Internal entanglement and external correlations of any form limit each other
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We show a relation between entanglement and correlations of any form. The internal entanglement of a bipartite system, and its correlations with another system, limit each other. A measure of correlations, of any nature, cannot increase under local operations. Examples are the entanglement monotones, the mutual information, that quantifies total correlations, and the Henderson-Vedral measure of classical correlations. External correlations, evaluated by such a measure, set a tight upper bound on the internal entanglement that decreases as they increase, and so does quantum discord.

Quantum entanglement is a useful resource for many tasks, such as cryptographic key distribution [1], state teleportation [2], or clock synchronization [3], to cite just a few. In more precise terms, it is a quantum resource that cannot be generated by local operations and classical communication [4-6]. The corresponding so-called free states, for which the resource vanishes, are the separable states, that are the mixtures of product states. Accordingly, entanglement is quantified by measures, termed entanglement monotones, which are non-negative functions of quantum states that vanish for separable states, and are nonincreasing under state transformations involving only local operations and classical communication.

Two real systems, whose entanglement is of interest, are never completely isolated from the surroundings. Consequently, a third system, which cannot be fully controlled, always comes into play. Using Hamiltonian models describing the influence of more or less realistic environments, different dynamic behaviours of the entanglement have been found, depending, for example, on whether the environment is in thermal equilibrium or not. For instance, an initial entanglement can vanish in finite time [7], or, on the contrary, entanglement can develop transiently [8, 9], or even be steady [10-12].

The impact of the surroundings on entanglement can also be approached by studying how entanglement is distributed between three systems in an arbitrary state. The amounts of entanglement between one of them and each of the two other ones constrain each other. This behaviour, known as entanglement monogamy, has first been shown for three two-level systems, and expressed as an inequality involving a particular entanglement monotone [13]. This inequality does not hold in general for familiar monotones such as the entanglement of formation, or the regularized relative entropy of entanglement. For these two measures, inequalities involving Hilbert space dimensions explicitly must be considered [14]. Relations have also been found between the amounts of entanglement for the three bipartitions of a tripartite system [15].

Recently, another restriction on the distribution of entanglement between three systems has been shown [16]. It is better understood by considering a finite-dimensional bipartite system, say A, and any other system, say B, which can be seen as the environment of A. It has been found that the internal entanglement, between the two subsystems of A, and the external entanglement, between A and B, limit each other. This relation is expressed by an inequality involving entanglement monotones and the Hilbert space dimensions of the subsystems of A. One may wonder whether this is a specific property of entanglement, or whether a similar relation exists between internal entanglement and external correlations of any kind.

In this Letter, we address this issue by using measures of external correlations, which we term correlation monotones. Such a measure C is a non-negative function of the state ρ shared by A and B, that vanishes for product states, and is nonincreasing under local operations, which do not affect either A or B. These are basic requirements for a measure of correlations, since correlations, whatever their nature, cannot increase when A and B evolve independently. Our main result relies essentially on them. To be more specific, our derivation does not require that C is a strict correlation monotone, but only that it is invariant under unitary local operations, and nonincreasing under operations performed on B. Examples of correlation monotones are the entanglement monotones, the mutual information, commonly used to quantify total correlations, and the Henderson-Vedral (HV) measure of classical correlations [17]. Quantum discords, on the other hand, are not correlation monotones [18]. However, the original quantum discord [19] as measured by A, satisfies the above mentioned properties [20, 21], and so our approach applies to it.

We show in the following that, for an arbitrary finite-dimensional bipartite system A and any system B, under an assumption of continuity usually fulfilled, C(ρ) and the internal entanglement of A are related to each other. More precisely, C(ρ) determines a tight upper bound on E(ρA), where ρA is the reduced density operator for A, and E is any convex entanglement monotone, that decreases as C(ρ) increases, see Figs. 1 and 2. As we will see, for familiar correlation monotones, this bound vanishes when C(ρ) equals to its maximum value, set by the Hilbert space dimension of A. Moreover, since our result holds when C is the HV measure, it implies that, even
when the external correlations are purely classical, they have a detrimental influence on internal entanglement.

In the following, \( \lambda(\omega) \) refers to the vector made up of the nonzero eigenvalues of the quantum state \( \omega \), in decreasing order. It is a probability vector, i.e., its components are positive and sum to unity. If \( \omega \) is a density operator on the Hilbert space \( \mathcal{H}_d \) of dimension \( d \), \( \lambda(\omega) \) belongs to the set \( \mathcal{E}_d \) of probability vectors of no more than \( d \) components. We call entropy any non-negative function of the probability vectors \( p \), which is nondecreasing with disorder, in the sense of majorization [22], and vanishes for \( p = 1 \) [23]. Any entropy has a largest value on \( \mathcal{E}_d \), reached for the equally distributed vector \( (1/d, \ldots, 1/d) \), which is majorized by any \( p \in \mathcal{E}_d \), and possibly also for other vectors.

To derive our main result, we use the following three Lemmas. The proofs of the first and third are given in the Supplemental Material [24]. The second is proved in Ref. [16].

**Lemma 1.** For any correlation monotone \( C \), there is a function \( f \) of the probability vectors with \( f(1) = 0 \), such that, for any global state \( \rho \),

\[
C(\rho) \leq f(\lambda(\rho_A)),
\]

with equality when \( \rho \) is pure.

We denote by \( c_d \) the supremum of \( f \) on \( \mathcal{E}_d \). Due to eq. [1], \( C(\rho) \) cannot exceed \( c_d \) when the Hilbert space of \( A \) is \( \mathcal{H}_d \). When \( C \) is an entanglement monotone, \( f \) is necessarily an entropy [30]. It is the Shannon entropy \( h \) for many familiar entanglement monotones and for the HV measure [6, 17, 31, 33]. For robustness and negativity, \( f \) is a function of the Rényi entropy [31, 37]. From the Araki-Lieb inequality \( S(\rho) \geq |S(\rho_B) - S(\rho_A)| \), where \( S \) is the von Neumann entropy [38], it follows that \( f = 2h \) for the mutual information \( S(\rho_A) + S(\rho_B) - S(\rho) \). As mentioned in the introduction, the quantum discord as measured by \( A \), though not a correlation monotone, has the required properties to satisfy Lemma 1; see the proof. The corresponding function \( f \) is \( h \) [21]. When \( f \) is an entropy, \( C \) coincides with an entanglement monotone for pure states [6, 39]. For all the correlation monotones mentioned above, \( f \) equals \( c_d \) for \( (1/d, \ldots, 1/d) \), and for no other vector of \( \mathcal{E}_d \).

This means that, on the set of the pure states \( |\psi\rangle \) of \( \mathcal{H}_d \otimes \mathcal{H}_d \), where \( d' \geq d \), the maximally entangled states are the only ones for which \( C(|\psi\rangle\langle\psi|) \) is maximum.

**Lemma 2.** For any convex entanglement monotone \( E \), and integers \( d_1 \geq 2 \) and \( d_2 \geq d_1 \), there are a positive number \( e_{d_1} \) and an entropy \( s_{d_1,d_2} \) such that the states \( \rho_A \) on \( \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \) satisfy

\[
E(\rho_A) \leq e_{d_1} - s_{d_1,d_2}(\lambda(\rho_A)),
\]

and such that, for any \( p \in \mathcal{E}_{d_1} \times \mathcal{E}_{d_2} \) and \( \eta > 0 \), there is \( \rho_A \) for which \( \lambda(\rho_A) = p \) and \( e_{d_1} - s_{d_1,d_2}(p) - E(\rho_A) < \eta \).

This Lemma expresses quantitatively how the mixedness of a quantum state limits its amount of entanglement [40]. In Ref. [16], \( e_{d_1} \) is obtained as the largest value of \( E(\rho_A) \) for pure states \( \rho_A \). Thus, it depends only on \( d_1 \) [5]. Inequality [4] shows that it is the maximum of \( E(\rho_A) \) on the set of all the density operators \( \rho_A \) on \( \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \). Contrary to \( e_{d_1} \), the entropy \( s_{d_1,d_2} \) can depend on both \( d_1 \) and \( d_2 \), see the Supplemental Material.

**Lemma 3.** For any positive integer \( d \), entropy \( s \), and nonnegative continuous function \( f \) of the probability vectors with \( f(1) = 0 \), there is a nondecreasing function \( g_d \) on \( I = [0, c_d] \), such that \( g_d(0) \) is the maximum of \( f \) on \( \mathcal{E}_d \), such that \( g_d(0) = 0 \), \( g_d \) is \( \leq s \) on \( \mathcal{E}_d \), and, for any \( x \in I \) and \( \eta > 0 \), there is \( p \in \mathcal{E}_d \), such that \( f(p) = x \), and \( s(p) = g_d(x) < \eta \).

If \( f(1/d, \ldots, 1/d) = c_d \) and \( f(p) < c_d \) for any other \( p \in \mathcal{E}_d \), then \( g_d(c_d) = s(1/d, \ldots, 1/d) \).

Using this Lemma with the function given by Lemma [1] and the entropy \( s_{d_1,d_2} \) given by Lemma [2] and defining \( \xi_{d_1,d_2} = e_{d_1} - g_d \), with \( d = d_1 d_2 \), we have the following result.

**Theorem.** Let \( \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \), with \( d_2 \geq d_1 \), be the Hilbert space of system \( A \), and \( d = d_1 d_2 \).

For a convex entanglement monotone \( E \), and a correlation monotone \( C \) such that \( f \) is continuous, \( C(\rho) \) and \( E(\rho_A) \) obey, for any global state \( \rho \),

\[
E(\rho_A) \leq \xi_{d_1,d_2}(C(\rho)),
\]

where \( \xi_{d_1,d_2} \) is a nonincreasing function on \( [0, c_d] \) with \( \xi_{d_1,d_2}(0) = e_{d_1} \). For any amount of correlations \( x \leq c_d \), there are states \( \rho \) such that \( C(\rho) = x \) and the two sides of inequality [3] are as close to each other as we wish.

If \( f(1/d, \ldots, 1/d) = c_d \) and \( f(p) < c_d \) for any other \( p \in \mathcal{E}_d \), then \( \xi_{d_1,d_2}(c_d) = 0 \).

Inequality [3] can be rewritten, in a more familiar form, as \( E(A_1 : A_2) \leq \xi_{d_1,d_2}(C(A_1 A_2 : B)) \), where \( A_1 \) and \( A_2 \) are the two subsystems of \( A \), see Fig. 1 [10]. For any \( x \in [0, c_d] \) and small \( \eta \), Lemmas [2] and [3] ensure that there is a local state \( \rho_A \) such that \( \xi_{d_1,d_2}(x) - E(\rho_A) < \eta \) and \( f(\lambda(\rho_A)) = x \). Due to Lemma [1], all the pure states \( \rho \) for which the reduced density operator for \( A \) is \( \rho_A \), are such that \( C(\rho) = x \). For such global states \( \rho \), \( E(\rho_A) \approx \xi_{d_1,d_2}(C(\rho)) \), and an increase of the correlations between \( A \) and \( B \) means a reduction of the internal entanglement of \( A \), and reciprocally. In general, the external correlations and the local entanglement limit each
other, see Fig. 2. For any amount of correlations \( x \leq \epsilon_d \), there is no state \( \rho \) such that \( C(\rho) = x \) and \( E(\rho_A) \) exceeds \( \xi_{d_1,d_2}(x) \). Similarly, for any amount of entanglement \( y \leq \epsilon_{d_1} \), there is no state \( \rho \) such that \( E(\rho_A) = y \) and \( C(\rho) \) is larger than the bound given by eq. (3). On the contrary, there are no positive lower bounds for \( E(\rho_A) \), for a given \( C(\rho) \), and for \( C(\rho) \), for a given \( E(\rho_A) \), whatever are the monotones \( E \) and \( C \).

For more than two systems, say \( A, B_1, B_2, \ldots \), different bounds on the entanglement \( E(\rho_A) \) can be obtained via eq. (3), depending on which systems \( B_n \) are taken into account. Let us first observe that only the systems sharing a state with genuine multipartite correlations matter \([12]\). Indeed, if the global state is of the form \( \rho = \tilde{\rho} \otimes \hat{\rho} \), where \( \tilde{\rho} \) is the state of \( A \) and some systems \( B_n \), and \( \hat{\rho} \) is the state of the other systems, then \( C(\rho) = C(\tilde{\rho}) \), where \( C \) measures the amount of correlations between \( A \) and the considered systems \( B_n \), since \( \rho \) and \( \tilde{\rho} \) can be transformed into each other by local operations. For a global state \( \rho \) with genuine multipartite correlations, as tracing out a system \( B_n \), a local operation \( \xi_{d_1,d_2} \) is a non-increasing function, the lowest bound on \( E(\rho_A) \) is given by eq. (3) with the state \( \rho \) of all the systems.

We now consider specific cases for which the boundary given by eq. (3) can be determined explicitly. A measure of total correlations can be defined as a minimal distance to the set of product states, i.e., \( C^{(D)}(\rho) = \inf_{\delta_A,\delta_B} D(\rho, \delta_A \otimes \delta_B) \), where the infimum is taken over all the density operators of \( A \) and \( B \), and \( D \) fulfills \( D[\Lambda(\omega), \Lambda'(\omega')] \leq D(\omega, \omega') \) for any quantum operation \( \Lambda \). Some possible choices for \( D \) are the relative entropy, the Bures distance \( D_B \), or the Hellinger distance \( D_H \) \([13–45]\). For the relative entropy, the above definition gives the mutual information \([44]\). For the monotones \( C^{(D_B)} \) and \( C^{(D_H)} \), an explicit expression for \( f \) can be obtained, see the Supplemental Material. For the entanglement of formation \( E_f \), the entropy \( s_{2,2} \) is known \([46]\). Using these results, we find

\[
E_{f,D_B}(x) = u \left( x^2 - \frac{x^4}{4} \right), \quad E_{f,D_H}(x) = u \left( \frac{x^2}{2} \right),
\]

where \( x \) varies from 0 to 1 for \( C^{(D_B)} \), and from 0 to \( \sqrt{3}/2 \) for \( C^{(D_H)} \). The expression of \( u \) is given in the Supplemental Material.

Figure 2 displays these two functions. They both vanish on a finite interval. As a consequence, for any state \( \rho \) such that \( C(\rho) \) exceeds a threshold value, the local entanglement \( E(\rho_A) \) necessarily vanishes, whereas, for any amount of correlations \( x \) below this threshold, there are states \( \rho \) such that \( C(\rho) = x \) and \( \rho_A \) is entangled. The existence of this threshold also implies that \( C(\rho) \) is at a finite distance from the maximum value \( c_d \) as soon as \( E(\rho_A) \) is not zero. As shown in the Supplemental Material, this feature is not specific to the particular cases considered above. Moreover, the threshold is the same for all the monotones \( E \) vanishing only for separable states.

As seen above, for some correlation monotones, \( C(\rho) = \epsilon_d \) ensures the vanishing of \( E(\rho_A) \). On the contrary, for any monotones \( C \) and \( E \), and dimension \( d_1 \), there are states \( \rho \) for which \( E(\rho_A) = \epsilon_{d_1} \) and \( C(\rho) \) is as high as we wish, provided \( d_2 \) is large enough. They are pure states \( \rho \) such that the reduced density operator \( \rho_A = \sum_j p_j |\phi_j\rangle \langle \phi_j| \) is a mixed maximally entangled state \([47]\). That is to say, the eigenvectors of \( \rho_A \) are of the form \( |\phi_j\rangle = \sum_{i=1}^{d_1} |j\rangle_1 |ij\rangle_2 / \sqrt{d_1} \), where \( |j\rangle_1 \) are orthonormal states of \( \mathcal{H}_{d_1} \), and \( |ij\rangle_2 \) of \( \mathcal{H}_{d_2} \), i.e., \( 2(i|i\rangle_{j'} \rangle_{j'}^\prime) = \delta_{ij} \delta_{j'j'} \).

As \( \rho \) is pure, \( C(\rho) = f(\rho) \), and, provided \( d_2/d_1 \) is large enough, there is a \( p \) such that \( C(\rho) \geq x \), where \( x \) is any amount of correlations. For any entanglement monotone \( E \), \( E(\rho_A) = E(|\phi_1\rangle \langle \phi_1|) = \epsilon_{d_1} \), since \( \rho_A \) and \( |\phi_1\rangle \langle \phi_1| \) can be transformed into each other by local operations, that do not affect one subsystem of \( A \) \([15]\). Note that, though \( E(\rho_A) = \epsilon_{d_1} \) does not imply \( C(\rho) = 0 \) in general, this is true for the entanglement of formation \( E_f \) and \( d_2 < 2d_1 \), since, for such dimensions, the only states \( \rho_A \) for which \( E_f(\rho_A) \) is maximum are pure \([47]\).

The above Theorem applies to many kinds of external correlations, as discussed below Lemma 1. When \( C \) is an entanglement monotone, it generalizes previously obtained results \([10]\). As mentioned above, \( C \) can also be a measure of total correlations, or the HV measure of classical correlations. For this last correlation monotone, equation (1) is an equality for some classical-classical states \( \rho = \sum_{i,j} p_{ij} |i\rangle_A \langle i| \otimes |j\rangle_B \langle j| \), where \( |i\rangle_A \) are orthonormal states of \( A \), \( |j\rangle_B \) of \( B \), and \( p_{ij} \) are probabilities summing to unity \([49–50]\). They are the strictly correlated classical-classical states, i.e., such that \( p_{ij} = p_i \delta_{ij} \) \([17]\). Consequently, there are not only pure states but also

![Figure 2: Maximum internal entanglement as a function of external correlations, for a system A consisting of two two-level systems, the entanglement of formation \( E_f \), and the measure of total correlations \( C^{(D_B)} \) (solid line). The maximum entanglement \( E_f(\rho_A) \) for classical-classical states \( \rho \) is given by the dashed line.](image-url)
classical-classical states close to the boundary given by eq.\((3)\), for any amount of correlations. Moreover, since \(\xi_{d_1,d_2}(c_d) = 0\), this shows that, even when external correlations are purely classical, the maximum accessible local entanglement decreases to zero as they increase.

In general, it can be proved that the classical-classical states \(\rho\) obey eq.\((1)\) with \(f\) replaced by a function \(\tilde{f}\) such that \(C(\rho) = \tilde{f}[\lambda(\rho_A)]\) when \(\rho\) is strictly correlated, see the Supplemental Material. Provided \(\tilde{f}\) is continuous, it follows that, for a classical-classical state \(\rho, E(\rho_A)\) and \(C(\rho)\) satisfy eq.\((3)\) with \(\xi_{d_1,d_2}\) replaced by an a priori different function \(\tilde{\xi}_{d_1,d_2}\). When \(C\) is an entanglement monotone, this is meaningless, since \(C(\rho) = 0\) for all classical-classical states \(\rho\). As seen above, for the HV measure, \(\xi_{d_1,d_2} = \xi_{d_1,d_2}\). For other correlation monotones, they obviously fulfill \(\xi_{d_1,d_2} \leq \xi_{d_1,d_2}\). For the measure of total correlations \(C^{(DB)}\) and the entanglement of formation \(E_f\), we find \(\xi_{d_1,d_2}^{C^{(DB)}} = \xi_{d_1,d_2}^{E_f}\), see the Supplemental Material and Fig.2. For the mutual information, \(\tilde{f}\) is the Shannon entropy \(h\). For this correlation monotone, inequality \((1)\) with \(h\) in place of \(f\), and hence \(E(\rho_A) \leq \xi_{d_1,d_2}[C(\rho)]\), is actually valid for all separable states \(\rho\), as \(S(\rho_{AB}) \leq S(\rho)\) for any separable state \(\rho\) [51], and, since \(f = 2\tilde{f} = 2h\), \(\xi_{d_1,d_2}(x) = \xi_{d_1,d_2}(2x)\) where \(x \in [0, \ln d]\), for any entanglement monotone \(E\).

We finally discuss the relations of other local properties to external correlations. A first natural question is whether \(E\) can be replaced by any correlation monotone in inequality \((3)\). Lemma 2 is not specific to entanglement monotones. It only requires that \(E\) is convex [16]. Many familiar entanglement monotones are convex, though this is not a basic requirement for such a measure [6]. For other correlation monotones, imposing convexity can lead to some difficulties. A convex correlation monotone is necessarily zero for all separable states. The measures of total correlations considered above do not vanish for all separable states, by construction, and are hence not convex. Consequently, the above derivation of eq.\((3)\) does not apply if \(E\) is replaced by anyone of these measures. Entanglement is not the only quantum resource for which there are measures that vanish only for free states and are convex. Other examples are the nonuniforality, which can be quantified by \(\ln d - S(\rho_A)\) for a system \(A\) of Hilbert space dimension \(d\) [52], and the coherence, which can be quantified by \(-\sum_i p_i \ln p_i - S(\rho_A)\), where \(p_i = \langle i | \rho_A | i \rangle\) and \(\{ | i \rangle \}\) is the basis with respect to which the incoherent states are defined [53]. In both these cases, inequality \((3)\) is satisfied with the above corresponding measure in place of \(E\), \(\ln d - x\) in place of \(\xi_{d_1,d_2}(x)\), and any correlation monotone \(C\) for which \(f = h\) [16]. A relation of the form of eq.\((3)\) can also be obtained for contextuality quantifiers [36, 41].

In summary, we have shown that internal entanglement and external correlations limit each other, whatever the nature of the correlations. For a given amount of external correlations \(C(\rho)\), the internal entanglement \(E(\rho_A)\) can approach but not exceed a value that decreases with increasing \(C(\rho)\), and reciprocally. For familiar correlation monotones, \(E(\rho_A)\) vanishes when the correlations are maximal. The entanglement can even be suppressed for lower values of \(C(\rho)\). In two particular cases, we have determined explicitly the tight upper bound on \(E(\rho_A)\) set by \(C(\rho)\), and found that the entanglement vanishes when the amount of correlations is above a threshold value. Such a threshold also exists for other entanglement and correlation monotones. On the contrary, a maximum internal entanglement does not always ensure that the external correlations vanish, due to the existence of mixed maximally entangled states [37]. If \(E\) is the entanglement of formation, for example, this is only true if none of the subsystems of \(A\) has a Hilbert space dimension larger, or equal, than twice that of the other one. As we have seen, the generalization of our result to other internal correlations is not obvious with the approach we have used. But it may be correct, and it would be of interest to determine whether this is indeed so.

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SUPPLEMENTAL MATERIAL

In this Supplemental Material, we prove the results mentioned in the main text.

Proof of Lemma 1

Consider any probability vector $p$, and any pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, with Schmidt coefficients $\sqrt{\rho_i}$, of the bipartite Hilbert spaces $H_1 \otimes \tilde{H}_1$ and $H_2 \otimes \tilde{H}_2$, respectively. The Hilbert spaces $H_1$ and $H_2$ can always be considered as subspaces of a larger Hilbert space $\mathcal{H}$, and similarly $\tilde{H}_1, \tilde{H}_2 \subset \mathcal{H}$. Moreover, there are unitary operators $U$ and $\tilde{U}$, on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, such that $|\psi_2\rangle = U \otimes \tilde{U} |\psi_1\rangle$. Thus, $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$ can be transformed into each other by local operations. Consequently, the amount of correlations $C(|\psi_i\rangle\langle\psi_i|)$ is the same for all the pure states $|\psi_i\rangle$ with Schmidt coefficients $\sqrt{\rho_i}$. We name it $f(p)$. For $p = 1$, $|\psi\rangle$ is necessarily a product state, and so $f(1) = 0$.

Consider any systems $A$ and $B'$, and any state $\rho$ of the composite system $AB'$, consisting of $A$ and $B'$, with Hilbert space $\mathcal{H}_{AB'}$. Denote its eigenvalues by $\mu_m$ and the corresponding eigenvectors by $|m\rangle$. Let us introduce a third system, say $B''$, which constitutes, together with $B'$, system $B$. Provided the dimension of $\mathcal{H}_{B''}$ is not smaller than that of $\mathcal{H}_{AB'}$, $\rho$ can be written as $\rho = \text{tr}_{B''} |\psi\rangle\langle\psi|$, where $\text{tr}_{B''}$ is the partial trace over $B''$, and $|\psi\rangle = \sum_m \sqrt{\mu_m} |m\rangle \tilde{m}_m$ is a pure state of system $AB$, with orthonormal states $|m\rangle \tilde{m}_m$ of $B''$. As $\text{tr}_{B''}$ is a local operation, performed on $B$, $C(\rho) \leq C(|\psi\rangle\langle\psi|)$. Since $\text{tr}_B |\psi\rangle\langle\psi| = \text{tr}_B \rho = \rho_A$, the Schmidt coefficients of $|\psi\rangle$, as a pure state of $\mathcal{H}_A \otimes \mathcal{H}_{B'}$, are $\sqrt{\lambda_i(\rho_A)}$, and hence $C(|\psi\rangle\langle\psi|) = f(\lambda(\rho_A))$, which finishes the proof.

Dependence on $d_1$ and $d_2$ of the entropy $s_{d_1,d_2}$

The entropy $s_{d_1,d_2}$ is defined by

$$s_{d_1,d_2}(p) = c_{d_1} - \sup_{\{i\}} E \left( \sum_i p_i |i\rangle \langle i| \right),$$

where the supremum is taken over the orthonormal basis sets $\{i\}$ of $H_{d_1} \otimes H_{d_2}$, for $p \in \mathcal{E}_{d_1,d_2}$, and by $s_{d_1,d_2}(p) = c_{d_1}$ otherwise \[16\]. First note that $s_{d_1,d_2}$ depends on $d_1$, since its maximum value is $c_{d_1}$. We now show that the entropies $s_{2,d}$ can be different from each other even on $\mathcal{E}_d$, where they are all given by the above expression involving $E$. Consider an entanglement monotone $E$ that vanishes only for separable states, e.g., the entanglement of formation. In this case, for any $p \in \mathcal{E}_{2d}$, $s_{2,d}(p) = e_2$ if and only if all the states $\rho_A$ on $H_2 \otimes H_2$ such that $\lambda(\rho_A) = p$ are separable. Consequently, for any $p \in \mathcal{E}_{2d}$, $s_{2,d}(p) = e_2$ if and only if $p_1 \leq p_{2d-1} + 2\sqrt{p_{2d-2}p_{2d}}$, where $p_i = 0$ for $i$ larger than the size of $p$. So, if $d \geq 3$ then $s_{2,d} < e_2$ on $\mathcal{E}_d$, whereas $s_{2,d}$ reaches $e_2$ on $\mathcal{E}_4$, e.g., for $(1/4,\ldots,1/4)$, and hence $s_{2,d} \neq s_{2,2}$ on $\mathcal{E}_4$.

Proof of Lemma 3

For any $p \in \mathcal{E}_d$, we define, for $\beta \geq 1$, the family of vectors $p^{(\beta)} \in \mathcal{E}_d$ as follows. We denote by $r$ the size of $p$. If $r = 1$ or $r = 2 < p_1$, the components of $p^{(\beta)}$ are given by $p^{(\beta)}_i = (p_i/p_1)^\beta$. Clearly, $p^{(\beta)}$ is continuous with $\beta$, $p^{(1)} = p$, and $p^{(\infty)} = 1$. If there is an index $j > 1$ such that $p_j = p_1$ and, if $j < r$, $p_{j+1} < 1$, consider the vectors $p^{(\eta)}$ given by $p^{(\eta)}_j = p_1 + \eta$, $p^{(\eta)}_i = p_1 - \eta/(j-1)$ for $i \in \{2,\ldots,j\}$, and, if $j < r$, by $p^{(\eta)}_i = p_1$ for $i > j$. There is $\eta > 0$ such that, for any $\eta \in [0,\eta^*]$, the components of $p^{(\eta)}$ are in decreasing order, and $p^{(\eta)} \in \mathcal{E}_d$. In this case, we define $p^{(\beta)}$ by $p^{(\beta)} = p^{(\beta-1)}$ for $\beta \in [1,1+\eta^*]$, and by $p^{(\beta)}_j = (p^{(\eta)}_j/p^{(\eta)}_i)^\beta - \eta^*$ for $\beta > 1 + \eta^*$. Here also $p^{(\beta)}$ is continuous with $\beta$, and $p^{(1)} = p$. Moreover, since $p^{(\eta)}_{2} < p^{(\eta)}_{1}$, $p^{(\infty)} = 1$.

We denote by $\mathcal{E}_d(x)$ the set of all $p \in \mathcal{E}_d$ such that $f(p) = x$, and define the function $g_d(x)$, on the set $I$ of the values $x$, by

$$g_d(x) = \inf_{p \in \mathcal{E}_d(x)} s(p).$$

By construction, $g_d(f) \leq s$ on $\mathcal{E}_d$, and there is $p \in \mathcal{E}_d(x)$ such that $s(p)$ and $g_d(x)$ are as close to each other as we wish. As $s(1) = f(1) = 0$, there is a set $\mathcal{E}_d(0)$ containing $p = 1$, and $g_d(0) = 0$. If $f(1/d,\ldots,1/d) = c_d$, and $f(p) < c_d$ for all other vectors $p \in \mathcal{E}_d$, then $\mathcal{E}_d(c_d)$ is the singleton $\{1/d,\ldots,1/d\}$, and hence $g_d(c_d) = s(1/d,\ldots,1/d)$.

Due to the extreme value theorem, the continuous function $f$ has a maximum, $c_d$, on the simplex $\mathcal{E}_d$. Let $q$ be a vector of $\mathcal{E}_d$ such that $f(q) = c_d$, and define $q^{(\beta)}$ as explained above. As $f$ is continuous, $f(q^{(\beta)})$ is a continuous function of $\beta$. It is equal to $c_d$ for $\beta = 1$, and to 0 for $\beta \to \infty$. So, due to the intermediate value theorem, for any $x \in [0,c_d]$, there is $\tilde{\beta}$ such that $f(q^{(\tilde{\beta})}) = x$. Thus, $I$ is equal to this interval. For any $p \in \mathcal{E}_d(x)$, $f(p^{(\beta)})$ is a continuous function of $\beta$, which is equal to $x$ for $\beta = 1$, and to 0 for $\beta \to \infty$. So, for any $y \in [0,x]$, there is $\tilde{\beta}$ such that $p^{(\tilde{\beta})} \in \mathcal{E}_d(y)$. Moreover, since $p$ is majorized by $p^{(\beta)}$, and $s$ is an entropy, $g_d(y) \geq s(p^{(\beta)}) \geq g_d(y)$.

Consequently, for any $y \leq x$, $g_d(y)$ is a lower bound of $s$ on $\mathcal{E}_d(x)$, which implies that $g_d$ is nondecreasing.

Expressions of $C(D_B)$ and $C(D_N)$ for specific states

The Bures distance is given by $D_B(\omega,\omega') = (2 - 2 \text{tr} \sqrt{\omega \omega'})^{1/2}$. The ensuing correlation monotone
\[ C^{(DB)}(\psi) = \inf_{\delta_A, \delta_B} \left( 2 - 2\langle \psi | \delta_A \otimes \delta_B | \psi \rangle \right)^{1/2}. \]

With the Schmidt form \( | \psi \rangle = \sum_i \sqrt{\rho_i} | i_A \rangle \otimes | i_B \rangle \), where \( | i_A \rangle \) are orthonormal states of \( A \), \( | i_B \rangle \) of \( B \), and the probabilities \( \rho_i \) are in decreasing order, one can write
\[ \langle \psi | \delta_A \otimes \delta_B | \psi \rangle = \sum_{i,j} \sqrt{\rho_i \rho_j} \langle i_A | \delta_A | j_A \rangle \langle i_B | \delta_B | j_B \rangle. \]

Using this expression, the Cauchy-Schwarz inequality, and \( p_i \leq p_1 \), leads to
\[ \langle \psi | \delta_A \otimes \delta_B | \psi \rangle^2 \leq p_1^2 \text{tr}(\delta_A^2) \text{tr}(\delta_B^2) \leq p_1^2. \]

For \( \delta_A = |1\rangle_A \langle 1| \) and \( \delta_B = |1\rangle_B \langle 1| \), the above inequalities are equalities, and hence
\[ C^{(DB)}(\psi) = \sqrt{2(1 - \sqrt{p_1})} = f^{(DB)}(\rho) \]

The Hellinger distance is given by \( D_H(\omega, \omega') = (2 - 2\sqrt{\omega \varpi})^{1/2} \), and thus
\[ C^{(DH)}(\psi) = \inf_{\delta_A, \delta_B} \left( 2 - 2\langle \psi | \sqrt{\delta_A \otimes \sqrt{\delta_B}} | \psi \rangle \right)^{1/2}. \]

Following the same steps as above, we obtain
\[ C^{(DH)}(\psi) = \sqrt{2(1 - p_1)} = f^{(DH)}(\rho). \]

For a strictly correlated classical-classical state \( \rho_{pc} = \sum_i p_i |i\rangle_AA |i\rangle_BB |i\rangle_BB |i\rangle \), with the probabilities \( p_i \) in decreasing order, one finds
\[ C^{(DH)}(\rho_{pc}) = \inf_{\delta_A, \delta_B} \left[ 2 - 2\sum_i \sqrt{\rho_i} \langle \sqrt{\delta_A} | i \rangle \langle \sqrt{\delta_B} | i \rangle \right]^{1/2}. \]

where \( \langle \sqrt{\delta_A} | i \rangle = |1\rangle_A |i\rangle_B |i\rangle_B |1\rangle \). As above, the infimum is a minimum reached for \( \delta_A = |1\rangle_A |1\rangle \) and \( \delta_B = |1\rangle_B |1\rangle \), and so
\[ C^{(DH)}(\rho_{pc}) = f^{(DH)}(\rho) = f^{(DH)}(\rho). \]

**Derivation of \( \xi_{2,2} \) for the entanglement of formation and the correlation monotones \( C^{(DB)} \) and \( C^{(DH)} \)**

For the entanglement of formation \( E_f \), the entropy \( s_{2,2} \) is given by
\[ s_{2,2}^{(E_f)}(\rho) = \ln 2 - v[ \max \{ 0, p_1 - p_3 - \sqrt{2p_1p_3} \}], \]
where \( v(y) = w_+(y) + w_-(y) \), with \( w_+(y) = -(1 \pm \sqrt{1 - y^2}) \ln[(1 \pm \sqrt{1 - y^2})/2]/2 \), and \( e_2^{(E_f)} = \ln 2 \). For the considered correlation monotones, the condition \( f(\rho) = x \) can be rewritten as \( p_1 = 1 - y \), where \( y = x^2 - x^4/4 \) for \( C^{(DB)} \), and \( y = x^2/2 \) for \( C^{(DH)} \). From this expression and \( \sum_i p_i = 1 \), it follows that the minimum value of \( s_{2,2}^{(E_f)}(\rho) \) under the constraint \( f(\rho) = x \), can be obtained by maximizing \( z(p_2, p_4) = 1 - 2y + (\sqrt{p_2} - \sqrt{p_4})^2 \), since \( v \) is an increasing function. The ordering of the probabilities \( p_i \), i.e., \( p_i \geq p_{i+1} \), imposes \( \delta_{p_0} z \geq 0 \), \( \delta_{p_0} z \leq 0 \), \( p_4 \geq 0 \), \( p_2 \leq 1 - y \), \( 2p_2 + p_4 \geq y \), and \( 2p_4 + p_2 \leq y \). Three cases must be distinguished: \( y \in [0, 1/2] \), \( y \in [1/2, 2/3] \), and \( y \in [2/3, 3/4] \). In the first, \( z \) reaches its maximum, \( 1 - y \), for \( (p_2, p_4) = (y, 0) \). In the second, \( z \) is maximum for \( (p_2, p_4) = (1 - y, 0) \), and it is equal to \( 2 - 3y \). In the last case, \( z \) is negative. The corresponding values for \( \xi_{2,2}(x) = u(y) \) are \( v(1 - y) \), \( v(2 - 3y) \), and \( v(0) = 0 \).

**Functions \( \xi_{d_1,d_2} \) vanishing on a finite interval**

If \( f = k \circ s_{Ra} \), where \( k \) is a strictly increasing function and \( s_{Ra} \) is the Rényi entropy of order \( \alpha \in (0, 1) \), then \( \xi_{d_1,d_2} \) vanishes on a finite interval for any \( d_1, d_2 \), and entanglement monotone \( E \).

**Proof.** Consider any vectors \( \rho \) and \( \eta \) of \( E_d \), where \( d \) is any positive integer, and pad them with zeros, if necessary, to make up the \( d \)-component vectors \( \tilde{\rho} \) and \( \tilde{\eta} \). The Rényi divergence \( D_\alpha \) of order \( \alpha \in (0, 1) \) is related to the total variation distance \( V(\tilde{\rho}, \tilde{\eta}) = \sum_i |\tilde{\rho}_i - \tilde{\eta}_i| \), by the generalized Pinsker’s inequality \( D_{\alpha}(\tilde{\rho}, \tilde{\eta}) \geq \alpha V(\tilde{\rho}, \tilde{\eta})^2/2 \), and hence \( D_{\alpha}(\tilde{\rho}, \tilde{\eta}) \geq 0 \). Since \( D_{\alpha}(\tilde{\rho}) \leq \alpha \sum_i (\tilde{\rho}_i - \tilde{\eta}_i)^2/2 \). Since \( D_{\alpha}(\tilde{\rho})(1/d, \ldots, 1/d) = \ln d - s_{Ra}(\rho) \), and \( k \) is increasing, the function \( f = k \circ s_{Ra} \) satisfies \( f(\rho) \leq k(\ln d - \alpha) \). Thus, due to Lemma 2, for any \( x \in E_d \) such that \( P(\rho) \leq 1/(d - 1) \), \( s_{d_1,d_2}(\rho) = e_{d_1,d_2} \). Consequently, as \( k \) is strictly increasing, for any \( \rho \in E_d \) with \( x \geq k(\ln d - \alpha) \), \( s(\rho) = e_{d_1,d_2} \), where \( s = s_{d_1,d_2} \), and thus, for any such \( x \), \( g_d(x) = e_{d_1,d_2} \), see the definition of \( g_d \). In other words, \( \xi_{d_1,d_2} = e_{d_1} - g_d \) vanishes on a finite interval.

Consider two entanglement monotones \( E \) and \( E' \) which are zero only for separable states. If \( \xi_{d_1,d_2}^{(E,C)} \) corresponding to \( E \) and the correlation monotone \( C \), vanishes on an interval \( J \) and is positive elsewhere, then \( \xi_{d_1,d_2}^{(E',C)} \) vanishes on \( J \) and is positive elsewhere.

**Proof.** With the entropy \( s_{d_1,d_2}^{(E)}(x) \) given by Lemma 2 with \( E \) and \( E' \), and \( f \) given by Lemma 1 with \( C \), define the function \( g_d(\rho_d) \), with \( d = d_1d_2 \), as above. Denote by \( J \) the maximal interval on which \( \xi_{d_1,d_2}^{(E,C)} = e_{d_1} - g_d \) vanishes. For any \( \rho \in E_d(x) \) with \( x \in J \), it follows from the definition of \( g_d \) and from the fact that \( s_{d_1,d_2} \) cannot exceed \( e_{d_1} \), that \( s_{d_1,d_2}(\rho) = e_{d_1} \). Thus, due to Lemma 2 and the assumption on \( E \), all the density operators \( \rho_A \)
for any classical-classical state $E$ monotone of system $\rho$. Consequently, $J$ is a subset of $J'$ the maximal interval on which $\zeta_{d_1,d_2}(E) = e_{d_1}' - g_{d_1}'$ vanishes. Swtiching the roles of $E$ and $E'$ in the above arguments leads to $J' = J$. □

Boundary for the classical-classical states

For any correlation monotone $C$, there is a function $\tilde{f}$ of the probability vectors with $\tilde{f}(1) = 0$, such that, for any classical-classical state $\rho$, $C(\rho) \leq \tilde{f}([\lambda(\rho_A)])$, with equality when $\rho$ is strictly correlated.

Let $H_{d_1} \otimes H_{d_2}$, with $d_2 \geq d_1$, be the Hilbert space of system $A$, and $d = d_1d_2$. For a convex entanglement monotone $E$, and a correlation monotone $C$ such that $\tilde{f}$ is continuous, $C(\rho)$ and $E(\rho_A)$ obey, for any classical-classical state $\rho$, $E(\rho_A) \leq \zeta_{d_1,d_2}[C(\rho)]$, where $\zeta_{d_1,d_2}$ is a nonincreasing function on $[0, \tilde{c}_d]$, with $\tilde{c}_d$ the maximum of $\tilde{f}$ on $E_d$, such that $\zeta_{d_1,d_2}(0) = e_{d_1}'$. For any $x \in [0, \tilde{c}_d]$, there are classical-classical states $\rho$ such that $C(\rho) = x$, and the two sides of the above inequality are as close to each other as we wish.

If $\tilde{f}(1/d, \ldots, 1/d) = \tilde{c}_d$ and $\tilde{f}(p) < \tilde{c}_d$ for any other $p \in E_d$, then $\zeta_{d_1,d_2}(\tilde{c}_d) = 0$.

Proof. Consider any probability vector $p$, and any strictly correlated classical-classical states $\rho_1$ and $\rho_2$, of the bipartite Hilbert spaces $H_1 \otimes \tilde{H}_1$ and $H_2 \otimes \tilde{H}_2$, respectively, such that $\rho_k = \sum_i p_i |i\rangle\langle i|_k \otimes |\rangle\langle \rangle_k$, where $|i\rangle_k$ are orthonormal states of $H_k$, and $|\rangle\langle \rangle_k$ of $\tilde{H}_k$. The Hilbert spaces $H_1$ and $H_2$ can always be considered as subspaces of a larger Hilbert space $H$, and similarly $\tilde{H}_1, \tilde{H}_2 \subset \tilde{H}$. Moreover, there are unitary operators $U$ and $\tilde{U}$, on $H$ and $\tilde{H}$, respectively, such that $\rho_2 = U \otimes \tilde{U} \rho_1 U^\dagger \otimes \tilde{U}^\dagger$. Thus, the amount of correlations $C(\rho)$ is the same for all the strictly correlated classical-classical states $\rho$ with eigenvalues $p_i$. We name it $\tilde{f}(p)$. For $p = 1$, $\rho$ is necessarily a product state, and so $\tilde{f}(1) = 0$.

Consider any systems $A$ and $B$, and any classical-classical state $\rho = \sum_i p_{ij} |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B$, where $\{|i\rangle_A\}_i$ is an orthonormal basis of $A$, $\{|i\rangle_B\}_i$ of $B$, and $p_{ij}$ are probabilities summing to unity. The corresponding reduced density operator for $A$ reads $\rho_A = \sum_i p_i |i\rangle\langle i|_A$. The local operation with Kraus operators $K_{ij} = \sqrt{p_{ij}/p_i I \otimes |j\rangle\langle j|_B}$, where $I$ is the identity operator of $A$, for $i$ such that $p_i \neq 0$ and any $j$, and $K_i = I \otimes |i\rangle\langle i|_B$ for $i$ such that $p_i = 0$, changes the strictly correlated classical-classical state $\tilde{\rho} = \sum_i p_i |i\rangle\langle i|_A \otimes |i\rangle\langle i|_B$ into $\rho$, and so $C(\rho) \leq C(\tilde{\rho})$. The state of $A$ is $\rho_A$ for both $\rho$ and $\tilde{\rho}$. From $[\lambda(\rho_A)] = p$, it follows that $C(\tilde{\rho}) = \tilde{f}([\lambda(\rho_A)])$. Using then Lemma 2 and Lemma 3 with $\tilde{f}$ and the entropy $s_{d_1,d_2}$ finishes the proof. □