1. Introduction

A path (cycle) in a graph passing each vertex is said to be a Hamilton path (cycle). A graph is hamiltonian if it consists of a Hamilton cycle and is Hamilton-connected if every pair of vertices are joined by a Hamilton path. A partition of the edge set $E(G)$ into Hamilton cycles (or plus a perfect matching when $G$ is $k$-regular with $k$ odd) is called a Hamiltonian decomposition of $G$. The Hamiltonian problems and related topics are classical in combinatorics and have been received extensive treatment. See [1–3] for more results on this area of investigation.

The current paper concentrates on the class of 3-arc graphs. This class of graphs is produced from a graph operation named 3-arc graph operation, which is similar to a line graph operation. This operation was initially introduced in [4] in exploring certain families of imprimitive symmetric graphs. It was also exploited in investigating several classes of symmetric graphs. See [4, 5] and references therein.

We consider only a graph with finite order and undirected without loops. The term multigraph is used if there are multiple edges connecting the same pair of vertices. An arc of $H = (V(H), E(H))$ is a directed edge. If $x, y$ are adjoined in $H$, we write $xy$ to denote the directed edge outgoing from $x$ to $y$, $yx$ ($\neq xy$) the directed edge from $y$ to $x$, and $\{x, y\}$ the edge connecting $x$ and $y$. A 4-tuple $(y, x, v, w)$ in a graph is a 3-arc if each of $(y, x, v)$ and $(x, v, w)$ is a path. The 3-arc graph of $H$ is the graph with vertex set all arcs of $H$ and edge set containing all edges joining $xy$ and $vw$ whenever $(y, x, v, w)$ is a 3-arc of $H$. A Hamilton cycle is a closed path meeting each vertex of a graph. A graph is hamiltonian if it contains a Hamilton cycle. A Hamiltonian and a Hamiltonian decomposition provided its edgeset admits a partition into disjoint Hamilton cycles (possibly with a single perfect matching). The current paper proves that every connected 3-arc graph consists of more than one Hamilton cycle. Since the 3-arc graph of a cubic graph is 4-regular, it further proves that each 3-arc graph of a cubic graph in a certain family has a Hamiltonian decomposition.
We illustrate the definition of this construction by depicting the 3-arc graph of complete graph $K_4$ on four vertices $a, b, c$ and $d$ (see Figure 1). For instance, both $(a, b, c, d)$ and $(a, b, c, a)$ are 3-arcs in $K_4$. The first one defines the adjacent relation between arcs $ba$ and $cd$, and the latter gives rise to the adjacency between $ba$ and $ca$.

It is not hard to see that $X(H)$ is not directed with $2|E(H)|$ vertices and $\sum_{\{x,y\}\in E(H)}(d(x)-1)(d(y)-1)$ edges. It is shown that $X(H)$ can actually be obtained from $L(H)$ of $H$ by the next procedures: divide every vertex $\{x, y\}$ of $L(H)$ into two, i.e., $xy$ and $yx$; for every pair of vertices $\{a, b\}$, $\{x, y\}$ of $L(H)$ that are at a distance 2 away in $L(H)$, say, $a$ and $x$ are adjoined in $H$, connect $ab$ and $xy$. For more recent work in this direction, see [6]. Specifically, in [7], it is established that all connected graphs contain a Hamilton cycle.

In 1975, Sheehan proposed the following famous conjecture:

**Conjecture 1 (Sheehan [8]).** Each Hamiltonian graph that is regular of degree 4 consists of no less than two Hamilton cycles.

Motivated by Sheehan’s Conjecture and the work [7], we study the Hamiltonian decomposition problem for 3-arc graphs. We show every connected 3-arc graph contains a second Hamilton cycle, and the 3-arc graphs of a family of cubic graphs have Hamiltonian decompositions. As a corollary, we demonstrate that the 3-arc graph of a bipartite cubic graph includes one Hamiltonian decomposition.

To confirm Sheehan’s conjecture for 3-arc graphs, we actually prove a stronger result which asserts that any connected 3-arc graph contains no less than two Hamilton cycles. As the 3-arc graph of a cubic graph is 4-regular. We shall show each 3-arc graph of a cubic graph in a certain family has a Hamiltonian decomposition.

The primary results are presented as follows.

**Theorem 1.** Let $G$ be with a maximum degree $\Delta$ and have a connected 3-arc graph $X(G)$. Then the number of Hamilton cycles of $X(G)$ is no less than the following:

\[
2, \quad \text{if } \Delta = 3 \text{ or } 4; \\
\left(\frac{(\Delta - 3)^{\Delta - 3}}{(\Delta - 2)^{\Delta - 1}}\right)^\Delta, \quad \text{if } \Delta \geq 5.
\]

(2)

**Remark 1.** Since $\left(\frac{(\Delta - 3)^{\Delta - 3}}{(\Delta - 2)^{\Delta - 1}}\right)^\Delta = \left(\frac{(\Delta - 2)1}{1 + 1/\Delta - 3}\right)^{\Delta - 3}$, $1 + 1/\Delta - 3$ is increasing and

\[
\lim_{\Delta \to \infty} \left(1 + \frac{1}{\Delta - 3}\right)^{\Delta - 3} = e.
\]

(3)

**Theorem 2.** Suppose $G$ is a connected cubic graph. If $G$ contains an even IS-C decomposition, then $X(G)$ has a Hamiltonian decomposition.

2. **Proof of Theorems 1 and 2**

We need the following notation and definitions.

A **cactus** is a graph in which any two cycles have at most one vertex in common. In particular, a tree is a special cactus. A cycle is said to be an odd (resp., even) cycle if it consists of an odd (resp., even) number of edges.

**Definition 2.** Let $(A, B)$ be a partition (i.e., $V(G) = A \cup B$ and $A \cap B = \emptyset$) of $V(G)$. $(A, B)$ is said to be an independent set-cactus (IS-C) decomposition of $G$ if in $G$ the set $A$ induces an empty graph and $B$ induces a (connected) cactus.

Note that not every graph has an IS-C decomposition. For example, the complete graph on five or more vertices has no IS-C decomposition.

Since trees are acyclic, as a convention, we define the length of each cycle (which does not exist) of a tree to be 0. A cactus is called even if the length of each cycle is even. We treat a tree as an even cactus. An IS-C decomposition is called even if every cycle in the cactus is even.

**Lemma 1.** Each connected cubic graph has an IS-C decomposition.

Proof. Suppose $G$ is a connected cubic graph. If every two cycles of $G$ share one or less common vertex, then $G$ itself is a cactus. Clearly, an IS-C decomposition, $(\emptyset, V(G))$, say, exists.

Suppose that $G$ has two distinct cycles sharing two or more common vertices. We construct an IS-C decomposition for $G$ as follows: choose any two cycles $C_1, C_2$ that share at least two common vertices. Denote the maximal common path occurring on both $C_1$ and $C_2$ by $P: x, \ldots, x': x, \ldots, x'$; that is, any vertex not on $P$ occurs on at most one of the two cycles $C_1$ and $C_2$. Then delete $x$ from $G$ and put it into the set $A$ (A takes the role of the independent set). Note that each neighbour of $x$ now has degree strictly less than 3, and hence occurs on at most one cycle in $G - x$. So, the resultant graph $G - x$ is
connected. To simplify notation, denote again by $G$ the graph resulting after deleting $x$. If $G$ still has two cycles $C_3, C_4$ that share at least two common vertices. Denote the maximal common path occurring on both $C_3$ and $C_4$ by $y, \ldots, y'$. Note that $y, y' \notin N(x)$. Delete $y$ and put it into $A$. Continue this process until the resultant graph becomes a cactus.

Since, at each stage, the set $A$ remains to be independent and the resulting graph stays to be connected, eventually $(A, V(G) - A)$ becomes a desired IS-C decomposition.

Though every cubic graph has an IS-C decomposition, not every cubic graph has an even IS-C decomposition. For instance, $K_3$ has no even IS-C decomposition.

A 2-trail $(u, x, v)$ with the middle term $x$ is said to be a visit to $x$. If $u \neq v$, then $(u, x, v)$ and $(v, x, u)$ are viewed as different visits to $x$. When we $(u, x, v)$, $(v, x, u)$ are not concerned about the directions of and, or its orientation is unknown, we denote $[u, x, v]$. The following operation regarding two parallel edges will be needed.

Definition 3. Let $e_1, e_2$ be parallel edges joining two adjacent $u, v$ of $G^*$, let $e_i$ be covered by a closed trail $C_i$ of length at least 4, $i = 1, 2$. (It may happen that $C_1 = C_2$.) Then one of $C_1(u) \cup C_2(u)$ and $C_1(v) \cup C_2(v)$, say $C_1(v) \cup C_2(v)$, contains two visits with one covering $e_1$ and the other covering $e_2$. Denote these two visits by $p_1 = [u, e_1, v, e_2, v_1]$ and $p_2 = [u, e_2, v, e_1, v_2]$, where $v_1, v_2$ are two neighbors of $v$ other than $u$ and, $e_1, e_2$ are edges between $v$ and $v_1, v_2$ respectively. Split the two 2-trails $p_1, p_2$ at $v$ and reconnect $e_1, e_2$ with $e_1, e_2$, respectively. We call this the edge-shift operation of $C_1, C_2$ with respect to $e_1, e_2$, and denote the resultant trail(s) by $C(C_1, C_2; e_1, e_2)$, or simply $C(C_1; e_1, e_2)$ if $C_1 = C_2$.

Remark 2. A few comments on Definition 3 are ready:

1. If $C_1 = C_2$, $C(C_1; e_1, e_2)$ remains to be a single closed trail if and only if the orientations of $e_1$ and $e_2$ are reverse to each other in $C_1$; in this case, this operation is, in fact, the edge version of the bow-tie operation (see, Definition 5 in [7]).

2. If $C_1 = C_2$, $C(C_1; e_1, e_2)$ is a set of two closed trails if and only if the orientations $e_1$ and $e_2$ are the same between $u$ and $v$ in $C_1$.

3. All edges covered by $C_1, C_2$ are covered by $C(C_1, C_2; e_1, e_2)$.

4. After the edge-shift operation above, the three terms involved in $p_1, p_2$ are swapped. That is, $p_1$ is transformed into $[u, v, v_1]$, and $p_2$ is transformed into $[u, v, v_1]$. All other visits induced are kept or orientation inverted.

Proof of Theorem 1. Suppose $G$ is a graph with $X(G)$ connected. Clearly, $\Delta(G) \geq 3$. Denote the set of degree-2 vertices of $G$ by $S_2$, the multigraph gained from $G$ by doubling its every edge by $G^*$.

Suppose that $\Delta = 3$. Since $X(G)$ is connected, by [7], $X(G)$ is Hamiltonian, and every vertex in $G$ has degree 2 or 3, $S_2$ is independent, and $G - S_2$ is connected. Further, it can be observed that $G - S_2$ contains at least 2 vertices. Suppose $u, v$ are two adjacent vertices in $G - S_2$, $u'$ is a neighbor of $u$ different from $v$, and $x, y$ are two distinct neighbors of $v$ not equal $u$ (it may occur that one of $x, y$ equals $u'$). Denote $P_1: u', u, v, x$ and $P_2: u', u, v, y$.

Let $C_1, C_2$ be two Eulerian tours of $G^*$ obtained without violating the condition (1) in Section 3 of [7], from extending $P_1, P_2$ to cover each edge of $G^*$, respectively. Apply the bow-tie operation (Definition 5 in [7]) when necessary to process $C_j$ in such a way that the bipartite graph $H_{C_j}(z)$ (Definition 3 in [7]) with respect to the resultant Eulerian tour $C_j$ has a perfect matching and the path $P_j$ is a segment of $C_j$, $j = 1, 2$. Note that both $C_1$ and $C_2$ exist. Consider $C_1$. If $H_{C_1}(v)$ has no perfect matching, then $C_1(v)$ contains twin visits by Lemma 1 in [7], that is, $C_1(v) = \{[u, v, x], [u, v, y], (y, y, y)\}$. Apply the bow-tie operation to $C_1$ with regard to $(y, y, y)$ and anyone of the twin visits $[u, v, x]$ that is not occurring on $P_1$, we get a new Eulerian tour $C_{1}' = C_1([u, v, x], (y, y, y))$, and one can observe that $H_{C_1}^*(v)$ is a perfect matching of 3 independent edges. The bow-tie operation can be performed similarly when necessary at $u$ to produce a new Eulerian tour $C_2$ such that $H_{C_2}(u)$ has a perfect matching and the path $P_j$ is maintained unchanged. It is analogous to showing that $C_2$ exists.

Let $T_1, T_2$ be two Hamilton cycles of $X(G)$ derived from $C_1', C_2'$, respectively. Denote the 3rd neighbor of $u$ other than $u', v$ by $w$. Then one can observe that $uw$ is connected to $u'z', yv$ in $T_1$, while connected to $u'z', vx$ in $T_2$, where $z, z' \in N(u') - \{u\}$ are not necessarily distinct. Since $[u'z, vy] \neq [u'z', vx]$, $T_1$ and $T_2$ are different Hamilton cycles of $X(G)$.

Suppose that $\Delta \geq 4$. Suppose $x$ is the maximum-degree vertex, $C$ an Eulerian tour of $G^*$ with the property that $H_{C}(z)$ contains a perfect matching for each $z$ (such a $C$ can be achieved by applying the bow-tie operations successively when needed). Denote $\hat{C}(x)$ the family of heavy visits to $x$ that is, each element of $\hat{C}(x)$ is of the form $[u, w, x, w]$, where $w$ is a neighbor of $x$). By performing a series of bow-tie operations, we will transform each visit of $\hat{C}(x)$ into a new visit containing three pairwise distinct terms.

To simplify notation we will still use $C$ to denote the new Eulerian tour produced by applying an operation. If $\hat{C}(x) \geq 2$, choose a pair of visits $p_1, p_2$ of $\hat{C}(x)$, and apply bow-tie operation to $C$ with regarding to $p_1, p_2$. Then in the new Eulerian tour $C(p_1, p_2)$, the pair $p_1, p_2$ is transformed into two visits with each containing three pairwise distinct terms. Thus, each time of applying the bow-tie operation $C$ with regard to two elements of $\hat{C}(x)$ deducts the number of $|\hat{C}(x)|$ by two. Apply this operation until $|\hat{C}(x)| \leq 1$. If $|\hat{C}(x)| = 0$, we are done. Suppose $\hat{C}(x) = \{[u, x, w]\}$. Let $p = [y, x, z]$ be an arbitrary visit outside $\hat{C}(x)$. Apply the same operation to $C$ with regard to $[u, x, w]$ and $p$. Again, $[u, x, w]$ can be eliminated without producing a new heavy visit. Eventually, we get an Eulerian tour, denoted again by $C$, of $G^*$ such that each element of $C(x)$ contains three pairwise distinct terms. Then in $H_{C}(x)$, each visit of $C(x)$ is adjacent to $\Delta - 2$ arcs of $A(x)$, and each arc of $A(x)$ is
adjacent to $Δ−2$ visits of $C(x)$. That is, $H_C(x)$ is a $(Δ−2)$-regular bipartite graph.

If $Δ = 4$, $H_C(x)$ is 2-regular, hence contains two distinct perfect matchings. Suppose that $Δ≥ 5$, and denote by $M_1, M_2, \ldots, M_t$ the set of all perfect matchings of $H_C(x)$. Schrijver [9] states that every bipartite graph regular of degree $k$ with $2n$ vertices contains at least $(k−1)^{k−1/(k−2)}n$ perfect matchings. Thus, we have the following:

$$i ≥ \left(\frac{(Δ−3)^{Δ−1}}{(Δ−2)^{Δ−4}}\right)^{Δ}. \tag{4}$$

For every vertex $u ≠ x$ in $G$, fix a perfect matching of $H_C(u)$. As in the Proof of Theorem 1 in [7], a Hamilton cycle of $X(G)$ can be derived from $C$ by using the fixed perfect matchings of all bipartite graphs $H_C(v)$ with $v ≠ x$, together with every single perfect matching $M_j$ of $H_C(x)$, $1 ≤ j ≤ t$. Note that corresponding to distinct $M_j, M_j'$ of $H_C(x)$, the derived Hamilton cycles of $X(G)$ are also distinct. Therefore, $X(G)$ has $i$ different Hamilton cycles and the result follows.

Suppose $G$ is a graph containing no isolates. If $X(G)$ is connected and 4-regular, then clearly $G$ has a maximum degree of no less than 3. Hence by Theorem 1, we have the next result, which verifies Sheehan’s conjecture for 3-arc graphs.

**Corollary 1.** Every 4-regular 3-arc graph has no less than two Hamilton cycles.

**Proof of Theorem 2.** Suppose $G$ is a connected cubic graph and $(A, B)$ is an even IS-C decomposition of $G$. To simply notation, we employ $B$ to represent the cactus induced by $B$.

Suppose $G^*$ is the multigraph achieved from $G$ by doubling its all edges, $B^*$ the multigraph achieved from $G^*$ by deleting all edges joining some vertex of $A$. Equivalently, $B^*$ is the multigraph obtained from $B$ by doubling each of its edges.

Then $B^*$ is Eulerian. Assume $C^*$ is an Eulerian tour of $B^*$, so that $C^*(x)$ contains no heavy visit for each $x ∈ B$ with 2 or 3 neighbors in $B$. Note that such a $C^*$ can be obtained from any Eulerian tour of $B^*$ by applying a succession of bow-tie operations when necessary.

We next extend $C^*$ to an Eulerian tour of $G^*$ as follows: for each vertex $x ∈ B$ with degree 1 in $B$, let $x'$ be the only neighbor of $x$ in $B$, and $x_1, x_2$ the other two distinct neighbors of $x$ in $A$. Then $C^*(x) = \{x', x, x'\}$. We extend $C^*$ at $x$ to cover the four edges between $x_1, x_2$ and $x$ such that the new visit decomposition at $x$ is $\{x', x_1, x_2, [x_1, x, x_2],[x_2, x, x']\}$. In other words, we insert the trail $x_1, x_2, x, x_2$ into $C^*$ via the midvertex of the visit $(x', x, x')$. For each vertex $x ∈ B$ with degree 2 in $B$, let $x_1$ be the unique neighbor of $x$ in $A$, and $x', x''$ the other two distinct neighbors of $x$ in $B$. Then $C^*(x) = \{[x', x, x''], [x', x, x'']\}$ by the assumption on $C^*$. We extend $C^*$ at $x$ to cover the two edges between $x_1$ and $x$ such that the new visit decomposition is $\{x_1, x_1, x_1\}$. Applying this extension to every vertex of $B$ that has degree 1 or 2, we obtain an Eulerian tour of $G^*$, denoted $C$.

By the way $C$ is extended, one may observe that $C(x)$ consists of no heavy visit if $x ∈ B$, and $C(x)$ contains only heavy visits if $x ∈ A$. In particular, the bipartite graph $H_C(x)$ is a set of three independent edges if $x ∈ B$, and $H_C(x)$ is $C_6$ if $x ∈ A$. Thus, in both cases $H_C(x)$ has a perfect matching, and, a Hamilton cycle of $X(x)$, denoted by $C^X$, can be derived from $C$.

Since $G$ is cubic, for each pair of adjacent $a, b$ of $G$, by Definition 1, the subgraph $X(a, b)$ induced by vertices in $A(a) ∪ A(b)$ in $X(G)$ is isomorphic to $K_{2,2}$.

For any two vertices $a, b$ adjacent in $B$, let $a_1, a_2$ be two distinct neighbors of $a$ other than $b$, and $b_1, b_2$ be two distinct neighbors of $b$ other than $a$. Since each of $C(a)$ and $C(b)$ contains no heavy visit, we have $C(a) = [[a_1, a, b], \left[a_1, a, a_2\right], \left[a_1, a, a_2\right]]$ and $C(b) = [[b_1, b, a], \left[b_1, b, a\right], \left[b_1, b, b_2\right]]$. Let $e_1, e_2$ be parallel between $a$ and $b$, and $b_3$ in $B^*$. Then each of $e_1, e_2$ be contained in a visit of $C(a)$ and a visit of $C(b)$. Assume w.l.o.g. that $[a_1, a, b], [b_1, b, a]$ contain $e_1$ and $[a_2, a, b], [b_1, b, a]$ contain $e_2$. Then each of $a_1, a_2, b_1$ and $a_2, a_1, b_2$ is a segment of length 3 of $C$. We may define each of these two segments as a visit to the edge $(a, b)$ of $B$, and denote them as $[a_1, a, b, b_1], [a_2, a, b, b_2]$, respectively. From the definition of $B^*$, each edge, and hence $[a, b]$, of $B$ is visited twice by $C(a, b)$. By $C(a, b)$ the set of two visits of $C$ to the edge $(a, b)$, i.e., $C((a, b)) = [[a_1, a, b, b_1], [a_2, a, b, b_2]]$.

Then, each element of $C((a, b))$ is a trail of length 3 which corresponds exactly to one edge of $X(a, b)$ covered by $C^X$. And, the two visits of $C((a, b))$ are in fact corresponding to two independent edges of $X(a, b)$, namely, $[a_1, a, b, b_1], [aa_1, bb_1]$, which are covered by $C^X$. This means that $C^X$ covers exactly one perfect matching (two independent edges) of $X(a, b)$ and leaves the other perfect matching uncovered by noting that $X(a, b) ≅ K_{2,2}$ contains two perfect matchings.

Apply the edge-shift operation (Definition 3) to $C$ with respect to the 2 parallel edges connecting $a$ and $b$, denote the resulted trail(s) by $C_0$. Then $C((a, b))$ is transformed into $C_0((a, b)) = [[a_1, a, b, b_1], [a_2, a, b, b_2]]$. If $C_0$ stays as an Eulerian tour of $G^*$, one can observe that any Hamilton cycle $C^X_0$, derived from $C_0$, will cover the pair of independent edges of $X(a, b)$ which are not covered by $C^X$.

The following claim shows that we can apply the edge-shift operations to every pair of parallel edges of $B^*$ once and still get an Eulerian tour of $G^*$.

**Claim.** There are a series of edge-shift operations such that to every pair of parallel edges of $B^*$, exactly one operation is performed and the resultant trail is an Eulerian tour of $G^*$.

**Proof of the Claim.** First consider an edge $[x, y]$ belonging to none of the cycles of $B$, then $[x, y]$ is a bridge of $B$. Let $e_1, e_2$ be the two parallel edges between $x$ and $y$ in $B^*$. Then the orientations of $e_1, e_2$ are opposite to each other in $C$, by (1) of Remark 2, $C(C, e_1, e_2)$ stays as an Eulerian tour of $B^*$. Thus, we can apply the edge-shift operation to each pair of parallel edges of $B^*$ joining two end-vertices of a bridge of $B$. To simplify notation, denote again by $C$ the Eulerian tour of $B^*$ after processing all such pairs of parallel edges.
Next we process edges on cycles in the cactus B. Since \( \Delta(B) \leq 3 \), any two distinct cycles in B are vertex- and edge-disjoint. We process these cycles one by one.

Let \( L = x_1, x_2, \ldots, x_l, x_1 \) be an arbitrary cycle of B, where \( l \geq 4 \) is even. Since every vertex has degree 3 in G, each \( x_i \) has a neighbor, denoted by \( x'_j \), where \( 1 \leq j \leq l \). Note that each edge \( \{x_j, x'_j\} \) is a bridge of B. Denote by \( L_i \) the component containing \( x_j \) after deleting the edge \( \{x_j, x'_j\} \). If C proceeds along some visit \( (x_{j_1}, x_{j_2}, x_{j_3}) \) into \( L_i \), C must return back to the component containing the cycle L via the visit \( (x_j, x_{j_1}, x_{j_2}) \) immediately after C finishing its all traverses in \( L_i \).

We can shrink the traverse of C in L, as a simple visit to \( x_j \), as follows: suppose C proceeds its traverses in \( L_j \) using the following trail:

\[
T': x_{j-1}, x_j, x'_j, \ldots, x_l, x_1, x_{j+1}.
\]

We can first shrink this trail into the visit \((x_{j-1}, x_j, x_{j+1})\) to \( x_j \) and this visit could be restored back to its original form \( T' \) above after the processing of the edges of L. After shrinking all such traverses of C in \( L_j \), we get an Eulerian tour \( C \) of the multigraph \( L^* \) (\( L^* \) is the multigraph obtained from the cycle by doubling each of its edges).

Equivalently, \( C \) is a single walk that covers each edge of the simple cycle L twice. Then, for each edge \( e_j = \{x_j, x_{j+1}\} \) of \( L \), denoted by \( e_j, e'_j \), the two parallel edges between \( x_j \) and \( x_{j+1} \).

Apply the edge-shift operation to \( C \) with respect to \( e_j, e'_j \). Since the orientations of \( e_j, e'_j \) in \( C \) are the same, \( C(e_j, e'_j) \) is a set of two closed trails, \( C_1, C_2 \), say. One can observe that the trail covering \( e_{j+1} \) and that covering \( e_{j-1} \) are distinct, where \( e_{j+1}, e_{j-1} \) are two parallel edges between \( x_{j+1} \) and \( x_{j-1} \). After this operation, we immediately apply the edge-shift operation to \( C_1, C_2 \) with respect to \( e_{j+1}, e_{j-1} \). Then, this operation will concatenate \( C_1 \) and \( C_2 \) together, that is, \( C(C_1, C_2; e_{j+1}, e_{j-1}) \) will become again an Eulerian tour of \( L^* \).

(Continue applying this process to the pairs \( (e_{j+2}, e_{j+1}), (e_{j-2}, e_{j-1}), \ldots, (e_{j-1}, e_{j-2}), (e_{j+1}, e_{j+2}), (e_{j+3}, e_{j+2}), \ldots, (e_{j+1}, e_{j+2}) \); we finally obtain an Eulerian tour of \( L^* \), denoted again by \( C \). Then we restore \( C \) back to an Eulerian tour of \( B^* \).

Using this procedure to process all cycles of B, we finally obtain an Eulerian tour of \( G^* \) after performing exactly one operation to every pair of parallel edges of \( B^* \). The claim follows.

Denote by \( \overline{C} \), the Eulerian tour of \( G^* \) produced after applying the edge-shift operation exactly once to each pair of parallel edges connecting two adjacent vertices of B.

Consider every edge \( e \) with one end-vertex \( x \in A \) (then \( e \) is not in B). Then all neighbours of \( x \) lie in B. Denote \( N(x) = \{x_1, x_2, x_3\} \), \( e = \{x, x_1\} \) and \( N(x_1) = \{x, z, z'\} \), where \( x_1, x_2, x_3, z \in B \). Then, by the construction of \( C \),

\[
C(x) = \{(x_1, x, x_1), (x_2, x, x_2), (x_3, x, x_3)\}.
\]

In \( X(x_1, x) \), such a heavy visit \((x_1, x, x_1)\) is exactly corresponding to two adjacent edges of \( X(x_1, x) \). For example, in this instance, if \( C \) enters \( x \) via \( z, x_1 \) and leaves via \( x_1, z' \) to complete the visit \((x_1, x, x_1, x_1, x, x_1)\); then, it 1-1 corresponds to the two edges \( \{x_1, z', x\} \), \{\( xt, x, x_1 \} \), which is a 2-path in \( X(x_1, x) \) (also in the Hamiltonian cycle \( C^X \)), where \( t \) is either \( x_2 \) or \( x_3 \). Note that the other two edges in \( X(x_1, x) \) uncovered by \( C^X \) will be available in the second Hamilton cycle \( C^X \).

We do not apply any operation on edges \( e = \{x, x_1\} \) with one end-vertex in A. We can just use the two uncovered edges in the new Hamilton cycle when constructing it. Equivalently, for each \( x \in A \), the perfect matching \( H^C(x) \) employed in deriving \( C^X \) is the one that shares no common edge with \( H^C(x) \).

Then, \( C^X \) is a Hamilton cycle sharing no edges with \( C^X \), the proof is completed. □

**Corollary 2.** Suppose \( G \) is cubic. If \( G \) has a stable set such that whose removal leaves \( G \) a tree, then \( X(G) \) has a Hamiltonian decomposition.

Since every cubic graph has an IS-C decomposition, every bipartite cubic graph has an even IS-C decomposition. We have the following.

**Corollary 3.** The 3-arc graph of any bipartite cubic graph has a Hamiltonian decomposition.

Let \( P \) be the Petersen graph. It is not difficult to find a stable set of \( P \) consisting of three vertices whose removal leaves \( P \) a tree. Let \( G \) be an \( n \)-prism, \( n \geq 3 \). Then, \( G \) is isomorphic to the Cartesian product \( C_n \boxtimes K_2 \). Denote \( V(G) = V(C_n) \cup V(C') \), where \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \), \( V(C') = \{v_1', v_2', \ldots, v_n'\} \), and, each of \( V(C_n) \) and \( V(C') \) induces an \( n \)-cycle. The edge set of \( G \) is \( E(C_n) \cup E(C') \cup \{\{v_i, v_i'\} : i = 1, 2, \ldots, n\} \). Set \( S = \{v_{2j} : j = 1, 2, \ldots, \lfloor n/2 \rfloor\} \). If \( n \) is even, then \( (S, V(G) - S) \) is an even IS-C decomposition of \( G \) with \( V(G) - S \) inducing a unicyclic graph; if \( n \) is odd, then \( (S \cup \{v_1', v_1\}, V(G) - (S \cup \{v_1\})) \) is an even IS-C decomposition of \( G \) with \( V(G) - (S \cup \{v_1\}) \) inducing a tree. By applying 2 and 3, we obtain the following example:

**Example 1.** (1) The 3-arc graph of Petersen graph has a Hamiltonian decomposition; (2) The 3-arc graph of the \( n \)-prism has a Hamiltonian decomposition.

**Remark 3.** The condition in Theorem 2 is sufficient but not necessary. For example, \( K_4 \) does not have an even IS-C decomposition but \( X(K_4) \) is still Hamiltonian decomposable. In \( X(K_4) \), see Figure 1, each of the two sets of bold edges (in color black) and thin edges (in color blue) forms a Hamilton cycle [10].

**Data Availability**

The findings of this study are supported by the rigorous proofs which are included within the paper.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
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