Free Vibrations of Plates and Shells with an Isogeometric RM Shell Element

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Abstract Free vibration analysis of plates and shells is carried out by using isogeometric approach. For this purpose, an isogeometric shell element based on Reissner-Mindlin (RM) shell theory is developed. Non-uniform rational B-spline surface (NURBS) definition is introduced to represent the geometry of shell and it is also used to derive all terms required in the isogeometric element formulation. New anchor positions are proposed to calculate the shell normal vector. Gauss integration rule is used for the formation of stiffness and mass matrices. The proposed shell element is then used to examine vibrational behaviours of plate and shell structures. From numerical results, it is found to be that reliable natural frequencies and associated mode shapes can be predicted by the present isogeometric RM shell element.

Keywords: Reissner-Mindlin, Shell element, Isogeometric Analysis, Free Vibration, NURBS

1. INTRODUCTION

The thorough understanding on the vibrational behaviour of curved shells is crucial in structural design process. Its importance naturally leads us to the researches on the shell vibrations. The first attempt was made by Love (1888) and Rayleigh (1894) and then Flügge (1934) provided numerical solution of shell vibrations using frequency equation. Numerical analysis techniques were introduced (Leissa, 1973; Qatu, 1992; Liew, 1995) and various types of shells with complex geometry have been solved. With more frequent appearance of shells in various engineering field, the finite element (FE) technologies are intensively used to find out the vibrational characteristics of shell.

Recently, the isogeometric analysis (IGA) concept has been introduced and it gradually applied into the vibrational problems. Elastic rod was analyzed to investigate the performance of NURBS based FE in structural vibration. The order of basis function and three refinement methods were tested in one dimensional situation (Hughes and Evans, 2010). The convergence rate of the isogeometric bar element was also investigated with different boundary conditions (Lee and Park, 2011).

More recently, the investigations on the vibrational behaviours of bending dominant structures such as beam and plate using IGA were also carried out by some researchers. Timoshenko beam theory was adopted to formulate the isogeometric beam element (Lee and Park, 2013) which can apply into both thin and thick beams and rotatory inertia effect was investigated with various beam problems to show the performance of isogeometric beam element. Reissner-Mindlin theory was used to formulate the isogeometric plate element (Lee and Kim, 2013; Kim and Lee) and it was extended into the element for laminated plates (Lee, 2014).

As a sequel of previous study on the bending dominant structures, we here propose an isogeometric RM shell element to calculate the natural frequencies and associated mode shapes for both thin and thick shells. The NURBS definition is used to represent the geometry of shell and all terms required in the derivation of shell element. New anchor positions are introduced to calculate the shell normal at each control points for shell vibrations. Gauss integration rules are used for the formation of stiffness and mass matrices. Some benchmark tests are carried out to see the accuracy of the RM shell element we provide here. Numerical results are then provided as the future reference solution based on isogeometric approach for the free vibration analysis of plate and shell structures.

2. B-SPLINES

2.1 Knot vector

A knot vector $\Xi$ is a set of non-decreasing real values that constitutes a set of coordinates in the parametric space:

$$\Xi = [\xi_1, \xi_2, \xi_3, \ldots, \xi_{n+p+1}]$$

where $n$ is the number of basis functions and $p$ is the order of the B-spline. A knot vector is said to be uniform if its knots are uniformly spaced and non-uniform otherwise. Moreover, a knot vector is said to be open if its first and last knots are repeated.
Basis functions formed from open knot vectors are interpolatory at the ends of the parametric interval \([\xi_i, \xi_{i+p+1}]\) but are not, in general, interpolatory at interior knots. It should be noted that we would employ open knot vectors throughout the analysis of plate and shell structures.

### 2.2 Basis functions

B-spline basis function (De Boor, 1978) is defined recursively starting with \(p = 0\) as:

\[
N_{i,0}(\xi) = \begin{cases} \frac{1}{\xi_t - \xi} & \text{if } \xi_t \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}
\]

For \(p \geq 1\),

\[
N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi).
\]

### 2.3 B-spline curves

We construct \(n\) basis functions with the order of a B-spline and an appropriately defined knot vector. The piecewise polynomial B-spline curve \(S(\xi)\) of order \(p\) can be obtained by taking a linear combination of basis function and control points:

\[
S(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) C_i
\]

where \(C_i\) is the \(i^{th}\) control point. The piecewise linear interpolation of the control points defines the control net.

### 2.4 B-spline surface

We construct a B-spline surface using tensor product spline curve in the direction of \(\xi\) and \(\eta\):

\[
S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi, \eta) C_{i,j}
\]

### 2.5 NURBS

NURBS can be defined as

\[
S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi, \eta) C_{i,j}
\]

where \(R_{i,j}^{p,q}\) can be written as

\[
R_{i,j}^{p,q}(\xi, \eta) = \frac{N_{i,p}(\xi) N_{j,q}(\eta) w_{i,j}}{\sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi) N_{j,q}(\eta) w_{i,j}}
\]

and \(w_{i,j}\) is the weight associated with the control point.

---

Figure 1. Shell configuration
3. KINEMATICS OF SHELL

3.1 Kinematics

In three dimensional space ($\mathbb{R}^3$), the geometry of the shell element can be defined by two vectors at each nodal point as shown in Figure 1. One vector $\mathbf{x}(\xi_1, \xi_2)$ expresses the positions of shell mid-surface which could be denoted as $[\xi_1, \xi_2, \mathbf{x} = \mathbb{R}^3]$ and the other vector $\mathbf{d}$, which is called as the unit normal vector, expresses any position between the top surface and the bottom surface of the shell in thickness direction $[\xi_3 \rightarrow \mathbb{R}]$.

After the element discretization on the mid-surface of the continuum shell, the initial configuration of the shell element having constant thickness $h$ can be written as

$$\mathbf{x}(\xi_1, \xi_2, \xi_3) = \sum_{a=1}^{n \times m} R_a(\xi_1, \xi_2) \left[ \mathbf{x}^a + \frac{\xi_3}{2} h a \mathbf{d}^a \right]$$  \hspace{1cm} (7)

where $R_a$ is the NURBS basis function associated with the control point $a$. Note that the one-dimensional index $a$ as a pointer can be expressed as $a = \mathbf{m}(l - 1) + j$ and therefore the $R_a$ of (7) can be interpreted as $R_{lj}$ of (6).

If the $n$-control point isogeometric shell element is used in this study, the position vectors $\mathbf{x}^a$ which have three Cartesian components $\mathbf{x}^a_i (i = 1, 2, 3)$ and unit normal vectors $\mathbf{d}^a$ will be set up at $n$ control points for the present shell element. These unit normal vectors, which are normal to the mid-surface, at the control points are evaluated as

$$\mathbf{d}^a = \frac{\mathbf{n}^a}{|\mathbf{n}^a|}$$  \hspace{1cm} (8)

in which $\mathbf{n}^a = [\mathbf{x}^a_1 \times \mathbf{x}^a_2]$.

3.2 Displacement definition

The displacement field $\mathbf{u}$ used in the present shell element having five degrees of freedom per node (Ahmad, 1970) can be defined as

$$\mathbf{u}(\xi_1, \xi_2, \xi_3) = \sum_{a=1}^{n \times m} R_a(\xi_1, \xi_2) \left[ \mathbf{u}^a + \xi_3 h \mathbf{v}^a \alpha^a \right]$$  \hspace{1cm} (9)

where, the transformation matrix is $\mathbf{v}^a = [\mathbf{v}^a_1, \mathbf{v}^a_2, \mathbf{v}^a_3]$, the translational displacement vector is $\mathbf{u}^a = [\mathbf{u}^a_1, \mathbf{u}^a_2, \mathbf{u}^a_3]$ and the rotational displacement vector is $\alpha^a = [\alpha^a_1, \alpha^a_2]$.

3.3 The position of shell normal

The isogeometric approach uses the control points which play the same roles as the nodal points in the FE method. However, the control points are not always located on the physical surface so that the establishment of nodal coordinates system at the control points could be troublesome. Therefore, the so-called anchor has to be determined for the establishment of nodal coordinates system. Each anchor on the physical curved surface is directly associated with each control points. In other words, the number of anchor is the same as the number of control points. The most popular way to determine the location of anchor points is the anchor equation (Hughes et al., 2005). However, in this study, the refinement (or knot insertion) algorithm suggested by Boehm and Prautzsch (Piegl and Tiller, 1997) is introduced to produce new anchor positions for the present isogeometric shell element.

Figure 2 illustrates the locations of the anchor points (×) produced by the present method together with the anchor points (□) produced by the anchor equation (Hughes et al., 2005). The both methods produce a similar anchor point distribution but there is distinctive difference in the ends of the parametric space which means the points close to 0 or 1 in the parametric space. The anchors illustrated in Figure 2 are calculated by using equally spaced knot vector in 1-dimensional parametric space. It will be then used to calculate the anchors located in real curved surface. Note that both set of anchor points produced by the anchor equation and the present method are identical when the order $p = 2$ is used. The numerical values of anchor positions are summarized in Table 1.

| $\xi_3$ | Anchor equation | Present (h-refinement) |
|---|---|---|
| 1 | 0.0000 | 0.0000 | 0.0000 |
| 2 | 0.0000 | 0.0000 | 0.0000 |
| 3 | 0.0000 | 0.03125 | 0.0625 |
| 4 | 0.0625 | 0.09375 | 0.1250 |
| 5 | 0.1250 | 0.15625 | 0.1875 |
| 6 | 0.1875 | 0.21875 | 0.2500 |
| 7 | 0.2500 | 0.28125 | 0.3125 |
| 8 | 0.3125 | 0.34375 | 0.3750 |
| 9 | 0.3750 | 0.40625 | 0.4375 |
| 10 | 0.4375 | 0.46875 | 0.5000 |
| 11 | 0.5000 | 0.53125 | 0.5625 |
| 12 | 0.5625 | 0.59375 | 0.6250 |
| 13 | 0.6250 | 0.65625 | 0.6875 |
| 14 | 0.6875 | 0.71875 | 0.7500 |
| 15 | 0.7500 | 0.78125 | 0.8125 |
| 16 | 0.8125 | 0.8375 | 0.8750 |
| 17 | 0.8750 | 0.90625 | 0.9375 |
| 18 | 0.9375 | 0.96875 | 1.0000 |
| 19 | 1.0000 | 1.0000 | 1.0000 |
| 20 | 1.0000 | 1.0000 | 1.0000 |
| 21 | 1.0000 | 1.0000 | 1.0000 |
To see the clear difference between the present method and the anchor equation, the present technique is utilized in 2-dimensional parametric space and the results are illustrated together with those produced by the anchor equation in Figure 3.

After four successive refinements, all the control points are mostly being located on the physical curve by using both methods. The present technique produces 21 anchors which are close to the position of control points. But the anchor equation (Hughes et al., 2005) can produce only 17 unique anchor points. Therefore the anchor equation cannot properly represent the physical situation of normal vectors on the curves.

![Image](image.png)

Figure 3. The comparison between the anchor equation (Hughes et al., 2005) and the present technique in 2-dimensional parametric space

4. RM SHELL ELEMENT FORMULATION

4.1 Basic assumptions

The present isogeometric shell element is formulated by using the following Reissner-Mindlin (RM) assumptions:

1. The normal to the mid-surface remains straight after deformation but not necessarily normal to the deformed mid-surface.
2. The normal transverse stress is negligible as in Kirchhoff-Love theory.

Numerical integration through the thickness direction is facilitated by the first assumption and the constitutive equation is simplified by the second assumption. Note that the above assumptions can be more effectively used with a specific local coordinate system.

4.2 Strain definition

The linear strain terms of shell is usually defined in local coordinate systems ($x_i'$):

$$
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_{i}'}{\partial x_j'} + \frac{\partial u_{j}'}{\partial x_i'} \right) = \frac{1}{2} \left( u_{i,j}' + u_{j,i}' \right). \quad (10)
$$

The deformation gradient in local coordinate system can be derived in the following way:

$$
\begin{bmatrix}
  u_{1,1}' & u_{2,1}' & u_{3,1}' \\
  u_{1,2}' & u_{2,2}' & u_{3,2}' \\
  u_{1,3}' & u_{2,3}' & u_{3,3}'
\end{bmatrix} = \Theta^T \begin{bmatrix}
  u_{1,1} & u_{2,1} & u_{3,1} \\
  u_{1,2} & u_{2,2} & u_{3,2} \\
  u_{1,3} & u_{2,3} & u_{3,3}
\end{bmatrix} \Theta \quad (11)
$$

where and $\Theta = [\theta_{i,j}]_{3 \times 3}$ and $\theta_{i,j} = \partial x_i / \partial x_j$. The derivatives of the displacements with respect to the local coordinates are given by

$$
\begin{bmatrix}
  u_{2,1} & u_{2,1} & u_{3,1} \\
  u_{1,2} & u_{2,2} & u_{3,2} \\
  u_{1,3} & u_{2,3} & u_{3,3}
\end{bmatrix} = J^{-1} \begin{bmatrix}
  \tilde{u}_{1,1} & \tilde{u}_{2,1} & \tilde{u}_{3,1} \\
  \tilde{u}_{1,2} & \tilde{u}_{2,2} & \tilde{u}_{3,2} \\
  \tilde{u}_{1,3} & \tilde{u}_{2,3} & \tilde{u}_{3,3}
\end{bmatrix} \quad (12)
$$

where $\tilde{u}_{i,j} = \partial u_i / \partial x_j$ and the Jacobi matrix $J$ is

$$
J = \begin{bmatrix}
  \tilde{x}_{1,1} & \tilde{x}_{2,1} & \tilde{x}_{3,1} \\
  \tilde{x}_{1,2} & \tilde{x}_{2,2} & \tilde{x}_{3,2} \\
  \tilde{x}_{1,3} & \tilde{x}_{2,3} & \tilde{x}_{3,3}
\end{bmatrix} \quad (13)
$$

in which $\tilde{x}_{i,j} = \partial x_i / \partial x_j$.

4.3 Constitutive equation

The constitutive equation for isotropic material can be written as

$$
\sigma_f = D_f e_f; \quad \sigma_s = D_s e_s \quad (14)
$$

where the rigidity matrices $D_f$ and $D_s$ are

$$
D_f = \begin{bmatrix}
  \lambda + 2\mu & \lambda & 0 \\
  \lambda & \lambda + 2\mu & 0 \\
  0 & 0 & \mu
\end{bmatrix} \quad \text{sym.} \quad D_s = \begin{bmatrix}
  k_s & 0 \\
  0 & \kappa_{s,\mu}
\end{bmatrix} \quad (15)
$$

where $k_s$ is a shear correction factor, $\mu$ is the shear modulus and $\lambda, \mu = \nu E / (1 - \nu^2)$ is the reduced Lamé constant. $E$ is the modulus of elasticity and $\nu$ is Poisson ratio.

4.4 Equilibrium equation

In the absence of external load and damping effects, the dynamic equilibrium equation based on principle of virtual work can be written as

$$
\int_{\Omega} \delta \epsilon_p D_p e_p \, d\Omega + \int_{\Omega} \delta \epsilon_s D_s e_s \, d\Omega = \int_{\Omega} \delta u^T \rho \ddot{u} \, d\Omega \quad (16)
$$

where $u$ is the displacement, $\ddot{u}$ is the acceleration, $\rho$ is the density of material and the notation $\delta$ denotes that the terms are virtual.

The relevant derivation takes place in finite-dimensional subspace to turn the above virtual statement of the problem into a system of algebraic equations. In this study, the subspaces are defined by
using the NURBS basis:

\[
\mathbf{u} = \sum_{a=1}^{n \times m} N_a(\xi_1, \xi_2, \xi_3) \mathbf{u}_a
\]

\[
\ddot{\mathbf{u}} = \sum_{a=1}^{n \times m} N_a(\xi_1, \xi_2, \xi_3) \ddot{\mathbf{u}}_a , \tag{17}
\]

where \(n \times m\) is the total number of the control point in the discretized domain and the virtual terms associated with the displacement and the acceleration are

\[
\delta \mathbf{u} = \sum_{a=1}^{n \times m} N_a(\xi_1, \xi_2, \xi_3) \delta \mathbf{u}_a
\]

\[
\delta \ddot{\mathbf{u}} = \sum_{a=1}^{n \times m} N_a(\xi_1, \xi_2, \xi_3) \delta \ddot{\mathbf{u}}_a , \tag{18}
\]

Using (17), the strains of (10) can be rewritten in the form of the strain-displacement relation matrix \(B\) as follows

\[
\epsilon_p = \sum_{a=1}^{n \times m} B_p^a \mathbf{u}_a^a ; \quad \epsilon_s = \sum_{a=1}^{n \times m} B_s^a \mathbf{u}_a^a \tag{19}
\]

Substituting (17), (18) and (19) into (16) yields

\[
\delta \mathbf{u}^T[Ku - \ddot{M}\mathbf{u}] = 0 . \tag{20}
\]

Since the virtual displacement \(\delta \mathbf{u}\) is arbitrary, the above equation may be written as

\[
Ku - \ddot{M}\mathbf{u} = 0 . \tag{21}
\]

A general solution of (21) may be written

\[
\mathbf{u} = \Phi_k e^{\omega_k t} . \tag{22}
\]

Substituting (22) into (21) yields

\[
[K - \omega_k^2M] \Phi_k = 0 \tag{23}
\]

where \(\Phi_k\) is a set of displacement-type amplitude at the control points otherwise known as the mode vector. \(\omega_k\) is the natural frequency associated with the \(k^{th}\) mode and \(K\) and \(M\) are global stiffness and mass matrices which contain contributions from element stiffness and mass matrices.

The structural stiffness and mass matrices in (23) can be written as

\[
K = K^{ab} + K_s^{ab} = \int_{\Omega} B_p^a D_p B_p^b \, d\Omega + \int_{\Omega} B_s^a D_s B_s^b \, d\Omega , \tag{24}
\]

\[
M = M^{ab} = \int_{\Omega} N_a^T \rho N_b \, d\Omega . \tag{25}
\]

Let the span is assumed to be as an isogeometric element and then the above equation can be written in the knot coordinate system as follows

\[
\begin{align*}
&\sum_{e=1}^{n_{el}} \int_{k_e}^{k_{e+1}} \left[K^{ab}\right]_{e} \, d\xi \, d\eta \\
&\quad = \sum_{e=1}^{n_{el}} \int_{k_e}^{k_{e+1}} \left[B_k^a D B_k^b\right] \det(J) \, d\xi \, d\eta \\
&\quad + \sum_{e=1}^{n_{el}} \int_{k_e}^{k_{e+1}} \left[B_k^a D B_k^b\right] \det(J) \, d\xi \, d\eta, \tag{26}
\end{align*}
\]

\[
\begin{align*}
&\sum_{e=1}^{n_{el}} \int_{k_e}^{k_{e+1}} \left[M^{ab}\right]_{e} \, d\xi \, d\eta \\
&\quad = \sum_{e=1}^{n_{el}} \int_{k_e}^{k_{e+1}} \left[N_a^T M N_b\right] \det(J) \, d\xi \, d\eta. \tag{27}
\end{align*}
\]

where \(U\) is the FE assembly operator (Hughes, 1987), \(n_{el} (= n_{span} \times n_{span} )\) is the number of element, \([K_e k_{e+1}]\) is knot interval for integration in the \(\xi\) and \(\eta\). \(J\) is Jacobian matrix between \(x\) and \(\xi\), \(a, b\) is the basis function number associated with the target element and the matrix \(\mathbf{m}\) is

\[
\mathbf{m} = \begin{bmatrix} \mathbf{m} & 0 \\
0 & \mathbf{m} \end{bmatrix}
\tag{28}
\]

where the submatrices are

\[
\mathbf{m} = \begin{bmatrix} \rho & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & \rho \end{bmatrix} ; \quad \mathbf{m} = \begin{bmatrix} \rho & 0 \\
0 & \rho \end{bmatrix} . \tag{29}
\]

The element mass matrix \(M^{ab(e)}\) can be written as

\[
M^{ab(e)} = \begin{bmatrix} \mathbf{m}^{ab(e)} & 0 \\
0 & \mathbf{m}^{ab(e)} \end{bmatrix} \tag{30}
\]

where the transitional mass matrix and the rotational mass matrix linking control points \(a\) and \(b\) can be written as

\[
\begin{align*}
\mathbf{m}^{ab(e)} &= \int_{V(e)} \begin{bmatrix} \mathbf{R}^{ab} & 0 & 0 \\
0 & \mathbf{R}^{ab} & 0 \\
0 & 0 & \mathbf{R}^{ab} \end{bmatrix} \, dV \tag{31}
\end{align*}
\]

\[
\mathbf{m}^{ab(e)} = \int_{V(e)} \begin{bmatrix} \sum_{i=1}^{3} q_{i1}^{ab} & 0 & 0 \\
0 & \sum_{i=1}^{3} q_{i1}^{ab} & 0 \\
0 & 0 & \sum_{i=1}^{3} q_{i1}^{ab} \end{bmatrix} \, dV \tag{32}
\]

where \(q_{i1}^{ab}\) and \(q_{i1}^{ab}\) are the \(i^{th}\) components of the vectors \(\mathbf{q}_i^a\) and \(\mathbf{q}_i^b\) respectively. \(\mathbf{R}^{ab} = R_a R_b\) and \(q_{i1}^{ab} = v_{i1}^{a} v_{i1}^{b} \xi_3 h^a h^b \mathbf{R}^{ab} / 4\).

5. NUMERICAL EXAMPLES

In this section, we analyze the plates and shells by using the isogeometric RM shell element. The lowest frequencies of shells are calculated by using subspace iteration method and the results are compared to the existing reference solutions (Kanok-nukulchai, 1979; Lee and Han, 2001; Lee et al., 1981; Leissa, 1969; Liew, 1992; Liew et al., 1991; Huang and Hinton, 1986).

5.1 Circular plate

A circular plate with clamped boundaries is analyzed. The thickness-span ratios \(h/2r = 0.01\) and 0.2 are used in this study.
For analysis, the entire plate is discretized with the mesh of 256 isogeometric elements illustrated in Figure 4.

In the analysis, three orders of basis function \( p = q = 3, 4, 5 \) are used. All the control points are located in the physical domain of structure and the control points on the circumference of the circle are fixed.

The resulting frequencies are presented in the dimensionless form:

\[
\lambda_n = \omega_n r^2 \left( \frac{h}{p} \right)^{\frac{1}{p}}
\]  

(33)

where \( r \) is the radius of the circular plate, \( \rho \) is the density of the material and the \( D = E h^3/12(1 - \nu^2) \) is the flexural rigidity of the plate in which \( E \) is the elastic modulus and \( \nu = 0.3 \) is the Poisson ratio. From numerical test, asymmetric and axisymmetric vibration modes are detected. In the axisymmetric modes, multiple frequencies are detected since the circular plate has symmetric geometry in radial direction. The frequencies obtained from the present shell elements show a very good agreement with the exact solutions in overall modes with 256 isogeometric elements. We illustrate the first nine mode shapes in Figure 5.

The natural frequencies are calculated in the dimensionless form of (33) and provided in Tables 2 and 3 for \( h/2r = 0.01 \) and 0.2 respectively.

**Table 2.** The non-dimensionalized natural frequencies \( \lambda_n \) of a clamped circular plate with \( h/2r = 0.01 \) and 256 elements.

| Mode | Present : order of basis function | Ref1 | Ref2 | Ref3 |
|------|----------------------------------|------|------|------|
| 1    | \( p=q=3 \)                      | 10.199 | 10.199 | 10.199 | 10.213 | 10.214 | 10.216 |
| 2    | \( p=q=4 \)                      | 21.198 | 21.195 | 21.195 | 21.231 | 21.234 | 21.260 |
| 3    | \( p=q=5 \)                      | -     | -     | -     | 34.792 | 34.803 | -     |
| 4    |                                 | 34.717 | 34.712 | 34.712 | 34.782 | 34.793 | 34.880 |
| 5    |                                 | 34.731 | -     | -     | 34.792 | 34.803 | -     |
| 6    |                                 | 39.584 | 39.566 | 39.565 | 39.677 | 39.680 | 39.771 |
| 7    |                                 | 50.734 | 50.690 | 50.689 | 50.835 | 50.859 | 51.040 |
| 8    |                                 | -     | -     | -     | -     | -     | -     |
| 9    |                                 | 60.453 | 60.369 | 60.368 | 60.676 | 60.680 | 60.820 |
| 10   |                                 | -     | -     | -     | -     | -     | -     |

Note: Ref1: assumed natural strain shell element (Lee, 2001); Ref2: nine-node shell element (Kanok-nukulchai, 1979); Ref3: analytical solutions (Leissa, 1969). – multiple frequencies associated with axisymmetric modes.
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Table 3. The non-dimensionalized natural frequencies $\lambda_n$ of a clamped circular plate with $h/2r=0.2$ and 256 elements.

| Mode | Present : order of basis function | Ref1 | Ref2 | Ref3 |
|------|----------------------------------|------|------|------|
|      | $p=q=3$                         |      |      |      |
| 1    | 7.275                           | 7.275| 7.274| 7.494|
| 2    | 12.340                          | 12.340| 13.040| 13.040|
|      | 17.245                          | 17.245| 18.623| 18.623|
|      | 19.137                          | 19.137| 20.521| 20.521|
|      | 22.135                          | 22.135| 24.291| 24.291|
|      | 24.092                          | 24.092| 27.350| 27.350|
|      | 27.017                          | 27.016| 29.985| 29.985|
|      | 27.929                          | 27.929| 33.923| 33.923|
|      | 30.502                          | 30.501| 35.126| 35.127|
|      | 30.646                          | 30.646| 35.650| 35.651|
|      | 24.092                          | 24.092| 27.350| 27.351|
| 7    | 27.017                          | 27.016| 29.985| 29.985|
|      | 27.929                          | 27.929| 33.923| 33.923|
|      | 30.502                          | 30.501| 35.126| 35.127|
|      | 30.646                          | 30.646| 35.650| 35.651|

Note: Ref1: assumed natural strain shell element (Lee, 2001); Ref2: nine-node shell element (Huang and Hinton, 1986); Ref3: nine-node shell element (Kanok-nukulchai, 1979). – multiple frequencies associated with axisymmetric modes

5.2 Elliptical plate

The elliptical plate is analyzed. The geometry of the plate is presented in Figure 6.

The plate has the aspect ratio $a/b = 4$ and the thickness-span ratio is taken as $2h/b = 0.01$. For analysis, 256 isogeometric elements are used. All units are assumed to be consistent.

The results are presented in the dimensionless form

$$\lambda_n = \frac{w_n a^2}{D} \left( \frac{h}{D} \right)^{1/2}$$

where $a$ is the radius of the elliptical plates in the $x$ direction, $p$ is the density of the material and the $D = Eh^3/12(1 - \nu^2)$ is the flexural rigidity of the plate in which $E$ is the elastic modulus and $\nu$ = 0.3 is the Poisson ratio.

The present solutions are compared with the reference solution produced by Liew et al. (1991), the FE solutions obtained by using the shell element S9R5 of the ABAQUS and assumed natural strained shell element (Lee and Han, 2001). The natural frequencies and mode shapes of the elliptical plates are presented in Table 4 and Figure 7 respectively.

Table 4. The non-dimensionalized natural frequencies $\lambda_n$ of clamped elliptical plate with $a/b=4$.

| Mode | Present : order of basis function | Ref1 | Ref2 | Ref3 |
|------|----------------------------------|------|------|------|
|      | $p=q=3$                         |      |      |      |
| 1    | 96.126                          | 96.129| 96.138| 97.598|
| 2    | 113.593                         | 113.554| 113.583| 115.608|
| 3    | 134.563                         | 134.456| 134.465| 137.269|
| 4    | 159.290                         | 158.906| 158.883| 164.325|
| 5    | 188.351                         | 187.111| 186.998| 195.340|
| 6    | 222.794                         | 219.350| 218.916| 255.095|
| 7    | 252.335                         | 252.357| 252.368| 250.668|
| 8    | 264.752                         | 256.323| 254.852| 250.668|
| 9    | 281.173                         | 281.091| 281.163| 281.112|
| 10   | 313.063                         | 299.665| 295.354| 289.620|
| 11   | 318.023                         | 312.806| 312.859| 310.332|
| 12   | 348.333                         | 347.498| 341.956| 333.060|
| 13   | 388.041                         | 352.620| 347.443| 343.824|
| 14   | 388.058                         | 385.497| 385.114| 380.760|
| 15   | 433.872                         | 420.386| 398.511| 381.084|

Note: Ref1: analytical solution (Liew, 1992); Ref2: FE solution using Abaqus; Ref3: assumed natural strain shell element (Lee and Han, 2001).

From Table 4, it is found to be that the present solutions have an excellent agreement with theoretical solutions. But there are some discrepancies in higher modes with FE solutions. Mostly, two FE reference solutions produces lower natural frequencies compare to theoretical solution (Liew et al., 1991) and the present isogeometric solutions. In particular, the solution produced by the ABAQUS shows rather flexible frequencies than other solutions.

From this example, we can see the difference between the elliptical plate and the circular plate tested in Section 5.1. With the use of the aspect ratio $a/b = 4$, the elliptical plate has oblique shape and therefore the axisymmetric vibrational modes and multiple
frequencies appeared in the circular plate are all disappeared. The elliptical plate produces more winking in x-direction with higher modes. The intermediate modes, which are dominant in y-direction, also appear.

5.3 Curved fan blade

The vibration of curved fan blade has been frequently studied. Lee et al. (1981) investigated the behaviour of this structure using the Ritz method based on classical thin shallow shell theory in which the different polynomial terms are used. The shell geometry is presented in Figure 8.

The aspect ratio (a/b = 5), shallowness ratio (b/R = 0.5) and width-thickness ratio (b/h = 100) are used in the analysis. Note that Lim and Liew (1994) also investigated the same problem and we used it as reference solutions in this example. The initial location of control point with exact geometry of fan blade section is illustrated in Figure 9.
After 4 times of h-refinement with the orders of basis function \( p = q = 3, 4, 5 \) in the direction of \( x \) and \( y \) directions, 256 isogeometric elements is produced and is used to analyze the entire shell. The control polygons for three cases \( (p = q = 3, p = q = 4 \) and \( p = q = 5) \) are illustrated in Figure 10. Since three orders of basis function are used in this example, the total degrees of freedoms are different for each case although total number of element is the same value of 256 elements for all three cases.

The result is expressed in non-dimensional form:

\[
\lambda_n = w_{nt} ab \left( \frac{\rho h}{D} \right)^{\frac{1}{2}}
\]

where, the flexural rigidity is \( D = E h^3/12(1 - v^2) \).

From the test, the natural frequencies and the mode shapes are calculated and it is presented in Table 5 and Figure 11.

### Table 5. The non-dimensionalized natural frequencies of a curved fan blade with the aspect ratio \( a/b=4 \).

| Mode | Present : order of basis function | Ref1 | Ref2 | Ref3 |
|------|----------------------------------|------|------|------|
|      | \( p=q=3 \) | \( p=q=4 \) | \( p=q=5 \) |      |
| 1    | 4.438  | 4.438  | 4.438 | 4.322 | 4.441 | 4.437 |
| 2    | 7.106  | 7.106  | 7.099 | 7.175 | 7.122 | 7.117 |
| 3    | 22.740 | 22.735 | 22.713 | 22.899 | 22.797 | 22.774 |
| 4    | 26.744 | 26.740 | 26.720 | 26.995 | 26.675 | 26.644 |
| 5    | 42.302 | 42.289 | 42.254 | 42.394 | 42.409 | 42.340 |
| 6    | 65.973 | 65.954 | 65.949 | 65.217 | 65.933 | 65.895 |
| 7    | 66.728 | 66.731 | 66.673 | 65.281 | 66.570 | 66.478 |
| 8    | 67.326 | 67.166 | 67.078 | 67.064 | 67.415 | 67.209 |
| 9    | 98.351 | 97.380 | 97.270 | –      | –      | –      |
| 10   | 103.152| 103.080| 103.003| –      | –      | –      |
| 11   | 108.801| 108.706| 108.678| –      | –      | –      |
| 12   | 117.019| 116.856| 116.834| –      | –      | –      |
| 13   | 133.113| 132.249| 132.006| –      | –      | –      |
| 14   | 136.448| 132.399| 132.259| –      | –      | –      |
| 15   | 140.221| 139.699| 139.703| –      | –      | –      |

Note: Ref1: analytical solution (Lee et al., 1981); Ref2: nine-node shell element (Huang and Hinton, 1986); Ref3: assumed natural strain shell element (Lee and Han, 2001).

The present results are compared with the solution obtained by the Ritz method where six and five polynomial terms are used in the \( x \) and \( y \) directions respectively. In addition, it is also compared with FE solutions (Lee and Han, 2001). With the elevation of the order of basis function from \( p=q=3 \) to \( p=q=5 \), the natural frequencies produced by using the present isogeometric RM shell element are readily converged to the analytical and FE solutions. Notably, the present solutions are most flexible compared to other solutions.

### 6. CONCLUSIONS

A RM shell element is formulated by using isogeometric concept and used to investigate the behaviours of plate and shell structures under free vibration. Non-uniform rational B-spline surface
(NURBS) definition is consistently used to derive all terms required in the present RM shell element formulation. In particular, new anchor positions are proposed to calculate the shell normal vector. Numerical examples are carried out to demonstrate the accuracy of the present isogeometric RM shell element. From numerical results, the present isogeometric shell element can produce reliable natural frequencies and associated mode shapes of shells with enough accuracy. We identify that it can be used to predict vibrational behaviours of both thin and thick plate and shell structures. The present isogeometric solutions described in this paper are provided as future reference solutions on the structural vibrations of plate and shell structures. Since the present shell element can represent the exact geometry of structures using NURBS definition, we readily see that there is also a great possibility of using the present isogeometric shell element in the analysis of shell surfaces with arbitrary shapes.

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