The Linearly Independent Non Orthogonal yet Energy Preserving (LINOEP) vectors

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Abstract

It is well known that, in any inner product space, a set of linearly independent (LI) vectors can be transformed to a set of orthogonal vectors, spanning the same space, by the Gram-Schmidt Orthogonalization Method (GSOM). In this paper, we propose a transformation from a set of LI vectors to a set of LI non orthogonal yet energy (square of the norm) preserving (LINOEP) vectors in an inner product space and we refer it as LINOEP method. We also show that there are various solutions to preserve the square of the norm.

Index terms— The Gram-Schmidt Orthogonalization Method (GSOM), Linearly Independent Non Orthogonal yet Energy Preserving (LINOEP) vectors, Empirical mode decomposition (EMD).

1 Introduction

The Gram-Schmidt Orthogonalization Method (GSOM) is a process for obtaining a set of orthogonal vectors, from a set of linearly independent (LI) vectors in an inner product space. The old set of LI vectors and the new set of orthogonal vectors have the same linear span. Let $Y = \{y_1, y_2, \ldots, y_n\}$ be a set of $n$ LI vectors. A set of orthogonal vectors $S = \{s_1, s_2, \ldots, s_n\}$ is generated from the set $Y$ as follows (for $k = 1, 2, \ldots, n$):

$$s_k = y_k - \sum_{i=1}^{k-1} c_{ki} s_i \iff \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ c_{21} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$ (1)

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The $c_{ki}$ is obtained by using inner product $\langle s_k, s_i \rangle = 0$, for $k \neq i$, i.e. $c_{ki} = \frac{\langle y_k, s_i \rangle}{\langle s_i, s_i \rangle}$ for $i = 1, 2, \ldots, n$, and $k \geq i$. By taking sum of all the $n$ equations of (1) along with some simple algebraic manipulations, it can be shown that

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} c_i s_i$$

(2)

where $c_i = \sum_{k=i}^{n} c_{ki}$ is sum of $i^{th}$ column of the coefficient matrix of (1). It can be easily shown that $c_{ki} = 1$, if $k = i$. From (2), it is easy to show (Plancherel/Parseval Equality)

$$\left\| \sum_{i=1}^{n} y_i \right\|^2 = \left\| \sum_{i=1}^{n} c_i s_i \right\|^2 = \sum_{i=1}^{n} |c_i|^2 \|s_i\|^2$$

(3)

From set $Y$, there are $n$ choices for selecting first vector, $n - 1$ choices for second vector, $n - 2$ choices for third vector and 1 choice for last vector, that means there are $n!$ permutations of the set $Y$, and the GSOM would produce $n!$ orthogonal sets of vectors from a set of $n$ LI vectors.

The empirical mode decomposition (EMD) is an adaptive signal analysis algorithm, introduced in [1], for the analysis of nonlinear and non stationary time series. The various variants of EMD algorithm are proposed in literature [2–4] and the orthogonal property of intrinsic mode functions (IMFs) are discussed in [5,6]. The LI non orthogonal yet energy (square of the norm) preserving (LINOEP) class of vectors and the following theorem are proposed, in [7], for the development of energy preserving EMD (EPEMD) algorithm.

**Theorem 1.** Let $H$ be a Hilbert space over the field of complex numbers, and let $\{x, x_1, \cdots, x_n\}$ be a set of vectors satisfying the following conditions:

$(i)$ \quad $x_i \perp \sum_{j=i+1}^{n} x_j$ \hspace{1cm} (4)

$(ii)$ \quad $x = \sum_{i=1}^{n} x_i$ \hspace{1cm} (5)

Then in the representation, given in (5), the square of the norm, and hence energy is preserved, i.e.

$$\|x\|^2 = \left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

(6)

In this result, pairwise orthogonality is not required and only last two vectors (i.e. $x_{n-1}$ and $x_n$) are orthogonal. It is the Pythagoras’s theorem for $n = 2$, and it is the Parseval’s theorem when all the basis vectors $x_i$ are orthogonal. We use the LINOEP class of vectors, the above theorem and the GSOM for development of LINOEP method in next section.
2 The LINOEP Method

There is the GSOM for transformation of a set of linearly independent (LI) vectors to a set of orthogonal vectors in an inner product space. We, motivated by the GSOM, propose the transformation from a set of LI vectors to a set of LINOEP vectors in an inner product space.

Let \( Y = \{y_1, y_2, \ldots, y_n\} \) be a set of LI vectors. This method generates a set of LI non orthogonal yet energy preserving (LINOEP) vectors \( S = \{c_1, c_2, \ldots, c_n\} \) from a set \( Y \) as follows (for \( k = 1, 2, \ldots, n - 1 \)):

\[
   c_k = y_k - \alpha_k \sum_{i=k+1}^{n} c_i, \quad c_n = y_n
\]

The values of \( \alpha_k \) are obtained by equation:

\[
   \alpha_k = \frac{\langle y_k; \sum_{i=k+1}^{n} c_i \rangle}{\langle \sum_{i=k+1}^{n} c_i; \sum_{i=k+1}^{n} c_i \rangle} \quad k = n - 1, \ldots, 2, 1
\]

where \( c_i \perp \sum_{j=i+1}^{n} c_j \), i.e., inner product \( \langle c_i, \sum_{j=i+1}^{n} c_j \rangle = 0 \) for \( i = 1, 2, \ldots, n - 1 \). The generated vectors \( c_i \) are LINOEP, thus,

\[
   \left\| \sum_{i=1}^{n} c_i \right\|^2 = \sum_{i=1}^{n} \|c_i\|^2.
\]

By taking sum of all the \( n \) equations of (7), along with algebraic manipulation, it can be easily shown that \( \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} c_i + \sum_{i=1}^{n-1} \beta_i c_{i+1} \), where \( \beta_i = \sum_{j=1}^{i} \alpha_j \), for \( i = 1, \ldots, n - 1 \). Let, \( p_2 = \sum_{i=1}^{n} c_i \) and \( p_1 = \sum_{i=1}^{n-1} \beta_i c_{i+1} \), and further let, \( z_1 = p_2 \) and \( z_2 = p_1 - \gamma z_1 \), where \( \gamma = \frac{\langle p_2, z_1 \rangle}{\|p_2, z_1\|} \) such that \( z_1 \) and \( z_2 \) are orthogonal. Through the addition of \( z_1 \) and \( z_2 \), we obtain \( p_1 + p_2 = (1 + \gamma)z_1 + z_2 \), and, \( \sum_{i=1}^{n} y_i = (1 + \gamma)z_1 + z_2 \). It can be easily shown that

\[
   \sum_{i=1}^{n} y_i = \sum_{i=1}^{n+1} d_i,
\]

where \( d_i = (1 + \gamma)c_i \) for \( i = 1, \ldots, n \) and \( d_{n+1} = z_2 \). From (9), (11), we obtain

\[
   \left\| \sum_{i=1}^{n} y_i \right\|^2 = \left\| \sum_{i=1}^{n+1} d_i \right\|^2 = \sum_{i=1}^{n+1} \|d_i\|^2
\]

There are \( n! \) permutations of the set \( Y \), and hence this method would produce \( n! \) sets of LINOEP vectors from a set of \( n \) LI vectors. In this method, we calculate \( n \) coefficients (i.e., \( n - 1 \) values of \( \alpha_i \) and one value of \( \gamma \)) and the only \( x = \delta_1 c_1 + \delta_2 c_2 + c_3 + \cdots + c_n \) linear combinations are energy preserving and all other linear combinations are not energy preserving.
We observe that if vectors $\mathbf{y}_i \in \mathbb{C}^m$ and $m \geq n$, and all the vectors $\{\mathbf{y}_i\}_{i=1}^n$ are LI, then $(n)$ out of $(n+1)$ generated vectors $\{\mathbf{d}_i\}_{i=1}^n$ are LI, and last vector $\mathbf{d}_{n+1}$ is linear combination of $\{\mathbf{d}_i\}_{i=1}^n$ vectors, and the complete set is non orthogonal yet energy preserving (NOEP).

![Figure 1](image1.png)

**Figure 1:** Three non orthogonal vectors $c_1, c_2, c_3$ such that $c_1 \perp (c_2 + c_3)$ and vector $s = c_1 + c_2 + c_3$ in 3-D.

![Figure 2](image2.png)

**Figure 2:** Three non orthogonal vectors $c_1, c_2, c_3$ such that $\langle c_2, c_3 \rangle + \langle c_1, c_2 \rangle = -\langle c_1, c_3 \rangle$ and vector $s = c_1 + c_2 + c_3$ in 3-D.
3 Various solutions to preserve the square of norm

It is easy to show that

$$\left\| \sum_{i=1}^{n} c_i \right\|^2 = \sum_{i=1}^{n} \|c_i\|^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \langle c_i, c_j \rangle$$

and to preserve the energy (square of the norm), cross terms are set to zero, i.e.,

$$\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \langle c_i, c_j \rangle = 0.$$

Assuming the underlying field to be $\mathbb{R}$, we write

$$\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \langle c_i, c_j \rangle = 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \langle c_i, c_j \rangle = 0.$$

There are various possible solution to make the cross term zero in Eq. (14):

1. vectors are pairwise orthogonal, i.e., $\langle c_i, c_j \rangle = 0$ for $i \neq j$.
2. use Eq. (4), i.e.,
   $$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \langle c_i, c_j \rangle = 0.$$ 
   There are $n$ choices for selecting first vector, $n - 1$ choices for second vector, $n - 2$ choices for third vector and 3 choices for last three vectors, hence, this gives $\frac{n!}{2}$ ways to make cross term zero.
3. There are so many other ways to make the cross term zero which can be obtained.

Example 1. For $n = 2$, there is only one solution to make the cross term zero and this is $\langle c_1, c_2 \rangle = 0$.

Example 2. For $n = 3$, there are 7 possible solutions to make the cross term zero:

1. Obvious solution is $\langle c_1, c_2 \rangle = 0, \langle c_2, c_3 \rangle = 0, \langle c_1, c_3 \rangle = 0$, and this gives one solution.
2. use Eq. (11) and this gives three solutions: (a) $\langle c_1, c_2 + c_3 \rangle = 0$ and $\langle c_2, c_3 \rangle = 0$ (e.g. see Figure 1). (b) $\langle c_2, c_1 + c_3 \rangle = 0$ and $\langle c_1, c_3 \rangle = 0$. (c) $\langle c_3, c_1 + c_2 \rangle = 0$ and $\langle c_1, c_2 \rangle = 0$.
3. obtain vectors such that inner product of two vectors cancel all others, i.e., $\langle c_1, c_2 \rangle + \langle c_1, c_3 \rangle + \langle c_2, c_3 \rangle = 0$, this gives three solutions: (a) $\langle c_1, c_2 \rangle + \langle c_1, c_3 \rangle = -\langle c_2, c_3 \rangle$. (b) $\langle c_1, c_3 \rangle + \langle c_2, c_3 \rangle = -\langle c_1, c_2 \rangle$. (c) $\langle c_2, c_3 \rangle + \langle c_1, c_2 \rangle = -\langle c_1, c_3 \rangle$ (e.g. see Figure 2).

Similarly, when we assume the underlying field to be $\mathbb{C}$, various possible solutions can be obtained to make the cross term zero in Eq. (13).

4 Conclusions

We have proposed the transformation from a set of linearly independent (LI) vectors to a set of LI non orthogonal yet energy (square of the norm) preserving (LINOEP) and non orthogonal yet energy preserving (NOEP) vectors in an inner product space. We have also shown that there are various solutions to preserve the square of the norm.
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References

[1] Huang N. E., Shen Z., Long S., Wu M., Shih H., Zheng Q., Yen N., Tung C., and Liu H. 1998 The empirical mode decomposition and Hilbert spectrum for non-linear and non-stationary time series analysis. Proc. R. Soc. A, 454, 903-995.

[2] Wu Z. and Huang N. E. 2009 Ensemble Empirical Mode Decomposition: a noise-assisted data analysis method. Advances in Adaptive Data Analysis, Vol. 1, No. 1, pp. 1-41.

[3] Rehman N. and Mandic D. P. 2010 Multivariate empirical mode decomposition. Proc. R. Soc. A, 466, 1291-1302, (doi:10.1098/rspa.2009.0502).

[4] Singh P., Srivastavay P.K., Patney R.K., Joshi S.D. and Saha K. 2013 Nonpolynomial Spline Based Empirical Mode Decomposition. 2013 International Conference on Signal Processing and Communication, pp. 435-440.

[5] Huang T., Ren W. and Lou M. 2008 The orthogonal Hilbert-Huang transform and its application in earthquake motion recording analysis. The 14th World Conference on Earthquake Engineering October 12-17, Beijing, China.

[6] Singh P., Patney R.K., Joshi S.D. and Saha K., “Some studies on nonpolynomial interpolation and error analysis,” Applied Mathematics and Computation, (2014), vol. 244, pp. 809-821.

[7] Singh P., Patney R.K., Joshi S.D. and Saha K., “The Hilbert spectrum and the Energy Preserving Empirical Mode Decomposition,” Signal Processing, submitted.