FORCED OSCILLATION OF VISCOUS BURGERS’ EQUATION
WITH A TIME-PERIODIC FORCE

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ABSTRACT. This paper is concerned about the existence of periodic solutions of the viscous Burgers’ equation when a forced oscillation is prescribed. We establish the existence theory by contraction mapping in $H^s[0,1]$ with $s \geq 0$. Asymptotical periodicity is obtained as well, and the periodic solution is achieved by selecting a suitable function as initial data to generate a solution and passing time limit to infinity. Moreover, uniqueness and global stability is achieved for this periodic solution.

1. Introduction. In this paper, we establish the existence of time-periodic solution for viscous Burgers’ equation with initial boundary data on bounded region $\Omega = [0,1]$, which reads:

\[
\begin{aligned}
    u_t - \nu u_{xx} + uu_x &= f(x,t), \\
    u(0,t) &= u(1,t) = 0, \\
    u(x,0) &= \phi(x). 
\end{aligned}
\]

(1)

$\theta$-time-periodic external force $f(x,t)$ is applied, i.e., for any location $x \in [0,1]$:

\[f(x,t) = f(x,t + \theta).\]

(2)

Positive constant $\nu$ is the viscosity of the fluid. For simplicity of analysis, we stipulate $\nu = 1$.

Existence of time-periodic solutions with their dynamical behaviors is considered as a classic topic in differential equations for over a century. For PDEs, questions about existence of periodic solutions, can be either that of the equation under external periodic forces (forced oscillation / vibration), or that without forces (i.e., free oscillation / vibrations). Theoretical efforts for the former cases can be carried out to models with the forces built in the equation, or applied on part of spatial boundary. When the nonlinear terms are present in ODEs, and if they are small,
Lindstedt and Poincaré method predicts the existence of periodic orbits via perturbation expansions (see, e.g., Grimshaw’s book [9]). Similar methodology can be used to obtain periodic solutions for free vibration cases of some PDEs such as nonlinear wave equations (see, e.g., J. Keller and L. Ting’s [10]), if eigenvalues of principal differential operators are simple, for instance $\Box = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}$.

In general, resonance occurs if the frequency of forced oscillation accords with the natural frequency. This is observed in ODEs and wave equations to cause infinite solution and energy in a long run. It brings analysis difficulty as well, i.e., small denominators arise when inverting the principal operator $\Box$ and invalidate the approximation of periodic solutions of PDEs. To avoid this, the force is selected to be small and orthogonal to the null space of $\Box$ and some further constraints about nonlinear terms in the equation and functions in the null space of $\Box$ were satisfied; When PDEs are posed on 1D interval $[0, \pi]$ with homogeneous Dirichlet condition, the solutions established must have periods, numerically rational multiples of $\pi$. Moreover, variational methods can be invoked to pursue similar periodic solutions, which are in fact minimizers of functionals. Irrational-$\pi$-multiple periods can’t be approached via these techniques in that small denominators will still appear. However, both rational and irrational multiples can be achieved for ODEs via Lyapunov-Schmidt method, with which some efforts used in wave equations, the KAM (Kolmogorov-Arnold-Moser) theory, to inverting an equivalent form of the original equation and constructing solutions with irrational-$\pi$-multiple periods. We refer readers for more detailed discussions about applying these techniques to H. Brézis, L. Nirenburg, P. Rabinowitz, W. Craig, E. Wayne’s seminal works [4, 5, 7, 12, 13, 19] and references therein.

Plenty of literatures already studied the forced oscillations for other types of PDEs (see Vejvoda’s monograph [18] for review of a whole set of results). Closer to topic in current paper, G. Lukaszewicz et al. invoked Galerkin’s decomposition to obtain strong periodic solutions for a set of abstract dissipative PDEs with a nonlinear term, which can lead to concrete fluid models such as incompressible magnetomicrocular fluid model, and incompressible Navier-Stokes equation. In their paper [11], they turned to establishing existence of periodic solution of an ODE whose solution space is spanned by finite number of basis from the decomposition. Furthermore, the solution of the ODE was proven to weakly converge to a solution of the abstract PDE. When higher regularity of this ODE ($H^2$-spatial-regularity) was obtained, the strong solution is reached by limit passage to an infinite sum from the compactness argument. In their proof, amplitude of force is small. In fact, this is a generalized situation of no periodic force. That is, periodic solution is trivially 0 if $f \equiv 0$, while periodic solution (referred as limit cycle) is unique but still close to zero, if $f$ is sufficiently small in some sense. Recently, S. Chen et al. extended this analysis to viscous Burgers’ equation (1) (see, e.g., [6]) under similar smallness assumption about external force, and show the numerical results if the amplitude of force is large enough. Their results show that when amplitude passes a critical value, more than one periodic solutions arise and some of them become unstable, leaving rigorous analysis still open. Readers may refer to R. Temam’s book [15] for Galerkin’s decomposition and corresponding wellposedness framework applied to dissipative equations. Besides, KdV (Korteweg-de Vries) equations are proved to possess periodic solutions by one of the authors, if similar assumption that the force has small amplitude holds (see, e.g., [16, 17, 20]). Instead of Galerkin’s method as solution of KdV equation doesn’t have form of Galerkin-type decomposition, the
The proof is mainly based on one of local wellposedness results in a particular Sobolev space $Y_{t,T}$ (defined on time interval $[t, t+\theta]$ with a particular $T$ including smoothing effects; see next section) in J. Bona, S. Sun, and B. Zhang [1, 2, 3]. Existence of periodic solutions of damped KdV equation on half line is proved with similar fashion in a variant of the $Y_{t,T}$ as well in [2]. Moreover, local and global stability of periodic solutions of KdV or KdV-Burgers equation posed on finite domains were obtained by one of us in [16, 17, 20]. The outline of the proof is, separating the problem into two parts: linear and nonlinear, and linear subproblem is done immediately while one is able to employ contraction mappings to obtain boundedness of solution in $Y_{t,T}$ for the nonlinear subproblem with small bound, then asymptotic periodicity with exponential decay can be proved. Exact periodic solution is achieved by selecting right initial data, and stability is achieved by boundedness of solutions. In [11, 6, 16, 17, 20], authors all proved limit cycle is stable, and in fact it is a stable global attractor. For this topic of dissipative equations, we refer readers to R. Temam’s book [15] again. For damped forced KdV equations, J. Ghidaglia, G. Sell, and Y. You proved existence of stable global attractors (see, e.g., [8, 14]).

The current paper will use similar techniques in $Y_{t,T}$ to obtain existence theory for viscous Burgers’ equation, instead of Galerkin method and compactness argument. This way turns out to bringing periodicity result in $L^2(\Omega)$ as we separate the regularity issue of solution in another round of discussion, while [11] and [6] directly established it in $H^2(\Omega)$, hence ours hold under a weaker smallness assumption of $f$. That is, only requirement about $L^2(\Omega)$ norm of $f(x,t)$ for any $t$. In their papers, the $L^2(\Omega)$ norm of not only $f(x,t)$’s at any time but that of first-order time derivative $f_t(x,t)$’s are required to be sufficiently small, as they need the regularity estimates of $u$ from smallness of $f_t$ to apply compactness argument. Both works pursue global stability of periodic solutions, and in our paper, the local stability is achieved first via an iteration technique. Our work is the first paper invoking idea from [1, 3, 16, 17, 20] to parabolic equations. We also pursue results in $H^s(\Omega)$ for any positive $s$ in the latter part of this paper, after modifying smallness conditions on norms of derivatives of $f(x,t)$ instead of $f$’s $L^2$ norm besides initial data is in $H^s(\Omega)$. We prove periodicity when $s = 2$ as demonstration for that of any positive $s$ as counterpart of [11] and [6]’s achievement.

The rest of this paper is organized as follows: notations and the main results will be stated in Section 2; Section 3 will deal with the preliminary lemmas; In Section 4, we shall prove theorems on the existence of periodic solutions in $L^2(\Omega)$ (see next section) sequentially; In Section 5, we will show similar result in $H^s(\Omega)$.

2. Main results. The notations in this paper are shown as follows:

$\|\cdot\|$ denotes $L^2$ norm on interval $\Omega = [0,1]$, while $\|\cdot\|_X$ means the canonical norm of Banach space $X$. Though $u(x,t)$ is bivariate, if it is modeled by mild solution form, we prefer using $u(t)$ like ODE fashion. We use $\|u(t)\|_X$ to represent $\|u(\cdot,t)\|_X$. In this article, we pay attention to estimates in a special Banach space

$$Y_{t,T} = \left\{ u : \sup_{t \leq s \leq t+T} \|u(s)\| + \left( \int_t^{t+T} \|u_x(s)\|^2 \, ds \right)^{\frac{1}{2}} < \infty \right\}$$

for given time point $t$ and length of time span $T$, endowed with norm

$$\|u\|_{Y_{t,T}} = \sup_{t \leq s \leq t+T} \|u(s)\| + \left( \int_t^{t+T} \|u_x(s)\|^2 \, ds \right)^{\frac{1}{2}}.$$
If \( u \in Y_{t,T} \) for all \( t \geq 0 \), we denote \( u \in Y_T \). Throughout the paper, \( T \) is given and fixed.

Now we can state our main results as follows:

**Theorem 2.1.** For given positive \( T \) and \( \tau \), there exists a positive number \( \delta^* \), when
\[
\|u\|_{Y_{\tau,T}} \leq C(\|\phi\| + \sup_{t \in [0,\infty)} \|f(t)\|)
\]
with constant \( C \) depending on \( T \).

**Theorem 2.2.** If the assumption in Theorem 2.1 holds, solution \( u(x,t) \) of equation (1) possesses asymptotical periodicity. In detail, that is, given positive \( T \) and \( \tau \), there exist positive constants \( C \) and \( \rho \) such that
\[
\|u(\cdot, \cdot + \theta) - u(\cdot, \cdot)\|_{Y_{\tau,T}} \leq C \exp(-\rho\tau).
\]

**Theorem 2.3.** If the assumption in Theorem 2.1 holds, equation (1) possesses a periodic solution with period \( \theta \) in \( L^2(\Omega) \), when initial data \( \phi \) is properly selected. And this periodic solution is locally stable.

**Theorem 2.4.** There exists a \( \delta > 0 \) such that \( \sup_{t \in [0,\infty)} \|f\| \leq \delta \), then the periodic solution \( \tilde{u}(x,t) \) found in Theorem 2.3 of equation (1) is in fact globally stable. That is, any \( u(x,t) \) will decay exponentially to \( \tilde{u}(x,t) \) when \( t \to \infty \). Moreover, this periodic solution is unique.

To obtain theorem 2.1, we would separate discussion into two subproblems: one for a linear equation, and one for a nonlinear equation. Our aim will show results for these separate ones hold, then results for the original equation hold by superposing.

3. Preliminaries. Let \( u(x,t) = v(x,t) + z(x,t) \), where \( v(x,t) \) satisfies linear equation
\[
\begin{aligned}
v_t - v_{xx} &= f(x,t), \\
v(0,t) &= v(1,t) = 0, \\
v(x,0) &= \phi(x),
\end{aligned}
\]
and \( z(x,t) \) satisfies the nonlinear equation with homogeneous boundary and initial datum
\[
\begin{aligned}
z_t - z_{xx} &= -z z_x - (zv)_x - vv_x, \\
z(0,t) &= z(1,t) = 0, \\
z(x,0) &= 0.
\end{aligned}
\]

We will prove the boundedness result of solution of the linear equation.

**Lemma 3.1.** For given \( T \), there exists a constant \( C > 0 \) independent of \( T \), linear equation (5) has estimate
\[
\|v\|^2_{Y_{0,T}} \leq C \left[ \|\phi\|^2 + \left( \int_0^T \|f(s)\| ds \right)^2 \right],
\]
Furthermore, for $\tau > 0$, there exists $C$ independent of $\tau$ and $T$ such that
\[
\|v\|^2_{Y_{\tau,T}} \leq C \left[ \|v(\tau)\|^2 + \left( \int_{\tau}^{\tau+T} \|f(s)\|ds \right)^2 \right].
\] (8)

Moreover, there exists $C > 0$ dependent of $T$ such that
\[
\|v\|^2_{Y_{\tau,T}} \leq C \left[ \|\phi\|^2 + \sup_{t \in [0,\infty)} \|f(t)\|^2 \right]
\] (9)
for any $\tau \geq 0$.

Proof. We formulate the mild solution for $t > 0$
\[
v(t) = e^{\Delta t} \phi + \int_0^t e^{\Delta (t-s)} f(s)ds
\] (10)
with Laplace operator $\Delta$ which generates analytic semigroup $\{e^{\Delta t}\}_{t \geq 0}$. According to semigroup theory for Laplacian as a generator, there holds estimate $\|e^{\Delta t}\| \leq e^{-\omega t}$ for a known $\omega > 0$, whence
\[
\|v(t)\| \leq e^{-\omega t} \|\phi\| + \int_0^t e^{-\omega (t-s)} \|f(s)\|ds
\]
\[
\leq \|\phi\| + \int_0^t \|f(s)\|ds.
\]

To estimate term $\int_0^t \|v_x\|^2 ds$, we multiply $v$ on both sides of the equation (3.1) and integrate over $(x,t) \in [0,1] \times [0,t]$, then
\[
\int_0^t \|v_x(s)\|^2 ds \leq \int_0^T \|f(s)\|\|v(s)\|ds + \frac{1}{2} \|\phi\|^2
\]

Given upper bound $T > 0$ of time $t$,
\[
\int_0^T \|v_x(s)\|^2 ds \leq \sup_{t \in [0,T]} \|v(t)\| \int_0^T \|f(s)\|ds + \frac{1}{2} \|\phi\|^2
\]
\[
\leq \frac{3}{2} \left( \int_0^T \|f(s)\|ds \right)^2 + \|\phi\|^2.
\]

Combining these two above estimates of $\|v\|$ and $\int_0^T \|v_x\|^2 ds$, we reach (7).

For any $\tau > 0$, we have similar mild solution formula:
\[
v(\tau + T) = e^{\Delta T} v(\tau) + \int_\tau^{\tau+T} e^{\Delta (\tau+s)} f(s)ds
\]
whence
\[
\|v\|^2_{Y_{\tau,T}} \leq \|v(\tau)\|^2 + \left( \int_\tau^{\tau+T} \|f(s)\|ds \right)^2
\]
in light of similar argument pursuing (7).

To prove (9) for any $\tau > 0$, we note the mild solution formulation
\[
v(\tau) = e^{\Delta \tau} \phi + \int_0^\tau e^{\Delta (\tau-s)} f(s)ds,
\]
then
\[ \|v(\tau)\| \leq e^{-\omega\tau}\|\phi\| + \int_0^\tau e^{\Delta(t-s)}\|f(s)\|ds \]
\[ \leq e^{-\omega\tau}\|\phi\| + \sup_{t\in[0,\tau]}\|f(t)\| \times \frac{1}{\omega}(1 - e^{-\omega\tau}) \]
\[ \leq e^{-\omega\tau}\|\phi\| + \frac{1 - e^{-\omega\tau}}{\omega} \sup_{t\in[0,\infty)}\|f(t)\| \]

For \( \int_\tau^{\tau+T}\|v_x(s)\|^2ds \), by multiplying \( v \) to the equation and integrate over \([\tau, \tau + T] \times [0, 1] \), we infer from the above estimate of \( \|v(\tau)\| \) for any \( \tau \), and Poincaré and Young inequalities, that
\[ \int_\tau^{\tau+T}\|v_x(s)\|^2ds = \frac{1}{2}\|v(r)\|^2 - \frac{1}{2}\|v(T)\|^2 + \int_\tau^{\tau+T}f(x,s,v(x,s))dxds \]
\[ \leq \frac{1}{2}\|v(r)\|^2 + \frac{1}{2}\|v(T)\|^2 + \int_\tau^{\tau+T}\|f(s)\|\|v(s)\|ds \]
\[ \leq C(\|\phi\|^2 + \sup_{t\in[0,\infty)}\|f(s)\|^2) + \frac{1}{2}\int_{\tau}^{\tau+T}\|v_x(s)\|^2ds + C\int_\tau^{\tau+T}\|f(s)\|^2ds \]
\[ \leq C(\|\phi\|^2 + \sup_{t\in[0,\infty)}\|f(s)\|^2) + \frac{1}{2}\int_{\tau}^{\tau+T}\|v_x(s)\|^2ds + CT\sup_{t\in[0,\infty)}\|f(s)\|^2. \]

These two estimates give (9). The proof is done. \( \square \)

For the nonlinear equation (6), we have the following local wellposedness result:

**Lemma 3.2.** For given \( T \), there exists a sufficiently small \( \delta > 0 \), \( \|v\|_{Y_0,T} \leq \delta \), then \( \|z\|_{Y_0,T} \leq \|v\|_{Y_0,T} \).

Moreover, For given \( \tau \), \( T > 0 \), there exists \( \delta' > 0 \) such that if \( \|z(\tau)\| + \|v\|_{Y_\tau,T} \leq \delta' \), then \( z(t) \) satisfies
\[ \|z\|_{Y_{\tau,T}} \leq 2(\|z(\tau)\| + \|v\|_{Y_{\tau,T}}). \] (11)

Before we prove this wellposedness result, we would borrow a bilinear estimate from [3], which is useful in contraction mapping formulations.

**Lemma 3.3.** There exists constant \( C \) which doesn’t depend on \( T \) for \( u, v \in Y_{0,T} \) such that
\[ \int_0^T \|(uv)_x\|ds \leq C(T^{1/2} + T^{1/3})\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}. \] (12)

When \( T > 1 \), it is trivial to denote majorization constant as \( CT^{1/2} \) in the inequality.

The following is the proof of Lemma 3.2.

**Proof.** We would give out the proof of estimate of \( \|z\|_{Y_{0,T}} \), and that of \( \|z\|_{Y_{\tau,T}} \) with nonzero \( \tau \) will follow by slight modification of the current one.

We formulate
\[ z(t) = -\int_0^t e^{\Delta(t-s)}(zz_x + vz_x + vv_x)ds. \]

Define the map \( q \mapsto \Gamma(q) \) referring to \( v \in Y_{0,T} \):
\[ \Gamma(q;v) = -\int_0^t e^{\Delta(t-s)}(qq_x + (qv)_x + vv_x)ds. \]

If we succeed to prove \( \Gamma(q;v) \) has a fix point in \( Y_{0,T} \), we obtain the existence of mild solution \( z(t) \) in \( Y_{0,T} \).
Define constant $M$ and a bounded subset in $Y_{0,T}$

$$M := \|v\|_{Y_{0,T}},$$

$$S_M = \{ q \in Y_{0,T}, \|q\|_{Y_{0,T}} \leq M \},$$

where $M$ will be selected properly.

First, we prove the boundedness of the map in $S_M$ given $q \in S_M$. Since $\Gamma(q)$ satisfies equation (6), we use estimate (8) with $f = -qq_x - (qv)_x - vv_x$ and Lemma 3.2 to infer that

$$\|\Gamma(q;v)\|_{Y_{0,T}} \leq \int_0^T e^{\Delta(T-s)} \|qq_x + (qv)_x + vv_x\| ds$$

$$\leq \int_0^T \left( \|q\|_{Y_{0,T}}^2 + \|q\|_{Y_{0,T}} \|v\|_{Y_{0,T}} + \|v\|_{Y_{0,T}}^2 \right) ds$$

$$\leq C(T^{1/2} + T^{1/3}) \left( \|q\|_{Y_{0,T}}^2 + \|v\|_{Y_{0,T}}^2 \right)$$

$$\leq 2CTM^2,$$

where $CT = C(T^{1/2} + T^{1/3})$. If $M \leq \frac{1}{2CT}$, then the boundedness is achieved:

$$\|\Gamma(q;v)\|_{Y_{0,T}} \leq M.$$

Second, we prove the Lipschitz continuity of map $\Gamma$ with respect to $q$ with Lipschitz constant less than 1. Pick any two distinct $q_1, q_2 \in S_M$, via Lemma 3.2,

$$\|\Gamma(q_1;v) - \Gamma(q_2;v)\|_{Y_{0,T}} \leq \int_0^T \left( \|q_1\|_{Q_{0,T}}^2 + \|q_2\|_{Q_{0,T}}^2 \right) ds$$

$$\leq C(T^{1/2} + T^{1/3}) \left( \|q_1\|_{Q_{0,T}} + \|q_2\|_{Q_{0,T}} \right)$$

$$\leq C(T^{1/2} + T^{1/3}) M \times 3M$$

$$\leq \frac{1}{2} \|q_1 - q_2\|_{Y_{0,T}}.$$

Therefore, we can define $\delta := M$ for any fixed $M \leq \frac{1}{2CT}$ to reach the above two properties, such that the fixed-point theorem leads to wellposedness of $q$ in subset $S_M$. It is noticed that if $T \uparrow \infty$, $\delta \downarrow 0$.

When $\tau > 0$, similar fixed-point type argument on subset $S_M$ of $Y_{\tau,T}$ is valid when one can apply to

$$\Gamma(q;v) = z(\tau) - \int_{\tau}^{\tau+T} e^{\Delta(T-s)} (qq_x + (qv)_x + vv_x)(s) ds,$$

with $M' = 2(\|z(\tau)\| + \|v\|_{Y_{\tau,T}})$.

In fact,

$$\|\Gamma(q;v)\|_{Y_{\tau,T}} \leq \|z(\tau)\| + C_T \left( \|q\|_{Y_{\tau,T}}^2 + \|q\|_{Y_{\tau,T}} \|v\|_{Y_{\tau,T}} + \|v\|_{Y_{\tau,T}}^2 \right)$$

$$\leq \|z(\tau)\| + \frac{3CT}{2} \left( \|q\|_{Y_{\tau,T}}^2 + \|v\|_{Y_{\tau,T}}^2 \right)$$

$$\leq \frac{M'}{2} + \frac{3CT}{2} \times 5M'^2.$$

If $M' \leq \frac{4}{15CT}$, we obtain $\Gamma(q;v) \leq M'$. The desired Lipschitz continuity holds exactly as before, if $M' \leq \frac{1}{3CT}$. It turns out that $\delta' \leq \frac{4}{15CT}$. The proof is complete. □
Next, we will derive similar a priori estimate for equation (6).

**Lemma 3.4.** For $T > 0$, there exist a constant $C > 0$ depending on $T$, and small constant $\delta'' > 0$ such that if $\|v\|_{Y_T} \leq \delta''$, then

$$\|z\|_{Y_{\tau,T}} \leq C\|v\|^2_{Y_T}$$

(13)

for any $\tau \geq 0$.

**Proof.** Note mild solution holds for equation (6) from time $\tau$ to $\tau + t$ when $t$ is positive:

$$z(\tau + t) = e^{\Delta t}z(\tau) - \int_\tau^{\tau+t} e^{\Delta(t-s)}(zz_x + (zv)_x)(s)ds - \int_\tau^{\tau+t} e^{\Delta(t-s)}vu_x(s)ds$$

$$=: W(t)z(\tau) + Q(\tau + t, \tau, z(\tau), v) + P(\tau + t, \tau, v)$$

Denote $z_k = z(kT)$ for $k \in \mathbb{N}$. It holds iteration formula from mild solution at times $\{kT\}$:

$$z_k = W(T)z_{k-1} + Q(kT, (k-1)T, z_{k-1}, v) + P(kT, (k-1)T, v)$$

and $z_0 = z(0) = 0$.

We will use this iteration to prove (13) when all $\tau$’s are at $kT$’s, then prove when $\tau$’s are between $kT$’s.

Therefore, from Lemma 3.3,

$$\|z_k\| \leq \|W(T)\|\|z_{k-1}\| + C_T \left( \|z\|_{Y_{(k-1)T,T}}^2 + \|z\|_{Y_{(k-1)T,T}} \|z\|_{Y_{(k-1)T,T}} + \|v\|_{Y_{(k-1)T,T}}^2 \right)$$

$$\leq e^{-\omega T}\|z_{k-1}\| + C_T\left(\|z_{k-1}\| + \|v\|_{Y_{(k-1)T,T}}^2 \right)$$

by Young’s inequality.

If $\|z_j\| \leq \eta$ for $j \in \{1, 2, \cdots, k-1\}$, let $\xi = e^{-\omega T} + \frac{5}{2}\eta C_T$. We have iterative formula

$$\|z_k\| \leq \xi\|z_{k-1}\| + \frac{\eta C_T}{2}\|v\|_{Y_{(k-1)T,T}}^2.$$

According to Lemma 3.3, for $\eta > 0$, if $\|z_{j-1}\| + \|v\|_{Y_{(j-1)T,T}} \leq \frac{\eta}{2}$, then $\|z_j\| \leq \eta$. We can let $\eta$ be sufficiently small such that $\xi := e^{-\omega T} + \frac{5}{2}C_T\eta < 1$.

Therefore, the following holds:

$$\|z_k\| \leq \xi\|z_{k-2}\| + \frac{\eta C_T}{2}\|v\|_{Y_{(k-2)T,T}}^2 + \frac{\eta C_T}{2}\|v\|_{Y_{(k-1)T,T}}^2$$

$$\vdots$$

$$\leq \xi^k\|z_0\| + \frac{\eta C_T}{2} \sum_{i=0}^{k-1} \xi^i\|v\|_{Y_{(k-i-1)T,T}}^2$$

$$\leq \frac{\eta C_T}{2}\|v\|_{Y_T}^2 \sum_{i=0}^{\infty} \xi^i = \frac{9C_T}{2(1-\xi)}\|v\|_{Y_T}^2.$$
since \( y_0 = 0 \). We can find in (13) majorization constant 
\[ C = \frac{9C_T}{2^{2(T-1)}} \]
depending on \( T \). And \( \delta'' < \frac{9}{2} \).

On the other hand, for \( \tau = kT \), it holds
\[ \|z\|_{Y_{kT,T}} \leq 2(\|z(kT)\| + \|v\|_{Y_{kT,T}}) \]
in (11) of Lemma 3.2, to warrant the desired estimate for \( \|z\|_{Y_{kT,T}} \).

When \( \tau \in ((k-1)T, kT) \) for \( k \in \mathbb{N} \), i.e., \( \tau = (k-1)T + \alpha T, 0 \leq \alpha \leq 1 \), similarly as before, we have estimate
\[ \|z(\tau)\| \leq \|z((k-1)T)\| + \frac{3C_T}{2}(\|z\|_{(k-1)T, \alpha T}^2 + \|v\|_{Y_{\tau, \alpha T}}^2) \]
which implies the estimate of \( \|z\|_{Y_{\tau,T}} \) by Lemma 3.2 or (11).

The following lemma will be used to prove asymptotical periodicity. Proof is an energy estimate.

**Lemma 3.5.** Given positive \( \tau \) and \( T \), \( a \) is a variable coefficient, if \( \|a\|_{Y_T} \) is bounded, the linear equation
\[
\begin{align*}
 w_t - w_{xx} + (aw)_x &= 0, \\
 w(0, t) &= w(1, t) = 0, \\
 w(x, 0) &= \psi(x),
\end{align*}
\]
has boundedness estimates
\[ \|w\|_{Y_{\tau,T}} \leq C\|\psi\|, \]
\[ \|w\|_{Y_{\tau,T}} \leq C\|w(\tau)\| \]
with majorization constant \( C \) depending on \( \|a\|_{Y_T} \).

**Proof.** We multiply \( w \) to both sides of (14) and integrate it over \( \Omega = [0, 1] \). Due to integration by parts,
\[
\frac{1}{2} \int_0^1 \frac{d}{d\tau} \|w(\tau)\|^2 + \|w_x(\tau)\|^2 = \frac{1}{2} \int_0^1 a_x w^2 dx.
\]  (17)

We infer from the right hand side that
\[
\frac{1}{2} \int_0^1 |a_x w^2| dx \leq \frac{1}{2} \|w(\tau)\|_{L^\infty(\Omega)} \int_0^1 |a_x w| dx
\]
\[
\leq \frac{1}{2} \int_0^1 |w_x(\tau)| dx \times \left( \int_0^1 a_x^2 dx \right)^{\frac{1}{2}} \left( \int_0^1 w^2 dx \right)^{\frac{1}{2}}
\]
\[
\leq \frac{1}{4} \|w_x(\tau)\|^2 + \frac{1}{4} \|a_x(\tau)\|^2 \|w(\tau)\|^2.
\]

Combining with equation (17), term \( \frac{1}{4} \|w_x\|^2 \) can be absorbed to the left, and we arrive at
\[
\frac{d}{d\tau} \|w(\tau)\|^2 \leq C\|a_x(\tau)\|^2 \|w(\tau)\|^2.
\]

Gronwall inequality directly warrants the boundedness
\[ \|w(\tau)\|^2 \leq C\|\psi\|^2 \]
with constant \( C \) depending on size of \( \|a\|_{Y_T} \).
The estimate about $\int_{\tau}^{\tau+T} \|w_x(s)\|^2 ds$ can be approached by looking at the inequality
\[
\frac{1}{2} \frac{d}{d\tau} \|w(\tau)\|^2 + 3 \frac{1}{4} \|w_x(\tau)\|^2 \leq \frac{1}{4} \|a_x(\tau)\|^2 \|w(\tau)\|^2.
\]
Integrate it over $[\tau, \tau+T]$
\[
\frac{3}{4} \int_\tau^{\tau+T} \|w_x(s)\|^2 ds \leq -\frac{1}{2} \|w(\tau+T)\|^2 + \frac{1}{2} \|w(\tau)\|^2 + \frac{1}{4} \sup_{\tau \leq s \leq \tau+T} \|w(s)\|^2 \int_\tau^{\tau+T} \|a_x(s)\|^2 ds. \quad (18)
\]

We apply the boundedness estimate of $\|w(\tau)\|^2$ to $\|w(\cdot)\|^2$ terms involved, and we are done for (3.11). (3.12) is achievable by same manner. \(\square\)

4. **Proof of theorems.** In this section, we will prove our main results. They are, the boundedness, the asymptotical $\theta$-time-periodicity of equation (1), and the existence of its $\theta$-periodic solution.

4.1. **Proof of Theorem 2.1.** From Lemma 3.1–3.4, it is true that when $\|\phi\| + \sup_{t \in [0, \infty)} \|f(t)\|$ is sufficiently small, $\|v\|_T$ is small, and further $\|z\|_T$ is small. Therefore, for $\tau \geq 0$
\[
\|u\|_{Y_{\tau,T}} \leq C \left( \|\phi\| + \sup_{t \in [0, \infty)} \|f(t)\| \right),
\]
with constant $C$ dependent of $T$.

Rigorously, $\delta^* = \frac{1}{T} \min\{\delta^2, \delta'^2, \delta''^2\}$ where $C$ is the majorizing constant in inequality (9). \(\square\)

4.2. **Proof of Theorem 2.2.** Consider $w(x, t) = u(x, t + \theta) - u(x, t)$. $w$ satisfies the initial boundary value problem (14) with initial data $\psi(x) = u(x, \theta) - \phi(x)$ and function $a(x, t) = \frac{1}{2} (u(x, t) + u(x, t + \theta))$.

According to the theorem statement, it suffices to prove $\|w\|_{Y_{\tau,T}}$ decays exponentially.

For any $\tau > 0$, we have mild solution formulation at time $\tau + t$
\[
w(\tau + t) = W(t)w(\tau) - \int_0^t W(t-s)(aw)_x(s+\tau)ds. \quad (19)
\]
For all $k \in \mathbb{N}$ and $T > 0$, we define
\[
w_k(x) = w(x, kT),
\]
i.e.,
\[
w_0(x) = w(x, 0) = \psi(x),
w_1(x) = w(x, T),
\]
\[\vdots\]
We define
\[ R = W(T), \quad S(T, w_{k-1}) = - \int_0^T W(T - s)(aw)_x((k - 1)T + s)ds \]

Directly from (19),
\[ w_k = Rw_{k-1} + S(T, w_{k-1}). \]

Owing to Lemma 3.5, \( S \) has estimate
\[ \|S(T, w_{k-1})\| = \|\int_0^T W(T - s)(aw)_x(s)ds\| \leq C_T \|a\|_{Y_{(k-1)T,T}} \|w\|_{Y_{(k-1)T,T}} \leq C \|a\|_{Y_{(k-1)T,T}} \|w_{k-1}\| \]
where \( C \) is dependent of \( C_T \) and \( \|a\|_{Y_{(k-1)T,T}} \).

Therefore,
\[ \|w_k\| \leq (\|R\| + C\|a\|_{Y_{(k-1)T,T}})\|w_{k-1}\|. \]

Since \( \|W(T)\| < 1 \) and \( \|a\|_{Y_{(k-1)T,T}} \) is small such that
\[ \mu = \|R\| + C\|a\|_{Y_{(k-1)T,T}} < 1 \]
and
\[ \|w_k\| \leq \mu\|w_{k-1}\| \leq \cdots \leq \mu^k\|w_0\| = \exp(k \ln(\mu))\|w_0\|, \]
which leads to exponential decay as \( \ln(\mu) < 0 \) when \( \tau \)'s are at \( kT \)'s, with decay rate
\[ \rho = -\frac{\ln(\mu)}{T}. \]

Now if \( \tau = kT + t \) with \( t \in (0, T) \) and \( k \in \mathbb{N} \), note \( k = \frac{\tau}{T} - 1 \), then
\[ \|w(\tau)\| \leq C \mu^{\frac{\tau - 1}{T}}\|\psi\| \leq C e^{(\frac{T - 1}{T})\ln(\rho)}\|\psi\|. \]

Note \( \rho = -\frac{\ln(\mu)}{T} \), then we obtain
\[ \|w(\tau)\| \leq C e^{-\rho T}\|\psi\| \]
for \( \tau \in \mathbb{R}_+ \).

As to estimate \( \int_{\tau + T}^{\tau + T} \|w_x(s)\|^2 ds \), we can follow the exactly same procedure in Lemma 3.5 to reach (17).

Note that exponential decay of \( \|w(\tau)\| \) obtained in the first half of this proof, and the boundedness of \( \|a\|_{Y_{\tau,T}} \) is implied by that of \( \|u\|_{Y_{\tau,T}} \) from Theorem 2.1, the proof is complete. \( \square \)

**Remark 1.** Note proof of theorem 2.2 indicates a more general fact that any \( w \) satisfying (14) with a sufficiently small \( \|a\|_{Y_T} \) possesses an exponential stability.

**4.3. Proof of theorem 2.3.** The idea of proving theorem 2.3 is to passing the time limit of asymptotically periodic solution, then invoking this limit as initial data prescribed to (1) to acquire periodic solution.

Let
\[ u_k = u(x, k\theta) \]
with \( k \in \mathbb{N} \). According to previous discussion, \( u_k \in L^2(\Omega) \). For any \( n, m \in \mathbb{N} \),
\[
\|u_{n+m} - u_n\| = \| \sum_{l=0}^{m-1} (u_{n+l+1} - u_{n+l}) \| \\
\leq \sum_{l=0}^{m-1} \|u_{n+l+1} - u_{n+l}\| \\
= \sum_{l=0}^{m-1} \|w((n + l)\theta)\| \\
\leq C \sum_{l=0}^{m-1} e^{-\rho(n + l)\theta} \\
\leq C e^{-\rho\theta} \frac{1}{1 - e^{-\rho\theta}} \downarrow 0
\]
as \( n \uparrow \infty \). I.e., \( \{u_k\} \) is a Cauchy sequence in \( L^2(\Omega) \). Denote its limit as \( \bar{\phi}(x) \) and \( \tilde{\phi} \in L^2(\Omega) \).

\( \tilde{\phi} \) will generate a precisely periodic solution if \( \bar{\phi} \) is the initial value of (1). In fact, if \( \tilde{\phi} \) generates \( \tilde{u}(x, t) \), then
\[
\|\tilde{u}(., \theta) - \bar{u}(., 0)\| = \|\tilde{u}(., \theta) - w(., (n + 1)\theta) + w(., (n + 1)\theta) - \tilde{u}(., 0)\| \\
\leq \|\tilde{u}(., (n + 1)\theta) - w(., (n + 1)\theta)\| + \|w(., (n + 1)\theta) - \tilde{u}(., 0)\| \\
\to 0,
\]
as each of three terms on the right hand side converges to 0 when \( n \to \infty \). We can obtain this due to the fact that \( \lim_{k \to \infty} u_k = \bar{u} \) in \( L^2(\Omega) \).

Local stability can be achieved by checking the exponential stability of function \( u(x, t) - \tilde{u}(x, t) \). That is, if let \( w(x, t) = u(x, t) - \tilde{u}(x, t) \) with \( w(x, 0) = \phi(x) - \tilde{\phi}(x) \), then \( w \) satisfies (14) with \( a(x, t) = \frac{1}{2} (u(x, t) + \tilde{u}(x, t)) \). One can take \( \delta^* \) in theorem 2.1 small enough so that \( \|(u, \bar{u})\|_{Y_T} \) is sufficiently small, then Remark 1 directly guarantees the exponential stability.

4.4. Proof of theorem 2.4. Note, the uniqueness can be inferred by exponential stability of (14). In fact, let \( w(x, t) = \tilde{u}(x, t) - \tilde{v}(x, t) \) where \( \tilde{u} \) and \( \tilde{v} \) are two periodic solutions generated by same initial value satisfying homogeneous Dirichlet condition. One can check that \( w \) satisfies (14) with \( a = \frac{1}{2} (\tilde{u} + \tilde{v}) \).

One can argue that, if \( \|(\tilde{u}, \tilde{v})\|_{Y_T} \) is sufficiently small, then Remark 1 concludes \( \|w(t)\| \leq C e^{-rt} \) holds for \( t \geq t_0 \) and a positive rate \( r \). However, since \( w \) is periodic, \( w \) has to be 0, hence the global uniqueness of the periodic solution is achieved.

Therefore, it suffices to prove this uniqueness is unique. I.e., for any non-periodic solution \( u \), it will converge to the same limit cycle. To reach that, we shall prove global absorbing property:
\[
\|u(t)\| \leq e^{-rt} \|\phi\| + C\delta, \quad (20)
\]
under condition \( \sup_{t \in [0, \infty)} \|f\| \leq \delta \). This is attainable via energy estimate of (1) by multiplying \( u \) on both sides.

One can arrive at
\[
\frac{d}{dt} \|u(t)\|^2 + \|u_x(t)\|^2 \leq -\frac{1}{2} \|u(t)\|^2 + C\|f\|^2
\]
by Poincaré and Young’s inequalities.
Gronwall inequality brings out
\[ \|u(t)\|^2 \leq e^{-\frac{t}{2}}\|\phi\|^2 + C \sup_{t \in [0,\infty)} \|f\|^2, \]
which leads to the desired estimate for any \( u(x,t) \in L^2(\Omega) \).

Upon this estimate, there exist a time \( t_0 \) and sufficiently small \( \delta \), such that there is a small constant \( \alpha \), \( \|u(t_0)\| < \alpha \) and \( \alpha^2 + \delta^2 \leq \delta^* \) which is the smallness condition of Theorem 2.1. Local stability in previous theorem guarantees the exponential decay for \( t \geq t_0 \), which is global. \( \square \)

5. Periodic solutions in \( H^s(\Omega) \). In this section, we will consider the existence of a period solution in \( H^s(\Omega) \) for any \( s > 0 \) following same fashion in Section 3 and 4. That is, to find a suitable initial data \( \phi \in H^s(\Omega) \) so that the solutions is time-periodic and in \( H^s \).

It suffices to prove the case of \( s = 2 \), as existence is still valid when \( s \in (0,2) \) if it holds for \( s = 2 \), obtained by Marcinkiewicz interpolation applied to solution map \( \phi \mapsto u(x,t) \). This is applicable to any positive \( s \). In the following, we will prove the boundedness and existence of periodic solutions for equation of \( u_t \) in \( Y_{\tau,T} \) under assumption \( \|\phi\|_{H^2} + \sup_{t \in [0,\infty)} \|f_t(t)\| \) is small, as similar theorems are claimed:

**Theorem 5.1.** For given positive \( T \) and \( \tau \), there exists a positive number \( \delta^{**} \), when \( \|\phi\|^2_{H^2} + \sup_{t \in [0,\infty)} \|f_t(t)\|^2 \leq \delta^{**} \), solution \( u(x,t) \) of equation (1) satisfies
\[ \|u_t\|_{Y_{\tau,T}} \leq C(\|\phi\|_{H^2} + \sup_{t \in [0,\infty)} \|f_t(t)\|) \tag{21} \]
with constant \( C \) depending on \( T \).

**Theorem 5.2.** If the assumption in Theorem 5.1 holds, solution \( u(x,t) \) of equation (1) possesses asymptotical periodicity. In detail, that is, given positive \( T \) and \( \tau \), there exist positive constants \( C \) and \( \rho \) such that
\[ \|u_t(\cdot, \cdot + \theta) - u_t(\cdot, \cdot)\|_{Y_{\tau,T}} \leq C \exp(-\rho \tau). \tag{22} \]

**Theorem 5.3.** If the assumption in Theorem 5.1 holds, equation (1) possesses a periodic solution with period \( \theta \) in \( H^2(\Omega) \), when initial data \( \phi \) is properly selected. Moreover, this periodic solution is locally stable.

**Theorem 5.4.** There exists a \( \delta > 0 \), such that \( \sup_{t \in [0,\infty)} \|f_t\| \leq \delta \), then the periodic solution \( \tilde{u}(x,t) \) found in Theorem 5.3 of equation (1) is globally stable and unique in \( H^2(\Omega) \). That is, any \( u(x,t) \) will decay exponentially to \( \tilde{u}(x,t) \) in \( H^2(\Omega) \) when \( t \to \infty \).

In the rest of paper, \( C_T \) will be a universal one related to constant \( T \).

5.1. Preliminaries. We consider the subproblems about \( v_t(x,t) \) and \( z_t(x,t) \). They are, \( v(x,t) \) satisfies
\[
\begin{align*}
v_{tt} - v_{txx} &= f_t(x,t), \\
v_t(0,t) &= v_t(1,t) = 0, \\
v_t(x,0) &= \zeta(x),
\end{align*}
\]
with \( \zeta(x) = u_t(x,0) \) satisfies \( u_t(x,0) - u_{xx}(x,0) + u(x,0)u_x(x,0) = f(x,t) \); \( z(t,x) \) satisfies the nonlinear equation with homogeneous boundary and initial datum
\[
\begin{cases}
  z_t - z_{xx} = -v_tv_x - v_tv_{tx} - v_tz_x - v_tz_{tx} - z_tv_t - z_tz_x - z_tz_{tx}, \\
  z(x,0) = z_k(1,0) = 0, \\
  z_t(x,0) = 0.
\end{cases}
\tag{24}
\]

Straight calculation exactly as that in lemma 3.1 gives the following lemma about \( v_t \) in space \( Y_{\tau,T} \).

**Lemma 5.5.** For given \( T \), there exists a constant \( C > 0 \) independent of \( T \), \( v_t \) has estimate for \( \tau \geq 0 \), there exists \( C \) independent of \( \tau \) and \( T \) such that
\[
\|v_t\|^2_{Y_{\tau,T}} \leq C \left[ \|v_t(\tau)\|^2 + \left( \int_\tau^{\tau+T} \|f_t(s)\|^2 \, ds \right)^{\frac{1}{2}} \right].
\tag{25}
\]

And, there exists \( C > 0 \) dependent of \( T \) such that
\[
\|v_t\|^2_{Y_{\tau,T}} \leq C \left[ \|\zeta\|^2 + \sup_{t \in [0,\infty)} \|f_t(t)\|^2 \right]
\tag{26}
\]
for any \( \tau \geq 0 \).

For \( z_t \), we have the following lemmas about equation (24) with sketched proof similar as counterparts in section 3.

**Lemma 5.6.** For given \( \tau \geq 0 \), \( T > 0 \), there exists \( \delta_1 \geq 0 \) such that if \( \|z_t(\tau)\| + \|z\|_{Y_{\tau,T}} + \|v\|_{Y_{\tau,T}} + \|v_t\|_{Y_{0,T}} \leq \delta_1 \), then \( z_t \) satisfies
\[
\|z_t\|_{Y_{\tau,T}} \leq C(\|z_t(\tau)\| + \|v_t\|_{Y_{\tau,T}} + \|v\|_{Y_{\tau,T}} + \|z\|_{Y_{\tau,T}}),
\tag{27}
\]
where \( C \) is independent of \( \tau \) and \( T \).

**Proof.** It suffices to prove the case when \( \tau = 0 \).

Consider the mapping \( \Gamma \), \( q_t \mapsto \Gamma(q_t) \):
\[
\Gamma(q_t;\tau,v_t,z) = -\int_0^t e^{\Delta(t-s)} (v_tv_x + v_tv_{tx} + v_tz_x + v_tv_z + zv_t + q_tv_x + q_tv_z + q_tv_t) ds
\]
in a subset of \( Y_{0,T}, S_M \), with
\[
S_M = \{q_t \in Y_{0,T}||q_t||_{Y_{0,T}} \leq M \}
\]
with \( M := \max\{\|v_t\|_{Y_{0,T}}, \|v\|_{Y_{0,T}}, \|z\|_{Y_{0,T}}\} \).

First, we infer that
\[
\|\Gamma(q_t)\|_{Y_{0,T}} \leq C_T (\|v\|_{Y_{0,T}} + \|v_t\|_{Y_{0,T}} + \|v\|_{Y_{0,T}} + \|z\|_{Y_{0,T}} + \|v_t\|_{Y_{0,T}} + \|z\|_{Y_{0,T}})
\leq C_T M^2.
\]

Take \( M \leq \frac{1}{C_T} \), then \( \|\Gamma(q_t)\|_{Y_{0,T}} \leq M \).

For distinc \( q_{t1} \) and \( q_{t2} \), we have
\[
\|\Gamma(q_{t1}) - \Gamma(q_{t2})\|_{Y_{0,T}} \leq C_T \|v\|_{Y_{0,T}} \|q_{t1} - q_{t2}\|_{Y_{0,T}} + C_T \|z\|_{Y_{0,T}} \|q_{t1} - q_{t2}\|_{Y_{0,T}}
\leq \frac{1}{2} \|q_{t1} - q_{t2}\|_{Y_{0,T}}
\]
if $M \leq \frac{1}{2C_T}$.

Similarly as Lemma 3.2, contracting mapping holds to lead to the conclusion, if $M$ is sufficiently small and $M = \delta_1$. \hfill $\Box$

**Lemma 5.7.** For $T > 0$, there exists a small constant $\delta_2 > 0$ such that if $\| (v_t, v) \|_{Y_T} \leq \delta_2$, then

$$\| z_t \|_{Y_{T, T}} \leq C_T (\| v_t \|_{Y_T}^2 + \| v \|_{Y_T}^2)$$

for any $T \geq 0$.

**Proof.** Note for any $T \geq 0$,

$$z_t(\tau + t) = e^{\Delta t} z_t(\tau) - \int_{\tau}^{\tau + t} e^{\Delta (t-s)} (v_t z_x + v z_{tx} + z_t v_x + z v_{tx} + z_t z_x + z z_{tx}) ds$$

$$= W(t) z_t(\tau) + Q(t + \tau, \tau, z_t, v, v_t, z) + P(\tau + t, \tau, v_t, v)$$

Let $z_{t,k} = z_t(kT)$ for $k \in \mathbb{N}$, and then $z_{t,0} = z_t(0) = 0$. Hence

$$z_{t,k} = W(t) z_{t,k-1} + Q(kT, (k-1)T, z_{t,k-1}, v, v_t, z) + P(kT, (k-1)T, v_t, v),$$

and it holds

$$\| Q(kT, (k-1)T, z_{t,k-1}, v, v_t, z) \|_{Y_{(k-1)T, T}} \leq C_T (2 \| z_t \|_{Y_{(k-1)T, T}} \| v_t \|_{Y_{(k-1)T, T}}$$

$$+ 2 \| z_t \|_{Y_{(k-1)T, T}} \| v \|_{Y_{(k-1)T, T}} + 2 \| v \|_{Y_{(k-1)T, T}} \| z_t \|_{Y_{(k-1)T, T}}),$$

$$\| P(kT, (k-1)T, v_t, v) \|_{Y_{(k-1)T, T}} \leq 2C_T \| v \|_{Y_{(k-1)T, T}} \| v_t \|_{Y_{(k-1)T, T}}.$$

Therefore, from Lemma 5.5 and 3.4, and Young’s inequality,

$$\| z_{t,k} \| \leq e^{-\omega T} \| z_{t,k-1} \| + C_T (\| v \|_{Y_{(k-1)T, T}}^2 \| v_t \|_{Y_{(k-1)T, T}} + (\| z_t \|_{Y_{(k-1)T, T}} + \| v \|_{Y_{(k-1)T, T}}$$

$$+ \| v_t \|_{Y_{(k-1)T, T}} + \| v \|_{Y_{(k-1)T, T}}) \| v \|_{Y_{(k-1)T, T}} + (\| z_t \|_{Y_{(k-1)T, T}} + \| v \|_{Y_{(k-1)T, T}}$$

$$+ \| v_t \|_{Y_{(k-1)T, T}} + \| v \|_{Y_{(k-1)T, T}}) \| v \|_{Y_{(k-1)T, T}} + \| v \|_{Y_{(k-1)T, T}}^2$$

$$\leq (e^{-\omega T} + C_T \| v \|_{Y_{(k-1)T, T}} + C_T \| v \|_{Y_{(k-1)T, T}}^2) \| z_t \|_{Y_{(k-1)T, T}} + C_T (\| v \|_{Y_{(k-1)T, T}}^2$$

$$+ \| v \|_{Y_{(k-1)T, T}}^2 + \| v_t \|_{Y_{(k-1)T, T}}^2).$$

Let $\xi = e^{-\omega T} + C_T \| v \|_{Y_{(k-1)T, T}} + C_T \| v \|_{Y_{(k-1)T, T}}^2$ and $\xi < 1$ as $\| (v, v_t) \|_{Y_{(k-1)T, T}}$ is small. Following iteration with geometric series fashion, and other argument for smoothing term in Lemma 3.4, we obtain the result. \hfill $\Box$

Take derivative of equation (14) with respect to $t$, we reach equation about $u_t$ (i.e., (29)), and we have following lemma:

**Lemma 5.8.** Given positive $\tau$ and $T$, $u_t$ satisfying

$$\begin{cases}
    w_{tt} - w_{xx} + (aw)_{tx} = 0, \\
    w_t(0, t) = w_t(1, t) = 0, \\
    w_t(x, 0) = \sigma(x),
\end{cases}$$

has estimate for a $C$ dependent of $\| a \|_{Y}$ and $\| a_t \|_{Y}$ such that

$$\| u_t \|_{Y_{\tau, T}} \leq C \| \sigma \|,$$

$$\| u_t \|_{Y_{\tau, T}} \leq C (\| u_t(\tau) \| + \| u \|_{Y_{\tau, T}}).$$
Proof. Equation (29) is
\[ w_{tt} - w_{txx} = -(a_t w)_x - (aw)_x. \]

We will prove \( \| w_t(\tau) \| \leq C \| \sigma \|, \) and the rest follows as similar estimates in proof of Lemma 3.5.

Multiply \( w_t \) on both sides, and integrate by parts on \( \Omega \) for time \( t = \tau \), we obtain
\[
\frac{d}{d\tau} \| w_t(\tau) \|^2 + \| w_{tx}(\tau) \|^2 \leq C(\| a_x(\tau) \| \| w_1(\tau) \|^2 + \| w_x(\tau) \|^2 \| a_t(\tau) \|^2).
\]

Gronwall inequality yields
\[
\| w_1(\tau) \|^2 \leq \| \sigma \|^2 \exp \left( \int_0^\tau \| a_x(s) \|^2 ds \right) \int_0^\tau \| w_x(s) \|^2 ds \times \sup_{s \in [0,\tau]} \| a_t(s) \|^2 \leq C \| \sigma \|^2,
\]
which is resulted from Lemma 5.5 and 5.7, and we are done.

5.2. Proof of theorems.

5.2.1. Proof of Theorem 5.1. Lemma 5.5, 5.6, 5.7 immediately lead to the theorem, if \( \| \phi \|_{H^*} + \sup_{t \geq 0} \| f_t \| \) is sufficiently small. \( \square \)

5.2.2. Proof of Theorem 5.2. Note
\[
w_{t,k}(\tau + t) = W(t)w_t(\tau) - \int_0^t W(t - s) \left[ (a_t w)_x + (aw)_x \right](s + \tau) ds.
\]

Define \( w_{t,k}(x) = w_t(x, kT) \) for \( k \in \mathbb{N} \), and \( U(T, w) = -\int_0^T W(T - s)(a_t w)_x(s) ds \), then
\[
w_{t,k} = Rw_{t,k-1} + U(T, w) + S(T, w_{t,k-1}).
\]

Therefore,
\[
\| w_{t,k} - U(T, w) \| \leq \mu \| w_{t,k-1} \|.
\]

Due to the fact that \( \| U \| \) follows an exponential decay (Theorem 2.2), the theorem is proved from same argument with \( \mu < 1 \) as Theorem 2.2. \( \square \)

5.2.3. Proof of Theorem 5.3. We only need to replace \( u_k \) in proof of theorem 2.3 by \( u_{t,k} = u_t(x, k\theta) \), and \( u_{t,k} \in L^2(\Omega) \) owing to previous Theorem 5.1, and proof follows similarly, and so do the local exponential stability. As analogy of local stability in theorem 2.3, local stability in \( H^2 \) is derived from same calculation in proof of theorem 5.2. \( \square \)

5.2.4. Proof of Theorem 5.4. It suffices to prove \( \| u_t(t) \| \) can be small when \( t \) is sufficiently large. The estimate would be close to (20).

Let \( v = u_t \), and take derivative of (1) with respect to \( t \), we have
\[
v_t - v_{xx} + (vw)_x = f_t.
\]

Multiplying \( v \) to its both sides, then
\[
\frac{1}{2} \frac{d}{dt} \| v(t) \|^2 + \| v_x(t) \|^2 - \int_0^1 wv_x dx = \int_0^1 f_tv dx.
\]
Note $\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{1}^{\frac{1}{2}} \|u_x\|_{1}^{\frac{1}{2}}$, and $\|u\|_{H^2(\Omega)}$ can be sufficiently small, then by Poincaré and Young’s inequalities like in proof of theorem 2.4, one can derive

$$\|v\| \leq e^{-rt} \|\zeta\| + \sup_{t \in [0, \infty)} \|f_t\|,$$

for a positive rate $r$.

The rest of proof will follow similarly as in theorem 2.4.

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