Production of Axions by Cosmic Magnetic Helicity

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We investigate the effects of an external magnetic helicity production on the evolution of the cosmic axion field. It is shown that a helicity larger than \((\text{few} \times 10^{-11} \text{G})^2 \text{Mpc}\), if produced at temperatures above a few GeV, is in contradiction with the existence of the axion, since it would produce too much of an axion relic abundance.

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Recently, the topic of cosmic magnetic helicity generation and phenomenology has been widely discussed (see, e.g.,\textsuperscript{4,5,6,7,8}). An in depth study of this very peculiar quantity could help in understanding the nature of the cosmic magnetic field itself and its generation. Magnetic helicity is defined as

\[
H_B(t) = \frac{1}{V} \int_V d^3x \mathbf{A} \cdot \nabla \times \mathbf{A},
\]

where \(A^\mu = (A^0, \mathbf{A})\) is the electromagnetic field. In a flat universe described by a Robertson-Walker metric, 
\(ds^2 = dt^2 - R^2 dx^2\), where \(R(t)\) is the expansion parameter normalized so that at the present time \(t_0\), \(R(t_0) = 1\), the electric and magnetic fields are defined as \(\mathbf{E} = -R^{-1} \mathbf{A}\) and \(\mathbf{B} = R^{-1} \nabla \times \mathbf{A}\), where a dot indicates the derivative with respect to the cosmic time \(t\). Hence, the helicity can be written as \(H_B(t) = (1/V) R^2 \int_V d^3x \mathbf{A} \cdot \nabla \times \mathbf{A}\). It should be noted that in the literature the definition of \(H_B\) is usually given without the factor \(R^2\). In any case, the two definitions coincide at the present time. However we note that, since the early universe is a very good conductor, magnetic fields are frozen into the plasma and then \(B\) scales in time as \(B \propto R^{-2}\). Therefore, magnetic helicity, as defined in Eq. (1), remains constant (after its generation).

The magnetic helicity is related to the topological properties of the magnetic field, it is a \(CP\) – odd pseudo-scalar quantity and, if different from zero, would reveal a macroscopic \(P\) and \(CP\) violation in the universe.

Let us introduce the magnetic energy and magnetic helicity spectra \(E_B(k,t) = (2\pi/V) k^2 \mathbf{B}(k) \cdot \mathbf{B}(-k)\) and \(\mathcal{H}_B(k,t) = (4\pi/V) R^2 k^2 \mathbf{A}(k) \cdot \mathbf{B}(-k)\), respectively, where \(\mathbf{B}(k)\) and \(\mathbf{A}(k)\) are the magnetic field and the vector potential in Fourier space and \(k = |k|\). In terms of the spectra, the magnetic energy, \(E_B(t) = (1/2V) \int_V d^3x \mathbf{B}^2(x,t)\), and the magnetic helicity are \(E_B = \int dk E_B\) and \(H_B = \int dk \mathcal{H}_B\). From the above definitions it follows that any magnetic field configuration satisfies the inequality \(|\mathcal{H}_B(k,t)| \leq 2R^3k^{-1}E_B(k,t)\) \([4]\). A straightforward integration in \(k\) leads to the so-called “realizability” condition, \(|H_B| \leq H_B^{\text{max}} = 2R^3\xi_B E_B\), which fixes the maximal helicity associated with a given magnetic field configuration. Here, \(\xi_B(t) = 2\pi R \int dk k^{-1}E_B(k,t)/E_B\) is the magnetic field correlation length. Therefore, bounds on the magnetic field can be directly translated into bounds on magnetic helicity. Before discussing this point more deeply, it is useful to introduce the following parametrization for the helicity produced at the temperature \(T_\text{s}\): \(H_B(T_\text{s}) = \pm 3.3 \times 10^{-19} \frac{g_s(T_\text{s})^{1/2}}{g_s(T_\text{s})} \frac{\text{GeV}}{T_\text{s}} \frac{\text{G}^2\text{Mpc}}{}\), (2) where \(g_s\) and \(g_{s,s}\) count the total numbers of effectively massless degrees of freedom referring to the energy and entropy density of the universe, respectively. (From now on, \(g_s\) and \(g_{s,s}\) will be considered equal.) The choice \(r = 1\) gives the maximal helicity associated with a magnetic field correlated on the Hubble scale, \(\xi_B = H^{-1}\) (\(H\) is the Hubble parameter), and whose energy density equals the total radiation energy density at \(T_\text{s}\). Thus, the parameter \(r \leq 1\) measures the fraction of helicity with respect to the maximum possible at a certain time.

It is known from the Big Bang Nucleosynthesis (BBN) argument that a magnetic field \(B\) at the temperature \(T \simeq 0.1\text{MeV}\), correlated on the Hubble scale at that time, cannot be more intense than \(10^{11}\text{G}\) \([4]\). This yields the upper limit \(r \lesssim r_{\text{BBN}} \simeq 82 g_s(T_\text{s})^{1/2}/(T_\text{s}/\text{GeV})\). On the other hand, the analysis of the observed anisotropy of the Cosmic Microwave Background (CMB) radiation requires the bound \(10^{-9}\text{G}\) on the scales of 1Mpc \([4]\) for the magnetic field (today), which leads to \(r \lesssim r_{\text{CMB}} \simeq 0.2 g_s(T_\text{s})^{1/2}/(T_\text{s}/\text{GeV})\). Observe that for magnetic helicity produced before \(1\text{GeV}\) neither of these limits give any interesting constraints, since \(r\) is always expected to be less than 1.

In this paper we consider the effects of magnetic helicity generation on the cosmic axion field, showing how such generation could considerably amplify the axion’s expected relic abundance \(\Omega_a\). In particular, we will show that the request \(\Omega_a \leq \Omega_{\text{matter}}\) sets, in most cases, a bound on \(r\) of many orders of magnitude smaller that the above \(r_{\text{BBN}}\) and \(r_{\text{CMB}}\).

The axion field \(a\) emerges in the so called Pecce-Quinn (PQ) mechanism \([3]\) for the solution of the strong CP problem (for a review see, e.g., \([10]\)). In most models, it is identified with the phase \(\Theta\) of a complex scalar field.
\(a = f_a \Theta\), where \(f_a\) is a phenomenological scale usually referred to as the PQ- or axion-constant, and presently constrained in the very narrow region \(10^9 \lesssim f_a \lesssim 10^{12}\text{GeV}\) by astrophysical and cosmological considerations \(^1\).  

For temperatures above the PQ-scale the total Lagrangian is to be invariant under a constant shift of the phase \(\Theta\) (PQ-symmetry). However, for lower temperatures, the symmetry is (spontaneously) broken and the angle \(\Theta\) is fixed on a precise value. We indicate this initial value with \(\Theta_i\). This phase evolves following the equation \(^1\)

\[
\ddot{\Theta} + 3H\dot{\Theta} + \frac{\partial V}{\partial \Theta} = 0, \tag{3}
\]

where \(V(\Theta, T) = m_a^2(T)(1 - \cos \Theta)\) is the instanton induced potential. The temperature-dependent axion mass is \(^2\)

\[
m_a(T) \simeq 0.1m (\Lambda/T)^{3.7} \quad \text{for} \quad T \gg \Lambda \quad \text{and} \quad m_a(T) = m \quad \text{for} \quad T \ll \Lambda \quad \text{with} \quad m \simeq 6.2 \times 10^{-6}\text{eV}/f_{12} \quad \text{[}f_{12} = f_a/(10^{12}\text{GeV})\], and \(\Lambda \sim 200\text{MeV}\) is the QCD scale.

As the curvature (mass) term in Eq. (3) becomes dominant over the friction (Hubble) term, \(\Theta\) begins to oscillate with the frequency \(m_a(T)\). This happens at about the temperature \(T = T_1\) defined by the equation \(m_a(T_1) = 3H(T_1)\). Approximately \(T_1 \simeq 0.9f_{12}^{-0.175}\Lambda_{1200}^{0.65}\text{GeV with} \Lambda_{200} = \Lambda/(200\text{MeV}).\)

During this period of coherent oscillations the number of axions in a comoving volume remains constant, \(n_a R^3 = \text{const}\), where \(n_a(T)\) is the axion number density. Thus, the axion relic abundance today is

\[
\Omega_a \simeq 0.2 \Lambda_{200}^{-0.65} [\Theta_1^2 + (\dot{\Theta}_1/3H_1)^2] f_{12}^{1.175}, \tag{4}
\]

where \(\Theta_1, \dot{\Theta}_1\) and \(H_1\) are respectively \(\Theta(T_1), \dot{\Theta}(T_1)\) and \(H(T_1)\). If the axion field is not interacting in the region \(T > T_1\), and \(T_1 < f_a\), then the axion kinetic energy at \(T = T_1\) is reliably negligible, \(\Theta_1 \propto (T_1/f_a)^{3} \ll 1\), and consequently \(\Theta_1 \simeq \Theta_i\). Thus, Eq. (4) reduces to

\[
\Omega_a = \Omega_{a0} \simeq 0.2 \Lambda_{200}^{-0.606} \Theta_i f_{12}^{1.175}. \tag{5}
\]

Assuming the natural value \(\Theta_i \sim 1\), this gives \(\Omega_{a0} \simeq 0.3\) (the expected value for the dark matter abundance) for \(f_{12} \sim 1\). Much larger values of \(f_{12}\) would cause too much axion production and are therefore excluded.

However, if the field evolution took place in the presence of an external magnetic field, the previous analysis needs to be modified. It is easy to show the importance of the magnetic helicity in this case. In fact, in the presence of a magnetic field, the right hand side of Eq. (3) is modified as \(\Theta\). However, if the field evolution took place in the presence of an external magnetic field \(^3\) and therefore its evolution is modified, the effect is negligible since the helicity produced by the axion itself is small. However, if another mechanism were responsible for the production of a significant amount of helicity, the consequences on the axion relic abundance today could be enormous.

Let us assume, first, that a large amount of helicity is produced at the temperature \(T_s \gg T_1\) and that the time scale of its production is short compared to the time scale \(t_1\). Moreover, taking into account that after its generation the helicity can be considered a (quasi)-conserved quantity (see, e.g., \(^4\)), we can safely approximate it as a step function, \(H_B(t) = H_B(t_s)\theta(t - t_s)\).

When the potential energy is small compared to the kinetic energy and/or the friction, Eq. (5) can be solved giving \(\dot{\Theta}(T) = \alpha H_B(T)/(4\pi f_a^2 R^3)\), that is, the kinetic energy is a decreasing function of time. Let us indicate with \(T_{eq}\) the temperature at which the kinetic energy equals the potential: \((\dot{\Theta}_s^2/2)\big|_{T_{eq}} = (V(\Theta))/\big|_{T_{eq}}\), where for the sake of simplicity we are considering the root mean square value of the potential \((V(\Theta)) = m_a^2(T)(1 - 1/\sqrt{2})\).

It is clear that if \(T_{eq} \geq T_1\) will the oscillations start at the temperature \(T_1\), where the friction becomes small. In this case, the standard analysis applies and we find

\[
\Omega_a \simeq 0.1 f_{12}^{1.175} \Lambda_{200}^{0.65} \left[1 + \left(r/r_a\right)^2\right], \quad \text{if} \quad r \leq r_a, \tag{6}
\]

where \(r \leq r_a \simeq 1.5 \times 10^{-11}\text{eV}/(\text{GeV})\); \(f_{12}^{-0.175}\Lambda_{200}^{0.65}\) corresponds to \(T_{eq} \geq T_1\). However, if \(T_{eq} < T_1\) the oscillations cannot start at \(T = T_1\), the kinetic energy being still too large, but will be delayed and start at \(T = T_{eq}\). The resulting axion relic abundance is

\[
\Omega_a \simeq 0.2 f_{12}^{1.175} \Lambda_{200}^{0.65} \left(r/r_a\right), \quad \text{if} \quad r > r_a. \tag{7}
\]

In Fig. 1 we show the cosmologically allowed region, \(\Omega_a < \Omega_{\text{matter}} \simeq 0.3\), for the PQ-constant \(f_a\) and \(r\) parameter. Observe that, since we are now considering helicity to be generated before the axion oscillations, \(T_s > T_1 \sim 1\text{GeV}\), the BBN and CMB arguments do not set any consistent limit on the magnetic helicity. On the other hand, the limit on \(r\) from the axion abundance is

\(^1\) For the so-called hadronic axion models, whose interaction with electrons is suppressed, a small region around \(f_a \sim 10^{9}\text{GeV}\) is also allowed \(^2\); \(^3\). For other possibilities of reconsidering the limits on the PQ-constant see, for example, Ref. \(^4\).

\(^2\) This is not a very strong assumption. For example, in the \(\alpha\)-dynamo mechanism this time scale is expected to be \(\Delta t_s \lesssim t_s \lesssim t_1\).
Let us suppose that the helicity is produced during the finite interval of time \((t_A, t_B)\). Since before \(t_A\) and after \(t_B\) the evolution of \(\varphi\) is that of a free oscillator, and since for \(r \lesssim r_{\text{CMB}}\) it results \(|\Theta| < 1\) if \(T_* \lesssim 1\)MeV, we can apply the standard procedure to find the axion relic abundance today. This reads \(\Omega_a = \Omega_{a0} + 2\Omega_{a0}^2 \text{Re} \frac{B}{2} + |z|^2\), where \(z = \left( f_a/\sqrt{2} \rho_c \right) f_a^T dt F(t) e^{it\rho}\), \(\rho_c\) is the present critical density, and \(\Omega_{a0}\) is the expected axion relic abundance (today) in the absence of magnetic helicity [see the discussion below Eq. (4)]. The time profile of the magnetic helicity in the expanding universe is conveniently parameterized by the following “rapidity” function, \(f(t) = (R_*/r)^{3/2} t_* H_B/H_B(t_*),\) where \(t_*\) is a reference time between \(t_A\) and \(t_B\). Therefore, the forcing term can be written as \(f(t) = -(\alpha/4\pi f_2^2) [H_B(T_*)/R_*^{3/2}] f(t)\) and, using Eq. (2), \(z\) becomes

\[
z \simeq \pm 3 \times 10^7 A_{200}^{1/2} f_{12}^{-1} r (T_*/\Lambda)^{1/2} \hat{f}(m),
\]

where \(\hat{f}(\omega) = \int_{-\infty}^{+\infty} dt f(t) e^{i\omega t}\) is the Fourier transform of \(f(t)\). The axion relic abundance can be exactly calculated once the rapidity function \(f(t)\) is known. In the case of slow \((\Delta t \gg m)\) helicity production it results \(\hat{f}(m) \approx 0\) and, consequently, \(\Omega_a \approx \Omega_{a0}\). On the other hand, if helicity production is fast \((\Delta t \ll m)\) we have \(|\hat{f}(m)| \approx \text{Re}[\hat{f}(m)] \approx 1\). To give a simple estimate of \(\Omega_a\) we observe that for \(f_{12} < 1\) it results that \(\Omega_{a0} \ll \Omega_a\), and therefore \(\Omega_a \sim |z|^2\). This yields the upper bound \(r \lesssim 10^{-8} A_{200} f_{12} (\Lambda/T_*)^{1/2}\) which translates into a bound on the present magnetic helicity

\[
H_B \lesssim 10^{-27} A_{200} g_*(T_*)^{-1/2} f_{12} \left(\frac{\text{GeV}}{T_*}\right)^{3/2} \text{Mpc}. \tag{10}
\]

This limit is always stronger than the one from the BBN and, for \(T_* \gtrsim \text{few} f_{12}^{3/2}\text{keV},\) it also results that \(r < r_{\text{CMB}}\) (see Fig. 4). In this figure, we have included the region above the cosmological limit \(f_a > 10^{12}\text{GeV}\) (see the continuous line), since the analysis is straightforward in this case. Simply note that the minus sign in Eq. (9) corresponds to a reduction of the axion relic abundance.

To conclude, we have pointed out that a strong connection between magnetic helicity and axion cosmology exists. In particular, we have studied the influence of primordial magnetic helicity generation on the present axion relic abundance. The result is that, allowing for possible helicity generation in the early universe, the knowledge of the PQ-constant would not be sufficient to determine the axion relic abundance today. Indeed, suppose axions were detected with the PQ-constant strictly under \(10^{12}\text{GeV}\). This would not exclude the possibility that they represent the dark matter component of our universe, if an appropriate amount of helicity has been generated (see Fig. 1). We focused our analysis on two different cases corresponding to helicity production before or during the axion coherent oscillations.

FIG. 1: Allowed region for the magnetic helicity and the PQ-constant in the case of helicity production before the axion coherent oscillations. The area under the curve represents the parameter region \(\Omega_a < \Omega_{\text{matter}} \simeq 0.3\) [13].
Let us consider the former, and more interesting, case first. We look at two different possibilities within this case: (i) first, if axions were detected as cold dark matter particles this would severely constrain the allowed amount of magnetic helicity in our universe, according to Eq. (8). On the other hand, (ii) if a helicity greater than \( (\text{few} \times 10^{-15} \text{G})^2 \text{Mpc} \) were revealed at the present time this would be in contradiction with the existence of the invisible axion itself, unless it is assumed that helicity production took place at a temperature above the PQ-scale. It should be noted that the bounds found are considerably stronger than the ones deduced from the BBN or CMB analysis.

Finally, if helicity production took place during the primordial axion oscillations, the constraints on the helicity depend on the time scale of its production, and are indeed relevant only if this is much smaller than the typical axion oscillation time \( 1/m_a \). Also in this case, the bounds found, Fig. 2 and Eq. (11), are relevant compared to the ones from the BBN and CMB.

We consider these results of interest in view of next generation experiments devoted to the axion search and future experiments on CMB, which are expected to be sensitive enough for the detection of a possible helical magnetic field \[ \text{[7, 21]} \].

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FIG. 2: Allowed region for the magnetic helicity and the PQ-constant in the case of fast helicity production during the axion coherent oscillations. The areas enclosed by the dashed and continuous curves represent the parameter region \( \Omega_a < \Omega_{\text{matter}} \simeq 0.3 \) for the plus and minus sign in Eq. (9), respectively. The areas under the horizontal lines represent the region allowed by the CMB analysis for three different temperatures.

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