On the slit motion obeying chordal Komatu–Loewner equation with finite explosion time

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Abstract

This paper studies the behavior of solutions near the explosion time to the chordal Komatu–Loewner equation for slits, motivated by the preceding studies by Bauer and Friedrich (2008) and by Chen and Fukushima (2018). The solution to this equation represents moving slits in the upper half-plane. We show that the distance between the slits and driving function converges to zero at its explosion time. We also prove a probabilistic version of this asymptotic behavior for stochastic Komatu–Loewner evolutions under some natural assumptions.

Keywords: Komatu–Loewner equation, stochastic Komatu–Loewner evolution, SLE, explosion time, kernel convergence

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1 Introduction

In the theory of conformal mappings on the complex plane, it is often useful to consider the evolution of a one-parameter family of conformal maps \( \{ g_t \}_{t \geq 0} \) or, equivalently, regions \( \{ D_t \}_{t \geq 0} \) that are domains or ranges of these maps. One of the main tools to describe such an evolution is the Loewner differential equation, from which some sharp estimates are obtained on the Taylor coefficients of univalent functions, such as Bieberbach–de Branges’ theorem. See [13] or [6] for this direction. These days, this equation is well known also in probability theory, especially in the context of stochastic Loewner evolution (SLE) defined by Schramm [14]. This random process was introduced to find the scaling limits of several two-dimensional discrete random processes on lattices, and actually a lot of results have been established so far.

Basically, the Loewner equation concerns maps on simply connected planar domains, such as the unit disk \( D \) (radial case) or upper half-plane \( \mathbb{H} \) (chordal case). However, recent studies [2,4,5] generalize this equation to a standard slit domain of the form \( D = \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j \), where \( C_j \), \( 1 \leq j \leq N \), are mutually disjoint horizontal slits (i.e., line segments parallel to the real axis). The resulting differential equation is called the chordal Komatu–Loewner equation [2,4]. In this case, the ranges of the conformal maps \( \{ g_t \} \) are specified in terms of moving slits \( \{ C_j(t) \} \) whose dynamics is described by the Komatu–Loewner equation for the slits [2,3]. See Figure [1]
In the Loewner theory on simply connected domains, this slit motion does not appear. Thus, there are few results known on the behavior of the solution to the Komatu–Loewner equation for the slits. In particular, the explosion of this solution is a new obstacle of the theory. Motivated by such a background, we focus on the asymptotic behavior of the slit motion around its explosion time $\zeta$ in this paper. Assuming $\zeta < \infty$, we observe that the distance between the slits and a moving point $\xi(t)$ on the real axis, called the driving function below, converges to zero as $t \to \zeta$. Moreover, we prove a probabilistic version of this asymptotic behavior for stochastic Komatu–Loewner evolutions, which was introduced by Bauer and Friedrich [2] and by Chen and Fukushima [3] to generalize SLE.

In order to provide a mathematical detail and an appropriate intuition on the asymptotic behavior of the slits, we now briefly recall the concrete form of the chordal Komatu–Loewner equations.

Let us consider a typical case where $F_t$ in Figure 1 is given by the trace $\gamma(0, t]$ of a simple curve $\gamma : [0, t_\gamma) \to D$ satisfying $\gamma(0) \in \partial H$ and $\gamma(0, t_\gamma) \subset D$. Then for each $t \in [0, t_\gamma)$, there exists a unique pair of a standard slit domain $D_t$ and conformal map $g_t : D \setminus \gamma(0, t] \to D_t$ with the hydrodynamic normalization $g_t(z) = z + a_t/z + o(z^{-1})$ ($z \to \infty$). The image $g_t(z)$ satisfies the chordal Komatu–Loewner equation

$$
\frac{d}{dt} g_t(z) = -\pi \dot{a}_t \Psi_{D_t}(g_t(z), \xi(t)), \quad g_0(z) = z \in D, \tag{1.1}
$$

where $\dot{a}_t$ stands for the $t$-derivative of $a_t$. The dynamics of the range $D_t$ is also described by the Komatu–Loewner equation for the slits

$$
\frac{d}{dt} z_j(t) = -\pi \dot{a}_t \Psi_{D_t}(z_j(t), \xi(t)), \quad \frac{d}{dt} z_j^r(t) = -\pi \dot{a}_t \Psi_{D_t}(z_j^r(t), \xi(t)), \tag{1.2}
$$

where $z_j(t)$ (resp. $z_j^r(t)$) is the left (resp. right) endpoint of the $j$-th slit $C_j(t)$ of $D_t$.

In the equations (1.1) and (1.2), the driving function $\xi(t)$ is given by $g_t(\gamma(t)) = \lim_{z \to \gamma(t)} g_t(z) \in \partial H$, and the kernel $\Psi_{D_t}$ is the complex Poisson kernel of Brownian motion with darning (BMD) for the domain $D_t$ [4] Lemma 4.1. If there are no slits (i.e., $D = H$) and if $a_t = 2t$ holds, then the equation (1.2) does not appear, and (1.1) reduces to the celebrated chordal Loewner equation

$$
\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z \in H. \tag{1.3}
$$

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$$
In the previous paragraph, we start at a given trace \( F_t = \gamma(t, 0) \) and then obtain the driving function \( \xi(t) \) and equations \( (1.1) \) and \( (1.2) \). In turn, given a driving function \( \xi \in C([0, \infty); \mathbb{R}) \), we consider the initial value problem of \( (1.1) \) and \( (1.2) \). In this case, let \([0, \xi_t)\) be the maximal time interval of existence of a unique solution \( g_t(z) \) to \((1.1)\) for each \( z \in D \). Then it can be checked that the solutions \( \{g_t(z) : z \in D\} \) constitute a conformal map \( g_t : D \setminus F_t \to D_t \) hydrodynamically normalized, where \( F_t \) is given by \( F_t := \{ z \in D ; t_j \leq t \} \). Though \( F_t \) is not the trace of a simple curve in general, it is at least a (compact \( \mathbb{H}\)-) hull in \( D \) as in Figure 1. Here, a hull means a non-empty, bounded and relatively closed subset of \( \mathbb{H} \) whose complement in \( \mathbb{H} \) is simply connected. We call \( \{g_t\} \) the (decreasing) Komatu–Loewner chain and \( \{F_t\} \) the Komatu–Loewner evolution driven by \( \xi(t) \) in this article. In particular, the stochastic Loewner evolution with parameter \( \kappa > 0 \), abbreviated as \( \mathrm{SLE}_\kappa \), is defined by putting \( \xi(t) = \sqrt{\kappa} B_t \) in the Loewner equation \((1.3)\) where \( B_t \) is the one-dimensional standard Brownian motion.

In the no slit case \( D = \mathbb{H} \), the Loewner evolution \( \{F_t\} \) is defined on the entire time interval \([0, \infty)\) if so is the driving function \( \xi(t) \). However, the Komatu–Loewner evolution \( \{F_t\} \) is not necessarily defined on \([0, \infty)\) even if \( \xi(t) \) is defined there, because the ranges \( \{D_t\} \) in the right-hand side of \((1.1)\) is determined by the slit motion that solves \((1.2)\). Thus, \( g_t \) and \( F_t \) are defined only up to the explosion time \( \zeta \) of the solution to \((1.2)\). This is a major difference between the Loewner and Komatu–Loewner equations, and hence the explosion of the solution to \((1.2)\) is the main theme of this paper as mentioned above. In particular, our interests are the following two points:

- the asymptotic behavior of the slits \( C_j(t) \) of \( D_t \),
- the relation between the asymptotic behaviors of \( C_j(t) \) and of \( F_t \).

To give a natural outlook on these two questions, let us formally discuss some possibilities of finite time explosion. The first possibility is a situation where \( \{F_t\} \) touches or swallows a certain slit \( C_j \) at time \( \zeta < \infty \). Here, we say that \( \{F_t\} \) swallows a point \( z \in \mathbb{H} \) if \( z \) is not in the union \( \bigcup_{j \leq \xi} F_t \) but in a bounded component of \( \mathbb{H} \setminus \bigcup_{t < \xi} F_t \). In this case, the unbounded component of \( \mathbb{D} \setminus \bigcup_{t < \xi} F_t \) no longer has \( N \) boundary slits. Hence the equation \((1.2)\) cannot have a solution representing disjoint \( N \) slits at \( \zeta \). The second one is the case where \( F_t \) becomes unbounded in finite time. This situation, however, does not seem to happen if \( \xi(t) \) is defined on the entire time interval \([0, \infty)\). Since the ‘preimage’ of \( \xi(t) \) by \( g_t \) is, loosely speaking, the ‘tip’ of \( F_t \), the driving function \( \xi(t) \) should diverge if \( F_t \) becomes unbounded. As a consequence, we are led to a guess that only the former case occurs when \( \zeta < \infty \) and that, if the slit \( C_j \) is touched or swallowed by \( F_t \), then the corresponding slit \( C_j(t) \) approaches \( \xi(t) \).

We now state our main results that are based on our observations above. Needless to say, it is difficult to verify all of these observations. However, we can prove that

\[
\lim_{t \to \zeta} \min_{1 \leq j < N} \operatorname{dist}(C_j(t), \xi(t)) = 0
\]  

(1.4)

assuming that \( \zeta < \infty \) (Theorem 4.1). We note that \((1.4)\) immediately implies that \( \lim_{t \to \zeta} \Im \gamma_j(t) = 0 \) for some \( j \), which justifies the comment in [2, Theorem 4.1]. Moreover, we can establish the property \((1.3)\) for the stochastic Komatu–Loewner evolution as well. Let us recall that, motivated by [2, Chen
and Fukushima [3] introduced SKLE\(\alpha,b\) by the following stochastic differential equation (SDE) for the driving function:

\[ d\xi(t) = \alpha(\xi(t), D_t) dB_t + b(\xi(t), D_t) dt. \]  

(1.5)

Under mild conditions on \(\alpha\) and \(b\), the property (1.4) still holds almost surely for the solution to the system (1.2) and (1.5) (Theorem 3.2). These two results, Theorems 3.1 and 3.2, are the main results of this paper.

In the proof of (1.4), we need to transform a Komatu–Loewner chain \(\{g_t\}\) into a Loewner one \(\{g^0_t\}\). Such a transformation method was originally established by Chen, Fukushima and Suzuki [5] and then generalized by the author [11]. See the paragraph after Theorem 2.3 in Section 2 for the background on this transformation method. In the paper [11], a version of Carathéodory’s kernel theorem, which is well known in complex analysis, was formulated and used extensively to establish the general transformation method. This kernel theorem will be used in the proof of (1.4) as well.

The rest of this paper is organized as follows: Section 2 is devoted to a short review on the previous results of [3, 11]. Section 3 is devoted to the formulation and proof of the property (1.4). We formulate (1.4) as Theorem 3.1 and its probabilistic version as Theorem 3.2 in Section 3.1. A key lemma, Lemma 3.4, is also established in the same subsection. Then we prove Theorem 3.1 through Sections 3.2, 3.3 and 3.4. The proof of Theorem 3.2 is given in Section 3.5 based on the proof of Theorem 3.1.

2 Preliminaries

Let \(A_1, \ldots, A_N\) be disjoint compact continua in \(\mathbb{H}\). Here, by a continuum we mean a connected closed sets in \(\mathbb{C}\) having more than one point. We work on a domain of the form \(D := \mathbb{H} \setminus \bigcup_{j=1}^N A_j\) throughout this paper. A basic fact is that, for any hull (or empty set) \(F \subset D\), the canonical map \(f_F : D \setminus \overline{F} \to \tilde{D}\) exists by [11, Proposition 2.3]. This means that \(f_F\) is a conformal map onto a standard slit domain \(\tilde{D}\) with the hydrodynamic normalization \(\lim_{z \to \infty} (f_F(z) - z) = 0\), and that the pair \((f_F, \tilde{D})\) is unique. After taking Schwarz’s reflection, the canonical map \(f_F\) has the Laurent expansion

\[ f_F(z) = z + \frac{\text{hcap}\mathbb{H}(F)}{z} + o(z^{-1}) \quad \text{as } z \to \infty. \]

The positive constant \(\text{hcap}\mathbb{H}(F)\) is called the half-plane capacity of \(F\) relative to \(D\).

Another basic fact that is used later is a variant of Carathéodory’s kernel theorem. For a sequence of subdomains \(D_n\) of \(\mathbb{H}\), we define the kernel of \(\{D_n\}\) [11, Definition 3.7] as the largest unbounded domain such that its every compact subset is included by \(D_n\) for all sufficiently large \(n\). Under the assumption that

(K.1) all \(D_n\) contain \(\mathbb{H} \cap \Delta(0, L)\) for some fixed \(L > 0\),

the kernel exists uniquely. Here \(\Delta(a, r) := \{z \in \mathbb{C} : |z - a| > r\}\) for \(a \in \mathbb{C}\) and \(r > 0\). We say that \(\{D_n\}\) converges to its kernel in the sense of kernel convergence if all subsequences of \(\{D_n\}\) have the same kernel. We consider, on such domains \(D_n\), a sequence of univalent functions \(f_n : D_n \to \mathbb{H}\) such that
(K.2) \( \lim_{z \to \infty} (f_n(z) - z) = 0 \);
(K.3) \( \lim_{z \to z_0} \exists f_n(z) = 0 \) for all \( z_0 \in \partial \mathbb{H} \cap \Delta(0, L) \).

**Lemma 2.1** ([11, Lemma 3.9]). Under Assumptions (K.1)–(K.3), the sequence of the ranges \( D_n := f_n(D_n) \) also satisfies Condition (K.1) with the constant \( L \) in (K.1) replaced by \( 2L \).

**Theorem 2.2** ([11, Theorem 3.8]). Suppose that, under Assumptions (K.1)–(K.3), the sequence \( \{D_n\} \) converges to a domain \( D = \mathbb{H} \setminus \bigcup_{j=0}^{N} A_j \) in the sense of kernel convergence, where \( A_0 \) is a hull or an empty set, each \( A_j \) for \( 1 \leq j \leq N \) is a connected compact subset whose complement in \( \mathbb{H} \) is simply connected, and all \( A_j \)'s are disjoint. Then the following are equivalent:

(i) \( \{f_n\} \) converges to a univalent function \( f : D \to \mathbb{H} \) locally uniformly on \( D \);

(ii) \( \{\tilde{D}_n\} \) converges to a domain \( \tilde{D} \) in the sense of kernel convergence.

If one of these holds, then \( \tilde{D} = f(D) \) and \( f_n^{-1} \to f^{-1} \) locally uniformly on \( D \).

Note that the locally uniform convergence of \( \{f_n\} \) makes sense since every compact subset of \( D \) is eventually included by \( D_n \). (In [11] the abbreviation ‘u.c.’ is used to indicate ‘uniform convergence on compacta’ following [6], but in this paper we avoid using it for the sake of readability.)

Keeping these two basic facts in mind, we proceed to the correspondence between driving functions and families of continuously growing hulls via the Komatu–Loewner equations, which was established in [3, 11]. We regard [12] as an ordinary differential equation (ODE) on the open subset

\[ \text{Slit} = \{s = (s_l)_{l=1}^{3N} = (y_1, \ldots, y_N, x_1, \ldots, x_N, x_1', \ldots, x_N') \in \mathbb{R}^{3N} \}
\]

\[ ; y_j > 0, x_j < x'_j, \text{ either } x'_j < x_k \text{ or } x'_k < x_j \text{ whenever } y_j = y_k, j \neq k \}

of \( \mathbb{R}^{3N} \) as follows: For a vector \( s = (y_1, \ldots, y_N, x_1, \ldots, x_N, x_1', \ldots, x_N') \in \text{Slit} \), the segment is denoted by \( C_j(s) \) whose endpoints are \( z_j := x_j + iy_j \) and \( z'_j := x'_j + iy_j \). We also put \( D(s) := \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j(s) \). The functions

\[ b_l(\xi_0, s) = \begin{cases} -2\pi \Im \Psi_{D(s)}(z_l, \xi_0) & (1 \leq l \leq N) \\
-2\pi \Re \Psi_{D(s)}(z_{l-N}, \xi_0) & (N+1 \leq l \leq 2N) \\
-2\pi \Re \Psi_{D(s)}(z'_{l-2N}, \xi_0) & (2N+1 \leq l \leq 3N) \end{cases} \]

are locally Lipschitz continuous on \( \mathbb{R} \times \text{Slit} \) by [3, Lemma 4.1] (see also [11, Section 2.2]). By utilizing these notations, we can write (1.2) in the form

\[ \frac{d}{dt}s_l(t) = \frac{\dot{\alpha}_l}{2} b_l(\xi(t), s(t)), \quad 1 \leq l \leq 3N. \tag{2.1} \]

Let \( \xi(t) \) be a continuous function on a fixed interval \([0, t_0] \) and \( a_l \) be a strictly increasing and differentiable function on this interval with \( a_0 = 0 \). Since the right-hand side of (2.1) satisfies the local Lipschitz condition, there exists a unique solution \( s(t) \) with arbitrary initial value in \( \text{Slit} \) up to its explosion time \( \zeta \). The time \( \zeta \) may be strictly less than \( t_0 \). For this solution \( s(t) \), the equation (1.1) is written as

\[ \frac{d}{dt}g_l(z) = -\pi \dot{\alpha}_l \Psi_{s(t)}(g_l(z), \xi(t)), \quad g_0(z) = z \in D(s(0)), \tag{2.2} \]
where we put $\Psi_s := \Psi_{D(s)}$ for $s \in \text{Slit}$. For each point $z$ in $D := D(s(0))$, this equation has a unique solution $g_t(z)$ up to the time $t_z := \zeta \wedge \sup\{t > 0; |g_t(z) - \xi(t)| > 0\}$ by [3] Theorem 5.5 (i). The sets $F_t := \{z \in D; t \leq t_z\}$, $t < \zeta$, constitute a family of growing (i.e. strictly increasing) hulls in $D$, and the function $g_t: D \setminus F_t \to D(s(t))$ is the canonical map for $F_t$. See [3] Section 5 for further detail.

While we have seen in Section 11 that the Komatu–Loewner equations were obtained for the canonical map induced from a simple curve, we now notice that these equations should be established even if we start at a nice family of growing hulls. To explain this fact precisely, let $\{F_t\}_{t \in [0, t_0]}$ be a family of growing hulls in a standard slit domain $D$ (of $N$ slits) and $g_t: D \setminus F_t \to D_t$ be the canonical map. We say that

- $\{F_t\}$ is **continuous** if $\{D \setminus F_t\}$ is continuous in the sense of kernel convergence [11] Definition 4.2;
- $\xi: [0, t_0) \to \mathbb{R}$ is the **driving function** of $\{F_t\}$ if, for each $t \in [0, t_0)$,
  $$\bigcap_{\delta > 0} g_t(F_t + \delta \setminus F_t) = \{\xi(t)\}. \quad (2.3)$$

Suppose that $\{F_t\}$ is continuous. Then the range $\{D_t\}$ is continuous in the sense of kernel convergence by [11] Lemma 4.4. The half-plane capacity $\text{hcap}^D(F_t)$ is also continuous and strictly increasing. Hence we can take a continuous Slit-valued function $s(t)$ satisfying $D_t = D(s(t))$ and reparametrize $\{F_t\}$ so that $\text{hcap}^D(F_t)$ is differentiable in $t$. The next theorem is, therefore, general enough to discuss what kind of hulls induce the canonical maps satisfying the chordal Komatu–Loewner equations.

**Theorem 2.3** ([11] Theorem 4.6). Let $a_t \in C^1([0, t_0); \mathbb{R})$ be strictly increasing with $a_0 = 0$ and $\xi(t) \in C([0, t_0]; \mathbb{R})$. The following are equivalent:

(i) $\{F_t\}_{t \in [0, t_0]}$ is a family of continuously growing hulls in $D$, its driving function is $\xi(t)$, and $\text{hcap}^D(F_t) = a_t$.

(ii) The slits $s(t)$ and map $g_t(z)$ solve (2.3) and (2.2) with $\zeta \geq t_0$.

The condition (i) in Theorem 2.3 is stable under conformal transformation. More precisely, let $V$ be a subdomain of $D$ with $\bigcup_{t \in [0, t_0]} F_t \subset V$, $\tilde{D}$ be another slit domain of a possibly different number of slits, and $h$ be a univalent function from $V$ into $\tilde{D}$. By [11] Theorem 4.8, the family of the images $\{h(F_t)\}_{t \in [0, t_0]}$ by $h$ is again a family of continuously growing hulls in $\tilde{D}$. Let $\tilde{g}_t$ be the canonical map for $h(F_t)$ and $h_t := \tilde{g}_t \circ h \circ g_t^{-1}$. The driving function of $\{h(F_t)\}$ is $h_t(\xi(t))$, and $d/dt \text{hcap}^{\tilde{D}}(h(F_t)) = h_t(\xi(t))^2 \partial_t h_t$ holds. The case where $\tilde{D} = \mathbb{H}$ was examined in [3] to reduce the analysis of SKLE to that of SLE. After that, the general case was proven in [11] to give a full comprehension of the locality of $\text{SKLE}_{\sigma, -b_{\text{map}}} \quad [11]$ Theorem 4.9] and to investigate the existing SLE-type processes on multiply connected domains via the Komatu–Loewner equations. The study [12] on the relation between SKLE$_{\alpha, \beta}$ and the Laplacian-$b$ motion [10] illustrates the latter motivation well.

Finally, we note that all the results summarized in this section are also the case for the chordal Loewner equation on $\mathbb{H}$ by defining $\Psi_{\mathbb{H}}(z, \xi_0) = \pi^{-1}(z - \xi_0)^{-1}$ except that we do not need to consider the equation for the slits.
Remark 2.4. $\Im \Psi_D(z, \xi_0) = K_D^*(z, \xi_0)$ is always positive because $K_D^*$ is the Poisson kernel of BMD. Hence both (2.4) and (2.2) yield downward flows, and the hull $F_t$ consists of the points $z$ whose images $g_t(z)$ eventually reach the point $\xi(t)$ on $\partial H$. The flow of $g_t(z)$ and the continuity of $\xi(t)$ thus strongly affect the shape of $F_t$. As for the chordal Loewner equation in $H$, visual and detailed expositions on this relation can be found in some literature, for example, in [8, Chapter 2]. Our observations in Section 1 also comes from such a visual comprehension.

3 Main results and proof

3.1 Asymptotic behavior of the slit motion

Thoughout this section, we fix a standard slit domain $D$ of $N(\geq 1)$ slits. The ODEs (2.1) and (2.2) under the half-plane capacity parametrization $a_t = 2t$ are written as follows:

\[
\frac{d}{dt}s_l(t) = b_l(\xi(t), s(t)), \quad 1 \leq l \leq 3N, \quad (3.1)
\]

\[
\frac{d}{dt}g(z) = -2\pi \Psi(s(t))(g(z), \xi(t)), \quad g_0(z) = z \in D(s(0)). \quad (3.2)
\]

We formulate the asymptotic behavior (1.4) in terms of the slit vector $s(t)$. We define a function $R(\xi_0, s)$ on $\mathbb{R} \times \text{Slit}$ by

\[
R(\xi_0, s) := \min_{1 \leq j \leq N} \text{dist}(C_j(s), \xi_0).
\]

This function is clearly invariant under horizontal translation, that is, $R(\xi_0, s) = R(0, s - \hat{\xi}_0)$. Here, $\hat{\xi}_0 \in \mathbb{R}^{3N}$ stands for the vector whose first $N$ entries are zero and last $2N$ entries are $\xi_0$. The functions $b_l$ on the right-hand side of (3.1) are also invariant under horizontal translation by [3, Eq. (3.29)]. For later use, we adopt the notation $f(s) := f(0, s)$ when a function $f$ on $\mathbb{R} \times \text{Slit}$ has this invariance. We have, for example, $f(\xi_0, s) = f(s - \hat{\xi}_0)$ under this notation. The main result in this section is now stated as follows:

**Theorem 3.1.** Suppose that $\xi \in C([0, \infty); \mathbb{R})$ and $s^{\text{int}} \in \text{Slit}$ with $D(s^{\text{int}}) = D$ are given. Let $\zeta$ denote the explosion time of the solution $s(t)$ to (3.1) driven by $\xi(t)$ with initial value $s^{\text{int}}$.

(i) If $\zeta$ is finite, then it holds that

\[
\lim_{t \uparrow \zeta} R(\xi(t), s(t)) = 0. \quad (3.3)
\]

(ii) The inequality $\zeta \geq 2y_0^2$ holds, where $y_0 := \min_{1 \leq i \leq N} s_i^{\text{int}}$.

In the proof of this and the next theorems, we consider the following condition for a function $f$ on $\mathbb{R} \times \text{Slit}$:

(B) $f(\xi_0, s)$ is bounded on the set $\{ (\xi_0, s) \in \mathbb{R} \times \text{Slit}; R(\xi_0, s) > r \}$ for each $r > 0$. 

7
If $f$ is invariant under horizontal translation, then this condition is equivalent to the one that

$$(B') \ f(s) = f(0, s)$$

is bounded on the set \( \{ s \in \text{Slit}; R(s) = R(0, s) > r \} \) for each \( r > 0 \).

Since most of the functions appearing in this paper is invariant under horizontal translation, the latter form $(B')$ is more convenient to our argument. We shall observe in Lemma 3.4 that the functions $b$ enjoy Condition (B).

We now provide a probabilistic version of Theorem 3.1. Let $\alpha$ be a non-negative function on $\text{Slit}$ homogeneous with degree 0 and $b$ be a function on the same space homogeneous with degree $-1$, both of which are supposed to enjoy the local Lipschitz continuity. Here, a function $f(s)$ of $s \in \text{Slit}$ is said to be homogeneous with degree $\delta \in \mathbb{R}$ if

$$f(cs) = c^\delta f(s)$$

holds for all $c > 0$ and $s \in \text{Slit}$.

The stochastic Komatu–Loewner evolution SKLE $[3, \text{Section 5.1}]$ is defined as the family $\{ F_t \}$ of continuously random growing hulls in $D$ produced by (3.2) and the system of SDEs (3.1) and

$$d\xi(t) = \alpha(\xi(t), s(t)) \ dB_t + b(\xi(t), s(t)) \ dt. \quad (3.4)$$

Here, $(B_t)_{t \geq 0}$ is a one-dimensional standard Brownian motion, and the coefficients in (3.4) are defined by

$$\alpha(\xi_0, s) := \alpha(s - \hat{\xi}_0) \text{ and } b(\xi_0, s) := b(s - \hat{\xi}_0).$$

By definition, these coefficients are invariant under horizontal translation. Since the local Lipschitz condition is assumed, the system of SDEs (3.1) and (3.4) has a unique strong solution that may blow up $[3, \text{Theorem 4.2}]$.

**Theorem 3.2.** Suppose that functions $\alpha \geq 0$ and $b$ on $\text{Slit}$ satisfy the local Lipschitz continuity, homogeneity with degree 0 and $-1$, respectively, and Condition (B). For $\xi_{\text{int}}, s_{\text{int}} \in \text{Slit}$ with $D(s_{\text{int}}) = D$, let $\zeta$ be the explosion time of the solution $W_t = (\xi(t), s(t))$ to the SDEs (3.1) and (3.4) with initial value $w_{\text{int}} = (\xi_{\text{int}}, s_{\text{int}})$.

(i) The property $(3.3)$ holds almost surely on the event $\{ \zeta < \infty \}$.

(ii) The inequality $\zeta \geq 2y_0^2$ holds almost surely, where $y_0 := \min_{1 \leq l \leq N} s_{l, \text{int}}$.

We shall discuss a non-trivial example that satisfies the assumptions of Theorem 3.2 in the forthcoming paper [12].

**Remark 3.3.** We have assumed that the driving function $\xi(t)$ is defined on the infinite time interval $[0, \infty)$ in Theorem 3.1. Although we have not assumed it in Theorem 3.2, we shall observe in Section 5.3 that, under Condition (B), the process $\xi(t)$ can be extended continuously as long as the slit vector $s(t)$ does not blow up. If we consider the situation where $\xi(t)$ diverges in finite time, then the conclusion of Theorem 3.1 may change as discussed in Section 4. We cannot tell whether the condition (3.3) holds or not, and the hulls $\{ F_t \}$ may “creep along to infinity very close to the real axis” $[2, \text{Section 5.1}]$ in this case.

The proof of Theorems 3.1 and 3.2 takes several steps. The first one is to prove the following key lemma on Condition (B):
Lemma 3.4. The functions $b_l(\xi_0, s), 1 \leq l \leq 3N,$ and

$$b_{\text{BMD}}(\xi_0, s) := 2\pi \lim_{z \to \xi_0} \left( \Psi_s(z, \xi_0) + \frac{1}{\pi} \frac{1}{z - \xi_0} \right)$$

all satisfy Condition (B).

We call $b_{\text{BMD}}(s) := b_{\text{BMD}}(0, s)$ the BMD domain constant of the domain $D(s)$ \cite{3} Section 6.1. By \cite{3} Lemma 6.1, $b_{\text{BMD}}$ is invariant under horizontal translation, homogeneous with degree $-1$ and locally Lipschitz continuous.

Proof of Lemma 3.4. It suffices to prove Condition (B') as mentioned above. We use classical estimates on the family

$$S := \{ f : D \to \mathbb{C}; f \text{ is univalent on } D, f(0) = 0 \text{ and } f'(0) = 1 \},$$

where $D$ stands for the unit disk centered at the origin.

Recall from \cite{11} Section 2.1 that the function

$$H_s(z, \xi_0) = \Psi_s(z, \xi_0) + \frac{1}{\pi} z - \xi_0,$$

defined in \cite{11} Eq. (2.2) is holomorphic in $z \in D(s) \cup \Pi D(s) \cup \partial \mathbb{H}$ after taking Schwarz's reflection. Here, $\Pi$ stands for the mirror reflection with respect to the real axis. It follows from definition that $b_{\text{BMD}}(\xi_0, s) = 2\pi H_s(\xi_0, \xi_0).$ Accordingly we can check by using \cite{4} Theorem 11.2 that $\Psi_s(z, \xi_0)$ defines a conformal map from $D(s) \cup \Pi D(s) \cup \partial \mathbb{H} \cup \{ \infty \} \text{ onto } \tilde{D} \cup \Pi \tilde{D} \cup \partial \mathbb{H} \cup \{ \infty \}$, where $\tilde{D}$ is another standard slit domain. Its Laurent expansion around $\xi_0$ is given by

$$\Psi_s(z, \xi_0) = -\frac{1}{\pi} \frac{1}{z - \xi_0} + \frac{1}{2\pi} b_{\text{BMD}}(\xi_0, s) + \left( H_s(z, \xi_0) - \frac{1}{2\pi} b_{\text{BMD}}(\xi_0, s) \right) = -\frac{1}{\pi} \frac{1}{z - \xi_0} + \frac{1}{2\pi} b_{\text{BMD}}(\xi_0, s) + o(1), \quad z \to \xi_0. \quad (3.5)$$

Now assume that $R(s) = R(0, s) > r$ for some $r > 0$. Since $T(z) := -1/z$ is a linear fractional transformation that maps $0$ to $\infty$ and $\infty$ to $0$, the function $h(z) := (\pi r)^{-1} (T \circ \Psi_s)(rz, 0)$ is univalent on $D$. By the expansion (3.5), we have

$$h(z) = -\frac{1}{\pi r} \left( -\frac{1}{\pi rz} + \frac{1}{2\pi} b_{\text{BMD}}(s) + o(1) \right)^{-1} = z \cdot \frac{1}{1 - (r/2)b_{\text{BMD}}(s)z + o(z)} = z + \frac{r}{2} b_{\text{BMD}}(s) z^2 + o(z^2), \quad z \to 0,$$

which yields $h \in S$. Thus we can apply Bieberbach’s theorem (see \cite{11} Proposition 3.4] and references therein) to $h$ to obtain

$$|b_{\text{BMD}}(s)| \leq \frac{4}{r}.$$

To show Condition (B') for $b_l$, we use Koebe’s one-quarter theorem:

$$f(D) \supset B(0, 1/4) \text{ for any } f \in S. \quad (3.6)$$
If \( (3.9) \) holds but \( \limsup_{t \to t^*} R(\xi(t), s(t)) \) goes to zero before \( Y(t) \) goes to zero, where \( Y(t) \) is the solution to the ODE
\[
\frac{dY(t)}{dt} = -\frac{1}{2Y(t)} \\
Y(0) = y_0.
\]
It is easy to check that \( Y(t) \) satisfies \( t = 2(y_0^2 - Y(t)^2) \). Hence Theorem 3.1 (ii) follows by letting \( Y(t) \to 0 \).

3.2 Outline of the proof of Theorem 3.1

Suppose that \( \xi \in \mathcal{C}[0, \infty); \mathbb{R} \) and \( s^{\text{int}} \in \text{Slit with } D(s^{\text{int}}) = D \) are given, and let \( \zeta \) denote the explosion time of the solution \( s(t) \) to (3.1) driven by \( \xi(t) \) with initial value \( s^{\text{int}} \). Moreover we suppose that \( \zeta \) is finite.

Proposition 3.5. If
\[
\liminf_{t \to \zeta} R(\xi(t), s(t)) = 0 \tag{3.9}
\]
holds, then \( \text{(3.3) holds.} \)

Proof. Suppose that (3.9) holds but \( \limsup_{t \to \zeta} R(\xi(t), s(t)) \geq 5r \) for some \( r > 0 \). There are then two increasing sequences \( \{t_n\}_{n=1}^\infty \) and \( \{t'_n\}_{n=1}^\infty \) both converging to \( \zeta \) such that \( R(\xi(t_n), s(t_n)) > 4r \) and \( R(\xi(t'_n), s(t'_n)) \leq r \). Taking their subsequences if necessary, we may and do assume \( t_n < t'_n < t_{n+1} \) for \( n \in \mathbb{N} \) without loss of generality. By this assumption \( \lim_{n \to \infty} (t'_n - t_n) = \zeta - \zeta = 0 \), but, in fact, we can show \( \inf_{n} (t'_n - t_n) > 0 \) as follows: Let
\[
\delta := \min\{|s - t|; s, t \in [0, \zeta], |\xi(s) - \xi(t)| \geq r\}.
\]
\[
M := \sup\{|b_0(\xi_0, s)|; (\xi_0, s) \in \mathbb{R} \times \text{Slit, } R(\xi_0, s) > 2r\},
\]
\[
I := \{t \in [0, \zeta); R(\xi(t), s(t)) > 2r\}.
\]
The constant $\delta$ is positive, $M$ is finite by Lemma 3.4, and
\[
\max_{1 \leq i \leq 3N} |s_i(s) - s_i(t)| \leq M|s - t|
\]
by (3.1) if $s$ and $t$ belong to the same subinterval of $I$. Thus it follows from the definition of $\{t_n\}_n$ and $\{t'_n\}_n$ that
\[
|t'_n - t_n| \geq \delta \land \frac{r}{M}.
\]
Since the right-hand side is independent of $n$, we have $\inf_n |t'_n - t_n| > 0$, which contradicts $\lim_{n \to \infty} (t'_n - t_n) = 0$.

The proof of (3.9) is rather complicated. We assume to the contrary that
\[
\inf_{t < \zeta} R(\xi(t), s(t)) > r
\]
holds for some $r > 0$.

**Proposition 3.6.** Under the assumption (3.10), the slit vector $s(t)$ converges to an element $s(\zeta) \in \text{Slit} \subset \mathbb{R}^{3N}$ as $t \nearrow \zeta$.

**Proof.** By (3.10) and Lemma 3.4, we have
\[
\sup_{t \in [0, \zeta)} |b_l(\xi(t), s(t))| \leq \sup_{(\xi_0, s) \in \mathbb{R} \times \text{Slit}, R(\xi_0, s) > r} |b_l(\xi_0, s)| < \infty.
\]
Hence the right-hand side of (3.1) is integrable in $t$ over the interval $[0, \zeta)$. By Proposition 3.6, the range $D_t := D(s(t))$ converges to a domain $D_\zeta$ as $t \nearrow \zeta$ in the sense of kernel convergence, and the limit domain $D_\zeta$ is of the form $\mathbb{H} \setminus \bigcup_{j=1}^{N} C_{j, \zeta}$, where $C_{j, \zeta}$ denotes the $j$-th 'slit' corresponding to $s(\zeta)$. The segment $C_{j, \zeta}$ may degenerate to a point or be a subset of $\partial \mathbb{H}$ for some $j$. Our goal is to show that actually $s(\zeta) \in \text{Slit}$, a contradiction to our assumption that $\zeta$ is the explosion time of the solution $s(t)$ to the ODE (3.1) on $\text{Slit}$.

For this purpose, we extend the associated Komatu–Loewner evolution $\{F_t\}_{t < \zeta}$ driven by $\xi(t)$ in $D$ continuously beyond $\zeta$ by regarding it as a Loewner evolution in $\mathbb{H}$ by means of [11, Theorem 4.8]. Let $\iota: D \hookrightarrow \mathbb{H}$ be the inclusion map and $g_0^\iota: \mathbb{H} \setminus F_t \to \mathbb{H}$ be the canonical map for $F_t$ in $\mathbb{H}$. We define (by Schwarz’s reflection)
\[
\iota_t := g_0^\iota \circ \iota \circ g_t^{-1}: D_t \cup \Pi D_t \cup \partial \mathbb{H} \hookrightarrow \mathbb{C}.
\]
As explained in Section 2 [5, Theorem 2.6] or [11, Theorem 4.8] implies that $\{F_t\}_{t < \zeta}$ is produced by a generalized chordal Loewner equation
\[
\frac{d}{dt} g_t^0(z) = 2\pi i \iota'_t(\xi(t))^2 \psi_H(g_t^0(z), \iota_t(\xi(t))), \quad z \in \mathbb{H}.
\]
In other words, its half-plane capacity and driving function in $\mathbb{H}$ are given by
\[
a_t^0 := \text{hcap}_{\mathbb{H}}(F_t) = 2 \int_0^t \iota'_s(\xi(s))^2 \, ds \quad \text{and} \quad U(t) := \iota_t(\xi(t)),
\]
respectively. The following three assertions hold under the assumption (3.10):
Proposition 3.7. There exist an open interval \( J \) and constants \( t_1 \in (0, \zeta) \) and \( A > 1 \) such that \( \xi([t_1, \zeta)) \subset J \) and

\[
\frac{1}{2A} \leq \xi'(\xi_0) \leq \frac{3A}{2}, \quad \xi_0 \in J, \ t \in [t_1, \zeta).
\]

Corollary 3.8. The monotone limit \( a_{0_-}^\zeta := \lim_{t \searrow \zeta} a_{0_t}^\zeta \) is finite.

Proposition 3.9. The driving function \( U(t) \) converges as \( t \nearrow \zeta \).

Corollary 3.8 immediately follows from (3.12) and Proposition 3.7. The proof of Propositions 3.7 and 3.9 is postponed to Sections 3.3 and 3.4.

Proof. For \( \xi \in \partial J \) the time-change, the equation (3.11) is reduced to the usual Loewner equation (1.3), and the evolution (3.9) is now produced by (1.3) driven by \( \xi(t) \) for \( t < \zeta \) by Proposition 3.9, the inclusion \( \hat{F}_t \subset D \) holds. Since \( h_t^1 \in S \) holds by (3.11), Koebe’s one-quarter theorem (3.6) applies to \( h_t^1 \).

Proposition 3.10. Under the assumption (3.10), the inclusion \( \hat{F}_t \subset D \) holds. In particular, the image \( \hat{g}_t^0(D \setminus \hat{F}_t) = \mathbb{H} \setminus \bigcup_{j=1}^N \hat{g}_t^0(C_j) \) is a non-degenerate \((N+1)\)-connected domain. ('Non-degenerate' means that none of the boundary components of \( \hat{g}_t^0(D \setminus \hat{F}_t) \) is a singleton.)

Proof. For \( t \in [0, \zeta) \), we set

\[
h_t^1(z) := (r t'(\xi(t)))^{-1} (\mu(rz + \xi(t)) - \eta(\xi(t))).
\]

Since \( h_t^1 \in S \) holds by (3.11), Koebe’s one-quarter theorem (3.6) applies to \( h_t^1 \). Combined with Proposition 3.7, the theorem implies that

\[
\min_{1 \leq j \leq N} \text{dist}(g_t^0(C_j), U(t)) \geq \frac{r}{2A}
\]

for \( t < \zeta \). By passing to the limit as \( t \nearrow \zeta \), we have

\[
\min_{1 \leq j \leq N} \text{dist}(g_0^0(C_j), \bar{U}(\zeta)) > 0,
\]

which yields \( \hat{F}_t \cap \bigcup_{j=1}^N C_j = \emptyset \). \( D \setminus \hat{F}_t \) is thus a non-degenerate \((N+1)\)-connected domain. Since the non-degeneracy of a finitely multiply connected domain is preserved under conformal maps (cf. [8], Exercise 15.2.1), the proposition follows.

Proposition 3.11. The slit domain \( D_\zeta = \mathbb{H} \setminus \bigcup_{j=1}^N C_{j, \zeta} \) is non-degenerate and \((N+1)\)-connected.

Proof. We consider the two families of domains

\[
\hat{D}_t := D_{(a_{0_t}^\zeta)^{-1}(2t)}, \ t < \zeta \quad \text{and} \quad \hat{D}_\zeta^0 := g_\zeta^0(D \setminus \hat{F}_\zeta), \ t \leq \zeta.
\]
We have seen just after Proposition 3.6 that the former family converges to $D_\zeta$ as $t \nrightarrow \zeta$ in the sense of kernel convergence. The latter one also converges to $D^0_\zeta$ at the same time by Theorem 2.2 since $\hat{\eta}^0_{t,\zeta} \to \hat{\eta}^0_{\zeta}$ locally uniformly on $\mathbb{H} \backslash \mathbb{F}_\zeta$. By Theorem 2.2 again, there exists a conformal map $\eta^{-1}_{\zeta} : D^0_\zeta \to D_\zeta$, which proves the proposition due to Proposition 3.10.

The claim of Proposition 3.11 is equivalent to $s(\zeta) \in \text{Slit}$, as was to be proven.

### 3.3 Proof of Proposition 3.7

The aim of this subsection is to prove Proposition 3.7 under the assumption 3.10. By Proposition 3.6, there is a constant $L > 0$ so that $\xi([0, \zeta]) \cup \bigcup_{t \in [0, \zeta]} C_{j,t} \subset B(0, L)$, where $C_{j,t} := C_j(s(t))$. Since the conformal map $\eta_t$ is the composite of three maps hydrodynamically normalized, it satisfies

$$
\eta_t(z) = z + \frac{c_t}{z} + o(z^{-1}) \ (z \to \infty), \ \ z \in \Delta(0, L),
$$

for some constant $c_t$. We define a normalized function $f_t$ on $D^* := \Delta(0, 1)$ by $f_t(z) := L^{-1}\eta_t(Lz)$. The function $f_t$ is an element of the set

$$
\Sigma := \{ f : D^* \to \mathbb{C}; f \text{ is univalent, } f(\infty) = \infty \text{ and } \text{Res}(f, \infty) = 1\}.
$$

Hence we have $\mathbb{C} \setminus f_t(D^*) \subset \overline{B(0, 2)}$ by [11] Lemma 3.5. In terms of $\eta_t$, this means that

$$
\eta_t(B(0, L)) \subset \mathbb{C} \setminus \eta_t(\Delta(0, L)) \subset \overline{B(0, 2L)}. \quad (3.14)
$$

If $D_t$ had no slits, then the boundedness of $\eta_t'(\eta(t))$ would follow from (3.14) combined with elementary tools in complex analysis such as Schwarz’s lemma. These tools, however, do not work on multiply connected domains. For this reason, we employ the boundary Harnack principle instead:

**Proposition 3.12** ([11] Theorem 8.7.14). *Let $G \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $V \subset \mathbb{R}^2$ be an open set, $K$ be a compact subset of $V$, and $z_0 \in G$. Then there exists a constant $A > 1$ such that, for any two harmonic functions $h_1$ and $h_2$ on $G$ taking value zero on $V \cap \partial G$, it holds that

$$
\frac{h_1(x)}{h_2(x)} h_2(z_0) \leq A, \quad x \in K \cap G.
$$

We shall apply this proposition to the harmonic functions $h_1(z) = \Im \eta_t(z)$ and $h_2(z) = \Im z$. The sets $G$, $V$, $K$ and point $z_0$ in the assumption are chosen as follows (see Figure 2): By the assumption (3.10) and Proposition 3.6, there exist a constant $t_1 \in [0, \zeta)$ and finite open subinterval $J$ of $\partial \mathbb{H}$ such that

$$
\xi([t_1, \zeta)) \subset J \quad \text{and} \quad \mathcal{J} \cap \bigcup_{t \in [t_1, \zeta]} C_{j,t} = \emptyset.
$$

For this interval $J$, there exist a relatively compact open set $O$ and an open set $V$ such that

$$
J \subset O \subset \overline{O} \subset V \quad \text{and} \quad \mathcal{J} \cap \bigcup_{t \in [t_1, \zeta]} C_{j,t} = \emptyset.
$$

13
For this set $V$ and an arbitrary fixed point $z_0 \in D$ with $\Im z_0 \geq 6L$, we can take a bounded domain $G \subset D$ with smooth boundary so that

$$z_0 \in G, \quad V \cap \partial \mathbb{H} \subset \partial G \quad \text{and} \quad G \cap \bigcup_{t \in [t_1, \zeta]} C_{j,t} = \emptyset.$$  

Now we apply Proposition 3.12 to $h_1$ and $h_2$ with $G$, $V$, $K := \overline{O}$ and $z_0$ chosen in this way to obtain

$$A^{-1} \leq \frac{\Im \imath_t(z)}{\Im z_0} \leq A, \quad z \in G \cap K, \ t \in [t_1, \zeta), \tag{3.15}$$

for a constant $A > 1$ independent of $z$ and $t$.

On the other hand, we can observe from (3.14) that

$$|\Im \imath_t(z_0) - \Im z_0| = \left| \mathbb{E}_{z_0}^{\mathbb{H}} \left[ \Im \imath_t(Z_{\sigma_{C_t}}) - \Im Z_{\sigma_{C_t}}^{\mathbb{H}} ; \sigma_{C_t} < \infty \right] \right| \leq \mathbb{E}_{z_0}^{\mathbb{H}} \left[ \Im \imath_t(Z_{\sigma_{C_t}}^{\mathbb{H}}) + \Im Z_{\sigma_{C_t}}^{\mathbb{H}} ; \sigma_{C_t} < \infty \right] \leq 3L.$$  

Here, $Z^{\mathbb{H}}$ is an absorbing Brownian motion in $\mathbb{H}$, $\sigma_{C_t}$ is the hitting time of $Z^{\mathbb{H}}$ to $C_t := \bigcup_{j=1}^N C_{j,t}$, and $\mathbb{E}_{z_0}^{\mathbb{H}}$ stands for the expectation with respect to $Z^{\mathbb{H}}$ starting at $z_0$. Hence we have

$$\left| \frac{\Im \imath_t(z_0)}{\Im z_0} - 1 \right| \leq \frac{3L}{\Im z_0} \leq \frac{1}{2}, \quad \text{i.e.,} \quad \frac{1}{2} \leq \frac{\Im \imath_t(z_0)}{\Im z_0} \leq \frac{3}{2}. \tag{3.16}$$

Substituting (3.16) into (3.15) yields that

$$\frac{1}{2A} \leq \frac{\Im \imath_t(z)}{\Im z} \leq \frac{3A}{2}, \quad z \in G \cap K, \ t \in [t_1, \zeta). \tag{3.17}$$

Since the function $\imath_t$ is defined across $\partial \mathbb{H}$ by Schwarz’s reflection, it is easily checked that $\lim_{z \to \xi_0} \Im \imath_t(z)/\Im z = \imath'_t(\xi_0)$ for $\xi_0 \in \partial \mathbb{H}$. Thus by taking the limit
as \( z \) goes to \( \xi_0 \in J \) in (3.17), we have
\[
\frac{1}{2A} \leq \nu_t'(\xi_0) \leq \frac{3A}{2}, \quad \xi_0 \in J,
\]
which proves Proposition 3.7.

3.4 Proof of Proposition 3.9

The aim of this subsection is to prove Proposition 3.9 under the assumption 3.10. To this end, we approximate the continuous function \( \xi \) by \( \xi^\varepsilon \in C^1[0,\infty) \) so that
\[
\sup_{t \in [t_1,\zeta]} |\xi(t) - \xi^\varepsilon(t)| < \varepsilon \quad \text{and} \quad \{\xi^\varepsilon(t); t \in [t_1,\zeta]\} \subset J
\]
hold for \( \varepsilon \in (0, r/2) \). Here, the constant \( t_1 \) and interval \( J \) are those in Proposition 3.7.

**Lemma 3.13.** \( \nu_t(\xi^\varepsilon(t)) \) converges as \( t \nearrow \xi \) for each fixed \( \varepsilon \in (0, r/2) \).

**Proof.** \( \nu_t(\xi^\varepsilon(t)) \) is represented as
\[
\nu_t(\xi^\varepsilon(t)) = \nu_{t_1}(\xi^\varepsilon(t_1)) + \int_{t_1}^{t} \frac{d}{ds} \nu_{s}(\xi^\varepsilon(s)) \, ds
\]
\[
= \nu_{t_1}(\xi^\varepsilon(t_1)) + \int_{t_1}^{t} \{ (\partial_{s} \nu_{s})(\xi^\varepsilon(s)) + \nu'_s(\xi^\varepsilon(s))\dot{\xi}^\varepsilon(s) \} \, ds
\]
for \( t \in [t_1, \zeta] \). By Proposition 3.7, we have
\[
\sup_{s \in [t_1, \zeta]} |\nu'_s(\xi^\varepsilon(s))\dot{\xi}^\varepsilon(s)| \leq \frac{3A}{2} \, \max_{s \in [t_1, \zeta]} |\dot{\xi}^\varepsilon(s)| < \infty.
\]
Thus it suffices to prove that \( \sup_{s \in [t_1, \zeta]} |(\partial_{s} \nu_{s})(\xi^\varepsilon(s))| < \infty \) in order to establish the lemma.

We begin with the computation of \( \partial_{s} \nu_{t}(z) \) for \( z \in D \). By the definition of \( \nu_{t} \) and \( H_{s(t)} \), we have
\[
\nu_{t}(z) = g_{t}^0(t \circ g_{t}^{-1}(z)) + (g_t^0)'(t \circ g_{t}^{-1}(z))\partial_{t}g_{t}^{-1}(z)
\]
\[
= \frac{2t_{4}(\xi(t))^{2}}{t_{4}(z) - U(t)} - (g_t^0)'(t \circ g_{t}^{-1}(z))(g_{t}^{-1})'(z)\dot{g}_{t}(g_{t}^{-1}(z))
\]
\[
= \frac{2t_{4}(\xi(t))^{2}}{t_{4}(z) - \nu_{t}(\xi(t))} + 2\pi i \nu_{t}(z)\Psi_{s(t)}(z, \xi(t))
\]
\[
= \frac{2t_{4}(\xi(t))^{2}}{t_{4}(z) - \nu_{t}(\xi(t))} + 2\pi i \nu_{t}(z)\Psi_{s(t)}(z, \xi(t)).
\]

We denote the first two terms in (3.18) by \( \Theta_{t}(z) \). Since \( \nu_{t} \) is holomorphic on the disk \( B(\xi(t), r) \), so is \( \Theta_{t} \) on the punctured disk \( B(\xi(t), r) \setminus \{\xi(t)\} \). In fact, \( \xi(t) \) is a removable singularity of \( \Theta_{t} \) because
\[
\Theta_{t}(z) = \frac{2t_{4}(\xi(t))^{2}}{t_{4}(z) - \nu_{t}(\xi(t))} - \frac{2\nu_{t}'(z)}{z - \xi(t)} \rightarrow -3\nu_{t}'(\xi(t)), \quad z \rightarrow \xi(t),
\]
15
by [9, Proposition 4.40]. Consequently, the expression (3.18) is valid for all $z \in D \cup \partial D \cup \partial \mathbb{H}$.

We now give a closer look at $\Theta_t(z)$. Since the function $2h^1_t(z/2)$ with $h^1_t$ defined by (3.13) belongs to $S$, we have

$$|\iota_t(z) - \iota_t(\xi(t))| \geq \frac{r^1_t(\xi(t))}{8} \geq \frac{r}{16.4}, \quad z \in \partial B\left(\xi(t), \frac{r}{2}\right), \quad t \in [t_1, \zeta), \quad (3.19)$$

by Proposition 3.7 and Koebe’s one-quarter theorem 3.3. Moreover, we utilize the distortion theorem (see [6, Theorem 14.7.9 (a)] or [13, Theorem 1.6 (11)]):

$$\frac{1 - |z|}{(1 + |z|)^2} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^2}, \quad z \in D, \quad f \in S. \quad (3.20)$$

The inequality (3.20) with $f = h^1_t$ and Proposition 3.7 yield, for $z \in B(\xi(t), \rho)$ and $t \in [t_1, \zeta)$,

$$|\iota_t'(z)| \leq r^1_t(\xi(t)) \frac{1 + r^{-1}|z - \xi(t)|}{(1 - r^{-1}|z - \xi(t)|)^2} \leq \frac{3A r^4 + r^3|z - \xi(t)|}{2} \left(\frac{1}{r} - \frac{1}{|z - \xi(t)|}\right)^3. \quad (3.21)$$

In addition, it follows from (3.14) that

$$\sup_{t \in [t_1, \zeta]} |\iota_t(\xi(t))| \leq 2L \quad \text{and} \quad \sup_{t \in [t_1, \zeta]} |\iota_t(\xi^\infty(t))| \leq 2L. \quad (3.22)$$

By (3.19), (3.21) and (3.22), there exists a constant $M_1 > 0$ such that

$$\sup_{t \in [t_1, \zeta]} \max_{z \in \partial B(\xi(t), r/2)} |\Theta_t(z)| \leq M_1. \quad (3.23)$$

The maximal value principle for $\Theta_t$ then implies that

$$\sup_{t \in [t_1, \zeta]} \sup_{z \in B(\xi(t), r/2)} |\Theta_t(z)| \leq M_1. \quad (3.24)$$

Hence it holds that

$$\sup_{s \in [t_1, \zeta]} |\Theta_s(\xi^\infty(s))| \leq M_1. \quad (3.25)$$

It remains to estimate $(\partial_{s^1} \Theta_s)(\xi^\infty(s)) - \Theta_s(\xi^\infty(s)) = 2\pi \iota_s(\xi^\infty(s))H_s(\xi^\infty(s), \xi(s))$. By (3.7) and (3.10), we have

$$|H_{s^1}(z, \xi(t))| = \left|\Psi_{s^1}(z, \xi(t)) + \frac{1}{\pi z - \xi(t)}\right| \leq \frac{5}{4\pi r}$$

for $z \in \partial B(\xi(t), r)$ and $t \in [0, \zeta)$. By the maximal value principle for $H_{s^1}(\cdot, \xi(t))$, we obtain

$$\sup_{s \in [0, \zeta]} \sup_{z \in B(\xi(t), r)} |H_{s^1}(z, \xi(t))| \leq \frac{5}{4\pi r}. \quad (3.26)$$

(3.22) and (3.26) yield

$$\sup_{s \in [t_1, \zeta]} |2\pi \iota_s(\xi^\infty(s))H_s(\xi^\infty(s), \xi(s))| \leq \frac{5L}{r}, \quad (3.27)$$

It follows from (3.18), (3.23) and (3.27) that $\sup_{s \in [t_1, \zeta]} |(\partial_{s^1} \Theta_s)(\xi^\infty(s))| < \infty$, which is the desired conclusion.
Recall that \( \sup_{t \in [t_\epsilon, 1]} |\xi(t) - \xi^\epsilon(t)| < \epsilon \) is assumed at the beginning of this subsection. It holds that
\[
\limsup_{t \uparrow \zeta} t_\epsilon(\xi(t)) - \liminf_{t \uparrow \zeta} t_\epsilon(\xi(t)) \leq |\limsup_{t \uparrow \zeta} t_\epsilon(\xi(t)) - t_\epsilon(\xi^\epsilon(t))| - |\liminf_{t \uparrow \zeta} t_\epsilon(\xi(t)) - t_\epsilon(\xi^\epsilon(t))| \\
\leq \frac{3\epsilon}{2} + \frac{3\epsilon}{2} = 3\epsilon
\]
by Lemma 3.13 and Proposition 3.7. By letting \( \epsilon \to 0 \) in this inequality and taking \( \epsilon \) into account, we observe that \( U(t) = t_\epsilon(\xi(t)) \) converges as \( t \not
\zeta \). The proof of Proposition 3.9 and thus of Theorem 3.1 (i) is now complete.

### 3.5 Proof of Theorem 3.2

This subsection is devoted to the proof of Theorem 3.2, which proceeds along lines similar to those in Section 3.2. Suppose that functions \( \alpha \geq 0 \) and \( b \) on \( \text{Slit} \) satisfy the assumption of Theorem 3.2. We denote by \( \mathbb{P}_w \) the law of the solution \( W_t = (\xi(t), s(t)) \) to the SDEs (3.1) and (3.4) with initial value \( W_0 = w \in \mathbb{R} \times \text{Slit} \). We write \( \mathbb{P}_w \) simply as \( \mathbb{P} \). As mentioned in Chapter IV, Section 6 of [17], the solution \( W = (W_t, \mathbb{P}_W) \) becomes a diffusion process on the state space \( (\mathbb{R} \times \text{Slit})_\infty := (\mathbb{R} \times \text{Slit}) \cup \{ w_{\infty} \} \), where \( w_{\infty} \) is the cemetery, with respect to the augmented filtration \( (\mathcal{F}_t)_{t \geq 0} \) of the Brownian motion \( (B_t)_{t \geq 0} \) in (3.3). We denote the life time of \( W \) by \( \zeta \). (This is a slight abuse of notation, but there should be no risk of confusion.)

We define an operator \( \Lambda_r : C(\mathbb{R} \times \text{Slit}) \to C((\mathbb{R} \times \text{Slit})_\infty) \) for \( r > 0 \) by
\[
\Lambda_r f(w) := \begin{cases} 
f(w) & \text{if } w \in \mathbb{R} \times \text{Slit} \text{ and } R(w) \geq r \\
0 & \text{otherwise.}
\end{cases}
\]

Using this operator, we define a process \( W^*_r = (\xi^*(t), s^*(t)) \) by
\[
W^*_r := W_0 + \int_0^t \Lambda_r \alpha(W_s) dB_s + \int_0^t \Lambda_r b(W_s) ds, \quad t \geq 0.
\]
The functions \( \Lambda_r \alpha \) and \( \Lambda_r b \) are bounded by Condition (B). Hence \( (W^*_r)_{t \geq 0} \) is a continuous semimartingale whose local martingale part is a square-integrable martingale. Let \( \tau_r := \inf \{ t > 0 ; W_t = w_{\infty} \text{ or } R(W_t) < r \} \).

**Proposition 3.14.** For any starting point \( w \in \mathbb{R} \times \text{Slit} \) and \( r \in (0, R(w)) \), it holds that \( W_t = W^*_r \) for all \( t \in [0, \tau_r) \) \( \mathbb{P}_w \)-almost surely. In particular, \( W_t \) converges in \( \mathbb{R} \times \text{Slit} \) as \( t \not\tau_r \) \( \mathbb{P}_w \)-almost everywhere on \( \{ \tau_r < \infty \} \).

**Proof.** Since \( \alpha(W_t) = \Lambda_r \alpha(W_t) \) and \( b(W_t) = \Lambda_r b(W_t) \) hold for \( t < \tau_r \), the conclusion follows from [17] Proposition II.2.2 (iv)] and the localization by an appropriate sequence of stopping times. \( \square \)

For \( r > 0 \), we define stopping times \( \{ \tau_{r,n} \}_{n=0}^{\infty} \), \( \{ \tau'_{r,n} \}_{n=0}^{\infty} \) and \( \{ \sigma_{r,n} \}_{n=0}^{\infty} \) recursively by \( \tau'_{r,0} := 0 \) and
\[
\tau_{r,n} := \inf \{ t > \tau'_{r,n} ; W_t = w_{\infty} \text{ or } R(W_t) < r \}, \\
\sigma_{r,n} := \inf \{ t > \tau_{r,n} ; W_t = w_{\infty} \text{ or } |\xi(t) - \xi(\tau_{r,n})| \geq r \}, \\
\tau'_{r,n+1} := \inf \{ t > \tau_{r,n} ; W_t = w_{\infty} \text{ or } R(W_t) \geq 4r \},
\]

and events $E_r$ and $E_{r,n}, n = 1, 2, \ldots, \infty$ by

$$E_r := \{\zeta < r^{-1}, \liminf_{t \uparrow \zeta} R(W_t) = 0 \text{ and } \limsup_{t \uparrow \zeta} R(W_t) \geq 5r\},$$

$$E_{r,n} := E_r \cap \{\sigma_{r,n} < \tau_{r,n}\}.$$  

Moreover, we have

$$\mathbb{P}(E_r) = \liminf_n E_{r,n} = \limsup_n E_{r,n}.$$  

**Lemma 3.15.** It holds that $E_r = \liminf_n E_{r,n} = \limsup_n E_{r,n}$.  

**Proof.** Assume that there are a sample $\omega \in E_r$ and an increasing sequence $(n_k(\omega))_{k=1}^{\infty}$ of natural numbers such that $\tau_{r,n_k(\omega)}(\omega) \geq \tau_{r,n_{k+1}(\omega)}(\omega)$ holds for all $k$. It follows from definition that $\tau_{r,n_k(\omega)}(\omega) - \tau_{r,n_{k+1}(\omega)}(\omega) \geq r/M$, where $M$ is the constant in the proof of Proposition 3.16. By this inequality, however, we have

$$r^{-1} > \zeta(\omega) > \sum_{k=1}^{\infty}(\tau_{r,n_k(\omega)}(\omega) - \tau_{r,n_{k+1}(\omega)}(\omega)) = \infty,$$

a contradiction. Therefore, it holds that $E_r \subseteq \liminf_n E_{r,n}$. Since it is obvious that $\liminf_n E_{r,n} \subset \limsup_n E_{r,n} \subset E_r$, the lemma follows.  

**Proposition 3.16.** The event

$$E := \{\zeta < \infty, \liminf_{t \uparrow \zeta} R(W_t) = 0 \text{ and } \limsup_{t \uparrow \zeta} R(W_t) > 0\}$$

is a $\mathbb{P}$-null set.  

**Proof.** We fix an arbitrary $r \in (0, R(\mathbb{w}^{\text{inf}}))$. It follows from the strong Markov property of $W$ that

$$r^2 \mathbb{P}(E_{r,n}) = \mathbb{E}\left[(\zeta(\sigma_{r,n}) - \zeta(\tau_{r,n}))^2 1_{E_{r,n}}\right]$$

$$\leq \mathbb{E}\left[(\zeta(\sigma_{r,n}) - \zeta(\tau_{r,n}))^2 1_{E}1_{(\zeta(\sigma_{r,n}) < \tau_{r,n})}\right]$$

$$= \mathbb{E}_{W_{r,n}}\left[(\zeta(\sigma_{r,0}) - \zeta(0))^2 1_{E}1_{(\zeta(\sigma_{r,0}) < \tau_{r,n})}\right]$$

$$= \mathbb{E}_{W_{r,n}}\left[(\zeta(0))^2 1_{E}1_{(\zeta(0) < \tau_{r,n})}\right].$$  

Moreover, we have

$$\mathbb{E}_{W_{r,n}}\left[(\zeta(r^{-1} \wedge \sigma_{r,0} \wedge \tau_r) - \zeta(0))^2\right]$$

$$= \mathbb{E}_{W_{r,n}}\left[\left(\int_0^{r^{-1} \wedge \sigma_{r,0} \wedge \tau_r} \Lambda_r \alpha(W_s) \, dB_s + \int_0^{r^{-1} \wedge \sigma_{r,0} \wedge \tau_r} \Lambda_r b(W_s) \, ds\right)^2\right]$$

$$\leq 2\mathbb{E}_{W_{r,n}}\left[\left(\int_0^{r^{-1} \wedge \sigma_{r,0} \wedge \tau_r} \Lambda_r \alpha(W_s) \, dB_s\right)^2 + \left(\int_0^{r^{-1} \wedge \sigma_{r,0} \wedge \tau_r} \Lambda_r b(W_s) \, ds\right)^2\right]$$

$$\leq 2\mathbb{E}_{W_{r,n}}\left[\int_0^{r^{-1} \wedge \sigma_{r,0} \wedge \tau_r} \Lambda_r \alpha(W_s)^2 \, ds + r^{-1} \int_0^{r^{-1} \wedge \sigma_{r,0} \wedge \tau_r} \Lambda_r b(W_s)^2 \, ds\right]$$

$$\leq 2(1 + r^{-1}) M_r \mathbb{E}_{W_{r,n}}\left[r^{-1} \wedge \sigma_{r,0} \wedge \tau_r\right].$$  

(3.27)
where \( M_r := \sup_w (\Lambda_r \alpha(w))^2 \vee \Lambda_r b(w)^2 \) < \( \infty \). Substituting (3.27) into (3.26) yields

\[
r^2 \mathbb{P}(E_{r,n}) \leq 2(1 + r^{-1}) M_r \mathbb{E} \left[ E_{W_{t,n}} \left[ (r^{-1} \wedge \sigma_{r,0} \wedge r_r) \mathbf{1}_{\{\tau_{r,n} < (r^{-1} \wedge \zeta)\}} \right] \right] = 2(1 + r^{-1}) M_r \mathbb{E} \left[ (r^{-1} + \tau'_{r,n}) \wedge \sigma_{r,n} \wedge r_r - \tau'_{r,n} \wedge \sigma_{r,n} \right] \mathbf{1}_{\{\tau_{r,n} < (r^{-1} \wedge \zeta)\}} = 2(1 + r^{-1}) M_r \mathbb{E} \left[ (2r^{-1}) \wedge \sigma_{r,n} \wedge r_r - (2r^{-1}) \wedge \tau'_{r,n} \right].
\]

Hence we have

\[
\sum_{n=0}^{\infty} \mathbb{P}(E_{r,n}) \leq 2r^{-2}(1 + r^{-1}) M_r \sum_{n=0}^{\infty} \mathbb{E} \left[ (2r^{-1}) \wedge \sigma_{r,n} \wedge r_r - (2r^{-1}) \wedge \tau'_{r,n} \right] \leq 4r^{-3}(1 + r^{-1}) M_r < \infty.
\]

It follows from the first Borel–Cantelli lemma that \( \mathbb{P}(\limsup_n E_{r,n}) = 0 \), which implies \( \mathbb{P}(E_r) = 0 \) by Lemma 3.15. Since \( E = \bigcup_k E_{1/k} \) holds, we obtain \( \mathbb{P}(E) = 0 \).

By Proposition 3.16, we can establish Theorem 3.2 (i) if we prove that the event

\[
E' := \{ \zeta < \infty, \ \liminf_{t < \zeta} R(W_t) > 0 \} = \{ \zeta < \infty, \ \liminf_{t < \zeta} R(W_t) > 0 \}
\]

is a \( \mathbb{P} \)-null set. To do this, we denote by \( \{ F_t \}_{t < \zeta} \) the SKLE\(_{a,b} \) driven by \( \xi(t) \) and take over the notations in Section 3.2 such as \( g_0^0, t_0 \), and so on. The relation (3.12) is valid also in this case. For a moment, we fix a constant \( \rho \in (0, R(W_{\text{int}})) \).

**Proposition 3.17.** There exist a random open interval \( J(\omega) \) and constants \( t_1(\omega) \in (0, \tau_r(\omega)) \) and \( A(\omega) > 1 \) such that \( \xi([t_1, \tau_r]) \subset J \) and

\[
\frac{1}{2A} \leq t'_r(\xi_0) \leq \frac{3A}{2}, \quad \xi_0 \in J, \ t \in [t_1, \tau_r),
\]

hold \( \mathbb{P} \)-almost everywhere on \( \{ \tau_r < \infty \} \).

**Proof.** For \( \mathbb{P} \)-a.a. \( \omega \in \{ \zeta = \tau_r < \infty \} \), it holds that \( \inf_{t < \zeta} R(W_t(\omega)) \geq r \). Hence the conclusion follows from Propositions 3.14 and 3.7. For \( \omega \in \{ \tau_r < \zeta \} \), the conclusion is trivial.

**Corollary 3.18.** The monotone limit \( d_{r,r-}^0 := \lim_{r \nearrow \tau_r} d_{r}^0 \) is \( \mathbb{P} \)-almost everywhere on \( \{ \tau_r < \infty \} \).

**Proposition 3.19.** The process \( U(t) = \xi(\xi(t)) \) converges as \( t \not\to \tau_r \) \( \mathbb{P} \)-almost everywhere on \( \{ \tau_r < \infty \} \).

**Proof.** While this proposition follows from Proposition 3.9, we can give a shorter proof in this case by using Itô’s formula. By [5, Theorem 2.8] or [11, Eq. (4.7)], it holds that

\[
U(t) = \xi(t) + \int_0^t \gamma'_r(\xi(s)) \alpha(W_s) dB_s + \int_0^t \gamma'_r(\xi(s)) (b_{\text{BMD}}(W_s) - b(W_s)) dt + \frac{1}{2} \int_0^t \gamma''_r(\xi(s)) (\alpha(W_s)^2 - 6) dt
\]

(3.28)
for $t < \zeta$ almost surely under $\mathbb{P}$. Here, $b_{\text{BMD}}$ is the BMD domain constant appearing in Lemma 3.4.

We set
\[
\mu_t := \iota_t'(\xi(t))1_{\{t < \tau\}} \quad \text{and} \quad \nu_t := \iota_t''(\xi(t))1_{\{t < \tau\}}.
\]
By Proposition 3.17, we can regard $\mu_t$ as a progressively measurable process that is bounded on every compact subinterval of $[0, \infty)$ a.s. We apply Bieberbach’s theorem to the function $h_t$ defined in (3.13) and use Proposition 3.17 again to obtain
\[
\frac{|\iota_t''(\xi(t))|}{2|\iota_t'(\xi(t))|} \leq 2, \quad \text{i.e.,} \quad |\iota_t''(\xi(t))| \leq \frac{4}{\tau} |\iota_t'(\xi(t))| \leq \frac{3A}{2r}
\]
for $t < \tau_r$ a.e. on $\{\tau_r < \infty\}$. Hence $\nu_t$ is also progressively measurable and bounded on every compact subinterval of $[0, \infty)$ a.s. In this way, we observe that
\[
\int_0^t \left\{ \left( \mu_t \Lambda_r \alpha(W_s) \right)^2 + |\mu_t \Lambda_r(b_{\text{BMD}} - b)(W_s)| + |\nu_t(\Lambda_r \alpha(W_s)^2 - 6)| \right\} \, ds < \infty
\]
for all $t \in [0, \infty)$ a.s., which implies that a process
\[
U_r(t) := \xi_{\text{int}}^{\text{inc}} + \int_0^t \mu_s \Lambda_r \alpha(W_s) \, dB_s + \int_0^t \mu_s \Lambda_r(b_{\text{BMD}} - b)(W_s) \, dt
\]
\[
+ \frac{1}{2} \int_0^t \nu_s \left( \Lambda_r \alpha(W_s)^2 - 6 \right) \, dt
\]
is a continuous semimartingale on $[0, \infty)$. We can check in a way similar to the proof of Proposition 3.14 that $U(t) = U_r(t)$ holds for all $t < \tau_r$ a.s. In particular, $U(t)$ converges as $t \nearrow \tau$ on $\{\tau_r < \infty\}$.

Let $E'_r := \{ \zeta < \infty, \inf_{t < \zeta} R(W_t) \geq r \}$. It holds that $\tau_r = \zeta < \infty$ on $E'_r$. From Propositions 3.17 and Corollary 3.18 it follows that Propositions 3.10 and 3.11 hold for $\mathbb{P}$-a.a. $\omega \in E'_r$. Hence we have $W'_r(\omega) \in \mathbb{R} \times \text{Slit}$ for $\mathbb{P}$-a.a. $\omega \in E'_r$, which yields $\mathbb{P}(E'_r) = 0$ by the definition of $\zeta$. Since $E'_r = \bigcup_k E'_{1/k}$ holds, we have $\mathbb{P}(E'_r) = 0$, which finishes the proof of Theorem 3.2.

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