Abstract.

We derive explicit local transport relations for the global gyrokinetic formalism at arbitrary wavelength. This is an extension of the analysis in Scott et. al. 2010 where this was examined in the long-wavelength limit. Deriving a local expression for the fluxes requires that the gyroaveraging operator is symmetric, so that if point B is on the gyroring around A, point A is on the gyroring around B, for the same value of the magnetic moment. An algorithm for constructing a symmetric gyroring in a global code is described. Finally, using a simple 2D gyrokinetic code, we demonstrate the application of the momentum transport relation in a model problem with the full gyroaveraging operator without any long-wavelength approximation.

Symmetries and the related conservation properties underpin almost every domain of basic physics. Approximate models of a physical system which recapitulate the complete system’s conservation properties are forced to embody the qualitative features of Hamiltonian dynamics. In turbulence physics, we are directly interested in the dynamics seen through the frame of these conserved quantities; we are primarily concerned with particle, heat and momentum transport through a system.

Global conservation of several important physical quantities, like particle number, momentum, and energy, is guaranteed in gyrokinetic formalisms based on a Lagrangian form[1]. In the full underlying Vlasov-Maxwell system, we are also able to write down transport equations which determine local time variation of these quantities in terms of fluxes evaluated using local values and gradients of the distribution function and electromagnetic fields; this is generally the case for systems with bounded speed for information propagation. It is a priori unclear whether we can do the same for gyrokinetic systems which involve nonlocal interactions across the gyroradius, due to rapid response on timescales which play an implicit role in the model.

Scott and Smirnov[2] give a general formalism for toroidal momentum transport, but only derive specific local transport relations for certain field dependences of the Lagrangian, with bilinear dependence on field gradients, appropriate for long-wavelength expansions of the
gyrokinetic Poisson equation. Abiteboul et. al. provide an explicit, local, form of this toroidal momentum relation for the formalism used in Gysela, based on a Padé expansion of the gyroaveraging operator. Many codes (like GYRO, GENE and NEMORB) use the full gyroaveraging operator at some point in their formalism. We demonstrate here how an explicit polarisation term and gyro-momentum can be constructed when the exact gyroaveraging operator is used.

As might be expected, the finite-range non-locality due to gyroaveraging introduces a nonlocality into the flux-surface averaged transport relations, so that fluxes and toroidal momentum density need to be defined in terms of gyroaveraged quantities.

We perform calculations to evaluate local conservation laws for the exact arbitrary wavelength gyroaveraging terms in the standard gyrokinetic equation, which were not evaluated explicitly in terms of local quantities in refs [2; 3]. Although the existence of a globally conserved toroidal momentum immediately allows the local momentum evolution to be written as the divergence of a global integral, this is not particularly useful for analysis. By deriving explicit, local, expressions for the fluxes, we show that the nonlocality in the fluxes is over the gyroradius length scale.

The existence of a local 1-D momentum transport equation is particularly vital for global gyrokinetic codes because they guarantee that the slow evolution of flows and temperature profiles at long length scales is consistent with the divergences of fluxes in the systems. We might otherwise be concerned that non-conservation would result in small non-local transfers of energy, which might overwhelm these divergences, which are also small due to the long length and timescales involved.

1. Polarisation vector

The total toroidal momentum depends both on the toroidal component of the flow along the field line and the bulk $E \times B$ velocity. The dependence on the potential $\phi$, is related via the gyrokinetic Poisson equation to the charge density $n_i$. We can use the conservation of charge relation to determine the dynamics of the $E \times B$ component of the toroidal momentum in terms of the time derivative of the polarisation vector.

We choose an electrostatic Lagrangian typically used for gyrokinetic simulations, which is close to that of ref. [4], but using a time-constant background density $f_0$ in the coupling to the squared potential (this leads to a linearised polarisation equation). We have

$$L = \int d^2Z L_{pl} f + L_{pn} f_0$$  \hspace{1cm} (1)

with

$$L_{pl} = \frac{e}{c} (A + p_z b) \cdot \ddot{R} + \frac{mc}{e} \mu \dot{\phi} - \frac{P_z^2}{2m} - \mu B - e < \phi >$$  \hspace{1cm} (2)

and

$$L_{pn} = - \frac{em}{2qB} \frac{\partial}{\partial \mu} < \tilde{\phi}^2 >.$$  \hspace{1cm} (3)

Here, $L$ is the Lagrangian density, which is integrated over the Lie-transformed phase space to form the Lagrangian functional. The particle coordinates, $Z_p = R, p_z, \mu$, are the gyrocentre position, parallel canonical momentum and magnetic moment. $\alpha$ is the ignorable sixth coordinate representing the gyroangle, $e$ and $m$ are the species charge and mass, $A$ and $b$ the magnetic vector potential and unit vector, and $B = \nabla \times A$ the magnetic field. The angle brackets around the electrostatic potential $\phi$ denote a gyroaveraging about a ring $\hat{R} + \rho(\alpha,\mu)$. The standard gyrocentre transform uses $\rho = (2\mu m/B)^{1/2}/q$, and $\rho/\rho = cos(\alpha)e_1 + sin(\alpha)e_2$, with $e_1$ and $e_2$ unit vectors defined perpendicular to the magnetic field. We also have:

$$\tilde{\phi} = \phi - < \phi >.$$  \hspace{1cm} (4)
Note that there is usually a sum over multiple species involved in the Lagrangian, which we suppress here to simplify notation.

We then follow Scott et. el., defining $H_p$ via

$$H(f) = (H_0 f + q\phi f) + H_p(f)$$

with $H_0$ collecting all the particle terms not dependent on $\phi$. The definition of $H_p$ implies

$$H_p(f) = \left[q < \phi > f - q\phi f + \frac{e}{2\Omega_{cs}} \frac{\partial}{\partial \mu} < \phi^2 > f_0\right]$$

We define a polarisation,

$$\nabla \cdot P = \frac{\partial}{\partial \phi} H_p(f),$$

which will be used to define charge fluxes through the system.

Let us separately consider $P_0$ associated with the first two terms of the eq. 6, and $P_1$ associated with the coupling to the background. We have

$$\nabla \cdot P_0 = e \int d^6Z \frac{\partial}{\partial \phi} (\phi - \phi)H_P(f) = e \int d^6Z \left[\delta (R - R' + \rho(R, \mu, \theta)) - \delta (R - R')\right] H_P(f, \phi[R']),$$

where the last expression exhibits the implicit dependence of the Hamiltonian on position.

We can insert a Greens’ function-type solution to find an integral form for $P$ which has the divergence specified above. Because we wish for $P$ to be near-local, we choose a function which is non-zero only on the gyrodisc. The polarisation term can be constructed via ‘lines of field’ connecting the charge at the gyrocentre, with a ring of charge on the gyrocircle: this accounts for the gyroaveraging which was not included in the definition of $H_0$. We can define a $P_0$ which satisfies this equation by introducing an integration from one end of this ‘line of charge’ to the other, for parameter $g \in [0, 1]$, via

$$P_0 = e \int d^3 R d\mu dv d\gamma dB \parallel (R + \rho)\rho\delta (R + g\rho - R')f(R + \rho).$$

To perform this calculation for $P_1$ we need to define $f_0$, which will be chosen as $f_0 = n(x)\exp(-2\mu/B + T + mv^2/2T)$. To explicitly calculate $P_1$ as an integro-differential function of $\phi$, we use the weak form of the definition of the polarisation $P_1$,

$$\int d^6 Z \nabla \cdot P_1 \delta \phi = \frac{\partial}{\partial \phi} \frac{eB}{2\Omega_{cs}} \int d^6 Z f_0 \frac{\partial}{\partial \mu} \langle \phi - \phi \rangle^2 > \delta \phi$$

and integrate by parts by parts, using the $\mu$-independence of the phase space volume element, to find

$$\int d^6 Z \nabla \cdot P_1 \delta \phi = \frac{\partial}{\partial \phi} \frac{eB}{2\Omega_{cs}} \int d^6 Z f_0 \langle \phi - \phi \rangle^2 > \delta \phi$$

Inserting explicit gyroaverages results in

$$\int d^6 Z \nabla \cdot P_1 \delta \phi (X)$$

$$= \frac{\partial}{\partial \phi} \frac{eB}{2\Omega_{cs}} \int d^6 Z f_0(X)|d\alpha_1 d^3 X'\delta(X - X' - \rho[\alpha_1, Z])\phi(X')^2$$

$$- d^3 X' d^3 X'' d\alpha_1 d\alpha_2 \delta(X - X' - \rho[\alpha_1, Z])\delta(X - X'' - \rho[\alpha_2, Z])\phi(X')\phi(X'')\delta \phi(X)$$
The right hand side is the variation of \( \phi \) by an infinitesimal \( \delta \phi \), which allows us to evaluate this equation as

\[
\int d^6 Z' \nabla \cdot \mathbf{P}_1 \delta \phi (\mathbf{X}') = \frac{eB}{T_{\Omega_{ci}}} \int d^6 Z d^6 Z' d^6 Z'' [\partial \alpha_1 \partial \alpha_2 (\mathbf{X}' - \mathbf{X}'' - \mathbf{X})] \delta (\mathbf{X} - \mathbf{X}') \delta (\mathbf{X} - \mathbf{X}'') \delta (\phi (\mathbf{X}')) f_0 (\mathbf{X}) \tag{16}
\]

where we have changed the dummy variable on the LHS to allow us to identify the divergence of the polarisation

\[
\nabla \cdot \mathbf{P}_1 (\mathbf{X}) = \frac{eB}{T_{\Omega_{ci}}} \int d^6 Z d^6 Z' \partial \alpha_1 \partial \alpha_2 (\mathbf{X}' - \mathbf{X}'' - \mathbf{X}) \delta (\mathbf{X} - \mathbf{X}'') \delta (\phi (\mathbf{X}')) f_0 (\mathbf{X}) \tag{18}
\]

A polarisation vector satisfying this equation which is a local function of the field can be written using the ‘lines of charge’ described above approach,

\[
\mathbf{P}_1 (\mathbf{X}) = -\frac{eB}{T_{\Omega_{ci}}} \int d^6 Z d^6 Z' \partial \alpha_1 \partial \alpha_2 g f_0 (\mathbf{X} - \mathbf{Y}'' - \mathbf{Y}) \partial \phi (\mathbf{X}'') f_0 (\mathbf{X}) \tag{20}
\]

where \( \partial \phi (\mathbf{X}'') \) and the integration domain for \( g \) is [0, 1].

2. Momentum equation

Following ref. [2], with the polarisation defined, the toroidal momentum equation can be written in the form

\[
-\frac{\partial}{\partial t} \left( \frac{1}{c} \mathbf{P} \cdot \nabla \mathbf{A}_\psi \right) + \mathbf{P} \frac{\partial H}{\partial \mathbf{A}_\psi} = 0 \tag{21}
\]

We evaluate the last term which contains the derivative of the Hamiltonian with respect to the toroidal component.

\[
f \frac{\partial H}{\partial \psi} = f \frac{\partial H}{\partial \phi} \frac{\partial \phi}{\partial \psi} + f \frac{\partial H}{\partial \phi^2} \frac{\partial \phi}{\partial \psi} \tag{22}
\]

\[
= -f \frac{\partial \phi}{\partial \psi} - \frac{eB}{T_{\Omega_{ci}}} f_0 \phi \frac{\partial H}{\partial \phi} \frac{\partial \phi}{\partial \psi} + \frac{eB f_0}{T_{\Omega_{ci}}} \phi \frac{\partial \phi}{\partial \psi} \tag{23}
\]

This needs to be written as the divergence of a flux to yield a local transport relation. The terms can be refactored as

\[
f \frac{\partial H}{\partial \psi} = < f > \frac{\partial \phi}{\partial \psi} - \frac{eB}{T_{\Omega_{ci}}} (\phi_0 \phi) \frac{\partial \phi}{\partial \psi} \tag{24}
\]

\[
- \left[ < f > \frac{\partial \phi}{\partial \psi} - f \frac{\partial \phi}{\partial \psi} \right] \tag{25}
\]

\[
+ \left[ \frac{eB}{T_{\Omega_{ci}}} \phi_0 \phi \frac{\partial \phi}{\partial \psi} - \frac{eB}{T_{\Omega_{ci}}} f_0 \phi \frac{\partial \phi}{\partial \psi} \right] \tag{26}
\]

\[
+ \left[ \frac{eB}{T_{\Omega_{ci}}} \phi_0 \phi \frac{\partial \phi}{\partial \psi} - \frac{eB}{T_{\Omega_{ci}}} f_0 \phi \frac{\partial \phi}{\partial \psi} \right] \tag{27}
\]

As in Scott and Smirnov, the first two terms are the functional derivative of \( Hf \) with respect to the potential, which is the Poisson equation, and therefore vanishes. The last two pairs of terms are of the form \( f < g > -g < f > \):

\[
\int d^6 Z' f (\mathbf{Z}) g (\mathbf{Z}') - f (\mathbf{Z}) G (\mathbf{Z}) \delta (\mathbf{R} - \mathbf{R}') \tag{28}
\]
In order to produce a near-local conservation relation, this need to be written in the form of a divergence. As before, we will relate this to a line integral

$$\nabla \cdot \int d\mathbf{g} d\mathbf{Z}' f(\mathbf{Z}') g(\mathbf{Z}' + \rho) \delta(g(\mathbf{R}' + \rho) + (1 - g)\mathbf{R}' - \mathbf{R}) \mathbf{\hat{p}}$$

(29)

where $\rho = \rho(\mathbf{R}, \alpha)$, as long as the gyroaveraging procedure has ‘pairing symmetry’, so that

$$\mathbf{R}' = \mathbf{R} + \rho[\mathbf{R}, \alpha] \implies \mathbf{R} = \mathbf{R}' + \rho[\mathbf{R}', -\alpha].$$

(30)

This relation is trivially true for standard flux tube codes, where $\rho$ is defined to be on the radial and binormal plane ($\rho \cdot z \equiv 0$), and is only a function of the direction along the field line, $z$ ($\rho = \rho(z)$).

For global codes the gyroaveraging procedure is typically asymmetric, because $\rho$ is a function of the magnetic field strength and direction, which varies across the simulation domain. Although the gyroaveraging will still be symmetric to first order in $\rho^*$, there is a small component which is non-local, which will be associated with non-local energy and momentum transport in the model. Abiteboul (ref [3]) notes that this non-local component will be zero as long as the gyroaveraging operator is Hermitian: the pairing symmetry requirement is sufficient to ensure that the gyroaveraging is Hermitian.

Before we explain how to construct pair-symmetric gyroaverages in a global model, we will evaluate these terms in the momentum balance equation for a simple physical problem.

3. A demonstration of FLR polarisation terms in a simple gyrokinetic model.

To demonstrate the practical calculation of the FLR terms in the momentum relation, we set up a simple gyrokinetic model and solve it numerically.

We consider one of the simplest possible gyrokinetics models: the solution of the 2D GK-Vlasov Poisson equation in the plane perpendicular to the magnetic field. To simplify further, no gradients are assumed to exist in the system, and the perturbed distribution function is taken to be $\delta f = \delta(\mu - \mu_0) \delta(v_\parallel)$. The background distribution $f_0$ is assumed Maxwellian. The magnetic field is taken to be of the form $\mathbf{B} = B(x) \hat{z}$. Parallel dynamics are absent in the model, and only the spatial advection is non-zero, so the equations of motion are simply

$$\frac{d^3 \mathbf{R}}{dt} = (mv_\perp^2/qB) B \times \nabla B / B^2 - B \times \nabla \phi / B^2.$$ 

(31)

To emphasise the FLR effects, electrons are assumed to act as a neutralising background, and do not respond adiabatically to the field. The Poisson equation, which describes the balance between the ion polarisation charge, and the gyrodensity is of the form

$$\int <f> d^6Z = \int d^6 Z (<\phi> - \phi) \delta(\mathbf{R} - \mathbf{R}' + \rho) f_0 d\mathbf{Z}.$$ 

(32)

At large scales, this model is equivalent to the short-wavelength limit of the Hasegawa-Mima equation.

Periodic boundary conditions are chosen in $x$ and $y$, with $x \in [0, L_x]$, $y \in [0, L_y]$. The distribution function is described using the PIC method, and the Poisson equation is discretised via a finite element method using quadratic splines. Fourth order Runge-Kutta timestepping is used to solve the resulting coupled Vlasov-Poisson system. For the example problem treated here, 256 grid points here chosen in each direction, and $2^{20}$ markers were loaded into the simulation domain. Because the problem is symmetric in the $y$-direction, the Poisson solution is performed
in Fourier space, as in [5], allowing variations of field strength in the \(x\) direction which will be considered in further work.

For an initial demonstration, a scenario with a homogeneous magnetic field was chosen. As the system does not contain spatial gradients in \(f_0\), there are no linear instabilities. To produce interesting dynamics, initial states with substantial flows must be specified. We choose

\[
\delta f = A \cdot \text{sign}[\sin k_y y]
\]

(33)
to produce an initial state with banded flow zones. We took \(k_y = 4\pi / L_y\).

This situation is unstable to a Kelvin-Helmholtz instability, which develops nonlinearly to produce filamentation at short scales and the transfer of the initial energy to long scales. The predominant mode in the end state of the simulation is the largest which fits in the box, with \((N_x, N_y) = (1, 0)\).

Figure 1. Color plot of gyrocentre density on the xy spatial domain early in the nonlinear period of the 2D gyrokinetic simulation.

For this situation, the momentum transport and vorticity transport equations are equivalent. We instrumented the code with the appropriate terms in the momentum balance equation. For ease of implementation, instead of directly evaluating the polarisation via the Green’s function approach described above, the divergence was calculated instead, and the inverse of the divergence operator was found via a Fourier solver.

The terms in the momentum transport equation, and the overall numerical imbalance, are shown in fig. 3. During the development of the instability, the conventional Reynolds stress and the additional non-local FLR correction to the Reynolds stress are of similar magnitude, so that this test problem is a strong test of accurate conservation: the RMS error is around 0.5% at this late stage of the simulation, where the absolute error is largest.

This provides a clear demonstration that vorticity transport remains a practical approach to understanding turbulent dynamics, even when the transport is determined by somewhat nonlocal operators on the gyroscale. The same principles should be applicable to typical 3D codes with inhomogeneous magnetic fields, which are required for global tokamak modelling.
Figure 2. Color plot of gyroaveraged density on the xy spatial domain early in the nonlinear period of the 2D gyrokinetic simulation.

Figure 3. Terms in the 1D momentum balance equation well into the nonlinear regime in the simple 2D simulation.

4. Pair-symmetric gyroaveraging for global gyrokinetics
In global gyrokinetic codes, gyroaveraging is typically defined on planes not exactly perpendicular to the field line: usually these are surfaces of constant toroidal or poloidal angle.
For example, NEMORB defines gyroaveraging on poloidal planes, whereas the global GENE code uses surfaces of constant poloidal angle. Numerical gyroaveraging is an integral around an ellipse on these planes, which is the intersection of the plane with a cylinder around a line tangent to the magnetic field at the gyrocentre (for the case of poloidal planes, these are approximately circles). This is, to leading order in $\rho^*$, equivalent to using gyroaveraging rings perpendicular to the field lines, because the perturbations are flute-like. The size of the ellipses is dependent on the magnetic moment $\mu$ and the field strength $B$ at the gyrocentre $R$.

In general, this leads to gyroaveraging points which are not conjugate. However, it is possible to redefine the gyroradius $\rho(\theta, \mu, R)$ so that it has ‘pairing symmetry’, but is equal to the usual gyroradius at first order.

We define a pair $R, R'$ to be gyroconjugate if they lie at generalised distance $\rho$ on straight lines on a plane $K$, with the distance defined by a metric $dR'^2 = BdR^2 - B(dR \cdot b)^2$ in terms of the usual Euclidean distance $R$. In algorithmic terms, this means we integrate from the gyrocentre along a straight line until $\int dR' = (2\mu m)^{1/2}/q$.

Because the magnetic field scale length is long compared to the gyroradius, approximate expressions can be easily generated for numerical evaluation. For example, by iteratively evaluating the metric one-half of a gyroradius from the initial gyrocentre, gyro-conjugate points can be generated without extensive coding.

As an example, we construct a symmetrised gyroring around a single point in a strongly spatially varying magnetic field, and show that the gyrorings of points on this curve are tangent to the initial point. Figure 4 shows these symmetrised gyrorings, which are somewhat distorted circles, for a case where the ratio of the gyroradius to the magnetic field scale length is 0.7. A large ratio is chosen for the purposes of visually demonstrating the distortion, and the conjugacy of the gyropoints (although clearly the gyrokinetic ordering is only valid when this ratio is small).

![Figure 4](image_url)

**Figure 4.** Five points on a symmetrised gyroring around the point (1, 1) (addition symbols), and the five conjugate gyrorings around these points.
5. Discussion

We have demonstrated that local conservation principles can be applied in practice to gyrokinetic simulations capturing physics all the way from the gyroscale to the system scale. This serves to reinforce in a more explicit form the statements of recent papers[2; 3; 6] on momentum conservation in gyrokinetic systems.

It is as yet unclear what the implications of the slight non-Hermicity of gyroaveraging in some gyrokinetic simulations is. Although distortions in the gyroring to allow the gyroaveraging to remain Hermitian in finite-size tokamaks would be small, small modifications in polarisation due to system size effects are known to be significant in certain situations. For example, a detailed code comparison between the codes ORB5 and GENE showed that the zonal response of the Poisson operator was subtly different for system-scale perturbations because GENE treated some small $\rho*$ effects differently to ORB5[7].

References
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