Noise effects on purity and quantum entanglement in terms of physical implementability

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Quantum decoherence due to imperfect manipulation of quantum devices is a key issue in the noisy intermediate-scale quantum (NISQ) era. Standard analyses in quantum information and quantum computation use error rates to parameterize quantum noise channels. However, there is no explicit relation between the decoherence effect induced by a noise channel and its error rate. In this work, we propose to characterize the decoherence effect of a noise channel by the physical implementability of its inverse, which is a universal parameter quantifying the difficulty to simulate the noise inverse with accessible quantum channels. We establish two concise inequalities connecting the decrease of the state purity and logarithmic negativity after a noise channel to the physical implementability of the noise inverse, which is required to be decomposed as mutually orthogonal unitaries or product channels respectively. Our results are numerically demonstrated on several commonly adopted two-qubit noise models. We believe that these relations contribute to the theoretical research on the entanglement properties of noise channels and provide guiding principles for quantum circuit design.

ARTICLE

INTRODUCTION
Quantum entanglement is an important resource in quantum computers1,2, empowering the establishment of quantum supremacy3,4. The characterization and detection of quantum entanglement5–7 in physical systems have been the primary concerns of quantum information and computation for decades. On the other hand, imperfect control of quantum systems in the noisy intermediate-scale quantum (NISQ) era may induce errors8 into quantum circuits composed of unitary gates, which can be described by general quantum channels.

How to characterize the change of quantum entanglement after the implementation of a quantum channel arises as an interesting problem. Roughly speaking, entangled unitary gates can generate entanglement, while noise channels destroy entanglement. The balance between these two parts gives the critical point of quantum information and computation for decades. On the other hand, imperfect control of quantum systems in the noisy intermediate-scale quantum (NISQ) era may induce errors into quantum circuits composed of unitary gates, which can be described by general quantum channels.

Here, we try to characterize how much purity and quantum entanglement a noise channel $E$ destroys with a universal parameter. We take its inverse $E^{-1}$ into consideration, which generally is not a physical quantum channel but can recover the quantum entanglement destroyed by $E$ mathematically since $E^{-1} \circ E = I$. Intuitively, the harder to implement the noise inverse, the more destructive the noise itself. Therefore, we believe that for an invertible noise channel $E$, the physical implementability of its inverse $E^{-1}$, which represents the sampling cost to implement a linear map $E^{-1}$, is a prime candidate. Such a sampling cost measure is constructed from quantum resource theories18–20, characterizing the distance between a non-physical linear map and the set of physical quantum channels. In particular, we establish two concise and universal inequalities bounding the decrease of the state purity and logarithmic negativity21–23 under noise channels with this measure. The first inequality is verified by several analytical examples, while the second is numerically demonstrated on four typical two-qubit noise channels.

RESULTS
Physical implementability
We adopt the Choi operator $\Lambda_N$24, see Supplementary Note 1 to represent a quantum linear map $N$, from which one can construct the output density operator

$$N(\rho) = Tr_2[(\rho \otimes I)\Lambda_N].$$

(1)

For an invertible completely positive (CP) and trace-preserving (TP) map $T$, its inverse is Hermitian-preserving (HP) and TP15,25. The sampling cost for implementing an HPTP map $N$ with the Monte Carlo method is characterized by its physical implementability15,17, defined as

$$\nu(N) := \log_2 \min_{T \in \text{CPTP}} \left\{ \sum_i |q_i| \middle| N = \sum_i q_i T, q_i \in \mathbb{R} \right\}.$$

(2)

From the perspective of quantum resource theories, such a quantity measures the distance of an HPTP map $N$ from the set of CPTP channels $\{T_i\}$. Inspired by the fact that the inverse of a quantum channel is still CPTP iff it is unitary15, we expect that the physical implementability of $E^{-1}$ can characterize the deviation of the noise channel $E$ from unitary maps, which preserve purity and are capable of increasing quantum entanglement.

Physical implementability and purity
We begin by connecting the decrease of purity to the physical implementability of the noise inverse. The purity of a quantum

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state $\rho$ is defined as $\mathcal{P}(\rho) := \text{Tr}[\rho^2]^{1/4}$. We decompose the density operator by a set of Hermitian, complete, and orthonormal basis $\{O_i\}$ containing the identity $O_0 = I$ as

$$\rho = I + \sum_{i=1}^{d^2-1} r_i O_i, \quad (3)$$

where $r_i$ is defined by the expectation value of $\{O_i\}$, i.e., $r_i = \text{Tr}[\rho O_i]$. Then the purity is related to the length of the vector $\mathbf{r}$ by

$$\mathcal{P}(\rho) := \text{Tr}[\rho^2] = \text{Tr} \left[ \rho \left( I + \sum_{i=1}^{d^2-1} r_i O_i \right) \right] = 1 + |\mathbf{r}|^2 / d, \quad (4)$$

which enables us to calculate the purity change via the transformation of $\mathbf{r}$.

It was proved in ref. 15 that if an HPTP map $\mathcal{N}$ on a $d$-dimensional Hilbert space is the superposition of mutually orthogonal unitaries, such a decomposition is optimal (in the sense of the minimization in Eq. (2)), thus the physical implementability of $\mathcal{N}$ is determined by the trace norm of its Choi operator, i.e.,

$$2^{\chi(\mathcal{N})} = \sum |q_i| = \|\Lambda_N\|_1 / d, \quad (5)$$

from which we derive the following theorem.

**Theorem 1.** For a mixed unitary map $\mathcal{N}$ decomposed by a set of mutually orthogonal unitaries, the purity of the input state $\rho_0$ and the output state $\rho$ satisfy

$$\log_2 \left( \frac{\mathcal{P}(\rho)}{\mathcal{P}(\rho_0)} d - 1 \right) \leq 2\chi(\mathcal{N}), \quad (6)$$

where $d$ is the dimension of the Hilbert space.

**Proof:** consider the unitary decomposition $\mathcal{N}(\cdot) = \sum_i q_i U_i().U_i^\dagger$, since unitary transformations leave $|\mathbf{r}|$ unchanged, we reach

$$|\mathbf{r}(\rho)| = \left| \mathbf{r} \left( \sum_i q_i U_i(\rho) U_i^\dagger \right) \right| = \left| \sum_i q_i r_i (U_i(\rho) U_i^\dagger) \right| \leq \sum_i |q_i| \left| \mathbf{r} (U_i(\rho) U_i^\dagger) \right| = \sum_i |q_i| |\mathbf{r}(\rho_0)| = 2^{\chi(\mathcal{N})} |\mathbf{r}(\rho_0)|, \quad (7)$$

where we have used Eq. (5). Finally, we build the relationship between the physical implementability of $\mathcal{N}$ and the ratio of purity, i.e.,

$$\log_2 \left( \frac{\mathcal{P}(\rho)}{\mathcal{P}(\rho_0)} d - 1 \right) = \log_2 \left| \frac{|\mathbf{r}(\rho)|^2}{|\mathbf{r}(\rho_0)|^2} \right| \leq 2\chi(\mathcal{N}). \quad (8)$$

From Theorem 1 we can directly derive the following corollary, which is one of the main results of this work.

**Corollary 1.** For a noise channel $\mathcal{E}$, if both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are mixed unitary maps decomposed by mutually orthogonal unitaries, the purity of the input state $\rho_0$ and the output state $\rho$ satisfy

$$-2\chi(\mathcal{E}^{-1}) \leq \log_2 \left( \frac{\mathcal{P}(\rho)}{\mathcal{P}(\rho_0)} d - 1 \right) \leq 2\chi(\mathcal{E}) = 0, \quad (9)$$

where $d$ is the dimension of the Hilbert space.

The last equality in Eq. (9) follows from the fact that $\chi(T) = 0$ for any CPTP map $T$ by definition. We note that several commonly used noise models, such as the (multiqubit) Pauli noise, depolarizing noise, and dephasing noise, all belong to this category (see Methods and Supplementary Note 2). It can be easily verified that both sides of this inequality can be reached for the single-qubit Pauli noise. As for a more generic multiqubit noise, such as the $n$-qubit dephasing noise, we discuss the bounds in Supplementary Note 3, where we conclude that the equality may hold when $n = 1$, while the lower bounds can be further tightened for $n \geq 2$.

### Physical implementability and logarithmic negativity

Now we turn to consider the noise effects on quantum entanglement, which limit the potential power of quantum computers. There are many entanglement measures for bipartite mixed states, such as concurrence, entanglement of formation, entanglement of assistance, etc. Here we choose the logarithmic negativity to measure the state entanglement, which characterizes the violation of the well-known positive partial transpose (PPT) criterion. For a quantum state $\rho$ on a bipartite system $A \otimes B$, its logarithmic negativity is defined as

$$E_{\text{L}}(\rho) := \log_2 \|\rho_{A}^{T_B}\|_1, \quad (10)$$

where $\rho_{A}^{T_B}$ denotes the partial transpose of subsystem $B$.

To fully understand this issue, we need to analyze the entanglement property of a quantum channel itself. Similar to the previous case, we wish to decompose the noise inverse into product quantum channels, inspired by a general property of any entanglement measure $E(\rho)$, namely that $E(\rho)$ does not increase under local operations and classical communication (LOCC). The Choi-Jamiolkowski isomorphism between linear maps and density operators motivates us to define separable (entangled) quantum maps, which is a generalization of separable (entangled) quantum states.

**Definition 1.** Let $\mathcal{N}$ be an HPTP map on a bipartite system $A \otimes B$. We say that $\mathcal{N}$ is separable, if there exist $q_i \in \mathbb{R}$ and product channels $T^A_i \otimes T^B_i$ such that

$$\mathcal{N}(\cdot) = \sum_i q_i (T^A_i \otimes T^B_i)(\cdot). \quad (11)$$

Otherwise, we call it an entangled map.

We note that not all HPTP maps are separable even if negative coefficients $(q_i)$ are allowed in decomposition, which can be proved with the following idea (see “Methods” section for a complete proof). The Choi operator of any HPTP map can be decomposed with the computational basis

$$\Lambda_N = \sum_{i=1}^d \sum_{j=1}^d |i\rangle\langle j| \otimes O_{ij}. \quad (12)$$

In this way, if a bipartite HPTP map is separable, we have

$$\sum_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l| \otimes O_{ijkl} = \sum_{m} q_m \sum_{ijkl} |ijkl\rangle\langle kijkl| \otimes O_{ijkl} = \sum_{m} (q_m)_{ijkl} \otimes O_{ijkl}. \quad (13)$$

The TP condition for subsystem B will give $\text{Tr}[\rho_{A}^{T_B} O_{ijkl}] = 0$ for $i \neq j$. In fact, it results in a much stronger condition than what we have for the original map $\text{Tr}[\rho_{A}^{T_B} O_{ijkl}] = 0$. Therefore, a necessary condition for an HPTP map to be separable is that it preserves the trace of two subsystems simultaneously. In other words, for those entangled unitary gates, such as the CNOT gate and the SWAP gate, subsystem $B$ becomes an open system if we take the partial trace of subsystem $A$ and thus the probability is not conserved.

To study the influence of noise on the state negativity, we try to connect the partial transpose of output and input states. The following theorem provides a general bound for the output state negativity concerning the optimal decomposition given in Eq. (11).

**Theorem 2.** For a separable HPTP map $\mathcal{N}$ on a bipartite system $A \otimes B$, the logarithmic negativity of the input state $\rho_0$ and the output state $\rho$ satisfy

$$E_{\text{L}}(\rho) - E_{\text{L}}(\rho_0) \leq \eta(\mathcal{N}) \equiv \log_2 \min \left\{ \sum_{q_i} |q_i| \mathcal{N} = \sum_{q_i} T^A_i \otimes T^B_i, q_i \in \mathbb{R} \right\}, \quad (14)$$
where $\rho^{A(B)}$ are quantum channels on subsystem $A(B)$.

Proof: For any decomposition $\mathcal{N} = \sum q_i T_i^A \otimes T_i^B$, we can bound the trace norm of $\rho^\#$ as

$$\|\rho^\#\|_1 = \left\| \sum q_i [T_i^A \otimes T_i^B(\rho^\#)^{\frac{1}{2}}] \right\|_1 \leq \sum q_i \|T_i^A \otimes T_i^B(\rho^\#)^{\frac{1}{2}}\|_1 \leq \sum q_i \|\rho^\#\|_1.$$  \hfill (15)

The above theorem indicates that $\eta(\mathcal{N})$ characterizes the potential of a separable HPT map $\mathcal{N}$ to increase entanglement, satisfying $\eta(\mathcal{N}) \geq \nu(\mathcal{N})$ by definitions. Generally $\eta(\mathcal{N})$ is hard to evaluate. However, if the decomposition by product channels is still optimal in the sense of Eq. (2), we can just adopt $\nu(\mathcal{N})$ as the upper bound in Theorem 2. For example, a noise channel $\mathcal{E}$ generally satisfies that $\eta(\mathcal{E}) = \nu(\mathcal{E}) = 0$ and cannot increase entanglement. On the other hand, we believe that the inverse of the noise channel can counteract the noise effect and recover coherence. It helps us characterize how much entanglement the noise channel destroys. With the above theorem, we can derive another important conclusion for our study.

Corollary 2. For a noise channel $\mathcal{E}$, if both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are separable with $\eta(\mathcal{E}^{-1}) = \nu(\mathcal{E}^{-1})$, the logarithmic negativity of the input state $\rho_0$ and the output state $\rho$ satisfy

$$-\nu(\mathcal{E}^{-1}) \leq \Delta_{\mathcal{E}} \equiv E_{\mathcal{N}}(\rho) - E_{\mathcal{N}}(\rho_0) \leq \nu(\mathcal{E}) = 0.$$  \hfill (16)

We note that several typical noise models\textsuperscript{37} (see “Methods” section) all fall into this category, which we believe is a general property for typical quantum noise originating from the coupling between system and environment and thus cannot induce extra entanglement between two subsystems.

Numerical simulations

In the following, we numerically verify Eq. (16) with these noise models applied to two-qubit quantum states. In Fig. 1a we detect the change of logarithmic negativity between two qubits $\Delta_{\mathcal{E}} \equiv E_{\mathcal{N}}(\rho) - E_{\mathcal{N}}(\rho_0)$. For each type of noise, we randomly sample 10,000 two-qubit pure states as input states, i.e., $\rho_0 = |\psi\rangle\langle\psi|$, where $E_{\mathcal{N}}(\rho_0)$ approximately follow a Gaussian distribution within (0.1). It is demonstrated that the bounds given by $\nu(\mathcal{E}^{-1})$ (solid lines) are about twice as large as the maximal values of $\Delta_{\mathcal{E}}$. We also provide numerical results in Supplementary Note 4, where we obtain similar results.

These bounds are mathematically tight, but maybe not physically. For example, if we consider the dephasing noise and let the output state be a product state $\rho^{(n)} = |+\rangle\langle+| + \cdots + |\rangle\langle\rangle$ with $E_{\mathcal{N}}(\rho^{(n)}) = 0$, then the input state can be reconstructed as

$$\rho_0^{(n)} = \frac{1}{1 - \epsilon} |+\rangle\langle+| + \cdots + |\rangle\langle\rangle - \frac{\epsilon}{2(1 - \epsilon)} \sum_{i=1}^{n} |i\rangle\langle i|,$$

(17)

whose logarithmic negativity is $E_{\mathcal{N}}(\rho_0) = \log_2 \frac{\pi^2}{\pi^2 - 1}$ in this case, we have $\Delta_{\mathcal{E}} = \nu(\mathcal{E}^{-1})$. However, both $\rho_0$ and $\rho_0^{(n)}$ are not positive, which is not a physical situation. To further demonstrate this point, we provide numerical results in Fig. 1b, where we choose mixture of pure states as input states $\rho_0 = \lambda_1 |\psi_1\rangle\langle\psi_1| + \lambda_2 |\psi_2\rangle\langle\psi_2|$. Here $|\psi_1\rangle$ and $|\psi_2\rangle$ are randomly chosen orthonormal two-qubit pure states, while $\lambda_1$ and $\lambda_2$ are randomly chosen from $[-1, 1]$ and then normalized as $\lambda_1 + \lambda_2 = 1$, thus maybe $\rho_0$ are not physical. It is shown that the variation range of the negativity decrease in Fig. 1b is larger than that in Fig. 1a for each type of noise due to the violation of the positivity restrictions on $\rho_0$. Therefore, we expect that the bounds in Corollary 2 may be further tightened for specific noise models with much more careful analyses combined with physical requirements, e.g., the positive conditions on $\rho_0$ and $\rho$.

Alternatively, we can use the root-mean-square of $|q_i|$ in Eq. (15) as an estimation value, i.e.,

$$\|\rho^\#\|_1 = \left\| \sum q_i [T_i^A \otimes T_i^B(\rho^\#)^{\frac{1}{2}}] \right\|_1 \approx \sqrt{\sum |q_i|^2 \left\| \left[ T_i^A \otimes T_i^B(\rho^\#)^{\frac{1}{2}} \right] \right\|_1^2} \leq \sqrt{\sum |q_i|^2 \|\rho^\#\|_1^2} = \sqrt{\sum |q_i|^2 \|\rho^\#\|_1}.$$  \hfill (18)
If we denote $\mu(\mathcal{N}) = \log_2 \sqrt{\sum q_i^2}$, then the above relation can be expressed in a concise form

$$E_{\mathbb{N}}(\rho) - E_{\mathbb{N}'}(\rho_0) \leq \mu(\mathcal{N}).$$

We simultaneously plot $\mu(\mathcal{E}^{-1})$ in Fig. 1, where we surprisingly find that $\mu(\mathcal{E}^{-1})$ (dashed lines) appear to upper-bound the decrease of the state negativity $|\Delta E|_{\mathbb{N}}$ for physical situations.

If there is only one positive coefficient in the decomposition $q_i \approx 1 + be$, and all other negative coefficients have the same order of magnitude as $e$, (all four types of noise models fall into this category), then to the first order of $e$ we have

$$2\nu(\mathcal{N}) = \sum_i |q_i| \approx 1 + 2be,$$

$$2\nu(\mathcal{N}) = \sum_i q_i^2 \approx 1 + be.$$  

As a result, for small $e$ we have $\mu(\mathcal{N}) \approx \frac{1}{2} \nu(\mathcal{N})$, which are demonstrated by the dashed and solid lines in Fig. 1.

Meanwhile, we note that the probability distributions $P(|\Delta E|_{\mathbb{N}})$ vary with different noise models, which are plotted in the insets of Fig. 1 for a fixed error rate $e = 0.01$ without loss of generality. The total probability (i.e. the area of shadow) is normalized to 1 for each subfigure. For example, there exists a sharp peak in Fig. 1b for the depolarizing noise. This phenomenon can be explained by the special property of the depolarizing noise, which can be written as

$$\mathcal{E}_i^{\mathbb{N}}(\rho_0) = (1 - e)\rho_0 + \frac{e}{2} |i\rangle \langle i|.$$  

It implies that the spectrum of $\rho^{\mathbb{N}}$ is directly related to that of $\rho_0^{\mathbb{N}}$

$$\lambda_i = (1 - e)\lambda_0 + \frac{e}{2}\,,$$

for $i = 1, 2, \ldots 2^n$. Therefore, if half of the eigenvalues of $\rho_0^{\mathbb{N}}$ are positive and half are negative, and $e$ is small which does not change the sign of $\lambda_i$, the trace norm of the output state can be analytically derived, i.e.,

$$||\rho^{\mathbb{N}}||_1 = (1 - e)||\rho_0^{\mathbb{N}}||_1,$$

which gives the decrease of logarithmic negativity $|\Delta E|_{\mathbb{N}} = \log_2 \sqrt{\frac{1}{|\Delta E|_{\mathbb{N}}}}$. Meanwhile, we analytically derive $|\Delta E|_{\mathbb{N}}$ for the maximally entangled state in Supplementary Note 5 and show them as the dash-dot lines in Fig. 1. For the depolarizing and the phase flip noise, they serve as the supremum and the infimum of $|\Delta E|_{\mathbb{N}}$ respectively in Fig. 1a, while there is no apparent feature for the amplitude damping and the dephasing noise. These properties reveal another aspect of the decoherence effect, namely the variety of $|\Delta E|_{\mathbb{N}}$ for different input states, which we leave for further study.

### DISCUSSION

In this work, we build two concise and essential inequalities connecting the output state to the input state, where the physical implementability of the noise inverse upper-bounds the decrease of the purity and logarithmic negativity of quantum states. Central to this is the optimal decomposition of the noise inverse via mutually orthonormal unitaries or product channels, which applies to several commonly-adopted noise models. Specifically, the former condition is satisfied by the depolarizing, dephasing, and phase flip noise, while the latter one is additionally satisfied by the amplitude damping noise. These relations imply that the physical implementability of the noise inverse, which is originally proposed to describe the sampling cost for error mitigation and the distance of the noise inverse away from the set of CPTP channels, is a better characterization for the decoherence effect of a noise channel than the commonly used error rate.

Compared with previous works on the entanglement or coherence properties of quantum channels$^{12,23,24}$ that describe the potential of a quantum channel to generate entanglement, our study provides a characterization of how destructive a noise channel is, which has applications in benchmarks of quantum hardware. For instance, when combined with quantum gate set tomography$^{39,40}$ to obtain a full characterization for noise models of quantum gates, one may estimate whether a quantum device is capable of generating the highly-entangled states required for quantum supremacy instead of directly detecting quantum entanglement in the output state, which is generally a difficult task with exponentially increasing experimental cost$^7$. Another interesting problem is to apply our results to the tensor network representation of quantum noise$^{41}$, which naturally captures the correlation of different qubits involved in the noise channel. Therefore, we believe that our work enables the theoretical and experimental research of quantum noise from a different perspective, namely, entanglement properties of noise channels.

### METHODS

#### Physical implementability

The core physical quantity in our work, the physical implementability of an HPTP map, derives from the quasi-probability method$^{42,43}$ for quantum error mitigation and its variants$^{25,41,46,45}$ which involve the simulation of the inverse of noise channels with physically implementable quantum channels. The sampling cost for implementing an HPTP map $\mathcal{N}$ is characterized by its physical implementability$^{15,17}$, defined as

$$\nu(\mathcal{N}) := \log_2 \min_{\mathbb{T}_i \in \text{CPTP}} \left\{ \sum_i |q_i| : \mathcal{N} = \sum_i q_i T_i, q_i \in \mathbb{R} \right\}.$$  

During the proof of our main results, we take advantage of the fact that the state purity remains unchanged under unitary channels, while the state negativity is non-increasing under product channels. It indicates that if the optimal decomposition of an HPTP map gives unitary (product) channels, we can bound the increase of purity (negativity) with its physical implementability. On the other hand, for a mixed unitary map, if it is decomposed by a set of mutually orthogonal unitary channels, such a decomposition is optimal.

Numerically, the physical implementability can be calculated via semidefinite programming (SDP)$^{13}$. Alternatively, we provide the upper and lower bounds for the physical implementability in terms of the maximum and minimum eigenvalues of the Choi matrix in Supplementary Note 6, which may inspire efficient estimation methods for the physical implementability with numerical approaches, such as the tensor network representation$^{46,48}$ of a general noise channel$^{51}$, instead of solving the entire optimization problem.

#### Noise models and physical implementability

In the following, we summarize four commonly used noise models and the physical implementability of their inverse, which enables the application of our results on these noise channels. We will show that Corollary 1 applies to the multiqubit Pauli noise, depolarizing noise, and dephasing noise, while Corollary 2 holds for the multiqubit amplitude damping noise additionally.

We define the multiqubit Pauli noise as

$$\mathcal{E}_i^{\mathbb{N}}(\rho^{\mathbb{N}}) = (1 - e)\rho^{\mathbb{N}} + e \left( \bigotimes_{i=1}^{n} \sigma_i \right) \rho^{\mathbb{N}} \left( \bigotimes_{i=1}^{n} \sigma_i \right).$$  

(25)
where $\sigma^i_\alpha$ represents the Pauli matrix $\sigma_\alpha$ applied on the $i$-th site. The inverse of this noise is analytically derived as
\begin{equation}
E^{n^{-1}}(\rho^{[n]}) = \frac{1 - \varepsilon}{1 - 2\varepsilon} (1 - \varepsilon) \rho^{[n]} - \frac{\varepsilon}{1 - 2\varepsilon} \left( \bigotimes_{i=1}^{n} \sigma^i_\alpha \right) \left( \bigotimes_{i=1}^{n} \sigma^i_\alpha \right)^\dagger,
\end{equation}
which provides the optimal decomposition, with the physical implementability being $\nu(E^{n^{-1}}) = \log_2 \left( \frac{1}{1 + \frac{1}{e}} \right)$. Each term in the decomposition is a unitary and product channel, hence both Corollary 1 and 2 hold. The two-qubit phase flip noise in Fig. 1(c) corresponds to taking $\alpha_1 = \alpha_2 = 3$ here, i.e., $E^{[2]}(\rho^{[2]}) = (1 - \varepsilon)\rho^{[2]} + \varepsilon (\sigma^1_3 \otimes \sigma^2_3) \rho^{[2]} (\sigma^1_3 \otimes \sigma^2_3)^\dagger$.

The $n$-qubit depolarizing noise is defined as
\begin{equation}
E^{[n]}(\rho^{[n]}) = (1 - \varepsilon)\rho^{[n]} + \frac{\varepsilon}{n} \rho^{[n]},
\end{equation}
where the indices $\alpha_i$ are summed from 0 to 3. Its inverse can be directly calculated with the first equality
\begin{equation}
E^{n^{-1}}(\rho^{[n]}) = \frac{1 - \varepsilon}{1 - 2\varepsilon} \rho^{[n]} - \frac{\varepsilon}{1 - 2\varepsilon} \sum_{\alpha_1} \left( \bigotimes_{i=1}^{n} \sigma^i_{\alpha_1} \right) \left( \bigotimes_{i=1}^{n} \sigma^i_{\alpha_1} \right)^\dagger,
\end{equation}
which is decomposed by unitary and product channels. The unitaries in the decomposition of $E^{[n]}$ (and $E^{[n^{-1}]}$) are also mutually orthogonal
\begin{equation}
\text{Tr} \left[ \left( \bigotimes_{i=1}^{n} \sigma^i_{\alpha_1} \right) \left( \bigotimes_{i=1}^{n} \sigma^i_{\beta_i} \right) \right] = 2^n \prod_{i=1}^{n} \delta_{\alpha_i \beta_i},
\end{equation}
allowing us to analytically calculate the physical implementability as
\begin{equation}
\nu(E^{n^{-1}}) = \log_2 \left( \frac{1 - \varepsilon}{1 - 2\varepsilon} \right) + (4^n - 1) \varepsilon \left( \frac{1}{1 - \varepsilon} - 1 \right) = \log_2 \left( 1 + \frac{1}{1 - \varepsilon} \right).
\end{equation}

The $n$-qubit dephasing noise is defined as
\begin{equation}
E^{[n]}(\rho^{[n]}) = (1 - \varepsilon)\rho^{[n]} + \frac{\varepsilon}{2^n \prod_{i=1}^{n} (i)} \sum_{\alpha_1 \in [0,3]} \left( \bigotimes_{i=1}^{n} \sigma^i_{\alpha_1} \right) \left( \bigotimes_{i=1}^{n} \sigma^i_{\alpha_1} \right)^\dagger,
\end{equation}
where the summation only contains $\sigma_0$ and $\sigma_3$. We assume that the inverse of $E^{[n]}$ takes a similar form
\begin{equation}
E^{n^{-1}}(\rho^{[n]}) = \Lambda \rho^{[n]} - B \sum_{\alpha_1 \in [0,3]} \left( \bigotimes_{i=1}^{n} \sigma^i_{\alpha_1} \right) \left( \bigotimes_{i=1}^{n} \sigma^i_{\alpha_1} \right)^\dagger,
\end{equation}
and solve the undetermined coefficients $A$ and $B$. Finally, we obtain
\begin{equation}
E^{n^{-1}}(\rho^{[n]}) = \frac{1 - \varepsilon}{1 - 2\varepsilon} \rho^{[n]} - \frac{\varepsilon}{1 - 2\varepsilon} \sum_{\alpha_1 \in [0,3]} \left( \bigotimes_{i=1}^{n} \sigma^i_{\alpha_1} \right) \left( \bigotimes_{i=1}^{n} \sigma^i_{\alpha_1} \right)^\dagger,
\end{equation}
where each term is product channels and mutually orthonormal unitaries. Similarly, we can derive the physical implementability of the noise inverse
\begin{equation}
\nu(E^{n^{-1}}) = \log_2 \left( \frac{1 - \varepsilon}{1 - 2\varepsilon} \right) + (2^n - 1) \varepsilon \left( \frac{1}{1 - \varepsilon} - 1 \right) = \log_2 \left( 1 + \frac{1 - \varepsilon}{1 - 2\varepsilon} \right).
\end{equation}
Therefore, we need to find the decomposition of \( O_{ijkl}^{\text{pt}} \):
\[
O_{ijkl}^{\text{pt}} = \sum_m q_m O_{mij}^{\text{tr}} \otimes O_{mkj}^{\text{pt}},
\]
(46)
satisfying that
\[
\text{Tr} \left[ O_{mij}^{\text{tr}} \right] = \delta_{ij}, \quad \text{Tr} \left[ O_{mkj}^{\text{pt}} \right] = \delta_{ij}.
\]
(47)
Such a decomposition cannot always be found. For example, for \( i \neq j \), the above conditions give \( \text{Tr} \left[ O_{mij}^{\text{tr}} \right] = 0 \). Then after taking the partial trace of Eq. (46), we require that
\[
\text{Tr}_\mathcal{A} \left[ O_{ijkl}^{\text{pt}} \right] = 0,
\]
(48)
which is a much stronger condition than what we have for the original map \( \text{Tr} \left[ O_{ijkl}^{\text{pt}} \right] = 0 \). We note that some commonly encountered two-qubit quantum gates, such as the CNOT gate and the SWAP gate, are not separable due to the above argument.

**Partial transpose of linear maps**

When evaluating the change of logarithmic negativity after the implementation of a noise channel, one has to connect the partial transpose of the output state and the input state. In the following lemma, we prove that such a connection can be built in terms of the partial transpose of the linear map.

**Lemma.** For a linear map \( N \) on a bipartite system \( \mathcal{A} \otimes \mathcal{B} \), the partial transpose of the output operator satisfies
\[
\rho^\text{pt} = N^\text{Ts} (\rho^\text{Ts}),
\]
(49)
where the (partial) transpose of a linear map \( N \) is defined by the (partial) transpose of its Choi operator, i.e.,
\[
\Lambda_{N^\text{ts}} = \Lambda_N^{\text{ts}}.
\]
(50)

**Proof:** we first take the Schmidt decomposition of operators
\[
\rho_0 = \sum_k \alpha_k \rho_k^A \otimes \rho_k^B, \quad \Lambda_N = \sum_k \beta_k \Lambda_k^A \otimes \Lambda_k^B,
\]
then the partially transposed output operator is calculated in terms of the Choi operator as
\[
\rho^\text{pt} = \text{Tr}_\mathcal{A} \left[ \left( \rho^\text{Ts} \otimes I_B \right) \Lambda_N^{\text{Ts}} \right] = \text{Tr}_\mathcal{A} \left[ \left( \sum_i \alpha_i \rho_i^A \otimes \rho_i^B \right) \left( \Lambda_i^A \otimes \Lambda_i^B \right) \right]^{\text{Ts}} = \sum_i \alpha_i \beta_i \left( \rho_i^\text{Ts} \otimes \rho_i^\text{Ts} \right) \left( \Lambda_i^\text{Ts} \otimes \Lambda_i^\text{Ts} \right) = \sum_i \alpha_i \beta_i \left( \rho_i^A \otimes I_B \right) \left( \Lambda_i^A \otimes I_B \right) = \Lambda_N^\text{Ts} \left( \rho_0^\text{Ts} \right),
\]
(52)
For the third equality, the partial transpose will introduce a permutation between \( \rho_i^A \otimes I_B \) and \( \Lambda_i^A \), while the permutation in the fourth equality comes from the partial trace of \( \alpha \) part and the identity of \( \tau \) part.

**DATA AVAILABILITY**

The datasets generated and analyzed during the current study are available from the corresponding author upon reasonable request.

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AUTHOR CONTRIBUTIONS

Y.G. conceived, designed, and performed the numerical experiments. Y.G. and S.Y. analyzed the data and wrote the paper. S.Y. contributed analysis tools.

COMPETING INTERESTS

The authors declare no competing interests.

ADDITIONAL INFORMATION

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