The Index Theorem and Universality Properties of the Low-lying Eigenvalues of Improved Staggered Quarks

E. Follana,1 A. Hart,2 and C.T.H. Davies1

(HPQCD and UKQCD collaborations)

1Department of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, U.K.
2School of Physics, University of Edinburgh, King’s Buildings, Edinburgh EH9 3JZ, U.K.

We study various improved staggered quark Dirac operators on quenched gluon backgrounds in lattice QCD generated using a Symanzik-improved gluon action. We find a clear separation of the spectrum into would-be zero modes and others. The number of would-be zero modes depends on the topological charge as expected from the Index Theorem, and their chirality expectation value is large (≈ 0.7). The remaining modes have low chirality and show clear signs of clustering into quartets and approaching the random matrix theory predictions for all topological charge sectors. We conclude that improvement of the fermionic and gauge actions moves the staggered quarks closer to the continuum limit where they respond correctly to QCD topology.

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INTRODUCTION

It has been widely held that lattice staggered quarks are insensitive to the topology of the underlying gauge fields. The low-lying spectrum of the Dirac operator has neither shown the number of chiral (near-) zero modes anticipated from the Index Theorem, nor have the eigenvalues lain on the expected universal distributions.

Here we present evidence that this is not a generic failing of staggered quarks, but simply a problem of discretisation errors, and that the use of improved staggered Dirac operators clarifies the situation and points to the correct continuum behaviour. This allows the use of improved staggered quarks in lattice QCD to study topologically sensitive states, such as those associated with the axial anomaly (principally the η′ meson). It also has a bearing on establishing the effect of taking the fourth root of the staggered determinant to represent one flavour of staggered sea quarks.

We begin by reviewing our understanding in the continuum. The eigenmodes of the (anti–Hermitian, gauge covariant) massless Dirac operator are given by

\[ \mathcal{D} f_s = i \lambda_s f_s , \quad \lambda_s \in \mathbb{R} . \]

where we use orthonormalised eigenvectors, \( f_s^\dagger f_t \equiv \delta_st \). As \( \{ \mathcal{D}, \gamma_5 \} = 0 \), the spectrum is symmetric about zero: if \( \lambda_s \neq 0 \), then \( \gamma_5 f_s \) is also an eigenvector with eigenvalue \( -i \lambda_s \), and chirality \( \chi_s \equiv f_s^\dagger \gamma_5 f_s = 0 \). The zero modes, \( \lambda_s = 0 \), can be chosen with definite chirality: \( \chi_s = \pm 1 \). In general there are \( n_\pm \) such modes, whose relative number is fixed by the (gluonic) topological charge

\[ Q = \frac{1}{32\pi^2} \int d^4x \, \epsilon_{\mu\nu\sigma\tau} \text{Tr} \, F_{\mu\nu}(x) F_{\sigma\tau}(x) \]

via the Atiyah–Singer Index Theorem \[1, 2\]:

\[ Q = m \text{Tr} \left( \frac{\gamma_5}{\mathcal{D} + m} \right) = n_+ - n_- , \]

where \( m \) is the quark mass. Based on \[3\], it has been suggested that (for a sufficiently large volume) the non-zero low-lying eigenmodes take values from a universal distribution \[4\] scaled by a QCD-specific quantity (the chiral condensate). The universality class is determined by the chiral symmetries of QCD with separate predictions for each sector of fixed topological charge. The distributions can be derived from any theory in the correct universality class, such as ensembles of random matrices \[5 \& 6\] (for a review of other theories, see \[7\]).

The four-dimensional lattice staggered massless Dirac operator is anti-Hermitian, with a purely imaginary spectrum \( i \lambda_s \). It represents \( N_f = 4 \) “tastes” of fermions that interact via highly-virtual gluon exchange at finite lattice spacing, causing taste-symmetry violations \[8 \& 9\]. These vanish in the continuum limit (as \( a^2 \)) and we then expect to recover a four-fold degeneracy in the spectrum. A remnant of continuum chiral symmetry in the staggered action gives a local and taste-non-singlet \( \gamma_5 \) operator that guarantees that the spectrum is symmetric about zero, as in the continuum. The \( \gamma_5 \) operator, \( \gamma_5^* \), relevant to the index theorem must be a taste-singlet one, however, since only this can couple to the vacuum correctly \[9\]. As \( \{ \mathcal{D}, \gamma_5^* \} \neq 0 \), there is no exact Index Theorem and all eigenmodes in principle contribute to Eqn. \[8 \& 10\].

If the gauge field is sufficiently close to the continuum limit, however, we expect to see the continuum features developing. There should be \( 2|Q| \) near-zero modes on either side of zero, whose chiralities are close to unity. The taste-singlet \( \gamma_5^* \) operator is not conserved so we expect a renormalisation to achieve a value of 1 in the continuum. The remaining modes should have chirality near zero, and come in approximately degenerate quartets on either side of zero. The values of the eigenvalue quartets should be described by the same universal distribution as continuum QCD, up to a renormalisation of the chiral condensate.
For thermalised lattices at finite lattice spacing, the fluctuations in the gauge field can lead to a breakdown of this picture. This happens for the simplest, “one-link” (naïve) staggered operator with the unimproved, Wilson gauge action at values of the gauge coupling used in present-day simulations. The breakdown is seen in two ways. Firstly, there is no clear separation (in eigenvalue or chirality) of near-zero modes of topological origin from the remainder of the spectrum [10, 11, 12, 13, 14]. In addition, the low-lying eigenvalues do not follow the predictions from universality. In fact, the eigenvalues in each sector of charge \(Q \neq 0\) all follow the distribution corresponding to the sector with \(Q = 0\) [13, 14, 17, 18, 19]. (It should be noted that the method we follow here, of grouping the eigenvalues into quartets, was not followed because this feature of the spectrum was not evident.)

This failure can be ascribed to taste-changing interactions and the lack of a good continuum chiral symmetry in the one-link staggered Dirac operator: good agreement with predictions for all topological charge sectors has been seen for Dirac operators obeying the Ginsparg-Wilson relation [21, 22, 23, 24, 25].

Over the past few years considerable advances have been made in lattice QCD phenomenology through the use of so-called “improved staggered” fermion formulations [26]. The goal of this programme is to systematically reduce the lattice artefact taste interactions, and it is reasonable to expect that this will improve the continuum-like chiral properties of the fermions. The variation of the topological susceptibility with the sea quark mass, in particular, shows that this is so: whilst it was largely insensitive to the presence of one-link staggered sea quarks [27, 28, 29, 30], the improved operator gives a variation with \(m\) that agrees well with theoretical expectations [30, 31]. It is thus pertinent to study in more detail the spectrum of the improved staggered Dirac operator, and we shall show here that this also now gives signs of converging to the expected results, in contrast to previous studies.

**RESULTS**

On a Euclidean lattice with lattice spacing \(a\), the one-link massless staggered operator is

\[
\mathcal{D}(x, y) = \frac{1}{2a} \sum_{\mu=1}^{4} \eta_\mu(x) [U_\mu(x) \delta_{x+\hat{\mu}, y} - H.c.] ,
\]

with \(\eta_1 = 1\) and \(\eta_\mu = (-1)^{x_\mu} \eta_{\mu-1}\). We study also three improved operators: the \textsc{asqtad} [32, 33, 34], the HYP [35] and the Highly Improved Staggered Quark (\textsc{hisq}) [36, 37]. These operators use “smeared” gauge fields in place of the \(U\) field above, obtained by multiplying \(U\) fields along combinations of bent paths from the start to end points of the original link. This reduces the coupling to highly virtual gluons and suppresses the taste-changing interactions. The \textsc{asqtad} action uses a “\textsc{fat7}” smearing, which includes paths made of up to seven links, and tadpole improvement. The \textsc{hyp} action uses an hypercubic blocking procedure, involving renunitarization back onto \(SU(3)\), and the \textsc{hisq} action uses two applications of the \textsc{fat7} smearing, and also includes renunitarization. The taste-changing interactions are most suppressed for the \textsc{hyp} and \textsc{hisq} cases. The \textsc{hisq} and \textsc{asqtad} cases are completely \(a^2\) improved at leading order by additional terms which correct for errors in the simple derivative above.

We calculate the eigenmodes for these Dirac operators on an ensemble of quenched (no sea quarks) \(SU(3)\) gauge configurations. The gauge action is Symanzik-improved at tree-level with tadpole improvement so that remaining discretisation errors from the gluon field are a small number times \(\alpha_s a^2\). The lattice spacing is \(0.093\) fm, representing a standard ensemble in present-day lattice simulations [37]. The majority of our results are from 1000 configs on a periodic lattice with \(16^4\) sites, which should be large enough that finite volume effects on the low-lying spectrum are negligible [24]. We have also studied a larger volume, \(24^4\). On each configuration we determine the topological charge by two standard methods that involve cooling the gauge fields [31]. The calculation of gluonic topological charge always involves ambiguities and we discard those configurations (10% of the ensemble) for which the two cooling methods do not agree, to leave a sample for which we are confident that we have a robust estimate of \(Q\). We stress at this point that cooling is used solely to determine the gluonic topological charge, and that all the Dirac eigenmode calculations are carried...
out on fully thermalised, “raw” configurations.

Fig. 1 compares the low-lying modes of the spectra of the various staggered quark formulations on a typical background with \( Q = 2 \). We show the eigenvalue \( \lambda_n \) and chirality for the upper half of the spectrum. We see that the Index Theorem is well approximated for the more improved quark formulation. Specifically, there is a clear delineation between the near-zero modes (small eigenvalues and large chirality) and the rest of the spectrum. The number of near-zero modes is \( 2|Q| \), as expected. The other eigenvalues have small chirality, and divide clearly into quartets. The taste-singlet \( \gamma_5^e \) operator used for the chirality is a point-split 4-link operator. We measure it here by inserting \( U \) fields to make it gauge-invariant. We will report elsewhere on the dependence of the chirality on the operator used.

As the near-zero modes for the more improved actions clearly separate and have well defined chirality, we may define an index \( \bar{Q} = n_+ - n_- \) on each configuration. \( \bar{Q} \) is then strongly correlated with \( Q \). If we count eigenmodes with absolute value of the chirality above 0.65 in \( n_\pm \), for example, we find that in about 90% of configurations \( \bar{Q} \) and \( Q \) are the same, satisfying the Index Theorem. Indeed, measuring \( Q \) from the Index Theorem then becomes as reliable as measuring it from gluonic methods.

In Fig. 2 we show a scatter plot of the absolute value of the chirality versus the (absolute value of) eigenvalues, for different operators. We can see the formation of a gap between modes of small and large chirality as we improve the staggered operators, as well as the overall increase in chirality of the large chirality modes.

We turn now to the non–zero modes. Subtracting the \( 2|Q| \) near-zero modes from the spectrum, we group the other eigenvalues, ordered by size, into sets of four, as indicated in the HISQ and HYP cases in Fig. 1. We call the average of these sets \( \Lambda_{1,2,\ldots} \). In Fig. 3 we plot the ratios \( \langle \Lambda_n \rangle_Q / \langle \Lambda_i \rangle_Q \) (denoted by “s/t”), where the expectation values \( \langle \cdot \rangle_Q \) are over configurations with gluonic topological charges \( \pm Q \) only. Also shown are the universal predictions. There is a clear dependence of the ratios on \( Q \), in marked contrast with previous results, which showed a precise agreement with the \( Q = 0 \) predictions for all sectors.

The results are systematically slightly lower than the theoretical predictions, especially for the ratios involving higher eigenvalues. This would be consistent with finite volume effects as in [25]. There is also a small but systematic difference between the one-link and the improved actions, with the improved results showing a better agreement with the theoretical values.

An important point to make here is that it is necessary to group the eigenvalues as explained above to get sensible results. If one ignores the near zero modes, or does not group in quartets, ratios which are close to one or very large will result. This is strong evidence that the four tastes are showing up in the spectrum, even where it is not directly evident in the spectrum itself. We are undertaking further analysis to understand why previous results with naive staggered quarks on unimproved gluon fields were in agreement with universal distributions for \( Q = 0 \) for all \( Q \) values when this procedure was not followed. It seems clear from our analysis here that results on unimproved gluon fields with a fine enough lattice spacing should show \( Q \)-dependent universal distributions when zero modes and quartets are taken into account. Preliminary results confirm this expectation.

**CONCLUSIONS AND OUTLOOK**

Improved staggered fermions are not blind to the topology, but in fact reproduce well the predictions of the Index Theorem, and the universality of ratios of eigenvalues as a function of topological sector. This means we can have confidence in using them to attack the questions arising from the axial anomaly in QCD.

We also remark that the fact that the 4-fold taste degeneracy of staggered quarks is becoming clear in the spectrum is encouraging for the programme of establishing the effect of taking the fourth root of the staggered determinant to represent one flavor of staggered sea quarks. This programme requires an analysis in the taste basis and progress towards this is now possible.

More extensive studies of finite volume and lattice spacing effects and analysis of the eigenvectors are underway and will be reported elsewhere.
FIG. 3: The ratios of expectation values of small eigenvalues (see text for notation) compared with the predictions based on a universal distribution (horizontal lines) for topological charge sectors 0, 1 and 2.

In the later stages of this study we became aware of work on a related topic [38].

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