A Four-Qubits Code that is a Quantum Deletion Error-Correcting Code with the Optimal Length

Manabu HAGIWARA∗ Ayumu NAKAYAMA†

Abstract

This paper provides a new instance of quantum deletion error-correcting codes. This code can correct any single quantum deletion error, while our code is only of length 4. This paper also provides an example of an encoding quantum circuit and decoding quantum circuits. It is also proven that the length of any single deletion error-correcting codes is greater than or equal to 4. In other words, our code is optimal for the code length.

1 Introduction

Similar to classical error-correcting codes, quantum error-correcting codes are an essential factor for implementing practical quantum communication and quantum computation.

In recent classical coding theory, deletion codes have attracted researcher’s attention. For example, while only three papers on deletion codes were presented at the symposium ISIT 2015, more than ten papers were presented at ISIT 2017, 2018 and 2019. Furthermore, two technical sessions for deletion codes were organized at the last ISIT. Classical deletion error-correcting codes have applications to reliable communication for synchronization error [10,16], error-correction for DNA storage [3], error-correction for racetrack memory [5], etc.

∗Department of Mathematics and Informatics, Graduate School of Science, Chiba University 1-33 Yayoi-cho, Inage-ku, Chiba City, Chiba Pref., JAPAN, 263-0022
†Department of Mathematics and Informatics, Graduate School of Science and Engineering, Chiba University 1-33 Yayoi-cho, Inage-ku, Chiba City, Chiba Pref., JAPAN, 263-0022
On the other hand, only a few studies on quantum deletion correcting codes have been published. In 2019, Leahy et.al. introduced quantum insertion/deletion channel \[13\] and a technique to reduce quantum deletion error to quantum erasure error under the specific assumption. The first quantum binary deletion codes under the general scenario was constructed in \[15\]. The code length was 8.

Deletion error-correction is a problem that to determine the original information from a partial information. The difference from erasure error-correction is that we are not given the information on the error positions. For example, a bit-sequence 00111000 is changed to 001?1000 by a single erasure error but is changed to 0011000 by a single deletion error. The symbol “?” tells the position where error occurred. An erasure bit-sequence 001?1000 can be transformed to 0011000 by deleting the symbol “?”. Hence deletion error-correction is more difficult than erasure error-correction.

In quantum information theory, quantum deletion error-correction is a problem to determine the quantum state in the entire quantum system from a quantum state in a partial system. Therefore it is related to various topics, e.g., quantum erasure error-correcting codes \[9\], partial trace, quantum secret sharing \[6,11\], purification of quantum state \[12\], quantum cloud computing \[2\] and etc.

This paper provides a new and a shorter length single deletion error-correcting code. The code space is not an instance of previously known quantum error-correcting codes, e.g., CSS codes \[4,17\], stabilizer codes \[8\], surface codes \[7\], and etc. Remark that our code length is only 4. This is the same length to the optimal shortest code length of quantum erasure error-correcting codes \[9\]. In fact, we show that 4 is also the optimal length of quantum deletion error-correcting codes.

This paper also provides examples of one encoding circuit and two different decoding circuits. The depth of the circuits are small.

The readers are assumed to be familiar with quantum information theory and coding theory, in particular, quantum error-correcting codes and classical deletion error-correcting codes.

2 Deletion Errors and Code Construction

2.1 Deletion Error and Deletion Error-Correcting Codes

Set \(|0\rangle, |1\rangle \in \mathbb{C}^2\) as

\[|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}\]
respectively. For a binary sequence $x = x_1 x_2 \ldots x_n \in \{0, 1\}^n$, $|x\rangle$ denotes $|x_1\rangle \otimes |x_2\rangle \otimes \ldots |x_n\rangle \in \mathbb{C}^{2^n}$.

We denote the set of all density matrices of order $N$ by $S(\mathbb{C}^N)$. An element of $S(\mathbb{C}^N)$ is called a quantum state in this paper. We also use a complex vector for representing a quantum state if the state is pure.

For an integer $1 \leq i \leq n$ and a square matrix

$$A = \sum_{x, y \in \{0, 1\}^n} a_{x, y} \cdot |x_1\rangle \langle y_1| \otimes \cdots \otimes |x_n\rangle \langle y_n|$$

with $a_{x, y} \in \mathbb{C}$, define the map $\text{Tr}_i : S(\mathbb{C}^{2^n}) \to S(\mathbb{C}^{2^{(n-1)}})$ as

$$\text{Tr}_i(A) := \sum_{x, y \in \{0, 1\}^n} a_{x, y} \cdot \text{Tr}(|x_i\rangle \langle y_i|) \cdot |x_1\rangle \langle y_1| \otimes \cdots \otimes |x_{i-1}\rangle \langle y_{i-1}| \otimes |x_{i+1}\rangle \langle y_{i+1}| \otimes \cdots \otimes |x_n\rangle \langle y_n|.$$ 

The map $\text{Tr}_i$ is called a partial trace.

Recall that in the classical coding theory, a single deletion error is defined as an operator that maps a sequence $x_1 x_2 \ldots x_i \ldots x_n$ to a short sequence $x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_n$ for some $i$.

**Definition 2.1 (Deletion Error $D_i$).** For an integer $1 \leq i \leq n$, we call $\text{Tr}_i$ a single deletion error $D_i$, i.e.,

$$D_i(\rho) := \text{Tr}_i(\rho),$$

where $\rho \in S(\mathbb{C}^{2^n})$ is a quantum state.

If a quantum state $\rho$ is corresponding to $n$-photons $p_1, p_2, \ldots, p_n$, the state $D_i(\rho)$ is corresponding to $n-1$-photons $p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$.

**Definition 2.2 (Deletion Error-Correcting Code).** We call a vector space $Q \subset \mathbb{C}^{2^n}$ an $[n, k]$ single deletion error-correcting code if

- there exists a complex linear bijection $\text{Enc} : \mathbb{C}^{2^k} \to Q$,
- there exists a map $\text{Dec} : S(\mathbb{C}^{2^{(n-1)}}) \to \mathbb{C}^{2^k}$ such that for any $|\phi\rangle \in \mathbb{C}^{2^k}$ and for any $1 \leq i \leq n$, $\text{Dec} \circ D_i \circ \text{Enc}(|\phi\rangle) = |\phi\rangle$,

where $\circ$ is the composite for operations. In other words, there exist an encoder $\text{Enc}$ and a decoder $\text{Dec}$ that correct any single deletion errors.

Comparing to erasure errors, deletion errors do not tell the position where the information is deleted. Hence to correct deletion errors is more difficult than to correct erasure errors.
2.2 Code Construction

Most famous classical error-correcting codes for single deletion errors are “non-linear” codes that are called VT codes \([18]\) discovered by Levenshtein \([14]\). One of the reasons why “non-linear” codes are preferable for deletion error-correction is unveiled in \([1]\). It is stated that a code rate of any single deletion error-correcting code cannot exceed \(1/2\) if a code is linear. This implies that if a CSS code is constructed from two classical linear deletion error-correcting codes then the code rate of the CSS code is 0.

**Definition 2.3** (\(E_n_4\) and \(Q_4\)). Let us define a linear map \(E_n_4 : C^2 \rightarrow C^{2 \otimes 4}\). For a quantum state \(|\phi\rangle = \alpha|0\rangle + \beta|1\rangle \in C^2\), \(E_n_4\) maps the state \(|\phi\rangle\) to the following state \(|\Phi\rangle\),

\[
|\Phi\rangle := \alpha \frac{\sqrt{2}}{\sqrt{2}} (|0000\rangle + |1111\rangle) + \beta \frac{\sqrt{6}}{\sqrt{6}} (|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle).
\]

Set \(Q_4\) as the image of \(E_n_4\) for quantum messages, i.e.,

\[
Q_4 := \{E_n_4(|\phi\rangle) \mid |\phi\rangle \in C^2, |\phi\rangle \langle \phi| \in S(C^2)\}.
\]

**Remark 2.4.** The QEC code with 4 qubits \(Q_4\) is closely related to our code \(Q_4\). We can say that more symmetry is introduced to our code for correcting error at the unknown position.

In fact, we will show that \(Q_4\) is a \([4, 1]\) single deletion error-correcting code with the encoder \(E_n_4\).

We can characterize this encoding in the following manner. Let \(A\) be the set of four bit sequences with Hamming weight 0 or 4 and \(B\) the set of four bit sequences with Hamming weight 2, i.e.,

\[
A = \{0000, 1111\},
B = \{0011, 0101, 0110, 1001, 1010, 1100\}.
\]

Then the codeword \(|\Phi\rangle\) is

\[
\frac{\alpha}{\sqrt{2}} \sum_{a \in A} |a\rangle + \frac{\beta}{\sqrt{6}} \sum_{b \in B} |b\rangle.
\]

By this encoding, \(|0\rangle\) and \(|1\rangle\) are encoded to quantum states which are superpositions of two and six orthonormal states. Hence this encoding is neither an instance of CSS codes nor stabilizer codes.
Definition 2.5 \((\text{De}_4)\). Let us define a map \(\text{De}_4\) from \(S(\mathbb{C}^{2\otimes 3})\) to \(\mathbb{C}^2\). The map \(\text{De}_4\) consists of the following steps:

(Step 1) Perform the measurement \(\{P_0, P_1\}\) to a quantum state in \(S(\mathbb{C}^{2\otimes 3})\) and obtain the outcome \(i \in \{0, 1\}\), where \(P_0\) (resp. \(P_1\)) is the projection from \(\mathbb{C}^{2\otimes 3}\) to the linear space \(V_0\) (resp. \(V_1\)) spanned by \(|000\rangle, |011\rangle, |101\rangle\) and \(|110\rangle\) (resp. \(|111\rangle, |100\rangle, |010\rangle\) and \(|001\rangle\)). Hence, the outcome is \(i\) if the quantum state in \(S(\mathbb{C}^{2\otimes 3})\) is changed into a state in \(S(V_i)\).

(Step 2) Only if the outcome is \(1\), act the quantum operation \(F: V_1 \rightarrow V_0\) to \(S\), where \(F\) is defined as

\[
F(a |111\rangle + b |100\rangle + c |010\rangle + d |001\rangle) := a |000\rangle + b |011\rangle + c |101\rangle + d |110\rangle.
\]

(Step 3) Act the quantum operation \(G: V_0 \rightarrow \mathbb{C}^{2\otimes 3}\) to the state, where \(G\) is defined as

\[
G(a |000\rangle + b |100\rangle + c |010\rangle + d |001\rangle) := a |000\rangle + b |100\rangle + c |010\rangle + d |101\rangle,
\]

and where

\[
|100\rangle = \frac{1}{\sqrt{3}} (|011\rangle + |101\rangle + |110\rangle),
\]

\[
|010\rangle = \frac{1}{\sqrt{3}} (|011\rangle + \omega |101\rangle + \omega^2 |110\rangle),
\]

\[
|001\rangle = \frac{1}{\sqrt{3}} (|011\rangle + \omega^2 |101\rangle + \omega |110\rangle).
\]

(Step 4) Finally, delete the 3rd and the 2nd qubits by the partial traces \(\text{Tr}_3\) and \(\text{Tr}_2\).

2.3 Proof for Error-Correcting Property

In this subsection, we prove our code \(Q_4\) corrects any single deletion errors \(D_1, D_2, D_3,\) and \(D_4\).

Let us set

\[
A_0 := \{000\},
\]

\[
A_1 := \{111\},
\]

\[
B_0 := \{011, 101, 110\},
\]

\[
B_1 := \{001, 010, 100\}.
\]
Lemma 2.6. Let $|\Phi\rangle := \alpha \sum_{a \in A} |a\rangle + \beta \sum_{b \in B} |b\rangle$ and $\rho := |\Phi\rangle \langle \Phi|$. For any $1 \leq i \leq 4$,

$$D_i(\rho) = \frac{1}{2}|\Phi_0\rangle \langle \Phi_0| + \frac{1}{2}|\Phi_1\rangle \langle \Phi_1|,$$

where $|\Phi_0\rangle = \alpha \sum_{a \in A_0} |a\rangle + \frac{\beta}{\sqrt{3}} \sum_{b \in B_0} |b\rangle$ and $|\Phi_1\rangle = \alpha \sum_{a \in A_1} |a\rangle + \frac{\beta}{\sqrt{3}} \sum_{b \in B_1} |b\rangle$.

Proof. At first, we show a case where $i = 1$.

By using $A_0, A_1, B_0$ and $B_1$, we can rewrite $|\Phi\rangle$ as

$$|\Phi\rangle = |0\rangle \left( \frac{\alpha}{\sqrt{2}} \sum_{a \in A_0} |a\rangle + \frac{\beta}{\sqrt{6}} \sum_{b \in B_0} |b\rangle \right) + |1\rangle \left( \frac{\alpha}{\sqrt{2}} \sum_{a \in A_1} |a\rangle + \frac{\beta}{\sqrt{6}} \sum_{b \in B_1} |b\rangle \right).$$

Hence,

$$\rho = |\Phi \rangle \langle \Phi|$$

$$= |0\rangle \langle 0| \otimes$$

$$\left( \frac{\alpha}{\sqrt{2}} \sum_{a,a' \in A_0} |a\rangle \langle a'| + \frac{\beta}{\sqrt{6}} \sum_{a \in A_0, b' \in B_0} |a\rangle \langle b'| \right)$$

$$+ \frac{\beta}{\sqrt{2}} \sum_{b \in B_0, a' \in A_0} |b\rangle \langle a'| + \frac{\beta}{\sqrt{6}} \sum_{b \in B_0, b' \in B_0} |b\rangle \langle b'| \right)$$

$$+ |1\rangle \langle 1| \otimes$$

$$\left( \frac{\alpha}{\sqrt{2}} \sum_{a,a' \in A_1} |a\rangle \langle a'| + \frac{\beta}{\sqrt{6}} \sum_{a \in A_1, b' \in B_1} |a\rangle \langle b'| \right)$$

$$+ \frac{\beta}{\sqrt{2}} \sum_{b \in B_1, a' \in A_1} |b\rangle \langle a'| + \frac{\beta}{\sqrt{6}} \sum_{b \in B_1, b' \in B_1} |b\rangle \langle b'| \right)$$

$$+ |0\rangle \langle 1| \otimes \rho' + |1\rangle \langle 0| \otimes \rho'',$$

for some matrices $\rho'$ and $\rho''$.

Note that $\text{Tr}(|0\rangle \langle 0|) = \text{Tr}(|1\rangle \langle 1|) = 1$ and $\text{Tr}(|0\rangle \langle 1|) = \text{Tr}(|1\rangle \langle 0|) = 0$. 
Proof. Since with the encoder $E_n$ the code holds. By the definition of the partial trace, 
\[
\text{Tr}_1(\rho) = \left( \frac{\alpha}{\sqrt{2}} \sqrt{2} \right) \sum_{a,a' \in A_0} |a\rangle \langle a'| + \frac{\alpha}{\sqrt{2}} \sqrt{6} \sum_{a \in A_0, b' \in B_0} |a\rangle \langle b'|
\]
\[
+ \frac{\beta}{\sqrt{6}} \sum_{b \in B_0, a' \in A_0} |b\rangle \langle a'| + \frac{\beta}{\sqrt{6}} \sum_{b, b' \in B_0} |b\rangle \langle b'|
\]
\[
+ \left( \frac{\alpha}{\sqrt{2}} \sqrt{2} \right) \sum_{a,a' \in A_1} |a\rangle \langle a'| + \frac{\alpha}{\sqrt{2}} \sqrt{6} \sum_{a \in A_1, b' \in B_1} |a\rangle \langle b'|
\]
\[
+ \frac{\beta}{\sqrt{6}} \sum_{b \in B_1, a' \in A_1} |b\rangle \langle a'| + \frac{\beta}{\sqrt{6}} \sum_{b, b' \in B_1} |b\rangle \langle b'|
\]
\[
= \frac{1}{2} \left( \alpha \sum_{a,a' \in A_0} |a\rangle \langle a'| + \frac{\alpha}{\sqrt{3}} \sum_{a \in A_0, b' \in B_0} |a\rangle \langle b'|
\]
\[
+ \frac{\beta}{\sqrt{3}} \sum_{b \in B_0, a' \in A_0} |b\rangle \langle a'| + \frac{\beta}{\sqrt{3}} \sum_{b, b' \in B_0} |b\rangle \langle b'|
\]
\[
+ \frac{\alpha}{\sqrt{2}} \sqrt{2} \sum_{a,a' \in A_1} |a\rangle \langle a'| + \frac{\alpha}{\sqrt{2}} \sqrt{6} \sum_{a \in A_1, b' \in B_1} |a\rangle \langle b'|
\]
\[
+ \frac{\beta}{\sqrt{6}} \sum_{b \in B_1, a' \in A_1} |b\rangle \langle a'| + \frac{\beta}{\sqrt{6}} \sum_{b, b' \in B_1} |b\rangle \langle b'|
\]
\[
= \frac{1}{2} |\Phi_0\rangle \langle \Phi_0| + \frac{1}{2} |\Phi_1\rangle \langle \Phi_1|.
\]

By the symmetry of $|\Phi\rangle$, 
\[
D_1(\rho) = D_2(\rho) = D_3(\rho) = D_4(\rho)
\]
holds. 

**Theorem 2.7.** The code $Q_4$ is a $[4,1]$ single deletion error-correcting code with the encoder $E_{n_1}$ and the decoder $D_{e_4}$.

**Proof.** Since $|\Phi_0\rangle \in V_0$ and $|\Phi_1\rangle \in V_1$, by the Step 1, The obtained quantum state $s$ is $|\Phi_0\rangle \langle \Phi_0|$ or $|\Phi_1\rangle \langle \Phi_1|$. 

At the Step 2, if the outcome is 1, the state is changed to $|\Phi_0\rangle \langle \Phi_0|$ by the operation $F$. Hence after the Step 2, obtained quantum state is $|\Phi_0\rangle \langle \Phi_0|$. 

Since $G(\Phi_0) = \alpha |00\rangle + \beta |10\rangle = |\phi\rangle \otimes |00\rangle$, by the Step 3, the quantum state is changed to $|\phi\rangle \langle \phi\rangle \otimes |00\rangle \langle 00|$. 

Hence at the Step 4, the obtained state is $|\phi\rangle \langle \phi|$, i.e. the original quantum state.
3 Encoding Circuit and Decoding Circuit

3.1 Encoding Circuit

Figure 1: Encoder

Figure 1 shows an example of an encoder for our four qubits code $Q_4$. The gate $H$ is the Hadamard matrix, i.e.,

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

and the gate $X$ is the bit-flip matrix, i.e.,

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

The black dot is the control gate for the connected gate. For example, the final gate of Figure 1 is the C-NOT gate with the 3rd qubit as the control qubit and the 1st qubit as the target bit.

The gates $U$ and $V$ are defined as

$$U := \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, V := \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

respectively.

The input of the circuit is a single qubit $|\phi\rangle \in \mathbb{C}^2$. The input position is settled to the left-most position. For the other three positions, initialized three qubits $|000\rangle$ are input.
By the first two depth operations, i.e., the controlled $U$ gate, the controlled $V$ gate and the Hadamard gate, a quantum state $|0000\rangle$ is changed to
\[ \frac{1}{\sqrt{2}} (|0000\rangle + |0010\rangle). \]
and a quantum state $|1000\rangle$ is changed to
\[ \frac{1}{\sqrt{6}} (|1000\rangle + |1010\rangle + |0100\rangle + |0110\rangle + |1100\rangle + |1110\rangle). \]

By the remaining four controlled $X$ gates, i.e. C-NOTs, a quantum state $|x_1 x_2 x_3 0\rangle$ is changed to $|(x_1 + x_3)(x_2 + x_3)(x_3)(x_1 + x_2 + x_3)\rangle$, e.g.,
\[
\begin{align*}
|0000\rangle &\rightarrow |0000\rangle, \\
|0010\rangle &\rightarrow |1111\rangle, \\
|1000\rangle &\rightarrow |1001\rangle, \\
|1010\rangle &\rightarrow |0110\rangle, \\
|0100\rangle &\rightarrow |0101\rangle, \\
|0110\rangle &\rightarrow |1010\rangle, \\
|1100\rangle &\rightarrow |1100\rangle, \\
|1110\rangle &\rightarrow |0011\rangle.
\end{align*}
\]

Hence $(\alpha |0\rangle + \beta |1\rangle) \otimes |000\rangle$ is encoded to
\[ \alpha \left( \frac{1}{\sqrt{2}} \sum_{a \in A} |a\rangle \right) + \beta \left( \frac{1}{\sqrt{6}} \sum_{b \in B} |b\rangle \right). \]

### 3.2 Decoding Circuits

Figure 2 shows a quantum circuit that changes a pure state $(|011\rangle + |101\rangle + |110\rangle) / \sqrt{3}$ to a pure state $|100\rangle$ and keeps a pure state $|000\rangle$. Hence the circuit changes a pure state $|\Psi_0\rangle = \alpha \sum_{a \in A_0} |a\rangle + \beta \sum_{b \in B_0} |b\rangle$ to a pure state $|\phi\rangle = \alpha |0\rangle + \beta |1\rangle$.

Remember that the step 1 of the decoding algorithm changes a single deleted quantum state $D_1(\rho)$ to $|\Psi_0\rangle$. Hence the circuit is available as a decoding circuit after step 1.

Let us provide another quantum circuit that is depicted as Figure 3. The quantum circuit consists of two parts. The first part has six C-NOT and the last part is the same as the circuit defined by Figure 2. The first part changes a quantum pure state $|x_1 x_2 x_3\rangle$ to $|x_1 x_2 x_3 0\rangle$ if the Hamming weight of $x_1 x_2 x_3$
is even and to $|(x_1 + 1)(x_2 + 1)(x_3 + 1)|$ if the weight is odd. This implies that the first part changes a single deleted quantum state $D_i(\rho)$ to

$$|\Psi_0\rangle \langle \Psi_0| \otimes \left( \begin{array}{cc} 1/2 & 0 \\ 0 & 1/2 \end{array} \right).$$

Since the last part is the same as the circuit corresponding to Figure 2, the state is changed to

$$|\phi\rangle \langle \phi| \otimes |0\rangle \otimes |0\rangle \langle 0| \otimes \left( \begin{array}{cc} 1/2 & 0 \\ 0 & 1/2 \end{array} \right).$$

In other words, the state at the first position of output is a pure state $|\phi\rangle$.

4 Generalization

4.1 Number of Information Qubits

We generalize our $[4, 1]$ single deletion error-correcting code $Q_4$ to a $[2^{k+2} - 4, k]$ single quantum deletion error-correcting code for any positive integer $k$.

Let $l$ be a positive integer. For $0 \leq i \leq l - 1$, let us define

$$\mathcal{A}_i := \{x \in \{0, 1\}^{4(l-1)} | \text{wt}(x) = 2i \text{ or } 4(l - 1) - 2i \}$$

and

$$\text{En}_{4(l-1)} (|\phi_i\rangle) := \sum_{0 \leq i \leq l-1} c_i \left( \sum_{x \in \mathcal{A}_i} |x\rangle \right),$$

where $|\phi_i\rangle := \sum_{0 \leq i \leq l-1} c_i |i\rangle$ is a quantum pure state of level $l$, i.e. $|\phi_i\rangle \in \mathbb{C}^l$, and $|0\rangle, |1\rangle, \ldots, |l-1\rangle$ is the standard orthonormal basis of $\mathbb{C}^l$. 
We claim that the image of $\text{En}_{4(l-1)}$ is a single quantum deletion error-correcting code for any $l \geq 2$. A decoding algorithm $\text{De}_{4(l-1)}$ can be defined similar to $\text{De}_4$. Due to the limit of page numbers, we only explain how to define its measurement $\{\mathcal{P}_0, \mathcal{P}_1\}$.

$\mathcal{P}_0$ (resp. $\mathcal{P}_1$) is the projection from $\mathbb{C}^{2 \otimes 4(l-1)}$ to the linear space $\mathcal{V}_0$ (resp. $\mathcal{V}_1$) spanned by

$$\{|x\rangle | x \in \{0, 1\}^{4(l-1)-1}, \text{ the Hamming weight of } x \text{ is even (resp. odd) } \}.$$ 

The outcome is $i \in \{0, 1\}$ if the quantum state in $S(\mathbb{C}^{2 \otimes 4(l-1)})$ is changed into a state in $S(\mathcal{V}_i)$.

For the case $l = 2^k$, we obtain a $[2^{k+2} - 4, k]$ single quantum deletion error-correcting code.

### 4.2 Permutation of The Received Qubits

Our code word has symmetry for position permutations. In other words, any permutation for four qubits does not change the codeword. Additionally, any received word after any single deletion has also symmetry for position permutation.
This means that even if we permute the three input to the quantum circuits corresponding to Figure 2, we can obtain the original quantum information \(|\phi\rangle\) at the left most position of output.

5 There is no Quantum Deletion Codes with less than 4 Qubits

Lemma 5.1. There is no single deletion error-correcting code of length 2.

Proof. Let us assume that there exist a code of length 2. Then we have an encoder \(E_{n_2} : \mathbb{C}^2 \rightarrow \mathbb{C}^{2 \odot 2}\) and a decoder \(D_{e_2} : S(\mathbb{C}^2) \rightarrow \mathbb{C}^2\). Let \(\rho \in \mathbb{C}^2\) and encode it to \(E_{n_2}(\rho)\). Assume that the encoded state is corresponding to two photons \(p_1\) and \(p_2\). The quantum states of these photons are \(D_1 \circ E_{n_2}(\rho)\) and \(D_2 \circ E_{n_2}(\rho)\) respectively. Perform the decoder \(D_{e_2}\) to the photons \(p_1\) and \(p_2\) simultaneously. Then the states for \(p_1\) and \(p_2\) are changed to \(D_2 \circ D_{e_2} \circ E_{n_2}(\rho) = \rho\) and \(D_2 \circ D_{e_2} \circ E_{n_2}(\rho) = \rho\) respectively. It contradicts to non-cloning theorem.

The following is the remarkable result by Grassl et.al.

Fact 5.2 (Theorem 5 [9]). There is no quantum error-correcting code of length three that can correct one erasure and encodes one qubit.

A similar result to deletion error-correcting codes holds.

Lemma 5.3. There is no single deletion error-correcting code of length 3.

Proof. Assume that there exists a single deletion error-correcting code of length 3. Let us explain that this code can be regarded as a quantum erasure error-correcting code of length 3. If we have a received word with a single quantum erasure error, we can find the position where the error occurs. Then delete the state at the position of the received word. In other words, we can convert an erasure error to a deletion error. By the assumption, we can correct the error by deletion error-correcting decoder. This contradicts to Fact 5.2.

Theorem 5.4. The shortest length of single quantum deletion error-correcting codes is 4.
6 Conclusion

This paper provided a single quantum deletion error-correcting code of optimal length, i.e. 4. The construction of this code is far from known quantum error-correcting codes, e.g., CSS codes [4,17], stabilizer codes [8], surface codes [7], and etc.

Acknowledgment

This paper is partially supported by KAKENHI 18H01435. The authors thank to Professor Mikio Nakahara, Professor Akinori Kawachi, and Professor Akihisa Tomita for valuable discussion.

References

[1] Khaled AS Abdel-Ghaffar, Hendrik C Ferreira, and Ling Cheng. Correcting deletions using linear and cyclic codes. IEEE Transactions on Information Theory, 56(10):5223–5234, 2010.

[2] Jacob Biamonte, Peter Wittek, Nicola Pancotti, Patrick Rebentrost, Nathan Wiebe, and Seth Lloyd. Quantum machine learning. Nature, 549(7671):195–202, 2017.

[3] Tilo Buschmann and Leonid V Bystrykh. Levenshtein error-correcting barcodes for multiplexed dna sequencing. BMC bioinformatics, 14(1):272, 2013.

[4] A Robert Calderbank and Peter W Shor. Good quantum error-correcting codes exist. Phys. Rev. A, 54(2):1098–1105, 1996. doi:10.1103/PhysRevA.54.1098

[5] Yeow Meng Chee, Han Mao Kiah, Alexander Vardy, Van Khu Vu, and Eitan Yaakobi. Coding for racetrack memories. arXiv preprint arXiv:1701.06874, 2017.

[6] Richard Cleve, Daniel Gottesman, and Hoi-Kwong Lo. How to share a quantum secret. Physical Review Letters, 83(3):648, 1999.

[7] Austin G Fowler, Matteo Mariantoni, John M Martinis, and Andrew N Cleland. Surface codes: Towards practical large-scale quantum computation. Physical Review A, 86(3):032324, 2012.
[8] Daniel Gottesman. Stabilizer codes and quantum error correction. *arXiv preprint quant-ph/9705052*, 1997.

[9] Markus Grassl, Th Beth, and Thomas Pellizzari. Codes for the quantum erasure channel. *Physical Review A*, 56(1):33, 1997.

[10] Albertus Stephanus Jacobus Helberg. *Coding for the correction of synchronization errors*. PhD thesis, University of Johannesburg, 1993.

[11] Mark Hillery, Vladimír Bužek, and André Berthiaume. Quantum secret sharing. *Physical Review A*, 59(3):1829, 1999.

[12] Lane P Hughston, Richard Jozsa, and William K Wootters. A complete classification of quantum ensembles having a given density matrix. *Physics Letters A*, 183(1):14–18, 1993.

[13] Janet Leahy, Dave Touchette, and Penghui Yao. Quantum insertion-deletion channels. *arXiv preprint arXiv:1901.00984*, 2019.

[14] V. I. Levenshtein. Binary codes capable of correcting deletions, insertions, and reversals. *Soviet Physics Dokl.*, 10(8):707–710, 1966.

[15] Ayumu Nakayama and Manabu Hagiwara. The first quantum error-correcting code for single deletion errors. *IEICE Communications Express*, (2019XBL0154), 2020. doi: 10.1587/comex.2019XBL0154.

[16] Frederic Sala, Ryan Gabrys, Clayton Schoeny, and Lara Dolecek. Exact reconstruction from insertions in synchronization codes. *IEEE Transactions on Information Theory*, 63(4):2428–2445, 2017.

[17] Andrew Steane. Multiple-particle interference and quantum error correction. *Proc. R. Soc. Lond. A*, 452(1954):2551–2577, 1996. doi: 10.1098/rspa.1996.0136.

[18] GM Tenengolts and RR Varshamov. Code correcting single asymmetric errors. *Avtomat. i Telemekh.*, 26:288–292, 1965.