Lower limits for distributions of randomly stopped sums

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Abstract

We study lower limits for the ratio $F^{*\tau}(x)/F(x)$ of tail distributions where $F^{*\tau}$ is a distribution of a sum of a random size $\tau$ of i.i.d. random variables having a common distribution $F$, and a random variable $\tau$ does not depend on summands.

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1. Introduction. Let $\xi, \xi_1, \xi_2, \ldots$ be independent identically distributed random variables. We assume that their common distribution $F$ is unbounded from the right, that is, $F(x) \equiv F(x, \infty) > 0$ for all $x$. Put $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n = 1, 2, \ldots$.

Let $\tau$ be a counting random variable which does not depend on $\{\xi_n\}_{n \geq 1}$. Denote by $F^{*\tau}$ the distribution of a random sum $S_\tau = \xi_1 + \ldots + \xi_\tau$. In this paper we study lower limits (as $x \to \infty$) for the ratio $F^{*\tau}(x)/F(x)$.

We distinguish two types of distributions, heavy- and light-tailed. A random variable $\eta$ has a heavy-tailed distribution if $E e^{\varepsilon \eta} = \infty$ for all $\varepsilon > 0$, and light-tailed otherwise.

We consider only non-negative random variables and, in the case of heavy-tailed $F$, study conditions for

$$\liminf_{x \to \infty} \frac{F^{*\tau}(x)}{F(x)} = E\tau$$

(1)

to hold. This problem has been given a complete solution in [5] for $\tau = 2$, and then in [3] for $\tau$ with a light-tailed distribution and for heavy-tailed summands. In the present work, we generalise results of [3] onto classes of distributions of $\tau$ which include all light-tailed distributions and also some heavy-tailed distributions. With each heavy-tailed distribution $F$, we associate a corresponding class of distributions of $\tau$. For earlier studies on lower limits and on a related problem of justifying a constant $K$ in the equivalence $F^{*\tau}(x) \sim K F(x)$, see e.g. [1, 2, 4, 7, 8] and further references therein.

Since the inequality “$\geq$” in (1) is valid for non-negative $\{\xi_n\}$ without any further assumptions (see, e.g., [2] or [3]), we immediately get the equality if $E\tau = \infty$. Therefore, in the rest of the paper, we consider the case $E\tau < \infty$ only. Our first result is

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**Theorem 1.** Assume that $\xi \geq 0$ is heavy-tailed and $E\xi < \infty$. Let, for some $c > E\xi$,

$$P\{c\tau > x\} = o(F(x)) \quad \text{as } x \to \infty. \quad (2)$$

Then (1) holds.

The proof of Theorem 1 is based on a study of moments $Ee^{f(\xi)}$ for appropriately chosen concave function $f$. More precisely, we deduce Theorem 1 from the following general result which explores some ideas from [9, 5, 3].

**Theorem 2.** Assume that $\xi \geq 0$ is heavy-tailed and $E\xi < \infty$. Let there exists a function $f : R^+ \to R$ such that

$$Ee^{f(\xi)} = \infty, \quad (3)$$

and, for some $c > E\xi$,

$$Ee^{f(c\tau)} < \infty. \quad (4)$$

If $f(x) \geq \ln x$ for all sufficiently large $x$ and if the difference $f(x) - \ln x$ is an eventually concave function, then (1) holds.

In particular, the equality (1) is valid provided $E\xi^k = \infty$ and $E\tau^k < \infty$ for some $k \geq 1$; it is sufficient to consider the function $f(x) = k \ln x$. Earlier this was proved in [3 Theorem 1] by a more simple method.

If we consider instead the function $f(x) = \gamma x$, $\gamma > 0$, then we obtain the equality (1) provided $\xi$ is heavy-tailed but $\tau$ is light-tailed. This is Theorem 2 from [3].

Finally, the equality (1) is valid if $F$ is a Weibull distribution with parameter $\beta \in (0, 1)$, $F(x) = e^{-x^\beta}$ and $f(x) = x^\beta$ or, more generally, $f(x) = x^\beta - c \ln x$ for $x \geq 1$ where $c \leq \beta$ is any fixed constant.

The counterpart of Theorem 1 in the light-tailed case is stated next. But first we need some notations. By the Laplace transform of $F$ at the point $\gamma \in R$ we mean

$$\varphi(\gamma) = \int_0^\infty e^{\gamma x} F(dx) \in (0, \infty].$$

Put

$$\hat{\gamma} = \sup\{\gamma : \varphi(\gamma) < \infty\} \in [0, \infty].$$

Note that the function $\varphi(\gamma)$ is monotone continuous in the interval $(-\infty, \hat{\gamma})$, and $\varphi(\hat{\gamma}) = \lim_{\gamma \to \hat{\gamma}} \varphi(\gamma) \in [1, \infty]$.

**Theorem 3.** Let $\hat{\gamma} \in (0, \infty]$, so that $\varphi(\hat{\gamma}) \in (1, \infty]$. If (2) holds and, for any fixed $y > 0$,

$$\lim inf_{x \to \infty} \frac{F(x - y)}{F(x)} \geq e^{\hat{\gamma}y}, \quad (5)$$

then

$$\lim inf_{x \to \infty} \frac{F(\tau x)}{F(x)} = E\tau e^{\tau-1(\hat{\gamma})}.$$
The paper is organised as follows. In Section 2, we formulate and prove a general result on characterisation of heavy-tailed distributions on the positive half-line. Section 3 is devoted to the estimation of the functional $Eg(S_n)$ for a concave function $h$. Sections 4 and 5 contain proofs of Theorems 2 and 1 respectively. Section 6 is devoted to the proof in light-tailed case.

2. Characterisation of heavy-tailed distributions. It was proved in [3, Lemma 2] that, for any heavy-tailed random variable $\xi \geq 0$ and for any real $\delta > 0$, there exists an increasing concave function $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $E\exp(h(\xi)) \leq 1 + \delta$ and $E\xi \exp(h(\xi)) = \infty$. In the present section, we obtain some generalisation of it.

**Lemma 1.** Let $\xi \geq 0$ be a random variable with a heavy-tailed distribution. Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a concave function such that

\[ E\exp(f(\xi)) = \infty. \]  

Let a function $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ be such that $g(x) \rightarrow \infty$ as $x \rightarrow \infty$. Then there exists a concave function $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $h \leq f$ and

\[ E\exp(h(\xi)) < \infty, \quad E\exp(h(\xi) + g(\xi)) = \infty. \]

**Proof.** Without loss of generality assume $f(0) = 0$. We will construct a function $h(x)$ on the successive intervals. For that we introduce two positive sequences, $x_n \uparrow \infty$ as $n \rightarrow \infty$ and $\varepsilon_n \in (0, 1]$. We put $x_0 = 0, h(0) = f(0) = 0, h'(0) = f'(0)$, and

\[ h(x) = h(x_{n-1}) + \varepsilon_n \min(h'(x_{n-1})(x - x_{n-1}), f(x) - f(x_{n-1})) \quad \text{for} \quad x \in (x_{n-1}, x_n]; \]

here $h'$ is the left derivative of the function $h$. The function $h$ is increasing, since $\varepsilon_n > 0$ and $f$ is increasing. Moreover, this function is concave, due to $\varepsilon_n \leq 1$ and concavity of $f$. Since $h(x) - h(x_{n-1}) \leq f(x) - f(x_{n-1})$ for $x \in (x_{n-1}, x_n]$, we have $h \leq f$.

Now proceed with the very construction of $x_n$ and $\varepsilon_n$. By conditions $g(x) \rightarrow \infty$ and (6), we can choose $x_1$ so large that $e^{g(x)} \geq 2^1$ for all $x \geq x_1$ and

\[ E\left\{ e^{\min(h'(0)x_1, f(x_1))}; \xi \in (x_0, x_1]\right\} + e^{\min(h'(0)x_1, f(x_1))}F(x_1) > F(x_0) + 1. \]

Choose $\varepsilon_1 \in (0, 1]$ so that

\[ E\left\{ e^{\varepsilon_1 \min(h'(0)x_1, f(x_1))}; \xi \in (x_0, x_1]\right\} + e^{\varepsilon_1 \min(h'(0)x_1, f(x_1))}F(x_1) = F(x_0) + 1. \]

Put $h(x) = \varepsilon_1 \min(x, f(x))$ for $x \in (0, x_1]$. Then the latter equality is equivalent to

\[ E\left\{ e^{h(\xi)}; \xi \in (x_0, x_1]\right\} + e^{h(x_1)}F(x_1) = e^{h(x_0)}F(x_0) + 1/2, \]

By induction we construct an increasing sequence $x_n$ and a sequence $\varepsilon_n \in (0, 1]$ such that $e^{g(x)} \geq 2^n$ for all $x \geq x_n$, and

\[ E\left\{ e^{h(\xi)}; \xi \in (x_{n-1}, x_n]\right\} + e^{h(x_n)}F(x_n) = e^{h(x_{n-1})}F(x_{n-1}) + 1/2^n \]

for any $n \geq 1$. For $n = 1$ this is already done. Make the induction hypothesis for some $n \geq 2$. For any $x > x_n$, denote

\[ \delta(x, \varepsilon) \equiv e^{h(x_n)} \left( E\left\{ e^{\varepsilon \min(h'(x_n)(\xi-x_n), f(\xi)-f(x_n))}; \xi \in (x_n, x]\right\} + e^{\varepsilon \min(h'(x_n)(x-x_n), f(x)-f(x_n))}F(x) \right). \]
By the convergence $g(x) \to \infty$, by heavy-tailedness of $\xi$, and by the condition (6), there exists $x_{n+1}$ so large that $e^{g(x)} \geq 2^{n+1}$ for all $x \geq x_{n+1}$ and
\[
\delta(x_{n+1}, 1) > e^{h(x_n)}F(x_n) + 1.
\]

Note that the function $\delta(x_{n+1}, \varepsilon)$ is continuously decreasing to $e^{h(x_n)}F(x_n)$ as $\varepsilon \downarrow 0$. Therefore, we can choose $\varepsilon_{n+1} \in (0, 1]$ so that
\[
\delta(x_{n+1}, \varepsilon_{n+1}) = e^{h(x_n)}F(x_n) + 1/2^{n+1}.
\]

Then
\[
\mathbb{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1})\} + e^{h(x_{n+1})}F(x_{n+1}) = e^{h(x_n)}F(x_n) + 1/2^{n+1}.
\]

Our induction hypothesis now holds with $n + 1$ in place of $n$ as required.

Next, for any $N$,
\[
\mathbb{E}\{e^{h(\xi)}; \xi \leq x_{N+1}\} = \sum_{n=0}^{N} \mathbb{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1})\} = \sum_{n=0}^{N} (e^{h(x_n)}F(x_n) - e^{h(x_{n+1})}F(x_{n+1}) + 1/2^{n+1}) \leq e^{h(x_0)}F(x_0) + 1,
\]

so that $\mathbb{E} e^{h(\xi)}$ is finite. On the other hand, since $e^{g(x)} \geq 2^k$ for all $x \geq x_k$,
\[
\mathbb{E}\{e^{h(\xi)} + g(\xi); \xi > x_n\} \geq 2^n \left( \mathbb{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1})\} + e^{h(x_{n+1})}F(x_{n+1}) \right) = 2^n \left( e^{h(x_{n+1})}F(x_{n+1}) + 1/2^{n+1} \right).
\]

Then, for any $n$, $\mathbb{E}\{e^{h(\xi)} + g(\xi); \xi > x_n\} \geq 1/2$, which implies $\mathbb{E} e^{h(\xi) + g(\xi)} = \infty$. The proof is complete.

**Lemma 2.** Let $\xi \geq 0$ be a random variable with a heavy-tailed distribution. Let $f_1 : \mathbb{R}^+ \to \mathbb{R}$ be any measurable function and $f_2 : \mathbb{R}^+ \to \mathbb{R}$ a concave function such that
\[
\mathbb{E} e^{f_1(\xi)} < \infty \quad \text{and} \quad \mathbb{E} e^{f_1(\xi) + f_2(\xi)} = \infty.
\]

Let a function $g : \mathbb{R}^+ \to \mathbb{R}$ be such that $g(x) \to \infty$ as $x \to \infty$. Then there exists a concave function $h : \mathbb{R}^+ \to \mathbb{R}$ such that $h \leq f_2$ and
\[
\mathbb{E} e^{f_1(\xi) + h(\xi)} < \infty \quad \text{and} \quad \mathbb{E} e^{f_1(\xi) + h(\xi) + g(\xi)} = \infty.
\]

**Proof.** Consider a new governing probability measure $\mathbb{P}^*$ defined in the following way:
\[
\mathbb{P}^*\{d\omega\} = \frac{e^{f_1(\xi(\omega))} \mathbb{P}\{d\omega\}}{\mathbb{E} e^{f_1(\xi)}}.
\]

Then
\[
\mathbb{E}^* e^{f_2(\xi)} = \frac{\mathbb{E} e^{f_1(\xi) + f_2(\xi)}}{\mathbb{E} e^{f_1(\xi)}} = \infty.
\]
In particular, \( \xi \) is heavy-tailed against the measure \( P^* \). Now it follows from Lemma[1] that there exists a concave function \( h : \mathbb{R}^+ \to \mathbb{R} \) such that \( h \leq f_2, h(x) = o(x), E^* e^{h(\xi)} < \infty, \) and \( E^* e^{h(\xi) + g(\xi)} = \infty. \) Equivalently,

\[
E e^{f_1(\xi) + h(\xi)} = E e^{f_1(\xi)} E^* e^{h(\xi)} < \infty
\]

and

\[
E e^{f_1(\xi) + h(\xi) + g(\xi)} = E e^{f_1(\xi)} E^* e^{h(\xi) + g(\xi)} = \infty.
\]

The proof is complete.

3. Growth rate of sums in terms of generalised moments. According to the Law of Large Numbers, the sum \( S_n \) grows like \( nE \xi \). In the following lemma we provide conditions on a function \( h(x) \), guaranteeing an appropriate rate of growth for the functional \( E e^{h(S_n)} \).

**Lemma 3.** Let \( \xi \) be a non-negative random variable. Let \( h : \mathbb{R}^+ \to \mathbb{R} \) be a non-decreasing eventually concave function such that \( h(x) = o(x) \) as \( x \to \infty \) and \( h(x) \geq \ln x \) for all sufficiently large \( x \). If \( E e^{h(\xi)} < \infty \), then, for any \( c > E \xi \), there exists a constant \( K(c) \) such that \( E e^{h(S_n)} \leq K(c) e^{h(nc)} \), for all \( n \).

To prove this lemma, we need the following assertion, which generalises the corresponding estimate from [6]:

**Lemma 4.** Let \( \eta \) be a random variable with \( E \eta < 0 \). Let \( h : \mathbb{R} \to \mathbb{R} \) be a non-decreasing and eventually concave function such that \( h(x) = o(x) \) as \( x \to \infty \) and \( h(x) \geq \ln x \) for all sufficiently large \( x \). If \( E e^{h(\eta)} < \infty \), then there exists \( x_0 \) such that the inequality \( E e^{h(x + \eta)} \leq e^{h(x)} \) holds for all \( x > x_0 \).

**Proof.** Since \( h \) is increasing, without loss of generality we may assume that \( \eta \) is bounded from below, that is, \( \eta \geq M \) for some \( M \). Also, we may assume that \( h \) is non-negative and concave on the whole half-line \([0, \infty)\).

Since \( h \) is concave, \( h'(x) \) is non-increasing function. With necessity \( h'(x) \to 0 \) as \( x \to \infty \), otherwise the condition \( h(x) = o(x) \) is violated. If ultimately \( h'(x) = 0 \), then \( h \) is ultimately a constant function and the proof of the theorem is obvious.

Consider now the case \( h'(x) \to 0 \) as \( x \to \infty \) but \( h'(x) > 0 \) for all \( x \). Put \( g(x) \equiv 1/h'(x) \), then \( g(x) \uparrow \infty \) as \( x \to \infty \). Since \( E \eta < 0 \), we can choose sufficiently large \( A \) such that

\[
\varepsilon \equiv E \{ \eta; \eta \in [M, A] \} + e E \{ \eta; \eta > A \} < 0.
\]

By concavity of \( h \), for any \( x \) and \( y \in \mathbb{R} \) we have the inequality \( h(x+y) - h(x) \leq h'(y) \). Hence,

\[
E e^{h(x+y)-h(x)} \leq E \{ e^{h'(y)\eta}; \eta \in [M, A] \} + E \{ e^{h'(y)\eta}; \eta \in (A, g(x)] \}
+ E \{ e^{h(x+y)-h(x)}; \eta > g(x) \}
\equiv E_1 + E_2 + E_3.
\]

Since \( h'(x) \to 0 \), the Taylor’s expansion for the exponent up to the linear term implies, as \( x \to \infty \),

\[
E_1 = P \{ \eta \in [M, A] \} + h'(x) E \{ \eta; \eta \in [M, A] \} + o(h'(x)).
\]

On the event \( \eta \in (A, g(x)] \) we have \( h'(x) \eta \leq 1 \) and, thus, \( e^{h'(x)\eta} \leq 1 + eh'(x)\eta \). Then

\[
E_2 \leq P \{ \eta \in (A, g(x)] \} + eh'(x) E \{ \eta; \eta \in (A, g(x)] \}.
\]

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We have
\[ E_3 = \mathbb{E}\{ e^{h(y)} e^{h(x+y)-h(x)-h(y)} ; \eta > g(x) \}. \] (11)

By concavity of \( h \), for \( x > 0 \), the difference \( h(x + y) - h(y) \) is non-increasing in \( y \). Therefore, for any \( y > g(x) \),
\[
\begin{align*}
  h(x + y) - h(x) - h(y) &\leq h(x + g(x)) - h(x) - h(g(x)) \\
 &\leq h'(x)g(x) - h(g(x)) \\
 &= 1 - h(g(x)) \\
 &\leq 1 - \ln g(x),
\end{align*}
\]
due to the condition \( h(x) \geq \ln x \) for all sufficiently large \( x \). This estimate and (11) imply
\[
E_3 \leq \mathbb{E}\{ e^{h(y)} ; \eta > g(x) \} e^{1-\ln g(x)}
= o(1)/g(x) = o(h'(x)) \quad \text{as } x \to \infty,
\] (12)
by the condition \( \mathbb{E}e^{h(\eta)} < \infty \). Substituting (9), (10) and (12) into (8) and taking into account the choice (7) of \( A \), we get
\[
\mathbb{E}e^{h(x+\eta)} = e^{h(x)} \mathbb{E}e^{h(x+\eta)-h(x)} \\
\leq e^{h(x)} (1 + h'(x)\varepsilon + o(h'(x))) \quad \text{as } x \to \infty.
\]

Since \( \varepsilon < 0 \), the latter estimate implies \( \mathbb{E}e^{h(x+\eta)} < e^{h(x)} \) for all sufficiently large \( x \). The proof is complete.

**Proof of Lemma 3** Put \( \eta_n = \xi_n - c \). We have \( \mathbb{E}\eta_n < 0 \) and \( \mathbb{E}e^{h(\eta_n)} < \infty \). By Lemma 4, there exists \( x_0 > 0 \) such that \( \mathbb{E}e^{h(x+\eta_n)} \leq \mathbb{E}e^{h(x)} \) for \( x > x_0 \). Then, by monotonicity of \( h(x) \) and by non-negativity of \( S_{n-1} \),
\[
\mathbb{E}e^{h(S_n)} \leq \mathbb{E}e^{h(S_{n+x_0})} = \mathbb{E}e^{h(S_{n-1}+x_0+c+\eta_n)} \leq \mathbb{E}e^{h(S_{n-1}+x_0+c)}.
\]

Now, by the induction arguments, \( \mathbb{E}e^{h(S_n)} \leq e^{h(cn+x_0)} \leq e^{h(cn)} e^{h(x_0)} \). The proof is complete.

**4. Proof of Theorem 2** Before starting the proof of Theorem 2, we formulate the following proposition from [3, Corollary 1]:

**Proposition 1.** Let there exist a concave function \( r : \mathbb{R}^+ \to \mathbb{R} \) such that \( \mathbb{E}e^{r(\xi)} < \infty \) and \( \mathbb{E}e^{r(S_{r-1})} = \infty \). If \( F \) is heavy-tailed and \( \mathbb{E}e^{r(S_{r-1})} < \infty \), then (1) holds.

We also need two auxiliary technical results.

**Lemma 5.** Let \( \chi \geq 0 \) be any random variable. Then there exists a differentiable concave function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \), \( g(0) = 0 \), such that \( g'(x) \leq 1 \) for all \( x \), \( g(x) \to \infty \) as \( x \to \infty \), and \( \mathbb{E}e^{g(\chi)} < \infty \).

**Proof.** Consider an increasing sequence \( \{x_n\} \) such that \( x_0 = 0, x_1 = 1, x_{n+1} > x_n \), and \( \mathbb{P}\{ \chi > x_n \} \leq e^{-n} \). Put \( g_1(x_n) = n/2 \) and continuously linear between these points. Then, for any \( x \in (x_n, x_{n+1}) \) and \( y \in (x_{n+1}, x_{n+2}) \) we have
\[
g_1'(x) = \frac{1}{2(x_{n+1} - x_n)} > \frac{1}{2(x_{n+2} - x_{n+1})} = g_1'(y),
\]
where we have
so that \( g_1 \) is concave. By the construction, \( g_1(x) \uparrow \infty \) as \( x \to \infty \) and \( g_1'(x) \leq 1 \) where the derivative exists. Finally,

\[
Ee^{g_1(x)} \leq \sum_{n=0}^{\infty} e^{g_1(x_{n+1})} P\{ \chi > x_n \} \leq \sum_{n=0}^{\infty} e^{(n+1)/2} e^{-n} < \infty.
\]

A procedure of smoothing, say \( g(x) = \int_x^{x+1} g_1(y)dy - \int_0^1 g_1(y)dy \), completes the proof.

**Lemma 6.** Let \( \chi \geq 0 \) be a random variable such that, for some concave function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \), \( Ee^{f(\chi)} = \infty \). Then there exists a concave function \( f_1 : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( f_1 \leq f \), \( f_1(x) = o(x) \) as \( x \to \infty \), and \( Ee^{f_1(\chi)} = \infty \).

**Proof.** Take \( x_1 \) so large that \( E\{e^{\min(\chi,f(x))}; \chi \leq x_1 \} \geq 1 \) and put \( f_1(x) = \min(x,f(x)) \) for \( x \in [0,x_1] \). Then by induction, for any \( n \), we can choose \( x_{n+1} \) such that

\[
E\{e^{f_1(x_n)+\min(n^{-1}f_1'(x_n)(\chi-x_n);f(\chi)-f(x_n))}; \chi \in (x_n,x_{n+1}] \} \geq 1.
\]

Let \( f_1(x) = f_1(x_n) + \min(n^{-1}f_1'(x_n)(x-x_n),f(x)-f(x_n)) \) for \( x \in (x_n,x_{n+1}] \). By construction, \( f_1 \) is concave, \( f_1 \leq f \), and \( f_1'(x_{n+1}) \leq f_1'(x_n)/n \to 0 \) as \( n \to \infty \).

**Proof of Theorem 2** Without loss of generality, assume that \( f(x) \geq \ln x \) for all \( x \) and that \( f_2(x) \equiv f(x) - \ln x \) is concave on the whole positive half-line. By Lemma 6 and by measure change arguments like in the proof of Lemma 2 we may assume from the very beginning that

\[
f(x) = o(x) \quad \text{as} \quad x \to \infty.
\]

Next we state the existence of a concave function \( g : \mathbb{R}^+ \to \mathbb{R} \) such that \( g(x) \to \infty \) as \( x \to \infty \), \( g(x) \leq \ln x \) for all sufficiently large \( x \), the difference \( \ln x - g(x) \) is a non-decreasing function, and

\[
Ee^{f(\tau)+g(\tau)} < \infty.
\]

Indeed, by Lemma 5 and again measure change technique, there exists a differentiable concave function \( g_1 : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( g_1(0) = 0 \), \( g_1(x) \uparrow \infty \), \( g_1'(x) \leq 1 \), and \( Ee^{f(\tau)+g_1(\tau)} < \infty \). Put \( g(x) = g_1(\ln(x+1)) - 1 \). Then \( g \) is a monotone function increasing to infinity and \( g(x) \leq \ln x \) for all sufficiently large \( x \) in addition,

\[
(\ln x - g(x))' = 1/x - g_1'(\ln(x+1))/(x+1) \geq 0,
\]

so that the difference \( \ln x - g(x) \) is a non-decreasing function as needed.

Since the function \( f_2(x) \) is concave, by Lemma 2 with \( f_1(x) = \ln x \), there exists a concave function \( h \) such that \( h \leq f_2 \), \( h(x) = o(x) \), \( E\xi e^{h(\xi)} < \infty \) and \( E\xi e^{h(\xi)+g(\xi)} = \infty \). Since \( \ln x + h(x)+g(x) \leq f(x)+g(x) \), by (4) and by the choice of \( g \),

\[
E\tau e^{h(\tau)+g(\tau)} < \infty.
\]

(13)

The concave function \( r(x) = h(x) + g(x) \) satisfies all conditions of Proposition 1. Indeed, due to the inequality \( g(x) \leq \ln x \) for all sufficiently large \( x \), we have \( Ee^{r(\xi)} < \infty \) because \( E\xi e^{h(\xi)} < \infty \). It remains to check that \( E\tau e^{r(S_{\tau-1})} < \infty \). Since, by (13),

\[
E\{\tau e^{r(S_{\tau})}; S_{\tau} \leq \tau\} \leq E\tau e^{r(\tau)} < \infty,
\]
it suffices to prove that
\[ \mathbb{E}\{\tau e^{r(S_{\tau})}; S_{\tau} > c\tau\} < \infty. \]

We proceed in the following way:
\[
\mathbb{E}\{c\tau e^{r(S_{\tau})}; S_{\tau} > c\tau\} = \sum_{n=1}^{\infty} \mathbb{P}\{\tau = n\}cn\mathbb{E}\{e^{r(S_n)}; S_n > cn\}
\]
\[
= \sum_{n=1}^{\infty} \mathbb{P}\{\tau = n\}e^{g(cn)+\ln(cn)-g(cn)}\mathbb{E}\{e^{h(S_n)+g(S_n)}; S_n > cn\}.
\]

By the monotonicity of the difference \(\ln x - g(x)\), we obtain the following estimate
\[
\mathbb{E}\{c\tau e^{r(S_{\tau})}; S_{\tau} > c\tau\} \leq \sum_{n=1}^{\infty} \mathbb{P}\{\tau = n\}e^{g(cn)}\mathbb{E}\{e^{\ln S_n+h(S_n)}; S_n > cn\},
\]

Since the function \(\ln x + h(x)\) is concave and \(\ln x + h(x) \geq \ln x\), by Lemma 3,
\[
\mathbb{E}e^{\ln S_n+h(S_n)} \leq K(e^{\ln(nc)+h(cn)})
\]
for some \(K(e) < \infty\). Therefore,
\[
\mathbb{E}\{c\tau e^{r(S_{\tau})}; S_{\tau} > c\tau\} \leq K(e)\sum_{n=1}^{\infty} \mathbb{P}\{\tau = n\}e^{g(cn)}e^{\ln(nc)+h(nc)}
\]
\[
= K(e)e^{\tau h(c\tau)+g(c\tau)} < \infty,
\]
from (13). The proof of Theorem 2 is complete.

5. Proof of Theorem 1

Denote by \(G\) the distribution function of \(c\tau\).

We will construct an increasing concave function \(f : \mathbb{R}^+ \rightarrow \mathbb{R}\) such that
\[
\mathbb{E}\xi e^{f(\xi)} = \infty \quad \text{and} \quad \mathbb{E}\tau e^{f(c\tau)} < \infty.
\]

Then the desired relation (14) will follow by applying Theorem 2

If \(G\) is light-tailed then one can take \(f(x) = \lambda x\) for a sufficiently small \(\lambda > 0\). From now on we assume \(G\) to be heavy-tailed.

Consider new random variables \(\xi_*\) and \(\tau_*\) with the following distributions:
\[
\mathbb{P}\{\xi_* \in dx\} = \frac{xF(dx)}{E_{\xi}} \quad \text{and} \quad \mathbb{P}\{\tau_* = n\} = \frac{n\mathbb{P}\{\tau = n\}}{E_{\tau}}.
\]

Denote by \(F_*\) and \(G_*\) the distributions of \(\xi_*\) and \(c\tau_*\) respectively. Then both \(F_*\) and \(G_*\) are heavy-tailed and
\[
\overline{G}_*(x) = o(\overline{F}_*(x)) \quad \text{as} \quad x \to \infty.
\]

The heavy-tailedness of \(G_*\) is equivalent to the following condition: for any \(\varepsilon > 0\),
\[
\int_{1}^{\infty} \overline{G}_*(e^{-1}\ln x)dx \equiv \int_{0}^{\infty} e^{\varepsilon\overline{G}_*(x/\varepsilon)}dx = \infty.
\]

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In terms of new distributions $F_*$ and $G_*$, conditions (14) may be reformulated as follows: we need to construct an increasing concave function $f$ such that $\mathbb{E}e^{f(\xi_*)} = \infty$ and $\mathbb{E}e^{f(c\tau_*)} < \infty$, or, equivalently,

$$
\int_1^\infty F_*(f^{-1}(\ln x))dx = \infty \quad \text{and} \quad \int_1^\infty G_*(f^{-1}(\ln x))dx < \infty. \quad (17)
$$

The concavity of $f$ is equivalent to the convexity of its inverse, $h = f^{-1}$. So, conditions (17) may be rewritten as: we have to present an increasing convex function $h$ such that

$$
\int_0^\infty e^x F_*(h(x))dx = \infty \quad \text{and} \quad \int_0^\infty e^x G_*(h(x))dx < \infty. \quad (18)
$$

We will construct $h(x)$ as a piece-wise linear function. For this, we will introduce two increasing sequences, say $x_n \uparrow \infty$ and $a_n \uparrow \infty$, and let

$$
h(x) = h(x_n) + a_n(x - x_n) \quad \text{for } x \in (x_n, x_{n+1}].
$$

Then the convexity of $f$ will follow from the increase of $\{a_n\}$.

Put $x_0 = 0$ and $f(x_0) = 0$. Due to (15) and (16), we can choose $x_1$ so large that

$$
\frac{F_*(y)}{G_*(y)} \geq 2^1
$$

for all $y > x_1$ and

$$
\int_0^{x_1} e^x G_*(h(x_0) + 1 \cdot (x - x_0))dx \geq 1.
$$

Then there exists a sufficiently large $a_0 \geq 1$ such that

$$
\int_0^{x_1} e^x G_*(h(x_0) + a_0(x - x_0))dx = 1.
$$

Now we use the induction argument to construct increasing sequences $\{x_n\}$ and $\{a_n\}$ such that

$$
\frac{F_*(y)}{G_*(y)} \geq 2^{n+1} \quad (19)
$$

for all $y > x_{n+1}$ and

$$
\int_{x_n}^{x_{n+1}} e^x G_*(h(x))dx = 2^{-n}.
$$

For $n = 0$ this is already done. Make the induction hypothesis for some $n \geq 1$. For any $x > x_{n+1}$, denote

$$
\delta(x, a) \equiv \int_{x_{n+1}}^x e^y G_*(h(x_{n+1} + a(y - x_{n+1})))dy.
$$
Due to (15) and (16), we can choose \( x_{n+2} \) so large that
\[
\frac{F_*(y)}{G_*(y)} \geq 2^{n+2}
\]
for all \( y > x_{n+2} \) and
\[
\delta(x_{n+2}, a_n) \geq 1.
\]
Since the function \( \delta(x_{n+2}, a) \) continuously decreases to 0 as \( a \uparrow \infty \), we can choose \( a_{n+1} > a_n \) such that
\[
\delta(x_{n+2}, a_{n+1}) = 2^{-(n+1)}.
\]
Then
\[
\int_{x_{n+1}}^{x_{n+2}} e^x G_*(h(x)) dx = 2^{-(n+1)}.
\]
Our induction hypothesis now holds with \( n + 1 \) in place of \( n \) as required.

Now the inequalities (18) follow since, from the construction of function \( h \),
\[
\int_0^\infty e^x G_*(h(x)) dx = \sum_{n=0}^{\infty} \int_{x_n}^{x_{n+1}} e^x G_*(h(x)) dx
\]
\[
= \sum_{n=0}^{\infty} 2^{-n} < \infty.
\]
and, by (19),
\[
\int_0^\infty e^x F_*(h(x)) dx = \sum_{n=0}^{\infty} \int_{x_n}^{x_{n+1}} e^x F_*(h(x)) dx \geq \sum_{n=0}^{\infty} 2^n \int_{x_n}^{x_{n+1}} e^x G_*(h(x)) dx
\]
\[
= \sum_{n=0}^{\infty} 2^n 2^{-n} = \infty.
\]
The proof of Theorem 1 is complete.

6. Proof of Theorem 3. We apply the exponential change of measure with parameter \( \hat{\gamma} \) and consider the distribution \( G(du) = e^{\hat{\gamma}u} F(du) / \phi(\hat{\gamma}) \) and the stopping time \( \nu \) with the distribution \( P\{\nu = k\} = \phi^k(\hat{\gamma})P\{\tau = k\} / E\phi^\gamma(\hat{\gamma}) \). Then it was proved in [3, Lemma 3] that
\[
\liminf_{x \to \infty} \frac{G^{su}(x)}{G(x)} \geq \frac{1}{E\phi^{\gamma-1}(\hat{\gamma})} \liminf_{x \to \infty} \frac{F^{w}(x)}{F(x)}.
\] 
\[
(20)
\]
From the definition of \( \hat{\gamma} \), the distribution \( G \) is heavy-tailed. Let us prove that
\[
P\{c\nu > x\} = o(G(x)) \text{ as } x \to \infty.
\] 
\[
(21)
\]
Indeed, put \( \lambda \equiv \ln \varphi(\hat{\gamma}) > 0 \); then
\[
P\{c\nu > x\} = \frac{1}{E\varphi^\tau(\hat{\gamma})} \sum_{k > x/c} e^{\lambda k} P\{\tau = k\}
\]
\[
\leq \frac{1}{E\varphi^\tau(\hat{\gamma})} \int_{x/c}^{\infty} e^{\lambda y} P\{\tau \in dy\}.
\]  
(22)

Integration by parts implies
\[
\int_{x/c}^{\infty} e^{\lambda y} P\{\tau \in dy\} = -e^{\lambda y} P\{\tau > y\} \bigg|_{x/c}^{\infty} + \lambda \int_{x/c}^{\infty} e^{\lambda y} P\{\tau > y\} dy
\]
\[
= e^{\lambda x/c} P\{c\tau > x\} + \frac{\lambda}{c} \int_{x}^{\infty} e^{\lambda y/c} P\{c\tau > y\} dy,
\]
because \( E\varphi^\tau(\hat{\gamma}) < \infty \) and, thus, \( e^{\lambda y} P\{\tau > y\} \to 0 \) as \( y \to \infty \). Now applying the condition (2) we obtain that the latter sum is of order
\[
o\left( e^{\lambda x/c} F(x) + \frac{\lambda}{c} \int_{x}^{\infty} e^{\lambda y/c} F(y) dy \right) = o\left( \int_{x}^{\infty} e^{\lambda y/c} F(dy) \right) \text{ as } x \to \infty.
\]
Together with (22) it implies (21). Therefore, by Theorem 1 we have the equality
\[
\liminf_{x \to \infty} \frac{G^\nu(x)}{G(x)} = E\nu = \frac{E\tau \varphi^\tau(\hat{\gamma})}{E\varphi^\tau(\hat{\gamma})},
\]
and, due to (20),
\[
\liminf_{x \to \infty} \frac{F^\tau(x)}{F(x)} \leq E\tau \varphi^{\tau-1}(\hat{\gamma}).
\]  
(23)

The result now follows from Lemma .

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