Further results for a subclass of univalent functions related with differential inequality

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Abstract Let $\Omega$ denote the class of functions $f$ analytic in the open unit disc $\Delta$, normalized by the condition $f(0) = f'(0) - 1 = 0$ and satisfying the inequality

$$|zf'(z) - f(z)| < \frac{1}{2} \quad (z \in \Delta).$$

The class $\Omega$ was introduced recently by Peng and Zhong (Acta Math Sci 37B(1):69–78, 2017). Also let $\mathcal{U}$ denote the class of functions $f$ analytic and normalized in $\Delta$ and satisfying the condition

$$\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1 \quad (z \in \Delta).$$

In this article, we obtain some further results for the class $\Omega$ including, an extremal function and more examples of $\Omega$, inclusion relation between $\Omega$ and $\mathcal{U}$, the radius of starlikeness, convexity and close-to-convexity and sufficient condition for function $f$ to be in $\Omega$. Furthermore, along with the settlement of the coefficient problem and the Fekete–Szegö problem for the elements of $\Omega$, the Toeplitz matrices for $\Omega$ are also discussed in this article.

Keywords Univalent · Starlike · Convex · Close-to-convex · Fekete–Szegö problem · Coefficient estimates · Toeplitz determinant

Mathematics Subject Classification 30C45

1 Introduction

Let $\mathcal{A}$ denote the family of functions $f$ of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \]
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta), \tag{1.1} \]

which are analytic and normalized by the condition \( f(0) = f'(0) - 1 = 0 \) in the open unit disc \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathcal{S} \) denote its subclass of univalent functions. We say that the function \( f \in \mathcal{A} \) is starlike in \( \Delta \) if \( f(\Delta) \) is a set that starlike with respect to the origin. In other words, the straight line joining any point in \( f(\Delta) \) to the origin lies in \( f(\Delta) \). This means that \( tz_0 \in f(\Delta) \) when \( z_0 \in f(\Delta) \) and \( 0 \leq t \leq 1 \). We denote this set of functions by \( \mathcal{S}^* \). The well–known analytic description of starlike functions in terms of functions with positive real part states that \( f \in \mathcal{S}^* \) if, and only, if

\[
\text{Re}\left\{ \frac{z f''(z)}{f'(z)} \right\} > 0 \quad (z \in \Delta).
\]

We say that a set \( \Lambda \) is convex if the line segment joining any two points in \( \Lambda \) lies in \( \Lambda \). This means that \( tz_0 + (1-t)z_1 \in \Lambda \) where \( z_0, z_1 \in \Lambda \) and \( 0 \leq t \leq 1 \). A function \( f \in \mathcal{A} \) is called convex if \( f(\Delta) \) is a convex set. The set of convex functions in \( \Delta \) is denoted by \( \mathcal{K} \). Analytically, \( f \in \mathcal{K} \) if, and only, if

\[
\text{Re}\left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > 0 \quad (z \in \Delta).
\]

The classes \( \mathcal{S}^* \) and \( \mathcal{K} \) were introduced by Robertson, see [17]. We have \( \mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \), see [3]. A function \( f \in \mathcal{A} \) is said to be close–to–convex function, if there exists a convex function \( g \) such that

\[
\text{Re}\left\{ \frac{f'(z)}{g'(z)} \right\} > 0 \quad (z \in \Delta).
\]

By the Noshiro–Warschawski theorem [3, Theorem 2.16], every close–to–convex function is univalent. Also, this theorem is one of the important criterion for univalence. Let \( \mathcal{U} \) denote the class of all functions \( f \in \mathcal{A} \) satisfying the following inequality

\[
\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1 \quad (z \in \Delta).
\]

It is well–known that \( \mathcal{U} \subset \mathcal{S} \), see [1]. It’s worth mentioning that the Koebe function \( k(z) = z/(1 - z)^2 \) belongs to the class \( \mathcal{U} \) although functions in \( \mathcal{U} \) are not necessarily starlike in \( \Delta \), see [4, 14]. For more details about the class \( \mathcal{U} \) one can refer to [6, 10–13, 15].

Lately, Peng and Zhong [16], introduced and discussed a new subclass of analytic functions as follows

\[
\Omega := \left\{ f \in \mathcal{A} : |zf''(z) - f(z)| < \frac{1}{2}, z \in \Delta \right\}.
\]

The class \( \Omega \) is a subclass of the starlike functions [16, Theorem 3.1]. The main motive for defining the class \( \Omega \) is the relationship between the class \( \Omega \) and the class \( \mathcal{U} \), see for more details [16]. Also, they have investigated some properties for the class \( \Omega \), such as

- growth and distortion theorem;
- \( \Omega \) is a subset of the starlike functions;
- the radius of convexity;
- if \( f, g \in \Omega \), then \( f \ast g \in \Omega \), where “\( \ast \)” is the well–known Hadamard product;
- \( \Omega \) is a closed convex subset of \( \mathcal{A} \);
- and properties support point and extreme point of \( \Omega \).

Peng and Zhong estimated the coefficients of function \( f \) of the form (1.1) belonging to the class \( \Omega \), but there was no mention of the proof and its accuracy. In this article we give sharp estimates for the coefficients of functions \( f \) belonging to the class \( \Omega \).

Very recently, also Obradović and Peng (see [9]) studied the class \( \Omega \) and obtained two sharp sufficient conditions for the function \( f \) to be in the class \( \Omega \) as follows:

- if \( |f''(z)| \leq 1 \), then \( f \in \Omega \);
- if \( |z^2f''(z) + zf'(z) - f(z)| \leq 3/2 \), then \( f \in \Omega \).

\[ \square \]
Following, we give another sufficient condition for functions \( f \) to be in the class \( \Omega \).

The function \( f \in \mathcal{A} \) is subordinate to the function \( g \in \mathcal{A} \), written as \( f(z) < g(z) \) or \( f \prec g \), if there exists an analytic function \( w \), known as a Schwarz function, with \( w(0) = 0 \) and \( |w(z)| \leq |z| \), such that \( f(z) = g(w(z)) \) for all \( z \in \Delta \). Moreover, if \( g \in \mathbb{S} \), then we have the following equivalence (c.f. [8])

\[
    f(z) < g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).
\]

The structure of the paper is the following. In Section 2 we give an extremal function for the class \( \Omega \) and solve an open question related to the inclusion relation between \( \Omega \) and \( \mathcal{U} \), partially. In Section 3 some radius problems for the function \( f \in \Omega \) are obtained. In Section 4 we present two conditions for functions \( f \) to be in the class \( \Omega \). In Section 5 we study the coefficients of the function \( f \) of the form (1.1) belonging to the class \( \Omega \). Finally, in Section 6, the Fekete-Szego problem and Toeplitz matrices are investigated.

## 2 Extremal function and inclusion relation

First, we give an example for the class \( \Omega \) which is an extremal function for several problems.

**Example 2.1** Let

\[
\tilde{f}_n(z) := z + \frac{1}{2(n-1)} z^n \quad (n = 2, 3, \ldots, z \in \Delta).
\]  

(2.1)

It is clear that \( \tilde{f}_n \in \mathcal{A} \) and

\[
\tilde{f}_n'(z) - \tilde{f}_n(z) = \frac{1}{2} n^2 (n-1) z^{n-1} \quad (n = 2, 3, \ldots, z \in \Delta).
\]

Since \( z \in \Delta \), thus \( |\tilde{f}_n'(z) - \tilde{f}_n(z)| = |z^n/2| < 1/2 \) and consequently \( \tilde{f}_n \in \Omega \). We remark that the function \( \tilde{f}_n \) is univalent in \( \Delta \) for \( n = 2, 3, \ldots \). The function \( \tilde{f}_n \) is an extremal function for several problems such as, coefficient estimates, the radius of convexity and starlikeness in the class \( \Omega \) as will be seen later.

Whether \( \mathcal{U} \subset \Omega \) or \( \Omega \subset \mathcal{U} \) is an open question for us. The following Example 2.2 shows that \( \Omega \subset \mathcal{U} \) and \( \mathcal{U} \not\subset \Omega \).

**Example 2.2** Let \( \tilde{f}_n \) \((n = 2, 3, \ldots)\) be defined by (2.1). Then by Example 2.1 we have \( \tilde{f}_n \in \Omega \) for \( n = 2, 3, \ldots \). In particular, if we take \( n = 2 \), then the bounded analytic function \( \tilde{f}_2 = z + z^2/2 \) belongs to \( \mathcal{U} \), see [10, p. 175]. Also, the function \( \tilde{f}_3 = z + z^3/4 \) belongs to \( \mathcal{U} \), too. Because

\[
\left( \frac{z}{\tilde{f}_3(z)} \right)^2 \tilde{f}_3''(z) = \frac{1 + 3z^2/4}{1 + z^2/4}^2
\]

and for all \( z \in \Delta \) we have

\[
0 \leq \left| \left( \frac{z}{\tilde{f}_3(z)} \right)^2 \tilde{f}_3''(z) - 1 \right| = \left| \frac{1 + 3z^2/4}{1 + z^2/4}^2 - 1 \right| < 0.56 < 1.
\]  

(2.2)

Now we consider the function \( f_1 \) as follows

\[
\frac{z}{f_1(z)} = 1 + \frac{1}{2} z + \frac{1}{2} z^3 \quad (z \in \Delta).
\]

It is easy to see that \( z/f_1(z) \neq 0 \) in \( \Delta \) and

\[
\left( \frac{z}{f_1(z)} \right)^2 f_1''(z) - 1 = -z^3.
\]

Since \( z \in \Delta \), thus \( f_1 \) belongs to \( \mathcal{U} \). A simple calculation gives us
Now, if we take $z_0 = -2/3 \in \Delta$, then

$$|z_0f'_1(z_0) - f_1(z_0)| = \left| -\frac{z_0^2 + 3z_0^4}{2 + z_0 + z_0^3}\right| = 1 > 1/2.$$  

This shows that $f_1 \notin \Omega$. Therefore $\mathcal{U} \notin \Omega$.

### 3 Radius problems

Peng and Zhong [16, Theorem 3.4] showed that the radius of convexity for the class $\Omega$ is 1/2. Here, by use of the function (2.1), we show that the result of Peng and Zhong is sharp.

**Example 3.1** The function $\tilde{f}_n$ shows that the members of the class $\Omega$ are convex in the open disc $|z| < r$ where $r < 1/2$. Thus the result of Theorem 3.4 of [16] is sharp.

**Proof** Let $\tilde{f}_n$ be given by (2.1). With a simple calculation, we get

$$1 + \frac{zf''}{f'(z)} = 1 + \frac{n}{2} \left( 1 + \frac{1}{2} \frac{n}{2(n-1)} \frac{z}{z^n} \right) (n = 2, 3, \ldots).$$

Using the analytic definition of convexity, the radius $r$ of convexity is the largest number $0 < r < 1$ for which

$$\min_{|z|=r} \text{Re} \left\{ 1 + \frac{zf''}{f'(z)} \right\} \geq 0.$$

Now, for every $r \in (0, 1)$, we have

$$\text{Re} \left\{ 1 + \frac{zf''}{f'(z)} \right\} = \text{Re} \left\{ 1 + \frac{n}{2} \left( 1 + \frac{1}{2} \frac{n}{2(n-1)} \frac{z}{z^n} \right) \right\}
\geq 1 - \frac{n}{2} \left| 1 + \frac{1}{2} \frac{n}{2(n-1)} \frac{z}{z^n} \right|
\geq 1 - \frac{n}{2} \frac{|z|^{n+1}}{2(n-1)|z|^n}
= 1 - \frac{n}{2} \frac{r^{n+1}}{2(n-1)r^n} =: \phi(r) \quad (|z| = r < 1).$$

It is easy to see that $\phi(r) > 0$ if and only if

$$r < \left( \frac{2(n-1)}{n^2} \right)^{1/n} =: r_0 \quad (n = 2, 3, \ldots).$$

We note that if we put $n = 2$, then $r_0$ becomes 1/2 and if $n \to \infty$, then $r_0 \to 1$. This is the end of proof. \(\square\)

In the next result, with other proof we show that $\Omega \subset \mathcal{S}^*$.  

**Lemma 3.1** Every function $f \in \Omega$ is a starlike univalent function in the open unit disc $\Delta$.  

**Proof** By [16, Eq. (3.4)], $f$ belongs to the class $\Omega$ if, and only if,

$$f(z) = z + \frac{1}{2} z \int_0^z \varphi(\zeta) d\zeta,$$

where $\varphi \in \mathcal{A}$ and $|\varphi(z)| \leq 1$ ($z \in \Delta$). Now from (3.1), we have
Therefore by the analytic definition of starlikeness, we get
\[
\min_{|z|=r} \Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq 1 - \frac{1}{2} \left| \frac{z^2 \varphi(z)}{z + \frac{1}{2} z^2 \int_0^1 \varphi(zt) dt} \right| 
\geq 1 - \frac{r}{2 - r} \quad (|z|=r<1).
\]
It is easy to see that \(1 - \frac{r}{2 - r} > 0\) when \(0 < r < 1\) and concluding the proof. \(\square\)

In the sequel, we will show that the class \(\Omega\) is a subclass of close–to–convex functions.

**Lemma 3.2** Every function \(f \in \mathcal{A}\) which belongs to the class \(\Omega\) is close–to–convex in \(\Delta\).

**Proof** Let the function \(f \in \mathcal{A}\) belongs to the class \(\Omega\). Then by (3.1), we get
\[
f'(z) = 1 + \frac{1}{2} z \varphi(z) + \frac{1}{2} z \int_0^1 \varphi(zt) dt \quad (0 \leq t \leq 1).
\]
Since \(|\varphi(z)| \leq 1\), we have
\[
\Re \{f'(z)\} = \Re \left\{ 1 + \frac{1}{2} z \varphi(z) + \frac{1}{2} z \int_0^1 \varphi(zt) dt \right\} 
\geq 1 - \frac{1}{2} z \varphi(z) + \frac{1}{2} z \int_0^1 \varphi(zt) dt 
\geq 1 - r \quad (|z|=r<1).
\]
The last inequality is non–negative if \(r<1\) and concluding the proof. \(\square\)

## 4 Conditions for functions \(f\) to be in \(\Omega\)

First, we give a sufficient condition for functions \(f\) of the form (1.1) to be in the class \(\Omega\). We remark that since \(\Omega\) is a subclass of the close–to–convex univalent functions, the following lemma also is a sufficient condition for univalence.

**Lemma 4.1** Let \(f \in \mathcal{A}\). If
\[
\left| \left( \frac{f(z)}{z} \right)' \right| < \frac{1}{2} \quad (z \in \Delta, z \neq 0),
\]
then \(f \in \Omega\). The number \(1/2\) is the best possible.

**Proof** Let \(f \in \mathcal{A}\) satisfies the inequality (4.1). Since \(z \in \Delta\) and consequently \(|z|^2 < 1\), thus by the inequality (4.1), we get
\[
\left| z^2 \left( \frac{f(z)}{z} \right)' \right| < \frac{1}{2} \quad (z \in \Delta, z \neq 0).
\]
Now the assertion follows from the following identity
\[
z^2 \left( \frac{f(z)}{z} \right)' = zf'(z) - f(z) \quad (z \in \Delta, z \neq 0)
\]
and concluding \(f \in \Omega\). For the sharpness, consider the function \(\tilde{f}_\lambda(z) = z + \lambda z^2\) where \(|\lambda| < 1/2\) and \(z \in \Delta\). A simple calculation gives that
Applying Lemma 4.1, we present a sufficient condition for the function $b$

\[ f_n(z) = z + a_n z^n \quad (n = 2, 3, \ldots) \]

Then $f_n \in \Omega$. It is easy to see that if $\lambda > 1/2$, then $\widehat{f}_\lambda \not\in \Omega$. Also, since $f'_n(z) = 1 + 2\lambda z$ vanish at $-\frac{1}{2\lambda}$, we conclude that $\widehat{f}_\lambda$ is not univalent in $\Delta$ when $\lambda > 1/2$. This is the end of proof.

As an application of the Lemma 4.1 we give another example for the class $\Omega$.

**Example 4.1** Define $f_{\gamma, \beta}(z) = z + \gamma z^2 + \beta z^3$, where $\gamma$ and $\beta$ are two complex numbers. If $|\gamma| + 2|\beta| < 1/2$, then $f_{\gamma, \beta}(z) \in \Omega$. In particular, the function $\ell(z) = z + z^2/5 + z^3/8$ belongs to the $\Omega$. We note that the function $\ell$ is univalent in the unit disc $\Delta$. The Figure 1(a) shows the image of $\Delta$ under the function $\ell(z)$.

Applying Lemma 4.1, we present a sufficient condition for the function $f_n(z) = z + a_n z^n$ ($n = 2, 3, \ldots$) to be in the class $\Omega$.

**Example 4.2** Consider the function $f_n(z) = z + a_n z^n$ where $z \in \Delta$. If

\[ |a_n| < \frac{1}{2(n-1)} \quad (n = 2, 3, \ldots), \]

then $f_n \in \Omega$.

**Proof** Let $f_n(z) = z + a_n z^n$ and the inequality (4.3) holds. From (4.3), we get

\[ (n-1)|a_n||z|^{n-2} < \frac{1}{2} \quad (z \in \Delta, n = 2, 3, \ldots). \]

Since

\[ \left( \frac{f_n(z)}{z} \right)' = (n-1)a_n z^{n-2} \quad (z \in \Delta, n = 2, 3, \ldots), \]

the inequalities (4.4) and (4.5), imply that

\[ \left| \left( \frac{f_n(z)}{z} \right)' \right| < \frac{1}{2} \quad (z \in \Delta, n = 2, 3, \ldots). \]

Now the desired result follows from the Lemma 4.1. \qed

In the sequel, we recall from [8, p. 24], the function $q_M(z)$ given by

\[ q_M(z) = M \frac{Mz + a}{M + az} \quad (z \in \Delta), \]

where $M > 0$, $|a| < M$. We have $q_M(0) = a$ and $q_M(\Delta) = \{ \zeta : |\zeta| < M \} = \Delta_M$. By applying the function (4.6) and by the subordination, we present a sufficient and necessary condition for functions $f$ to be in the class $\Omega$.

**Lemma 4.2** Let $f \in A$. Then $f \in \Omega$ if, and only if,

\[ zf'(z) - f(z) < q_{1/2}(z) \quad (z \in \Delta), \]

where $q_{1/2}(z)$ is defined by (4.6) when $M = 1/2$ and $a = 0$.

**Proof** If $f \in \Omega$, then by definition we have

\[ |zf'(z) - f(z)| < \frac{1}{2} \quad (z \in \Delta). \]

Thus, by (4.8), $zf'(z) - f(z)$ lies in the open disc $\Delta_{1/2}$ and it is clear that $q_{1/2}(\Delta) = \Delta_{1/2}$. Because $q_{1/2}$ is univalent, thus by the subordination principle, we get (4.7). Indeed, since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $zf'(z) - f(z) = \sum_{n=2}^{\infty} (n-1)a_n z^n$, we have $a = 0$ in (4.6). This is the end of proof. \qed
5 On coefficients

The first result of this section is the following.

**Theorem 5.1** Let \( \phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) be an analytic function in \( \Delta \) that satisfies the coefficient condition
\[
\sum_{n=1}^{\infty} n|c_n| < \frac{1}{2}.
\]

Then the function \( f(z) = z \phi(z) \) belongs to the class \( \Omega \).

**Proof** Let \( f \) be given by \( f(z) = z \phi(z) \) where \( \phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \). We have
\[
\frac{f(z)}{z} = \phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.
\]

Now by differentiating the above relation (5.2) and multiplying by \( z^2 \), we get
\[
z^2 \left( f(z) \right)' = \sum_{n=1}^{\infty} n c_n z^{n+1}.
\]

Therefore using the coefficient condition (5.1) and the identity (4.2), we deduce that
\[
|zf'(z) - f(z)| = \left| z^2 \left( f(z) \right)' \right| = \left| \sum_{n=1}^{\infty} n c_n z^{n+1} \right| < \sum_{n=1}^{\infty} n |c_n| < \frac{1}{2}
\]
and concluding the proof.

![Fig. 1](a): The image of \( \Delta \) under the function \( \ell(z) = z + z^2/5 + z^3/8 \) (b): The image of \( \Delta \) under the function \( \phi_1(z) = z - z^2/5 - z^3/8 \)
Theorem 5.1 allows us to find many examples that belong to $\Omega$. For example, consider $\phi_1(z) = 1 - z/5 - z^2/8$. We have $c_1 = -1/5$, $c_2 = -1/8$ and $c_3 = c_4 = \cdots = 0$. Thus the coefficients of $\phi_1$ satisfy the condition (5.1) and we conclude that the univalent function $f(z) = z\phi_1(z) = z - z^2/5 - z^3/8$ belongs to the class $\Omega$. The Figure 1(b) shows the image of $\Delta$ under the function $\phi_1(z)$.

Lemma 5.1

Let $q(z) = \sum_{n=1}^{\infty} Q_n z^n$ be analytic and univalent in $\Delta$ such that maps $\Delta$ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} P_n z^n$ is analytic in $\Delta$ and satisfies the subordination $p(z) < q(z)$, then $|P_n| \leq |Q_1|$ where $n = 1, 2, \ldots$.

Theorem 5.2

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ be in the class $\Omega$. Then

$$|a_n| \leq \frac{1}{2(n-1)} \quad (n \geq 2).$$

The result is sharp.

Proof. Let $f$ be of the form (1.1) belongs to the class $\Omega$. Then by Lemma 4.2, we have

$$zf'(z) - f(z) = \sum_{n=2}^{\infty} (n-1)a_n z^n < \frac{1}{2} z =: q_{1/2}(z) \quad (z \in \Delta).$$

Since $q_{1/2}$ is convex univalent, thus applying the Lemma 5.1 we get

$$|(n-1)a_n| \leq \frac{1}{2} \quad (n \geq 2).$$

Therefore the inequality (5.3) holds. It is easy to see that the result is sharp for the function $\tilde{f}_n$, where $\tilde{f}_n$ is defined in (2.1). This completes the proof. \qed

6 Fekete–Szegö problem and Toeplitz matrices

In recent years, the problem of finding sharp upper bounds for the Fekete–Szegö coefficient functional associated with the $k$–th root transform has been studied by many scholars (see for example [2, 5, 19]). For a univalent function $f$ of the form (1.1), the $k$–th root transform is defined by

$$F_k(z) := (f(z^k))^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1} \quad (z \in \Delta).$$

A simple calculation gives that, for $f$ given by (1.1),

$$(f(z^k))^{1/k} = z + \frac{1}{k} a_2 z^{k+1} + \left( \frac{1}{k} a_3 - \frac{1}{2} \frac{1-k}{k^2} a_2 \right) z^{2k+1} + \cdots.$$  \hspace{1cm} (6.2)

Equating the coefficients of (6.1) and (6.2), we have

$$b_{k+1} = \frac{1}{k} a_2 \quad \text{and} \quad b_{2k+1} = \frac{1}{k} a_3 - \frac{1}{2} \frac{1-k}{k^2} a_2^2.$$  \hspace{1cm} (6.3)

In the sequel, we obtain this problem for the class $\Omega$. Further we denote by $\mathcal{P}$ the well–known class of analytic functions $p$ with $p(0) = 1$ and $\text{Re} \{p(z)\} > 0$ where $z \in \Delta$. Functions in $\mathcal{P}$ are called Carathéodory functions. The following lemma due to Keogh and Merkes [7] will be useful in this section.

Lemma 6.1

Let the function $p(z)$ given by

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots,$$

be in the class $\mathcal{P}$. Then, for any complex number $\mu$
\[ |p_2 - \mu p_1^2| \leq 2 \max \{1, |2\mu - 1|\} . \]

The result is sharp.

**Theorem 6.1** Let the function \( f \) of the form (1.1) belongs to the class \( \Omega \). Then for any complex number \( \mu \) and \( k \in \{1, 2, 3, \ldots \} \), we have

\[ |b_{2k+1} - \mu b_{k+1}^2| \leq \frac{1}{4k} \max \left\{ 1, \left| \frac{2\mu + k - 1}{2k} \right| \right\}. \]  

(6.4)

where \( b_{2k+1} \) and \( b_{k+1} \) are defined in (6.3). The result is sharp.

**Proof** If \( f \in \Omega \), then by Lemma 4.2 and definition of subordination there exits a Schwarz function \( w(z) \) such that

\[ zf'(z) - f(z) = \frac{1}{2} zw(z) \quad (z \in \Delta). \]  

(6.5)

If we define the function \( p \) as follows

\[ p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \Delta), \]  

(6.6)

thus \( p(z) \) is a analytic function in \( \Delta \) and \( p(0) = 1 \). A simple calculation gives us

\[ w(z) = \frac{1}{2} p_1 z + \frac{1}{2} \left( p_2 - \frac{1}{2} p_1^2 \right) z^2 + \frac{1}{2} \left( p_3 - p_1 p_2 + \frac{1}{4} p_1^3 \right) z^3 + \cdots \quad (z \in \Delta). \]  

(6.7)

From (6.5)–(6.7), equating coefficients gives, after simplification,

\[ a_2 = \frac{1}{4} p_1, \]  

(6.8)

\[ 2a_3 = \frac{1}{4} \left( p_2 - \frac{1}{2} p_1^2 \right). \]  

(6.9)

Replacing (6.8) and (6.9) into (6.3), we get

\[ b_{k+1} = \frac{1}{4k} p_1 \quad \text{and} \quad b_{2k+1} = \frac{1}{8k} \left( p_2 - \frac{1}{2} p_1^2 \right) - \frac{1}{32} \frac{k - 1}{k^2} p_1^2. \]

Thus

\[ b_{2k+1} - \mu b_{k+1}^2 = \frac{1}{8k} \left( p_2 - \frac{1}{2} \left( \frac{3k + 2\mu - 1}{2k} \right) p_1^2 \right). \]

If we let \( \mu' = \frac{1}{4} \left( \frac{3k + 2\mu - 1}{2k} \right) \), then as an application of the Lemma 6.1, we get the desired inequality (6.4). \( \square \)

Putting \( k = 1 \) in the Theorem 6.1, we have.

**Theorem 6.2** (Fekete–Szegö problem) Let \( f \) be of the form (1.1) belongs to the class \( \Omega \). Then we have the following sharp inequality

\[ |a_3 - \mu a_2^2| \leq \frac{1}{4} \max \{1, |\mu|\} \quad (\mu \in \mathbb{C}). \]

Since every function \( f \) belongs to the class \( \Omega \) is univalent, and every univalent function has an inverse \( f^{-1} \), which is defined by \( f^{-1}(f(z)) = z \ (z \in \Delta) \) and

\[ f(f^{-1}(w)) = w \quad (|w| < r_0; \ r_0 \geq 1/4), \]

where

\[ f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots, \]  

(6.10)

thus it is natural to consider the following result.
Corollary 6.1  Let $f \in A$ be in the class $\Omega$. Also let the function $f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n$ be inverse of $f$. Then we have the following sharp inequalities
\[ |b_i| \leq \frac{1}{2} \quad (i = 2, 3) \]
and $|b_4| \leq 19/24$.

Proof  Relation (6.10) gives us
\[ b_2 = -a_2, \quad b_3 = 2a_2^2 - a_3 \quad \text{and} \quad b_4 = -(5a_2^2 - 5a_2a_3 + a_4). \]
Thus, by putting $n = 2$ in (5.3) we can get
\[ |b_2| = |a_2| \leq 1/2. \]
For estimate of $|b_3|$, it suffices in Theorem 6.2, we put $\mu = 2$. Finally, since $b_4 = 5a_2(a_3 - a_2^2) - a_4$ by the Fekete–Szegö problem (Theorem 6.2) for $\mu = 1$, we get
\[ |b_4| = |5a_2(a_3 - a_2^2) - a_4| \]
\[ \leq 5|a_2||a_3 - a_2^2| + |a_4| \]
\[ \leq \frac{5}{2} \times \frac{1}{4} + \frac{1}{6} = \frac{19}{24} \]
and concluding the proof. □

Following, we recall the symmetric Toeplitz determinant
\[ T_q(n) := \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_n & \ldots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \ldots & a_n \end{vmatrix}, \]
where $n, q \in \{1, 2, 3, \ldots\}$ and $a_1 = 1$. Toeplitz matrices are one of the most well–studied and understood classes of structured matrices. Also, they have many applications in all branches of pure and applied mathematics (see for more details Ye and Lim [20, Section 2]). In the next result, we obtain sharp bounds for the coefficient body $|T_q(n)|$, $q = 2, 3$ and $n = 1, 2$ where the entries of $T_q(n)$ are the coefficients of functions $f$ of form (1.1) that are in the class $\Omega$.

Theorem 6.3  Let $f$ be of the form (1.1) belongs to the class $\Omega$. Then we have
\begin{enumerate}
  \item $|T_2(n)| \leq \frac{1}{4(n-1)^2} + \frac{1}{4n^2} \quad (n \geq 2)$.
  \item $|T_3(1)| \leq \frac{1}{4n^2}$.
  \item $|T_3(2)| \leq \frac{1}{32n^2}$.
\end{enumerate}
All the inequalities are sharp.

Proof  (1) If $f \in \Omega$, then by Theorem 5.2 and by the definition of the symmetric Toeplitz determinant $T_q(n)$ we have
\[ |T_2(n)| = |a_n^2 - a_{n+1}^2| \leq |a_n|^2 + |a_{n+1}|^2 \leq \frac{1}{4(n-1)^2} + \frac{1}{4n^2}. \]
(2) It is clear that $|T_3(1)| = |1 - 2a_2^2 + 2a_3a_2^2 - a_3^2|$. Thus
\[ |T_3(1)| \leq 1 + 2|a_2|^2 + |a_3||2a_2^2 - a_3| \]
\[ \leq 1 + 2(1/4) + (1/4)(1/2) = 13/8. \]
Note that the Fekete–Szegö problem is used, (see Theorem 6.2 with $\mu = 2$).
(3) We have
Thus we get $T_3(2) = (a_2 - a_3)(a_2^2 - 2a_3^2 + a_2a_4)$. For any $f \in \Omega$, we have $|a_2 - a_3| \leq |a_2| + |a_4| \leq 1/2 + 1/6 = 2/3$. Now, it is enough to obtain the maximum of $|a_2^2 - 2a_3^2 + a_2a_4|$ when $f \in \Omega$. By (6.8), (6.9) and since

$$3a_4 = \frac{1}{4} \left( p_3 - p_1p_2 + \frac{1}{3}p_1^3 \right),$$

we get

$$|a_2^2 - 2a_3^2 + a_2a_4| = \left| \frac{1}{16}p_1^2 - \frac{1}{8} \left( p_2 - \frac{1}{2}p_1^2 \right)^2 + \frac{1}{48}p_1 \left( p_3 - p_1p_2 + \frac{1}{3}p_1^3 \right) \right|$$

$$\leq \frac{1}{16} |p_1|^2 + \frac{1}{8} \left| p_2 - \frac{1}{2}p_1^2 \right|^2 + \frac{1}{48} |p_1| |p_3 - p_1p_2| + \frac{1}{144} |p_1|^4$$

$$\leq \frac{4}{16} + \frac{1}{8} + \frac{(2)(2)}{48} + \frac{16}{144} = \frac{329}{366}. $$

Because $|p_3 - p_1p_2| \leq 2$ and $|p_2 - \frac{1}{2}p_1^2| \leq 4$ (by Lemma 6.1). Thus $|T_3(2)| \leq \frac{2 \times 329}{366} = \frac{329}{183}$. Here, the proof ends. \hfill \Box

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**References**

1. L. A. Aksentiev, Sufficient conditions for univalence of regular functions, *Izv. Vyssh. Uchebn. Zaved. Mat.*, 3(1958), 3–7.
2. R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, The Fejér–Szegö coefficient functional for transforms of analytic functions, *Bull. Iranian Math. Soc.*, 2009, 119–142.
3. P. L. Duren, *Univalent Functions*, Springer–Verlag, New York (1983).
4. R. Fournier, and S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, *Complex Var. Elliptic Equ.*, 52(2007), 1–8.
5. R. Kargar, A. Ebadian, and J. Sokół, On Booth lemniscate and starlike functions, *Anal. Math. Phys.*, 9(2019), 143–154.
6. R. Kargar, A. Ebadian, and J. Sokół, On subordination of some analytic functions, *Sib. Math. J.*, 57(2016), 599–604.
7. F. R. Keogh, and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, 20(1969), 8–12.
8. S. S. Miller, and P. T. Mocanu, *Differential Subordinations, Theory and Applications*, Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York/Basel (2000).
9. M. Obradović, and Z. Peng, Some new results for certain classes of univalent functions, *Bull. Malays. Math. Sci. Soc.*, 41(2018), 1623–1628.
10. M. Obradović, and S. Ponnusamy, New criteria and distortion theorems for univalent functions, *Compl. Var. Theory Appl.*, 44(2001), 173–191.
11. M. Obradović, and S. Ponnusamy, On a class of univalent functions, *Appl. Math. Lett.*, 25(2012), 1373–1378.
12. M. Obradović, and S. Ponnusamy, Product of univalent functions, *Math. Comput. Model.*, 57(2013), 793–799.
13. M. Obradović, and Ponnusamy (2012) Radius of univalence of certain combination of univalent and analytic functions, *Bull. Malays. Math. Sci. Soc.*, 35, 325–334.
14. M. Obradović, and S. Ponnusamy, Univalence and starlikeness of certain products of univalent functions defined by convolution, *J. Math. Anal. Appl.*, 336(2007), 758–767.
15. S. Ozaki, and M. Nunokawa, The Schwarzian derivative and univalent functions, *Proc. Amer. Math. Soc.*, 33(1972), 392–394.
16. Z. Peng, and G. Zhong, Some properties for certain classes of univalent functions defined by differential inequalities, *Acta Math. Sci.*, 37B(2017), 69–78.
17. M. S. Robertson, On the theory of univalent functions, *Ann. Math.*, 37(1936), 374–408.
18. W. Rogosinski, On the coefficients of subharmonic functions, *Proc. London. Math. Soc.*, 2(1945), 48–82.
19. H. M. Srivastava, D. Răducanu, and P. Zaprawa, A certain subclass of analytic functions defined by means of differentiation subordination, *Filomat*, 30(2016), 3743–3757.
20. K. Ye, and L. Lek–Heng, Every matrix is a product of Toeplitz matrices, *Found. Comput. Math.*, 16(2016), 577–598.