Nice formulas, $xyx$-formulas, and palindrome patterns

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Abstract. We characterize the formulas that are avoided by every $\alpha$-free word for some $\alpha > 1$. We study the avoidability index of formulas whose fragments are of the form $XYX$. The largest avoidability index of an avoidable palindrome pattern is known to be at least 4 and at most 16. We make progress toward the conjecture that every avoidable palindrome pattern is 4-avoidable.

1 Introduction

A \emph{pattern} $p$ is a non-empty finite word over an alphabet $\Delta = \{A, B, C, \ldots\}$ of capital letters called \emph{variables}. An \emph{occurrence} of $p$ in a word $w$ is a non-erasing morphism $h : \Delta^* \to \Sigma^*$ such that $h(p)$ is a factor of $w$ (a morphism is \emph{non-erasing} if the image of every letter is non-empty). The \emph{avoidability index} $\lambda(p)$ of a pattern $p$ is the size of the smallest alphabet $\Sigma$ such that there exists an infinite word over $\Sigma$ containing no occurrence of $p$. Since there is no risk of confusion, $\lambda(p)$ will be simply called the index of $p$.

A variable that appears only once in a pattern is said to be \emph{isolated}. Following Cassaigne \cite{Cassaigne}, we associate a pattern $p$ with the \emph{formula} $f$ obtained by replacing every isolated variable in $p$ by a dot. The factors between the dots are called \emph{fragments}.

An \emph{occurrence} of a formula $f$ in a word $w$ is a non-erasing morphism $h : \Delta^* \to \Sigma^*$ such that the $h$-image of every fragment of $f$ is a factor of $w$. As for patterns, the index $\lambda(f)$ of a formula $f$ is the size of the smallest alphabet allowing the existence of an infinite word containing no occurrence of $f$. Clearly, if a formula $f$ is associated with a pattern $p$, every word avoiding $f$ also avoids $p$, so $\lambda(p) \leq \lambda(f)$. Recall that an infinite word is \emph{recurrent} if every finite factor appears infinitely many times and that any infinite factorial language contains a recurrent word \cite[Proposition 5.1.13]{Cassaigne}. If there exists an infinite word over $\Sigma$ avoiding $p$, then there exists an infinite recurrent word over $\Sigma$ avoiding $p$. This recurrent word also avoids $f$, so that $\lambda(p) = \lambda(f)$. Without loss of generality, a

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formula is such that no variable is isolated and no fragment is a factor of another fragment.

Let us define the types of formulas we consider in this paper. A pattern is doubled if it contains every variable at least twice. Thus it is a formula with only one pattern. A formula \( f \) is nice if for every variable \( X \) of \( f \), there exists a fragment of \( f \) that contains \( X \) at least twice. Notice that a doubled pattern is a nice pattern. A formula is an \( xyx \)-formula if every fragment is of the form \( XYX \), i.e., the fragment has length 3 and the first and third variable are the same. A formula is hybrid if every fragment has length 2 or is of the form \( XYX \). Thus, an \( xyx \)-formula is a hybrid formula.

In Section 3, we consider the avoidance of nice formulas. In Section 4, we find some formulas \( f \) such that every recurrent word avoiding \( f \) over \( \Sigma_{\lambda(f)} \) is equivalent to a well-known morphic word. In Section 5, we consider the avoidance of \( xyx \)-formulas and hybrid formulas. In Section 6, we consider the avoidance of patterns that are palindromes.

2 Preliminaries

The Zimin function associates to a pattern \( p \) the pattern \( Z(p) = pXp \) where \( X \) is a variable that is not contained in \( p \). Notice that a recurrent word avoids \( Z(p) \) if and only if it avoids \( p \). In particular, \( \lambda(p) = \lambda(Z(p)) \).

We say that a formula \( f \) is divisible by a formula \( f' \) if \( f \) does not avoid \( f' \), that is, there is a non-erasing morphism \( h \) such that the image of any fragment of \( f' \) by \( h \) is a factor of a fragment of \( f \). If \( f \) is divisible by \( f' \), then every word avoiding \( f' \) also avoids \( f \) and \( \lambda(f') \geq \lambda(f) \). Let \( \Sigma_k = \{0, 1, \ldots, k-1\} \) denote the \( k \)-letter alphabet. We denote by \( \Sigma_k^n \) the \( k \) words of length \( n \) over \( \Sigma_k \).

The operation of splitting a formula \( f \) on a fragment \( \phi \) consists in replacing \( \phi \) by two fragments, namely the prefix and the suffix of length \( |\phi| - 1 \) of \( \phi \). A formula \( f \) is minimally avoidable if splitting any fragment of \( f \) gives an unavoidable formula. The set of every minimally avoidable formula with at most \( n \) variables is called the \( n \)-avoidance basis.

The adjacency graph \( AG(f) \) of the formula \( f \) is the bipartite graph such that

- for every variable \( X \) of \( f \), \( AG(f) \) contains the two vertices \( X_L \) and \( X_R \),
- for every (possibly equal) variables \( X \) and \( Y \), there is an edge between \( X_L \) and \( Y_R \) if and only if \( XY \) is a factor of \( f \).

We say that a set \( S \) of variables of \( f \) is free if for all \( X, Y \in S \), \( X_L \) and \( Y_R \) are in distinct connected components of \( AG(f) \). A formula \( f \) is said to reduce to \( f' \) if it is obtained by deleting all the variables of a free set from \( f \), discarding any empty word fragment. A formula is reducible if there is a sequence of reductions to the empty formula. Finally, a locked formula is a formula having no free set.

**Theorem 1 ([3]).** A formula is unavoidable if and only if it is reducible.

Let us define here the following well-known pure morphic words. To specify a morphism \( m : \Sigma_s \to \Sigma_e \), we use the notation \( m = m(0)/m(1)/\cdots/m(s-1) \).
Assuming a morphism $m : \Sigma_s \rightarrow \Sigma_s$ is such that $m(0)$ starts with 0, the fixed point of $m$ is the right infinite word $m^\omega(0)$.

- $b_2$ is the fixed point of $01/02$.  
- $b_3$ is the fixed point of $012/02/1$.  
- $b_4$ is the fixed point of $01/03/21/23$.  
- $b_5$ is the fixed point of $01/23/4/21/0$

We also consider the morphic words $v_3 = M_1(b_5)$ and $w_3 = M_2(b_5)$, where $M_1 = 012/1/02/12/e$ and $M_2 = 02/1/0/12/e$. The languages of each of these words have been studied in the literature. Let us first recall the following characterization of $b_3$, $v_3$, and $w_3$. We say that two infinite words are equivalent if they have the same set of factors.

Theorem 2 ([1]).

- Every ternary square-free recurrent word avoiding $010$ and $212$ is equivalent to $b_3$.  
- Every ternary square-free recurrent word avoiding $010$ and $020$ is equivalent to $v_3$.  
- Every ternary square-free recurrent word avoiding $121$ and $212$ is equivalent to $w_3$.

Interestingly, these three words can be characterized in terms of a forbidden distance between consecutive occurrences of one letter.

Theorem 3.

- Every ternary square-free recurrent word such that the distance between consecutive occurrences of 1 is not 3 is equivalent to $b_3$.  
- Every ternary square-free recurrent word such that the distance between consecutive occurrences of 0 is not 2 is equivalent to $v_3$.  
- Every ternary square-free recurrent word such that the distance between consecutive occurrences of 0 is not 4 is equivalent to $w_3$.

The word $b_4$ is also known to avoid large families of formulas.

Theorem 4 ([2]). Every locked formula is avoided by $b_4$.

Theorem 5 ([5, Proposition 1.13]). If every fragment of an avoidable formula $f$ has length 2, then $b_4$ avoids $f$.

Let us give here a result that will be needed in various parts of the paper.

Lemma 6. $ABA.ACA.ABCA.ACBA.ABCBA \preceq AA$.

Proof. Indeed, $Z^2(AA) = AABAACAABAA$ contains the occurrence $A \rightarrow A$, $B \rightarrow ABA$, $C \rightarrow ACA$ of $ABA.ACA.ABCA.ACBA.ABCBA$.  

\[\square\]
Thus, if \( w \) is a recurrent word that avoids a formula dividing \( ABA.ACA.ABCA.ACBA.ABCBA \), then \( w \) is square-free.

Recall that the repetition threshold \( RT(n) \) is the smallest real number \( \alpha \) such that there exists an infinite \( \alpha \)-free word over \( \Sigma_n \). The proof of Dejean’s conjecture established that \( RT(2) = 2 \), \( RT(3) = \frac{7}{2} \), \( RT(4) = \frac{7}{4} \), and \( RT(n) = \frac{n}{n-1} \) for every \( n \geq 5 \). An infinite \( RT(n)^\dagger \)-free word over \( \Sigma_n \) is called a Dejean word.

### 3 Nice formulas

All the nice formulas considered so far in the literature are also 3-avoidable. This includes doubled patterns [11], circular formulas [8], the nice formulas in the 3-avoidance basis [8], and the minimally nice ternary formulas in Table 1 [14].

**Theorem 7 ([8,14]).** Every nice formula with at most 3 variables is 3-avoidable.

We have a risky conjecture that would generalize both Theorem 7 and the 3-avoidability of doubled patterns.

**Conjecture 8.** Every nice formula is 3-avoidable.

Theorem 19 in Section 5 shows that there exist infinitely many nice formulas with index 3. It means that Conjecture 8 would be best possible and it contrasts with the case of doubled patterns, since we expect that there exist only finitely many doubled patterns with index 3 [11,12]. In this section, we make progress toward Conjecture 8 by proving that every nice formula is avoidable and we explain how to get an upper bound on the index of a given nice formula.

#### 3.1 The avoidability exponent

Let us consider a useful tool in pattern avoidance that has been defined in [11] and already used implicitly in [10]. The **avoidability exponent** \( AE(p) \) of a pattern \( p \) is the largest real \( \alpha \) such that every \( \alpha \)-free word avoids \( p \). We extend this definition to formulas.

Let us show that \( AE(ABCBA.CBABC) = \frac{4}{3} \). Suppose for contradiction that a \( \frac{4}{3} \)-free word contains an occurrence \( h \) of \( ABCBA.CBABC \). We write \( y = |h(Y)| \) for every variable \( Y \). The factor \( h(ABCBA) \) is a repetition with period \( |h(ABCB)| \). So we have \( \frac{a+b+c+b+a}{a+b+c+b} < \frac{4}{3} \). This simplifies to \( 2a < 2b+c \). Similarly, \( CBABC \) gives \( 2c < a+2b \), \( BAB \) gives \( 2b < a \), and \( BCB \) gives \( 2b < c \). Summing up these four inequalities gives \( 2a+4b+2c < 2a+4b+2c \), which is a contradiction. On the other hand, the word \( 0123420156786834201234 \) is \( \left( \frac{4}{3} \right)^\dagger \)-free and contains the occurrence \( A \rightarrow 01, B \rightarrow 2, C \rightarrow 34 \) of \( ABCBA.CBABC \).

As a second example, we obtain that \( AE(ABCDBACBD) = 1.246266172 \ldots \). When we consider a repetition \( uvu \) in an \( \alpha \)-free word, we derive that \( \frac{|uvu|}{|uv|} < \alpha \), which gives \( \beta|u| < |v| \) with \( \alpha = 1 + \frac{1}{\beta+1} \). We consider an occurrence \( h \) of the
pattern. The maximal repetitions in \(ABCDBACBD\) are \(ABCDBA, BCDB, BACB, CDBAC,\) and \(DBACBD\). They imply the following inequalities.

\[
\begin{align*}
\beta a &\leq 2b + c + d \\
\beta b &\leq c + d \\
\beta b &\leq a + c \\
\beta c &\leq a + b + d \\
\beta d &\leq a + 2b + c
\end{align*}
\]

We look for the smallest \(\beta\) such that this system has no solution. Notice that \(a\) and \(d\) play symmetric roles. Thus, we can set \(a = d\) and simplify the system.

\[
\begin{align*}
\beta a &\leq a + 2b + c \\
\beta b &\leq a + c \\
\beta c &\leq 2a + b
\end{align*}
\]

Then \(\beta\) is the largest eigenvalue of the matrix \[
\begin{bmatrix}
1 & 2 & 1 \\
1 & 0 & 1 \\
2 & 1 & 0
\end{bmatrix}
\]that corresponds to the latter system. So \(\beta = 3.060647027\ldots\) is the largest root of the characteristic polynomial \(x^3 - x^2 - 5x - 4\). Then \(\alpha = 1 + \frac{1}{\beta + 1} = 1.246266172\ldots\)

This matrix approach is a convenient trick to use when possible. It was used in particular for some doubled patterns such that every variable occurs exactly twice [11]. It may fail if the number of inequalities is strictly greater than the number of variables or if the formula contains a repetition \(uvu\) such that \(|u| \geq 2\).

In any case, we can fix a rational value to \(\beta\) and ask a computer algebra system whether the system of inequalities is solvable. Then we can get arbitrarily good approximations of \(\beta\) (and thus \(\alpha\)) by a dichotomy method.

Of course, the avoidability exponent is related to divisibility.

**Lemma 9.** If \(f \preceq g\), then \(AE(f) \leq AE(g)\).

The avoidability exponent depends on the repetitions induced by \(f\). We have \(AE(f) = 1\) for formulas such as \(f = AB.BA.AC.CA.BC\) or \(f = AB.BA.AC.BC.CDA.DCD\) that do not have enough repetitions. That is, for every \(\epsilon > 0\), there exists a \((1 + \epsilon)\)-free word that contains an occurrence of \(f\).

Let us investigate formulas with non-trivial avoidability exponent, that is, \(AE(f) > 1\). To show that a nice formula has a non-trivial avoidability exponent (see Lemma 10), we first introduce a notion of minimality for nice formulas similar to the notion of minimally avoidable for general formulas. A nice formula \(f\) is **minimally nice** if there exists no nice formula \(g\) such that \(v(g) \leq v(f)\) and \(g \preceq f\). Alternatively, splitting a minimally nice formula on any of its fragments leads to a non-nice formula. The following property of every minimally nice formula is easy to derive. If a variable \(V\) appears as a prefix of a fragment \(\phi\), then

- \(V\) is also a suffix of \(\phi\) (since otherwise we can split on \(\phi\) and obtain a nice formula),
- \(\phi\) contains exactly two occurrences of \(V\) (since otherwise we can remove the prefix letter \(V\) from \(\phi\) and obtain a nice formula),
- $V$ is neither a prefix nor a suffix of any fragment other than $\phi$ (since otherwise we can remove this prefix/suffix letter $V$ from the other fragment and obtain a nice formula).
- Every fragment other than $\phi$ contains at most one occurrence of $V$ (since otherwise we can remove the prefix letter $V$ from $\phi$ and obtain a nice formula).

**Lemma 10.** If $f$ is a nice formula, then $AE(f) \geq 1 + 2^{1-v(f)}$.

**Proof.** Notice that $AE(AA) = 2$ and $AE(ABA.BAB) = \frac{3}{2}$, which settles the case $v(f) \leq 2$. Suppose that $f$ contradicts the lemma. Since $1 + 2^{1-v(f)}$ is decreasing with $v(f)$, we can assume that $f$ is a minimally nice formula by Lemma 9.

Then there exists a $(1 + 2^{1-v(f)})$-free word $w$ containing an occurrence $h$ of $f$. Let $X$ be a variable of $f$ such that $|h(X)| \geq |h(Y)|$ for every variable $Y$. Thus, for every sequence $s$ of variables, $|h(s)| \leq |s| \times |h(X)|$. Since $f$ is nice, $f$ contains a factor of the form $XzX$. By minimality, $z$ does not contain $X$, so that $v(z) \leq v(f) - 1$.

If $|z| \geq 2^{v(z)}$, then $z$ contains a doubled pattern with at most $v(z)$ variables [12, Claim 3]. This contradicts the minimality of $f$.

If $|z| \leq 2^{v(z)} - 1$, then the exponent of $h(XzX)$ in $w$ is $\frac{|h(XzX)|}{|h(X)|} = 1 + \frac{|h(X)|}{|h(X)|} \geq 1 + \frac{1}{2^{v(z)} - 1} \geq 1 + \frac{1}{2^{v(f) - 1}} = 1 + 2^{1-v(f)}$. This contradicts that $w$ is $(1 + 2^{1-v(f)})$-free.

The circular formulas studied in [8] show that $AE(f)$ can be as low as $1 + (v(f))^{-1}$. Moreover, our example $AE(ABCDBACBD) = 1.246266172\ldots$ shows that lower avoidability exponents exist among nice formulas with at least 4 variables. However, the bound in Lemma 10 is probably very far from optimal.

We will describe below a method to construct infinite words avoiding a formula. This method can be applied if and only if the formula $f$ satisfies $AE(f) > 1$. So we are interested in characterizing the formulas $f$ such that $AE(f) > 1$. By Lemmas 9 and 10, if $f$ is a formula such that there exists a nice formula $g$ satisfying $g \preceq f$, then $AE(f) > 1$. Now we prove that the converse also holds, which gives the following characterization.

**Theorem 11.** A formula $f$ satisfies $AE(f) > 1$ if and only if there exists a nice formula $g$ such that $g \preceq f$.

**Proof.** What remains to prove is that for every formula $f$ that is not divisible by a nice formula and for every $\varepsilon > 0$, there exists an infinite $(1 + \varepsilon)$-free word $w$ containing an occurrence of $f$, such that the size of the alphabet of $w$ only depends on $f$ and $\varepsilon$.

First, we consider the equivalent pattern $p$ obtained from $f$ by replacing every dot by a distinct variable that does not appear in $f$. We will actually construct an occurrence of $p$. Then we construct a family $f_i$ of pseudo-formulas as follows. We start with $f_0 = p$. To obtain $f_{i+1}$ from $f_i$, we choose a variable that appears
at most once in every fragment of \( f_i \). This variable is given the alias name \( V_i \) and every occurrence of \( V_i \) is replaced by a dot. We say that \( f_i \) is a pseudo-formula since we do not try to normalize \( f_i \), that is, \( f_i \) can contain consecutive dots and \( f_i \) can contain fragments that are factors of other fragments. However, we still have a notion of fragment for a pseudo-formula. Since \( f \) is not divisible by a nice formula, this process ends with the pseudo-formula \( f_{v(p)} \) with no variable and \(|p|\) consecutive dots. The goal of this process is to obtain the ordering \( V_0, V_1, \ldots, V_{v(p)−1} \) on the variables of \( p \).

The image of every \( V_i \) is a finite factor \( w_i \) of a Dejean word over an alphabet of \( \lfloor ε−1 \rfloor + 2 \) letters, so that \( w_i \) is \((1 + ε)\)-free. The alphabets are disjoint: if \( i \neq j \), then \( w_i \) and \( w_j \) have no common letter. Finally, we define the length of \( w_i \) as follows: \(|w_{v(p)−1}| = 1 \) and \(|w_i| = \lfloor ε−1 \rfloor \times |p| \times |w_{i+1}| \) for every \( i \) such that \( 0 \leq i \leq v(p) − 2 \). Let us show by contradiction that the constructed occurrence \( h \) of \( p \) is \((1 + ε)\)-free. Consider a repetition \( xyzx \) of exponent at least \( 1 + ε \) that is maximal, that is, which cannot be extended to a repetition with the same period and larger exponent. Since every \( w_i \) is \((1 + ε)\)-free and since two matching letters must come from distinct occurrences of the same variable, then \( x = h(x′) \) and \( y = h(y′) \) where \( x′ \) and \( y′ \) are factors of \( p \). Our ordering of the variables of \( p \) implies that \( y′ \) contains a variable \( V_i \) such that \( i < j \) for every variable \( V_j \) in \( x′ \). Thus, \(|y| \geq |w_i| = \lfloor ε−1 \rfloor \times |p| \times |w_{i+1}| \geq \lfloor ε−1 \rfloor \times |x| \), which contradicts the fact that the exponent of \( xyzx \) is at least \( 1 + ε \).

To obtain the infinite word \( w \), we can insert our occurrence of \( p \) into a bi-infinite \((1 + ε)\)-free word over an alphabet of \( \lfloor ε−1 \rfloor + 2 \) new letters. So \( w \) is an infinite \((1 + ε)\)-free word over an alphabet of \( v(p) \left( \lfloor ε−1 \rfloor + 2 \right) + 1 \) letters which contains an occurrence of \( f \). 

By Lemma 10, every nice formula is avoidable since it is avoided by a Dejean word over a sufficiently large alphabet. Thus, if a formula is nice and minimally avoidable, then it is minimally nice. This is the case for every formula in the 3-avoidance basis, except \( AB.AC.BACA.CB \). However, a minimally nice formula is not necessarily minimally avoidable. Indeed, we have shown [14] that the set of minimally nice ternary formulas consists of the nice formulas in the 3-avoidance basis, together with the minimally nice formulas in Table 1 that can be split to \( AB.AC.BACA.CB \).

\[\begin{align*}
- & ABA.BCB.CAC \\
- & ABCA.BCAB.CBAC \text{ and its reverse} \\
- & ABCA.BAB.CAC \\
- & ABCA.BAB.CBC \text{ and its reverse} \\
- & ABCA.BAB.CBAC \text{ and its reverse} \\
- & ABCBA.CABC \text{ and its reverse} \\
- & ABCBA.CAC \\
\end{align*}\]

Table 1. The minimally nice ternary formulas that are not minimally avoidable.
3.2 Avoiding a nice formula

Recall that a nice formula \( f \) is such that \( AE(f) > 1 \). We consider the smallest integer \( s \) such that \( RT(n) < AE(f) \). Thus, every Dejean word over \( \Sigma_s \) avoids \( f \), which already gives \( \lambda(f) \leq s \). Recall that a morphism is \( q \)-uniform if the image of every letter has length \( q \). Also, a uniform morphism \( h : \Sigma_s^* \to \Sigma_s^* \) is synchronizing if for any \( a, b, c \in \Sigma_s \) and \( v, w \in \Sigma_s^* \), if \( h(ab) = vh(c)w \), then either \( v = \varepsilon \) and \( a = c \) or \( w = \varepsilon \) and \( b = c \). For increasing values of \( q \), we look for a \( q \)-uniform morphism \( h : \Sigma_s^* \to \Sigma_s^* \) such that \( h(w) \) avoids \( f \) for every \( RT(s)^+ \)-free word \( w \in \Sigma_s^\ell \), where \( \ell \) is given by Lemma 12 below. Recall that a word is \( (\beta^+, n) \)-free if it contains no repetition with exponent strictly greater than \( \beta \) and period at least \( n \).

Lemma 12. [10] Let \( \alpha, \beta \in \mathbb{Q} \), \( 1 < \alpha < \beta < 2 \) and \( n \in \mathbb{N}^* \). Let \( h : \Sigma_s^* \to \Sigma_s^* \) be a synchronizing \( q \)-uniform morphism (with \( q \geq 1 \)). If \( h(w) \) is \( (\beta^+, n) \)-free for every \( \alpha^+ \)-free word \( w \) such that \( |w| < \max \left( \frac{2q\beta}{n}, \frac{2q(\beta-1)}{(\beta+1)} \right) \), then \( h(w) \) is \( (\beta^+, n) \)-free for every (finite or infinite) \( \alpha^+ \)-free word \( w \).

Given such a candidate morphism \( h \), we use Lemma 12 to show that for every \( RT(s)^+ \)-free word \( w \in \Sigma_s^* \), the image \( h(w) \) is \( (\beta^+, n) \)-free. The pair \( (\beta, n) \) is chosen such that \( RT(s) < \beta < AE(f) \) and \( n \) is the smallest possible for the corresponding \( \beta \). If \( \beta < AE(f) \), then every occurrence \( h \) of \( f \) in a \( (\beta^+, t) \)-free word \( w \) is such that the length of the \( h \)-image of every variable of \( f \) is upper bounded by a function of \( n \) and \( f \) only. Thus, the \( h \)-image of every fragment of \( f \) has bounded length and we can check that \( f \) is avoided by inspecting a finite set of factors of words of the form \( h(w) \).

3.3 The number of fragments of a minimally avoidable formula

Interestingly, the notion of (minimally) nice formula is helpful in proving the following.

Theorem 13. The only minimally avoidable formula with exactly one fragment is \( AA \).

Proof. A formula with one fragment is a doubled pattern. Since it is minimally avoidable, it is a minimally nice formula. By the properties of minimally nice formulas discussed above, the unique fragment of the formula is either \( AA \) or is of the form \( ApA \) such that \( p \) does not contain the variable \( A \). Thus, \( p \) is a doubled pattern such that \( p < ApA \), which contradicts that \( ApA \) is minimally avoidable. \( \square \)

By contrast, the family of two-birds formulas, which consists of \( ABA.BAB \), \( ABCBA.CBABC \), \( ABCDCBA.DCBABCD \), and so on, shows that there exist infinitely many minimally avoidable formulas with exactly two fragments. Every two-birds formula is nice. Let us check that every two-birds formula \( AB \cdots X \cdots BA.X \cdots A \cdots X \) is minimally avoidable. Since the two fragments
play symmetric roles, it is sufficient to split on the first fragment. We obtain the formula \( AB \cdots X \cdots B.B \cdots X \cdots BA.X \cdots A \cdots X \) which divides the pattern \( B \cdots X \cdots BAB \cdots X \cdots B = Z(B \cdots X \cdots B) \). This pattern is equivalent to \( B \cdots X \cdots B \), which is unavoidable. Thus, every two-birds formula is indeed minimally avoidable.

Concerning the index of two-birds formulas, we have seen that \( \lambda(ABA.BAB) = 3 \) and \( \lambda(ABCBA.CBABC) = 2 \) \[8\]. Computer experiments suggest that larger two-birds formulas are easier to avoid.

**Conjecture 14.** Every two-birds formula with at least 3 variables is 2-avoidable.

### 4 Characterization of some famous morphic words

Our next result gives characterizations of \( w_3 \), up to renaming, that use just one formula. Then we give similar characterizations of \( b_3 \) and \( b_2 \). Let \( \sigma = 1/2/0 \) be the morphism that cyclically permutes \( \Sigma_3 \).

**Theorem 15.** Let \( f \) be a ternary formula such that \( ABA.BCB.ACA \preceq f \preceq ABA.ABCBA.ACA.ACB.BCA \). Every ternary recurrent word avoiding \( f \) is equivalent to \( w_3, \sigma(w_3) \), or \( \sigma^2(w_3) \).

**Proof.** Using Cassaigne’s algorithm \[4\], we have checked that \( w_3 \) avoids \( ABA.BCB.ACA \). By divisibility, \( w_3 \) avoids \( f \).

Let \( w \) be a ternary recurrent word avoiding \( f \). By Lemma 6, \( w \) is square-free.

Let \( v = 210201202101201021 \). A computer check shows that no infinite ternary word avoids \( ABA.ABCBA.ACA.ACB.BCA \), squares, \( v \), \( \sigma(v) \), and \( \sigma^2(v) \). So, without loss of generality, \( w \) contains \( v \). If \( w \) contains \( 121 \), then \( w \) contains the occurrence \( A \rightarrow 1, B \rightarrow 2, C \rightarrow 0 \) of \( ABA.ACA.ABCA.ACBA.ABCBA \). Similarly, if \( w \) contains \( 212 \), then \( w \) contains the occurrence \( A \rightarrow 2, B \rightarrow 1, C \rightarrow 0 \) of \( ABA.ACA.ABCA.ACBA.ABCBA \). Thus, \( w \) avoids squares, \( 121 \), and \( 212 \). By Theorem 2, \( w \) is equivalent to \( w_3 \).

By symmetry, every ternary recurrent word avoiding \( f \) is equivalent to \( w_3, \sigma(w_3) \), or \( \sigma^2(w_3) \). \( \Box \)

**Theorem 16.** Let \( f \) be such that

- \( ABCA.ABA.ACA \preceq f \preceq ABCA.ABA.ACA.ACB.CBA \),
- \( ABCA.ABA.BCB.AC \preceq f \preceq ABCA.ABA.ABCBA.ACB \), or
- \( ABCA.ABA.BCB.CBA \preceq f \preceq ABCA.ABA.ABCBA.ACB \).

Every ternary recurrent word avoiding \( f \) is equivalent to \( b_3, \sigma(b_3) \), or \( \sigma^2(b_3) \).

**Proof.** Using Cassaigne’s algorithm \[4\], we have checked that \( b_3 \) avoids \( ABCA.ABA.ACA \), \( ABCA.ABA.BCB.AC \), and \( ABCA.ABA.BCB.CBA \). By divisibility, \( b_3 \) avoids \( f \).

Let \( w \) be a ternary recurrent word avoiding \( f \). By Lemma 6, \( w \) is square-free.

Let \( v = 202101201021020120 \). A computer check shows that no infinite ternary word avoids \( ABCA.ABA.ACA.ACB.CBA \) (resp. \( ABCA.ABA.ABCBA.ACB \)), squares, \( v \), \( \sigma(v) \), and \( \sigma^2(v) \).
So, without loss of generality, \( w \) contains \( v \). If \( w \) contains \( 010 \), then \( w \) contains the occurrence \( A \to 0, B \to 1, C \to 2 \) of \( ABA.ACA.ABCA.ACBA.ABCBA \). Similarly, if \( w \) contains \( 212 \), then \( w \) contains the occurrence \( A \to 2, B \to 1, C \to 0 \) of \( ABA.ACA.ABCA.ACBA.ABCBA \). Thus, \( w \) avoids squares, \( 010 \), and \( 212 \). By Theorem 2, \( w \) is equivalent to \( b_3 \).

By symmetry, every ternary recurrent word avoiding \( f \) is equivalent to \( b_3 \), \( \sigma(b_3) \), or \( \sigma^2(b_3) \). \( \square \)

Notice that Theorem 16 is a complement to [14, Theorem 2] in which we gave a disjoint set of formulas with the same property. The difference between Theorem 16 and [14, Theorem 2] is that a different occurrence of \( f \) shows that \( f \) divides \( Z^n(AA) \).

**Theorem 17.** Let \( f_h = AABCAA.BCB, f_e = AABCAAB.AABCAB.AABCB \), and let \( f \) be such that \( f_h \preceq f \preceq f_e \). Every binary recurrent word avoiding \( f \) is equivalent to \( b_2 \).

**Proof.** Using Cassaigne’s algorithm [4], we have checked that \( b_2 \) avoids \( f_h \). First, \( f_e \preceq AAA \) because \( Z(AAA) = AAAABAAA \) contains the occurrence \( A \to A, B \to A, C \to B \) of \( f_e \). Second, \( f_e \preceq ABABA \) because \( Z(ABABA) = ABABACABABA \) contains the occurrence \( A \to AB, B \to A, C \to C \) of \( f_e \).

Thus, every recurrent word avoiding \( f_e \) also avoids \( AAA \) and \( ABABA \), which means that it is overlap-free. Finally, it is well-known that every binary recurrent word that is overlap-free is equivalent to \( b_2 \). \( \square \)

## 5 \( xyx \)-formulas

Recall that every fragment of an \( xyx \)-formula is of the form \( XYX \). We associate to an \( xyx \)-formula \( F \) the directed graph \( G \) such that every variable corresponds to a vertex and \( G \) contains the arc \( XY \) if and only if \( F \) contains the fragment \( XYX \). We will also denote by \( G \) the underlying simple graph of \( \overrightarrow{G} \).

**Lemma 18.** Let \( F_1 \) and \( F_2 \) be \( xyx \)-formulas associated to \( G_1 \) and \( G_2 \). If there exists a homomorphism \( \overrightarrow{G_1} \to \overrightarrow{G_2} \), then \( F_1 \preceq F_2 \).

**Proof.** Since both digraph homomorphism and formula divisibility are transitive relations, we only need to consider the following two cases. If \( G_1 \) is a subgraph of \( G_2 \), then \( F_1 \) is obtained from \( F_2 \) by removing some fragments. So every occurrence of \( F_2 \) is also an occurrence of \( F_1 \) and thus \( F_1 \preceq F_2 \). If \( G_2 \) is obtained from \( G_1 \) by identifying the vertices \( u \) and \( v \), then \( F_2 \) is obtained from \( F_1 \) by identifying the variables \( U \) and \( V \). So every occurrence of \( F_2 \) is also an occurrence of \( F_1 \) and thus \( F_1 \preceq F_2 \). \( \square \)

For every \( i \), let \( T_i \) be the \( xyx \)-formula corresponding to the directed circuit \( G_i \) of length \( i \), that is, \( T_1 = AAA, T_2 = ABA.BAB, T_3 = ABA.BCB.CAC, T_4 = ABA.BCB.CDC.DAD \), and so on. More formally, \( T_i \) is the formula with
$i$ variables $A_0,\ldots, A_{i-1}$ which contains the $i$ fragments of length three of the form $A_jA_{j+1}A_j$ such that the indices are taken modulo $i$. Notice that $T_i$ is a nice formula.

**Theorem 19.** For every $i \geq 2$, $\lambda(T_i) = 3$

**Proof.** We use Lemma 12 to show that the image of every $(7/4^\ast)$-free word over $\Sigma_4$ by the following 58-uniform morphism is $(3/2,3)$-free.

$$
\begin{align*}
0 &\rightarrow 0012211002201021120022100112201002211200112201102211002211201022 \\
1 &\rightarrow 001221002201021120011022110021122010021122001102211002211201022 \\
2 &\rightarrow 00112210022010221120011022110021122010021122001102211002211201022 \\
3 &\rightarrow 0011221002201021120011022110021122010021122001102211002211201022
\end{align*}
$$

In these words, the factor $010$ is the only occurrence $m$ of $ABA$ such that $|m(A)| \geq |m(B)|$. This implies that these ternary words avoid $T_i$ for every $i \geq 1$, so that $\lambda(T_i) \leq 3$.

To show that $T_i$ is not 2-avoidable, we consider the $xyz$-formula $H = ABA.BAB.ACA.CBC$ associated to the directed graph $\overrightarrow{D_3}$ on 3 vertices and 4 arcs that contains a circuit of length 2 and a circuit of length 3. Standard backtracking shows that $\lambda(H) > 2$, and even the stronger result that $\lambda(ABA.BAB.ACA.CBC) > 2$.

For every $i \geq 2$, the circuit $C_i$ admits a homomorphism to $\overrightarrow{D_3}$. By Lemma 18, this means that $T_i \preceq H$, which implies that $\lambda(T_i) \geq \lambda(H) \geq 3$.

**Theorem 20.** For every $i \geq 1$, $b_4$ avoids $T_i$.

**Proof.** Suppose for contradiction that there exist $i$ and $n$ such that $m^n(0)$ contains an occurrence $h$ of $T_i$. Further assume that $n$ is minimal. Notice that in $b_4$, every even (resp. odd) letter appears only at even (resp. odd) positions. Thus, for every fragment $XYX$ of $T_i$, the period $|h(XY)|$ of the repetition $h(XX)$ must be even. This implies that $|h(X)|$ and $|h(Y)|$ have the same parity. By contagion, the lengths of the images of all the variables of $T_i$ have the same parity. Now we proceed to a case analysis.

- Every $|h(X)|$ is even.
  - Every $h(X)$ starts with 0 or 2. By taking the pre-image by $m$ of every $h(X)$, we obtain an occurrence of $T_i$ that is contained in $m^{n-1}(0)$. This contradicts the minimality of $n$.
  - Every $h(X)$ starts with 1 or 3. Notice that in $b_4$, the letter 1 (resp. 3) is in position 1 (mod 4) (resp. 3 (mod 4)). $m^n(0)$ contains the occurrence $h'$ of $T_i$ such that $h'(X)$ is obtained from $h(X)$ by adding to the right the letter 1 or 3 depending on its position modulo 4 and by removing the first letter. Since is also contained in $m^n(0)$ and every $h'(X)$ starts with 0 or 2, $h'$ satisfies the previous subcase.

- Every $|h(X)|$ is odd. It is not hard to check that every factor $uvw$ in $b_4$ with $|v| = 1$ satisfies $v \in \{1,3\}$ and $u \in \{0,2\}$. So $|h(X)| \geq 3$ for every variable $X$ of $T_i$. Let $X_1, \ldots, X_i$ be the variables of $T_i$. Up to a shift of indices, we can assume that $j$ and the first and last letters of $h(X_j)$ have the same parity.
We construct the occurrence $h'$ of $T_i$ as follows. If $j$ is odd, then $h'(X_j)$ is obtained by removing the first letter of $h(X_j)$. If $j$ is even, then $h'(X_j)$ is obtained by adding to the right the letter 1 or 3 depending on its position modulo 4. Since $h'$ is also contained in $m^n(0)$ and every $|h'(X)|$ is even, $h'$ satisfies the previous case.

Our next result generalizes Theorems 5 and 20. Recall that every fragment of a hybrid formula has length 2 or is of the form $XYX$.

**Theorem 21.** Every avoidable hybrid formula is avoided by $b_4$.

**Proof.** Let $f$ be a hybrid formula. If $f$ contains a locked formula or a formula $T_i$, then $b_4$ avoids $f$ by Theorems 4 and 20. If $f$ contains neither a locked formula nor a formula $T_i$, then we show that $f$ is unavoidable. By induction and by Theorem 1 it is sufficient to show that $f$ is reducible to a hybrid formula containing neither a locked formula nor a formula $T_i$. Since $f$ is not locked, $f$ contains a free set of variables and thus $f$ has a free singleton $\{X\}$. If $f$ contains a fragment $XYX$, then $\{Y\}$ is also a free singleton of $f$. Using this argument iteratively, we end up with a free singleton $\{Z\}$ such that $f$ contains no fragment $TZT$, since $f$ contains no formula $T_i$.

So we can assume that $f$ contains a free singleton $\{Z\}$ and no fragment $TZT$. Thus, deleting every occurrence of $Z$ from $f$ gives an hybrid sub-formula containing neither a locked formula nor a formula $T_i$. By induction, $f$ is unavoidable. □

So the index of an avoidable $xyx$-formula is at most 4 and we have seen examples of $xyx$-formulas with index 3 in Theorems 15 and 19. The next results give an $xyx$-formula with index 2 and an $xyx$-formula with index 2 that is not divisible by AAA.

**Theorem 22.** $\lambda(ABA.BCB.DCD.DED.AEA) = 4$.

**Proof.** By Theorem 21, $ABA.BCB.DCD.DED.AEA$ is 4-avoidable. Notice that $ABA.BCB.DCD.DED.AEA \preceq ABA.BCB.ACA$ via the homomorphism $A \rightarrow A$, $B \rightarrow B$, $C \rightarrow C$, $D \rightarrow B$, $E \rightarrow C$. Moreover, $w_3$ contains the occurrence $A \rightarrow 0$, $B \rightarrow 1$, $C \rightarrow 02$, $D \rightarrow 01$, $E \rightarrow 2$ of $ABA.BCB.DCD.DED.AEA$. By Theorem 15, the formula is not 3-avoidable. □

**Theorem 23.** The fixed point of 001/011 avoids the $xyx$-formula associated to the directed graph on 4 vertices with all the 12 arcs.

**Proof.** We use again Cassaigne’s algorithm. □

6 Palindrome patterns

Mikhailova [9] has considered the index of an avoidable pattern that is a palindrome and proved that it is at most 16. She actually constructed a morphic word over $\Sigma_{16}$ that avoids every avoidable palindrome pattern.
We make a distinction between the largest index $P_w$ of an avoidable palindrome pattern and the smallest alphabet size $P_s$ allowing an infinite word avoiding every avoidable palindrome pattern. We obtained [14] the lower bound $\lambda(ABCAACBA) = \lambda(ABCAACBA) = 4$, so that $4 \leq P_w \leq P_s \leq 16$.

The following result is a slight improvement to $\lambda(ABCAACBA) = 4$ that is not related to palindromes.

**Theorem 24.** $\lambda(ABCAACBAABCBA) = 4$.

**Proof.** By Lemma 6, every recurrent word avoiding $ABCAACBAABCBA$ is square-free. A computer check shows that no infinite ternary square-free word avoids the occurrences $h$ of $ABCAACBAABCBA$ such that $|h(A)| = 1$, $|h(B)| \leq 2$, and $|h(C)| \leq 3$.

Let us give necessary conditions on a palindrome pattern $P$ so that $5 \leq \lambda(P) \leq 16$.

1. The length of $P$ is odd and the central variable of $P$ is isolated. Indeed, otherwise $P$ would be a doubled pattern and thus 3-avoidable [11].

2. No variable of $P$ appears both at an even and an odd position. Indeed, if $P$ had a variable that appears both at an even and an odd position, then $P$ would be divisible by a formula in the family $AA$, $ABCAACBA$, $ABCDEA.AEDCBA$, $ABCDEF GA.AGF EDCBA$, . . . Such formulas (with an odd number of variables) are locked and thus are avoided by $b_4$ by Theorem 4. So $P$ would be 4-avoidable.

We have found three patterns/formulas satisfying these conditions (see Theorem 25), but they seem to be 2-avoidable. We use again Cassaigne’s algorithm with simple pure morphic words to ensure that they are 4-avoidable. Let $z_3$ be the fixed point of $01/2/20$.

**Theorem 25.**

1. $ADBDCDAD.DADCDBDA$ is avoided by $b_4$.
2. $ABCDADC.CDADCBA$ is avoided by $z_3$.
3. $ABACDBAC.CABDCABA$ is avoided by $z_3$ and $b_4$.

7 Discussion

Let us briefly mention the things that we have attempted to do in this paper, without success.

– Improve the bound in Lemma 10.
– Improve Theorem 23 by showing that some $xyx$-formula on 4 variables and fewer fragments is 2-avoidable.
– Show that the $xyx$-formula associated to the transitive tournament on 5 vertices is 2-avoidable.
References

1. G. Badkobeh and P. Ochem. Characterization of some binary words with few squares. *Theor. Comput. Sci.* 588 (2015), 73–80.
2. K. A. Baker, G. F. McNulty, and W. Taylor. Growth problems for avoidable words. *Theoret. Comput. Sci.*, 69(3):319–345, 1989.
3. D. R. Bean, A. Ehrenfeucht, and G. F. McNulty, Avoidable patterns in strings of symbols. *Pac. J. of Math.* 85 (1979), 261-294
4. J. Cassaigne. *An Algorithm to Test if a Given Circular HD0L-Language Avoids a Pattern.* IFIP Congress, pages 459–464, 1994.
5. J. Cassaigne. *Motifs évitables et régularité dans les mots.* PhD thesis, Université Paris VI, 1994.
6. R. J. Clark. *Avoidable formulas in combinatorics on words.* PhD thesis, University of California, Los Angeles, 2001. Available at [http://www.lirmm.fr/~ochem/morphisms/clark_thesis.pdf](http://www.lirmm.fr/~ochem/morphisms/clark_thesis.pdf)
7. Pytheas Fogg. Substitutions in Dynamics, Arithmetics and Combinatorics. Springer Science & Business Media, 2002.
8. G. Gamard, P. Ochem, G. Richomme, and P. Séébold. Avoidability of circular formulas. *Theor. Comput. Sci.*, 726:1–4, 2018.
9. I. Mikhailova. On the avoidability index of palindromes. *Matematicheskie Zametki.*, 93(4):634–636, 2013.
10. P. Ochem. A generator of morphisms for infinite words. *RAIRO - Theoret. Informatics Appl.*, 40:427–441, 2006.
11. P. Ochem. Doubled patterns are 3-avoidable. *Electron. J. Combinatorics.*, 23(1):#P1.19, 2016.
12. P. Ochem and A. Pinlou. Application of entropy compression in pattern avoidance. *Electron. J. Comb.* 21(2) (2014), #P2.7.
13. P. Ochem and M. Rosenfeld. Avoidability of formulas with two variables. *Electron. J. Combin.*, 24(4):#P4.30, 2017.
14. P. Ochem and M. Rosenfeld. On some interesting ternary formulas. *Electron. J. Combin.*, 26(1):#P1.12, 2019.