DISSIPATIVE SOLUTIONS AND THE INCOMPRESSIBLE INVISCID LIMITS OF THE COMPRESSIBLE MAGNETOHYDRODYNAMIC SYSTEM IN UNBOUNDED DOMAINS

EDUARD FEIREISL
Institute of Mathematics of the Academy of Sciences of the Czech Republic
Zitná 25, 115 67 Praha 1, Czech Republic

ANTONIN NOVOTNY
IMATH, EA 2134, Université du Sud Toulon-Var
BP 132, 83957 La Garde, France

YONGZHONG SUN
Department of Mathematics, Nanjing University
Nanjing, Jiangsu 210093, China

Abstract. We consider the compressible Navier-Stokes system coupled with the Maxwell equations governing the time evolution of the magnetic field. We introduce a relative entropy functional along with the related concept of dissipative solution. As an application of the theory, we show that for small values of the Mach number and large Reynolds number, the global in time weak (dissipative) solutions converge to the ideal MHD system describing the motion of an incompressible, inviscid, and electrically conducting fluid. The proof is based on frequency localized Strichartz estimates for the Neumann Laplacean on unbounded domains.

1. Introduction. We consider the motion of a compressible, viscous, and conducting fluid confined to an unbounded domain \( \Omega \subset \mathbb{R}^3 \). The commonly accepted system of MHD equations, describing the time evolution of the mass density \( \varrho = \varrho(t,x) \), the fluid velocity \( \mathbf{u} = \mathbf{u}(t,x) \), and the magnetic field \( \mathbf{B} = \mathbf{B}(t,x) \) reads:

\[
\begin{align*}
\partial_t \varrho + \text{div}_x (\varrho \mathbf{u}) &= 0, \\
\partial_t (\varrho \mathbf{u}) + \text{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) &= \text{div}_x \left( \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\mu} \mathbf{B} \otimes \mathbf{B} - \frac{1}{2\mu} |\mathbf{B}|^2 \mathbf{I} \right), \\
\partial_t \mathbf{B} + \text{curl}_x (\mathbf{B} \times \mathbf{u}) + \text{curl}_x (\lambda \text{curl} \mathbf{B}) &= 0, \quad \text{div}_x \mathbf{B} = 0,
\end{align*}
\]

where the viscous stress \( \mathbb{S} \) is given by Newton’s law

\[
\mathbb{S}(\nabla_x \mathbf{u}) = \nu \left( \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \text{div}_x \mathbf{u} \mathbf{I} \right),
\]

and where the transport coefficients \( \mu, \nu, \) and \( \lambda \) are positive constants. The system is supplemented by the no-stick boundary conditions for the velocity field:

\[
\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega}, \quad \mathbf{n} \otimes \mathbf{n}|_{\partial \Omega} = 0,
\]

2010 Mathematics Subject Classification. Primary: 35Q30, 35Q60; Secondary: 35B25.

Key words and phrases. Compressible MHD system, inviscid limit, incompressible limit.
together with the assumption that $\partial \Omega$ is a perfect conductor, specifically,
\[
\mathbf{B} \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \text{curl}(\mathbf{B}) \times \mathbf{n}|_{\partial \Omega} = 0,
\]
where $\mathbf{n}$ denotes the outer normal vector to $\partial \Omega$. Moreover, since our model domain is unbounded, we prescribe the far field behavior:
\[
\varrho \to \varrho_0 > 0, \quad \mathbf{u}, \mathbf{B} \to 0 \text{ as } |x| \to \infty.
\]
We are interested in the inviscid, incompressible limit of the above system, meaning in the situation when
\[
p(\varrho) = p_\varepsilon(\varrho) = \frac{1}{\varepsilon^2} \varrho^\gamma, \quad \gamma = \frac{5}{3},
\]
and
\[
\varrho \to 0.
\]

Note that we have deliberately omitted the bulk viscosity component in (1.4), and, similarly, we fix $\gamma = \frac{5}{3}$ in (1.8), where both assumptions reflect the characteristic properties of a monoatomic gas relevant in the context of plasma motion. The methods we use adapt easily to the more general case
\[
\varrho(\nabla_x \mathbf{u}) = \nu_1 \left( \nabla_x \mathbf{u} + \nabla^T_x \mathbf{u} - \frac{2}{3} \text{div}_x \mathbf{u} \mathbf{I} \right) + \nu_2 \text{div}_x \mathbf{u} \mathbf{I}, \quad p_\varepsilon = \frac{1}{\varepsilon^2} \varrho^\gamma, \quad \gamma > 3/2.
\]

Our aim is to study the asymptotic behavior of solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon\}$ for $\varepsilon \to 0$. The same problem in the whole space case $\Omega = \mathbb{R}^N$ and in the periodic case, $N = 2, 3$ was studied by Jiang, Ju, and Li [21], [22]. In particular, they identified the limit problem - the ideal MHD system - in the form:
\[
\text{div}_x \mathbf{v} = 0,
\]
\[
\varrho (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v}) + \nabla_x \Pi = \text{div}_x \left( \frac{1}{\mu} \mathbf{H} \otimes \mathbf{H} - \frac{1}{2\mu} |\mathbf{H}|^2 \mathbf{I} \right),
\]
\[
\partial_t \mathbf{H} + \text{curl}(\mathbf{H} \times \mathbf{v}) = 0, \quad \text{div}_x \mathbf{H} = 0.
\]
In the presence of a physical boundary, the system (1.10 - 1.12) must be supplemented by the relevant boundary conditions
\[
\mathbf{v} \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \mathbf{H} \cdot \mathbf{n}|_{\partial \Omega} = 0,
\]
and, in accordance with (1.7),
\[
\mathbf{v}, \mathbf{H} \to 0 \text{ as } |x| \to \infty.
\]

Our main goal is to extend the result of Jiang et al. [22] to fluids confined to a physical domain with a boundary. Since our method is based on dispersive estimates eliminating the effect of acoustic waves, we need the domain $\Omega \subset \mathbb{R}^3$ to be unbounded. In contrast with the viscous incompressible limit, where the absence of proper eigenvalues of the acoustic generator is sufficient (and necessary) to provide the desired result (see [12]), the inviscid incompressible limit requires certain uniformity of the decay of acoustic waves, typically provided by the Strichartz estimates if $\Omega = \mathbb{R}^3$. The validity of Strichartz estimates imposes severe restrictions on the geometric properties of the domain. The crucial observation exploited in the present paper is that it is enough to show frequency localized Strichartz estimates (see Section 5) valid for a considerably larger class of domains. We believe that this observation may be of independent interest.
As a model example, we consider a perturbed half space in $R^3$, specifically,

$$
\begin{cases}
\Omega \subset R^3 \text{ a domain with smooth boundary,} \\
\Omega \cap \{ |x| > R \} = \{ (x_1, x_2, x_3) \mid x_3 > 0, \ |x| > R \}. 
\end{cases}
$$

(1.15)

The propagation of acoustic waves is described by the scaled wave equation

$$
\varepsilon \partial_t s + \rho \Delta \Phi = 0, \\varepsilon \partial_t \Phi + \overline{\alpha} \nabla_x s = 0, \ \nabla_x \Phi \cdot n|_{\partial \Omega} = 0, \ \overline{\alpha} = \frac{p'(\varrho)}{\varrho} = \frac{5}{3} \varrho^{2/3} > 0, \ \varepsilon > 0,
$$

(1.16)

where $\Phi$ is called  acoustic potential. Using the results of Edward and Pravica [9] on the spectral properties of the Neumann Laplacian $\Delta_N$, we show that

$$
\left\| G(-\Delta_N) \exp \left( \pm i \sqrt{-\Delta_N} t \right) [h] \right\|_{L^p(0,\infty; L^q(\Omega))} \leq \| h \|_{H^1(\Omega)}, \ \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty
$$

(1.17)

for any $G \in \mathcal{C}^\infty(0, \infty)$, which, scaled in time, yields the desired decay estimates for the acoustic potential $\Phi$, see Section 5. Note that the Neumann Laplacian $\Delta_N$ in a perturbed half-space admits, in general, a sequence of resonances in the complex plane approaching asymptotically the essential spectrum (see Edward and Pravica [9, Corollary 1]); whence (1.17) is probably optimal.

Our approach is based on the concept of suitable weak or dissipative solutions for the system (1.1 - 1.6) proposed, in the context of the Euler system by DiPerna and Lions, see [30]. The dissipative solutions, introduced for the barotropic Navier-Stokes system in [16] and for the full Navier-Stokes-Fourier system in [14], satisfy the so-called relative entropy inequality representing a suitable extension of the standard energy inequality to a class of suitable “test” functions. In particular, a dissipative solution coincides with a (hypothetical) strong solution as long as the latter exists, see [14]. The idea of exploiting the effect of mechanical energy dissipation imposed in continuum mechanics by the Second law of thermodynamics goes back to the seminal paper by Dafermos [4]. More recently, Berthelin and Vasseur [1], Desjardins [7], and Germain [19] adapted this approach to problems in fluid mechanics. Notably, Germain [19] proved the weak-strong uniqueness for the barotropic Navier-Stokes system in the class of “more regular” weak solutions.

The paper is organized as follows. In Section 2, we recall the standard definition of weak solutions to the compressible MHD system (1.1 - 1.7) and introduce the relative entropy, together with the related concept of dissipative solutions. We also show that any finite-energy weak solution to (1.1 - 1.7) is a dissipative solution, in particular, the property of weak-strong uniqueness holds for the compressible MHD system - a result that may be of independent interest, see Section 2, Theorem 2.1. With all the preliminary material at hand, we state our main result concerning the singular limit in Section 3. The rest of the paper is devoted to the proof of convergence of solutions of the primitive system (1.1 - 1.7) to the target system (1.10 - 1.14). Using the relative entropy inequality we establish the necessary uniform bounds on the family of solutions $\{ \varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon \}_{\varepsilon > 0}$ in Section 4. Section 5 is devoted to acoustic waves, in particular, we show the key estimate (1.17). The proof of convergence is completed in Section 6 by means of another application of the relative entropy inequality, where the “ansatz” is inspired by the pioneering paper of Masmoudi [32]. Possible extensions as well as limitations of the method are briefly discussed in Section 7.
To conclude this introduction, we point out that our basic framework is the theory of weak solutions and its application to singular limits in the spirit of the work by Lions and Masmoudi [31], see also the survey by Masmoudi [33] and the references cited therein. We also mention the papers by Hu and Wang [20], Kukučka [28], Kwon and Trivisa [29], where the method is applied to the incompressible limits for the MHD system. There is an alternative and more classical approach based on strong solutions proposed in the seminal paper by Klainerman and Majda [26] and later adopted and developed by many authors, the complete list of which goes beyond the scope of the present contribution. The interested reader may consult the surveys by Danchin [5], Gallagher [18], or Schochet [34].

2. Weak and dissipative solutions. We start with the standard definition of weak solutions to the compressible MHD system $(1.1 - 1.7)$, supplemented with the initial conditions:

$$\rho(0, \cdot) = \rho_0, \ (\rho\mathbf{u})(0, \cdot) = (\rho\mathbf{u})_0, \ \mathbf{B}(0, \cdot) = \mathbf{B}_0,$$

(2.1)

cf. [8].

- The functions $\rho, \mathbf{u}, \mathbf{B}$ belong to the class:

$$\rho \geq 0 \text{ in } (0, T) \times \Omega, \ (\rho - \overline{\rho}) \in C_{\text{weak}}([0, T]; (L^2 + L^{5/3})(\Omega)),$$

$$\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega; R^3)), \ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \sqrt{\rho}\mathbf{u} \in C_{\text{weak}}([0, T]; L^2(\Omega; R^3)),$$

$$\mathbf{B} \in C_{\text{weak}}([0, T]; L^2(\Omega; R^3)) \cap L^2(0, T; W^{1,2}(\Omega; R^3)),$$

$$\mathbf{B} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \operatorname{div}_x \mathbf{B} = 0 \text{ a.a. in } (0, T) \times \Omega.$$

- The equation of continuity (1.1) is replaced by a family of integral identities

$$\int_{\Omega} \left( \rho \varphi(\tau, \cdot) - \rho_0 \varphi(0, \cdot) \right) \, dx = \int_0^\tau \int_{\Omega} \left( \rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla_x \varphi \right) \, dx \, dt \quad (2.2)$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C^\infty_c([0, T] \times \overline{\Omega}).$

- The weak formulation of the momentum equation (1.2) reads

$$\int_{\Omega} \left( \rho \mathbf{u} \cdot \varphi(\tau, \cdot) - (\rho\mathbf{u})_0 \cdot \varphi(0, \cdot) \right) \, dx = \int_0^\tau \int_{\Omega} \left( \rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla_x \varphi + p(\rho) \operatorname{div}_x \varphi \right) \, dx \, dt \quad (2.3)$$

$$= \int_0^\tau \int_{\Omega} \left( \rho \mathbf{u} \cdot \partial_t \varphi + \rho (\mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mu} \nabla_x \varphi + \frac{1}{2\mu} |\mathbf{B}|^2 \operatorname{div}_x \varphi \right) \, dx \, dt$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C^\infty_c([0, T] \times \overline{\Omega}, R^3), \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0.$

- The Maxwell equation (1.3) is satisfied as a family of integral identities

$$\int_{\Omega} \left( \mathbf{B} \cdot \varphi(\tau, \cdot) - \mathbf{B}_0 \cdot \varphi(0, \cdot) \right) \, dx = \int_0^\tau \int_{\Omega} \left( \mathbf{B} \cdot \partial_t \varphi - (\mathbf{B} \times \mathbf{u} + \lambda \operatorname{curl} \mathbf{B}) \cdot \operatorname{curl} \varphi \right) \, dx \, dt \quad (2.4)$$

for any $\tau \in [0, T]$ and any test function in $C^\infty_c([0, T] \times \overline{\Omega}; R^3).$
2.1. Relative entropies and dissipative solutions. Developed in Lions [30] and [15], cf. also [8]. To the compressible MHD system (1.1 - 1.7), (2.1) can be shown by the methods introduced the relative entropy explicitly by means of (2.2 - 2.4), specifically, by taking inequality (2.5). On the other hand, the remaining integrals can be computed where the first integral on the right-hand side may be controlled by the energy inequality

\[ E(\varrho, \varpi) = P(\varrho) - P'(\varpi)(\varrho - \varpi) - P(\varpi), \quad P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, dz. \]  

(2.6)

For the weak solutions to exist globally in time, the initial data must satisfy certain compatibility conditions:

\[ \int_\Omega \left( \frac{1}{2} \frac{\varrho}{\varrho_0} |u|^2 + \frac{1}{2\mu} |B|^2 + E(\varrho, \varpi) \right) \, dx < \infty, \quad \text{in particular } (\varrho u)_0 = 0 \text{ a.a. on the set } \{ \varrho_0 = 0 \}, \]  

(2.7)

and

\[ \int_\Omega E(\varrho_0, \varpi) \, dx < \infty, \]  

(2.8)

where the latter bound holds if, for instance,

\[ \varrho_0 \in L^\infty(\Omega), \quad \varrho_0 - \varpi \in L^2(\Omega). \]

Under the hypotheses (2.7), (2.8), the existence of global-in-time weak solutions to the compressible MHD system (1.1 - 1.7), (2.1) can be shown by the methods developed in Lions [30] and [15], cf. also [8].

2.1. Relative entropies and dissipative solutions. Motivated by [16], we introduce the relative entropy in the form

\[ E(\varrho, u, B|R, U, b) = \int_\Omega \left( \frac{1}{2} \varrho |u - U|^2 + \frac{1}{2\mu} |B - b|^2 + E(\varrho, r) \right) \, dx, \]  

(2.9)

where \( U, b, \) and \( r \) are smooth satisfying

\[ U \in C^\infty_c([0,T] \times \Omega; R^3), \quad U \cdot n_{|\partial\Omega} = 0, \]  

(2.10)

\[ b \in C^\infty_c([0,T] \times \Omega; R^3), \quad \text{div}_x b = 0, \quad b \cdot n_{|\partial\Omega} = 0, \]  

(2.11)

and

\[ r > 0 \text{ in } [0,T] \times \Omega, \quad (r - \varpi) \in C^\infty_c([0,T] \times \Omega). \]  

(2.12)

Our next goal is to derive a relation for

\[ \left[ E(\varrho, u, B|R, U, b) \right]_{t=0}^{t=\tau} = \left[ \int_\Omega \left( \frac{1}{2} \varrho |u|^2 + \frac{1}{2\mu} |B|^2 + E(\varrho, \varpi) \right) \, dx \right]_{t=0}^{t=\tau} + \left[ \int_\Omega \left( \frac{1}{2} \varrho |U|^2 - \varrho u \cdot U + \frac{1}{2\mu} |b|^2 - \frac{1}{\mu} \varpi \cdot b \right) \, dx \right]_{t=0}^{t=\tau} + \left[ \int_\Omega \left( P(\varpi) - P(r) + \left( P'(\varpi) - P'(r) \right) \varrho + P'(r) \varpi - P'(\varpi) \varrho \right) \, dx \right]_{t=0}^{t=\tau}, \]

where the first integral on the right-hand side may be controlled by the energy inequality (2.5). On the other hand, the remaining integrals can be computed explicitly by means of (2.2 - 2.4), specifically, by taking \(|U|^2\) and \(P'(r) - P'(\varpi)\), \(U, b,\) and \(r\).
and \( b \) as test functions in (2.2), (2.3), and (2.4), respectively. After a bit tedious but straightforward manipulation, we obtain the relative entropy inequality:

\[
\left[ E \left( \varrho, \mathbf{u}, \mathbf{B} \Big| r, \mathbf{U}, \mathbf{b} \right) \right]_{t=0}^{t=T} \leq \int_0^T \int_{\Omega} \left( S(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) + \frac{\lambda}{\mu} |\text{curl}(\mathbf{B} - \mathbf{b})|^2 \right) \, dx \, dt
\]

(2.13)

\[
\leq \int_0^T \mathcal{R} \left( \varrho, \mathbf{u}, \mathbf{B}, r, \mathbf{U}, \mathbf{b} \right) \, dt,
\]

where the remainder \( \mathcal{R} \) reads

\[
\mathcal{R} \left( \varrho, \mathbf{u}, \mathbf{B}, r, \mathbf{U}, \mathbf{b} \right) = \int_{\Omega} \varrho \left( \partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) : (\mathbf{U} - \mathbf{u}) \, dx
\]

(2.14)

\[
+ \int_{\Omega} S(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, dx + \frac{\lambda}{\mu} \int_{\Omega} \text{curl} \, \mathbf{b} \cdot (\text{curl} \, \mathbf{b} - \text{curl} \, \mathbf{B}) \, dx
\]

\[
+ \int_{\Omega} \left( (r - \varrho) \partial_t P'(r) + \nabla_x P'(r) \cdot (r \mathbf{U} - \varrho \mathbf{u}) \right) \, dx - \int_{\Omega} \left( p(\varrho) - p(r) \right) \text{div}_x \mathbf{U} \, dx
\]

\[
+ \int_{\Omega} \frac{1}{\mu} \left( \partial_t \mathbf{b} \cdot (\mathbf{b} - \mathbf{B}) - (\text{curl} \, \mathbf{b} \times \mathbf{u}) \cdot \mathbf{B} - (\text{curl} \, \mathbf{B} \times \mathbf{B}) \cdot \mathbf{U} \right) \, dx.
\]

A trio \( \varrho, \mathbf{u}, \mathbf{B} \) is called a dissipative solution of the compressible MHD system (1.1 - 1.7), (2.1) if the relative entropy inequality (2.13) holds for all test functions \( r, \mathbf{U}, \mathbf{b} \) satisfying (2.10 - 2.12).

It can be shown, by means of the methods developed in [16], that any weak solution of the compressible MHD system (1.1 - 1.7) is a dissipative solution.

2.2. Weak-strong uniqueness. Given the regularity and integrability properties of a weak solution \( \varrho, \mathbf{u}, \mathbf{B} \), validity of the relative entropy inequality (2.13) can be extended to a larger class of test functions \( r, \mathbf{U}, \mathbf{b} \). In particular, we can establish the weak-strong uniqueness property for the compressible MHD system. Following the arguments of [16], we suppose that \( \varrho, \mathbf{u}, \mathbf{B} \) is a weak and \( r, \mathbf{U}, \mathbf{b} \) a sufficiently smooth solution of the compressible MHD system, both emanating from the same initial data. Here “sufficiently smooth” means that the relative entropy inequality (2.13) holds for \( r, \mathbf{U}, \mathbf{b} \). In particular, the strong solutions considered in Theorem 2.1 below fall in this category. Similarly to [16], we want to use a Gronwall type argument to conclude that, in fact, \( \varrho \equiv r, \mathbf{u} \equiv \mathbf{U}, \) and \( \mathbf{B} \equiv \mathbf{b} \) as long as the smooth solution exists.

**Theorem 2.1. [Weak-strong uniqueness]** Let \( \Omega \subset \mathbb{R}^3 \) be a perturbed halfspace specified in (1.15). Let \( \varrho, \mathbf{u}, \mathbf{B} \) be a weak (dissipative) solution to the compressible MHD system (1.1 - 1.7) emanating from (smooth) initial data

\[
\varrho(0, \cdot) = \varrho_0, \, \varrho_0 > 0, \, \mathbf{u}(0, \cdot) = \mathbf{u}_0, \, \mathbf{B}(0, \cdot) = \mathbf{B}_0, \, \text{div}_x \mathbf{B}_0 = 0.
\]

Let \( (r, \mathbf{U}, \mathbf{b}) \) be a smooth solution of the same system, belonging to the class

\[
r \in C^1([0,T]; W^{m,2}(\Omega)), \, \mathbf{U} \in C^1([0,T]; W^{m+2,2}(\Omega; R^3)), \, \mathbf{b} \in C^1([0,T]; W^{m+2,2}(\Omega; R^3))
\]

for a certain \( m \geq 3, \)

\[
0 < \tau \leq r \leq T < \infty,
\]

and emanating from the same initial data.

Then

\[
\varrho = r, \, \mathbf{u} = \mathbf{U}, \, \mathbf{B} = \mathbf{b} \text{ in } (0,T) \times \Omega.
\]
Remark. Of course, the conclusion of Theorem 2.1 holds for a large class of domains, in particular, for bounded domains in $\mathbb{R}^3$ with sufficiently smooth boundary. We refer to [16] for details.

In what follows, we use the following structural properties of the function $E$ introduced in (2.6): For $r \in [r, \bar{r}] \subset (0, \infty)$, we have

$$c_1|q - r|^2 \leq E(q, r) \leq c_2|q - r|^2 \text{ for all } \frac{1}{2} \bar{r} < q < 2\bar{r}, \quad (2.15)$$

$$c_1|q - r|^{5/3} \leq E(q, r) \leq c_2|q - r|^{5/3} \text{ for all } \frac{1}{2} \bar{r} \leq q \leq 2\bar{r}, \quad (2.16)$$

where $c_1, c_2$ are positive constants depending only on $\bar{r}, \bar{r}$. In particular, relation (2.16) yields

$$E(q, r) \geq c_3 \text{ whenever } q \leq \frac{1}{2}\bar{r} \text{ or } q \geq 2\bar{r}. \quad (2.17)$$

To prove Theorem 2.1, we use the equations satisfied by the smooth solution $(r, U, b)$ to replace the time derivatives in the relative entropy inequality. In comparison with [16], we get the following extra terms in (2.14) resulting from the presence of the magnetic field:

$$\frac{1}{\mu} \int_{\Omega} \frac{\partial b}{\partial t} \cdot (U - U) \, dx$$

$$+ \frac{1}{\mu} \int_{\Omega} \left( \partial_i b \cdot (b - B) - (\nabla b \times u) \cdot B - (\nabla b \times B) \cdot U \right) \, dx$$

$$+ \frac{\lambda}{\mu} \int_{\Omega} \nabla b \cdot (\nabla b - \nabla B) \, dx$$

$$= \frac{1}{\mu} \int_{\Omega} \frac{\partial b}{\partial t} \cdot (U - U) \, dx + \frac{1}{\mu} \int_{\Omega} (\nabla b \times b) \cdot (U - u) \, dx$$

$$+ \frac{1}{\mu} \int_{\Omega} \left( \nabla(U \times b) \cdot (b - B) - (\nabla b \times u) \cdot B - (\nabla b \times B) \cdot U \right) \, dx$$

$$= \frac{1}{\mu} \int_{\Omega} \frac{\partial b}{\partial t} \cdot (U - U) \, dx - \frac{1}{\mu} \int_{\Omega} (\nabla b \times b) \cdot u \, dx$$

$$- \frac{1}{\mu} \int_{\Omega} \left( \nabla(U \times b) \cdot B + (\nabla b \times u) \cdot B + (\nabla b \times B) \cdot U \right) \, dx$$

$$= \frac{1}{\mu} \int_{\Omega} \frac{\partial b}{\partial t} \cdot (U - U) \, dx$$

$$+ \int_{\Omega} (B \times (B - B)) \cdot \nabla b \, dx + \int_{\Omega} (B \times (B - B)) \cdot \nabla b \, dx$$

$$= \frac{1}{\mu} \int_{\Omega} \frac{\partial b}{\partial t} \cdot (U - U) \, dx$$

$$+ \int_{\Omega} (U \times (B - b)) \cdot \nabla b \, dx + \int_{\Omega} (U \times (B - b)) \cdot \nabla b \, dx.$$
Following the line of arguments of Germain [19], we may control the first term by
\[ c(\varepsilon)\|\text{curl } b \times b\|_{L^2(\Omega; \mathbb{R}^3)}^2 \int_\Omega E(\varrho, r) \, dx + \varepsilon \|U - u\|^2_{L^6(\Omega; \mathbb{R}^3)}, \]
where we have used (2.15), (2.17). On the other hand, in accordance with (2.16), we have \(|\varrho - r| \leq c\varrho^{1/6}E(\varrho, r)^{1/2}\) for \(\{\varrho > 2\sigma\}\). Consequently, by virtue of Hölder inequality,
\[ \int_{\varrho > 2\sigma} |\varrho - r|\|\text{curl } b \times b\|\|U - u\|dx \]
\[ \leq c \int_{\varrho > 2\sigma} \varrho^{1/6}E^{1/2}(\varrho, r)|\text{curl } b \times b|\|U - u\|dx \]
\[ \leq c\|E(\varrho, r)^{1/2}\|_{L^2(\Omega)}\|U - u\|_{L^6(\Omega; \mathbb{R}^3)}^{2/3}\|\varrho^{1/2}(u - U)\|_{L^2(\Omega; \mathbb{R}^3)}^{1/3}\|\text{curl } b \times b\|_{L^2(\Omega; \mathbb{R}^3)}^{1/2} \]
\[ \leq c(\varepsilon)\|\text{curl } b \times b\|_{L^2(\Omega; \mathbb{R}^3)}^{1/2} \left( \|E(\varrho, r)\|_{L^1(\Omega)} + \|\varrho^{1/2}(u - U)\|_{L^2(\Omega)}^2 \right) \]
\[ + \varepsilon\|U - u\|^2_{L^6(\Omega; \mathbb{R}^3)}. \]
Therefore we may conclude that
\[ \left| \frac{1}{\mu} \int_\Omega \frac{\varrho - r}{r} (\text{curl } b \times b) \cdot (U - u) \, dx \right| \]
\[ \leq c(\varepsilon) \left( \|E(\varrho, r)\|_{L^1(\Omega)} + \|\varrho^{1/2}(u - U)\|_{L^2(\Omega; \mathbb{R}^3)}^2 \right) \]
\[ + \varepsilon\|\nabla_x (u - U)\|^2_{L^2(\Omega; \mathbb{R}^3)}. \]
Finally, by virtue of (2.17), we have
\[ \left| \left\{ \varrho < \frac{1}{2\varepsilon} \right\} \right| < \infty, \]
in particular, we may use a generalized version of Korn’s inequality to obtain
\[ \|\nabla_x (u - U)\|^2_{L^2(\Omega; \mathbb{R}^3)} \]
\[ \leq c \left( \int_\Omega |\varrho||u - U|^2 \, dx + \int_\Omega S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx \right). \]

Thus, combining the previous estimates with those obtained in [16, Section 4.1] we may infer that
\[ |R(\varrho, u, B, r, U, b)| \]
\[ \leq c \int_\Omega S(\nabla_x u - \nabla_x U) : (\nabla_x u - \nabla_x U) \, dx + c(\varepsilon)\varrho (\varrho, u, B \mid r, U, b), \]
which, by means a straightforward application of Gronwall’s lemma to (2.13), completes the proof of Theorem 2.1.
3. **Singular limit, the main result.** Before stating our main result, we briefly discuss the question of existence of solutions to the target problem (1.10 - 1.14). Here, the standard framework for local existence of smooth solutions is the scale of Sobolev spaces $W^{m,2}$, specifically,

$$\left\{ \begin{array}{l} v, \ H \in C([0, T]; W^{m,2}(\Omega; R^3)), \\
\partial_t v, \partial_t H, \nabla_x \Pi \in C([0, T]; W^{m-1,2}(\Omega; R^3)), \end{array} \right. \quad (3.1)$$

where $m > \frac{5}{2}$ is a positive integer.

Local existence of solutions on bounded domains have been established by many authors, see Kozono [27], Secchi [35], among others. The local existence results are easy to extend on “large” domains, in particular the perturbed half-space, by the method of invading domains, specifically, by replacing $\Omega$ by $\Omega_r$

$$\Omega_r = \Omega \cap \left\{ x \in R^3 \mid |x| < r \right\}, \quad r \to \infty.$$ 

The problem then reduces to finding suitable a priori bounds yielding the existence time $T_{max}$ independent of the size of the invading domain, cf. also Kato and Lai [24] and the remarks in Kato’s paper [23]. We recall that, unlike the Euler system, where solutions are regular in the 2D-physical space, the solutions of the ideal MHD system do not, or at least are not known to, enjoy the same property so the local solutions are relevant also in the 2D-setting.

Let $H$ denote the standard Helmholtz projection onto the space of solenoidal functions in $\Omega$, specifically,

$$v = H[v] + \nabla_x \Psi,$$

where the gradient component is the unique solution of the Neumann problem

$$\Delta \Psi = \text{div}_x v \text{ in } \Omega, \quad \nabla_x \Psi \cdot n = v \cdot n|_{\partial \Omega}, \quad \Psi \to 0 \text{ as } |x| \to \infty.$$

We refer to the monograph by Galdi [17] concerning the construction and the basic properties of $H$.

We are ready to formulate the main result of the present paper:

**Theorem 3.1.** Let $\Omega \subset R^3$ be a perturbed half-space specified in (1.15). Let the pressure $p = p_e$ and the transport coefficients $\nu_e, \lambda_e$ satisfy (1.8), (1.9), with

$$0 < a < \frac{4}{3}, \quad b > 0. \quad (3.2)$$

Suppose that \{qe, ue, Be\}_{e>0} is a family of the global-in-time weak (dissipative) solutions to the compressible MHD system (1.1 - 1.7), with the initial data

$$q_e(0, \cdot) = q_{0,e} = \overline{q} + \varepsilon r_{0,e}, \quad \|r_{0,e}\|_{L^2(\Omega)} \leq c, \quad r_{0,e} \to r_0 \text{ in } L^2(\Omega), \quad (3.3)$$

$$\begin{array}{l}
(qu)_e(0, \cdot) = q_{0,e} u_{0,e}, \quad u_{0,e} \to u_0 \text{ in } L^2(\Omega; R^3), \\
B_e(0, \cdot) = B_{0,e} \to B_0 \text{ in } L^2(\Omega; R^3), \quad \text{div}_x B_{0,e} = 0,
\end{array} \quad (3.4)$$

for some integer $m > \frac{5}{2}$.

Finally, let $[0, T_{max})$ be the time interval on which the ideal MHD system (1.10 - 1.14) admits the unique smooth solution $(v, H)$ in the class (3.1) emanating from the initial data

$$v(0, \cdot) = v_0 = H[u_0], \quad H(0, \cdot) = B_0.$$ 

Then

$$\sup_{t \in [0, T]} \|q_e(t, \cdot) - \overline{q}\|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon^{6/5} c, \quad (3.6)$$
\[
\begin{IEEEeqnarray*}
\sqrt{\varepsilon} \mathbf{u}_\varepsilon &\to \sqrt{\varepsilon} \mathbf{w} \quad \left\{ \begin{array}{ll}
\text{weakly-(\ast)} & \text{in } L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)), \\
\text{(strongly)} & \text{in } L^2(0,T; L^2_{\text{loc}}(\Omega; \mathbb{R}^3)),
\end{array} \right. \\
\mathbf{B}_\varepsilon &\to \mathbf{H} \quad \left\{ \begin{array}{ll}
\text{weakly-(\ast)} & \text{in } L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)), \\
\text{(strongly)} & \text{in } L^2(0,T; L^2_{\text{loc}}(\Omega; \mathbb{R}^3)),
\end{array} \right.
\end{IEEEeqnarray*}
\]

for any \( 0 < T < T_{\text{max}}. \)

The rest of the paper is devoted to the proof of Theorem 3.1.

4. Uniform bounds. The proof of Theorem 3.1 is based on uniform energy bounds independent of the singular parameter \( \varepsilon \). Similarly to \([11]\), it is convenient to introduce the following notation:

\[
\Omega_{\text{ess}}(t) = \left\{ x \in \Omega \mid \frac{1}{2} \varepsilon \leq \varrho_\varepsilon(t, x) \leq 2 \varrho \right\}, \quad \Omega_{\text{res}}(t) = \Omega \setminus \Omega_{\text{ess}}(t),
\]

and, for a measurable function \( f(t,x) \),

\[
[f]_{\text{ess}} = 1_{\Omega_{\text{ess}}(t)} f, \quad [f]_{\text{res}} = 1_{\Omega_{\text{res}}(t)} f.
\]

4.1. Energy bounds. As a consequence of (2.15 - 2.17), we get

\[
E(\varrho, \varpi) = P(\varrho) - P'(\varpi)(\varrho - \varpi) - P(\varpi) \geq c \left( [\varrho - \varpi]_{\text{ess}}^2 + [\varrho - \varpi]_{\text{res}}^2 \right).
\]

Consequently, the energy inequality (2.5), with \( E_\varepsilon(\varrho, \varpi) = \frac{1}{\varepsilon} E(\varrho, \varpi) \), and the bounds imposed through the hypotheses (3.3-3.5) yield immediately

\[
\left\| \left[ \frac{\varrho_\varepsilon - \varpi}{\varepsilon} \right]_{\text{ess}} \right\|_{L^\infty(0,T; L^2(\Omega))}^2 \leq c,
\]

\[
\left\| [\varrho_\varepsilon]_{\text{res}} \right\|_{L^\infty(0,T; L^2(\Omega))} \leq \varepsilon^\frac{3}{2} c, \quad |\Omega_{\text{res}}(t)| \leq \varepsilon^2 c,
\]

and

\[
\left\| \varrho_\varepsilon \mathbf{u}_\varepsilon \right\|_{L^\infty(0,T; L^1(\Omega))}^2 \leq c,
\]

\[
\left\| \mathbf{B}_\varepsilon \right\|_{L^\infty(0,T; L^2(\Omega; \mathbb{R}^3))}^2 \leq c.
\]

Moreover, we record the “dissipative” estimates

\[
\varepsilon^b \int_0^T \left\| \nabla_x \mathbf{B}_\varepsilon \right\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 \, dt \leq c,
\]

and, by virtue of Korn’s inequality,

\[
\varepsilon^a \int_0^T \left\| \nabla_x \mathbf{u}_\varepsilon \right\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 \, dt \leq c.
\]
4.2. Convergence, part I. The first convergence statement (3.6) in Theorem 3.1 follows immediately from the estimates (4.4) and (4.5).

Furthermore, in accordance with (4.6), there exists $v \in L^\infty(0, T; L^2(\Omega; R^3))$ such that, at least for a suitable subsequence,

$$
\sqrt{\varrho_\varepsilon} u_\varepsilon \to \sqrt{\varrho} u \text{ weakly-(*) in } L^\infty(0, T; L^2(\Omega; R^3)); \tag{4.10}
$$

therefore

$$
[\varrho_\varepsilon]_{\text{res}} u_\varepsilon \to \varrho u \text{ weakly-(*) in } L^\infty(0, T; L^2(\Omega; R^3)), \tag{4.11}
$$

and, according to (4.5) and (4.6),

$$
[\varrho_\varepsilon]_{\text{res}} u_\varepsilon = \sqrt{[\varrho_\varepsilon]_{\text{res}}} \sqrt{[\varrho_\varepsilon]_{\text{res}}} u_\varepsilon \to 0 \text{ in } L^\infty(0, T; L^{\frac{5}{4}}(\Omega; R^3)). \tag{4.12}
$$

Passing to the limit in the weak formulation of the equation of continuity (2.2), we deduce that

$$
\text{div}_x u = 0. \tag{4.13}
$$

Finally, by virtue of the uniform bounds on $B_\varepsilon$, there exists $B \in L^\infty(0, T; L^2(\Omega; R^3))$ such that

$$
B_\varepsilon \to B \text{ weakly-(*) in } L^\infty(0, T; L^2(\Omega; R^3)). \tag{4.14}
$$

In order to prove Theorem 3.1, one needs to show the strong convergence (3.7)-(3.8) identifying the limit system of equations.

5. Acoustic waves. We examine the acoustic equation (1.16), in particular, we show the frequency localized Strichartz estimates (1.17). Let $\Delta_N$ denote the standard $L^2$ realization of the Neumann Laplacean on the domain $\Omega$ specified in (1.15),

$$
D(\Delta_N) = \left\{ v \in L^2(\Omega) \ \middle| \ \nabla_x v \in L^2(\Omega; R^3), \int_\Omega \nabla_x v \cdot \nabla_x \varphi \, dx = \int_\Omega g \varphi \, dx \right\}
$$

for any $\varphi \in C_0^\infty(\overline{\Omega})$ and a certain $g \in L^2(\Omega)$.

Since we assume that $\partial \Omega$ is smooth, we have

$$
D(\Delta_N) = \{ v \in W^{2,2}(\Omega) \mid \nabla_x v \cdot n|\partial\Omega = 0 \}, \Delta_N[v] = \Delta v.
$$

It can be shown, see Edward and Pravica [9], that $-\Delta_N$ is a non-negative self-adjoint operator on the Hilbert space $L^2(\Omega)$, with the essential spectrum $[0, \infty)$. Moreover, by virtue of Rellich’s lemma (cf. Eidus [10, Theorem 2.1]), the point spectrum of $-\Delta_N$ is empty. Indeed, supposing that

$$
-\Delta v = \lambda v, \ v \in L^2(\Omega), \ \lambda > 0,
$$

we deduce

$$
-\Delta \tilde{v} = \lambda \tilde{v} \text{ for } |x| > R,
$$

where

$$
\tilde{v}(x_1, x_2, x_3) = v(x_1, x_2, x_3) \text{ for } x_3 \geq 0, \ \tilde{v}(x_1, x_2, x_3) = v(x_1, x_2, -x_3) \text{ for } x_3 < 0.
$$

Thus a direct application of Rellich’s lemma yields $\tilde{v}(x) = 0$ for $|x| > R$; whence, by means of the unique continuation principle, $v \equiv 0$ in $\Omega$. 

5.1. Limiting absorption principle and local energy decay. Since the point spectrum of \(-\Delta_N\) is empty, the result of Dermenjian and Guillot [6] implies that \(-\Delta_N\) satisfies the limiting absorption principle (LAP), specifically, the cut-off resolvent operator
\[
(1 + |x|^2)^{-s/2} \circ [-\Delta_N - \mu \pm i\delta]^{-1} \circ (1 + |x|^2)^{-s/2}, \ \delta > 0, \ s > 1
\]
can be extended as a bounded linear operator on \(L^2(\Omega)\) for \(\delta \to 0\) and \(\mu\) belonging to compact subintervals of \((0, \infty)\). As observed by Edward and Pravica [9, Corollary 1], the norm of this extension need not be bounded uniformly for \(\mu \in (0, \infty)\) for certain domains \(\Omega\).

As shown in [11, Section 5.5], the validity of LAP yields the local energy decay estimates in the form
\[
\int_{-\infty}^{\infty} \| \varphi G(-\Delta_N) \exp\left(\pm i \sqrt{-\Delta_N} t\right) [h] \|_{L^2(\Omega)}^2 \, dt \leq c(\varphi, G) \| h \|_{L^2(\Omega)}^2
\]
for any \(\varphi \in C_c^\infty(\Omega)\) and any \(G \in C_c^\infty(0, \infty)\).

5.2. Dispersive estimates for the free Laplacean. We recall the standard Strichartz estimates for the free Laplacean \(\Delta\) in \(\mathbb{R}^3\),
\[
\int_{-\infty}^{\infty} \| \exp\left(\pm i \sqrt{-\Delta} t\right) [h] \|_{L^p(\mathbb{R}^3)}^p \, dt \leq \| h \|_{H^{1,2}(\mathbb{R}^3)}^p, \ \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty,
\]
where \(H^{1,2}\) denotes the homogeneous Sobolev space, see Keel and Tao [25], Strichartz [37].

In addition, the free Laplacean satisfies the local energy decay in the form
\[
\int_{-\infty}^{\infty} \| \varphi \exp\left(\pm i \sqrt{-\Delta} t\right) [h] \|_{H^{\alpha, 2}(\mathbb{R}^3)}^2 \, dt \leq c(\varphi) \| h \|_{H^{\alpha, 2}(\mathbb{R}^3)}^2, \ \alpha \leq \frac{3}{2},
\]
see Smith and Sogge [36, Lemma 2.2].

Clearly, the estimates (5.2), (5.3) remain valid for the Neumann Laplacean defined on the half-space \(\{x_3 > 0\}\).

5.3. Frequency localized Strichartz estimates for the Neumann Laplacean. Our goal is to show (1.17), specifically,
\[
\int_{-\infty}^{\infty} \| G(-\Delta_N) \exp\left(\pm i \sqrt{-\Delta_N} t\right) [h] \|_{L^p(\Omega)}^p \leq c(G) \| h \|_{H^{1,2}(\mathbb{R}^3)}^p, \ \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty
\]
for any \(G \in C_c^\infty(0, \infty)\). To this end, we adapt the method developed by Burq [2], Smith and Sogge [36].

We decompose the function
\[
U(t, \cdot) = G(-\Delta_N) \exp\left(\pm i \sqrt{-\Delta_N} t\right) [h] = \exp\left(\pm i \sqrt{-\Delta_N} t\right) G(-\Delta_N)[h]
\]
as
\[
U = v + w, \ v = \chi U, \ w = (1 - \chi) U,
\]
where \(\chi \in C_c^\infty(\mathbb{R}^3), \ 0 \leq \chi \leq 1, \ \chi\) even in \(x_3, \ \chi(x) = 1\) for \(|x| \leq R\).

Accordingly,
\[
w = w^1 + w^2,
\]
where \(w^1\) solves the homogeneous wave equation
\[
\partial_{tt}^2 w^1 - \Delta w^1 = 0 \text{ in the half-space } \{x_3 > 0\}, \ \partial_{x_3} w^1_{\{x_3=0\}} = 0.
\]
supplemented with the initial conditions
\[ w^1(0) = (1 - \chi)G(-\Delta_N)[\chi], \quad \partial_t w^1(0) = \pm i(1 - \chi)\sqrt{-\Delta_N}G(-\Delta_N)[\chi], \]
while
\[ \partial^2_{t,x}w^2 - \Delta w^2 = F \text{ in the half-space } \{x_3 > 0\}, \quad \partial_{x_3} w^2|_{x_3=0} = 0, \]
\[ w^2(0) = \partial_t w^2(0) = 0, \]
with
\[ F = -\nabla_x \chi \nabla_x U - U \Delta \chi. \]

**Step 1.** As a direct consequence of the standard Strichartz estimates (5.2), we get
\[ \int_{-\infty}^{\infty} \|w^1\|_{L^p_t(R^3)}^p \, dt \leq c(G) \|h\|_{H^1,2(R^3)}^p, \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \quad q < \infty. \]

**Step 2.** Using Duhamel’s formula, we obtain
\[ w^2(\tau, \cdot) = \frac{1}{2\sqrt{-\Delta}} \left[ \exp \left(i\sqrt{-\Delta} \tau \right) \int_0^\tau \exp \left(-i\sqrt{-\Delta}s \right) \left[ \eta^2 F(s) \right] \, ds \right] \\
- \frac{1}{2\sqrt{-\Delta}} \left[ \exp \left(-i\sqrt{-\Delta} \tau \right) \int_0^\tau \exp \left(i\sqrt{-\Delta}s \right) \left[ \eta^2 F(s) \right] \, ds \right], \]
with
\[ \eta \in C_c^\infty(R^3), \quad 0 \leq \eta \leq 1, \eta \text{ even in } x_3, \eta = 1 \text{ on } \text{supp}[F]. \]

At this stage, similarly to [2], we use the following result of Christ and Kiselev [3]:

**Lemma 5.1.** Let \( X \) and \( Y \) be Banach spaces and assume that \( K(t,s) \) is a continuous function taking its values in the space of bounded linear operators from \( X \) to \( Y \). Set
\[ T[f](t) = \int_a^b K(t,s)f(s) \, ds, \quad W[f](t) = \int_t^\infty K(t,s)f(s) \, ds, \]
where
\[ 0 \leq a \leq b \leq \infty. \]

Suppose that
\[ \|T[f]\|_{L^p(a,b; Y)} \leq c_1\|f\|_{L^r(a,b; X)} \]
for certain
\[ 1 \leq r < p \leq \infty. \]

Then
\[ \|W[f]\|_{L^p(a,b; Y)} \leq c_2\|f\|_{L^r(a,b; X)}, \]
where \( c_2 \) depends only on \( c_1, p, \) and \( r \).

We apply Lemma 5.1 to
\[ X = L^2(\{x_3 > 0\}), \quad Y = L^q(\{x_3 > 0\}), \quad q < \infty, \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \quad r = 2, \]
and
\[ f = F, \quad K(t,s)[F] = \frac{1}{\sqrt{-\Delta}} \exp \left( \pm i\sqrt{-\Delta}(t-s) \right) \left[ \eta^2 F \right]. \]

Writing
\[ \int_0^\infty K(t,s)F(s) \, ds = \exp \left( \pm i\sqrt{-\Delta}t \right) \frac{1}{\sqrt{-\Delta}} \int_0^\infty \exp \left( \mp i\sqrt{-\Delta}s \right) \left[ \eta^2 F(s) \right] \, ds, \]
we have to show, in accordance with the Strichartz estimates (5.2), that
\[ \left\| \int_0^\infty \exp \left( \pm i \sqrt{-\Delta} s \right) [\eta^2 F(s)] \, ds \right\|_{L^2(x_3>0)} \leq c \|F\|_{L^2(0,\infty;L^2(x_3>0))}. \] (6.6)

However,
\[ \left\| \int_0^\infty \exp \left( \pm i \sqrt{-\Delta} s \right) [\chi^2 F(s)] \, ds \right\|_{L^2(x_3>0)} \]
\[ = \sup_{\|v\|_{L^2(x_3>0)}} \int_0^\infty \left\langle \exp \left( \pm i \sqrt{-\Delta} s \right) [\chi^2 F(s)]; v \right\rangle \, ds \]
\[ = \sup_{\|v\|_{L^2(x_3>0)}} \int_0^\infty \left\langle \chi F(s); \chi \exp \left( -i \sqrt{-\Delta} s \right) [v] \right\rangle \, ds; \]
whence the desired conclusion (6.6) follows from the local energy decay estimates (5.3). As the norm of \( F \) is bounded in view of (5.1), we may infer that
\[ \int_{-\infty}^\infty \|u^2\|_{L^p(R^3)}^p \, dt \leq c(G)\|h\|_{H^{1,2}(\Omega)}^2, \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \quad q < \infty. \] (6.7)

**Step 3.** Finally, since \( v = \chi U \) is compactly supported, we deduce from (5.1) combined with the standard elliptic regularity for \(-\Delta_N\) that
\[ \int_0^\infty \|v\|_{L^1(\Omega)}^\infty \, dt \leq c(G)\|h\|_{H^{1,2}(\Omega)}^2; \] (6.8)
while, by virtue of the standard energy estimates,
\[ \sup_{t>0} \|v(t, \cdot)\|_{L^2(\Omega)} \leq c(G)\|h\|_{H^{1,2}(\Omega)}, \] (6.9)
where \( q < \infty \) is the same as in (5.4).

Interpolating (5.8), (5.9) and combining the result with (5.5), (5.7), we get (5.4). Finally, let us remark that (5.4) can be “strengthened” to
\[ \int_{-\infty}^\infty \left\| G(-\Delta_N) \exp \left( \pm i \sqrt{-\Delta_N t} \right) [h] \right\|_{L^p(\Omega)}^p \leq c(G)\|h\|_{L^2(\Omega)}^p, \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \quad q < \infty \] (5.10)
for any \( G \in C^\infty_c(0, \infty). \)

6. **Convergence, part II.** In this section, we complete the proof of Theorem 3.1 by showing the strong convergence of the velocity field claimed in (3.7), (3.8).

6.1. **Oscillatory component of the velocity.** We start by introducing the functions \([s_{\epsilon, \delta}, \Phi_{\epsilon, \delta}]\) - the unique finite energy solution of the acoustic equation (1.16) emanating from the initial data
\[ s_{\epsilon, \delta}(0, \cdot) = [r_{\epsilon, \delta}], \quad \Phi_{\epsilon, \delta}(0, \cdot) = [\Phi_{0, \epsilon}]\delta, \quad \nabla_x \Phi_{0, \epsilon} = u_{0, \epsilon} - \mathcal{H}[u_{0, \epsilon}], \] (6.1)
where the \([\cdot]\) denotes a regularization defined as follows:
\[ [v]_{\delta} = G_{\delta}(\sqrt{-\Delta_N})[\psi_{\delta} v], \] (6.2)
with
\[ G_{\delta} \in C^\infty_c(R), \ 0 \leq G_{\delta} \leq 1, \ G_{\delta}(-z) = G_{\delta}(z), \]
\[ G_{\delta}(z) = 1 \text{ for } z \in \left( -\frac{1}{\delta}, -\frac{\delta}{2} \right) \cup \left( \frac{1}{\delta}, \frac{\delta}{2} \right), \]
\[ G_{\delta}(z) = 0 \text{ for } z \in \left( -\infty, -\frac{2}{\delta} \right) \cup \left( -\frac{\delta}{2}, \frac{\delta}{2} \right) \cup \left( \frac{2}{\delta}, \infty \right), \]
and
\[ \psi_\delta \in C^\infty_c(\Omega) , \; 0 \leq \psi_\delta \leq 1 , \; \psi_\delta(x) = 1 \text{ for } |x| < \frac{1}{\delta} , \; \psi_\delta(x) = 0 \text{ for } |x| > \frac{2}{\delta} . \]

The quantity \( \nabla_x \Phi_{\varepsilon,\delta} \) represents the (regularized) oscillatory component of the velocity field. Setting, for the sake of simplicity, \( a = 1 \) in (1.16), we recall the standard Duhamel’s formula:

\[ \Phi_{\varepsilon,\delta}(t) = \frac{1}{2} \exp \left( i \sqrt{-\Delta} \frac{t}{\varepsilon} \right) \left[ [\Phi_{0,\varepsilon}]_\delta - \frac{i}{\sqrt{-\Delta}} [r_{0,\varepsilon}]_\delta \right] \]
\[ + \frac{1}{2} \exp \left( -i \sqrt{-\Delta} \frac{t}{\varepsilon} \right) \left[ [\Phi_{0,\varepsilon}]_\delta + \frac{i}{\sqrt{-\Delta}} [r_{0,\varepsilon}]_\delta \right] , \]
\[ s_{\varepsilon,\delta}(t) = \frac{1}{2} \exp \left( i \sqrt{-\Delta} \frac{t}{\varepsilon} \right) \left[ i \sqrt{-\Delta} \left[ [\Phi_{0,\varepsilon}]_\delta \right] + [r_{0,\varepsilon}]_\delta \right] \]
\[ + \frac{1}{2} \exp \left( -i \sqrt{-\Delta} \frac{t}{\varepsilon} \right) \left[ -i \sqrt{-\Delta} \left[ [\Phi_{0,\varepsilon}]_\delta \right] + [r_{0,\varepsilon}]_\delta \right] . \]

In particular, we have the energy equality

\[ \frac{d}{dt} \int_{\Omega} \left( \| (\nabla_x)^1/2 \Phi_{\varepsilon,\delta} \|_{L^2(\Omega)}^2 + \| s_{\varepsilon,\delta} \|_{L^2(\Omega)}^2 \right) \, dx = 0. \]

As a direct consequence of (6.5), regularity of the initial data, and the standard Sobolev embedding relation

\[ W^{m+2,2}(\Omega) \hookrightarrow W^{m,\infty}(\Omega) , \]

we deduce that

\[ \sup_{t \in [0,T]} \left( \| \nabla_x \Phi_{\varepsilon,\delta} \|_{W^{k,2}(\Omega;R^3)} + \| s_{\varepsilon,\delta} \|_{W^{k,2}(\Omega;R^3)} \right) \]
\[ \leq c(k, \delta) \left( \| \nabla_x \Phi_{0,\varepsilon} \|_{L^2(\Omega;R^3)} + \| r_{0,\varepsilon} \|_{L^2(\Omega)} \right) \]
for any \( k = 0, 1, \ldots \). Moreover, as shown in [13, Section 5.3],

\[ |x|^s |\partial_x^k [h_\delta(x)]| \leq c(s, k, \delta) \| h \|_{L^2(R^3)} \]
\[ \text{for all } x \in \Omega , \; s \geq 0 , \; k \geq 0 , \]

therefore the functions \( \Phi_{\varepsilon,\delta} , s_{\varepsilon,\delta} \) decay fast for \( |x| \to \infty \) as soon as \( \delta > 0 \) is fixed.

Finally, it follows from the frequency localized Strichartz estimates (5.10) that

\[ \int_0^T \left( \| \nabla_x \Phi_{\varepsilon,\delta} \|_{W^{k,\infty}(\Omega;R^3)} + \| s_{\varepsilon,\delta} \|_{W^{k,\infty}(\Omega;R^3)} \right) \, dt \]
\[ \leq \omega(\varepsilon, \delta, k) \left( \| \nabla_x \Phi_{0,\varepsilon} \|_{L^2(\Omega;R^3)} + \| r_{0,\varepsilon} \|_{L^2(\Omega)} \right) \]
where

\( \omega(\varepsilon, \delta, k) \to 0 \) as \( \varepsilon \to 0 \) for any fixed \( \delta > 0 , \; k \geq 0 . \)
Another application of the relative entropy inequality. We exploit the relative entropy inequality (2.13), this time for
\[ r = r_{\varepsilon, \delta} = \tilde{r} + \varepsilon s_{\varepsilon, \delta}, \quad U = U_{\varepsilon, \delta} = v + \nabla_x \Phi_{\varepsilon, \delta}, \quad b = H. \]

In accordance with (2.13), (2.14), we obtain
\[ \left[ \mathcal{E}_\varepsilon \left( \varrho_{\varepsilon}, u_{\varepsilon}, B_{\varepsilon} \big| r_{\varepsilon, \delta}, U_{\varepsilon, \delta}, H \right) \right]_{t=0}^{t=\tau} \]
\[ + \int_0^T \int_\Omega \left( S_\varepsilon \left( \nabla_x u_{\varepsilon} - \nabla_x U_{\varepsilon, \delta} \right) : \left( \nabla_x u_{\varepsilon} - \nabla_x U_{\varepsilon, \delta} \right) + \frac{\lambda_{\varepsilon}}{\mu} |\text{curl}(B_{\varepsilon} - H)|^2 \right) dx dt \]
\[ \leq \int_0^T \mathcal{R} \left( \varrho_{\varepsilon}, u_{\varepsilon}, B_{\varepsilon}, r_{\varepsilon, \delta}, U_{\varepsilon, \delta}, H \right) dt, \]
with the remainder
\[ \mathcal{R} \left( \varrho_{\varepsilon}, u_{\varepsilon}, B_{\varepsilon}, r_{\varepsilon, \delta}, U_{\varepsilon, \delta}, H \right) = \int_\Omega \varrho_{\varepsilon} \left( \partial_t U_{\varepsilon, \delta} + u_{\varepsilon} \cdot \nabla_x U_{\varepsilon, \delta} \right) : \left( U_{\varepsilon, \delta} - u_{\varepsilon} \right) dx \]
\[ + \int_\Omega S_\varepsilon (\nabla_x U_{\varepsilon, \delta}) : (\nabla_x U_{\varepsilon, \delta} - \nabla_x u_{\varepsilon}) dx + \frac{\lambda_{\varepsilon}}{\mu} \int_\Omega \text{curl} H \cdot (\text{curl} H - \text{curl} B_{\varepsilon}) dx \]
\[ + \frac{1}{\varepsilon^2} \int_\Omega \left( (r_{\varepsilon, \delta} - \varrho_{\varepsilon}) \partial_t P'(r_{\varepsilon, \delta}) + \nabla_x P'(r_{\varepsilon, \delta}) \cdot (r_{\varepsilon, \delta} U_{\varepsilon, \delta} - \varrho_{\varepsilon} u_{\varepsilon}) \right) dx \]
\[ - \frac{1}{\varepsilon^2} \int_\Omega \left( p(\varrho_{\varepsilon}) - p(r_{\varepsilon, \delta}) \right) \text{div}_x U_{\varepsilon, \delta} dx \]
\[ + \int_{\Omega} \frac{1}{\mu} \left( \partial_t H \cdot (H - B_{\varepsilon}) - (\text{curl} H \times u_{\varepsilon}) \cdot B_{\varepsilon} - (\text{curl} B_{\varepsilon} \times B_{\varepsilon}) \cdot U_{\varepsilon, \delta} \right) dx. \]

By virtue of a variant of Korn’s inequality and (4.5), we have
\[ \int_0^T \int_\Omega S_\varepsilon (\nabla_x u_{\varepsilon} - \nabla_x U_{\varepsilon, \delta}) : (\nabla_x u_{\varepsilon} - \nabla_x U_{\varepsilon, \delta}) dx dt + \varepsilon^a \int_0^T \int_\Omega |u_{\varepsilon} - U_{\varepsilon, \delta}|^2 dx dt \]
\[ \geq \varepsilon^a c \int_0^T \|u_{\varepsilon} - U_{\varepsilon, \delta}\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 dt, \]
and, similarly,
\[ \frac{\lambda_{\varepsilon}}{\mu} \int_0^T \int_\Omega |\text{curl}(B_{\varepsilon} - H)|^2 dx dt + \varepsilon^b \int_0^T \int_\Omega |B_{\varepsilon} - H|^2 dx dt \]
\[ \geq \varepsilon^b c \int_0^T \|B_{\varepsilon} - H\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 dt. \]
6.3. Estimates. Similarly to the proof of weak-strong uniqueness, we conclude the proof of Theorem 3.1 by means of Gronwall type arguments. In what follows, we denote by \( g_{\varepsilon,\delta} = g_{\varepsilon}(t), \ h_{\varepsilon,\delta} = h_{\varepsilon,\delta}(t) \) generic functions such that

\[
\|g_{\varepsilon,\delta}\|_{L^1(0,T)} \leq c(\delta), \ \|h_{\varepsilon,\delta}\|_{L^1(0,T)} \leq c, \ g_{\varepsilon,\delta} \to 0 \text{ weakly in } L^1(0,T) \text{ as } \varepsilon \to 0.
\]

We proceed in several steps:

**Step 1.** We have

\[
\left| \int_{\Omega} S_\varepsilon(\nabla_x U_{\varepsilon,\delta}) : (\nabla_x U_{\varepsilon,\delta} - \nabla_x u_\varepsilon) \, dx \right|
\]

\[
= \left| \int_{\Omega} \nabla_x U_{\varepsilon,\delta} : S_\varepsilon(\nabla_x U_{\varepsilon,\delta} - \nabla_x u_\varepsilon) \, dx \right|
\]

\[
\leq \varepsilon \lambda \|\nabla_x U_{\varepsilon,\delta}\|_{L^2(\Omega; R^{3 \times 3})}^2 + \frac{1}{2} \varepsilon \int_{\Omega} S_\varepsilon(\nabla_x U_{\varepsilon,\delta} - u_\varepsilon) : (\nabla_x U_{\varepsilon,\delta} - u_\varepsilon) \, dx.
\]

Estimating the term

\[
\frac{\lambda_{\varepsilon}}{\mu} \int_{\Omega} \text{curl} \ H \cdot \text{curl}(H - B_\varepsilon) \, dx
\]

in a similar manner we reduce the relative entropy inequality (6.9) to

\[
\left[ C_\varepsilon \left( \varrho_\varepsilon, u_\varepsilon, B_\varepsilon \right| r_{\varepsilon,\delta}, U_{\varepsilon,\delta}, H \right) \right]_{t=0}^{t=\tau} (6.13)
\]

\[
+ \frac{1}{2} \int_0^\tau \int_{\Omega} \left( S_\varepsilon(\nabla_x u_\varepsilon - \nabla_x U_{\varepsilon,\delta}) : (\nabla_x U_{\varepsilon,\delta} - \nabla_x u_\varepsilon) + \frac{\lambda_{\varepsilon}}{\mu} |\text{curl}(B_\varepsilon - H)|^2 \right) \, dx \, dt
\]

\[
\leq \int_0^\tau \int_{\Omega} \varrho_\varepsilon \left( \partial_t U_{\varepsilon,\delta} + u_\varepsilon \cdot \nabla_x U_{\varepsilon,\delta} \right) \cdot (U_{\varepsilon,\delta} - u_\varepsilon) \, dx \, dt
\]

\[
+ \frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega} \left( (r_{\varepsilon,\delta} - \varrho_\varepsilon) \partial_t P''(r_{\varepsilon,\delta}) + \nabla_x P''(r_{\varepsilon,\delta}) \cdot (r_{\varepsilon,\delta} U_{\varepsilon,\delta} - \varrho_\varepsilon u_\varepsilon) \right) \, dx \, dt
\]

\[
- \frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega} \left( \partial_t (p(\varrho_\varepsilon) - p(r_{\varepsilon,\delta})) \right) \cdot \nabla_x U_{\varepsilon,\delta} \, dx \, dt
\]

\[
+ \int_0^\tau \int_{\Omega} \frac{1}{\mu} \left( \partial_t H \cdot (H - B_\varepsilon) - (\text{curl} \ H \times u_\varepsilon) \cdot B_\varepsilon - (\text{curl} B_\varepsilon \times B_\varepsilon) \cdot U_{\varepsilon,\delta} \right) \, dx \, dt
\]

\[
+ \int_0^\tau g_{\varepsilon,\delta}(t) \, dt.
\]

**Step 2.** Furthermore,

\[
\int_{\Omega} \varrho_\varepsilon \left( \partial_t U_{\varepsilon,\delta} + u_\varepsilon \cdot \nabla_x U_{\varepsilon,\delta} \right) \cdot (U_{\varepsilon,\delta} - u_\varepsilon) \, dx
\]

\[
= \int_{\Omega} \varrho_\varepsilon \left( \partial_t U_{\varepsilon,\delta} + U_{\varepsilon,\delta} \cdot \nabla_x U_{\varepsilon,\delta} \right) \cdot (U_{\varepsilon,\delta} - u_\varepsilon) \, dx
\]

\[
+ \int_{\Omega} \varrho_\varepsilon \nabla_x U_{\varepsilon,\delta} \cdot (u_\varepsilon - U_{\varepsilon,\delta}) \cdot (U_{\varepsilon,\delta} - u_\varepsilon) \, dx,
\]

where

\[
\left| \int_{\Omega} \varrho_\varepsilon \nabla_x U_{\varepsilon,\delta} \cdot (u_\varepsilon - U_{\varepsilon,\delta}) \cdot (U_{\varepsilon,\delta} - u_\varepsilon) \, dx \right| \leq \int_{\Omega} \|\nabla_x U_{\varepsilon,\delta}(t, \cdot)\|_{L^\infty(\Omega)} \mathcal{E} \left( \varrho_\varepsilon, u_\varepsilon, B_\varepsilon \right| r_{\varepsilon,\delta}, U_{\varepsilon,\delta}, H \right) \, dx
\]

(6.14)
can be “absorbed” by the left-hand side of (6.9) by means of a Gronwall type argument. More specifically, making use of (6.8), we have

\[ \int_\Omega \| \nabla x U_{\epsilon,\delta}(t, \cdot) \|_{L^\infty(\Omega)} E \left( \varrho_{\epsilon}, u_{\epsilon}, B_{\epsilon} \middle| r_{\epsilon,\delta}, U_{\epsilon,\delta}, H \right) \, dx \]

(6.15)

On the other hand,

\[ \int_0^T \int_\Omega \varrho_{\epsilon} \left( \partial_t U_{\epsilon,\delta} + U_{\epsilon,\delta} \cdot \nabla x U_{\epsilon,\delta} \right) \cdot (U_{\epsilon,\delta} - u_{\epsilon}) \, dx \, dt \]

(6.16)

where, by virtue of the dispersive estimates (6.6), (6.8), the last three integrals vanish for \( \epsilon \to 0 \).

As for the remaining two integrals, we have

\[ \int_0^T \int_\Omega \varrho_{\epsilon} \left( \partial_t U_{\epsilon,\delta} + U_{\epsilon,\delta} \cdot \nabla x U_{\epsilon,\delta} \right) \cdot (U_{\epsilon,\delta} - u_{\epsilon}) \, dx \, dt \]

(6.17)

Next, employing once more the dispersive estimates (6.6), (6.8), we obtain

\[ \int_0^T \int_\Omega \varrho_{\epsilon} (v + \nabla \Phi_{\epsilon,\delta}) \cdot \nabla x \Pi \, dx \, dt = \int_0^T g_{\epsilon,\delta} \, dt. \]

Finally, we get

\[ \int_0^T \int_\Omega \varrho_{\epsilon} (U_{\epsilon,\delta} - u_{\epsilon}) \cdot \partial_t \nabla \Phi_{\epsilon,\delta} \, dx \, dt \]
\[
= - \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \nabla_x \Phi_{\varepsilon,\delta} \, dx \, dt + \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{v} \cdot \partial_t \nabla_x \Phi_{\varepsilon,\delta} \, dx \, dt \\
+ \frac{1}{2} \int_0^\tau \int_\Omega \varrho_\varepsilon \partial_\varepsilon |\nabla_x \Phi_{\varepsilon,\delta}|^2 \, dx \, dt,
\]

where, in accordance with (1.16),
\[
\int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{v} \cdot \partial_t \nabla_x \Phi_{\varepsilon,\delta} \, dx \, dt = \varepsilon \int_0^\tau \int_\Omega \frac{\varrho_\varepsilon - \mathcal{Q}}{\varepsilon} \mathbf{v} \cdot \partial_t \nabla_x \Phi_{\varepsilon,\delta} \, dx \, dt \\
= - \int_0^\tau \int_\Omega \frac{\varrho_\varepsilon - \mathcal{Q}}{\varepsilon} \mathbf{v} \cdot \nabla_x \varepsilon \, dx \, dt.
\]

Similarly,
\[
\frac{1}{2} \int_0^\tau \int_\Omega \varrho_\varepsilon \partial_\varepsilon |\nabla_x \Phi_{\varepsilon,\delta}|^2 \, dx \, dt \\
= \varepsilon \int_0^\tau \int_\Omega \frac{\varrho_\varepsilon - \mathcal{Q}}{\varepsilon} \partial_t |\nabla_x \Phi_{\varepsilon,\delta}|^2 \, dx \, dt + \frac{1}{2} \int_0^\tau \int_\Omega \partial_\varepsilon |\nabla_x \Phi_{\varepsilon,\delta}|^2 \, dx \, dt,
\]

where
\[
\varepsilon \int_0^\tau \int_\Omega \frac{\varrho_\varepsilon - \mathcal{Q}}{\varepsilon} \partial_t |\nabla_x \Phi_{\varepsilon,\delta}|^2 \, dx \, dt = - \int_0^\tau \int_\Omega \frac{\varrho_\varepsilon - \mathcal{Q}}{\varepsilon} \nabla_x \Phi_{\varepsilon,\delta} \cdot \nabla_x \varepsilon \, dx \, dt.
\]

Summarizing the previous discussion we may rewrite (6.13) in the form
\[
\left[ \mathcal{E}_e \left( \varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon \right|_{r_\varepsilon,\delta}, \mathbf{U}_{r_\varepsilon,\delta}, \mathbf{H} \right) \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left( \mathcal{E}_e \left( \varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon \right|_{r_\varepsilon,\delta}, \mathbf{U}_{r_\varepsilon,\delta}, \mathbf{H} \right) \right) \, dx \, dt \\
+ \frac{1}{2} \int_0^\tau \int_\Omega \left( \mathcal{S}_e (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}_{r_\varepsilon,\delta}) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}_{r_\varepsilon,\delta}) + \frac{\lambda_2}{\mu} |\text{curl} \mathbf{B}_\varepsilon - \mathbf{H}|^2 \right) \, dx \, dt \\
\leq \left[ \frac{1}{2} \int_0^\tau \left( \frac{\mathcal{S}_e}{\varepsilon} |\nabla_x \Phi_{r_\varepsilon,\delta}|^2 \right) \, dx \right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \nabla_x \Phi_{r_\varepsilon,\delta} \, dx \, dt \\
+ \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left( \left( r_\varepsilon,\delta - \varrho_\varepsilon \right) \partial_\varepsilon \mathbf{P}'(r_\varepsilon,\delta) + \nabla_x \mathbf{P}'(r_\varepsilon,\delta) \cdot \left( r_\varepsilon,\delta \mathbf{U}_{r_\varepsilon,\delta} - \varrho_\varepsilon \mathbf{u}_\varepsilon \right) \right) \, dx \, dt \\
- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left( p(\varrho_\varepsilon) - p(r_\varepsilon,\delta) \right) \text{div}_x \mathbf{U}_{r_\varepsilon,\delta} \, dx \, dt \\
+ \int_\Omega \left( \frac{1}{\mu} (\partial_\varepsilon \mathbf{H} \cdot (\mathbf{H} - \mathbf{B}_\varepsilon) - (\text{curl} \mathbf{H} \times \mathbf{u}_\varepsilon) \cdot \mathbf{B}_\varepsilon - (\text{curl} \mathbf{B}_\varepsilon \times \mathbf{B}_\varepsilon) \cdot \mathbf{U}_{r_\varepsilon,\delta}) \right) \, dx \, dt \\
+ \int_\Omega \left( \frac{1}{\mu} \text{curl} \mathbf{H} \times \mathbf{H} \right) \cdot (\mathbf{U}_{r_\varepsilon,\delta} - \mathbf{u}_\varepsilon) \, dx \, dt \\
+ \int_\Omega (\varrho_\varepsilon + \eta_\delta) \mathcal{E} \left( \varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon \right|_{r_\varepsilon,\delta}, \mathbf{U}_{r_\varepsilon,\delta}, \mathbf{H} \right) \, dt + \int_0^\tau \varrho_\varepsilon(t) \, dt.
\]

**Step 3.** We start with a simple observation
\[
\int_\Omega \left( \nabla_x \mathbf{P}'(r_\varepsilon,\delta) \cdot r_\varepsilon,\delta \mathbf{U}_{r_\varepsilon,\delta} + p(\varrho_\varepsilon) \text{div}_x \mathbf{U}_{r_\varepsilon,\delta} \right) \, dx = 0. \quad (6.19)
\]

Furthermore,
\[
\frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \nabla_x \mathbf{P}'(r_\varepsilon,\delta) \cdot \varrho_\varepsilon \mathbf{u}_\varepsilon \, dx \, dt = \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \mathbf{P}''(r_\varepsilon,\delta) \nabla_x \varrho_\varepsilon \mathbf{u}_\varepsilon \, dx \, dt \\
= \int_\Omega \int_\Omega \mathbf{P}''(\varrho + \varepsilon s_{\varrho,\delta}) - \mathbf{P}''(\varrho) \nabla_x s_{\varrho,\delta} \cdot \varrho_\varepsilon \mathbf{u}_\varepsilon \, dx \, dt \\
+ \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \frac{\mathbf{P}''(\varrho)}{\varepsilon} \nabla_x \varrho_\varepsilon \mathbf{u}_\varepsilon \, dx \, dt.
\]
where the former integral on the right-hand side tends to zero as \( \varepsilon \to 0 \) as a consequence of the dispersive estimates (6.6), (6.8), while the latter reads

\[
\frac{1}{\varepsilon} \int_0^T \int_\Omega \left( \frac{p'(|\Phi|)}{\Phi} \nabla_x s_{\varepsilon, \delta} \cdot \varrho \varepsilon \mathbf{u}_\varepsilon \right) dx \, dt = - \int_0^T \int_\Omega \varrho \varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \nabla_x \Phi_{\varepsilon, \delta} dx \, dt.
\]

Next, we have

\[
\frac{1}{\varepsilon^2} \int_0^T \int_\Omega \left( (\varrho_{\varepsilon, \delta} - \varrho_\varepsilon) \partial_t P'_{\varepsilon, \delta} - p(\varrho_\varepsilon) \text{div}_x U_{\varepsilon, \delta} \right) dx \, dt
\]

\[
= \frac{1}{\varepsilon} \int_0^T \int_\Omega \left( \frac{\varrho - \varrho_\varepsilon}{\varepsilon} P''_{\varepsilon, \delta} \partial_t s_{\varepsilon, \delta} \right) dx \, dt - \int_0^T \int_\Omega \left( \frac{p(\varrho_\varepsilon) - p(\varrho)}{\varepsilon^2} \Delta \Phi_{\varepsilon, \delta} \right) dx \, dt
\]

\[
= \int_0^T \int_\Omega \frac{\varrho - \varrho_\varepsilon}{\varepsilon} P''_{\varepsilon, \delta} \partial_t s_{\varepsilon, \delta} dx \, dt + \int_0^T \int_\Omega s_{\varepsilon, \delta} P''_{\varepsilon, \delta} \partial_t s_{\varepsilon, \delta} dx \, dt
\]

\[
+ \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \frac{p'(\varrho)}{\varrho} \left( \varrho_{\varepsilon, \delta} - \varrho \right) \Delta \Phi_{\varepsilon, \delta} dx \, dt
\]

\[
- \int_0^T \int_\Omega \frac{p(\varrho_\varepsilon) - p(\varrho)}{\varepsilon^2} \Delta \Phi_{\varepsilon, \delta} dx \, dt,
\]

where, by virtue of (6.6), (6.8), the last integral tends to zero for \( \varepsilon \to 0 \). Similarly, neglecting small terms, we have

\[
\int_0^T \int_\Omega \frac{\varrho - \varrho_\varepsilon}{\varepsilon} P''_{\varepsilon, \delta} \partial_t s_{\varepsilon, \delta} dx \, dt
\]

\[
= \int_0^T \int_\Omega \frac{\varrho - \varrho_\varepsilon}{\varepsilon} P''_{\varepsilon, \delta} \partial_t s_{\varepsilon, \delta} dx \, dt + \int_0^T g_{\varepsilon, \delta}(t) dx,
\]

and

\[
\int_0^T \int_\Omega s_{\varepsilon, \delta} P''_{\varepsilon, \delta} \partial_t s_{\varepsilon, \delta} dx \, dt = \frac{1}{\varepsilon^2} \left[ \frac{1}{2} \int_\Omega |s_{\varepsilon, \delta}|^2 dx \right]_{t=0}^T + \int_0^T g_{\varepsilon, \delta}(t) dt.
\]

Consequently, relation (6.18) takes the form

\[
\left[ \mathcal{E}_\varepsilon \left( \varrho_{\varepsilon, \delta}, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon \left| \mathbf{r}_{\varepsilon, \delta}, \mathbf{U}_{\varepsilon, \delta}, \mathbf{H} \right. \right) \right]_{t=0}^{t=\tau} = \frac{1}{2} \int_0^T \int_\Omega \left( \frac{\varrho}{\varepsilon} \right) (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}_{\varepsilon, \delta}) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}_{\varepsilon, \delta}) + \frac{\lambda_c}{\mu} |\text{curl} \mathbf{B}_\varepsilon \mathbf{H}|^2 dx \, dt
\]

\[
\leq \int_0^T \int_\Omega \frac{1}{\mu} \left( \partial_t \mathbf{H} \cdot (\mathbf{H} - \mathbf{B}_\varepsilon) - (\text{curl} \mathbf{H} \times \mathbf{u}_\varepsilon) \cdot \mathbf{B}_\varepsilon - (\text{curl} \mathbf{B}_\varepsilon \times \mathbf{B}_\varepsilon) \cdot \mathbf{U}_{\varepsilon, \delta} \right) dx \, dt
\]

\[
+ \int_0^T \int_\Omega \left( \frac{\varrho}{\varepsilon} \right) \left( \frac{1}{\mu} \text{curl} \mathbf{H} \times \mathbf{H} \right) \cdot (\mathbf{U}_{\varepsilon, \delta} - \mathbf{u}_\varepsilon) dx \, dt
\]

\[
+ \int_0^T \left( g_{\varepsilon, \delta} + h_{\varepsilon, \delta} \right) \mathcal{E} \left( \varrho_{\varepsilon, \delta}, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon \left| \mathbf{r}_{\varepsilon, \delta}, \mathbf{U}_{\varepsilon, \delta}, \mathbf{H} \right. \right) dt + \int_0^T g_{\varepsilon, \delta}(t) dt.
\]

**Step 4.** In this final step, we handle the remaining terms containing the magnetic field. Replacing \( \partial_t \mathbf{H} \) by \( \text{curl}(\mathbf{v} \times \mathbf{H}) \) we get

\[
\int_\Omega \left( \partial_t \mathbf{H} \cdot (\mathbf{H} - \mathbf{B}_\varepsilon) - (\text{curl} \mathbf{H} \times \mathbf{u}_\varepsilon) \cdot \mathbf{B}_\varepsilon - (\text{curl} \mathbf{B}_\varepsilon \times \mathbf{B}_\varepsilon) \cdot \mathbf{U}_{\varepsilon, \delta} \right) dx
\]

\[
= \int_\Omega (\mathbf{B}_\varepsilon - \mathbf{H}) \times (\text{curl} \mathbf{H} \cdot (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon, \delta}) - (\text{curl} (\mathbf{U}_{\varepsilon, \delta} \times (\mathbf{B}_\varepsilon - \mathbf{H}))) \cdot (\mathbf{B}_\varepsilon - \mathbf{H}) dx
\]

\[
+ \int_\Omega ((\mathbf{U}_{\varepsilon, \delta} - \mathbf{v}) \times \mathbf{H}) \cdot (\text{curl} (\mathbf{H} - \mathbf{B}_\varepsilon) dx - \int_\Omega (\text{curl} \mathbf{H} \times \mathbf{H}) \cdot (\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon, \delta}) dx.
\]
Furthermore,
\[
\int_\Omega (B_\varepsilon - H) \times \mathbf{curl} H \cdot (u_\varepsilon - U_{\varepsilon,\delta}) \, dx =
\]
\[
\int_\Omega (B_\varepsilon - H) \times \mathbf{curl} H \cdot [u_\varepsilon - U_{\varepsilon,\delta}]_{\text{res}} \, dx + \int_\Omega (B_\varepsilon - H) \times \mathbf{curl} H \cdot [u_\varepsilon - U_{\varepsilon,\delta}]_{\text{ess}} \, dx,
\]
where the first integral on the right-hand side can be “absorbed” by Gronwall’s argument, while, by virtue of (4.5),
\[
\left| \int_\Omega (B_\varepsilon - H) \times \mathbf{curl} H \cdot [u_\varepsilon - U_{\varepsilon,\delta}]_{\text{res}} \, dx \right| 
\leq |\Omega_{\text{res}}(t)|^{1/3} \| \mathbf{curl} H \|_{L^\infty(\Omega; R^3)} \| u_\varepsilon - U_{\varepsilon,\delta} \|_{L^6(\Omega; R^3)} \| B_\varepsilon - H \|_{L^2(\Omega; R^3)}
\leq \varepsilon^{2/3} \| \mathbf{curl} H \|_{L^\infty(\Omega; R^3)} \| u_\varepsilon - U_{\varepsilon,\delta} \|_{L^6(\Omega; R^3)} \| B_\varepsilon - H \|_{L^2(\Omega; R^3)}.
\]
The expression on the right-hand side can be controlled by the integrals on the left-hand side of (6.20) provided
\[
a \leq \frac{4}{3}.
\]
Next, we compute
\[
\int_\Omega \mathbf{curl} (U_{\varepsilon,\delta} \times (B_\varepsilon - H)) \cdot (B_\varepsilon - H) \, dx = -\int_\Omega \int \frac{1}{2} \text{div}_x U_{\varepsilon,\delta} [B_\varepsilon - H]^2 + (B_\varepsilon - H) \cdot \nabla_x U_{\varepsilon,\delta} \cdot (B_\varepsilon - H) \, dx \leq c \int \| B_\varepsilon - H \|^2 \, dx
\]
and observe that
\[
\int \mathbf{curl} (U_{\varepsilon,\delta} - v) \times H \cdot \mathbf{curl} (H - B_\varepsilon) \, dx = \int \mathbf{curl} (H - B_\varepsilon) \cdot (\mathbf{curl} (H - B_\varepsilon) \, dx = g_{\varepsilon,\delta}.
\]
Thus we are left with
\[
\int_0^\tau \int_\Omega \left( \frac{\varrho_\varepsilon}{\varrho} - 1 \right) (\mathbf{curl} H \times H) \cdot (u_\varepsilon - U_{\varepsilon,\delta}) \, dx \, dt
= \int_0^\tau \int_\Omega \left[ \frac{\varrho_\varepsilon}{\varrho} - 1 \right]_{\text{ess}} (\mathbf{curl} H \times H) \cdot (u_\varepsilon - U_{\varepsilon,\delta}) \, dx \, dt
+ \int_0^\tau \int_\Omega \left[ \frac{\varrho_\varepsilon}{\varrho} - 1 \right]_{\text{res}} (\mathbf{curl} H \times H) \cdot (u_\varepsilon - U_{\varepsilon,\delta}) \, dx \, dt,
\]
where the first integral tends to zero for \( \varepsilon \to 0 \). Finally, by virtue of (4.5),
\[
\left| \int_0^\tau \int_\Omega \left[ \frac{\varrho_\varepsilon}{\varrho} - 1 \right]_{\text{res}} (\mathbf{curl} H \times H) \cdot (u_\varepsilon - U_{\varepsilon,\delta}) \, dx \, dt \right|
\leq c |\Omega_{\text{res}}(t)|^{7/30} \| \varrho_\varepsilon - \bar{\varrho} \|_{L^{5/3}(\Omega)} \| u_\varepsilon - U_{\varepsilon,\delta} \|_{L^6(\Omega; R^3)}
\leq \varepsilon^{5/3} \| u_\varepsilon - U_{\varepsilon,\delta} \|_{L^6(\Omega; R^3)};
\]
whence we need \( a < 10/3 \).
We therefore conclude that
\[
\left[ \mathcal{E}_\varepsilon \left( g_{\varepsilon,\delta}, u_\varepsilon, B_\varepsilon; r_{\varepsilon,\delta}, U_{\varepsilon,\delta}, H \right) \right]_{\varepsilon=0}^{\varepsilon=\tau} \leq \int_0^\tau (g_{\varepsilon,\delta} + h_{\varepsilon,\delta}) \mathcal{E} \left( g_{\varepsilon,\delta}, u_\varepsilon, B_\varepsilon; r_{\varepsilon,\delta}, U_{\varepsilon,\delta}, H \right) \, dt + \int_0^\tau g_{\varepsilon,\delta}(t) \, dt.
\]
Thus letting first \( \varepsilon \to 0 \) and then \( \delta \to 0 \) completes the proof of Theorem 3.1.
7. Concluding remarks. The restrictions imposed on the "diffusion" coefficients
\[ \nu_\varepsilon \approx \varepsilon^a, \quad \lambda_\varepsilon \approx \varepsilon^b, \quad a, b > 0, a \leq \frac{4}{3} \]
are probably not optimal. As a matter of fact, we do not need any restriction on \( \lambda_\varepsilon \) but the restrictions imposed on \( \nu_\varepsilon \) are more restrictive than in [22].

On the other hand, the restriction on \( \nu_\varepsilon \) can be relaxed if \( b \) satisfies certain bounds. Thus, for instance, replacing (6.21) by

\[
\left| \int_{\Omega} (B_\varepsilon - H) \times \text{curl} \ H \cdot [u_\varepsilon - U_{\varepsilon,\delta}]_{\text{res}} \, dx \right|
\leq |\Omega_{\text{res}}(t)|^{2/3} \|	ext{curl} \ H\|_{L^\infty(\Omega;R^3)} \|u_\varepsilon - U_{\varepsilon,\delta}\|_{L^6(\Omega;R^3)} \|B_\varepsilon - H\|_{L^6(\Omega;R^3)};
\]
whence we need
\[ 0 < a + b < \frac{8}{3}. \]

In particular, this improves the result in [22].

Finally, we remark that the same result can be obtained on exterior domains as well as in other geometries that admit the relevant dispersive estimates for acoustic waves.

Acknowledgments. Eduard Feireisl is supported by Grant 201/09/0917 of GA ČR as a part of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan RVO: 67985840. Yongzhong Sun is supported by NSF of China(Grant No. 11171145) and 10931007 and the PAPD of Jiangsu Higher Education Institutions.

REFERENCES

[1] F. Berthelin and A. Vasseur, From kinetic equations to multidimensional isentropic gas dynamics before shocks, SIAM J. Math. Anal., 36 (2005), 1807–1835.
[2] N. Burq, Global Strichartz estimates for nontrapping geometries: About an article by H. F. Smith and C. D. Sogge: “Global Strichartz estimates for nontrapping perturbations of the Laplacian,” Comm. Partial Differential Equations, 28 (2003), 1675–1683.
[3] M. Christ and A. Kiselev, Maximal functions associated to filtrations, J. Funct. Anal., 179 (2001), 409–425.
[4] C. M. Dafermos, The second law of thermodynamics and stability, Arch. Rational Mech. Anal., 70 (1979), 167–179.
[5] R. Danchin, Low Mach number limit for viscous compressible flows, M2AN Math. Model Numer. Anal., 39 (2005), 459–475.
[6] Y. Dermedjian and J.-C. Guillot, Scattering of elastic waves in a perturbed isotropic half space with a free boundary. The limiting absorption principle, Math. Methods Appl. Sci., 10 (1988), 87–124.
[7] B. Desjardins, Regularity of weak solutions of the compressible isentropic Navier-Stokes equations, Commun. Partial Differential Equations, 22 (1997), 977–1008.
[8] B. Ducomet and E. Feireisl, The equations of magnetohydrodynamics: On the interaction between matter and radiation in the evolution of gaseous stars, Commun. Math. Phys., 266 (2006), 595–629.
[9] J. Edward and D. Pravica, Bounds on resonances for the Laplacian on perturbations of half-space, SIAM J. Math. Anal., 30 (1999), 1175–1184.
[10] D. M. Éidus, The principle of limiting amplitude, (in Russian) Usp. Mat. Nauk, 24 (1969), 91–156.
[11] E. Feireisl, Incompressible limits and propagation of acoustic waves in large domains with boundaries, Commun. Math. Phys., 294 (2010), 73–95.
[12] E. Feireisl, *Local decay of acoustic waves in the low Mach number limits on general unbounded domains under slip boundary conditions*, Commum. Partial Differential Equations, 36 (2011), 1778–1796.

[13] E. Feireisl, *Low Mach number limits of compressible rotating fluids*, J. Math. Fluid Mechanics, 14 (2012), 61–78.

[14] E. Feireisl and A. Novotný, *Weak-strong uniqueness property for the full Navier-Stokes-Fourier system*, Arch. Rational Mech. Anal., 204 (2012), 683–706.

[15] E. Feireisl, A. Novotný and H. Petzeltová, *On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids*, J. Math. Fluid Mech., 3 (2001), 358–392.

[16] E. Feireisl, A. Novotný and Y. Sun, *Suitable weak solutions to the Navier-Stokes equations of compressible viscous fluids*, Indiana Univ. Math. J., 60 (2011), 611–631.

[17] G. P. Galdi, “An Introduction to the Mathematical Theory of the Navier-Stokes Equations. I,” Springer-Verlag, New York, 1994.

[18] I. Gallagher, *Résultats récents sur la limite incompressible*, Séminaire Bourbaki. Vol. 2003/2004, Astérisque, Exp. No. 926, vii, (2005), 29–57.

[19] P. Germain, *Weak-strong uniqueness for the isentropic compressible Navier-Stokes system*, J. Math. Fluid Mech., 13 (2011), 137–146.

[20] X. P. Hu and D. H. Wang, *Low Mach number limit of viscous compressible magnetohydrodynamic flows*, SIAM J. Math. Anal., 41 (2009), 1272–1294.

[21] S. Jiang, Q. Ju and F. Li, *Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions*, Comm. Math. Phys., 297 (2010), 371–400.

[22] S. Jiang, Q. Ju and F. Li, *Incompressible limit of the compressible magnetohydrodynamic equations with vanishing viscosity coefficients*, SIAM J. Math. Anal., 42 (2010), 2539–2553.

[23] T. Kato, *On classical solutions of the two-dimensional nonstationary Euler equation*, Arch. Rational Mech. Anal., 25 (1967), 188–200.

[24] T. Kato and C. Y. Lai, *Nonlinear evolution equations and the Euler flow*, J. Funct. Anal., 56 (1984), 15–28.

[25] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math., 120 (1998), 955–980.

[26] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math., 34 (1981), 481–524.

[27] H. Kozono, *Weak and classical solutions of the two-dimensional magnetohydrodynamic equations*, Tohoku Math. J. (2), 41 (1989), 471–488.

[28] P. Kukučka, *Singular limits of the equations of magnetohydrodynamics*, J. Math. Fluid Mech., 13 (2011), 173–189.

[29] Y.-S. Kwon and K. Trivisa, *On the incompressible limits for the full magnetohydrodynamics flows*, J. Differential Equations, 251 (2011), 1990–2023.

[30] P.-L. Lions, “Mathematical Topics in Fluid Dynamics. Vol. 2. Compressible Models,” Oxford Lecture Series in Mathematics and its Applications, 10, Oxford Science Publication, The Clarendon Press, Oxford University Press, New York, 1998.

[31] P.-L. Lions and N. Masmoudi, *Incompressible limit for a viscous compressible fluid*, J. Math. Pures Appl. (9), 77 (1998), 585–627.

[32] N. Masmoudi, *Incompressible, inviscid limit of the compressible Navier-Stokes system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 18 (2001), 199–224.

[33] N. Masmoudi, *Examples of singular limits in hydrodynamics*, in “Handbook of Differential Equations: Evolutionary Equations. Vol. III” (eds. C. Dafermos and E. Feireisl), Elsevier/North-Holland, Amsterdam, 2007.

[34] S. Schochet, *The mathematical theory of low Mach number flows*, M2ANMath. Model Numer. Anal., 39 (2005), 441–458.

[35] P. Secchi, *On the equations of ideal incompressible magnetohydrodynamics*, Rend. Sem. Univ. Padova, 90 (1993), 103–119.

[36] H. F. Smith and C. D. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. Partial Differential Equations, 25 (2000), 2171–2183.

[37] R. S. Strichartz, *A priori estimates for the wave equation and some applications*, J. Functional Analysis, 5 (1970), 218–235.

Received August 2012 for publication.

E-mail address: feireisl@math.cas.cz; novotny@univ-tln.fr; sunyz@nju.edu.cn