Asymptotic Analysis of Risk Premia Induced by Law-Invariant Risk Measures

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Abstract

We analyze the limiting behavior of the risk premium associated with the Pareto optimal risk sharing contract in an infinitely expanding pool of risks under a general class of law-invariant risk measures encompassing rank-dependent utility preferences. We show that the corresponding convergence rate is typically only $n^{1/2}$ instead of the conventional $n$, with $n$ the multiplicity of risks in the pool, depending upon the precise risk preferences.

Keywords: Risk premium; Risk sharing; Pareto optimality; Large risk pools; Law invariance; Probabilistic sophistication; Rank-dependent utility.

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1 Introduction

Sharing of risk, or risk exchange by redistributing risk among economic agents, is at the heart of economics, insurance and finance. Its potential benefits and welfare implications have been analyzed in a large literature that starts with Borch [5, 6], Arrow [2], Wilson [27] and DuMouchel [11]; see also the early Arrow [1]. As the benefits of risk sharing often grow with the multiplicity of risks, there are clear incentives for the formation of large pools of risk. In large risk pools, by the Law of Large Numbers (whenever valid), the average risk is close to its expected value. That is, by redistributing and subdividing risks in a sufficiently large pool, a nearly riskless situation can be established.\footnote{See Samuelson [25] for an insightful perspective.}

This paper explicitly derives the limiting behavior of the risk premium associated with the Pareto optimally shared risk in an infinitely expanding pool of risks under a general class of law-invariant risk measures encompassing rank-dependent utility preferences. While the convergence of the shared risk to its expected value already follows from the Law of Large Numbers, with corresponding convergence rate $n$, the limiting behavior of the risk premium is much more delicate, and its convergence rate can be $n$ or only $n^{1/2}$ depending upon the agent’s precise risk preferences. In the former case, a Law of Large Numbers suffices, in the latter case a Central Limit Theorem is required to establish formal convergence proofs. This dichotomy can be linked to the notion of first- and second-order risk aversion introduced by Segal and Spivak [26].

This paper fits to the rapidly growing literature on the problem of risk sharing under general preferences; see e.g., Carlier and Dana [7], Heath and Ku [17], Barrieu and El Karoui [3, 4], Dana and Scarsini [9], Jouini, Schachermayer and Touzi [18], Filipović and Svindland [13], Dana [10], Laeven and Stadje [21], Ravanelli and Svindland [24], and the references therein. To our best knowledge, except in trivial cases, these papers do not establish the asymptotic behavior of the associated risk premia.

2 Main Result

We fix a probability space $(\Omega, \mathcal{F}, P)$ and denote by $E[\cdot]$ the expectation operator with respect to the reference probability measure $P$. For any random variable $X$ defined on this probability space, let

$$q_X(t) := \inf\{m \in \mathbb{R} | P[X \leq m] \geq t\}, \quad t \in [0, 1],$$

be its (left-continuous) quantile function, where $\inf \emptyset = \infty$ by convention.

We consider, on $L^\infty(\Omega, \mathcal{F}, P) =: L^\infty(P)$, a general class of preferences given by a numerical representation constructed from building blocks of the form

$$\mathcal{U}_\lambda(X) := \frac{1}{\lambda} \int_0^\lambda q_X(t) \, dt,$$

where $\lambda \in (0, 1]$. Note that $\mathcal{U}_1(\cdot) = E[\cdot]$, and we set $\mathcal{U}_0(X) := \essinf X$ by convention. As is well-known (e.g., Föllmer and Schied [14], Section 4.4), $\mathcal{U}_\lambda(\cdot)$ admits the dual representation

$$\mathcal{U}_\lambda(X) = \min_{Q \in \mathcal{Q}_\lambda} E_Q[X],$$

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with
\[ Q_\lambda := \left\{ Q \ll P \mid \frac{dQ}{dP} \leq \frac{1}{\lambda} \right\}, \quad \lambda \in (0, 1], \quad Q_0 := \{ Q \ll P \}. \tag{2.4} \]

Under law invariance, any superadditive comonotonic risk measure can be represented as a mixture of building blocks \(2.2\) as follows (Föllmer and Schied [14], Section 4.7, with the appropriate sign conventions):
\[ U_\mu (X) := \int_0^1 U_\lambda (X) \mu(d\lambda). \tag{2.5} \]

Here, \( \mu \) is a probability measure supported on \([0, 1] \). We refer to e.g., Ravanelli and Svindland [24] for the link between law invariance and probabilistic sophistication.

Now let
\[ V_\mu(X) := u^{-1} (U_\mu (u (X))), \tag{2.6} \]
with \( u : \mathbb{R} \to \mathbb{R} \) assumed to be continuously differentiable, strictly increasing and concave\(^2\), and \( U_\mu \) as defined above.

In the following, we derive the precise limiting behavior, and the corresponding convergence rate, of the risk premium induced by \(2.6\) associated with the average risk given by
\[ S_n/n, \tag{2.7} \]
where \( S_n = \sum_{i=1}^n X_i, \ n \in \mathbb{N} \). The Pareto optimality of the equal risk sharing contract \(2.7\) for economic agents with risk preferences of the form \(2.6\) follows from Knispel et al. [19]. We assume that the risks \( X_i \) are i.i.d. under the reference measure \( P \).

More precisely, we analyze
\[ \sqrt{n} \pi(v, S_n/n), \tag{2.8} \]
as \( n \to \infty \), where the risk premium \( \pi(v, S_n/n) \) is given by
\[ \pi(v, S_n/n) := v + E[X_1] - V_\mu(v + S_n/n), \tag{2.9} \]
with \( v \) the agent’s initial wealth level. Indeed, to understand \(2.9\), one first solves for \( \bar{\pi}(v, S_n/n) \) in the equivalent utility equation
\[ U_\mu(u(v + S_n/n)) = u(v - \bar{\pi}(v, S_n/n)), \tag{2.10} \]
yielding
\[ \bar{\pi}(v, S_n/n) = v - V_\mu(v + S_n/n), \tag{2.11} \]
and next considers the risk premium
\[ \pi(v, S_n/n) = E[S_n/n] + \bar{\pi}(v, S_n/n) = v + E[X_1] - V_\mu(v + S_n/n), \tag{2.12} \]
which agrees with \(2.9\). As seen from \(2.10\)–\(2.12\), the risk premium is such that the economic agent is indifferent between bearing the shared risk and paying the risk premium minus the expectation of the shared risk. Henceforth, we take \( v \equiv 0 \) without losing generality. (Indeed, one may re-define \( \bar{X}_i := v + X_i, \ i \in \mathbb{N}, \) and \( \bar{S}_n/n := v + S_n/n, \) respectively.) Thus, we analyze
\[ \sqrt{n} (E[X_1] - V_\mu(S_n/n)). \tag{2.13} \]

\(^2\)We note that concavity is not required for Theorem 2.1 below.
Note that $\mathcal{V}_\mu(S_n/n) \leq E[X_1]$ for all $n \in \mathbb{N}$ by Jensen’s inequality (and the dual representation and the monotonicity of $u^{-1}$). In general, $\mathcal{V}_\mu(S_n/n)$ may not converge to $E[X_1]$. We restrict attention to $\mu$ supported on $(0,1]$ and satisfying
\[
\int_0^1 \log \left( \frac{1}{\lambda} \right) \mu(d\lambda) < \infty; \tag{2.14}
\]
see Föllmer and Knispel [15]. In that case, $\mathcal{U}_\mu$ is continuous with respect to $L^1$-convergence.

We state the following theorem:

**Theorem 2.1** Assume that $u$ and $u^{-1}$ are continuously differentiable and strictly increasing. Furthermore, assume condition (2.14). Let $\sigma_{X_1} := \sqrt{\text{Var}[X_1]}$. Then,
\[
\lim_{n \to \infty} \sqrt{n} (E[X_1] - \mathcal{V}_\mu(S_n/n)) = -\sigma_{X_1} \mathcal{U}_\mu(Z) = -\sigma_{X_1} \int_0^1 \mathcal{U}_\lambda(Z) \mu(d\lambda) = \sigma_{X_1} \int_0^1 \frac{1}{\lambda} \phi(\Phi^{-1}(\lambda)) \mu(d\lambda), \tag{2.15}
\]
where $Z$ is a standard normal random variable under $P$ and $\phi$ and $\Phi$ are its probability density and cumulative distribution function (under $P$), respectively.

**Proof.** We may assume that $\sigma_{X_1} > 0$ since the assertion is trivial otherwise. We start by computing a Taylor expansion of $u(S_n/n)$ around $E[X_1]$ up to the first order:
\[
u(S_n/n) = u(E[X_1]) + u'(Y_n) \left( \frac{S_n}{n} - E[X_1] \right). \tag{2.16}
\]
Here, $Y_n$ is a random variable taking values between $\frac{S_n}{n}$ and $E[X_1]$.

Next, by translation invariance of $\mathcal{U}_\mu$ and invoking another first-order Taylor expansion (of $u^{-1}$),
\[
\mathcal{V}_\mu(S_n/n) = u^{-1} \left( \mathcal{U}_\mu \left( u \left( \frac{S_n}{n} \right) \right) \right) \\
= u^{-1} \left( \mathcal{U}_\mu \left( u(E[X_1]) + u'(Y_n) \left( \frac{S_n}{n} - E[X_1] \right) \right) \right) \\
= u^{-1} \left( u(E[X_1]) + \mathcal{U}_\mu \left( u'(Y_n) \left( \frac{S_n}{n} - E[X_1] \right) \right) \right) \\
= u^{-1} \circ u(E[X_1]) + (u^{-1})'(y_n) \mathcal{U}_\mu \left( u'(Y_n) \left( \frac{S_n}{n} - E[X_1] \right) \right) \\
= E[X_1] + (u^{-1})'(y_n) \left( \mathcal{U}_\mu \left( u'(Y_n) \left( \frac{S_n}{n} - E[X_1] \right) \right) \right),
\]
with $y_n$ taking values between $u(E[X_1])$ and $u(E[X_1]) + \mathcal{U}_\mu \left( u'(Y_n) \left( \frac{S_n}{n} - E[X_1] \right) \right)$.

Hence, by positive homogeneity of $\mathcal{U}_\mu$,
\[
\sqrt{n} (\mathcal{V}_\mu(S_n/n) - E[X_1]) = \sqrt{n} (u^{-1})'(y_n) \left( \mathcal{U}_\mu \left( u'(Y_n) \left( \frac{S_n}{n} - E[X_1] \right) \right) \right) \\
= \sigma_{X_1} (u^{-1})'(y_n) \left( \mathcal{U}_\mu \left( u'(Y_n) \left( \frac{S_n}{\sqrt{n}\sigma_{X_1}} - \frac{nE[X_1]}{\sqrt{n}\sigma_{X_1}} \right) \right) \right) \\
= \sigma_{X_1} (u^{-1})'(y_n) \left( \mathcal{U}_\mu \left( u'(Y_n) S_n^* \right) \right), \tag{2.17}
\]
with

$$S^*_n := \frac{S_n}{\sqrt{n\sigma_{X_1}}} - \frac{nE[X_1]}{\sqrt{n\sigma_{X_1}}}.$$  

In view of the proof of Theorem 3.1 in Föllmer and Knispel [15], under condition (2.14),

$$U_\mu (S^*_n) \to U_\mu (Z),$$  

(2.18)

with $Z$ a standard normal random variable. By superadditivity,

$$U_\mu (u'(Y_n)S^*_n) - U_\mu (u'(E[X_1])S^*_n) \geq U_\mu ((u'(Y_n) - u'(E[X_1]))S^*_n) \to 0,$$

since $Y_n \to E[X_1]$ almost surely and in $L^1$, and by continuity of $u'$ and $L^1$-continuity of $U_\mu$. Similarly,

$$U_\mu (u'(E[X_1])S^*_n) - U_\mu (u'(Y_n)S^*_n) \geq U_\mu ((u'(E[X_1]) - u'(Y_n))S^*_n) \to 0.$$  

Therefore, and by positive homogeneity and (2.18),

$$\lim_{n \to \infty} U_\mu (u'(Y_n)S^*_n) = \lim_{n \to \infty} u'(E[X_1])U_\mu (S^*_n) = u'(E[X_1])U_\mu (Z).$$

(2.19)

Furthermore,

$$\left( u^{-1} \right)' (y_n) = \frac{1}{u'(u^{-1} (y_n))} \to \frac{1}{u'(E[X_1])},$$

This, with the product rule for limits of functions, proves the stated result. \hfill \Box

**Remark 2.2** Note that $u(\cdot)$ does not appear on the right-hand side of (2.15), i.e., the limit is independent of the function $u(\cdot)$.

Of course, the convergence of the equal risk sharing contract to its expected value already follows from the Law of Large Numbers, with corresponding rate of convergence $n$. The limiting behavior of the risk premium is, however, much more delicate. Its convergence rate can be $n$, as in Knispel et al. [19] under the expected utility model (which occurs when $\mu$ has full mass on $\{1\}$, hence $U_\mu (\cdot) = E [\cdot]$), or only $n^{1/2}$, as typically in Theorem 2.1 of the present paper, depending upon the agent’s precise risk preferences. In the former case a Law of Large Numbers already suffices to establish the convergence proof, in the latter case a Central Limit Theorem is required.

As $U_\mu$ corresponds to a concave distortion risk measure (Föllmer and Schied [14], Section 4.6) up to a sign change, (2.6) corresponds to a certainty equivalent in the popular rank-dependent utility (RDU) model (Quiggin [23]). The RDU model encompasses Yaari’s [28] dual theory of choice under risk and the expected utility model as special cases (when $u(\cdot)$ is affine and $\mu$ has full mass on $\{1\}$, respectively).

Theorem 2.1 jointly with the results of Knispel et al. [19] reveal that the risk premium under the RDU model features a “first-order” term in the limit unless one considers the plain expected utility model without distortion in which case the limit features only a “second-order” term.

This dichotomy can be linked to the notions of first- and second-order risk aversion introduced by Segal and Spivak [26]; see also Lang [22] and Eeckhoudt and Laeven [12]. Indeed, the insightful analysis of Segal and Spivak [26] shows that the risk premia have distinct limiting
behavior for "small" risks under RDU and under expected utility: under the RDU model risk aversion is a first-order phenomenon, while under the expected utility model risk aversion is only a second-order phenomenon; see also Eeckhoudt and Laeven [12], Section 5.3.

More generally, consider

$$U_M(X) := \inf_{\mu \in \mathcal{M}} U_{\mu}(X),$$

(2.20)

where $\mathcal{M} \subset \mathcal{M}_1((0,1])$ with $\mathcal{M}_1((0,1])$ the class of probability measures on $(0,1]$. From Kusuoka [20] (see also Dana [8] and Frittelli and Rosazza Gianin [16]) we know that, upon a sign change, law-invariant coherent risk measures are of the form (2.20).

Furthermore, let

$$\mathcal{V}_M(X) := u^{-1}(U_M(u(X))),$$

(2.21)

with $u(\cdot)$ and $U_M$ as above. We now analyze

$$\sqrt{n} (E[X_1] - \mathcal{V}_M(S_n/n)),$$

(2.22)

as $n \to \infty$. We assume that, for $M$, $\sup_{\mu \in M} \int_0^1 \log \left( \frac{1}{\lambda} \right) \mu(d\lambda) < \infty.$

(2.23)

We state the following theorem, our main result:

**Theorem 2.3** Assume that $u$ and $u^{-1}$ are continuously differentiable and strictly increasing. Furthermore, assume condition (2.23). Then,

$$\lim_{n \to \infty} \sqrt{n} (E[X_1] - \mathcal{V}_M(S_n/n)) = -\sigma X_1 U_M(Z) = -\sigma X_1 \inf_{\mu \in \mathcal{M}} \int_0^1 U_\mu(Z) \mu(d\lambda)$$

$$= \sigma X_1 \sup_{\mu \in \mathcal{M}} \int_0^1 \frac{1}{\lambda} \phi \left( \Phi^{-1}(\lambda) \right) \mu(d\lambda),$$

(2.24)

where $Z$ is a standard normal random variable under $P$ and $\phi$ and $\Phi$ are its probability density and cumulative distribution function (under $P$), respectively.

**Proof.** Since $U_M(X) = \inf_{\mu \in \mathcal{M}} U_\mu(X)$ is translation invariant and positively homogeneous, by invoking two first-order Taylor expansions, we obtain

$$\sqrt{n} (\mathcal{V}_M(S_n/n) - E[X_1]) = \sigma X_1 \left( u^{-1} \right)'(y_n) \left( U_M(u'(Y_n) S_n^*) \right),$$

with $Y_n$ and $S_n^*$ defined as in the proof of Theorem 2.1 and with $y_n$ defined analogously (mutatis mutandis).

It remains to verify the uniform convergence

$$\lim_{n \to \infty} U_M(u'(Y_n) S_n^*) = \lim_{n \to \infty} u'(E[X_1]) U_M(S_n^*) = u'(E[X_1]) U_M(Z),$$

(2.25)

where $Z$ is a standard normally distributed random variable. Indeed, condition (2.23) ensures that

$$\lim_{n \to \infty} U_M(S_n^*) = U_M(Z)$$

(cf. the proof of Theorem 3.2 in Föllmer and Knispel [15]), hence, in view of superadditivity, similar arguments as in the proof of Theorem 2.1 yield (2.25).
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