Counting with 3-valued truth tables of bracketed formulae connected by implication

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Abstract

In this paper we investigate the combinatorical structure of the Kleene type truth tables of all bracketed formulae with n distinct variables connected by the binary connective of implication.

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1 Notations

- \( p_1, \ldots, p_n \), and \( \phi, \psi \) are all distinct propositional variables.
- ‘True’ will be denoted by 1
- ‘False’ will be denoted by 0
- ‘Unknown’ will be denoted by 2
- The set of counting numbers is denoted by \( \mathbb{N} \)
- \( \nu \) is the valuation function:
  \[ \nu(\phi) = 1 \text{ if } \phi \text{ is true}, \nu(\phi) = 0 \text{ if } \phi \text{ is false}, \text{ and } \nu(\phi) = 2 \text{ if } \phi \text{ is unknown.} \]
- \( \wedge, \vee \) are the conjunction and disjunction operators.
- \( \Rightarrow \) the implication operator
- \( \neg \) the negation operator
- \#c denotes the case number in \( t^n \)
- For the coefficient of \( x^n \) in \( G(x) = \sum_{n \geq 1} g_n x^n \) is \( g_n \)

2 Preface

We have written in former papers about counting in truth tables in 2012. This paper was written in 2013, but never been published due to time constraints and changes in my living conditions. Now during the pandemic period, I have some time in my hands to return to my incomplete work. For 2-valued truth tables counting arguments one can refer back to my paper ‘General combinatorical structure of truth tables of bracketed formulae connected with implication’, [1].

3 Intro

Recall that the number of bracketings of a product of \( n \) terms is the Catalan number:

\[ C_n = \frac{1}{n} \binom{2n}{n-1} \]

and its generating function

\[ C(x) = \frac{1 - \sqrt{1 - 4x}}{2} \]

Here we define the implication operator in the following way, “Kleene’s way”

\[
\begin{array}{ccc}
\Rightarrow & 1 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
2 & 1 & 2
\end{array}
\]

Proposition 1. Let \( g_n \) be the total number of rows in all Kleene truth tables for bracketed implication with \( n \) distinct variables \( p_1, \ldots, p_n \). Then

\[ g_n = \sum_{i=1}^{n-1} g_i g_{n-i}, \quad g_1 = 3 \]

Proof.

\[ g_n = 3^n C_n = 3^n \sum_{i=1}^{n-1} C_i C_{n-i} = \sum_{i=1}^{n-1} (3^i C_i)(3^{n-i} C_{n-i}) = \sum_{i=1}^{n-i} g_i g_{n-i} \]

Thus it has the following generating function.

\[ G(x) = \frac{1 - \sqrt{1 - 12x}}{2} \] (1)
Proposition 2. Let $f_n$, $t_n$ and $u_n$ be the number of rows with the value “false”, “true”, and “unknow” in the Kleene truth tables of all bracketed formulae with $n$ distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of implication. Then $u_n$ has the following recurrence relation, and generating function $U(x)$.

\[ u_n = \sum_{i=1}^{n-1} u_i g_{n-i} = \sum_{i=1}^{n-1} u_i 3^{n-i} C_{n-i}, \quad u_1 = 1 \]

\[ U(x) = \frac{1 - \sqrt{1 - 12x}}{6} \]

Proof.

\[ u_n = t_1 u_{n-1} + u_i f_{n-i} + u_i u_{n-i} = t_i u_{n-i} + u_i (g_{n-i} - t_{n-i}) \]

Summing over $n$, gives us

\[ U(x) = x + U(x) G(x) \]

Solving this for $U(x)$ gives us the required result.

⋆

Corollary 3. The number of unknown entries in bracketed Kleene’s truth table connected by the implication is given by

\[ u_n = \frac{3^{n-1}}{n} \binom{2n-2}{n-1} \quad (2) \]

Proposition 4. $f_n$ has the following recurrence relation,

\[ f_n = \sum_{i=1}^{n-1} f_i \left( 2C_{n-i} 3^{n-i-1} - f_{n-i} \right), \quad f_1 = 1 \]

and $f_n$ has the following generating function

\[ F(x) = \frac{-2 - \sqrt{1 - 12x} + \sqrt{5 + 24x + 4\sqrt{1 - 12x}}}{6} \]

Proof. Consider

\[ f_n = f_1 t_{n-i} \]
\[ = f_1 (g_{n-i} - f_{n-i} - u_{n-i}) \]
\[ = f_1 3^{n-i} C_{n-i} - f_1 f_{n-i} - f_1 3^{n-i-1} C_{n-i} \]
\[ = f_1 (2C_{n-i} 3^{n-i-1} - f_{n-i}) \]

Summing over $n$, gives us

\[ F(x) = 2F(x) U(x) - F(x)^2 + x. \]

Solving it for $F(x)$ gives us the required result.

⋆

Corollary 5. $t_n$ has the following generating function

\[ T(x) = \frac{4 - \sqrt{1 - 12x} - \sqrt{5 + 24x + 4\sqrt{1 - 12x}}}{6} \]

4 Asymptotic 1

Theorem 6. Let $f_n$, $t_n$ and $u_n$ be number of rows with the value false, true and unknown in the Kleene truth tables of all the bracketed implications with $n$ variables. Then we have the following asymptotics

\[ f_n \sim \left( \frac{7 - 2\sqrt{7}}{21} \right) \frac{12^{n-1} 3}{\sqrt{\pi n^3}} \]
\[ t_n \sim \left( \frac{7 + 2\sqrt{7}}{21} \right) \frac{12^{n-1} 3}{\sqrt{\pi n^3}} \]
\[ u_n \sim \left( \frac{1}{3} \right) \frac{12^{n-1} 3}{\sqrt{\pi n^3}} \]
\[ g_n \sim \frac{12^{n-1} 3}{\sqrt{\pi n^3}} \]
Proof. Recall
\[ F(x) = \frac{-2 - \sqrt{1 - 12x} + \sqrt{5 + 24x + 4\sqrt{1 - 12x}}}{6} \]
By using the asymptotic techniques that we have discussed in [1], we have \( r = \frac{1}{12} \) and \( F(\frac{1}{12}) \neq 0 \). So let \( A(x) = F(x) - F(\frac{1}{12}) \).
\[
\lim_{x \to \frac{1}{12}} A(x) = \lim_{x \to \frac{1}{12}} \frac{-\sqrt{1 - 12x} + \sqrt{5 + 24x + \sqrt{1 - 12x}}}{6\sqrt{1 - 12x}} - \frac{7 - 2\sqrt{7}}{12} \]
Therefore
\[
f_n \sim \frac{7 - 2\sqrt{7}}{21} \left( n - \frac{3}{2} \right) \frac{1}{12} ^{n} \sim \frac{7 - 2\sqrt{7}}{21} \frac{12^{n-1}3}{\sqrt{\pi n^3}} \]
With similar arguments we have the above asymptotics for \( t_n \) and \( u_n \).

**Corollary 7.** The number of rows with unknown in the Kleene truth tables is the average of the number of rows with true and false.
\[
\frac{f_n + t_n}{2} = u_n, \quad \forall n \geq 1. \]

Proof. Since \( u_n = 3^{n-1}C_n \) \( \forall n \geq 1 \), we have
\[
f_n + t_n = \frac{2}{3} g_n, \quad 3(f_n + t_n) = 2g_n, \quad f_n + t_n = 2g_n - 2f_n - 2t_n, \quad f_n + t_n = 2u_n, \quad u_n = \frac{f_n + t_n}{2}. \]

Note here
\[
\frac{7 - 2\sqrt{7}}{21} \approx 0.0813570180, \quad \frac{7 + 2\sqrt{7}}{21} \approx 0.5853096486, \quad \frac{1}{3} \approx 0.333333333 \]
\[
\frac{7 - 2\sqrt{7}}{21} + \frac{7 + 2\sqrt{7}}{21} = \frac{2}{3} \]
The below table shows the sequences which we have discussed so far, up to \( n = 9 \).

|   | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|   | 10n   | 15n   | 30n   | 50n   | 120n  | 198n  | 294n  | 402n  | 519n  |
|   | 1     | 1     | 6     | 41    | 330   | 2882  | 26604 | 255313 | 2521986 |
|   | 1     | 3     | 18    | 135   | 1134  | 10206 | 96228 | 938223 | 9382230 |
|   | 3     | 9     | 54    | 405   | 3402  | 30618 | 28684 | 2814699 | 28146990 |

5 Generation of truth table sequences

Since \( g_n = t_n + f_n + u_n, \forall n \geq 1 \), we have
\[
g_n = \sum g_n g_{n-i} = \sum (t_i + f_i + u_i)(t_{n-i} + f_{n-i} + u_{n-i}) = \sum t_i t_{n-i} + \sum t_i f_{n-i} + \sum t_i u_{n-i} + \sum f_i t_{n-i} + \sum f_i f_{n-i} + \sum f_i u_{n-i} + \sum u_i t_{n-i} + \sum u_i f_{n-i} + \sum u_i u_{n-i} \]
We can generate nine more sequences: $t_n^{#1}$, $t_n^{#2}$, $t_n^{#3}$, $t_n^{#4}$, $t_n^{#5}$, $f_n$, $u_n^{#1}$, $u_n^{#2}$, and $u_n^{#3}$, except from $f_n$, all other sequences equals to 0 when $n = 1$. Each of these sequences, (and their generating functions) counts different rows of the corresponding truth table. E.g. Let $\phi$ and $\psi$ be propositional variables, then

$$\nu(\phi \Rightarrow \psi) = 1 : (\nu(\phi) = 1 = \nu(\psi))$$

$$\nu(\phi \Rightarrow \psi) = 1 : (\nu(\phi) = 0 \land \nu(\psi) = 1)$$

$$\nu(\phi \Rightarrow \psi) = 1 : (\nu(\phi) = 0 = \nu(\psi))$$

$$\nu(\phi \Rightarrow \psi) = 1 : (\nu(\phi) = 0 = \nu(\psi) = 2)$$

$$\nu(\phi \Rightarrow \psi) = 1 : (\nu(\phi) = 2 \land \nu(\psi) = 1).$$

In each case we are interested in formulae obtained from $p_1 \Rightarrow \ldots \Rightarrow p_n$ by inserting brackets such that the valuation of the first $i$ bracketing and the rest $(n-i)$ bracketing both give 1, ‘true’; such that the valuation of the first $i$ bracketing is 0 and the rest $(n-i)$ bracketing gives 1; such that the valuation of the first $i$ bracketing and the rest $(n-i)$ bracketing both give 0; such that the valuation of the first $i$ bracketing is 0 and the rest $(n-i)$ bracketing gives 2; such that the valuation of the first $i$ bracketing is 2 and the rest $(n-i)$ bracketing gives 1, respectively. To get the corresponding generating function for $t_n^{#1}$ we can make the following calculations

$$t_n^{#1} = \sum_{n=1}^{n-1} t_i t_{n-1}$$

Summing over $n$ gives us $T_1(x) = T(x)^2$. Using the same method we can obtain the following generating functions: $T_2(x) = F(x)T(x)$, $T_3(x) = F(x)^2$, $T_4(x) = F(x)U(x)$, and $T_5(x) = U(x)T(x)$.

A few terms for these fresh sequences:

\[
\begin{align*}
(t_n^{#1})_{n>0} &= 0, 1, 10, 85, 758, 7066, 68180, 675725, 6840190, 70431982, 735446924, \ldots \\
(t_n^{#2})_{n>0} &= 0, 1, 6, 43, 330, 2882, 26604, 255313, 2521986, 25473638, 261898548, \ldots \\
(t_n^{#3})_{n>0} &= 0, 1, 2, 13, 94, 778, 6916, 64613, 625478, 6219070, 63138652, \ldots \\
(t_n^{#4})_{n>0} &= 0, 1, 4, 27, 212, 1830, 16760, 159963, 1573732, 15846354, 162518600, \ldots \\
(t_n^{#5})_{n>0} &= 0, 1, 8, 63, 544, 4974, 47392, 465519, 4681088, 47952810, 498672736, \ldots \\
(t_n)_{n>0} &= 1, 5, 30, 229, 1938, 17530, 165852, 1621133, 16242174, 165923854, 1721675460, \ldots
\end{align*}
\]

Note that $\forall n \geq 1 \ t_n = \sum_{i=1}^{5} t_i^{#i}$. With similar arguments we can get the rest of the generating functions and their corresponding sequences.

$$\nu(\phi \Rightarrow \psi) = 0 : (\nu(\phi) = 1 \land \nu(\psi) = 0)$$

$$\nu(\phi \Rightarrow \psi) = 2 : (\nu(\phi) = 2 \land \nu(\psi) = 0)$$

$$\nu(\phi \Rightarrow \psi) = 2 : (\nu(\phi) = 2 = \nu(\psi))$$

$F(x)$ and $U(x)$ have been studied in former chapters. Here we want to get more sequences from the original sequence $u_n$, i.e. we want to break $u_n$ into $u_n^{#1}$, $u_n^{#2}$, and $u_n^{#3}$. Moreover the following corresponding generating function, and their sequences exist: $U_1(x) = T(x)U(x)$, $U_2(x) = U(x)F(x)$, and $U_3(x) = U(x)^2$.

\[
\begin{align*}
(u_n^{#1})_{n>0} &= 0, 1, 8, 63, 544, 4974, 47392, 465519, 4681088, 47952810, 498672736, \ldots \\
(u_n^{#2})_{n>0} &= 0, 1, 4, 27, 212, 1830, 16760, 159963, 1573732, 15846354, 162518600, \ldots \\
(u_n^{#3})_{n>0} &= 0, 1, 6, 45, 378, 3402, 32076, 312741, 3127410, 31899582, 330595668, \ldots \\
(u_n)_{n>0} &= 1, 3, 18, 135, 1134, 10206, 96228, 938223, 9382230, 95698746, 9917870040, \ldots
\end{align*}
\]

Consequently, the following observation is essential.

**Corollary 8.**

$$g_n = \sum_{i=1}^{5} t_i^{#i} + \sum_{i=1}^{3} u_i^{#i} + f_n \sum_{i=1}^{3} t_i^{#i} + 2(t_n^{#4} + t_n^{#5}) + u_n^{#3} + f_n$$
6  Asymptotics 2

In this part we will be exploring the asymptotics of the sequences which we have seen in the former chapter.

Lemma 9. Consider the sequences that we have discussed in the former chapter: \( t^{#1}_n \), \( t^{#2}_n \), \( t^{#3}_n \), \( t^{#4}_n \), \( t^{#5}_n \), and \( u^{#3}_n \), then we have the following asymptotics

\[
\begin{align*}
  t^{#1}_n &\sim \left(\frac{14 + \sqrt{7}}{63}\right) \frac{12^{n-1}3}{\sqrt{\pi n}}, \\
  t^{#2}_n &= f_n = t^{#2}_n \sim \left(\frac{7 - 2\sqrt{7}}{21}\right) \frac{12^{n-1}3}{\sqrt{\pi n}}, \\
  t^{#3}_n &\sim \left(\frac{11\sqrt{7} - 28}{63}\right) \frac{12^{n-1}3}{\sqrt{\pi n}^3}, \\
  u^{#1}_n &= t^{#5}_n \sim \left(\frac{35 - 5\sqrt{7}}{126}\right) \frac{12^{n-1}3}{\sqrt{\pi n}^3}, \\
  u^{#2}_n &= u^{#2}_n \sim \left(\frac{5\sqrt{7} - 7}{126}\right) \frac{12^{n-1}3}{\sqrt{\pi n}^3}, \\
  u^{#3}_n &\sim \left(\frac{1}{9}\right) \frac{12^{n-1}3}{\sqrt{\pi n}^3}.
\end{align*}
\]

where

\[
\left(\frac{14 + \sqrt{7}}{63}\right) \approx 0.2642182748, \quad \left(\frac{14 + \sqrt{7}}{42}\right) \approx 0.0813570180
\]

\[
\left(\frac{11\sqrt{7} - 28}{63}\right) \approx 0.0175121337, \quad \left(\frac{5\sqrt{7} - 7}{126}\right) \approx 0.04943457584
\]

\[
\left(\frac{35 - 5\sqrt{7}}{126}\right) \approx 0.1727876464, \quad \left(\frac{1}{9}\right) \approx 0.1111111111.
\]

Proof. Proofs are similar to the proof of theorem 6. No need to repeat the same type of calculations.

6.1 comparing asymptotics with 2 valued truth tables

\[
\begin{align*}
  t^{#1}_n &\sim \frac{14 + \sqrt{7}}{63} & \text{decreased } \approx 47.2\% \\
  t^{#2}_n &= f_n = \frac{3 - \sqrt{3}}{6} & \text{decreased } \approx 61.5\% \\
  t^{#3}_n &\sim \frac{2\sqrt{3} - 3}{6} & \text{decreased } \approx 77.4\% \text{ decreased}
\end{align*}
\]

References

[1] Volkan Yildiz. General Combinatorical Structure of Truth Tables of Bracketed Formulae Connected by Implication. Arxiv: https://arxiv.org/abs/1205.5595

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