Quasilinear and Hessian Lane-Emden type systems with measure data

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Abstract
We study nonlinear systems of the form \(-\Delta_p u = v^{q_1} + \mu, -\Delta_p v = u^{q_2} + \eta \) and \( F_k[-u] = v^{s_1} + \mu, F_k[-v] = u^{s_2} + \eta \) in a bounded domain \( \Omega \) or in \( \mathbb{R}^N \) where \( \mu \) and \( \eta \) are nonnegative Radon measures, \( \Delta_p \) and \( F_k \) are respectively the \( p \)-Laplacian and the \( k \)-Hessian operators and \( q_1, q_2, s_1 \) and \( s_2 \) positive numbers. We give necessary and sufficient conditions for existence expressed in terms of Riesz or Bessel capacities.

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Key words: \( p \)-Laplacian, \( k \)-Hessian, Bessel and Riesz capacities, measures, maximal functions.

1 Introduction and Main results

Let \( \Omega \subset \mathbb{R}^N \) be either a bounded domain or the whole \( \mathbb{R}^N \), \( p > 1 \) and \( k \in \{1, 2, ..., N\} \). We denote by

\[
\Delta_p u := \text{div} (|\nabla u|^{p-2} \nabla u)
\]

the \( p \)-Laplace operator and by

\[
F_k[u] = \sum_{1 \leq j_1 < j_2 < ... < j_k \leq N} \lambda_{j_1} \lambda_{j_2} ... \lambda_{j_k}
\]

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the k-Hessian operator where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of the Hessian matrix $D^2 u$. In the work [20], Phuc and Verbitsky obtained necessary and sufficient conditions for existence of nonnegative solutions to the following equations

$$-\Delta_p u = u^q + \mu \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega, \quad (1.1)$$

and

$$F_k[-u] = u^q + \mu \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega. \quad (1.2)$$

Their conditions involve the continuity of the measures with respect to Bessel or Riesz capacities and Wolff potentials estimates. For example, if $\Omega$ is bounded and $\mu$ has compact support in $\Omega$, they proved that it is equivalent to solve (1.1), or to have

$$\mu(E) \leq c_1 \text{Cap}_{\mathbb{R}^N, \frac{q}{q-1}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.3)$$

for some constant $c_1 > 0$ where $\text{Cap}_{\mathbb{R}^N, \alpha}$ is a Bessel capacity, or to have

$$\int_B (W_{R, \alpha} (B(x)) - \mu(B)) \, dx \leq c_2 \mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp } \mu \neq \emptyset, \quad (1.4)$$

for some constant $c_2 > 0$, where $R = 2 \text{ diam}(\Omega)$ and $W_{R, \alpha} (B(x))$ denotes the R-truncated Wolff potential of the measure $\mu_B = \chi_B \mu$. Concerning the k-Hessian operator in a bounded $(k-1)$-convex domain $\Omega$, they proved that if $\mu$ has compact support, the problem (1.2) with $q > k$ admits a nonnegative solution if and only if

$$\mu(E) \leq c_3 \text{Cap}_{\mathbb{R}^N, \frac{q}{q-k+1}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.5)$$

for some $c_3$. In turn this condition is equivalent to

$$\int_B (W_{R, \alpha} (B(x)) - \mu(B)) \, dx \leq c_4 \mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp } \mu \neq \emptyset, \quad (1.6)$$

for some $c_4 > 0$. The results concerning the linear case $p = 2$ and $k = 1$, can be found in [2, 3, 27].

The natural counterpart of equation (1.1) and (1.2) for systems:

$$-\Delta_p u = u^q + \mu \quad \text{in } \mathbb{R}^N$$

$$-\Delta_p v = u^q + \eta \quad \text{in } \mathbb{R}^N$$

$$u = v = 0 \quad \text{on } \partial \Omega, \quad (1.7)$$

and

$$F_k[-u] = u^q + \mu \quad \text{in } \mathbb{R}^N$$

$$F_k[-v] = u^q + \eta \quad \text{in } \mathbb{R}^N$$

$$u = v = 0 \quad \text{on } \partial \Omega, \quad (1.8)$$

where $q_1, q_2 > p - 1, s_1, s_2 > k$ and $\mu, \eta$ are Radon measures. If $\Omega = \mathbb{R}^N$, we consider the same equations, except that the boundary conditions are replaced by $\inf_{\mathbb{R}^N} u = \inf_{\mathbb{R}^N} v = 0$ and our statements involve the Riesz potentials and their associated capacities $\text{Cap}_{\mathbb{R}^N, \alpha}$. Our main results are the following.

**Theorem A** Let $1 < p < N$, $q_1, q_2 > 0$ and $q_2 q_1 > (p - 1)^2$. Let $\mu, \eta$ be nonnegative Radon measures in $\mathbb{R}^N$. If the following system

$$-\Delta_p u = u^{q_1} + \mu \quad \text{in } \mathbb{R}^N$$

$$-\Delta_p v = u^{q_2} + \eta \quad \text{in } \mathbb{R}^N, \quad (1.9)$$

The natural counterpart of equation (1.1) and (1.2) for systems:
Corollary B Assume that

\[ \frac{p(q_1 q_2 + (p-1) \max\{q_1, q_2\})}{q_1 q_2 - (p-1)^2} \geq N. \]

Any nonnegative p-superharmonic solution \((u, v)\) of inequalities

\[
\begin{align*}
-\Delta_p u &\geq v^{q_1} & \text{in } \mathbb{R}^N \\
-\Delta_p v &\geq u^{q_2} & \text{in } \mathbb{R}^N, 
\end{align*}
\]

is trivial, i.e. \(u = v = 0\).

Classical Liouville results for one equation or inequality, are proved in [4], [5], [11], [22].

When \(\Omega\) is bounded domain, we have a similar result in which we denote by \(d\) the distance function to the boundary \(x \mapsto d(x) = \text{dist}(x, \partial \Omega)\).

Theorem C Let \(1 < p < N\), \(q_1, q_2 > 0\) and \(q_2 q_1 > (p-1)^2\). Let \(\Omega \subset \mathbb{R}^N\) be a bounded domain and \(\mu, \eta\) nonnegative Radon measures in \(\Omega\). If the following problem

\[
\begin{align*}
-\Delta_p u &= v^{q_1} + \mu & \text{in } \Omega \\
-\Delta_p v &= u^{q_2} + \eta & \text{in } \Omega \\
u &= v = 0 & \text{on } \partial \Omega, 
\end{align*}
\]

admits a nonnegative renormalized solution \((u, v)\), then then for any compact set \(K \subset \Omega\), there exists a positive constant \(c_{10}\) depending on \(N, p, q_1, q_2\) and \(\text{dist}(K, \partial \Omega)\) such that

\[ \eta(E) + \int_E \left( W_{1,p}(x) \right)^{q_2} dx \leq c_{10} \text{Cap}_{\frac{p(q_1 q_2 + (p-1) \max\{q_1, q_2\})}{q_1 q_2 - (p-1)^2}}(E) \quad \text{for all Borel sets } E \subset K. \]
Conversely, let $\mu$ and $\eta$ be bounded with the property that there exists $c_{11} > 0$ depending on $N, p, q_1, q_2$ and $R = 2\text{diam} \ (\Omega)$ such that if $0 < q_1 < \frac{N(p-1)}{N-p}$ and
\[
\eta(K) + \int_K (W_{1,p}^{2R}[\mu])^{q_2} \, dx \leq c_{11} \text{Cap}_{G^{(q_1+q_2-1)}}(K),
\] (1.16)
for all compact set $K \subseteq \Omega$, then (1.13) admits a nonnegative renormalized solution $(u, v)$ satisfying
\[
v \leq c_{13} W_{1,p}^R[\omega], \quad u \leq c_{14} [W_{1,p}^R(\omega)]^{q_1} + c_{12} W_{1,p}^R[\mu] \quad \text{in } \Omega,
\] (1.17)
in $\Omega$, where $d\omega = (W_{1,p}^R[\mu])^{q_2} \, dx + d\eta$.

It is known that
\[
\text{Cap}_{G^{q_2}}(\{x_0\}) > 0
\]
if and only if $\alpha \beta > N$. Thus, as an application in a partially subcritical case we have,

**Corollary D** Let the assumptions on $p, q_1, q_2, \Omega$ and $R$ of Theorem C be satisfied, $x_0 \in \Omega$, $a > 0$ and $\mu$ be a nonnegative Radon measures in $\Omega$. If the following problem
\[
-\Delta_p u = v^\alpha + \mu \quad \text{in } \Omega
\]
\[-\Delta_p v = u^\beta + a\delta_{x_0} \quad \text{in } \Omega
\]
\[u = v = 0 \quad \text{on } \partial \Omega,
\] (1.18)
admits a nonnegative renormalized solution $(u, v)$, then there exist positive constants $c_i = c_{15}(N, p, q_1, q_2, d(x_0))$ and, for any compact subset $K$ of $\Omega$, $c_i = c_{16}(N, p, q_1, q_2, \text{dist } (K, \partial \Omega))$, such that
\[
\begin{align*}
(i) \quad & N < \frac{pq_2(q_1 + p - 1)}{q_1q_2 - (p - 1)^2}, \\
(ii) \quad & a \leq c_{15}, \\
(iii) \quad & \int_K (W_{1,p}^{2R}[\mu])^{q_2} \, dx \leq c_{15}.
\end{align*}
\] (1.19)

Conversely, assuming that $\mu$ is bounded, there exist positive constants $c_{17} = c_{17}(N, p, q_1, q_2, d(x_0))$, $c_{18} = c_{18}(N, p, q_1, q_2)$ such that if $0 < q_1 < \frac{N(p-1)}{N-p}$ and (1.19) holds with $c_{15}$ and $c_{16}$ replaced respectively by $c_{17}$ and $c_{18}$, then there exists a nonnegative renormalized solution $(u, v)$ of (1.18) satisfying
\[
v \leq c_{21} W_{1,p}^R[\omega], \quad u \leq c_{22} W_{1,p}^R[(W_{1,p}^R[\omega])^{q_1}] + c_{20} W_{1,p}^R[\mu] \quad \text{in } \Omega,
\] (1.20)
in $\Omega$, where
\[
W_{1,p}[\omega] = W_{1,p}^R[(W_{1,p}^R[\omega])^{q_2}] + u^{\frac{p-1}{p}} \left( \frac{N-p}{r} \right)^{-\frac{N-p}{r-p}} - R^{-\frac{N-p}{r-p}}.
\]

Concerning the $k$-Hessian operator we recall some notions introduced by Trudinger and Wang \cite{23, 21, 25}, and we follow their notations. For $k = 1, \ldots, N$ and $u \in C^2(\Omega)$ the $k$-Hessian operator $F_k$ is defined by
\[
F_k[u] = S_k(\lambda(D^2u)),
\]
where $\lambda(D^2u) = \lambda(\lambda_1, \lambda_2, \ldots, \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives $D^2u$ and $S_k$ is the $k$-th elementary symmetric polynomial that is
\[
S_k(\lambda) = \sum_{1 \leq i_1 < \ldots < i_k \leq N} \lambda_{i_1} \ldots \lambda_{i_k}.
\]
Since $D^2 u$ is symmetric, it is clear that

$$F_k[u] = [D^2 u]_k,$$

where we denote by $[A]_k$ the sum of the $k$-th principal minors of a matrix $A = (a_{ij})$. In order that there exists a smooth $k$-admissible function which vanishes on $\partial \Omega$, the boundary $\partial \Omega$ must satisfy a uniformly ($k$-1)-convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial \Omega,$$

for some positive constant $c_0$, where $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{n-1})$ denote the principal curvatures of $\partial \Omega$ with respect to its inner normal. We also denote by $\Phi^k(\Omega)$ the class of upper-semicontinuous functions $\Omega \to [-\infty, \infty)$ which are $k$-convex, or subharmonic in the Perron sense (see Definition 5.1). In this paper we prove the following theorem (in which expression $E[q]$ is the largest integer less or equal to $q$)

**Theorem E** Let $2k < N, s_1, s_2 > 0, s_1 s_2 > k^2$. Let $\Omega$ be a bounded uniformly ($k$-1)-convex domain in $\mathbb{R}^N$ with diameter $R$. Let $\mu = \mu_1 + f$ and $\eta = \eta_1 + g$ be nonnegative Radon measures where $\mu_1, \eta_1$ has compact support in $\Omega$ and $f, g \in L^1(\Omega)$ for some $l > \frac{N}{2k}$. If the following problem

$$F_k[-u] = u^{s_1} + \mu \quad \text{in } \Omega$$

$$F_k[-v] = u^{s_2} + \eta \quad \text{in } \Omega$$

$$u = v = 0 \quad \text{on } \partial \Omega,$$

admits a nonnegative solutions $(u, v)$, continuous near $\partial \Omega$, with $-u$ and $-v$ elements of $\Phi^k(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant $c_{23}$ depending on $N, k, s_1, s_2$ and $\text{dist}(K, \partial \Omega)$ such that there holds

$$\eta(E) + \int_E \left( W_{\frac{1}{s_1+1}} \right)^{s_2} dx \leq c_{23} \text{Cap}_G \left( 2^{k+1} k \frac{1}{s_1 s_2 - 2} \right)(E) \quad \forall E \subset K, E \text{ Borel}.$$  \hspace{1cm} (1.21)

Conversely, if $\mu$ and $\eta$ are bounded, there exist a positive constant $c_{23}$ depending on $N, k, s_1, s_2$ and $\text{diam}(\Omega)$ such that, if $k \leq s_1 < \frac{N k}{N - 2k}$, and

$$\eta(K) + \int_K \left( W_{\frac{2R}{s_1+1}} \right)^{s_2} dx \leq c_{23} \text{Cap}_G \left( 2^{k+1} k \frac{1}{s_1 s_2 - 2} \right)(K)$$  \hspace{1cm} (1.22)

for all Borel set $K \subset \Omega$, then (1.21) admits a nonnegative solution $(u, v)$, continuous near $\partial \Omega$, with $-u, -v \in \Phi^k(\Omega)$ satisfying

$$v \leq c_{28} W_{\frac{2R}{s_1+1}}[\omega], \quad u \leq c_{29} W_{\frac{2R}{s_1+1}}[\left( W_{\frac{2R}{s_1+1}}[\omega] \right) \frac{1}{s_1} + c_{27} W_{\frac{2R}{s_1+1}}[\mu]] \quad \text{in } \Omega$$  \hspace{1cm} (1.23)

in $\Omega$ for some constants $c_j$ ($j = 27, 28, 29$) depending on $N, k, s_1, s_2,$ and $\text{diam}(\Omega)$. If $\Omega$ is replaced by the whole space we prove,

**Theorem F** Let $2k < N, s_1, s_2 > 0, s_1 s_2 > k^2$. Let $\mu, \eta$ be a nonnegative Radon measures in $\mathbb{R}^N$. If the following problem

$$F_k[-u] = u^{s_1} + \mu \quad \text{in } \mathbb{R}^N$$

$$F_k[-v] = u^{s_2} + \eta \quad \text{in } \mathbb{R}^N,$$  \hspace{1cm} (1.25)

admits a nonnegative solutions $(u, v)$ with $-u$ and $-v$ belonging to $\Phi^k(\mathbb{R}^N)$, then there exists a positive constant $c_{30}$ depending on $N, k, s_1, s_2$ such that there holds

$$\eta(E) + \int_E \left( W_{\frac{1}{s_1+1}} \right)^{s_2} dx \leq c_{30} \text{Cap}_G \left( 2^{k+1} k \frac{1}{s_1 s_2 - 2} \right)(E) \quad \forall E \text{ Borel}.$$  \hspace{1cm} (1.26)
Conversely, if $\mu$ and $\eta$ are bounded, there exists positive constant $c_{31}$ depending on $N, k, s_1, s_2$ such that, if $0 < s_1 < \frac{N}{k+2}$ and (1.26) holds with $c_{30}$ instead of $c_{31}$, then (1.25) admits a nonnegative solution $(u, v)$ with $-u$ and $-v$ in $\Phi^k(\mathbb{R}^N)$ satisfying

\[ v \leq c_{33} W_{\frac{2k}{s_1}, k+1}[\omega], \quad u \leq c_{34} W_{\frac{2k}{s_2}, k+1}[\left(W_{\frac{2k}{s_1}, k+1}[\omega]\right)^{s_1}] + c_{32} W_{\frac{2k}{s_2}, k+1}[\mu] \]  

(1.27)
in $\mathbb{R}^N$, where the $c_j \ (j = 32, 33, 34)$ depend on $N, k, s_1, s_2$.

As in the p-Laplace case, we have a Liouville property for Hessian systems.

**Corollary G** Assume that \( \frac{2k(s_2 s_1 + k \max\{s_1, s_2\})}{s_1 s_2 - k^2} \geq N \). (1.28)

Any nonnegative solution $(u, v)$ of inequalities

\[ F_k[-u] \geq v^{s_1} \quad \text{in} \quad \mathbb{R}^N \]
\[ F_k[-v] \geq u^{s_2} \quad \text{in} \quad \mathbb{R}^N, \]  

(1.29)

with $-u$ and $-v$ in $\Phi^k(\mathbb{R}^N)$ is trivial.

## 2 Estimates on potentials

Throughout this article $c_j, j=1,2,\ldots$, denote structural positive constants and $c_N$ is the volume of the unit ball in $\mathbb{R}^N$. The following inequality will be used several times in the sequel.

**Lemma 2.1** Let $\kappa, \gamma, \theta \in \mathbb{R}$, such that $\kappa, \gamma > 0$. Let $h : (0, \infty) \rightarrow (0, \infty)$ be nondecreasing. Then,

\[ \int_0^R t^\kappa \left( \int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} \leq c_{35} \int_0^{2R} t^{\kappa + \gamma \theta} h^\gamma(t) \frac{dt}{t} \quad \forall R \in (0, \infty], \]  

(2.1)

for some $c_{35} > 0$ depending on $\kappa, \gamma, \theta$.

**Proof.** Case 1: $\gamma \leq 1$. Since there holds

\[ \left( \sum_{j=0}^\infty a_j \right)^\gamma \leq \sum_{j=0}^\infty a_j^\gamma \quad \forall a_j \geq 0, \]

we deduce

\[ \left( \int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \leq c_{\gamma, \theta} \left( \sum_{j=0}^{j_0} h(2^{j+1} t)/(2^{j+1} t)^\theta \right)^\gamma \]
\[ \leq c_{\gamma, \theta} \sum_{j=0}^{j_0} \left( h^\gamma(2^{j+1} t)/(2^{j+1} t)^\theta \right)^\gamma \]
\[ \leq c_{\gamma, \theta} \int_t^{2R} h^\gamma(r) r^\theta \frac{dr}{r}. \]
where $c_{\gamma,\theta} = 2^{\frac{2j_0}{N}} \max\{1, 2^{-\frac{j_0}{N}} \}$ and $2^{\frac{j_0}{N}} t < R \leq 2^{\frac{j_0+1}{N}} t$ if $R < \infty$ and $j_0 = \infty$ if $R = \infty$. By Fubini’s theorem,

$$\int_0^R t^\kappa \left( \int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} \leq c_{\gamma,\theta} \int_0^R t^\kappa \int_t^{2R} h^\gamma(r) r^\theta \frac{dr}{r} \frac{dt}{t} \leq c_{\gamma,\theta} \int_0^{2R} t^\kappa + \theta \gamma h^\gamma(t) \frac{dt}{t},$$

which is (2.2).

**Case 2: $\gamma > 1$.** Since

$$\left( \int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \leq \left( \int_t^R r^{-\frac{\gamma}{\gamma - 1}} \frac{dr}{r} \right)^{-1} \int_t^R h^\gamma(r) r^{\gamma(1+\theta)} \frac{dr}{r},$$

we obtain

$$\int_0^R t^\kappa \left( \int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} \leq c_{\gamma,\theta} \int_0^{2R} t^\kappa + \theta \gamma h^\gamma(t) \frac{dt}{t},$$

by Fubini’s theorem, which completes the proof. □

We recall that if $\alpha > 0$, $1 < \beta < \frac{N}{\alpha}$ and $\mu$ belongs to the set of positive Radon measures in $\mathbb{R}^N$ that we denote $\mathcal{M}^+(\mathbb{R}^N)$, the Wolff potential of $\mu$ is defined by

$$W_{\alpha, \beta}[\mu](x) = \int_0^\infty \left( \frac{\mu(B_t(x))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\gamma - 1}} \frac{dt}{t}, \quad (2.2)$$

and if $R > 0$, the $R$-truncated Wolff potential of $\mu$ is

$$W_{\alpha, \beta}^R[\mu](x) = \int_0^R \left( \frac{\mu(B_t(x))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\gamma - 1}} \frac{dt}{t}. \quad (2.3)$$

If $\mu$ is a Radon measure on a Borel set $G$, it’s Wolff potential (or truncated Wolff potential) is the potential of its extension by 0 in $C^0$. We start with the following composition estimate on Wolff potentials.

**Lemma 2.2** Let $1 < \beta < N/\alpha$. Then for any $q > 0$ and $\mu \in \mathcal{M}^+(\mathbb{R}^N)$ we have

$$W_{\alpha, \beta}^{q(\beta-1)\frac{q-1}{2} + 1}[\mu] \leq c_{36} W_{\alpha, \beta}[(W_{\alpha, \beta}[\mu])^q], \quad (2.4)$$

in $\mathbb{R}^N$ for some $c_{36} > 0$ depending on $\alpha, \beta, N, q$. Moreover, if $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$, there holds

$$W_{\alpha, \beta}[(W_{\alpha, \beta}[\mu])^q](x) \leq c_{37} W_{\alpha, \beta}^{q(\beta-1)\frac{q-1}{2} + 1}[\mu], \quad (2.5)$$

in $\mathbb{R}^N$, where $c_{37} > 0$ depends on $\alpha, \beta, N, q$.

**Proof.** For any $x \in \mathbb{R}^N$, using the fact if $y \in B_t(x)$ then $B_t(x) \subset B_{2t}(y)$, we have

$$W_{\alpha, \beta}[(W_{\alpha, \beta}[\mu])^q](x) = \int_0^\infty \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \int_0^\infty \left( \frac{\mu(B_t(y))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\gamma - 1}} \frac{dy}{y} \right)^q \frac{dt}{t} \right)^{\frac{1}{\gamma - 1}} \frac{dx}{x} \geq c_{38} \int_0^\infty \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \frac{\mu(B_{2t}(y))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\gamma - 1}} \frac{dy}{y} \right)^{\frac{1}{\gamma - 1}} \frac{dt}{t} \geq c_{38} \int_0^\infty \left( \frac{\mu(B_t(x))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\gamma - 1}} \frac{dt}{t} = c_{38} W_{\alpha, \beta}^{q(\beta-1)\frac{q-1}{2} + 1}[\mu](x),$$

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where \(c_m = \chi_{\Omega}(\alpha, \beta, N, q) > 0\), which proves (2.1).

In order to prove (2.4) we recall the following estimate on Wolff potentials [7]

\[
\|W_{\alpha, \beta}[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq c_m \left(\omega(\mathbb{R}^N)^\frac{1}{\alpha \beta} \right)^{\frac{1}{(\alpha \beta)^2}} \quad \forall \omega \in \mathcal{M}^+_{q+1}(\mathbb{R}^N),
\]

where \(L^{(\frac{q-1}{N-\alpha \beta})}_\infty\) denotes the weak-\(L^{(\frac{q-1}{N-\alpha \beta})}_\infty\) space. In particular, since \(0 < q < \frac{N(\delta-1)}{N-\alpha \beta}\),

\[
\int_{B_r(x)} (W_{\alpha, \beta}[\omega])^q \, dy \leq c_m r^N \left(\frac{\omega(\mathbb{R}^N)}{r^{N-\alpha \beta}}\right)^{\frac{1}{\alpha \beta}} \quad \forall r > 0.
\]

Applying this inequality to \(\omega = \chi_{B_{2r}(x)}\mu\) yields

\[
\int_{B_r(x)} (W_{\alpha, \beta}[\mu])^q \, dy \leq c_m r^N \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha \beta}}\right)^{\frac{1}{\alpha \beta}} \quad \forall r > 0.
\]

We claim that

\[
I := \int_0^\infty \left(\frac{1}{r^{N-\alpha \beta}}\right) \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\mu(B_r(y))}{r^{N-\alpha \beta}}\right)^{\frac{1}{\alpha \beta}} \frac{dr}{r}\right)^q \, dy \, dt \leq c_m t^N \left(\int_t^\infty \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha \beta}}\right)^{\frac{1}{\alpha \beta}} \frac{dr}{r}\right)^q.
\]

Since \(B_r(y) \subset B_{2r}(x)\) for any \(y \in B_t(x), r \geq t\), we have

\[
\int_{B_t(x)} \left(\int_t^\infty \left(\frac{\mu(B_r(y))}{r^{N-\alpha \beta}}\right)^{\frac{1}{\alpha \beta}} \frac{dr}{r}\right)^q \, dy \leq \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha \beta}}\right)^{\frac{1}{\alpha \beta}} \frac{dr}{r}\right)^q \, dy \leq c_m t^N \left(\int_t^\infty \left(\frac{\mu(B_r(y))}{r^{N-\alpha \beta}}\right)^{\frac{1}{\alpha \beta}} \frac{dr}{r}\right)^q.
\]

Hence,

\[
I \leq c_m \int_0^\infty t^{\frac{\alpha \beta}{\alpha \beta + (\delta-1)^2}} \left(\int_t^\infty \left(\frac{\mu(B_r(x))}{r^{N-\alpha \beta}}\right)^{\frac{1}{\alpha \beta}} \frac{dr}{r}\right)^q \, dt \leq c_m W_{\alpha, \beta}^{\delta + (\delta-1)^2 + 1}[\mu](x),
\]

which completes the proof. \(\square\)

The following is a version of Lemma 2.2 for truncated Wolff potentials,

**Lemma 2.3** Let \(1 < \beta < N/\alpha\) and \(q > 0\). If \(\delta \in (0, 1)\) there holds for any \(\mu \in \mathcal{M}^+(\mathbb{R}^N)\)

\[
W_{\alpha, \beta}^{\delta + (\delta-1)^2 + 1}[\mu](x) \leq c_{\alpha, \beta} \left(W_{\alpha, \beta}[\mu]\right)^q \mu(x)
\]

in \(\Omega\). Moreover, if \(0 < q < \frac{N(\delta-1)}{N-\alpha \beta}\), there holds for any \(\mu \in \mathcal{M}^+(\mathbb{R}^N)\),

\[
W_{\alpha, \beta}^R[\mu](x) \leq c_{\alpha, \beta} W_{\alpha, \beta}^{\delta + (\delta-1)^2 + 1}[\mu](x)
\]

in \(\mathbb{R}^N\).
Proof. For any \( x \in \Omega \),

\[
W_{\alpha,\beta}^{\delta d(x)} \left[ \left( W_{\alpha,\beta}^{\delta d(.)} [\mu(.)] \right)^q \right] (x) = \int_0^{\delta d(x)} \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_{r}(x)} \left( \frac{\mu(B_{r}(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \frac{dy}{r} \right)^q \frac{dt}{t}.
\]

Since \( \delta d(y) \geq \frac{2^4}{3} d(x) \) for all \( y \in B_{\frac{2}{3}}(x) \), provided \( 0 < t < \delta d(x) \),

\[
\int_{B_{r}(x)} \left( \int_0^{\delta d(y)} \left( \frac{\mu(B_{r}(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \frac{dy}{r} \right)^q \frac{dt}{t} \geq \int_{B_{r}(x)} \left( \int_0^{\frac{2^4}{3} d(x)} \left( \frac{\mu(B_{r}(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \frac{dy}{r} \right)^q \frac{dt}{t} \geq c_{44} \int_{B_{r}(x)} \left( \mu(B_{\frac{2}{3}}(x)) \right)^{\frac{1}{\alpha\beta}} \frac{dt}{t} \geq c_{44} t^N \left( \frac{\mu(B_{\frac{2}{3}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} dt.
\]

Hence

\[
W_{\alpha,\beta}^{\delta d(x)} \left[ \left( W_{\alpha,\beta}^{\delta d(.)} [\mu(.)] \right)^q \right] (x) \geq c_{46} \int_0^{\delta d(x)} \left( t^{\alpha\beta} \left( \frac{\mu(B_{\frac{2}{3}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \right) \frac{dt}{t},
\]

which implies (2.10).

Because of (2.8), it is sufficient to prove that there holds

\[
\int_0^{R} \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_{r}(x)} \left( \int_t^{R} \left( \frac{\mu(B_{r}(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \frac{dy}{r} \right)^q \frac{dt}{t} \right) \frac{dr}{r} \leq c_{47} W_{\alpha,\beta}^{\delta d(\rho+y)} dt
\]

in order to obtain (2.11). Since \( B_{\rho}(y) \subset B_{2\rho}(x) \) for any \( y \in B_{r}(x), \rho \geq r \), we have

\[
\int_{B_{r}(x)} \left( \int_t^{R} \left( \frac{\mu(B_{r}(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \frac{dy}{r} \right)^q \frac{dt}{t} \leq \int_{B_{r}(x)} \left( \int_t^{R} \left( \frac{\mu(B_{2\rho}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \frac{dy}{r} \right)^q \frac{dt}{t} \leq c_N t^N \left( \int_t^{R} \left( \frac{\mu(B_{2\rho}(x))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \frac{dt}{t} \right).
\]

Therefore

\[
\int_0^{R} \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_{r}(x)} \left( \int_t^{R} \left( \frac{\mu(B_{r}(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \frac{dy}{r} \right)^q \frac{dt}{t} \right) \frac{dr}{r} \leq c_{47} \int_0^{R} \left( t^{\alpha\beta} \left( \int_t^{R} \left( \frac{\mu(B_{2\rho}(x))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\alpha\beta}} \frac{dt}{t} \right)^q \right) \frac{dt}{t},
\]

which completes the proof.
We infer (2.12) by Lemma 2.1, which completes the proof. □

The next two propositions link Wolff potentials of a measure with Riesz capacicities (in the case of whole space) and truncated Wolff potentials with Bessel capacicities (in the bounded domain case). Their proof can be found in [20, 21] (and [8] with a different method).

**Proposition 2.4** Let $1 < \beta < N/\alpha$, $q > \beta - 1$, $\nu \in M^+(\mathbb{R}^N)$. Then, the following statements are equivalent:

(a) The inequality

$$\nu(K) \leq c_{48} \text{Cap}_{1, \beta, \frac{q}{\beta - 1}}(K)$$

holds for any compact set $K \subset \mathbb{R}^N$, for some $c_{48} > 0$.

(b) The inequality

$$\int_{\mathbb{R}^N} (W_{\alpha, \beta}^{(\chi_{B_t(x)})}(y))^q dy \leq c_{49} \nu(B_t(x))$$

holds for any ball $B_t(x) \subset \mathbb{R}^N$, for some $c_{49} > 0$.

(c) The inequality

$$W_{\alpha, \beta}[(W_{\alpha, \beta}[^q \nu])^q] \leq c_{50} W_{\alpha, \beta}[^q \nu] < \infty \ a.e \ in \ \mathbb{R}^N$$

holds for some $c_{50} > 0$.

**Proposition 2.5** Let $1 < \beta < N/\alpha$, $q > \beta - 1$, $R > 0$ and $\nu \in M^+_0 (B_R(x_0))$ for some $x_0 \in \mathbb{R}^N$. Then, the following statements are equivalent:

(a) The inequality

$$\nu(K) \leq c_{51} \text{Cap}_{G, \beta, \frac{q}{\beta - 1}}(K)$$

holds for any compact set $K \subset \mathbb{R}^N$, for some $c_{51} = c_{48}(R) > 0$.

(b) The inequality

$$\int_{\mathbb{R}^N} (W_{4R}^{(\chi_{B_t(x)})}(y))^q dy \leq c_{52} \nu(B_t(x))$$

holds for any ball $B_t(x) \subset \mathbb{R}^N$, for some $c_{52} = c_{51}(R) > 0$.

(c) The inequality

$$W_{4R}^{(\chi_{B_t(x)})}[W_{4R}^{(\chi_{B_t(x)})}[^q \nu]]^q \leq c_{53} W_{4R}^{(\chi_{B_t(x)})}[\nu] < \infty \ a.e \ in \ B_{2R}(x_0)$$

holds for some $c_{53} = c_{53}(R) > 0$.

In the following statement we obtain capacitary estimates on combinations of measures.

**Proposition 2.6** Let $\eta, \mu$ be in $M^+(\mathbb{R}^N)$. Assume that $0 < q < \frac{N(\beta - 1)}{N - \alpha \beta}$ and $qs > (\beta - 1)^2$.

(i) If there holds

$$\eta(K) + \int_K (W_{\alpha, \beta}[^q \mu])^q dx \leq \text{Cap}_{1, \alpha, \beta, \frac{q}{\beta - 1}}(K),$$

for any compact set $K \subset \mathbb{R}^N$, then

$$W_{\alpha, \beta}[(W_{\alpha, \beta}[^q \nu])^q^q] \leq c_{54} W_{\alpha, \beta}[^q \nu] < \infty \ a.e \ in \ \mathbb{R}^N,$$
where \( \omega = (W_{\alpha,\beta}[\mu])^s + \eta \).

(ii) If there holds
\[
\eta(K) + \int_K (W_{\alpha,\beta}^{2r}[\mu])^s \, dx \leq \text{Cap}_{\mathbb{R}^N}^{\frac{\alpha}{\alpha + \beta - 1}}(K), \tag{2.21}
\]
for any compact set \( K \subset \mathbb{R}^N \), then
\[
W_{\alpha,\beta}^{2r} \left[ \left( W_{\alpha,\beta}^{2r} \left[ (W_{\alpha,\beta}[\omega])^q \right] \right)^s \right] \leq c_{33} W_{\alpha,\beta}^{2r}[\omega] < \infty \text{ a.e in } B_R(x_0), \tag{2.22}
\]
where \( \omega = \chi_{B_R(x_0)} \left( W_{\alpha,\beta}^{2r}[\mu] \right)^s + \chi_{B_R(x_0)} \eta \).

**Proof.** Statement (i): We assume that (2.19) holds. Put \( \omega = (W_{\alpha,\beta}[\mu])^s + \eta \) and apply (2.19) to \( K = \overline{B}_\rho(x) \). Since by homogeneity
\[
\text{Cap}_{\mathbb{R}^N}^{\frac{\alpha}{\alpha + \beta - 1}}(\overline{B}_\rho(0)) = \rho^{N - \frac{\alpha(q + \beta - 1)}{q + (\beta - 1)^2}} \text{Cap}_{\mathbb{R}^N}^{\frac{\alpha}{\alpha + \beta - 1}}(\overline{B}_1(0)),
\]
we deduce from (2.19) that
\[
\omega(B_\rho(x)) \leq c_{33} \rho^{N - \frac{\alpha(q + \beta - 1)}{q + (\beta - 1)^2}} \forall \rho > 0,
\]
which is equivalent to
\[
\rho^{\frac{\alpha}{\beta - 1}} \left( \frac{\omega(B_\rho(x))}{\rho^{N - \frac{\alpha(q + \beta - 1)}{q + (\beta - 1)^2}}} \right)^{\frac{\alpha}{\beta - 1}} \leq c_{33} \left( \frac{\omega(B_\rho(x))}{\rho^{N - \alpha \beta}} \right)^{\frac{1}{\beta - 1}} \forall \rho > 0. \tag{2.23}
\]
We apply Proposition 2.4 to \( \nu = \omega \) with \( (\alpha, \beta, q) = \left( \frac{\alpha(q + \beta - 1)}{q + (\beta - 1)^2}, \frac{(\beta - 1)^2}{q + (\beta - 1)^2}, 1, s \right) \). (2.19) implies
\[
\int_{\mathbb{R}^N} \left( W_{\alpha,\beta}^{\frac{\alpha}{\beta - 1}} \left[ \left( W_{\alpha,\beta}^{\frac{\alpha}{\beta - 1}} \left[ (W_{\alpha,\beta}[\omega])^q \right] \right)^s \right] \right) \, dy \leq c_{33} \omega(B_t(x)). \tag{2.24}
\]
By Lemma 2.2 (2.20) is equivalent to
\[
W_{\alpha,\beta} \left[ \left( W_{\alpha,\beta}^{\frac{\alpha}{\beta - 1}} \left[ (W_{\alpha,\beta}[\omega])^q \right] \right)^s \right] \leq c_{33} W_{\alpha,\beta}[\omega] < \infty \text{ a.e } \mathbb{R}^N. \tag{2.25}
\]
Therefore, it is enough to show that (2.23) and (2.24) imply (2.25). In fact, since for \( t > 0 \)
\[
\int_{B_t(x)} \left( W_{\alpha,\beta}^{\frac{\alpha}{\beta - 1}} \left[ (W_{\alpha,\beta}[\omega])^q \right] \right)^s \, dy = \int_{B_t(x)} \left( W_{\alpha,\beta}^{\frac{\alpha}{\beta - 1}} \left[ (W_{\alpha,\beta}[\omega])^q \right] \right)^s \, dy,
\]
we apply (2.24) and obtain
\[
\int_{B_t(x)} \left( W_{\alpha,\beta}^{\frac{\alpha}{\beta - 1}} \left[ (W_{\alpha,\beta}[\omega])^q \right] \right)^s \, dy \leq c_{33} \omega(B_{2t}(x)).
\]
So, it is enough to show that
\[
I := \int_0^\infty \left( \frac{1}{t^{N-\alpha \beta}} \int_{B_t(x)} \left( \int_t^\infty \left( \frac{\omega(B_r(y))}{r^{N-\alpha \beta}} \right)^{\frac{\alpha}{\beta - 1}} \, dr \right)^s \, dy \right)^{\frac{\beta - 1}{s}} \, dt \leq c_{33} W_{\alpha,\beta}[\omega](x). \tag{2.26}
\]
Since $B_r(y) \subset B_{2r}(x)$ for any $y \in B_t(x), r \geq t$, we have

$$I \leq c_N \int_0^\infty \left( \int_t^\infty \frac{\omega(B_{2r}(x))}{r^{N-\frac{\alpha\beta}{q}(\beta-1)}} \frac{dr}{r} \right)^\frac{1}{\alpha\beta} \frac{dt}{t} \leq c_{N6} \int_0^\infty \left( \frac{\omega(B_{2r}(x))}{t^{N-\alpha\beta}} \right)^\frac{1}{\alpha\beta} \frac{dt}{t},$$

which is (2.20).

**Statement (ii):** We assume that (2.21) holds. Put $d\omega = \chi_\alpha (W_{\alpha,\beta}[\mu])^s + \chi_\alpha \eta$, then

$$\omega(B_\rho(x)) \leq c_{\rho} \rho^{-\frac{\alpha\beta}{q}(\beta-1)} \forall 0 < \rho < 2R.$$

As in the proof of statement (i), the above inequality is equivalent to

$$\rho^{\alpha\beta} \left( \frac{\omega(B_\rho(x))}{\rho^{-\frac{\alpha\beta}{q}(\beta-1)} \forall 0 < \rho < 2R} \leq c_{\rho1} \left( \frac{\omega(B_\rho(x))}{\rho^{-\alpha\beta}} \right)^\frac{1}{\alpha\beta} \forall 0 < \rho < 2R. \quad (2.27)$$

Applying Proposition 2.22 with $\nu = \omega$ and $(\alpha, \beta, q) = \left( \frac{\alpha\beta(q+\beta-1)}{q+\beta-1}, \frac{(\beta-1)^2}{q}, 1, s \right)$,

$$\int_{\mathbb{R}^N} \left( W_{\alpha,\beta}^{4R} \left[ \frac{\omega(B_{2r}(x))}{\rho^{-\frac{\alpha\beta}{q}(\beta-1)}} \right]^{\frac{1}{\alpha\beta}} \right)^s \leq c_{\rho2} \omega(B_t(x)). \quad (2.28)$$

By Lemma 2.25 (2.28) is equivalent to

$$W_{\alpha,\beta}^{4R} \left[ \left( \frac{\omega(B_{2r}(x))}{\rho^{-\frac{\alpha\beta}{q}(\beta-1)}} \right)^\frac{1}{\alpha\beta} \right] \leq c_{\rho3} W_{\alpha,\beta}^{4R} \omega \ \text{a.e. in } B_R(x). \quad (2.29)$$

Therefore, it is sufficient to prove that (2.27) and (2.28) imply (2.29). Actually, since

$$\int_{B_t(x)} \left( W_{\alpha,\beta}^{4R} \left[ \frac{\omega(B_{2r}(x))}{\rho^{-\frac{\alpha\beta}{q}(\beta-1)}} \right]^{\frac{1}{\alpha\beta}} \right)^s \leq c_{\rho4} \omega(B_{2r}(x))$$

for all $0 < t < 4R$, thus applying (2.28), we obtain

$$\int_{B_t(x)} \left( W_{\alpha,\beta}^{4R} \left[ \frac{\omega(B_{2r}(x))}{\rho^{-\frac{\alpha\beta}{q}(\beta-1)}} \right]^{\frac{1}{\alpha\beta}} \right)^s \leq c_{\rho4} \omega(B_{2r}(x)).$$

So, it is sufficient to show that for any $x \in B_R(x_0)$

$$II := \int_0^{4R} \left( \int_{B_t(x)} \left( \int_t^{4R} \frac{\omega(B_{2r}(x))}{r^{N-\frac{\alpha\beta}{q}(\beta-1)}} \frac{dr}{r} \right)^s \frac{dy}{t} \right)^\frac{1}{\alpha\beta} \frac{dt}{t} \leq c_{\rho5} W_{\alpha,\beta}^{4R} \omega(x). \quad (2.30)$$

Since $B_r(y) \subset B_{2r}(x)$ for any $y \in B_t(x)$ with $r \geq t$, we have

$$II \leq c_N \int_0^{4R} \left( \int_t^{4R} \frac{\omega(B_{2r}(x))}{r^{N-\frac{\alpha\beta}{q}(\beta-1)}} \frac{dr}{r} \right)^\frac{1}{\alpha\beta} \frac{dt}{t},$$

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Combining this with Lemma 2.1 and (2.27) yields

\[ 11 \leq c_{\theta\alpha} W_{0,\alpha}^{16R}(\omega)(x). \]

Therefore, (2.29) follows since \( W_{0,\alpha}^{16R}(\omega) \leq c_{\theta\alpha} W_{0,\alpha}^{4R}(\omega) \) in \( B_R(x_0). \)

**Proposition 2.7** Let \( \eta, \mu \) be in \( \mathcal{M}(\mathbb{R}^N) \). Assume that \( 0 < q < \frac{N(\beta - 1)}{N - \alpha \beta} \) and \( qs > (\beta - 1)^2 \).

Let \( (u_m, v_m) \) be nonnegative measurable functions in \( \mathbb{R}^N \) verifying, for all \( m \geq 0 \),

\[ u_{m+1} \leq c^* W_{\alpha,\beta}[v_m^q + \mu], \quad v_{m+1} \leq c^* W_{\alpha,\beta}[u_m^s + \eta] \quad \text{a.e. in } \mathbb{R}^N, \]

for some \( c^* > 0 \) and \( (u_0, v_0) = 0 \). Then, there exists a constant \( M^* > 0 \) depending only on \( N, \alpha, \beta, q, s, c^* \) such that if the measure \( d\omega = (W_{\alpha,\beta}[\mu])^{\rho} dx + d\eta \) satisfies

\[ \omega(K) \leq M^* \text{Cap}_{\mu, \frac{\alpha\beta}{(N - \alpha \beta)} - 1, \frac{\alpha\beta}{(N - \alpha \beta)} - 1}(K), \]

(2.31)

for any compact \( K \subset \mathbb{R}^N \), then

\[ v_m \leq c_{\theta\alpha} W_{\alpha,\beta}[\omega], \quad u_m \leq c_{\theta\alpha} W_{\alpha,\beta}[(W_{\alpha,\beta}[\omega])^q] + c_{\theta\alpha} W_{\alpha,\beta}[\mu] \quad \forall \ m \geq 0, \]

(2.32)

for some constants \( c_{\theta\alpha}, c_{\theta\beta}, c_{\theta\alpha} \) depending only on \( N, \alpha, \beta, q, s \) and \( c^* \).

**Proof.** By Proposition 2.6 (2.31) implies

\[ W_{\alpha,\beta}[(W_{\alpha,\beta}[\omega])^q] \leq c_{\theta\alpha} M^* \text{Cap}_{\mu, \frac{\alpha\beta}{(N - \alpha \beta)} - 1, \frac{\alpha\beta}{(N - \alpha \beta)} - 1}(K), \]

(2.33)

We set

\[ c_{\theta\alpha} = c^* 2^{\frac{1}{q-1}}, \]

\[ c_{\theta\beta} = c^* 2^{1 + \frac{1}{q-1}} (c_{\theta\alpha} 2^{s-1} + 1) \frac{1}{q-1}, \]

\[ c_{\theta\alpha} = c^* 2^{\frac{1}{q-1}} (c_{\theta\beta})^{\frac{2}{2-q}} \]

and choose \( M^* > 0 \) such that

\[ c^* 2^{\frac{1}{q-1}} (c_{\theta\beta} 2^{s-1})^{\frac{1}{q-1}} c_{\theta\alpha} M^* \text{Cap}_{\mu, \frac{\alpha\beta}{(N - \alpha \beta)} - 1, \frac{\alpha\beta}{(N - \alpha \beta)} - 1}(K) = \frac{c_{\theta\alpha}}{2}. \]

We claim that

\[ v_m \leq c_{\theta\alpha} W_{\alpha,\beta}[\omega], \quad u_m \leq c_{\theta\alpha} W_{\alpha,\beta}[(W_{\alpha,\beta}[\omega])^q] + c_{\theta\alpha} W_{\alpha,\beta}[\mu] \quad \forall \ m \geq 0. \]

(2.34)

Clearly, by definition of \( c_{\theta\alpha}, c_{\theta\beta} \) and \( c_{\theta\alpha} \), we have (2.31) for \( m = 0, 1 \). Next we assume that (2.34) holds for all integer \( m \leq l \) for some \( l \in \mathbb{N}^*_+ \), then

\[ u_{l+1} \leq c^* W_{\alpha,\beta}[v_l^q + \mu] \]

\[ \leq c^* 2^{\frac{1}{q-1}} c_{\theta\alpha} W_{\alpha,\beta}[(W_{\alpha,\beta}[\omega])^q] + c^* 2^{\frac{1}{q-1}} W_{\alpha,\beta}[\mu] + c_{\theta\alpha} W_{\alpha,\beta}[\mu], \]

and

\[ v_{l+1} \leq c^* W_{\alpha,\beta}[(c_{\theta\alpha} W_{\alpha,\beta}[(W_{\alpha,\beta}[\omega])^q] + c_{\theta\alpha} W_{\alpha,\beta}[\mu])^s + \eta] \]

\[ \leq c^* W_{\alpha,\beta}[(c_{\theta\alpha} 2^{s-1} (W_{\alpha,\beta}[(W_{\alpha,\beta}[\omega])^q]))^s + c_{\theta\alpha} 2^{s-1} (W_{\alpha,\beta}[\mu])^s + \eta] \]

\[ \leq c^* 2^{\frac{1}{q-1}} (c_{\theta\alpha} 2^{s-1})^{\frac{1}{q-1}} W_{\alpha,\beta}[(W_{\alpha,\beta}[\omega])^q] + c^* 2^{\frac{1}{q-1}} (c_{\theta\alpha} 2^{s-1})^{\frac{1}{q-1}} W_{\alpha,\beta}[\mu] \]

\[ = \frac{c_{\theta\alpha}}{2} W_{\alpha,\beta}[\omega] + \frac{c_{\theta\alpha}}{2} W_{\alpha,\beta}[\omega], \]

\[ = c_{\theta\alpha} W_{\alpha,\beta}[\omega]. \]
Thus, (2.34) holds true for $m = l + 1$. Hence, (2.34) is valid for all $l \geq 0$. 

The next result is an adaptation of Proposition 2.7 to truncated Wolff potentials.

**Proposition 2.8** Let $\eta, \mu$ be in $\mathcal{M}_0^+(B_R(x_0))$. Assume that $0 < q < \frac{N(\beta - 1)}{N - \alpha}$ and $qs > (\beta - 1)^2$. Let $(u_m, v_m)$ be nonnegative measurable functions in $\mathbb{R}^N$ such that for all $m \geq 0$

$$u_{m+1} \leq c_2 W_{\alpha, \beta}^R \chi_{E_R(x_0)}^{q/m} + \mu, \quad v_{m+1} \leq c_2 W_{\alpha, \beta}^R \chi_{E_R(x_0)}^{s/m} + \eta$$
a.e. in $B_R(x_0)$, and $(u_0, v_0) = 0$. If we set $d\omega = \left(W_{\alpha, \beta}^R[\mu]\right)^s dx + d\eta$, there exists a constant $M_* > 0$ depending only on $N, \alpha, \beta, q, s, R$ and $c_*$ such that if

$$\omega(K) \leq M_* \text{Cap}_G \frac{\alpha + \beta - 1}{\alpha - \beta - 1} \frac{qs}{(\beta - 1)^2} (K), \quad (2.35)$$

for any compact set $K \subset \mathbb{R}^N$, then

$$v_m \leq c_3 W_{\alpha, \beta}^{2R}[\omega], \quad u_m \leq c_4 W_{\alpha, \beta}^{2R} \left([W_{\alpha, \beta}^{2R}[\omega]]^q + c_2 W_{\alpha, \beta}^{2R}[\mu]\right) \quad \forall k \geq 0 \quad (2.36)$$
in $B_R(x_0)$ for some constants $c_{72}, c_{73}, c_{74}$ depending only on $N, \alpha, \beta, q, s, R$ and $c_*$. 

**Proof.** The proof is similar to the one of Proposition 2.7 and we omit the details. 

**Proposition 2.9** Let $1 < \beta < N/\alpha$ and $q, s > 0$ such that $qs > (\beta - 1)^2$.

(i) Assume that $\eta$ and $\mu$ belong to $\mathcal{M}_0^+(\mathbb{R}^N)$ and $(u, v)$ are nonnegative measurable functions satisfying

\begin{align*}
(i) \quad W_{\alpha, \beta}[u^q] + W_{\alpha, \beta}[\mu] & \leq c_{75} u, \\
(ii) \quad W_{\alpha, \beta}[u^s] + W_{\alpha, \beta}[\eta] & \leq c_{75} v \quad \text{a.e. in } \mathbb{R}^N, \quad (2.37)
\end{align*}

for some $c_{75} > 0$. Then there exists a constant $c_{76} > 0$ depending only on $N, \alpha, \beta, q, s$ and $c_{75}$ such that

$$\eta(K) + \int_K (W_{\alpha, \beta}[\mu](x))^s \, dx \leq c_{76} \text{Cap}_G \frac{\alpha + \beta - 1}{\alpha - \beta - 1} \frac{qs}{(\beta - 1)^2} (K), \quad (2.38)$$

for any compact set $K \subset \mathbb{R}^N$.

(ii) Assume that $\eta$ and $\mu$ belong to $\mathcal{M}_0^+(\Omega)$ and $(u, v)$ are nonnegative functions satisfying

\begin{align*}
(i) \quad W_{\alpha, \beta}[u^q] + W_{\alpha, \beta}[\mu] & \leq c_{77} u, \\
(ii) \quad W_{\alpha, \beta}[u^s] + W_{\alpha, \beta}[\eta] & \leq c_{77} v \quad \text{a.e. in } \Omega, \quad (2.39)
\end{align*}

for some $c_{77} > 0$. Then for any $\Omega' \subset \subset \Omega$, there exists a constant $c_{78} > 0$ depending only on $n, \alpha, \beta, q, s, c_{77}$ and dist$(\Omega', \partial \Omega)$ such that

$$\eta(K) + \int_K (W_{\alpha, \beta}[\mu](x))^s \, dx \leq c_{78} \text{Cap}_G \frac{\alpha + \beta - 1}{\alpha - \beta - 1} \frac{qs}{(\beta - 1)^2} (K), \quad (2.40)$$

for any compact set $K \subset \Omega'$.

**Proof.** (i): Set $\omega = u^s + \eta$, then

$$\omega \geq u^s \geq (W_{\alpha, \beta}[u^q])^s \geq c_{79} (W_{\alpha, \beta}[(W_{\alpha, \beta}[\omega])^q])^s.$$ 

By (2.34) in Lemma 2.2 we get

$$\omega \geq c_{80} \left(W_{\alpha, \beta}^{(q + \beta - 1)} \frac{\alpha - \beta - 1}{\alpha - \beta} \chi_{\omega} \right)^s,$$
which implies
\[
\int_{\mathbb{R}^N} \left( W_{\alpha\beta, \frac{q+\beta-1}{q+(-\beta-1)^2}}^{d(B_t(x)) \frac{(\beta-1)^2}{q} + 1} [\chi_{B_t(x)}] \right)^s \, dy \leq c_{s_1} \omega(B_t(x)) \quad \forall \, x \in \mathbb{R}^N, \, \forall \, t > 0.
\]
Applying Proposition 2.3 to \( \mu = \omega \) with \( (\alpha, \beta, q) = \left( \frac{\alpha\beta(q+\beta-1)}{q+(-\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1, s \right) \), we get (2.38).

(ii) We define \( \omega \) as above and we have

\[
\omega \geq u^* \geq \left( W_{\alpha, \beta}^{\delta d, \frac{q}{q+(-\beta-1)^2}} \right)^s \geq c_{s_2} \left( W_{\alpha, \beta}^{\delta d, \left[\left(W_{\alpha, \beta}^{\frac{\delta d}{\delta d}} \omega \right)\right]^s} \right) \quad \text{a.e. in } \Omega,
\]
which leads to

\[
\omega \geq c_{s_3} \left( W_{\alpha, \beta}^{\delta d, \frac{q}{q+(-\beta-1)^2}} \right)^s \quad \text{a.e. in } \Omega,
\]
by inequality (2.10) in Lemma 2.3. Let \( M_\omega \) denote the centered Hardy-Littlewood maximal function which is defined for any \( f \in L^1_{\text{loc}}(\mathbb{R}^N, d\omega) \) by

\[
M_\omega f(x) = \sup_{t > 0} \frac{\omega(B_t(x))}{\omega(B_t(x))} \int_{B_t(x)} \left| f \right| \, d\omega.
\]
Let \( K \subset \Omega \) be compact. Set \( r_K = \text{dist}(K, \partial \Omega) \) and \( \Omega_K = \{ x \in \Omega : d(x, K) < r_K/2 \} \). Then, for any Borel set \( E \subset K \),

\[
c_{s_4} \int_{\Omega} \left( M_\omega \chi_E \right)^{\frac{q}{q-1} - 1} \left( W_{\alpha, \beta}^{\delta d, \frac{q}{q+(-\beta-1)^2}} \right)^s \, dx \leq \int_{\Omega} \left( M_\omega \chi_E \right)^{\frac{q}{q-1} - 1} \, dx.
\]
Since \( M_\omega \) is a bounded linear map on \( L^p(\mathbb{R}^N, d\omega) \) for any \( p > 1 \) and

\[
(M_\omega \chi_E)^{\frac{q}{q-1} - 1} \left( W_{\alpha, \beta}^{\delta d, \frac{q}{q+(-\beta-1)^2}} \right)^s \geq \int_0^{\frac{\delta d(x)}{\delta d(y)}} \left( \omega(B_t(x) \cap E) \right) \frac{\omega(B_t(x))}{\omega(B_t(x))} \frac{\omega(B_t(x))}{t^{N-\frac{2\alpha\beta(q+\beta-1)}{q+(-\beta-1)^2}}} \, \frac{1}{t},
\]
we obtain

\[
\int_{\Omega} \left( W_{\alpha, \beta}^{\delta d, \frac{q}{q+(-\beta-1)^2}} \right)^s \, dx \leq c_{s_5} \omega(E),
\]
where \( \omega_E = \chi_{\omega^c} \). Note that if \( x \in \Omega \) and \( d(x) \leq r_K/8 \), then \( B_1(x) \subset \Omega \setminus \Omega_K \) for all \( t \in (0, \frac{\delta d(x)}{2}) \); indeed, for all \( y \in B_1(x) \)

\[
d(y, \partial \Omega) \leq d(x, \partial \Omega) + |x - y| < (1 + \delta)d(x, \partial \Omega) < \frac{1}{4} r_K,
\]
thus

\[
d(y, K) \geq d(K, \partial \Omega) - d(y, \partial \Omega) > \frac{3}{4} r_K > \frac{1}{2} r_K,
\]
which implies \( y \notin \Omega_K \). We deduce that

\[
W_{\alpha, \beta}^{\delta d, \frac{q}{q+(-\beta-1)^2}} \left[ \omega_E \right] \geq W_{\alpha, \beta}^{\delta d, \frac{q}{q+(-\beta-1)^2}} \left[ \omega_E \right] \quad \forall x \in \Omega,
\]
and

\[
W_{\alpha, \beta}^{\delta d, \frac{q}{q+(-\beta-1)^2}} \left[ \omega_E \right] = 0 \quad \forall x \in \Omega^c.
\]
Hence we obtain

\[
\int_{\mathbb{R}^N} \left( W_{\alpha, \beta}^{\delta d, \frac{q}{q+(-\beta-1)^2}} \right)^s \, dx \leq c_{s_5} \omega(E) \quad \forall E \subset K, \ E \text{ Borel.} \quad (2.41)
\]
Applying Proposition 2.3 with \( \mu = \chi_{K \cap \Omega^c} \) and \( \omega \) we get (2.41), which completes the proof. \( \square \)
3 Quasilinear Dirichlet problems

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). If \( \mu \in \mathcal{M}_b(\Omega) \), we denote by \( \mu^+ \) and \( \mu^- \) respectively its positive and negative parts in the Jordan decomposition. We denote by \( \mathcal{M}_c(\Omega) \) the space of measures in \( \Omega \) which are absolutely continuous with respect to the Lebesgue measure \( \lambda \). We denote by \( \mathcal{M}_b(\Omega) \) the space of measures in \( \Omega \) with support on a set of zero \( \mathcal{C}^1 \)-capacity.

We also denote \( \mathcal{M}_c(\Omega) \) the space of measures in \( \Omega \) with support on a set of zero \( \mathcal{C}^1 \)-capacity. Classically, any \( \mu \in \mathcal{M}_b(\Omega) \) can be written in a unique way under the form \( \mu = \mu_0 + \mu_+ \) where \( \mu_0 \in \mathcal{M}_0(\Omega) \cap \mathcal{M}_b(\Omega) \) and \( \mu_+ \in \mathcal{M}_c(\Omega) \cap \mathcal{M}_b(\Omega) \). It is well known that any \( \mu_0 \in \mathcal{M}_0(\Omega) \cap \mathcal{M}_b(\Omega) \) can be written under the form \( \mu_0 = f - \text{div} \ g \) where \( f \in L^1(\Omega) \) and \( g \in L^p(\Omega, \mathbb{R}^N) \).

For \( k > 0 \) and \( s \in \mathbb{R} \) we set \( T_k(s) = \max\{\min\{s, k\}, -k\} \). If \( u \) is a measurable function defined in \( \Omega \), finite a.e. and such that \( T_k(u) \in W^{1,p}_{\text{loc}}(\Omega) \) for any \( k > 0 \), there exists a measurable function \( v : \Omega \to \mathbb{R}^N \) such that \( \nabla T_k(u) = \chi_{(|\nabla u| \leq k)} v \) a.e. in \( \Omega \) and for all \( k > 0 \).

We define the gradient a.e. \( \nabla u \) of \( u \) by \( v = \nabla u \). We recall the definition of a renormalized solution given in [12].

**Definition 3.1** Let \( \mu = \mu_0 + \mu_+ \in \mathcal{M}_b(\Omega) \). A measurable function \( u \) defined in \( \Omega \) and finite a.e. is called a renormalized solution of

\[
\begin{align*}
-\Delta_p u &= \mu & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

if \( T_k(u) \in W^{1,p}_{\text{loc}}(\Omega) \) for any \( k > 0 \), \( |\nabla u|^{p-1} \in L^r(\Omega) \) for any \( 0 < r < \frac{N}{N-p} \), and \( u \) has the property that for any \( k > 0 \) there exist \( \lambda^+_k \) and \( \lambda^-_k \) belonging to \( \mathcal{M}_c^{+}(\Omega) \cap \mathcal{M}_b(\Omega) \), respectively concentrated on the sets \( u = k \) and \( u = -k \), with the property that \( \mu_+ \rightarrow \lambda^+_k \) and \( \mu^- \rightarrow \lambda^-_k \) in the narrow topology of measures and such that

\[
\int_{\{|u|<k\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda^+_k - \int_{\Omega} \varphi d\lambda^-_k,
\]

for every \( \varphi \in W^{1,p}_{\text{loc}}(\Omega) \). \( \mathcal{L}^\infty(\Omega) \).

**Remark 3.2** We recall that if \( u \) is a renormalized solution to problem (3.1), then \( \frac{|\nabla u|^p}{|\nabla u|+1} \in L^1(\Omega) \) for all \( r > 1 \). Furthermore, \( u \geq 0 \) a.e. in \( \Omega \) if \( \mu \in \mathcal{M}_c^{+}(\Omega) \).

The following general stability result has been proved in [12] Th 4.1.

**Theorem 3.3** Let \( \mu = \mu_0 + \mu_+ - \mu_- \), with \( \mu_0 = F - \text{div} \ g \in \mathcal{M}_b(\Omega) \) and \( \mu_+ - \mu_- \) belonging to \( \mathcal{M}_c^{+}(\Omega) \). Let \( \mu_0 = F_n - \text{div} \ g_n + \rho_n - \eta_n \) with \( F_n \in L^1(\Omega) \), \( g_n \in \text{Lip}(\Omega) \), and \( \rho_n, \eta_n \) belonging to \( \mathcal{M}_b^+(\Omega) \). Assume that \( \{F_n\} \) converges to \( F \) weakly in \( L^1(\Omega) \), \( \{g_n\} \) converges to \( g \) strongly in \( \text{Lip}(\Omega) \), and \( \{\rho_n\} \) converges to \( \mu^+ \) and \( \{\eta_n\} \) to \( \mu^- \) in the narrow topology. If \( \{u_n\} \) is a sequence of renormalized solutions of (3.1) with data \( \mu_n \), then, up to a subsequence, it converges a.e. in \( \Omega \) to a renormalized solution \( u \) of problem (3.1). Furthermore, \( T_k(u_n) \) converges to \( T_k(u) \) in \( W^{1,p}_{\text{loc}}(\Omega) \) for any \( k > 0 \).

We also recall the following estimate [20] Th 2.1.

**Proposition 3.4** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \). Then there exists a constant \( C > 0 \), depending on \( p \) and \( N \) such that if \( \mu \in \mathcal{M}_c^{+}(\Omega) \) and \( u \) is a nonnegative renormalized solution of problem (3.1) with data \( \mu \), there holds

\[
\frac{1}{c_{\text{iso}}} W_{1,p}^{d_{\text{iso}}(\Omega)}[\mu](x) \leq u(x) \leq c_{\text{iso}} W_{1,p}^{d_{\text{iso}}(\Omega)}[\mu](x) \quad \text{a.e. in } \Omega.
\]
Proof of Theorem C. The condition is necessary. Assume that \(1.14\) admits a nonnegative renormalized solutions \((u, v)\). By Proposition 2.9 there holds
\[
\begin{align*}
\frac{d}{dx} (\frac{d}{dx} p) & \geq c_{s}\frac{d}{dx} p \frac{d}{dx} p [v^{q} + \mu](x) \\
v(x) & \geq c_{s}\frac{d}{dx} p [u^{q} + \mu](x) \quad \text{a.e. in } \Omega.
\end{align*}
\]
Hence, we infer \(1.15\) from Proposition 2.9 (ii).

Sufficient conditions. Let \(\{\{u_{m}, v_{m}\}\}_{m \in \mathbb{N}}\) be a sequence of nonnegative renormalized solutions of the following problems for \(m \in \mathbb{N}\),
\[
\begin{align*}
\Delta_{p} u_{m+1} & = v_{m}^{q} + \mu \quad \text{in } \Omega \\
\Delta_{p} v_{m+1} & = u_{m}^{q} + \eta \quad \text{in } \Omega \\
u_{m+1} & = v_{m+1} = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
with initial condition \((u_{0}, v_{0}) = 0\). The sequences \(\{u_{m}\}\) and \(\{v_{m}\}\) can be constructed in such a way that they are nondecreasing (see e.g. \([21]\)). By Proposition 3.4 we have
\[
\omega(K) \leq M_{s} \text{Cap}_{\mathbb{R}^{N}} \left(\frac{\mu + \eta^{q} + \mu^{q}}{q^{q} - (p - 1)}\right)(K) \quad \text{for any compact set } K \subset \mathbb{R}^{N}
\]
and
\[
u_{m} \leq c_{3} W^{R}_{1,p}[\omega], \quad u_{m} \leq c_{4} W^{R}_{1,p} [W^{R}_{1,p}[\omega]] + c_{5} W^{R}_{1,p}[\mu] \quad \forall \; k \geq 0
\]
in \(\Omega\), and
\[
W^{R}_{1,p}[\omega] \in L^{q^{2}}(\Omega), \quad W^{R}_{1,p} [W^{R}_{1,p}[\omega]] + W^{R}_{1,p}[\mu] \in L^{q^{2}}(\Omega).
\]
This implies that \(\{u_{m}\}, \{v_{m}\}\) are well defined and nondecreasing. Thus \(\{u_{m}, v_{m}\}\) converges a.e in \(\Omega\) to some functions \((u, v)\) which satisfies \(1.17\) in \(\Omega\). Furthermore, we deduce from \(3.6\) and the monotone convergence theorem that \(u_{m}^{q} \to u^{q}\) and \(v_{m}^{q} \to v^{q} in \ L^{1}(\Omega)\). Finally we infer that \(u\) is a renormalized solution of \(1.14\) by Theorem 3.3. \(\square\)

4 p-superharmonic functions and quasilinear equations in \(\mathbb{R}^{N}\)

We recall some definitions and properties of p-superharmonic functions (see e.g. \([13, 14, 15]\) for general properties and \([28]\) for a simple presentation).

Definition 4.1 A function \(u\) is said to be \(p\)-harmonic in \(\mathbb{R}^{N}\) if \(u \in W^{1,p}_{loc}(\mathbb{R}^{N})\) and \(-\Delta_{p} u = 0\) in \(\mathcal{D}'(\mathbb{R}^{N})\); it is always \(C^{1}\). A function \(u\) is called a \(p\)-super solution in \(\mathbb{R}^{N}\) if \(u \in W^{1,p}_{loc}(\mathbb{R}^{N})\) and \(-\Delta_{p} u \geq 0\) in \(\mathcal{D}'(\mathbb{R}^{N})\).

Definition 4.2 A lower semicontinuous (l.s.c) function \(u : \mathbb{R}^{N} \to (-\infty, \infty]\) is called \(p\)-superharmonic if \(u\) is not identically infinite and if, for all open \(D \subset \subset \mathbb{R}^{N}\) and all \(v \in C(\overline{D})\), \(p\)-harmonic in \(D\), \(v \leq u\) on \(\partial D\) implies \(v \leq u\) in \(D\).
Let $u$ be a $p$-superharmonic in $\mathbb{R}^N$. It is well known that $u \wedge k := \min\{u, k\} \in W^{1,p}_{lo}^N(\mathbb{R}^N)$ is a $p$-supersolution for all $k > 0$ and $u < \infty$ a.e in $\mathbb{R}^N$; thus, $u$ has a gradient (see the previous section). We also have $|\nabla u|^{p-1} \in L^1_{loc}(\mathbb{R}^N)$, $(-\Delta_p)^{r/2} u \in L^1_{loc}(\mathbb{R}^N)$ and $u \in L^p_{loc}(\mathbb{R}^N)$ for $1 \leq q < \frac{N}{N-1}$ and $r > 1$, $1 \leq s < \frac{N(p-1)}{N-p}$ (see [13, Theorem 7.46]). Thus for any $0 \leq \varphi \in C^1_c(\Omega)$, by the dominated convergence theorem,

$$
\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u. \nabla \varphi dx = \lim_{k \to \infty} \int_{\mathbb{R}^N} |\nabla (u \wedge k)|^{p-2} \nabla (u \wedge k). \nabla \varphi \geq 0.
$$

Hence, by the Riesz Representation Theorem, there is a nonnegative Radon measure denoted by $\mu[u]$, called the Riesz measure, such that $-\Delta_p u = \mu[u]$ in $\mathcal{D}'(\mathbb{R}^N)$.

The following weak convergence result for Riesz measures proved in [20] will be used to obtain the existence of $p$-superharmonic solutions to quasilinear equations.

**Proposition 4.3** Suppose that $\{u_n\}$ is a sequence of nonnegative $p$-superharmonic functions in $\mathbb{R}^N$ that converges a.e to a $p$-superharmonic function $u$. Then the sequence of measures $\{\mu[u_n]\}$ converges to $\mu[u]$ in the weak sense of measures.

The proof of the next result can be found in [20].

**Proposition 4.4** Let $\mu$ be a measure in $\mathcal{M}^+(\mathbb{R}^N)$. Suppose that $\mathcal{W}_{1,p}[\mu] < \infty$ a.e. Then there exists a nonnegative $p$-superharmonic function $u$ in $\mathbb{R}^N$ such that $-\Delta_p u = \mu$ in $\mathcal{D}'(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} u = 0$ and

$$
\frac{1}{c_{ss}} \mathcal{W}_{1,p}[\mu](x) \leq u(x) \leq c_{ss} \mathcal{W}_{1,p}[\mu](x),
$$

for almost all $x$ in $\mathbb{R}^N$, where the constant $c_{ss}$ is the one of Proposition 3.4. Furthermore any $p$-superharmonic function $u$ in $\mathbb{R}^N$, such that $\inf_{\mathbb{R}^N} u = 0$ satisfies (4.1) with $\mu = -\Delta_p u$.

**Proof of Theorem A.** The condition is necessary. Assume that (4.4) admits a nonnegative $p$-superharmonic functions $(u, v)$. By Proposition 4.3 there holds

$$
\begin{align*}
\quad & u(x) \geq c_{s7} \mathcal{W}_{1,p}[u^q] + \mu(x), \\
\quad & v(x) \geq c_{s7} \mathcal{W}_{1,p}[u^q] + \eta(x)
\end{align*}
$$

for almost all $x \in \Omega$.

Hence, we obtain (4.10) from Proposition 2.9(i).

The condition is sufficient. Let $\{(u_m, v_m)\}_{m \in \mathbb{N}}$ be a sequence of nonnegative $p$-superharmonic solutions of the following problems for $m \in \mathbb{N}$,

$$
\begin{align*}
-\Delta_p u_{m+1} &= u_{m+1}^q + \mu & \text{in } \mathbb{R}^N \\
-\Delta_p v_{m+1} &= u_{m+1}^q + \eta & \text{in } \mathbb{R}^N \\
\inf_{\mathbb{R}^N} u_{m+1} &= \inf_{\mathbb{R}^N} v_{m+1} = 0,
\end{align*}
\quad (4.2)
$$

with $(u_0, v_0) = (0, 0)$. As in the proof of Theorem C we can assume that $\{u_m\}$ and $\{v_m\}$ are nondecreasing. By Proposition 4.2 we have

$$
\begin{align*}
u_{m+1} &\leq c_{ss} \mathcal{W}_{1,p}[u_m^q] + \mu(x) \\
v_{m+1} &\leq c_{ss} \mathcal{W}_{1,p}[u_m^q] + \eta(x)
\end{align*}
$$

for all $x \in \Omega$.

Thus, by Proposition 2.7 there exists a constant $c > 0$ depending only on $N, p, q_1, q_2$ such that, if

$$
\omega(K) \leq M^* \Cap_{\mathcal{P}_{1+\frac{p(q_1+q_2)}{q_1(q_2+1)}}} \frac{q_1 q_2}{(q_2+1)(p-1)^2}(K)
\quad (4.3)
$$
for any compact set $K \subset \mathbb{R}^N$ with $d\omega = (W_{1,p}[\mu])^{q_2} \, dx + d\eta$, then there holds in $\Omega$,  
\[ v_m \leq c_{\omega_{n}} W_{1,p}[\omega], \quad u_m \leq c_{\omega_{n}} W_{1,p}[(W_{1,p}[\omega])^{q_1}] + c_{\omega_{n}} W_{1,p}[\mu] \quad \text{for all } m \geq 0, \] (4.4) 
and  
\[ W_{1,p}[\omega] \in L^{q_2}_{\text{loc}}(\mathbb{R}^N), \quad W_{1,p}[(W_{1,p}[\omega])^{q_1}] + W_{1,p}[\mu] \in L^{q_1}_{\text{loc}}(\mathbb{R}^N). \] (4.5) 
This implies that $\{u_n\}, \{v_m\}$ are well defined and nondecreasing. Thus $\{(u_m, v_m)\}$ converges a.e in $\mathbb{R}^N$ to some functions $(u, v)$ which satisfies (1.17) in $\mathbb{R}^N$. Furthermore, we infer from (3.6) and the monotone convergence theorem that $u^{q_2}_m \rightarrow u^{q_2}$, $v^{q_1}_m \rightarrow v^{q_1}$ in $L^{q_1}_{\text{loc}}(\mathbb{R}^N)$. By Proposition 4.3 we deduce that $(u, v)$ are nonnegative $p$-superharmonic solutions of (1.17). 

5 Hessian equations

In this section $\Omega \subset \mathbb{R}^N$ is either a bounded domain with a $C^2$ boundary or the whole $\mathbb{R}^N$. For $k = 1, \ldots, N$ and $u \in C^2(\Omega)$ the $k$-hessian operator $F_k$ is defined by  
\[ F_k[u] = S_k(\lambda(D^2 u)), \] 
where $\lambda(D^2 u) = \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivative $D^2 u$ and $S_k$ is the $k$-th elementary symmetric polynomial that is  
\[ S_k(\lambda) = \sum_{1 \leq i_1 < \ldots < i_k \leq N} \lambda_{i_1} \ldots \lambda_{i_k}. \] 
We can see that  
\[ F_k[u] = [D^2 u]_k, \] 
where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the $k$-th principal minors. We assume that $\partial \Omega$ is uniformly $(k-1)$-convex, that is  
\[ S_{k-1}(\kappa) \geq c_0 > 0 \quad \text{on} \quad \partial \Omega, \] 
for some positive constant $c_0$, where $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{n-1})$ denote the principal curvatures of $\partial \Omega$ with respect to its inner normal.

**Definition 5.1** An upper-semicontinuous function $u : \Omega \rightarrow [-\infty, \infty)$ is k-convex (k-subharmonic) if, for every open set $\Omega' \subset \overline{\Omega} \subset \Omega$ and for every function $v \in C^2(\Omega') \cap C(\overline{\Omega'})$ satisfying $F_k[v] \leq 0$ in $\Omega'$, the following implication is true  
\[ u \leq v \text{ on } \partial \Omega' \quad \implies \quad u \leq v \text{ in } \Omega'. \]

We denote by $\Phi^k(\Omega)$ the class of all $k$-subharmonic functions in $\Omega$ which are not identically equal to $-\infty$.

The following weak convergence result for $k$-Hessian operators proved in [24] is fundamental in our study:

**Proposition 5.2** Let $\Omega$ be either a bounded uniformly $(k-1)$-convex in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$. For each $u \in \Phi^k(\Omega)$, there exists a nonnegative Radon measure $\mu_k[u]$ in $\Omega$ such that  
1. $\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$.
2. If $\{u_n\}$ is a sequence of $k$-convex functions which converges a.e to $u$, then $\mu_k[u_n] \rightharpoonup \mu_k[u]$ in the weak sense of measures.
As in the case of quasilinear equations with measure data, precise estimates of solutions of $k$-Hessian equations with measures data are expressed in terms of Wolff potentials. The next results are proved in [24, 17, 20].

**Theorem 5.3** Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^2$, uniformly $(k-1)$-convex domain. Let $\mu$ be a nonnegative Radon measure in $\Omega$ which can be decomposed under the form

$$\mu = \mu_1 + f,$$

where $\mu_1$ is a measure with compact support in $\Omega$ and $f \in L^q(\Omega)$ for some $q > \frac{N}{k}$ if $k \leq \frac{N}{2}$, or $p = 1$ if $k > \frac{N}{2}$. Then there exists a nonnegative function $u$ in $\Omega$, continuous near $\partial \Omega$, such that $-u \in \Phi^k(\Omega)$ and $u$ is a solution of the problem

$$F_k[-u] = \mu \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega.$$

Furthermore, any nonnegative function $u$ such that $-u \in \Phi^k(\Omega)$ which is continuous near $\partial \Omega$ and is a solution of above equation, satisfies

$$\frac{1}{c_{ss}} W_{\frac{d(x,\Omega)}{N-k+1},k+1}[\mu] \leq u(x) \leq c_{ss} W_{\frac{d(x,\Omega)}{N-k+1},k+1}[\mu](x), \quad (5.1)$$

where $c_{ss}$ is a positive constant independent of $x, u$ and $\Omega$.

**Theorem 5.4** Let $\mu$ be a measure in $\mathfrak{M}^+(\mathbb{R}^N)$ and $2k < N$. Suppose that $W_{\frac{d(x,\Omega)}{N-k+1},k+1}[\mu] < \infty$ a.e. Then there exists $u, -u \in \Phi^k(\mathbb{R}^N)$ such that $\inf_{\mathbb{R}^N} u = 0$ and $F_k[-u] = \mu$ in $\mathbb{R}^N$ and

$$\frac{1}{c_{ss}} W_{\frac{d(x,\Omega)}{N-k+1},k+1}[\mu] \leq u(x) \leq c_{ss} W_{\frac{d(x,\Omega)}{N-k+1},k+1}[\mu](x), \quad (5.2)$$

for all $x$ in $\mathbb{R}^N$. Furthermore, if $u$ is a nonnegative function such that $\inf_{\mathbb{R}^N} u = 0$ and $-u \in \Phi^k(\mathbb{R}^N)$, then $(5.2)$ holds with $\mu = F_k[-u]$.

**Proof of Theorem E.** The condition is necessary. Assume that $(1.21)$ admits a nonnegative solution $(u, v)$, continuous near $\partial \Omega$, such that $-u, -v \in \Phi^k(\Omega)$ and $u^{s_2}, v^{s_1} \in L^1(\Omega)$. Then by Theorem 5.3 we have

$$u(x) \geq \frac{1}{c_{ss}} W_{\frac{d(x,\Omega)}{N-k+1},k+1}[v^{s_1} + \mu](x)$$

$$v(x) \geq \frac{1}{c_{ss}} W_{\frac{d(x,\Omega)}{N-k+1},k+1}[u^{s_2} + \eta](x) \quad \text{for almost all } x \in \Omega.$$

Using the part 2 of Proposition 2.11 we conclude that $(1.22)$ holds.

The condition is sufficient. We define a sequence of nonnegative functions $u_m, v_m$, continuous near $\partial \Omega$ and such that $-u_m, -v_m \in \Phi^k(\Omega)$, by the following iterative scheme for $m \geq 0$,

$$F_k[-u_{m+1}] = v^{s_1}_m + \mu \quad \text{in } \Omega,$$

$$F_k[-v_{m+1}] = u^{s_2}_m + \eta \quad \text{in } \Omega,$$

$$u_{m+1} = v_{m+1} = 0 \quad \text{on } \partial \Omega. \quad (5.3)$$

Clearly, we can assume that $\{u_m\}$ is nondecreasing as in [24]. By Theorem 5.3 we have

$$u_{m+1} \leq c_{ss} W_{\frac{d(x,\Omega)}{N-k+1},k+1}[v^{s_1}_m + \mu], \quad v_{m+1} \leq c_{ss} W_{\frac{d(x,\Omega)}{N-k+1},k+1}[u^{s_2}_m + \mu] \quad \text{in } \Omega, \quad (5.4)$$

where $R = 2 \text{diam } (\Omega)$.

Then, by Proposition 2.8 there exists a constant $M_* > 0$ depending only on $N, q_1, q_2, R$ such that if

$$\omega(K) \leq M_* \text{Cap}_{G_{\frac{2(k+1)}{N-k+1},k+1}}(K)$$

then $\omega(K) \leq \frac{M_*}{C_{ss}} W_{\frac{d(x,\Omega)}{N-k+1},k+1}[\mu](x)$.
for any compact set $K \subset \mathbb{R}^N$ with $d\omega = \left( W_{\frac{2k}{r+1}, k+1}[\mu] \right)^{s_2} dx + d\eta$, then there holds,

$$v_m \leq c_{m_{\omega}} W_{\frac{2k}{r+1}, k+1}[\omega], \quad u_m \leq c_{m_{\omega}} W_{\frac{2k}{r+1}, k+1}\left( \left( W_{\frac{2k}{r+1}, k+1}[\omega] \right)^{s_1} \right) + c_{m_{\omega}} W_{\frac{2k}{r+1}, k+1}[\mu]$$

in $\Omega$, for all $m \in \mathbb{N}$, for some positive constants $c_{m_{\omega}}$, $c_{m}$ depending only on $N, k, s_1, s_2, R$.

Note that we can write

$$u_{s_1} = \mu + \chi_{\Omega_{s_1}} u_{s_1} + \left( (1 - \chi_{\Omega_{s_1}}) v_{s_1} - f \right),$$

and

$$u_{s_2} + \eta = \left( \eta_{1} + \chi_{\Omega_{s_2}} u_{s_2} + \left( (1 - \chi_{\Omega_{s_2}}) v_{s_2} + g \right) \right),$$

where $\Omega_{s_1} = \left\{ x \in \Omega : d(x, \partial \Omega) > \delta \right\}$ and $\delta > 0$ is small enough and since $u_m$ is continuous near $\partial \Omega$, then $v_{m_{s_1}} + \mu$, $u_{m_{s_2}} + \eta$ satisfy the assumptions of the data in Theorem 5.3. Therefore the sequence $\{u_m\}$ is well defined and nondecreasing. Thus, $\{u_m\}$ converges a.e in $\Omega$ to some function $u$ which satisfies (1.22) in $\Omega$. Furthermore, by the monotone convergence theorem there holds $v_{m_{s_1}} \to v, u_{m_{s_2}} \to u$ in $L^1(\Omega)$. Finally, by Proposition 5.2 we infer that (1.21) admits a nonnegative solutions $u, v$, continuous near $\partial \Omega$, with $-u, -v \in \Phi^k(\Omega)$ satisfying (1.24). □

**Proof of Theorem F** The condition is necessary. Assume that (1.24) admits nonnegative solution $(u, v)$, such that $-u, -v \in \Phi^k(\mathbb{R}^N)$ and $u_{s_2}, v_{s_1} \in L^1_{loc}(\mathbb{R}^N)$. Then by Theorem 5.3 we have

$$u(x) \geq 1 W_{\frac{2k}{r+1}, k+1}[v_{s_1} + \mu](x)$$

$$v(x) \geq 1 W_{\frac{2k}{r+1}, k+1}[u_{s_2} + \eta](x) \quad \text{for almost all } x \in \mathbb{R}^N.$$

Using Proposition 2.8(ii), we conclude that (1.22) holds.

The condition is sufficient. We defined a sequence of nonnegative functions $u_m, v_m$, continuous near $\partial \Omega$ and such that $-u_m, -v_m \in \Phi^k(\Omega)$, by the following iterative scheme for $m \geq 0$,

$$F_k[-u_{m+1}] = v_{s_1} + \mu \quad \text{in } \mathbb{R}^N,$$

$$F_k[-v_{m+1}] = u_{s_2} + \eta \quad \text{in } \mathbb{R}^N,$$

$$\inf_{\mathbb{R}^N} u_{m+1} = \inf_{\mathbb{R}^N} v_{m+1} = 0.$$

As in the previous proofs $\{u_m\}$ is nondecreasing. By Theorem 5.3 we have

$$u_{m+1} \leq c_{m_{n}} W_{\frac{2k}{r+1}, k+1}[v_{s_1} + \mu]$$

$$v_{m+1} \leq c_{m_{n}} W_{\frac{2k}{r+1}, k+1}[u_{s_2} + \mu] \quad \text{a.e. in } \mathbb{R}^N.$$

Then, by Proposition 2.7, there exists a constant $M^* > 0$ depending only on $N, p, q_1, q_2, R$ such that if

$$\omega(K) \leq M^* \text{Cap}_1 \left( 1^{\frac{k(q_1+k)}{q_1}} \right)^{\frac{k}{2}} \left( K \right)$$

for any compact set $K \subset \mathbb{R}^N$ with $d\omega = \left( W_{\frac{2k}{r+1}, k+1}[\mu] \right)^{s_2} dx + d\eta$, then

$$v_m \leq c_{m_{\omega}} W_{\frac{2k}{r+1}, k+1}[\omega], \quad u_m \leq c_{m_{\omega}} W_{\frac{2k}{r+1}, k+1}\left( \left( W_{\frac{2k}{r+1}, k+1}[\omega] \right)^{s_1} \right) + c_{m_{\omega}} W_{\frac{2k}{r+1}, k+1}[\mu]$$

in $\Omega$, for all $m \in \mathbb{N}$, where $c_{m_{\omega}}, c_{m}$ and $c_{m}$ depend on $N, k, s_1, s_2, R$. Therefore the sequence $\{u_m\}$ is well defined and nondecreasing. Thus, $\{u_m\}$ converges a.e in $\Omega$ to some function $u$ for which (1.24) is satisfied in $\mathbb{R}^N$. Furthermore, by the monotone convergence theorem we have $v_{m_{s_1}} \to v, u_{m_{s_2}} \to u$ in $L^1_{loc}(\mathbb{R}^N)$. Finally, by Proposition 5.2 we obtain that (1.21) admits a nonnegative solutions $u, v$ with $-u, -v \in \Phi^k(\mathbb{R}^N)$ satisfying (1.24). □
6 Further results

The method exposed in the previous sections, can be applied to types of problems. We give below an example for a semilinear system in $\mathbb{R}^N_+ = \{x = (x', x_N), x' \in \mathbb{R}^{N-1}, x_N > 0\}$.

\begin{equation}
\begin{align*}
-\Delta u &= v^{q_1} & \text{in } \mathbb{R}^N_+ \\
-\Delta v &= u^{q_2} & \text{in } \mathbb{R}^N_+ \\
\quad \text{on } \partial \mathbb{R}^N_+ \cap \mathbb{R}^{N-1}, \\
\quad \text{where we have identified } \partial \mathbb{R}^N_+ \text{ and } \mathbb{R}^{N-1}. \text{ We denote by } P \text{ (resp. } G) \text{ the Poisson kernel in } \mathbb{R}^N_+ \text{ (resp the Green kernel in } \mathbb{R}^N). \text{ The Poisson potential and the Green potential, } P[\cdot] \text{ and } G[\cdot], \text{ associated to } -\Delta \text{ are defined respectively by}
\end{align*}
\end{equation}

\begin{equation}
P[\sigma](y) = \int_{\partial \mathbb{R}^N_+^+} P(y, z) d\sigma(z), \quad G[f](y) = \int_{\partial \mathbb{R}^N_+^+} G(y, x) f(x) dx,
\end{equation}

see [18]. We set $\rho(x) = x_N$ and define the capacity $\text{Cap}_{\alpha, s}^\rho(K)$ by

\begin{equation}
\text{Cap}_{\alpha, s}^\rho(K) = \inf \left\{ \int_{\mathbb{R}^N_+} f^\alpha \rho(x) dx : f \geq 0, L_\alpha[\rho] \leq K \right\},
\end{equation}

for all Borel set $K \subset \mathbb{R}^N$, where $L_\alpha$ is the Riesz kernel of order $\alpha$ in $\mathbb{R}^N$.

**Theorem 6.1** Let $1 \leq q_1 < \frac{N}{N-1}, q_1 q_2 > 1$. If there exists a constant $\tilde{c} > 0$ such that if

\begin{equation}
\begin{align*}
(i) \quad \int_K \rho(x) (P[\sigma_1])^{q_2} dx &\leq \tilde{c} \text{Cap}_{\frac{q_1+1}{q_1 q_2}, \frac{q_1+1}{q_2}}(K), \\
(ii) \quad \sigma_2(G) &\leq \tilde{c} \text{Cap}_{\frac{q_1+1}{q_1 q_2}, \frac{q_1+1}{q_2}}(G),
\end{align*}
\end{equation}

for all Borel sets $K \subset \mathbb{R}^N_+ \text{ and } G \subset \mathbb{R}^{N-1}$, then the problem (6.1) admits a solution.

All solutions in above theorem are understood in the usual very weak sense: $u \in L_1^1(\mathbb{R}^N \cap B), u^{q_1}, v^{q_2} \in L^1_{\text{loc}}(\mathbb{R}^N \cap B)$ for any ball $B$ and

\begin{equation}
\begin{align*}
\int_{\mathbb{R}^N_+} u(-\Delta \xi) dx &= \int_{\mathbb{R}^N_+} v^{q_1} \xi dx - \int_{\partial \mathbb{R}^N_+^+} \frac{\partial \xi}{\partial n} d\sigma_1, \\
\int_{\mathbb{R}^N_+} v(-\Delta \xi) dx &= \int_{\mathbb{R}^N_+} u^{q_2} \xi dx - \int_{\partial \mathbb{R}^N_+^+} \frac{\partial \xi}{\partial n} d\sigma_2,
\end{align*}
\end{equation}

for any $\xi \in C^2(\mathbb{R}^N_+) \cap C_0(\mathbb{R}^N)$ with $\xi = 0$ on $\partial \mathbb{R}^N_+$. It is well-known that such a solution $u$ satisfies

\begin{equation}
\begin{align*}
u = G[u] + P[\sigma_1], \quad v = G[u] + P[\sigma_2] \quad \text{a.e. in } \mathbb{R}^N_.
\end{align*}
\end{equation}

To prove Theorem 6.1 we need the following basic estimate,

**Lemma 6.2** Assume that $0 < q_1 < \frac{N}{N-1}$. Then for any $\omega \in \mathfrak{M}_0(\mathbb{R}^N)$,

\begin{equation}
\int_\mathbb{R}^N [(I[\omega])^{q_1}] \leq c_{q_1} W_{\frac{q_1+1}{q_1}, \frac{q_1+1}{q_1}} [\omega] \quad \text{a.e. in } \mathbb{R}^N,
\end{equation}

where $c_{q_1} > 0$ depends on $q_1, q_2$ and $N$.

**Proof.** The proof of Lemma 6.2 is similar to the one of Lemma 2.2 and details are omitted. Note that if $\omega \in \mathfrak{M}_0(\mathbb{R}^N)$ it is extended by 0 in $\mathbb{R}^N_.$
Remark 6.3 The condition \(0 < q_1 < \frac{N}{1-q_2} \) is a necessary and sufficient condition in order \((I_1[\nu])^{\lambda_1} \) be locally integrable in \(\mathbb{R}^N \) for any \(\omega \in \mathfrak{M}_b^+(\mathbb{R}^N)\).

Theorem 6.4 Let \(q_1 \geq 1, q_1 q_2 > 1\) and \(\omega \in \mathfrak{M}_b(\mathbb{R}^N)\). If

\[
\omega(K) \leq c_{\infty} \text{Cap}^{q_1+2}_{q_1+q_2} (K) \quad \forall K \subset \mathbb{R}^N, \ K \text{ Borel},
\]

for some \(c_{\infty} > 0\), then

\[
I_1 \left( \left( \frac{W_{q_1+2, q_1}}{q_1} \right)^{\frac{q_2}{q_1}} \rho \chi_{\omega} \right) \leq c_{\infty} I_1[\omega] \quad \text{a.e. in } \mathbb{R}^N. \tag{6.4}
\]

**Proof.** Step 1. For any compact \(K \subset \left\{ x \in \mathbb{R}^N : I_{q_1+2} [f \rho \chi_{\omega}] (x) > \lambda \right\} \), we have

\[
\omega(K) \leq c_{\infty} \text{Cap}^{q_1+2}_{q_1+q_2} (K) \leq c_{\infty} \lambda^{-q_1-2\epsilon} \int_{\mathbb{R}^N} f_{q_1+2} \rho dx
\]

by assumption and the definition of the capacity. Hence,

\[
\lambda^{q_1+2\epsilon} \omega \left( \left\{ I_{q_1+2} [f \rho \chi_{\omega}] > \lambda \right\} \right) \leq c_{\infty} \int_{\mathbb{R}^N} f_{q_1+2} \rho dx \quad \forall \lambda > 0.
\]

This implies an estimate in Lorentz space,

\[
\left\| I_{q_1+2} [f \rho \chi_{\omega}] \right\|_{L^{q_1+2, \infty} (\mathbb{R}^N, d\omega)} \leq \left\| f \right\|_{L^{q_1+2, \infty} (\mathbb{R}^N, \rho dx)} \quad \forall f \geq 0. \tag{6.5}
\]

Step 2. Since, for any \(g \in C_c(\mathbb{R}^N)\),

\[
\int_{\mathbb{R}^N} I_{q_1+2} [g \omega] f \rho dx = \int_{\mathbb{R}^N} I_{q_1+2} [f \rho \chi_{\omega}] g d\omega,
\]

we infer, using duality between \(L^{p,1}\) and \(L^{p',\infty}\), Holder’s inequality therein and \((6.5)\), that

\[
\int_{\mathbb{R}^N} I_{q_1+2} [g \omega] f \rho dx \leq \left\| I_{q_1+2} [f \rho \chi_{\omega}] \right\|_{L^{q_1+2, \infty} (\mathbb{R}^N, d\omega)} \left\| g \right\|_{L^{q_1+2, 1} (\mathbb{R}^N, d\omega)} \quad \forall f, g \geq 0.
\]

Therefore,

\[
\left\| I_{q_1+2} [g \omega] \right\|_{L^{q_1+2} (\mathbb{R}^N, \rho dx)} \leq \left\| g \right\|_{L^{q_1+2, 1} (\mathbb{R}^N, d\omega)}. \tag{6.6}
\]

Step 3. Taking \(g = \chi_{B_{\rho}(x)}\) and since for \(q_1 \geq 1\)

\[
W_{q_1+2, q_1} [\nu] (x) = \int_0^{\infty} \left( \frac{\mu(B_{\rho}(x))}{\rho^{N-q_1-2\epsilon}} \right) dx
\]

\[
\leq c_{\infty} \left( \int_0^{\infty} \frac{\mu(B_{\rho}(x))}{\rho^{N-q_1-2\epsilon}} dx \right) \quad \forall \nu \in \mathfrak{M}_b^+(\mathbb{R}^N), \ \forall x \in \mathbb{R}^N,
\]

\[
= c_{\infty} \left( I_{q_1+2} [\nu] (x) \right) \quad \forall \nu \in \mathfrak{M}_b^+(\mathbb{R}^N), \ \forall x \in \mathbb{R}^N.
\]
we deduce that for almost all $x \in \mathbb{R}^N_+$,
\[
\int_{\mathbb{R}^N_+} \left( \mathbf{W}_{\frac{n+2}{n_1}, \frac{1}{n_1}} [\chi_{B_t(x)} w] \right)^{q_2} \rho \, dy \leq c_m \omega(B_t(x)),
\]
from (6.6), which implies
\[
\omega(B_t(x)) \leq c_{a_1} \frac{\mu_t^{(N-\frac{n+2}{n_1})} \chi_{B_t(x)}}{\int_{B_2(x)} \chi_{\mathbb{R}^N_+} \rho \, dy} \leq c_{a_2} \frac{\mu_t^{(N-\frac{n+2}{n_1})} \chi_{\mathbb{R}^N_+} \rho \, dy}{(\max\{x_n, t\})^{\frac{n+2}{n_1}}},
\] (6.7)
since $\int_{B_t(x)} \chi_{\mathbb{R}^N_+} \rho \, dy \approx r^N \max\{x_N, r\}$ for any $x \in \mathbb{R}^N_+, r > 0$ where the symbol $\approx$ is defined by
\[
A \approx B \iff \frac{1}{c} B \leq A \leq c B \quad \text{for some constant } c > 0.
\]
It implies also
\[
\int_{B_t(x)} \left( \mathbf{W}_{\frac{n+2}{n_1}, \frac{1}{n_1}} [\omega] \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dy \leq c_m \omega(B_t(x)),
\] (6.8)
from which follows
\[
\int_0^\infty \frac{1}{t^{N-1}} \int_{B_t(x)} \left( \mathbf{W}_{\frac{n+2}{n_1}, \frac{1}{n_1}} [\omega] \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dy \, dt \leq c_m I_1[\omega](x).
\]
Therefore, if the following inequality holds
\[
\int_0^\infty \frac{1}{t^{N-1}} \int_{B_t(x)} \left( \mathbf{W}_{\frac{n+2}{n_1}, \frac{1}{n_1}} [\omega] \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dy \, dt \leq c_m I_1[\omega](x),
\] (6.9)
it will imply (6.4).

**Step 4.** We claim that (6.9) holds. Since $B_r(y) \in B_{2r}(x)$, $y \in B_t(x), r \geq t$,
\[
\int_0^\infty \frac{1}{t^{N-1}} \int_{B_t(x)} \left( \mathbf{W}_{\frac{n+2}{n_1}, \frac{1}{n_1}} [\omega(B_t(y))] \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dy \, dt
\]
\[
\leq \int_0^\infty \frac{1}{t^{N-1}} \int_{B_t(x)} \chi_{\mathbb{R}^N_+} \rho \, dy \left( \int_t^\infty \frac{\omega(B_{2r}(x))}{r^{N-\frac{n+2}{n_1}}} \frac{dr}{r} \right)^{q_2} \, dt
\]
\[
\approx \int_0^\infty \max\{x_n, t\} \left( \int_t^\infty \frac{\omega(B_{2r}(x))}{r^{N-\frac{n+2}{n_1}}} \frac{dr}{r} \right)^{q_2} \, dt.
\]
By integration by part,
\[
\int_0^\infty \frac{1}{t^{N-1}} \int_{B_t(x)} \left( \mathbf{W}_{\frac{n+2}{n_1}, \frac{1}{n_1}} [\omega(B_t(y))] \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dy \, dt
\]
\[
= \int_0^\infty \int_0^t \max\{x_n, s\} \, ds \left( \int_t^\infty \frac{\omega(B_{2r}(x))}{r^{N-\frac{n+2}{n_1}}} \frac{dr}{r} \right)^{q_2-1} \left( \frac{\omega(B_{2r}(x))}{t^{N-\frac{n+2}{n_1}}} \right)^{q_1} \, dt
\]
\[
= \int_0^\infty \int_0^t \max\{x_n, s\} \, ds \left( \int_t^\infty \frac{\omega(B_{2r}(x))}{r^{N-\frac{n+2}{n_1}}} \frac{dr}{r} \right)^{q_2-1} \left( \frac{\omega(B_{2r}(x))}{t^{N-\frac{n+2}{n_1}}} \right)^{q_1} \, dt
\]
We have
\[
\int_0^t \max\{x_N, s\} ds \geq t \max\{x_N, t\},
\]
and we obtain (6.9).

Lemma 6.5 Let \( \alpha > 0, s > 1 \) such that \( \alpha + \frac{2}{s} < N - 1 \) where \( s' = \frac{s}{s-1} \). For all \( \eta \in \mathcal{M}^+(\mathbb{R}^{N-1}) \), there holds
\[
\int_{\mathbb{R}^N} (I_\alpha[\eta \otimes \delta_{\{x_N=0\}}])^{s'} x_N dx \leq \int_{\mathbb{R}^{N-1}} \left( \int_0^\infty \frac{\eta(B'_r(x')) dt}{t^{N-1-\alpha - \frac{2}{s'}}} \right)^{s'} dx',
\]
where \( I_\beta \) is the Riesz potential of order \( \beta \) in \( \mathbb{R}^{N-1} \). As a consequence, we have
\[
\text{Cap}_{\alpha,s}(E \times \{x_N = 0\}) \approx \text{Cap}_{\alpha+2/s'-1,s}(E) \quad \forall E \subset \mathbb{R}^{N-1}, \text{ E Borel.}
\]

Proof. We have
\[
\int_{\mathbb{R}^N} (I_\alpha[\eta \otimes \delta_{\{x_N=0\}}])^{s'} x_N dx \geq \int_{\mathbb{R}^N} \left( \int_{2x_N}^{4x_N} \frac{(\eta \otimes \delta_{\{x_N=0\}})(B_r(x)) dr}{r^{N-\alpha}} \right)^{s'} x_N dx
\geq c_\eta \int_{\mathbb{R}^N} \left( \frac{\eta(B'_{x_N}(x'))}{x_N^{-\alpha}} \right)^{s'} x_N dx
\geq c_\eta \int_{\mathbb{R}^{N-1}} \left( \sup_{t > 0} \frac{\eta(B'_{t}(x'))}{t^{N-1-\alpha - \frac{2}{s'}}} \right) dx'.
\]
By using Lemma 2.1 we obtain
\[
\int_{\mathbb{R}^N} (I_\alpha[\eta \otimes \delta_{\{x_N=0\}}])^{s'} x_N dx \leq \int_{\mathbb{R}^N} \left( \int_{x_N}^{\infty} \frac{\eta(B'_r(x')) dr}{r^{N-\alpha}} \right)^{s'} dx_N dx'
\leq c_\eta \int_{\mathbb{R}^{N-1}} \int_{0}^{\infty} \left( \frac{\eta(B'_r(x'))}{r^{N-1-\alpha - \frac{2}{s'}}} \right)^{s'} \frac{dt}{t} dx'.
\]
On the other hand, by [20, Proposition 5.1], there holds

\[
\int_{\mathbb{R}^{N-1}} \left( \sup_{t>0} \eta(B_t'(x')) \right) dx' \sim \int_{\mathbb{R}^{N-1}} \int_0^\infty \frac{\eta(B_t'(x'))}{t^{N-1-\frac{\alpha}{2}}} \frac{dt}{t} dx' \\
\lesssim \int_{\mathbb{R}^{N-1}} \left( \int_0^\infty \frac{\eta(B_t'(x'))}{t^{N-1-\frac{\alpha}{2}}} \frac{dt}{t} \right)^{\frac{\alpha}{2}} dx'.
\]

(6.14)

Combining (6.12), (6.13) and (6.14) we obtain (6.10). Moreover, we deduce (6.11) from (6.10) and [1, Theorem 2.5.1], which ends the proof. □

**Proof of Theorem 6.1**

The following estimates are classical

\[
G(x, y) \asymp \frac{x_N y_N}{|x - y|^{N-2}} \max\{|x - y|, x_N, y_N\}^2 \leq c_{100} \frac{y_N}{|x - y|^{N-1}},
\]

(6.15)

\[
P(x, z) = c_{101} \frac{x_N}{|x - z|^{N-1}} \leq c_{101} \frac{1}{|x - z|^{N-1}}.
\]

(6.16)

Thus,

\[
G ([P[\sigma_1])^{q_1} + P[\sigma_2] \leq c_{102} I_1[\omega],
\]

(6.17)

where \(\omega(x) = \rho(P[\sigma_1])^{q_2} + \sigma_2\) in \(\mathbb{R}^N\). Therefore, we infer that if

\[
I_1 \left( I_2 [I_1[\omega])^{q_2} \frac{X_{\mathbb{R}^N}}{r} \right) \leq c_{103} I_1[\omega] \text{ in } \mathbb{R}^N
\]

(6.18)

for some \(c_{103} > 0\) small enough, then (6.19) admits a positive solution \((u, v)\). On the other hand, we deduce (6.18) from Lemma 6.2 and Theorem 6.4. The proof is complete. □

**Remark 6.6**

The system

\[
- \Delta u = v^{q_1} + \epsilon_1 \mu \quad \text{in } \Omega \\
- \Delta v = u^{q_2} + \epsilon_2 \eta \quad \text{in } \Omega \\
u = \epsilon_3 \sigma_1, \ v = \epsilon_4 \sigma_2 \quad \text{in } \partial \Omega,
\]

(6.19)

where \(d(.)\mu, d(.)\lambda\) belong to \(\mathcal{M}^+\) (\(\Omega\)), \(\sigma_1, \sigma_2\) to \(\mathcal{M}^+(\partial \Omega)\) and the \(\epsilon_j\) are positive numbers, is analyzed in [10, Th 4.6]. Therein it is proved that if

\[
\int_\Omega (G[\mu] + P[\lambda])^\max(q_1, q_2) \ d(x) dx < \infty,
\]

(6.20)

which is equivalent to a capacitary estimate, and

\[
\min \left\{ q_1 \frac{q_1 + 1}{q_2 + 1}, q_2 \frac{q_2 + 1}{q_1 + 1} \right\} < \frac{N + 1}{N - 1}
\]

(6.21)

and if the \(\epsilon_j\) are small enough, then (6.19) admits a positive solution. Now condition (6.21) is a subcriticality assumption (for at least one of the two exponents \(q_j\)) since there is no condition on the boundary measures.

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