On weakly equational Noetherian direct powers of groups and semigroups

A N Shevlyakov

1 Sobolev Institute of Mathematics, Pevtsova 13, Omsk, 644099, Russia
Omsk State Technical University, pr. Mira 11, Omsk, 644050, Russia
E-mail: a_shev10@mail.ru

Abstract. We study Diophantine equations over direct powers of groups and semigroups and prove that any direct power of a finite semigroup (group) with zero (resp. with the equational domain property) is weakly equationally Noetherian.

1. Introduction
There are many obvious connections between the equations over a certain algebraic structure $A$ and equations over its direct power $\Pi A$. Namely, the solution set of an equation over $\Pi A$ is a product of the solutions sets over the isomorphic copies of $A$.

However, properties of infinite systems of equations may be completely different over $\Pi A$ and $A$. In particular, any infinite system of equations over a finite algebraic structure $A$ is equivalent to a finite system of equations (the weakly equationally Noetherian property), but there are infinite systems of equations $S$ over the direct power $\Pi A$ such that $S$ is not equivalent to any finite system over $\Pi A$.

In the current paper we study weakly equationally Noetherian direct powers of finite algebraic structures and prove that any direct power of a finite semigroup (group) with zero (resp. with the equational domain property) is weakly equationally Noetherian (see Theorems 3.4 and 4.2).

2. Basic notions
Let $\mathcal{L}$ be a language. Below we will consider the following languages:

$$
\mathcal{L}_s = \{\cdot\} \text{ semigroup language},
\mathcal{L}_g = \{\cdot, -1, 1\} \text{ group language},
\mathcal{L}_r = \{+, -, \cdot, 0\} \text{ ring language}.
$$

An equation over $\mathcal{L}$ ($\mathcal{L}$-equation) is an atomic formula of $\mathcal{L}$. Let us give the examples of equations in various languages:

$$
x_1 x_2 = x_2 x_1, \quad x_2 x_1 = x_1 x_3 \text{ language } \mathcal{L}_s,
x_1^{-1} x_2^{-1} x_1 x_2 = 1, \quad x_1^{-1} x_2 x_1 = x_3 \text{ language } \mathcal{L}_g,
x_1^2 + x_2^2 = 0, \quad x_1 - x_1 x_2 - x_3 x_2 x_1 = 0 \text{ language } \mathcal{L}_r.
$$

1 Present address: Sobolev Institute of Mathematics, Pevtsova 13, Omsk, 644099, Russia
A system of \( \mathcal{L} \)-equations (system, for shortness) is an arbitrary set of equations. The solution set of a system \( \mathcal{S} \) over an algebraic structure \( \mathcal{A} \) is denoted by \( \mathcal{V}_\mathcal{A}(\mathcal{S}) \). An algebraic structure \( \mathcal{A} \) of a language \( \mathcal{L} \) is weakly equationally Noetherian iff any infinite system \( \mathcal{S} \) is equivalent to an appropriate finite system \( \mathcal{S}' \). \( \mathcal{L} \)-systems \( \mathcal{S}_1, \mathcal{S}_2 \) are called equivalent over an \( \mathcal{L} \)-algebraic structure \( \mathcal{A} \) if \( \mathcal{V}_\mathcal{A}(\mathcal{S}_1) = \mathcal{V}_\mathcal{A}(\mathcal{S}_2) \). A set \( \mathcal{Y} \subseteq \mathcal{A}^n \) is algebraic if there exists an \( \mathcal{L} \)-system \( \mathcal{S} \) with \( \mathcal{V}_\mathcal{A}(\mathcal{S}) = \mathcal{Y} \).

Let us define the language extension as follows: let \( \mathcal{L} \) be a language and \( \mathcal{A} \) be an algebraic structure of the language \( \mathcal{L} \). Then

\[
\mathcal{L}(\mathcal{A}) = \mathcal{L} \cup \{ a \mid a \in \mathcal{A} \}
\]

is the new language extended by constants that correspond to all elements of \( \mathcal{A} \). Thus, one can consider the languages \( \mathcal{L}_s(\mathcal{S}), \mathcal{L}_s(G), \mathcal{L}_s(R) \) (where \( S, G, R \) is a semigroup, group, ring respectively) and give the examples of equations over such languages:

\[
\begin{align*}
  x_1x_2 &= x_2x_1, & x_2x_1 &= x_1x_3 \text{ language } \mathcal{L}_s, \\
  x_1^{-1}x_2^{-1}x_1x_2 &= 1, & x_1^{-1}x_2x_1 &= x_3 \text{ language } \mathcal{L}_g, \\
  x_1^2 + x_2^2 &= 0, & x_1 - x_1x_2 - x_3x_2x_1 &= 0 \text{ language } \mathcal{L}_r.
\end{align*}
\]

Let

\[
\Pi \mathcal{A} = \prod_{i \in I} \mathcal{A}
\]

be a direct power of an \( \mathcal{L} \)-structure \( \mathcal{A} \), i.e. \( \Pi \mathcal{A} \) consists of the series \( [a_i \mid i \in I] \) with the coordinate-wise operations. For example, if a language \( \mathcal{L} \) contain a multiplication, then this operation is defined over \( \Pi \mathcal{A} \) as follows:

\[
[a_i \mid i \in I] \cdot [b_i \mid i \in I] = [a_ib_i \mid i \in I].
\]

Let us take an \( \mathcal{L}(\Pi \mathcal{A}) \)-system \( \mathcal{S} = \{ E_j(X) \mid j \in J \} \) (\( E_j(X) \) are equations of the language \( \mathcal{L}(\Pi \mathcal{A}) \)). The \( i \)-th projection \( \pi_i(\mathcal{S}) \) is the \( \mathcal{L}(\mathcal{A}) \)-system \( \{ \pi_i(E_j(X)) \mid j \in J \} \). In particular, if one of the projections \( \pi_i(\mathcal{S}) \) is inconsistent, so is \( \mathcal{S} \).

One can directly prove the following result.

**Lemma 2.1.** An \( \mathcal{L}(\Pi \mathcal{A}) \)-system \( \mathcal{S} = \{ E_j(X) \mid j \in J \} \) over \( \Pi \mathcal{A} \) is consistent iff all projections \( \pi_i(\mathcal{S}) \) are consistent.

**Proof.** It is easy to see that the solution set of \( \mathcal{S} \) over \( \Pi \mathcal{A} \) is the product of the solution sets of all projections. Thus, \( \mathcal{V}_{\Pi \mathcal{A}}(\mathcal{S}) \neq \emptyset \) implies \( \mathcal{V}_\mathcal{A}(\pi_i(\mathcal{S})) \neq \emptyset \) for each \( i \in I \).

The aim of the current paper is the study of weakly equationally Noetherian property for direct powers of semigroups, groups and rings in the languages \( \mathcal{L}(\mathcal{S}), \mathcal{L}(\mathcal{G}), \mathcal{L}(\mathcal{R}) \) respectively.

### 3. Direct powers of finite semigroups

Let \( t(X), s(X) \) be terms of the semigroup language \( \mathcal{L}_s(\mathcal{S}) \). We write \( t(X) \approx s(X) \) if the terms \( t(X), s(X) \) are reduced to the common constant-free term \( t_0(X) \) by the deletion of all constants from \( t(X), s(X) \). For example,

\[
\begin{align*}
  s_1x_1s_2x_2 &\approx s_3x_1s_4x_2 \approx x_1s_5x_2 \approx x_1x_2s_6 \approx x_1x_2.
\end{align*}
\]

for any \( s_i \in \mathcal{S} \).

Two \( \mathcal{L}_s(\mathcal{S}) \)-equations \( t_1(X) = s_1(X), t_2(X) = s_2(X) \) are \( \approx \)-equivalent if one of the following conditions holds:
\[(i) \ t_1(X) \approx t_2(X) \ \text{and} \ s_1(X) \approx s_2(X); \]
\[(ii) \ t_1(X) \approx s_2(X) \ \text{and} \ s_1(X) \approx t_2(X); \]

The proof of the following lemma is straightforward.

**Lemma 3.1.** Let \( S \) be a finite semigroup and \( X \) be a finite set of variables. Then there exists a finite set of \( \mathcal{L}_a(S) \)-equations \( M = \{ t_i(X) = s_i(X) \mid 1 \leq i \leq m \} \) such that any \( \mathcal{L}_a(S) \)-equation \( t(X) = s(X) \) is \( \approx \)-equivalent over \( S \) to some equation from \( M \).

**Lemma 3.2.** Suppose a semigroup \( S \) contains zero, then any \( \mathcal{L}_a(S) \)-equation \( t(X) = s(X) \) is \( \approx \)-equivalent to an equation \( \tau(X) = \sigma(X) \) such that \( V_S(\tau(X) = \sigma(X)) \) is a nilpotent, i.e. any point \( P \in S^n \) satisfies the equation \( \tau(X) = \sigma(X) \). Moreover,

**Proof.** One should put \( \tau(X) = 0 \cdot t(X), \ \sigma(X) = 0 \cdot s(X). \)

**Lemma 3.3.** Let \( S \) be a semigroup with zero, \( I \) be the set of indexes for the direct power \( \Pi S \) and \( \Sigma = \{ t_i(X) = s_i(X) \mid i \in J \} \) \( (J \subseteq I, \ X = \{ x_1, \ldots, x_n \}) \) be a system of \( \mathcal{L}_a(S) \)-equations. The system \( \Sigma \) is the coordinate-wise projection of an \( \mathcal{L}(\Pi S) \)-equation \( t(X) = s(X) \) over \( \Pi S \) iff any two equations in \( \Sigma \) are \( \approx \)-equivalent.

**Proof.** Suppose any two equations in \( \Sigma \) are \( \approx \)-equivalent, i.e. the left (right) part of any equation from \( \Sigma \) is reduced to the constant-free term \( t_0(X) \) (respectively, \( s_0(X) \)). For any index from \( i \in I \setminus J \) we add to \( \Sigma \) the trivial equation \( t_i(X) = s_i(X) \) with \( t_i(X) = 0 \cdot t_0(X), \ s_i(X) = 0 \cdot s_0(X). \) Finally, all equations from \( S \) may be wrapped into a single \( \mathcal{L}_a(\Pi S) \)-equation over the direct power \( \Pi S \).

**Theorem 3.4.** Let \( S \) be a finite semigroup with zero. Then \( \Pi S \) is weakly equationally Noetherian.

**Proof.** Let \( S \) be an infinite system of equations over \( \Pi S \) and \( S_i = \pi_i(S) \) \( (i \in I) \) be the coordinate-wise projections of \( S \). By Lemma 3.1, there exists a finite set \( M \) of \( \mathcal{L}_a(S) \)-equations such that each \( S_i \) is equivalent to a system \( S'_i \subseteq M \).

For each \( t(X) = s(X) \in M \) we do the following operations.

Let \( I'_1 \subseteq I \) be the set of all indexes with \( t(X) = s(X) \in S'_i \), and \( I'_0 = I \setminus I'_1 \). Consider the following set of \( \mathcal{L}_a(S) \)-equations:

\[ \{ t(X) = s(X) \mid i \in I'_1 \} \cup \{ 0 \cdot t(X) = 0 \cdot s(X) \mid i \in I'_0 \} \]

Obviously the set of equations above is wrapped into a single equation \( t(X) = s(X) \) over \( \Pi S \).

Thus, the processing of the set \( M \) gives the finite set of \( \mathcal{L}_a(\Pi S) \)-equations \( S' \) such that \( S' \) is equivalent to \( S \) over \( \Pi S \).

**4. Direct powers of finite groups**

Let us give the necessary definitions for the next theorem. A group \( G \) of the language \( \mathcal{L}_d(G) \) is called an equationally domain if any union \( Y_1 \cup Y_2 \) of algebraic sets \( Y_1, Y_2 \) is algebraic.

There are enough examples of equationally domains in the class of groups. In particular, all finite non-abelian groups and non-abelian free groups are equationally domains. There exists a simple criterion (see [1]) for a group to be an equationally domain:

**Theorem 4.1.** A group of the language \( \mathcal{L}(G) \) is an equationally domain iff the set

\[ \{ (x, y) \mid x = 1 \ \text{or} \ y = 1 \} \]
is algebraic.

Obviously, any point \( P = (p_1, p_2, \ldots, p_n) \in G^n \) is algebraic over \( G \) in the language \( \mathcal{L}(G) \), since

\[ P = V_G(\{x_1 = p_1, x_2 = p_2, \ldots, x_n = p_n\}). \]

By the definition of an equational domain, any finite set is also algebraic. Hence, any subset \( Y \subseteq G^n \) is algebraic over a finite group \( G \). The last fact will be used in the following theorem.

**Theorem 4.2.** Let \( G \) be a finite group and \( G \) is an equational domain. Then \( \Pi G \) is weakly equationally Noetherian.

**Proof.** Let \( S \) be a \( \mathcal{L}_q(G) \)-system in variables \( X = \{x_1, x_2, \ldots, x_n\} \) over \( \Pi G \) and \( \pi_i(S) \) be the coordinate-wise projections of \( S \). Let \( Y_i \subseteq G^n \) be the solution set of the system \( \pi_i(S) \). Since \( G \) is finite and equational domain, for each point \( P \in G^n \) there exists a term \( t_P(X) \) such that \( V_G(t_P(X)) = G^n \setminus P \). Then

\[ Y_i = \bigcap_{P \not\in Y_i} V_G(t_P(X) = 1). \]

Let us prove that the system of equations

\[ \{t_P(X) = 1 \mid P \not\in Y_i\} \]

is equivalent to a single equation

\[ w_i(X_i) = \prod_{P \not\in Y_i} [t_P(X), a_i^{-1} t_P(X) a_i P] = 1, \quad (1) \]

where the constants \( a_i P \in G \) are defined as follows: since \( G \) is an equational domain, for the element \( t_P(P) \in G \setminus \{1\} \) there exists an element \( a_i P \) with

\[ t_P(P), a_i^{-1} t_P(P) a_i P \neq 1. \]

Let us prove \( V_G(w(X_i)) = Y_i \). Indeed, if \( Q \not\in Y_i \) then

\[ w_i(Q) = \prod_{P \not\in Y_i} [t_P(Q), a_i^{-1} t_P(Q) a_i P] = [t_Q(Q), a_i^{-1} t_Q(Q) a_i Q] \neq 1. \]

Otherwise \( (Q \in Y_i) \):

\[ w_i(Q) = \prod_{P \not\in Y_i} [t_P(Q), a_i^{-1} t_P(Q) a_i P] = \prod_{P \not\in Y_i} 1 = 1. \]

Obviously, any equation \( w(X_i) \) (1) is equivalent to

\[ w_i(X_i) = \prod_{P \in G^n} [t_P(X_i), a_i^{-1} t_P(X_i) a_i P] = 1, \quad (2) \]

if we put \( a_i P = 1 \) for each \( P \in Y_i \).

Then the set of equations \( \{w_i(X_i) \mid i \in I\} \) may be wrapped into the \( \mathcal{L}_q(G) \)-equation over \( \Pi G \)

\[ w(X) = \prod_{P \in G^n} [t_P(X), a_i^{-1} t_P(X) a_i P] = 1, \]

where \( a_i P = [a_i P \mid i \in I] \).

Thus, the system \( S \) is equivalent to the equation \( w(X) = 1 \), and the group \( \Pi G \) is weakly equationally Noetherian.
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