Morse theory for a fourth order elliptic equation with exponential nonlinearity

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Abstract

Given a Hilbert space \((H, \langle \cdot, \cdot \rangle)\), \(\Lambda \subset (0, +\infty)\) an interval and \(K \in C^2(H, \mathbb{R})\) whose gradient is a compact mapping, we consider the family of functionals of the type:

\[
I(\lambda, u) = \frac{1}{2} \langle u, u \rangle - \lambda K(u), \quad (\lambda, u) \in \Lambda \times H.
\]

As already observed by many authors, for the functionals we are dealing with the (PS) condition may fail under just these assumptions. Nevertheless, by using a recent deformation Lemma proven by Lucia in [17], we prove a Poincaré-Hopf type theorem. Moreover by using this result, together with some quantitative results about the formal set of barycenters, we are able to establish a direct and geometrically clear degree counting formula for a nonlinear scalar field equation on a bounded and smooth \(C^\infty\) region of the four dimensional Euclidean space in the flavor of [19]. We remark that this formula has been proven with complete different methods in [13] by using blow-up type estimates.

Key Words: Scalar field equations, Geometric PDE's, Morse Theory, Leray-Schauder degree.

AMS subject classification: 35B33, 53A30, 53C21, 58E05.

Introduction

Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space whose associated norm will be denoted by \(\| \cdot \|\). Given an interval \(\Lambda\) of \((0, \infty)\) and \(K\) such that

\[
K \in C^2(H, \mathbb{R}), \quad \nabla K : H \to H \text{ compact,}
\]

let us consider the functionals which are of the form:

\[
I(\lambda, u) = \frac{1}{2} \langle u, u \rangle - \lambda K(u), \quad (\lambda, u) \in \Lambda \times H.
\]

We observe that the conditions (1)-(2) are not enough to ensure the (PS)-condition which is known to hold only for bounded sequences. (See, [17, Lemma 2.3]). Therefore the classical flow defined by the vector-field \(-\nabla_u I(\lambda, u)\) is not suitable to derive a deformation lemma. However, by using a recent deformation result proven by [17, Proposition 1.1], we prove the following result.

**Theorem 1** Let \(I(\lambda, \cdot)\) be a family of functionals satisfying (23)-(24) and fix \(\bar{I}(\cdot) := I(\bar{\lambda}, \cdot)\) for some \(\bar{\lambda} \in \Lambda\). Given \(\varepsilon > 0\), let \(\Lambda' := [\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon]\) be a (compact) subset of \(\Lambda\) and consider \(a, b \in \mathbb{R}\) (a < b), so that all the critical points \(\bar{u}\) of \(I(\lambda, \cdot)\) for \(\lambda \in \Lambda'\) satisfy \(\bar{I}(\bar{u}) \in (a, b)\). If

\[
\bar{I}^a_b := \{ u \in H : a \leq \bar{I}(u) \leq b \},
\]

and assuming that \(\bar{I}\) has no critical points at the levels \(a, b\), we have

\[
\deg_{LS}(\nabla \bar{I}, \bar{I}^a_b, 0) = \chi(\bar{I}^b, \bar{I}^a).
\]

where we denoted by \(\deg_{LS}\) the Leray-Schauder degree of the compact vector field \(\nabla \bar{I}\).

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Now, let $\Omega \subset \mathbb{R}^4$ be a bounded connected and open set having smooth $C^\infty$ boundary, and let us consider the following boundary value problem

\[
\begin{aligned}
\Delta^2 u &= \tau \frac{h(x)e^u}{\int_{\Omega} h(x)e^u \, dx} \quad \text{in } \Omega \\
u &= \Delta u = 0 \quad \text{on } \partial \Omega
\end{aligned}
\]  

(4)

where $h$ is a $C^{2,\alpha}$ positive function for $\alpha \in (0,1)$, and $\tau \in \mathbb{R}^+$. As already observed by many authors, the importance of this equation is related to its physical meaning. In fact, it arises in mathematical physics as a mean field equation of Euler flows or for the description of self-dual condensates of some Chern-Simons-Higgs model. (See [15, 16, 17, 8, 18], for further details). If $\mathcal{H}$ denotes the space of all functions of Sobolev class $H^2(\Omega) \cap H_0^2(\Omega)$ endowed with the equivalent norm $\|u\|_{\mathcal{H}} := \|\Delta u\|_2$, than problem (4) has a variational structure and for each fixed constant $\tau$, the (weak) solutions can be found as critical points of the functional

\[
I_\tau(u) := \frac{1}{2} \|u\|_{\mathcal{H}}^2 - \tau \log \left( \frac{1}{|\Omega|} \int_{\Omega} h(x)e^u \, dx \right) \quad \forall u \in \mathcal{H},
\]  

(5)

where we denoted by $|\cdot|$ the Lebesgue measure in $\mathbb{R}^4$. The key analytic fact which we need in order to classify the critical points of (5) is a version for higher order operators of the Moser-Trudinger inequality. As a direct consequence of this inequality, it follows that the functional (5) is coercive for $\tau < 64\pi^2$ and thus it is possible to find the solutions of (4), by using the direct method of the calculus of variation. If $\tau > 64\pi^2$, the functional $I_\tau$ is unbounded both from below and from above and hence the solutions have to be found by other methods, for instance as saddle points, by using some min-max scheme. A general feature of the problem is a compactness property if $\tau$ is not integer multiple of $64\pi^2$ as proven by Lin & Wei in [10].

If $\tau < 64\pi^2$ or $\tau \in (64k\pi^2, 64(k+1)\pi^2)$, $k \in \mathbb{N}$, by elliptic regularity and by taking into account the compactness result proven in [12, Theorem 1.2], it is possible to define the Leray-Schauder degree for the boundary value problem (4), fixing a large ball $B_R \subset \mathcal{H}$ centered at 0 and containing all the solutions. In fact, let us consider the family of compact operators $T_\tau : B_R \to \mathcal{H}$, defined by

\[
T_\tau(u) := \tau \Delta^{-2} \frac{he^u}{\int_{\Omega} he^u};
\]

then the Leray-Schauder degree

\[
d_\tau := \text{deg}_{LS}(I - T_\tau, B_R, 0)
\]

is well-defined for $\tau \neq 64k\pi^2$, $k \in \mathbb{N}$.

**Notation 1** For any two integers $k_1, k_2$, we use the notation $\binom{k_1}{k_2}$ to denote

\[
\binom{k_1}{k_2} := \begin{cases} \frac{k_1(k_1-1)\ldots(k_1-k_2+1)}{k_2!} & \text{if } k_1 > 0 \\
1 & \text{if } k_1 = 0,
\end{cases}
\]

and $\mathbf{k}$ to denote the set $\{1, \ldots, k\}$.

By applying Theorem 3.1 together with a precise homological properties of the formal set of barycenters obtained in [5] we can reprove the following result.

**Theorem 2** ([13]) For $\tau \in (64k\pi^2, 64(k+1)\pi^2)$, and $k \in \mathbb{N}$, the Leray-Schauder degree $d_\tau$ of (4) is given by

\[
d_\tau = \binom{k - \chi(\Omega)}{k},
\]

where $\chi(\Omega)$ denotes the Euler characteristic of the domain $\Omega$.

As direct consequence if $\chi(\Omega) \leq 0$ then the problem (4) possesses a solution provided that $\tau \neq 64k\pi^2$, $k \in \mathbb{N}$.
In the rest of the section we briefly describe the method and the main ideas of the proof. As already observed for $\tau > 64\pi^2$, the functional $I_\tau$ is unbounded both from above and below due to the so-called *bubbling phenomenon* which often occurs in geometric problems. More precisely, for a given point $x \in \Omega$ and for $\lambda > 0$, we consider the following function

$$\varphi_{\lambda,x}(y) = \frac{1}{4} \log \left( \frac{2\lambda}{1 + \lambda^2 \text{dist}(y,x)^2} \right)^4$$

where $\text{dist}(.,:)$ denotes the metric distance on $\Omega$. For large $\lambda$, one has $e^{\varphi_{\lambda,x}} \rightarrow \delta_x$ (the Dirac mass at $x$) and moreover one can show that $I(\tau, \varphi_{\lambda,x}) \rightarrow -\infty$ as $\lambda \rightarrow +\infty$. Similarly, if $\tau > 64\pi^2$ it is possible to construct a function $\varphi$ of the above form (near at each $x_i$) with $e^{\varphi_{\lambda,x}} \rightarrow \sum_{i=1}^k t_i \delta_{x_i}$, and on which $I_\tau$ still attains large negative values. A crucial observation, as proven in [8], is that the constant in Moser-Trudinger inequality can be divided by the number of regions where $e^u$ is supported. From this argument we see that one is led naturally to consider the family of barycenters of $\bigcup_{i=1}^k \{ t_i \delta_{x_i} \}$, which often occurs in geometric problems. More precisely, for each $\lambda \in \mathbb{R}$, let $I_{\lambda,x} : \mathcal{H} \rightarrow \mathbb{R}$ be the functional

$$I_{\lambda,x}(\varphi) = \int_{\Omega} \varphi + \frac{\lambda}{2} \int_{\Omega} \varphi^2 - \int_{\Omega} F(\varphi).$$

We will apply this result to $X = \mathcal{H}$ and $\mathcal{G} = \nabla I_\tau$. Since both the map $\mathcal{G}$ and its Fréchet derivative are of the form $\text{Id} - K$ where $K$ is a compact operator, than the assumptions of theorem 1.1 are fulfilled.

Now let $\Gamma$ be an open subset of $X$ and let $\mathcal{F} : \Gamma \rightarrow X$ be a strict $\alpha$-contraction, meaning that $\alpha(\mathcal{F}(B)) < k\alpha(B)$ for some fixed $k \in [0,1)$, where $B \subset \Omega$ is a bounded subset and where $\alpha$ denotes the *Kuratowski measure of non-compactness*. If $y \notin (\text{Id} - \mathcal{F})(\partial \Omega)$ and $(\text{Id} - \mathcal{F})^{-1}(\{y\})$ is compact, we can define the *generalized degree* $\deg$, in such a way that if $\text{Id} - \mathcal{F}$ is a compact vector field and $\Gamma$ is a bounded subset it enjoys all the properties of the Leray-Schauder degree.

**Formal set of barycenters.** The aim of this paragraph is to recall some facts about the formal set of barycenters. Following [3], these spaces are defined by

$$\Sigma_k := \left\{ \sum_{i=1}^n t_i \delta_{x_i} \mid (x_1, \ldots, x_n) \in \Omega^n, (t_1, \ldots, t_n) \in \Delta_{n-1} \right\}$$

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## 1 Preliminaries

The aim of this section is to recall some abstract results from degree theory for $\alpha$-contractions, Sard’s lemma for Fredholm maps and to recall some topological and homological properties of the so-called *formal set of barycenters*. Our main references will be [3, 7, 8, 9, 10].

**The Sard-Smale theorem and Kuratowski non-compactness measure.** We start this section with the classical Sard-Smale theorem stated in a form suitable for our purposes. See [7, pag.91].

**Theorem 1.1 (Sard-Smale)** Let $\Gamma$ be an open subset of a Hilbert space $X$. Suppose that $\mathcal{G} \in C^1(\Gamma, X)$ is proper when restricted to any closed bounded subset of $\Gamma$ and that $\nabla \mathcal{G}(x) = \text{Id} - K(x)$ where for every $x \in \Gamma$, $K(x)$ is a compact operator. Then the set of regular values of $\mathcal{G}$ is dense in $X$.

We will apply this result to $X = \mathcal{H}$ and $\mathcal{G} = \nabla I_\tau$. Since both the map $\mathcal{G}$ and its Fréchet derivative are of the form $\text{Id} - K$ where $K$ is a compact operator, than the assumptions of theorem 1.1 are fulfilled.

Now let $\Gamma$ be an open subset of $X$ and let $\mathcal{F} : \Gamma \rightarrow X$ be a strict $\alpha$-contraction, meaning that $\alpha(\mathcal{F}(B)) < k\alpha(B)$ for some fixed $k \in [0,1)$, where $B \subset \Omega$ is a bounded subset and where $\alpha$ denotes the *Kuratowski measure of non-compactness*. If $y \notin (\text{Id} - \mathcal{F})(\partial \Omega)$ and $(\text{Id} - \mathcal{F})^{-1}(\{y\})$ is compact, we can define the *generalized degree* $\deg$, in such a way that if $\text{Id} - \mathcal{F}$ is a compact vector field and $\Gamma$ is a bounded subset it enjoys all the properties of the Leray-Schauder degree.
where $\delta_x$ is the true Dirac mass at the point $x$ and $\Delta_{n-1}$ is the $(n-1)$-simplex of all tuples $(t_1, \ldots, t_n)$ such that $t_i \geq 0$ and $\sum_{i=1}^n t_1 = 1$. We observe that the set $\Sigma_k$ is provided by the weak convergence of measures. In order to give a more topological insight on these spaces, some definitions are in order.

Let us denote by $J_k$ the $k$-fold join of $\Omega$. We recall that a point $x \in J_k$ is specified by:

(i) $k$ real numbers $t_1, \ldots, t_k$ satisfying $t_i \geq 0$, $\sum_{i=1}^k t_i = 1$, and

(ii) a point $x_i \in \Omega$ for each $i \in k$ such that $t_i \neq 0$.

Such a point will be denoted by the symbol $\oplus_{i=1}^k t_i x_i$, where the elements $x_i$ may be chosen arbitrarily or omitted whenever the corresponding $t_i$ vanishes. Furthermore we will endow this space with the strongest topology such that the coordinate functions are continuous. Now, if $\Sigma^k$ denotes the symmetric group over $k$ elements, we assume that $\Sigma^k$ acts on $J_k$ by permuting factors, namely

$$\forall \sigma \in \Sigma^k: \sigma(t_1 x_1 \oplus \cdots \oplus t_k x_k) := (t_{\sigma(1)} x_{\sigma(1)} \oplus \cdots \oplus t_{\sigma(k)} x_{\sigma(k)}).$$

Thus, the $k$-th symmetric join of $\Omega$, say $SJ_k$ is defined as the quotient of $J_k$ with respect to $\Sigma^k$.

**Definition 1.2** ([9 Definition 5.1]) The $k$-th barycenter space $\Sigma_k$ can be defined as the quotient of the symmetric join $SJ_k$ under the equivalence relation $\sim$:

$$t_1 x_1 \oplus t_2 x_1 \oplus \cdots \oplus t_k x_k \sim (t_1 + t_2)x_1 \oplus \cdots \oplus t_k x_k.$$ 

That is a point in $\Sigma_k$ is a formal abelian sum with the topology that when $t_i = 0$ the entry $0x_i$ is discarded from the sum, and when $x_i$ moves in coincidence with $x_j$, one identifies $t_i x_i + t_j x_i$ with $(t_i + t_j)x_i$. It is possible to show that we have the embeddings

$$\Omega \hookrightarrow \Sigma_2 \hookrightarrow \cdots \hookrightarrow \Sigma_{k-1} \hookrightarrow \Sigma_k$$

and each factor is contractible in the next. Let $P$ be the projection on $\mathcal{H}$ (i.e. $P \varphi = \varphi - h$ with $\Delta^2 h = 0$ in $\Omega$ and $h = \varphi$ and $\Delta h = \Delta \varphi$ on $\partial \Omega$), $\Sigma \subset \mathcal{H}$ be the unit sphere and finally let

$$R : \mathcal{H}\backslash \{0\} \to \Sigma : u \mapsto R(u) := u/||u||_{\mathcal{H}}.$$ 

Let $g_k : SJ_k \to \Sigma$ be the map defined as: $g_k((x_1, \ldots, x_k), (t_1, \ldots, t_k)) := R(\sum_{i=1}^k t_i P \varphi_{x_i \lambda})$, where $\lambda > 0$ is fixed and $\varphi_{x_i \lambda}$, are given by

$$\varphi_{x_i \lambda}(y) = \frac{1}{4} \log \left( \frac{2\lambda}{1 + \lambda^2 \text{dist}(y, x)^2} \right)^4.$$ 

We observe that since two elements in $SJ_k$ equivalent for the relation introduced in definition 1.2 have the same image through $g_k$, this implies that $g_k$ is well-defined on the quotient. Denoting by $\Omega^k$ the $k$-fold product of copies of $\Omega$ and by $\Delta_k$ the collision set $\bigcup_{i,j=1}^k \Delta_{i,j}$, where

$$\Delta_{i,j} := \{(x_1, \ldots, x_k) \in \Omega^k | x_i = x_j, i \neq j, \text{ for } i, j \in k\},$$

we define the configuration space $\mathcal{X}_k := \Omega^k \backslash \Delta_k$. Let us consider the fibration

$$\mu : \mathcal{X}_k \to \mathcal{X}_{k-1},$$ 

defined by $\mu(x_1, \ldots, x_k) := (x_1, \ldots, x_{k-1})$, it is easy to observe that each fiber $\mu^{-1}((x_1, \ldots, x_{k-1})) = \Omega \backslash \{x_1, \ldots, x_{k-1}\}$ is homeomorphic to each other. Thus by using the classical Hopf theorem for fibrations (see, for instance Spanier [21], for further details), the Euler characteristic of $\mathcal{X}_k$ can be computed through the fiber $\Omega \backslash \{x_1, \ldots, x_{k-1}\}$ and $\mathcal{X}_{k-1}$. By an easy calculations it follows that

$$\chi(\mathcal{X}_k) = \chi(\Omega)(\chi(\Omega) - 1) \cdots (\chi(\Omega) - k + 1).$$
**Lemma 1.3** (well-known) The set $\Sigma_k \setminus \Sigma_{k-1}$ is an open smooth manifold of dimension $5k - 1$ for each $k \in \mathbb{N}$.

**Proof.** The case $k = 1$ is trivial, since $\Sigma_1 = \Omega$ and $\Omega$ is a four-dimensional manifold being an open subset of $\mathbb{R}^4$. For $k \geq 2$, the join $J_k$ is a smooth manifold. Since the action of the symmetric group on $J_k$ is free of fixed points than the symmetric join is a smooth manifold. Moreover, since $\Sigma_{k-1}$ is the boundary of $\Sigma_k$, than $\Sigma_k \setminus \Sigma_{k-1}$ is a smooth open manifold in which the elements in $\Sigma_k \setminus \Sigma_{k-1}$ are smoothly parameterized by $4k$ coordinates locating the points $x_i$ and $k - 1$ coordinates identifying the numbers $t_i$’s. The conclusion immediately follows. q.e.d.

**Lemma 1.4** (well-known) For any $k \geq 1$, the set $\Sigma_k$ is a non contractible stratified set.

**Proof.** (Sketch). It can be proved by arguing as follows. The case $k = 1$ is trivial. For $k \geq 2$ even if the set $\Sigma_{k-2}$ is not a smooth manifold (actually it is a stratified set) however it is an ENR which implies that there exists a non trivial (mod 2) orientation class with respect to its boundary. However by using the Čech-cohomology, and by taking into account that it is isomorphic to the singular homology, the thesis follows by using the exactness of the pair once it is proven that

\[ H_{5k-1}(\Sigma_k; \mathbb{Z}_2) \simeq H_{5k-1}(\Sigma_k \Sigma_{k-1}; \mathbb{Z}_2). \]

(See, for instance, [3] Lemma 3.7, for further details). q.e.d.

By using the same arguments as in [19] Proposition 5.1, it can be proven the following result.

**Lemma 1.5** Let $\eta > 0$ be smaller than the injectivity radius of $\Omega$ and let $G : (0, +\infty) \to (0, +\infty)$ be the non-increasing function satisfying:

\[ G(t) = \frac{1}{t} \text{ for } t \in (0, \eta] \quad G(t) = \frac{1}{2\eta} \text{ for } t > 2\eta. \]

If $d(x_i, x_j) := \text{dist}(x_i, x_j)$ and $F^* : \Sigma_k \setminus \Sigma_{k-1} \to \mathbb{R}$ as follows

\[ F^*(\sum_{i=1}^{k} t_i \delta_{x_i}) := - \sum_{i \neq j} G(d(x_i, x_j)) - \sum_{i=1}^{k} \frac{1}{t_i(1 - t_i)}. \tag{9} \]

Then we have

\[ \sum_{i=1}^{5k-1} c_i = (-1)^{k-1} \frac{\chi(\Sigma_k)}{k!} \tag{10} \]

where $c_i$ denotes the number of critical points of index $i$.

The following result will be crucial in order to compute the Leray-Schauder degree of our result.

**Proposition 1.6** For any natural number $k$ we have:

\[ \chi(\Sigma_k) = 1 - \binom{k - \chi(\Omega)}{k}. \]

**Proof.** The proof is given by induction over $k$. The case $k = 1$ is trivial being $\Sigma_1$ homeomorphic to $\Omega$. For $k > 1$ we consider the pair $(\Sigma_k, \Sigma_{k-1})$ and we remark that the Euler characteristic is additive. Thus $\chi(\Sigma_k) = \chi(\Sigma_k, \Sigma_{k-1}) + \chi(\Sigma_{k-1})$.

**Claim 1.** The following formula holds for any natural number $k$

\[ \chi(\Sigma_k, \Sigma_{k-1}) = (-1)^{k-1} \binom{\chi(\Omega) - k}{k}. \tag{11} \]

Once this is done the proposition easily follows. By Lemma 1.3 the space $\Sigma_k \setminus \Sigma_{k-1}$ is an open manifold of dimension $5k - 1$ with boundary $\Sigma_{k-1}$ and by the definition of $F^*$, Palais-Smale condition holds.

Observe that $\Sigma_{k-1}$ is a deformation retract of the sublevel $F^*_L := \{ F^* \leq -L \} \cup \Sigma_k$ for $L$ sufficiently
large and positive (simply by taking the limit for $L \to +\infty$). Thus denoting by $\tilde{F}^*$: $\{F^* \geq -L\} \to \mathbb{R}$ a non-degenerate function $C^2$-close to the restriction of $F^*$ to the subset $\{F^* \geq -L\}$, by excision of the sublevel $F^*_{-L} := \{F^* < -L\}$ and by the classical Poincaré-Hopf theorem it holds

$$\chi(\Sigma_k, \Sigma_{k-1}) = \sum_{i=1}^{5k-1} (-1)^i c_i.$$  

The thesis follows by formula (10) and (9).

**q.e.d.**

**Improved Moser-Trudinger inequality.** Let $C^\infty_c(\Omega)$ be the set of all smooth functions with compact support in $\Omega$, and let $\mathcal{H}$ be the completion with respect to the norm $\|u\|_{\mathcal{H}} := \|\Delta u\|_2$. The space $\mathcal{H}$ is a Hilbert space with respect to the scalar product $\langle u, v \rangle_{\mathcal{H}} := \int_\Omega \Delta u \Delta v dx$ for all $u, v \in \mathcal{H}$, and, by the local regularity theorem and by the Poincaré inequality, it follows that $\mathcal{H}$ agrees with the space of all functions on $\Omega$ of Sobolev class $H^2(\Omega) \cap H^1_0(\Omega)$. As immediate consequence of [12, Theorem 1.2] and the Schauder estimates, the following crucial compactness results hold.

**Proposition 1.7** ([12, Theorem 1.2]) Let $h: \Omega \to \mathbb{R}$ be a positive $C^{2,\alpha}$ function and $\tau \not\in \{64k\pi^2\}$ for $k \in \mathbb{N}$. Then the solutions of (4) are bounded in $C^{4,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$.

**Proposition 1.8** ([10, Lemma 2.1]) Let $u$ be a solution of (4) with $\tau \leq c$, for some constant $c$. Then there exists a $\delta > 0$ such that

$$u(x) \leq c, \quad \forall x \in \Omega_\delta,$$

where $\Omega_\delta := \{x \in \Omega : d(x, \partial \Omega) \leq \delta\}$.

We remark that proposition [10, Lemma 2.1] excludes boundary bubbles.

**Lemma 1.9** There exists a constant $C_\Omega$ depending only on $\Omega$ such that for all $u \in \mathcal{H}$ one has:

$$\log \left( \frac{1}{|\Omega|} \int_\Omega e^{(u-a)} dx \right) \leq C_\Omega + \frac{1}{128\pi^2} \|u\|_{\mathcal{H}}^2$$  

(12)

where $\bar{u} := \frac{1}{|\Omega|} \int_\Omega u dx$ stands for the average of $u$ over $\Omega$.

**Proof.** In fact by [11, Theorem 1], there exists $C'_\Omega > 0$ depending only on $\Omega$ such that for all $u \in C^2_c(\Omega)$ it holds

$$\frac{1}{|\Omega|} \int_\Omega e^{\frac{32\pi^2(u-b)^2}{|\Delta u|^2}} dx \leq C'_\Omega, \quad \forall u \in \mathcal{H}.$$

Since for any $a, b \in \mathbb{R}$, we have $(8\pi a - \frac{1}{b})^2 \geq 0$ is $2ab \leq \frac{1}{64\pi^2} b^2 + 64\pi^2 a^2$, by setting $a := u - \bar{u}$ and $b = \|u\|_{\mathcal{H}}^2$, and exponentiating, we have

$$\frac{1}{|\Omega|} \int_\Omega e^{(u-a)} dx \leq e^{\frac{1}{128\pi^2}\|u\|_{\mathcal{H}}^2} \frac{1}{|\Omega|} \int_\Omega e^{\frac{32\pi^2(u-b)^2}{|\Delta u|^2}} dx \leq e^{\frac{1}{128\pi^2}\|u\|^{5}_{\mathcal{H}}} C'_\Omega, \quad \forall u \in \mathcal{H}.$$

Taking the logarithm of this last inequality the conclusion follows by setting $C_\Omega := \log C'_\Omega$. **q.e.d.**

In order to study how the function $e^u$ is spread over $\Omega$ we need some quantitative results. In fact, we will show that if $e^u$ has integral bounded from below on $(l+1)$-regions, the constant $\frac{1}{128\pi^2}$ can be basically divided by $(l + 1)$. The proof of the proposition [11, Theorem 1.10] is up to minor modifications, an adaptation of the arguments given in [8, Lemma 2.2]; we will reproduce it for the sake of completeness.

**Proposition 1.10** For any fixed integer $l$, let $\Omega_1, \ldots, \Omega_{l+1}$ be subsets of $\Omega$ satisfying $\operatorname{dist}(\Omega_i, \Omega_j) \geq \delta_0$, for $i \neq j$, where $\delta_0$ be positive real number, and let $\gamma_0 \in (0, 1/(l+1)]$. Then for any $\bar{c} > 0$ there exists a constant $\tilde{C} := \tilde{C}(\bar{c}, \delta_0, \gamma_0)$ such that

$$\log \left( \frac{1}{|\Omega|} \int_\Omega e^{(u-a)} dx \right) \leq -\frac{1}{128(l+1)\pi^2 - \bar{c}} \|u\|_{\mathcal{H}}^2 + \tilde{C},$$

(11)

where $\bar{u} := \frac{1}{|\Omega|} \int_\Omega u dx$ stands for the average of $u$ over $\Omega$. **q.e.d.**
for all functions $u \in \mathcal{H}$ satisfying
\[
\frac{\int_{\Omega} e^u \, dx}{\int_{\Omega} e^v \, dx} \geq \gamma_0 \quad \forall \ i \in 1 + 1.
\] (13)

Proof. We consider the $(l + 1)$ smooth cut-off functions $g_1, \ldots, g_{l + 1}$, satisfying the following properties:
\[
\begin{align*}
&g_{1}(x) \in [0, 1] \quad \text{for every } x \in \Omega; \\
g_{i}(x) = 1 \quad \text{for every } x \in \Omega_i, i \in 1 + 1; \\
g_{l}(x) = 0 \quad \text{if } \text{dist}(x, \Omega_1) \geq \frac{\delta_0}{4};
\end{align*}
\] (14)

where $C_{\delta_0}$ depends only on $\delta_0$. By interpolation, (see, for instance, [13 Prop. 4.1]), for any $\varepsilon > 0$, there exists $C_{\varepsilon, \delta_0}$, such that for any $u \in \mathcal{H}$, and for any $i \in 1 + 1$ there holds
\[
\|g_i u\|_{C^l(\Omega)}^2 \leq C_{\delta_0},
\]
\[
\|g_i v\|_{C^l(\Omega)}^2 := \int_{\Omega} |\Delta(g_i v)|^2 \, dx \leq \int_{\Omega} g_i^2 |\Delta v|^2 \, dx + \varepsilon \int_{\Omega} |\Delta v|^2 \, dx + C_{\varepsilon, \delta_0} \int_{\Omega} v^2 \, dx.
\] (15)

Let $u - \bar{u} = u_1 + u_2$ with $u_1 \in L^\infty(\Omega)$, then from our assumptions we deduce
\[
\int_{\Omega} e^{u_2} \, dx \geq e^{-\|u_1\|_{L^\infty(\Omega)}} \int_{\Omega} e^{(u-\bar{u})} \, dx \geq e^{-\|u_1\|_{L^\infty(\Omega)}} \gamma_0 \int_{\Omega} e^{(u-\bar{u})} \, dx \quad i \in 1 + 1.
\]

By invoking inequality (12) in lemma 1.9 together with the last two inequalities, it follows that
\[
\log \left( \frac{1}{|\Omega|} \int_{\Omega} e^{(u-\bar{u})} \, dx \right) \leq \log \frac{1}{\gamma_0} + \|u_1\|_{L^\infty(\Omega)} + \log \left( \frac{1}{|\Omega|} \int_{\Omega} e^{g_1 u_2} \, dx \right) + C_\Omega
\]
\[
\leq \log \frac{1}{\gamma_0} + \|u_1\|_{L^\infty(\Omega)} + \frac{1}{128 \pi^2} \|g_1 u\|_{C^l(\Omega)}^2 + C_\Omega.
\] (16)

We choose i such that $\int_{\Omega} |\Delta(g_i u)|^2 \, dx \leq \int_{\Omega} |\Delta(g_i u)|^2 \, dx$, for each $j \in 1 + 1$. Since the functions $g_j$ have disjoint supports, the last formula and (15) implies that
\[
\log \left( \frac{1}{|\Omega|} \int_{\Omega} e^{v_2} \, dx \right) \leq \log \frac{1}{\gamma_0} + \|u_1\|_{L^\infty(\Omega)} + C_{\varepsilon, \delta_0} + \left( \frac{1}{128 (l + 1) \pi^2} + \varepsilon \right) \|u_2\|_{C^l(\Omega)}^2 + C_{\varepsilon, \delta_0} \int_{\Omega} v^2 \, dx.
\]

Now let $\lambda_{\varepsilon, \delta_0}$ to be an eigenvalue of $-\Delta^2$ such that $\frac{C_{\varepsilon, \delta_0}}{\lambda_{\varepsilon, \delta_0}} < \varepsilon$, and we set
\[
u_1 := P_{V_{\varepsilon, \delta_0}} (u - \bar{u}); \quad \nu_2 := P_{V_{\varepsilon, \delta_0}^l} (u - \bar{u}).
\]
Here $V_{\varepsilon, \delta_0}$ is the direct sum of the eigenspaces of $-\Delta^2$ with Navier boundary conditions and having eigenvalues less or equal than $\lambda_{\varepsilon, \delta_0}$ and $P_{V_{\varepsilon, \delta_0}}$, $P_{V_{\varepsilon, \delta_0}^l}$ the orthogonal projections onto $V_{\varepsilon, \delta_0}$ and $V_{\varepsilon, \delta_0}^l$, respectively. Since $V_{\varepsilon, \delta_0}$ is finite dimensional, the $L^2$ norm and $L^\infty$ norm of $u - \bar{u}$ on $V_{\varepsilon, \delta_0}$ are equivalent; then, by using the Poincaré-Wirtinger inequality (cfr. [14] pag. 308), there holds:
\[
\|u_1\|_{L^\infty(\Omega)} \leq C_{\varepsilon, \delta_0} \|u_1\|_{H^2(\Omega)} \leq C_{\varepsilon, \delta_0} \|u_1\|_{H^2(\Omega)} \leq C_{\varepsilon, \delta_0} \|u_1\|_{C^l(\Omega)}^2
\]
where $C_{\varepsilon, \delta_0}$ is another constant depending only on $\varepsilon$ and $\delta_0$. Furthermore
\[
C_{\varepsilon, \delta_0} \int_{\Omega} u_2^2 \, dx \leq C_{\varepsilon, \delta_0} \int_{H^2(\Omega) \cap H^1_0(\Omega)} u_2^2 \, dx \leq \varepsilon \|u_2\|_{H^2(\Omega) \cap H^1_0(\Omega)}^2 \leq \varepsilon C_{\Omega} \|u_2\|_{C^l(\Omega)}^2
\]
where $C_{\varepsilon, \delta_0}$ is a constant depending only on $\Omega$. Hence the last formulas imply
\[
\log \left( \frac{1}{|\Omega|} \int_{\Omega} e^{(u-\bar{u})} \, dx \right) \leq \log \frac{1}{\gamma_0} + C_{\varepsilon, \delta_0} \|u_1\|_{C^l(\Omega)}^2 + C_{\Omega} + \left( \frac{1}{128 (l + 1) \pi^2} + \varepsilon \right) \|u_2\|_{C^l(\Omega)}^2 + \varepsilon C_{\Omega} \|u_2\|_{C^l(\Omega)}^2
\]
where $C_{\varepsilon, \delta_0}$ depends only on $\varepsilon$ and $\delta_0$ (and $l$ which is fixed). This conclude the proof. q.e.d.

In the next Lemma we show a criterion which implies the situation described in the first condition in (13).
Lemma 1.11 ([8, Lemma 2.3]) Let \( l \) be a positive integer, and suppose that \( \varepsilon \) and \( r \) are positive numbers. Suppose that for a non-negative function \( f \in L^1(\Omega) \) with \( \|f\|_1 = 1 \) there hold
\[
\int_{U_{\varepsilon,r} \cap B_r(p_i)} fdx < 1 - \varepsilon, \quad \forall \ l = t \geq p_1, \ldots, p_l \in \Omega.
\]

Then there exists \( \varepsilon > 0 \) and \( r > 0 \), depending on \( \varepsilon, r, l \) and \( \Omega \) (but not on \( f \)), and \( l + 1 \) points \( p_1, \ldots, p_{l+1} \in \Omega \) satisfying
\[
\int_{B_r(p_i)} fdx \geq \varepsilon, \ldots, \int_{B_r(p_{l+1})} fdx \geq \varepsilon; \quad B_{2r}(p_i) \cap B_{2r}(p_j) = \emptyset \quad \text{for} \ i \neq j.
\]

Proof. To prove the thesis, we argue by contradiction. Thus, there exist \( \varepsilon, r > 0 \) and \( l > 0 \) such that for any \( \varepsilon, r, l \) and \( \Omega \) (but not on \( f \)), and \( l + 1 \) points \( p_1, \ldots, p_{l+1} \in \Omega \) satisfying
\[
\int_{B_r(p_i)} fdx \geq \varepsilon, \ldots, \int_{B_r(p_{l+1})} fdx \geq \varepsilon; \quad B_{2r}(p_i) \cap B_{2r}(p_j) = \emptyset \quad \text{for} \ i \neq j.
\]

Lemma 1.12 If \( \tau \in (64k\pi^2, 64(k+1)^2\pi^2) \) with \( k \geq 1 \), the following property holds. For any \( \varepsilon > 0 \) and any \( r > 0 \) there exists a large positive \( L = L(\varepsilon, r) \) such that for every \( u \in H^1(\Omega) \) with \( \frac{1}{|\Omega|} \int_{\Omega} e^u dx = 1 \) and \( I_\tau(u) \leq -L \) there exist \( k \) points \( p_1, u, \ldots, p_k, u \in \Omega \) such that
\[
\frac{1}{|\Omega|} \int_{\Omega \setminus (\cup_{k=1}^k B_r(p_k))} e^u dx < \varepsilon.
\]

Proof. To prove the thesis, we argue by contradiction. Thus, there exist \( \varepsilon, r > 0 \) and a sequence \( (u_n)_n \in H^1(\Omega) \) with \( \frac{1}{|\Omega|} \int_{\Omega} e^u dx = 1 \) and \( I_\tau(u_n) \to -\infty \) such that for every \( k \)-tuple \( p_1, \ldots, p_k \) in \( \Omega \) we have \( \frac{1}{|\Omega|} \int_{\Omega \setminus (\bigcup_{k=1}^k B_r(p_k))} e^u dx \geq \varepsilon \). Now applying Lemma 1.11 with \( l = k, f = e^u \) and finally with \( \delta_0 = 2r, \Omega_j = B_r(p_j) \) and \( \gamma_0 = \varepsilon \) for \( j \in k \) and where the symbols \( \delta_0, \Omega_j, \gamma_0 \) were defined in Lemma 1.9 and \( \tilde{r}, B_{\tilde{r}}(p_j), \varepsilon, (p_j)_j \) were defined in Lemma 1.11. By this it follows that, for any given \( \varepsilon > 0 \) there exists a constant \( C > 0 \) depending on \( \varepsilon, \varepsilon \) and on \( r \) such that
\[
I_\tau(u_n) \geq \frac{1}{2} \|u_n\|^2_{H^1} - C\tau = \frac{\tau}{64(k+1)^2\pi^2 - \varepsilon} \|u\|^2_{H^1},
\]
where the constant \( C \) does not depend on \( n \). Now since \( \tau < 64(k+1)^2\pi^2 \), we can choose \( \varepsilon > 0 \) small enough that the number \( \frac{\tau}{64(k+1)^2\pi^2 - \varepsilon} = \delta > 0 \). Therefore the inequality (17) reduces to
\[
I_\tau(u_n) \geq \frac{\delta}{2} \|u_n\|^2_{H^1} - C\tau \geq -K,
\]
where \( K \) is a positive constant independent of \( n \). This violates our contradiction assumption, and conclude the proof.

q.e.d.

Given a non-negative \( L^1 \) function \( \Omega \) on \( \Omega \), we define the distance of \( f \) from \( \Sigma_k \) as
\[
\text{dist}(f, \Sigma_k) := \sup \left\{ \left| \int_{\Omega} f\sigma dx - \langle \sigma, \psi \rangle \right| : \sigma \in \Sigma_k, \quad \|\psi\|_{L^1(\Omega)} \leq 1 \right\},
\]
where we denoted by \( \langle \cdot, \cdot \rangle \) the usual duality product. We also define the set
\[
\mathcal{D}_{\varepsilon,k} = \{ f \in L^1(\Omega) : f \geq 0, \|f\|_{L^1(\Omega)} = 1, d(f, \Sigma_k) < \varepsilon \}.
\]

With this notation in mind, by Lemma 1.1 we deduce the following.

Lemma 1.13 Suppose \( \tau \in (64k^2, 64(k+1)^2) \) with \( k \geq 1 \). Then for any \( \varepsilon > 0 \) there exists a large positive \( L = L(\varepsilon) \) such that for all \( u \in H^1(\Omega) \) with \( I(\tau, u) \leq -L \) and \( \frac{1}{|\Omega|} \int_{\Omega} e^u dx = 1 \), we have \( \text{dist}(e^u, \Sigma_k) < \varepsilon \).

We remark that as a direct consequence of [12, Theorem 1.2,(ii)], the blow-up points \( \tilde{p}_j, u \) at which the local-mass is concentrated cannot lie on the boundary of \( \Omega \).
2 A topological argument

The aim of this section is to show that an image of the $\Sigma_k$ can be mapped into very negative sublevels of the Euler functional $I_\tau$. Moreover this map is non-trivial in the sense that it carries some homology. The goal of this section is to sketch the proof of the following result which is given along the lines of [18].

**Proposition 2.1**

For any $k \in \mathbb{N}$ and $\tau \in (64k\pi^2, 64(k+1)\pi^2)$, there exists $L > 0$ such that the sublevel $\mathcal{K}^{-L}$ has the same homology as $\Sigma_k$.

The proof of the Proposition [2.1] will follows from the homotopy invariance of the homology groups once the following facts will be established.

**Mapping $\Sigma_k$ into very low sublevels of $I_\tau$.** To do so, for $\eta > 0$ small enough, consider the smooth non-decreasing cut-off function $\chi_\eta : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:

\[
\begin{cases}
\chi_\eta(t) = t, & \text{for } t \in [0, \eta]; \\
\chi_\eta(t) = 2\eta, & \text{for } t \geq 2\eta; \\
\chi_\eta(t) \in [\eta, 2\eta], & \text{for } t \in [\eta, 2\eta].
\end{cases}
\]

Then given $\sigma \in \Sigma_k$, and $k > 0$, we define the family of maps $\phi_\lambda : \Sigma_k \to H^2(\Omega)$ as $\phi_\lambda(\sigma) = \varphi_{\lambda,\sigma}(\cdot)$ where the function $\varphi_{\lambda,\sigma} : \Omega' \to \mathbb{R}$ is defined by

\[
\varphi_{\lambda,\sigma}(y) = \frac{1}{4} \log \sum_{i=1}^k t_i \left( \frac{2\lambda}{1 + \lambda^2 \chi_\eta^2(d_i(y))} \right)^4, \quad y \in \Omega
\]

where we set $d_i(y) = d(y, x_i)$, for $x_i \in \Omega$. Let

\[
\varphi_{\lambda,\sigma}(y) := P(\varphi_{\lambda,\sigma})(y), \quad \forall y \in \Omega.
\]

Then given $\sigma \in \Sigma_k$, and $k > 0$, we define the family of maps $\phi_\lambda : \Sigma_k \to H^2(\Omega)$ as $\phi_\lambda(\sigma) = \varphi_{\lambda,\sigma}(\cdot)$ where the function $\varphi_{\lambda,\sigma} : \Omega' \to \mathbb{R}$ is defined by

\[
\varphi_{\lambda,\sigma}(y) = \frac{1}{4} \log \sum_{i=1}^k t_i \left( \frac{2\lambda}{1 + \lambda^2 \chi_\eta^2(d_i(y))} \right)^4, \quad y \in \Omega
\]

where $x$ is a fixed point in $\Omega$. It is easy to verify that $\varphi_{\lambda,\sigma}(y) \to \delta_x$ for $\lambda \to +\infty$. Then (i) follows from the explicit expression of $\varphi_{\lambda,\sigma}$.

In order to prove (ii), we evaluate separately each term of $I_\tau$, and claim that the following estimates hold

\[
\frac{1}{2} \| \varphi_{\lambda,\sigma} \|^2_{\mathcal{K}} \leq (128k\pi^2 + o_1(1)) \log \lambda + C_\epsilon \quad (\text{uniformly in } \sigma \in \Sigma_k),
\]

where $o_1(1) \to 0$ as $\epsilon \to 0$ and where $C_\epsilon$ is a constant independent of $(x_i)$. The proof of (21) it is easy and it follows by integrating over $\Omega$. The proof of (22) is much more involved and it follows by Lemma 4.2 in [8].

\[q.e.d.\]
**Mapping very low sublevels of $I$ into $\Sigma_k$ and an homotopy inverse.** The main idea is to construct a non-trivial continuous map $\psi: \mathcal{H} \to \Sigma_k$ from the sublevels of the Euler functional into $\Sigma_k$ such that the composition $\psi \circ \phi_\lambda$ is homotopic to identity on $\Sigma_k$.

**Proposition 2.3** Suppose that $\tau \in (64k^2, 64(k + 1)\pi^2)$ with $k \geq 1$. Then there exists $L > 0$ and a continuous projection $\psi: \mathcal{H} \to \Sigma_k$ with the following properties.

(i) If $(u_n)_n \subset \mathcal{H}^{-L}$ is such that $e^{u_n} \to \sigma$, for some $\sigma \in \Sigma_k$, then $\psi(u_n) \to \sigma$;

(ii) if $\psi\lambda, \sigma$ is as in (20), then for any $\lambda$ sufficiently large the map $\sigma \mapsto \psi(\psi\lambda, \sigma)$ is homotopic to the identity on $\Sigma_k$.

**Proof.** First of all we observe that item (i) follows directly from item (ii) and Proposition 2.2.

The non-trivial part is the construction of the global continuous projection map $\psi$ which is a left homotopy inverse has proven in [5, Section 3].

We close this section by observing that, up to minor modifications, the above defined map $\psi$ is also a right inverse homotopy as proven in [10, Appendix]. Thus summing up we conclude that

**Corollary 2.4** Given $L$ sufficiently large the topological spaces $\mathcal{H}^{-L}$ and $\Sigma_k$ are equivalent, up to homotopy.

### 3 A Poincaré-Hopf Theorem without (PS)

The aim of this section is to prove an analogous of the Poincaré-Hopf theorem for a special class of functionals. To do so, let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space whose associated norm will be denoted by $\| \cdot \|$. Given an interval $\Lambda$ of $(0, \infty)$ and a map $K$ such that

$$K \in C^2(\mathcal{H}, \mathbb{R}), \quad \text{with } \nabla K : \mathcal{H} \to \mathcal{H} \text{ compact},$$

let us consider the functionals which are of the form:

$$I(\lambda, u) = \frac{1}{2} \langle u, u \rangle - \lambda K(u), \quad (\lambda, u) \in \Lambda \times \mathcal{H}. \quad (23)$$

It is well-known (see, for instance, [17, Lemma 2.3]) that the conditions (23) and (24) could do not enough to ensure the (PS)-condition which is known to hold only for bounded sequences. Now by using the deformation Lemma proven in [17, Proposition 1.1], we are in position to derive the following result.

**Theorem 3.1** (A Poincaré-Hopf theorem) Let $I(\lambda, \cdot)$ be a family of functionals satisfying (23) and (24) and fix $I(\cdot) := I(\lambda, \cdot)$ for some $\lambda \in \Lambda$. Given $\varepsilon > 0$, let $\Lambda' := [\lambda - \varepsilon, \lambda + \varepsilon]$ be a (compact) subset of $\Lambda$ and consider $a, b \in \mathbb{R}$ $(a < b)$, so that all the critical points $\bar{u}$ of $I(\lambda, \cdot)$ for $\lambda \in \Lambda'$ satisfy $I(\bar{u}) \in (a, b)$. Assuming that $I$ has no critical points at the levels $a, b$, we have

$$\deg_{LS}(\nabla I, \bar{u}^0, 0) = \chi(\bar{I}^0, \bar{I}^a). \quad (25)$$

The proof of this result will be given into two main steps. In the first step we will assume that all the critical points are non-degenerate; in the second step we will remove this assumption.

**Proof.** First step: non-degenerate case. We let $\mathcal{H}$ denote the set of critical points of $I$ which is compact by hypothesis. By compactness and non-degeneracy assumptions, $I$ has only finitely-many critical levels each of whose consists only of finitely-many critical points. Let $R$ be so large that all the critical points of $I_\lambda$ for $\lambda \in \Lambda'$ are in $B_R(0)$. Then we can define the cut-off function $\theta : \mathcal{H} \to [0, 1]$ satisfying

$$\theta(u) = 1 \text{ for } u \in B_R(0); \quad \theta(u) = 0 \text{ for } u \in \mathcal{H}\backslash B_R(0).$$

Following Lucia in [17], let $Z \in C^1(\mathcal{H}, \mathcal{H})$ be defined by:

$$Z(u) := -||\nabla K(u)||\nabla I(u) + ||\nabla I(u)||\nabla K(u)|,$$
and choose $\omega_{\epsilon} \in C^\infty(\mathbb{R})$ such that

$$0 \leq \omega_{\epsilon} \leq 1, \quad \omega_{\epsilon}(\zeta) = 0 \text{ for all } \zeta \leq \epsilon, \quad \omega_{\epsilon}(\zeta) = 1 \text{ for all } \zeta \geq 2\epsilon.$$ 

Finally we can define

$$W(u) := -\omega_{\epsilon}\left(\frac{\lvert \nabla \tilde{I}(u)\rvert}{\lvert \nabla K(u)\rvert}\right) \nabla \tilde{I}(u) + Z(u),$$

where $\omega_{\epsilon}\left(\lvert \nabla \tilde{I}(u)\rvert/\lvert \nabla K(u)\rvert\right)$ is understood to be equal 1 when $\nabla K(u) = 0$. Given the vector field:

$$\tilde{W}(u) := -\theta(u)\nabla \tilde{I}(u) + (1 - \theta(u))W(u),$$

we observe that it decreases $\tilde{I}$ in the complement of $\mathcal{K}$. We consider the local flow $\eta = \eta(t,u_0)$ defined by the Cauchy problem:

$$\frac{du}{dt} = \tilde{W}(u), \quad u(0) = u_0.$$

**Claim 1.** If $\tilde{I}$ has no critical levels inside some interval $[\bar{a}, \bar{b}]$, then the sub-level $\tilde{I}^a$ is a deformation retract of $\tilde{I}^b$.

To prove this, we arguing as follows. Given $u_0 \in \tilde{I}^b$, we can prove that

$$\tilde{I}(\eta(t,u_0)) \leq -c^2t + \tilde{I}(u_0)$$

(26)

Thus there exists a $t$ such that $\tilde{I}(\eta(t,u_0)) \leq \bar{a}$. Then we define:

$$t_a(u_0) := \left\{ \begin{array}{ll} \inf\{t \geq 0 : \tilde{I}(\eta(t,u_0)) \in \tilde{I}^\bar{a}\} & \text{if } \tilde{I}(u_0) > \bar{a} \\ 0 & \text{if } \tilde{I}(u_0) \leq \bar{a}. \end{array} \right.$$

Now the map

$$\tilde{\eta} : [0,1] \times \mathcal{H} \rightarrow \mathcal{H}, \quad (s,u_0) \mapsto \eta(st_a(u_0), u_0),$$

is a deformation retraction, as required.

Now let $\tilde{c}_i$ be the number of critical points of $\tilde{I}$ of index $i$. By classical Morse-theoretical arguments as in [3] Theorem 3.2, 3.3, pagg. 100-103, by excising $\{\tilde{I} < \bar{a}\}$, it follows that

$$\deg_{LS}(\nabla \tilde{I}, \tilde{I}^b, 0) = \sum_i (-1)^i\tilde{c}_i = \chi(\tilde{I}^b, \tilde{I}^a).$$

This conclude the proof in the non degenerate case.

**Second step: degenerate case.** We reduce ourselves to the non-degenerate case. To do so, fix a small $\delta > 0$ so that $dist(\mathcal{X}, \tilde{I}^b_\delta) > 4\delta$, and define the set $\mathcal{X}_\delta = \{ u \in \mathcal{H} : dist(u, \mathcal{X}) < \delta \}$. We next choose a smooth cut-off function $p$ such that

$$p(u) = 1 \text{ for every } u \in \mathcal{X}_\delta; \quad p(u) = 0 \text{ for every } u \in \mathcal{H}\setminus \mathcal{X}_{2\delta}.$$ 

We can also choose $p$ such that $0 \leq p(u) \leq 1$ for all $u \in \mathcal{H}$ and having uniformly bounded derivative in $\mathcal{X}_{2\delta}$. Now let $\mathcal{G} := \nabla \tilde{I}|_{\mathcal{X}_\delta} : \mathcal{X}_\delta \rightarrow \mathcal{H}$. Since the map $\mathcal{G}$ is a compact perturbation of the identity, by applying the Sard-Smale theorem (see theorem [14]), the set of regular values of $\mathcal{G}$ is dense in $\mathcal{H}$. This implies that we can find an arbitrarily small $u_0$ such that $\nabla \mathcal{G}(p)$ is non-degenerate for each $p \in \mathcal{G}^{-1}(u_0)$ which is equivalent to say that $\nabla^2 \tilde{I}$ is non-degenerate on the set

$$\Gamma(u_0) := \{ u \in \mathcal{H} : \nabla \tilde{I}(u) = u_0 \} \cap \mathcal{X}_\delta.$$ 

Moreover we observe that $\lVert \nabla \tilde{I} \rVert \geq \gamma_\delta > 0$ on $\mathcal{X}_{2\delta}\setminus \mathcal{X}_\delta$ for some constant $\gamma_\delta$. Now let us consider the function

$$\tilde{I}(u) := \tilde{I}(u) + p(u)(u_0, u).$$

It can be shown that the following facts hold:

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1The proof of this inequality is the most involved part of this claim and it can be proven up to minor modifications reading word by word the arguments given in [12] pagg. 121-122.
(i) \( \tilde{I} \) coincides with \( I \) in \( \mathcal{H} \setminus \mathcal{K}_{25} \);

(ii) \( \tilde{I} \) has the same critical points as \( I(\tau, \cdot) \) in \( \mathcal{H} \setminus \mathcal{K}_{5} \);

(iii) \( \tilde{I} \) is non-degenerate in \( \tilde{I}^{b} \).

Item (i) is trivial. To prove (ii) we observe that since \( \tilde{I} \) and \( \tilde{I} \) coincides out of \( \mathcal{K}_{25} \), it is enough to prove the claim for \( u \in \mathcal{K}_{25} \setminus \mathcal{K}_{5} \). By differentiating, we have

\[
\langle \nabla \tilde{I}(u), v \rangle = \langle \nabla I(u) + \nabla p(u)\rangle(u, u_{0}) + p(u)u_{0}, v), \quad \forall v \in \mathcal{H}.
\]

Thus, by recalling that \( u \in \mathcal{K}_{25} \setminus \mathcal{K}_{5} \), it follows that

\[
||\nabla \tilde{I}(u)|| \geq ||\nabla I(u)|| - \left( ||u, u_{0}||\nabla p(u) - p(u)||u_{0}|| \right) \geq \gamma_{5} - ||u_{0}||\nabla p(u)||u|| + 1 > 0,
\]

where the last inequality follows since \( p \) has uniformly bounded derivatives and \( u_{0} \) can be chosen arbitrarily small. To prove (iii) we argue as follows. Since all the critical points of \( I \) are in \( \mathcal{K}_{5} \), let us assume by contradiction that \( \tilde{I} \) is degenerate at some critical point \( \tilde{u} \). Now since \( \tilde{u} \notin \mathcal{K} \), it follows that \( \tilde{u} \in \mathcal{K}_{5} \setminus \mathcal{K} \). Moreover \( \nabla \tilde{I}(\tilde{u}) = 0 \) is equivalent to say that \( \nabla I(\tilde{u}) = u_{0} \) and therefore \( \tilde{u} \in \Gamma(u_{0}) \). But this is contradict the fact that \( \nabla^{2} \tilde{I}(p) \) is non-degenerate on \( p \in \Gamma(u_{0}) \).

Now, for \( ||u_{0}|| \) sufficiently small the map \( \nabla I - Id \) is a strict \( \alpha \)-contraction. (See Section 4) and since \( (\nabla \tilde{I})^{-1}(\{u_{0}\}) = \mathcal{K} \), the generalized degree \( \text{Deg}(\nabla \tilde{I}, \tilde{I}^{b}, u_{0}) \) is well-defined; moreover it coincides with \( \text{Deg}(\nabla \tilde{I}, \tilde{I}^{b}, 0) \) since it is locally constant. With the above choice for \( R \) and by using the excision property and the homotopy invariance of the generalized degree, (see, for instance, [7] for further details), we have

\[
\text{deg}_{LS}(\nabla \tilde{I}, \mathcal{B}_{R}, 0) = \text{Deg}(\nabla \tilde{I}, \mathcal{B}_{R}, 0).
\]

Now choosing a possibly larger \( R \) in such a way \( \mathcal{K}_{25} \subset \mathcal{B}_{R/2} \), the conclusion readily follows by the first step, simply by replacing \( I \) with \( \tilde{I} \). q.e.d.

**Corollary 3.2** If \( \tau \in (64k\pi^{2}, 64(k+1)\pi^{2}) \) for some \( k \in \mathbb{N} \) and if \( b \) is sufficiently large positive, the sub-level \( \mathcal{H}^{b} \) is a deformation retract of \( \mathcal{H} \) and hence it has the homology of a point.

**Proof.** This result follows, by using the deformation constructed in the proof of the Poincaré-Hopf theorem. See, for instance [19], Corollary 2.8.

(q.e.d.)

Setting

\[
J(u) := \log \left( \frac{1}{||u||} \int_{\Omega} h(x)e^{u}dx \right)
\]

the functional \([5] \) can be put in the following form: \( I(\tau)(u) = \frac{1}{2}||u||_{\mathcal{H}^{b}}^{2} - \tau J(u) \).

**Proof of Theorem 2.** Proof. In order to prove 2 it is enough to apply theorem [5] to the functional \([4] \) for \( \lambda = \tau \), \( A = (64k\pi^{2}, 64(k+1)\pi^{2}) \) for \( k \geq 1 \), \( \mathcal{H} = \mathcal{K} \) and finally \( K(u) = J(u) \) where \( J \) was given in \([24] \). The only thing it should be noted, is that all the critical points \( \tilde{u} \) of \( I_{\tau} \) for \( \tau \in [\tau - \varepsilon, \tau + \varepsilon] \subset (64k\pi^{2}, 64(k+1)\pi^{2}) \) satisfy \( \tilde{I}(\tilde{u}) \in (a, b) \). This is a consequence of proposition [17] and of the boundedness of \( J \) which is consequence of the Moser-Trudinger inequality. Now the conclusion follows choosing \( a = -L \) as in proposition [21] and \( b \) as in corollary [3.2]. In fact by using theorem [5] we have that

\[
d_{\tau} = \chi(\tilde{I}^{b}, \tilde{I}) = \chi(\tilde{I}^{b}) - \chi(\tilde{I}^{a}) = \chi(\mathcal{K}) - \chi(\Sigma_{k}) = 1 - \chi(\Sigma_{k}).
\]

The conclusion follows by invoking proposition [17].

(q.e.d.)

**Remark 3.3** We observe that the Leray-Schauder degree in the Sobolev space \( \mathcal{K} \) coincides with the degree in every Hölder space \( C^{2,\alpha}(\Omega), \alpha \in (0,1) \). See for instance [17] Part I, Appendix B, Theorem B.1-B.2.
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