2+\beta)(9-4\nu-\beta)}, \\
\kappa &\leq \frac{1}{2} \cdot 0.1 < \frac{1}{10}, \\
\kappa &\leq \frac{1}{2} \frac{1}{4(3\nu+d)} < \frac{1}{2(2\nu+d)}.
\end{align*}
\]
Therefore, (S.49) is satisfied, and we have that uniformly for all \( \alpha \in [\underline{\alpha}_n, \alpha_0] \), \( \|w(\alpha)\|_1 + 2\|w(\alpha)\|_\infty z = o(1) \).

Based on the analysis of two situations above, we can conclude that there exists a constant \( \tau \in (0, 1/2) \) that depends only on \( \nu, d, T, \alpha_0 \), such that \( n^{-\tau} \) is strictly larger in order than the maximum of the right-hand sides of (S.40), (S.44), (S.46), and (S.48), depending on the values of \( \nu \) and \( d \). In other words, this constant \( \tau \in (0, 1/2) \) satisfies the following:

If \( 2\nu + d - 2 \geq 0 \), then with \( a = 0.01 \) and \( \beta = 3.9 - d \),

\[
n^{-\tau} > \frac{\alpha_n^{4\nu+5d/2+\beta/2} \log n}{n^{\beta/(8a+4d+2\beta)}} + \frac{\alpha_n^{2\nu+d} \log^2 n}{\sqrt{n}}
= n^{(4\nu+5d/2+(3.9-d)/2)/2} \|w(\alpha)\|_1 + n^{(2\nu+d)\beta/2} \|w(\alpha)\|_\infty
= n^{(2\nu+d)\pi-1/2} \log^2 n,
\]

If \( -1 < 2\nu + d - 2 < 0 \ (d = 1 \text{ and } \nu \in (0, 1/2)) \), then with \( a = 0.01 \) and \( \beta = 3.9 - d \),

\[
n^{-\tau} > \frac{\alpha_n^{6-2\nu-d/2+\beta/2} \log n}{n^{\beta/(8a+4d+2\beta)}} + \frac{\alpha_n^{2\nu+d} \log^2 n}{\sqrt{n}}
= n^{(6-2\nu-d)/2} \|w(\alpha)\|_1 + n^{(2\nu+1)\pi-1/2} \log^2 n,
\]

An equivalent way is to set \( \tau \) such that

\[
0 < \tau < \min \left\{ \frac{3.9 - d}{2(d + 3.94)} - \left[ \frac{4\nu + 5d/2 + 3.9 - d/2}{2} \right], \frac{3.9 - d}{2(d + 3.94)} - \left[ \frac{4\nu + 3d/2 - 3.9 - d/2}{2} \right], \frac{145}{394} - (3.05 - \nu)\pi, \frac{145}{394} - (3.05 - 2\nu)\pi, \frac{1}{2} - (5 - 4\nu)\pi, \frac{1}{2} - 2(3\nu + d)\kappa \right\} < \frac{1}{2},
\]

With this \( \tau \), we have shown that uniformly for all \( \alpha \in [\underline{\alpha}_n, \overline{\alpha}_n] \), there exists a large integer \( N_1 \) that depends only on \( \nu, d, T, \alpha_0, a = 0.01, \beta = 3.9 - d \), such that for all \( n > N_1 \),

\[
|w(\alpha)|_1 + 2\|w(\alpha)\|_\infty \log n + 2\|w(\alpha)\|_\infty \log^2 n \leq n^{-\tau} = o(1).
\]

In particular, we apply (S.36) and Lemma S.6 to the two end-points \( \alpha = \overline{\alpha}_n \) and \( \alpha = \underline{\alpha}_n \), to obtain that for all \( n > N_1 \),

\[
\Pr \left( \sqrt{n} \left| \bar{\theta}_{\alpha_n} - \bar{\theta}_{\alpha_0} \right| > \theta_0 n^{-\tau} \right)
= \Pr \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left| \lambda_{i,n} (\overline{\alpha}_n)^{-1} - 1 \right| Y_{i,n} (\overline{\alpha}_n)^2 > n^{-\tau} \right)
\]

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\[
\leq \Pr \left( \sum_{i=1}^{n} w_i(\alpha_n)Y_{i,n}(\alpha_n)^2 > |w(\alpha_n)|_1 + 2\|w(\alpha_n)\| \log n + 2|w(\alpha_n)|_\infty \log^2 n \right) \\
\leq e^{-(\log n)^2},
\]

and similarly,
\[
\Pr \left( \sqrt{n} |\tilde{\alpha}_n - \tilde{\alpha}_0| > \theta_0 n^{-\tau} \right)
= \Pr \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |\lambda_{i,n}(\alpha_n)^{-1} - 1| Y_{i,n}(\alpha_n)^2 > n^{-\tau} \right)
\leq \Pr \left( \sum_{i=1}^{n} w_i(\alpha_n)Y_{i,n}(\alpha_n)^2 > |w(\alpha_n)|_1 + 2\|w(\alpha_n)\| \log n + 2|w(\alpha_n)|_\infty \log^2 n \right)
\leq e^{-(\log n)^2}.
\]

(S.53)

Now, based on Lemma S.5, we have that for any \( \alpha \in [\alpha_n, \pi_n] \), \( \tilde{\alpha}_n \leq \tilde{\alpha} \leq \tilde{\pi}_n \). Therefore, for \( \alpha \in [\alpha_n, \alpha_0] \),
\[
\sup_{\alpha \in [\alpha_n, \alpha_0]} |\tilde{\alpha} - \tilde{\alpha}_0| = |\tilde{\alpha}_n - \tilde{\alpha}_0|,
\]
and that from (S.54),
\[
\Pr \left( \sup_{\alpha \in [\alpha_n, \alpha_0]} \sqrt{n} |\tilde{\alpha} - \tilde{\alpha}_0| > \theta_0 n^{-\tau} \right)
= \Pr \left( \sqrt{n} |\tilde{\alpha}_n - \tilde{\alpha}_0| > \theta_0 n^{-\tau} \right) \leq e^{-(\log n)^2},
\]
which implies (S.33). Similarly, for \( \alpha \in [\alpha_0, \pi_n] \),
\[
\sup_{\alpha \in [\alpha_0, \pi_n]} |\tilde{\alpha} - \tilde{\alpha}_0| = |\tilde{\pi}_n - \tilde{\alpha}_0|,
\]
and that from (S.53),
\[
\Pr \left( \sup_{\alpha \in [\alpha_0, \pi_n]} \sqrt{n} |\tilde{\alpha} - \tilde{\alpha}_0| > \theta_0 n^{-\tau} \right)
= \Pr \left( \sqrt{n} |\tilde{\pi}_n - \tilde{\alpha}_0| > \theta_0 n^{-\tau} \right) \leq e^{-(\log n)^2},
\]
which implies (S.34). Finally, since
\[
\sup_{\alpha \in [\alpha_n, \pi_n]} |\tilde{\alpha} - \tilde{\alpha}_0| \leq \max \left( \left| \tilde{\alpha}_n - \tilde{\alpha}_0 \right|, \left| \tilde{\pi}_n - \tilde{\alpha}_0 \right| \right),
\]
we conclude that for all \( n > N_1 \),
\[
\Pr \left( \sup_{\alpha \in [\alpha_n, \pi_n]} \sqrt{n} |\tilde{\alpha} - \tilde{\alpha}_0| > \theta_0 n^{-\tau} \right)
= \Pr \left( \sqrt{n} |\tilde{\alpha}_n - \tilde{\alpha}_0| > \theta_0 n^{-\tau} \right) \text{ or } \sqrt{n} |\tilde{\pi}_n - \tilde{\alpha}_0| > \theta_0 n^{-\tau}
\leq \Pr \left( \sqrt{n} |\tilde{\alpha}_n - \tilde{\alpha}_0| > \theta_0 n^{-\tau} \right) + \Pr \left( \sqrt{n} |\tilde{\pi}_n - \tilde{\alpha}_0| > \theta_0 n^{-\tau} \right)
\leq 2e^{-(\log n)^2}.
\]

Hence the conclusion follows. \( \square \)
Lemma S.8. There exists a large integer $N_2$ that only depends on $\nu, d, T, \alpha_0$, such that for all $n > N_2$,
\[
\Pr \left( \sqrt{n} \left| \bar{\theta}_{\alpha_0} - \theta_0 \right| \leq 4\theta_0 \log n \right) \geq 1 - 2 \exp(-\log^2 n). \tag{S.55}
\]

Proof of Lemma S.8. By definition,
\[
\Pr \left( \sqrt{n} \left| \bar{\theta}_{\alpha_0} - \theta_0 \right| > 4\theta_0 \log n \right)
= \Pr \left( \sqrt{n} \left| 2\nu X_n^\top R_{\alpha_0}^{-1} X_n - \theta_0 \right| > 4\theta_0 \log n \right)
= \Pr \left( \sqrt{n} \left| \theta_0 X_n^\top (\sigma_0^2 R_{\alpha_0})^{-1} X_n - \theta_0 \right| > 4\theta_0 \log n \right)
= \Pr \left( \sqrt{n} \left| \frac{X_n^\top (\sigma_0^2 R_{\alpha_0})^{-1} X_n}{\theta_0} - 1 \right| > 4 \log n \right)
= \Pr \left( \sqrt{n} \left| \frac{\sum_{k=1}^n Z_{k,n}^2}{\theta_0} - 1 \right| > 4 \log n \right), \tag{S.56}
\]
where $Z_n = \sigma_0^{-1} R_{\alpha_0}^{-1/2} X_n = (Z_{1,n}, \ldots, Z_{n,n})^\top \sim \mathcal{N}(0, I_n)$. Now in Lemma S.6, we set $w_i = 1/n$ for all $i = 1, \ldots, n$, such that $\|w\|_1 = 1, \|w\|_2 = 1/\sqrt{n}$, and $\|w\|_\infty = 1/n$. We set $z = \log^2 n$. For sufficiently large $n$, we have that $2\|w\|\sqrt{z} + 2\|w\|_\infty z = 2 \log n/\sqrt{n} + 2 \log^2 n/n \leq 4 \log n/\sqrt{n}$. From (S.56), we can obtain that
\[
\Pr \left( \sqrt{n} \left| \bar{\theta}_{\alpha_0} - \theta_0 \right| > 4\theta_0 \log n \right)
= \Pr \left( \sum_{k=1}^n w_k Z_{k,n}^2 - 1 > 4n^{-1/2} \log n \right)
\leq \Pr \left( \left| \sum_{k=1}^n Z_{k,n}^2 - \|w\|_1 \right| > 2\|w\|\sqrt{z} + 2\|w\|_\infty z \right)
\leq 2e^{-z} = 2 \exp(-\log^2 n),
\]
which proves the result. \hfill \square

S3 Technical Lemmas for Profile Likelihood
We derive some useful results for the profile likelihood $\bar{L}_n(\alpha)$ defined in (5).

Lemma S.9. Let $\tau \in (0, 1/2)$ be the positive constant specified in Lemma S.7. Then with probability at least $1 - 2 \exp(-\log^2 n)$, for all sufficiently large $n$,
\[
\inf_{\alpha \in [\alpha_0, 2^{(2/\nu+d)}\alpha_0]} \exp \left\{ \bar{L}_n(\alpha) - \bar{L}_n(\alpha_0) \right\} \geq \exp \left( -n^{1/2-\tau} \right). \tag{S.57}
\]

Proof of Lemma S.9. According to (S.6) and (S.7) in Lemma S.1, we have that for all $k = 1, \ldots, n$ and all $\alpha \in [\alpha_0, 2^{(2/\nu+d)}\alpha_0]$,
\[
1 \geq \lambda_{k,n}(\alpha) \geq \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d} \geq 1/2. \tag{S.58}
\]
Let $\bar{X}_n(\alpha) = \left\{ \prod_{k=1}^n \lambda_{k,n}(\alpha) \right\}^{1/n}$. (S.58) implies that $\bar{X}_n(\alpha) \leq 1$. For any $\alpha > 0$, let $Y_n(\alpha) = U_n^\top X_n = (Y_{1,n}(\alpha), \ldots, Y_{n,n}(\alpha))^\top$ with $U_n$ given in (S.1). Then from the definition in (5), we have that
\[
\bar{L}_n(\alpha) - \bar{L}_n(\alpha_0)
\]
Therefore, we apply the second inequality in Lemma S.6 directly to the
\[
\sum_{k=1}^{n} \frac{\lambda_k, n(\alpha)^{-1} Y_k, n(\alpha)^2}{\sum_{k=1}^{n} Y_k, n(\alpha)^2} - \frac{1}{2} \sum_{k=1}^{n} \log \lambda_k, n(\alpha). \tag{S.59}
\]
For \( \alpha \in [\alpha_0, 2^{1/(2\nu+d)}\alpha_0] \) and for all \( n > N_1 \), we have that
\[
\exp \left\{ \mathcal{L}_n(\alpha) - \mathcal{L}_n(\alpha_0) \right\} = \left[ \sum_{k=1}^{n} \frac{\lambda_k, n(\alpha)^{-1} Y_k, n(\alpha)^2}{\sum_{k=1}^{n} Y_k, n(\alpha)^2} \cdot \left\{ \prod_{k=1}^{n} \lambda_k, n(\alpha) \right\}^{1/n} \right]^{-n/2} \geq \left[ \sum_{k=1}^{n} \frac{\lambda_k, n(\alpha)^{-1} Y_k, n(\alpha)^2}{\sum_{k=1}^{n} Y_k, n(\alpha)^2} \right]^{-n/2} \geq \left[ 1 + \sum_{k=1}^{n} \left\{ \lambda_k, n(\alpha)^{-1} - 1 \right\} \frac{Y_k, n(\alpha)^2}{\sum_{k=1}^{n} Y_k, n(\alpha)^2} \right]^{-n/2}. \tag{S.60}
\]
In the proof of Lemma S.7 from the derivation of (S.36), we can see that
\[
\sum_{k=1}^{n} \left\{ \lambda_k, n(\alpha)^{-1} - 1 \right\} Y_k, n(\alpha)^2 = \frac{n}{\theta_0} \left( \tilde{\theta}_\alpha - \tilde{\theta}_{\alpha_0} \right), \tag{S.61}
\]
where the right-hand side is an increasing function in \( \alpha \) according to Lemma S.5. Therefore, for the \( \tau \) given in Lemma S.7 (which satisfies \( \tau \in (0, 1/2) \)), we can use S.43 to obtain that for all sufficiently large \( n \),
\[
\Pr \left( \sup_{\alpha \in [\alpha_0, 2^{1/(2\nu+d)}\alpha_0]} \sum_{k=1}^{n} \left\{ \lambda_k, n(\alpha)^{-1} - 1 \right\} Y_k, n(\alpha)^2 > n^{-\tau + 1/2} \right) \leq \Pr \left( \sup_{\alpha \in [\alpha_0, 2^{1/(2\nu+d)}\alpha_0]} \sqrt{n}\theta_0^{-1} \left| \tilde{\theta}_\alpha - \tilde{\theta}_{\alpha_0} \right| > n^{-\tau} \right) \leq \Pr \left( \sup_{\alpha \in [\alpha_0, 2^{1/(2\nu+d)}\alpha_0]} \sqrt{n}\theta_0^{-1} \left| \tilde{\theta}_\alpha - \tilde{\theta}_{\alpha_0} \right| > n^{-\tau} \right) \leq e^{-\left(\log n\right)^2}. \tag{S.62}
\]
On the other hand, using Lemma S.13 we have that \( \sum_{k=1}^{n} Y_k, n(\alpha)^2 = Y_n(\alpha)^\top Y_n(\alpha) = Z_n^\top Z_n = \sum_{k=1}^{n} Z_{k,n}^2 \) for arbitrary \( \alpha > 0 \), where \( Z_n = \sigma_0^{-1} R_{\alpha_0}^{-1/2} X_n \equiv (Z_1, \ldots, Z_n)^\top \sim \mathcal{N}(0, I_n) \). Therefore, we apply the second inequality in Lemma S.6 directly to the \( \chi_2^2 \) random variables of \( \{Z_{k,n} : k = 1, \ldots, n\} \) with \( z = (\log n)^2 \) and obtain that for all sufficiently large \( n \),
\[
\Pr \left( \inf_{\alpha \in [\alpha_0, 2^{1/(2\nu+d)}\alpha_0]} \sum_{k=1}^{n} Y_k, n(\alpha)^2 \leq n - 2\sqrt{n}\log n \right) = \Pr \left( \sum_{k=1}^{n} Z_{k,n}^2 \leq n - 2\sqrt{n}\log n \right) \leq e^{-\left(\log n\right)^2}. \tag{S.63}
\]
Lemma S.10. Let \( \tau \in (0, 1/2) \) be the positive constant specified in Lemma S.7. Then with probability at least \( 1 - 2e^{-\log^2 n} \), for all sufficiently large \( n \),
\[
\sup_{\alpha \in [\alpha_n, \alpha_0]} \exp \left\{ \tilde{L}_n(\alpha) - \tilde{L}_n(\alpha_0) \right\} < \exp \left( 2n^{1/2-\tau} \right). 
\]  
(S.65)

Proof of Lemma S.10. According to (S.6) and (S.7) in Lemma S.1, we have that for all \( k = 1, \ldots, n \) and all \( \alpha \in [\alpha_n, \alpha_0] \),
\[
1 \leq \lambda_k, n(\alpha) \leq \left( \frac{\alpha_0}{\alpha} \right)^{2\nu+d} \leq \left( \frac{\alpha_0}{\alpha_n} \right)^{2\nu+d}. 
\]  
(S.66)

Let \( \bar{\lambda}_n(\alpha) = \left\{ \prod_{k=1}^n \lambda_k, n(\alpha) \right\}^{1/n} \) (S.66) implies that \( \bar{\lambda}_n(\alpha) \geq 1 \). For any \( \alpha > 0 \), let \( Y_n(\alpha) = U_\alpha^\top X_n = (Y_1, n(\alpha), \ldots, Y_n, n(\alpha))^\top \) with \( U_\alpha \) given in (S.1). Then from (S.59), we have that
\[
\exp \left\{ \mathcal{L}_n(\alpha) - \mathcal{L}_n(\alpha_0) \right\} 
\leq \left[ \frac{\sum_{k=1}^n \lambda_k, n(\alpha)^{-1} Y_k, n(\alpha)^2}{\sum_{k=1}^n Y_k, n(\alpha)^2} \right]^{-n/2} 
\leq \left[ \frac{\sum_{k=1}^n \lambda_k, n(\alpha)^{-1} Y_k, n(\alpha)^2}{\sum_{k=1}^n Y_k, n(\alpha)^2} \right]^{-n/2} 
= 1 + \sum_{k=1}^n \left\{ \lambda_k, n(\alpha)^{-1} - 1 \right\} Y_k, n(\alpha)^2 \right\} \right\}^{-n/2}. 
\]  
(S.67)

We then invoke the result from Lemma S.7. For the \( \tau \in (0, 1/2) \) given in Lemma S.7 we can use (S.33) to obtain that for all sufficiently large \( n \),
\[
\Pr \left( \sup_{\alpha \in [\alpha_n, \alpha_0]} \left\{ \sum_{k=1}^n \left\{ \lambda_k, n(\alpha)^{-1} - 1 \right\} Y_k, n(\alpha)^2 \right\} > n^{1/2-\tau} \right) 
\]  
(S.68)
\[
\leq \Pr \left( \sup_{\alpha \in [a_n, a_0]} \sqrt{n} \theta_0^{-1} \left| \tilde{\theta}_n - \tilde{\theta}_0 \right| > n^{-\tau} \right) \leq e^{-(\log n)^2}. \tag{S.68}
\]

Since \( \lambda_{k,n}^{-1} - 1 \leq 0 \) for all \( k = 1, \ldots, n \), (S.68) implies that uniformly for all \( \alpha \in [a_n, a_0] \), for all sufficiently large \( n \), with probability at least \( 1 - \exp(-\log^2 n) \),
\[
\sum_{k=1}^{n} \left\{ \lambda_{k,n}(\alpha)^{-1} - 1 \right\} Y_{k,n}(\alpha)^2 \geq -n^{1/2-\tau}. \tag{S.69}
\]
(S.67), (S.69), and (S.63) together imply that uniformly for all \( \alpha \in [a_n, a_0] \), for all sufficiently large \( n \), with probability at least \( 1 - 2\exp(-\log^2 n) \),
\[
\exp \left\{ \tilde{\mathcal{L}}_n(\alpha) - \tilde{\mathcal{L}}_n(\alpha_0) \right\} \\
\leq \left[ 1 - \frac{n^{1/2-\tau}}{\inf_{\alpha \in [a_n, a_0]} \sum_{k=1}^{n} Y_{k,n}(\alpha)^2} \right]^{-n/2} \leq \left( 1 - \frac{n^{1/2-\tau}}{n - 2\sqrt{n \log n}} \right)^{-n/2} \\
\leq \left( 1 - \frac{2n^{1/2-\tau}}{n} \right)^{-n/2} = \left\{ 1 - \frac{2}{n^{1/2+\tau}} \right\}^{-n^{1/2-\tau}} < \exp \left( 2n^{1/2-\tau} \right), \tag{S.70}
\]
where in the last step, we use the fact that the function \((1-x^{-1})^x\) is continuous and monotonely increasing to \(1/e\) for \( x > 1 \), so \((1-x^{-1})^x > 1/e^2\) for \( x = n^{1/2+\tau} \) given that \( n \) is sufficiently large. \(\square\)

**Lemma S.11.** Suppose that the sequence \( \{w_i : i = 1, \ldots, n\} \) satisfies \( \sum_{i=1}^{n} w_i \geq n - c_1 n^{b_1} \), \( \max_{1 \leq i \leq n} w_i \leq 1 \) and \( \min_{1 \leq i \leq n} w_i \geq c_2 n^{-b_2} \), where \( 0 < b_2 < b_1 < 1 \), \( c_1 > 0 \), and \( c_2 > 0 \) are all constants. Then \( \prod_{i=1}^{n} w_i \geq \exp \left( -4b_2 c_1 n^{b_1} \log n \right) \) for all \( n > \max \left\{ c_2^{-1/b_2}, (2c_2)^{1/b_2} \right\} \).

**Proof of Lemma S.11.** Given the constraints in the lemma, minimizing \( \prod_{i=1}^{n} w_i \) is equivalent to choosing as many \( w_i \)'s to reach the lower bound of \( c_2 n^{-b_2} \) as possible. On the other hand, the constraints \( \sum_{i=1}^{n} w_i \geq n - c_1 n^{b_1} \) and \( \max_{1 \leq i \leq n} w_i \leq 1 \) imply that the number of \( w_i \)'s that attain the lower bound cannot be too large. Suppose that out of \( n \) terms of \( w_i \)'s, \( w_1 = \ldots = w_k = c_2 n^{-b_2} \), where \( k \) is an integer between 1 and \( n \). Then \( k \) must satisfy the relation (since all \( w_i \)'s satisfy \( w_i \leq 1 \)):
\[
kc_2 n^{-b_2} + (n - k) \cdot 1 \geq n - c_1 n^{b_1},
\]
which implies that \( k \leq c_1 n^{b_1}/(1 - c_2 n^{-b_2}) \). Therefore,
\[
\prod_{i=1}^{n} w_i \geq (c_2 n^{-b_2})^k \cdot 1^{n-k} \geq (c_2 n^{-b_2})^{c_1 n^{b_1}/(1 - c_2 n^{-b_2})}.
\]
Finally, for all \( n > \max \left\{ c_2^{-1/b_2}, (2c_2)^{1/b_2} \right\} \), we have that \( c_2 > n^{-b_2} \) and \( 1 - c_2 n^{-b_2} < 1/2 \). Hence the conclusion follows. \(\square\)

**Lemma S.12.** There exist constants \( \kappa_1 \in (0, 1) \) and \( C_3 > 0 \) that depends on \( \nu, d, T, \alpha_0 \), such that for all sufficiently large \( n \),
\[
\sup_{\alpha \in [a_0, \alpha_n]} \exp \left\{ \tilde{\mathcal{L}}_n(\alpha) - \tilde{\mathcal{L}}_n(\alpha_0) \right\} < \exp \left( C_3 n^{\kappa_1} \log n \right). \tag{S.71}
\]
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Proof of Lemma S.12. According to (S.6) and (S.7) in Lemma S.1, we have that for all \( k = 1, \ldots, n \) and all \( \alpha \in [\alpha_0, \overline{\alpha}_n] \),
\[
1 \geq \lambda_{k,n}(\alpha) \geq \left( \frac{\alpha_0}{\overline{\alpha}_n} \right)^{2\nu+d} \geq \frac{\alpha_0^{2\nu+d}}{n^{2(\nu+d)\delta}}. \tag{S.72}
\]

Let \( \overline{\lambda}_n(\alpha) = \left\{ \prod_{k=1}^{n} \lambda_{k,n}(\alpha) \right\}^{1/n} \). In Lemma S.4, we can choose \( a = 0.01, \beta = 3.9 - d, \) and \( \varepsilon_n = n^{-1/(4a+2d+\beta)} \), such that for a large constant \( C_2 > 0 \), for all \( \alpha \in [\alpha_0, \overline{\alpha}_n] \), and for all sufficiently large \( n \),
\[
\sum_{k=1}^{n} \{ 1 - \lambda_{k,n}(\alpha) \} \leq C_1 \max(\alpha_0^6, \alpha^6) \max \left\{ \frac{\alpha_0^{3(2\nu+d-2)}, \alpha^{3(2\nu+d-2)}}{\alpha^{4\nu+3d/2-\beta/2}} \right\} \sqrt{n\varepsilon_n^{\beta/2}} + C_2 \max(\alpha_0^6, \alpha^6) \max \left\{ \frac{\alpha_0^{3(2\nu+d-2)}, \alpha^{3(2\nu+d-2)}}{\alpha^{2(\nu+d)}} \right\}. \tag{S.73}
\]

If \( 2\nu + d - 2 \geq 0 \), then for all \( \alpha \in [\alpha_0, \overline{\alpha}_n] \), and for all sufficiently large \( n \),
\[
\sum_{k=1}^{n} \{ 1 - \lambda_{k,n}(\alpha) \} \leq \alpha_n^{2\nu+3d/2+\beta/2} \cdot n^{(2a+d)/(4a+2d+\beta)} + \alpha_n^{2\nu+d} \cdot (2a+d)/(4a+2d+\beta) + \alpha_n^d \leq n^{\overline{\pi}(2\nu+d+1.95)/(d+0.02)/(d+3.94)} + n^{\overline{\pi} \delta}. \tag{S.74}
\]

Given the definition of \( \overline{\pi} \) in (9) and \( d \geq 1 \), we have that
\[
\overline{\pi}(2\nu+d+1.95) \cdot (2\nu+d+1.95) + \frac{1}{4(2\nu+d+2)} \cdot (2\nu+d+1.95) + \frac{d}{4} < 1.
\]

Therefore, (S.74) implies that \( \sum_{k=1}^{n} \{ 1 - \lambda_{k,n}(\alpha) \} < C_2 n^{\kappa_1} \) for some constants \( \kappa_1 \in (0, 1) \) and \( C_2 > 0 \). If \( -1 < 2\nu+d-2 < 0 \) \((d = 1 \text{ and } \nu \in (0, 1/2))\), then for all \( \alpha \in [\alpha_0, \overline{\alpha}_n] \), and for all sufficiently large \( n \), (S.73) implies that
\[
\sum_{k=1}^{n} \{ 1 - \lambda_{k,n}(\alpha) \} \leq \alpha_n^{6-4\nu-3d/2+\beta/2} \cdot n^{(2a+d)/(4a+2d+\beta)} + \overline{\alpha}_n^{2\nu+d} \cdot n^{(2a+d)/(4a+2d+\beta)} + \overline{\alpha}_n^d \leq n^{(5.95-4\nu)\overline{\pi}+51/247} + n^{(2\nu+1)\overline{\pi}+51/247} + n^{\overline{\pi} \delta}. \tag{S.75}
\]

Given the definition of \( \overline{\pi} \) in (9), we have that
\[
(5.95-4\nu)\overline{\pi} + \frac{51}{247} < 6 \cdot 0.02 + \frac{1}{2} < 1,
\]
\[
(2\nu+1)\overline{\pi} + \frac{51}{247} < \frac{2\nu+1}{4(2\nu+d+2)} + \frac{1}{2} < 1.
\]

Therefore, (S.75) also implies that \( \sum_{k=1}^{n} \{ 1 - \lambda_{k,n}(\alpha) \} < C_2 n^{\kappa_1} \) for some constants \( \kappa_1 \in (0, 1) \) and \( C_2 > 0 \). As a result of (S.74) and (S.75), we have that for all sufficiently large \( n \),
\[
\sum_{k=1}^{n} \{ 1 - \lambda_{k,n}(\alpha) \} \leq C_2 n^{\kappa_1}, \quad \text{or} \quad \sum_{k=1}^{n} \lambda_{k,n}(\alpha) \geq n - C_2 n^{\kappa_1}. \tag{S.76}
\]
Now in Lemma S.11 we set \( w_i = \lambda_i n, \) \( c_1 = C_2, \) \( b_1 = \kappa_1, \) \( c_2 = \alpha_0^{2\nu + d}, \) \( b_2 = (2\nu + d)\pi, \) and use (S.72) and (S.76) to obtain that for all sufficiently large \( n, \)
\[
\inf_{\alpha \in [\alpha_0, \pi n]} \bar{\lambda}_n(\alpha) \geq \exp\left\{ -4C_2(2\nu + d)\pi n^{\nu_1 - 1} \log n \right\}. \tag{S.77}
\]
On the other hand, (S.72) implies that \( \sum_{k=1}^{n} \{ \lambda_{k,n}(\alpha)^{-1} - 1 \} Y_{k,n}(\alpha)^2 \geq 0. \) Therefore,
\[
\sup_{\alpha \in [\alpha_0, \pi n]} \exp \left\{ \bar{\lambda}_n(\alpha) - \bar{\lambda}_n(\alpha_0) \right\}
= \sup_{\alpha \in [\alpha_0, \pi n]} \bar{\lambda}_n(\alpha)^{-n/2} \left[ 1 + \sum_{k=1}^{n} \{ \lambda_{k,n}(\alpha)^{-1} - 1 \} Y_{k,n}(\alpha)^2 \right]^{-n/2}
\leq \sup_{\alpha \in [\alpha_0, \pi n]} \bar{\lambda}_n(\alpha)^{-n/2} \cdot 1^{-n/2}
\leq \exp \left\{ 2C_2(2\nu + d)\pi n^{\nu_1 - 1} \log n \right\}, \tag{S.78}
\]
where the last step follows from (S.77). The conclusion follows by taking \( C_3 = 2C_2(2\nu + d)\pi. \)

**Lemma S.13.** For any \( \alpha > 0, \) let \( Y_n(\alpha) = U_\alpha^T X_n, \) where \( U_\alpha \) is given in (S.1). Let \( Z_n = \sigma_0^{-1} R_{\alpha_0}^{-1/2} X_n. \) Then \( Y_n(\alpha)^T Y_n(\alpha) = Z_n^T Z_n. \)

**Proof of Lemma S.13.** By our model setup, \( X_n \sim \mathcal{N}(0, \sigma_0^2 R_{\alpha_0}). \) Therefore, \( Z_n \sim \mathcal{N}(0, I_n). \)
(S.1) implies that \( Y_n(\alpha) \sim \mathcal{N}(0, \sigma_0^2 U_\alpha^T R_{\alpha_0} U_\alpha) = \mathcal{N}(0, I_n). \) Since \( Y_n(\alpha) = U_\alpha^T X_n = U_\alpha^T (\sigma_0 R_{\alpha_0}^{-1/2}) Z_n \) and \( Z_n \sim \mathcal{N}(0, I_n), \) this implies that we must have
\[
U_\alpha^T (\sigma_0 R_{\alpha_0}^{-1/2}) (\sigma_0 R_{\alpha_0}^{-1/2}) U_\alpha = I_n
\]
\[
\implies \sigma_0^{-2} R_{\alpha_0} = (U_\alpha^T)^{-1} U_\alpha^{-1} = (U_\alpha U_\alpha^T)^{-1}
\]
\[
\implies \sigma_0^{-2} R_{\alpha_0}^{-1} = U_\alpha U_\alpha^T.
\]
Therefore,
\[
Y_n(\alpha)^T Y_n(\alpha) = X_n^T U_\alpha U_\alpha^T X_n = X_n^T (\sigma_0^{-2} R_{\alpha_0}^{-1}) X_n
= (\sigma_0^{-1} R_{\alpha_0}^{-1/2} X_n)^T (\sigma_0^{-1} R_{\alpha_0}^{-1/2} X_n) = Z_n^T Z_n.
\]

**Lemma S.14.** The profile log-likelihood function defined in (5) satisfies that for any \( 0 < \alpha_1 < \alpha_2 < \infty, \) for all possible value of \( X_n, \)
\[
\left( \frac{\alpha_1}{\alpha_2} \right)^{(\nu + d)/2} < \exp \left\{ \bar{\lambda}_n(\alpha_2) - \bar{\lambda}_n(\alpha_1) \right\} < \left( \frac{\alpha_2}{\alpha_1} \right)^{(\nu + d)/2}.
\]

**Proof of Lemma S.14.** We use the similar argument to the proof of Lemma 1 in Kaufman and Shaby \cite{38}. The key idea is to construct monotone spectral density functions. First using (5), we can write the exponential difference as
\[
\exp \left\{ \bar{\lambda}_n(\alpha_2) - \bar{\lambda}_n(\alpha_1) \right\} = \left( \frac{X_n^T R_{\alpha_1}^{-1} X_n}{X_n^T R_{\alpha_2}^{-1} X_n} \right)^{n/2} \cdot \left( \frac{R_{\alpha_1}}{R_{\alpha_2}} \right)^{1/2}. \tag{S.79}
\]
We bound the two terms in (S.79) separately. For the first term in (S.79), we define the matrix \( \Omega^\dagger = \alpha_1^{-2\nu} R_{\alpha_1} - \alpha_2^{-2\nu} R_{\alpha_2}. \) Then the entries of \( \Omega^\dagger \) can be expressed in terms of a function \( \tilde{\Omega}^\dagger : \mathbb{R}^d \to \mathbb{R}, \) with
\[
\Omega_{ij}^\dagger = \tilde{\Omega}^\dagger (x_i - x_j) = \alpha_1^{-2\nu} K_{\alpha_1,ij} (x_i - x_j) - \alpha_2^{-2\nu} K_{\alpha_2,ij} (x_i - x_j),
\]

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for \( i, j \in \{1, \ldots, n\} \). The matrix \( \Omega^\dagger \) is positive definite if \( \tilde{K}_{\Omega^\dagger} \) is a positive definite function. We compute the spectral density of \( \tilde{K}_{\Omega^\dagger} \):

\[
\begin{align*}
\tilde{f}_{\Omega^\dagger}(\omega) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\omega^\top \mathbf{x}} \tilde{K}_{\Omega^\dagger}(\mathbf{x}) d\mathbf{x} \\
&= \frac{1}{(2\pi)^d} \left\{ \alpha_1^{-2\nu} \int_{\mathbb{R}^d} e^{-i\omega^\top \mathbf{x}} K_{\alpha_1,\nu}(\mathbf{x}) d\mathbf{x} - \alpha_2^{-2\nu} \int_{\mathbb{R}^d} e^{-i\omega^\top \mathbf{x}} K_{\alpha_2,\nu}(\mathbf{x}) d\mathbf{x} \right\} \\
&= \frac{\Gamma(\nu + d/2)}{\pi^{d/2} \Gamma(\nu)} \left\{ \frac{\alpha_1^{-2\nu}}{(\alpha_1^2 + ||\omega||^2)^{\nu+d/2}} - \frac{\alpha_2^{-2\nu}}{(\alpha_2^2 + ||\omega||^2)^{\nu+d/2}} \right\} \\
&> 0, \text{ for all } \omega \in \mathbb{R}^d,
\end{align*}
\]  

(S.80)

where the last step follows because \( 0 < \alpha_1 < \alpha_2 \). This has shown that \( \tilde{K}_{\Omega^\dagger} \) is indeed a positive definite function. Therefore, \( \Omega^\dagger \) is positive definite. Since both \( R_{\alpha_1} \) and \( R_{\alpha_2} \) are positive definite matrices, and

\[
\Omega^\dagger = \alpha_1^{-2\nu} R_{\alpha_1} - \alpha_2^{-2\nu} R_{\alpha_2}
\]

it follows that \( \alpha_2^{-2\nu} R_{\alpha_2} - \alpha_1^{-2\nu} R_{\alpha_1} \) is a positive definite matrix. For any value of \( X_n \),

\[
X_n^\top \left( \alpha_2^{-2\nu} R_{\alpha_2} - \alpha_1^{-2\nu} R_{\alpha_1} \right) X_n = \alpha_2^{-2\nu} X_n^\top R_{\alpha_2}^{-1} X_n - \alpha_1^{-2\nu} X_n^\top R_{\alpha_1}^{-1} X_n > 0
\]

\[
\implies \left( \frac{X_n^\top R_{\alpha_1}^{-1} X_n}{X_n^\top R_{\alpha_2}^{-1} X_n} \right)^{\nu/2} < \left( \frac{\alpha_2}{\alpha_1} \right)^\nu.
\]  

(S.81)

Furthermore, for the determinant, we have

\[
|\alpha_2^{-2\nu} R_{\alpha_2}^{-1}| > |\alpha_1^{-2\nu} R_{\alpha_1}^{-1}| \implies \left( \frac{R_{\alpha_1}}{R_{\alpha_2}} \right)^{1/2} > \left( \frac{\alpha_1}{\alpha_2} \right)^\nu
\]  

(S.82)

Now we define the matrix \( \Omega^\dagger_{ij} = \alpha_2^d R_{\alpha_2} - \alpha_1^d R_{\alpha_1} \). Then the entries of \( \Omega^\dagger \) can be expressed in terms of a function \( \tilde{K}_{\Omega^\dagger} : \mathbb{R}^d \to \mathbb{R} \), with

\[
\Omega^\dagger_{ij} = \tilde{K}_{\Omega^\dagger}(x_i - x_j) = \alpha_2^d K_{\alpha_2,\nu}(x_i - x_j) - \alpha_1^d K_{\alpha_1,\nu}(x_i - x_j),
\]

for \( i, j \in \{1, \ldots, n\} \). The matrix \( \Omega^\dagger \) is positive definite if \( \tilde{K}_{\Omega^\dagger} \) is a positive definite function. We compute the spectral density of \( \tilde{K}_{\Omega^\dagger} \):

\[
\begin{align*}
\tilde{f}_{\Omega^\dagger}(\omega) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\omega^\top \mathbf{x}} \tilde{K}_{\Omega^\dagger}(\mathbf{x}) d\mathbf{x} \\
&= \frac{1}{(2\pi)^d} \left\{ \alpha_2^d \int_{\mathbb{R}^d} e^{-i\omega^\top \mathbf{x}} K_{\alpha_2,\nu}(\mathbf{x}) d\mathbf{x} - \alpha_1^d \int_{\mathbb{R}^d} e^{-i\omega^\top \mathbf{x}} K_{\alpha_1,\nu}(\mathbf{x}) d\mathbf{x} \right\} \\
&= \frac{\Gamma(\nu + d/2)}{\pi^{d/2} \Gamma(\nu)} \left\{ \frac{\alpha_2^d}{(\alpha_2^2 + ||\omega||^2)^{\nu+d/2}} - \frac{\alpha_1^d}{(\alpha_1^2 + ||\omega||^2)^{\nu+d/2}} \right\} \\
&> 0, \text{ for all } \omega \in \mathbb{R}^d,
\end{align*}
\]  

(S.83)
where the last step follows because $0 < \alpha_1 < \alpha_2$. This has shown that $\tilde{K}_{\alpha\beta}$ is indeed a positive definite function. Therefore, $\Omega^\dagger = \alpha_2^dR_{\alpha_2} - \alpha_1^dR_{\alpha_1}$ is positive definite. This implies that for the determinant,

$$
\left|\alpha_2^dR_{\alpha_2}\right| > \left|\alpha_1^dR_{\alpha_1}\right| \implies \alpha_2^{nd}|R_{\alpha_2}| > \alpha_1^{nd}|R_{\alpha_1}|
$$

$$
\implies \left(\frac{|R_{\alpha_1}|}{|R_{\alpha_2}|}\right)^{1/2} < \left(\frac{\alpha_2}{\alpha_1}\right)^{nd/2}.
$$

(S.84)

And for any value of $X_n$,

$$
X_n^\top \left(\alpha_1^{-d}R_{\alpha_1}^{-1} - \alpha_2^{-d}R_{\alpha_2}^{-1}\right) X_n = \alpha_1^{-d}X_n^\top R_{\alpha_1}^{-1}X_n - \alpha_2^{-d}X_n^\top R_{\alpha_2}^{-1}X_n > 0
$$

$$
\implies \left(\frac{X_n^\top R_{\alpha_1}^{-1}X_n}{X_n^\top R_{\alpha_2}^{-1}X_n}\right)^{n/2} > \left(\frac{\alpha_1}{\alpha_2}\right)^{nd/2}.
$$

(S.85)

The first inequality in Lemma S.14 follows from (S.79), (S.82), and (S.85). The second inequality in Lemma S.14 follows from (S.79), (S.81), and (S.84). □

**Lemma S.15.** Suppose that Assumptions (A.1) and (A.3) hold. Then the profile posterior distribution of $\alpha$ given by $\tilde{\pi}(\alpha|X_n)$ in (17) is a proper posterior for any given $n$ and data $X_n$.

**Proof of Lemma S.15.** We fix $n$ and $X_n$. Since the Matérn covariance function is continuous in $\alpha \in \mathbb{R}_+$, $R_\alpha$ is also continuous in $\alpha \in \mathbb{R}_+$, and so is the profile likelihood $\exp\{\tilde{L}_n(\alpha)\}$. Furthermore, both $\pi(\theta_0|\alpha)$ and $\pi(\alpha)$ are continuous functions in $\alpha \in \mathbb{R}_+$ by Assumptions (A.1) and (A.3). As a result, the profile posterior in (17) is well defined as long as the function $\exp\{\tilde{L}_n(\alpha)\} \pi(\theta_0|\alpha)\pi(\alpha)$ is integrable as $\alpha \to +\infty$ and $\alpha \to 0+$.

As $\alpha \to +\infty$, $R_\alpha \to I_n$ elementwise. Therefore, the profile likelihood

$$
\exp\{\tilde{L}_n(\alpha)\} \to \exp\left\{-\frac{n}{2} \log \frac{X_n^\top X_n}{n}\right\} = \left(\frac{X_n^\top X_n}{n}\right)^{-n/2},
$$

which is a finite positive number for given $n$ and $X_n$. Since Assumption (A.3) says that $\int_0^\infty \pi(\theta_0|\alpha)\pi(\alpha)d\alpha < \infty$ and $\exp\{\tilde{L}_n(\alpha)\}$ is continuous in $\alpha$, it follows that the integral of $\exp\{\tilde{L}_n(\alpha)\} \pi(\theta_0|\alpha)\pi(\alpha)$ on $\alpha \in [1, +\infty)$ is finite.

Then we consider the case when $\alpha \to 0+$. The property of the Matérn covariance function as $\alpha \to 0+$ has been analyzed in Berger et al. [7] and Gu et al. [27]. Lemma 3.3 of Gu et al. [27] implies that for given $n$ and $X_n$, the profile likelihood function converges to zero as $\alpha \to 0+$ with the following rates:

$$
\exp\{\tilde{L}_n(\alpha)\} \leq \begin{cases} 
C(n, X_n)\alpha^\nu, & \text{if } \nu \in (0, 1), \\
C(n, X_n)\alpha \log(1/\alpha), & \text{if } \nu = 1, \\
C(n, X_n)\alpha, & \text{if } \nu > 1,
\end{cases}
$$

where $C(n, X_n)$ is a positive quantity that depends on $n$ and $X_n$ but not $\alpha$. In all three cases, $\exp\{\tilde{L}_n(\alpha)\} \to 0$ as $\alpha \to 0+$.

Together with $\int_0^\infty \pi(\theta_0|\alpha)\pi(\alpha)d\alpha < \infty$ from Assumption (A.3), we conclude that the integral of $\exp\{\tilde{L}_n(\alpha)\} \pi(\theta_0|\alpha)\pi(\alpha)$ on $\alpha \in (0, 1)$ is also finite. Therefore, $\int_0^\infty \exp\{\tilde{L}_n(\alpha)\} \pi(\theta_0|\alpha)\pi(\alpha)d\alpha < \infty$, and the profile posterior defined in (17) is a proper posterior for any given $n$ and $X_n$. □

**S4 Proof of Lemma 3**

**Proof of Lemma 3** We decompose the integral in (33) into three parts:

$$
\int_\mathbb{R} |g_n(t; \alpha)|dt = \int_{A_1} |g_n(t; \alpha)|dt + \int_{A_2} |g_n(t; \alpha)|dt + \int_{A_3} |g_n(t; \alpha)|dt,
$$

(S.86)
where \( A_1 = \{ t \in \mathbb{R} : |t| \geq D_1 \sqrt{n} \} \), \( A_2 = \{ t \in \mathbb{R} : \theta_n \leq |t| < D_1 \sqrt{n} \} \), and \( A_3 = \{ t \in \mathbb{R} : |t| < \theta_n \} \), with the constant \( D_1 = \theta_0/4 \) and the sequence \( \theta_n \) as specified in the lemma.

We have the following relation for the difference of log-likelihoods.

\[
\mathcal{L}_n(\alpha^{-2\nu}, \alpha) - \mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}) = -\frac{n}{2} \log \frac{\theta}{\tilde{\alpha}} + \frac{n(\theta - \tilde{\alpha})}{2\theta} \quad \text{(S.87)}
\]

\[
= -\frac{n}{2} \log \left( 1 + \frac{t}{\sqrt{n}\theta} \right) + \frac{\sqrt{nt}}{2(\tilde{\alpha} + \frac{t}{\sqrt{n}})} \quad \text{(S.88)}
\]

Bound the first term in \[\text{(S.86)}\]: We have

\[
\int_{A_1} |\theta_n(t; \alpha)| dt \leq \int_{A_1} \exp \left\{ \mathcal{L}_n(\alpha^{-2\nu}, \alpha) - \mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}) \right\} \frac{\pi (\tilde{\alpha} + \frac{t}{\sqrt{n}} | \alpha))}{\pi(\theta_0 | \alpha)} dt
\]

\[
+ \int_{A_1} e^{-\frac{t^2}{4\theta_0}} dt.
\]

The second term in \[\text{(S.89)}\] can be bounded by

\[
\int_{A_1} e^{-\frac{t^2}{4\theta_0}} dt = \pi(\theta_0 | \alpha) \cdot \int_{A_1} e^{-\frac{t^2}{4\theta_0}} dt
\]

\[
\leq 2\sqrt{\pi} \theta_0 \cdot \int_{|t| \geq D_1 \sqrt{n}} \frac{1}{\sqrt{2\pi \cdot 2\theta_0^2}} e^{-\frac{t^2}{4\theta_0}} dt
\]

\[
\leq 2\sqrt{\pi} \theta_0 \exp \left\{ -\frac{nD_1^2}{4\theta_0^2} \right\}, \quad \text{(S.90)}
\]

where the last inequality follows from the tail bounds for a normal random variable: if \( Z \sim \mathcal{N}(0,1) \), then for any \( z > 0 \),

\[
\Pr(|Z| > z) \leq e^{-z^2/2}. \quad \text{(S.91)}
\]

For the first term in \[\text{(S.89)}\], we note that \( \theta \) is a linear transformation of \( t \). We use the relation \[\text{(S.87)}\] and obtain that

\[
\int_{A_1} \exp \left\{ \mathcal{L}_n(\alpha^{-2\nu}, \alpha) - \mathcal{L}_n(\alpha^{-2\nu}, \tilde{\alpha}) \right\} \frac{\pi (\tilde{\alpha} + \frac{t}{\sqrt{n}} | \alpha))}{\pi(\theta_0 | \alpha)} dt
\]

\[
= \int_{|t| \geq D_1 \sqrt{n}} \exp \left\{ -\frac{n}{2} \log \frac{\theta}{\tilde{\alpha}} + \frac{n(\theta - \tilde{\alpha})}{2\theta} \right\} \frac{\pi (\tilde{\alpha} + \frac{t}{\sqrt{n}} | \alpha))}{\pi(\theta_0 | \alpha)} dt
\]

\[
\leq \sqrt{n} \int_{|\tilde{\alpha} - \theta_0| \geq D_1} \frac{\pi (\tilde{\alpha} + \frac{t}{\sqrt{n}} | \alpha))}{\pi(\theta_0 | \alpha)} \cdot \exp \left\{ -\frac{n}{2} \varphi \left( \frac{\tilde{\alpha}}{\theta_0} \right) \right\} d\tilde{\alpha}. \quad \text{(S.92)}
\]

For any constant \( \epsilon > 0 \), define the event \( \mathcal{E}_1(\epsilon, \alpha) = \{|\tilde{\alpha} - \theta_0| < \epsilon\} \). Let \( D_1 = \theta_0/4 \) and \( 0 < \epsilon_{1n} < D_1 \), where \( \epsilon_{1n} \downarrow 0 \) and its order will be determined later. Then, on the event \( \mathcal{E}_1(\epsilon_{1n}, \alpha) \) and \( \{|\theta - \tilde{\alpha}| \geq D_1\} \), we consider two cases: If \( \theta > \tilde{\alpha} + D_1 \), then

\[
1 - \frac{\tilde{\alpha}}{\theta} = 1 - \frac{\tilde{\alpha}}{\theta - \tilde{\alpha} + \theta} \geq 1 - \frac{\tilde{\alpha}}{D_1 + \theta}
\]

\[
= \frac{D_1}{D_1 + \theta} \geq \frac{D_1}{D_1 + \theta_0 + \epsilon_{1n}} = \frac{1}{4} \theta_0 + \frac{1}{4} \theta_0 + \frac{1}{4} \theta_0 = \frac{1}{6}.
\]

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If $\theta < \bar{\theta}_\alpha - D_1$, then

$$\frac{\bar{\theta}_\alpha}{\theta} - 1 = \frac{\bar{\theta}_\alpha}{\theta - \bar{\theta}_\alpha + \theta_\alpha} - 1 \geq \frac{\bar{\theta}_\alpha}{-D_1 + \theta_\alpha} - 1 = \frac{D_1}{\bar{\theta}_\alpha - D_1} > \frac{D_1}{\theta_0 + \epsilon_{1n} - D_1} = \frac{\frac{1}{2} \theta_0}{\theta_0 + \frac{1}{2} \theta_0 - \frac{1}{2} \theta_0} = \frac{1}{4}.$$

This implies that on the event $\mathcal{E}_1(\epsilon_{1n}, \alpha)$ and $\{|\theta - \bar{\theta}_\alpha| \geq D_1\}$, we must have either $\frac{\bar{\theta}_\alpha}{\theta} < \frac{5}{6}$ or $\frac{\bar{\theta}_\alpha}{\theta} > \frac{5}{4}$. Since the function $\varphi(u) = u - \log u - 1$ is monotonely decreasing on $(0,1)$ and monotonely increasing on $[1, +\infty)$, we have that on the event $\mathcal{E}_1(\epsilon_{1n}, \alpha)$ and $\{|\theta - \bar{\theta}_\alpha| \geq D_1\}$, either $\varphi(\theta_0/\theta) > \min\{\varphi(5/6), \varphi(5/4)\} > 0.15$. Therefore, from (S.92), we obtain that with the choice $D_1 = \theta_0/4$, on the event $\mathcal{E}_1(\epsilon_{1n}, \alpha)$,

$$\int_{A_1} \exp \left\{ \mathcal{L}(\alpha^{-2\nu} \theta, \alpha) - \mathcal{L}(\alpha^{-2\nu} \bar{\theta}_\alpha, \alpha) \right\} \frac{\pi \left( \theta_0 + \frac{\alpha}{\sqrt{n}} - \alpha \right)}{\pi(\theta_0|\alpha)} \, d\theta \leq \frac{\sqrt{n}}{\pi(\theta_0|\alpha)} \exp \left\{ -0.015n \right\} \, d\theta < \frac{\sqrt{n}}{\pi(\theta_0|\alpha)} e^{-0.007n},$$

where in the last inequality, we use the fact that $\pi(\theta_0|\alpha)$ is a proper prior density. Thus, combining (S.89), (S.90) and (S.93) yields that on the event $\mathcal{E}_1(\epsilon_{1n}, \alpha)$ with $D_1 = \theta_0/4$,

$$\int_{A_1} |\theta_0(t; \alpha)| \, dt \leq 2\sqrt{n} \exp \left( -\frac{n}{64} \right) + \frac{\sqrt{n}}{\pi(\theta_0|\alpha)} e^{-0.007n}.$$

Bound the second term in (S.86): On the event $\mathcal{E}_1(\epsilon_{1n}, \alpha)$ and $\{|\theta - \bar{\theta}_\alpha| < D_1\}$ with $D_1 = \theta_0/4$ and $0 < \epsilon_{1n} < D_1$, if $\theta \geq \bar{\theta}_\alpha$, then

$$1 - \frac{\bar{\theta}_\alpha}{\theta} = 1 - \frac{\bar{\theta}_\alpha}{\theta - \bar{\theta}_\alpha + \theta_\alpha} < 1 - \frac{\bar{\theta}_\alpha}{-D_1 + \theta_\alpha} = \frac{D_1}{D_1 + \theta_\alpha} \leq \frac{D_1}{D_1 + \theta_0 - \epsilon_{1n}} < \frac{\frac{1}{2} \theta_0}{\theta_0} = \frac{1}{4}.$$

If $\theta \leq \bar{\theta}_\alpha$, then

$$\frac{\bar{\theta}_\alpha}{\theta} - 1 = \frac{\bar{\theta}_\alpha}{\theta - \bar{\theta}_\alpha + \theta_\alpha} - 1 < \frac{\bar{\theta}_\alpha}{-D_1 + \theta_\alpha} - 1 = \frac{D_1}{\bar{\theta}_\alpha - D_1} < \frac{D_1}{\theta_0 - \epsilon_{1n} - D_1} < \frac{\frac{1}{2} \theta_0}{\theta_0 - \frac{1}{2} \theta_0 - \frac{1}{2} \theta_0} = \frac{1}{2}.$$

Hence on the event $\mathcal{E}_1(\epsilon_{1n}, \alpha)$ and $\{|\theta - \bar{\theta}_\alpha| < D_1\}$, $\frac{\bar{\theta}_\alpha}{\theta} \in (\frac{3}{4}, \frac{3}{2})$. For any $u \in (\frac{3}{4}, \frac{3}{2})$, by simple calculus, we have

$$\left| \varphi(u) - \frac{1}{2} \left( \frac{1}{u} - 1 \right) \right|^2 \leq \frac{6}{5} \left| \frac{1}{u} - 1 \right|^3.$$

Let

$$g_n(t) = \frac{1}{n} \left[ \mathcal{L}(\alpha^{-2\nu}(\bar{\theta}_\alpha + \frac{1}{\sqrt{n}}), \alpha) - \mathcal{L}(\alpha^{-2\nu}, \alpha) \right] - \frac{t^2}{2n \bar{\theta}_\alpha^2}.$$
We continue to use the bound in (S.95) and (S.96) for $E$ on the event $g_{n}(t)$, then $\frac{1}{2} (1 - 1)^2 = \frac{t^2}{2n\theta_n^2}$. Thus, we can obtain that on the event $E_1(\epsilon_n, \alpha)$ and $t \in A_2$ (so that $|\theta - \tilde{\theta}_\alpha| < D_1$),

$$
|g_n(t)| = \left| \varphi \left( \frac{1}{1 + \frac{t}{\sqrt{n} \theta_n}} \right) - \frac{t^2}{2n\theta_n^2} \right| \leq \frac{12|\theta - \tilde{\theta}_\alpha|^2}{5\theta_n^2} + \frac{2.4D_1}{\theta_0 - \epsilon_n} \cdot \frac{n|\theta - \tilde{\theta}_\alpha|^2}{2n\theta_n^2} = \frac{2t^2}{5n\theta_n^2}.
$$  \hspace{1cm} (S.96)

Therefore, on the event $E_1(\epsilon_n, \alpha)$ with $0 < \epsilon_n < D_1 = \theta_0/4$,

$$
\int_{A_2} |g_n(t; \alpha)|dt \leq \int_{A_2} \exp \left\{ -\frac{n}{2} \varphi(\tilde{\theta}_\alpha/\theta) \right\} \frac{\pi(\tilde{\theta}_\alpha + \frac{\epsilon}{\sqrt{n}})}{\pi(\theta_0/\alpha)} dt + \int_{A_2} e^{-\frac{t^2}{4\theta_n^2}} dt
$$

$$
\leq \int_{A_2} \exp \left\{ -\frac{t^2}{40\theta_n^2} + \frac{n}{2} |g_n(t)| \right\} \frac{\pi(\tilde{\theta}_\alpha + \frac{\epsilon}{\sqrt{n}})}{\pi(\theta_0/\alpha)} dt + \int_{A_2} e^{-\frac{t^2}{4\theta_n^2}} dt
$$

\begin{enumerate}
  \item \quad \leq \sup_{|\theta - \tilde{\theta}_\alpha| < D_1} \frac{\pi(\theta/\alpha)}{\pi(\theta_0/\alpha)} \cdot \int_{A_2} \exp \left\{ -\frac{t^2}{20\theta_n^2} \right\} dt + \int_{A_2} e^{-\frac{t^2}{4\theta_n^2}} dt
  \item \quad \leq \sup_{\theta \in (\frac{1}{2}\theta_0, \frac{3}{2}\theta_0)} \frac{\pi(\theta/\alpha)}{\pi(\theta_0/\alpha)} \cdot \int_{|t| > s_n} \exp \left\{ -\frac{t^2}{20\theta_n^2} \right\} dt + \int_{|t| > s_n} e^{-\frac{t^2}{4\theta_n^2}} dt
  \item \quad \leq \sup_{\theta \in (\frac{1}{2}\theta_0, \frac{3}{2}\theta_0)} \frac{\pi(\theta/\alpha)}{\pi(\theta_0/\alpha)} \cdot \frac{2\sqrt{5\pi\theta_0} \exp \left( -\frac{s_n^2}{20\theta_n^2} \right) + 2\sqrt{\pi\theta_0} \exp \left( -\frac{s_n^2}{4\theta_0^2} \right) + \frac{5}{2} \sqrt{5\pi\theta_0} \exp \left( -\frac{4s_n^2}{125\theta_0^2} \right)}{\pi(\theta_0/\alpha)}.
  \end{enumerate}

where (i) is from the upper bound of $g_n(t)$ in (S.96); (ii) is based on the relation $|\theta - \theta_0| \leq |\theta - \tilde{\theta}_\alpha| + |\tilde{\theta}_\alpha - \theta_0| < D_1 + \epsilon_n < \theta_0/2$; (iii) follows from the normal tail inequality (S.91); (iv) is based on the relation $\tilde{\theta}_\alpha \leq \theta_0 + \epsilon_n < \theta_0 + D_1 < \frac{5}{4} \theta_0$.

Bound the third term in (S.86): We continue to use the bound in (S.95) and (S.96) for $t \in A_3$ on the event $E_1(\epsilon_n, \alpha)$ and obtain that

$$
|g_n(t)| \leq \frac{6|t|^3}{5n^{3/2}\theta_n^3} \leq \frac{6s_n^3}{5n^{3/2}\theta_n^3},
$$  \hspace{1cm} (S.98)
Therefore,
\[
\int_{A_3} |g_n(t; \alpha)| \, dt \\
= \int_{A_3} \left| \exp \left\{ -\frac{n}{2} \left( \bar{\theta}/\theta \right) \right\} \frac{\pi \left( \bar{\theta} + \frac{t}{\sqrt{n}} \right) \alpha}{\pi(\theta_0 | \alpha)} - e^{-\frac{t^2}{4n}} \right| \, dt \\
\leq \int_{A_3} \left| \exp \left( -\frac{t^2}{4\theta_0^2} - \frac{n}{2} g_n(t) \right) \frac{\pi \left( \bar{\theta} + \frac{t}{\sqrt{n}} \right) \alpha}{\pi(\theta_0 | \alpha)} - e^{-\frac{t^2}{4\theta_0^2}} \right| \, dt \\
\leq \int_{A_3} \left| \exp \left( -\frac{t^2}{4\theta_0^2} - \frac{n}{2} g_n(t) \right) - \exp \left( -\frac{t^2}{4\theta_0^2} \right) \right| \cdot \frac{\pi \left( \bar{\theta} + \frac{t}{\sqrt{n}} \right) \alpha}{\pi(\theta_0 | \alpha)} \, dt \\
+ \int_{A_3} e^{-\frac{t^2}{4n}} \left| \frac{\pi \left( \bar{\theta} + \frac{t}{\sqrt{n}} \right) \alpha}{\pi(\theta_0 | \alpha)} - 1 \right| \, dt \\
\leq \sup_{|t| < s_n} \left\{ \frac{t^2}{4} \left( \theta_0^{-2} - \bar{\theta}^{-2} \right) - \frac{n}{2} g_n(t) \right\} - 1 \cdot \sup_{|t| < s_n} \frac{\pi \left( \bar{\theta} + \frac{t}{\sqrt{n}} \right) \alpha}{\pi(\theta_0 | \alpha)} \\
+ \sup_{|t| < s_n} \left| \frac{\pi \left( \bar{\theta} + \frac{t}{\sqrt{n}} \right) \alpha}{\pi(\theta_0 | \alpha)} - 1 \right| \cdot \int_{|t| < s_n} e^{-\frac{t^2}{4n}} \, dt \\
\leq 2\sqrt{n} \theta_0 \cdot \sup_{|t| < s_n} \left\{ \frac{t^2}{4} \left( \theta_0^{-2} - \bar{\theta}^{-2} \right) - \frac{n}{2} g_n(t) \right\} - 1 \cdot \sup_{|t| < s_n} \frac{\pi \left( \bar{\theta} + \frac{t}{\sqrt{n}} \right) \alpha}{\pi(\theta_0 | \alpha)} \\
+ 2\sqrt{n} \theta_0 \cdot \sup_{|t| < s_n} \left| \frac{\pi \left( \bar{\theta} + \frac{t}{\sqrt{n}} \right) \alpha}{\pi(\theta_0 | \alpha)} - 1 \right|. 
\]  
(S.99)

For the first term in (S.99), we can choose \( \epsilon_{1n} \downarrow 0 \) as \( n \to \infty \) and \( \epsilon_{1n} < D_1 = \theta_0/4 \), such that on the event \( \mathcal{E}_1(\epsilon_{1n}, \alpha) \), for all \( |t| < s_n \), using (S.98), we have
\[
\left| \frac{t^2}{4} \left( \theta_0^{-2} - \bar{\theta}^{-2} \right) - \frac{n}{2} g_n(t) \right| \\
\leq \frac{s_{1n}^2}{4} \left| \frac{\bar{\theta}^2 - \theta_0^2}{\theta_0^2 \theta_0^2} \right| + \frac{n}{2} g_n(t) \\
\leq \frac{s_{1n}^2}{4} \left| \frac{\bar{\theta} + \theta_0}{\theta_0^2 \theta_0^2} \right| + \frac{n}{2} g_n(t) \\
\leq \frac{s_{1n}^2}{4} \left| \frac{\bar{\theta} + \theta_0}{\theta_0^2 \theta_0^2} \right| + \frac{n}{2} g_n(t) \\
< \frac{s_{1n}^2}{4} \theta_0^3 + \frac{2s_{1n}^3}{\sqrt{n} \theta_0^3}. 
\]  
(S.100)

We choose sufficiently large \( n \) that satisfies \( \epsilon_{1n} \leq \frac{\theta_0}{2s_{1n}^2} \) and \( n \geq \frac{16s_{1n}^6}{\theta_0^6} \), such that the upper bound in (S.100) is smaller than 1. Then we can apply the inequality \( |e^u - 1| \leq 2|u| \) for all \( |u| \leq 1 \) and obtain that
\[
\sup_{|t| < s_n} \left| \exp \left\{ \frac{t^2}{4} \left( \theta_0^{-2} - \bar{\theta}^{-2} \right) - \frac{n}{2} g_n(t) \right\} - 1 \right| \\
\leq 2 \left| \frac{t^2}{4} \left( \theta_0^{-2} - \bar{\theta}^{-2} \right) - \frac{n}{2} g_n(t) \right| < \frac{2s_{1n}^2}{\theta_0^3} + \frac{4s_{1n}^3}{\sqrt{n} \theta_0^3}. 
\]  
(S.101)
Furthermore, we can choose $n \geq \frac{16s_n^2}{\theta_0^2}$ such that for all $|t| < s_n$, on the event $\mathcal{E}_1(\epsilon_1, \alpha)$, $\bar{\theta}_n + t/\sqrt{n} \leq \theta_0 + \epsilon_1 + s_n/\sqrt{n} < \frac{3}{2}\theta_0$ and $\bar{\theta}_\alpha + t/\sqrt{n} > \theta_0 - \epsilon_1 > \frac{3}{4}\theta_0$. Then from Assumption (A.1) (ii), we have that on the interval $(\frac{3}{4}\theta_0, \frac{3}{2}\theta_0)$,

$$\sup_{|t| < s_n} \frac{\pi\left(\bar{\theta}_\alpha + \frac{t}{\sqrt{n}}\right)}{\pi(\theta_0)} \leq \sup_{\theta \in (\frac{3}{4}\theta_0, \frac{3}{2}\theta_0)} \frac{\pi(\theta/\alpha)}{\pi(\theta_0)}.$$  \hfill (S.102)

For the second term in (S.100), by Assumption (A.1) and the fact that $\epsilon_1 \to 0, s_n/\sqrt{n} \to 0$, we have that on the event $\mathcal{E}_1(\epsilon_1, \alpha)$, for all sufficiently large $n$,

$$\sup_{|t| < s_n} \left| \frac{\pi\left(\bar{\theta}_\alpha + \frac{t}{\sqrt{n}}\right)}{\pi(\theta_0)} \cdot 1 - \frac{\pi(\theta/\alpha)}{\pi(\theta_0)} \right| \leq \sup_{\theta \in (3\theta_0/4, 3\theta_0/2)} \left| \frac{\partial \log \pi(\theta/\alpha)}{\partial \theta} \right| \cdot \sup_{\theta \in (3\theta_0/4, 3\theta_0/2)} \frac{\pi(\theta/\alpha)}{\pi(\theta_0)} \cdot \sup_{|t| < s_n} \left| \bar{\theta}_\alpha + \frac{t}{\sqrt{n}} - \bar{\theta}_0 \right| \leq \sup_{\theta \in (3\theta_0/4, 3\theta_0/2)} \left| \frac{\partial \log \pi(\theta/\alpha)}{\partial \theta} \right| \sup_{\theta \in (3\theta_0/4, 3\theta_0/2)} \frac{\pi(\theta/\alpha)}{\pi(\theta_0)} \cdot \left( \epsilon_1 + \frac{s_n}{\sqrt{n}} \right).$$ \hfill (S.103)

Therefore, (S.99), (S.101), (S.102), and (S.103) together yield that on the event $\mathcal{E}_1(\epsilon_1, \alpha)$, with $\epsilon_1 \leq \min\left(\frac{\theta_0}{2s_n}, \frac{\theta_0}{4}\right)$ and $n \geq \max\left(\frac{16s_n^6}{\theta_0^2}, \frac{16s_n^6}{\theta_0^2}\right)$,

$$\int_{A_3} |\varrho_n(t; \alpha)| \, dt \leq 4\sqrt{\pi} \theta_0 \left( s_n^2 \epsilon_1 + \frac{2s_n^3}{\sqrt{n}} \right) \cdot \sup_{\theta \in (3\theta_0/4, 3\theta_0/2)} \frac{\pi(\theta)}{\pi(\theta_0)} + 2\frac{\sqrt{\pi}\theta_0}{s_n} \sup_{\theta \in (3\theta_0/4, 3\theta_0/2)} \left| \frac{\partial \log \pi(\theta/\alpha)}{\partial \theta} \right| \sup_{\theta \in (3\theta_0/4, 3\theta_0/2)} \frac{\pi(\theta/\alpha)}{\pi(\theta_0)} \cdot \left( \epsilon_1 + \frac{s_n}{\sqrt{n}} \right). \hfill (S.104)$$

Finally, we combine (S.94), (S.97), and (S.104) to conclude that on the event $\mathcal{E}_1(\epsilon_1, \alpha)$ with $D_1 = \theta_0/4, \epsilon_1 \leq \min\left(\frac{\theta_0}{2s_n}, \frac{\theta_0}{4}\right)$ and $n \geq \max\left(\frac{16s_n^6}{\theta_0^2}, \frac{16s_n^6}{\theta_0^2}\right)$,

$$\int_{\mathbb{R}} |\varrho_n(t; \alpha)| \, dt \leq 2\sqrt{\pi}\theta_0 e^{-n/64} + \frac{\sqrt{n}}{\pi(\theta_0)} e^{-0.007n} + \sup_{\theta \in (3\theta_0/4, 3\theta_0/2)} \frac{\pi(\theta)}{\pi(\theta_0)} \cdot 5 \cdot \frac{2\sqrt{\pi}\theta_0}{\pi(\theta_0)} e^\left(-\frac{4s_n^2}{125\theta_0^2}\right) + 2\sqrt{\pi}\theta_0 e^\left(-\frac{s_n^2}{4\theta_0^2}\right) + 4\sqrt{\pi} \theta_0 \left( s_n^2 \epsilon_1 + \frac{2s_n^3}{\sqrt{n}} \right) \cdot \sup_{\theta \in (3\theta_0/4, 3\theta_0/2)} \frac{\pi(\theta)}{\pi(\theta_0)} \cdot \left( \epsilon_1 + \frac{s_n}{\sqrt{n}} \right).$$ \hfill (S.105)

By adjusting the constants to be slightly larger, we obtain the bound in (33). \hfill \Box

The proof of Theorem 4 has used on the following lemmas.
Lemma S.16. For two nonnegative functions \(f\) and \(g\), define the integrals \(F = \int f\) and \(G = \int g\), then
\[
\frac{\left|f - g\right|}{\int f} \leq \frac{2\left|f - g\right|}{G}.
\]
Proof of Lemma S.16.
\[
\frac{\left|f - g\right|}{\int f} = \frac{\left|f - g\right|}{\int f} \leq \frac{\left|f - g\right|}{\int f} \leq \frac{\left|f - g\right|}{\int f} = \frac{2\left|f - g\right|}{G}.
\]

For two probability measures \(P\) and \(Q\), let \(d_{TV}(P, Q) = \sup_A |P(A) - Q(A)|\) be their total variation distance, where \(A\) is taken over all measurable sets.

Lemma S.17. For two univariate normal distributions \(\mathcal{N}(\mu_1, \sigma^2)\) and \(\mathcal{N}(\mu_2, \sigma^2)\) on \(\mathbb{R}\), their total variation distance is given by
\[
d_{TV}\{\mathcal{N}(\mu_1, \sigma^2), \mathcal{N}(\mu_2, \sigma^2)\} = 2\left\{2\Phi\left(\frac{\mu_1 - \mu_2}{2\sigma}\right) - 1\right\},
\]
where \(\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz\) is the standard normal cdf.

Proof of Lemma S.17. Let \(f_i(x)\) be the normal density of \(\mathcal{N}(\mu_i, \sigma^2)\), \(i = 1, 2\). Suppose that \(\mu_1 < \mu_2\) without loss of generality. Then it is clear that \(f_1(x) > f_2(x)\) if \(x < (\mu_1 + \mu_2)/2\) and \(f_1(x) < f_2(x)\) if \(x > (\mu_1 + \mu_2)/2\). Therefore,
\[
d_{TV}\{\mathcal{N}(\mu_1, \sigma^2), \mathcal{N}(\mu_2, \sigma^2)\}
= \int_{-\infty}^{(\mu_1 + \mu_2)/2} \{f_1(x) - f_2(x)\} dx + \int_{(\mu_1 + \mu_2)/2}^{+\infty} \{f_2(x) - f_1(x)\} dx
= \Phi\left(\frac{\mu_2 - \mu_1}{2\sigma}\right) - \Phi\left(\frac{\mu_1 - \mu_2}{2\sigma}\right) + 1 - \Phi\left(\frac{\mu_1 - \mu_2}{2\sigma}\right) - \Phi\left(\frac{\mu_2 - \mu_1}{2\sigma}\right)
= 2\left\{2\Phi\left(\frac{\mu_2 - \mu_1}{2\sigma}\right) - 1\right\}.
\]

S5 Proof of Propositions 1 and 2

Proof of Proposition 1. (i) Since \(\pi(\theta|\alpha) = \pi(\theta)\) and does not depend on \(\alpha\), we have that \(\frac{\partial \log \pi(\theta|\alpha)}{\partial \theta} = \pi'(\theta)/\pi(\theta)\). Since \(\pi(\theta) > 0\) and \(\pi'(\theta) = d\pi(\theta)/d\theta\) is continuous on \(\mathbb{R}_+\), (11) is satisfied for all sufficiently large \(n\) since
\[
\sup_{\alpha \in [\underline{\alpha}, \overline{\alpha}]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \left| \frac{\partial \log \pi(\theta|\alpha)}{\partial \theta} \right| \leq \sup_{\theta \in (\theta_0/2, 2\theta_0)} \pi'(\theta) < nC_{*1},
\]
for arbitrary \(C_{*1} > 0\).

\(\pi(\theta)\) has finite supremum and positive infimum on \((\theta_0/2, 2\theta_0)\). (12) is satisfied for all sufficiently large \(n\) since
\[
\sup_{\alpha \in [\underline{\alpha}, \overline{\alpha}]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \frac{\pi(\theta|\alpha)}{\pi(\theta)} \leq \sup_{\theta \in (\theta_0/2, 2\theta_0)} \pi(\theta) < nC_{*2},
\]
for arbitrary \(C_{*2} > 0\).
for arbitrary $C_{\pi,2} > 0$. Since $C_{\pi,1}$ and $C_{\pi,2}$ can be arbitrarily small, $C_{\pi,1} + C_{\pi,2} < 1/2$ is satisfied.

(13) is satisfied for all sufficiently large $n$ since $\pi(\theta_0) > 0$ and for all sufficiently large $n$,

$$\inf_{\alpha \in [\alpha, \alpha_n]} \log \pi(\theta_0 | \alpha) = \log \pi(\theta_0) > -nC_{\pi,3},$$

for arbitrarily small $C_{\pi,3} > 0$.

(ii) If $\pi(\alpha)$ is supported on a compact interval $[\alpha_1, \alpha_2]$, then all $\sup_{\alpha \in [\alpha, \alpha_n]}$ can be replaced by $\sup_{\alpha \in [\alpha_1, \alpha_2]}$. Based on the conditions, for all sufficiently large $n$,

$$\sup_{\alpha \in [\alpha_1, \alpha_2]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \left| \frac{\partial \log \pi(\theta | \alpha)}{\partial \theta} \right| < nC_{\pi,1},$$

for arbitrary $C_{\pi,1} > 0$.

Since $\pi(\theta | \alpha) > 0$ for all $(\theta, \alpha) \in \mathbb{R}^2_+$, for all sufficiently large $n$,

$$\sup_{\alpha \in [\alpha_1, \alpha_2]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \left| \frac{\partial \log \pi(\theta | \alpha)}{\partial \theta} \right| < nC_{\pi,2},$$

for arbitrary $C_{\pi,2} > 0$. Since $C_{\pi,1}$ and $C_{\pi,2}$ can be arbitrarily small, $C_{\pi,1} + C_{\pi,2} < 1/2$ is satisfied.

Since $\pi(\theta | \alpha) > 0$ is continuous in $\alpha \in \mathbb{R}_+$, for all sufficiently large $n$,

$$\inf_{\alpha \in [\alpha_1, \alpha_2]} \log \pi(\theta_0 | \alpha) = \inf_{\alpha \in [\alpha_1, \alpha_2]} \log \pi(\theta_0 | \alpha) > -nC_{\pi,3},$$

for arbitrarily small $C_{\pi,3} > 0$.

(iii) If the prior of $\sigma^2$ is independent of $\alpha$, then by the relation $\theta = \alpha^2 \sigma^{2\nu}$, the prior of $\theta$ given $\alpha$ is $\pi(\theta | \alpha) = \pi_{\sigma^2}(\theta/\alpha^{2\nu})/\alpha^{2\nu}$, where we use $\pi_{\sigma^2}(-)$ to denote the prior density of $\sigma^2$. Therefore, $
\frac{\partial \log \pi(\theta | \alpha)}{\partial \theta} = \frac{\pi'_{\sigma^2}(\theta/\alpha^{2\nu})}{\alpha^{2\nu} \pi_{\sigma^2}(\theta/\alpha^{2\nu})}$. For the transformed beta family density, the derivative is

$$\pi'_{\sigma^2}(\sigma^2) = \frac{\Gamma(\gamma_1 + \gamma_2)}{\Gamma(\gamma_1) \Gamma(\gamma_2)} \left\{ \frac{\alpha^2}{\nu} \right\}^{\gamma_2/\gamma - 2} \left[ \gamma_2 - \gamma - (\gamma_1 + \gamma) \left( \frac{\alpha^2}{\nu} \right)^{1/\gamma} \right].$$

Therefore, for all sufficiently large $n$,

$$\sup_{\alpha \in [\alpha, \alpha_n]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \left| \frac{\partial \log \pi(\theta | \alpha)}{\partial \theta} \right| \leq \sup_{\alpha \in [\alpha, \alpha_n]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \left| \frac{\gamma_2 - \gamma - (\gamma_1 + \gamma) \left( \frac{\theta}{b \alpha^{2\nu}} \right)^{1/\gamma}}{\alpha^{2\nu} \left( \frac{\theta}{b \alpha^{2\nu}} \right)} [1 + \left( \frac{\theta}{b \alpha^{2\nu}} \right)^{1/\gamma}] \right|$$

$$\leq \frac{2b |\gamma_2 - \gamma|}{\theta_0} + \frac{2b (\gamma_1 + \gamma)}{\theta_0} < nC_{\pi,1},$$

for arbitrary $C_{\pi,1} > 0$.  

$$\sup_{\alpha \in [\alpha, \alpha_n]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \frac{\pi(\theta | \alpha)}{\pi(\theta_0 | \alpha)} \leq \sup_{\alpha \in [\alpha, \alpha_n]} \sup_{\theta \in (\theta_0/2, 2\theta_0)} \left( \frac{\theta}{\theta_0} \right)^{\gamma_2/\gamma - 1} \left[ \frac{b^{1/\gamma} \alpha^{2\nu/\gamma} + \theta_0^{1/\gamma}}{b^{1/\gamma} \alpha^{2\nu/\gamma} + \theta^{1/\gamma}} \right]^{\gamma_1 + \gamma_2}$$

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From (S.106) and (S.107), we can see that (18) will be satisfied if for all sufficiently large \( n \),

\[
\left\{ \left( \frac{\theta}{\theta_0} \right)^{\gamma_2/n^{\gamma-1}}, \left( \frac{\theta}{\theta_0} \right)^{-\gamma_1/n^{\gamma-1}} \right\} < n^{C_{\pi,2}},
\]

for arbitrary \( C_{\pi,2} > 0 \). Since \( C_{\pi,1} \) and \( C_{\pi,2} \) can be arbitrarily small, \( C_{\pi,1} + C_{\pi,2} < 1/2 \) is satisfied.

\[
\inf_{\alpha \in [\pi_n, \pi_n]} \log \pi(\theta_0 | \alpha) \\
\geq \inf_{\alpha \in [\pi_n, \pi_n]} \left\{ -2\nu \log \alpha - \log \frac{\Gamma(\gamma_1 + \gamma_2)}{b\gamma \Gamma(\gamma_1)\Gamma(\gamma_2)} + \left( \frac{\gamma_2}{\gamma} - 1 \right) \log \left( \frac{\theta_0}{b\alpha^2\nu} \right) \\
- (\gamma_1 + \gamma_2) \log \left[ 1 + \left( \frac{\theta_0}{b\alpha^2\nu} \right)^{1/\gamma} \right] \right\} \\
\geq -2\nu \pi \log n - \log \frac{\Gamma(\gamma_1 + \gamma_2)}{b\gamma \Gamma(\gamma_1)\Gamma(\gamma_2)} + \left( \frac{\gamma_2}{\gamma} - 1 \right) \log \left( \frac{\theta_0}{b} \right) \\
- \left| \frac{\gamma_2}{\gamma} - 1 \right| \cdot 2\nu(\pi + \kappa) \log n - (\gamma_1 + \gamma_2) \log \left[ 1 + \left( \frac{\theta_0 n^{2\nu\pi}}{b} \right)^{1/\gamma} \right] \\
\geq -\log n > -n^{C_{\pi,3}},
\]

for arbitrarily small \( C_{\pi,3} > 0 \).

**Proof of Proposition [2]**. We will verify only (18) with \( 0 < \pi < (\nu + d/2)\pi \) for each conditions in the list. The verification of (19) with \( 0 < \pi < (\nu + d/2)\pi \) is similar and omitted.

For \( p(\alpha) \) that satisfies (i), we use the change of variable \( u = \alpha^{1/\delta_1} \) to obtain that

\[
\int_{\pi_n}^{\infty} \alpha^{n(\nu+d/2)} p(\alpha) \, d\alpha \\
\leq \int_{\pi_n}^{\infty} \alpha^{n(\nu+d/2)} \exp \left( -\alpha^{\delta_1} \right) \, d\alpha \\
\leq \frac{1}{\delta_1} \int_{\pi_n^{\delta_1}}^{\infty} u^{n(\nu+d/2)+1}/\delta_1 \cdot e^{-u} \, du < \frac{1}{\delta_1} \int_{0}^{\infty} u^{n(\nu+d/2)+1}/\delta_1 \cdot e^{-u} \, du \\
= \frac{1}{\delta_1} \Gamma \left( \delta_1^{-1} \{ n(\nu + d/2) + 1 \} \right),
\]

where \( \Gamma(x) = \int_{0}^{\infty} u^{x-1}e^{-u} \, du \) is the gamma function. Using the Stirling’s approximation for gamma functions (\( \Gamma(x) < 2\sqrt{2\pi x}(x/e)^x \) for any \( x > 0 \)), we have that for sufficiently large \( n \),

\[
\Gamma \left( \delta_1^{-1} \{ n(\nu + d/2) + 1 \} \right) < 2\sqrt{2\pi \delta_1^{-1} \{ n(\nu + d/2) + 1 \}} \left( e^{-\delta_1^{-1} \{ n(\nu + d/2) + 1 \}} \right)^{\delta_1^{-1} \{ n(\nu + d/2) + 1 \}}.
\]

From (S.106) and (S.107), we can see that (18) will be satisfied if for all sufficiently large \( n \),

\[
2\delta_1^{-1} \sqrt{2\pi \delta_1^{-1} \{ n(\nu + d/2) + 1 \}} \left( e^{-\delta_1^{-1} \{ n(\nu + d/2) + 1 \}} \right)^{\delta_1^{-1} \{ n(\nu + d/2) + 1 \}} < \exp(\pi_\alpha n \log n).
\]

A comparison of the orders in \( n \) on both sides immediately shows that this relation holds for all sufficiently large \( n \), as long as \( \delta_1^{-1}(\nu + d/2) < \pi_\alpha \). Since \( \pi_\alpha \) can be chosen as any constant between 0 and \((\nu + d/2)\pi \), it suffices to have \( \delta_1^{-1}(\nu + d/2) < (\nu + d/2)\pi \), or equivalently \( \delta_1 > 1/\pi \).
For $p(\alpha)$ that satisfies (ii), we use the change of variable $u = n^{\delta_2} \alpha$ and the Stirling’s approximation to obtain that

$$\int_{\pi}^{\infty} \frac{\alpha^{\nu+d/2}}{\pi} p(\alpha) d\alpha \leq \int_{\pi}^{\infty} \alpha^{\nu+d/2} n^{\delta_1} \exp\left(-n^{\delta_2} \alpha\right) d\alpha$$

$$\leq n^{\delta_1-\delta_2} n^{\nu+d/2+1} \int_{\pi n}^{\infty} u^{\nu+d/2} e^{-u} du$$

$$< n^{\delta_1-\delta_2} n^{\nu+d/2+1} \int_{0}^{\pi n} u^{\nu+d/2} e^{-u} du$$

$$= n^{\delta_1-\delta_2} n^{\nu+d/2+1} \Gamma(n(\nu + d/2) + 1)$$

$$\leq n^{\delta_1-\delta_2} n^{(\nu+d/2)+1} \cdot 2\sqrt{2\pi n(\nu + d/2) + 1}$$

$$\times \left(e^{-1} n(\nu + d/2) + 1\right)^{n(\nu+d/2)+1}.$$

From the last display, (18) will be satisfied if for all sufficiently large $n$,

$$n^{\delta_1-\delta_2} n^{(\nu+d/2)+1} \cdot 2\sqrt{2\pi n(\nu + d/2) + 1}$$

$$\times \left(e^{-1} n(\nu + d/2) + 1\right)^{n(\nu+d/2)+1} < \exp(\pi n \log n).$$

A comparison of the orders in $n$ on both sides immediately shows that this relation holds for all sufficiently large $n$, as long as $-\delta_2(\nu + d/2) + (\nu + d/2) < \pi$. Since $\pi$ can be chosen as any constant between 0 and $(\nu + d/2)\pi$, it suffices to have $(1 - \delta_2)(\nu + d/2) < (\nu + d/2)\pi$, or equivalently $\delta_2 > 1 - \pi$. \(\square\)

### S6 Proof of Theorem \(3\)

Recall that for the 1-dimensional Ornstein-Uhlenbeck process $X$ on the grid $s_i = i/n$, $i = 1, \ldots, n$, we have defined

$$A_1 = \sum_{i=2}^{n-1} X(s_i)^2, \quad A_2 = \sum_{i=1}^{n-1} X(s_i)X(s_{i+1}), \quad A_3 = \sum_{i=1}^{n} X(s_i)^2.$$

**Lemma S.18.** Under the conditions of Theorem \(3\) we have the following results:

(i) $A_1 + A_3 - 2A_2 > 0$ a.s. $P(\sigma_0^2, \alpha_0)$;

(ii) $A_1 + A_3 - 2A_2 \asymp 1$ as $n \to \infty$ a.s. $P(\sigma_0^2, \alpha_0)$;

(iii) $|A_1 - A_2| \leq \log n$ as $n \to \infty$ a.s. $P(\sigma_0^2, \alpha_0)$;

(iv) $A_1/n \asymp 1$ as $n \to \infty$ a.s. $P(\sigma_0^2, \alpha_0)$;

(v) $|u_*| = n|A_1 - A_2|/A_1 = O_p(1)$ as $n \to \infty$ in $P(\sigma_0^2, \alpha_0)$-probability, and $|u_*| \leq \log^2 n$ as $n \to \infty$ a.s. $P(\sigma_0^2, \alpha_0)$;

(vi) $v_* = n(A_1 - 2A_2 + A_3)/A_1 \asymp 1$ as $n \to \infty$ a.s. $P(\sigma_0^2, \alpha_0)$;

(vii) Uniformly over all $\alpha \in [0, n^{1/6}]$,

$$\left| \frac{A_1 e^{-2\alpha/n} - 2A_2 e^{-\alpha/n} + A_3}{A_1 \left(\frac{n}{n} - \frac{A_1 - A_3}{A_1}\right)^2 + A_1 + A_3 - 2A_2} - \frac{\left(A_1 \left(\frac{n}{n} - \frac{A_1 - A_3}{A_1}\right)^2 + A_1 + A_3 - 2A_2\right)}{A_1 \left(\frac{n}{n} - \frac{A_1 - A_3}{A_1}\right)^2 + A_1 + A_3 - 2A_2} \right| = O\left(n^{-3/2}\right),$$

as $n \to \infty$ a.s. $P(\sigma_0^2, \alpha_0)$;
(viii) Uniformly over all \( \alpha \in [0, n^{1/6}] \),

\[
\sqrt{1 - e^{-2\alpha/n}} = \sqrt{\frac{2\alpha}{n} \left[ 1 + O(n^{-5/12}) \right]},
\]

as \( n \to \infty \) a.s. \( P(\sigma_0^2, \alpha_0) \).

**Proof of Lemma S.18**

(i) By definition, \( A_1 + A_3 - 2A_2 = \sum_{i=1}^{n-1} [X(s_{i+1}) - X(s_i)]^2 > 0 \) almost surely \( P(\sigma_0^2, \alpha_0) \).

(ii) Let \( W_{i,n} = \frac{[X(s_i) - e^{-\alpha_0/n}X(s_{i-1})]}{\sqrt{\sigma_0^2(1 - e^{-2\alpha_0/n})}} \) for \( i = 2, \ldots, n \). Then by the property of Ornstein-Uhlenbeck process, \( W_{i,n} \)’s are i.i.d. \( \mathcal{N}(0, 1) \) random variables. \( W_{i,n} \) is independent of \( X(s_{i-1}) \), and \( X(s_i) = e^{-\alpha_0/n}X(s_{i-1}) + \sqrt{\sigma_0^2(1 - e^{-2\alpha_0/n})}W_{i,n} \), for \( i = 2, \ldots, n \). We can derive that

\[
A_1 + A_3 - 2A_2 = \sum_{i=1}^{n-1} \left[ X(s_{i+1}) - e^{-\alpha_0/n}X(s_i) - (1 - e^{-\alpha_0/n})X(s_i) \right]^2
\]

\[
= \sum_{i=1}^{n-1} \left[ X(s_{i+1}) - e^{-\alpha_0/n}X(s_i) \right]^2 + \sum_{i=1}^{n-1} \left( 1 - e^{-\alpha_0/n} \right)^2 X(s_i)^2
\]

\[+ 2 \sum_{i=1}^{n-1} \left( 1 - e^{-\alpha_0/n} \right)X(s_i) \left[ X(s_{i+1}) - e^{-\alpha_0/n}X(s_i) \right]. \tag{S.108}
\]

The first term in (S.108) is

\[
\sum_{i=1}^{n-1} \left[ X(s_{i+1}) - e^{-\alpha_0/n}X(s_i) \right]^2 = \sum_{i=2}^{n} \sigma_0^2(1 - e^{-2\alpha_0/n}) W_{i,n}^2 = \frac{\sigma_0^2 \alpha_0 [1 + o(1)]}{n} \sum_{i=2}^{n} W_{i,n}^2,
\]

using a Taylor expansion of \( 1 - e^{-x} \) around \( x = 0 \). Since \( W_{i,n} \)’s are i.i.d. \( \mathcal{N}(0, 1) \) random variables, we have that \( n^{-1} \sum_{i=2}^{n} W_{i,n}^2 \to 1 \) as \( n \to \infty \) almost surely \( P(\sigma_0^2, \alpha_0) \).

The second term in (S.108) is

\[
\sum_{i=1}^{n-1} \left( 1 - e^{-\alpha_0/n} \right)^2 X(s_i)^2 \leq \frac{\alpha_0^2}{n} \sup_{s \in [0, 1]} X(s)^2.
\]

For the Ornstein-Uhlenbeck process, \( \sup_{s \in [0, 1]} X(s)^2 < \infty \) almost surely \( P(\sigma_0^2, \alpha_0) \). Therefore, \( \sum_{i=1}^{n-1} (1 - e^{-\alpha_0/n})^2 X(s_i)^2 = O(1/n) \) almost surely \( P(\sigma_0^2, \alpha_0) \).

The third term in (S.108)

\[
2 \sum_{i=1}^{n-1} \left( 1 - e^{-\alpha_0/n} \right)X(s_i) \left[ X(s_{i+1}) - e^{-\alpha_0/n}X(s_i) \right]
\]

\[
= 2 \sum_{i=1}^{n-1} \sqrt{\sigma_0^2(1 - e^{-2\alpha_0/n})(1 - e^{-\alpha_0/n})}X(s_i)W_{i+1,n}
\]

\[= \frac{2\sqrt{2}\sigma_0 \alpha_0^{3/2}[1 + o(1)] \sum_{i=1}^{n-1} X(s_i)W_{i+1,n}}{n^{3/2}}
\]

\[\leq \frac{2\sqrt{2}\sigma_0 \alpha_0^{3/2}[1 + o(1)] \sum_{i=1}^{n-1} X(s_i)^2 W_{i+1,n}^2}{n}
\]

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\[
\left| A_1 - A_2 \right| \leq \frac{1}{2} |A_1 + A_3 - 2A_2| + \frac{1}{2} [X(s_1^2) + X(s_n^2)].
\]

Since \( X(s_1) \sim \mathcal{N}(0, \sigma_0^2) \), \( X(s_n) \sim \mathcal{N}(0, \sigma_0^2) \), we have \( X(s_1) = O_p(1) \) and \( X(s_n) = O_p(1) \) in \( P(\sigma_0, \alpha_0) \)-probability. By the Borel-Cantelli lemma, \( X(s_1) \leq \log n \) and \( X(s_n) \leq \log n \) as \( n \to \infty \) almost surely \( P(\sigma_0, \alpha_0) \). Then the conclusion follows by combining these relations with Part (ii).

(iii)

\[
\left| A_1 - A_2 \right| \leq \frac{1}{2} |A_1 + A_3 - 2A_2| + \frac{1}{2} [X(s_1^2) + X(s_n^2)].
\]

Now if we replace \( 1 - e^{-\alpha/n} \) with \( \alpha/n \) for all \( \alpha \in [0, n^{1/6}] \), then the difference would be

\[
\begin{align*}
& \left| (A_1 e^{-\alpha/n} - 2A_2 e^{-\alpha/n} + A_3) - \left[ A_1 \left( \frac{\alpha}{n} - \frac{A_1 - A_2}{A_1} \right)^2 + A_1 + A_3 - 2A_2 \right] \right| \\
& = A_1 \left[ (1 - e^{-\alpha/n}) - \frac{A_1 - A_2}{A_1} \right]^2 - \left( \frac{\alpha}{n} - \frac{A_1 - A_2}{A_1} \right)^2 \\
& = A_1 \left| 1 - e^{-\alpha/n} + \frac{\alpha}{n} + \frac{2(A_1 - A_2)}{A_1} \right| \cdot |1 - e^{-\alpha/n} - \frac{\alpha}{n}| \\
& \leq \left( A_1 \frac{n^{1/6}}{n} + |A_1 - A_2| \right) \frac{n^{1/3}}{n^2},
\end{align*}
\]

where in the last inequality, we used the fact that \( 1 - e^{-x} \leq x \) and \( |x - (1 - e^{-x})| \leq x^2/2 \) for all \( x > 0 \). This implies that

\[
\begin{align*}
& \left| (A_1 e^{-\alpha/n} - 2A_2 e^{-\alpha/n} + A_3) - \left[ A_1 \left( \frac{\alpha}{n} - \frac{A_1 - A_2}{A_1} \right)^2 + A_1 + A_3 - 2A_2 \right] \right| \\
& \leq \frac{n^{1/3}}{n^2} \cdot \left( A_1 \frac{n^{1/6}}{n} + |A_1 - A_2| \right).
\end{align*}
\]
Using Parts (ii), (iii), and (iv) together with the definition of \( \bar{\pi}_n \), we observe that

\[
\frac{n^{1/3}}{n^2} \cdot \left( \frac{(A_{n^{1/6}} + |A_1 - A_2|)}{A_1 + A_3 - 2A_2} \right) \leq O \left( n^{-3/2} \right),
\]

as \( n \to \infty \) almost surely \( P_{(\sigma_0^2, a_0)} \). Hence the conclusion follows.

(viii) For \( \alpha \in [0, n^{1/6}] \), \( \alpha / n \leq n^{-5/6} \to 0 \) as \( n \to \infty \). With the Taylor expansion of \( 1 - e^{-x} \) around \( x = 0 \), as \( n \to \infty \) almost surely \( P_{(\sigma_0^2, a_0)} \),

\[
\sqrt{1 - e^{-2\alpha/n}} = \frac{2\alpha}{n} \left[ 1 + O(n^{-5/6}) \right] = \frac{2\alpha}{n} \left[ 1 + O(n^{-5/12}) \right]
\]

and the o(1) term is uniformly over all \( \alpha \in [0, n^{1/6}] \). \( \square \)

\textbf{Lemma S.19.} Define a normalized log profile likelihood function

\[
\tilde{L}_n(\alpha) = L_n(\alpha^{-2\nu} \bar{\theta}, \alpha) + \frac{n}{2} \log(A_1 + A_3 - 2A_2) + \frac{1}{2} \log \frac{n}{2}
\]

\[
= -\frac{n}{2} \log \left( A_1 e^{-2\alpha/n} - 2A_2 e^{-\alpha/n} + A_3 \right) + \frac{1}{2} \log(1 - e^{-2\alpha/n})
\]

\[
+ \frac{n}{2} \log(A_1 + A_3 - 2A_2) + \frac{1}{2} \log \frac{n}{2}.
\]

Then, under the setup of Theorem 3 and Assumptions (A.1), (A.2), and (A.3), the integrals

\[
\int_0^\infty \exp \left\{ \tilde{L}_n(\alpha) \right\} \pi(\theta_0 | \alpha) \pi(\alpha) d\alpha, \text{ and } \int_0^\infty \exp \left\{ \tilde{L}_n(\alpha) \right\} \pi(\alpha) d\alpha
\]

are lower bounded by positive constants in \( P_{(\sigma_0^2, a_0)} \)-probability. Furthermore, the following convergence relations hold

\[
\int_0^\infty \exp \left\{ \tilde{L}_n(\alpha) \right\} - \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} \pi(\theta_0 | \alpha) \pi(\alpha) d\alpha \to 0, \quad (S.110)
\]

\[
\int_0^\infty \exp \left\{ \tilde{L}_n(\alpha) \right\} - \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} \pi(\alpha) d\alpha \to 0, \quad (S.111)
\]

\[
\int_0^\infty |\tilde{\pi}(\alpha | X_n) - \pi_*(\alpha | X_n)| d\alpha \to 0, \quad (S.112)
\]

as \( n \to \infty \) in \( P_{(\sigma_0^2, a_0)} \)-probability, for \( \tilde{\pi}(\alpha | X_n) \) given in Theorem 2 and \( \pi_*(\alpha | X_n) \) given in Theorem 3.

\textbf{Proof of Lemma S.19.} We first prove the convergence in \( P_{(\sigma_0^2, a_0)} \)-probability in (S.110), and that \( \int_0^\infty \exp \{ \tilde{L}_n(\alpha) \} \pi(\theta_0 | \alpha) \pi(\alpha) d\alpha \) is lower bounded by positive constant in \( P_{(\sigma_0^2, a_0)} \)-probability. Note that the only difference between (S.110) and (S.111) is that \( \pi(\theta_0 | \alpha) \pi(\alpha) \) is replaced by \( \pi(\alpha) \). The integral condition (22) in Assumption (A.3) guarantees that in the following derivation, all \( \pi(\theta_0 | \alpha) \pi(\alpha) \) can be replaced by \( \pi(\alpha) \). Therefore, in the derivation below, we will only prove for the integrals involving \( \pi(\theta_0 | \alpha) \pi(\alpha) \), and the proof of (S.111) and lower boundedness of \( \int_0^\infty \exp \{ \tilde{L}_n(\alpha) \} \pi(\alpha) d\alpha \) follow similarly.

Define the following quantities

\[
N_1 = \int_{(0, n^{1/6})} \exp \left\{ \tilde{L}_n(\alpha) \right\} - \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} \pi(\theta_0 | \alpha) \pi(\alpha) d\alpha,
\]

\[
N_2 = \int_{n^{1/6}}^\infty \exp \left\{ \tilde{L}_n(\alpha) \right\} \pi(\theta_0 | \alpha) \pi(\alpha) d\alpha,
\]

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\[ N_3 = \int_{n^{1/6}}^{\infty} \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} \pi(\theta_0|\alpha)\pi(\alpha)d\alpha, \]

\[ D = \int_{0}^{\infty} \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} \pi(\theta_0|\alpha)\pi(\alpha)d\alpha. \]

Uniformly for all \( \alpha \in [0, n^{1/6}] \), as \( n \to \infty \) almost surely \( P(\sigma_0^2, \alpha_0) \), we have that

\[ \left| \exp \left\{ \tilde{L}_s(\alpha) \right\} - \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} \right| \]

\[ = \frac{(A_1 + A_3 - 2A_2)^{n/2} \sqrt{\frac{n}{2}[1 - e^{-2\alpha/n}]} (A_1e^{-2\alpha/n} - 2A_2e^{-\alpha/n} + A_3)^{n/2} - \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} )}{(A_1 + A_3 - 2A_2)^{n/2} \sqrt{\alpha} \left[ 1 + O(n^{-5/12}) \right]} \]

\[ \leq O(n^{-5/12}) \cdot \sqrt{\alpha} \left[ 1 + \frac{1}{n} \left( \frac{\alpha - u_s)^2}{v_s} \right)^{n/2} + \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} \right] \]

\[ \leq O(n^{-5/36} + \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} \cdot \exp \left\{ \frac{n}{2} \left( \frac{(\alpha - u_s)^2}{nv_s} \right)^{1/6} \right\} \right] \]

\[ \leq O(n^{-5/36} + \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_s)^2}{2v_s} \right\} \cdot \frac{n}{2} \left( \frac{(\alpha - u_s)^2}{nv_s} \right)^{11/6} \]

\[ \leq O(n^{-5/36}). \]

In the derivations above, (i) follows from Lemma \( \text{S.18} \) (vii) and (viii); (ii) follows from the fact that \( [1 + O(n^{-3/2})]^{-n/2} = 1 + O(n^{-1/2}) \) and the definitions of \( u_s \) and \( v_s \); (iii) follows from the triangle inequality; (iv) follows from Lemma \( \text{S.18} \) (v), (vi), and the fact that \( \alpha \in [0, n^{1/6}] \), hence \( (\alpha - u_s)^2/(2v_s) \leq n^{1/3} \), and the inequality \( 0 < x - \log(1 + x) \leq x^{1/6} \) for all \( x > 0 \); (v) follows from the inequality \( e^x - 1 \leq 2x \) for \( x \in (0, 1) \) and for sufficiently large \( n \); (vi) follows from a comparison of orders. As a result, we have that there exists a constant \( C_1 > 0 \) such that as \( n \to \infty \) almost surely \( P(\sigma_0^2, \alpha_0) \),

\[ N_1 \leq C_1 n^{-5/36} \int_{0}^{\infty} \pi(\theta_0|\alpha)\pi(\alpha)d\alpha. \quad (\text{S.113}) \]

For \( N_2 \), we have that

\[ N_2 = \int_{n^{1/6}}^{\infty} \exp \left\{ \tilde{L}_s(\alpha) \right\} \pi(\theta_0|\alpha)\pi(\alpha)d\alpha \]

\[ = \int_{n^{1/6}}^{\infty} (A_1 + A_3 - 2A_2)^{n/2} \sqrt{\frac{n}{2}[1 - e^{-2\alpha/n}]} (A_1e^{-2\alpha/n} - 2A_2e^{-\alpha/n} + A_3)^{n/2} \pi(\theta_0|\alpha)\pi(\alpha)d\alpha. \]
which follows from (S.110) and (S.116). Hence the proof for Lemma S.16 is complete.

\[ \inf_{C_0, \text{ as } n \to \infty} \text{inf}_{[1,2]} \quad \]

The proof follows the same process

\[ \text{Proof of (38):} \]

For \( N_3 \), by Lemma S.18 (v) and (vi), we have that for some constant \( C_2 > 0 \), as \( n \to \infty \) almost surely \( P(\sigma_{\alpha_0}^2) \).

\[ N_3 = \int_{n^{1/6}}^\infty \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_*)^2}{2v_*} \right\} \pi(\theta_0) \pi(\alpha) d\alpha \]

\[ \leq \exp \left\{ -\frac{(n^{1/6} - u)^2}{2v_*} \right\} \int_{n^{1/6}}^\infty \sqrt{\alpha} \pi(\theta_0) \pi(\alpha) d\alpha \]

\[ \leq \exp(-C_2n^{1/3}) \int_{n^{1/6}}^\infty \sqrt{\alpha} \pi(\theta_0) \pi(\alpha) d\alpha \to 0. \]  

Hence, (S.110) follows by combining (S.113), (S.114), and (S.115) using the triangle inequality.

To show the lower boundedness of \( D \), by Lemma S.18 (v), for any given \( \epsilon > 0 \) and \( C_3 > 0 \), as \( n \to \infty \), the \( P(\sigma_{\alpha_0}^2) \) probability \( \Pr(|u_*| \leq C_3) > 1 - \epsilon \). By Lemma S.18 (vi), \( v_* > C_4 \) for some \( C_4 > 0 \) as \( n \to \infty \) almost surely \( P(\sigma_{\alpha_0}^2) \). By Assumptions (A.1) and (A.3'), \( \inf_{\alpha \in [1,2]} \pi(\theta_0) \pi(\alpha) \geq C_5 > 0 \) for some constant \( C_5 \). This implies that there exists a constant \( C_6 > 0 \), such that with \( P(\sigma_{\alpha_0}^2) \) probability at least \( 1 - \epsilon \),

\[ D \geq \int_1^2 \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_*)^2}{2v_*} \right\} \pi(\theta_0) \pi(\alpha) d\alpha \]

\[ \geq C_5 \int_1^2 \exp \left\{ -\frac{(\alpha - u_*)^2}{2v_*} \right\} d\alpha \]

\[ \geq C_5 \int_1^2 \exp \left\{ -\frac{(a + C_3)^2}{2C_4} \right\} d\alpha \equiv C_6 > 0. \]

Based on the definitions of \( \bar{\pi}(\alpha|X_n) \) and \( \pi_*(\alpha|X_n) \), by Lemma S.16, the convergence in (S.112) holds true if the following relation holds as \( n \to \infty \), in \( P(\sigma_{\alpha_0}^2) \) probability,

\[ \frac{\int_0^\infty |\bar{\pi}(\alpha) - \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_*)^2}{2v_*} \right\} \pi(\theta_0) \pi(\alpha) d\alpha}{\int_0^\infty \sqrt{\alpha} \exp \left\{ -\frac{(\alpha - u_*)^2}{2v_*} \right\} \pi(\theta_0) \pi(\alpha) d\alpha} \to 0, \]  

which follows from (S.110) and (S.116). Hence the proof for Lemma S.16 is complete.

\[ \text{Proof of Theorem 3:} \]

We first prove the convergence in (23). The proof follows the same process in the proof of Theorem 2 with some differences due to the new Assumption (A.3'). The conclusion of Theorem 2 is proved by showing (38) and (39). We show them respectively under the new Assumption (A.3').

Proof of (38):

Using the same notation as in the proof of Theorem 2, we define \( N_1 \), \( N_2 \), \( N_3 \), and \( D \) as in (43) and (42). The first step of showing \( N_1/D \to 0 \) is exactly the same as in the proof of Theorem 2.
since this step only relies on Assumptions \([A.1]\) and \([A.2]\) which are both assumed in Theorem 3 as well. The main differences lie in the next two steps of showing \(N_2/D \to 0\) and \(N_3/D \to 0\).

Proof of \(N_2/D \to 0\):

Using the upper bound of \(N_2\) in (47), together with the definition of \(D\) in (42), we have that

\[
\frac{N_2}{D} \leq 2 \int_0^{2\pi} e^{L_n(\alpha-2\theta_0,\alpha)} \pi(\alpha) \, d\alpha + \frac{4\theta_0 \sqrt{n}}{\sqrt{n}} \int_0^{2\pi} e^{L_n(\alpha-2\theta_0,\alpha)} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha
\]

\[
= \sqrt{n} \int_0^{2\pi} \exp \{ \tilde{L}_n(\alpha) \} \pi(\alpha) \, d\alpha + \int_0^{2\pi} \exp \{ \tilde{L}_n(\alpha) \} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha + \int_0^{2\pi} \exp \{ \tilde{L}_n(\alpha) \} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha,
\]

(S.118)

where \(\tilde{L}_n(\alpha)\) is the normalized log profile likelihood defined in (S.109).

We now show the first term in (S.118) converges to zero in probability. For the numerator, by the definition of \(\tilde{L}_n(\alpha)\), we have that

\[
\sqrt{n} \int_0^{2\pi} \exp \{ \tilde{L}_n(\alpha) \} \pi(\alpha) \, d\alpha
\]

\[
= \sqrt{n} \int_0^{2\pi} \frac{(A_1 - 2A_2 + A_3)^{n/2}}{(A_1 e^{-2\theta_0/n} - 2A_2 e^{-\alpha/n} + A_3)^{n/2}} \sqrt{n/2} \left( 1 - e^{-2\alpha/n} \right) \pi(\alpha) \, d\alpha
\]

\[
= \sqrt{n} \int_0^{2\pi} \frac{(A_1 - 2A_2 + A_3)^{n/2}}{A_1 \left( 1 - e^{-\alpha/n} - \frac{A_2}{A_1} \right) + (A_1 - 2A_2 + A_3)} \left( 1 - e^{-2\alpha/n} \right) \pi(\alpha) \, d\alpha
\]

\[
\leq \sqrt{n} \int_0^{2\pi} \sqrt{\alpha} \pi(\alpha) \, d\alpha,
\]

(S.119)

where in the last step, the first ratio in the integral is less than 1 and we have used \(1 - e^{-x} \leq x\) for all \(x > 0\). By (22) in Assumption \([A.3']\), we have that this upper bound goes to zero as \(n \to \infty\). Therefore, \(\sqrt{n} \int_0^{2\pi} \exp \{ \tilde{L}_n(\alpha) \} \pi(\alpha) \, d\alpha \to 0\) as \(n \to \infty\) almost surely \(P_{(\sigma_0^2,\alpha_0)}\). Since the denominator \(\theta_0 \sqrt{n} \int_0^{2\pi} \exp \{ \tilde{L}_n(\alpha) \} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha\) is lower bounded by positive constant in \(P_{(\sigma_0^2,\alpha_0)}\)-probability according to Lemma S.19, we have that the first term in (S.118) converges to zero as \(n \to \infty\) in \(P_{(\sigma_0^2,\alpha_0)}\)-probability.

We then show the second term in (S.118) converges to zero in probability. For the numerator, similar to (S.119), we have that

\[
\int_0^{2\pi} \exp \{ \tilde{L}_n(\alpha) \} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha \leq \int_0^{2\pi} \sqrt{\alpha} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha,
\]

which converges to zero as \(n \to \infty\) since \(\alpha_0 \to 0\) as \(n \to \infty\) and \(\int_0^{2\pi} \sqrt{\alpha} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha\) is finite according to Assumption \([A.3']\). Therefore, with the lower bounded denominator, the second term in (S.118) also converges to zero as \(n \to \infty\) in \(P_{(\sigma_0^2,\alpha_0)}\)-probability. This together with (S.118) has shown that \(N_2/D \to 0\) as \(n \to \infty\) in \(P_{(\sigma_0^2,\alpha_0)}\)-probability.

Proof of \(N_3/D \to 0\):

Using the upper bound of \(N_3\) in (52), together with the definition of \(D\) in (42), we have that

\[
\frac{N_3}{D} \leq 2 \int_0^{\infty} e^{L_n(\alpha-2\theta_0,\alpha)} \pi(\alpha) \, d\alpha + \frac{4\theta_0 \sqrt{n}}{\sqrt{n}} \int_0^{\infty} e^{L_n(\alpha-2\theta_0,\alpha)} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha
\]

\[
= \frac{\sqrt{n}}{\theta_0 \sqrt{n}} \int_0^{\infty} \exp \{ \tilde{L}_n(\alpha) \} \pi(\alpha) \, d\alpha + \int_0^{\infty} \exp \{ \tilde{L}_n(\alpha) \} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha + \int_0^{\infty} \exp \{ \tilde{L}_n(\alpha) \} \pi(\theta_0|\alpha) \pi(\alpha) \, d\alpha,
\]

(S.120)
For both terms in (S.120), the denominators are lower bounded by positive constants in \( P_{(a_0^2, a_0)} \)-probability by Lemma S.19. Using the same derivation as in (S.119), the numerator in the first term of (S.120) can be upper bounded by
\[
\sqrt{n} \int_{\pi_n}^{\infty} \exp \{ \tilde{L}_*(\alpha) \} \pi(\alpha) d\alpha \leq \sqrt{n} \int_{\pi_n}^{\infty} \sqrt{\alpha} \pi(\alpha) d\alpha,
\]
which converges to zero as \( n \to \infty \) by (22) in Assumption (A.3'). The numerator in the second term of (S.120) also converges to zero since \( \pi_n \to \infty \) as \( n \to \infty \) and \( \int_{0}^{\infty} \sqrt{\alpha} \pi(\theta_0(\alpha)) \pi(\alpha) d\alpha \) is finite according to Assumption (A.3'). Therefore, it follows that \( N_j / D \to 0 \) as \( n \to \infty \) in \( P_{(a_0^2, a_0)} \)-probability. Thus, the convergence in (38) happens as \( n \to \infty \) in \( P_{(a_0^2, a_0)} \)-probability.

Proof of (39):

Compared to the proof of (39) in the proof of Theorem 2, the upper bounds in (56) and (57) still hold. We only need to show the convergence in (58) and (59) using the new Assumption (A.3'). In particular, using the definition of \( \tilde{L}_*(\alpha) \) in (S.109), we have
\[
2 \int_{0}^{\infty} \tilde{\pi}(\alpha | X_n) d\alpha \leq 2 \int_{0}^{\infty} \exp \{ \tilde{L}_*(\alpha) \} \pi(\theta_0(\alpha)) \pi(\alpha) d\alpha \int_{0}^{\infty} \exp \{ \tilde{L}_*(\alpha) \} \pi(\theta_0(\alpha)) \pi(\alpha) d\alpha
\]
which converges to zero in \( P_{(a_0^2, a_0)} \)-probability as already shown above in the proof of \( N_j / D \to 0 \). Similarly, \( 2 \int_{0}^{\infty} \tilde{\pi}(\alpha | X_n) d\alpha \to 0 \) in \( P_{(a_0^2, a_0)} \)-probability as shown in the proof of \( N_3/D \to 0 \). Therefore, the convergence in (39) happens as \( n \to \infty \) in \( P_{(a_0^2, a_0)} \)-probability. This completes the proof of the convergence in (23).

For the proof of the convergence in (24), we notice that
\[
\int_{0}^{\infty} \int_{\mathbb{R}} \left| \frac{\sqrt{n}}{2\sqrt{n}} - \frac{n(\theta - \theta_0(\alpha))^2}{4\theta_0(\alpha)} \cdot \tilde{\pi}(\alpha | X_n) - \frac{\sqrt{n}}{2\sqrt{n}} e^{-\frac{n(\theta - \theta_0(\alpha))^2}{4\theta_0(\alpha)}} \cdot \pi_*(\alpha | X_n) \right| d\theta d\alpha
\]
\[
= \int_{0}^{\infty} \left| \tilde{\pi}(\alpha | X_n) - \pi_*(\alpha | X_n) \right| d\alpha \to 0,
\]
as \( n \to \infty \) in \( P_{(a_0^2, a_0)} \)-probability, by (S.112) of Lemma S.19. Then (24) follows from (23) and the triangle inequality.

\[\square\]

S7 Proof of Theorems 4, 5 and 6

Proof of Theorem 4: Proof of Part (i):

Recall our reparametrization says that \( \theta = \sigma^2 \alpha^{2\nu} \), so \( \sigma^2 = \theta / \alpha^{2\nu} \). For abbreviation, let \( v_n(s^*; \theta, \alpha) = E_{\theta/\alpha^{2\nu}} \{ e_n(s^*; \alpha)^2 \} \) for any \( s^* \in S / S_n \) and \( (\theta, \alpha) \in \mathbb{R}_+^2 \). We first notice the relation
\[
\frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \theta_0, \alpha)} = \frac{v_n(s^*; \tilde{\theta}_0, \alpha)}{v_n(s^*; \theta_0, \alpha)}.
\]

To prove Part (i), we first calculate the two ratios using (26):
\[
\begin{align*}
\frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \theta_0, \alpha)} &= \frac{\theta}{\alpha^{2\nu}} \{ 1 - r_\alpha(s^*)^T R_\alpha^{-1} r_\alpha(s^*) \} = \frac{\theta}{\alpha}, \\
\frac{v_n(s^*; \tilde{\theta}_0, \alpha)}{v_n(s^*; \theta_0, \alpha)} &= \frac{\tilde{\theta}_0}{\alpha^{2\nu}} \{ 1 - r_\alpha(s^*)^T R_\alpha^{-1} r_\alpha(s^*) \} = \tilde{\theta}_0.
\end{align*}
\]
both of which do not depend on $s^*$.

Define the event $\mathcal{E}_7 = \{ |\theta/\tilde{\theta}_n - 1| > n^{-1/2} \log n \}$. From the proof of Theorem 1, we can see that the total variation difference in $\mathbb{I}$ in Theorem 1 converges to zero on the event $\mathcal{E}_1(5\theta_0 n^{-1/2} \log n, \alpha) = \{ |\theta_n - \theta_0| < 5\theta_0 n^{-1/2} \log n \}$, and $\Pr \{ \mathcal{E}_1(5\theta_0 n^{-1/2} \log n, \alpha) \} \geq 1 - 4 \exp(-\log^2 n)$ for all sufficiently large $n$. This convergence in total variation norm implies that as $n \to \infty$, almost surely $P(\sigma^*_0, \alpha_0)$,

$$
\left| \mathbb{I} \left( \left| \frac{\theta}{\tilde{\theta}_n} - 1 \right| > n^{-1/2} \log n \middle| X_n, \alpha \right) - \int_{\mathcal{E}_7} \frac{\sqrt{n}}{2\sqrt{\pi} \theta_0} e^{-\frac{n(\theta - \tilde{\theta}_n)^2}{4\theta_0^2}} d\theta \right| \to 0. \quad (S.121)
$$

Let $Z \sim \mathcal{N}(0, 1)$. On the event $\mathcal{E}_1(5\theta_0 n^{-1/2} \log n, \alpha) \cap \mathcal{E}_7$, for all sufficiently large $n$,

$$
\left| \theta - \tilde{\theta}_n \right| > \tilde{\theta}_n n^{-1/2} \log n > (\theta_0 - 5\theta_0 n^{-1/2} \log n)n^{-1/2} \log n > \theta_0 n^{-1/2} \log n/2.
$$

Using the normal tail inequality [S.91], the integral in (S.121) can be bounded by

$$
\begin{align*}
&\int_{\mathcal{E}_7} \frac{\sqrt{n}}{2\sqrt{\pi} \theta_0} e^{-\frac{n(\theta - \tilde{\theta}_n)^2}{4\theta_0^2}} d\theta \leq \int_{|\theta - \tilde{\theta}_n| > \theta_0 n^{-1/2} \log n/2} \frac{\sqrt{n}}{2\sqrt{\pi} \theta_0} e^{-\frac{n(\theta - \tilde{\theta}_n)^2}{4\theta_0^2}} d\theta \\
&= \Pr \left( \left| Z \right| > \frac{\log n}{2\sqrt{2}} \right) \leq \exp \left( -\log^2 n/16 \right) \to 0, \text{ as } n \to \infty. \tag{S.122}
\end{align*}
$$

Therefore, by combining [S.121] and (S.122) and noticing that $\mathcal{E}_1(5\theta_0 n^{-1/2} \log n, \alpha)$ happens almost surely $P(\sigma^*_0, \alpha_0)$ as $n \to \infty$ by the Borel-Cantelli lemma, we have that

$$
\mathbb{I} \left( \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{v_n(s^*; \tilde{\theta}_n, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| > n^{-1/2} \log n \middle| X_n, \alpha \right) \to 0, \text{ a.s. } P(\sigma^*_0, \alpha_0). \tag{S.123}
$$

Using the definition of $\mathcal{E}_1(5\theta_0 n^{-1/2} \log n, \alpha)$, we have that as $n \to \infty$,

$$
\begin{align*}
&\mathbb{I} \left( \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{v_n(s^*; \tilde{\theta}_n, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| > 5n^{-1/2} \log n \middle| X_n, \alpha \right) \\
= &\mathbb{I} \left( \left| \frac{\tilde{\theta}_n}{\theta_0} - 1 \right| > 5n^{-1/2} \log n \middle| X_n, \alpha \right) = 0, \text{ a.s. } P(\sigma^*_0, \alpha_0). \tag{S.124}
\end{align*}
$$

For $n$ sufficiently large, $5n^{-1/2} \log n < 1/2$. Hence $|\tilde{\theta}_n/\theta_0 - 1| < 5n^{-1/2} \log n < 1/2$ and $\tilde{\theta}_n/\theta_0 < 3/2$. For such $n$, we combine (S.123) and (S.124) to obtain that

$$
\begin{align*}
&\mathbb{I} \left( \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{v_n(s^*; \tilde{\theta}_n, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| > 7n^{-1/2} \log n \middle| X_n, \alpha \right) \\
= &\mathbb{I} \left( \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{v_n(s^*; \tilde{\theta}_n, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| > 7n^{-1/2} \log n \middle| X_n, \alpha \right) \\
\leq &\mathbb{I} \left( \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left\{ \left| \frac{v_n(s^*; \tilde{\theta}_n, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| \cdot \frac{v_n(s^*; \tilde{\theta}_n, \alpha)}{v_n(s^*; \theta_0, \alpha)} \right\} + \frac{v_n(s^*; \tilde{\theta}_n, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right) \\
&> 7n^{-1/2} \log n \middle| X_n, \alpha \right) \\
\leq &\mathbb{I} \left( \frac{3}{2} \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{v_n(s^*; \tilde{\theta}_n, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| + \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{v_n(s^*; \tilde{\theta}_n, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| > 7n^{-1/2} \log n \middle| X_n, \alpha \right)
\end{align*}
$$
This has proved Part (i).

Proof of Part (ii):

First, we show the existence of the sequence \( \zeta_n(\alpha) \). Since the two Gaussian measures \( \text{GP}(0, (\theta_0/\alpha^{2\nu})K_{\alpha,\nu}) \) and \( \text{GP}(0, \sigma^2_{\theta_0}K_{\alpha_0,\nu}) \) are equivalent, by Assumption [A.4] Equation (3.4) in Stein [58] implies that there exists a positive sequence \( \zeta_{1n}(\alpha) \to 0 \) as \( n \to \infty \), such that

\[
\sup_{s^* \in S \setminus S_n} \frac{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \}}{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \} - 1} < \frac{1}{2} \zeta_{1n}(\alpha).
\]

Notice that for a small \( \epsilon \in (0, 1/2) \), \(|a/b - 1| < \epsilon\) implies that \( a/b \geq 1 - \epsilon \) and hence \(|b/a - 1| \leq |b/a - 1|/|a/b| \leq \epsilon/(1 - \epsilon) < 2\epsilon\). Therefore, for sufficiently large \( n \), \( \zeta_{1n}(\alpha) < 1/2 \) and

\[
\sup_{s^* \in S \setminus S_n} \frac{\mathbb{E}_{\theta_0/\alpha^{2\nu}, \alpha} \{ e_n(s^*; \alpha)^2 \}}{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \} - 1} < \zeta_{1n}(\alpha).
\]

Equation (S.125) further shows that there exists a positive sequence \( \zeta_{2n}(\alpha) \to 0 \) as \( n \to \infty \), such that

\[
\sup_{s^* \in S \setminus S_n} \frac{\mathbb{E}_{\theta_0/\alpha^{2\nu}, \alpha} \{ e_n(s^*; \alpha)^2 \}}{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \} - 1} < \zeta_{2n}(\alpha).
\]

Therefore, we can set \( \zeta_n(\alpha) = \max\{\zeta_{1n}(\alpha), \zeta_{2n}(\alpha)\} \) and \( \zeta_n(\alpha) \to 0 \) as \( n \to \infty \).

For abbreviation, let \( \epsilon_{2n}(\alpha) = \max\{8n^{-1/2} \log n, \zeta_n(\alpha)\} \). Then based on (S.125) and the result in Part (i), we have that

\[
\begin{align*}
\Pi \left( \sup_{s^* \in S \setminus S_n} \frac{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \}}{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \} - 1} > 2\epsilon_{2n}(\alpha) \right) X_n, \alpha) \\
\leq \Pi \left( \sup_{s^* \in S \setminus S_n} \frac{\mathbb{E}_{\theta_0/\alpha^{2\nu}, \alpha} \{ e_n(s^*; \alpha)^2 \}}{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \} - 1} \right) \cdot \frac{\mathbb{E}_{\theta_0/\alpha^{2\nu}, \alpha} \{ e_n(s^*; \alpha)^2 \}}{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \} - 1} \geq \epsilon_{2n}(\alpha) \right) X_n, \alpha) \\
+ \Pi \left( \sup_{s^* \in S \setminus S_n} \frac{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \}}{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \} - 1} \geq \epsilon_{2n}(\alpha) \right) X_n, \alpha) \quad .
\end{align*}
\]

Equation (S.127) shows that the first term on the right-hand side of (S.127) is zero, due to (S.125) and \( \epsilon_{2n}(\alpha) \geq \zeta_n(\alpha) \). In the first term on the right-hand side of (S.127), using (S.125) and the fact that \( \zeta_{1n}(\alpha) < 1/7 \) for sufficiently large \( n \), we have from (S.127) that

\[
\begin{align*}
\Pi \left( \sup_{s^* \in S \setminus S_n} \frac{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \}}{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \} - 1} > 2\epsilon_{2n}(\alpha) \right) X_n, \alpha) \\
\leq \Pi \left( \sup_{s^* \in S \setminus S_n} \frac{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \}}{\mathbb{E}_{\sigma^2_{\theta_0}, \alpha_0} \{ e_n(s^*; \alpha)^2 \} - 1} > \frac{7}{8} \epsilon_{2n}(\alpha) \right) X_n, \alpha).
\end{align*}
\]
Using the normal tail inequality (S.91), the integral in (S.128) can be bounded by
\[ \sup_{s^* \in S \setminus S_n} \left\{ \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right\} > 7n^{-1/2} \log n \bigg| X_n, \alpha \bigg] \to 0, \text{ a.s.} \ P_{(\sigma^2_0, \alpha_0)}, \]
following the result in Part (i). This has proved the first convergence in Part (ii). The proof of the second convergence is similar, by instead using (S.126) and replacing all \( E_{\sigma^2_0, \alpha_0} \{ e_n(s^*; \alpha)^2 \} \) in the display above by \( E_{\sigma^2_0, \alpha_0} \{ e_n(s^*; \alpha_0)^2 \} \).

Proof of Theorem 4. The proof is similar to that of Theorem 3 and relies on the Bernstein-von Mises theorem of \((\theta, \alpha)\) in Theorem 2. Recall from the proof of Theorem 2 that the event \( \mathcal{E}_4(\varepsilon) = \{ |\tilde{\theta}_{\alpha_0} - \theta_0| < \varepsilon \} \) and \( \Pr \{ \mathcal{E}_4(4\theta_0n^{-1/2} \log n)^c \} \leq 2 \exp(-\log^2 n) \).

Proof of Part (i):

Using the same notation as in the proof of Theorem 3, we have the following decomposition of ratio
\[ \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \theta_0, \alpha)} = \frac{v_n(s^*; \tilde{\theta}_{\alpha_0}, \alpha)}{v_n(s^*; \theta_0, \alpha)}. \]

It follows that
\[ \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \tilde{\theta}_{\alpha_0}, \alpha)} = \frac{\theta}{\tilde{\theta}_{\alpha_0}}, \]
\[ \frac{v_n(s^*; \tilde{\theta}_{\alpha_0}, \alpha)}{v_n(s^*; \theta_0, \alpha)} = \frac{\tilde{\theta}_{\alpha_0}}{\theta_0}. \]

Let \( \mathcal{E}_8 = \{ |\theta/\tilde{\theta}_{\alpha_0} - 1| > n^{-1/2} \log n \} \). Then by Theorem 2 as \( n \to \infty \), almost surely \( P_{(\sigma^2_0, \alpha_0)} \),
\[ \left| \Pi (\mathcal{E}_8 | X_n) - \int_0^\infty \int_{\mathcal{E}_8} \frac{\sqrt{n}}{2\sqrt{\pi} \theta_0} e^{-\frac{(\theta - \tilde{\theta}_{\alpha_0})^2}{4\theta_0^2}} \cdot \tilde{\pi}(\alpha | X_n) d\theta d\alpha \right| \to 0. \quad (S.128) \]

Let \( Z \sim N(0, 1) \). On the event \( \mathcal{E}_4(4\theta_0n^{-1/2} \log n) \cap \mathcal{E}_8 \), for all sufficiently large \( n \),
\[ |\theta - \tilde{\theta}_{\alpha_0}| > \tilde{\theta}_{\alpha_0} n^{-1/2} \log n > (\theta_0 - 4\theta_0n^{-1/2} \log n)n^{-1/2} \log n > \theta_0 n^{-1/2} \log n/2. \]

Using the normal tail inequality (S.91), the integral in (S.128) can be bounded by
\[ \int_0^\infty \int_{\mathcal{E}_8} \frac{\sqrt{n}}{2\sqrt{\pi} \theta_0} e^{-\frac{(\theta - \tilde{\theta}_{\alpha_0})^2}{4\theta_0^2}} \cdot \tilde{\pi}(\alpha | X_n) d\theta d\alpha \leq \int_{|\theta - \tilde{\theta}_{\alpha_0}| > \theta_0 n^{-1/2} \log n/2} \frac{\sqrt{n}}{2\sqrt{\pi} \theta_0} e^{-\frac{(\theta - \tilde{\theta}_{\alpha_0})^2}{4\theta_0^2}} d\theta \cdot \int_0^\infty \tilde{\pi}(\alpha | X_n) d\alpha \]
\[ = \Pr \left( Z > \frac{\log n}{2\sqrt{2}} \right) \leq \exp \left( -\log^2 n/16 \right) \to 0, \text{ as } n \to \infty. \quad (S.129) \]

Therefore, by combining (S.128) and (S.129) and noticing that \( \mathcal{E}_4(4\theta_0n^{-1/2} \log n, \alpha) \) happens almost surely \( P_{(\sigma^2_0, \alpha_0)} \) as \( n \to \infty \) by the Borel-Cantelli lemma, we have that
\[ \Pi \left( \sup_{s^* \in S \setminus S_n} \left\{ \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right\} > n^{-1/2} \log n \bigg| X_n \right) \to 0, \text{ a.s.} \ P_{(\sigma^2_0, \alpha_0)}. \quad (S.130) \]
The definition and almost sure property of $E_4(4\theta_0 n^{-1/2} \log n)$ also implies that

$$\Pi \left( \sup_{s^* \in S \setminus S_n} \left| \frac{v_n(s^*; \tilde{\theta}_{00}, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| > 4n^{-1/2} \log n \big| X_n \right)$$

$$= \Pi \left( \left| \frac{\tilde{\theta}_{00}}{\theta_0} - 1 \right| > 4n^{-1/2} \log n \big| X_n \right) = 0, \text{ a.s. } P_{(\sigma^2_0, \alpha)}.$$  (S.131)

For $n$ sufficiently large, we have $4n^{-1/2} \log n < 1/2$. Hence, $|\tilde{\theta}_{00}/\theta_0 - 1| < 1/2$ and $\tilde{\theta}_{00}/\theta_0 < 3/2$ almost surely $P_{(\sigma^2_0, \alpha)}$. We combine (S.130) and (S.131) to obtain that

$$\Pi \left( \sup_{s^* \in S \setminus S_n} \left| \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \tilde{\theta}_{00}, \alpha)} - 1 \right| \leq \frac{3}{2} \sup_{s^* \in S \setminus S_n} \left| \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \tilde{\theta}_{00}, \alpha)} - 1 \right| + \sup_{s^* \in S \setminus S_n} \left| \frac{v_n(s^*; \tilde{\theta}_{00}, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| > 6n^{-1/2} \log n \big| X_n \right)$$

$$\leq \Pi \left( \sup_{s^* \in S \setminus S_n} \left| \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \tilde{\theta}_{00}, \alpha)} - 1 \right| + \sup_{s^* \in S \setminus S_n} \left| \frac{v_n(s^*; \tilde{\theta}_{00}, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| > 6n^{-1/2} \log n \big| X_n \right)$$

$$\leq \Pi \left( \sup_{s^* \in S \setminus S_n} \left| \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \tilde{\theta}_{00}, \alpha)} - 1 \right| > 6n^{-1/2} \log n \big| X_n \right)$$

$$\rightarrow 0, \text{ a.s. } P_{(\sigma^2_0, \alpha)}.$$  

This has proved Part (i).

Proof of Part (ii):

Let $\epsilon_{3n} = \max(8n^{-1/2} \log n, \zeta_n)$. Let $E_9 = \{ \alpha \in [\alpha_n, \pi_n] \}$. By Assumption (A.5) for all sufficiently large $n$, on the event $E_9$,

$$\sup_{s^* \in S \setminus S_n} \left| \frac{E_{\theta_0/\alpha^2, \alpha} \{ e_n(s^*; \alpha)^2 \}}{E_{\sigma^2_{0, \alpha}} \{ e_n(s^*; \alpha)^2 \}} - 1 \right| \leq \zeta_n < 1/7;$$

$$\sup_{s^* \in S \setminus S_n} \left| \frac{E_{\theta_0/\alpha^2, \alpha} \{ e_n(s^*; \alpha)^2 \}}{E_{\sigma^2_{0, \alpha}} \{ e_n(s^*; \alpha_0)^2 \}} - 1 \right| \leq \zeta_n < 1/7.$$  

Therefore, based on the result of Part (i), we have that

$$\Pi \left( \sup_{s^* \in S \setminus S_n} \left| \frac{E_{\sigma^2, \alpha} \{ e_n(s^*; \alpha)^2 \}}{E_{\sigma^2_{0, \alpha}} \{ e_n(s^*; \alpha)^2 \}} - 1 \right| > 2\epsilon_{3n}, E_9 \big| X_n \right)$$

$$= \Pi \left( \sup_{s^* \in S \setminus S_n} \left| \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \theta_0, \alpha)} \cdot \frac{E_{\theta_0/\alpha^2, \alpha} \{ e_n(s^*; \alpha)^2 \}}{E_{\sigma^2_{0, \alpha}} \{ e_n(s^*; \alpha)^2 \}} - 1 \right| > 2\epsilon_{3n}, E_9 \big| X_n \right)$$

$$\leq \Pi \left( \sup_{s^* \in S \setminus S_n} \left| \frac{E_{\theta_0/\alpha^2, \alpha} \{ e_n(s^*; \alpha)^2 \}}{E_{\sigma^2_{0, \alpha}} \{ e_n(s^*; \alpha)^2 \}} \cdot \frac{v_n(s^*; \theta, \alpha)}{v_n(s^*; \theta_0, \alpha)} - 1 \right| > \epsilon_{3n}, E_9 \big| X_n \right)$$
+ \Pi \left( \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{E_{\theta_0/\alpha^{2\nu},\alpha} \{e_n(s^*;\alpha)\} - 1}{E_{\sigma_0^2,\alpha^0} \{e_n(s^*;\alpha)\} - 1} \right| > \epsilon_{3n}, \mathcal{E}_9 \mid X_n \right) \\
\leq \Pi \left( \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{v_n(s^*;\theta_0,\alpha)}{v_n(s^*;\theta_0,\alpha)} - 1 \right| > 7n^{-1/2} \log n, \mathcal{E}_9 \mid X_n \right) \\
+ \Pi \left( \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{E_{\theta_0/\alpha^{2\nu},\alpha} \{e_n(s^*;\alpha)\} - 1}{E_{\sigma_0^2,\alpha^0} \{e_n(s^*;\alpha)\} - 1} \right| > \epsilon_{3n}, \mathcal{E}_9 \mid X_n \right) \\
\leq \Pi \left( \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{v_n(s^*;\theta_0,\alpha)}{v_n(s^*;\theta_0,\alpha)} - 1 \right| > 7n^{-1/2} \log n \mid X_n \right) \rightarrow 0, \text{ a.s.} \ P_{(\sigma_0^2,\alpha^0)}. \quad (S.132)

On the other hand, for the event \mathcal{E}_9, Theorem 2 implies that on the event

\[ \mathcal{E}_{10} = \left\{ \sup_{s^* \in \mathcal{S} \setminus \mathcal{S}_n} \left| \frac{E_{\sigma_0^2,\alpha} \{e_n(s^*;\alpha)\} - 1}{E_{\sigma_0^2,\alpha^0} \{e_n(s^*;\alpha)\} - 1} \right| > \max \left( 16n^{-1/2} \log n, 2\epsilon_n \right) \right\} \cap \mathcal{E}_9, \]

as \( n \rightarrow \infty \), almost surely \( P_{(\sigma_0^2,\alpha^0)} \),

\[ \Pi (\mathcal{E}_{10} \mid X_n) - \int_{\mathcal{E}_{10}} \frac{\sqrt{n}}{2 \sqrt{\pi \theta_0}} e^{-\frac{n(\theta - \theta_0)^2}{\theta_0}} \cdot \tilde{\pi}(\alpha \mid X_n) d\theta d\alpha \rightarrow 0. \quad (S.133) \]

But from (58) and (59) in the proof of Theorem 2 it follows that as \( n \rightarrow \infty \), almost surely \( P_{(\sigma_0^2,\alpha^0)} \),

\[ \int_{\mathcal{E}_{10}} \frac{\sqrt{n}}{2 \sqrt{\pi \theta_0}} e^{-\frac{n(\theta - \theta_0)^2}{\theta_0}} \cdot \tilde{\pi}(\alpha \mid X_n) d\theta d\alpha \leq \int_{\mathcal{E}_9} \tilde{\pi}(\alpha \mid X_n) d\theta d\alpha \rightarrow 0. \quad (S.134) \]

Therefore, (S.133) and (S.134) imply that \( \Pi (\mathcal{E}_{10} \mid X_n) \rightarrow 0 \) almost surely \( P_{(\sigma_0^2,\alpha^0)} \) as \( n \rightarrow \infty \). The first convergence in Part (ii) follows by combining this with (S.132). The second convergence follows from the similar argument as above by replacing all \( E_{\sigma_0^2,\alpha^0} \{e_n(s^*;\alpha)\} \) by \( E_{\sigma_0^2,\alpha^0} \{e_n(s^*;\alpha^0)\} \).

Define KL(\( P_1, P_2 \)) = \int \log (dP_1/dP_2) dP_1 \) to be the Kullback-Leibler divergence between two measures \( P_1 \) and \( P_2 \), where \( dP_1/dP_2 \) is the Radon-Nikodym derivative of \( P_1 \) with respect to \( P_2 \). For two mean zero Gaussian processes with Matérn covariance functions \( \sigma_i^2 K_{\alpha_i,\nu} \) \( (i = 1, 2) \), let \( P_{(\sigma_i^2,\alpha_i)}^{(n)} \) be the joint Gaussian distribution of the observations \( X(s_1), \ldots, X(s_n) \). Then one can show that

\[ \text{KL} \left( \left( P_{(\sigma_1^2,\alpha_1)}^{(n)}, P_{(\sigma_2^2,\alpha_2)}^{(n)} \right) \right) = \frac{1}{2} \left\{ \log \left( \frac{\sigma_2^2 R_{a_2}}{\sigma_1^2 R_{a_1}} \right) - n + \frac{\sigma_1^2}{\sigma_2^2} \text{tr} \left( R_{a_2}^{-1} R_{a_1} \right) \right\}. \]

Now consider two equivalent Gaussian measures with Matérn covariance functions \( \sigma_0^2 K_{\alpha_0,\nu} \) and \( \sigma_2^2 K_{\alpha_0,\nu} \), such that \( \sigma_0^2 \alpha^{2\nu} = \theta_0 = \sigma_2^2 \alpha^{2\nu} \). Let

\[ r_n(\alpha) = \text{KL} \left( P_{(\sigma_0^2,\alpha_0)}^{(n)}, P_{(\sigma_2^2,\alpha_2)}^{(n)} + \text{KL}(P_{(\sigma_2^2,\alpha_2), \sigma_0^2,\alpha_0}^{(n)} \right) + \text{KL}(P_{(\sigma_0^2,\alpha_0), \sigma_2^2,\alpha_2}^{(n)} \right), \]

where \( \text{KL}(P_{(\sigma_0^2,\alpha_0), \sigma_2^2,\alpha_2}^{(n)}) \) and \( \text{KL}(P_{(\sigma_0^2,\alpha_0), \sigma_2^2,\alpha_2}^{(n)}) \) are the limits of \( \text{KL}(P_{(\sigma_0^2,\alpha_0), \sigma_2^2,\alpha_2}^{(n)}) \) and \( \text{KL}(P_{(\sigma_0^2,\alpha_0), \sigma_2^2,\alpha_2}^{(n)}) \) as \( n \rightarrow \infty \) (Kullback et al. [12]). See Section 3 of Stein [59].

The following lemma is a result from Stein [59].
Lemma S.20. Consider two mean zero Gaussian processes with Matérn covariance functions $\sigma_0^2 K_{\alpha,\nu}$ and $\sigma_0^2 K_{\alpha,\nu}$, where $\sigma_0^2 \alpha^{2\nu} = \theta_0 = \sigma_0^2 \alpha^{2\nu}$ and $\alpha > 0$ is given. If $r_n(\cdot)$ is defined as in (S.135), then

\[
\sup_{s^* \in S \setminus S_n} \left| \frac{E_{\sigma_0^2,\alpha_0} \{e_n(s^*; \alpha)^2\}}{E_{\sigma_0^2,\alpha} \{e_n(s^*; \alpha)^2\}} - 1 \right| \leq 4 \left[ r(\alpha) - r_n(\alpha) \right],
\]

(S.136)

\[
\sup_{s^* \in S \setminus S_n} \left| \frac{E_{\sigma_0^2,\alpha_0} \{e_n(s^*; \alpha_0)^2\}}{E_{\sigma_0^2,\alpha_0} \{e_n(s^*; \alpha)^2\}} - 1 \right| \leq 18 \left[ r(\alpha) - r_n(\alpha) \right].
\]

(S.137)

Proof of Lemma S.20. Using similar notation to Stein [50], we let

\[
a_n(s^*; \alpha) = \frac{E_{\sigma_0^2,\alpha_0} \{e_n(s^*; \alpha)^2\}}{E_{\sigma_0^2,\alpha} \{e_n(s^*; \alpha)^2\}} - 1, \quad \bar{a}_n(s^*; \alpha) = \frac{E_{\sigma_0^2,\alpha} \{e_n(s^*; \alpha_0)^2\}}{E_{\sigma_0^2,\alpha} \{e_n(s^*; \alpha)^2\}} - 1,
\]

\[
b_n(s^*; \alpha) = \frac{E_{\sigma_0^2,\alpha} \{e_n(s^*; \alpha_0)^2\}}{E_{\sigma_0^2,\alpha} \{e_n(s^*; \alpha)^2\}} - 1, \quad \bar{b}_n(s^*; \alpha) = \frac{E_{\sigma_0^2,\alpha} \{e_n(s^*; \alpha)^2\}}{E_{\sigma_0^2,\alpha} \{e_n(s^*; \alpha)^2\}} - 1.
\]

(S.136) follows from Lemma 2 and Section 3 in Stein [50]. In fact, Lemma 2 in Stein [50] implies that

\[
\sup_{s^* \in S \setminus S_n} |b_n(s^*; \alpha)| \leq 4 \left[ r(\alpha) - r_n(\alpha) \right], \quad \sup_{s^* \in S \setminus S_n} \left| \bar{b}_n(s^*; \alpha) \right| \leq 4 \left[ r(\alpha) - r_n(\alpha) \right].
\]

Furthermore, as $n \to \infty$, $r_n(\alpha)$ increases to $r(\alpha)$. Using the relation

\[
[1 + a_n(s^*; \alpha)][1 + \bar{a}_n(s^*; \alpha)] = [1 + b_n(s^*; \alpha)][1 + \bar{b}_n(s^*; \alpha)],
\]

and the fact that $\bar{a}_n(s^*; \alpha) \geq 0$, we can obtain that

\[
\sup_{s^* \in S \setminus S_n} a_n(s^*; \alpha)
\]

\[
\leq \sup_{s^* \in S \setminus S_n} \left[ b_n(s^*; \alpha) + \bar{b}_n(s^*; \alpha) + b_n(s^*; \alpha)\bar{b}_n(s^*; \alpha) \right]
\]

\[
\leq \sup_{s^* \in S \setminus S_n} \left| b_n(s^*; \alpha) \right| + \sup_{s^* \in S \setminus S_n} \left| \bar{b}_n(s^*; \alpha) \right| + \sup_{s^* \in S \setminus S_n} \left| b_n(s^*; \alpha)\bar{b}_n(s^*; \alpha) \right|
\]

\[
\leq 8 \left[ r(\alpha) - r_n(\alpha) \right] + 8 \left[ r(\alpha) - r_n(\alpha) \right]^2
\]

\[
\leq 9 \left[ r(\alpha) - r_n(\alpha) \right],
\]

since $r(\alpha) - r_n(\alpha) \leq 1/8$ as $n \to \infty$. In this case, $\sup_{s^* \in S \setminus S_n} \left| \bar{b}_n(s^*; \alpha) \right| \leq 1/2$, and $1 \leq a_n(s^*; \alpha) + 1 \leq 9/8$, which implies that

\[
\sup_{s^* \in S \setminus S_n} \left| \frac{E_{\sigma_0^2,\alpha_0} \{e_n(s^*; \alpha)^2\}}{E_{\sigma_0^2,\alpha} \{e_n(s^*; \alpha)^2\}} - 1 \right| = \sup_{s^* \in S \setminus S_n} \left| \frac{a_n(s^*; \alpha) + 1}{1 + b_n(s^*; \alpha)} - 1 \right|
\]

\[
\leq \sup_{s^* \in S \setminus S_n} a_n(s^*; \alpha) + \sup_{s^* \in S \setminus S_n} \left| \frac{a_n(s^*; \alpha) + 1}{1 + b_n(s^*; \alpha)} \right|
\]

\[
\leq 9 \left[ r(\alpha) - r_n(\alpha) \right] + \frac{(9/8) \cdot 4 \left[ r(\alpha) - r_n(\alpha) \right]}{1/2}
\]

\[
= 18 \left[ r(\alpha) - r_n(\alpha) \right].
\]

This has proved (S.137). \hfill \square
Proof of Theorem 6. We verify Assumption 5 for this special case. We can calculate that
\[
  r_n(\alpha) = \frac{\alpha}{2\alpha_0} \text{tr} \left( R_{n}^{-1} R_{\alpha_0} \right) + \frac{\alpha_0}{2\alpha} \text{tr} \left( R_{\alpha_0}^{-1} R_{\alpha} \right) - n
  = \frac{\alpha}{2\alpha_0} \left[ n + \frac{(n-1)\alpha - e^{-\alpha/n}(e^{-\alpha/n} - e^{-\alpha_0/n})}{\alpha_0} \right]
  \quad + \frac{\alpha_0}{2\alpha} \left[ n + \frac{(n-1)\alpha_0 - e^{-\alpha_0/n}(e^{-\alpha_0/n} - e^{-\alpha/n})}{\alpha} \right] - n.
\] (S.138)

The Taylor series expansion of the first term in (S.138) over all \( \alpha \in [\alpha_0, \alpha_0] \) gives
\[
  \frac{\alpha}{2\alpha_0} \left[ n + \frac{(n-1)\alpha - e^{-\alpha_0/n}(e^{-\alpha_0/n} - e^{-\alpha/n})}{\alpha_0} \right]
  = \frac{n}{2} + \frac{(\alpha - \alpha_0)(\alpha + \alpha_0 + 2)}{4\alpha_0} + \frac{(\alpha_0^2 - \alpha^2)(\alpha_0 + 3)}{12\alpha_0 n}
  \quad + \frac{(\alpha_0^2 - \alpha^2)(\alpha_0^2 + 4\alpha_0 - \alpha_0^2)}{48\alpha_0 n^2} + O \left( \frac{1}{n^{5/2}} \right).
\] (S.139)

The order of the remainder is at most \( O(n^{-5/2}) \) since \( \alpha_0 \leq n^{0.02} \) and \( \alpha_0 \geq n^{-0.05} \).

By symmetry, for the second term in (S.138), we have
\[
  \frac{\alpha_0}{2\alpha} \left[ n + \frac{(n-1)\alpha_0 - e^{-\alpha_0/n}(e^{-\alpha_0/n} - e^{-\alpha_0/n})}{\alpha} \right]
  = \frac{n}{2} + \frac{(\alpha - \alpha_0)(\alpha + \alpha_0 + 2)}{4\alpha} + \frac{(\alpha_0^2 - \alpha^2)(\alpha_0 + 3)}{12\alpha n}
  \quad + \frac{(\alpha_0^2 - \alpha^2)(\alpha_0^2 + 4\alpha - \alpha_0^2)}{48\alpha_0 n^2} + O \left( \frac{1}{n^{5/2}} \right).
\] (S.140)

Therefore, (S.138), (S.139), and (S.140) together imply that
\[
  r_n(\alpha) = \frac{(\alpha - \alpha_0)^2(\alpha + \alpha_0 + 2)}{4\alpha_0} - \frac{(\alpha - \alpha_0)^2(\alpha + \alpha_0)}{4\alpha_0 n}
  - \frac{(\alpha - \alpha_0)^2(\alpha + \alpha_0)^3}{48\alpha_0 n^2} + O \left( \frac{1}{n^{5/2}} \right),
\]
and
\[
  r(\alpha) = \lim_{n \to \infty} r_n(\alpha) = \frac{(\alpha - \alpha_0)^2(\alpha + \alpha_0 + 2)}{4\alpha_0}.
\]

Therefore, uniformly over all \( \alpha \in [\alpha_0, \alpha_0] \),
\[
  r(\alpha) - r_n(\alpha) = \frac{(\alpha - \alpha_0)^2(\alpha + \alpha_0)}{4\alpha_0 n} + \frac{(\alpha - \alpha_0)^2(\alpha + \alpha_0)^3}{48\alpha_0 n^2} + O \left( \frac{1}{n^{5/2}} \right).
\] (S.141)

By (S.136) in Lemma 20 and the uniformity over all \( \alpha \in [\alpha_0, \alpha_0] \), we obtain that for sufficiently large \( n \),
\[
  \sup_{\alpha \in [\alpha_0, \alpha_0], \sigma \in S \setminus \sigma_s} \sup_{s \in S \setminus \sigma_s} \left[ E_{\sigma,0/\alpha_0} \{ e_{\sigma}^2(s^*; \alpha) \} + E_{\theta_0/\alpha_0,\alpha} \{ e_{\sigma}^2(s^*; \alpha) \} - 1 \right]
  \leq \sup_{\alpha \in [\alpha_0, \alpha_0]} \frac{2(\alpha - \alpha_0)^2(\alpha + \alpha_0)}{n\alpha_0}.
\]

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\[
\leq \sup_{\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n]} \frac{2(\alpha + \alpha_0)}{n} \cdot \left( \frac{\alpha}{\alpha_0} + \frac{\alpha_0}{\alpha} - 2 \right)
\]
\[
\leq \frac{2(\tilde{\alpha}_n + \alpha_0)}{n} \cdot \max \left\{ \frac{(\tilde{\alpha}_n - \alpha_0)^2}{\tilde{\alpha}_n \alpha_0}, \frac{(\alpha_0 - \alpha)^2}{\alpha_0 \alpha} \right\}
\]
\[
\leq \frac{3\tilde{\alpha}_n \sup \left( \frac{\tilde{\alpha}_n}{\alpha_0} - \frac{\alpha_0}{\tilde{\alpha}_n} \right)}{n}.
\]
Since \( E_{\sigma_0^2, \alpha_0} \{ e_n(s^*; \alpha)^2 \} \geq E_{\sigma_0^2, \alpha} \{ e_n(s^*; \alpha)^2 \} \), it follows that
\[
\sup_{\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n]} \sup_{s^* \in S \setminus S_n} E_{\Theta_0/\alpha^2, \alpha} \left\{ e_n(s^*; \alpha)^2 \right\} - 1
\]
\[
= \sup_{\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n]} \sup_{s^* \in S \setminus S_n} \left[ 1 - \frac{E_{\Theta_0/\alpha^2, \alpha} \{ e_n(s^*; \alpha)^2 \}}{E_{\sigma_0^2, \alpha} \{ e_n(s^*; \alpha)^2 \}} \right]
\]
\[
= \sup_{\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n]} \sup_{s^* \in S \setminus S_n} \frac{E_{\sigma_0^2, \alpha_0} \{ e_n(s^*; \alpha)^2 \}}{E_{\Theta_0/\alpha^2, \alpha} \{ e_n(s^*; \alpha)^2 \}} - 1
\]
\[
\leq \sup_{\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n]} \sup_{s^* \in S \setminus S_n} \left[ \frac{E_{\sigma_0^2, \alpha_0} \{ e_n(s^*; \alpha)^2 \}}{E_{\Theta_0/\alpha^2, \alpha} \{ e_n(s^*; \alpha)^2 \}} - 1 \right]
\]
\[
\leq 3\tilde{\alpha}_n \max \left( \frac{\tilde{\alpha}_n}{\alpha_0}, \frac{\alpha_0}{\tilde{\alpha}_n} \right).
\]
From (S.137) in Lemma (S.20) we obtain that for sufficiently large \( n \),
\[
\sup_{\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n]} \sup_{s^* \in S \setminus S_n} \left| \frac{E_{\Theta_0/\alpha^2, \alpha} \{ e_n(s^*; \alpha)^2 \}}{E_{\sigma_0^2, \alpha_0} \{ e_n(s^*; \alpha_0)^2 \}} - 1 \right|
\]
\[
\leq \sup_{\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n]} \frac{5(\alpha - \alpha_0)^2}{n} \cdot \left( \frac{\alpha}{\alpha_0} + \frac{\alpha_0}{\alpha} - 2 \right)
\]
\[
\leq \sup_{\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n]} \frac{5(\alpha + \alpha_0)}{n} \cdot \left( \frac{\alpha}{\alpha_0} + \frac{\alpha_0}{\alpha} - 2 \right)
\]
\[
\leq \frac{5(\tilde{\alpha}_n + \alpha_0)}{n} \cdot \max \left\{ \frac{(\tilde{\alpha}_n - \alpha_0)^2}{\tilde{\alpha}_n \alpha_0}, \frac{(\alpha_0 - \alpha)^2}{\alpha_0 \alpha} \right\}
\]
\[
\leq \frac{6\tilde{\alpha}_n \max \left( \frac{\tilde{\alpha}_n}{\alpha_0}, \frac{\alpha_0}{\tilde{\alpha}_n} \right)}{n} \leq 7n^{2\pi + \varepsilon - 1}.
\]
Therefore, Assumption (A.5) is satisfied with \( \varsigma_n = 7n^{2\pi + \varepsilon - 1} \).
\]

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