ON DEFORMATIONS OF TORIC VARIETIES

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Abstract. Let $X$ be a smooth complete toric variety. We describe the Altmann-Ilten-Vollmert equivariant deformations of toric varieties in the language of Cox rings. More precisely we construct one parameters families $\pi: \mathcal{X} \to \mathbb{A}^1$ of deformations of $X$, such that the total space $\mathcal{X}$ of the deformation is a $T$-variety of complexity one, defined by a trinomial equation, and the map $\pi$ is equivariant with respect to the torus action. Moreover we show that the images of all these families via the Kodaira-Spencer map form a basis of the vector space $H^1(X, T_X)$.

Introduction

The topic of deformations of toric varieties has been studied by K. Altmann in [4] and A. Mavlyutov in [11]. In the affine case they describe toric deformations in a combinatorial way via polyhedral decompositions of linear sections of the defining cone of the toric variety. The theory of polyhedral divisors is later developed in [2] and [3] as a generalization of toric varieties to $T$-varieties, i.e. varieties coming with a torus action. N. Ilten and R. Vollmert make use of the language of polyhedral divisors in [9] to describe deformations of $T$-varieties of complexity one. Their method involves decomposing the polyhedral data of the varieties, similar to what Altmann did for the affine case. Moreover, in the case of smooth toric varieties they also prove that such deformations are in correspondence with a generating set of the space of infinitesimal deformations of the starting variety. In [6], we are presented with an explicit way to compute the Cox ring of a $T$-variety, starting from its polyhedral representation. This serves as the main connection between the work of N. Ilten and R. Vollmert and the present paper.

This paper is devoted to studying deformations of smooth toric varieties from the point of view of Cox rings, in a similar spirit of [11]. Starting from a toric variety $X$ and some extra combinatorial data, we describe the Cox ring of a complexity one variety $\mathcal{X}$ which fits into to a one-parameter deformation $\mathcal{X} \to \mathbb{A}^1$ of $X$, as shown in Theorem 2.1. After that, we proceed to study the corresponding Kodaira-Spencer map in Theorem 2.3.

The variety $\mathcal{X}$ turns out to be the same variety introduced in [9] and described with the language of polyhedral divisors. Moreover, and much like what was done for polyhedral divisors, we show that the deformations we describe generate the space of infinitesimal deformations of $X$. As applications of this theory we study deformations of scrolls and deformations of hypersurfaces of smooth toric varieties.

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The paper is organized as follows: Section 1 of this work covers some concepts and properties of deformation theory and toric geometry which will be used throughout the rest of the article. Section 2 is the central section of this paper, where we explain how to construct deformations of a toric variety $X$, as well as the fact that the images of these deformations, under the Kodaira-Spencer map, generate $H^1(X, T_X)$. Section 3 is a summary on the language of polyhedral divisors that was mentioned above. We also mention Ilten’s and Vollmert’s way of finding deformations of T-varieties and note that these deformations are equivalent the ones we give in the previous section. Lastly, in Section 4, we apply our results to the study of rational scrolls over $\mathbb{P}^1$, finding exactly those which are rigid and proving that every scroll deforms to a rigid one, and to deformations of hypersurfaces of toric varieties.

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1. Preliminaries

We recall here some basic facts about deformation theory and toric varieties. For the rest of this article, $\mathbb{K}$ will be an algebraically closed field of characteristic 0.

1.1. Generalities on deformations.

**Definition 1.1.** Let $X$ be a scheme over $\mathbb{K}$. A deformation of $X$ over a scheme $S$ is a flat surjective morphism of schemes $\pi : \mathcal{X} \to S$ that fits in a cartesian diagram

\[
\begin{array}{c}
\Spec(\mathbb{K}) \\
\downarrow s \\
\Spec(\mathbb{K})
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow \pi \\
\rightarrow \\
\rightarrow
\end{array} \quad \begin{array}{c}
\mathcal{X} \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\]

If $S$ is algebraic, then for each rational point $t \in S$, the scheme-theoretic fiber $\mathcal{X}(t)$ is also called a deformation of $X$. Given another deformation $\pi' : \mathcal{X}' \to S$ of $X$, we say that $\pi$ and $\pi'$ are isomorphic if there is a morphism $\phi : \mathcal{X} \to \mathcal{X}'$ inducing the identity over $X$ and such that the diagram

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow \phi \\
\downarrow \\
\downarrow \\
\Spec(\mathbb{K})
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array} \quad \begin{array}{c}
\mathcal{X}' \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\]

is commutative. We denote the set of isomorphism classes of deformations by $\text{Def}_X(S)$.

**Theorem 1.2.** [13, Theorem 2.4.1] If $X$ is a smooth scheme, there is an isomorphism of vector spaces

\[
\kappa : \text{Def}_X(\mathbb{K}[t]/(t^2)) \xrightarrow{\sim} H^1(X, T_X),
\]

where $T_X$ is the tangent sheaf of $X$. 
Let $X$ be a smooth algebraic variety and consider a deformation $\pi : X \rightarrow S$ of $X$. Giving $\varphi \in T_{S,s}$ is equivalent to giving a morphism $\varphi : \text{Spec}(K[t]/(t^2)) \rightarrow S$ with image $s$. Pulling back the deformation by $\varphi$, we obtain a deformation

$$
\begin{align*}
X & \xrightarrow{\pi} X \times_S \text{Spec}(K[t]/(t^2)) \\
\text{Spec}(K) & \xrightarrow{\pi'} \text{Spec}(K[t]/(t^2))
\end{align*}
$$

which, by Theorem 1.2, corresponds to an element of $H^1(X, TX)$.

**Definition 1.3.** The above construction gives a map

$$T_{S,s} \rightarrow H^1(X, TX)$$

called the *Kodaira-Spencer map* of the deformation $\pi$.

Note that in the case $S = \text{Spec} K[x]$, the morphism $\varphi$ above is uniquely defined up to scalar multiplication, so the image of the Kodaira-Spencer map is determined by a single element in $H^1(X, TX)$.

1.2. Toric varieties. A variety $X$ is called a *toric variety* if it contains an $n$-dimensional torus as a Zariski open subset in a way such that the action of the torus on itself extends to an action of the torus on $X$. Torus varieties can be completely described in a purely combinatorial way, via the concept of toric fans. Information about this topic can all be found in [5]. We will very briefly go through the concept. Let $N$ be a lattice of rank $n$ and let $M$ be its dual. A fan $\Sigma$ in $N \otimes \mathbb{Q}$ is a collection of cones that is closed under intersection and cone faces. Once a toric fan $\Sigma$ is given, we can define a toric variety $X_\Sigma$ by gluing the affine toric varieties $\text{Spec} \mathbb{K}[\sigma^\vee \cap M]$ as $\sigma$ runs through $\Sigma$.

Let $X$ be an irreducible normal variety with finitely generated divisor class group $\text{Cl}(X)$ and $\Gamma(X, O_X^*) \simeq K$, i.e. the only global invertible regular functions are the constants. The *Cox ring* of $X$ is [1]:

$$\mathcal{R}(X) := \bigoplus_{[D] \in \text{Cl}(X)} \Gamma(X, O_X(D)).$$

In the case that $X$ is also a toric variety, let $D_1, \ldots, D_r$ be its prime invariant divisors. It can be shown (cf. [1, Ch. II, §1.3]) that

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r], \quad \deg(T_i) = [D_i].$$

Elements in this ring are said to be *in Cox coordinates*.

1.3. The tangent sheaf of a toric variety. Let $X$ be a smooth complete toric variety with defining fan $\Sigma \subseteq N_\mathbb{Q}$ and character group $M$. By the Euler exact sequence for the tangent sheaf $T_X$ of $X$, the cohomology group of $T_X$ are graded by $M$. In particular

$$H^1(X, T_X) = \bigoplus_{m \in M} H^1(X, T_X)_m.$$

**Definition 1.4.** (cf. [8, §2.1]) Let $m \in M$ be such that there exists $\varrho \in \Sigma(1)$ with $m(\varrho) = -1$, where with abuse of notation we identify the one dimensional cone $\varrho$ with its primitive generator. Define the graph $\Gamma_\varrho(m)$ whose set of vertices is

$$\text{Vertices}(\Gamma_\varrho(m)) := \{ \varrho' \in \Sigma(1) \, \backslash \, \{ \varrho \} : m(\tau) < 0 \},$$
and two vertices are joined by an edge if and only if they are rays of a common cone of $\Sigma$. If $C$ is a proper component of $\Gamma_g(m)$ we say that the triple $(m, s, C)$ is admissible.

To any admissible triple $(m, s, C)$ one can associate a cocycle of $H^1(X, T_X)_m$ in the following way. Define a derivation $\partial_{m, s} \in \text{Der}(\mathbb{K}[M], \mathbb{K}[M])$ by

$$\partial_{m, s} : \mathbb{K}[M] \to \mathbb{K}[M] \quad \chi^u \mapsto u(s)\chi^{u+m}.$$ 

The announced cocycle is

$$(1.1) \quad \xi(m, s, C) = \{ \alpha(\sigma, \tau) : \sigma, \tau \in \Sigma(n) \} \in H^1(X, T_X)_m$$

where $\alpha(\sigma, \tau)$ equals 1 if $\sigma(1) \cap C$ is non-empty and $\tau(1) \cap C$ is empty, it equals $-1$ if the roles of $\sigma$ and $\tau$ are exchanged and it equals 0 otherwise.

**Proposition 1.5.** [9, Thm 6.5] The cocycles $\xi(m, s, C)$ span the vector space $H^1(X, T_X)$.

## 2. Deformations of smooth toric varieties

In what follows, $X$ is a smooth toric variety. Our aim now is to show how to associate to any admissible triple a one parameter deformation $\pi : \mathcal{X} \to \mathbb{A}^1$ such that the image of the Kodaira-Spencer map $T_{\mathcal{X}^1} \to H^1(X, T_X)$ associated to $\pi$ is the original admissible triple.

### 2.1. The deformation space.

Let $X$ be a smooth toric variety and let $(m, s, C)$ be an admissible triple. The element $m \in \text{Hom}(\mathbb{N}, \mathbb{Z})$ is a homomorphism $N \to \mathbb{Z}$ with kernel $K$. We let $\gamma : \mathbb{Z} \to N$ be the section of $m$ defined by $\gamma(-1) = m$ and let $\pi : N \to K$ be the corresponding projection. The above maps are encoded in the following exact sequence:

$$(2.1) \quad 0 \longrightarrow K \xrightarrow{\pi} N \xrightarrow{m} \mathbb{Z} \longrightarrow 0.$$ 

Let $\tilde{N} = \mathbb{Z}^2 \oplus K \oplus \mathbb{Z}$ and $\tilde{M}$ its dual. We define the map

$$(2.2) \quad i : N \to \tilde{N} \quad v \mapsto (m(v), m(v), \pi(v), 0).$$

Let $\varrho_1, \ldots, \varrho_r$ be the primitive generators of the one-dimensional cones of the fan $\Sigma$, let $a_i = m(\varrho_i)$ for any $i$ and let

$$U_1 = \{(1, i) : a_i > 0\} \quad U_2 = \{(2, i) : a_i < 0 \text{ and } \varrho_i \in C \cup \{s\}\} \quad U_3 = \{(3, i) : a_i < 0 \text{ and } \varrho_i \notin C\}$$

and let $U$ be the union $U_1 \cup U_2 \cup U_3 \cup U_4$. For any $(j, i) \in U$ we define the row vector $v_j = [a_i : (j, i) \in U_j]$. Define the matrix $A_j = [\pi(\varrho_i) : (j, i) \in U_j]$ whose columns are the vectors $\pi(\varrho_j)$. Finally we define the following block matrix

$$(2.3) \quad P(m, s, C) := \begin{bmatrix}
1 & v_1 & v_2 & 0 & 0 \\
1 & v_1 & 0 & v_3 & 0 \\
0 & A_1 & A_2 & A_3 & A_4 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix},$$

where a 0 represents a zero matrix of adequate dimensions, whereas a 1 is simply the number one $(1 \times 1$ matrix).

From here on, given a one dimensional ray $\varrho_s$ we denote by $i(\varrho_s)$ its index $s$ and by $k(\varrho_s) \in \{1, 2, 3, 4\}$ the index of the set $U_k$ corresponding to the sign of $a_s$. Given
a maximal cone \( \sigma \in \Sigma_X \) we define the cone indices of \( \bar{\sigma} \), such that the cones \( \{ \bar{\sigma} \}_{\sigma \in \Sigma} \) define the ambient toric variety \( \bar{X} \) where \( X \) is embedded. For every \( g_i \in \sigma(1) \) we add every possible \((k, i) \in U\) as a cone index for \( \bar{\sigma} \). We also add, if it is not added already, the index \((2, i(g))\) if \( \sigma(1) \cap C = \emptyset \) or the index \((3, i(g))\) if \( \sigma(1) \cap C \neq \emptyset \). Lastly, we always add 1 as an index for \( \bar{\sigma} \). All this can be summarized as follows:

- if \( \sigma(1) \cap C \) is empty then the cone indices for \( \bar{\sigma} \) are:
  \[
  \{1, (2, i(g))\} \cup \{(1, s) : g_s \in \sigma(1) \text{ and } a_s > 0\}
  \]
  \[
  \cup \{(4, s) : g_s \in \sigma(1) \text{ and } a_s = 0\} \cup \{(3, s) : g_s \in \sigma(1) \text{ and } a_s < 0\}
  \]

- if \( \sigma(1) \cap C \) is non-empty then the cone indices for \( \bar{\sigma} \) are:
  \[
  \{1, (3, i(g))\} \cup \{(1, s) : g_s \in \sigma(1) \text{ and } a_s > 0\}
  \]
  \[
  \cup \{(4, s) : g_s \in \sigma(1) \text{ and } a_s = 0\} \cup \{(2, s) : g_s \in \sigma(1) \text{ and } a_s < 0\}.
  \]

Denote by \( \bar{X} \) the toric variety whose fan \( \Sigma_{\bar{X}} \), defined on \( \bar{N} \), is given by the cones \( \bar{\sigma} \) for every \( \sigma \in \Sigma_X \). We then define \( X = X(m, \varrho, C) \) as the \( T \)-variety of complexity one embedded in \( \bar{X} \) whose equation in Cox coordinates \( T_1, T_2 \) with \((i, j) \in U\), is the following trinomial

\[
T_1 \prod_{(1, j) \in U_1} T_{ij}^{a_{ij}} - \prod_{(2, j) \in U_2} T_{2j}^{-a_{2j}} + \prod_{(3, j) \in U_3} T_{3j}^{-a_{3j}}.
\]

We denote by \( \tilde{X} \) the affine subvariety defined by the above trinomial equation in Cox coordinates.

**Theorem 2.1.** Let \((m, \varrho, C)\) be an admissible triple and let \( X = X(m, \varrho, C) \) be the \( T \)-variety of Construction 2.1. The inclusion \( \mathbb{K}[T_1] \rightarrow \mathbb{K}[\tilde{X}] \) defines a \( T \)-equivariant morphism \( \pi: X \rightarrow \mathbb{A}^1 \) which is a one-parameter deformation of \( X \), i.e. \( X \) is isomorphic to the fiber of \( \pi \) over \( 0 \in \mathbb{A}^1 \).

**Proof.** To see that \( \pi \) is indeed a morphism, recall that \( X \) is embedded in a toric variety \( \bar{X} \) whose toric fan \( \Sigma_{\bar{X}} \) has the columns of \( P \) as ray generators and maximal cones given by \( \{ \bar{\sigma} \}_{\sigma \in \Sigma_{\bar{X}}} \). By taking the projection onto the last coordinate, we map every ray of \( \Sigma_{\bar{X}} \) to 0, except for the one corresponding to \( T_1 \), which is mapped onto \( \mathbb{Q}_{\geq 0} \). Thus, we have a morphism of toric varieties

\[
X \rightarrow \mathbb{A}^1.
\]

Let \( \mathcal{X}_0 \) be the fiber of \( \pi \) resulting by setting \( T_1 = 0 \). The trinomial in (2.4) becomes a binomial \( \chi^{v_1} - \chi^{v_2} \), with \( v_1, v_2 \in \bar{N} \). Let \( v = v_1 - v_2 \) and let \( u \in \bar{M} \) be such that \( P^*(u) = v \). Now, \( \mathcal{X}_0 \) admits an action of the subtorus defined by \( u^\perp \). Recall that \( T_1 = 0 \), so for the action to be effective we must take the subtorus \( N_0 := u^\perp \cap (e_{n+2}^*)^\perp \). Since \( N_0 \) has the same dimension as \( \mathcal{X}_0 \), this fiber can be seen as a toric variety having

\[
\Sigma_{\mathcal{X}_0} := \Sigma_{\bar{X}} \cap N_0
\]
as fan. It can be shown that \( \iota(N) = N_0 \): Indeed, a vector \([a, b] \oplus w \oplus [d] \in \bar{N} \) belongs to \( N_0 \) if and only if \( a = b \) and \( d = 0 \), so it is clear that \( \iota(N) \subseteq N_0 \). Conversely, if the vetor is of the form \([a, b] \oplus w \oplus [0] \in \bar{N} \), then it is equal to \( \iota(w + \gamma(a)) \). We now wish to prove that the following equality holds

\[
\Sigma_X = \Sigma_{\mathcal{X}_0}.
\]
Take a cone \( \sigma \in \Sigma_X \) and a ray \( \tau \in \sigma(1) \) and let \( v_\tau \) be its primitive generator. If \( \tau \in U_1 \) or \( \tau \in U_4 \), then \( v(\tau) \in \sigma \) because it is a column of \( P(m, \varrho, C) \). Otherwise, \( v(\tau) \) is a linear combination of columns of \( P(m, \varrho, C) \), one of index \((j_1, i(\tau))\) and one of index \((j_2, i(\varrho))\) with \( \{j_1, j_2\} = \{2, 3\} \), thus we still have \( v(\tau) \in \sigma \) in this case. We conclude that \( v(\tau) \in \sigma \cap \sigma(N) = \sigma \cap N_0 \). Due to the completeness of the fans, the fact that \( v(\sigma) \in \sigma \cap N_0 \) implies that \( \Sigma_X = \Sigma_{X_0} \) as claimed. \( \square \)

2.2. The central fiber. We now describe the embedding \( X \to \mathcal{X} \) at the level of Cox rings. We define the following homomorphism of polynomial rings

\[
\eta: \mathbb{K}[T_{ij} : (i, j) \in U] \to \mathbb{K}[S_1, \ldots, S_r] \quad T_{ij} \mapsto \begin{cases} \prod_{(3, j) \in U_2} S_j^{-a_j} & \text{if } i = 2 \text{ and } g_j = \varrho \\ \prod_{(2, j) \in U_3} S_j^{-a_j} & \text{if } i = 3 \text{ and } g_j = \varrho \\ S_j & \text{otherwise} \end{cases}
\]

And \( T_1 \mapsto 0 \). Observe that the variable \( T_1 \) is the variable which gives the coordinate on the base \( A^1 \) of the deformation.

**Proposition 2.2.** The homomorphism of polynomial rings \( \eta \) induces an isomorphism \( \eta': \mathbb{K}[\mathcal{X}]/(T_1) \to \mathcal{R}(X) \) which induces the inclusion \( X \to \mathcal{X} \) in Cox coordinates.

**Proof.** First of all we observe that the binomial \( \prod_{(2, j) \in U_2} T_j - \prod_{(3, j) \in U_3} T_j \) is contained in the kernel of \( \eta \). Moreover since the kernel is a prime principal ideal we conclude that it is generated by the above binomial. Thus, after identifying \( \mathbb{K}[T_{ij} : (i, j) \in U] \) with \( \mathbb{K}[\mathcal{X}]/(T_1) \), the homomorphism \( \eta \) induces an isomorphism \( \eta': \mathbb{K}[\mathcal{X}]/(T_1) \to \mathcal{R}(X) \) as claimed. Observe that \( \eta' \) is a graded map with respect to the \( \text{Cl}(\mathcal{X}) \)-grading on the domain and the \( \text{Cl}(X) \)-grading on the codomain. Denote by

\[
\tilde{P} := \begin{bmatrix} v_1 & v_2 & 0 & 0 \\ v_1 & 0 & v_3 & 0 \\ A_1 & A_2 & A_3 & A_4 \end{bmatrix}.
\]

the matrix obtained by removing the first column and the first row from \( P(m, \varrho, C) \).

Define the homomorphism

\[
\psi: \mathbb{Z}^r \to \mathbb{Z}^{r+1} \quad e_j \mapsto \begin{cases} e_{(1,j)} & \text{if } a_j > 0 \\ e_{(2,j)} - a_j e_{(3,\varrho)} & \text{if } g_j \in C \cup \{\varrho\} \\ e_{(3,j)} - a_j e_{(2,\varrho)} & \text{if } g_j \in (\Gamma_\varrho(m) \setminus C) \cup \{\varrho\} \\ e_{(4,j)} & \text{if } a_j = 0 \end{cases}
\]

Observe that \( \psi \) fits in the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}^r & \xrightarrow{\psi} & \mathbb{Z}^{r+1} \\
\downarrow P_X & & \downarrow \rho \\
\mathbb{Z}^n & \xrightarrow{(a_1, \ldots, a_n) \to (-a_n, a_1, \ldots, a_{n-1})} & \mathbb{Z}^{n+1}
\end{array}
\]

where \( P_X \) is the \( P \)-matrix of the Cox construction of \( X \). Moreover \( \psi \) maps the positive orthant of \( \mathbb{Z}^r \) into the positive orthant of \( \mathbb{Z}^{r+1} \), it maps cones of \( X \) into cones of \( \mathcal{X} \) and it induces \( \eta' \). The statement follows. \( \square \)
2.3. The Kodaira Spencer map.

**Theorem 2.3.** Let \((m, g, C)\) be an admissible triple and let \(\pi: X \to \mathbb{A}^1\) be the corresponding one-parameter family. The image of \(\pi\) via the Kodaira-Spencer map is the cohomology \(H^1(X, T_X)\) defined in (1.1).

**Proof.** The complexity one variety \(X\) is canonically embedded into the toric variety \(\tilde{X}\) and the morphism \(\pi: X \to \mathbb{A}^1\) is induced by a toric morphism \(\tilde{X} \to \mathbb{A}^1\). Let \(\Sigma\) be the fan of the toric variety \(X\). We denote by \(\Sigma \subseteq N_\mathbb{Z}\) the fan of \(\tilde{X}\). Given a cone \(\sigma \in \Sigma\) we denote by \(\vec{\sigma}\) the corresponding cone of \(\tilde{\Sigma}\), that is \(\vec{\sigma} \cap i(N) = \sigma\), where the map \(i\) is the one defined in (2.2). Let \(\tilde{M}\) be the dual of \(\tilde{N}\). The trinomial (2.4) is locally described in \(K[\vec{\sigma}^\vee \cap \tilde{M}]\) by a polynomial of the form

\[
\chi u_4^1 - \chi u_2^1 + \chi u_3^1,
\]

where \(u_4 = [0, \ldots, 0, 1]\), so that \(\varepsilon = \chi u_4\). We denote by \(g_\sigma\) the primitive generator of the extremal ray of the cone \(\vec{\sigma}\) which is one of the column of the \(P\)-matrix (2.3) whose index is \((2, i(\sigma))\) if \(\sigma \cap C\) is non-empty and it is \((3, i(\sigma))\) otherwise. Assume we are in the first case then the following equation holds

\[
P^*(u_2) = v_2,
\]

where \(T^v\) is the monomial in Cox coordinates which corresponds to the character \(\chi u_2\). The monomial \(T^v\) does not contain any variable \(T_{k,i}\) such that \(g_1 \in \sigma(1)\) with the only exception of the variable \(T_{(2,i(\sigma))}\) which appears with exponent 1. This implies that \(u_2\) has scalar product 0 with each column of the \(P\)-matrix of index \((k,i)\) when \(g_1 \in \sigma(1) \setminus \{g\}\) and it has scalar product 1 with the column of index \((2,i(\sigma))\). In particular \(u_2\) generates an extremal ray of the smooth cone \(\vec{\sigma}\) and then \(\chi u_2\) is a variable of the polynomial ring \(K[\vec{\sigma}^\vee \cap \tilde{M}]\). Analogously, if \(\sigma \cap C\) is empty, the character \(\chi u_3\) is a variable of the ring. Both cases establish an isomorphism

\[
\frac{K[\vec{\sigma}^\vee \cap \tilde{M}]}{(\chi u_4^1 - \chi u_2^1 + \chi u_3^1)} \to K[\vec{\sigma}^\vee \cap \tilde{M} \cap g_\sigma^1].
\]

For the rest of this proof, we fix two cones \(\sigma, \tau \in \Sigma\). The isomorphism above leads to the following diagram

\[
\frac{K[(\vec{\sigma} \cap \vec{\tau})^\vee \cap \tilde{M}]}{(\chi u_4, \chi u_4 + u_2, \chi u_3)} \cong \frac{K[(\vec{\sigma} \cap \vec{\tau})^\vee \cap \tilde{M} \cap g_\tau^1]}{(\chi u_4)} \cong \frac{K[(\sigma \cap \tau)^\vee \cap M]}{K[\varepsilon]} \otimes K[\varepsilon]
\]

where the map \(\beta\) is defined by the composition \(\beta^* : M \cap g_\tau^1 \to M \to M/\langle u \rangle\) of the inclusion with the projection and observing that \(\beta^*(\vec{\sigma}) = \sigma\) and \(\beta^*(\vec{\tau}) = \tau\). Moreover \(\beta^*\) is an isomorphism being \(u(g_\sigma) = \pm 1\). We have thus constructed an isomorphism

\[
\varphi : K[(\sigma \cap \tau)^\vee \cap M] \otimes K[\varepsilon] \to K[(\sigma \cap \tau)^\vee \cap M] \otimes K[\varepsilon].
\]
Let $a \in (\tilde{\sigma} \cap \tilde{\tau})^\vee \cap M$. We will assume that $\chi^{u_2}$ is a variable in $\mathbb{K}[\tilde{\sigma}^\vee \cap \tilde{M}]$ and $\chi^{u_3}$ is a variable in $\mathbb{K}[\tilde{\tau}^\vee \cap \tilde{M}]$. The other cases work similarly so analyzing only this case is enough. Since $\tilde{\tau}^\vee$ is smooth, we can write $a = v + a(\phi_r) u'_3$ where $v$ is a linear combination of the rays of $\tilde{\tau}^\vee$ different from $u'_3$. Hence the following hold
\[
\beta(\chi^a) = \beta(\chi^v, \chi^{a(\phi_r)u'_3}) = \beta(\chi^v (\varepsilon \chi^{u'_3} + \chi^{u_3})^n) = \chi^{\ast(a)} + a(\phi_r) \varepsilon \chi^{\ast(a) + \ast(u'_3 - u'_3)}
\]
where the last equality is due to the fact that the argument of $\beta$ does not contain $\chi^{u_3}$. Now, observe that $\chi^{u_3 - u_3}$ is the monomial $T^w$ where $w$ is the difference between the second and last row of the matrix $P$. This means $u_1 - u_3 = [0,1,0,\ldots,0,-1]$ and therefore $\ast(u_1 - u_3) = m$. By setting $u = \ast(a)$, this shows that $\varphi$ is defined as
\[
\varphi(\chi^a) = a(\phi_r) \chi^{u+m}.
\]
The only thing left to prove is that $a(\phi_r) = u(\phi)$. Simply notice that $a(\phi_\sigma) = 0$, so
\[
a(\phi_r) = a(\phi_r + \phi_\sigma) = a(\ast(\phi)) = \ast(a) = u(\phi).
\]
The coefficient $a(\sigma, \tau)$ from (1.1) equals 1 in this case and is easily seen to appear when checking the other cases. \hfill $\square$

## 3. Polyhedral description

In this section we describe deformations of toric varieties as shown in [9]. Their results are very closely related to the ones found in Section 2 of this paper, but they use a completely different language. Namely, the language of polyhedral divisors, which we summarize here.

### 3.1. $T$-varieties

Let $X$ be an algebraic variety over $\mathbb{K}$ having an action of $T := (\mathbb{K}^*)^n$ (which is called the $n$-dimensional torus). This action is called effective if the only $t \in T$, for which $t \cdot x = x$ holds for all $x \in X$, is the identity of $T$. A $T$-variety is a normal algebraic variety $X$ coming with an effective $(\mathbb{K}^*)^n$-action. The complexity of $X$ is the difference $\dim X - n$. These varieties admit a polyhedral description given by K. Altmann and J. Hausen for the affine case in [2] and later, together with H. Suß, in [3] for the non-affine case. In the following section, we briefly recall this construction.

### 3.2. Polyhedral divisors

Let $N$ be a lattice of rank $n$ and $M = \text{Hom}(N, \mathbb{Z})$ its dual. We denote by $N_\mathbb{Q} := N \otimes \mathbb{Q}$ and by $M_\mathbb{Q} := M \otimes \mathbb{Q}$ the rational vector spaces. A polyhedron in $N_\mathbb{Q}$ is an intersection of finitely many affine half spaces in $N_\mathbb{Q}$. If we require the supporting hyperplane of any half space to be a linear subspace, the polyhedron is called a cone. If $\sigma$ is a cone in $N_\mathbb{Q}$, its dual cone is defined as
\[
\sigma^\vee := \{ u \in M_\mathbb{Q} : u(v) \geq 0 \text{ for all } v \in N_\mathbb{Q} \}.
\]
Let $\Delta \subseteq N_\mathbb{Q}$ be a polyhedron. The set
\[
\sigma := \{ v \in N_\mathbb{Q} : tv + \Delta \subseteq \Delta, \forall t \in \mathbb{Q} \}
\]
is a cone called the tailcone of $\Delta$ and $\Delta$ is called a $\sigma$-polyhedron. Let $Y$ be a normal variety and $\sigma$ a cone. A polyhedral divisor on $Y$ is a formal sum

$$D := \sum_P \Delta_P \otimes P,$$

where $P$ runs over all prime divisors of $Y$ and the $\Delta_P$ are all $\sigma$-polyhedrons such that $\Delta_P = \sigma$ for all but finitely many $P$. We admit the empty set as a valid $\sigma$-polyhedron too. Let $D := \sum \Delta_P \otimes P$ be a polyhedral divisor on $Y$, with tailcone $\sigma$. For every $u \in \sigma^\vee$ we define the evaluation

$$D(u) := \sum_{P \in Y, v \in \Delta_P} \min_{\Delta_P \subset P} u(v) \otimes P \in WDiv_Q(Loc \ D)$$

where $Loc \ D := Y \setminus (\cup_{\Delta_P = \emptyset} P)$ is the locus of $D$.

**Definition 3.1.** Let $Y$ be a normal variety. A proper polyhedral divisor, also called a pp-divisor is a polyhedral divisor $D$ on $Y$, such that

(i) $D(u)$ is Cartier and semiample for every $u \in \sigma^\vee \cap M$.

(ii) $D(u)$ is big for every $u \in (\text{relint } \sigma^\vee) \cap M$.

Now, let $\mathcal{D}$ be a pp-divisor on a semiprojective (i.e. projective over some affine variety) variety $Y$, $\mathcal{D}$ having tailcone $\sigma \subseteq N_Q$. This defines an $M$-graded algebra

$$A(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Loc \mathcal{D}, \mathcal{O}(D(u))).$$

The affine scheme $X(\mathcal{D}) := \text{Spec } A(\mathcal{D})$ comes with a natural action of $\text{Spec } K[M]$. Definition 3.1 is mainly motivated by the following result [2, Theorem 3.1 and Theorem 3.4].

**Theorem 3.2.** Let $\mathcal{D}$ be a pp-divisor on a normal variety $Y$. Then $X(\mathcal{D})$ is an affine $T$-variety of complexity equal to $\dim Y$. Moreover, every affine $T$-variety arises like this.

### 3.3. Divisorial fans

Non-affine $T$-varieties are obtained by gluing affine $T$-varieties coming from pp-divisors in a combinatorial way as specified in Definition 3.3. Consider two polyhedral divisors $\mathcal{D} = \sum \Delta_P \otimes P$ and $\mathcal{D}' = \sum \Delta'_P \otimes P$ on $Y$, with tailcones $\sigma$ and $\sigma'$ respectively and such that $\Delta_P \subseteq \Delta'_P$ for every $P$. We then have an inclusion

$$\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Loc \mathcal{D}, \mathcal{O}(D(u))) \subseteq \bigoplus_{u \in \sigma'^\vee \cap M} \Gamma(Loc \mathcal{D}, \mathcal{O}(D'(u))),$$

which induces a morphism $X(D') \to X(D)$. We say that $D'$ is a face of $D$, denoted by $D' \prec D$, if this morphism is an open embedding.

**Definition 3.3.** A divisorial fan on $Y$ is a finite set $S$ of pp-divisors on $Y$ such that for every pair of divisors $\mathcal{D} = \sum \Delta_P \otimes P$ and $\mathcal{D}' = \sum \Delta'_P \otimes P$ in $S$, we have $\mathcal{D} \cap \mathcal{D}' \in S$ and $\mathcal{D} \supseteq \mathcal{D} \cap \mathcal{D}' \prec \mathcal{D}'$, where $\mathcal{D} \cap \mathcal{D}' := \sum (\Delta_P \cap \Delta'_P) \otimes P$.

This definition allows us to glue affine $T$-varieties via

$$X(\mathcal{D}) \leftarrow X(\mathcal{D} \cap \mathcal{D}') \rightarrow X(D'),$$

thus resulting in a scheme $X(S)$. For the following theorem see [3, Theorem 5.3 and Theorem 5.6].
Theorem 3.4. The scheme $X(\mathcal{S})$ constructed above is a $T$-variety of complexity equal to $\dim Y$. Every $T$-variety can be constructed like this.

3.4. Deformations via polyhedral decompositions. In [9, §6], Ilten and Vollmert define one-parameter deformations of smooth toric varieties denoted by $\pi = \pi(m, \varrho, C)$, where $m$ is a lattice vector, $\varrho$ is a ray in a fan and $C$ is a connected component of some graph. The construction is as follows.

Let $\Sigma$ be a smooth complete fan giving rise to a toric variety $X = X_\Sigma$. By choosing a $m \in M$ and intersecting the hyperplanes $\{v \in N : m(v) = -1\}$ and $\{v \in N : m(v) = 1\}$ with $\Sigma$ we get two polyhedral subdivisions corresponding to the slices $\mathcal{S}_0$ and $\mathcal{S}_\infty$ of a divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$, describing $X$ as a variety of complexity one. Now, choose $\varrho \in \Sigma(1)$ such that $m(\varrho) = -1$ and recall Definition 1.4 of the graph $\Gamma_\varrho(m)$, whose set of vertices is

$$\{\tau \in \Sigma(1) : \tau \neq \varrho, m(\tau) < 0\}$$

and whose edges join two vertices whose corresponding one-dimensional rays lie in a common cone. Assume $\Gamma_\varrho(m)$ has at least two connected components, and let $C$ be one of them. This choice induces a one parameter deformation on $X$ as follows.

Each polyhedron $\Delta \in \mathcal{S}_0$ will be decomposed as $\Delta = \Delta^0 + \Delta^1$. If $\Delta$ contains a vertex coming from a ray in $C$, take $\Delta^0 = \text{tail } \Delta$ and $\Delta^1 = \Delta$. If $\Delta$ contains no such vertex, take $\Delta^0 = \Delta$ and $\Delta^1 = \text{tail } \Delta$. The sets $\{\Delta^0\}_{\Delta \in \mathcal{S}_0}$ and $\{\Delta^1\}_{\Delta \in \mathcal{S}_0}$ define new polyhedral subdivisions $\mathcal{S}^0_0$ and $\mathcal{S}^1_0$ such that $\mathcal{S} = \mathcal{S}^0_0 + \mathcal{S}^1_0$. Let $\mathcal{S}$ be the divisorial fan on $\mathbb{A}^1 \times \mathbb{P}^1$ whose only non-trivial slices are $\mathcal{S}^0_0$ at $V(y)$, $\mathcal{S}^1_0$ at $V(y - x)$ and $\mathcal{S}_\infty$ at $V(y^{-1})$, where we are using coordinates $(x, y) \in \mathbb{A}^1 \times \mathbb{P}^1$. Then $\mathcal{X} := X(\mathcal{S})$ comes with a morphism $\pi : \mathcal{X} \to \mathbb{A}^1$ which is a one-parameter deformation of $X$. This deformation is called $\pi(m, \varrho, C)$.

Proposition 3.5. The deformation $\pi(m, \varrho, C)$ is the same as the one described in Theorem 2.1.

Proof. We consider the $\mathbb{K}^*$-action on $\mathbb{A}^1 \times \mathbb{P}^1$ given by $t \cdot (x, y) = (tx, ty)$. This allows us to describe this surface with a divisorial fan $\mathcal{Z}$ on $\mathbb{P}^1$ whose tailfan is given by a single ray on the positive axis and whose only non-trivial slice has vertices in $0$ and $1$. By applying [10, Prop 2.1], we describe $\mathcal{X}$ with a new divisorial fan $\mathcal{S}'$ on $\mathbb{P}^1$, having three non-trivial slices. One non-trivial slice of $\mathcal{S}'$ contains $\mathcal{S}^0_0$ at height 0 and a single vertex at height 1, whereas the other two non-trivial slice of $\mathcal{S}'$ are simply $\mathcal{S}^1_0$ and $\mathcal{S}_\infty$ embedded in the corresponding higher dimensional space. Thus, by [6, Corollary 4.9] we see that the cox ring of $\mathcal{X}$ is given precisely by (2.4). The matrix $P(m, \varrho, C)$ can be obtained from [6, Pop 4.7].

4. Applications

We use the language developed in subsection 1.3 and in section 2 to study deformations of scrolls and deformations of hypersurfaces of smooth toric varieties.

4.1. Deformations of scrolls. Let $n > 1$ be an integer and let $a_1, \ldots, a_n$ be integers. We denote by $\mathbb{F}(a_1, \ldots, a_n)$ the $\mathbb{P}^{n-1}$-bundle (i.e. a scroll) over $\mathbb{P}^1$ associated to the sheaf $\mathcal{O}_{\mathbb{P}^1}(a_1) + \cdots + \mathcal{O}_{\mathbb{P}^1}(a_n)$. It can be defined as the quotient of the space $(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^n \setminus 0)$ by the following $(\mathbb{K}^*)^2$-action.
The action on the first two coordinates gives \( \mathbb{F}(a_1, \ldots, a_n) \) a morphism over \( \mathbb{P}^1 \) by projecting on the first factor:

\[
\begin{array}{ccc}
(\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^n \setminus \{0\}) & \longrightarrow & \mathbb{F}(a_1, \ldots, a_n) \\
\downarrow & & \downarrow \\
(\mathbb{A}^2 \setminus \{0\}) & \longrightarrow & \mathbb{P}^1
\end{array}
\]

**Remark 4.1.** It can be shown (cf. [12, Ch. 2]) that \( \mathbb{F}(a_1, \ldots, a_n) \cong \mathbb{F}(b_1, \ldots, b_n) \) if and only if there exists \( c \in \mathbb{Z} \) and a permutation \( \sigma \in S_n \) such that for every \( i \) we have \( a_i = b_{\sigma(i)} + c \).

**Proposition 4.2.**
(a) A scroll over \( \mathbb{P}^1 \) is rigid if and only if it is isomorphic to \( \mathbb{F}(a_1, \ldots, a_n) \) where \( \{a_i\}_{i=1}^n \subseteq \{0, 1\} \).
(b) Let \( X = \mathbb{F}(a_1, \ldots, a_n) \), such that \( d := a_1 - a_2 > 2 \). For any \( d' < d \), the scroll \( X \) admits a deformation to \( \mathbb{F}(a_1 - d', a_2 + d', a_3, \ldots, a_n) \).

**Proof.** If \( n = 2 \), we have a Hirzebruch surface and these results are well known. They can be found for example in [8, §3]. Therefore, we will assume \( n \geq 3 \). The degree matrix of \( \mathbb{F}(a_1, \ldots, a_n) \) is:

\[
Q = \begin{bmatrix}
1 & 1 & -a_1 & \cdots & -a_n \\
0 & 0 & 1 & \cdots & 1
\end{bmatrix}
\]

Since the components of the irrelevant ideal are \( (1, 2) \) and \( (3, \ldots, n + 2) \) and \( n \geq 3 \), it is clear that \( (\varrho_1, \varrho_2) \) is the only pair of rays in \( \Sigma \) that are not in a common cone. Thus, if we choose an admissible triple \( (m, \varrho, C) \) we must have that:

- \( m(\varrho_1) < 0 \) and \( m(\varrho_2) < 0 \).
- \( m(\varrho_k) = -1 \) for some \( 3 \leq k \leq n + 2 \). This \( \varrho_k \) will be \( \varrho \).
- \( m(\varrho_i) \geq 0 \) for every \( i = 3, \ldots, n + 2, i \neq k \).

since this conditions are the only way to ensure that \( \Gamma_\varrho(m) \) has at least two connected components. Now, define \( u_i := m(\varrho_i) \) and form the column vector:

\[
u := (u_1, \ldots, u_{n+2})^t.
\]

The conditions above become:

\[
u_1, \nu_2 < 0; \quad \nu_k = -1; \quad \nu_i \geq 0, i \notin \{1, 2, k\}.
\]

From the Gale duality (cf. [1, Ch. II, §1.2]) we see that \( Q(u) = 0 \), which when written as a system of equations is equivalent to:

\[
u_1 + \nu_2 - \sum_{i=3}^{n+2} u_i a_{i-2} = 0
\]

\[
\sum_{i=3}^{n+2} u_i = 0
\]
From (4.1) and (4.3) we deduce there exists \(3 \leq \ell \leq n + 2\), with \(\ell \neq k\), such that
\[
(4.4) \quad u_{\ell} = 1 \text{ and } u_i = 0 \text{ for } i \notin \{1, 2, k, \ell\}.
\]

Additionally, by (4.2),
\[
(4.5) \quad a_{k-2} - a_{\ell - 2} = -u_1 - u_2 \geq 2.
\]

Thus, \(F(a_1, \ldots, a_n)\) has an admissible triple if and only if two of the \(a_i\) have distance at least 2, proving part (a).

Assume now that we are in the case where the admissible triple \((m, \varrho, C)\) exists. We will set \(C := \{q_1\}\). From here on, to simplify notation without loss of generality, let \(k = 3\) and \(\ell = 4\). Recall that \(u_i = m(q_i)\). Therefore, (4.1) and (4.4) imply that the trinomial of the Cox ring of the total deformation space \(X\) is
\[
(4.6) \quad T_1T_{(1,4)} - T_{(2,3)}T_{(2,1)}^{-u_1} + T_{(3,3)}T_{(3,2)}^{-u_2}
\]

The irrelevant ideal \(I\) of \(X\) is given by the following components
\[
I_1 = \langle T_{(2,1)}, T_{(3,2)} \rangle \quad I_2 = \langle T_{(2,1)}, T_{(2,3)}, T_{(1,4)} \rangle + \langle T_{(4,5)}, T_{(4,6)}, \ldots, T_{(4,n + 2)} \rangle \\
I_3 = \langle T_{(3,2)}, T_{(3,3)}, T_{(1,4)} \rangle + \langle T_{(4,5)}, T_{(4,6)}, \ldots, T_{(4,n + 2)} \rangle \\
I_4 = \langle T_{(2,3)}, T_{(3,3)}, T_{(1,4)} \rangle + \langle T_{(4,5)}, T_{(4,6)}, \ldots, T_{(4,n + 2)} \rangle.
\]

In the general fiber of the deformation, we put \(T_1 = t \in K^*\), so by (4.6), the variable \(T_{(1,4)}\) can be replaced by the other variables. This reduces \(I\) to
\[
\langle T_{(2,1)}, T_{(3,2)} \rangle, \quad \langle T_{(2,3)}, T_{(3,3)} \rangle + \langle T_{(4,5)}, T_{(4,6)}, \ldots, T_{(4,n + 2)} \rangle.
\]

By Proposition 2.2, and using the same notation, we have
\[
\deg(T_{(2,3)}) = \deg(S_2^{-u_2}S_3); \quad \deg(T_{(3,3)}) = \deg(S_1^{-u_1}S_3).
\]

This means that both the irrelevant ideal and the degree matrix of the general fiber of the deformation match that of \(F(a_1 + u_1, a_1 + u_2, a_3, \ldots, a_n)\). Then (b) follows after noticing that (4.5) implies \(a_1 + u_2 = a_2 - u_1\). \(\square\)

**Proposition 4.3.** The scroll \(F(a_1, \ldots, a_n)\) can be deformed to
\[
\mathcal{F}(1,1,\ldots,1,0,0\ldots,0)
\]
where
\[
r \equiv \sum_{i=1}^{n} a_i \pmod{n}.
\]

**Proof.** By Remark 4.1, we can assume that the sequence \(a_1, \ldots, a_n\) is decreasing and non-negative, with \(a_n = 0\). We proceed by induction over \(a_1\). The cases \(a_1 = 0\) and \(a_1 = 1\) are trivial. Assume now that \(a_1 \geq 2\). Let
\[
M = \#\{i : a_i = a_1\}, \quad m = \#\{i : a_i = 0\}.
\]

If \(M < m\), then by Proposition 4.2 the scroll can be deformed by subtracting 1 from each \(a_1, \ldots, a_M\) and adding 1 to \(M\) of the \(a_i\) that equal 0.

If \(M \geq m\), the scroll can be deformed by subtracting 1 from each \(a_1, \ldots, a_m\) and adding 1 to every \(a_i\) that equals 0. Then we subtract 1 from every \(a_i\) (recall that this does not change the variety).
Note that in both cases, and after just a permutation of indices, we have deformed the original scroll to $F(b_1, \ldots, b_n)$ where $b_i \geq b_{i+1}$ for every $i$, the $b_i$ are all non-negative and $b_n = 0$. Furthermore $\sum a_i \equiv \sum b_i \pmod{n}$ and $b_1 < a_1$ so the induction is complete. □

4.2. Deformation of hypersurfaces. Let $X$ be a toric variety and let $\mathbb{K}[S_1, \ldots, S_r]$ be its Cox ring. Choose an admissible triple $(m, g, C)$ and construct the corresponding deformation as explained in section 2.1. The map $\eta$ given in section 2.2 is defined by a semigroup homomorphism $\nu_+ : \mathbb{Z}_r^{r+1} \to \mathbb{Z}_r^{r}$ which can be naturally extended to a group homomorphism $\nu : \mathbb{Z}_r^{r+1} \to \mathbb{Z}_r$. Notice that $\nu$ is the transpose of $\psi$ defined in (2.5). A homogeneous polynomial $f \in \mathbb{K}[S_1, \ldots, S_r]$ can be written as a sum

$$f = c_1 m_1 + \ldots + c_k m_k$$

where $c_i \in \mathbb{K}$ and $m_i$ is a monomial for all $i$. A homogeneous polynomial $\tilde{f} \in \mathbb{K}[T_1, T_{ij}]$ such that $f = \eta(\tilde{f})$ will exist if and only if the exponent vector of each $m_i$ is in the image of $\nu_+$. In this case, if we let $g \in \mathbb{K}[T_1, T_{ij}]$ be the trinomial (2.4) corresponding to $(m, g, C)$, the subvariety

$$V(\tilde{f}, g) \subset \tilde{X}$$

defines a one-parameter deformation of $X$. Observe that if $\tilde{f}' \in \mathbb{K}[T_1, T_{ij}]$ is another lifting of $f$, i.e. $\eta(\tilde{f}') = f$, then $\tilde{f}' - \tilde{f} \in \langle g, T_1 \rangle$ so that the equality $V(\tilde{f}', g, T_1) = V(\tilde{f}, g, T_1)$ holds.

Let $Q_X : \mathbb{Z}_r \to \text{Cl}(X)$ be the grading map of the toric variety $X$, i.e. $Q_X$ maps $e \in \mathbb{Z}_r$ to the class of the divisor $\sum_{i=1}^{r} e_i D_i$, where $D_i$ is the $i$-th invariant prime divisor of $X$. Given a class $w \in \text{Cl}(X)$ and an equivariant divisor $D$ of $X$ such that $[D] = w$, a monomial basis of the Riemann-Roch space of $D$ is in bijection with the set

$$Q_X^{-1}(w) \cap \mathbb{Z}_r^{r+1}.$$ 

The subset of monomials that can be lifted to monomials of $\mathbb{K}[T_1, T_{ij}]$ via $\eta$ is in bijection with

$$\text{im}(\nu_+) \cap Q_X^{-1}(w) \cap \mathbb{Z}_r^{r+1}.$$ 

**Proposition 4.4.** The set $\text{im}(\nu_+)$ is the Hilbert basis of the rational polyhedral cone that it generates.

**Proof.** Let $A_\nu$ be the matrix associated to the map $\nu$ and let $j_\rho$ be the index such that $S_{j_\rho}$ corresponds to the ray $\rho$ in $\Sigma_X$. Due to the way $\eta$ is defined, it is clear that by removing the $(2, j_\rho)$-th column from $A_\nu$, and after an adequate rearrangement of its columns, we obtain a matrix with the following properties:

- All the entries in the diagonal are 1.
- Only one column has non-zero entries outside of the diagonal.

It is easy to see that such a matrix has determinant equal to 1.

Similarly, we can remove the $(3, j_\rho)$-th column from $A_\nu$ to get a matrix with determinant 1. This shows that the cone generated by the columns of $A_\nu$ is the union of two smooth cones (in the sense of toric geometry), which proves the statement. □

Let $P_X$ be the matrix whose columns are the generators of the rays of $\Sigma_X$. Let $\tilde{P}$ be the the minor of $P(m, g, C)$ resulting from removing the leftmost column
and bottom row. Let $\tilde{Q}$ be the cokernel of $\tilde{P}^*$, i.e. the grading matrix of $\tilde{X}$ after removing the null vector column corresponding to $T_1$. From the Cox construction, we get the following commutative diagram of group homomorphisms with exact rows

$$
\begin{array}{cccccc}
0 & \rightarrow & M & \rightarrow & P^* & \rightarrow & \mathbb{Z}^{r+1} & \rightarrow & \tilde{Q} & \rightarrow & \text{Cl}(\tilde{X}) & \rightarrow & 0 \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M & \rightarrow & P^*_\xi & \rightarrow & \mathbb{Z}^r & \rightarrow & Q_X & \rightarrow & \text{Cl}(X) & \rightarrow & 0 \\
\end{array}
$$

where the square on the left is the dual of (2.6) and $\bar{\nu}$ is uniquely defined by $\nu$. Denote the exponent vector of a monomial $m$ by $v(m)$. Then we have

$$\ker \nu = v \left( \prod_{(i,j) \in U_2} T_{2j}^{-a_j} \right) - v \left( \prod_{(i,j) \in U_3} T_{3j}^{-a_j} \right) \subseteq \ker \tilde{Q},$$

which together with the surjectivity of $\xi$ and $\nu$, imply that $\bar{\nu}$ is an isomorphism.

**Example 4.5.** We now turn our attention to the case of Hirzebruch surfaces, i.e. $X = \mathbb{F}_n$. The fan $\Sigma_X$ has four rays $\rho_1, \rho_2, \rho_3, \rho_4$, generated respectively by

$$v_1 = (1, 0), \ v_2 = (0, 1), \ v_3 = (-1, n), \ v_4 = (0, -1).$$

Let $D_1, D_2, D_3, D_4$ be the corresponding invariant divisors. In this case we have $\text{Cl}(X) \cong \mathbb{Z}^2$ generated by $[D_1]$ and $[D_2]$, along with

$$P_X = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & n & -1 \end{bmatrix}, \quad Q_X = \begin{bmatrix} 1 & 0 & 1 & n \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

plus a section $s$ for $Q$ and a projection $\pi$ for $P^*$ given by

$$s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \pi = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$
which shows that the trapezoid is contained in im(ν+). This means that when we deform Hirzebruch surfaces, every function in the Riemann-Roch space of the class ω can be lifted via η.

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