BOUNDARY DYNAMICS OF THE REPLICATOR EQUATIONS
FOR NEUTRAL MODELS OF CYCLIC DOMINANCE

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Abstract. We study the replicator equations, also known as mean-field equations, for a simple model of cyclic dominance with any number \( m \) of strategies, generalizing the rock-paper-scissors model which corresponds to the case \( m = 3 \). Previously the dynamics were solved for \( m \in \{3, 4\} \) by consideration of \( m - 2 \) conserved quantities. Here we show that for any \( m \), the boundary of the phase space is partitioned into heteroclinic networks for which we give a precise description. A set of \( \lfloor m/2 \rfloor \) conserved quantities plays an important role in the analysis. We also discuss connections to the well-mixed stochastic version of the model.

1. Introduction. Cyclic dominance is any interaction scenario with \( m \geq 3 \) strategies, in which strategy \( i \) is dominant over strategy \( i - 1 \) modulo \( m \), for every \( i \in \{0, \ldots, m - 1\} \). The simplest example is the rock-paper-scissors game, in which rock beats scissors, which beats paper, which in turn beats rock. Models of cyclic dominance are of interest not only mathematically but also in population ecology and evolutionary game theory – see [12] for a fairly recent review. In particular, they comprise a class of examples with a simple interaction structure that gives rise to complex patterns, and is fruitful for the study of coexistence. Cyclic dominance is a phenomenon that can emerge spontaneously; this is discussed for public goods games in [13] [14], and for neutral models of cooperation, in [15].

The earliest apparent study of cyclic dominance [10] is in the context of ordinary differential equations (ODEs). The authors consider a model with \( m = 3 \) and balanced strategies, i.e., such that the dynamics is equivariant with respect to cyclic permutation. They give conditions on parameters for stability of either the coexistence equilibrium, or the boundary of the phase space, and also describe the neutral case in which the interior of the phase space is mostly foliated by periodic orbits. A later work [8] explores orbit shapes for various \( m \). The model is generalized somewhat in [16] to show robustness of the heteroclinic cycle, and persistence is studied in [6]. These studies focus mostly on the case \( m = 3 \).

On a separate front, 14 years later, a stochastic individual-based model of cyclic dominance was studied on the lattice \( \mathbb{Z}^d \) [2]. The authors show that for any \( d \geq 1 \)

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and \( m \in \{3,4\} \), every site changes strategy infinitely often, while for \( d = 1 \) and \( m \geq 5 \), every site eventually adopts a strategy that it never changes. For \( m \in \{3,4\} \), the question of local coexistence, i.e., whether or not two nearby sites are likely to differ in strategy after a long time, was settled only recently for \( d = 1 \) in [7] with a negative answer, and remains open for \( d = 2 \) even in the simplest non-trivial case \( m = 3 \).

With the increasing popularity of evolutionary game theory, models of cyclic dominance began to appear in a wide variety of different guises. We shall not discuss all of these, but instead note that most formulations begin at the individual level, by specifying a spatial/network structure, and rules governing local interactions. Interactions are such that a type \( i \) site can invade a neighbouring type \( i - 1 \) region, and may also include random mutation and movement.

Usually, the underlying model is stochastic, and deterministic versions are obtained through a large-population or high-density limit. For example, if interactions between pairs of individuals are well-mixed, i.e., occur uniformly at random, the fraction of individuals following each strategy, for large population values, follows solutions to the classical ODE models, which are often called replicator equations, or sometimes mean-field equations.

Generally speaking, before studying a complex spatial or network model it is helpful to first understand the corresponding well-mixed model and its replicator equations. A comparatively recent study [5] gives a complete description of the replicator equations, for \( m \in \{3,4\} \) and all possible pairwise interactions equivariant under cyclic permutation, which includes models of cyclic dominance. The well-mixed stochastic model with \( m = 3 \) is studied in [11] in the case of balanced strategies plus mutation, and in [3] for unbalanced strategies in the form of unequal conversion rates. In [3] they discover a law of survival of the weakest, i.e., in the long run, the strategy with the slowest conversion rate is the one that tends to survive. For four strategies, a well-mixed stochastic model is explored in some detail in [4], where they find that long-term survival is favoured for the non-interacting pair of strategies for which the product of the initial population densities is higher.

So far, there are no detailed studies of either the replicator equations or the well-mixed stochastic model for \( m \geq 5 \). Here we begin this program by studying the replicator equations, for any \( m \), in the simplest case: balanced strategies with no mutation, and no random copying. With the help of some conserved quantities, we are able to completely describe the orbits on the boundary of the phase space, which consist of equilibria and heteroclinic orbits arranged in a complex network. In addition, we discover an interesting checkering pattern for the limit points of non-equilibrium boundary trajectories.

Since our results hold for every \( m \), this represents an important first step towards a complete understanding of the replicator equations for this model, and for variations that include asymmetry, mutation and copying. Moreover, we believe that our results can be used to study the long-term behaviour of the well-mixed stochastic model for any \( m \), and that the conserved quantities and the checkering pattern that we found will play a key role in the analysis. We explore these connections, and directions for future work, in the discussion section at the end of the article.

The article is laid out as follows. In the next section we introduce the model and the important conserved quantities, and give a simplified statement of the main results. In Section 3 we give a concise proof of existing results for \( m \in \{3,4\} \). In
Section 4 we establish Theorem 2.3, the main result and give a precise description of heteroclinic orbits, for any $m$.

2. Model and main results. Fix a positive integer $m \geq 2$, and let $[m]$ denote $\{0, \ldots, m-1\}$, with $i \in [m]$ understood modulo $m$. Define the continuous time dynamical system on $\mathbb{R}^m$ by

$$u' = F(u), \quad \text{where } F_i(u) = u_i(u_{i-1} - u_{i+1}) \text{ for each } i \in [m].$$

We call this the cyclic model with $m$ strategies. If $m = 2$ then $u'_0 = u'_1 = 0$ no matter the initial value, so the model is only interesting if $m \geq 3$. Let $\phi(t, u)$ denote the solution flow, i.e., $\phi(t, u)$ is the value at time $t$ of the solution with initial condition $u$. Since $F$ is locally Lipschitz, uniqueness and local existence hold. As shown in the Appendix, the unit simplex $S_m = \{u \in \mathbb{R}^m: \sum_i u_i = 1\}$ is invariant, that is, if $u \in S_m$ then $\phi(t, u)$ exists and has $\phi(t, u) \in S_m$ for all $t \in \mathbb{R}$.

If each $i \in [m]$ is a possible strategy and $u_i$ is the fraction of individuals in a population following each strategy, then (1) gives the approximate time evolution of $(u_i)_{i \in [m]}$ when the population size is large, interactions occur uniformly at random between pairs of individuals and strategy $i$ converts strategy $i - 1$ for every $i \in [m]$. In other words, (1) are the replicator equations corresponding to an underlying well-mixed stochastic model, which is described in the Appendix. If $m = 3$ this cyclic arrangement of winning strategies is reminiscent of the game rock-paper-scissors.

The version we consider is the simplest model of cyclic dominance. For the sake of comparison, in the Appendix we describe some variations including unequal conversion rates, mutation, and random copying, and derive the corresponding replicator equations. We do not study those in detail here.

Let $\partial S_m$ and $S_m^o$ denote the boundary and interior of $S_m$, respectively, relative to the affine space $\{u \in \mathbb{R}^m: \sum_i u_i = 1\}$. The orbit of a point $u \in S_m$ is the set $O(u) = \{\phi(t, u): t \in \mathbb{R}\}$. Our main results are a complete description of the orbits

1. on $S_m$, for $m \in \{3, 4\}$, and
2. on $\partial S_m$, for $m \geq 5$.

Item 1 is covered for $m = 3$ and partially covered for $m = 4$ by Theorems 4.2(iii) and 5.14 in [5], respectively. For completeness and to indicate the challenges present for larger $m$, we give a self-contained proof of item 1. Item 2 occupies most of our effort.

We begin with an easily stated result. Let $\phi^\pm(u) = \lim_{t \to \pm\infty} \phi(t, u)$, if either exists. An orbit $O(u)$ is periodic if $\phi(s, u) \neq \phi(t, u) = u$ for some $s, t \neq 0$, convergent if both $\phi^\pm(u)$ exist, homoclinic if $\phi^-(u) = \phi^+(u) \neq u$ and heteroclinic if $\phi^-(u) \neq \phi^+(u)$. A heteroclinic cycle of length $k \geq 2$ is a finite union of heteroclinic orbits $O(u_1), \ldots, O(u_k)$ and equilibria $w_1, \ldots, w_k$ satisfying $w_i = \phi^+(u_i) = \phi^-(u_{i+1})$, with the index modulo $k$.

**Theorem 2.1.** Let $\partial S_m, S_m^o$ denote the boundary and interior of $S_m$, respectively, relative to the affine space $\{u \in \mathbb{R}^m: \sum_i u_i = 1\}$.

1. If $u \in S_m^o$, $m \in \{3, 4\}$ and $u$ is not an equilibrium then $O(u)$ is periodic.
2. If $u \in \partial S_m$, $m \geq 3$ and $u$ is not an equilibrium then $O(u)$ is heteroclinic.

**Proof.** Statements 1 and 2 are easy corollaries of Theorem 3.3 and Proposition 1, respectively.

We suspect, but cannot prove, that the behaviour on $S_m^o$ is the same for $m \geq 5$ as for $m \in \{3, 4\}$.
Conjecture 1. If \( u \in S_m^o \), \( m \geq 5 \) and \( u \) is not an equilibrium then \( O(u) \) is periodic.

It should be noted that, even if Conjecture 1 is true, the increasing complexity of orbits for larger \( m \) makes them appear aperiodic. This is illustrated in Figure 1.

I interpret the graphs here:

(i) For \( m = 4 \), periodicity is clear and the graph is almost sinusoidal in shape.
(ii) For \( m = 5 \), the pattern appears to repeat itself in reverse, beginning a bit before \( t = 300 \).
(iii) For \( m = 6 \), the shape resembles a product of two sinusoids at different frequencies.
(iv) For \( m = 7 \), the pattern appears almost to repeat itself in reverse beginning near \( t = 700 \).
(v) For \( m = 8 \), the pattern appears to nearly repeat itself in reverse, beginning near \( t = 600 \).
(vi) For \( m = 9 \), no clear period can be observed on the time interval \([0, 1000]\).

Conserved quantities play a central role in our results. Certain homogeneous polynomials in the entries \( (u_i) \) that are invariant with respect to cyclic permutation – and two that are not – are conserved by the flow. Moreover, on sets where certain entries of \( (u_i) \) are zero, these polynomials factor into further conserved quantities.

Before stating the result, we give the necessary definitions.

Let \( \text{supp}(u) = \{ i : u_i \neq 0 \} \) denote the support of a point \( u \in S_m \) and for any \( i, j \in [m] \) let \( [i, j] = \{ i, i+1, \ldots, j \} \), wrapping modulo \( m \) if \( i > j \). Say that \( [i, j] \subseteq A \) is a component of \( A \subseteq [m] \) if \( i - 1 \notin A \) and \( j + 1 \notin A \). Every \( A \subseteq [m] \) has a unique partition into components. Say that \( A \) is independent if its components are all singletons. For fixed \( m \), let \( \mathcal{I}_k(i,j) \) be the set of independent subsets of \( [i, j] \) of cardinality \( k \), and let \( \mathcal{I}_k \) be the set of independent subsets of \( [m] \) of cardinality \( k \).

Lemma 2.2 (Conserved quantities). The following functions are constant along solutions of (1).

(i) The full product \( P(u) = \prod_i u_i \),
(ii) For even \( m \), the even and odd products
\[
P_e(u) = \prod_{\text{even } i} u_i \quad \text{and} \quad P_o(u) = \prod_{\text{odd } i} u_i,
\]
(iii) For each \( k \in \{1, \ldots, \lfloor m/2 \rfloor \} \), the sum of products over independent sets of cardinality \( k \):
\[
P_k(u) = \sum_{A \in \mathcal{I}_k} \prod_{i \in A} u_i.
\]
(iv) For each component \( [i, j] \) of \( \text{supp}(u) \) and \( 1 \leq k \leq \lfloor (j - i + 1)/2 \rfloor \), the sum of products over independent subsets of \( [i, j] \) of cardinality \( k \):
\[
P^{(i,j)}_k(u) = \sum_{A \in \mathcal{I}_k(i,j)} \prod_{\ell \in A} u_\ell.
\]

Note that \( P_1 \) is just the sum of entries. Also, for even \( m \), \( P = P_e P_o \). Moreover, as shown in Lemma 4.3, \( \text{supp}(u(t)) \) does not depend on \( t \), so for given \( u \), \( P_k(u(t)) \) is a polynomial in the quantities \( P^{(i,j)}_\ell(u(t)) \), where \( \ell \leq k \) and \( [i, j] \) range over components of \( \text{supp}(u) \). The proof of Lemma 2.2 is in the Appendix.
Figure 1. Simulation output of $u_0(t)$ from (1) for various $m$, with initial data $u(0) = (0.01, 0.99/(m - 1), \ldots, 0.99/(m - 1))$, chosen to be close to $\partial S_m$.

For $m \in \{3, 4\}$ the proof mostly consists in indentifying the level sets of $P$ and $(P_e, P_o)$, respectively. In both cases each level set is either an equilibrium, a periodic orbit, or a heteroclinic cycle. For $m \geq 5$ the story is similar: each level set of $(P_1, \ldots, P_{\lfloor m/2 \rfloor})$ is either an equilibrium or a heteroclinic network. Let us be more precise about the last point. By Theorem 2.1, $\partial S_m$ is made up of equilibria and heteroclinic orbits, so we can define a directed graph $G = (V, E)$, that we call the orbit graph, by

$$V = \{u \in \partial S_m : u \text{ is an equilibrium}\} \quad \text{and}$$

$$E = \{(u, v) \in V \times V : (u, v) = (\phi^- (w), \phi^+ (w)) \text{ for some } w \in \partial S_m\}.$$ 

Let $\Pi : \partial S_m \to G$ denote the projection map given by $\Pi(u) = u$ if $u$ is an equilibrium and $\Pi(u) = (\phi^- (u), \phi^+ (u))$ otherwise. Let $\mathcal{C}$ denote the strongly
connected components of $G$ and define the heteroclinic components on $\partial S_m$ by $H = \{\Pi^{-1}(H) : H \in C\}$. Then we have the following result, true for any $m \geq 3$.

**Theorem 2.3.** Let $(P_i)_{i=1}^{\lfloor m/2 \rfloor}$ be as in Lemma 2.2 and let $P = (P_1, \ldots, P_{\lfloor m/2 \rfloor})$. For $p = (p_1, \ldots, p_{\lfloor m/2 \rfloor}) \in \mathbb{R}^{\lfloor m/2 \rfloor}$ let $L(p) = \{u \in \partial S_m : P(u) = p\}$. For even $m$ let $[m]_e = \{0, 2, \ldots, m-2\}$ denote even indeces and let $[m]_o$ denote odd indeces. If $L(p)$ is non-empty and

(i) $m$ is odd or $p_{m/2} = 0$ then $L(p)$ is a heteroclinic component on $\partial S_m$,

(ii) $m$ is even and $p_{m/2} > 0$ then $L(p)$ consists of two heteroclinic components:

$L(p) \cap \{u : [m]_e \subset \text{supp}(u)\}$ and $L(p) \cap \{u : [m]_o \subset \text{supp}(u)\}$.

More detailed results than Theorem 2.3 are given in Section 4. In particular, it follows from Theorem 4.1 that from an initial condition of the form $u(0) = (0, 1/(m-1), \ldots, 1/(m-1))$, $\lim_{t \to \infty} u(t)$ is an equilibrium point supported on $[m/2]$ distinct colours. This is illustrated in Figure 2 (note that for $m = 10$, one of the colour persists at a rather low level).

An immediate take-away observation from Theorem 2.3 is that the complexity of heteroclinic components grows quickly with $m$. For example, suppose $m$ is even and $u$ is a point with supp$(u) = [m]_e$, all of whose non-zero values are distinct. Since
Theorem 3.3. The following describes all orbits in $S$.

Most of the result is covered by Theorems 4.2(iii) and 5.14 in [5]. What is new here is a complete description of the dynamics on $\partial S$. The maximum number of equilibria on a single heteroclinic component grows factorially with $m$. This begs the question whether, on $S_m$, there exist trajectories that visit a neighbourhood of a significant fraction of the equilibria on a given component. At the moment, I do not know the answer.

3. Dynamics on $S_m$ for $m \in \{3, 4\}$. Our goal in this section is to give a detailed description of all the orbits in the cases $m = 3$ and $m = 4$, which is contained in Theorem 3.3. We begin by describing the equilibria, for any $m$.

Lemma 3.1. Suppose $u \in S_m$ is an equilibrium of (1).

- If $m$ is odd, then $u$ is either
  1. Central: $u_i = 1/m$ for all $i$,
  2. Isolated: $u_i u_{i+1} = 0$ for all $i$.
- If $m$ is even, then $u$ is either central, isolated or
  3. Alternating: $u_0 \neq u_1$, $u_{2k} = u_0 > 0$ and $u_{1+2k} = u_1 > 0$ for all $k$.

Proof of Lemma 3.1. It is easy to check the above are equilibria. Suppose $u \in S_m$ is a non-isolated equilibrium, i.e., $u_i u_{i+1} \neq 0$ for some $i$. Since $u_{i+1} = 0$, it follows that $u_{i+2} = u_i$ and $u_{i+1} u_{i+2} \neq 0$. Continuing in this way, we find that $u_{i+2k} = u_i > 0$ and $u_{i+1+2k} = u_{i+1} > 0$ for all $k$. If $m$ is odd then $u_i = u_j$ for all $i, j$, whose only solution in $S_m$ is the central equilibrium. If $m$ is even and $u_i \neq u_{i+1}$ then $u$ is an alternating equilibrium.

We also need the following result, which is an easy consequence of the am-gm inequality and the fact that $\sum_i u_i = 1$.

Lemma 3.2. Let $P, P_c, P_o$ be as in Lemma 2.2.

- $P(u) \leq (1/m)^m$ with equality iff $u$ is the central equilibrium.
- $P_c(u)^{2/m} + P_o(u)^{2/m} \leq 2/m$ for even $m$, with equality iff $u_0 = u_{2k}$ and $u_1 = u_{1+2k}$ for all $k$.

The following theorem gives a detailed description of the dynamics for $m \in \{3, 4\}$. Most of the result is covered by Theorems 4.2(iii) and 5.14 in [5]. What is new here is a complete description of the dynamics on $\partial S_m$ in the case $m = 4$, as well as a more thorough proof establishing periodic orbits in $S_m^o$ for $m = 4$.

Theorem 3.3. The following describes all orbits in $S_m$ for $m \in \{3, 4\}$.

For $p \in [0, 1/9]$ let $L(p) = \{u \in S_3; P(u) = p\}$. Then,

(i) $L(1/9)$ is the central equilibrium,
(ii) $L(0) = \partial S_3$ is a heteroclinic cycle with three equilibria, and
(iii) for each $p \in (0, 1/9)$, $L(p)$ is a periodic orbit in $S_3^o$.

For $p_c, p_0 \geq 0$ such that $\sqrt{p_c} + \sqrt{p_0} \leq 1/2$ let $L(p_c, p_0) = \{u \in S_4; (P_c(u), P_o(u)) = (p_c, p_0)\}$. Then,

(i) if $p_c p_0 > 0$, $\sqrt{p_c} + \sqrt{p_0} = 1/2$ and $p_c = p_0$ then $p_c = p_0 = 1/16$ and $L(p_c, p_0)$ is the central equilibrium,
(ii) if $p_c p_0 > 0$, $\sqrt{p_c} + \sqrt{p_0} = 1/2$ and $p_c \neq p_0$ then $L(p_c, p_0)$ is an alternating equilibrium,
(iii) if $p_c p_0 > 0$ and $\sqrt{p_c} + \sqrt{p_0} < 1/2$ then $L(p_c, p_0)$ is a periodic orbit in $S_4^o$. 


(iv) if \( p_e, p_0 = 0 \) and \( \sqrt{p_e} + \sqrt{p_0} = 1/2 \) then \( L(p_e, p_0) = \{ (1/2, 0, 1/2, 0), (0, 1/2, 0, 1/2) \} \),
(v) if \( p_e, p_0 = 0 \) and \( 0 < \sqrt{p_e} + \sqrt{p_0} < 1/2 \) then \( L(p_e, p_0) \) is a heteroclinic cycle of length 2, and
(vi) if \( p_e = p_0 = 0 \) then \( L(p_e, p_0) \) is a heteroclinic cycle of length 4. \( S_4^0 \) corresponds to cases (i)-(iii) and \( \partial S_4 \) to cases (iv)-(vi).

Proof.

**Dynamics on \( S_3 \).** The level sets are as follows:

(i) \( L(1/9) \) is the central point \((1/3, 1/3, 1/3)\),
(ii) \( L(0) \) is the boundary of \( S_3 \), and
(iii) for \( 0 < p < 1/9 \), \( L(p) \) is the smooth closed curve \( \gamma_p \) defined by

\[
\gamma_p = \{ (u_0, u_1, u_2) \in \mathbb{R}^3_+ : \sum_i u_i = 1, \prod_i u_i = p \}.
\]

The central point is an equilibrium. By Lemma 3.1 the only other equilibria are the isolated equilibria \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\), which are on the boundary of \( S_3 \). In particular, \( F(u) \neq 0 \) for all \( u \in \bigcup_i \gamma_p \), which implies that each \( \gamma_p \) is a periodic orbit. Aside from the isolated equilibria, the boundary contains three line segments \( \ell_i \) of the form \( u_{i+1} = 0 \) and \( u_{i-1} = 1 - u_i \in (0, 1) \). On \( \ell_i \), \( u_i' = u_{i-1} u_i = (1 - u_i) u_i \), so for any value \( u_i(0) \in (0, 1) \), \( u_i(t) \to 1 \) as \( t \to \infty \) and \( u_i(t) \to 0 \) as \( t \to -\infty \). Thus the corners and the sides of \( S_3 \) form a heteroclinic cycle.

**Dynamics on \( S_4 \).** Partition \( S_4 \) according to the level sets

\[
L(p_e, p_0) = \{ u \in S_4 : (P_e(u), P_0(u)) = (p_e, p_0) \},
\]

where \( p_e, p_0 \geq 0 \) and by Lemma 3.2, \( \sqrt{p_e} + \sqrt{p_0} \leq 1/2 \).

Cases (i), (ii) and (iv). If \( \sqrt{p_e} + \sqrt{p_0} = 1/2 \) then by Lemma 3.2, \( u_0 = u_2 \) and \( u_1 = u_3 \), from which (i) and (ii) are immediate. To obtain (iv), note that either \( u_0 = u_2 = 1/2 \) or \( u_1 = u_3 = 1/2 \); since \( \sum_i u_i = 1 \) the only points are \((1/2, 0, 1/2, 0)\) or \((0, 1/2, 0, 1/2)\).

Case (v). In this case either \( p_e = 0 \) or \( p_0 = 0 \); if \( p_0 = 0 \) then \( 0 < \sqrt{p_e} < 1/2 \).

For any \( s \geq 2\sqrt{p_e} \) the equations \( u_0 + u_2 = s \), \( u_0 u_2 = p_e \) and \( (u_1, u_3) = (1-s, 0) \) have a unique solution \( u^-(s) \) with \( 0 < u_0 \leq u_2 \) and a unique solution \( u^+(s) \) with \( u_0 \geq u_2 > 0 \). Together with conservation of \((P_e, P_0)\) this implies the set

\[
\{ u^-(s) : s \in [2\sqrt{p_e}, 1] \} \cup \{ u^+(s) : s \in [2\sqrt{p_e}, 1] \}
\]

is a single heteroclinic orbit \( O(w) \), together with \( \phi^-(w) \) and \( \phi^+(w) \). The same is true if we require instead that \( (u_3, u_1) = (1-s, 0) \). The same is true if \( p_e = 0 \) and the index is shifted by 1.

Case (vi). In this case, up to a shift in the index we must have \( u_2 = u_3 = 0 \) and so \( u_1 = 1 - u_0 \). This gives four equilibria connected by four line segments, that form a heteroclinic cycle.

Case (iii). First we show that if \((p_e, p_0)\) satisfies (iii) then \( L(p_e, p_0) \) has four balance points, i.e., points satisfying either \( u_0 = u_2 \) or \( u_1 = u_3 \). Let \( u_0 = u_2 = \sqrt{p_e} \).

If \( u \in L(p_e, p_0) \) then \( u_1, u_3 \geq 0 \). If \( u_1 u_3 = p_0 > 0 \) and \( u_1 + u_3 = 1 - 2\sqrt{p_e} \). A unique solution with \( u_1 < u_3 \) exists if \( 0 < (u_1 u_3)^{1/2} < (u_1 + u_3)/2 \) or equivalently, \( 0 < \sqrt{p_0} < 1/2 - \sqrt{p_e} \), which is satisfied by (iii). A second solution is obtained by imposing \( u_1 < u_3 \) and there are no solutions with \( u_1 = u_3 \). Applying the same argument with the index shifted by 1 mod 4 gives the other two points.

Next we show that all four balance points lie on a common periodic orbit that contains all points in \( L(p_e, p_0) \). Thus, given \((p_e, p_0)\) satisfying (iii) let \( u \) denote the
unique point in $L(p_c, p_0)$ with $u_0 = u_2$ and $u_1 < u_3$ and let $u(t) = \phi(t, u)$. Let 
$\tau = \inf\{ t: u_1(t) = u_3(t) \}$, then by continuity $\tau > 0$ and $u_1(t) < u_3(t)$ for $t \in [0, \tau)$.
Since $u_0 = u_2 > 0$ it follows that $0 < u_0(s) - u_2(s) < u_0(t) - u_2(t)$ for $0 < s < t \leq \tau$, which implies that $u_1(s) < u_2(t)$ and $u_3(s) > u_3(t)$ for $0 \leq s < t \leq \tau$, and a fortiori that $0 < F_1(s) < F_1(t)$ and $0 > F_3(s) > F_3(t)$ for $0 < s < t \leq \tau$. Since $\tau > 0$, letting $0 < s < \tau$ it follows that

$$u_3(t) - u_1(t) \leq u_3(s) - u_1(s) - (t - s)(F_1(s) - F_3(s)) \leq u_3 - u_1 - c(t - s)$$

where $c = F_1(s) - F_3(s) > 0$. It follows that $\tau \leq s + (u_3 - u_1)/c$, and in particular, $\tau < \infty$. Moreover, $u_2(\tau) < u_0(\tau)$ and $u_1(\tau) = u_3(\tau)$, so $u(\tau)$ is a different balance point. Repeating the argument three more times, the solution started at $u$ visits all four balance points and then returns to $u$, which means it is periodic. To check that $L(p_c, p_0)$ contains no other points, suppose $u \in L(p_c, p_0)$ with $u_0 \neq u_2$ and $u_1 \neq u_3$. It suffices to show that $u(t)$ is a balance point for some $t > 0$. If $u_0 > u_2$ and $u_1 < u_3$ then defining $\tau$ as above, the same argument applies, and $u(\tau)$ is a balance point. If $u_0 > u_2$ and $u_1 > u_3$ then shifting the values left by 1 mod 4, we have $u_0 > u_2$ and $u_1 < u_3$ as before. If $u_0 < u_2$ then shifting the values left or right by 2 mod 4, we have $u_0 > u_2$. Since the dynamics is equivariant under cyclic shift, we obtain a balance point in every case. \hfill \Box

4. Dynamics on $\partial S_m$ for $m \geq 5$. In this section our goal is to prove Theorem 2.3. Although we say it’s for $m \geq 5$, everything below is true for any $m \geq 3$. We will use the definitions given in the Introduction. Half the theorem is easy, since $P = \{P_1, \ldots, P_{\lfloor m/2 \rfloor}\}$ is constant on heteroclinic components. The other half is proved by showing that the orbit graph has enough edges. This is done in two steps:

(i) in Theorem 4.1 we show that all non-equilibrium orbits on $\partial S_m$ are heteroclinic, and we describe precisely the relationship between the forward and backward limit points, then

(ii) in Theorem 4.2 we show that most of the viable candidates for edges $(u, v)$ on the orbit graph, i.e., those satisfying the relations outlined in Theorem 4.1, are in fact edges.

We give a couple of required definitions, state Theorems 4.1 and 4.2, use them to prove Theorem 2.3, then prove Theorems 4.1 and 4.2. Note that in this section all points under consideration belong to $\partial S_m$, and that if $w \in \partial S_m$ then $\text{supp}(w) \neq [m]$, so $\text{supp}(w)$ is partitioned into components.

Let $I = [i, j]$ be an interval, let $d = \lfloor (j - i)/2 \rfloor$, and define the checkering operations $\text{chk}^\pm$ on $I$ by

- $\text{chk}^+(I) = I \cap \{ j - 2k: 0 \leq k \leq d \}$
- $\text{chk}^-(I) = I \cap \{ i + 2k: 0 \leq k \leq d \}$

In words, $\text{chk}^+(I)$ includes every second point in $I$ starting at $j$ and looking left, and $\text{chk}^-(I)$ includes every second point in $I$ starting at $i$ and looking right. $\text{chk}^+$ and $\text{chk}^-$ extend to arbitrary subsets $A \subseteq [m]$ by applying the operation on each component and then taking the union of the resulting sets. Also, given a finite list $a = (a_0, \ldots, a_k)$, define the reflection $b = \text{refl}(a)$ by $b_i = a_{k-i}$ for $i \in \{0, \ldots, k\}$, and for $u \in S_m$ let $u_{[i,j]}$ denote the list $a$ with entries $a_k = u_{i+k}$ for $k \in \{0, \ldots, j - i\}$. 
Theorem 4.1 (non-equilibrium orbits). Suppose $w \in \partial S_m$ is not an equilibrium. Then, $O(w)$ is heteroclinic, $\text{supp}(\phi^\pm(w)) = \text{ch}^\pm(\text{supp}(w))$, and on each component $[i,j]$ of $\text{supp}(w)$, $\phi^+(w)_{[i,j]} = \text{refl}(\phi^-(w)_{[i,j]})$ and

$$
\phi^-(w)_{[i,j]} = \begin{cases} 
(x_0,0,x_1,0,\ldots,0,x_d,0) & \text{if } j-i \text{ is odd} \\
(x_0,0,x_1,0,\ldots,0,x_d) & \text{if } j-i \text{ is even,}
\end{cases}
$$

where $d = [(j-i)/2]$, $x_0 \geq x_2 \geq \cdots \geq x_d > 0$ are the roots of the polynomial

$$
Q(x) = x^{d+1} + \sum_{k=1}^{d+1} (-1)^k P_k(w)x^{d+1-k}.
$$

Theorem 4.2 (edges on the orbit graph). Let $u,v \in \partial S_m$ be equilibria. Then $(u,v)$ is an edge on the orbit graph if there is a set $A \subseteq [m]$ with components $I_1,\ldots,I_k$ such that

1. $\text{supp}(u), \text{supp}(v) \subseteq A$, and
2. for each $\ell \in \{1,\ldots,k\}$, denoting $I_\ell = [i,j]$ and $d = [(j-i)/2]$, there exist $x_0 > x_2 > \cdots > x_d > 0$ such that $v_{[i,j]} = \text{refl}(u_{[i,j]})$ and

$$
u_{[i,j]} = \begin{cases} 
(x_0,0,x_1,0,\ldots,0,x_d,0) & \text{if } j-i \text{ is odd} \\
(x_0,0,x_1,0,\ldots,0,x_d) & \text{if } j-i \text{ is even.}
\end{cases}
$$

It should be noted that in Theorem 4.2 we require that the $x_i$ are strictly decreasing with $i$, whereas Theorem 4.1 only imposes that they are non-increasing, and not all equal. We believe the latter condition suffices for an edge to exist. However, as shown below, to prove Theorem 2.3 we don’t need to find all the edges on the orbit graph; we only need to find enough of them. In fact, it would have been enough to prove Theorem 4.2 in the special case $d = 1$; the reason we have gone a bit further in Theorem 4.2 is because it wasn’t hard, and adds a bit of colour to the results.

Proof of Theorem 2.3. Let $P = (P_1,\ldots,P_{\lfloor m/2 \rfloor})$, where $P_k$, $1 \leq k \leq \lfloor m/2 \rfloor$ are as in Lemma 2.2, let $G = (V,E)$ denote the orbit graph and let $\Pi : \partial S_m \to G$ denote the projection map defined in the Introduction. It is enough to show that

(i) $P$ is constant on every heteroclinic component,

(ii) if $m$ is even, $[m]_{e} \subset \text{supp}(u)$ and $[m]_{o} \subset \text{supp}(v)$ then $u$ and $v$ belong to different components, and

(iii) if $P(u) = P(v)$ and either

(a) $m$ is odd or $P_{m/2}(u) = 0$, or

(b) $m$ is even and either $[m]_{e}$ or $[m]_{o}$ belongs to both $\text{supp}(u)$ and $\text{supp}(v)$ then $u$ and $v$ belong to the same heteroclinic component.

First we show (i). Note that if $w$ is not an equilibrium then since $P$ is constant on $O(w)$ and is continuous it follows that $P(\phi^-(w)) = P(w) = P(\phi^+(w))$. Thus, if $(u,v) \in E$, $P(u) = P(v)$ since there exists $w$ such that $\phi^-(w) = u$ and $\phi^+(w) = v$. If $u,v$ are equilibria on the same component, by definition there exists a finite path $(u,u_1),(u_1,u_2),\ldots,(u_k,v)$ of edges in $E$ and so $P(u) = P(u_1) = \cdots = P(v)$. If one or both of $u,v$ is not an equilibrium, argue in the same way except with $\phi^+(u)$ and/or $\phi^-(v)$ in place of $u,v$, noting that $P(\phi^+(u)) = P(u)$ and/or $P(\phi^-(v)) = P(v)$, respectively.

Next we show (ii). Let $w \in \partial S_m$. By Lemma 4.3, $\text{supp}(z) = \text{supp}(w)$ for all $z \in O(w)$. Let $[i,j]$ be a component of $\text{supp}(w)$. If $w$ is not an equilibrium and $[m]_{e} \subset \text{supp}(w)$ then by definition of $[i,j]$, $i-1,j+1 \notin \text{supp}(w)$, so $i,j$
are even. Using this and \([m]_e \subset \text{supp}(w)\) again it follows from Theorem 4.1 that \(\text{supp}(\phi^+(w)) = \text{supp}(\phi^-(w)) = [m]_e\). Arguing as above, it follows that the property \([m]_o \subset \text{supp}(w)\) is constant on heteroclinic components, i.e., if the property holds at one point then it holds at every point on that component. By shift invariance the same is true with \([m]_o\) in place of \([m]_e\). Since \(\text{supp}(w) \not\subseteq [m] = [m]_e \cup [m]_o\) for every \(w \in \partial S_m\), at most one of the two properties is true on each component, which implies (ii).

Now we show (iii). We will show that if \(u, v\) are equilibria satisfying the conditions of (iii), then there is a path of edges \((u, u_1), \ldots, (u_{k-1}, v)\) in \(E\). To cover the case where \(u\) and/or \(v\) is not an equilibrium, note that if \([m]_e \subset \text{supp}(u), \text{supp}(v)\) then \([m]_e = \text{supp}(\phi^+(u)), \text{supp}(\phi^-(v))\) and it suffices to find a path of edges connecting \(u\) or \(\phi^+(u)\) to \(v\) or \(\phi^-(v)\). We can view Theorem 4.2 as specifying a set of moves that takes \(w\) to a point \(z\) satisfying \((w, z) \in E\). Our goal can thus be rephrased as follows: given equilibria \(u, v\) satisfying the above conditions, we seek a finite sequence of moves given by Theorem 4.2 that takes \(u\) to \(v\). This task will be further broken down as follows:

(a) if \(P(u) = P(v)\) then \(u \sim v\) modulo permutation of entries, and
(b) if \(u \sim v\) and either \(m\) is odd, \(P_{m/2}(u) = 0\) or \(\text{supp}(u) = \text{supp}(v) \subseteq \{[m]_e, [m]_o\}\), there is a sequence of moves taking \(u\) to \(v\).

To obtain (a), note that if \(u\) is an equilibrium on \(\partial S_m\) then since \(\text{supp}(u)\) is an independent set, in the definition of each \(P_k\), the summation runs over all subsets of \(\text{supp}(u)\) of cardinality \(k\). Thus, \(P_k(u)\) depends only on the cardinality of \(\text{supp}(u)\) and on the multiset of entries \(\{u_i : i \in \text{supp}(u)\}\), which gives (a).

To obtain (b), we use the following notation to describe the allowed moves:

\[ a_1 \ldots a_k \rightarrow b_1 \ldots b_k \]

means that if \(u, v\) are equilibria satisfying \(u_{i+\ell} = a_\ell\) and \(v_{i+\ell} = b_\ell\) for some \(i \in [m]\) and every \(\ell \in \{1, \ldots, k\}\) where \(k \leq m\), and \(u_j = v_j\) for all other \(j\), then \((u, v) \in E\). The only moves we will use are the slide \(0 a 00 \rightarrow 00 a 0\) where \(a > 0\) and the exchange \(0 a 0 b 0 \rightarrow 0 b 0 a 0\), where \(a > b > 0\). First we describe how to perform a cyclic shift – by one if \(m\) is odd or \(\text{supp}(u) \neq [m]_e, [m]_o\) and by two otherwise – then how to reorder \(u\) so it is equivalent to \(v\) modulo cyclic shift. Composing the two operations then gives (b).

**Cyclic shift.** Since \(\text{supp}(u)\) is independent, if \(m\) is odd or if \(\text{supp}(u)\) is not equal to \([m]_e\) or \([m]_o\) then it has a gap of length 2, i.e., \(u_i = u_{i+1} = 0\) for some \(i\). Since \(\sum_i u_i = 1\) it has a non-zero entry, so the pattern \(0a00\) appears and can be slid to \(00a0\). This leaves a gap of length 2 to the left of \(a\), so the next non-zero entry to the left of \(a\) can be slid, and so on. This sequence of moves shifts all values to the right by one. If \(m\) is even and \(\text{supp}(u) = [m]_e\), then \(u\) has a non-empty set of entries \(J_{\max}\) satisfying \(u_i = u_{\max} = \max_i u_j\) for every \(i \in J_{\max}\). For each \(i \in J_{\max}\) such that \(i + 2 \notin J_{\max}\), let \(j\) be the next entry in \(J_{\max}\) to the right of \(i\), and perform exchange on the pattern \(0u_{\max}000\) until the entry originally at \(i\) is at index \(j - 2\). The result on the interval \([i, j]\) is that the previous value at every entry \(k\) with \(i + 2 \leq k < j\) is shifted to the left by 2, and the new value at entry \(j - 2\) is \(u_{\max}\), so the values on \([i + 2, j]\) are effectively shifted to the left by two onto \([i, j - 2]\). If \(i, i + 2 \in J_{\max}\) the value at \(i\) is unchanged, so is also equal to the previous value at \(i + 2\). Thus all non-zero values are shifted to the left by two.

**Reordering.** There are two steps: ordering non-zero elements, and adjusting gap lengths. In this part it is helpful to denote points as \(a_1 b_1 \ldots a_k b_k\) modulo cyclic shift, where \(a_1, \ldots, a_k\) are the non-zero entries and \(b_1, \ldots, b_k\) are positive integer gap lengths. So, for example, \(a_1 3 a_2 2\) denotes the point with entries \(a_1 000 a_2 a_2 00\).
Ordering non-zero elements. In this step we use only the following basic sequence of moves: if \( a_i > a_{i+1} \) then by sliding the pattern 0a00 → 00a0 until \( b_i = 1 \), then performing the exchange 0a0a_{i+1} → 0a_{i+1}0a0, we obtain the point
\[
a_1b_1 \ldots a_{i-1}b_{i-1}a_{i+1}b_{i+1}a_i \ldots a_kb_k,
\]
where \( b_{i-1} = b_{i-1} + (b_i - 1) \) and \( b_i' = 1 \). Since we adjust gap lengths in the next step, we will ignore the values of the gap lengths and simply notate this sequence of moves by \( a_1 \ldots a_ia_{i+1} \ldots a_k \to a_1 \ldots a_{i+1}a_i \ldots a_k \) and refer to it as pushing \( a_i \). The goal is thus to achieve an arbitrary arrangement of the values \( a_1 \ldots a_k \), by pushing larger entries past smaller ones. The way to do this is by arranging entries in ascending order of their values.

We first demonstrate the approach using a toy example. Suppose we have entries 112334 that we want to arrange as 121343. Entries with value 1 are already trivially ordered relative to each other. The unique entry with value 2 is also ordered relative to the 1s, since the order is cyclic. To arrange the 3s among the 1s and 2s, we desire the order 1213, which is the same as 1332. A 3 can be pushed past 1s and 2s by exchange, but not past another 3, or a 4. Thus, we need to move the 4 as well. We do so as
\[
11234 \to 141233 \to 134123 \to 133412.
\]
Ignoring the 4, the arrangement is now 13312 as desired. Finally we can push the 4 into place.

Let \( c_1 < \cdots < c_\ell \) denote the distinct values in the set \( \{a_i\}_{i=1}^k \) and for \( j \leq \ell \) let \( C_j = \{c_1, \ldots, c_j\} \). All entries with value \( c_1 \) are, trivially, ordered, and if \( \ell = 1 \) there is nothing else to arrange. Suppose for some \( j \leq \ell \) that all entries with values in \( C_{j-1} \) are correctly ordered relative to one another. To arrange entries with value \( c_j \) among them, first choose an entry with value \( c_j \). Take the nearest entry with value \( c_j \) to its left, and push it until the two are adjacent. Continue in this way, looking left for the next entry and pushing, until all entries with value \( c_j \) are adjacent. Then, take the nearest entry to the left with value \( c_{\ell-1} \) and do the same, and so on for entries with values \( c_{\ell-2}, \ldots, c_{j+1}, c_j \), in that order. All entries except those with values in \( C_{j-1} \) are now adjacent and arranged in increasing order. Then, push the entries with value \( c_{\ell} \), one at a time, all the way around until they are to the immediate left of the entries with values \( c_j \). Do the same for all entries with values \( c_{\ell}, c_{\ell-1}, \ldots, c_{j+1} \), but not \( c_j \). Now, all entries with values larger than \( c_j \) are to the immediate left of all entries with value \( c_j \). The entries with values \( c_j \) can now be pushed among the entries with values in \( C_{j-1} \), filling spaces from right to left.

Adjusting gap lengths. Suppose we have two points \( a_1b_1 \ldots a_kb_k \) and \( a_1c_1 \ldots akc_k \). Then, clearly \( \sum_i b_i = \sum_i c_i \), so unless all gap lengths coincide, \( b_i > c_i \) for some \( i \). If \( b_i > c_i \) then since \( c_i \geq 1 \), \( a_i \) can be slid to the right until the \( i^{th} \) gap length is equal to \( c_i \). To continue, we will use the following algorithm that operates on \( a_1b_1 \ldots a_kb_k \) to reach the fixed target \( a_1c_1 \cdots akc_k \), keeping track of the current gap index \( I \) and the set of corrected gaps \( G \).

\[
I = 1, \ G = \{ \}
\]
while \( G \neq [m] \) do
\[
I \leftarrow (\text{nearest value } i \text{ equal or to the left of } I, \text{ not currently in } G, \text{ such that } b_i \geq c_i )
\]
Slide \( a_I \) to the right until the \( I^{th} \) gap length is equal to \( c_I \)
\[
G \leftarrow G \cup \{I\}
\]
\[
I \leftarrow I - 1
\]
while $I \in G$ do
  Slide $a_I$ to the right until the $I^{th}$ gap length is equal to $c_I$
  $I \leftarrow I - 1$
end while

To explain: the initialization of $I$ is arbitrary. The algorithm proceeds by searching for gaps that are too long and shortening them. So long as we have not determined that all gaps have the correct length, we look at index $I$ and to its left, stopping at the first gap that is either the correct length, or too long. We add the index of that gap to $G$ and and shorten it if necessary. Since sliding to shorten a gap lengthens the gap to its immediate left, we pass through and make sure we re-shorten any gaps already belonging to $G$. We then continue to look left for gaps not belonging to $G$ until there are none. Since upon each iteration of the outer while loop, all gaps in $G$ have the correct length and $|G|$ increases by 1, the algorithm terminates after at most $m$ iterations and all gaps have the correct length. \hfill $\Box$

4.1. Non-equilibrium orbits. It is easy to see the dynamics of (1) are equivariant under the cyclic shift defined by $\sigma(u)_i = u_{i-1}$, that is, $\phi_i(t, \sigma(u)) = \phi_{i-1}(t, u)$ for $i \in [m]$, $u \in S_m$ and $t \in \mathbb{R}$, and also under simultaneous time and index reversal, that is, $\phi_i(t, u) = \phi_{-i}(-t, u)$. We refer to these respectively as shift and reversal symmetry.

The proof of Theorem 4.1 is broken up into two pieces.

**Proposition 1** (support of orbit limits). If $u \in S_m$ then $u$ belongs to a heteroclinic orbit of (1) iff $\text{supp}(u)$ is not equal to $[m]$ and is not an independent set. Equivalently, the boundary of $S_m$ consists of heteroclinic orbits and isolated equilibria. Moreover, if $u$ is on a heteroclinic orbit then

$$\text{supp}(\phi^+(u)) = \text{chk}^+(\text{supp}(u)) \quad \text{for} \quad \bullet \in \{+, -\}.$$  

**Proposition 2** (value of orbit limits). Suppose $w \in \partial S_m$ is not an equilibrium. Then on each component $[i, j]$ of $\text{supp}(w)$, $\phi^+(w)_{[i, j]} = \text{refl}(\phi^-(w)_{[i, j]})$ and

$$\phi^-(w)_{[i, j]} = \begin{cases} (x_0, 0, x_1, 0, \ldots, 0, x_d, 0) & \text{if } j - i \text{ is odd} \\ (x_0, 0, x_1, 0, \ldots, 0, x_d) & \text{if } j - i \text{ is even}, \end{cases}$$

where $d = \lfloor (j - i)/2 \rfloor$, $x_0 \geq x_2 \geq \cdots \geq x_d > 0$ are the roots of the polynomial

$$Q(x) = x^{d+1} + \sum_{k=1}^{d+1} (-1)^k P_k^{(i,j)}(w)x^{d+1-k}.$$  

We begin with a simple result about the support.

**Lemma 4.3.** If $u \in S_m$ and $t \in \mathbb{R}$ then $\text{supp}(\phi(t, u)) = \text{supp}(u)$.

If $\phi^+(u)$ exists, then $\text{supp}(\phi^+(u)) \subseteq \text{supp}(u)$ and similarly for $\phi^-(u)$.

**Proof.** The first half is a restatement of (b) from Lemma 5.1. For the second half, if $i \notin \text{supp}(u)$ then $u_i = 0$ so $\phi_i(t, u) = 0$ for $t \in \mathbb{R}$ and $\phi^+_i(u(0)) = \phi^-_i(u(0)) = 0$, which implies $i \notin \text{supp}(\phi^\pm(u))$. \hfill $\Box$

**Lemma 4.4.** Suppose that $\text{supp}(u) \neq [m]$. Then, $\phi^+(u)$ exists.
Lemma 4.4. \( \phi_i = \int_0^t u_i(s) u_{i+1}(s) ds \), so that \( u_i(t) - u_i(0) = m_{i-1}(t) - m_i(t) \). In particular, \( |m_i(t) - m_{i-1}(t)| \leq 1 \) for each \( i \). Since \( m_i(t) \) is non-decreasing, \( m_i(\infty) \) exists, though it may be infinite. If \( \text{supp}(u) \neq |m| \) then let \( j \notin \text{supp}(u) \); \( m_j(t) = 0 \) for all \( t \), which implies \( m_i(\infty) < \infty \) for every \( i \). In turn, this implies \( u_i(\infty) \) exists for every \( i \).

Lemma 4.5. Suppose that \( \{0, 1, 2\} \subseteq \text{supp}(u) \neq |m| \) and that \( \phi_1^+(u) = \lim_{t \to \infty} \phi_2(t, u) \) exists and is positive. Then \( \phi_1^+(u) = 0 \) and \( \phi_0^+(u) > 0 \).

Proof. Let \( u(t), m(t) \) be as in the proof of Lemma 4.4, and let \( a = u_2(\infty) > 0 \). Since \( \text{supp}(u) \neq |m|, u(\infty), m(\infty) \) exist and are finite.

Let \( t_0 \) be large enough that \( u_2(t) \geq a/2 \) for \( t \geq t_0 \). Then \( u_2(\infty) = 0 \), for if \( u_1(t) \geq c \) infinitely often, since \( |u_1| \leq 1 \) it is not hard to show that \( m_1(\infty) - m_1(t_0) = \int_{t_0}^\infty u_1(s) u_2(s) ds \geq (a/2) \int_{t_0}^\infty u_1(s) ds = \infty \), which contradicts \( m_1(\infty) < \infty \).

It remains to show \( u_0(\infty) > 0 \). Since \( u'_0 = u_0(u_{-1} - u_1) \), write
\[
\frac{u_0(\infty)}{u_0(0)} = \exp\left(\int_0^\infty (u_{-1}(s) - u_1(s)) ds\right) \geq \exp\left(-\int_0^\infty u_1(s) ds\right).
\]
Suppose \( u_0(\infty) \leq a/2 \), and let \( t_1 \) be large enough that \( |u_0(\infty) - u_0(t)| \leq a/8 \) and \( |u_2(\infty) - u_2(t)| \leq a/8 \) for \( t \geq t_1 \). Then for \( t \geq t_1 \),
\[
u_1' = -u_1(u_2 - u_0) \leq -u_1(a/4).
\]
Thus \( u_1(t)/u_1(t_1) \leq e^{-a(t-t_1)/4} \), so \( \int_0^\infty u_1(s) ds < \infty \) and so \( u_0(\infty) > 0 \).

Proof of Proposition 1. Since the right-hand side of (1) is continuous, if \( \phi_1^+(u) \) or \( \phi_1^-(u) \) exists then it is an equilibrium. It follows that \( u \) is on a heteroclinic orbit iff
\begin{enumerate}
\item \( u \) is not an equilibrium,
\item \( \phi_1^+(u) \) both exist, and
\item \( \phi_1^+(u) \neq \phi_1^-(u) \).
\end{enumerate}

Case 1: \( \text{supp}(u) \) independent. Then, \( u \) is an isolated equilibrium, so it fails (a).

Case 2: \( \text{supp}(u) = |m| \). Defining \( P \) as in Lemma 2.2, it follows that \( P(u) > 0 \). Since \( P \) is conserved and continuous, if \( \phi_1^+(u) \) exists then \( P(\phi_1^+(u)) = P(u) \). If \( \phi_1^+(u) \) is the central equilibrium then \( P(u) = P(\phi_1^+(u)) = (1/m)^m \) and Lemma 3.2 implies \( u \) is the central equilibrium and fails (a). If \( \phi_1^+(u) \) is an alternating equilibrium, using again Lemma 3.2 we find that \( P_1(u)^2/m + P_2(u)^2/m = P_1(\phi_1^+(u))^{2^2/m} + P_2(\phi_1^+(u))^{2^2/m} = 2/m \), which implies that \( u \) is an alternating equilibrium and again fails (a). Since isolated equilibria have \( P = 0 \), \( \phi_1^+(u) \) cannot be an isolated equilibrium. Thus if (a) holds, then \( \phi_1^+(u) \) is not an equilibrium, so it does not exist, and (b) fails.

Case 3: \( \text{supp}(u) \) not equal to \( |m| \) and not independent. To see that (a) holds, note that if \( u \) is a central or alternating equilibrium then \( \text{supp}(u) = |m| \), and if \( u \) is an isolated equilibrium then \( \text{supp}(u) \) is independent. Since \( \text{supp}(u) \neq |m| \), by Lemma 4.4, \( \phi_1^+(u) \) exists, and by reversal symmetry, \( \phi_1^-(u) \) exists, i.e., (b) holds.

It remains to show (c) and the checkering property. Let \( u(t) = \phi(t, u) \) and let \( I = [i, j] \) be a component of \( \text{supp}(u) \). By Lemma 4.3, \( [i, j] \subseteq \text{supp}(u(t)) \) and \( i - 1, j + 1 \notin \text{supp}(u(t)) \) for \( t \in \mathbb{R} \).
The result for $\phi_u$ and continuity, supp$(\phi^+(u)) \cap I = \text{chk}^+(\text{supp}(u)) \cap I$.

- If $i = j$ it follows that $u_{i-1}(t) = u_{i+1}(t) = 0$, and thus $u'_i(t) = 0$, for $t \in \mathbb{R}$, so $u_i(t) = u_i(0)$ for $t \in \mathbb{R}$ and so $\phi^+_i(u) = \lim_{t \to \infty} u_i(t) = u_i(0) > 0$, which implies (trivially) that supp$(\phi^+(u)) \cap I = \text{chk}^+(\text{supp}(u)) \cap I$.
- If $i < j$ then $u_{i-1}(t) > 0$ and $u_{j+1}(t) = 0$ for $t \in \mathbb{R}$, so $u'_i(t) = u_{i-1}(t)u_j(t) > 0$ for $t \in \mathbb{R}$ and so $u_j(\infty) > u_j(0) > u_j(-\infty) \geq 0$. Since $u_j(\infty) > u_j(-\infty)$, $\phi^+(u) \neq \phi^-(u)$. Since $u_j(\infty) > 0$, repeated application of Lemma 4.5 shows that

\[
\begin{align*}
    u_{j-1-2k}(\infty) &= 0 \quad \text{for} \quad k = 0, \ldots, \lfloor (j - i)/2 \rfloor \quad \text{and} \\
    u_{j-2k}(\infty) &> 0 \quad \text{for} \quad k = 0, \ldots, \lfloor (j - i)/2 \rfloor,
\end{align*}
\]

which shows that supp$(\phi^+(u)) \cap I = \text{chk}^+(\text{supp}(u)) \cap I$.

Since $A$ is not an independent set, there is at least one component $I = [i, j]$ with $i < j$, as shown above this implies (c). By Lemma 4.3, if $i \notin \text{supp}(u)$ then $i \notin \text{supp}(\phi^+(u))$. Since we have shown that supp$(\phi^+(u)) \cap I = \text{chk}^+(\text{supp}(u)) \cap I$ for all components $I$, it follows by definition that supp$(\phi^+(u)) = \text{chk}^+(\text{supp}(u))$. The result for $\phi^-(u)$ follows by reversal symmetry.

Next we prove Proposition 2, again in a few steps. We begin by showing limit equilibria are spatially ordered. For a point $u \in S_m$, let $O(u) = \{\phi(t, u) : t \in \mathbb{R}\}$ denote the orbit of $u$.

**Lemma 4.6.** Suppose $O(u)$ is heteroclinic and let $I = [i, j]$ be a component of supp$(u)$. Then,

\[\phi^+_j(u) \geq \phi^+_{j-2}(u) \geq \cdots \geq \phi^+_2([j-i)/2](u).\]

**Proof.** Let $u(t) = \phi(t, u)$ and let $k \in [i, j]$ be such that $j - k$ is odd. Since $k \in \text{supp}(u)$, $u_k > 0$, and by Proposition 1, $\lim_{t \to \infty} u_k(t) = 0$. Using the representation of $u_k(t)$ from the Appendix and letting $t \to \infty$,

\[
0 = \lim_{t \to \infty} u_k(t) = u_k \lim_{t \to \infty} \exp \left( \int_0^t (u_{k-1}(s) - u_{k+1}(s))ds \right),
\]

which implies that $\lim_{t \to \infty} \int_0^t (u_{k+1}(s) - u_{k-1}(s))ds = \infty$. Since $\lim_{t \to \infty} (u_{k+1}(s) - u_{k-1}(s))$ exists, it must be non-negative, which is the desired result. \hfill $\square$

**Proof of Proposition 2.** If $w \in \partial S_m$ is not an equilibrium then $O(w)$ is heteroclinic. Let $I = [i, j]$ be a component of supp$(w)$. Let $I_k^+$ denote the set of subsets of chk$(I)$ of cardinality $k$ and let $d = \lfloor (j - i)/2 \rfloor$ so that $d + 1$ is the cardinality of chk$(I)$. Since chk$(I)$ is independent, for $k \in \{1, \ldots, d + 1\}$ we find that

\[
P_k^{(i,j)}(\phi^+(w)) = \sum_{A \in I_k^+} \prod_{i \in A} \phi^+_i(w). \tag{3}
\]

Let $y = \phi^+(w)$ and define $Q^+(x) = \prod_{i \in \text{chk}^+(I)} (x - y_i)$. From (3) and the formula for the coefficients of a polynomial in terms of its roots, we find that

\[
Q^+(x) = x^{d+1} + \sum_{k=1}^{d+1} (-1)^k P_k^{(i,j)}(\phi^+(w)) x^{d+1-k} \tag{4}
\]

The above is also true with $\phi^-$, chk$^-$ etc. in place of $\phi^+$, chk$^+$. By Lemma 2.2 and continuity, $P_k^{(i,j)}(\phi^+(w)) = P_k^{(i,j)}(\phi^-(w))$, so the polynomials $Q^+$ and $Q^-$ coincide. Since polynomials and their roots are in 1:1 correspondence, it follows that $(\phi^+_i(w))_{i \in \text{chk}^+(I)}$ and $(\phi^-_i(u))_{i \in \text{chk}^-(I)}$ coincide up to a permutation of the entries.
Lemma 4.6 then fixes their order, and the reflection property and the formula for \( \phi^\pm(w) \) follow. \( \square \)

### 4.2. Edges on the Orbit Graph

To prove Theorem 4.2 we need to produce heteroclinic orbits connecting points that satisfy the given conditions. First we break that task down over components.

**Lemma 4.7** (Componentwise construction). Let \( A \subseteq [m] \) and let \( (V, E) \) denote the orbit graph. Let \( I_1, \ldots, I_t \) denote the components of \( A \). Suppose that for each \( k \in \{1, \ldots, t\} \) there are \( u^{(k)}, v^{(k)} \) with \( \text{supp}(u^{(k)}), \text{supp}(v^{(k)}) \subseteq I_k \) and \( (u^{(k)}, v^{(k)}) \in E \).

Let \( u = \sum_{k=1}^t u^{(k)} \) and \( v = \sum_{k=1}^t v^{(k)} \). Then \( (u, v) \in E \).

**Proof.** For each \( k \) let \( w^{(k)} \) be such that \( \phi^-(w^{(k)}) = u^{(k)} \) and \( \phi^+(w^{(k)}) = v^{(k)} \), and let \( w = \sum_{k=1}^t w^{(k)} \). Then by (c) of Lemma 5.1 it follows easily that \( \phi^-(w) = u \) and \( \phi^+(w) = v \), which implies that \( (u, v) \in E \). \( \square \)

**Lemma 4.8** (edges on the graph). Let \( u, v \) have \( \text{supp}(u), \text{supp}(v) \subseteq [i, j] \). Suppose that \( u^{[i, j]} = (x_0, 0, \ldots, x_d) \) or \( u^{[i, j]} = (x_0, 0, \ldots, x_d, 0) \) with \( x_0 > x_1 > \ldots x_d > 0 \) and \( v^{[i, j]} = \text{refl}(u^{[i, j]}) \). Then \( (u, v) \in E \).

**Proof.** Suppose \( i - j \) is even so \( u^{[i, j]} = (x_0, 0, \ldots, x_d) \), for the other case is analogous. For \( z = (z_0, \ldots, z_d) \in \mathbb{R}^{d+1} \) and \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d \) let \( w(z,y) \) be the point with \( \text{supp}(w(z,y)) \subseteq [i, j] \) and

\[
  w(z,y)^{[i,j]} = (z_0, y_1, z_1, y_2, \ldots, z_{d-1}, y_d, z_d).
\]

Define \( G(z,y): \mathbb{R}^{d+1} \times \mathbb{R}^d \to \mathbb{R}^{d+1} \) by \( G_k(z,y) = P_k(w(z,y)) \) for \( k \in \{1, \ldots, d+1\} \).

Then, note that

\[
  G_k(z,0) = \sum_{B \in C_k} \prod_{n \in B} z_n
\]

where \( C_k \) is the collection of subsets of \( \{1, \ldots, d+1\} \) of cardinality \( k \). Letting \( x = (x_0, \ldots, x_d) \),

\[
  \partial_{x_\ell} G_k(x,0) = \sum_{B \in C_{k-1}(\ell)} \prod_{n \in B} x_n,
\]

where \( C_k(\ell) \) is the collection of subsets of \( \{1, \ldots, d+1\} \setminus \{\ell\} \) of cardinality \( k \), \( C_0(\ell) = \emptyset \) and the empty product is equal to 1. Letting \( M_{k\ell} = \partial_{x_\ell} G_k(x,0) \), we see that \( (-1)^{k-1} M_{k\ell} \) is the \( a^{d+1-k} \) coefficient in the degree \( d \) polynomial given by

\[
  Q_\ell(a) = \prod_{n \in \{1, \ldots, d+1\} \setminus \{\ell\}} (a - x_n).
\]

The entries \( M_{k\ell} \) form the \( (d+1) \times (d+1) \) matrix \( M = \partial_x G(x,0) \) with columns \( M_\ell = (M_{k\ell})_{k=1}^{d+1} \). Let constants \( (c_\ell)_{\ell=1}^{d+1} \) be such that the linear combination \( \sum_\ell c_\ell M_\ell \) is the zero vector. By the above correspondence, \( \sum_\ell c_\ell Q_\ell(a) \) is the zero polynomial. Since the roots \( (x_\ell) \) are distinct by assumption, \( Q_\ell(x_\ell) = 0 \) iff \( k \neq \ell \), so evaluating this polynomial at \( a = x_\ell \) gives \( c_\ell = 0 \), i.e., only the trivial combination is zero. Thus, the columns \( \{M_\ell\}_{\ell=0}^d \) are linearly independent and \( \det(M) \neq 0 \).

By the implicit function theorem, there is a \( C^1 \) function \( z(y) \), defined in a neighbourhood of \( y = 0 \) and satisfying \( z(0) = x \), such that \( G(z(y), y) = G(x,0) \). Let \( y = (\epsilon, \epsilon, \ldots, \epsilon) \) for small \( \epsilon > 0 \) to obtain a point \( w = w(z(y), y) \) with \( P_k(w) = P_k(u) \).
for $k \in \{1, \ldots, d + 1\}$. Since $\text{supp}(w) = [i, j]$, $\phi_k^-(w) = 0$ for $k \notin [i, j]$, and by Theorem 4.2, $\phi^-(w)_{[i,j]} = (r_0, 0, r_1, 0, \ldots, r_d)$ where $r_0 \geq \cdots \geq r_d$ are the roots of the polynomial

$$Q(x) = x^{d+1} + \sum_{k=1}^{d+1} (-1)^k P_k^{(i,j)}(w)x^{d+1-k}.$$ 

Since $\text{supp}(w) = [i, j]$, $P_k^{(i,j)}(w) = P_k(w)$ and $P_k(w) = P_k(u)$ for $i \in \{1, \ldots, d + 1\}$. Since each $(-1)^k P_k(u)$ is the $x^{d+1-k}$ coefficient in the polynomial with roots $x_0 > x_1 > \cdots > x_d$, $\phi^-(w) = u$ as desired. By Theorem 4.2, $\phi^+(w)_{[i,j]} = \text{refl}(u_{[i,j]}) = v_{[i,j]}$ by assumption, and as above, $\phi_k^+(w) = 0$ for $k \notin [i, j]$ so $\phi^+(w) = v$. 

Proof of Theorem 4.2. The result follows directly from Lemmas 4.7 and 4.8. 

5. Discussion and future work. In this article we studied the replicator equations (1) of a neutral cyclic dominance model with no mutation and no copying. With the help of the conserved quantities described in Lemma 2.2, we were able to give a complete description of dynamics on the boundary of the simplex. We found that all boundary orbits are either equilibria or heteroclinic orbits, and that each level set of the given conserved quantities forms either a single heteroclinic network, or else two identical networks differing by a cyclic shift, with the latter occurring only for even $m$ and only when the support is maximal in the sense described in Theorem 2.3. Moreover, we noticed an interesting pattern: the forward or backward limit point of a non-equilibrium boundary trajectory is supported on strategies that form a checkered pattern relative to the support of the initial value.

There are at least three productive directions for future work:

(i) solve the dynamics of this model in the interior of the simplex,

(ii) study the stability of the given heteroclinic networks, both structurally and with respect to the interior dynamics, for variations that include asymmetry, mutation or copying, and

(iii) use the boundary dynamics of the replicator equations to deduce results about the well-mixed stochastic model described in the Appendix.

Direction (i). This could begin by either proving Conjecture 1 or else finding a counterexample. The conserved quantities $P_k$, for $k = 1, \ldots, [m/2]$, together with the full product $P$, suffice to tease apart the individual orbits for $m \in \{3,4\}$ but fail short for larger $m$, as there are only $[m/2] + 1$ of them and we require, in principle, $m - 1$ to identify 1-dimensional sets in $m$-dimensional space. It might be that additional quantities can be found. Another approach might be to start near boundary equilibria, use the results of this article, and attempt some kind of geometric continuation of solutions.

Direction (ii). This could begin by establishing conserved quantities similar to the ones from this article, for variations on our simple model. There is a precedent suggesting this is possible: in [5] the full product $P$ is, for many variations, either conserved or is a Lyapunov function, and is key to the analysis for $m \in \{3,4\}$. It can also be modified to work for unequal conversion rates, as follows. The replicator equations in that case are derived in the Variations section of the Appendix:

$$u_i' = u_i(p_iu_{i-1} - p_{i+1}u_{i+1}), \quad i \in [m].$$
If a product of the form \( P(u) = \prod_i u_i^{c_i} \) is conserved, then matching coefficients of \( u_i \) in the equation \( \frac{d}{dt} \log(P(u(t))) = 0 \) gives the set of constraints
\[
c_{i+1} = c_{i-1} \frac{p_i}{p_{i+1}}.
\]
If \( m = 3 \) these are satisfied by taking \( c_1 = p_{t-1}, \) as observed in [3] and probably in earlier references. If \( m \) is odd, then taking \( c_0 = 1 \) and defining \( c_2, c_4, \ldots \) recursively all the way back to
\[
c_0 = c_0 \prod_i \frac{p_i}{p_{i+1}} = c_0,
\]
we find the constraints are satisfied, and \( P \) is conserved. If \( m \) is even, we instead look for products \( P_e(u) = \prod_{\text{even } i} u_i^{c_i} \) and \( P_o(u) = \prod_{\text{odd } i} u_i^{c_i} \). Imposing \( c_0 = c_1 = 1 \) and doing the same, we find, interestingly, that constraints are satisfiable in both cases if and only if
\[
\prod_{\text{even } i} p_i = \prod_{\text{odd } i} p_i.
\]
In any case, this suggests that it is promising that studying cyclic sums of products of the type defined in Lemma 2.2, or variations thereof, might make it possible to study variations on our simple model.

**Direction (iii).** The long-term behaviour of the well-mixed stochastic version of our model can likely be fairly well understood from our results; we describe in a short story what we expect to occur. The boundary equilibria are the absorbing states for the stochastic model. Suppose the model is started from an initial configuration. The boundary equilibria are the absorbing states for the stochastic model. Eventually, some strategy \( i \) will vanish due to fluctuations. At this point, no one following strategy \( i - 1 \) will ever change strategy. The deterministic approximation then shows that a checkering pattern supported on strategies \( i - 1, i - 3, \ldots \) quickly takes hold, and before long, \( u_{i-1} > u_{i-3} \). At this point, point strategy \( i - 2 \) has a negative net replacement rate, so within \( O(\log(N)) \) time where \( N \) is total population size, strategy \( i - 2 \) vanishes. This continues for strategy \( i - 4, \ldots \), finishing in an absorbing state supported on \( \lfloor m/2 \rfloor \) non-interacting strategies.

**Appendix.**

**Basic properties of solutions.**

**Lemma 5.1.** If \( u \in S_m \), there is a unique solution \( u(t) \) of (1) with \( u(0) = u \), defined for \( t \in \mathbb{R} \) and satisfying \( u(t) \in S_m \) for all \( t \in \mathbb{R} \). Moreover,
\begin{enumerate}[(a)]
  \item for each \( i \), \( u_i(t) = u_i \exp(\int_0^t (u_{i-1}(s) - u_{i+1}(s))ds) \),
  \item if \( u_i = 0 \) then \( u_i(t) = 0 \) for all \( t \in \mathbb{R} \), and if \( u_i > 0 \) then \( u_i(t) > 0 \) for all \( t \in \mathbb{R} \), and
  \item if \( u_{i-1} = u_{j+1} = 0 \) for some \( i, j \) then \( (u_k(t))_{k \in [i, j], t \in \mathbb{R}} \) does not depend on \( (u_{\ell})_{\ell \in [i-1, j+1]} \).
\end{enumerate}

**Proof.** See for example [1] for the basic ODE theory used here. Since the right-hand side of (1) is a polynomial it is locally Lipschitz, so local existence and uniqueness holds, and solutions are defined until they diverge. Thus, given \( u \in \mathbb{R}^m \) and solution \( u(t) \) with \( u(0) = u \), let \( \tau = \sup_{M > 0} \inf \{ t : \sum_i |u_i(t)| = M \} \), so that \( \tau > 0 \) and \( u(t) \) is the unique solution on time interval \([0, \tau)\) with initial value \( u \). Since
\[
\sum_i u_i' = \sum_i u_i u_{i-1} - \sum_i u_i u_{i+1} = 0,
\]
and doing the same, we find, interestingly, that constraints are satisfiable in both cases if and only if
\[
\prod_{\text{even } i} p_i = \prod_{\text{odd } i} p_i.
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In any case, this suggests that it is promising that studying cyclic sums of products of the type defined in Lemma 2.2, or variations thereof, might make it possible to study variations on our simple model.

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\begin{enumerate}[(a)]
  \item for each \( i \), \( u_i(t) = u_i \exp(\int_0^t (u_{i-1}(s) - u_{i+1}(s))ds) \),
  \item if \( u_i = 0 \) then \( u_i(t) = 0 \) for all \( t \in \mathbb{R} \), and if \( u_i > 0 \) then \( u_i(t) > 0 \) for all \( t \in \mathbb{R} \), and
  \item if \( u_{i-1} = u_{j+1} = 0 \) for some \( i, j \) then \( (u_k(t))_{k \in [i, j], t \in \mathbb{R}} \) does not depend on \( (u_{\ell})_{\ell \in [i-1, j+1]} \).
\end{enumerate}

**Proof.** See for example [1] for the basic ODE theory used here. Since the right-hand side of (1) is a polynomial it is locally Lipschitz, so local existence and uniqueness holds, and solutions are defined until they diverge. Thus, given \( u \in \mathbb{R}^m \) and solution \( u(t) \) with \( u(0) = u \), let \( \tau = \sup_{M > 0} \inf \{ t : \sum_i |u_i(t)| = M \} \), so that \( \tau > 0 \) and \( u(t) \) is the unique solution on time interval \([0, \tau)\) with initial value \( u \). Since
\[
\sum_i u_i' = \sum_i u_i u_{i-1} - \sum_i u_i u_{i+1} = 0,
\]
it follows that \( \sum_i u_i(t) = \sum_i u_i \) for \( t \in [0, \tau) \). Solving for \( u_i(t) \) in (1) as a single-variable inhomogeneous linear equation, we find that

\[
u_i(t) = u_i \exp \left( \int_0^t (u_{i-1}(s) - u_{i+1}(s)) ds \right)
\]

for \( t \in [0, \tau) \). Combining the two observations, if \( u_i \geq 0 \) for each \( i \) and \( \sum_i u_i = 1 \) then for all \( t \in [0, \tau) \), \( u_i(t) \geq 0 \) and \( \sum_i u_i(t) = 1 \), and thus \( \int_0^t (u_{i-1}(s) - u_{i+1}(s)) ds \leq t \) and so \( u_i(t) \leq u_i e^t \). If \( \tau < \infty \) we obtain a contradiction since \( \sum_i |u_i(t)| \leq e^t \sum_i |u_i| = e^t \) for all \( t < \tau \), so \( \tau = \infty \). By an obvious symmetry of (1), given \( u \in S_m \), if \( u(t) \) is a solution for \( t > 0 \) with initial value satisfying \( u_i(0) = u_{-i} \) for each \( i \) modulo \( m \) then the function \( v(t) \) with \( v_i(t) = u_{-i}(-t) \) is a solution for \( t < 0 \) with final value \( v(0) = u \). This establishes existence for \( t < 0 \) as well as reversal symmetry, and together with the above shows that if \( u \in S_m \) there is a unique solution \( u(t) \) defined for \( t \in \mathbb{R} \), satisfying \( u(0) = u \) and \( u(t) \in S_m \) for all \( t \in \mathbb{R} \).

We’ve already shown (a), the exponential formula for \( u_i(t) \). In turn, (a) easily gives (b). To obtain (c), if \( u_{i-1} = u_{j+1} = 0 \) then using (b), \( u_{i-1}(t) = u_{j+1}(t) = 0 \) for all \( t \in \mathbb{R} \), so \( (u_k(t))_{k \in [i,j], t \in \mathbb{R}} \) solves the system

\[
u_k = F_k(u), \quad k \in [i,j], \quad u_{i-1} = u_{j+1} = 0
\]

with \( u_{i-1}, u_{j+1} \) understood as fixed constants. Existence and uniqueness of solutions to (5) follows in the same way as shown above for (1). Since \( F_k(u) \) depends only on \( u_{k-1}, u_k, u_{k+1} \), if \( w \) is any point with \( w_k = u_k \) for \( k \in [i,j] \) then for the corresponding solution \( w(t) \), \( (w_k(t))_{k \in [i,j], t \in \mathbb{R}} \) also solves (5). By uniqueness, \( w_k(t) = u_k(t) \) for all \( k \in [i,j] \) and \( t \in \mathbb{R} \), which gives (c).

**Well-mixed stochastic model.** Here we discuss the underlying stochastic model for which (1) is the large-population limit. Suppose we have a population of fixed size \( N \), and each individual follows one of the strategies \( i \in [m] \). At some rate \( r_N \), a pair of individual is selected uniformly at random, and they interact. If, before interaction, one of them follows strategy \( i - 1 \) while the other follows \( i \) for some \( i \), then after interaction both follow strategy \( i \); otherwise, nothing happens. The rate \( r_N \) should be chosen so that as \( N \to \infty \), each individual interacts with others at a roughly constant rate, that we can suppose is equal to 1. Since each individual is chosen with probability \( 2/N \) during each selection of a pair, let us set \( r_N = N/2 \).

For each \( i \), let \( U_i \) denote the number of individuals following strategy \( i \). Let \( e_i \) denote the \( i^{th} \) unit vector in \( \mathbb{R}^m \). At each selection, with probability \( 2(U_{i-1}/N)(U_i/N) \) a pair is selected with one individual following strategy \( i - 1 \) and one following strategy \( i \). Thus, \( U = (U_i)_{i \in [m]} \) is a Markov chain with transitions

\[
U \to U + e_i - e_{i-1} \quad \text{at rate} \quad 2r_N \frac{U_{i-1} U_i}{N} = \frac{N U_{i-1} U_i}{N}.
\]

Let \( u = U/N \), then

\[
u \to u + (e_i - e_{i-1})/N \quad \text{at rate} \quad N u_{i-1} u_i.
\]

In particular, \( u \) is a density-dependent Markov chain in the sense of [9], with drift \( F(u) \) given by the right-hand side of (1). For each \( N \) let \( u(t; N) \) denote the Markov chain corresponding to \( u \), and let \( \phi_t \) denote the flow of (1). It follows that if \( u(0; N) \to u_0 \) as \( N \to \infty \) then for any \( \epsilon, T > 0 \),

\[
\lim_{N \to \infty} \mathbb{P}(\sup_{t \leq T} |u(t; N) - \phi_t(u_0)| \geq \epsilon) \to 0,
\]
which is a precise way of expressing the fact that (1) is the large-population deterministc limit of the well-mixed stochastic model just described.

Variations. The above stochastic model has only one kind of transition, namely, strategy $i$ converts strategy $i-1$, and the rates are invariant with respect to cyclic permutation of strategies. Several variations can be introduced while still maintaining an interaction structure of cyclic dominance. We shall discuss three such variations: unequal conversion rates, mutation, and random copying.

Unequal conversion rate. This is implemented by assigning to each $i$ a probability $p_i$ of conversion at each interaction between strategies $i$ and $i-1$. The above model assumes $p_i = 1$ for all $i$. Making this adjustment gives

$$u \to u + \left( e_i - e_{i-1} \right)/N \text{ at rate } Np_i u_{i-1} u_i,$$

and the corresponding system of replicator equations is

$$u'_i = u_i (p_i u_{i-1} - p_{i+1} u_{i+1}), \quad i \in [m].$$

Mutation. This is implemented by having each individual select a new strategy uniformly at random at some rate $\alpha$. It adds the transitions

$$u \to u + \left( e_i - e_j \right)/N \text{ at rate } \alpha N u_j/m \text{ for every } i, j \in [m],$$

and combining with unequal conversion rates, gives the replicator equations

$$u'_i = u_i (p_i u_{i-1} - p_{i+1} u_{i+1}) + \alpha (1/m - u_i).$$

Random copying. This is implemented by having each individual copy another individual chosen uniformly at random, at some rate $\beta$. It adds the transitions

$$u \to u + \left( e_i - e_j \right)/N \text{ at rate } \alpha \beta N u_i u_j \text{ for every } i, j \in [m], i \neq j.$$

Notice that copying is symmetric, i.e., the total rate $\beta N u_i u_j$ that an individual of strategy $i$ copies an individual of strategy $j$ is the same with $i$ and $j$ reversed. So, this modification has actually no effect on the replicator equations.

Conserved quantities. For convenience we recall the statement before proving. Refer to the Introduction for definitions relating to the quantities given below.

**Lemma 2.2.** The following functions are constant along solutions of (1).

1. The full product $P(u) = \prod_i u_i$,
2. For even $m$, the even and odd products
   $$P_e(u) = \prod_{\text{even } i} u_i \quad \text{and} \quad P_o(u) = \prod_{\text{odd } i} u_i,$$
3. For each $k \in \{1, \ldots, [m/2]\}$, the sum of products over independent sets of cardinality $k$:
   $$P_k(u) = \sum_{A \in \mathcal{I}_k} \prod_{i \in A} u_i.$$
4. For each component $[i, j]$ of supp$(u)$ and $1 \leq k \leq \lfloor (j - i + 1)/2 \rfloor$, the sum of products over independent subsets of $[i, j]$ of cardinality $k$:
   $$P_k^{(i,j)}(u) = \sum_{A \in \mathcal{I}_k(i,j)} \prod_{\ell \in A} u_{\ell}.$$

**Proof of Lemma 2.2.** We treat each part as listed above.
(i) If \( P(u(0)) = 0 \) then \( u_i(0) = 0 \) for some \( i \) which implies \( u_i(t) = 0 \), and thus \( P(u(t)) = 0 \), for all \( t \in \mathbb{R} \). If \( P(u(0)) \neq 0 \) then since \( \log P(u) = \sum_i \log(u_i) \),

\[
\frac{d}{dt} P(u(t)) = \sum_i (\log u_i(t))' = \sum_i u_{i-1}(t) - \sum_i u_{i+1}(t) = 0.
\]

(ii) The proof that \( P_c \) and \( P_o \) are conserved is similar, so we omit it.

(iii) Recall that for fixed \( m \) and \( k \in \{1, \ldots, \lfloor m/2 \rfloor \} \), \( I_k \) denotes the set of independent subsets of \( [m] \) of cardinality \( k \). Given \( u \), let \( u(t) = \phi(t,u) \) denote the solution. For a monomial \( \prod_i u_i^{a_i} \), we compute

\[
\frac{d}{dt} \prod_i u_i(t)^{a_i} = \sum_j a_j u_j(t)^{a_j-1} F_j(u(t)) \prod_{i \neq j} u_i(t)^{a_i} \\
= \sum_j a_j u_j(t)^{a_j} (u_j(t) - u_{j+1}(t)) \prod_{i \neq j} u_i(t)^{a_i} \\
= \prod_i u_i(t)^{a_i} \left( \sum_j a_j (u_j(t) - u_{j+1}(t)) \right) \\
= \prod_i u_i(t)^{a_i} \left( \sum_j u_j(t)(a_{j+1} - a_{j-1}) \right).
\]

Viewing \( a = (a_1, \ldots, a_m) \) as a vector in \( \mathbb{N}^m \), letting \( \delta_j \) denote the \( j^{th} \) unit vector and letting \( M(a) \) denote \( \prod_i u_i^{a_i} \), the above can be formally viewed as a linear operation \( D \) on polynomials, defined by

\[
D(M(a)) = \sum_j (a_{j+1} - a_{j-1}) M(a + \delta_j).
\]

For \( A \subseteq [m] \), let \( 1(A) = \sum_{j \in A} \delta_j \). Then, \( P_k(u) = \sum_{A \in I_k} M(1(A)) \), so \( P_k \) is conserved if

\[
\sum_{A \in I_k} D(M(1(A))) = 0.
\]

(7)

Fix \( A \in I_k \) and let \( a = 1(A) \). Since \( A \) is independent, if \( j \in A \) then \( j-1, j+1 \notin A \) so

\[
a_{j+1} - a_{j-1} = \begin{cases} 
0 & \text{if } j \in A \text{ or if } \{j-1, j+1\} \subseteq A, \\
1 & \text{if } j+1 \in A, j-1 \notin A, \text{ and}
\end{cases}
\]

\[-1 & \text{if } j-1 \in A, j+1 \notin A.
\]

Let \( J_k \) denote the set of subsets of \( [m] \) having \( k-1 \) components of cardinality 1, and one component of cardinality 2. Then, let

\[
r(A) = \{ B \in J_k : A \subset B \text{ and if } [j, j+1] \in B \text{ then } j \in A \} \quad \text{and}
\]

\[
\ell(A) = \{ B \in J_k : A \subset B \text{ and if } [j, j+1] \in B \text{ then } j+1 \in A \}.
\]

It follows that

\[
D(M(1(A))) = \sum_{B \in \ell(A)} M(1(B)) - \sum_{B \in r(A)} M(1(B)).
\]

(8)

For each \( B \in J_k \) there is a unique \( A \) such that \( B \in r(A) \). Moreover, if \( A, A' \in I_k \) with \( A \neq A' \) then \( r(A) \cap r(A') = \emptyset \). Thus \( \{r(A) : A \in I_k \} \) is a
partition of \( J_k \). Since the same is true with \( \ell \) in place of \( r \), summing over \( A \in \mathcal{I}_k \) in (8), we obtain (7).

(iv) Let \([i, j]\) be a component of \( \text{supp}(u) \) and let \( w \) have \( w_k = u_k \) for \( k \in [i, j] \) and \( w_k = 0 \) elsewhere. By definition, \( u_{i-1} = u_{j+1} = 0 \), so using Lemma 5.1, \( \phi_k(t, w) = \phi_k(t, u) \) for all \( k \in [i, j] \) and \( t \in \mathbb{R} \). It is easy to show that for any \( k \) and \( t \in \mathbb{R} \), \( P_k^{[i,j]}(\phi(t, u)) = P_k(\phi(t, u)) \), using the fact that \( \text{supp}(\phi(t, w)) = [i, j] \) for all \( t \in \mathbb{R} \), which follows again from Lemma 5.1. Since \( P_k \) is conserved, it follows that \( P_k^{[i,j]} \) is as well.

\[ \square \]

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