Supersymmetric Higgs Bosons at the Limit

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Abstract

Using a combination of renormalisation group and effective potential methods, we discuss the bound on the lightest CP-even Higgs boson mass $m_h$ in the next-to-minimal supersymmetric standard model. We find $m_h \leq 146, 139, 149 \text{ GeV}$ for $m_t = 90, 140, 190 \text{ GeV}$.  

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1. Introduction

1.1. Triviality

The question of the origin of electroweak mass is one of the most urgent questions of present day particle physics. The discovery of a particle which resembles the Higgs boson of the minimal standard model, and the measurement of its mass, will provide clues as to the nature of new physics beyond the standard model. In this report we shall be concerned with the question of how heavy the lightest neutral CP-even supersymmetric Higgs boson, $h^0$, can be within the framework of supersymmetric grand unified theories (SUSY GUTs) \[1\]. In SUSY GUTs all the Yukawa couplings are constrained to remain perturbative in the region $M_{\text{SUSY}} \sim 1$ TeV to $M_{\text{GUT}} \sim 10^{16}$ GeV. This constraint provides a maximum value at low energies for those Yukawa couplings which are not asymptotically-free, and is obtained from the renormalisation group (RG) equations together with the boundary conditions that the couplings become non-perturbative at $M_{\text{GUT}}$ – the so-called “triviality limit”. In the minimal supersymmetric standard model (MSSM) \[2\], the triviality limits provide a useful bound on the top quark mass $m_t$. The upper bound on the $h^0$ mass, $m_h$, in the MSSM, including radiative corrections, has recently been the subject of much discussion\[3, 4, 5, 6, 7, 8, 9\]. However the MSSM is not the most general low energy manifestation of SUSY GUTs. It is possible that SUSY GUTs give rise to a low energy theory which contains an additional gauge singlet field, the so called next-to-minimal supersymmetric standard model (NMSSM) \[10, 11, 12\]. Here we shall concentrate on the question of the upper-bound on $m_h$ in the NMSSM which is obtained from triviality limits of Yukawa couplings.

1.2. The NMSSM

The NMSSM provides an elegant extension of MSSM which eliminates the $\mu$–problem, the appearance of a mass–scale in the superpotential. The NMSSM differs
from the MSSM by the presence of a gauge singlet field whose vacuum expectation value (vev) plays the role of the mass parameter $\mu$ in the MSSM, and is defined by the superpotential

$$W = h_t Q H_2 t^c + \lambda N H_1 H_2 - \frac{1}{3} k N^3 + \ldots,$$

(1)

where the superfield $Q^T = (t_L, b_L)$ contains the left-handed top and bottom quarks, and $t^c$ contains the charge conjugate of the right-handed top quark; $H_1$ and $H_2$ are the usual Higgs doublet superfields and $N$ is the Higgs gauge singlet superfield; the ellipsis represents terms whose relatively small couplings play no role in our analysis (in particular, we assume that the bottom quark Yukawa coupling, $h_b$, satisfies $h_b \ll h_t$).

The fields $H_1^T = (H_1^0, H_1^-)$, $H_2^T = (H_2^+, H_2^0)$ and $N$ develop vevs which may be assumed to be of the form

$$< H_1 > = \left( \begin{array}{c} \nu_1 \\ 0 \end{array} \right), \quad < H_2 > = \left( \begin{array}{c} 0 \\ \nu_2 \end{array} \right), \quad < N > = x,$$

(2)

where $\nu_1$, $\nu_2$ and $x$ are real, and $\sqrt{\nu_1^2 + \nu_2^2} = \nu = 174$ GeV. The low energy physical spectrum of the Higgs scalars consists of 3 CP-even neutral states, 2 CP-odd neutral states, and 2 charged scalars. A third CP-odd state is a Goldstone mode which becomes the longitudinal component of the $Z^0$, while a further two charged scalars become those of the $W^\pm$'s.

An upper bound on the lightest neutral CP-even scalar $h^0$ in the NMSSM may be obtained from the real symmetric $3 \times 3$ neutral scalar mass squared matrix, by using the fact that the smallest eigenvalue of such a matrix must be smaller than the smallest eigenvalue of its upper $2 \times 2$ block. The resulting bound at tree-level is

$$m_h^2 \leq M_{Z^0}^2 + (\lambda^2 \nu^2 - M_{Z^0}^2) \sin^2 2\beta.$$

(3)

where $\tan \beta \equiv \frac{\nu_2}{\nu_1}$, and $\lambda$ is regarded as a running parameter evaluated at $M_{SUSY}$. The upper bound on $m_h$ is determined by the maximum value of $\lambda(M_{SUSY})$, henceforth
denoted $\lambda_{\text{max}}$. The value of $\lambda_{\text{max}}$ is obtained by solving the SUSY RG equations for the Yukawa couplings $h_t$, $\lambda$ and $k$ in the region $M_{\text{SUSY}} = 1 \text{ TeV}$ to $M_{\text{GUT}} = 10^{16} \text{ GeV}$ \[13, 14\]. If the Yukawa couplings $h_t$, $\lambda$ and $k$ are all initially large at $M_{\text{GUT}}$ then they approach low energy fixed point ratios \[13\]. However, if the boundary condition at $M_{\text{GUT}}$ is $\lambda \gg k$, then larger values of $\lambda(M_{\text{SUSY}})$ can be achieved. We have repeated the calculation of ref.\[14\] and found that for $h_t(M_{\text{SUSY}}) = 0.5-1.0$, $\lambda_{\text{max}} = 0.87-0.70$ and for $h_t(M_{\text{SUSY}}) \to 1.06$, $\lambda_{\text{max}} \to 0$ (with $k = 0$ always). Radiative corrections to the tree-level bound in Eq.(2) have been considered in refs.\[15, 16, 17\]. In ref.\[16\] these were estimated from a low energy RG analysis of the Higgs sector of the model between $M_{\text{SUSY}}$ and a lower scale $\mu$, assuming that only one Higgs boson has a mass below $M_{\text{SUSY}}$.

In Section 2 \[18\] we shall discuss a more general RG analysis in which both Higgs doublets and the Higgs singlet may be lighter than $M_{\text{SUSY}}$, making a simple approximation of hard decoupling below $M_{\text{SUSY}} = 1\text{ TeV}$ of the superpartners. In Section 3 \[19\] we shall go on to consider the more general case in which the squarks have a more general mass spectrum, using the effective potential approach. Our numerical results for the bound are also presented in this section. Section 4 contains our concluding remarks.

2. Renormalisation Group Approach \[18\]

The basic assumption of this approach is that the squarks are degenerate at $M_{\text{SUSY}} = 1 \text{ TeV}$ while the Higgs bosons and top quarks have masses $\mu \approx 150 \text{ GeV}$. The effective theory below $M_{\text{SUSY}}$ is just the standard model with two light Higgs doublets and a light Higgs singlet. Thus the Higgs potential at some low energy scale

\[1\] $h_t(M_{\text{SUSY}}) \leq 1.06$ is the triviality bound, which, together with $m_t = h_t(m_t)v\sin\beta$, where $h_t(m_t) \leq 1.12$, implies the bound $m_t \leq 195 \text{ GeV}$.
\( \mu < M_{\text{SUSY}} \) is given by the general expression

\[
V_0(\mu) = \frac{1}{2} \lambda_1 (H_1^\dagger H_1)^2 + \frac{1}{2} \lambda_2 (H_2^\dagger H_2)^2 + (\lambda_3 + \lambda_4) (H_1^\dagger H_1)(H_2^\dagger H_2) - \lambda_4 |H_2^\dagger H_1|^2 + \lambda_5 |N|^2 |H_1|^2 + \lambda_6 |N|^2 |H_2|^2 + \lambda_7 (N^* H_1 H_2 + \text{H.c.}) + \lambda_8 |N|^4 + m_1^2 |H_1|^2 + m_2^2 |H_2|^2 + m_3^2 |N|^2 - m_4 (H_1 H_2 N + \text{H.c.}) - \frac{1}{3} m_5 (N^3 + \text{H.c.}).
\]

The running quartic couplings \( \lambda_i(\mu) \) and the mass parameters \( m_i(\mu) \) must satisfy the following boundary conditions at \( M_{\text{SUSY}} \)

\[
\begin{align*}
\lambda_1 &= \lambda_2 = \frac{1}{4} (g^2_1 + g_1^2), \quad \lambda_3 = \frac{1}{4} (g^2_2 - g_1^2) \\
\lambda_4 &= \lambda^2 - \frac{1}{2} g^2_2, \quad \lambda_5 = \lambda_6 = \lambda^2, \quad \lambda_7 = -\lambda k, \quad \lambda_8 = k^2, \\
m_1 &= m_{H_1}, \quad m_2 = m_{H_2}, \quad m_3 = m_N, \\
m_4 &= \lambda A_\lambda, \quad m_5 = k A_k,
\end{align*}
\]

where \( A_\lambda \) and \( A_k \) are soft parameters associated with the trilinear Higgs couplings in Eq.(1), and we have deferred a discussion of squark effects to Section 3. At energy scales \( \mu \) below \( M_{\text{SUSY}} \), the values of the quartic couplings may be obtained by solving the RG equations given in ref. [18]. The minimisation conditions implied by \( \frac{\partial V_{\text{Higgs}}}{\partial v_i} = 0 \) and \( \frac{\partial V_{\text{Higgs}}}{\partial x} = 0 \) allow us to eliminate the low energy parameters \( m_1, m_2, m_3 \). The remaining masses \( m_4 \) and \( m_5 \) are related to the parameters \( A_\lambda \) and \( A_k \) at \( M_{\text{SUSY}} \) by Eq.(5). Below this scale we shall regard \( m_4 \) and \( m_5 \) as free parameters.

The neutral CP-even (scalar) mass squared symmetric matrix in the basis 
\( 1, 2, 3 = H_1, H_2, N \) is

\[
M^2 = \begin{pmatrix}
2\lambda_1 v_1^2 & 2(\lambda_3 + \lambda_4) v_1 v_2 & 2\lambda_5 v_1 x \\
2(\lambda_3 + \lambda_4) v_1 v_2 & 2\lambda_2 v_2^2 & 2\lambda_6 v_2 x \\
2\lambda_5 v_1 x & 2\lambda_6 v_2 x & 4\lambda_8 x^2 - m_5 x \\
\end{pmatrix} + \begin{pmatrix}
\tan \beta [m_4 x - \lambda_7 x^2] & -[m_4 x - \lambda_7 x^2] & -\frac{\mu_3}{x} [m_4 x - 2\lambda_7 x^2] \\
-\frac{\mu_3}{x} [m_4 x - \lambda_7 x^2] & \cot \beta [m_4 x - \lambda_7 x^2] & -\frac{\mu_4}{x} [m_4 x - 2\lambda_7 x^2] \\
-\frac{\mu_4}{x} [m_4 x - 2\lambda_7 x^2] & -\frac{\mu_4}{x} [m_4 x - 2\lambda_7 x^2] & \frac{\mu_5}{x} [m_4 x]
\end{pmatrix}
\]
The neutral CP-odd (pseudoscalar) mass squared symmetric matrix is
\[
\tilde{M}^2 = \begin{pmatrix}
\tan \beta [m_4 x - \lambda_7 x^2] & [m_4 x - \lambda_7 x^2] & \frac{\nu_2}{x}[m_4 x + 2\lambda_7 x^2] \\
[m_4 x - \lambda_7 x^2] & \cot \beta [m_4 x - \lambda_7 x^2] & \frac{\nu_4}{x}[m_4 x + 2\lambda_7 x^2] \\
\frac{\nu_2}{x}[m_4 x + 2\lambda_7 x^2] & \frac{\nu_4}{x}[m_4 x + 2\lambda_7 x^2] & 3m_5 x + \frac{\nu_4^2}{x^2}[m_4 x - 4\lambda_7 x^2]
\end{pmatrix}.
\] (7)

In ref. [18] we obtained an upper bound on the mass of the lightest neutral scalar Higgs boson \(h^0\), by using the fact that \(m_h^2\) must not exceed the lower eigenvalue of the upper \(2 \times 2\) block matrix. The resulting upper bound is a complicated function of \(m_c^2\). It is easy to show that the bound reaches a maximum asymptotically for \(m_c \to \infty\) [18]. It is easy to show that the bound is maximised for \(\lambda > \lambda_{\text{max}}\) and \(k = 0\), and so we shall use \(\lambda_{\text{max}}\) and \(k = 0\), as in the case of the tree level bound discussed previously. With this information in hand, a bound may be calculated from the values of \(\delta \lambda_i = \lambda_i(\mu) - \lambda_i(M_{\text{SUSY}})\) obtained from numerically integrating the RG equations, decoupling the top quark below its mass and choosing \(\mu = 150\) GeV. For each value of \(m_t\), we have determined the value of \(h_t(M_{\text{SUSY}})\) and the corresponding value of \(\lambda_{\text{max}}\) which maximises the bound from a numerical analysis of the triviality condition as discussed previously. We defer a discussion of this bound until after squark effects have been considered.

3. Squark Contributions

In order the calculate the shifts in the mass–squared matrices due to squarks, we employ the full one–loop effective potential. Much of our analysis of the squark spectrum is similar to that of the MSSM in Ref. [3], whose notation we follow closely. A similar approach has also been followed by Ellwanger [15]. The one–loop effective potential is given by
\[
V_1(Q) = V_0(Q) + \Delta V_1(Q),
\]
\[
\Delta V_1(Q) = \frac{1}{64\pi^2} \text{Str} \mathcal{M}^4 \left( \ln \frac{\mathcal{M}^2}{Q^2} - \frac{3}{2} \right),
\] (8)
where \(V_0(Q)\) is the tree–level potential at the arbitrary \(\overline{\text{MS}}\) scale \(Q\), \text{Str} denotes
the usual supertrace, and $\mathcal{M}^2$ is the field–dependent mass–squared matrix. We shall take $Q = M_{SUSY} = 1$ TeV. $V_0$ is restricted to the pure Higgs part of the tree–level potential. Since the RG approach of Section 2 has already included the logarithmic effects of top quark and Higgs boson loops we exclude these particles from the supertrace. The logarithmic contributions to $\Delta V_1(Q)$ from the top quark and Higgs bosons may be absorbed into $V_0(Q)$ where $Q = 1$ TeV, to yield $V_0(\mu)$ where $\mu = 150$ GeV, as in Eq. (4). It only remains to calculate $\Delta V_1(Q)$, where $Q = 1$ TeV, involving squark contributions.

To calculate the shifts in the Higgs mass matrices as a result of squark effects we require the field–dependent mass–squared matrices of the squarks. Taking the contribution to the soft SUSY breaking potential involving squarks to be

$$\Delta V_{soft} = h_t A_t (\bar{Q} H_2 \bar{t} + h.c.) + m_Q^2 |\bar{Q}|^2 + m_T^2 |\bar{t}^c|^2 + m_B^2 |\bar{b}^c|^2,$$

where $b^c$ contains the charge conjugate of the right–handed bottom quark, and tildes denote the scalar components of the superfields, together with the superpotential in Eq. (1), leads to the field–dependent squark mass–squared matrix (ignoring contributions proportional to gauge couplings and $h_b$)

$$\mathcal{M}^2 = \begin{pmatrix}
    m_Q^2 + h_t^2 |H_2^0|^2 & \lambda t_N H_1^0 + h_t A_t \tilde{H}_2^0 & -h_t^2 H_2^0 H_2^+ & 0 \\
    \lambda t_N H_1^0 + h_t A_t \tilde{H}_2^0 & m_T^2 + h_t^2 (|H_2^0|^2 + H_2^+ H_2^-) & \lambda t_N H_1^+ - h_t A_t H_2^- & m_Q^2 + h_t^2 H_2^+ H_2^- \\
    -h_t^2 H_2^0 H_2^- & \lambda t_N H_1^- - h_t A_t H_2^+ & m_Q^2 & m_B^2 \\
    0 & 0 & m_Q^2 & m_B^2
\end{pmatrix},$$

in the basis $\{\bar{t}_L, \bar{t}_R^+, \bar{b}_L, \bar{b}_R^+\}$ where a bar denotes complex conjugation. Manifestly, one eigenvalue is field–independent and may be discarded. Only the upper $2 \times 2$ submatrix contributes to the CP-even and CP-odd mass–squared matrices, since the charged fields have zero vevs in order not to break QED, whereas the upper $3 \times 3$ submatrix contributes to the charged mass–squared matrix. By differentiating $\Delta V_1$
once with respect to the fields we may obtain the shifts in the minimisation conditions
induced by squark loops, and twice will yield the shifts in the mass matrices.  

In the basis \(\{H_1, H_2, N\}\) we have the following mass-squared matrices, after
ensuring that the full one–loop potential is correctly minimised. The couplings \(\lambda_i\)
are those obtained from the potential in Eq.(4) renormalised at the scale \(\mu = 150\)
GeV. For notational simplicity we drop the tildes. Moreover, we work in the basis of
mass eigenstates \(\{t_1, t_2, b_1, b_2\}\) (though, since \(h_b\), the bottom quark Yukawa coupling,
is taken to be zero, \(b_2\) never contributes to the mass–squared matrices), where the
mass of the \(t_1\) squark is \(m_{t_1}\), and so on for the other squarks. The CP-even (scalar)
mass–squared matrix is

\[
M_s^2 = M^2 + \delta M^2, \tag{11}
\]

where \(M^2\) is given in Eq.(6), and

\[
\delta M^2 = \left( \begin{array}{ccc}
\Delta_{11}^2 & \Delta_{12}^2 & \Delta_{13}^2 \\
\Delta_{12}^2 & \Delta_{22}^2 & \Delta_{23}^2 \\
\Delta_{13}^2 & \Delta_{23}^2 & \Delta_{33}^2 \\
\end{array} \right) + \left( \begin{array}{ccc}
\tan \beta & -1 & -\frac{\nu_1}{x} \\
-1 & \cot \beta & -\frac{\nu_2}{x} \\
-\frac{\nu_1}{x} & -\frac{\nu_2}{x} & \frac{\nu_1 \nu_2}{x^2} \\
\end{array} \right) \Delta_p^2. \tag{12}
\]

The \(\Delta_{ij}^2\) and \(\Delta_p^2\) are given by

\[
\Delta_{ij}^2 = \frac{3}{16 \pi^2} h_i^4 \nu_j^2 (\lambda x). A_i. f(m_{t_1}^2, m_{t_2}^2),
\]

\[
\Delta_{11}^2 = \frac{3}{8 \pi^2} h_1^4 \nu_2^2 (\lambda x)^2. \left( \frac{A_t + \lambda x \cot \beta}{m_{t_2}^2 - m_{t_1}^2} \right)^2 g(m_{t_1}^2, m_{t_2}^2),
\]

\[
\Delta_{22}^2 = \frac{3}{8 \pi^2} h_1^4 \nu_2^2 \left( \ln \frac{m_{t_1}^2 m_{t_2}^2}{M_{3USY}^2} + 2A_t(A_t + \lambda x \cot \beta) \ln \frac{m_{t_2}^2}{m_{t_1}^2} \right)
+ \frac{3}{8 \pi^2} h_1^4 \nu_2^2. \left( \frac{A_t(A_t + \lambda x \cot \beta)}{m_{t_2}^2 - m_{t_1}^2} \right)^2 g(m_{t_1}^2, m_{t_2}^2),
\]

\[
\Delta_{33}^2 = \frac{3}{8 \pi^2} h_1^4 \nu_1^2 (\lambda x)^2. \left( \frac{A_t + \lambda x \cot \beta}{m_{t_2}^2 - m_{t_1}^2} \right)^2 g(m_{t_1}^2, m_{t_2}^2),
\]

\[
\Delta_{13}^2 = \frac{3}{8 \pi^2} h_1^4 \nu_2^2 (\lambda x). \left( \frac{A_t + \lambda x \cot \beta}{m_{t_2}^2 - m_{t_1}^2} \right) \left( \ln \frac{m_{t_2}^2}{m_{t_1}^2} + \frac{A_t(A_t + \lambda x \cot \beta)}{m_{t_2}^2 - m_{t_1}^2} g(m_{t_1}^2, m_{t_2}^2) \right),
\]

\[
\Delta_{23}^2 = \frac{3}{8 \pi^2} h_1^4 \nu_2^2 (\lambda x). \left( \frac{A_t + \lambda x \cot \beta}{m_{t_2}^2 - m_{t_1}^2} \right) \left( \ln \frac{m_{t_2}^2}{m_{t_1}^2} + \frac{A_t(A_t + \lambda x \cot \beta)}{m_{t_2}^2 - m_{t_1}^2} g(m_{t_1}^2, m_{t_2}^2) \right),
\]

\[
\Delta_{12}^2 = \frac{3}{8 \pi^2} h_1^4 \nu_2^2 (\lambda x). \left( \frac{A_t + \lambda x \cot \beta}{m_{t_2}^2 - m_{t_1}^2} \right) \left( \ln \frac{m_{t_2}^2}{m_{t_1}^2} + \frac{A_t(A_t + \lambda x \cot \beta)}{m_{t_2}^2 - m_{t_1}^2} g(m_{t_1}^2, m_{t_2}^2) \right).
\]

\[\text{In fact, this is only approximately true due to Higgs self-energy corrections; these are expected to be small for the lightest Higgs bosons.}\]
\[ \Delta_{13}^2 = \frac{3}{8\pi^2} h_1^2 \nu_2^2 (\lambda x) \cdot (\lambda \nu_1) \cdot \left( \frac{A_t + \lambda x \cot \beta}{m_{t_2}^2 - m_{t_1}^2} \right)^2 g(m_{t_1}^2, m_{t_2}^2) \]
\[ - \frac{3}{8\pi^2} h_1^2 \nu_2^2 (\lambda x) \cdot (\lambda \nu_1) \cdot f(m_{t_1}^2, m_{t_2}^2), \]
\[ \Delta_{23}^2 = \frac{\nu_1}{x} \Delta_{12}^2, \]  
(13)

where the functions \( f \) and \( g \) are defined by
\[ f(m_{t_1}^2, m_{t_2}^2) = \frac{1}{m_{t_2}^2 - m_{t_1}^2} \left[ \frac{m_{t_1}^2}{M_{SU SY}^2} \ln \frac{m_{t_1}^2}{M_{SU SY}^2} - m_{t_1}^2 - m_{t_2}^2 \ln \frac{m_{t_2}^2}{M_{SU SY}^2} + m_{t_2}^2 \right], \]
\[ g(m_{t_1}^2, m_{t_2}^2) = \frac{1}{m_{t_1}^2 - m_{t_2}^2} \left[ (m_{t_1}^2 + m_{t_2}^2) \ln \frac{m_{t_2}^2}{m_{t_1}^2} + 2(m_{t_1}^2 - m_{t_2}^2) \right]. \]  
(14)

If the squarks are degenerate in mass, then Eq.(10) implies that \( A_t + \lambda x \cot \beta = 0 \). Furthermore, if \( m_{t_1} = m_{t_2} = M_{SU SY} \), then Eqs.(12-14) imply that the squark contribution to the CP-even mass–squared matrix vanishes. This is the limit of our previous analysis in Section 2 [18].

The CP-odd (pseudoscalar) mass–squared matrix is
\[ M_p^2 = \tilde{M}^2 + \delta \tilde{M}^2, \]  
(15)

where \( \tilde{M}^2 \) is given in Eq.(7) and
\[ \delta \tilde{M}^2 = \begin{pmatrix} \tan \beta & 1 & \nu_2 \nu_1 x^2 \\ \cot \beta & \frac{\nu_1}{x} & \frac{\nu_2}{x} \\ \frac{\nu_1}{x} & \frac{\nu_2}{x} & \frac{\nu_1 \nu_2}{x^2} \end{pmatrix} \Delta_{p}^2. \]  
(16)

Finally, the charged mass–squared matrix is
\[ M_c^2 = \begin{pmatrix} \tan \beta & 1 \end{pmatrix} \left( m_4 x - \lambda_7 x^2 - \lambda_4 \nu_1 \nu_2 + \Delta_c^2 \right), \]  
(17)

where
\[ \Delta_c^2 = \frac{3}{16\nu_1^2} \sum_{m_{a} \in \{s_{t_{1_{1}}, m_{s_{t_{2}}, m_{b_{1}}}} \}} m_{a}^2 \left( \ln \frac{m_{a}^2}{M_{SU SY}^2} - 1 \right) \frac{\partial^2 m_{a}^2}{\partial H_1^2 \partial H_2^2} \bigg|_{vevs}, \]  
(18)

and
\[ \frac{\partial^2 m_{a}^2}{\partial H_1^2 \partial H_2^2} \bigg|_{vevs} = - \frac{h_1^4 \nu_2^2 (\lambda x)^2 \cdot \cot \beta}{(m_{t_1}^2 - m_{t_2}^2)(m_{t_1}^2 - m_{b_1}^2)} - \frac{h_2^2 (\lambda x) A_t}{m_{t_1}^2 - m_{t_2}^2}, \]
\[ \frac{\partial^2 m_{a}^2}{\partial H_1^2 \partial H_2^2} \bigg|_{vevs} = - \frac{h_1^4 \nu_2^2 (\lambda x)^2 \cdot \cot \beta}{(m_{t_2}^2 - m_{b_1}^2)(m_{t_2}^2 - m_{t_1}^2)} + \frac{h_2^2 (\lambda x) A_t}{m_{t_1}^2 - m_{t_2}^2}, \]
\[ \frac{\partial^2 m_{a}^2}{\partial H_1^2 \partial H_2^2} \bigg|_{vevs} = - \frac{h_1^4 \nu_2^2 (\lambda x)^2 \cdot \cot \beta}{(m_{b_1}^2 - m_{t_1}^2)(m_{b_1}^2 - m_{t_2}^2)}. \]  
(19)
The bound on the lightest CP-even Higgs mass is a consequence of the fact that the minimum eigenvalue of $M_s^2$ is bounded by the minimum eigenvalue of the upper $2 \times 2$ submatrix of $M_s^2$. Using Eq. (17) to obtain the physical charged Higgs mass squared, $m_c^2$, in terms of $m_4$,

$$m_4 x - \lambda_7 x^2 = \frac{1}{2}(m_c^2 + \lambda_4 \nu^2) \sin 2\beta - \Delta_c^2,$$

we may eliminate $m_4$ from $M_s^2$ in favour of $m_c^2$, and thus write the upper $2 \times 2$ submatrix in the form

$$M_s'^2 = M'^2 + \delta M'^2,$$

where

$$M'^2 = \begin{pmatrix} 2\lambda_1 \nu_1^2 & 2(\lambda_3 + \lambda_4)\nu_1 \nu_2 \\ 2(\lambda_3 + \lambda_4)\nu_1 \nu_2 & 2\lambda_2 \nu_2^2 \end{pmatrix} + \begin{pmatrix} \tan \beta & -1 \\ -1 & \cot \beta \end{pmatrix} \frac{1}{2}(m_c^2 + \lambda_4 \nu^2) \sin 2\beta,$$

and

$$\delta M'^2 = \begin{pmatrix} \Delta_{11}^2 & \Delta_{12}^2 \\ \Delta_{12}^2 & \Delta_{22}^2 \end{pmatrix} + \begin{pmatrix} \tan \beta & -1 \\ -1 & \cot \beta \end{pmatrix} (\Delta_p^2 - \Delta_c^2).$$

It is clear from the form of the second term on the right-hand-side of Eq. (23) that the factor $\Delta_p^2 - \Delta_c^2$ does not change the bound at all, but merely serves, for a fixed $\tan \beta$, to shift the bound, when plotted as a function of $m_c$, to the left or right. We may then drop this term, since its presence may be re-parametrised by a shift in the free parameter $m_c$. In this way $m_{bi}$ is eliminated.

From Eq. (22) and Eq. (23) it is a simple matter to determine the shift in the minimum eigenvalue of $M'^2$ due to squark effects. Defining $A = (M'^2)_{11}$, $B = (M'^2)_{12}$ and $C = (M'^2)_{22}$, the shift is given by

$$\frac{1}{2} \left[ \Delta_{11}^2 + \Delta_{22}^2 - \frac{(A - C)(\Delta_{11}^2 - \Delta_{22}^2) + 4B\Delta_{12}^2}{\sqrt{(A - C)^2 + 4B^2}} \right].$$

Because this shift is a one loop effect, the couplings $\lambda_i$ in $A$, $B$ and $C$ may be evaluated at any renormalisation point, since the difference between a coupling evaluated at two
different renormalisation points is also a one loop effect, thus giving an overall error at the two loop level, which we neglect in our approximation.

In our previous analysis \[18\] we used triviality limits on the couplings $h_t$, $\lambda$ and $k$ \[13, 14\] to determine the bound on the lightest CP-even Higgs boson mass $m_h^0$ not including general squark effects, that is, with $\delta M^2 \equiv 0$. Let the triviality limit on $\lambda$ for a given $h_t$ be $\lambda_{\text{max}}$. In table \[1\] we reproduce the values of $h_t$ and $\lambda_{\text{max}}$ which generated the bound $m_h^0$ of our previous analysis — all other values, for a given top quark mass, resulted in smaller values of the bound. It transpires that $k = 0$ in all cases. In order to maximise the bound resulting from Eq. (22) we take the limit $m_c = \infty$, thus eliminating $m_c$ as a parameter. As a function of $m_c$ the bound approaches its maximum asymptotically as $m_c \to \infty$. The approach is rapid, with $m_c \sim 200$ GeV being a good approximation to $m_c = \infty$. Thus, the bound resulting from the purely formal procedure of taking the limit $m_c = \infty$ is not unphysical \[18\]. Had this not been the case, then certainly we would have an upper bound, but one perhaps not capable of realisation in an actual spectrum.

We use the values of $h_t$ and $\lambda_{\text{max}}$ in table \[1\] to calculate the shift in the bound on the lightest CP-even Higgs boson mass–squared given by Eq. (24). The shift in the bound on the mass is denoted by $\delta m_h$. This is a hideously complicated function of many parameters, including the squark masses $m_{t_1}$ and $m_{t_2}$, the soft SUSY breaking parameter $A_t$, the vev of the gauge singlet field $x$, and $m_c$. As a function of $m_c$ the shift appears, numerically, to be maximised for $m_c$ small, typically between 100 GeV and 200 GeV, with the difference between the shifts at these two points being less than 1 GeV. Thus, we set $m_c = 200$ GeV in the general squark contributions to the bound. Doing this enables us to retain the bound from our previous analysis derived from Eq. (22) in the limit $m_c = \infty$, and simply maximise the

\footnote{Taking $k = 0$ gives rise to an axion in the physical spectrum. However, a small, non-zero value of $k$ is sufficient to give the would-be axion a mass of several tens of GeV. Such a value of $k$ does not invalidate our calculations.}
Table 1: Lightest CP-even Higgs mass bound in the NMSSM including general squark effects. In row 1 is the top mass, $m_t$, in GeV; in rows 2 and 3 those values of $h_t(M_{\text{SUSY}})$ and $\lambda_{\text{max}}(M_{\text{SUSY}})$, which produce the bound in our previous analysis, this being in row 4, in GeV; in row 5 is the contribution to the bound resulting from a general squark mass spectrum rather than degenerate squarks with mass $M_{\text{SUSY}}$, in GeV; in row 6 is the lightest CP-even Higgs mass bound including general squark effects, in GeV; in rows 7 and 8 are those value of $m_{t_1}$ and $A_t$, in GeV, which generate the bound in row 6; $m_{t_2} = 1$ TeV.

| $m_t$   | 90  | 100 | 110 | 120 | 130 | 140 | 150 | 160 | 170 | 180 | 190 |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $h_t$   | 0.60| 0.67| 0.73| 0.79| 0.84| 0.88| 0.92| 0.95| 0.98| 1.00| 1.03|
| $\lambda_{\text{max}}$ | 0.87| 0.85| 0.83| 0.81| 0.79| 0.77| 0.74| 0.71| 0.67| 0.63| 0.50|
| $m_h^0$ | 145 | 143 | 140 | 137 | 135 | 132 | 129 | 126 | 124 | 123 | 126 |
| $\delta m_h$ | 1   | 2   | 3   | 4   | 6   | 8   | 10  | 13  | 16  | 20  | 23  |
| $m_{h_1}$ | 146 | 145 | 143 | 141 | 140 | 139 | 139 | 139 | 140 | 143 | 149 |
| $m_{t_1}$ | 800 | 770 | 750 | 720 | 700 | 670 | 650 | 630 | 610 | 590 | 570 |
| $A_t$   | 940 | 1000| 1040| 1130| 1160| 1270| 1330| 1410| 1500| 1630| 1820|

Squark contributions separately and add them to our previous bound to yield the new bound $m_h = m_h^0 + \delta m_h$. The $x$ dependence is not too strong, with the ratio $r = x/\nu$ typically taking values close to 10 in order to maximise the bound. Where this is not the case, the difference between the bound at its maximum, as a function of $x$, and that at $r = 10$ is less than 1 GeV. Thus, we set $r = 10$, this giving a reliable indication of the maximum shift. We calculate the shift for various ranges of squark masses and a range of values of $A_t$, and record the maximum value obtained. The squark masses are allowed to vary between 20 GeV and 1000 GeV in steps of 10 GeV. (1000 GeV is the upper limit since we take $M_{\text{SUSY}} = 1000$ GeV.) Given this restriction, the value of $A_t$ which maximises the general squark contribution to the bound never exceeds 2 TeV. We impose the constraint $2h_t\nu_2(A_t + \lambda x \cot \beta) \leq |m_{t_1}^2 - m_{t_2}^2|$ which follows from the form of the squark mass matrix in Eq. (10).

In table 1 we show the bound on the lightest CP-even Higgs mass $m_h$ including general squark effects (and $m_h^0$ for comparison). Table 1 also shows the values of $m_{t_1}$ and $A_t$ used to generate the bound. As $m_t$ increases, $m_{t_1}$ monotonically decreases,
while $A_t$ monotonically increases. The other squark mass, $m_{t_2}$, takes the value $1$ TeV over the whole range of top quark masses — that one squark mass takes on its maximum permitted value to maximise the general squark contribution to the bound can be seen analytically. It will be noticed that these squark masses and $A_t$ are somewhat larger than typically expected from GUT scenarios. However, we feel it wise not to restrict ourselves too excessively to particular prejudices regarding physics as yet unknown (though, of course, the notion of triviality does require the assumption of a SUSY desert up to the unification scale).

Our calculations suggest that the universal upper bound on the lightest CP-even Higgs mass is $149$ GeV. Table 1 reveals that for large values of $m_t$ squark effects may contribute up to $\delta m_h = 23$ GeV, but for small $m_t$ the squark effects are small, as expected. We emphasise that we have not re-maximised the complete one loop corrected bound, but rather made the assumption that the values of $h_t$ and $\lambda_{max}$ which generated the bound of our previous analysis are not significantly modified by the inclusion of a general spectrum of squark masses.

4. Conclusions

We have combined an RG analysis of the Higgs sector of the NMSSM [18] with a calculation of general squark effects [19], and determined a bound on the mass of the lightest CP-even state of $m_h \leq 146, 139, 149$ GeV for $m_t = 90, 140, 190$ GeV. Our calculations indicate that the effects of a general squark spectrum can be very significant for large top quark masses. One reason why, for large top quark masses, squark effects are important is that there exist finite, one-loop diagrams with vertices containing the uncontrolled soft SUSY breaking parameter $A_t$. These diagrams can give large contributions to the Higgs boson masses. We close by indicating that in the NMSSM there are similar diagrams involving loops of Higgs bosons, again
with vertices containing soft SUSY breaking parameters, this time $A_\lambda$ and $A_k$. The existence of these diagrams is directly due to the gauge singlet field $N$; they do not occur in the MSSM. We see no reason in principle why these new diagrams should not give rise to similarly large effects, except for obvious factors of 3 due to colour. We are currently in the process of estimating these effects [20].

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