DG COALGEBRAS AS FORMAL STACKS

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1. Introduction

1.1. In this paper we provide the category of unital (dg, unbounded) coalgebras $\text{dgc}(k)$ over a field $k$ of characteristic zero with a structure of a simplicial closed model category (SCMC=simplicial CMC) — see 2.3.

This structure generalizes the one defined by Quillen $[Q2]$ in 1969 for 2-reduced unital coalgebras. The major difference is that our notion of weak equivalence is strictly stronger (see the example in 9.1.2) that that of quasi-isomorphism.

The structure of the proof is close to that of Quillen — we use the pair of adjoint functors connecting $\text{dgc}(k)$ with the category $\text{dg Lie}(k)$ of Lie dg algebras over $k$ and the existing SCMC structure on $\text{dg Lie}(k)$ — see $[H2]$. The proof is technically more difficult since the spectral sequence arguments do not work here and one should carefully work in the tensor category of filtered complexes instead.

The main theorems 3.1 and 3.2 are proven in Sections 3 – 7.

In the rest of the paper (and of the Introduction) we try to persuade the reader that unital dg coalgebras provide a proper context to talk about formal deformations in characteristic zero.

In this approach formal deformation problems over a field $k$ of characteristic zero are described (up to an equivalence) by functors

$$F : \text{dagr}^{\leq 0}(k) \to \Delta^0\text{Ens}$$

from the category of non-positively graded artinian local dg $k$-algebras to the category of simplicial sets. These functors are represented (in a homotopy category sense) by unital coalgebras from $\text{dgc}(k)$ — see 8.1. Such a coalgebra plays role of the coalgebra of distributions concentrated at a point and it is defined uniquely up to a weak equivalence in $\text{dgc}(k)$.

In Section 8 we study properties of the functors (1) which appear as nerves of dg Lie algebras. In Section 9 we calculate some elementary examples. In particular, we identify in 9.7 the coalgebra of the formal stack of deformations of a principal $G$-bundle $P$ on a scheme $X$ with the standard complex of the Lie algebra $\mathfrak{R}(X, \mathfrak{g}_P)$. Note that this answer is well-known (see, for instance, $[K]$ for a similar problem); however it was not so clear (at least for the author) what did this answer describe.

1.2. Tangent Lie algebra. We suggest to interpret a unital dg coalgebra as the coalgebra of distributions concentrated at a point of a would-be-a-space.
1.2.1. The Quillen functor $\mathcal{L} : \text{dgcu}(k) \to \text{dglie}(k)$ to the category of dg Lie algebras (see [Q2, App. B or 2.2 below) is then interpreted as the tangent Lie algebra functor. Theorem 3.2 claims that the adjoint pair $(\mathcal{L}, \mathcal{C})$ of functors establishes an equivalence between the homotopy categories of $\text{dgcu}(k)$ and of $\text{dglie}(k)$. This means that a formal dg stack can be (uniquely up to weak equivalence) reconstructed from the homotopy type of its tangent Lie algebra.

1.2.2. Formal schemes define, of course, coalgebras concentrated in degree zero. The corresponding tangent Lie algebra is concentrated in strictly positive degrees. More generally, a formal stack $X \in \text{dgcu}(k)$ satisfying $H^i(\mathcal{L}(X)) = 0$ for $i \leq 0$, is called a formal space. Equivalently, this means that $X$ is weakly equivalent to a coalgebra concentrated in non-negative degrees. Formal spaces have also a description in terms of the functor on Artin rings they represent — see 1.3.2 below.

1.3. **Functors on Artin rings.** Thus, we consider unital coalgebras as “the most general” formal (dg) stacks concentrated in a point. It is reasonable to describe them as functors on formal spaces — as one defines stacks using functors on affine schemes. One can go even further and take into account that any formal space is a filtered colimit of finite dimensional ones which now take form $A^*$ where $A \in \text{dgart}^{\leq 0}(k)$.

1.3.1. Any formal stack $X \in \text{dgcu}(k)$ gives rise to a deformation functor

$$\tilde{X} : \text{dgart}^{\leq 0}(k) \to \Delta^0\text{Ens}$$

which is defined up to homotopy equivalence. This corresponds to the usual description of stacks as 2-functors from affine schemes to groupoids. The deformation functor enjoys nice exactness properties (see §1.3). Given the tangent Lie algebra $g = \mathcal{L}(X)$, the functor $\tilde{X}$ can be described as the nerve $\Sigma_g$ of the dg Lie algebra $g$ (see [H] and §1.1) defined by the formula

$$\Sigma_g((A, m))_n = \text{MC}(m \otimes \Omega_n \otimes g)$$

for $(A, m) \in \text{dgart}^{\leq 0}(k)$, where $\text{MC}(\cdot)$ denotes the collection of Maurer-Cartan elements of a dg Lie algebra and $\Omega_n$ is the algebra of polynomial differential forms on the standard $n$-simplex.

The nerve of a dg Lie algebra is homotopy equivalent to the Deligne groupoid (cf. [GM1], [H]) if $(m \otimes g)^i = 0$ for $i < 0$.

1.3.2. If $X$ is a formal space, the restriction of $\tilde{X}$ to the category $\text{art}(k)$ of artinian $k$-algebras concentrated in degree zero, is a functor to discrete simplicial sets (i.e., essentially, to $\text{Ens}$) — see §0.3.2.

This means in particular, that the restriction of $\tilde{X}$ to $\text{art}(k)$ is representable in usual sense by $H^0(X)$.

1.4. **Rational spaces.** A very “non-geometric” class of unital coalgebras is the Quillen’s category $\text{dgcu}_2(\mathbb{Q})$ of 2-reduced unital coalgebras which is one of the models for simply connected rational homotopy types.

One can easily calculate the deformation functor defined by a simply connected rational homotopy type. Let $g \in \text{dglie}(\mathbb{Q})$ be the Lie algebra model for it. This is the tangent Lie algebra of the corresponding unital coalgebra. One has $g^i = 0$ for $i \geq 0$. 
Let \((A, m) \in \text{dgart}^{\leq 0}(\mathbb{Q})\). Then \(\Sigma_0(A)\) is a simply-connected rational space and its homotopy type corresponds to the Lie algebra \(m \otimes g\) — see §4.

1.5. **Coarse moduli.** Let \(g \in \text{dglie}(k)\). One might be willing to consider the functor \(\Pi_g : (A, m) \mapsto \pi_0(\Sigma_0(A))\) on the category \(\text{art}(k)\) as the “coarse moduli” space for the deformation problem defined by \(g\). Usually this functor is not representable (except for the case described in §3.2). However, it admits a hull in the sense of [Sc] which can be easily constructed using a dg Lie subalgebra \(h\) which is a 1-truncation of \(g\) as in [GM2]. A 1-truncation \(h\) being chosen, the hull of the functor \(\Pi_g\) can be described as \(H^0(C(h))\) — see §3.4.

The choice of the truncation \(h\) is not unique though the resulting coalgebra is unique up to a non-canonical isomorphism by a general result of [Sc]. One might ask whether the 1-truncation \(h\) of \(g\) is unique up to a quasi-isomorphism. This is obviously so in the case of [GM2] where \(H^i(g) = 0\) for \(i \leq 0\). We doubt this is true in general.

1.6. **Two general ideas**

**Idea 1:** Any reasonable formal deformation problem in characteristic zero can be described by Maurer-Cartan elements of an appropriate dg Lie algebra

**Idea 2:** Moduli spaces should admit a natural sheaf of dg commutative algebras as a structure sheaf

have been spelled out by different people during the last years (Drinfeld [D], Feigin, Deligne, Kontsevich [Ko]). In this paper we tried to show that these two claims are essentially equivalent to the one saying that any reasonable formal deformation problem can be described by a representable functor on dg artinian rings with values in simplicial sets.

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2. **Preliminaries**

In this Section we fix some standard notations and definitions.

Throughout this Section \(k\) is a commutative ring containing \(\mathbb{Q}\).

2.1. **Unital coalgebras.** Let \(X\) be a cocommutative dg coalgebra \(X\) over \(k\) with comultiplication \(\Delta : X \to X \otimes X\) and counit \(\epsilon : X \to k\).

Recall that an element \(1 \in X\) is called a **group-like element** if

1. \(d(1) = 0\);
2. \(\Delta(1) = 1 \otimes 1\);
3. \(\epsilon(1) = 1\).

A choice of a group-like element \(1 \in X\) defines a decomposition

\[ X = k \cdot 1 \oplus \overline{X} \]

\(\text{translated to the language of coalgebras}\)
where $\overline{X} = \ker(\epsilon)$. This defines an increasing filtration on $X$ by the formula

$$X_n = \ker(X \xrightarrow{\Delta_n} X^\otimes (n+1) \to \overline{X}^\otimes (n+1))$$

(2)

where $\Delta_n$ is the $n$-th iteration of $\Delta$.

2.1.1. **Definition.** (see [HS2]) A pair $(X, 1)$ consisting of a dg cocommutative coalgebra $X$ and a group-like element $1$ is called a **unital coalgebra** if the filtration (2) is exhausting.

The group-like element defining a unital algebra $X$ is called the **unit of $X$**. The filtration (2) of a unital coalgebra is called the **canonical filtration**.

Note that if $k$ is a field then unital coalgebras are just connected coalgebras of Quillen (see [Q2], App. B). In this case the unit element is unique and it is preserved by any coalgebra map.

2.1.2. The category of unital coalgebras over $k$ will be denoted by $\text{dgcu}(k)$. The morphisms in it are supposed to preserve the units (this is automatically fulfilled when $k$ is a field).

2.1.3. Let $\Omega$ be a commutative dg algebra over $k$. Similarly to the above, one defines $\Omega$-coalgebras as cocommutative coalgebras in the category of $\Omega$-modules. Furthermore, one defines unital $\Omega$-coalgebras as pairs $(X, 1)$ with a $\Omega$-coalgebra $X$ and a group-like element $1$ of $X$ such that the filtration (2) is exhausting. The category of unital $\Omega$-coalgebras is denoted $\text{dgcu}(\Omega)$.

2.2. **Quillen functors.** Recall the definition of the couple of adjoint functors

$$\mathcal{L} : \text{dgcu}(k) \rightleftarrows \text{dglie}(k) : \mathcal{C}$$

(3)

defined by Quillen in [Q2], App. B.

2.2.1. Let $X \in \text{dgcu}(k)$, $\overline{X} = \ker \epsilon$ in the standard notation. The dg Lie algebra $\mathcal{L}(X)$ is defined as follows. As a graded Lie algebras, this is the free Lie algebra $F(\overline{X}[-1])$. The differential in $\mathcal{L}(X)$ is the sum of two parts: the one generated by the differential of $\overline{X}[-1]$, and the second defined to be the only derivation of the free Lie algebra $F(\overline{X}[-1])$ whose restriction on $\overline{X}[-1]$ is given by the map

$$\Delta - 1 \otimes \text{id} - \text{id} \otimes 1 : \overline{X} \to \overline{X} \otimes \overline{X}.$$

2.2.2. Let $\mathfrak{g} \in \text{dglie}(k)$. The unital coalgebra $\mathcal{C}(\mathfrak{g})$ is defined as follows. As a graded coalgebra, this is the cofree cocommutative coalgebra $S(\mathfrak{g}[1])$. The differential in $\mathcal{C}(\mathfrak{g})$ is the sum of two parts: the one generated by the differential in $\mathfrak{g}[1]$, and the second defined by its 1-component given by the Lie bracket $[,] : \wedge^2 \mathfrak{g} \to \mathfrak{g}$.

2.2.3. Let $\mathfrak{g} \in \text{dglie}(k)$. Recall that an element $x \in \mathfrak{g}^1$ is called a Maurer-Cartan element if $dx + \frac{1}{2}[x, x] = 0$. The set of Maurer-Cartan elements of $\mathfrak{g}$ is denoted by $\text{MC}(\mathfrak{g})$. 
2.2.4. If $X \in \text{dgcu}(k)$, $g \in \text{dglie}(k)$, then the complex $\text{Hom}(X, g)$ admits a natural structure of a dg Lie algebra given by the formula

$$[f, g] = \mu(f \otimes g)\Delta$$

where $\Delta$ is the comultiplication in $X$ and $\mu$ is the bracket in $g$. In particular the set $\text{MC}(X, g) := \text{MC}(\text{Hom}(X, g))$ is defined. This is the set of twisting functions from $X$ to $g$.

2.2.5. **Theorem.** The functors $\mathcal{L}$ and $\mathcal{C}$ (3) are adjoint. More precisely, for $X \in \text{dgcu}(k)$, $g \in \text{dglie}(k)$ one has natural bijections

$$\text{Hom}(\mathcal{L}(X), g) = \text{Hom}(X, \mathcal{C}(g)) = \text{MC}(X, g).$$

See [Q2], App. B6.

2.3. **Simplicial closed model categories.** We use the axioms (CM1)–(CM5) of [Q2] for the definition of closed model category (CMC).

A simplicial category $\mathcal{C}$ is a collection of objects $\text{Ob}\mathcal{C}$ together with a collection of simplicial sets $\text{Hom}(X, Y)$ assigned to each pair $(X, Y)$ of objects, with strictly associative compositions.

A simplicial category $\mathcal{C}$ with a CMC structure will be called a simplicial CMC (or SCMC) if the following Quillen's axiom (SM7) [Q1] is fulfilled

(SM7) Let $i : A \to B$ be a cofibration and $p : X \to Y$ be a fibration in $\mathcal{C}$. Then the map of simplicial sets

$$\text{Hom}(B, X) \to \text{Hom}(A, X) \times_{\text{Hom}(A, Y)} \text{Hom}(B, X)$$

is a Kan fibration. If, moreover, either $i$ or $p$ is a weak equivalence, then (4) is an acyclic Kan fibration.

*Note that we do not include Quillen's axiom (SM0) claiming the existence of cylinder and path objects — see [Q1] — in our definition of simplicial closed model category.*

2.4. **Models for dg Lie algebras.** Recall [H2] that the category $\text{dglie}(k)$ of dg Lie algebras over a commutative ring $k \supseteq \mathbb{Q}$ admits a simplicial model structure. More precisely, one has the following

**Theorem.** (see [H2], 4.1.1 and 4.8) The category $\text{dglie}(k)$ admits a simplicial CMC structure with surjective maps as fibrations and quasi-isomorphisms as weak equivalences. The simplicial structure on $\text{dglie}(k)$ is defined by the formula

$$\text{Hom}_n(g, h) = \text{Hom}_{\text{dglie}(k)}(g, \Omega_n \otimes h).$$

(5)

where $\Omega_n$ is the algebra of polynomial differential forms on the standard $n$-simplex.
2.5. Operad notations. Throughout the paper we will sometimes use the language of operads — see e.g. [HS1]. In particular, \( \text{COM} \) denotes the operad governing commutative algebras, \( \text{LIE} \) the one for Lie algebras and \( S \) denotes the standard Lie operad of [HS1] governing “strongly homotopy Lie algebras”. If \( \mathcal{O} \) is an operad and \( A \) is an \( \mathcal{O} \)-algebra then \( U(\mathcal{O}, A) \) denotes the corresponding enveloping algebra.

We will use sometimes different base tensor categories. If \( \mathcal{C} \) is a tensor (= symmetric monoidal) category, \( \text{Op}(\mathcal{C}) \) denotes the category of operads over \( \mathcal{C} \).

3. Main theorems

Now we are ready to formulate the main results of the paper.

3.1. Theorem. The category \( \text{dgcu}(k) \) of unital coalgebras over a field \( k \) of characteristic zero admits a simplicial CMC structure. Cofibrations in it are just injective maps and weak equivalences are the maps \( f \) in \( \text{dgcu}(k) \) such that \( L(f) \) is a quasi-isomorphism. The simplicial structure on \( \text{dgcu}(k) \) is given by the condition

\[
\text{Hom}_n(X, Y) = \text{Hom}_{\text{dgcu}(\Omega_n)}(\Omega_n \otimes X, \Omega_n \otimes Y)
\]

where as in [5] \( \Omega_n \) is the algebra of polynomial differential forms on the standard \( n \)-simplex.

Note that the property of a morphism \( f \) of \( \text{dgcu}(k) \) to be a weak equivalence is strictly stronger then that of being a quasi-isomorphism — see the counter-example in 9.1.2.

3.2. Theorem. The adjoint functors \( L, C \) induce an equivalence of the corresponding homotopy categories

\[
L \mathcal{L}: \text{Ho}(\text{dgcu}(k)) \rightleftarrows \text{Ho}(\text{dglie}(k)) : R \mathcal{C}.
\]

The proof of the theorems is given in Sections 3 – 7.

3.3. Functors \( \mathcal{C} \) and \( \mathcal{L} \). Let us study some basic properties of the adjoint functors \( \mathcal{L} \) and \( \mathcal{C} \) defined above. We start with the following Lemma whose proof can be found in [H2] (note that the general claim of [HS1], 3.6.12 contains an error).

3.3.1. Lemma. (see [H2], 6.8.5) Let \( \mathfrak{g} \) be a dg Lie algebra over a commutative ring \( k \supseteq \mathbb{Q} \). It can be obviously considered as a strong homotopy Lie algebra, i.e., a \( S \)-algebra where \( S \) is the standard Lie operad (see [HS1] and also [2.4]). Suppose that \( \mathfrak{g} \) is \( k \)-flat. Then the natural map

\[
U(S, \mathfrak{g}) \rightarrow U(\text{LIE}, \mathfrak{g})
\]

is a quasi-isomorphism.

Proof. Recall shortly the reasoning. The functor \( U(S, \_ ) \) carries quasi-isomorphisms of flat dg Lie algebras into quasi-isomorphisms since \( S \) is a cofibrant operad.

The functor \( U(\text{LIE}, \_ ) \) carries quasi-isomorphisms of flat dg Lie algebras into quasi-isomorphisms by PBW (see [Q2], App. B). Finally, the map \( U(S, \mathfrak{g}) \rightarrow U(\text{LIE}, \mathfrak{g}) \) is a quasi-isomorphism for a cofibrant \( \mathfrak{g} \) by the Comparison theorem [H2], 5.5.1.
Taking into account that cofibrant dg Lie algebras are flat, we get the result.

The following Proposition 3.3.2 has a very important filtered analog — see 4.4.3 below.

3.3.2. Proposition.

1. The functor $C$ preserves quasi-isomorphisms.
2. The adjunction maps $i_X : X \to CL(X)$ and $p_\mathfrak{g} : CL(\mathfrak{g}) \to \mathfrak{g}$ are quasi-isomorphisms.
3. The restriction of $L$ to the subcategory $\text{dgcu}^{\geq 0}(k)$ of non-negatively graded coalgebras preserves quasi-isomorphisms.

Proof. Step 1. Let us prove first that the map $p_\mathfrak{g} : CL(\mathfrak{g}) \to \mathfrak{g}$ is a quasi-isomorphism. Consider the Lie algebra $\mathfrak{g}$ as an algebra over the operad $S$ as above. According to Lemma 3.3.1, the natural map of the enveloping algebras $U(S, \mathfrak{g}) \to U(\mathfrak{g})$ is a quasi-isomorphism. Now, one has an isomorphism $U(S, \mathfrak{g}) = U(CL(\mathfrak{g}))$ and then the adjunction map $p_\mathfrak{g}$ is quasi-isomorphism by the PBW theorem [Q2], App. B.

Step 2. Now we check that $C$ preserves quasi-isomorphisms. The coalgebra $C(\mathfrak{g})$ admits an increasing filtration natural in $\mathfrak{g}$ so that the associated graded pieces are $S^n(\mathfrak{g}[1])$. This clearly implies the claim.

Step 3. Now we can prove that the map $i_X : X \to CL(X)$ is a quasi-isomorphism for any $X \in \text{dgcu}(k)$. In fact, the map $i_X$ admits a natural splitting $q_X : CL(X) \to X$ as a map of complexes: $Y = CL(X)$ as a graded vector space takes form

$$Y = S(F(X[-1])[1])$$

where $S$ is the symmetric algebra and $F$ is the free Lie algebra functor. This defines the projection $q_X : Y \to F(X[-1])[1] \to X$ which is compatible with the differentials and splits $i : X \to Y$. Now consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i_X} & CL(X) & \xrightarrow{q_X} & X \\
\downarrow{i_X} & & \downarrow{CL(i_X)} & & \downarrow{i_X} \\
CL(X) & \xrightarrow{i_{CL(X)}} & CLCL(X) & \xrightarrow{q_{CL(X)}} & CL(X)
\end{array}
\]

The left square in it is commutative by the general nonsense of adjoint functors; the right square is commutative since $q_X$ is functorial in $X$. The map $CL(i_X)$ is a quasi-isomorphism since it is split by the quasi-isomorphism $p_{CL(X)}$ by Step 1. Then, by Step 2, the map $CL(i_X)$ is also a quasi-isomorphism, and therefore its retract $i_X$ is a quasi-isomorphism as well.

Step 4. The claim (3) follows by a standard spectral sequence argument.

□
4. Filtered world and graded world

In this Section we prove a filtered analog of Lemma 3.3.1 and of Proposition 3.3.2 — see Propositions 4.3.7, 4.4.3 below. For this we need a number of new categories and functors and a well-known Rees trick which allows one to reduce some filtered objects to graded objects over the polynomial ring — see 4.3.

4.1. Filtered world.

4.1.1. Definitions. Here $k$ is a base commutative ring. A filtered $k$-module $V$ is a collection $V = \{V_i\}$, $i \in \mathbb{Z}$ with $V_i \subseteq V_{i+1}$ and $V = \bigcup V_i$. The category of filtered $k$-modules is denoted $\text{modf}(k)$.

A filtered complex is a complex in $\text{modf}(k)$. The category of filtered complexes will be denoted in the sequel $\text{CF}(k)$ instead of $\text{C}(\text{modf}(k))$.

The category $\text{modf}(k)$ admits a tensor structure given by the formula

$$ (X \otimes Y)_n = \sum_{p+q=n} \text{Im}(X_p \otimes Y_q \to X \otimes Y). $$

This tensor structure induces a tensor structure on $\text{CF}(k)$.

The functor $\# : \text{CF}(k) \to \text{C}(k)$ forgetting the filtration preserves the tensor structure.

4.1.2. There is an obvious functor

$$ \tau : \text{C}(k) \to \text{CF}(k) $$
given by $\tau(X)_{-1} = 0$; $\tau(X)_n = X$, for $n \geq 0$. The functor $\tau$ preserves the tensor structure. Thus $\tau$ induces a functor $\tau : \text{Op}(\text{C}(k)) \to \text{Op}(\text{CF}(k))$. For an operad $O \in \text{Op}(\text{C}(k))$ we denote by $\text{Alg}_f(O)$ (instead of $\text{Alg}(\tau(O))$) the category of filtered $O$-algebras.

4.1.3. Example. We write $\text{dgf}(k)$ instead of $\text{Alg}(\text{LIE})$ for the category of filtered dg Lie algebras. Explicitly, such an algebra is a filtered complex $g = \{g_i\}$ with a Lie bracket satisfying $[g_i, g_j] \subseteq g_{i+j}$. Similarly, we write $\text{dgcf}$ for the category of filtered cocommutative coalgebras. Its objects are filtered complexes $X = \{X_i\}$ endowed with a cocommutative comultiplication satisfying

$$ \Delta(X_n) \subseteq \sum_{p+q=n} X_p \otimes X_q. $$

4.1.4. Fix $O \in \text{Op}(\text{C}(k))$ and let $A \in \text{Alg}_f(O)$. The filtration on $A$ induces a natural filtration on the enveloping algebra $U(O, A^\#)$ defined as follows. Recall that $U(O, A^\#)$ is a quotient of the “$O$-tensor algebra”

$$ T(O, A^\#) = \bigoplus_{n \geq 0} O(n+1) \otimes_{\Sigma_n} (A^\#)^{\otimes n}. $$
We endow $T(O, A^\#)$ with the tensor product filtration and $U(O, A^\#)$ with the quotient filtration. One easily sees that this defines a filtered associative dg algebra which will be denoted in the sequel by $U(O, A)$.

4.2. **Graded world.** Let $R$ be a graded commutative (not super-commutative!) $k$-algebra and let $R^\#$ be the corresponding commutative algebra with forgotten grading.

Denote by $\text{modg}(R)$ the category of graded $R$-modules. It has an obvious tensor structure with the isomorphism

$$M \otimes_R N \rightarrow N \otimes_R M$$

given by the formula $m \otimes n \mapsto n \otimes n$ (no signs involved).

One has the obvious forgetful functor $\#: \text{modg}(R) \rightarrow \text{mod}(R^\#)$ which also preserves the tensor structure.

Denote $CG(R) = C(\text{modg}(R))$. The forgetful functor defines a tensor functor

$$\#: CG(R) \rightarrow C(R^\#).$$

4.2.1. Tensoring by $R$ defines a tensor functor

$$\tau: \text{mod}(k) \rightarrow \text{modg}(R).$$

This allows one, for any operad $O \in \text{Op}(C(k))$, consider the category of $\tau(O)$-algebras. This latter will be denoted $\text{A1gg}(O, R)$ or just $\text{A1gg}(O)$ (this will not lead to a confusion).

The enveloping algebra $U(O, A)$ of $A \in \text{A1gg}(O)$ is defined in a standard way as in [HS1] using the tensor structure on $CG(R)$.

We will need the following graded analog of Lemma 3.3.1.

4.2.2. **Proposition.** Let $\mathfrak{g}$ be a flat graded dg Lie algebra over $R$. Then the natural map

$$U(S, \mathfrak{g}) \rightarrow U(\text{LIE}, \mathfrak{g})$$

is a graded quasi-isomorphism.

*Proof.*** Since the forgetful functor $\#: \text{modg}(R) \rightarrow \text{mod}(R^\#)$ is exact, and since a map $f: X \rightarrow Y$ is a graded quasi-isomorphism iff $f^\#$ is a quasi-isomorphism, the result immediately follows from Lemma 3.3.1. \Box

4.3. **Rees functor.**

*From now on* $k$ is a field of characteristic zero and $R = k[t]$ with $\text{deg}(t) = 1$.

The Rees functor

$$\rho: \text{modf}(k) \longrightarrow \text{modg}(R)$$

is defined by the formula

$$\rho(V) = \sum V_i t^i \subseteq \tau(V) = V \otimes R.$$
4.3.1. **Lemma.** 1. Rees functor preserves the tensor structure.

2. One has $\rho \tau = \tau$ (two different $\tau$, from 4.1 and from 4.2, are involved).

**Proof.** Straightforward. \hfill \qed

4.3.2. **Corollary.** The Rees functor induces a functor

$$\rho : \text{Algf}(O) \to \text{Algg}(O).$$

4.3.3. The Rees functor $\rho$ identifies the category $\text{modf}(k)$ with the full subcategory of $\text{modg}(R)$ consisting of graded torsion-free (=flat) $R$-modules. The functor $\rho$ admits a left adjoint functor $\phi : \text{modg}(R) \to \text{modf}(k)$ defined by the formulas

$$\phi(M) = \lim_{\to} M_n = M/(1 - t)M; \quad \phi(M)_n = \text{Im}(M_n \to \phi(M)).$$

4.3.4. **Proposition.** Let $O \in \text{Op}(C(k))$, $A \in \text{Algf}(O)$. The filtered enveloping algebra can be calculated by the formula

$$U(O, A) = \phi(U(O, \rho(A))).$$

**Proof.** The total space of $\phi(U(O, \rho(A)))$ is equal to

$$U(O, \rho(A)) \otimes_R R/(1 - t)R = U(O, \rho(A)) \otimes_R R/(1 - t)R = U(O, A).$$

To identify the filtration, recall that $U(O, \rho(A))_n$ is the image of the $n$-th component of the tensor algebra $T(O, \rho(A))$ which is an image of

$$\bigoplus_{i_1 + \ldots + i_k = n} O(k + 1) \otimes A_{i_1} \otimes \ldots \otimes A_{i_k}.$$

But this coincides with the definition of the filtration on $U(O, A)$ as in 4.1.4. \hfill \qed

4.3.5. **Corollary.** Let $O = \text{LIE}$ or $\text{S}$. Then for any $A \in \text{Algf}(O)$ one has

$$\rho(U(O, A)) = U(O, \rho(A)).$$

**Proof.** Having in mind Proposition 4.3.4, it is enough to check that $U(O, \rho(A))$ is torsion-free for $O = \text{LIE}$ or $\text{S}$.

If $O = \text{LIE}$ the claim follows from the PBW theorem. In the second case $O = \text{S}$ is semi-free, so that the corresponding enveloping algebra is a tensor algebra which has no torsion. \hfill \qed
4.3.6. Note the following nice (surely well-known) generalization of the PBW.

**Corollary.** Let \( \mathfrak{g} \) be a filtered Lie algebra. Then the associated graded of the filtered enveloping algebra \( \tilde{U}(\mathfrak{g}) \) is isomorphic to the enveloping algebra of the associated graded Lie algebra.

*Proof.* The passage to associated graded module is the composition of the Rees functor with the base change \( R \rightarrow R/(t) \).

4.3.7. Comparing Corollary 4.3.5 with Proposition 4.2.2 we get the following filtered version of 3.3.1.

**Proposition.** Let \( \mathfrak{g} \) be a filtered dg Lie algebra over \( k \). The natural map
\[
U(S, \mathfrak{g}) \rightarrow U(\text{LIE}, \mathfrak{g})
\]
is a filtered quasi-isomorphism.

4.3.8. Note also the following filtered version of PBW.

**Lemma.** Let \( \mathfrak{g} \) be a filtered dg Lie algebra over \( k \). The symmetrization map \( S(\mathfrak{g}) \rightarrow U(\text{LIE}, \mathfrak{g}) \) is a filtered isomorphism.

*Proof.* Use the usual PBW for the dg \( R \)-Lie algebra \( \rho(\mathfrak{g}) \).

4.4. **A filtered version of Proposition 3.3.2.** Let \( \mathfrak{g} \) be a filtered dg Lie algebra. The coalgebra \( C(\mathfrak{g}) \) endowed with the induced filtration is a filtered unital coalgebra. *Note* that filtered unital coalgebras admit two filtrations, the one being the given filtration, and the second being defined by the unit. In the same way the functor \( \mathcal{L} \) sends filtered unital coalgebras to filtered Lie algebras.

4.4.1. **Definition.** 1. A unital filtered coalgebra \( X = \{X_i\} \in \text{dgcf}(k) \) is called admissible (or, in other words, \( \{X_i\} \) is an admissible filtration on \( X \)) if \( X_{-1} = 0 \) and \( X_0 = k \cdot 1 \).

2. A filtered dg Lie algebra \( \mathfrak{g} = \{\mathfrak{g}_i\} \in \text{dglf}(k) \) is admissible if \( \mathfrak{g}_0 = 0 \).

4.4.2. **Note.** If \( X \) is an admissible coalgebra, one has
\[
\Delta(x) - 1 \otimes x - x \otimes 1 \in X_n
\]
whenever \( x \in X_{n+1} \). This follows from the formula
\[
(1 \otimes \epsilon)\Delta = (\epsilon \otimes 1)\Delta = \text{id}.
\]

The category of admissible filtered coalgebras is denoted by \( \text{dgca}(k) \), and the category of admissible dg Lie algebras is \( \text{dgla}(k) \).
4.4.3. **Proposition.** 1. The functors $L$ and $C$ define a pair of adjoint functors
\[ L : \mathsf{dgca}(k) \rightleftarrows \mathsf{dgl}(k). \]

2. The functor $C$ preserves filtered quasi-isomorphisms.

3. The adjunction maps $i_X : X \to \mathcal{C}(X)$ and $p_g : \mathcal{C}(g) \to g$ are filtered quasi-isomorphisms.

**Proof.** For $X \in \mathsf{dgca}(k)$ and $g \in \mathsf{dgl}(k)$ the sets $\text{Hom}(L(X), g)$ and $\text{Hom}(X, \mathcal{C}(g))$ coincide with the collection of filtration preserving twisted cocycles from $X$ to $g$. This proves the first claim.

Now we can proceed as in the proof of 3.3.2.

**Step 1.** Let $g \in \mathsf{dgl}(k)$. To check that the adjunction map $p_g : \mathcal{C}(g) \to g$ is a filtered quasi-isomorphism, note that $U(S, g) = U(\text{LIE, } \mathcal{C}(g))$ as filtered algebras so that Proposition 4.3.7 together with 4.3.8 give what we need.

**Step 2.** Exactly as in the proof of 3.3.2, $C$ preserves filtered quasi-isomorphisms since the coalgebras $\mathcal{C}(g)$ admits an increasing filtration natural in $g$ with the associated graded pieces $S^n(g[1])$.

**Step 3.** Consider now the diagram of the Step 3 of the proof of 3.3.2. Since all the maps involved preserve the filtrations, and since the map $p_{L(X)}$ is a filtered quasi-isomorphism by Step 1, the map $\mathcal{L}(i_X)$ and, therefore, its retract $i_X$, are also filtered quasi-isomorphisms.

This proves the Proposition.

Filtered quasi-isomorphisms of admissible coalgebras are useful because of the following

4.4.4. **Proposition.** Let $f : X \to Y$ be a filtered quasi-isomorphism of admissible coalgebras. Then $\mathcal{L}(f)$ is a quasi-isomorphism.

**Proof.** Since the functor $\mathcal{L}$ commutes with colimits and passing to cohomology commutes with filtered colimits, the claim can be proven by induction. Suppose, by the inductive hypothesis, that the map $\mathcal{L}(f_n) : \mathcal{L}(X_n) \to \mathcal{L}(Y_n)$ is a quasi-isomorphism. Denote $M = X_{n+1}/X_n$ and $N = Y_{n+1}/Y_n$. Choosing compatible splittings for the short exact sequences $X_n \to X_{n+1} \to M$ and $Y_n \to Y_{n+1} \to N$, we get compatible maps $\alpha : M[-2] \to \mathcal{L}(X_n)$ and $\beta : N[-2] \to \mathcal{L}(Y_n)$. Since $M$ and $N$ are quasi-isomorphic, and $\mathcal{L}(X_{n+1}) = \mathcal{L}(X_n)(M, \alpha)$, $\mathcal{L}(Y_{n+1}) = \mathcal{L}(Y_n)(N, \beta)$ (notation of [H2], Sect. 1), Proposition follows from the following lemma in the spirit of [H2], 5.3.3.

4.4.5. **Lemma.** Let $O$ be an operad in $C(k)$. Suppose a commutative square
\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & X \\
g \downarrow & & \downarrow f \\
N & \xrightarrow{\beta} & Y
\end{array}
\]
is given so that \( f : X \to Y \) is a quasi-isomorphism of cofibrant \( O \)-algebras and \( g : M \to N \) is a quasi-isomorphism of complexes. Then the induced map of \( O \)-algebras \( X(M, \alpha) \to Y(N, \beta) \) is a quasi-isomorphism.

**Proof.** One easily reduces the claim to the case \( M = N = k \cdot e \) is generated by an only element \( e \). Since \( X \) and \( Y \) are cofibrant, \( f \) is homotopy equivalence. This means that there exists a map \( g : Y \to X \) homotopically inverse to \( f \). Let \( x = \alpha(e), y = \beta(e) = f(x) \). The difference \( x - g(y) \) is obviously a boundary in the complex \( X \); write it as \( x - g(y) = du, u \in X \).

Choose a path diagram \( X \xrightarrow{i} X^I \xrightarrow{\Phi} X \) so that \( X^I = X \coprod F(V) \) with a contractible \( V \) and a map \( \Phi : X \to X^I \) which realizes a homotopy between \( \text{id}_X \) and \( gf : X \to X \). The map
\[
i' : X\langle e; de = x \rangle \to X^I\langle e; de = i(x) \rangle
\]
is a quasi-isomorphism since \( X^I = X \coprod F(V) \). Since \( i(x) \) and \( \Phi(x) \) represent one and the same cohomology class in \( X^I \), there is an obvious isomorphism between \( X^I\langle e; de = i(x) \rangle \) and \( X^I\langle e; de = \Phi(x) \rangle \). This implies that the arrows \( p_1' \) in the diagram below are quasi-isomorphisms.

\[
\begin{array}{ccc}
X\langle e; de = x \rangle & \xrightarrow{\Phi'} & X^I\langle e; de = \Phi(x) \rangle \\
\downarrow & & \downarrow \\
X\langle e; de = x + du \rangle & \xrightarrow{p_1'} & X\langle e; de = x \rangle
\end{array}
\]

Thus, the map \( \Phi' \), and therefore the composition
\[
p_1'\Phi' : X\langle e; de = x \rangle \to X\langle e; de = x + du \rangle
\]
induced by the map \( gf : X \to X \) is a quasi-isomorphism. The same is of course true for the morphism \( fg : Y \to Y \). This implies that the map
\[
f' : X\langle e; e = x \rangle \to Y\langle e; e = y \rangle
\]
is a quasi-isomorphism. \( \square \)

5. **Proofs of the theorems**

5.1. The proof of Theorem 3.1 is based on the following Key Lemma whose proof we postpone until Section 6.

5.1.1. **Key Lemma.** Given \( X \in \text{dgcu}(k) \), let \( f : g \to \mathcal{L}(X) \) be a surjective map in \( \text{dglie}(k) \). Consider the cartesian diagram
\[
\begin{array}{ccc}
Z & \xrightarrow{j} & \mathcal{C}(g) \\
\downarrow & & \downarrow \mathcal{C}(f) \\
X & \xrightarrow{i_X} & \mathcal{C}\mathcal{L}(X)
\end{array}
\]
in \( \text{dgcu}(k) \). Then \( \mathcal{L}(j) : \mathcal{L}(Z) \to \mathcal{L}(g) \) is a quasi-isomorphism.
The lemma is very close to [Q2], II.5.6. Its proof in our context uses the technicalities of Section 4.

5.2. **Proof of Theorem 3.1.** To prove Theorem 3.1 we use Lemma 5.1.1 and follow the proof of Theorem II.5.2 of [Q2].

5.2.1. **Limits and colimits in \( \text{dgcu}(k) \).** The functor \( \# : \text{dgcu}(k) \to C(k) \) defined by the formula \( \#(X) = X \), commutes with colimits. This gives a obvious construction of arbitrary colimits in \( \text{dgcu}(k) \).

Finite products in \( \text{dgcu}(k) \) correspond to tensor products of the underlying complexes; also kernels of a pair of maps in \( \text{dgcu}(k) \) are the same as in \( C(k) \). This proves the property CM1 — see [Q2], II.1.

5.2.2. **Cofibrations in \( \text{dgcu}(k) \).**

**Lemma.** Let \( f : X \to Y \) be a cofibration (resp., an acyclic cofibration) in \( \text{dgcu}(k) \). Then \( \mathcal{L}(f) \) is a cofibration (resp., an acyclic cofibration) in \( \text{dglie}(k) \).

**Proof.** Let \( f : X \to Y \) be injective, \( \{Y_i\} \) be the canonical filtration of \( Y \), \( Z_i = f(X) + Y_i \subseteq Y \) are subcoalgebras in \( Y \). Since \( Y = \lim Z_i \) and \( \mathcal{L} \) commutes with colimits, it is enough to check that \( \mathcal{L}(Z_i) \to \mathcal{L}(Z_{i+1}) \) is a cofibration. Actually, since \( Z_i/Z_{i+1} \) is primitive, the Lie algebra \( \mathcal{L}(Z_{i+1}) \) is obtained from \( \mathcal{L}(Z_i) \) by “joining variables to kill cycles” procedure.

The claim about acyclic cofibrations follows from the above and from the definition of weak equivalences in \( \text{dgcu}(k) \).

5.2.3. **Fibrations in \( \text{dgcu}(k) \).**

**Lemma.** Let \( f : \mathfrak{g} \to \mathfrak{h} \) be a surjective map (resp., a surjective quasi-isomorphism) in \( \text{dglie}(k) \). Then \( \mathcal{C}(f) \) is a fibration (resp., an acyclic fibration) in \( \text{dgcu}(k) \).

**Proof.** If \( f \) is surjective, \( \mathcal{C}(f) \) is a fibration by Lemma 5.2.2 and the adjointness of \( \mathcal{L} \) and \( \mathcal{C} \). If, moreover, \( f \) is a quasi-isomorphism, \( \mathcal{C}(f) \) is a weak equivalence by Proposition 3.3.2(2).

5.2.4. The properties (CM2), (CM3) are obvious. Also the lifting property (CM4)(ii) is valid by definition of fibrations in \( \text{dgcu}(k) \).

(CM5)(i). Given a map \( f : X \to Y \) in \( \text{dgcu}(k) \) let \( \mathcal{L}(f) = pi \) be a decomposition of \( \mathcal{L}(f) \) with a cofibration \( i : \mathcal{L}(X) \to \mathfrak{g} \) and an acyclic fibration \( p : \mathfrak{g} \to \mathcal{L}(Y) \).

Let \( Z = Y \times_{\mathcal{L}(Y)} \mathcal{C}(\mathfrak{g}) \). According to Lemma 5.1.1 and Proposition 3.3.2, the map \( Z \to Y \) is a weak equivalence. It is also a fibration since it is obtained by a base-change from the fibration \( \mathcal{C}(p) \) in \( \text{dgcu}(k) \). Now, the induced map \( X \to Z \) being obviously injective, we get (CM5)(i).

(CM4)(i). If \( f : X \to Y \) is an acyclic fibration, the already proven property (CM5)(i) gives a decomposition \( f = qj \) where \( j \) is an acyclic cofibration and \( q \) is obtained by a base-change from a map \( \mathcal{C}(p) \) with \( p \) an acyclic fibration in \( \text{dglie}(k) \). Adjointness of \( \mathcal{L} \) and \( \mathcal{C} \) immediately gives that \( \mathcal{C}(p) \) has a RLP with respect to all cofibrations. Therefore \( q \) admits the same property.
Since \( j \) admits a LLP with respect to \( f \), we get that \( f \) is a retract of \( q \) and therefore, it also satisfies RLP with respect to all cofibrations.

(CM5)(ii). Let \( f : X \to Y \) and let \( \mathcal{L}(f) = p \) be a decomposition with a fibration \( p : \mathfrak{g} \to \mathcal{L}(Y) \) and an acyclic cofibraton \( i : \mathcal{L}(X) \to \mathfrak{g} \). According to Lemma 5.1.1, the map \( j : Z \to C(\mathfrak{g}) \) is a weak equivalence and the map \( Z \to Y \) is a fibration where \( Z = Y \times_{C(\mathcal{L}(Y))} C(\mathfrak{g}) \). This immediately implies that the map \( X \to Z \) is an acyclic cofibration.

Therefore, we proved \( \mathsf{dgcu}(k) \) admits a CMC structure. The simplicial structure on \( \mathsf{dgcu}(k) \) and the proof of the axiom (SM7) will be provided in Section 7.

5.3. **Proof of Theorem 3.2.** Now Theorem 3.2 follows immediately from the general Theorem II.1.4 of \( \mathbb{Q}^2 \) and from Proposition 3.3.2.

6. **Proof of the Key Lemma 5.1.1**

In this Section we prove the Key Lemma 5.1.1.

Endow \( X \) with the canonical filtration and \( \mathcal{L}(X) \) with the induced filtration. Let \( \alpha = \ker(f : \mathfrak{g} \to \mathcal{L}(X)) \). Define an admissible filtration on \( \mathfrak{g} \) by setting \( \mathfrak{g}_n = f^{-1}(\mathcal{L}(X)_n) \) for \( n > 0 \). This induces admissible filtrations on \( C(\mathfrak{g}) \) and on \( C(\mathcal{L}(X)) \).

According to Proposition 4.4.3, \( i_X : X \to \mathcal{L}(X) \) is a filtered quasi-isomorphism. Define finally a filtration on \( Z \) by the formula \( Z_n = j^{-1}(C(\mathfrak{g})_n) \).

According to Proposition 4.4.4, it is enough to check that \( j : Z \to C(\mathfrak{g}) \) is a filtered quasi-isomorphism.

Let us describe more explicitly the filtrations involved. Forget about the differentials. Choose a graded Lie algebra splitting \( s : \mathcal{L}(X) \to \mathfrak{g} \) of \( f \). This defines isomorphisms (not preserving the differentials)

\[
C(\mathfrak{g}) \cong C(\mathcal{L}(X) \otimes C(\alpha))
\]

and

\[
Z \cong X \otimes C(\alpha)
\]

of filtered coalgebras, the filtration on \( C(\mathfrak{g}) \) being the standard one. In fact, the first isomorphism obviously preserves filtrations, and the second one preserves the filtrations because of the equality \( X_n = i_X^{-1}(\mathcal{L}(X)_n) \).

The isomorphism (8) can be rewritten as

\[
Z_n \cong \bigoplus_r X_{n-r} \otimes S^r(\alpha[1])
\]

and similarly

\[
C(\mathfrak{g})_n \cong \bigoplus_r \mathcal{L}(X)_{n-r} \otimes S^r(\alpha[1]).
\]

Unfortunately, the isomorphisms (8), (10) are not compatible with the differentials. To overcome this minor difficulty, we define a double filtration on the complexes involved so that the associated graded guys will be already isomorphic as complexes. We will write formulas only for the
filtration on $Z$ and on $Z_n$, the formulas for $C(\mathfrak{g})$ being obtained by substitution of $X_n$ with $CL(X)_n$. Here are the formulas.

$$F_p^q = \bigoplus_{r \geq q} X_p \otimes S^r(a[1])$$  \hspace{1cm} (11)

$$F_p^q(Z_n) = F_p^q \cap Z_n = \bigoplus_{n \geq r \geq q} X_{\min(p,n-r)} \otimes S^r(a[1])$$  \hspace{1cm} (12)

The filtrations are increasing on $p$ and decreasing on $q$. The filtration (12) is finite. Its $(p, q)$-graded piece vanishes for $p + q > n$ and is otherwise isomorphic to $X_p/X_{p+1} \otimes S^q(a[1])$ as a complex.

Associated graded pieces of the corresponding filtration for $C(\mathfrak{g})$ have form $CL(X)_p/CL(X)_{p+1} \otimes S^q(a[1])$.

The $(p, q)$-graded piece of the map $j_n : Z_n \to C(\mathfrak{g})_n$ takes form

$$X_p/X_{p+1} \otimes S^q(a[1]) \longrightarrow CL(X)_p/CL(X)_{p+1} \otimes S^q(a[1])$$

which obviously a quasi-isomorphism by Proposition 4.4.3.

Key Lemma is proven.

7. Simplicial structure on $\text{dgcu}(k)$

In this Section we define a simplicial structure on the category $\text{dgcu}(k)$ of dg unital coalgebras over a field $k$ of characteristic zero and check the axiom (SM7) — see Introduction.

7.1. Functional spaces for unital coalgebras. Recall (cf. [BG]) that the functor of polynomial differential forms

$$\Omega : \Delta^0\text{Ens} \to \text{dga}(k)$$  \hspace{1cm} (13)

is the one defined uniquely by its values on the standard simplices

$$\Omega(\Delta^n) = \Omega_n = k[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/(\sum t_i - 1, \sum dt_i)$$

and by the property that $\Omega$ commutes with colimits.

7.1.1. For any commutative dg algebra $\Omega \in \text{dga}(k)$ tensoring by $\Omega$ defines a functor

$$\Omega \otimes - : \text{dgcu}(k) \to \text{dgcu}(\Omega).$$  \hspace{1cm} (14)

Therefore, the following definition makes sense.
7.1.2. **Definition.** Let $X, Y \in \text{dgcu}(k)$. The simplicial set $\text{Hom}(X,Y)$ is defined by the formula

$$\text{Hom}(X,Y)_n = \text{Hom}_{\text{dgcu}(\Omega_n)}(\Omega_n \otimes X, \Omega_n \otimes Y),$$

the faces and the degeneracies being defined in an obvious way.

Note the following

7.1.3. **Lemma.** The functor (14) commutes with colimits and with finite limits

*Proof.* The claim about colimits is obvious. In order to prove that (14) commutes with finite limits we check separately the case of a product of two coalgebras and that of kernel of a couple of maps. This follows from the description of limits in 5.2.1. □

Lemma 7.1.3 immediately implies the following

7.1.4. **Corollary.** 1. The functor $\text{Hom}(X,-) : \text{dgcu}(k) \to \Delta^0\text{Ens}$ commutes with finite limits. 2. The functor $\text{Hom}(-, Y) : \text{dgcu}(k)^{\text{op}} \to \Delta^0\text{Ens}$ carries arbitrary colimits to limits.

One has the following standard fact.

7.1.5. **Lemma.** (see [BC], Lemma 5.2, [H2], 4.8.3) There is a natural in $S \in \Delta^0\text{Ens}$ morphism

$$\Phi(S) : \text{Hom}_{\text{dgcu}(\Omega(S))}(\Omega(S) \otimes X, \Omega(S) \otimes Y) \sim \text{Hom}(S, \text{Hom}(X,Y))$$

which is a bijection provided $S$ is finite.

*Proof.* The proof of [H2], 4.8.3 is applicable here. □

7.1.6. **Lemma.** The adjoint functors $C$ and $L$ induce an isomorphism

$$\text{Hom}(X, C(g)) \sim \text{Hom}(L(X), g)$$

of simplicial sets for every $X \in \text{dgcu}(k), g \in \text{dg lie}(k)$.

*Proof.* Repeats the standard argument of Theorem 2.2.5 substituting the base category $C(k)$ with $\text{mod}(\Omega_n)$. □

7.2. **Property (SM7).**

7.2.1. **Proposition.** Let $i : A \to B$ be a cofibration and $p : X \to Y$ be a fibration in $\text{dgcu}(k)$. Then the map of simplicial sets

$$\pi(i,p) : \text{Hom}(B,X) \to \text{Hom}(A,X) \times_{\text{Hom}(A,Y)} \text{Hom}(B,Y)$$

is a Kan fibration. If, moreover, either $i$ or $p$ is a weak equivalence, then $\pi(i,p)$ is an acyclic Kan fibration.
In what follows we will say that a pair of maps \((i : A \to B, p : X \to Y)\) satisfies (SM7) if the map \(\pi(i, p)\) from (15) is a Kan fibration, acyclic if one of \((i, p)\) is a weak equivalence. To prove Proposition, we will show step by step that any pair \((i : A \to B, p : X \to Y)\) such that \(i\) is a cofibration and \(p\) is a fibration, satisfies (SM7).

**Step 1.** Suppose that \(p = C(f)\) where \(f : g \to h\) is a surjective map of dg Lie algebras. Then any pair \((i, p)\) satisfies (SM7) by 7.1.6, 5.2.2 and [H2], 4.8.4.

**Step 2.** Suppose that a pair \((i, p)\) satisfies (SM7) and let a map \(q : Z \to T\) be obtained from \(p : X \to Y\) by a base change \(a : T \to Y\).

Using Corollary 7.1.4, we easily see that \(\pi(i, q)\) is obtained by a base change from \(\pi(i, p)\). Therefore the pair \((i, q)\) also satisfies (SM7).

**Step 3.** Suppose that a pair \((i, p)\) satisfies (SM7) and let a map \(q : Z \to T\) be a retract of \(p : X \to Y\). Then the map \(\pi(i, q)\) is a retract of \(\pi(i, p)\) and therefore, the pair \((i, q)\) also satisfies (SM7).

Now Proposition follows from the following lemma.

**Lemma.** 1. Any fibration in \(\text{dgcu}(k)\) can be obtained, using the operations of retraction and base change, from a map \(L(f)\) where \(f\) is a surjective map of dg Lie algebras.

2. Any acyclic fibration in \(\text{dgcu}(k)\) can be obtained, using the operations of retraction and base change, from a map \(L(f)\) where \(f\) is a surjective quasi-isomorphism of dg Lie algebras.

**Proof.** 1. Let \(f : X \to Y\) be a fibration. Using the maps \(i : Y \to CL(Y)\) and \(CL(f) : CL(X) \to CL(Y)\), define \(Z = Y \times_{CL(Y)} CL(X)\) and let \(j : Z \to CL(X), k : X \to Z\) and \(g : Z \to Y\) be the obviously defined maps. According to the Key Lemma, \(j\) is an acyclic cofibration, and therefore, \(k\) is an acyclic cofibration as well. Since \(f\) is a fibration, the map \(k\) splits over \(g\) and this gives a presentation of \(f\) as a retract of \(g\) which is obtained by a base change from \(C(L(f))\).

2. If, moreover, \(f\) is an acyclic fibration, then \(C(L(f))\) is a surjective quasi-isomorphism, and it is nothing to prove.

8. The nerve of a dg Lie algebra

8.1. The nerve and its properties. Let \(X \in \text{dgcu}(k)\). Choose a fibrant resolution \(X \to F\) and define a functor

\[
\tilde{X} : \text{dgart}^{\leq 0}(k) \to \Delta^0\text{Ens}
\]

by the formula

\[
\tilde{X}(A) = \text{Han}(A^*, F)
\]

where \(A^*\) is the unital coalgebra with the unit \(A \to k\). The resulting functor \(\tilde{X}\) does not depend, up to a homotopy, on the choice of the resolution. One can get a specific representative for \(\tilde{X}\) as follows.

Let \(g = L(X)\) be the tangent Lie algebra of \(X\). Choose \(L(g)\) to be a fibrant resolution of \(X\). This allows one to easily express the functor \(\tilde{X}\) through the tangent Lie algebra \(g\).
8.1.1. **Definition.** Let $g \in \text{dglie}(k)$. The nerve of $g$ is the functor

$$\Sigma_g : \text{dgart}^{\leq 0}(k) \to \Delta^0\text{Ens}$$

defined by the formula

$$\Sigma_g((A, m))_n = \text{MC}(m \otimes \Omega_n \otimes g).$$

One has immediately the following

8.1.2. **Proposition.** For $X \in \text{dgcu}(k)$ the functor $\tilde{X} : \text{dgart}^{\leq 0}(k) \to \Delta^0\text{Ens}$ is homotopy equivalent to the nerve $\Sigma_{\mathcal{L}(X)}$.

**Proof.** According Lemma 7.1.6 one has for any $g \in \text{dglie}(k)$, $A \in \text{dgart}^{\leq 0}(k)$

$$\Sigma_g(A)_n = \text{MC}(\Omega_n \otimes m \otimes g) = \text{Hom}_{\text{dgcu}(\Omega_n)}(\Omega_n \otimes A^*, \Omega_n \otimes C(g)) = \text{Ham}_n(A^*, C(g)).$$

(16)

8.1.3. **Proposition.**

1. A quasi-isomorphism $f : g \to h$ of dg Lie algebras induces a homotopy equivalence for every $A \in \text{dgart}^{\leq 0}(k)$

$$\Sigma_f : \Sigma_g(A) \to \Sigma_h(A).$$

2. For each $g \in \text{dglie}(k)$ the functor $\Sigma_g$ carries quasi-isomorphisms in $\text{dgart}^{\leq 0}(k)$ to homotopy equivalences.

3. For each $g \in \text{dglie}(k)$ the functor $\Sigma_g$ carries surjective maps to Kan fibration. In particular, $\Sigma_g(A)$ is always a Kan simplicial set.

4. $\Sigma_g$ commutes with finite projective limits.

**Proof.** The claims 1,3,4 follow from Theorem 9.5.1 and Lemma 7.1.6. By Proposition 3.3.4 (3), a quasi-isomorphism $f : A \to B$ in $\text{dgart}^{\leq 0}(k)$ defines a weak equivalence $f^* : B^* \to A^*$ in $\text{dgcu}(k)$. Then the induced map $\Sigma_g(f)$ is a weak equivalence of Kan simplicial sets, hence a homotopy equivalence.

8.2. **Some calculations.** Here we provide some explicit calculations which help to better understand how the nerve of a dg Lie algebra looks like. We will use them below in 9.7.

In this subsection $g$ is a nilpotent dg Lie algebra (it substitutes $m \otimes g$ from 8.1.1), and we denote by $\Sigma(g)$ the simplicial set

$$\Sigma(g)_n = \text{MC}(g_n)$$

where the simplicial dg Lie algebra $g_* = \{g_n\}$ is defined by the formula $g_n = \Omega_n \otimes g$. 

8.2.1. **Deligne groupoid.** Recall (cf. [GM1]) that for a nilpotent dg Lie algebra \( g \) one defines *Deligne groupoid* \( \Gamma(g) \) as follows.

The Lie algebra \( g^0 \) acts on \( \text{MC}(g) \) by vector fields:
\[
\rho(y)(x) = dy + [x, y]
\]
for \( y \in g^0, x \in g^1 \).

This defines the action of the nilpotent group \( G = \exp(g^0) \) on the set \( \text{MC}(g) \). Then the groupoid \( \Gamma(g) \) is defined by the formulas
\[
\text{Ob } \Gamma = \text{MC}(g), \\
\text{Hom}_\Gamma(x, x') = \{ g \in G | x' = g(x) \}.
\]

8.2.2. **Maurer-Cartan elements of \( g^{(1)} \).** Let us explicitly describe the set \( \text{MC}(g^{(1)}) \). Since \( g^{(1)} = k[t, dt] \otimes g \) we will iterate the calculation to get the description of \( \text{MC}(g^{(n)}) \).

Write an element \( z \in g^{(1)} \) in the form
\[
z = x + dt \cdot y
\]
with \( x \in g^1[t], y \in g^0[t] \). Then the Maurer-Cartan equation is easily seen to be equivalent to the differential equation
\[
\frac{dx(t)}{dt} = dy(t) + [x(t), y(t)] \\
x(0) = x_0
\]
where \( x_0 \) is an element of \( \text{MC}(g) \).

An element \( y \in g[t] \) defines a unique polynomial path \( g(t) \) in the Lie group \( G = \exp(g^0) \) satisfying the differential equation \( \dot{g}(t) = g(t)(y(t)) \) with the initial condition \( g(0) = 1 \).

Let \( \mathfrak{t} \) be the Lie subalgebra \( t g^0[t] \subseteq g^{(1)} \) and let \( K = \exp(\mathfrak{t}) \). The above consideration proves the following

8.2.3. **Lemma.** An element \( x \) of \( \text{MC}(g^{(1)}) \) can be uniquely represented in the form
\[
x = g(x_0)
\]
where \( x_0 \in \text{MC}(g) \subseteq \text{MC}(g^{(1)}) \), \( g \in K \subseteq \exp(g^{(1)}) \) and the action is defined as in 8.2.1 — for the nilpotent Lie algebra \( g^{(1)} \).

8.2.4. Iteratively using Lemma 8.2.3 we can describe the set of Maurer-Cartan elements of \( g^{(n)} \) as follows.

Let \( \mathfrak{t}_i = t_i g^0_{i-1}[t_i] \) for \( i > 0 \) be the Lie subalgebra of \( g^{(1)} \). Denote \( K_i = \exp(\mathfrak{t}_i) \). These are subgroups of \( \exp(g^{(1)}) \).

**Lemma.** Let \( 0 \leq i \leq j \leq n \). Then \( K_i \) normalizes \( K_j \).

**Proof.** This immediately follows from the inclusion
\[
[\mathfrak{t}_i, \mathfrak{t}_j] \subseteq \mathfrak{t}_j.
\]
Define $G_n = \exp(g_0^n)$ and let $T_n = K_n \cdot K_{n-1} \cdots K_1$ be the subgroup in $G_n$. The lemma above implies that this group is the exponent of the Lie algebra $\oplus_{i \geq 1} t_i$. Then one has the following

8.2.5. **Proposition.** Any element of $\text{MC}(g(n))$ can be uniquely presented as $g(x_0)$ where $x_0 \in \text{MC}(g)$ and $g \in T_n$.

Note that the simplicial group $G_\bullet = \{G_n\}$ acts on the nerve $\Sigma(g)$. Proposition 8.2.3 implies that the restriction

$$G_\bullet \times \text{MC}(g) \to \Sigma(g)$$

(17)

is surjective. One has the following stronger

8.2.6. **Corollary.** The map (17) admits a pseudo-cross section (see [M], §18).

**Proof.** The pseudo-cross section is given as the composition

$$\Sigma_n(g) \sim \to T_n \times \text{MC}(g) \to G_n \times \text{MC}(g).$$

This map obviously commutes with the faces $d_i$ for $i > 0$ and with all degeneracies.\[\square\]

8.2.7. **Note.** Using the explicit description of $\Sigma(g)$ above, one can easily get the property (3) of 8.1.3 independently on Theorems 3.1, 3.2.

9. Remarks and applications

In this Section we provide some examples, definitions, calculations and remarks.

9.1. **Homology of Lie algebras.** Let us give another description of the homology functor $\# \circ C : \text{dglie}(k) \to C(k)$ where the functor $\# : \text{dgcu}(k) \to C(k)$ is given by the formula

$$\#(C) = C.$$

9.1.1. **Proposition.** One has $\# \circ C = \text{LAb}$ where $\text{LAb} : \text{dglie}(k) \to C(k)$ is defined as

$$\text{LAb}(g) = g/[g,g].$$

**Proof.** It suffices to check that the map of complexes $\overline{C}(g) \to g/[g,g]$ is a quasi-isomorphism provided $g$ is standard cofibrant. Consider $C(g)$ as a bicomplex so that the horizontal differential $d'$ is defined by the Lie bracket in $g$ and the vertical differential $d''$ is induced by the differential in $g$. Forget for a moment the differential in $g$. Then

$$g = F(V) = \oplus F^n(V)$$

(18)

is a free Lie algebra over a graded vector space $V$. The differential $d'$ preserves the grading which comes from the presentation (18). Since $H(C,d') = V$ and each homogeneous component of $C$ is finite, the proposition follows.\[\square\]

The above description allows one to construct easily an example of a acyclic coalgebra $X$ such that $\mathcal{L}(X)$ has non-trivial cohomology (another example is given in Kontsevich’s lectures [KC]).
9.1.2. Example. Let $\mathfrak{g}$ be the cofibrant Lie algebra having generators $e, f, h$ of degree 0, $x, y, z$ of degree $-1$ with the differential given by
\[
de = dh = df = 0;\ dx = [h, e] - 2e;\ dy = [h, f] + 2f;\ dz = [e, f] - h.
\]
According to Proposition 9.1.1, $C(\mathfrak{g})$ is acyclic though $\mathfrak{g}$ is not (one has $H^0(\mathfrak{g}) = \mathfrak{sl}_2(\mathbb{k})$). One can equally set $X = C(\mathfrak{g})$ and get a non-contractible $\mathcal{L}(X)$. This counter-example means that there are quasi-isomorphisms in $dgcu(\mathbb{k})$ which are not weak equivalences.

9.2. Infinitesimals. Look at the first infinitesimal deformations corresponding to a dg Lie algebra $\mathfrak{g}$.

9.2.1. Dual numbers. For each $n = 0, 1, \ldots$ define
\[
A_n = k[\varepsilon; \deg \varepsilon = -n]/(\varepsilon^2) \in dgart_{\leq 0}(\mathbb{k}).
\]
This is a $k$-vector space object in the category $dgart_{\leq 0}(\mathbb{k})$.

Let us calculate the simplicial vector space $\Sigma_{\mathfrak{g}}(A_n)$. Its $i$-simplices are the Maurer-Cartan elements of $\varepsilon \cdot \Omega_i \otimes \mathfrak{g}$ which is a dg Lie algebra with zero multiplication. Therefore,
\[
\Sigma_{\mathfrak{g}}(A_n)_i = Z^0(\Omega_i \otimes \mathfrak{g}[1+n]).
\]
Using the Dold-Puppe equivalence of categories and the fact that the cosimplicial complex $\{\Omega_i\}_{i \in \mathbb{N}}$ is homotopy equivalent to the cosimplicial complex of cochains $\{C^\ast(\Delta^i)\}_{i \in \mathbb{N}}$, we obtain the following

9.2.2. Claim. $\Sigma_{\mathfrak{g}}(A_n)$ is homotopy equivalent to the simplicial abelian group corresponding to the complex $\tau_{\leq 0}(\mathfrak{g}[1+n])$.

Note the following

9.2.3. Corollary. If a map $f : \mathfrak{g} \to \mathfrak{h}$ in $dglie(\mathbb{k})$ induces an equivalence of the nerve functors, then $f$ is itself a quasi-isomorphism.

9.3. Formal spaces.

9.3.1. Definition. A formal stack $X \in dgcu(\mathbb{k})$ is called a formal space if it is weakly equivalent to a coalgebra $Y \in dgcu(\mathbb{k})$ satisfying $Y^i = 0$ for $i < 0$.

An equivalent condition: $X$ is a formal space if $H^i(\mathcal{L}(X)) = 0$ for $i \leq 0$. According to 9.2.2, this property is equivalent to the one saying that $\Sigma_{\mathfrak{g}}(A_0)$ is discrete. By an obvious artinian induction we get

9.3.2. Lemma. $X \in dgcu(\mathbb{k})$ is a formal space iff for each $A \in art(\mathbb{k})$ the simplicial set $\tilde{X}(A)$ is discrete.
9.3.3. Now the two ideas mentioned in the Introduction about formal deformations in characteristic zero can be formulated as follows.

Any formal deformation problem in characteristic zero can be described by a representable functor

$$F : \text{dgart}^{\leq 0}(k) \to \Delta^0 \text{Ens}. $$

Classical deformation problems are often not representable, since in the classical picture we see only the $\pi_0$ of the genuine deformation functor.

Definition. Let $X \in \text{dgcu}(k)$. The classical part of $\tilde{X}$ is the functor

$$\tilde{X}_{cl} : \text{art}(k) \to \text{Ens}$$

defined by $\tilde{X}_{cl}(A) = \pi_0(\tilde{X}(A))$.

Let $\mathfrak{g} = \mathcal{L}(X)$. Suppose first that $X$ is a formal space. Put $Y = H^0(\mathcal{C}(\mathfrak{g}))$. Then for any $A \in \text{art}(k)$ one has

$$X_{cl}(A) = \pi_0(\tilde{X}(A)) = \tilde{X}(A) = \text{Hom}(A^*, X) = \text{Hom}(A^*, Y).$$

This means that the classical part $\tilde{X}_{cl}$ of a formal space $X$ is representable by the coalgebra $Y$. For a general $X \in \text{dgcu}(k)$ one should not expect representability of the classical part. However, the functor $\tilde{X}_{cl}$ admits a hull in the sense of [Sc]. In fact, choose a complement $V$ in $\mathfrak{g}^1$ to the vector subspace $\text{Im}(d : \mathfrak{g}^0 \to \mathfrak{g}^1)$ and define a 1-truncation $\mathfrak{h}$ of $\mathfrak{g}$ by the formulas

$$\mathfrak{h}^i = \begin{cases} 
0, & i \leq 0 \\
V, & i = 1 \\
\mathfrak{g}^i, & i > 1.
\end{cases}$$

(19)

Put $Y = H^0(\mathcal{C}(\mathfrak{h}))$ and define $h_Y(A) = \text{Hom}(A^*, Y)$.

9.3.4. Lemma. The injection $\mathfrak{h} \to \mathfrak{g}$ induces a smooth morphism of functors $h_Y \to \tilde{X}_{cl}$ which is isomorphism on the tangent spaces.

Proof. This claim essentially belongs to Goldman-Millson [GM1], [GM2] (who considered however only the case of formal spaces). The tangent spaces to $h_Y$ and to $\tilde{X}_{cl}$ are isomorphic to $H^1(\mathfrak{h})$ and to $H^1(\mathfrak{g})$ respectively. To check the smoothness it is enough to prove that for any surjection $f : (B, n) \to (A, m)$ in $\text{art}(k)$ whose kernel $I$ is annihilated by $n$, the map

$$h_Y(B) \to h_Y(A) \times_{\tilde{X}_{cl}(A)} \tilde{X}_{cl}(B)$$

is surjective.

That is, let $x \in \text{MC}(m \otimes V)$ and $y \in \text{MC}(n \otimes \mathfrak{g}^1)$ have the same image in $\tilde{X}_{cl}(A)$. Then, first of all, one can substitute the element $y$ by an equivalent one, so that the images of $x$ and of $y$ in $\text{MC}(m \otimes \mathfrak{g}^1)$ coincide. Then the element $y$ belongs to $n \otimes V$, up to an element from $I \otimes \text{Im}(d : \mathfrak{g}^0 \to \mathfrak{g}^1)$ which can be killed by the action of $I \otimes \mathfrak{g}^0$. After this correction, the element $y$ already belongs to $n \otimes V$ and it automatically satisfies the MC equation. $\square$
Therefore, the coalgebra \( Y \) (or, its dual complete local algebra) is a hull of \( \tilde{X}_{cl} \).

We would like to have a direct proof of the uniqueness of \( Y \). For this it would be enough to prove that \( h \) does not depend, up to a quasi-isomorphism, on the choice of 1-truncation. This is of course so if (as in \([GM2]\)) one supposes that \( H^0(g) = 0 \). Unfortunately, we doubt this is true in general.

However, due to the general claim of \([S]\), the hull \( Y \) is unique up to a non-canonical isomorphism provided \( H^1(g) \) is finite-dimensional.

9.4. Simply connected rational spaces. Let \( S \) be a simply connected rational space. According to \([Q2]\), it has a dg Lie algebra model \( g \) which satisfies \( g^i = 0 \) for \( i \geq -1 \) (we keep using degree +1 differentials). Therefore, \( S \) should define a formal deformation in our general definition. It looks strangely a bit, since “usual” deformations are described by non-negatively graded Lie algebras. The classical part of such a deformation is trivial. However, one can easily calculate the corresponding to \( S \) deformation functor — in terms of dg Lie algebra models.

9.4.1. Proposition. Let \( S \) have a finite \( \mathbb{Q} \)-type. For any \( A \in \text{dgart}^{\leq 0}(\mathbb{Q}) \) the simplicial set \( \Sigma_g(A) \) is simply connected and rational. Its Lie algebra model is given by \( m \otimes g \).

Proof. Put \( h = m \otimes g \). We wish to check that \( h \) is a Lie algebra model for the simplicial set \( \Sigma(h) \). But this is clear: the coalgebra model of \( h \) is \( C(h) \), so the dg algebra model of \( h \) is the dual complex \( C^*(h) \) and the corresponding simplicial set is given according to \([BC]\), Thm. 9.4, by the formula

\[
\Sigma_n(h) = \text{Hom}(C^*(h), \Omega_n) = \text{MC}(\Omega_n \otimes h) = \Sigma_n(h)
\]

since \( h^i \) are finitely dimensional.

9.5. Example: Intersection of subschemes. A typical example of a formal space which is not a formal scheme is given by a non-transversal intersection of subschemes. Let \( X, Y \subseteq Z \) be closed subschemes in a noetherian scheme \( Z \), \( z \in X \cap Y \). We wish to describe the intersection of \( X \) and \( Y \) in \( Z \) near \( z \). Let \( A, B, C \) be the local rings of \( X, Y, Z \) respectively.

These rings (or the corresponding dual coalgebras \( A^*, B^*, C^* \)) represent functors

\[
F_A, F_B, F_C : \text{dgart}^{\leq 0}(k) \to \Delta^0\text{Ens}
\]

where \( k = k(z) \) is the base field.

The functors \( F_A, F_B, F_C \) being defined up to homotopy equivalence, the best thing we can do is to consider their homotopy fibre product.

Thus, define the homotopy intersection of \( X \) and \( Y \) in \( z \in Z \) to be the homotopy fibre product functor \( F \) of \( F_A \) and \( F_B \) over \( F_C \). In order to calculate it, one has to substitute a map \( F_A \to F_C \) (or the other one) with a fibration and take the usual fibre product.

For this it suffices to take a cofibrant resolution \( \tilde{A} \) for the \( C \)-algebra \( A \) and substitute \( F_A \) with \( F_{\tilde{A}} \). The result will be given by the dg algebra \( \tilde{A} \otimes_C B \) defined canonically in the corresponding homotopy category, concentrated in the nonpositive degrees. Its cohomology is given by the formula

\[
H^i(\tilde{A} \otimes_C B) = \text{Tor}^C_{-i}(A, B)
\]
9.5.1. **Remark.** It is unclear how to define a global object corresponding, say, to the intersection of two subschemes in a non-affine scheme. One should probably use a technique suggested by Hirschowitz-Simpson in [HiSi].

9.6. **Example: Quotient by a group action.** Let an algebraic group $G$ over $k$ acts on a dg Lie algebra $\mathfrak{h}$. Then $G$ acts on each simplicial set $\Sigma_{\mathfrak{h}}(A)$. Define the functor

$$F : \text{dgart}^{\leq 0}(k) \to \Delta^0\text{Ens}$$

as the homotopy quotient $F(A) = \Sigma_{\mathfrak{h}}(A)/\tilde{G}_1(A)$ where $\tilde{G}_1$ is the formal completion of $G$ at 1 so that $\tilde{G}_1(A) = \exp(m \otimes g)$.

9.6.1. **Proposition.** The functor $F$ is homotopy equivalent to the nerve of the semidirect product $g \rtimes \mathfrak{h}$.

**Proof.** According to Proposition 8.1.3, (3), (4), one has a fibration

$$f : \Sigma_{g \times \mathfrak{h}}(A) \to \Sigma_{\mathfrak{g}}(A)$$

with fibre $\Sigma_{\mathfrak{h}}(A)$. On the other hand, the map (17) gives in our case a fibration

$$\pi : G_{\bullet}(A) \to \Sigma_{\mathfrak{g}}(A)$$

with fibre $\tilde{G}_1(A)$ and contractible total space. Then the cartesian diagram

\[
\begin{array}{ccc}
X & \longrightarrow & \Sigma_{g \times \mathfrak{h}}(A) \\
\downarrow_{\tilde{G}_1(A)} & & \downarrow_{\Sigma_{\mathfrak{h}}(A)} \\
G_{\bullet}(A) & \longrightarrow & \Sigma_{\mathfrak{g}}(A)
\end{array}
\]

(all the arrows being fibrations marked by the corresponding fibres) presents $\Sigma_{g \times \mathfrak{h}}(A)$ as a homotopy quotient of $\Sigma_{\mathfrak{h}}(A)$ modulo $\tilde{G}_1(A)$. \qed

9.7. **Example: Moduli of $G$-torsors.** The last example of a formal stack will be that of moduli of principal $G$-bundles.

Let $G$ be an algebraic group over a field $k$ of characteristic zero, $S$ be a scheme over $k$, $P$ be a $S$-torsor under $G$. We wish to study formal deformations of $P$. For this we have to define a deformation functor

$$F_P : \text{dgart}^{\leq 0}(k) \to \Delta^0\text{Ens}$$

naturally generalizing the standard functor of formal deformations $\text{art}(k) \to \text{Grp}$.

We will proceed as follows. First of all, we define in 9.7.1 torsors with a (affine) dg base. Then (in 9.7.2) we construct a simplicial set describing the formal deformations of a torsor in the affine case.
Deformations of a torsor over a non-affine scheme are defined in 9.7.3 as an appropriate homotopy limit of the deformations over affine bases. After this we explicitly calculate in 9.7.4–9.7.6 the deformation functor describing deformations of a trivial torsor over an affine base. Finally, using the main result of [H1], we get the final answer — Theorem 9.7.7.

Fix some notations. Recall that \( \text{dga}(k) = \text{Alg} \left( \text{Com}(k) \right) \) is the category of commutative dg algebras over \( k \).

Let \( R \) be the Hopf algebra of regular functions on \( G \). Then \( g = \text{Der}(R, R) \) is the Lie algebra of \( G \).

9.7.1. **Definition.** Let \( B \in \text{dga}(k) \). A \( B \)-torsor under \( G \) is a morphism \( x : B \to X \) together with an associative (co)multiplication map \( \mu : X \to X \otimes R \) satisfying the following properties:

0. \( \mu x = (\text{id}_X \otimes 1)x \)

1. (pseudo-torsor) The multiplication \( \mu \) together with \( \text{id}_X \otimes 1 \) induce an isomorphism \( X \otimes_B X \to X \otimes R \).

2. (local triviality) The map \( x \) is faithfully flat (this property does not depend on the differential, see [H3]).

9.7.2. The definition above gives rise to a stack of groupoids on \( \text{dga}(k) \) in the topology generated by the faithfully flat maps. This is a (2-) functor \( \mathcal{C} : \text{dga}(k) \to \text{Grp} \) such that for \( B \in \text{dga}(k) \), \( \mathcal{C}(B) \) is the groupoid of \( B \)-torsors under \( G \).

For a fixed \( B \)-torsor \( P \) one defines a fibred category \( \mathcal{C}(P) \) over \( \text{dgart} \leq 0(k) \) by the formula

\[
\mathcal{C}(P, A) = \{ A \otimes B \text{-torsors } \tilde{P} \text{ with a trivialization } \tilde{P} \otimes_{A \otimes B} B \simto P \}
\]

This is not yet the deformation functor we need since one cannot expect that deformations with a dg base have no “higher morphisms” between them. Thus we define, for a given \( B \)-torsor \( P \) and \( A \in \text{dgart} \leq 0(k) \), the following simplicial category \( \mathcal{D}(P, A) \):

\[
\text{Ob} \mathcal{D}(P, A) = \text{Ob} \mathcal{C}(P, A); \\
\text{Hom}_{\mathcal{D}(P, A)}(x, y)_n = \text{Hom}_{\mathcal{C}(P, A)}(x_n, y_n).
\]

Here \( P_n \) is the torsor \( \Omega_n \otimes P \) over \( \Omega_n \otimes B \) and, similarly, \( x_n, y_n \) are the torsors over \( A \otimes \Omega_n \otimes B \).

Finally, we apply the simplicial nerve functor \( \mathcal{N} \) (see the Appendix) to get a simplicial set from the simplicial category \( \mathcal{D}(P, A) \):

\[ \mathcal{T}_P(A) := \mathcal{N}(\mathcal{D}(P, A)). \]

9.7.3. Let now \( S \) be an arbitrary scheme over \( k \) and \( P \) be a \( S \)-torsor under \( G \). We define then \( F_P(A), A \in \text{dgart} \leq 0(k) \), by the formula

\[ F_P(A) = \text{holim}_{i : U \to S} \mathcal{T}_{i^* P}(A), \]

the inverse homotopy limit being taken over all affine schemes over \( S \).
As a result of our computations we will see in particular that $\tilde{F}$ and $F$ are homotopy equivalent for affine $S$.

9.7.4. Let us make some calculations. Suppose that $P$ is a trivial torsor over $B$.

Fix $A \in \text{dgart}^{\leq 0}(k)$ and calculate the simplicial category $\mathcal{D}(A)$ — we omit $P$ from the notation since $P$ is supposed to be trivial.

The functor $\#$ forgetting the differential in dg objects, transforms $B$-torsors to $B^\#$-torsors.

**Lemma.** Let $P$ be a torsor over $B \otimes A$ trivial over $B$. Then $P^\#$ is a trivial $(B \otimes A)^\#$-torsor.

**Proof.** Similarly to the usual algebraic geometry, one defines a cotangent complex complex $L_{B/A}$ (see 

A map of graded (super) commutative algebras is *formally smooth* if it satisfies the left lifting property with respect to surjective maps having a nilpotent kernel. As in [Ill], III, 3.1.2, a $A$-algebra $B$ whose cotangent complex is represented by a projective module, is formally smooth.

Since the cotangent complex $L_{R/k}$ is finitely generated free, the base change property and faithfully flat descent give that $L_{P/B}$ is finitely generated projective for any $B$-torsor $P$ under $G = \text{Spec}(R)$. This implies that all graded torsors under $G$ are formally smooth.

Then the map $P \to P \otimes_{B \otimes A} B \to B$ can be lifted to a map $P \to B \otimes A$ splitting the structure map. This proves triviality of $P$.

9.7.5. **Corollary.** For a given $B \in \text{dga}(k)$, the groupoid $\mathcal{C}(A)$ is canonically (on $A$) equivalent to the following groupoid $\mathcal{C}(A)$

$$\text{Ob}\mathcal{C}(A) = \text{MC}(m \otimes B \otimes g)$$

$$\text{Hom}_{\mathcal{C}(A)}(x, y) = \{g \in \exp((m \otimes B)^0 \otimes g)|y = g(x)\}.$$

**Proof.** According to 9.7.4, any $A \otimes B$-torsor trivial over $B$ has form $A \otimes B \otimes R$ as a graded algebra; its differential is defined by its restriction on $R$ which is a derivation $\delta : R \to A \otimes B \otimes R$ trivialized by $A \to k$. This is given by a Maurer-Cartan element of $m \otimes B \otimes g$. Any automorphism of the graded torsor $(A \otimes B \otimes R)^\#$ is given by a $A \otimes B$-point of $G$. This one should map to the unit $B$-point of $G$ under $A \to k$. This gives the second formula of the claim.

9.7.6. **Proposition.** For a given commutative $k$-algebra $B$ the functor

$$F_{\text{triv}} : \text{dgart}^{\leq 0}(k) \to \Delta^0\text{Ens}$$

describing deformations of the trivial $B$-torsor under $G$, is naturally equivalent to the nerve of the Lie algebra $B \otimes g$.

**Proof.** Let $B$ be a commutative $k$-algebra. According to 9.7.5, one has $\text{Ob} \mathcal{D}(A) = \text{MC}(m \otimes B \otimes g)$ is a singleton since $m \otimes B \otimes g$ is non-positively graded, so $\mathcal{D}(A)$ is actually a simplicial group.
Re-denoting for simplicity $D(A)$ by $D$ and $m \otimes B \otimes g$ by $g$ we have

$$D_n = \{ u \in \exp(\Omega_n \otimes g) | u(0) = 0 \} = \text{Stab}_{G_n}(0)$$

in the notation of 8.2.

Now we have to find a natural equivalence from $\Sigma(g)$ to $\mathcal{N}(D)$. Since $MC(g) = \{0\}$, Corollary 8.2.6 furnishes us a principal fibration with the base $\Sigma(g)$, the total space $G_*$ with the group $\text{Stab}_{G_*}(0)$ and a canonical pseudo-cross section.

The simplicial set $G_*$ is isomorphic to

$$n \mapsto (\Omega_n \otimes g)^0$$

which is (a simplicial vector space and) a direct sum of simplicial vector spaces of form $\Omega^p$ which are all contractible by \cite{L}, p. 44.

Then Theorem 21.13 of \cite{M} provides a canonical homotopy equivalence $\Sigma(g) \to \overline{W}(D)$ — see Appendix. According to Lemma 10.3, this gives a canonical equivalence in question.

9.7.7. Now, using the faithfully flat descent (see, e.g., \cite{R}) and taking into account the main result of \cite{M}, we obtain the following

**Theorem.** Let $S$ be a scheme over a field $k$ of characteristic zero, $G$ be an affine algebraic group and $P$ be a $S$-torsor under $G$. Let $g$ be the Lie algebra of $G$ and $g_P$ be the induced by $P$ coherent sheaf of Lie algebras on $S$.

Then the formal stack of deformations of $P$ has form

$$\mathcal{C}(\mathcal{R}\Gamma(S, g_P))$$

where $\mathcal{R}\Gamma(S, g_P)$ is calculated using Thom-Sullivan Tot functor of \cite{HS2}.

9.7.8. A similar result holds in the context of \cite{Ka} where $G$-local systems are considered.

For this one defines torsors under $G$ over a “$dg$-ringed space” $(X, \mathcal{O}_X)$ where $X$ is a topological space endowed with a sheaf of commutative dg $k$-algebras $\mathcal{O}_X$, using the topology generated by surjective open covers of $X$. Then, given a good topological space $X$ and a torsor $P$ under $G$ over the ringed space $(X, k_X)$ ($\mathcal{O} = k_X$ is the constant sheaf), its deformations over a local dg artinian $k$-algebra $A$ are defined as $(X, A \otimes k_X)$-torsors under $G$ endowed with a trivialization over $(X, k_X)$. These deformations are governed locally by the sheaf of Lie algebras $g_P$ and, therefore, globally, by $\mathcal{R}\Gamma(X, g_P)$.

This formula coincides with Kapranov’s \cite{Ka}, 2.5.1. Our construction of the formal moduli seems to be more “honest” then the *ad hoc* definition 2.2.2 of \cite{Ka}: we start with a reasonable functor on artinian rings and look for a representing object. Of course, it is not absolutely “honest” since we knew the answer to obtain.

10. Appendix: Nerve of a simplicial category

Let $\mathcal{C}$ be a simplicial category. We present here two versions of the nerve of $\mathcal{C} = \mathcal{N}(\mathcal{C})$ and $\overline{W}(\mathcal{C})$ — and prove they are canonically homotopy equivalent under some assumptions which are fulfilled in the application we have in mind (see 9.7).
As usual, for \(x, y \in \text{Ob} \mathcal{C}\) we denote by \(\text{Han}(x, y)\) the simplicial set of arrows from \(x\) to \(y\).

10.1. One can consider \(\mathcal{C}\) as a simplicial object \(\{\mathcal{C}_n\}\) in the category \(\text{Cat}\) of categories; thus, applying the standard nerve construction to each one of the \(\mathcal{C}_n\), we get a bisimplicial set; its diagonal is denoted by \(\mathcal{N}(\mathcal{C})\).

More explicitly, \(n\)-simplices of \(\mathcal{N}(\mathcal{C})\) are the sequences
\[
[f_1 | \ldots | f_n], \ f_i \in \text{Han}_n(v_{i-1}, v_i) \text{ where } v_0, \ldots, v_n \in \text{Ob} \mathcal{C}.
\]
The faces and the degeneracies are given by the standard formulas.

10.2. Another construction is a minor generalization of the \(\overline{\text{W}}\) functor, see [M], chapter IV. The \(n\)-simplices of \(\overline{\text{W}}(\mathcal{C})\) are the sequences
\[
(g_1 | \ldots | g_n), \ g_i \in \text{Han}_{n-i}(v_{i-1}, v_i) \text{ where } v_0, \ldots, v_n \in \text{Ob} \mathcal{C}.
\]
The faces and the degeneracies are given by the following formulas
\[
\begin{align*}
d_i(g_1 | \ldots | g_n) &= (d_{i-1}g_1 | \ldots | d_1g_{i-1} | g_i | d_0g_{i+1} | \ldots | g_n) \quad \text{for } i \neq 0, n \\
d_0(g_1 | \ldots | g_n) &= (g_2 | \ldots | g_n) \\
d_n(g_1 | \ldots | g_n) &= (d_{n-1}g_1 | \ldots | d_1g_n) \\
s_0(g_1 | \ldots | g_n) &= (id | g_1 | \ldots | g_n) \\
s_i(g_1 | \ldots | g_n) &= (s_{i-1}g_1 | \ldots | s_0g_i | id | g_{i+1} | \ldots | g_n) \quad \text{for } i > 0.
\end{align*}
\]

10.3. One has the following canonical maps connecting \(\mathcal{N}(\mathcal{C})\) and \(\overline{\text{W}}(\mathcal{C})\)
\[
\pi : \mathcal{N}(\mathcal{C}) \xrightarrow{\cong} \overline{\text{W}}(\mathcal{C}) : \rho
\]
given by the formulas
\[
\begin{align*}
\pi[f_1 | \ldots | f_n] &= (d_0f_1 | d_0^2f_2 | \ldots | d_0^nf_n) \\
\rho(g_1 | \ldots | g_n) &= [s_0g_1 | s_0^2g_2 | \ldots | s_0^ng_n].
\end{align*}
\]

**Lemma.** Suppose that the simplicial category \(\mathcal{C}\) satisfies the following properties
\(\text{(a)}\) for each \(v, w \in \text{Ob} \mathcal{C}\) the simplicial set \(\text{Han}(v, w)\) is Kan fibrant.
\(\text{(b)}\) for each \(n\) the category \(\mathcal{C}_n\) is a groupoid.

Then the maps \(\pi, \rho\) are homotopy equivalences.

**Proof.** We will directly check that under the said assumptions the map \(\pi\) satisfies the RLP with respect to the maps \(\partial \Delta^n \to \Delta^n\).

A map \(\Delta^n \to \overline{\text{W}}(\mathcal{C})\) is given by a collection
\[
g = (g_1 | \ldots | g_n), \ g_i \in \text{Han}_{n-i}(v_{i-1}, v_i).
\]
A map \(\partial \Delta^n \to \mathcal{N}(\mathcal{C})\) is given by a compatible collection
\[
x^i = [x^i_1 | \ldots | x^i_{n-1}] \text{ with } \deg x^i_j = n - 1,
\]
the compatibility conditions being the conditions (A), (B) below.

(A) The condition \(d_i(x^k) = d_{k-1}(x^i)\) for \(i < k\) amounts to the system
\[ d_i(x^k_j) = d_{k-1}(x^i_j) \text{ for } j \leq i - 1 \text{ or } j \geq k + 1 \]
\[ d_i(x^k_{i+1} \circ x^k_i) = d_{k-1}(x^i_j) \text{ for } i \leq k - 2 \]
\[ d_i(x^k_{j+1}) = d_{k-1}(x^i_j) \text{ for } i < j \leq k - 2 \]
\[ d_i(x^k_j) = d_{k-1}(x^i_j \circ x^k_{k-1}) \text{ for } i \leq k - 2 \]
\[ d_{k-1}(x^k_j \circ x^k_{k-1}) = d_{k-1}(x^{k-1}_k \circ x^{k-1}_{k-1}) \]

(B) The compatibilities of \( x^i \) with \( g \) say that \( \pi(x^i) = d_i(g) \) which gives
\[
d^i_0 x^j_0 = \begin{cases} d_{i-1}g_j & \text{for } j \leq i - 1 \\ g_{i+1} \circ d_0g_i & \text{for } j = i \\ g_{j+1} & \text{for } j > i. \end{cases}
\]

(C) Now, we have to construct a map \( \Delta^n \rightarrow \mathcal{N}(\mathcal{C}) \) i.e. a collection \( [f_1|...|f_n] \) satisfying the conditions \( d_i f = x^i, \pi(f) = g \) which can be rewritten as a system

1. \( d^i_0(f_j) = g_j \)
2. \( d_i(f_j) = \begin{cases} x^i_j, & i \geq j + 1 \\ x^{i-1}_{j-1}, & i \leq j - 2 \end{cases} \)
3. \( d_{j-1}(f_j) \circ d_{j-1}(f_{j-1}) = x^{j-1}_{j-1} \)

One is looking for \( f_1 \) by induction. For \( j = 1 \) we have prescribed values for \( d_i(f_1), i \neq 1 \). Thus, checking their compatibility and using the condition (a) of the Lemma, we deduce that the system admits a solution \( f_1 \).

Suppose that \( f_i \) have already been found for \( i < j \) so that the equations above are satisfied. One checks first of all that for \( j > 1 \) the equation (C.1) follows from (C.2). Afterwards, one finds the value for \( d_{j-1}(f_j) \) using the condition (b) of the Lemma. Then we have the prescribed values for \( d_i(f_j) \) for all \( i \neq j \) and we have only to check using the compatibility conditions (A), (B) that these prescriptions given by (C.2), (C.3), are compatible.

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