On the quantization of isomonodromic deformations on the torus

D.A. Korotkin* and J.A.H. Samtleben

II. Institut für Theoretische Physik, Universität Hamburg,
Lurupfer Chaussee 149, 22761 Hamburg, Germany
E-mail: korotkin@x4u2.desy.de, jahsamt@x4u2.desy.de

Abstract

The quantization of isomonodromic deformation of a meromorphic connection on the torus is shown to lead directly to the Knizhnik-Zamolodchikov-Bernard equations in the same way as the problem on the sphere leads to the system of Knizhnik-Zamolodchikov equations. The Poisson bracket required for a Hamiltonian formulation of isomonodromic deformations is naturally induced by the Poisson structure of Chern-Simons theory in a holomorphic gauge fixing. This turns out to be the origin of the appearance of twisted quantities on the torus.

1 Introduction

The fundamental role of the Knizhnik-Zamolodchikov (KZ) equations is well-known. Originally discovered in conformal field theory — resulting from the Ward-identities as differential equations for correlation functions in WZW models [1] — further appearance in Chern-Simons theory [2]–[8] shows these equations to represent underlying links between different physical areas in the fundamental mathematical framework of affine algebras and quantum groups. In the context of geometrical quantization of Chern-Simons theory [4, 5], KZ equations arise as conditions of invariance of the quantization procedure with respect to a holomorphic change of polarization.

In [9] it was realized the link of KZ equations with a classical mathematical object — the Schlesinger equations on the Riemann sphere [10]. Playing in particular an important role in the theory of integrable systems [11], Schlesinger equations describe monodromy preserving deformations of matrix differential equations with rational coefficients on the sphere. The procedure of canonical quantization of the Schlesinger equations in the framework of “multi-time” Hamiltonian formalism derived in [12] leads (in the Schrödinger picture) to the KZ equations. Being rewritten in the Heisenberg picture [13] the KZ equations turn out to coincide with the Schlesinger system for higher dimensional matrices supplied with additional commutation relations. It is interesting to note that the Schlesinger equations, being the classical counterpart of the KZ equations, in general case do not admit solutions in terms of quadratures, although the KZ system does [14]. In this sense the quantized system (KZ) turns out to be simpler then the classical one (Schlesinger).

In [15] a similar “two-time” quantization procedure was applied to the stationary axisymmetric Einstein equations and allowed to express the explicit solutions of related Wheeler-deWitt equations...
in terms of the integral representation for solutions of KZ equations found in [14].

The main result of the present paper is the existence of a similar link between KZ and the Schlesinger system on the torus. The KZ equations on the torus — usually referred to as Knizhnik-Zamolodchikov-Bernard (KZB) equations — were first derived in the framework of conformal field theory on higher genus Riemann surfaces [14] and further studied by several authors (see [17] and references therein). We show them to arise from quantization of isomonodromic quantization on the torus. As a by-product of our construction we derive the explicit form of the Schlesinger equations on the torus, which also seems to be new.

The main differences to the case of the Riemann sphere are the following:

• The basic variables, meromorphic Lie-algebra valued one-forms, are non-singlevalued (“twisted”) on the torus. The origin of the twist becomes clear, if one considers this one-form as the holomorphic gauge of a general (single-valued) flat connection on the torus: the gauge transformation which leads to holomorphic gauge turns out to be non-invariant with respect to translation along the basic periods in general.

In WZW models on the torus, twisted quantities were introduced by hand to achieve completeness of the Ward identities by defining a proper action of affine zero modes on character valued correlation functions, which led to the KZB equations [16].

• In addition to the deformation with respect to the positions of the poles of the connection on the torus, the Schlesinger system on the torus contains a deformation equation with respect to the moduli (i.e. \(b\)-period) of the torus. Thus the set of the deformation parameters coincides with the full set of coordinates on the moduli space of punctured Riemann surfaces.

The multi-time Hamiltonian formulation of the genus one Schlesinger equations with respect to these deformation parameters reveals the link with Chern-Simons theory on the torus — the relevant Poisson bracket turns out to be the Dirac bracket of Chern-Simons Poisson bracket [2] with respect to the flatness and gauge fixing constraints.\(^\dagger\) Canonical quantization of this Hamiltonian formulation is shown to give the KZB equations in the form of [17].

In Section 2 we discuss the links between all these objects for the case of the sphere. In Section 3 we show how to extend this treatment on the torus with the main focus on the features which arise due to topological non-triviality. Section 4 contains some further comments.

2 Holomorphic Poisson bracket and isomonodromic quantization on the sphere

We consider the space of holomorphic Lie-algebra valued one-forms on the punctured Riemann sphere, that are meromorphic with simple poles on the whole sphere. These forms may be viewed as connections on a trivial bundle. To simplify notation and without any loss of generality, in the explicit expressions we restrict to the case of \(\mathfrak{g} = \mathfrak{su}(2)\).

Introducing local coordinates on the sphere by marking a point \(\infty\), an element \(A(z)dz\) of this space is uniquely determined by its poles \(z_j\) and the corresponding residues \(A_j\) taking values in \(\mathfrak{g}\):

\[
A(z) = \sum_{j=1}^{N} \frac{A_j}{z - z_j} \tag{2.1}
\]

\(^1\)In hidden form these equations were exploited, for example, in the investigation of the XYZ Landau-Lifshitz equations [18], where the associated spectral problem lives on the curve of genus one.

\(^2\)The related fact on the sphere was noticed without proof in [19].
Holomorphic behavior at infinity is ensured by

\[ Q := \sum_j A_j = 0 \]  

(2.2)

There is a natural Poisson structure on the space of holomorphic connections on the punctured complex plane, that may be formulated in the equivalent expressions:

\[ \{ A^a_i, A^b_j \} = 2 \delta_{ij} \varepsilon^{abc} A^c_i \]  

(2.3)

\[ \Leftrightarrow \{ A^a(z), A^b(w) \} = -\varepsilon^{abc} A^c(z) \frac{A^a(w)}{z-w} \]

\[ \Leftrightarrow \{ A(z) \otimes A(w) \} = [r(z-w), A(z) \otimes I + I \otimes A(w)] \]

with a classical \( r \)-matrix \( r(z) = \frac{\Pi^z}{z} \), \( \Pi \) being the \( 4 \times 4 \) permutation operator.

The condition (2.2) that restricts the connections to live on the sphere, transforms as a first-class constraint under this bracket: \( \{ Q^a, Q^b \} = \varepsilon^{abc} Q^c \)

### 2.1 Holomorphic bracket from gauge fixed Chern-Simons theory

Let us shortly describe the relation of the bracket (2.3) to the fundamental Atiyah-Bott symplectic structure, that was claimed by Fock and Rosly [19].

The space of smooth connections on a Riemann surface is endowed with the natural symplectic form [20]:

\[ \Omega = \text{tr} \int \delta A \wedge \delta A, \]

that leads to the Poisson bracket

\[ \{ A^a_\bar{z}(z), A^b_\bar{z}(w) \} = \delta^{ab} \delta^{(2)}(z-w), \]  

(2.4)

where the connection \( A \) is split into \( A_z dz + A_{\bar{z}} d\bar{z} \) and the \( \delta \)-function is understood as a real two-dimensional \( \delta \)-function: \( \delta^{(2)}(x+i y) \equiv \delta(x) \delta(y) \).

The condition of flatness is \( F = dA + A \wedge A = 0 \) and transforms as a first-class constraint under the bracket:

\[ \{ F^a(z), F^b(w) \} = \varepsilon^{abc} F^c(z) \delta^{(2)}(z-w) \]

This constraint generates gauge transformations

\[ A \mapsto g A g^{-1} + dgg^{-1} \]  

(2.5)

that leave the symplectic structure invariant.

These brackets and constraints arise naturally from the Chern-Simons action [2]. They may be extended to punctured Riemann surfaces if the singularities of the connection restrict to first order poles, leading to \( \delta \)-function-like singularities of the curvature [2, 3].

We may now fix the gauge freedom by choosing the gauge \( A_{\bar{z}} = 0 \) that makes flatness turn into holomorphy. Whereas there is certainly no problem to achieve this gauge on the complex plane, in the case of the punctured sphere a few comments are in order:

- Poles in \( A_{\bar{z}} \) may be removed by gauge transformations of the local form \((z-z_j)^{A_j}(\bar{z}-\bar{z}_j)^{A_j}\). Even though these gauge transformations are singular in \( z_j \) they should yield no severe problems, as they do not change the residue of the pole, but shift it into a pole of the surviving \((1,0)\)-form \( A_z dz \).
A more subtle problem is the non-existence of appropriate global gauge-transformations in general. The space of smooth Lie-algebra-valued \((0,1)\)-forms on the sphere is naturally isomorphic to the space of holomorphic bundles, the pure gauge forms corresponding to the trivial bundle. This means, in general gauging away \(A_z\) may be interpreted as \(A_z dz\) becoming a connection on a nontrivial bundle on the sphere. An additional degree of freedom, measuring the non-triviality of the transition functions would have to be introduced, the \(\mathbb{Z}\)-valued Chern number in the case of the sphere. We will work out an example of only locally defined gauge transformations in the next section on the torus.

However, as on the sphere the space of pure gauge \((0,1)\)-forms builds a dense subset in the space of smooth \((0,1)\)-forms \([20, 7]\), we restrict the following considerations to this subspace. A closer investigation of this restriction determines the subspace of physical Chern-Simons states \([7, 8]\). In our framework this identification should be achieved in terms of the monodromy algebra of the connection \(A\), see last section.

The bracket between constraints and gauge-fixing condition is of the form:

\[
\{F^a(z), A^b_z(w)\} = -\delta^{ab} \delta^{(2)}(z - w) + \varepsilon^{abc} A^c_z(z) \delta^{(2)}(z - w) \quad (2.6)
\]

Following the Dirac procedure of gauge fixing \([21]\), the bracket (2.6) has to be modified to make it consistent with the now second-class constraints. The matrix of brackets (2.6) can be inverted using \(\partial_z \frac{1}{z} = \delta^{(2)}(z)\) and yields exactly the bracket (2.3). Notice that in the framework of geometric quantization the variables \(A_z\) and \(A\) are, according to (2.2), considered as canonically conjugated coordinate and momentum, respectively. After the holomorphic gauge fixing the surviving variable \(A(z) \equiv A_z(z)\) looks, according to (2.3), more like a combination of angular momenta.

Note that because of the appearance of \(\partial_z\) in (2.6), the holomorphic part of the constraints \(F^a(z)\) survives as a first-class constraint. As holomorphic functions on the sphere are constants, the remaining flatness conditions become

\[
\int F^a(z) dz d\bar{z} = \int \partial_z A^a(z) dz d\bar{z} = \sum_j A^a_j = Q^a
\]

The holomorphic bracket (2.3) is therefore induced by restricting the fundamental Poisson structure on the space of smooth connections to the space of flat connections and fixing holomorphic gauge. The first-class constraint (2.2) ensuring \(A(z)\) to live on the sphere, arises naturally as surviving flatness condition, generating the constant gauge transformations.

### 2.2 Hamiltonian formulation of isomonodromic deformation

We now describe isomonodromic deformation on the sphere in terms of the introduced holomorphic Poisson structure. Consider the system of linear differential equations:

\[
\partial \Psi(z) = A(z) \Psi(z) \quad (2.7)
\]

As \(A(z)\) has simple poles, the Lie-group-valued function \(\Psi(z)\) lives on a covering of the punctured sphere. Let \(\Psi\) be normalized to \(\Psi(\infty) = I\), thereby marking one of the points \(\infty\) on this covering. In the neighborhood of the points \(z_i\), the function \(\Psi\) is given by:

\[
\Psi(z) = G_i \Psi_i(z)(z - z_i)^T_i C_i \quad (2.8)
\]

with \(\Psi_i(z) = I + \mathcal{O}(z - z_i)\) being holomorphic and invertible. The relation to the residues of the connection is given by \(A_i = G_i T_i G_i^{-1}\). 

The local behavior (2.8) also yields explicit expressions for the monodromies around the singularities:

\[ \Psi(z) \mapsto \Psi(z)M_i \quad \text{for } z \text{ encircling } z_i \]

with

\[ M_i = C_i^{-1} \exp(2\pi iT_i)C_i \]

Note that the normalization \( \Psi(\infty) = I \) couples the freedom of right multiplication in the linear system (2.7) to the left action of constant gauge transformations (2.5) on \( \Psi \) to \( g \Psi g^{-1} \) and therefore \( M_i \mapsto gM_ig^{-1} \).

The aim of isomonodromic deformation [11] is the investigation of a family of linear systems (2.7) parameterized by the choice of singular points \( z_i \), that have the same monodromies. In other words, one studies the change of the connection data \( A_i \) with respect to a change in the parameters of the Riemann surface that is required to keep the monodromy data constant. Treating \( A_i(z) \) and \( \Psi(z) \) as functions of \( z \) and \( z_i \), these isomonodromy conditions impose a formal condition of \( z_i \)-independence of the monodromy data \( T_i \) and \( C_i \). This ensures the function \( \partial_{z_i} \Psi^{-1}(z) \) to have a simple pole in \( z_i \):

\[ \partial_{z_i} \Psi(z) = \frac{-A_i}{z - z_i} \Psi(z) \quad (2.9) \]

Compatibility of these equations with the system (2.7) yields the classical Schlesinger equations [10]

\[ \partial_{z_i} A_j = \frac{[A_i, A_j]}{z_i - z_j} \quad \text{for } j \neq i, \quad \partial_{z_i} A_i = -\sum_{j \neq i} \frac{[A_i, A_j]}{z_i - z_j} \quad (2.10) \]

A Hamiltonian description of this dependence is given by [12]

\[ H_i = \sum_{j \neq i} \frac{\text{tr}(A_i A_j)}{z_i - z_j} \quad (2.11) \]

in the described holomorphic Poisson-bracket (2.3).

As was first noticed by Reshetikhin [9] and realized in a simpler form in [13], quantization of this system leads directly to the Knizhnik-Zamolodchikov equations, that are known as differential equations for correlation functions in conformal field theory [1].

Quantization of the system is performed straight-forward by replacing the Poisson structure by commutators. Shifting the \( z_i \)-dependence of the operators \( A_i^a \) (2.10) into the states on which these operators act, corresponds to a transition from the Heisenberg picture to the Schrödinger picture in ordinary quantum mechanics. In the Schrödinger representation the quantum states \( |\omega\rangle \) then are sections of a holomorphic \( V \equiv \bigotimes_j V_j \) vector bundle over \( X_0 \equiv \mathbb{C}^N \setminus \{ \text{diagonal hyperplanes} \} \). The \( z_i \)-independent operator-valued coordinates of \( A_i \) are realized as

\[ A_i^a = i\hbar \, I \otimes \ldots \otimes t_i^a \otimes \ldots \otimes I \quad (2.12) \]

where \( t_i^a \) acts in the representation \( V_i \).

In this Schrödinger picture the quantum states \( |\omega\rangle \) obey the following multi-time \( z_i \)-dynamics:

\[ \partial_{z_i} |\omega\rangle = H_i |\omega\rangle = i\hbar \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} |\omega\rangle \quad (2.13) \]

Here, \( \Omega_{ij} = \text{tr}(t_i \otimes t_j) \) is the Casimir operator of the algebra acting non-trivially only on \( V_i \) and \( V_j \).
System (2.13) may be equivalently rewritten in the Heisenberg picture introducing the multi-time evolution operator $U(\{z_i\})$ by

$$\partial_{z_i} U = H_i U, \quad U(\{z_i = 0\}) = \text{id}$$

Then in terms of the variables

$$\hat{A}^a_i \equiv U A^a_i U^{-1}$$

the quantum equations of motion give rise to higher-dimensional Schlesinger equations with the matrix entries $A^a_i$ being operators in $V$. These equations turn out to be a very special case of the general $(\dim g \times \dim V)$-dimensional classical Schlesinger system since all operators $\hat{A}^a_i$ may be simultaneously transformed to (2.12) by some similarity transformation (2.14).

The system (2.13) that defines horizontal sections on the bundle of quantum states is the famous Knizhnik-Zamolodchikov system [1] that arises here in a different context.

3 Quantization of Schlesinger equations in genus one

We are going to repeat the analysis of the last section for the case of the torus now, which will show the link between isomonodromic deformation and the KZB-equations on the torus. The conceptual novelty of twisted functions, that is introduced more or less by hand in WZW conformal field theories on the torus in order to get a proper description of the action of inserted affine zero modes in the correlation functions [16], enters the game in a very natural way in our treatment.

3.1 Holomorphic gauge fixing

We start again from a smooth $\mathfrak{su}(2)$-valued one-form $A$ on the torus. In the explicit formulae we will use standard Chevalley generators $t^3, t^\pm$. Denote the periods of the torus by 1 and $\tau$.

Holomorphic gauge $A_\bar{z} = 0$ can not be achieved in general. However, taking into account our remarks from the previous section, the essential fact is [8], that a dense subspace of smooth (0,1)-forms can be gauged into constants of the form

$$A_\bar{z} = \frac{2\pi i \lambda}{\tau - \bar{\tau} \sigma_3}, \quad \lambda \in \mathbb{C}$$

(3.1)

The holomorphic gauge condition would require an additional gauge transformation of the kind $g = \exp(2\pi i \lambda \frac{z}{\tau - \bar{\tau} \sigma_3})$. This is obviously multi-valued on the torus, having a multiplicative twist: $g \mapsto \exp(2\pi i \lambda \sigma_3)g$ for $z$ encircling the fundamental $(0, \tau)$-cycle. The result of a gauge transformation of this kind is a twist in the remaining holomorphic $(1,0)$-form $A(z)$:

$$A(z + 1) = A(z) \quad A(z + \tau) = e^{2\pi i \lambda \sigma_3} A(z)$$

(3.2)

In components this reads:

$$A^3(z + \tau) = A^3(z) \quad A^\pm(z + \tau) = e^{\pm 4\pi i \lambda} A^\pm(z)$$

Even though in general gauge transformations must be defined globally single-valued in order to conserve physics, in this case our proceeding is justified by the fact, that the non-gauge-trivial part of $A_\bar{z}$ survives as an arising twist of the holomorphic connection.

This is how the holomorphic gauge causes the appearance of twisted quantities in a very natural way.
3.2 Some meromorphic functions on the torus

Before we start to investigate isomonodromic quantization on the torus, let us collect some simple facts about twisted meromorphic functions on the torus. A basic ingredient to describe functions of this kind, is Jacobi’s theta-function:

$$\theta(z) := \sum_{n \in \mathbb{Z}} e^{2\pi i (\frac{1}{2} n^2 \tau + n z)},$$

which is holomorphic, twisted as: $$\theta(z+1) = \theta(z), \quad \theta(z+\tau) = e^{-i\pi(\tau+2z)}\theta(z)$$ and has simple zeros for $$z \in \frac{1}{2}(\tau+1) + \mathbb{Z} + \tau\mathbb{Z}.$$ 

Standard combinations are the functions \([17]\):

$$\rho(z) := \theta'(z) - \frac{1}{2}(\tau+1) + i\pi \quad \text{and} \quad \sigma_\lambda(z) := \frac{\theta(\lambda - z - \frac{1}{2}(\tau+1))\theta'(\frac{1}{2}(\tau+1))}{\theta(z + \frac{1}{2}(\tau+1))\theta(\lambda - \frac{1}{2}(\tau+1))} (3.3)$$

which have simple poles with normalized residue in $$z = 0$$ and additive and multiplicative twist respectively:

$$\rho(z+\tau) = \rho(z) - 2\pi i; \quad \sigma_\lambda(z+\tau) = e^{2\pi i \lambda} \sigma_\lambda(z)$$

We should still mention the properties

$$\rho(-z) = -\rho(z); \quad \sigma_\lambda(z) = -\sigma_{-\lambda}(-z)$$

and the identity

$$\partial_\lambda \sigma_\lambda(x-y) = \sigma_\lambda(x-y)\left(\rho(z-x) - \rho(z-y)\right) - \sigma_{-\lambda}(z-x)\sigma_\lambda(z-y)$$

They can be proved checking residues and twist properties. All the following calculations rely on the fact, that meromorphic functions on the torus with simple poles are uniquely determined by their residues if they are multiplicatively twisted, whereas functions with additive or vanishing twist are determined only up to constants. Note further that in generic situation there are no holomorphic twisted functions on the torus.

3.3 Isomonodromic deformation

Equipped with these tools we can now start to describe the twisted meromorphic connection $$A(z)$$. Because of its twist properties \((3.3)\), $$A(z)$$ is of the form:

$$A^\pm(z) = \sum_j A_j^\pm \sigma_{\pm2\lambda}(z-z_j) \quad (3.4)$$

$$A^3(z) = \sum_j A_j^3 \rho(z-z_j) - B^3$$

Define again $$\Psi$$ by the linear system

$$\partial \Psi(z) = A(z)\Psi(z) \quad (3.5)$$

The function $$\Psi$$ will get monodromies $$M_i$$ and $$M_{(0,1)}$$ from the right hand side, if $$z$$ encircles $$z_i$$ or the $$(0,1)$$ cycle of the torus. If $$z$$ runs along the $$(0,\tau)$$ cycle, $$\Psi$$ will exhibit an additional left monodromy due to the twist of $$A$$:

$$\Psi(z) \mapsto e^{2\pi i \lambda \sigma_3} \Psi(z) M_{(0,\tau)} \quad (3.6)$$
Under isomonodromic deformation we will understand the invariance of the right hand side monodromy data under the change of the parameters of the punctured torus, which are the singular points $z_i$ and the period $\tau$. The connection data in this case are the residues $A_i$, the additive constant $B^3$ and the twist $\lambda$.

Let us first investigate their $z_i$-dependence. In addition to the residues of $\partial_z \Psi \Psi^{-1}$ we have to determine its twist around $(0, \tau)$ from isomonodromy conditions. Equation (3.6) yields:

$$
\left( \partial_z \Psi \Psi^{-1} \right)(z) \mapsto e^{2\pi i \lambda} \text{ad}_{\sigma_3} \left( \partial_z \Psi \Psi^{-1} \right)(z) + 2\pi i \partial_z \lambda \sigma_3
$$

This determines the form of the $z_i$-dependence of $\Psi$ to:

$$
\left( \partial_z \Psi \Psi^{-1} \right)^{\pm}(z) = -A_{i}^{\pm} \sigma_{\pm2\lambda}(z - z_i)
$$

$$
\left( \partial_z \Psi \Psi^{-1} \right)^3(z) = -A_{i}^{3} \rho(z - z_i) + B_{i}^{3}
$$

and further on yields the $z_i$-dependence of the twist parameter $\lambda$:

$$
\partial_i \lambda = A_{i}^{3}
$$

We can now proceed as in the previous section. Compatibility of the equations (3.3) and (3.7) implies the following Schlesinger equations on the torus:

$$
\partial_z A_j^3 = -A_i^+ A_j^- \sigma_{2\lambda}(z_j - z_i) + A_i^- A_j^+ \sigma_{-2\lambda}(z_j - z_i); \quad j \neq i
$$

$$
\partial_z A_i^3 = \sum_{j \neq i} A_i^+ A_j^- \sigma_{2\lambda}(z_j - z_i) - \sum_{j \neq i} A_i^- A_j^+ \sigma_{-2\lambda}(z_j - z_i)
$$

$$
\partial_z A_j^\pm = \pm 2A_i^\pm A_j^3 \sigma_{\pm2\lambda}(z_j - z_i) \mp 2A_i^\pm A_j^3 \rho(z_j - z_i) \mp 2B_i^3 A_j^\pm; \quad j \neq i
$$

$$
\partial_z A_i^\pm = \pm 2 \sum_{j \neq i} A_i^\pm A_j^3 \rho(z_i - z_j) \pm 2 \sum_{j \neq i} A_i^\pm A_j^3 \sigma_{\pm2\lambda}(z_i - z_j) \mp 2B_i^3 A_i^\pm
$$

$$
\partial_i B_i^3 = \frac{1}{2} \sum_{j \neq i} \left( A_i^- A_j^+ \partial_{\lambda} \sigma_{2\lambda}(z_i - z_j) - A_i^+ A_j^- \partial_{\lambda} \sigma_{2\lambda}(z_j - z_i) \right)
$$

and a curvature condition on the constants $B_i$:

$$
\partial_i B_j^3 - \partial_j B_i^3 = \frac{1}{2} A_i^- A_j^+ \partial_{\lambda} \sigma_{2\lambda}(z_i - z_j) - \frac{1}{2} A_i^+ A_j^- \partial_{\lambda} \sigma_{2\lambda}(z_j - z_i)
$$

As on the sphere, it is possible to formulate this dependence as a multi-time Hamiltonian structure. The Hamiltonians can be written down as

$$
H_i = \sum_{j \neq i} \left( 2A_i^3 A_j^3 \rho(z_i - z_j) + A_i^+ A_j^- \sigma_{-2\lambda}(z_i - z_j) + A_i^- A_j^+ \sigma_{2\lambda}(z_i - z_j) \right) - 2B_i^3 A_i^3 + 2B_i^3 \sum_j A_j^3
$$

in the Poisson structure:

$$\{A_i^a, A_j^b\} = 2\delta_{ij} \epsilon^{abc} A_c^c$$

$$\{\lambda, B^3\} = \frac{1}{2}$$
We strongly suppose that this structure arises from holomorphic gauge-fixing of the original bracket (2.4) in the same way, as does the bracket (2.3) on the sphere. In particular, remembering the origin of \( \lambda \) (3.1), the second equation may be viewed as a reminiscent of (2.4) for the constant terms of \( A_z \) and \( A_{\bar{z}} \).

In analogy with (2.3) this Poisson structure admits a generalized \( r \)-matrix formulation

\[
\{ A(z) \otimes A(w) \} = [r(z - w), A(z) \otimes I + I \otimes A(w)] - \partial_\lambda r(z - w) \left( \sum_j A_j^3 \right)
\]

(3.13)

with the twisted \( r \)-matrix

\[
r(z) = \frac{1}{2} \rho(z)(t^3 \otimes t^3) + \sigma_2 \lambda(z)(t^+ \otimes t^-) + \sigma_{-2} \lambda(z)(t^- \otimes t^+)
\]

(3.14)

that in some sense restricts to a classical \( r \)-matrix formulation on the constraint surface (3.15) below. Validity of the Jacobi identities is expressed by a twisted version of the classical Yang-Baxter equation. Notice that the twisted \( r \)-matrix (3.14) reminds the elliptic \( r \)-matrix arising in the Hamiltonian formulation of the XYZ Landau-Lifshitz equation [22], which, however, has different twist properties and satisfies standard (non-twisted) Yang-Baxter equation.

In addition to (3.9), compatibility of the linear systems implies

\[
\sum_j A_j^3 = 0
\]

(3.15)

In the holomorphic Poisson-structure this simply generates the remaining constant gauge transformations compatible with (3.1). In contrast to the sphere, the gauge has been fixed more rigorously on the torus in order to diagonalize the twist around the \((0, \tau)\)-cycle. Ref. [23] treats the weaker gauge of arbitrary twist that accordingly leads to a stronger constraint. In this gauge it is still possible to achieve a normalization \( \Psi(z_0) = I \) of \( z_i \)-independence of \( \Psi \) at a fixed point \( z_0 \) like on the sphere, which turns out to be inconsistent with our restrictive choice of gauge.

A consequence of this restrictive gauge on the torus is the appearance of the undetermined constants \( B_j \) in the Schlesinger equations (3.9) on which we still have to spend some comments. The way they arise in the Hamiltonians (3.11) proves them to generate gauge transformations. This suggests to simply skip these terms from the Hamiltonians, as is in fact done in the sequel, leading to the KZB equations. However, this is certainly not compatible with the curvature condition (3.10) which in turn implies that these truncated Hamiltonians only commute up to (3.15), meaning that the generated \( z_i \) dynamics of the connection data produces isomonodromic deformation only up to certain shifts in the gauge orbit. Nevertheless, this seems to be the most elegant way to treat these gauge constants.

We will finally study isomonodromic deformation with respect to a change in the period \( \tau \) of the torus. This can be done in complete analogy with the just treated case. From (3.6) the twist of \( \partial_\tau \Psi \Psi^{-1} \) around \((0, \tau)\) turns out to be

\[
\left( \partial_\tau \Psi \Psi^{-1} \right)(z) \mapsto e^{2\pi i \lambda \mathrm{ad} \sigma_3} \left( \partial_\tau \Psi \Psi^{-1} \right)(z) - e^{2\pi i \lambda \mathrm{ad} \sigma_3} A(z) + 2\pi i \partial_\tau \lambda \sigma_3
\]

which leads to the following \( \tau \)-dependence of the function \( \Psi \):

\[
2\pi i \left( \partial_\tau \Psi \Psi^{-1} \right)^{\pm}(z) = \mp \frac{1}{2} \sum_j A_j^\pm \partial_\lambda \sigma_{\pm 2} \lambda(z - \hat{z}_j)
\]

(3.16)

\[
2\pi i \left( \partial_\tau \Psi \Psi^{-1} \right)^3(z) = \frac{1}{2} \sum_j A_j^3 \left( \rho(z - \hat{z}_j)^2 - \varphi(z - \hat{z}_j) \right) + B_j^3
\]
and determines the \( \tau \)-dependence of the twist parameter

\[
\partial_\tau \lambda = -\frac{1}{2\pi i} B^3
\]  

(3.17)

Compatibility of (3.5) and (3.16) now yields additional Schlesinger-type equations:

\[
2\pi i \partial_\tau A_i^3 = -\frac{1}{2} \sum_j A_i^+ A_j^- \partial_\lambda \sigma_{-2\lambda}(z_i - z_j) - \frac{1}{2} \sum_j A_i^- A_j^+ \partial_\lambda \sigma_{2\lambda}(z_i - z_j)
\]

(3.18)

\[
2\pi i \partial_\tau A_i^\pm = \pm \sum_j A_i^\pm A_j^3 \left( \rho(z_i - z_j)^2 - \wp(z_i - z_j) \right)
\]

\[
+ \sum_j A_i^\pm A_j^3 \partial_\lambda \sigma_{\pm 2\lambda}(z_i - z_j) \pm 2B_3^3 A_i^\pm
\]

\[
2\pi i \partial_\tau B_3 = -\frac{1}{8} \sum_{i,j} \left( A_i^+ A_j^- \partial_\lambda^2 \sigma_{-2\lambda}(z_i - z_j) - A_i^- A_j^+ \partial_\lambda^2 \sigma_{2\lambda}(z_i - z_j) \right)
\]

(3.19)

together with a curvature condition for \( \frac{1}{2\pi i} \partial_\tau B_3^3 - \partial_\tau B_i^3 \).

This dependence is described by the Hamiltonian

\[
2\pi i H_\tau = \frac{1}{2} \sum_{i,j} A_i^3 A_j^3 \left( \rho(z_i - z_j)^2 - \wp(z_i - z_j) \right)
\]

\[
+ \frac{1}{4} \sum_{i \neq j} \left( A_i^+ A_j^- \partial_\lambda \sigma_{-2\lambda}(z_i - z_j) - A_i^- A_j^+ \partial_\lambda \sigma_{2\lambda}(z_i - z_j) \right) + B_3^3 B_3^3 + 2B_3^3 \sum_j A_j^3
\]

(3.20)

where again we will skip \( B_3^3 \) under the above remarks.

### 3.4 KZB-equations

Quantization is again performed straightforward with (3.12) being replaced by

\[
[A_i^a, A_j^b] = 2i\hbar \delta_{ij} \varepsilon^{abc} A_c^c
\]

\[
[\lambda, B^3] = \frac{1}{2} i\hbar
\]

In the \( z_i \)-independent Schrödinger representation of the operators they can be realized as:

\[
A_i^a = i\hbar I \otimes \ldots \otimes t_i^a \otimes \ldots \otimes I
\]

\[
B^3 = -\frac{i}{2} \hbar \partial_\lambda
\]

acting on quantum states \( |\omega\rangle \) that are \( \lambda \)-dependent sections of a \( V \equiv \bigotimes_j V_j \) bundle over \( X_1 \equiv \{ \text{fundamental domain of } \tau \} \otimes \mathbb{C}^N \setminus \{ \text{diagonal hyperplanes} \} \).

The quantization of (3.9) and (3.18) in the Schrödinger picture provides this bundle with the horizontal connection:

\[
\partial_{z_i} |\omega\rangle = H_i |\omega\rangle = \frac{1}{2} i\hbar t_i^3 \partial_\lambda |\omega\rangle + i\hbar \sum_{j \neq i} \Omega_{ij}^3(z_i - z_j, \tau, \lambda) |\omega\rangle
\]

(3.20)

\[
2\pi i \partial_\tau |\omega\rangle = 2\pi i H_\tau |\omega\rangle = \frac{1}{4} i\hbar \partial_\lambda^2 |\omega\rangle + i\hbar \sum_{i,j} \Omega_{ij}^3(z_i - z_j, \tau, \lambda) |\omega\rangle
\]
with
\[
\Omega^\tau_{ij}(z, \tau, \lambda) = \frac{1}{2} \rho(z)(t^3_i \otimes t^3_j) + \sigma_{-2\lambda}(z) (t_i^+ \otimes t_j^-) + \sigma_{2\lambda}(z) (t_i^- \otimes t_j^+)
\]
\[
\Omega^\tau_{ij}(z, \tau, \lambda) = \frac{1}{8} (\rho(z)^2 - \varphi(z)) (t^3_i \otimes t^3_j) + \frac{1}{4} \partial_\lambda \sigma_{-2\lambda}(z) (t_i^+ \otimes t_j^-) - \frac{1}{4} \partial_\lambda \sigma_{2\lambda}(z) (t_i^- \otimes t_j^+)
\]
acting non-trivially only on \(V_i\) and \(V_j\).

This is the KZB connection, found in Ref. [16] as differential equations for character-valued correlation functions. The form (3.20) coincides exactly with the form presented in Ref. [17]. In particular, the term that includes the derivative with respect to the twist parameter \(\lambda\) is the explicit analogue of the action of affine zero modes on correlation functions in WZW models. We have thereby proved the fundamental role of the KZB connection in the context of isomonodromic deformation on the torus.

We stress again that in contrast to the KZ-system on the sphere these Hamiltonians only commute up to the constraint (3.15) which implies the fact that the KZB-connection is flat only as a connection on the subbundle of states annihilated by \(\sum_j t^3_j\), see [17].

4 Concluding Remarks

We have studied isomonodromic deformation on the torus and shown that quantization of the Schlesinger system on the torus leads to the KZB system of differential equations. The result suggests further that the quantization procedure of isomonodromic deformations on higher genus Riemann surfaces should lead to the corresponding higher KZB equations [16, 23].

Let us close with some general remarks.

- There obviously seem to be two fundamental links between quantization of Chern-Simons theory and the system of KZ(B) equations, that arise from geometric quantization [4, 5, 6] and isomonodromic quantization, respectively. The idea may be roughly sketched as follows:

```
| Classical Chern-Simons theory | KZ(B) connection |
|--------------------------------|------------------|
| \{A^a(z), A^b(w)\} = g^{ab}(z-w) | polarization invariant geometric quantization of the moduli space |
| holomorphic gauge fixing | (Twisted) Lie-Poisson algebra of holomorphic connections |
| \{A(z) \otimes A(w)\} = \{r(z-w), A(z) \otimes I + I \otimes A(w)\} + \sum \{\text{constraints}\} \partial_{\{\text{twist}\}} r(z-w) \quad \text{ (for genus } g \geq 1) |
```

- As both approaches lead directly to the KZ(B) connection, preserving some structure on the bundle of quantum states, they should be regarded as fundamental and probably somehow equivalent. In particular, one is tempted to consider further properties of KZ(B) equations, that were discovered in [7] and [8] — namely that these connections preserve the space of physical Chern-Simons states by preserving a certain fusion-rule-like condition — to be a corollary of this more fundamental structure, the more, as the KZ(B) systems prove to be sufficient but by no means unique in that context.
• The physical subspace of Chern-Simons theory may be isolated in the framework of isomonodromic quantization in the same way as in the combinatorial quantization of Chern-Simons theory [24], which we have not depicted in the diagram. The basic variables here are the monodromies of the connection $A$; their quantized algebra then turns out to be a certain quantum group. As this approach yields the same dimensions (Verlinde indices) of the physical spaces, on which these operators are nontrivially represented, that are known from geometrical quantization, it seems to be essentially equivalent. The link to KZ(B) equations in this framework should be provided by their appearance in the theory of quantum groups [25].

• In spite of the close relation between the Poisson structures, classically the Chern-Simons and Schlesinger systems are essentially different: the dynamics is trivial in the first case and nontrivial in the second one. “Times” of the Schlesinger system on a Riemann surface may be chosen as coordinates of the the moduli space of punctured Riemann surfaces which play in (quantum) Chern-Simons theory the role of formal deformation parameters. The natural question is: what is the role of Schlesinger system in classical Chern-Simons theory?

In this context one also has to notice an obvious link to 2+1 gravity: in the approach of Witten [26] 2+1 gravity is treated as a Chern-Simons theory, which suggests relevance of Schlesinger equations in this context; this agrees with the results of [27] where Schlesinger equations arise directly in 2+1 gravity treated in the framework of canonical Arnowitt-Deser-Misner formulation.

Acknowledgments We thank V. Schomerus and J. Teschner for useful discussions. D. K. is supported by Deutsche Forschungsgemeinschaft. H. S. thanks Studienstiftung des deutschen Volkes for financial support.

References

[1] V. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83.
[2] E. Witten, Comm. Math. Phys. 121 (1989) 351.
[3] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. B326 (1989) 104.
[4] S. Axelrod, E. Witten and S. Della Pierra, J. Diff. Geom. 33 (1991) 787.
[5] N. Hitchin, Comm. Math. Phys. 131 (1990) 347.
[6] M.A. Olshanetsky, Generalized Hitchin systems and Knizhnik-Zamolodchikov-Bernard equation on elliptic curves, preprint SISSA Ref. 125/95/EP, ITEP-TH3/95 (1995), hep-th/9510143.
[7] K. Gawedzki and A. Kupiainen, Comm. Math. Phys. 135 (1991) 531.
[8] F. Falceto and K. Gawedzki, Comm. Math. Phys. 159 (1994) 549.
[9] N. Reshethikin, Lett. Math. Phys. 26 (1992) 167.
[10] L. Schlesinger, J. Reine u. Angew. Math. 141 (1912) 96.
[11] M. Jimbo, T. Miwa and K. Ueno, Physica 2D (1981) 306.
[12] M. Jimbo, T. Miwa, Y. Môri and M. Sato, Physica 1D (1980) 80.
[13] J. Harnad, Quantum isomonodromic deformations and the Knizhnik-Zamolodchikov equations, preprint CRM-2890 (1994), hep-th/9406073.
[14] V.V. Schechtman and A.N. Varchenko, Lett. Math. Phys. 20 (1990) 279.
[15] D. Korotkin and H. Nicolai, Phys. Rev. Lett. 74 (1995) 1272; Phys. Lett. 356B (1995) 211.

[16] D. Bernard, Nucl. Phys. B303 (1988) 77; Nucl. Phys. B309 (1988) 145.

[17] G. Felder and C. Wieczerkowski, in: Mathematical Quantum Theory I: Field Theory and Many-Body Theory, CRM Proceedings & Lecture notes, vol. 7, ed. by J. Feldman, R. Froese and L.M. Rosen (1994);
G. Felder and C. Wieczerkowski, Conformal blocks on elliptic curves and the Knizhnik-Zamolodchikov-Bernard equations, preprint [hep-th/9411004] (1994).

[18] R.F. Bikbaev and A.R. Its, Theor. Math. Phys. 76 (1988) 665.

[19] V.V. Fock and A.A. Rosly, Poisson structures on moduli of flat connections on Riemann surfaces and r-matrices, preprint ITEP 72-92 (1992), Moscow.

[20] M.F. Atiyah and R. Bott, Phil. Trans. R. Soc. Lond. A308 (1982) 523.

[21] P.A.M. Dirac, Lectures on quantum mechanics, Yeshiva University (1967), Academic Press, New York.

[22] L.D. Faddeev and L.A. Takhtadjan, Hamiltonian methods in the theory of solitons, 1987, Springer, Berlin.

[23] D. Ivanov, Int. J. Mod. Phys. A10 (1995) 2507.

[24] A.Y. Alekseev and V. Schomerus, Representation theory of Chern-Simons observables, preprint HUTMP-95-B-342 (1995), [q-alg/9503016].

[25] V.G. Drinfeld in: Problems of modern quantum field theory, Proceedings Alushta 1989, Research reports in physics, 1989, Springer, Heidelberg;
A. Varchenko, Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups, Advanced series in mathematical physics. 21, 1995, World Scientific, Singapore;
C. Kassel, Quantum groups, 1995, Springer, New York.

[26] E. Witten, Nucl. Phys. B368 (1988) 46.

[27] M. Welling, Gravity in 2 + 1 dimensions as a Riemann-Hilbert problem, preprint THU 95/24 (1995), [hep-th/9510060].
A. Bellini, M. Ciafaloni and P. Valtancoli, Phys. Lett. B357 (1995) 532.