A FUZZY INVENTORY MODEL FOR WEIBULL DETERIORATING ITEMS UNDER COMPLETELY BACKLOGGED SHORTAGES

Deepak Kumar Nayak¹, Sudhansu Sekhar Routray², Susanta Kumar Paikray¹∗ and Hemen Dutta³

¹ Department of Mathematics, Veer Surendra Sai University of Technology Burla 768018, Odisha, India
² Department of Mathematics, College of Engineering and Technology Bhubaneswar 751029, Odisha, India
³ Department of Mathematics, Gauhati Guwahati 781014, India

Abstract. In this paper, a fuzzy stock replenishment policy implemented for inventory items that follows linear demand and Weibull deterioration under completely backlogged shortages. Moreover, to minimize the aggregate expense per unit time, the fuzzy optimal solution is obtained using general mathematical techniques by considering hexagonal fuzzy numbers and graded mean preference integration strategy. Finally, the complete exposition of the model is provided by numerical examples and sensitivity behavior of the associated parameters.

1. Introduction. Inventory models for deteriorating items have been an all around well-studied problem, and various ideal and heuristic methodologies were developed for demonstrating and taking care of many varieties of such issues. Among all the researchers, Whitin [32] was the first to discuss the inventory problem for deteriorating items. In later years, Ghere and Schrader [9] proposed a model for exponentially decay items, Covert and Philip [5] figured a model with a variable rate of deterioration with two parameters Weibull distributions which were further stretched out by Shah [28]. Barik et al. [2] developed an inventory model for deteriorating items follows a time varying request. In subsequent years, many inventory problems for deteriorating items were considered by different researchers.

In many inventory systems, shortages occur commonly due to different reasons, such as an unexpected increase in demand, under-ordering due to lack of investment capital, deterioration of items, limited items delivery by supplier per cycle and so on. Many authors included shortages in their proposed models. In this context, we may refer to works of Yao and Lee [34], Jain and Kumar [12], Tomba and Brojendro [29], Yang [33], Raju et al. [24], Mishra et al. [18], Agarwal et al. [1], Sanni and Chukwu [26], Jaggi et al. [11], Dinagar and Kannan [6], Barik et al. [3], Viji and Karthikeyan [31] and Chakraborty et al. [4] and many others.

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∗ Corresponding author: Susanta Kumar Paikray.
Moreover, many researchers worked on inventory problems for items having deterioration and/or shortages by incorporating one or more constraint like two-warehouses, inflation, trade credit financing and so on. Sarbjit and Raj [27] studied the effect of trade credit. Tripathy et al. [30] developed EPQ model by including the variable holding cost. Mishra et al. [20] and [18] incorporated inflation in their models. Agarwal et al. [1] considered two-warehouse system. Jaggi et al. [11] considered a stock model involving two-warehouses. Pal et al. [22] incorporated inflation in their model. Chakraborty et al. [4] included inflation and trade credit in a two-warehouse model.

Furthermore, there are different sorts of instabilities and imprecision inherent in the genuine stock issue which are traditionally displayed utilizing the methodologies from the likelihood hypothesis. However, there are uncertainties that cannot be suitably treated by regular probabilistic models. In this manner, it turns out to be more advantageous for arrangement of such issues with the fuzzy set hypothesis rather than probability theory.

After the introduction of fuzzy set concepts by Zadeh [35], many authors from various dimensions of research areas shown their interest to study their problems by incorporating fuzziness. Zimmerman [36] published a paper in linear programming by considering fuzzy techniques. Kacprzyk and Stanieski [13] proposed a model on long term strategy settling on through fuzzy-decision making model. Park [23] proposed a model on fuzzy set theoretical interpretation of monetary request amount stock issue. In recent years, many authors published their papers which include fuzzy concepts and techniques. Najafi et al. [21] proposed an efficient method to solve fully fuzzy linear programming (FFLP) problem with unrestricted fuzzy coefficients and fuzzy variables based on crisp nonlinear programming technique. Faddel et al. [8] used fuzzy optimization technique in his Electric Vehicle Parking Lots problem. Lamata et al. [15] explored the decision making problem to obtain best possible decision in the context of fuzzy. Mezei and Nikou [17] applied fuzzy optimization technique to improve mobile health and wellness recommendation systems. Djordjevic et al. [7] developed fuzzy linear programming model for aggregated production planning (APP) in the automotive industry.

Lee and Yao [16] first used the fuzziness in their inventory problem, then Yao and Lee [34] proposed a fuzzy inventory model without backordering, where they considered the fuzzy request amount as the trapezoidal fuzzy number. Kao and Hsu [14] proposed a single period Stock model with fuzzy interest. Dinagar [6] proposed a fuzzy stock model with reasonable shortage. Moreover, for different Fuzzy inventory models one may refer to the recent works [19], [10] and [25].

Motivated essentially by the above mentioned works, here we propose an inventory model for Weibull deteriorating items having linear increasing demand with shortages at the end of the cycle under fuzzy environment. Our objective is to obtain a fuzzy optimal solution that minimizes the total cost per unit time. The parameters associated with this model are the hexagonal fuzzy numbers, and the Graded Mean Integration Representation (GMIR) method is used for defuzzification.

2. Definitions and preliminaries.

**Definition 2.1.** A fuzzy set $\tilde{A}$ in the universe of discourse $X$ is expressed as $\tilde{A} = \{(x, \mu_{\tilde{A}}) : x \in X\}$, where the mapping $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ is a called the membership value of $x \in X$. 
Definition 2.2. A fuzzy set $[a_\alpha, b_\alpha]$, where $0 \leq \alpha \leq 1$ and $a < b$ defined on $\mathbb{R}$ is known as the level of a fuzzy interval, if its membership function is

$$
\mu_{[a_\alpha, b_\alpha]} = \begin{cases} 
\alpha & (a \leq x \leq b); \\
0 & \text{otherwise.}
\end{cases}
$$

Definition 2.3. A hexagonal fuzzy number $\tilde{A} = (a, b, c, d, e, f)$ is represented with the membership function $\mu_{\tilde{A}}$ as:

$$
\mu_{\tilde{A}} = \begin{cases} 
L_1(x) = \frac{1}{2} \left( \frac{x-a}{b-a} \right), & a \leq x \leq b \\
L_2(x) = \frac{1}{2} + \frac{1}{2} \left( \frac{x-b}{b-a} \right), & b \leq x \leq c \\
1, & c \leq x \leq d \\
R_1(x) = 1 - \frac{1}{2} \left( \frac{x-c}{b-c} \right), & d \leq x \leq e \\
R_2(x) = \frac{1}{2} \left( \frac{x-e}{f-c} \right), & e \leq x \leq f \\
0 & \text{otherwise.}
\end{cases}
$$

Definition 2.4. The $\alpha$-cut of $\tilde{A} = (a, b, c, d, e, f)$, $0 \leq \alpha \leq 1$ is $A(\alpha) = [A_L(\alpha), A_R(\alpha)]$, where

$$
\begin{align*}
A_{L_1}(\alpha) &= a + (b-a)\alpha = L_1^{-1}(\alpha), \\
A_{L_2}(\alpha) &= b + (c-b)\alpha = L_2^{-1}(\alpha), \\
A_{R_1}(\alpha) &= e + (e-d)\alpha = R_1^{-1}(\alpha), \\
A_{R_2}(\alpha) &= f + (f-c)\alpha = R_2^{-1}(\alpha),
\end{align*}
$$

and,

$$
\begin{align*}
L^{-1}(\alpha) &= \frac{L_1^{-1}(\alpha) + L_2^{-1}(\alpha)}{2} = \frac{a+b+(c-a)\alpha}{2}; \\
R^{-1}(\alpha) &= \frac{R_1^{-1}(\alpha) + R_2^{-1}(\alpha)}{2} = \frac{e+f+(d-f)\alpha}{2}.
\end{align*}
$$

Definition 2.5. If $\tilde{A} = (a, b, c, d, e, f)$ is a hexagonal fuzzy number, then the graded mean integration representation (GMIR) of $\tilde{A}$ is defined as:

$$
P(\tilde{A}) = \frac{\int_{h=0}^{W_A} \tilde{A} \left( L^{-1}(h) + R^{-1}(h) \right) dh}{\int_{h=0}^{W_A} dh}, 0 \leq h \leq W_A \\
\text{such that } P(\tilde{A}) = \frac{1}{12} \left( a + 3b + 2c + 2d + 3e + f \right).
$$

Definition 2.6. Suppose $\tilde{X} = (x_1, x_2, x_3, x_4, x_5, x_6)$ and $\tilde{Y} = (y_1, y_2, y_3, y_4, y_5, y_6)$ are two hexagonal fuzzy numbers, and $x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6$ are all real numbers, then the arithmetical operations under function principle are as follows:

(i) $\tilde{X} \oplus \tilde{Y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5, x_6 + y_6)$

(ii) $\tilde{X} \odot \tilde{Y} = (x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_6y_6)$

(iii) $-\tilde{Y} = (-y_1, -y_2, -y_3, -y_4, -y_5, -y_6)$

(iv) $\frac{1}{\tilde{Y}} = \tilde{Y}^{-1} = \left( \frac{1}{y_1}, \frac{1}{y_2}, \frac{1}{y_3}, \frac{1}{y_4}, \frac{1}{y_5}, \frac{1}{y_6}\right)$

(v) $\tilde{X}/\tilde{Y} = (\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \frac{x_4}{y_4}, \frac{x_5}{y_5}, \frac{x_6}{y_6})$

(vi) Let $\alpha \in R$, then $\alpha \odot \tilde{X} = \left\{ \alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5, \alpha x_6 \right\}, \alpha \geq 0$, and $\left\{ \alpha x_6, \alpha x_5, \alpha x_4, \alpha x_3, \alpha x_2, \alpha x_1 \right\}, \alpha < 0$.

3. Assumptions and notations. The proposed inventory model has been developed on the basis of the following assumptions and notations:
3.1. **Assumptions.** Suppositions are made for the proposed model are as follows:

(i) single stock will be utilized

(ii) lead time is zero

(iii) the demand takes after the linear request pattern

(iv) shortages are permitted and are totally backlogged

(v) replenishment rate is vast, however; size is limited

(vi) the time horizon is limited

(vii) there is no repair of crumbled things occurring during the cycle

(viii) the second and higher powers are neglected in this examination of the model.

3.2. **Notations.** We have considered different notations for the proposed model as follows:

\[ Q(t) \]: On hand inventory level at any time \( t, t \geq 0 \)

\[ D(t) \]: Demand rate (linear; \( \alpha + \beta t \)) at time \( t \)

\( \vartheta \): Two-parameter weibull distribution deterioration rate (\( \eta \gamma t^{\gamma-1} \)) per unit time, where \( 0 < \eta < 1 \) is the scale parameter, and \( \gamma > 0 \) is the shape parameter

\( W1 \): Total amount of replenishment in the beginning of each cycle

\( W \): Inventory at time \( t = 0 \)

\( T \): Duration of a cycle

\( S \): No of shortage units during the cycle

\( \lambda \): The holding cost per unit item

\( \mu \): The deterioration cost per unit item

\( \xi \): The shortage cost per unit item

\( Z(t) \): The total average cost of the system.

4. **Formulation of mathematical model.** The aim of the model is to decide the ideal request amount with a specific end goal to keep the aggregate pertinent expense as low as could reasonably be expected. The optimality is resolved for deficiency of items. Taking \( W1 \) as the aggregate sum of renewal at the start of every cycle, and in the wake of satisfying back orders, let \( W \) be the level of stock at the beginning. In the period \( (0, \tau) \), the stock level steadily diminishes because of increase in demand and deterioration rate.

At \( \tau \), the level of stock achieves zero and after that, the deficiencies are permitted to happen amid the interval \( [\tau, T] \), which are completely backlogged. Just the accumulating things are supplanted in the subsequent cycles. The conduct of stock amid the period \( (0, T) \) is portrayed in the accompanying stock time diagram. Here we have taken the aggregate span as settled steady. The target here is to decide the ideal request amount with a specific end goal to keep the aggregate important expense as low as could reasonably be expected.
If \( Q(t) \) be the stock on-hand at time \( t \geq 0 \), then at time \( t + \Delta t \), the inventory on-hand in the interval \([0, \tau]\) will be \( Q(t + \Delta t) = Q(t) - \vartheta(t)Q(t)\Delta t - \mathbb{D}(t)\Delta t \).

Dividing both sides by \( \Delta t \) and thereafter taking limit as \( \Delta t \to 0 \), we obtain

\[
\frac{dQ(t)}{dt} + \eta \gamma t^{\gamma-1}Q(t) = -(\alpha + \beta t) \quad (0 \leq t \leq \tau). \tag{1}
\]

Next, for the subsequent interval \([\tau, T]\), where the shortages are allowed, we find

\[
Q(t + \Delta t) = Q(t) - \mathbb{D}(t)\Delta t.
\]

Dividing both sides by \( \Delta t \) and thereafter taking limit as \( \Delta t \to 0 \), we obtain

\[
\frac{dQ(t)}{dt} = -(\alpha + \beta t) \quad (\tau \leq t \leq T). \tag{2}
\]

The boundary conditions are \( Q(0) = W \) and \( Q(\tau) = 0 \).

Solving equation (1) under boundary condition \( Q(\tau) = 0 \), we get

\[
Q(t) = e^{-\eta t^\gamma} \left[ a(\tau - t) + \frac{a \eta}{\gamma + 1} (\tau^{\gamma+1} - t^{\gamma+1}) + \frac{\beta \eta}{\gamma + 2} (\tau^{\gamma+2} - t^{\gamma+2}) \right] \quad (0 \leq t \leq \tau). \tag{3}
\]

Solving equation (2) with boundary condition \( Q(\tau) = 0 \), we obtain

\[
Q(t) = a(\tau - t) + \frac{\beta}{2} (\tau^2 - t^2) \quad (\tau \leq t \leq T). \tag{4}
\]

Substituting \( Q(0) = W \) in equation (3), we find

\[
W = a \tau + \frac{a \eta}{\gamma + 1} \tau^{\gamma+1} + \frac{\beta}{2} \tau^2 + \frac{\beta \eta}{\gamma + 2} \tau^{\gamma+2}. \tag{5}
\]
The total deteriorated units in \(0 \leq t \leq \tau\) is given by,

\[
\int_0^\tau \eta t^{\gamma-1}Q(t) dt = \eta \gamma \left[ \frac{W}{\gamma} \tau^\gamma - \frac{\eta W}{2\gamma} \tau^{2\gamma} - \frac{\alpha}{\gamma + 1} \tau^{\gamma+1} - \frac{\beta}{2(\gamma + 2)} \tau^{\gamma+2} \right.
\]
\[
+ \frac{\alpha \eta \gamma}{(2\gamma + 1)(\gamma + 1)} \tau^{3\gamma+1} + \frac{\beta \eta \gamma}{(\gamma + 1)(3\gamma + 1)} \tau^{3\gamma+2} \left. \right] .
\]

The following elements are associated with the total cost function:

(i) holding cost per cycle

\[
\lambda \int_0^\tau Q(t) dt = \lambda \left[ W\tau - \frac{\alpha \tau^2}{2} - \frac{\beta \tau^3}{3} - \frac{\eta W}{\gamma + 1} \frac{\tau^{\gamma+1}}{\gamma + 1} + \frac{\alpha \eta \gamma}{(\gamma + 1)(\gamma + 2)} \tau^{\gamma+2} \right.
\]
\[
+ \frac{\eta \beta \gamma}{2(\gamma + 2)(\gamma + 3)} \tau^{\gamma+3} \left. \right] .
\]

(ii) deterioration cost per cycle

\[
\mu \int_0^\tau \eta t^{\gamma-1}Q(t) dt = \eta \gamma \mu \left[ \frac{W}{\gamma} \tau^\gamma - \frac{\eta W^{2\gamma}}{2\gamma} - \frac{\alpha \tau^{\gamma+1}}{\gamma + 1} - \frac{\beta \tau^{\gamma+2}}{2(\gamma + 2)} \right.
\]
\[
+ \frac{\alpha \eta \gamma^{2\gamma+1}}{(2\gamma + 1)(\gamma + 1)} + \frac{\beta \eta \gamma^{2\gamma+2}}{4(\gamma + 1)(\gamma + 2)} \left. \right] .
\]

(iii) shortage cost per cycle

\[
\xi \int_\tau^T Q(t) dt = \xi \left[ \alpha \tau T - \frac{\alpha T^2}{2} + \frac{\beta \tau^3 T}{2} - \frac{\beta T^3}{6} - \frac{\alpha \tau^2}{2} - \frac{\beta \tau^3}{3} \right] .
\]

Considering the associated costs as mentioned above, the total average cost per unit time of the present system is given by,

\[
Z(W, \tau) = \frac{1}{T} \left[ \text{Holding Cost} + \text{Deterioration Cost} - \text{Shortage Cost} \right].
\]

This implies,

\[
Z(W, \tau)
\]
\[
= \frac{1}{T} \left\{ \lambda \left[ W\tau - \frac{\alpha \tau^2}{2} - \frac{\beta \tau^3}{3} - \frac{\eta W}{\gamma + 1} \frac{\tau^{\gamma+1}}{\gamma + 1} + \frac{\alpha \eta \gamma \tau^{\gamma+2}}{2(\gamma + 2)(\gamma + 3)} \right]
\]
\[
+ \eta \gamma \mu \left[ \frac{W}{\gamma} \tau^\gamma - \frac{\eta W^{2\gamma}}{2\gamma} - \frac{\alpha \tau^{\gamma+1}}{\gamma + 1} - \frac{\beta \tau^{\gamma+2}}{2(\gamma + 2)} \right.
\]
\[
+ \frac{\alpha \eta \gamma^{2\gamma+1}}{(2\gamma + 1)(\gamma + 1)} + \frac{\beta \eta \gamma^{2\gamma+2}}{4(\gamma + 1)(\gamma + 2)} \left. \right] \right. 
\]
\[
- \xi \left[ \alpha \tau T - \frac{\alpha T^2}{2} + \frac{\beta \tau^3 T}{2} - \frac{\beta T^3}{6} - \frac{\alpha \tau^2}{2} - \frac{\beta \tau^3}{3} \right].
\]
Using equation (5), and eliminating $W$ from equation (9), we get

$$
Z(\tau) = \frac{1}{T} \left\{ \lambda \left[ -\frac{\alpha\tau^2}{2} - \frac{\beta\tau^3}{3} + \frac{\alpha\eta\tau^{\gamma+2}}{(\gamma + 1)(\gamma + 2)} + \frac{\eta\beta\gamma\tau^{\gamma+3}}{2(\gamma + 2)(\gamma + 3)} \right] \\
+ \left( \tau - \frac{\eta\tau^{\gamma+1}}{\gamma + 1} \right) \left( \alpha\tau + \frac{\alpha\eta\tau^{\gamma+1}}{\gamma + 1} + \frac{\beta\tau^2}{2} + \frac{\beta\eta\tau^{\gamma+2}}{\gamma + 2} \right) \right\} \\
+ \eta\gamma\mu \left[ -\frac{\alpha\tau^2}{2} - \frac{\beta\tau^3}{3} + \frac{\alpha\eta\tau^{\gamma+2}}{(\gamma + 1)(\gamma + 2)} + \frac{\eta\beta\gamma\tau^{\gamma+3}}{2(\gamma + 2)(\gamma + 3)} \right] \\
+ \frac{\tau^\gamma}{\gamma} \left( \alpha\tau + \frac{\alpha\eta\tau^{\gamma+1}}{\gamma + 1} + \frac{\beta\tau^2}{2} + \frac{\beta\eta\tau^{\gamma+2}}{\gamma + 2} \right) \\
- \xi \left[ \alpha\tau^2 - \frac{\alpha\eta\tau^{\gamma+1}}{\gamma + 1} + \frac{\beta\tau^3}{2} \right] \right\}. \quad (10)
$$

Now in view of minimizing the cost, we need to solve equation (10), but it is difficult to obtain the on hand solution by deriving a closed equation of the solution of equation (10). Thus, we use Mathematica (11.1.6) software to determine optimal $\tau^*$, and thereafter the optimal cost $Z(\tau^*)$ can be calculated. Also, the initial inventory level $W^*$ and the total amount of replenishment $W^1$ in the beginning of each cycle can be achieved.

5. Fuzzy model. In the above created crisp model, the costs associated with inventory were taken as constant. But these costs couldn’t be accepted in real situations as they are imprecision in nature. To overcome this problem, we consider deteriorating cost, holding cost and shortage cost as fuzzy numbers $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\xi}$ respectively.

Now the corresponding total average cost in fuzzy environment is

$$
\tilde{Z}(\tau) = \frac{1}{T} \left\{ \tilde{\lambda} \left[ -\frac{\alpha\tau^2}{2} - \frac{\beta\tau^3}{3} + \frac{\alpha\eta\tau^{\gamma+2}}{(\gamma + 1)(\gamma + 2)} + \frac{\eta\beta\gamma\tau^{\gamma+3}}{2(\gamma + 2)(\gamma + 3)} \right] \\
+ \left( \tau - \frac{\eta\tau^{\gamma+1}}{\gamma + 1} \right) \left( \alpha\tau + \frac{\alpha\eta\tau^{\gamma+1}}{\gamma + 1} + \frac{\beta\tau^2}{2} + \frac{\beta\eta\tau^{\gamma+2}}{\gamma + 2} \right) \right\} \\
+ \eta\gamma\tilde{\mu} \left[ -\frac{\alpha\tau^2}{2} - \frac{\beta\tau^3}{3} + \frac{\alpha\eta\tau^{\gamma+2}}{(\gamma + 1)(\gamma + 2)} + \frac{\eta\beta\gamma\tau^{\gamma+3}}{2(\gamma + 2)(\gamma + 3)} \right] \\
+ \left( \frac{\tau^\gamma}{\gamma} - \frac{\eta\tau^{2\gamma}}{2\gamma} \right) \left( \alpha\tau + \frac{\alpha\eta\tau^{\gamma+1}}{\gamma + 1} + \frac{\beta\tau^2}{2} + \frac{\beta\eta\tau^{\gamma+2}}{\gamma + 2} \right) \\
- \tilde{\xi} \left[ \alpha\tau^2 - \frac{\alpha\eta\tau^{\gamma+1}}{\gamma + 1} + \frac{\beta\tau^3}{3} \right] \right\}. \quad (11)
$$

Suppose $\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$, $\tilde{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6)$ and $\tilde{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$ are the non negative hexagonal fuzzy numbers (HFN).

Let $\tilde{Z}_i(\tau)$ be the corresponding total average cost obtained by replacing $\tilde{\lambda}$ by $\lambda_i$, $\tilde{\mu}$ by $\mu_i$ and $\tilde{\xi}$ by $\xi_i$ in equation (11) for $i=1, 2, 3, 4, 5, 6$. 




Then the defuzzified total average cost obtained by GMIR method is

\[ GZ(\tau) = \frac{1}{12T} \left\{ \lambda_1 \left[ \left( \tau - \frac{\eta_T^{\gamma+1}}{\gamma + 1} \right) \left( \alpha T + \frac{\alpha \eta^{\gamma+1}_T}{\gamma + 1} + \frac{\beta T^2}{2} + \frac{\beta \eta^{\gamma+2}_T}{\gamma + 2} \right) \right] - \frac{\alpha^2 T^2}{2} - \frac{\beta T^3}{6} + \frac{\eta \gamma \mu_1}{3} \right\} \]

and simplifying, we get the defuzzified total average cost as

\[ GZ(\tau) = \frac{1}{12T} \left\{ \lambda_1 \left[ \left( \tau - \frac{\eta_T^{\gamma+1}}{\gamma + 1} \right) \left( \alpha T + \frac{\alpha \eta^{\gamma+1}_T}{\gamma + 1} + \frac{\beta T^2}{2} + \frac{\beta \eta^{\gamma+2}_T}{\gamma + 2} \right) \right] - \frac{\alpha^2 T^2}{2} - \frac{\beta T^3}{6} + \frac{\eta \gamma \mu_1}{3} \right\} \]
Solution procedure.

6.1. Crisp model. Our aim is to find an optimal $\tau^*$ for $\tau$ such that the total average cost $Z(\tau)$ is minimized. For this, it is required to solve $\frac{dZ(\tau)}{d\tau} = 0$ for $\tau$ such that $\frac{d^2Z(\tau)}{d\tau^2} > 0$.

6.2. Fuzzy model. In the similar lines of (6.1), we can find an optimal $\tau^*$ for $\tau$ such that the total average cost $GZ(\tau)$ is minimized.

7. Examples. The results in the following examples are obtained with the help of 
Mathematica (11.1.6) software.

Example 1. (A) Crisp Model: Let the values of the different parameters involved in the model are $\alpha = 200$, $\beta = 20$, $\eta = 0.02$, $\gamma = 4$, $T = 2$, $\lambda = 60$, $\mu = 80$ and $\xi = 100$.

Solution. Following the solution procedure and using equation (10) and (5), we obtained the optimal solution as $\tau^* = 1.23489$ years, $Z(\tau^*) = 58290.67$, $W = 264.76$ units, $S = 177.773$ units and $W1 = 442.534$ units.

(B) Fuzzy Model: Let the values of different parameters are $\alpha = 200$, $\beta = 20$, $\eta = 0.02$, $\gamma = 4$, $T = 2$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (20, 30, 40, 50, 60, 70)$, $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (20, 40, 60, 80, 90, 100)$ and $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) = (50, 60, 70, 80, 90, 100)$.
Example 2. (A) Crisp Model: Let the values of the different parameters involved in the model be $\alpha = 200$, $\beta = 20$, $\eta = 0.02$, $\gamma = 4$, $T = 1.5$, $\lambda = 860$, $\mu = 80$, and $\xi = 100$.

Solution. Following the solution procedure and using equation (12) and (5), we obtained the optimal solution as $\tau^* = 1.23281$ years, $GZ(\tau^*) = \$6225.45$, $W = 264.273$ units, $S = 178.239$ units and $W1 = 442.512$ units.

Solution. Following the solution procedure and using equation (10) and (5), we obtained the optimal solution as $\tau^* = 0.940398$ years, $Z(\tau^*) = \$6021.1$, $W = 197.558$ units, $S = 125.577$ units and $W1 = 323.134$ units.

Example 3. (A) Crisp Model: Let the values of the different parameters involved in the model be $\alpha = 400$, $\beta = 30$, $\eta = 0.001$, $\gamma = 1$, $T = 1$, $\lambda = 40$, $\mu = 70$, and $\xi = 100$.

Solution. Following the solution procedure and using equation (10) and (5), we obtained the optimal solution as $\tau^* = 0.719109$ years, $Z(\tau^*) = \$5894.03$, $W = 295.508$ units, $S = 119.6$ units and $W1 = 415.107$ units.

(B) Fuzzy Model: Let the values of different parameters be $\alpha = 400$, $\beta = 30$, $\eta = 0.001$, $\gamma = 1$, $T = 1$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (20, 30, 40, 50, 60, 70)$, $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (20, 40, 60, 80, 90, 100)$, and $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) = (50, 60, 70, 80, 90, 100)$.

Solution. Following the solution procedure and using equation (12) and (5), we obtained the optimal solution as $\tau^* = 0.629908$ years, $GZ(\tau^*) = \$5804.05$, $W = 257.997$ units, $S = 157.085$ units and $W1 = 415.082$ units.

8. Sensitivity analysis. Here we consider Example 1 for sensitivity of different parameters involved in the model.

Effect of Changes in Parameter $\alpha$ on Optimal Inventory

(i) From the Table 1 and Figure 2, we observe that as the value of $\alpha$ increases, the span of the positive inventory becomes decreases (that is, the value of $\tau$ decreases).

(ii) From the Table 1 and Figure 3, we observe as the value of $\alpha$ increases, the cost of the inventory ($GZ(\tau)$) increases.

(iii) From the Table 1 and Figure 4, we observe that as the value of $\alpha$ increases, the total replenishment amount $W1$ increases.

Effect of Changes in Parameter $\beta$ on Optimal Inventory

(i) From the Table 2 and Figure 5, we observe that as the value of $\beta$ increases, the span of the positive inventory becomes increase (that is, the value of $\tau$ increases).
(ii) From the Table 2 and Figure 6, we observe that as the value of $\beta$ increases, the cost of the inventory $(GZ(\tau))$ increases.

(iii) From the Table 2 and Figure 7, we observe that as the value of $\beta$ increases, the total replenishment amount $W1$ increases.
Effect of Changes in Parameter $\eta$ on Optimal Inventory

(i) From the Table 3 and Figure 8, we observe that as the value of $\eta$ increases, the span of the positive inventory becomes decrease (that is, the value of $\tau$ decreases).

(ii) From the Table 3 and Figure 9, we observe that as the value of $\eta$ increases, the total cost of the inventory $(GZ(\tau))$ increases.

(iii) From the Table 3 and Figure 10, we observe that as the value of $\eta$ increases, the total replenishment amount $W1$ increases.

Effect of Changes in Parameter $\gamma$ on Optimal Inventory

(i) From the Table 4 and Figure 11, we observe that as the value of $\gamma$ increases, the span of the positive inventory becomes decrease (that is, the value of $\tau$ decreases).

(ii) From the Table 4 and Figure 12, we observe that as the value of $\gamma$ increases, the total cost of the inventory $(GZ(\tau))$ increases.

(iii) From the Table 4 and Figure 13, we observe that as the value of $\gamma$ increases, the total replenishment amount $W1$ decreases.

Sensitivity effect of Parameter $T$ on Inventory

(i) From the Table 5 and Figure 14, we observe that as the value of $T$ increases, the span of the positive inventory becomes increases (that is, the value of $\tau$ increases).

(ii) From the Table and Figure 15, we observe that as the value of $T$ increases, the total cost of the inventory $(GZ(\tau))$ increases.

(iii) From the Table and Figure 16, we observe that as the value of $T$ increases, the total replenishment amount $W1$ increases.

![Figure 2: Effect of parameter $\alpha$ on $\tau$](image)
A FUZZY INVENTORY MODEL FOR WEIBULL DETERIORATING ITEMS

Figure 3. Effect of parameter $\alpha$ on $GZ(\tau)$

Figure 4. Effect of parameter $\alpha$ on $W1$

Figure 5. Effect of parameter $\beta$ on $\tau$
Figure 6. Effect of parameter $\beta$ on $GZ(\tau)$

Figure 7. Effect of parameter $\beta$ on $W1$

Figure 8. Effect of parameter $\eta$ on $\tau$
Figure 9. Effect of parameter $\eta$ on $GZ(\tau)$

Figure 10. Effect of parameter $\eta$ on $W1$

Figure 11. Effect of parameter $\gamma$ on $\tau$
Figure 12. Effect of parameter $\gamma$ on $G_Z(\tau)$

Figure 13. Effect of parameter $\gamma$ on $W_1$

Figure 14. Effect of parameter $T$ on $\tau$
A FUZZY INVENTORY MODEL FOR WEIBULL DETERIORATING ITEMS

Figure 15. Effect of parameter $T$ on $GZ(\tau)$

Figure 16. Effect of parameter $T$ on $W_1$

9. **Conclusion.** Inventory control of products having linear demand with Weibull deterioration rate under shortages is quite relevant in many business organizations. In this paper a stock model is determined for perishable things with linear request design having Two-parameter Weibull deterioration rate and allowance of backlogged shortages. The proposed model is created in both crisp and fuzzy situations. In fuzzy environment, all related stock parameters were considered as hexagonal fuzzy numbers. Graded mean representation technique is used for defuzzification of the model. The minimum cost of the model is achieved by using optimization techniques. Different illustrative examples are provided to justify the work. Also, sensitivity analysis has been studied, and the outcomes are presented in the form of tables providing the nature of the model with slight changes of the different parameters.
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E-mail address: deepakmca52@gmail.com
E-mail address: sudhansu31@gmail.com
E-mail address: skpaikray_math@vssut.ac.in
E-mail address: hemen.dutta08@rediffmail.com