Characterizing Strongly First Order Dependencies: The Non-Jumping Relativizable Case

Pietro Galliani
Free University of Bozen-Bolzano, Italy
Pietro.Galliani@unibz.it

Abstract
Team Semantics generalizes Tarski’s Semantics for First Order Logic by allowing formulas to be satisfied or not satisfied by sets of assignments rather than by single assignments. Because of this, in Team Semantics it is possible to extend the language of First Order Logic via new types of atomic formulas that express dependencies between different assignments.

Some of these extensions are much more expressive than First Order Logic proper; but the question of which such atoms can instead be added to First Order Logic without increasing its expressive power is still unanswered.

In this work, I provide an answer to this question under the additional assumptions (true of most atoms studied so far) that the dependency atoms are relativizable and non-jumping. Furthermore, I show that the Boolean disjunction connective can be added to any strongly first order family of dependencies without increasing the expressive power, but that the same is not true in general for not strongly first order dependencies.

2012 ACM Subject Classification Theory of computation → Logic; Theory of computation → Higher order logic

Keywords and phrases Team Semantics, Dependence Logic, Strongly First Order Dependencies, Second Order Logic, First Order Logic

1 Introduction
Team Semantics [20, 30] generalizes Tarski’s Semantics for First Order Logic by letting formulas be satisfied or not satisfied by sets of assignments (called teams) rather than just by single assignments. This semantics was originally developed by Hodges in [20] in order to provide a compositional semantics for Independence-Friendly Logic [18, 27], an extension of First Order Logic that generalizes its game-theoretic semantics by allowing agents to have imperfect information regarding the current game position; but, as observed by Väänänen [30], Team Semantics is a logical framework that is deserving of study in its own right.

Team Semantics is a natural generalization of Tarskian Semantics for First Order Logic with deep connections to its game-theoretical semantics (very briefly, sets of assignments in Team Semantics correspond precisely to sets of possible plays in the corresponding semantic game). In the case of First Order Logic itself, this semantics is equivalent and reducible to the usual Tarskian semantics, but the higher order nature of its satisfaction relation makes it possible to extend it in new ways. This is of considerable theoretical interest: indeed, Team Semantics may then be seen as a tool to describe and classify novel fragments of Second Order Logic, an issue of great importance – and deep connections, via Descriptive Complexity Theory, to the theory of computation – regarding which much is still not known; and it is also of more practical interest, particularly because of the connections between Team

---

1 In [2], a combinatorial argument was used to show that a compositional semantics cannot exist for Independence Friendly Logic if we require satisfaction (with respect to a model M) to be a relation between single assignments and formulas. In [7], this result was extended to the case of infinite models.

2 “Teamified” versions of other semantics, however, exist and have been studied as well. See in particular Modal Team Semantics [17, 29, 31, 55] and Propositional Team Semantics [53, 55].
Semantics and Database Theory (see for instance [16, 24]). It is worth mentioning here also that probabilistic variants of Team Semantics have recently gathered some interest (see for instance [5, 4, 13]).

Much of the initial wave of research in this area focused on specific Team Semantics-based extensions of First Order Logic, in particular Dependence Logic [30] and later Independence Logic [14] and Inclusion Logic [8, 13]; but there are still relatively few general results regarding the effects of extending First Order Logic via Team Semantics. The simplest way of doing so, for instance, is by introducing generalized dependency atoms $D_x$ that express dependencies between different assignments in the current set of assignments; and it is a consequence of the higher order nature of Team Semantics that, even if $D$ itself is first order definable as a property of relations, the logic $\text{FO}(D)$ obtained by adding it to First Order Logic (with Team Semantics) may well be much more expressive than First Order Logic.

A natural question would then be: can we find necessary and sufficient conditions for that not to happen? Or, in other words, for which dependency atoms $D$ or families of dependency atoms $\mathcal{D}$ do we have that every sentence of $\text{FO}(D)$ (resp. $\text{FO}(\mathcal{D})$) is equivalent to some first order sentence? An answer to this would be of clear theoretical interest, as part of the before-mentioned programme of using Team Semantics to describe and classify fragments of Second Order Logic; and it would also be of more practical interest, as it would allow us to find out which families of dependencies are expressively “safe”.

This question, however, has not been answered yet. In [9], a fairly general family of dependencies was found that does not increase the expressive power of First Order Logic if added to it; but it is an open question whether any dependency that is “safe” for First Order Logic (the term used for this is “Strongly First Order”) is also definable in terms of dependencies in that family.

Building on recent work in [11] on the classification of downwards closed dependencies, this work provides a partial answer to this under two additional assumptions, namely that such a dependency is relativizable (Definition 15) and non-jumping (Definition 17). These are natural properties that are true of essentially all the dependency atoms studied so far, and of most types of dependencies that are of interest; and thus, for those dependencies, the results of this work completely answer the above question. Additionally, a simple result concerning Boolean Disjunction in Team Semantics will be proved along the way – as a necessary tool for the main result – that may be seen as a preliminary step towards the study of such questions in the more general case of operators (rather than mere atoms) in Team Semantics.

## 2 Preliminaries

In Team Semantics, formulas are satisfied or not satisfied by sets of assignments (called teams) rather than by single assignments as in Tarskian semantics.

The following definitions provide the basic framework of this semantics:

\begin{definition}[Team; Relation corresponding to Team; Supplementation; Duplication] Let $\mathfrak{M}$ be a first order model with domain $M$ and let $V$ be a set of variable symbols. Then a team $X$ with domain $\text{Dom}(X) = V$ is a set of assignments $s: V \rightarrow M$.
\end{definition}

3 Examples of results of this type can be found for instance in [25], which studies the complexity of the finite decidability problem in First Order Logic plus generalized dependency atoms.
Definition 2 (Relation Corresponding to a Team). Given a team $X$ and a tuple $v = v_1 \ldots v_k$ of variables occurring in its domain, we write $X(v)$ for the $k$-ary relation $\{(s(v_1) \ldots s(v_k)) : s \in X\}$.

Definition 3 (Team Duplication). Given a team $X$ over $\mathcal{M}$ and a tuple of pairwise distinct variables $y = y_1 \ldots y_k$ (which may or may not occur already in the domain of $X$), we write $X[M/y]$ for the team with domain $\text{Dom}(X) \cup \{y_1 \ldots y_k\}$ defined as

$$X[M/y] = \{s[m_1 \ldots m_k/y_1 \ldots y_k] : s \in X, (m_1 \ldots m_k) \in M^k\}$$

where, as usual, $s[m_1 \ldots m_k/y_1 \ldots y_k]$ is the result of extending/modifyng $s$ by assigning $m_1 \ldots m_k$ to $y_1 \ldots y_k$.

Definition 4 (Team Supplementation). Given a team $X$ over $\mathcal{M}$, a tuple of distinct variables $y = y_1 \ldots y_k$ (which may or may not occur already in the domain of $X$) and a function $H : X \to \mathcal{P}(M)^k \setminus \{\emptyset\}$ assigning to each $s \in X$ a nonempty set of tuples of elements of $\mathcal{M}$, we write $X[H/y]$ for the team with domain $\text{Dom}(X) \cup \{y_1 \ldots y_k\}$ defined as

$$X[H/y] = \{s[m_1 \ldots m_k/y_1 \ldots y_k] : s \in X, (m_1 \ldots m_k) \in H(s)\}.$$

As a special case of supplementation, if $a = a_1 \ldots a_k$ is a tuple of elements of the model we write $X[a/y]$ for $\{s[a_1 \ldots a_k/y_1 \ldots y_k] : s \in X\}$.

Definition 5 (Team Semantics for First Order Logic). Let $\mathcal{M}$ be a first order model with at least two elements, let $\phi$ be a First Order formula over its signature in Negation Normal Form, and let $X$ be a team over $\mathcal{M}$ with domain containing the free variables of $\phi$. Then we say that $\phi$ is satisfied by $X$ in $\mathcal{M}$, and we write $\mathcal{M} \models_X \phi$, if this is a consequence of the following rules:

- **TS-lit:** For all first order literals $\alpha$, $\mathcal{M} \models_X \alpha$ if and only if, for all $s \in X$, $\mathcal{M} \models_s \alpha$ in the usual sense of Tarskian Semantics;

- **TS-\lor:** For all $\psi_1$ and $\psi_2$, $\mathcal{M} \models_X \psi_1 \lor \psi_2$ iff there exist teams $Y_1, Y_2 \subseteq X$ such that $X = Y_1 \cup Y_2$ and $\mathcal{M} \models_{Y_1} \psi_1$ and $\mathcal{M} \models_{Y_2} \psi_2$;

- **TS-\land:** For all $\psi_1$ and $\psi_2$, $\mathcal{M} \models_X \psi_1 \land \psi_2$ iff $\mathcal{M} \models_X \psi_1$ and $\mathcal{M} \models_X \psi_2$;

- **TS-\exists:** For all $\psi$ and all variables $v$, $\mathcal{M} \models_X \exists v \psi$ iff there exists some function $H : X \to \mathcal{P}(M \setminus \{\emptyset\})$ such that $\mathcal{M} \models_{X[H/v]} \psi$;

- **TS-\forall:** For all $\psi$ and all variables $v$, $\mathcal{M} \models_X \forall v \psi$ iff $\mathcal{M} \models_{X[M/v]} \psi$.

Given a sentence $\phi$ and a model $\mathcal{M}$ whose signature contains that of $\phi$, we say that $\phi$ is true in Team Semantics if and only if $\mathcal{M} \models_{\{\epsilon\}} \phi$, where $\{\epsilon\}$ is the team containing the only assignment $\epsilon$ over the empty set of variables.

As mentioned in the Introduction, with respect to First Order Logic proper Team Semantics is equivalent and reducible to Tarskian Semantics. More precisely, it can be shown by structural induction that

---

4 We need at least two elements in our model in order to encode disjunctions in terms of existential quantifications in Proposition 20 and Theorem 29. The case in which only one element exists is in any case trivial, and may be dealt with separately if required.

5 As is common in the study of Team Semantics, we will generally assume that all expressions are in Negation Normal Form.

6 We do not require $Y_1$ and $Y_2$ to be disjoint.
Proposition 6. For all first order formulas φ, models M and teams X, M ⊨ X φ if and only if, for all assignments s ∈ X, M ⊨ s φ according to Tarskian Semantics.

In particular, if φ is a sentence then φ is true in M in the sense of Team Semantics if and only if it is true in M in the sense of Tarskian Semantics.

What is then the point of Team Semantics? In brief, Team Semantics allows us to extend First Order Logic in new ways, like for instance by adding new types of atoms describing dependencies between different assignments.

Definition 7 (Generalized Dependency). Let k ∈ N. A k-ary generalized dependency D is a class of models M = (M, R) over the signature {R}, where R is a k-ary relation symbol, that is closed under isomorphisms (that is, if M_1 and M_2 are isomorphic and M_1 ∈ D then M_2 ∈ D as well). Given a family of such dependencies D_1, D_2, ..., we write FO(D_1, D_2, ...) for the language obtained by adding atoms of the form D_iy_i to First Order Logic, where the y_i range over all tuples of variables of the same arity as D_i, with the satisfaction rules

TS-D_i: M ⊨ X D_iy_i if and only if (M, X(y_i)) ∈ D.

A case of particular interest is the one in which the class of models describing the semantics of a generalized dependency is itself first order definable:

Definition 8 (First Order Generalized Dependency). A generalized dependency D is first order if and only if there exists a first order sentence D(R), where R is a relation symbol of the same arity as D, such that (M, R) ∈ D ⇔ (M, R) ⊨ D(R) for all models M = (M, R).

A peculiar aspect of Team Semantics is that, due to the second order existential quantification implicit in its rules for disjunction and existential quantification, first order generalized dependencies can still increase considerably the expressive power of First Order Logic when added to it. For example, the Team Semantics-based logics that have been most studied so far are Dependence Logic [30], Independence Logic [14] and Inclusion Logic [8], that add to First Order Logic respectively

 Functional Dependence Atoms: For all tuples of variables x and y, M ⊨ x=(x, y) iff any two s, s′ ∈ X that agree on the value of x also agree on the value y;

 Independence Atoms: For all tuples of variables x, y and z, M ⊨ x ⊥ yz iff for any two s, s′ ∈ X that agree on y there is some s″ ∈ X that agrees with s on x and y and with s′ on y and z;

 Inclusion Atoms: For all tuples of variables x and y of the same length, M ⊨ x ≤ y iff for all s ∈ X there exists some s′ ∈ X with s(x) = s′(y).

It is easy to see that these three types of dependency atoms are all first order in the sense of Definition 8. However, (Functional) Dependence Logic FO((=,·)) is as expressive as full Existential Second Order Logic, and so is Independence Logic FO(⊥), whereas Inclusion Logic is equivalent to the positive fragment of Greatest Fixed Point Logic [13] (and hence, by [21, 32], captures PTIME over finite ordered models).

Does this imply that (Functional) Dependence Logic and Independence Logic are equivalent to each other and strictly contain Inclusion Logic? This is not as unambiguous a question as it may seem. It certainly is the case that every Inclusion Logic sentence is equivalent to some Independence Logic sentence, that every Dependence Logic sentence is equivalent to some Independence Logic sentence, and that every Independence Logic sentence is equivalent to each other and strictly contain Inclusion Logic? This is not as unambiguous a question as it may seem. It certainly is the case that every Inclusion Logic sentence is equivalent to some Independence Logic sentence, that every Dependence Logic sentence is equivalent to some Independence Logic sentence, and that every Independence Logic sentence...
sentence is equivalent to some Dependence Logic sentence; but on the other hand, it is not the case that every Inclusion Logic formula, or every Independence Logic one, is equivalent to some Dependence Logic formula. This follows at once from the following classification:

**Definition 9 (Empty Team Property, Closure Properties).** Let \( D \) be a generalized dependency. Then

- \( D \) has the **Empty Team Property** iff \((M, \emptyset) \in D\) for all \( M \);
- \( D \) is **Downwards Closed** iff whenever \((M, R) \in D\) and \( R' \subseteq R\) then \((M, R') \in D\);
- \( D \) is **Union Closed** iff whenever \( \{ R_i : i \in I \} \) is a family of relations over some \( M \) such that \((M, R_i) \in D\) for all \( i \in I \), then \((M, \bigcup_i R_i) \in D\);
- \( D \) is **Upwards Closed** iff whenever \((M, R) \in D\) and \( R \subseteq R'\) then \((M, R') \in D\).

The first three of the above properties are preserved by Team Semantics:

**Proposition 10** (Empty Team Property, Downwards Closure and Union Closure are preserved by Team Semantics). Let \( D \) be a family of dependencies, let \( \phi(v) \in FO(D) \) be a formula with free variables in \( v \) and let \( \mathcal{M} \) be a first order model. Then

- If all \( D \in D \) have the Empty Team Property then \( \mathcal{M} \models \phi \);
- If all \( D \in D \) are downwards closed and \( \mathcal{M} \models \phi \) then \( \mathcal{M} \models \phi \) for all \( X' \subseteq X \);
- If all \( D \in D \) are union closed and \( \mathcal{M} \models \phi \) for all \( i \in I \), then \( \mathcal{M} \models \bigcup_i X_i \models \phi \).

From these facts – that are proven easily by structural induction – it follows at once that functional dependence atoms (which are downwards closed, but not union closed) cannot be used to define inclusion atoms (which are union closed, but not downwards closed) or independence atoms (which are neither downwards closed nor union closed). Additionally, since all these three types of dependencies have the Empty Team Property we have at once that none of them, even together, can be used to define for instance the nonemptiness atom \( NE = \{(M, P) : P \neq \emptyset \} \), such that \( \mathcal{M} \models X(v) \neq \emptyset \).

Differently from functional dependence atoms, inclusion atoms and independence atoms, some types of generalized dependencies do not increase the expressive power of First Order Logic when added to it: this is the case, for example, of the \( NE \) dependency just introduced. More in general, it was shown in 9 that if \( D^\uparrow \) is the set of all **upwards closed** first order dependencies and \( =_(\cdot) \) is the constancy atom such that \( \mathcal{M} \models X =_\cdot(v) \) iff \( |X(v)| \leq 1 \)
then every sentence of \( FO(D^\uparrow, =_(\cdot)) \) is equivalent to some first order sentence. In other words, we have that \( D^\uparrow \cup \{ =_\cdot \} \) is strongly first order according to the following definition:

**Definition 11 (Strongly First Order Dependencies).** A dependency \( D \), or a family of dependencies \( D \), is said to be strongly first order iff every sentence of \( FO(D) \) (resp. \( FO(D) \)) is equivalent to some first order sentence.

Additionally, it is clear that any dependency \( E \) that is definable in \( FO(D^\uparrow, =_(\cdot)) \), in the sense that there exists some formula \( \phi(v) \in FO(D^\uparrow, =_(\cdot)) \) over the empty signature such that \( \mathcal{M} \models X : Ev \leftrightarrow \mathcal{M} \models X \phi(v) \), is itself strongly first order. This can be used, as discussed in 9, to show that for instance the negated inclusion atoms

\[
\mathcal{M} \models X \not\subseteq Y \text{ iff } X(x) \not\subseteq X(y)
\]

\[8\] The choice of the variable \( v \) is of course irrelevant here, and we could have defined \( NE \) as a 0-ary dependency instead; but treating it as a 1-ary dependency is formally simpler.

\[9\] That is, \( \mathcal{M} \models X =_\cdot(v) \) iff for all \( s, s' \in X \), \( s(v) = s'(v) \).

\[10\] It is worth pointing out here that if \( D \) is strongly first order then it is first order in the sense of Definition 9 because \((M, R) \in D \leftrightarrow (M, R) \models \forall x (\neg Rx \lor (Rx \land Dx))\). The converse is however not true in general.
are strongly first order, as they can be defined in terms of upwards closed first order dependencies and constancy atoms; and as mentioned in [10], the same type of argument can be used to show that all first order dependencies $\mathcal{D}(R)$ where $R$ has arity one are also strongly first order.

No strongly first order dependency has been found yet that is not definable in $\text{FO}^{\uparrow}(\cdot)$. This led to the following

**Conjecture 1.** Every strongly first order dependency $\mathcal{D}(R)$ is definable in terms of upwards closed dependencies and constancy atoms.

We also recall here the following slight generalization of the notion of strongly first order dependency:

**Definition 12 (Safe Dependencies).** Let $\mathcal{D}$ and $\mathcal{E}$ be two families of dependencies. Then we say that $\mathcal{D}$ is safe for $\mathcal{E}$ iff any sentence of $\text{FO}(\mathcal{D}, \mathcal{E})$ is equivalent to some sentence of $\text{FO}(\mathcal{E})$.

Clearly, a dependency is strongly first order if and only if it is safe for the empty set of dependencies. However, as shown in [12], a strongly first order dependency is not necessarily safe for all families of dependencies: in particular, the constancy atom is not safe for the unary inclusion atom $v_1 \subseteq v_2$, in which $v_1$ and $v_2$ must be single variables (rather than tuples of variables). On the other hand, in [11] it was shown that strongly first order dependencies are safe for any family of downwards closed dependencies:

**Theorem 13.** Let $\mathcal{D}$ be a family of strongly first order dependencies and let $\mathcal{E}$ be a family of downwards closed dependencies. Then every sentence of $\text{FO}(\mathcal{D}, \mathcal{E})$ is equivalent to some sentence of $\text{FO}(\mathcal{E})$.

It is also worth mentioning here that the same notions of safety and strong first orderness can be easily generalized to operators. For example, in [12] it was shown that the possibility operator

$$\mathcal{M} \models X \diamond \phi \text{ iff } \exists Y \subseteq X, Y \neq \emptyset, \text{ s.t. } \mathcal{M} \models Y \phi$$

is safe for any collection of dependencies $\mathcal{D}$, in the sense that every sentence of $\text{FO}(\mathcal{D}, \diamond)$ is equivalent to some sentence of $\text{FO}(\mathcal{D})$. In the next section, we will instead see an example of an operator that is safe for any strongly first order collection of dependencies, but that is not safe for some other (non strongly first order, albeit still first order) dependency families.

### 3 On the Safety (and Unsafty) of Boolean Disjunction

A connective often added to the language of Team Semantics is the **Boolean disjunction**

$$\text{TS-}\sqcup: \mathcal{M} \models X \phi \sqcup \psi \text{ if and only if } \mathcal{M} \models X \phi \text{ or } \mathcal{M} \models X \psi.$$  

---

11 This is a binary first order dependency, defined by the sentence $\mathcal{D}(R) = \forall xy (Rxy \rightarrow \exists z Rzx)$. The term “unary” is used here because each “side” of the dependency may have only one variable.

12 As a quick aside, similar phenomena occur in the study of the theory of second-order generalized quantifiers [22]. This suggests the existence of interesting – and, so far, largely unexplored – connections between the theory of second order generalized quantifiers and that of generalized dependency atoms.
This is a different connective than the disjunction $\lor$ of Definition\ref{def:team-operands} for example, a team $X$ of the form $\{(v:0,w:0),(v:0,w:1)\}$ does not satisfy $v = w \lor v \neq w$, although it satisfies $v = w \lor v \neq w$.

It is well known in the literature that, as long as the Empty Team Property holds in our language and the model contains at least two elements, this connective can be expressed in terms of constancy atoms as

$$\phi \lor \psi \equiv \exists pq (=(p)\land =(q)\land ((p = q \land \phi) \lor (p \neq q \land \psi))),$$

where $p$ and $q$ are two new variables not occurring in $\phi$ or $\psi$. However, this is not enough to guarantee that this connective will not affect the expressive power of a language based on Team Semantics if added to it, because of two reasons:

1. The empty team property does not necessarily apply to all logics $\mathcal{L}$.

2. As shown in \cite{P. Galliani 7}, the constancy atom itself is not safe for all families of dependencies.

As we will now show, the following result nonetheless holds:

\begin{proposition}[Boolean Disjunction is Safe for Strongly First Order dependencies] Let $\mathcal{D}$ be any strongly first order family of dependencies, and let $\mathcal{L}(\mathcal{D}, \lor)$ be the logic obtained by adding to $\mathcal{L}(\mathcal{D})$ the $\lor$ connective with the semantics given above. Then every sentence of $\mathcal{L}(\mathcal{D}, \lor)$ is equivalent to some first order sentence.
\end{proposition}

\begin{proof}
Let $\phi$ be any sentence of $\mathcal{L}(\mathcal{D}, \lor)$. Then apply iteratively the following, easily verified transformations

- $$(\phi \lor \psi) \\lor \theta \equiv \theta \lor (\phi \lor \psi) \equiv (\phi \lor \theta) \lor (\psi \lor \theta);$$
- $$(\phi \land \psi) \land \theta \equiv \theta \land (\phi \land \psi) \equiv (\phi \land \theta) \land (\psi \land \theta);$$
- $$\exists \phi \lor \psi \equiv (\exists \phi) \lor (\exists \psi);$$
- $$\forall \phi \lor \psi \equiv (\forall \phi) \lor (\forall \psi)$$

until we obtain an expression $\phi'$, equivalent to $\phi$, of the form $\lor_i \psi_i$, where each $\psi_i$ is a sentence of $\mathcal{L}(\mathcal{D})$. But since $\mathcal{D}$ is strongly first order, every such $\psi_i$ is equivalent to some first order sentence $\theta_i$; and, therefore, $\phi$ itself is equivalent to the first order sentence $\lor_i \theta_i$.\end{proof}

Thus, whenever we have a family of strongly first order dependencies $\mathcal{D}$ we can freely add the Boolean disjunction connective $\lor$ to our language without increasing its expressive power. This is a deceptively simple result: in particular, it is not immediately obvious whether $\lor$ is similarly “safe” for families of dependencies $\mathcal{D}$ that are not strongly first order. In fact, this is not the case! Consider, indeed, the two (first order, but not strongly first order) dependencies

**TS-LO2:** $\mathcal{M} \models \text{LO2}(x,y,z)$ if and only if $X(xy)$ describes a total linear order with endpoints over $M$ and $X(z)$ does not contain the first element of this order, contains the second and the last, and whenever it contains an element it does not contain its successor (in the linear order) but it contains its successor’s successor.

**TS-LO3:** $\mathcal{M} \models \text{LO3}(x,y,z)$ if and only if $X(xy)$ describes a total linear order with endpoints over $M$ and $X(z)$ does not contain the first or the second elements of this order, contains the third and the last, and whenever it contains an element it does not contain its successor (in the linear order) or its successor’s successor but it contains its successor’s successor’s successor.

Then the $\mathcal{L}(\text{LO2, LO3}, \lor)$ sentence $(\exists xy \land \text{LO2}(x,y,z)) \lor (\exists xy \land \text{LO3}(x,y,z))$ is easily seen to hold in a model $\mathcal{M}$ if and only if $|M|$ is a multiple of two or of three (or is infinite).
However, there is no formula $\phi$ of $\text{FO}(\text{LO}_2, \text{LO}_3)$ that is true if and only if this property holds. Indeed, suppose that such a $\phi$ existed. Then the $\text{LO}_2$ dependency cannot appear in it: indeed, $\phi$ must be true of a model with exactly three elements, and in such a model any occurrence of $\text{LO}_2$ would be false of every team (including the empty team) and hence, by the rules of Team Semantics (Definition 5), would make $\phi$ itself false. Similarly, the $\text{LO}_3$ dependency cannot occur in $\phi$, because $\phi$ must be true in a model with exactly two elements.

Therefore $\phi$ must be first order; and a standard back-and-forth argument shows that there is no first order formula over the empty signature that is true in a model if and only if its size is divisible by two or by three (or is infinite).

In conclusion, even though we may add the $\sqcup$ operator “for free” as long as we are only working with strongly first order dependencies, this is not necessarily the case if we are working with more expressive types of dependencies.

4 Non-Jumping, Relativizable Dependencies

In this section we will prove a restricted version of Conjecture 4 under two additional (and commonly true) conditions. The first condition that we will assume will be that the dependencies we are discussing are relativizable in the sense of the following definition.

Definition 15 (Relativized Dependencies, Relativizable dependencies). Let $\mathcal{D}$ be a family of dependencies, and let $P$ be a unary predicate. Then the language $\text{FO}(\mathcal{D}(P))$ adds to First Order Logic the relativized dependence atoms $\mathcal{D}(P)y$, and the corresponding semantics (for models whose signature contains $P$) is given by

$$\mathfrak{M} \models X\mathcal{D}(P)y \iff (P^{\text{mr}}, X(y)) \in \mathcal{D}$$

where $P^{\text{mr}}$ is the interpretation of $P$ in $\mathfrak{M}$.

A dependency $\mathcal{D}$, or a family of dependencies $\mathcal{D}$, is said to be relativizable if any sentence of $\text{FO}(\mathcal{D}(P))$ (resp. $\text{FO}(\mathcal{D}(P))$) is equivalent to some sentence of $\text{FO}(\mathcal{D})$ (resp. $\text{FO}(\mathcal{D})$).

Essentially all the dependencies studied in the context of Team Semantics thus far are relativizable. Most of them have even the stronger property of being universe independent in the sense of [23]; in other words, whether $\mathfrak{M} \models X\mathcal{D}y$ or not depends only on the value of $X(y)$ (and not on the domain $M$ of $\mathfrak{M}$), from which relativizability follows trivially.

As was pointed out to the author by Fausto Barbero in a personal communication, a counting argument shows that there exist generalized dependencies that are not relativizable. A concrete example is the unary dependency $I_\infty = \{(M, P) : M \text{ is infinite}\}$. Of course this is not a first order dependency, and it is an unusual dependency in that whether $\mathfrak{M} \models X I_\infty v$ or not does not depend on $X(v)$ but only on $M$; but it is nonetheless a perfectly legitimate generalized dependency, and it is not relativizable. Indeed, the class of models $\mathcal{C} = \{(M, P) : P \text{ is infinite}\}$ is defined by the $\text{FO}(I_\infty(P))$ sentence $\exists v I_\infty(P)v$; however, the same class of models is not defined by any $\text{FO}(I_\infty)$ sentence, because in any infinite model any occurrence of $I_\infty$ can be replaced by the trivially true literal $\top$ and the class $\mathcal{C}$ is not first order definable. The author does not however know of any strongly first order generalized dependency that

---

13 This definition is related, but not identical, to the notion of relativization of formulas in Team Semantics discussed by Rönnholm in §3.3.1 of [28].
14 In this work, when no ambiguity is possible we will generally write the relation symbols $P$, $R$, $S$ instead of the corresponding interpretations $P^{\text{mr}}$, $R^{\text{mr}}$, $S^{\text{mr}}$. 
is not relativizable. The following conjecture is, therefore, open and – if true – would allow
us to remove the relativizability requirement:

\[ \text{Conjecture 2. Every strongly first order generalized dependency is relativizable.} \]

In order to describe the second condition we need the following definition:

\[ \text{Definition 16 (D_{max}). Let D be any generalized dependency. Then D_{max} is the dependency} \]
\[ \{(M, R) : (M, R) \in D \text{ and } \forall S \supseteq R, (M, S) \not\in D\}. \]

In general, for D first order there is no guarantee that whenever \((M, R) \in D\) there is
some \(S \supseteq R\) at all such that \((M, S) \in D_{\text{max}}\); but, as we will see soon, if D is strongly first
order this is indeed the case, and moreover \(D_{\text{max}}\) itself is also strongly first order.

\[ \text{Definition 17. A dependency D is non-jumping if, for all sets of elements M and all} \]
\[ \text{relations R over M of the same arity as D, if } (M, R) \in D \text{ then there exists some } R' \supseteq R \text{ such that} \]
\[ 1. (M, R') \in D_{\text{max}}; \]
\[ 2. \text{For all relations } S, \text{ if } R \subseteq S \subseteq R' \text{ then } (M, S) \in D. \]

In other words, a dependency D is non-jumping if whenever it holds of some R we can “en-
large” R to some R’ that is maximal among those that satisfy D and such that, furthermore,
any relation S between R and R’ satisfies also D. It is easy to see that all dependencies
discussed in this work thus far are non-jumping. It is possible to find examples of jumping
dependencies, like for instance \(D = \{(M, P) : |P| \neq 1\}\); but non-jumping dependencies
clearly constitute a natural and general category of dependencies.

We now need to generalize the two following results from [11] to the case of dependencies
that are not necessarily downwards closed:

\[ \text{Proposition 18. Let D be a downwards closed strongly first order dependency. Then} \]
\[ D_{\text{max}} \text{ is also strongly first order, and whenever } (M, R) \in D \text{ there is some } R' \supseteq R \text{ such that} \]
\[ (M, R') \in D_{\text{max}}. \]

\[ \text{Theorem 19. Let D be a downwards closed, strongly first order, relativizable depen-
dency. Then there are first order formulas } \theta_1(x, z) \ldots \theta_n(x, z) \text{ over the empty signa-
ture such that, for all models } \mathfrak{M} = (M, R), \]
\[ \mathfrak{M} \models D_{\text{max}}(R) \Rightarrow \mathfrak{M} \models \bigvee_{i=1}^n (\exists z \forall x (Rx \leftrightarrow \theta_i(x, z))) \]

To do so, it suffices to observe the following:

\[ \text{Proposition 20. Let D be a strongly first order, relativizable dependency. Then there exists} \]
\[ \text{a downwards closed strongly first order relativizable dependency F such that } F_{\text{max}} = D_{\text{max}}. \]

\[ ^{15} \text{For this dependency, we have that } \mathfrak{M} \models \text{Dv if and only if } X = \emptyset \text{ or } |X(v)| \geq 2. \text{ In other words, Dv is} \]
\[ \text{equivalent to } \bot \sqcup \neq (v), \text{ where } \neq (v) \text{ is the non-constancy atom which is true in a team iff } |X(v)| \geq 2. \]

\[ ^{16} \text{The empty team property is not required, because in the proof of Theorem 4.5 of [11] this property} \]
\[ \text{was necessary only to translate a Boolean disjunction into } \mathbb{FO}(= (\cdot)) \text{ and not to find the } \theta_i \text{ via the} \]
\[ \text{Chang-Makkai Theorem.} \]
Proposition 21. Let $D$ be a strongly first order dependency. Then $D_{\text{max}}$ is also strongly first order, and whenever $(M, R) \in D$ there is some $R' \supseteq R$ such that $(M, R') \in D_{\text{max}}$.

Theorem 25. Let $D$ be a strongly first order, relativizable dependency. Then there are first order formulas $\theta_1(x, z) \ldots \theta_n(x, z)$ over the empty signature such that, for all models $\mathfrak{M} = (M, R)$,

$$\mathfrak{M} \models D_{\text{max}}(R) \Rightarrow \mathfrak{M} \models \bigvee_{i=1}^n (\exists z \forall x (Rx \rightarrow \theta_i(x, z)))$$

For our next lemma, we need some model-theoretic machinery:

Definition 23 (\(\omega\)-big models). A model $\mathfrak{A}$ is \(\omega\)-big if for all finite tuples $a$ of elements in $\mathfrak{A}$ and for all models $\mathfrak{B}$ with the same signature as $\mathfrak{A}$, if $b$ is such that $(\mathfrak{A}, a) \equiv (\mathfrak{B}, b)$ and $S$ is a relation over $\mathfrak{B}$ then we can find a new relation $R$ over $\mathfrak{A}$ such that $(\mathfrak{A}, R, a) \equiv (\mathfrak{B}, S, b)$.

Definition 24 (\(\omega\)-saturated models). A model $\mathfrak{M}$ is \(\omega\)-saturated if it realizes all complete 1-types with respect to $\mathfrak{M}$ over a finite parameter set.

The three following results can be found in [19]:

Theorem 25 ([19], Theorem 8.1.2). If a model $\mathfrak{M}$ is \(\omega\)-big then it is \(\omega\)-saturated.

Theorem 26 ([19], Theorem 8.2.1). Every model has a \(\omega\)-big elementary extension.

Theorem 27 ([19], Lemma 8.3.4). Let $\mathfrak{A}$ and $\mathfrak{B}$ be \(\omega\)-saturated structures over a finite signature (containing some relation symbol $R$) such that, for all first order sentences $\psi^+$ in which $R$ occurs only positively, $\mathfrak{A} \models \psi^+ \Rightarrow \mathfrak{B} \models \psi^+$. Then there are elementary substructures $\mathfrak{C}$ and $\mathfrak{D}$ of $\mathfrak{A}$ and $\mathfrak{B}$ respectively and a bijective homomorphism $h : \mathfrak{C} \rightarrow \mathfrak{D}$ that fixes all symbols except $R$.

Lemma 28. Let $D$ be a strongly first order, relativizable, non-jumping dependency, and suppose that $(M, R)$ is a \(\omega\)-big model such that $(M, R) \in D$. Then there exist a formula $\psi^+(R, z)$ over the signature $\{R\}$, positive in $R$, and a formula $\theta(x, z)$ over the empty signature such that

\[\chi^{(P)} \] is the usual relativization of the first order sentence $\chi$ with respect to $P$.\[\]
1. \((M, R) \models \exists z (\psi^+(R, z) \land \forall x (Rx \rightarrow \theta(x, z)))\);
2. \(\exists z (\psi^+(R, z) \land \forall x (Rx \rightarrow \theta(x, z))) \models D(R)\).

**Proof.** Since \(D\) is non-jumping, we can find a \(R'\) such that \(R \subseteq R', \) \((M, R') \in D_{\text{max}}\), and \((M, R') \in D\) for all \(R'\) such that \(R \subseteq R'' \subseteq R'\). But then by Theorem \[22\] there exist a first order formula \(\theta(x, z)\) over the empty signature and a tuple of elements \(a\) such that \(R' = \{ m \in M^k : (M, R) \models \theta(m, a) \}\).

Now consider \(\Psi = \{ \psi^+(R, a) : R \text{ occurs only positively in } \psi^+ \text{ and } (M, R) \models \psi^+(R, a) \}\).

I state that \(\Psi \cup \{ \forall x (Rx \rightarrow \theta(x, a)), \neg D(R) \}\) is unsatisfiable. Indeed, suppose that it is satisfiable, and let \(B = (B, S, b)\) be a model that satisfies it. By Theorems \[26\] and \[24\] we can assume that \(B\) is \(\omega\)-saturated.

Now, since every formula positive in \(R\) that is true of \((M, R, a)\) is also true of \((B, S, b)\) and both are \(\omega\)-saturated, by Theorem \[27\] we have that there exist elementary substructures \((A, T, a)\) and \((C, K, b)\) of \((M, R, a)\) and \((B, S, b)\) respectively such that \(C\) is the image of a bijective homomorphism \(h : (A, T, a) \rightarrow (C, K, b)\) that sends \(a\) into \(b\). Now let \(T'' = h^{-1}(C)\) be the inverse image of \(C\) under this bijective homomorphism: then \((A, T'', a)\) is isomorphic to \((C, K, b)\), and thus \((A, T'', a) \models \forall x (T''x \rightarrow \theta(x, a)) \land \neg D(T'')\); and furthermore, since \(h\) is a homomorphism, we have at once that \(T \subseteq T''\).

Therefore, the model \((A, T, a)\) can be expanded to a model \((A, T, T'', a)\) such that \(T \subseteq T''\), \(\forall x (T''x \rightarrow \theta(x, a))\) and \(\neg D(T'')\). But \((A, T, a)\) is elementarily equivalent to \((M, R, a)\), which is \(\omega\)-big. Therefore \((M, R, a)\) can also be expanded to some \((M, R', a)\) which is elementarily equivalent to \((A, T, T'', a)\) and in which thus \(R''\) likewise contains \(R\), contains only tuples \(m\) such that \(\theta(m, a)\) (and, thus, is contained in \(R'\)), and does not satisfy \(D(R'')\). This is however impossible, because we said that no such \(R''\) exists; and therefore \(\Psi \cup \{ \forall x (Rx \rightarrow \theta(x, a)), \neg D(R) \}\) is indeed unsatisfiable.

By compactness, this implies that there exists a finite \(\Psi_0 \subseteq \Psi\) such that, for \(\psi^+ = \bigwedge \Psi_0\),

1. \(M, R, a \models \psi^+(R, a) \land \forall x (Rx \rightarrow \theta(x, a))\);
2. \(\exists z (\psi^+(R, z) \land \forall x (Rx \rightarrow \theta(x, z))) \models D(R)\).

This concludes the proof.

Then the following result follows at once by compactness and Theorems \[26\] and \[24\].

**Theorem 29.** Let \(D\) be a strongly first order, relativizable, non-jumping dependency. Then there exists a formula \(\psi^+(R, z)\) over the signature \([R]\), positive in \(R\), and a formula \(\theta(x, z)\) over the empty signature such that, for all \(M\) and \(R\),

\[(M, R) \in D \iff (M, R) \models \exists z (\psi^+(R, z) \land \forall x (Rx \leftrightarrow \theta(x, z))).\]

**Proof.** For all countable \(M\) and all \(R \subseteq M^k\) such that \((M, R) \in D\), let \((M_1, R_1)\) be a \(\omega\)-big elementary extension of it. Then \((M_1, R_1) \in D\) as well, and therefore by Lemma \[23\] there exist some formulas \(\psi^+_{M, R}\) and \(\theta_{M, R}\) (\(R\) positive in \(\psi^+\), not appearing in \(\theta\)) such that

1. \((M_1, R_1)\) (and therefore \((M, R)\) as well, since it is elementarily equivalent to it) satisfies \(\exists z (\psi^+_{M, R}(R, z) \land \forall x (Rx \leftrightarrow \theta_{M, R}(x, z)))\);
2. \(\exists z (\psi^+_{M, R}(R, z) \land \forall x (Rx \leftrightarrow \theta_{M, R}(x, z))) \models D(R)\).

\[\text{It is trivial to see that if } A \text{ is } \omega\text{-big or } \omega\text{-saturated and } a \text{ is a finite tuple of constants then so is } (A, a).\]
Now consider the theory
\[ T = \{ D(R) \cup \{ \neg \exists z (\psi^+_M(R, z) \land \forall x (Rx \to \theta_{M,R}(x, z))) \} : (M, R) \in D, M \text{ countable} \}. \]
This theory is unsatisfiable: indeed, otherwise by Löwenheim-Skolem it would have a countable model \((M, R)\), and since \((M, R) \in D\) we would have that \((M, R) \models \exists z (\psi^+_M(R, z) \land \forall x (Rx \to \theta_{M,R}(x, z)))\). Thus \((M, R)\) would not be a model of \(T\), contradicting our hypothesis.

So by compactness there exist formulas \(\psi^+_1(R, z_i), \theta_i(x, z_i), i = 1 \ldots n\) such that
\[ D(R) \equiv \bigwedge_{i=1}^{n} \exists z_i (\psi^+_i(R, z_i) \land \forall x (Rx \to \theta_i(x, z_i))). \]
But then \(D(R)\) is also equivalent to
\[ \exists q_1 \ldots q_n p z_1 \ldots z_n \left( \psi^+_0(R, q, p, z_1 \ldots z_n) \land \forall x (R_x \to \theta_0(x, q, p, z_1 \ldots z_n)) \right) \]
for \(\psi^+_0 = \left( \bigwedge_{i \neq j} q_i \neq q_j \land \bigvee_j p = q_i \land \bigwedge_i (p = q_i \rightarrow \psi^+_i(R, z_i)) \right)\) and \(\theta_0 = \bigwedge_i (p = q_i \rightarrow \theta_i(x, z_i))\), where \(p\) and all \(q_i\) are tuples of distinct, new variables of length \([\log_2(n)]\).

\begin{corollary}
Every strongly first order, relativizable, non-jumping dependency is definable in terms of first order upwards closed dependencies and constancy atoms.
\end{corollary}

\begin{proof}
Let \(D\) be such a dependency. By the previous theorem, \(D(R)\) is equivalent to some expression of the form \(\exists z (\psi^+(R, z) \land \forall x (Rx \to \theta(x, z)))\), where \(R\) occurs only positively in \(\psi^+\) and not at all in \(\theta\). Now, if \(k\) is the arity of \(D\) and \(l\) is the length of \(z\), consider the \((k+l)\)-ary first order upwards closed dependency \(E\) such that \(M \models_x Exz \iff \exists a \in X(z)\) s.t. \(\psi^+(X(x), a)\), defined formally as
\[ (M, R) \in E \iff (M, R) \models Exy (Rxy \land \psi^+(\exists z R_{\_z} y)) \]
where \(|x| = k\), \(|y| = l\), and \(\psi^+(\exists z R_{\_z}/Rt)\) is obtained from \(\psi^+(R, a)\) by replacing every \(a_i\) with the corresponding \(y_i\), and by replacing every occurrence \(Rt\) of \(R\) with \(\exists z R_{\_t}\).

\(E\) is upwards closed and first order, and thus strongly first order; and \(D\) is definable in \(FO(E, =)\) as \(\exists z (= z) \land Exz \land \theta(x, z)\), where \(z\) is a tuple of new variables disjoint from those of \(x\).
\end{proof}

The above result provides a full characterization of strongly first order dependencies that are relativizable and non-jumping. I suspect that this result may be further generalized to jumping dependencies as a consequence of the following

\begin{conjecture}
Every strongly first order dependency \(D(R)\) can be expressed as a disjunction \(\bigvee_i D_i(R)\) of dependencies \(D_i\) that are strongly first order and non-jumping.
\end{conjecture}

Furthermore, if Conjecture \ref{conjecture} holds, the requirement of relativizability may also be disposed of. This however, is left for future work.

\section{Conclusions and Further Work}

In this work I provided a full characterization of strongly first order dependencies in Team Semantics under the two (commonly true) additional assumptions that these dependencies
are relativizable and non-jumping. The obvious next step consists in trying to find ways to remove or weaken these assumptions, for instance by proving Conjectures \cite{2} and/or \cite{3}.

Another research direction worth investigating at this point is to generalize this approach to the study of operators (rather than just dependencies) in Team Semantics, building on the work on generalized quantifiers in Team Semantics of \cite{1} \cite{5} \cite{25}. Finally, it would be interesting to study axiomatizations for First Order Logic plus strongly first order dependencies.

The procedure to translate from $\text{FO}(D,\cdot)$ to $\text{FO}$ described in \cite{9} is deterministic and ends in finitely many steps for all formulas, and therefore if Conjecture \cite{1} holds then it should be possible to extend the proof system for logics based on Team Semantics of \cite{26} or of \cite{6} to deal with all such dependencies.

\begin{thebibliography}{10}
\bibitem{1} Fausto Barbero. Some observations about generalized quantifiers in logics of imperfect information. \textit{arXiv preprint arXiv:1709.07301}, 2017.
\bibitem{2} Peter Cameron and Wilfrid Hodges. Some Combinatorics of Imperfect Information. \textit{The Journal of Symbolic Logic}, 66(2):673–684, 2001.
\bibitem{3} Arnaud Durand, Miika Hannula, Juha Kontinen, Arne Meier, and Jonni Virtema. Approximation and dependence via multiteam semantics. \textit{Annals of Mathematics and Artificial Intelligence}, 83(3-4):297–320, 2018.
\bibitem{4} Arnaud Durand, Miika Hannula, Juha Kontinen, Arne Meier, and Jonni Virtema. Probabilistic team semantics. In \textit{International Symposium on Foundations of Information and Knowledge Systems}, pages 186–206. Springer, 2018.
\bibitem{5} Fredrik Engström. Generalized quantifiers in dependence logic. \textit{Journal of Logic, Language and Information}, 21(3):299–324, 2012. \texttt{doi:10.1007/s10795-012-9162-4}
\bibitem{6} Fredrik Engström, Juha Kontinen, and Jouko Väänänen. Dependence logic with generalized quantifiers: Axiomatizations. \textit{Journal of Computer and System Sciences}, 88:90–102, 2017.
\bibitem{7} Pietro Galliani. Sensible semantics of imperfect information. In Mohua Banerjee and Anil Seth, editors, \textit{Logic and Its Applications}, volume 6521 of \textit{Lecture Notes in Computer Science}, pages 79–89. Springer Berlin / Heidelberg, 2011.
\bibitem{8} Pietro Galliani. Inclusion and exclusion dependencies in team semantics: On some logics of imperfect information. \textit{Annals of Pure and Applied Logic}, 163(1):68 – 84, 2012. \texttt{doi:10.1016/j.apal.2011.08.005}
\bibitem{9} Pietro Galliani. Upwards closed dependencies in team semantics. \textit{Information and Computation}, 245:124–135, 2015.
\bibitem{10} Pietro Galliani. On strongly first-order dependencies. In \textit{Dependence Logic}, pages 53–71. Springer, 2016.
\bibitem{11} Pietro Galliani. Characterizing downwards closed, strongly first order, relativizable dependencies. \textit{Journal of Symbolic Logic}, 2018. Accepted with corrections. URL: \texttt{https://arxiv.org/abs/1809.00179}
\bibitem{12} Pietro Galliani. Safe dependency atoms and possibility operators in team semantics. In \textit{Proceedings Ninth International Symposium on Games, Automata, Logics, and Formal Verification, GandALF 2018, Saarbrücken, Germany, 26-28th September 2018.}, pages 58–72, 2018. URL: \texttt{https://doi.org/10.4204/EPTCS.277.5} \texttt{doi:10.4204/EPTCS.277.5}
\bibitem{13} Pietro Galliani and Lauri Hella. Inclusion Logic and Fixed Point Logic. In Simona Ronchi Della Rocca, editor, \textit{Computer Science Logic 2013 (CSL 2013)}, volume 23 of \textit{Leibniz International Proceedings in Informatics (LIPIcs)}, pages 281–295, Dagstuhl, Germany, 2013. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: \texttt{http://drops.dagstuhl.de/opus/volltexte/2013/4203} \texttt{doi:http://dx.doi.org/10.4230/LIPIcs.CSL.2013.281}
\bibitem{14} Erich Grädel and Jouko Väänänen. Dependence and independence. \textit{Studia Logica}, 101(2):399–410, 2013. \texttt{doi:10.1007/s11225-013-9479-2}.
\end{thebibliography}
Miika Hannula, Åsa Hirvonen, Juha Kontinen, Vadim Kulikov, and Jonni Virtema. Facets of distribution identities in probabilistic team semantics. arXiv preprint arXiv:1812.05873, 2018.

Miika Hannula and Juha Kontinen. A finite axiomatization of conditional independence and inclusion dependencies. Information and Computation, 249:121–137, 2016.

Lauri Hella, Kerkko Luosto, Katsuhiko Sano, and Jonni Virtema. The expressive power of modal dependence logic. In Advances in Modal Logic 10, invited and contributed papers from the tenth conference on 'Advances in Modal Logic,' held in Groningen, The Netherlands, August 5–8, 2014, pages 294–312, 2014. URL: http://www.aiml.net/volumes/volume10/Hella-Luosto-Sano-Virtema.pdf

Jaakko Hintikka and Gabriel Sandu. Informational independence as a semantic phenomenon. In J.E Fenstad, I.T Frolov, and R. Hilpinen, editors, Logic, methodology and philosophy of science, pages 571–589. Elsevier, 1989. doi:10.1016/S0049-237X(08)70066-1.

Wilfrid Hodges. A Shorter Model Theory. Cambridge University Press, 1997.

Wilfrid Hodges. Compositional Semantics for a Language of Imperfect Information. Journal of the Interest Group in Pure and Applied Logics, 5(4):539–563, 1997. doi:10.1093/jigpal/5.4.539.

Neil Immerman. Relational queries computable in polynomial time. In Proceedings of the fourteenth annual ACM symposium on Theory of computing, pages 147–152. ACM, 1982.

Juha Kontinen. Definability of second order generalized quantifiers. Archive for Mathematical Logic, 49(3):379–398, 2010.

Juha Kontinen, Antti Kuusisto, and Jonni Virtema. Decidable fragments of logics based on team semantics. CoRR, abs/1410.5037, 2014. URL: http://arxiv.org/abs/1410.5037.

Juha Kontinen, Sebastian Link, and Jouko Väänänen. Independence in database relations. In Logic, Language, Information, and Computation, pages 179–193. Springer, 2013.

Antti Kuusisto. A double team semantics for generalized quantifiers. Journal of Logic, Language and Information, 24(2):149–191, Jun 2015. URL: https://doi.org/10.1007/s10849-015-9217-4.

Martin Lück. Axiomatizations of team logics. Ann. Pure Appl. Logic, 169(9):928–969, 2018. URL: https://doi.org/10.1016/j.apal.2018.04.010.

Allen L. Mann, Gabriel Sandu, and Merlijn Sevenster. Independence-Friendly Logic: A Game-Theoretic Approach. Cambridge University Press, 2011. doi:10.1017/CBO9780511981418.

Raine Rönnholm. Arity Fragments of Logics with Team Semantics. PhD thesis, Tampere University, 2018. URL: https://tampub.uta.fi/handle/10024/105699.

Merlijn Sevenster. Model-theoretic and computational properties of modal dependence logic. Journal of Logic and Computation, 19(6):1157–1173, 2009.

Jouko Väänänen. Dependence Logic. Cambridge University Press, 2007. doi:10.1017/CBO9780511611193.

Jouko Väänänen. Modal Dependence Logic. In Krzysztof R. Apt and Robert van Rooij, editors, New Perspectives on Games and Interaction. Amsterdam University Press, Amsterdam, 2008.

Moshe Y Vardi. The complexity of relational query languages. In Proceedings of the fourteenth annual ACM symposium on Theory of computing, pages 137–146. ACM, 1982.

Fan Yang. Modal dependence logics: axiomatizations and model-theoretic properties. Logic Journal of the IGPL, 25(5):773–805, 2017.

Fan Yang and Jouko Väänänen. Propositional logics of dependence. Annals of Pure and Applied Logic, 167(7):557–589, 2016.

Fan Yang and Jouko Väänänen. Propositional team logics. Annals of Pure and Applied Logic, 168(7):1406–1441, 2017.