A Rigidity Theorem for Affine Kähler-Ricci Flat Graph

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Abstract: It is shown that any smooth strictly convex global solution on $\mathbb{R}^n$ of

$$\det \left( \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) = \exp \left\{ - \sum_{i=1}^{n} d_i \frac{\partial u}{\partial \xi_i} - d_0 \right\},$$

where $d_0, d_1, ..., d_n$ are constants, must be a quadratic polynomial. This extends a well-known theorem of Jörgens-Calabi-Pogorelov.

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Key words: Pogorelov Theorem; Kähler-Ricci Flat; Monge-Ampère equation.

§1. Introduction

A well-known theorem of Jörgens ($n = 2$ [J]), Calabi ($n \leq 5$ [Ca]), and Pogorelov ($n \geq 2$ [P]) states that any smooth strictly convex solution of

$$(1.1) \quad \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) = 1 \quad \text{on} \quad \mathbb{R}^n$$

must be a quadratic polynomial. In [C-Y] Cheng and Yau gave an analytical proof. Recently Caffarelli and Li [C-L] extended the result for classical solution to viscosity solution.

In this paper we study the following PDE

$$(1.2) \quad \frac{\partial^2}{\partial x_i \partial x_j} \left( \log \det \left( \frac{\partial^2 f}{\partial x_k \partial x_l} \right) \right) = 0,$$

or

$$(1.3) \quad \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) = \exp \left\{ \sum_{i=1}^{n} d_i x_i + d_0 \right\},$$

where $d_0, d_1, ..., d_n$ are constants. Obviously, all solutions of (1.1) satisfy (1.2). Introduce the Legendre transformation of $f$

$$\xi_i = \frac{\partial f}{\partial x_i}, \quad i = 1, 2, \ldots, n,$$

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\[ u(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} - f(x). \]

In terms of \( \xi_1, \ldots, \xi_n, u(\xi_1, \ldots, \xi_n) \), the PDE (1.3) can be written as

\[ (1.4) \quad \det \left( \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) = \exp \left\{ - \sum_{i=1}^{n} d_i \frac{\partial u}{\partial \xi_i} - d_0 \right\}. \]

Note that, under the Legendre transformation, the PDE (1.1) reads

\[ (1.1)' \quad \det \left( \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) = 1 \quad \text{on} \quad \mathbb{R}^n. \]

Given any smooth, bounded convex domain \( \Omega \subset \mathbb{R}^n \) and any smooth boundary value \( \phi \), the existence of the solution of the boundary problem

\[ (1.4)' \quad \det \left( \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) = \exp \left\{ - \sum_{i=1}^{n} d_i \frac{\partial u}{\partial \xi_i} - d_0 \right\} \quad \text{in} \quad \Omega, \quad u = \phi \quad \text{on} \quad \partial \Omega \]

is well-known. So there are many locally solutions to the PDE (1.4). In this paper we prove the following theorem

**Main Theorem.** Let \( u(\xi_1, \ldots, \xi_n) \) be a \( C^\infty \) strictly convex function defined on whole \( \mathbb{R}^n \). If \( u(\xi) \) satisfies the PDE (1.4), then \( u \) must be a quadratic polynomial.

The PDE (1.2) arises naturally in the construction of Ricci flat Kähler-affine metric for affine manifolds. An affine manifold is a manifold which can be covered by coordinate charts so that the coordinate transformations are given by invertible affine transformations. Let \( M \) be an affine manifold. A Kähler affine metric or Hessian metric \( G \) on \( M \) is a Riemannian metric on \( M \) such that locally, for affine coordinates \((x_1, x_2, \ldots, x_n)\), there is a potential \( f \) such that

\[ G_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}. \]

The pair \((M, G)\) is called a Kähler affine manifold or a Hessian manifold, and \( G \) is called Kähler affine metric. Kähler affine metric was first studied by Cheng and Yau in [C-Y-1]. For more details about Hessian manifolds please see [Sh]. Following Cheng and Yau we introduce the concepts of the Kähler Ricci curvature and the Kähler scalar curvature of \( G \) on \( M \). It is easy to see that the tangent bundle \( TM \) is a complex manifold with a natural complex structure in the following way. For coordinate chart \((x_1, x_2, \ldots, x_n)\), we can consider a tube over the coordinate neighborhood with complex coordinate system
The Hessian metric $G$ was naturally extended to be a Kähler metric of the complex manifold $TM$. The Ricci curvature tensor and the scalar curvature of this Kähler metric are given by respectively

$$R_{ij} = -\frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})), \quad R = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f^{ij}_{kl} \frac{\partial^2 (\log \det (f_{kl}))}{\partial x_i \partial x_j}.$$  

It is obvious that the restrictions of $R_{ij}$ and $R$ to $M$ are tensors of $M$. We also call $R_{ij}$ and $R$ the Kähler Ricci curvature and the Kähler scalar curvature of $G$ on $M$. We say that the metric $G$ is Kähler-Ricci flat if (1.2) holds on $M$ everywhere. In this geometric language, our Main Theorem can be stated as

**Main Theorem.** Let $M$ be a graph given by a smooth strictly convex function $x_{n+1} = f(x_1, ..., x_n)$ defined in a domain $\Omega$. If the Hessian metric of $M$ is Kähler-Ricci flat and the image of $M$ under the normal mapping is whole $\mathbb{R}^n$, then $f$ must be a quadric.

**Remark 1.** In [J-L] the authors have proved that

**Theorem.** Let $M$ be a Kähler affine manifold. If the Hessian metric of $M$ is Kähler-Ricci flat and complete, then $M$ must be $\mathbb{R}^n/\Gamma$, where $\Gamma$ is a subgroup of isometries which acts freely and properly discontinuously on $\mathbb{R}^n$.

**Remark 2.** From our proof of the Main Theorem the following stronger version is also true:

**Main Theorem'.** Let $u(\xi_1, ..., \xi_n)$ be a $C^\infty$ strictly convex function defined in a convex domain $\Omega \subset \mathbb{R}^n$. If $u(\xi)$ satisfies the PDE (1.4) and if $u(p) \rightarrow \infty$ as $p \rightarrow \partial \Omega$, then $u$ must be a quadratic polynomial.

**Remark 3.** The global solution of the PDE (1.3) on the $x$-coordinate plane $\mathbb{R}^n$ is not unique. For example,

$$f(x_1, ..., x_n) = \sum_{i=1}^{n} x_i^2, \quad \text{and} \quad f(x_1, ..., x_n) = \exp\{x_1\} + \sum_{i=2}^{n} x_i^2$$

are global solutions of the PDE (1.3).

**Remark 4.** Our study in this paper is based on the following differential inequality for $\Phi$ (for details see Proposition 3.1 below)

$$\Delta \Phi \geq \frac{n}{n-1} \frac{\|\nabla \Phi\|^2}{\Phi} + \frac{n^2 - 3n - 10}{2(n-1)} \langle \nabla \Phi, \nabla \log \rho \rangle + \frac{(n+2)^2}{n-1} \Phi^2.$$  

This type of differential inequality for $\Phi$ first appeared in [L-J-1], in which Li and Jia announced that they solved the Chern’s conjecture for 2-dimension and 3-dimension. While
Trudinger and Wang solved Chern’s conjecture for 2-dimension in [T-W]. Li and Jia’s method, which is quite different from that of Trudinger and Wang, is to estimate $\Phi$ and $\|\nabla f\|$ based on the differential inequality:

$$
\Delta^B \Phi \geq \frac{n}{2(n-1)} \frac{\|\nabla \Phi\|_{G^B}^2}{\Phi} - \frac{n^2 - n - 2}{2(n-1)} (\nabla \Phi, \nabla \log \rho)_{G^B} \\
+ \left( 2 - \frac{(n-2)^2(n-1)}{8n} - \frac{n^2 - 2}{2(n-1)} \right) \frac{\Phi^2}{\rho},
$$

where $G^B$ is the Blaschke metric and $\Delta^B$ is the Laplacian with respect to $G^B$.

However, Li later found a gap in their proof, so the full research paper is not published. In [L-J-2] the author use the similar differential inequality to prove Bernstein properties for some more general fourth order nonlinear PDE for 2 dimension. As a corollary, they fix the gap to 2 dimensional Chern’ conjecture. So far the 3 dimensional Chern’ conjecture is open.

§2. Preliminaries

Let $f(x_1, ..., x_n)$ be a $C^\infty$ strictly convex function defined on a domain $\Omega \subset \mathbb{R}^n$. Denote $M := \{(x, f(x))| x_{n+1} = f(x_1, ..., x_n), (x_1, ..., x_n) \in \Omega\}$. We choose the canonical relative normalization $Y = (0, 0, ..., 1)$. Then, in terms of language of the relative affine differential geometry, $G$ is the relative metric with respect to the normalization $Y$. Denote by $y = (x_1, ..., x_n, f(x_1, ..., x_n))$, the position vector of $M$. We have

$$
y_{ij} = \sum A_{ij}^k y_k + f_{ij} Y.
$$

The conormal field $U$ is given by

$$
U = (-f_1, ..., -f_n, 1).
$$

We recall some fundamental formulas for the graph $M$ without proof, for details see [P-1]. The Levi-Civita connection with respect to the metric $G$ is

$$
\Gamma_{ij}^k = \frac{1}{2} \sum f^{kl} f_{ijl},
$$

The Fubini-Pick tensor $A_{ijk}$ and the Weingarten tensor are given by

$$
A_{ijk} = -\frac{1}{2} f_{ijk}, \quad B_{ij} = 0.
$$
The relative Pick invariant is
\[ J = \frac{1}{4n(n-1)} \sum f^{il} f^{jm} f^{kn} f_{ij} f_{lmn}. \]

The Gauss equations and the Codazzi equations read
\[ R_{ijkl} = \sum f^{mh} (A_{jkm} A_{hil} - A_{ikm} A_{hji}), \]
\[ A_{ijk,l} = A_{ijl,k}. \]

From (2.6) we have
\[ R_{ik} = \sum f^{mh} f^{ij} (A_{iml} A_{hjk} - A_{imk} A_{hlj}). \]

Denote
\[ \rho = [\det(f_{ij})]^{1/(n+2)}, \quad \Phi = \frac{\|\nabla \rho\|^2}{\rho^2}. \]

Let \( \Delta \) be the laplacian with respect to the Calabi metric, which is defined by
\[ \Delta = \frac{1}{\sqrt{\det(G_{kl})}} \sum \frac{\partial}{\partial x_i} \left( G^{ij} \sqrt{\det(G_{kl})} \frac{\partial}{\partial x_j} \right). \]

By a direct calculation from (2.10) we have
\[ \Delta = \sum f^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \frac{n+2}{2} \frac{1}{\rho} \sum f^{ij} \frac{\partial \rho}{\partial x_j} \frac{\partial}{\partial x_i} \]
\[ = \sum w^{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j} - \frac{n+2}{2} \frac{1}{\rho} \sum w^{ij} \frac{\partial \rho}{\partial \xi_j} \frac{\partial}{\partial \xi_i}, \]
\[ \Delta f = n + \frac{n+2}{2} \frac{1}{\rho} \langle \nabla \rho, \nabla f \rangle, \]
\[ \Delta u = n - \frac{n+2}{2} \frac{1}{\rho} \langle \nabla \rho, \nabla u \rangle. \]
§3. Calculation of $\Delta \Phi$

The following proposition is proved in [J-L], however, we include here for the reader’s convenience.

**Proposition 3.1** Let $f(x_1, \ldots, x_n)$ be a $C^\infty$ strictly convex function satisfying the PDE (1.3). Then the following estimate holds

$$
\Delta \Phi \geq \frac{n}{n-1} \frac{\|\nabla \Phi\|^2}{\Phi} + \frac{n^2 - 3n - 10}{2(n-1)} \langle \nabla \Phi, \nabla \log \rho \rangle + \frac{(n+2)^2}{n-1}\Phi^2.
$$

**Proof.** From the PDE (1.4) we have

$$
0 = \frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})) = -(n+2) \left( \frac{\rho_{ij}}{\rho} - \frac{\rho_i \rho_j}{\rho} \right),
$$

where $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\rho_{ij} = \frac{\partial^2 \rho}{\partial x_i \partial x_j}$. It follows that

$$
\Delta \rho = \frac{n+4}{2} \frac{\|\nabla \rho\|^2}{\rho}.
$$

Let $p \in M$, we choose a local orthonormal frame field of the metric $G$ around $p$. Then

$$
\Phi = \frac{\sum (\rho_{ij})^2}{\rho^2}, \quad \Phi_{,i} = 2 \sum \frac{\rho_j \rho_{j,i}}{\rho^2} - 2 \rho_{,i} \sum \frac{(\rho_j)^2}{\rho^3},
$$

$$
\Delta \Phi = 2 \sum \frac{(\rho_{ji})^2}{\rho^2} + 2 \sum \frac{\rho_{j} \rho_{jii}}{\rho^2} - 8 \sum \frac{\rho_{j} \rho_{i} \rho_{ji}}{\rho^3} - (n-2) \left( \frac{\sum (\rho_{j})^2}{\rho^4} \right)^2,
$$

where we used (3.2). In the case $\Phi(p) = 0$, it is easy to get, at $p$,

$$
\Delta \Phi \geq 2 \frac{\sum (\rho_{ij})^2}{\rho^2}.
$$

Now we assume that $\Phi(p) \neq 0$. Choose a local orthonormal frame field of the metric $G$ around $p$ such that $\rho_{,1}(p) = \|\nabla \rho\|(p) > 0$, $\rho_{,i}(p) = 0$ for all $i > 1$. Then

$$
\Delta \Phi = 2 \sum \frac{(\rho_{ij})^2}{\rho^2} + 2 \sum \frac{\rho_{j} \rho_{jii}}{\rho^2} - 8 \sum \frac{\rho_{j} \rho_{i} \rho_{ji}}{\rho^3} - (n-2) \left( \frac{\rho_{1}}{\rho^4} \right)^2.
$$

Applying an elementary inequality

$$
a_1^2 + a_2^2 + \cdots + a_{n-1}^2 \geq \frac{(a_1 + a_2 + \cdots + a_{n-1})^2}{n-1}
$$


and (3.2), we obtain

\[(3.4) \quad 2\sum \frac{(\rho_{,ij})^2}{\rho^2} \geq 2\frac{(\rho_{,11})^2}{\rho^2} + 4\sum_{i>1} \frac{(\rho_{,ii})^2}{\rho^2} + 2\sum_{i>1} \frac{(\rho_{,ii})^2}{\rho^2} \geq 2\frac{(\rho_{,11})^2}{\rho^2} + 4\sum_{i>1} \frac{(\rho_{,ii})^2}{\rho^2}\]

\[+ \frac{2}{n-1} \frac{(\Delta \rho - \rho_{,11})^2}{\rho^2} \geq \frac{2n}{n-1} \frac{(\rho_{,11})^2}{\rho^2} + 4\sum_{i>1} \frac{(\rho_{,ii})^2}{\rho^2} - \frac{2n + 4 (\rho_{,11})^2 + (n + 4)^2 (\rho_{,1})^4}{2(n-1) \rho^4}.
\]

An application of the Ricci identity shows that

\[(3.5) \quad \frac{2}{\rho^2} \sum \rho_j \rho_{,j,ii} = \frac{2}{\rho^2} (\Delta \rho)_{,1} \rho_{,1} + 2R_{11} \frac{(\rho_{,1})^2}{\rho^2}
\]

\[= 2(n+4) \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^4} - (n+4) \frac{(\rho_{,1})^4}{\rho^4} + 2R_{11} \frac{(\rho_{,1})^2}{\rho^2}.
\]

Substituting (3.4) and (3.5) into (3.3) we obtain

\[(3.6) \quad \Delta \Phi \geq \frac{2n}{n-1} \frac{(\rho_{,11})^2}{\rho^2} + \left(2n - 2 \frac{n + 4}{n-1}\right) \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} + 2R_{11} \frac{(\rho_{,1})^2}{\rho^2}
\]

\[+ \left(\frac{(n+4)^2}{2(n-1)} - 2(n+1)\right) (\rho_{,1})^4 + 4\sum_{i>1} \frac{(\rho_{,ii})^2}{\rho^2}.
\]

Note that

\[(3.7) \quad \sum \frac{(\Phi_{,i})^2}{\Phi} = 4 \sum \frac{(\rho_{,11})^2}{\rho^2} - 8 \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} + 4 \frac{(\rho_{,1})^4}{\rho^4}.
\]

Then (3.6) and (3.7) together give us

\[(3.8) \quad \Delta \Phi \geq \frac{n}{2(n-1)} \frac{\sum (\Phi_{,i})^2}{\Phi} + \left(2n - 8 \frac{n}{n-1} + 2n\right) \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3}
\]

\[+ 2R_{11} \frac{(\rho_{,1})^2}{\rho^2} + \left[\frac{(n+4)^2}{2(n-1)} - 2(n+1) - \frac{2n}{n-1}\right] (\rho_{,1})^4.
\]

From (3.1) we easily obtain

\[\rho_{,ij} = \rho_{,ij} + A_{ij1} \rho_{,1} = \frac{\rho_{,i} \rho_{,j}}{\rho} + A_{ij1} \rho_{,1}.
\]

Thus we get

\[(3.9) \quad \Phi_{,i} = \frac{2 \rho_{,1} \rho_{,11}}{\rho^2} - 2 \frac{\rho_{,i} (\rho_{,1})^2}{\rho^3} = 2A_{111} \frac{(\rho_{,1})^2}{\rho^2}, \quad \sum \Phi_{,i} \frac{\rho_{,i}}{\rho} = 2 \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} - 2 \frac{(\rho_{,1})^4}{\rho^4}.
\]

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By the same method as deriving (3.4) we get
\begin{equation}
\sum_{i} (A_{ml})^2 \geq \sum_{i=1}^{n-1} (A_{i1})^2 + \frac{2}{n-1} \sum_{i=1}^{n-1} (A_{i1})^2 + \frac{1}{n-1} \left( \sum A_{i1} - A_{111} \right)^2,
\end{equation}

Then inserting (3.12) and (3.9) into (3.8) we have
\begin{equation}
\Delta \Phi \geq \frac{n}{n-1} \sum_{i} (\Phi_{,i})^2 + \frac{n^2 - 3n - 10}{2(n-1)} \sum_{i} \Phi_{,i}^2 + \frac{(n+2)^2}{n-1} \Phi^2. \quad \square
\end{equation}

§4. Proof of Main Theorem for $n \leq 4$

In the case $n \leq 4$ the proof of the Main Theorem is relatively simple, we first consider this case.

We shall show that $\Phi = 0$ on $M$ everywhere, namely, $\det \left( \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) = \text{const}$. Therefore the main Theorem follows by J-C-P Theorem. By a coordinate translation transformation and by subtracting a linear function we may suppose that
\[ u(0) = 0, \quad u(\xi) \geq 0. \]

Then for any $C > 0$ the set
\[ S_u(0, C) := \{ \xi \in \mathbb{R}^n | u(\xi) \leq C \} \]
is compact. Consider the function
\[ L = \exp \left\{ -\frac{m}{C - u} \right\} \Phi \]
defined on \( \bar{S}_u(0, C) \), where \( m \) is a positive constant to be determined later. Clearly, \( L \) attains its supremum at some interior point \( p^* \). Then, at \( p^* \),

(4.1) \[ \frac{\Phi_{,i}}{\Phi} - hu_{,i} = 0, \]

(4.2) \[ \frac{\Delta \Phi}{\Phi} - \sum_{i} (\Phi_{,i})^2 - h' \sum_{i} (u_{,i})^2 - h\Delta u \leq 0, \]

where and later we denote
\[ h = \frac{m}{(C - u)^2}, \quad h' = \frac{2m}{(C - u)^3}, \]
and ",," denotes the covariant derivatives with respect to the metric \( G \). Inserting (3.13) (2.13) and (4.1) into (4.2) we get

(4.3) \[ \frac{(n + 2)^2}{n - 1} \Phi + \left( \frac{1}{n - 1} h^2 - h' \right) \sum_{i} (u_{,i})^2 - nh + \frac{(n + 2)(n - 3)}{(n - 1)} h \sum_{i} \frac{\rho_{,i} u_{,i}}{\rho} \leq 0. \]

By the Schwarz’s inequality
\[ \frac{(n + 2)(n - 3)}{(n - 1)} h \sum_{i} \frac{\rho_{,i} u_{,i}}{\rho} \leq \frac{1}{2(n - 1)} h^2 \sum_{i} (u_{,i})^2 + \frac{(n + 2)^2(n - 3)^2}{2(n - 1)} \Phi. \]
Therefore

(4.4) \[ \frac{(n + 2)^2(2 - (n - 3)^2)}{2(n - 1)} \Phi + \left( \frac{1}{2(n - 1)} h^2 - h' \right) \sum_{i} (u_{,i})^2 - nh \leq 0. \]

In the case \( n \leq 4 \) we have

(4.5) \[ \frac{(n + 2)^2}{2(n - 1)} \Phi + \left( \frac{1}{2(n - 1)} h^2 - h' \right) \sum_{i} (u_{,i})^2 - nh \leq 0. \]

We choose \( m = 8(n - 1)C \), then \( \frac{1}{2(n - 1)} h^2 - h' \geq 0 \). It follows that, at \( p^* \),

(4.6) \[ \exp \left\{ -\frac{8(n - 1)C}{C - u} \right\} \Phi \leq n \exp \left\{ -\frac{m}{C - u} \right\} h \leq \frac{b}{C}, \]

where \( b \) is a constant depending only on \( n \). In the calculation of (4.6) and later we often use the fact that \( \exp \left\{ -\frac{m}{C - u} \right\} \frac{m^2}{(C - u)^2} \) has a universal upper bound. Since \( L \) attains its supremum at \( p^* \), (4.6) holds everywhere in \( \bar{S}_u(0, C) \). For any fixed point \( p \), we let \( C \to \infty \) then \( \Phi(p) = 0 \). Therefore \( \Phi = 0 \) everywhere on \( M \). □
§5. Estimate for $\sum \left( \frac{\partial u}{\partial \xi_i} \right)^2$

For general dimensions ($n > 4$) the proof of the Main Theorem is much more difficult than $n \leq 4$, it needs more estimates. In this section we estimate $\sum \left( \frac{\partial u}{\partial \xi_i} \right)^2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. It is well-known (see [G]) that there exists a unique ellipsoid $E$, which attains the minimum volume among all the ellipsoids that contain $\Omega$ and that are centered at the center of mass of $\Omega$, such that

$$n^{-\frac{3}{2}}E \subset \Omega \subset E,$$

where $n^{-\frac{3}{2}}E$ means the $n^{-\frac{3}{2}}$-dilation of $E$ with respect to its center. Let $T$ be an affine transformation such that $T(E) = B(0,1)$, the unit ball. Put $\tilde{\Omega} = T(\Omega)$. Then

(5.1) \hspace{1cm} B(0,n^{-\frac{3}{2}}) \subset \tilde{\Omega} \subset B(0,1).

A convex domain $\Omega$ is called normalized if it satisfies (5.1). Let $u$ be a smooth strictly convex function defined on $\Omega$ such that

(5.2) \hspace{1cm} \inf_{\Omega} u(\xi) = u(p) = 0, \ u|_{\partial \Omega} = 1.

A strictly convex function defined on $\Omega$ is called normalized at $p$ if (5.2) holds.

**Lemma 5.1** Let $\Omega_k$ be a sequence of smooth and normalized convex domains, $u^{(k)}$ be a sequence of smooth strictly convex functions defined on $\Omega_k$, normalized at $p_k$. Then there are constants $d > 1$, $b > 0$ independent of $k$ such that

$$\sum_i \left( \frac{\partial u^{(k)}}{\partial \xi_i} \right)^2 (0) \leq b, \quad k = 1, 2, \ldots \quad \text{on} \quad \tilde{\Omega}_k.$$

**Proof.** We may suppose by taking subsequence that $\Omega_k$ converges to a convex domain $\Omega$ and $u^{(k)}$ converges to a convex function $u^\infty$, locally uniformly in $\Omega$. Obviously, we have the uniform estimate

(5.3) \hspace{1cm} \sum \left( \frac{\partial u^{(k)}}{\partial \xi_i} \right)^2 (0) \leq 4n^3.

For any $k$, let

(5.4) \hspace{1cm} \tilde{u}^{(k)} = u^{(k)} - \sum \frac{\partial u^{(k)}}{\partial \xi_i}(0)\xi_i - u^{(k)}(0).
Then
\[ \tilde{u}^{(k)}(0) = 0, \quad \tilde{u}^{(k)}(\xi) \geq 0, \quad \tilde{u}^{(k)}|_{\partial \Omega_k} \leq C_0, \]
where \( C_0 \) is a constant depending only on \( n \). As \( B(0, n^{-\frac{3}{2}}) \subset \Omega_k \), we have
\[ | \nabla \tilde{u}^{(k)} |^2 \leq \frac{C_0^2}{\text{dist}(B(0, 2^{-1}n^{-\frac{3}{2}}), \partial \Omega_k)^2} \leq 4n^3 C_0^2 \]
on \( B(0, 2^{-1}n^{-\frac{3}{2}}) \), where \( \tilde{f}^{(k)} \) is the Legendre transformation of \( \tilde{u}^{(k)} \) relative to 0. For any \( p \in \tilde{\Omega}_k \setminus B(0, 2^{-1}n^{-\frac{3}{2}}) \), we may suppose that \( p = (\xi_1, 0, \ldots, 0) \) with \( \xi_1 > 0 \) by an orthonormal transformation. Then, at \( p \),
\[ C_0 + \tilde{f}^{(k)} \geq \tilde{u}^{(k)} + \tilde{f}^{(k)} = \frac{\partial \tilde{u}^{(k)}}{\partial \xi_1} \xi_1. \]
It follows that
\[ \frac{\left( \frac{\partial \tilde{u}^{(k)}}{\partial \xi_1} \right)^2}{(C_0 + \tilde{f}^{(k)})^2} < \frac{1}{\xi_1^2} < 4n^3. \]
Therefore there exist constants \( \tilde{d} > 1, \tilde{b} > 0 \) depending only on \( n \) such that
\[ \frac{\left( \frac{\partial \tilde{u}^{(k)}}{\partial r} \right)^2}{(\tilde{d} + \tilde{f}^{(k)})^2} < \tilde{b}, \]
where \( \frac{\partial}{\partial r} \) denotes the radial derivative. Note that
\[ \frac{\partial u^{(k)}}{\partial \xi_i} = \frac{\partial u^{(k)}}{\partial \xi_i} - \frac{\partial u^{(k)}}{\partial \xi_i}(0), \quad \tilde{f}^{(k)} = f^{(k)} + u^{(k)}(0). \]
It follows from (5.3) and (5.4) that
\[ \left( \frac{\partial u^{(k)}}{\partial r} \right)^2 \leq 2 \left( \frac{\partial \tilde{u}^{(k)}}{\partial r} \right)^2 + 8n^3. \]
Then
\[ \frac{\left( \frac{\partial u^{(k)}}{\partial r} \right)^2}{(d' + f^{(k)})^2} < b', \]
for some constants \( d' > 1, b' > 0 \) independent of \( k \). Note that
\[ | \nabla u^{(k)}(p) | = \frac{1}{\cos \alpha_k} \left| \frac{\partial u^{(k)}}{\partial r}(p) \right|, \]
where $\alpha_k$ is the angle between vectors $\nabla u^{(k)}(p)$ and $\frac{\partial u^{(k)}}{\partial r}(p)$. Since $u^{(k)} = 1$ on $\partial \Omega_k$, $\nabla u^{(k)}(p)$ is perpendicular to the boundary of $\Omega_k$ at any $p \in \partial \Omega_k$. As $\Omega$ is convex and $0 \in \Omega$, it follows that $\frac{1}{\cos \alpha_k}$ have a uniform upper bound. Then the Lemma 5.1 follows. □

**Remark 5.2** We may choose $d$ in Lemma 5.1 such that the following holds for any $k$

\[(5.10)\quad \frac{|u + f^{(k)}|}{d + f^{(k)}} \leq 1.\]

§6. Estimates of $\rho$, $\rho^\alpha \Phi$ and $\sum u_{ii}$

From now on we assume that $n \geq 5$. In this section we prove some estimates which we need in the next section. Suppose that $p \in \Omega$ and $u$ is normalized at $p$. For any positive number $C \leq 1$, denote

$S_u(p, C) = \{ \xi \in \Omega | u(\xi) < C \}$, $\bar{S}_u(p, C) = \{ \xi \in \Omega | u(\xi) \leq C \}$.

Introduce notations:

$A := \max_{p \in S_u(p, C)} \left\{ \exp \left\{ -\frac{m}{C - u} \right\} \rho^\alpha \Phi \frac{(d + f)^{2n}}{(d + f)^2} \right\}$,

$B := \max_{p \in \bar{S}_u(p, C)} \left\{ \exp \left\{ -\frac{m}{C - u} + H \right\} \frac{(h + 2\alpha)^{2n}}{(d + f)^{2n+2}} \right\}$,

where

$\alpha = \frac{(n + 2)(n - 3)}{2} + \frac{n - 1}{4}$, $m = 32(n + 2)C$, $H = \epsilon \frac{\sum x_k^2}{(d + f)^2}$.

From Lemma 5.1, we always choose small enough constant $\epsilon$ such that $H < \frac{1}{30}$ in this section.

We prove the following lemmas, which play important role in the proof of the Main Theorem.

**Lemma 6.1** Let $u$ be a smooth and strictly convex function defined in $\Omega$ which satisfies the equation $(1.4)$. Suppose that $u$ is normalized at 0 and the section $\bar{S}_u(p, C)$ is compact. And assume that there are constants $b_1 \geq 0$, $d > 1$ such that

$\frac{\sum x_k^2}{(d + f)^2} \leq b_1$

on $\bar{S}_u(p, C)$. Then there is a constant $d_1 > 0$, depending only on $n$, $b_1$ and $C$, such that

$A \leq d_1, \quad B \leq d_1$. 

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Proof. Firstly, we show \( A \leq 10B \). To this end, consider the following function

\[
F = \exp \left\{ -\frac{m}{C - u} \right\} \frac{\rho^\alpha \Phi}{(d + f)^{n+2}}
\]
defined on \( S_u(p, C) \). Clearly, \( F \) attains its supremum at some interior point \( p^* \) of \( S_u(p, C) \). Thus, at \( p^* \),

\[
(6.1) \quad \frac{\Phi}{\Phi} = \frac{\alpha \rho^i}{\rho} - \frac{2n\alpha}{n+2} \frac{f^i}{d + f} - hu^i = 0,
\]

\[
(6.2) \quad \frac{\Delta \Phi}{\Phi} - \frac{(\Phi, i)^2}{\Phi^2} + \frac{n+2}{(n-1)} \frac{\Phi}{\Phi} + \frac{2n\alpha}{n+2} \frac{\Delta f}{(d + f)^2} + \frac{2n\alpha}{n+2} \frac{\sum(f, i)^2}{(d + f)^2} - h' \sum(u, i)^2 - h \Delta u \leq 0,
\]

where \( , \ldots \) denotes the covariant derivatives with respect to the metric \( G \). In the calculation of \( (6.2) \) we used \( (3.2) \). Inserting \( (2.12), (2.13) \) and \( (3.13) \) into \( (6.2) \) we get

\[
\left[ \frac{(n+2)\alpha}{2(n-1)} + \frac{(n+2)^2}{n-1} \right] \Phi + \frac{1}{(n-1)} \frac{\sum(\Phi, i)^2}{\Phi^2} + \frac{n+2}{2} \frac{\sum u, i\rho^i}{\rho} - n\alpha \frac{\sum f, i\rho^i}{(d + f)^2} + \frac{2n\alpha}{n+2} \frac{\sum(f, i)^2}{(d + f)^2} - h' \sum(u, i)^2 - nh - \frac{2n\alpha}{n+2} \frac{n}{d + f} \leq 0.
\]

Using \( (6.1) \) yields

\[
(6.3) \quad \frac{1}{(n-1)} \sum \left[ hu^i + \frac{2n\alpha}{n+2} \frac{f^i}{d + f} - \frac{\alpha \rho^i}{\rho} \right]^2 + \frac{2(n+2)}{n-1} \alpha + \frac{(n+2)^2}{n-1} \right] \Phi \\
+ \frac{(n+2)(n-3)}{n-1} \frac{h\sum u, i\rho^i}{\rho} - \frac{4n\alpha}{n-1} \frac{\sum f, i\rho^i}{\rho} + \frac{2n\alpha}{n+2} \frac{\sum(f, i)^2}{(d + f)^2} - h' \sum(u, i)^2 - nh - \frac{2n^2\alpha}{n+2} \leq 0.
\]

Note that

\[
(6.4) \quad \left| \sum u, i f^i \right| = \left| \sum \frac{\partial u}{\partial \xi^j} \frac{\partial f}{\partial x^k} u_{kj} u^{ij} \right| = \left| \sum \xi \frac{\partial u}{\partial \xi^j} \right| \frac{d + f}{d + f} = \frac{|u + f|}{d + f} \leq 1.
\]

Inserting \( (6.4) \) into \( (6.3) \), we have

\[
(6.5) \quad \frac{1}{(n-1)} h^2 \sum(u, i)^2 + \left[ \frac{4n^2\alpha^2}{(n+2)^2(n-1)} + \frac{2n\alpha}{n+2} \right] \frac{\sum(f, i)^2}{(d + f)^2} - \frac{1}{2} \frac{h\sum u, i\rho^i}{\rho} \\
+ \frac{(n-1)(2n+5)^2}{16} \Phi - \frac{n(2n+5)\alpha}{n+2} \frac{\sum f, i\rho^i}{(d + f)^2} - h' \sum(u, i)^2 - 10B \leq 0.
\]
\[-\left(n + \frac{4n\alpha}{(n-1)(n+2)}\right) h - \frac{2n^2\alpha}{n+2} \leq 0.\]

As \(\alpha = \frac{(n+2)(n-3)}{2} + \frac{n-1}{4}\), it is easy to check that

\[
\frac{4n^2\alpha^2}{(n+2)^2(n-1)} + \frac{2n\alpha}{n+2} = \frac{4n^2\alpha^2}{(n+2)^2(n-1)} \left(1 + \frac{(n+2)(n-1)}{2n\alpha}\right) > \frac{4n^2\alpha^2}{(n+2)(n^2-1)}.
\]

Using the Schwarz’s inequality we get

\[
\frac{1}{2} h \sum \frac{u_i \rho_i}{\rho} \leq \frac{1}{2(n-1)} h^2 \sum (u_i)^2 + \frac{n-1}{8} \Phi,
\]

\[
\frac{n(2n+5)\alpha}{n+2} \sum \frac{f_i \rho_i}{(d+f)\rho} \leq \frac{4n^2\alpha^2}{(n+2)(n^2-1)} \sum (f_i)^2 + \frac{(2n+5)(n^2-1)}{16(n+2)} \Phi.
\]

Note that \(\frac{1}{2(n-1)} h^2 \geq h'\), we get from (6.5)

\[
\frac{(n+2)(n-1)}{4} \Phi - \left(n + \frac{4n\alpha}{(n-1)(n+2)}\right) h - \frac{2n^2\alpha}{n+2} \leq 0.
\]

It follows that

(6.6) \quad \mathcal{A} \leq 10\mathcal{B}.

Secondly, we consider the following function

\[
\tilde{F} = \exp \left\{ -\frac{m}{C-u} + H \right\} \frac{(h+2\alpha)\rho^\alpha}{(d+f)^{\frac{2\alpha}{n+2}}}
\]

defined on \(S_u(p, C)\). Clearly, \(\tilde{F}\) attains its supremum at some interior point \(q^*\) of \(S_u(p, C)\). Thus, at \(q^*\),

(6.7) \quad -hu_j + \frac{h' u_j}{h+2\alpha} + H_j + \alpha \frac{\rho_j}{\rho} - \frac{2n\alpha}{n+2} \frac{f_j}{d+f} = 0,

(6.8) \quad \left(\frac{h''}{h+2\alpha} - \frac{h'^2}{(h+2\alpha)^2} - h'\right) \sum (u_j)^2 + \left(\frac{h'}{h+2\alpha} - h\right) \Delta u

+ \Delta H + \frac{n+2}{2} \alpha \Phi - \frac{2n\alpha}{n+2} \left(\frac{\Delta f}{d+f} - \frac{\sum (f_i)^2}{(d+f)^2}\right) \leq 0
where $h'' = \frac{6n}{(2-\alpha)^4}$. By (2.11) and the Schwarz inequality

\begin{equation}
\sum H_i^2 = \sum \left(\epsilon \frac{2x_i}{(d+f)^2} - 2\epsilon \frac{\sum x_k^2}{(d+f)^3} \delta_{i,j} \right)^2 \leq 8\epsilon H \frac{\sum f_{ii}}{(d+f)^2} + 8H^2 \frac{\sum (f_{ii})^2}{(d+f)^2},
\end{equation}

\begin{equation}
\Delta H = \frac{\epsilon}{(d+f)^2} \left[ 2 \sum f_{ii} + \frac{n+2}{2} \langle \nabla \log \rho, \nabla (\sum x_k^2) \rangle - 4\epsilon \frac{\sum x_k^2 \sum f_{ii}}{d+f} \right]
+ 6\epsilon \sum x_k^2 \sum (f_{ii})^2 \frac{n+2}{(d+f)^3} - (n+2)\epsilon \sum x_k^2 \langle \nabla \log \rho, \nabla f \rangle \frac{n+2}{(d+f)^3}
\geq \frac{\epsilon}{(d+f)^2} \sum f_{ii} - 27H \frac{\sum (f_{ii})^2}{(d+f)^2} - \frac{3(n+2)^2}{4} H\Phi - 2nH.
\end{equation}

Note that $\frac{(n+2)^2}{2} > n + 2 \geq \frac{n+2}{2}$ and

\begin{equation}
\Phi = \frac{1}{\alpha^2} \sum \left( -hu_i + \frac{h' u_i}{h + 2\alpha} + H_i - \frac{2n\alpha}{n+2} \frac{f_{ii}}{d+f} \right)^2
\geq \frac{1}{2\alpha^2} \sum \left( -hu_i + \frac{h' u_i}{h + 2\alpha} - \frac{2n\alpha}{n+2} \frac{f_{ii}}{d+f} \right)^2 - \frac{1}{\alpha^2} \sum (H_i)^2
\geq \frac{h^2}{4\alpha^2} \sum (u_i)^2 + \frac{n^2}{(n+2)^2} \frac{\sum (f_{ii})^2}{(d+f)^2} - \frac{1}{2\alpha^2} \frac{h^2}{(h+2\alpha)^2} - \frac{1}{\alpha^2} \frac{\sum (H_i)^2}{4} - 4h,
\end{equation}

where we use the fact (6.4). Inserting (2.12), (2.13), (6.9) and (6.10) into (6.8) and using the Schwartz inequality we have

\begin{equation}
\frac{\epsilon}{2} \sum f_{ii} - a_0 \Phi - a_1 h - 3n\alpha \leq 0
\end{equation}

for some constant $a_0 > 0$, $a_1 > 0$ depending only on $n$. Since $\sum f_{ii} \geq n\rho^{\frac{n+2}{n}}$, we get

\begin{equation}
\frac{\rho^{\frac{n+2}{n}}}{(d+f)^2} \leq \frac{a_0}{\epsilon} \Phi + \frac{2a_1}{\epsilon} h + \frac{6\alpha}{\epsilon},
\end{equation}

It follows that

\begin{equation}
\mathcal{B}^{1+\frac{n+2}{na}} \leq a_2 A + a_3 B,
\end{equation}

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for some positive constants $a_2$ and $a_3$, where we used the fact that $\exp \left\{ -\frac{m}{C-u} \right\} h^\gamma$ has a universal upper bounded for any $\beta > 0$, $\gamma > 0$. By (6.6), we have

(6.14) \[ B \leq d_1, \quad A \leq d_1 \]

for some $d_1$ depending only on $C$, $n$ and $b_1$. Thus the proof of Lemma 6.1 is complete.$\square$

In the following we estimate $\sum u_{ii}$. To this end we first derive a general formula which we need later.

**Lemma 6.2** Let $u(\xi)$ be a smooth strictly convex function defined in $\Omega \subset \mathbb{R}^n$. Assume that

\[ \inf_{\Omega} u = 0, \quad u|_{\partial \Omega} = C. \]

Consider the function

(6.15) \[ F = \exp \left\{ -\frac{m}{C-u} + H \right\} Q\|\nabla K\|^2, \]

where $Q > 0$, $H > 0$ and $K$ are smooth functions defined on $\overline{\Omega}$. $F$ attains its supremum at an interior point $p^*$. We choose a local orthonormal frame field on $M$ such that, at $p^*$, $K_1 = \|\nabla K\|$, $K_{i} = 0$, for all $i > 1$. Then at the point $p^*$ we have the following estimates

(6.16) \[
2 \left( \frac{1}{n-1} - \delta - 1 \right) (K_{,11})^2 + 2 \sum K_{,j}(\Delta K)_{,j} \\
+ 2(1 - \delta) \sum A_{ml}^2 (K_{,1})^2 - \frac{(n+2)^2}{8\delta} \Phi(K_{,1})^2 - \frac{2}{\delta(n-1)^2}(\Delta K)^2 \\
+ \left[ -h' \sum (u_{i})^2 - h\Delta u + \Delta H + \frac{\Delta Q}{Q} - \frac{\sum (Q_{,i})^2}{Q^2} \right] (K_{,1})^2 \leq 0,
\]

for any small positive number $\delta$.

**Proof.** We can assume that $\|\nabla K\|(p^*) > 0$. Then, at $p^*$,

(6.17) \[ F_{,i} = 0, \]

(6.18) \[ \sum F_{,ii} \leq 0. \]

By calculating both expressions (6.17) and (6.18) explicitly, we have

(6.19) \[
\left( -hu_{,i} + H_{,i} + \frac{Q_{,i}}{Q} \right) \sum (K_{,j})^2 + 2 \sum K_{,j} K_{,ji} = 0,
\]
(6.20) \[ 2 \sum (K_{,ij})^2 + 2 \sum K_{,j}K_{,jii} + 2 \sum \left( -hu_{,i} + H_{,i} + \frac{Q_{,i}}{Q} \right) K_{,j}K_{,ji} \]
\[ + \left[ -h' \sum (u_{,i})^2 - h\Delta u + \Delta H + \frac{\Delta Q}{Q} - \frac{\sum (Q_{,i})^2}{Q^2} \right] (K_{,1})^2 \leq 0. \]

Let us simplify (6.20). From (6.19)

(6.21) \[ 2K_{,1i} = \left( hu_{,i} - H_{,i} - \frac{Q_{,i}}{Q} \right) K_{,1}. \]

Applying the Schwarz inequality yields

(6.22) \[ 2 \sum (K_{,ij})^2 \geq 2(K_{,11})^2 + \frac{2}{n-1} (\Delta K - K_{,11})^2 + 4 \sum_{i>1} (K_{,1i})^2 \]
\[ \geq 2 \left( \frac{n}{n-1} - \delta \right) (K_{,11})^2 + 4 \sum_{j>1} (K_{,1j})^2 - \frac{2}{\delta(n-1)^2} (\Delta K)^2 \]
for any \( \delta > 0 \). Inserting (6.21) and (6.22) into (6.20) we get

(6.23) \[ 2 \left( \frac{1}{n-1} - \delta - 1 \right) (K_{,11})^2 + 2 \sum K_{,j}K_{,jii} - \frac{2}{\delta(n-1)^2} (\Delta K)^2 \]
\[ + \left[ -h' \sum (u_{,i})^2 - h\Delta u + \Delta H + \frac{\Delta Q}{Q} - \frac{\sum (Q_{,i})^2}{Q^2} \right] (K_{,1})^2 \leq 0. \]

An application of the Ricci identity shows that

(6.24) \[ 2 \sum K_{,j}K_{,jii} = 2 \sum K_{,j}(\Delta K)_{,j} + 2R_{11}(K_{,1})^2 \]
\[ = 2 \sum K_{,j}(\Delta K)_{,j} + 2 \sum A_{ml1}^2 (K_{,1})^2 - (n + 2) \sum A_{11k} \frac{\rho_k}{\rho} (K_{,1})^2 \]
\[ \geq 2 \sum K_{,j}(\Delta K)_{,j} + 2(1 - \delta) \sum A_{ml1}^2 (K_{,1})^2 - \frac{(n + 2)^2}{8\delta} \Phi(K_{,1})^2. \]

Consequently, inserting (6.24) into (6.23) we get (6.16). \( \square \)

**Lemma 6.3** Let \( u \) be a smooth and strictly convex function defined in \( \Omega \) which satisfies the equation (1.4). Suppose that \( u \) is normalized at \( p \) and the section \( \tilde{S}_u(p,C) \) is compact. And assume that there are constants \( b_2 \geq 0, d > 1 \) such that

\[ \frac{\sum x_k^2}{(d + f)^2} \leq b_2, \quad \frac{\rho^\alpha}{(d + f)^{\frac{2\alpha}{n+2}}} \leq b_2, \quad \frac{\rho^\alpha \Phi}{(d + f)^{\frac{2\alpha}{n+2}}} \leq b_2 \]
on $S_u(p, C)$. Then there is a constant $d_2 > 0$, depending only on $n$, $b_2$ and $C$, such that

$$\exp \left\{ - \frac{64(n-1)C}{C-u} \right\} \frac{\rho^u \sum u_{ii}}{(d+f)^{\frac{2n+6}{n^2+2}} \leq d_2}$$
on $S_u(p, C)$, where $\alpha = \frac{(n+2)(n-3)}{2} + \frac{n-1}{4}$.

**Proof.** Put

$$H = \epsilon (d+f)^2, \quad K = x_1, \quad Q = \frac{\rho^u}{(d+f)^{\frac{2n+6}{n^2+2}}}$$
in (6.15). Now we first calculate $2 \sum K_j(\Delta K)_j + 2(1-\delta) \sum A_{mli}A_{mlj}K_iK_j$. By (2.11) we have in this case

$$\Delta K = \frac{n+2}{2} \langle \nabla \log \rho, \nabla K \rangle,$$

$$2 \sum K_j(\Delta K)_j = (n+2) \frac{\rho_{11}}{\rho} (K,_{1})^2 - (n+2) \frac{(\rho_{1})^2}{\rho^2} (K,_{1})^2 + (n+2) \sum K,_{1i}K,_{1j} \frac{\rho_{ij}}{\rho}$$

$$\geq (n+2) \sum \frac{\rho_{ij}}{\rho} K,_{1i}K,_{1j} - \delta \sum (K,_{1i})^2 - \frac{(n+2)^2 + 1}{4\delta} \Phi(K,_{1})^2$$

for $\delta \leq \frac{1}{4(n+2)}$. We use the coordinates $\xi_1, \ldots, \xi_n$ to calculate $\sum (K,_{ij})^2$ and $\sum A_{mli}^2(K,_{1})^2$. Note that the Levi-Civita connection is given by $\Gamma^k_{ij} = \frac{1}{2} \sum u^{kl}u_{ijl}$. Then

$$K,_{ij} = u_{1ij} - \frac{1}{2} \sum u_{1kl}u^{kl}u_{ij} = \frac{1}{2} u_{1ij},$$

$$\sum (K,_{ij})^2 = \frac{1}{4} \sum u^{ik}u^{j}u_{1ij}u_{1kl},$$

(6.25) $\sum (A_{mli})^2(K,_{1})^2 = \frac{1}{4} \sum u^{ik}u^{j}u_{1ij}u_{1jp}u_{klq}u^{pr}u_{1r}u^{qs}u_{1s} = \sum (K,_{ij})^2$.

In the coordinates $x_1, \ldots, x_n$ we have (see (3.1))

$$\frac{\rho_{ij}}{\rho} = \frac{\rho_{k}}{\rho} \frac{\rho_{ij}}{\rho}, \quad \frac{\rho_{ij}}{\rho} = \frac{\rho_{k}}{\rho} \frac{\rho_{ij}}{\rho} + \sum A_{ij}^k \frac{\rho_{ik}}{\rho},$$

It follows that

(6.26) $(n+2) \sum \frac{\rho_{ij}}{\rho} K,_{i}K,_{j} \leq \delta \sum (K,_{ij})^2 + \frac{(n+2)^2 + 1}{4\delta} \Phi(K,_{1})^2.$

Then

(6.27) $2 \sum K_j(\Delta K)_j + 2(1-\delta) \sum A_{mli}A_{mlj}K_iK_j$
\[\geq (2 - 4\delta) \sum (K_{ij})^2 - \frac{(n + 2)^2 + 1}{2\delta} \Phi(K, i)^2.\]

A direct calculation yields

\[\frac{\Delta Q}{Q} - \sum \frac{(Q_{ij})^2}{Q^2} \geq -\frac{(n\alpha + n + 2)(n + 2)}{8} \Phi - 2n(\alpha + 1).\]

From (6.21) we obtain

\[\sum (K_{1i})^2 = \frac{1}{4} \sum \left[ hu_{i} - \alpha \frac{p_{i}}{\rho} + \left( \frac{2n\alpha}{n + 2} + 2 \right) \frac{f_{i}}{d + f} - H_{i} \right]^2 (K, i)^2 \]

\[\geq \frac{1}{16} \left[ h^2 \sum (u_{i})^2 + \frac{4n^2\alpha^2}{(n + 2)^2} \frac{(f_{i})^2}{(d + f)^2} \right] (K, i)^2 - \frac{1}{8} \alpha^2 \Phi(K, i)^2 - \frac{1}{4} (K, i)^2 \sum (H_{i})^2 \]

\[\geq \frac{1}{16} \left[ h^2 \sum (u_{i})^2 + \frac{4n^2\alpha^2}{(n + 2)^2} \frac{(f_{i})^2}{(d + f)^2} \right] (K, i)^2 - \frac{1}{8} \alpha^2 \Phi(K, i)^2 - \frac{1}{4} (K, i)^2 \sum (H_{i})^2 - a_{4}h(K, i)^2,\]

where we used (6.4), for some positive constant \(a_{4}\). Choose \(\delta = \frac{1}{6(n+2)}\) and \(m = 64(n - 1)C\). Inserting (6.9), (6.10), (6.27), (6.28) and (6.29) into (6.16) and using the Schwarz inequality we get

\[\epsilon \frac{\sum fii}{2 (d + f)^2} - a_{5} \Phi - a_{6}h - a_{7} \leq 0,\]

In the above \(a_{4} = -a_{7}\) denote constants depending only on \(n\). Note that

\[\sum f^{ii} \geq u_{11} = (K, i)^2.\]

It follows that

\[\exp \left\{ \frac{m}{C - u} \right\} \frac{\rho^{n}u_{11}}{(d + f)^{\frac{2n\alpha}{n + 2} + 2}} \leq d_{2}\]

for some constant \(d_{2}\) depending only on \(n, b_{2}\) and \(C\). Similar inequalities for \(u_{ii}\) remain true. Thus the proof of Lemma 6.3 is complete. \(\Box\)

\section*{§7. Proof of Main Theorem}

Let \(u(\xi_{1}, ..., \xi_{n})\) be a locally strongly convex function defined on whole \(\mathbb{R}^{n}\) such that its Legendre function \(f\) satisfying

\[\frac{\partial^2}{\partial x_{i} \partial x_{j}} (\log \det (f_{kl})) = 0.\]
Let $p \in \mathbb{R}^n$ be any point. By a coordinate translation transformation and by subtracting a linear function we may suppose that $u$ satisfying

$$u(\xi) \geq u(p) = 0, \quad \forall \xi \in \mathbb{R}^n.$$  

Choose a sequence $\{C_k\}$ of positive numbers such that $C_k \to \infty$ as $k \to \infty$. For any $C_k$ the level set $S_u(p, C_k) = \{u(\xi) < C_k\}$ is a bounded convex domain. Let

$$u^{(k)}(\xi) = \frac{u(\xi)}{C_k}, \quad k = 1, 2, \ldots$$

There exists the unique minimum ellipsoid $E$ of $S_u(p, C_k)$ centered at $q_k$, the center of mass of $S_u(p, C_k)$, such that

$$n^{-\frac{3}{2}}E \subset S_u(p, C_k) \subset E.$$ 

Let

$$T_k : \tilde{\xi}_i = \sum a^i_j \xi_j + b_i$$ 

be a linear transformation such that

$$T_k(q_k) = 0, \quad T_k(E) = B(0, 1).$$

Then

$$B(0, n^{-\frac{3}{2}}) \subset \Omega_k := T_k(S_u(p, C_k)) \subset B(0, 1).$$

Thus we obtain a sequence of convex functions

$$\tilde{u}^{(k)}(\tilde{\xi}) := u^{(k)}\left(\sum b^i_j (\tilde{\xi}_j - b_j), \ldots, \sum b^j_n (\tilde{\xi}_j - b_j)\right)$$

where $(b^i_j) = (a^i_j)^{-1}$.

In the following we will use the coordinates $\xi$ to denote the $\tilde{\xi}$ and $u^{(k)}$ to denote $\tilde{u}^{(k)}$ to simplify the notations. We may suppose by taking subsequences that $\Omega_k$ converges to a convex domain $\Omega$ and $u^{(k)}(\xi)$ converges to a convex function $u^\infty(\xi)$, locally uniformly in $\Omega$. Consider the Legendre transformation relative to $u^{(k)}$:

$$x_i = \frac{\partial u^{(k)}}{\partial \xi_i},$$

$$f^{(k)}(x_1, \ldots, x_n) = \sum \xi_i \frac{\partial u^{(k)}}{\partial \xi_i} - u^{(k)}(\xi_1, \ldots, \xi_n), \quad (\xi_1, \ldots, \xi_n) \in \Omega_k.$$ 

Put $\Omega^{(k)*} = \{(x_1, \ldots, x_n) | x_i = \frac{\partial u^{(k)}}{\partial \xi_i}\}$. Obviously, $f^{(k)}$ satisfies (7.1), therefore there are constants $d^{(k)}_1, \ldots, d^{(k)}_n, d^{(k)}_0$ such that

$$\text{(7.2)} \quad \det \left(\frac{\partial^2 f^{(k)}}{\partial x_i \partial x_j}\right) = \exp \left\{\sum d^{(k)}_i x_i + d^{(k)}_0\right\}.$$
We use Lemmas 5.1, 6.1 and 6.3 for each \(u^{(k)}\) with \(C = 1\) to get the following uniform estimates

\[
\frac{\rho^{(k)}}{(d + f^{(k)}) \frac{2n}{n+2}} \leq d_3, \quad \frac{\rho^{(k)} \alpha \Phi^{(k)}}{(d + f^{(k)}) \frac{2n}{n+2}} \leq d_3, \quad \frac{\rho^{(k)} \alpha \sum u^{(k)}_{ii}}{(d + f^{(k)}) \frac{2n}{n+2}} \leq d_3
\]

on \(S_{u^{(k)}}(T^k(p), \frac{1}{2})\) for some constant \(d_3 > 0\), where \(\alpha = \frac{(n+2)(n-3)}{2} + \frac{n-1}{4}\).

Let \(B_R(0)\) be a Euclidean ball such that \(S_{u^{(k)}}(T^k(p), \frac{1}{2}) \subset B_{R/2}(0)\), for all \(k\). The comparison theorem for the normal mapping (see [G] or [L-J-3]) yields

\[
B^*_r(0) \subset \Omega^{(k)*}
\]

for every \(k\), where \(r = \frac{1}{2R} + 1\) and \(B^*_r(0) = \{x|x_1^2 + ... + x_n^2 \leq r^2\}\). Note that \(u^k(T^k(p)) = 0\) and its image under normal mapping is \((x_1, ..., x_n) = 0\). Restricting to \(B^*_r(0)\), we have

\[
-R' \leq f^{(k)} = \sum \xi_i x_i - u^{(k)} \leq R',
\]

where \(R' = \frac{1}{R} + 1\). Therefore \(f^{(k)}\) locally uniformly converges to a convex function \(f^\infty\) on \(B^*_r(0)\) and there are uniform estimates

\[
\rho^{(k)} \leq d_4, \quad (\rho^{(k)} \alpha \Phi^{(k)}) \leq d_4, \quad (\rho^{(k)} \alpha \sum u^{(k)}_{ii}) \leq d_4
\]

on \(B^*_r(0)\) for some constant \(d_4 > 0\).

**Lemma 7.1** Let \(f(x)\) be a smooth strictly convex function defined in \(B^*_\delta(0)\) satisfying

\[
-R' \leq f \leq R'.
\]

Then there exists a point \(p^* \in B^*_\delta(0)\) such that at \(p^*\)

\[
\frac{1}{\rho} < \left(\frac{4R'}{\delta^2}\right)^{\frac{n}{n+2}} 2^{\frac{n+1}{n+2}} := d_5.
\]

**Proof.** If Lemma 7.1 does not hold, we would have

\[
\frac{1}{\rho} \geq d_5 \quad \text{on} \quad B^*_\delta(0).
\]

It follows that

\[
\det(f_{ij}) \geq d_5^{n+2} \quad \text{on} \quad B^*_\delta(0).
\]
Define a function

\[ F(x) = \left( \frac{d_n^{n+2}}{2n+1} \right)^{\frac{1}{2n+1}} \left( \sum x_i^2 - \delta^2 \right) + 2R' \text{ on } B^*_\delta(0). \]

Then

\[ \det(F_{ij}) = \frac{d_n^{n+2}}{2} < \det(f_{ij}) \text{ in } B^*_\delta(0), \]

\[ F(x) \geq f(x) \text{ on } \partial B^*_\delta(0). \]

By the comparison principle, we have

\[ F(x) \geq f(x) \text{ on } B^*_\delta(0). \]

On the other hand, note that

\[ F(0) = - \left( \frac{d_n^{n+2}}{2n+1} \right)^{\frac{1}{2}} \delta^2 + 2R' = -2R' < f(0). \]

This is a contradiction. □

From Lemma 7.1 and (7.3), for any \( B^*_\delta(0) \) we have a point \( p_k \in B^*_\delta(0) \) such that \( \rho^{(k)}, \frac{1}{\rho^{(k)}}, \Phi^{(k)} \) and \( \sum u_{ii}^{(k)} \) are uniformly bounded at \( p_k \). Therefore there are constants \( 0 < \lambda \leq \Lambda < \infty \) independent of \( k \) such that the following estimates hold

\[ \lambda < \text{the eigenvalues of } \left( f_{ij}^{(k)} \right)(p_k) < \Lambda. \]

Since \( f^{(k)} \) satisfies (7.2),

\[ \Phi^{(k)} = \frac{1}{(n+2)^2} \sum f^{(k)ij} d_i^{(k)} d_j^{(k)}. \]

It follows that

\[ \sum (d_i^{(k)})^2 \leq d_6 \]

for some constant \( d_6 > 0 \). Thus

\[ \| \nabla \log \rho^{(k)} \|^2_E = \sum \left( \frac{\partial \log \rho^{(k)}}{\partial x_i} \right)^2 = \frac{1}{(n+2)^2} \sum (d_i^{(k)})^2 \leq d_6, \]

(7.4) where \( \| \cdot \|_E \) denotes the norm of a vector with respect to the Euclidean metric. Then for any unit speed geodesic starting from \( p_k \),

\[ \frac{d \log \rho^{(k)}}{ds} \leq \| \nabla \log \rho^{(k)} \|_E \leq d_6. \]

(7.5)
Thus for any \( q \) we have

\[
\rho^{(k)}(p_k) \exp\{ -|q - p_k|d_6 \} \leq \rho^{(k)}(q) \leq \rho^{(k)}(p_k) \exp\{ |q - p_k|d_6 \}.
\]

In particular, we choose \( q \) be the point \( x_i = 0 \) for all \( i \geq 1 \). It follows from (7.3) that

\[
\Phi^{(k)}(q) \leq d_7
\]

for some constant \( d_7 > 0 \) independent of \( k \). On the other hand, if \( \Phi(p) \neq 0 \), by a direct calculation yields

\[
\Phi^{(k)}(q) = C_k \Phi(p) \to \infty, \quad \text{as} \quad k \to \infty.
\]

This contradicts to (7.7). Thus

\[
\Phi(p) = 0.
\]

Since \( p \) is arbitrary we conclude that \( \Phi = 0 \) everywhere. Consequently

\[
\det \left( \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) = \text{const.} > 0.
\]

This means that \( M \) is an affine complete parabolic affine hypersphere. By the J-C-P Theorem we conclude that \( M \) must be elliptic paraboloid. This complete the proof of the Main Theorem. \( \square \)

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