A Study of Complex Dombi Fuzzy Graph With Application in Decision Making Problems

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ABSTRACT A complex fuzzy set (CFS) is a generalization of a fuzzy set (FS) in which a limit of degrees occurs on the complex plane with unit disc. The averaging operators are a key part of turning all the data into one value. Dombi operators have exceptional flexibility with operating factors, and they are particularly efficient in decision-making problems. In this paper, we establish a complex dombi fuzzy graph (CDFG). We implement dombi operators on CFSs to extend graph nomenclature. We define the complement of CDFG with an example. The idea of self-complementary in CDFG is discussed. The concepts of homomorphism, isomorphism, weak isomorphism, and co-weak isomorphism of two CDFGs are discussed. We define regular and entirely regular graphs with sufficient elaboration and examine their key properties. Furthermore, significant characteristics are used to explain the edge regularity of CDFG. Lastly, we establish an application of CDFG in decision making problems.

INDEX TERMS CDFG, complement, self complementary of CDFG, homomorphism, isomorphism, weak isomorphism, co-weak isomorphism, regular and totally regular CDFG, application.

I. INTRODUCTION

Due to the existence of unclear data, Zadeh [20] created the concept of a FS, which is an extension of the crisp set theory. A FS consists of a true membership function that belongs to a closed interval [0, 1]. The FS has many applications in the area of science.

Menger [12] introduced triangle norms and conorms in the context of probabilistic metric spaces, later defined and analysed by Schweizer and Sklar [13]. Numerous additional researchers have proposed alternative T-operators [5], [10]. Zadeh’s conventional T-operators, min and max, are widely employed in fuzzy logic, especially in decision-making and fuzzy graph theory. It is commonly recognised that alternative T-operators function better in specific contexts, especially in decision-making procedures. Examples of preferable operators include products [6]. When choosing T-operators for a certain application, one must evaluate their features, model applicability, simplicity, software and hardware implementation, etc. As the study of these operators has grown, more alternatives for selecting T-operators have emerged.

Graphs have many applications in the field of operational research and computer science. A graph is a visual representation of links between several items that is useful for elaborating on information. However, haziness turns a graph into a fuzzy graph. Fuzzy graphs are intended to portray as a matter of degree structures of connections (in the form of edges) between tangible objects (nodes). Fuzzy graphs have a wide variety of applications, including decision-making, database theory, cluster analysis, and network optimization. Kaufman [11] firstly proposed the concept of fuzzy graph. After that, Rosenfeld [14] studied fuzzy relations on fuzzy sets and used max and min operations to construct the structure of fuzzy graphs, resulting in analogues of numerous graph-theoretical ideas. Bhattacharya [4] made some comments on fuzzy graphs. Reference [16] described new
operations on picture fuzzy graph. Recently, a few researchers are contributing their efforts in filed of fuzzy theory [7, 8], [9], [17], [18], [19].

Fuzzy graph theory makes it simple to structure and model uncertain decision-making issues. In the discipline of graph theory, only a small amount of work is put towards using the Dombi operator. As a result, Ashraf et al. [3] proposed the Dombi fuzzy graph (DFG). Ramot et al. [15] presented the concept of CFS in which a range of degrees occurs in the complex plane with unit disk. Akram and Khan [1] studied complex pythagorean fuzzy graph in decision making problem.

The following is a summary of the motivation for this paper:

- When faced with one-dimensional phenomena of imprecise and intuitive knowledge, a CFS is capable of coping with the situation effectively. There is no information loss due to the phase term of the CFS.
- As a result of incorporating the qualities of numerous frequently used operators, Dombi operators have a broader range of applications and are extremely efficient in decision-making.

The following are the key points of this paper:

- The notion of CDFG is initiated.
- The concept of the degree and total degree of a node in both phase terms and amplitude terms are discussed with examples.
- We define complement, self-complementary, homomorphism, isomorphism, weak isomorphism and co-weak isomorphism with their properties.
- We define strong CDFG and complete CDFG.
- We introduce regular and totally regular graphs with appropriate elaboration, and their pivotal properties are discussed.

The following is the structure of this paper:

We presented some basic definitions which will help to understand the paper in Section II. In section III, we study the notion of CDFG, the degree and total degree of a node, complement, self-complementary, homomorphism, isomorphism, weak isomorphism, co-weak isomorphism, strong CDFG, complete CDFG, regular and totally regular graphs with appropriate elaboration, and their pivotal properties are discussed. In Section IV, application of CDFG is discussed. At the end, we write the conclusion and some future plans in Section V.

II. PRELIMINARIES

Definition 1 [1]: A FS on a universe ψ is an object of the following form C = {< g, ςC(g) > | g ∈ ψ}, where ςA : ψ → [0, 1] denotes the membership value of C.

Definition 2 [1]: A FS on ψ × ψ is called fuzzy relation on ψ, denoted by F = {< xy, ςF(xy) > | xy ∈ ψ × ψ}, where ςA : ψ × ψ → [0, 1] denotes the membership value of F.

Definition 3 [1]: A fuzzy graph on ψ ̸= φ is a pair τ = (C, F) with C a FS on ψ and F a fuzzy relation on ψ such that ςF(xy) ≤ ςC(x) ∧ ςC(y) for all x, y ∈ ψ.

Definition 4 [1]: A : [0, 1] × [0, 1] → [0, 1] binary function known as triangular norm (t-norm) if for all x, y, r ∈ [0, 1], it follows the following axioms:

1. A(x, 1) = x.
2. A(x, y) = A(y, x).
3. A(x, A(y, r)) = A(A(x, y), r).
4. A(x, y) ≤ A(r, s) if x ≤ r and y ≤ s.

Interchanging 1 by 0 in axiom (1), we get the idea of triangular conorm (t-conorm).

Following are some popular t-norms:

- M(x, y) = min(x, y). (minimum operator M)
- P(x, y) = xy. (product operator P)
- V(x, y) = max(x + y − 1, 0). (Łukasiewicz’s t-norm V)
- D(x, y) = \( \frac{1}{1+[(\frac{x}{λ})^{2}+(\frac{y}{λ})^{2}]^{1/2}} : λ > 0 \). (Dombi’s t-norm D)
- By putting λ = 1 in dombi’s t-norm, we obtain one more T-operator that is T(x, y) = \( \frac{x+y}{x+y} \).

The corresponding t-conorms are as follows:

- M+(x, y) = max(x, y). (maximum operator M+)
- P+(x, y) = x + y − xy. (probalistic sum P+)
- V+(x, y) = min(x + y, 1). (Łukasiewicz’s t-conorm V+)
- D+(x, y) = \( \frac{1}{1+[(\frac{x}{λ})^{2}+(\frac{y}{λ})^{2}]^{1/2}} : λ > 0 \). (Dombi’s t-

Definition 5 [1]: A DFG on ψ (underlying set) is an ordered pair τ = (C, F), where F : ψ × ψ → [0, 1] is a symmetric fuzzy relation on ψ, ψ : ψ → [0, 1] is a fuzzy subset in ψ such that

\[ ςF(gh) ≤ \frac{ςC(g)ςC(h)}{ςC(g) + ςC(h) − ςC(g)ςC(h)} \]

for all g,h ∈ ψ, where ςC and ςF denotes the membership values of C and F respectively.

Definition 6 [1]: A complex fuzzy set (CFS) on a universe ψ is an object of the form C = {< g, ςC(g)e\( ϑC(g) \) > | g ∈ ψ}, i = \( \sqrt{-1} \), where ςC : ψ → [0, 1] is a real valued function represents the membership value and \( ϑC(g) ∈ [0, 2\pi] \), for all g ∈ ψ. Note that ςC(g) is called amplitude term and \( ϑC(g) \) is called phase term.

Definition 7 [1]: A CFS on ψ × ψ is said to be complex fuzzy relation (CFR) denoted by F = {< gh, ςF(gh)e\( ϑF(gh) \) > | gh ∈ ψ × ψ}, i = \( \sqrt{-1} \), where ςF : ψ × ψ → [0, 1] represents the membership value and \( ϑF(gh) \) ∈ [0, 2\pi], for all gh ∈ ψ. Note that ςF(gh) is called amplitude term and \( ϑF(gh) \) is called phase term.

III. CDFG

Definition 8: A CDFG on a universe ψ is an ordered pair τ = (C, F), where C = (g, ςC(e\( ϑC(g) \)) : ψ → {z : z ∈ C, |Z| ≤ 1} is a CFS subset in ψ and F = (gh, ςF(e\( ϑF(gh) \)) : ψ × ψ → {z : z ∈ C, |Z| ≤ 1} is a complex fuzzy relation (CFR) on C such that for amplitude term

\[ ςF(gh) ≤ \frac{ςC(g)ςC(h)}{ςC(g) + ςC(h) − ςC(g)ςC(h)} \]
The degree of a node $g \in \psi$ for a phase term is expressed by $\mathcal{T}D_{e\phi}(g) = \mathcal{T}D_{e\phi}(g)$, where

$$\mathcal{T}D_{e\phi}(g) = \sum_{g, h \notin \psi} \frac{\partial_{c}(g)\partial_{c}(h)}{\partial_{c}(g) + \partial_{c}(h) - \partial_{c}(g)\partial_{c}(h)} + \partial_{A}(g).$$

**Example 2:** From above example, we have
- The degree of nodes in $\tau$ are as follows:
  - $\mathcal{D}_{r}(g) = 0.63e^{2\pi(0.74)}$.
  - $\mathcal{D}_{r}(h) = 0.37e^{2\pi(0.39)}$.
  - $\mathcal{D}_{r}(r) = 0.3e^{2\pi(0.25)}$.
  - $\mathcal{D}_{r}(s) = 0.3e^{2\pi(0.5)}$.

- The total degree of nodes in $\tau$ are as follows:
  - $\mathcal{T}D_{r}(g) = 1.13e^{2\pi(1.14)}$.
  - $\mathcal{T}D_{r}(h) = 0.77e^{2\pi(0.69)}$.
  - $\mathcal{T}D_{r}(r) = 0.75e^{2\pi(0.75)}$.
  - $\mathcal{T}D_{r}(s) = 0.6e^{2\pi(1.1)}$.

**Definition 10:** Let $\tau = (C, F)$ be a CDFG on a graph $\tau^* = (\psi, A)$, for all $g \in CDFG$, then

- The complement of $\tau$ for amplitude term is determined by:
  1. $\zeta_{C}(g) = \zeta_{C}(g)$.
  2. $\zeta_{F}(gh) = \left\{\begin{array}{ll}
  \zeta_{C}(h) + \frac{\zeta_{C}(g) + \zeta_{C}(h) - \zeta_{C}(g)\zeta_{C}(h)}{\zeta_{C}(g)\zeta_{C}(h)} \\
  \zeta_{C}(g) + \zeta_{C}(h) - \zeta_{C}(g)\zeta_{C}(h)
  \end{array}\right.$ if $\frac{\zeta_{F}(gh)}{\zeta_{C}(g)\zeta_{C}(h)} = 0$.

Similarly the complement of $\tau$ for phase term is determined by:

- The total degree of a CDFG $\tau$ is denoted by $\bar{\tau} = (\bar{C}, \bar{F})$.

**Definition 11:** A homomorphism $Z : \tau \rightarrow \tau'$ of two CDFGs $\tau = (C, F)$ and $\tau' = (C', F')$ is a mapping $Z : \psi \rightarrow \psi'$ satisfying

1. $\zeta_{C}(g) \leq \zeta_{C}'(Z(g))$, $\zeta_{F}(gh) \leq \zeta_{F}'(Z(g)Z(h))$, for all $g \in \psi$.
2. $\zeta_{F}(gh) \leq \zeta_{F}'(Z(g)Z(h))$, $\zeta_{F}(gh) \leq \zeta_{F}'(Z(g)Z(h))$, for all $gh \in A$.

**Definition 12:** An isomorphism $Z : \tau \rightarrow \tau'$ of two CDFGs $\tau = (C, F)$ and $\tau' = (C', F')$ is a bijective mapping $Z : \psi \rightarrow \psi'$ satisfying

1. $\zeta_{C}(g) = \zeta_{C}'(Z(g))$, $\zeta_{F}(gh) = \zeta_{F}'(Z(g)Z(h))$, for all $g \in \psi$.
2. $\zeta_{F}(gh) = \zeta_{F}'(Z(g)Z(h))$, $\zeta_{F}(gh) = \zeta_{F}'(Z(g)Z(h))$, for all $gh \in A$.

**Definition 13:** A weak isomorphism $Z : \tau \rightarrow \tau'$ of two CDFGs $\tau = (C, F)$ and $\tau' = (C', F')$ is a bijective mapping $Z : \psi \rightarrow \psi'$ satisfying
1. $Z$ is a homomorphism.

2. $\sigma(g) = \sigma'(Z(g))$, $\theta_c(g) = \theta_c'(Z(g))$ for all $g \in \psi$.

Definition 14: An co-weak isomorphism $Z: \tau \rightarrow G'$ of two CDFGs $\tau = (C, F)$ and $G' = (C', F')$ is a bijective mapping $Z: \psi \rightarrow \psi'$ satisfying

1. $Z$ is a homomorphism.

2. $\sigma_Z(gh) = \sigma_{Z'}(Z(g)Z(h))$, $\theta_Z(gh) = \theta_{Z'}(Z(g)Z(h))$ for all $gh \in A$.

Definition 15: A CDFG $\tau = (C, F)$ is called self complementary if $\bar{\tau} \cong \tau$.

Proposition 1: If $\tau = (C, F)$ is a self complementary CDFG, then

$$\sum_{g \neq h} \sigma_Z(gh) = \frac{1}{2} \sum_{g \neq h} \sigma_c(g)\sigma_c(h) - \frac{1}{2} \sigma_{Z'}(Z(g)Z(h)),$$

and

$$\sum_{g \neq h} \theta_Z(gh) = \frac{1}{2} \sum_{g \neq h} \theta_c(g)\theta_c(h).$$

Proof: Suppose that $\tau$ is a self complementary CDFG, then there occurs an isomorphism $Z: \psi \rightarrow \psi$ such that

$$\sigma_{Z'}(Z(g)) = \sigma_c(g), \quad \theta_{Z'}(Z(g)) = \theta_c(g)$$

for all $g \in \psi$. Then

$$\sum_{g \neq h} \sigma_Z(gh) = \frac{1}{2} \sum_{g \neq h} \sigma_c(g)\sigma_c(h) - \sigma_{Z'}(Z(g)Z(h)),$$

and

$$\sum_{g \neq h} \theta_Z(gh) = \frac{1}{2} \sum_{g \neq h} \theta_c(g)\theta_c(h).$$

Proposition 2: If a CDFG $G = (A, B)$ on an underlying graph $\tau^* = (\psi, A)$ satisfy the following:

$$\sigma_Z(gh) = \frac{1}{2} \left( \sigma_c(g)\sigma_c(h) + \sigma_c(h)\sigma_c(g) - \sigma_{Z'}(Z(g)Z(h)) \right),$$

$$\theta_Z(gh) = \frac{1}{2} \left( \theta_c(g)\theta_c(h) - \theta_{Z'}(Z(g)Z(h)) \right),$$

for all $g, h \in \psi$, then $\tau$ is self complementary.

Proof: Consider $\tau$ is CDFG that satisfies

$$\sigma_Z(gh) = \frac{1}{2} \left( \sigma_c(g)\sigma_c(h) + \sigma_c(h)\sigma_c(g) - \sigma_{Z'}(Z(g)Z(h)) \right),$$

for all $g, h \in \psi$, then the identity mapping $I: \psi \rightarrow \psi$ is an isomorphism from $\tau$ to $\bar{\tau}$ that satisfies the following conditions:

$$\sigma_{Z'}(I(g)) = \sigma_c(g), \quad \theta_{Z'}(I(g)) = \theta_c(g)$$

for all $g \in \psi$. Since the membership value of an edge set $gh$ is given by

$$\sigma_Z(gh) = \frac{1}{2} \left( \sigma_c(g)\sigma_c(h) + \sigma_c(h)\sigma_c(g) - \sigma_{Z'}(Z(g)Z(h)) \right),$$

for all $g, h \in \psi$. We have

$$\sigma_Z(\bar{\tau}(gh)) = \sigma_{Z'}(I(\bar{\tau}(gh))).$$

Similarly the phase term condition of isomorphism

$$\theta_{Z'}(I(\bar{\tau}(gh))) = \theta_{Z'}(gh).$$

is satisfied by $I$. Hence $G = (A, B)$ is self complementary. \[\square\]

Proposition 3: Let $\tau = (C, F)$ and $G' = (C', F')$ be two CDFGs, then $\tau \cong \bar{\tau}$ if $\bar{\tau} \cong \tau$. Proof: Suppose that $\tau$ and $G'$ are two isomorphic CDFGs. Then by definition of isomorphism, there occur a bijective mapping $Z: \psi \rightarrow \psi'$ that satisfies

$$\sigma_c(g) = \sigma_c'(Z(g)),$$

$$\theta_c(g) = \theta_c'(Z(g))$$

for all $g \in \psi$. Then

$$\sigma_Z(gh) = \sigma_{Z'}(Z(g)Z(h)),$$

$$\theta_Z(gh) = \theta_{Z'}(Z(g)Z(h))$$

for all $gh \in A_1$.

By using definition of complement, the membership of an edge $gh$ is

$$\sigma_Z(gh) = \frac{1}{2} \left( \sigma_c(g)\sigma_c(h) + \sigma_c(h)\sigma_c(g) - \sigma_{Z'}(Z(g)Z(h)) \right),$$

$$\theta_Z(gh) = \frac{1}{2} \left( \theta_c(g)\theta_c(h) - \theta_c'(Z(g)Z(h)) \right).$$
to the complement of \( \bar{\tau} \).

Similarly for the phase term,

\[
\vartheta(h) = \frac{\vartheta_1(h) - \vartheta_0(h)}{\vartheta_1(h) + \vartheta_0(h)} - \vartheta_f(h).
\]

We conclude that the complement of \( \tau \) is isomorphic to the complement of \( \bar{\tau} \). Similarly, we can prove its converse part.

**Proposition 4:** Let \( \tau = (C, F) \) and \( \bar{\tau} = (C', F') \) be two weak isomorphic CDFGs, then \( \bar{\tau} \) and \( \bar{G}' \) are also weak isomorphic to each other.

**Proof:** Suppose that \( \tau \) and \( \bar{G}' \) are two weak isomorphic CDFGs. Then utilizing the definition of weak isomorphism, there exist a bijective mapping \( Z : \psi \to \bar{\psi}' \) that satisfies

\[
\varphi(C) = \varphi(C), \quad \varphi(g) = \varphi(C)(Z(g)) \text{ for all } g \in \psi,
\]

and

\[
\varphi(h) \leq \varphi(Z(g)Z(h)), \quad \varphi(h) \leq \varphi(Z(g)Z(h)) \text{ for all } gh \in A_1.
\]

For the membership value of an edge, we have

\[
\varphi_f(g) \leq \varphi_f(Z(g)Z(h))
\]

\[
- \varphi_f(g) \geq - \varphi_f(Z(g)Z(h))
\]

\[
T(\varphi_C(g), \varphi_C(h)) - \varphi_f(g)
\]

\[
\geq T(\varphi_C(g), \varphi_C(h)) - \varphi_f(Z(g)Z(h))
\]

\[
\geq T(\varphi_C(Z(g)), \varphi_C(Z(h))) - \varphi_f(Z(g)Z(h))
\]

\[
\varphi_f(h) \geq \varphi_f(Z(g)Z(h)).
\]

Similarly for the phase term

\[
\vartheta_f(g) \leq \vartheta_f(Z(g)Z(h))
\]

\[
- \vartheta_f(g) \geq - \vartheta_f(Z(g)Z(h))
\]

\[
T(\vartheta_0(g), \vartheta_0(h)) - \vartheta_f(g)
\]

\[
\geq T(\vartheta_0(g), \vartheta_0(h)) - \vartheta_f(Z(g)Z(h))
\]

\[
\geq T(\vartheta_0(Z(g)), \vartheta_0(Z(h))) - \vartheta_f(Z(g)Z(h))
\]

\[
\vartheta_f(h) \geq \vartheta_f(Z(g)Z(h)).
\]

Hence we conclude that complement of \( \tau \) is weak isomorphic to the complement of \( \bar{\tau} \).
Definition 20: A CDFG $\tau = (C, F)$ on a graph $\tau^* = (\psi, A)$ is called totally regular of degree $T_1 e^{R_1^*}$ or $T_1 e^{T_1^*}$, totally regular if its each node has same total degree, i.e.,

$$TD_\tau(g) = \sum_{g, h \not\in \psi} \varsigma_F(gh) + \varsigma_C(g) = T_1,$$

$$TD_{\psi}(g) = \sum_{g, h \not\in \psi} \vartheta_F(gh) + \vartheta_C(g) = T_1^*,$$

for all $g \in \psi$.

Example 4: Let $\tau = (C, F)$ be a CDFG on $\sigma^* = (\psi, A)$ as shown in Figure 3, where $\psi = \{g, h, r, s\}$ and $A = \{gh, qr, ps, rs\}$. The set of nodes $C$ and set of edges $F$ of $\tau$ are defined on $\psi$ and $A$, respectively.

$$C = \begin{pmatrix} g & h \\ 0.45 e^{2\pi(0.35)} & 0.45 e^{2\pi(0.35)} \end{pmatrix},$$

$$F = \begin{pmatrix} gh & qr & ps & rs \\ 0.25 e^{2\pi(0.2)} & 0.25 e^{2\pi(0.2)} & 0.15 e^{2\pi(0.2)} & 0.15 e^{2\pi(0.2)} \end{pmatrix} > 0.85 e^{2\pi(0.75)}$$

By calculations, one can see that $\tau = (C, F)$ is a totally regular CDFG.

Theorem 1: Consider a CDFG $\tau = (C, F)$ which is isomorphic to another CDFG $\mathcal{G}' = (C', F')$:

1. If $\tau$ is regular CDFG, then $\mathcal{G}'$ is also regular CDFG.
2. If $\tau$ is totally regular CDFG, then $\mathcal{G}'$ is also totally regular CDFG.

Proof: 1. Suppose that $\tau$ is isomorphic to $\mathcal{G}'$ and $\tau$ is $R_1 e^{R_1^*}$-regular CDFG, therefore degree of each node of $\tau$ is given by

$$D_\tau(g) = D_{\psi}(g),$$

$$D_\tau(g) = \sum_{g, h \not\in \psi} \varsigma_F(gh)e^{i \vartheta_F(gh)}.$$
Also for phase term,
\[ TD_\tau(g) = TD_{\tau^0}(g) \]
\[ = \sum_{g \in A} \vartheta_\tau(gh) + \vartheta(c) \]
\[ = \sum_{g, h \in A} \frac{\vartheta_c(g) \vartheta_c(h)}{\vartheta_c(g) + \vartheta_c(h) - \vartheta_c(g) \vartheta_c(h)} + \vartheta_c(g) \]
\[ = \frac{1}{T_1}. \]

Since \( \tau \equiv G' \), we must have
\[ \mathcal{T}_1^* = TD_{\tau^0}(g) \]
\[ = \sum_{g \in A} \vartheta_\tau(gh) + \vartheta(c) \]
\[ = \sum_{g, h \in A} \frac{\vartheta_c(g) \vartheta_c(h)}{\vartheta_c(g) + \vartheta_c(h) - \vartheta_c(g) \vartheta_c(h)} + \vartheta_c(g) \]
\[ = \sum_{g, h \in A} \frac{\vartheta_c(Z(g)) \vartheta_c(Z(h))}{\vartheta_c(Z(g)) + \vartheta_c(Z(h)) - \vartheta_c(Z(g)) \vartheta_c(Z(h))} + \vartheta_c(Z(g)) \]
\[ = \sum_{g \in A} \vartheta_\tau(Z(g)Z(h)) + \vartheta_c(Z(g)) \]
\[ = TD_{\tau^0}(g) \]
\[ = TD_\tau(g) \]
\[ = TD_{\tau^0}(g) \]
\[ = TD_{\tau^0}(g) \]
\[ = TD_{\tau^0}(g) \]
\[ = TD_{\tau^0}(g) \]
\[ = TD_{\tau^0}(g) \]
\[ + \vartheta_c(g) = T_1, \]
\[ \mathcal{T}_1^* - c_1 = R_1. \]

So \( \tau \) is a \( R_1^*e^{R_1^*} \)-regular CDFG.

**Theorem 2:** Suppose that \( \tau = (C, F) \) is a CDFG on a graph \( \tau^* = (\psi, A) \) with \( \vartheta_c^0 \) as a constant function, then \( \tau = (C, F) \) is a regular CDFG if and only if \( \tau \) is totally regular CDFG. 

**Proof:** Suppose that \( \vartheta_c^0 \) is a constant function, i.e., \( \vartheta_c(g) = c_1 \) is a constant function for all \( g \in \psi \), where \( c_1 \) is constant.

Suppose that \( \tau = (C, F) \) is \( R_1e^{R_1^*} \)-regular CDFG, then
\[ D_\tau(g) = \sum_{g, h \in A} \vartheta_c(g) \vartheta_c(h) = R_1. \]
\[ D_{\tau^0}(g) = \sum_{g, h \in A} \vartheta_c(g) \vartheta_c(h) = R_1^*. \]

The total degree of a node is given by
\[ TD_\tau(g) = \sum_{g, h \in A} \frac{\vartheta_c(g) \vartheta_c(h)}{\vartheta_c(g) + \vartheta_c(h) - \vartheta_c(g) \vartheta_c(h)} + \vartheta_c(g) = R_1 + c_1, \]
\[ TD_{\tau^0}(g) = \sum_{g, h \in A} \frac{\vartheta_c(g) \vartheta_c(h)}{\vartheta_c(g) + \vartheta_c(h) - \vartheta_c(g) \vartheta_c(h)} + \vartheta_c(g) = R_1^* + c_1. \]

Hence, \( \tau \) is \( (R_1 + c_1)e^{R_1^*} \)-totally regular CDFG.

Conversely, suppose that \( \tau = (C, F) \) is \( T_1e^{T_1^*} \)-totally regular CDFG, then
\[ TD_\tau(g) = \sum_{g, h \in A} \frac{\vartheta_c(g) \vartheta_c(h)}{\vartheta_c(g) + \vartheta_c(h) - \vartheta_c(g) \vartheta_c(h)} + \vartheta_c(g) = T_1, \]
\[ \mathcal{T}_1^* = c_1 = R_1. \]

**Theorem 3:** Suppose that \( \tau = (C, F) \) is a CDFG on a graph \( \tau^* = (\psi, A) \). If \( \tau \) is both \( R_1e^{R_1^*} \)-regular and \( T_1e^{T_1^*} \)-totally regular CDFG, then \( \vartheta_c^0 \) is a constant function.

**Proof:** Suppose that \( \tau = R_1e^{R_1^*} \)-regular and \( T_1e^{T_1^*} \)-totally regular CDFG. Then, the degree of a node is given by
\[ D_\tau(g) = \sum_{g, h \in A} \frac{\vartheta_c(g) \vartheta_c(h)}{\vartheta_c(g) + \vartheta_c(h) - \vartheta_c(g) \vartheta_c(h)} = R_1, \]
\[ D_{\tau^0}(g) = \sum_{g, h \in A} \frac{\vartheta_c(g) \vartheta_c(h)}{\vartheta_c(g) + \vartheta_c(h) - \vartheta_c(g) \vartheta_c(h)} = R_1^*. \]

The total degree of a node is given by
\[ TD_\tau(g) = \sum_{g, h \in A} \frac{\vartheta_c(g) \vartheta_c(h)}{\vartheta_c(g) + \vartheta_c(h) - \vartheta_c(g) \vartheta_c(h)} + \vartheta_c(g) = T_1, \]
\[ TD_{\tau^0}(g) = \sum_{g, h \in A} \frac{\vartheta_c(g) \vartheta_c(h)}{\vartheta_c(g) + \vartheta_c(h) - \vartheta_c(g) \vartheta_c(h)} + \vartheta_c(g) = T_1^* \]

It follows that
\[ TD_\tau(g) = R_1 + \vartheta_c(g) = T_1, \]
\[ \vartheta_c(g) = T_1 - R_1. \]
\[ TD_{\tau^0}(g) = R_1^* + \vartheta_c(g) = T_1^*, \]
\[ \vartheta_c(g) = T_1^* - R_1^*. \]

Hence, \( \vartheta_c^0 = (T_1 - R_1)e^{(T_1^* - R_1^*)} \) is a constant function.

**Remark 2:** Converse of above theorem need not to be true as given in the following example.

**Example 5:** Let \( \tau = (C, F) \) be a CDFG on \( \tau^* = (\psi, A) \) as shown in Figure 4, where \( \psi = \{g, h, r, s\} \) and \( A = \)
\[ \{gh, ps, qr, rs\}. \] The set of nodes \( C \) and set of edges \( F \) of \( \tau \) are defined on \( \psi \) and \( A \), respectively.

\[ C = \left\{ \frac{g}{0.5e^{2\pi(0.42)}}, \frac{h}{0.5e^{2\pi(0.42)}}, \frac{r}{0.5e^{2\pi(0.42)}}, \frac{s}{0.5e^{2\pi(0.42)}} \right\} > \]

and

\[ F = \left\{ \frac{gh}{0.25e^{2\pi(0.12)}}, \frac{ps}{0.12e^{2\pi(0.17)}}, \frac{qr}{0.24e^{2\pi(0.23)}}, \frac{rs}{0.23e^{2\pi(0.24)}} \right\} > \]

Here \( \varsigma C^{\theta} \) for \( gh, rs \) is a constant function. But

\[ \mathcal{D}_r(g) = 0.37e^{2\pi(0.29)} \neq 0.47e^{2\pi(0.47)} = \mathcal{D}_r(r) \]

and

\[ \mathcal{T}\mathcal{D}_c(g) = 0.87e^{2\pi(0.71)} \neq 0.97e^{2\pi(0.89)} = \mathcal{T}\mathcal{D}_c(r). \]

Hence, \( \tau = (C, F) \) is neither regular nor totally regular CFG.

Definition 21: Let \( F = \{gh, \varsigma F(gh)\} \) be the set of edges in CFG \( \tau \), then

\[ \mathcal{D}_c(gh) = \sum_{pr \in A, r \neq h} \varsigma F(pr) + \sum_{qr \in A, g \neq r} \varsigma F(qr) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - 2\varsigma F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \frac{\varsigma c(g)\varsigma c(h)}{\varsigma c(g) + \varsigma c(h) - \varsigma c(g)\varsigma c(h)}. \]

\[ \mathcal{D}_c(gh) = \sum_{pr \in A, r \neq h} \theta F(pr) + \sum_{qr \in A, g \neq r} \theta F(qr) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - 2\theta F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \frac{\theta c(g)\theta c(h)}{\theta c(g) + \theta c(h) - \theta c(g)\theta c(h)}. \]

\[ \mathcal{D}_c(gh) = \sum_{pr \in A, r \neq h} \varsigma F(pr) + \sum_{qr \in A, g \neq r} \varsigma F(qr) + \varsigma F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \varsigma F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \frac{\varsigma c(g)\varsigma c(h)}{\varsigma c(g) + \varsigma c(h) - \varsigma c(g)\varsigma c(h)}. \]

\[ \mathcal{T}\mathcal{D}_c(gh) = \sum_{pr \in A, r \neq h} \theta F(pr) + \sum_{qr \in A, g \neq r} \theta F(qr) + \theta F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \varsigma F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \frac{\varsigma c(g)\varsigma c(h)}{\varsigma c(g) + \varsigma c(h) - \varsigma c(g)\varsigma c(h)}. \]

\[ \mathcal{T}\mathcal{D}_c(gh) = \sum_{pr \in A, r \neq h} \varsigma F(pr) + \sum_{qr \in A, g \neq r} \varsigma F(qr) + \varsigma F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \varsigma F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \frac{\varsigma c(g)\varsigma c(h)}{\varsigma c(g) + \varsigma c(h) - \varsigma c(g)\varsigma c(h)}. \]

\[ \mathcal{T}\mathcal{D}_c(gh) = \sum_{pr \in A, r \neq h} \theta F(pr) + \sum_{qr \in A, g \neq r} \theta F(qr) + \theta F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \varsigma F(gh) \]

\[ = \mathcal{D}_c(g) + \mathcal{D}_c(h) - \frac{\varsigma c(g)\varsigma c(h)}{\varsigma c(g) + \varsigma c(h) - \varsigma c(g)\varsigma c(h)}. \]
Theorem 4: Suppose $\tau = (C, F)$ is $R_1e^{R_1^*}$-regular CDFG. If $\zeta_\tau e^{io_\tau}$ is a constant function, then $\tau$ is $L_1e^{L_1^*}$-edge regular CDFG.

Proof: Suppose that $\tau = (C, F)$ is a $R_1e^{R_1^*}$-regular CDFG, then

$$D_\zeta e^{io_\zeta}(g) = \sum_{g,h \neq \psi} \frac{\zeta_c(g)\zeta_c(h)}{\zeta_c(g) + \zeta_c(h) - \zeta_c(g)\zeta_c(h)} \times e^{\zeta_\tau e^{io_\tau}} = R_1e^{R_1^*}.$$

Now $\zeta_\tau e^{io_\tau}$ is a constant function, therefore, $\zeta_\tau e^{io_\tau}(gh) = c_1 e^{i c_1}$ for all $gh \in A$.

Since the degree of an edge $gh \in A$ is given by $D_{\tau}(gh) = D_\zeta e^{io_\zeta}(gh)$, where

$$D_\zeta(g) = D_{\zeta C}(g) + D_{\zeta C}(h) - 2(\zeta_c(g)\zeta_c(h)/\zeta_c(g) + \zeta_c(h) - \zeta_c(g)\zeta_c(h)) = R_1 + R_1 - 2c_1 = 2(R_1 - c_1) = L_1.$$

$$D_{\zeta e^{io_\zeta}}(gh) = D_{\zeta C}(g) + D_{\zeta C}(h) - 2\frac{\zeta_c(g)\zeta_c(h)}{\zeta_c(g) + \zeta_c(h) - \zeta_c(g)\zeta_c(h)} = 2R_1^* - 2c_1 = 2(R_1^* - c_1) = L_1^*.$$

Hence $\tau$ is $L_1e^{L_1^*}$-edge regular CDFG. $\square$

Theorem 5: Suppose a CDFG $\tau$ is $L_1e^{L_1^*}$-edge regular and $K_1e^{K_1^*}$-totally edge regular, then $\zeta_\tau e^{io_\tau}$ is a constant function.

Proof: Suppose that $\tau$ is $L_1e^{L_1^*}$-edge regular CDFG, then the degree of its every arc is

$$D_\zeta(g) = D_{\zeta C}(g) + D_{\zeta C}(h) - 2(\zeta_c(g)\zeta_c(h)/\zeta_c(g) + \zeta_c(h) - \zeta_c(g)\zeta_c(h)) = L_1.$$

$$D_{\zeta e^{io_\zeta}}(gh) = D_{\zeta C}(g) + D_{\zeta C}(h) - 2\frac{\zeta_c(g)\zeta_c(h)}{\zeta_c(g) + \zeta_c(h) - \zeta_c(g)\zeta_c(h)} = 2R_1^* - 2c_1 = 2(R_1^* - c_1) = L_1^*.$$

Also $\tau$ is $K_1e^{K_1^*}$-totally edge regular CDFG, then the degree of its each edge is

$$TD_\zeta(g) = D_{\zeta C}(g) + D_{\zeta C}(h) - (\zeta_c(g)\zeta_c(h)/\zeta_c(g) + \zeta_c(h) - \zeta_c(g)\zeta_c(h)) = K_1.$$

$$TD_{\zeta e^{io_\zeta}}(gh) = D_{\zeta C}(g) + D_{\zeta C}(h) - 2\frac{\zeta_c(g)\zeta_c(h)}{\zeta_c(g) + \zeta_c(h) - \zeta_c(g)\zeta_c(h)} = K_1.$$

Further, it follows that

$$TD_\zeta(gh) = K_1$$

$$TD_{\zeta e^{io_\zeta}}(gh) = K_1$$

$$TD_{\zeta}(gh) = K_1$$

$$TD_{\zeta e^{io_\zeta}}(gh) = K_1$$

$$TD_\zeta(gh) = K_1 - R_1.$$
\[ = \sum_{r \in A, r \neq h} c_i^s + \sum_{r \in A, r \neq r} c_i^t + c_i^1 \]
\[ = c_i^t(R - 1) + c_i^1(R - 1) + c_i^s \]
\[ = c_i^t(2R - 1). \]

Hence, \( \tau \) is \( c_1(2R - 1)e^{d_i^t((2R - 1))} \), totally edge regular CDFG. So \( \tau \) is both regular-CDFG and totally edge regular-CDFG.

Conversely, Let \( \mathcal{L}_t e^{d_i^t} \)-edge regular and \( \mathcal{K}_t e^{K_i^t} \)-totally edge regular CDFG. Furthermore, the total degree of an edge is given by

\[
\mathcal{T}D_1(gh) = \mathcal{T}D_{\mathcal{D}_1}(gh), \text{ where} \\
\mathcal{T}D_1(gh) = \mathcal{D}_1(gh) + \mathcal{D}_1(h) - \mathcal{F}(gh) \\
\mathcal{K}_1 = \mathcal{R}_1 + \mathcal{R}_1 - \mathcal{F}(gh) \\
\mathcal{F}(gh) = 2R - \mathcal{K}_1. \\
\mathcal{T}D_1(gh) = \mathcal{D}_1(gh) + \mathcal{D}_1(h) - \mathcal{F}(gh) \\
\mathcal{K}_1^* = \mathcal{R}_1^* + \mathcal{R}_1^* - \mathcal{F}(gh) \\
\mathcal{F}(gh) = 2R_1 - \mathcal{K}_1^*. \\
\text{for all } gh \in A.
\]

Hence, \( \mathcal{F}e^{d_i^t} \) is a constant function.

**IV. APPLICATION**

In this part, we present an algorithm and resolve a problem of decision making to choose the best spot to set up an internet office in a city. This situation may help us to understand the proposed methodology.

**A. ALGORITHM**

The algorithm to determine a suitable location or place for an internet office in a city is as follows:

**INPUT:** A distinct collection of suitable options \( P = \{P_1, P_2, \ldots, P_n\} \) in certain conditions in order to reach the goal of construction of complex fuzzy preference relation (CFPR) \( Q = (b \times h)_{n \times n} \).

**OUTPUT:** The decision of an appropriate choice.

1. Consider \( d_{kq} = \mathcal{F}e^{d_i^t} \) (k, h = 1, 2, \ldots, n) and collection of choices \( P = \{P_1, P_2, \ldots, P_n\} \).
2. Aggregate all \( d_{kq} = \mathcal{F}e^{d_i^t} \) (k, h = 1, 2, \ldots, n) corresponding to the choice \( P_k \) and obtain the complex fuzzy element (CFE) \( d_k \) of the choice \( P_k \) over all other choices by using Complex Dombi Fuzzy operator.

\[
d_k = \text{CDFoperator}(d_{k1}, d_{k2}, \ldots, d_{kn}) \\
d_k = (1 - \frac{1}{1 + \left[ \sum_{h=1}^{n} \frac{1}{n} \left( \frac{d_{kq}}{d_{kq}} \right)^{1/k} \right]^{1/k}} \\
id \in (0, 1), \sum_{h=1}^{n} \frac{1}{n} \left( \frac{d_{kq}}{d_{kq}} \right)^{1/k} \]

**TABLE 1. CFPR of experts.**

| \( Q \) | \( B_1 \) | \( B_2 \) | \( B_3 \) |
|---|---|---|---|
| \( B_1 \) | 0.6e^{12\pi(0.5)} | 0.7e^{12\pi(0.3)} | 0.4e^{12\pi(0.2)} |
| \( B_2 \) | 0.5e^{12\pi(0.5)} | 0.6e^{12\pi(0.5)} | 0.3e^{12\pi(0.4)} |
| \( B_3 \) | 0.7e^{12\pi(0.8)} | 0.5e^{12\pi(0.6)} | 0.6e^{12\pi(0.5)} |

**FIGURE 6. CDFG directed network.**

3. The formula of score functions is given by:

\[
s(d_k) = s + \frac{1}{2\pi} \theta
\]

4. Compute the score function \( s(d_k) \) of the combined overall preference value \( d_k \) (k = 1, 2, \ldots, n) by using the formula of score function.

5. Rank all the choices \( P_k \) (k = 1, 2, \ldots, n) on the basis of score function \( s(d_k) \) (k = 1, 2, \ldots, n).

6. Output the appropriate option based on the score functions derived in step 4 of the procedure.

**B. SELECTION OF SUITABLE PLACE TO ESTABLISH AN INTERNET OFFICE**

Telecommunication plays an important role in the developmental level of any country. Developed countries have strong telecommunication system. A variety of factors like social interaction, employee, economic growth, job creation, business productivity are all dependent on telecommunication system. There are many ways of telecommunication including smart phones features, skype, whatsapp, imo, snapchat, facebook etc. It plays an important role in globalization. The telecommunication through these softwares is dependent on high speed internet. A private internet company decided to build their office in a city for the convenience of its service to the public. They decided three place in a city \( P_k \) (k=1,2,3). The company make a pairwise comparison in these three places to build an internet office. Following are some parameters that are to be observed.

- a desirable location to open an office.
- Any internet office.
- Available resources.
- Expenditures and outcomes.
- Facilitation for public.

The specialists of company give their preference information in the form of CFPR \( Q = (d_{kq})_{3 \times 3} \) as shown in Table 1, where \( d_{kq} = \mathcal{F}e^{d_i^t} \) is a complex fuzzy element (CFE) preferred by the expert. Consider 0.8e^{12\pi(0.7)}. For the value 0.8, the amplitude term shows that eighty percent of the specialist says \( P_1 \) is best choice to establish an office.
over place $P_2$. Now the phase term 0.7 represents that seventy percent of the specialists conclude that $P_1$ location will create more time profit for the company over location $P_2$. The CFPR $Q = (d_{kq})_{3 \times 3}$ is given in Table 1.

The directed network of CFPR $Q$ represented in Table 1 and is shown as in Figure 6.

To evaluate $d_{kq} = \xi_{kq}^{d_{kq}} (k, h = 1, 2, 3)$ of the place $P_k$ over all other places, we use complex dombi fuzzy operator (CDFO). We have taken $\xi = 1$. The combined overall preference value $d_k (k=1,2,3)$ follows:

\[ d_1 = 0.6e^{2\pi(0.0533)} \]
\[ d_2 = 0.4939e^{2\pi(0.0796)} \]
\[ d_3 = 0.617e^{2\pi(0.1012)} \]

The score function $s(d_k) (k=1,2,3)$ is calculated by using $s(d_k) = \zeta + \frac{1}{2\pi} \vartheta$ which is given below:

\[ s(d_1) = 0.6085 \]
\[ s(d_2) = 0.5065 \]
\[ s(d_3) = 0.6331 \]

We get the ranking order of the four terminals $P_k$ from the score functions as follows:

$P_3 > P_1 > P_2$

The ranking leads to the conclusion that $P_3$ is best place to establish an internet office.

V. CONCLUSION

In order to represent information visually, graphs are quite useful. They are also used to model interactions between different objects. Graphical models can be found everywhere, for example, in manufacturing, communications network diagnosis, and a variety of social, biological, and physical systems, among other applications. They are extremely important since they play a critical role in changing the data received from diverse sources in order to establish the outcomes of decision-making difficulties. In this paper, the idea of CDFG is introduced. A CDFG is extension of DFG. The flexibility and comparability of CDFG is much higher. The concept of complement of CDFG is defined. The concepts of homomorphism, isomorphism, weak isomorphism, and co-weak isomorphism of two CDFGs are discussed in details with different results. Regular and totally regular CDFGs are discussed. At the end, we wrote the application of CDFG. In the future work, we will describe some operation on CDFG. Energy of CDFG may be discuss.

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COMPETING INTERESTS

All authors are here with confirm that there are no competing interests between them.
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