Zigzag structure of complexes

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Abstract

Inspired by Coxeter’s notion of Petrie polygon for $d$-polytopes (see [Cox73]), we consider a generalization of the notion of zigzag circuits on complexes and compute the zigzag structure for several interesting families of $d$-polytopes, including semiregular, regular-faced, Wythoff Archimedean ones, Conway’s 4-polytopes, half-cubes, folded cubes.

Also considered are regular maps and Lins triality relations on maps.

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1 Introduction

The notion of zigzag was introduced for plane graphs in [Sh75] (as a left-right path) and for regular polytopes in [Cox73] (as a Petrie polygon). We focus here on generalization of zigzags for higher dimension.

Zigzags can be also defined for maps on orientable surface; see, for example, on Figure 3 typical zigzags for dual Klein map $\{7,3\}$ and dual Dyck map $\{8,3\}$. Moreover, this notion, being local, is defined even for non-oriented maps. See Section 6 on maps. Also, the notion of zigzag extends naturally on infinite plane graphs.

We use for polytopes notations and terminology of [Cox73]; for example, $\alpha_d, \beta_d, \gamma_d$ and $\frac{1}{2}\gamma_d$ denote $d$-dimensional simplex, cross-polytope, cube and half-cube, respectively. Their 1-skeleton graphs are denoted by $K_{d+1}, K_{d\times2}, H_d$ and $\frac{1}{2}H_d$, respectively. We use also Schlafli notation from [Cox73] in Tables 1 and 2. By Prism$_m$...

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and $APrism_m$ are denoted semiregular $m$-gonal prism and $m$-gonal antiprism, respectively.

The medial of a polytope $P$, denoted by $Med(P)$, is the polytope formed by the convex hull of the midpoints of all edges of $P$. It can also be defined combinatorially on maps on surfaces by taking as vertices the edge of the original map, by taking as edges the pair of edges sharing an incident vertex and an incident face and by taking as faces the vertices and faces of the original map. This notion of medial can also be defined combinatorially on $d$-dimensional complexes, including maps, i.e. the case $d = 2$.

## 2 Zigzags for $d$-dimensional complexes

We extend here the definitions of zigzags to any complex. A chain of length $k$ in a partially ordered set is a sequence $(x_0, \ldots, x_k)$, such that $x_i < x_{i+1}$. A chain $C$ is a subchain of another chain $C'$ if it is obtained by removing some elements in $C'$. A chain is maximal if it is not a subchain of another chain. The rank $\text{rank}(x)$ of an element $x$ is the maximal length of chains, beginning at the lowest elements 0 and terminating at $x$. A partially ordered set is called ranked if there is a lowest element 0 and a greatest element 1 and if, given two elements $x < y$ with no elements $z$ satisfying to $x < z < y$, one has $\text{rank}(y) = 1 + \text{rank}(x)$.

A partially ordered set is called a lattice if for any two elements $x$ and $y$, there are an unique smallest element $s$ and an unique greatest element $t$, such that $x \leq s$ and $x \geq t$, $y \leq s$ and $x \geq t$.

The dimension of an element is defined as $\text{rank}(x) - 1$.

A $d$-dimensional complex $K$ is a finite partially ordered set, such that it holds:

(i) $K$ has a smallest 0 and highest element 1,

(ii) $K$ is ranked and all maximal chains have length $d + 2$,

(iii) given two elements $x$ and $y$ with $x \leq y$ and $\text{dim}(y) = 2 + \text{dim}(x)$, there are exactly two elements $u, u'$, such that $x \leq u \leq y$ and $x \leq u' \leq y$.

A $d$-dimensional complex is called simplicial if for every element $x$ of dimension $d$, there is exactly $d + 1$ elements of dimension 0 contained in it.

In a $d$-dimensional complex, a maximal chain is called a flag; it necessarily begins at 0 and terminates at 1.

Using (iii), one can define the following permutation operator on flags. For $1 \leq i \leq d + 1$, denote by $\sigma_i$ the operator transforming $(0, x_1, \ldots, x_i, \ldots, x_{d+1}, 1)$ into the flag $(0, x_1, \ldots, x'_i, \ldots, x_{d+1}, 1)$ with $x'_i$ being the unique element satisfying to $x'_i \neq x_i$ and $x_{i-1} \leq x'_i \leq x_{i+1}$. One has $\sigma_i^2 = 1$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $i < j - 1$.

### Definition 2.1

Let $K$ be a $d$-dimensional complex, then:

(i) denote by $\mathcal{F}(K)$ the set of flags of $K$,

(ii) denote by $\mathcal{G}(K)$ the graph having, as vertex-set, $\mathcal{F}(K)$, with two flags being adjacent if they are obtained one from the other by a permutation $\sigma_i$,

(iii) the complex $K$ is said to be orientable if $\mathcal{G}(K)$ is bipartite; an orientation of $K$ consists in selecting one of the two connected components.
In the case $d = 2$, the elements of dimension 0, 1 and 2 are called *vertices*, *edges* and *faces*, respectively.

The definition of orientability, given above, corresponds to the fact that, given an orientation on a cell complex and a maximal chain $(f_1, \ldots, f_{d-1})$ of faces, one can find the last face $f_d$ that makes it a flag.

A $(d+1)$-polytope $P$ is defined as the convex hull of a set of points in $\mathbb{R}^{d+1}$. The set of faces of $P$ defines a lattice and so, a $d$-dimensional complex, which is a lattice, since the boundary of a $d$-polytope is homeomorphic to $S^{d-1}$.

Call a $d$-dimensional complex *regular* if its symmetry group is transitive on the set of flags.

**Theorem 2.2** Let $\mathcal{K}$ be a $d$-dimensional complex and $x = (0, x_1, \ldots, x_{d+1}, 1)$ be a flag in $\mathcal{K}$.

Then there exists an unique sequence of faces $(x_{i,j})_{1 \leq i \leq d+1, 1 \leq j \leq d+2-i}$, namely:

\[
\begin{align*}
x_{1,1}, \ldots, x_{1,j}, \ldots, x_{1,d+1} \\
x_{2,1}, \ldots, x_{2,j}, \ldots, x_{2,d} \\
\vdots \\
x_{d-1,1}, x_{d-1,2}, x_{d-1,3} \\
x_{d,1}, x_{d,2} \\
x_{d+1,1}
\end{align*}
\]

such that it holds:

(i) $x_{i,1} = x_i$,

(ii) $\dim(x_{i,j}) = i - 1$,

(iii) $x_{i,j} \leq x_{i+1,j}$ for $1 \leq i \leq d$ and $1 \leq j \leq d + 1 - i$,

(iv) $x_{i,j} \leq x_{i+1,j-1}$ for $1 \leq i \leq d$ and $2 \leq j \leq d + 2 - i$.

Moreover, if $\mathcal{K}$ is a lattice, then the elements $(x_{i,j})$ are uniquely defined by the vertex sequence $(x_{1,j})_{1 \leq j \leq d+1}$.

**Proof.** Using property (iii), one can find successively, $x_{1,2}, \ldots, x_{d,2}$, then $x_{1,3}$ and so on.

If $\mathcal{K}$ is a lattice, then $x_{i,j}$ can be characterized as the smallest element greater than $x_{i-1,j}$ and $x_{i-1,j+1}$. 

**Definition 2.3** Let $\mathcal{K}$ be a $d$-dimensional complex.

(i) Denote by $T = \sigma_{d+1}\sigma_d \ldots \sigma_1$ the translation operator of $\mathcal{K}$.

(ii) A zigzag in $\mathcal{K}$ is a circuit $(f_1, \ldots, f_l)$ of flags, such that $f_{j+1} = T(f_j)$; $l$ denotes the length of the zigzag.

(iii) Given a flag $f$, the reverse $f^t$ of $f$ is defined as $(0, x_{1,d+1}, x_{2,d}, \ldots, x_{d,2}, x_{d+1,1}, 1)$ with $(x_{i,j})$ as in Theorem 2.2.

(iv) The reverse of a zigzag $(f_1, \ldots, f_l)$ is the zigzag $(f_l^t, f_{l-1}^t, \ldots, f_1^t)$.

The above notion (central in this paper), for the special case of an $d$-polytope, essentially coincides with the following notion on page 223 of [Cox73]: “A Petrie polygon
of an $d$-dimensional polytope or of an $(d-1)$-dimensional honeycomb, is a skew polygon, such that any $(d-1)$ consecutive sides but no $d$, belong to a Petrie polygon of a cell."

The choice of a zigzag $(f_1, \ldots, f_t)$ over its reverse $(f_t^l, \ldots, f_1^l)$ amounts to choosing an orientation on the zigzag. In the sequel a zigzag is identified with its reverse.

Note that if $\mathcal{K}$ is a $d$-dimensional simplicial complex with $f$ facets, then one has $|\mathcal{F}(\mathcal{K})| = (d + 1)!f$. Note also that the stabilizer of a flag is trivial and so, if $\mathcal{K}$ has $p$ orbits of flags, then $|\mathcal{F}(\mathcal{K})| = p|\text{Sym}(\mathcal{K})|$.

**Proposition 2.4** If the complex $\mathcal{K}$ is oriented and of even dimension, then the length of any zigzag is even.

**Proof.** Since $\mathcal{K}$ is oriented, the set $\mathcal{F}(\mathcal{K})$ is splitted in two parts, $\mathcal{F}_1$ and $\mathcal{F}_2$. Since $d$ is even, the translation $T = \sigma_{d+1}\sigma_d \ldots \sigma_1$ of all its flags has an odd number of components; so, it interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$. On Section 5.91 of [Cox73] the evenness of the length of zigzags was obtained for complexes arising from Coxeter groups of dimension 3; there was given the formula $g = h(h + 2)$ with $g$ being the size of the group and $h$ the length of the zigzag.

**Definition 2.5** Take a zigzag $Z = (f_1, \ldots, f_t)$ and its reverse $Z^t = (f_t^l, \ldots, f_1^l)$.

(i) Given a flag $f_j$, if $\sigma_1(f_j)$ belongs to $Z$, then self-intersection is called of type I, while if $\sigma_1(f_j)$ belongs to $Z^t$, then it is called of type II.

(ii) The signature of the zigzag $Z$ is the pair $(n_I, n_{II})$ with $n_I$ being the number of self-intersections of type I and $n_{II}$ the number of self-intersections of type II. The signature does not change if one interchanges $Z$ and $Z^t$.

(iii) Take two zigzags $Z_1$ and $Z_2$ with associated circuits $(f_{1,1}, \ldots, f_{1,l}), (f_{1,1}^l, \ldots, f_{1,1}^l)$ and $(f_{2,1}, \ldots, f_{2,l}), (f_{2,1}^l, \ldots, f_{2,1}^l)$. If $f_{1,1}^l$ belongs to $Z_2$, then it is called an intersection of type I, while if it belongs to $Z_2^t$, it is called an intersection of type II.

(iv) The signature $(n_I, n_{II})$ is the pair enumerating such intersections. If $Z_2$ and $Z_2^t$ are interchanged, then the types of intersections are interchanged also.

The $z$-vector of a complex $\mathcal{K}$ is the vector enumerating the lengths of all its zigzags with their signature as subscript. The simple zigzags are put in the beginning, in increasing order of length, without their signature $(0, 0)$, and separated by a semicolon from others. Self-intersecting zigzags are also ordered by increasing lengths. If there are $m > 1$ zigzags of the same length $l$ and the same signature $(\alpha_1, \alpha_2) \neq (0, 0)$, then we write $l_{\alpha_1, \alpha_2}^m$. It turns out, that Snub Cube, Snub Dodecahedron, $\text{Py}r(\beta_{d-1})$ and $B\text{Py}r(\beta_{d-1})$ are the only polytopes in Tables of this paper, having self-intersecting zigzags.

Given two zigzags $Z$ and $Z'$, their normalized signature is the pair $(n_I, n_{II})$ enumerating intersection of type I and II with orientation chosen so that $n_I \leq n_{II}$. For a zigzag $Z$, its intersection vector $\text{Int}(Z) = \ldots, (c_{k,1}, c_{k,II})m_k, \ldots$ is such that $(\ldots, (c_{k,1}, c_{k,II}), \ldots)$ is a sequence $(c_{k,1}, c_{k,II})$ of its non-zero normalized signature with all others zigzags, and $m_k$ denote respective multiplicities. If the zigzag has signature $(n_1, n_{II})$, then its length $l$ satisfies to

$$l = 2(n_I + n_{II}) + \sum_k m_k(c_{k,1} + c_{k,II}).$$

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The dual $K^*$ of a $d$-dimensional complex $K$ is the complex with the same elements as $K$, but with $x \leq y$ in $K$ being equivalent to $y \leq x$ in $K$.

**Theorem 2.6** Every zigzag $Z$ in $K$ corresponds to an unique zigzag $Z^*$ in $K^*$ with the same length.

**Proof.** Given a flag $f = (0, x_1, \ldots, x_{d+1}, 1)$ of $K$, one can associate to it a flag $f' = (1, x_{d+1}, \ldots, x_1, 0)$ of $K^*$. Denote by $\sigma_i'$ the operator on $K^*$, which acts by changing the $i$-th element. It is easy to see that its action on $f'$ corresponds to the action of $\sigma_{d+2-i}$ on $f$. So, one has $T'(f') = (T^{-1} f)'$ and every zigzag $(f_1, \ldots, f_l)$ of $K$ corresponds to a zigzag $(f'_1, \ldots, f'_1)$ of $K^*$.

In the case of maps (i.e. for $d = 2$), every intersection in $K$ corresponds to an intersection in $K^*$ with type I or II interchanged. This is not, a priori, the case of complexes of dimension $d > 2$.

A $d$-dimensional complex $K$ is said to be $z$-transitive if its symmetry group $\text{Sym}(K)$ is transitive on zigzags. It is said to be $z$-knotted if it has only one zigzag. Note that the stabilizer of a flag is necessarily the trivial group, i.e., every orbit of flags has the size $|\text{Sym}(K)|$.

Denote by $Z(K)$ the graph formed by the set of zigzags of a complex $K$ with two zigzags being adjacent if the signature of their intersection is different from $(0, 0)$. In the case of a 2-dimensional complexes, we prove (see Section 6) that $Z(K)$ is connected. In the case of complexes of dimension $d > 2$, there is no reason to think that connectivity will still hold.

**Proposition 2.7** If a $d$-dimensional complex $K$ is regular, then:

(i) $K$ is z-transitive,

(ii) if $Z(K)$ is connected, then either zigzags have no self-intersections, or $K$ is z-knotted.

**Proof.** The transitivity on zigzags is obvious. If a zigzag has a self-intersection, then, by transitivity, all flags correspond to a self-intersection of zigzags. Since $Z(K)$ is connected, it means that there is only one zigzag. □

**Conjecture 2.8** The signature of any zigzag in any odd-dimensional complex is $(0, 0)$.

The above conjecture is strange and we do not see why it would be true. Nevertheless, we did not find a single example violating it.

### 3 Some generalizations of regular $d$-polytopes

Remind, that a *regular $d$-polytope* is one whose symmetry group is transitive on flags.

A *regular-faced $d$-polytope* is one having only regular facets. A *semiregular $d$-polytope* is a regular-faced $d$-polytope whose symmetry group is transitive on vertices. All semiregular, but not Platonic, 3-polytopes (i.e. 13 Archimedean 3-polytopes and
Prism\textsubscript{m}, APrism\textsubscript{m} for any \( m \geq 3 \) were discovered by Kepler \([Ke1619]\). The list of all 7 semiregular, but not regular, \( d \)-polytopes with \( d \geq 4 \) was given by Gosset in 1897 \([Gos00]\), but proofs were never published; see also \([BlBl91]\). This list consists of 5 polytopes, denoted by \( n_{21} \) (where \( n \in \{0, 1, 2, 3, 4\} \)) of dimension \( n + 4 \), and two exceptional ones (both 4-dimensional): snub 24-cell \( s(3, 4, 3) \) and octicosahedric polytope \( 0_{21}, 24 \)-cell, \( s(3, 4, 3) \) and the octicosahedric polytope are the medials of \( \alpha_4, \beta_4, 24 \)-cell and 600-cell, respectively (see also Section 5 and Table 5 for the notion of Wythoff Archimedean). \( s(3, 4, 3) \) is obtained also by eliminating some 24 vertices of 600-cell \([Cox73]\). \( 1_{21} \) is \( \gamma_5 \); \( 2_{21} \) and \( 3_{21} \) are Delaunay polytopes of the root lattices \( E_6 \) and \( E_7 \). The skeleton of \( 4_{21} \) is the root graph of all 240 roots of the root system \( E_8 \).

The pyramid operation \( \text{Pyr}(K) \) (respectively, bipyramid operation \( \text{BPyr}(K) \)) on a \( d \)-dimensional complex \( K \) is the \( (d+1) \)-dimensional complex obtained by adding one (respectively, two) new vertices, connected to all vertices of the original complex.

All 92 Johnson solids, i.e. regular-faced 3-polytopes were found in \([Jo66]\). All regular-faced, but not semiregular, \( d \)-polytopes, \( d \geq 4 \) are known also \([BlBl80]\). This list consists of two infinite families of \( d \)-polytopes \( \text{Pyr}(\beta_{d-1}) \) and \( \text{BPyr}(\alpha_{d-1}) \), three particular 4-polytopes \( \text{Pyr}(\text{Ico}), \text{BPyr}(\text{Ico}) \) and the union of \( 0_{21} + \text{Pyr}(\beta_3) \), where \( \beta_3 \) is a facet of \( 0_{21} \) and, finally, any 4-polytope (except of snub 24-cell), arising from 600-cell by the following special cut of vertices. If \( E \) is a subset of the 120 vertices of 600-cell, such that any two vertices in \( E \) are not adjacent, then this polytope is the convex hull of all vertices of 600-cell, except those in \( E \).

Conway \([Con67]\) enumerated all Archimedean 4-polytopes, i.e. those having a vertex-transitive group of symmetry and whose cells are regular or Archimedean polyhedra and prisms or antiprisms with regular faces. The list consists of:

1. 45 polytopes obtained by Wythoff’s kaleidoscope construction from regular 4-polytopes (see Table 5 and, more generally, Section 5);
2. 17 prisms on Platonic, other than Cube, and Archimedean solids (see Table 2);
3. prisms on \( APrism_m \) for any \( m > 3 \) (see Conjecture 4.6);
4. a doubly infinite set of 4-polytopes, which are direct products \( C_p \times C_q \) of two regular polygons (if one of polygons is a square, then one gets prisms on \( Prism_m \)) (see Conjecture 4.5);
5. the snub 24-cell \( s(3, 4, 3) \) (see Table 1);
6. a 4-polytope, called in \([Con67]\) Grand Antiprism; it has 100 vertices (all from 600-cell), 300 cells \( \alpha_3 \) and 20 cells \( APrism_5 \) (those antiprisms form two interlocking tubes).

**Remark 3.1** The Grand Antiprism has \( z \)-vector \( 30^{20}, 50^{40}, 90^{20} \). The corresponding intersection vectors are \( (0, 1)^{10}, (0, 2)^{10} \) and \( (0, 1)^{10}, (0, 2)^{20} \) and \( (0, 1)^{10}, (0, 2)^{10}, (0, 4)^5, (4, 4)^5 \).
| dimension | complex | $z$-vector | int. vectors |
|-----------|---------|------------|-------------|
| $d - 1$  | $d$-simplex $\alpha_d = \{3^{d-1}\}$ | $(d + 1)^{d/2}$ | $(0, 1)^{d+1}$ if $d \geq 4$ $(1, 1)^2$ if $d = 3$ $(0, 2)^d$ $(0, 2)^5$ $(0, 2)^3$ $(0, 1)^5$ |
| $d - 1$  | cross-$d$-polytope $= \beta_d = \{3^{d-2}, 4\}$ | $(2d)^{2d-2(d-1)!}$ | $10^6$ $6^{10}$ $5^6$ |
| 2        | Dodecahedron $= \{5, 3\}$ | $10^6$ | $(0, 2)^5$ |
| 2        | Great Dodecahedron $= \{5, \frac{5}{2}\}$ | $5^6$ | $(0, 2)^3$ |
| 2        | Petersen graph on $P^2$ | | |
| 3        | 600-cell $= \{3, 3, 5\}$ | $30^{240}$ | $(0, 2)^{15}$ |
| 3        | 24-cell $= \{3, 4, 3\}$ | $12^{48}$ | $(0, 2)^6$ |
| 3        | snub 24-cell $= s(3, 4, 3)$ | | |
| 3        | octicosahedric polytope | $20^{144}$ | $(1, 1)^4, (0, 2)^1, (0, 4)$ |
| 3        | $0_{21} = \text{Med}(\alpha_4)$ | $45^{480}$ | $(0, 1)^{15}, (0, 2)^{15}$ |
| 4        | $1_{21} = \frac{1}{2}\gamma_5 = \text{Med}(\beta_5)$ | $15^{12}$ | $(1, 2)^5$ |
| 4        | $2_{21} = \text{Schläfi polytope (in } E_6\text{)}$ | $12^{240}$ | $(0, 1)^8, (0, 2)^2$ |
| 5        | $3_{21} = \text{Gosset polytope (in } E_7\text{)}$ | $18^{4320}$ | $(0, 1)^6, (0, 2)^6$ |
| 5        | $4_{21}$ (240 roots of $E_8$) | $90^{48384}$ | $(0, 2)^{15}, (0, 4)^{15}$ |
| 6        | $3_{21}$ (240 roots of $E_8$) | $36^{29030400}$ | $(0, 1)^{24}, (0, 4)^3$ |
| 7        | 92 Johnson solids | See Remark 3.2 | |
| 3        | $\text{Pyr(Icosahedron)}$ | $25^{12}$ | $(0, 10), (0, 3)^5$ |
| 3        | $\text{BPyr(Icosahedron)}$ | $40^{12}$ | $(0, 20), (0, 4)^5$ |
| 3        | $0_{21} + \text{Pyr}(\beta_3)$ | $42^6$ | $(1, 1), (8, 8), (12, 12)$ |
| $d - 1$  | special cuts of 600-cell | See Remark 3.3 | |
| $d - 1$  | $\text{Pyr}(\beta_{d-1})$ | See Conjecture 4.4 | |
| $d - 1$  | $\text{BPyr}(\alpha_{d-1})$ | See Conjecture 4.4 | |
| 3        | 45 Wythoff Archimedean 4-polytopes | See Table 5 | |
| 3        | 17 prisms on Platonic and Archimedean solids | See Table 2 | |
| 3        | Grand Antiprism | See Remark 3.1 | |
| 3        | $C_p \times C_q$ | See Conjecture 4.5 | |
| 3        | prisms on $\text{APrism}_m$ | See Conjecture 4.6 | |

Table 1: $z$-structure of regular, semiregular, regular-faced $d$-polytopes and Conway’s 4-polytopes
| polyhedron $P$                  | $P$       | $Prism(P)$ |
|---------------------------------|----------|-----------|
| **z**                           | int. vectors | $z$       | int. vectors |
| Tetrahedron                     | $4^4$    | $(1,1)^2$ | $16^6$     | $(3,3)^2$, $(0,4)$ |
| Octahedron                      | $6^4$    | $(0,2)^3$ | $8^{24}$   | $(0,2)^4$          |
| Dodecahedron                    | $10^6$   | $(0,2)^5$ | $40^{12}$  | $(0,6)^5$, $(0,10)$|
| Icosahedron                     | $10^6$   | $(0,2)^5$ | $40^{12}$  | $(0,6)^5$, $(0,10)$|
| Cuboctahedron                   | $8^6$    | $(0,2)^4$ | $32^{12}$  | $(0,6)^4$, $(0,8)$ |
| Icosidodecahedron               | $10^{12}$| $(0,2)^5$ | $40^{24}$  | $(0,6)^5$, $(0,10)$|
| Truncated Tetrahedron           | $12^3$   | $(3,3)^2$ | $16^{18}$  | $(0,3)^4$, $(0,4)$ |
| or $(3,3)^2$, $(0,4)$          |          |           |           |                    |
| Truncated Octahedron            | $12^6$   | $(0,4)$, $(0,2)^3$ | $16^{36}$  | $(0,2)^4$, $(0,4)^2$ |
| or $(2,4)^3$, $(0,6)$          |          |           |           |                    |
| or $(2,4)^3$, $(0,6)$          |          |           |           |                    |
| Truncated Icosahedron           | $18^{10}$| $(0,2)^9$ | $24^{60}$  | $(0,2)^9$, $(0,6)$ |
| or $(2,4)^5$, $(0,10)$         |          |           |           |                    |
| or $(2,4)^5$, $(0,10)$         |          |           |           |                    |
| Truncated Dodecahedron          | $30^6$   | $(2,4)^5$ | $40^{36}$  | $(0,2)^5$, $(0,4)^5$, $(0,10)$ |
| or $(2,4)^5$, $(0,10)$         |          |           |           |                    |
| or $(2,4)^5$, $(0,10)$         |          |           |           |                    |
| Snub Cube                       | $30^{4}_{3,0}$ | $(4,4)^3$ | $40^{24}$  | $(0,2)^4$, $(2,2)^4$, $(0,16)$ |
| Snub Dodecahedron               | $50^{6}_{5,0}$ | $(4,4)^5$ | $200^{12}$ | $(12,12)^5$, $(0,80)$ |

Table 2: $z$-structure of prisms on Platonic and Archimedean solids

| dimension | half-$d$-cube | $z$-vector | int. vectors |
|-----------|---------------|------------|--------------|
| 2         | $\frac{1}{2} \gamma_3 = \alpha_3$ | $4^3$     | $(1,1)^2$ |
| 3         | $\frac{1}{2} \gamma_4 = \beta_4$ | $8^{24}$ | $(0,2)^4$ |
| 4         | $\frac{1}{2} \gamma_5 = Med(\beta_5)$ | $12^{240}$ | $(0,1)^8$, $(0,2)^2$ |
| 5         | $\frac{1}{2} \gamma_6$ | $32^{1440}$ | $(0,2)^4$, $(0,3)^8$ |
| 6         | $\frac{1}{2} \gamma_7$ | $120^{6720}$ | $(0,3)^{24}$, $(0,12)^4$ |
| 7         | $\frac{1}{2} \gamma_8$ | $36^{430080}$ | $(0,2)^{12}$, $(0,4)^3$ |
| 8         | $\frac{1}{2} \gamma_9$ | $84^{3870720}$ | $(0,4)^6$, $(0,5)^{12}$ |
| 9         | $\frac{1}{2} \gamma_{10}$ | $192^{38707200}$ | $(0,5)^{24}$, $(0,12)^6$ |
| 10        | $\frac{1}{2} \gamma_{11}$ | $216^{851558400}$ | $(0,3)^{48}$, $(0,18)^4$ |
| 11        | $\frac{1}{2} \gamma_{12}$ | $160^{30656102400}$ | $(0,6)^8$, $(0,7)^{16}$ |
| 12        | $\frac{1}{2} \gamma_{13}$ | $880^{159411732480}$ | $(0,7)^{80}$, $(0,40)^8$ |

Table 3: $z$-structure of half-$d$-cubes for $d \leq 13$
Schläfli symbol of $P$ & $|\text{Aut}(P)|$ & $z$-vector \\
\{\frac{5}{2}, 5\} & 120 & 6^{10} \\
\{\frac{5}{2}, 3\} & 120 & 10^6 \\
\{\frac{5}{2}, 5, 3\} & 14400 & 20^{360} \\
\{\frac{5}{3}, \frac{5}{2}, 5\} & 14400 & 14^{180} \\
\{\frac{5}{3}, 3, 5\} & 14400 & 12^{300} \\
\{\frac{5}{2}, 5, \frac{5}{2}\} & 14400 & 15^{180} \\
\{\frac{3}{2}, \frac{5}{2}, 5\} & 14400 & 20^{360} \\
\{\frac{5}{2}, 3, 3\} & 14400 & 30^{240} \\

Table 4: $z$-structure of non-convex regular 3- and 4-polytopes (adapted from pages 292 and 294 of [Cox73])

**Remark 3.2** Complete information on $z$-structure of 92-Johnson polyhedra is available from [Dut04]. We found 25 $z$-uniform ones.

**Remark 3.3** The number of polytopes, obtained by special cuts, is unknown but it is finite. By special cutting with 1, . . . , 7 vertices, one obtains, respectively, 1, 7, 436, 4776, 45775, 334380 polytopes. We expect that for 24 vertices, there is only one possible special cut, which yields semiregular snub 24-cell. For more than 25 vertices, there is no special cut possible (i.e. the skeleton of 600-cell has independence number 24, see page 82 of [Mar94]). Due to the difficulty of the computation and very large size of data, we computed the $z$-structure of special cuts of 600-cell only up to 3 vertices. Results are available from [Dut04].

**Remark 3.4** In Table 4 note that:
(i) Amongst those eight polytopes only $\{5, \frac{5}{2}, 5\}$ and $\{\frac{5}{2}, 5, \frac{5}{2}\}$ are self-dual.
(ii) In the case of Great Stellated Dodecahedron $\{\frac{5}{2}, 3\}$, the item $h$ in Table 1 on page 292 of [Cox73] (corresponding to the length of a zigzag) was $\frac{10}{3}$, while in Table 4 we put the value 10. In fact, our notion is combinatorial, while Coxeter define Petrie polygon as a skew polygon (see Figure 6.1A on page 93 of [Cox73]).

### 4 General results on $z$-structure of some generalizations of regular polytopes

**Proposition 4.1** For infinite series of regular polytopes we have:
(i) $z(\alpha_d) = (d+1)^{d/2}$ with $\text{Int} = (0, 1)^{d+1}$ for $d \geq 4$ and $(1, 1)^2$ for $d = 3$.
(ii) $z(\beta_d) = (2d)^{2d-2(d-1)!}$ with $\text{Int} = (0, 2)^d$.

**Proof.** Both polytopes are regular polytopes. Therefore, they are $z$-uniform. In order to know the length of a zigzag, one needs to compute the successive images of a flag under $T = \sigma_d \ldots \sigma_2 \sigma_1$. 
Denote by \{0, \ldots, d\} the vertices of \(\alpha_d\). It is easy to see that the image of the flag \(f = (\{0\}, \{0,1\}, \ldots, \{0,\ldots, d-1\})\) is \(\{1\}, \{1,2\}, \ldots, \{1,\ldots, d\}\), i.e., it is the image of \(f\) under a cycle of length \(d + 1\). Therefore, its length is \(d + 1\) and there is no self-intersection; hence, the \(z\)-vector is as in (i). Also, one can check that two different zigzags intersect at most once if \(d \geq 4\). Hence, the intersection vector is \((0,1)^{d+1}\). The case \(d = 3\) is trivial.

Denote by \(\pm e_i\) with \(1 \leq i \leq d\) the vertices of \(\beta_d\). It is easy to see that the image of the flag \(f = (\{e_1\}, \{e_1, e_2\}, \ldots, \{e_1, \ldots, e_d\})\) is the flag \(f' = (\{e_2\}, \{e_2, e_3\}, \ldots, \{e_2, \ldots, e_d\}, \{-e_1, e_2, \ldots, e_d\})\). Denote by \(\phi\) the composition of the cycle \((1, \ldots, d)\) on the coordinates with the symmetry \((x_1, \ldots, x_d) \mapsto (-x_1, x_2, \ldots, x_d)\). The order of \(\phi\) is \(2d\) and \(\phi(f) = f'\). Therefore, all zigzags have length \(2d\) and there is no self-intersection. If two zigzags are intersecting, then they, moreover, intersect twice, since \(\phi^d = -Id\) and one gets \(Int = (0,2)^d\).

In Table 3 are given \(z\)-structure of half-\(d\)-cubes for \(d \leq 13\); note that the length of any zigzag there divides \(2(d - 2)\).

**Proposition 4.2** For half-\(d\)-cube it holds:

(i) There are \(d!2^{d-1}(d - 2)\) flags, forming one orbit for \(d = 3, 4\) and \(d - 2\) orbits for \(d \geq 5\).

(ii) It is \(z\)-uniform.

**Proof.** Let us write the set of vertices of \(\frac{1}{2}\gamma_d\) as \(\{S \subset \{1, \ldots, d\} \mid |S|\text{ even}\}\). One has \(\frac{1}{2}\gamma_3 = \alpha_3\) and \(\frac{1}{2}\gamma_4 = \beta_4\), which are regular polytopes and whose structure is known. Therefore, one can assume \(d \geq 5\). The list of facets of \(\frac{1}{2}\gamma_d\) consists of:

1. \(2d\) facets \(x_i = 0\) and \(x_i = 1\) (those facets are incident to \(2^{d-2}\) vertices of \(\frac{1}{2}\gamma_d\), which form a polytope \(\frac{1}{2}\gamma_{d-1}\)).

2. \(2^{d-1}\) simplex facets generated by vertices \(\{S_1, \ldots, S_d\}\) with \(|S_i \Delta S_j| = 2\) if \(i \neq j\).

From the above list of facets, one can easily deduce the list of \(i\)-faces of \(\frac{1}{2}\gamma_d\); they are:

1. all \(\frac{1}{2}\gamma_i\) with \(4 \leq i \leq d - 1\) and

2. all \(k\)-sets \(\{S_1, \ldots, S_k\}\) with \(|S_i \Delta S_j| = 2\) if \(i \neq j\).

The first kind of faces is obtained by intersecting hyperplanes \(x_i = 0, 1\), while the second is obtained by taking any subset of a simplex face of \(\frac{1}{2}\gamma_d\). The symmetry group of \(\frac{1}{2}\gamma_d\) has size \(2^{d-1}d!\). It is generated by permutations of \(d\) coordinates and operation \(S \mapsto S_0 \Delta S\) for a fixed \(S_0 \in \frac{1}{2}\gamma_d\). There is one orbit of \(k\)-dimensional faces if \(k \leq 2\) and two orbits, otherwise.

Take a flag \(F_0 \subset F_1 \subset \cdots \subset F_{d-1}\). If \(F_i\) is a simplex face, then all faces, contained in it, are also simplexes. Therefore, the orbit, to which a flag belongs, is determined by the highest index \(i\), for which it is still a simplex. Since \(2 \leq i \leq d - 1\), this makes \(d - 2\) orbits. This yields (ii), since the stabilizer of a flag is trivial.

Let us denote by \(O_i\) with \(2 \leq i \leq d - 1\), the orbit formed by all flags, whose highest index is \(i\). One has \(\sigma_4(O_2) \subset O_3\) and \(\sigma_k(O_2) \subset O_2\) for \(k \neq 2\). If \(i = d - 1\),
then $\sigma_d(O_{d-1}) \subset O_{d-2}$, while $\sigma_k(O_{d-1}) \subset O_{d-1}$ if $k \neq d$. If $2 < i < d$, then one has $\sigma_{i+2}(O_i) \subset O_{i+1}$ and $\sigma_{i+1}(O_i) \subset O_{i-1}$; for other $k$, one has $\sigma_k(O_i) \subset O_i$.

Recalling $T = \sigma_d \sigma_{d-1} \ldots \sigma_1$, one obtains $T(O_i) \subset O_{i-1}$ if $i > 2$ and $T(O_2) \subset O_{d-1}$. Therefore, all orbits of flags are touched by any zigzag of $\frac{1}{2} \gamma_d$. This proves $z$-uniformity.

\begin{proposition}
For $\text{Pyr}(\beta_{d-1})$, it holds:

(i) there are $(d+1)(d-1)!2^{d-2}$ flags partitioned into $d+1$ orbits.

(ii) it is $z$-uniform.
\end{proposition}

\begin{proof}
Denote by $v$ the vertex, on which we do the pyramid construction. Take a flag $(F_1, \ldots, F_d)$ of $\text{Pyr}(\beta_{d-1})$. The sequence of faces $(F_1 \cap \beta_{d-1}, \ldots, F_d \cap \beta_{d-1})$ cannot be a flag for three possible reasons:

1. $F_1 \cap \beta_{d-1} = \emptyset$, it means that $F_1 = \{v\}$.
2. $F_i \cap \beta_{d-1} = F_{i+1} \cap \beta_{d-1}$, it means that $F_{i+1} = \text{conv}(F_i, v)$.
3. $F_d \cap \beta_{d-1} = \beta_{d-1}$, it means that $F_d = \beta_{d-1}$.

This implies, since $\beta_{d-1}$ is regular, that $\text{Pyr}(\beta_{d-1})$ has the following orbits of flags:

1. $O_i$, with $1 \leq i \leq d$, being the orbit of flags of $\text{Pyr}(\beta_{d-1})$, whose first face containing $v$ is in position $i$;

2. the orbit $O_{d+1}$ of flags obtained by adding $\beta_{d-1}$ to a flag of $\beta_{d-1}$.

The operator $\sigma_i$ with $1 \leq i \leq d$, which acts on the flag $(F_1, \ldots, F_d)$ by exchanging the term $F_i$, acts on the orbit by permuting the orbits $O_i$ and $O_{i+1}$ and leaving the others preserved. Hence, the product $T$ acts on the set of orbits $O_i$ as the cycle $(1, 2, \ldots, d+1)$. So, $\text{Pyr}(\beta_{d-1})$ is $z$-uniform. \hfill \Box

\begin{conjecture}
(i) For $z$-structure of $\text{Pyr}(\beta_{d-1})$ it holds:

(i.1) $z$-vector is:

\[
\begin{dcases}
(d^2 - 1)(d-2)!2^{d-2} & \text{for } d \text{ even}, \\
2(d^2 - 1)(d-2)!2^{d-3} & \text{for } d \text{ odd and } d > 3, \\
16_{8,8} & \text{for } d = 3.
\end{dcases}
\]

(i.2) Intersection vectors are:

\[
\begin{dcases}
(0, d-1)^{d-1}, (0, 2d-2) & \text{for } d \text{ even and } d \geq 4, \\
(0, 2d-2)^{d-1} & \text{for } d \text{ odd}.
\end{dcases}
\]

(ii) For $z$-structure of $\text{BPyr}(\alpha_{d-1})$ it holds:
\end{conjecture}
(ii.1) z-vector is:

\[
\begin{cases}
(d^2)^{\frac{(d-1)!}{2}} & \text{for } d \text{ even and } d \geq 4, \\
(2d^2d_0)^{\frac{(d-1)!}{2}} & \text{for } d \text{ odd and } d > 3, \\
(18_{6,3}) & \text{for } d = 3.
\end{cases}
\]

(ii.2) Intersection vectors are:

\[
\begin{cases}
(0, 2d), (0, d - 2)^d & \text{for } d \text{ even and } d > 4, \\
(0, 2d - 4)^d & \text{for } d \text{ odd and } d > 3, \\
(0, 8), (2, 2)^2 & \text{for } d = 4,
\end{cases}
\]

Clearly, \(Pyr(\beta_2)\) and \(BPyr(\alpha_2)\) are just square pyramid and dual Prism\(_3\), respectively. Above conjecture was checked for \(n \leq 10\).

**Conjecture 4.5** Let \(t\) denote \(gcd(p, q)\) and \(s\) denote \(\frac{pq}{t^2}\). Then for z-structure of the direct product \(C_p \times C_q\) holds:

(i) If \(p, q\) are both even, then
\(z = (2ts)^t\) with \(\text{Int} = (0, 2s)^t\) for all zigzags.

(ii) If exactly one of \(p, q\) is odd, then
\(z = (2ts)^t\) with \(\text{Int} = (0, s)^{2t}\) for 4t zigzags and \(\text{Int} = (s, s)^t\) for the remaining 2t zigzags.

(iii) If \(p, q\) are both odd, then
\(z = (2ts)^t, (4ts)^t\) with \(\text{Int} = (s, s)^t\) for zigzags of length 2ts and \(\text{Int} = (2s, 2s)^t\) for zigzags of length 4ts.

The above conjecture was checked for \(p, q \leq 15\).

For any zigzag of Prism\((P)\) with \(z(P) = a^b\) and \(P\) being Platonic or Archimedean 3-polytope, one has \(z = (\frac{4a}{gcd(a, 3)})^{2gcd(a, 3)b}\). In general, \(z = (\frac{da}{gcd(a, d-1)})^{2gcd(a, d-1)b}\). Cube is not included in Table 2 because the prism on it is just \(\gamma_4\). Above relation works also for prisms on antiprisms.

**Conjecture 4.6** For z-structure of prism on AP prism, it holds:

\(z = (\frac{8m}{gcd(m, 3)})^{8gcd(m, 3)}\) with \(\text{Int} = (\frac{2m}{3})^4\) if \(gcd(m, 3) = 3\) and, otherwise, two zigzags have \(\text{Int} = (0, 2m)^4\), two zigzags have \(\text{Int} = (0, 2m), (2m, 4m)\) and four zigzags have \(\text{Int} = (0, 2m)^2, (0, 4m)\).

The above conjecture was checked for \(m \leq 15\).

Denote by \(I(Z_1, Z_2) = (n_I, n_{II})\) the pair of intersection numbers between two zigzags, \(Z_1\) and \(Z_2\), corresponding to intersections of type I and II. Given a map \(f\) acting on a complex \(K\) without any fixed face, the folded complex \(\tilde{K}\) is defined as the quotient space of \(K\) under \(f\); it is not always a lattice.

**Proposition 4.7** Let \(K\) be a complex and \(f\) a fixed-point free involution on \(K\); then one has:

(i) For any zigzag \(Z\) of \(K\), such that \(f(Z) = Z\), the length and the signature of its image \(\tilde{Z}\) in \(\tilde{K}\) are the half of the length and the signature, respectively, of \(Z\).
The above conjecture was checked up to $d$-intersection vectors are \((1,2)\) and \((0,3)\). Exist a partition \((\alpha_1, \ldots, \alpha_t)\) if the sizes of parts in the corresponding partition are either \(d\) or \(d-1\).

Conjecture 4.9 \(i\) A simplicial complex of type \([3,4]\) has the following $z$-structure: 

- (iv.1) $\left\lfloor \frac{d}{2} \right\rfloor$ orbits, each zigzag has length $6d$ and intersection vector $(d,0)^6$ (intersection vectors are $(12,6)$ for $d=3$ and $(0,8),(2,2)^2$ for $d=4$).
- (iv.2) For odd $d$, all orbits have $2^{d-3}(d-2)!$ zigzags. For even $d$, one orbit has size $2^{d-4}(d-2)!$ and $\frac{d-2}{2}$ orbits have size $2^{d-3}(d-2)!$.

The above conjecture was checked up to $d=8$. 

Proof. Every zigzag of $K$ corresponds to a zigzag of $\beta_d$; hence, by the proof of Proposition 4.1, the zigzags of $\gamma_d$ are centrally symmetric. By applying Theorem 4.7 one obtains $z = d^{2d-2(d-1)!}$. Furthermore, one can prove easily that zigzags of $\gamma_d$ have Int = $(0,2)^d$; hence, the intersection vector of $\Delta_d$ is $(0,1)^d$. 

A $(d-1)$-dimensional complex $K$ is said to be of type $[3,4]$ if every $(d-2)$-dimensional face is contained in 3 or 4 faces of dimension $d-1$. Those simplicial complexes are classified in terms of partitions: given such a simplicial complex, there exist a partition \((P_1, \ldots, P_t)\) of \([1, \ldots, d]\), such that $K$ is isomorphic to $\Delta_1 \times \Delta_2 \times \cdots \times \Delta_t$, with $\Delta_i$ being the simplex of dimension $|P_i|$; see [DDS04] for details.

Conjecture 4.9 \(i\) A simplicial complex of type $\{3,4\}$ is not $z$-uniform if and only if the sizes of parts in the corresponding partition are either $\left\lfloor \frac{d}{2} \right\rfloor$ or $\left\lceil \frac{d}{2} \right\rceil$, or all even (except simplex).

(ii) If $Z_0 = f(Z_1)$ with $Z_0 \neq Z_1$, then we put compatible orientation on $Z_1$ and $Z_2$. The zigzags $Z_1$ and $Z_2$ are mapped to a zigzag $\tilde{Z}$ of $\tilde{K}$ with its signature being equal to the signature of $Z_1$ plus $\frac{1}{2}I(Z_1, Z_2)$.

Concerning intersection vectors, one has:

(i) Two zigzags of $K$, which are invariant under $f$, are mapped to zigzags of $\tilde{K}$ with halved intersection.

(ii) Take an invariant zigzag $Z$ of $\tilde{K}$ and $Z_2 = f(Z_1)$ two equivalent zigzags of $K$. They are mapped to $\tilde{Z}$ and $\tilde{Z}'$ and one has $I(\tilde{Z}, \tilde{Z}') = I(Z, Z_1)$.

(iii) Take two pairs $(Z_1, Z_1')$ and $(Z_2, Z_2')$ with $Z_i f(Z_i)$. They are mapped to $\tilde{Z}_1$ and $\tilde{Z}_2$ and their intersection $I(\tilde{Z}_1, \tilde{Z}_2)$ is equal to $I(Z_1, Z_2) + I(Z_1, Z_2)$.

For example, Petersen graph, embedded on the projective plane, is a folding of the Dodecahedron by central inversion. Another example is a map on torus, which is folded onto the Klein bottle.

The folded cube $\Box_d$ is obtained from $d$-cube by folding, i.e., by identifying opposite faces of $\gamma_d$. Obtained complex is $(d-1)$-dimensional, like $\gamma_d$, but it is not a lattice, which imply that this complex does not admit a realization as polyhedral complex.

Proposition 4.8 For $\Box_d$ one has $z = d^{2d-2(d-1)!}$ with Int = $(0,1)^d$.

Proof. Every zigzag of $\beta_d$ corresponds to a zigzag of $(\beta_d)^* = \gamma_d$; hence, by the proof of Proposition 4.1, the zigzags of $\gamma_d$ are centrally symmetric. By applying Theorem 4.7 one obtains $z = d^{2d-2(d-1)!}$. Furthermore, one can prove easily that zigzags of $\gamma_d$ have Int = $(0,2)^d$; hence, the intersection vector of $\Box_d$ is $(0,1)^d$. 

A $(d-1)$-dimensional complex $K$ is said to be of type $[3,4]$ if every $(d-2)$-dimensional face is contained in 3 or 4 faces of dimension $d-1$. Those simplicial complexes are classified in terms of partitions: given such a simplicial complex, there exist a partition $(P_1, \ldots, P_t)$ of $[1, \ldots, d]$, such that $K$ is isomorphic to $\Delta_1 \times \Delta_2 \times \cdots \times \Delta_t$, with $\Delta_i$ being the simplex of dimension $|P_i|$; see [DDS04] for details.

Conjecture 4.9 (i) A simplicial complex of type $\{3,4\}$ is not $z$-uniform if and only if the sizes of parts in the corresponding partition are either $\left\lfloor \frac{d}{2} \right\rfloor$, or $\left\lceil \frac{d}{2} \right\rceil$, or all even (except simplex).

(ii) In special case $\{1, \ldots, \left\lfloor \frac{d}{2} \right\rfloor\}, \{\left\lceil \frac{d}{2} \right\rceil + 1, \ldots, d\}$ one has max $(l_i) = \frac{d(d+1)}{2}$ and min $(l_i) = d + 2$. In other extreme case $\{1,2\}, \ldots, \{d-1, d\}$ one has max $(l_i) = 3d$ and min $(l_i) = \frac{3d}{2}$.

(iii) For partition $\{1\}, \{2, \ldots, d\}$ the simplicial complex of type $\{3,4\}$ is, in fact, BPyr$(\alpha_{d-1})$.

(iv) For partition $\{1\}, \{2\}, \ldots, \{d-2\}, \{d-1, d\}$ the simplicial complex of type $\{3,4\}$ has the following $z$-structure:

- (iv.1) $\left\lfloor \frac{d}{2} \right\rfloor$ orbits, each zigzag has length $6d$ and intersection vector $(d,0)^6$ (intersection vectors are $(12,6)$ for $d=3$ and $(0,8),(2,2)^2$ for $d=4$).
- (iv.2) For odd $d$, all orbits have $2^{d-3}(d-2)!$ zigzags. For even $d$, one orbit has size $2^{d-4}(d-2)!$ and $\frac{d-2}{2}$ orbits have size $2^{d-3}(d-2)!$.

The above conjecture was checked up to $d=8$. 

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| Wythoff 4-polytope | z-vector | intersection vectors |
|--------------------|----------|----------------------|
| $\alpha_4=\alpha_4(0)$ | $5^2$ | $(0,1)^5$ |
| $\alpha_4(0,1)$ | $2012$ | $(0,4)^5$ |
| $\alpha_4(0,1,2)$ | $2036$ | $(0,1)^5, (0,3)^5$ or $(1,3)^5$ |
| $\alpha_4(0,1,3)$ | $2072$ | $(0,2)^10$ |
| $\alpha_4(0,2)$ | $4820$ | $(0,5)^6, (3,15)^3$ |
| $\alpha_4(0,3)$ | $4512$ | $(4,5)^5$ |
| $\alpha_4(1)$ | $10^{12}, 3012$ | $(0,2)^5$ or $(0,6)^5$ |
| $\alpha_4(2)$ | $1512$ | $(1,2)^5$ |
| $\alpha_4(3)$ | $10^{12}, 2012$ | $(0,2)^5$ or $(0,4)^5$ |
| $\beta_4(0)$ | $824$ | $(0,2)^4$ |
| $\beta_4(1)$ | $1648$ | $(0,1)^8, (0,4)^2$ |
| $\beta_4(2)$ | $2496$ | $(0,2)^9, (0,6)^2$ |
| $\beta_4(3)$ | $32144$ | $(0,2)^16$ |
| $\beta_4(4)$ | $6448$ | $(0,2)^{24}$ |
| $\beta_4(5)$ | $10^{12}$ | $(0,2)^{24}, (0,4)^{24}$ |
| $\gamma_4(0)$ | $16248$ | $(0,2)^8$ or $(0,6)^8$ |
| $\gamma_4(1)$ | $24^{32}$ | $(0,2)^{24}$ |
| $\gamma_4(2)$ | $32^{72}$ | $(0,2)^{40}$ |
| $\gamma_4(3)$ | $64^{48}$ | $(0,2)^{64}$ |
| $\gamma_4(4)$ | $128^{48}$ | $(0,2)^{128}$ |
| $\beta_4(1)$ | $24^{32}$ | $(0,2)^{12}$ |
| $\beta_4(2)$ | $48^{144}$ | $(0,2)^{12}, (0,4)^6$ |
| $\beta_4(3)$ | $96^{96}$ | $(0,2)^{24}$ |
| $\beta_4(4)$ | $192^{192}$ | $(0,2)^{48}$ |
| $\beta_4(5)$ | $24^{48}$ | $(0,2)^{12}$ or $(0,4)^{12}$ |
| $\beta_4(6)$ | $30^{240}$ | $(0,2)^{15}$ |
| $\beta_4(7)$ | $36^{96}$ | $(0,2)^{30}$ |
| $\gamma_4(0)$ | $60^{240}$ | $(0,2)^{20}$ |
| $\gamma_4(1)$ | $24^{240}, 80^{560}$ | $(0,2)^{20}, (0,4)^{10}$ |
| $\gamma_4(2)$ | $120^{240}$ | $(0,2)^{30}$ |
| $\gamma_4(3)$ | $120^{240}$ | $(0,2)^{60}$ |
| $\gamma_4(4)$ | $30^{240}$ | $(0,2)^{15}$ |
| $\gamma_4(5)$ | $30^{240}$ | $(0,2)^{30}$ |
| $\gamma_4(6)$ | $30^{240}$ | $(0,2)^{60}$ |

Table 5: z-structure of Wythoff Archimedean 4-polytopes
5 Wythoff kaleidoscope construction

Wythoff construction is defined for any $d$-dimensional complex $\mathcal{K}$ and non-empty subset $V$ of $\{0, \ldots, d\}$. It was introduced in \[Wy07\] and \[Cox35\].

The set of all partial flags $(f_0, \ldots , f_m)$, with $f_i \subset f_{i+1}$ and $i_j \in V$, is the vertex-set of a complex, which we denote by $\mathcal{K}(V)$ and call Wythoff construction with respect to the complex $\mathcal{K}$ and the set $V$.

In general, one has $\mathcal{K}(V) = \mathcal{K}^*(d-V)$ with $d-V$ denoting the set of all $d-i$, $i \in V$. If a complex $\mathcal{K}$ is self-dual, then one has $\mathcal{K}(V) = \mathcal{K}(d-V)$. One has, in general, $\mathcal{K}(\{0\}) = \mathcal{K}$, $\mathcal{K}(\{d\}) = \mathcal{K}^*$ and $\mathcal{K}(\{1\}) = Med(\mathcal{K})$. Dual $\mathcal{K}(\{0, \ldots, n\})$ is a simplicial $(d-1)$-complex called order-complex (\[St97\]).

Easy to see that a general $d$-dimensional complex admits at most $2^{d+1} - 1$ non-isomorph Wythoff constructions, while a self-dual $d$-dimensional complex admits at most $2^d + 2\left\lceil\frac{d-1}{2}\right\rceil - 1$ such non-isomorph constructions. Curiously, in the regular complexes considered, we obtain exactly $2^{d+1} - 1$ and $2^d + 2\left\lceil\frac{d-1}{2}\right\rceil - 1$ non-isomorph complexes.

If $\mathcal{K}$ is a 2-dimensional complex, then it is easy to see that $\mathcal{K}(V)$ with $V$=\{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 2\}, \{1, 2\}, \{1\}$ and \{2\} correspond, respectively, to following maps: original map $\mathcal{M}$, truncated $\mathcal{M}$, truncated $Med(\mathcal{M})$, $Med(Med(\mathcal{M}))$, truncated $\mathcal{M}^*$, $Med(\mathcal{M})$ and $\mathcal{M}^*$ (dual $\mathcal{M}$).

Call Wythoff Archimedean any Wythoff construction with respect to some regular $d$-polytope. By applying the Wythoff construction to the three 3-valent Platonic solids (Tetrahedron, Cube and Dodecahedron) one obtains all Archimedean 3-polytopes, except Snub Cube and Snub Dodecahedron; their $z$-structure is indicated in columns 2 and 3 of Table 2. See in Table 5 the $z$-structure of Wythoff Archimedean 4-polytopes.

6 Lins triality

In the case of maps on surfaces, flags are triples $(v, e, f)$ with $v \in e \subset f$, where $v$, $e$ and $f$ are incident vertex, edge, and face, respectively. Denote by $a$, $b$ and $c$ the three mappings $\sigma_1$, $\sigma_2$ and $\sigma_3$. Vertex, edge and face are identified with the set of flags
containing them; therefore, with orbits on flags of the groups \(\langle b, c \rangle\), \(\langle a, c \rangle\) and \(\langle a, b \rangle\). Zigzags were defined in Section 2 above as circuits of flags \((f_i)_{1 \leq i \leq l}\) with \(f_{i+1} = cba f_i\).

It is easy to see that this correspond to orbits of the group \(\langle ac, b \rangle\).

Let \(v = (v_i)_{i \geq 1}\), \(p = (p_j)_{j \geq 1}\) and \(z = (z_k)_{k \geq 1}\) are, respectively, \(v\)-, \(p\)- and \(z\)-vectors of a map. Then the number \(\sum_{i \geq 1} iv_i = \sum_{j \geq 1} jp_j = \sum_{k \geq 1} k z_k\) is the double of the number of edges.

One can reconstruct the map from the flag-set and the triple \((a, b, c)\) of operations acting on it by using the representation of vertices, edges and faces as orbits. The only restriction, that applies to \(a\), \(b\) and \(c\) is \(a^2 = b^2 = c^2 = (ac)^2 = 1\). If one changes the triple \((a, b, c)\) to \((c, b, a)\), then the map is changed to its dual.

Other operations were introduced in [Li82]: mapping \((a, b, c)\) to \((a, b, ac)\) or \((ac, b, c)\) produce the maps called phial(\(\mathcal{M}\)) and skew(\(\mathcal{M}\)). In [JoTh87] it is proved that there is no other “good” notions of dualities for maps on surfaces than the six ones given in Table 6. The skeleton graph of a map (i.e. the graph of its vertices and edges) is connected. It is well-known that the dual graph of any connected map on a surface is connected also. By using operation phial, we see that, moreover, the graph of zigzags \(Z(\mathcal{M})\) is connected.

The six operations depicted in Table 6 form a group isomorphic to \(\text{Sym}(3)\). In particular, each of operations dual, skew and phial is a reflexion.

Denote the Euler characteristic of \(\mathcal{M}\), phial(\(\mathcal{M}\)) and skew(\(\mathcal{M}\)) by \(\chi(\mathcal{M})\), \(\chi_p(\mathcal{M})\) and \(\chi_s(\mathcal{M})\), respectively.
Conjecture 6.1 (i) For Lins triality for Prism$_m$ it holds:

(i.1) $\chi_s(Prism_m) = \gcd(m, 4) - m$ and skew(Prism$_m$) is oriented if and only if $m$ is even,

(i.2) $\chi_p(Prism_m) = 2 + \gcd(m, 4) - 2m = \chi(Prism_m) + \chi_s(Prism_m) - m$ and phial($\mathcal{M}$) is non-oriented.

(ii) For Lins triality for AP prism$_m$ it holds:

(ii.1) $\chi_s(AP prism_m) = 1 + \gcd(m, 3) - 2m$ and skew($\mathcal{M}$) is non-oriented,

(ii.2) $\chi_p(AP prism_m) = 3 + \gcd(m, 3) - 2m = \chi(AP prism_m) + \chi_s(AP prism_m)$ and skew(AP prism$_m$) is oriented.

The above conjecture was checked up to $n = 100$.

The phial(Tetrahedron) is the complex obtained by taking the octahedron and identifying opposite points, while skew(Tetrahedron) is the complex obtained by taking Cube and identifying opposite points; see Figure 2.

The complex skew(Cube) is a 3-valent map on the torus with 8 vertices and 4 hexagonal faces (twisted construction); see Figure 2.

A vertex of a graph, embedded in an orientable surface, is called twisted if the clockwise order of its adjacent vertices is the reversal, with respect of original clockwise order, given by the original embedding.

Conjecture 6.2 Let $\mathcal{M}$ be a map on an oriented surface, such that its skeleton $G(\mathcal{M})$ is bipartite, then skew($\mathcal{M}$) is a map on an oriented surface and $G(\text{skew}(\mathcal{M})) = G(\mathcal{M})$. The orientation of surface induces, for each vertex $x$ of $G(\mathcal{M})$, a cyclic order on vertices, to which $x$ is adjacent; then the maps $\mathcal{M}$ and skew($\mathcal{M}$) differ only by the twisting of the vertices of one part of the bipartition of $G(\mathcal{M})$.

In particular, Skew(Prism$_{2k}$) is Prism$_{2k}$ with $k$ independent vertices being twisted, if above conjecture is true; we checked it for $k \leq 4$. Also, we checked above conjecture for two following cases:

(i) (phial($\mathcal{M}$))$^*$ (i.e. skew o dual) of AP prism$_4$ is dual AP prism$_4$ with exactly five independent vertices (i.e. a part of this bipartition of two parts of size 5) being twisted.

(ii) (phial(Cuboctahedron))$^*$ is dual Cuboctahedron with exactly one part (eight 3-valent vertices) of this bipartite graph (eight 3-valent and six 4-valent vertices) being twisted.
Table 7: z-structure of some regular 3-valent maps and of their Goldberg-Coxeter GC\(_{k,l}\) construction (see \[DuDe03\])

In Table 7 are presented several regular maps. The group of dual Dyck map is denoted by \(^4O\), because \(O\) is a subgroup of index 4 of it; by the same reason, this group (of order 96) is called tetrahisoctahedral; it is generated by two elements \(R, S\) subject to the relations \(R^3 = S^8 = (RS)^2 = (S^2R^{-1})^3 = 1\). Now, \(PSL(2, p)\) for \(p = 5, 7, 11\) are denoted by \(^5T\), \(^7O\), \(^{11}I\), respectively, and called, respectively, pentakistetrahedral, heptakioctohedral, undecakisicosahedral. A well-known result of Evariste Galois is that they are the only ones amongst all \(PSL(2, p)\), which act transitively on less than \(p+1\) elements.

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