Abstract. In this paper, new advances on the compactifications of topological spaces, especially on the Stone-Čech and Alexandroff compactifications have been made. Among the main results, it is proved that the minimal spectrum of the direct product of a family of integral domains indexed by a set $X$ is the Stone-Čech compactification of the discrete space $X$. Dually, it is proved that the maximal spectrum of the direct product of a family of local rings indexed by $X$ is also the Stone-Čech compactification of the discrete space $X$. The Alexandroff (one-point) compactification of a discrete space is constructed by a new method. Next, we proceed to give a natural and quite simple way to construct ultra-rings. Then this new approach is used to obtain several new results on the Stone-Čech compactification.

1. Introduction

Compactification is one of the main topics which is investigated in this paper from a purely algebraic perspective. Among various compactifications, the Stone-Čech compactification of a discrete space $X$ is particularly important. One of the main reasons of its importance is that it admits a semigroup structure whenever $X$ is a semigroup, and this semigroup structure has vast and interesting applications in diverse fields of mathematics especially in combinatorial number theory, Ramsey theory, topological dynamics and Ergodic theory. An accessible concrete description of this compactification often remains elusive. For instance the semigroup $\beta\mathbb{N}$, the Stone-Čech compactification of the natural numbers, is amazingly complicated and there are some unanswered questions about its semigroup structure. For example, whether or not $\beta\mathbb{N}$ contains any elements of finite order which are not idempotent still remains a challenging open problem. See [12] and [26] and their rich bibliography for further studies. Perhaps as another main reason for the importance of the Stone-Čech compactification of a discrete space is its vital role in proving Theorem [15] which asserts that every topological space admits the Stone-Čech compactification.

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Classically, the Stone-Čech compactification of a discrete space is usually constructed via the ultrafilters of that space. In this paper, we find two new and interesting ways to construct this compactification using only the standard and elementary methods of commutative algebra. In fact in Theorem 3.5, we prove that the minimal spectrum of the direct product of a family of integral domains indexed by a set \( X \) is the Stone-Čech compactification of the discrete space \( X \). In Theorem 5.4, it is shown that the maximal spectrum of the direct product of a family of local rings indexed by \( X \) is the Stone-Čech compactification of the discrete space \( X \). These results improve all of the former constructions of the Stone-Čech compactification of a discrete space, and also show that this compactification is independent of choosing of integral domains and local rings. In particular, we get that \( \beta X = \text{Spec} \mathcal{P}(X) \). The classical construction is also recovered (see Remark 3.9). Throughout this paper, \( \beta X \) denotes the Stone-Čech compactification of the discrete space \( X \). These results allow us to understand the number of prime ideals of the infinite direct products of integral domains and local rings more precisely. As another application, the Stone-Čech compactification of an arbitrary topological space \( X \) is deduced from the Stone-Čech compactification of the discrete space \( X \) by passing to a certain quotient (see Theorem 4.5). It is worth mentioning that our results considerably generalize several related results in the literature (see e.g. [2]).

In §6, using ultra-rings and Theorems 3.5 and 5.4, then we obtain new results on the Stone-Čech compactification (see Theorems 6.2 and 6.3).

We introduce a new way to build the Alexandroff (one-point) compactification of a discrete space, see Corollary 7.3. This result tells us that for any set \( X \), then \( \alpha X = \text{Spec}(\mathcal{R}) \). Here \( \alpha X \) denotes the Alexandroff compactification of the discrete space \( X \) and \( \mathcal{R} \) is a certain subring of \( \mathcal{P}(X) \). Then in Theorem 5.2 we show that every totally disconnected compactification of a discrete space \( X \) is precisely of the form \( \text{Spec}(\mathcal{R}') \) where the ring \( \mathcal{R}' \) satisfies in the extensions of rings \( \mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{P}(X) \). After proving this result, we were informed that it is also proved in [18, Theorems 2.2 and 2.3] by another approach using Boolean algebras. In summary, our result shows that all of the totally disconnected compactifications of a discrete space are in the scope of the Zariski topology. In particular this class, up to isomorphisms, forms a set and the extensions of the corresponding rings put a partial order over this set in a way that the Alexandroff compactification is the minimal one and the Stone-Čech compactification is the maximal one.

It is well known that if the discrete space \( X \) is also a (commutative) semigroup then its operation can be extended uniquely to an operation on \( \beta X \) which forms a semigroup structure as well, see [12, Theorems 4.1 and 4.4]. This result opens new horizons to explore the basic and also sophisticated properties of the semigroup \( \beta X \). Although some of them have been done
in the literature over the years (see [12] and its bibliography), there is a pressing need for new constructions to aid the development and the understanding the algebraic structure of this semigroup specially $\beta \mathbb{N}$ more deeply. We have made very little contributions to this subject but the results are sufficiently general (Theorems 9.1, 9.2 and 9.3). Indeed, in Theorem 9.1 we reformulate this important result into a more standard form and then it is proven by a new approach. Then in Theorems 9.2 and 9.3 various aspects of the semigroup $\beta X$ are investigated, specially it is shown that this semigroup structure is actually functorial. Finally, in Section 10, the absolutely flatness of the total ring of fractions is investigated.

2. Preliminaries

In this paper, all rings are commutative. If $\varphi : R \rightarrow R'$ is a morphism of rings then the induced map $\text{Spec}(R') \rightarrow \text{Spec}(R)$ given by $p \rightsquigarrow \varphi^{-1}(p)$ is denoted by $\text{Spec}(\varphi)$, (sometimes it is also denoted by $\varphi^*$). Let $X$ be a set and $I$ an ideal of the power set ring $\mathcal{P}(X)$. If $A, B \in I$ then $A \cup B = A + B + A \cap B \in I$. Also if $A \in I$ and $B \subseteq A$, then $B \in I$. For the definition of power set ring see e.g. [23, §2]. By $\text{Fin}(X)$ we mean the set of all finite subsets of $X$, it is an ideal of $\mathcal{P}(X)$. By $\text{Clop}(X)$ we mean the set of clopen (both open and closed) subsets of a topological space $X$ (for more information see [23, §3]). By a compact space we mean a quasi-compact and Hausdorff topological space.

Let $R$ be a ring. The set of minimal primes of $R$ is denoted by $\text{Min}(R)$ and the set of maximal ideals of $R$ is denoted by $\text{Max}(R)$. Note that the induced Zariski topology over $\text{Min}(R)$ is not necessarily quasi-compact. The Jacobson radical of $R$ is denoted by $\mathfrak{J}$.

**Definition 2.1.** The Stone-Čech compactification of a topological space $X$ is the pair $(\beta X, \eta)$ where $\beta X$ is a compact space and $\eta : X \rightarrow \beta X$ is a continuous map such that the following universal property holds. For each such pair $(Y, \varphi)$, i.e. $Y$ is a compact space and $\varphi : X \rightarrow Y$ is a continuous map, then there exists a unique continuous map $\bar{\varphi} : \beta X \rightarrow Y$ such that $\varphi = \bar{\varphi} \circ \eta$.

All of the remaining undefined notions such as the flat topology, retraction, mp-ring and etc can be found in [1, 19, 21, 22, 24] and [25].

3. Minimal spectrum as Stone-Čech compactification

The main result of this section (Theorem 3.5) asserts that the minimal spectrum of the direct product of a family of integral domains indexed by a set $X$ is the Stone-Čech compactification of the discrete space $X$.

We start with the following result which generalizes [7, p. 460].
Proposition 3.1. Consider the canonical ring map \( \pi : R \to S^{-1}R \) where \( S \) is a multiplicative subset of a ring \( R \), and let \( f \in R \). Then \( f \in \bigcap_{p \in \text{Im} \, \pi^*} p \) if and only if there exists some \( g \in S \) such that \( fg \) is nilpotent.

Proof. If \( f \in \bigcap_{p \in \text{Im} \, \pi^*} p \) then \( f/1 \in \bigcap_{p \in \text{Im} \, \pi^*} S^{-1}p = \bigcap_{q \in \text{Spec}(S^{-1}R)} q = \sqrt{0} \). Thus there exist a natural number \( n \geq 1 \) and some \( g \in S \) such that \( f^n g = 0 \). Hence, \( fg \) is nilpotent. The reverse implication is easy.

Corollary 3.2. ([11], Lemma 1.1 and [13], Lemma 3.1) Let \( p \) be a prime ideal of a ring \( R \). Then \( p \) is a minimal prime of \( R \) if and only if for each \( f \in p \) there exists some \( g \in S(R) \backslash p \) such that \( fg \) is nilpotent.

Proof. It is an immediate consequence of Proposition 3.1.

Theorem 3.3. Let \( R \) be a ring. Then the induced Zariski topology over \( \text{Min}(R) \) is finer than the induced flat topology. These two topologies over \( \text{Min}(R) \) are the same if and only if \( \text{Min}(R) \) is Zariski compact.

Proof. Let \( f \in R \). If \( p \in W = \text{Min}(R) \cap V(f) \) then by Corollary 3.2 there exists some \( g \in R \backslash p \) such that \( fg \) is nilpotent. This yields that \( p \in \text{Min}(R) \cap D(g) \subseteq W \). Therefore \( W \) is a Zariski open of \( \text{Min}(R) \). Hence the Zariski topology over \( \text{Min}(R) \) is finer than the flat topology. For any ring \( R \), then by [11] Lemma 3.2, \( \text{Min}(R) \) is Zariski Hausdorff. Also \( \text{Min}(R) \) is flat quasi-compact. Therefore if these two topologies over \( \text{Min}(R) \) are the same then \( \text{Min}(R) \) is Zariski compact. Conversely, suppose \( \text{Min}(R) \) is Zariski compact. In the above we observed that \( U = \text{Min}(R) \cap D(f) \) is a Zariski clopen of \( \text{Min}(R) \). Every closed subspace of a quasi-compact space is quasi-compact. Thus there exist finitely many elements \( g_1, \ldots, g_n \in R \) such that \( U^c = \text{Min}(R) \setminus U = \bigcup_{i=1}^n \text{Min}(R) \cap D(f_i) \). It follows that \( U = \text{Min}(R) \cap V(I) \) where \( I = (g_1, \ldots, g_n) \) is a finitely generated ideal of \( R \). Thus \( U \) is a flat open of \( \text{Min}(R) \).

Throughout this paper, \( \Lambda = \prod_{x \in X} R_x \) where each \( R_x \) is an integral domain. For each \( f = (f_x) \in \Lambda \), the set \( \text{Supp}(f) = \{ x \in X : f_x \neq 0 \} \) is simply denoted by \( S(f) \). Clearly \( S(fg) = S(f) \cap S(g) \) for all \( f, g \in \Lambda \).

Corollary 3.4. The space \( \text{Min}(\Lambda) \) is Zariski compact.

Proof. By Theorem 3.3 it suffices to show that for each \( f \in \Lambda \) then \( U = \text{Min}(\Lambda) \cap D(f) \) is a flat open of \( \text{Min}(\Lambda) \). Consider the sequence \( e = (e_x) \in \Lambda \) where \( e_x \) is either 0 or 1, according as \( x \in S(f) \) or \( x \notin S(f) \). Then clearly \( ef = 0 \) and \( g = ge \) for all \( g \in \text{Ann}(f) \). Hence \( \text{Ann}(f) \) is generated by the sequence \( e \). Now let \( p \in \text{Min}(\Lambda) \cap V(e) \). If \( f \in p \) then by Corollary 3.2 there exists some \( h \in \Lambda \setminus p \) such that \( fh \) is nilpotent. But \( \Lambda \) is a reduced ring. Hence \( h \in \text{Ann}(f) \). Thus \( h = he \in p \). But this is a contradiction. This shows that \( U = \text{Min}(\Lambda) \cap V(e) \) is a flat open of \( \text{Min}(\Lambda) \).
For each $x \in X$ then $p_x := \text{Ker} \pi_x$ is a minimal prime of $\Lambda$ and it is generated by the sequence $1 - \Delta_x$ where $\pi_x : \Lambda \rightarrow R_x$ is the canonical projection, $\Delta_x = (\delta_{x,y})_{y \in X}$ and $\delta_{x,y}$ is the Kronecker delta.

Now we are ready to prove the main result of this section:

**Theorem 3.5.** The space $\text{Min}(\Lambda)$ together with the canonical map $\eta : X \rightarrow \text{Min}(\Lambda)$ given by $x \mapsto p_x$ is the Stone-Čech compactification of the discrete space $X$.

**Proof.** By Corollary 3.4 the space $\text{Min}(\Lambda)$ is compact. It remains to check the universal property of the Stone-Čech compactification. Let $Y$ be a compact topological space and $\varphi : X \rightarrow Y$ a function. We shall find a continuous function $\tilde{\varphi} : \text{Min}(\Lambda) \rightarrow Y$ such that $\varphi = \tilde{\varphi} \circ \eta$ and then we show that such function is unique. If $p \in \text{Min}(\Lambda)$ then the subsets $S(f)$ with $f \in \Lambda \setminus p$ have the finite intersection property. It follows that the subsets $\varphi(S(f))$ and so their closures $\overline{\varphi(S(f))}$ with $f \in \Lambda \setminus p$ have the finite intersection property. This yields that $\bigcap_{f \in \Lambda \setminus p} \overline{\varphi(S(f))} \neq \emptyset$ because $Y$ is quasi-compact. We claim that this intersection has exactly one point. If $y$ and $y'$ are two distinct points of the intersection then there exist disjoint opens $U$ and $V$ in $Y$ such that $y \in U$ and $y' \in V$. Then consider the sequence $f \in \Lambda$ where $f_x$ is either 0 or 1, according as $x \in \varphi^{-1}(U)$ or $x \notin \varphi^{-1}(U)$. Then we have either $f \in p$ or $1 - f \in p$ since $f$ is an idempotent. If $f \in p$ then $\varphi^{-1}(V) \cap S(1-f) \neq \emptyset$. So we may choose some $x$ in this intersection. Thus $x \notin \varphi^{-1}(U)$, hence $f_x = 1$. But this is a contradiction since $x \in S(1-f)$. If $1 - f \in p$ then $\varphi^{-1}(U) \cap S(f) \neq \emptyset$, but this is again a contradiction. Hence, there exists a unique point $y_p \in Y$ such that

$$\bigcap_{f \in \Lambda \setminus p} \overline{\varphi(S(f))} = \{y_p\}.$$  

This establishes the claim. Then we define the map $\tilde{\varphi} : \text{Min}(\Lambda) \rightarrow Y$ as $p \mapsto y_p$. It is easy to see that $\varphi(x) \in \bigcap_{f \in \Lambda \setminus p_x} \varphi(S(f))$ for all $x \in X$. Therefore $\varphi = \tilde{\varphi} \circ \eta$. Now we show that $\tilde{\varphi}$ is continuous. Let $U$ be an open of $Y$ and let $p \in (\tilde{\varphi})^{-1}(U)$. There exists an open neighborhood $V$ of $y_p$ such that $\overline{V} \subseteq U$, because it is well known that every compact space is a normal space. Let $h \in \Lambda$ be a sequence which is defined as $h_x = 1$ or $h_x = 0$, according as $x \in \varphi^{-1}(V)$ or $x \notin \varphi^{-1}(V)$. Then $p \in D(h)$, since if $h \in p$ then $1 - h \notin p$ and so $\varphi^{-1}(V) \cap S(h) = \emptyset$, which is impossible. To conclude the continuity of $\tilde{\varphi}$ we show that $\text{Min}(\Lambda) \cap D(h) \subseteq (\tilde{\varphi})^{-1}(U)$. Suppose there exists some $q \in \text{Min}(\Lambda) \cap D(h)$ such that $y_q \notin U$. Thus $y_q \in W := Y \setminus \overline{U}$. It follows that $W \cap \varphi(S(h)) \neq \emptyset$. But this is impossible since $S(h) = \varphi^{-1}(V)$ and so $W \cap \varphi(S(h)) \subseteq W \cap V = \emptyset$. Therefore $\tilde{\varphi}$ is continuous. If $\text{Min}(\Lambda) \cap D(f)$ is non-empty then $f \neq 0$ and so there exists some $x \in X$ such that $p_x \in D(f)$. This shows that $\eta(X)$ is a dense subspace of $\text{Min}(\Lambda)$, hence the uniqueness of $\tilde{\varphi}$ is deduced from the basic fact that if
two continuous maps into a Hausdorff space agree on a dense subspace of the domain, they are equal.

Lemma 3.6. If each $R_x$ is a field, then every prime ideal of $\Lambda$ is a maximal ideal.

Proof. Let $p$ be a prime ideal of $\Lambda$ and $f \in \Lambda \setminus p$. Then consider the sequence $g = (g_x) \in \Lambda$ where $g_x$ is 1 or $1/f(x)$, according as $f_x = 0$ or $f_x \neq 0$. Then it is obvious that $f(1 - fg) = 0 \in p$. This yields that $1 - fg \in p$. Therefore $\Lambda/p$ is a field. As a second proof, the assertion is also deduced from the fact that $\Lambda$ is an absolutely flat ring.

Corollary 3.7. The space $\text{Spec}(\Lambda)$ together with the canonical map $\eta : X \to \text{Spec}(\Lambda)$ is the Stone-\v{C}ech compactification of the discrete space $X$ if and only if each $R_x$ is a field.

Proof. If each $R_x$ is a field then the assertion is deduced from Theorem 3.5 and Lemma 3.6. Conversely, if $m$ is a maximal ideal of $R_x$ then $\pi_x^{-1}(m) = \pi_x^{-1}(0)$ because $\text{Spec}(\Lambda)$ is Hausdorff and so every prime ideal of $\Lambda$ is a maximal ideal. But $\pi_x$ is surjective and so the induced map $\pi_x^*$ is injective. Therefore the zero ideal of $R_x$ is a maximal ideal and so it is a field.

Corollary 3.8. The space $\text{Spec} \mathcal{P}(X)$ together with the canonical map $\eta : X \to \text{Spec} \mathcal{P}(X)$ given by $x \sim m_x = \mathcal{P}(X \setminus \{x\})$ is the Stone-\v{C}ech compactification of the discrete space $X$.

Proof. The map $\mathcal{P}(X) \to \prod_{x \in X} \mathbb{Z}_2$ given by $A \sim \chi_A$ is an isomorphism of rings where $\chi_A$ is the characteristic function of $A$ and $\mathbb{Z}_2 = \{0, 1\}$. Then apply Corollary 3.7.

Remark 3.9. Here we establish a bridge that allows us to translate all of the theory of Boolean algebras into the standard language of commutative algebra (and vice versa). For instance, the classical approach to construct the Stone-\v{C}ech compactification of a discrete space $X$ is easily recovered. Indeed, if $X$ is a set then one can easily check that the map $M \sim \mathcal{P}(X) \setminus M = \{A \in \mathcal{P}(X) : A^c \in M\}$ is a homeomorphism from $\text{Spec} \mathcal{P}(X)$ onto $\mathcal{F}(X)$, the space of ultrafilters on $X$ equipped with the Stone topology. Recall that the collection of $d(A) = \{F \in \mathcal{F}(X) : A \in F\}$ with $A \in \mathcal{P}(X)$ forms a base for the opens of the Stone topology. The space $\mathcal{F}(X)$ is called the Stone space of the Boolean algebra $\mathcal{P}(X)$. Note that the above identification can be generalized to any Boolean ring $R$. In fact, the map $M \sim R \setminus M$ is a homeomorphism from $\text{Spec}(R)$ onto the Stone space of the corresponding Boolean algebra of $R$.

Remember that for any two objects $X$ and $Y$ of a category $\mathcal{C}$, by $\text{Mor}_{\mathcal{C}}(X, Y)$ we mean the set of all morphisms of $\mathcal{C}$ from $X$ to $Y$.

Corollary 3.10. For any two sets $X$ and $Y$ then we have the following canonical bijections:

$$\text{Mor}_{\text{Set}}(X, \beta Y) \simeq \text{Mor}_{\text{Top}}(\beta X, \beta Y) \simeq \text{Mor}_{\text{Ring}}(\mathcal{P}(Y), \mathcal{P}(X)).$$
Proof. The first bijection follows from Corollary 3.8, and the second bijection is an immediate consequence of [23, Theorem 5.6]. □

4. The Stone-Čech compactification of an arbitrary space

In this section we give a new proof to the fact that every topological space admits the Stone-Čech compactification. To realize this goal, we first obtain some results which are interesting in their own right. We should mention that these results are not new and can be found in the literature which are obtained by using the theory of ultrafilters. Maybe the only novelty is that we will use only the ring-theoretical methods, it seems that this approach is much simpler than the theory of ultrafilters.

We begin with the following key definition (this notion is due to Henri Cartan, we interpreted it into the language of commutative algebra).

Definition 4.1. Let $X$ be a topological space, $x \in X$ and $M$ a maximal ideal of $\mathcal{P}(X)$. We say that $M$ is convergent to the point $x$ if whenever $U$ is an open subset of $X$ containing $x$, then $M \in D(U)$.

Lemma 4.2. Let $X$ be a set. If $M$ is a maximal ideal of $\mathcal{P}(X)$ then $\mathcal{P}(\eta)^*(M)$ is convergent to the point $M \in \beta X = \text{Spec} \mathcal{P}(X)$.

Proof. Let $U$ be an open of $\beta X$ such that $M \in U$. If $U \in \mathcal{P}(\eta)^*(M)$ then $\eta^{-1}(U) \in M$. But there exists some $A \in \mathcal{P}(X)$ such that $M \in D(A) \subseteq U$. If $x \in A$ then $\eta(x) = m_x \in D(A)$ and so $x \in \eta^{-1}(U)$. This shows that $A \subseteq \eta^{-1}(U)$. Thus $A \in M$. But this is a contradiction. □

Lemma 4.3. Let $X$ be a topological space and let $A$ be a subset of $X$ with the property that $D(A)$ contains every maximal ideal of $\mathcal{P}(X)$ which is convergent to a point of $A$. Then $A$ is an open subset of $X$.

Proof. Take $x \in A$ and let $\mathcal{S}$ be the set of all opens of $X$ which are containing $x$. Then by the hypothesis, the ideal of $\mathcal{P}(X)$ generated by $A$ and the elements $U^c = X \setminus U$ with $U \in \mathcal{S}$ is the whole ring. Thus we may find a finite number $U_1,\ldots,U_n$ of elements of $\mathcal{S}$ such that $X = A \cup (\bigcup_{i=1}^n U_i^c)$. It follows that $x \in \bigcap_{i=1}^n U_i \subseteq A$. Hence, $A$ is an open of $X$. □

Note that the converse of the above lemma holds trivially.

Let $\varphi : X \to Y$ be a continuous map of topological spaces. If a maximal ideal $M$ of $\mathcal{P}(X)$ converges to some point $x \in X$, then clearly $\mathcal{P}(\varphi)^*(M)$ is convergent to $\varphi(x)$. In the following result we establish its converse.

Corollary 4.4. Let $\varphi : X \to Y$ be a function between topological spaces with the property that $\mathcal{P}(\varphi)^*(M)$ is convergent to $\varphi(x)$ whenever a maximal ideal $M$ of $\mathcal{P}(X)$ converges to some point $x \in X$. Then $\varphi$ is continuous.
Proof. It is easily deduced from Lemma 4.3. □

Now we establish the main result of this section. This result shows that the Stone-Čech construction can be performed for any topological space $X$, but in that case the canonical map from $X$ to its Stone-Čech compactification need not be a homeomorphism onto its image (and sometimes is not even injective).

**Theorem 4.5.** Every topological space $X$ admits the Stone-Čech compactification.

Proof. Consider the equivalence relation $\sim$ on $\beta X = \text{Spec} \mathcal{P}(X)$ defined as $M \sim N$ if $\varphi : X \to Y$ is a continuous function to a compact space $Y$ then $\tilde{\varphi}(M) = \tilde{\varphi}(N)$ where $\tilde{\varphi} : \beta X \to Y$ is the unique continuous function such that $\varphi = \tilde{\varphi} \circ \eta$, see the proof of Theorem 3.5. Now to prove that the pair $(X', \pi \circ \eta)$ is the Stone-Čech compactification of the space $X$ it suffices to show that $\pi \circ \eta : X \to X'$ is continuous where $\pi : \beta X \to X' = \beta X/\sim$ is the canonical map and $X'$ is equipped with the quotient topology. To prove the continuity of $\pi \circ \eta$, by Corollary 4.4, it will be enough to show that if a maximal ideal $M$ of $\mathcal{P}(X)$ converges to some point $x \in X$ then $\mathcal{P}(\pi \circ \eta)^*(M)$ is convergent to the point $(\pi \circ \eta)(x)$. We have $\mathcal{P}(\pi \circ \eta)^*(M) = \mathcal{P}(\pi)^*(\mathcal{P}(\eta)^*(M))$. By Lemma 4.2, $N := \mathcal{P}(\eta)^*(M)$ is convergent to the point $M \in \beta X$. Thus $\mathcal{P}(\pi)^*(N)$ is convergent to the point $\pi(M)$ since $\pi$ is continuous. Then we show that $M \sim m_x$. Because take $A \in \mathcal{P}(X) \setminus M$ and let $V$ be an open of a compact space $Y$ such that $\varphi(x) \in V$ where $\varphi : X \to Y$ is a continuous map. Then $\varphi^{-1}(V) \notin M$. Note that $S(A) = A$. Now if $V \cap \varphi(A) = \emptyset$ then $A \in M$, a contradiction. Hence, $\varphi(x) \in \varphi(S(A))$. Thus by the definition of $\tilde{\varphi}$, see the proof of Theorem 3.5, we get that $\varphi(x) = \tilde{\varphi}(M)$ and so $M \sim m_x$. Therefore $\mathcal{P}(\pi \circ \eta)^*(M)$ is convergent to the point $\pi(M) = (\pi \circ \eta)(x)$. Note that during to verify the universal property of the Stone-Čech compactification for the pair $(X', \pi \circ \eta)$, the uniqueness is deduced from the fact that $(\pi \circ \eta)(X)$ is a dense subspace of $X'$.

Recall that by a compactification of a topological space $X$ we mean a compact space $\tilde{X}$ together with an open embedding (a continuous injective open map) $\eta : X \to \tilde{X}$ such that $\eta(X)$ is a dense subspace of $\tilde{X}$. If moreover, $\tilde{X} \setminus \eta(X)$ consisting only a single point then $\tilde{X}$ is called the one-point or the Alexandroff compactification of $X$ and it is often denoted by $\alpha X$, and this single point is called the point at infinity of $X$.

**Remark 4.6.** Note that the map $\eta : X \to \text{Min}(\Lambda)$ given by $x \sim p_x$ is an open embedding, because $\{p_x\} = \text{Min}(\Lambda) \cap D(\Delta_x)$ for all $x \in X$. Hence, $\text{Min}(\Lambda)$ is also a compactification of the discrete space $X$ in the above sense. But it is important to notice that the Stone-Čech compactification of an arbitrary topological space need not be a compactification in the above sense. In fact, it is well known that the canonical map from a topological
space $X$ to its Stone-Čech compactification induces a homeomorphism onto its image if and only if $X$ is a Tychonoff space. Thus for general space $X$, this map need not be injective. It is also well known that the canonical map from a topological space $X$ to its Stone-Čech compactification is an open embedding if and only if $X$ is locally compact.

5. Maximal spectrum as Stone-Čech compactification

The main result of this section (Theorem 5.4) asserts that the maximal spectrum of the direct product of a family of local rings indexed by a set $X$ is the Stone-Čech compactification of the discrete space $X$. Then some applications are also given.

Let $R$ be a ring and $f \in R$. If $m \in U = \text{Max}(R) \cap D(f)$ then there exist some $g \in m$ and $h \in R$ such that $1 = fh + g$. This yields that $m \in \text{Max}(R) \cap V(g) \subseteq U$. Thus $U$ is a flat open of $\text{Max}(R)$. Therefore the induced flat topology over $\text{Max}(R)$ is finer than the induced Zariski topology.

**Proposition 5.1.** For a ring $R$ the following statements are equivalent.
(i) $R/\mathfrak{J}$ is a zero dimensional ring.
(ii) The induced Zariski and flat topologies over $\text{Max}(R)$ are the same.
(iii) $\text{Max}(R)$ is flat compact.

**Proof.** (i) $\Rightarrow$ (ii) : If $f \in R$ then there exists some $g \in R$ such that $f(1 - fg) \in \mathfrak{J}$, because $R/\mathfrak{J}$ is reduced and so it is absolutely flat. It follows that $\text{Max}(R) \cap V(f) = \text{Max}(R) \cap D(1 - fg)$.
(ii) $\Rightarrow$ (iii) : The subset $\text{Max}(R)$ is Zariski quasi-compact and flat Hausdorff.
(iii) $\Rightarrow$ (i) : See [22, Theorem 4.5]. □

**Lemma 5.2.** Let $R$ be a ring such that $R/\mathfrak{J}$ is a zero dimensional ring. Then the clopens of $\text{Max}(R)$ are precisely of the form $\text{Max}(R) \cap V(f)$ where $f \in R$.

**Proof.** By Proposition 5.1, the Zariski and flat topologies over $\text{Max}(R)$ are the same. If $f \in R$ then we observed that $\text{Max}(R) \cap V(f)$ is a clopen of $\text{Max}(R)$. Conversely, let $U$ be a clopen of $\text{Max}(R)$. It is easy to see that every closed subspace of a quasi-compact space is quasi-compact. Hence, we may write $U = \bigcup_{k=1}^{n} \text{Max}(R) \cap V(I_k)$ where each $I_k$ is a (finitely generated) ideal of $R$. This yields that $U = \text{Max}(R) \cap V(I)$ where $I = I_1...I_n$. Similarly we get that $U^c = \text{Max}(R) \setminus U = \text{Max}(R) \cap V(J)$ where $J$ is a (finitely generated) ideal of $R$. It follows that $I + J = R$. Thus there exist some $f \in I$ and $g \in J$ such that $f + g = 1$. This implies that $U = \text{Max}(R) \cap V(f)$. □

Throughout this paper, $\Gamma = \prod_{x \in X} R_x$ where each $R_x$ is a local ring with the maximal ideal $m_x$. For each $x \in X$ then $\mathfrak{M}_x := \pi_x^{-1}(m_x)$ is a maximal ideal of $\Gamma$, because the ring map $\Gamma/\mathfrak{M}_x \rightarrow R_x/m_x$ induced by the canonical projection $\pi_x : \Gamma \rightarrow R_x$ is an isomorphism. If $f = (f_x) \in \Gamma$ then we define
there exists a unique point $y$ satisfying the property. Thus by a similar argument as applied in the proof of Theorem 3.5, it follows that $f$ is invertible in $\Gamma$ if and only if $\Omega(f) = X$. It is also easy to see that $\Omega(fg) = \Omega(f) \cap \Omega(g)$ for all $f, g \in \Gamma$.

**Lemma 5.3.** Let $f \in \Gamma$. Then $\Omega(f) = \emptyset$ if and only if $f \in \mathfrak{J}$.

**Proof.** If $\Omega(f) = \emptyset$ then $f_x \in \mathfrak{m}_x$ for all $x$. This yields that $\Omega(1 + fg) = X$ for all $g \in \Gamma$. Thus $f \in \mathfrak{J}$. Conversely, if $f \in \mathfrak{J}$ then $f \in \mathfrak{M}_x$ for all $x$. So $\Omega(f)$ is empty. \hfill $\square$

**Theorem 5.4.** The space $\text{Max}(\Gamma)$ together with the canonical map $\eta : X \rightarrow \text{Max}(\Gamma)$ given by $x \mapsto \mathfrak{M}_x$ is the Stone-Čech compactification of the discrete space $X$.

**Proof.** If $f \in \Gamma$ then consider the sequence $g = (g_x) \in \Gamma$ such that $g_x$ is either $0$ or $f_x^{-1}$, according as $f_x \in \mathfrak{m}_x$ or $f_x \notin \mathfrak{m}_x$. Then $\Omega(1 + fh(1 - fg)) = X$ for all $h \in \Gamma$. Hence, $f(1 - fg) \in \mathfrak{J}$. Thus $\Gamma/\mathfrak{J}$ is absolutely flat. Therefore by Proposition 5.1, the space $\text{Max}(\Gamma)$ is compact. Then we verify the universal property of the Stone-Čech compactification. Let $Y$ be a compact topological space and $\varphi : X \rightarrow Y$ a function. If $M \in \text{Max}(\Gamma)$ then by Lemma 5.3, the subsets $\Omega(f)$ with $f \in \Gamma \setminus M$ have the finite intersection property. Thus by a similar argument as applied in the proof of Theorem 3.5, there exists a unique point $y_M \in Y$ such that $\bigcap_{f \in \Gamma \setminus M} \varphi(\Omega(f)) = \{y_M\}$. Then we define the map $\tilde{\varphi} : \text{Max}(\Gamma) \rightarrow Y$ as $M \mapsto y_M$. Again exactly like the proof of Theorem 3.5 it is shown that $\varphi = \tilde{\varphi} \circ \eta$ and $\tilde{\varphi}$ is continuous. Finally, to prove the uniqueness of $\tilde{\varphi}$ it suffices to show that $\eta(X)$ is a dense subspace of $\text{Max}(\Gamma)$. The space $\text{Max}(\Gamma)$ is totally disconnected, see [22, Proposition 4.4]. It is well known that in a compact totally disconnected space, the collection of clopens is a base for the opens. Using this and Lemma 5.2, the collection of $\text{Max}(\Gamma) \cap V(f)$ with $f \in \Gamma$ forms a base for the opens of $\text{Max}(\Gamma)$. Now if $\text{Max}(\Gamma) \cap V(f)$ is non-empty then $\Omega(f) \neq X$. Hence there exists some $x \in X$ such that $\mathfrak{M}_x \in \text{Max}(\Gamma) \cap V(f)$. Therefore $\eta(X)$ is a dense subspace of $\text{Max}(\Gamma)$. \hfill $\square$

**Example 5.5.** If $p$ is a prime number then for each $n \geq 1$, $\mathbb{Z}/p^n\mathbb{Z}$ is a local zero dimensional ring (its prime spectrum is a singleton), but the direct product ring $\prod_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z}$ has infinite Krull dimension. This ring also has a huge number of prime ideals. In fact by Theorem 5.4, the cardinality of its maximal ideals equals $2^c$ where $c$ is the cardinality of the continuum.

**Remark 5.6.** The canonical map $\eta : X \rightarrow \text{Max}(\Gamma)$ given by $x \mapsto \mathfrak{M}_x$ is an open embedding. In fact, $\{\mathfrak{M}_x\} = \text{Max}(\Gamma) \cap D(\Delta_x)$ for all $x \in X$. To see this let $M \in \text{Max}(\Gamma) \cap D(\Delta_x)$ and $f \in M$. If $f \notin \mathfrak{M}_x$ then $\Omega(1 - \Delta_x + \Delta_x f) = X$ and so $1 - \Delta_x + \Delta_x f$ is invertible in the ring $\Gamma$. But this is a contradiction because $1 - \Delta_x + \Delta_x f \in M$. Therefore $M \subseteq \mathfrak{M}_x$ and so $M = \mathfrak{M}_x$.

**Corollary 5.7.** There exists a unique homeomorphism:

$$\text{Min}(\Lambda) \xrightarrow{\sim} \text{Max}(\Gamma)$$
such that \( p_x \) is mapped into \( \mathfrak{M}_x \) for all \( x \in X \).

**Proof.** It is deduced from the universal property of the Stone-Čech compactification by taking into account Theorems 3.5 and 5.4. \( \square \)

In the next section, we will precisely determine the rule of isomorphism of Corollary 5.7.

**Corollary 5.8.** Let \( R \) be a ring and let \( X \) be a subset of \( \text{Spec}(R) \). Then the following spaces are canonically isomorphic (up to a unique isomorphism).

(i) \( \text{Min}(\prod_{p \in X} R/p) \).

(ii) \( \text{Spec}(\prod_{p \in X} \kappa(p)) \).

(iii) \( \text{Max}(\prod_{p \in X} R_p) \).

**Proof.** It is an immediate consequence of Corollary 5.7. \( \square \)

If \( X \) is a set with the cardinality \( \kappa \) and \( \widetilde{X} \) is a compactification of the discrete space \( X \), then by [5, Tag 0909] and assuming the generalized continuum hypothesis, we have \( |\widetilde{X}| \in \{\kappa, 2^\kappa, 2^{2\kappa}\} \).

**Corollary 5.9.** If \( X \) is an infinite set with the cardinality \( \kappa \), then \( |\text{Min}(\Lambda)| = |\text{Max}(\Gamma)| = |\text{Spec}P(X)| = 2^{2^\kappa} \).

**Proof.** It follows from Corollaries 3.8 and 5.7 and the fact that the cardinality of the Stone-Čech compactification of the infinite discrete space \( X \) is equal to \( 2^{2^\kappa} \). To see the proof of this fact please consider [12, Theorem 3.58] or [26, Theorem on page 71]. \( \square \)

**Corollary 5.10.** Let \( X \) and \( Y \) be two sets with the cardinalities \( \kappa \) and \( \lambda \), respectively. Then the number of all ring maps \( P(X) \rightarrow P(Y) \) is either \( \kappa^\lambda \) or \( 2^{\lambda2^\kappa} \), according as \( X \) is finite or infinite.

**Proof.** It is deduced from Corollaries 3.10 and 5.9. \( \square \)

If \( \kappa \) is an infinite cardinal, then \( \lambda2^\kappa = \max\{\lambda, 2^\kappa\} \). To see its proof apply Cantor’s theorem and [51, p. 162, Lemma 6R] which states that \( \kappa\kappa = \kappa \).

**Corollary 5.11.** Let \( X \) be a set with the cardinality \( \kappa \). Then the number of all ring maps \( P(X) \rightarrow P(X) \) is either \( \kappa^\kappa \) or \( 2^{2^\kappa} \), according as \( \kappa \) is finite or infinite.

**Proof.** It is an immediate consequence of Corollary 5.10. \( \square \)

6. Ultra-rings and their applications in compactification

In this section, we introduce a new way to construct the ultraproduct of rings which considerably simplifies the existence method in the literature (see e.g. [4], [6], [9] and [17]). Then we use this new approach to determine precisely the isomorphisms whose rules are already obtained in an implicit
way (see e.g. Corollary 5.7).

Let \((R_x)\) be a family of rings indexed by a set \(X\) and let \(R = \prod_{x \in X} R_x\) be their direct product ring. Let \(M\) be a maximal ideal of \(\mathcal{P}(X)\). Then it can be easily seen that \(M^* = \{f \in R : S(f) \in M\}\) is an ideal of \(R\), because clearly \(S(0) = 0 \in M\) and so \(0 \in M^*\), also \(S(f + g) \subseteq S(f) \cup S(g)\) and \(S(fg) \subseteq S(f) \cap S(g)\) for all \(f, g \in R\). We call the quotient ring \(R/M^*\) the ultraproduct (or, ultra-ring) of the family \((R_x)\) with respect to \(M\).

It is interesting to notice the map \(\varphi : R \to \mathcal{P}(X)\) given by \(f \sim S(f)\) is not a morphism of rings, since it is not additive, in fact \(S(f) + S(g) \subseteq S(f + g)\).

For a ring \(R\), the following assertions hold.

**Theorem 6.1.** For a ring \(R = \prod_{x \in X} R_x\) the following assertions hold.

(i) If each \(R_x\) is a field, then \(R/M^*\) is a field.

(ii) If each \(R_x\) is an integral domain, then \(R/M^*\) is an integral domain.

(iii) If each \(R_x\) is a local ring, then \(R/M^*\) is a local ring.

(iv) If each \(K_x\) is the fraction field of an integral domain \(R_x\), then the ultra-ring of the family \((K_x)\) with respect to \(M\) is the fraction field of \(R/M^*\).

(v) If each \(R_x\) is a local ring with the residue field \(K_x\), then the ultra-ring of the family \((K_x)\) with respect to \(M\) is the residue field of \(R/M^*\).

**Proof.**

(i) : Take \(f \in R \setminus M^*\) and consider the sequence \(g = (g_x) \in R\) where each \(g_x\) is either \(f_x^{-1}\) or 1, according as \(x \in S(f)\) or \(x \notin S(f)\). Then clearly \(S(1 - fg) \subseteq S(f)^c \in M\). Thus \(S(1 - fg) \in M\) and so \(1 - fg \in M^*\).

(ii) : Suppose \(fg \in M^*\) for some \(f, g \in R\). Then clearly \(S(f) \cap S(g) \subseteq S(fg) \in M\). Thus \(S(f) \cap S(g) \in M\). It follows that either \(f \in M^*\) or \(g \in M^*\).

(iii) : Clearly \(M^b = \{f \in R : \Omega(f) \in M\}\) is a proper ideal of \(R\) and \(M^* \subseteq M^b\), since \(\Omega(f) = \{x \in X : f_x \notin \mathfrak{m}_x\} \subseteq S(f)\) for all \(f \in R\) where \(\mathfrak{m}_x\) is the maximal ideal of \(R_x\). If \(f \in R \setminus M^b\) then \(S(1 - fg) \subseteq \Omega(f)^c \in M\) where \(g = (g_x)\) and each \(g_x\) is either \(f_x^{-1}\) or 1, according as \(x \in \Omega(f)\) or \(x \notin \Omega(f)\). Therefore \(S(1 - fg) \in M\) and so \(1 - fg \in M^*\). Hence, \(M^b/M^*\) is the only maximal ideal of \(R/M^*\). The proof of (iv) is easy and left as an exercise to the reader.

(v) : It suffices to show that the map \(R/M^b \to R'/M^*\) given by \(f + M^b \sim \overline{f} + M^*\) is an isomorphism of rings where \(R' = \prod_{x \in X} K_x\), \(M^* = \{g \in R' : S(g) \in M\}\) and \(\overline{f} = (f_x + \mathfrak{m}_x)\) with \(\mathfrak{m}_x\) is the maximal ideal of \(R_x\). If
Let $f \in M^\flat$ then $S(f) \subseteq \Omega(f) \in M$ and so $S(f) \in M$. Hence, the above map is well-defined. Clearly it is also an isomorphism.

**Theorem 6.2.** The map $\varphi : \text{Spec} \mathcal{P}(X) \to \text{Min}(\Lambda)$ given by $M \mapsto M^*$ is a homeomorphism.

**Proof.** First we need to show that $M^*$ is a minimal prime of $\Lambda$. By Theorem 6.1 (ii), $M^*$ is a prime ideal of $\Lambda$. Suppose there exists a prime ideal $p$ of $\Lambda$ such that $p \subseteq M^*$. If $f \in M^*$ then consider the sequence $g = (g_x) \in \Lambda$ where each $g_x$ is either 1 or 0, according as $x \in S(f)$ or $x \notin S(f)$. Then clearly $S(g) = S(f) \in M$ and so $g \in M^*$. Moreover $f(1 - g) = 0$. This yields that $f \in p$. Hence, $M^*$ is a minimal prime of $\Lambda$. The map $\varphi$ is continuous, since $\varphi^{-1}(\text{Min}(\Lambda) \cap D(f)) = D(S(f))$ for all $f \in \Lambda$. Clearly $m^*_x = p_x$ for all $x \in X$, where $m_x = \mathcal{P}(X \setminus \{x\})$ and for $p_x$ see the above of Theorem 6.1. This shows that $\eta = \varphi \circ \eta'$ where $\eta : X \to \text{Min}(\Lambda)$ and $\eta' : X \to \text{Spec} \mathcal{P}(X)$ are the canonical maps. Therefore, by the universal property of the Stone-Čech compactification and by taking into account Theorem 3.8 and Corollary 3.8 we deduce that $\varphi$ is a homeomorphism.

**Theorem 6.3.** The map $\psi : \text{Spec} \mathcal{P}(X) \to \text{Max}(\Gamma)$ given by $M \mapsto M^\flat$ is a homeomorphism.

**Proof.** By the proof of Theorem 6.1 (iii), $M^\flat$ is a maximal ideal of $\Gamma$. Hence, the above map is well-defined. It is also continuous, since $\psi^{-1}(\text{Max}(\Gamma) \cap D(f)) = D(\Omega(f))$ for all $f \in \Gamma$. Moreover $m^*_x = \mathfrak{m}_x$ for all $x \in X$, where $m_x = \mathcal{P}(X \setminus \{x\})$ and for $\mathfrak{m}_x$ see Theorem 5.4. This shows that $\eta = \psi \circ \eta'$ where $\eta : X \to \text{Max}(\Gamma)$ and $\eta' : X \to \text{Spec} \mathcal{P}(X)$ are the canonical maps. Thus, by the universal property of the Stone-Čech compactification and by taking into account Corollary 3.8 and Theorem 5.4 we deduce that $\psi$ is a homeomorphism.

### 7. Alexandroff compactification

Let $\mathcal{R}$ be the set of all subsets of a set $X$ which are either finite or cofinite (i.e. its complement is finite). Then clearly $\mathcal{R}$ is a subring of the power set ring $\mathcal{P}(X)$. Recall that if $x \in X$ then $m_x = \mathcal{P}(X \setminus \{x\})$ is a maximal ideal of $\mathcal{P}(X)$. In the following result the maximal ideals of $\mathcal{R}$ are characterized.

**Theorem 7.1.** Let $X$ be an infinite set. Then the maximal ideals of $\mathcal{R}$ are precisely $\text{Fin}(X)$ or of the form $m_x \cap \mathcal{R}$ where $x \in X$.

**Proof.** First we have to show that $\text{Fin}(X)$ is a maximal ideal of $\mathcal{R}$. Clearly $\text{Fin}(X) \neq \mathcal{R}$ since $X$ is infinite. If there exists an ideal $I$ of $\mathcal{R}$ strictly containing $\text{Fin}(X)$ then we may choose some $A \in I$ such that $A \notin \text{Fin}(X)$. It follows that $A^c \in \text{Fin}(X)$ and so $1 = A + A^c \in I$. Hence, $\text{Fin}(X)$ is a maximal ideal of $\mathcal{R}$. Conversely, let $M$ be a maximal ideal of $\mathcal{R}$ such that $M \neq m_x \cap \mathcal{R}$ for all $x \in X$. It follows that $A_x := X \setminus \{x\} \in \mathcal{R} \setminus M$ for all $x \in X$. But $\{x\}, A_x = 0 \in M$. Therefore $\{x\} \in M$ for all $x \in X$. This yields that $\text{Fin}(X) \subseteq M$ and so $\text{Fin}(X) = M$. \qed
Remark 7.2. Let $X$ be an infinite set. Here we give a second proof to show that $\text{Fin}(X)$ is a maximal ideal of $\mathcal{R}$. There exists a maximal ideal $M$ of $\mathcal{P}(X)$ such that $\text{Fin}(X) \subseteq M$ since $\text{Fin}(X) \neq \mathcal{P}(X)$. We have then $\text{Fin}(X) = M \cap \mathcal{R}$. Thus $\text{Fin}(X)$ is a maximal ideal of $\mathcal{R}$.

Corollary 7.3. If $X$ is an infinite set, then $\text{Spec}(\mathcal{R})$ is the Alexandroff compactification of the discrete space $X$.

Proof. The space $\text{Spec}(\mathcal{R})$ is compact. The map $\eta : X \to \text{Spec}(\mathcal{R})$ given by $x \mapsto m_x \cap \mathcal{R}$ is an open embedding. Because by Theorem 7.1, $D(\{x\}) = m_x \cap \mathcal{R}$ for all $x \in X$. Now if $A$ is a subset of $X$ then $\eta(A) = \bigcup_{x \in A} D(\{x\})$. If $U$ is an open neighborhood of $\text{Fin}(X)$ in $\text{Spec}(\mathcal{R})$ then $U^c$ is a finite set. Hence, $\eta(X)$ is a dense subspace of $\text{Spec}(\mathcal{R})$. □

Clearly $\text{Fin}(X)$ is the point at infinity of the (infinite) discrete space $X$.

Note that, unlike the Stone-Čech compactification which exists for any topological space, the Alexandroff compactification does not necessarily exist for any space.

8. Totally disconnected compactifications

In this section it is shown that every totally disconnected compactification of a discrete space $X$ is precisely of the form $\text{Spec}(\mathcal{R}')$ where the ring $\mathcal{R}'$ satisfies in the extensions of rings $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{P}(X)$. For $\mathcal{R}$ see §7.

Lemma 8.1. Let $f : X \to Y$ be a continuous map of topological spaces such that $f(X)$ is a dense subspace of $Y$. Then the induced map $\text{Clop}(f) : \text{Clop}(Y) \to \text{Clop}(X)$ is injective.

Proof. Let $A$ be a clopen of $Y$ such that $f^{-1}(A) = \emptyset$. If $A$ is non-empty then $A \cap f(X)$ is non-empty. But this is a contradiction.

As a second proof, let $D_1$ and $D_2$ be two clopens of $Y$ such that $f^{-1}(D_1) = f^{-1}(D_2)$. Suppose there exists some $y \in D_1$ such that $y \notin D_2$. It follows that $(D_1 \cap D_2) \cap f(X) \neq \emptyset$. Hence there exists some $x \in X$ such that $f(x) \in D_1 \cap D_2$. But this is a contradiction. Therefore $D_1 = D_2$. □

Theorem 8.2. Every totally disconnected compactification of a discrete space $X$ is precisely of the form $\text{Spec}(\mathcal{R}')$ where the ring $\mathcal{R}'$ satisfies in the extensions of rings $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{P}(X)$.

Proof. It is easy to see that for any such ring $\mathcal{R}'$ then $\text{Spec}(\mathcal{R}')$ together with the canonical open embedding $\eta : X \to \text{Spec}(\mathcal{R}')$ which sends each point $x \in X$ into $m_x \cap \mathcal{R}'$ is a totally disconnected compactification of the discrete space $X$. Conversely, let $(\tilde{X}, \eta)$ be a totally disconnected compactification of a discrete space $X$. By [23, Corollary 5.4], the space $\tilde{X}$ is homeomorphic to $\text{Spec}(R)$ where $R = \text{Clop}(\tilde{X})$. By Lemma 8.1 the induced map $\text{Clop}(\eta) : R \to \text{Clop}(X) = \mathcal{P}(X)$ is an injective ring map. So the ring $R$ is isomorphic
to \( \mathcal{R}' \), the image of \( \text{Clop}(\eta) \). It remains to show that \( \mathcal{R} \subseteq \mathcal{R}' \). Take \( A \in \mathcal{R} \).

If \( A \) is finite then \( D := \eta(A) = \bigcup_{x \in A} \{\eta(x)\} \) is a closed subset of \( \tilde{X} \) and so \( D \in \text{Clop}(\tilde{X}) \). Therefore \( A = \eta^{-1}(D) \in \mathcal{R}' \). But if \( A \) is cofinite then the above argument shows that \( A^c \in \mathcal{R}' \), and so \( A = 1 - A^c \in \mathcal{R}' \).

\[ \square \]

**Remark 8.3.** If \( (\tilde{X}, \eta) \) is an arbitrary compactification of a discrete space \( X \) then by [23, Theorem 5.2], the space of connected components \( \pi_0(\tilde{X}) \) is homeomorphic to \( \text{Spec}(\mathcal{R}') \) where \( \mathcal{R}' = \text{Clop}(\tilde{X}) \). Also \( \mathcal{R}' \), via the ring map \( \text{Clop}(\eta) \), can be viewed as a subring of \( \mathcal{P}(X) \) and containing \( \mathcal{R} \). Note that there are compactifications of a discrete space which are not totally disconnected.

### 9. Semigroup structure on \( \beta X \)

In this section, \( \beta X = \text{Spec} \mathcal{P}(X) \) together with the canonical map \( \eta : X \to \beta X \) denotes the Stone-Čech compactification of the discrete space \( X \). If \( f : X \to Y \) is a function then by Corollary [3.8.5] there exists a unique continuous function \( \beta f : \beta X \to \beta Y \) such that \((\beta f)(m_x) = m_{f(x)} \) for all \( x \in X \). This yields that \( \beta f = \mathcal{P}(f)^* \). In particular, if \( f : X \to Y \) is injective then \( \beta f : \beta X \to \beta Y \) is as well.

Let \((S, *)\) be a semigroup such that \( S \) is a topological space. Then equip \( S \times S \) with the product topology. If the operation \( * : S \times S \to S \) is continuous, then \((S, *)\) is called a topological semigroup. If the operation \( * \) is not continuous, then this leads us to a weaker notion. Indeed, the pair \((S, *)\) is called a left topological semigroup if the operation \( * \) is left semi-continuous. That is, for each \( p \in S \) then the map \( \ell_p : S \to S \) given by \( x \mapsto p \ast x \) is continuous. The right topological semigroup is defined dually. Obviously every topological semigroup is both right topological and left topological semigroup. We have then the following interesting result.

**Theorem 9.1.** The operation of every commutative semigroup \((X, \cdot)\) can be extended uniquely to an operation \( * \) on \( \beta X \) such that: \((\beta X, *)\) is a left topological semigroup, the canonical map \( \eta : X \to \beta X \) is a morphism of semigroups and \( m_x \ast M = M \ast m_x \) for all \( M \in \beta X \) and \( x \in X \). If moreover \( e \) is the identity of \( X \), then \( m_e \) is the identity of \( \beta X \).

**Proof.** If \( x \in X \) then by Theorem [5.7] there exists a unique continuous function \( \varphi_x : \beta X \to \beta X \) such that \( \varphi_x(m_y) = m_{x \cdot y} \) for all \( y \in X \). For a fixed \( M \in \beta X \), again by Theorem [3.7], there exists a unique continuous map \( \theta_M : \beta X \to \beta X \) such that \( \theta_M(m_x) = \varphi_x(M) \) for all \( x \in X \). Now we define the operation \( * \) on \( \beta X \) as \( M \ast N = \theta_M(N) \). Then we show this operation is associative. To prove this it suffices to show that \( \theta_M \circ \theta_N = \theta_L \) for every \( M, N \in \beta X \) with \( L = \theta_M(N) \). To see this it will be enough to show that \( \theta_M \circ \varphi_x = \varphi_x \circ \theta_M \) for all \( M \in \beta X \) and \( x \in X \). But to see the latter it suffices to show that \( \theta_M \circ \varphi_x \) and \( \varphi_x \circ \theta_M \) agree on \( \eta(X) \), (recall that if two continuous
maps into a Hausdorff space agree on a dense subspace of the domain, they are equal). This reduces to show that \( \varphi_x \circ \varphi_y = \varphi_{xy} \) for all \( x, y \in X \). Finally, to see this it suffices to show that \((\varphi_x \circ \varphi_y)(m_z) = \varphi_{xy}(m_z) \) for all \( z \in X \). But the latter obviously holds since the operation of \( X \) is associative. Clearly \( \ell_M = \theta_M \) for all \( M \in \beta X \). Hence, \((\beta X, *)\) is a left topological semigroup. The map \( \eta \) is a morphism of semigroups since \( \varphi_x = \theta_{m_x} \) for all \( x \in X \). This also yields that \( m_x * M = M * m_x \) for all \( M \in \beta X \) and \( x \in X \). To see the uniqueness of \( * \), suppose there is another operation \( *' \) on \( \beta X \) such that \((\beta X, *')\) is a left topological semigroup, the canonical map \( \eta : X \to (\beta X, *') \) is a morphism of semigroups and \( m_x *' M = M *' m_x \) for all \( M \in \beta X \) and \( x \in X \). Then clearly for each \( x \in X \), the maps \( \ell_{m_x} \) and \( \ell'_{m_x} \) agree on \( \eta(X) \), hence they are equal. It follows that for each \( M \in \beta X \), then \( \ell_M \) and \( \ell'_M \) agree on \( \eta(X) \), hence they are equal. The latter implies that \( * = *' \). Finally, if \( e \) is the identity element of \( X \) then \( \varphi_e \) is the identity map. It follows that \( m_e \) is the identity element of \( \beta X \).

Note that the operation \( * \) of Theorem 9.1 is not necessarily commutative. Hence, we may define a new operation on \( \beta X \) as \( M *' N := \theta_N(M) = N * M \). Then it is easy to see that \((\beta X, *')\) is a right topological semigroup. Therefore we may consider \( \beta X \) as left topological or right topological semigroup, depending on the preferred construction, but never both (specically when \( X \) is an infinite set).

In the proof of Theorem 9.1 we have \( \varphi_x = \mathcal{P}(f_x)^* \) for all \( x \in X \) where the function \( f_x : X \to X \) is defined by \( f_x(y) = x.y \). By [23, Theorem 5.6], there exists a (unique) morphism of rings \( h_M : \mathcal{P}(X) \to \mathcal{P}(X) \) such that \( \theta_M = \text{Spec}(h_M) \). In the following result, the rule of this morphism is determined explicitly.

**Theorem 9.2.** Let \((X, \cdot)\) be a commutative semigroup and \( M \) a maximal ideal of \( \mathcal{P}(X) \). Then the map \( \zeta_M : \mathcal{P}(X) \to \mathcal{P}(X) \) given by \( A \rightsquigarrow \{ x \in X : f_x^{-1}(A) \notin M \} \) is a morphism of rings and \( \theta_M = \text{Spec}(\zeta_M) \).

**Proof.** It is not hard to see that the map \( \zeta_M \) is actually a morphism of rings. To see \( \theta_M = \text{Spec}(\zeta_M) \) it suffices to show that \( \theta_M(m_x) = \zeta_M^{-1}(m_x) \) for all \( x \in X \). By the category of left topological monoids we mean a category whose objects are the left topological monoids and whose morphisms are the continuous morphisms of monoids.

**Theorem 9.3.** The assignments \( X \rightsquigarrow \beta X \) and \( h \rightsquigarrow \beta f \) form a faithful covariant functor from the category of commutative monoids to the category of left topological monoids.

**Proof.** By the universal property of the Stone-Čech compactification, it is a functor provided that we could prove that if \( f : X \to Y \) is a morphism of commutative monoids then \( \beta f : \beta X \to \beta Y \) is a morphism of monoids.
Clearly $\beta f$ preserves the identities. It remains to show that $(\beta f) \circ \theta_M = \theta_{M'} \circ (\beta f)$ for all $M \in \text{Spec } P(X)$ with $M' = (\beta f)(M)$, for the notations see the proof of Theorem 10.1. To see this it suffices to show that these functions agree on $\eta(X)$. To see the latter it will be enough to show that $(\beta f) \circ \varphi_x = \varphi_{f(x)} \circ (\beta f)$ for all $x \in X$. Clearly these maps agree on $\eta(X)$, hence they are equal. □

10. Absolutely flatness of the total ring of fractions

Theorems 10.1 and 10.9 provide new and simple proofs to the main results of [10, Theorem 2.9], [13, Chap I, Theorem 4.5], [15, Proposition 1.4] and [16, Proposition 9]. In the following results, $T(R)$ denotes the total ring of fractions of a ring $R$.

**Theorem 10.1.** Let $R$ be a ring. Then $T(R)$ is absolutely flat if and only if the following two conditions hold.

(i) $R$ is reduced and $\text{Min}(R)$ is Zariski compact.

(ii) Every finitely generated and faithful ideal of $R$ contains a non zero-divisor of $R$.

**Proof.** Assume $T(R)$ is absolutely flat. Then $R$ is reduced, since every absolutely flat ring and so each subring are reduced. By [24, Lemma 3.4], $\text{Min}(R)$ is Zariski compact. If $I = (f_1, \ldots, f_n)$ is a finitely generated and faithful ideal of $R$ then for each $i$, there exists a non zero-divisor $g_i$ of $R$ such that $f_i (g_i - f_i h_i) = 0$ for some $h_i \in R$. It follows that $(g_1 - f_1 h_1) \cdots (g_n - f_n h_n) = 0$ and so $g_1 \cdots g_n \in I$. Conversely, if $f \in R$ then it will be enough to find a non zero-divisor $g$ of $R$ such that $fg = f^2 h$ for some $h \in R$. Setting $X = \{ p \in \text{Min}(R) : f \in p \}$. If $p \in X$ then there exists some $x_p \in R \setminus p$ such that $fx_p = 0$. It follows that $\text{Min}(R) \subseteq D(f) \cup \bigcup_{p \in X} D(x_p)$. Using the quasi-
compactness of $\text{Min}(R)$, then we may write $\text{Min}(R) \subseteq D(f) \cup \left( \bigcup_{i=1}^n D(x_i) \right)$ and that $fx_i = 0$ for all $i$. Therefore $I = (f, x_1, \ldots, x_n)$ is a faithful ideal of $R$, because suppose $r I = 0$, if $p \in \text{Min}(R)$ then $r \in p$ and so $r \in \bigcap_{p \in \text{Min}(R)} p = \sqrt{0} = 0$. Hence, $I$ contains a non zero-divisor $g$ of $R$. Thus we may write $g = fh + \sum_{i=1}^n r_i x_i$ where $h, r_1, \ldots, r_n \in R$. This yields that $fg = f^2 h$. □

**Remark 10.2.** Let $R$ be a ring. It is easy to see that if at least one of the coefficients of a polynomial $f \in R[x]$ is a non zero-divisor of $R$, then $f$ is a non zero-divisor of $R[x]$. But the converse does not hold. As an example, take $R = \mathbb{Z}/6\mathbb{Z}$ then $f = 2 + 3x$ is a non zero-divisor of $R[x]$, but all of its coefficients are zero-divisors of $R$. This observation shows that if $T(R[x])$ is zero dimensional (or, an absolutely flat ring) then the same assertion does not necessarily hold for $T(R)$. 
Corollary 10.3. Let \( R \) be a reduced ring such that \( \text{Min}(R) \) is a finite set. Then \( T(R) \) is absolutely flat.

Proof. Let \( I = (f_1, \ldots, f_n) \) be a faithful ideal of \( R \) and setting \( S := R \setminus Z(R) \). If \( I \cap S = \emptyset \) then there exists a prime ideal \( p \) of \( R \) such that \( I \subseteq p \) and \( p \cap S = \emptyset \). It follows that \( p \subseteq Z(R) = \bigcup_{q \in \text{Min}(R)} q \). Thus by the Prime Avoidance Lemma (cf. [20, Theorem 2.2]), \( p \in \text{Min}(R) \). So for each \( i \), there exists some \( g_i \in R \setminus p \) such that \( f_i g_i = 0 \). Therefore \( gI = 0 \) where \( g = g_1 \ldots g_n \). But this is a contradiction. Hence, \( I \) admits a non zero divisor of \( R \). Therefore by Theorem 10.1, \( T(R) \) is absolutely flat.

The following two results are easily deduced from the above corollary.

Corollary 10.4. Let \( R \) be a reduced ring such that \( \text{Spec}(R) \) is a noetherian space with respect to the Zariski topology. Then \( T(R) \) is absolutely flat.

Corollary 10.5. Let \( R \) be a reduced and noetherian ring. Then \( T(R) \) is absolutely flat.

Corollary 10.6. Let \( R \) be a ring. Then \( T(R[x]) \) is absolutely flat if and only if \( R \) is reduced and \( \text{Min}(R) \) is Zariski compact.

Proof. It is interesting to notice that for any ring \( R \), then every finitely generated and faithful ideal of \( R[x] \) contains a non zero divisor of \( R[x] \). Then apply Theorem 10.1. \( \square \)

Corollary 10.7. Let \( R \) be a ring. If \( T(R) \) is absolutely flat, then \( T(R[x]) \) is as well.

Remark 10.8. Here we prove the reverse implication of Theorem 10.1 by an alternative approach. Clearly a ring \( R \) is reduced if and only if \( T(R) \) is reduced. To prove that \( T(R) \) is zero dimensional it will be enough to show that every prime ideal \( q \) of \( R \) which does not meet \( R \setminus Z(R) \), then it is a minimal prime. To see this it suffices to show that \( q \subseteq p \) for some \( p \in \text{Min}(R) \). Suppose \( q \nsubseteq p \) for all \( p \in \text{Min}(R) \), then there exists some \( x_p \in q \) such that \( x_p \notin p \). So \( \text{Min}(R) \subseteq \bigcup_{p \in \text{Min}(R)} D(x_p) \). By the quasi-compactness of the minimal spectrum, we may write \( \text{Min}(R) \subseteq \bigcup_{i=1}^n D(x_i) \) where \( x_i := x_{p_i} \) for all \( i \). Then \( I = (x_1, \ldots, x_n) \subseteq q \subseteq Z(R) \). So \( I \) is not a faithful ideal, i.e., \( \text{Ann}(I) \neq 0 \). Thus we may choose some nonzero \( a \in \text{Ann}(I) \). Now if \( p \in \text{Min}(R) \) then \( x_i \notin p \) for some \( i \). But \( ax_i = 0 \). Thus \( a \in p \). So \( a \in \bigcap_{p \in \text{Min}(R)} p = 0 \) which is a contradiction.

Theorem 10.9. For a reduced ring \( R \) the following assertions are equivalent.

(i) \( T(R) \) is absolutely flat.
(ii) If \( f \in R \) there exists some \( g \in R \) such that \( fg = \text{Ann}(Rf + Rg) = 0 \).
(iii) If an ideal \( I \) of \( R \) is contained in \( Z(R) \), then \( I \subseteq p \) for some \( p \in \text{Min}(R) \).
Proof. (i) ⇒ (ii) : There exists a non zero-divisor \( s \in R \) such that \( fs = f^2h \) for some \( h \in R \). Then \( g := fh - s \) is the desired element.

(ii) ⇒ (i) : It suffices to show that \( h := f - g \) is a non zero-divisor of \( R \). Suppose \( rh = 0 \) and \( p \) is a minimal prime ideal of \( R \) such that \( r \not\in p \).

Then \( f, g \in p \) and so there exist \( f', g' \in R \setminus p \) such that \( ff' = gg' = 0 \).

This yields that \( f'g' \in \text{Ann}(Rf + Rg) = 0 \) which is a contradiction. Hence, \( r \in \bigcap_{p \in \text{Min}(R)} p = \sqrt{0} = 0 \).

(i) ⇒ (iii) : Suppose \( I \not\subseteq p \) for all \( p \in \text{Min}(R) \), then we may choose some \( x_p \in I \setminus p \). Using the quasi-compactness of \( \text{Min}(R) \), then we may write \( \text{Min}(R) \subseteq \bigcup_{i=1}^n D(x_i) \) where \( x_i \in I \) for all \( i \). It follows that \( J = (x_1, ..., x_n) \) is a faithful ideal of \( R \). Thus by Theorem [10.1], \( J \) admits a non zero-divisor which is a contradiction.

(iii) ⇒ (i) : Suppose \( \text{Min}(R) \subseteq \bigcup_{i \in S} D(f_i) \) where \( f_i \in R \) for all \( i \). Then by the hypothesis, the ideal \( (f_i : i \in S) \) admits a non zero-divisor \( g \) of \( R \). So there exists a finite subset \( S' \) of \( S \) such that \( g = \sum r_i f_i \) where \( r_i \in R \) for all \( i \in S' \). This yields that \( \text{Min}(R) \subseteq \bigcup_{i \in S'} D(f_i) \), since otherwise we may find some \( p \in \text{Min}(R) \) such that \( g \notin p \), but this is impossible since \( R \subseteq Z(R) \). Hence, \( \text{Min}(R) \) is quasi-compact. Now let \( I \) be a finitely generated and faithful ideal of \( R \). If \( I \subseteq Z(R) \) then \( I \) is contained in a minimal prime ideal \( p \) of \( R \). Thus we may find some \( s \in R \setminus p \) such that \( sI = 0 \), which is a contradiction. So \( I \) admits a non zero-divisor of \( R \). Therefore by Theorem [10.1], \( T(R) \) is absolutely flat. 

\[ \square \]

**Proposition 10.10.** Let \( R \) be a ring. If \( T(R[x]) \) is a zero dimensional ring, then \( \text{Min}(R) \) is Zariski compact.

Proof. For any ring \( R \), the minimal prime ideals of \( R[x] \) are precisely of the form \( p[x] \) where \( p \) is a minimal prime ideal of \( R \). Hence, the map \( \varphi : \text{Min}(R) \to \text{Min}(R[x]) \) given by \( p \mapsto p[x] \) is bijective. This map is continuous, because if \( f = \sum_{i=0}^n f_i x^i \in R[x] \) with the \( f_i \in R \), then \( \varphi^{-1}(U) = \bigcup_{i=0}^n U_i \) where \( U = \text{Min}(R[x]) \cap D(f) \) and \( U_i = \text{Min}(R) \cap D(f_i) \). The converse of \( \varphi \) is also continuous, because it is induced by the ring extension \( R \subseteq R[x] \). Therefore \( \varphi \) is a homeomorphism. By [24, Lemma 3.4], \( \text{Min}(R[x]) \) is Zariski compact. Hence, \( \text{Min}(R) \) is Zariski compact. 

\[ \square \]

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