Truly Random Number Generation Based on Measurement of Phase Noise of Laser

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We present a simple approach to realize truly random number generation based on measurement of the phase noise of a single mode vertical cavity surface emitting laser (VCSEL). The true randomness of the quantum phase noise originates from the spontaneous emission of photons and the random bit generation rate is ultimately limited only by the laser linewidth. With the final bit generation rate of 20 Mbit/s, the physically guaranteed truly random bit sequence passes the three standard random tests. Moreover, for the first time, a continuously generated random bit sequence up to 14 Gbit is verified by two additional criteria for its true randomness.

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Random number generator (RNG) has wide applications in statistical sampling [1], computer simulations [2], randomized algorithm [3] and cryptography [4]. Traditionally, pseudorandom number generator (PRNG) based on computational algorithms is adopted to generate random bits and is competent in many fields. However, it cannot produce truly random (unpredictable and irreproducible) bit sequence, and so may result in potential dangers in security related applications, say, in quantum cryptography [5]. Actually, the unconditional security of quantum key distribution can ONLY be guaranteed when a truly random number generator (TRNG), based on quantum mechanical process instead of the intractability assumption with classical algorithms [6], is available.

Distinct from PRNG, a TRNG can only be realized by a physical way, instead of an algorithm-based way; however, a physical way does not sufficiently guarantee the true randomness. The physically random processes, such as radioactive decay [7], electric noise in circuits [8], frequency jitter of electric oscillator [9], and those based on laser (photon) emission/detection [10–12], can ensure the inability of pre-estimation on random numbers and so can be adopted as candidates to implement TRNG. In particular, those based on the detection of laser field attracted tremendous interests in recent decade. Recently, chaotic laser, with ultra-wide bandwidth, becomes a promising candidate for GHz random bit generation [13, 14]. However, since the signal of chaotic laser has a periodicity originated from the photon round trip time, it is essentially NOT a truly random source. Also, we know, the chaotic systems are deterministic that looks random but without inherent randomness [15, 16]. Hence, we coin this kind of physically-based, rather than algorithm-based, pseudo RNG as physically pseudorandom number generator (PPRNG). Thus, the true randomness guaranteed by physical principle(s), instead of its generation rate, should be firstly pursued for a TRNG, otherwise, even though with ultrahigh rate, it is just a pseudo RNG; on the other hand, the TRNGs based on the above-mentioned physical mechanisms [7–12] cannot offer the bit generation rate as high as PPRNG based on chaotic lasers [13, 14]. So far, the typical maximal random bit generation rate is around 10 Mbit/s for electric oscillator jitter measurement scheme [9] and 4 Mbit/s for photon detection scheme [11]. Also, it should be noted that in these schemes, the statistical bias and correlation for long random bit sequence were not investigated.

In this Letter, we propose a new and simple TRNG scheme based on the true randomness of the quantum phase noise, which is a Gaussian random variable [17, 18], of a single-mode vertical cavity surface emitting laser (VCSEL). The true randomness of the quantum phase noise is originated from the random nature of spontaneous emission and is guaranteed by physical principle. It will, in the following, be shown that the random bit generation rate of this TRNG is ultimately limited only by the laser linewidth. In our experiment, the high final bit generation rate reaches 20 Mbit/s with guaranteed true randomness. Further, the true randomness is not only guaranteed in physical principle and standard tests, but is also verified by two additional criteria (statistical bias and correlation) for the long random bit sequence up to 14 Gbit, for the first time.

The schematic setup is shown in Fig. 1 and the delayed self-homodyne method is used to measure the
amplitude) noise of the laser field is observed with (without) the beat signal. (b) Autocorrelation function of the beat signal versus time interval. In our experiment, the sampling interval of 25 ns (40 MHz sampling rate) is chosen.

phase noise of the VCSEL. In this case, the output alternative current (AC) voltage of the avalanche photodetector (APD) detecting the beat signal is $V_{ph} \propto \text{AC}[\phi(t)E(t + \tau)\exp(-\tau/\tau_{coh})]$, where the phase fluctuation within 200 MHz (the gap is about 20 dB). Using Wiener-Khintchine theorem \[18\] and the phase difference $\Delta \phi(t) = \phi(t) - \phi(t + \tau)$ is a Gaussian random variable, and thus the autocorrelation function of the electric field of the laser is eliminated [17], i.e.,

$$\langle E^* (t) E(t + \tau) \rangle \propto \exp(-|\tau|/\tau_{coh}) \to 0, \quad (1)$$

where $\tau_{coh} = (\pi \Delta \nu_{laser})^{-1}$ [18], and $\Delta \nu_{laser}$ is the laser linewidth. This indicates that the electric field amplitudes of the laser at different time are mutually independent, if time interval is much longer than the coherence time of the laser. Further, similar calculation procedure can be applied to obtain the autocorrelation function of the beat signal $[\phi(t)E(t + \tau)]$ as

$$\langle E_{\text{beat}}^*(t)E_{\text{beat}}(t + \Delta t) \rangle,$$

where $\Delta t$ is the sampling interval for original random bit generation. Using $E_{\text{beat}}(t) = E(t) + E(t + \tau)$ and Eq. (1), it is evident that when the sampling time $\Delta t$ meets $\Delta t \gg \tau + \tau_{coh}$, the autocorrelation of the beat signal will also be eliminated. Thus, the bits extracted from the beat signal are mutually independent and can be adopted to implement TRNG.

In experiment (Fig. 1), a 785 nm VCSEL laser works at 1.5 mA, a little above the threshold current 1.0 mA. The laser linewidth $\Delta \nu_{laser} = 200 \text{ MHz}$ ($\tau_{coh} = 1.59 \text{ ns}$) of laser is inversely proportional to the laser power, while the classical noises are independent on it [25]. Due to working just above threshold, the quantum phase noise of laser dominates over the classical amplitude noise to ensure the true randomness of generated bits. The delay time $\tau$ is set to be about 10 ns (corresponds to 3.0 m space delay) in order to fulfill $\tau \gg \tau_{coh}$. So, the self-homodyne method with delay time $\tau$ is used to obtain the beat signal with 3 dB linewidth of 400 MHz (detected by an APD) and its power spectral density is shown in the inset of Fig. 2 (a). It can be seen, from Fig. 2 (a), that the classical amplitude fluctuation is negligible compared to the quantum phase fluctuation within 200 MHz (the gap is about 20 dB). Using Wiener-Khintchine theorem [19, 20], i.e.,

$$R_{\text{beat}}(t) = \int_{-\infty}^{+\infty} P_{\text{beat}}(\omega) \exp(-i\omega t) d\omega, \quad (2)$$

the autocorrelation function $[R_{\text{beat}}(t)]$ of the beat signal is obtained from the power spectral density of the phase noise $[P_{\text{beat}}(\omega)]$ in Fig. 2 (a) and illustrated in Fig. 2 (b). It can be seen from Fig. 2 (b) that the autocorrelation of the beat signal can be ignored, if the sampling interval is set as $\Delta t \gg \tau + \tau_{coh}$. In our experiment, the sampling
rate is chosen as 40 MHz accordingly, i.e., $\Delta t = 25$ ns, so the bits extracted from these sampled voltages are mutually independent. These sampled voltages are digitized by an 8-bit analog-digital-converter (ADC) shown as the red dots in Fig. 3 and further we have confirmed that the distribution of voltages is symmetric. Hence, we take the least significant bit (LSB) of each sampled 8-bit voltage as the original random bit, i.e., the parity of this 8-bit binary number, which represents whether the voltage falls in an even or odd bin of the total 256 bins. For the nonideal distribution of these voltages, the probability of all the even and odd bins of the total 256 bins are not perfectly equal, and the bit sequence shows a statistical bias of the order of $10^{-3}$. For much lower bias, we perform a subtraction between two consecutive sampled voltages to obtain a sequence of $N/2$ 8-bit binary derivatives as $V_2 - V_1, V_4 - V_3, \ldots, V_N - V_{N-1}$, where $N$ is the total number of the original sampled voltages. In this process, each voltage is used only once, and thus no correlation is introduced. After that, we adopt the LSB of the 8-bit binary derivatives to generate the final random bits.

Therefore, we directly obtained the final random bit at generation rate of 20 Mbit/s with a software-based post-processing. Note that, the post-processing enhances the performance of the random bits sequence by lowering the statistical bias while not introducing any additional correlations. For a TRNG, both the statistical bias and the absolute value of the first-order correlation coefficient of the final random bit sequence are expected to be smaller than three standard deviations (3$\sigma$ = $1.5/\sqrt{N}$ for statistical bias [Fig. 4(a)], and 3$\sigma_2$ = $3/\sqrt{N}$ for correlation coefficient [Fig. 4(b)]) with the probability of 99.7%. In our case, both criteria are well satisfied for the final random bit sequence up to 14 Gbit.

We continuously record a final random bit sequence of 1 Gbit, which passed three standard random tests, i.e., ENT [21], Diehard [22] and STS [23]. The ENT results are: Entropy = 1.000000 bit per bit (the optimum compression would reduce the bit file by 0%). $\chi^2$ distribution is 0.53 (randomly would exceed this value by 46.62% of the times). Arithmetic mean value of data bits is 0.5000. Monte Carlo value for $\pi$ is 3.141725650. Serial correlation coefficient is −0.000017. The Diehard and STS test results are shown in Tables I and II, respectively.

![FIG. 4: (color online). (a) The statistical bias (B) of the final random bit sequence. It can be seen that B < 1.5/$\sqrt{N}$ always holds and converges to zero for large bit sequence, where p(1) is the probability of ones in sequence. (b) The absolute value of the first-order correlation coefficient |a_1| of the final random bit sequence. It can be seen that |a_1| < 3/$\sqrt{N}$ always holds and |a_1| converges to zero for large bit sequence.](image)

![TABLE I: Results of Diehard statistical test suite. Data sample containing 100 Mbits is used for the Diehard test. For the cases of multiple p-values, a Kolmogorov-Smirnov (KS) test is used to obtain a final P-value, which measures the uniformity of the multiple p-values. The test is considered successful if all final P-values satisfy 0.01 ≤ P ≤ 0.99.](table)

| Statistical test                      | P-value | Result |
|--------------------------------------|---------|--------|
| Birthday spacings                    | 0.910531 [KS] | Success |
| Overlapping permutations              | 0.294899 | Success |
| Ranks of 31 x 31 matrices             | 0.322213 | Success |
| Ranks of 32 x 32 matrices             | 0.482575 | Success |
| Ranks of 6 x 8 matrices               | 0.749427 [KS] | Success |
| Monkey tests on 20-bit words          | 0.019887 [KS] | Success |
| Monkey test OPSO                      | 0.079864 [KS] | Success |
| Monkey test OQSO                      | 0.725649 [KS] | Success |
| Monkey test DNA                       | 0.293543 [KS] | Success |
| Count 1’s in stream of bytes          | 0.244463 | Success |
| Count 1’s in specific bytes           | 0.062188 [KS] | Success |
| Parking lot test                      | 0.806989 [KS] | Success |
| Minimum distance test                 | 0.326209 [KS] | Success |
| Random spheres test                   | 0.902946 [KS] | Success |
| Squeeze test                          | 0.815876 [KS] | Success |
| Overlapping sums test                 | 0.806025 [KS] | Success |
| Runs test (up)                        | 0.817356 | Success |
| Runs test (down)                      | 0.805323 | Success |
| Craps test No. of wins                | 0.502035 | Success |
| Craps test throws/game                | 0.403322 | Success |

It should be noted that, for a nonuniform distribution of the probability of 256 8-bit binary derivatives, if more
than 1 bit are extracted from each 8-bit binary derivatives in order to improve the random bit generation rate (see, e.g., 5 LSBs are adopted in [14]), an additive correlation in the final random bit sequence will be introduced, even though this additive correlation is not so significant to fail the random tests. Taking 5 LSBs for an instance, every set of the 5 LSBs possesses a different probability (due to the nonuniform distribution) and thus these 5 bits from the same set are correlated to some extent. However, with this additive correlation within the same set, both the random bit sequence of extracting 5 LSBs (with the sampling rate of 2.5 GHz in [14]) and 4 LSBs (with the sampling rate of 40 MHz in our case) from an 8-bit binary number both successfully pass the three standard random tests. This fact also indicates that the standard random tests are only a way to examine whether the random bit stream is “sufficiently” random, but not to judge whether it is truly random.

We propose a new and simple approach to realize a high-speed TRNG, which is compact and convenient to implement. The randomness of our TRNG is physically guaranteed by the *intrinsic* random nature of the quantum phase noise originated from the spontaneous emission of photons. Moreover, for the first time, the true randomness is verified by both the statistical bias and the correlation coefficient for long random bit sequence up to 14 Gbit. Note that, the long random bit sequence is even more important than generation rate, because it is the length of the random bit sequence that is required in most applications and essentially, it is a metric for qualifying the true randomness. Compared to the chaotic laser, the intrinsic phase noise of a free-running laser is confirmed in true randomness, which only depends on its inherent quantum mechanical properties and does not need the external optical feedback to laser thereby introducing photon round trip period. Although the random bit generation rate is not as high as that of chaotic laser scheme [13, 14], its physically guaranteed true randomness and high generation rate, together with its simplicity and compactness, are attractive for applications which need true randomness. Also, a higher generation rate is attainable using a laser with larger linewidth and faster data acquisition hardware.

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| Statistical test          | P-value | Proportion | Result |
|---------------------------|---------|------------|--------|
| Frequency                 | 0.679846| 0.9916     | Success|
| Block frequency           | 0.248571| 0.9897     | Success|
| Cumulative sums           | 0.858032| 0.9888     | Success|
| Runs                      | 0.816029| 0.9907     | Success|
| Longest run               | 0.648795| 0.9935     | Success|
| Rank                      | 0.609895| 0.9860     | Success|
| Nonperiodic               | 0.569334| 0.9823     | Success|
| Overlapping               | 0.565500| 0.9916     | Success|
| Universal                 | 0.143336| 0.9888     | Success|
| Approximate               | 0.590520| 0.9879     | Success|
| Random excursions         | 0.016388| 0.9880     | Success|
| Random variant            | 0.029796| 0.9865     | Success|
| Serial                    | 0.946683| 0.9916     | Success|
| Linear complexity         | 0.732979| 0.9915     | Success|

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