HYPERCURRENTS

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Abstract. We introduce the notion of a protocol, which consists of a space whose points are labeled by real numbers indexed by the set of cells of a fixed CW complex in prescribed degrees, where the labels are required to vary continuously. If the space is a one-dimensional manifold, then a protocol determines a continuous time Markov chain.

When a homological gap condition is present, we associate to each protocol a ‘characteristic’ cohomology class which we call the hypercurrent. The hypercurrent comes in two flavors: one algebraic topological and the other analytical. For generic protocols we show that the analytical hypercurrent tends to the topological hypercurrent in the low temperature limit. We also exhibit examples of protocols having nontrivial hypercurrent.

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1. Introduction

1.1. Background. This paper is a result of our ongoing investigation of the stochastic motion of cellular cycles in a finite CW complex [1], [2], [3]. In [1], we introduced the notion of a driving protocol, which is a one-parameter family of potentials labeling the vertices and edges of a connected finite topological graph \( X \). We explained how a driving protocol determines a continuous time Markov chain whose state diagram is the double of \( X \). Associated with such a Markov chain, we defined an average current which is a 1-dimensional chain with real coefficients that measures of the net flow of probability across the edges of \( X \). When the protocol is periodic of period \( \tau \), we showed in the adiabatic slow driving limit \( \tau \to \infty \) that the resulting current tends to a 1-dimensional real homology class. After taking a second limit, the ‘low temperature limit,’ we showed that, for generic parameters, the average current quantizes to an integer homology class. Furthermore, we showed that the integer homology class admits an algebraic topological description.

In [3] we allowed \( X \) to be an arbitrary finite CW complex of dimension \( d \). In this case a driving protocol consists of a one-parameter family of potentials labeling the cells of \( X \) in the contiguous dimensions \( d - 1 \) and \( d \). When the driving protocol is periodic, the average current in the adiabatic limit was defined as a real \( d \)-dimensional homology class. In this generality, we extended the principal results of [1]. In particular, we showed that for generic parameters the average current in the low temperature limit fractionally quantizes to a homology class with coefficients in \( \mathbb{Z}[\frac{1}{\delta}] \), for some positive integer \( \delta \) that is a combinatorial invariant of \( X \).

In the current paper we consider a further generalization: we introduce driving protocols in higher dimensions. We are tempted to conjecture that an \( n \)-dimensional protocol determines an ‘\( n \)-dimensional Markov process’ with time substituted by an \( n \)-manifold. However, we are not in a position to state a conjecture since we do not know what an ‘\( n \)-dimensional Markov process’ should be. Nevertheless, in the presence of a homological gap condition, we are able to associate an observable to any higher dimensional protocol, called the hypercurrent, which is a kind of secondary current.

We will exhibit several different descriptions of the hypercurrent, some algebraic topological as well as an analytical one defined using differential forms. We axiomatically characterize the hypercurrent and exhibit some examples. Lastly, we prove a quantization result relating
the analytical definition to the topological one; this a kind of index theorem.

1.2. Motivation. As this paper is written with algebraic topologists in mind, it may be prudent to outline how continuous time Markov chains relate to driving protocols.

1.2.1. Markov chains. A continuous time Markov chain consists of:

- a directed finite simple graph\(^1\)

\[ \Gamma = (\Gamma_0, \Gamma_1), \]

called a state diagram;
- an assignment of a continuous function

\[ k_{ij} : \mathbb{R} \to [0, \infty), \]

to each edge \((i, j) \in \Gamma_1 \subset \Gamma_0 \times \Gamma_0\).

The function \(k_{ij}(t)\) is called the transition rate of the edge \((i, j)\). It is to be interpreted as the instantaneous rate of change of probability in jumping from state \(i\) to state \(j\) at time \(t\).

The rates define a square matrix \(H\) with entries are

\[
H_{ij} = \begin{cases} 
  k_{ij}, & (i, j) \in \Gamma_1; \\
  0, & (i, j) \notin \Gamma_1, i \neq j; \\
  -\sum_{\ell \neq j} k_{\ell j}, & i = j. 
\end{cases}
\]

For any initial probability distribution \(p_0 : \Gamma_0 \to \mathbb{R}\), the unique solution \(p(t)\) to the initial value problem

\[
\dot{p} = Hp, \quad p(0) = p_0
\]

describes the time evolution of probability. The equation (1) is called the Kolmogorov equation or master equation of the process. In the case when \(k_{ij} = 1\) for all edges, \(H\) is the graph Laplacian and (1) is the heat (diffusion) equation.

1.2.2. Driving protocols. A convenient way to obtain a Markov chain is to fix a finite, connected, simple undirected graph

\[ X = (X_0, X_1). \]

We denote edges in this case by (unordered) pairs of vertices \(\{i, j\}\). Consider the real vector space \(\mathcal{M}_X\) consisting of pairs \((E, W)\) in which \(E : X_0 \to \mathbb{R}\) and \(W : X_1 \to \mathbb{R}\) are functions. Thus \(E\) and \(W\) equip the vertices and edges of the graph with real number weights. For a vertex

\(^1\)The condition that the graph be finite and simple is only made to keep the exposition non-technical.
i, we let $E_i$ denote the value of $E$ at a vertex $i$, and we let $W_{ij}$ denote the value of $W$ at an edge $\{i, j\}$. Note that $W_{ij} = W_{ji}$.

As in [1], a driving protocol is a continuous map

$$\gamma: \mathbb{R} \to \mathcal{M}_X,$$

i.e., a one-parameter family of weights $(E(t), W(t))$.

A driving protocol $\gamma$ determines a continuous time Markov chain whose state diagram is the “double” of $X$. The latter is the directed graph $\Gamma = (\Gamma_0, \Gamma_1)$ in which $\Gamma_0 = X_0$ and $\Gamma_1$ consists of the ordered pairs $(i, j) \in X_0 \times X_0$ such that $\{i, j\} \in X_1$. The transition rate at $(i, j)$ is defined by

$$(2) \quad k_{ij}(t) := e^{E_i(t) - W_{ij}(t)}.$$ 

We call a Markov chain Arrhenius if its rates can be written in the form (2), i.e., the Markov chain arises from a driving protocol.

Conversely, a Markov chain $(\Gamma, \{k_{ij}\})$ is Arrhenius if

- $\Gamma$ is the double of an undirected finite simple graph $X$;
- $k_{ij}(t) > 0$ for all $(i, j) \in \Gamma_1$ and all $t \in \mathbb{R}$;
- for each $i \in \Gamma_0$ there there is a continuous function $\kappa_i(t) > 0$ such that

$$(3) \quad \kappa_i k_{ij} = \kappa_j k_{ji}$$

for $(i, j) \in \Gamma_1$.

To verify this, set $E_i := -\ln \kappa_i$ and $W_{ij} := -\ln(\kappa_i k_{ij})$.

1.3. Generalized protocols. We now turn to the main object of this study. Let $X$ be a finite CW complex and let $X_k$ denote its set of $k$-cells. For natural numbers $p \leq q$, let $\mathcal{M}_{p,q}(X)$ be the vector space of functions (weights)

$$W_\bullet: \prod_{k=p}^q X_k \to \mathbb{R}.$$ 

We let $W_j: X_j \to \mathbb{R}$ denote the restriction of $W_\bullet$ to $X_j$.

**Definition 1.1.** Let $\Sigma$ be a topological space. A $\Sigma$-protocol is a continuous map

$$\gamma: \Sigma \to \mathcal{M}_{p,q}(X).$$

*Remark 1.2.* Hereafter, we refer to these as protocols. When $X$ is understood, we abbreviate notation $\mathcal{M}_{p,q} = \mathcal{M}_{p,q}(X)$. There is an evident projection

$$\mathcal{M}_{p,q} \to \mathcal{M}_{r,s}$$

whenever $p \leq r \leq s \leq q$. Given a protocol $\gamma$ and a map of spaces $f: \Sigma' \to \Sigma$, the composition $\gamma' := \gamma \circ f$ defines a protocol.
Given \( \gamma \) and \( b \in \Sigma \), we write \( W_j(b) : X_j \to \mathbb{R} \) for the \( j \)-th component of \( \gamma(b) \), i.e.,
\[
\gamma(b) = W_\bullet(b) := (W_p(b), W_{p+1}(b), \ldots, W_q(b))
\]

### 1.4. Homological hypercurrents.

As \( \mathcal{M}_{p,q} \) is contractible, there is at this point no algebraic topology that can be extracted from a protocol \( \Sigma \to \mathcal{M}_{p,q} \). Algebraic topology enters the picture by imposing restrictions on the kinds of families that are allowed.\(^2\) The first type of restriction that we will consider are “good” protocols:

**Definition 1.3 (Good Protocols).** A protocol \( \Sigma \to \mathcal{M}_{p,q} \) is good if for each \( b \in \Sigma \), there exists an integer \( j = j(b) \) satisfying \( p \leq j \leq q \) such that the weight \( W_j(b) : X_j \to \mathbb{R} \) is one-to-one.\(^3\)

Let
\[
\tilde{\mathcal{M}}_{p,q} \subset \mathcal{M}_{p,q}
\]
be the (open, dense) subspace of \( \mathcal{M}_{p,q} \) consisting of those weights \( W_\bullet \) such that \( W_j : X_j \to \mathbb{R} \) is one-to-one for some \( p \leq j \leq q \). Then a good protocol is nothing more than a map \( \Sigma \to \tilde{\mathcal{M}}_{p,q} \).

Collectively, the good protocols are objects of the category
\[
\text{Top}_{/\tilde{\mathcal{M}}_{p,q}},
\]
i.e., the category of spaces over \( \tilde{\mathcal{M}}_{p,q} \). When the structure map is understood we drop it from the notation: an object \( \Sigma \to \tilde{\mathcal{M}}_{p,q} \) is then referred to as \( \Sigma \).

**Definition 1.4 (Gap Condition).** A gap in \( X \) is a pair of non-negative integers \( p, q \) such that \( p \leq q \) and the Betti numbers \( \beta_j(X) \) are trivial for \( p < j < q \). Denote the gap by the closed interval \([p, q]\).

**Remark 1.5.** For intervals of the form \([p, p + 1]\), the gap condition is automatically satisfied.

Let \( \mathbb{F} \) be a field of characteristic zero. In what follows, chain complexes and their homology will always be taken with coefficients in \( \mathbb{F} \), even though almost always the coefficients are suppressed from the notation.

\(^2\)This is akin to what one does with the space of smooth functions \( C^\infty(M, \mathbb{R}) \) on a compact smooth manifold \( M \). The space \( C^\infty(M, \mathbb{R}) \) is contractible. Algebraic topology materializes by restricting to a generic subspace of functions such as the space of Morse functions.

\(^3\)In §8, we relax the condition by considering so-called robust protocols.
Given a gap \([p, q]\) and a good protocol \(\gamma : \Sigma \to \tilde{\mathcal{M}}_{p,q}\), we will construct a linear transformation of vector spaces

\[(4) \quad J_{p,q} : H_{q-p}(\Sigma) \otimes H_p(X) \to H_q(X)\]
called the hypercurrent homomorphism (we reiterate that homology is taken with coefficients in \(F\)). The following result enumerates some of its basic properties.

**Theorem A.** The hypercurrent homomorphism (4) exhibits the following properties:

1. (Functoriality). \(J_{p,q}\) is contravariantly functorial in \(\Sigma\);
2. (Homotopy Invariance). \(J_{p,q}\) is homotopy invariant: if there is homotopy from \(\gamma\) to \(\gamma'\) in the space of good protocols, then the associated hypercurrent homomorphisms coincide.
3. (Initial Condition). When \(q = p\), the homomorphism
   \[J_{p,p} : H_0(\Sigma) \otimes H_p(X) \to H_p(X)\]
is the identity.
4. (Non-triviality). When \(q > p\), there exists a finite connected CW complex \(X\) with gap \([p, q]\) and a good protocol \(\Sigma \to \tilde{\mathcal{M}}_{p,q}(X)\) such that \(J_{p,q}\) is nontrivial.

**Remarks 1.6.** (1). The hypercurrent homomorphism is also natural in the field \(F\) by extension of scalars. In particular, it suffices to consider the case when \(F = \mathbb{Q}\) is the rational numbers.

(2). When \(\dim X = 1\), the homomorphism
   \[J_{0,1} : H_1(\Sigma) \otimes H_0(X) \to H_1(X)\]
coincides with the average current of [1] evaluated at the generator of \(H_0(X)\) given by any zero cell.

(3). When \(\dim X = d > 1\), the homomorphism \(J_{d-1,d}\) coincides with the average current homomorphism of [3] evaluated at the homology class of the higher Boltzmann cycle \([\rho^B]\) \(\in H_p(X)\) (cf. [2, defn. 1.12]).

(4). Set \(H^q_\Sigma(X) := H^p(X; H_q(X))\). Since \(J_{p,q}\) is functorial in \(\Sigma\), we tend to view \(J_{p,q}\), or equivalently, its adjoint
   \[J_{p,q}^* \in H^{q-p}(\Sigma; H^p_q(X))\],
as a kind of characteristic class for spaces \(\Sigma\) equipped with a good protocol: the operation \(\Sigma \mapsto H^{q-p}(\Sigma; H^p_q(X))\) defines a presheaf on the category of good protocols and the hypercurrent homomorphism defines a global section of this presheaf.
The following hints that the hypercurrent homomorphism measures in some way the difference between the non-triviality of the boundary operator in the cellular chain complex of $X$ and the triviality of the homology in the gap degrees.

**Addendum B.** Assume $X$ has a gap $[p,q]$. If there is a $j \in [p,q-1]$ such that

- the cellular boundary operator $\partial: C_{j+1}(X) \to C_j(X)$ is trivial, or
- there is a $j \in [p,q]$ such that $|X_j| \leq 1$,

then the hypercurrent homomorphism $\mathcal{J}_{p,q}$ is trivial.

1.5. **The hypercurrent chain map**. Let $C_*(X)$ be the cellular chain complex over $\mathbb{F}$ of the CW complex $X$. Then $C_j(X)$ is the vector space over $\mathbb{F}$ with basis $X_j$. Let $\bar{C}_* = \bar{C}_*(X)$ be the chain complex given by

$$\bar{C}_j := C_{j+p}(X^{(q)}, X^{(p-1)})$$

i.e., the cellular chain complex of the pair $(X^{(q)}, X^{(p-1)})$ shifted by $p$. There is an evident surjection of vector spaces

$$\text{hom}(H_0(\bar{C}_*), H_{q-p}(\bar{C}_*)) \to \text{hom}(H_p(X), H_q(X))$$

(cf. Remark 2.2 below). The homological hypercurrent homomorphism will be induced by a chain map

$$\mathcal{J}: I_*(\Sigma) \otimes \bar{C}_* \to \bar{C}_*,$$

where $I_*(\Sigma)$ is the chain complex over $\mathbb{F}$ freely generated by the set of “small” singular simplices $\sigma: \Delta^k \to \Sigma$. Here “small” means that there is a $j \in [p,q]$ such that $W_j(b): X_j \to \mathbb{R}$ is one-to-one for all $b \in \sigma(\Delta^k)$. A subdivision argument shows that inclusion of $I_*(\Sigma)$ into the full total singular complex $S_*(\Sigma)$ is a quasi-isomorphism.

**Theorem C.** The hypercurrent chain map (5) exists and is well-defined up to contractible choice.

Theorem C is proved using Quillen model category machinery applied to a certain functor category. With the intent of clarifying the way in which the current work relates to [1], we discuss two cases: (i) dimension one and (ii) the general case.

1.6. **The graph case.** Consider the case $[p,q] = [0,1]$, where $X = (X_0, X_1)$ is a graph. In this instance, the chain map we seek arises from a map of spaces. We will sketch below a form of the construction using the methods of [1].
If $X$ is a graph, then
\[ \Sigma = \Sigma(0) \cup \Sigma(1), \]
where $W_j(b): X_j \to \mathbb{R}$ is one-to-one for $b \in \Sigma(j)$. If $b \in \Sigma(0)$, then $W_0(b): X_0 \to \mathbb{R}$ has a unique minimum $L_b \in X_0$, which is a zero cell of $X$. If $b \in \Sigma(1)$, then the greedy algorithm applied to $W_1(b): X_1 \to \mathbb{R}$ produces a minimal spanning tree $T_b$. Thus to every point of $\Sigma$ the assignment
\[ b \mapsto \begin{cases} L_b & \text{if } b \in \Sigma(0); \\ T_b & \text{otherwise.} \end{cases} \]
codifies a quasifibration $N \to \Sigma$ with contractible fibers, where $N$ is the space of pairs $(b, x)$ with $b \in \Sigma$ and where $x$ is either $L_b$ or a point of $T_b$. It follows that the map $N \to \Sigma$ is a weak homotopy equivalence. Second factor projection defines a map $N \to X$. Consequently, modulo technical details, we have defined a weak map of spaces
\[ \Sigma \hookrightarrow N \to X \]
which induces a space level version of the desired chain map.

1.7. The general case. When $q - p > 1$, the above approach doesn’t generalize. Let $I_{\Sigma}$ be the poset of small singular simplices partially ordered with respect to facial inclusion. The map (5) is defined by a local construction in sense that it is the induced map of homotopy colimits associated with functors appearing in a canonical chain of natural transformations of chain complex valued functors $I_{\Sigma} \to \mathcal{C}$. The chain of natural transformations has the form
\[ \chi \to H_*(\tau) \hookrightarrow \tau \to \chi, \]
in which
- the functor $\chi$ is the constant functor with value $\bar{C}_*$;
- the functor $\tau$, called the tree functor, is defined in terms of higher dimensional spanning tree and co-tree data in the CW complex $X$ (cf. §4 and Definition 5.4);
- the functor $H_*(\tau)$ is given by taking the homology of $\tau$ objectwise, where the homology of a chain complex is considered as a chain complex with trivial boundary operator.
- The natural transformation $\tau \to \chi$ is induced by objectwise inclusion, and the natural transformation $\tau \to H_*(\tau)$ exists as the functor $\tau$ is objectwise acyclic in positive degrees.
- The natural transformation $\chi \to H_*(\tau)$ is determined by specifying its value in degree 0, the latter which we take to be the projection onto 0-dimensional homology.
The above leads to a description of $\mathcal{J}$ as arising from a map in the homotopy category of functors. To obtain an actual map in the functor category itself, one has to work a bit harder, appealing to model category machinery.

1.8. Examples. For $q \geq 1$, let $X^q = S^q$ be the sphere of dimension $q$ equipped with the CW composition in which there are two $j$-cells $e^j_-$ in each dimension $j \leq q$ given by the upper and lower hemispheres of $S^j \subset S^q$. Let $\Sigma := \mathcal{M}_{0,q}(X^q)$ be the (universal) good protocol for $X^q$. By Proposition 6.1 below, there is a homotopy equivalence $\Sigma \simeq S^q$.

Let $Y^q$ be the CW complex obtained from $X^{q-1}$ by attaching two additional $q$-cells. One of the $q$-cells is attached using the identity map and the other is attached using the constant map to a 0-cell. Then $\tilde{\mathcal{M}}(Y^q) = \tilde{\mathcal{M}}(X^q)$. In particular, $\Sigma$ is also a good protocol for $Y^q$. Furthermore, $Y^q$ is homotopy equivalent to $X^q$ (in fact, $Y^q$ is homeomorphic to the wedge $D^q \vee S^q$). Consequently, $X^q$ and $Y^q$ are a pair of homotopy equivalent CW complexes having the same number of cells in each dimension. The following result shows that $\mathcal{J}$ distinguishes the cell structures.

**Theorem D.** (a). The hypercurrent homomorphism $J_{0,q} : H_q(\Sigma) \otimes H_0(X^q) \to H_q(X^q)$ is an isomorphism of vector spaces.
(b). The hypercurrent homomorphism $J_{0,q} : H_q(\Sigma) \otimes H_0(Y^q) \to H_q(Y^q)$ is trivial.

**Remark 1.7.** The case $q = 1$ was discussed in [1, ex. 7.9].

1.9. Analytical hypercurrents. In what follows, we work over the field $\mathbb{F} = \mathbb{R}$ of real numbers. The space $\Sigma$ will be a compact smooth manifold, possibly with boundary and $X$ will be a finite connected CW complex with gap $[p, q]$.

We will describe a map

$$J^{\text{an}} : \mathcal{M}_{p,q}^\Sigma \to \Omega^*(\Sigma; \text{end}(\bar{\mathcal{C}})),$$

$$\gamma \mapsto J^{\text{an}}(\gamma)$$

where the source of (7) is the the function space $C^\infty(\Sigma, \mathcal{M}_{p,q})$, i.e., the space of smooth protocols $\gamma : \Sigma \to \mathcal{M}_{p,q}$ (not necessarily good). The target of (7) is the total complex of the de Rham complex of $\Sigma$ with coefficients in the chain complex

$$\text{end}(\bar{\mathcal{C}}), \partial$$
of endomorphisms of $\bar{C}$, with boundary operator $\delbar$ (cf. §2).

The map $\theta$ is natural with respect to smooth maps $\Sigma \to \Sigma'$. The form $\mathcal{J}^\text{an}(\gamma)$ is of total degree zero. We will characterize it by three axioms. The characterization requires some preparation.

**Notation 1.8.** When $\gamma \in \mathcal{M}^\Sigma_{p,q}$ is understood and there is no ambiguity, we drop the argument and write $\mathcal{J}^\text{an}$ in place of $\mathcal{J}^\text{an}(\gamma)$ to avoid notational clutter.

**Definition 1.9** (De Rham Complex). If $V$ is a finite dimensional real vector space, let

$$\Omega^*(\Sigma; V) = \Omega^*(\Sigma) \otimes \mathbb{R} V,$$

where $\Omega^*(\Sigma)$ is the de Rham complex of smooth differential forms on $\Sigma$. Note that the operator $d \otimes \text{id}_V$, makes $\Omega^*(\Sigma; V)$ into a cochain complex, where $d$ is the usual exterior derivative on differential forms.

Note that $\Omega^k(\Sigma; V)$ is identified with the space of smooth sections of the bundle over $\Sigma$ whose fiber at $b \in \Sigma$ is the space of linear maps $\Lambda^k T_b \Sigma \to V$.

**Notation 1.10.** If $\phi: V_* \to V_*$ is a graded map, then it induces in the evident way a map $\Omega^*(\Sigma; V_*) \to \Omega^*(\Sigma; V_*)$, which we also denote by $\phi$.

**Definition 1.11** (Modified Inner Product). For $W_* \in \mathcal{M}_{p,q}$, the modified inner product on $\bar{C}_k$ is defined on basis elements $x, y \in X_{k+p}$ by

$$\langle x, y \rangle_W := \epsilon_k^{W_{k+p}(x)} \delta_{xy}$$

where $\delta_{xy}$ is the Kronecker delta of $x$ and $y$. Similarly, if $\gamma: \Sigma \to \mathcal{M}_{p,q}$ is a protocol, then we obtain a family of modified inner products on $\bar{C}_k$ parametrized by $\Sigma$.

**Definition 1.12.** Let

$$\bar{D}_{W_*} \subset \bar{C}_*$$

be the graded vector subspace defined as follows:

- If $j \notin [0, q-p]$, then we take $\bar{D}_{W_j}$ to be trivial.
- If $j \in (0, q-p]$, then $\bar{D}_{W_j}$ is defined to be the orthogonal complement to the subspace of $j$-cycles $\bar{Z}_j$ in the modified inner product.
If \( j = 0 \), then \( \bar{D}_{W_0} \) is defined to be the orthogonal complement to the subspace of 0-boundaries \( B_0 \) in the modified inner product.

We return to the problem of characterizing \( J^{an} \). By the dimensional constraints on \( \bar{C} \), the form is a finite direct sum

\[
J^{an} = \sum_{\ell=0}^{q-p} J^{an}_\ell,
\]

in which \( J^{an}_\ell \in \Omega^\ell(\Sigma; \text{end}(\bar{C})_\ell) \).

The forms \( J^{an}_\ell \) are subject to the following axioms:

(A1) (Continuity Equation). For \( \ell \geq 1 \), the identity

\[
\bar{\partial}J^{an}_\ell = dJ^{an}_{\ell-1}
\]

is satisfied, where \( \bar{\partial} \) is given by (8).

(A2) (Orthogonality Condition). The form

\[
J^{an}_\ell \in \Omega^\ell(\Sigma; \text{hom}(\bar{C}, \bar{D}_W)_\ell)
\]

lies in the subspace \( \Omega^\ell(\Sigma; \text{hom}(\bar{C}, \bar{D}_W)_\ell) \).

Axiom A1 implies that the image of \( J^{an}_0 : \Sigma \to \text{end}(\bar{C})_0 \) lies in subspace of chain maps \( \bar{C} \to \bar{C} \). By taking 0-th homology, we infer that \( J^{an}_0 \) induces a 0-form

\[
\Sigma \to \text{hom}(H_0(\bar{C}), H_0(\bar{C})).
\]

(A3) (Initial Value). The form (9) is constant with value the identity map.

**Theorem E.** There exists precisely one map (7) satisfying axioms A1-A3.

1.10. **Quantization.** Our conventions for our last main result are as follows: \( \mathbb{F} = \mathbb{R} \) will be the real numbers and \( \Sigma \) will be a smooth manifold. The protocol \( \gamma : \Sigma \to M_{p,q} \) will be a smooth map. We let \( I_\gamma(\Sigma) \) denote the chain complex of small smooth singular simplices in \( \Sigma \). For any real number \( \beta > 0 \), one has an associated protocol \( \beta \gamma \) defined by pointwise scalar multiplication in \( M_{p,q} \).

Let \( R_* \subset I_\gamma(\Sigma) \) be any chain subcomplex generated by small smooth singular simplices of dimension \( \leq q - p \). We define

\[
J^{an}(\beta \gamma)_* : R_* \to \text{end}(\bar{C})
\]

by the formula

\[
J^{an}(\beta \gamma)_* (\Delta^j \xrightarrow{q} \Sigma) := \int_{\Delta^j} \sigma^*(J^{an}(\beta \gamma)).
\]

The parameter \( \beta \) is to be regarded as inverse temperature.
In §9 we will show that $\mathcal{J}^\text{an}(\beta\gamma)_2$ is a chain map. The following result pins down the relationship between the analytical and topological hypercurrent maps (compare [1, thm. A] and [2, thm. C]).

**Theorem F (Quantization).** In the low temperature limit $\beta \to \infty$, the chain homotopy class of $\mathcal{J}^\text{an}(\beta\gamma)_2$ converges to the chain homotopy class of the restriction to $R_*$ of the hypercurrent chain map $\mathcal{J}$.

**Remark 1.13.** For a more detailed statement, see Theorem 10.6. Theorem F is to be viewed as a ‘fractional’ quantization result because $\mathcal{J}$ is defined over the rational numbers (and extended by scalars to the reals) whereas the analytical current is not. It can also be thought of as an index theorem relating an analytically defined invariant to a topologically defined one.

1.11. **Summary.** Suppose that $\gamma: \Sigma \to \mathcal{M}_{p,q}(X)$ is a smooth protocol, with $\Sigma$ a closed Riemannian manifold of dimension $n := q - p$. The paper [1] deals with the case $p = 0$ and $n = 1$. We showed there that $\gamma$ determines a continuous time Markov chain whose state diagram is the double of the 1-skeleton of $X$. The evolution of the system in this case is governed by the Kolmogorov equation which is a first order ODE acting on distributions (0-chains). The average current was an observable associated with the adiabatic limit of the system (the adiabatic limit is given by rescaling the metric on $\Sigma$ so that the total length tends to $\infty$). In [3] we generalized [1] to $p > 0$ with $q = p + 1$. In this case the double of the 1-skeleton of $X$ is replaced by the cycle incidence graph of the $(p + 1)$-skeleton of $X$. The latter is the (possibly infinite) graph whose vertices are the cellular $p$-cycles of $X$ in a given homology class in which an edge is determined by an elementary homology between cycles, where an elementary homology is given by taking a suitable scalar multiple of a $(p + 1)$-cell. In this case the process is a continuous time biased random walk on the cycle incidence graph. To avoid the technical problems of working with the Kolmogorov equation acting on infinite dimensional Hilbert spaces, we worked with a dynamical equation which acts on the finite dimensional vector space of cellular $p$-cycles in $X$. By contrast, when $q - p > 1$ we do not know of any physical or dynamical system which gives rise to the hypercurrent as an observable. When $p = 0$ and $q - p > 1$, then we suspect that the hypercurrent is a measurement on an as yet to be defined statistical field theory based on $(q - p)$-branes. The above discussion is summarised in the following table:
| $p$ | $q - p$ | Process | States | State Diagram | Observable |
|-----|--------|---------|--------|---------------|------------|
| 0   | 1      | biased random walk | vertices of $X$ | double of $X^{(1)}$ | average current |
| $>0$ | 1      | biased random walk | $p$-cycles of $X$ | cycle-incidence graph of $X^{(q-p)}$ | average current |
| 0   | $>1$   | $(q-p)$-brane theory? | ? | ? | hypercurrent |

**Outline.** In Section 2 we review the standard properties of chain complexes over a field of characteristic zero. Section 3 introduces the projective model structure on the category of chain complex valued functors from a suitable small category. We also prove a kind of acyclic models result which will enable us to construct the hypercurrent map. Section 4 reviews spanning trees and spanning co-trees in higher dimensions; this material is lifted from [4] and [2]. In section 5 we construct the hypercurrent chain map. In section 6, we show that the space of good weights has the homotopy type of a wedge of spheres. Section 7 contains proofs of Theorems A and D. In section 8 we determine the homotopy type of the space of robust weights and extend the hypercurrent homomorphism to robust protocols. In Section 9 we define and axiomatically characterize the analytical hypercurrent. In section 10 we provide a proof of the Quantization Theorem. Lastly, in section 11 we construct space level hypercurrent maps in certain cases.

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## 2. Preliminaries

### 2.1. Chain complexes.** Let $\mathbb{F}$ be a field of characteristic zero. Let $C$ be a chain complex over $\mathbb{F}$ with boundary operator $\partial$. In what follows all chain complexes are unbounded in the sense that they indexed over $\mathbb{Z}$. Recall that a quasi-isomorphism $C \rightarrow D$ is a chain map which induces an isomorphism on homology. A morphism $C \rightarrow D$ is a quasi-isomorphism if and only if it is a chain homotopy equivalence, since we are working over a field. Additionally for any chain complex $C$, there is a quasi-isomorphism

$$(C, \partial) \rightarrow (H_*(C), 0)$$

which induces the identity map in homology, where the target is the homology of $C$ with trivial boundary operator. The quasi-isomorphism is constructed as follows: equip $C$ with an inner product in each degree and decompose $C_k$ orthogonally as

$$B_k \oplus H_k \oplus B^*_k$$

where $B_k$ is the vector space of $k$-boundaries and $H_k$ is the orthogonal complement of $B_k$ in the vector space of $k$-cycles $Z_k$. With respect
to this decomposition, the boundary operator is trivial on the first two summands and gives an isomorphism $B_k^* \cong B_{k-1}$. The desired quasi-isomorphism is then defined by sending a vector $(x \oplus y \oplus z)$ to $[x] \in H_k(C)$, where $[x]$ denotes the homology class of the cycle $x \in \mathcal{H}_k$.

Suppose $f : C \to D$ is a chain map. If $f$ is null homotopic, then $f$ induces the trivial map in homology. The converse is also true as is easily seen by replacing $C$ and $D$ with the chain map $f^* : (H_*(C), 0) \to (H_*(D), 0)$.

Let $\text{hom}(C, D)$ be the internal hom-complex in which

$$\text{hom}(C, D)_n = \prod_{i \in \mathbb{Z}} \text{hom}(C_i, D_{n+i}),$$

with boundary operator

$$\partial(f) = \partial_D f - (-1)^n f \partial_C,$$

$f \in \text{hom}(C, D)_n$.

**Notation 2.1.** When $C = D$ we write

$$\text{end}(C) = \text{hom}(C, C).$$

**2.2. Model structure.** Let $\mathcal{C}$ denote the category of unbounded chain complexes over $\mathbb{F}$ equipped with the projective model structure [5, §2.3]. A weak equivalence in $\mathcal{C}$ is a quasi-isomorphism. A morphism is a fibration if and only if it is surjective in every degree. Every object of $\mathcal{C}$ is fibrant.

A morphism is a cofibration if it has the left lifting property with respect to the trivial (acyclic) fibrations. An equivalent characterization of cofibrations is given by attaching cells: for $j \in \mathbb{Z}$, we let $D^j_*$ be the chain complex which is given by $\mathbb{F}$ in degrees $j$ and $j - 1$ and trivial otherwise, where the boundary operator in degree $j$ is given by the identity map. Let $S^{j-1}_*$ be the chain complex which is $k$ in degree $j - 1$ and trivial otherwise; one has an inclusion $S^{j-1}_* \to D^j_*$. Then the cofibrations of $\mathcal{C}$ are generated by these maps in the following sense: if $C_*$ is a chain complex and $f : S^{j-1}_* \to C_*$ is a chain map, one can form the algebraic mapping cone

$$M(f)_* := C_* \oplus f D^j_*$$

which is defined as the pushout of the diagram $C_* \leftarrow S^{j-1}_* \subset D^j_*$. With respect to the inclusion $C_* \to M(f)_*$, one says that $M(f)_*$ is obtained from $C_*$ by attaching a $j$-cell. A cofibration of $\mathcal{C}$ is a map $C_* \to D_*$ which is the result of iteratively attaching (possibly transfinitely many) cells, or is a retract thereof.

Every bounded below object is cofibrant [5, lem. 2.3.6]. A morphism is a cofibration if and only if it is degree-wise injective with cofibrant
cokernel [5, prop. 2.3.9]. In particular, a degreewise injective morphism of bounded below objects is a cofibration.

2.3. **Truncation/shift.** Recall from the introduction that $\bar{C}_*$ is the chain complex given by $\bar{C}_j = C_{j+p}(X^{(q)}, X^{(p-1)})$, in other words

$$\bar{C}_* = C_*((X^{(q)}, X^{(p-1)}))[p]$$

is the $p$-fold desuspension of the cellular chain complex of the CW pair $(X^{(q)}, X^{(p-1)})$.

**Remark 2.2.** For any $X$, there are evident homomorphisms

$$H_p(X) \to H_0(\bar{C}) \quad \text{and} \quad H_{q-p}(\bar{C}) \to H_q(X),$$

where the former is injective and the latter is surjective. In particular, projection/restriction defines a surjective homomorphism of vector spaces

$$\text{hom}(H_0(\bar{C}), H_{q-p}(\bar{C})) \to \text{hom}(H_p(X), H_q(X)).$$

This last map shows $\bar{C}_*$ can be used in place of the cellular chain complex $C_*(X)$ when attempting to define the hypercurrent homomorphism $J_{p,q}$.

3. **Functor categories of chain complexes**

Let $I$ be a partially ordered set such that any element $x \in I$ has finite degree in the sense that the longest chain of strictly increasing objects less than $x$ is finite. Call the length of this chain $\text{deg}(x)$. The function

$$\text{deg}: I \to \mathbb{N}$$

is a morphism of partially ordered sets, i.e., it is a functor.

**Example 3.1.** Let $S_B$ denote the poset of singular simplices in a space $B$. An element of $S_B$ is just a singular simplex $\Delta^j \to B$ and the partial ordering is defined by facial inclusion. Then every element of $S_B$ has finite degree.

As in §2, $\mathcal{C}$ will denote the category of unbounded chain complexes over $\mathbb{F}$. By [5, th. 5.1.3], the functor category

$$\mathcal{C}^I,$$

forms a model category in which the weak equivalences/fibrations are the objectwise equivalences/fibrations and the cofibrations are defined to be those maps satisfying the left lifting property with respect to the acyclic fibrations.
We now give an explicit characterization of the cofibrations. Suppose if $X : I \to C$ is a functor and $r \in I$ is an object. Define the latching object

$$L_r X := \colim_{s < r} X(s).$$

The assignment $X \mapsto L_r X$ defines a functor $L_r : I \to C$, and one has a natural latching map $L_r(X) \to X(r)$. Then $X$ is cofibrant if and only if the latching map is a cofibration of $C$ for all objects $r$. More generally, a morphism $A \to X$ of $C^I$ is a cofibration if and only if for every $r \in I$ the relative latching map

$$L_r X \oplus_{L_r A} A(r) \to X(r)$$

is a cofibration of $C$, where the domain of this map is the pushout of $L_r X \leftarrow L_r A \to A(r)$.

**Example 3.2.** Fix $E \to B$, a Serre fibration of spaces. Define a functor $F : S_B \to C$ by the rule $F(\sigma: \Delta^j \to B) = S_\ast(\sigma^* E)$, where $\sigma^* E$ denotes the pullback of $E \to B$ along $\sigma$ and $S_\ast(\sigma^* E)$ is its singular chain complex over $\mathbb{F}$. Then the functor $S_\ast$ is cofibrant.

Recall that a chain complex $C_\ast$ is bounded below by 0 if $C_k = 0$ for $k < 0$.

**Definition 3.3.** A functor $F : I \to C$ is bounded below by 0 if the chain complex $F(r)$ is is bounded below by 0 for all $r \in I$.

A functor $G : I \to C$ is positively acyclic if

- $G$ is bounded below by 0, and
- for all objects $r \in I$, the homology groups $H_\ast G(r)$ are trivial in degrees $\ast \neq 0$.

Recall that cofibrant approximation of an object $F \in C^I$ consists of a cofibrant object $F^c$ equipped with a weak equivalence $F^c \xrightarrow{\sim} F$.

**Example 3.4.** If $G$ is positively acyclic, then the evident natural transformation

$$G \to H_\ast G,$$

is an acyclic fibration (i.e., both a fibration and a weak equivalence).

The next result is similar in spirit to the classical acyclic model theorem, although the technical assumptions are somewhat different.
Proposition 3.5 ("Acyclic Models"). Let \( F, G : I \to C \) be functors where \( F \) is bounded below by 0 and \( G \) is positively acyclic. Let \( F^c \to F \) be a cofibrant approximation. Then any natural transformation \( \alpha_* : H_* F \to H_* G \) is induced by a natural transformation
\[
F^c \to G .
\]
In particular, when \( F \) is cofibrant there is a natural transformation \( F \to G \) that induces \( \alpha_* \).

Proof. The function \( \text{deg} \) equips \( I \) with the structure of a direct category in the sense of [5, defn. 5.1.1]. Let \( F^c_0 \) be the degree zero part of \( F^c \), and let \( u : F^c_0 \to H_0 F \) be the evident map, . Consider the a lifting problem
\[
\begin{array}{ccc}
F^c & \xrightarrow{\alpha} & H_* G \\
\downarrow & & \downarrow \\
H_* F & \xrightarrow{\alpha_* \circ u} & H_* G,
\end{array}
\]
where \( \alpha : F^c \to H_* G \) is the natural transformation defined by
\[
\begin{cases}
\alpha_* \circ u & \text{in degree 0} \\
0 & \text{otherwise}.
\end{cases}
\]
The existence of a lift follows directly from the model category lifting axiom since \( F^c \) is cofibrant and \( G \to H_* G \) is an acyclic fibration.

\]

Corollary 3.6 (Rigidity). Assume \( F, G \) and \( \alpha_* \) satisfy the conditions of Proposition 3.5. Then a natural transformation \( F^c \to G \) inducing \( \alpha_* \) exists and is unique up to contractible choice.

Proof. Once basepoint \( F^c \to G \) in the affine space of lifts has been chosen, the set of lifts is in bijection with the set of lifts of the trivial map. The latter has the structure of the vector space of zero chains of the kernel of the map of complexes \( \text{hom}(F^c, G) \to \text{hom}(F^c, H_* G) \). But this kernel is contractible because the natural transformation \( G \to H_* G \) is both a weak equivalence and a fibration.

4. Spanning trees and co-trees

Let \( X \) be a finite connected CW complex. In the following definitions, \( \beta_j(X) = \dim_k H_k(X) \) denotes the \( j \)th Betti number of \( X \), and \( X^{(j)} \) denotes the \( j \)-skeleton of \( X \).

Definition 4.1 ([4, defn. 1.2]). A subcomplex \( T \subset X \) is a spanning tree in degree \( d \) if
\[
\bullet \ H_d(T) = 0,
\]
\[ \beta_{d-1}(T) = \beta_{d-1}(X), \] and
\[ X^{(d-1)} \subset T \subset X^{(d)}. \]

**Example 4.2.** If \( d = 1 \), then the above coincides with the usual definition of a spanning tree for the 1-skeleton of \( X \), considered as a graph.

**Definition 4.3** ([2, defn. 1.9]). A spanning co-tree in degree \( d \) for \( X \) is a subcomplex \( L \subset X \) such that
1. The inclusion \( L \subset X \) induces an isomorphism of vector spaces
   \[ H_d(L) \overset{\cong}{\to} H_d(X); \]
2. \( \beta_{d-1}(L) = \beta_{d-1}(X); \)
3. \( X^{(d-1)} \subset L \subset X^{(d)}. \)

**Example 4.4.** A spanning co-tree in degree 0 is just a 0-cell of \( X \).

**Example 4.5.** If \( H_d(X) = 0 \) then a spanning tree in degree \( d \) is the same thing as a spanning co-tree in degree \( d \).

**Lemma 4.6** ([4, lem. 2.3,2.4], [2, lem. 2.2,2.3]). Let \( X \) be a connected finite CW complex. Then \( X \) has a spanning tree and spanning co-tree in every degree \( d \geq 0 \) and in each case there are only finitely many.

**Definition 4.7.** Let \([p, q]\) be a gap. If \( p \leq d \leq q \) then a \( d \)-tree for \( X \) is subcomplex \( T \subset X \) such that
1. if \( d = p \), then \( T \) is a spanning co-tree;
2. if \( d > p \), then \( T \) is a spanning tree.

5. **Chain level hypercurrents**

Assume \( \gamma: \Sigma \to \mathcal{M}_{p,q} \) is a good protocol. Recall that
\[ \gamma(b) = (W_p(b), \ldots, W_q(b)) \]
where \( W_j(b): X_j \to \mathbb{R} \).

**Definition 5.1** (Stratification of \( \Sigma \)). Let \( \Sigma_j \) be the set of \( b \in B \) such that the function \( W_j(b) \) is one-to-one. Then \( \Sigma_j \subset \Sigma \) is open and \( \Sigma = \coprod \Sigma_j \) as sets.

**Definition 5.2** (Small simplices). A singular simplex \( \sigma: \Delta^j \to \Sigma \) is small if its image is contained in \( \Sigma_j \) for some index \( j \).

Let
\[ I_\Sigma \]
be the poset of small singular simplices in \( \Sigma \). An object of \( I_\Sigma \) is a small singular simplex and a morphism is an inclusion of faces. The \( I_\Sigma \) is a poset in which every element has finite degree.
Notation 5.3. If \( \sigma: \Delta^j \to \Sigma \) is a (small) singular simplex we set
\[
j_\sigma := j.
\]

Definition 5.4 (Tree functor). For a small simplex \( \sigma: \Delta^j \to \Sigma \), let
\[
k_\sigma \in \mathbb{N}
\]
be the smallest integer such that \( \sigma(\Delta^j) \subset \Sigma_{k_\sigma} \). By the greedy algorithm (cf. [2, p. 9]) there is a preferred \( k_\sigma \)-tree \( T_\sigma \) associated with \( s \).

The assignment \( \sigma \mapsto \bar{C}(T_\sigma) \) defines a functor
\[
\tau: I_\Sigma \to C.
\]
Call this the tree functor. (We remind the reader that \( \bar{C}(T_\sigma) \) is the chain complex with \( \bar{C}_j(T_\sigma) = C_{p+j}(T_\sigma^{(q)}, T_\sigma^{(p-1)}) \).)

5.1. The pre-hypercurrent map. Let \( \Delta_j^i \) denote the simplicial chain complex of the standard \( j \)-simplex. Consider the following three functors \( I_\Sigma \to C \):

1. The constant functor \( \chi \) given by
   \[
   \sigma \mapsto \bar{C} := \bar{C}(X).
   \]

2. The functor \( \chi^c \) given by
   \[
   \sigma \mapsto \bar{C} \otimes \Delta^j_\sigma.
   \]

3. The tree functor \( \tau \) as defined above.

There are evident natural transformations
\[
u: \chi^c \to \chi \quad \text{and} \quad v: \tau \to \chi.
\]

Lemma 5.5. The follow properties hold:

- The natural transformation \( u \) is a cofibrant approximation to \( \chi \).
- The functor \( \tau \) is positively acyclic.
- Both \( \chi^c \) and \( \tau \) are globally bounded below by 0.
- The natural transformation \( v: \tau \to \chi \) induces an isomorphism on homology in degree 0 and is trivial in positive degrees.

Proof. The only non-trivial thing to check is that \( \chi^c \) is cofibrant. This amounts to the statement that for \( j \geq 0 \) the inclusion
\[
\bar{C} \otimes \partial\Delta^j_\sigma \to \bar{C} \otimes \Delta^j_\sigma
\]
is a cofibration of \( C \), where \( \partial\Delta^j_\sigma \) is the simplicial chain complex of \( \partial\Delta^j \).

But this is clear. \( \square \)
Definition 5.6. A pre-hypercurrent map is a natural transformation
\[ j : \chi^c \to \tau \]
such that the induced homomorphisms in zero dimensional homology
\[ (vj)_* : H_0\chi^c \to H_0\chi \]
coincide.

Proposition 5.7. A pre-hypercurrent map \( j \) exists and is unique up to contractible choice.

Proof. Since \( v_* \) is an isomorphism in degree 0, the homological condition can be rephrased as the following equation of maps \( H_* (\chi^c) \to H_* (\tau) \):
\[ j_* = v_*^{-1} \circ u_* . \]
In particular, the value of \( j_* \) is fixed. The result then follows immediately by application of Proposition 3.5 and Corollary 3.6, where the assumptions are verified using Lemma 5.5. \( \square \)

5.2. The hypercurrent map. We construct the hypercurrent map as a colimit. Let \( S_\Sigma \) be the full singular simplex category of \( \Sigma \) (see Ex. 3.1). By a subdivision argument, the inclusion of posets
\[ I_\Sigma \subset S_\Sigma \]
induces a weak equivalence on realizations. In particular, the corresponding map of (singular) chain complexes
\[ I_* (\Sigma) \to S_* (\Sigma) \]
is a chain homotopy equivalence.

The following is a special case of a well-known result.

Lemma 5.8. Let \( c : I_* (\Sigma) \to C \) be a constant functor with value \( A_* \). Then
\[ \text{hocolim}_{I_\Sigma} c \simeq I_* (\Sigma) \otimes A_* , \]
Proof. The displayed homotopy colimit is just
\[ \text{colim}_{I_\Sigma} (\Delta^j_\sigma \otimes A_*) \simeq (\text{colim}_{\sigma \in I_\Sigma} \Delta^j_\sigma) \otimes A_* \]
where \( \sigma \) ranges through elements of \( I \). But
\[ \text{colim}_{\sigma \in I} \Delta^j_\sigma \]
is just the singular chain complex of the nerve of \( I_\Sigma \). \( \square \)

Theorem 5.9. A hypercurrent map
\[ \mathcal{J} : I_* (\Sigma) \otimes \check{C}_* \to \check{C}_* \]
exists and is unique up to contractible choice.
Proof. Apply the colimit operation to the composed natural transformations

\[ \chi^c \xrightarrow{\iota} \tau \xrightarrow{\nu} \chi \]

(where the pre-hypercurrent \(j\), which is unique up to contractible choice, is obtained from Proposition 5.7). This gives a morphism of chain complexes

\[ \colim_{I_{\Sigma}} \chi^c \rightarrow \colim_{I_{\Sigma}} \chi. \]

By Lemma 5.8 the source of this last map is identified with \(I_\ast(\Sigma) \otimes \bar{C}_\ast\), whereas the target is identified with \(\bar{C}_\ast\). □

5.3. Variants. We describe variants of Theorem 5.9 which are useful for the proof of the Quantization Theorem.

Definition 5.10. Let \(\mathcal{P}\) be a partially ordered set. A subset \(R \subset \mathcal{P}\) is closed if for all \(x \in R\) and \(y \in \mathcal{P}\), then \(y < x\) implies \(y \in T\).

Clearly, a closed subset of \(\mathcal{P}\) is again a poset.

In what follows, we set \(\mathcal{P} := I_{\Sigma}\), and let \(R \subset I_{\Sigma}\) be any closed subset. Clearly, the model category machinery applies equally to the functor category \(\mathcal{C}^R\).

Applying Definition 5.6 to the restrictions to \(T\) of the functors \(\chi\) and \(\tau\), we obtain a restricted pre-hypercurrent map

\[ j|_R : \chi^c \rightarrow \tau \]

which is unique up to contractible choice. In particular, we obtain the following variant of Theorem 5.9:

Corollary 5.11. Restriction to a closed subset \(R \subset I_{\Sigma}\) yields a restricted hypercurrent chain map

\[ \mathcal{J}_R : R_\ast \otimes \bar{C} \rightarrow \bar{C} \]

which is unique up to contractible choice, where \(R_\ast\) is the chain complex over \(\mathbb{F}\) is freely generated in degree \(j\) by the objects \(\sigma : \Delta^j \rightarrow \Sigma\) of \(R\), with boundary operator defined by taking the alternating sum of the codimension one faces of \(\Delta^j\).

5.4. A derived category approach. An inspection of the above recipe yields a direct construction a pre-hypercurrent map in the homotopy category of functors \(I \rightarrow \mathcal{C}\). The idea is that the canonical chain of natural transformations

\[ \chi \rightarrow H_\ast(\tau) \leftarrow \tau \xrightarrow{\varphi} \chi \]

defines an endomorphism \([\mathcal{J}] : \chi \rightarrow \chi\) in the homotopy category. It is trivial to check that the pre-hypercurrent map \(\mathcal{J}\) descends to this endomorphism.
5.5. The small cell approach. We briefly describe an alternative
construction of a hypercurrent map when $\Sigma$ has the structure of finite
CW complex. We say that a good protocol $\Sigma \rightarrow \check{M}_{p,q}$ is cellularly
small if each characteristic map $\chi: D^j \rightarrow \Sigma$ has image in some $\Sigma(k)$. This
condition can usually be arranged: if $\Sigma$ is a regular CW complex
equipped with a good protocol, then there is always a subdivision of $\Sigma$
such that the protocol becomes cellularly small. To avoid technicalities,
we will restrict ourselves to the case when $\Sigma$ is regular, which for us is
the case of interest.

Recall that a regular CW complex $Y$ is a CW complex if every char-
acteristic map $D^j \rightarrow Y$ is an embedding. Let $\mathcal{P}_Y$ be the poset given by
the set of cells of $Y$ partially ordered as follows: $e \leq e'$ iff and only if $e$
is contained in the closure of $e'$. The realization $|\mathcal{P}_Y|$ is the barycentric
subdivision of $Y$ and there is a cellular map $Y \rightarrow |\mathcal{P}_Y|$ which is also a
homeomorphism [6, thms. 1.7, 2.1]

Assume that $\Sigma$ is a regular CW complex with respect to a cellularly
small protocol $\Sigma \rightarrow \check{M}_{p,q}$. By replacing $I_\Sigma$ with $\mathcal{P}_\Sigma$ in the construction
of the pre-hypercurrent map, we obtain a natural transformation

$$j^{\text{cw}}: \chi^c \rightarrow \chi$$

just as above, but where now the source category for $\chi$ and $\chi^c$ is the
poset $\mathcal{P}_\Sigma$. Taking colimits we obtain the cellular hypercurrent map

$$\mathcal{J}^{\text{cw}}: C_\ast(\Sigma) \otimes \check{C}_\ast \rightarrow \check{C}_\ast,$$

in which $C_\ast(\Sigma)$ is the cellular chain complex of $\Sigma$. When considered in
the derived category, the maps $\mathcal{J}^{\text{cw}}$ and $\mathcal{J}$ coincide. In particular they
agree on the level of homology.

6. Homotopy type of the space of good weights

For spaces $A$ and $B$, recall that the topological join $A \ast B$ is defined to be
the quotient space of $A \times B \times [0, 1]$ in which for all points $a, a' \in A$ and
$(b, b') \in B$, the point $(a, b, 0)$ is identified with $(a, b', 0)$ and the point
$(a, b, 1)$ is identified with $(a', b, 1)$. It is often convenient to identify the
equivalence class of $(a, b, t)$ with the affine combination $ta + (1 - t)b$.
Another description of $A \ast B$ is

$$(CA \times B) \cup (A \times CB)$$

where $CA$ is the cone on $A$ and the union is amalgamated over $CA \times CB$.

The homotopy type of $\check{M}_{p,q}$ depends only on the number of cells in
each dimension $\mathcal{J}$ where $j \in [p, q]$. In particular, it is independent of
the choices of the attaching maps of the cells of $X$. Recall that $X_j$ denotes the number of $j$-cells of $X$. Let

$$E(X_j, \mathbb{R})$$

be the space of one-to-one functions $X_j \to \mathbb{R}$.

**Proposition 6.1.** There a preferred homotopy equivalence

$$\mathcal{M}_{p,q} \simeq E(X_p, \mathbb{R}) \ast E(X_{p+1}, \mathbb{R}) \ast \cdots \ast E(X_q, \mathbb{R}).$$

**Remark 6.2.** In particular, if some $X_j$ has cardinality $\leq 1$, then $\mathcal{M}_{p,q}$ is contractible.

**Proof of Proposition 6.1.** If $q - p = 0$, then $\mathcal{M}_{p,q}$ is just $E(X_p, \mathbb{R})$.

Assume next that $q - p = 1$. Then $\mathcal{M}_{p,q}$ decomposes as

$$E(X_p, \mathbb{R}) \times F(X_q, \mathbb{R}) \cup F(X_p, \mathbb{R}) \times E(X_q, \mathbb{R}),$$

where the union is amalgamated over $E(X_p, \mathbb{R}) \times E(X_q, \mathbb{R})$ and $F(X, \mathbb{R})$ denotes the function space of maps $X \to \mathbb{R}$. As the latter is a convex space, there is a preferred homotopy equivalence of pairs

$$(F(X, \mathbb{R}), E(X, \mathbb{R})) \simeq (C(E(X, \mathbb{R})), E(X, \mathbb{R})).$$

Hence, the union (13) is identified with the join $E(X_p, \mathbb{R}) \ast E(X_q, \mathbb{R})$. This completes the case $q - p = 1$.

The general case now follows from a straightforward induction on $q - p$ which we leave to the reader. □

**Corollary 6.3.** If $X$ has at most one $j$-cell for some $j \in [p, q]$, then the topological hypercurrent map $\mathcal{J} : H_{p-q}(\Sigma) \otimes H_p(X) \to H_q(X)$ is trivial for any good protocol $\Sigma$.

**Proof.** If we take $\Sigma = \mathcal{M}_{p,q}$, then $\Sigma$ is contractible by Proposition 6.1 (since at least one of the factors $E(X_j, \mathbb{R})$ is contractible for some $j$). This shows gives the result in the universal case. The general case follows by functoriality. □

Let $n_j$ be the cardinality $X_j$ and set

$$c_{p,q} := \prod_{j=p}^q (n_j! - 1)$$

**Corollary 6.4.** Assume $n_j \geq 2$ for $j \in [p, q]$, then there is a preferred homotopy equivalence

$$\tilde{\mathcal{M}}_{p,q} \simeq \bigvee_{c_{p,q}} S^{q-p},$$

i.e., $\tilde{\mathcal{M}}_{p,q}$ has the homotopy type of a $c_{p,q}$-fold wedge of $(q - p)$-spheres.
Proof. The proof uses the following elementary fact: let $A$ and $B$ be well-pointed based spaces. Then the join $A \ast B$ has the homotopy type of $\Sigma A \wedge B$, that is, the suspension of the smash product.

Let $L(X_j)$ denote the set of linear orderings on the set $X_j$. Then the obvious map $E(X_j, \mathbb{R}) \to L(X_j)$ is a deformation retraction. In particular, $E(X_j, \mathbb{R})$ has the homotopy type of a finite set of cardinality $n_j!$. Assuming that $n_j \geq 2$, a choice of basepoint in $E(X_j, \mathbb{R})$ gives an identification up to homotopy $E(X_j, \mathbb{R}) \simeq \bigvee_{n_j!-1} S^0$, since the right side has cardinality $n_j!$. The result then follows from Proposition 6.1, using the fact that smash products distribute over wedges.

6.1. **Explicit description of the summands.** We describe maps $S^{q-p} \to \tilde{M}_{p,q}$ that correspond to the summands in the decomposition (14). For this we fix a basepoint $*_j \in E(X_j, \mathbb{R})$ for each $j \in [p, q]$. Note that the automorphism (permutation) group of $X_j$ acts freely on $E(X_j, \mathbb{R})$. For each non-trivial automorphism $\sigma_j : X_j \to X_j$, one has a map $\alpha_{\sigma_j} : S^0 \to E(X_j, \mathbb{R})$ in which $\alpha_{\sigma_j}(-1) = \sigma \cdot *_j$ and $\alpha_{\sigma_j}(+1) = *_j$. For each $J$ we choose such a $\sigma_j$ and join the maps $\alpha_{\sigma}$ together. This produces a map $S^{q-p} \simeq S^0 \ast \ldots \ast S^0 \xrightarrow{\alpha_{\sigma_p} \ast \ldots \ast \alpha_{\sigma_q}} E(X_p, \mathbb{R}) \ast \ldots \ast E(X_q, \mathbb{R}) \simeq \tilde{M}_{p,q}$ yielding a summand in the decomposition. By varying the choices of the automorphisms, every summand in the decomposition arises in this way.

7. **Proof of Theorems A and D**

Let $X = S^q$ with the CW structure having two $j$-cells $e^j_\pm$ for $0 \leq j \leq q$. The attaching maps $\partial e^j_\pm \to S^{j-1} = X^{(j-1)}$ are given by the identity map, where $0 \leq j \leq q$. Similarly, $Y = D^q \vee S^q$, where the CW structure is such that $Y^{(q-1)} = X^{(q-1)} = S^{q-1}$ and the two $q$-cells $e^q_\pm$ are attached respectively by the identity and the constant map. Note that $X$ and $Y$ have the same number of cells in each degree and have the same homotopy type.
Proof of Theorem D. Consider first the case \( q = 1 \). For convenience, we change notation and denote the weights by \( E = (E_-, E_+) \) and \( W = (W_-, W_+) \), where \( E_\pm \) is a real number associated with \( e^\pm_0 \) and \( W_\pm \) is a real number associated with \( e^\pm_1 \). A spanning 0-tree is then a 0-cell and a spanning 1-tree is a 1-cell. The greedy algorithm says in this case that we select the 0-tree \( e_0^+ \) if \( E^+ < E^- \) and \( e_0^- \) if \( E^+ > E^- \). Similarly we select the 1-tree \( e_1^+ \) if \( W^+ < W^- \) and we select the 1-tree \( e_1^- \) if \( W^- < W^+ \). Suppose we can find a map \( f : S^1 \rightarrow \hat{\mathcal{M}}_{0,1} \), with \( f(e^{i\theta}) = (E_-(\theta), E_+(\theta), W_-(\theta), W_+(\theta)) \), having the properties indicated by Fig. 1, where we are considering \( S^1 \) as the boundary of the square using the homeomorphism \((x, y) \mapsto (x,y) \parallel \| (x,y) \| \) for \((x, y) \) lying in the boundary of the square \([-1, 1] \times 2 \). The contribution to the current on the green edges of the square is trivial, since along each such edge there is a preferred 0-tree. The top edge of the square gives a contribution of \( e^+_1 \) to the current and the bottom edge contributes \( e^-_1 \). It follows that the current generated by the square is the 1-cycle \( e^+_1 + e^-_1 \). Hence \( \mathcal{J}^{cw} : H_1(S^1) \otimes H_0(S^1) \rightarrow H_1(S^1) \) is non-trivial if we can find such a map \( f \). A concrete example is given by \( f(e^{i\theta}) = ((0, \cos \theta), (0, \sin \theta)) \). This takes care of the case \( q = 1 \).

Next consider the case \( q = 2 \). We identify \( S^2 \) with the boundary of the cube \([-1, 1] \times 3 \) and suppose we can find a map \( f : S^2 \rightarrow \hat{\mathcal{M}}_{0,2} \) with the properties appearing in Fig. 2. Then an unravelling of the definition of \( \mathcal{J}^{cw} \) shows the contribution to the hypercurrent is trivial along all faces of the cube except the top face, which contributes \( e^-_2 \) and the bottom face, which contributes \( -e^+_2 \). Hence the value of the hypercurrent on the entire boundary of the cube is given by the cycle \( e^-_2 - e^+_2 \), which generates homology of \( X = S^2 \) in degree 2. An example of \( f : S^2 \rightarrow \hat{\mathcal{M}}_{0,2} \) with the above properties is

\[
f(x_0, x_1, x_2) = (0, x_0, 0, x_1, 0, x_2),
\]
where $\sum x_i^2 = 1$, and $W_{i+} = x_i, W_{i-} = 0$. This shows that the hypercurrent map $F^{cw}: H_2(S^2) \otimes H_0(S^2) \rightarrow H_2(S^2)$ is non-trivial in this case.

The case $q > 2$ is completely analogous: we take the boundary of a $q$-dimensional cube and label its $q + 1$ pairs of opposing faces by $W_{j-} < W_{j+}$ and $W_{j+} < W_{j-}$ for $0 \leq j \leq q$. The map $f: S^q \rightarrow \tilde{M}_{0,q}$ is given by

\begin{equation}
\begin{aligned}
f(x_0, \ldots, x_q) &= (0, x_0, 0, x_1, \ldots, 0, x_q).
\end{aligned}
\end{equation}

Again, the hypercurrent map will be non-trivial.

We remark that $f$ in the above example is a homotopy equivalence.

Consider now the case of $Y = D^q \vee S^q$ with the CW structure having the same $(q-1)$-skeleton as $X$, i.e., $S^{q-1}$. There are two two $q$-cells $e^q_+$ and $e^q_-$, where $e^q_-$ is attached via the identity map and $e^q_+$ is attached using the constant map whose value is a 0-cell. Then there is exactly one $q$-tree $T$ given by the summand $D^q \subset Y$. It follows that the hypercurrent is necessarily trivial since it factors through the homology of $D^q$. \hfill \Box

\textbf{Proof of Theorem A.} Statements (1)-(3) of Theorem A follow directly from the construction of the pre-hypercurrent map. It will therefore suffice to establish statement (4). When $p = 0$ an example satisfying (4) is given by Theorem D. Therefore, it will suffice to consider the case when $p > 0$.

Let $Z = X/X^{(p-1)}$, where $X = S^q$ is provided with the CW structure of Theorem D. In addition to a unique 0-cell, $Z$ has two cells in each dimension $j \in [p, q]$. Furthermore, $Z$ has the homotopy type of $S^p \vee S^q$. Hence, $[p, q]$ is a gap (for both $X$ and $Z$). There is a homeomorphism

\[ \tilde{M}_{0,q}(S^{q-p}) \rightarrow \tilde{M}_{p,q}(Z), \]

where $S^{q-p}$ is regarded as the $(q - p)$-skeleton of $X$, which is given by the evident bijections on the set of cells. The homotopy equivalence
$S^{(q-p)} \xrightarrow{f} \tilde{\mathcal{M}}_{0,q}(S^{q-p})$ defined as in (15), followed by this homeomorphism gives a good protocol

$$\Sigma := S^{q-p} \xrightarrow{\sim} \hat{\mathcal{M}}_{p,q}(Z)$$

The rest of the argument follows by the argument of the proof of Theorem D(1).

\section{Robust weights}

\textbf{Definition 8.1.} The \textit{discriminant} $D_{p,q}$ is the complement of the space good parameters from $\mathcal{M}_{p,q}$, i.e.,

$$D_{p,q} := \mathcal{M}_{p,q} \setminus \hat{\mathcal{M}}_{p,q}.$$

Let $\mathcal{M}_{p,q}^+$ be the one-point compactification of $\mathcal{M}_{p,q}$ and similarly, let $D_{p,q}^+$ be the one-point compactification of $D_{p,q}$. Then $\mathcal{M}_{p,q}^+$ is a sphere of dimension $\sum j n_j$.

\textbf{Theorem 8.2.} $\mathcal{M}_{p,q}^+$ has the structure of a regular CW complex in which the subspace $D_{p,q}^+$ is a subcomplex of codimension $q - p + 1$.

The proof of Theorem 8.2 is an adaptation of the argument of [1, prop. 7.1].

By \textit{height data} $h_\bullet$ we mean a partial ordering $<_i$ on $X_i$ for $i = p, \ldots, q$. The partial ordering defines an equivalence relation $\sim_i$ defined by $x \sim_i y$ if and only if the length of a maximal chain terminating in $x$ is the same as the length of a maximal chain terminating in $y$. Note that the set $X_i/\sim_i$ comes equipped with a canonical linear order defined by $<_i$.

Here is the main example:

\textbf{Example 8.3.} For $W_\bullet := (W_p, \ldots, W_q) \in \mathcal{M}_{p,q}$ we define $x <_i y$ if and only if $W_i(x) < W_i(y)$ for $x, y \in X_i$.

\textbf{Definition 8.4.} Given height data $h_\bullet$, let

$$C(h_\bullet) \subset D_{p,q}$$

be the subspace consisting of the set of weights $W_\bullet$ whose associated height data is $h_\bullet$ in the sense of Example 8.3.

Let $D(h_\bullet)$ denote the closure of $C(h_\bullet)$ in $\mathcal{M}_{p,q}$.

\textbf{Proposition 8.5.} The one-point compactification $D(h_\bullet)^+$ of $D(h_\bullet)$ is homeomorphic to a disk of dimension $\sum_i d_i$, where $d_i$ is the cardinality of $X_i/\sim_i$. 
Proof. There is an evident homeomorphism
\[ D(h_\bullet) \cong D(h_p) \times \cdots \times D(h_q), \]
where \( D(h_i) \) is the closure of the set of those \( W_i : X_i \to \mathbb{R} \) in the function space \( F(X_i, \mathbb{R}) \) that induce the equivalence relation \( \sim_i \). Hence, it will be enough to show that \( D(h_i)^+ \) is homeomorphic to a disk of dimension \( d := d_i \).

We first show that \( D(h_i) \) is homeomorphic to the subspace space \( H \subset \mathbb{R}^d \) consisting of \( d \)-tuples of real numbers \((x_1, \ldots, x_d)\) such that \( x_1 \leq x_2 \leq \cdots \leq x_d \). The homeomorphism is defined as follows: a function \( W_i : X_i \to \mathbb{R} \) in the interior \( C(h_i) \) of \( D(h_i) \) defines a \( d \)-tuple of increasing real numbers \( x_1 < \cdots < x_d \) by taking the image of \( W_i \). This assignment defines an embedding \( C(h_i) \to \mathbb{R}^d \) which extends to a homeomorphism \( D(h_i) \to H \). On the other hand, it is easy to show that \( H^+ \) is homeomorphic to the disk \( D^d \). \( \square \)

Proof of Theorem 8.2. The inclusions \( D(h_\bullet)^+ \subset M_{p,q}^+ \) define characteristic maps for a cell structure on \( M_{p,q} \). It is straightforward to check that cells of a given dimension are attached on top of cells of lower dimension, so we obtain regular CW structure on \( M_{p,q}^+ \). The latter is a sphere of dimension \( \sum_i n_i \), where \( n_i = |X_i| \) for \( p \leq i \leq q \).

Clearly, \( D_{p,q}^+ \subset M_{p,q}^+ \) is a subcomplex, since a weight system \( W_\bullet \) lies in the discriminant precisely when no \( W_i : X_i \to \mathbb{R} \) is one-to-one (this means the associated partial ordering is not linear). A straightforward dimension count then shows that the top dimensional cells of \( D_{p,q}^+ \) have dimension
\[ \sum_{i=p}^q (n_i - 1). \]

It follows that the codimension of \( D_{p,q}^+ \) is \( q - p + 1 \). \( \square \)

8.1. The space of robust weights. Consider a top-dimensional cell \( D \subset D_{p,q}^+ \). Choose a small closed disk \( E^{q-p+1} \subset M_{p,q} \) which is transversal to \( D \) such that \( D \cap E \) is a single point. Then we obtain an inclusion
\[ S^{q-p} := \partial E \to \hat{M}_{p,q} . \]

Consequently, the hypercurrent
\[ J_{p,q}^* \in H^{q-p}(\partial E; H^p(X; H_q(X))) \cong \text{hom}(H_p(X), H_q(X)) \]
is defined, and it doesn’t depend on the choice of \( E \). To emphasize its dependence on the cell \( D \), we denote the above hypercurrent by \( J_D \).
Definition 8.6. The cell $D$ is inessential if $J_D$ is trivial. It is essential if $J_D$ is non-trivial.

Definition 8.7. Let $\mathcal{D}_{p,q} \subset \mathcal{D}_{p,q}$ be the subspace obtained by removing the interiors of every inessential cell. The space of robust weights $\mathcal{M}_{p,q}$ is defined as the complement of $\mathcal{D}_{p,q}$ in $\mathcal{M}_{p,q}$, i.e.,

$$\mathcal{M}_{p,q} := \mathcal{M}_{p,q} \setminus \mathcal{D}_{p,q}.$$  

A robust protocol is a map of spaces $\Sigma \rightarrow \mathcal{M}_{p,q}$.

Proposition 8.8. There is a preferred extension

$$\mathcal{J}_{p,q} : H_{q-p}(\mathcal{M}_{p,q}) \otimes H_p(X) \rightarrow H_q(X)$$

of the universal topological hypercurrent homomorphism $\mathcal{J}$.

Proof. This is a direct consequence of obstruction theory, since the relative homology groups $H_*(\mathcal{M}_{p,q}, \mathcal{M}_{p,q})$ are non-trivial in the single degree $q-p$ and are freely generated by the homology class of the maps $(E, \partial E) \rightarrow (\mathcal{M}_{p,q}, \mathcal{M}_{p,q})$, with one such map for each inessential cell $D$. Since each $J_D$ is trivial it follows that the extension exists. The extension is preferred because the relative homology group vanishes in degree $q-p+1$. $\square$

Corollary 8.9. A robust protocol $\Sigma \rightarrow \mathcal{M}_{p,q}$ induces a hypercurrent homomorphism

$$\mathcal{J}_{p,q} : H_{q-p}(\Sigma) \otimes H_p(X) \rightarrow H_q(X).$$

8.2. Homotopy type of $\mathcal{M}_{p,q}$. An explicit closed disk $E \subset \mathcal{M}_{p,q}$ is obtained as follows: let $W_* \in \mathcal{D}_{p,q}^+$ be a point in the interior of a top dimensional closed cell. Then for each $i$ with $p \leq i \leq q$, there are a unique pair of elements $x_i, y_i \in W_i$ such that $W_i(x_i) = W_i(y_i)$. Then construct a transversal cube of dimension $q-p+1$ in $\mathcal{M}_{p,q}$ with center $W_*$, we resolve the $W_i$ as follows: consider the set of weights $V_* = (V_p, \ldots, V_q)$ such that for a sufficiently small choice of $\epsilon > 0$,

- $V_i(u) = W_i(u)$ for $u \neq y_i$,
- $|V_i(y_i) - W_i(y_i)| \leq \epsilon$.

(compare Fig. 2). Then the set of such $V_*$ forms a transversal $(q-p+1)$-cube. Let $\tau : S^{q-p} \rightarrow \mathcal{M}_{p,q}$ denote the restriction to the boundary.

Applying the above procedure to each inessential cell, we obtain a map

$$\beta : S^{q-p} \times T \rightarrow \mathcal{M}_{p,q},$$

where $T$ denotes the set of inessential cells of $\mathcal{D}_{p,q}$. By Alexander duality, we infer
Lemma 8.10. There is a homotopy equivalence
\[ \mathcal{M}_{p,q} \simeq \mathcal{M}_{p,q} \cup_{\beta} (D^{q-p+1} \times T). \]

We now identify the map \( \tau \) in terms of the decomposition of \( \mathcal{M}_{p,q} \) into a wedge of spheres. Observe that \( \tau \) is just the iterated join of maps
\[ S^0 \ast \cdots \ast S^0 \xrightarrow{\tau_1 \ast \cdots \ast \tau_q} E(X_p, \mathbb{R}) \ast \cdots \ast E(X_q, \mathbb{R}), \]
where \( \tau_j : S^0 \to E(X_j, \mathbb{R}) \) is defined by \( \tau_j(\pm 1) = W_j^\pm \), with \( W_j^\pm(x) = W_j(x) \) for \( x \neq y_j \) and \( W_j^\pm(y_j) = W_j(y_j) \pm \epsilon \).

The map \( \tau_j \) is generally not pointed: it is determined up to contractible choice by two nontrivial permutations, given by the changes in the ordering from the basepoint component to the components of \( \tau_i(\pm 1) \). Hence, in terms of the identification \( \mathcal{M}_{p,q} \) with a wedge of \((q-p)\)-spheres, \( \tau \) is identified with the following composite
\[ S^{q-p} \xrightarrow{\text{pinch}} \bigvee_{2^{q-p}} S^{q-p} \subset \bigvee_{c_{p,q}} S^{q-p}, \]
where the first of these is the pinch map and the second is an inclusion of summands obtained from the choice of two non-trivial permutations of \( X_j \) for each \( j \), with \( j \in [p, q] \).

Corollary 8.11. There is a homotopy equivalence
\[ \mathcal{M}_{p,q} \simeq \bigvee_{d_{p,q}} S^{q-p}, \]
with \( d_{p,q} = c_{p,q} - u \), where \( u \) is the number of inessential cells.

Proof. \( \mathcal{M}_{p,q} \) is obtained by attaching \((q-p+1)\)-cells a wedge of \( c_{p,q} \) copies of the sphere \( S^{q-p} \), where each cell is attached by a map of the form (16).

The mapping cone of (16) is the wedge of the mapping cone of the pinch map with a wedge of spheres \( S^{q-p} \) indexed over \( c_{p,q} - 2^{q-p} \). Hence, by induction it suffices to show that the mapping cone of the above pinch map is a wedge of \( 2^{q-p} - 1 \) copies of \( S^{q-p} \). But this follows directly from Lemma 8.12 below.

The above proof used the following assertion about the pinch map.

Lemma 8.12. The mapping cone of the pinch map
\[ S^k \to \bigvee_m S^k. \]
is a wedge of \((m - 1)\)-copies of \( S^k \).
Proof. For the sake of completeness we first define the pinch map. Let $T \subset S^1$ be the set of $m$-th roots of unity. Suspend the quotient map $S^1 \to S^1/T$ $(k-1)$-times to obtain a map $S^k \to \Sigma^{k-1}(S^1/T)$. The latter is the pinch map once we identify $\Sigma^{k-1}(S^1/T)$ with an $m$-fold wedge of $k$-spheres (this is clear, since $S^1/T$ is a bouquet of $m$ circles).

We now proceed with the proof. Consider the commutative diagram

$$
\begin{array}{c}
* \\
\downarrow \\
S^k \text{ pinch} \\
\downarrow \\
S^k \\
\end{array}
\quad
\begin{array}{c}
\lor_{m-1} S^k \\
\downarrow i_1 \\
\lor_m S^k \\
\downarrow p_1 \\
C \\
\downarrow \\
* \\
\end{array}
$$

whose horizontal and vertical rows are homotopy cofiber sequences, where $p_1$ is projection onto the first summand and $i_1$ is the inclusion away from the first summand. The upper left square is homotopy cocartesian, so the map $\lor_{m-1} S^k \to C$ is a homotopy equivalence. $\square$

9. Analytical hypercurrents

In this section we define an analytical version of the hypercurrent. In what follows, $\mathbb{F}$ is taken to be the field of real numbers $\mathbb{R}$.

9.1. Motivation. Suppose $\Sigma = \mathbb{R}$ and $X$ is a connected CW complex of dimension one. Let $C_\ast(X)$ be the real cellular chain complex of $X$. Fix a smooth protocol $\gamma : \mathbb{R} \to \mathcal{M}_X$. The master equation (or Kolmogorov equation) is a first order linear differential equation

$$
\dot{\rho} = -H\rho
$$

where $H = H(t) = \partial E,W$ is the biased graph Laplacian, and $\rho(t) \in C_0(X)$ is a one-parameter family of zero chains, i.e., distributions. The operator $\partial E,W : C_0(X) \to C_1(X)$ is the (time-dependent) biased formal adjoint to the boundary operator $\partial : C_1(X) \to C_0(X)$ that uses $\gamma(t) = (E(t),W(t))$ to modify the standard inner product structure on $C_\ast(X)$ (for details, see [1] and [2]). Equation (17) describes biased diffusion on $X$.

The analytical current $\gamma$ is the quantity

$$
-\partial E,W \rho \in C_1(X)
$$

where $\rho$ is a formal solution to (17). Then (18) coincides with

$$
\partial E,W \dot{\rho} = \partial E,W d\rho \mu
$$
where $\partial_{E,W}^\dagger$ is the Moore-Penrose pseudoinverse of $\partial$ (using the modified inner product structure), $d$ is exterior derivative, $\mu = \frac{d}{dx}$ is the co-volume form of $\mathbb{R}$ and $\cdot$ denotes contraction.

Since the operation $\phi \mapsto \phi \cdot \mu$ equates 1-forms with 0-forms, we could have instead defined the analytical current as the 1-form

$$J^{an} := \partial_{E,W}^\dagger d\rho \in \Omega^1(\mathbb{R}; C_1(X)) \tag{20}$$

The main advantage of the reformulation (20) is that it will generalize to higher dimensions.

**Remark 9.1.** In [1] we studied the case when $\gamma$ is periodic. After taking adiabatic (slow driving) limit, we showed that the formal solution $\rho$ to (17) is also periodic. In this case, we can consider the analytical current as a 1-form on the circle. Furthermore, the “adiabatic theorem” implies that can use the Boltzmann distribution in place of the formal solution $\rho$ to define $J^{an}$. The Boltzmann distribution is the unique 0-form $\rho^B : S^1 \to C_0(X)$ such that $\partial_{E,W}^\dagger \rho^B = 0$.

The fundamental theorem of calculus implies that integration of this one form defines a homology class $q \in H_1(X; \mathbb{R})$ called the average current. In what follows below, the periodicity condition is replaced by the assumption the domain $\Sigma$ of $\gamma$ is a closed manifold.

9.2. **Higher dimensions.** The input for the construction is a triple

$$(X, \Sigma, \gamma) ,$$

in which $X$ is a finite connected CW complex, and $\Sigma$ is a compact smooth manifold and $\gamma : \Sigma \to \prod_j \mathbb{R}^{X_j}$ is a smooth protocol with respect to a gap $[p, q]$. As usual, we often write $\gamma(b) = W_b(b)$. We do not require $\gamma$ to be good.

The analytical hypercurrent takes values in homology with real coefficients. We will define a smooth differential form

$$\mathcal{J}^{an} \in \Omega^*(\Sigma; \text{end}(\bar{C}))$$

of total degree zero.

9.3. **Generalized inverses.** We digress to describe formulas for the pseudoinverse of a linear transformation in two different cases.

Suppose that $A : V \to W$ is a one-to-one linear transformation of finite dimensional real vector spaces, where $W$ is equipped with an inner product. The adjoint can be considered as the linear map $A^* : W \to V^*$ characterized by the equation $A^*(w)(v) = \langle w, Av \rangle$. Then the composition $A^*A$ is invertible and the pseudoinverse of $A$ in this case can be defined as the left inverse

$$A^\dagger := (A^*A)^{-1} A^* : W \to V .$$
In this instance one can characterize $A^\dagger$ as follows: for a given vector $w \in W$, the vector $v := A^\dagger(w) \in V$ is the unique vector which minimizes the norm
\[ |Av - w| . \]

Similarly, suppose a linear transformation $D: V \to U$ is onto, and assume that $V$ is equipped with an inner product. Consider the adjoint $D^*: U^* \to V$ that is characterized by $\langle D^*(\delta), v \rangle = \delta(D(v))$. In this case $DD^*$ is invertible and we define the pseudoinverse as the right inverse
\[ D^\dagger := (DD^*)^{-1}: U \to V . \]

Then $D^\dagger$ is characterized by the following property: for any vector $u \in U$, the vector $v = D^\dagger(u) \in V$ is the unique vector satisfying $Dv = u$ such that the norm
\[ |v| \]
is a minimum.

9.4. Some operators. Fix a smooth protocol $\Sigma \to M_{p,q}$. For each $b \in \Sigma$ one has a smooth weight $W_k(b): X_k \to \mathbb{R}$ for $p \leq k \leq q$. The weight $W_k(b)$ defines a modified inner product on $\bar{C}_k$ which is defined on basis elements by $(e, e')_W := W_k(b)\delta_{ee'}$ where $e, e' \in X_{k-p}$ and $\delta_{ee'}$ is Kronecker delta.

Using the modified inner product, we form the pseudoinverse of the surjection $\partial: \bar{C}_k \to \bar{B}_{k-1}$, where $B_j$ denotes the vector space of $j$-boundaries. If we let $b \in B$ vary, the construction just outlined gives a 0-form
\[ \partial_W^\dagger: \Sigma \to \text{hom}(\bar{B}_{k-1}, \bar{C}_k) . \]
The latter induces a homomorphism of the same name
\[ (21) \quad \partial_W^\dagger: \Omega^*(\Sigma; \text{hom}(U, \bar{B}_{k-1})) \to \Omega^*(\Sigma; \text{hom}(U, \bar{C}_k)) \]
for any finite dimensional vector space $U$. The map (21) can be described pointwise at $b \in \Sigma$ by the operation which sends $\phi_b: \Lambda^*T_b\Sigma \to \text{hom}(U, \bar{B}_{k-1})$ to the composite
\[ \Lambda^*T_b\Sigma \xrightarrow{\phi_b} \text{hom}(U, \bar{B}_{k-1}) \xrightarrow{\partial_W^\dagger(b)\circ} \text{hom}(U, \bar{C}_k) , \]
where the second displayed map is given by $f \mapsto \partial_W^\dagger(b) \circ f$.

Similarly, using the modified inner product, we can form the pseudoinverse of the inclusion $i: \bar{B}_k \to \bar{C}_k$ to obtain a 0-form
\[ (22) \quad i_W^\dagger: \Sigma \to \text{hom}(\bar{C}_k, \bar{B}_k) . \]
9.5. **Procedure.** For each integer \( \ell \geq 0 \) we construct an \( \ell \)-form

\[
J^\text{an}_\ell \in \Omega^\ell(\Sigma; \hom(\overline{C}_0, \overline{C}_\ell)).
\]

**Step (1):** When \( \ell = 0 \), define

\[
J^\text{an}_0 : \Sigma \to \operatorname{end}(\overline{C}_0, \overline{C}_0) = I - i^\dagger W : \Sigma \to \hom(\overline{C}_0, \overline{B}_0),
\]

where, \( I \) denotes the constant map whose value is the identity.

**Step (2):** When \( \ell = 1 \), note that

\[
dJ^\text{an}_0 = dI - di^\dagger W = -di^\dagger W
\]

lies in the subspace \( \Omega^1(\Sigma; \hom(\overline{C}_0, \overline{B}_0)) \). Define

\[
J^\text{an}_1 := \partial_W dJ^\text{an}_0,
\]

where \( \partial_W \) is as in (21). Note that \( J^\text{an}_1 \) lies in \( \Omega^1(\Sigma; \hom(\overline{C}_0, \overline{C}_1)) \).

**Step (3):** Assume

\[
J^\text{an}_{\ell-1} \in \Omega^{\ell-1}(\Sigma; \hom(\overline{C}_0, \overline{C}_{\ell-1}))
\]

has been defined for some \( \ell \) satisfying \( 2 \leq \ell \leq q - p \). We will show how to define \( J^\text{an}_\ell \).

First apply exterior derivative to \( J^\text{an}_{\ell-1} \), giving

\[
dJ^\text{an}_{\ell-1} \in \Omega^\ell(\Sigma; \hom(\overline{C}_0, \overline{C}_{\ell-1})).
\]

**Lemma 9.2.** Assume \( 2 \leq \ell \leq q - p \). Then \( dJ^\text{an}_{\ell-1} \) lies in the subspace \( \Omega^\ell(\Sigma; \hom(\overline{C}_0, \overline{Z}_{\ell-1})) \).

**Proof.** For any chain \( c \in \overline{C}_s \) we write \( J^\text{an}_{\ell-1}(c) \in \Omega^{\ell-1}(\Sigma; \overline{C}_{s+\ell-1}) \) for the evaluation of \( J^\text{an}_{\ell-1} \) at \( c \) arising from the linear transformation

\[
\operatorname{end}(\overline{C})_{\ell-1} \to \overline{C}_{s+\ell-1}
\]

given by \( f \mapsto f(c) \).

It will be enough to prove that \( \partial dJ^\text{an}_{\ell-1}(c) = 0 \) for all \( c \), where the boundary operator \( \partial \) in this case is the one for \( \overline{C}_s \). The latter assertion follows directly from the fact that the operators \( \partial \) and \( d \) commute and the fact that \( \partial \partial_W^\dagger \) is the identity. \( \square \)

Since \([p, q]\) is a gap, we have \( \overline{B}_{\ell-1} = \overline{Z}_{\ell-1} \) for \( 2 \leq \ell \leq q - p \). Hence,

**Corollary 9.3.** If \( 2 \leq \ell \leq q - p \), then the form \( dJ^\text{an}_{\ell-1} \) lies in the subspace

\[
\Omega^\ell(\Sigma; \hom(\overline{C}, \overline{B})_{\ell-1}).
\]
Using the corollary, we are entitled to apply
\[ \partial^\dagger_W : \Omega^\ell(\Sigma; \hom(\overline{C}_0, \overline{B}_{\ell-1})) \to \Omega^\ell(\Sigma; \hom(\overline{C}_0, \overline{C}_\ell)) \]
to the form \( dJ_{\ell-1}^\an \).

**Definition 9.4.** For \( 2 \leq \ell \leq q - p \) we set
\[
(23) \quad J^\an_{\ell} := \partial^\dagger_W dJ_{\ell-1}^\an \in \Omega^\ell(\Sigma; \hom(\overline{C}_0, \overline{C}_\ell)).
\]
The above defines \( J^\an_{\ell} \) for \( 0 \leq \ell \leq q - p \). For all other \( \ell \) we set \( J^\an_{\ell} = 0 \). Taking the sum over \( \ell \) then defines the form
\[
J^\an = \sum_{\ell} J^\an_{\ell} \in \Omega^*(\Sigma; \text{end}(\overline{C})).
\]

**Lemma 9.5.** For all \( \ell \) we have
\[
\partial J^\an_{\ell} = dJ^\an_{\ell-1}.
\]

*Proof.* The statement is obvious for \( \ell = 0 \). For \( \ell > 0 \) let \( c \in \overline{C}_0 \) be a 0-chain. We apply the boundary operator \( \partial \) for the chain complex \( \overline{C} \) to the equation \( J^\an_{\ell}(c) = \partial^\dagger_W dJ^\an_{\ell-1}(c) \). This results in the identity
\[
\partial J^\an_{\ell}(c) = dJ^\an_{\ell-1}(c).
\]
On the other hand
\[
(\partial J^\an_{\ell})(c) = \partial J^\an_{\ell}(c)
\]
since \( \partial c = 0 \).

**Corollary 9.6.** The form \( J^\an \) satisfies the identity
\[
\delta J^\an = dJ^\an_{q-p},
\]
where \( \delta \) is the coboundary operator for the complex \( \Omega^*(\Sigma; \text{end}(\overline{C})) \).

*Proof.* We have
\[
\delta J^\an = \sum_{\ell} \delta J^\an_{\ell},
\]
\[
= \sum_{\ell} dJ^\an_{\ell} + (-1)^\ell dJ^\an_{\ell-1},
\]
\[
= dJ^\an_{q-p},
\]
where the second line of the display uses the definition of \( \delta \) as well as the identity \( \delta J^\an_{\ell} = dJ^\an_{\ell-1} \).

*Proof of Theorem E.* To prove the existence part of Theorem E we need to show that the forms \( J^\an_{\ell} \) that were constructed above satisfy axioms A1-A3. Axiom A1 was established in Lemma 9.5. Axiom A2 holds when \( \ell = 0 \) since \( J^\an_{0} = I - \delta^\dagger_W \) is the orthogonal projection onto \( D_{W_0} \).
in the modified inner product. For \( \ell > 0 \) axiom A2 follows from the definition

\[ J^{\text{an}}_{\ell} = \partial_{W} d J^{\text{an}}_{\ell-1} \]

and the formula for the pseudoinverse of \( \partial \) given by

\[ \partial^{\dagger}_{W} := \partial^{\ast}_{W} (\partial \partial^{\ast}_{W})^{-1}. \]

It follows that \( J^{\text{an}}_{\ell} \) lies in the subspace \( \Omega^{\ell}(\Sigma; \text{hom}(\bar{C}_{0}, \bar{D}_{W\ell})) \). The verification of axiom A3 is trivial.

We next show that the axioms uniquely determine the forms \( J^{\text{an}}_{\ell} \). Set \( \Phi_{\ell} = J^{\text{an}}_{\ell} \) and suppose that \( \Psi_{\ell} \) is another collection of forms satisfying the axioms.

Set \( \alpha_{\ell} = \Phi_{\ell} - \Psi_{\ell} \). It suffices to prove that \( \alpha_{\ell} \) is trivial. The forms \( \alpha_{\ell} \) satisfy axiom A2 x the axioms

(B1): \( \delta \alpha_{\ell} = 0; \)

(B3): \( \alpha_{0}: \Sigma \to \text{end}(H_{0}(\bar{C})) \) is trivial.

For the argument we rename Axiom A2 by B2.

For \( \ell = 0 \) axiom B1 implies that \( \alpha_{0}: \Sigma \to \text{end}(\bar{C}_{0}) \) takes values in chain maps. Project \( \alpha_{0} \) onto the \( j \)-th component of \( \text{end}(\bar{C}_{0}) \) to obtain a map \( \alpha_{0,j}: \Sigma \to \text{end}(\bar{C}_{j}) \) and let \( b \in \Sigma \) be any point. Define \( f_{0,j}: \bar{C}_{j} \to \bar{C}_{0} \) by \( f_{0,j} = \alpha_{0,j}(b) \).

Axiom B2 shows that \( f_{0,0} \) factors as

\[ \bar{C}_{0} \to \bar{D}_{W0} \subset \bar{C}_{0} \]

where \( \bar{D}_{W0} \) is the orthogonal complement of \( B_{0} \) in the modified inner product. Axiom B3 implies that the first map in the composite is trivial. If follows that \( f_{0,0} \) is trivial. A similar argument shows that \( f_{0,j} \) is trivial for \( j > 0 \). Consequently, \( \alpha_{0} = 0 \). This establishes the basis step of the induction on \( \ell \).

For the inductive step, assume \( \alpha_{j} = 0 \) for \( 0 \leq j \leq \ell - 1 \). Let \( f_{\ell,k}: \bar{C}_{k} \to \bar{C}_{\ell+k} \) be the \( k \)-th component of the evaluation of \( \alpha_{\ell} \) at any vector \( v \in \Lambda^{\ell} T_{0} \Sigma \) for some \( b \in \Sigma \). In other words, \( \alpha_{\ell}(v) = \oplus_{k} f_{\ell,k}. \)

Then Axiom B1 implies that the diagram

\[ \begin{array}{ccc}
\bar{C}_{k} & \xrightarrow{f_{\ell,k}} & \bar{C}_{\ell+k} \\
\partial & & \partial \\
\bar{C}_{k-1} & \xrightarrow{f_{\ell,k-1}} & \bar{C}_{\ell+k-1}
\end{array} \]

commutes up to sign. It follows that \( \partial f_{\ell,k} = \pm f_{\ell,k-1} \partial \). We next make subsidiary induction in \( k \). An argument like the one in the previous paragraph shows that \( f_{\ell,0} \) is trivial. Assume that \( f_{\ell,k-1} \) is trivial. Then by axiom B2, the image of \( f_{\ell,k} \) lies in \( \bar{D}_{W_{\ell+k}} \cap \bar{Z}_{\ell+k} = 0 \). It follows
that \( f_{\ell,k} \) is trivial, establishing the inductive step indexed for the pair \((\ell, k)\). It follows that \( \alpha_\ell = 0 \), completing the induction in \( \ell \).

9.6. **An explicit formula.** Let 0-forms \( \alpha_j \in \Omega^0(\Sigma; \text{end}(\bar{C})_0) \) be defined by

\[
\alpha_0 := I - i^t_W, \quad \alpha_j := \partial_j^t W \quad \text{for} \quad 0 < j \leq q - p.
\]

Set

\[
\beta_j := d\alpha_j, \quad 0 \leq j \leq q - p.
\]

Then \( \beta_j \) is a 1-form.

**Corollary 9.7.** For \( 0 \leq \ell \leq q - p \), the following identity is satisfied:

\[
J_{\text{an}}^{\ell} = \begin{cases} 
\alpha_0 & \text{if} \quad \ell = 0, \\
\alpha_\ell \wedge \beta_{\ell-1} \wedge \cdots \wedge \beta_0 & \text{otherwise}.
\end{cases}
\]

**Proof.** The forms on the right side satisfy axioms A1-A3. Consequently, the result follows from Theorem E. \( \square \)

**10. The Quantization Theorem**

In this section we carefully formulate and prove Theorem F. In what follows we assume \( \mathbb{F} = \mathbb{R} \).

10.1. **An analytical cochain.** Let

\[
R \subset I_\Sigma
\]

be a closed subset. The following are some examples of interest:

1. \( R = R_\Sigma \) is the poset of smooth small singular simplices.
2. \( R = R_\Sigma^{q-p} \) is the poset of smooth small singular simplices of dimension at most \( q - p \).
3. \( R = R_\sigma \) consists of a smooth small singular simplex \( \sigma : \Delta^j \to \Sigma \) together with all of its faces, where \( j \leq q - p \).

Let \( R^* \) be the chain complex over \( \mathbb{R} \) that is freely generated in each degree by the objects of \( R \). Then integration along the elements of \( R^* \) defines a map of cochain complexes

\[
\Omega^*(\Sigma; \text{end}(\bar{C})) \to \text{hom}(R^*, \text{end}(\bar{C})),
\]

\[
\phi \mapsto (\sigma \mapsto \int_{\Delta^j} \sigma^* \phi),
\]

called the **Stokes map.** It is a quasi-isomorphism when \( R = R_\Sigma \).

**Definition 10.1.** The **restricted analytical hypercurrent cochain**

\[
J_{\text{an}}^{\ell} \in \text{hom}(R^*, \text{end}(\bar{C}))
\]

is given by the image of the analytical hypercurrent current form \( J_{\text{an}}^{\ell} \) with respect to the Stokes map (24).
Lemma 10.2. If $R = R^q_{\Sigma}$ or $R = R_\sigma$, then $\mathcal{J}^{an}_{|R}$ is a cocycle.

Proof. By Corollary 9.6,

$$\delta \mathcal{J}^{an} = d \mathcal{J}^{an}_{q-p} \in \Omega^{q-p+1}(\Sigma; \text{hom}(\bar{C}_0, \bar{C}_{q-p})).$$

The result then follows by observing that the pullback of $d \mathcal{J}^{an}_{q-p}$ along a small smooth singular simplex of dimension $\leq q - p$ is trivial for dimensional reasons. \qed

Let $\mathcal{K}^*(R)$ denote the right side of (24). Then there is a canonical isomorphism of vector spaces

$$\text{hom}_{\text{ho} \mathcal{C}^R}(\chi, \chi) \cong H^0(\mathcal{K}^*(R)), $$

where the left side denotes hom taken in the homotopy category of $\mathcal{C}^R$ (cf. Definition 5.10) and $\chi$ is the constant functor with value $\bar{C}$.

Definition 10.3. (1). Let $\mathcal{U}_R$ be the vector space of 0-cochains of the complex $\mathcal{K}^*(R)$. Let

$$\mathcal{B}_R \subset \mathcal{U}_R \quad \text{and} \quad \mathcal{Z}_R \subset \mathcal{U}_R$$

be the space of 0-coboundaries and 0-cocycles respectively.

(2). Set

$$\mathcal{U}^*_R := \mathcal{U}_R / \mathcal{B}_R.$$ 

For an affine subspace $V \subset \mathcal{U}_R$ we let $V^* \subset \mathcal{U}^*_R$ be the image of $V$ under the quotient map $\mathcal{U}_R \to \mathcal{U}^*_R$. When $V = \mathcal{Z}_R$ we set

$$\mathcal{H}_R := \mathcal{Z}^*_R = \mathcal{Z}_R / \mathcal{B}_R.$$

(3). For $\xi \in \text{end}(H_0(\bar{C}))$, let

$$\mathcal{Z}_R(\xi) \subset \mathcal{Z}_R$$

be the subspace of 0-cocycles $c: R_* \to \text{end}(\bar{C})$ such for $\sigma \in R$, the chain map $c(\sigma): \bar{C} \to \bar{C}$ induces $\xi$ in homology in degree 0, and is trivial in homology in higher degrees. We set

$$\mathcal{H}_R(\xi) := (\mathcal{Z}_R(\xi))^*,$$

i.e., the image of $\mathcal{Z}_R(\xi)$ under the projection $\mathcal{Z}_R \to \mathcal{Z}^*_R$.

(4). Let

$$\mathcal{T}_R \subset \mathcal{U}_R$$

be the subspace of 0-cochains $c: R_* \to \text{end}(\bar{C})_*$ such that if $\sigma \in R$, then $c(\sigma)$ lies in $\text{hom}(\bar{C}, \tau(\sigma))$, where $\tau(\sigma) = \bar{C}(T_\sigma)$ is the value of the tree functor at the small simplex $\sigma$. 
Let
\[ T_R^\sharp \subset U_R^\sharp \]
denote the image of \( T_R \) under the quotient \( U_R \to U_R^\sharp \).

Observe that \( H_R(\xi) \subset U_R^\sharp \) is an affine subspace, whereas \( T_R^\sharp \subset U_R^\sharp \) is a vector subspace.

**Proposition 10.4.** Let \( R \) be any of the posets of interest and suppose \( \xi \in \text{end}(H_0(\bar{C})) \) is the identity. Then the intersection
\[ H_R(\xi) \cap T_R^\sharp \]
consists of a single element, namely, the affine subspace
\[ J_{|R} + B_R \]
associated with the restricted topological hypercurrent chain map \( J_{|R} \).

**Proof.** An unravelling of the definitions shows the assertion to be a reformulation of Corollary 5.11. \( \square \)

10.2. **Topologizing \( U_R^\sharp \).** The affine spaces considered above are subspaces of \( U_R^\sharp \), the latter which is typically infinite dimensional when \( R = R_\Sigma \) or \( R_p^{q-p} \). We will topologize \( U_R^\sharp \) as a quotient of \( U_R^\sharp \). We will topologize the latter as subspace of the product
\[ U_R \subset \bigprod_{\sigma \in R} \text{end}(\bar{C})_{j_\sigma} \]
We are reduced to describing a topology on the displayed product. Note that the real vector space \( \text{end}(\bar{C})_{j_\sigma} \) is finite dimensional. We topologize the displayed product using the box topology.

Let \( \pi: U_R \to U_R^\sharp \), denote the quotient map. The following is a trivial consequence of the definitions.

**Lemma 10.5.** Let \( \hat{x}: \mathbb{N} \to U_R \) be a sequence. Then the composite sequence
\[ x: \mathbb{N} \xrightarrow{\hat{x}} U_R \xrightarrow{\pi} U_R^\sharp \]
converges to \( y \in U_R^\sharp \) if and only if for any open neighborhood \( V \) of \( y \), there is an \( N \geq 0 \) such that \( \hat{x}_n \) is contained in \( \hat{V} := \pi^{-1}(V) \) for \( n > N \).
10.3. **The low temperature limit.** The form $J_{an}$ depends in a crucial way on the choice of smooth protocol $\gamma: \Sigma \to M_{p,q}$. If $\beta \in \mathbb{R}_+$, let $\gamma_\beta: \Sigma \to M_{p,q}$ be the protocol given by
\[ \gamma_\beta(x) := \beta \gamma(x), \]
where the right side is given by scalar multiplication. By replacing $\gamma$ with $\gamma_\beta$ in the construction of the analytical hypercurrent we similarly obtain
$$J_{an,\beta} \in \Omega^*(\Sigma; \text{end}(\bar{C})).$$
If $R$ is one of the posets of interest, then the corresponding restricted analytical hypercurrent cochain is denoted by
$$J_{an,\beta}^R.$$ By mild abuse of notation we identify the latter with its coset $J_{an,\beta}^R + B_R$ in $U^\sharp$.

**Theorem 10.6** (Quantization). Let $R \subset R_{\Sigma}^{q-p}$ be closed. Then in $U^\sharp$ we have
$$\lim_{\beta \to \infty} J_{an,\beta}^R = J_R,$$
where the limit is taken in the topology described above.

**Remark 10.7.** The limit $\beta \to \infty$ is known as the low temperature limit. Theorem 10.6 says that the low temperature limit of the restricted analytical hypercurrent cochain coincides with its topological counterpart.

10.4. **The Kirchhoff decomposition.** In this subsection, we extend the generalized Kirchhoff and Boltzmann distributions of [4],[2] to deduce a decomposition of the form $J_{an}^p$. This decomposition is subsequently used in the proof of the Quantization Theorem. In what follows we fix a pair
$$(\gamma, \beta),$$
where $\gamma: \Sigma \to M_{p,q}(X)$ is a protocol and $\beta \in \mathbb{R}_+$ is a choice of inverse temperature $\beta \in \mathbb{R}_+$. Recall that $\gamma_\beta(x) := \beta \gamma(x)$.

Let $T$ be a $d$-tree of $X$, where $d \in [p, q]$ (cf. Definition 4.7) If $d > p$, then $T$ determines a preferred right inverse
$$\partial_T^+: \bar{B}_{d-p-1} \to \bar{C}_{d-p}$$
for the boundary operator $\partial: \bar{C}_{d-p} \to \bar{B}_{d-p-1}$. The right inverse is obtained by inverting the composition
$$\bar{C}_{d-p}(T) \xrightarrow{\cap} \bar{C}_{d-p}(X) \xrightarrow{\partial} \bar{B}_{d-p-1}(X),$$
the latter which is an isomorphism.
If \( d = p \), then a \( p \)-co-tree \( T \) determines a right inverse \(-i_T^+\) to \(-i: \bar{B}_0 \to \bar{C}_0\), where \( i \) denotes the inclusion. In this case, it will be convenient to abuse notation slightly by setting
\[
\partial_T^+ := -i_T^+: \bar{C}_0 \to \bar{B}_0,
\]

For a \( d \)-tree \( T \) with \( d > p \), let \( \tau_T \) be the order of the torsion subgroup of \( H_{d-1}(T; \mathbb{Z}) \). When \( j = p \) we define \( \tau_T \) to be the order of the torsion subgroup of \( C_p(X; \mathbb{Z})/B_p(T; \mathbb{Z}) \). Let \( W_T: \Sigma \to \mathbb{R} \) be defined by \( W_T(x) = \sum_{b \in T} W_d(x)(b) \). Set
\[
(25) \quad \rho_T = \frac{1}{\Delta} \tau_T^2 e^{-\beta W_T}, \quad \Delta = \sum_T \tau_T^2 e^{-\beta W_T},
\]
where the sum is over all \( d \)-trees.

**Lemma 10.8** (cf. \[4],[2]). With respect to the modified inner product defined by \( \beta W_j \), the Moore-Penrose pseudoinverse to the boundary operator \( \partial: \bar{C}_j \to \bar{B}_{j-1} \) (respectively to the inclusion \(-i: \bar{B}_0 \to \bar{C}_0 \) if \( j = 0 \)) is given by the expression
\[
(26) \quad \partial_j^\dagger := \sum_T \rho_T \partial_T^+,
\]
where the sum is over all \((j+p)\)-trees \( T \).

**Remark 10.9.** The function \( \rho_T: \Sigma \to \mathbb{R}_+ \) and the operator \( \partial_j^\dagger \) depend on the value of \( \beta W_{j+p} \) even though the notation fails to indicate it. The operator \( \partial_T^+ \), which depends on \( T \), is independent of \( \beta W_{j+p} \).

**Definition 10.10.** Let \( \ell \in [0, q-p] \) be an integer. An \( \ell \)-orchard for \( X \) is an ordered \((\ell+1)\)-tuple
\[
\omega_\bullet = (\omega_0, \omega_1, \ldots, \omega_\ell),
\]
in which \( \omega_j \) is a \((j+p)\)-tree of \( X \).

To build a Kirchhoff decomposition for the form \( J^\omega_\bullet \), we arbitrarily fix, for each \( j \in (0, \ell) \), a left inverse
\[
\zeta_j: \bar{C}_j \to \bar{B}_j
\]
to the inclusion \( \bar{B}_j \to \bar{C}_j \); denote these data by \( \zeta_\bullet \). For an \( \ell \)-orchard \( \omega_\bullet \), let \( f_\ell(\omega_\bullet): \bar{C}_0 \to \bar{C}_\ell \) be the linear transformation defined by
\[
(26) \quad f_\ell(\omega_\bullet) = \partial_{\omega_\ell}^+ \zeta_{\ell-1} \partial_{\omega_{\ell-1}}^+ \zeta_{\ell-2} \partial_{\omega_{\ell-2}}^+ \cdots \zeta_1 \partial_{\omega_1}^+ \partial_{\omega_0}^+.
\]
Define an \( \ell \)-form \( g_{\ell}(\omega_\bullet) \in \Omega^\ell(\Sigma; \mathbb{R}) \) by
\[
(27) \quad g_{\ell}(\omega_\bullet) := \rho_{\omega_\ell} d\rho_{\omega_{\ell-1}} \wedge \cdots \wedge d\rho_{\omega_0}.
\]
Hence, \( f_\ell \) depends on the pair \((\omega_\bullet, \zeta_\bullet)\), whereas \( g_{\ell} \) depends on the triple \((\omega_\bullet, \gamma, \beta)\).
**Proposition 10.11** (Kirchhoff Decomposition). *The form* \( \mathcal{J}^\text{an}_\ell \) *is given by*

\[
\sum_{\omega} f_\ell(\omega) \varrho_\ell(\omega),
\]

*where the sum is indexed over all \( \ell \)-orchards of \( X \).*

**Remark 10.12.** *In particular, Proposition 10.11 implies that the sum of the terms* \( f_\ell(\omega) \varrho_\ell(\omega) \) *is independent of the choice of* \( \zeta \).

**Proof of Proposition 10.11.** *Observe that* \( \zeta_{j-1} d\partial^j_{j-1} \) *coincides with* \( d\partial^j_{j-1} \), *since the image of* \( d\partial^j_{j-1} \) *is contained in* \( \bar{B}_{j-1} \). *Then*

\[
\mathcal{J}^\text{an}_\ell = \partial^\dagger_\ell d\partial^\dagger_{\ell-1} \wedge \cdots \wedge d\partial^\dagger_1 \wedge d\partial^\dagger_0,
\]

\[
= \partial^\dagger_j \zeta_{j-1} d\partial^\dagger_{\ell-1} \wedge \cdots \wedge \zeta_1 d\partial^\dagger_1 \wedge d\partial^\dagger_0,
\]

\[
= \sum_{\omega} \rho_{\omega_\ell} \partial_{\omega_\ell}^+ \sum_{\omega_{\ell-1}} d\rho_{\omega_{\ell-1}} \zeta_{\ell-1} \partial_{\omega_{\ell-1}}^+ \wedge \cdots \wedge \sum_{\omega_1} d\rho_{\omega_1} \zeta_1 \partial_{\omega_1}^+ \wedge \sum_{\omega_0} \rho_{\omega_0} \partial_{\omega_0}^+,
\]

\[
= \sum_{\omega} f_\ell(\omega) \varrho_\ell(\omega),
\]

*where the first line uses Lemma 9.7 and the transition from line two to line three uses Lemma 10.8.*

10.5. **Proof of the Quantization Theorem.** *With respect to Definition 10.3, set*

\[
\mathcal{D}^\sharp_R := \mathcal{U}^\sharp_R / \mathcal{T}^\sharp_R.
\]

*Then we have an short exact sequence of vector spaces*

\[
0 \to \mathcal{T}^\sharp_R \xrightarrow{i} \mathcal{U}^\sharp_R \xrightarrow{p} \mathcal{D}^\sharp_R \to 0
\]

*in which each term is finite dimensional when* \( R = R_\sigma \).

**Lemma 10.13.** *For any of the posets of interest, the composition*

\[
\mathcal{H}_R(0) \to \mathcal{U}^\sharp_R \xrightarrow{p} \mathcal{D}^\sharp_R
\]

*is a monomorphism of vector spaces.*

**Proof.** *The proof is almost the same as the proof of Proposition 10.4 where* \( \xi \) *is now replaced by 0. The details will be left to the reader.*

*The Quantization Theorem will follow from the next two results.*

**Lemma 10.14.** *If the Quantization Theorem holds for* \( R = R_\sigma \), *then it also holds for all closed subsets of* \( R^{P-\Sigma}_\Sigma \).

**Proof.** *By definition, convergence in* \( \mathcal{U}^\sharp_R \) *is defined objectwise in* \( \sigma \).
Proposition 10.15. Let $R = R_\sigma$. Then
\[
\lim_{\beta \to \infty} p(J_{an}^{R,\beta}) = 0.
\]

Proof of Theorem 10.6 assuming Proposition 10.15. By Lemma 10.14, it is enough to consider the case $R = R_\sigma$. Consider the difference
\[
z_\beta := J_{an}^{R,\beta} - J^R
\]
Then $z_\beta \in \mathcal{H}_R(0)$ for all $\beta$. We need to show that $z_\beta$ tends to zero in $U^2_R$.

Observe that $p(z_\beta) = p(J_{an}^{R,\beta})$. Using Proposition 10.15, we infer that $z_\beta$ tends to the subspace $T^2_R$ as $\beta$ tends to $\infty$. Hence, $z_\beta$ tends to the intersection $\mathcal{H}_R(0) \cap T^2_R$. But Lemma 10.13 says this intersection is just the zero vector. It follows that $z_\beta$ tends to $0$. \qed

Proof of Proposition 10.15. We begin by reviewing the relevant definitions and introducing some notation. In what follows

- $\gamma: \Sigma \to M_{p,q}$ is a good protocol;
- $\beta > 0$ is a real number;
- For $t \in \Sigma$, and $p \leq n \leq q$, the function $W_n(t): X_\ell \to \mathbb{R}$ is the $n$-th component of $\gamma(t)$.
- $\sigma: \Delta^j \to \Sigma$ is a small smooth singular simplex;
- $k := k(\sigma)$ is the smallest integer such that $\sigma(\Delta^j) \subset \Sigma_k$, where $\Sigma_k$ is the set of points $t \in \Sigma$ in which $W_k(t): X_k \to \mathbb{R}$ is one-to-one;
- $T_\sigma$ is the preferred $k$-tree associated with $\sigma$;
- for an $n$-tree $T$, $p \leq n \leq q$, we define $W_T = \sum_{e \in T_n} W_e$.

For this proof only, it will also be convenient to use the notation
\[
\mathcal{J}^{an}(\sigma, \beta) := J_{an}^{R,\beta}.
\]

Then we have a canonical decomposition
\[
(28) \quad \mathcal{J}^{an}(\sigma, \beta) = \mathcal{J}^{an}_T(\sigma, \beta) + \mathcal{J}^{an}_D(\sigma, \beta),
\]
uniquely defined by the requirement that
\[
\mathcal{J}^{an}_T(\sigma, \beta) \in \text{hom}(\bar{C}_0(X), \bar{C}_j(T_\sigma)) \subset \text{hom}(\bar{C}_0, \bar{C}_j)
\]
as well as the requirement that the projection (in the basis defined by the cells of $X_{p+j}$) of $\mathcal{J}^{an}_D(\sigma, \beta)$ onto the subspace $\text{hom}(\bar{C}_0, \bar{C}_j(T_\sigma))$ is trivial. It will suffice to show that
\[
\lim_{\beta \to 0} \mathcal{J}^{an}_D(\sigma, \beta) = 0.
\]
The proof of the latter reduces to two cases.

**Case 1:** \( j < k \). In this case, by the defining properties of trees, \( F^m(\sigma, \beta) \) lies in the subspace \( \text{hom}(\hat{C}_0, \hat{C}_j(T_\sigma)) \) which immediately implies that \( F^m(\sigma, \beta) = 0 \).

**Case 2:** \( j > k \). Let \( \omega_\bullet = (\omega_0, \ldots, \omega_\ell) \) be an \( \ell \)-orchard (cf. Defn. 10.10). Then one has, analogous to (28), the expression

\[
\ell(\omega_\bullet, \beta) = \ell^T(\omega_\bullet, \beta) + \ell^D(\omega_\bullet, \beta),
\]

where \( \ell(\omega_\bullet, \beta) \) is defined in (27) (here we are emphasizing its dependence on the parameter \( \beta \)). It will be enough to establish the following.

**Claim:** There are positive constants \( C = C(\sigma), E = E(\sigma) \) such that

\[
|\ell^D(\omega_\bullet, \beta)| < C\beta^j e^{-\beta E},
\]

where the norm on the left is defined using the basis defined by the cells of \( X \).

To establish the claim, we recall the definition (27)

\[
\ell(\omega_\bullet, \beta) := \rho_{\omega_\ell}(\beta) d\rho_{\omega_{\ell-1}}(\beta) \wedge \cdots \wedge d\rho_{\omega_0}(\beta),
\]

where the functions \( \rho_{\omega_j} : \Sigma \to \mathbb{R} \) are defined in (25). By direct computation, we have

\[
d\rho_{\omega_\ell}(\beta) = \beta \sum_\alpha \eta_{\omega_\ell}(\alpha, \beta) dW_\alpha,
\]

with \( \alpha \) ranging over all \((p + \ell)\)-trees and the functions \( \eta_{\omega_\ell}(\alpha, \beta) \) satisfy the identity

\[
\eta_{\omega_\ell}(\alpha, \beta) = \begin{cases} 
\rho_{\omega_\ell}(\beta) \rho_\alpha(\beta), & \alpha \neq \omega_\ell; \\
-\rho_{\omega_\ell}(\beta)(1 - \rho_{\omega_\ell}(\beta)), & \alpha = \omega_\ell.
\end{cases}
\]

In particular, using (27) we infer that

\[
|\eta_{\omega_\ell}(\alpha, \beta)| \leq 1
\]

for all \((p + \ell)\)-trees \( \alpha \).

Set

\[
E := \inf_{\alpha \neq T_\sigma, t \in \Delta} (W_\alpha(\sigma(t)) - W_{T_\sigma}(\sigma(t)))
\]

Then \( E > 0 \) by the defining property of \( T_\sigma \).

In the case when \( \ell = k = k(\sigma) \), it is straightforward to check that one has a tighter bound

\[
|\eta_{\omega_k}(\alpha, \beta)| < e^{-\beta E},
\]

where in this case \( \alpha \) ranges of \( k \)-trees distinct from \( T_\sigma \).
Let \( d\mu \) be the usual volume form for \( \Delta^j \), normalized so that \( \Delta^j \) has unit volume. Let \( g: \Delta^j \to \mathbb{R} \) be the unique function such that
\[
gd\mu = \sigma^*dW_{\omega_{j-1}} \wedge \cdots \wedge dW_{\omega_0}.
\]
Combining the bounds (30) and (31) leads to the inequality
\[
|\varphi^\beta(\omega_\bullet, \beta)| \leq A\beta^je^{-\beta E} \int_{\Delta^j} gd\mu,
\]
where \( A \) is the number of \((p + j)\)-trees. Setting \( C = A\int_{\Delta^j} gd\mu \) establishes the claim.

**Case 2: \( j = k \).** This case is sufficiently similar to Case (2) and is left to the reader to verify. \( \square \)

### 11. Space level hypercurrents

The goal of this section is to show that, in certain cases, the hypercurrent map lifts to a space level construction.

**11.1. The graph case.** In the graph case, a hypercurrent map amounts to the current map in the sense of [1] and can be defined over the ring of integers. We outline the construction using model category language.

Equip the category of topological spaces \( \text{Top} \) with Quillen model structure so that the weak equivalences are the weak homotopy equivalences, the fibrations are the Serre fibrations and the (Serre) cofibrations are defined by the left lifting property with respect to the acyclic fibrations. One can characterize the cofibrations as the (retracts of) relative cell complexes.

For an indexing poset \( I \) whose objects have finite degree, the functor category \( \text{Top}^I \) can then be equipped with the projective model structure, in which the weak equivalences and a fibrations are defined objectwise and the cofibrations satisfy the relative latching condition. In this model structure every object is fibrant.

Let \( X \) be a finite connected topological graph (i.e., a CW complex of dimension one) and let \( \Sigma \) be a good protocol for \( X \). Let \( F \in \text{Top}^I \Sigma \) be the constant functor with value \( X \) and let \( * \in \text{Top}^I \Sigma \) be the constant functor with value the one-point space. Let \( T \in \text{Top}^I \Sigma \) be the tree functor.

The functor \( F^c: I_\Sigma \to \text{Top} \) given by \( F(\sigma) = X \times \Delta^j \) for \( \sigma: \Delta^j \to \Sigma \) is a cofibrant approximation to \( F \). In particular, the lifting problem
\[
\begin{array}{ccc}
T & \xrightarrow{\sim} & \ast \\
\downarrow & & \downarrow \\
F^c & \xrightarrow{T} & * \\
\end{array}
\]
admits a solution (which is also unique up to contractible choice). Consequently, we have a space level pre-hypercurrent map

\[ j : F^c \to T. \]

Consider the composition

\[ F^c \xrightarrow{1} T \xrightarrow{\subseteq} F. \]

Taking colimits, we obtain a weak map of spaces

\[ \mathcal{J} : \Sigma \times X \to X. \]

defining a space level version hypercurrent map. A geometric variant of this construction appeared in [1, defn. 5.8].

11.2. The case \( p = 0, q > 1 \). In this case, \( X \) is a finite connected CW complex with gap \( [0, q) \) for \( q > 0 \). Without loss in generality we can assume \( X \) has dimension \( q \).

Equip \( \text{Top} \) with the localization model category structure defined by the rationalization functor \( X \mapsto X_\mathbb{Q} \) [7, th. 4.7] (cf. [8, ch. 2] in the case of based spaces). If \( f : X \to Y \) is a map of nilpotent spaces, then \( f \) is a weak equivalence with respect to this model structure if and only if \( f \) induces an isomorphism on homology with rational coefficients. A map \( X \to Y \) is a cofibration if and only if it is a Serre cofibration. A fibration is a map that satisfies the right lifting property with respect to the maps which are both Serre cofibrations and weak equivalences. An object \( Y \) is fibrant if and only if the rationalization map \( Y \to Y_\mathbb{Q} \) is a weak homotopy equivalence, i.e., \( Y \) is a rational space.

The construction of a space level hypercurrent map parallels the \( q = 1 \) case, with one difference: the tree functor is no longer fibrant. We should therefore replace it with a fibrant approximation \( T^f \) given by its rationalization \( T^f(s) = T(s)_\mathbb{Q} \). In what follows \( F \) is the constant functor with value \( X \). As in the \( q = 1 \) case, we obtain a space level pre-hypercurrent map

\[ j : F^c \to T^f. \]

The space level hypercurrent map is associated with the composition

\[ F^c \xrightarrow{1} T^f \xrightarrow{\subseteq} F^f, \]

where \( F^f \) is the constant functor with value \( X_\mathbb{Q} \). Applying the homotopy colimit construction to the each functor of (35) and composing, we obtain a map

\[ \Sigma \times X \to \Sigma \times X_\mathbb{Q}. \]
Composing the latter with the second factor projection $\Sigma \times X_\mathbb{Q} \to X_\mathbb{Q}$ gives a space level hypercurrent map
\[
\mathcal{J} : \Sigma \times X \to X_\mathbb{Q}.
\]

11.3. **The case** $p > 0$. In this case, there is generally an obstruction to defining a space level pre-hypercurrent map. However, as we will see, we can perform the construction certain cases.

Let $X$ be a finite connected CW complex $X$ having gap $[p, q]$ with $p > 1$. Without loss in generality, we can assume that $X$ has been $(p, q)$-adjusted, i.e., $X = X^q_p$. In particular $X$ is a based space. In this case we use in the localization model category structure on $\text{Top}_\ast$ defined by rationalization.

Set $A := H_p(X; \mathbb{Q})$ and let $M := M(A, p)$ be the Moore space having reduced homology $A$ in degree $p$ and trivial otherwise.

**Lemma 11.1.** If either
- $p < 2q$, or
- $p, q$ are both odd,

there is a based map $X \to M$ which induces an identity map in rational homology in degree $p$.

**Proof.** There is already a preferred map $X^{(p)} \to M$ which induces a projection on rational homology in degree $p$. The obstructions to extending this map to $X$ lie in the cohomology group $H^q(X/X^{(p)}; \pi_{q-1}(M))$. If either of the conditions hold, then $\pi_{q-1}(M)$ is trivial. \qed

In what follows, we fix such a map $X \to M$. To construct a space level pre-hypercurrent map, we let $F$ be the constant functor with value $X$ and we let $H$ be the constant functor with value $M$. Using the model category factorization axioms, the natural transformations $T \to F \to H$ fit into a commutative diagram
\[
\begin{array}{ccc}
T' & \to & T'' & \to & H \\
\downarrow & & \downarrow & & \downarrow \\
F' & \to & F'' & \to & H
\end{array}
\]
in which the left horizontal maps are acyclic cofibrations and the right maps are fibrations. It is automatic that the map $T' \to H$ is a weak equivalence. It is also automatic that both $T'$ and $F'$ are fibrant since $H$ is.
In particular, the lifting problem

\[
\begin{array}{c}
T' \\
\sim \\
\downarrow \\
F' \\
\downarrow \\
H,
\end{array}
\]

admits a solution \( F^c \to T' \), where \( F^c \) is the functor which assigns to a small simplex \( \Delta^j \to \Sigma \) the space \( \Delta^j \times X \). Taking homotopy colimits of the composition

\[
F^c \to T' \to F'
\]

we obtain a map

\[
\Sigma \times X \to \Sigma \times X_Q.
\]

Composing with the second factor projection, we obtain a space level hypercurrent map

\[
\mathcal{J} : \Sigma \times X \to X_Q.
\]

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