Singular Behavior of Harmonic Maps Near Corners

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ABSTRACT
For a harmonic map \( \mathcal{F} : Z \overset{\text{harm}}{\rightarrow} W \) transforming the contour of a corner of the boundary \( \partial Z \) into a rectilinear segment of the boundary \( \partial W \), the behavior near the vertex of the specified corner is investigated. The behavior of the inverse map \( \mathcal{F}^{-1} : W \rightarrow Z \) near the preimage of the vertex is investigated as well. In particular, we prove that if \( \varphi \) is the value of the exit angle from the vertex of the reentrant corner for a smooth curve \( \mathcal{L} \) and \( \theta \) is the value of the exit angle from the vertex image for the image \( \mathcal{F}(\mathcal{L}) \) of the specified curve, then the dependence of \( \theta \) on \( \varphi \) is described by a discontinuous function.

Thus, such a behavior of the harmonic map qualitatively differs from the behavior of the corresponding conformal map: for the latter one, the dependence \( \theta(\varphi) \) is described by a linear function.

KEYWORDS
Harmonic maps, quasiconformal maps, map asymptotics near corners of planar domains

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1. Introduction

We say that a homeomorphic map of domains \( Z \) and \( W \) located in the complex planes \( z = x + iy \) and \( w = u + iv \) respectively is harmonic (see, e.g., [3], [4]) and denote it by the symbol

\[
\mathcal{F} : Z \overset{\text{harm}}{\rightarrow} W \tag{1}
\]

if the corresponding complex-valued function \( \mathcal{F}(z) = u(x, y) + iv(x, y) \) is harmonic. Note that, unlike conformal maps, we do not require the scalar harmonic functions \( u \) and \( v \) to obey the Cauchy–Riemann conditions.

If \( Z \) and \( W \) are Jordan domains, then map (1) is frequently constructed (see, e.g.,
as a harmonic extension of the given homeomorphism $\mathcal{B} : \partial \mathcal{Z} \xrightarrow{\text{Hom}} \partial \mathcal{W}$ of their boundaries into the domain $\mathcal{Z}$, i.e., as a solution of the following Dirichlet problem for the Laplace equation:

$$\Delta \mathcal{F}(z) = 0, \quad z \in \mathcal{Z}; \quad \mathcal{F}(z) = \mathcal{B}(z), \quad z \in \partial \mathcal{Z}. \quad (2)$$

However, it is not guaranteed that the map constructed that way is one-sheeted; this is easily confirmed by the corresponding counterexamples (see, e.g., [8]). Note that, for conformal maps (unlike harmonic ones), the homeomorphism of the boundaries of Jordan domains implies the homeomorphism of the (closed) domains themselves (see the Caratheodory theorem in [9]).

Harmonic maps are broadly investigated (see [1]–[4], [8], [10]–[12]). The interest to this direction is caused by its great theoretical importance. In particular, harmonic maps play a substantial role in differential geometry and the theory of minimal surfaces (see [13], [14]), in the geometric function theory (see [15]–[20]), and in the geometric theory of quasiconformal maps (see [21]–[29]), including the extension problem of the Riemann theorem for those maps (see [3], [30]–[33]).

On the other hand, harmonic maps have numerous applications. In particular, they are applied for the constructing of computational meshes in complicated domains $\mathcal{Z}$, based on the Winslow approach (see [34]–[45]). Under such an approach, the desired mesh (in a Jordan domain) is constructed as follows: we take the Cartesian mesh natural for the square $\Pi := \{ u \in (-1/2, 1/2), v \in (0, 1) \}$ and transform it to the domain $\mathcal{Z}$ by means of the map $\mathcal{F}^{-1}(w) = x(u, v) + iy(u, v) : \Pi \xrightarrow{\text{harm}^{-1}} \mathcal{Z}$, which is inverse to a harmonic map. To find the specified map, we use numerical methods to solve the following Dirichlet problem in the square $\Pi$ for the “inverse vector” Laplace equation:

$$\Delta^{-1} \mathcal{F}^{-1}(w) = 0, \quad w \in \Pi; \quad \mathcal{F}^{-1}(w) = \mathcal{B}^{-1}(w), \quad w \in \partial \Pi. \quad (3)$$

The expanded form of the said equation is the following quasilinear system of equations with respect to the component pair $x(u, v), y(u, v)$:

$$\begin{align*}
(x^2 + y^2) x_{uu} - 2(x_u x_v + y_u y_v) x_{uv} + (x^2 + y^2) x_{vv} &= 0, \\
(x^2 + y^2) y_{uu} - 2(x_u x_v + y_u y_v) y_{uv} + (x^2 + y^2) y_{vv} &= 0;
\end{align*}$$

here, subscripts denote the differentiating with respect to the corresponding variables (that expanded form can be found, e.g., in [34], [40]–[44]).

To pose the boundary-value condition for problem (3), a homeomorphism $\mathcal{B}^{-1}$ between the boundaries $\partial \Pi$ and $\partial \mathcal{Z}$ is taken such that the following property takes place: if a point $w$ moves along the square side with a constant velocity, then its image $z = \mathcal{F}^{-1}(w)$ moves along the boundary $\partial \mathcal{Z}$ with a constant velocity as well. This requirement is imposed to ensure the desired quality of the computational mesh (see [40], [41], [43]).

It is convenient to use the map $\mathcal{F}^{-1}(w)$ to construct the mesh in a complicated domain $\mathcal{Z}$ because a computational scheme for the resolving of problem (3) is rather easily constructed on the Cartesian mesh of the square $\Pi$ (see, e.g., [35], [37], [40]–[44]); on the other hand, the homeomorphism of the map is guaranteed by the Radó–Kneser–Choquet theorem (see [3], [5]–[7]) because the square is a convex domain.
The described approach is broadly propagated (see [34]-[45]), but it occurs that its practical applications face difficulties for the numerically obtained map: it even might lose the homeomorphism (see, e.g., [37],[40]-[41]), contradicting the above theoretical aspects. In particular, such a difficulty arises if the domain $Z$ has reentrant corners, i.e., with angles exceeding $\pi$ (see [41],[45]). To overcome that obstacle, one has (primarily) to understand its nature. To do that, one has to consider harmonic maps $F$ transforming the contour of a corner $\partial Z$ of the boundary into a rectilinear segment of the boundary $\partial \Pi$ and investigate its behavior near the corner vertex (this is provided in Sec. 4) as well as the behavior of the inverse map near the vertex preimage (see Sec. 3). To obtain those results, we use the asymptotic behavior of the map $F$ near the corner vertex, found in Sec. 2. Also, we compare the harmonic map $F$ with the corresponding conformal map (see the final part of Sec. 4).

2. Harmonic maps near corner vertices: asymptotic behavior

Let $\Omega$ be a domain of the complex plane $z = r e^{i\varphi}$. Let its boundary $\partial \Omega$ contain an angle of value $\pi \beta$ (measured with respect to the domain), $\beta \in (0,2)$, such that its vertex is the co-ordinate origin and its sides are $L_\pm = \{ r \in (0,R) \}, \varphi = \pm \pi \beta/2 \}$. Then the domain $\Omega$ contains a sector

$$S = \{ r \in (0,R), \varphi \in \left( -\pi \beta/2, \pi \beta/2 \right) \}$$

(4)

of radius $R$, adjoining the boundary $\partial \Omega$ along the corner contour $L = L_- \cup \{0\} \cup L_+$ (see Fig. 1a). Further, let $\mathcal{W}$ be a domain of the complex plane $w = \rho e^{i\theta}$ such that its boundary contains a segment $I \ni \{0\}$ of the real axis $\mathbb{R}$ (see Fig. 1b) and the domain $\mathcal{W}$ itself contains a subdomain $\mathcal{H}$ of the upper half-plane $\mathbb{H}$, adjoining the segment $I$. Finally, let a harmonic function $w = F(z)$ maps $\Omega$ onto $\mathcal{W}$ such that the mapping of the sector $S$ into $\mathcal{H}$ is one-sheeted and the homeomorphism $w = B(z)$ between $L$ and $\mathcal{I}$ (see Fig. 1) acts as follows:

$$u(z) = -\sigma_- r, \quad z \in L_-, \quad u(z) = \sigma_+ r, \quad z \in L_+, \quad v(z) = 0, \quad z \in L,$$

(5)

where $\sigma_-$ and $\sigma_+$ are positive numbers. Note that the required rule corresponds to the coordination requirement for the “movement velocities” of the point $z$ on $L$ and its image.

![Figure 1](image-url)
Homeomorphism \([5]\) preserves the orientation of the boundaries: if the contour \(L\) of the corner is passed through its points \(z\) to leave the domain \(Z\) on the left, then the contour \(L\) is passed through points \(w = B(z)\) such that \(H\) is left on the left as well. This implies that \(B(L_+) = : L_+ \subset \mathbb{R}_+\), \(B(L_-) = : L_- \subset \mathbb{R}_-\), and \(B(0) = 0\) (see Fig. 1).

Introduce the linear function

\[
Q(z) := -\mu r \sin(\varphi - \varphi^*)
\]

such that its parameters \(\mu\) and \(\varphi^*\) are defined by the angle value \(\pi\beta\) and the “velocities” \(\sigma_-\) and \(\sigma_+\) of homeomorphism \([5]\) as follows:

\[
\mu := |\sin\pi\beta|^{-1} \sqrt{\sigma_+^2 + \sigma_-^2 + 2\sigma_+\sigma_-\cos\pi\beta} \quad \text{and} \quad \varphi^* := \arctan\left(\frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} \tan\frac{\pi\beta}{2}\right).
\]

The imposed positivity of \(\sigma_+\) and \(\sigma_-\) implies that \(\mu > 0\) and \(\varphi^* \in (-\pi\beta/2, \pi\beta/2)\).

One can verify that function \([6]\) satisfies Condition \([5]\) on the contour \(\Gamma\) of the corner, i.e.,

\[
Q(re^{i\varphi}) = \begin{cases}
\sigma_+ r, & \varphi = -\pi\beta/2, \\
-\sigma_- r, & \varphi = \pi\beta/2.
\end{cases}
\]

Therefore, the function \(F - Q\), which is harmonic in sector \([4]\), vanishes on \(L\) and, therefore, on the set \(S \cup L\), it is represented by a series

\[
F(z) - Q(z) = \sum_{n=1}^{\infty} (a_n + ib_n) \text{Im}(i z^{1/\beta})^n,
\]

where \(a_k\) and \(b_k\) are real numbers and series \([8]\) converges on the set \(S \cup L\). Since \(F\) is a one-sheeted map preserving its orientation, it follows that

\[
a_1 \neq 0 \quad \text{and} \quad b_1 > 0.
\]

Move the function \(Q\) defined by relation \([6]\) to the right-hand part of \([8]\) and truncate its series after the third term. This yields the asymptotic behavior of the considered harmonic map near the corner vertex \(z = 0\), uniform with respect to \(\varphi\) from \([-\pi\beta/2, \pi\beta/2]\). For \(\beta \in (0, 1)\), the specified asymptotic behavior is as follows:

\[
F(re^{i\varphi}) = -\mu r \sin(\varphi - \varphi^*) + (a_1 + ib_1)r^{1/\beta} \cos\frac{\varphi}{\beta}
\]

\[
+ (a_2 + ib_2)r^{2/\beta} \sin\frac{2\varphi}{\beta} + O(r^{3/\beta}), \quad r \to 0.
\]

If \(\beta \in (1, 2)\), then the order of the second term is greater than the order of the first one; therefore, they are to interchange their places.

3. The map inverse to the harmonic one: behavior near the preimage of the corner vertex

To investigate the behavior of the map

\[
z = F^{-1}(w) : \mathcal{H}^{\text{harm}^{-1}} \rightarrow S
\]
inverse to the map $\mathcal{F}$ near the point $w = 0$, which is the image $\mathcal{F}(0)$ of the corner vertex (see Sec. 2), we introduce the curves $L_\theta$ on the plane $z$ as follows: they are the images (under the inverse map) of the rays

$$\Lambda_\theta := \{H \ni w : |w| > 0, \arg w = \theta\}$$

starting from the point $w = 0$ under the angle $\theta$, i.e., $L_\theta := \mathcal{F}^{-1}(\Lambda_\theta)$ (see Fig. 1); recall that $(\rho, \theta)$ are the polar coordinates on the plane $w$.

Let $\varphi = \varphi_\theta(r)$ be the equation of the curve $L_\theta$ in the polar coordinates $(r, \varphi)$. Let us find the asymptotic behavior of the function $\varphi_\theta(r)$ as $r \to 0$. Since the curve $L_\theta$ satisfies the relation $\Im\left[\mathcal{F}(\Lambda_\theta) e^{-i\varphi}\right] = 0$ by definition, it follows that the function $\varphi_\theta(r)$ satisfies the relation

$$\Im\left[\mathcal{F}(re^{i\varphi_\theta(r)}) e^{-i\varphi}\right] = 0.$$

In that last relation, take into account the asymptotic behavior of the map $\mathcal{F}$, given by (10). We obtain the following (asymptotic) equation for the desired function $\varphi_\theta(r)$:

$$\mu r \sin(\varphi_\theta(r) - \varphi^*) = (a_1 - b_1 \cot \theta) r^{1/\beta} \cos \frac{\varphi_\theta(r)}{\beta} + (a_2 - b_2 \cot \theta) r^{2/\beta} \sin \frac{2 \varphi_\theta(r)}{\beta} + \mathcal{O}(r^{3/\beta}), \quad r \to 0.$$  \hspace{1cm} (11)

First, consider the special case where the expression in brackets at the right-hand part of (11) vanishes, i.e., the following relation is satisfied:

$$a_1 - b_1 \cot \theta = 0.$$  \hspace{1cm} (12)

Denote the angle corresponding to that relation as follows:

$$\theta^* = \arctan \frac{b_1}{a_1}.$$  \hspace{1cm} (13)

The curve $L_\theta$ corresponding to that case is denoted by $L^*$. The equation for that curve is denoted by $\varphi = \varphi^*(r)$. Its geometric properties are different from the ones of the curve $L_\theta$ for $\theta \neq \theta^*$.

Substitute (12) in relation (11) and divide both parts of the obtained relation by $\mu r$; this yields the following (asymptotic) equation for $\varphi^*(r)$:

$$\sin(\varphi^*(r) - \varphi^*) = \frac{(a_2 - b_2 \cot \theta)}{\mu} r^{2/\beta - 1} \sin \frac{2 \varphi^*(r)}{\beta} + \mathcal{O}(r^{3/\beta - 1}), \quad r \to 0.$$  \hspace{1cm} (14)

Taking into account that its right-hand part tends to zero as $r \to 0$ provided that $\beta \in (0, 2)$, we obtain the desired asymptotic behavior in the form

$$\varphi^*(r) = \varphi^* + E_1^* r^{2/\beta - 1} + \mathcal{O}(r^{3/\beta - 1}), \quad r \to 0, \quad \theta = \theta^*, \quad \beta \in (0, 1),$$  \hspace{1cm} (14)

where

$$E_1^* = \frac{\mu^{-1} (a_2 - b_2 \cot \theta) \sin \frac{2 \varphi^*}{\beta}}.$$
Now, let $\theta \neq \theta^*$. Then the following two cases are to be selected:

\begin{align}
\text{(I) } & \beta \in (0, 1) \quad \text{and} \quad \text{(II) } \beta \in (1, 2),
\end{align}

(15)

In the second case, we say that the corner is reentrant. This case is the most interesting from the viewpoint of applications and the difficulty treated in Sec. 1 arises in this case. However, we start our investigation from the first case.

(I) Let $\beta \in (0, 1)$. Then, dividing both parts of (11) by $\mu r$ we obtain the equation

\begin{align}
\sin(\varphi_\theta(r) - \varphi^*) = & \mu^{-1}(a_1 - b_1 \cot \theta) r^{1/\beta - 1} \cos \frac{\varphi_\theta(r)}{\beta} \\
& + \mu^{-1}(a_2 - b_2 \cot \theta) r^{2/\beta - 1} \sin \frac{2 \varphi_\theta(r)}{\beta} + O(r^{3/\beta - 1}), \quad r \to 0.
\end{align}

(16)

Its right-hand part tends to zero as $r \to 0$. Taking this fact into account, we find the following asymptotic behavior for the considered case (I):

\begin{align}
\varphi_\theta(r) = \varphi^* + E_1 r^{1/\beta - 1} + o(r^{1/\beta - 1}), \quad r \to 0, \quad \theta \in (0, \pi), \quad \beta \in (0, 1),
\end{align}

(17)

where

$$E_1 = \mu^{-1} (a_1 - b_1 \cot \theta) \cos \frac{\varphi^*}{\beta}.$$ 

If $\theta = 0$ or $\theta = \pi$, then we have the following exact relations instead of (17), which is an asymptotic one:

\begin{align}
\varphi_0(r) = -\frac{\pi \beta}{2} \quad \text{and} \quad \varphi_\pi(r) = \frac{\pi \beta}{2},
\end{align}

(18)

this obviously follows from Conditions (5). Thus, the dependence $\varphi(\theta)$, where $\theta$ is the exit angle of the ray $\Lambda_\theta$ leaving the preimage $w = 0$ of the corner vertex, while $\varphi$ is the exit corner of the image $L_\theta = \mathcal{F}(\Lambda_\theta)$ of the ray, leaving the corner vertex $z = 0$ itself, is as follows (provided that $\beta \in (0, 1)$):

\begin{align}
\beta \in (0, 1) : \quad \varphi(\theta) = \begin{cases} 
-\pi \beta/2, & \theta = 0, \\
\varphi^*, & \theta \in (0, \pi), \\
\pi \beta/2, & \theta = \pi.
\end{cases}
\end{align}

(19)

Figure 2 displays the graph of dependence (19). Figure 3a provides the polar mesh

$$\Xi = \left\{ \rho = \frac{n}{5}, n = 1, 5; \quad \theta = \frac{\pi n}{7}, n = 0, 7 \right\},$$

(20)

while Fig. 3b displays its image $\mathcal{F}^{-1}(\Xi)$ on the plane $z$ under the inverse map (for the considered case I).

(II) Now, assume that the second condition of (13) is satisfied and $\theta \neq \theta^*$. To obtain the asymptotic behavior of the function $\varphi_\theta(r)$ describing the curve $L_\theta$, we take into account that $(a_1 - b_1 \cot \theta) r^{1/\beta}$ is different from zero and divide Eq. (11) by the specified value. This allows one to represent the specified equation as follows:

\begin{align}
\cos \frac{\varphi_\theta(r)}{\beta} = F_1 r^{1-1/\beta} \sin(\varphi_\theta(r) - \varphi^*) + O(r^{1/\beta}), \quad r \to 0; \quad \theta^* \neq \theta \in (0, \pi),
\end{align}

(21)
Figure 2.

Figure 3.
where

\[ F_1 = \mu (a_1 - b_1 \cot \theta)^{-1}. \]

Since the right-hand part of relation (21) tends to zero as \( r \to 0 \), it follows that the first term of the asymptotic representation of \( \varphi_\theta(r) \) is either \( \pi \beta/2 \) or \( -\pi \beta/2 \). The detailed analysis of Eq. (21) and relation (14) yields the following result:

\[
\varphi_\theta(r) = \begin{cases} 
-\pi \beta/2 - \left[ \beta F_1 \sin \left( \frac{\pi \beta}{2} + \varphi^* \right) \right] r^{1-1/\beta} + O(r^\gamma), & \theta \in (0, \theta^*), \\
\varphi^* + E^*_1 r^{2/\beta-1} + o(r^{2/\beta-1}), & \theta = \theta^*, \quad r \to 0, \\
\frac{\pi \beta}{2} - \left[ \beta F_1 \sin \left( \frac{\pi \beta}{2} - \varphi^* \right) \right] r^{1-1/\beta} + O(r^\gamma), & \theta \in (\theta^*, \pi),
\end{cases}
\]

where \( \gamma = \min \{1/\beta, 2(1 - 1/\beta)\} \). This implies that if \( \beta \in (1, 2) \), then the dependence \( \varphi(\theta) \), where \( \theta \) is the exit angle of the ray \( \Lambda_\theta \) leaving the preimage \( w = 0 \) of the corner vertex, while \( \varphi \) is the exit angle of the image \( L_\theta = \mathcal{F}^{-1}(\Lambda_\theta) \) of the ray, leaving the corner vertex \( z = 0 \) itself, is described by the relation

\[
\beta \in (1, 2) : \quad \varphi(\theta) = \begin{cases} 
-\pi \beta/2, & \theta \in (0, \theta^*), \\
\varphi^*, & \theta = \theta^*, \\
\pi \beta/2, & \theta \in (\theta^*, \pi).
\end{cases}
\]

Figure 4 displays the graph of dependence (23). Figure 3c displays \( \mathcal{F}^{-1}(\Xi) \) on the plane \( z \), i.e., the image under the inverse map for the polar mesh defined by (20) (for case II).

Combining relation (22) with the similar result for case I, we arrive at the following assertion.

**Theorem 3.1.** The function \( \varphi = \varphi_\theta(r) \) describing the curve \( L_\theta := \mathcal{F}^{-1}(\Lambda_\theta) \) in the polar coordinated obeys the asymptotic relation given by (17)\)-(18) provided that \( \beta \in (0, 1) \) and obeys the asymptotic relation given by (22) provided that \( \beta \in (1, 2) \).
4. Behavior of harmonic maps near vertices of corners

To investigate the behavior of the map $F: S^{harm} \to H$ near the vertex $w = 0$ of a corner, introduce curves $l_\varphi$ on the plane $w = \rho e^{i \theta}$ such that they are the images of the rays

$$\lambda_\varphi := \{S \ni z : |z| > 0, \arg z = \varphi\},$$

starting from the specified vertex, i.e., $l_\varphi := F(\lambda_\varphi)$. Similarly to the investigation of the inverse map $F^{-1}$ (see Sec. 3), to investigate the behavior of $F$ near the point $z = 0$, one has to consider cases (13) separately.

(I) Let $\beta \in (0, 1)$. To find the form of the curve $l_\varphi$ in the Cartesian coordinates $(u, v)$, one has to find the parametric dependence of those coordinates on the coordinates on the plane $z$, i.e., to find the functions $u_\varphi = u(r)$ and $v_\varphi = v(r)$. To do that, separate the real and imaginary part in the asymptotic relation given by (10). We obtain that

$$u_\varphi(r) = -\mu r \sin(\varphi - \varphi^*) + a_1 r^{1/\beta} \cos \frac{\varphi}{\beta} + a_2 r^{2/\beta} \sin \frac{\varphi}{\beta} + O(r^{3/\beta}), \quad r \to 0,$$

and

$$v_\varphi(r) = b_1 r^{1/\beta} \cos \frac{\varphi}{\beta} + b_2 r^{2/\beta} \sin \frac{\varphi}{\beta} + O(r^{3/\beta}), \quad r \to 0.$$

From the former relation, express $r$ via $u$ and $\varphi$. Then substitute it in the latter one. We obtain the following asymptotic behavior of the function $v_\varphi(u)$ as $\varphi \neq \varphi^*$:

$$v_\varphi(u) = \frac{b_1 \cos(\varphi/\beta)}{(\mu \sin(\varphi - \varphi^*))^{1/\beta}} |u|^{1/\beta} + o(|u|^{1/\beta}), \quad |u| \to 0,$$

$$\beta \in (0, 1), \quad \varphi^* \neq \varphi \in \left(-\frac{\pi \beta}{2}, \frac{\pi \beta}{2}\right).$$

It is substantially simplified if $\varphi = \varphi^*$:

$$v_\varphi(u) = \frac{b_1}{a_1} u + o(u), \quad u \to 0, \quad \beta \in (0, 1), \quad \varphi = \varphi^*. \quad (24)$$

(II) Let $\beta \in (1, 2)$. Introduce the Cartesian coordinate system rotated to the angle $\theta^*$ with respect to the original one. Then the new coordinates $(U, V)$ are related to the original coordinates $(u, v)$ by the relation

$$U + i V = (u + iv) e^{-i \theta^*},$$

where $\theta^*$ is the value of the angular coordinate $\theta$ such that the first relation of (11) is satisfied for $\theta = \theta^*$. Let $V = V_\varphi(U)$ denote the equation of the curve $l_\varphi$ in the rotated Cartesian coordinates $(U, V)$.

Using asymptotics (10) for the map $F$, we obtain the desired asymptotic behavior for the function $V_\varphi(U)$. For $\varphi = \varphi^*$, we have the asymptotic relation

$$V^*(U) = \frac{2(a_1 b_2 - a_2 b_1) \tan \frac{\theta^*}{2}}{(a_1^2 + b_1^2)^{3/2}} U^2 + O(U^3), \quad U \to 0,$$

$$\varphi = \varphi^*, \quad (26)$$

$$\varphi = \varphi^*, \quad (25)$$
where \( V^\ast \) denotes \( V_{\phi^\ast} \). For other values of \( \phi \), we have the following asymptotic relation:

\[
V_{\phi}(U) = \frac{\mu b_1 \sin(\phi - \phi^\ast)}{(a_1^2 + b_1^2)^{\beta/2} \cos^{\beta} \frac{\phi}{\beta}} U^\beta + O(U^{2\beta - 1}), \quad U \to 0, \quad \phi^\ast \neq \phi \in \left(-\frac{\pi \beta}{2}, \frac{\pi \beta}{2}\right). \tag{27}
\]

Let \( \theta = \theta_{\phi}(\rho) \) be the equation of the curve \( l_{\phi} \) in the polar coordinates of the plane \( w \). Represent relations (16)-(17) as asymptotic relations for \( l_{\phi} \) in the polar coordinates near the image \( w = 0 \) of the vertex of the reentrant corner. For \( \phi = \phi^\ast \), we have the asymptotic relation

\[
\theta^*(\rho) = \theta^* + \frac{2(a_1 b_2 - a_2 b_1) \tan \frac{\phi^\ast}{\beta}}{(a_1^2 + b_1^2)^{3/2}} \rho + O(\rho^2), \quad \rho \to 0,
\]

where \( \theta^*(\rho) \) denotes \( \theta_{\phi^\ast}(\rho) \). For other values of \( \phi \), we have the asymptotic relation

\[
\theta_{\phi}(\rho) = \theta^* + \frac{\mu b_1 \sin(\phi - \phi^\ast)}{(a_1^2 + b_1^2)^{\beta/2} \cos^{\beta} \frac{\phi}{\beta}} \rho^{\beta - 1} + O(\rho^{2\beta - 2}), \quad \rho \to 0, \quad \phi^\ast \neq \phi \in \left(-\frac{\pi \beta}{2}, \frac{\pi \beta}{2}\right).
\]

Combining the results of the present section, we arrive at the following assertion.

**Theorem 4.1.** If \( \beta \in (0, 1) \), then the asymptotic behavior of the function \( v(u) \) describing the curve \( l_{\phi} = F(\lambda_{\phi}) \) in the Cartesian coordinates is expressed by relations (24)-(25). If \( \beta \in (1, 2) \), then the specified curve is described by means of asymptotics (26)-(27) in the
It follows that the dependence $\theta(\varphi)$, under which the ray $\lambda_\varphi$ leaves the vertex $z = 0$ of the corner, and the angle $\theta$, under which its image $l_\varphi := F^{-1}(\lambda_\varphi)$ leaves the preimage of the vertex $w = 0$, is described for $\beta \in (0, 1)$ by the formula

$$
\beta \in (0, 1) : \quad \theta(\varphi) = \begin{cases} 
0, & \varphi \in \left[-\pi \beta / 2, \varphi^*\right), \\
\theta^*, & \varphi = \varphi^*, \\
\pi, & \varphi \in \left(\varphi^*, \pi \beta / 2\right].
\end{cases}
$$

and when $\beta \in (1, 2)$ — by the following formula:

$$
\beta \in (1, 2) : \quad \theta(\varphi) = \begin{cases} 
0, & \varphi = -\pi \beta / 2, \\
\theta^*, & \varphi \in \left(-\pi \beta / 2, \pi \beta / 2\right), \\
\pi, & \phi = \pi \beta / 2.
\end{cases}
$$

In Fig. 5a is a polar grid

$$
T = \left\{ r = \frac{n}{5}, n = 1, 5; \quad \varphi = \frac{\pi \beta}{2} \left(\frac{2n}{7} - 1\right), n = 0, 7 \right\},
$$

and in Fig. 5b and 5c show its images $F(T)$ on the plane $w$, respectively for $\beta \in (0, 1)$ and $\beta \in (1, 2)$.

Theorems 3.1 and 4.1 admit the following obvious adherence.

**Theorem 4.2.** Let the boundary of the domain $Z$ contain the contour of the angle $L = L_+ \cup M \cup L_+$ of the solution $\pi \beta \in (0, 2 \pi)$ with vertex $M$, $W$ contains rectilinear segment $I = I_+ \cup N \cup I_+$, and the map $F : Z \to W$, univalent and preserving orientation near $L$, corresponds to the following conditions: $F(L) = I$, $F(M) = N$, and for the uniform motion of a point $z \in L_+$ the speed of its image on $L_+$ is equal to the constant $\sigma_\pi$.

Further, let $Z \ni \lambda_\varphi$ be a smooth Jordan curve, leaving the vertex $M$ at an angle $\varphi$, $\theta$ is the angle at which its image leaves the point $N$, and $\theta(\varphi)$ is the correspondence between these angles; let it go $W \ni \Lambda_\varphi$ is a smooth Jordan curve issuing from the point $N$ at an angle $\theta$, $\varphi$ is the angle by which its image $F^{-1}(\Lambda_\theta)$ leaves the vertex $M$, and $\varphi(\theta)$ — the correspondence between these angles.

Then the dependence $\theta(\varphi)$ for $\beta \in (0, 1)$ is given by the formula (28), and for $\beta \in (1, 2)$ by the formula (29); the dependence $\varphi(\theta)$ for $\beta \in (0, 1)$ is given by the formula (17), and for $\beta \in (1, 2)$, by the formula (23).

Thus, the dependencies $\theta(\varphi)$ and $\varphi(\theta)$ are discontinuous for a harmonic mapping of a domain with an angle, in contrast to a similar conformal imaging $K : S^{conf} \to H$, $K(0) = 0$, for which these dependences are linear, $\theta(\varphi) = \varphi/\beta + \pi/2$, $\varphi(\theta) = \beta \theta - \pi \beta / 2$.

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