Black holes in a cavity: Heat engine and Joule-Thomson expansion

Yihe Cao1 · Hanwen Feng1 · Jun Tao1 · Yadong Xue1

Received: 27 January 2022 / Accepted: 31 August 2022 / Published online: 15 September 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
We consider the charged d-dimensional black holes in the cavity in extended phase space and investigate the heat engine and the Joule-Thomson (JT) expansion. Since the phase structure of black holes in the cavity is similar to anti-de-sitter (AdS) cases, we take black holes in a cavity as the working substance in the heat engine and calculate their efficiency in Carnot cycle and rectangular cycle. Also, we discuss whether the JT expansion of charged black holes in the cavity is consistent with AdS cases and find the charged black hole in a cavity always cools down during the isenthalpic process with the decreasing pressure.

Keywords Black hole · Heat engine · Joule-Thomson expansion

Contents
1 Introduction ............................................. 1
2 Black holes in a cavity ........................................ 3
3 Black holes as heat engines in a cavity ....................... 4
   3.1 Schwarzschild black holes in a cavity ..................... 4
   3.2 RN black holes in a cavity ................................ 7
4 The JT expansion of RN black hole in a cavity ............... 8
5 Conclusions ............................................. 12
References ................................................ 12

1 Introduction
The thermal quantum radiance of black holes reveals the connection among gravity, quantum theory and black hole thermodynamics [1–3]. The idea to place black holes in a cavity is proposed to solve the problem of investigating the thermodynamically

Jun Tao
taojun@scu.edu.cn

1 Center for Theoretical Physics, College of Physics, Sichuan University, Chengdu 610065, China
unstable black hole in asymptotically flat space, which attracts interests in the study of black hole thermodynamics and quantum gravity [4]. The temperature of the surface of the cavity is fixed with a heat bath surrounding. According to the semi-classical asymptotic relationship between the on-shell Euclidean action and the partition function, we can obtain the expression of free energy. Furthermore, we define the volume of cavity as the one of the system. In this framework, the phase transition, the thermodynamic geometry and the critical behaviour of vary BHs in the cavity are discussed in [5–13]. The Schwarzschild-AdS black holes inside a spherical reflecting cavity was recently discussed in [14]. When the AdS radius tends to infinity, the situation goes back to the Schwarzschild black hole in a cavity. Another boundary condition to make the black hole thermodynamically stable is to put it in the AdS space. The AdS black hole thermodynamics have something in common with general thermodynamics. Hawking and Page first studied the phase transition between the Schwarzschild black hole and the thermal AdS space, which is now called the Hawking-Page phase transition [15]. In [16, 17], the author found that the charged AdS black hole has a van der Waals-like phase transition, and subsequent studies have concluded that many other black holes also show van der Waals-like phase transition. Many study on black hole thermodynamics correspond to general thermodynamics, such as van der Waals-like phase transitions, reentrant phase transitions, holographic heat engines, triple points and P-V critical phenomena. And related work reveals the thermodynamic behaviours of AdS-BHs are similar to those of BHs in cavities [6, 15–22].

Due to this fact, we can investigate its properties as a heat engine like AdS-BH, which is to define a cycle in the extended thermodynamical space and the AdS black holes are considered as the working substance, and the related work can be found in [23–38]. Moreover, a cavity has explicit boundaries, which makes the correspondence between the classic heat engine and the black hole one more reasonable. We considered charged black holes in a cavity as the heat engine which works in the Carnot cycle and rectangular cycle. The Carnot cycle consists of two isothermal paths and two adiabatic paths, whose efficiency is always the theoretically highest according to the second law of thermodynamics, and thus it provides an upper bound for us to check the calculation. The rectangular cycle is made up of two isochoric paths and two isobaric paths, which can be taken as the smallest unit of a more complicated cycle [36].

As an important property of van der Waals system, Joule-Thomson expansion of van der Waals fluids occurs during the throttling process, which means that the fluid flows from the high-pressure side to the low-pressure side through the porous plug under adiabatic conditions. It shows that the temperature of the fluid changes on both sides of the porous plug before and after the throttling process. The effect is called Joule-Thomson expansion. It is introduced to AdS-BHs and firstly investigated in [39], which shows both cooling and heating states of the RN-AdS black hole exist during the isenthalpic process. Furthermore, it’s found that different black holes have similar isenthalpic and inversion curves [40–70], which means isenthalpic curves divide the heating and cooling regions and the extreme points are coincident with the inversion curves. Moreover, [71] presents different types of isenthalpic curves when studying the JT expansion of the FRW universe. We are curious about the JT expansion of black holes in a cavity and hope to enrich the research of black hole thermodynamics. There were studies that compared the phase transition, the thermodynamic geometry,
the validities of the second thermodynamic law and the weak cosmic censorship of AdS black holes with that of black holes in a cavity [8–10, 72]. The result shows the dissimilarities caused by the two different boundary conditions. And these findings prompted us to further explore the JT expansion of black holes in a cavity. In this paper, we prove the JT expansion of black holes in a cavity does exist, while it differs from AdS-BHs, and the JT coefficient always keeps positive, which means the temperature decrease during the adiabatic expansion. In other words, the black hole can only become colder after the isenthalpic process, which is different from the situations ever investigated before. And we calculated the JT coefficient of the charged black hole in a cavity and plotted the temperature versus pressure during the process.

In sect. 2, we derive the thermodynamic quantities of the d-dimensional charged BH in the cavities. We calculate the heat engine efficiency of 4-dimensional Schwarzschild black holes and RN black holes in a cavity in sect. 3. In sect. 4, the JT expansion of the RN black holes and the higher dimensional cases in a cavity are investigated. We summarize our results in sect. 5.

2 Black holes in a cavity

The d-dimensional Einstein-Maxwell action according to the ref is given by [73, 74]

\[
I = I_{\text{bulk}} + I_{\text{surf}} = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{d}x \sqrt{-g} \mathcal{R} + \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-\gamma} (K - K_{0}) + \frac{1}{16\pi} \int_{\mathcal{M}} d^{d}x \sqrt{-g} F^{2} + \frac{1}{16\pi} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-\gamma} n_{\nu} F^{\mu\nu} A_{\mu},
\]

(1)

where \( n_{\nu} \) is the unit normal vector of the boundary, \( \gamma \) is the metric on the boundary, \( K \) is the trace of the extrinsic curvature, and \( K_{0} \) is a subtraction term to ensure the Gibbons-Hawking-York term vanish in flat space-time. The metric of the spherical d-dimensional charged BH reads [75]

\[
ds^{2} = -f(r)dt^{2} + \frac{1}{f(r)}dr^{2} + r^{2}d\omega_{d-2}^{2}, \quad A_{i}dt = -\sqrt{\frac{d-2}{2d-6}} \frac{Q}{r^{d-3}} dt.
\]

(2)

By varying the action Eq.(1), we obtain the metric function as

\[
f(r) = 1 + \frac{Q^{2}}{r^{2(d-3)}} - \frac{Q^{2}}{r^{d-3}r_{+}^{d-3}} - \frac{r^{d-3}}{r_{+}^{d-3}},
\]

(3)

where the radius of the event horizon is defined as \( f(r_{+}) = 0 \) and the parameter \( Q \) is the charge of the black hole. Then we can derive the cavity temperature \( T \), which is related to the Hawking temperature \( T_{H} \).

\[
T = \frac{T_{H}}{\sqrt{f(r_{B})}} = \frac{(d-3)(1 - Q^{2}r_{+}^{6-2d})}{4\pi r_{+}\sqrt{f(r_{B})}},
\]

(4)
where $r_B$ is the radius of a cavity. Furthermore, the Euclidean action corresponds to the action Eq.(1) by $I_E = iI$. The Euclidean time $\tau$ is related to $t$ by $\tau = it$ and the metric becomes positive infinite [4, 8]. In this way, we derive the Euclidean action by substituting Euclidean metric into the action, which gives

$$I_E = \frac{d - 2}{8\pi} \frac{\omega_{d-2} r_B^{d-3}}{T}(1 - \sqrt{f(r_B)}) - S, \quad (5)$$

where $S = \frac{\omega_{d-2}}{4} r^d_{+}$ is the entropy of the black hole, and $\omega_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$ is the volume of the unit $d$-sphere where $\Gamma$ represents the gamma function. The free energy is related to the Euclidean action in an approximately semi-classical way as $F = -T \ln Z = TI_E$ [7], and thus the thermal energy of the black hole in a cavity is $E = -T^2 \frac{\partial F}{\partial T}$, which yields as

$$E = \frac{(d - 2) r_B^{d-3}}{8\pi} \omega_{d-2} \left(1 - \sqrt{f(r_B)}\right). \quad (6)$$

The thermodynamic volume $V$ of the system is defined as the volume of the cavity

$$V = \frac{r_B^{d-1} \omega_{d-2}}{d - 1}, \quad (7)$$

and the conjugate thermodynamic pressure naturally arises according to $P = -\frac{\partial E}{\partial V}$. In this new extended phase space, the enthalpy $H$ can be derived by $H = E + PV$.

### 3 Black holes as heat engines in a cavity

For the purpose of comparing the heat engine properties of black hole in a cavity with AdS-BH, we consider a cavity as the container and take black holes as the working substance in the pressure-volume space.

#### 3.1 Schwarzschild black holes in a cavity

In the following sections, the black hole is set in thermodynamical cycles. For simplicity, we focus on the case of $d = 4$. Four-dimentional space-time metric of RN black hole yields

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (8)$$

where $f(r)$ is expressed as

$$f(r) = (1 - \frac{r_+}{r})(1 - \frac{Q_h^2}{r_+ r}), \quad (9)$$
and the effective potential satisfies $A = A_0 dt = -\frac{Q_b}{r} dt$. The event horizon radius is represented by $r_+$, and meanwhile the charge of the black hole is represented by $Q_b$. Subsequently the temperature of this system with radius $r = r_B > r_+$ had a definition in [5], and for $d = 4$,

$$T(r_B, x, Q) = \frac{T_H}{\sqrt{f(r_B)}} = \frac{1 - \frac{Q^2}{r_B^2 x^2}}{4\pi r_B x \sqrt{f(r_B)}},$$  \hspace{1cm} (10)$$

where $Q_b = Q$, $r_B$ is the radius of a cavity and $T_H = \frac{1}{4\pi} f'(r_+)$ is the Hawking temperature of the black hole. It is worth noting that the physical range of the event horizon is constrained as $\frac{r_+}{r_B} \leq x \equiv \frac{r}{r_B} \leq 1$ in which $r_+ = Q$ is the horizon radius of the extremal RN-BH.

First, we start from the Schwarzschild black holes, which is the form of RN black holes reduce to when $Q = 0$. Under these conditions, Eq.(10) takes the form as

$$T(r_B, x) = \frac{T_H}{\sqrt{f(r_B)}} = \frac{1}{4\pi r_B x \sqrt{1 - x}}.$$

(11)

and the conjugate pressure is

$$P(r_B, x) = \frac{2r_B^2 x - r_B^2 x^2}{8\pi r_B^2 x \sqrt{1 - x}} - \frac{1}{4\pi r_B^2}.$$

(12)

We can derive the expression for $x$ in terms of $r_B$ and $P$ with Eq.(12),

$$x = 4 \left( \left( 4\pi P r_B^3 + r_B \right) \sqrt{2\pi P \left( 2\pi P r_B^2 + 1 \right)} - 4\pi P r_B^2 \left( 2\pi P r_B^2 + 1 \right) \right).$$

(13)

With this result, the temperature can be rewritten as

$$T(r_B, P) = \left( 16\pi r_B \Gamma \sqrt{1 + 4\Gamma} \right)^{-1},$$

$$\Gamma = \sqrt{2\pi P \left( 2\pi P r_B^2 + 1 \right)} \left( 4\pi P r_B^3 + r_B \right)^2 - 4\pi P r_B^2 \left( 2\pi P r_B^2 + 1 \right).$$

(14)

We denote the heat absorbed as $Q_H$, and the heat delivered as $Q_C$, so that the mechanical work is $W = Q_H - Q_C$. The efficiency is the ratio of mechanical work $W$ to heat absorbed $Q_H$

$$\eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H}.$$  \hspace{1cm} (16)$$

The Carnot cycle consists of two isothermal paths and two adiabatic paths which means that the Carnot heat engine is between two different temperatures. We define the high
The Carnot cycle efficiency of Schwarzschild black hole heat engine with varying radius $R$, where we set $r_{B1} = 1$, $p_1 = 1$, $p_4 = 4$

temperature as $T_H(r_{B2}, p_2)$ and the cold one as $T_L(r_{B1}, p_1)$, which are connected through the isochoric paths. The efficiency yields as

$$\eta = 1 - \frac{T_L(r_{B1}, p_1)}{T_H(r_{B4}, p_4)},$$

by which we plot how it varies with respect to the radius of a cavity in Fig. 1.

Moreover, we define a rectangular cycle with two isochoric and two isobaric paths and compare its efficiency with that of Carnot cycle, and the efficiency for the rectangle cycle can be calculated by a formula deduced in [76, 77]

$$\eta = \frac{W}{Q} = \frac{(p_1 - p_4)(V_4 - V_1)}{Q},$$

where

$$Q = \Delta E + P \Delta V.$$  

Thus the Eq.(18) yield as

$$\eta = \frac{4\pi (p_1 - p_4) (r_{B1}^3 - r_{B4}^3)}{r_{B4} (-4\pi p_1 r_{B4}^2 + 3\Delta_4 - 3) + 4\pi p_1 r_{B1}^3 - 3r_{B1} (\Delta_1 - 1)},$$

where we define

$$\Delta_\alpha = \sqrt{16\pi p_\alpha r_{B\alpha}^2 \beta - 4 \sqrt{2\pi} \sqrt{p_\alpha \beta (4\pi p_\alpha r_{B\alpha}^3 + r_{B\alpha})^2 + 1}}.$$  

here $\beta = (2\pi p_\alpha r_{B\alpha}^2 + 1)$ and $\alpha = 1, 4$.

In this way, we can show how the efficiency changes as well Fig. 2.
Then we discuss a more complicated situation. Different from the case of Schwarzschild, we can’t analytically derive the expression of $x$ in terms of $r_B$ and $P$. Alternatively, we calculate the efficiency numerically.

In the 4-dimension, the conjugate thermodynamic pressure is

$$P(r_B, x, Q) = -\frac{\partial E}{\partial V} = \frac{2r_B^2x - Q^2 - r_B^2x^2}{8\pi r_B^4x \sqrt{f(r_B)}} - \frac{1}{4\pi r_B^2}. \quad (22)$$

For every pressure we have fixed, we can solve the corresponding $x$ by Eq.(22) with conditions $0 < x \leq 1$ and $x > \frac{Q^2}{R^2}$ which guarantee $P$ and $T$ are real, and then we can obtain the temperature with $x$ to derive the efficiency of Carnot cycle. Similarly, we can plot the efficiency curve in the rectangular cycle, as shown in Fig. 3.

**3.2 RN black holes in a cavity**

![Fig. 2](image1.png)

The rectangular cycle efficiency in the Schwarzschild case with varying radius $R$, where we set $r_{B1} = 1$, $P_1 = 1$, $P_4 = 4$.

![Fig. 3](image2.png)

Left Plane: The Carnot cycle efficiency of RN black hole heat engine; Right Plane: The rectangular cycle efficiency of RN black hole heat engine. Here we set $r_{B1} = 1$, $P_1 = 0.01$, $P_4 = 0.02$.
4 The JT expansion of RN black hole in a cavity

Since the thermodynamic laws of black holes have been established, many study reveals black hole thermodynamics can correspond to classical thermodynamics, such as van der Waals-like phase transitions, reentrant phase transitions, holographic heat engines, triple points, and P-V critical phenomena. The first study that introduces JT expansion into black hole thermodynamics is the Joule–Thomson expansion of the charged AdS black holes by Ö. Ökcü [39], which discusses thermodynamic properties of van der Waals equation, as well as the JT expansion of van der Waals fluids and charged AdS black holes. As a characteristic of the van der Waals system, the JT expansion takes place when the system undergoes an isenthalpic process, and thus the enthalpy is the key to our work. In our research, we investigate the JT expansion of the charged black holes in a cavity.

First of all, we show how the four dimensional temperature $T$ varies with respect to $x$ in Fig. 4 according to Eq. (4). It shows the temperature diverges when $x$ approaches 1, which is different from the RN-AdS black holes [17].

In the following, we derive the relating thermodynamic quantities for further study. The thermal energy of the system can be obtained from Eq. (9) as discussed in [78], which yields

$$E(r_B, x, Q) = r_B(1 - \sqrt{f(r_B)}). \quad (23)$$

And the volume of the system is related to the cavity radius,

$$V = \frac{4}{3} \pi r_B^3, \quad (24)$$

![Fig. 4](image-url) We set $Q = 0.035, 0.1, 0.2$ and 0.35 from top to the bottom. And $r_B = 1$ while $x$ have a range of $[0, 1]$ in $T - x$ plane.
Fig. 5 Isenthalpic curves in $T - P$ plane. $H = 1, 2, 3$ and $5$. $Q = 0.035$, $d = 4$, and $r_B$ varies from 0.2 to 10

whose conjugate thermodynamic pressure naturally gives

$$P(r_B, x, Q) = -\frac{\partial E}{\partial V} = \frac{2r_B^2 x - Q^2 - r_B^2 x^2}{8\pi r_B^4 x \sqrt{f(r_B)}} - \frac{1}{4\pi r_B^2}. \quad (25)$$

With Eqs. (23), (24) and (25), we can obtain the enthalpy of the system by $H = E + PV$,

$$H(r_B, x, Q) = -r_B^2 x \left(6 f(r_B) - 4\sqrt{f(r_B)} + x - 2\right) - \frac{Q^2}{6\sqrt{f(r_B)}r_B x}. \quad (26)$$

Then we plot isenthalpic curves to study the JT expansion of RN black hole in a cavity from Eqs. (10), (25) and (26) numerically in Fig. 5.

As we can see, there are no extreme points in the curves and the temperature increases with higher pressure. Furthermore, the left plane shows larger enthalpy in the case of high pressure, which is the opposite when the pressure is sufficiently small as the left part of the right plane presents. By focusing on the situation of lower temperature and pressure, we find that the part where larger enthalpy leads to lower temperature corresponds to small region of $x$. And the orange line coincides with the orange one in Fig. 7 which represents the four dimensional cases when the enthalpy equals 1.

We can investigate the project further. For d-dimensional cases, the explicit form of $P(r_+, r_B, q, d)$ and $H(r_+, r_B, q, d)$ takes the form as

$$P = -\frac{\partial E}{\partial V} = \frac{(d - 2)(d - 3)}{8\pi} \Xi, \quad (27)$$

where

$$\Xi = -\frac{q^2}{2\sqrt{f(r_B)}r_B^{3d-6}r_+^{-d-3}} - \frac{1}{2\sqrt{f(r_B)}r_B^{d-1}r_+} + \frac{1}{\sqrt{f(r_B)}r_B^2} - \frac{1}{r_B^2}. \quad (28)$$
The enthalpy $H$ is derived by $H = E + PV$, which yields

$$H = \frac{(d-2) r_B^{d-3} \omega_d}{8\pi} (1 - \sqrt{f(r_B)} + \frac{(d-3)r_B^{2}}{d-1} \Xi). \quad (28)$$

We obtain the isenthalpic curves in numerically way with Eqs.(4), (27) and (28), and then we plot how the enthalpy varies from 1 to 5 when $d = 5$, $Q = 0.035$ in Fig. 6.

The temperature decreases with pressure as shown in Fig. 6, which shows that the process only cools the system. The intersections of the curves can divide the graph into two parts: with the left part representing the low pressure region and the right one indicating the high pressure region. Particularly, the temperature increases with decreasing enthalpy in the low pressure region and does the opposite in the high pressure region. Similar to the four-dimensional case, in the low pressure region, the ratio of $r_+$ and $r_B$ is small.

After studying the four and five dimensional cases, we would like to consider the impact of dimensions on the isenthalpic curves.

The picture Fig. 7 presents different dimensional cases where the red and the orange curve coincides with the orange one in Fig. 6 and the orange one in Fig. 5 respectively. We can see from the picture that, in the low pressure area, higher dimensions lead to higher temperatures, which is the opposite in the high pressure region. In conclusion, the dimension parameter $d$ and the enthalpy $H$ have opposite influence on the temperature during the throttling process.

As we know, the JT coefficient of classical ideal gas always equals 0, which means the isnenthalpic process doesn’t change the temperature of ideal gas, while the process can cool or heat the classical van der Waals gas only depending on the initial temperature and pressure. With Eqs.(10) and (25), we derive JT coefficient for 4 dimensional case by

$$\mu = \left( \frac{\partial T}{\partial P} \right)_H, \quad (29)$$
Fig. 7 The isenthalpic curves of charged black holes in a cavity for high dimensional cases. We set the dimension $d$ equals 4, 5, 6 and 7 respectively, the enthalpy $H$ equals 1, and the charge $Q$ equals 0.035

Fig. 8 The Joule-Thomson coefficient of the charged RN black hole in a cavity for $d = 4, 5, 6$ and 7. The enthalpy $H = 1$, $r_B$ varies from 0.5 to 2.5, and $Q = 0.035$ and show how it changes with $r_B$ in Fig. 8 with the red line. For comparison, we calculate higher dimensional JT coefficients $\mu$ with Eqs. (4) and (27) and take $d = 5, 6, 7$ as shown in Fig. 8. When $\mu = \left( \frac{\partial T}{\partial P} \right)_H = 0$, the transition between cooling and heating region occurs. But for $d$-dimensional charged black holes in a cavity, the Joule-Thomson coefficient is positive.

We conclude Joule-Thomson expansion is characterized by the invariance of enthalpy and Joule-Thomson coefficient $\mu$ determines the final change of temperature during the process. One can use the sign of $\mu$ to divide the heating and cooling zone. We can see from Fig. 8 that $\mu$ is always positive, which means no transition occurs. Comparing with the isenthalpic curves above, it’s obvious that the coefficient remains positive and always corresponds to the cooling phase.
5 Conclusions

Based on the thermodynamics of black holes in a cavity, we derive the temperature, pressure and other thermodynamic quantities in d-dimension to discuss the heat engine and the Joule-Thomson expansion of charged black holes.

As a cavity defines an explicit boundary for the working substance, i.e. black holes, it’s more rational to consider a heat engine in cavities rather than in AdS space, whose scale corresponds to the pressure in the extended space. Therefore, we derive the efficiency of Schwarzschild and RN black hole heat engines in the Carnot cycle and rectangular cycle to show how it changes with respect to the radius of a cavity.

Since previous study shows the charged AdS black hole always cools above the inversion curve during JT expansion, which is different from the JT expansion of van der Waals fluids, we intend to put the charged black hole into a cavity instead of in AdS space. And the result shows the dissimilarities caused by the two different boundary conditions. In particular, we find the Joule-Thomson expansion of charged black holes exist in a cavity but differ from the AdS cases, as the energy, temperature and pressure are defined with different boundary conditions. We plot the figure of isenthalpic curves for different enthalpy and dimensions, as well as the graph of Joule-Thomson coefficient with different dimensions. In this way, we conclude the charged black hole in a cavity always cools down during the isenthalpic process with the decreasing pressure, while the JT coefficient $\mu$ keeps positive.

Acknowledgements We are grateful to Yuchen Huang and Peng Wang for useful discussions and valuable comments. This work is supported by NSFC (Grant No.11947408 and 12047573).

Data Availability Statement The datasets generated during and analysed during the current study are available from the corresponding author on reasonable request.

References

1. York, J.W., Jr.: Dynamical origin of black hole radiance. Phys. Rev. D 28, 2929 (1983)
2. Hawking, S.W.: Particle creation by black holes. Commun. Math. Phys. 43, 199–220 (1975)
3. Bekenstein, J.D.: Statistical black hole thermodynamics. Phys. Rev. D 12, 3077–3085 (1975)
4. York, J.W., Jr.: Black hole thermodynamics and the Euclidean Einstein action. Phys. Rev. D 33, 2092–2099 (1986)
5. Braden, H.W., Brown, J.D., Whiting, B.F., York, J.W., Jr.: Charged black hole in a grand canonical ensemble. Phys. Rev. D 42, 3376–3385 (1990)
6. Wang, P., Wu, H., Yang, H., Yao, F.: Extended phase space thermodynamics for black holes in a cavity. JHEP 09, 154 (2020)
7. Wang, P., Yang, H., Ying, S.: Thermodynamics and phase transition of a Gauss-Bonnet black hole in a cavity. Phys. Rev. D 101(6), 064045 (2020)
8. Wang, P., Wu, H., Yang, H.: Thermodynamics and phase transition of a nonlinear electrodynamics black hole in a cavity. JHEP 07, 002 (2019)
9. Wang, P., Wu, H., Yang, H.: Thermodynamic geometry of AdS black holes and black holes in a cavity. Eur. Phys. J. C 80(3), 216 (2020)
10. Wang, P., Wu, H., Yang, S.: Validity of thermodynamic laws and weak cosmic censorship for AdS black holes and black holes in a cavity. Chin. Phys. C 45(5), 055105 (2021)
11. Zhao, W.-B., Liu, G.-R., Li, N.: Hawking-Page phase transitions of the black holes in a cavity. Eur. Phys. J. Plus 136, 981 (2021)
12. Simovic, F., Mann, R.B.: Critical phenomena of Born-Infeld-de sitter black holes in cavities. JHEP 05, 136 (2019)
13. Dias, O.J.C., Masachs, R.: Charged black hole bombs in a Minkowski cavity. Class. Quant. Grav. 35(18), 184001 (2018)
14. Marolf, D., Santos, J.E.: The Canonical Ensemble Reloaded: The Complex-Stability of Euclidean quantum gravity for Black Holes in a Box. arXiv:2202.11786
15. Hawking, S.W., Page, D.N.: Thermodynamics of black holes in anti-De sitter space. Commun. Math. Phys. 87, 577 (1983)
16. Chamblin, A., Emparan, R., Johnson, C.V., Myers, R.C.: Charged AdS black holes and catastrophic holography. Phys. Rev. D 60, 064018 (1999)
17. Chamblin, A., Emparan, R., Johnson, C.V., Myers, R.C.: Holography, thermodynamics and fluctuations of charged AdS black holes. Phys. Rev. D 60, 104026 (1999)
18. Cal黛relli, M.M., Cognola, G., Klemm, D.: Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories. Class. Quant. Grav. 17, 399–420 (2000)
19. Padmanabhan, T.: Classical and quantum thermodynamics of horizons in spherically symmetric spacetimes. Class. Quant. Grav. 19, 5387–5408 (2002)
20. Kastor, D., Ray, S., Traschen, J.: Enthalpy and the mechanics of AdS black holes. Class. Quant. Grav. 26, 195011 (2009)
21. Kubiznak, D., Mann, R.B.: P-V criticality of charged AdS black holes. JHEP 07, 033 (2012)
22. He, S., Li, L.-F., Zeng, X.-X.: Holographic Van der Waals-like phase transition in the Gauss-Bonnet gravity. Nucl. Phys. B 915, 243–261 (2017)
23. Mo, J.-X., Lan, S.-Q.: Phase transition and heat engine efficiency of phantom AdS black holes. Eur. Phys. J. C 78(8), 666 (2018)
24. Rajani, K.V., Ahmed Rizwan, C.I., Naveena Kumara, A., Vaid, D., Ajith, K.M.: Regular Bardeen AdS black hole as a heat engine. Nucl. Phys. B 960, 115166 (2020)
25. Deb Nath, U.: The General Class of Accelerating, Rotating and Charged Plebanski-Demianski Black Holes as Heat Engine. arXiv:2006.02920
26. Johnson, C.V.: de Sitter Black Holes, Schottky Peaks, and Continuous Heat Engines. arXiv:1907.05883
27. Balart, L., Fernando, S.: Non-linear black holes in 2+1 dimensions as heat engines. Phys. Lett. B 795, 638–643 (2019)
28. Yerra, P.K., Chandrasekhar, B.: Heat engines at criticality for nonlinearly charged black holes. Mod. Phys. Lett. A 34(27), 1950155 (2019)
29. Ghaffarnejad, H., Yerra, P.K., Farsam, M., Bamba, K.: Hairy black holes and holographic heat engine. Nucl. Phys. B 952, 114941 (2020)
30. Zhang, J., Li, Y., Yu, H.: Accelerating AdS black holes as the holographic heat engines in a benchmarking scheme. Eur. Phys. J. C 78(8), 645 (2018)
31. Zhang, J., Li, Y., Yu, H.: Thermodynamics of charged accelerating AdS black holes and holographic heat engines. JHEP 02, 144 (2019)
32. Yerra, P.K., Bhamidipati, C.: Critical heat engines in massive gravity. Class. Quant. Grav. 37(20), 205020 (2020)
33. Fernando, S.: Massive gravity with Lorentz symmetry breaking: black holes as heat engines. Mod. Phys. Lett. A 33(31), 1950157 (2018)
34. Hendi, S.H., Eslam Panah, B., Panahiyan, S., Liu, H., Meng, X.H.: Black holes in massive gravity as heat engines. Phys. Lett. B 781, 40–47 (2018)
35. Chakraborty, A., Johnson, C.V.: Benchmarking black hole heat engines, I. Int. J. Mod. Phys. D 27(16), 1950006 (2018)
36. Chakraborty, A., Johnson, C.V.: Benchmarking black hole heat engines, II. Int. J. Mod. Phys. D 27(16), 1950006 (2018)
37. Chandrasekhar, B., Yerra, P.K.: Heat engines for dilatonic Born-Infeld black holes. Eur. Phys. J. C 77(8), 534 (2017)
38. Chandrasekhar, B., Yerra, P.K.: A note on Gauss-Bonnet black holes at criticality. Phys. Lett. B 772, 800–807 (2017)
39. Ökçü, O., Aydiner, E.: Joule-Thomson expansion of the charged AdS black holes. Eur. Phys. J. C 77(1), 24 (2017)
40. Cao, Y., Feng, H., Hong, W., Tao, J.: Joule-Thomson expansion of RN-AdS black hole immersed in perfect fluid dark matter. Commun. Theor. Phys. 73(9), 095403 (2021)
41. Bi, S., Du, M., Tao, J., Yao, F.: Joule-Thomson expansion of Born-Infeld AdS black holes. Chin. Phys. C 45(2), 025109 (2021)
42. Guo, S., Han, Y., Li, G.-P.: Joule-Thomson expansion of a specific black hole in $f(R)$ gravity coupled with Yang-Mills field. Class. Quant. Grav. 37(8), 085016 (2020)
43. Hegde, R., Naveena Kumara, A., Rizwan, C.L., M., A.K., Ali, M.S.: Thermodynamics, Phase Transition and Joule Thomson Expansion of novel 4-D Gauss Bonnet AdS Black Hole. arXiv:2003.08778
44. Rajani, K.V., Rizwan, C.L.A., Naveena Kumara, A., Ali, M.S., Vaid, D.: Joule-Thomson expansion of regular Bardeen AdS black hole surrounded by static anisotropic matter field. Phys. Dark Univ. 32, 100825 (2021)
45. Rostami, M., Sadeghi, J., Miraboutalebi, S., Masoudi, A.A., Pourhassan, B.: Charged accelerating AdS black hole of $f(R)$ gravity and the Joule-Thomson expansion. Int. J. Geom. Meth. Mod. Phys. 17(09), 2050136 (2020)
46. Nam, C.H.: Heat engine efficiency and Joule-Thomson expansion of nonlinear charged AdS black hole in massive gravity. Gen. Rel. Grav. 53(3), 30 (2021)
47. Mahdavian Yekta, D., Hadikhani, A., Ökcü, O.: Joule-Thomson expansion of charged AdS black holes in Rainbow gravity. Phys. Rev. D 98(8), 084014 (2018)
48. Ghaffarnejad, H., Yaraie, E., Farsam, M.: Quintessence Reissner Nordström Anti de Sitter Black Holes and Joule Thomson effect. Int. J. Theor. Phys. 57(6), 1671–1682 (2018)
49. Pu, J., Guo, S., Jiang, Q.-Q., Zu, X. -T.: Joule-Thomson expansion of the regular(Bardeen)-AdS black hole. Chin. Phys. C 44(3), 035102 (2020)
50. Mo, J. -X., Li, G.-Q.: Effects of Lovelock gravity on the Joule-Thomson expansion. Class. Quant. Grav. 37(4), 045009 (2020)
51. Cisterna, A., Hu, S.-Q., Kuang, X.-M.: Joule-Thomson expansion in AdS black holes with momentum relaxation. Phys. Lett. B 795, 521–527 (2019)
52. Li, C., He, P., Li, P., Deng, J.-B.: Joule-Thomson expansion of the Bardeen-AdS black holes. Gen. Rel. Grav. 52(5), 50 (2020)
53. Ahmed Rizwan, C.L., Naveena Kumara, A., Vaid, D., Ajith, K.M.: Joule-Thomson expansion in AdS black hole with a global monopole. Int. J. Mod. Phys. A 33(35), 1850210 (2019)
54. Chabab, M., El Moumni, H., Iraoui, S., Masmar, K., Zhizeh, S.: Joule-Thomson Expansion of RN-AdS Black Holes in $f(R)$ gravity. LHEP 02, 05 (2018)
55. Mo, J.-X., Li, G.-Q., Lan, S.-Q., Xu, X.-B.: Joule-Thomson expansion of $d$-dimensional charged AdS black holes. Phys. Rev. D 98(12), 124032 (2018)
56. Ökcü, O., Aydner, E.: Joule-Thomson expansion of Kerr–AdS black holes. Eur. Phys. J. C 78(2), 123 (2018)
57. Haldar, A., Biswas, R.: Joule-Thomson expansion of five-dimensional Einstein-Maxwell-Gauss-Bonnet-AdS black holes. EPL 123(4), 40005 (2018)
58. Feng, Z.-W., Zhou, X., He, G., Zhou, S.-Q., Yang, S.-Z.: Joule-Thomson expansion of higher dimensional nonlinearly AdS black hole with power Maxwell invariant source. Commun. Theor. Phys. 73(6), 065401 (2021)
59. Nam, C.H.: Effect of massive gravity on Joule-Thomson expansion of the charged AdS black hole. Eur. Phys. J. Plus 135(2), 259 (2020)
60. Meng, Y., Pu, J., Jiang, Q.-Q.: P-V criticality and Joule-Thomson expansion of charged AdS black holes in the Rastall gravity. Chin. Phys. C 44(6), 065105 (2020)
61. Zhang, M., Zhang, C.-M., Zou, D.-C., Yue, R.-H.: P-V criticality and Joule-Thomson expansion of Hayward-AdS black holes in 4D Einstein-Gauss-Bonnet gravity. Nucl. Phys. B 973, 115608 (2021)
62. Liang, J., Lin, W., Mu, B.: Joule-Thomson expansion of the torus-like black hole. Eur. Phys. J. Plus 136(11), 1169 (2021)
63. Liang, J., Mu, B., Wang, P.: Joule-Thomson expansion of lower-dimensional black holes. Phys. Rev. D 104(12), 124003 (2021)
64. Graça, J.P.M., Capossoli, E.F., Boschí-Filho, H.: Joule-Thomson expansion for quantum corrected AdS-Reissner-Nordstrom black holes in Kiselev spacetime. arXiv:2105.04689
65. Mirza, B., Naemipour, F., Tavakoli, M.: Joule-Thomson expansion of the quasitopological black holes. Front. in Phys. 9, 33 (2021)
66. Yin, R., Liang, J., Mu, B.: Joule-Thomson expansion of Reissner–Nordström-Anti-de Sitter black holes with cloud of strings and quintessence. Phys. Dark Univ. 34, 100884 (2021)
68. Zhang, C.-M., Zhang, M., Zou, D.-C.: Joule-Thomson Expansion of Born-Infeld AdS Black Holes in 4D Einstein-Gauss-Bonnet gravity. arXiv:2106.00183
69. Biswas, A.: Joule-Thomson expansion of AdS black holes in Einstein Power-Yang-mills gravity. Phys. Scripta 96(12), 125310 (2021)
70. Graça, J.P.M., Capossoli, E.F., Boschi-Filho, H.: Joule-Thomson expansion for noncommutative uncharged black holes. EPL 135(4), 41002 (2021)
71. Abdusattar, H., Kong, S.-B., You, W.-L., Zhang, H., Hu, Y.-P.: Joule-Thomson Expansion and Heat Engine of the FRW Universe. arXiv:2108.09407
72. Liang, K., Wang, P., Wu, H., Yang, M.: Phase structures and transitions of Born-Infeld black holes in a grand canonical ensemble. Eur. Phys. J. C 80(3), 187 (2020)
73. York, J.: Boundary terms in the action principles of general relativity. Found. Phys. 16, 249–257 (1986)
74. Lundgren, A.P.: Charged black hole in a canonical ensemble. Phys. Rev. D 77, 044014 (2008)
75. Chabab, M., El Moumni, H., Iraoui, S., Masmar, K.: Behavior of quasinormal modes and high dimension RN–AdS black hole phase transition. Eur. Phys. J. C 76(12), 676 (2016)
76. Johnson, C.V.: An exact efficiency formula for Holographic heat engines. Entropy 18, 120 (2016)
77. Rosso, F.: Holographic heat engines and static black holes: a general efficiency formula. Int. J. Mod. Phys. D 28(02), 02 (2018)
78. Carlip, S., Vaidya, S.: Phase transitions and critical behavior for charged black holes. Class. Quant. Grav. 20, 3827–3838 (2003)
79. York, J.W., Jr.: Black hole in thermal equilibrium with a scalar field: the back reaction. Phys. Rev. D 31, 775 (1985)
80. Peca, C.S., Lemos, J.P.S.: Thermodynamics of Reissner-Nordstrom anti-de Sitter black holes in the grand canonical ensemble. Phys. Rev. D 59, 124007 (1999)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.