Gallai–Ramsey Number for the Union of Stars

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Abstract  Given a graph $G$ and a positive integer $k$, define the Gallai–Ramsey number to be the minimum number of vertices $n$ such that any $k$-edge coloring of $K_n$ contains either a rainbow (all different colored) triangle or a monochromatic copy of $G$. In this paper, we obtain exact values of the Gallai–Ramsey numbers for the union of two stars in many cases and bounds in other cases. This work represents the first class of disconnected graphs to be considered as the desired monochromatic subgraph.

Keywords  Ramsey theory, Gallai–Ramsey number, union of stars

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1  Introduction

In this work, we consider only edge-colorings of graphs. A coloring of a graph is called rainbow if no two edges have the same color.

Colorings of complete graphs that contain no rainbow triangle have very interesting and rather surprising structure. In 1967, Gallai [6] first examined this structure using the terminology transitive orientations. The result was reproven in [9] in the terminology of graphs and can also be traced to [1]. For the following statement, a trivial partition is a partition into only one part.

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Theorem 1.1 ([1, 6, 9]) In any coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition of the vertices (that is, with at least two parts) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.

For ease of notation, we refer to a colored complete graph with no rainbow triangle as a Gallai-coloring and the partition provided by Theorem 1.1 as a Gallai-partition. The induced subgraph of a Gallai colored complete graph constructed by selecting a single vertex from each part of a Gallai partition is called the reduced graph of that partition. By Theorem 1.1, the reduced graph is a 2-colored complete graph.

Given two graphs $G$ and $H$, let $R(G,H)$ denote the 2-color Ramsey number for finding a monochromatic $G$ or $H$, that is, the minimum number of vertices $n$ needed so that every red-blue coloring of $K_n$ contains either a red copy of $G$ or a blue copy of $H$. Similarly let $R_k(H)$ denote the $k$-color Ramsey number for finding a monochromatic copy of $H$ (in any color), that is the minimum number of vertices $n$ needed so that every $k$-coloring of $K_n$ contains a monochromatic copy of $H$. Although the reduced graph of a Gallai partition uses only two colors, the original Gallai-colored complete graph could certainly use more colors. With this in mind, we consider the following generalization of the Ramsey numbers. Given two graphs $G$ and $H$, the general $k$-colored Gallai–Ramsey number $gr_k(G : H)$ is defined to be the minimum integer $m$ such that every $k$-coloring of the complete graph on $m$ vertices contains either a rainbow copy of $G$ or a monochromatic copy of $H$. With the additional restriction of forbidding the rainbow copy of $G$, it is clear that $gr_k(G : H) \leq R_k(H)$ for any graph $G$.

We refer the interested reader to [14] for a dynamic survey of small Ramsey numbers and [5] for a dynamic survey of rainbow generalizations of Ramsey theory, and a book [13] on the Gallai–Ramsey numbers. One may notice that all of the results contained in the dynamic survey regarding Gallai–Ramsey numbers consider monochromatic subgraphs that are connected. In this work, we consider the Gallai–Ramsey numbers for finding either a rainbow triangle or a monochromatic copy of $K_{1,n} \cup K_{1,m}$, this being the first examination of disconnected monochromatic subgraphs.

The star was one of the first monochromatic graphs to be considered in the context of Gallai–Ramsey numbers.

Theorem 1.2 ([8]) For $m \geq 3$ and $k \geq 2$,

$$gr_k(K_3 : K_{1,m}) = \begin{cases} \frac{5m - 6}{2} & \text{if } m \text{ is even,} \\ \frac{5m - 3}{2} & \text{if } m \text{ is odd.} \end{cases}$$

Our first result, provided in Section 2, provides an analysis of the stability of Theorem 1.2.

For the union of two stars, Grossman [7] obtain the classical Ramsey number in the following result.

Theorem 1.3 ([7]) Let $n, m$ be two integers with $n \geq m \geq 1$, and let $K_{1,n}, K_{1,m}$ be two stars. Then

$$R(K_{1,n} \cup K_{1,m}) = \max\{n + 2m, 2n + 1, n + m + 3\}.$$
In Section 3, we prove the exact value of the Gallai–Ramsey number for the union of two stars $K_{1,n} \cup K_{1,m}$, where $m \leq \frac{n-8}{6}$.

**Theorem 1.4** Let $n \geq 38$, $m \geq 5$ and $k \geq 3$ be three integers with $m \leq \frac{n-14}{8}$. Then

$$\text{gr}_k(K_3 : K_{1,n} \cup K_{1,m}) = \begin{cases} \frac{5n-6}{2} + k - 3 & \text{if } n \text{ is even,} \\ \frac{5n-3}{2} + k - 3 & \text{if } n \text{ is odd.} \end{cases}$$

In Section 4, we prove the exact value of the Gallai–Ramsey number for the union of two equal stars $K_{1,n} \cup K_{1,n}$.

**Theorem 1.5** For $k \geq 3$ and $n \geq 2$,

$$\text{gr}_k(K_3 : K_{1,n} \cup K_{1,n}) = 3n + k - 1.$$ 

Finally, in Section 5, we get the upper and lower bounds for the union of two stars for general $n, m$.

**Theorem 1.6** Let $n \geq 9$, $n > m \geq 2$ and $k \geq 3$ be three integers with $m \geq \frac{n-2}{6}$. Then

$$\max \left\{ 2n + m + k - 5, \frac{5n-6}{2} + k - 3 \right\} \leq \text{gr}_k(K_3 : K_{1,n} \cup K_{1,m}) \leq 3n + 3m + k - 3$$

if $n$ is even,

$$\max \left\{ 2n + m + k - 4, \frac{5n-3}{2} + k - 3 \right\} \leq \text{gr}_k(K_3 : K_{1,n} \cup K_{1,m}) \leq 3n + 3m + k - 2$$

if $n$ is odd.

**2 Star Lemma**

First a helpful lemma regarding the stability of Theorem 1.4.

**Lemma 2.1** For positive integers $n$ and $r$ with $n \geq 5$ and $4 \leq r \leq \frac{n-9}{4}$, if $G$ is a Gallai coloring of a complete graph of order $\frac{5n-r}{2}$ which contains no monochromatic copy of $K_{1,n}$ on edges between parts of a Gallai partition, then there are exactly 5 parts in any Gallai partition, each of order at least $\frac{n-r+3}{4}$, and every vertex has at least $n-r+3$ incident edges to other parts of the Gallai partition in each of two colors (the two colors appearing between parts of the Gallai partition).

**Proof** Let $G$ be a Gallai coloring of a complete graph of order $\frac{5n-r}{2}$ and suppose that $G$ contains no monochromatic copy of $K_{1,n}$. By Theorem 1.1, there is a Gallai partition of $G$, say using red and blue on edges between parts of the partition. Choose such a partition $H_1, H_2, \ldots, H_t$ with the smallest number of parts $t$. In order to avoid a vertex of degree $n$ in red or blue, certainly no part of the Gallai partition can have order at least $n$. On the other hand, if we choose a vertex $v$ in a part of order at most $\frac{n-r+2}{2}$, then there are at least

$$\left\lfloor \frac{|G| - \frac{n-r+2}{2}}{2} \right\rfloor = \left\lfloor \frac{\frac{5n-r}{2} - \frac{n-r+2}{2}}{2} \right\rfloor \geq n$$

edges of one color incident to $v$, which means that every part of the Gallai partition has order at least $\frac{2-r+3}{2}$.

If $2 \leq t \leq 3$, then by minimality of $t$, we may assume that $t = 2$, but then there is a part of the partition of order at least $\frac{|G|}{2} > n$, a contradiction. We may therefore assume that $t \geq 4$. 


Additionally, since each part has order at least \( \frac{n-r+3}{2} \) and \( r \leq \frac{n+9}{3} \), if any vertex has edges of all one color to 3 different parts, then there would be a monochromatic copy of \( K_{1,n} \). Thus, to avoid a monochromatic copy of \( K_{1,3} \) in the reduced graph, we have \( t \leq 5 \).

If \( t = 4 \), then there exists a “big” part of order at least \( \frac{|G|}{4} = \frac{5n-r}{8} \). By the minimality of \( t \), the four parts must have the structure that there is a red path \( H_1 H_3 H_4 H_2 \) in the reduced graph. Since \( |H_2| + |H_3| \geq n \) or \( |H_1| + |H_4| \geq n \), it follows that the induced subgraph by the edges from \( H_2 \cup H_3 \) to \( H_1 \cup H_4 \) contains a blue star \( K_{1,n} \), a contradiction.

Finally assume \( t = 5 \). Then in order to avoid a monochromatic copy of \( K_{1,3} \) in the reduced graph, the reduced graph must be the unique 2-coloring of \( K_5 \) containing no monochromatic triangle. Thus, every vertex of \( G \) has edges to exactly 2 parts of the Gallai partition in red and 2 parts of the partition in blue. This means that every vertex has red degree and blue degree at least \( 2 \cdot \frac{n-r+3}{2} = n - r + 3 \).

\[ \square \]

3 For Small \( m \) and Large \( n \)

In this section we give a proof for Theorem 1.4.

For small \( m \) and large \( n \), we first give the lower bound on the Gallai–Ramsey number for \( K_{1,n} \cup K_{1,m} \).

**Lemma 3.1**  Let \( n \geq 3 \), \( m \geq 1 \) and \( k \geq 3 \) be three integers. Then

\[
gr_k(K_3 : K_{1,n} \cup K_{1,m}) \geq \begin{cases} 
\frac{5n-6}{2} + k - 3, & \text{if } n \text{ is even}, \\
\frac{5n-3}{2} + k - 3, & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proof**  We prove this result by inductively constructing a coloring of \( K_N \) where

\[
N = \begin{cases} 
\frac{5n-8}{2} + k - 3, & \text{if } n \text{ is even}, \\
\frac{5n-2}{2} + k - 3, & \text{if } n \text{ is odd},
\end{cases}
\]

which contains no rainbow triangle and no monochromatic copy of \( K_{1,n} \cup K_{1,m} \).

For odd \( n \), we construct \( G_3^e \) by making five copies of \( K_{\frac{n+1}{2}} \) each colored entirely with color 1, and then inserting edges of colors 2 and 3 between the copies to form a blow-up of the unique 2-colored \( K_5 \) which contains no monochromatic triangle. For even \( n \), we construct \( G_3^e \) by making one copy of \( K_{\frac{n}{2}} \) and four copies of \( K_{\frac{n+1}{2}} \) each colored entirely with color 1, and then inserting edges of colors 2 and 3 between the copies to form a blow-up of the unique 2-colored \( K_5 \) which contains no monochromatic triangle. This coloring clearly contains no rainbow triangle and, since no vertex has at least \( n \) incident edges in any one color, there can be no monochromatic copy of \( K_{1,n} \cup K_{1,m} \).

To this base graph, for each \( i \) with \( 4 \leq i \leq k \) in sequence, we add a vertex \( v_i \) with all edges to the new vertex \( v_i \) having color \( i \). The resulting colored complete graph has order \( N \) and contains no rainbow triangle or monochromatic \( K_{1,n} \cup K_{1,m} \), completing the construction. \( \square \)
Proposition 3.2 Let $n \geq 22$, $m \geq 5$ and $k \geq 3$ be three integers with $m \leq \frac{n-14}{8}$. Then
\[ \text{gr}_k(K_3 : K_{1,n} \cup K_{1,m}) \leq \begin{cases} \frac{5n-6}{2} + k - 3 & \text{if } n \text{ is even,} \\
\frac{5n-3}{2} + k - 3 & \text{if } n \text{ is odd.} \end{cases} \]

Proof Let $G$ be a Gallai-coloring of $K_N$ where
\[ N = \begin{cases} \frac{5n-6}{2} + k - 3 & \text{if } n \text{ is even,} \\
\frac{5n-3}{2} + k - 3 & \text{if } n \text{ is odd.} \end{cases} \]
We only give the proof for the case that $n$ is odd, and the proof of the case that $n$ is even can be proved similarly.

Since $G$ is a Gallai-coloring, by Theorem 1.1, there is a Gallai-partition of $G$. Suppose red and blue are the two colors appearing in the partition. Let $t$ be the number of parts in this partition and choose such a partition where $t$ is minimized.

We consider the case when $n$ is odd since the case where $n$ is even is similar. Let $r$ be the number of parts of the Gallai-partition with order at least $\frac{n-1}{2}$, say with $|H_1| \geq |H_2| \geq \cdots \geq |H_r| \geq \frac{n-1}{2}$ and $|H_{r+1}|, |H_{r+2}|, \ldots, |H_t| \leq \frac{n-3}{2}$.

The overall structure of the proof is by induction on $k$. We consider cases based on the values of $k$ and $t$. For the base of the induction, we first suppose that $k = 3$ and further break into two cases based on the value of $t$.

Case 1 $k = 3$ and $2 \leq t \leq 3$.

Since $2 \leq t \leq 3$, by the minimality of $t$, we may assume $t = 2$. Let $H_1$ and $H_2$ be the corresponding parts. Suppose all edges from $H_1$ to $H_2$ are red. If $|H_1| \geq 2$ and $|H_2| \geq 2$, then there is a red $K_{1,n} \cup K_{1,m}$ since $|H_1| + |H_2| = N = \frac{5n-3}{2}$ and so $|H_1| \geq \frac{5n-3}{4} = n + \frac{n-3}{2} \geq n + m$, a contradiction since we can build a red copy of $K_{1,n} \cup K_{1,m}$ with both stars centered in $H_2$.

Suppose then that $|H_2| = 1$. Then $|H_1| = \frac{5n-5}{2}$. Recall that red is the color appearing in the edges from $H_1$ to $H_2$. Let $H_2 = \{v_1\}$.

If there is a vertex $w \in H_1$ with $m + 1$ incident red edges, then there is a red copy of $K_{1,n} \cup K_{1,m}$ centered at $w$ and $v_1$. We may therefore assume that every vertex in $H_1$ has at most $m$ incident red edges. Choose a vertex $v_2 \in H_1$ with the smallest number of incident red edges. Let $S \subseteq H_1$ be the set of vertices with red edges to $v_2$, $T \subseteq H_1$ be the set of vertices with blue edges to $v_2$, and $U \subseteq H_1$ be the set of vertices with green edges to $v_2$. Then we have assumed $|S| \leq m - 1$ and suppose $|T| \geq |U|$.

Claim 1 $|U| \leq |T| \leq n + m$.

Proof Assume, to the contrary, that $|T| \geq n + m + 1$. If there is a vertex $w \in S \cup T \cup U$ with a total of at least $m + 1$ incident blue edges, then there is a blue copy of $K_{1,n} \cup K_{1,m}$ centered at $w$ and $v_2$, so every vertex in $S \cup T \cup U$ has at most $m$ incident blue edges. This means that every vertex in $S \cup T \cup U$ has at least
\[ |G| - 1 - 2m \geq \frac{5n-3}{2} - 1 - 2m \geq n + m + 1 \]
incident green edges. This means that any two vertices in $S \cup T \cup U$ form the centers of a green copy of $K_{1,n} \cup K_{1,m}$, a contradiction. \qed
From Claim 1, we have $|U| \leq |T| \leq n + m$. Since $|S| + |T| + |U| = N - 2$, we have $|T| \geq \frac{N - 2 - (m - 1)}{2} \geq n + 1$ and $|U| \geq N - n - 2m - 1$. If a vertex $w \in T$ has at least $m$ blue edges to $U$, then there is a blue copy of $K_{1,n} \cup K_{1,m}$ with centers $v_2$ and $w$. Thus, each vertex $w \in T$ has at most $m - 1$ blue edges to $U$. Since $G$ is a Gallai-coloring, there is no red edges between $T$ and $U$. Thus $w$ has at least $|U| - (m - 1) \geq N - n - 3m \geq m$ green edges to $U$. Note that $|U| \geq N - n - 2m - 1 \geq n + m$. Thus there is a green $K_{1,n} \cup K_{1,m}$.

**Case 2** $k = 3$ and $t \geq 4$.

First a claim about the orders of parts in the Gallai partition.

**Claim 2** Every part has order at least $\frac{n - 5}{2}$.

**Proof** First suppose there is a part $H_i$ with $2 \leq |H_i| \leq \frac{n - 4m - 6}{2}$. Let $A$ be the set of vertices in $G \setminus H_i$ with red edges to $H_i$ and $B$ be the set of vertices in $G \setminus H_i$ with blue edges to $H_i$ and suppose that $|A| \geq |B|$. Then $|A| \geq \frac{N - |H_i|}{2} \geq n + m$. This means that any two vertices of $H_i$ form the centers of a red copy of $K_{1,n} \cup K_{1,m}$, a contradiction. This means we may assume that every part has order either 1 or at least $\frac{n - 4m - 5}{2}$.

Suppose there is a part $H_0$ of order 1, say with $H_0 = \{v_1\}$. Since $G$ cannot be colored entirely with 2 colors, there must also be a part $H_i$ with $|H_i| \geq \frac{n - 4m - 5}{2} \geq m + 1$. Let $A$ be the set of vertices with red edges to $v_1$ and $B$ be the set of vertices with blue edges to $v_1$, again assuming that $|A| \geq |B|$. In particular, this means that $|A| \geq \frac{(5n - 3)/2 - 1}{2} \geq n + m + 1$ so every vertex of $G$ other than $v_1$ has at most $m$ incident red edges. This means that every part of order at least $m + 1$ has all blue edges to other parts (except to $v_1$). If $H_1$ is in $B$, then to avoid a red copy of $K_{1,n} \cup K_{1,m}$, the edges from $A$ to $H_1$ are blue, and hence there is a blue copy of $K_{1,n} \cup K_{1,m}$, a contradiction. So $H_1$ is in $A$. In order to avoid making a blue copy of $K_{1,n} \cup K_{1,m}$ centered at one vertex in $H_1$ and one vertex in $G \setminus (H_1 \cup \{v_1\})$, we must have $|G \setminus (H_1 \cup \{v_1\})| < n + 1$ but then $|H_i| \geq n + m + 1$, in order to avoid making a blue copy of $K_{1,n} \cup K_{1,m}$ centered at two vertices of $G \setminus (H_1 \cup \{v_1\})$, we must have $|G \setminus (H_1 \cup \{v_1\})| = 1$, which contradicts to the fact $t \geq 4$. This means that every part has order at least $\frac{n - 4m - 5}{2} \geq m + 1$.

Finally suppose there is a part $H_i$ with $m + 1 \leq |H_i| \leq \frac{n - 7}{2}$. Then letting $A$ be the set of vertices with red edges to $H_i$ and $B$ be the set of vertices with blue edges to $H_i$, we see that one of $A$ or $B$ (suppose $A$) has order at least $\frac{N - |H_i|}{2} \geq n + 1$. Then there is a red copy of $K_{1,n} \cup K_{1,m}$ centered at one vertex in $H_i$ and another in $A$, a contradiction. \qed

Let $r$ be the number of parts of order at least $\frac{n - 7}{2}$ and call these parts “big”. Call any remaining parts “small”. By Claim 2, there are at most 5 parts in this Gallai partition of $G$.

We distinguish the following subcases to complete the proof of this case.

**Subcase 2.1** $r = 5$.

To avoid a pair of vertices having all one color on edges to three big parts, the reduced graph on the parts $H_1, H_2, H_3, H_4, H_5$ must be the unique 2-coloring of $K_5$ with no monochromatic triangle, say with $H_1H_2H_3H_4H_5H_1$ and $H_1H_3H_5H_2H_4H_1$ making two complementary monochromatic cycles with in red and blue respectively. Since $\sum_{i=1}^{5} |H_i| = \frac{5n - 3}{2}$, it follows that there exists a big part, say $H_1$, such that $|H_1| \geq \frac{n - 1}{2}$. Choose $v_2 \in H_2$ and $v_4 \in H_4$. Then the edges from $v_2$ to $H_1 \cup H_3$ contain a red copy of $K_{1,n}$, and the edges from $v_4$ to $H_5$ contain a red copy of $K_{1,m}$, and so there is a red $K_{1,n} \cup K_{1,m}$, a contradiction.
Subcase 2.2 \( r = 4 \).

To avoid a part having edges of all one color to three of the big parts, by symmetry, the four big parts must form one of the following two structures:

Type 1: There is a red cycle \( H_1H_4H_3H_2H_1 \) and a blue 2-matching \( \{H_1H_3, H_2H_4\} \) in the reduced graph, or

Type 2: There is a red path \( H_3H_2H_1H_4 \) and a blue path \( H_1H_3H_4H_2 \) in the reduced graph.

First suppose that \( t = 4 \). Since \( \sum_{i=1}^{t} |H_i| = \frac{n-3}{2} \), it follows that there exists a big part, without loss of generality (regardless of Type 1 or Type 2) say \( H_1 \), such that \( |H_1| \geq \frac{n+3}{2} \). Choose \( v_1 \in H_1 \) and \( v_2 \in H_2 \). Then the edges from \( v_2 \) to \( (H_1 \setminus \{v_1\}) \cup H_3 \) contain a red copy of \( K_{1,n} \) and the edges from \( v_1 \) to \( H_4 \) contain a red copy of \( K_{1,m} \), so there is a red copy of \( K_{1,n} \cup K_{1,m} \), a contradiction.

Thus, we may assume that \( t = 5 \). Let \( H_1, H_2, H_3, H_4, H_5 \) be the big parts, and \( H_5 \) be the (small) part of order at least \( \frac{n-5}{2} \). If the reduced graph does not consist of two monochromatic 5-cycles, then there exist three parts, say \( H_2, H_3, H_5 \), adjacent to the part \( H_1 \) by a single color. Since \( |H_2| + |H_3| + |H_5| = n - 1 + \frac{n-5}{2} = n + \frac{n-7}{2} \geq n + m \), this structure contains a monochromatic copy of \( K_{1,n} \cup K_{1,m} \), a contradiction. Thus, suppose that the reduced graph is two 5-cycles \( H_1H_2H_3H_4H_5H_1 \) and \( H_1H_3H_5H_2H_4H_1 \) say in red and blue respectively. By the same argument used in Subcase 2.1, where \( r = t = 5 \), there is a monochromatic copy of \( K_{1,n} \cup K_{1,m} \), a contradiction.

Subcase 2.3 \( r \leq 3 \).

Since \( t \geq 4 \), there is at least one small part, say \( H_1 \), so \( \frac{n-5}{2} \leq |H_1| \leq \frac{n+3}{2} \). Let \( A \) be the set of vertices with red edges to \( H_1 \) and let \( B \) be the set of vertices with blue edges to \( H_1 \). If \( |H_1| = \frac{n-5}{2} \), then one of \( |A| \) or \( |B| \) is large, with say \( |A| \geq \left\lceil \frac{N - \frac{n-5}{2}}{2} \right\rceil \geq n + 1 \) so there is a red copy of \( K_{n+1,m+1} \) between \( H_1 \) and \( A \), which contains a red copy of \( K_{1,n} \cup K_{1,m} \), a contradiction. Thus, all small parts have order \( \frac{n-3}{2} \). In order to avoid the same construction, we must have \( |A| = |B| = n \).

Also since \( t \leq 5 \), there is at least one big part.

The following facts are also immediate from the restrictions on small parts.

Fact 1 One of the following holds:

1. \( A \) contains only one big part with \( |A| = n \), or
2. \( A \) contains a small part \( H_{1,A} \) with \( |H_{1,A}| = \frac{n-3}{2} \) and a big part \( H_{2,A} \) with \( |H_{2,A}| = \frac{n+3}{2} \),
3. \( A \) contains a small part \( H_{1,A} \) with \( |H_{1,A}| = \frac{n-1}{2} \) and a big part \( H_{2,A} \) with \( |H_{2,A}| = \frac{n+1}{2} \).

Fact 2 One of the following holds:

4. \( B \) contains only one big part with \( |B| = n \), or
5. \( B \) contains a small part \( H_{1,B} \) with \( |H_{1,B}| = \frac{n-3}{2} \) and a big part \( H_{2,B} \) with \( |H_{2,B}| = \frac{n+3}{2} \),
6. \( B \) contains a small part \( H_{1,B} \) with \( |H_{1,B}| = \frac{n-1}{2} \) and a big part \( H_{2,B} \) with \( |H_{2,B}| = \frac{n+1}{2} \).

If (1) and (4) both hold, then there are only three parts \( H_1, A, B \) which contradicts the assumption that \( t \geq 4 \).

First suppose (without loss of generality) that (1) and (5) hold. If the edges from \( A \) to \( H_{1,B} \) (or \( H_{2,B} \)) are blue, then for \( u, v \in H_{1,B} \) (respectively \( H_{2,B} \)), the edges from \( \{u, v\} \) to \( H_1 \cup A \) contain a blue copy of \( K_{1,n} \cup K_{1,m} \), a contradiction. Thus, the edges from \( A \) to \( H_{1,B} \cup H_{2,B} \)
must be all red. Then if we choose \( u, v \in A \), then the edges from \( \{u, v\} \) to \( H_1 \cup B \) contain a red copy of \( K_{1,n} \cup K_{1,m} \), a contradiction. Similarly, we can get a contradiction for (1) and (6).

Finally assume that (2) and (5) hold. If all of the edges from \( H_{i,B} \) (with \( i \in \{1, 2\} \)) to \( A \) are blue, then there is a blue copy of \( K_{1,n} \cup K_{1,m} \) centered at two vertices of \( H_{i,B} \), a contradiction. Thus, we may assume that for each part \( H_{i,B} \), there is a part \( H_{j,A} \) such that the edges in between these two parts are red where \( i, j \in \{1, 2\} \). Similarly from the opposite perspective, for each part \( H_{j,A} \), there is a part \( H_{i,B} \) such that the edges between these two parts are blue where \( i, j \in \{1, 2\} \). From the above arguments, without loss of generality, we may assume that

- the edges from \( H_{1,A} \) to \( H_{1,B} \) and the edges from \( H_{2,A} \) to \( H_{2,B} \) are red;
- the edges from \( H_{1,A} \) to \( H_{2,B} \) and the edges from \( H_{2,A} \) to \( H_{1,B} \) are blue;

Now let \( v \in H_{2,A} \) and let \( u \in H_{1,A} \). Then the edges from \( u \) to \( H_{1,B} \) contain a red copy of \( K_{1,m} \), and the edges from \( v \) to \( H_1 \cup H_{2,B} \) contain a red copy of \( K_{1,n} \), a contradiction.

It is clear that (3) and (6) do not hold. Suppose that (2) and (6) both hold. Similarly to the proof of (2) and (5), we can get a contradiction.

**Case 3** \( k \geq 4 \).

Let \( T \) be a largest set of vertices in \( G \) such that each vertex in \( T \) has edges of all one color to \( G \setminus T \) with the added restriction that \( |G \setminus T| \geq n + m \). For each \( i \) with \( 1 \leq i \leq k \), if we let \( T_i \) be the set of vertices in \( T \) with all edges of color \( i \) to \( G \setminus T \), then in order to avoid a monochromatic copy of \( K_{1,n} \cup K_{1,m} \), we have \( |T_i| \leq 1 \) for all \( i \). This means that \( |T| \leq k \). Let \( G' = G \setminus T \) so \( |G'| \geq \frac{5n-9}{2} \).

Within \( G' \), there is no vertex with degree at least \( m \) in a color \( i \) for which \( T_i \neq \emptyset \) to avoid creating a copy of \( K_{1,n} \cup K_{1,m} \) in color \( i \). We first claim that there are at least two colors not appearing on edges from \( T \) to \( G' \).

**Claim 3** \( |T| \leq k - 2 \).

**Proof** First if \( |T| = k \), then every vertex in \( G' \) has color degree at most \( m - 1 \) within \( G' \) in every color. This contradicts Theorem 1.2 since there must be a vertex with degree at least \( \frac{2|G'|}{5} \geq m \) in some color.

Next if \( |T| = k - 1 \), then there is again a vertex with degree at least \( m \) in some color (say red) but this means that red must be the color not represented on edges between \( T \) and \( G' \). Consider a Gallai partition of \( G' \), say with the smallest possible number of parts. If this partition has only 2 parts, then red must be the color between the parts and in order to avoid creating a red copy of \( K_{1,n} \cup K_{1,m} \), one part must have order 1. This part can be moved to \( T \), contradicting the maximality of \( |T| \). Thus, we may assume that there are at least 4 parts in the Gallai partition of \( G' \), say with red and blue appearing on edges between the parts. By minimality of the number of parts in this partition, both red and blue must induce connected subgraphs of the reduced graph. This means that in order to avoid having a vertex with at least \( m \) edges in blue, all parts of this partition must have order at most \( m - 1 \). Every vertex \( v \in G' \) must then have at least \( |G'| - 2(m - 1) > n + m + 2 \) incident red edges. This means that there is a red copy of \( K_{1,n} \cup K_{1,m} \) centered at any pair of vertices within \( G' \), a contradiction. \( \square \)

Choose a Gallai partition of \( G' \) with the smallest number of parts, say \( q \), and let red and blue be the colors that appear in between the parts of this partition. Note that from the argument
above, red and blue do not appear on the edges coming from vertices of $T$. Let $s = |T|$.

If $2 \leq q \leq 3$, then by the minimality of $q$, we may assume that $q = 2$. Then if $I_1$ and $I_2$ are the parts of this partition, say with $|I_1| \geq |I_2|$, we have

$$|I_1| + |I_2| = |G| - |T| \geq \frac{5n - 6}{2} + (k - 3) - s \geq 2n + \frac{n - 8}{2} + (k - s - 2),$$

which means that $|I_1| \geq n + \frac{6 - n}{2} \geq n + m$. Then any two vertices of $I_2$ form the centers of a monochromatic copy of $K_{1,m} \cup K_{1,n}$, a contradiction. Thus, we may assume that $q \geq 4$.

If $s \leq k - 3$, then we may apply the same argument as in the case $k = 3$ (Case 2). Thus, suppose $s = k - 2$. This means that $|G'| \geq \frac{mn - 8}{2}$.

**Claim 4** Every part has order at least $m + 1$.

**Proof** First suppose there is a part $H_0$ of order 1. Let $A$ be the set of vertices with red edges to $H_0$ and let $B$ be the set of vertices with blue edges to $H_0$, say with $|A| \geq |B|$. This means $|A| \geq n + 2m + 1$ so there can be no vertex anywhere other than the vertex of $H_0$ with at least $m$ red edges. This means that every vertex from $B$ has at least $n + m + 1$ blue edges to $A$, producing a blue copy of $K_{1,n} \cup K_{1,m}$.

Thus, suppose there is a part $H$ of order $r$ with $2 \leq r \leq m$. Then each vertex in $H$ has at least $\frac{|G\setminus H|}{2} \geq n + m$ edges in a single color to $G\setminus H$ (since $m \leq \frac{n - 8}{6}$). Choosing two vertices from $H$ produces a monochromatic copy of $K_{1,n} \cup K_{1,m}$, a contradiction.

If there is a vertex $v$ with at least $n + 1$ incident red (or blue) edges to other parts of the Gallai partition, then if we let $u$ be a vertex with a red edge to $v$, then $v$ is the center of a red copy of $K_{1,n}$ avoiding $u$ and $u$ is the center of a disjoint red copy of $K_{1,m}$ since $v$ is in a part of the Gallai partition with order at least $m + 1$ (by Claim 4). Thus, there is no vertex $v \in G'$ with red or blue degree at least $n$ to other parts of the Gallai partition.

Applying Lemma 2.1 with $r = 8$, we see that the Gallai partition of $G'$ has exactly 5 parts, each of order at least $\frac{6n - 8}{5}$, and every vertex of $G'$ has at least $n - 5$ incident edges in red and at least $n - 5$ incident edges in blue to other parts of the Gallai partition.

Let $H_1$ be a largest part of this Gallai partition of $G'$. Since $H_1$ contains no rainbow triangle, by Theorem 1.2, there is a monochromatic star on at least $\frac{2|H_1|}{5} \geq m$ edges within $H_1$. If this star has some color $i$ for which $T_i \neq \emptyset$, then there is a monochromatic copy of $K_{1,n} \cup K_{1,m}$ in color $i$ so this must be either red or blue, say red. This along with the structure we have already shown implies the existence of a red copy of $K_{1,n} \cup K_{1,m}$, a contradiction. This completes the proof of Proposition 3.2. \[\square\]

**4 For Equal $m$ and $n$**

In this section we give a proof for Theorem 1.5.

For $m = n$, we first give a lower bound for the Gallai Ramsey number of $K_{1,n} \cup K_{1,n}$.

**Lemma 4.1** For $k \geq 3$,

$$\text{gr}_k(K_3 : K_{1,n} \cup K_{1,n}) \geq 3n + k - 1.$$
Proof We prove this result by inductively constructing a coloring $G_k$ of $K_t$ where $t = 3n+k-2$ which contains no rainbow triangle and no monochromatic copy of $K_{1,n} \cup K_{1,n}$.

Let $G_3$ be a graph constructed by steps as follows:

- Let $F_1$ be a complete graph $K_{2n-1}$ edge-decomposed into an $(n-1)$-regular graph of order $2n-1$ with color 1 and an $(n-1)$-regular graph of order $2n-1$ with color 2.
- Let $F_2$ be a complete graph $K_{3n-1}$ obtained from the graph $F_1$ and a complete graph $K_n$ colored with color 1 by adding all edges from $F_1$ to $K_n$ with color 3.
- Let $G_3$ be a complete graph $K_{3n+1}$ obtained from $F_2$ by adding two new vertices $v, w$ and adding the edges from $v$ to $F_2$ with color 1, and the edges from $w$ to $F_2 \cup \{v\}$ with color 2.

In order to construct $G_{i+1}$, we add a vertex $v_{i+1}$ to $G_i$ with all edges from $v_{i+1}$ to $G_i$ having color $i+1$ for $3 \leq i \leq k-1$. This coloring certainly contains no rainbow triangle or monochromatic copy of $K_{1,n} \cup K_{1,n}$ and has order $3n + k - 2$, completing the construction.

Lemma 4.2 For $k \geq 3$ and $n \geq 4$,

$$\text{gr}_k(K_3 : K_{1,n} \cup K_{1,n}) \leq 3n + k - 1.$$ 

Proof We will assume $n$ is odd since the even case can be proved similarly. Suppose $k \geq 3$ and let $G$ be a Gallai coloring of $K_{3n+k-1}$.

Let $T$ be a maximal set of vertices where each vertex of $T$ has all edges in a single color to $G \setminus T$ with the additional assumption that $|G \setminus T| \geq 2n$. Then in order to avoid a monochromatic copy of $K_{1,n} \cup K_{1,n}$, there is at most one vertex in $T$ with edges of each color to $G \setminus T$, meaning that $|T| \leq k$. Let $G' = G \setminus T$ so $|G'| = |G| - |T| \geq 3n - 1$.

Since $G'$ contains no rainbow triangle, by Theorem 1.1, there is a Gallai partition of $G'$, say using red and blue on edges between the parts of this partition. In order to avoid a monochromatic copy of $K_{1,n} \cup K_{1,n}$, if there is a vertex $v_i \in T$ with color $i$ on all edges to $G'$, there is no vertex with at least $n$ incident edges in a single color $i$ within $G'$. By Lemma 2.1, this means that red and blue cannot appear on edges between $T$ and $G'$ so $|T| \leq k - 2$ and $|G'| \geq 3n + 1$.

Let $H_1, H_2, \ldots, H_t$ be the parts of the Gallai partition of $G'$ chosen so that $t$ is minimized, say with $|H_i| \geq |H_{i+1}|$ for all $i$. If $2 \leq t \leq 3$, then by minimality of $t$, we may assume $t = 2$, say with red edges between $H_1$ and $H_2$. In this case, if $|H_2| = 1$, then the vertex of $H_2$ can be moved to $T$, contradicting the maximality of $T$. Thus, $|H_2| \geq 2$. In order to avoid a red copy of $K_{1,n} \cup K_{1,n}$, we have $|H_1| \leq 2n - 1$, so $|H_2| \geq (3n + 1) - (2n - 1) = n + 2$. We then find a red copy of $K_{1,n} \cup K_{1,n}$ centered on one vertex from each of $H_1$ and $H_2$, a contradiction. This means that we may assume that $t \geq 4$.

Since all edges between any pair of parts have a single color and a monochromatic copy of $K_{n+1,n+1}$ contains a monochromatic copy of $K_{1,n} \cup K_{1,n}$, we immediately see that there is at most one part with at least $n + 1$ vertices.

Case 1 There is a part $H_1$ with $|H_1| \geq n + 1$.

For any other part $H_i$ with $2 \leq i \leq t$, let $A_i$ be the set of vertices with red edges to $H_i$ and let $B_i$ be the set of vertices with blue edges to $H_i$. Certainly $H_1$ is in either $A_i$ or $B_i$, so the next claim shows the opposite set is small.

Claim 5 If $H_1 \subseteq A_i$ (or $H_1 \subseteq B_i$), then $|B_i| \leq n - 1$ (resp., $|A_i| \leq n - 1$).
Proof Without loss of generality, suppose $H_1 \subseteq A_i$ and for a contradiction, suppose that $|B_i| \geq n$. First we additionally assume that $|B_i| \geq n + 1$. Then by minimality of $t$, there is a part with blue edges to $H_1$, say containing a vertex $v$. Then choosing any vertex $u \in H_i$, we have a blue copy of $K_{1,n}$ centered at $u$ with edges to $B_i \setminus \{v\}$ (since $v$ might be in $B_i$) and a blue copy of $K_{1,n}$ centered at $v$ with edges to $H_1$, a contradiction. Thus, we may assume that $|B_i| = n$.

Next suppose that $A_i = H_1$. If $|H_i| = 1$, then $|H_1| = |G'| - 1 - n \geq 2n$. Since $|B_i| = n \geq 3$, there are two vertices $u, v \in B_i$ with all one color on their edges to $H_1$. These form the centers of a monochromatic copy of $K_{1,n} \cup K_{1,n}$, so we may assume that $|H_i| \geq 2$. Since $|H_i \cup H_1| = |G'| - |B_i| \geq 2n + 1$, there can be at most one vertex in $B_i$ with blue edges to $H_1$. On the other hand, by minimality of $t$, there must then be exactly one vertex in $B_i$ with blue edges to $H_1$. This means there are exactly $n - 1$ vertices in $B_i$ with all red edges to $H_1$. Then if we choose one vertex $u \in H_i$ and one vertex $v \in H_1$, then there is a red copy of $K_{1,n} \cup K_{1,n}$ centered at $u$ with edges to $H_1 \setminus \{v\}$ and centered at $v$ with edges to $B \cup H_i$. Thus, we may assume that $A_i$ contains at least one other (smaller) part in addition to $H_1$.

Let $H_j$ be a part in $A_i \setminus H_1$. Then in order to avoid creating a blue copy of $K_{1,n} \cup K_{1,n}$ centered at a vertex in $H_1$ and a vertex in $H_j$, all edges from $H_j$ to $H_1$ must be red. Then we have the following claim.

Claim 6 There is a part $H_j \subseteq A_i$ with some red edges to a part $H_\ell \subseteq B_i$.

Proof Assume, to the contrary, that for any part $H_j$ ($j \neq 1$) in $A_i$, the edges from $A_i \setminus H_1$ to $B_i$ are blue. Since $|H_i \cup A_i| = |G'| - |B_i| \geq 2n + 1$, there can be at most one vertex in $B_i$ with blue edges to $H_1$. This means that there are at least $n - 1$ vertices in $B_i$ with all red edges to $H_1$. Then if we choose one vertex $u \in H_i$ and one vertex $v \in H_1$, then there is a red copy of $K_{1,n} \cup K_{1,n}$ centered at $u$ with edges to $A_i \setminus \{v\}$ and centered at $v$ with edges to $B \cup H_i$, a contradiction.

From Claim 6, there is a part $H_j \subseteq A_i$ with some red edges to a part $H_\ell \subseteq B_i$. Choose a vertex $v \in H_j$ and a vertex $u \in H_i$. Choose $n$ red neighbors of $v$ by first selecting all of $H_\ell$, then vertices from $H_i \setminus \{u\}$, and finally some vertices of $H_1$ as needed. Let $S$ be the set of vertices in this red copy of $K_{1,n}$ centered at $v$. Since $|B_i| = n$, we have $|B_i \setminus H_\ell| \leq n - 1$. Since $|G'| - |S| \geq 2n$, there are at least $2n - 1 - (n - 1) = n$ remaining red neighbors of $u$, to form a second disjoint red copy of $K_{1,n}$, a contradiction.

If we choose $i = 1$, then in order to avoid a monochromatic copy of $K_{n+1,n+1}$, we immediately see that $|A_1|, |B_1| \leq n$. It turns out that we can say a bit more.

Claim 7 The reduced graph of $A_1$ is a blue complete graph and the reduced graph of $B_1$ is a red complete graph.

Proof We show that the reduced graph of $A_1$ is a blue complete graph since the other proof is symmetric. Suppose not, so there are red edges between a pair of parts $H_i$ and $H_j$ within $A_1$. Choose one vertex from each of these parts, say $v_i \in H_i$ and $v_j \in H_j$. Then there is a red copy of $K_{1,n}$ centered at $v_i$ with $n$ edges to all of $(H_j \setminus v_j)$ and part of $H_1$. Let $S$ be the vertices of this star. Since the edges from $H_1$ to $H_j$ are red, by Claim 5, we have $|B_j| \leq n - 1$. In $G' \setminus S$, there are at least $|G'| - |S| - |B_j| - 1 \geq (3n + 1) - (n + 1) - (n - 1) - 1 = n$ red edges incident
to $v_j$. This star, along with $S$, forms a red copy of $K_{1,n} \cup K_{1,n}$, a contradiction. □

Let $H_x$ be a smallest part within $A_1 \cup B_1$, say with $H_x \subseteq A_1$. Let $B'_1 \subseteq B$ be the union of the parts with red edges to $H_x$ and let $B''_1 \subseteq B$ be the union of the parts with blue edges to $H_x$. To avoid a blue $K_{1,n} \cup K_{1,n}$, there exist a part $H_y$ in $B''_1$ and a part $H_z$ in $A_1$ such that the edges from $H_y$ to $H_z$ are red. To avoid a red $K_{1,n} \cup K_{1,n}$, we have $|B'_1| + |H_y| + |H_y| < 2n$, and hence $|B''_1| + |H_y| \geq n + 1$. Since $H_x$ is a smallest part, it follows that $|B''_1| + |A_1| - |H_x| \geq n + 1$. Choose $u \in H_x$ and $v \in B'_1$. Then the edges from $u$ to $(B''_1 \cup A_1) - H_x$ and the edges $v$ to $H_1$ form a blue $K_{1,n} \cup K_{1,n}$, a contradiction.

**Case 2** Every part of the Gallai partition of $G'$ has order at most $n$.

Again for each part $H_i$ of the Gallai partition of $G'$, let $A_i$ be the set of vertices with red edges to $H_i$ and let $B_i$ be the set of vertices with blue edges to $H_i$. For any pairwise disjoint sets of vertices $V_1, V_2, \ldots, V_t$ and a disjoint vertex $v$, we say that we choose-in-order $n$ neighbors of $v$ from $V_1, V_2, \ldots, V_t$ if we choose all neighbors of $v$ from $V_1$, then from $V_2$, and so on until we have $n$ neighbors of $v$ by using up all of each set before moving on to the subsequent set. Obviously, this assumes that $|V_1 \cup V_2 \cup \cdots \cup V_t| \geq n$.

**Claim 8** There exists a part $H_i$ such that one of $|A_i| \leq n$ or $|B_i| \leq n$.

**Proof** Suppose not, so $|A_i| \geq n + 1$ and $|B_i| \geq n + 1$ for all $i$ with $1 \leq i \leq t$. Choose an arbitrary index $i$.

Suppose further that there is a vertex $v \in A_i$ with at most $n - 1$ blue edges to $B_i$. Let $H_j \subseteq A_i$ be the part containing $v$ and choose $u \in H_j$. Since $|A_j| \geq n + 1$, there are at least $n + 1$ red neighbors of $v$. Choose-in-order $n$ neighbors of $v$ using red edges from the sets $H_i \setminus \{u\}, B_i, A_i$. Let $S$ be the resulting red copy of $K_{1,n}$. Since we have assumed $|S| = n + 1$, $|H_i| \leq n$, and $|B_i \setminus S| \leq n - 1$, we have that $|A_i \setminus S| \geq n$. This means that there remains in $G' \setminus S$ a red copy of $K_{1,n}$ centered at $u$, a contradiction. Thus, we may assume that every vertex in $A_i$ has at least $n$ blue edges to $B_i$ and similarly every vertex in $B_i$ has at least $n$ red edges to $A_i$.

Then the number of red edges plus the number of blue edges between $A_i$ and $B_i$ is at least $n \cdot |A_i| + n \cdot |B_i| \geq n(3n + 1 - |H_i|)$. On the other hand, the total number of edges between $A_i$ and $B_i$ is $|A_i||B_i| \leq (\frac{3n + 1 - |H_i|}{2})^2 < n(3n + 1 - |H_i|)$, a contradiction. □

By Claim 8, there is a part $H_j$ such that either $|A_j| \leq n$ or $|B_j| \leq n$, say $|B_j| \leq n$. We now consider cases based on the value of $|B_j|$.

**Subcase 2.1** $|B_j| \leq n - 1$.

First a claim about the red edges incident to a vertex in $A_j$.

**Claim 9** For each vertex $v \in A_j$, there are at most $n$ red edges incident to $v$.

**Proof** Suppose, for a contradiction, that there is a vertex $v \in A_j$ with at least $n + 1$ incident red edges and let $u$ be any vertex in $H_j$. Then choose-in-order $n$ vertices with red edges to $v$ from $H_j \setminus \{u\}, B_j, A_j$ and let $S$ be the resulting red copy of $K_{1,n}$. Then since $|B_j| \leq n - 1$, we have $|A_j \setminus S| \geq n$ so there is a second red copy of $K_{1,n}$ centered at $u$ with edges to $A_j \setminus S$, a contradiction. □

By Claim 9, every vertex in $A_j$ has at most $n$ incident red edges and so at least $n + 1$
incident blue edges. Next a claim about the reduced graph restricted to $A_j$.

**Claim 10** The blue edges in the reduced graph of $G'$ restricted to $A_j$ form a union of cliques.

**Proof** Suppose, for a contradiction, that there are three parts within $A_j$, say $H_{i_1}, H_{i_2}$, and $H_{i_3}$ such that the edges from $H_{i_1}$ to $H_{i_2}$ are red while the edges from $H_{i_3}$ to $H_{i_1} \cup H_{i_2}$ are blue. Note that $H_{i_2} \subseteq A_{i_1}$ and $H_{i_3} \subseteq B_{i_1}$. By Claim 9, we have $|A_{i_1}| \leq n$.

Since $|A_{i_1}| \leq n$ and $|H_{i_1}| \leq n$, we have $|B_{i_3}| \geq n + 1$ (so each vertex in $H_{i_1}$ has at least $n + 1$ blue edges to other parts). Let $u \in H_{i_1}$ and $v \in H_{i_3}$ and choose-in-order $n$ vertices with blue edges to $v$ from $H_{i_1} \setminus \{u\}, H_{i_2}, B_{i_3}$ and let $S$ be the resulting blue copy of $K_{1,n}$. Since $H_{i_2} \subseteq A_{i_1}$, we have $|A_{i_1} \setminus S| \leq n - 1$. This means that $u$ has at least $|G' \setminus (H_{i_1} \cup A_{i_1})| \geq n$ blue edges to $G' \setminus S$. This produces a second disjoint blue copy of $K_{1,n}$, for a contradiction.

By Claim 10, the blue edges in the reduced graph of $G'$ restricted to $A_j$ form a union of cliques, say $J_1, J_2, \ldots, J_p$. If $p \geq 2$, then all edges between pairs of these cliques must be red. By Claim 9 (considering a vertex in $J_1$), $|(A_j \setminus J_1) \cup H_j| \leq n$ and similarly (by considering a vertex in $J_2$) $|J_1| < n$ so $|A_j| + |H_j| \leq 2n - 1$. Since $|B_j| \leq n - 1$, this means that $|G'| \leq 3n - 2$, a contradiction, meaning that $p = 1$ so we arrive at the following fact.

**Fact 3** The reduced graph restricted to $A_j$ is a single blue clique.

**Claim 11** For any three parts $X_1, X_2 \subseteq A_j$ and $Y \subseteq B_j$, if the edges from $Y$ to $X_1$ are red, then the edges from $Y$ to $X_2$ are also red. Symmetrically, if the edges from $Y$ to $X_1$ are blue, then the edges from $Y$ to $X_2$ are also blue.

**Proof** Suppose, for a contradiction, that there exist parts $X_1, X_2 \subseteq A_j$ and $Y \subseteq B_j$ with red edges from $Y$ to $X_1$ and blue edges from $Y$ to $X_2$. By Fact 3, the edges from $X_1$ to $X_2$ are blue. Then the same argument as in the proof of Claim 10 produces a contradiction.

By minimality of $t$, $A_j$ is a single part of the partition, but since $|A_j| > n$, this is a contradiction, completing the proof in this subcase.

**Subcase 2.2** $|B_j| = n$.

We have the following claim.

**Claim 12** The blue edges in the reduced graph of $G'$ restricted to $A_j$ form a union of cliques.

**Proof** Assume, to the contrary, that the subgraph induced by blue edges is not a union of cliques. Then there exist three parts $H_{i_1}, H_{i_2}, H_{i_3}$ such that the edge from $H_{i_1}$ to $H_{i_2}$ is red and the edges from $H_{i_3}$ to $H_{i_1} \cup H_{i_2}$ are blue. Note that the part $H_{i_2}$ belongs to $A_{i_1}$, and the part $H_{i_3}$ belongs to $B_{i_1}$. The following fact is immediate.

**Claim 13** $|B_{i_1}| \geq n + 1$ and $|B_{i_3}| \geq n + 1$.

**Proof** If the edges from $H_{i_1}$ to $B_j$ are all blue, then $|B_{i_1}| \geq n + 1$, since $|B_j| = n$ and the edges from $H_{i_3}$ to $H_{i_1}$ are blue. Then there exists a part $Y \subseteq B_j$ such that the edges from $H_{i_1}$ to $Y$ are red. Then the number of blue edges from $H_{i_1}$ to $G - H_{i_1}$ is at most $n$, that is, $|A_{i_1}| \leq n$. Then $|B_{i_1}| \geq n + 1$. Similarly, we have $|B_{i_3}| \geq n + 1$.

From Claim 13, we have $|B_{i_1}| \geq n + 1$ and $|B_{i_3}| \geq n + 1$. We choose $n$ red edges from $v$ to $H_{i_1} - u, H_{i_2}$ in order and forms a red $K_{1,n}$, say $X$. The edges from $u$ to $B_{i_1}$ forms another red $K_{1,n}$, a contradiction.

By Claim 12, the blue edges in the reduced graph of $G'$ restricted to $A_j$ form a union of
cliques, say $J_1, J_2, \ldots, J_p$. If $p \geq 2$, then all edges between pairs of these cliques must be red. By the same argument as the proof of Claim 10, we see that for every part $X \subseteq J_i$ and for every part $Y \subseteq J_j$, if the edges from $X$ to $Y$ are red, then all the edges from $Y$ to $J_i \setminus X$ are red. By minimality of $t$, this means that each set $J_i$ is a single part of the Gallai partition of $G'$, meaning that the reduced graph restricted to $A_j$ is one red clique.

Note that $|A_j \cup H_j| = |G'| - n \geq 2n + 1$ so since every part has order at most $n$, for every part $H_i \subseteq A_j$ we have $|A_i| \geq n$. By minimality of $t$, there is at least one part $H_i \subseteq A_j$ with red edges to a part $H_k \subseteq B_j$. Choose two vertices $u \in H_i$ and $v \in H_j$ and choose-in-order $n$ vertices with red edges to $u$ from $H_j \setminus \{v\}, H_i, A_i$ and let $S$ be this red star. Then since $|H_i| \geq 1$, $v$ has at least $n$ remaining incident red edges in $G' \setminus S$, making a disjoint red copy of $K_{1,n}$, a contradiction completing the proof. \hfill $\square$

5 For General $m$ and $n$

In this section we give a proof for Theorem 1.6.

We first give a general lower bound on the Gallai–Ramsey number for $K_{1,n} \cup K_{1,m}$.

**Lemma 5.1** For $k \geq 3$, $m \leq n$,

$$\text{gr}_k(K_3 : K_{1,n} \cup K_{1,m}) \geq \begin{cases} 2n + m + k - 5 & \text{if } n \text{ is even}, \\ 2n + m + k - 4 & \text{if } n \text{ is odd}. \end{cases}$$

**Proof** For odd $n$, we prove this result by inductively constructing a coloring of $K_t$ where $t = 2n + m + k - 5$ which contains no rainbow triangle and no monochromatic copy of $K_{1,n} \cup K_{1,m}$. We construct $G_3$ by making four copies of $K_{\frac{m}{3}+1}$ in color 1 and one copy of $K_m$ in color 1, and then inserting edges of colors 2 and 3 between the copies to form a blow-up of the unique 2-colored $K_5$ which contains no monochromatic triangle. This coloring clearly contains no rainbow triangle and no monochromatic copy of $K_{1,n} \cup K_{1,m}$, completing the base construction.

For even $n$, we prove this result by inductively constructing a coloring of $K_t$ where $t = 2n + m + k - 6$ which contains no rainbow triangle and no monochromatic copy of $K_{1,n} \cup K_{1,m}$. We construct $G_3$ by making three copies of $K_{\frac{m}{2}+1}$ in color 1 and one copy of $K_m$ in color 1 and one copy of $K_{\frac{m}{2}}$ in color 1, and then inserting edges of colors 2 and 3 between the copies to form a blow-up of the unique 2-colored $K_5$ which contains no monochromatic triangle. This coloring clearly contains no rainbow triangle and no monochromatic copy of $K_{1,n} \cup K_{1,m}$, completing the base construction.

For $i$ with $3 \leq i \leq k-1$, given $G_i$, we construct $G_{i+1}$ by adding a single vertex with all edges to $G_i$ having color $i + 1$. This coloring certainly contains no rainbow triangle or monochromatic copy of $K_{1,n} \cup K_{1,m}$ and has the desired order, completing the construction. \hfill $\square$

The lower bounds in Theorem 1.6 can be derived from Lemmas 3.1 and 5.1. To complete the proof of Theorem 1.6, we provide the following upper bound.

**Lemma 5.2** For $k \geq 3$, $m \leq n$,

$$\text{gr}_k(K_3 : K_{1,n} \cup K_{1,m}) \leq \begin{cases} 3n + 3m + k - 3 & \text{if } n \text{ is even}, \\ 3n + 3m + k - 2 & \text{if } n \text{ is odd}. \end{cases}$$
Proof We only give the proof of the case when $n$ is odd since the even case can be proved similarly. Suppose $k \geq 3$ and let $G$ be a Gallai coloring of $K_N$ where

$$N = \begin{cases} 3n + 3m + k - 3 & \text{if } n \text{ is even}, \\ 3n + 3m + k - 2 & \text{if } n \text{ is odd}. \end{cases}$$

Let $T$ be a maximal set of vertices where each vertex of $T$ has all edges in a single color to $G \setminus T$ with the additional assumption that $|G \setminus T| \geq n + m$. Then in order to avoid a monochromatic copy of $K_{1,n} \cup K_{1,m}$, there is at most one vertex in $T$ with edges of each color to $G \setminus T$, meaning that $|T| \leq k$. Let $G' = G \setminus T$ so $|G'| = |G| - |T| \geq 3n + 3m - 2$.

Since $G'$ contains no rainbow triangle, by Theorem 1.1, there is a Gallai partition of $G'$. Suppose that there is one color, say red, appearing in the edges between the parts of this partition. Under this partition, we have two parts, say $H_1, H_2$. From the maximality of $T$, we have $|H_i| \geq 2$ for $i = 1, 2$. Furthermore, $|H_1| \geq n + m$ or $|H_2| \geq n + m$. Then there is a monochromatic copy of $K_{1,n} \cup K_{1,m}$, contradiction.

Suppose that there are two colors, say red and blue, on edges between the parts of this partition. In order to avoid a monochromatic copy of $K_{1,n} \cup K_{1,m}$, if there is a vertex $v_i \in T$ with color $i$ on all edges to $G'$, there is no vertex with at least $m$ incident edges in a single color $i$ within $G'$. By Lemma 2.1, this means that red and blue cannot appear on edges between $T$ and $G'$ so $|T| \leq k - 2$ and $|G'| \geq 3n + 3m$.

If $2 \leq t \leq 3$, then by minimality of $t$, we may assume that $t = 2$. Then one part, say $H_1$, has $|H_1| \geq \frac{|G'|}{2} > n + m$. If there are at least 2 vertices in $H_2$, then there is a monochromatic copy of $K_{1,n} \cup K_{1,m}$ centered at these two vertices with edges to $H_1$ so $|H_2| = 1$. We may then move that vertex in $H_2$ to $T$, contradicting the maximality to $|T|$. We may therefore assume that $t \geq 4$. By the same argument, we arrive at the following fact.

Fact 4 For each $i$ with $1 \leq i \leq t$, we have $|H_i| \leq n + m - 1$.

By Theorem 1.3, we know that $R(K_{1,n} \cup K_{1,m}) = \max\{n + 2m, 2n + 1, n + m + 3\} < 3m + 3n \leq |G'|$, and hence there exists a part, say $H_1$, with $|H_1| \geq 2$. Let $A$ be the set of vertices with red edges to $H_1$ and let $B$ be the set of vertices with blue edges to $H_1$. Since $|G'| \geq 3n + 3m$, it follows that $|H_1| \leq n + m - 1$, and $|A| \geq n + m$ or $|B| \geq n + m$. Without loss of generality, suppose $|A| \geq n + m$. Then since $|H_1| \geq 2$, these two vertices form the centers of a red copy of $K_{1,n} \cup K_{1,m}$, a contradiction, completing the proof.

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