Algebraic and qualitative remarks about the family

\[ yy' = (\alpha x^{m+k-1} + \beta x^{m-k-1}) y + \gamma x^{2m-2k-1} \]

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Abstract. The aim of this paper is the analysis, from algebraic and qualitative point of view, of the 5-parametric family of differential equations

\[ yy' = (\alpha x^{m+k-1} + \beta x^{m-k-1}) y + \gamma x^{2m-2k-1}, \quad y' = \frac{dy}{dx} \]

where \(a, b, c \in \mathbb{C}, m, k \in \mathbb{Z}\) and
\[ \alpha = a(2m+k), \quad \beta = b(2m-k), \quad \gamma = -(a^2mx^{2k} + cx^{2k} + b^2m). \]

This family is very important because include Van Der Pol equation. Moreover, this family seems to appear as exercise in the celebrated book of Polyanin and Zaitsev. Unfortunately, the exercise presented a typo which does not allow to solve correctly it. We present the corrected exercise, which corresponds to the title of this paper. We solve the exercise and afterwards we make algebraic and qualitative studies to this family of differential equations.

Keywords and Phrases. Critical points, integrability, Gegenbauer equation, Legendre equation, Lienard equation.

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Introduction

Dynamical systems is a topic of interest for a large number of theoretical physicists and mathematicians due to the seminal works of H. Poincaré. It is well known that any dynamical system is a system which evolves in the time. H. Poincaré introduced the qualitative approach to study dynamical systems, which has been useful to study theoretical aspects and applications to biology, chemistry, physics, among others, see \([9, 10, 11, 13, 14, 15, 17]\).

On another hand, E. Picard and E. Vessiot introduced an algebraic approach to study linear differential equations based on the Galois theory for polynomials, see \([2, 6, 7, 8, 12, 18]\). Combination of dynamical systems with differential Galois theory is a recent topic which started with the works of J.J Morales-Ruiz (see \([12]\) and references therein) and with the works of J.-A. Weil (see \([19]\) ). Further works about applications of differential Galois theory include \([11, 3, 5]\).
The Handbook of Exact Solutions of Ordinary Differential Equations, see [16], is one important reference for scientists and engineers interested in solving explicitly ordinary differential equations. This book contains around 3,000 nonlinear ordinary differential equations with solutions, as well as exact, symbolic, and numerical methods for solving nonlinear equations. Nonlinear equations and systems with first-, second-, third-, fourth-, and higher-order are considered there.

Inspired by a previous version of the paper [4], we analysed the Exercise 11 in [16, §1.3.3], which corresponds to a five parametric family of differential equations. We discovered a typo (also corrected by us), which was corrected in the final version of [4] to study from differential Galois Theory point of view the integrability of the dynamical system proposed in such exercise.

We call as Polyanin-Zaitsev vector field to the vector field associated to this system of differential equations that comes from the corrigendum of the Exercise 11 in [16, §1.3.3]. Moreover, we study integrability aspects using differential Galois theory, following [4, 6] as well qualitative aspects due to the foliation associated to Polyanin-Zaitzev vector field is a Lienard equation, which is closely related to a Van Der Pol equation.

This paper not only present the corrigendum and complete solution of the Polyanin-Zaitsev exercise mentioned above, it also extends the results given in [4] concerning the Polyanin-Zaitsev vector field. From algebraic point of view we give conditions over the parameters to have polynomial vector field, while from qualitative point of view we obtain the critical points for some particular cases and we describe their behavior.

The results of this paper were obtained, but not published, during the seminar Algebraic Methods in Dynamical Systems in 2013 developed by the first author and in the master thesis of the second author in 2014 (supervised by the first and third author).

1. Preliminaries

In this section we provide the necessary theoretical background to understand the rest of the paper.

A planar polynomial system of degree $n$ is given by

\begin{align*}
\dot{x} &= P(x, y) \\
\dot{y} &= Q(x, y),
\end{align*}

being $P, Q \in \mathbb{C}[x, y]$ and $n = \max(\deg P, \deg Q)$. By $X := (P, Q)$ we denote the polynomial vector field associated to the system (1.1). The planar polynomial vector field $X$ can be also written in the form

$$X = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.$$
A foliation of a polynomial vector field of the form (1.1) is given by
\[
\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}
\]

Following [15], we present the following theorem, which allows us to characterize the critical points.

**Theorem 1.1.** Let \( X, Y \) be analytic functions with polynomial part containing terms of degree greater than 1. Consider the planar differential system
\[
\begin{align*}
\dot{x} &= y + X(x,y) \\
\dot{y} &= Y(x,y)
\end{align*}
\]
being the origin an isolated critical point. Assume
\[
y = F(x) = a_2x^2 + a_3x^3 + \ldots
\]
as the solutions of \( y + X(x,y) = 0 \) near to \((0, 0)\). Suppose
\[
f(x) = Y(x, F(x)) = ax^\alpha(1 + \ldots), \quad a \neq 0, \quad \alpha \geq 2
\]
and also
\[
\Phi(x) = \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right)|_{(x, F(x))} = bx^\beta(1 + \ldots), \quad b \neq 0, \quad \beta \geq 1.
\]

Then the following statements hold:

a) If \( \alpha \) is even and \( \alpha > 2\beta + 1 \), then \((0, 0)\) is a saddle node.
If \( \alpha \) is even and \( \alpha < 2\beta + 1 \) or \( \Phi(x) \equiv 0 \), then the flow near of \((0,0)\) have two hyperbolic sectors.

b) If \( \alpha \) is odd and \( a > 0 \), then \((0, 0)\) is a saddle point.

c) If \( \alpha \) is odd and \( a < 0 \), several cases can occur:

- \( c_1 \) \( \alpha > 2\beta + 1 \), \( \beta \) even
- \( c_2 \) \( \alpha = 2\beta + 1 \), \( \beta \) even, \( b^2 + 4a(\beta + 1) \geq 0 \)

In this case for \( b < 0 \) the critical point \((0,0)\) is a stable node, while for \( b > 0 \) the critical point \((0,0)\) is unstable node.

- \( c_3 \) \( \alpha > 2\beta + 1 \), \( \beta \) odd
- \( c_4 \) \( \alpha = 2\beta + 1 \), \( \beta \) odd, \( b^2 + 4a(\beta + 1) \geq 0 \)

In this case the flow near to the critical point \((0,0)\) is topologically conformed by an elliptic sector joint with an hyperbolic sector.

- \( c_5 \) \( \alpha = 2\beta + 1 \quad b^2 + 4a(\beta + 1) < 0 \)
- \( c_6 \) \( \alpha < 2\beta + 1 \), or, \( \Phi(x) \equiv 0 \)

In this case the critical point \((0,0)\) is focus or center.
2. Corrigendum to the problem

The original Exercise 11, section 1.3.3 of the book of Polyanin-Zaitsev (see [16] §1.3.3.11) was presented as follows:

\[ yy' = (a(2m + k)x^{2k} + b(2m - k))x^{m-k-1}y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1}. \]

The transformation \( z = x^k; \ y = x^m(t + ax^k + bx^{-k}) \) leads to a Riccati equation with respect to \( z = z(t) \):

\[ (-mt^2 + 2mab - c) \frac{dz}{dt} = bk + tkz + akz^2 \quad (1). \]

The substitution \( z = \frac{mt^2 + ca \cdot w'}{w} \), where \( c_0 = c - 2abm \), reduces equation (1) to a second order linear equation:

\[ (mt^2 + c_0)w'' + (2m + k)t(mt^2 + c_0)w' + abk^2w = 0. \quad (2) \]

The transformation \( \xi = \frac{t}{\sqrt{t^2 + c_0/m}} \), \( u = (1 - \xi^2)^{n/2}w \) where \( \mu = -\frac{m+k}{2m} \) bring equation (2) to the Legendre equation 2.1.2.226:

\[ (1 - \xi^2)u'' - 2\xi u' + [\nu(\nu + 1) - \mu^2(1 - \xi^2)^{-1}]u = 0 \]

where \( \nu \) is a root of the quadratic equation \( \nu^2 + \nu + \frac{m^2-k^2}{4m^2} - \frac{abk^2}{mcm} = 0 \).

A typo in this exercise does not allow its solving. The correction of the problem is presented in the following proposition:

**Proposition 2.1.** Given the family of Lienard equations of the form:

\[ yy' = (a(2m + k)x^{m+k-1} + b(2m - k)x^{m-k-1})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1}, \]

the change of variables \( w = x^k \) and \( y = x^m(t + ax^k + bx^{-k}) \) allow to transform any equation of this family to a Riccati equation.

**Proof.** The system of equations, associated with this Lienard equation is:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= (a(2m + k)x^{m+k-1} + b(2m - k)x^{m-k-1})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1}
\end{align*}
\]

Now, applying the transformation

\[
z = x^k, \quad y = az^\frac{m+k}{k} + tz^\frac{m}{k} + bz^\frac{m-k}{k}
\]

and differentiating we have that:

\[
\begin{align*}
\frac{dx}{dt} &= \frac{1}{k} \frac{dz}{dt}^\frac{k}{m+k} \\
\frac{dy}{dt} &= z^\frac{m}{k} \frac{dz}{dt} + (t^\frac{m}{k}z^\frac{m+k}{k} + a^\frac{m+k}{k}z^\frac{m}{k} + b\frac{m-2k}{k}z^{\frac{m-k}{k}})dz
\end{align*}
\]
In this way, the associated foliation has the form \( ydy = (f(x)y - g(x))dx \). Now we compute each part of this equality, thus we obtain the left side as:

\[
kydy = k(az^{m+k} + tz^m + bz^{m-k})z^m dt + k(t^2 \frac{2}{k} z^{2m-k} + at \frac{m-k}{k} z^{2m-k}) + bt \frac{2m-k}{k} z^{2m-k} + ab \frac{m-k}{k} z^{2m-k} + ab^{m-k} z^{2m-k} + b^2 z^{2m-k} \]

and the right side as

\[
(f(x)y - g(x))dx = ((a(2m+k)x^{m+k-1} + b(2m-k)x^{m-k-1}) + (a^2mx^{2k} + cz^{2k} + b^2m)x^{2m-2k-1})dx
\]

\[
= ((a(2m+k)z^{m+k-1} + b(2m-k)z^{m-k-1})(az^{m+k} + tz^m + bz^{m-k}) + (a^2mz^{2k} + cz^{2k} + b^2m)z^{2m-2k-1})z^{1-k} \frac{1}{z} dz
\]

\[
= ((a(2m+k)z^{m+k-1} + b(2m-k)z^{m-k-1})(az^{m+k} + tz^m + bz^{m-k}) + (a^2mz^{2k} + cz^{2k} + b^2m)z^{2m-2k-1})z^{1-k} \frac{1}{z} dz
\]

\[
= (at(2m+k)z^{2m-k} + a^2(2m+k)z^{2m-k} + ab(2m+k)z^{2m-k} + bt(2m-k)z^{2m-k} + ab(2m-k)z^{2m-k} + b^2(2m-k)z^{2m-k} + a^2mz^{m-k} + cz^{m-k} + b^2mz^{m-k})z^{1-k} \frac{1}{z} dz
\]

To achieve the proposed purpose, we organize the terms with respect to \( dz \), that is:

\[
(at(2m+k)z^{2m-k} + a^2(2m+k)z^{2m-k} + ab(2m+k)z^{2m-k} + bt(2m-k)z^{2m-k} + ab(2m-k)z^{2m-k} + b^2(2m-k)z^{2m-k} + a^2mz^{m-k} + cz^{m-k} + b^2mz^{m-k})z^{1-k} \frac{1}{z} dz
\]

\[
= (akz^2 + tkz^b + bk)z^{2m-k} \frac{1}{z} dt
\]

Now organizing the terms again we have:

\[
((ab(2m+k) + ab(2m-k) - c - t^2m - ab(m-k) - ab(m+k))z^{2m-k} + (a^2(2m+k) - a^2m - a^2(m+k))z^{2m-k} + (at(2m+k) - at(m+k) - atm)z^{2m-k} dt)
\]

\[
= (akz^2 + tkz + bk)
\]

Thus, we obtain the Riccati equation:

\[
(2.1) \quad (-mt^2 + 2mab - c) \frac{dz}{dt} = bk + tkz + akz^2
\]
and we conclude the proof.

For the rest of transformations proposed in the Exercise of Polyanin-Zaitsev we need the results concerning the transformations, which will be given in the next section.

3. Some transformations

In this section we study some transformations that allow us to complete the exercise stated by Polyanin-Zaitsev above.

**Lemma 3.1.** If \( R = a_1 x^2 + a_1 x + a_0, \) \( S = b_1 x + b_0, \) with \( a_2, b_1 \neq 0. \) The differential equation

\[ R^2 \partial_x^2 y + SR \partial_x y + Cy = 0 \]

is transformed in the equation

\[ Q^2 \partial_x^2 y + LQ \partial_x y + \lambda y = 0, \quad Q = \tau^2 + q_0, \quad L = l_1 \tau + l_0, \quad \tau = x + \frac{a_1}{2a_2}. \]

**Proof.** \( R^2 \partial_x^2 y + SR \partial_x y + Cy = 0 \) then replacing

\[ (a_1 x^2 + a_1 x + a_0)^2 \partial_x^2 y + (b_1 x + b_0)(a_1 x^2 + a_1 x + a_0) \partial_x y + Cy = 0 \]

We Divide all by \( a_2^2, \) thus we get:

\[ \left( x^2 + \frac{a_1}{a_2} x + \frac{a_0}{a_2} \right)^2 \partial_x^2 y + \left( \frac{b_1}{a_2} x + \frac{b_0}{a_2} \right) \left( x^2 + \frac{a_1}{a_2} x + \frac{a_0}{a_2} \right) \partial_x y + \frac{C}{a_2^2} y = 0 \]

Now in \( R \) we complete the square

\[ \left( x^2 + \frac{a_1}{a_2} x + \left( \frac{a_1}{2a_2} \right)^2 + \frac{a_0}{a_2} - \left( \frac{a_1}{2a_2} \right)^2 \right)^2 \partial_x^2 y + \]

\[ \left( \frac{b_1}{a_2} x + \frac{b_0}{a_2} \right) \left( x^2 + \frac{a_1}{a_2} x + \left( \frac{a_1}{2a_2} \right)^2 + \frac{a_0}{a_2} - \left( \frac{a_1}{2a_2} \right)^2 \right) \partial_x y + \frac{C}{a_2^2} y = 0 \]

Then:

\[ \left( x + \frac{a_1}{2a_2} \right)^2 + \frac{a_0}{a_2} - \left( \frac{a_1}{2a_2} \right)^2 \right)^2 \partial_x^2 y + \]

\[ \left( \frac{b_1}{a_2} x + \frac{b_0}{a_2} \right) \left( x + \frac{a_1}{2a_2} \right)^2 + \frac{a_0}{a_2} - \left( \frac{a_1}{2a_2} \right)^2 \right) \partial_x y + \frac{C}{a_2^2} y = 0. \]

Assume

\[ \tau = x + \frac{a_1}{2a_2}, \quad q_0 = \frac{a_0}{a_2} - \left( \frac{a_1}{2a_2} \right)^2, \]

then

\[ (x + \frac{a_1}{2a_2})^2 + \frac{a_0}{a_2} - \left( \frac{a_1}{2a_2} \right)^2 = \tau^2 + q_0 = Q(\tau). \]

Replacing \( x \) in the polynomial \( S \) in term of \( \tau \) we get:

\[ \frac{b_1}{a_2} x + \frac{b_0}{a_2} = \frac{b_1}{a_2} \left( \tau - \frac{a_1}{2a_2} \right) + \frac{b_0}{a_2} = \frac{b_1}{a_2} \tau - \frac{b_1 a_1}{2a_2^2} + \frac{b_0}{a_2} \]

then if \( l_1 = \frac{b_1}{a_2} \) and \( l_0 = -\frac{b_1 a_1}{2a_2^2} + \frac{b_0}{a_2} \) we get \( l_1 x + l_0 = L(\tau) \)
Now if $\lambda = \frac{C}{\sigma^2}$, the differential equations will be:

$$Q^2 \hat{\tau}^2 \hat{y} + LQ\hat{\tau} \hat{y} + \lambda \hat{y} = 0$$

where $\hat{y} = y(x(\tau))$, and the transformation $\tau = x + \frac{\sigma}{2\sigma^2}$ send $\partial_x$ on $\partial_\tau$.

\[ \square \]

**Remark 3.1.** The differential equation of the form

$$(1 - x^2)\partial_x^2 y + (\hat{b} - \hat{a} - (\hat{a} + \hat{b} + 2)x)\partial_x y + \lambda y = 0$$

with $\lambda = n(n + 1 + \hat{a} + \hat{b})$ and $n \in \mathbb{N}$, is known as Jacobi equation (in general form).

It is a particular case of the hypergeometric equation, but the solutions include Jacobi polynomials. If we take $\hat{a} = \hat{b}$ and $\lambda = n(n + 2\hat{a})$ with $n \in \mathbb{N}$, we get a Gegenbauer equation (or ultraspherical case):

$$(1 - x^2)\partial_x^2 y - 2(\hat{a} + 1)x\partial_x y + \hat{\lambda} y = 0.$$  

Now we study a special transformation, in the next theorem:

**Theorem 3.2.** The differential equation

$$a_n z^{(n)} + (\hat{k} + a_{n-1})z^{(n-1)} + a_{n-2}z^{(n-2)} + \ldots + a_1 z^{(1)} + a_0 z = 0,$$

with $a_i \in \mathbb{C}(x)$ can be transformed into the differential equation

$$y^{(n)} + \tilde{k}y^{(n-1)} + b_{n-2}y^{(n-2)} + b_{n-3}y^{(n-3)} + \ldots + b_1 y^{(1)} + b_0 y = 0$$

with $b_i \in \mathbb{C}(x)$.

**Proof.** Following the Lemma [5.1] and taking the implicit transformation $z = \varepsilon(t)y + \mu(t)$ with $\varepsilon(t), \mu(t) \in \mathbb{C}(x)$, we compute the first equation applying the change of variable

$$(\varepsilon y)^{(n)} + \mu^{(n)} + \sum_{i=0}^{n-2} b_i (\varepsilon y + \mu)^{(i)} = 0.$$  

Then, computing in general form the Leibniz rule we get:

$$\varepsilon^{(n)} y + \binom{n}{1} \varepsilon^{(n-1)} y^{(1)} + \binom{n}{2} \varepsilon^{(n-2)} y^{(2)} + \binom{n}{3} \varepsilon^{(n-3)} y^{(3)} + \ldots + \binom{n}{n-1} \varepsilon^{(1)} y^{(n-1)} + \binom{n}{n} \varepsilon^{(0)} y^{(n)},$$

$$\tilde{k} a_{n-1} \left[ \varepsilon^{(n)} y + \binom{n}{1} \varepsilon^{(n-1)} y^{(1)} + \binom{n}{2} \varepsilon^{(n-2)} y^{(2)} + \binom{n}{3} \varepsilon^{(n-3)} y^{(3)} + \ldots + \binom{n}{n-1} \varepsilon^{(1)} y^{(n-1)} + \binom{n}{n} \varepsilon^{(0)} y^{(n)} \right]$$

$$a_{n-2} \left[ \varepsilon^{(n-2)} y + \binom{n-2}{1} \varepsilon^{(n-3)} y^{(1)} + \binom{n-2}{2} \varepsilon^{(n-4)} y^{(2)} + \binom{n-2}{3} \varepsilon^{(n-5)} y^{(3)} + \ldots + \binom{n-2}{n-3} \varepsilon^{(1)} y^{(n-3)} + \binom{n-2}{n-2} \varepsilon^{(0)} y^{(n-2)} \right]$$

$$a_{n-3} \left[ \varepsilon^{(n-3)} y + \binom{n-3}{1} \varepsilon^{(n-4)} y^{(1)} + \binom{n-3}{2} \varepsilon^{(n-5)} y^{(2)} + \binom{n-3}{3} \varepsilon^{(n-6)} y^{(3)} + \ldots + \binom{n-3}{n-4} \varepsilon^{(1)} y^{(n-4)} + \binom{n-3}{n-3} \varepsilon^{(0)} y^{(n-3)} \right]$$

$$a_{n-m} \left[ \varepsilon^{(n-m)} y + \binom{n-m}{1} \varepsilon^{(n-m-1)} y^{(1)} + \binom{n-m}{2} \varepsilon^{(n-m-2)} y^{(2)} + \binom{n-m}{3} \varepsilon^{(n-m-3)} y^{(3)} + \ldots + \binom{n-m}{n-m-1} \varepsilon^{(1)} y^{(n-m-1)} + \binom{n-m}{n-m} \varepsilon^{(0)} y^{(n-m)} \right]$$

Continuing of this form, we divide all equation by $\varepsilon$. Then, we use the same method of the indeterminate coefficients to calculate $\varepsilon$ and the $b_i$ coefficient:

If we take $y^{(n-1)}$:

$$\binom{n}{n-1} \varepsilon^{(1)} + \tilde{k} a_{n-1} = \tilde{k}.$$
then:

\[ \frac{a_{(n-1)}}{n} \varepsilon^{(1)} + a_{(n-1)} = 0 \]

From this differential equation we get an appropriate \( \varepsilon \) value, and with it we obtain the coefficient \( b_i \).

If we take \( y^{(n-2)} \):

\[
\left( \frac{n}{n-2} \right) \varepsilon^{(2)} + a_{(n-1)} \left( \frac{n-1}{n-2} \right) \varepsilon^{(1)} + a_{n-2} \left( \frac{n-2}{n-2} \right) = b_{n-2}
\]

If we take \( y^{(n-3)} \):

\[
\left( \frac{n}{n-3} \right) \varepsilon^{(3)} + a_{(n-1)} \left( \frac{n-1}{n-3} \right) \varepsilon^{(2)} + a_{n-2} \left( \frac{n-2}{n-3} \right) \varepsilon^{(1)} + a_{n-3} \left( \frac{n-3}{n-3} \right) = b_{n-3}
\]

Continuing of this form we see, that for any \( k \in \mathbb{N} \) the recurrent formulae will be:

\[
\left( \frac{n}{n-k} \right) \varepsilon^{(k)} + ... + a_{n-k+3} \left( \frac{n-k+3}{n-k} \right) \varepsilon^{(3)} + a_{n-k+2} \left( \frac{n-k+2}{n-k} \right) \varepsilon^{(2)} + a_{n-k+1} \left( \frac{n-k+1}{n-k} \right) \varepsilon^{(1)} = b_{n-k}
\]

If in the previous theorem, we consider \( \tilde{k} = 0 \), we get the next corollary:

**Corollary 3.3.** *the differential equation*

\[ a_n z^{(n)} + a_{n-1} z^{(n-1)} + a_{n-2} z^{(n-2)} + ... + a_1 z^{(1)} + a_0 z = 0 \]

*with \( a_i \in \mathbb{C}(x) \) can transformed in the differential equation*

\[ y^{(n)} + b_{n-2} y^{(n-2)} + b_{n-3} y^{(n-3)} + ... + b_1 y^{(1)} + b_0 y = 0 \]

*where \( b_i \in \mathbb{C}(x) \)*

**Example:** Applying the transformation over the general second order differential equation \( z'' + a_1 z' + a_0 z = 0 \):

Using the Theorem 3.2 we get \( \frac{2a_1}{\varepsilon} + a_1 = 0 \), then \( \varepsilon' = -\frac{2a_1}{\varepsilon} \). Now, through derivatives and dividing by \( \varepsilon \), we get \( \varepsilon'' = -\frac{a_1}{\varepsilon} - \frac{2a_1^2}{\varepsilon^2} \), but \( \frac{a_1}{\varepsilon} = -\frac{a_1}{2} \). Thus we obtain

\[ \varepsilon'' = -\frac{a_1}{2} - \frac{a_1^2}{2}. \]

In this way we arrive to \( b = -\frac{a_1}{2} + a_0 \), for the differential equation \( y'' + b = 0 \).

In the following theorem we recall that a Hamiltonian Change of Variable \( z = z(x) \) is a change of variable in where \( (z(x), z'(x)) \) is a solution curve of a Hamiltonian system of one degree of freedom. The new derivation is given by \( \hat{\partial}_z = \sqrt{\alpha} \partial_x \), being \( \alpha = (\partial_x z)^2 \), see \[ \text{[11, 13, 15]} \] and references therein.

**Theorem 3.4.** Let \( Q \) and \( L \) be as in \[ \text{[3.1]} \], with \( a_1 = b_1 = 0 \) or \( b_0 = \frac{a_1 b_1}{2a_2} \). Through the Hamiltonian change of variable \( \hat{\xi} = \partial_x \sqrt{Q} \), the differential equation

\[ Q^2 \partial_t^2 w + LQ \partial_t w + \lambda w = 0, \]
is transformed in the equation

\[(1 - \xi^2)\partial_\xi^2 \hat{w} + \left(l_1 - \frac{3}{q_0}\right) \xi \partial_\xi \hat{w} + \frac{\lambda}{q_0} \hat{w} = 0.\]

Owing to \(\lambda = n(n + 1 + \tilde{\alpha} + \tilde{b})\), we have the Jacobi equation with \(\tilde{\alpha} = \tilde{b}\). Moreover, if \(\lambda = n(n + 2\tilde{\alpha})\) then we obtain a Gegenbauer equation.

**Proof. Case 1:**

If we assume \(a_1 = b_0 = 0\), we obtain \(l_0 = 0, l_1 = \frac{b_0}{a_1}, q_0 = \frac{a_1}{a_2}\). Then:

\[(\tau^2 + q_0)^2 \partial_\tau^2w + \xi \tau^2 + q_0) \partial_\tau \hat{w} + \lambda w = 0.\]

Now, by hypothesis \(\xi = \partial_\tau \sqrt{Q}\), this is

\[\xi = \frac{\partial_\tau Q}{2\sqrt{Q}} = \frac{\partial_\tau (\tau^2 + q_0)}{2\sqrt{\tau^2 + q_0}}\]

Furthermore, due to \(\xi = \partial_\tau \sqrt{Q}\) then \(\alpha\) we arrive to:

\[\sqrt{\alpha} = \partial_\tau \xi = \partial_\tau^2 \sqrt{Q} = \frac{1}{\sqrt{\alpha(1 - \xi^2) - \xi^2 \sqrt{1 - \xi^2}}} = \frac{1}{\sqrt{\alpha(1 - \xi^2) - \xi^2 \sqrt{1 - \xi^2}}} = \sqrt{\frac{(1 - \xi^2)^3}{q_0}}\]

i.e \(\alpha = \frac{(1 - \xi^2)^3}{q_0}\).

Now \(\hat{\xi} = \sqrt{\frac{(1 - \xi^2)^3}{q_0}} \partial_\xi\), and

\[\hat{\xi}^2 = \alpha \partial_\xi^2 + \frac{1}{2} \partial_\xi \alpha \partial_\xi = \frac{(1 - \xi^2)^3}{q_0} \partial_\xi^2 - \frac{3(1 - \xi^2)^2}{q_0} \partial_\xi\]  

We compute all elements of the Hamiltonian change of variable \(\hat{Q} = Q(\tau(\xi)) = \frac{q_0^2}{1 - \xi^2}, \hat{L} = l_1 \frac{\xi \sqrt{q_0}}{\sqrt{1 - \xi^2}}, \hat{w}(\xi(\tau)) = w(\tau)\).

Then:

\[Q^2 \partial_\tau^2 w \rightarrow \hat{Q} \hat{\xi}^2 \hat{w} = (\frac{q_0^2}{1 - \xi^2})^2 \left(\frac{(1 - \xi^2)^3}{q_0} \partial_\xi^2 \hat{w} - \frac{3(1 - \xi^2)^2}{q_0} \partial_\xi \hat{w}\right) = q_0(1 - \xi^2) \partial_\xi^2 \hat{w} - 3\xi \partial_\xi \hat{w}\]

\[LQ \partial_\tau w \rightarrow \hat{L} \hat{Q} \hat{\xi} \hat{w} = (l_1 \frac{\xi \sqrt{q_0}}{\sqrt{1 - \xi^2}})(\frac{q_0^2}{1 - \xi^2}) \frac{(1 - \xi^2)^3}{q_0} \partial_\xi \hat{w} = l_1 q_0 \xi \partial_\xi \hat{w}\]

If we replace, in the transformed differential equation, we obtain:

\[q_0(1 - \xi^2) \partial_\xi^2 \hat{w} + (l_1 q_0 - 3) \xi \partial_\xi \hat{w} + \lambda \hat{w} = 0\]

It is equivalent to Gegenbauer equation with \(\hat{\lambda} = \frac{\lambda}{q_0}, -2\hat{\alpha} + 1 = l_1 - \frac{3}{q_0}\) i.e \(\hat{\alpha} = \frac{1}{q_0} - l_1 - 1\) then:

\[(1 - \xi^2) \partial_\xi^2 \hat{w} + \left(l_1 - \frac{3}{q_0}\right) \xi \partial_\xi \hat{w} + \frac{\lambda}{q_0} \hat{w} = 0\]

Finally if \(\frac{\lambda}{q_0} = n(n + 1 + \tilde{\alpha} + \tilde{b})\) with \(n \in \mathbb{N}\), then the solutions of this equation are Ultraspherical Gegenbauer polynomial.

**Case 2:**

If \(b_0 = \frac{b_0}{2a_2}\) then:

\[Q(\tau) = \tau + q_0 \text{ with } q_0 = \frac{a_1}{a_2} - \left(\frac{a_1}{2a_2}\right)^2, L(\tau) = l_1 \tau + l_0, l_1 = \frac{b_1}{a_2}, \text{ and } l_0 = -\frac{b_0}{2a_2} + \frac{b_0}{a_2} = \]
\[ -\frac{b_1 a_1}{2\alpha^2} + \frac{b_2 a_2}{2\alpha^2} = 0 \]

i.e.

\[(\tau^2 + q_0)^2 \partial_{\tau}^2 \hat{w} + \tilde{l}_1 \tau (\tau^2 + q_0) \partial_\tau \hat{w} + \lambda \tilde{w} = 0 \]

for instance, the same differential equation of the previous case.

Now we transform the Gegenbauer equation into an Hypergeometric equation.

**Lemma 3.5.** The Gegenbauer equation
\[
(1 - x^2) \partial_x^2 y - 2(\mu + 1)x \partial_x y + (\nu - \mu)(\nu + \mu + 1)y = 0
\]

is transformed into an hypergeometric equation
\[
z(1 - z) \partial_z^2 y + (c - (a + b + 1)z) \partial_z y - aby = 0,
\]

where \( a = \mu - \nu, \ b = \nu + \mu + 1 \) and \( c = \mu + 1 \).

**Proof.** For this transformation we will use a Hamiltonian change of variable, over the independent variable of the Gegenbauer equation \( x = 1 - 2z \). Then \( \sqrt{\alpha} = \partial_z z = -\frac{1}{2} \), that is, \( \alpha = \frac{1}{4}, \ \partial_z = \frac{1}{2} \partial_z \) and \( \partial_z^2 = \frac{1}{4} \partial_z^2 \).

Substituting in the Gegenbauer equation we obtain
\[
(1 - x^2) \partial_x^2 y = (1 - 4z + 4z^2)\frac{1}{4} \partial_z^2 \hat{y} = z(1 - z) \partial_z^2 \hat{y}
\]

Thus, we obtain the equation
\[
z(1 - z) \partial_z^2 \hat{y} + (\mu + 1)(1 - 2z) \partial_z \hat{y} - (\mu - \nu)(\nu + \mu + 1) \hat{y} = 0.
\]

We know that Hypergeometric equation is of the form
\[
z(1 - z) \partial_z^2 y + (c - (a + b + 1)z) \partial_z y - aby = 0.
\]

Then we compute the parameters values \( a, b, c \) as follows:

\[
ab = (\mu - \mu)(\mu + \nu + 1)
\]

\[
c - (a + b + 1)z = (\mu + 1) - 2(\mu + 1)z
\]

Therefore
\[
c = \mu + 1, \ a = \mu - \nu, \ b = \mu + \nu + 1,
\]

which concludes the proof.

**Proposition 3.6.** Through the Hamiltonian change of variables \( x = 1 - 2\xi \) and \( y = (x^2 - 1)^\frac{\xi}{2} \), we can transform the Hypergeometric equation
\[
\xi(\xi - 1) \partial_\xi^2 w + (\mu + 1)(1 - 2\xi) \partial_\xi w + (\nu - \mu)(\nu + \mu + 1)w = 0
\]

into the Legendre equation
\[
(1 - x^2) \partial_x^2 y - 2x(1 - x^2) \partial_x y + \nu(\nu + 1)(1 - x^2) - \mu^2 y = 0.
\]

**Proof.** Firstly we transform the Legendre equation into the Hypergeometric equation:

If \( y = (x^2 - 1)^\frac{\xi}{2} \) then
\[
\partial_x y = \mu x(x^2 - 1)^{\frac{\xi}{2} - 1}w + (x^2 - 1)^{\frac{\xi}{2}} \partial_x w
\]
and
\[ \partial_x^2 y = \mu[(x^2 - 1)^{2\mu - 1} + 2(\frac{\mu}{2} - 1)x(x^2 - 1)^{2\mu - 2}]w + 2\partial x(x^2 - 1)^{2\mu - 1}\partial_x w + (x^2 - 1)^{2\mu} \partial_x^2 w \]

Now dividing by \((x^2 - 1)^{2\mu - 2}\) and replacing in each terms of the Legendre equation we get:
\[ (x^2 - 1)^2 \partial_x^2 y = \mu(x^2 - 1)w + 2\mu(\frac{\mu}{2} - 1)x(x^2 - 1)w + 2\mu x(x^2 - 1)\partial_x w + (x^2 - 1)^2 \partial_x^2 w \]
\[ 2x(x^2 - 1)\partial_x y = 2\mu x^2 w + 2x(x^2 - 1)\partial_x w \]
\[ \left[ v(1 + v) - \mu^2 \right] y = \left[ -v(1 + v)(x^2 - 1) - \mu^2 \right] w \]

Now we obtain
\[ (x^2 - 1)^2 \partial_x^2 w + 2[2\mu x + 2x]((x^2 - 1)\partial_x w + (-v(1 + v)(x^2 - 1) - \mu^2 + 2\mu x^2 + \mu(x^2 - 1) + 2\mu(\frac{\mu}{2} - 1)x)w = 0 \]
\[ \Rightarrow \]
\[ (x^2 - 1)^2 \partial_x^2 w + 2x(\mu + 1)(x^2 - 1)\partial_x w + (-v(1 + v)(x^2 - 1) - \mu^2 + 2\mu x^2 + \mu(x^2 - 1) + \mu x^2 - 2\mu x^2) = 0 \]
\[ \Rightarrow \]
\[ (x^2 - 1)^2 \partial_x^2 w + 2x(\mu + 1)(x^2 - 1)\partial_x w + (-v(1 + v)(x^2 - 1) + \mu(x^2 - 1))w = 0 \]
\[ \Rightarrow \]
\[ (x^2 - 1)^2 \partial_x^2 w + 2x(\mu + 1)\partial_x w + [\mu^2 - v^2 + \mu - v]w = 0 \]
\[ \Rightarrow \]
\[ (x^2 - 1)^2 \partial_x^2 w + 2x(\mu + 1)\partial_x w + (\mu - v)(\mu + v + 1)w = 0 \]

Now applying the Hamiltonian change of variable \( x = 1 - 2\xi \frac{\xi}{4} \), we obtain \( \partial_x \xi = \frac{-1}{2} \sqrt{\alpha} \), being \( \alpha = \frac{1}{4} \). Thus, we obtain \( \partial_x \xi = -\frac{1}{2} \partial_t \xi \); then \( x^2 = 1 - 4\xi + 4\xi^2 = 1 = 4\xi(\xi - 1) \). Now, replacing we obtain:
\[ \xi(\xi - 1)\partial_t \xi \hat{w} - (\mu + 1)(1 - 2\xi)\partial_t \xi \hat{w} + (\mu - v)(\mu + v + 1)\hat{w} = 0 \]

Following exactly the reversed process we can transform a Hypergeometric equation into Legendre equation.

\[ \square \]

**Example.** Transform the next equation on ultraespheric form.
\[ (mt^2 + c_0)^2 \hat{w} + t(2m + k)(mt^2 + c_0)\hat{w} + abk^2 w = 0. \]
Now \( R = mt^2 + c_0, S = (2m + k)t \tau = t + \frac{\xi}{2m}, q_0 = \frac{m}{a}, l_0 = 0, l_1 = \frac{2m + k}{m} y \lambda = \frac{abk_0 m}{a bk^2 m^2}. \)

applying the previous lemma we have the equation:
\[ (\tau^2 + \frac{c_0}{m})^2 \partial_x^2 \hat{w} + \frac{2m + k}{m} \tau(\tau^2 + \frac{c_0}{m})\partial_x \hat{w} + \frac{abk^2 m^2}{a} \hat{w} = 0. \]

Now applying the previous theorem we get:
\[ (1 - \xi^2)\partial_x^2 \hat{u} + (\frac{2m + k}{m} - \frac{3m}{c_0})\xi \partial_x \hat{u} + \frac{abk^2}{c_0 m} \hat{u} = 0 \]
with \( \hat{u}(\xi) = \hat{w}(\tau(\xi)) \)
Remark 3.2. If we have our equation in the Legendre form, we apply the proposition 3.6 and therefore we can study it as in [4] to conclude the integrability or non-integrability, of the Lienard equation. Moreover, such as we will see in the next section, through equation (4.1) in Legendre’s form we can apply the Kimura table, see [4].

4. Polyanin-Zaitsev vector field

The associated system of the Polyanin-Zaitsev vector field, with \(a, b, c, m, k \in \mathbb{R}\), is given by:

\[
\dot{x} = y \\
\dot{y} = (\alpha x^{m+k-1} + \beta x^{m-k-1})y - \gamma x^{2m-2k-1}.
\]

with \(\alpha = a(2m + k), \beta = b(2m - k)\) \(\gamma = (a^2 mx^{4k} + cx^{2k} + b^2 m)\),

where the Polyanin-Zaitsev vector field is given by \(X = \langle P, Q \rangle\), being,

\[
(P, Q) := (y, (a(2m+k)x^{m+k-1}+b(2m-k)x^{m-k-1})y-(a^2 mx^{4k}+cx^{2k}+b^2 m)x^{2m-2k-1}).
\]

The next proposition can illustrate the cases in which the Polyanin-Zaitsev vector field is formed by non trivial polynomial functions.

Proposition 4.1. The system (4.1) is a not null differential polynomial system if it is equivalently to one of the next families:

\[
\dot{x} = y \\
\dot{y} = \bigl[a(\frac{3s+p+4}{2})x^s + b(\frac{3s+p+4}{2})x^p\bigr]y - a^{s+p+2}x^{2s+1} - cx^{s+p+1} - b^{2s+p+2}x^{2p+1}
\]

\[
\dot{x} = y \\
\dot{y} = bx^{s+p+4}y^2 - cx^{r} - b^{2s+p+2}x^{2p+1}
\]

\[
\dot{x} = y \\
\dot{y} = \bigl[a(\frac{3s+p+4}{2})x^s + b(\frac{3s+p+4}{2})x^p\bigr]y - a^{s+p+2}x^{2s+1} - b^{2s+p+2}x^{2p+1}
\]

\[
\dot{x} = y \\
\dot{y} = -cx^{s+p+1}
\]

\[
\dot{x} = y \\
\dot{y} = a(\frac{3s+p+4}{2})y^s - a^{s+p+1}x^{2s+1} - cx^r
\]

\[
\dot{x} = y \\
\dot{y} = +b(m + p + 1)y^p - b^2 mx^{2p+1}
\]

\[
\dot{x} = y \\
\dot{y} = a(m + s + 1)y^s - amx^{2s+1}
\]
Proof. The system (4.1) is a polynomial system if \( Q \) is a polynomial function, that is, the exponents of each term must be non-negative integer. Furthermore, we need to consider the values of the constants \( a, b \) and \( c \). Now we consider the different possibilities for these constants:

Case 1. For \( a \neq 0, b \neq 0, c \neq 0 \), it must be satisfied:

\[
\begin{align*}
m + k - 1 &= s \\
m - k - 1 &= p \\
2m + 2k - 1 &= 2s + 1 \\
2m - 1 &= r \\
2m - 2k - 1 &= 2p + 1
\end{align*}
\]

being \( s, p, r \in \mathbb{Z}^+ \). Thus \( m = \frac{r+1}{2} \) and \( m = \frac{s+p+2}{2} \), which means that \( r = s + p + 1 \). Therefore we obtain the following system associated with the family (4.1):

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= [a \frac{s+p+2}{2} x^s + b \frac{s+p+2}{2} x^p] y - a \frac{s+p+2}{2} x^{s+1} - cx^{s+p+1} - b^2 \frac{s+p+2}{2} x^{2p+1}.
\end{align*}
\]

Case 2. For \( a = 0, b \neq 0, c \neq 0 \), the system (4.1) is reduced to:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= b(2m - k)yx^{m-k-1} - b^2 mx^{2m-2k-1} - cx^{2m-1}
\end{align*}
\]

Since the exponents must be non-negative integers, then:

\[
\begin{align*}
m - k - 1 &= p \\
2m - 1 &= r \\
2m - 2k - 1 &= 2p + 1
\end{align*}
\]

For instance, we arrive to the system:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= b \frac{s+p+4}{2} yx^p - cx^r - b^2 \frac{r+1}{2} x^{2p+1}
\end{align*}
\]

Case 3. For \( a \neq 0, b \neq 0, c = 0 \), the system (4.1) is reduced to:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= a(2m + k)yx^{m+k-1} + b(2m - k)yx^{m-k-1} - a^2 mx^{2m+2k-1} - b^2 mx^{2m-2k-1}
\end{align*}
\]

Again the exponents must be non-negative integers, therefore:

\[
\begin{align*}
m + k - 1 &= s \\
m - k - 1 &= p \\
2m + 2k &= 2s + 1 \\
2m - 2k - 1 &= 2p + 1
\end{align*}
\]

Thus, \( m = \frac{s+p}{2} \) and \( k = \frac{s-p}{2} \), which lead us to the following system:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= b \frac{s+p+4}{2} yx^p - cx^r - b^2 \frac{r+1}{2} x^{2p+1}
\end{align*}
\]
Case 4. For \( a = 0, b = 0, c \neq 0 \), the system \((4.1)\) is reduced to:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= cx^{2m-1}
\end{align*}
\]

Due to the exponents must be non-negative integers, we arrive to

\[
2m - 1 = r \in \mathbb{N}, \ \text{that is,} \ \ m = \frac{r+1}{2}.
\]

Case 5. For \( a \neq 0, b = 0, c \neq 0 \), the system \((4.1)\) is reduced to:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -cx^{s+p+1}
\end{align*}
\]

Case 6. For \( a = 0, b \neq 0, c = 0 \), the system \((4.1)\) is reduced to:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= b(2m - k)y^{m-k-1} - b^2m^2x^{2m-2k-1}
\end{align*}
\]

Case 7. For \( a \neq 0, b = 0, c = 0 \), the system \((4.1)\) is reduced to:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= a(2m + k)y^{m+k-1} - a^2m^2x^{2m+2k-1}
\end{align*}
\]
\[ \dot{x} = y \]
\[ \dot{y} = a(m + s + 1)yx^{s} - amx^{2s+1} \]

\[ \square \]

5. Critical Points

In this section, we present an analysis of the existence of critical points and the stability for each family associated to the Polyanin-Zaitsev vector field.

**Proposition 5.1.** For the family of systems (4.2), the following statements hold:

i. If \( c > 0 \), then \((0,0)\) is the only one critical point of the family.

a. If \( k = 0 \) and \( m \geq 1 \), then \((0,0)\) is an stable critical point.

b. If \( m + k - 1 \) is even and \( a(2m + k) > 0 \), then \((0,0)\) is an unstable node.

c. If \( m + k - 1 \) is odd, then \((0,0)\) is the union of one elliptic sector with one hyperbolic sector.

ii. If \( c < 0 \) then exist five critical points.

**Proof.** For this proof we take the family (4.2) in form (4.1). That is, we have

Now completing squares in the polynomial:

\[ y = 0 \]

\[ (a(2m + k)x^{m+k-1} + b(2m - k)x^{m-k-1})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1} = 0 \]

If \( y = 0 \), then \( (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1} = 0 \), with \( 2m - 2k - 1 \geq 1 \), then for the product equal to 0 it must be fulfilled that \( x = 0 \) or \( (a^2mx^{4k} + cx^{2k} + b^2m) = 0 \).

In the first case we obtain that \( x = 0 \) and we conclude \((x,y) = (0,0)\).

Now completing squares in the polynomial:

\[ a^2mx^{4k} + cx^{2k} + b^2m \]

\[ = a^2m(x^{4k} + \frac{cx^{2k}}{a^2m} + \frac{b^2m}{a^2m}) \]

\[ = a^2m(x^{4k} + \frac{cx^{2k}}{a^2m} + \frac{c^2}{4a^2m^2} + \frac{b^2m}{a^2m} - \frac{c^2}{4a^2m^2}) \]

\[ = a^2m(x^{2k} + \frac{c}{2a^2m})^2 + a^2m(b^m - \frac{c^2}{4a^2m}) \]

\[ = a^2m(x^{2k} + \frac{c}{2a^2m})^2 - (\frac{c^2 - 4a^2b^2m^2}{4a^2m}) \]

\[ = a^2m(x^{2k} + \frac{c}{2a^2m})^2 - (\frac{c^2 - 4a^2b^2m^2}{4a^2m}) \]

We can see that if \( c > 0 \) for the equations \( x^{2k} + \frac{c + \sqrt{c^2 - 4a^2b^2m^2}}{2a^2m} \) = 0 or \( x^{2k} + \frac{c - \sqrt{c^2 - 4a^2b^2m^2}}{2a^2m} = 0 \), then there are not real roots. That is, the only one critical
that:

\[ V + 4 \]

We remember, in this case \( V \) is a first critical point. But there are other solutions for equations \( a^2m^4 + c^2 + b^2m = 0 \),

\[ x_1^k = \frac{-c + \sqrt{c^2 - 4a^2b^2m^2}}{2a^2m} \quad \quad x_2^k = \frac{-c - \sqrt{c^2 - 4a^2b^2m^2}}{2a^2m} \]

We remember, in this case \( c < 0 \) then \( x_1^k \) and \( x_2^k \) is always positive, it is to say that:

\[ x_1^k = \pm \sqrt{-c + \sqrt{c^2 - 4a^2b^2m^2}} \quad \quad x_2^k = \pm \sqrt{-c - \sqrt{c^2 - 4a^2b^2m^2}} \]

Now we have to consider two cases:

\( k \in \mathbb{Z} \): Then the critical points for \( (4.2) \) are

\( (0,0), \quad \left( \pm \frac{2c}{a}, 0 \right) \), \quad \left( \pm \frac{2c}{a}, 0 \right) \).

\( k \notin \mathbb{Z} \): The critical points are

\( (0,0), \quad \left( \pm \sqrt{-c + \sqrt{c^2 - 4a^2b^2m^2}}, 0 \right) \), \quad \left( \pm \sqrt{-c - \sqrt{c^2 - 4a^2b^2m^2}}, 0 \right) \).

For the next step we consider \( c > 0 \) and the Liapunov function

\[ V = c_1x^{2m-2k} + c_2x^{m-k}y + c_3y^2 \]

The next stage is to find the conditions in which this function can be positive:

\[ V = c_1x^{2m-2k} + c_2x^{m-k}y + c_3y^2 \]

\[ = c_1\left( x^{2m-2k} + \frac{c_2x^{m-k}}{c_1} + \frac{c_3y^2}{c_1} \right) \]

\[ = c_1\left( x^{2m-2k} + \frac{c_2x^{m-k}}{c_1} + \frac{c_3y^2}{4c_1} - \frac{c_3y^2}{4c_1} \right) \]

\[ = c_1\left( x^{m-k} + \frac{c_2x^{m-k}}{c_1} \right)^2 + \left( \frac{4c_1c_3 - c_3^2}{4c_1} \right)y^2 \]

Therefore, the function \( V \) is positive for \( c_1 > 0 \) and \( 4c_1c_3 - c_3^2 \geq 0 \).

The derivative of \( V \) is

\[ V' = 2(m-k)c_1x^{2m-2k-1} + c_2(m-k)y^{m-k-1} + c_2x^{m-k}y' + 2c_3yy' \]

Now for the family \( (4.4) \) \( \dot{y} = 2m(a+b)x^{m-1}y - (a^2m+c+b^2m)x^{2m-1} \) we have that

\[ V' = 2mc_1(a+b)x^{2m-1} + c_2my^2x^{m-1} + c_2my^2x^{2m-1} - c_2(a^2m+c+b^2m)x^{3m-1} + 4mc_3(a+b)x^{m-1}y^2 - 2c_3(a^2m+c+b^2m)y^2x^{2m-1} \]

Arranging the right side we get:

\[ V' = yx^{2m-1}(2mc_1(a+b)+mc_2-2c_3(a^2m+c+b^2m)) + my^2x^{m-1}(c_2+4c_3(a+b)) - c_2(a^2m+c+b^2m) \]

We can observe that two cases should be considered, the first one corresponds to \( m \) is odd. Thus, the critical point \((0,0)\) is stable whenever:
a. $a^2m + c + b^2m = 0$  

b. $2mc_1(a + b) + mc_2 = 0$: $c_2 + 4c_3(a + b) < 0$

If $m$ is even, for $(0, 0)$ to be stable, it is required that:

a. $a^2m + c + b^2m = 0$  

b. $2mc_1(a + b) + mc_2 = 0$: $c_2 + 4c_3(a + b) < 0$

Now over the conditions of (1.1):

$(0, 0)$ is an isolated critical point.

$$X(x, y) = 0$$

$$Y(x, y) = (a(2m + k)x^{m+k} + b(2m - k)x^{m-k})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k}$$

The degree of $Y(x, y)$ should be greater than 1.

$$y = F(x) = 0$$

$$f(x) = Y(x, F(x)) - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k} = -a^2mx^{2m+2k} - 1 + \frac{c}{a^2m}x^{-2k} + \frac{b^2}{a^2}x^{4k}$$

$$\phi(x) = \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)(x, F(x)) = a(2m + k)x^{m+k} - 1 + \frac{b(2m-k)}{(2m+k)x^{2k}}$$

Then

$$\alpha = 2m + 2k - 1$$

$$\beta = m + k - 1$$

$$\bar{a} = -a^2m$$

$$\bar{b} = a(2m + k)$$

Now checking the conditions of the theorem we have:

$\alpha$ is odd, $\bar{a} < 0$,

$\bar{b}^2 + 4\bar{a}(\beta + 1) = a^2(2m + k) + 4a^2m(m + k) > 0$, we have the conditions of item c) (1.1).

If $\beta$ is even and $\bar{b} > 0$, then $(0, 0)$ is an unstable node. On the other hand, if $\beta$ is odd, then there exists the union of an elliptical sector and with an hyperbolic sector. Thus, we conclude the proof.

PROPOSITION 5.2. For the system (1.3) and (1.6) there are three critical points

PROOF. If

$$b^{x+3p+4}y^{x}-cx^{r} - b^{2x+1}x^{2p+1} = 0$$

then $\deg(Q) = \max\{r, 2p + 1\}$, that is, we should consider two cases.

- If $\deg(Q) = 2p + 1$ then $x^{r}(c + b^{2}(r+1)x^{2p+1-r}) = 0$. This implies that

$$x = 0 \quad \text{or} \quad x^{\gamma} = \frac{-2c}{b^{2}(r+1)}$$

being $\gamma = 2p - r + 1$. If $\gamma$ is even then it is necessary that $c < 0$, and therefore the critical points are $(0, 0)$, $\left(\sqrt[2p(r+1)]{-2c}, 0\right)$ and $\left(-\sqrt[2p(r+1)]{-2c}, 0\right)$. If $\gamma$ is odd then the critical points are $(0, 0)$ and $\left(\sqrt[2p(r+1)]{-2c}, 0\right)$.
• If $\deg(Q) = r$ analogously $x^{2p+1}((c)x^{r-2p-1} + \frac{b^2}{2}(r+1)) = 0$. If $r - 2p - 1 = \gamma$ is even then the critical points for the system (4.3) are $(0, 0)$, \((\sqrt{-\frac{2c}{a(r+1)}}, 0)\). On the other hand, if $\gamma$ is odd, it is necessary that $c < 0$ and for instance the critical points are $(0, 0)$, \((\sqrt{\frac{-2c}{a(r+1)}}, 0)\) and \((\sqrt{\frac{-2c}{a(r+1)}}, 0)\).

Now for the family (4.6), we have that
$$y = 0$$
\[a^{\frac{3s+p+1}{2}}y x^s - a^{\frac{r+1}{2}}x^{2s+1} - cx^r = 0.\]
Then $\deg(Q) = \max\{r, 2s + 1\}$, again we have to consider two cases.

• If $\deg(Q) = 2s + 1$ then $x^r \left( a \left(\frac{r+1}{2}\right) x^{2s+1-r} + c \right) = 0$. This implies that $x = 0$ or $x^3 = \frac{-2c}{a(r+1)}$, $\gamma = 2s - r + 1$.

If $\gamma$ is even then it is necessary that $c < 0$, and for instance the critical points are $(0, 0)$, \((\sqrt{\frac{-2c}{a(r+1)}}, 0)\) and \((\sqrt{\frac{-2c}{a(r+1)}}, 0)\).

If $\gamma$ is odd then the critical points are $(0, 0)$ and \((\sqrt{\frac{-2c}{a(r+1)}}, 0)\).

• If $\deg(Q) = r$ analogously $x^{2s+1}((c)x^{r-2s-1} + \frac{a(r+1)}{2}) = 0$. If $r - 2s - 1 = \gamma$ is even then the critical points for the system (4.6) are $(0, 0)$, \((\sqrt{\frac{-2c}{a(r+1)}}, 0)\). On the other hand, if $\gamma$ is odd, it is necessary that $c < 0$ and for instance the critical points are $(0, 0)$, \((\sqrt{\frac{-2c}{a(r+1)}}, 0)\) and \((\sqrt{\frac{-2c}{a(r+1)}}, 0)\).

\[\square\]

**Proposition 5.3.** For systems of the form (4.4), (4.5), (4.7) and (4.8), the point $(0,0)$ is the only critical point.

**Proof.** We can see that the common characteristic in these families is that $c = 0$. Then for the family (4.4)
$$y = 0$$
\[\left(a^{\frac{3s+p+1}{2}}x^s + b^{\frac{3p+4}{2}}x^p\right) y - a^{\frac{s+p+2}{2}}x^{2s+1} - b^{2s+p+2}x^{2p+1} = 0.\]
Now we have two cases. If $\deg(Q) = 2s + 1$ then $x^{2p+1} \left(\frac{s+p+2}{2}\right) (x^{2s-2p} + b^2) = 0$, where $s, p \in \mathbb{Z}^+$ and $2(s - p)$ is even. Therefore, we can conclude that the only solution for the systems under the conditions given above is $(0, 0)$. It follows analogously when $\deg(Q) = 2p + 1$.

For the family (4.5)
$$y = 0$$
\[-c x^{s+p+1} = 0\]
Then we can see that $(0,0)$ is the only critical point. For the family (4.7)
$$y = 0$$
\[b(m + p + 1)yx^p - b^2m x^{2p+1} = 0.\]
Then $b^2 mx^{2p+1} = 0$. Again $(0,0)$ is the only critical point. For the family \[1.8\]

$$a(m + s + 1)yx^s - amx^{2s+1} = 0$$

then $amx^{2s+1} = 0$, therefore $(0,0)$ is the only critical point. 

\[\square\]

6. Final Remarks

In this paper we studied from algebraic and qualitative point of view the five parametric family of linear differential systems that came from the corrigendum of Exercise 11 in [16, §1.3.3], which we called Polyanin-Zaitsev vector field. We solved the corrected exercise through a series of transformations using Hamiltonian changes of variables. A qualitative analysis was also developed to find critical points and their behavior.

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