A Construction of Solutions to Reflection Equations for Interaction-Round-a-Face Models

Roger E. Behrend

Department of Mathematics, University of Melbourne
Parkville, Victoria 3052, Australia

and

Paul A. Pearce

Physikalische Institut der Universität Bonn
Nußallee 12, D-53115 Bonn, Germany

Abstract

We present a procedure in which known solutions to reflection equations for interaction-round-a-face lattice models are used to construct new solutions. The procedure is particularly well-suited to models which have a known fusion hierarchy and which are based on graphs containing a node of valency 1. Among such models are the Andrews-Baxter-Forrester models, for which we construct reflection equation solutions for fixed and free boundary conditions.

1. Introduction

Boundary weights which satisfy reflection equations are important in the study of solvable interaction-round-a-face (IRF) lattice models with non-periodic boundary conditions [1]–[8]. More specifically, such boundary weights lead to families of commuting transfer matrices and hence integrability. In [1, 3, 8], boundary weights were obtained by directly solving the IRF reflection equations, while in [4] they were obtained using intertwiners together with known boundary weights for a related vertex model.

Here, we present a procedure in which known boundary weights for an IRF model—together with auxiliary face weights, generally obtained from a fusion hierarchy—are used to construct new boundary weights for that model. This procedure takes two forms, one which leads to weights for fixed boundary conditions and the other which leads to weights for free, or at least quasi-free, boundary conditions. In each case, the resulting boundary weights contain an arbitrary parameter.

Our procedure is particularly effective for models, such as the Andrews-Baxter-Forrester (ABF) models [9], which are based on graphs containing a node of valency 1, since there then exist trivial weights which can be used as the known, starting weights. In this paper, we apply our procedure to the ABF models and obtain weights for fixed boundary conditions which match those of [1], as well as weights for free boundary conditions.

1E-mail: reb@maths.mu.oz.au
2E-mail: pap@maths.mu.oz.au. On leave from University of Melbourne.
2. General Procedure

We are considering an IRF model on a square lattice, and we assume that there are restrictions on the spins allowed on any adjacent lattice sites, as specified by an adjacency matrix

\[ A_{ab} = \begin{cases} 
0, & \text{spins } a \text{ and } b \text{ may not be adjacent} \\
1, & \text{spins } a \text{ and } b \text{ may be adjacent} 
\end{cases} \]

For such models, we associate a Boltzmann weight with each set of spins \( a, b, c, d \) that are allowed to be adjacent around a face—\( i.e. \) for which \( A_{ab} A_{bc} A_{cd} A_{da} = 1 \). These weights are denoted

\[
W\left( \begin{array}{c}
\bar{a} \\
c \\
b \\
\end{array} \right) = a \left( \begin{array}{c}
d \\
_\backslash \\
c \\
\end{array} \right) (2.1)
\]

where \( u \) is the spectral parameter.

2.1 Fixed Boundary Conditions

We now consider a boundary containing a fixed spin \( \bar{a} \). In this case we associate a boundary weight with each spin \( a \) which is allowed to be adjacent to \( \bar{a} \)

\[
\bar{B}_a( a \mid u ) = a \left( \begin{array}{c}
u \\
\bar{a} \\
a \\
\end{array} \right) (2.2)
\]

It can be shown \([1]\) that families of commuting transfer matrices can be obtained if the boundary weights (2.2), together with the face weights (2.1), satisfy the fixed-boundary reflection equations for \( \bar{a} \). There is one such equation for each set of spins \( b, c, d \) satisfying \( A_{ab} A_{bc} A_{cd} A_{da} = 1 \),

\[
\sum_f W\left( \begin{array}{c}
c \\
b \\
\bar{a} \\
\end{array} \right) u-v W\left( \begin{array}{c}
d \\
\bar{a} \\
c \\
\end{array} \right) \mu - v \bar{B}_a( f \mid u ) \bar{B}_a( d \mid v ) =
\sum_f W\left( \begin{array}{c}
\bar{a} \\
d \\
c \\
\end{array} \right) u-v W\left( \begin{array}{c}
f \\
\bar{a} \\
c \\
\end{array} \right) \mu - v \bar{B}_a( f \mid u ) \bar{B}_a( b \mid v ) (2.3)
\]
Here, $\mu$ is an arbitrary fixed parameter and the sums are over all spins $f$ which are allowed to be adjacent to $\bar{a}$. We note that if the face weights satisfy the symmetry

$$W\left(\begin{array}{c} d \\ a \\ b \end{array} \bigg| u \right) = W\left(\begin{array}{c} b \\ a \\ d \end{array} \bigg| u \right)$$

(2.4)

then (2.3) is automatically satisfied whenever $b = d$. Furthermore, in the case in which there is only one spin $a$ allowed to be adjacent to $\bar{a}$—ie $\bar{a}$ has a valency of 1—we must have $b = d = f = a$ in (2.3) implying that the equation is always satisfied and that the single boundary weight $B_{\bar{a}}(a | u)$ may be assigned to any function of $u$.

### 2.2 Free Boundary Conditions

For the case of free, or at least quasi-free, boundary conditions, we associate a boundary weight with each set of spins $a, b, c$ satisfying

$$A_{ab}A_{bc} = 1,
B\left(\begin{array}{c} b \\ c \\ a \end{array} \bigg| u \right) =
\begin{array}{c} b \\ c \\ a \end{array}$$

(2.5)

In this case, families of commuting transfer matrices can be obtained if the boundary weights (2.5), together with the face weights (2.1), satisfy the free-boundary reflection equations. There is one such equation for each set of spins $a, b, c, d, e$ satisfying $A_{ab}A_{bc}A_{cd}A_{de} = 1$,

$$\sum_{fg} W\left(\begin{array}{c} c \\ f \\ a \end{array} \bigg| u-v \right) W\left(\begin{array}{c} d \\ g \\ f \end{array} \bigg| \mu-u-v \right) B\left(\begin{array}{c} f \\ g \\ a \end{array} \bigg| u \right) B\left(\begin{array}{c} e \\ c \\ g \end{array} \bigg| v \right) =$$

$$\sum_{fg} W\left(\begin{array}{c} e \\ f \\ d \end{array} \bigg| u-v \right) W\left(\begin{array}{c} f \\ g \\ c \end{array} \bigg| \mu-u-v \right) B\left(\begin{array}{c} f \\ g \\ c \end{array} \bigg| u \right) B\left(\begin{array}{c} b \\ e \\ g \end{array} \bigg| v \right)$$

(2.6)

Here, the sum on the left side is over all spins $f, g$ satisfying $A_{af}A_{cf}A_{df}A_{gd} = 1$ and that on the right side is over all spins $f, g$ satisfying $A_{cf}A_{ef}A_{fg}A_{gb} = 1$.

### 2.3 Construction of New Boundary Weights

Our construction of new boundary weights requires that there exist an auxiliary adjacency matrix $\bar{A}$ and, for each set of spins $a, b, c, d$ satisfying $\bar{A}_{ab}A_{bc}\bar{A}_{cd}A_{da} = 1$, an auxiliary face
These weights, together with the fundamental face weights (2.1), are assumed to satisfy the generalised Yang-Baxter equations. There is one such equation for each set of spins \(a, b, c, d, e, f\) satisfying \(A_{ab}A_{bc}A_{cd}A_{de}A_{ef}A_{fa} = 1\),

\[
\sum_{g} W\left(\begin{array}{c|c}
    f & g \\
    \hline
    a & b \\
\end{array}\right) W\left(\begin{array}{c|c}
    g & d \\
    \hline
    b & c \\
\end{array}\right) W\left(\begin{array}{c|c}
    f & e \\
    \hline
    g & d \\
\end{array}\right) = 1,
\]

and

\[
\sum_{g} W\left(\begin{array}{c|c}
    a & g \\
    \hline
    f & e \\
\end{array}\right) W\left(\begin{array}{c|c}
    e & d \\
    \hline
    a & g \\
\end{array}\right) W\left(\begin{array}{c|c}
    e & d \\
    \hline
    a & g \\
\end{array}\right) = 1.
\]

In practice, the auxiliary face weights can generally be constructed using fusion with an appropriate row of fundamental face weights.

Our construction of new boundary weights takes two forms. In the first form, we obtain new weights for a boundary with fixed spin \(\bar{a}\) using known weights for a boundary with fixed spin \(\bar{b}\), where we assume that, with respect to \(\bar{A}\), \(\bar{a}\) is the only spin allowed to be adjacent to \(\bar{b}\). The new weights depend on an arbitrary parameter \(\bar{\chi}\) and, for each spin \(a\) allowed to be adjacent to \(\bar{a}\), are defined as

\[
\bar{B}_{\bar{a}}'(a | u) = \sum_{b} W\left(\begin{array}{c|c}
    a & b \\
    \hline
    \bar{a} & \bar{b} \\
\end{array}\right) W\left(\begin{array}{c|c}
    \bar{a} & \bar{b} \\
    \hline
    a & b \\
\end{array}\right) W\left(\begin{array}{c|c}
    a & b \\
    \hline
    u+\bar{\chi} & \mu-u+\bar{\chi} \\
\end{array}\right) \bar{B}_{\bar{b}}(b | u)
\]

where the sum is over all spins \(b\) satisfying \(\bar{A}_{ab}A_{b\bar{b}} = 1\). We note that we suppress the dependence of these weights on the the spin \(\bar{b}\) and the parameter \(\bar{\chi}\). It is straightforward to show that the new weights (2.9) satisfy the fixed boundary reflection equations for \(\bar{a}\), using the assumptions that the known weights satisfy the fixed boundary reflection equations for \(\bar{b}\), that the auxiliary face weights satisfy (2.8), and that \(\bar{b}\) has valency 1 with respect to \(\bar{A}\).

In the second form of our construction of new boundary weights, we obtain certain weights for free boundary conditions using known weights for a boundary with fixed spin
The new weights depend on an arbitrary parameter \( \chi \) and, for each set of spins \( a, b, c \) satisfying \( \bar{A}_{\bar{a}a} A_{ab} A_{bc} \bar{A}_{c\bar{a}} = 1 \), are defined as

\[
B\left( \begin{array}{c} b \\ c \\ a \end{array} \mid u \right) = \sum_d W\left( \begin{array}{c} b \\ a \end{array} \mid u+\chi \right) \cdot W\left( \begin{array}{c} c \\ b \end{array} \mid \mu-u+\chi \right) \cdot \bar{B}_{\bar{a}}(d \mid u) \tag{2.10}
\]

where the sum is over all spins \( d \) satisfying \( \bar{A}_{bd} A_{da} = 1 \). Again we suppress the dependence of these weights on the spin \( \bar{a} \) and the parameter \( \chi \). The new weights (2.10) satisfy the free boundary reflection equations for each set of spins \( a, b, c, d, e \) in (2.6) which satisfy \( \bar{A}_{\bar{a}a} A_{ab} A_{bc} \bar{A}_{cd} A_{de} \bar{A}_{c\bar{a}} = 1 \). This follows straightforwardly from the assumptions that the known weights satisfy the fixed boundary reflection equations for \( \bar{a} \), and that the auxiliary face weights satisfy (2.8).

### 3. ABF Models

We now consider the Andrews-Baxter-Forrester (ABF) models [9]. There is one such model for each integer \( L \geq 3 \), with the spins \( a \) in this model taking the values

\[
a \in \{1, 2, \ldots, L\} \tag{3.1}
\]

The adjacency matrix is defined by the condition that \( A_{ab} = 1 \) if and only if

\[
|a - b| = 1 \tag{3.2}
\]

There is a fixed crossing parameter

\[
\lambda = \frac{\pi}{L + 1} \tag{3.3}
\]

and the face weights are given by

\[
W\left( \begin{array}{c} a+1 \\ a \\ a \end{array} \mid u \right) = \frac{\theta(\lambda-u)}{\theta(\lambda)}
\]

\[
W\left( \begin{array}{c} a \\ a+1 \\ a \end{array} \mid u \right) = \sqrt{\frac{\theta((a-1)\lambda) \cdot \theta((a+1)\lambda)}{\theta(a\lambda)^2} \cdot \frac{\theta(u)}{\theta(\lambda)}} \tag{3.4}
\]

\[
W\left( \begin{array}{c} a \\ a\pm1 \\ a \end{array} \mid u \right) = \frac{\theta(a\lambda\pm u)}{\theta(a\lambda)}
\]

where \( \theta \) is the standard elliptic theta-1 function of fixed nome.
For these models, an auxiliary adjacency matrix and auxiliary face weights which satisfy (2.8) are provided by the level $n$ fused adjacency matrix and the $n \times 1$ fused face weights $[10, 11, 12]$
\[
\bar{A} = A^n, \quad \bar{W} = W^{n,1}
\] (3.5)

where
\[
n \in \{0, 1, \ldots, L-1\}
\] (3.6)

The level $n$ fused adjacency matrix is defined by the condition that
\[
a - b \in \{-n, -n+2, \ldots, n-2, n\}
\] (3.7)

and
\[
a + b \in \{n+2, n+4, \ldots, 2L-n-2, 2L-n\}
\] (3.8)

We note that $A^1 = A$. The $n \times 1$ fused face weights are defined in terms of rows of $n$ fundamental face weights (3.4) and, after appropriate normalisation and symmetrisation, are given by
\[
W^{n,1}(d c a b u) = (3.9)
\]

where $\epsilon_a$ are factors whose required properties are
\[
(\epsilon_a)^2 = 1, \quad \epsilon_a \epsilon_{a+2} = -1
\] (3.10)

We note that the fused weights (3.9) reduce to (3.4) for $n = 1$.

### 3.1 Weights for Fixed Boundary Conditions

Since, for the ABF models, the spin 1 has valency 1 with respect to $A$, and the face weights satisfy the symmetry (2.4), the boundary weight $\bar{B}_1(2 \mid u)$ can be set to an arbitrary function of $u$. Furthermore, it follows from (3.7) and (3.8) that the spin 1 has valency 1 with respect to any $A^{a-1}$, the only allowed neighbour being the spin $\bar{a}$. It is therefore possible to construct new weights for a boundary with fixed spin $\bar{a}$ using an arbitrary weight for a boundary with fixed spin 1. Accordingly, we apply (2.9) with $A = A^{a-1}$, $\bar{W} = W^{a-1,1}$, $b = 1$, $\mu = \lambda$, $\tilde{\chi} = -\lambda - \xi$ and $\bar{B}_1(2 \mid u) = \epsilon_1 \epsilon_2 \epsilon_a \epsilon_{a-1} \sqrt{\theta(2\lambda)/\theta(\lambda)} g(u)$, which gives
\[
\bar{B}'_{\bar{a}}(\bar{a} \pm 1 \mid u) = g(u) \sqrt{\frac{\theta((\bar{a} \pm 1)\lambda)}{\theta(\bar{a}\lambda)}} \frac{\theta(u \pm \xi)}{\theta(u \mp \bar{a}\lambda \mp \xi)} \frac{\theta(u \pm \xi)}{\theta(u \mp \bar{a}\lambda \mp \xi)}
\] (3.11)

where $\xi$ is an arbitrary constant and $g$ is an arbitrary function. It can be seen that these weights exactly match those obtained in [1] by directly solving the reflection equations.
3.2 Weights for Free Boundary Conditions

We now consider the construction of ABF weights for free boundary conditions using (2.10) together with (3.5) and (3.11). We shall associate with any ABF weight \( B(\frac{b}{a} \mid u) \) either odd or even parity, according to the parity of \( b \) which gives the same sign. Together with (3.5) and (3.11), we find that the unique values \( B(\frac{b}{a} \mid u) \) generated for each \( b \) of the appropriate parity. However, by examining (3.7) and (3.8), we find that the unique values

\[
 n = \begin{cases} 
 \frac{L-1}{2}, & \text{L odd} \\
 \frac{L}{2}, & \text{L even} 
\end{cases} \\
\bar{a} = \begin{cases} 
 n, & \text{odd weights} \\
 n+1, & \text{even weights}
\end{cases} 
\]

do generate a full set of boundary weights of a given parity.

We now apply (2.10) with \( \mu = \lambda, \chi = \xi + \frac{n-1}{2} \lambda \), \( g(u) \mapsto \epsilon_{\bar{a}} \epsilon_{a-1} g(u) \) and \( \bar{\xi} \mapsto \bar{\xi} - \frac{n-1}{2} \lambda \), which gives

\[
 B\left(\frac{a \pm 1}{a \mp 1} \mid u\right) = g(u) \sqrt{\frac{\theta(a \lambda)}{\theta((a \pm 1) \lambda)}} \times \] (3.13)

\[
\left[ \frac{\theta\left(\frac{n+1}{2} \lambda \right) \theta\left(\frac{n+1}{2} \lambda\right)}{\theta(a \lambda)^2} \frac{\theta\left(\frac{n+1}{2} \lambda\right) \theta\left(\frac{n+1}{2} \lambda\right)}{\theta(\lambda)^2} \right] \frac{\theta\left(\frac{n}{2} \lambda - \bar{\xi} + \xi\right)}{\theta(\lambda)} \frac{\theta\left(\frac{n}{2} \lambda + \bar{\xi} + \xi\right)}{\theta(\lambda)} \frac{\theta(2u)}{\theta(2u)}
\]

\[
 B\left(\frac{a \pm 1}{a \mp 1} \mid u\right) = g(u) \sqrt{\frac{\theta(a \lambda)}{\theta((a \pm 1) \lambda)}} \times \] (3.14)

\[
\left( \frac{\theta\left(\frac{n+1}{2} \lambda \right) \theta\left(\frac{n+1}{2} \lambda\right)}{\theta(a \lambda) \theta(\bar{a} \lambda) \theta(\lambda)^4} \frac{\theta\left(\frac{n+1}{2} \lambda \right) \theta\left(\frac{n+1}{2} \lambda\right)}{\theta(\lambda)^4} \right)
\]

Here, the two terms which led to (3.13) were combined using a standard elliptic identity, and a common factor \( \epsilon_{\bar{a}} \epsilon_{a-1} \) in (3.13) and (3.14) was eliminated since, for a given \( \bar{a} \), the allowed values of \( a \) must all have the same parity implying that this factor always produces the same sign.
4. Discussion

We have presented a general procedure for obtaining boundary weights for IRF models and have applied this to the ABF models. Our method should be useful for determining classes of IRF models for which solutions of the reflection equations exist and contain arbitrary parameters. In particular, our method implies the existence of such solutions for the standard A-D-E models, since these are all based on graphs containing a node of valency 1, and have known fusion hierarchies.

In future work, we plan to construct the weights for the D and E series within the standard A-D-E models, and to study further the weights obtained here for the ABF models, which form the A series. In particular, we intend to investigate the relationship between the ABF weights for free boundary conditions found using our method, and those obtained using intertwiners or by directly solving the reflection equations. We also hope to be able to show that these weights can be used to obtain genuine free boundary conditions at the isotropic point, and that the associated transfer matrices satisfy functional equations with the same form as in the case of fixed and periodic boundary conditions.

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