SOME PROPERTIES OF THE SOLUTIONS OF OBSTACLE PROBLEMS WITH MEASURE DATA

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Abstract

We study some properties of the obstacle reactions associated with the solutions of unilateral obstacle problems with measure data. These results allow us to prove that, under very weak assumptions on the obstacles, the solutions do not depend on the components of the negative parts of the data which are concentrated on sets of capacity zero. The proof is based on a careful analysis of the behaviour of the potentials of two mutually singular measures near the points where both potentials tend to infinity.
1. Introduction

Given a regular bounded open set $\Omega$ of $\mathbb{R}^N$, $N \geq 2$, and a linear elliptic operator $A$ of the form

$$Au = -\sum_{i,j=1}^{N} D_i(a_{ij}D_ju),$$

with $a_{ij} \in L^\infty(\Omega)$, we study some properties of the solution of the obstacle problem for the operator $A$ in $\Omega$ with homogeneous Dirichlet boundary conditions on $\partial \Omega$, when the datum $\mu$ is a bounded Radon measure on $\Omega$ and the obstacle $\psi$ is an arbitrary function on $\Omega$. According to [7], a function $u$ is a solution of this problem, which will be denoted by $OP(\mu, \psi)$, if $u$ is the smallest function with the following properties: $u \geq \psi$ in $\Omega$ and $u$ is a solution in the sense of Stampacchia [18] of a problem of the form

$$\begin{cases}
Au = \mu + \lambda & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

for some bounded Radon measure $\lambda \geq 0$. The measure $\lambda$ which corresponds to the solution of the obstacle problem is called the obstacle reaction.

Existence and uniqueness of the solution of $OP(\mu, \psi)$ have been proved in [7], provided that there exists a measure $\lambda$ such that the solution of (1.2) is greater than or equal to $\psi$. These results have been extended to the non-linear case in [14], when $\mu$ vanishes on all sets with capacity zero. For a different approach to obstacle problems for non-linear operators with measure data see [5], [3], [4], [15], and [16].

If the measure $\mu$ belongs to the dual $H^{-1}(\Omega)$ of the Sobolev space $H^1_0(\Omega)$, and if there exists a function $w \in H^1_0(\Omega)$ above the obstacle $\psi$, then the solution of the obstacle problem $OP(\mu, \psi)$ according to the previous definition coincides with the solution $u$ of the variational inequality

$$\begin{cases}
u \in H^1_0(\Omega), \\
\langle Au, v - u \rangle \geq \langle \mu, v - u \rangle & \forall v \in H^1_0(\Omega), v \geq \psi,
\end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^1(\Omega)$ and $H^1_0(\Omega)$. In this case the obstacle reaction $\lambda$ belongs to $H^{-1}(\Omega)$. It is concentrated on the contact set $\{u = \psi\}$ if $\psi$ is continuous, or, more in general, quasi upper semicontinuous.

An important role in this problem is played by the space $\mathcal{M}^0_0(\Omega)$ of all bounded Radon measures on $\Omega$ which are absolutely continuous with respect to the harmonic
capacity. If the datum \( \mu \) belongs to \( \mathcal{M}_b^0(\Omega) \), so does the obstacle reaction, provided that there exists a measure \( \lambda \in \mathcal{M}_b^0(\Omega) \) such that the solution of (1.2) is greater than or equal to \( \psi \) (see [7], Theorem 7.5). In this case the obstacle reaction is concentrated on the contact set \( \{ u = \psi \} \), whenever the obstacle \( \psi \) is quasi upper semicontinuous (see [14], Theorem 2.9). Example 2.3, which is a variant of an example proposed by L. Orsina and A. Prignet, shows that this is not always true when \( \mu \) is not absolutely continuous with respect to the harmonic capacity.

Using the linearity of the operator \( A \), it is easy to see that the obstacle reaction belongs to \( \mathcal{M}_b^0(\Omega) \) and is concentrated on the contact set \( \{ u = \psi \} \), whenever \( \psi \) is quasi upper semicontinuous and just the negative part \( \mu^- \) of \( \mu \) belongs to \( \mathcal{M}_b^0(\Omega) \). Therefore we concentrate our attention on the case \( \mu^- \notin \mathcal{M}_b^0(\Omega) \). Then \( \mu^- \) can be decomposed as \( \mu^- = \mu^-_a + \mu^-_s \), where \( \mu^-_a \in \mathcal{M}_b^0(\Omega) \) and \( \mu^-_s \) is concentrated on a set of capacity zero. We assume that the obstacle \( \psi \) satisfies the estimates \(-v - w \leq \psi \leq v\), where \( w \in H^1(\Omega) \) and \( v \) is the solution in the sense of Stampacchia of a problem of the form

\[
\begin{aligned}
  & A v = \nu \quad \text{in } \Omega, \\
  & v = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

with \( \nu \in \mathcal{M}_b^0(\Omega) \). We prove (Theorem 4.3) that the obstacle problems \( OP(\mu, \psi) \) and \( OP(\mu^+ - \mu^-_a, \psi) \) have the same solution \( u \), while the corresponding obstacle reactions \( \lambda \) and \( \lambda_0 \) satisfy \( \lambda = \lambda_0 + \mu^-_s \). This shows that, under these assumptions, the solution \( u \) of \( OP(\mu, \psi) \) does not depend on \( \mu^-_s \), while the obstacle reaction has the form \( \lambda_0 + \mu^-_s \), where \( \lambda_0 \) is a non-negative measure in \( \mathcal{M}_b^0(\Omega) \). This measure is concentrated on the contact set \( \{ u = \psi \} \) whenever the obstacle \( \psi \) is quasi upper semicontinuous (Theorem 4.5).

These results will be used in a forthcoming paper [6] to study the dependence of the solutions on the obstacles. Their proof relies on a variant (Lemma 3.5) of the following result, which has an intrinsic interest. Let \( u_\mu \) and \( u_\nu \) be the solutions of (1.4) corresponding to the measures \( \mu \) and \( \nu \), which are not assumed to belong to \( \mathcal{M}_b^0(\Omega) \). Suppose that \( \mu^+ \perp \nu \) and \( u_\mu \leq u_\nu \). Then \( \mu^+ \in \mathcal{M}_b^0(\Omega) \). This result is obtained by investigating the behaviour of the potentials of two mutually singular measures near their singular points (Lemmas 3.3 and 3.4).
2. Notation and preliminary results

Let us fix a bounded open set $\Omega$ in $\mathbb{R}^N$, $N \geq 2$. We assume that $\Omega$ satisfies the following regularity condition, considered by Stampacchia in [18]: there exists a constant $\alpha > 0$ such that
\[
\text{meas}(B_r(x) \setminus \Omega) \geq \alpha \text{meas}(B_r(x)),
\]
for every $x \in \partial \Omega$ and for every $r > 0$, where $B_r(x)$ denotes the open ball with centre $x$ and radius $r$.

Let $A$ be the linear elliptic operator introduced in (1.1), where $(a_{ij})$ is an $N \times N$ matrix of functions in $L^\infty(\Omega)$, and, for a suitable constant $\beta > 0$,
\[
\sum_{i,j=1}^{N} a_{ij}(x)\xi_i \xi_j \geq \beta |\xi|^2,
\]
for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

In order to include in our analysis also the case of thin obstacles, it is convenient to introduce the notions of capacity and of quasi continuous representative of a Sobolev function. Given a set $E \subseteq \Omega$, its capacity with respect to $\Omega$ is defined by
\[
\text{cap}(E) = \inf \int_{\Omega} |\nabla v|^2 \, dx,
\]
where $v$ runs over all functions $v \in H^1_0(\Omega)$ such that $v \geq 1$ a.e. in a neighbourhood of $E$. We say that a property holds quasi everywhere (abbreviated as q.e.) when it holds everywhere except on a set of capacity zero. A function $v: \Omega \to \mathbb{R}$ is quasi continuous (resp. quasi upper semicontinuous) if, for every $\varepsilon > 0$, there exists a set $E \subseteq \Omega$, with $\text{cap}(E) < \varepsilon$, such that $v|_{\Omega \setminus E}$ is continuous (resp. upper semicontinuous) in $\Omega \setminus E$. We recall also that, if $u$ and $v$ are quasi continuous functions and $u \leq v$ a.e. in $\Omega$, then $u \leq v$ q.e. in $\Omega$.

Every function $u \in H^1_0(\Omega)$ has a quasi continuous representative, i.e., a quasi continuous function $\tilde{u}$ which is equal to $u$ a.e. in $\Omega$. We shall always identify $u$ with its quasi continuous representative $\tilde{u}$, which is uniquely defined quasi everywhere in $\Omega$. A self-contained presentation of all these notions can be found, for instance, in Chapters 4 of [8] and [10].

Let us fix a function $\psi: \Omega \to \mathbb{R}$, and the corresponding convex set
\[
\mathcal{K}_\psi(\Omega) := \{z \text{ quasi continuous in } \Omega : z \geq \psi \text{ q.e. in } \Omega\}.
\]
In their natural setting, obstacle problems are part of the theory of Variational Inequalities (for which we refer to the books [2], [12], and [19]). For any \(\mu \in H^{-1}(\Omega)\) the variational inequality with obstacle \(\psi\)

\[
\begin{cases}
  u \in K_{\psi}(\Omega) \cap H^1_0(\Omega), \\
  \langle Au, v - u \rangle \geq \langle \mu, v - u \rangle \\
  \forall v \in K_{\psi}(\Omega) \cap H^1_0(\Omega),
\end{cases}
\]

which will be indicated by \(VI(\mu, \psi)\), has a unique solution \(u\), whenever the set \(K_{\psi}(\Omega) \cap H^1_0(\Omega)\) is nonempty, i.e.,

there exists \(w \in H^1_0(\Omega)\) such that \(w \geq \psi\) q.e. in \(\Omega\). \hspace{1cm} (2.2)

In this case we say that the obstacle is \(VI\)-admissible.

Among all classical results, we recall that the solution of \(VI(\mu, \psi)\) is also characterized as the smallest function \(u \in H^1_0(\Omega)\) such that

\[
\begin{cases}
  Au - \mu \geq 0 \text{ in } \mathcal{D}'(\Omega), \\
  u \geq \psi \text{ q.e. in } \Omega.
\end{cases}
\]

Then \(\lambda := Au - \mu\) is a non-negative measure, that is called the obstacle reaction associated with \(u\).

Let \(\mathcal{M}_b(\Omega)\) be the space of all bounded Radon measures on \(\Omega\), and let \(\mathcal{M}_b^0(\Omega)\) be the subspace of all measures of \(\mathcal{M}_b(\Omega)\) which vanish on all sets of capacity zero. The corresponding cones of non-negative measures will be denoted by \(\mathcal{M}_b^+(\Omega)\) and \(\mathcal{M}_b^{0,+}(\Omega)\), respectively. Recall that \(H^{-1}(\Omega) \not\subseteq \mathcal{M}_b(\Omega)\), but \(H^{-1}(\Omega) \cap \mathcal{M}_b(\Omega) \subseteq \mathcal{M}_b^0(\Omega)\). Any measure \(\mu \in \mathcal{M}_b(\Omega)\) can be decomposed as \(\mu = \mu_a + \mu_s\), where \(\mu_a \in \mathcal{M}_b^0(\Omega)\) and \(\mu_s\) is concentrated on a set of capacity zero (see [9]).

When the datum is a measure, equations and inequalities can not be studied in the variational framework, and the usual notion of solution in the sense of distributions does not guarantee uniqueness when the coefficients are discontinuous, as shown by a celebrated counterexample due to J. Serrin [17]. To overcome these difficulties, Stampacchia introduced in [18] the following notion of solution, obtained by duality.
Definition 2.1. For every $\mu \in \mathcal{M}_b(\Omega)$, the solution $u_\mu$ in the sense of Stampacchia of the problem

\[
\begin{aligned}
&Au_\mu = \mu \quad \text{in } \Omega, \\
&u_\mu = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\tag{2.4}
\]
is the unique function $u_\mu \in L^1(\Omega)$ such that

\[
\int_{\Omega} u_\mu g \, dx = \int_{\Omega} u^*_g \, d\mu, \quad \text{for every } g \in L^\infty(\Omega),
\]
where $u^*_g$ is the solution of

\[
\begin{aligned}
&A^* u^*_g = g \quad \text{in } H^{-1}(\Omega), \\
&u^*_g \in H^1_0(\Omega),
\end{aligned}
\]
and $A^*$ is the adjoint of $A$.

Existence and uniqueness of $u_\mu$ are proved in [18]. Let $T_k(s) := (-k) \vee (s \wedge k)$ be the usual truncation function. It is easy to prove that

\[
T_k(u_\mu) \in H^1_0(\Omega) \quad \text{and} \quad \int_{\Omega} |DT_k(u_\mu)|^2 \, dx \leq k |\mu|(\Omega),
\tag{2.5}
\]
for any $k > 0$. These facts imply that $u_\mu$ has a quasi continuous representative which is finite q.e. in $\Omega$. If $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, then the solution in the sense of Stampacchia coincides with is a the usual variational solution in $H^1_0(\Omega)$.

In the rest of the paper, for every $\mu \in \mathcal{M}_b(\Omega)$ we shall use the notation $u_\mu$ to indicate the quasi continuous representative of the solution of (2.4), which is uniquely defined quasi everywhere in $\Omega$.

The Green’s function $G^A_{\Omega}(x, y)$ relative to the operator $A$ in $\Omega$ is defined as the solution in the sense of Stampacchia of the equation

\[
\begin{aligned}
&AG^A_{\Omega}(\cdot, y) = \delta_y \quad \text{in } \Omega, \\
&G^A_{\Omega}(\cdot, y) = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\tag{2.6}
\]
where $\delta_y$ is the unit mass concentrated at $y \in \Omega$. In [18] it is proved that $G^A_{\Omega}: \Omega \times \Omega \to [0, +\infty]$ is continuous and satisfies the following estimates: for every compact set $K \subseteq \Omega$ there exist four constants $c_1 > 0$, $c_2 > 0$, $d_1 \geq 0$, and $d_2 \geq 0$ ($d_1 = d_2 = 0$ if $N \geq 3$), such that

\[
c_1 G(|x - y|) - d_1 \leq G^A_{\Omega}(x, y) \leq c_2 G(|x - y|) + d_2,
\tag{2.7}
\]
for every $x, y \in K$, where $G(|x|)$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^N$, i.e.,

$$G(|x|) = \begin{cases} 
\frac{1}{(N-2)\sigma_{N-1}} \frac{1}{|x|^{N-2}}, & \text{if } N > 2, \\
\frac{1}{2\pi} \log \left( \frac{1}{|x|} \right), & \text{if } N = 2,
\end{cases} \quad (2.8)$$

with $\sigma_{N-1}$ equal to the $(N-1)$-dimensional measure of the boundary of the unit ball in $\mathbb{R}^N$. As proved in [18], the solution of (2.4) satisfies

$$u_\mu(x) = \int_{\Omega} G^A_{\Omega}(x,y) d\mu(y), \quad \text{for a.e. } x \in \Omega. \quad (2.9)$$

The following notion of solution for obstacle problems with measure data has been introduced in [7].

**Definition 2.2.** Let $\mu \in \mathcal{M}_b(\Omega)$. We say that a function $u$ is a solution of the obstacle problem with datum $\mu$ and obstacle $\psi$ (shortly $OP(\mu, \psi)$) if the following conditions are satisfied:

(a) $u \in \mathcal{K}_\psi(\Omega)$ and there exists $\lambda \in \mathcal{M}_b^+(\Omega)$ such that $u = u_\mu + u_\lambda$ q.e. in $\Omega$;

(b) $u \leq v$ q.e. in $\Omega$ for every $v \in \mathcal{K}_\psi(\Omega)$ such that $v = u_\mu + u_\nu$ q.e. in $\Omega$, with $\nu \in \mathcal{M}_b^+(\Omega)$.

Existence and uniqueness of the solution of $OP(\mu, \psi)$ are proved in [7], assuming that the obstacle $\psi$ satisfies the following natural hypothesis, which replaces (2.2):

there exists $\rho \in \mathcal{M}_b(\Omega)$ such that $u_\rho \geq \psi$ q.e. in $\Omega$.

In this case we shall say that $\psi$ is $OP$-admissible.

The non-negative measure $\lambda$ which appears in condition (a) of Definition 2.2 is uniquely determined by the solution $u$ and is called the obstacle reaction associated with $u$. It is possible to prove that $\lambda$ belongs to $\mathcal{M}_b^0(\Omega)$ if the datum $\mu$ belongs to $\mathcal{M}_b^0(\Omega)$ and

there exists $\sigma \in \mathcal{M}_b^0(\Omega)$ such that $u_\sigma \geq \psi$ q.e. in $\Omega$.

When the last condition is satisfied, we shall say that $\psi$ is $OP^o$-admissible. Notice that, if the datum $\mu$ is in $\mathcal{M}_b^0(\Omega)$, but the obstacle is only $OP$-admissible, then the reaction
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λ may not belong to $\mathcal{M}_b^0(\Omega)$. For instance, if $\mu = 0$ and $\psi = u_\delta y$ for some $y \in \Omega$, then the solution of $OP(0, \psi)$ is $u_\delta y$, and hence $\lambda = \delta y \notin \mathcal{M}_b^0(\Omega)$.

If the obstacle $\psi$ is continuous, or, more in general, quasi upper semicontinuous, then the solution of the variational inequality (2.1) must touch the obstacle at all points where it is not solution of the equation $Au = \mu$. Indeed, under these assumptions on $\psi$, the obstacle reaction $\lambda$ of the solution of (2.1) with $\mu \in H^{-1}(\Omega)$ is concentrated on the coincidence set $\{x \in \Omega : u(x) = \psi(x)\}$; in other words, $u = \psi$ $\lambda$-a.e. in $\Omega$. When $\psi$ is continuous, this result is well known and can be found in the books mentioned above; the quasi upper semicontinuous case is discussed, e.g., in Section 3 of [1].

The same properties are true for the solutions of $OP(\mu, \psi)$ when $\mu \in \mathcal{M}_b^0(\Omega)$ and $\psi$ is $OP^0$-admissible and quasi upper semicontinuous (see [14]), but they do not hold for an arbitrary $\mu \in \mathcal{M}_b(\Omega)$, as shown by the following example, which is a variant of an example studied by L. Orsina and A. Prignet.

Example 2.3. Let $\mu \in \mathcal{M}_b^+(\Omega)$ be a non-negative measure concentrated on a set of capacity zero. Suppose that there exists a constant $k > 0$ such that $-k \leq \psi \leq 0$ q.e. in $\Omega$. Let $u = u - \mu + u_\lambda$ be the solution of $OP(-\mu, \psi)$. We want to show that $u = 0$ q.e. in $\Omega$ and $\lambda = \mu$ in $\Omega$.

Taking $\nu = \mu$ in condition (b) of Definition 2.2, we obtain $u \leq 0$ q.e. in $\Omega$. As $u \geq -k$ q.e. in $\Omega$, we have $u = T_k(u)$ q.e. in $\Omega$, and hence $u \in H^1_0(\Omega)$ by (2.5). This implies that the measure $-\mu + \lambda$ belongs to $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, which is contained in $\mathcal{M}_b^0(\Omega)$. In other words $\lambda = \mu + \lambda_0$, with $\lambda_0 \in \mathcal{M}_b^0(\Omega)$. Since $\lambda$ is non-negative and $\mu \perp \lambda_0$ (recall that $\mu$ is concentrated on a set of capacity zero), the measure $\lambda_0$ is non-negative. As $u = u_\lambda$, by the maximum principle we have $u \geq 0$ q.e. in $\Omega$. Therefore $u = 0$ q.e. in $\Omega$ and, consequently, $\lambda = \mu$ in $\Omega$.

In particular, if $\mu = \delta_y$ for some $y \in \Omega$, and $\psi = -k$, we have an example of a continuous obstacle for which the solution $u$ of $OP(-\mu, \psi)$ does not touch $\psi$, although $u$ is not the solution of the equation $Au = -\mu$, since the obstacle reaction is not zero.

In Section 3 we will show that, when the obstacle is controlled from above and from below in an appropriate way (see Theorem 4.1), it is possible to “isolate” the effect of the singular negative part of the data. Namely, the reaction $\lambda$ will be written as $\lambda = \lambda_0 + \mu_s^-$, where $\lambda_0$ belongs to $\mathcal{M}_b^{0,+}(\Omega)$. Moreover the “regular part” $\lambda_0$ is concentrated on the coincidence set $\{x \in \Omega : u(x) = \psi(x)\}$ whenever $\psi$ is quasi upper semicontinuous, and a complementarity condition holds (Theorem 4.5).
The proof of these facts will be based on some new results in Potential Theory, which are obtained in the next section.

3. Some results in Potential Theory

We will prove some results concerning the potential of a measure. The first two lemmas characterize the measures of $M^0_0(\Omega)$ in terms of the sets where their potentials are infinite. The main result of this section is Lemma 3.3 on the behaviour of the potentials of two mutually singular measures near the points where both potentials tend to infinity. It allows us to study the solutions of two equations of the form (2.4) corresponding to mutually singular data. In particular we will compare these solutions near their singular points (Lemma 3.4).

For every $\mu \in M^+_b(\Omega)$ we consider the potentials $G_\mu$ and $G^A_{\Omega_\mu}$ defined by

$$G_\mu(x) = \int_{\Omega} G(|x-y|) \, d\mu(y), \quad \text{for } x \in \mathbb{R}^N,$$

$$G^A_{\Omega_\mu}(x) = \int_{\Omega} G^A_{\Omega}(x,y) \, d\mu(y), \quad \text{for } x \in \Omega,$$

where $G$ and $G^A_{\Omega}$ are defined in (2.8) and (2.6). Note that $-\Delta G_\mu = \mu$ in the sense of distributions in $\Omega$. By (2.9) $G^A_{\Omega_\mu}$ coincides almost everywhere with the solution $u_\mu$ of (2.4).

**Lemma 3.1.** Let $\mu \in M^+_b(\Omega)$. Then

$$\mu \in M^0_0(\Omega) \iff G_\mu < +\infty \quad \mu\text{-a.e. in } \Omega.$$  

**Proof.** One implication is easy: by a classical result (see, e.g., Theorem 7.33 in [11]) $G_\mu$ is finite q.e. in $\Omega$, and hence $\mu$-a.e. in $\Omega$ if $\mu \in M^0_0(\Omega)$.

Let us prove the converse in the case $N > 2$, so that $G \geq 0$. We start by proving that $\mu^*(\{x \in \Omega : G_\mu(x) < +\infty\}) = 0$. For every $t > 0$, let $E_t := \{x \in \mathbb{R}^N : G_\mu(x) \leq t\}$, and let $\mu_t$ be the measure defined by $\mu_t(B) := \mu(B \cap E_t)$ for every Borel set $B \subseteq \Omega$. Note that $E_t$ is closed since $G_\mu$ is lower semicontinuous. As $\mu_t \leq \mu$, we have $G_{\mu_t} \leq G_\mu$ (recall that $G \geq 0$). In particular $G_{\mu_t} \leq t$ in $E_t$. By the maximum principle (see, e.g., Theorem 1.10 in [13]) we obtain $G_{\mu_t} \leq t$ in $\mathbb{R}^N$. Since $G_{\mu_t}$ is superharmonic and
bounded, it belongs to $H^1_{\text{loc}}(\mathbb{R}^N)$ (see, e.g., Corollary 7.20 in [10]). As $\mu_t = -\Delta G \mu_t$ in the sense of distributions in $\Omega$, we have $\mu_t \in H^{-1}(\Omega)$, and hence $\mu_t \in \mathcal{M}_b^{0,+}(\Omega)$.

Let us consider a Borel set $B \subseteq \{x \in \Omega : G \mu(x) < +\infty\}$ with $\text{cap}(B) = 0$. Then $B$ is the union of the sets $E_t \cap B$, for $t > 0$, and hence

$$
\mu(B) = \sup_{t \in \mathbb{R}^+} \mu(E_t \cap B) = \sup_{t \in \mathbb{R}^+} \mu_t(B) = 0.
$$

Consequently $\mu^s(\{x \in \Omega : G \mu(x) < +\infty\}) = 0$. Therefore, if $\mu^s$ were not identically zero, it would be $\mu^s(\{x \in \Omega : G \mu(x) = +\infty\}) > 0$, and this would contradict the assumption $G \mu < +\infty$ $\mu$-a.e. in $\Omega$.

The case $N = 2$ can be dealt with by adding a suitable constant $c$ to $G$ so that $G + c \geq 0$ in $\Omega$. The proof is the same with minor modifications, among which we point out the use of the maximum principle for logarithmic potentials (see, e.g., Theorem 1.6 in [13]).

Using (2.7) we can now extend Lemma 3.1 to the general case of the operator $A$.

**Lemma 3.2.** Let $\mu \in \mathcal{M}_b^+(\Omega)$. Then

$$
\mu \in \mathcal{M}_b^{0,+}(\Omega) \iff G^A_{\Omega} \mu < +\infty \text{ $\mu$-a.e. in } \Omega.
$$

**Proof.** Thanks to (2.7) it is easy to prove that for every $x \in \Omega$

$$
G^A_{\Omega}(x) < +\infty \iff G^A_{\Omega} \mu(x) < +\infty,
$$

so the thesis follows from Lemma 3.1. □

The mean value of an integrable function $f$ on a measurable set $B$ with positive measure is defined by

$$
\int_B f \, dx := \frac{1}{\text{meas}(B)} \int_B f \, dx.
$$

In the next lemma we compare the mean values of the potentials of two mutually singular measures on small balls centered at a point where both potentials are infinite.
Lemma 3.3. Let \( \mu, \nu \in M^+_b(\Omega) \), with \( \mu \perp \nu \), and let
\[
E := \{ x \in \Omega : G_\mu(x) = G_\nu(x) = +\infty \}.
\]
Then
\[
\lim_{r \to 0^+} \frac{\int_{B_r(x)} G_\nu \, dy}{\int_{B_r(x)} G_\mu \, dy} = 0, \quad \text{for } \mu\text{-a.e. } x \in E. \tag{3.2}
\]

Proof. Let \( R > 0 \) be such that \( \Omega \subseteq B_R(0) \). Observing that \( \Omega \subseteq B_{2R}(x) \) for every \( x \in \Omega \), we have
\[
\int_{B_r(x)} G_\nu \, dy = \int_{B_{2R}(x)} G_r(|x - z|) \, d\nu(z),
\]
where
\[
G_r(|x - z|) := \int_{B_r(x)} G(|y - z|) \, dy,
\]
and \( \nu \) is defined for every Borel set \( B \subseteq \mathbb{R}^N \) by \( \nu(B) = \nu(B \cap \Omega) \). As \( G(|x|) \) is superharmonic in \( \mathbb{R}^N \) and harmonic for \( x \neq 0 \), we obtain
\[
G_r(s) \begin{cases} 
= G(s), & \text{for } s \geq r, \\
\leq G(s), & \text{for } s < r,
\end{cases}
\]
and \( G_r(s) \not\nearrow G(s) \) as \( r \searrow 0 \).

It is easy to prove that
\[
\int_{B_{2R}(x)} G_r(|x - z|) \, d\nu(z) = G_r(2R) \nu(\Omega) - \int_0^{2R} G'_r(s) \nu(B_s(x)) \, ds; \tag{3.3}
\]
the proof can be obtained by using polar coordinates if \( \nu \) is absolutely continuous with respect to the Lebesgue measure, and an easy approximation argument extends the result to the general case. Note that \( \nu(\Omega) < +\infty \) and that \( G_r(2R) = G(2R) \) for \( r \) small enough. Since the left hand side of (3.3) tends to \( G_\nu(x) = +\infty \), the last term tends to infinity for every \( x \in E \).
The same argument can be developed for the denominator, so the limit in (3.2) is equal to

$$\lim_{r \to 0^+} \frac{\int_0^{2R} G'_r(s) \nu(B_s(x)) \, ds}{\int_0^{2R} G'_r(s) \mu(B_s(x)) \, ds}$$

for every $x \in E$. Given $\delta \in (0, 2R)$, the integrals between $\delta$ and $2R$ remain bounded as $r \to 0$, so that (3.4) is equal to

$$\lim_{r \to 0^+} \frac{\int_0^\delta G'_r(s) \nu(B_s(x)) \, ds}{\int_0^\delta G'_r(s) \mu(B_s(x)) \, ds}$$

for every $x \in E$. Since $\mu \perp \nu$, by the Besicovitch differentiation theorem (see, e.g., Chapter 1.6 in [8]), for $\mu$-a.e. $x \in \Omega$ we have

$$\lim_{r \to 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} = 0.$$ 

Let us fix $x \in E$ such that (3.6) holds. For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\nu(B_r(x)) < \varepsilon \mu(B_r(x)),$$

for all $r \in (0, \delta)$,

and since $G_r$ is decreasing in $s$, we have

$$-\int_0^\delta G'_r(s) \nu(B_s(x)) \, ds \leq -\varepsilon \int_0^\delta G'_r(s) \mu(B_s(x)) \, ds.$$ 

This shows that the limit in (3.5), and hence in (3.4), is less than or equal to $\varepsilon$. Since $\varepsilon$ is arbitrary, the limit in (3.4) is zero and the proof is complete.

Using (2.7) we can extend Lemma 3.3 to the general case of the operator $A$. 


Lemma 3.4. Let \( \mu, \nu \in M^+_b(\Omega) \), with \( \mu \perp \nu \), and let \( F \) be the set of all points \( x \in \Omega \) such that
\[
\lim_{r \to 0^+} \int_{B_r(x)} u_\mu \, dy = \lim_{r \to 0^+} \int_{B_r(x)} u_\nu \, dy = +\infty.
\] (3.7)
Then
\[
\lim_{r \to 0^+} \frac{\int_{B_r(x)} u_\nu \, dy}{\int_{B_r(x)} u_\mu \, dy} = 0, \quad \text{for } \mu\text{-a.e. } x \in F.
\] (3.8)

Proof. Let us fix \( x \in F \) and \( R > 0 \) such that \( B_R(x) \subset \subset \Omega \). By (2.9) we have
\[
\int_{B_r(x)} u_\nu \, dy = \int_{B_r(x)} \int_{B_R(x)} G^A_{\Omega,\Omega}(y,z) \, d\nu(z) \, dy
\]
\[
= \int_{\Omega \setminus B_R(x)} \int_{B_r(x)} G^A_{\Omega,\Omega}(y,z) \, d\nu(z) \, dy + \int_{B_R(x)} \int_{B_r(x)} G^A_{\Omega,\Omega}(y,z) \, d\nu(z) \, dy.
\]
The first term is bounded when \( r < R/2 \), so only the second one is relevant in the limit in (3.8). The same can be said of the denominator, so that it is enough to study the quotient
\[
\frac{\int_{B_r(x)} \int_{B_R(x)} G^A_{\Omega,\Omega}(y,z) \, d\nu(z)}{\int_{B_r(x)} \int_{B_R(x)} G^A_{\Omega,\Omega}(y,z) \, d\mu(z)}.
\]
Thanks to (2.7) this is smaller than or equal to
\[
\frac{c_2}{c_1} \int_{B_R(x)} \int_{B_r(x)} G(|y-z|) \, d\nu(z) \, dy + d_1 \nu(B_R(x))
\]
\[
\int_{B_R(x)} \int_{B_r(x)} G(|y-z|) \, d\mu(z) \, dy \cdot d_2 \mu(B_R(x))
\] (3.9)
By (3.7) and (2.7), for every \( x \in F \) we have
\[
\lim_{r \to 0^+} \int_{B_r(x)} G_\mu \, dy = \lim_{r \to 0^+} \int_{B_r(x)} G_\nu \, dy = +\infty.
\]
Since $G_\mu$ and $G_\nu$ are superharmonic, this implies $G_\mu(x) = G_\nu(x) = +\infty$ for every $x \in F$. Therefore Lemma 3.3 shows that (3.2) holds for $\mu$-a.e. $x \in F$.

Using once again the fact that the integrals over $\Omega \setminus B_R(x)$ remain bounded as $r \to 0^+$, from (3.2) we obtain that the quotient in (3.9) tends to zero as $r \to 0^+$ for $\mu$-a.e. $x \in F$.

Lemma 3.5. Let $\mu, \nu \in M_b(\Omega)$, let $\lambda \in M_0^b(\Omega)$, and let $w \in H^1(\Omega)$. Assume that $\nu \perp \mu^+$ and that $u_\mu \leq u_\nu + u_\lambda + w$ a.e. in $\Omega$. Then $\mu^+ \in M_b^{0,+}(\Omega)$.

Proof. First of all the measures $\nu$ and $\lambda$ can be assumed to be non-negative, replacing them with their positive parts. The function $w$ can be replaced by $v + h$, where $h$ is the solution of
\[
\begin{cases}
Ah = 0 & \text{in } H^{-1}(\Omega), \\
h - w^+ \in H_0^1(\Omega),
\end{cases}
\]
and $v = (w - h)^+$. Note that $h$ is a non-negative $A$-harmonic function and $v$ is a non-negative function of $H_0^1(\Omega)$, and we still have $u_\mu \leq u_\nu + u_\lambda + v + h$ a.e. in $\Omega$.

Step 1. Consider first the case $u_\mu \leq u_\nu$ a.e. in $\Omega$. Then $u_{\mu^+} \leq u_\nu + u_{\mu^-}$ a.e. in $\Omega$, and $\mu^+ \perp (\nu + \mu^-)$. Let $E$ be the set of all points $x \in \Omega$ such that
\[
\lim_{r \to 0^+} \int_{B_r(x)} u_{\mu^+} \, dy = +\infty.
\]
Note that $E$ coincides with the set $F$ of Lemma 3.4, relative to the non-negative measures $\mu^+$ and $\nu + \mu^-$. Consequently we have
\[
\lim_{r \to 0^+} \frac{\int_{B_r(x)} u_{(\nu + \mu^-)} \, dy}{\int_{B_r(x)} u_{\mu^+} \, dy} = 0, \quad \text{for } \mu^+\text{-a.e. } x \in E. \tag{3.10}
\]
Since $u_{\mu^+} \leq u_\nu + u_{\mu^-}$ a.e. in $\Omega$, the quotient in (3.10) is greater than or equal to 1. Therefore we conclude that $\mu^+(E) = 0$. As $G^{A}_\Omega \mu^+$ is lower semicontinuous, by (2.9) we have $G^{A}_\Omega \mu^+(x) < +\infty$ for $x \in \Omega \setminus E$, and this implies $\mu^+ \in M_b^{0,+}(\Omega)$ by Lemma 3.2.

Step 2. Assume that $u_\mu \leq u_\nu + h$ a.e. in $\Omega$. Since $h$ is $A$-harmonic, by De Giorgi’s theorem it is continuous, hence
\[
\lim_{r \to 0^+} \int_{B_r(x)} h \, dy = h(x) < +\infty, \quad \text{for every } x \in \Omega.
\]
Therefore, if we add this integral to the numerator of (3.10), we can repeat the argument of Step 1 and we obtain $\mu^+ \in \mathcal{M}_b^{0,+}(\Omega)$ in this case too.

**Step 3.** Assume that $u_\mu \leq u_\nu + h + u_\lambda$ a.e. in $\Omega$. As before we have $u_{\mu^+} \leq u_{(\nu+\mu^-)} + h + u_\lambda$ a.e. in $\Omega$, with $\mu^+ \perp (\nu + \mu^-)$. We write now $\mu^+ = \mu_1 + \mu_2$ in $\Omega$, with $\mu_i \in \mathcal{M}_b^{0,+}(\Omega)$, $\mu_1 \ll \lambda$, and $\mu_2 \perp \lambda$. Then $u_{\mu^+} \leq u_{(\lambda+\nu+\mu^-)} + h$ a.e. in $\Omega$, and $u_\mu \leq (\lambda+\nu+\mu^-)$. Therefore we have $u_\mu \leq \mu_2^+ \in \mathcal{M}_b^{0,+}(\Omega)$ by Step 2. As $\mu_1 \in \mathcal{M}_b^{0,+}(\Omega)$, being $\lambda \in \mathcal{M}_b^{0,+}(\Omega)$, we conclude that $\mu^+ \in \mathcal{M}_b^{0,+}(\Omega)$.

**Step 4.** Assume now that $u_\mu \leq u_\nu + h + u_\lambda + v$ a.e. in $\Omega$. Consider the obstacle $\psi_0 := u_\mu - u_\nu - h - u_\lambda$, which is bounded from above both by $v$ and by $u_\mu$, so that it is both VI- and OP-admissible. Then the solution $u_\tau$ of $OP(0, \psi_0)$ belongs to $\mathcal{H}_0^1(\Omega)$ (see Theorem 5.2 in [7]), hence $\tau \in \mathcal{M}_b^{0,+}(\Omega) \cap \mathcal{H}^1(\Omega) \subseteq \mathcal{M}_b^{0,+}(\Omega)$. So $u_\mu \leq u_\nu + h + u_{(\lambda+\tau)}$ a.e. in $\Omega$, and we conclude by means of Step 3.

**Corollary 3.6.** Let $\mu, \nu \in \mathcal{M}_b(\Omega)$, let $\lambda \in \mathcal{M}_b^0(\Omega)$, and let $w \in \mathcal{H}^1(\Omega)$. Assume that $\nu \perp \mu$ and that $|u_\mu| \leq u_\nu + u_\lambda + w$ a.e. in $\Omega$. Then $\mu \in \mathcal{M}_b^0(\Omega)$.

**Proof.** It is enough to apply Lemma 3.5 to $\mu$ and $-\mu$.

## 4. Interaction between obstacles and singular data

The next theorem is the main result of the paper. We prove that the component of $\mu^-$ which is singular with respect to the capacity is completely absorbed by the obstacle reaction $\lambda$, provided the obstacle $\psi$ satisfies very weak estimates from above and from below.

**Theorem 4.1.** Let $\mu \in \mathcal{M}_b(\Omega)$ and let $\mu_s^-$ be the part of $\mu^-$ which is concentrated on a set of capacity zero. Assume that the obstacle $\psi$ satisfies the estimates

$$-u_\tau - u_\sigma - w \leq \psi \leq u_\sigma \text{ q.e. in } \Omega, \quad (4.1)$$

where $w \in \mathcal{H}^1(\Omega)$, $\sigma \in \mathcal{M}_b^0(\Omega)$, and $\tau \in \mathcal{M}_b(\Omega)$, with $\tau \perp \mu_s^-$. Let $u = u_\mu + u_\lambda$ be the solution of $OP(\mu, \psi)$. Then $\lambda = \lambda_0 + \mu_s^-$ in $\Omega$, with $\lambda_0 \in \mathcal{M}_b^{0,+}(\Omega)$.

**Proof.** It is not restrictive to assume that $\sigma \geq 0$ in $\Omega$. Using the decomposition $\mu^- = \mu_a^- + \mu_s^-$, with $\mu_a^- \in \mathcal{M}_b^{0,+}(\Omega)$, we can write $u = u_{\mu^+} - u_{\mu_a^-} - u_{\mu_s^-} + u_\lambda$ q.e. in $\Omega$. 

As \( u_\mu + u_{(\mu^- + \sigma)} = u_\mu + u_\sigma \geq \psi \) q.e. in \( \Omega \), by Definition 2.2 we have \( u_\mu + u_\sigma \geq u \) q.e. in \( \Omega \), hence \( u_\lambda - u_{\mu^-} \leq u_\sigma + u_{\mu_a} \) q.e. in \( \Omega \). By Lemma 3.5 this implies \((\lambda - \mu_s^-)^+ \in M_0^b(\Omega)\).

On the other hand, \(-u_{\mu_a^-} + u_\lambda \geq -u_\mu + u_{\mu_a^-} \) q.e. in \( \Omega \), and hence \( u_{(\mu_s^- - \lambda)} \leq u_\mu + u_\sigma + w \) q.e. in \( \Omega \). Now \((\mu^+ + \tau) \perp (\mu_s^- - \lambda)^+\), since \( \mu^+ \perp \mu^-, \tau \perp \mu_s^-\), and \( \lambda \geq 0 \) in \( \Omega \). So \((\mu_s^- - \lambda)^+ \in M_0^b(\Omega)\) by Lemma 3.5.

As \((\mu_s^- - \lambda)^- = (\lambda - \mu_s^-)^+ \in M_0^b(\Omega)\), we conclude that \((\mu_s^- - \lambda) \in M_0^b(\Omega)\). Therefore \( \lambda = \lambda_0 + \mu_s^-\), with \( \lambda_0 \in M_0^b(\Omega)\). Since \( \lambda \geq 0 \) in \( \Omega \) and \( \lambda_0 \perp \mu_s^-\), we deduce that \( \lambda_0 \geq 0 \) in \( \Omega \).

\[\int_{\Omega} |D(\psi^+ \land k)|^2 dx \leq \int_{\Omega} |D\psi^+|^2 dx < +\infty,\]

the function \( \psi^+ \) is the limit of the increasing sequence \( \psi^+ \land k \), which is bounded in \( H_1^0(\Omega) \). This implies that \( \psi^+ \in H_1^0(\Omega) \), hence \( \psi \) is VI-admissible. Let \( u_\sigma \) be the solution of \( OP(0, \psi) \). Since \( u_\sigma \) is also the solution of \( VI(0, \psi) \) (see Theorem 5.2 of [7]), we have \( \sigma \in M_b(\Omega) \cap H^{-1}(\Omega) \subseteq M_0^b(\Omega) \). Then we can take \( w = -\psi \) and \( \tau = 0 \) in (4.1).

**Theorem 4.3.** Let \( \mu \in M_b(\Omega) \). Assume that the obstacle \( \psi \) satisfies hypothesis (4.1). Let \( u \) and \( u_0 \) be the solutions of \( OP(\mu, \psi) \) and \( OP(\mu^+ - \mu_a^-, \psi) \), and let \( \lambda \) and \( \lambda_0 \) be the corresponding obstacle reactions. Then \( u = u_0 \) q.e. in \( \Omega \) and \( \lambda = \lambda_0 + \mu_s^- \) in \( \Omega \). Moreover \( \lambda_0 \in M_0^b(\Omega) \).

**Proof.** The function \( u \) can be written as \( u_{(\mu^- + \mu_a^-)} + u_{(-\mu_s^- + \lambda)} \). Since \( u \geq \psi \) q.e. in \( \Omega \) and \( -\mu_s^- + \lambda \geq 0 \) in \( \Omega \) by Theorem 4.1, we have \( u \geq u_0 \) q.e. in \( \Omega \) by Definition 2.2. Similarly, we have \( u_0 = u_{(\mu^+ - \mu_a^-)} + u_{\lambda_0} = u_\mu + u_{(\mu_s^- + \lambda_0)} \) q.e. in \( \Omega \). Since \( u_0 \geq \psi \) q.e. in \( \Omega \) and \( \mu_s^- + \lambda_0 \geq 0 \) in \( \Omega \), we have \( u_0 \geq u \) q.e. in \( \Omega \) by Definition 2.2. Therefore \( u = u_0 \) q.e. in \( \Omega \) and, consequently, \( \lambda = \lambda_0 + \mu_s^- \) in \( \Omega \). Finally, \( \lambda_0 \in M_0^b(\Omega) \) by Theorem 4.1. \qed

We recall a theorem proved by C. Leone in [14].
Theorem 4.4. Let $\mu \in \mathcal{M}_b^0(\Omega)$ and let $\psi$ be a quasi upper semicontinuous $OP^o$-admissible obstacle. Then the following facts are equivalent:

(a) $u$ is the solution of $OP(\mu, \psi)$ and $\lambda$ is the corresponding obstacle reaction;
(b) $\lambda \in \mathcal{M}_b^{0,+}(\Omega)$, $u = u_\mu + u_\lambda$ q.e. in $\Omega$, $u \geq \psi$ q.e. in $\Omega$, and $u = \psi \lambda$ a.e. in $\Omega$.

The following theorem extends this result to the case of data in $\mathcal{M}_b(\Omega)$, provided the obstacle satisfies (4.1).

Theorem 4.5. Let $\mu \in \mathcal{M}_b(\Omega)$. Assume that the obstacle $\psi$ is quasi upper semicontinuous and satisfies hypothesis (4.1). Then the following facts are equivalent:

(a) $u$ is the solution of $OP(\mu, \psi)$ and $\lambda$ is the corresponding obstacle reaction;
(b) $\lambda = \lambda_0 + \mu_s^-$ in $\Omega$, with $\lambda_0 \in \mathcal{M}_b^{0,+}(\Omega)$, $u = u_\mu + u_\lambda$ q.e. in $\Omega$, $u \geq \psi$ q.e. in $\Omega$, $u = \psi \lambda_0$ a.e. in $\Omega$.

Proof. Step 1. First of all we consider the case $\mu^- \in \mathcal{M}_b^{0,+}(\Omega)$. Observe that $u_\mu + u_\lambda$ is the solution of $OP(\mu, \psi)$ if and only if $-u_\mu^- + u_\lambda$ is the solution of $OP(-\mu^-, \psi - u_\mu^+)$. By Theorem 4.4 this happens if and only if $\lambda \in \mathcal{M}_b^{0,+}(\Omega)$, $-u_\mu^- + u_\lambda \geq \psi - u_\mu^+$ q.e. in $\Omega$, and $-u_\mu^- + u_\lambda = \psi - u_\mu^+$ $\lambda$-a.e. in $\Omega$. The last two conditions are equivalent to $u_\mu + u_\lambda \geq \psi$ q.e. in $\Omega$ and $u_\mu + u_\lambda = \psi \lambda$ a.e. in $\Omega$.

Step 2. Let us consider the general case $\mu \in \mathcal{M}_b(\Omega)$. By Theorem 4.3 $u$ is the solution of $OP(\mu, \psi)$ and $\lambda$ is the corresponding obstacle reaction if and only if $u$ is the solution of $OP(\mu^+ - \mu^-_a, \psi)$ and $\lambda_0 = \lambda - \mu_s^-$ is the corresponding obstacle reaction. By Step 1 this happens if and only if $\lambda_0 \in \mathcal{M}_b^{0,+}(\Omega)$, $u = u_{\mu^+} - u_{\mu^-} + u_{\lambda_0} = u_\mu + u_\lambda$ q.e. in $\Omega$, $u \geq \psi$ q.e. in $\Omega$, and $u = \psi \lambda_0$ a.e. in $\Omega$. 

References

[1] Attouch H., Picard C.: Problèmes variationnels et théorie du potentiel non linéaire. Ann. Fac. Sci. Toulouse Math. 1 (1979), 89-136.
[2] Baiocchi C., Capelo A.: Disequazioni variazionali e quasivariazionali. Applicazioni a problemi di frontiera libera. Quaderni dell’Unione Matematica Italiana, Pitagora Editrice, Bologna, 1978, translated in Variational and quasivariational inequalities. Wiley, New York, 1984.
[3] Boccardo L., Cirmi G.R.: Nonsmooth unilateral problems. Nonsmooth optimization: methods and applications (Erice, 1991), F. Giannessi ed., 1-10, Gordon and Breach, Amsterdam, 1992.
[4] Boccardo L., Cirmi G.R.: Existence and uniqueness of solution of unilateral problems with $L^1$-data. To appear.

[5] Boccardo L., Gallouët T.: Problèmes unilatéraux avec données dans $L^1$. C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), 617-655.

[6] Dall’Aglio P.: Stability results for solutions of obstacle problems with measure data. In preparation.

[7] Dall’Aglio P., Leone C.: Obstacle problems with measure data. Preprint S.I.S.S.A., Trieste, 1997.

[8] Evans L.C., Gariepy R.F.: Measure theory and fine properties of functions. CRC Press, Boca Raton, 1992.

[9] Fukushima M., Sato K., Taniguchi S.: On the closable part of pre-Dirichlet forms and the fine supports of underlying measures. Osaka J. Math. 28 (1991), 517-535.

[10] Heinonen J., Kilpeläinen T., Martio O.: Nonlinear potential theory of degenerate elliptic equations. Clarendon Press, Oxford, 1993.

[11] Helms L.L.: Introduction to potential theory. Wiley, New York, 1969.

[12] Kinderlehrer D., Stampacchia G.: An introduction to variational inequalities and their applications. Academic Press, New York, 1980.

[13] Landkof N.S.: Foundations of potential theory. Springer Verlag, Berlin, 1972.

[14] Leone C.: Existence and uniqueness of solutions for nonlinear obstacle problems with measure data. Preprint S.I.S.S.A., Trieste, 1998.

[15] Oppezzi P., Rossi A.M.: Existence of solutions for unilateral problems with multivalued operators. J. Convex Anal. 2 (1995), 241-261.

[16] Oppezzi P., Rossi A.M.: Esistenza di soluzioni per problemi unilaterali con dato misura o $L^1$. Ricerche Mat., to appear.

[17] Serrin J.: Pathological solutions of elliptic differential equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 18 (1964), 385-387.

[18] Stampacchia G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier Grenoble 15 (1965), 189-258.

[19] Troianiello G.M.: Elliptic differential equations and obstacle problems. Plenum Press, New York, 1987.