A new class of solutions for the multi-component extended Harry Dym equation\(^1\)

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Abstract

We construct a point transformation between two integrable systems, the multi-component Harry Dym equation and the multi-component extended Harry Dym equation, that does not preserve the class of multi-phase solutions. As a consequence we obtain a new type of wave-like solutions, generalising the multi-phase solutions of the multi-component extended Harry Dym equation. Our construction is easily transferrable to other integrable systems with analogous properties.

Keywords: Harry Dym, invertible transformation, high-frequency limit, multi-phase solutions, Lax pair

2010 MSC: 35Q51, 37J35, 37K10, 37K40

PACS: 02.30.Ik, 02.30.Jr

1. Introduction

In a number of papers [2, 3, 4], integrable systems associated with the energy-dependent linear Schrödinger equation

\[
\left( \sum_{m=0}^{M} \epsilon_m \lambda^m \right) \psi_{xx} = \left( \sum_{m=0}^{M} v_m \lambda^m \right) \psi
\]

were investigated in details. In these papers, two main classes were selected by the conditions \(\epsilon_M = 0, v_M = 1\) (the so called “multi-component KdV systems”) and \(\epsilon_M = 0, v_0 = 1\) (the so called “multi-component extended Harry Dym systems” or “multi-component Camassa–Holm systems”). In the present paper we consider another class determined by a sole restriction \(v_M = 0\) (the so called “multi-component Harry Dym systems” (13) or “multi-component Hunter–Saxton equations”). We show that the multi-component Harry Dym equations are connected with the multi-component extended Harry Dym equations by the \(\kappa\)-transformation introduced below, see (15). The method presented here is applicable to any solutions. Without loss of generality and for simplicity we restrict our consideration to multi-phase solutions only. Applying the \(\kappa\)-transformation to the multi-phase solutions of multi-component Harry Dym systems we obtain a new class of solutions of multi-component extended Harry Dym systems,

\(^1\)Dedicated to the 80th birthday of A.B. Shabat
which we call the $\kappa$-deformed multi-phase solutions. This new class of solutions cannot be obtained as a reduction of multi-phase solutions. Whilst multi-phase solutions belong to the so called “solitonic sector” (i.e., the case of reflectionless potentials), the new class of solutions (the $\kappa$-deformed multi-phase solutions) has a rapidly increasing behaviour with respect to $x$.

**Example 1.** The extended Harry Dym equation ($\kappa$ is an arbitrary constant, $\kappa \neq 0$)

$$v_t = -\frac{1}{2}(\partial_x^2 - \kappa^2)(v^{-1/2})_x$$

possesses the one-phase solution ($c$ is an arbitrary constant)

$$v = \frac{1}{c^2 r^2},$$

where $r$ is a function of $\theta = x + ct$ determined implicitly by ($s_0, s_1$ are arbitrary constants)

$$\theta = \int^r \frac{\sqrt{\lambda}d\lambda}{\sqrt{4c^{-2} + s_1 \lambda + s_0 \lambda^2 + \kappa^2 \lambda^3}};$$

and simultaneously the $\kappa$-deformed one-phase solution

$$v = \frac{e^{2\kappa x}}{c^2 R^2},$$

where the function $R(\vartheta)$ is determined implicitly by

$$\frac{e^{\kappa x} - 1}{\kappa} + ct = \vartheta = \int^R \frac{\sqrt{\lambda}d\lambda}{\sqrt{4c^{-2} + s_1 \lambda + s_0 \lambda^2}}.$$

These solutions are significantly different: the first solution is essentially one-dimensional (see Figure 1) and is obtained by a regular procedure (one can look for a travelling wave reduction determined by the ansatz $v(\theta)$, where $\theta = x + ct$), while the second solution is two-dimensional (see Figure 2) and is obtained (see below) by means of an invertible point transformation between the extended Harry Dym equation and its high-frequency limit ($\kappa = 0$), which is the well-known Harry Dym equation.

![Figure 1: The unit-speed ($c = 1$) travelling wave $v(\theta)$ for $s_1 = 12$, $s_0 = 8$, $\kappa = 0.14$.](image)

Many integrable systems have such an invertible point transformation that connects them with their high-frequency limits. So, once a one-phase (or a multi-phase) solution is found, one can construct the so-called $\kappa$-deformed solution following our approach.
Moreover, in the particular case

\[ 4c^{-2} + s_1 \lambda + s_0 \lambda^2 = s_0 (\lambda - \lambda_1)^2, \]

the second solution can be found in elementary functions, i.e.

\[
\frac{e^{\kappa x} - 1}{\kappa} + ct = \vartheta = \frac{1}{\sqrt{s_0}} \int_{\lambda}^{R} \frac{\sqrt{\lambda} \, d\lambda}{\lambda - \lambda_1} = \frac{1}{\sqrt{s_0}} \left( 2 \sqrt{R} - 2 \sqrt{\lambda_1} \text{arctanh} \sqrt{\frac{R}{\lambda_1}} \right).
\]

In another particular case

\[ 4c^{-2} + s_1 \lambda + s_0 \lambda^2 + \kappa^2 \lambda^3 = \kappa^2 (\lambda - \lambda_2)(\lambda - \lambda_3)^2, \]

the first solution can be found in elementary functions, i.e.

\[
\theta = \frac{1}{\kappa} \int_{\lambda}^{r} \frac{\sqrt{\lambda} \, d\lambda}{(\lambda - \lambda_3) \sqrt{\lambda - \lambda_2}} = \frac{1}{\kappa} \left( \ln (r - \frac{1}{2} \lambda_2 + \sqrt{r} \sqrt{r - \lambda_2} + \frac{\sqrt{\lambda_3}}{\sqrt{\lambda_3} - \lambda_2} \ln \left( \frac{2 \lambda_3 - \lambda_2 (r + \lambda_3) + 2 \sqrt{\lambda_1 \lambda_3 - \lambda_2 r} \sqrt{r} \sqrt{r - \lambda_2}}{r - \lambda_3} \right) \right).
\]

Thus, both types of solutions have completely different behaviour.

In this paper we restrict our consideration to the simplest integrable system whose multi-phase solutions are associated with hyperelliptic Riemann surfaces: the multi-component extended Harry Dym equation.
2. A Special Class of Integrable Systems

The energy dependent linear Schrödinger equation

\[ \psi_{xx} = U \psi \]  

(1)

was recently [8] investigated for the special case (with respect to the spectral parameter \( \lambda \))

\[ U(x, t, \lambda) = \sigma + \frac{u^1}{\lambda} + \frac{u^2}{\lambda^2} + \frac{u^3}{\lambda^3} + ..., \]

(2)

where \( \sigma \) is an arbitrary constant. If \( \sigma \neq 0 \), then the parameter \( \sigma \) can be fixed to 1 by an appropriate scaling of independent variables and dependent functions without loss of generality.

Integrable systems associated with (1) can be obtained from the compatibility condition \((\psi_t)_{xx} = (\psi_{xx})_t\), where \( \psi_t = a\psi_x - \frac{1}{2}a_x \psi \).

(3)

The compatibility condition \((\psi_t)_{xx} = (\psi_{xx})_t\) yields the relationship

\[ U_t = \left( -\frac{1}{2} \partial^3_x + 2U \partial_x + U_x \right) a, \]

which leads to the dispersive integrable chain (\( \xi \) is an integration constant)

\[ u^k_t = u^{k+1}_x + a^1 u^k_x + 2u^k a^1_x, \quad k = 1, 2, ..., \quad u^1 + \xi = \frac{1}{2}a_{1,xx} - 2\sigma a_1, \]

(4)

where \( a = \lambda + a^1 \) and the function \( U \) is determined by (2). This dispersive integrable chain can be reduced to \( M \)-component integrable dispersive systems by simple reductions \( u^{M+1} = 0 \) for any natural number \( M \). This means that one should consider the linear problem (1), (3), where

\[ U(x, t, \lambda) = \sigma + \frac{u^1}{\lambda} + \frac{u^2}{\lambda^2} + ... + \frac{u^M}{\lambda^M} \]

(5)

instead of (2). If \( M = 1 \), one obtains the remarkable Camassa–Holm equation

\[ u^1_t = a^1 u^1_x + 2u^1 a^1_x, \quad u^1 + \xi = \frac{1}{2}a_{1,xx} - 2\sigma a_1, \]

if \( M > 1 \), the multi-component generalisation of the Camassa–Holm equation is

\[ u^k_t = u^{k+1}_x + a^1 u^k_x + 2u^k a^1_x, \quad k = 1, 2, ..., M - 1, \quad u^M_t = a^1 u^M_x + 2u^M a^1_x, \]

(6)

where again \( u^1 + \xi = \frac{1}{2}a_{1,xx} - 2\sigma a_1 \).

Below we also investigate the special case \( \sigma = 0 \) and discuss the relationship between integrable systems determined by both choices \( \sigma = 0 \) and \( \sigma \neq 0 \). The corresponding dispersive integrable chain reduces to the form (cf. (4))

\[ u^k_t = u^{k+1}_x + a^1 u^k_x + 2u^k a^1_x, \quad k = 1, 2, ..., \quad u^1 + \xi = \frac{1}{2}a_{1,xx}. \]

(7)
So, the main difference between (4) and (7) is a difference between the constraints
\[ u^1 + \xi = \frac{1}{2}a_{1,xx} - 2\sigma a_1 \]
and \[ u^1 + \xi = \frac{1}{2}a_{1,xx}. \] Again if \( M = 1 \), one can obtain the Hunter–Saxton equation
\[ u^1_t = a_1 u^1_x + 2u^1_{1,x}, \quad u^1 + \xi = \frac{1}{2}a_{1,xx}, \]
which is a high frequency limit of the Camassa–Holm equation (see detail in [7]). If \( M > 1 \), the multi-component generalisation of the Hunter–Saxton equation is (cf. (6))
\[ u^k_t = u^k_{x} + a^1 u^k_x + 2u^k_{1,x}, \quad k = 1, 2, ..., M - 1, \]
\[ u^M_t = a^1 u^M_x + 2u^M_{1,x}, \]
where again \( u^1 + \xi = \frac{1}{2}a_{1,xx} \). If instead of the choice \( a = \lambda + a_1 \) we consider the dependence \( a = a_{-1}/\lambda \), then
\[ u^1_t = -\frac{1}{2}(a_{-1})_{xxx}, \]
\[ u^k_t = a_{-1}u^k_x + 2u^k_{-1}(a_{-1})_x, \quad k = 2, ..., M, \]
\[ a_{-1} = (u^M)^{-1/2}. \] (8)
If \( M = 1 \), then this is well-known Harry Dym equation; if \( M > 1 \), then this system (8) will be called the multi-component Harry Dym equation.

Below we show that integrable systems (8) associated with the energy dependent linear Schrödinger equation (see (1) and (5) in the limit \( \sigma = 0 \))
\[ \psi_{xx} = \left( \frac{u^1}{\lambda} + \frac{u^2}{\lambda^2} + ... + \frac{u^M}{\lambda^M} \right) \psi \] (9)
can be interpreted as a high frequency limit of the so called multi-component extended Harry Dym equation (see detail below).

Indeed, one can consider the linear spectral problem (9) written in the form
\[ \psi_{zz} = \left( \frac{v^1}{\lambda} + \frac{v^2}{\lambda^2} + ... + \frac{v^M}{\lambda^M} \right) \psi. \] (10)
Then we apply the point transformation
\[ z = \frac{e^{\kappa x} - 1}{\kappa}, \] (11)
where \( \kappa \) is an arbitrary parameter. Then \( \partial_z \rightarrow e^{-\kappa x}\partial_x \). If \( \kappa \rightarrow 0 \), then \( z(x, \kappa) \rightarrow x \).

Under transformation (11) the linear spectral problem (10) becomes (see (1) and (2))
\[ \varphi_{xx} = \left( \sigma + \frac{v^1}{\lambda} + \frac{v^2}{\lambda^2} + ... + \frac{v^M}{\lambda^M} \right) \varphi, \] (12)
where \( \sigma = \kappa^2/4, \ v^k = u^k e^{2\kappa x} \) and \( \psi = \varphi \exp(\kappa x/2). \) The high frequency limit \( \kappa \rightarrow 0 \) reduces the above linear problem to (9). We illustrate this property for the multi-component extended Harry Dym equation
\[ v^1_t = -\frac{1}{2}(\partial^3_x - \kappa^2 \partial_x)\tilde{a}_{-1}, \]
\[ v^k_t = \tilde{a}_{-1}\tilde{v}^k_x - 2v^{k-1}(\tilde{a}_{-1})_x, \quad k = 2, ..., M, \]
\[ \tilde{a}_{-1} = (v^M)^{-1/2}. \] (13)
This system follows from the compatibility condition \((\varphi_t)_{xx} = (\varphi_{xx})_t\), where the function \(\varphi\) is a common solution of two linear equations, i.e. (12) and (cf. (3))

\[
\varphi_t = \frac{1}{\lambda} \left( \tilde{a}_{-1} \varphi_x - \frac{1}{2} (\tilde{a}_{-1})_x \varphi \right).
\]

The high frequency limit \(\kappa \to 0\) leads to the system (8). Now we apply point transformation (11) to system (8) written in the form (here we simply replaced \(x\) by \(z\))

\[
\begin{align*}
 u_1^t &= -\frac{1}{2} (a_{-1})_{zzz}, \\
 u_k^t &= a_{-1}u_{x}^{k-1} + 2u^{k-1}(a_{-1})_x, \quad k = 2, \ldots, M, \\
 a_{-1} &= (u^M)^{-1/2}
\end{align*}
\]  

Then we again obtain system (13), where \(a_{-1} = \tilde{a}_{-1} e^{\kappa x}\), \(v^k = u^k e^{2\kappa x}\). Thus integrable systems (13) and (14) are connected with each other by the point transformation

\[
\begin{align*}
 z &= \frac{e^{\kappa x} - 1}{\kappa}, \\
 u^k &= e^{-2\kappa x} v^k
\end{align*}
\]  

and simultaneously system (8) \(\equiv (14)\) is a high frequency limit of system (13).

3. Multi-Phase Solutions and a High Frequency Limit

To illustrate a difference between the general case \(\kappa \neq 0\) and its high-frequency limit \(\kappa = 0\), in this section we consider multi-gap solutions of the multi-component extended Harry Dym equation

\[
\begin{align*}
 v_1^t &= -\frac{1}{2} (\partial_x^3 - \kappa^2 \partial_x) \tilde{a}_{-1}, \\
 v_k^t &= \tilde{a}_{-1} v_{x}^{k-1} + 2v^{k-1}(\tilde{a}_{-1})_x, \quad k = 2, \ldots, M,
\end{align*}
\]  

where \(\tilde{a}_{-1} = (v^M)^{-1/2}\).

In this case linear problem (1), (3) reduces to the form

\[
\lambda^M (2\phi \phi_{xx} - \phi_x^2) = \left( \kappa^2 \lambda^M + 4 \sum_{m=1}^{M} v_m \lambda^{M-m} \right) \phi^2 - S(\lambda), \quad \phi_t = a\phi_x - a_x \phi,
\]  

where \(\phi = \psi \psi^+\) (here \(\psi\) and \(\psi^+\) are two linearly independent solutions), \(a = \tilde{a}_{-1}/\lambda\) and \(S(\lambda)\) is a polynomial expression with constant coefficients.

As usual, finite-gap solutions connected with hyperelliptic Riemann surfaces can be constructed in several steps:

1. We seek polynomial solutions (with respect to the spectral parameter \(\lambda\)) for the function \(\phi\) in the factorised form

\[
\phi = \prod_{m=1}^{N} (\lambda - r^m(x, t)),
\]  

where \(N\) is an arbitrary natural number.
2. Since function \( \phi \) is a polynomial of the degree \( N \), the dependence \( S(\lambda) \) is a polynomial of the degree \( 2N + M \), i.e.

\[
S(\lambda) = s_{2N+M-1} + s_{2N+M-2}\lambda + \cdots + s_1\lambda^{2N+M-2} + s_0\lambda^{2N+M-1} + s_{-1}\lambda^{2N+M},
\]

where \( s_{-1} = \kappa^2 \), while other \( s_k \) are “integration constants”.

3. Expanding \( \phi \) by virtue of (18) in the first equation of (17) with respect to the spectral parameter \( \lambda \), one can find expressions for field variables \( v_k \). Indeed, substituting (18) into (17), one obtains

\[
v^k = \frac{1}{4} \sum_{m=0}^{M-k} \frac{s_{2N+k-1+m}B_m}{\prod_{n=1}^N r^n}^2, \quad k = 1, \ldots, M,
\]

where

\[
B_0 = 1, \quad B_k = \sum_{k_1 \geq 0, \ldots, k_N \geq 0 \atop k_1 + \cdots + k_N = k} \prod (k_m + 1)(r^m)^{k-k_m}. \tag{21}
\]

4. Following B.A. Dubrovin [5, 6], we consider the limit \( \lambda \to r^t(x, t) \) of the two equations (17). This straightforward computation yields two autonomous systems

\[
r^t_z = \frac{1}{\prod_{m \neq t}(r^t - r^m)} \sqrt{S(r^t)/(r^t)^M}, \quad r^t_\xi = \frac{a^t(r)}{\prod_{m \neq t}(r^t - r^m)} \sqrt{S(r^t)/(r^t)^M}, \tag{22}
\]

where \( a^t(r) = a(\lambda, r)|_{\lambda = r^t} \). In our case (see (20), \( k = M \))

\[
a^t(\lambda, r) = \frac{\bar{a}_{-1}}{r^t} = \frac{(r^t)^{-1/2}}{r^t} = \frac{2 \prod_{m=1}^N r^m}{r^t \sqrt{s_{2N+M-1}}}.
\]

5. A straightforward integration of (22) implies multi-phase solutions of (13) written in the implicit form\(^2\)

\[
x = \sum_{m=1}^N \int r^m \lambda^{M/2+N-1} d\lambda, \quad t = \frac{\sqrt{s_{2N+M-1}}}{2} \sum_{m=1}^N \int \lambda^{M/2} d\lambda / \sqrt{S(\lambda)}, \tag{23}
\]

\[
0 = \sum_{m=1}^N \int r^m \lambda^{M/2+k} d\lambda / \sqrt{S(\lambda)}, \quad k = 1, \ldots, N - 2.
\]

Remark: If \( N = 1 \), then a one-phase solution is parameterised by a single function \( r(\theta) \). Namely,

\[
v_k = \frac{1}{4} \sum_{m=1}^{M-k+1} ms_{m+k}r^{-m-1},
\]

where the function \( r(\theta) \) is determined by the relationship (here \( s_{M+1} = 4c^{-2} \))

\[
x + ct = \theta = \int r \sqrt{s_{M+1} + s_M\lambda + \cdots + s_1\lambda^M + s_0\lambda^{M+1} + \kappa^2\lambda^{M+2}} \lambda^{M/2} d\lambda.
\]

\(^2\)Explicit formulae for more wide class of integrable systems, whose multi-phase solutions are associated with hyperelliptic Riemann surfaces, were obtained in [1]
3.1. Finite-Gap Solutions and $\kappa$-Transformation

Now we can compare finite-gap solutions for both cases $\kappa = 0$ and $\kappa \neq 0$. So in the case $\kappa = 0$ we have for system (14) multi-phase solutions (cf. (20) and (21))

$$u^k = \frac{1}{4} \sum_{m=0}^{M-k} \frac{s_{2N+k-1+m} \tilde{B}_m}{\left(\prod_{n=1}^{N} R_n\right)^{m+2}}, \quad k = 1, \ldots, M,$$

where

$$\tilde{B}_0 = 1, \quad \tilde{B}_k = \sum_{k_1 \geq 0, \ldots, k_N \geq 0} \prod_{m=1}^{N} (k_m + 1)(R_m)^{k-m}.$$

The dependencies $R^k(z, t)$ are presented in implicit form (cf. (23))

$$
\begin{align*}
  z &= \sum_{m=1}^{N} \int \frac{\lambda^{M/2+N-1} d\lambda}{\sqrt{P(\lambda)}}, \quad t = \frac{\sqrt{s_{2N+M-1}}}{2} \sum_{m=1}^{N} \int \frac{\lambda^{M/2} d\lambda}{\sqrt{P(\lambda)}}, \\
  0 &= \sum_{m=1}^{N} \int \frac{\lambda^{M/2+k} d\lambda}{\sqrt{P(\lambda)}}, \quad k = 1, \ldots, N-2,
\end{align*}
$$

where (cf. (19))

$$P(\lambda) = s_{2N+M-1} + s_{2N+M-2} \lambda + \cdots + s_1 \lambda^{2N+M-2} + s_0 \lambda^{2N+M-1}.$$

Under transformation (11) system (14) becomes (13), while multi-phase solutions of system (14) take the form (we remind the reader that $v^k = u^k e^{2\kappa x}$)

$$v^k = \frac{1}{4} e^{2\kappa x} \sum_{m=0}^{M-k} \frac{s_{2N+k-1+m} \tilde{B}_m}{\left(\prod_{n=1}^{N} R_n\right)^{m+2}}, \quad k = 1, \ldots, M,$$

where the dependencies $R^k(x, t)$ are presented in implicit form (cf. (23), (24))

$$
\begin{align*}
  \frac{e^{\kappa x} - 1}{\kappa} &= \sum_{m=1}^{N} \int \frac{\lambda^{M/2+N-1} d\lambda}{\sqrt{P(\lambda)}}, \quad t = \frac{\sqrt{s_{2N+M-1}}}{2} \sum_{m=1}^{N} \int \frac{\lambda^{M/2} d\lambda}{\sqrt{P(\lambda)}}, \\
  0 &= \sum_{m=1}^{N} \int \frac{\lambda^{M/2+k} d\lambda}{\sqrt{P(\lambda)}}, \quad k = 1, \ldots, N-2.
\end{align*}
$$

Thus, we found a new type of solutions of multi-component Harry Dym equation (13), which do not coincide with the corresponding multi-phase solutions (23).

3.2. The One-Phase Solution

In the particular case $N = 1$, the multi-component extended Harry Dym equation (16) has the one-phase solution

$$v_k = \frac{1}{4} \sum_{m=1}^{M-k+1} ms_{m+k} r^{-m-1},$$
where the function \( r(\theta) \) is determined by the relationship (here \( s_{M+1} = 4c^{-2} \))

\[
x + ct = \theta = \int r \frac{\lambda^{M/2} d\lambda}{\sqrt{s_{M+1} + s_M \lambda + \cdots + s_1 \lambda^M + s_0 \lambda^{M+1} + \kappa^2 \lambda^{M+2}}};
\]

and the \( \kappa \)-deformed one-phase solution

\[
v_k = \frac{1}{4} e^{2\kappa x} \sum_{m=1}^{M-k+1} ms_m R^{-m-1},
\]

where the function \( R(\vartheta) \) is determined by the relationship (here \( s_{M+1} = 4c^{-2} \))

\[
e^{\kappa x} - 1 + ct = \vartheta = \int R \frac{\lambda^{M/2} d\lambda}{\sqrt{s_{M+1} + s_M \lambda + \cdots + s_1 \lambda^M + s_0 \lambda^{M+1}}};
\]

3.3. The Extended Harry Dym Equation

Here we consider the particular case \( M = 1 \), i.e., the extended Harry Dym equation

\[
v_t = -\frac{1}{2} (\partial_x^3 - \kappa^2 \partial_x) v^{-1/2}.
\]

Its \( N \)-phase solutions are determined by

\[
v = \frac{s_{2N}}{4} \left( \prod_{n=1}^{N} r^n \right)^{-2}
\]

where

\[
x = \sum_{m=1}^{N} \int_{\lambda}^{\alpha} \frac{\lambda^{N-1/2} d\lambda}{\sqrt{S(\lambda)}}, \quad t = \frac{\sqrt{s_{2N}}}{2} \sum_{m=1}^{N} \int_{\lambda}^{\alpha} \frac{\lambda^{1/2} d\lambda}{\sqrt{S(\lambda)}},
\]

\[
0 = \sum_{m=1}^{N} \int_{\lambda}^{\alpha} \frac{\lambda^{k+1/2} d\lambda}{\sqrt{S(\lambda)}}, \quad k = 1, \ldots, N - 2,
\]

and

\[
S(\lambda) = s_{2N} + s_{2N-1} \lambda + \cdots + s_1 \lambda^{2N-1} + s_0 \lambda^{2N} + \kappa^2 \lambda^{2N+1}.
\]

A new class of solutions (\( \kappa \)-deformed \( N \)-phase solutions) is determined by

\[
v = \frac{s_{2N}}{4} e^{2\kappa x} \left( \prod_{n=1}^{N} R^n \right)^{-2},
\]

where

\[
e^{\kappa x} - 1 = \sum_{m=1}^{N} \int_{\lambda}^{\alpha} \frac{\lambda^{N-1/2} d\lambda}{\sqrt{P(\lambda)}}, \quad t = \frac{\sqrt{s_{2N}}}{2} \sum_{m=1}^{N} \int_{\lambda}^{\alpha} \frac{\lambda^{1/2} d\lambda}{\sqrt{P(\lambda)}},
\]

\[
0 = \sum_{m=1}^{N} \int_{\lambda}^{\alpha} \frac{\lambda^{k+1/2} d\lambda}{\sqrt{P(\lambda)}}, \quad k = 1, \ldots, N - 2,
\]

and

\[
P(\lambda) = s_{2N} + s_{2N-1} \lambda + \cdots + s_1 \lambda^{2N-1} + s_0 \lambda^{2N}.
\]

The case \( N = 1 \) for the extended Harry Dym equation was considered in the Introduction.

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3 The simplest case \( N = 1 \) is presented in the Introduction.
4. Conclusion

Using the multi-component extended Harry Dym equation as an example, we studied integrable systems connected with their high-frequency limits (\( \kappa = 0 \)) by an invertible point transformation and obtained a new class of their solutions. Applying transformation (11), the multi-phase solutions of the high-frequency limits could be recalculated into a new kind of solutions for the original systems. As a future perspective, one can apply the \( \kappa \)-transformation to, e.g., multi-peakon solutions of the Extended Harry Dym equation to obtain a new class of solutions for its high-frequency limit, well-known as the Hunter–Saxton equation (see again [7]), etc.

Acknowledgements

MM gratefully acknowledges the support from GAČR under project P201/12/G028. MVP’s work was partially supported by the grant of Presidium of RAS “Fundamental Problems of Nonlinear Dynamics” and by the RFBR grant 14-01-00012. MVP thanks V.E. Adler, L.V. Bogdanov, E.V. Ferapontov, V.G. Marikhin, A.I. Zenchuk for important discussions.

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