Entanglement and second quantization in the framework of the fermionic projector

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Abstract

A method is developed for realizing entangled states and general second-quantized fermionic and bosonic fields in the framework of the fermionic projector.

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1. Introduction

In [3, 9], it was proposed to formulate physics based on a new action principle in spacetime. One difference of this approach to standard quantum field theory is that a many-particle state no longer corresponds to a vector in the fermionic Fock space, but instead it is described by the so-called fermionic projector, an operator which acts on the one-particle Hilbert space (or more generally on an indefinite inner product space spanned by the one-particle wavefunctions). Another difference is that the bosonic fields obtained in the so-called continuum limit are only classical. Due to these differences, it is not at all obvious whether the fermionic projector can account for all quantum effects observed in nature. More specifically, is it possible to describe entanglement? Can one reproduce the effects of second-quantized fields?

In this paper, we analyze these questions in detail. We show that it is indeed possible to describe entangled states as well as general second-quantized bosonic and fermionic fields in the framework of the fermionic projector. This is achieved by introducing the physical concept of a microscopic mixing of decoherent subsystems. The physical picture is that spacetime is not smooth on the microscopic scale (typically thought of as the Planck scale), but has a non-trivial microstructure. Homogenizing this microstructure, we obtain an effective description of the system by a vector in the Fock space. To make this picture precise, we use the fact that in the framework of the fermionic projector, the usual topological and causal structure of Minkowski space is not a priori given, but it is induced on the spacetime points by the states of the fermionic projector. Thus, by bringing the wavefunctions between certain pairs of spacetime points ‘out of phase’, we obtain decoherence effects which result in a
decomposition of the whole system into subsystems between which the usual causal relations are no longer valid. This makes it possible to realize many independent physical systems simultaneously in one spacetime, in such a way that homogenizing on the microscopic scale leads to an effective ‘superposition’ of the subsystems. For technical simplicity, we describe the microscopic mixing by localizing the subsystems in disjoint spacetime regions (see figure 2). But one can also think of the subsystems as being delocalized, similar as if one combines several images in a single hologram (see section 5.3).

In order to make the paper self-contained and easily accessible, we begin in section 2 with a brief outline of the fermionic projector approach. In section 3, we recall the basics on entanglement and Fock spaces and work out the connection to projectors in the one-particle Hilbert space. In section 4 we describe entanglement using the concept of microscopic mixing and by introducing a decoherence between spacetime regions. As is worked out in section 5, this notion of decoherence also makes it possible to describe second-quantized bosonic fields.

2. An outline of the fermionic projector approach

In this section we give a brief introduction to the framework of the fermionic projector, outlining a few ideas, methods and results. For details we refer to [3, 6, 8] or to the review papers [9, 10].

2.1. An action principle for fermion systems in discrete spacetime

In the fermionic projector approach, the physical equations are formulated intrinsically in a discrete spacetime. To introduce the basic framework, we let \((H, \langle \cdot | \cdot \rangle)\) be a finite-dimensional complex vector space endowed with an indefinite inner product (thus \(\langle \cdot | \cdot \rangle\) is non-degenerate, but not positive definite). Next, we let \(M = \{1, \ldots, m\}\) be a finite set. With every point \(x \in M\) we associate a projector \(E_x\) (a projector in \(H\) is defined just as in Hilbert spaces as a linear operator which is idempotent and self-adjoint). We assume that these projectors are orthogonal and complete in the sense that

\[
E_x E_y = \delta_{xy} E_x \quad \text{and} \quad \sum_{x \in M} E_x = 1.
\]  

Furthermore, we assume that the images \(E_x(H) \subset H\) of these projectors are non-degenerate subspaces of \(H\), which all have the same signature \((2, 2)\). The points \(x \in M\) are called discrete spacetime points and the corresponding projectors \(E_x\) are the spacetime projectors. The structure \((H, \langle \cdot | \cdot \rangle, (E_x)_{x \in M})\) is called discrete spacetime. The particles of our system are described by one more projector \(P\) in \(H\), the so-called fermionic projector, which has the property that its image \(P(H)\) is a negative definite subspace of \(H\). The resulting system \((H, \langle \cdot | \cdot \rangle, (E_x)_{x \in M}, P)\) is referred to as a fermion system in discrete spacetime.

Let us briefly discuss these definitions and introduce a convenient notation. The vectors in the image of \(P\) have the interpretation as the occupied fermionic states of our system, and thus the rank of \(P\) gives the number of particles \(f := \dim P(H)\). The spacetime projectors \(E_x\) can be used to project vectors of \(H\) to the subspace \(E_x(H) \subset H\). Using a more graphic notion, we also refer to this projection as the localization at the spacetime point \(x\). For the localization of a vector \(\psi \in \mathcal{H}\) we use the short notation

\[
\psi(x) := E_x \psi,
\]

and refer to \(\psi(x)\) as the corresponding wavefunction. Having the connection to relativistic quantum mechanics in mind (see section 2.3 below), we refer to \(E_x(H)\) as the spinor space at
x and denote it by $S_x$. It is endowed with the inner product $\langle \cdot | \cdot \rangle$ of signature $(2, 2)$, which we also denote by $\prec \cdot | \cdot \succ$. Using relations (2.1), we can then write

$$
\langle \psi | \phi \rangle = \sum_{x \in M} \prec \psi(x) | \phi(x) \succ.
$$

(2.3)

The localization of the fermionic projector is denoted by $P(x, y) := E_x P E_y$. This operator maps $S_y \subset H$ to $S_x$, and we usually regard it as a mapping between these subspaces:

$$
P(x, y) = E_x P E_y : S_y \rightarrow S_x.
$$

Again using (2.1), we can write the wavefunction corresponding to $P \psi$ as follows:

$$
(P \psi)(x) = E_x P \psi = \sum_{y \in M} E_x P E_y \psi = \sum_{y \in M} (E_x P E_y) (E_y \psi),
$$

and thus

$$
(P \psi)(x) = \sum_{y \in M} P(x, y) \psi(y).
$$

(2.4)

This relation resembles the representation of an operator with an integral kernel, and therefore we call $P(x, y)$ the discrete kernel of the fermionic projector. Finally, it is often useful to choose an orthonormal basis $\psi_1, \ldots, \psi_f$ of $P(H)$ (i.e. $\langle \psi_i | \psi_j \rangle = -\delta_{ij}$). Then the fermionic projector and its discrete kernel can be written in bra/ket notation as

$$
P = -\sum_{j=1}^f |\psi_j \rangle \langle \psi_j| \quad \text{and} \quad P(x, y) = -\sum_{j=1}^f |\psi_j(x) \succ \prec \psi_j(y) |.
$$

(2.5)

In order to introduce our action principle, for any $x, y \in M$, we define the closed chain $A_{xy}$ by

$$
A_{xy} = P(x, y) P(y, x) : S_y \rightarrow S_x.
$$

(2.6)

We denote the eigenvalues of $A_{xy}$ counted with algebraic multiplicities by $\lambda_1, \ldots, \lambda_4$ and define the spectral weight $|A_{xy}|$ by

$$
|A_{xy}| = \sum_{j=1}^4 |\lambda_j|.
$$

Similarly, one can take the spectral weight of powers of $A_{xy}$. Our action principle is to minimize the action

$$
S[P] := \sum_{x, y \in M} |A_{xy}^2|
$$

under the constraint

$$
T[P] := \sum_{x, y \in M} |A_{xy}|^2 = \text{const},
$$

(2.7)

where we consider variations of the fermionic projector, keeping the number of particles as well as discrete spacetime fixed. In [6] it is shown that the minimizers of this nonlinear variational principle exist. For a discussion of the underlying physical principles see [4, section 2].

2.2. The correspondence to Minkowski space

At first sight, the above setting seems inappropriate for physical applications because important structures such as the notion of causality, topology and metric of spacetime, gauge fields, etc, are missing. However, the idea is that these additional structures arise as a consequence of a
self-organization of the states of the fermionic projector as described by our action principle. More specifically, for a given minimizing fermionic projector $P$, its discrete kernel $P(x, y)$ induces relations between the points $x$ and $y$, which can be used to introduce additional structures in spacetime. This mechanism is referred to as spontaneous structure formation. The first rigorous result on spontaneous structure formation was obtained in [5], where it is shown that the permutation symmetry of the spacetime points is spontaneously broken, giving rise to a non-trivial outer symmetry group. More detailed information is obtained by analyzing the eigenvalues of the closed chain.

**Definition 2.1 (causal structure).** Two spacetime points $x, y \in M$ are called timelike separated if the spectrum of $A_{xy}$ is real. The points are spacelike separated if the spectrum of $A_{xy}$ forms two complex conjugate pairs having the same absolute value. In all other cases, the two points are lightlike separated.

Furthermore, from the discrete kernel $P(x, y)$ one can deduce the objects of differential geometry like the spin connection and curvature (see [11]). Finally, the papers [7, 12] show that in the limit of an infinite number of particles and spacetime points, our action principle admits minimizers which are regularizations of vacuum Dirac sea structures in Minkowski space (see also [10]). In this limit, the wavefunction $\psi(x)$, (2.2), goes over to a four-component Dirac spinor. The spin scalar product becomes $\langle \psi | \phi \rangle = \psi \phi^{\dagger}$, where $\psi = \psi^{\dagger} \gamma^0$ is the usual adjoint spinor of Dirac theory. Discrete spacetime goes over to a spacetime continuum $M = \mathbb{R}^4$, and the sum in (2.3) becomes a spacetime integral:

$$\langle \psi | \phi \rangle := \int_M \langle \psi(x) | \phi(x) \rangle \, d^4x.$$  

In the simplest case of one sea, the discrete kernel corresponds to the Fourier integral of the lower mass shell:

$$P(x, y) = \int \frac{d^4k}{(2\pi)^4} (k \cdot \gamma + m) \delta(k^2 - m^2) \Theta(k^0) e^{-ik(x-y)},$$  

where $\Theta$ is the Heaviside function (more generally, one can take sums or direct sums of Dirac seas to describe different types of elementary particles; see [3, section 5.1]). Here the exponent $e^{-ik(x-y)}$ involves the Minkowski metric. Even more, (2.9) determines the Minkowski metric and can be used to define it. If this is done, the causal structure corresponding to the Minkowski metric indeed agrees with the spectral definition in definition 2.1 (for details see [4, section 3]). Moreover, one can introduce all the familiar objects of Dirac theory. For example, the non-negative quantity $\langle \psi | \gamma^0 \psi \rangle$ has the interpretation as the probability density of the Dirac particle. Polarizing and integrating over space yields the scalar product

$$\langle \psi | \phi \rangle = \int_{t = \text{const}} \langle \psi(t, \vec{x}) | \gamma^0 \phi(t, \vec{x}) \rangle \, d\vec{x}. $$

For the solutions of the Dirac equation, current conservation implies that this scalar product is time independent.

2.3. The continuum limit, a formulation of quantum field theory

The above correspondence to vacuum Dirac sea structures can also be used to analyze our action principle for interacting systems in the so-called continuum limit (for details see [3, chapter 4] and [8]). We now outline a few ideas and constructions needed later on. First, it is helpful to observe that the vacuum fermionic projector (2.9) is a solution of the Dirac equation $(i \gamma^j \partial_j - m) P(x, y) = 0$. To introduce the interaction, we replace the free Dirac operator
by a more general Dirac operator, for example involving gauge potentials or a gravitational
field. Thus, considering for simplicity an electromagnetic potential $A$, we demand that
\[(i\gamma^j(\partial_j - ieA_j) - m)P(x, y) = 0.\] (2.11)
Moreover, we introduce particles and anti-particles by occupying (suitably normalized)
positive-energy states and by removing the states of the sea:
\[P(x, y) = P_{\text{sea}}(x, y) - \frac{1}{2\pi} \sum_{k=1}^{n_f} |\psi_k(x)\rangle \langle \psi_k(y)| + \frac{1}{2\pi} \sum_{l=1}^{n_a} |\phi_l(x)\rangle \langle \phi_l(y)|.\] (2.12)
Using the so-called causal perturbation expansion and light-cone expansion, the fermionic
projector can be introduced via (2.11) and (2.12).

It is important that our setting so far does not involve the field equations; in particular, the
electromagnetic potential in the Dirac equation (2.11) does not need to satisfy the Maxwell
equations. Instead, the field equations should be derived from our action principle (2.7).
Indeed, analyzing the corresponding Euler–Lagrange equations, one finds that they are satisfied
only if the potentials in the Dirac equation satisfy certain constraints. Some of these constraints
are partial differential equations involving the potentials as well as the wavefunctions of the
particles and anti-particles in (2.12). In [8], such field equations are analyzed in detail for a
system involving an axial field. In order to keep the setting as simple as possible, we here
consider the analogous field equation for the electromagnetic field
\[\partial_{jk} A^k - \Box A_j = e \sum_{k=1}^{n_f} |\psi_k(x)\rangle \langle \gamma_j \psi_k| - e \sum_{l=1}^{n_a} |\phi_l(x)\rangle \langle \gamma_j \phi_l|.\] (2.13)
With (2.11) and (2.13), the interaction as described by the action principle (2.7) reduces in the
continuum limit to the coupled Dirac–Maxwell equations. The many-fermion state is again
described by the fermionic projector, which is built up of one-particle wavefunctions.
The electromagnetic field is merely a classical bosonic field.

For the considerations in sections 4 and 5, it is important to keep in mind that in the
framework of the fermionic projector, spacetime is not smooth on the microscopic scale,
but it has an underlying discrete structure. The dynamics is described intrinsically in
discrete spacetime by the action principle (2.7). A minimizing fermionic projector has a
rich microscopic structure from which one can deduce notions which have a correspondence
to macroscopic physics. In the continuum limit, the causal and metric structure of Minkowski
space can be recovered from the fermionic projector using the notion of definition 2.1. Thus,
we can say that the wavefunctions of the Dirac particles (also taking into account the states
of the Dirac sea) generate the causal and geometric structure of spacetime. This observation
is helpful in section 4, where by bringing the wavefunctions in different regions of spacetime
‘out of phase’, we are able to turn off causal influences between these regions.

3. Preliminaries on projectors, Fock spaces and entanglement

In this section we first recall the notions of the fermionic Fock space and entanglement, also
fixing our notation. Then we show that a projector in the one-particle Hilbert space uniquely
determines a Hartree–Fock state in the fermionic Fock space, making it impossible to describe
entangled states.

3.1. The fermionic Fock space and entanglement

In non-relativistic quantum mechanics, the one-particle states form a separable Hilbert space
($\mathcal{H}, \langle \cdot | \cdot \rangle$). Similarly, in Dirac theory we let $\mathcal{H}$ be the Hilbert space corresponding to the scalar
product (2.10). In each of these settings, a many-fermion state is usually described by a vector in the fermionic Fock space, which we now introduce (see also [16, section II.4] or [18, section I.1]). We let \( \mathcal{H}^n = \mathcal{H} \otimes \cdots \otimes \mathcal{H} \) be the \( n \)-fold tensor product, endowed with the natural scalar product

\[
\langle \psi_1 \otimes \cdots \otimes \psi_n | \phi_1 \otimes \cdots \otimes \phi_n \rangle := \langle \psi_1 | \phi_1 \rangle \cdots \langle \psi_n | \phi_n \rangle. \tag{3.1}
\]

Totally anti-symmetrizing the tensor product gives the wedge product

\[
\psi_1 \wedge \cdots \wedge \psi_n := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)}, \tag{3.2}
\]

(here \( S_n \) denotes the set of all permutations and \( \text{sign}(\sigma) \) is the sign of the permutation \( \sigma \)). The wedge product gives rise to a mapping

\[
\Lambda_n : \mathcal{H} \times \cdots \times \mathcal{H} \rightarrow \mathcal{H}^n : (\psi_1, \ldots, \psi_n) \mapsto \psi_1 \wedge \cdots \wedge \psi_n.
\]

We denote the image of this mapping by \( \mathcal{F}^\text{HF}_n \). The vectors in \( \mathcal{F}^\text{HF}_n \) are called \( n \)-particle Hartree–Fock states or factorizable states. These states in general do not form a vector space, as the following example shows, which in discussions of spin correlation experiments and Bell’s inequalities is often referred to as the EPR singlet state (see for example [1, section I.5]).

**Example 3.1 (The spatially separated singlet state).** We consider the one-particle Hilbert space \( \mathcal{H} = \mathbb{C}^2_\Lambda \oplus \mathbb{C}^2_\mu \) of two spins (in quantum information theory called ‘qubits’), located at the positions of two observers ALICE and BOB. Choosing in \( \mathbb{C}^2 \) the standard basis \( \psi^\uparrow = (1, 0) \) and \( \psi^\downarrow = (0, 1) \) yields the basis \( (\psi^\uparrow_\Lambda, \psi^\downarrow_\Lambda, \psi^\uparrow_\mu, \psi^\downarrow_\mu) \) of \( \mathcal{H} \). The spatially separated singlet state is the following linear combination of two-particle Hartree–Fock states:

\[
\Psi := \frac{1}{\sqrt{2}} (\psi^\uparrow_\Lambda \wedge \psi^\downarrow_\mu - \psi^\downarrow_\Lambda \wedge \psi^\uparrow_\mu). \tag{3.3}
\]

Let us verify in detail that this state is not factorizable. Thus, assume conversely that \( \Psi \) can be written as a product

\[
\Psi = \psi_1 \wedge \psi_2.
\]

Computing this wedge product in the basis representations

\[
\begin{align*}
\psi_1 &= \alpha^\uparrow_\Lambda \psi^\uparrow_\Lambda + \alpha^\downarrow_\Lambda \psi^\downarrow_\Lambda + \alpha^\downarrow_\mu \psi^\downarrow_\mu + \alpha^\uparrow_\mu \psi^\uparrow_\mu, \tag{3.4} \\
\psi_2 &= \beta^\uparrow_\Lambda \psi^\uparrow_\Lambda + \beta^\downarrow_\Lambda \psi^\downarrow_\Lambda + \beta^\downarrow_\mu \psi^\downarrow_\mu + \beta^\uparrow_\mu \psi^\uparrow_\mu, \tag{3.5}
\end{align*}
\]

the vanishing of the term \( -\psi^\downarrow_\Lambda \wedge \psi^\uparrow_\mu \) implies that the vectors \( \alpha^\uparrow_\Lambda \psi^\uparrow_\Lambda + \alpha^\downarrow_\Lambda \psi^\downarrow_\Lambda \) and \( \beta^\downarrow_\Lambda \psi^\downarrow_\Lambda + \beta^\uparrow_\Lambda \psi^\uparrow_\Lambda \) must be linearly dependent. Similarly, the vanishing of the term \( -\psi^\downarrow_\mu \wedge \psi^\uparrow_\mu \) implies that the vectors \( \alpha^\uparrow_\mu \psi^\uparrow_\mu + \alpha^\downarrow_\mu \psi^\downarrow_\mu \) and \( \beta^\downarrow_\mu \psi^\downarrow_\mu + \beta^\uparrow_\mu \psi^\uparrow_\mu \) are linearly dependent. Hence, \( \psi_2 \) can be written as

\[
\psi_2 = \beta_\Lambda (\alpha^\downarrow_\Lambda \psi^\downarrow_\Lambda + \alpha^\uparrow_\Lambda \psi^\uparrow_\Lambda) + \beta_\mu (\alpha^\downarrow_\mu \psi^\downarrow_\mu + \alpha^\uparrow_\mu \psi^\uparrow_\mu), \tag{3.6}
\]

with suitable complex coefficients \( \beta_\Lambda \) and \( \beta_\mu \). Taking the wedge product of (3.4) and (3.6) yields

\[
\Psi = \psi_1 \wedge \psi_2 = (\beta_\mu - \beta_\Lambda)(\alpha^\downarrow_\Lambda \psi^\downarrow_\Lambda + \alpha^\uparrow_\Lambda \psi^\uparrow_\Lambda) \wedge (\alpha^\downarrow_\mu \psi^\downarrow_\mu + \alpha^\uparrow_\mu \psi^\uparrow_\mu).
\]

Multiplying out and comparing with (3.3), one sees that the products \( \alpha^\downarrow_\Lambda \alpha^\downarrow_\mu \) and \( \alpha^\uparrow_\Lambda \alpha^\uparrow_\mu \) must be non-zero, and thus none of these four coefficients vanishes. But then the term \( -\psi^\uparrow_\Lambda \wedge \psi^\downarrow_\mu \) is non-zero, a contradiction.
We denote the vector space generated by the \( n \)-particle Hartree–Fock states by
\[
F_n = \langle \Lambda_n(\mathcal{H}^n) \rangle.
\]
Their direct sum is the fermionic Fock space,
\[
F = \bigoplus_{n=0}^{\infty} F_n.
\]
The non-factorizable vectors \( \Psi \in F_n \setminus \Lambda_n(\mathcal{H}^n) \) are called entangled states. The spatially separated singlet state is the standard example of an entangled state. Entanglement is a basic phenomenon of quantum physics with important potential applications in quantum computing.

### 3.2. Projectors and Hartree–Fock states

Let us examine in which sense a projector in the one-particle Hilbert space characterizes a many-particle quantum state. Thus, let \( P \) be a projector in the Hilbert space \( (\mathcal{H}, \langle \cdot | \cdot \rangle) \), for simplicity of finite rank \( f \), i.e.
\[
P^* = P = P^2 \quad \text{and} \quad \dim P(\mathcal{H}) = f.
\] (3.7)

In order to get a connection to the fermionic Fock space formalism, we choose an orthonormal basis \( \psi_1, \ldots, \psi_f \) of \( P(\mathcal{H}) \) and form the Hartree–Fock state
\[
\Psi := \psi_1 \wedge \cdots \wedge \psi_f \in \mathcal{F}_{HF}^f.
\] (3.8)
The choice of our orthonormal basis was unique only up to the unitary transformations
\[
\psi_i \to \tilde{\psi}_i = \sum_{j=1}^f U_{ij} \psi_j \quad \text{with} \quad U \in U(f).
\] (3.9)

Due to the anti-symmetrization, this transformation changes the corresponding Hartree–Fock state only by a phase factor
\[
\tilde{\psi}_1 \wedge \cdots \wedge \tilde{\psi}_f = \det U \ \psi_1 \wedge \cdots \wedge \psi_f.
\] (3.10)

Thus, we can indeed associate with the projector \( P \) a Hartree–Fock state, which is well defined up to a phase. As the phase of \( \Psi \) has no physical significance, the physical system is described equivalently by a projector \( P_f \) on the many-particle state \( \Psi \), i.e. in the bra/ket notation\(^1\)
\[
P_f = \frac{1}{\| \Psi \|_F^2} \langle \Psi | = f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \wedge \cdots \wedge \psi_f \rangle : \mathcal{F}_f \to \mathcal{F}_f.
\] (3.11)

Since the phase freedom drops out when forming the projector (3.11), this operator is well defined. The next proposition gives an alternative definition of \( P_f \) which does not involve a choice of basis.

**Proposition 3.2.** For any projector \( P \) in \( (\mathcal{H}, \langle \cdot | \cdot \rangle) \) of rank \( f \), the corresponding operator
\[
P_f : \mathcal{F}_f \to \mathcal{F}_f : \psi_1 \wedge \cdots \wedge \psi_f \to (P \psi_1) \wedge \cdots \wedge (P \psi_f)
\] (3.12)

\(^1\) Here the factor \( f! \) comes about because, according to our conventions (3.2) and (3.1),
\[
\langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \wedge \cdots \wedge \psi_f \rangle = \langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \otimes \cdots \otimes \psi_f \rangle
\]
\[
= \frac{1}{f!} \sum_{\sigma \in S_f} (-1)^{\text{sign} \sigma} \langle \psi_{\sigma(1)} | \psi_1 \rangle \cdots \langle \psi_{\sigma(f)} | \psi_f \rangle = \frac{1}{f!}.
\]
is a projector onto an \( f \)-particle Hartree–Fock state. The mapping \( P \mapsto P_f \) gives a one-to-one correspondence between projectors in \( \mathcal{H} \) and projectors on Hartree–Fock states in \( \mathcal{F} \).

**Proof.** It follows immediately from the definitions that \( P_f \) is symmetric and idempotent and is thus a projector. To compute the rank of \( P_f \), we choose an orthonormal basis \( \psi_1, \ldots, \psi_f \) of \( P(\mathcal{H}) \) and extend it to an orthonormal basis of \( \mathcal{H} \). As is obvious from (3.12), the operator \( P_f \) applied to any wedge product of basis vectors vanishes unless all basis vectors are elements of the set \( \{\psi_1, \ldots, \psi_f\} \). Hence, the vector \( \Psi := \psi_1 \wedge \cdots \wedge \psi_f \) is a basis of the image of \( P_f \). We conclude that \( P_f \) has indeed rank 1 and is thus a projector onto the Hartree–Fock state \( \Psi \).

Now suppose conversely that \( P_f \) is a projector onto a Hartree–Fock state. Representing this operator in the form (3.11), we let \( P \) be the projector in \( \mathcal{H} \) on the subspace \( \langle \psi_1, \ldots, \psi_f \rangle \). Then the operator \( P_f \) has the representation (3.12), concluding the proof. \( \square \)

### 3.3. Projectors and expectation values

We now consider how the expectation values of observables can be expressed in terms of the projectors \( P \) and \( P_f \). We begin with a one-particle observable \( \mathcal{O} \), being a self-adjoint operator in the one-particle Hilbert space \( \mathcal{H} \). By

\[
\mathcal{O}^F(\psi_1 \wedge \cdots \wedge \psi_n) := (\mathcal{O} \psi_1) \wedge \cdots \wedge \psi_n + \psi_1 \wedge (\mathcal{O} \psi_2) \wedge \cdots \wedge \psi_n + \cdots + \psi_1 \wedge \cdots \wedge \psi_{n-1} \wedge (\mathcal{O} \psi_n),
\]

(3.13)

we can define a corresponding operator \( \mathcal{O}^F \) on the Fock space \( \mathcal{F} \). This operator preserves the number of particles in the sense that it maps the \( n \)-particle subspace \( \mathcal{F}_n \) onto itself.

Suppose that an \( f \)-fermion state is described by a projector \( P \), (3.7). The next lemma shows how the expectation values of \( \mathcal{O}^F \) and of products of one-particle operators can be expressed in terms of traces involving the projector \( P \).

**Lemma 3.3.** Suppose that \( \mathcal{O} \) and \( \mathcal{O}_{12} \) are one-particle observables. Describing a many-fermion state by a projector \( P \) in \( (\mathcal{H}, \langle \cdot | \cdot \rangle) \), we have

\[
\langle \mathcal{O}^F \rangle = \text{Tr}_\mathcal{H}(P \mathcal{O}) \quad (3.14)
\]

\[
\langle \mathcal{O}_{1}^F \mathcal{O}_{2}^F \rangle = \text{Tr}_\mathcal{H}(P \mathcal{O}_1 \mathcal{O}_2) + \text{Tr}_\mathcal{H}(P \mathcal{O}_1)\text{Tr}_\mathcal{H}(P \mathcal{O}_2) - \text{Tr}_\mathcal{H}(P \mathcal{O}_1 P \mathcal{O}_2). \quad (3.15)
\]

**Proof.** The expectation values are obtained by taking the trace of the observables multiplied by the operator \( P_f \),

\[
\langle \mathcal{O}^F \rangle = \text{Tr}_{\mathcal{F}_f}(P_f \mathcal{O}^F), \quad \langle \mathcal{O}_{1}^F \mathcal{O}_{2}^F \rangle = \text{Tr}_{\mathcal{F}_f}(P_f \mathcal{O}_{1}^F \mathcal{O}_{2}^F).
\]

Representing \( P_f \) in the form (3.11), it follows that

\[
\langle \mathcal{O}^F \rangle = f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \mathcal{O}^F(\psi_1 \wedge \cdots \wedge \psi_f) \rangle
\]

\[
= f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \mathcal{O}^F(\psi_1 \otimes \cdots \otimes \psi_f) \rangle
\]

\[
= \sum_{\sigma \in S_n} \langle \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(f)} | \mathcal{O}^F(\psi_1 \otimes \cdots \otimes \psi_f) \rangle = \sum_{i=1}^{f} \langle \psi_i | \mathcal{O} \psi_i \rangle,
\]

where in the last step we applied (3.13) together with (3.1) and used the fact that the vectors \( \psi_1, \ldots, \psi_f \) are orthonormal. This proves (3.14).
With the same method, we obtain
\[
\langle O_1^1 O_2^2 \rangle = f!\langle \psi_1 \wedge \cdots \wedge \psi_f | O_1^1 O_2^2 (\psi_1 \otimes \cdots \otimes \psi_f) \rangle \\
= \sum_{k=1}^f f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \otimes \cdots \otimes \psi_{k-1} \otimes (O_1 O_2 \psi_k) \otimes \psi_{k+1} \cdots \otimes \psi_f \rangle \\
+ \sum_{k \neq l} f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \otimes \cdots (O_1 \psi_k) \cdots (O_2 \psi_l) \cdots \otimes \psi_f \rangle \\
= \sum_{k=1}^f \langle \psi_k | O_1 O_2 \psi_k \rangle + \sum_{k \neq l} \left( \langle \psi_k | O_1 \psi_l \rangle \langle \psi_l | O_2 \psi_k \rangle - \langle \psi_k | O_1 \psi_k \rangle \langle \psi_l | O_2 \psi_l \rangle \right). 
\]
The last sum can be extended to all \( k, l = 1, \ldots, f \), because the summands for \( k = l \) vanish. We thus obtain (3.15). □

The method of this lemma immediately extends to higher powers of one-particle observables.

More generally, one can consider many-particle observables, described by a self-adjoint operator \( O \) on the Fock space \( \mathcal{F} \). In this paper, we only consider observables which preserve the number of particles, i.e. which are invariant on the \( n \)-particle subspaces \( \mathcal{F}_n \),
\[
O : \mathcal{F}_n \to \mathcal{F}_n.
\tag{3.16}
\]
This assumption is justified by the physical law of the conservation of the baryon and lepton numbers, stating that the total number of fermions is preserved. Thus, by considering a system which is so large that no fermion enters or leaves it, we can arrange that all physical observables satisfy (3.16). For a many-particle observable satisfying (3.16), the expectation value is again expressed by a trace,
\[
\langle O \rangle = \text{Tr}_{\mathcal{F}} (P_f O). 
\tag{3.17}
\]

4. Microscopic mixing of decoherent spacetime regions

In this section we develop a method for describing entangled states by a projector in the one-particle Hilbert space. In section 4.1 we give a preliminary construction, which clarifies the difficulty in describing entangled states. In section 4.2 we overcome this difficulty on a rather formal and axiomatic level. The microscopic justification of the resulting formalism is given in sections 4.3–4.5 in the framework of the fermionic projector.

4.1. Microscopic mixing of the wavefunctions

Our idea for realizing entangled states is to give \( P \) a non-trivial microscopic structure, with the hope that ‘averaging’ this microstructure over macroscopic regions of spacetime will give rise to an effective kernel \( P(x, y) \) of a more general form which allows for the description of entanglement. As the relativistic generalization will not be quite straightforward, we begin for clarity in the non-relativistic setting. Thus, we consider the situation where space is subdivided into sets \( M_1, \ldots, M_L \),
\[
M = M_1 \cup \cdots \cup M_L \quad \text{and} \quad M_a \cap M_b = \emptyset \quad \text{if} \quad a \neq b, \tag{4.1}
\]
which are fine grained in the sense that every macroscopic region of spacetime intersects several of the sets \( M_a \). The sets \( M_a \) may be localized, but they can also be extended over a macroscopic region of spacetime, for example by forming ‘layers’ or ‘filaments’ connecting
the two observers in the spin correlation experiment of example 3.1 (see figure 1). The macroscopic physical objects are then introduced by homogenizing over the sets $M_a$. We refer to this technique as the method of microscopic mixing.

The partition (4.1) allows us to decompose $\mathcal{H}$ into an orthogonal direct sum of the Hilbert spaces $\mathcal{H}_a$ of square integrable wavefunctions in $M_a$,

$$\mathcal{H} = \bigoplus_{a=1}^{L} \mathcal{H}_a. \tag{4.2}$$

Thus, every wavefunction $\psi_i$ in the image of $P$, (3.8), has the unique decomposition

$$\psi_i = \sum_{a=1}^{L} \psi_i^{(a)} \quad \text{with} \quad \psi_i^{(a)} \in \mathcal{H}_a. \tag{4.3}$$

The simplest attempt is to realize a macroscopic local one-particle observable $O$ (such as a position or spin operator) by an operator on $\mathcal{H}$ being invariant on $\mathcal{H}_a$,

$$O : \mathcal{H}_a \rightarrow \mathcal{H}_a. \tag{4.4}$$

Then the corresponding one-particle expectation values split into a sum over the subsystems,

$$\langle \psi | O | \psi \rangle = \sum_{a=1}^{L} \langle \psi^{(a)} | O | \psi^{(a)} \rangle = \sum_{a=1}^{L} \int_{M_a} \overline{\psi(x)} O \psi(x) \, dx,$$

and this can be understood as an ‘averaging process’ over the subregions $M_a$.

Following the constructions in section 3.3, every one-particle operator induces a corresponding operator on the Fock space $\mathcal{F}$, and the products of such operators yield corresponding many-particle observables. Taking expectation values of such operators in the Fock space again involves an ‘averaging process’ over the subregions $M_a$. More specifically, describing the many-particle system by a projector $P$ in $\mathcal{H}$, the expectation value of the two-particle observables corresponding to ALICE and Bob is given by (see lemma 3.3)

$$\langle O_A^F O_B^F \rangle = \sum_{i,j=1}^{L} \sum_{a=1}^{L} \langle \psi_i^{(a)} | O_A^F O_B^F \psi_j^{(a)} \rangle \tag{4.5}$$

$$+ \sum_{i,j=1}^{L} \sum_{a,b=1}^{L} \left( \langle \psi_i^{(a)} | O_A^F | \psi_j^{(b)} \rangle \langle \psi_j^{(b)} | O_B^F | \psi_i^{(a)} \rangle - \langle \psi_i^{(a)} | O_A^F | \psi_j^{(b)} \rangle \langle \psi_j^{(b)} | O_B^F | \psi_i^{(a)} \rangle \right). \tag{4.6}$$
In (4.6), an ‘averaging process’ takes place at each of the observers. However, it is important to note that there is no averaging over correlations between the two observers, as would be the case for an expression like

$$\sum_{i,j=1}^{f} \sum_{a=1}^{L} \langle \psi_{i}^{(a)} | O_{A} \psi_{i}^{(a)} \rangle \langle \psi_{j}^{(a)} | O_{B} \psi_{j}^{(a)} \rangle.$$  (4.7)

We conclude that the above method does not make it possible to realize general entangled states.

4.2. A formalism for the description of entanglement

The previous construction did not take into account that the measurement process is a result of an interaction of the system with the measurement device. Assuming that the subsystems have an independent dynamics (an assumption which is justified in section 4.4 below), also the measurement process should take place independently in the subsystems. Following this idea makes it possible to describe entanglement, as we now explain.

For the one-particle observables, the assumption of an independent dynamics of the subsystems was already taken into account in (4.4) by the assumption that $O$ should be invariant on the subspaces $H_{a}$. However, for a many-particle observable, it was too naive to simply take the product of the one-particle operators (see (4.5) and (4.6)). Thinking of the situation in figure 1, ALICE is built up of fermionic wavefunctions. Thus, considering her as part of the physical system, we should replace the corresponding measurement operator $O_{A}$ by the separate operators $O_{A}^{(a)}$ for each of the subsystems. Proceeding similarly for BOB, in the subsystem $M_{a}$, measurements are to be carried out with the operators $O_{A}^{(a)}$ and $O_{B}^{(a)}$. For a correlation measurement in $M_{a}$, we should extend the one-particle observables $O^{(a)}$ to the operators $O_{F}^{(a)}$ defined on the Fock space $F_{n}^{(a)}$ of the subsystem given by

$$F_{n}^{(a)} = (\lambda_{1}^{a} H_{a}) \subset F_{n},$$  (4.8)

(as explained after (3.16), we again restrict attention to observables which preserve the number of particles) and consider the corresponding two-particle observable

$$O_{A}^{F_{n}^{(a)}} O_{B}^{F_{n}^{(a)}} : F_{n}^{(a)} \rightarrow F_{n}^{(a)}.$$  

More generally, we make the following assumption.

(A) The observables correspond to the operators $O$ which are invariant on $F_{n}^{(a)}$,

$$O : F_{n}^{(a)} \rightarrow F_{n}^{(a)}, \quad a = 1, \ldots, L.$$  

Similar to what is explained in section 3.2 for the Fock space $F_{f}$, we can get a simple connection between a projector $P$ in $H$ and the Fock spaces $F_{n}^{(a)}$. Namely, choosing again an orthonormal basis $\psi_{1}, \ldots, \psi_{f}$ of $P(H)$ and decomposing each of the one-particle wavefunctions according to (4.3), we can construct the Hartree–Fock states $\Psi^{(a)}$ in the $f$-particle Fock spaces of the subsystems,

$$\Psi^{(a)} := \psi_{1}^{(a)} \wedge \cdots \wedge \psi_{f}^{(a)} \in F_{f}^{(a)}.$$  (4.9)

Exactly as in (3.10), one sees that these vectors are unique up to a common phase,

$$\Psi^{(a)} \rightarrow e^{i\varphi} \Psi^{(a)} \quad \text{with} \quad \varphi \in \mathbb{R} \ \text{independent of} \ a.$$  (4.10)

The setting so far is not sufficient for determining the expectation value of a measurement, because for computing an expectation value we need to take an ‘average’ over the subsystems. This process can be described conveniently by the so-called measurement scalar product $\langle \cdot | \cdot \rangle$. 


which we now introduce (for a microscopic derivation of the measurement scalar product and the measurement process see section 4.5 below). It is defined on the one-particle Hilbert space \((\mathcal{H}, \langle \cdot | \cdot \rangle)\) as a positive semi-definite inner product

\[(\cdot | \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C},\]

(4.11)

with respect to which the direct sum decomposition (4.2) need not be orthogonal. The fact that this inner product is only semi-definite models the fact that the measurement process may involve a homogenization process on the microscopic scale, so that the fluctuations of the wavefunctions on the small scale might not enter the measurement process. The measurement scalar product induces on the Fock spaces the bilinear form

\[\langle \cdot | \cdot \rangle_{(a,b)} : F_n(a) \times F_n(b) \to \mathbb{C} : \psi(a)_1 \wedge \cdots \wedge \psi(a)_n, \psi(b)_1 \wedge \cdots \wedge \psi(b)_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} (-1)^{\text{sign}(\sigma)} (\psi(a)_{\sigma(1)} | \psi(b)_1) \cdots (\psi(a)_{\sigma(n)} | \psi(b)_n).\]

We now specify how expectation values are to be computed and state the assumptions which ensure that these expectation values are real.

(B) The expectation value of the measurement of the observable \(\mathcal{O}\) is given by

\[\langle \mathcal{O} \rangle = \frac{\sum_{a,b=1}^{L} \langle \psi(a) | \mathcal{O} | \psi(b) \rangle_{(a,b)}(a,b)}{\sum_{a,b=1}^{L} \langle \psi(a) | \psi(b) \rangle_{(a,b)}}.\]

(4.12)

(C) The observables are symmetric possibly up to a microscopic error, meaning that

\[\langle \psi(a) | \mathcal{O} | \psi(b) \rangle_F = \langle \psi(a) | \mathcal{O} | \psi(b) \rangle_F + \mathcal{O}(\epsilon),\]

where \(\epsilon\) is the length scale of microscopic mixing.

Finally, we need to specify how a state changes in a measurement process. In order to ensure that a repeated measurement of the same observable yields identical results, one usually asserts that after the measurement, the state should be an eigenstate of the observable. In our setting, the situation is a bit more involved because the measurement process may change the number of subsystems, and only the wavefunction after the homogenization should be an eigenstate of the observable. This is made precise by the following construction. We take the direct sum of the vector spaces \(F_n(a)\),

\[\mathcal{G} := \bigoplus_{a=1}^L F_n(a),\]

and on these spaces we introduce the inner product

\[(\cdot | \cdot)_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \to \mathbb{C} : ((\psi(a))_{a=1,\ldots,L}, (\psi(b))_{a=1,\ldots,L}) \mapsto \sum_{a,b=1}^{L} \langle \psi(a) | \psi(b) \rangle_{(a,b)}.\]

This inner product is positive semi-definite, but it need not be definite. Dividing out the null space and taking the completion,

\[F_n^{\text{eff}} := \overline{\mathcal{G}} / \mathcal{G}_0, \quad \text{where} \quad \mathcal{G}_0 = \{u \in \mathcal{G} \text{ with } (u | u)_{\mathcal{G}} = 0\},\]

(4.13)

we obtain a Hilbert space, which we can regard as the effective \(n\)-particle Fock space obtained by homogenization over the subsystems. We denote the natural projection operator by \(\pi_n:\)

\[\pi_n : \bigoplus_{a=1}^L F_n(a) \to F_n^{\text{eff}}.\]

(4.14)
Using linearity together with assumption (C) above, every observable $O$ induces an operator
\[ O_{\text{eff}} : \mathcal{F}_{\text{eff}}^n \rightarrow \mathcal{F}_{\text{eff}}^n, \tag{4.15} \]
uniquely defined possibly up to a microscopic error. With the projector $P$ in $\mathcal{H}$ we associate a corresponding state
\[ \Psi_{\text{eff}} = \pi_f(\Psi^{(1)}, \ldots, \Psi^{(L)}) \in \mathcal{F}_{\text{eff}}^n \]  
(with $\Psi^{(a)}$ as in (4.9)). According to (4.10), this state is well defined up to an irrelevant phase.

(D) After a measurement of the observable $O$, the one-particle projector $P$ takes such a form that the corresponding state $\Psi_{\text{eff}} \in \mathcal{F}_{\text{eff}}^n$ defined by (4.16) is an eigenstate of the operator $O_{\text{eff}}$, (4.15).

Similar to that in the Copenhagen interpretation or the formulation of the measurement process by von Neumann [19, section V.1], the above assumptions (A)–(D) are merely working rules to determine the results of measurements. For a conceptually convincing treatment, these assumptions should be derived from the physical equations.

We now verify that the above setting indeed makes it possible to realize the EPR singlet state.

**Example 4.1 (The spatially separated singlet state).** We choose a microscopic length scale $\varepsilon > 0$ and subdivide the position space $M = \mathbb{R}^3$ into two subregions $M_1$ and $M_2$ which form layers of width $\varepsilon$,
\[ M_1 = [\bar{x} \in \mathbb{R}^3 \mid [x_1, \varepsilon] \subset 2\mathbb{Z}], \quad M_2 = [\bar{x} \in \mathbb{R}^3 \mid [x_1, \varepsilon] \subset 2\mathbb{Z} + 1] \]
(where $[x] = \text{min} \{n \in \mathbb{Z} \mid n \geq x\}$ is the Gauss bracket). We introduce the wavefunctions
\[ \psi_1(\bar{x}) = \psi_A^1(\bar{x}) \chi_{M_1}(\bar{x}) + \psi_B^1(\bar{x}) \chi_{M_2}(\bar{x}), \]
\[ \psi_2(\bar{x}) = \psi_B^1(\bar{x}) \chi_{M_1}(\bar{x}) - \psi_A^1(\bar{x}) \chi_{M_2}(\bar{x}), \]
where $\psi_{A/B}^1$ are smooth one-particle wavefunctions supported near Alice or Bob (and where $\chi$ is the characteristic function defined by $\chi_N(x) = 1$ if $x \in N$ and $\chi_N(x) = 0$ otherwise). Defining $P$ as the projector on the subspace spanned by $\psi_1$ and $\psi_2$, the corresponding two-particle wavefunctions of the subsystems are
\[ \Psi^{(1)} = c(\psi_A^1 \wedge \psi_B^1) \chi_{M_1 \times M_1}, \quad \text{and} \quad \Psi^{(2)} = -c(\psi_A^1 \wedge \psi_B^1) \chi_{M_2 \times M_2} \]
with $c$ a normalization constant.

In order to realize a suitable mixing of the subregions in the measurement process, we introduce the measurement scalar product as
\[ (\psi \mid \phi) = \int_M \psi(\bar{x}) \overline{\phi(\bar{x})} \, d\bar{x} + \frac{1}{2} \int_M (\psi(\bar{x} + \varepsilon e_1) \overline{\phi(\bar{x})} + \overline{\psi(\bar{x})} \phi(\bar{x} + \varepsilon e_1)) \, d\bar{x}. \]

The spin operators are symmetric with respect to this inner product, whereas the position operators are symmetric up to an error of order $\varepsilon$.
\[ \langle \bar{x} \psi \mid \phi \rangle - (\psi \mid \bar{x} \phi) = \varepsilon \int_M x_1 (\overline{\psi(\bar{x} + \varepsilon e_1) \phi(\bar{x})} - \overline{\psi(\bar{x}) \phi(\bar{x} + \varepsilon e_1)}) \, d\bar{x}. \]

Thus, the general observables introduced according to (A) indeed satisfy condition (C). The expectation values of the spin operators can now be calculated by applying the rule (B). More precisely, the inner products in (4.12) are computed by
\[
\begin{align*}
(\Psi^{(1)}_1 \mid O \Psi^{(1)}_1)_{(1,1)} &= (\Psi^{(1)}_1 \mid O \Psi^{(1)}_1) \\
(\Psi^{(2)}_1 \mid O \Psi^{(2)}_1)_{(1,2)} &= (\Psi^{(2)}_1 \mid O \Psi^{(2)}_1) \\
(\Psi^{(1)}_1 \mid O \Psi^{(2)}_1)_{(1,2)} &= \frac{1}{2} ((\Psi^{(1)}_1 \mid O \Psi^{(2)}_1) + (\Psi^{(2)}_1 \mid O \Psi^{(1)}_1)) \\
(\Psi^{(2)}_1 \mid O \Psi^{(1)}_1)_{(2,1)} &= \frac{1}{2} ((\Psi^{(2)}_1 \mid O \Psi^{(1)}_1) + (\Psi^{(2)}_1 \mid O \Psi^{(1)}_1)).
\end{align*}
\]
where on the right-hand side the scalar product on $\mathcal{F}$ as defined by (3.1) appears, and the subscript + denotes that both spatial arguments of the corresponding two-particle wavefunction have been shifted by $\varepsilon e_1$. Note that all these inner products involve integrals over $M_1 \times M_1$ or $M_2 \times M_2$. Since the wavefunctions $\psi_{\Lambda/B}^{\uparrow\downarrow}$ are all smooth, we can extend the two-particle wavefunctions of the subsystems to smooth functions on $M \times M$,

$$
\Psi^{(1)}_{\text{eff}} = c(\psi_{\Lambda}^{\uparrow} \wedge \psi_{\Lambda}^{\downarrow}) \quad \text{and} \quad \Psi^{(2)}_{\text{eff}} = c(\psi_{\Lambda}^{\downarrow} \wedge \psi_{\Lambda}^{\uparrow}).
$$

(4.17)

Shifting the arguments changes these smooth wavefunctions only by a term of order $\varepsilon$. Also, extending the integration range in the above integrals from $M_1 \times M_1$ or $M_2 \times M_2$ to $M \times M$ changes the values of the integrals only by a factor of 4, again up to the contributions of order $\varepsilon$. We thus obtain

$$
\sum_{a,b=1}^{L} (\Psi^{(a)} | O \Psi^{(b)})^{(a,b)} = 4\langle (\Psi^{(1)}_{\text{eff}} + \Psi^{(2)}_{\text{eff}}) | O (\Psi^{(1)}_{\text{eff}} + \Psi^{(2)}_{\text{eff}}) \rangle + O(\varepsilon).
$$

We conclude that in the limit $\varepsilon \searrow 0$, the expectation values as defined by (B) indeed coincide with the expectation values of the spin singlet state.

Moreover, it is straightforward to verify that the space $\mathcal{F}^{\text{eff}}_{2}$ as defined by (4.13) can be identified with the ordinary Fock space $\mathcal{F}_{2}$, and that under this identification, the state $\Psi^{\text{eff}}$ as given by (4.16) goes over to the state $\Psi^{(1)}_{\text{eff}} + \Psi^{(2)}_{\text{eff}}$ with the wavefunctions $\Psi^{(1)}_{\text{eff}}$ as in (4.17).

### 4.3. Microscopic mixing in the framework of the fermionic projector

We now generalize and adapt the method of microscopic mixing to the framework of the fermionic projector. For the extension to the relativistic setting, we decompose the Minkowski space $M$ into two disjoint subregions

$$
M = M_1 \cup M_2 \quad \text{with} \quad M_1 \cap M_2 = \emptyset,
$$

(4.18)

which may be fine grained as depicted in figure 2 (for simplicity, we only consider two subregions; the generalization to a finite number of subsystems is straightforward).

If one prefers, one can in addition replace the spacetime continuum by a discrete set of points, but the distinction between continuum and discrete spacetime will not be of relevance for the following considerations. We again consider a family of wavefunctions $\psi_1, \ldots, \psi_f$, where in view of the fact that we also count the states of the Dirac sea, the number $f$ of particles is very large (see section 2.3). As in (2.5), the fermionic projector takes the form

$$
P(x, y) = -\sum_{j=1}^{f} |\psi_j(x)\rangle \langle \psi_j(y)| \quad \text{with} \quad x, y \in M_1 \cup M_2.
$$

(4.19)
Splitting up the wavefunctions similar to (4.3) as
\[ \psi_j = \psi_j^{(1)} + \psi_j^{(2)} \quad \text{with} \quad \psi_j^{(\alpha)} = \psi_j \chi_{M_\alpha} \quad (\alpha = 1, 2), \quad (4.20) \]
with each subsystem we can associate similar to (4.9) the many-particle wavefunction
\[ \Psi^{(\alpha)} = \psi_1^{(\alpha)} \wedge \cdots \wedge \psi_f^{(\alpha)}. \quad (4.21) \]

At this point, we need to discuss the significance of the inner products (2.8) and (2.10). In discrete spacetime, the underlying inner product involves the sum over all spacetime points (2.3). Likewise, in the continuum limit, the sum is to be replaced by a spacetime integral (2.8). The notions of symmetry and idempotence of the fermionic projector refer to this indefinite inner product (for details see [3, section 2.6]). However, as the fermionic projector is built up of negative definite states, we may always restrict attention to a negative definite subspace, on which \( \langle \cdot | \cdot \rangle := -\langle \cdot | \cdot \rangle \) is a scalar product. After completion, we again get a separable Hilbert space \( (\mathcal{H}, \langle \cdot | \cdot \rangle) \), so that we are back to the setting of section 3. It is important to keep in mind that the inner product (2.8) is not suitable for computing the expectation value of a measurement, which usually takes place at a fixed time. Thus in the measurement process, it is natural to work with a different scalar product, which can be regarded as the relativistic analog of the measurement scalar product (4.11). In view of the fact that the integrand of (2.10) has the interpretation as the probability density, the scalar product (2.10) seems to be the correct choice.

Let us consider how to implement the scalar product (2.10) as the measurement scalar product in the presence of a microscopic mixing of two subsystems. First, taking into account that realistic measurements take place in a finite time interval, it seems a good idea to replace the spatial integral in (4.18) by an integral over a strip of width \( \Delta t \) (as shown in figure 2). Moreover, the measurement should involve some kind of homogenization process on the microscopic scale. This is clear empirically because the expectation value must be a computable quantity which involves taking averages over the subsystems. The homogenization can also be understood microscopically from the fact that the measurement devices are themselves formed of quantum mechanical wavefunctions which are spread out in spacetime. A typical example for an idealized measurement device is the operator \( |\eta\rangle \langle \eta| \), where \( \eta \) is a wavefunction which is supported inside a strip of width \( \Delta t \) as shown in figure 2. The simplest way to take into account the effect of such a measurement device would be to consider instead of (2.10) the measurement scalar product
\[ \langle \psi | \phi \rangle = \frac{1}{(\Delta t)^2} \int d\vec{x} \int_{t_1}^{t_2+\Delta t} dt_1 \int_{t_1}^{t_2+\Delta t} dt_2 \left< \psi(t_1, \vec{x}) | \gamma^0 \phi(t_2, \vec{x}) \right>, \quad (4.22) \]
where we take the average over a small time interval before computing the spatial inner product. A more realistic measurement device is of course much more complicated, but fortunately the details are of no relevance. All that matters is that the measurement scalar product involves a homogenization process, with the effect that the direct summands in (4.2) are in general not orthogonal with respect to \( \langle \cdot | \cdot \rangle \) and that the null space \( \mathcal{G}_0 \) in (4.13) becomes non-trivial.

4.4. Justification of an independent dynamics of the subsystems

Following the concept of discrete spacetime in section 2.1, spacetime is not smooth on the microscopic scale, but it should have a non-trivial microstructure. In order to explain the possible consequences of such a microstructure in a simple setting, we consider what happens
if we choose the wavefunctions $\psi_j^{(a)}$ in the two subregions differently. More specifically, we transform the wavefunctions in the second subsystem by a unitary matrix of determinant 1,

$$
\psi_j^{(1)} \rightarrow \psi_j^{(1)}, \quad \psi_j^{(2)} \rightarrow \tilde{\psi}_j^{(2)} := \sum_{k=1}^f U_{jk} \psi_k^{(2)} \quad \text{with} \quad U \in SU(f). \quad (4.23)
$$

This transformation has the advantage that it has no effect on the many-particle wavefunctions (4.21) because (exactly as in (3.10))

$$
\Psi^{(2)} \rightarrow \Psi^{(2)} = \psi_1^{(2)} \wedge \ldots \wedge \psi_f^{(2)} = \det U \psi_1^{(2)} \wedge \ldots \wedge \psi_f^{(2)} = \Psi^{(2)}. \quad (4.24)
$$

Furthermore, the fermionic projector does not change if its two arguments are in the same subsystem, because in the case $x, y \in M_2$, the unitarity of $U$ yields that

$$
P(x, y) \rightarrow - \sum_{j=1}^f |\tilde{\psi}_j^{(2)}(x)\rangle \langle \psi_j^{(2)}(y)| = - \sum_{j,k=1}^f (UU_\dagger)_{jk} |\psi_j^{(2)}(x)\rangle \langle \psi_k^{(2)}(y)|
$$

$$
= - \sum_{j=1}^f |\psi_j^{(2)}(x)\rangle \langle \psi_j^{(2)}(y)| = P(x, y).
$$

However, if the two arguments of the fermionic projector are in different spacetime regions, the operator $U$ does not drop out. For example, if $x \in M_2$ and $y \in M_1$,

$$
P(x, y) \rightarrow - \sum_{j=1}^f |\tilde{\psi}_j^{(2)}(x)\rangle \langle \psi_j^{(1)}(y)| = - \sum_{j,k=1}^f U_{jk} |\psi_j^{(2)}(x)\rangle \langle \psi_k^{(1)}(y)|. \quad (4.25)
$$

In the special case when $U$ is a diagonal matrix whose entries are phase factors,

$$
U = \text{diag}(e^{\i \varphi_1}, \ldots, e^{\i \varphi_f}) \quad \text{with} \quad \sum_{j=1}^f \varphi_j = 0 \mod 2\pi,
$$

the summations in (4.25) reduce to one sum involving the phase factors,

$$
P(x, y) \rightarrow - \sum_{j=1}^f e^{\i \varphi_j} |\psi_j^{(2)}(x)\rangle \langle \psi_j^{(1)}(y)|.
$$

If the angles $\varphi_j$ are chosen stochastically, the phases of the summands are random. As a consequence, there will be cancellations in the sum, and keeping in mind that the number of summands is very large, we conclude that $P(x, y)$ will be very small. More generally, we find that if $U$ is a random matrix, $P(x, y)$ becomes small if $x$ and $y$ lie in different subsystems (this argument is quantified in section 4.6 below by integrating over the space of unitary matrices).

From the physical point of view, the above consideration can be understood using the notion of decoherence. If the one-particle wavefunctions $\psi_j^{(1)}$ and $\psi_j^{(2)}$ are coherent or ‘in phase’, then the fermionic projector $P(x, y)$ has the usual form, no matter whether $x$ and $y$ are in the same subsystem or not. If however the wavefunctions in the subregions are decoherent or ‘out of phase’, then the fermionic projector $P(x, y)$ will be very small if $x$ and $y$ are in different subregions. We refer to this effect as the decoherence between spacetime regions. It should be carefully distinguished from the decoherence of the many-particle wavefunction (see for example [13]). Namely, as we saw in (4.24), in our case the many-particle wavefunctions remain unchanged. Thus they remain coherent, no decoherence in the sense of [13] appears. But the one-particle wavefunctions become decoherent (4.23), having an influence on the fermionic projector (4.19).
We next consider the influence of the decoherence between spacetime regions on the dynamics of our system. We begin by discussing the extreme case where $P(x, y)$ vanishes identically for $x$ and $y$ in different subsystems. Then the action (2.7) splits into the sum of the actions of the two subsystems, so that the interaction takes place independently in the two subsystems. In other words, the subsystems decouple. By restricting the two different systems in Minkowski space to $M_1$ respectively $M_2$, one can apply the methods of section 2.3 to both subsystems separately. Then each subsystem is described by an independent continuum limit in terms of a Dirac equation (2.11) coupled to a classical field (2.13). This explains the assumption of an independent dynamics of the subsystems made in section 4.2. We point out that, following the concept that the wavefunctions generate the causal and geometric structure of spacetime (see the last paragraph in section 2.3), the decoupling of the subsystems even implies that between the subsystems, the usual causal and topological structure of Minkowski space ceases to exist.

If we merely assume that $P(x, y)$ is small for $x$ and $y$ in different subsystems, our action principle (2.7) does describe a coupling of the two subsystems, which however should be weak. Keeping in mind that the causal structure of Minkowski space is related to the singularities of distributions like (2.9) on the light cone, and that this singular structure will be destroyed by decoherence, we know that the coupling of the two subsystems cannot be described by causal equations formulated in Minkowski space. Although we have a precise mathematical framework (2.7), describing the coupling of the subsystems quantitatively seems very difficult and goes beyond the scope of this paper. But from the mathematical structure of our action it is already clear that we do not get contributions from the boundaries of the two subregions. Therefore, instead of considering the ‘layers’ in figure 1, it seems more appropriate to draw in each subsystem as many disconnected ‘bubbles’ in spacetime as shown in figure 2. In view of the continuum limit, each system has an underlying smooth structure inherited from a corresponding system in Minkowski space, as indicated in figure 2 by the two ‘smooth spacetime sheets’ $M_1^{cont}$, $M_2^{cont}$. But of course, this picture should be considered only as a vain attempt to illustrate an unknown and probably very complicated microscopic structure of spacetime (maybe a more realistic picture is outlined in section 5.3).

4.5. Justification of the superposition of Fock states

We now want to verify that expectation values computed with respect to the measurement scalar product indeed involve superpositions of Fock states. For simplicity, we again consider the situation for two subsystems $M_1$ and $M_2$ and assume that the fermions in each subsystem are described by an $n$-particle Hartree–Fock state

$$\psi_1^{(a)} \wedge \cdots \wedge \psi_n^{(a)} ,$$

whereas the remaining $f-n$ particles describe the Dirac sea. Thus, we choose the one-particle wavefunctions before microscopic mixing as $\psi_1^{(a)}, \ldots, \psi_n^{(a)}, \psi_{n+1}^{(a)}, \ldots, \psi_{f}^{(a)}$, where the first $n$ wavefunctions describe the particles, whereas the other wavefunctions form the sea. We again introduce a decoherence between the subsystems by a unitary transformation of all states in the second subsystem (4.23).

Expectation values $(\psi | \tilde{\psi})$ of the one-particle wavefunctions with respect to the measurement scalar product involve a homogenization process, with the result that wavefunctions which differ only by microscopic fluctuations should be identified. More specifically, we should not distinguish between the sea states of the two subsystems. Thus, introducing on $\mathcal{F}$ the equivalence relation

$$\psi \cong \tilde{\psi} \iff (\psi - \tilde{\psi}|\psi - \tilde{\psi}) = 0,$$
we assume that
\[ \psi^{(1)}_j \cong \psi^{(2)}_j \quad \text{for all } j = n + 1, \ldots, f. \]

For ease in notation, we make this identification clear simply by omitting the corresponding superscripts \((1)\) or \((2)\).

Under this identification, the many-particle wavefunction of the whole system becomes
\[ \Psi = \psi^{(1)}_1 \land \cdots \land \psi_f \]
\[ = (\psi^{(1)}_1 + \tilde{\psi}^{(2)}_1) \land \cdots \land (\psi^{(1)}_n + \tilde{\psi}^{(2)}_n) \land (\psi_{n+1} + \tilde{\psi}_{n+1}) \land \cdots \land (\psi_f + \tilde{\psi}_f). \]
(4.26)

Multiplying out, we obtain many contributions. One of them corresponds to the many-particle wavefunction of the first subsystem
\[ \psi^{(1)}_1 \land \cdots \land \psi^{(1)}_n \land \psi_{n+1} \land \cdots \land \psi_f, \]
(4.27)
and another is the many-particle wavefunction of the second subsystem
\[ \tilde{\psi}^{(2)}_1 \land \cdots \land \tilde{\psi}^{(2)}_n \land \tilde{\psi}_{n+1} \land \cdots \land \tilde{\psi}_f = \det U \psi^{(2)}_1 \land \cdots \land \psi^{(2)}_n \land \psi_{n+1} \land \cdots \land \psi_f \]
(4.28)

All the other contributions involve matrix elements of the unitary operator \(U\). Similar to what is explained after (4.25), all these contributions become small if \(U\) is a random matrix.

We conclude that the measurement process involves the sum of the many-particle wavefunctions (4.27) and (4.28) of the two subsystems. This justifies assumption (B) in section 4.2. Moreover, this consideration explains why for measurements one should work in the Fock space \(F^{\text{eff}}_n\) defined by (4.13).

### 4.6. Describing Fock superpositions with random matrices

In order to make the consideration of the previous section more precise, we now reformulate its mathematical core in terms of random matrices. Note that \(SU(f)\) is a compact Lie group, on which we consider a probability measure \(d\mu\). Then taking the average over a random matrix \(U \in SU(f)\) corresponds to integrating \(U\) with respect to \(d\mu\). The simplest choice for \(d\mu\) is the normalized Haar measure \(d\mu_{\text{Haar}}\) (see for example [2, section I.5]), but other choices are possible, as will be discussed below.

We first observe that certain products of matrix elements of \(U\) vanish on average. For simplicity, we consider the Haar measure.

**Lemma 4.2.** Suppose that for any \(p\) in the range \(1 \leq p \leq f - 1\), we choose the indices \(i_1, \ldots, i_p\) and \(j_1, \ldots, j_p\) with \(i_1 < \cdots < i_p\). Then
\[ \int_{SU(f)} U_{i_1j_1} \cdots U_{i_pk} \, d\mu_{\text{Haar}} = 0. \]

**Proof.** We let \(k\) be an index which is not contained in \([i_1, \ldots, i_p]\) and let \(V\) be the diagonal matrix which has entries 1, except that \(V_{i_1i_1} = V_{kk} = -1\). Then \(V \in SU(f)\), and thus a variable transformation shows that the above integral is invariant under the replacement \(U \to VU\). But this transformation flips the sign of the integrand. \(\square\)

Applying this lemma to expression (4.25), we see that the fermionic projector \(P(x, y)\) indeed vanishes for \(x\) and \(y\) in different subregions, if the mean value over \(SU(f)\) is taken. The lemma also applies to the contributions obtained by multiplying out (4.26). It shows that all contributions vanish on average, except for the many-particle wavefunctions (4.27) and (4.28) of the two subsystems.
Since the expectation value of a measurement involves the absolute square of the wavefunctions, we also need to integrate the absolute square of the many-particle wavefunction \((4.26)\) over \(SU(f)\). We begin with a simple integral involving the absolute square of one matrix element of \(U\).

**Lemma 4.3.** For any \(j, k \in \{1, \ldots, f\}\),
\[
\int_{SU(f)} |U_{jk}|^2 \, d\mu_{\text{Haar}} = \frac{1}{f},
\]  
(4.29)

**Proof.** By multiplying with suitable unitary operators from the left or the right, we can arbitrarily change the values of the indices \(j\) and \(k\), without changing the integral. Thus,
\[
\int_{SU(f)} |U_{jk}|^2 \, d\mu_{\text{Haar}} = \frac{1}{f^2} \int_{SU(f)} \sum_{j,k=1}^{f} |U_{jk}|^2 \, d\mu_{\text{Haar}} = \frac{1}{f^2} \int_{SU(f)} \text{Tr}(U^*U) \, d\mu_{\text{Haar}} = \frac{1}{f},
\]
concluding the proof. \(\Box\)

Applying this result to \((4.25)\), we see that decoherence typically scales the kernel of the fermionic projector down by a factor \(f^{-1/2}\). This quantifies that \(P(x, y)\) really becomes small if \(x\) and \(y\) lie in different subregions.

Lemma 4.3 could be generalized to integrals involving the absolute squares of \(n\) matrix elements, giving the result
\[
\int_{SU(f)} |U_{j_1 k_1}|^2 \cdots |U_{j_n k_n}|^2 \, d\mu_{\text{Haar}} \sim \frac{1}{f^n} \text{ if } n \ll f.
\]
This shows that every fixed summand obtained by multiplying out \((4.26)\) except for \((4.27)\) and \((4.28)\) vanishes in the limit \(f \to \infty\). Unfortunately, this is not quite good enough, because the number of summands becomes large if \(f\) increases. Thus, in order to estimate the whole sum of terms, we need to use a different method, which we now explain.

To describe the combinatorics of the wavefunctions \(\psi_1^{(1)}\) and \(\psi_2^{(2)}\), we consider a subset \(I^{(1)} \subset \{1, \ldots, n\}\) and take its complement \(I^{(2)} = \{1, \ldots, n\} \setminus I^{(1)}\). As the case of no particles is trivial, we may assume that \(n \geq 1\). Suppose that we are interested in the contribution to \((4.26)\) of the form
\[
\Psi \asymp c(U) \left( \bigwedge_{i \in I^{(1)}} \psi_i^{(1)} \right) \wedge \left( \bigwedge_{j \in I^{(2)}} \psi_j^{(2)} \right) \wedge \psi_{n+1} \wedge \cdots \wedge \psi_f
\]
with a complex prefactor \(c(U)\). Multiplying out only the first \(n\) factors in \((4.26)\) and using the definition of \(\tilde{\psi}_i^{(2)}\) in \((4.23)\), we find that
\[
c(U) = \text{sign}(I^{(1)}) \det(X^{(1)}) + UX^{(2)},
\]
where \(X^{(a)}\) are the diagonal matrices
\[
(X^{(a)})^{(i)} = \delta_{j}^{i} \chi_{I^{(a)} \cup \{n+1, \ldots, f\}}(i)
\]
(here \(\chi\) is the characteristic function, and \(\text{sign}(I)\) is defined by
\[
\text{sign}(I) = (-1)^{i_1 + \cdots + i_g + \sum_{k=1}^{g} i_k}
\]
if we consider \(I\) as the ordered set \(I = (i_1, \ldots, i_g)\) with \(1 \leq i_1 < i_2 < \cdots < i_g \leq f\)). Thus, to take the average of \(\Psi\) and \(|\Psi|^2\), we need to compute the integrals
\[
\int_{SU(f)} \det(X^{(1)} + UX^{(2)}) \, d\mu \quad \text{and} \quad \int_{SU(f)} |\det(X^{(1)} + UX^{(2)})|^2 \, d\mu,
\]
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respectively. We get agreement with the Fock space formalism if and only if the following identities hold:

\[
\lim_{f \to \infty} \int_{SU(f)} \det(X^{(1)} + UX^{(2)}) \, d\mu = \delta_{I^{(1)}, \emptyset} + \delta_{I^{(2)}, \emptyset} \quad (4.30)
\]

\[
\lim_{f \to \infty} \int_{SU(f)} \frac{\det(X^{(1)} + UX^{(2)}) \det(\tilde{X}^{(1)} + U \tilde{X}^{(2)})}{\det(\tilde{X}^{(1)} + U \tilde{X}^{(2)})} \, d\mu = c \delta_{I^{(1)}, \tilde{I}^{(1)}} (\delta_{I^{(1)}, \emptyset} + \delta_{I^{(2)}, \emptyset}) \quad (4.31)
\]

Here the parameter \( c > 0 \) is an overall constant (to be chosen independent of \( f \)), which can be absorbed into the definition of the scalar product on the Fock space \( \mathcal{F}^{\text{eff}}_n \). In the above integrals we worked with a general probability measure \( d\mu \). Let us discuss how it is to be chosen. Ultimately, this measure should be determined by analyzing the statistics of the decoherent subsystems which form according to the action principle (2.7). Since such an analysis is not yet available, we must rely on heuristic considerations. One guide line is that \( d\mu \) should respect all symmetries of the underlying framework. The simplest measure which meets this requirement is the Haar measure \( d\mu_{\text{Haar}} \). Writing the above determinant as

\[
\det(11 + UX^{(2)})
\]

one sees that the identity matrix is distinguished, and thus we can choose \( d\mu \) more generally as any measure which is formed of \( U \) and 11. For example, one could choose \( d\mu \) to be a constant times the measure

\[
|\text{Tr}(11 + U)|^\alpha d\mu_{\text{Haar}}
\]

with \( \alpha \in \mathbb{R} \), but many other choices are possible (see for example [15]). The question for which choices of \( d\mu \) relations (4.30) and (4.31) hold is an open problem which will be considered elsewhere.

We finally remark that the above considerations immediately generalize to more than two subsystems, if one replaces the term \( \det(X^{(1)} + UX^{(2)}) \) by

\[
\det(X^{(1)} + U_2X^{(2)} + \ldots + U_LX^{(L)})
\]

and integrates over all random matrices \( U_2, \ldots, U_L \in SU(f) \).

5. Second quantization of the bosonic field

In section 4.4 we considered two decoherent subsystems \( M_1 \) and \( M_2 \) and saw that by analyzing each subsystem in the continuum limit, we could describe the dynamics by the Dirac equation coupled to a classical field. Taking a finite number of such decoherent subsystems, the whole dynamics is described by several classical fields, one for each subsystem. In this section we show that the resulting framework indeed allows for the description of second-quantized bosonic fields.

For simplicity, we merely consider an electromagnetic field (the generalization to other bosonic fields is straightforward). We subdivide Minkowski space into \( L \) disjoint regions \( M_1, \ldots, M_L \), which are again assumed to be fine grained. Similar to (4.19) and (4.20), the fermionic projector can be written as

\[
P(x, y) = -\sum_{j=1}^{f} |\psi_j(x)\rangle\langle\psi_j(y)| \quad \text{with} \quad \psi_j = \sum_{a=1}^{L} \psi_j^{(a)}, \quad \psi_j^{(a)} := \psi_j \chi_{M_a}, \quad (5.1)
\]

where \( f \) is a large number which tends to infinity if the ultraviolet regularization is removed. As in section 4.4, we arrange by unitary transformations of the form (4.23) that the subsystems
are decoherent. Considering each subsystem in the continuum limit, we obtain similar to (2.11) and (2.13) the Dirac–Maxwell equations
\[
(\gamma^j (\partial_j - ieA^{(a)}_j) - m) P^{(a)}(x, y) = 0
\]
\[
\partial_j A^{(a)}_j - \Box A^{(a)}_j = e \sum_{k=1}^{N_q} \langle \psi^{(a)}_k | \gamma_j \psi^{(a)}_k \rangle - e \sum_{l=1}^{N_q} \langle \phi^{(a)}_l | \gamma_j \phi^{(a)}_l \rangle.
\]
(5.2)

We note for clarity that according to (4.20), the wavefunctions \( \psi^{(a)}_k \) and \( \phi^{(a)}_l \) are obtained by restriction to a subregion \( M_a \subset M \) of spacetime. But it is reasonable to assume that these wavefunctions are macroscopic in the sense that they can be extended smoothly to the whole Minkowski space. Similarly, we assume that the potentials \( A^{(a)} \) and the fermionic projectors \( P^{(a)} \) are defined on a whole sheet \( M^\text{com}_a \) of Minkowski space (see figure 2).

5.1. Describing a second-quantized free bosonic field

In order to get a simple connection to standard textbooks like \([14, 22]\), we begin with a free electromagnetic field (i.e. the situation where no fermionic particles or anti-particles are present). Furthermore, to avoid the technical issues involved in taking an infinite volume limit, we restrict attention to the situation in finite spatial volume by considering a three-dimensional box of length \( \ell \) with periodic boundary conditions. Working in the Coulomb gauge \( \text{div} \vec{A} = 0 \), the Maxwell equations reduce to the ordinary wave equation for each component of the vector potential,
\[
\Box \vec{A}(t, \vec{x}) = 0,
\]
whereas the electric potential \( A^0 \) can be arranged to vanish identically. Decomposing \( \vec{A} \) into the Fourier modes of momentum \( \vec{k} \in (2\pi \mathbb{Z}/\ell)^3 \),
\[
\vec{A}(t, \vec{x}) = \sum_{\vec{k}} (\vec{a}(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}} + \overline{\vec{a}}(t, \vec{k}) e^{-i\vec{k} \cdot \vec{x}}),
\]
the Maxwell equations are solved by
\[
\vec{a}(t, \vec{k}) = \overline{\vec{a}}(\vec{k}) e^{-i\omega t} \quad \text{with} \quad \omega := |\vec{k}|,
\]
whereas the Coulomb gauge gives rise to the transversality condition \( \vec{k} \cdot \overline{\vec{a}}(\vec{k}) = 0 \) (see [14, chapter I, section 2]). The two linearly independent solutions of this transversality condition correspond to the two polarizations of the electromagnetic wave; we denote them by an index \( \beta = 1, 2 \). Introducing the canonical field variables
\[
q_{\beta}(\vec{k}) = \frac{1}{4\pi} (a_\beta(\vec{k}) + \overline{a}_\beta(\vec{k})), \quad p(\vec{k}) = \frac{d}{dt} q_{\beta}(\vec{k}) = -\frac{i\omega}{4\pi} (a_\beta(\vec{k}) - \overline{a}_\beta(\vec{k})),
\]
the energy \( H \) of the classical electromagnetic field becomes (for details see [14, chapter I, section 2])
\[
H = \sum_{\vec{k} \in (2\pi \mathbb{Z}/\ell)^3} \sum_{\beta = 1, 2} \frac{1}{2} \left( p_{\beta}(\vec{k})^2 + \omega^2 q_{\beta}(\vec{k})^2 \right).
\]
(5.3)

Here each summand is the Hamiltonian of a harmonic oscillator. Thus, we have rewritten the classical electromagnetic field as an infinite collection of classical harmonic oscillators.

The second quantization of the electromagnetic field corresponds to quantizing each harmonic oscillator as in standard quantum mechanics (see for example \([22, \text{part I, section 1.2}]\)). We proceed by discussing the connection between the classical and the quantum
dynamics in detail, for simplicity for a single harmonic oscillator of frequency $\omega$. Thus, our starting point is the classical Hamiltonian

$$h(p, q) = \frac{1}{2}(p^2 + \omega^2 q^2). \quad (5.4)$$

Here $q$ and $p$ are the canonical variables, which together form the classical phase space $\mathcal{P} = \{(p, q) \text{ with } p, q \in \mathbb{R}\}$. The classical dynamics is described by Hamilton’s equations

$$\frac{dq}{dt} = \frac{\partial h}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial h}{\partial q} = -\omega^2 q.$$

A solution $(p(t), q(t))$ describes a classical trajectory. Solving Hamilton’s equations, the classical dynamics describes a rotation in phase space,

$$\begin{pmatrix} p(t) \\ \omega q(t) \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} p(0) \\ \omega q(0) \end{pmatrix}. \quad (5.5)$$

In order to get a setting similar to that in quantum theory, we next consider on phase space complex-valued functions $\psi(p, q)$, referred to as ‘classical wavefunctions’. Introducing the scalar product

$$\langle \psi | \phi \rangle_{\text{class}} = \int_{\mathbb{R} \times \mathbb{R}} \overline{\psi(p, q)} \phi(p, q) \, dp \, dq, \quad (5.6)$$

the classical wavefunctions form a Hilbert space $(\mathcal{H}_{\text{class}}, \langle \cdot | \cdot \rangle_{\text{class}})$. The phase flow (5.5) induces a flow on $\mathcal{H}$, which is described most conveniently by the time evolution operator $U_{\text{class}}$ defined by

$$U_{\text{class}}(t)\psi(p(t), q(t)) = U_{\text{class}}(0)\psi(p(0), q(0)). \quad (5.7)$$

It is a unitary operator on $\mathcal{H}_{\text{class}}$. Before going on, we remark that in classical physics one usually works instead of complex functions with positive functions or densities on phase space. Working with complex-valued functions and the scalar product (5.6) seems unusual but will be very useful for the following considerations. In a somewhat different context, the Hilbert space $(\mathcal{H}_{\text{class}}, \langle \cdot | \cdot \rangle_{\text{class}})$ is also used in geometric quantization for the so-called prequantization (see [21, section 5.2]).

For the quantization of the oscillator, one replaces the canonical variables $p$ and $q$ by the self-adjoint operators $\hat{P}$ and $\hat{Q}$ which act on a complex Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ and satisfy the canonical commutation relations $[\hat{P}, \hat{Q}] = -i$. The physical system is now characterized by a state $\Psi \in \mathcal{H}$. The dynamics is described by the Schrödinger equation

$$i\hbar \frac{d}{dt} \Psi = \hat{H} \Psi \quad \text{with} \quad \hat{H} = \frac{1}{2}(\hat{P}^2 + \omega^2 \hat{Q}^2). \quad (5.8)$$

It is most common to represent $\mathcal{H}$ as the space of square integrable functions with the inner product

$$\langle \Psi | \Phi \rangle = \int_{\mathbb{R}} \overline{\Psi(q)} \Phi(q) \, dq, \quad (5.9)$$

and to choose the operators $\hat{Q}$ and $\hat{P}$ as

$$\langle \hat{Q} \psi(q) | q \psi(q) \rangle \quad \text{and} \quad \hat{P} = -i \frac{d}{dq}.$$ Integrating the Schrödinger equation gives rise to the unitary time evolution operator

$$U(t) = e^{-i\hat{H}t} : \mathcal{H} \rightarrow \mathcal{H} : \Psi(0, q) \mapsto \Psi(t, q). \quad (5.10)$$

With the above formulation we expressed both the classical and the quantum dynamics by a unitary time evolution operator acting on a Hilbert space (see (5.7), (5.6) and (5.10), (5.9)). But the time evolution operators have a completely different form. Furthermore,
the Hilbert spaces are different because the ‘classical wavefunctions’ depend on both \( q \) and \( p \). In the quantized theory, however, the Heisenberg uncertainty principle prevents \( P \) and \( Q \) from being simultaneously measurable, as is reflected mathematically by the fact that they correspond to non-commuting operators. Since in the classical theory, position and momentum can be chosen independently, there is much more freedom to choose the initial wavefunction \( \psi(p, q) \) than in quantum theory, where choosing \( \Psi_1(q) \) automatically determines the corresponding wavefunction in momentum space. This raises the question if for a given quantum wavefunction \( \Psi(q) \) we can choose a corresponding classical wavefunction \( \psi(p, q) \) such that the classical dynamics of \( \psi \) as described by (5.7) coincides with the time evolution of the quantum wavefunction (5.10). While the general answer to this question is no, it turns out that for the harmonic oscillator this correspondence can indeed be made.

**Lemma 5.1** (Correspondence between classical and quantum dynamics). Consider the classical harmonic oscillator (5.4) with dynamics (5.5) and (5.7), and the corresponding quantum harmonic oscillator with the dynamics described by the Schrödinger equations (5.8) and (5.10). Then there is an isometric embedding \( \iota : \mathcal{H} \to \mathcal{H}_{\text{class}} \) which maps the quantum evolution onto a corresponding classical evolution, in the sense that

\[
U_{\text{class}}(t) \iota = \iota U(t) e^{i\omega t/2} \quad \text{for all} \quad t \in \mathbb{R}.
\]

Moreover, there are differential operators \( H_{\text{class}}, P_{\text{class}} \) and \( Q_{\text{class}} \) in \( \mathcal{H}_{\text{class}} \) such that

\[
H_{\text{class}} = \iota H, \quad P_{\text{class}} = \iota P \quad \text{and} \quad Q_{\text{class}} = \iota Q.
\]

We point out that the factor \( e^{i\omega t/2} \) in (5.11) corresponds to the zero point energy of the quantum harmonic oscillator. This factor modifies the wavefunctions only by a joint global phase, without an influence on any observations or expectation values.

**Proof of lemma 5.1.** We choose an orthonormal eigenvector basis \( \Psi_n \) of the Hamiltonian in (5.8) (see for example [17, section 3.1])

\[
H \Psi_n = \left(n + \frac{1}{2}\right) \omega \Psi_n, \quad n = 0, 1, \ldots
\]

Writing the Hamiltonian as \( H = \omega(a^\dagger a + \frac{1}{2}) \) with the annihilation and creation operators

\[
a = \frac{1}{\sqrt{2\omega}} \left( \omega q + \frac{d}{dq} \right), \quad a^\dagger = \frac{1}{\sqrt{2\omega}} \left( \omega q - \frac{d}{dq} \right),
\]

the eigenvectors can be obtained by acting with the creation operators on the ground state,

\[
\Psi_0 = c_0 \exp \left(-\frac{\omega q^2}{2}\right) \quad \text{and} \quad \Psi_n = c_n (a^\dagger)^n \Psi_0,
\]

where the \( c_n \) are positive normalization constants. From (5.10) it follows immediately that

\[
U(t) e^{i\omega t/2} \Psi_n = e^{-i\omega t} \Psi_n.
\]

In order to define the mapping \( \iota \), it suffices to associate with every eigenfunction \( \Psi_n \) the corresponding classical wavefunctions \( \psi_n \in \mathcal{H}_{\text{class}} \) (then \( \iota \) is determined uniquely by linearity and continuity). First, in order to write the classical dynamics in a simpler form, we rescale the momentum variable by introducing the new phase space variables

\[
x = q \quad \text{and} \quad y = \frac{p}{\omega}.
\]

Setting \( z = x + iy \), the time evolution operator (5.7) becomes

\[
(U_{\text{class}}(t) \psi)(z) = \psi(e^{i\omega t} z).
\]
We now define the ‘classical annihilation and creation operators’ on $H_{\text{class}}$ by
\[
a_{\text{class}} = \frac{1}{\sqrt{2}} (a_{x} + ia_{y}), \quad a_{\text{class}}^{\dagger} = \frac{1}{\sqrt{2}} (a_{x}^{\dagger} + ia_{y}^{\dagger}),
\]
where $a_{x}$ and $a_{\dagger x}$ are given in analogy to (5.12) by
\[
a_{x} = \frac{1}{\sqrt{2\omega}} \left( \omega x + \frac{d}{dx} \right), \quad a_{x}^{\dagger} = \frac{1}{\sqrt{2\omega}} \left( \omega x - \frac{d}{dx} \right),
\]
whereas the subscript $y$ refers similar to the variable $y$. We introduce the wavefunctions $\psi_{n}$ as
\[
\psi_{0} = c_{0}^{2} \exp\left( -\frac{\omega(x^{2} + y^{2})}{2} \right) \quad \text{and} \quad \psi_{n} = c_{n}(a_{\text{class}}^{\dagger})^{n} \psi_{0}.
\] (5.16)

Let us verify that the resulting mapping $\iota$ has the required properties. First, it is obvious from their definition (5.16) that the functions $\psi_{n}$ are orthonormal in $H_{\text{class}}$, and thus $\iota$ is indeed an isometric embedding. Using a polar decomposition $z = r e^{i\phi}$, a short calculation shows that
\[
[i \partial \phi, a_{\text{class}}^{\dagger}] = a_{\text{class}}^{\dagger}.
\]

Applying this relation in (5.16) and using that $\psi_{0}$ is radially symmetric, we obtain
\[
i \partial \phi \psi_{n} = n \psi_{n}, \quad \text{and thus} \quad \psi_{n}(z) = e^{-in\phi} \phi_{n}(r),
\]
with the radially symmetric functions $\phi_{n}$. Thus, the classical dynamics (5.15) implies that
\[
U_{\text{class}}(t) \psi_{n} = e^{-i\omega nt} \psi_{n},
\]
Comparing with (5.14) proves (5.11).

In order to construct the operators $H_{\text{class}}, P_{\text{class}}$ and $Q_{\text{class}}$, we first note that both classical and quantum annihilation and creation operators satisfy the canonical commutation relations
\[
[a_{\text{class}}^{\dagger}, a_{\text{class}}] = 1 \quad \text{and} \quad [a^{\dagger}, a] = 1,
\]
and in view of (5.13) and (5.16) they correspond to each other in the sense that
\[
a_{\text{class}}^{\dagger} = \iota a \quad \text{and} \quad a_{\text{class}}^{\dagger} = \iota a^{\dagger}.
\]

Thus, expressing the operators in $\mathcal{H}$ in terms of $a$ and $a^{\dagger}$, we obtain the corresponding ‘classical’ operators simply by adding subscripts. Thus, we set
\[
H_{\text{class}} = \omega (a_{\text{class}}^{\dagger} a_{\text{class}} + \frac{1}{2}),
\]
and
\[
Q_{\text{class}} = \frac{1}{\sqrt{2\omega}} (a_{\text{class}}^{\dagger} + a_{\text{class}}^{\dagger}), \quad P_{\text{class}} = -i \sqrt{\frac{\omega}{2}} (a_{\text{class}} - a_{\text{class}}^{\dagger}).
\]
This concludes the proof. $\square$

We remark that the mapping $\iota$ appears in the mathematical physics literature as the so-called Bargmann transform (see [20, section 4.3]). But to our knowledge, it has not been used to get a connection between the classical and quantum dynamics.

The above lemma shows that by choosing the ‘classical wavefunction’ $\phi \in H_{\text{class}}$ appropriately, we can arrange that the classical dynamics reproduces any quantum dynamics. In simpler terms, the quantum dynamics of the harmonic oscillator can be recovered as a special case of the classical dynamics. However, for making this correspondence, we had to take a somewhat unusual point of view and work on classical phase space with complex-valued functions and the scalar product (5.6). To us, lemma 5.1 is useful because it makes it possible to approximate a quantum state by a finite number of classical trajectories, if with every classical trajectory we associate a complex number. This can be seen as follows. Suppose that we want to describe a quantum state $\Psi \in \mathcal{H}$. According to lemma 5.1, this state has the
same dynamics as the classical wavefunction \( \psi := i\Psi \in \mathcal{H}_{\text{class}} \). For any \( L \in \mathbb{N} \) and an index \( a = 1, \ldots, L \), we now choose the points \( (p^{(a)}, q^{(a)}) \) in phase space together with the complex coefficients \( \phi(a) \) which approximate \( \psi \) in the sense that

\[
\sum_{a=1}^{L} \phi(a) \delta(p - p^{(a)}) \delta(q - q^{(a)}) \xrightarrow{L \to \infty} \psi(p, q),
\]

(5.17)

with convergence in the distributional sense. For these discrete configurations, we can make sense of the scalar product (5.6) by setting

\[
\langle (p^{(a)}, q^{(a)}, \phi(a)) | (\tilde{p}^{(b)}, \tilde{q}^{(b)}, \tilde{\phi}(b)) \rangle = \sum_{a,b} \delta_{p^{(a)}, \tilde{p}^{(b)}} \delta_{q^{(a)}, \tilde{q}^{(b)}} \phi(a) \tilde{\phi}(b).
\]

Thus, by choosing \( L \) sufficiently large, we can approximate the quantum dynamics of \( \Psi \) by a complex-valued function \( \phi \) defined on a finite number of classical trajectories.

Before going on, we point out that the scalar product (5.6) is invariant under the local phase transformations

\[
\phi(p, q) \to e^{i\psi(p, q)} \phi(p, q),
\]

(5.18)

where \( \psi \) is a real-valued function on phase space. Thus, the phase of the functions in \( \mathcal{H}_{\text{class}} \) has no physical relevance. What counts is only the relative phase when taking the superpositions of two wavefunctions \( \psi, \phi \in \mathcal{H}_{\text{class}} \). Similarly, in the discrete approximations in (5.17), the phase of the function \( \phi(a) \) can be changed by

\[
\phi(a) \to e^{i\psi(a)} \phi(a) \quad \text{under the constraint} \quad (p^{(a)}, q^{(a)}) = (p^{(b)}, q^{(b)}) \implies \psi(a) = \psi(b).
\]

In the remainder of this section, we extend the above considerations on the harmonic oscillator to the Hamiltonian of the electromagnetic field (5.3). By taking tensor products, the result of lemma 5.1 immediately extends to a collection of harmonic oscillators as in (5.3). It then states that by considering suitable complex-valued functions on the set of all classical field configurations, one can reproduce the full dynamics of the free second-quantized field. Using an approximation argument similar to (5.17), it suffices to consider a finite number of classical field configurations. Thus, the remaining task is to associate with every classical field configuration a complex number \( \phi(a) \). Let us return to the setting of decoherent subsystems in the continuum limit (5.2). Then every subsystem involves a classical electromagnetic potential \( A^{(a)} \).

In the considered case without fermions, the field equations reduce to the free Maxwell equations, i.e. again in the Coulomb gauge

\[
\Box \tilde{A}^{(a)}(t, \tilde{x}) = 0,
\]

(5.19)

and \( (A^{(a)})^{0}(t, \tilde{x}) = 0 \). Moreover, we have the Dirac equation for the fermionic projector, which according to (2.12) consists only of the sea states,

\[
(i\gamma^j (\partial_j - ieA_j^{(a)}) - m) P^{\text{sea}}(x, y) = 0 \quad \text{if} \quad x \in M_a.
\]

(5.20)

Having an ultraviolet regularization in mind, the number \( f \) of sea states is finite, so that \( P^{\text{sea}} \) can be written in the form (5.1). For a given solution \( \tilde{A} \) of the free Maxwell equations, we now introduce the following reference system. The causal perturbation expansion distinguishes a subspace of the solution space of the Dirac equation as being formed by the sea states. Selecting the \( f \)-dimensional subspace of the sea states which is compatible with our regularization and choosing an orthonormal basis \( \tilde{\psi}_1, \ldots, \tilde{\psi}_f \) of this subspace, we can introduce the many-particle wavefunction \( \tilde{\Psi} \) of our reference system as

\[
\tilde{\Psi} = \tilde{\psi}_1 \wedge \cdots \wedge \tilde{\psi}_f.
\]

As in (3.10), the freedom in choosing the orthonormal basis implies that \( \tilde{\Psi} \) is determined only up to a phase. Now suppose that \( \tilde{A} \) coincides with the electromagnetic potential \( A_a \) in one of our
subsystems. Then the wavefunctions $\psi_1^{(a)}, \ldots, \psi_f^{(a)}$ obtained by restricting the wavefunctions of the fermionic projector of the whole system to the subsystem $M_a$ span the same subspace of the Dirac solutions as the vectors $\psi_1, \ldots, \psi_f$ (probably after suitably modifying the solutions on the microscopic scale or modifying the regularization; a technical issue which for simplicity we ignore here). Hence the corresponding many-particle wavefunction (4.21) coincides with $\tilde{\Psi}$ up to a complex number,

$$\Psi^{(a)} = \phi(a)\tilde{\Psi} \quad \text{with} \quad \phi(a) \in \mathbb{C}. \quad (5.21)$$

In this way, we have associated with the field configuration $A_a$ a complex number $\phi(a)$.

Let us consider the phase freedom. As noted above, the phase of the wavefunction $\tilde{\Psi}$ depends on the choice of the basis $\psi_1, \ldots, \psi_f$. Similarly, by transforming the orthonormal basis $\psi_1, \ldots, \psi_f$ of the image of $P$, we can also change the phase of $\Psi^{(a)}$ arbitrarily. Thus, (5.21) is well defined only up to a phase. Now suppose that $A$ also coincides with the electromagnetic potential $A_b$ of another subsystem. Then writing the many-particle wavefunction of the new subsystem as $\tilde{\Psi}^{(b)} = \phi(b)\tilde{\Psi}$, transforming the bases $\psi_1, \ldots, \psi_f$ or $\psi_1^{(a)}, \ldots, \psi_f^{(a)}$ changes the phase of both $\phi(a)$ and $\phi(b)$ in the same way. Thus, the relative phase of $\phi(a)$ and $\phi(b)$ is well defined. In other words, the complex-valued function $\phi$ is uniquely defined up to the transformations

$$\phi(a) \rightarrow e^{i\phi(a)}\phi(a) \quad \text{under the constraint} \quad A_a = A_b \implies \psi(a) = \psi(b). \quad (5.22)$$

These transformations can be regarded as local phase transformations on the classical field configurations, just as explained after (5.18) for one harmonic oscillator on phase space.

We conclude that the above construction indeed yields a complex-valued wavefunction $\phi(a)$, $a = 1, \ldots, L$, defined on the classical field configurations $\{A^{(1)}, \ldots, A^{(L)}\}$ of the subsystems. It is uniquely determined up to the local phase transformations (5.22). These results make it possible to approximate a general state of the bosonic Fock space by our decoherent subsystems, as the following consideration shows. According to lemma 5.1, the dynamics of a given bosonic Fock state can be described by a complex-valued wavefunction $\phi$ on the classical field configurations. By considering similar to (5.17) a sequence of systems where the number of decoherent subsystems tends to infinity, we can approximate $\phi$ by a finite collection of classical field configurations $\{A^{(1)}, \ldots, A^{(L)}\}$ and a corresponding complex-valued functions $\phi(a)$. By suitably adjusting the phases of the sea states $\psi_1^{(a)}, \ldots, \psi_f^{(a)}$ (for the given reference systems $\tilde{\Psi}$), we can arrange that the function $\phi(a)$ satisfies (5.21). Then the wavefunctions $\psi_1, \ldots, \psi_f$ of the whole system encode the classical potentials $\{A^{(1)}, \ldots, A^{(L)}\}$ as well as the complex-valued function $\phi(a)$, which together approximate the given bosonic Fock state.

5.2. Describing a second-quantized fermion–boson system

We now combine the considerations of the previous section with the constructions of section 4 to obtain a unified framework for describing second-quantized fermions and bosons. We again consider $L$ decoherent subsystems in the continuum limit (5.2). According to (2.12), we can split up the fermionic projector into the particle and anti-particle as well as the sea states. We begin for clarity in the situation without pair creation where the numbers $n_a$ and $n_f$ of particles and anti-particles are constant and coincide in all subsystems (this constraint will be removed below). Then setting $n = n_a - n_f$, the corresponding many-particle wavefunctions of the subsystems (4.21) can be decomposed as

$$\Psi^{(a)} = (\psi_1^{(a)} \wedge \cdots \wedge \psi_{n_f}^{(a)} \wedge \phi_1^{(a)} \wedge \cdots \wedge \phi_{n_a}^{(a)}) \wedge [\psi_{n+1} \wedge \cdots \wedge \psi_f]. \quad (5.23)$$
Here the round brackets can be regarded as the fermionic wavefunction of the particles and anti-particles. As explained in section 4.5, measurements involve superpositions of these many-particle wavefunctions, so that it is reasonable to regard the round brackets in (5.23) as a vector in the fermionic Fock space \( \mathcal{F}^{\text{eff}} \). Likewise, the square brackets in (5.23) describe the sea. The construction (5.21) yields a corresponding complex wavefunction \( \phi(a) \) on the classical field configurations, which can be used to describe the dynamics of a second-quantized bosonic field. In this way, we have extracted from the fermionic projector both a fermionic and a bosonic quantum field.

We now give a construction which avoids the splitting of the many-particle wavefunction into the particle/anti-particle component and the sea component. Apart from being simpler and cleaner, this construction has the advantage of working just as well for fully interacting systems, including pair creation or annihilation processes. Recall that in (5.21) we compared the many-particle wavefunction \( \Psi^{(a)}(a) \) of our subsystem with the wavefunction \( \tilde{\Psi} \) of a ‘reference system’ having the same classical field configuration. The proportionality factor \( \phi(a) \) then gave us the desired complex-valued function \( \phi \) on the classical field configurations. Giving up the requirement that the vector space \( \mathcal{H}_{\text{class}} \) should be represented by complex-valued functions, one can work instead of \( \phi(a) \) with the corresponding vector \( \Psi^{(a)}(a) \in \mathcal{F}^{\text{eff}} \). This has no effect on superpositions, because the complex coefficients \( \phi(a) \) and \( \phi(b) \) can be linearly combined only if the corresponding classical field configurations \( A^{(a)} \) and \( A^{(b)} \) coincide. But then the corresponding Fock vectors \( \Psi^{(a)}, \Psi^{(b)} \in \mathcal{F}^{\text{eff}} \) are linearly dependent, so that taking their linear combination has the same effect as taking the linear combinations of the coefficients \( \phi(a) \) and \( \phi(b) \). This leads us to replace the complex-valued function \( \phi(a) \) constructed in (5.21) by a mapping with values in \( \mathcal{F}^{\text{eff}} \),

\[
\phi : \{1, \ldots, L\} \rightarrow \mathcal{F}^{\text{eff}} : a \mapsto \psi^{(a)}_1 \wedge \cdots \wedge \psi^{(a)}_f .
\]  

(5.24)

In the setting involving particles and anti-particles (5.23), this mapping has the nice property that it involves at the same time the fermionic wavefunctions of the particles and anti-particles. In free field theory, it can be thought of as the tensor product of a fermionic and a bosonic Fock state. As desired, two such tensor states are linearly dependent only if both the fermionic and bosonic parts are. Superpositions of these tensor states can be justified exactly as explained in section 4.5. In a fully interacting system, the mapping (5.24) can no longer be decomposed into a fermionic and a bosonic part, in agreement with the fact that in interacting quantum systems the bosonic and fermionic Fock spaces are coupled together and ‘mixed’ by the Hamiltonian. Even in this in general very complicated situation, the mapping \( \phi \) gives a conceptually simple description of the whole system.

5.3. Remarks and outlook

To avoid confusion, we point out that the constructions in this section are not equivalent to the canonical quantization of the bosonic field. In particular, we do not get the physical equations for second-quantized fields. Instead, we merely show that the dynamics of free second-quantized bosonic fields can be mimicked by an ensemble of decoherent subsystems, each with a classical dynamics. However, we do not get a justification nor an explanation for the physical assumption that electromagnetic wave modes should behave like quantum mechanical oscillators. But we show that this assumption is not in conflict with the framework of the fermionic projector. In particular, it is possible to describe entangled bosonic states. In order to explain why we do not even attempt to reproduce the physical equations for second-quantized fields, we now briefly outline how interacting quantum field theory should be formulated in the framework of the fermionic projector. Recently, this formulation of quantum
field theory has been worked out in detail for a system involving an axial field [8]. The general strategy is as follows. Instead of quantizing the classical field equations, we describe the interaction and the dynamics of the system by the action principle (2.7). In the continuum limit, the Euler–Lagrange equations corresponding to this action principle give rise to the Dirac equation coupled to classical bosonic field equations (cf (2.11) and (2.13)). Treating this coupled system of nonlinear partial differential equations in a perturbation expansion gives rise to all the Feynman diagrams of perturbative quantum field theory (see [8, section 8.4]). In particular, this gives agreement with the high-precision tests of quantum field theory. We remark that we get additional small corrections to the field equations which are absent in perturbative quantum field theory; the interested reader is referred to [8, section 8.2 and 8.3].

Since the quantitative aspects are respected, it remains to explain the particular effects of quantized fields. This paper is concerned with entanglement and shows that entangled fermionic and bosonic states can be described in the framework of the fermionic projector. For other quantum effects related to the measurement problem and the wave-particle duality see also [4, section 4]. Putting these results together, it seems to us that the framework of the fermionic projector is in agreement with all effects and predictions of quantum field theory (except for the additional corrections discussed in [8, sections 8.2 and 8.3]). But this agreement cannot be stated in terms of a mathematical equivalence, partly because standard quantum field theory at present has no fully convincing mathematical formulation. Also, many difficulties of quantum field theory clearly remain unsettled in our approach. Thus, many conceptual and technical issues need to be debated in the future.

We finally point out that the agreement with free quantized fields in section 5.1 is obtained only in the limit when the number of subsystems tends to infinity. Thus, even for describing the quantum oscillations of the harmonic oscillator corresponding to one mode of the electromagnetic field, one needs to consider a large number of decoherent subsystems. Although this seems possible in principle, it seems hard to imagine that decoherence should really lead to a ‘fragmentation’ of spacetime into many disjoint regions with an independent dynamics. This raises the question whether the microscopic mixing of decoherent subsystems might not be a too simple picture for understanding the mechanisms of spacetime on a small scale. Indeed, it might be more appropriate to replace this picture by a more general concept which we now explain in words. Recall that in section 4.4 the decoherence of subsystems was introduced by inserting a unitary transformation into the fermionic projector, 

\[ P(x, y) = \sum_{j,k=1}^{f} U_{jk} \langle \psi_j^{(2)}(x) | \psi_k^{(1)}(y) \rangle, \]  

(5.25)

if \( x \) and \( y \) are in different subsystems (see (4.25)). Since \( P(x, y) \) is a 4 \times 4-matrix, a dimensional argument shows that there is a large class of operators \( U \in SU(f) \) which do not affect the form of \( P(x, y) \). In order to make use of this additional freedom, we replace \( U \) by a family of local unitary transformations \( U(x) \in SU(f) \), which brings the fermionic projector to the more general form

\[ P(x, y) = \sum_{j,k,l=1}^{f} U_{jk}(x) U_{kl}^{-1}(y) \langle \psi_j(x) | \psi_l(y) \rangle. \]  

(5.26)

By dividing \( M \) into subregions and choosing \( U(x) \) to be constant on each subregion, we get back to the setting of section 4.4. But if \( U(x) \) is not a piecewise constant function, the situation is more involved. Namely, for any spacetime points \( x \) and \( y \), it is possible that the transformation \( U_{jk}(x) U_{kl}^{-1}(y) \) has no effect on \( P(x, y) \) (similar to that discussed in (5.25)); in this case the pair \( (x, y) \) is said to be coherent. Another possibility is that the transformation
$U_{jk}(x) U_{kl}^{-1}(y)$ leads to cancellations in the sum so that $P(x, y)$ is very small (similar to that explained after (4.25)), in which case the pair $(x, y)$ is said to be decoherent. This notion of decoherence again gives a relation between spacetime points. But in contrast to the situation in section 4.4, this relation is no longer transitive; for example, it is possible that the pairs $(x, y)$ and $(y, z)$ are coherent, but the pair $(x, z)$ is decoherent. As a consequence, decoherence no longer gives rise to a decomposition of spacetime into subregions. But for any fixed spacetime point $x$, one can form the set $M(x)$ of all spacetime points which are coherent to $x$. This set can then be divided into the subsets $M_j(x)$ by the condition that any two points $y, z \in M_j(x)$ should be coherent to each other. On the sets $M_j(x)$, one can then again consider the continuum limit to obtain for example the Dirac–Maxwell system (2.11), (2.13). Thus, on the coherent spacetime points one again gets a description involving classical field equations. Decoherent pairs of spacetime points, on the other hand, are not connected by our action principle. We remark that it is also conceivable that two spacetime points are partially decoherent in the sense that there are cancellations in the sums (5.26), but without $P(x, y)$ being very small. We expect that such a partial decoherence would yield a large contribution to the action and should thus be avoided by our action principle. Then the resulting structure resembles the situation in section 4.4 in that we obtain decoherent subsystems with an independent dynamics. The main difference is that the subsystems are no longer localized in disjoint regions of spacetime. Instead, they are all delocalized, and only when picking a pair of spacetime points $(x, y)$, the phases in the sum (5.26) determine to which subsystem the pair belongs. Due to the obvious analogy to a hologram, we refer to this concept as the holographic superposition of subsystems (but it does not seem to be directly related to 't Hooft’s holographic principle). The main advantage of a holographic superposition is that a large number of subsystems nolotr leading to a ‘fragmentation’ of spacetime into disjoint spacetime regions. On the other hand, all the effects considered in this paper can be described just as well by decoherent spacetime regions. Therefore, the holographic superposition is not essential for our purposes, and we shall not enter the detailed constructions here.

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