An Invariant of Surfaces Embedded in the 3-Sphere

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Abstract

A closed connected orientable surface embedded in the 3-sphere $S^3$ splits it into two
submanifolds. On the other hand, Heegaard splittings of each submanifold give us a pair
of handlebody-knots (a two-component handlebody-link). In this paper, we construct an
invariant of such surfaces by using an invariant of handlebody-knots derived from quandles.
We compute our invariant in the case of bi-knotted surfaces.

1 Introduction

Throughout this paper, we assume that all surfaces are closed connected orientable and smooth
surface unless otherwise stated. Consider a surface embedded in the 3-sphere $S^3$. We say two
surfaces in $S^3$ are equivalent if there is an isotopy of $S^3$ which maps one surface onto the other.
In this paper, we construct an invariant of such embedded surfaces up to the equivalent relation
given by isotopies of $S^3$. By the Alexander duality theorem, the exterior of an embedded surface
consists of two compact connected manifolds with a common connected boundary. Then, we can
classify surfaces embedded in $S^3$ into three cases by considering compact connected manifolds
of the exterior of a surface as follows: Let us consider a surface that gives a Heegaard splitting
of $S^3$. Then, both of manifolds are homeomorphic to handlebodies. The second case is that
one of the manifolds is homeomorphic to a handlebody and the other is not homeomorphic to a
handlebody. Moreover, as the final case, Homma \cite{1} constructed a surface in $S^3$ such that none
of the connected components is homeomorphic to a handlebody.

After that, Tsukui \cite{12} and Suzuki \cite{10} studied a decomposition problem concerning the
isotopy sum of surfaces, and Suzuki \cite{11} studied complexity of a surface in $S^3$.

On the other hand, invariants of handlebody-knots have been actively studied (see e.g. \cite{2},
\cite{3}, \cite{4}). Every handlebody-knot can be represented by a spatial trivalent graph by using its
trivalent spine. A diagram of a handlebody-knot is obtained by projecting a spatial trivalent
graph of the handlebody-knot onto a plane. We say two handlebody-knots are equivalent if
there is an isotopy of $S^3$ which maps one to the other. Ishii \cite{2} introduced moves on diagrams
of handlebody-knots up to this equivalent relation given by isotopies of $S^3$. Furthermore, by
applying a $kei$, which is an algebraic system, Ishii constructed an invariant of handlebody-knots.
After that, Ishii–Iwakiri–Jang–Oshiro \cite{3} constructed invariants of handlebody-knots.

In this paper, we construct an invariant of surfaces embedded in $S^3$ by combining a Heegaard
splitting of a 3-manifold and an invariant of handlebody-knots derived from a $G$-family of
quandles. Here, a $G$-family of quandles is an algebraic system introduced by Ishii–Iwakiri–Jang–
Oshiro \cite{3}, which is motivated by handlebody-knot theory (defined in Section 2). We construct
our invariant of surface embedded in $S^3$ in the following way. Given a surface embedded in
$S^3$, we obtain a splitting of $S^3$. Then connected components of the exterior of the surface consist of two compact connected 3-manifolds with common connected boundary. Then, by using
Heegaard splittings of 3-manifolds, we can decompose each of two 3-manifolds into a handlebody and a compression body. Hence, we can obtain two mutually disjoint handlebody-knots (that is, two-component handlebody-link) from the given surface embedded in $S^3$. Generally, since Heegaard splittings are not unique, the pair of handlebody-knots corresponding to the given surface is not uniquely determined by the given embedded surface. However, by using Reidemeister–Singer theorem [8], [9] (see Section 2), any two Heegaard splittings of a compact 3-manifold with boundary become equivalent after applying a finite number of stabilizations. Therefore, we construct our invariant of surfaces embedded in $S^3$ in terms of Heegaard splittings up to stabilizations.

This paper is organized as follows. In the first half of Section 2, we will introduce stabilization of a Heegaard splitting of a 3-manifold, which is used when we construct an invariant of surfaces embedded in $S^3$, and several known results related to stabilization. In the second half of Section 2, we prepare several terminologies and notations of handlebody-knots. After that, we review quandles and a $G$-family of quandles, and introduce an invariant of handlebody-knots derived from a $G$-family of quandles. In Section 3, we construct an invariant of surface embedded in $S^3$ based on preparation in Section 2. In Section 4, we calculate our invariant for two surfaces whose connected components of the exterior are not homeomorphic to handlebodies. In Section 5, we give another invariant of surfaces embedded in $S^3$ by using the same way used in Section 3.

2 Preliminaries

In this section, we will introduce a Heegaard splitting of a compact connected 3-manifold possibly with boundary. To state the definition of a Heegaard splitting of such a 3-manifold, we need several terminologies. Let us start from preparation for them.

2.1 Heegaard splittings of 3-manifolds

**Definition 2.1.** ([6], [7]) A compression body $C$ is a compact 3-manifold constructed by the following procedures. Let $F$ be a closed orientable surface. First, we consider the product manifold $F \times [0, 1]$. Then, attach 1-handles on $F \times \{1\}$. The resulting manifold $C$ is called a compression body. We denote $F \times \{0\}$ by $\partial^- C$ and $\partial C \setminus \partial^- C$ by $\partial^+ C$.

There is the dual construction of a compression body $C$. Consider the product manifold $F \times [0, 1]$. Then attach 2-handles along $F \times \{0\}$, and cap off any resulting 2-sphere components by 3-handles. In this case, we denote $F \times \{1\}$ by $\partial^+ C$ and $\partial C \setminus \partial^+ C$ by $\partial^- C$.

**Remark 2.2.** A compression body $C$ is called a handlebody if $\partial^- C = \emptyset$.

**Definition 2.3.** The genus of a handlebody is the genus of its boundary.

**Definition 2.4.** Let $M$ and $N$ be a 3-manifold and a submanifold of $M$, respectively. $N$ is said to be properly embedded in $M$ if $\partial N \subset \partial M$ and $\text{int}(N) \subset \text{int}(M)$.

**Definition 2.5.** Let $M$ be a connected 3-manifold. A properly embedded disk $D$ is inessential if there is a 2-disk $D'$ in $\partial M$ such that $\partial D = \partial D'$ and $D \cup D'$ is the boundary of a 3-ball in $M$. A properly embedded disk $D$ is essential if $D$ is not inessential.

**Definition 2.6.** Let $M$ be a connected 3-manifold. An essential disk $D$ is separating if $D$ cuts $M$ into two parts. Otherwise, $D$ is non-separating.
Lemma 2.7. Let $H_g$ be a handlebody of genus $g$. Then, a separating disk $D$ of $H_g$ cuts $H_g$ into two handlebodies $H_{g_1}$ and $H_{g_2}$ such that $\text{cl}(H_g \setminus N(D)) = H_{g_1} \sqcup H_{g_2}$, where $g = g_1 + g_2$ and $N(D)$ is a closed regular neighborhood of $D$ ($N(D) \approx D \times [0,1]$).

We introduce the notion of Heegaard splittings of 3-manifolds. Originally, Heegaard splittings were introduced to represent a connected closed 3-manifold by the union of two handlebodies along their boundaries. In the context of knot theory, a Heegaard surface of the Heegaard genus (Definition 2.9) of the exterior $E(K)$ of a given knot $K$ is related to the tunnel number of $E(K)$.

Definition 2.8. (Heegaard Splittings) Let $M$ be a compact connected 3-manifold possibly with boundary. Fix a partition of $\partial M$ as $\partial M = \partial_1 M \sqcup \partial_2 M$. If $M$ admits a decomposition consists of two compression bodies $C_1$ and $C_2$ such that $M = C_1 \sqcup_S C_2$, $\partial_+ C_1 = \partial_+ C_2 = S$, $\partial_- C_1 = \partial_- M$ ($i = 1, 2$) we call $(M, S)$ a Heegaard splitting of $M$ and $S$ a Heegaard surface of $M$.

Definition 2.9. Let $M$ be a compact connected 3-manifold possibly with boundary. Among the Heegaard splittings of $M$, the minimal genus of the Heegaard surfaces is called the Heegaard genus.

Regarding Heegaard splittings of 3-manifolds, the following theorem is known as Moise’s theorem.

Theorem 2.10. ([8], [9]) Every compact connected 3-manifold possibly with boundary admits a Heegaard splitting.

Next, we define a parallel arc in a compression body. This parallel arc is used when we define stabilization of a Heegaard splitting of a 3-manifold.

Definition 2.11. Let $C$ be a compression body. A properly embedded arc $\alpha$ in $C$ is parallel to an arc $\beta$ in $\partial C$ if there is a properly embedded disk $D$ in $C$ such that $\partial D = \alpha \cup \beta$.

Definition 2.12. (Stabilization) Let $M$ be a compact connected 3-manifold possibly with boundary. Let $(M, S)$ be a Heegaard splitting of $M$ with a Heegaard surface $S$. The following procedure to construct a new Heegaard splitting $(M, S')$ from $(M, S)$ is called stabilization: $M$ can be represented in the form of $M = C_1 \sqcup_S C_2$ with two compression bodies $C_1$ and $C_2$, $\partial_+ C_1 = S = \partial_- C_2$. Take a parallel arc $\alpha$ in $C_2$ to an arc $\beta$ in $\partial_+ C_2$. Then, we remove a tubular neighborhood $N(\alpha)$ of $\alpha$ from $C_2$ and take a closure, and add it to $C_1$, namely, we consider $C'_1 := C_1 \cup_S N(\alpha)$ and $C'_2 := \text{cl}(C_2 \setminus N(\alpha))$. We can see that $C'_1$ and $C'_2$ are also compression bodies and $C'_1 \cup_{S'} C'_2 = M$, where $S' := \partial C'_1 = \partial C'_2$.

Hence we obtained a new Heegaard splitting $(M, S')$ of $M$ from a given Heegaard splitting $(M, S)$. We denote by $g(F)$ the genus of a closed connected orientable surface $F$. Then, $g(S') = g(S) + 1$.

Example 2.13. (The trivial Heegaard splitting) Let $H_g$ be a handlebody of genus $g$. Set $\partial H_g = F$. Consider the product manifold $C = F \times [0,1]$. Then, $C$ is a compression body with $\partial_+ C = F \times \{1\} = F \times \{0\} = \partial_- C$. Moreover, $C \cup_F H_g$ gives a Heegaard splitting of the handlebody $H_g$.

The following theorem is known related to the Heegaard splittings of the handlebodies of any genus $g$. 

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Theorem 2.14. ([7]) All Heegaard splittings of a handlebody of genus \( g \) are standard: They are obtained from the trivial Heegaard splitting by applying a finite number of stabilizations (defined in Definition 2.12).

We define an equivalence of two Heegaard splittings.

Definition 2.15. Let \((M, S)\) and \((M, S')\) be Heegaard splittings of a compact connected 3-manifold \( M \) possibly with boundary. Two Heegaard splittings are equivalent if there exists an isotopy \( f_t, t \in [0, 1] \), of \( M \) such that \( f_0 = \text{id}_M, f_1(S) = S' \).

The following theorem, known as Reidemeister–Singer theorem, plays an important role when we construct an invariant for surfaces embedded in \( S^3 \).

Theorem 2.16. ([8], [9]) Let \( M \) be a compact connected 3-manifold. Then, any two Heegaard splittings of \( M \) become equivalent after a finite number of stabilizations.

Also, the following theorem is known as Waldhausen’s theorem.

Theorem 2.17. ([13]) The Heegaard splittings of \( S^3 \) of the same genus of Heegaard surface are unique up to ambient isotopy of \( S^3 \).

2.2 Quandles and \( G \)-family of quandles

Let us move to an introduction of quandles and a \( G \)-family of quandles. A lot of invariants of knots using quandles have been studied. Moreover, a \( G \)-family of quandles can be used for studies of handlebody-knots and gives plenty invariants of handlebody-knots (refer to [3]). We start from the definition of quandles.

Definition 2.18. Let \( X \) be a non-empty set with a binary operation \( \ast : X \times X \to X \). The pair \((X, \ast)\) is a quandle if for any \( x, y, z \in X \), \( \ast \) satisfies

(i) \( x \ast x = x \).
(ii) The map \( S_x : X \to X \) defined by \( S_x(y) = y \ast x \) is bijective.
(iii) \( (x \ast y) \ast z = (x \ast z) \ast (y \ast z) \).

Let \( G \) be a group. Consider a family of binary operations parametrized by the elements of \( G \). Then, it enables us to construct the following \( G \)-family of quandles (refer to [3]).

Definition 2.19. Let \( G \) be a group, and \((X, \{\ast_g\}_{g \in G})\) be a \( G \)-family of quandles, respectively.

Set \( Q = X \times G \). We define a binary operation \( \ast : Q \times Q \to Q \) by \((x, g) \ast (y, h) = (x \ast_h y, h^{-1}gh)\). Then, the pair \((Q, \ast)\) is a quandle called the associated quandle of \( X \).
Let \((X, \ast)\) be a quandle. For \(x, y \in X\), we set \(x \ast_i y := S^i_y(y) = (\cdots (y \ast x) \ast x) \cdots \ast x\). Then, we define a positive integer \(m\) as follows:

\[
    m := \min \{ \ i \in \mathbb{N} \mid S^i_x(x) = x \ \text{for all} \ x, y \in X \}.
\]

If such a positive integer does not exist, we regard \(m\) as \(\infty\). We call \(m\) the type of \(X\), and denote it by \(\text{Type}(X)\). If \(X\) is a finite set, then \(\text{Type}(X) < \infty\).

Suppose that \(X\) is a finite set. Given a quandle \((X, \ast)\), by using \(\text{Type}(X)\) we can construct a \(G\)-family of quandles in the following way (refer to \([3]\)).

**Proposition 2.21.** Let \((X, \ast)\) be a quandle. Set \(G = \mathbb{Z}/(\text{Type}(X))\mathbb{Z}\), and define a binary operation \(\ast_i : X \times X \to X\) by \(x \ast_i y = S^i_y(x)\). Then, the pair \((X, \{\ast_i\}_{i \in G})\) is a \(G\)-family of quandles.

**Proof.** By the definition of the binary operation \(\ast_i\), we can see that \(x \ast_i x = x\) for any \(x \in X\) and \(i \in G = \mathbb{Z}/(\text{Type}(X))\mathbb{Z}\). Then, the axiom (i) holds. Since \(G\) is an Abelian group, \(x \ast_{i+j} y = S^{i+j}_y(x) = S^i_y \circ S^j_y(x) = (x \ast_i y) \ast_j y\) for any \(x, y \in X\) and \(i, j \in G\). Also, \(x \ast_0 y = S^0_y(x) = \text{id}_X(x) = x\). Therefore, the axiom (ii) holds. The axiom (iii) is shown by induction concerning \(i\) and \(j\). \(\square\)

### 2.3 Handlebody-knots and a \(G\)-family of quandles

Now, we introduce handlebody-knots and related terminologies of them.

**Definition 2.22.** A handlebody-knot of genus \(g\) is a handlebody of genus \(g\) embedded in \(S^3\).

**Definition 2.23.** Two handlebody-knots \(H_1\) and \(H_2\) are equivalent if one can be sent to the other by an ambient isotopy of \(S^3\).

A handlebody-knot \(H\) is represented by a spatial trivalent graph \(K\) if a regular neighborhood of \(K\) is \(H\). Here, a spatial trivalent graph is a finite graph embedded in \(S^3\) such that each vertex is of valence three. It is known that any handlebody-knot can be represented by a spatial trivalent graph (see \([2]\)). A diagram \(D\) of a handlebody-knot \(H\) is a diagram of a spatial trivalent graph \(K\) of \(H\) obtained by projecting onto a plane.

Using a \(G\)-family of quandles and an oriented diagram \(D\) of a handlebody-knot, we introduce an \(X\)-coloring of such a diagram \(D\). We denote by \(\mathcal{A}(D)\) the set of arcs of \(D\). The normal orientation of an arc is given by rotating an orientation of the arc counterclockwise by \(\pi/2\). The normal orientation of an arc is represented by an arrow on the arc (see Figure [1]).

**Definition 2.24.** Let \((X, \{\ast_g\}_{g \in G})\) a \(G\)-family of quandles. Let \(D\) be an oriented diagram of a handlebody-knot \(H\). A map \(C : \mathcal{A}(D) \to Q\) is an \(X\)-coloring of \(D\) if \(C\) satisfies the following conditions.

- At each crossing of \(D\), the map \(C\) satisfies \(C(\alpha_2) = C(\alpha_1) \ast C(\alpha_3)\).
- At each vertex of \(D\), the map \(C\) satisfies

\[
\begin{align*}
    p_X \circ C(\alpha_1) &= p_X \circ C(\alpha_2) = p_X \circ C(\alpha_3) \\
    (p_G \circ C(\alpha_1))^{e(\omega, \alpha_1)} \cdot (p_G \circ C(\alpha_2))^{e(\omega, \alpha_2)} \cdot (p_G \circ C(\alpha_3))^{e(\omega, \alpha_3)} &= e
\end{align*}
\]
where, \(\alpha_1, \alpha_2\) and \(\alpha_3\) are assigned as Figure 1. \(p_X : Q \to X\) and \(p_G : Q \to G\) are projections with respect to \(X\) and \(G\), respectively. Also, \(\varepsilon(\omega, \alpha_i)\) is the sign of arc \(\alpha_i\) which is defined as

\[
\varepsilon(\omega, \alpha_i) = \begin{cases} 
1, & \text{If the orientation of } \alpha_i \text{ points a vertex } \omega. \\
-1, & \text{Otherwise.}
\end{cases}
\]

![Figure 1: Crossing \(\chi\)](image1)

![Figure 2: Vertex \(\omega\)](image2)

We denote by \(\text{Col}_X(D)\) the set of \(X\)-colorings of \(D\).

Regarding \(\text{Col}_X(D)\), the following theorem is known.

**Theorem 2.25.** Let \((X, \{g\})\) be a \(G\)-family of quandles. Let \(D\) be a diagram of an oriented spatial trivalent graph of a handlebody-knot. Then, the cardinality \(#\text{Col}_X(D)\) is an invariant of a handlebody-knot.

**Remark 2.26.** Since after an orientation of a diagram \(D\) of a handlebody-knot is given \(X\)-colorings of \(D\) are defined, the cardinality \(#\text{Col}_X(D)\) depends on an orientation of the diagram \(D\) of a handlebody-knot.

**Lemma 2.27.** Let \(G\) and \(X\) be a finite group and a finite set, respectively. We denote by \((X, \{g\})\) a \(G\)-family of quandles. Let \(F\) be a closed connected orientable surface embedded in \(S^3\). We denote by \(V_F\) and \(W_F\) the two connected components of the exterior of \(F\). Let \((V_F, F_1)\) and \((W_F, F_2)\) be Heegaard splittings of \(V_F\) and \(W_F\), respectively. Let \((V_F, F'_1)\) and \((W_F, F'_2)\) be Heegaard splittings obtained from \((V_F, F_1)\) and \((W_F, F_2)\) by applying stabilization, respectively. We denote by \(H_{F_1}, H_{F_2}, H_{F'_1}\), and \(H_{F'_2}\) handlebody-knots obtained from the Heegaard splittings \((V_F, F_1), (W_F, F_2), (V_F, F'_1),\) and \((W_F, F'_2)\), respectively. Also, we denote by \(D_{F_1}\) and \(D_{F_2}\) oriented diagrams of \(H_{F_1}\) and \(H_{F_2}\), respectively, and denote by \(D_{F'_1}\) and \(D_{F'_2}\) diagrams of \(H_{F'_1}\) and \(H_{F'_2}\) with induced orientations from \(D_{F_1}\) and \(D_{F_2}\), respectively. Then, \(#\text{Col}_X(D_{F'_1}) = #\text{Col}_X(D_{F'_2})\) and \(#\text{Col}_X(D_{F'_1}) = #\text{Col}_X(D_{F'_2})\).

**Proof.** By the definition of stabilization and using an isotopy of \(S^3\), we can assume that the diagram \(D_{F'_1}\) is obtained from \(D_{F_1}\) by attaching an edge \(e_0\) and an \(S^1\) component on one of the outermost arcs \(\alpha_0\) of \(D_{F_1}\). Then we give arbitrary orientations to the edge \(e_0\) and the \(S^1\) component. For other arcs, orientations are induced from that of \(D_{F_1}\) as shown in the following Figure 2. Let \(C\) be an \(X\)-coloring of \(D_{F_1}\). Suppose that \(C(\alpha_0) = (x, g) \in X \times G\). Then,
$C$ is extended to an $X$-coloring of $D_{F_1}'$ by setting $C(e_0) = (x, e)$ and $C(\beta) = (x, h)$ where $e$ is the unit element of $G$, $h \in G$ and $\beta$ is a diagram of $S^1$ component. Hence, we obtain $\#\text{Col}_X(D_{F_1}') = \#\text{Col}_X(D_F) \cdot \#(G)$. The same holds for $D_{F_2}'$ and $D_{F_2}$.

Figure 3: Change of an outermost arc

3 Main Result

Let $F$ be a closed connected orientable surface embedded in the 3-sphere $S^3$. By Alexander duality theorem, $F$ splits $S^3$ into two submanifolds $V_F$ and $W_F$, namely $S^3 \setminus \text{int}(N(F)) = V_F \cup W_F$, where $N(F)$ is a regular neighborhood of $F$. Let us start from the definition of equivalence of surfaces embedded in $S^3$.

**Definition 3.1.** Two surfaces $F_1$ and $F_2$ embedded in $S^3$ are equivalent if there exists an ambient isotopy $f_t : S^3 \to S^3$, $t \in [0, 1]$, such that $f_0 = \text{id}_{S^3}$, and $f_1(F_1) = F_2$.

Considering submanifolds $V_F$ and $W_F$, we can define three classes of such a surface.

**Definition 3.2.** Let $F$ be a closed connected orientable surface embedded in $S^3$.

- $F$ is called an unknotted surface if both $V_F$ and $W_F$ are homeomorphic to handlebodies.
- $F$ is called a knotted surface if exactly one of $V_F$ or $W_F$ is homeomorphic to a handlebody.
- $F$ is called a bi-knotted surface if neither $V_F$ nor $W_F$ is homeomorphic to a handlebody.

The following theorem is the main result of this paper.

**Theorem 3.3.** Let $G$ and $X$ be a finite group and a non-empty finite set, respectively. We denote by $(X, \{g\}_{g \in G})$ a $G$-family of quandles. Let $F$ be a closed connected orientable surface in $S^3$, and $V_F$ and $W_F$ be the two connected components of the exterior of $F$, respectively. Let $F_V$ and $F_W$ be Heegaard surfaces of $V_F$ and $W_F$ respectively. We denote by $H_V$ and $H_W$ the corresponding handlebodies, respectively, and by $D_V$ and $D_W$ diagrams of oriented spatial trivalent graphs of $H_V$ and $H_W$, respectively. Then, the unordered pair

$$\left(\frac{\#\text{Col}_X(D_V)}{\#(G)^{g(F_V)}}, \frac{\#\text{Col}_X(D_W)}{\#(G)^{g(F_W)}}\right)$$

of rational numbers is an invariant of $F$. 

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Proof. Let \((V_F,F_V)\) and \((V_F',F_V')\) be Heegaard splittings of \(V_F\), respectively. Then, we have pairs of a handlebody-knot and a compression body (in \(S^3\)). We denote by \((H_V,C_V)\) and \((H'_V,C'_V)\) such pairs. By Reidemeister–Singer theorem, two Heegaard splittings \((V_F,F_V)\) and \((V_F',F_V')\) become equivalent after a finite number of stabilizations. Let such integers be \(n\) and \(m\), and suppose that \((V_F,F_V)\) and \((V_F',F_V')\) are equivalent to a Heegaard splitting \((V_F,F_V')\) (namely, \(V_F = \overline{H}_V \cup \overline{C}_V\)) after \(n\)-times of stabilizations and \(m\)-times of stabilizations. Let \(D_V, D'_V\) and \(D_V, D'_V\) be diagrams of \(H_V, H'_V\) and \(\overline{H}_V, \overline{H}_V\), respectively. We observed a variation of the cardinality of the set of \(X\)-coloring after applying stabilization (Lemma 2.27). Then, we have

\[
\#\text{Col}_X(D_V) = \#\text{Col}_X(D'_V) \cdot (\#(G))^n = \#\text{Col}_X(D'_V) \cdot (\#(G))^m.
\]

On the other hand, we also have \(g(F_V) = g(F'_V) + n = g(F'_V) + m\). Hence we can see that for a Heegaard splitting of \(V_F\), \(\#\text{Col}_X(D_V)/(\#(G))^{g(F_V)}\) is an invariant of a surface \(F\) up to stabilization. The same holds for \(W_F\).

Moreover, suppose that two surfaces \(F\) and \(\tilde{F}\) embedded in \(S^3\) are equivalent. Then, there is an isotopy \(f_t, t \in [0,1]\), of \(S^3\) such that \(f_0 = \text{id}_{S^3}, f_1(F) = \tilde{F}\). Since \(F\) splits \(S^3\) into to two submanifolds \(V_F\) and \(W_F\), and \(\tilde{F}\) splits \(S^3\) into to two submanifolds \(\tilde{V}_F\) and \(\tilde{W}_F\), we can assume that \(f_t(V_F) = \tilde{V}_F\). Especially, since \(f_t\) is a homeomorphism for any \(t \in [0,1]\), \(f_1(H_V)\) and \(f_1(C_V)\) are a handlebody-knot and a compression body in \(\tilde{V}_F \subset S^3\). Since \(f_1(H_V)\cup f_1(C_V) = \tilde{V}_F, \partial(f_1(H_V))\) gives a Heegaard surface of \(\tilde{V}_F\). The same holds for \(W_F\). Therefore, the unordered pair of rational numbers in Theorem 3.3 is an invariant of a surface \(F\).

\[\square\]

Remark 3.4. Once a surface \(F\) embedded in \(S^3\) is given, we have the unique representative of a two-component handlebody-link (or a two-component spatial graph) up to stabilization of a Heegaard splitting of both \(V_F\) and \(W_F\). Therefore, invariants of two-component handlebody-links can be used to construct invariants of surfaces embedded in \(S^3\) if we can grasp a variation of invariants by stabilization. For example, for a two-component handlebody-link, its linking number is an invariant (refer to [3]). We have a handlebody-link by considering Heegaard splittings of \(V_F\) and \(W_F\). Also, we can see that linking numbers do not change after applying stabilization of both \(V_F\) and \(W_F\). Then, linking number of a two-component handlebody-link obtained from \(F\) is also an invariant of \(F\).

4 Examples

We compute our invariant by using the following examples of bi-knotted surfaces. Through both examples we set \(X = SL(2;\mathbb{Z}/2\mathbb{Z})\), \(G = \mathbb{Z}/3\mathbb{Z}\), \(g \ast_i h := h^{-1}gh^i\) for any \(g, h \in SL(2;\mathbb{Z}/2\mathbb{Z})\) and \(i \in \mathbb{Z}/3\mathbb{Z}\).

Figure 4: Bi-knotted surface \(F\)  
Figure 5: Bi-knotted surface \(F'\)
Example 4.1. For a surface $F$ depicted in Figure 4, connected components of the exterior $V_F$ and $W_F$ of $F$ are given in the following Figure 6 and Figure 7.

$$\text{Figure 6: Exterior } V_F \quad \text{Figure 7: Exterior } W_F$$

Also, handlebody-knots $H_V$ and $H_W$ obtained from Heegaard splittings of $V_F$ and $W_F$ are depicted in the following Figure 8 and Figure 9.

$$\text{Figure 8: Handlebody-knot } H_V \quad \text{Figure 9: Handlebody-knot } H_W$$

Finally, we compute $\#\text{Col}_X(D_{V'})$ and $\#\text{Col}_X(D_{W'})$ where $D_V$ and $D_W$ are diagrams of handlebody-knots $H_V$ and $H_W$ of genus 3, respectively. Then we have

$$\frac{\text{Col}_X(D_V)}{\#(\mathbb{Z}/3\mathbb{Z})^3} = 8, \quad \frac{\text{Col}_X(D_W)}{\#(\mathbb{Z}/3\mathbb{Z})^3} = 8.$$ 

Example 4.2. We consider a surface $F'$ depicted in Figure 5. In this case $V_{F'}, W_{F'}, H_{V'}$ and $H_{W'}$ are given as the following figures.

$$\text{Figure 10: Exterior } V_{F'} \quad \text{Figure 11: Exterior } W_{F'}$$
Then we have
$$\frac{\text{Col}_X(D_{V'})}{\#(\mathbb{Z}/3\mathbb{Z})^3} = 8, \quad \frac{\text{Col}_X(D_{W'})}{\#(\mathbb{Z}/3\mathbb{Z})^3} = 6$$
where $D_{V'}$ and $D_{W'}$ are diagrams of handlebody-knots $H_{V'}$ and $H_{W'}$ of genus 3, respectively.

Through the above examples, we can see that two surfaces $F$ and $F'$ embedded in $S^3$ are not equivalent by using our invariant.

**Remark 4.3.** In case of a surface $F$ embedded in $S^3$ is an unknotted surface of genus $g$, we can easily calculate our invariant by using theorem 2.17.

## 5 Further results

By using the same method in theorem 3.3 we can also construct another invariant of surfaces embedded in $S^3$. We introduce an $X$-set $Y$ and $X_Y$-coloring of a diagram of a handlebody-knot (refer to 3).

**Definition 5.1.** Let $(X, \{\ast_g\}_{g \in G})$ be a $G$-family of quandles, and let $Y$ be a non-empty set with a family of maps $\pi_g : Y \times X \rightarrow Y$. The pair $(Y, \{\pi_g\}_{g \in G})$ is called an $X$-set if $\pi_g$ satisfies

(i) $y\pi_{gh}x = (y\pi_gx)\pi_hx$ and $y\pi_e x = y$. (Here, $e$ is the unit element of $G$).

(ii) $(y\pi_gx)\pi_{hx'} = (y\pi_{hx'})\pi_{h^{-1}gh}(x \ast_h x')$.

For any $y \in Y$, $x, x' \in X$, and any $g, h \in G$.

Let $D$ be an oriented diagram of a spatial trivalent graph. We denote by $\mathcal{R}(D)$ the set of complementary regions of $D$. We set $y \ast (x, g) = y\pi_gx$ for $y \in Y$, $x \in X$, and $g \in G$.

**Definition 5.2.** Let $(X, \{\ast_g\}_{g \in G})$ and $(Y, \{\pi_g\}_{g \in G})$ be a $G$-family of quandles and an $X$-set, respectively. Let $D$ be an oriented diagram of a handlebody-knot. An $X_Y$-coloring of $D$ is a map $C : \mathcal{A}(D) \cup \mathcal{R}(D) \rightarrow Q \cup Y$ satisfying

C1. $C(\mathcal{A}(D)) \subset Q$ and $C(\mathcal{R}(D)) \subset Y$

C2. The restriction of $C$ on $\mathcal{A}(D)$ is an $X$-coloring of $D$.

C3. For an over arc $\alpha$ and complementary regions $\alpha_1$ and $\alpha_2$, $C$ satisfies $C(\alpha_2) = C(\alpha_1) \ast C(\alpha)$ (see Figure 14).
We denote by $\text{Col}_X(D)_Y$ the set of $X_Y$-colorings. Using a $G$-family of quandles and an Abelian group $A$, the chain complex, denoted $C_*(X)_Y$, and the cochain complex, denoted $C^*(X;A)_Y := \text{Hom}(C_*(X)_Y, A)$, can be defined. Then, we can consider homology group and cohomology group (refer to [3]). For an $X_Y$-coloring $C$ and a crossing $\chi$ of $D$, we define the weight of the crossing $\chi$, denoted $w(\chi; C)$, as $w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi), C(\chi_3))$ where $R_\chi$, $\chi_1$, and $\chi_3$ are a complementary region and arcs assigned as shown in Figure 15 and $\epsilon(\chi)$ is the sign of the crossing $\chi$.

Related to homology and cohomology with a $G$-family of quandles, the following lemma and theorem are known (see [3]).

**Lemma 5.3.** Let $(X, \{*_g\}_{g \in G})$ and $(Y, \{*_g\}_{g \in G})$ be a $G$-family of quandles and an $X$-set, respectively. Let $D$ be an oriented diagram of a spatial trivalent graph. Let $C$ be an $X_Y$-coloring of $D$. Then, sum of the weights $W(D; C) := \sum_{\chi \in D} w(\chi; C)$ is a 2-cycle.

Let $A$ be an Abelian group. Let $\theta$ be a 2-cocycle of the cochain complex $C^*(X;A)_Y$. We define a multiset as follows:

$$\Phi_\theta(D) := \{\theta(W(D;C)) \in A \mid C \in \text{Col}_X(D)_Y\}.$$  

**Theorem 5.4.** Let $(X, \{*_g\}_{g \in G})$ and $(Y, \{*_g\}_{g \in G})$ be a $G$-family of quandles and an $X$-set, respectively. Let $H$ be a handlebody-link and $D$ be an oriented diagram of $H$. Let $A$ and $\theta$ be an Abelian group and a 2-cocycle of the cochain complex $C^*(X;A)_Y$. Then, $\Phi_\theta(D)$ is an invariant of a handlebody-link $H$.

Since $\Phi_\theta(D)$ is a multiset, we can write $\Phi_\theta(D)$ in the form of

$$\Phi_\theta(D) = \{(a_1)_{l_1}, \ldots, (a_m)_{l_m}\}$$

where $l_j$ is a multiplicity of $a_j \in A$, and $(a_j)_{l_j}$ represents $a_j, \ldots, a_j$. Using these notations and a natural number $N$, we define a set as follows:

$$\Phi_\theta(D)_N := \{(a_1, l_1/N), \ldots, (a_m, l_m/N) \mid (a_i, l_i/N) \in A \times \mathbb{Q}\}.$$

Let $F$ be a surface embedded in $S^3$. Then we have two handlebody-knots $H_V$ and $H_W$ obtained from Heegaard splittings of connected components $V_F$ and $W_F$ of the exterior of $F$. Let $D_V$ and $D_W$ be diagrams of $H_V$ and $H_W$, respectively. We denote by $D'_V$ and $D'_W$ diagrams of handlebody-knots obtained from $H_V$ and $H_W$ by applying stabilization. From lemma 2.27, we have $\#\text{Col}_X(D'_V)_Y = \#\text{Col}_X(D_V)_Y \cdot \#G$ and $\#\text{Col}_X(D'_W)_Y = \#\text{Col}_X(D_W)_Y \cdot \#G$. On the other hand, the numbers of crossings of $D_V$ and $D_W$ do not change. Then we have the following theorem using the same way in theorem 3.3.
Theorem 5.5. Let $G$ be a group. Let $(X, \{*_{g}\}_{g \in G})$ and $(Y, \{\pi_{g}\}_{g \in G})$ be a $G$-family of quandles and an $X$-set, respectively. Let $F$ be a closed connected orientable surface embedded in $S^3$, and $V_F$ and $W_F$ be the two connected components of the exterior, respectively. Let $F_V$ and $F_W$ be Heegaard surfaces of $V_F$ and $W_F$. Let $H_V$ and $H_W$ be handlebody-knots obtained from each Heegaard splitting of $V_F$ and $W_F$, respectively, and $L_F$ be a two-component handlebody-link consists of $H_V$ and $H_W$. Let $D_F$ an oriented diagram of $L_F$. Then, $\Phi(\theta(D_F)) \# G^{(F_V)} + \# G^{(F_W)}$ is an invariant of a surface $F$.

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