The development of projective metric method for analyzing star positions

E Y Kostina¹, A O Andreev², Y A Nefedev² and N Y Demina²
¹Moscow State University, Moscow, 119899 Russia
²Kazan Federal University, Institute of Physics, Kazan, 420008 Russia

E-mail: kostina_elena_1955@mail.ru

Abstract. In this work, the projective geometry method was used for analyzing star clusters. When carrying out the calculation procedures, it was considered that non-linear distortion factors had been removed from the measured stars’ coordinates. Determination of stars’ proper motions is of great practical importance, as the inertial coordinate system relies on catalogues of star positions, and it is necessary to be aware of the stellar reference marks’ time shift. In the practical part of the work, the breadboard simulation of the use of the proposed method for determining stars’ proper motions is performed. At the same time, it is supposed that at 90°/mm breadboard image scale the absolute values of proper motions do not exceed 0.050′ over a period of 50 years. As result, determined that the standard deviation of the calculated proper motions $\mu_\alpha$, $\mu_\beta$ from their true value is 0.0065 arcseconds for the first model (when the proper motions of the reference “stars” are negligible and equal to 0) and 0.0072 arcseconds or the second model (when the reference stars do have real proper motions). These values indicate the high accuracy of the used method.

1. Introduction
The main purpose of the Global Astrometric Interferometer for Astrophysics (GAIA) mission [1], successor of the Hipparcos project and Hipparcos Celestial Reference Frame (HCRF) system [2], is creation of highly accurate map of star distribution in the Milky Way galaxy. The GAIA scanning system allows producing star positions with an accuracy of 10 mas [3]. However, the problem of determining proper motions of stars has not yet been completely solved [4]. This work focuses on the use of projective metric method [5, 6] for the determination of star positions and proper motions produced on the basis of positional observations. The novelty of the method lies in the fact that this approach allows analyzing images of multiple systems of reference star positions having separate proper motions. This projective geometry apparatus may have a wide application for studying multi-parameter dynamic coordinate systems and building model of the clusters under consideration [7]. As is known, projective geometry studies invariant projective transformations [8]. The main feature of projective geometry is the principle of duality, which adds elegant symmetry to many designs [9]. While the properties of the figures that Euclidean geometry deals with are metric [10], but there are figures that can be converted into one another by movement while maintaining the metric properties, that is, there are more complex properties of geometric figures, it is necessary to use the corresponding methods [11].

Some of the main methods available for analyzing complex astronomical systems are the robust methods [7–11]. It is necessary to take into account that such methods cannot be used in all cases, as
not all celestial structures are of stochastic nature (including shape, physical parameters etc.). To study complex systems, fractal analysis can be used. For complex objects, the fractal method allows to determine values of fractal dimension (FD) and self-similarity coefficients. On the basis of these parameters it is possible to study connections between a celestial body’s evolutionary parameters and its structure. Therefore, the study of celestial objects by means of fractal geometry methods is the relevant and modern task. Both the structure of Venus and its gravitational field relate to complex multiparameter systems. For studying such systems, it is necessary to use the theory of complex physics, which includes the fractal geometry methods [3]. In this work, the regression model of Venusian structure was created on the basis of altimetry data of “Magellan” mission (NASA) [5]. The general aim of “Magellan” mission was the study of Venusian chemical parameters, its inner and outer structure, and planetary properties [5]. Using methods of projective geometry, particularly projective coordinates, allows to create a strict method with which it is possible to analyze the images of multiple systems of the stars positions that have their proper motions, independent of each other. To define the relative proper motions of a large numbers of stars it is convenient to use the method which based on the comparison of the same coordinate domain, where the explored objects, obtained in different time eras: \( T_0 \) and \( T_1 \) \( (T_0 < T_1) \) are situated, herewith the stars positions are considered as the coordinate points.

The problem of analyzing the independent proper motions of the stars positions cluster can be solved in 2 stages. At the first stage as a result of projective images transformation of the explored coordinate points from \( T_1 \)-era image field to \( T_0 \)-era image field \( (T_0 < T_1) \), linear displacements, caused by the explored points position change, are found \( (\Delta x, \Delta y)_i \) \( (i = 1,2\ldots k) \). At the second stage displacement equalizations \( (\Delta x, \Delta y) \) are taken as “measured values”, which directly connected with the proper motions of the explored coordinate points [12].

The use of this projective geometry formalism may have wide application for multiparametric dynamical coordinate systems exploration and building models of the considered cluster [13]. In this paper the described method was used for analyzing images of star cluster and it was considered that the field of the measured coordinates had been previously cleared of non-linear distorting parameters using the data of observations made with telescopes in EAO [14]. It was also supposed that defining the proper motions of the stars has a great practical value as an inertial reference system is based on the stars positions catalogues and the time displacement of the star reference points should be known.

2. The projective metric method

The transformation from image field \( F_1 \), \( (T_1\text{-era}) \), to image \( F_0 \) \( (T_0\text{-era}) \) corresponds to detecting the proper star motions in the form of linear displacement of these objects images. The following steps were performed to solve the problem:

I. There are \( n \) images of the same stars, considered as reference, both in the first and in the second fields. \( m \) reference groups are formed from these reference stars, there are \( 4 \) reference stars in each group. Thus, there are \( m \) systems of projective coordinates \( (m \geq 1) \) set in each field.

II. The reference groups do not contain any common stars, i.e. the reference groups contain only independent reference points

III. The measurements of images coordinates are independent.

Since we take measured coordinates of an image of an object in the field \( F_0 \) as preliminary coordinates, from the solution of the system of normal equations we are going to obtain a linear displacement of an image of an object caused by its proper motion. The solution obtained with the least square method (LSM) allows to get a covariance matrix \( K \) of the images’ linear displacement caused by proper motion of the object, and an error of unit weight \( \mu_0 \). The covariance matrix is equal to the inverse matrix of normal equations: \( K = Q^{-1} \).

It is obvious that while performing the transformations we are forced to take images of an object, that has its proper motion, as the reference points. That is why it is desirable to take the objects, which have small proper motion and which are not capable of distorting the results. These stars \textit{a priori} cannot be specified.
The reference has its own proper motion although it is small, so that leads to changes in mutual arrangements of the reference stars in the field \( F_1 \). Let’s pretend that there is a fixed point on the images in \( F_0 \) and \( F_1 \), relative to which the displacement of the reference is considered. Then projective coordinates of this star in the field \( F_0 \) will not be equal to its projective coordinates in the field \( F_1 \). Theoretically, from projective geometry point of view, it is inadmissible. It does not seem possible to take the proper motions into account directly in coordinates. It is quite obvious that this effect will cause a certain component \( \Delta \bar{t} \) in the defined image displacements. The component will not be the same for all the objects being explored because, as the results of numerical experiments show, the precision of projective transformation depends on mutual arrangement of reference points and the transformed point. The value of the \( \Delta \bar{t} \) – vector for each of the objects, being defined, should be estimated and taken into account at the second stage of the calculations. For practical reasons, let’s pretend that \( \Delta \bar{t} = \bar{t}_1 = \bar{t}_2 = \ldots = \bar{t}_k = \bar{t} \). In this case the problem can be the first stage of solving the problem a number of coordinates being transformed include \( l \) reference stars with well-known coordinates and the displacements of their images are being calculated. In this case the vector \( \bar{t} \) – a result of the first stage – can be represented in the form of 2 vectors \( \bar{t} \) and \( \bar{t} \) of \( 2l \) dimension respectively, so that \( t + l = k \).

Similarly, one may explore Venusian gravity anomalies \([18, 19]\). In general case to define the best vector \( \Delta \) estimation we should process the set of \( \Delta s \) vectors as a series of interdependent vectors. In particular case it can be assumed that the proper motions of the reference points do not correlate with each other. The assumption is not strict enough from the theoretical point of view, but it may be justified in practical terms, since the preliminary covariance matrices of the proper motions of the objects are missing. In this case the covariance matrix of the set of the vectors \( \delta_{ij} \) values will be in a block-diagonal form. There will be \( K \) blocks \((2m, 2m)\) sized on the diagonal which are covariance matrices of \( m \) realizations of \( s \)-th reference point image displacement.

Then the procedure of defining \( \Delta \) estimation can be described as follows: Then the procedure of defining \( \Delta \) estimation can be described as follows:

I. Separately for each \( s \)-th reference star an estimation of vector of image displacement is defined. For this purpose, processing of the set of the vectors \( \delta_{js} \) is carried out. The relationship between the vectors is characterized by the covariance matrix \( K_s \).

II. For each reference star the vector \( \Delta s \) (characterizing the impact of proper motions of the reference stars of \( m \) reference groups on the image displacements being defined) is determined.

III. Value of vector \( \Delta \) is defined as a generalized midpoint from all \( \Delta s \) \((s = 1, 1, \ldots, l)\).

In further calculations components of the vector \( \Delta \) are taken into account in image displacement of the objects, whose proper motions are being defined. The components of the vector \( \delta \) with the covariance matrix \( Q^{-1} \), free the influence of proper motions of reference points, constructing reference groups, are accepted as determined values on the second stage of equalizing. Projective coordinates \((\xi_1, \xi_2, \xi_3)\) in this case will be defined as some ratios of mixed products, constructed from the unit vectors to the reference points \( R_0^0, R_1^0, R_2^0, R_3^0 \) and \( R_{0i}^0 \) unit vector to the star. Differential changes of space coordinates will induce corresponding changes of stars images coordinates. Proposing that the proper motions of stars are small, we can accept \( d\xi_0 = \mu_0 \), \( d\xi_0 = \mu_0 \). If \( B_{ij} \) – matrix of partial derivatives. Coefficients of the matrix \( B_{ij} \) are calculated by the measured coordinates of images of 4 stars from the reference group, their space coordinates, as well as by the images coordinates and space coordinates of the explored stars and will be non-dimensional. Summation of the blocks \( Q_{ij}, L_j \) is performed over numbers of the reference groups. Estimate of precision of equalizing results is provided according to the principles of LSM applied to dependent measurements.

3. Results of the projective metric method analysis

Practical application of the algorithm written above was implemented for deriving proper motions of the stars and was performed according to the model images of the same reference and explored “stars”, for which the proper motions were simulated. For the reference “stars” the simulation was performed in the time interval of 50 years assuming that the proper motions of the reference “stars” do
not exceed 0.050" per year. The established value of variance in the image coordinates for each model picture is 2 micron, variances of the “stars” spherical coordinates are 0.15 arcseconds. The scale of the model images corresponds to 90”/mm.

In the Table 1 the values of proper motions, obtained with assuming that the proper motions of the reference “stars” are negligible and equal to 0, are represented. For the first stage this assumption is equivalent to the presence of special marks, depicted in both pictures. Mean square errors of the proper motions determination are given in thousandths of arcseconds (columns 6, 7). The equalizing is performed at 4 reference groups.

In the Table 2 the results of calculations for the case, that most consistent with reality – the reference stars do have proper motions, are represented. To take account of systematic component $\Delta$, arising due to the reference stars displacement, the controlling “stars” were used. In this experiment the “proper motions” of the controlling “stars” were defined as well. In Table 2 and Table 3 the controlling “stars” are marked with *. The corresponding changes of coordinates of the images are represented in the Table 3 (columns 6, 7).

The standard deviation of the calculated proper motions $\mu_{\alpha}$, $\mu_{\delta}$ from their true value is 0.0065 arcseconds for the first experiment and 0.0072 arcseconds – for the second.

Table 1. Results of equalizing in the absence of reference system of points motion.

| N  | Control Values | Equalizing Results |
|----|----------------|--------------------|
|    | $\mu_{\alpha}$ (s) | $\mu_{\delta}$ (") | $\mu_{\alpha}$ | $\mu_{\delta}$ | $m_{\alpha}$ | $m_{\delta}$ |
| 1  | 2              | 3                  | 4     | 5      | 6      | 7      |
| 71 | 0.0000         | 0.0000             | 0.0002 | 0.012  | 1.3    | 1.3    |
| 82 | -0.0008        | 0.100              | -0.0005 | 0.092  | 1.3    | 1.2    |
| 84 | -0.0036        | 0.117              | -0.0039 | 0.126  | 1.3    | 1.2    |
| 109| 0.0067         | 0.012              | 0.0066  | 0.015  | 1.3    | 1.3    |
| 120| 0.0029         | 0.039              | -0.0027 | 0.039  | 1.3    | 1.3    |
| 126| -0.0027        | 0.005              | -0.0020 | -0.011 | 1.3    | 1.3    |

Unit weight error $\mu_0 = 0.0028$ mm

Table 2. Results of equalizing in the presence of reference system of points motion.

| N  | Control Values | Equalizing Results |
|----|----------------|--------------------|
|    | $\mu_{\alpha}$ (s) | $\mu_{\delta}$ (") | $\mu_{\alpha}$ | $\mu_{\delta}$ | $m_{\alpha}$ | $m_{\delta}$ |
| 1  | 2              | 3                  | 4     | 5      | 6      | 7      |
| 71*| 0.0000         | 0.0000             | 0.0001 | 0.007  | 1.4    | 1.4    |
| 82 | -0.0008        | 0.100              | -0.0006 | 0.084  | 1.4    | 1.4    |
| 84 | -0.0036        | 0.117              | -0.0038 | 0.119  | 1.4    | 1.4    |
The Table 4 represents values of components of the vectors $\Delta_1$ for the reference stars and components of the vector $\Delta$, which was taken into account in displacement of images during the second stage of equalizing.

| N   | Control Values | Reference Stars Immovable | Reference Stars Movable | With Considering Systematic Component |
|-----|----------------|---------------------------|-------------------------|---------------------------------------|
|     | $dx$           | $dy$                      | $dx$                    | $dy$                                  |
| 71* | 0.0067         | 0.012                     | 0.0065                  | 0.011                                 |
| 120*| −0.0029        | 0.039                     | −0.0028                 | 0.033                                 |
| 126 | −0.0027        | −0.005                    | −0.0021                 | −0.016                                |
|     | 1.4            | 1.4                       | 1.4                     | 1.4                                   |

Unit weight error $\mu_0 = 0.0030$ mm

Table 3. Changes of the coordinates of the images (micron).

Table 4. Systematic components in displacements (mm).

4. Summary and conclusions
The analysis of the numerical experiments results allows to conclude that there is the possibility of defining proper motions of independent objects, included in space clusters, with the method based on the vector interpretation of the projective geometry invariants – projective coordinates [15]. The proposed method has the following positive features [16]:

I. The proper motions of the stars are obtained directly from the solution of the set of equations with the LSM (least square method) with the estimate of precision of the obtained parameters. The solution can also be found with the robust statistical procedure [17].

II. The proper motions of the objects are defined strictly without inner orientation of a filming camera [18]. The knowledge of the external orientation angles of an image is not required either. So this method is convenient for processing of images, which were obtained by multi-scale filming systems [19].

The precision of defining the proper motions of the objects will depend on a number of factors, one of which is geometry and dynamics of the reference system of points [20].

Acknowledgments
This work was partially supported by Russian Science Foundation, grants no. 20-12-00105 (according to the grant, the method for data analysis was created and the numerical calculations were carried out). This work is performed according to the Russian Government Program of Competitive Growth of
Kazan Federal University. This work was partially supported by a scholarship of the President of the Russian Federation to young scientists and post-graduate students SP-3225.2018.3, the Russian Foundation for Basic Research grant no. 19-32-00024 Aspirants and the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS”.

References
[1] Prusti T et al. 2016 Astron. Astrophys. 595 A2
[2] Malkin Z 2016 Mon. Not. R. Astron. Soc. 461 1937–42
[3] Brown A et al. 2018 Astron. Astrophys. 616 A1
[4] Zacharias N, Finch C and Frouard J 2017 Astron. J. 153 166
[5] Guo R, Shi X and Wang Z 2019 J. Electron. Imaging 28 023032
[6] Lynchenko A, Sheshkus A and Arlazarov V 2018 Proc. SPIE 11041 110411K
[7] Struve H and Struze R 2010 J. Geom. 98 151–70
[8] Nefedyev Y, Bezmenov V, Demin S, Andreev A and Demina N 2016 Nonlinear Phenom. Complex Syst. 19 102–06
[9] Uddin W and Sozer Y 2017 IEEE Trans. Ind. Appl. 53 4431–40
[10] Andreev A, Demina N, Demin S, Nefedyev Y and Churkin K O 2016 Nonlinear Phenom. Complex Syst. 19 271–77
[11] Bauer A, Schiesser E and Rolland J 2018 Nat. Commun. 9 1–11
[12] Demin S, Nefedyev Y, Andreev A, Demina N and Timashev S 2018 Adv. Space Res. 61 639–44
[13] Tang Y and Yin J 2017 Extreme Mech. Lett. 12 77–85
[14] Nefedyev Y, Andreev A, Petrova N, Demina, N and Zagidullin A 2018 Astron. Rep. 62 1016–20
[15] Neto J, Junior E, Moreno S, Ayala H, Mariani V and dos Santos Coelho L 2018 Energy 162 645–58
[16] Zhang X and Xue Y 2019 IOP Conf. Ser.: Mater. Sci. Eng. 573 012100
[17] Churkin K, Andreev A, Nefedyev Y, Petrova N and Demina N 2018 Astron. Rep. 62 1042–49
[18] Andreev A, Zagidullin A, Demina N, Petrova N, Nefedyev Y and Demin S 2018 J. Phys. Conf. Ser. 1135 012002
[19] Demina N, Andreev A, Nefedyev Y, Akhmededshina E and Demin S 2019 J. Phys. Conf. Ser. 1400 022019
[20] Vieira R, Petry A, Rocha L, Isoldi L and Dos Santos E 2017 Renew. Energ. 109 222–34