AN EULER-MACLAURIN FORMULA FOR POLYGONAL SUMS

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Abstract. We prove an Euler-Maclaurin formula for double polygonal sums and, as a corollary, we obtain approximate quadrature formulas for integrals of smooth functions over polygons with integer vertices. Our Euler-Maclaurin formula is in the spirit of Pick’s theorem on the number of integer points in an integer polygon and involves weighted Riemann sums, using tools from Harmonic analysis. Finally, we also exhibit a classical trick, dating back to Huygens and Newton, to accelerate convergence of these Riemann sums.

1. Introduction

One motivation for this work comes from an elegant, elementary result discovered in 1899 by G. Pick: If $P$ is a simple polygon in the Cartesian plane with vertices with integer coordinates, if $|P|$ is its area, and if $I$ and $B$ are the number of integer points in the interior and on the boundary, then

$$|P| = I + \frac{1}{2}B - 1.$$

There are many simple proofs of this beautiful result, see e.g. [6]. In particular, in [3] we presented a proof based on harmonic analysis techniques. One of the goals of this paper is to show how Pick’s theorem can be extended to more general results. The area is an integral, and the enumeration of integer points is a Riemann sum. Pick’s theorem may therefore be thought of as a particular case of a quadrature rule, as well as a particular case of an Euler-Maclaurin formula for double sums. Such ideas have been pursued before ([2], [5], [7], [9], [10], [11]) but here we emphasize solid angle weights at all integer points, thereby getting a new type of weighted Euler-Maclaurin summation formula and quadrature formula, with lower-order error terms.

Let $P$ be an integer polytope in $\mathbb{R}^d$, meaning that all vertices of $P$ have integer coordinates. The normalized angle at a point $x$ is defined by the proportion of $P$ in a small ball centered at $x$:

$$\omega_P(x) = \lim_{\varepsilon \to 0^+} \left\{ \frac{|P \cap \{ y : |x - y| < \varepsilon \}|}{|\{ y : |y| < \varepsilon \}|} \right\}.$$

In particular, in dimension $d = 1$, $P$ is a segment, $\omega_P(x) = 0$ if $x$ is outside $P$, $\omega_P(x) = 1$ if $x$ is inside $P$, $\omega_P(x) = 1/2$ at the extremes. In dimension $d = 2$, $P$ is a polygon, $\omega_P(x) = 0$ if $x$ is outside $P$, $\omega_P(x) = 1$ if $x$ is inside $P$, $\omega_P(x) = 1/2$ if $x$ is on a side but it is not a vertex, $\omega_P(x) = \theta/2\pi$ if $x$ is a vertex with interior angle $\theta$.

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Our goal is to compare the integral of a smooth function \( g(x) \) over \( P \),

\[
\int_P g(x) \, dx
\]

with the weighted Riemann sums over the lattice \( N^{-1} \mathbb{Z}^d \),

\[
N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_P (N^{-1} n) g \chi_P (N^{-1} n).
\]

The Fourier transform of an integrable function \( f(x) \) is defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi x} \, dx.
\]

Let \( \varphi(x) \) be a radial smooth function with compact support and integral 1 and, for \( \varepsilon > 0 \), define \( \varphi_{\varepsilon}(x) = \varepsilon^{-d} \varphi(\varepsilon^{-1} x) \). Then one easily verify that

\[
\lim_{\varepsilon \to 0^+} \left\{ \varphi_{\varepsilon} \ast (g \chi_P)(x) \right\} = \omega_P(x) g(x).
\]

Hence, by the Poisson summation formula,

\[
N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_P (N^{-1} n) g \chi_P (N^{-1} n) = \int_P g(x) \, dx + \lim_{\varepsilon \to 0^+} \left\{ \sum_{n \in \mathbb{Z}^d - \{0\}} \varphi(\varepsilon n) \hat{g}(\varepsilon n) \chi_P(n) \right\}.
\]

It should be emphasized that this application of the Poisson summation formula and the existence of the limit are easy to prove, although not entirely trivial. The function \( g(x) \chi_P(x) \) is discontinuous when \( g(x) \neq 0 \) on the boundary of \( P \), hence the Fourier transform \( \hat{g}(\varepsilon n) \chi_P(n) \) is not absolutely integrable. The convolution with a test function allows us to bypass the problem.

The main result of this paper is an asymptotic expansion for above weighted Riemann sums, in dimension \( d = 2 \),

\[
N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_P (N^{-1} n) g \chi_P (N^{-1} n) = \int_P g(x) \, dx + \frac{\alpha}{N^2} + \frac{\beta}{N^4} + \frac{\gamma}{N^6} + \cdots.
\]

See Theorem 4 below. Our main tools are the Poisson summation formula together with an asymptotic expansion for the Fourier transform of the function \( g(x) \chi_P(x) \).

First we claim that \( \hat{g}(\varepsilon x) \chi_P(n) \) has an asymptotic expansion. Let us explain this point in some more detail. See [4] for the case \( g(x) = 1 \). We offer the following outline of a possible proof. In dimension 2, we shall give a rigorous proof, following a different argument.

Claim. If \( P \) is an integer polyhedron and \( g(x) \) is a smooth function, then there exist functions \( \{A_j(n)\} \) homogeneous of degree \(-j\), odd when \( j \) is odd and even when \( j \) is even, such that the Fourier transform of \( g(x) \chi_P(x) \) has for every \( n \in \mathbb{Z}^d - \{0\} \) an asymptotic expansion

\[
\hat{g}(\varepsilon x) \chi_P(n) = \sum_{j=1}^{+\infty} A_j(n).
\]
To see why the claim above holds, suppose that $\nabla$ is the gradient and $n(x)$ is the outward unit normal to $\partial P$ at the point $x$, then for every $n \neq 0$ the divergence theorem gives

$$
\hat{\chi}_P g(n) = \int_P g(x) e^{-2\pi i n \cdot x} \, dx
= \int_P \text{div} \left( \left( \frac{g(x) e^{-2\pi i n \cdot x}}{-2\pi i |n|^2} \right) \right) \, dx - \int \frac{n \cdot \nabla g(x)}{|n|^2} \cdot e^{-2\pi i n \cdot x} \, dx
= \int_{\partial P} \frac{n \cdot n(x) g(x)}{-2\pi i |n|^2} e^{-2\pi i n \cdot x} \, dx - \int \frac{n \cdot \nabla g(x)}{|n|^2} \cdot e^{-2\pi i n \cdot x} \, dx.
$$

Observe that the last integral is formally analog to the first one that defines $g \hat{\chi}_P (n)$, but with $g(x)$ replaced by $(n \cdot \nabla g(x)) / \left(2\pi i |n|^2\right)$, which is smaller by a factor $|n|^{-1}$ when $|n| \to +\infty$. Moreover, if $\{F\}$ are the $d-1$ dimensional faces and $\{n(F)\}$ the outward normals to these faces, then

$$
\int_{\partial P} \frac{n \cdot n(x) g(x)}{-2\pi i |n|^2} e^{-2\pi i n \cdot x} \, dx = \sum_{F} \frac{n(F) \cdot n}{|n|^2} \int_{F} g(x) e^{-2\pi i n \cdot x} \, dx.
$$

Fix a face $F$. Two cases are possible. If $n$ is orthogonal to the face, then $e^{-2\pi i n \cdot x} = 1$ for every $x$ in the face, because of the assumption that $n$ is an integer point and $P$ is an integer polyhedron, and the integral over this $d-1$ dimensional face give a term homogeneous of degree $-1$ in the asymptotic expansion. If $n$ is not orthogonal to the face, then $e^{-2\pi i n \cdot x}$ oscillates, and in a suitable coordinate system one can apply the divergence theorem again. Again there are two possible cases. If $n$ is orthogonal to a $d-2$ dimensional face, the integral gives a term homogeneous of degree $-2$ in the asymptotic expansion. If $n$ is not orthogonal to a $d-2$ dimensional face, then we can keep going . . .

We also conjecture that an asymptotic expansion for the Fourier transform gives an asymptotic expansion for the weighted Riemann sums.

**Conjecture.** If $P$ is an integer polyhedron and $g(x)$ is a smooth function, then there exist constants $\{\delta(h)\}$ such that for every integer $N$ one has an asymptotic expansion

$$
N^{-d} \sum_{n \in Z^d} \omega_P \left( N^{-1} n \right) g \left( N^{-1} n \right) = \int_P g(x) \, dx + \sum_{h=1}^{+\infty} \frac{\delta(h)}{N^{2h}}.
$$

A possible approach to a proof is as follows. By the Poisson summation formula, the asymptotic expansion of $g \hat{\chi}_P (Nn)$, and the homogeneity $A_j(Nn) = N^{-j} A_j(n)$, one has

$$
N^{-d} \sum_{n \in Z^d} \omega_P \left( N^{-1} n \right) g \left( N^{-1} n \right)
= \int_P g(x) \, dx + \lim_{\varepsilon \to 0^+} \left\{ \sum_{n \in Z^d \setminus \{0\}} \hat{\varphi}(\varepsilon n) g \hat{\chi}_P (Nn) \right\}
$$
\[ \int g(x) \, dx + \sum_{j=1}^{+\infty} N^{-j} \lim_{\varepsilon \to 0^+} \left\{ \sum_{n \in \mathbb{Z}^d - \{0\}} \hat{\varphi}(\varepsilon n) A_j(n) \right\}. \]

By the symmetry \( A_j(-n) = (-1)^j A_j(n) \), the terms of odd homogeneity sum to zero,
\[ \sum_{n \in \mathbb{Z}^d - \{0\}} \hat{\varphi}(\varepsilon n) A_{2h+1}(n) = 0. \]

Moreover, the terms of even homogeneity are summable to finite limits,
\[ \lim_{\varepsilon \to 0^+} \left\{ \sum_{n \in \mathbb{Z}^d - \{0\}} \hat{\varphi}(\varepsilon n) A_{2h}(n) \right\} = \delta(h). \]

The existence of the limit is not obvious, since the limit series with \( 2h \leq d \) are not absolutely convergent, they are only summable with the multiplier \( \hat{\varphi}(\varepsilon n) \).

See Corollary 5.3 in [12] for a formula for \( \delta(1) \) in the case \( g(x) = 1 \). In what follows we shall show that this conjecture is true at least in dimensions 1 and 2. Our main result is Theorem 4, and we include the already known theorems 2 and 3, for the sake of comparison with Theorem 4.

2. One-dimensional Euler-Maclaurin summation

The following Fourier analytic proof of the one dimensional Euler-Maclaurin summation formula is essentially the one of Poisson. See Chapter XII in [8]. Recall that there are two possible definitions of Bernoulli polynomials, which differ by a factorial. Here the Bernoulli polynomials in the interval \( 0 \leq t \leq 1 \) are defined recursively by
\[ B_0(x) = 1, \quad \frac{d}{dx} B_{j+1}(x) = B_j(x), \quad \int_0^1 B_{j+1}(x) \, dx = 0. \]

In particular, \( B_0(x) = 1, B_1(x) = x - 1/2, B_2(x) = x^2/2 - x/2 + 1/12, B_3(x) = x^3/6 - x^2/4 + x/12, \ldots \) A direct computation shows that the periodization of \( B_1(x) \) has the Fourier expansion
\[ x - [x] - 1/2 = \sum_{n \in \mathbb{Z}} \left( \int_0^1 (y - 1/2) e^{-2\pi iny} \, dy \right) e^{2\pi inx} = -\sum_{n \in \mathbb{Z} - \{0\}} \frac{e^{2\pi inx}}{2\pi in}. \]

A repeated integration term by term of this series gives the Fourier expansions of the other \( B_j(x) \),
\[ B_j(x - [x]) = -\sum_{n \in \mathbb{Z} - \{0\}} \frac{e^{2\pi inx}}{(2\pi in)^j}. \]

The following elementary computation will be used here and in the next section.

Lemma 1. If the function \( g(x) \) is smooth, then for every \( y \neq 0 \),
\[ \int_a^b g(x) e^{-2\pi ixy} \, dx = \sum_{j=0}^{w} (2\pi y)^{-j-1} \left( e^{-2\pi iay} \frac{d^j}{dx^j} g(a) - e^{-2\pi iby} \frac{d^j}{dx^j} g(b) \right) \]
Hence, by the Poisson summation formula,
\[ (2\pi iy)^{-w-1} \int_a^b \frac{d^{w+1}}{dx^{w+1}} g(x) e^{-2\pi ixy} dx. \]

Proof. It follows by an iterated integration by parts. \[ \square \]

**Theorem 2.** If \( g(x) \) is a smooth function on \( R \), and \( a \) and \( b \) and \( N \) are integers, then for every integer \( w \),

\[
\frac{1}{N} \left( \frac{1}{2} g(a) + \sum_{n=Na+1}^{Nb-1} g(n/N) + \frac{1}{2} g(b) \right)
= \int_a^b g(x) dx + \sum_{j=0}^{w} N^{-j-1} B_{j+1} (0) \left( g^{(j)} (b) - g^{(j)} (a) \right)
+ (-1)^w N^{-w-1} \int_a^b B_{w+1} (Nx - [Nx]) g^{(w+1)} (x) dx.
\]

Finally, since \( B_{j+1} (0) = 0 \) when \( j \) is even, in the last sum only odd \( j \) are involved.

Proof. The formula with \( N \neq 1 \) follows from the one with \( N = 1 \). It suffices to replace \( g(x) \) with \( N^{-1} g(N^{-1}x) \) and \( a \) and \( b \) with \( Na \) and \( Nb \). Hence there is no loss of generality in assuming \( N = 1 \). If \( \varphi(t) \) is a smooth even function with compact support and integral 1, then,

\[
\lim_{\varepsilon \to 0^+} \{ \varphi \varepsilon * (g\chi_{[a,b]}) (x) \} = \begin{cases} 
    g(x) & \text{if } a < x < b, \\
    g(x)/2 & \text{if } x = a \text{ or } x = b, \\
    0 & \text{if } x < a \text{ or } x > b.
\end{cases}
\]

Hence, by the Poisson summation formula,

\[
\frac{1}{2} g(a) + \sum_{n=Na+1}^{b-1} g(n) + \frac{1}{2} g(b) = \sum_{n=-\infty}^{+\infty} \lim_{\varepsilon \to 0^+} \{ \varphi \varepsilon * (g\chi_{[a,b]}) (n) \}
= \lim_{\varepsilon \to 0^+} \left\{ \sum_{n=-\infty}^{+\infty} \varphi \varepsilon * (g\chi_{[a,b]}) (n) \right\} = \lim_{\varepsilon \to 0^+} \left\{ \sum_{n=-\infty}^{+\infty} \widehat{\varphi} (\varepsilon n) \widehat{g\chi_{[a,b]}} (n) \right\}.
\]

The interchange of sum and limit is justified since the convolution \( \varphi \varepsilon * (g\chi_{[a,b]}) (n) \) is bounded with uniformly bounded support, hence the sum has only a finite and bounded number of nonzero terms. And also the application of the Poisson summation formula is legitimate, since it has been applied to a mollification of the discontinuous function \( g(x) \chi_{[a,b]} (x) \). Indeed all series in the above formulas are absolutely convergent. By Lemma 11 the Fourier transform of \( g(x) \chi_{[a,b]} (x) \) has the asymptotic expansion

\[
\widehat{g\chi_{[a,b]}} (n) = \int_a^b g(x) e^{-2\pi inx} dx
= \begin{cases} 
    \int_a^b g(x) dx & \text{if } n = 0, \\
    \sum_{j=0}^{w} \frac{g^{(j)} (a) - g^{(j)} (b)}{(2\pi in)^{j+1}} + \frac{1}{(2\pi in)^{w+1}} \int_a^b g^{(w+1)} (x) e^{-2\pi inx} dx & \text{if } n \neq 0.
\end{cases}
\]
We used the assumptions that $a$, $b$, and $n$, are integers, so that $e^{-2\pi ina} = e^{-2\pi inb} = 1$. Hence,

$$\frac{1}{2}g(a) + \sum_{n=a+1}^{b-1} g(n) + \frac{1}{2}g(b) = \lim_{\varepsilon \to 0^+} \left\{ \sum_{n\in\mathbb{Z}} \hat{\varphi}(\varepsilon n) \hat{g}_{[a,b]}(n) \right\}$$

$$= \int_a^b g(t) \, dt + \sum_{j=0}^{w} \lim_{\varepsilon \to 0^+} \left\{ \sum_{n\in\mathbb{Z} - \{0\}} \left( \frac{\hat{\varphi}(\varepsilon n)}{(2\pi i n)^{j+1}} \right) \left( g^{(j)}(a) - g^{(j)}(b) \right) \right\}$$

$$+ \int_a^b \lim_{\varepsilon \to 0^+} \left\{ \sum_{n\in\mathbb{Z} - \{0\}} \left( \frac{\hat{\varphi}(\varepsilon n)}{(2\pi i n)^{w+1}} e^{-2\pi i n x} \right) g^{(w+1)}(x) \right\} dx.$$ 

The limits as $\varepsilon \to 0^+$ give the Bernoulli numbers and polynomials,

$$\frac{1}{2}g(a) + \sum_{n=a+1}^{b-1} g(n) + \frac{1}{2}g(b)$$

$$= \int_a^b g(x) \, dx + \sum_{j=0}^{w} B_{j+1}(0) \left( g^{(j)}(b) - g^{(j)}(a) \right)$$

$$+ (-1)^w \int_a^b B_{w+1}(x - [x]) g^{(w+1)}(x) \, dx.$$ 

Finally observe that, by the symmetry of the sums that define $B_{j+1}(0)$, one has $B_{j+1}(0) = 0$ when $j$ is even. Hence in the last sum only odd $j$ are involved. \[\square\]

Observe that if in the Euler-Maclaurin summation formula one disregards the terms with $j \geq 1$ and the remainder, then one obtains the trapezoidal rule for approximating integrals,

$$\left| \int_a^b g(t) \, dt - \frac{1}{N} \left( \frac{1}{2}g(a) + \sum_{n=Na+1}^{Nh-1} g(n/N) + \frac{1}{2}g(b) \right) \right| \leq \frac{C}{N^2}.$$ 

3.\ TWO-DIMENSIONAL EULER-MACLAURIN SUMMATION

A two-dimensional generalization of the Euler-Maclaurin summation formula is the following, known result. We include this result here for the sake of comparison with the main result, Theorem 4.

**Theorem 3.** If $P$ is an open integer polygon, or a closed integer polygon, and if $g(x)$ is a smooth function, then there exist constants $\{\gamma(j)\}$ such that for every positive integers $w$ and $N$,

$$N^{-2} \sum_{n \in \mathbb{Z}^2, \; N^{-1} n \in P} g(N^{-1} n) = \int_P g(x) \, dx + \sum_{j=1}^{w} \frac{\gamma(j)}{N^j} + \frac{R(w, N)}{N^{w+1}}.$$ 

The remainder $R(w, N)$ can be bounded by a constant $C$ which does not depend on $N$. 

Theorem 4.
This theorem is already known, and not only in dimension two. When \( g(x) = 1 \) it is a celebrated result of Ehrhart. See Chapter 3 in [1]. When \( g(x) \) is not constant, see [2], [7], [9], [10], [11] and Chapter 12 in [1]. An alternative proof follows from the next theorem, the main result of this paper, which compares the integral \( \int_{P} g(x) \, dx \) with the weighted Riemann sum 
\[ \frac{1}{N} \sum_{n \in \mathbb{Z}^2} \omega_P \left( (N-1)n \right) g \left( (N-1)n \right). \]

**Theorem 4.** If \( P \) is an integer polygon in the Cartesian plane, and if \( g(x) \) is a smooth function, then there exist computable constants \( \{\delta(j)\}_{j=1}^{\infty} \) with the property that for every positive integers \( w \) and \( N \) there exists \( R(w, N) \) such that
\[
N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_P \left( (N-1)n \right) g \left( (N-1)n \right) = \int_{P} g(x) \, dx + \sum_{j=1}^{w} \frac{\delta(j)}{N^{2j}} + \frac{R(w, N)}{N^{2w+2}}.
\]

The constants \( \delta(j) \) depend on the derivatives \( \frac{\partial^{[|\alpha|]} g(x)}{\partial x^\alpha} \) of order \( 2j - 2 \) and \( 2j - 1 \) evaluated at the boundary of the polygon. The remainder \( R(w, N) \) depends on the derivatives of order \( 2w + 1 \) and \( 2w + 2 \) inside the polygon. Moreover, for every \( w \) there exists \( C \) such that \( |R(w, N)| \leq C \) for every \( N \).

The particular case \( g(x) = 1 \) of this theorem is due to Macdonald. See Chapter 13 of [1]. Compare the statements of Theorem 3 and Theorem 4: the asymptotic expansion of the sums without the weights \( N^{-2} \sum_{n \in \mathbb{Z}^2} g \left( (N-1)n \right) \) may contain all powers of \( 1/N \), while the expansion of the weighted sums
\[
N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_P \left( (N-1)n \right) g \left( (N-1)n \right)
\]
contains only even powers. The unweighted sums approximate the integral \( \int_{P} g(x) \, dx \) to an order \( 1/N \), while the weighted sums give an approximation to an order \( 1/N^2 \). The weighted sums have another advantage. While the unweighted sum of open and closed polygons are different, the weighted sums are the same. Moreover, they are additive with respect to the polygons. If \( P \) and \( Q \) are integer polygons with disjoint interiors, then
\[
N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_{P \cup Q} \left( (N-1)n \right) g \left( (N-1)n \right) = N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_P \left( (N-1)n \right) g \left( (N-1)n \right) + N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_Q \left( (N-1)n \right) g \left( (N-1)n \right).
\]
In particular, the theorem for triangles implies the theorem for all other polygons. For all these reasons, the weights \( \omega_P \left( (N-1)n \right) \) are quite natural.
4. Proofs of Theorem 3 and Theorem 4

Proof of Theorem 3. Theorem 3 is a corollary of Theorem 4 and of the one-dimensional Euler-Maclaurin summation formula. The idea is the following. The difference between weighted sums in the theorem and unweighted sums is due to the points $N^{-1} n$ on the boundary of the polygon and, by the one-dimensional Euler-Maclaurin summation formula, the contribution of these points is asymptotically equal to $\frac{\vartheta}{N} + \ldots$, for some constant $\vartheta$. Hence, if the asymptotic expansions of the weighted sums contain only even powers of $1/N$, the expansions of the unweighted sums may contain also some odd powers. The details of the proof are as follows. Assume that $P$ is closed, and denote by $\{P_j\}$ and $\{L_j\}$ the vertices and the sides of $P$, each side with both vertices included. The difference between weighted and unweighted Riemann sums is due to the sampling points on the sides and the vertices of polygon,

$$
N^{-2} \sum_{n \in \mathbb{Z}^2, N^{-1} n \in P} g \left( N^{-1} n \right)
= N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_P \left( N^{-1} n \right) g \left( N^{-1} n \right) + (2N)^{-1} \sum_j \left( N^{-1} \sum_{n \in \mathbb{Z}^2, N^{-1} n \in L_j} g \left( N^{-1} n \right) \right)
- N^{-2} \sum_j \omega_P \left( P_j \right) g \left( P_j \right).
$$

A similar formula holds for an open polygon. By theorem 3,

$$
N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_P \left( N^{-1} n \right) g \left( N^{-1} n \right) = \alpha + \frac{\beta}{N^2} + \frac{\gamma}{N^4} + \ldots
$$

By the one-dimensional Euler-Maclaurin summation formula, for every $j$,

$$
N^{-1} \sum_{n \in \mathbb{Z}^2, N^{-1} n \in L_j} g \left( N^{-1} n \right) = \delta + \frac{\varepsilon}{N} + \frac{\zeta}{N^2} + \ldots
$$

Finally,

$$
- N^{-2} \sum_j \omega_P \left( P_j \right) g \left( P_j \right) = \frac{\eta}{N^2}.
$$

Putting together these three asymptotic expansions one obtains the theorem. □

Proof of Theorem 4. The proof is in principle similar to the one of Theorem 2. Here it is convenient to adopt a more explicit notation. Instead of the vector notation $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$, we write $(x, y)$ in $\mathbb{R}^2$ and $(m, n)$ in $\mathbb{Z}^2$. Moreover, since the dependence of the coefficients $\{\delta (j)\}$ and the remainder $R (w, N)$ from the polygon $P$ and the function $g (x, y)$ is more complicated than in one variable, we keep the parameter $N$. We denote by $\hat{g\chi_P} (m, n)$ the Fourier transform of $g (x, y) \chi_P (x, y)$,

$$
\hat{g\chi_P} (m, n) = \int \int_{\mathbb{R}^2} g (x, y) \chi_P (x, y) e^{-2\pi i (mx + ny)} dx dy.
$$
Moreover, we denote by $\varphi(x, y)$ a smooth radial function with compact support and integral 1. Then,

$$
\sum_{(m,n) \in \mathbb{Z}^2} \omega_P \left( N^{-1}m, N^{-1}n \right) N^{-2} g \left( N^{-1}m, N^{-1}n \right)
= \lim_{\varepsilon \to 0^+} \left\{ \sum_{(m,n) \in \mathbb{Z}^2} \varphi_{\varepsilon} \ast (g \chi_P)_{N^{-1}} (m, n) \right\}.
$$

By the Poisson summation formula,

$$
\sum_{(m,n) \in \mathbb{Z}^2} \varphi_{\varepsilon} \ast (g \chi_P)_{N^{-1}} (m, n) = \sum_{(m,n) \in \mathbb{Z}^2} \hat{\varphi} (\varepsilon m, \varepsilon n) \hat{g} \chi_P (Nm, Nn).
$$

Observe that the application of the Poisson summation formula is legitimate. The first series is finite since both $g(x, y) \chi_P(x, y)$ and $\varphi(x, y)$ have compact support, and the second series is absolutely convergent since $\hat{g} \chi_P (m, n)$ is bounded and $\hat{\varphi} (m, n)$ has fast decay at infinity. Hence,

$$
\sum_{(m,n) \in \mathbb{Z}^2} \omega_P \left( N^{-1}m, N^{-1}n \right) N^{-2} g \left( N^{-1}m, N^{-1}n \right)
= \int \int g(x, y) \, dx \, dy + \lim_{\varepsilon \to 0^+} \left\{ \sum_{(m,n) \neq (0,0)} \hat{\varphi} (\varepsilon m, \varepsilon n) \hat{g} \chi_P (Nm, Nn) \right\}.
$$

The Euler-Maclaurin summation formula is a quite straightforward consequence of this version of the Poisson summation formula, and of an asymptotic expansion of the Fourier transform $\hat{g} \chi_P (Nm, Nn)$. Observe that both Fourier transforms and weighted Riemann sums are additive with respect to integer polygons with disjoint interiors. Since polygons with more than three sides have at least two ears, see [13], or simply have interior diagonals, an integer polygon can be decomposed into integer triangles, and since with an affine change of variables one can transform a triangle into the simplex $\{0 \leq x, y, x+y \leq 1\}$, it suffices to compute the asymptotic expansion of the Fourier transform of this simplex. An iterated application of Lemma [1] gives an asymptotic expansion of the Fourier transform of a smooth function restricted to the simplex $\{0 \leq x, y, x+y \leq 1\}$. In this asymptotic expansion there is a difference between directions orthogonal to the sides of the simplex, and generic directions non orthogonal to the sides. Then the proof of Theorem [3] follows from a few technical lemmas.

We now proceed to develop some technical lemmas which will allow us to prove very precise EM-formulas for polygons, as well as quadrature-type formulas for polygons. We begin with the simplest right triangle in the plane.

**Lemma 5.** Assume that the function $g(x, y)$ is smooth and let

$$
T = \{0 \leq x, y, x+y \leq 1\}.
$$

1. There exist constants $\{\alpha(j)\}$, $\{\beta(j)\}$, $\{\gamma(j)\}$, such that for every non zero integer $n$ and every $w$,

$$
\hat{g} \chi_T (n, 0) = \int_0^1 \left( \int_0^{1-x} g(x, y) \, dy \right) e^{-2\pi i nx} \, dx = \sum_{j=0}^w \frac{\alpha(j)}{n^{j+1}} + \frac{C_1(w, n)}{n^{w+1}},
$$

where $C_1(w, n)$ is a constant independent of $n$. We will prove these lemmas separately.
\[ g_{\hat{X}}(0, n) = \int_0^1 \left( \int_0^{1-y} g(x, y) \, dx \right) e^{-2\pi iny} \, dy = \sum_{j=0}^{w} \frac{\beta(j)}{n^{j+1}} + C_2(w, n) n^{w+1}. \]

\[ g_{\hat{X}}(n, n) = \int_0^1 \left( \int_0^{1-x} g(x, y) \, dx \right)e^{-2\pi inx} \, dy = \sum_{j=0}^{w} \frac{\gamma(j)}{n^{j+1}} + \frac{C_3(w, n)}{n^{w+1}}. \]

The constants \( \alpha(j), \beta(j), \gamma(j) \) depend on the partial derivatives of \( g(x, y) \) of order \( j-1 \) and \( j \) evaluated on the boundary of the triangle with vertices \((0, 0), (1, 0), (0, 1)\), and the remainders \( C_1(w, n), C_2(w, n) \), \( C_3(w, n) \) depend on the partial derivatives of \( g(x, y) \) of order \( w \) and \( w+1 \) in this triangle. Moreover, for some constant \( C \),

\[ \sum_{n \in \mathbb{Z} - \{0\}} |C_1(w, n)|^2 \leq C, \quad \sum_{n \in \mathbb{Z} - \{0\}} |C_2(w, n)|^2 \leq C, \quad \sum_{n \in \mathbb{Z} - \{0\}} |C_3(w, n)|^2 \leq C. \]

(2) There exist constants \( \{\alpha(h, k)\} \) and \( \{\beta(h, k)\} \), such that for every non zero integers \( m \) and \( n \), with \( m \neq n \), and every \( w \),

\[ g_{\hat{X}}(m, n) = \int_0^1 \left( \int_0^{1-x} g(x, y) \, dx \right)e^{-2\pi inx} \, dx = \sum_{j=0}^{w} \left( \sum_{h+k=j} \frac{\alpha(h, k)}{m^{h+1}n^{k+1}} + \sum_{h+k=j} \frac{\beta(h, k)}{m-n}^{h+1}n^{k+1} \right) + R(w, m, n). \]

The constants \( \alpha(h, k) \) and \( \beta(h, k) \) depend on the partial derivatives of \( g(x, y) \) of order \( h + k \) evaluated at the points \((0, 0), (1, 0), (0, 1)\). The remainder \( R(w, m, n) \) has the form

\[ R(w, m, n) = \sum_{k=0}^{w} \frac{A_k(w, m)}{m^{w-k+1}n^{k+1}} + \sum_{k=0}^{w} \frac{B_k(w, m-n)}{(m-n)^{w-k+1}n^{k+1}} + \frac{\Gamma_1(w, n)}{mn^{w+1}} + \frac{\Gamma_2(w, m-n)}{mn^{w+1}} + \frac{\Gamma_3(w, m, n)}{mn^{w+1}}. \]

The functions \( A_k(w, m), B_k(w, m-n), \Gamma_1(w, n), \Gamma_2(w, m-n), \Gamma_3(w, m, n) \) depend on the partial derivatives of \( g(x, y) \) of order \( w+1 \) in the triangle with vertices \((0, 0), (1, 0), (0, 1)\). Moreover, for some constant \( C \),

\[ \sum_{m \in \mathbb{Z} - \{0\}} |A_k(w, m)|^2 \leq C, \quad \sum_{n \in \mathbb{Z} - \{0\}} |B_k(w, n)|^2 \leq C, \quad \sum_{n \in \mathbb{Z} - \{0\}} |\Gamma_1(w, n)|^2 \leq C, \quad \sum_{n \in \mathbb{Z} - \{0\}} |\Gamma_2(w, n)|^2 \leq C, \quad \sum_{m \in \mathbb{Z} - \{0\}} \sum_{n \in \mathbb{Z} - \{0\}} |\Gamma_3(w, m, n)|^2 \leq C. \]

**Proof.** (1) is the asymptotic expansion of the Fourier transform in directions orthogonal to the sides of the simplex. The first two expansions follows directly from Lemma 1 and the same for the third one, but after a change of variables. Let us consider this last one,

\[ \hat{g}_{\hat{X}}(n, n) = \int_0^1 \left( \int_0^{1-x} g(x, y) \, dx \right)e^{-2\pi iny} \, dy \]
\[
= \int_0^1 \left( \int_0^t g(s, t-s) \, ds \right) e^{-2\pi i t} \, dt.
\]

The constants \( \gamma(j) \) and the remainder \( C_3(w, n) \) in the asymptotic expansion of this integral can be written explicitly in terms of the function

\[
G(t) = \int_0^t g(s, t-s) \, ds.
\]

Indeed, by Lemma 1,

\[
\int_0^1 \left( \int_0^t g(s, t-s) \, ds \right) e^{-2\pi i t} \, dt = \sum_{j=0}^w (2\pi i n)^{-j-1} \left( G^{(j)}(0) - G^{(j)}(1) \right) + (2\pi i n)^{-w-1} \int_0^1 G^{(w+1)}(t) \, e^{-2\pi i t} \, dt.
\]

One has

\[
\frac{d^j}{dt^j} G(t) = \frac{d^j}{dt^j} \left( \int_0^t g(s, t-s) \, ds \right)
\]

\[
= \frac{d^{j-1}}{dt^{j-1}} \left( g(t, 0) + \int_0^t \frac{\partial}{\partial y} g(s, t-s) \, ds \right)
\]

\[
= \frac{d^{j-2}}{dt^{j-2}} \left( \frac{\partial}{\partial x} g(t, 0) + \frac{\partial}{\partial y} g(t, 0) + \int_0^t \frac{\partial^2}{\partial y^2} g(s, t-s) \, ds \right) = ...
\]

Hence \( \gamma(j) \) is a sum of derivatives of \( g(x, y) \) of order \( j-1 \) evaluated at the points \((0, 0)\) and \((1, 0)\), and an integral of derivatives of \( g(x, y) \) of order \( j \) along the side between \((1, 0)\) and \((0, 1)\). Similarly, the remainder \( C_3(w, n) \) depend on the partial derivatives of \( g(x, y) \) of order \( w \) and \( w+1 \) in the triangle with vertices \((0, 0)\), \((1, 0)\), \((0, 1)\). Finally, by Bessel’s inequality,

\[
\sum_{w \in \mathbb{Z} \setminus \{0\}} |C_3(w, n)|^2 \leq (2\pi)^{-2w-2} \int_0^1 \left| G^{(w+1)}(t) \right|^2 \, dt.
\]

(2) is the asymptotic expansion of the Fourier transform in generic directions non orthogonal to the sides of the simplex. By Lemma \( \text{[I]} \) for every non zero integers \( m \) and \( n \), with \( m \neq n \),

\[
\int_0^1 \left( \int_0^1 -x \, g(x, y) \, e^{-2\pi i n y} \, dy \right) \, e^{-2\pi i m x} \, dx
\]

\[
= \sum_{k=0}^w (2\pi i n)^{-k-1} \int_0^1 \frac{\partial^k}{\partial y^k} g(x, 0) \, e^{-2\pi i m x} \, dx
\]

\[
- \sum_{k=0}^w (2\pi i n)^{-k-1} \int_0^1 \frac{\partial^k}{\partial y^k} g(x, 1-x) \, e^{-2\pi i (m-n) x} \, dx
\]

\[
+ (2\pi i n)^{-w-1} \int_0^1 \left( \int_0^1 -x \, \frac{\partial^{w+1}}{\partial y^{w+1}} g(x, y) \, e^{-2\pi i n y} \, dy \right) \, e^{-2\pi i m x} \, dx.
\]

Again by Lemma \( \text{[I]} \) the first sum is

\[
\sum_{k=0}^w (2\pi i n)^{-k-1} \int_0^1 \frac{\partial^k}{\partial y^k} g(x, 0) \, e^{-2\pi i m x} \, dx
\]
Integrating by parts we have

\[
\frac{\partial^{h+k}}{\partial x^h \partial y^k} g(0, 0) - \frac{\partial^{h+k}}{\partial x^h \partial y^k} g(1, 0)
\]

\[
+ \sum_{k=0}^{w} \sum_{h=0}^{w-k} (2\pi im)^{-h-1} (2\pi in)^{-k-1} \int_0^1 \frac{\partial^{w+1}}{\partial x^{w-k+1} \partial y^k} g(x, 0) e^{-2\pi imx} dx.
\]

The terms in the double sum define \( \alpha (h, k) m^{-h-1} n^{-k-1} \). The terms in the last sum define the remainders \( A_k (w, m) m^{k-w-1} n^{-k-1} \) and, by Bessel’s inequality,

\[
\sum_{m \in \mathbb{Z} - \{0\}} |A_k (w, m)|^2 \leq (2\pi)^{-2w-4} \int_0^1 \left| \frac{\partial^{w+1}}{\partial x^{w-k+1} \partial y^k} g(x, 0) \right|^2 dx.
\]

Similarly,

\[
- \sum_{k=0}^{w} \sum_{h=0}^{w-k} (2\pi i)^{w-2} \frac{(2\pi i)^{w-2}}{(m-n)^{w+1-k} n^{k+1}} \times \left( \frac{\partial^h}{\partial x^h} \left( \frac{\partial^k}{\partial y^k} g(x, 1-x) \right) \right|_{x=1} - \frac{\partial^h}{\partial x^h} \left( \frac{\partial^k}{\partial y^k} g(x, 1-x) \right) \left|_{x=0} \right) \\
- \sum_{k=0}^{w} \sum_{h=0}^{w-k} \frac{(2\pi i)^{w-2}}{(m-n)^{w+1-k} n^{k+1}} \int_0^1 \frac{\partial^{w-k+1}}{\partial x^{w-k+1} \partial y^k} \frac{\partial^k}{\partial y^k} g(x, 1-x) e^{-2\pi imx} dx.
\]

The terms in the double sum define \( \beta (h, k) (m-n)^{-h-1} n^{-k-1} \). The terms in the last sum define the remainders \( B_k (w, m-n) (m-n)^{k-w-1} n^{-k-1} \) and, by Bessel’s inequality,

\[
\sum_{n \neq 0} |B_k (w, n)|^2 \leq (2\pi)^{-2w-4} \int_0^1 \left| \frac{\partial^{w-k+1}}{\partial x^{w-k+1} \partial y^k} \left( \frac{\partial^k}{\partial y^k} g(x, 1-x) \right) \right|^2 dx.
\]

It remains to consider

\[
(2\pi im)^{-w-1} \int_0^1 \left( \int_0^{1-x} \frac{\partial^{w+1}}{\partial y^{w+1}} g(x, y) e^{-2\pi i ny} dy \right) e^{-2\pi imx} dx.
\]

Integrating by parts we have

\[
(2\pi im)^{-w-1} \int_0^1 \left( \int_0^{1-x} \frac{\partial^{w+1}}{\partial y^{w+1}} g(x, y) e^{-2\pi i ny} dy \right) e^{-2\pi imx} dx
\]

\[
= (2\pi im)^{-1} (2\pi im)^{-w-1} \int_0^1 \frac{\partial^{w+1}}{\partial y^{w+1}} g(0, y) e^{-2\pi i ny} dy
\]

\[
+ (2\pi im)^{-1} (2\pi im)^{-w-1} \int_0^1 \frac{\partial}{\partial x} \left( \int_0^{1-x} \frac{\partial^{w+1}}{\partial y^{w+1}} g(x, y) e^{-2\pi i ny} dy \right) e^{-2\pi imx} dx
\]

\[
= (2\pi im)^{-1} (2\pi im)^{-w-1} \int_0^1 \frac{\partial^{w+1}}{\partial y^{w+1}} g(0, y) e^{-2\pi i ny} dy
\]

\[
- (2\pi im)^{-1} (2\pi im)^{-w-1} \int_0^1 \frac{\partial^{w+1}}{\partial y^{w+1}} g(x, 1-x) e^{-2\pi i (m-n)x} dx
\]
\begin{equation*}
+ (2\pi i m)^{-1} (2\pi i n)^{-w-1} \int_0^1 \left( \int_0^{1-x} \frac{\partial^{w+2}}{\partial x \partial y^{w+1}} g(x,y) e^{-2\pi i ny} dy \right) e^{-2\pi i mx} dx.
\end{equation*}

Observe that the integral at the top depends only on \( \frac{\partial^{w+1}}{\partial y^{w+1}} g(x,y) \) in the triangle. In particular, if this derivative vanishes, then also the sum of the three integrals at the bottom vanishes. These three integrals define the remainders \( \Gamma_1(w,n) m^{-1} n^{-w-1}, \Gamma_2(w,m-n) m^{-1} n^{-w-1}, \Gamma_3(w,m) w^{-1} n^{-w-1} \). Finally, by Bessel’s inequality,

\begin{equation*}
\sum_{n \neq 0} |\Gamma_1(w,n)|^2 \leq (2\pi)^{-2w-4} \int_0^1 \left| \frac{\partial^{w+1}}{\partial y^{w+1}} g(0,y) \right|^2 dy,
\end{equation*}

\begin{equation*}
\sum_{n \neq 0} |\Gamma_2(w,n)|^2 \leq (2\pi)^{-2w-4} \int_0^1 \left| \frac{\partial^{w+1}}{\partial y^{w+1}} g(x,1-x) \right|^2 dx,
\end{equation*}

\begin{equation*}
\sum_{m \neq 0|n \neq 0} |\Gamma_3(w,m,n)|^2 \leq (2\pi)^{-2w-4} \int_0^1 \left( \int_0^{1-x} \left| \frac{\partial^{w+2}}{\partial x \partial y^{w+1}} g(x,y) \right|^2 dy \right) dx.
\end{equation*}

\[\square\]

**Lemma 6.** If \( P \) is the triangle with vertices \((p,q),(p+a,q+b),(p+c,q+d)\), and \( T \) is the triangle with vertices \((0,0),(1,0),(0,1)\), then

\begin{equation*}
\int_P g(x,y) e^{-2\pi i (mx+ny)} dxdy = e^{-2\pi i (pm+qn)} |ad-bc| \times \int_T g(p+as+ct,q+bs+dt) e^{-2\pi i (am+bn)s+(cm+dn)t} dsdt.
\end{equation*}

**Proof.** This follows by a change of variables. \[\square\]

By the above lemmas, if \( P \) is the triangle with vertices \((p,q),(p+a,q+b),(p+c,q+d)\), then the asymptotic expansion of

\begin{equation*}
\lim_{\varepsilon \to 0^+} \left\{ \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \hat{\varphi}(\varepsilon m,\varepsilon n) \hat{g}_{X^P}(Nm,Nn) \right\}
\end{equation*}

is a sum of several terms. By part (1) of Lemma 5 there are terms of the form

\begin{equation*}
N^{-h-1} \lim_{\varepsilon \to 0^+} \left\{ \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}, cm+dn=0} \frac{\hat{\varphi}(\varepsilon m,\varepsilon n)}{(am+bn)^{h+1}} \right\}.
\end{equation*}

By part (2) of Lemma 5 there are terms of the form

\begin{equation*}
N^{-h-k-2} \lim_{\varepsilon \to 0^+} \left\{ \sum_{am+bn \neq 0, cm+dn \neq 0, cm+fn \neq 0} \frac{\hat{\varphi}(\varepsilon m,\varepsilon n)}{(am+bn)^{h+1}(cm+dn)^{k+1}} \right\}
\end{equation*}

\begin{equation*}
= N^{-h-k-2} \lim_{\varepsilon \to 0^+} \left\{ \sum_{am+bn \neq 0, cm+dn \neq 0} \frac{\hat{\varphi}(\varepsilon m,\varepsilon n)}{(am+bn)^{h+1}(cm+dn)^{k+1}} \right\}
\end{equation*}
\[-N^{-h-k-2} \lim_{\varepsilon \to 0^+} \left\{ \sum_{(m,n) \neq (0,0), \varepsilon m + f n = 0} \frac{\tilde{\varphi}(\varepsilon m, \varepsilon n)}{(am + bn)^{h+1} (cm + dn)^{k+1}} \right\}.\]

There are also remainder terms of similar forms. $a, b, c, d, e, f$ are integers, and $am + bn = 0$, $cm + dn = 0$, $em + fn = 0$ are distinct lines. In our case $e = a - c$ and $f = b - d$. By the homogeneity of these expressions, one can also assume that $(a, b) = 1$, $(c, d) = 1$, $(e, f) = 1$. Observe the similarity of the above expansions with the trigonometric expansions of the periodized Bernoulli polynomials:

\[B_k(x - [x]) = \left( \frac{-1}{2\pi i} \right)^k \sum_{n \in \mathbb{Z} - \{0\}} \frac{e^{2\pi i n x}}{n^k}.\]

**Lemma 7.** (1) If $h$ is even,

\[\sum_{(m,n) \neq (0,0), \varepsilon m + f n = 0} \frac{\tilde{\varphi}(\varepsilon m, \varepsilon n)}{(am + bn)^{h+1}} = 0.\]

(2) If $ad + bc \neq 0$ with $c$ and $d$ coprime, and if $h$ is odd,

\[\lim_{\varepsilon \to 0^+} \left\{ \sum_{(m,n) \neq (0,0), \varepsilon m + f n = 0} \frac{\tilde{\varphi}(\varepsilon m, \varepsilon n)}{(am + bn)^{h+1}} \right\} = (-1)^{(h-1)/2} 2^{h+1} n^{h+1} B_{h+1}(0) (ad - bc)^{-h-1}.\]

(3) If $h + k$ is odd,

\[\sum_{(m,n) \neq (0,0), \varepsilon m + f n = 0} \frac{\tilde{\varphi}(\varepsilon m, \varepsilon n)}{(am + bn)^{h+1} (cm + dn)^{k+1}} = 0.\]

(4) If $af - be \neq 0$, $cf - de \neq 0$, with $e$ and $f$ coprime, and if $h + k$ is even,

\[\lim_{\varepsilon \to 0^+} \left\{ \sum_{(m,n) \neq (0,0), \varepsilon m + f n = 0} \frac{\tilde{\varphi}(\varepsilon m, \varepsilon n)}{(am + bn)^{h+1} (cm + dn)^{k+1}} \right\} = \frac{(-1)^{(h+k)/2} 2^{h+k+2} \pi^{h+k+2} B_{h+k+2}(0)}{(af - be)^{h+1} (cf - de)^{k+1}}.\]

**Proof.** (1) If $h$ is even, since $\tilde{\varphi}(\xi, \eta)$ is radial, then $\tilde{\varphi}(\varepsilon m, \varepsilon n) (am + bn)^{-h-1}$ is odd and the sum vanishes.

(2) Assume $h$ odd. If $(c, d) = 1$, the non zero integer points on the line $cm + dn = 0$ are $(m,n) = j (d, -c)$, with $j \in \mathbb{Z} - \{0\}$. Hence, by dominated convergence,

\[\lim_{\varepsilon \to 0^+} \left\{ \sum_{(m,n) \neq (0,0)} \frac{\tilde{\varphi}(\varepsilon m, \varepsilon n)}{(am + bn)^{h+1}} \right\} = (ad - bc)^{-h-1} \sum_{j \in \mathbb{Z} - \{0\}} j^{-h-1} = -(2\pi i)^{h+1} B_{h+1}(0) (ad - bc)^{-h-1}.\]

The proof of (3) is the same as (1), and the proof of (4) is the same as (2).

The following lemma is the analogous of the previous one for double series.
Lemma 8. Set
\[ B_{h,k}(x,y) = \begin{cases} B_h(x)B_k(y) & \text{if } 0 \leq x, y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \]
Assume that \( a \) and \( b \) are coprime, that \( c \) and \( d \) are coprime, and that \( ad - bc \neq 0 \), and denote by \( R \) the parallelogram
\[ R = \{(x,y) \in \mathbb{R}^2, \ 0 \leq \frac{dx - cy}{ad - bc} \leq 1, \ 0 \leq \frac{-bx + ay}{ad - bc} \leq 1\} \]

1. If \( h + k \) is odd,
\[ \sum_{am+bn\neq 0, \ cm+dn\neq 0} \varphi(\varepsilon m, \varepsilon n)(am + bn)^{-h-1}(cm + dn)^{-k-1} = 0. \]

2. If \( h + k \) is even,
\[ \lim_{\varepsilon \to 0^+} \left\{ \sum_{am+bn\neq 0, \ cm+dn\neq 0} \varphi(\varepsilon m, \varepsilon n)(am + bn)^{-h-1}(cm + dn)^{-k-1} \right\} \]
\[ = (-1)^{(h+k+2)/2} 2^{h+k+2} \pi^{h+k+2} |ad - bc|^{-1} \]
\[ \times \sum_{(m,n) \in \mathbb{Z}^2} \omega_R(m,n) B_{h+1,k+1} \left( \frac{dn - cn}{ad - bc}, \frac{-bn + an}{ad - bc} \right). \]

Proof. (1) If \( h + k \) is odd, then the sum vanishes by symmetry.

(2) Assume \( h + k \) even. The Fourier transform of
\[ B_{h,k} \left( \frac{dx - cy}{ad - bc}, \frac{-bx + ay}{ad - bc} \right) \]
evaluated at the integers \( (m,n) \), with \( am + bn \neq 0 \) and \( cm + dn \neq 0 \), is
\[ \int_{\mathbb{R}^2} B_{h,k} \left( \frac{dx - cy}{ad - bc}, \frac{-bx + ay}{ad - bc} \right) e^{-2\pi i(mx+ny)} dxdy \]
\[ = |ad - bc| \int_{\mathbb{R}^2} B_{h,k}(s,t) e^{-2\pi i(m(as+ct)+n(bs+dt))} dsdt \]
\[ = |ad - bc| \left( \int_0^1 B_k(t)e^{-2\pi i(am+bn)t} dt \right) \left( \int_0^1 B_h(s)e^{-2\pi i(cm+dn)s} ds \right) \]
\[ = |ad - bc| \left( 2\pi i (am + bn) \right)^{-h} \left( 2\pi i (cm + dn) \right)^{-k}. \]

Then
\[ \sum_{am+bn\neq 0, \ cm+dn\neq 0} \ |ad - bc| \left( 2\pi i (am + bn) \right)^{-h} \left( 2\pi i (cm + dn) \right)^{-k-1} \]
is formally the sum over \( \mathbb{Z}^2 \) of the Fourier coefficients of the function
\[ B_{h,k} \left( \frac{dx - cy}{ad - bc}, \frac{-bx + ay}{ad - bc} \right). \]
Hence (2) follows from the Poisson summation formula. The factors \( \omega_R(m,n) \) in the formula come from the fact that the function \( B_{h,k} \left( \frac{dx - cy}{ad - bc}, \frac{-bx + ay}{ad - bc} \right) \) may
be discontinuous on the boundary of $R$. Observe that the sum
\[ \sum_{(m,n) \in \mathbb{Z}^2} \omega_R(m,n) B_{h,k} \left( \frac{dm - cn}{ad - bc}, \frac{bm + an}{ad - bc} \right) \]
is finite. Hence limit of the series
\[ \lim_{\varepsilon \to 0^+} \sum_{am + bn \neq 0, cm + dn \neq 0} \hat{\varphi}(\varepsilon m, \varepsilon n) (am + bn)^{-h-1} (cm + dn)^{-k-1} \]
can be computed explicitly in a finite number of steps. □

It remains to estimate the remainders.

**Lemma 9.** Assume $w > 0$ and let $am + bn = 0$, $cm + dn = 0$, $em + fn = 0$ be distinct lines.

(1) If $n^{-w-1}C(w, n)$ is one of the remainders in part (1) of Lemma 5, then for some constant $C$ independent of $\varepsilon$ and $N$,
\[ \left| \sum_{(m,n) \neq (0,0), cm+dn=0} \hat{\varphi}(\varepsilon m, \varepsilon n) \frac{C(w, N(am + bn))}{(N(am + bn))^{w+1}} \right| \leq CN^{-w-1}. \]

(2) If $R(w, m, n)$ is the remainder in part (2) of Lemma 5, then for some constant $C$ independent of $\varepsilon$ and $N$,
\[ \left| \sum_{am+bn\neq 0, cm+dn\neq 0, cm+fn\neq 0} \hat{\varphi}(\varepsilon m, \varepsilon n) R(w, N(am + bn), N(cm + dn)) \right| \leq CN^{-w-2}. \]

**Proof.** (1) follows from the fact that $C(w, N(am + bn))$ is bounded. The proof of (2) is a bit more complicated, since $R(w, m, n)$ is sum of many terms, and not all the series involved are absolutely convergent. We shall consider just two of them. A term in $R(w, m, n)$ is
\[ A_0(w, N(am + bn)) (N(am + bn))^{-w-1} (N(cm + dn))^{-1}. \]

After factorizing $N^{-w-2}$, one has to estimate the series
\[ \sum_{am+bn\neq 0, cm+dn\neq 0} \hat{\varphi}(\varepsilon m, \varepsilon n) A_0(w, N(am + bn)) (am + bn)^{-w-1} (cm + dn)^{-1}. \]

Observe that the corresponding series without the cutoff $\hat{\varphi}(\varepsilon m, \varepsilon n)$ does not converge absolutely. As in the previous lemma define
\[ F_N(x, y) = \begin{cases} \sum_{m \neq 0} A_0(w, Nm) e^{2\pi imx/m^{w+1}} - \sum_{n \neq 0} e^{2\pi iny}/n & \text{if } 0 \leq x, y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \]

The first series converges absolutely and it can be bounded independently on $N$. And also the second series defines a bounded function, which is up to a constant the degree one Bernoulli polynomial. Hence $F_N(x, y)$ is a bounded function. Moreover,
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as in Lemma 8, the Fourier transform of \( F_N \left( \frac{dx - cy}{ad - bc}, \frac{-bx + ay}{ad - bc} \right) \) evaluated at the integers is

\[
\int \int_{\mathbb{R}^2} F_N \left( \frac{dx - cy}{ad - bc}, \frac{-bx + ay}{ad - bc} \right) e^{-2\pi i(mx + ny)} \, dx \, dy
\]

\[
= |ad - bc| \int \int_{\mathbb{R}^2} F_N(s, t) e^{-2\pi i(m(as + ct) + n(bs + dt))} \, ds \, dt
\]

\[
= |ad - bc| A_0(w, N (am + bn)) (am + bn)^{-w-1} (cm + dn)^{-1}.
\]

Then, by the Poisson summation formula,

\[
|ad - bc|^{-1} \sum \sum_{(m, n) \in \mathbb{Z}^2} \int \int_{\mathbb{R}^2} \varphi_\varepsilon(x, y) F_N \left( \frac{dm - cn}{ad - bc} - x, \frac{-bn + an}{ad - bc} - y \right) dx \, dy
\]

\[
= \sum \sum_{(m, n) \in \mathbb{Z}^2} \hat{\varphi}(\varepsilon m, \varepsilon n) \hat{F}_N(am + bn, cm + dn)
\]

\[
= \sum \sum_{am + bn \neq 0, cm + dn \neq 0} \hat{\varphi}(\varepsilon m, \varepsilon n) \frac{A_0(w, N (am + bn))}{(am + bn)^{w+1} (cm + dn)^{-1}}.
\]

The first series is indeed a finite sum that can be bounded independently of \( \varepsilon \) and \( N \). Hence also the last series is bounded independently of \( \varepsilon \) and \( N \).

Another term in \( R(w, m, n) \) is

\[
A_w(w, N (am + bn)) (N (am + bn))^{-1} (N (cm + dn))^{-w-1}.
\]

After factorizing \( N^{-w-2} \) one has to estimate the series

\[
\sum \sum_{am + bn \neq 0, cm + dn \neq 0} \hat{\varphi}(\varepsilon m, \varepsilon n) \frac{A_w(w, N (am + bn))}{(am + bn)^{w+1} (cm + dn)^{-1}}.
\]

This series is absolutely convergent and it can be bounded independently of \( \varepsilon \) and \( N \),

\[
\sum \sum_{am + bn \neq 0, cm + dn \neq 0} \left| \hat{\varphi}(\varepsilon m, \varepsilon n) \frac{A_w(w, N (am + bn))}{(am + bn)^{w+1} (cm + dn)^{-1}} \right|
\]

\[
\leq \sup_{(\xi, \eta) \in \mathbb{R}^2} \{|\hat{\varphi}(\xi, \eta)| \sum_{j \neq 0} |j|^{-w-1} \left\{ \sum \sum_{am + bn \neq 0, cm + dn = j} |am + bn|^{-2} \right\}^{1/2}
\]

\[
\times \left\{ \sum \sum_{am + bn \neq 0, cm + dn = j} |A_w(w, N (am + bn))|^2 \right\}^{1/2}.
\]

All the other terms in \( R(w, m, n) \) can be estimated in a similar way. We remark about the notation that the \( w \)'s in part (1) and (2) of the above lemma are arbitrary, not necessarily equal, and not equal to the \( w \) in the theorem. This completes the proof of Theorem 4. \( \square \)
5. Quadrature Formulas for Polygons

In this section, we give quadrature formulas whose nodes are all of the integer points. We return to our use of compact vector notation $x \in \mathbb{R}^2$ and $n \in \mathbb{Z}^2$. The constants $\{\delta(j)\}$ in Theorem 4 can be computed explicitly in a finite number of steps, but they are composed of many pieces, and the final result is not as clean as it is in $\mathbb{R}^1$. However, disregarding these terms in the asymptotic expansion, one recognizes an analog of the trapezoidal rule for approximating integrals.

**Corollary 10.** If $P$ is an open integer polygon, and if $g(x)$ is a smooth function, then for every positive integer $N$,

$$\left| \int_P g(x) \, dx - \left( \frac{1}{N^2} \sum_{N^{-1}n \in P} g(N^{-1}n) + \frac{1}{2N^2} \sum_{N^{-1}n \in \partial P} g(N^{-1}n) \right) \right| \leq \frac{C}{N^2}.$$

**Proof.** By Theorem 4, it suffices to observe that on the boundary of the polygon we have $\omega_P(N^{-1}n) = 1/2$, except at the vertices, where the contribution is of the order of $N^{-2}$, hence negligible. \qed

In the above corollary there are no weights $\omega_P(P_j)$ at the vertices $P_j$. When $P$ is an integer polygon, then $\omega_P(P_j) = (2\pi)^{-1} \arctan(a/b)$, with $a$ and $b$ suitable integers, and it can be proved that either this weight is an integer multiple of $1/8$, or it is irrational. See Corollary 3.12 in [14]. Hence Corollary 10 is, so to speak, a rational approximation of Theorem 4. Anyhow, by an elementary trick suggested by Huygens and Newton, one can accelerate the convergence of the Riemann sums and obtain a result which is better than the one above. Huygens in "De circuli magnitudine inventa" (1654) proved the following:

*The circumference of a circle is larger than the perimeter of an inscribed equilateral polygon plus one third of the difference between the perimeter of this polygon and the perimeter of an inscribed polygon with half number of sides.*

The latter statement reduces to the trigonometric inequality

$$\pi > 2N \sin \left( \frac{\pi}{2N} \right) + \frac{1}{3} \left( 2N \sin \left( \frac{\pi}{2N} \right) - N \sin \left( \frac{\pi}{N} \right) \right).$$

Observe that this inequality gives a better approximation to $\pi$ than the inequality $\pi > 2N \sin \left( \frac{\pi}{2N} \right)$ used by Archimedes in the "Dimensio circuli". Newton in his correspondence with Leibniz through Oldenburg, ("Epistola Prior" 13/6/1676) explained the theorem of Huygens in term of the power series expansion of the sine function:

$$\sin(x) = x - x^3/6 + x^5/120 - \ldots,$$

$$\frac{4 \sin(x)}{x} - \frac{1}{3} \frac{\sin(2x)}{2x} = 1 - x^4/30 + \ldots$$

By applying this trick to our Riemann sums one can accelerate their convergence from $N^{-2}$ to $N^{-4}$, or $N^{-6}$, or in principle to an arbitrary speed.
Corollary 11. If $P$ is an integer polygon, and if $g(x)$ is a smooth function, set

$$S(N) = N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_P(N^{-1}n) g(N^{-1}n).$$

Then there exists $C$ such that for every positive integer $N$,

$$\left| \int g(x) \, dx - \left( -\frac{1}{3} S(N) + \frac{4}{3} S(2N) \right) \right| \leq \frac{C}{N^4}.$$

Similarly, there exists $C$ such that for every positive integer $N$,

$$\left| \int g(x) \, dx - \left( \frac{1}{45} S(N) - \frac{20}{45} S(2N) + \frac{64}{45} S(4N) \right) \right| \leq \frac{C}{N^6}.$$

And so on...

Proof. By the theorem, there exist $\alpha, \beta, \gamma, \ldots$ such that

$$S(N) = \alpha + \beta \frac{N}{N^2} + \gamma \frac{1}{N^4} + \ldots$$

Multiply $S(N)$ and $S(2N)$ by some constants $x$ and $y$, and add,

$$xS(N) + yS(2N) = (x + y) \alpha + (x + y/4) \beta \frac{N}{N^2} + (x + y/16) \gamma \frac{1}{N^4} + \ldots$$

Then, if $x = -1/3$ and $y = 4/3$,

$$-\frac{1}{3} S(N) + \frac{4}{3} S(2N) = \alpha - \frac{\gamma}{4N^4} + \ldots$$

Similarly, with a suitable linear combination $xS(N) + yS(2N) + zS(4N)$ one obtains an approximation of $\alpha$ of order $N^{-6}$, and this process may be continued. Observe that the matrix associated to the linear system is a Vandermonde matrix. □

Observe that the sampling points $N^{-1}\mathbb{Z}^2 \cap P$ that appear in the sum $S(N)$ are a subset of the sampling points in $S(2N)$, and these are a subset of the ones in $S(4N)$, ... In the computation of $xS(N) + yS(2N) + zS(4N) + \ldots$ one can collect equal sampling points, and multiply by suitable weights. In particular the computation of $xS(N) + yS(2N) + zS(4N) + \ldots$ has the same complexity of the computation of the last summand $S(2^nN)$. As an explicit example, observe that

$$-\frac{1}{3} S(N) + \frac{4}{3} S(2N) = -\frac{1}{3} N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_P(N^{-1}n) g(N^{-1}n)$$

$$+ \frac{4}{3} (2N)^{-2} \sum_{n \in \mathbb{Z}^2} \omega_P((2N)^{-1}n) g((2N)^{-1}n)$$

$$= \frac{1}{3} N^{-2} \sum_{n \in \mathbb{Z}^2-2\mathbb{Z}^2} \omega_P((2N)^{-1}n) g((2N)^{-1}n).$$

The vertices of the polygon do not appear in the last sum, since if $(2N)^{-1}n$ is an integer vertex, then $n \in 2\mathbb{Z}^2$. Hence in the last sum $\omega_P((2N)^{-1}n)$ takes only the values 0, 1/2, 1,

$$-\frac{1}{3} S(N) + \frac{4}{3} S(2N)$$
\[
\frac{1}{3N^2} \sum_{n \in \mathbb{Z}^2 \setminus 2\mathbb{Z}^2} g((2N)^{-1}n) + \frac{1}{6N^2} \sum_{n \in \mathbb{Z}^2 \setminus 2\mathbb{Z}^2} g((2N)^{-1}n).
\]

The statement of the corollary seems paradoxical. One throws away one fourth of the grid \((2N)^{-1} \mathbb{Z}^2 \cap \mathcal{P})\), and the order of approximation increases.

This trick of Huygens and Newton works in every dimension. In particular, in dimension one the weighted sum
\[
-\frac{1}{3} S(N) + \frac{4}{3} S(2N)
\]
decreases to the Kepler, or Cavalieri, or Simpson quadrature rule,
\[
-\frac{1}{6} \left[ g(a) + \frac{4}{3} g\left(a + \frac{1}{2N}\right) + 2 g\left(a + \frac{2}{2N}\right) \right] + \left[ g\left(a + \frac{1}{2N}\right) + 2 g\left(a + \frac{2}{2N}\right) + \ldots \right].
\]

"What has been will be again, what has been done will be done again; there is nothing new under the sun."

Let us go back to the theorem. When \(g(x)\) is a polynomial only a finite number of coefficients \(\{\delta(h)\}\) are nonzero, and the asymptotic formula becomes exact.

**Corollary 12.** For a homogeneous polynomial \(g(x)\) of degree \(\alpha\), we have:
\[
\sum_{n \in \mathbb{Z}^2} \omega_{N \mathcal{P}}(n) g(n) = N^{\alpha+2} \int_{\mathcal{P}} g(x) \, dx + N^{\alpha+2} \sum_{j=1}^{w} \frac{\delta(j)}{N^{2j}}.
\]

**Proof.** In Theorem 1 the reminder \(R(w, N)\) vanishes provided that all derivatives of order \(2w + 1\) vanish. Hence, for every \(w > (\alpha - 1)/2\),
\[
N^{-2} \sum_{n \in \mathbb{Z}^2} \omega_{N \mathcal{P}}(N^{-1}n) g(N^{-1}n) = \int_{\mathcal{P}} g(x) \, dx + \sum_{j=1}^{w} \frac{\delta(j)}{N^{2j}}.
\]

Because \(g(x)\) is homogeneous of degree \(\alpha\), we have
\[
\omega_{\mathcal{P}}(N^{-1}n) g(N^{-1}n) = N^{-\alpha} \omega_{N \mathcal{P}}(n) g(n).
\]

The case \(g(x) = 1\) of Corollary 12 is the celebrated MacDonald solid-angle polynomial in two dimensions, from which the classical Pick’s formula follows easily.

**Corollary 13.** (Pick’s formula) If \(\mathcal{P}\) is a simply connected integer polygon in the plane, with \(I\) interior integer points, \(B\) boundary integer points, and area \(|\mathcal{P}|\), then
\[
|\mathcal{P}| = I + \frac{B}{2} - 1.
\]
Proof. By Corollary 12 with $g(x) = 1$ and $N = 1$,

$$\int_P dx = \sum_{n \in \mathbb{Z}^2} \omega_P(n).$$

If $n$ is an integer point inside $P$, then $\omega_P(n) = 1$. If $n$ is on a side but it is not a vertex, then $\omega_P(n) = 1/2$. If $P$ has $r$ vertices, then

$$\sum_{n \text{ vertex of } P} \omega_P(n) = \frac{(r-2)\pi}{2\pi} = \frac{r}{2} - 1.$$

It follows that

$$\sum_{n \in \mathbb{Z}^2} \omega_P(n) = I + \frac{B}{2} - 1.$$

□

In [3], we presented a harmonic analysis proof of Pick’s theorem, together with one possible conjecture for an extension to higher dimensions. See [15] for a very recent counterexample to this conjecture.

6. Appendix: A numerical example

Let us test Corollary 10 and Corollary 11 on an explicit example. Take $P = \{y > x/2, y < 3 - x, y < 2x\}$, $g(x, y) = x^3y^3$, and $N = 2$, that is a grid of side 1/4. The exact value of the integral is

$$\int \int_{\{y > x/2, y < 3 - x, y < 2x\}} x^3y^3 dxdy = \frac{423}{140} = 3.021...$$

The Riemann sum in Corollary 10 runs over 31 sampling points, and it is a rough approximation to the integral:

$$\frac{1}{16} \sum_{(m,n) \in P} \left(\frac{m}{4}\right)^2 \left(\frac{n}{4}\right)^3 + \frac{1}{32} \sum_{(m,n) \in \partial P} \left(\frac{m}{4}\right)^2 \left(\frac{n}{4}\right)^3 = 54335 \frac{1}{16384} = 3.316...$$

The Riemann sum in Corollary 11 runs over 21 sampling points, and it gives a much better approximation to the integral above:

$$\frac{1}{12} \sum_{(m,n) \in \mathbb{Z}^2 \setminus 2\mathbb{Z}^2} \left(\frac{m}{4}\right)^2 \left(\frac{n}{4}\right)^3 + \frac{1}{24} \sum_{(m,n) \in \mathbb{Z}^2 \setminus 2\mathbb{Z}^2} \left(\frac{m}{4}\right)^2 \left(\frac{n}{4}\right)^3 = 37295 \frac{1}{12288} = 3.035...$$
Figure 1. Small and large sampling points in $4^{-1} \mathbb{Z}^2$ in Corollary 10 and large sampling points in $4^{-1} \mathbb{Z}^2 - 2^{-1} \mathbb{Z}^2$ in Corollary 11.

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