REPRESENTATIONS OF THE $q$-DEFORMED ALGEBRA $U_q'(so_4)$

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Abstract

We study the nonstandard $q$-deformation $U_q'(so_4)$ of the universal enveloping algebra $U(so_4)$ obtained by deforming the defining relations for skew-symmetric generators of $U(so_4)$. This algebra is used in quantum gravity and algebraic topology. We construct a homomorphism $\phi$ of $U_q'(so_4)$ to the certain nontrivial extension of the Drinfeld–Jimbo quantum algebra $U_q'(sl_2)^{\otimes2}$ and show that this homomorphism is an isomorphism. By using this homomorphism we construct irreducible finite dimensional representations of the classical type and of the nonclassical type for the algebra $U_q'(so_4)$. It is proved that for $q$ not a root of unity each irreducible finite dimensional representation of $U_q'(so_4)$ is equivalent to one of these representations. We prove that every finite dimensional representation of $U_q'(so_4)$ for $q$ not a root of unity is completely reducible. It is shown how to construct (by using the homomorphism $\phi$) tensor products of irreducible representations of $U_q'(so_4)$. (Note that no Hopf algebra structure is known for $U_q'(so_4)$.) These tensor products are decomposed into irreducible constituents.

1. INTRODUCTION

In [1] it was introduced a $q$-deformation of the universal enveloping algebra $U(so_n)$ which uses a realization of the Lie algebra $so_n$ by skew-symmetric matrices (rather than by root elements and the diagonal Cartan subalgebra). If $I_{ij} = E_{ij} - E_{ji}$ are the skew-symmetric matrices of $so_n$, where $(E_{ij})_{ij} := \delta_{ij}\delta_{jr}$, then $so_n$ is generated by the matrices $I_{21}, I_{32}, \cdots, I_{n,n-1}$ (other basis matrices are obtained by taking commutators of these matrices). The following Serre type theorem is true (see [2]): The universal enveloping algebra $U(so_n)$ is isomorphic to the associative algebra generated by (abstract) elements $I_{21}, I_{32}, \cdots, I_{n,n-1}$ satisfying the relations

$$I_{i,i-1}^2 I_{i+1,i} - 2I_{i,i-1} I_{i+1,i} I_{i,j-1} + I_{i+1,i} I_{i,j-1}^2 = -I_{i+1,i},$$

$$I_{i,i-1}^2 I_{i+1,i} - 2I_{i+1,i} I_{i,i-1} I_{i+1,i} + I_{i+1,i} I_{i,i-1}^2 = -I_{i,i-1},$$

$$I_{i,j-1} I_{i,j-1} = I_{j,j-1} I_{i,j-1} \text{ for } |i - j| > 1.$$  

Now we $q$-deform these relations by $2 \rightarrow [2]^q \equiv q + q^{-1}$. As a result, we obtain the associative algebra generated by elements $I_{21}, I_{32}, \cdots, I_{n,n-1}$ satisfying the relations

$$I_{i,i-1}^2 I_{i+1,i} - (q + q^{-1})I_{i,i-1} I_{i+1,i} I_{i,i-1} + I_{i+1,i} I_{i,i-1}^2 = -I_{i+1,i},$$

$$I_{i,i-1}^2 I_{i+1,i} - (q + q^{-1})I_{i+1,i} I_{i,i-1} I_{i+1,i} + I_{i+1,i} I_{i,i-1}^2 = -I_{i,i-1},$$

$$I_{i,j-1} I_{i,j-1} = I_{j,j-1} I_{i,j-1} \text{ for } |i - j| > 1.$$  

We denote this algebra by $U_q'(so_n)$. Associative algebras isomorphic to $U_q'(so_n)$ appear in quantum gravity [3-5], in discrete Schrödinger equation [6], in algebraic topology [7-8], in the theory
of \(q\)-orthogonal polynomials [9] and in the theory of \(q\)-Laplace operators and \(q\)-harmonic polynomials [10]. For this reason, studying the algebra \(U_q'(\mathfrak{so}_n)\) (especially for small numbers \(n\)) is of great importance. There are several problems which have to be solved. The most important are the following ones:

(a) relation of \(U_q'(\mathfrak{so}_n)\) to Drinfeld–Jimbo quantum algebras;
(b) structure of the algebra \(U_q'(\mathfrak{so}_n)\) (including explicit form of the center, Casimir elements, automorphism group, etc);
(c) construction and classification of irreducible finite dimensional representations.

Part of the main problems are solved. For example, it is shown (see [11] and [12]) that \(U_q'(\mathfrak{so}_n)\) can be embedded as a subalgebra into the Drinfeld–Jimbo quantum algebra \(U_q(\mathfrak{sl}_n)\). (Remind that the Drinfeld–Jimbo quantum algebra \(U_q(\mathfrak{sl}_n)\) is not contained in \(U_q(\mathfrak{sl}_n)\)) The main classes of irreducible finite dimensional representations of \(U_q'(\mathfrak{so}_n)\) are constructed for \(q\) not a root of unity (see [13] and [14]) and for \(q\) a root of unity (see [12] and [15]). But the corresponding classification theorem for the representation theory of \(U_q'(\mathfrak{so}_n)\) is not proved. Casimir elements were constructed for \(q\) not a root of unity (see [16] and [17]) and for \(q\) a root of unity (see [18]). But it is not known whether they generate the center.

Most problems are solved [19-22] for the algebra \(U_q'(\mathfrak{so}_3)\). In particular, it was shown that \(U_q'(\mathfrak{so}_3)\) can be embedded into a certain extension of the Drinfeld–Jimbo algebra \(U_q(\mathfrak{sl}_2)\) (but \(U_q'(\mathfrak{so}_3)\) is not isomorphic to \(U_q(\mathfrak{sl}_2)\)). The classification theorem for representation theory of \(U_q'(\mathfrak{so}_3)\) were proved [22]. It was shown that the automorphism group of \(U_q'(\mathfrak{so}_3)\) contains a group isomorphic to the modular group \(SL(2,\mathbb{Z})\).

The aim of this paper is to solve the main problems for the algebra \(U_q'(\mathfrak{so}_4)\) naturally appearing in algebraic topology [8]. (Note that unlike to the classical case, the algebra \(U_q'(\mathfrak{so}_4)\) is not isomorphic to the product of two copies of \(U_q'(\mathfrak{so}_3)\)). The algebra \(U_q'(\mathfrak{so}_4)\) and its irreducible finite dimensional representations were studied in several papers [23-25]. Nevertheless, the main problems were not solved. The main results of this paper are the following:

(I) We construct a homomorphism from \(U_q'(\mathfrak{so}_4)\) to some extension of the Drinfeld–Jimbo quantum algebra \(U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2)\). It is shown that this homomorphism is injective. Thus, \(U_q'(\mathfrak{so}_4)\) is embedded into this extension of \(U_q(\mathfrak{sl}_2)^{\otimes 2}\). This solves for \(U_q'(\mathfrak{so}_4)\) the problem (a).

(II) It is proved the theorem classifying irreducible finite dimensional representations of \(U_q'(\mathfrak{so}_4)\), when \(q\) is not a root of unity. According to this theorem, irreducible representations of the classical type (\(q\)-anologue of the irreducible representations of the Lie algebra \(\mathfrak{so}_4\)) and irreducible representations of the nonclassical type (these representations do not have any classical analogue) exhaust all irreducible finite dimensional representations of \(U_q'(\mathfrak{so}_4)\). This solves for \(U_q'(\mathfrak{so}_4)\), when \(q\) is not a root of unity, the problem (c).

(III) It is proved that if \(q\) is not a root of unity, then any finite dimensional representation of \(U_q'(\mathfrak{so}_4)\) is completely reducible.

(IV) It is shown how to construct tensor products of irreducible finite dimensional representations of \(U_q'(\mathfrak{so}_4)\). (Note that no Hopf algebra structure is known on \(U_q'(\mathfrak{so}_4)\)).

In sections 2–4 \(q\) is any complex number different from \(\pm 1\). In other sections it is assumed that \(q\) is not a root of unity.

2. THE ALGEBRA \(U_q'(\mathfrak{so}_4)\)

We first define the \(q\)-deformed algebra \(U_q'(\mathfrak{so}_3)\) which is a subalgebra of \(U_q'(\mathfrak{so}_4)\). The algebra \(U_q'(\mathfrak{so}_3)\) is obtained [26] by a \(q\)-deformation of the standard commutation relations

\[
[I_{21}, I_{32}] = I_{31}, \quad [I_{32}, I_{31}] = I_{21}, \quad [I_{31}, I_{21}] = I_{32}
\]
of the Lie algebra \(\mathfrak{so}_3\). So, the algebra \(U'_q(\mathfrak{so}_3)\) is the complex associative algebra (with unit element) generated by the elements \(I_{21}, I_{32}, I_{31}\) satisfying the defining relations

\[
[I_{21}, I_{32}] := q^{1/2}I_{21}I_{32} - q^{-1/2}I_{32}I_{21} = I_{31},
\]
\[
[I_{32}, I_{31}] := q^{1/2}I_{32}I_{31} - q^{-1/2}I_{31}I_{32} = I_{21},
\]
\[
[I_{31}, I_{21}] := q^{1/2}I_{31}I_{21} - q^{-1/2}I_{21}I_{31} = I_{32}.
\]

Note that by (1) the element \(I_{31}\) is not independent: it depends on the elements \(I_{21}\) and \(I_{32}\). Substituting the expression (1) for \(I_{31}\) into (2) and (3) we obtain the relations

\[
I_{21}I_{32}^2 - (q + q^{-1})I_{32}I_{21}I_{32} + I_{32}^2I_{21} = -I_{21},
\]
\[
I_{32}I_{21}^2 - (q + q^{-1})I_{21}I_{32}I_{21} + I_{21}^2I_{32} = -I_{32}.
\]

The relations (4) and (5) restore the relations (2) and (3) if to introduce the element \(I_{31}\) defined by (1). The algebra \(U'_q(\mathfrak{so}_3)\) can be defined as the associative algebra generated by the elements \(I_{21}\) and \(I_{32}\) satisfying the defining relations (4) and (5).

Starting from the definition of \(U'_q(\mathfrak{so}_3)\) by relations (4) and (5), we give the following definition of the \(q\)-deformed algebra \(U'_q(\mathfrak{so}_4)\). It is an associative algebra (with unit element) generated by the elements \(I_{21}, I_{32}, I_{43}\) satisfying the defining relations

\[
I_{21}I_{32}^2 - (q + q^{-1})I_{32}I_{21}I_{32} + I_{32}^2I_{21} = -I_{21},
\]
\[
I_{32}I_{21}^2 - (q + q^{-1})I_{21}I_{32}I_{21} + I_{21}^2I_{32} = -I_{32},
\]
\[
I_{32}I_{43}^2 - (q + q^{-1})I_{43}I_{32}I_{43} + I_{43}^2I_{32} = -I_{32},
\]
\[
I_{43}I_{32}^2 - (q + q^{-1})I_{32}I_{43}I_{32} + I_{32}^2I_{43} = -I_{43},
\]
\[
I_{21}I_{43} - I_{43}I_{21} = 0.
\]

It is clear that \(U'_q(\mathfrak{so}_4)\) contains at least two subalgebras isomorphic to \(U'_q(\mathfrak{so}_3)\). The first one is generated by \(I_{21}\) and \(I_{32}\), and the second one by \(I_{32}\) and \(I_{43}\).

We can introduce in \(U'_q(\mathfrak{so}_4)\) the elements \(I_{31}, I_{42}\) and \(I_{41}\). They are defined as in (1):

\[
I_{31} := [I_{21}, I_{32}], \quad I_{42} := [I_{32}, I_{43}], \quad I_{41} := [I_{21}, I_{42}].
\]

where \([A, B] := q^{1/2}AB - q^{-1/2}BA\) is the \(q\)-commutator of \(A\) and \(B\). Then the elements \(I_{ij}\), \(4 \geq i > j \geq 1\), satisfy the relations [13]

\[
[I_{21}, I_{32}] = I_{31}, \quad [I_{32}, I_{31}] = I_{21}, \quad [I_{31}, I_{21}] = I_{32},
\]
\[
[I_{32}, I_{43}] = I_{42}, \quad [I_{43}, I_{42}] = I_{32}, \quad [I_{42}, I_{32}] = I_{43},
\]
\[
[I_{31}, I_{43}] = I_{41}, \quad [I_{43}, I_{41}] = I_{31}, \quad [I_{41}, I_{31}] = I_{43},
\]
\[
[I_{21}, I_{42}] = I_{41}, \quad [I_{42}, I_{41}] = I_{21}, \quad [I_{41}, I_{21}] = I_{42},
\]
\[
[I_{21}, I_{43}] = 0, \quad [I_{32}, I_{41}] = 0, \quad [I_{42}, I_{31}] = (q - q^{-1})(I_{21}I_{43} - I_{32}I_{41}).
\]

As in the case of the algebra \(U'_q(\mathfrak{so}_3)\), the relations (11)–(15) are equivalent to the relations (6)–(10). Note that the relations (11)–(15) define the algebra appearing in algebraic topology [8].

Four sets of relations (11)–(14) gives four subalgebras of \(U'_q(\mathfrak{so}_4)\) isomorphic to \(U'_q(\mathfrak{so}_3)\). They are generated by triples

\[
(I_{21}, I_{32}, I_{31}), \quad (I_{32}, I_{43}, I_{42}), \quad (I_{31}, I_{43}, I_{41}), \quad (I_{21}, I_{42}, I_{41}),
\]
respectively.

The Poincaré–Birkhoff–Witt theorem is true for $U'_q(\mathfrak{so}_4)$. It can be formulated as: The elements

$$I_{31}^{m_{31}} I_{32}^{m_{32}} I_{41}^{m_{41}} I_{42}^{m_{42}} I_{43}^{m_{43}} I_{21}^{m_{21}}, \quad m_{ij} = 0, 1, 2, \cdots,$$

form a basis of $U'_q(\mathfrak{so}_4)$. This theorem is proved by means of the diamond lemma (see [27], subsection 4.1.5). As in the case of ordinary simple Lie algebra, the same theorem holds for any other ordering of the six generators.

We shall need Casimir elements of $U'_q(\mathfrak{so}_4)$. In order to give them we introduce also the elements

$$I_{31} = [I_{21}, I_{32}]_{q^{-1}}, \quad I_{42} = [I_{32}, I_{43}]_{q^{-1}}, \quad I_{41} = [I_{31}, I_{43}]_{q^{-1}},$$

where $[A, B]_{q^{-1}} := q^{-1/2}AB - q^{1/2}BA$. Then (see [16])

$$C_4 = q^{-1}I_{21}I_{43} - I_{31}I_{42} + qI_{32}I_{41},$$

$$C'_4 = q^{-2}I_{21}^2 + I_{32}^2 + q^2(I_{43}^2 + I_{21}^2 + I_{31}^2) - (q - q^{-1})q^{-3/2}(I_{32}I_{43}^2 + I_{31}I_{41}I_{43}) -$$

$$(q - q^{-1})q^{1/2}(I_{32}I_{42}I_{41} + I_{41}I_{42}I_{21}) + (q - q^{-1})^2I_{33}I_{43}I_{21}.$$

3. THE ALGEBRA $U_q(\mathfrak{sl}_2) \otimes_{\mathbb{C}} \mathbb{C}$

Let $e_1, f_1, q^{H_1}$ and $e_2, f_2, q^{H_2}$ be generating elements of two copies of the quantum algebra $U_q(\mathfrak{sl}_2)$ satisfying the relations

$$q^{H_1}e_i = qe_iq^{H_1}, \quad q^{H_1}f_i = q^{-1}f_iq^{H_1}, \quad [e_i, f_i] = \frac{q^{2H_i} - q^{-2H_i}}{q - q^{-1}}.$$

The expressions

$$c_i = e_if_i + \frac{q^{2H_i-1} + q^{-2H_i+1}}{(q - q^{-1})^2}, \quad i = 1, 2,$$

give Casimir elements of these algebras $U_q(\mathfrak{sl}_2)$. The comultiplication $\Delta$ is given in $U_q(\mathfrak{sl}_2)$ by the formulas

$$\Delta(q^{\pm H_i}) = q^{\pm H_i} \otimes q^{\pm H_i}, \quad \Delta(e_i) = e_i \otimes q^{H_i} + q^{-H_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{H_i} + q^{-H_i} \otimes f_i.$$

Let us consider the polynomials

$$p_i(x_i) = q^{-1}x_i^4 - c_i(q - q^{-1})^2x_i^2 + q,$$

where $c_i$ are the Casimir elements. They are irreducible in $U_q(\mathfrak{sl}_2)$, that is, there exists no element $a \in U_q(\mathfrak{sl}_2)$ such that $p_i(a) = 0$. Therefore, we can define the quadric algebraic extension $\hat{U}_q(\mathfrak{sl}_2)$ of the algebra $U_q(\mathfrak{sl}_2)$ by means of the element $x_i$ commuting with all elements of $U_q(\mathfrak{sl}_2)$:

$$\hat{U}_q(\mathfrak{sl}_2) = \{a_3x_i^3 + a_2x_i^2 + a_1x_i + a_0 \mid a_j \in U_q(\mathfrak{sl}_2)\},$$

assuming that $p_i(x_i) = 0$, that is, $x_i^4 = qc_i(q - q^{-1})^2x_i^2 - q^2$. This equation is equivalent to the following one

$$c_i = \frac{x_i^2q^{-1} + x_i^{-2}q}{(q - q^{-1})^2}.$$

(16)
Note that the element $x_i$ has an inverse in $\hat{U}_q(\mathfrak{sl}_2)$ since

$$x_i(-x_i^3 q^{-1} + c_i (q - q^{-1})^2 x_i)q^{-1} = 1,$$

that is,

$$x_i^{-1} = (-x_i^3 q^{-1} + c_i (q - q^{-1})^2 x_i)q^{-1}. \quad (17)$$

We consider two algebras $\hat{U}_q(\mathfrak{sl}_2)$ generated by the elements $e_1, f_1, q^{\pm H_1}, x_1$ and by the elements $e_2, f_2, q^{\pm H_2}, x_2$, respectively. Let $\hat{U}_q(\mathfrak{sl}_2)_{\otimes 2}$ be the tensor product of these algebras. Then we extend (in the sense of [28]) this algebra $\hat{U}_q(\mathfrak{sl}_2)$ by adding to it the commuting elements

$$(q^{H_1} q^{H_2} q^j + q^{-H_1} q^{-H_2} q^{-j})^{-1}, \quad (q^{H_1} q^{H_2} q^j + q^{-H_1} q^{H_2} q^{-j})^{-1}, \quad j = 0, \pm 1, \pm 2, \cdots. \quad (18)$$

This extended algebra will be denoted by $\hat{U}_q(\mathfrak{sl}_2)_{\otimes 2, \text{ext}}$. It is the associative algebra generated by the elements $e_i, f_i, q^{\pm H_i}, x_i$, $i = 1, 2$, and by elements (18) such that $e_1, f_1, q^{\pm H_1}, x_1$ and $e_2, f_2, q^{\pm H_2}, x_2$ satisfy the relations determined in the algebra $\hat{U}_q(\mathfrak{sl}_2)$, each of the elements $e_1, f_1, q^{\pm H_1}, x_1$ commute with each of the elements $e_2, f_2, q^{\pm H_2}, x_2$, each of the elements $q^{\pm H_1}$ and $q^{\pm H_2}$ commutes with each of elements (18), and

$$(q^{H_1 + H_2 + j} + q^{-H_1 - H_2 - j})^{-1} e_i = e_i (q^{H_1 + H_2 + j + 1} + q^{-H_1 - H_2 - j - 1}),$$

$$(q^{H_1 - H_2 + j} + q^{-H_1 + H_2 - j})^{-1} e_i = e_i (q^{H_1 - H_2 + j + 1} + q^{-H_1 + H_2 - j - 1}),$$

$$(q^{H_1 - H_2 + j} + q^{-H_1 + H_2 - j})^{-1} f_i = f_i (q^{H_1 - H_2 + j - 1} + q^{-H_1 + H_2 - j + 1}),$$

$$(q^{H_1 + H_2 + j} + q^{-H_1 - H_2 - j})^{-1} f_i = f_i (q^{H_1 + H_2 + j - 1} + q^{-H_1 - H_2 - j + 1}),$$

$$(q^{H_1 + H_2 + j} + q^{-H_1 - H_2 - j})^{-1} (q^{H_1 + H_2 + j} + q^{-H_1 - H_2 - j}) = 1,$$

$$(q^{H_1 - H_2 + j} + q^{-H_1 + H_2 - j})^{-1} (q^{H_1 - H_2 + j} + q^{-H_1 + H_2 - j}) = 1,$$

where $\varepsilon = 1$ if $i = 1$ and $\varepsilon = -1$ if $i = 2$.

Let us find irreducible finite dimensional representations of the algebra $\hat{U}_q(\mathfrak{sl}_2)_{\otimes 2, \text{ext}}$ for $q$ not a root of unity. For these values of $q$ the algebra $\hat{U}_q(\mathfrak{sl}_2)$ has finite dimensional irreducible representations $T_l \equiv T_l^{(1)}(q^H), T_l^{(-1)}(q^H), l = 0, 1, 2, 3, \cdots$, acting on the vector space $\mathcal{H}_l$ with basis $|l, m\rangle$, $m = -l, -l + 1, \cdots, l$. These representations are given by the formulas

$$T_l^{(1)}(q^H)|l, m\rangle = q^m|l, m\rangle, \quad T_l^{(1)}(e)|l, m\rangle = [l - m]|l, m + 1\rangle, \quad (19)$$

$$T_l^{(1)}(f)|l, m\rangle = [l + m]|l, m - 1\rangle, \quad (20)$$

where numbers in square brackets mean $q$-numbers determined by

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}},$$

and by the formulas

$$T_l^{(-1)}(q^H)|l, m\rangle = -q^H|l, m\rangle, \quad T_l^{(-1)}(X) = T_l^{(1)}(X), \quad X = e, f, \quad (21)$$

$$T_l^{(i)}(q^H)|l, m\rangle = iq^H|l, m\rangle, \quad T_l^{(i)}(e) = T_l^{(1)}(e), \quad T_l^{(i)}(f) = -T_l^{(1)}(f), \quad (22)$$

$$T_l^{(-i)}(q^H)|l, m\rangle = -iq^H|l, m\rangle, \quad T_l^{(-i)}(e) = T_l^{(1)}(e), \quad T_l^{(-i)}(f) = -T_l^{(1)}(f), \quad (23)$$

(see, for example, [27], chapter 3). The representations $T_l^{(1)}, T_l^{(-1)}, T_l^{(i)}, T_l^{(-i)}, l = 0, 1, 2, 3, \cdots$, are pairwise nonequivalent and any irreducible finite dimensional representation of $\hat{U}_q(\mathfrak{sl}_2)$ is...
equivalent to one of these representations. Values of the Casimir element $c$ on these representations are given by

$$
T^{(1)}_l(c) = T^{(-1)}_l(c) = \frac{q^{2l+1} + q^{-2l-1}}{(q - q^{-1})^2}, \quad T^{(i)}_l(c) = T^{(-i)}_l(c) = -\frac{q^{2l+1} + q^{-2l-1}}{(q - q^{-1})^2}.
$$

Since the Casimir element of $U_q(\mathfrak{sl}_2)$ is multiple to the unit operator on the space $\mathcal{H}_l$, then each of the representations $T^{(1)}_l$, $T^{(-1)}_l$, $T^{(i)}_l$, $T^{(-i)}_l$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, can be extended to a representation of $\hat{U}_q(\mathfrak{sl}_2)$. In order to determine these extensions we have to determine the operators $\hat{T}^{(\varepsilon)}_l(x)$, $\varepsilon = \pm 1, \pm i$, corresponding to the element $x$ from (16). It follows from (16) that

$$
\hat{T}^{(\varepsilon)}_l(c) = \frac{T^{(\varepsilon)}_l(x) q^{-1} + T^{(\varepsilon)}_l(x) - 2q}{(q - q^{-1})^2}.
$$

If some operator $\hat{T}^{(\varepsilon)}_l(x)$ is a solution of this equation, then the operators

$$
\hat{T}^{(\varepsilon)}_l(x) = -T^{(\varepsilon)}_l(x), \quad T^{(\varepsilon)}_l(x) - 1q, \quad -T^{(\varepsilon)}_l(x) - 1q
$$

are also its solutions. Each of them can be taken for extension of the representation $\hat{T}^{(\varepsilon)}_l$ of $U_q(\mathfrak{sl}_2)$. Since the element $x$ commute with all elements of $U_q(\mathfrak{sl}_2)$, then different extensions of $\hat{T}^{(\varepsilon)}_l$ (obtained by using different solutions of the above equation) do not essentially differ from each other. For this reason, we shall use only the solution $\hat{T}^{(\varepsilon)}_l(x) = q^{-1}I$, where $I$ is the unit operator. We denote the extended representations of $\hat{U}_q(\mathfrak{sl}_2)$, extended by using the solution $\hat{T}^{(\varepsilon)}_l(x) = q^{-1}I$, by the same symbols $\hat{T}^{(\varepsilon)}_l$, $\hat{T}^{(-1)}_l$, $\hat{T}^{(i)}_l$, $\hat{T}^{(-i)}_l$.

It is clear that irreducible finite dimensional representations of $\hat{U}_q(\mathfrak{sl}_2)^{\otimes 2}$ are equivalent to the following ones:

$$
T^{(\varepsilon)}_l \otimes T^{(\varepsilon')}_l, \quad l, l' = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad \varepsilon, \varepsilon' = 1, -1, i, -i,
$$

where $T^{(\varepsilon)}_l$ and $T^{(\varepsilon')}_l$ are irreducible representations of two copies of the algebra $\hat{U}_q(\mathfrak{sl}_2)$, respectively. Now we wish to extend these representations of $\hat{U}_q(\mathfrak{sl}_2)^{\otimes 2}$ to representations of the algebra $\hat{U}_q(\mathfrak{sl}_2)^{\otimes 2, \text{ext}}$ by using the relation

$$
(T^{(\varepsilon)}_l \otimes T^{(\varepsilon')}_l) \left( (q^{H_1 + H_2} + q^{-H_1 - H_2})^{-1} \right) =
$$

$$
= \left( q^j T^{(\varepsilon)}_l(q^{H_1}) \otimes T^{(\varepsilon')}_l(q^{H_2}) + q^{-j} T^{(\varepsilon)}_l(q^{-H_1}) \otimes T^{(\varepsilon')}_l(q^{-H_2}) \right)^{-1},
$$

$$
(T^{(\varepsilon)}_l \otimes T^{(\varepsilon')}_l) \left( (q^{H_1 - H_2} + q^{-H_1 + H_2})^{-1} \right) =
$$

$$
= \left( q^j T^{(\varepsilon)}_l(q^{H_1}) \otimes T^{(\varepsilon')}_l(q^{-H_2}) + q^{-j} T^{(\varepsilon)}_l(q^{-H_1}) \otimes T^{(\varepsilon')}_l(q^{H_2}) \right)^{-1}.
$$

Clearly, only those irreducible representations $T^{(\varepsilon)}_l \otimes T^{(\varepsilon')}_l$ of $\hat{U}_q(\mathfrak{sl}_2)^{\otimes 2}$ can be extended to representations of $\hat{U}_q(\mathfrak{sl}_2)^{\otimes 2, \text{ext}}$ for which all the operators

$$
q^j T^{(\varepsilon)}_l(q^{H_1}) \otimes T^{(\varepsilon')}_l(q^{H_2}) + q^{-j} T^{(\varepsilon)}_l(q^{-H_1}) \otimes T^{(\varepsilon')}_l(q^{-H_2}), \quad j = 0, \pm 1, \ldots,
$$

$$
q^j T^{(\varepsilon)}_l(q^{H_1}) \otimes T^{(\varepsilon')}_l(q^{-H_2}) + q^{-j} T^{(\varepsilon)}_l(q^{-H_1}) \otimes T^{(\varepsilon')}_l(q^{H_2}), \quad j = 0, \pm 1, \ldots,
$$

are invertible. From formulas (19)–(23) it follows that these operators are always invertible for the representations $T^{(\varepsilon)}_l \otimes T^{(\varepsilon')}_l$ such that $\varepsilon, \varepsilon' = 1, -1$ or $\varepsilon, \varepsilon' = i, -i$, and also for all the
representations $T_l^{(ε)} \otimes T_{l'}^{(ε')} \otimes T_{l''}^{(ε'')}$ such that $l + l'$ is half-integral (but not integral) number and $ε = ±1$, $ε' = ±i$ or $ε = ±i$, $ε'' = ±1$. For the representations $T_l^{(ε)} \otimes T_{l'}^{(ε')} \otimes T_{l''}^{(ε'')}$ with $ε = ±1$, $ε' = ±i$ or $ε = ±i$, $ε'' = ±1$ and $l + l'' \in Z$ some of these operators are not invertible since they have zero eigenvalue. Denoting the extended representations by the same symbols, we can formulate the following assertion:

**Theorem 1.** If $q$ is not a root of unity, then the algebra $\hat{U}_q(sl_2) \otimes Z^{2,ext}$ has irreducible finite dimensional representations

$$T_l^{(ε)} \otimes T_{l'}^{(ε')} \quad l, l' = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad ε, ε' = ±1 \text{ or } ε, ε' = ±i,$$

$$T_l^{(ε)} \otimes T_{l'}^{(ε')} \quad l + l' \in \frac{1}{2} Z, \quad l + l' \notin Z, \quad ε = ±1, \quad ε' = ±i \text{ or } ε = ±i, \quad ε' = ±1$$

(all four combinations of signs are possible). Up to values of the operators corresponding to the elements $x^k \otimes x_s^k$, $k, s = 1, 2, 3$, any irreducible finite dimensional representation of $\hat{U}_q(sl_2) \otimes Z^{2,ext}$ is equivalent to one of these representations.

4. THE ALGEBRA HOMOMORPHISM $U'_q(so_4) \rightarrow \hat{U}_q(sl_2) \otimes Z^{2,ext}$

The aim of this section is to give in an explicit form the algebra homomorphism of $U'_q(so_4)$ to $\hat{U}_q(sl_2) \otimes Z^{2,ext}$. This homomorphism is given by the following theorem:

**Theorem 2.** There exists a unique algebra homomorphism $φ : U'_q(so_4) \rightarrow \hat{U}_q(sl_2) \otimes Z^{2,ext}$ such that

$$φ(I_{21}) = i[H_1 + H_2]_q, \quad φ(I_{43}) = i[H_1 - H_2]_q,$$

$$φ(I_{32}) = -\frac{x_1^{-1}q^{-H_2+1} + x_1q^{H_2-1}}{(q^{H_1+H_2-1} + q^{-H_1+H_2+1})}E_2 + \frac{x_1^{-1}q^{H_2+1} + x_1^{-1}q^{-H_2-1}}{(q^{H_1+H_2+1} + q^{-H_1+H_2-1})}F_2 + \frac{x_2^{-1}q^{-H_1+1} + x_2q^{H_1-1}}{(q^{H_1+H_2-1} + q^{-H_1+H_2+1})}E_1 - \frac{x_2^{-1}q^{H_1+1} + x_2q^{-H_1-1}}{(q^{H_1+H_2+1} + q^{-H_1+H_2-1})}F_1.$$  

**Proof.** We have to show that three elements $φ(I_{21})$, $φ(I_{32})$ and $φ(I_{43})$ from (24) and (25) satisfy the defining relations (6)–(10). It is made by direct verification. Namely, we substitute the expressions (24) and (25) for $φ(I_{21})$, $φ(I_{43})$, $φ(I_{32})$ into (6)–(10) and then permute the generating elements $(q^{H_1})^{±1}, e_i, f_i$ in numerators (using the defining relations of the algebra $U_q(sl_2)$) reducing them to the form

$$(q^{H_1})^ry(q^{H_2})^se_1^{a_1}e_1^{b_1}f_1^{a_2}f_1^{b_2}, \quad r, s \in Z, \quad a_1, b_1, a_2, b_2 \in Z_+.$$ 

Then it is directly seen that the relations (7), (8) and (10) are fulfilled. So, we have to prove the relations (6) and (9). We cancel in these relations separately terms ending with $e_1^{a_1}, e_2^{a_2}, f_1^{b_1}$ and $f_2^{b_2}$. Now in the relation (6) we cancel terms ending with $e_1e_2$, $f_1f_2$, $e_1e_2$ and in the relation (9) terms ending with $e_1f_2, f_1e_2$. Then we multiply both sides of (6) by

$$(q^{H_1+H_2-1} + q^{-H_1-H_2+1})^{-1}(q^{H_1+H_2+1} + q^{-H_1-H_2-1})^{-1}$$

and both sides of (9) by

$$(q^{H_1-H_2-1} + q^{-H_1+H_2+1})^{-1}(q^{H_1-H_2+1} + q^{-H_1+H_2-1})^{-1}.$$
After that we cancel in (6) terms ending with $e_1 f_2$ and in (9) terms ending with $e_1 e_2$ and $f_1 f_2$. Now we have in (6) and (9) only the terms ending with $e_1 f_1$, $e_2 f_2$ and terms without any $e_j f_k$, $j \neq k$. We replace $e_1 f_1$, $e_2 f_2$ by the expressions following from the expression for the Casimir elements, that is, by $c_i - (q^{2H_i - 1} + q^{-2H_i + 1})/(q - q^{-1})^2$, respectively, and multiply both sides of the both relations by

\[(q^{H_1 + H_2 - 1} + q^{-H_1 - H_2 + 1})(q^{H_1 + H_2 + 1} + q^{-H_1 - H_2 - 1})(q^{H_1 + H_2} + q^{-H_1 - H_2})\]

\[\times(q^{H_1 - H_2 - 1} + q^{-H_1 + H_2 + 1})(q^{H_1 - H_2 + 1} + q^{-H_1 + H_2 - 1})(q^{H_1 - H_2} + q^{-H_1 + H_2}).\]

We obtain the relations both sides of which are cancelled. Theorem is proved.

5. PROPERTIES OF REPRESENTATIONS OF $U'_q(so_4)$

We assume everywhere below that $q$ is not a root of unity.

Our aim is to obtain irreducible finite dimensional representations of $U'_q(so_4)$ by using the homomorphism of Theorem 2. But before we need some statements on such representations of $U'_q(so_4)$.

Let $U'_q(so_3)_1$ and $U'_q(so_3)_2$ denote the subalgebras of $U'_q(so_4)$ generated by $I_{21}, I_{32}$ and by $I_{32}, I_{43}$, respectively. It is known that the restriction of a finite dimensional representation $T$ of $U'_q(so_4)$ to any of these subalgebras is a completely reducible representation since any finite dimensional representation of $U'_q(so_3)$ for $q$ not a root of unity is completely reducible (see [29]). Let

\[T \downarrow_{U'_q(so_3)_1} = R_1 \oplus R_2 \oplus \cdots \oplus R_k, \quad (26)\]

\[T \downarrow_{U'_q(so_3)_2} = R'_1 \oplus R'_2 \oplus \cdots \oplus R'_{k'}. \quad (27)\]

Below we shall prove some assertions characterizing these decompositions. Using the classification of irreducible finite dimensional representations of $U'_q(so_3)$ for $q$ not a root of unity (see [22]), we can state that each of irreducible representations $R_i$ and $R'_j$ in (26) and (27) is a representation of the classical type or a representation of the nonclassical type.

**Proposition 1.** The decomposition (26), as well as the decomposition (27), contains only irreducible representations of the classical type or only irreducible representations of the nonclassical type.

**Proof.** Let us prove our proposition for the decomposition (26). It follows from the results of [30] that the operators $T(I_{41})$, $T(I_{42})$, $T(I_{43})$ form a tensor operator transforming under the vector (3-dimensional) representation of $U'_q(so_3)_1$. It is known from [21] that tensor product of a classical (nonclassical) type irreducible representation by the vector representation contains in the decomposition classical (respectively nonclassical) type representations. For this reason, if, for example, the representation $R_1$ in (26) is of the classical type and $|l_1, m_1\rangle$, $m_1 = -l_1, -l_1 + 1, \ldots, l_1$, are basis elements of its representation subspace, then according to Wigner–Eckart theorem (see [30]) for vector tensor operator $\{T(I_{41}), T(I_{42}), T(I_{43})\}$ the vectors $T(I_{4i})|l_1, m_1\rangle$, $i = 1, 2, 3$, are linear combinations of vectors belonging to subspaces of irreducible representations of $U'_q(so_3)_1$ of the classical type. The same assertion is true for vectors belonging to subspaces of representations of the nonclassical type: that is, if $R_1$ is of the nonclassical type, then the vectors $T(I_{4i})|l_1, m_1\rangle$ are linear combinations of vectors belonging to subspaces of irreducible representations of the nonclassical type. Thus, in the decomposition (26) there exists an irreducible representation of the classical type, then acting upon vectors of the corresponding subspace by the operators $T(I_{4i})$, $i = 1, 2, 3$, we obtain vectors belonging to subspaces on which irreducible representations of the classical type are realized. Since the representation $T$ of $U'_q(so_4)$ is irreducible, then in this case the decomposition (26) contains only irreducible representations of the classical type. If the decomposition (26) does not contain an
irreducible representation of the classical type, then all representations in this decomposition are of the nonclassical type. Proposition is proved.

**Proposition 2.** Both decompositions (26) and (27) contain irreducible representations of the same type (classical or nonclassical).

Proof. In order to prove this proposition we note (see [22]) that eigenvalues of the operator $R_i(I_{21})$ of an irreducible representation $R$ of $U_q'(so_3)$ are of the form $i[m]$, $i[m+1], \cdots$ if $R$ is of the classical type, and of the form $\pm [m]_+^+, \pm [m+1]_+^+$ if $R$ is of the nonclassical type, where

$$[m]_+^+ = \frac{q^m + q^{-m}}{q - q^{-1}}. \quad (28)$$

Let the decomposition (26) consist of irreducible representations of the classical type. Then eigenvalues of the operators $R_i(I_{21})$ are of the form $i[m]$, $i[m+1], \cdots$. We state that then the operators $R_i(I_{32})$ can be diagonalized and their eigenvalues are also of this form. Really, the algebra $U_q'(so_3)_1$ has the automorphism $\tau$ such that $\tau(I_{21}) = I_{32}$ and $\tau(I_{32}) = I_{21}$. Since $\text{Tr} R_i(I_{21}) = \text{Tr} R_i(I_{32}) = 0$ (this equality characterizes (see [21]) irreducible representations of the classical type), then the representations $R_i' = R_i \circ \tau$ are of the classical type. Moreover, $R_i' \sim R_i$ (since up to equivalence there exists a single irreducible representation of the classical type with a fixed dimension). Then the operators $R_i'(I_{21}) = R_i'(I_{32})$ are diagonalizable and the spectrum of $R_i(I_{32})$ coincides with that of $R_i(I_{21})$. Our statement concerning the operator $R_i(I_{32})$ is proved.

Now we consider the decomposition (27). The operator $T(I_{32}) = \sum_i \oplus R_i(I_{32})$ coincides with the operator $T(I_{32}) = \sum_i \oplus R_i'(I_{32})$. We have found a form of the spectrum of the operator $T(I_{32})$. Now we can conclude that the operators $R_i'(I_{32})$ have eigenvalues of the form $i[m]$, $i[m+1], \cdots$. This means that the irreducible representations $R_i'$ in (27) are of the classical type.

If the decomposition (26) consists of irreducible representations of the nonclassical type, the decomposition (27) consists of representations of the same type. Really, if the decomposition (27) would consist of representations of the classical type, then conducting the above reasoning in the converse order we would conclude that the decomposition (26) consists of representations of the classical type. Proposition is proved.

**Corollary.** If $T$ is an irreducible finite dimensional representation of the algebra $U_q'(so_4)$, then both operators $T(I_{21})$ and $T(I_{43})$ can be simultaneously diagonalized and eigenvalues of both these operators are of the form $i[m]$, $i[m+1], \cdots$ or of the form $[m]_+^+, [m+1]_+^+, \cdots$.

Proof. The operators $T(I_{21})$ and $T(I_{43})$ can be simultaneously diagonalized since the operators $R_i(I_{21})$ and $R_j'(I_{43})$ in the decompositions (26) and (27) can be simultaneously diagonalized. (Note that elements $I_{21}$ and $I_{43}$ are commuting in $U_q'(so_4)$.) Since decompositions (26) and (27) consist only of irreducible representations of the classical type or only irreducible representations of the nonclassical type, then the second assertion of the corollary follows. Corollary is proved.

6. IRREDUCIBLE REPRESENTATIONS OF $U_q'(so_4)$ OF THE CLASSICAL TYPE

If $T$ is a representation of the algebra $\hat{U}_q(sl_2)^{\otimes 2,\text{ext}}$ on a finite dimensional linear space $\mathcal{H}$, then the mapping

$$R : U_q'(so_4) \to \mathcal{L}(\mathcal{H}) \quad (29)$$

(where $\mathcal{L}(\mathcal{H})$ is the space of linear operators on $\mathcal{H}$) defined by the composition $R = T \circ \phi$ (where $\phi$ is the homomorphism of Theorem 2) is a representation of $U_q'(so_4)$. Let us consider the representations

$$R_{j,j'} \equiv R_{j,j'}^{(1,1)} = (T_j^{(1)} \otimes T_{j'}^{(1)}) \circ \phi, \quad j, j' = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \quad (30)$$
of $U'_q(\text{so}_4)$, where $T^{(1)}_j \otimes T^{(1)}_{j'}$ are irreducible representations of $\hat{U}_q(\text{sl}_2)^{\otimes 2,\text{ext}}$ from Theorem 1.

Using formulas for the representations $T^{(1)}_j \otimes T^{(1)}_{j'}$ of $\hat{U}_q(\text{sl}_2)^{\otimes 2,\text{ext}}$ from Theorem 1 and the expressions (24) and (25) for $\phi(I_{i,i-1})$, $i = 2, 3, 4$, we find that

$$R_{ij'}(I_{21})|k,l\rangle = i|k + l\rangle|k,l\rangle, \quad (31)$$

$$R_{ij'}(I_{43})|k,l\rangle = i|k - l\rangle|k,l\rangle, \quad (32)$$

$$R_{ij'}(I_{32})|k,l\rangle = \frac{1}{(q^{k+l} + q^{k-l})(q^{k+l} + q^{-k+l})} \times$$

$$\times \{- (q^{j-l} + q^{-j+l})[j-l]|k,l\rangle + (q^{j+l} + q^{-j-l})[j+l]|k,l\rangle +$$

$$+ (q^{j-k} + q^{-j+k})[j-k]|k,l\rangle + (q^{j+k} + q^{-j-k})[j+k]|k,l\rangle\}, \quad (33)$$

where numbers in square brackets are corresponding $q$-numbers and $|k,l\rangle$ denote the basis vector

$$|k,l\rangle = |j,k\rangle \otimes |j',l\rangle$$

of the space $\mathcal{H}_j \otimes \mathcal{H}_{j'}$ of the representation $T^{(1)}_j \otimes T^{(1)}_{j'}$ of $\hat{U}_q(\text{sl}_2)^{\otimes 2,\text{ext}}$.

**Remark:** Taking instead of $T^{(1)}_j \otimes T^{(1)}_{j'}$ the irreducible representations with other values of the operators corresponding to the elements $x^k_s \otimes x^s_t$, $k, s = 1, 2, 3$ (see Theorem 1), we would obtain representations of $U'_q(\text{so}_4)$ equivalent to $R_{ij'}$.

The representation $R_{ij'}$ of $U'_q(\text{so}_4)$ is equivalent to the representation $T_{rs}$, $r = j + j'$, $s = j - j'$, from [24] which in the $U'_q(\text{so}_3)$ basis

$$|j'', m\rangle, \quad |s| \leq j'' \leq r, \quad m = -j'', -j'' + 1, \ldots, j'',$$

is given by the formulas

$$T_{rs}(I_{21})|j'', m\rangle = i[m]|j'', m\rangle, \quad (34)$$

$$T_{rs}(I_{32})|j'', m\rangle = \frac{1}{q^m + q^{-m}}(|j'' - m\rangle|j'', m + 1\rangle - |j'' + m\rangle|j'', m - 1\rangle), \quad (35)$$

$$T_{rs}(I_{43})|j'', m\rangle = |[r + 1|s]|m|j'', m\rangle + |[r - j'']|s|j'' + 1\rangle |j'', m\rangle -$$

$$- |[r + j'' + 1]|j'' - m\rangle |j'' - m|j'', m\rangle - [j'' + 1]_{2j'' + 1} |j'', m\rangle \quad (36)$$

(note that our basis elements $|j'', m\rangle$ differ from the basis elements in formula (19) of [24] by the appropriate multipliers; for this reason, formula (36) differs from formula (19) in [24]). It was shown in [24] that under diagonalization of the operator $T_{rs}(I_{43})$ we obtain a new basis $\{|x, m\rangle\}$ on which the operators $T_{rs}(I_{21})$, $T_{rs}(I_{43})$, $T_{rs}(I_{32})$ are given by formulas

$$T_{rs}(I_{21})|x, m\rangle = i[m]|x, m\rangle, \quad T_{rs}(I_{43})|x, m\rangle = i[x]|x, m\rangle \quad (37)$$

and by formula (33) in [24]. Under the renotation of the basis elements and multiplying them by the appropriate multipliers we obtain formulas for the representation $R_{ij'}$ obtained above. This proves the equivalence stated above.

The irreducible representations $R_{ij'}$ are called representations of the classical type (see [14]). If $q \to 1$, then the operators $R_{ij'}(I_{i,i-1})$, $i = 2, 3, 4$, tend to the corresponding operators of the irreducible representations of the Lie algebra $\text{so}_4$.

7. IRREDUCIBLE REPRESENTATIONS OF $U'_q(\text{so}_4)$ OF THE NONCLASSICAL TYPE
Now we apply the method of the previous section to the irreducible representations $T_j^{(-i)} \otimes T_{j'}^{(1)}$ of the algebra $\hat{U}_q(\mathfrak{sl}_2)^{\otimes 2,\text{ext}}$ with $j$ half-integral and $j'$ integral or with $j$ integral and $j'$ half-integral. Then we obtain the representations
\[ R_{jj'}^{(-i,1)} = (T_j^{(-i)} \otimes T_{j'}^{(1)}) \circ \phi, \tag{38} \]
of $U'_q(\mathfrak{so}_4)$ with
\[ j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \quad \text{and} \quad j' = 0, 1, 2, \ldots \quad \text{or} \quad j = 0, 1, 2, \ldots \quad \text{and} \quad j' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots. \tag{39} \]

Using formulas for the representations $T_j^{(-i)}$ and $T_{j'}^{(1)}$ of $U_q(\mathfrak{sl}_2)$ and the expressions (24) and (25) for $\phi(I_{i,i-1})$, $i = 2, 3, 4$, we find that
\[ R_{jj'}^{(-i,1)}(I_{21})|k,l\rangle = |k+l|_+|k,l\rangle, \tag{40} \]
\[ R_{jj'}^{(-i,1)}(I_{43})|k,l\rangle = |k-l|_+|k,l\rangle, \tag{41} \]
\[ R_{jj'}^{(-i,1)}(32)|k,l\rangle = \frac{1}{[k+l][k-l](q-q^{-1})} \times \]
\[ \times \left\{ -i[j-l][j'-l]|k,l+1\rangle + i[j+l][j'+l]|k,l-1\rangle - 
- i[j'-k][j-k]|k+1,l\rangle + i[j'+k][j+k]|k-1,l\rangle \right\}, \tag{42} \]
where the numbers in square brackets are the corresponding $q$-numbers, $[a]_+$ are defined in (38), and $|k,l\rangle$ denote the basis vectors
\[ |k,l\rangle = |j,k\rangle \otimes |j',l\rangle \tag{43} \]
of the space $\mathcal{H}_j \otimes \mathcal{H}_{j'}$ of the representation $T_j^{(-i)} \otimes T_{j'}^{(1)}$ of $\hat{U}_q(\mathfrak{sl}_2)^{\otimes 2,\text{ext}}$. Note that both $j + j'$ and $k + l$ are half-integral.

In this case we have the equalities $[k+l]_+ = [-k-l]_+$, $[k-l]_+ = [-k+l]_+$ and for this reason the operators $R_{jj'}^{(-i,1)}(I_{21})$ and $R_{jj'}^{(-i,1)}(I_{43})$ have multiple eigenvalues. Namely, all pairs of the vectors $|k,l\rangle$ and $|-k,-l\rangle$ have the same eigenvalues:
\[ R_{jj'}^{(-i,1)}(I_{21})|k,l\rangle = |k+l|_+|k,l\rangle, \quad R_{jj'}^{(-i,1)}(I_{43})|k,l\rangle = |k-l|_+|k,l\rangle, \]
\[ R_{jj'}^{(-i,1)}(I_{21})|-k,-l\rangle = |k+l|_+|-k,-l\rangle, \quad R_{jj'}^{(-i,1)}(I_{43})|-k,-l\rangle = |k-l|_+|-k,-l\rangle. \]
The representation $R_{jj'}^{(-i,1)}$ is reducible. In order to show this we distinguish two cases:

(a) $j$ is half-integral and $j'$ integral;
(b) $j$ is integral and $j'$ half-integral.

We consider first the case (a). In order to decompose $R_{jj'}^{(-i,1)}$ into irreducible constituents we choose a new basis in the representation space consisting of the vectors
\[ |k,l\rangle^+ = |k,l\rangle + (-1)^l|k,-l\rangle, \quad k > 0, \]
\[ |k,l\rangle^- = |k,l\rangle + (-1)^{l+1}|k,-l\rangle, \quad k > 0. \]

Using formulas (40)–(42) we easily find that
\[ R_{jj'}^{(-i,1)}(I_{21})|k,l\rangle^\pm = |k+l|_+|k,l\rangle^\pm, \quad R_{jj'}^{(-i,1)}(I_{43})|k,l\rangle^\pm = |k-l|_+|k,l\rangle^\pm, \tag{44} \]
The operator \( R_{j j'}^{(-i,1)}(I_{32}) \) acts on the vectors \(|k, l\rangle\) by formulas (45) and (46) if to replace all \(|k, l\rangle^+\) by the corresponding \(|k, l\rangle^-\) and in (46) to replace \((-1)^l\) by \((-1)^{l+1}\).

Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be the subspaces of the representation space of \( R_{j j'}^{(-i,1)} \) spanned by the vectors \(|k, l\rangle^+\) and by the vectors \(|k, l\rangle^-\), respectively. We see from the above formulas that these subspaces are invariant with respect to the representation \( R_{j j'}^{(-i,1)} \). We denote the restrictions of \( R_{j j'}^{(-i,1)} \) to \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) by \( R_{j j'}^{(+,+,+)} \) and \( R_{j j'}^{(+,+,-)} \), respectively. Note that

\[
R_{j j'}^{(+,+,+)}(I_{32})|1/2, 0\rangle^+ = \frac{1}{[1/2]^2(q-q^{-1})} \left\{ -i[j][j']|1/2, 1\rangle^+ + i[j][j']|1/2, -1\rangle^- - i[j' - 1/2][j - 1/2]|3/2, 0\rangle^+ + i[j' + 1/2][j + 1/2]|1/2, 0\rangle^+ \right\},
\]

that is, the operator \( R_{j j'}^{(+,+,+)}(I_{32}) \) has nonzero diagonal element

\[
\langle 1/2, 0 | R_{j j'}^{(+,+,+)}(I_{32}) | 1/2, 0 \rangle^+ = i \frac{|j' + 1/2|j + 1/2}{[1/2]^2(q-q^{-1})}.
\]

In the same way it is shown that

\[
-\langle 1/2, 0 | R_{j j'}^{(+,+,-)}(I_{32}) | 1/2, 0 \rangle^- = -i \frac{|j' + 1/2|j + 1/2}{[1/2]^2(q-q^{-1})}.
\]

Now we consider the case (b). The new basis of the representation space is

\[
|k, l\rangle^+ = |k, l\rangle + (-k)^l|k, l\rangle^- - l > 0,
\]

\[
|k, l\rangle^- = |k, l\rangle + (-1)^{l+1}|k, l\rangle^+, \quad l > 0.
\]

Then formulas for \( R_{j j'}^{(-i,1)}(I_{21}) \) and \( R_{j j'}^{(-i,1)}(I_{43}) \) are the same as in case (a). Formula (45) is the same, but we have consider it for \( l \neq \frac{1}{2} \). Instead of (46) we have the formula

\[
R_{j j'}^{(-i,1)}(I_{32})|k, 1/2\rangle^+ = \frac{1}{[k + 1/2]|k - 1/2|(q-q^{-1})} \times \left\{ -i[j - 1/2][j' - 1/2]|k, 3/2\rangle^+ + i[j + 1/2][j' + 1/2]|(-1)^k - k, 1/2\rangle^+ - i[j' - k][j - k]|k + 1, 1/2\rangle^+ + i[j' + k][j + k]|k - 1, 1/2\rangle^+ \right\}.
\]

The operator \( R_{j j'}^{(-i,1)}(I_{32}) \) acts on the vectors \(|k, l\rangle^-\) by formulas (45) and (48) if to replace all \(|k, l\rangle^+\) by \(|k, l\rangle^-\), respectively, and in (48) to replace \((-1)^k\) by \((-1)^{k+1}\).
We denote the subspaces spanned by the vectors \(|k,l\rangle^+\) and by the vectors \(|k,l\rangle^-\) as \(\mathcal{H}_1\) and \(\mathcal{H}_2\), respectively. As we see from above formulas, these subspaces are invariant with respect to the representation \(R_{jj'}^{(-i,-1)}\). We denote the corresponding subrepresentations by \(R_{jj'}^{(+,+,-)}\) and \(R_{jj'}^{(+,+,+)}\), respectively. In particular, we have

\[
R_{jj'}^{(+,+,-)}(I_{32})|0, 1/2\rangle^+ = -\frac{1}{[1/2]^2(q-q^{-1})}\left\{-i[j - 1/2][j' - 1/2]|0, 3/2\rangle^+ + i[j + 1/2][j' + 1/2]|0, 1/2\rangle^+ - i[j'][j]|1, 1/2\rangle^+ + i[j'][j]|1, 3/2\rangle^+\right\},
\]

that is, the operator \(R_{jj'}^{(+,+,-)}(I_{32})\) has nonzero diagonal element

\[
\langle 0, 1/2 | R_{jj'}^{(+,+,-)}(I_{32}) | 0, 1/2 \rangle^+ = -i\frac{[j' + 1/2][j + 1/2]}{[1/2]^2(q-q^{-1})}.
\]

For the operator \(R_{jj'}^{(+,+,+)}(I_{32})\) we have

\[
-\langle 0, 1/2 | R_{jj'}^{(+,+,+)}(I_{32}) | 0, 1/2 \rangle^- = i\frac{[j' + 1/2][j + 1/2]}{[1/2]^2(q-q^{-1})}.
\]

Now we consider in the same way the representations

\[
R_{jj'}^{(-i,-1)} = (T_j^{(-i)} \otimes T_{j'}^{(-1)}) \circ \phi
\]

of \(U'_q(so_4)\). As a result, we obtain for these representations formulas (40)–(42) in which right hand sides of (40) and (41) are multiplied by \(-1\) and the right hand side of (42) is left without any change. These representations are reducible and we have

\[
R_{jj'}^{(-i,-1)} = R_{jj'}^{(-,-,+)} \oplus R_{jj'}^{(-,-,-)},
\]

where the representations \(R_{jj'}^{(-,-,+)}\) and \(R_{jj'}^{(-,-,-)}\) act on such subspaces as in the previous case, with those difference that

\[
R_{jj'}^{(-,-,\pm)}(I_{21}) = -R_{jj'}^{(+,+,\pm)}(I_{21}), \quad R_{jj'}^{(-,-,\pm)}(I_{43}) = -R_{jj'}^{(+,+,-,\pm)}(I_{43}),
\]

that is,

\[
R_{jj'}^{(-,-,\pm)}(I_{21})|k,l\rangle^\pm = -[k + l]_+|k,l\rangle^\pm, \quad R_{jj'}^{(-,-,\pm)}(I_{43})|k,l\rangle^\pm = -[k - l]_+|k,l\rangle^\pm.
\]

The operators \(R_{jj'}^{(-,-,-,\pm)}(I_{32})\) coincide with the corresponding operators \(R_{jj'}^{(+,+,-,\pm)}(I_{32})\).

Similarly, the representation \(R_{jj'}^{(i,1)} = (T_j^{(i)} \otimes T_{j'}^{(1)}) \circ \phi\) of \(U'_q(so_4)\) is reducible and decomposes as

\[
R_{jj'}^{(i,1)} = R_{jj'}^{(-,-,+)} \oplus R_{jj'}^{(-,-,-)},
\]

where \(R_{jj'}^{(-,-,\pm)}\) are such as above, and the representation \(R_{jj'}^{(i,-1)} = (T_j^{(i)} \otimes T_{j'}^{(-1)}) \circ \phi\) of \(U'_q(so_4)\) is reducible and decomposes as

\[
R_{jj'}^{(i,-1)} = R_{jj'}^{(+,+,-)} \oplus R_{jj'}^{(+,+,+)}.
\]

Let us consider the representation

\[
R_{jj'}^{(1,i)} = (T_j^{(1)} \otimes T_{j'}^{(i)}) \circ \phi
\]
of $U_q'(so_4)$ (for convenience we take first the index $j'$ and then $j$). We have
\[
R_{jj'}^{(1,i)}(I_{21})|l, k\rangle = -[k + l]_+|l, k\rangle, \quad R_{jj'}^{(1,i)}(I_{43})|l, k\rangle = [k - l]_+|l, k\rangle,
\]
\[
R_{jj'}^{(1,i)}(I_{32})|l, k\rangle = \frac{1}{[k + l][l - k](q - q^{-1})} \times
\times \{ -i[j - l][j' - l]|l + 1, k\rangle + i[j + l][j' + l]|l - 1, k\rangle -
- i[j' - k][j - k]|l, k + 1\rangle + i[j' + k][j + k]|l, k - 1\rangle \}.
\]
In order to have a similarity with formulas (40)–(42) we denote the vectors $|l, k\rangle$ by $|k, l\rangle$ and the representation $\hat{R}_{jj'}^{(1,i)}$ by $\hat{R}_{jj'}^{(1,i)}$.

The operators $\hat{R}_{jj'}^{(1,i)}(I_{21})$ and $\hat{R}_{jj'}^{(1,i)}(I_{43})$ have multiple common eigenvectors $|k, l\rangle$ and $|k, -l\rangle$. The representation $\hat{R}_{jj'}^{(1,i)}$ is reducible and decomposes into irreducible constituents in the same way as in the previous case. The representation $\hat{R}_{jj'}^{(1,i)}$ is analysed as the representation $R_{jj'}^{(-i,1)}$ above and we have the following result. The representation $\hat{R}_{jj'}^{(1,i)}$ is reducible and decomposes into irreducible subrepresentations as
\[
\hat{R}_{jj'}^{(1,i)} = R_{jj'}^{(-1,+,-)} \oplus R_{jj'}^{(-1,-,+)}.
\]
where the representations $R_{jj'}^{(-1,+,-)}$ differ from the representations $R_{jj'}^{(+1,1)}$, respectively, only by the operator $R_{jj'}^{(-1,+,-)}(I_{21})$ and
\[
R_{jj'}^{(-1,+,-)}(I_{21}) = -R_{jj'}^{(+1,1)}(I_{21}).
\]
Similarly, the representation
\[
R_{jj'}^{(1,-i)} = \hat{R}_{jj'}^{(-1,1)} = (T_j^{(1)} \otimes T_j^{(-1)}) \circ \phi
\]
of $U_q'(so_4)$ is reducible and decomposes into irreducible components as
\[
\hat{R}_{jj'}^{(-1,1)} = R_{jj'}^{(+1,-,+) \oplus R_{jj'}^{(+1,-,-)},
\]
where the representations $R_{jj'}^{(+1,-,\pm)}$ differ from the representations $R_{jj'}^{(+1,\pm,\pm)}$, respectively, only by the operator $R_{jj'}^{(+1,-,\pm)}(I_{43})$ and
\[
R_{jj'}^{(+1,-,\pm)}(I_{43}) = -R_{jj'}^{(+1,\pm,\pm)}(I_{43}).
\]
We do not consider other representations $R_{jj'}^{(1,\pm1,\pm)} = (T_j^{(\pm1)} \otimes T_j^{(\pm1)})$ since they do not give new irreducible representations of $U_q'(so_4)$.

Thus, for every values of $j$ and $j'$ such that
\[
\begin{align*}
j = 0, 1, 2, \ldots, \quad & j' = 1, 2, 3, 5, 7, \ldots \quad \text{or} \quad j = 0, 1, 2, 3, 5, 7, \ldots, \quad & j' = 0, 1, 2, \ldots
\end{align*}
\]
we constructed 8 representations $R_{jj'}^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}$, $\varepsilon_i = \pm$. These representations act on the linear space $\mathcal{H}$ with the basis
\[
|k, l\rangle, \quad k = j, j - 1, \ldots, \frac{1}{2}, \quad l = j', j' - 1, \ldots, -j',
\]
}\]

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if \( j' \) is integral and with the basis
\[
|k, l\rangle, \quad k = j, j-1, \ldots, -j, \quad l = j', j'-1, \ldots, \frac{1}{2},
\]
if \( j \) is integral. The representations are given by the formulas
\[
R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)}(I_{21})|k, l\rangle = \epsilon_1 [k + l]|k, l\rangle, \quad (50)
\]
\[
R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)}(I_{43})|k, l\rangle = \epsilon_2 [k - l]|k, l\rangle, \quad (51)
\]
\[
R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)}(I_{32})|k, l\rangle = \frac{1}{(k + l)[k - l](q - q^{-1})}\{-i[j'-l][j-l]|k, l+1\} +
+i[j' + l][j + l][k, l-1] - i[j'-k][j-k][k+1, l] + i[j' + k][j + k][k-1, l]\}, \quad (52)
\]
where \( k \neq \frac{1}{2} \) if \( j \) is half-integral and \( l \neq \frac{1}{2} \) if \( j' \) is half-integral, and by
\[
R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)}(I_{32})|1/2, l\rangle = \frac{1}{[l + 1/2][l - 1/2](q - q^{-1})}\{-i[j'-l][j'-l]|1/2, l+1\} +
+i[j + l][j' + l][1/2, l-1] - i[j'-l/2][j-l/2]|3/2, l\} + i[j' + l/2][j + l/2]j_3(-1)^l|1/2, l\} \quad (53)
\]
if \( j \) is half-integral and by
\[
R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)}(I_{32})|k, 1/2\rangle = \frac{1}{[k + 1/2][k - 1/2](q - q^{-1})}\{-i[j - l][j'-l]|k, 3/2\} +
+i[j + l][j' + l]j_3(-1)^k|k, 1/2\} - i[j'-k][j-k][k+1, 1/2] + i[j' + k][j + k][k-1, 1/2]\} \quad (54)
\]
if \( j' \) is half-integral.

It is seen from formulas for representations that the representations \( R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)} \) and \( R_{jj'}^{(\epsilon_1, -\epsilon_2, \epsilon_3)} \) are equivalent. The equivalence operator \( A \) is given by the formula \( A|k, l\rangle = |l, k\rangle \). For this reason, we consider the representations \( R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)} \) only for \( j \geq j' \).

**Theorem 3.** The representations \( R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)}, j \geq j', \) are irreducible and pairwise nonequivalent.

**Proof.** Irreducibility of these representations will be proved in section 8. In order to prove their pairwise nonequivalence we note that finite dimensional representations \( T \) and \( T' \) of \( U'_q(\mathfrak{so}_4) \) cannot be equivalent if at least for one pair \( T(I_{i,i-1}), T'(I_{i,i-1}), i = 2, 3, 4 \), we have \( \text{Tr} T(I_{i,i-1}) \neq \text{Tr} T'(I_{i,i-1}) \). For this reason, the representations \( R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)} \) and \( R_{ss'}^{(\epsilon_1, \epsilon_2, \epsilon_3)} \) with \( (\epsilon_1, \epsilon_2, \epsilon_3) \neq (\epsilon'_1, \epsilon'_2, \epsilon'_3) \) or and with \( (j, j') \neq (s, s') \) are not equivalent.

Irreducible representations \( R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)} \) are called **representations of the nonclassical type.** They have no classical analogue. Their main property is that the operators \( R_{jj'}^{(\epsilon_1, \epsilon_2, \epsilon_3)}(I_{i,i-1}), i = 2, 3, 4, \) have a nonzero trace. Note that there exist 8 nontrivial one-dimensional representations of the nonclassical type. They coincide with the representations \( R_{j_0}^{(\epsilon_1, \epsilon_2, \epsilon_3)} \).

8. THE CLASSIFICATION THEOREM

The main aim of this section is to prove that constructed above irreducible representations of the classical type and of the nonclassical type exhaust all irreducible finite dimensional representations of the algebra \( U'_q(\mathfrak{so}_4) \). We also prove that pairs of the irreducible representations of Theorem 3 are not equivalent. But first we study some auxiliary operators.
Let $R$ be a finite dimensional representation of $U_q'(so_4)$ on a linear vector space $\mathcal{H}$. We suppose that the operators $R(I_{21})$ and $R(I_{43})$ have eigenvalues only of the classical type, that is, of the form $i[m]$, where $[m]$ means a $q$-number. Let $|k, l\rangle$ be an eigenvector such that

$$R(I_{21})|k, l\rangle = i[k + l]|k, l\rangle, \quad R(I_{43})|k, l\rangle = i[k - l]|k, l\rangle.$$  

We associate with this eigenvector the operators

$$X_1^{(k,l)} = -R(I_{41}) + q^{-2k}R(I_{32}) - iq^{-k-l+1/2}R(I_{42}) - iq^{-k+l-1/2}R(I_{31}),$$  

$$X_2^{(k,l)} = -R(I_{41}) + q^{2k}R(I_{32}) + iq^{k+l+1/2}R(I_{42}) + iq^{k-l-1/2}R(I_{31}),$$  

$$X_3^{(k,l)} = R(I_{41}) + q^{-2l}R(I_{32}) + iq^{-k-l+1/2}R(I_{42}) - iq^{k-l-1/2}R(I_{31}),$$  

$$X_4^{(k,l)} = R(I_{41}) + q^{2l}R(I_{32}) - iq^{k+l+1/2}R(I_{42}) + iq^{-k+l-1/2}R(I_{31}).$$

Note that below we shall use explicit form of the operators $X_1^{(k,l)}, X_2^{(k,l)}, X_3^{(k,l)}, X_4^{(k,l)}$ under action on the vector $|k, l\rangle$. In this case we may consider formulas (55)–(58) as a system of linear equations with unknown vectors $R(I_{21})|k, l\rangle$, $R(I_{31})|k, l\rangle$, $R(I_{42})|k, l\rangle$, $R(I_{41})|k, l\rangle$. The determinant of this system can be easily calculated:

$$\det\begin{pmatrix} -1 & q^{-2k} & -iq^{-k-l+1/2} & -iq^{-k+l-1/2} \\ -1 & q^{2k} & iq^{k+l+1/2} & iq^{k-l-1/2} \\ 1 & q^{-2l} & iq^{-k-l+1/2} & -iq^{k-l-1/2} \\ 1 & q^{2l} & -iq^{k+l+1/2} & iq^{-k+l-1/2} \end{pmatrix} = (q^{k+l} + q^{-k-l})(q^{k-l} + q^{-k+l}).$$

If $q$ is not a root of unity, then this determinant does not vanish for any integral or half-integral $k$ and $l$. This means that the system of above equations can be solved and we can find how the operators $R(I_{32})$, $R(I_{31})$, $R(I_{42})$ and $R(I_{41})$ act on the vector $|k, l\rangle$. This reasoning will be used below.

**Lemma 1.** The vectors $X_i^{(k,l)}|k, l\rangle$ are eigenvectors of the operators $R(I_{21})$ and $R(I_{43})$:

$$R(I_{21})(X_1^{(k,l)}|k, l\rangle) = i[k + l + 1](X_1^{(k,l)}|k, l\rangle), \quad R(I_{43})(X_1^{(k,l)}|k, l\rangle) = i[k - l - 1](X_1^{(k,l)}|k, l\rangle),$$  

$$R(I_{21})(X_2^{(k,l)}|k, l\rangle) = i[k + l - 1](X_2^{(k,l)}|k, l\rangle), \quad R(I_{43})(X_2^{(k,l)}|k, l\rangle) = i[k - l + 1](X_2^{(k,l)}|k, l\rangle),$$  

$$R(I_{21})(X_3^{(k,l)}|k, l\rangle) = i[k + l + 1](X_3^{(k,l)}|k, l\rangle), \quad R(I_{43})(X_3^{(k,l)}|k, l\rangle) = i[k - l + 1](X_3^{(k,l)}|k, l\rangle),$$  

$$R(I_{21})(X_4^{(k,l)}|k, l\rangle) = i[k + l - 1](X_4^{(k,l)}|k, l\rangle), \quad R(I_{43})(X_4^{(k,l)}|k, l\rangle) = i[k - l - 1](X_4^{(k,l)}|k, l\rangle).$$

**Proof.** The lemma is proved by direct calculation using the defining relations for the elements $I_{ij} \in U_q'(so_4)$, $i > j$. For example, by using relations (11)–(14) we have

$$R(I_{21})(X_1^{(k,l)}|k, l\rangle) = R(-qI_{41} + q^{-2k-1}I_{32} - iq^{-k-l+1/2}I_{42} - iq^{k+l+1/2}I_{31})R(I_{21})|k, l\rangle +$$

$$+ R(iq^{-k-l}I_{41} - iq^{-k}I_{32} + q^{1/2}I_{42} + q^{-2k-1/2}I_{31})|k, l\rangle = i[k + l + 1](X_1^{(k,l)}|k, l\rangle).$$

Lemma is proved.

Lemma 1 means that the operators $X_1^{(k,l)}$ and $X_3^{(k,l)}$ increase $k$ and $l$ in the eigenvalues of $R(I_{21})$ and $R(I_{43})$, respectively, and the operators $X_2^{(k,l)}$ and $X_4^{(k,l)}$ decrease these numbers in these eigenvalues. Symbolically we write this in the form

$$X_1: \quad l \to l + 1, \quad X_2: \quad l \to l - 1,$$

$$X_3: \quad k \to k + 1, \quad X_4: \quad k \to k - 1.$$
Lemma 2. The operators (55)–(58) have the properties

\[ X_3^{(k,l+1)} X_1^{(k,l)} |k,l\rangle = X_1^{(k+1,l)} X_3^{(k,l)} |k,l\rangle, \quad X_4^{(k,l-1)} X_2^{(k,l)} |k,l\rangle = X_2^{(k-1,l)} X_4^{(k,l)} |k,l\rangle, \]

\[ X_4^{(k,l+1)} X_1^{(k,l)} |k,l\rangle = X_1^{(k-1,l)} X_4^{(k,l)} |k,l\rangle, \quad X_3^{(k,l-1)} X_2^{(k,l)} |k,l\rangle = X_2^{(k+1,l)} X_3^{(k,l)} |k,l\rangle. \]

Proof. The first of these relations is proved as follows. Using the expressions (55) and (57) for \( X_1 \) and \( X_3 \) we express (by using relations (11)–(15)) the elements \( X_3^{(k,l+1)} X_1^{(k,l)} \) and \( X_1^{(k+1,l)} X_3^{(k,l)} \) as a linear combination of the basis elements from Poincaré–Birkhoff–Witt theorem. As a result, we receive the first relation. Other relations are proved in the same way. Lemma is proved.

Lemma 2 means that the pairs of operators \( X_1 \) and \( X_3 \), \( X_2 \) and \( X_4 \), \( X_1 \) and \( X_4 \), \( X_2 \) and \( X_3 \) (with appropriate upper indices) commute under action on the vector \( |k,l\rangle \).

Lemma 3. The operators (55)–(58) have the properties

\[ X_2^{(k,l+1)} X_1^{(k,l)} |k,l\rangle = (C'_4 - (q^{2l+1} + q^{-2l-1})C_4 + [2l][2(l + 1)])|k,l\rangle, \]

\[ X_1^{(k,l-1)} X_2^{(k,l)} |k,l\rangle = (C'_4 - (q^{2l-1} + q^{-2l+1})C_4 + [2l][2(l - 1)])|k,l\rangle, \]

\[ X_4^{(k+1,l)} X_3^{(k,l)} |k,l\rangle = (C'_4 + (q^{2k+1} + q^{-2k-1})C_4 + [2k][2(k + 1)])|k,l\rangle, \]

\[ X_3^{(k-1,l)} X_4^{(k,l)} |k,l\rangle = (C'_4 + (q^{2k-1} + q^{-2k+1})C_4 + [2k][2(k - 1)])|k,l\rangle, \]

where \( C_4 \) and \( C'_4 \) are the Casimir elements of \( U'_4(\mathfrak{so}_4) \) from section 2.

Proof is given in the same way as that of Lemma 2, taking into account expressions for the Casimir elements.

Lemma 3 can be used for evaluation of eigenvalues of Casimir elements \( C_4 \) and \( C'_4 \) on irreducible representations \( R \) when the operators \( R(I_{21}) \) and \( R(I_{43}) \) have eigenvalues of the classical type. An eigenvectors \( |k,l\rangle \) of the operators \( R(I_{21}) \) and \( R(I_{43}) \) are called weight vectors of the representation \( R \). A weight vector \( |j,j'\rangle \) is called a highest weight vector if

\[ X_1^{(j,j')} |j,j'\rangle = 0, \quad X_3^{(j,j')} |j,j'\rangle = 0. \]

If \( R \) is an irreducible representation with classical type eigenvalues of the operators \( R(I_{21}) \) and \( R(I_{43}) \), then we apply both sides of the first and the third relations of Lemma 3 to the vector of highest weight \( |j,j'\rangle \). The left hand sides send this vector to zero, and then the right hand sides (equating to zero) gives the following eigenvalues for \( C_4 \) and \( C'_4 \): \( C_4 = [j + j' + 1][j' - j]I \), \( C'_4 = \{(q^{2j+1} + q^{-2j-1})[j - j'][j + j' + 1] - [2j][2j + 2]\}I \), where \( I \) is the unit operator in the representation space. In particular, such eigenvalues have Casimir operators of the classical type representation \( R_{j,j'} \).

Now let \( R' \) be a finite dimensional representation of \( U'_4(\mathfrak{so}_4) \) on a linear space \( \mathcal{H}' \). Suppose that the operators \( R'(I_{21}) \) and \( R'(I_{43}) \) have eigenvalues only of the nonclassical type, that is, of the form \( \pm[m]_+ \), where \( [m]_+ = (q^m + q^{-m})/(q - q^{-1}) \) and \( m \) are half-integral. If \( |k,l\rangle \) is an eigenvector such that

\[ R'(I_{21})|k,l\rangle = \varepsilon_1[k + l]_+|k,l\rangle, \quad R'(I_{43})|k,l\rangle = \varepsilon_2[k - l]_+|k,l\rangle, \]

then we associate with it the operators

\[ X_1^{(k,l)} = R'(I_{41}) + q^{-2k}R'(I_{32}) - q^{-k-l+1/2}R'(I_{42}) - q^{-k+l-1/2}R'(I_{31}), \quad (59) \]
\[ X_2^{(k,l)} = R'(I_{41}) + q^{2k} R'(I_{32}) - q^{k+l+1/2} R'(I_{42}) - q^{k-l-1/2} R'(I_{31}), \]
\[ X_3^{(k,l)} = -R'(I_{41}) - q^{-2l} R'(I_{32}) + q^{-k-l+1/2} R'(I_{42}) + q^{-k+l-1/2} R'(I_{31}), \]
\[ X_4^{(k,l)} = -R'(I_{41}) - q^{2l} R'(I_{32}) + q^{k+l+1/2} R'(I_{42}) + q^{-k+l-1/2} R'(I_{31}). \]

Below we shall consider relations (59)–(62) as a system of linear equations. Determinant of the matrix of this system is equal to
\[
\begin{vmatrix}
1 & q^{-2k} & -q^{-k-l+1/2} & -q^{-k+l-1/2} \\
1 & q^{2k} & -q^{k+l+1/2} & -q^{-k-l-1/2} \\
-1 & -q^{-2l} & q^{-k-l+1/2} & q^{k-l+1/2} \\
-1 & -q^{2l} & q^{k+l+1/2} & q^{-k+l-1/2}
\end{vmatrix} = (q^{k+l} - q^{-k-l})(q^{k-l} - q^{-k+l}).
\]

If $q$ is not a root of unity, then this determinant does not vanish for any half-integral $k \pm l$.

Hence, the system of above equations can be solved and we can find how the operators $R'(I_{32}), R'(I_{31}), R'(I_{42})$ and $R'(I_{41})$ act on the vector $|k, l\rangle$.

Below we formulate three lemmas for these operators analogous to Lemma 1–3. Proofs of these lemmas are the same as in the case of Lemmas 1–3 and we omit them.

**Lemma 4.** The vectors $X_i^{(k,l)}|k,l\rangle$ are eigenvectors of the operators $R'(I_{21})$ and $R'(I_{43})$:
\[
R'(I_{21})(X_1^{(k,l)}|k,l\rangle) = \varepsilon_1[k + l + 1]_+(X_1^{(k,l)}|k,l\rangle), \quad R'(I_{43})(X_1^{(k,l)}|k,l\rangle) = \varepsilon_2[k - l - 1]_+(X_1^{(k,l)}|k,l\rangle),
\]
\[
R'(I_{21})(X_2^{(k,l)}|k,l\rangle) = \varepsilon_1[k + l - 1]_+(X_2^{(k,l)}|k,l\rangle), \quad R'(I_{43})(X_2^{(k,l)}|k,l\rangle) = \varepsilon_2[k - l + 1]_+(X_2^{(k,l)}|k,l\rangle),
\]
\[
R'(I_{21})(X_3^{(k,l)}|k,l\rangle) = \varepsilon_1[k + l + 1]_+(X_3^{(k,l)}|k,l\rangle), \quad R'(I_{43})(X_3^{(k,l)}|k,l\rangle) = \varepsilon_2[k - l + 1]_+(X_3^{(k,l)}|k,l\rangle),
\]
\[
R'(I_{21})(X_4^{(k,l)}|k,l\rangle) = \varepsilon_1[k + l - 1]_+(X_4^{(k,l)}|k,l\rangle), \quad R'(I_{43})(X_4^{(k,l)}|k,l\rangle) = \varepsilon_2[k - l - 1]_+(X_4^{(k,l)}|k,l\rangle).
\]

**Lemma 5.** The operators (59)–(62) have the properties
\[
X_3^{(k,l+1)}X_1^{(k,l)}|k,l\rangle = X_1^{(k+1,l)}X_3^{(k,l)}|k,l\rangle, \quad X_4^{(k,l-1)}X_2^{(k,l)}|k,l\rangle = X_2^{(k-1,l)}X_4^{(k,l)}|k,l\rangle,
\]
\[
X_4^{(k,l+1)}X_1^{(k,l)}|k,l\rangle = X_1^{(k-1,l)}X_4^{(k,l)}|k,l\rangle, \quad X_3^{(k,l-1)}X_2^{(k,l)}|k,l\rangle = X_2^{(k+1,l)}X_3^{(k,l)}|k,l\rangle.
\]

**Lemma 6.** For the operators (59)–(62) we have
\[
X_2^{(k,l+1)}X_1^{(k,l)}|k,l\rangle = (C_4' - (q^{2l+1} + q^{-2l-1})C_4 + [2l][2(l + 1)])|k,l\rangle,
\]
\[
X_1^{(k,l-1)}X_2^{(k,l)}|k,l\rangle = (C_4' - (q^{2l-1} + q^{-2l+1})C_4 + [2l][2(l - 1)])|k,l\rangle,
\]
\[
X_4^{(k,l+1)}X_3^{(k,l)}|k,l\rangle = (C_4' - (q^{2k+1} + q^{-2k-1})C_4 + [2k][2(k + 1)])|k,l\rangle,
\]
\[
X_3^{(k,l-1)}X_4^{(k,l)}|k,l\rangle = (C_4' - (q^{2k-1} + q^{-2k+1})C_4 + [2k][2(k - 1)])|k,l\rangle,
\]
where $C_4$ and $C_4'$ are the Casimir elements of $U'_q(\mathfrak{so}_4)$ from section 2.

Lemma 6 can be used for evaluation of eigenvalues of Casimir elements $C_4$ and $C_4'$ on irreducible representations $R'$ when the operators $R(I_{21})$ and $R(I_{43})$ have eigenvalues of the non-classical type. Eigenvectors $|k,l\rangle$ of the operators $R'(I_{21})$ and $R'(I_{43})$ are called weight vectors of the representation $R'$. A weight vector $|j,j'\rangle$ is called a highest weight vector if
\[
X_1^{(j,j')}|j,j'\rangle = 0, \quad X_3^{(j,j')}|j,j'\rangle = 0.
\]
If \( R' \) is an irreducible representation with nonclassical type eigenvalues of the operators \( R'(I_{21}) \) and \( R'(I_{43}) \), then applying both sides of the first and the third relations of Lemma 6 to the vector of highest weight \( |j,j'\rangle \) we derive that

\[
C_4 = |j + j' + 1|_+ [j - j']_+ I,
\]
\[
C'_4 = \{ (q^{2j+1} + q^{-2j-1})[j - j']_+ [j + j' + 1]_+ - [2j][2j + 2] \} I,
\]

where \( I \) is the identity operator on the representation space. In particular, such eigenvalues have Casimir operators of the nonclassical type representation \( R'^{e_1,e_2,e_3}_{jj'} \).

**Proof of the first part of Theorem 3.** Let \( R'^{e_1,e_2,e_3}_{jj'} \) be a representation of the nonclassical type on the vector space \( \mathcal{H} \). Then the commuting operators \( R'^{e_1,e_2,e_3}_{jj'}(I_{21}) \) and \( R'^{e_1,e_2,e_3}_{jj'}(I_{43}) \) are simultaneously diagonalized. Since \( q \) is not a root of unity, the eigenvalues \( (\varepsilon_1[k + l]_+, \varepsilon_2[k - l]_+) \) for the corresponding vector \( |k,l\rangle \) are of multiplicity 1. Let \( \mathcal{H}' \) be a nontrivial invariant subspace of the representation space \( \mathcal{H} \), and let \( \sum_{k,l} \alpha_{k,l} |k,l\rangle \) be a nonzero vector from \( \mathcal{H}' \). Since eigenvalues \( (\varepsilon_1[k + l]_+, \varepsilon_2[k - l]_+) \) are of multiplicity 1, then each \( |k,l\rangle \) from this linear combination belongs to \( \mathcal{H}' \). Let \( |k',l'\rangle \) be one of these vectors. Applying the operators \( X_1 \) and \( X_3 \) (with appropriate upper indices) to \( |k',l'\rangle \) we obtain the vector of highest weight \( |j,j'\rangle \) of the representation \( R'^{e_1,e_2,e_3}_{jj'} \). Applying to \( |j,j'\rangle \) the operators \( X_2 \) and \( X_4 \) (with appropriate indices) we obtain all basis vectors of the space \( \mathcal{H} \). Hence, the representation \( R'^{e_1,e_2,e_3}_{jj'} \) is irreducible. Theorem is proved.

Now we can prove the theorem on classification of irreducible finite dimensional representations of \( U'_q(so_4) \).

**Theorem 4.** If \( q \) is not a root of unity, then each irreducible finite dimensional representation \( R \) of \( U'_q(so_4) \) is equivalent to one of the irreducible representations of the classical type or to one of the irreducible representations of the nonclassical type.

**Proof.** Let us first prove the following assertion: if eigenvalues of the operators \( R(I_{21}) \) and \( R(I_{43}) \) of an irreducible finite dimensional representation \( R \) of \( U'_q(so_4) \) are of the classical type (that is, of the form \( i[m], m \in \mathbb{Z}/2\mathbb{Z} \), then \( R \) is equivalent to one of the irreducible representations of the classical type. We diagonalize both operators \( R(I_{21}) \) and \( R(I_{43}) \) and represent their eigenvectors in the form \( |k,l\rangle \), where

\[
R(I_{21})|k,l\rangle = i[k + l]|k,l\rangle, \quad R(I_{43})|k,l\rangle = i[k - l]|k,l\rangle.
\]

(These eigenvectors are called weight vectors.) Due to Lemmas 1 and 2, there exists an eigenvector of highest weight (we denote it by \( |j,j'\rangle \)), that is such that

\[
R(I_{21})|j,j'\rangle = i[j + j']|j,j'\rangle, \quad R(I_{43})|j,j'\rangle = i[j - j']|j,j'\rangle.
\]
\[
X_1^{(j,j')}|j,j'\rangle = 0, \quad X_3^{(j,j')}|j,j'\rangle = 0.
\]

Applying the first and the third relations of Lemma 3 to the vector \( |j,j'\rangle \) we find eigenvalues of the Casimir operators \( R(C_4) \) and \( R(C'_4) \) on the representation \( R \):

\[
R(C_4) = |j + j' + 1| j' - j],
\]
\[
R(C'_4) = (q^{2j+1} + q^{-2j-1})[j - j']_+ [j + j' + 1]_+ - [2j][2j + 2].
\]

Applying the first and the third relations of Lemma 3 to the vector \( |j,j'\rangle \) we find eigenvalues of the Casimir operators \( R(C_4) \) and \( R(C'_4) \) on the representation \( R \):

\[
|j,j' - s\rangle = X_2^{(j,j' - s + 1)} \cdots X_2^{(j,j' - 1)} X_2^{(j,j')} X_2^{(j,j')} |j,j'\rangle, \quad s = 0, 1, 2, \ldots.
\]
Since the representation \( R \) is finite dimensional and (by Lemma 1) these vectors have different eigenvalues, there exists smallest positive integer \( n \) such that
\[
|j, j' - n\rangle := X_2^{(j, j' - n + 1)}|j, j' - n + 1\rangle = 0. \tag{65}
\]

Similarly, between vectors
\[
|j - r, j'\rangle := X_4^{(j - r + 1, j')} \cdots X_4^{(j - 1, j')} X_4^{(j, j')}|j, j'\rangle, \quad r = 0, 1, 2, \ldots,
\]
there exists a nonzero vector with smallest positive integer \( m \) such that
\[
|j - m, j'\rangle := X_4^{(j - m + 1, j')}|j - m + 1, j'\rangle = 0. \tag{66}
\]

Then using the second and the fourth relations of Lemma 3, we find from (63)–(66) that
\[
X_1^{(j, j' - n)} X_2^{(j, j' - n + 1)}|j, j' - n + 1\rangle
= [n][n - 2j' - 1](q^{j' - n} + q^{-j' + n})(q^{j' - n + 1} + q^{-j' + n - 1})|j, j' - n + 1\rangle = 0,
\]
\[
X_3^{(j - m, j')} X_4^{(j - m + 1, j')}|j - m + 1, j'\rangle
= [m][m - 2j - 1](q^{m - j' - 1} + q^{-m + j' + 1})(q^{m - j' + 1} + q^{-m + j' - 1})|j - m + 1, j'\rangle = 0.
\]

Therefore, \([n - 2j' - 1] = 0\) and \([m - 2j - 1] = 0\), that is
\[
m = 2j + 1, \quad n = 2j' + 1.
\]

Now we act successively on the vectors \(|j, j'\rangle, |j, j' - 1\rangle, |j, j' - 2\rangle, \ldots, |j, j'\rangle\) by the operators \( X_4 \) with the appropriate upper indices. As a result, we construct the vectors
\[
|j, j'\rangle, \quad |j - 1, j'\rangle, \quad \ldots \quad |j, j' - 1\rangle,
|j, j' - 1\rangle, \quad |j - 1, j' - 1\rangle, \quad \ldots \quad |j, j' - 1\rangle,
\]
\[
\ldots \quad \ldots \quad \ldots \quad \ldots
|j, j' - 1\rangle, \quad |j - 1, j' - 1\rangle, \quad \ldots \quad |j, j' - 1\rangle
\]
for which
\[
R(I_{21})|k, l\rangle = i|k + l\rangle|k, l\rangle, \quad R(I_{43})|k, l\rangle = i|k - l\rangle|k, l\rangle.
\]

We can find how the operators \( X_i, i = 1, 2, 3, 4 \), with appropriate indices act on these vectors:
\[
X_2^{(k, l)}|k, l\rangle = |k, l - 1\rangle, \quad X_4^{(k, l)}|k, l\rangle = |k - 1, l\rangle,
\]
\[
X_1^{(k, l)}|k, l\rangle = X_1^{(k, l)}X_2^{(k, l + 1)}|k, l + 1\rangle = (R(C'_4) - (q^{2l+1} + q^{-2l-1})R(C_4) + [2l + 2][2l])|k, l + 1\rangle,
\]
\[
X_3^{(k, l)}|k, l\rangle = X_3^{(k, l)}X_4^{(k, 1 + l)}|k + 1, l\rangle = (R(C'_4) + (q^{2k+1} + q^{-2k-1})R(C_4) + [2k + 2][2k])|k + 1, l\rangle.
\]

Putting here the explicit expression for the Casimir operators, substituting these expressions for \( X_i^{(k, l)}|k, l\rangle \) to (55)–(58) and considering (55)–(58) as a system of linear equations with unknown \( R(I_{32})|k, l\rangle, R(I_{31})|k, l\rangle, R(I_{32})|k, l\rangle, R(I_{41})|k, l\rangle \) we solve this system and find that
\[
R(I_{32})|k, l\rangle = \frac{1}{(q^{k+l} + q^{-k-l})(q^{k-l} + q^{-k+l})}
\]
\[
\left[ (q^{j-l} + q^{j+l}) (q^{j+l+1} + q^{-j-l-1}) [j' - l][j' + l + 1][k, l + 1] -
(q^{j' - k} + q^{j+k})(q^{j'+k+1} + q^{-j'-k-1}) [j - k][j + k + 1][k + 1, l]\right].
\]
Thus, the vectors \(|k, l\), \(-j \leq k \leq j, -j' \leq l \leq j'\), constitute a basis of the representation space \(\mathcal{H}\). Introducing a new basis \(\{|k, l\}'\) such that

\[
|k, l\> = (-1)^{k+l} \prod_{r=-j}^{k} (q^{j'+r} + q^{-j'-r})^{-1} [j + r]^{-1} \prod_{s=-j'}^{k} (q^{j'+s} + q^{-j'-s})^{-1} [j' + s]^{-1} |k, l\>
\]

we shall obtain for \(R(I_{21})\), \(R(I_{32})\), \(R(I_{43})\) the operators of the irreducible representation \(R_{j,j'}\) of the classical type from section 4. Thus, in this case Theorem is proved.

Now we prove the second part of the theorem which can be formulated as follows: if eigenvalues of the operators \(R(I_{21})\) and \(R(I_{43})\) of an irreducible finite dimensional representation \(R\) are of the nonclassical type, that is of the form \(\pm [m]_+\), \(m \in \frac{1}{2} \mathbb{Z}, m \notin \mathbb{Z}\), then \(R\) is equivalent to one of the irreducible representations of the nonclassical type.

We first prove that if eigenvalues of the operators \(R(I_{21})\) and \(R(I_{43})\) are of the form \([m]_+\) (only sign + is taken), then \(R\) is equivalent to one of the irreducible representations of the nonclassical type. A proof is similar to that of the previous case and for this reason we do not give details.

Due to Lemmas 4 and 5, there exists an eigenvector of highest weight \(|j, j'\rangle\) such that

\[
R(I_{21})|j, j'\rangle = [j + j']_+|j, j'\rangle, \quad R(I_{43})|j, j'\rangle = [j - j']_+|j, j'\rangle.
\]

Applying relations of Lemma 6 to \(|j, j'\rangle\) we find eigenvalues of the Casimir operators \(R(C_4)\) and \(R(C_4')\):

\[
R(C_4) = [j + j' + 1]_+[j - j]_+,
\]

\[
R(C_4') = (q^{2j+1} + q^{-2j-1})[j - j']_+[j + j' + 1]_+ - [2j][2j + 2].
\]

Now we construct recursively the vectors

\[
|j - r, j' - s\rangle := X_{4}^{(j-j'+1,j'-s)}\cdots X_{4}^{(j,j'-s)}X_{2}^{(j,j'-s+1)}\cdots X_{2}^{(j,j'-1)}X_{2}^{(j,j')}|j, j'\rangle,
\]

\(r, s = 0, 1, 2, \ldots\).

By Lemma 4, for these vectors we have

\[
R(I_{21})|k, l\rangle = [k + l]_+|k, l\rangle, \quad R(I_{43})|k, l\rangle = [k - l]_+|k, l\rangle.
\]

These vectors satisfy the relations

\[
X_{2}^{(k,l)}|k, l\rangle = |k, l - 1\rangle, \quad X_{4}^{(k,l)}|k, l\rangle = |k - 1, l\rangle,
\]

\[
X_{1}^{(k,l)}|k, l\rangle = X_{1}^{(k,l+1)}|k, l - 1\rangle
\]

\[
= (R(C_4) - (q^{2l+1} + q^{-2l-1})R(C_4) + [2l + 2][2l])|k, l + 1\rangle,
\]

\[
X_{3}^{(k,l)}|k, l\rangle = X_{3}^{(k,l+1)}|k, l + 1\rangle
\]

\[
= (R(C_4') - (q^{2k+1} + q^{-2k-1})R(C_4') + [2k + 2][2k])|k + 1, l\rangle.
\]

Since the operators \(X_{1}^{(k,l)}, X_{2}^{(k,l)}, X_{3}^{(k,l)}, X_{4}^{(k,l)}\) acting on the vector \(|k, l\rangle\) determine the action of the operators \(R(I_{32}), R(I_{31}), R(I_{42}), R(I_{41})\) on this vector, then the vectors

\[
|j, j'\rangle, \quad |j - 1, j'\rangle, \quad |j - 2, j'\rangle, \ldots,
\]

\[
|j, j' - 1\rangle, \quad |j - 1, j' - 1\rangle, \quad |j - 2, j' - 1\rangle, \ldots,
\]

(72)
Rewriting the relations (69)–(71) for this new basis we obtain a system of linear equations with

\[ |j, j' - 2), |j - 1, j' - 2), |j - 2, j' - 2), \cdots, \]
\[ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \]

span an invariant subspace in the representation space \( \mathcal{H} \). Since the representation \( R \) is irreducible, they span the whole space \( \mathcal{H} \). It follows from (68) that only pairs of vectors \( |k, l) \) and \( | − k, − l) \) have the same eigenvalues for the operators \( R(I_{21}) \) and \( R(I_{43}) \).

In order to determine which possibilities exist for the representation \( R \), we make as follows. For definiteness we suppose that \( j \) is half-integral. Using formula (67) we first create the set of all possible vectors \( |k, l) \), which does not contains pairs \( |k, l) \) and \( | − k, − l) \). For example, we create all the vectors \( |k, l) \) with \( k > 0 \). This set contains the vector \( |\frac{1}{2}, 0) \). There are two possibilities:

(a) The vectors \( |\frac{1}{2}, 0) \) and \( | − \frac{1}{2}, 0) \) are linearly dependent, that is, \( X_{4}^{(1/2, 0)}|\frac{1}{2}, 0) = a|\frac{1}{2}, 0) \); 
(b) the vectors \( |\frac{1}{2}, 0) \) and \( | − \frac{1}{2}, 0) \) are linearly independent.

In the case (a) all pairs \( |k, l) \) and \( | − k, − l) \) consist of linearly dependent vectors. The reason of this is that

\[ X_{1}^{(k, l)} = X_{2}^{(−k, −l)}, \quad X_{3}^{(k, l)} = X_{4}^{(−k, −l)} \tag{73} \]

(as it follows from expressions (59)–(62) for the operators \( X_{i}, \quad i = 1, 2, 3, 4 \). Therefore, for every positive integrals \( r \) and \( s \) the vectors

\[ X_{3}^{(r + 1/2, s)} \cdots X_{3}^{(r + 1/2, 0)} X_{1}^{(r − 1/2, 0)} \cdots X_{1}^{(1/2, 0)}|1/2, 0), \]
\[ X_{4}^{(−r − 1/2, −s)} \cdots X_{4}^{(−r − 1/2, 0)} X_{2}^{(−r + 1/2, 0)} \cdots X_{2}^{(−1/2, 0)}|−1/2, 0) \]

are linear dependent with the same constant \( a \). The constant \( a \) can be explicitly calculated. Namely, since \( X_{4}^{(1/2, 0)}|1/2, 0) = |−1/2, 0) = a|1/2, 0) \) and \( X_{3}^{(−1/2, 0)} = X_{4}^{(1/2, 0)} \), we have

\[ X_{3}^{(−1/2, 0)}|1/2, 0) = X_{3}^{(−1/2, 0)}X_{4}^{(1/2, 0)}|1/2, 0) = R(C_{4}' − 2C_{4} − 1)|1/2, 0) = \]
\[ = X_{3}^{(−1/2, 0)}a|1/2, 0) = aX_{4}^{(1/2, 0)}|1/2, 0) = a^{2}|1/2, 0), \]

that is, \( a^{2} = R(C_{4}' − 2C_{4} − 1) \). This means that \( a \) is determined up to a sign. Using expressions for values of the Casimir elements of \( U_{q}^{'}(so_{4}) \) on irreducible representations with highest weight vector \( |j, j') \), we find that

\[ a = \varepsilon_{3}(q − q^{-1})[j + 1/2][j' + 1/2], \]

where \( \varepsilon_{3} \) takes one of the values \( ±1 \).

Thus, we received the following set of linear independent vectors of the representation space \( \mathcal{H} \):

\[ |j, j'), |j − 1, j'), \cdots, |1/2, j'), \]
\[ |j, j' − 1), |j − 1, j' − 1), \cdots, |1/2, j' − 1), \]
\[ \cdots \cdots \cdots \cdots \cdots \cdots \]
\[ |j, −j'), |j − 1, −j'), \cdots, |1/2, −j'). \]

These vectors constitute a basis of the space \( \mathcal{H} \). We introduce a new basis \( \{|k, l)\} \) such that

\[ |k, l) = \prod_{r=1/2}^{k} (q^{j' + r} − q^{−j' − r})^{−1}[j + r]^{−1} \prod_{s=−j'}^{k} (q^{j + s} − q^{−j − s})^{−1}[j' + s]^{−1}|k, l)'. \]

Rewriting the relations (69)–(71) for this new basis we obtain a system of linear equations with unknown \( R(I_{32})|k, l)' \), \( R(I_{42})|k, l)' \), \( R(I_{41})|k, l)' \), \( R(I_{31})|k, l)' \). Solving this system we find that
the operator $R(I_{32})$ acts on the basis vectors by the formulas for the irreducible nonclassical type representation $R^{(+,+,+)}_{jk'}$ or the irreducible nonclassical type representation $R^{(+,+,−)}_{jk'}$.

Now we consider case (b). Due to the relation (73) we conclude that the vectors

$$\langle k, l \rangle, \quad k = j, j - 1, j - 2, \cdots, -j, \quad l = j', j' - 1, j' - 2, \cdots, -j',$$

are linearly independent and constitute a basis of the representation space $\mathcal{H}$. Solving the system of linear equations (69)–(71) we obtain a representation of $U'_q(sl_2)$ equivalent to one of the representations $R^{(\pm,\pm)}_{jj'}$ or $R^{(\pm,\pm)}_{jj'}$ from section 6. These representations are reducible. So, case (b) is not possible for our irreducible representation $R$.

We have considered the case when all eigenvalues of the operators $R(I_{21})$ and $R(I_{43})$ are of the form $[m]_+$ (with sign +). However, due to automorphisms $\psi_1$ and $\psi_2$, mapping $I_{21} \to -I_{21}$ and $I_{43} \to -I_{43}$, respectively, and concerning all other generating elements in $\{I_{21}, I_{32}, I_{43}\}$, to every such irreducible representation $R$ there correspond the representations $R^{(+,−)} = R \circ \psi_1$, $R^{(−,−)} = R \circ \psi_2$ such that

$$R^{(+,−)}(I_{21})|k, l\rangle = [k + l]_+|k, l\rangle, \quad R^{(+,−)}(I_{43})|k, l\rangle = −[k − l]_+|k, l\rangle, \quad (74)$$

$$R^{(−,+)}(I_{21})|k, l\rangle = −[k + l]_+|k, l\rangle, \quad R^{(−,+)}(I_{43})|k, l\rangle = [k − l]_+|k, l\rangle, \quad (75)$$

$$R^{(−,−)}(I_{21})|k, l\rangle = −[k + l]_+|k, l\rangle, \quad R^{(−,−)}(I_{43})|k, l\rangle = −[k − l]_+|k, l\rangle. \quad (76)$$

Conversely, to any of the representations $R^{(+,−)}$, $R^{(−,+)}$, $R^{(−,−)}$ with these properties there corresponds a unique representation $R$ such that

$$R(I_{21})|k, l\rangle = [k + l]_+|k, l\rangle, \quad R(I_{43})|k, l\rangle = [k − l]_+|k, l\rangle \quad (77)$$

for all eigenvectors $|k, l\rangle$. This means that the classification of irreducible representations $R$ with property (77) automatically leads to the classification of irreducible representations with any of the properties (74)–(76) and vise versa. Therefore, any of irreducible representations of $U'_q(sl_2)$ with one of the properties (74)–(76) is equivalent to one of the irreducible representations of the nonclassical type. Theorem is proved.

In section 4 we constructed the homomorphism $\phi : U'_q(sl_2) \to \hat{U}_q(sl_2)^{\otimes 2, \text{ext}}$ (see Theorem 2). Now we are able to prove more strong assertion:

**Corollary.** If $q$ is not a root of unity, then the homomorphism $U'_q(sl_2) \to \hat{U}_q(sl_2)^{\otimes 2, \text{ext}}$ of Theorem 2 is injective.

**Proof.** If the assertion of Corollary is not true, then there exists nonzero element $a \in U'_q(sl_2)$ such that $\phi(a) = 0$. Then for any finite dimensional representation $T$ of the algebra $\hat{U}_q(sl_2)^{\otimes 2, \text{ext}}$ we have $T(\phi(a)) = 0$. Taking the representations $T^{(\pm,\pm)}_{jj'}$, $T^{(\pm,\pm)}_{jj'}$, and $T^{(\pm,\pm)}_{jj'}$, where $T^{(\pm,\pm)}_{jj'} = T^{(\pm,\pm)}_{jj'} \otimes T^{(\pm,\pm)}_{jj'}$, (see sections 6 and 7), as a representation $T$ we obtain

$$T^{(\pm,\pm)}_{jj'}(\phi(a)) = 0, \quad T^{(\pm,\pm)}_{jj'}(\phi(a)) = 0, \quad T^{(\pm,\pm)}_{jj'}(\phi(a)) = 0$$

for all admissible values of $j$ and $j'$. As we have seen above, any irreducible finite dimensional representation of $U'_q(sl_2)$ is equivalent to the representation $T^{(1,1)}_{jj'} \circ \phi$ with appropriate values of $j$ and $j'$ or to one of irreducible constituents of the representations $T^{(1,1)}_{jj'} \circ \phi$ and $T^{(1,1)}_{jj'} \circ \phi$. This means that $R(a) = 0$ for any irreducible finite dimensional representation of $U'_q(sl_2)$. But it was shown in [13] (see also [2]) that irreducible finite dimensional representations of $U'_q(sl_2)$ separate elements of this algebra. Thus, for our element $a$ these exists an irreducible finite dimensional representation $R$ such that $R(a) \neq 0$. This contradiction proves Corollary.
9. COMPLETE REDUCIBILITY OF FINITE DIMENSIONAL REPRESENTATIONS

It was proved in [29] that if \( q \) is not a root of unity, then every finite dimensional representation of the algebra \( U'_q(\text{so}_3) \) is completely reducible. The aim of this section is to prove the corresponding theorem for the algebra \( U'_q(\text{so}_4) \).

**Theorem 5.** If \( q \) is not a root of unity, then each finite dimensional representation of \( U'_q(\text{so}_4) \) is completely reducible.

**Proof.** In order to prove this theorem it is enough to show that every finite dimensional representation \( R \) of \( U'_q(\text{so}_4) \), containing only two irreducible constituents, is completely reducible.

There are three possibilities for two irreducible constituents:

(a) both representations are of the classical type;
(b) both representations are of the nonclassical type;
(c) representations belong to different types.

Each case will be proved separately.

**Case (a).** We shall use in the proof the following properties of the operators \( X_i^{(k,l)} \) from (55)–(58), which are derived from Lemmas 1–3:

(A) Let \( |k,l\rangle \) be such as in Lemma 1. Acting on \( |k,l\rangle \) by the operators \( X_i^r, i = 1, 2, 3, 4 \), with appropriate upper indices, we can obtain a vector \( |k,l\rangle' \) with eigenvalues of the operators \( R(I_{21}) \) and \( R(I_{43}) \) coinciding with those of the vector \( |k,l\rangle \). Then \( |k,l\rangle' = a|k,l\rangle \) for some complex number \( a \).

(B) Let \( |k,l\rangle \) be such as in Lemma 1. If \( X_1^{(k,l)}|k,l\rangle = |k,l+1\rangle = 0 \) and \( |k',l\rangle \) is another weight vector of the operators \( R(I_{21}) \) and \( R(I_{43}) \) obtained by action of the operators \( X_i, i = 1, 2, 3, 4 \), with appropriate upper indices, then \( X_1^{(k',l)}|k',l\rangle = |k',l+1\rangle = 0 \). The same assertion is valid for the vectors \( X_2^{(k,l)}|k,l\rangle, X_3^{(k,l)}|k,l\rangle \) and \( X_4^{(k,l)}|k,l\rangle \). This means that by acting on \( |k,l\rangle \) by the operators \( X_i, i = 1, 2, 3, 4 \), with appropriate upper indices, we obtain the set of nonzero vectors \( |k',l\rangle \) such that their values \( (k',l') \) constitute a parallelogram.

First let us consider the subcase when two constituents are equivalent. We denote them by \( R_{jj'} \) and \( R'_{jj'} \). Since restriction of the representation \( R \) to the subalgebra \( U'_q(\text{so}_3) \) is completely reducible, there exists a basis in the space of the representation \( R \) consisting of eigenvectors for both operators \( R(I_{21}) \) and \( R(I_{43}) \). In this basis there exist exactly two vectors of highest weight. Let \( |j,j\rangle \) and \( |j,j\rangle' \) be these vectors. We create two sets of vectors

\[
X_2^rX_4^s|j,j\rangle, \quad r, t = 0, 1, 2, \cdots, \quad \text{and} \quad X_2^rX_4^s|j,j\rangle', \quad r, t = 0, 1, 2, \cdots,
\]

where the operators \( X_2 \) and \( X_4 \) are taken with the appropriate upper indices. Due to the properties of the operators \( X_2 \) and \( X_4 \), these two sets span two subspaces \( V_1 \) and \( V_2 \) which are invariant with respect to the operators \( R(I_{21}), R(I_{32}) \) and \( R(I_{43}) \). Moreover, we have \( V = V_1 \oplus V_2 \) and the theorem is proved in this case.

Now suppose that two constituents (denote them by \( R_{jj'} \) and \( R_{ss'} \)) of our reducible representation \( R \) are not equivalent and that \( R_{jj'} \) is realized in an invariant subspace \( V_1 \). As in the previous subcase, the operators \( R(I_{21}) \) and \( R(I_{43}) \) can be simultaneously diagonalized. We represent the whole representation space \( V \) in the form \( V = V_1 \oplus V_2 \), where \( V_2 \) have a basis consisting of eigenvectors of the operators \( R(I_{21}) \) and \( R(I_{43}) \). Two subcases are possible in this case:

(I) highest weight of the representation \( R_{ss'} \) is not a weight of \( R_{jj'} \);
(II) highest weight of \( R_{ss'} \) is a weight of \( R_{jj'} \).
In the first subcase let \(|s, s'|\)' be a vector of highest weight for the representation \(R_{ss'}\) in the subspace \(V_2\) (its eigenvalue is of multiplicity 1). We create the set of vectors
\[
X_2^r X_4^t |s, s'|', \quad r, t = 0, 1, 2, \cdots,
\]
where \(X_2\) and \(X_4\) are taken with the appropriate upper indices. Then these vectors span a subspace invariant with respect to action of the operators \(X_i, i = 1, 2, 3, 4\), taken with appropriate indices. Therefore, this subspace is invariant with respect to the operators \(R(I_{21}), R(I_{32}), R(I_{43})\), and no of these basis vectors belong to \(V_1\). This means that the representation \(R\) is completely reducible in this subcase.

In the subcase (II) we consider the eigenspace \(V_{k,l}\) of the operators \(R(I_{21})\) and \(R(I_{43})\) with eigenvalues \(k + l\) and \(k - l\), respectively, such that
\[
\dim V_{k,l} = 2, \quad \dim V_{k,l+1} < 2, \quad \dim V_{k+1,l} < 2.
\]
(This means that the highest weight of the representation \(R_{ss'}\) is \((k + l, k - l)\), that is, \(k = s, l = s'\).) Then \(\dim V_{k,l+1} = 1\) or/and \(\dim V_{k+1,l} = 1\). Let \(\dim V_{k+1,l} = 1\) (if \(\dim V_{k,l+1} = 1\), then a proof is the same). Then in \(V_{k,l}\) there exists a vector \(|k, l\)\)' such that \(X_3^{(k,l)}|k, l\)' = 0. Let us show that \(X_3^{(k,l)}|k, l\)' = 0. Suppose that \(X_3^{(k,l)}|k, l\)' = \(\alpha|k, l + 1\)'\), \(\alpha \neq 0\). Then we act successively on the vector \(|k, l + 1\)'\) by the operators \(X_i, i = 1, 2, 3, 4\), with the appropriate upper indices and obtain the set of vectors \(|k', l\)'\) which span invariant subspace \(V'\). Due to the property (B) of the operators \(X_i\), this set of vectors does not contain vectors \(|k + 1, l\)'\) with some values of \(l\). This means that \(V'\) is a non-trivial subspace of the representation space. A highest weight vector of the subspace \(V'\) does not coincide with \((s, s')\) and \((j, j')\). Thus, the whole representation \(R\) contains the third irreducible constituent. It is a contradiction which show that \(X_3^{(k,l)}|k, l\)' = 0.

We constructed the vector \(|k, l\)'\) such that \(X_3^{(k,l)}|k, l\)' = 0 and \(X_3^{(k,l)}|k, l\)' = 0. Let \(V_2\) be a subspace of the representation space \(V\) spanned by the vectors
\[
X_2^r X_4^t |k, l\)'', \quad r, t = 0, 1, 2, \cdots.
\]
According to the properties of the operators \(X_2\) and \(X_4\), then \(V_2\) is invariant subspace for the operators \(R(I_{21}), R(I_{32}), R(I_{43})\) and \(V = V_1 \oplus V_2\), where \(V_1\) is the subspace of the irreducible representation \(R_{jj'}\). Theorem is proved in this subcase.

**Case (b).** We distinguish here the following subcases:

(I) Two constituents are equivalent;

(II) two constituents are of the form \(R_{jj'}^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)}\) and \(R_{jj'}^{(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)}\), where \(\varepsilon_1 \neq \varepsilon'_1\) or/and \(\varepsilon_2 \neq \varepsilon'_2\);

(III) two constituents are of the form \(R_{jj'}^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)}\) and \(R_{jj'}^{(\varepsilon_1, \varepsilon_2, -\varepsilon_3)}\);

(IV) two constituents are \(R_{jj'}^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)}\) and \(R_{ss'}^{(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)}\), \((j, j') \neq (s, s')\).

For subcase (I) a proof is such as in the first part of case (a). Let us consider subcase (II). Let \(\varepsilon_1 \neq \varepsilon'_1\). In the representation space \(V\), there exist two linearly independent vectors \(|j, j'\)\) and \(|j, j'\)'\) which are of highest weights, that is,
\[
X_1^{(j,j')}|j, j\)' = X_1^{(j,j')}|j, j\)' = 0, \quad X_3^{(j,j')}|j, j\)' = X_3^{(j,j')}|j, j\)' = 0,
\]
and such that
\[
R(I_{21})|j, j\)' = \varepsilon_1|j + j\)' + |j, j\)'', \quad R(I_{21})|j, j\)' = \varepsilon'_1|j + j\)' + |j, j\)''.
\]
We create two sets of vectors
\[ X^r_i X^l_j |j, j', \rangle, \quad r, t = 0, 1, 2, \ldots, \quad \text{and} \quad X^r_i X^l_j |j, j', \rangle', \quad r, t = 0, 1, 2, \ldots. \]

No nonzero vector of the first set is multiple of some vector of the second set (since otherwise these two sets span the same vector subspace of the representation space \( V \)). These two sets of vectors span two invariant linear subspaces \( V_1 \) and \( V_2 \) of \( V \). Since \( V = V_1 \oplus V_2 \), then the representation \( R \) is a direct sum of the subrepresentations \( R^{(\epsilon_1, \epsilon_2, \epsilon_3)}_{jj'} \) and \( R^{(\epsilon_1, \epsilon_2, \epsilon_3)}_{jj'} \).

Let us prove the theorem in subcase (III). We suppose that the representation \( R \) has two such sets span the same vector subspace of the representation space \( V \). No nonzero vector of the first set is multiple of some vector of the second set (since otherwise these two sets span two invariant linear subspaces \( V_1 \) and \( V_2 \) of \( V \). Since \( V = V_1 \oplus V_2 \), then the representation \( R \) is a direct sum of the subrepresentations \( R^{(\epsilon_1, \epsilon_2, \epsilon_3)}_{jj'} \) and \( R^{(\epsilon_1, \epsilon_2, \epsilon_3)}_{jj'} \).

The subcase (IV) is proved in the same way as the second part of the previous case.

**Case (c).** First we note that we can diagonalize simultaneously the operators \( T(I_{21}) \) and \( T(I_{43}) \). Let \( V = V_1 \oplus V_2 \) be the decomposition of the representation space into the direct sum of subspaces \( V_1 \) and \( V_2 \) spanned by eigenvectors of the classical types (with eigenvalues of the type \([k + l] \) and \([k - l] \)) and of the nonclassical type (with eigenvalues of the type \([l + k] \) and \([k - l] \)), respectively. Now we take a vector of highest weight in the subspace \( V_1 \) (denote it by \([j, j']\)) and a vector of highest weight in the subspace \( V_2 \) (denote it by \([s, s']\)) and then create two sets of vectors
\[ X^r_i X^l_j |j, j', \rangle, \quad r, t = 0, 1, 2, \ldots, \quad \text{and} \quad X^r_i X^l_j |s, s', \rangle, \quad r, t = 0, 1, 2, \ldots. \]

In fact, they span the vector subspaces \( V_1 \) and \( V_2 \), respectively. Using properties of the operators \( X_2 \) and \( X_4 \), we conclude that the subspaces \( V_1 \) and \( V_2 \) are invariant with respect to the operators \( R(I_{21}), R(I_{32}) \) and \( R(I_{43}) \). Theorem is proved.

**10. TENSOR PRODUCTS OF REPRESENTATIONS**

As in the case of the algebra \( U'_q (so_3) \) (see [21]), the homomorphism of Theorem 2 allows us to determine tensor products of irreducible finite dimensional representations of the algebra \( U'_q (so_4) \) and decompose them into irreducible constituents.

Let us explain this on the example of the classical type irreducible representations of \( U'_q (sl_2) \). Let \( T^{(1,1)}_{jj'} = T^{(1)}_{jj'} \otimes T^{(1)}_{jj'} \) be the irreducible representation of \( U_q (sl_2) \otimes \). Then the tensor
product $T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)}$ is well defined representation of the algebra $U_q(sl_2)^{\otimes 2}$. Thus, we have to determine the operators $(T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)})(x_i)$, $i = 1, 2$. For this we use the determined operators $(T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)})(c_i)$, $i = 1, 2$, where $c_i$ are Casimir elements from section 3. We define the operators $(T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)})(x_i)$ as solutions of equations

$$q^{-1}\{(T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)})(x_i)\}^4 - (T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)})(c_i)(q-q^{-1})^2\{(T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)})(x_i)\}^2 + q = 0 \quad (78)$$

(see the equation for the elements $x_i$ in section 3). In order to find these solutions we may diagonalize the operators $(T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)})(c_i)$. Then solutions of equations (78) can be easily calculated. Composing the representation $T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)}$ of $U_q(sl_2)^{\otimes 2,ext}$ with the homomorphism $\phi$ from Theorem 2 we obtain the representation

$$R_{jj'} \otimes R_{ss'} = \{T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)}\} \circ \phi$$

which is treated as the tensor product of irreducible representations $R_{jj'}$ and $R_{ss'}$ of the algebra $U_q^f(so_4)$.

For the representation $T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)}$ of $U_q(sl_2)^{\otimes 2,ext}$ we have the decomposition

$$T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)} = \sum_{k=|j-s|}^{j+s} \sum_{k'=|j'-s'|}^{j'+s'} \oplus T_{kk'}^{(1,1)}.$$ 

Composing the left and the right hand sides of this relation with the homomorphism $\phi$ from Theorem 2:

$$(T_{jj'}^{(1,1)} \otimes T_{ss'}^{(1,1)}) \circ \phi = \left(\sum_k \sum_{k'} \oplus T_{kk'}^{(1,1)}\right) \circ \phi$$

we obtain the decomposition of the tensor product $R_{jj'} \otimes R_{ss'}$:

$$R_{jj'} \otimes R_{ss'} = \sum_{k=|j-s|}^{j+s} \sum_{k'=|j'-s'|}^{j'+s'} \oplus R_{kk'}.$$ 

As in the case of the algebra $U_q^f(so_3)$ in [21], we can also determine in the similar way tensor products of irreducible representations of the classical and the nonclassical types and tensor products of irreducible representations of the nonclassical type.

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