**Abstract**

We conjecture a closed form expression for the simplest class of multiplicity-free quantum $6j$-symbols for $U_q(\mathfrak{sl}_N)$. The expression is a natural generalization of the quantum $6j$-symbols for $U_q(\mathfrak{sl}_2)$ obtained by Kirillov and Reshetikhin. Our conjectured form enables computation of colored HOMFLY polynomials for various knots and links carrying arbitrary symmetric representations.

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**Keywords:** Quantum $6j$-symbols

1 Introduction

From the beginning of the twentieth century, we have witnessed the mutual interactive developments between quantum physics and representation theory. Right from the birth of quantum mechanics, applications of representation theory to quantum physics have turned out to be indispensable to the study of symmetries inherent in a quantum system. It is well-known that the Clebsch-Gordan coefficients ($3j$-symbols) in decomposition of tensor product of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ naturally appear in quantum theory of angular momenta. Furthermore, in the study of atomic spectroscopy \cite{19}, the Racah coefficients were defined as linear combinations of products of four Clebsch-Gordan coefficients. Around the same time, by studying algebra the $3j$-symbols satisfy, Wigner independently introduced (classical) $6j$-symbols \cite{25, 26}. The $6j$-symbols are of importance in all situations where the recoupling of angular momenta is involved.

Inspired by ideas coming from quantum physics, quantum deformation of universal enveloping algebra of a semi-simple Lie algebra, a.k.a. a quantum group, was introduced by Drinfel’d and Jimbo \cite{4, 10}, which captures the symmetry behind the Yang-Baxter equations. Meanwhile, on a separate track, Jones constructed a new polynomial invariant of links using von Neumann algebra \cite{11}. It soon became apparent that the basic algebraic structures of the polynomial invariants of links could be described by quantum groups. This viewpoint leads to a wide class of polynomial invariants of links \cite{23, 21, 22}. Furthermore, quantum groups are also related to the two-dimensional Wess-Zumino-Novikov-Witten (WZNW) model. The actions of braid groups on conformal blocks of WZNW model turn out to be equivalent to the braid group representation obtained from the universal $R$-matrix of the quantum group \cite{14, 5, 1, 2}. Hence, the polynomials invariants of links can be obtained from monodromy representations along solutions of the Knizhnik-Zamolodchikov equations.
It has been proven that the quantum deformations of the Clebsch-Gordan coefficients and the 6j-symbols which naturally appear in the representation theory of quantum groups connect these areas of mathematics and physics in a beautiful way \[13\]. However, the generalization of the quantum 6j-symbols for \( U_q(\mathfrak{sl}_2) \) to higher ranks has remained a challenging open problem. In this paper, we shall conjecture a closed form expression of the simplest class of the quantum 6j-symbols for \( U_q(\mathfrak{sl}_N) \).

2 Quantum 6j-symbols

Let us denote the spin-\( j \) representation of \( U_q(\mathfrak{sl}_2) \) by \( V_j \) whose highest weight is \( \lambda = 2j \) (\( j \in \frac{1}{2}\mathbb{Z} \)). The space of four-point conformal blocks is the space of linear maps \( \text{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \otimes V_{j_4}, \mathbb{C}) \) invariant under the diagonal action of \( U_q(\mathfrak{sl}_2) \) and, most importantly, it is a finite-dimensional vector space. By associativity of the tensor product, we can take a basis in two ways;

\[
\text{Hom}_{U_q(\mathfrak{sl}_2)}((V_{j_1} \otimes V_{j_2})_{j_{12}} \otimes V_{j_3}, V_{j_4}) = \bigoplus_{j_{12}} \mathbb{C} \begin{array}{ccc}
\lambda \\
\lambda_i \\
\lambda_j 
\end{array} ,
\]

and

\[
\text{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_1} \otimes (V_{j_2} \otimes V_{j_3})_{j_{23}}, V_{j_4}) = \bigoplus_{j_{23}} \mathbb{C} \begin{array}{ccc}
\lambda_i \\
\lambda_j \\
\lambda_k 
\end{array} ,
\]

where \( j_{12} \) and \( j_{23} \) satisfy the quantum Clebsch-Gordan condition (the fusion rule); \( j_1 + j_2 + j_{12} \in \mathbb{Z} \) and \( |j_1 - j_2| \leq j_{12} \leq j_1 + j_2 \). (The same condition for \( j_{23} \).) Then, the quantum 6j-symbols \( \left\{ \begin{array}{ccc}
j_1 & j_2 & j_{12} \\
j_3 & j_4 & j_{23} \end{array} \right\} \) for \( U_q(\mathfrak{sl}_2) \) appear in the transformation matrix of the two bases

\[
\begin{array}{ccc}
\lambda_i \\
\lambda_j \\
\lambda_k 
\end{array} = \sum_{j_{23}} a_{j_{12},j_{23}} \left| \begin{array}{ccc}
j_1 & j_2 & j_{12} \\
j_3 & j_4 & j_{23} \end{array} \right| \begin{array}{ccc}
\lambda_i \\
\lambda_j \\
\lambda_k 
\end{array} ,
\]

where

\[
a_{j_{12},j_{23}} \frac{1}{\begin{array}{ccc}
j_1 & j_2 & j_{12} \\
j_3 & j_4 & j_{23} \end{array}} = (-1)^{j_1+j_2+j_3+j_4} \sqrt{|2j_{12}+1||2j_{23}+1|} \left| \begin{array}{ccc}
j_1 & j_2 & j_{12} \\
j_3 & j_4 & j_{23} \end{array} \right| .
\]

Here, the square bracket defines a \( q \)-number

\[
[\alpha] = \frac{q^{\alpha/2} - q^{-\alpha/2}}{q^{1/2} - q^{-1/2}} ,
\]

and the transformation matrix \( a_{j_{12},j_{23}} \) is usually called the fusion matrix. The rigorous derivation of a closed form expression of the quantum 6j-symbols for \( U_q(\mathfrak{sl}_2) \) was first given by Kirillov and Reshetikhin \[13\]. Later, Masbaum and Vogel provided another derivation based on linear skein theory \[15\].

The generalization to higher ranks requires replacement of a spin \( j \) by a highest weight \( \lambda \) of a representation of \( U_q(\mathfrak{sl}_N) \). A representation of \( U_q(\mathfrak{sl}_N) \) with a highest weight \( \lambda \) can be equivalently specified by a Young tableau \( \{ \ell_i \}_{1 \leq i \leq N-1} \) with \( \ell_1 \geq \cdots \geq \ell_{N-1} \). If one writes the highest weight as \( \lambda = \sum_{i=1}^{N-1} \lambda_\ell \) where \( \{ \lambda_\ell \}_{1 \leq i \leq N-1} \) are the fundamental weights, the relation to the Young tableau can be read off by \( \ell_\ell = \lambda_1 + \lambda_{i+1} + \cdots + \lambda_{N-1} \). In what follows, we identify a highest weight with a Young tableau by this dictionary. For general \( N \), it is necessary to introduce the conjugate representation \( V^\ast_\lambda \) of the representation with the
We note that the highest weight of the symmetric representation $\lambda^*$ is $\sum_{i=1}^{N-1} \lambda_{N-i} \omega_i$. Then, the fusion rule of quantum 6j-symbols for $U_q(\mathfrak{sl}_N)$

$$\left\{ \frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_4}, \frac{\lambda_{12}}{\lambda_{23}} \right\}$$

is that $V_{\lambda_2} \in (V_{\lambda_1} \otimes V_{\lambda_2}) \cap (V_{\lambda_3} \otimes V_{\lambda_4})$ and $V_{\lambda_2} \in (V_{\lambda_2} \otimes V_{\lambda_3}) \cap (V_{\lambda_4} \otimes V_{\lambda_5})$. From the construction, we expect the quantum 6j-symbols to satisfy the following symmetries:

$$\begin{align*}
\left\{ \frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_4}, \frac{\lambda_{12}}{\lambda_{23}} \right\} &= \left\{ \frac{\lambda_3}{\lambda_1}, \frac{\lambda_4}{\lambda_2}, \frac{\lambda_{23}}{\lambda_{12}} \right\} \\
\left\{ \frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_4}, \frac{\lambda_{12}}{\lambda_{23}} \right\} &= \left\{ \frac{\lambda_3}{\lambda_1}, \frac{\lambda_4}{\lambda_2}, \frac{\lambda_{23}}{\lambda_{12}} \right\}
\end{align*}$$

(6)

In addition, the relationship between the fusion matrix and the quantum 6j-symbols for $U_q(\mathfrak{sl}_N)$ is generalized to

$$a_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \left\{ \frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_4} \right\} = \epsilon(\lambda) \sqrt{\text{dim}_q V_{\lambda_2} \text{dim}_q V_{\lambda_3} \text{dim}_q V_{\lambda_4}} \left\{ \frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_4}, \frac{\lambda_{12}}{\lambda_{23}} \right\},$$

(7)

where $\epsilon(\lambda) = \pm 1$ and $\text{dim}_q V_{\lambda}$ is the quantum dimension of the representation $V_{\lambda}$ with highest weight $\lambda$.

Unlike the representations of $U_q(\mathfrak{sl}_2)$, there are serious technical difficulties for $U_q(\mathfrak{sl}_N)$ in the fact that the decompositions of tensor products involve multiplicity structure in general. Specifically, isomorphic irreducible constituents will arise more than once in the decomposition of a tensor product. However, there are special cases which decompose in a multiplicity-free way. Among them, we shall restrict ourselves to the simplest class which can be regarded as the natural extension of the case for $U_q(\mathfrak{sl}_2)$ [13]: the tensor products of two symmetric representations, and the tensor products of a symmetric representation and a representation conjugate to a symmetric representation. To obey the fusion rule, two of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ need to be symmetric representations, and the other two must be conjugate to symmetric representations. Using the symmetries [20], it turns out that multiplicity-free quantum 6j-symbols for $U_q(\mathfrak{sl}_N)$ of this class amount to the following two types:

- **Type I**

$$\begin{align*}
\left\{ \frac{n_1 \omega_1}{n_3 \omega_1}, \frac{n_2 \omega_N-1}{n_4 \omega_N-1}, \frac{(n_1 - n_3 + k_1) \omega_1 + k_1 \omega_N-1}{(n_3 - n_2 + k_2) \omega_1 + k_2 \omega_N-1} \right\}
\end{align*}$$

(8)

where $n_2 \leq n_1 \leq n_3, k_1 \leq n_2$ and $k_2 \leq n_1$. The fusion rule requires $n_1 + n_3 = n_2 + n_4$.

- **Type II**

$$\begin{align*}
\left\{ \frac{n_1 \omega_1}{n_3 \omega_N-1}, \frac{n_2 \omega_1}{n_4 \omega_N-1}, \frac{(n_1 + n_2 - 2k_1) \omega_1 + k_1 \omega_N-1}{(n_1 - n_3 + k_2) \omega_1 + k_2 \omega_N-1} \right\}
\end{align*}$$

(9)

where $n_1 \leq n_2, k_2 \leq \min(n_1, n_3)$ and $k_1 \leq \min(n_1, n_3, n_4)$. The fusion rule requires $n_1 + n_2 = n_3 + n_4$.

We note that the highest weight of the symmetric representation $\frac{n}{n}$ is $n \omega_1$ and that of the conjugate representation $\frac{n}{n}$ is $n \omega_N-1$ where $\square$ represents $(N - 1)$ vertical boxes.
For low representations \((n_i \leq 2)\) we have determined the quantum 6j-symbols \([20, 27]\). Extension to higher representations \((n_i \leq 4)\) is achieved through the following route: In our recent paper \([17]\), we have colored HOMFLY polynomials for a class of knots called twist knots \(K_p\) where \(p\) denotes the number of full-twists. Using the identities obeyed by quantum 6j symbols and equating the colored HOMFLY polynomials of these twist knots with the \(SU(N)\) Chern-Simons invariant involving quantum 6j-symbols (see (C.1) in \([17]\), we determined the quantum 6j symbols for \(n_i = 3\) and \(n_i = 4\). We recognized a pattern from the data on quantum 6j symbols \((n_i \leq 4)\) motivating us to attempt a closed form expression for arbitrary \(n_i\)’s.

**Conjecture** The closed form expression for quantum 6j-symbols of type I and type II is as follows:

\[
\begin{bmatrix}
\lambda_1 & \lambda_2 & \lambda_{12} \\
\lambda_3 & \lambda_4 & \lambda_{23}
\end{bmatrix} = \Delta(\lambda_1, \lambda_2, \lambda_{12})\Delta(\lambda_3, \lambda_4, \lambda_{12})\Delta(\lambda_1, \lambda_4, \lambda_{23})\Delta(\lambda_2, \lambda_3, \lambda_{23})
\times [N-1]! \sum_{z \in \mathbb{Z}_{\geq 0}} (-1)^z [z + N - 1]! C_z(\{\lambda_i\}, \lambda_{12}, \lambda_{23})
\times \left\{ \left[ z - \frac{1}{2} (\lambda_1 + \lambda_2 + \lambda_{12}, \alpha^\vee_1 + \alpha^\vee_{N-1}) \right]! \times \left[ z - \frac{1}{2} (\lambda_3 + \lambda_4 + \lambda_{12}, \alpha^\vee_1 + \alpha^\vee_{N-1}) \right]! \times \left[ z - \frac{1}{2} (\lambda_1 + \lambda_4 + \lambda_{23}, \alpha^\vee_1 + \alpha^\vee_{N-1}) \right]! \times \left[ z - \frac{1}{2} (\lambda_2 + \lambda_3 + \lambda_{12}, \alpha^\vee_1 + \alpha^\vee_{N-1}) \right]! \times \left[ \frac{1}{2} (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \alpha^\vee_1 + \alpha^\vee_{N-1}) - z \right]! \times \left[ \frac{1}{2} (\lambda_1 + \lambda_3 + \lambda_2 + \lambda_{12}, \alpha^\vee_1 + \alpha^\vee_{N-1}) - z \right]! \right\}^{-1},
\]

(10)

where

\[
\Delta(\lambda_1, \lambda_2, \lambda_3) = \left\{ \left[ \frac{1}{2} (-\lambda_1 + \lambda_2 + \lambda_3, \alpha^\vee_1 + \alpha^\vee_{N-1}) \right]! \times \left[ \frac{1}{2} (\lambda_1 - \lambda_2 + \lambda_3, \alpha^\vee_1 + \alpha^\vee_{N-1}) \right]! \times \left[ \frac{1}{2} (\lambda_1 + \lambda_2 - \lambda_3, \alpha^\vee_1 + \alpha^\vee_{N-1}) \right]! \right\}^{1/2}
\times \left\{ \left[ \frac{1}{2} (\lambda_1 + \lambda_2 + \lambda_3, \alpha^\vee_1 + \alpha^\vee_{N-1}) + N - 1 \right]! \right\}^{-1/2}.
\]

(11)

Here, we define the \(q\)-factorial by \([n]! = [n][n-1] \cdots [3][2][1]\) and take \([0]! = 1\). We use the fact that \(\alpha^\vee_i\) are duals of simple roots which form a basis of coroots, and the paring with the fundamental weights provides \((\omega_i, \alpha^\vee_j) = \delta_{ij}\).

It is important to stress that only finite number of positive integers \(z\) in the summation will give non-zero contribution. The factors \(C_z(\{\lambda_i\}, \lambda_{12}, \lambda_{23})\) in (10) for type I are

\[
C_z \left( \begin{array}{c}
\frac{n_1 \omega_1}{n_2 \omega_{N-1}} \\
\frac{n_3 \omega_1}{n_4 \omega_{N-1}}
\end{array} \right)
= \left\{ \begin{array}{ll}
\delta_{z, z_{\text{min}}+i} & \left( N - 2 + k_2 - i \right)^{-1} \text{ for } k_1 > k_2, \\
\delta_{z, z_{\text{min}}+i} & \left( N - 2 + k_1 - i \right)^{-1} \text{ for } k_1 \leq k_2,
\end{array} \right.
\]

(12)

where \(z_{\text{min}}\) is the smallest integer \(z\) in the summation in (10) which gives a non-trivial value. Similarly, for type II, the factors \(C_z(\{\lambda_i\}, \lambda_{12}, \lambda_{23})\) are

\[
C_z \left( \begin{array}{c}
\frac{n_1 \omega_1}{n_3 \omega_{N-1}} \\
\frac{n_2 \omega_1}{n_4 \omega_{N-1}}
\end{array} \right)
= \left\{ \begin{array}{ll}
\delta_{z, z_{\text{max}}-i} & \left( N - 2 + k_2 - i \right)^{-1} \text{ for } k_1 > k_2, \\
\delta_{z, z_{\text{max}}-i} & \left( N - 2 + k_1 - i \right)^{-1} \text{ for } k_1 \leq k_2,
\end{array} \right.
\]

(13)
where $z_{\text{max}}$ is the largest integer $z$ in the summation in (10) which gives a non-trivial value. We denote the $q$-binomial by

$$\left[ \frac{p}{q} \right] = \frac{[p]!}{[q]![p-q]!}.$$  

(14)

Although there are the square roots in the expression (11), the $6j$-symbols (10) are actually rational functions with respect to $q^{-1/2}$. Obviously, it is easy to see that the expression (10) reduces to the form of $U_q(\mathfrak{sl}_2)$ provided by Kirillov and Reshetikhin [13] when we take $N = 2$.

In addition, we have checked that (10) satisfy the orthogonal property

$$\sum_{\lambda_{12}} \dim_q V_{\lambda_{12}} \dim_q V_{\lambda_{23}} \left\{ \lambda_1, \lambda_2, \lambda_{12} \right\} \left\{ \lambda_3, \lambda_4, \lambda_{12} \right\} = \delta_{\lambda_{12} \lambda_{23}},$$  

(15)

and the Racah identity

$$\sum_{\lambda_{12}} \epsilon_{(\lambda_{12}, \lambda_{23}, \lambda_{24})} q^{\frac{C_{23}}{2}} \dim_q V_{\lambda_{12}} \left\{ \lambda_1, \lambda_2, \lambda_{12} \right\} \left\{ \lambda_3, \lambda_4, \lambda_{12} \right\} = \left\{ \lambda_3, \lambda_4, 0 \right\} q^{\frac{C_{23}+C_{24}}{2}} q^{\frac{C_{23}+C_{24}+C_{4}}{2}},$$  

(16)

as well as the identity

$$\sum_{\mu_1} \epsilon_{(\mu_1, \kappa_1)} \dim_q V_{\mu_1} \left\{ \kappa_1, \lambda_3, \lambda_4 \right\} \left\{ \mu_1, \lambda_5, \mu_2 \right\} = \epsilon_{(\lambda_1, \mu_1)} \frac{\delta_{\lambda_1, \lambda_2} \delta_{\lambda_3, \lambda_4}}{\sqrt{\dim_q V_{\lambda_1} \dim_q V_{\lambda_3}}}.$$  

(17)

Here $C_i$ is the quadratic Casimir invariant for the representation of highest weight $\lambda_i$. The sign $\epsilon = \pm 1$ can be easily read off by comparing it with the results for $U_q(\mathfrak{sl}_2)$.

We expect that the quantum $6j$-symbols for $U_q(\mathfrak{sl}_N)$ obey the pentagon (Biedenharn-Elliott) identity

$$\sum_{\mu_1} \epsilon_{(\mu_1, \kappa_1)} \dim_q V_{\mu_1} \left\{ \kappa_1, \lambda_3, \lambda_4 \right\} \left\{ \mu_1, \lambda_5, \mu_2 \right\} = \epsilon_{(\lambda_1, \mu_1)} \frac{\delta_{\lambda_1, \lambda_2} \delta_{\lambda_3, \lambda_4}}{\sqrt{\dim_q V_{\lambda_1} \dim_q V_{\lambda_3}}}.$$  

(18)

However, this property cannot be checked unless expressions beyond the multiplicity-free ones are obtained.

With our conjectured quantum $6j$-symbols (10), we can verify that they reproduce the HOMFLY polynomials colored by the symmetric representation of many non-torus knots [8, 9], the Whitehead link and the Borromean rings [12, 7] up to 4 boxes. Moreover, we compute colored HOMFLY polynomials of many knots and links which we tabulate in the companion paper [16]. However, it is not clear, at present, whether the proof [13, 15] can be extended to our conjectured $\mathfrak{sl}_N$ quantum $6j$ symbols mainly due to the presence of $C_z(\{\lambda_i, \lambda_{12}, \lambda_{23}\})$ in (10).

3 Discussion

In this paper, we proposed the closed form expression of the quantum $6j$-symbols for $U_q(\mathfrak{sl}_N)$. However, the structure behind $6j$-symbols is by far richer and we only scratch the surface of this topic. Firstly, a rigorous derivation of (10) by quantum groups still remain an open problem. In addition to this, further study needs to be undertaken to obtain the expressions for other multiplicity-free cases as well as general cases. For this purpose, it is necessary
to investigate explicit expressions for the quantum 3j-symbols and their relation to the quantum 6j-symbols.

There is an important property of quantum 6j-symbols for \( U_q(\mathfrak{sl}_N) \) which will be useful for evaluating quantum 6j-symbols beyond symmetric representations. We can observe that a quantum 6j-symbol involving anti-symmetric representations and their conjugate representations can be obtained by changing \([N + k] \leftrightarrow [N - k]\) in the quantum 6j-symbol for symmetric representations related by transposition (mirror reflection across the diagonal). For example, we can find the following 6j-symbols using properties (15), (16) and (17):

\[
\begin{bmatrix}
\begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\nu_1 & \nu_2 & \nu_3 \\
\mu_1 & \mu_2 & \mu_3
\end{array}
\end{bmatrix} = \frac{[2]^2}{[N + 1][N][N - 1][N - 2]^2} \left( \frac{[3][N - 3][N - 4] - [N][N + 1]}{[N][3][N - 4]} \right).
\]  

The quantum 6j-symbol related by transposition is

\[
\begin{bmatrix}
\begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\nu_1 & \nu_2 & \nu_3 \\
\mu_1 & \mu_2 & \mu_3
\end{array}
\end{bmatrix} = \frac{[2]^2}{[N - 1][N][N + 1][N + 2]^2} \left( \frac{[3][N + 3][N + 4] - [N][N - 1]}{[N][3][N + 4]} \right).
\]  

It is easy to see the symmetry \([N + k] \leftrightarrow [N - k]\) between the two quantum 6j-symbols. In fact, this explains the symmetry between the colored HOMFLY polynomials colored by symmetric representation \(S^r\) and anti-symmetric representations \(\Lambda^r\):

\[
P_{S^r}(K; a, q) = P_{\Lambda^r}(K; a, q^{-1}).
\]  

Following [5], we call this property the mirror symmetry of quantum 6j-symbols. In this way, every multiplicity-free quantum 6j-symbol involving anti-symmetric representations can be obtained from its mirror dual 6j-symbol which can be explicitly evaluated by [10]. Nevertheless, a closed form expression using highest weights as in (10) is still lacking in anti-symmetric representations. Certainly, it would be intriguing to see whether the mirror symmetry holds beyond multiplicity-free quantum 6j-symbols.

Another important aspect of quantum 6j-symbols is their relationship with \(q\)-hypergeometric functions [3]. The quantum 6j-symbols for \( U_q(\mathfrak{sl}_2) \) can be expressed as the balanced hypergeometric function \( {}_4\phi_3 \) [13]. However, for \( U_q(\mathfrak{sl}_N) \), the coefficients \(C_{ij}(\{\lambda_i\}, \lambda_{12}, \lambda_{23})\) prevents us from writing the expression (10) in terms of the balanced hypergeometric function \( {}_4\phi_3 \). Therefore, it is important to study the connection to generalized \(q\)-hypergeometric functions [17]. Besides, it is well-known that there are many different ways to express the quantum 6j-symbols for \( U_q(\mathfrak{sl}_2) \). Hence, it would be worthwhile to find the other expressions for the quantum 6j-symbols for \( U_q(\mathfrak{sl}_N) \).

Furthermore, there is a geometric interpretation of the quantum 6j-symbols. One can associate the quantum 6j-symbols to a tetrahedron whose edges are colored by representations of \( U_q(\mathfrak{sl}_N) \) [24]. Although it is necessary to have quantum 6j-symbols for arbitrary representations to obtain invariants of 3-manifolds [24] in the context of \( U_q(\mathfrak{sl}_N) \), the expression (10) is suitable to study the large color behavior [13]. Therefore, it would be interesting to explore the large color behavior of quantum 6j-symbols and their relation to the geometry of the complement of a tetrahedron in \( S^3 \).

As we have seen, quantum 6j-symbols are very interesting in their own right and contain remarkable mathematical structure. Despite their long history, they are indeed among the least understood quantities in mathematical physics. While in this paper we focus on the simplest class of multiplicity-free quantum 6j-symbols for \( U_q(\mathfrak{sl}_N) \), we hope that our results will serve as a stepping stone towards the study of general quantum 6j-symbols.

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