Exponential Consistency of the M-estimators of Regression Coefficients with Multivariate Responses

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Abstract

In this brief note, we present the exponential consistency of the M-estimators of regression coefficients for models with multivariate responses. We first prove an exponential tail bound for the $\ell_2$-norm of the M-estimator from the true value of the regression coefficients under suitable assumption, which directly leads to the exponential consistency result for the M-estimators. We are working on to apply this general results for some particular M-estimators, including the maximum likelihood estimator, under the special set-ups of multivariate linear regression models and linear mixed-effects models.

Keywords: Exponential consistency; M-estimator; Multivariate regression.

1 Introduction

Consider general regression model with multivariate response $y \in \mathcal{Y} \subseteq \mathbb{R}^m$ and a set of linear predictor $X^T\beta$, where $X = [X_1, \cdots, X_m]$ for some explanatory variables $X_j \in \mathcal{X} \subseteq \mathbb{R}^q$, $j = 1, \ldots, m$, and the regression coefficient $\beta \in \mathbb{R}^q$, so that $X \in \mathcal{X}^m$. In most applications, $X_j$s are IID copies of one underlying $q$-dimensional explanatory variables. A regression model is assumed yielding the distribution of $y$, given $X$, for which the expectation of $y$ is a known function of the linear predictor $X^T\beta$. The variance of $y_n$ is assumed to be $\Sigma(\eta, Z)$ which may depend on some known or unknown parameter $\eta \in \mathbb{R}^d$ and may even depend on some covariates $Z \subset X$ in more general case. Examples are normal multivariate linear regression, multivariate generalized linear models and linear or generalized linear mixed models, panel data models and many others.

Suppose that we have IID observations $(y_i, X_i)$, $i = 1, \ldots, n$, as independent realizations of $(y, X)$ and we want to estimate the unknown parameters $\theta = (\beta, \eta)$ based on these $n$ sample observations. Note that the total effective sample size is $N = mn$ which is assumed to be larger than $q + d$. The popular general class of M-estimators may defined as the minimizer of the empirical average of a suitable loss function $l(y, X^T\beta; \eta)$ based on the specified regression model of $y$ on $X$, i.e., the M-estimator of $\theta$ is given by

$$\hat{\theta} = (\hat{\beta}, \hat{\eta}) = \arg \min_{\theta} \mathbb{P}_n \left[ l(y, X^T\beta; \eta) \right],$$

where $\mathbb{P}_n$ denotes the empirical average based on a sample of $n$ observations, i.e., $\mathbb{P}_n \left[ f(y, X) \right] = n^{-1} \sum_{i=1}^n f(y_i, X_i)$ for any function $f$. Note that, the above definition of M-estimators is slightly stringent than the estimating equation based definition of M-estimators and cover many important estimators, e.g., MLE and minimum
divergence estimators, as special cases (for MLE, the loss function $l$ is the negative log-likelihood). We will restrict our attention to the above-mentioned class of M-estimators for which there is an appropriate (convex) loss-function.

Let us define the corresponding best fitting (population) parameter value as

$$
\theta_0 = (\beta_0, \eta_0) = \arg\min_\theta E\left[ l(y, X^T \beta; \eta) \right].
$$

(2)

This best fitting parameter should coincide with the true parameter value when the true data generating distribution belongs to the model family. It is known that the M-estimator $\theta$ is consistent for $\theta_0$ as $n \to \infty$ and $m$ remains fixed. In this note, we will show its exponential consistency assuming $m$ can also vary, satisfying $m n \to 0$ as $n \to \infty$. The dimension of the parameter vector is assumed to be fixed.

## 2 Exponential Consistency Results for the Regression Coefficients

Let us first assume that the variance parameter $\eta$ is known or consistently estimated so that it does not affect the asymptotic properties of $\hat{\beta}$. So, we can consider the loss function $l = l(y, X^T \beta)$ to be a function of $\beta$ only and accordingly prove the exponential consistency of the M-estimator $\hat{\beta}$ of the regression coefficient $\beta$. In this case, note that, $\hat{\beta} = \arg\min_\beta P_n l(y, X^T \beta)$ and $\beta_0 = \arg\min_\beta E l(y, X^T \beta)$.

### 2.1 Regularity Conditions

In the following, let us denote the gradient with respect to $\beta$ by $\nabla$ and the $\ell_2$ and $\ell_\infty$ norms by $\| \cdot \|$ and $\| \cdot \|_\infty$, respectively. Also, let $1_m$ denotes the $m$-vector will all entry one. For any set $S$ of points $(y, X)$, we denote the indicator function of this set by $1_S(X, y)$ which takes the value one if $(y, X) \in S$ and zero otherwise.

**A.1** Suppose that, for a sufficiently large $B > 0$, $\beta$ is an interior point of a compact and convex set

$$
\mathcal{B} = \{\beta : \|\beta - \beta_0\| < B\}.
$$

**A.3** There exists a positive constant $C_q$, depending only on $q$, such that $E\|X1_m\|^2 \leq mC_q$.

**A.2** The loss function $l(y, X^T \beta)$ is convex in $\beta$ and the semi-marginal pseudo-information matrix

$$
I(\beta) = E \left[ \left( \nabla l(y, X^T \beta) \right) \left( \nabla l(y, X^T \beta) \right)^T \right]
$$

is finite and positive definite at $\beta = \beta_0$. Moreover, $\|I(\beta)\|$ is bounded from above.

**A.6** There exists a positive constant $V_n$ such that, for all $\beta \in \mathcal{B}$, we have

$$
E \left[ l(y, X^T \beta) - l(y, X^T \beta_0) \right] \geq V_n \|\beta - \beta_0\|^2.
$$

**A.5** The function $l(y, X^T \beta)$ satisfies the Lipschitz condition with a Lipschitz constant $k_n > 0$, i.e., for any $\beta, \beta' \in X$, we have

$$
\|l(y, X^T \beta) - l(y, X^T \beta')\|_{\Lambda_n} \leq k_n \|1_m^T X^T \beta - 1_m^T X^T \beta'\|_{\Lambda_n} \leq k_n \|1_m^T X^T \beta\|_{\Lambda_n}.
$$
where \( \Lambda_n = \{(y, X) : \|y\|_\infty \leq K_n^*, \|X\|_\infty \leq K_n\} \) for sufficiently large constants \( K_n, K_n^* > 0 \).

A.4 Consider \( V_n \) as defined in Condition A.6, and \( k_n \) and \( \Lambda_n \) as defined in Condition A.5. Then, there exists a sufficiently large constant \( C > 0 \) such that, with \( b_n = Ck_n(mC_q/n)^{1/2}/V_n \), we have

\[
\sup_{\beta \in B_n} \left| E[l(y, X^T \beta) - l(y, X^T \beta_0)](1 - \mathbb{I}_{\Lambda_n}(X, y)) \right| \leq o(mC_q/n).
\]

Note that the above assumptions are just direct extensions of the required conditions in Fan and Song (2010) from the cases with univariate responses to our present case with multivariate responses. We will later verify that these assumptions indeed hold in most practical situations under the particular set-ups of multivariate linear regression models and linear mixed-effects models.

2.2 The Main Result

**Theorem 1 (Exponential tail bound)** Assume that Conditions A.1–A.6 hold. Then, for any \( t > 0 \) and large enough \( n \), we have

\[
Pr \left( \sqrt{n} \left\| \hat{\beta} - \beta_0 \right\| \geq 16k_nC_q^{-1/2}(1 + t) \right) \leq \exp \left( -\frac{2C_q^2t^2}{K_n^2} \right) + nPr(\Lambda_n^c).
\]

**Proof:**
Here we refer to Lemmas 2–4 in Fan and Song (2010) for the proof. The idea of the proof is to bound the quantity \( G_1(B) \) (defined below) and then use this to bound \( \|\hat{\beta} - \beta_0\| \). Let us define

\[
G_1(B) = \sup_{\beta \in B} \left| (\mathbb{P}_n - E)\{(l(y, X^T \beta) - l(y, X^T \beta_0))\mathbb{I}_{\Lambda_n}(X, y)\} \right|.
\]

Step 1:
By an application of Lemma 2 of Fan and Song (2010) (Symmetrization theorem), we get

\[
E(G_1(B)) \leq 2E \left[ \sup_{\beta \in B} \left| \mathbb{P}_n\varepsilon\{l(y, X^T \beta) - l(y, X^T \beta_0)\}\mathbb{I}_{\Lambda_n}(X, y) \right| \right],
\]

where \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_N) \) is a Rademacher sequence. By Lemma 3 Fan and Song (2010) and Condition A.5, we have that \( E(G_1(B)) \) is further bounded above by

\[
4k_nE \left[ \sup_{\beta \in B} \left| \mathbb{P}_n\varepsilon_1 X_m^T(\beta - \beta_0)\mathbb{I}_{\Lambda_n}(X, y) \right| \right].
\]

Next, by the Cauchy-Schwarz inequality, we get that

\[
E \left[ \sup_{\beta \in B} \left| \mathbb{P}_n\varepsilon_1 X_m^T(\beta - \beta_0)\mathbb{I}_{\Lambda_n}(X, y) \right| \right] \leq E\|\mathbb{P}_n\varepsilon X_1^T\mathbb{I}_{\Lambda_n}(X, y)\| \sup_{\beta \in B} \|\beta - \beta_0\| \leq E\|\mathbb{P}_n\varepsilon X_1^T\mathbb{I}_{\Lambda_n}(X, y)\|_B.
\]

And, by Jensen’s inequality, we have that

\[
[(E\|\mathbb{P}_n\varepsilon X_1^T\mathbb{I}_{\Lambda_n}(X, y)\|)^2]^{1/2} \leq [E(\|\mathbb{P}_n\varepsilon X_1^T\mathbb{I}_{\Lambda_n}(X, y)\|)^2]^{1/2} = [E\|X_1^T\|^2\mathbb{I}_{\Lambda_n}(X, y)/n]^{1/2}.
\]
But, from Condition A.3, we have that
\[ E\|X_1\|^2 1_{\Lambda_n}(X, y) \leq E\|X\|^2 \leq mC_q, \]
and hence, we finally get
\[ E(G_1(B)) \leq 4k_n B(mC_q/n)^{1/2}. \]

Now, on the set \( \Lambda_n \), we have from Condition A.5 and the Cauchy-Schwarz inequality that
\[ |l(y, X^T \beta) - l(y, X^T \hat{\beta})| \leq k_n|1_m^TX^T(\beta - \hat{\beta})| \leq k_n\|X_1\|\|\beta - \beta^*\| \leq k_n m^{1/2} K_n B. \]

So, we can apply Lemma 4 of [Fan and Song (2010)] with \( L^2 = 4k_n^2 mK_n^2 B^2 \) to get
\[ \Pr \left( G_1(B) \geq 4k_n B(mC_q/n)^{1/2} (1 + t) \right) = \exp \left( -2C_q t^2 / K_n^2 \right). \tag{4} \]

**Step 2:**

Now, in order to bound \( \|\hat{\beta} - \beta_0\| \), we first define a convex combination \( \beta_s = s\hat{\beta} + (1 - s)\beta_0 \), where \( s = (1 + \|\beta - \beta_0\| / B)^{-1} \). By definition, \( \|\beta_s - \beta_0\| = s\|\beta - \beta_0\| \leq B \), i.e. \( \beta_s \in B(B) \). Due to the convexity, we have that
\[ \mathbb{P}_n l(y, X^T \beta_s) \leq s\mathbb{P}_n l(y, X^T \hat{\beta}) + (1 - s)\mathbb{P}_n l(y, X^T \beta_0) \leq \mathbb{P}_n l(y, X^T \beta). \]

Since \( \beta^* \) minimizes \( l(y, X^T \beta) \), we have
\[ E[l(y, X^T \beta_s) - l(y, X^T \beta_0)] \geq 0, \]
if we regard \( \beta_s \) as a parameter in the expectation. Combining the above two results we have that
\[ E[l(y, X^T \beta_s) - l(y, X^T \beta_0)] \leq (E - \mathbb{P}_n)[l(y, X^T \beta_s) - l(y, X^T \beta_0)] \leq G(B), \]
where
\[ G(B) = \sup_{\beta \in B(B)} |(\mathbb{P}_n - E)\{l(y, X^T \beta) - l(y, X^T \beta_0)\}|. \]

By Condition A.6, we have
\[ \|\beta_s - \beta^*\| \leq G(B)^{1/2}. \]

Next, for any \( x \), we have that \( \Pr(\|\beta_s - \beta^*\| \geq x) \leq \Pr(G(B) \geq V_n x^2) \). In particular, letting \( x = B/2 \), we get
\[ \Pr(\|\beta_s - \beta^*\| \geq B/2) \leq \Pr(G(B) \geq V_n B^2 / 4). \]

By the definition of \( \beta_s \), \( \Pr(\|\beta_s - \beta^*\| \geq B/2) = \Pr(\|\hat{\beta} - \beta^*\| \geq B) \). By setting \( B = 4a_n(1 + t)/V_n \), with
\[ a_n = 4k_n \sqrt{mC_n/n}, \] we get

\[ \Pr(\|\hat{\beta} - \beta^*\| \geq B) \leq \Pr[G(B) \geq Ba_n(1 + t)], \]

and

\[ \Pr[G(B) \geq Ba_n(1 + t)] \leq \Pr[G(\hat{\beta}) \geq Ba_n(1 + t), \Lambda_n] + \Pr(\Lambda_n^c). \quad (5) \]

Finally, on the set \( \Lambda_n \), we have

\[ \sup_{\beta \in B} \mathbb{P}_n[l(y, X^T \beta) - l(y, X^T \beta_0)](1 - \mathbb{I}_{\Lambda_n}(X, y)) = 0. \]

Hence, by the triangle inequality,

\[ G(B) \leq G_1(B) + \sup_{\beta \in B} |\mathbb{E}[l(y, X^T \beta) - l(y, X^T \beta_0)](1 - \mathbb{I}_{\Lambda_n}(X, y))|. \]

Then, since Condition A.4 holds for all columns in \( X \), it follows that (5) is bounded above by

\[ \Pr[G_1(B) \geq Ba_n(1 + t) + o(mC_n/n)] + \Pr(\Lambda_n^c). \]

Finally, combining the above with the bound (4), we get the desired results. \( \square \)

Theorem 1 also yields the exponential consistency of the M-estimator \( \hat{\beta} \) of the regression coefficient \( \beta \) if we additionally assume that the tail probabilities of the response and covariates go to zero at an exponential rate. The result is summarized in the following theorem.

**Theorem 2 (Exponential consistency)** Along with Conditions A.1–A.6, let us assume that the distributions of the response variable and the covariates are light tailed so that tail probability of \( X^T \beta \) converges to zero at an exponential rate. Then, as \( n \to \infty \) with \( m/n \to 0 \), the M-estimator \( \hat{\beta} \) of the regression coefficient \( \beta \) converges to \( \beta_0 \), in probability, at an exponential rate of convergence.

**References**

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