3D Exact Analytical Solutions of Two-fluid Plasma, Magnetohydrodynamics, and Neutral Fluid Equations for the Creation of Ordered Structures as well as Jet-like Flows

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Abstract

The 3D exact analytical solutions of ideal two-fluid plasma, single-fluid plasma, and neutral fluid equations have been found using physically justifiable assumptions. Surprisingly these solutions satisfy all nonlinearities in the systems. It is pointed out that these solutions explain the fundamental mechanism behind the creation of a vast variety of ordered structures in plasmas and fluids. In the limiting case of 2D dependence of fields, the theoretical model for plasma is applied to explain the formation of spicules in the solar chromosphere. It is pointed out that the main contribution of electron (ion) baroclinic vectors is to produce vorticity in the plasma, and that magnetic field generation is coupled with the flow of both electrons and ions.

Key words: Solar atmosphere

1. Introduction

Ordered structures appear in both laboratory (Dnestrovskij et al. 2005; Kugland et al. 2012) and astrophysical (Zweibel & Heiles 1997; Widrow 2002; Ryu et al. 2008) plasmas. Self-organization is a natural process for producing global structures on a macroscopic scale as the complex system evolves from a microscopic turbulent state (Yoshida & Mahajan 2002; Yoshida & Mahajan 2014). Astrophysical jets were observed long ago (Curtis 1918) and later several similar observations were reported in astrophysical environments such as the material outflows from young stellar objects (YSOs), accretion disks, and active galactic nuclei (AGNs; Jennison & Gupta 1953; Mestel 1961; Ferrari 1998). Solar spicules, coronal loops, coronal holes, surges, prominences, flares are examples of ordered structures emerging from highly nonlinear solar plasma dynamics (Priest 1982; Woo 1996; Aschwanden et al. 1999; De Pontieu et al. 2007; Aschwanden & Peter 2017; Klimchuk & DeForest 2020). Material ejection in several different forms in perpendicular direction to the solar plasma is very common in its atmosphere, which is divided into three regions: surface (photosphere), chromosphere, and corona. The solar photosphere is a thin layer of plasma, having a thickness of about 500 km, and above it lies the chromosphere with a thickness of 2500 km. The solar atmosphere is not smooth, rather it contains neutral atoms and is collisional as well. In the chromosphere, the electron density in the photosphere is \( 3 \times 10^{13} \text{ cm}^{-3} \); the temperature is low, therefore the plasma contains neutral atoms and is collisional as well. In the chromosphere, the electron density is in the range \( 3 \times (10^{10} - 10^{11}) \text{ cm}^{-3} \) but the temperature is higher than the photosphere, and the concentration of neutral atoms is negligible. Beyond the chromosphere is the solar corona which extends to infinity, and there the plasma density is very small \( n_e \approx 10^6 \text{ cm}^{-3} \). In between the chromosphere and the corona lies a thin transition region of thickness 500 km. Temperature varies in the vertical direction in a very peculiar manner. First, it decreases from 6600 K at the surface to 4400 K at a height of 500 km, and then it starts increasing up to \( 10^6 \text{ K} \) in the lower chromosphere. After that, it increases abruptly from \( 10^6 \text{ K} \) to \( (2 - 8) \times 10^7 \text{ K} \) in the transition region and attains the value of \((1 - 2) \times 10^6 \text{ K} \) in the lower corona (Priest 1982; Slemzin et al. 2014; Narain & Ulmschneider 1990; Klimchuk 2006; Tian 2017; Tian et al. 2018). In our opinion, the inhomogeneous density at the plasma plane and the vertical variation of temperature produce a thermal force to accelerate the plasma in the perpendicular direction. This phenomenon can occur in two-fluid plasma, magnetohydrodynamics (MHD), and neutral fluids. Charged particles also flow in an upward direction against Earth’s gravity in the ionosphere (Amatucci 1999). Recently, a new phenomenon named “campfires” has been observed in the solar corona by the Solar Orbiter, the joint solar mission of ESA and NASA launched on 10 February 2020 (Antolin et al. 2021). The solar corona contains many different types of magnetic tubes with hot plasma (Klimchuk & DeForest 2020; Litwin & Rosner 1993; Harra et al. 1992; Moses et al. 1997; Schrijver et al. 2019). Abrupt coronal plasma heating is a mystery for astrophysicists, and a great deal of work in this direction has appeared in the literature (Manobini et al. 2000; Xie et al. 2017; Hollweg & Sterling 1984; Ilonson 1983; Steinolfson & Davila 1993; Ofman et al. 1995; Poedts et al. 1989; Poedts et al. 1990; Poedts et al. 1994; Poedts & Goedbloed 1997; Vranjes & Poedts 2009; Vranjes & Poedts 2010; Saleem et al. 2012; Saleem et al. 2021). Plasma filaments in corona have different sizes (De Pontieu et al. 2007) and scale sizes of the order of 1 km in corona have been discussed (Woo 1996). The coronal loop structures are highly nonuniform and consist of threads and strands (Woo 1996; De Pontieu et al. 2007; Aschwanden & Peter 2017; De Pontieu et al. 2017; Goddard et al. 2017; Klimchuk & DeForest 2020).

Horizontal and vertical flows are very common in the solar atmosphere (Brekke et al. 1997a, 1997b; Pike & Harrison 1997; Chae et al. 1998; Peter & Judge 1999; Schrijver et al. 1999; Brekke 2000; Qiu et al. 2000; Zacharias et al. 2018; Johnston & Bradshaw 2019). Jet-like cylindrical plasma structures (spicules) flowing vertically are observed at any time in the chromosphere with a diameter of \((500-1000) \text{ km} \), heights of the order of \( 10^4 \text{ km} \), and upward velocities in the range \((20-30) \text{ km s}^{-1} \). The average number of spicules observed at solar disk any time is about half a million \((0.5) \times 10^6 \)
which becomes larger than the gravitational constant of electron limiting 2D version of the two-dimensional upward acceleration in the plasma slab of area.

Several years ago, an effort was made to find out an exact analytical solution of two-fluid plasma equations (Saleem 2010). It was shown that if the plasma density has an exponential-like profile in the $xy$-plane, and the temperature has inhomogeneity in the $yz$-plane, then the longitudinally uniform flow in plasma is created along with the magnetic field. The flow and magnetic fields become functions of $(x, y, t)$ coordinates. In Saleem 2010, the two-dimensional (2D) analytical solution of ideal classical plasma equations was presented to explain the generation of seed magnetic field and plasma vorticity by thermal forces using Cartesian geometry. In addition to numerical studies, it is very interesting to find out an exact three-dimensional (3D) analytical solution of two-fluid plasma and MHD equations in which the flow and magnetic fields can be expressed as functions of $(x, y, z, t)$ coordinates.

No standard mathematical procedure is available to obtain an exact nontrivial 3D solution of two-fluid plasma equations. However, one can still find out a 3D exact analytical solution of the set of nonlinear partial differential equations keeping in view the often-observed spatial dependence of density function, which has approximately exponential form. The Biermann battery effect was used to explain the generation of seed magnetic field in stars (Biermann 1950), galaxies (Widrow 2002; Lazarian 1992), as well as in classical laser plasmas (Brueckner & Jorna 1974) using a one-dimensional exponential density profile. In the basic mechanism, ions were assumed to be static, and nonparallel density and temperature gradients of electrons were considered to be the source for the generation of the magnetic field. However, computer simulations have taken into account the ion dynamics and several other effects as well. Here we present an exact 3D solution of two-fluid plasma equations to explain the fundamental mechanism behind the creation of velocity, magnetic field, and ordered structures. This solution is also applicable to the single-fluid plasma model (MHD) for the generation of vorticity.

The formation of solar spicules is also discussed using the limiting 2D version of the two-fluid plasma model. Exponential-like density profile in the $xy$-plane, and the linear variation of electron–ion temperatures along the $z$-axis, produce an upward acceleration in the plasma slab of area $xy$ and height $h$ which becomes larger than the gravitational constant $g_{\odot}$ at the Sun’s atmosphere, giving rise to spicules in the solar chromosphere. A more general 3D exact solution of neutral fluid equations is presented as well.

2. Exact 3D Solutions of Plasma and Fluid Equations

2.1. Two-fluid Plasma

Let us consider the two-fluid ideal plasma in which $\Omega_e = \frac{eB}{m_e c}$ is the electron gyro frequency, $\omega_{pe} = \left(\frac{4\pi e^2 n_e}{m_e}\right)^{1/2}$ is the electron plasma oscillation frequency, and $\lambda_{De} = \left(\frac{T_i}{4\pi n_e e^2}\right)^{1/2}$ is the electron Debye length. Our aim is twofold. First is to find out the spatial dependence of density $n$ and temperatures $T_j$ where $j = e, i$ on $(x, y, z)$ coordinates which should be the exact analytical solution of the two-fluid plasma equations and it should also determine the structural forms and time evolution of flow $v_j$ and magnetic $B$ fields. Second is to use the 2D density spatial profile and 1D spatial dependence of electron (ion) temperatures $T_j(T_i)$ which give an upward acceleration to a plasma slab to produce spicules in the solar chromosphere. The latter will be discussed in Section 3. The two-fluid momentum conservation equations for electrons and singly charged ions in the gravitational field $g$ can be expressed as

$$m_e n_e (\partial_i + v_i \cdot \nabla) v_i = -en_e \left(\frac{E}{c} + \frac{1}{c} v_i \times B\right) - \nabla p_e + m_e n_e g$$

$$m_i n_i (\partial_i + v_i \cdot \nabla) v_i = en_i \left(\frac{E}{c} + \frac{1}{c} v_i \times B\right) - \nabla p_i + m_i n_i g$$

where all the physical quantities have standard definitions. Continuity equations for electrons and ions are

$$\partial_i n_j + \nabla \cdot (n_j v_j) = 0.$$

We assume that the plasma has time-independent gradients of density and temperatures at $t = 0$ and evolves with time creating electromagnetic and flow fields. The pressure gradient forces on the right-hand sides of Equations (1) and (2) are the source terms for the generation of vorticities and the magnetic field. We will find a 3D exact analytical solution of these equations which can be used to explain a number of plasma phenomena without dropping any term in Equations (1) and (2). In addition, we will also show that in the case of 2D plasma, the baroclinic vectors $[\nabla n \times (\nabla T_e + \nabla T_i)]$ will produce acceleration to the ions in the direction perpendicular to the 2D plasma disk, and the electron baroclinic vector will be responsible for generating the magnetic field $\nabla n \times \nabla T_e$. Both of these phenomena are coupled. The flow velocities of electrons and ions are also coupled through Ampere’s law.

Before discussing the 3D and 2D exact analytical solutions of Equations (1) and (2), it seems important to describe the Biermann battery mechanism, which is widely used to investigate the seed magnetic field generation in stars and galaxies as well as in laser-produced classical plasmas using two-fluid equations. The same mechanism is also described by using the equations of single-fluid MHD which will be discussed in Section 2.3.

In 1950, Biermann proposed a mechanism for seed magnetic field generation in stars assuming the plasma has time-independent nonuniform profiles of density and temperatures at time $t = 0$ and the magnetic field is produced during $t: 0 \rightarrow \tau$. It was assumed that $\tau$ lies in between the ion and electron oscillation time periods, i.e., $\omega_{pe}^{-1} \ll \tau \ll \omega_{pi}^{-1}$, where $\omega_{pi} = \left(\frac{4\pi e^2 n_e}{m_i}\right)^{1/2}$ is the plasma oscillation frequency of the jth species. Under this approximation, the electron inertia is ignored and ions are assumed to be static. The forces on the right-hand side of Equation (2) are assumed to balance each other to maintain ion flow at zero. In the limit $m_e \rightarrow 0$, the electron equation of motion becomes

$$0 = -en_e (v_e \times B) + \nabla p_e.$$
pressure, \( p_e = n_e T_e \), along with Faraday’s law, the curl of Equation (4) yields
\[
\partial_t \mathbf{B} = \frac{c}{e} \left( \frac{\nabla T_e \times \nabla n_e}{n_e} \right).
\] (5)
Integrating for \( t: 0 \rightarrow \tau \), one obtains
\[
\mathbf{B}(t) = \frac{c}{e} \left( \frac{\nabla T_e \times \nabla n_e}{n_e} \right) \tau.
\] (6)
Using Equation (6), the magnitude of seed magnetic fields in galaxies has been estimated as follows. Let the galactic seed field be \( \Delta \mathbf{B} \), generation time \( \Delta t \), the thickness of the clump of gas cloud \( h = 10^2 \) pc, and galaxy radius \( R = 10^7 \) pc. Here pc denotes parsec = \( (3.09) \times 10^{18} \) km. Then Equation (6) can be expressed as (Lazarian 1992)
\[
\Delta B = \frac{e k_B}{
\ln n}
\tau (\Delta T_e \Delta t)
\] (7)
where \( S = h \times R \) is the area of the cloud, \( \Delta T_e \) is the temperature difference from one end to the other end of the gas clump, and \( k_B \) is the Boltzmann constant. The author used \( \Delta T_e \approx 10^6 \) K, and the scale length of exponential density variation was \( \Sigma n / n = h^{-1} \).
In Equation (7), the temperature has units in Kelvin. Approximating \( \nabla T_e \sim \frac{\Delta T_e}{R} \) and \( \tau = 10^5 \) yr, Equation (7) gives \( |B| \approx 3 \times 10^{-17} \) G (Lazarian 1992).
Long ago in Brueckner & Jorna (1974), Equation (4) was also applied to estimate the magnetic field generation in classical laser plasma experiments. Assuming \( \nabla n_e = \hat{x} \frac{dn_e}{dx} \), \( \nabla T_e = \hat{y} \frac{dT_e}{dx} \), and the scale lengths of gradients to be \( L_n = \frac{dn_e}{n_e dx}^{-1} \) and \( L_T = \frac{dT_e}{T_e dx}^{-1} \), Equation (6) can be written as
\[
\mathbf{B} = \left\{ \frac{c}{e} \left( \frac{T_e}{L_n L_T} \right) \right\}^\xi.
\] (8)
Here \( \xi \), \( \hat{y} \), \( \hat{z} \) are unit vectors. In the initial classical laser plasma experiments, the parameters were \( T_e = 1eV \), \( n_e = 10^{20} \) cm\(^{-3} \), and ion sound speed \( c_s = 3 \times 10^7 \) cm s\(^{-1} \). Then assuming \( L_n \approx L_T \approx 0.005 \) cm, the magnitude of the magnetic field turned out to be \( |B| \approx 0.6 \times 10^6 \) G, which was in agreement with the observations of Brueckner & Jorna (1974).
Note that in the experiment \( \omega_p^3 \approx 10^{-13} \) while \( \tau = (1.66) \times 10^{-11} \) s. This is a contradiction to the initial assumption that \( \tau < \omega_p \). Furthermore \( \tau \) has been \( 10^9 \) yr in estimating the magnitude of seed magnetic field in galaxies. Considering the aforementioned weaknesses in the application of the Biermann battery mechanism, it was suggested that ion dynamics must be included in the theory of magnetic field generation (Saleem 1996, 2010).
In the present theoretical model, we will keep all the terms and find out an exact 3D analytical solution of Equations (1–3) with the help of Maxwell’s equations. The baroclinic vectors \( \nabla \psi \times \nabla T_e \) become the source terms for the generation of magnetic field and vorticities. The interesting point is that all the nonlinear terms vanish due to the chosen spatial profiles of \( \psi \) and \( T_e \). In the case of 2D plasma, the baroclinic vectors produce an acceleration in the direction perpendicular to the plasma disk and give rise to a jet-like flow. The gradients of density and temperatures create time-dependent magnetic \( \mathbf{B} \) and flow fields \( \mathbf{v} \). We assume longitudinally uniform flows of electrons and ions \( \nabla \cdot \mathbf{v} = 0 \) along with \( \partial n = 0 \). Quasi-neutrality \( n_e \approx n_e = n \) is used under assumption \( |\nabla n / n| \ll 1 \), and electron inertia is ignored under the assumption \( |\partial n / n| < \omega_p \). The dynamics of the two-fluid plasma can be described by taking curls of electron and ion equations of motion expressed, respectively, as Saleem (2010),
\[
\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \left( \frac{c}{4\pi e} \right) |\nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B})| + \frac{c}{4\pi e} |\nabla \psi \times ((\nabla \times \mathbf{B}) \times \mathbf{B})| - \frac{c}{e} (\nabla \psi \times \nabla T_e)
\] (9)
and
\[
\frac{e}{m_i c} \partial_t \mathbf{B} + \partial_t (\nabla \times \mathbf{v}_i) = \nabla \times [\mathbf{v}_i \times (\nabla \times \mathbf{v}_i)] + \frac{e}{m_i c} \nabla \times (\mathbf{v}_i \times \mathbf{B}) + \frac{1}{m_i} (\nabla \psi \times \nabla T_e).
\] (10)
where \( \psi = \ln \bar{n}, \bar{n} = \frac{e}{N_0}, \) and \( N_0 \) is the arbitrary constant density. The equation of state for ideal classical gas is \( p_j = n_j T_j \) where temperature \( T_j \) has units of energy. Also, the mass conservation requires, \( \nabla \psi \cdot \mathbf{v}_j = 0 \) and Ampere’s law gives us
\[
\mathbf{v}_e = \mathbf{v}_i - \frac{c}{4\pi e} \left( \frac{\nabla \times \mathbf{B}}{n} \right).
\] (11)
Electron velocity is coupled with ion velocity and also depends upon the curl of the magnetic field and the density in Equation (11). Thus for longitudinally uniform flows, we also require
\[
\nabla \psi \cdot (\nabla \times \mathbf{B}) = 0.
\] (12)
If all nonlinear terms of Equations (9) and (10) vanish then they reduce, respectively, to
\[
\partial_t \mathbf{B} = -\frac{c}{e} (\nabla \psi \times \nabla T_e)
\] (13)
and
\[
\frac{e}{m_i c} \partial_t \mathbf{B} + \partial_t (\nabla \times \mathbf{v}_i) = \frac{1}{m_i} (\nabla \psi \times \nabla T_e).
\] (14)
Equations (13) and (14) give an expression for the generation of ion vorticity by baroclinic vectors
\[
\partial_t (\nabla \times \mathbf{v}_i) = \frac{1}{m_i} (\nabla \psi \times (\nabla T_e + \nabla T_i)).
\] (15)
Equation (14) indicates that \( \mathbf{B} \) and \( \nabla \times \mathbf{v}_i \) can be parallel to each other and our solutions will show that this is true. Therefore, for mathematical convenience, we assume
\[
\mathbf{B} = b_0 (\nabla \times \mathbf{v}_i)
\] (16)
where \( b_0 \) is a constant. Let us look for an exact 3D analytical solution of the two-fluid plasma equations. If we find a suitable function \( \mathbf{v}_i(x, y, z, t) \) which gives
\[
\nabla \times [\mathbf{v}_i \times (\nabla \times \mathbf{v}_i)] = 0
\] (17)
then due to Equation (16), the following expression must also be true:
\[ \nabla \times [v_i \times B] = 0. \]  
(18)

Equations (13) and (14) are valid if, in addition to Equations (17) and (18), the following conditions also hold:
\[ \nabla \times [B \times (\nabla \times B)] = 0 \]  
(19)
and
\[ \nabla \psi \times [B \times (\nabla \times B)] = 0. \]  
(20)

Conditions Equations (19) and (20) are satisfied if
\[ \nabla^2 v_i = \eta v_i \]  
(21)
where \( \eta \) is a constant.

Interestingly all the conditions on plasma fields in Equations (17–21) are satisfied if \( \psi(x, y, z) \) and \( T_f(x, y, z) \) have the following spatial dependence on \( (x, y, z) \) coordinates (see Appendix A):
\[ \psi(x, y, z) = \psi_0 [e^{\xi (x-y+z)} + e^{\xi (x+y+z)}] \]  
(22)
\[ T_f(x, y, z) = T_{0f} (x + y + z) + T_{00j} \]  
(23)
where \( T_{0j} = \frac{dT_0}{dx} = \frac{dT_0}{dy}, \psi_0, T_{00j} \) and \( \xi \) are assumed to be constant. Let us define the time evolution of the system by function
\[ f = C_0 f(t) = C_0 t \]  
(24)
where \( C_0 \) is a constant and we use \( C_0 = 1 \) for simplicity. Then ion velocity is determined by Equation (15) as a function of \( (x, y, z, t) \) (see Appendix A).
\[ v_i(x, y, z, t) = a_0 \psi_0 [e^{\xi (x-y+z)} + e^{\xi (x+y+z)} + e^{\xi (x-y+z)}] (1, 1, 1) t = a t = a_0 (\psi, \psi, \psi) t \]  
(25)
and the magnetic field is given by Equation (13),
\[ B(x, y, z, t) = \left( \frac{\lambda_0}{c} \right) T_{0j} \psi_0 [(-2e^{\xi (x-y+z)}) + e^{\xi (x+y+z)} + e^{\xi (x+y+z)} + e^{\xi (x+y+z)}] (1, 1, 1) t \]  
(26)
where \( a_0 = \left( T_{0j} \psi_0 / m_e \right) \) is constant. Note that \( \nabla \psi \cdot (\nabla \times B) = 0 \) and \( \nabla \cdot B = 0 \) are also satisfied. Thus we notice that the highly nonlinear set of coupled partial differential equations is solved exactly if density function \( \psi \) and temperatures \( T_f \) have the spatial profiles given by Equations (22) and (23), respectively.

2.2. Neutral Fluid

Since ideal neutral fluid dynamics are governed by a simpler set of partial differential equations, a more general form of density function gives an exact 3D solution. Let us consider the following equations:
\[ \rho (\partial_t + v \cdot \nabla) v = -\nabla p \]  
(27)
and
\[ \partial_t \rho + \nabla \cdot (\rho v) = 0 \]  
(28)
where \( \rho = \text{mn}, n \) is the number density, and \( m \) is the mass of the fluid particle. We again assume the flow to be longitudinally uniform \( \nabla \cdot v = 0 \) with \( \partial_t \rho = 0 \) and define the density function \( \psi = \ln \rho \), where \( \rho = \frac{\rho_0}{\rho_0} \) and \( \rho_0 \) is some constant density. The curl of Equation (27) is
\[ \partial_t (\nabla \times v) = \frac{1}{m} (\nabla \psi \times \nabla T) \]  
(29)
The condition \( \partial_t \rho = 0 \) also demands \( \nabla \psi \cdot v = 0 \). Let \( \alpha = (-\lambda_1 x - \mu_1 y + v_1 z), \beta = (-\lambda_2 x - \mu_2 y + v_2 z), \) and \( \gamma = (-\lambda_3 x - \mu_3 y + v_3 z) \) where \( \nu_k = \lambda_k + \mu_k \) and \( k = 1, 2, 3 \). If \( \psi = \psi(x, y, z) \) and \( T = T(x, y, z) \) have the following forms in 3D space:
\[ \psi(x, y, z) = \psi_0 (e^{\alpha} + e^{\beta} + e^{\gamma}) \]  
(30)
\[ T(x, y, z) = T_0 (x + y + z) + T_00 \]  
(31)
where \( T_0' \) and \( T_{00} \) are constants, then the nonlinear term of Equation (29) vanishes (see Appendix B) and it reduces to
\[ \partial_t (\nabla \times v) = \frac{1}{m} (\nabla \psi \times \nabla T). \]  
(32)
Here \( \psi_0, \lambda_k, \mu_k, \nu_k \) and \( T_0' = \frac{dT_0}{dx} = \frac{dT_0}{dy}, \frac{dT_0}{dz} \) are constants.

The following form of the velocity field is generated:
\[ v(x, y, z, t) = a_0 \psi (1, 1, 1) f(t) \]  
(33)
which satisfies Equation (32) where \( a_0 = \frac{T'}{m} \) is constant.

2.3. Magnetohydrodynamics

The momentum conservation equation for a single-fluid plasma, the MHD, is
\[ \rho (\partial_t + v \cdot \nabla) v = -\frac{1}{c} \times \frac{B}{B} - \nabla p + \rho g. \]  
(34)
Here \( v \) is the bulk plasma velocity, \( \rho = \text{nm} \), and \( m = m_v + m_p \). Generalized Ohm’s law is:
\[ E + \frac{1}{c} v \times B = \frac{1}{\sigma} \frac{1}{en} \left( \frac{1}{c} \times B - \nabla p \right). \]  
(35)
Assuming collisionless plasma, the first term on the right-hand side of Equation (35) is ignored. The Biermann battery mechanism for the generation of a magnetic field can also be considered using MHD equations (Boyd & Sanderson 2003). The terms containing \( B \) are neglected for \( B = 0 \) at \( t = 0 \). Plasma bulk motion is also neglected assuming that the forces on the right-hand side of Equation (34) balance each other. Then Ohm’s law Equation (35) reduces to Equation (4) which has been obtained through the two-fluid plasma equations. But the Biermann mechanism does not take into account the creation of plasma vorticity. Here we show that the MHD equations also predict the generation of bulk plasma flows with nonzero...
vorticity $\nabla \times \mathbf{v} = 0$. The curl of Equation (34) gives
\[
\frac{1}{4\pi \rho} \nabla \times \left[ \mathbf{v} \times (\nabla \times \mathbf{v}) \right] - \frac{1}{4\pi \rho} \nabla \times \left[ (\nabla \times \mathbf{B}) \times \mathbf{B} \right] + \frac{1}{4\pi \rho} \nabla \psi \times (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\rho} (\nabla \psi \times \nabla \rho).
\] (36)

Nonlinear terms in Equation (36) vanish if $\psi$ is given by Equation (22) and $T$ is given by Equation (31). Then Equation (36) reduces to
\[
\frac{\partial}{\partial t} (\nabla \times \mathbf{v}) = \frac{1}{\rho} (\nabla \psi \times \nabla \rho).
\] (37)

If we use the form of $\psi$ defined by Equation (22) along with the temperature profile given by Equation (31) then Equation (37) yields the following form of plasma flow:
\[
\mathbf{v}(x, y, z, t) = \left( \frac{T_0}{\alpha T_{00}} \right) \psi_0 \left( e^{\mathcal{E}(y+z)} + e^{\mathcal{E}(-x+z)} \right) + e^{\mathcal{E}(-y+z)}(1, 1, 1) t.
\] (38)

Thus we conclude that magnetic field and plasma vorticity are generated by the baroclinic vector simultaneously in a single-fluid plasma model as well.

### 3. Creation of Solar Spicules

Plasma in these structures has the predominant flow in the upward direction perpendicular to the surface. For $\alpha = (\mu x + \nu y)$, $\beta = (\mu x - \nu y)$, and neglecting the $\gamma$ term, the $\psi = \psi(x, y)$ function becomes independent of the $z$ coordinate:
\[
\psi(x, y) = \psi_0 (e^{\mu x + \nu y} + e^{\mu x - \nu y}).
\] (39)

Let us assume that temperatures $T_j$ vary linearly only along the $z$-axis as
\[
T_j(z) = T'_0(z) + T_00 j
\] (40)
where $T_00$ and $T'_0 = \frac{dT_0}{dz}$ are constants. Use of the above profiles of $\psi$ and $T_j$ in Equation (13) yields
\[
\mathbf{B}(x, y) = -\mathbf{e} \int_0^{z} \frac{\partial}{\partial \psi} (\partial \psi, -\partial \psi, 0) f(t) = B_g f(t)
\] (41)
where $B_g = \mathbf{B}(x, y)$ is the spatial part of the generated field, which has the following form:
\[
B_g(x, y) = \frac{\mathbf{eT}_0}{\mu} \frac{\mu}{\mu} \left[ \frac{\nu}{\mu} \left( e^{\mu x + \nu y} - e^{\mu x - \nu y} \right) \times (e^{\mu x + \nu y} + e^{\mu x - \nu y}) \right] 0.\]
(42)

Then Equation (15) gives ion plasma vorticity
\[
\frac{\partial}{\partial t} (\nabla \times \mathbf{v}_i) = a_0 \psi_0 \mu \left[ \frac{\nu}{\mu} \left( e^{\mu x - \nu y} - e^{\mu x + \nu y} \right) \times (e^{\mu x - \nu y} + e^{\mu x + \nu y}) \right] 0.
\] (43)

Here $a_0$ has the same definition as has been mentioned below Equation (25). Equation (43) determines the unidirectional flow in $z$-direction perpendicular to the plasma surface given by
\[
\mathbf{v}_i(x, y) = [(a_0 \psi)(x, y)] \hat{z} f(t) = at f(t) = at
\] (44)
where $a(x, y) = a_0 \psi(x, y)$ is the upward ion acceleration. Let us assume that the spicules are formed by many small slabs created one by one due to gradients of density and temperatures at the bottom of these structures. These slabs move upward due to the acceleration produced by gradients, and as they reach upper regions the temperature gradient approaches zero and velocity becomes constant, as observed. Since spicules have a strong ambient magnetic field as well, therefore we add a constant magnetic field to the weak magnetic field generated by the baroclinic vectors and write the total field within the structures $\mathbf{B}_{st}$ as
\[
\mathbf{B}_{st} = B_g(x, y, t) + \mathbf{B}_0.
\] (45)

The $\mathbf{B}_0 = B_0 \hat{z}$ is the constant external magnetic field produced at the footprints of the structures by physical mechanisms operative at the solar surface or below. Note that the 2D theoretical model remains valid when we express the total magnetic field in the form of Equation (45).

We will show now that for a suitable choice of numerical values in our model, the acceleration of the plasma can be larger than the solar surface gravity and that the height of the solar spicules turns out to be closer to the heights observed. For the plasma slab we assume $x: 0 \rightarrow x_m, y: 0 \rightarrow y_m$ and choose height $h = z_2 - z_1$ in the direction perpendicular to the 2D plasma in the $xy$-plane. Forms of $\psi(x, y)$ and $n(x, y)$ for our choice of numerical values $x_m = 0.5$ and $y_m = 0.7$ are shown in Figure 1. Let $T_e \sim T_i \sim T$ and consider the slab formed at the bottom of the spicule with dimensions $x_m = 3 \times 10^7 \text{cm}$, $y_m = 4 \times 10^7 \text{cm}$, and $h = 3 \times 10^7 \text{cm}$. We consider $T = 3 \times 10^7 \text{K}$ in the upper region of the solar surface ($z = z_1$) and $T = 5 \times 10^7 \text{K}$ in the lower chromosphere ($z = z_2$). Approximating $T' = \frac{\int (T_0 - T_00)}{h}$, we find $\frac{a_0}{\mathbf{e}} = \frac{a_0}{\mathbf{e}} = \frac{3.3}{4} \times 10^4 \text{cm}^{-2}$ and hence $g_0 < a_0$. Velocities at different points can be expressed as $v(x, y) = \frac{1}{2} \left( a(x, y) - g_0 \right) \tau$ where $\tau$ is the evolution time of the inhomogeneous plasma slab. Choosing $\tau = 10 \text{ s}$, the magnitude of initial velocities turns out to be $v(x_m, y_m) = 7.5 \times 10^4 \text{cm}^{-1} \text{s}^{-1}$ and $\tau(0, 0) = 3.63 \times 10^4 \text{cm}^{-1} \text{s}^{-1}$. Later the solar attraction produces deceleration and the structure height is limited by this process. If we consider the constant velocity magnitude to be $5 \times 10^7 \text{ s}^{-1}$, then during the lifetime $\tau_L = 300 \text{ s}$, the spicule attains the height $H = 1.4 \times 10^8 \text{ cm}$. The magnetic field produced by baroclinic vectors is too small of the order of $10^{-7} \text{ G}$ compared to the ambient magnetic field $B_0 = 2 \text{ G}$ which is produced at the footprints of the structure on the solar surface. The form of velocity function in the solar spicule is shown in Figure 2, and the formation of spicules is schematically shown in Figure 3. The baroclinic vectors $\nabla \psi \times (\nabla T_i + \nabla T_s)$ act as the source for producing ions acceleration in the upward direction $\mathbf{a}$. Ions can flow in the upward direction when $|\mathbf{g} \cdot \mathbf{e}| < |\mathbf{a}|$. Since ions velocity and the curl of the magnetic field are parallel, Equation (11) shows that electrons follow the ion motion along the $z$-axis.

### 4. Summary

The exact 3D analytical solutions of ideal two-fluid plasma, MHD, and neutral fluid equations have been presented using Cartesian geometry. We have assumed an exponential-like spatial profile for density in 3D and linear dependence of ion
and electron temperatures on \((x, y, z)\) coordinates with \(T_i \neq T_e\). The velocity \(v_j\) and magnetic \(B_g\) fields evolve with time and become functions of \((x, y, z, t)\) coordinates. One can possibly obtain several similar solutions by varying signs of exponential functions in the definition of \(\psi\). The formation of solar spicules has been discussed using 2D version of this model. These structures disappear at higher altitudes where the temperature gradient vanishes.

It is very interesting that similar spicule structures can also be obtained using MHD Equation (37) or neutral fluid Equation (32), but in these cases the very weak magnetic field \(B_g\) will not be associated with the vorticity generation. This indicates that the nonparallel density and temperature gradients \(\nabla \psi \times \nabla T_j \neq 0\) mainly produce an acceleration in both plasma and neutral fluid. This acceleration turns out to be in the direction perpendicular to the 2D material when temperatures vary only in a vertical direction. The Biermann battery effect has a small contribution of electron baroclinic vector while it creates large acceleration when dynamics of the whole system of two-fluid plasma is considered in detail. If \(\nabla \times v_j = 0\) is assumed, then Equations (13) and (14) show that in this case either plasma attains equilibrium \((T_i = T_e)\) with \(\partial_j B_g = 0\) or \(\nabla T_e = -\nabla T_i\). The forms of \(\psi\) and \(T_j\) given in Equations (14) and (15), respectively, are also applicable to ideal MHD.

The 3D and 2D models discussed above can be applied to explain the formation of several structures emerging in natural and laboratory environments. The formation of solar spicules, solar coronal loops, and prominences can all be explained on the basis of this theoretical model. Furthermore, this model is also useful for understanding the mechanisms of coronal mass ejection and the formation of jets ejected from astrophysical objects consisting of classical plasmas and neutral fluids.

Appendix A

Two-fluid Plasma: \(\nabla \times [v_i \times (\nabla \times v_j)] = 0\)

Here we show that with the spatial functions \(\psi\) and \(T_j\) defined in Equations (22) and (23), the ions velocity and magnetic field are forced to have the forms given in Equations (25) and (26), respectively, and consequently the
nonlinear term $\nabla \times [v_i \times (\nabla \times v_i)]$ vanishes. To write equations in a simpler form, let us define $S_1 = (-y + z)$, $S_2 = (-x + z)$, and $S_3 = (-x + y)$. Then the function $\psi$ can be expressed as,

$$\psi = \psi_0 [e^{S_1} + e^{S_2} + e^{S_3}].$$  \hfill (A1)

The cross product of $\nabla \psi$ and $\nabla T_j$ becomes

$$(\nabla \psi \times \nabla T_j) = T_0' \left( (\partial_\psi \psi - \partial_\psi \psi) \right) \times \left( (\partial_\psi \psi - \partial_\psi \psi) \right),$$

or

$$(\nabla \psi \times \nabla T_j) = (\xi \psi_0 T_0') \left[ (-2e^{S_1} + e^{S_2} + e^{S_3}), \times (2e^{S_2} + e^{S_3} + e^{S_3}), \times (-2e^{S_1} - e^{S_2} + e^{S_3}) \right].$$  \hfill (A2)

Using Equation (A3) in Equation (13), we obtain,

$$B = \left( -\frac{c}{e} \xi \psi_0 T_0' \right) \left[ (-2e^{S_1} + e^{S_2} + e^{S_3}), \times (2e^{S_2} + e^{S_3} + e^{S_3}), \times (-2e^{S_1} - e^{S_2} + e^{S_3}) \right] f(t).$$  \hfill (A4)

If the ion flow velocity is expressed as $\nu = a_0 (\psi \tilde{x} + \psi \tilde{y} + \psi \tilde{z}) f(t)$ then one can easily show that $\nabla \cdot \nu = 0$. Ion vorticity becomes

$$(\nabla \times \nu) = a_0 \left[ (\partial_\psi \psi - \partial_\psi \psi), (\partial_\psi \psi - \partial_\psi \psi) \right] f(t).$$  \hfill (A5)

It shows that ion vorticity $\nabla \times \nu$ is parallel to $B$ which is parallel to the electron borusvector $\nabla \times \nabla T_0$. Note that $\nabla^2 \psi = 2\xi^2 \psi$, which yields $\nabla^2 \nu = 2\xi^2 \nu$, and hence $\nabla \times \nu = -b_0 (2\xi^2 \nu)$. Thus $\nu$ turns out to be parallel to $\nabla \times B$. Equation (22) yields

$$\psi^2 = \psi_0^2 \left[ (2e^{S_2} + e^{S_3} + e^{S_2}), \times (2e^{S_3} + e^{S_3} + e^{S_3}) \right] \right].$$  \hfill (A6)

which gives

$$[\psi \times \nabla \times \psi] = \frac{1}{2} \left( a_0 T_0 \right) \left[ (2\partial_\psi \psi^2 - \partial_\psi \psi^2 - \partial_\psi \psi^2), \times (2\partial_\psi \psi^2 - \partial_\psi \psi^2 - \partial_\psi \psi^2), \times (2\partial_\psi \psi^2 - \partial_\psi \psi^2 - \partial_\psi \psi^2) \right].$$  \hfill (A7)

Using (A6) in (A7) and following straightforward but cumbersome algebra, one can easily show that $\nabla \times \left[ \psi \times (\nabla \times \nu) \right] = 0$.

**Appendix B**

**Neutral Fluid:** $\nabla \times [v \times (\nabla \times v)] = 0$

Equation (30) gives

$$\nabla \psi = \psi_0 (-\lambda_1 e^\alpha - \lambda_2 e^\beta - \lambda_3 e^\gamma),$$

$$\times (-\mu_1 e^\alpha - \mu_2 e^\beta - \mu_3 e^\gamma),$$

$$\times (\nu_1 e^\alpha + \nu_2 e^\beta + \nu_3 e^\gamma).$$  \hfill (B1)

Equations (B1) and (31) yield

$$(\nabla \psi \times \nabla T) = \psi_0 T_0' \left[ (-\mu_1 + \nu_1) e^\alpha - (\mu_2 + \nu_2) e^\beta$$

$$- (\mu_3 + \nu_3) e^\gamma, (\nu_1 + \lambda_1) e^\alpha$$

$$+ (\nu_2 + \lambda_2) e^\beta + (\nu_3 + \lambda_3) e^\gamma, \times (\mu_1 - \lambda_1) e^\alpha + (\mu_2 - \lambda_2) e^\beta + (\mu_3 - \lambda_3) e^\gamma \right].$$  \hfill (B2)

Then using Equation (33), we obtain

$$\nabla \cdot \nu = a_0 f(t) \psi_0 \left[ (-\lambda_1 - \mu_1 + \nu_1) e^\alpha$$

$$+ (-\lambda_2 - \mu_2 + \nu_2) e^\beta + (-\lambda_3 - \mu_3 + \nu_3) e^\gamma \right]$$

and

$$\nabla \cdot \nu = \psi_0 a_0 f(t) \left[ e^\alpha + e^\beta + e^\gamma \right] \left[ (\nu_1 - \lambda_1 - \mu_1) e^\alpha$$

$$+ (\nu_2 - \lambda_2 - \mu_2) e^\beta + (\nu_3 - \lambda_3 - \mu_3) e^\gamma \right].$$  \hfill (B3)

For $\nu_k = \lambda_k + \mu_k$ where $k = 1, 2, 3$, we have $\nabla \cdot \nu = 0$ and $\nabla \cdot \nu = 0$. Form of velocity Equation (33) gives

$$\nabla \times \nu = a_0 f(t) \left[ (-\mu_1 + \nu_1) e^\alpha$$

$$- (\mu_2 + \nu_2) e^\beta - (\mu_3 + \nu_3) e^\gamma \right]$$

$$\left[ (\nu_1 + \lambda_1) e^\alpha + (\nu_2 + \lambda_2) e^\beta + (\nu_3 + \lambda_3) e^\gamma \right],$$

$$\left[ (\mu_1 - \lambda_1) e^\alpha + (\mu_2 - \lambda_2) e^\beta - (\mu_3 + \lambda_3) e^\gamma \right].$$  \hfill (B5)

Let $e_1 = (e^\alpha + e^{(\alpha + \beta)} + e^{(\alpha + \gamma)}), e_2 = (e^\beta + e^{(\alpha + \beta)} + e^{(\beta + \gamma)})$, $e_3 = (e^\gamma + e^{(\alpha + \gamma)} + e^{(\beta + \gamma)})$. Then we can write

$$\nu \times (\nabla \times \nu) = 3 a_0^2 \left[ (-\lambda_1 e_1 - \lambda_2 e_2 - \lambda_3 e_3), \times (-\mu_1 e_2 - \mu_2 e_2 - \mu_3 e_3), \times (\nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3) \right].$$  \hfill (B6)

Again following a little laborious but straightforward algebra, we obtain from Equation (33)

$$\nabla \times (\nabla \times \nu) = 0.$$  \hfill (B7)

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