LYAPUNOV OPERATOR $\mathcal{L}$ WITH DEGENERATE KERNEL AND GIBBS MEASURES

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Abstract. In this paper we'll give a connection between four competing interactions (external field, nearest neighbor, second neighbors and triples of neighbors) of models with uncountable (i.e. $[0,1]$) set of spin values on the Cayley tree of order two and Lyapunov integral equation. Also we'll study fixed points of Lyapunov operator with degenerate kernel which each fixed point of the operator is correspond to a translation-invariant Gibbs measure.

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1. Preliminaries

Spin systems on lattices are a large class of systems considered in statistical mechanics. Some of them have a real physical meaning, others are studied as suitably simplified models of more complicated systems [6], [11].

The various partial cases of Ising model have been investigated in numerous works, for example, the case $J_3 = \alpha = 0$ was considered in [8] and [9], the exact solutions of an Ising model with competing restricted interactions with zero external field was presented. In [10] it is proved that there are two translation-invariant and uncountable number of distinct non-translation-invariant extreme Gibbs measures. In [5] the phase transition problem was solved for $\alpha = 0, J \cdot J_1 \cdot J_3 \neq 0$ and for $J_3 = 0, \alpha \cdot J \cdot J_1 \neq 0$ as well. In [4] it's considered Ising model with four competing interactions (i.e., $J \cdot J_1 \cdot J_3 \cdot \alpha \neq 0$) on the Cayley tree of order two. Mainly these papers are devoted to models with a finite set of spin values and in [13] given other important results on a Cayley tree. In [3] the Potts model with a countable set of spin values on a Cayley tree is considered and it was showed that the set of translation-invariant splitting Gibbs measures of the model contains at most one point, independently on parameters of the Potts model with countable set of spin values on the Cayley tree.

It has been considering Gibbs measures for models with uncountable set of spin values for last five years. Until now it has been considered models with nearest-neighbor interactions ($J_3 = J = \alpha = 0, J_1 \neq 0$) and with the set $[0,1]$ of spin values on a Cayley tree and gotten following results: "Splitting Gibbs measures" of the model on a Cayley tree of order $k$ is described by solutions of a nonlinear integral equation. For $k = 1$ it’s shown that the integral equation has a unique solution (i.e., there is a unique Gibbs measure). For periodic splitting Gibbs measures it was found a sufficient condition under which the measure is unique and was proved existence of phase transitions on a Cayley tree of order $k \geq 2$ (see [1], [2], [12]).

In [13] it’s considered splitting Gibbs measures for four competing interactions i.e. ($J \cdot J_1 \cdot J_3 \cdot \alpha \neq 0$) of models with uncountable set of spin values on the Cayley tree of order two. In this paper we'll give a connection between Gibbs measures for a given model and solutions of
Lyapunov integral equations. Also we’ll study fixed points of Lyapunov operator with degenerate kernel. Each fixed point of the operator is correspond to a translation-invariant Gibbs measure.

A Cayley tree $\Gamma^k = (V, L)$ of order $k \in \mathbb{N}$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k + 1$ edges incident to each vertices. Here $V$ is the set of vertices and $L$ that of edges (arcs). Two vertices $x$ and $y$ are called nearest neighbors if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y \rangle$. The distance $d(x, y), x, y \in V$ on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d| x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \text{ such that the pairs }$$

$$< x_0, x_1 >, ..., < x_{d-1}, x_d > \text{ are neighboring vertices}\}.$$

Let $x^0 \in V$ be a fixed and we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \leq n\},$$

$$L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$$

The set of the direct successors of $x$ is denoted by $S(x)$, i.e.

$$S(x) = \{y \in W_{n+1} \mid d(x, y) = 1\}, \quad x \in W_n.$$ We observe that for any vertex $x \neq x^0$, $x$ has $k$ direct successors and $x^0$ has $k + 1$. The vertices $x$ and $y$ are called second neighbor which is denoted by $> x, y <$, if there exist a vertex $z \in V$ such that $x, z$ and $y, z$ are nearest neighbors. We will consider only second neighbors $> x, y <$, for which there exist $n$ such that $x, y \in W_n$. Three vertices $x, y$ and $z$ are called a triple of neighbors and they are denoted by $< x, y, z >$, if $< x, y >, < y, z >$ are nearest neighbors and $x, z \in W_n$, $y \in W_{n-1}$, for some $n \in \mathbb{N}$.

Now we consider models with four competing interactions where the spin takes values in the set $[0, 1]$. For some set $A \subset V$ an arbitrary function $\sigma_A : A \rightarrow [0, 1]$ is called a configuration and the set of all configurations on $A$ we denote by $\Omega_A = [0, 1]^A$. Let $\sigma(\cdot)$ belong to $\Omega_V = \Omega$ and $\xi_1 : (t, u, v) \in [0, 1]^3 \rightarrow \xi_1(t, u, v) \in R$, $\xi_i : (u, v) \in [0, 1]^2 \rightarrow \xi_i(u, v) \in R$, $i \in \{2, 3\}$ are given bounded, measurable functions. Then we consider the model with four competing interactions on the Cayley tree which is defined by following Hamiltonian

$$H(\sigma) = -J_3 \sum_{<x,y,z>} \xi_1(\sigma(x), \sigma(y), \sigma(z)) - J \sum_{x,y<} \xi_2(\sigma(x), \sigma(z))$$

$$- J_1 \sum_{<x,y>} \xi_3(\sigma(x), \sigma(y)) - \alpha \sum_{x \in V} \sigma(x),$$

(1.1)

where the sum in the first term ranges all triples of neighbors, the second sum ranges all second neighbors, the third sum ranges all nearest neighbors and $J, J_1, J_3, \alpha \in R \setminus \{0\}$. Let $h : [0, 1] \times V \setminus \{x^0\} \rightarrow \mathbb{R}$ and $|h(t, x)| = |h_{t,x}| < C$ where $x_0$ is a root of Cayley tree and $C$ is a constant which does not depend on $t$. For some $n \in \mathbb{N}$, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and $Z_n$ is the corresponding partition function we consider the probability distribution $\mu^{(n)}$ on $\Omega_{V_n}$ defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \left(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x}\right),$$

(1.2)
We shall find positive continuous solutions to (2.1) i.e. such that $f$ and corresponding distributions $\mu$ are compatible if $\mu^{(n)}$ satisfies the following condition:

$$
\int_\Omega^{(n)} \mu_{\sigma_{n-1} \vee \omega_n} (\lambda_{W_n} \times \lambda_{W_n}) (d\omega_n) = \mu_{\sigma_{n-1}}^{(n-1)}.
$$

Let $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega_n \in \Omega_{V_n}$ is the concatenation of $\sigma_{n-1}$ and $\omega_n$. For $n \in \mathbb{N}$ we say that the probability distributions $\mu^{(n)}$ are compatible if $\mu^{(n)}$ satisfies the following condition:

$$
\int_\Omega^{(n)} \mu_{\sigma_{n-1} \vee \omega_n} (\lambda_{W_n} \times \lambda_{W_n}) (d\omega_n) = \mu_{\sigma_{n-1}}^{(n-1)}.
$$

By Kolmogorov's extension theorem there exists a unique measure $\mu$ on $\Omega_V$ such that, for any $n$ and $\sigma_n \in \Omega_{V_n}$, $\mu \left( \{ \sigma \mid V_n = \sigma_n \} \right) = \mu^{(n)}(\sigma)$. The measure $\mu$ is called splitting Gibbs measure corresponding to Hamiltonian (1.1) and function $x \mapsto h_x$, $x \neq x^0$.

Denote

$$
K(u, t, v) = \exp \{ J_3 \beta \xi_1 (t, u, v) + J_2 \xi_2 (u, v) + J_1 \beta (\xi_3 (t, u) + \xi_3 (t, v)) + \alpha \beta (u + v) \},
$$

and

$$
f(t, x) = \exp (h_{t, x} - h_{0, x}), \quad (t, u, v) \in [0, 1]^3, \quad x \in V \setminus \{x^0\}.
$$

The following statement describes conditions on $h_x$ guaranteeing compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

**Theorem 1.1.** The measure $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$ satisfies the consistency condition (1.4) iff for any $x \in V \setminus \{x^0\}$ the following equation holds:

$$
f(t, x) = \prod_{y \leq z \in S(x)} \frac{\int_0^1 \int_0^1 K(t, u, v) f(u) f(v) dudv}{\int_0^1 \int_0^1 K(0, u, v) f(u) f(v) dudv},
$$

where $S(x) = \{y, z\}$, $<y, z, x>$ is a ternary neighbor and $du = \lambda(du)$ is the Lebesgue measure.

2. **Lyapunov's Operator L with Degenerate Kernel**

Now we consider the case $J_3 \neq 0$, $J = J_1 = \alpha = 0$ for the model (1.1) in the class of translational-invariant functions $f(t, x)$ i.e $f(t, x) = f(t)$, for any $x \in V$. For such functions equation (1.1) can be written as

$$
f(t) = \frac{\int_0^1 \int_0^1 K(t, u, v) f(u) f(v) dudv}{\int_0^1 \int_0^1 K(0, u, v) f(u) f(v) dudv},
$$

where $K(t, u, v) = \exp \{ J_3 \beta \xi_1 (t, u, v) + J_2 \xi_2 (u, v) + J_1 \beta (\xi_3 (t, u) + \xi_3 (t, v)) + \alpha \beta (u + v) \}$, $f(t) > 0$, $t, u \in [0, 1]$.

We shall find positive continuous solutions to (2.1) i.e. such that $f \in C^+[0, 1] = \{ f \in C[0, 1] : f(x) \geq 0 \}$.

Define a nonlinear operator $H$ on the cone of positive continuous functions on $[0, 1]$

$$
(Hf)(t) = \frac{\int_0^1 \int_0^1 K(t, s, u) f(s) f(u) dsdu}{\int_0^1 \int_0^1 K(0, s, u) f(s) f(u) dsdu}.
$$
We’ll study the existence of positive fixed points for the nonlinear operator $H$ (i.e., solutions of the equation (2.1)). Put $C_0^+[0,1] = C^+[0,1] \setminus \{\theta \equiv 0\}$. Then the set $C^+[0,1]$ is the cone of positive continuous functions on $[0,1]$.

We define the Lyapunov integral operator $L$ on $C[0,1]$ by the equality (see [7])

$$
L f(t) = \int_0^1 K(t,s,u)f(s)f(u)dsdu.
$$

Put

$$
M_0 = \{ f \in C^+[0,1] : f(0) = 1 \}.
$$

Denote by $N_{fix,p}(H)$ and $N_{fix,p}(L)$ are the set of positive numbers of nontrivial positive fixed points of the operators $N_{fix,p}(H)$ and $N_{fix,p}(L)$, respectively.

**Theorem 2.1.** [13]

i) The equation

$$
H f = f, \quad f \in C_0^+[0,1]
$$

has a positive solution iff the Lyapunov equation

$$
L g = \lambda g, \quad g \in C^+[0,1]
$$

has a positive solution in $M_0$ for some $\lambda > 0$.

ii) The equation $H f = f$ has a nontrivial positive solution iff the Lyapunov equation $L g = g$ has a nontrivial positive solution.

iii) The equation

$$
L f = \lambda f, \quad \lambda > 0
$$

has at least one solution in $C_0^+[0,1]$.

iv) The equation (2.2) has at least one solution in $C_0^+[0,1]$.

v) The equality $N_{fix,p}(H) = N_{fix,p}(L)$ is hold.

Let $\varphi_1(t), \varphi_2(t)$ and $\psi_1(t), \psi_2(t)$ are positive functions from $C_0^+[0,1]$. We consider Lyapunov’s operator $L$

$$
(Lf)(t) = \int_0^1 (\psi_1(t)\varphi_1(u) + \psi_2(t)\varphi_2(v))f(u)f(v)dudv.
$$

and quadratic operator $P$ on $\mathbb{R}^2$ by the rule

$$
P(x,y) = (\alpha_{11} x^2 + \alpha_{12} xy + \alpha_{22} y^2, \beta_{11} x^2 + \beta_{12} xy + \beta_{22} y^2).
$$

$$
\alpha_{11} = \int_0^1 \int_0^1 \psi_1(u)\psi_1(v)\varphi_2(v)dudv, \quad \alpha_{12} = \int_0^1 \int_0^1 (\psi_1(v)\psi_2(u) + \psi_1(u)\psi_2(v))\varphi_2(v)dudv
$$

$$
\alpha_{22} = \int_0^1 \int_0^1 \psi_2(u)\psi_2(v)\varphi_2(v)dudv, \quad \beta_{11} = \int_0^1 \int_0^1 \psi_1(u)\psi_1(v)\varphi_1(u)dudv,
$$

$$
\beta_{12} = \int_0^1 \int_0^1 (\psi_1(u)\psi_2(v) + \psi_1(v)\psi_2(u))\varphi_1(u)dudv, \quad \beta_{22} = \int_0^1 \int_0^1 \psi_2(u)\psi_2(v)\varphi_1(u)dudv.
$$

**Lemma 2.2.** The Lyapunov’s operator $L$ has a nontrivial positive fixed point iff the quadratic operator $P$ has a nontrivial positive fixed point, moreover $N_{fix}^+(H_k) = N_{fix}^+(P)$. 

Proof. a) Put
\[ R^+_2 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}, \quad R^-_2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}. \]
Let \( f(t) \in C^+_0 [0, 1] \) be a nontrivial positive fixed point of \( L \). Let
\[ c_1 = \int_{0}^{1} \varphi_1(u)f(u)f(v)dudv, \quad c_2 = \int_{0}^{1} \varphi_2(u)f(u)f(v)dudv. \]
Clearly, \( c_1 > 0, \ c_2 > 0 \) and \( f(t) = c_1 \psi_1(t) + c_2 \psi_2(t) \). If we put \( f(t) = c_1 \psi_1(t) + c_2 \psi_2(t) \) to the equation \((2.5)\) we’ll get
\[ c_1 = \alpha_{11}c_1^2 + \alpha_{12}c_1c_2 + \alpha_{22}c_2^2, \quad c_2 = \beta_{11}c_1^2 + \beta_{12}c_1c_2 + \beta_{22}c_2^2. \]
Therefore, the point \((c_1, c_2)\) is fixed point of the quadratic operator \( P \).

b) Assume, that the point \((x_0, y_0)\) is a nontrivial positive fixed point of the quadratic operator \( P \), i.e. \((x_0, y_0) \in \mathbb{R}^+_2 \setminus \{0\}\) and numbers \(x_0, y_0\) satisfies following equalities
\[ a_{11}x_0^2 + a_{12}x_0y_0 + a_{22}y_0^2 = x_0, \quad \beta_{11}x_0^2 + \beta_{12}x_0y_0 + \beta_{22}y_0^2 = y_0. \]
Similarly, we can prove that the function \( f_0(t) = x_0\psi_1(t) + y_0\psi_2(t) \) is a fixed point of the operator \( L \) and \( f_0(t) \in C^+_0 [0, 1] \). This completes the proof. \( \square \)

We define positive quadratic operator \( Q \):
\[ Q(x, y) = (a_{11}x^2 + a_{12}xy + a_{22}y^2, \ b_{11}x^2 + b_{12}xy + b_{22}y^2). \]

**Proposition 2.3.**

i) If \( \omega = (x_0, y_0) \in \mathbb{R}^+_2 \) is a positive fixed point of \( Q \), then \( \lambda_0 = \frac{x_0}{y_0} \) is a root of the following equation
\[ a_{11}\lambda^3 + (a_{12} - b_{11})\lambda^2 + (a_{22} - b_{12})\lambda - b_{22} = 0. \]

ii) If the positive number \( \lambda_0 \) is a positive root of the equation \((2.6)\), then the point \( \omega_0 = (\lambda_0y_0, y_0) \) is a positive fixed point of \( Q \), where \( y_0^{-1} = a_{11} + a_{12}\lambda_0 + a_{22}\lambda_0^2 \).

**Proof.** i) Let the point \( \omega = (y_0, x_0) \in \mathbb{R}^+_2 \) be a fixed point of \( Q \). Then
\[ a_{11}x_0^2 + a_{12}x_0y_0 + a_{22}y_0^2 = x_0, \quad b_{11}x_0^2 + b_{12}x_0y_0 + b_{22}y_0^2 = y_0 \]
Using the equality \( \frac{x_0}{y_0} = \lambda_0 \) we obtain
\[ a_{11}\lambda_0^2y_0^2 + a_{12}\lambda_0y_0^2 + a_{22}y_0^2 = \lambda_0y_0, \quad b_{11}\lambda_0^2y_0^2 + b_{12}\lambda_0y_0^2 + b_{22}y_0^2 = y_0. \]
Thus we get
\[ \frac{a_{11}\lambda_0^2 + a_{12}\lambda_0 + a_{22}}{b_{11}\lambda_0^2 + b_{12}\lambda_0 + b_{22}} = \lambda_0. \]

Consequently,
\[ a_{22} + (a_{12} - b_{22})\lambda_0 + (a_{11} - b_{12})\lambda_0^2 - b_{11}\lambda_0^3 = 0. \]

ii) Let \( \lambda_0 > 0 \) is a root of the cubic equation \((2.6)\). Put \( x_0 = \lambda_0y_0 \), where
\[ x_0 = \frac{\lambda_0}{a_{11}\lambda_0^2 + 2a_{12}\lambda_0 + a_{22}}. \]
Since
\[ a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 = \frac{1}{a_{11}\lambda_0^2 + 2a_{12}\lambda_0 + a_{22}}, \]
we get
\[ a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 = y_0. \]
By the other hand
\[ a_{22} + (a_{12} - b_{22})\lambda_0 + (a_{11} - b_{12})\lambda_0^2 - b_{11}\lambda_0^3 = 0. \]
Then we get
\[ b_{11}\lambda_0^2 + b_{12}\lambda_0 + b_{22} = \lambda_0(a_{11}\lambda_0^2 + a_{12}\lambda_0 + a_{22}). \]
From the last equality we get
\[ \frac{\lambda_0}{a_{11}\lambda_0^2 + a_{12}\lambda_0 + a_{22}} = \frac{b_{11}\lambda_0^2 + b_{12}\lambda_0 + b_{22}}{(a_{11}\lambda_0^2 + a_{12}\lambda_0 + a_{22})^2} = b_{11}x_0^2 + 2b_{12}x_0y_0 + b_{22}y_0^2 = y_0. \]
This completes the proof. \(\square\)

Denote
\[ P(\lambda) = \alpha_{11}\lambda^2 + (\alpha_{12} - \beta_{11})\lambda^2 + (\alpha_{22} - \beta_{12})\lambda - \beta_{22} = 0, \mu_0 = \alpha_{11}, \mu_1 = \alpha_{12} - \beta_{11}, \mu_2 = \alpha_{22} - \beta_{12}, \mu_3 = \beta_{22}, \]
\[ P_3(\xi) = \mu_0\xi^3 + \mu_1\xi^2 + \mu_2\xi - \mu_3, \quad (2.7) \]
\[ D = \mu_1^2 - 3\mu_0\mu_2, \quad \alpha = -\frac{\mu_1 + \sqrt{D}}{3\mu_0}, \quad \beta = -\frac{\mu_1 - \sqrt{D}}{3\mu_0}. \]

**Theorem 2.4.** Let \( Q \) satisfies one of the following conditions

i) \( D \leq 0; \)

ii) \( D > 0, \beta \leq 0; \)

iii) \( D > 0, \alpha \leq 0, \beta > 0; \)

iv) \( D > 0, \alpha > 0, P_3(\alpha) < 0; \)

v) \( D > 0, \alpha > 0, P_3(\alpha) > 0, P_3(\beta) > 0, \) then \( Q \) has a unique nontrivial positive fixed point.

**Proof.** The proof of Theorem 2.4 is basis on monotonous property of the function \( P_3(\xi). \) Clearly,
\[ (P_3(\xi))' = 3\mu_0\xi^2 + 2\mu_1\xi + \mu_2. \quad (2.8) \]
and \( P_3'(\alpha) = P_3'(\beta) = 0. \) Moreover,

i) In the case \( D \leq 0, \) by the equality (2.8) the function \( P_3(\xi) \) is an increasing function on \( \mathbb{R} \) and \( P_3(0) = -b_{11} < 0. \) Therefore, the polynomial \( P_3(\xi) \) has a unique positive root.

ii) Let \( D > 0 \) and \( \beta \leq 0. \) For the case \( D > 0 \) the function \( P_3(\xi) \) is an increasing function on \( (-\infty, \alpha) \cup (\beta, \infty) \) and it is a decreasing function on \( (\alpha, \beta) \). Hence from the inequality \( P_3(0) < 0 \)
the polynomial \( P_3(\xi) \) has a unique positive root.

iii) Let \( D > 0, \alpha \leq 0 \) and \( \beta > 0. \) Since the function \( P_3(\xi) \) is decreasing on \( (\alpha, \beta) \) and increasing on \( (\beta, \infty) \) the polynomial \( P_3(\xi) \) has a unique positive root as \( P_3(0) < 0. \)

iv) Let \( D > 0, \alpha > 0 \) and \( P_3(\alpha) < 0. \) Then \( \max_{\xi \in (-\infty, \beta)} P_3(\xi) = P_3(\alpha) < 0. \) Consequently,
by the function \( P_3(\xi) \) is increasing on \( (\beta, \infty) \) the polynomial \( P_3(\xi) \) has a unique positive root \( \xi_0 \in (\beta, \infty). \)

v) Let \( D > 0, \alpha > 0, P_3(\alpha) > 0 \) and \( P_3(\beta) > 0. \) Then \( \min_{\xi \in (\alpha, \infty)} P_3(\xi) = P_3(\beta) > 0. \) From
the function \( P_3(\xi) \) on \( (-\infty, \alpha), \) \( P_3(\xi) \) \( P_3(\xi) \) has a unique positive root \( \xi_0 \in (0, \alpha), \) as \( P_3(0) < 0 \)
and \( P_3(\alpha) > 0. \)

From the upper analysis and by Lemmas 2.3 it follows that the Theorem 2.4 \( \square \)

**Theorem 2.5.** Let be \( D > 0. \) If \( Q \) satisfies one of the following conditions

i) \( \alpha > 0, P_3(\alpha) = 0, P_3(\beta) < 0; \)

ii) \( \alpha > 0, P_3(\alpha) > 0, P_3(\beta) = 0, \) then \( QO \) has two nontrivial positive fixed points and \( N_{fix}^+(Q) = N_{fix}(Q) = 2. \)
Proof. i) Let $\alpha > 0$, $P_3(\alpha) = 0$ and $P_3(\beta) < 0$. Then $\max_{\xi \in (-\infty, \beta)} P_3(\xi) = P_3(\alpha) = 0$ and $\xi_1 = \alpha$ is the root of the polynomial $P_3(\xi)$. By the increase property on $(\beta, \infty)$ of the function $P_3(\xi)$ the polynomial $P_3(\xi)$ has a root $\xi_1 \in (0, \alpha)$. By the other hand $\min_{\xi \in (\alpha, \infty)} P_3(\xi) = P_3(\beta) = 0$ and the number $\xi_2 \in (0, \alpha)$ is the second positive root of the polynomial $P_3(\xi)$. The polynomial $P_3(\xi)$ has not another roots. From above and by Lemmas 2.3 we get Theorem 2.6.

**Theorem 2.6.** Let be $D > 0$. If $Q$ satisfies one of the following conditions

i) $\alpha > 0$, $P_3(\alpha) = 0$ and $P_3(\beta) < 0$,

ii) $\alpha > 0$, $P_3(\alpha) > 0$, $P_3(\beta) = 0$,

then $Q$ has two nontrivial positive fixed points and $N^+_{fix}(Q) = N^+_{fix}(Q) = 2$.

Proof. i) Let $\alpha > 0$, $P_3(\alpha) = 0$ and $P_3(\beta) < 0$. Then $\max_{\xi \in (-\infty, \beta)} P_3(\xi) = P_3(\alpha) = 0$ and $\xi_1 = \alpha$ is the root of the polynomial $P_3(\xi)$. By the increase property on $(\beta, \infty)$ of the function $P_3(\xi)$ the polynomial $P_3(\xi)$ has a root $\xi_2 \in (\beta, \infty)$, as $\beta > 0$ and $P_3(\beta) < 0$. There is not any other positive roots of the polynomial $P_3(\xi)$.

ii) Let $\alpha > 0$, $P_3(\alpha) > 0$ and $P_3(\beta) = 0$. Then by the increase property on $(-\infty, \alpha)$ of the function $P_3(\xi)$ the polynomial $P_3(\xi)$ has a root $\xi_1 \in (0, \alpha)$. By the other hand $\min_{\xi \in (\alpha, \infty)} P_3(\xi) = P_3(\beta) = 0$ and the number $\xi_2 = \alpha$ is the second positive root of the polynomial $P_3(\xi)$. The polynomial $P_3(\xi)$ has not another roots. From above and by Lemmas 2.3 we get Theorem 2.6.

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