MINIMUM MEAN-SQUARED-ERROR AUTOCORRELATION PROCESSING IN COPRIME ARRAYS

ABSTRACT

Coprime arrays enable Direction-of-Arrival (DoA) estimation of an increased number of sources. To that end, the receiver estimates the autocorrelation matrix of a larger virtual uniform linear array (coarray), by applying selection or averaging to the physical array’s autocorrelation estimates, followed by spatial-smoothing. Both selection and averaging have been designed under no optimality criterion and attain arbitrary (suboptimal) Mean-Squared-Error (MSE) estimation performance. In this work, we design a novel coprime array receiver that estimates the coarray autocorrelations with Minimum-MSE (MMSE), for any probability distribution of the source DoAs. Our extensive numerical evaluation illustrates that the proposed MMSE approach returns superior autocorrelation estimates which, in turn, enable higher DoA estimation performance compared to standard counterparts.

Keywords Coprime arrays · DoA estimation · Mean-squared-error · Sparse arrays · Spatial smoothing

1 Introduction

Coprime arrays (CAs) are non-uniform linear arrays with element locations determined by a pair of distinct coprime numbers. CAs are a special class of sparse arrays [3,4] which are often preferred due to their desirable bearing properties (e.g., enhanced degrees of freedom and closed-form expressions for element-locations). CAs have attracted significant research interest over the past years and have been successfully employed in applications such as Direction-of-Arrival (DoA) estimation [5,17], beamforming [18,20], interference localization and mitigation in satellite systems [21],
and space-time adaptive processing \cite{22}, to name a few. More recently, scholars have explored/employed CAs for underwater localization \cite{23,24}, channel estimation in MIMO communications via tensor decomposition \cite{25}, receivers on moving platforms which further increase degrees of freedom \cite{26,27}, and receivers capable of two-dimensional DoA estimation.

In standard DoA with CAs \cite{3}, the receiver conducts a series of intelligent processing steps and assembles an autocorrelation matrix which corresponds to a larger virtual Uniform Linear Array (ULA), known as the coarray. Accordingly, CAs enable the identification of more sources than physical sensors compared to equal-length ULAs. Processing at a coprime array receiver commences with the estimation of the nominal (true) physical-array autocorrelations based on a collection of received-signal snapshots. The receiver processes the estimated autocorrelations so that each coarray element is represented by one autocorrelation estimate. Next, the processed autocorrelations undergo spatial smoothing, forming an autocorrelation matrix estimate which corresponds to the coarray. Finally, a DoA estimation approach, such as the MUltiple SIgnal Classification (MUSIC) algorithm, can be applied on the resulting autocorrelation matrix estimate for identifying the source directions.

At the autocorrelation processing step, the estimated autocorrelations are commonly processed by selection combining \cite{3}, retaining only one autocorrelation sample for each coarray element. Alternatively, an autocorrelation estimate for each coarray element is obtained by averaging combining \cite{11} all available sample-estimates corresponding to a particular coarray element. The two methods coincide in Mean-Squared-Error (MSE) estimation performance when applied on the nominal physical-array autocorrelations—which the receiver could only estimate with asymptotically large number of received-signal snapshots. In practice, due to a finite number of received-signal snapshots available at the receiver and the fact that these methods have been designed under no optimality criterion, the estimated autocorrelations diverge from the nominal ones and attain arbitrary MSE performance. In this case, the two methods no longer coincide in MSE estimation performance. It was recently shown in \cite{1} that averaging combining attains superior estimation performance compared to selection combining with respect to the MSE metric.

Motivated by prior works which treat angular variables as statistical random variables \cite{28,29}, in this work, we make the mild assumption that the DoAs are independent and identically distributed random variables and design a novel coprime array receiver equipped with a linear autocorrelation combiner which is designed under the Minimum-MSE (MMSE) optimality criterion. The proposed MMSE combiner minimizes, in the mean (i.e., for any configuration of DoAs), the error in estimating the physical-array autocorrelations with respect to the MSE metric. Moreover, we review the MSE expressions of selection and averaging combining of \cite{1} and, for the first time, offer formal mathematical proofs for these expressions. Finally, we conduct extensive numerical studies and compare the performance of the proposed MMSE combiner to existing counterparts, with respect to autocorrelation estimation error and DoA estimation.

The rest of this paper is organized as follows. In Section 2, we present the signal model and state the problem of interest. In Section 3 we review existing selection and averaging autocorrelation combining methods for coprime arrays, providing their closed-form MSE expressions \cite{11}, and offering formal mathematical proofs for these expressions. We present the proposed MMSE autocorrelation combining approach in Section 4. Next, in Section 5 we conduct extensive numerical performance evaluations of the proposed combining approach and compare against existing counterparts. Conclusions are drawn in Section 6.

2 Signal Model

Consider coprime naturals \((M, N)\) such that \(M < N\). A coprime array equipped with \(L = 2M + N - 1\) antenna elements is formed by overlapping a ULA with \(N\) antenna elements at positions \(p_{M,i} = (i - 1)Md, i = 1, 2, \ldots, N\), and a ULA equipped with \(2M - 1\) antenna elements at positions \(p_{N,i} = iNd, i = 1, 2, \ldots, 2M - 1\). The reference unit-spacing \(d\) is typically set to one-half wavelength at the operating frequency. The positions of the \(L\) elements of the coprime array are described by the element-location vector \(p := \text{sort}([p_{M,1}, \ldots, p_{M,N}, p_{N,1}, \ldots, p_{N,2M-1}]^\top)\), where \(\text{sort}(\cdot)\) sorts the entries of its vector argument in ascending order and the superscript ‘\(\top\)’ denotes matrix
transpose. We assume that narrowband signals impinge on the array from $K < MN + M$ sources with propagation speed $c$ and carrier frequency $f_c$. Assuming far-field conditions, a signal from source $k \in \{1, 2, \ldots, K\}$ impinges on the array from direction $\theta_k \in (-\pi/2, \pi/2]$ with respect to the broadside. The array response vector for source $k$ is $s(\theta_k) := [v(\theta_k)|^{(p_1)}, \ldots, v(\theta_k)|^{(p_L)}]^\top \in \mathbb{C}^{L \times 1}$, with $v(\theta) := \exp\left(-\frac{j 2 \pi L}{c} \sin(\theta)\right)$ for every $\theta \in (-\pi/2, \pi/2]$. Accordingly, the $q$th collected received-signal snapshot is of the form

$$y_q = \sum_{k=1}^{K} s(\theta_k) \xi_{q,k} + n_q \in \mathbb{C}^{L \times 1},$$

where $\xi_{q,k} \sim \mathcal{CN}(0, d_k)$ is the $q$th symbol for source $k$ (power-scaled and flat-fading-channel processed) and $n_q \sim \mathcal{CN}(0, \sigma^2 I_L)$ models Additive White Gaussian Noise (AWGN). We make the common assumptions that the random variables are statistically independent across different snapshots and symbols from different sources are independent of each other and of every entry of $n_q$. The received-signal autocorrelation matrix is given by

$$R_y := \mathbb{E}\{y_q y_q^H\} = S \text{ diag}(d) S^H + \sigma^2 I_L,$$

where $d := [d_1, d_2, \ldots, d_K]^\top \in \mathbb{R}^{K \times 1}$ is the source-power vector and $S := [s(\theta_1), s(\theta_2), \ldots, s(\theta_K)] \in \mathbb{C}^{L \times K}$ is the array-response matrix. We define

$$r := \text{vec}(R_y) = \sum_{i=1}^{K} a(\theta_i) d_i + \sigma^2 I_L \in \mathbb{C}^{L^2 \times 1},$$

where vec($\cdot$) returns the column-wise vectorization of its matrix argument, $a(\theta) := s(\theta)^* \otimes s(\theta)$, $I_L := \text{vec}(I_L) \in \mathbb{R}^{L^2 \times 1}$, the superscript `$*$' denotes complex conjugate, and `$\otimes$' is the Kronecker product operator [39]. By coprime number theory [3], for every $n \in \{-L' + 1, -L' + 2, \ldots, L' - 1\}$ with $L' := MN + M$, there exists a well-defined set of indices $\mathcal{J}_n \subset \{1, 2, \ldots, L^2\}$, such that

$$[a(\theta)]_j = v(\theta)^n \ \forall j \in \mathcal{J}_n,$$

for every $\theta \in (-\pi/2, \pi/2]$. We henceforth consider that $\mathcal{J}_n$ contains all $j$ indices that satisfy (4). In view of (4), a coprime array receiver assembles a linear combining matrix $E \in \mathbb{R}^{L^2 \times 2L'-1}$ and forms a length-$2L' - 1$ autocorrelation-vector $r_{co}$, each element of which corresponds to a single set $\mathcal{J}_n$, for every $n \in \{-L', 2 - L', \ldots, L' - 1\}$, by conducting linear processing (e.g., $E^\top r$) on the autocorrelations in $r$. That is, there exists linear combiner $E$ such that

$$r_{co} = E^\top r = \sum_{k=1}^{K} a_{co}(\theta_k) d_k + \sigma^2 e_{L', 2L'-1},$$

with $a_{co} := [v(\theta)^{1-L'}, v(\theta)^{2-L'}, \ldots, v(\theta)^{L'-1}]^\top$ for any $\theta \in (-\pi/2, \pi/2]$, and, for any $p \leq P \in \mathbb{N}_+$, $e_{p,P}$ is the $p$th column of $I_P$. Thereafter, the receiver applies spatial-smoothing to organize the sampled autocorrelations as the matrix

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$^2$Existing coprime array autocorrelation processing methods in the literature are reviewed in section 3.
where it holds that $Z = S_{\text{co}} \operatorname{diag}(d) S_{\text{co}}^H + \sigma^2 I_{L'}$, 

$$Z := F(I_{L'} \otimes r_{co}) \in \mathbb{C}^{L' \times L'},$$

where $F := [F_1, F_2, \ldots, F_{L'}]$ and, for every $m \in \{1, 2, \ldots, L'\}$, $F_m := [0_{L' \times (L' - m)}, I_{L'}, 0_{L' \times (m - 1)}]$. Importantly, $Z$ coincides with the autocorrelation matrix of a length-$L'$ ULA with antenna elements at locations $\{0, 1, \ldots, L' - 1\}d$. That is,

$$Z = S_{\text{co}} \operatorname{diag}(d) S_{\text{co}}^H + \sigma^2 I_{L'},$$

where it holds that $[S_{\text{co}}]_{m,k} = v(\theta_k)^{m-1}$ for every $m \in \{1, 2, \ldots, L'\}$ and $k \in \{1, 2, \ldots, K\}$. Standard MUSIC DoA estimation can be applied on $Z$. Let the columns of $U \in \mathbb{C}^{L' \times K}$ be the dominant left-hand singular vectors of $Z$, corresponding to its $K$ highest singular values, acquired by means of singular-value-decomposition (SVD). Defining $v(\theta) = [1, v(\theta), \ldots, v(\theta)^{L'-1}]^T$, we can accurately decide that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ belongs in $\Theta := \{\theta_1, \theta_2, \ldots, \theta_K\}$ if $(I_{L'} - U U^H) v(\theta) = 0_{L'}$ is satisfied for some $\theta$. Equivalently, we can resolve the angles in $\Theta$ by the $K$ (smallest) local minima of the MUSIC spectrum

$$P_{\text{MUSIC}}(\theta) = \left\| (I_{L'} - U U^H) v(\theta) \right\|_2^2. $$

In practice, $R_y$ in (2) is unknown to the receiver and sample-average estimated by a collection of $Q$ received-signal snapshots in $Y = [y_1, y_2, \ldots, y_Q]$ by

$$\hat{R}_y = \frac{1}{Q} \sum_{q=1}^{Q} y_q y_q^H. $$

Accordingly, the physical-array autocorrelation-vector $r$ in (3) is estimated by

$$\hat{r} := \text{vec}(\hat{R}_y) = \frac{1}{Q} \sum_{q=1}^{Q} y_q^* \otimes y_q. $$

The receiver then conducts linear combining on the estimated autocorrelation vector $\hat{r}$ to obtain an estimate of $r_{co}$, $\hat{r}_{co} = E \hat{r}$. The estimation error $\|r_{co} - \hat{r}_{co}\|_2$ depends on how well the linear combiner $E$ estimates the nominal physical-array autocorrelations. Accordingly, $Z$ in (7) is estimated by

$$\hat{Z} := F(I_{L'} \otimes \hat{r}_{co}) \in \mathbb{C}^{L' \times L'}. $$

Finally, MUSIC DoA estimation can be applied using the $K$ dominant left-hand singular vectors of $\hat{Z}$ instead of those of $Z$. Of course, in practice, there is an inherent DoA estimation error due to the mismatch between $Z$ and $\hat{Z}$. A schematic illustration of the coprime array processing steps presented above is offered in Fig. 1. In the sequel, we review the most commonly considered autocorrelation combining approaches in the coprime array literature and conduct a formal MSE analysis.

3 Technical Background on Autocorrelation Combining

3.1 Selection Combining [3]

The most commonly considered autocorrelation combining method is selection combining based on which the receiver selects any single index $j_n \in J_n$, for $n \in \{-L' + 1, \ldots, L' - 1\}$, and builds the $L^2 \times (2L' - 1)$ selection matrix

$$E_{sel} := [e_{j_{-L' \times L', L^2}}, e_{j_{-L' \times L', L^2}}, \ldots, e_{j_{L' \times L', L^2}}], $$

where $e_{jk} := \delta_{jk}$.
by which it processes the autocorrelations in \( r \), discarding by selection all duplicates (i.e., every entry with index in \( J_n \setminus j_n \), for every \( n \)), to form the length-(\( 2L' - 1 \)) autocorrelation vector

\[
\hat{r}_{sel} := E_{sel}^T r.
\]  

(13)

Importantly, when the nominal entries of \( r \) are known to the receiver, \( r_{sel} \) coincides with \( r_{co} \) in (5), thus, applying spatial smoothing on \( r_{sel} \) yields the exact coarray autocorrelation matrix \( Z = F(I_{L'} \otimes r_{sel}) \). In contrast, when \( r \) is unknown to the receiver and estimated by \( \hat{r} \) in (10), \( r_{sel} \) in (13) is estimated by

\[
\hat{r}_{sel} = E_{sel}^T \hat{r}.
\]  

(14)

Accordingly, the coarray autocorrelation matrix is estimated as

\[
Z_{sel} := F(I_{L'} \otimes \hat{r}_{sel}) \in \mathbb{C}^{L' \times L'}.
\]  

(15)

3.2 Averaging Combining

Instead of selecting a single index in \( J_n \) by discarding duplicates, averaging combining conducts averaging on all autocorrelation estimates that correspond to \( J_n \) for every \( n \in \{1 - L', 2 - L', \ldots, L' - 1\} \). That is, the receiver assembles the averaging combining matrix \( E_{avg} \), where, for every \( i \in \{1, \ldots, 2L' - 1\} \),

\[
[E_{avg}]_{i} := \frac{1}{|J_{i-L'}|} \sum_{j \in J_{i-L'}} e_{j,L'^2}.
\]  

(16)

where \(| \cdot |\) denotes the cardinality of its argument. Then, it processes the autocorrelation vector \( r \) to obtain

\[
r_{avg} := E_{avg}^T r.
\]  

(17)

By (4) and the fact that \([I_{L'}]_j\) equals 1, if \( j \in J_0 \) and 0 otherwise, it holds that, for any \( n \in \{-L' + 1, \ldots, L' - 1\} \), \([r]_j = e^T_{j,L'^2} r \) takes a constant value for every \( j \in J_n \). Thus, \( r_{avg} \) coincides with \( r_{sel} \) and \( r_{co} \). Therefore, similar to the selection combining approach, when \( r \) is known to the receiver, applying spatial smoothing on \( r_{avg} \) yields \( Z = F(I_{L'} \otimes r_{avg}) \). In practice, when \( R_y \) in (2) is estimated by \( \hat{R}_y \) in (9), \( r_{avg} \) is estimated by

\[
\hat{r}_{avg} := E_{avg}^T \hat{r},
\]  

(18)

and, accordingly, \( Z \) is estimated by

\[
\hat{Z}_{avg} := F(I_{L'} \otimes \hat{r}_{avg}) \in \mathbb{C}^{L' \times L'}.
\]  

(19)

3.3 Closed-form MSE Expressions for Selection and Averaging Combining

In general, estimates \( \hat{r}_{sel} \) and \( \hat{r}_{avg} \) diverge from \( r_{co} \) and attain MSE \( e(\hat{r}_{sel}) := E\{||r_{co} - \hat{r}_{sel}\|^2\} \) and \( e(\hat{r}_{avg}) := E\{||r_{co} - \hat{r}_{avg}\|^2\} \), respectively. Accordingly, \( \hat{Z}_{sel} \) and \( \hat{Z}_{avg} \) diverge from the nominal \( Z \) and attain MSE \( e(\hat{Z}_{sel}) := E\{||Z - \hat{Z}_{sel}\|^2\} \) and \( e(\hat{Z}_{avg}) := E\{||Z - \hat{Z}_{avg}\|^2\} \), respectively. Closed-form MSE expressions for the errors above were first presented in the form of Lemmas and Propositions (proofs omitted) in (11). For completeness purposes, we present again the MSE expressions in the form of Lemmas and Propositions, this time, accompanied by formal mathematical proofs.

For any sample support \( Q \), the following Lemma 1 and Lemma 2 express in closed-form the MSE attained by \( \hat{r}_{sel} \).

**Lemma 1.** For any \( n \in \{-L' + 1, -L' + 2, \ldots, L' - 1\} \) and \( j \in J_n \), it holds \( e = E\{||r|j| - \hat{r}|j|^2\} = \frac{1}{Q} \frac{(\sum d + s^2)^2}{Q} \).

**Lemma 2.** \( \hat{r}_{sel} \) attains MSE \( e(\hat{r}_{sel}) = E\{||r_{co} - \hat{r}_{sel}\|^2\} = (2L' - 1) e. \)
We focus on the autocorrelation combining step of coprime-array processing (see Fig. 1) where the receiver applies linear combining matrix $E$ to the estimated autocorrelations of the physical array. Arguably, a preferred receiver will attain consistently (i.e., for any possible configuration of DoAs, $\Theta$) low squared-estimation error $\|r_{co} - E^T \hat{r}\|_2$. If this requires $\Theta$.

\[ e_n = \mathbb{E} \left\{ \left\| r_{co} - \frac{1}{|J_n|} \sum_{j \in J_n} [\hat{r}]_j \right\|^2 \right\} = \frac{1}{Q} \left( 2\sigma^2 1_K^T d + \sigma^4 \right) + \sum_{i \in J_n} \sum_{j \in J_n} |z_{i,j}^H d|^2 |J_n|^2 . \]  

(20)

Lemma 4. $\hat{r}_{avg}$ attains MSE $e(\hat{r}_{avg}) = \mathbb{E} \left\{ \|r_{co} - \hat{r}_{avg}\|_2 \right\} = \sum_{n=1}^{L'-1} e_n$. 

By Lemma 4 the following Proposition naturally follows.

Proposition 2. $\hat{Z}_{avg}$ attains MSE $e(\hat{Z}_{avg}) = \mathbb{E} \left\{ \|Z - \hat{Z}_{avg}\|_F \right\} = \sum_{n=1}^{L' - m} e_n$. 

Similar to selection combining, as the sample-support $Q$ grows asymptotically, $e_n$, $e(\hat{r}_{avg})$, and $e(\hat{Z}_{avg})$ converge to zero. Complete proofs for the above Lemmas and Propositions are offered for the first time in the Appendix.

3.3.1 Remarks on selection and averaging sampling

In view of Proposition 2, the following Proposition follows in a straightforward manner.

Proposition 1. $\hat{Z}_{sel}$ attains MSE $e(\hat{Z}_{sel}) = \mathbb{E} \left\{ \|Z - \hat{Z}_{sel}\|_F \right\} = L'^2 e$. 

Expectedly, as the sample-support $Q$ grows asymptotically, $e$, $e(\hat{r}_{sel})$, and $e(\hat{Z}_{sel})$ converge to zero.

For any sample support $Q$, the following Lemma 3 and Lemma 4 express in closed-form the MSE attained by $\hat{r}_{avg}$, where $\hat{p} := \hat{p} \otimes 1_L$, $\hat{\omega}_{i,j} := [\hat{p}]_i - [\hat{p}]_j$, and $z_{i,j} := \left[ v(\theta_1)\hat{\omega}_{i,j} \ v(\theta_2)\hat{\omega}_{i,j} \ ... \ v(\theta_K)\hat{\omega}_{i,j} \right]^H$.

Lemma 3. For any $n \in \{-L' + 1, \ldots, L' - 1\}$ and $j_n \in J_n$, it holds that

\[ e_n = \mathbb{E} \left\{ \left\| [r]_{j_n} - \frac{1}{|J_n|} \sum_{j \in J_n} [\hat{r}]_j \right\|^2 \right\} = \frac{1}{Q} \left( 2\sigma^2 1_K^T d + \sigma^4 \right) + \sum_{i \in J_n} \sum_{j \in J_n} |z_{i,j}^H d|^2 |J_n|^2 . \]  

(20)

Lemma 4. $\hat{r}_{avg}$ attains MSE $e(\hat{r}_{avg}) = \mathbb{E} \left\{ \|r_{co} - \hat{r}_{avg}\|_2 \right\} = \sum_{n=1}^{L'-1} e_n$. 

By Lemma 4 the following Proposition naturally follows.

Proposition 2. $\hat{Z}_{avg}$ attains MSE $e(\hat{Z}_{avg}) = \mathbb{E} \left\{ \|Z - \hat{Z}_{avg}\|_F \right\} = \sum_{n=1}^{L' - m} e_n$. 

Similar to selection combining, as the sample-support $Q$ grows asymptotically, $e_n$, $e(\hat{r}_{avg})$, and $e(\hat{Z}_{avg})$ converge to zero. Complete proofs for the above Lemmas and Propositions are offered for the first time in the Appendix.

4 Proposed Minimum Mean-Squared-Error Autocorrelation Combining

We focus on the autocorrelation combining step of coprime-array processing (see Fig. 1) where the receiver applies linear combining matrix $E$ to the estimated autocorrelations of the physical array. Arguably, a preferred receiver will attain consistently (i.e., for any possible configuration of DoAs, $\Theta$) low squared-estimation error $\|r_{co} - E^T \hat{r}\|_2$. If this...
error is exactly equal to zero, then at the fourth step of coprime-array processing, MUSIC will identify the exact DoAs in \( \Theta \).

In view of the above, in this work, we treat the DoAs in \( \Theta \) as independent and identically distributed (i.i.d.) random variables and focus on designing a coprime array receiver equipped with linear combiner \( E \) such that \( \|r_{co} - E^T \hat{r}\|^2 \) is minimized in the mean. We assume that, for any \( k, \theta_k \in \Theta \) is a random variable with probability distribution \( D(a, b) \) (e.g., uniform, truncated normal, or, other) where \( a \) and \( b \) denote the limits of the support of \( D \) and seek the minimum mean-squared-error combining matrix \( E \) which minimizes \( \mathbb{E}\{\|r_{co} - E^T \hat{r}\|^2\} \). In fact, probability distributions of angular variables is not a new concept. Angular variables have been modeled by the von Mises Probability Density Function (PDF) which can include (or, nearly approximate) standard distributions such as Uniform, Gaussian, and wrapped Gaussian, to name a few, by tuning a parameter in the PDF expression \( [28][29] \). Angular distributions have also been considered for Bayesian-based beamforming for millimeter wave channel tracking \( [32] \).

In the most general case and in lieu of any pertinent prior information at the receiver, the DoAs in \( \Theta \) can, for instance, be assumed to be the uniformly distributed in \( (-\frac{\pi}{2}, \frac{\pi}{2}) \) i.e., \( D(a, b) = U(-\frac{\pi}{2}, \frac{\pi}{2}) \). In the sequel, we derive the minimum mean-squared-error combining matrix for any continuous probability distribution \( D(a, b) \) with \(-\frac{\pi}{2} < a < b \leq \frac{\pi}{2} \).

First, we introduce new notation on the problem statement and formulate the MSE minimization problem. By defining

\[
A := \left[ S \, \text{diag}([\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_K}]), \sigma I_L \right] \in \mathbb{C}^{L \times K+L},
\]

the autocorrelation matrix \( R_q \) in (2) is factorized as \( R_q = AA^H \). Accordingly, \( r \) in (3) can be expressed as \( r = (A^* \otimes A) \text{vec}(I_{K+L}) \). Moreover, we define

\[
V := A^* \otimes A \in \mathbb{C}^{L^2 \times (K+L)^2},
\]

and \( i := \text{vec}(I_{K+L}) \in \mathbb{R}^{(K+L)^2} \), where \( I_{K+L} \) is the \((K + L)\)-size identity matrix. Then, \( r \) takes the form

\[
r = Vi.
\]

It follows that \( r_{co} \) (or, \( r_{sel} \) and \( r_{avg} \) in [13] and [17], respectively) can be expressed as \( E_{sel}^T Vi = E_{avg}^T Vi \). Next, we observe that for every \( q \in \{1, 2, \ldots, Q\} \), there exists a vector \( x_q \sim \mathcal{CN}(0_{K+L}, I_{K+L}) \), pertinent to \( y_q \), such that \( y_q = Ax_q \). From the signal model, \( x_q \) is statistically independent from \( x_p \) for any pair \( (p, q) \in \{1, 2, \ldots, Q\}^2 \) with \( p \neq q \). Then, the physical-array autocorrelation matrix estimate \( \hat{R}_q \) in (5), is expressed as \( \hat{R}_q = AWA^H \), where \( W := \frac{1}{Q} \sum_{q=1}^{Q} x_q x_q^H \). Moreover, by defining \( w := \text{vec}(W) = \frac{1}{Q} \sum_{q=1}^{Q} x_q^\otimes x_q \), the estimate \( \hat{r} \) in (10) takes the form

\[
\hat{r} = Vw.
\]

In view of (24) and (25), we propose to design a MMSE linear combiner, \( E_{MMSE} \), by formulating and solving the MSE-minimization problem

\[
\arg \min_{E \in \mathbb{C}^{L^2 \times (K+L-1)^2}} \mathbb{E}_{w} \left\{ \|E^H Vw - E_{sel}^T V_i\|^2 \right\}.
\]

Of course, if we replace \( E_{sel} \) by \( E_{avg} \) in (26), the resulting problem will be equivalent to (26). In the sequel, we show that a closed-form solution for (26) exists for any finite value of sample support \( Q \) and present a step-by-step solution. We commence our solution by defining \( B := Vww^H \in \mathbb{C}^{L^2 \times L^2} \) and \( H := Vw^H \in \mathbb{C}^{L^2 \times L^2} \). Then, the problem in (26) simplifies to

\[
\arg \min_{E \in \mathbb{C}^{L^2 \times (K+L-1)^2}} \mathbb{E}_{w} \left\{ \text{Tr} (E^H GE) - 2 \Re \left\{ \text{Tr} (E^H HE_{sel}) \right\} \right\}.
\]
where \(\Re\{\cdot\}\) extracts the real part of its argument and \(\Tr(\cdot)\) returns the sum of the diagonal entries of its argument. Furthermore, we define \(\mathbf{G}_E := \mathbb{E}_{\theta \sim \mathcal{D}}\{\mathbf{G}\}\) and \(\mathbf{H}_E := \mathbb{E}_{\theta \sim \mathcal{D}}\{\mathbf{H}\}\). Then, (27) takes the equivalent form

\[
\arg\min_{\mathbf{E} \in \mathbb{C}^{2L \times 2L}} \left\{ \Tr(\mathbf{E}^H \mathbf{G}_E \mathbf{E}) - 2\Re\{\Tr(\mathbf{E}^H \mathbf{H}_E \mathbf{E})\} \right\}.
\]

(28)

Next, we focus on deriving closed-form expressions for \(\mathbf{G}_E\) and \(\mathbf{H}_E\) that will allow us to solve (28) and obtain the minimum-MSE linear combiner \(\mathbf{E}_{\text{MMSE}}\). At the core of our developments lies the observation that, for any \(\theta \sim \mathcal{D}(a, b)\) with PDF \(f(\theta)\) and scalar \(x \in \mathbb{R}\), it holds that

\[
\mathbb{E}_{\theta} \{ v(\theta)^x \} = \int_a^b f(\theta) \exp \left( -j x \frac{2\pi f_c}{c} \sin \theta \right) d\theta := \mathcal{I}(x).
\]

(29)

The integral \(\mathcal{I}(x)\) can be approximated within some numerical error tolerance with numerically efficient vectorized methods [33]. In Fig. 2, we offer visual illustration examples of \(f(\theta)\) when \(\theta \sim \mathcal{D}(a, b)\), i.e., \(\theta \sim \mathcal{U}(a, b)\), or, a truncated normal distribution in \((a, b)\) with mean \(\mu\) and variance \(\sigma^2\), i.e., \(\theta \sim \mathcal{T}\mathcal{N}(a, b, \mu, \sigma^2)\). More specifically,

\[
f(\theta) = \begin{cases} 
\frac{1}{\sqrt{\pi} b - a} \exp \left( -\frac{(\theta - \mu)^2}{2\sigma^2} \right), & \theta \sim \mathcal{T}\mathcal{N}(a, b, \mu, \sigma^2), \\
\frac{1}{b - a}, & \theta \sim \mathcal{U}(a, b), 
\end{cases}
\]

(30)

where \(\text{erf}(\cdot)\) denotes the Error Function [34]. In the special case that \(\mathcal{D}(a, b) \equiv \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})\), \(\mathcal{I}(x)\) coincides with \(J_0(x^{2\pi f_c/c})\): the 0-th order Bessel function of the first kind [35] for which there exist look-up tables. Next, we define the indicator function \(\delta(x)\) which equals 1 if \(x = 0\) and assumes a value of zero otherwise. We provide in the following Lemma the statistics of the random variable \(w\) which appears in the closed-form expressions of \(\mathbf{G}_E\) and \(\mathbf{H}_E\).

**Lemma 5.** The first- and second-order statistics of the random variable \(w\) are given by \(\mathbb{E}_w \{ w \} = i \in \mathbb{R}^{(K+L)^2}\) and \(\mathbb{E}_w \{ ww^H \} = i i^\top + \frac{1}{Q} \mathbf{I}_{(K+L)^2} \in \mathbb{R}^{(K+L)^2 \times (K+L)^2}\), respectively. \(\square\)

A proof for Lemma 5 is offered in the Appendix. Next, we define \(\hat{p} := \mathbf{1}_L \otimes p\) and \(\omega_i := |\hat{p}|_i - |\hat{p}|_i\). In view of Lemma 5, we present an entry-wise closed-form expression for \(\mathbf{H}_E\) in the following Lemma.

---

**Figure 2:** Probability density function \(f(\theta)\) for different distributions and support sets.
A proof for Lemma 7 is offered in the Appendix. We differentiate (28) with respect to $V$.

A complete proof for Lemma 6 is also provided in the Appendix. Hereafter, we focus on deriving a closed-form expression for matrix $G$. Accordingly, the closed-form expression for matrix $G$ is given by

$$E_{\Theta} = H_{\Theta} + \frac{1}{Q} \tilde{V}_E.$$  \(\square\)

A proof for Lemma 7 is offered in the Appendix. We differentiate (28) with respect to $E$, set its derivative to zero, and obtain

$$\left( H_E + \frac{1}{Q} \tilde{V}_E \right) E_{\text{MMSE}} = H_E E_{\text{sel}}.$$  \(\square\)

### Table 1: Entry-wise closed-form expression for matrix $V$ defined in (23).

| $[V]_{i,j}$ | Condition on $(i,j)$ |
|-------------|----------------------|
| $\sqrt{d}[\tilde{u}_j^T \bar{d}[\tilde{u}_j^T v(\Theta_{[\tilde{u}_j])} [p_i] v(\Theta_{[\tilde{u}_j]})]}^2$ | $[\tilde{u}]_j, [\tilde{u}]_j \leq K$ |
| $\sigma \sqrt{d}[\tilde{u}_j^T v(\Theta_{[\tilde{u}_j]})] [p_i]$ | $[\tilde{u}]_j \leq K$ and $[\tilde{u}]_j = [\tilde{v}]_i + K$ |
| $\sigma^2 \sqrt{d}[\tilde{u}_j^T v(\Theta_{[\tilde{u}_j]})] [p_i]$ | $[\tilde{u}]_j = [\tilde{v}]_i + K$ and $[\tilde{u}]_j = [\tilde{v}]_i + K$ |
| 0 | otherwise |

Auxiliary variables used in the above conditions/expressions:

$s_x = [1, 2, \ldots, x]^T$, $\tilde{u} = 1_{K+L} \otimes s_{K+L}$, $\tilde{v} = s_{K+L} \otimes 1_{K+L}$, $\tilde{v} = s_{L} \otimes 1_{L}$.  \(\square\)

### Lemma 6. For any $(i, m) \in \{1, 2, \ldots, L^2\}$,

$$[H_E]_{i,m} = \|d\|_2^2 \mathcal{I}(\omega_i - \omega_m) + \sigma^4 \delta(\omega_i) \delta(\omega_m) + \mathcal{I}(\omega_i) \mathcal{I}(-\omega_m) \left( 1_{K}^T d \right)^2 - ||d||_2^2$$  \(31\)

$$+ \sigma^2 \left( 1_{K}^T d \right) \left( \delta(\omega_i) \mathcal{I}(-\omega_m) + \mathcal{I}(\omega_i) \delta(-\omega_m) \right).$$  \(32\)

A complete proof for Lemma 6 is also provided in the Appendix. Hereafter, we focus on deriving a closed-form expression for $G_E$. First, we define $\tilde{V} := VV^H$ whose expectation, $\tilde{V}_E := E_{\Theta}\{\tilde{V}\}$, appears in $G_E$. We observe that each entry of $\tilde{V}$ can be expressed as a linear combination of the entries of $V$. That is, for any $(i, m) \in \{1, 2, \ldots, L^2\}$ and $j \in \{1, 2, \ldots, (K+L)^2\}$, it holds

$$[\tilde{V}]_{i,m} = \sum_{j=1}^{(K+L)^2} [V]_{i,j} [V^*]_{m,j}. $$  \(33\)

An entry-wise closed-form for $V$, in terms of $\Theta$, is offered in Table 1. Then, for any triplet $(i, m, j)$ such that $(i, m) \in \{1, 2, \ldots, L^2\}$ and $j \in \{1, 2, \ldots, (K+L)^2\}$, we derive a closed-form expression for

$$\gamma_{j}^{(i,m)} := [V]_{i,j} [V^*]_{m,j},$$  \(34\)

in Table 2.

Accordingly, $E_{\Theta}\{\gamma_{j}^{(i,m)}\}$ is offered in Table 3 based on which, the $(i, m)$-th entry of $\tilde{V}_E$ is computed as

$$[\tilde{V}_E]_{i,m} = \sum_{j=1}^{(K+L)^2} E_{\Theta}\{\gamma_{j}^{(i,m)}\}. $$  \(35\)

The closed-form expression for matrix $G_E$ is provided in the following Lemma.

### Lemma 7. Matrix $G_E$ is given by $G_E = H_E + \frac{1}{Q} \tilde{V}_E$. \(\square\)
We observe that (36) is, in practice, a collection of solution and a Least Squares (LS) approach can be followed by solving $N$

$$\sigma^2 \|d\|_{\mathbf{u},j} \mathbb{I}(\omega_{m,i} + \omega_{i,m}) \] \]$$

$$\sigma^4 \|d\|_{\mathbf{u},j} \mathbb{I}(\omega_{m,i}) \] \]$$

$$\sigma^4 \|d\|_{\mathbf{u},j} \mathbb{I}(\omega_{i,m}) \] \]$$

Table 2: Closed-form expression for $\gamma_{j}^{(i,m)}$ defined in (34).

| $\gamma_{j}^{(i,m)}$ | Condition on $(i,m,j)$ |
|----------------------|--------------------------|
| $[d\mathbf{u}_j, [d\mathbf{u}_j, v(\theta_{u,j})][p]_{m} - [p]_{m}] \mathbb{I}(\omega_{m,i} + \omega_{i,m}) \] \]$$

$$\sigma^2 \|d\|_{\mathbf{u},j} \mathbb{I}(\omega_{m,i}) \] \]$$

$$\sigma^4 \|d\|_{\mathbf{u},j} \mathbb{I}(\omega_{i,m}) \] \]$$

Auxiliary variables used in the above conditions/expressions:

$\hat{\mathbf{p}} = \mathbf{p} \otimes 1_L, \hat{\mathbf{p}}_i = 1_L \otimes \mathbf{p}, \mathbf{s}_x = [1, 2, \ldots, x] \top, \hat{\mathbf{u}} = 1_{K+L} \otimes \mathbf{s}_{K+L}, \hat{\mathbf{v}} = \mathbf{s}_{K+L} \otimes 1_{K+L}, \hat{\mathbf{v}} = 1_L \otimes \mathbf{s}_L, \hat{\mathbf{v}} = \mathbf{s}_L \otimes 1_L.$

Table 3: closed-form expression for $\mathbb{E}_{\Theta} \{\gamma_{j}^{(i,m)}\}$.

$\mathbb{E}_{\Theta} \{\gamma_{j}^{(i,m)}\}$

| $\gamma_{j}^{(i,m)}$ | Condition on $(i,m,j)$ |
|----------------------|--------------------------|
| $[d\mathbf{u}_j, [d\mathbf{u}_j, v(\theta_{u,j})][p]_{m} - [p]_{m}] \mathbb{I}(\omega_{m,i} + \omega_{i,m}) \] \]$$

$$\sigma^2 \|d\|_{\mathbf{u},j} \mathbb{I}(\omega_{m,i}) \] \]$$

$$\sigma^4 \|d\|_{\mathbf{u},j} \mathbb{I}(\omega_{i,m}) \] \]$$

Auxiliary variables used in the above conditions/expressions:

$\hat{\mathbf{p}} = \mathbf{p} \otimes 1_L, \hat{\mathbf{p}}_i = 1_L \otimes \mathbf{p}, \hat{\omega}_{i,j} = [\hat{\mathbf{p}}]_{i} - [\hat{\mathbf{p}}]_{j}, \hat{\omega}_{i,j} = [\hat{\mathbf{p}}]_{i} - [\hat{\mathbf{p}}]_{j}, \mathbf{s}_x = [1, 2, \ldots, x] \top, \hat{\mathbf{u}} = 1_{K+L} \otimes \mathbf{s}_{K+L}, \hat{\mathbf{v}} = \mathbf{s}_{K+L} \otimes 1_{K+L}, \hat{\mathbf{v}} = 1_L \otimes \mathbf{s}_L, \hat{\mathbf{v}} = \mathbf{s}_L \otimes 1_L.$

We observe that (36) is, in practice, a collection of $(2L' - 1)$ systems of linear equations. Let $\mathbb{E}_{\text{MMSE}} = [\mathbf{e}_1, \ldots, \mathbf{e}_{2L'-1}]$ and $\mathbf{c}_i = [\mathbf{H}_2 \mathbb{E}_{\text{mmse}}]_{i,1} \top \forall i \in \{1, 2, \ldots, 2L'-1\}$. Solving (36) is equivalent to solving, for every $i$,

$$\mathbf{G}_\mathbb{E} \mathbf{e}_i = \mathbf{c}_i.$$ (37)

For any $i$ such that $\mathbf{c}_i \in \text{span}(\mathbf{G}_\mathbb{E})$, (37) has at least one exact solution $\mathbf{e}_i = \mathbf{V} \Sigma^{-1} \mathbf{U}^H \mathbf{c}_i + \mathbf{b}_i$, where $\mathbf{G}_\mathbb{E}$ admits SVD $\mathbf{U}_{L' \times \rho} \Sigma_{\rho \times \rho} \mathbf{V}^H_{\rho \times L'}$, $\rho = \text{rank}(\mathbf{G}_\mathbb{E})$, and $\mathbf{b}_i$ is an arbitrary vector in the nullspace of $\mathbf{G}_\mathbb{E}$ which is denoted by $\mathcal{N}(\mathbf{G}_\mathbb{E})$. In the special case that $\rho = L^2$, that is, $\mathbf{G}_\mathbb{E}$ has full-rank, then $\mathcal{N}(\mathbf{G}_\mathbb{E}) = \mathbf{0}_{L^2}$ and there exists a unique solution $\mathbf{e}_i = \mathbf{V} \Sigma^{-1} \mathbf{U}^H \mathbf{c}_i$. If, on the other hand, $\exists i$ such that $\mathbf{c}_i \notin \text{span}(\mathbf{G}_\mathbb{E})$, then (37) does not have an exact solution and a Least Squares (LS) approach can be followed by solving $\min_{\mathbf{e}_i} \|\mathbf{G}_\mathbb{E} \mathbf{e}_i - \mathbf{c}_i\|_2^2$. Interestingly, it is easy to show that the LS solution is the same as before, i.e., $\mathbf{e}_i = \mathbf{V} \Sigma^{-1} \mathbf{U}^H \mathbf{c}_i + \mathbf{b}_i$, where $\mathbf{b}_i \in \mathcal{N}(\mathbf{G}_\mathbb{E})$. In every case, each column of $\mathbb{E}_{\text{MMSE}}$ can be computed in closed-form as

$$\mathbf{e}_i = \mathbf{V} \Sigma^{-1} \mathbf{U}^H \mathbf{c}_i + \mathbf{b}_i, \quad \mathbf{b}_i \in \mathcal{N}(\mathbf{G}_\mathbb{E}).$$ (38)
In view of the above, we propose to process the autocorrelations in \( \hat{r} \) by the linear combiner \( \mathbf{E}_{\text{MMSE}} \) to obtain the MMSE estimate of \( r_{co} \).

\[
\hat{r}_{\text{MMSE}} := \mathbf{E}_{\text{MMSE}}^T \hat{r}.
\]  

(39)

In turn, we propose to minimum-MSE estimate \( Z \) in (6) by

\[
\hat{Z}_{\text{MMSE}} := \mathbf{F}(\mathbf{I}_{L'} \otimes \hat{r}_{\text{MMSE}}) \in \mathbb{C}^{L' \times L'}.
\]  

(40)

The proposed linear combiner \( \mathbf{E}_{\text{MMSE}} \) depends on \( \mathbf{H}_S \) and \( \mathbf{V}_E \) which, in turn, depend on the powers \( d_1, d_2, \ldots, d_K, \sigma^2 \) associated to the source DoAs \( \theta_1, \theta_2, \ldots, \theta_K \) and noise, respectively. In general, the receiver has no prior knowledge of these powers. Thus, it cannot compute \( \mathbf{E}_{\text{MMSE}} \) directly. This drawback can be addressed with one of the following ways. First, similar to the DoAs \( \{e\text{g., uniform}\} \). In this case, the proposed linear combiner \( \mathbf{E}_{\text{MMSE}} \) will cease to depend on \( \{d_k\}_{k=1}^K, \sigma^2 \) and can be deterministically formed at the receiver, independently from the DoAs \( \{\theta_k\}_{k=1}^K \), their respective powers \( \{d_k\}_{k=1}^K \), and noise power \( \sigma^2 \). Alternatively, we can estimate the powers associated with the DoAs \( \{\theta_k\}_{k=1}^K \) and noise in order to compute the linear combiner \( \mathbf{E}_{\text{MMSE}} \). A simple manner in which to estimate \( \{d_k\}_{k=1}^K \) is the Minimum Variance Distortion-Less Response (MVDR) spectrum, also known as Capon spectrum \([36,39]\), or, its robust version \([40]\). More specifically, we can estimate the DoAs \( \{\theta_k\}_{k=1}^K \) by applying MUSIC on \( \hat{Z}_{\text{avg}} \), obtaining \( \{\hat{\theta}_k\}_{k=1}^K \). Then, the power \( d_k \) of the source with DoA \( \theta_k \) can be estimated by \( d_k = (v^H(\hat{\theta}_k) \hat{Z}_{\text{avg}}^{-1} v(\hat{\theta}_k))^{-1} \), for every \( k = 1, 2, \ldots, K \). The noise power can be estimated by the square root of the smallest eigenvalue of \( \hat{Z}_{\text{avg}} \hat{Z}_{\text{avg}}^H \), yielding \( \hat{\sigma}^2 \). The estimates \( \{d_k\}_{k=1}^K \) and \( \hat{\sigma}^2 \) can then be used for approximating \( \mathbf{E}_{\text{MMSE}} \). Of course, there exist more sophisticated algorithms for estimating the source powers (e.g., \([41]\) estimated the powers \( \{d_k\}_{k=1}^K \) based on a Vandermonde decomposition of \( \sqrt{\frac{1}{L'} \hat{Z}_{\text{sel}} \hat{Z}_{\text{avg}}^H} \)). Interestingly enough, however, the receiver needs not know the exact powers \( \{d_k\}_{k=1}^K, \sigma^2 \). In fact, knowledge of the power ratios suffices to form the exact MMSE combiner. Without loss of generality, we can assume the ratios \( \frac{d_2}{d_1}, \frac{d_3}{d_1}, \ldots, \frac{d_K}{d_1}, \frac{\sigma^2}{d_1} \) to be known and then, \( \mathbf{E}_{\text{MMSE}} \) can be computed exactly. It is easy to see that substituting \( d_2, d_3, \ldots, d_K, \sigma^2 \) in the closed-form expressions of the columns of \( \mathbf{E}_{\text{MMSE}} \) by the corresponding ratios above does not result in any change.

## 5 Numerical Studies

We consider comprime naturals \( M, N \) with \( M < N \) and form an \( L \)-element physical coprime array, which yields a length-(\( L' = MN + M \)) ULA as the virtual coarray. Signals from \( K \) sources impinge on the array with equal transmit power \( d_1 = d_2 = \ldots = d_K = \alpha^2 = 10 \) dB. The noise variance is fixed at \( \sigma^2 = 0 \) dB. Accordingly, the signal-to-noise-ratio (SNR) is equal to \( 10 \) dB for every DoA signal source.

We commence our studies by computing the empirical Cumulative Distribution Function (CDF) of the Normalized-MSE (NMSE) in estimating \( Z \) for a given DoA collection \( \Theta = \{\theta_1, \ldots, \theta_K\} \) such that the DoAs in \( \Theta \) are i.i.d., i.e., \( \theta_k \sim \mathcal{D}(a, b) \) \( \forall k \). More specifically, we consider fixed sample-support \( Q = 10 \) and for each estimate \( \hat{Z} \in \{\hat{Z}_{\text{avg}}, \hat{Z}_{\text{MMSE}}\} \), we compute the estimation error

\[
\text{MNSE} = \left\| Z - \hat{Z} \right\|_F^2 / \left\| Z \right\|_F^2.
\]  

(41)

We repeat this process over 4000 statistically independent realizations of \( \Theta \) and noise and collect 4000 NMSE measurements. Based on these NMSE measurements, we plot in Fig. 5 the empirical CDF of the NMSE in estimating \( Z \) fixing \( (M, N) = (2, 3), K \in \{5, 7\} \), and \( \mathcal{D}(a, b) = \{\mathcal{U}(-\frac{a}{2}, \frac{a}{2}), \mathcal{U}(-\frac{a}{2}, \frac{a}{2}), \mathcal{T}N(-\frac{a}{2}, \frac{a}{2}, 0, 1)\} \). We observe that the proposed MMSE combining approach attains superior MSE in estimating \( Z \) for any distribution and support set

\[\text{Recall that } v(\theta) \text{ is defined in Section } 3 \text{ for any } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).\]
Figure 3: Empirical CDF of the MSE in estimating $Z$ for $(M, N) = (2, 3)$, SNR $= 10$ dB, $Q = 10$, $K = 5$ (top), and $K = 7$ (bottom). \( \forall k, \theta_k \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2}) \) (left), \( \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6}) \) (center), \( \mathcal{T}_N(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1) \) (right).

We illustrate the new CDFs in Fig. 4. Similar observations as in Fig. 3 are made. The proposed MMSE combining approach clearly outperforms its standard counterparts for any distribution assumption and support set for \( \theta_k \forall k \).

We repeat the last study for $(M, N) = (2, 5)$, $K \in \{7, 9\}$, and \( D(a, b) \in \{ \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2}), \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6}), \mathcal{T}_N(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1) \} \).

For $K = 7$ and every other parameter same as above, we plot the NMSE (averaged over 4000 realizations) versus sample support $Q \in \{1, 10, 100, 1000, 10000\}$ in Fig. 5. Consistent with the observations above, we note that selection attains the highest NMSE while the proposed attains, expectedly, the lowest NMSE in estimating $Z$ across the board. The performance gap between the proposed MMSE combining estimate and the estimates based on existing combining approaches decreases as the sample support $Q$ increases. Nonetheless, it remains superior, in many cases, even for high values of $Q$ (e.g., $Q = 10^4$).

Moreover, in the first two subplots (uniform DoA distribution), we notice that the performance gap between the MMSE combining approach and the averaging approach is wider when the range of the support set $(a, b)$ is narrower.

We conclude the performance evaluations with measuring the Root-MSE (RMSE) in estimating the DoAs in $\Theta$ versus sample support $Q$. We first evaluate the MUSIC spectrum on each estimate $\hat{Z} \in \{ \hat{Z}_{sel}, \hat{Z}_{avg}, \hat{Z}_{MMSE} \}$ over the probability distribution support set $(a, b)$ and obtain DoA estimates $\hat{\Theta} = \{ \hat{\theta}_1, \ldots, \hat{\theta}_K \}$. Then, we compute the
Finally, we compute the RMSE by taking the square root of the MSE computed over 4000 statistically independent realizations of $\Theta$ and noise. The resulting RMSE curves are depicted in Fig. 6. For every DoA distribution (even the most general case of uniform distribution in $(-\frac{\pi}{2}, \frac{\pi}{2})$) and every sample support (even as high as $10^4$), the proposed method attains the lowest RMSE.

6 Conclusions

We proposed a novel coprime array receiver that attains minimum MSE in coarray autocorrelation estimation, for any probability distribution of the source DoAs. Moreover, we offered formal mathematical proofs for the closed-form MSE expressions of selection and averaging, which were first presented in [1]. Extensive numerical studies on various DoA distributions demonstrate that the proposed MMSE combining method consistently outperforms its existing counterparts in autocorrelation estimation performance with respect to the MSE metric. In turn, the proposed MMSE combiner enables lower RMSE in DoA estimation.
Figure 5: NMSE in estimating $\mathbf{Z}$ versus sample support $Q$. $(M, N) = (2, 5)$, $\text{SNR} = 10\text{dB}$, and $K = 7$. $\forall k, \theta_k \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$ (left), $U(-\frac{\pi}{4}, \frac{\pi}{6})$ (center), $\mathcal{N}(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1)$ (right).

Figure 6: RMSE (degrees) in estimating the DoA set $\Theta$, versus sample support, $Q$. $(M, N) = (2, 5)$, $\text{SNR} = 10\text{dB}$, and $K = 7$. $\forall k, \theta_k \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$ (left), $U(-\frac{\pi}{4}, \frac{\pi}{6})$ (center), $\mathcal{N}(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1)$ (right).

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7 Appendix

7.1 Proof of Lemma 1

Proof. We define $\mathbf{b}_q := \mathbf{y}_q^* \otimes \mathbf{y}_q$, for every $q = 1, 2, \ldots, Q$. Utilizing the auxiliary variables $\mathbf{v} = \mathbf{1}_L \otimes \mathbf{s}_L$, $\mathbf{p} = \mathbf{p} \otimes \mathbf{1}_L$, $\mathbf{v} = \mathbf{s}_L \otimes \mathbf{1}_L$, and $\mathbf{p} = \mathbf{1}_L \otimes \mathbf{p}$, we find that $[\mathbf{b}_q]_{jn} = [\mathbf{y}_q^*]_{jn}[\mathbf{y}_q]_{jn}$ for any $n \in \{1 - L', 2 - L', \ldots, L' - 1\}$.

Recall that for any $x \in \mathbb{N}_+$, $s_x = [1, 2, \ldots, x]^T$. 
and \( j_n \in \mathcal{J}_n \), with
\[
[Y_q | \varphi]_{j_n} = \sum_{k=1}^K v(\theta_k) [\tilde{p}]_{j_n} \xi_{q,k} + [n_q | \varphi]_{j_n},
\]
(43)
\[
[Y_q | \varphi]_{j_n} = \sum_{k=1}^K v(\theta_k) [\tilde{p}]_{j_n} \xi_{q,k} + [n_q | \varphi]_{j_n}.
\]
(44)

Next, we compute \( \mathbb{E} \{ [b_q]_{j_n} \} = \sum_{k=1}^K v(\theta_k) d_k + \sigma^2 [i_L]_{j_n} = [r]_{j_n} \). The latter implies that \( \mathbb{E} \{ \tilde{r} \} = r \). We also define \( b_{q,p}^{(n)}(i,j) := [b^*_q]_i [b_p]_j \), for \( (i,j) \in \mathcal{J}_n \) and \( n \in \{1 - L', 2 - L', \ldots, L' - 1\} \) and compute
\[
\mathbb{E} \left\{ b_{q,p}^{(n)}(i,j) \right\} = \left| g_n^H d + \delta(n) \sigma^2 \right|^2 + \delta(q - p) [\xi_{i,j}]^2 \left| d + \delta(i - j) \sigma^2 \right|^2,
\]
(45)
where \( g_n := \begin{bmatrix} v(\theta_1)^n & v(\theta_2)^n & \ldots & v(\theta_K)^n \end{bmatrix}^\top \). Next, we proceed as follows.

\[
e = \mathbb{E} \left\{ \left| [r]_{j_n} - [\tilde{r}]_{j_n} \right|^2 \right\}
\]
(46)
\[
= \left| [r]_{j_n} \right|^2 + \mathbb{E} \left\{ [r]_{j_n} [\tilde{r}]_{j_n}^* \right\} - 2 \mathbb{E} \left\{ [r]_{j_n} [\tilde{r}]_{j_n}^* \right\}
\]
(47)
\[
= \mathbb{E} \left\{ [r]_{j_n} [\tilde{r}]_{j_n}^* \right\} - \left| [r]_{j_n} \right|^2
\]
(48)
\[
= \frac{1}{Q^2} \sum_{q=1}^Q \sum_{p=1}^Q [b_q]_{j_n} [b_p]_{j_n} - \left| [r]_{j_n} \right|^2
\]
(49)
\[
= \frac{1}{Q^2} \sum_{q=1}^Q \sum_{p=1}^Q \mathbb{E} \left\{ b_{q,p}^{(n)}(j_n,j_n) \right\} - \left| [r]_{j_n} \right|^2
\]
(50)
\[
\overset{(45)}{=} \frac{(1 - \delta) d + \sigma^2)^2}{Q}.
\]
(51)

\[\square\]

### 7.2 Proof of Lemma 2

**Proof.** By Lemma 1 \( e(\tilde{r}_{sel}) = \mathbb{E} \left\{ \| r_{co} - \tilde{r}_{sel} \|^2 \right\} = \sum_{n=1-L'}^{L'-1} \mathbb{E} \left\{ \left| [r]_{j_n} - [\tilde{r}]_{j_n} \right|^2 \right\} = (2L' - 1) e. \) \( \square \)

### 7.3 Proof of Proposition 1

**Proof.** Notice that \( Z = F(I_{L'} \odot r_{co}) = [F_1 r_{co} \ F_2 r_{co} \ \ldots \ F_L r_{co}] \). By its definition, \( F_m \) is a selection matrix that selects the \( \{L' - (m - 1), L' - (m - 2), \ldots, 2L' - m\} \)-th entries of the length-\( (2L' - 1) \) vector it multiplies, for every \( m \in \{1, 2, \ldots, L'\} \). That is, \( F_m r_{co} = [r_{co}]_{L' - (m - 1):2L' - m} \). Similarly, \( Z_{sel} = [F_1 \tilde{r}_{sel} \ F_2 \tilde{r}_{sel} \ \ldots \ F_L \tilde{r}_{sel}] \)

\[\overset{5}{\text{Recall that for any } i \in \{1, 2, \ldots, L\}, \omega_i = [\tilde{p}]_i - [\hat{p}]_i.} \]

15
with $F_m \hat{r}_{sel} = [\hat{r}_{sel}]_{L'-(m-1):2L'-m}$. In view of the above,
\[
e(\hat{Z}_{\text{sel}}) = E \left\{ \left\| Z - \hat{Z}_{\text{sel}} \right\|_F^2 \right\}
\]
\[= E \left\{ \sum_{m=1}^{L'} \left\| F_m r_{co} - F_m \hat{r}_{sel} \right\|_2^2 \right\}
\]
\[= \sum_{m=1}^{L'} E \left\{ \left( \left\| r_{co} \right\|_{L'-(m-1):2L'-m} - [\hat{r}_{sel}]_{L'-(m-1):2L'-m} \right) \right\}^2 \]
\[= \sum_{m=1}^{L'} \sum_{n=1-m}^{L'-m} E \left\{ \left| r_{co} \right|_{L'+n} - [\hat{r}_{sel}]_{L'+n} \right\}^2 \]
\[= \sum_{m=1}^{L'} \sum_{n=1-m}^{L'-m} e \]
\[= L'^2 e. \]  

\[\square\]

7.4 Proof of Lemma 3

Proof.
\[
e_n = E \left\{ \left( r_{j,n} - \frac{1}{|J_n|} \sum_{j \in J_n} [\hat{r}_j] \right)^2 \right\}
\]
\[= \left| r_{j,n} \right|^2 + \frac{1}{|J_n|^2} E \left\{ \left| \sum_{j \in J_n} [\hat{r}_j] \right|^2 \right\} - 2E \left\{ \Re \left\{ \left( \frac{1}{|J_n|} \sum_{j \in J_n} [\hat{r}_j] \right) r_{j,n} \right\} \right\}
\]
\[= \left| r_{j,n} \right|^2 + \frac{1}{|J_n|^2} \sum_{j \in J_n} \sum_{i \in J_n} \frac{1}{Q^2} \sum_{q=1}^{Q} \sum_{p=1}^{Q} E \left\{ h_{q,i}(i,j) \right\} - 2\Re \left\{ \left( \frac{1}{|J_n|} \sum_{j \in J_n} [\hat{r}_j] \right) r_{j,n} \right\}
\]
\[= 1 \left( \frac{2\sigma^2 \hat{r}_j d + \sigma^4}{|J_n|} + \sum_{i \in J_n} \frac{|a_{i,j}|^2}{|J_n|^2} \right). \]

7.5 Proof of Lemma 4

Proof. By Lemma 3, $e(r_{avg}) = E \left\{ \left\| r_{co} - r_{avg} \right\|_2^2 \right\} = \sum_{n=1-L'} E \left\{ \left| r_{j,n} \right| - \frac{1}{|J_n|} \sum_{j \in J_n} [\hat{r}_j] \right\}^2 = \sum_{n=1-L'} e_n$.

7.6 Proof of Proposition 2

Proof. We know that $Z = [F_1 r_{co} \ldots F_{L'} r_{co} ]$. Similarly, $\hat{Z}_{avg} = [F_1 \hat{r}_{avg} \ldots F_{L'} \hat{r}_{avg} ]$. By the definition of $F_m$, for every $m \in \{1, 2, \ldots, L'\}$ it holds that $F_m r_{co} = [r_{co}]_{L'-(m-1):2L'-m}$ and $F_m \hat{r}_{avg} =$
We recall that

\[
[F_{\text{avg}}]_{L'-(m-1):2L'-m}.
\]

In view of the above,

\[
e(\bar{Z}_{\text{avg}}) = \mathbb{E} \left\{ \left\| Z - \bar{Z}_{\text{avg}} \right\|_F^2 \right\}
\]

\[= \mathbb{E} \left\{ \sum_{m=1}^{L'} \left\| F_m r_c - F_m \bar{Z}_{\text{avg}} \right\|_2^2 \right\}
\]

\[= \sum_{m=1}^{L'} \mathbb{E} \left\{ \left\| [r_c]_{L'-(m-1):2L'-m} - [\bar{Z}_{\text{avg}}]_{L'-(m-1):2L'-m} \right\|_2^2 \right\}
\]

\[= \sum_{m=1}^{L'} \sum_{n=1-m}^{L'-m} \mathbb{E} \left\{ \left\| [r_c]_{L'+n} - [\bar{Z}_{\text{avg}}]_{L'+n} \right\|_2^2 \right\}
\]

\[= \sum_{m=1}^{L'} \sum_{n=1-m}^{L'-m} e_n.
\]

\[7.7 \text{ Proof of Lemma } 5\]

We recall that \( w = \frac{1}{\sigma} \sum_{q=1}^{Q} x_q \otimes x_q \). Next, we notice that by utilizing the auxiliary variables \( u = 1_{K+L} \otimes s_{K+L} \) and \( \bar{u} = s_{K+L} \otimes 1_{K+L} \), we obtain

\[
[w]_i = \frac{1}{\sigma} \sum_{q=1}^{Q} [x_q^*]_i [x_q]_i [x_q^*]_m [x_p]_i [x_p]_m.
\]

Then, we define \( \mathcal{I} := \{ i \in \{1, 2, \ldots, (K + L)^2 \} : [i]_i = 1 \} \), and observe that \( \mathbb{E} \{ [x_q^*]_i [x_q]_i \} = \delta ([i]_i - [\bar{u}]_i) \) which is equal to 1, if \( i \in \mathcal{I} \) and 0 if \( i \notin \mathcal{I} \). The latter implies that

\[
\mathbb{E} \{ w \} = i.
\]

Next, for \( (i, m) \in \{1, 2, \ldots, (K + L)^2 \} \), we define \( \eta_{i,m} := [x_q^*]_i [x_q]_i [x_q^*]_m [x_p]_i [x_p]_m \). It holds that

\[
[w]_i [w^*]_m = \frac{1}{\sigma^2} \sum_{q=1}^{Q} \sum_{p=1}^{Q} \eta_{i,m}.
\]

Exploiting the 2-nd and 4-th order moments of zero-mean independent normal variables, we find that \( \mathbb{E} \{ \eta_{i,m} \} \) is equal to 1 + \( \delta (p-q) \delta (i-m) \) if \( (i, m) \in \mathcal{I} \), and 0 otherwise. The latter implies that \( \mathbb{E} \{ [w]_i [w^*]_m \} = \frac{1}{\sigma^2} \sum_{q=1}^{Q} \sum_{p=1}^{Q} \mathbb{E} \{ \eta_{i,m} \} \) is equal to 1 + \( \frac{1}{\sigma^2} \delta (i,m) \), if \( (i, m) \in \mathcal{I} \) and 0 otherwise. Altogether, we have

\[
\mathbb{E}_w \{ ww^T \} = ii^T + \frac{1}{Q} I_{K+L}.
\]

\[7.8 \text{ Proof of Lemma } 6\]

Proof. For \( (i, m) \in \{1, 2, \ldots, L^2 \} \), \( [H]_{i,m} = [Vw]_i [(Vi)^*]_m = \sum_{j=1}^{(K+L)^2} [V]_{i,j} [w]_j [V^*]_{m,j} [i]_i \). Accordingly, \( [H]_{i,m} = \sum_{j=1}^{(K+L)^2} \mathbb{E}_\Theta \mathbb{E}_w \{ [V]_{i,j} [w]_j [V^*]_{m,j} [i]_i \} \). Considering that the random variables \( w \) and \( V \) are statistically independent from each other and that \( \mathbb{E}_w \{ w \} = i \) (see Lemma 5), we obtain

\[
[H]_{i,m} = \mathbb{E}_\Theta \{ [V]_{i,i} [(Vi)^*]_m \} = \mathbb{E}_\Theta \{ [r]_i [r^*]_m \}.\]

Then, we substitute \( \hat{r}_{ij} = \sum_{k=1}^{K} v(\theta_k) \omega_i d_k + \sigma^2 [l]_i \) in \( \mathbb{E}_\Theta \{ [r]_i [r^*]_m \} \) and perform plain algebraic operations, obtaining

\[
[H]_{i,m} = \|d\|^2 \mathcal{I}(\omega_i - \omega_m) + \sigma^4 \delta(\omega_i) \delta(\omega_m) + \sigma^2 \left(1_K^T d\right) \left(\delta(\omega_i) \mathcal{I}(-\omega_m) + \mathcal{I}(\omega_i) \delta(\omega_m)\right) + \mathcal{I}(\omega_i) \mathcal{I}(-\omega_m) \left(1_K^T d^2 - \|d\|^2\right).
\]

\[\square\]

\[\text{Recall that for any } x \in \mathbb{N}_+, s_x = [1, 2, \ldots, x]^T.\]

\[\text{Recall that for any } i \in \{1, 2, \ldots, L^2\}, \omega_i = [\bar{p}]_i - [\bar{p}]_i.\]
7.9 Proof of Lemma}\ref{lemma:proof}

Proof. For $(i, m) \in \{1, 2, \ldots, L^2\}$ it holds $[G]_{i,m} = [Vw]_i([Vw]^*)_m = \sum_{j=1}^{(K+L)^2} [V]_{i,j} [w]_j [V^*]_{m,l} [w^*]_l$. Accordingly, $[G_E]_{i,m} = \sum_{j=1}^{(K+L)^2} E_{\Theta} \{ [V]_{i,j} E_w \{ [w]_j [V^*]_{m,l} [w^*]_l \} \}$. Next, we recall that the random variables $\Theta$ and $w$ are statistically independent from each other. Thus, $[G_E]_{i,m} = \sum_{j=1}^{(K+L)^2} E_{\Theta} \{ [V]_{i,j} E_w \{ [w]_j [V^*]_{m,l} [w^*]_l \} \}$. The latter is equivalent to $G_E = \sum_{j=1}^{(K+L)^2} E_{\Theta} \{ [V]_{i,j} E_w \{ [w]_j [w^*]_l \} [V^*]_{m,l} \}$. By Lemma\ref{lemma:proof} we find that $G_E = H_E + \frac{1}{Q} \tilde{V}_E$. □

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