A geometric procedure for the reduced-state-space quantisation of constrained systems

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Abstract

We propose in this paper a new method for the quantisation of systems with first-class constraints. This method is a combination of the coherent-state-path-integral quantisation developed by Klauder, with the ideas of reduced state space quantisation. The key idea is that the physical Hilbert space may be defined by a coherent-state path-integral on the reduced state space and that the metric on the reduced state space that is necessary for the regularisation of the path-integral may be computed from the geometry of the classical reduction procedure. We provide a number of examples—notably the relativistic particle. Finally we discuss the quantisation of systems, whose reduced state space has orbifold-like singularities.

1 Introduction

There exist many approaches for the quantisation of systems with (first-class) constraints. They fall essentially into two categories, according to whether they implement the constraints before or after quantisation. In the first category lie schemes that attempt to quantise directly the reduced state space of a constrained system. Schemes of the second category are exemplified by Dirac quantisation: one quantises the theory before the imposition of the constraints and identifies the Hilbert space for the physical degrees of freedom as the zero-eigenspace of a quantum operator that represents the constraint.

The present paper proposes a variation of the Reduced State Space Quantisation scheme. It is based upon coherent-state-path-integral quantisation scheme method, which has been developed by Klauder [1][2]. In this method the only input necessary for the quantisation of a classical symplectic manifold $\Gamma$ is the
introduction of a Riemannian metric $ds^2$ on $\Gamma$. The metric allows the definition of a Wiener process on $\Gamma$, which plays the role of a regulator for the rigorous definition of a path-integral for the coherent state matrix elements. The key suggestion of our proposal is that a Riemannian metric on the reduced state space may be always identified, by exploiting the **geometry** associated to the classical reduction procedure.

2 Dirac vs Reduced state space quantisation

A Hamiltonian system with first-class constraints is described by a symplectic manifold $\Gamma$, equipped with a symplectic form $\Omega$ and a number of constraint functions $f^i : \Gamma \rightarrow \mathbb{R}$, such that $\{f^i, f^j\} = c^{ij}_k f^k$, for some structure functions $c^{ij}_k$. One defines the constraint surface $C$ as the submanifold of $\Gamma$ defined by the conditions $f^i = 0$. The restriction of the symplectic form $\Omega$ on $C$ is degenerate. The reduced state space $\Gamma_{\text{red}}$ is defined as the space of all orbits on the constraint surface generated by the constraints through the Poisson bracket.

In other words, $\Gamma_{\text{red}}$ is the quotient $C/\sim$, where $\sim$ is an equivalence relation, defined as follows: $z \sim z'$, if $z'$ can be reached from $z$ through a canonical transformation generated by the constraints.

We define the projection map $\pi : C \rightarrow \Gamma_{\text{red}}$ as $\pi(z) = [z]$, where $[z]$ the orbit, in which the point $z \in C$ belongs then. $\Gamma_{\text{red}}$ is equipped with a symplectic form $\tilde{\Omega}$, such that $\pi^* \tilde{\Omega} = \Omega|_C$.

The physical idea reduced state space quantisation (RSSQ) is that the rules of quantisation are only meaningful, when applied to the true physical degrees of freedom and are rather ambiguous when applied to the gauge degrees of freedom. It is suggested therefore that one should construct the quantum theory for the reduced state space $\Gamma_{\text{red}}$. In general, $\Gamma_{\text{red}}$ has a non-trivial topological structure, and generalisations of the canonical quantisation scheme is needed to pass into the quantum theory. The implementation of RSSQ, however, necessitates that one is able to fully solve the constraints classically, something that is very difficult if not impossible for interesting physical systems (e.g. general relativity). So for such systems not even the first step in the quantisation procedure can be implemented, leaving us in a complete blank about the properties of various quantum mechanical kinematical variables.

Moreover, the reduced state space of field systems characterised by first-class constraints generically consists of non-local variables: the true degrees of freedom are not fields themselves. For this reason the spacetime character of the theory is not explicitly manifest in the reduced state space. In a reduced state space quantisation the quantum mechanical variables one constructs are not fields. Furthermore, there exist quantities of potential interest quantum mechanically that do not commute with the constraints—such as the area and volume operators in loop quantum gravity [3]. There is no way to describe such objects in the RSSQ scheme.

In many systems of interest, such as General Relativity, the reduced state space is not a manifold but an orbifold: a singular structure arises as a result
of taking the equivalence classes with respect to the action of the constraints. This creates additional technical problems in the quantisation of such systems.

Dirac quantisation on the other hand involves the construction a kinematical Hilbert space $H$, in which the basic observables of $\Gamma$ are represented by self-adjoint operators. One then constructs self-adjoint operators $\hat{f}$ to represent the constraints. The physical Hilbert space $H_{\text{phys}}$ is the subspace of $H$ consisting of all vectors $|\psi\rangle \in H$, such that $\hat{f}|\psi\rangle = 0$. Typically variations of this theme are employed for the construction of the physical Hilbert space, because constraint operators do not have continuous spectrum around zero. Hence ‘wave functions’ that solve the quantum constraints are not square-integrable. This issue together with the normal-ordering ambiguities associated with the definition of constraint operators are the major technical problems of Dirac quantisation.

The method we propose here solves the constraints at the classical level, but it does allow for the existence of a kinematical Hilbert space (like Dirac quantisation) and of a map between that Hilbert space and the one corresponding to the physical degrees of freedom.

3 Geometric reduction and coherent state path integrals

3.1 Coherent states and path-integrals

Our prescription for RSSQ is based upon Klauder’s scheme of coherent state quantisation. The Hilbert space $H$ associated to a classical symplectic manifold $\Gamma$ with symplectic form $\Omega$ is a suitable subspace of the space $L^2(\Gamma, dz)$, where $dz$ is a shorthand for the Liouville form $\Omega \times \ldots \times \Omega$ on $\Gamma$. This subspace corresponds to a projection operator $E$ on $L^2(\Gamma, dz)$, whose matrix elements define a complex-valued function $K(z, z')$, $z, z' \in \Gamma$, such that

$$\int dz' K(z, z')K(z', z'') = K(z, z''),$$

$$K(z, z') = K^*(z', z).$$

Moreover, the Hilbert space $H$ is equipped with a family of generalised coherent states $|z\rangle$, $z \in \Gamma$, such that

$$\langle z | z' \rangle = K(z, z').$$

Conversely, given any complex-valued function on $\Gamma \times \Gamma$ that satisfies and is positive definite, one may construct uniquely a Hilbert space $H$, equipped with a family of generalised coherent states $|z\rangle$, such that holds–see [1] for a detailed exposition. Positivity in this context is defined as the requirement that

$$\sum_{i, i'} c_i c_{i'}^* K(z_i, z_{i'}) \geq 0,$$
for any finite set of complex number $c_l$ and state space points $z_l$, and where equality in (3.4) holds only for $c_l = 0$.

It is important to note that two kernels $K_1$ and $K_2$ related by a phase transformation

$$K_1(z, z') = e^{i\theta(z) - i\theta(z')}K_2(z, z'),$$

(3.5)

define the same Hilbert space, since the corresponding family of coherent states differ only with respect to the phase $e^{i\theta}$.

The overlap kernel may be computed by means of path-integrals, in a procedure developed by Klauder. The key ingredient is a Riemannian metric $ds^2$ on the classical phase space. This metric is employed for the purpose of regularisation of the path-integral expressions, which cannot otherwise be defined.

The metric $ds^2$ defines a Wiener measure on the space of paths $\Pi$, which consists of suitable continuous functions (in general non-differentiable) from a subset of $\mathbb{R}$ to $\Gamma$.

$$d\mu^\nu[z(\cdot)] = Dz^\nu(\cdot)e^{-\frac{1}{2\nu}\int_0^T d\lambda(\frac{d\lambda}{\nu})^2}d\lambda,$$

(3.6)

where $\nu$ is a diffusion coefficient for the Wiener process, which plays the role of a regularisation parameter.

The overlap kernel is then obtained as

$$\langle zf | z_i \rangle = \lim_{\nu \to \infty} N_\nu \int d\mu^\nu[z(\cdot)]e^{i\int A[z(\cdot)]}.$$

(3.7)

In this expression the summation is over paths $z(\cdot)$, such that $z(0) = z_i$ and $z(T) = z_f$. The Wiener measure is conditioned on these boundary values. The object $A_a$ is a $U(1)$ connection, which is related to the symplectic form of the manifold $\Gamma$ by means of the relation $\Omega = dA$. In the coordinates that $\Omega = dp_a \wedge dq^a$, the connection $A$ reads $A = p_a dq^a$. The parameter $N_\nu$ is a normalisation constant, which may be fixed by the condition that $\langle z|z \rangle = 1$.

The propagator has also a path integral expression

$$\langle zf | e^{-iHT} | z_i \rangle = \lim_{\nu \to \infty} N_\nu \int d\mu^\nu[z(\cdot)]e^{i\int A_a(z)dz^a - i\int_0^T d\lambda h(z)},$$

(3.8)

where $h(z)$ is the classical Hamiltonian function corresponding to the Hamiltonian operator $H$.

Klauder extended this procedure to deal with systems with constraints (either first-class or second-class) \[4, 5\]. His scheme involved a generalisation of the Dirac method – he showed that the projection operator into the zero eigenspace of the constraint operators may be expressed as a path-integral over the space of paths on $\Gamma$.

In this paper we suggest that it is not necessary to define the constraints at the quantum level. Rather the reduction may be implemented at the classical level, in a way that allows the definition of integral of the form (3.7), with paths on the reduced state space.
3.2 Path-integrals on the reduced state space

Our proposal starts from the remark that the path-integral (3.7) may be employed to also define coherent states on the reduced state space $\Gamma_{\text{red}}$. The symplectic potential $A$ on the reduced state space is easily obtained, since the reduced state space $\Gamma_{\text{red}}$ is also a symplectic manifold. The problem is to identify the correct metric $ds^2_{\text{red}}$ on $\Gamma_{\text{red}}$, through which to define the path-integral.

In unconstrained systems the metric is chosen by symmetry requirements. It is usually a homogeneous metric, whose isometries corresponding to the transitive action of a group on the state space. For classical mechanics on $\mathbb{R}^{2n}$, the corresponding group is the Weyl group, for the sphere $S^2$ that corresponds to a spin-system it is the group $SU(2)$ etc. This group—referred to as the canonical group—provides a symmetry of the kinematical description [6]. Conversely, the requirement of symmetry in the kinematical description allows one to select a homogeneous Riemannian metric to define the path-integral.

But a symmetry of the kinematical description does not, in general, leave the dynamics or the constraints invariant. (We know in standard quantum mechanics that no physically interesting Hamiltonian commutes with both position and momentum that form the generators of the Weyl group.) The metric $ds^2_{\text{red}}$ on $\Gamma_{\text{red}}$ therefore cannot be easily identified from the 'symmetries' of the original theory. Still $ds^2_{\text{red}}$ may be computed, if one has chosen a Riemannian metric $ds^2$ on the unconstrained state space $\Gamma$, which can itself be chosen by invoking symmetry arguments.

A metric $ds^2$ on $\Gamma$ defines by restriction a metric on the constraint surface $C$—hence a distance function $d(z, z')$ between points $z, z' \in C$. By definition, a point $\zeta$ of $\Gamma_{\text{red}}$ is a collection of points on $C$ generated by the action of the constraint functions. Hence a point $\zeta$ is identified with a submanifold of $C$. Moreover the submanifolds corresponding to different points of the reduced state space are disjoint, as different equivalence classes are always disjoint. This implies that one may exploit the distance function on $C$ to define a distance function on $\Gamma_{\text{red}}$, namely

$$d_{\text{red}}(\zeta, \zeta') = \inf_{z \in \zeta, z' \in \zeta'} d(z, z').$$  \hspace{1cm} (3.9)

From $d_{\text{red}}$ one may define a metric on $\Gamma_{\text{red}}$.

To summarise, given a metric on $\Gamma$—which defines a quantisation of the system prior to the imposition of the constraints— one may construct uniquely a metric on $\Gamma_{\text{red}}$, through which one may construct the Hilbert space for the true degrees of freedom.

There is a point of ambiguity in the present construction. The definition of distance in the reduced state space Eq. (3.9) involves the infimum of the distances between the points of these orbits. The infimum involves taking a limit, and it may turn out that the distance between specific orbits $\zeta$ and $\zeta'$ vanishes—even though these orbits are disjoint. In that case one has to identify the points $\zeta$ and $\zeta'$ in the reduced state space—the coherent states cannot distinguish between them. In many cases this point turns out to be a benefit, rather
than a disadvantage. We shall later show that it is usually in singular points (ones that render $\Gamma_{red}$ into an orbifold) that the distance function vanishes.

There is a physical interpretation of the reduction procedure we propose, which has to do with the uncertainty principle. A Riemannian metric on state space may be employed for a geometric expression of the uncertainty principle. Two points on the classical state space may be distinguished quantum mechanically only if their distance $\delta s^2$ is greater than one (in units such that $\hbar = 1$). The presence of a first-class constraint implies that points in the same orbit cannot be physically distinguished. To see whether one can distinguish between two orbits, the distance between any two points of each orbit should be larger than one (in terms of the original metric). This suggests the definition of the statistical distance between orbits—and consequently the metric on $\Gamma_{red}$—as the minimum of distance between its points.

### 3.3 Relation to Dirac quantisation

An important feature of the construction described above is that it preserves some of the advantages of Dirac quantisation, namely the existence of quantum operators for variables that do not commute with the constraints. Indeed, one may employ the metric $ds^2$ on $\Gamma$ to construct a kinematical Hilbert space $H$, spanned by a coherent states family $|z\rangle$, $z \in \Gamma$. We may consider then the subspace of $H$, spanned by all finite linear combinations $\sum_{l=1}^{n} c_l |z_l\rangle$, where the points $z_l \in C$. On the other hand, the physical Hilbert space $H_{phys}$ is constructed from the metric $ds^2_{red}$ and is spanned by coherent states $|\zeta\rangle$. The natural projection map $\pi : C \to \Gamma_{red}$ defined as $\pi(z) = [z]$, induces a map $i_\pi$ between $H_C$ and $H_{phys}$

$$i_\pi|z\rangle = ||z||,$$  \hspace{1cm} (3.10)

which by linearity can be extended to the whole of $H_C$. Moreover, if we denote by $E$ the projection map from $H$ to $H_C$, any vector $|\psi\rangle$ on $H$ may be mapped to a vector on $H_{phys}$, by mapping its projection $P|\psi\rangle$ on $H_C$ to $H_{phys}$ through the map $i_\pi$. The map $i_\pi$ is then extended to a map between the kinematical Hilbert space and the physical Hilbert space. It is not a projection operator in general (or even a self-adjoint operator), as the inner products on $H$ and on $H_C$ cannot be simply related.\footnote{Note that even in conventional Dirac quantisation the mapping from $H$ and $H_{phys}$ is not implemented by a self-adjoint operator, if the constraints do not have a continuous spectrum at 0.} Still $i_\pi$ may be employed to distinguish the 'gauge invariant part' of any operator in $H$, namely the part of the operator that can be 'projected' to the physical Hilbert space. Thus we obtain some of the benefits of Dirac quantisation, without having to deal with issues such as operator ordering or the existence of a continuous spectrum for the constraints near zero.

The map between $H$ and $H_{phys}$ is hardly unique—the physical predictions of the theory remain the same, if one multiplies the right-hand-side of (3.10)
with a \( z \)-dependent phase \( e^{i\theta(z)} \). Each choice of a function \( e^{i\theta(z)} \) leads to a different linear map. But this arbitrariness is not problematic, because in any case the definition of \( H \) in most physical systems contains a large degree of arbitrariness—especially in field theories.

In a nutshell, our procedure proposes a way to pass from the symplectic form \( \Omega \) and Riemannian metric \( ds^2 \) on \( \Gamma \) to the symplectic form \( \Omega_{\text{red}} \) and Riemannian metric \( ds^2_{\text{red}} \) on \( \Gamma_{\text{red}} \). The knowledge of the symplectic form and the metric defines uniquely a Hilbert space equipped with a family of coherent states. We may then construct on such Hilbert space \( H \) for \( \Gamma \) constructed via path integrals from \( \Omega \) and \( ds^2 \) and one Hilbert space \( H_{\text{phys}} \) for \( \Gamma_{\text{red}} \) constructed from \( \Omega_{\text{red}} \) and \( ds^2_{\text{red}} \). The reduction procedure that takes us from \( (\Omega, ds^2) \) to \( (\Omega_{\text{red}}, ds^2_{\text{red}}) \) defines a map from \( H \) to \( H_{\text{phys}} \), in a way that mirrors the projection map from \( H \) to \( H_{\text{phys}} \) appearing in Dirac quantisation.

### 3.4 Relation with path-integrals on \( \Gamma \)

An important point that provides a clarification of our proposal is that the path integral over the reduced state space may be equivalently rewritten as one over paths on \( \Gamma \), similar (but not identical) to the one employed by Klauder in the Dirac quantisation scheme through the coherent-state-path-integral.

If we define the Wiener measure \( d\tilde{\mu} \) over paths in \( \Gamma_{\text{red}} \) through the metric \( ds^2_{\text{red}} \) and denote as \( \tilde{A} \) a \( U(1) \) connection on \( \Gamma_{\text{red}} \) corresponding to \( \Omega_{\text{red}} \), one writes the path-integral (3.7) on the reduced state space as

\[
\langle \zeta_f | \zeta_i \rangle = \lim_{\nu \to \infty} N_\nu \int d\tilde{\mu}_\nu[\zeta(\cdot)] e^{if \int \tilde{A}[\zeta(\cdot)]}. \tag{3.11}
\]

Assuming that \( \Gamma \) is a \( 2n \)-dimensional manifold, and that we have \( m < n \) first-class constraints, the constraint surface \( C \) is a \((2n - m)\) dimensional manifold, and \( \Gamma_{\text{red}} \) has dimension equal to \( 2(n - m) \). If the functions \( \zeta^i, i = 1, \ldots, 2(n - m) \) define a coordinate system on \( \Gamma_{\text{red}} \), their pull-back on \( C \) define a coordinate system on \( C \) together with \( m \) coordinates \( v^i, i = 1, \ldots, m \) that span each gauge orbit. The symplectic form on \( C \) does not depend on the coordinates \( \lambda^i \)—it is written as

\[
\Omega|_C = \Omega_{ij}(\zeta) d\zeta^i \wedge d\zeta^j \tag{3.12}
\]

If \( \tilde{A}^i \) is a \( U(1) \) connection on \( \Gamma_{\text{red}} \) that satisfies \( d\tilde{A} = \Omega \), the most general connection one-form on \( C \) that satisfies \( dA = \Omega|_C \) may be written locally as

\[
\tilde{A}_i(\zeta) d\zeta^i + d\theta(\zeta, v), \tag{3.13}
\]

in terms of a scalar function \( \theta \) on \( C \).

Let us now consider an integral over paths on \( C \)

\[
\int d\mu^\nu_C[z(\cdot)] e^{if \int A[z(\cdot)]} \tag{3.14}
\]
Let us, for the moment, refrain from specifying the metric on $C$ that determines the Wiener measure $d\mu_{\nu}[z(\cdot)]$, except for the fact that it is conditioned on the endpoints. The exponent in the path integral reads $i \int \tilde{A} dz^a + i(\theta(z_f, v_f) - \theta(z_i, v_i))$. The path integral then reads

$$e^{i\theta(z_f, v_f) - i\theta(z_i, v_i)} \int d\mu_{\nu}^{\ast}[z(\cdot)] e^{i \int \tilde{A} dz^a}, \quad (3.15)$$

since the integration measure is conditioned at the endpoints. The phases in front of the path-integral may be reabsorbed in a phase change of the coherent states, and may therefore be dropped out.

If we assume that the integration measure factorises into a piece along the orbits ($\mu_{\text{gauge}}$) and one across the orbits ($\mu_{\text{red}}$)

$$\int d\mu_{\nu}^{\ast}[z(\cdot)] = \int d\mu_{\text{red}}^{\ast}[\xi(\cdot)] \int d\mu_{\text{gauge}}^{\ast}[v(\cdot)] \quad (3.16)$$

then we see that

$$\int d\mu_{\nu}^{\ast}[z(\cdot)] e^{i \int \tilde{A} dz^a} = \int d\mu_{\text{red}}^{\ast}[\xi(\cdot)] e^{i \int \tilde{A} dz^a} \int d\mu_{\text{gauge}}^{\ast}[v(\cdot)] \quad (3.17)$$

If the two measures are separately normalised, the overlap kernel on the reduced phase space may be obtained by a path integral on the constraint surface,

$$\langle \xi_f | \xi_i \rangle = \lim_{\nu \to \infty} N_{\nu} \int d\hat{\nu}_{\nu}^{\ast}[z(\cdot)] e^{i \int \tilde{A} dz^a}, \quad (3.18)$$

The crucial point is that the measure should satisfy the factorisation condition

$$\int d\mu_{\nu}[z(\cdot)] = \prod_a \delta(f^a(z)) \quad (3.19)$$

which involves a suitable choice of metric on $C$. Recall that the metric is a bilinear functional on the tangent space $T_z C$ of the constraint surface. The degeneracy of the symplectic form $\Omega$ implies that each tangent space is split into the degeneracy subspace $D_z C$ of all tangent vectors $X$ such that $\Omega_C(X, \cdot) = 0$ and its complement $\bar{D}_z C$, which may be naturally identified with $T_{\xi} \Gamma_{\text{red}}$. Since the Wiener measure is of the form (3.16), the factorisation condition may be obtained if the metric is in a block-diagonal form with respect to the split $T_z C = D_z C \oplus \bar{D}_z C$. In other words, $g(X, Y) = 0$, if $X \in D_z C$ and $Y \in \bar{D}_z C$. Moreover, it is necessary that the restriction of the metric in $\bar{D}_z C$ does not depend on the variables $v$. The Hamiltonian vector fields generated by the constraints should leave the restriction of the metric on $\bar{D}_z C$ invariant.

The constraint surface $C$ is defined as the submanifold of $\Gamma$, in which the constraint functions $f^a, a = 1, \ldots, m$ vanish. Any ordinary integral over $C$, may be expressed as an integral over $\Gamma$ by the inclusion of the product of delta functions $\prod_a \delta(f^a(z))$. Similarly, we may turn the path integral on $C$ into a path integral over $\Gamma$ by the insertion of delta function at each time point of the discretisation.

We then write

$$\langle \xi_f | \xi_i \rangle = \lim_{\nu \to \infty} N_{\nu} \int d\hat{\nu}_{\nu}[z(\cdot)] \prod_a \Delta[f^a] e^{i \int \tilde{A} dz^a}, \quad (3.19)$$
where $\Delta[f^a] = \prod_t \delta(f^a(z(t)))$; the product refers to any discretisation of the paths in the path integral. The integration measure may be defined by any metric $ds^2$ on $\Gamma$, which reduces to a factorisable metric on $C$. Similarly, the connection on $\Gamma$ may be chosen arbitrarily as long as it satisfies $dA = \Omega$.

Exploiting the representation of the delta function as an integral $\delta(f) = \int dN e^{-iNF}$ we write formally

$$\Delta[f^a] = \int D N(\cdot) e^{-i\int N_\alpha(t)f^\alpha(t)}, \quad (3.20)$$

where $DN(\cdot)$ is a continuous limit of $\prod_t dN_t^\alpha$. We then write the formal expression

$$\langle \zeta f | \zeta i \rangle = \lim_{\nu \to \infty} N_\nu \int D N(\cdot) \int d\mu[z(\cdot)] e^{\int A_\alpha dz^\alpha - \int_0^T ds N_\alpha(s)f^\alpha(s)} \quad (3.21)$$

This expression is formally similar to the one employed by Klauder in his implementation of Dirac quantisation through the coherent state path-integral \cite{4}. The key difference is that the Wiener process in Klauder’s scheme is defined by means of a homogeneous metric on $\Gamma$, while in the RPSQ scheme the metric on $\Gamma$ is in general non-homogeneous and needs to satisfy the factorisation condition on the constraint surface we described earlier\footnote{Moreover, in Klauder’s version of Dirac quantisation the measure over $N(\cdot)$ is normalised to one, while here it is only a formal continuum limit of $\prod_t dN_t$.}. The difference in the metrics implies that the diffusion processes regularising the path-integral are different, hence the final expressions for the overlap kernels. Since, however, the metric only appears for regularisation purposes, and both the constraints and the symplectic form are the same, the two methods are expected to yield the same results at the semi-classical level.

4 Examples

4.1 A trivial example

We may consider a particle moving on the Euclidean three-space $\mathbb{R}^3$. The state space $\Gamma = \mathbb{R}^6$ is spanned by the coordinates $(x^i, p_i), i = 1, 2, 3$ and is equipped with the natural symplectic form. We then consider the constraint $x^3 = 0$, which can be trivially shown to imply that $\Gamma_{\text{red}} = \mathbb{R}^4$, spanned by the variables $(x_1, x_2, p_1, p_2)$.

The coherent states on $\Gamma$ are the standard Gaussian coherent states, with overlap kernel

$$\langle x, p | x', p' \rangle = \exp \left( ip_i x'^i - ip'_i x^i - \frac{1}{2} |x - x'|^2 - \frac{1}{2} |p - p'|^2 \right) \quad (4.1)$$

which correspond to the homogeneous Riemannian metric on $\mathbb{R}^6$:

$$ds^2 = \delta_{ij} dx^i dx^j + \delta^{ij} dp_i dp_j \quad (4.2)$$
Since the orbits of the constraint surface are the lines of constant \((x, y, p_x, p_y)\) with \(z = 0\), it is elementary to show that the distance between the lines corresponds to the reduced metric

\[
    ds^2_{\text{red}} = dx_1^2 + dx_2^2 + dp_1^2 + dp_2^2.
\]

which again corresponds to Gaussian coherent states for the reduced Hilbert space.

### 4.2 A spin system

We consider the state space \(\mathbb{R}^4\), with variables \((x_1, x_2, p_1, p_2)\), equipped with the standard symplectic form

\[
    \omega = dp_1 \wedge dx_1 + dp_2 \wedge dx_2
\]

and a constraint

\[
    \frac{1}{2}(p_1^2 + p_2^2 + x_1^2 + x_2^2) = k > 0.
\]

The constraint surface is the two sphere \(S^3\). Employing the coordinates \((\theta, \phi, \chi)\) on \(S^3\) through the definition

\[
    \frac{1}{\sqrt{2}}(x_1 - ip_1) = \sqrt{k} \cos \frac{\theta}{2} e^{i(\phi + \chi) / 2}
\]

\[
    \frac{1}{\sqrt{2}}(x_2 - ip_2) = \sqrt{k} \sin \frac{\theta}{2} e^{i(\phi - \chi) / 2},
\]

we obtain

\[
    \Omega_C = \frac{k}{2} \sin \theta d\theta \wedge d\phi,
\]

which implies that the degenerate direction corresponds to the vector field \(\frac{\partial}{\partial \chi}\). Its orbits define the usual Hopf fibration of \(S^3\), hence the reduced state space is \(S^2\) equipped with the standard symplectic form. It is the state space of a classical spin system. As is well-known, the single-valuedness of the \(U(1)\) connection \(\cos \theta d\phi\) that appears in the path-integral, implies that in the quantum theory \(k = 2n\), for integers \(n\).

The homogeneous metric on \(\mathbb{R}^4\) corresponding to Gaussian coherent states is

\[
    ds^2 = dx_1^2 + dx_2^2 + dp_1^2 + dp_2^2,
\]

reduces on the constraint surface \(S^3\) to

\[
    ds^2_{\text{red}} = \frac{k}{2}(d\theta^2 + d\phi^2 + d\chi^2 - 2 \cos \theta d\phi d\chi).
\]
It is easy to minimise this metric over the orbits of constant $(\theta, \phi)$ to obtain the Riemannian metric on $S^2$.

$$ds^2_{\text{red}} = \frac{k}{2}(d\theta^2 + \sin^2 \theta d\phi^2).$$

(4.11)

This is the *homogeneous* metric for a sphere of radius $k/2 = n$; for this metric and this connection one may obtain through the path-integral (3.7) the quantum description of spin in terms of the irreducible representations of $SU(2)$ for each value of $n$.

### 4.3 Spinless relativistic particle

The relativistic particle illustrate the strengths of the method, perhaps better than any other example. The reason is that the Poincaré symmetry of the system leaves few alternatives about the form of the Riemannian metric on the constraint surface. The resulting coherent states are therefore fully covariant.

The spinless relativistic particle of mass is described by the presymplectic manifold $C$ (the constraint surface), which is spanned by a unit, future-directed, timelike vector $I$ and an element $X$ of Minkowski spacetime. The topology of $C$ is then $V \times \mathbb{R}^4$, where $V$ the mass-shell for particles of mass $m$. The symplectic form then reads

$$\Omega_C = -mdI_\mu \wedge dX^\mu.$$  

(4.12)

Note that we employ a $(+---)$ signature convention. It is easy to see that the vector field $\frac{\partial}{\partial X^\mu}$ corresponds to the null direction of $\Omega_C$—it is the Hamiltonian vector field generated by the constraint $I^2 - 1 = 0$. The parameter $u = I \cdot X$ is the gauge degree of freedom along the orbits of the constraint, while the variables $I_\mu, Y^\mu = X^\mu - u I^\mu$ are projected on the reduced state space $\Gamma_{\text{red}}$. The reduced symplectic form reads

$$\Omega_{\text{red}} = -mdI_\mu \wedge dY^\mu.$$  

(4.13)

We next identify a Lorentz-invariant Riemannian metric on $C$. For this purpose we need to identify a Lorentz-invariant notion of distance between pairs $(I, X)$ and $(I', X')$. On the mass-shell $V$ there exists a natural Riemannian metric

$$ds_V^2 = -dI^\mu dI_\mu \geq 0,$$  

(4.14)

which may be employed to define distance between $I$ and $I'$. To define a (positive definite) distance between $X$ and $X'$, let us note that if $I = I'$, then $2I_\mu I_\nu + \eta_{\mu\nu}$ defines a Riemannian metric on $\mathbb{R}^4$, and the corresponding distance between $X$ and $X'$ equals

$$[2I_\mu I_\nu - \eta_{\mu\nu}](X^\mu - X'^\mu)(X^\nu - X'^\nu).$$  

(4.15)

If $I \neq I'$, we need to boost $X'$ to the frame where $I = I'$ and then employ expression (4.14) for distance. If we denote by $\Lambda$ the unique Lorentz transformation (boost) that takes $I$ into $I'$, we may define the distance function

$$[2I_\mu I_\nu - \eta_{\mu\nu}](X^\mu - \Lambda^\mu_\rho X'^\rho)(X^\nu - \Lambda^\nu_\sigma X'^\sigma).$$  

(4.16)
The construction above suggests the following definition of a Lorentz invariant metric on $C$

$$ds_C^2 = -c_1 dI^\mu dI_\mu + \frac{c_2}{m}[2I_\mu I_\nu - \eta_{\mu\nu}](dX^\mu - I^\mu X \cdot dI - dI^\mu I \cdot X) \times (dX^\nu - I^\nu X \cdot dI - dI^\nu I \cdot X), \quad (4. 17)$$

where $c_1, c_2$ are arbitrary positive numbers and the mass $m$ appears in the denominator to make the metric unit dimensionless ($\hbar = c = 1$). We shall choose for convenience $c_1 = c_2 = \frac{1}{2}$.

To construct a metric on $\Pi_{\text{red}}$, we write Eq. (4. 17) in terms of the coordinate $u$, together with the physical variables $I^\mu, I^\nu$

$$ds_C^2 = -\frac{1}{2} dI^\mu dI_\mu + \frac{m^2}{2}[I_\mu I_\nu - \eta_{\mu\nu}]dY^\mu dY^\nu + \frac{m^2}{2} du^2 \quad (4. 18)$$

It is then easy to find the distance between neighboring orbits, by minimising over $du$, thus obtaining a Lorentz-invariant metric on the reduced state space

$$ds_{\text{red}}^2 = -\frac{1}{2} dI^\mu dI_\mu + m^2[I_\mu I_\nu - \eta_{\mu\nu}]dY^\mu dY^\nu. \quad (4. 19)$$

**Particle in 2d** For simplicity we next consider the case of a particle in two-dimensional Minkowski spacetime. In terms of $I = I^1$ and $q = I^0 X^1 - I^1 X^0$, we may write $\Omega_{\text{red}}$ as

$$\Omega_{\text{red}} = md\xi \wedge dq, \quad (4. 20)$$

where

$$\xi = \int^1 dx \frac{dx}{\sqrt{1 + x^2}} = \log(1 + \sqrt{1 + I^2}). \quad (4. 21)$$

The coordinates $\xi$ and $q$ are global on $\Gamma_{\text{red}}$. Moreover, they define a set of Cartesian coordinates for the metric (4. 19), since

$$ds_{\text{red}}^2 = \frac{1}{2} d\xi^2 + \frac{1}{2m^2} dq^2 \quad (4. 22)$$

The metric and symplectic form on $\Gamma_{\text{red}} = \mathbb{R}^2$ correspond to that of the standard Gaussian coherent states. The path-integral then leads to the overlap kernel

$$\langle \xi, q | \xi', q' \rangle = \exp \left( im\xi' q - im\xi q' - \frac{1}{2}(\xi - \xi')^2 - \frac{m^2}{2}(q - q')^2 \right) \quad (4. 23)$$

Even though the definition of the parameters $\xi$ and $q$ involved the choice of a coordinate system, the coherent states constructed from the kernel (4. 23) are not. The reason is that $q$ has a covariant definition as $q = \epsilon_{\mu\nu} I^\mu X^\nu$, in terms of the alternating tensor $\epsilon_{\mu\nu}$ in two dimensions, while $\xi \rightarrow \xi + c$ under a Lorentz boost in two dimensions, leaving the kernel (4. 23) invariant up to a change of phase.
We may pull-back the kernel $\langle \xi, q | \xi', q' \rangle$ on the constraint surface and write it in terms of $x = X^1$, $t = X^0$ and $p = mI^1$, which are the natural variables in the canonical description of time evolution. We obtain

$$
\langle x, p, t | x', p', t' \rangle = \exp \left( \frac{i}{m} p' (\sqrt{m^2 + p^2} x - pt) - \frac{i}{m} p (\sqrt{m^2 + p'^2} x' - p't') \right) - \frac{1}{2} \left[ \log \frac{p + \sqrt{m^2 + p^2}}{p' + \sqrt{m^2 + p'^2}} \right]^2 - \frac{1}{2} \left( \sqrt{m^2 + p^2} x - pt - \sqrt{m^2 + p'^2} x' + p't' \right)^2
$$

(4.24)

In the non-relativistic limit $p << m$, the overlap kernel reduces to

$$
\langle x, p, t | x', p', t' \rangle = \exp \left( ip' (x - p/mt) - ip (x' - p'/mt') \right) - \frac{1}{2m^2} (p - p')^2 - \frac{m^2}{2} \left( x - p/mt - x' + p'/mt' \right)^2
$$

(4.25)

For $t = t' = 0$ this kernel defines a Gaussian family of coherent states of the Weyl group, typical in the description of non-relativistic particles. However, for $t \neq t'$, $\langle x, p, t | x', p', t' \rangle$ does not correspond to matrix elements $\langle x, p | e^{-iH(t-t')} | x', p' \rangle$ of any self-adjoint Hamiltonian $\hat{H}$.

The coherent states (4.23) are instantaneous: they are properly defined only on the reduced state space, in which the Hamiltonian vanishes due to constraints. For this reason the time-dependence in the arguments of (4.24) reflects solely the relation of the parameters on the reduced state space to a coordinate that is usually considered to play the role of time—see [9, 10] for an interpretation. The coherent states are covariant under the Poincaré group; for any element $g$ of the Poincaré group, one may define the unitary operator $\hat{U}(g)$ as

$$
\hat{U}(g) | z \rangle = | g \cdot z \rangle,
$$

(4.26)

where $z \in \Gamma_{\text{red}}$ and $g \cdot z$ denotes the symplectic action of the Poincaré group on $\Gamma_{\text{red}}$. For the one parameter subgroup of time-translations (the 0-direction) in particular, we obtain the transformation

$$
| \xi, q \rangle \rightarrow | \xi, q + \sin \xi s \rangle,
$$

(4.27)

which can be easily checked to be unitary. The matrix elements of the Hamiltonian in the coherent state basis have no relation to the pull-back of the coherent state overlap kernel on $C$, Eq. (4.24). The parameter $t = X^0$ cannot be identified with the time parameter of Schrödinger’s equation, a fact that is also responsible for the non-definability of a covariant position operator for relativistic particles [7].

Minisuperspace models Minisuperspace models are quantum cosmological models characterised by a canonical state space $\mathbf{R}^{2n}$ with configuration variables $q^a$ and conjugate momenta $p_a$ and a by constraints of the form

$$
\frac{1}{2} g_{ab}(q) p^a p^b + V(q) = 0,
$$

(4.28)
in terms of a metric $g$ on the configuration space with signature $-++\ldots+$. In many aspects their structure is similar to that of the relativistic particle. There exist, however, no spacetime covariance argument to determine a form of the metric. We have applied the present method of quantisation to minisuperspace models in Ref. [11], where we studied in detail a Robertson-Walker universe with a scalar field.

4.4 Divergent points and orbifold structure

In many systems with first-class constraints, there may exist exceptional points of the reduced state space that correspond to orbits of reduced dimensionality than the generic orbit. This is usually the case, when specific subsets of the constraint surface are invariant under some (or even all) the first-class constraints. The symplectic structure on such points is typically divergent, with the result that the reduced state space is not well defined as a classical symplectic manifold. This is, in fact, one of the reasons that Dirac quantisation is preferred over RSSQ—one does not have to deal with such singular orbits in the Dirac method.

We argue here that such divergent points do not pause any problem in the quantisation scheme through coherent states that is proposed here. The orbits corresponding to singular points in the reduced state space are absorbed in a redefinition of the reduced state space that is imposed by the consideration of the Riemannian structure.

We shall consider for simplicity the case of a system with a single first-class constraint. Each orbit of the constraint surface $C$ is characterised by coordinates $\zeta^i$ that are constant along the orbit—and thus project on the reduced state space—and a single gauge coordinate $\lambda$ distinguishing points along the orbit. Let us assume that there exist a specific point on $C$, determined by the coordinates $(\zeta_0^i, \lambda_0)$, that is left invariant under the action of the constraint. The corresponding orbit $\gamma_0$ consists then only of the single point $(\zeta_0^i, \lambda_0)$, while a generic orbit is characterised by varying $\lambda$, namely it corresponds to a line on $C$. There exist other orbits $\gamma_0'$ characterised by the same parameters $\zeta_0^i$ with $\gamma_0$: they correspond to values of $\lambda > \lambda_0$ and $\lambda < \lambda_0$. There will be two orbits if $\lambda$ runs to the full real axis and only one orbit if $\lambda$ is a periodic variable.

To compute the metric on the reduced state space, one has to calculate the distance function between different orbits. The determination of the distance between $\gamma_0$ and the other orbits $\gamma_0'$ of the same value of $\zeta^i$, is the infimum of the distance of the points of $\gamma_0'$ from the point $(\zeta^i, \lambda_0)$. The infimum will be always achieved for $\lambda \to \lambda_0$, implying that the distance between $\gamma_0$ and $\gamma_0'$ will be equal to zero. Consequently, the Wiener process constructed from the metric on $\Gamma_{\text{red}}$ will fail to distinguish between $\gamma_0$ and $\gamma_0'$. The path-integration will, therefore, treat all orbits with the same value of $\zeta^i$ as a single point. The coherent states, therefore, will only be parameterised by the regular points of $\Gamma_{\text{red}}$. Recalling the relation between the state space metric and the uncertainty relation, one may say that the quantum uncertainties essentially wash out any divergences related to accidental symmetries of the classical constraint surface.

The argument we provided here is very general and immediately generalises
to systems with more than one constraint functions. For a detailed example (a Robertson-Walker minisuperspace model) the reader is referred to [11].

5 Conclusions

We have presented here a variation of the reduced-state-space quantisation procedure that is based on the coherent-state-path-integral quantisation, developed by Klauder. The key point of our construction is that a metric $ds^2_{\text{red}}$ on the reduced state space may be constructed purely geometrically from a metric on the unconstrained state space $\Gamma$. The metric $ds^2_{\text{red}}$ may then be employed for the definition of a path integral for the degrees of freedom in the reduced state space.

The advantages of this method are the following:

i. While the method incorporates the constraints at the classical level, it can be easily related to Dirac quantisation. The identification of a metric on the reduced state space is constructed uniquely from a metric on $\Gamma$ (or on the constraint surface $C$), serves to define a map from a kinematical Hilbert space (related to the degrees of freedom on $\Gamma$) to a physical Hilbert space that corresponds to the true degrees of freedom. One may therefore enjoy all benefits of Dirac quantisation, for example the existence of quantum operators for kinematical variables, without any of its drawbacks.

ii. The singular points of the reduced state space are ‘smeared out’ in the quantum theory, and do not appear as parameters in the coherent states.

iii. Finally, the method is purely geometrical; the basic ingredient is the identification of the distance between constraint orbits. As such it is particularly suitable for theories, in which the state space involves geometric or combinatorial objects: for example, approaches to canonical quantum gravity that involve discretisation.

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