Heisenberg-picture approach to the exact quantum motion of a
time-dependent forced harmonic oscillator

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Abstract

In the Heisenberg picture, the generalized invariant and exact quantum motions are found for a time-dependent forced harmonic oscillator. We find the eigenstate and the coherent state of the invariant and show that the dispersions of these quantum states do not depend on the external force. Our formalism is applied to several interesting cases.

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I. INTRODUCTION

The quantization of time-dependent harmonic oscillator is very important when we treat the coherent states, the squeezed states, and many other branches of physics. For example, the propagator of the quantum mechanical system with friction can be treated as a harmonic oscillator with time dependent mass [1]. One of the most powerful method to find a quantum state of the oscillator is the generalized invariant method introduced by Lewis and Riesenfeld (LR) [2]. Recently, some of the authors obtained the most general form of the invariant and the general quantum states for the oscillator using the Heisenberg picture when the mass and the frequency are explicitly time dependent [3,4].

The time-dependent harmonic oscillator with a perturbative force was considered by Khandekar and Lawande [5] who found that the wave function and propagator depend only on the solution of a classical damped oscillator through a single function. The exact evolution operator for the harmonic oscillator subject to arbitrary force was obtained by Lo [6]. He obtained the quantum states by solving the differential equation satisfied by the evolution operator. He also investigated the time evolution of a charged oscillator with a time dependent mass and frequency in a time-dependent field.

We consider the forced harmonic oscillator, where the external force depends on time. Even though Lo found the solution in the general situation, the solutions are difficult to analyze the physical properties of the system. While, our formulations has an advantage in that point, since the classical solutions suffice to understand the corresponding quantum system. Our choice of quantum state is different from those of Ref. [3,5]. They choose the Fock space in terms of the eigenstates of the Hamiltonian with the external force term suppressed. However, its ground state cannot be the lowest-energy state since the external force lower the minimum of the potential, hence the zero-point energy. Further, the dispersions of \( p \) and \( q \) for those states are dependent on the external force.

On the other hand, we construct the Fock space in terms of the eigenstates of the generalized invariant for \textit{the forced oscillator}. These number-states can be set equal to the
ones of the Hamiltonian by choosing the parameter properly. It should be emphasized that the dispersions of $p$ and $q$ for these states are independent of the external forces.

In Sec. II we present the general formulation for finding the quantum solutions of the time-dependent forced harmonic oscillator using the LR invariant method. In Sec. III we find the quantum evolution of the Heisenberg operator, $q(t)$ and $p(t)$, and the time-evolution operator. In Sec. IV we take some examples for the applications and Sec. V is devoted to summary.

II. FORCED HARMONIC OSCILLATOR - GENERAL FORMULATION

The Hamiltonian for a time-dependent harmonic oscillator under a force $F(t)$ is

$$H_T(t) = H(t) + H_F(t) = \frac{p^2(t)}{2M(t)} + \frac{M(t)\omega^2(t)}{2}q^2(t) - M(t)F(t)q(t).$$

In the previous paper [4] we found the invariant $I(t)$ which satisfies

$$\frac{\partial}{\partial t}I(t) - i[I(t), H(t)] = 0.$$  

The invariant $I(t)$ is

$$I(t) = g_-(t)\frac{p^2(t)}{2} + g_0(t)\frac{p(t)q(t) + q(t)p(t)}{2} + g_+(t)\frac{q^2(t)}{2},$$

and $g_k(t)$ are given in Ref. [4] using two independent solutions of

$$\frac{d}{dt}\left[M(t)\frac{d}{dt}f_{1,2}(t)\right] + M(t)\omega^2(t)f_{1,2}(t) = 0.$$  

Introducing the creation and annihilation operators:

$$b(t) = \sqrt{\frac{\omega_I}{2g_-(t)}} + i\sqrt{\frac{1}{2\omega_1 g_-(t)}}g_0(t)q(t) + i\sqrt{\frac{g_-(t)}{2\omega_I}}p(t),$$

$$b^\dagger(t) = \sqrt{\frac{\omega_I}{2g_-(t)}} - i\sqrt{\frac{1}{2\omega_1 g_-(t)}}g_0(t)q(t) - i\sqrt{\frac{g_-(t)}{2\omega_I}}p(t),$$

we can write the invariant as

$$I(t) = \omega_I\left(b^\dagger(t)b(t) + \frac{1}{2}\right),$$
where \( \omega_I^2 = g_+(t)g_-(t) - g_0^2(t) \) is a constant of motion.

In the presence of the external force, \( I(t) \) is no longer the invariant. Therefore, we search for the invariant of the form

\[
I_T(t) = \omega_I \left( B^\dagger(t)B(t) + \frac{1}{2} \right),
\]

where

\[
B(t) = b(t) + \beta(t).
\]

Here \( \beta(t) \) is a c-function of time and \([B(t), B^\dagger(t)] = 1\).

The Heisenberg equation of motion for \( b(t) \) leads to

\[
\frac{d}{dt} b(t) = \frac{\partial}{\partial t} b(t) - i[b(t), H(t) + H_F(t)]
\]

\[
= -i \frac{\omega_I}{M(t)g_-(t)} b(t) + iM(t)F(t) \sqrt{\frac{g_-(t)}{2\omega_I}}.
\]

Then the Heisenberg equation for \( B(t) \) can be written as

\[
\frac{d}{dt} B(t) = -i \frac{\omega_I}{M(t)g_-(t)} B(t),
\]

where \( \beta(t) \) satisfies the differential equation

\[
\frac{d}{dt} \beta(t) + i \frac{\omega_I}{M(t)g_-(t)} \beta(t) = -iM(t)F(t) \sqrt{\frac{g_-(t)}{2\omega_I}}.
\]

Further its solution is easily found

\[
\beta(t) = e^{-i\Theta(t)} \beta_0 - ie^{-i\Theta(t)} \int_{t_0}^t dt \sqrt{\frac{g_-(t)}{2\omega_I}} M(t)F(t)e^{i\Theta(t)},
\]

where \( \beta_0 \) is an arbitrary parameter which will be fixed by the extra condition. For the later convenience, we define

\[
\mathcal{F}(t) = e^{-i\Theta(t)} \beta_0 - \beta(t).
\]

Now, the solutions of (14) are

\[
B^\dagger(t) = e^{i\Theta(t)} B^\dagger(t_0), \quad B(t) = e^{-i\Theta(t)} B(t_0),
\]
where
\[
\Theta(t) = \int_{t_0}^{t} dt \frac{\omega_I}{M(t)g_-(t)}.
\] (16)

Thus we have found the generalized invariant even in the presence of the external force. As seen from (3) and (3), the creation and annihilation operators of the invariant are constructed from shifting those of the corresponding Hamiltonian with no external force.

From Eq. (15), we can easily notice that (8) is indeed the invariant for \( H_T(t) \) and it can be rewritten as
\[
I_T(t) = \frac{1}{2} \omega_I [B(t)B^\dagger(t) + B^\dagger(t)B(t)]
= I(t) + \omega_I \text{Re}(\beta) \sqrt{\frac{2\omega_I}{g_-(t)}} q(t) + \omega_I \text{Im}(\beta) \sqrt{\frac{2g_-(t)}{\omega_I}} \left( p(t) + \frac{g_0(t)}{g_-(t)} q(t) \right) + \omega_I |\beta(t)|^2.
\] (17)

By setting the parameter constant \( \beta_0 \) in Eq. (13) as
\[
\beta_0 = -\frac{1}{2} \frac{M(t_0)}{\omega_I} \sqrt{\frac{2g_-(t_0)}{\omega_I}} F(t_0)
\] (18)
and choosing \( I(t) \) according to Eq. (3.4) of Ref. [7], the invariant becomes the instantaneous Hamiltonian at \( t_0 \) with the additional constant, i.e.
\[
I_T(t_0) = H_T(t_0) + \beta_0^2.
\] (19)

Let us define the creation and annihilation operator of the Hamiltonian with no external force by
\[
H(t) = \omega(t) \left( a^\dagger(t)a(t) + \frac{1}{2} \right).
\] (20)

Then the vacuum state \( |0, t\rangle_a \) is defined by
\[
a(t) |0\rangle_a = 0,
\] (21)
and the number-states are constructed as
\[
|n, t\rangle = \frac{a^{\dagger n}(t)}{\sqrt{n!}} |0, t\rangle.
\] (22)
These states were studied by Lo \cite{5}.

Here, we construct the Fock space as eigenstates of $I_T(t)$. The vacuum state $|0\rangle_B$ for the harmonic oscillator is defined by

$$B(t) |0\rangle_B = 0,$$

and the number-states by

$$|n, t\rangle_B = \frac{B^\dagger n(t)}{\sqrt{n!}} |0, t\rangle_B.$$  \hspace{1cm} (24)

It can be written as

$$|n, t\rangle_B = e^{i(n+1/2)\Theta(t)} |n, t_0\rangle_B,$$

with the phase of $|0\rangle_B$ fixed properly.

These are different from (21) and they are correlated through the following Bogoliubov transformation:

$$B(t) = v_1(t)a(t) + v_2(t)a^\dagger(t) + \beta(t),$$

where

$$v_1(t) = \frac{1}{2} \left[ \sqrt{\frac{M(t)g_-(t)\omega(t)}{\omega_I}} + \sqrt{\frac{\omega_I}{M(t)g_-(t)\omega(t)}} \left( 1 + i\frac{g_0(t)}{\omega_I} \right) \right],$$

$$v_2(t) = \frac{1}{2} \left[ -\sqrt{\frac{M(t)g_-(t)\omega(t)}{\omega_I}} + \sqrt{\frac{\omega_I}{M(t)g_-(t)\omega(t)}} \left( 1 + i\frac{g_0(t)}{\omega_I} \right) \right],$$

(27)

with $|v_1(t)|^2 - |v_2(t)|^2 = 1$. This transformation can be written as the unitary one

$$B(t) = S(t)D^\dagger(t)a(t)D(t)S(t).$$ \hspace{1cm} (28)

Here the squeezing operator is given by

$$S(t) = \exp(i\theta_1 a^\dagger a) \exp \left( \frac{1}{2} e^{i(\theta_2 - \theta_1)} \cosh^{-1} |v_1|^2 - H.C. \right),$$ \hspace{1cm} (29)

where $v_1 = |v_1| e^{i\theta_1}$, $v_2 = |v_2| e^{i\theta_2}$, and the displacement operator is
\[ D(t) = e^{\beta(t)a(t) - \beta^*(t)a(t)}. \]  

Thus the ground state of (24) are the displaced squeezed states of (22), and vice versa:

\[ |0, t\rangle_B = S^\dagger(t)D(t) |0, t\rangle_a. \]  

Furthermore, it should be noted that the vacuum state of (24) is the coherent state of the unforced oscillator:

\[ |0\rangle_B = e^{-|\beta(t)|^2/2} \sum_{n=0}^{\infty} \frac{|\beta(t)|^2}{n!} b^n |0\rangle_b, \]  

with

\[ b|0\rangle_b = 0. \]  

Now let us calculate the vacuum expectation value of \( H_T(t) \) for \( |0\rangle_a \) and \( |0, t\rangle_B \). The Hamiltonian can be rewritten by \( B(t), B^\dagger(t), \) and \( \beta(t) \) as:

\[ H_T(t) = H_Q(t) - d(t)B^\dagger(t) - d^\dagger(t)B(t) + H_d(t), \]  

where \( H_Q(t) \) is the quadratic term in \( B(t) \) and \( B^\dagger(t) \), and \( H_d(t) \) is quadratic in \( F(t) \).

\[ H_Q(t) = h_+(t) \frac{B^2(t)}{2} + h_0(t) \frac{B(t)B^\dagger(t) + B^\dagger(t)B(t)}{4} + h_-(t) \frac{B(t)^2}{2}; \]  

\[ H_d(t) = h_+(t) \frac{\beta^2(t)}{2} + h_0(t) \frac{\beta(t)\beta^\dagger(t) + \beta^\dagger(t)\beta(t)}{4} + h_-(t) \frac{\beta^2(t)}{2} \]  

\[ - M(t) \sqrt{\frac{g(t)}{2\omega_I}} F(t) \left[ \beta(t) + \beta^\dagger(t) \right], \]

where \( h_i \) is defined at the previous paper \([4]\) and the displacement \( d(t) \)

\[ d(t) = h_0(t)\beta(t) + h_-(t)\beta(t) + \sqrt{\frac{g(t)}{2\omega_I}} M(t)F(t), \]  

and \( H_d(t) \) are proportional to the identity operator. From (20) and (21) the vacuum expectation value of \( H_T(t) \) for \( |0\rangle_a \) is

\[ a \langle 0 | H_T(t) |0\rangle_a = \frac{\omega(t)}{2}. \]
On the contrary, if we take the instantaneous invariant states from (19) and (23), the vacuum expectation value for $|0\rangle_B$ becomes

$$B \langle 0, t_0 | H_T(t_0) | 0, t_0 \rangle_B = \frac{\omega(t_0)}{2} - \beta_0^2.$$  (39)

The state (21) cannot be the lowest energy state as expected; the minimum of the quadratic potential is lowered by the external force.

## III. QUANTUM EVOLUTIONS

Now let us find the quantum evolutions of $p$ and $q$ which are the only things we should to know in order to study the quantum mechanical system in the Heisenberg picture. Equating the Hermitian and anti-Hermitian parts of both sides of (15) and using (5) and (9), the time evolution of $q(t)$ and $p(t)$ is given by

$$q(t) = q(t_0) \sqrt{\frac{g_-(t)}{g_-(t_0)}} \left[ \cos \Theta(t) + \frac{g_0(t_0)}{\omega_I} \sin \Theta(t) \right] + p(t_0) \sqrt{\frac{g_-(t)g_-(t_0)}{\omega_I}} \sin \Theta(t)$$

$$+ \sqrt{\frac{g_-(t)}{2\omega_I}} [\mathcal{F}(t) + \mathcal{F}^{\dagger}(t)],$$  (40)

$$p(t) = q(t_0) \frac{1}{\sqrt{g_-(t)g_-(t_0)}} \left[ \{g_0(t_0) - g_0(t)\} \cos \Theta(t) - \left\{ \omega_I + \frac{g_0(t_0)g_0(t)}{\omega_I} \right\} \sin \Theta(t) \right]$$

$$+ p(t_0) \sqrt{\frac{g_-(t)}{g_-(t_0)}} \left\{ \cos \Theta(t) - \frac{g_0(t)}{\omega_I} \sin \Theta(t) \right\}$$

$$- i \sqrt{\frac{\omega_I}{2g_-(t)}} \left[ \left\{ 1 - i \frac{g_0(t)}{\omega_I} \right\} \mathcal{F}(t) - \left\{ 1 + i \frac{g_0(t)}{\omega_I} \right\} \mathcal{F}^{\dagger}(t) \right].$$  (41)

By direct calculation, one can easily check that (40) and (42) satisfies the equation of motion

$$\frac{dq(t)}{dt} = -i [q(t), H(t)],$$  (42)

$$\frac{dp(t)}{dt} = -i [p(t), H(t)].$$

The dispersions of $p$ and $q$ for a number state (24) are given by

$$B \langle n | [\Delta q(t)]^2 | n \rangle_B = (2n + 1) \frac{g_-(t)}{2\omega_I},$$  (43)

$$B \langle n | [\Delta p(t)]^2 | n \rangle_B = (2n + 1) \frac{\omega_I}{2g_-(t)} \left[ 1 + \frac{g_0^2(t)}{\omega_I^2} \right].$$  (44)
If we construct the coherent state of the forced oscillator as
\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{|\alpha|^n}{n!} B^n_{\ast} |0\rangle_B , \tag{45}
\]
with \(\alpha = |\alpha| e^{-i\delta}\), the dispersions in \(q\) and \(p\) for this state is given by (43) and (44) with \(n = 0\). It should be noted that the dispersions (43) and (44) do not depend on \(F(t)\) and they are the same as the ones of the eigenstate of the invariant (7) with no external force. Thus the shape of the wavepacket for the eigenstate of the invariant does not alter by the external force. The physical meaning of this result is clear and non-frightening: the force \(F(t)\) acts uniformly on the wavepacket so that the wavepacket is shifted with no distortion. In other words, the eigenstate of the invariant is the quantum state whose wavepacket does not alter its shape by the external force. However, a general state which is constructed by a linear combination of the eigenstates does not possess this property because of the time-dependent phase. This phase makes interferences between the eigenstates with different quantum numbers. However, the coherent state (45), although it is made by the linear combination of the eigenstates, has the same property as the eigenstate. This is a remarkable property of a coherent state.

Further the expectation value \(\langle q(t) \rangle\) and \(\langle p(t) \rangle\) becomes
\[
\langle \alpha | q(t) | \alpha \rangle = \sqrt{\frac{g_- (t)}{2\omega_I}} \left\{ 2|\alpha| \cos[\Theta(t) + \delta] - \mathcal{F}(t) - \mathcal{F}^\dagger(t) \right\} \tag{46}
\]
\[
\langle \alpha | p(t) | \alpha \rangle = \sqrt{\frac{\omega_I}{2g_- (t)}} \left\{ -2|\alpha| \sin[\Theta(t) + \delta] - \left( i + \frac{g_0 (t)}{\omega_I} \right) \mathcal{F}(t) - \left( i - \frac{g_0 (t)}{\omega_I} \right) \mathcal{F}^\dagger(t) \right\} . \tag{47}
\]
Thus the wavepacket of the coherent state moves backward and forward like a classical particle.

Finally we want to find the time evolution operator \(U_T(t, t_0)\). In the Heisenberg picture the position and momentum operators transform according to \(q(t) = U_T^\dagger(t, t_0) q(t_0) U_T(t, t_0)\), and \(p(t) = U_T^\dagger(t, t_0) p(t_0) U_T(t, t_0)\). By expressing these in terms of the creation and annihilation operators at \(t_0\) and using the Eqs. (3) and (15), we obtain
\[
U_T^\dagger(t, t_0) B(t_0) U_T(t, t_0) = -u_2^* (t) B^\dagger(t_0) e^{i\Theta(t)} + u_1 (t) B(t_0) e^{-i\Theta(t)} + d(t), \tag{48}
\]
\[
U_T^\dagger(t, t_0) B^\dagger(t_0) U_T(t, t_0) = u_1^* (t) B^\dagger(t_0) e^{-i\Theta(t)} - u_2 (t) B(t_0) e^{i\Theta(t)} + d^\ast(t), \tag{49}
\]
where $u_1(t)$ and $u_2(t)$ are given [4] and $d(t)$ is

$$
\begin{aligned}
d(t) &= \frac{1}{2} \left[ \left( 1 + \frac{i g_0(t_0)}{2 \omega t} \right) \sqrt{\frac{g_-(t)}{g_-(t_0)}} \{ \beta(t) + \beta^\dagger(t) \} \\
&\quad + \sqrt{\frac{g_-(0)}{g_-(t)}} \left[ \{ \beta(t) - \beta^\dagger(t) \} - \frac{i g_0(t)}{\omega t} \{ \beta(t) + \beta^\dagger(t) \} \right] \right] t_{t_0}.
\end{aligned}
$$

(50)

Then the time-evolution operator $U_T(t, t_0)$ is given by

$$
U_T(t, t_0) = U_d(t, t_0) U(t, t_0),
$$

(51)

where

$$
U_d(t, t_0) = e^{d^\dagger B(t_0) - d^\ast B(t_0)}
$$

(52)

is the displacement operator and $U(t, t_0)$ is given by

$$
U(t, t_0) = \exp \left( \frac{i}{2} \phi_1 \left\{ B^\dagger(t_0) B(t_0) + B(t_0) B^\dagger(t_0) \right\} \right) \\
\exp \left( \frac{\nu}{2} \left\{ e^{-i(\phi_1 - \phi_2)} B^2(t_0) - e^{i(\phi_1 - \phi_2)} B^2(t_0) \right\} \right),
$$

(53)

(for the definition of $\phi_1$, $\phi_2$, and $\nu$, see Ref. [4]). It is easily checked that (51) indeed satisfies (48) by using the following properties of the displacement operator

$$
U^\dagger_d(t, t_0) B(t_0) U_d(t, t_0) = B(t_0) + d(t).
$$

(54)

$$
U^\dagger_d(t, t_0) B^\dagger(t_0) U_d(t, t_0) = B^\dagger(t_0) + d^\ast(t).
$$

(55)

Thus we obtained the time-evolution operator for the forced harmonic oscillator. This operator is in agreement with that of Lo [3].

**IV. SOME EXACTLY SOLVABLE MODELS**

The simplest example is the harmonic oscillator under the constant driving force $F$. Let all quantities are time independent $M(t) = m$, and $\omega_0(t) = \omega$. After a little algebra, we get the following results:
\[ q(t) - \frac{F}{\omega^2} = \left( q(t_0) - \frac{F}{\omega^2} \right) \cos \omega(t - t_0) + \frac{p(t_0)}{m \omega} \sin \omega(t - t_0), \quad (56) \]
\[ p(t) = -\frac{\omega I}{m} \left( q(t_0) - \frac{F}{\omega^2} \right) \sin \omega(t - t_0) + p(t_0) \cos \omega(t - t_0). \quad (57) \]

As expected, the center of oscillation is shifted by \( F/\omega^2 \) in \( q \) space.

Next, let us consider the damped pulsating oscillator with an arbitrary driving force \( F(t) \), the mass \( M(t) = m_0 \exp[2(\gamma t + \mu \sin \nu t)] \), and the frequency \( \omega^2(t) = \Omega^2 + \frac{1}{\sqrt{M(t)}} \frac{d^2 \sqrt{M(t)}}{dt^2} \). This example was also studied by Lo [6]. Without loss of generality, we can set the initial time \( t_0 = 0 \). It is easy to find the classical solution: \( f(t) = e^{i \Omega t}/\sqrt{M(t)} \) and two real independent solutions, say \( f_1(t) \) and \( f_2(t) \), can be obtained by taking its real and imaginary parts, respectively. By setting the parameter constants in Eq. (7) of Ref. [4] as \( c_1 = 1 = c_3, c_2 = 0 \), we have

\[ g_-(t) = \frac{1}{M(t)}, \quad g_0(t) = \frac{1}{2} \frac{d}{dt} \ln M(t), \quad g_+(t) = \frac{M(t)}{4} \left( \left[ \frac{d}{dt} \ln M(t) \right]^2 + \Omega^2 \right). \quad (58) \]

Thus, we have fixed the invariant (3) with the frequency \( \omega^2_I = g_+g_- - g_0^2 = \Omega^2 \). For the ground state and the coherent state of this invariant, we have the dispersions in \( q \) and \( p \)

\[ \langle \Delta p \rangle^2 = \frac{\Omega M(t)}{2} \left[ 1 + \left( \frac{\gamma + \mu \nu \cos \nu t}{\Omega} \right)^2 \right], \quad (59) \]
\[ \langle \Delta q \rangle^2 = \frac{1}{2 \Omega M(t)}. \quad (60) \]

Note that these are not affected by the external force, as stated in the previous section. This property makes our quantum state be distinguishable from that of Lo – see Eqs. (89) and (90) in Ref. [3]. Furthermore, the quantum motions of \( q(t) \) and \( p(t) \) are obtained by inserting (58) into (40) and (42):

\[ q(t) = q(0) \sqrt{\frac{m_0}{M(t)}} \left[ \cos \Omega t + \frac{\gamma + \mu \nu}{\Omega} \sin \Omega t \right] + p(0) \frac{1}{\Omega \sqrt{m_0 M(t)}} \sin \Omega t \]
\[ + \frac{1}{\Omega \sqrt{M(t)}} \int_0^t dt' \sqrt{M(t')} F(t') \sin \Omega(t - t'), \quad (61) \]
\[ p(t) = q(0) \sqrt{\frac{m_0 M(t)}{m_0 \mu \nu}} \left[ \mu \nu (1 - \cos \nu t) \cos \Omega t - \left( \Omega + \frac{\gamma + \mu \nu}{\Omega} \right) (\gamma + \mu \nu \cos \nu t) \sin \Omega t \right] \]
\[ + p(0) \sqrt{\frac{M(t)}{m_0}} \left[ \cos \Omega t - \frac{1}{\Omega} (\gamma + \mu \nu \cos \nu t) \sin \Omega t \right]. \quad (62) \]
\[-\sqrt{M(t)} \left[ \int_0^t dt' \sqrt{M(t')} F(t') \cos \Omega(t' - t) \right. \\
+ \left. 2\gamma + \frac{\mu \nu}{\Omega} \cos \nu t \right] \int_0^t dt' \sqrt{M(t')} F(t') \sin \Omega(t' - t) \].

Let us see the time-evolutions of the coherent states with (a) and without (b) the external force in the phase space diagram (Fig. 1). It vividly shows that the dispersion does not depends on the external force, although their time-evolutions of the expectation values of \(p\) and \(q\) are completely different.
We choose the parameters to be $m_0 = 1 = \Omega$, $\gamma = 1/10$, $\mu = 4$, and $\nu = 1/3$.

(a) The damped-forced oscillator with the external force $F(t) = \sin(t)$.

(b) The damped oscillator without external force.

Each initial state is the coherent state with $\langle q(0) \rangle = 5$, $\langle p(0) \rangle = 0$. Each real line denotes the time evolution of $\langle q(t) \rangle$ and $\langle p(t) \rangle$. The ellipses are depicted for every four timing units to denote $\langle \Delta q(t) \rangle$ (horizontal axis) and $\langle \Delta p(t) \rangle$ (vertical axis). Note that two ellipses of (a) and (b), with the same time, have the same shapes.

V. SUMMARY

We considered the time-dependent harmonic oscillator with a driving force and found its quantum solutions. We found the LR invariant and constructed the Fock space as its eigenstates. The quantum states of Lo [1] can be constructed by squeezing and displacing...
our quantum states as seen from (31). Further, our ground state is the coherent state of the invariant with no external force – see (32). We showed that the ground state of the invariant has lower energy expectation value than the ground state of the instantaneous Hamiltonian.

Using the time evolution of the creation and annihilation operators for the LR invariant, we also found the solutions of the Heisenberg equation of motion for the position and momentum operator. The exact quantum motion of a forced time-dependent harmonic oscillator is described by the classical solution of the corresponding unforced oscillator. The dispersions of $q$ and $p$ for the eigenstates and the coherent state of the invariant do not depend on the external force. It was found that the external force merely shifts the wavepackets of the eigenstates of the invariant with no external force. These results were exemplified by a model.

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