A.C. Conductivity of a Disordered Metal

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Abstract

The degenerate free Fermi gas coupled to a random potential is used to compute a.c. conductivity in various dimensions. We first formally diagonalise the hamiltonian using an appropriate basis that is a functional of the disorder potential. Then we compute the a.c. conductivity at zero temperature using the Kubo formula. This a.c. conductivity is a functional of the disordered potential. The wavefunction of extended states is written as exponential of the logarithm. We use the cumulant expansion to compute the disordered averaged a.c. conductivity for Gaussian disorder. The formula is valid if a certain linearization approximation is valid in the long-wavelength limit.

1 Introduction

A simple theory of the Anderson transition is presented that directly computes measurable quantities such as a.c. conductivity. The relevant literature on this subject is vast and we shall not attempt to be exhaustive in surveying it. Anderson’s pioneering work on localization[1] was followed by the work of Abrahams et.al. [2] and later on a more rigorous formulation of the notion of disorder averaging was given by McKane and Stone[3]. This relates to a single electron in a disordered potential. The classic review of Lee and Ramakrishnan [4] includes many references on the literature concerning the degenerate electron gas in a disordered potential. A more recent review is by Abrahams and [5]. Belitz and Kirkpatrick[6] have a review that is aimed at theorists. The present write-up is intended to give a simple derivation of the Anderson transition in three dimensions.

The basic idea of this work is as follows. We formally diagonalise the hamiltonian of a single electron in a fixed disorder potential. The wavefunction of the electron then may be formally written as,

\[ \phi_i(x) = \frac{C_i}{\sqrt{V}} e^{i\theta_i(x)} R_i^+(x) \]

(1)
Here $C_i$ is an appropriate normalization constant. The novel feature involves rewriting the square of the amplitude $R_i(x)$ also as an exponential.

$$R_i(x) = e^{\tilde{\Lambda}_i(x)}$$  \hspace{1cm} (2)

Then one uses the observation that only extented states participate in the conduction. Thus we may write,

$$\phi_i(x) = \frac{C_i}{\sqrt{V}} e^{i k_i \cdot x + i \tilde{\theta}_i(x)} e^{i \tilde{\Lambda}_i(x)}$$  \hspace{1cm} (3)

Then we make the ansatz that $< \tilde{\theta}_i >_{dis} = < \tilde{\Lambda}_i >_{dis} = 0$. This ansatz is shown to be consistent in the main text. Then one uses the Kubo formula to evaluate the a.c. conductivity. One simply plugs in the form in Eq.(3) into this formula and uses the cumulant expansion to evaluate the disorder averaged conductivity. In the cumulant expansion, we encounter correlation functions such as $< \tilde{\theta}_i(x) \tilde{\theta}_j(x') >_{dis}$, $< \tilde{\theta}_i(x) \tilde{\Lambda}_j(x') >_{dis}$ and so on. These have been explicitly computed in the appendix. It can be shown that in some approximate sense, the higher order correlation functions all vanish. Hence we may deduce a formula for the disorder averaged a.c. conductivity provided it is legitimate to use only the linearization approximation. Our formalism does not have an arbitrarily chosen cutoff. A natural smooth cutoff emerges from a careful treatment of curvature effects by retaining double derivatives, in other words, by not linearizing the bare fermion dispersion. In passing we note that this formalism involves no bosonization or other advanced field theoretic ideas. It only involves the simple Kubo formula. The answer for the a.c. conductivity may be written down as follows for Gaussian disorder. It has been reduced to quadratures.
\[ Re[\sigma(\omega)] = \left( \frac{\pi e^2}{mV} \right) \frac{1}{\omega} \sum_{k, k_j} \langle J_{ji}(U_{dis}) \cdot J_{ij}(U_{dis}) \rangle \theta(\epsilon_F - \frac{k_j^2}{2m}) \theta(\omega - \frac{k_i^2}{2m} - \epsilon_F) \delta(\omega - \frac{k_i^2}{2m} + \frac{k_j^2}{2m}) \] (4)

where,
\[
\langle J_{ji}(U_{dis}) \cdot J_{ij}(U_{dis}) \rangle = -\frac{1}{V} \int d^d x e^{i(k_j - k_i) \cdot x} \frac{1}{2} <E' F(k_i, k_j) > (5)
\]
\[
\frac{1}{2} < E'^2 > = -\frac{m^2 \Delta^2}{V} \sum_q \left( \frac{1}{D_i(q)} - \frac{1}{D_j(q)} \right)^2 + \frac{m^2 \Delta^2}{V} \sum_q \left( \frac{1}{D_i(-q)} + \frac{1}{D_j(q)} \right)^2 e^{-i q \cdot x} (6)
\]
\[
F(k_i, k_j) = \frac{1}{2V} \sum_q \frac{2m^2 \Delta^2 q^2}{D_i(-q)D_j(q)} e^{-i q \cdot x}
\]
\[
+ [ik_i + \frac{1}{2V} \sum_q \frac{2m^2 \Delta^2}{D_i(q)D_j(q)} (-i q)] + \frac{1}{2V} \sum_q \left( \frac{2m^2 \Delta^2}{D_i(-q)D_j(q)} (i q) + \frac{2m^2 \Delta^2}{D_i(-q)D_j(q)} (i q) \right) e^{-i q \cdot x}
\]
\[
\cdot [ik_j + \frac{1}{2V} \sum_q \frac{2m^2 \Delta^2}{D_i(q)D_j(q)} (-i q)] + \frac{1}{2V} \sum_q \left( \frac{2m^2 \Delta^2}{D_i(-q)D_j(q)} (-i q) + \frac{2m^2 \Delta^2}{D_i(-q)D_j(q)} (-i q) \right) e^{-i q \cdot x}
\]
\[
D_i(q) = -k_i \cdot q + \frac{q^2}{2} (8)
\]

The presence of the \( q^2 / 2 \) makes the integrals finite at high momenta. This comes about by retaining double derivatives, in other words by taking into account the parabolic nature of the band. The remaining sections explain how these formulas are derived. Before we do this we would like to point out some general facts. Since,
\[
f(k_i, k_j) \equiv \langle J_{ji}(U_{dis}) \cdot J_{ij}(U_{dis}) \rangle \equiv \langle |J_{ij}(U_{dis})|^2 \rangle \geq 0 (9)
\]
We may sum over the angles first to write,
\[
\frac{1}{V} \hat{f}(|k_i|, |k_j|) \equiv \sum_{k, k_j} f(k_i, k_j) \geq 0 (10)
\]
\[ Re[\sigma(\omega)] = \left( \frac{\pi e^2}{m} \right) \frac{1}{\omega} \int_{0}^{\infty} dk_i k_i^{d-1} \int_{0}^{\infty} dk_j k_j^{d-1} \tilde{f}(k_i, k_j) \theta(\epsilon_F - \frac{k_j^2}{2m}) \theta(\omega - \frac{k_j^2}{2m} - \epsilon_F) \delta(\omega - \frac{k_i^2}{2m} + \frac{k_j^2}{2m}) \] (11)

For \( m\omega \ll k_F^2 \), we have,

\[ Re[\sigma(\omega)] \sim \tilde{f}(k_F(1 + m\omega/k_F^2), k_F) \] (12)

One really important question that we have to face is whether the zero frequency limit of the a.c. conductivity (obtained via Kubo formula) is the d.c. conductivity. According to the review by Lee and Ramakrishnan[4] this is true only in three dimensions. In one and two dimensions, it appears that according to the results presented, the d.c. conductivity may not be obtained by taking the zero frequency limit of the a.c. conductivity. They show that the a.c. conductivity diverges logarithimically for small frequencies in two dimensions whereas the d.c. conductivity is strictly zero. A frequency cutoff emerges in these dimensions that tells us that for frequencies smaller than this cutoff the formulas given there break down. According to the analysis given in Appendix C of this article, there should be no problem in any number of dimensions. The zero frequency limit of the a.c. conductivity is in fact the d.c. conductivity. In three dimensions we may expect to find the mobility edge while computing the d.c. conductivity.

\[ \sigma_{d.c.} \sim \tilde{f}(k_F, k_F) \] (13)

**Results So Far :**

So far the results have been very disappointing. In one dimension, we expect to see that the d.c. conductivity is zero. Instead we find that it is not although there appear to be many subtle cancellations. It is likely that the linearization is not adequate and retaining some nonlinear terms may be required. The main culprit seems to be an approximation that replaces a certain variable quadratic in the \( \tilde{\theta} \) and \( \tilde{\Lambda} \) by it disorder average in an effort to render the equations linear differential equations. However when evaluated this average is zero and hence it is a really bad idea. However the author had hoped that it would not be serious. But it is. Future attempts may try and address these problems, although there are many good ideas in this preprint.

2 **Some Technical Musings**

It appears that the mathematical literature on the subject of quantum particles in random potentials is vast[7]. It is possible, indeed likely that many mathematically rigorous results are known regarding this problem. But this does not prevent the authors from making some remarks that more knowledgeable readers may choose to critique. In particular, the author is uncomfortable
with the notion of disorder averaging. Nature chooses its potentials based on
the distribution of impurities, defects and so on. This potential is fixed and
well-defined for a particular distribution of these imperfections. The physicists’
ignorance of the precise nature of this potential is not a license to average over
these potentials. Nature does not average, people do. But are people justified in
averaging? In other words can averaging simplify the problem without washing
out essential physics? In order to answer this question we have to make the
following conjectures.

**Defn0**: Let \( \mathcal{U}_d \) be the set of all potentials \( U(x) \) in a fixed spatial dimension \( d \).

**Defn1**: Let \( \mathcal{F}_d \) be the set of all potentials \( U(x) \) in a fixed spatial dimension \( d \)
that has the following property. They all lead to the same exponent \( \delta \) for the
frequency dependence of the a.c. conductivity. In other words, each of these
potentials predicts that \( \text{Re}[\sigma(\omega)] \sim \omega^\delta \) (in some region of \( \omega \) with possibly some
additive part independent of \( \omega \)) with the same \( \delta \).

**Conjecture 1**: \( \mathcal{F}_d \) is dense in \( \mathcal{U}_d \).

If **Conjecture 1** is valid, then one may average over all these ‘sufficiently erratic’
potentials and expect to extract \( \delta \) which is all that physicists care about. It
is possible that \( \delta \) may be extracted from a numerical solution of the Schrodinger
equation using a specific \( U \) that belongs to the set \( \mathcal{F}_d \). But this would involve
using the computer for more than checking one’s email, and not everyone likes
that.

**Defn2**: Let \( \mathcal{M}_3 \) be the set of all potentials \( U(x) \) in spatial dimension \( d = 3 \)
that has the following property. They all lead to the same exponent \( \beta \) for the
mobility edge exponent. In other words, each of these potentials predict that
\( \sigma_{d.c.} \sim (E_F - E_c)^\beta \theta(E_F - E_c) \) with the same \( \beta \). However for different potentials,
\( E_c \) - the mobility edge, may be different.

**Conjecture 2**: \( \mathcal{F}_3 \) is dense in \( \mathcal{U}_3 \).

If **Conjecture 2** is valid, then one may average over all these ‘sufficiently erratic’
potentials and expect to extract \( \beta \).

Thus the validity of the process of averaging over potentials rests crucially
it seems, on all these sufficiently erratic potentials predicting the same exponents
and on these sufficiently erratic potentials spanning nearly all possible
potentials.

If both these are satisfied then one may average over all potentials and
extract the exponents, or, if one is better at programming, choose a particular
potential from this set, numerically solve the Schrodinger equation and extract
the exponent from there. In either case we should get the same answer. A final
conjecture seems appropriate.

**Conjecture 3**: Let \( \mathcal{M}_3 \) have an exponent \( \beta' \) and \( \mathcal{F}_3' \) have an exponent \( \delta' \),
then \( \beta = \beta' \) and \( \delta = \delta' \). In other words, these exponents are unique.

With powerful computers now available, purely analytical methods such as
this work may seem passé, but a closed formula for the a.c. conductivity that
one can stare at (and one that is hopefully right) and admire has a charm that
a cold data file on the hard disk is unable to duplicate. Besides, with Coulomb interaction, the problem becomes intractable numerically, however, one may expect to combine the sea-boson method with the present one to extract the exponents analytically.

3 A.C. Conductivity Using Kubo Formula

In this section, we derive a formula for the a.c. conductivity in terms of the total momentum-momentum correlation function. To derive this, observe that the relevant Hamiltonian that couples to external fields is of the form,

\[ H_{\text{ext}}(t) = -\frac{|e|}{m} \mathbf{A}_{\text{ext}}(t) \cdot \mathbf{P} \]  

(14)

where \( \mathbf{P} = \sum_k \mathbf{k} \ c_k^{\dagger} c_k \) is the total momentum operator. Thus we may define formally the net momentum in the presence of the external field in the imaginary time formalism in the interaction representation as

\[ \langle \mathbf{P}(t) \rangle = \frac{T S \hat{\mathbf{P}}(t)}{\langle T S \rangle} \]  

(15)

where the S-matrix is defined as,

\[ S = e^{i \int_{-\beta}^{0} dt \frac{|e|}{m} \mathbf{A}_{\text{ext}}(t) \cdot \hat{\mathbf{P}}(t)} \]  

(16)

Here \( \hat{\mathbf{P}}(t) \) evolves according to the time-independent Hamiltonian. The net current density in the system is given by \( \mathbf{J}(t) = \frac{|e|}{mV} \mathbf{P}(t) \), this is in units of charge flowing per unit area per unit time in three space dimensions. From Ohm’s law we expect,

\[ \langle \mathbf{J}(t) \rangle = \int_{0}^{-i\beta} dt' \hat{\mathbf{\sigma}}(t - t') \mathbf{E}_{\text{ext}}(t') \]  

(17)

The a.c. conductivity for complex frequencies is then given by,

\[ \sigma(iz) = \int_{0}^{-i\beta} dt \hat{\mathbf{\sigma}}(t) e^{-zt} \]  

(18)

The d.c. conductivity is then given by,

\[ \sigma_{d.c.} = \sigma(i0 + \epsilon) \]  

(19)

where, \( \epsilon = 0^+ \).

\[ \hat{\mathbf{\sigma}}(t - t') = \frac{|e|}{mV} \left( \frac{\delta \langle \mathbf{P}(t) \rangle}{\delta \mathbf{E}_{\text{ext}}(t')} \right)_{\mathbf{E}_{\text{ext}}=0} \]  

(20)
\[ \tilde{\sigma}(t - t') = \frac{|e|}{mV} \frac{\delta}{\delta E_{ext}(t')} \frac{\langle T \hat{S} \hat{P}(t) \rangle}{\langle T \hat{S} \rangle} \] (21)

Since \( E_{ext}(t) = -\partial A_{ext}(t)/\partial t \), we have,

\[ \frac{\delta A_{ext}(t)}{\delta E_{ext}(t')} = -\theta(t - t') \] (22)

\[ \frac{\delta}{\delta E_{ext}(t')} S = S \left( -i \int_{0}^{-i\beta} dt_1 \theta(t' - t_1) \frac{|e|}{m} \hat{P}(t_1) \right) \] (23)

\[ \tilde{\sigma}(t - t') = -\frac{i e^2}{m^2 V} \int_{0}^{-i\beta} dt_1 \theta(t' - t_1) \langle T \delta \hat{P}(t_1) \cdot \delta \hat{P}(t) \rangle \] (24)

\[ \langle T \delta \hat{P}(t_1) \cdot \delta \hat{P}(t) \rangle = \theta(t_1 - t) \sum_{kk'} (k \cdot k') N(k; t_1 - t; k', 0) + \theta(t - t_1) \sum_{kk'} (k \cdot k') N(k, t - t_1; k', 0) \] (25)

If we write for the dynamical number-number correlation function,

\[ N(k; t; k', 0) \equiv \langle n_k(t) n_{k'}(0) \rangle - \langle n_k(t) \rangle \langle n_{k'}(0) \rangle = \sum_{ij} e^{-i(i \epsilon_i - \epsilon_j)t} \tilde{N}(k, \epsilon_i, \epsilon_j; k', 0) \] (26)

where \( n_k = c_k^\dagger c_k \). We may write,

\[ \langle T \delta \hat{P}(t_1) \cdot \delta \hat{P}(t) \rangle = \theta(t_1 - t) \sum_{kk'} (k \cdot k') \sum_{ij} \tilde{N}(k, \epsilon_i, \epsilon_j; k', 0) e^{-i(i \epsilon_i - \epsilon_j)(t_1 - t)} \]

\[ + \theta(t - t_1) \sum_{kk'} (k \cdot k') \sum_{ij} \tilde{N}(k, \epsilon_i, \epsilon_j; k', 0) e^{-i(i \epsilon_i - \epsilon_j)(t - t_1)} \] (27)

From the above formulas retaining only the terms that do not violate causality and using \( i \varepsilon_n \rightarrow \omega - i0^+ \), we have at absolute zero,

\[ \text{Re}[\sigma(\omega, U_{dis})] = \left( \frac{\pi e^2}{m^2 V} \right) \frac{1}{\omega} \sum_{kk'} \sum_{i,j} (k \cdot k') \tilde{N}(k, \epsilon_i, \epsilon_j; k', 0) \delta(\omega - \epsilon_i + \epsilon_j) \] (28)

To compute this we have to first evaluate the dynamical number-number correlation function. Consider the hamiltonian,

\[ H = \sum_k c_k^\dagger c_k + \sum_q \frac{U_{dis}(q)}{\sqrt{V}} \sum_k c_{k+q/2}^\dagger c_{k-q/2} \] (29)

This may be diagonalised by the following formal transformation,

\[ c_k = \sum_i \varphi_i(k) d_i \] (30)
\[ \langle c_k^\dagger c_k \rangle = \sum_i |\varphi_i(k)|^2 \theta(\epsilon_F - \epsilon_i) \] (31)

\[ n_k(t) = \sum_{ij} \varphi_j^*(k)\varphi_i(k)d_j^\dagger d_i e^{-i(\epsilon_i - \epsilon_j)t} \] (32)

\[ n_k'(0) = \sum_{i'j'} \varphi_{i'}^*(k')\varphi_{i'}(k')d_{i'}^\dagger d_i e^{-i(\epsilon_i - \epsilon_{i'})t} \] (33)

\[ N(k, t; k', 0) = \sum_{i,j} \varphi_i^*(k)\varphi_i(k)\varphi_{i'}^*(k')\varphi_{i'}(k')\theta(\epsilon_F - \epsilon_j)\theta(\epsilon_i - \epsilon_F)e^{-i(\epsilon_i - \epsilon_{i'})t} \] (34)

This means,

\[ \sum_{kk'} (k, k') \tilde{N}(k, \epsilon_i, \epsilon_j; k', 0) = J_{ji}(U_{dis}) \cdot J_{ij}(U_{dis}) \theta(\epsilon_F - \epsilon_j)\theta(\epsilon_i - \epsilon_F) \] (35)

Here \( \phi_i(x) \) is the solution to the equation below, and \( \varphi_i(k) \) is its Fourier transform.

\[ \left( -\nabla^2 + U_{dis}(x) \right) \phi_i(x) = \epsilon_i \phi_i(x) \] (36)

\[ J_{ij}(U_{dis}) = \sum_k k \varphi_i^*(k)\varphi_j(k) = -\int d^d x \phi_i^*(x) i\nabla \phi_j(x) \] (37)

\[ \text{Re}[\sigma(\omega, U_{dis})] = \left( \frac{\pi e^2}{mV} \right) \frac{1}{\omega} \sum_{i,j} J_{ji}(U_{dis}) \cdot J_{ij}(U_{dis}) \theta(\epsilon_F - \epsilon_j)\theta(\omega + \epsilon_j - \epsilon_F) \delta(\omega - \epsilon_i + \epsilon_j) \] (38)

Till now the discussion has been at the formal level. Now we would like to compute the disorder averaged conductivity assuming a Gaussian disorder. In other words, the averages over the disordered potential have to be performed using the conditions,

\[ <U_{dis}(x)> = 0 \] (39)

\[ <U_{dis}(x)U_{dis}(x')> = \Delta^2 \delta^d(x - x') \] (40)

Thus the sum over all configurations of disordered potential keeps the sum \( \int d^d x U_{dis}^2(x) \) fixed and sums the conductivity obtained from each configuration and then computes the average. Several observations may be made regarding this. First, the energy \( \epsilon_i \) for each choice of \( U_{dis} \) may be discrete and negative (bound state) or positive and continuous, corresponding to Bloch waves. The delta-function forces \( \omega \) to be equal to the difference \( \epsilon_i - \epsilon_j \). For small \( \omega \), the difference \( \epsilon_i - \epsilon_j \) is likely to be comparable to \( \omega \) only for the Bloch states. That is to say, that only the extended states participate in conduction. Thus we may polar decompose the wavefunction \( \phi_i \) and assume that the magnitude is (roughly) independent of position, a feature characteristic of extended states. We have found that these simplifying assumptions though natural lead to some
divergences in the ultraviolet regime. Thus we shall have to retain the double
derivatives. This is done in the appendix and the correlations between the $\tilde{\Lambda}'s$
and the $\tilde{\theta}'s$(see below) and amongst themselves are derived. Thus we write,

$$\phi_i (x) = e^{i\theta_i (x)} R_i^2 (x)$$

(41)

For some phase $\theta_i$ and $R_i$ that is a function of the disordered potential. These
obey the following set of coupled equations.

$$- \nabla^2 \theta_i - \frac{\nabla \theta_i \cdot \nabla R_i}{2mR_i} = 0$$

(42)

$$\frac{(\nabla \theta_i)^2}{2m} - \frac{1}{8m} \left[ \frac{2\nabla^2 R_i}{R_i} - \frac{(\nabla R_i)^2}{R_i^2} \right] = \epsilon_i - U_{dis}$$

(43)

Now we set,

$$R_i (x) = C_i^0 e^{\tilde{\Lambda}_i (x)}$$

(44)

Here $C_i^0$ is a normalization constant. It may be determined as follows.

$$\left\langle \int d^d x \ |\phi_i (x)|^2 \right\rangle_{dis} = \int d^d x \langle R_i (x) \rangle_{dis} = \frac{C_i^0}{V} \int d^d x \langle e^{\tilde{\Lambda}_i (x)} \rangle = C_i^0 e^{\frac{1}{2}<\tilde{\Lambda}_i^2 (x)>_{dis}} = 1$$

(45)

Thus,

$$C_i^0 = e^{-\frac{1}{2}<\tilde{\Lambda}_i^2 (x)>_{dis}}$$

(46)

It will be shown in the appendices that $<\tilde{\Lambda}_i^2 (x)>$ is independent of position.
Further since only extended states participate in the conduction, we may write
$\theta_i (x) = k_i \cdot x + \tilde{\theta}_i (x)$. In the appendices, an approximate scheme is written down
that allows for a relatively simple computation of the average $<J_{ij} (U_{dis}) \cdot J_{ji} (U_{dis})>$.
From Eq. (38) it is clear that we would very much like to consider the eigenener-
gies $\epsilon_i$ as being independent of $U_{dis}$ in the sense that we may replace $\epsilon_i \equiv <\epsilon_i >_{dis}$.
This requires some justification. Also from the appendix we see that

$$\nabla^2 \tilde{\theta}_i + k_i \cdot \nabla \tilde{\Lambda}_i + \nabla \tilde{\theta}_i \cdot \nabla \tilde{\Lambda}_i = 0$$

(47)

$$\frac{k_i^2}{2m} + \frac{(\nabla \tilde{\theta}_i)^2}{2m} + \frac{k_i \cdot \nabla \tilde{\theta}_i}{m} - \frac{1}{8m} \left[ 2\nabla^2 \tilde{\Lambda}_i + (\nabla \tilde{\Lambda}_i)^2 \right] = \epsilon_i - U_{dis}$$

(48)

The above Eq. (47) and Eq. (48) are absolutely exact. The approximations arise
when we decide to linearize the above nonlinear partial differential equations
by replacing the quadratic parts with their disorder averages. Obviously, this
is justifiable only if we show that the fluctuations of the operators we have
averaged out are small, preferably zero. First we try and justify $\epsilon_i \equiv <\epsilon_i >_{dis}$.
For this we have to evaluate $< (\epsilon_i - <\epsilon_i >_{dis})^2 >$ and show that it is small
compared to $<\epsilon_i>_{\text{dis}}$. To do this we take recourse to perturbation theory. Using elementary perturbation theory up to second order we may write,

$$
\epsilon_i = \frac{k_i^2}{2m} + \int \frac{d^d x}{V} U_{\text{dis}}(x) + \frac{1}{V^2} \sum_{k_j} \frac{1}{k_j^2 - \frac{k_i^2}{2m}} \left| \int d^d x \ e^{i(k_i - k_j) \cdot x} U_{\text{dis}}(x) \right|^2
$$

Now,

$$
<\epsilon_i>_{\text{dis}} = \frac{k_i^2}{2m} + \frac{\Delta^2}{V} \sum_{k_j \neq k_i} \frac{1}{k_j^2 - \frac{k_i^2}{2m}}
$$

Therefore,

$$
<\epsilon_i^2>_{\text{dis}} - <\epsilon_i>_{\text{dis}}^2 = \frac{\Delta^2}{V} + \frac{\Delta^4}{V^2} \sum_{k_j \neq k_i} \frac{1}{k_j^2 - \frac{k_i^2}{2m}} \left( \frac{k_j^2 - (2k_i - k_j)^2}{2m} \right) + \frac{\Delta^4}{V^2} \sum_{k_j \neq k_i} \frac{1}{k_j^2 - \frac{k_i^2}{2m}}^2
$$

The sums over $k_j$ in Eq.(51) are finite in all three dimensions and hence the fluctuation $\epsilon_i$ is vanishingly small in the thermodynamic limit. Even otherwise, within this scheme the sum in Eq.(50) diverges, hence the fluctuation $<\epsilon_i^2>_{\text{dis}}$ is bound to be small compared to the mean. The proof that we may legitimately linearize the equations above (Eq.(47) and Eq.(48)) is given below. First we note that two random variables $O_i(x)$ and $O_j(x')$ may be replaced by their means if,

$$
< O_i(x)O_j(x') > = < O_i(x) > < O_j(x') >
$$

for $i, j = 1, 2$. Thus we would like to make the following identifications.

$$
\nabla \tilde{\theta}_i \cdot \nabla \tilde{\Lambda}_i \approx \left< \nabla \tilde{\theta}_i \cdot \nabla \tilde{\Lambda}_i \right>_{\text{dis}} = 0 \ (\text{from appendix})
$$

$$
\frac{(\nabla \tilde{\theta}_i)^2}{2m} - \frac{1}{8m} (\nabla \tilde{\Lambda}_i)^2 \approx \frac{(\nabla \tilde{\theta}_i)^2}{2m} - \frac{1}{8m} (\nabla \tilde{\Lambda}_i)^2_{\text{dis}}
$$

The first condition is satisfied if we ensure,

$$
\left< \left( \nabla_x \tilde{\theta}_i \cdot \nabla_x \tilde{\Lambda}_i \right) \left( \nabla_{x'} \tilde{\theta}_j \cdot \nabla_{x'} \tilde{\Lambda}_j \right) \right> \approx 0
$$

In other words,

$$
\nabla_x^m \nabla_{x'}^n \tilde{\theta}_i(x) \tilde{\theta}_j(x')_{\text{dis}} \nabla_x^m \nabla_{x'}^n \tilde{\Lambda}_i(x) \tilde{\Lambda}_j(x')_{\text{dis}}
$$

$$
+ \nabla_x^m \nabla_{x'}^n \tilde{\theta}_i(x) \tilde{\Lambda}_j(x')_{\text{dis}} \nabla_x^m \nabla_{x'}^n \tilde{\Lambda}_i(x) \tilde{\theta}_j(x')_{\text{dis}} \approx 0
$$

The second condition is obeyed if we ensure,

$$
[2\delta(\nabla_x \tilde{\theta}_i)^2 - \frac{1}{2}\delta(\nabla_x \tilde{\Lambda}_i)^2] [2\delta(\nabla_{x'} \tilde{\theta}_j)^2 - \frac{1}{2}\delta(\nabla_{x'} \tilde{\Lambda}_j)^2]
$$
\[
\begin{align*}
&= 8 \left( \nabla_x^m \nabla_x^n \left\langle \tilde{\theta}_i(x) \tilde{\theta}_j(x') \right\rangle \right) \left( \nabla_x^m \nabla_x^n \left\langle \tilde{\theta}_i(x) \tilde{\theta}_j(x') \right\rangle \right) \\
&+ \frac{1}{2} \left( \nabla_x^m \nabla_x^n \left\langle \tilde{\Lambda}_i(x) \tilde{\Lambda}_j(x') \right\rangle \right) \left( \nabla_x^m \nabla_x^n \left\langle \tilde{\Lambda}_i(x) \tilde{\Lambda}_j(x') \right\rangle \right) \\
&- 2 \left( \nabla_x^m \nabla_x^n \left\langle \tilde{\Lambda}_i(x) \tilde{\theta}_j(x') \right\rangle \right) \left( \nabla_x^m \nabla_x^n \left\langle \tilde{\Lambda}_i(x) \tilde{\theta}_j(x') \right\rangle \right) \\
&- 2 \left( \nabla_x^m \nabla_x^n \left\langle \tilde{\theta}_i(x) \tilde{\Lambda}_j(x') \right\rangle \right) \left( \nabla_x^m \nabla_x^n \left\langle \tilde{\theta}_i(x) \tilde{\Lambda}_j(x') \right\rangle \right) \approx 0 \quad (57)
\end{align*}
\]

Finally, we have to also ensure that the cross correlation functions are zero.

\[
\begin{align*}
&\frac{1}{m} \left( \nabla_x^m \nabla_x^n < \tilde{\theta}_i(x) \tilde{\theta}_j(x') > \right) \left( \nabla_x^m \nabla_x^n < \tilde{\Lambda}_i(x) \tilde{\Lambda}_j(x') > \right) \\
&\quad - \frac{1}{4m} \left( \nabla_x^m \nabla_x^n < \tilde{\theta}_i(x) \tilde{\Lambda}_j(x') > \right) \left( \nabla_x^m \nabla_x^n < \tilde{\Lambda}_i(x) \tilde{\Lambda}_j(x') > \right) \approx 0 \quad (58)
\end{align*}
\]

The three Eq. (56), Eq. (57) and Eq. (58) represent correlation functions of random variables that we have assumed may be replaced by their averages. Unfortunately, we have found that these conditions are too severe to be obeyed. Thus we have to be content at saying that the formulas are not exact, but are based on a plausible linearization assumption. This assumption is ‘self-consistent’ in the sense that no obvious inconsistency shows up. A hand-waving justification for at least Eq. (54) is that the right hand side diverges when evaluated thus the fluctuation of this quantity being finite is certainly small compared to the mean which is formally infinite. This infinity may be absorbed by a suitable redefinition of the Fermi energy. On the other hand, Eq. (53) is impossible to justify except that making it renders an elegant analytical solution possible since the equations are now linear.
4 Appendix A

\[
\frac{\nabla R_i}{R_i} = \nabla \ln |R_i| = \nabla \tilde{\Lambda}_i
\]  

We then have to solve,

\[
\nabla^2 \theta_i + \nabla \theta_i \cdot \nabla \tilde{\Lambda}_i = 0
\]  

\[
\frac{(\nabla \theta_i)^2}{2m} - \frac{1}{8m} \left[ 2 \nabla^2 \tilde{\Lambda}_i + (\nabla \tilde{\Lambda}_i)^2 \right] = \epsilon_i - U_{\text{dis}}
\]

We now decompose \( \theta_i \) as follows.

\[
\theta_i(x) = k_i \cdot x + \tilde{\theta}_i(x)
\]

The reduced system may be written as follows.

\[
\nabla^2 \tilde{\theta}_i + k_i \cdot \nabla \tilde{\Lambda}_i + \nabla \tilde{\theta}_i \cdot \nabla \tilde{\Lambda}_i = 0
\]

\[
\frac{k_i^2}{2m} + \frac{(\nabla \tilde{\theta}_i)^2}{2m} - \frac{1}{8m} \left( 2 \nabla^2 \tilde{\Lambda}_i + (\nabla \tilde{\Lambda}_i)^2 \right) = \epsilon_i - U_{\text{dis}}
\]

Now we make the assertion that \( \langle \nabla \tilde{\theta}_i \cdot \nabla \tilde{\Lambda}_i \rangle = 0 \) and \( \langle \nabla \tilde{\Lambda}_i \rangle = 0 \). By taking the average of the above equations we may deduce that

\[
\langle \nabla \tilde{\theta}_i \rangle = 0
\]

\[
\frac{k_i^2}{2m} + \langle (\nabla \tilde{\theta}_i)^2 \rangle - \frac{1}{8m} \langle (\nabla \tilde{\Lambda}_i)^2 \rangle = \epsilon_i
\]

From this we may also deduce,

\[
\nabla^2 \tilde{\theta}_i + k_i \cdot \nabla \tilde{\Lambda}_i \approx 0
\]

\[
\frac{k_i \cdot \nabla \tilde{\theta}_i}{m} - \frac{\nabla^2 \tilde{\Lambda}_i}{4m} \approx -U_{\text{dis}}
\]

The above two equations may be used to iteratively compute the various correlation functions. First we have,

\[
\nabla^2 < \tilde{\theta}_i(x)U_{\text{dis}}(x') > + k_i \cdot \nabla < \tilde{\Lambda}_i(x)U_{\text{dis}}(x') > \approx 0
\]

\[
\frac{k_i \cdot \nabla < \tilde{\theta}_i(x)U_{\text{dis}}(x') >}{m} - \frac{\nabla^2 < \tilde{\Lambda}_i(x)U_{\text{dis}}(x') >}{4m} \approx - < U_{\text{dis}}(x)U_{\text{dis}}(x') > = -\Delta^2 \delta^d(x-x')
\]

We solve this by Fourier transforms.

\[
< \tilde{\theta}_i(x)U_{\text{dis}}(x') > = \frac{1}{V} \sum_{q} F_{10}(q \tilde{q}) e^{-i \tilde{q}(x-x')}
\]
Thus we have,

\[ -q^2 F_{10}(q) - i (k_i \cdot q) F_{20}(q) = 0 \]  
\[ -i \frac{k_i \cdot q}{m} F_{10}(q) + \frac{q^2}{4m} F_{20}(q) = -\Delta^2 \]

\[ F_{10}(q) = \left( -\frac{q^4}{4m} + \frac{(k_i \cdot q)^2}{m} \right)^{-1} [-i(k_i \cdot q)\Delta^2] \]

\[ F_{20}(q) = \left( -\frac{q^4}{4m} + \frac{(k_i \cdot q)^2}{m} \right)^{-1} [q^2\Delta^2] \]

\[ \nabla^2 \hat{\tilde{\Lambda}}_i(x) > +k_i \cdot \nabla < \hat{\tilde{\Lambda}}_i(x) \Lambda_j(x') > \approx 0 \]

\[ \frac{k_i \cdot \nabla < \hat{\tilde{\Lambda}}_i(x) \Lambda_j(x') >}{m} - \frac{\nabla^2 \hat{\tilde{\Lambda}}_i(x) \Lambda_j(x') >}{4m} \approx - < U_{dis}(x) \Lambda_j(x') > \]

\[ < \hat{\tilde{\Lambda}}_i(x) \Lambda_j(x') > = \frac{1}{V} \sum_q F_{12}(q; ij) e^{-iq.(x-x')} \]

\[ < \hat{\tilde{\Lambda}}_i(x) \Lambda_j(x') > = \frac{1}{V} \sum_q F_{22}(q; ij) e^{-iq.(x-x')} \]

\[ F_{12}(q; ij) = \left( -\frac{q^4}{4m} + \frac{(k_i \cdot q)^2}{m} \right)^{-1} [-i(k_i \cdot q)F_{20}(-qj)] \]

\[ F_{22}(q; ij) = \left( -\frac{q^4}{4m} + \frac{(k_i \cdot q)^2}{m} \right)^{-1} [q^2F_{20}(-qj)] \]

\[ \frac{k_i \cdot \nabla < \hat{\tilde{\Lambda}}_i(x) \tilde{\theta}_j(x') >}{m} - \frac{\nabla^2 \hat{\tilde{\Lambda}}_i(x) \tilde{\theta}_j(x') >}{4m} \approx - < U_{dis}(x) \tilde{\theta}_j(x') > \]

\[ < \hat{\tilde{\Lambda}}_i(x) \tilde{\theta}_j(x') > = \frac{1}{V} \sum_q F_{11}(q; ij) e^{-iq.(x-x')} \]

\[ < \hat{\tilde{\Lambda}}_i(x) \tilde{\theta}_j(x') > = \frac{1}{V} \sum_q F_{21}(q; ij) e^{-iq.(x-x')} \]

\[ F_{11}(q; ij) = \left( -\frac{q^4}{4m} + \frac{(k_i \cdot q)^2}{m} \right)^{-1} [-i(k_i \cdot q)F_{10}(-qj)] \]
\[
F_{21}(q;ij) = \left( -\frac{q^4}{4m} + \frac{(k_i \cdot q)^2}{m} \right)^{-1} \left[ q^2 F_{10}(-qj) \right] \quad (88)
\]

From the above equations it is clear that,
\[
\nabla \bar{\theta}_i \cdot \nabla \tilde{\Lambda}_i = (\nabla_x \cdot \nabla_{x'})|_{x=x'} < \tilde{\theta}_i(x) \tilde{\Lambda}_i(x') > \\
= \frac{1}{V} \sum_q F_{12}(q;ij) q^2 = 0 \quad (89)
\]
\[
\epsilon_i = \frac{k_i^2}{2m} - \frac{\Delta^2}{2V} \sum_q q^2 \left( -\frac{q^4}{4m} + \frac{(k_i \cdot q)^2}{m} \right)^{-1} \quad (90)
\]

In one dimension, we have,
\[
\epsilon_i = \frac{k_i^2}{2m} + \frac{m\Delta^2}{2\pi k_i} \int_0^\infty dx \left( \frac{1}{x-k_i} - \frac{1}{x+k_i} \right) \quad (91)
\]

If we interpret the above integral as the principal part then we have,
\[
\epsilon_i = \frac{k_i^2}{2m} \quad (92)
\]

In two space dimensions, it appears that we have to be more careful. In particular, we have to introduce a large momentum cutoff that may not be easily dropped.
\[
\epsilon_i = \frac{k_i^2}{2m} + \frac{m\Delta^2}{\pi} \int_{\Delta} dx \sqrt{x^2 - \frac{1}{k_i^2}} \quad (93)
\]

We take the point of view that this may be absorbed by a suitable redefinition of the Fermi energy. Thus in all three dimensions, we take the liberty to set \( \epsilon_i = k_i^2/2m \).
5 Appendix B

Thus,

\[
\langle J_{ij}(U_{dis}) \cdot J_{ij}(U_{dis}) \rangle = -\int d^d x \int d^d y \delta^d(y - x) \int d^d x' \int d^d y' \delta^d(y' - x')
\]

\[
(\nabla_y \cdot \nabla_{y'}) \left( \phi^*_i(x) \phi_j(y) \phi^*_j(x') \phi_i(y') \right)
\]

\[
= -\frac{1}{V^2} \int d^d x \int d^d y \delta^d(y - x) \int d^d x' \int d^d y' \delta^d(y' - x')
\]

\[
(\nabla_y \cdot \nabla_{y'}) e^{ik \cdot (y' - x)} e^{ik \cdot (y - x')} \left( e^{i\theta_j(y) + \frac{1}{2} \hat{\Lambda}_j(y) e^{i\theta_i(y')} + \frac{1}{2} \hat{\Lambda}_i(x) e^{-i\theta_i(x)} + \frac{1}{2} \hat{\Lambda}_j(x') e^{-i\theta_j(x')} + \frac{1}{2} \hat{\Lambda}_i(x)} e^{-i\theta_j(y)} \right)
\]

Thus we have,

\[
\langle e^{i\theta_j(y) + \frac{1}{2} \hat{\Lambda}_j(y) e^{i\theta_i(y')} + \frac{1}{2} \hat{\Lambda}_i(x) e^{-i\theta_i(x)} + \frac{1}{2} \hat{\Lambda}_j(x')} e^{i\theta_j(y)} \rangle = \frac{1}{V^2} < E^2 >
\]

\[
< E^2 > = \left( i\dot{\theta}_j(y) + \frac{1}{2} \Lambda_j(y) + \frac{1}{2} \Lambda_i(y) - i\dot{\theta}_i(x) + \frac{1}{2} \Lambda_i(x) - i\dot{\theta}_j(x') + \frac{1}{2} \Lambda_j(x') \right)^2
\]

\[
= -\frac{1}{4} \Lambda^2_j(y) + \frac{1}{4} \Lambda^2_i(x) + \frac{1}{4} \Lambda^2_j(x') > - \frac{1}{4} \Lambda^2_j(y') >
\]

\[
+ i \Lambda_j(y) \Lambda_j(y') - 2 \Lambda_j(y) \frac{1}{2} \Lambda_i(y') + i \Lambda_j(y') \Lambda_i(y) > + 2 \Lambda_j(y) \frac{1}{2} \Lambda_i(y') + i \Lambda_j(y') \Lambda_i(y') >
\]

\[
+ 2 \Lambda_j(y) \Lambda_j(x') + i \Lambda_j(y) \Lambda_j(x') > + 2 \Lambda_j(y') \Lambda_i(x) + i \Lambda_j(y') \Lambda_i(y') >
\]

\[
- i \Lambda_j(y) \Lambda_i(x) + \frac{1}{2} \Lambda_j(y) \Lambda_i(x) > - \Lambda_j(x) \Lambda_i(x') + \frac{1}{2} \Lambda_j(x) \Lambda_j(x') >
\]

\[
E_0(i, j) + \sum_q A_{22'}(q, i, j) e^{-i q \cdot (y - y')} + \sum_q A_{2'}(q, i, j) e^{-i q \cdot (y' - x')}
\]

\[
+ \sum_q A_{21'}(q, i, j) e^{-i q \cdot (y' - x')} + \sum_q A_{21}(q, i, j) e^{-i q \cdot (y - x')}
\]

15
\[ \frac{1}{V} \sum_q A_{21}(q; i, j) e^{-iq \cdot (x-x')} + \frac{1}{V} \sum_q A_{11'}(q; i, j) e^{-iq \cdot (x-x')} \]  

\[ E_0(i, j) = -\frac{2}{V} \sum_q F_{11}(q; jj) + \frac{1}{2V} \sum_q F_{22}(q; jj) - \frac{2}{V} \sum_q F_{11}(q; ii) + \frac{1}{2V} \sum_q F_{22}(q; ii) \]  

\[ A_{22'}(q; i, j) = -2 F_{11}(q; ji) + i F_{12}(q; ji) + i F_{21}(q; ji) + \frac{1}{2} F_{22}(q; ji) \]  

\[ A_{21'}(q; i, j) = 2 F_{11}(q; jj) + i F_{12}(q; jj) - i F_{21}(q; jj) + \frac{1}{2} F_{22}(q; jj) \]  

\[ A_{22'}(q; i, j) = 2 F_{11}(q; ij) + i F_{12}(q; ij) - i F_{21}(q; ij) + \frac{1}{2} F_{22}(q; ij) \]  

\[ A_{21'}(q; i, j) = 2 F_{11}(q; ji) + i F_{12}(q; ji) - i F_{21}(q; ji) + \frac{1}{2} F_{22}(q; ji) \]  

\[ A_{11'}(q; i, j) = -2 F_{11}(q; ij) - i F_{12}(q; ij) - i F_{21}(q; ij) + \frac{1}{2} F_{22}(q; ij) \]  

\[ \langle J_{ij}(U_{dis}) \cdot J_{ji}(U_{dis}) \rangle = -\frac{C_0}{V^2} \int d^d x \int d^d x' e^{i(k_i - k_j) \cdot (x-x')} e^{\frac{\lambda}{2} <E^2> F(k_i, k_j)} \]  

\[ F(k_i, k_j) = \frac{1}{2V} \sum_q A_{22'}(q; ij) e^{-iq \cdot (x-x')} q^2 \]  

\[ \begin{aligned} &\left[ i k_i + \frac{1}{2V} \sum_q A_{22'}(q; ij)(-iq) + \frac{1}{2V} \sum_q (A_{22'}(q; ij)(iq) + A_{21'}(-q; ij)(iq)) e^{-iq \cdot (x-x')} \right] \\ &\times \left[ i k_j + \frac{1}{2V} \sum_q A_{21}(q; ij)(-iq) + \frac{1}{2V} \sum_q (A_{22'}(q; ij)(-iq) + A_{21'}(q; ij)(-iq)) e^{-iq \cdot (x-x')} \right] \end{aligned} \]  

\[ \frac{1}{2} <E^2> = -\frac{1}{V} \sum_q F_{11}(q; jj) + \frac{1}{4V} \sum_q F_{22}(q; jj) - \frac{1}{V} \sum_q F_{11}(q; ii) + \frac{1}{4V} \sum_q F_{22}(q; ii) + \frac{1}{2V} \sum_q A_{21}(q; i, j) + \frac{1}{2V} \sum_q A_{22'}(q; i, j) \]  

\[ + \frac{1}{2V} \sum_q [A_{22'}(q; i, j) + A_{21'}(-q; i, j) + A_{22'}(q; i, j) + A_{11'}(q; i, j)] e^{-iq \cdot (x-x')} \]  

\[ F_{12}(q; ij) = P_i(q) P_j(q)[-i(k_i, q)q^2 \Delta^2] \]  

\[ F_{22}(q; ij) = P_i(q) P_j(q)[q^4 \Delta^2] \]  

\[ F_{11}(q; ij) = P_i(q) P_j(q)[(k_i, q)(k_j, q) \Delta^2] \]
\[ F_{21}(q; ij) = P_i(q)P_j(q)[i(k_j, q)q^2\Delta^2] \]  
\[ P_i(q) = \left(-\frac{q^4}{4m} + \frac{(k_i, q)^2}{m}\right)^{-1} \]  
\[ D_i(q) = -k_i.q + \frac{q^2}{2} \]  
\[ A_{22'}(q; i, j) = A_{11'}(q; i, j) = \frac{2m^2\Delta^2}{D_i(q)D_j(q)} \]  
\[ A'_{21}(q; i, j) = \frac{2m^2\Delta^2}{D_i^2(q)} \]  
\[ A'_{21'}(q; i, j) = \frac{2m^2\Delta^2}{D_j^2(q)} \]  
\[ C^0_i = \exp\left[-\frac{m^2\Delta^2}{2V} \sum_q \left( \frac{1}{D_i(q)} + \frac{1}{D_i(-q)} \right)^2 \right] \]

Define,
\[ \frac{1}{2} < E'^2 > = \frac{1}{2} < E^2 > - \frac{m^2\Delta^2}{2V} \sum_q \left( \frac{1}{D_i(q)} + \frac{1}{D_i(-q)} \right)^2 - \frac{m^2\Delta^2}{2V} \sum_q \left( \frac{1}{D_j(q)} + \frac{1}{D_j(-q)} \right)^2 \]  
\[ \frac{1}{2} < E'^2 > = - \frac{m^2\Delta^2}{V} \sum_q \left( \frac{1}{D_i(q)} - \frac{1}{D_j(q)} \right)^2 + \frac{m^2\Delta^2}{V} \sum_q \left( \frac{1}{D_i(-q)} + \frac{1}{D_j(q)} \right)^2 \]  
\[ e^{-i\mathbf{q}.(\mathbf{x} - \mathbf{x}')} \]

The above equation leads to a finite result as we may see below.
\[ \frac{1}{2} < E'^2 > = \frac{m^2\Delta^2}{V} \sum_q \left( \frac{1}{D_i(-q)} + \frac{1}{D_i(q)} \right) \left( \frac{1}{D_j(-q)} + \frac{1}{D_j(q)} \right) \]  
\[ - \frac{m^2\Delta^2}{V} \sum_q \left( \frac{1}{D_i(-q)} + \frac{1}{D_j(q)} \right)^2 \left( 1 - \cos[q.(x - x')] \right) \]  
\[ -i \frac{m^2\Delta^2}{2V} \sum_q \left( \frac{1}{D_i(-q)} + \frac{1}{D_i(q)} + \frac{1}{D_j(q)} + \frac{1}{D_j(-q)} \right) \]  
\[ \times \left( \frac{1}{D_i(-q)} + \frac{1}{D_j(q)} - \frac{1}{D_i(q)} - \frac{1}{D_j(-q)} \right) \sin[q.(x - x')] \]  
\[ \]
Similarly, we may write for \( F(\mathbf{k}_i, \mathbf{k}_j) \) as follows,

\[
F(\mathbf{k}_i, \mathbf{k}_j) = \frac{1}{2V} \sum_{\mathbf{q}} \frac{2m^2 \Delta^2 \mathbf{q}^2}{D_i(-\mathbf{q})D_j(\mathbf{q})} e^{-i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')}
+ \frac{1}{2V} \sum_{\mathbf{q}} \frac{2m^2 \Delta^2}{D_i(-\mathbf{q})D_j(\mathbf{q})} e^{-i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} + \frac{1}{2V} \sum_{\mathbf{q}} \frac{2m^2 \Delta^2}{D_i(-\mathbf{q})D_j(\mathbf{q})} e^{-i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')}
\]

\[
\cdot [i\mathbf{k}_i + \frac{1}{2V} \sum_{\mathbf{q}} \frac{2m^2 \Delta^2}{D_i(-\mathbf{q})D_j(\mathbf{q})} e^{-i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')}
+ \frac{1}{2V} \sum_{\mathbf{q}} \frac{2m^2 \Delta^2}{D_i(-\mathbf{q})D_j(\mathbf{q})} e^{-i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')}
\]

\[
= \frac{1}{2V} \sum_{\mathbf{q}} \left( \frac{2m^2 \Delta^2}{D_i(-\mathbf{q})D_j(\mathbf{q})} + \frac{2m^2 \Delta^2}{D_i(-\mathbf{q})D_j(\mathbf{q})} \right) e^{-i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \]

Now we would like to solve for \( F(\mathbf{k}_i, \mathbf{k}_j) \) and \( \frac{1}{2} < E'^2 > \) in one dimension and three dimensions where the integrals are likely to be simple.

B.1: One Dimension:

\[
\frac{1}{2} < E'^2 >= -\frac{m^2 \Delta^2}{2} \left( \frac{1}{k_i} - \frac{1}{k_j} \right)^2 |x| + i \frac{m^2 \Delta^2}{k_F^2} \text{sgn}(x) \left( \frac{1}{k_i} - \frac{1}{k_j} \right) (120)
\]

\[
F(k_i, k_j) = -\frac{1}{2} \frac{m^2 \Delta^2}{k_i + k_j} \left( e^{-2i\mathbf{k}_i x} - e^{-2i\mathbf{k}_j x} \right) \text{sgn}(x)
\]

\[
+ [ik_i - \frac{m^2 \Delta^2}{2} \left( \frac{1}{k_i k_j} - \frac{e^{-2i\mathbf{k}_i x}}{k_j (k_j + k_i)} - \frac{e^{2i\mathbf{k}_i x}}{k_i (k_j + k_i)} \right) + \frac{1}{k_i k_j} e^{2i\mathbf{k}_i x} sgn(x)]
\]

\[
x \left[ ik_j + \frac{m^2 \Delta^2}{2} \left( \frac{1}{k_i k_j} - \frac{e^{-2i\mathbf{k}_j x}}{k_j (k_j + k_i)} - \frac{e^{2i\mathbf{k}_j x}}{k_i (k_j + k_i)} \right) + \frac{1}{k_i k_j} e^{-2i\mathbf{k}_j x} sgn(x) \right]
\]

\[
f(k_i, k_j) \equiv \langle \mathbf{J}_{ij}(U_{dis}) \cdot \mathbf{J}_{ji}(U_{dis}) \rangle = -\frac{1}{L} \int_{-\infty}^{+\infty} dx e^{i(k_j-k_i)x} e^{\frac{1}{2} < E'^2 > F(k_i, k_j)} (121)
\]

The final formula for the a.c. conductivity involves evaluating the following integral. For \( m\omega < k_F^2 \) we have,

\[
\text{Re} \left[ \sigma(\omega) \right] \sim \frac{1}{\omega} \left( \int_{-k_F}^{-k_F + \omega/v_F} dk_j + \int_{k_F - \omega/v_F}^{k_F} dk_j \right) \left[ f(k_j(1 + m\omega/k_F^2), k_j) + f(-k_j(1 + m\omega/k_F^2), k_j) \right]
\]

\[
\approx \frac{2}{v_F} \left[ f(k_F(1 + m\omega/k_F^2), k_F) + f(-k_F(1 + m\omega/k_F^2), k_F) \right]
+ f(-k_F(1 + m\omega/k_F^2), -k_F) + f(k_F(1 + m\omega/k_F^2), -k_F) (123)
\]

We would like to systematically evaluate this and show that in the zero frequency limit, the a.c. conductivity is proportional to \( \omega \) and vanishes for \( \omega = 0 \). The
\(\omega \to 0\) limit is quite subtle. In particular, we may not set \(\omega = 0\) at the outset. For then, \(f(k_F, k_F) = \infty\).

\[
f(k_F(1 + m\omega/k_F^2), k_F) \approx -\frac{1}{L} \int_{-\infty}^{\infty} dx \ e^{-im\omega x/k_F^2} e^{-i\frac{m^2 \Delta^2}{2k_F^2} \text{sgn}(x)} F(k_F, k_F)
\]

\[
F(k_F, k_F) = -i \frac{m^2 \Delta^2}{2k_F} (e^{-2ik_F x} - e^{2ik_F x}) \text{ sgn}(x)
\]

\[
+ [ik_F + \frac{m^2 \Delta^2}{2k_F^2} \{ - \frac{e^{2ik_F x}}{2k_F^2} + \frac{e^{-2ik_F x}}{2k_F^2} \} \text{sgn}(x)]^2
\]

and,

\[
f(-k_F(1 + m\omega/k_F^2), k_F) \approx f(-k_F, k_F) \approx
\]

\[
\approx -\frac{1}{L} \int_{-\infty}^{\infty} dx \ e^{2ik_F x} e^{-2m^2 \Delta^2 x/k_F^2} e^{-i\frac{m^2 \Delta^2}{k_F} \text{sgn}(x)} F(-k_F, k_F)
\]

\[
F(-k_F, k_F) = -(2m^2 \Delta^2) e^{-2ik_F x} x
\]

\[
- [ik_F + \frac{m^2 \Delta^2}{k_F^2} (-1 + e^{-2ik_F x}) \text{ sgn}(x)]^2
\]

\[
f(-k_F(1 + m\omega/k_F^2), -k_F) \approx -\frac{1}{L} \int_{-\infty}^{\infty} dx \ e^{i\frac{m^2 \Delta^2}{k_F} x} e^{i\frac{m^2 \Delta^2}{k_F} \text{sgn}(x)} F(-k_F, -k_F)
\]

\[
F(-k_F, -k_F) = F(k_F, k_F)
\]

and,

\[
f(k_F(1 + m\omega/k_F^2), -k_F) \approx f(k_F, -k_F) \approx
\]

\[
\approx -\frac{1}{L} \int_{-\infty}^{\infty} dx \ e^{-2ik_F x} e^{-2m^2 \Delta^2 x/k_F^2} e^{i\frac{m^2 \Delta^2}{k_F^2} \text{sgn}(x)} F(k_F, -k_F)
\]

\[
F(k_F, -k_F) = -(2m^2 \Delta^2) e^{2ik_F x} x
\]

\[
- [ik_F + \frac{m^2 \Delta^2}{k_F^2} (-1 + e^{2ik_F x}) \text{ sgn}(x)]^2
\]
\[ f(k_F(1 + m\omega/k_F^2), k_F) + f(-k_F(1 + m\omega/k_F^2), -k_F) \]

\[ \approx -2\left[ \frac{\Delta^4 k_F m^5 \omega}{k_F^4} + (8k_F^5 + \Delta^4 m^4)(16k_F^4 - m^2\omega^2) \right] \frac{\sin[\Delta^2 m^3\omega/k_F^5]}{k_F^4 (16k_F^4 - m^2\omega^2)} \]

These have been evaluated using Mathematica\textsuperscript{TM}. Immediately we see several problems. \( f(k_i, k_j) \geq 0 \) for all arguments. Yet the above equations tell us that they are negative sometimes and sometimes even complex!

This means that something is seriously wrong with our formalism. Perhaps replacing \( \nabla \tilde{\theta}_i \cdot \nabla \tilde{\Lambda}_i \) by the average (which is zero!) was not such a good idea after all.

What is as bad is that the final formula for the a.c. conductivity has a non-vanishing constant part (at least it is real!) this means that d.c. conductivity is not zero. This is clearly wrong. Maybe some readers of this preprint will offer to collaborate with the authors to fix this difficulty.

**B.2 : Two and Three Dimensions :**

In two and three dimensions, the integrals are substantially more complicated caused by the complicated angular parts. Work is in progress in collaboration with Shri. Chandradew Sharma it will be reported soon. Useful conversations with Prof. N.D. Haridass of IMSc. is gratefully acknowledged.

**6 Appendix C**

We would like to ascertain whether or not the zero frequency limit of the a.c. conductivity is the d.c. conductivity. First let us define d.c. conductivity. Consider a system of electrons coupled first to a uniform d.c. electric field. The interaction part of the hamiltonian may be written as,

\[ H_I = |e| \int d^d x \psi^\dagger(x) \left( E_{\text{ext}} \cdot \mathbf{x} \right) \psi(x) \]

(135)
The expectation value of the total momentum of the electrons may be written in the interaction representation as,

\[ \langle P(t) \rangle = \frac{\langle T S \hat{P}(t) \rangle}{\langle T S \rangle} \] (136)

\[ S = e^{-i \int_0^{-i\beta} dt \hat{H}_I(t)} \] (137)

The d.c. conductivity is then simply given by,

\[ \sigma_{d.c.} = \frac{|e|}{mV} \left( \frac{\delta}{\delta E_{ext}} \langle P(t) \rangle \right)_{E_{ext}=0} \] (138)

Since,

\[ \frac{\delta}{\delta E_{ext}} S = -i \int_0^{-i\beta} dt' |e| \int d^d x' \psi^\dagger(x',t') \vec{x}' \psi(x',t') \] (139)

Define,

\[ \hat{X}(t') = \int d^d x' \psi^\dagger(x',t') \vec{x}' \psi(x',t') \] (140)

\[ \sigma_{d.c.} = -i \frac{e^2}{m^2 V} \int_0^{-i\beta} dt' \langle T \hat{X}(t') \cdot \hat{P}(t) \rangle \] (141)

From the main text we see that the zero frequency limit of the a.c. conductivity is written as,

\[ \sigma_{a.c.}(0) \equiv \int_0^{-i\beta} dt' \tilde{\sigma}(t-t') \] (142)

Here \( \hat{P} = \sum_k \vec{k} \hat{c}_k^\dagger \hat{c}_k \). At first sight it appears that \( \sigma_{d.c.} \) of Eq.(141) is not equal to \( \sigma_{a.c.}(0) \) of Eq.(142). For the two expressions to be the same we must have,

\[ \hat{X}(t') = \frac{1}{m} \int_0^{-i\beta} dt_1 \theta(t' - t_1) \hat{P}(t_1) \] (143)

If we take the derivative with respect to \( t' \) we find,

\[ \frac{\partial}{\partial t} \hat{X}(t) = \frac{1}{m} \hat{P}(t) \] (144)

This is nothing but the definition of the momentum operator. It is a trivial kinematical result. Hence we may conclude that the zero frequency limit of the a.c. conductivity is in fact the d.c. conductivity.
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