We study numerically the zero temperature Random Field Ising Model on cubic lattices of various linear sizes $L$ in three dimensions. For each random field configuration we vary the ferromagnetic coupling strength $J$. We find that in the infinite volume limit the magnetization is discontinuous in $J$. The energy and its first $J$ derivative are continuous. The approach to the thermodynamic limit is slow, behaving like $L^{-p}$ with $p \sim 0.8$ for the gaussian distribution of the random field. We also study the bimodal distribution $h_i = \pm h$, and we find similar results for the magnetization but with a different value of the exponent $p \sim 0.6$. This raises the question of the validity of universality for the random field problem.

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I. INTRODUCTION

Despite its long history, the Random Field Ising Model (RFIM) is not yet fully understood [1]. It is known that in three dimensions and for weak disorder there is a paramagnetic to ferromagnetic phase transition but the nature of the transition is still unknown. What is most remarquable is that the RFIM is (together with branched polymers [2]) one of the very few cases where Perturbative Renormalization Group (PRG) can be analyzed to all orders of pertubation theory [3,4] and this analysis leads to wrong conclusions (while for the branched polymers the same analysis is correct!). It has been argued that this failure of PRG is due to replica symmetry breaking [5–7]. This failure raises the more general question of the validity of perturbative renormalization group for disordered systems.

In the present paper we study numerically the phase transition of the RFIM in three dimensions. Several numerical studies have already been performed [8–12] but the problem has been proven to be very difficult and, in our opinion, no definite conclusion about the nature of the transition has been drawn yet. We simulated much larger sizes than before (up to $90^3$) with much higher statistics and we establish new results on the phase transition of the RFIM. We find that the magnetization is discontinuous at the transition both for the gaussian and bimodal distribution. The energy and its derivative are continuous only for the gaussian distribution. The exponents are found to be non classical. Such a behaviour has already been suggested on the grounds of real space renormalization group [13,14]. (This result has been often overlooked in the past due to the approximations of the method.) We also find different exponents for different random field distributions, which is in contradiction with the PRG.

The Hamiltonian of the RFIM is given by

$$H = -J \sum_{<i,j>} \sigma_i \sigma_j - \sum_i h_i \sigma_i \quad (1)$$

where $\sum_{<i,j>}$ runs over neighbouring sites of the lattice (we have only considered three dimensional cubic lattices with periodic boundary conditions) and $\{h_i\}$ are independent random variables identically distributed with zero mean and variance one. For a given random field sample, one can vary both the ferromagnetic coupling $J$ and the temperature.
$T$, i.e. the phase boundary is a line in the $J, T$ plane. It is thought, in accordance with PRG, that the nature of the transition and the value of the exponents do not depend on the position on the transition line, nor on the direction one crosses it, and that this is true down to zero temperature. So it is advantageous to work at $T=0$ where it has been shown that the RFIM is equivalent to the problem of finding a maximum flow in a graph \cite{13}, for which very fast (polynomial) algorithms are known. Note that these algorithms provide the exact ground-states and therefore there is no thermalization problem. Simulations using such algorithms have already been performed in the past \cite{10,11}. In the present paper we use the latest version of the algorithm developed by Goldberg and Tarjan \cite{16}, which we optimized for the case of the cubic lattice. It has been shown \cite{16} that this algorithm converges to the ground state in a time $t < L^6 \ln L$ where $L$ is the linear size of the cubic lattice. We found experimentally that $t \sim L^4$ \cite{17}.

II. GAUSSIAN DISTRIBUTION OF RANDOM FIELD.

We first consider the case of a gaussian distribution of the random field with variance equal to one. It is customary to present the data as a function of $x = 1/J$. For every sample $\{h_i\}$, we chose equally spaced $x$’s with $\delta x = .0125$ and found the corresponding ground state. We considered lattices of 16 different linear sizes $L$, from $L = 7$ to $L = 90$. We studied between 750 samples for $L = 90$ and 20000 samples for $L = 8$ and $L = 7$.

A. Magnetization

We study the variation of the absolute value of the magnetization $m$ and of the energy as a function of $x$. We have found that there is a region in $x$ where there are large discontinuities of $|m|$ and that outside that region $m$ is a smooth function of $x$. The amplitude of the discontinuities is volume independent (this is shown in Fig. 1), while the width of the region in $x$ where they appear, shrinks as the volume increases. We analyze our data as follows. For every $\{h_i\}$ sample, we chose the $n_d$ largest variations of the magnetization between two successive values of $x$, $x_1 = 1/J_i$ and $x_{i+1}$. Let’s call $x_1 < x_2 < \cdots < x_{n_d}$ the values of $x$ at which they occur. The choice of $n_d$ is somehow arbitrary. We took $2 \leq n_d \leq 6$. It turns out that the $x_i$’s fluctuate from sample to sample and their probability distribution is well described by a gaussian with mean $\bar{x}_i(L)$ and variance $\sigma(L)^2$. The variance decreases with $L$ and is compatible with a power law decay:

$$\sigma(L) \sim \sigma_0 L^{-\delta}$$ (2)

We first made the ansatz that:

$$\bar{x}_i = x_\infty + \frac{c_i}{L^p}$$ \hspace{1cm} (3)

i.e. that there is a single discontinuity in the infinite volume limit and that the approach to the thermodynamic limit obeys a power law. We obtain a more accurate determination of $p$ by taking differences to eliminate $x_\infty$ from the fit

$$x_{ij} \equiv \bar{x}_i - \bar{x}_j = \frac{c_{ij}}{L^p}$$ \hspace{1cm} (4)

We have found that our results cannot be described by a single power for the whole range of sizes we have studied. We tried the following alternatives.

a) Analyze only the largest sizes that are compatible with Eq. 3. We found that sizes larger or equal to $24^3$ are well described by Eq. 3, with $x_\infty = 2.26 \pm .01$, $p = .80$ and a $\chi^2$ per degree of freedom $\chi^2 = 1.29$. With $90\%$ probability $$.76 \leq p \leq .83$$ We also have found the value of the exponent $\delta$ of Eq. 3: $.78 \leq \delta \leq .86$. The choices of the lower size cut-off and of the number of discontinuities $n_d$ are quite arbitrary and induce systematic uncertainties on $p$. In order to get an idea on these uncertainties, we tried different “reasonable” choices, i.e. choices with a reasonable $\chi^2$ per degree of freedom. Keeping only sizes larger or equal to $30^3$, we found that $p = .77$ and $\chi^2 = .56$. With $90\%$ probability $$.74 \leq p \leq .81$$. These results have been obtained by analyzing the $5$ largest discontinuities of the magnetization ($n_d = 5$). We also took $n_d = 3$ and, in this case, $.735 \leq p \leq .81$. We conclude that the dependence of $p$ on $n_d$ is small.

b) Consider also a subdominant power law correction:

$$x_i = x_\infty + \frac{c_i}{L^p} \left(1 + \frac{\bar{f}_i}{L^q} \right)$$ \hspace{1cm} (5)
This ansatz describes well the data down to a value of $L$ as small as $L = 7$ with a $\chi^2$ per degree of freedom $\chi^2 = 1.$ and yields $p = .66$ and $q = 1.4.$ There are many parameters in this fit, $p$ and $q$ are correlated and the uncertainty on $p$ and $q$ much larger than before. The best fit to the data is shown in Fig. 2, while the allowed region in $p$ and $q$ is in Fig. 3.

We conclude that the appearance of several discontinuities in the magnetization is a finite volume artifact and that our results are fully compatible with the hypothesis of a single discontinuity in the thermodynamic limit. The approach to this limit is a very slow power law. The value of this power depends essentially on the assumptions underlying the data analysis: a single power law or a power law behaviour with a subdominant correction. This discontinuity has not been seen in the previous simulations for the following reason. Only the average magnetization has been measured and for finite volumes, the position of the discontinuities fluctuates from sample to sample so that the average magnetization seems continuous.

B. Energy

Concerning now the energy, it is convenient to separate the energy into two terms, $H = -L^3(JH_1 + H_2),$ with

$$
H_1 = \frac{1}{L^3} \sum_{<i,j>} \sigma_i \sigma_j, \quad H_2 = \frac{1}{L^3} \sum_i h_i \sigma_i
$$

(6)

For a fixed sample, the value of $H_1 = E_1(J)$ in the ground state is the derivative of the energy per spin with respect to $-J.$ It can be shown that $E_1(J)$ is a non decreasing function of $J$, i.e., $-H$ is a convex function of $J.$ We observe large discontinuities of $H_1$ as a function of $J$, as in the case of the magnetization, but, in the present case, the amplitude of these discontinuities shrinks as the volume increases. Our data are compatible with the following ansatz:

$$
\Delta E_i \sim \frac{d_i}{L^a}
$$

(7)

where $\Delta E_i, \ i = 1, \ldots, n_L$ denote the $n_L$ largest discontinuities of $E_1.$ We find $a = .48 \pm .06$ Therefore $H_1,$ i.e. the $J$ derivative of the energy, becomes continuous at the thermodynamic limit. Despite the discontinuity of the magnetization, the energy and its first derivative are continuous and the exponents are non classical. Eq. 7 sheds some light to the long standing question of the existence of different spin configurations almost degenerate in energy. The reasoning goes as follows. For $J = 0,$ $E_1 \sim 0,$ while for $J > 1,$ $E_1 = 3.$ In general $E_1$ is a slow varying monotonous function of $J$ except in the region $2 < J < .5$ where most of the variation occurs. If the spin configuration $\sigma_i^{(1)}$ is a ground state for $J = J^{(1)}$ with energy per spin $-E^{(1)} = J^{(1)}E_1^{(1)} + E_2^{(1)}$ and another spin configuration $\sigma_i^{(2)}$ is a ground state for $J = J^{(2)} > J^{(1)}$ with energy per spin $-E^{(2)} = J^{(2)}E_1^{(2)} + E_2^{(2)}$ then for $J^{(3)} = (E_2^{(2)} - E_1^{(2)})/(E_1^{(1)} - E_2^{(1)}),$ the two configurations have exactly the same energy $-E^{(3)} = J^{(3)}E_1^{(1)} + E_2^{(1)}$ because of the convexity of $-H.$ If the largest discontinuity is of the order of $L^{-\alpha},$ it follows that in the interval $J^{(1)} < J < J^{(2)}$ there are at least $\sim (E_2^{(2)} - E_1^{(2)})L^\alpha$ different configurations with energy $E_2 \leq E \leq E_1,$ which are pairwise exactly degenerate for some value of $J.$ This argument is valid for any choice of $E_1, E_2,$ arbitrarily close to each other.

C. Disconnected susceptibility

In order to compare with previous simulations, we have considered the so called disconnected susceptibility $\chi_{dis} = L^3m^2$ i.e. the square of the magnetization averaged over the disorder. It has been assumed that at zero temperature

$$
\chi_{dis} \sim L^{4-\tilde{\eta}} f((x-x_c)L^{1/\nu})
$$

(8)

where $x_c$ is the location of the transition at infinite volume and $\nu$ the correlation length exponent. In [1] it was found that, for gaussian random fields at zero temperature, $\tilde{\eta} = 1.1 \pm .1$ and $\nu = 1 \pm .1.$ In fact Eq. 8 is compatible

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1 One could find the ground state energy for $J = J^{(3)},$ which is either identical to $E^{(3)}$ or lower. In the former case the ground state energy is exactly known in the entire range $J^{(1)} \leq J \leq J^{(2)}.$ In the latter case one could iterate this procedure and so it is possible to compute the ground state energy exactly as a function of $J.$ This will be exploited in a forthcoming publication.
with our results. To see this, we propose the following simplified model: $m = m_1$ for $x < x_1$ and $m = m_2$ for $x > x_1$, where $x_1$ is a gaussian random variable with mean $\bar{x} \sim x_{\infty} + c/L^p$ and standart deviation $\sigma \sim \sigma_0 L^{-\delta}$. A simple calculation gives, in this model, $\chi_{\text{dis}} \sim L^3 f((x - x_{\infty})L^\delta + L^{\delta-p})$. If $\delta = p$, this agrees with Eq. 8 provided $\bar{\eta} = 1$ (which is compatible with [11]) and $\delta = 1/\nu$. We found above (case a), that $\delta = 0.82 \pm 0.04$ and $p = 0.80 \pm 0.04$, compatible with $\delta = p$. This gives $\nu = 1.22 \pm 0.06$ which is slightly different from the value $\nu = 1 \pm .1$ of [11]. If we restrict our analysis to the same (smaller) sizes as in [11] we find $\bar{\eta} = 0.96 \pm 0.03$, i.e. $\nu = 1.04 \pm 0.03$, as in [11] and $p = 1.17 \pm 0.03$. Alternatively we also have fitted directly our data using Eq. 8. This provides a different method of analysis. Since, as already mentioned, a single power is not sufficient to describe the whole range of sizes, we have restricted our analysis to the largest sizes. We have found $\bar{\eta} = 0.98 \pm 0.01$ in agreement with our simple model and $x_{\infty} = 2.265 \pm .005$, a value compatible with the previously deduced value.

**III. BIMODEL DISTRIBUTION OF RANDOM FIELD.**

We consider next the case of a bimodal field distribution,

$$P(h_i) = (1/2) (\delta(h_i - 1) + \delta(h_i + 1))$$

which describes the finite size corrections and is usually identified with $1/\nu$ per degree of freedom $\chi^2$ for $\chi_{\text{dis}} \sim L^\delta f((x - x_{\infty})L^\delta + L^{\delta-p})$. With $90\%$ probability $57 \leq p \leq 63$. Keeping only sizes larger or equal to $30^4$, we found that $p = 0.59$ and $\chi^2 = 0.76$. With $90\%$ probability $55 \leq p \leq 63$. These results were obtained by analyzing the 4 largest discontinuities of the magnetization ($n_d = 4$).

b) Considering also a subdominant power correction the data are well described with a $\chi^2$ per degree of freedom $\chi^2 = 1.2$ and yields $p = 0.56$ and $q = 3$. Unfortunately the error bars are much larger than before. The allowed region in $p$ and $q$ is presented in Fig. [3].

For the energy, the situation is more complex. In addition to the type of singularities we observed in the gaussian case, there are new stronger singularities, at integer values of $x$, i.e. $x = 3, 4, 5, ...$. These singularities are present for all sizes and the amplitude of the discontinuity of $E_1$ at these points seems to be volume independent. Nothing in particular is happening to the magnetization at these points. While the presence of these singularities is intuitively easy to understand in the case of the bimodal distribution, the decoupling of the magnetization from the energy is less intuitive.

**IV. CONCLUSION AND DISCUSSION**

In this paper we have found that he exponent $p$, which describes the finite size corrections and is usually identified with $1/\nu$ ( $\nu$ is the correlation length exponent), seems to take different values for the gaussian or the bimodal distribution of the random field. This would signal the breaking of universality for the RFIM. It is therefore important to discuss a) the uncertainties on the determination of $p$ and b) the identification of $p$ with $1/\nu$.

a) The statistical errors on $p$ are rather small. The largest uncertainties are due to the assumptions made to analyze the data, namely single power law behaviour or inclusion of subdominant corrections in $L$. But whatever the assumptions are, one finds significantly different values, provided one uses the same assumptions for both random field distributions.

b) Let us remind the arguments which lead to the identification of $p$ with $1/\nu$. For finite systems of linear sizes $L$, one can define an effective critical temperature $T_{\text{eff}}(L)$. For a second order transition, $T_{\text{eff}}(L)$ could be the temperature for which the susceptibility is maximum. The finite size scaling hypothesis asserts that $T_{\text{eff}}(L)$ is shifted from $T_c$, the critical temperature of the infinite volume system: $T_{\text{eff}} = T_c + cL^{-1/\nu}$. Assume a second order transition for the RFIM and, in agreement with PRG, that the exponents do not depend on the position on the transition line in the $T J$ plane, or the direction one crosses it. Assume furthermore, that this is valid down to zero temperature. We then expect the locations of the magnetization discontinuities $x_i(L)$’s to be shifted from their infinite volume value $x_{\infty}$: $x_i(L) = x_{\infty} + c_i L^{-1/\nu}$, i.e. that the exponent $p$ defined above is $p = 1/\nu$ and therefore is universal. If $p$ is different for the gaussian or the bimodal random field distribution as we claimed above, one of the above generally accepted assumptions, which can be derived in the context of PRG, in not valid for the random field Ising model. After all it is already known that the $\epsilon$ expansion leads to incorrect results for the RFIM. The most conservative attitude would
be to assume (without any a priori reason) that the bimodal distribution is singular at $T = 0$ and that the general argument does not apply. But we would like to point out that, except from aesthetic appealing, the only theoretical argument for universality in disordered systems is based on (unbroken) replicas and perturbative renormalization group. And we know that this may be incorrect. We would like to point out that also in the spin-glass case, there is numerical evidence that universality may be broken \[18\]. We feel therefore that universality should be further checked for disordered systems. In particular it would be interesting to study other distributions of random fields.

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FIG. 1. Difference of magnetizations just before $x_1$ and just after $x_5$ as a function of $1/L$ (see text), averaged over random field configurations drawn from a gaussian distribution. The error bars are smaller than the symbol.
FIG. 2. Figure 2 Averages of the position differencies $\bar{x}_{51}$, $\bar{x}_{41}$, $\bar{x}_{52}$, $\bar{x}_{53}$ (see text) as a function of $1/L$ for the gaussian distribution of the random fields. Continuous lines are the best fit to the data. Statistical error bars are smaller than the symbol size.
FIG. 3. Figure 3. Allowed regions for the $p$ and $q$ exponents (see text) within a 90% confidence level. The upper left region is for a gaussian distribution and the lower right region is for the bimodal distribution of the random fields.