ON DUALITY TRIADS

Summary

In this paper we introduce a duality triads’ notion. These are dual recurrences \([1, 2]\) as used in dynamical data bases theory completed by a third pertinacious relation. After duality triads being defined several representative examples of them are given. One illustrates these examples with help of Pascal-like triangles which as a matter of fact contain in the illustrative way the clue information on the triad system. \(q\)-Gaussian triads as well as Fibonomial triangle \([3-8]\) and duality triads in finite operator calculus \([9-17]\) are presented in the paper that follows this one.

1. The notion of duality triads’ system

Let us consider a kind of random walking process on \(\mathbb{N} \cup \{0\}\) set of nonnegative integers. Let \(c_{n,k}\) be the number of ways to reach level \(k\) in \(n\) steps starting from the level \(0\). Then \([1, 2]\) :

\[
c_{n+1,k} = i_{k-1}c_{n,k-1} + q_kc_{n,k} + d_{k+1}c_{n,k+1},
\]

\[c_{0,0} = 1; \ c_{0,k} = 0 \text{ for } k > 0.\]

Here numbers \(i_k, q_k, d_k; k \geq 0\) – independent of “discrete time” parameter \(n\) are defined as:

\[i_k = N\text{pos}(I, k), \ q_k = N\text{pos}(Q, k), \ d_k = N\text{pos}(D, k); \ k \geq 0,\]

where \(N\text{pos}(O, k) = \text{number of possibilities to perform the operation } O \text{ on the file of size } k\). One of the most important enumerative object of the analysis of data structures \([1, 2]\) is the sequence \(\{H_{k,l,n}\}\), where \(H_{k,l,n} = \text{number of histories of length } n \text{ with initial height } k \text{ and final height } l\). While starting with an empty file
it is natural then to consider in this special situation the above introduced array of nonnegative integer numbers i.e. – say it again –

\[ c_{n,k} = H_{0,k,n} = \text{number of ways to reach level } k \text{ in } n \text{ steps starting from the level } 0. \]

With this in mind one recognizes that for the one-step transition from time \( n \) to \( n+1 \) the array of numbers \( c_{n,k} \) must be the unique solution of the recurrence Eq. (1).

Let now \( \{\Phi_n\}_{n \geq 0} \) be a polynomial sequence (i.e. \( \deg \Phi_n(x) = n \)) determined by the following recurrence equation “dual with respect to do (1)”:

\[
x \Phi_n(x) = d_n \Phi_{n-1}(x) + q_n \Phi_n(x) + i_n \Phi_{n+1}(x),
\]

\[ \Phi_0(x) = 1, \quad \Phi_{-1}(x) = 0; \quad n \geq 0. \]

In combinatorics one calls the recurrence Eq. (2) – the dual recurrence with respect to the recurrence Eq. (1) because from Eq. (1) equivalent to Eq. (2) one derives the so-called duality relations between polynomial solutions of Eq. (2) and monomials \( x^n \). Namely, one may see that (with \( i_k, q_k, d_k; k \geq 0 \) – independent of “discrete time” parameter \( n \)):

**Lemma 1.**

\[
x_n = \sum_{k \geq 0} c_{n,k} \Phi_k(x) \quad n \geq 0.
\]

**Proof.** The identity (3) is trivial for \( n = 0 \). Let it be then true for \( n \geq 0 \). Then

\[
x^{n+1} = x \sum_{k \geq 0} C_{n,k} \phi_k(x) = \text{see (3)} = \sum_{k \geq 0} C_{n,k} \{d_k \Phi_{k-1}(x) + q_k \Phi_k(x) + i_k \Phi_{k+1}(x)\} =:
\]

\[
= \left\{ \text{due to } \sum_{k \geq 0} C_{n,k} i_k \Phi_{k+1} = \sum_{k=0} C_{n,k-1} i_k \Phi_k \{i_{-1} = 0\} \text{ and due to} \right.
\]

\[
\sum_{k \geq 0} C_{n,k} d_k \Phi_{k-1} = \sum_{k \geq 1} C_{n,k} d_k \Phi_{k-1} \{d_0 = 0\} = \sum_{k \geq 0} C_{n,k+1} d_{k+1} \Phi_k \left. \right\} =
\]

\[
= \sum_{k \geq 0} \{d_{k+1} C_{n,k+1} + q_k C_{n,k} + i_{k-1} C_{n,k-1}\} \Phi_k = \sum_{k \geq 0} C_{n+1,k} \phi_k(x).
\]

**Definition 1.** \( \{\Phi_n\}_{n \geq 0} \) are then said to be triad polynomials.

The formula (3) allows us to interpret the counting sequence \( c_{n,k} \) as the sequence of expansion coefficients of monomials \( x_n \) in the basis of polynomials \( \{\Phi_n\}_{n \geq 0} \) as in the framework of umbral or finite operator calculus. The array of numbers \( c_{n,k} \) is interpreted as special connection constants’ lower triangle infinite array [10–12] and other triangles. (The Fibonomial [3–8] triangle no triad case is treated separately at the end).
Let us note that restoring either algorithms or formulas for connection constants is one of the central problems in finite operator calculus in its classical formulation [11, 12] or in its later on extended formulation (see [13–17] and references therein).

Remarks and Information I

Note however that from (3) it does not follow that numbers $c_{n,k}$ satisfy a second order recurrence (1). Compare

$$x^n = \sum_{k=0}^{n} \binom{n}{k} k^{n-k} A_k(x), \quad n \geq 0$$

where \{A_n\}_{n \geq 0} represents the Abel binomial polynomial sequence $A_n(x) = [x(x + n)^{n-1}]$, $n \geq 0$ (the recurrence for \{A_n\}_{n \geq 0} see: p. 73 in [16]). Neither it follows that corresponding polynomial sequence (i.e. \deg\Phi_k(x) = k) exists at all. Compare with Euler numbers $\langle \binom{n}{k} \rangle$ [18] for which

$$x^n = \sum_{k} \binom{n}{k} \left( \frac{x + k}{n} \right) \neq k \quad \text{for} \quad k \neq n,$$

for numbers $i_k, q_k, d_k; k \neq 0$ – dependent of “discrete time” parameter $n$. \{A_n\}_{n \geq 0} see: p. 73 in [16]). Neither it follows that corresponding polynomial sequence (i.e. \deg\Phi_k(x) = k) exists at all for numbers $i_k, q_k, d_k; k \geq 0$ – dependent of “discrete time” parameter $n$.

Compare with Euler numbers $\langle \binom{n}{k} \rangle$ [18] for which

$$x^n = \sum_{k} \binom{n}{k} \left( \frac{x + k}{n} \right) \neq k \quad \text{for} \quad k \neq n,$$

while the corresponding recurrence for Euler numbers $\langle \binom{n}{k} \rangle$ reads:

$$\binom{n+1}{k} = (k+1) \binom{n}{k} + (n+1-k) \binom{n}{k-1},$$

$k > 0$; $\binom{0}{0} = 1, \binom{0}{k} = 0, k > 0$.

There are many polynomial sequences of distinguished importance which are not solutions of recurrence of the type (2), for example Euler or (see: p. 203 in [11]) exponential polynomials are not triad polynomial sequences.

Existence of dual triad is then a quite specifically qualified property of corresponding objects. Let then \deg\Phi_k(x) = k.

**Definition 2.** The ensemble of (1), (2), (3) shall be called the duality triad system or just duality triad – for short. Polynomials \{\Phi_n\}_{n \geq 0} are then said to be triad polynomials.
Introducing the row vector iteration system the recurrences (6) may be written in the form of discrete-time dynamical under iteration of the “one step transition matrix” $X$:

$$X = \begin{pmatrix} q_0 & i_0 & 0 & 0 & 0 & \ldots \\ d_1 & q_1 & i_1 & 0 & 0 & \ldots \\ 0 & d_2 & q_2 & i_2 & 0 & \ldots \\ 0 & 0 & d_3 & q_3 & i_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \ldots \end{pmatrix} = x \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \ldots \end{pmatrix}.$$  

This transition matrix $X$ may be interpreted as the matrix of the difference equation (2), i.e. (2) may be rewritten in a matrix form and then be looked upon as the eigenvector equation:

$$X = \begin{pmatrix} q_0 & i_0 & 0 & 0 & 0 & \ldots \\ d_1 & q_1 & i_1 & 0 & 0 & \ldots \\ 0 & d_2 & q_2 & i_2 & 0 & \ldots \\ 0 & 0 & d_3 & q_3 & i_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \ldots \end{pmatrix} = x \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \ldots \end{pmatrix}.$$  

One also easily recognizes that for the one-step transition from time (or level) $n$ to $n + 1$:

$$c_{n+1,k} = i_{k-1}c_{n,k-1} + q_kc_{n,k} + d_{k+1}c_{n,k+1}$$

we have

$$c_{0,0} = 1; \quad c_{0,k} = 0 \quad \text{for} \quad k > 0.$$  

so the recurrence (6) in its array form should read as follows:

$$\begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} & c_{0,3} & \ldots \\ c_{1,0} & c_{1,1} & c_{1,2} & c_{1,3} & \ldots \\ c_{2,0} & c_{2,1} & c_{2,2} & c_{2,3} & \ldots \\ c_{3,0} & c_{3,1} & c_{3,2} & c_{3,3} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} q_0 & i_0 & 0 & 0 & 0 & \ldots \\ d_1 & q_1 & i_1 & 0 & 0 & \ldots \\ 0 & d_2 & q_2 & i_2 & 0 & \ldots \\ 0 & 0 & d_3 & q_3 & i_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \ldots \end{pmatrix} = x \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \ldots \end{pmatrix}.$$  

Introducing the row vector $C_n$ (“the state of a system”) as

$$C_n = (c_{n,0}, c_{n,1}, c_{n,2}, c_{n,3}, \ldots),$$

the recurrences (6) may be written in the form of discrete-time dynamical under iteration system

$$C_{n+1} = C_nX; \quad n \geq 0; \quad \text{hence} \quad C_n = C_0X^n; \quad n \geq 0,$$
where \( C_0 = (1, 0, 0, 0, \ldots) \) or

\[
\begin{pmatrix}
\vec{C}_0 \\
\vec{C}_1 \\
\vec{C}_2 \\
\vec{C}_3 \\
\vdots
\end{pmatrix}
\quad X =
\begin{pmatrix}
\vec{C}_1 \\
\vec{C}_2 \\
\vec{C}_3 \\
\vec{C}_4 \\
\vdots
\end{pmatrix},
\]

under the identification

\[
(c_{nk}) = C = \begin{pmatrix}
\vec{C}_0 \\
\vec{C}_1 \\
\vec{C}_2 \\
\vec{C}_3 \\
\vdots
\end{pmatrix}, \quad \vec{C}_n = (C_{n,0}, C_{n,1}, C_{n,2}, \ldots, C_{n,k}, \ldots).
\]

Here \((c_{nk}) = C\) constitutes the lower triangle infinite array of numbers \(c_{n,k}\) interpreted in the finite operator calculus as special connection constants\([10–12]\).

Naturally the recurrence (6) or (7) or (8) may be written also in the form (with transposed \(X^T\) dual to \(X\)) as compared to (10):

\[
C_{n+1}^T = X^T C_n^T; \quad n \geq 0 \quad \text{hence} \quad C_{n+1}^T = (X^T)^n C_0^T; \quad n \geq 0.
\]

Indeed, under the notation

\[
\Phi = \begin{pmatrix}
\Phi_0 \\
\Phi_1 \\
\Phi_2 \\
\Phi_3 \\
\vdots
\end{pmatrix}
\]

the recurrence (2) takes the form of an eigenvector and eigenvalue equation

\[
x \Phi = X \Phi.
\]

As for the iteration of the transition matrix \(X\) the following interpretation is pertinent [1]: the \(kl\)-th entry of \(X_n = \text{number of ways of going from the level } k \rightarrow \text{to} \rightarrow l \text{ in } n \text{ steps.}\)

2. First Examples of triads

Here come some classical and then new examples of triads.

2.1. Pascal triad

This is at the same time an example of one of the simplest Appell polynomials\([19, 20, 13, 19, 22]\) sequence of triad polynomials \(\Phi_n(x)_{n \geq 0}; \quad \Phi_n(x) = (x-1)^n.\) Namely:
the choice $i_k = 1, q_k = d_k = 0; k \geq 0$ leads to $c_{n,k} = \binom{n}{k}$ and consequently to the Pascal triangle and

$$c_{n+1,k} = i_{k-1}c_{n,k-1} + q_k c_{n,k} + d_{k+1}c_{n,k+1},$$

where $c_{0,0} = 1; c_{0,k} = 0$ for $k > 0$.

The Pascal triangle might be obtained while considering subsequent powers of

$$X = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \ldots \\
0 & 1 & 1 & 0 & 0 & \ldots \\
0 & 0 & 1 & 1 & 0 & \ldots \\
0 & 0 & 0 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

and the calculating $C_n = C_0 X^n; n \geq 0$. An easy generalization is obtained for connection constants $c_{n,k} = \binom{n}{k} s^{n-k}$, where $s$ is a parameter.

### 2.2. Stirling triad

The choice $i_k = 1, q_k = k; k \geq 0, d_k = 0; k > 0$ leads to $c_{n,k} = \binom{n}{k}$ i.e. Stirling numbers of the second type for which we have (3):

$$x_n = \sum_{k \geq 0} c_{n,k} \Phi_k(x), \quad n \geq 0.$$

where triad polynomials are $\Phi_k(x) = x^k = x(x - 1)(x - 2)\ldots(x - k + 1)$ and consequently, following (2) and (1),

$$x \Phi_n(x) = d_n \Phi_{n-1}(x) + q_n \Phi_n(x) + i_n \Phi_{n+1}(x),$$

$$\Phi_0(x) = 1, \quad \Phi_{-1}(x) = 0; \quad n \geq 0,$$

$$c_{n+1,k} = i_{k-1}c_{n,k-1} + q_k c_{n,k} + d_{k+1}c_{n,k+1},$$

where $c_{0,0} = 1; c_{0,k} = 0$ for $k > 0$. 

$$c_{0,0} = 1; c_{0,k} = 0$$
The above can be illustrated by the corresponding well known II-type – Stirling triangle

\[
\begin{array}{cccccc}
0 & 1 & & & & \\
0 & 1 & 1 & & & \\
0 & 1 & 3 & 1 & & \\
0 & 1 & 7 & 6 & 1 & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

which supplies in a compact form the full defining information on the triad. The II-Stirling triangle might be obtained while considering subsequent powers of

\[
X = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 2 & 1 & 0 & \cdots \\
0 & 0 & 0 & 3 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

and then calculating its subsequent rows \( C_n = C_0 X^n; \ n \geq 0 \).

Persistent roots polynomials \( \Phi_k(x) = x^k, k \geq 0 \), constitute a classical example of polynomials of binomial type which are basic polynomials for the forward difference calculus delta operator \( \Delta \equiv E - id \), where

\[
E^a \equiv \exp \left( a \frac{d}{dx} \right); \quad a \in \mathbb{R}
\]

denotes a shift operator, i.e.

\[
(E^a f)(x) = f(x + a); \quad \text{for } f : \mathbb{R} \to \mathbb{R}.
\]

Remarks and Information II

Recall: \((c_{nk}) = C\) constitutes the lower triangle infinite array of numbers \(c_{nk}\) interpreted in finite operator calculus as special connection constants [10–12], [23–25]. In the general case of two arbitrary \(\{q_n\}_{n \geq 0}\) and \(\{p_n\}_{n \geq 0}\) polynomial sequences i.e. \(\deg p_n = \deg q_n = n, n \geq 0\); connection constants \((L_{n,k})\) are defined via \(p_n(x) = \sum_{0 \leq k \leq n} L_{n,k} q_k(x), n \geq 0\) – using the notation of [23, 24]. In [23] one derives explicit formulas for connection constants \((L_{n,k})\) in terms of roots of monic polynomial sequences \(\{q_n\}_{n \geq 0}\) and \(\{p_n\}_{n \geq 0}\). In the special case \(\{q_n\}_{n \geq 0}\) of being any monic polynomial sequence with persistent roots [26] one states (see: Propositions 10 and 9 in [23]) that \(i_k = 1, q_k \neq 0; k \geq 0, d_k = 0; k > 0\) (as in Examples 1–3). Namely:

\[
L_{n+1,k} = L_{n,k-1} + (r_{k+1} - s_{n+1}) L_{n,k}, \
(11)
\]

\[
L_{0,0} = 1, \quad L_{0,-1} = 0, \quad n, k \geq 0,
\]
where for
\[ p_n(x) = \sum_{k \geq 0} L_{n,k} q_k(x), \quad n \geq 0, \]

\([r] \equiv \{r_0, r_1, r_2, \ldots, r_k, \ldots\}\) is the root sequence determining \(\{q_n\}_{n \geq 0}\) while \([s] \equiv \{s_0, s_1, s_2, \ldots, s_k, \ldots\}\) is the root sequence determining \(\{p_n\}_{n \geq 0}\). The authors of [23] call such connection constants \((L_{n,k})\) – the generalized Lah numbers because with the choice \([r] \equiv \{0, 1, 2, \ldots, k, \ldots\}\) and \([s] \equiv \{0, -1, 2, \ldots, -k, \ldots\}\) the Lah numbers and connection constants \(L_{n,k}\) coincide as can be seen from
\[ X^= = \sum_{1 \leq k \leq n} \binom{n-1}{k-1} \frac{n!}{k!} p_k, \quad n, k \geq 0, \]

where – recall –
\[ e_{nk} = \binom{n-1}{n-k} \frac{n!}{k!} \]
are Ivo Lah numbers.

As for the treatment of the Fibonacci and Lucas numbers as cumulative connection constants \(K_n = \sum_{0 \leq k \leq n} L_{n,k}\) (sums of \(n\)-th row numbers in a corresponding triangle), see [24].

**Remarks and Information III**

The requirement that persistent root polynomial sequences \(\{q_n\}_{n \geq 0}\) and \(\{p_n\}_{n \geq 0}\) should be monic, leads to (11). If not monic – as it is for example in the case with Newton-Gregory polynomials (see: (1.6) in [27]) \(\Phi_k(x) = \binom{x}{k}, k \geq 0\), one gets different recurrences: here – for Newton-Gregory polynomials the triad is given by
\[ x\Phi_k(x) = (k + 1)\Phi_{k+1}(x) + k\Phi_k(x), \]
(12)
\[ \Phi_0(x) = 1, \quad \Phi_{-1}(x) = 0; \quad n \geq 0, \]
and, consequently,
\[ L_{n+1,k} = kL_{n,k-1} + kL_{n,k}, \]
(13)
\[ L_{0,0} = 1, \quad L_{0,-1} = 0, \quad n, k \geq 0, \]
\[ x^n = \sum_{k \geq 0} k! \binom{n}{k} \Phi_k(x), \]
(14)
so the Newton-Gregory triangle is then of the form

\[
\begin{array}{cccccc}
& & & & 1 \\
& & 0 & 1 & & \\
& 0 & 1 & 2 & & \\
0 & 1 & 6 & 6 & & \\
0 & 1 & 14 & 36 & 24 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

This triangle can be obtained while considering subsequent powers of

\[
X = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 2 & 0 & 0 & \cdots \\
0 & 0 & 2 & 3 & 0 & \cdots \\
0 & 0 & 0 & 3 & 4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

and then calculating its subsequent rows \( C_n = C_0 X^n; \ n \geq 0. \)

3. **Stirling signed triad**

The choice \( i_k = 1, \ q_k = -k; \ k \geq 0, \ d_k = 0, k > 0 \) gives

\[
c_{n,k} = \binom{n}{k} (-1)^{n-k}, \quad k \geq 0,
\]

where \( \binom{n}{k} \) are Stirling numbers of the second type. Thus we have:

\[
x^n = \sum_{k \geq 0} \binom{n}{k} (-1)^{n-k} \Phi_k(x),
\]

where

\[
\Phi_k(x) = x^k = x(x+1)(x+2)\ldots(x+k-1)
\]

and, consequently,

\[
x \Phi_k(x) = \Phi_{k+1}(x) - k \Phi_k(x),
\]

\[
\Phi_0(x) = 1, \quad \Phi_{-1}(x) = 0; \quad n \geq 0,
\]

\[
c_{n+1,k} = c_{n,k-1} - kc_{n,k},
\]

\[
c_{0,0} = 1, \quad c_{0,k} = 0 \quad \text{for} \quad k > 0.
\]
The above might be illustrated by the corresponding II-Stirling signed triangle

\[
\begin{array}{cccccc}
 & & & 1 & & \\
 & & 0 & -1 & 1 & \\
0 & 1 & -3 & 1 & \\
0 & 1 & -15 & 7 & -6 & 1 \\
0 & 1 & -\cdots & -10 & 1 & \\
\end{array}
\]

Persistent roots polynomials \( \Phi_k(x) = x^k \), \( k \geq 0 \) are strictly related to Stirling numbers \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) of the first kind triangle (see the correspondent statement and formula 6.33 in [18]). Let then note that the persistent roots polynomial sequence \( \{x^k\}_{k \geq 0} \) constitutes simultaneously another classical example of polynomials of binomial type [11–17] which are now basic polynomials for backward difference calculus delta operator \( \nabla \):

\[
\nabla \equiv \text{id} - E^{-1}; \quad (\nabla f)(x) = f(x) - f(x - 1) \quad \text{for} \quad f : \mathbb{R} \to \mathbb{R}.
\]

4. Hermite triad

With the following choice of numbers:

\[
q_k = 0; \quad k \geq 0, \quad d_k = k; \quad k \geq 1, \quad i_k = 1; \quad k \geq 0,
\]

the recurrence for monic polynomials takes the form

\[
xH_k(x) = kH_{k-1}(x) + H_{k+1}(x), \quad k \geq 0,
\]

(18)

\[
H_0 = 1, \quad H_{-1} = 0.
\]

which is the recurrent definition of Hermite monic polynomials (monic: i.e. the coefficient of \( x^n \) is one; \( \text{deg} \Phi_n = n \)). This is the example of nontrivial, important Appell polynomial sequence [19, 20, 13, 21, 22]. Naturally Hermite classical orthogonal polynomials are solutions of the recurrence (18) which is dual to the recurrence (19):

\[
c_{n+1,k} = c_{n,k-1} + (k + 1)c_{n,k+1},
\]

(19)

\[
c_{0,0} = 1; \quad c_{0,k} = 0 \quad \text{for} \quad k > 0.
\]

Hence we also have

\[
x^n = \sum_{k \geq 0} c_{n,k}H_k(x) \quad n \geq 0.
\]

(20)
The above can be illustrated by the corresponding Hermite triangle

\[
\begin{array}{ccccccc}
& & & & & & 1 \\
& & & & 0 & 1 & \\
& & 1 & 0 & 1 & \\
& 0 & 3 & 0 & 1 & \\
3 & 0 & 6 & 0 & 1 & \\
0 & 15 & 0 & 10 & 0 & 1 & \\
\end{array}
\]

which supplies in a compact form the full defining information on the triad.

This Hermite triangle might be obtained while considering subsequent powers of

\[
X = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 2 & 0 & 1 & 0 & \cdots \\
0 & 0 & 3 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

and then calculating its subsequent rows \(C_n = C_0 X^n; \ n \geq 0\).

Let us take the opportunity to note that Hermite polynomials are known to be the so called “associated polynomials” to priority queue organization of data bases (see Propositions 4 and 6 in [28]).

5. Laguerre triad and Lah triangle

It well known that

\[
x^n = \sum_{1 \leq k \leq n} (n - 1k - 1) \frac{n!}{k!} (-1)^k L_k(x), \quad n, k \geq 0,
\]

where \(\{L_k(x)\}_{k \geq 0}\) represents self-inverse [11] Laguerre binomial polynomial sequence which at the same time is the unique basic sequence of the delta operator \(L = D/(D - 1)\) [11, 12] (see Chapter 4, Example 5 in [17] for extensions). It is also well known that the following choice of numbers:

\[
q_k = 2k; \ k \geq 0, \quad d_k = -k(k - 1); \ k \geq 1, \quad i_k = -1; \ k \geq 0,
\]

uniquely determines the following recurrence for basic, binomial Laguerre polynomials

\[
xL_k(x) = -L_{k+1}(x) + 2kL_k(x) - k(k - 1)L_{k-1}(x), \quad k \geq 0,
\]

\[
L_0(x) = 1, \quad L_{-1}(x) = 0.
\]

Connection constants from (21) are expressed by Lah numbers [31] :

\[
L_{n,k} = (-1)^k C_{n,k} = \binom{n - 1}{n - k} \frac{n!}{k!} = \binom{n}{k} \frac{(n - 1)!}{(k - 1)!}.
\]
Recall: $L_{n,k}x^k$ number of linear deployments of $n$ distinguishable objects in exactly $k$ cells out of $x \in \mathbb{N}$ distinguishable cells are available. We infer from (22) that $c_{n,k}$ satisfy accordingly

$$c_{n+1,k} = -c_{n,k-1} + 2k c_{n,k} - k(k+1)c_{n,k+1},$$

(23)

$$c_{0,0} = 1; \quad c_{0,k} = 0 \quad \text{for} \quad k > 0.$$ 

Hence for Ivo Lah numbers we have:

$$L_{n+1,k} = L_{n,k-1} + 2k L_{n,k} + k(k+1)L_{n,k+1},$$

(24)

$$L_{0,0} = 1; \quad L_{0,k} = 0 \quad \text{for} \quad k > 0.$$ 

The transition matrix is then of the form

$$X = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & \cdots \\
0 & 2 & -1 & 0 & 0 & \cdots \\
0 & 2 & 4 & -1 & 0 & \cdots \\
0 & 0 & -6 & 6 & -1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},$$

and the Laguerre triangle is obtained via calculating its subsequent rows $C_n = C_0 X^n; \ n \geq 0$. Thus we get: Laguerre triangle

$$\begin{array}{ccccccc}
1 & 0 & -1 & \\
0 & -2 & 1 & \\
0 & 8 & 32 & -12 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

and Lah triangle

$$\begin{array}{ccccccc}
1 & 0 & 1 & \\
0 & 2 & 1 & \\
0 & 4 & 6 & 1 & \\
0 & 10 & 34 & 12 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

with recurrence for the corresponding triad polynomials being of the form

$$x\Lambda_k(x) = \Lambda_{k+1}(x) + 2\Lambda_k(x) + k(k-1)\Lambda_{k-1}(x), \quad k \geq 0,$$

(25)

$$\Lambda_0(x) = 1, \quad \Lambda_{-1}(x) = 0.$$
6. Tchebychev triad

With the following choice of numbers [1, 29]:

\[ q_k = 0; \ k \geq 0, \quad d_k = 1; \ k \geq 1, \quad i_k = 1; \ k \geq 0, \]

the recurrence for corresponding triad polynomials takes the form

\[ x\Omega_k(x) = \Omega_{k-1}(x) + \Omega_{k+1}(x), \quad k \geq 0, \]

(26)

\[ \Omega_0 = 1, \quad \Omega_{-1} = 0. \]

which is the recurrent definition of \( \Omega_k = U_k(x/2) \) polynomials where \( U_k(x) \) represent classical orthogonal Tchebychev polynomials of the second kind. The Tchebychev triangle might be obtained while considering subsequent powers of

\[
X = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 1 & 0 & 1 & 0 & \cdots \\
0 & 0 & 1 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\]

and then calculating its rows \( C_n = C_0 X^n; \ n \geq 0; \) thus getting the Tchebychev triangle

\[
\begin{array}{ccccccc}
1 \\
& 0 & 1 \\
& & 1 & 0 & 1 \\
& & & 2 & 0 & 3 & 0 & 1 \\
& & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{array}
\]

The connection constants \( c_{n,k} \) from

(27)

\[ x^n = \sum_{k \geq 0} c_{n,k} \Omega_k(x), \quad n \geq 0, \]

are then determined by the recurrence dual to (26):

\[ c_{n+1,k} = c_{n,k-1} + c_{n,k+1}, \]

(28)

\[ c_{0,0} = 1; \quad c_{0,k} = 0 \text{ for } k > 0. \]

References

[1] P. Feinsilver and R. Schott, *Algebraic Structures and Operator Calculus*, Vol. II "Special Functions and Computer Science", Kluwer Academic Publishers, Dordrecht 1993.
[2] I. Jaroszewski and A. K. Kwaśniewski, *On the principal recurrence of data structure organization and orthogonal polynomials*, Integral Transforms and Special Functions 11 no. 1 (2001), 1–12.

[3] V. E. Hoggat Jr., *Fibonacci numbers and generalized binomial coefficients*, The Fibonacci Quarterly 5 no. 4 (1967), 383–400.

[4] H. W. Gould, *The bracket function and Fontené-Ward generalized binomial coefficients with applications to fibonacci coefficients*, The Fibonacci Quarterly 7 (1969), 23–40.

[5] V. E. Hoggat Jr., M. Bicknell and E. L. King, *Fibonacci and Lucas triangles*, The Fibonacci Quarterly 10 no.5 (1972), 555–560.

[6] H. Hosoya, *Fibonacci triangle*, The Fibonacci Quarterly 14 no.2 (1976), 173–179.

[7] D. L. Wells, *The Fibonacci and Lucas triangles modulo 2*, The Fibonacci Quarterly 32 no.2 (1994), 111–123.

[8] B. Wilson, *The Fibonacci triangle modulo p*, The Fibonacci Quarterly 36 no.3 (1998), 195–203.

[9] I. M. Sheffer, *Some properties of polynomial sets of type zero*, Duke Math. J. 5 (1939), 590–622.

[10] J. F. Steffenson, *The poweroid, an extension of the mathematical notion of power*, Acta Math. 73 (1941), 333–366.

[11] R. Mullin and G.-C. Rota, *On the Foundations of Combinatorial Theory, III . Theory of Binomial Enumeration*, in: Graph Theory and Its Applications, Ed. B. Harris, Academic Press, New York 1970, pp. 167–213.

[12] G.-C. Rota, *Finite Operator Calculus*, Academic Press, New York 1975.

[13] O. V. Viskov, *Operator characterization of generalized Appell polynomials*, Soviet Math. Dokl. 16 (1975), 1521–1524.

[14] O. V. Viskov, *On the basis in the space of polynomials*, Dokl. Akad. Nauk SSSR 239 no. 1 (1978): Soviet Math. Dokl. 19 (1978), 250–253.

[15] G. Markowsky, *Differential operators and theory of binomial enumeration*, J. Math. Anal. Appl. 63 (1978), 145–155.

[16] S. M. Roman, *The Umbral Calculus*, Academic Press, New York 1984.

[17] A. K. Kwaśniewski, *Towards y-extension of Finite Operator Calculus of Rota*, Rep. Math. Phys. 48 no. 3 (2001), 305–342.

[18] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics. A Foundation of Computer Science*, Addison-Wesley Publishing Company, 1994.

[19] P. Appell, *Une classe des polynômes*, Ann. Sci. École Norm. Sup. 9 no. 2 (1880), 119–144.

[20] W. C. Brenke, *On generating functions of polynomials systems*, Am. Math. Monthly 52 (1945), 297–301.
On duality triads

[21] O. V. Viskov, *A noncommutative identity for Appell Polynomials*, Matem. Zametki 64 (1998), 307–311.

[22] P. Feinsilver and R. Schott, *Appell systems on Lie groups*, Journal of Theoretical Probability 5 no. 2 (1992), 251–281.

[23] E. Damiani, O. D’Antona, and G. Naldi, *On the connection constants*, Studies in Appl. Math. 85 no. 4 (1991), 289–302.

[24] L. Colucci, O. D’Antona, and C. Mereghetti, *Fibonacci and Lucas numbers as cumulative connection constants*, The Fibonacci Quarterly 38 no. 2 (2000), 157–164.

[25] S. C. Milne, *Inversion properties of triangular arrays of numbers*, Analysis 1 (1981), 1–7.

[26] A. Di Bucchianico and D. Loeb, *Sequences of binomial type with persistent roots*, J. Math. Anal. Appl. 199 (1996), 39–58.

[27] H. W. Gould, *Stirling number representation problems*, Proc. Amer. Math. Soc. 11 (1960), 447–451.

[28] P. Flajolet, J. Françon, and J. Vuillemin, *Sequence of operations analysis for dynamic data structures*, Journal of Algorithms 1 (1980), 111–141.

[29] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.

[30] O. D’Antona, *Introduzione alla matematica discreta*, APOGEO, Milano 1999.

Institute of Computer Science
Bialystok University
Sosnowa 64, PL-15-887 Bialystok
Poland

Presented by Julian Lawrynowicz at the Session of the Mathematical-Physical Commission of the Lodz Society of Sciences and Arts on November 19, 2003

**O TRIADACH DUALNYCH**

**S t r e s z c z e n i e**

W pracy – z inspiracji opisu dynamicznych baz danych w modelach typu “random walk” [1, 2] – wprowadza się pojęcie tzw. triad dualnych. Są to układy dwu rekurencji dualnych dopelnione trzecią relacją o zazwyczaj ważnej interpretacji kombinatorycznej. Podano szereg przykładów takich triad ilustrując informację o nich odpowiednimi “trój-kątami” na podobieństwo trójkąta Pascala. Triady traktowane są jednocześnie jako układy dynamiczne z czasem dyskretnym.