THE BELLMAN FUNCTION OF THE DYADIC MAXIMAL OPERATOR RELATED TO KOLMOGOROV'S INEQUALITY

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Abstract. We precisely compute the Bellman function of two variables of the dyadic maximal operator in relation to Kolmogorov’s inequality. In this way we give an alternative proof of the results in [5]. Additionally, we characterize the sequences of functions that are extremal for this Bellman function. The proof for this is based on that is given in this paper for the Bellman function we are interested in.

1. Introduction

The dyadic maximal operator is defined on \( \mathbb{R}^n \) by

\[
M_d \phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(u)| \, du : x \in Q, \, Q \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\},
\]

(1.1)

for every \( \phi \in L^1_{\text{loc}}(\mathbb{R}^n) \) where the dyadic cubes are those formed by the grids \( 2^{-N} \mathbb{Z}^n \), for \( N = 0, 1, 2, \ldots \).

As it is well known it satisfies the following weak type (1,1) inequality

\[
\{ x \in \mathbb{R}^n : M_d \phi(x) > \lambda \} \leq \frac{1}{\lambda} \int_{\{M_d \phi > \lambda\}} |\phi(u)| \, du
\]

(1.2)

for every \( \phi \in L^1(\mathbb{R}^n) \) and every \( \lambda > 0 \), from which it is easy to get the following \( L^p \)-inequality:

\[
\|M_d \phi\|_p \leq \frac{p}{p-1} \|\phi\|_p
\]

(1.3)

for every \( \phi \in L^p(\mathbb{R}^n) \), \( p > 1 \).

It is easy to see that (1.2) is best possible. It has also been proved that (1.3) is sharp (see [1] and [2] for general martingales and [16] for dyadic ones).

A way of studying the dyadic maximal operator is by finding refinements of the above inequalities.

Concerning (1.2) refinements have been studied in [8] and [9]. About (1.3) the following function has been precisely computed in [3]:

\[
B^Q_p(f, F) = \sup \left\{ \frac{1}{|Q|} \int_Q (M_d \phi)^p : \phi \geq 0, \frac{1}{|Q|} \int_Q \phi(u) \, du = f, \frac{1}{|Q|} \int_Q \phi^p(u) \, du = F \right\}
\]

(1.4)

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where $Q$ is a fixed dyadic cube on $\mathbb{R}^n$ and $f, F$ variables satisfying: $0 < f^p \leq F$. It turns out that (1.4) is independent of the cube $Q$. Its exact value is given by

$$B_p^Q(f, F) = F \omega_p(f^p/F)^p$$

where $\omega_p : [0, 1] \to \left[1, \frac{p}{p-1}\right]$ is the inverse function of $H_p$, which is given by $H_p(z) = -(p-1)z^p + pz^{p-1}$, for $z \in \left[1, \frac{p}{p-1}\right]$. After working the case $p > 1$ it is interesting to search the case $p = q < 1$. This is connected with the following known as Kolmogorov's inequality

$$\int_E |\mathcal{M}_d \phi(u)|^q du \leq \frac{1}{1-q} |E|^{1-q} \left(\int_{\mathbb{R}^n} |\phi|\right)^q$$

for every $q \in (0, 1)$, $\phi \in L^1(\mathbb{R}^n)$ and $E$ measurable subset of $\mathbb{R}^n$ with finite measure.

This inequality connects the $L^q$ norm of $\mathcal{M}_d \phi$ upon subsets of $\mathbb{R}^n$ of finite measure with the $L^1$-norm of $\phi$. This inequality was studied in [5]. It is proved there that it is sharp. More precisely a stronger result than the above sharpness is proved, namely the exact evaluation of the following function of four variables $f, h, L, k$:

$$B_q(f, h, L, k) = \sup \left\{ \frac{1}{|Q|} \int_E (\mathcal{M}_d \phi)^q : \phi \geq 0, \frac{1}{|Q|} \int_Q \phi = f, \int_X \phi d\mu = h, E \subseteq Q \text{ measurable with } |E| = k \right\}$$

where $Q$ is a fixed dyadic cube, $Q'$ runs over all the dyadic cubes containing $Q$, $\phi \in L^1(Q)$, $0 < k \leq |Q|$ and $f, h, L$ satisfy $0 < f \leq L$, $h \leq f^q$.

It turns out that (1.4) is independent of $Q$ so we can consider $Q = [0, 1]^n$. More generally we consider a non-atomic probability measure space $(X, \mu)$ equipped with a tree structure $T$, which plays the role of the dyadic sets in our situation (see definition in Section 2).

Then the dyadic maximal operator $\mathcal{M}_T$ is defined by:

$$\mathcal{M}_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in T \right\}$$

for every $\phi \in L^1(X, \mu)$.

It is not difficult to see that (1.2) and (1.3) remain true even in this setting.

We define now

$$B'_q(f, h, L, k) = \sup \left\{ \int_E \max(\mathcal{M}_T \phi, L)^q d\mu, \phi \geq 0, \phi \in L^1(X, \mu), \int_X \phi d\mu = f, \int_X \phi^q d\mu = h, E \subseteq X \text{ measurable with } \mu(E) = k \right\}$$

where $L, f, h, k$ satisfy $L \geq f > 0$, $0 < h \leq f^q$, $0 < k \leq 1$.

Then $B'_q = B_q$ according to arguments given in [3].
The precise value of $B'_q$ has been found by working the respective Bellman function of two variables which is defined by,

$$B_q(f, h) = \sup \left\{ \int_X (\mathcal{M}_T \phi)^q d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = h \right\}$$

with $0 < h \leq f^q$.

Several calculus arguments and the use of the value of (1.9) in certain subsets of $X$ gives (1.8) as is done in [5]. We are thus interested in (1.9). The result is the following:

**Theorem 1.** It is true that:

$$B_q(f, h) = h \omega_q(f^q/h), \text{ where } \omega_q : [1, +\infty) \to [1, +\infty)$$

is defined by $\omega_q(z) = [H^{-1}_q(z)]^q$ where

$$H_q(z) = (1 - q)z^q + qz^{q-1}, \quad z \geq 1.$$  

Our first aim in this paper is to give an alternative proof of Theorem 1.

Our second aim is to characterize the extremal sequences of functions concerning (1.9). More precisely we will prove the following.

**Theorem 2.** Let $\phi_n : (x, \mu) \to \mathbb{R}^+$ be such that $\int_X \phi_n dh = f$ and $\int_X \phi_n^q d\mu = h$, for any $n \in \mathbb{N}$. Then the following are equivalent

i) $\lim_n \int_X (\mathcal{M}_T \phi_n)^q d\mu = h \omega_q(f^q/h)$

ii) $\lim_n \int_X |\mathcal{M}_T \phi_n - c \phi_n|^q d\mu = 0$, where $c = \omega_q(f^q/h)^{1/q}$.

That is $\phi_n$ behaves approximately in $L^q$ like eigenfunction of $\mathcal{M}_T$ for the eigenvalue $c$.

We also remark that there are several problems in Harmonic Analysis were Bellman functions arise. Such problems (including the dyadic Carleson embedding theorem and weighted inequalities) are described in [12] (see also [9], [7]) and also connections to Stochastic Optimal Control are provided, from which it follows that the corresponding Bellman functions satisfy certain nonlinear second-order PDEs. The exact evaluation of a Bellman function is a difficult task which is connected with the deeper structure of the corresponding Harmonic Analysis problem. Until now several Bellman functions have been computed (see [1], [2], [3], [4], [5], [7], [12], [13], [14], [15]). The exact evaluation of (1.9) for $q > 1$ has been also given in [11] by L. Slavin, A. Stokolos and V. Vasyunin which linked the computation of it to solving certain PDEs of the Monge-Ampère type and in this way they obtained an alternative proof of the results in [3] for the Bellman functions related to the dyadic maximal operator.

The paper is organized as follows.
In Section 2 we give some preliminary results and facts needed for use in the subsequent sections. In Section 3 we give a proof that the right side of \((1.10)\) is an upper bound for \(\int_X (M_T \phi)^q d\mu \). In Section 4 we give the sharpness of the result just mentioned. In Section 5 we prove Theorem 2. At last in Section 6 we discuss further properties of certain extremal sequences for \((??)\).

We need also to say that analogous results for the case \(q > 1\) are treated in [4] but for the Bellman function of three variables.

We proceed now to the next Section.

2. Preliminaries

Let \((X, \mu)\) be a non-atomic probability measure space. We give the following.

**Definition 2.1.** A set \(\mathcal{T}\) of measurable subsets of \(X\) will be called a tree if it satisfies

i) \(X \in \mathcal{T}\) and for every \(I \in \mathcal{T}\) we have that \(\mu(I) > 0\).

ii) For every \(I \in \mathcal{T}\) there corresponds a finite or countable subset \(C(I) \subseteq \mathcal{T}\) containing at least two elements such that

(a) the elements of \(C(I)\) are pairwise disjoint subsets of \(I\)

(b) \(I = \bigcup C(I)\).

iii) \(\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_m\) where \(\mathcal{T}_0 = \{X\}\) and \(\mathcal{T}_{m+1} = \bigcup_{I \in \mathcal{T}_m} C(I)\).

iv) We have that \(\lim_{m \to \infty} \sup_{I \in \mathcal{T}_m} \mu(I) = 0\). \(\square\)

Examples of trees are given in [3]. The most known is the one given by the family of dyadic subcubes of \([0,1]^n\).

The following has been proved in [10].

**Theorem 2.1.** For any \(g : (0,1) \to \mathbb{R}^+\) non-increasing, every increasing function \(G_1\) defined on \([0, +\infty)\) with non-negative values and every \(k \in (0,1]\) the following holds:

\[
\sup \left\{ \int_K G_1(M_T \phi) d\mu : \phi \geq 0, \phi^* = g, K \text{ measurable subset of } X \text{ with } \mu(K) = k \right\}
\]

\[
= \int_0^k G_1 \left( \frac{1}{t} \int_0^t g \right) dt.
\]

\(\square\)

Here by \(\phi^*\) we mean the decreasing rearrangement of \(\phi\) defined by

\[\phi^*(t) = \sup_{e \in X} \inf_{x \in e} |\phi(x)|, \quad 0 < t \leq 1.\]

Now given a tree on \((X, \mu)\) we define the associated dyadic maximal operator as follows

\[M_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\}\]
for every $\phi \in L^1(X, \mu)$.

3. The Bellman function

We are now able to prove the following

Lemma 3.1. For every $q$ such that $0 < q < 1$ and every $f, h$ such that $0 < h \leq f^q$ we have that

$$
\int_X (M_T \phi)^q d\mu \leq h \omega_q \left( \frac{f^q}{h} \right)
$$

for any $\phi \in L^1(X, \mu)$ such that $\phi \geq 0$, $\int_X \phi d\mu = f$, $\int_X \phi^q d\mu = h$.

Proof. We set $I = \int_X (M_T \phi)^q d\mu$. Then

$$
I = \int_{\lambda=0}^{+\infty} q\lambda^{q-1} \mu(\{x \in X : M_T \phi(x) > \lambda\}) d\lambda
$$

$$
= \int_{\lambda=0}^{f} q\lambda^{q-1} \mu(\{x \in X : M_T \phi(x) > \lambda\}) d\lambda + \int_{\lambda=f}^{+\infty} q\lambda^{q-1} \mu(\{x \in X : M_T \phi(x) > \lambda\}) d\lambda = II + III,
$$

where

$$
II = \int_{\lambda=0}^{f} q\lambda^{q-1} d\lambda = f^q, \quad \text{and}
$$

$$
II = \int_{\lambda=f}^{+\infty} q\lambda^{q-1} \mu(\{x \in X : M_T \phi(x) > \lambda\}) d\lambda.
$$

Now, because of the weak type inequality (1.2) for $M_T$ we have that

$$
III \leq \int_{\lambda=f}^{+\infty} q\lambda^{q-1} \mu(\{x \in X : M_T \phi(x) > \lambda\}) d\lambda
$$

$$
= \int_{\lambda=f}^{+\infty} q\lambda^{q-2} \mu(\{M_T \phi > \lambda\}) d\lambda \quad \text{(by Fubini’s theorem)}
$$

$$
= \int_X \phi(x) \frac{q}{q-1} \left[ \frac{\lambda^{q-1}}{M_T \phi(x)} \right]_{\lambda=f}^{+\infty} d\mu(x)
$$

$$
= \frac{q}{1-q} f^q - \frac{q}{1-q} \int_X \phi(x) [M_T \phi(x)]^{q-1} d\mu(x).
$$

Thus, we have that

$$
I = \int_X (M_T \phi)^q d\mu \leq \frac{1}{1-q} f^q - \frac{q}{1-q} IV,
$$

where

$$
IV = \int_X \phi(M_T \phi)^{q-1} d\mu.
$$

We now know from Holder’s inequality that the following is true for any $\phi_1, \phi_2$ such that $\phi_1 \in L^1(X, \mu)$, $\phi_2 \in L^{q/(1-q)}(X, \mu)$, where $q \in (0, 1)$

$$
\int_X (\phi_1 \phi_2)^q d\mu \leq \left( \int_X \phi_1 d\mu \right)^q \left( \int_X \phi_2^{q/(1-q)} d\mu \right)^{1-q}.
$$
We set $\phi_1 = \phi(\mathcal{M}_T \phi)^{q-1}$ and $\phi_2 = (\mathcal{M}_T \phi)^{1-q}$ in (3.3), and we conclude that

$$h = \int_X \phi^q d\mu \leq \left[ \int_X \phi(\mathcal{M}_T \phi)^{q-1} d\mu \right]^q \cdot \left[ \int_X (\mathcal{M}_T \phi)^q d\mu \right]^{1-q}$$

(3.4)

$$= IV^q \cdot I^{1-q} \Rightarrow IV \geq h^{\frac{q}{q}} I^{1-\frac{q}{q}},$$

(3.2) now in view of (3.4) gives

$$I \leq \frac{1}{1-q} f^q - \frac{q}{1-q} h \frac{1}{n} I^{1-\frac{q}{q}} \Rightarrow (1-q) \frac{I}{h} \leq f^q - q \left( \frac{I}{h} \right)^{1-\frac{1}{q}} \Rightarrow \left( u = \frac{I}{h} \right)$$

$$qu^{1-\frac{1}{q}} + (1-q)u \leq \frac{f^q}{h} \Rightarrow u \leq \omega_q \left( \frac{f^q}{h} \right)$$

in case where $u \geq 1$, while $u \leq 1 \leq \omega_q \left( \frac{f^q}{h} \right)$ in case where $u < 1$, because of the definition of $\omega_q(z)$, $z \geq 1$.

Thus, we proved that

$$I = \int_X (\mathcal{M}_T \phi)^q d\mu \leq h \omega_q \left( \frac{f^q}{h} \right).$$

Lemma 3.1 is now proved.

We will also need the following:

**Lemma 3.2.** For any $g : (0, 1] \to \mathbb{R}^+$ non-increasing, with $\int_0^1 g(u) du = f$, and any $q$ such that $0 < q < 1$, the following equality holds:

$$\int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt = \frac{1}{1-q} f^q - \frac{q}{1-q} \int_0^1 g(t) \left( \frac{1}{t} \int_0^t g \right)^{q-1} dt.$$

**Proof.** The proof is similar to that of Lemma 3.1.

We set

$$I = \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt = f^q + \int_0^{+\infty} q \lambda^{q-1} \left\{ t \in (0, 1] : \frac{1}{t} \int_0^t g > \lambda \right\} d\lambda.$$

We consider now for every $\lambda > f$ the unique real number on $(0, 1]$, $\beta(\lambda)$ such that

$$\frac{1}{\beta(\lambda)} \int_0^1 g = \lambda \quad \text{without loss of generality} \quad g(0+) = +\infty, \quad \text{the finite case is treated similarly}.$$

Then, because of the monotonicity of $g$, for any $\lambda > f$

$$\left\{ t \in (0, 1] : \frac{1}{t} \int_0^t g > \lambda \right\} = (0, \beta(\lambda)), \quad \text{so that}$$
\[ I = f^q + \int_{\lambda=f}^{+\infty} q\lambda^{q-1} \beta(\lambda) d\lambda = f^q + \int_{\lambda=f}^{+\infty} q\lambda^{q-1} \left( \frac{1}{\lambda} \int_0^{\beta(\lambda)} g(u) du \right) d\lambda \]

\[ = f^q + \int_{\lambda=f}^{+\infty} q\lambda^{q-2} \left( \int_{t}^{+\infty} \frac{1}{t} \int_{g>\lambda} \right) g(u) du d\lambda = f^q + \int_0^1 g(t) \frac{q}{q-1} \left( \lambda^{q-1} \right) \frac{1}{\lambda=f} d\lambda \]

\[ = \frac{1}{q-1} f^q - \frac{q}{1-q} \int_0^1 g(t) \left( \frac{1}{t} \int_0^t g \right)^{q-1} dt, \]

and Lemma 3.2 is proved. \( \square \)

4. Sharpness of Lemma 3.1

In the determination of the upper bound of \( \int (\mathcal{M}_T \phi)^q d\mu \) in Lemma 3.1 there are exactly two steps where inequalities are used.

The first is before we reach to the following inequality

\[ \int_X (\mathcal{M}_T \phi)^q d\mu \leq \frac{1}{1-q} f^q - \frac{q}{1-q} \int_X \phi (\mathcal{M}_T \phi)^{q-1} d\mu, \]

while by Lemma 3.2 we have equality in the respective inequality for the Hardy operator, this is

\[ \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt = \frac{1}{1-q} f^q - \frac{q}{1-q} \int_0^1 g(t) \left( \frac{1}{t} \int_0^t g \right)^{q-1} dt. \]

We now use Theorem 2.1 of Section 2 which states that

\[ \int_X (\mathcal{M}_T \phi)^q d\mu \leq \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt, \]

with \( \phi^* = g \), which is sharp when one considers all \( \phi \) such that \( \phi^* = g \).

What we are saying is that if we fix \( g \), and leave \( \phi \) run across all the rearrangements of \( g \) we attain equality in the first inequality which we meet in Lemma 3.1 As for the second we need to mention the following.

According (3.3) we have that

\[ \int_0^1 g(t) \left( \frac{1}{t} \int_0^t g \right)^{q-1} dt \geq h^{\frac{1}{q}} \left[ \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt \right]^{1-\frac{1}{q}}. \]

Now (4.2)-(4.4) give for any \( \phi : \phi^* = g \)

\[ \int_X (\mathcal{M}_T \phi)^q d\mu \leq \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt = I_g \leq \frac{1}{1-q} f^q - \frac{q}{1-q} h^{\frac{1}{q}} I_g \frac{1}{q-1}. \]

(4.5) now gives as in Lemma 3.1 that \( I_g \leq \omega_q(f^q/h) \). So the second step we use inequality is in Holder’s inequality: (4.3).

So, if we want to attain equality there for the function \( g \) we must have that

\[ \frac{1}{t} \int_0^t g = cg(t), \quad t \in (0, 1] \]
for some constant $c$. If additionally $\int_0^1 g = f$, $\int_0^1 g^q = h$ and $c = \omega_q \left( \frac{f^q}{h} \right)^{1/q}$, then in view of (4.3) we will have that

$$\sup_{\phi^* = g} \int_X (\mathcal{M}_T \phi)^q d\mu = \omega_q \left( \frac{f^q}{h} \right) \cdot \int_0^1 g^q = h \omega_q \left( \frac{f^q}{h} \right), \text{ for that } g.$$ 

Thus the following will give the sharpness of Lemma 3.1.

**Lemma 4.1.** For any $f, h$: $0 < h \leq f^q$ there exists $g : (0, 1] \rightarrow \mathbb{R}^+$ non-increasing, continuous such that for which $\int_0^1 g(u) du = f$, $\int_0^1 g^q(u) du = h$ and

$$\int_0^t g(u) du = \omega_q \left( \frac{f^q}{h} \right)^{1/q} g(t), \text{ } t \in (0, 1].$$

**Proof.** We define $g(t) = K t^{-1-q} = \frac{f^q}{h}$, $t \in (0, 1]$ where $c = \omega_q \left( \frac{f^q}{h} \right)^{1/q}$, thus it satisfies

$$(1-q)c^q + qc^{q-1} = \frac{f^q}{h}. $$

Let $K$ be such that

$$\int_0^1 g = f \iff K \int_0^1 t^{-1+q} dt = f \iff Kc = f \iff K = \frac{f}{c}.$$ 

For this $K$ we claim that $\int_0^1 g^q = h$. Indeed:

$$\int_0^1 g^q = K^q \int_0^1 t^{-q+\frac{q}{2}} dt = \frac{f^q}{c^q} \left( \frac{1}{1-q + \frac{q}{c}} \right) = \frac{f^q}{(1-q)c^q + qc^{q-1}} = \frac{f^q}{f^q/h} = h,$$

and Lemma 4.1 is proved. 

From all the above we conclude **Theorem 1**. 

We are now ready for the last

5. **Characterization of the extremal sequences**

**Proof of Theorem 2.** We consider $\phi_n : (X, \mu) \rightarrow \mathbb{R}^+$ such that the hypotheses of Theorem 2 are satisfied. That is

$$\int_X \phi_n d\mu = f, \int_X \phi_n^q d\mu = h \text{ and } \lim_n \int_X (\mathcal{M}_T \phi_n)^q d\mu = h \omega_q(\frac{f^q}{h}).$$

We will prove that $\lim_n \int_X |\mathcal{M}_T \phi_n - c \phi_n|^q d\mu = 0$, where $c = \omega_q(\frac{f^q}{h})^{1/q}$.

By setting $\Delta_n = \{\mathcal{M}_T \phi_n \geq c \phi_n\}$ and $\Delta'_n = X \setminus \Delta_n$, it is enough to prove that if $I_n$ and $J_n$ are defined as

$$I_n = \int_{\Delta_n} (\mathcal{M}_T \phi_n - c \phi_n)^q d\mu \text{ and } J_n = \int_{\Delta'_n} (c \phi_n - \mathcal{M}_T \phi_n)^q d\mu$$

then $I_n, J_n \rightarrow 0$, as $n \rightarrow \infty$. 

Define the following functions on \((X, \mu)\)
\[ g_n = \phi_n^q (\mathcal{M}_T \phi_n)^{q(q-1)} \quad \text{and} \quad h_n = (\mathcal{M}_T \phi_n)^{q(1-q)}. \]
Remember that in the proof of Theorem 1 in Section 3 it is used the inequality:
\[
\int_X \phi^q d\mu \leq \left[ \int_X \phi (\mathcal{M}_T \phi)^{q-1} d\mu \right]^q \cdot \left[ \int_X (\mathcal{M}_T \phi)^q d\mu \right]^{1-q},
\]
for every suitable \(\phi\).

Thus, since \((\phi_n)\) is extremal for (1.9) we must have equality in (5.1) in the limit if \(\phi\) is replaced by \(\phi_n\). We can write:
\[
\int_X g_n \cdot h_n d\mu \approx \left[ \int_X g_n^{1/q} d\mu \right] \cdot \left[ \int_X h_n^{1/(1-q)} d\mu \right]^{1-q}.
\]
We need now two lemmas before we proceed to the proof of Theorem 2.

**Lemma 5.1.** Under the above notation and hypotheses we have that:
\[
\int_{X_n} g_n h_n d\mu \approx \left[ \int_{X_n} g_n^{1/q} d\mu \right] \cdot \left[ \int_{X_n} h_n^{1/(1-q)} d\mu \right]^{1-q},
\]
where \(X_n\) may be replaced either by \(\Delta_n\) or \(\Delta'_n\).

**Proof.** Of course the following inequalities hold true, in view of Holder’s inequality. These are:
\[
\int_{\Delta_n} g_n h_n d\mu \leq \left[ \int_{\Delta_n} g_n^{1/q} d\mu \right] \cdot \left[ \int_{\Delta_n} h_n^{1/(1-q)} d\mu \right]^{1-q}, \tag{5.4}
\]
and
\[
\int_{\Delta'_n} g_n h_n d\mu \leq \left[ \int_{\Delta'_n} g_n^{1/q} d\mu \right] \cdot \left[ \int_{\Delta'_n} h_n^{1/(1-q)} d\mu \right]^{1-q}. \tag{5.5}
\]
We add then and we obtain:
\[
\int_X g_n h_n d\mu \leq \left[ \int_{\Delta_n} g_n^{1/q} d\mu \right] \cdot \left[ \int_{\Delta_n} h_n^{1/(1-q)} d\mu \right]^{1-q} + \left[ \int_{\Delta'_n} g_n^{1/q} d\mu \right] \cdot \left[ \int_{\Delta'_n} h_n^{1/(1-q)} d\mu \right]^{1-q}. \tag{5.6}
\]
We use now the following elementary inequality which proof is given below:
For every \(t, t' \geq 0, s, s' \geq 0\) such that \(t + t' = a > 0\) and \(s + s' = b > 0\) and any \(q \in (0, 1)\), we have that
\[
t^q \cdot s^{1-q} + (t')^q \cdot (s')^{1-q} \leq a^q b^{1-q}. \tag{5.7}
\]
Applying it on (5.6) we obtain that
\[
\int_X g_n h_n d\mu \leq \left[ \int_X g_n^{1/q} d\mu \right] \cdot \left[ \int_X h_n^{1/(1-q)} d\mu \right]^{1-q}
\]
which is equality in the limit. As a consequence we must have equality in the limit on (5.4) and (5.5), and Lemma 5.1 follows. It remains to prove the inequality (5.7).
Fix $t$ such that $0 < t < a$ and consider the function $F$ of the variable $s \in [0, b]$ defined by

$$F(s) = t^q \cdot s^{1-q} + (a - t)^q (b - s)^{1-q}.$$  

It can be easily seen that $F$ is strictly increasing on $[0, \frac{b}{a}]$ and strictly decreasing on $[\frac{b}{a}, b]$. Thus it attains its maximum value on $\frac{b}{a}$. This maximum value equals to $F\left(\frac{b}{a}\right) = a^q b^{1-q}$, and the inequality is proved. \hfill \Box

We state now the following:

**Lemma 5.2.** We suppose we are given $w_n : X_n \to \mathbb{R}^+$ where $X_n \subseteq X$ for any $n \in \mathbb{N}$ such that $w_n \geq w$ on $S_n$ where $w$ is defined on $X$ with non-negative values. Suppose also that $q \in (0, 1)$ and $\lim_n \int_{X_n} w_n^q d\mu = \lim_n \int_{X_n} w^q d\mu$. Then the following is true:

$$\lim_n \int_{X_n} (w_n - w)^q d\mu = 0.$$  

**Proof.** We set $z_n = w_n^q$ and $z = w^q$ defined on $X_n$ and $X$ respectively. We use now the inequality:

$$x^p - y^p \leq px^{p-1}(x - y), \quad \text{for} \quad x > y > 0, \quad \text{and} \quad p > 1$$  

which can be proved easily by the mean value theorem on derivatives.

We apply it in case where $p = 1/q$. Thus, we have that:

$$w_n - w = z_n^p - z^p \leq pz_n^{p-1}(z_n - z)$$

$$= \frac{1}{q} z_n^{\frac{q}{q-1}} (z_n - z), \quad \text{on} \quad X_n.$$  

This gives:

$$\int_{X_n} (w_n - w)^q d\mu \leq \left(\frac{1}{q}\right)^q \int_{X} z_n^{1-q}(z_n - z)^q d\mu$$

$$\leq \left(\frac{1}{q}\right)^q \left[ \int_{X_n} (z_n - z) d\mu \right]^q \cdot \left[ \int_{X_n} z_n \right]^{1-q},$$

which is obviously tending to 0, by the hypotheses of the Lemma. Note that in the last inequality we use Holder’s inequality with exponents $p = 1/q$ and $p' = \frac{1}{1 - q}$. Lemma 5.1 is now proved. \hfill \Box

We continue with the proof of Theorem 2.

We set $\lambda = \lim_n \left( \frac{h}{\int_X g_n^{1/q} d\mu} \right)^{1/(1-q)}$ or equivalently:

$$\lambda^{1-q} = \lim_n \frac{h}{\int_X g_n^{1/q} d\mu}.$$
In view of the equality \(5.2\) we must have that
\[
\lim_n \int_X g_n^{1/q} d\mu = \lim_n \int_X \phi_n (M_T \phi_n)^{q-1} d\mu = \frac{h^{1/q}}{h \omega_q(f^q/h)^{1/q}}.
\]
Thus: \(\lambda = \omega_q(f^q/h)^{1/q} = c\).

Remember that \(I_n = \int_{\Delta_n} (M_T \phi_n - c \phi_n)^q d\mu\), where \(\Delta_n = \{M_T \phi_n \geq c \phi_n\}\). Because of Lemma \(5.2\) we have that \(I_n \to 0\) if we are able to show that
\[
\lim_n \int_{\Delta_n} (M_T \phi_n)^q d\mu = c^q \lim_n \int_{\Delta_n} \phi_n^q d\mu.
\]
We suppose that (we pass to a subsequence if necessary) that
\[
\Delta = \lim \mu(\Delta_n) \in (0, 1).
\]
We discuss the alternative case \(\Delta = 0\) or \(1\) at the end of this section.

We set now
\[
\lambda_n = \frac{\int_{\Delta_n} \phi_n^q d\mu}{\int (M_T \phi_n)^q d\mu} \quad \text{and} \quad \mu_n = \frac{\int \Delta_n' \phi_n^q d\mu}{\int (M_T \phi_n)^q d\mu}.
\]
In view of \(5.9\) \(\lambda_n, \mu_n\) are well defined for all large \(n\) since \(M_T \phi_n \geq f > 0\) on \(X\).

We set \(\lambda_n = a_n/b_n\) and \(\mu_n = c_n/d_n\) with the obvious meaning on these parameters and suppose without loss of generality that \(a_n \to a, b_n \to b, c_n \to c\) and \(d_n \to d\). Then, according to \(5.9\) \(c, d > 0\).

Because of the definition of \(\Delta_n\) and \(\Delta_n'\) we see immediately that
\[
\lambda_n \leq \frac{1}{c^q} \leq \mu_n.
\]

In order to prove \(5.8\) and the respective equality in the case of \(J_n\) we need to prove that \(\lambda_n \to 1/c^q\) and \(\mu_n \to 1/c^q\). So we just need to prove that \(\mu_n - \lambda_n \to 0\) (we write \(\mu_n \approx \lambda_n\)).

We proceed to this proof:
In Section 3 we saw after replacing \(\phi\) by \(\phi_n\) that:
\[
I = \int_X (M_T \phi_n)^q d\mu \leq \frac{1}{1-q} f^q - \frac{q}{1-q} \int_X \phi_n (M_T \phi_n)^{q-1} d\mu
\]
\[
= \frac{1}{1-q} f^q - \frac{q}{1-q} \left[ \int_{\Delta_n} \int \phi_n (M_T \phi_n)^{q-1} d\mu \right].
\]
Because now of Lemma 5.1 and since \((\phi_n)_n\) is extremal for (1.9), we conclude that
\[
h\omega_q(f^q/h) \leq \frac{f^q}{1-q} - q \frac{\lim_{n} \left( \frac{\int_{\Delta_n} \phi_n d\mu}{(\int_{\Delta_n} (M_T\phi_n)^q d\mu)^{1/q-1}} \right)^{1/q}}{1-q} \left\{ \begin{array}{l}
\frac{\int_{\Delta_n} \phi_n d\mu}{(\int_{\Delta_n} (M_T\phi_n)^q d\mu)^{1/q-1}} \right. \\
\left. \frac{\int_{\Delta_n} \phi_n^q d\mu}{(\int_{\Delta_n} (M_T\phi_n)^q d\mu)^{1/q-1}} \right. \\
\left. + \frac{\int_{\Delta_n} (M_T\phi_n)^q d\mu}{(\int_{\Delta_n} (M_T\phi_n)^q d\mu)^{1/q-1}} \right. \\
\end{array} \right. 
\]  
(5.12)

We use now Holder’s inequality in its primitive form
\[
\frac{(x+y)^p}{(s+t)^{p-1}} \leq \frac{x^p}{s^{p-1}} + \frac{y^p}{t^{p-1}},
\]
for \(x, y \geq 0\) and \(s, t > 0, p > 1\), with equality only if \(x/s = y/t = k \in \mathbb{R}^+\).

We, thus, have for \(p = 1/q > 1\), that the expression in brackets in (5.12) is not less than
\[
\left( \frac{\int_X (M_T\phi_n)^q d\mu}{(\int_X (M_T\phi_n)^q d\mu)^{1/q-1}} \right)^{1/q-1}
\]
which tends to \(h\omega_q(f^q/h)^{1-\frac{1}{q}}\). So from (5.12) we obtain that:
\[
h\omega_q(f^q/h) \leq \frac{f^q}{1-q} - q \frac{\omega_q(f^q/h)^{1-\frac{1}{q}}}{1-q} \iff \]
\[
q\omega_q(f^q/h)^{1-\frac{1}{q}} + (1-q)\omega_q(f^q/h) \leq f^q/h. 
\]  
(5.13)

But by the definition of \(\omega_q(z), z \geq 1\) we have that (5.13) is equality. As a consequence of all the above we conclude that \(a/b = c/d \in \mathbb{R}^+,\) that is what exactly we wanted to show.

The case \(\mu(\Delta_n) \to 0\) is treated in a similar but more simple way since then
\[
\lim_{n} \int_{\Delta_n} (M_T\phi_n)^q d\mu = 0. 
\]  
(5.14)

This is true since if we define
\[
B_q(f, h, k) = \sup \left\{ \int_K (M_T\phi)^q d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = h, \phi \mu - \text{measurable with } \mu(K) = k \right\}
\]
for \(0 < h \leq f^q\) and \(k \in (0, 1]\), we easily see by its evaluation on [5] (which is based only on the evaluation of \(1.9\) and calculus arguments) that
\[
\lim_{k \to 0^+} B_q(f, h, k) = 0
\]
for any fixed \(f, h\) such that \(0 < h \leq f^q\).

Thus, we end the one direction of Theorem 2. For the other: ii) \(\Rightarrow\) i)
Since ii) holds we must have that:

\[
\lim_{n} \int_{\Delta_n} (M_T \phi_n - c \phi_n)^q d\mu = 0 \quad \text{and} \quad \lim_{n} \int_{\Delta'_n} (c \phi_n - M_T \phi_n)^q d\mu = 0,
\]

with \(\Delta_n\) and \(\Delta'_n\) defined as above. We use now the elementary inequality:

\[
0 < x^q - y^q \leq (x - y)^q \quad \text{for any} \quad x > h > 0 \quad \text{and} \quad q \in (0, 1).
\]

So by (5.15) we must have that

\[
\lim_{n} \int_{\Delta_n} (M_T \phi_n)^q d\mu = c^q \lim_{n} \int_{\Delta_n} \phi_n^q d\mu
\]

by passing if necessary to a subsequence, and analogously for \(\Delta'_n\).

Adding these two equalities we obtain i).

Theorem 2 is now proved.

6. Further properties of extremal sequences

In Theorem 2.1 we stated an equality which relates the dyadic maximal operator with the Hardy operator in an immediate way. This equality involves a free parameter which is the function \(G_1\). In this section we will prove the \(\leq\) of Theorem 2.1 for the case \(G_1(x) = x^q, q \in (0, 1)\), and we will use this proof and the statement of Theorem 2.1 to find another characterization of some extremal sequences of certain type for the Bellman function of the dyadic maximal operator in relation with Kolmogorov’s inequality. We proceed to it as follows. We are going to prove

**Lemma 6.1.** For any \(g : (0, 1] \to \mathbb{R}^+\) integrable and non-increasing for which the integral on the right hand side of the following inequality is finite, we have that:

\[
\int_X (M_T \phi)^q d\mu \leq \int_0^1 \left( \frac{1}{t} \int_0^t g(u) du \right)^q dt,
\]

for any \(\phi : (X, \mu) \to \mathbb{R}^+\) such that \(\phi^* = g\).

**Proof.** We have that

\[
I = \int_X (M_T \phi)^q d\mu = q \int_{\lambda=0}^{+\infty} \lambda^{q-1} \mu(\{M_T \phi \geq \lambda\}) d\lambda
\]

\[
= q \int_{\lambda=f}^{+\infty} \lambda^{q-1} d\lambda + q \int_{\lambda=f}^{+\infty} \lambda^{q-1} \mu(\{M_T \phi \geq \lambda\}) d\lambda,
\]

where \(f = \int_X g = \int_X \phi d\mu\), for any \(\phi\) such that \(\phi^* = g\).

Thus, \(I = f^q + q \int_{\lambda=f}^{+\infty} \lambda^{q-1} \beta(\lambda) d\lambda\), where \(\beta(\lambda) = \mu(\{M_T \phi \geq \lambda\})\).

From the weak type(1,1) inequality we see that

\[
\mu(\{M_T \phi \geq \lambda\}) = \beta(\lambda) \leq \frac{1}{\lambda} \int_{\{M_T \phi \geq \lambda\}} \phi d\mu, \quad \text{for any} \quad \lambda > f.
\]
Thus
\[
\frac{1}{\beta(\lambda)} \int_{\{M_T \phi \geq \lambda\}} \phi d\mu \geq \lambda.
\]
Since \(\phi^* = g\) is non-increasing we have that
\[
(6.2) \quad \int_{\{M_T \phi \geq \lambda\}} \phi d\mu \leq \int_0^{\beta(\lambda)} g(u) du.
\]
We, now, choose for any \(\lambda > f\) the unique \(a(\lambda) \in (0, 1]\) such that
\[
1 \int_0^{a(\lambda)} g = \lambda.
\]
Altogether we have that
\[
(6.3) \quad \frac{1}{\beta(\lambda)} \int_0^{\beta(\lambda)} g \geq \frac{1}{\beta(\lambda)} \int_{\{M_T \phi \geq \lambda\}} \phi d\mu \geq \frac{1}{a(\lambda)} \int_0^{a(\lambda)} g,
\]
and since \(g : (0, 1] \to \mathbb{R}^+\) is non-increasing we conclude that \(\beta(\lambda) \leq a(\lambda)\). Consequently, from (6.1) we produce
\[
I = \int_X (M_T \phi)^q d\mu \leq f^q + q \int_{\lambda = f}^{+\infty} \lambda^{q-1} a(\lambda) d\lambda
\]
\[
= f^q + \int_{\lambda = f}^{+\infty} q \lambda^{q-1} \left\{ t \in (0, 1]: \frac{1}{t} \int_0^t g \geq \lambda \right\} d\lambda,
\]
from the definition of \(a(\lambda)\). So (6.4) \Rightarrow
\[
I = \int_X (M_T \phi)^q d\mu \leq \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt,
\]
for any \(\phi\) such that \(\phi^* = g\), which is the result we needed to prove.

We will need also the following.

**Lemma 6.2.** Let \((\phi_n)_n\) be such that \(\phi_n : (x, \mu) \to \mathbb{R}^+\) are measurable rearrangements of \(g\) \((\phi_n^* = g)\), such that
\[
(6.5) \quad \lim_n \int_X (M_T \phi_n)^q d\mu = \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt,
\]
Then the following is true. For any \(k \in (0, 1]\)
\[
\lim_n \int_0^k [(M_T \phi_n)^*(t)]^q dt = \int_0^k \left( \frac{1}{t} \int_0^t g \right)^q dt.
\]
**Proof.** We suppose that (6.5) is true. Then in view of the proof of Lemma 6.1 we must have that
\[
(6.6) \quad \lim_n \int_{\lambda = f}^{+\infty} q \lambda^{q-1} \mu(\{M_T \phi_n \geq \lambda\}) d\lambda = \int_{\lambda = f}^{+\infty} q \lambda^{q-1} [a(\lambda)] d\lambda,
\]
where \(f = \int_0^1 g = \int_X \phi_n d\mu\) and \(a(\lambda)\) is as in the proof of Lemma 6.1.
This means that the following should be true:

\[
(6.7) \quad \lim_n \int_{\lambda=f}^{+\infty} q^q^{-1} |\{(\mathcal{M}_T \phi_n)^* \geq \lambda\}| d\lambda = \int_{\lambda=f}^{+\infty} q^q^{-1} [a(\lambda)] d\lambda,
\]

since \(\mu(\{\mathcal{M}_T \phi \geq \lambda\}) = |\{(\mathcal{M}_T \phi)^* \geq \lambda\}|,\) for any \(\lambda > 0,\) where \(| \cdot |\) denotes the Lesbegue measure on \((0, 1].\) Then for any \(k \in (0, 1]\) we have that

\[
I_n = \int_0^k [\{(\mathcal{M}_T \phi_n)^* (t)\}^q] dt = \int_{\lambda=0}^{+\infty} q^q^{-1} |\{t \in (0, k] : (\mathcal{M}_T \phi_n)^* (t) \geq \lambda\}| d\lambda
\]

\[
(6.8) = kf^q + \int_{\lambda=f}^{+\infty} q^q^{-1} |\{t \in (0, k] : (\mathcal{M}_T \phi_n)^* (t) \geq \lambda\}| d\lambda.
\]

We set now \((\mathcal{M}_T \phi_n)^* (k) = \lambda_k^{(n)} \in [f, +\infty),\) for any \(n \in \mathbb{N}.\) For any \(t \in (0, k] : (\mathcal{M}_T \phi_n)^* (t) \geq (\mathcal{M}_T \phi_n)^* (k) = \lambda_k^{(n)} \Rightarrow \forall \lambda \in [f, \lambda_k^{(n)}]\) we have that

\[
(6.9) \quad |\{t \in (0, k] : (\mathcal{M}_T \phi_n)^* (t) \geq \lambda\}| = |(0, k]| = k.
\]

By (6.8) and (6.9) we have that

\[
I_n = \int_0^k [\{(\mathcal{M}_T \phi_n)^* \}^q] dt = kf^q + \int_{\lambda=\lambda_k^{(n)}}^{+\infty} q^q^{-1} \cdot kd\lambda
\]

\[
+ \int_{\lambda=\lambda_k^{(n)}}^{+\infty} q^q^{-1} |\{t \in (0, k] : (\mathcal{M}_T \phi_n)^* (t) \geq \lambda\}| d\lambda
\]

\[
(6.10) = k(\lambda_k^{(n)})^q + \int_{\lambda=\lambda_k^{(n)}}^{+\infty} q^q^{-1} |\{t \in (0, 1] : (\mathcal{M}_T \phi_n)^* (t) \geq \lambda\}| d\lambda,
\]

by the definition of \(\lambda_k^{(n)}\). Thus

\[
I_n = k(\lambda_k^{(n)})^q + \int_{\lambda=\lambda_k^{(n)}}^{+\infty} q^q^{-1} |\{t \in (0, 1] : (\mathcal{M}_T \phi_n)^* (t) \geq \lambda\}| d\lambda
\]

\[
- \int_{\lambda=f}^{\lambda_k^{(n)}} q^q^{-1} |\{t \in (0, 1] : (\mathcal{M}_T \phi_n)^* (t) \geq \lambda\}| d\lambda
\]

\[
(6.11) = k(\lambda_k^{(n)})^q + I_1 - I_2, \quad \text{say.}
\]

Concerning \(I_1\) we have that

\[
(6.12) I_1 \to \int_{\lambda=f}^{+\infty} q^q^{-1} [a(\lambda)] d\lambda,
\]

as \(n \to \infty\) by the comments in the beginning of the proof of this Lemma. For \(I_2\) we have that

\[
I_2 = \int_{\lambda=f}^{\lambda_k^{(n)}} q^q^{-1} |\{t \in (0, 1] : (\mathcal{M}_T \phi_n)^* (t) \geq \lambda\}| d\lambda
\]

\[
= \int_{\lambda=f}^{\lambda_k^{(n)}} q^q^{-1} \beta_n(\lambda), \quad \text{where}
\]

\[
\beta_n(\lambda) = \mu(\{\mathcal{M}_T \phi_n \geq \lambda\}) \leq a(\lambda),
\]
since $\phi^*_n = q$ and the proof of Lemma 6.1 Thus

\[(6.14)\qquad I_2 \leq \int_{\lambda=f}^{\lambda_k(n)} q\lambda^{q-1}a(\lambda)d\lambda.\]

Thus

\[(6.15)\qquad \liminf_n J_n \geq \int_{\lambda=f}^{+\infty} q\lambda^{q-1}[a(\lambda)]d\lambda + \liminf_n \left[ k(\lambda_k(n))^q - \int_{\lambda=f}^{\lambda_k(n)} q\lambda^{q-1}a(\lambda)d\lambda \right].\]

Since now $k$ is fixed and $k > 0$, and since $\sup_n \int (\mathcal{M}_T \phi_n)^q d\mu < +\infty$ (use of Lemma 6.1), we conclude that $(\lambda_k(n))_n$ is bounded above. Thus, there exist a subsequence and a $\lambda_0 \geq f$ such that $\lambda_k(n_i) \to \lambda_0$, as $i \to \infty$. Thus

\[
\liminf_i J_{n_i} \geq \liminf_i J_n \geq \int_{\lambda=f}^{+\infty} q\lambda^{q-1}[a(\lambda)]d\lambda + k\lambda_0^q - \int_{\lambda=f}^{\lambda_0} q\lambda^{q-1}[a(\lambda)]d\lambda
\]
\[
= k\lambda_0^q + \int_{\lambda=\lambda_0}^{+\infty} q\lambda^{q-1}[a(\lambda)]d\lambda
\]
\[
= k\lambda_0^q + \int_{\lambda=\lambda_0}^{+\infty} q\lambda^{q-1}\left\{ t \in (0,1] : \frac{1}{t} \int_0^t g \geq \lambda \right\} d\lambda
\]
\[
\geq \int_{\lambda=0}^{\lambda_0} q\lambda^{q-1}kd\lambda + \int_{\lambda=\lambda_0}^{+\infty} \left\{ t \in (0,k] : \frac{1}{t} \int_0^t g \geq \lambda \right\} d\lambda
\]
\[
\geq \int_{\lambda=0}^{+\infty} q\lambda^{q-1}\left\{ t \in (0,k] : \frac{1}{t} \int_0^t g \geq \lambda \right\} d\lambda = \int_0^k \left( \frac{1}{t} \int_0^t g \right)^q dt.
\]

That is we proved that for any fixed $k \in (0,1]$ there is a subsequence of integers $(n_j)_j$ such that

\[
\lim_n \int_0^k [(\mathcal{M}_T \phi_{n_j})]^q dt \geq \int_0^k \left( \frac{1}{t} \int_0^t g \right)^q dt.
\]

This result, Lemma 6.1 and standard arguments about subsequences give the result we need. \qed

We are now able to prove the main theorem in this section.

**Theorem 6.1.** Let $g : (0,1] \to \mathbb{R}^+$ be an integrable, non-increasing function such that $\int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt < +\infty$, where $q \in [0,1)$, and $(\phi_n)$ be a sequence of $\mu$-measurable rearrangements of $g$ ($\phi_n^* = g$) such that

\[
\lim_n \int_X (\mathcal{M}_T \phi_n)^q d\mu = \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^q dt.
\]

Then the following equality is true:

\[
\lim_n \int_0^1 \left| (\mathcal{M}_T \phi_n)^*(t) - \frac{1}{t} \int_0^t g \right|^q dt = 0.
\]
Proof. We consider the set

\[ F_n = \left\{ t \in (0, 1] : (\mathcal{M}_T \phi_n)^*(t) > \frac{1}{t} \int_0^t g \right\} \]

and its complement in \((0, 1], F_n^c\).

We will prove that

\[
\lim_{n \to \infty} \left| \int_{X_n} [(\mathcal{M}_T \phi_n)^*(t)]^q \, dt - \int_{X_n} \left( \frac{1}{t} \int_0^t g \right)^q \, dt \right| = 0,
\]

where \(X_n\) is either \(F_n\), for every \(n \in \mathbb{N}\), or \(F_n^c\), \(\forall n \in \mathbb{N}\). If we have (6.17) in both cases for \(X_n\) and apply Lemma 5.2, then we have the result we need to prove. We will prove (6.17) only in the case where \(X_n = F_n\), \(\forall n \in \mathbb{N}\). The other one is treated in a similar way.

For every \(n \in \mathbb{N}\) we choose \(U_n\) open subset of \((0, 1]\) such that \(F_n \subseteq U_n\) and \(|U_n \setminus F_n| \leq \frac{1}{n}\). Then \(U_n\) can be written as \(U_n = \bigcup_k (a_k^{(n)}, b_k^{(n)})\), a disjoint union of open intervals on \((0, 1]\). By Lemma 6.2 and since the above union is disjoint we have that

\[
\lim_{n \to \infty} \left| \int_{U_n \setminus F_n} [(\mathcal{M}_T \phi_n)^*(t)]^q \, dt - \int_{U_n} \left( \frac{1}{t} \int_0^t g \right)^q \, dt \right| = 0.
\]

This is not difficult to prove even for an infinite sequence of pairwise disjoint intervals that decompose \(U_n\). Now

\[
\int_{U_n \setminus F_n} [(\mathcal{M}_T \phi_n)^*(t)]^q \, dt \leq \int_0^{[U_n \setminus F_n]} [(\mathcal{M}_T \phi_n)^*(t)]^q \, dt,
\]

for any \(n \in \mathbb{N}\), since \((\mathcal{M}_T \phi_n)^*\) is non-increasing on \((0, 1]\).

Additionally, by Theorem 2.1 it is easy to see that

\[
\int_0^{[U_n \setminus F_n]} [(\mathcal{M}_T \phi_n)^*(t)]^q \, dt \leq \int_0^{[U_n \setminus F_n]} [H(t)]^q \, dt, \quad \text{where} \quad H(t) = \frac{1}{t} \int_0^t g.
\]

The last integral now tends to zero since by our hypothesis \(H(t) \in L^q((0, 1])\). Further:

\[
\int_{U_n \setminus F_n} \left( \frac{1}{t} \int_0^t g \right)^q \, dt \to 0, \quad \text{as} \quad n \to \infty
\]

for the same reasons. Altogether, we conclude, by using also (6.18), Theorem 6.1. \(\square\)

Remark 6.1. By using the elementary inequality \(x^q - y^q < (x - y)^q\), for \(x > y > 0\), it is easy to see that the converse statement of Theorem 6.1 is true. That is any sequence satisfying

\[
\lim_{n \to \infty} \int_0^{1} [(\mathcal{M}_T \phi_n)^*(t)]^q - \frac{1}{t} \int_0^t g \, dt = 0,
\]

must also satisfy:

\[
\lim_{n \to \infty} \int_X (\mathcal{M}_T \phi_n)^q \, d\mu = \int_0^{1} \left( \frac{1}{t} \int_0^t g \right)^q \, dt.
\]
Corollary 6.1. Let $g$ be as in Section 4, Lemma 6.1. Then for any sequence $(\phi_n)$ of rearrangements of $g$ such that \[ \int_X (M_T \phi_n)^q d\mu \to h \omega_q(f^q/h), \] we must have that
\[ \int_0^1 |(M_T \phi_n)^*(t) - c \phi_n^*(t)|^q dt \to 0, \quad \text{as } n \to \infty \] where $c = \omega_q(f^q/h)^{1/q}$.

Proof. Immediate. \qed
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