Unified description of equation of state and transport properties of nuclear matter

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Abstract. Correlated basis function perturbation theory and the formalism of cluster expansions have been recently employed to obtain an effective interaction from a state-of-the-art nucleon nucleon potential model. The approach based on the effective interaction allows for a consistent description of the nuclear matter ground state and nucleon-nucleon scattering in the nuclear medium. This paper reports the results of numerical calculations of different properties of nuclear and neutron matter, including the equation of state and the shear viscosity and thermal conductivity transport coefficients, carried out using the effective interaction.

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INTRODUCTION

The theoretical description of neutron stars requires a quantitative understanding of both equilibrium and non equilibrium properties of cold nuclear matter at high density. While equilibrium properties, e.g. the equation of state (EOS), are generally obtained from realistic dynamical models, strongly constrained from nuclear systematics and nucleon-nucleon scattering data, the studies of non equilibrium behavior, based on the solution of the linearized Boltzmann equation, often resort to oversimplified models of the nucleon-nucleon (NN) interaction.

It has been recently suggested \cite{1, 2, 3} that the many body formalism based on correlated wave functions and cluster expansion techniques provides an ideal framework for the unified treatment of a variety of nuclear matter properties, including the EOS, the dynamic response, the spin susceptibility and the transport coefficients. The main element of the approach of Ref. \cite{1, 2, 3} is a well behaved effective interaction, derived from a state-of-the-art model of nuclear dynamics, suitable for use in standard perturbation theory.

In this paper, after reviewing the underlying assumptions of nonrelativistic nuclear many body theory (NMBT), we briefly discuss the derivation of the effective interaction within the Correlated Basis Function (CBF) approach, and focus on its application to the description of NN scattering in the nuclear medium. The results of numerical calculations of the shear viscosity and thermal conductivity of pure neutron matter, carried out using the in-medium NN cross section within the framework of Landau theory of normal Fermi liquids, are also reported.
FORMALISM

The paradigm of Nuclear Many Body theory

Within Nuclear Many Body Theory (NMBT), nuclear matter is viewed as a collection of pointlike protons and neutrons, whose dynamics are described by the Hamiltonian

\[ H = \sum_i \frac{p_i^2}{2m} + \sum_{j<i} v_{ij} + \ldots \]  

\( m \) and \( p_i \) being the nucleon mass and the momentum of the \( i \)-th particle, respectively. The NN potential \( v_{ij} \) reduces to the well known Yukawa one-pion exchange potential at large distances, while its short and intermediate range behavior is determined by fitting the available experimental data on the two-nucleon system (deuteron properties and 4000 NN scattering phase shifts). The state-of-the-art parametrization of Wiringa et al [4], generally referred to as Argonne \( v_{18} \), is written in the form

\[ v_{ij} = \sum_{n=1}^{18} v^n(r_{ij})O^n_{ij}; \]

where

\[ O^n_{ij} = [1; (\sigma_i \sigma_j ; S_{ij}) ; [1; (\tau_i \tau_j ) ] ]; \]

\( \sigma_i \) and \( \tau_i \) are Pauli matrices acting in spin and isospin space, respectively, and

\[ S_{ij} = \frac{3}{r_{ij}} (\sigma_i \tau_j ; i\gamma)(\sigma_j \tau_i ; i\gamma) \sigma_i \sigma_j : \]

The operators with \( n \leq 4 \) account for the dependence of the NN force on the total spin and isospin of the interacting pair, while \( S_{ij} \) produces the non-central interaction responsible for the nonvanishing electric quadrupole moment of the deuteron. Inclusion of the the operators with \( n = 6 \) is needed to obtain a reasonable description of the two-nucleon bound state. The operators corresponding to \( n = 7;\ldots;14 \) are associated with the non static components of the NN interaction, while those corresponding to \( p = 15;\ldots;18 \) account for charge symmetry violations. The minimal set required to describe NN scattering in \( S \) and \( P \) states (i.e. states of relative angular momentum \( L = 0;1 \)) consists of the operators corresponding to \( n = 8 \), with \( O^n_{ij} = L \cdot S [\tau_i (\tau_j \tau_i \gamma) \sigma_i \cdot \sigma_j \]  

The ellipses in Eq.(1) refer to the presence of interactions involving more than two nucleons. It is long known that the inclusion of a three-nucleon potential \( V_{ijk} \) is needed to reproduce the observed binding energies of the three-nucleon system, as well as the equilibrium density of symmetric nuclear matter [5]. Note that, while at nuclear density \( \sum_{k>j>i} V_{ijk} \) \( \sum_{j>i} v_{ij} \) in the density regime relevant to the description of the neutron-star core many body forces are expected to provide a sizable contribution to the energy.
Correlated Basis Functions

Due to the presence of a strongly repulsive core, the matrix elements of the NN interaction between eigenstates of the non interacting system (Fermi gas (FG) states in nuclear matter) are large. As a consequence, standard perturbation theory is not applicable. This difficulty can be overcome carrying out a resummation of the perturbative expansion, leading to the replacement of the bare interaction with the well behaved G-matrix (for a recent review see, e.g., Ref. \[6\]). An alternative approach, originally developed to study strongly correlated quantum liquids \[7\], is based on the replacement of the FG basis with a basis of correlated states, embodying the nonperturbative effects arising from the short range behavior of the NN force.

The correlated states of nuclear matter are obtained from FG states through

\[
j' \leftarrow n = F n_{FG} \; ;
\]

where the operator \( F \), carrying the correlation structure induced by the NN interaction, is written in the form

\[
F = \mathcal{S} \prod_{ij} f_{ij} \; ;
\]

\( \mathcal{S} \) being the symmetrization operator. The two-body correlation functions \( f_{ij} \), whose operatoral structure reflects the complexity of the NN potential, read

\[
f_{ij} = \sum_{n=1}^{6} f^n (r_{ij}) O^n_{ij} \; ;
\]

with the \( O^n_{ij} \) given by Eq.(3).

The radial functions \( f^n (r_{ij}) \) are obtained from functional minimization of the expectation value of the nuclear hamiltonian (1) in the correlated ground state \( [8] \).

The effective interaction

The effective interaction \( v_{\text{eff}} \) is defined by the relation [1]

\[
\hbar H \hat{\imath} = \frac{\hbar 0 H_{\hat{\imath}}}{\hbar 0 \hat{\imath}} = \hbar 0_{FG} K + v_{\text{eff}} \hat{\imath} \hbar 0_{FG} \hat{\imath} \; ;
\]

where \( H \) is the full nuclear hamiltonian of Eq.(1) and \( K \) is the kinetic energy operator.

In order to include interactions involving more than two nucleons, we have followed the approach originally proposed in Ref. [9], in which the main effect of three- and many-body forces is taken into account through a density dependent modification of the NN potential \( v_{ij} \) at intermediate range. Moreover, in view of the weak model dependence of our results [2], the full \( v_{18} \) potential has been replaced by its reduced form \( v_{8}^{0} [10] \), in which only the components with \( n \geq 8 \) are retained.

The cluster expansion technique [11] allows one to rewrite \( \hbar H \hat{\imath} \) in the form

\[
\hbar H \hat{\imath} = K_{FG} + \sum_{n \geq 2} \langle \Delta E \rangle_{n} \; ;
\]
where $K_{FG}$ is the Fermi gas kinetic energy and $(\Delta E)_n$ is the contribution to the energy arising from $n$-nucleon clusters.

To obtain $v_{\text{eff}}$ from Eq. (8), $\mathbf{H_{\perp}}$ is evaluated at the two-body level of the cluster expansion [1]. The resulting effective interaction reads

$$v_{\text{eff}} = \sum_{i<j} f_{ij}^\dagger \frac{1}{m} (\nabla^2 f_{ij}) + \frac{2}{m} (\nabla f_{ij}) \nabla \nabla v_{ij} f_{ij}.$$  (10)

For any given density, the radial functions $f^n(r_{ij})$ of Eq. (7) are solutions of a set of Euler-Lagrange equations satisfying the boundary conditions $f^1(r_{ij}d) = 1$, $f^n(r_{ij}d) = 0$, for $n = 2, 3$ and $4$, and $f^n(r_{ij}d) = 0$, for $i = 5, 6$ (see, e.g., Ref. [9]). Note that, as the non static terms in Eq. (10) yield a negligibly small contribution to the ground state energy [1], they have been neglected altogether.

The effective interaction of Eq. (10) was tested by computing the energy per particle of symmetric nuclear matter and pure neutron matter in first order perturbation theory using the FG basis. In Fig. 1 our results are compared to those of Refs. [8] and [12]. The calculations of Ref. [8] (solid lines) have been carried out using a variational approach based on the FHNC-SOC formalism, with a Hamiltonian including the Argonne $v_{18}$ NN potential and the Urbana IX three-body potential [5]. The results of Ref. [12] (dashed line of the lower panel) have been obtained using the $v_8^0$ and the same three-body potential within the framework of the Auxiliary Field Diffusion Monte Carlo (AFDMC) approach.

![Figure 1](image_url)

**FIGURE 1.** Energy per particle of symmetric nuclear matter (upper panel) and pure neutron matter (lower panel). The diamonds represent the results obtained using the effective interaction discussed in the text in first order perturbation theory with the FG basis, whereas the solid lines correspond to the results of Akmal, Pandharipande and Ravenhall [8]. The dashed line of the lower panel represents the results of the AFDMC approach or Ref. [12].

The results of Fig. 1 show that the effective interaction provides a fairly reasonable description of the EOS over a broad density range. Note that empirical equilibrium prop-
roperties of symmetric nuclear matter are accounted for without including the somewhat *ad hoc* density dependent correction of Ref. [8]. It should also be emphasized that our approach does not involve adjustable parameters. The correlation ranges $d$ and $d_t$ have been taken from Ref. [8], while the parameters entering the definition of the three-nucleon interaction (TNI) have been determined by the authors of Ref. [9] through a fit of nuclear matter equilibrium properties.

**Transport properties of interacting Fermi systems**

The theoretical description of transport properties of normal Fermi liquids is based on Landau theory [13]. Working within this framework and including the leading term in the low-temperature expansion, Abrikosov and Khalatnikov (AK) [14] obtained approximate expressions for the shear viscosity and the thermal conductivity. Let us consider viscosity, as an example. The AK result reads

$$
\eta_{\text{AK}} = \frac{1}{5} \rho m^2 v_F^2 \tau \frac{2}{\pi^2} \left( \frac{1}{\lambda_{\eta}} \right); \tag{11}
$$

where $\rho$ is the density, $v_F$ is the Fermi velocity and $m^2$ and $\tau$ denote the quasiparticle effective mass and lifetime, respectively. The latter can be written in terms of the angle-averaged scattering probability, $\langle \mathcal{W} \rangle$, according to

$$
\tau T^2 = \frac{8 \pi^4}{m^3} \frac{1}{\langle \mathcal{W} \rangle}; \tag{12}
$$

where $T$ is the temperature and

$$
\langle \mathcal{W} \rangle = \frac{3}{2 \pi} \int d\Omega \frac{W(\theta, \phi)}{\cos(\theta=\frac{\pi}{3})}; \tag{13}
$$

Note that the scattering process involves quasiparticles on the Fermi surface. As a consequence, for any given density $\rho$, $W$ depends only on the angular variables $\theta$ and $\phi$, the magnitude of all quasiparticle momenta being equal to the Fermi momentum $p_F = (3\pi^2 \rho)^{1/3}$. Finally, the quantity $\lambda_{\eta}$ appearing in Eq. (11) is defined as

$$
\lambda_{\eta} = \frac{\langle \mathcal{W} \rangle [1 - 3 \sin^4(\theta=\frac{\pi}{3}) \sin^2 \phi]}{\langle \mathcal{W} \rangle}; \tag{14}
$$

The exact solution of the equation derived in Ref. [14], obtained by Brooker and Sykes [15], reads

$$
\eta = \eta_{\text{AK}} \frac{1}{4} \frac{\lambda_{\eta}}{\sum_{k=0}^{\infty} \frac{4k+3}{(k+1)(2k+1)((k+1)(2k+1)} \left[ \lambda_{\eta} \right]};
$$

the size of the correction with respect to the result of Eq. (11) being $0.750 < \langle \eta=\eta_{\text{AK}} \rangle < 0.925$. 
Eqs. (11)-(15) show that the key element in the determination of the viscosity is the in-medium NN scattering cross section. In Ref. [16], the relation between NN scattering in vacuum and in nuclear matter has been analyzed under the assumption that the nuclear medium mainly affects the flux of incoming particles and the phase space available to the final state particles, while leaving the transition probability unchanged. Within this picture \( W(\theta, \phi) \) can be extracted from the NN scattering cross section measured in free space, \( d\sigma/d\Omega_{\text{vac}} \), according to

\[
W(\theta, \phi) = \frac{16\pi^2}{m^2} \frac{d\sigma}{d\Omega_{\text{vac}}}
\]

where \( m^2 \) is the nucleon effective mass and \( \theta \) and \( \phi \) are related to the kinematical variables in the center of mass frame through \( E_{\text{cm}} = \vec{p}_{E}^2 (1 - \cos \theta) = (2m) \eta m = \phi \) [17].

The above procedure has been followed in Ref. [18], whose authors have used the available tables of vacuum cross sections obtained from partial wave analysis [19]. In order to compare with the results of Ref. [18], and gauge the model dependence of our results, we have carried out a calculation of the viscosity using Eqs. (11)-(15) and the free space neutron-neutron cross section obtained from the Argonne \( v_{18}^{-} \) and \( v_{8}^{-} \) potentials.

![Figure 2](image-url)  
**FIGURE 2.** Differential neutron-neutron scattering cross section at \( E_{\text{cm}} = 100 \text{ MeV} \), as a function of the scattering angle in the center of mass frame. Solid line: cross section in vacuum, calculated with the \( v_{8}^{-} \) potential [10]. Dot-dash line: medium modified cross section obtained from the effective interaction described in the text at \( \rho = 0.08 \text{ fm}^{-3} \). Dashed line: same as the dot-dash line, but for \( \rho = 0.16 \text{ fm}^{-3} \).

In the upper left panel of Fig. 3, we show the quantity \( \eta T^2 \) as a function of density. Our results are represented by the solid line, while the dot-dash line corresponds to the results obtained from Eqs. (43) and (46) of Ref. [18] using the same effective masses, computed from the effective interaction discussed above. The differences between the two curves are likely to be ascribed to the extrapolation needed to determine the cross sections at small angles within the approach of Ref. [18].

To improve upon the approximation of Eq. (15) and include the effects of medium modifications of the NN scattering amplitude, we have replaced the bare NN potential with the CBF effective interaction. Knowing the effective interaction, the in-medium scattering probability can be readily obtained from Fermi’s golden rule. The corresponding cross section at momentum
transfer $q$ reads

$$\frac{d\sigma}{d\Omega} = \frac{m_{\text{eff}}^2}{16\pi^2} \hat{F}_{\text{eff}}(q)^2; \tag{16}$$

$\hat{F}_{\text{eff}}$ being the Fourier transform of the effective potential. The effective mass can also be extracted from the quasiparticle energies computed in Hartree-Fock approximation. For symmetric nuclear matter at equilibrium, we find $m_{\text{eff}}(p_F)=m=0.65$, in close agreement with the lowest order CBF result of Ref. [20].

In Fig. 2 the in-medium neutron-neutron cross section at $E_{\text{cm}}=100$ MeV obtained from the effective potential, with $\rho = \rho_0$ and $\rho_0=2$ ($\rho_0 = 0.16 \text{ fm}^3$ is the equilibrium density of nuclear matter), is compared to the corresponding free space result. As expected, screening of the bare interaction leads to an appreciable suppression of the scattering cross section.

**FIGURE 3.** Left Panel: neutron matter $\eta T^2$ as a function of density. (A). Solid line: results obtained from Eqs.(11)-(15) using the Argonne $v_{18}$ potential and $m_{\text{eff}}$ computed from the effective interaction described in the text. Dot-dash line: results obtained from Eqs.(43) and (46) of Ref. [18] using the same $m_{\text{eff}}$. Dashed line: same as the solid line, but with the Argonne $v_{18}$ replaced by its reduced form $v_{18}'$. (B). Solid line: results obtained using the effective interaction described in the text. Dashed line: $\eta T^2$ obtained from the free space cross section corresponding to the $v_{18}'$ potential. Right panel: neutron matter $\kappa T$ as a function of density. Solid line: calculation carried out using the approach of Ref. [2]; dashed line: results of Baiko, Hänsel & Yakovlev [22].

Replacing the cross section in vacuum with the one defined in Eq.(16), the medium modified scattering probability can be obtained from Eq.(15). The resulting $W(\theta, \varphi)$ can then be used to calculate $\eta T^2$ from Eqs.(11)-(15). Similar expression are found for the thermal conductivity.

The effect of using the medium modified cross section is illustrated in the lower left panel of Fig. 3. Comparison between the solid and dashed lines shows that inclusion of medium modifications leads to a large increase of the viscosity, ranging between a factor 2.5 at half nuclear matter density to a factor 10 at $\rho = 2\rho_0$. Such an increase is likely
to produce appreciable effects on the damping of neutron-star oscillations associated with emission of gravitational waves \[21\].

The right panel of figure 3 shows that medium modification of NN scattering also produce a large effect on the thermal conductivity, $\kappa$. The comparison between our results and those of Ref. \[22\], obtained from the free space NN cross sections, shows a difference in $\kappa T$ exceeding one order of magnitude at $\rho = 2\rho_0$. The impact of this result on neutron star cooling is currently being investigated \[23\].

**SUMMARY**

Working within the formalism based on correlated functions and cluster expansion techniques, we have derived an effective interaction from a realistic NN potential model. Our work improves upon the CBF effective interaction of Ref. \[1\] in that it includes the effects of many-nucleon forces, which become sizable, indeed dominant, in the high density region relevant to the studies of neutron star properties. The energy per nucleon of both symmetric nuclear matter and pure neutron matter, obtained from our effective interaction model, turns out to be in fairly good agreement with the results of highly refined many body calculations, based on similar dynamical models. The emerging picture suggests that our approach captures the relevant physics, allowing for a unified description of a number of different properties of neutron star matter based on standard perturbation theory in the FG basis.

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