Associated Lah numbers and $r$-Stirling numbers

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Abstract

We introduce the associated Lah numbers. Some recurrence relations and convolution identities are established. An extension of the associated Stirling and Lah numbers to the $r$-Stirling and $r$-Lah numbers are also given. For all these sequences we give combinatorial interpretation, generating functions, recurrence relations, convolution identities. In the sequel, we develop a section on nested sums related to binomial coefficient.

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1 Introduction

The Stirling numbers of the first and second kind, denoted respectively $\left[ \begin{array}{c} n \\ k \end{array} \right]$ and $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$, are defined by

\[
x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] x^k,
\]

and

\[
x^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x(x-1)\cdots(x-k+1).
\]

It is well known that $\left[ \begin{array}{c} n \\ k \end{array} \right]$ is the number of permutations of the set $Z_n := \{1, 2, \ldots, n\}$ with $k$ cycles and that $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the number of partitions of the set $Z_n$ into $k$ non empty subsets [17 Ch. 5], [23 Ch. 4].

The Lah numbers $\left[ \begin{array}{c} n \\ k \end{array} \right]$ (Stirling numbers of the third kind), see [19 pp. 44], are defined as the sum of products of the Stirling number of the first kind and the Stirling numbers of the second kind

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] = \sum_{j=k}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] \left\{ \begin{array}{c} j \\ k \end{array} \right\},
\]

and count the number of partitions of the set $Z_n$ into $k$ ordered lists. According to [1] and [2] they satisfy

\[
x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] x(x-1)\cdots(x-k+1),
\]
see for instance [3, eq 8].

Broder [12] gives a generalization of the Stirling numbers of the first and second kind the so-called r-Lah numbers of the first and second kind, denoted respectively \([n\!\!\,k]_r\) and \({n\!\!\,k}\}_r\), by adding restriction on the elements of \(Z_n\) : the \([n\!\!\,k]_r\) is the number of permutations of the set \(Z_n\) with \(k\) cycles such that the \(r\) first elements are in distinct cycles and the \({n\!\!\,k}\}_r\) is the number of partitions of the set \(Z_n\) into \(k\) subsets such that the \(r\) first elements are in distinct subsets. The \(r\-\)Lah numbers \([n\!\!\,k]_r\), see [3], count the number of partitions of the set \(Z_n\) into \(k\) ordered lists such that the \(r\) first elements are in distinct lists.

These three sequences satisfy respectively the following recurrence relations

\[
\begin{align*}
[n\!\!\,k]_r &= \binom{n-1}{k-1}_r + (n-1) \binom{n-1}{k}_r, \\
{n\!\!\,k}_r &= \binom{n-1}{k-1}_r + k \binom{n-1}{k}_r, \\
[n\!\!\,k]_r &= \binom{n-1}{k-1}_r + (n+k-1) \binom{n-1}{k}_r.
\end{align*}
\]

with \([n\!\!\,k]_r = {n\!\!\,k}_r = \binom{n\!\!\,k}{r} = \delta_{n,k}\) for \(k = r\), where \(\delta\) is the Kronecker delta, and \([n\!\!\,k]_r = {n\!\!\,k}_r = \binom{n}{k}_r = 0\) for \(n < r\).

For \(r = 1\) and \(r = 0\), these numbers coincide with the classical Stirling numbers of both kinds and with the classical Lah numbers.

Comtet [17, pp. 222] define an other generalization of the Stirling numbers of both kinds by adding a restriction on the number of elements by cycle or subset and call them, for \(s \geq 1\), the \(s\)-associated Stirling numbers of the first kind \(\gamma^s_k\) and of the second kind \(\{\!\!\,s\!\!\,k\}\). The \(\gamma^s_k\) is the number of permutations of the set \(Z_n\) with \(k\) cycles such that, each cycle has at least \(s\) elements. The \(\{\!\!\,s\!\!\,k\}\) is the number of partitions of the set \(Z_n\) into \(k\) subsets such that each subset has at least \(s\) elements. They have, each one, an explicit formula, see for instance [20, Eq 4.2, Eq 4.9]:

\[
\begin{align*}
[n\!\!\,k]^{(s)} &= \frac{n!}{k!} \sum_{i_1+i_2+\cdots+i_k=n} \frac{1}{i_1!i_2!\cdots i_k!}, \\
{n\!\!\,k}^{(s)} &= \frac{n!}{k!} \sum_{i_1+i_2+\cdots+i_k=n} \frac{1}{i_1!i_2!\cdots i_k!}.
\end{align*}
\]

The generating functions are respectively

\[
\begin{align*}
\sum_{n\geq sk} [n\!\!\,k]^{(s)} \frac{x^n}{n!} &= \frac{1}{k!} \left( -\ln(1-x) - \sum_{i=1}^{s-1} \frac{x^i}{i} \right)^k, \\
\sum_{n\geq sk} \{n\!\!\,k\}^{(s)} \frac{x^n}{n!} &= \frac{1}{k!} \left( \exp(x) - \sum_{i=0}^{s-1} \frac{x^i}{i!} \right)^k.
\end{align*}
\]

For \(s = 2\), these numbers are reduced to the specific associated Stirling numbers of both kinds, see for instance [23, pp. 73].

Note that, from (11) and (9), for \(n = sk\), we get

\[
[n\!\!\,k]^{(s)} = \frac{(sk)!}{k!s^k} \quad \text{and} \quad \{n\!\!\,k\}^{(s)} = \frac{(sk)!}{k!(s)!k}.
\]

Ahuja and Enneking [11] give a generalization of the Lah numbers called the associated Lah numbers using an analytic approach. In Sloane [24, A076126], we have a definition of the associated Lah numbers \([n\!\!\,k]^{(2)}\) as
the number of partitions of the set $Z_n$ into $k$ ordered lists such that each list has at least 2 elements. They satisfy the following explicit formula

\[ \binom{n}{k}^{(2)} = \frac{n!}{k!} \binom{n-k-1}{k-1}, \tag{12} \]

and have the double generating function

\[ \sum_{n \geq 2} \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{k}^{(2)} y^k x^n k! = \exp \left( y \frac{x^2}{1-x} \right) - 1, \tag{13} \]

they consider $k \geq 1$, which means there is at least one part.

Hsu and Shiue [22] defined a Stirling-type pair \{S1, S2\} as a unified approach to the Stirling numbers, this approach generalize degenerate Stirling numbers [13], Weighted Stirling numbers [14] [15], r-Whitney numbers [9] [10] and many other ones. The authors and Belkhir in [4] and the authors in [7] give a combinatorial approach to special cases of the Stirling-type pair. Howard [21] extend the associated generalization to the Weighted Stirling numbers. Note that the Stirling-type pair does not generalize the associated Stirling numbers. Motivated by this, we introduce and develop the s-associated Lah numbers and the s-associated r-Stirling numbers.

In section 2, we define the s-associated Lah numbers $\binom{n}{k}_{s}$, $n \geq sk$, by a combinatorial approach analogous to Comtet’s generalization. We derive an explicit formula, a triangular recurrence relation, a combinatorial identity and some generating functions. We study, in section 3, some nested sums related to binomial coefficients in order to develop, in section 4, a generalization of the Stirling numbers of the three kinds using the two restrictions (Broder’s and Comtet’s ones), we call them respectively the s-associated r-Stirling numbers of the first kind $\binom{n}{k}_{r}$, the s-associated r-Stirling numbers of the second kind $\binom{n}{k}_{r}^{(s)}$ and the s-associated r-Lah numbers $\binom{n}{k}_{r}^{(s)}$. We give some recurrence relations and combinatorial identities in sections 5 and 6. Cross recurrences and convolution identities are established in sections 7 and 8. In section 9, we propose some generating functions of the s-associated r-Stirling numbers.

2 The s-associated Lah numbers

We start by introducing the s-associated Lah numbers.

**Definition 1** The s-associated Lah number, denoted by $\binom{n}{k}_{s}$, is the number of partitions of $Z_n$ into $k$ order lists such that each list contains at least $s$ elements.

**Theorem 2** The s-associated Lah numbers obey to the following 'triangular' recurrence relation, for $n \geq sk$,

\[ \binom{n}{k}_{s} = \binom{n-1}{s-1}_{s} \binom{n-s}{k-1}_{s} + (n+k-1) \binom{n-1}{k}_{s}, \tag{14} \]

with $\binom{n}{0}_{s} = \delta_{n,0}$ for $k = 0$, where $\delta$ is the Kronecker delta, and $\binom{n}{k}_{s} = 0$ for $n < sk$

**Proof.** Let us consider the $n^{th}$ elements, if it belongs to a list containing exactly $s$ elements, so we have $\binom{n-1}{s-1}_{s}$ ways to choose the remaining $(s-1)$ elements and $s!$ ways to order them into the cited list, then distribute the $(n-s)$ remaining elements into the $(k-1)$ remaining lists such that each list have at least $s$ elements and we have $\binom{n-s}{k-1}_{s}$ ways to do it. Thus, we get $\binom{n-1}{s-1}_{s} \binom{n-s}{k-1}_{s}$ possibilities. Else, we consider all the possibilities of ordering $(n-1)$ elements into $k$ lists under the usual condition which can be done by $\binom{n-1}{k}_{s}$ ways, then add the $n^{th}$ elements next to an other and we have $n-1$ possibilities, or as head of each list and we have $k$ possibilities, this gives $(n+k-1) \binom{n-1}{k}_{s}$ possibilities. \[\square\]
For \( s = 1 \) and \( s = 2 \), we get Lah numbers and associated Lah numbers respectively.

For \( s = 3 \), we obtain the following table, for \( n \leq 15 \),

| \( n \backslash k \) | 1 | 2 | 3 | 4 | 5 |
|-----|----|----|----|----|----|
| 3   | 6  |    |    |    |    |
| 4   | 24 |    |    |    |    |
| 5   | 120|    |    |    |    |
| 6   | 720| 360|    |    |    |
| 7   | 5040| 5040|    |    |    |
| 8   | 40320| 60480|    |    |    |
| 9   | 362880| 725760| 60480|    |    |
| 10  | 3628800| 9072000| 1814400|    |    |
| 11  | 39916800| 119750400| 39916800|    |    |
| 12  | 479001600| 1676505600| 798336000| 199584000|    |
| 13  | 6227020800| 24908083200| 15567552000| 1037836800|    |
| 14  | 87178291200| 392302310400| 305124019200| 36324288000|    |
| 15  | 1307674368000| 6538371840000| 6102480384000| 1089728640000| 108972864000 |

The following result gives an explicit formula for the \( s \)-associated Lah numbers according to identities (17) and (15) for the \( s \)-associated Stirling numbers of both kinds.

**Theorem 3** Let \( s, k \) and \( n \) be nonnegative integers such that \( n \geq sk \), we have

\[
\binom{n}{k}^{(s)} = \frac{n!}{k!} \binom{n - (s - 1)k - 1}{k - 1}.
\]  

**(Proof 1.)** We order \( n \) elements on \( k \) ordered lists such that, each list contains at least \( s \) elements: first, we suppose that the lists are labeled \( 1, \ldots, k \) and for each list \( j \) we choose \( (i_j + s) \) \((0 \leq i_j \leq n - s)\) elements, we have \( \binom{n}{i_1 + s, i_2 + s, \ldots, i_k + s} \) possibilities to constitute the \( k \) groups. The arrangement of the \( j \)th subset gives \((i_j + s)!\) possibilities. It gets \( \sum_{i_1 + i_2 + \cdots + i_k = n - sk} \binom{n}{i_1 + s, i_2 + s, \ldots, i_k + s}(i_1 + s)!(i_2 + s)! \cdots (i_k + s)! = n! \binom{n - (s - 1)k - 1}{k - 1} \), we divide by \( k! \) to unlabeled the lists. \( \square \)

**Proof 2.** First we choose \( k \) elements to identify the \( k \) lists with \( \binom{n}{k} \) possibilities, then we choose \( k \) groups of \( s - 1 \) elements to get the condition of having \( s \) elements by list and we have \( \binom{n - k}{s - 1} \) possibilities for the first list, and \( \binom{n - k - (s - 1)}{s - 1} \) possibilities for the second one, and so on ... the last list have \( \binom{n - k - (s - 1)(k - 1)}{s - 1} \) possibilities. So we get \( \binom{n - k}{s - 1} \binom{n - k - (s - 1)}{s - 1} \cdots \binom{n - k - (s - 1)(k - 1)}{s - 1} = \binom{n - k}{s - 1} \cdots \binom{n - k}{s - 1} \binom{n - k}{s - 1} \) possibilities. We affect the remaining \( n - sk \) elements to the lists and we have \( k \) ways for the first element, \( k + 1 \) ways for the second one, and so on ... the last one have \( k + n - sk - 1 = n - (s - 1)k - 1 \). So, we get \( k(k + 1) \cdots (n - (s - 1)k - 1) = \binom{n - (s - 1)k - 1}{k - 1} \) possibilities. The result follows. \( \square \)

**Remark 4** For \( n = sk \), we get the following according to relations given by (11)

\[
\binom{sk}{k}^{(s)} = \frac{(sk)!}{k!}.
\]  

Comparing to (14), an other recurrence relation, with rational coefficients, can be deduced form the explicit formula (15) as follows.

**Theorem 5** The \( s \)-associated Lah numbers satisfy the following recurrence relation

\[
\binom{n}{k}^{(s)} = \frac{n!}{(n - s)!k} \binom{n - s}{k - 1}^{(s)} + n \binom{n - 1}{k}^{(s)}.
\]
Proof. Using Pascal’s formula and relation (15), we get the result. □

Note that for \( s = 1 \), we get the relation given by the authors [8, eq 7] when \( r = 0 \). The \( s \)-associated Lah numbers can be expressed as a Vandermonde type formula as follows.

\[
\begin{align*}
\binom{n}{k}^{(s)} &= n! \sum_{i=0}^{p} \frac{(k-i)!}{(n-p-(s-1)i)!} \left( \binom{n-(s-1)i-p}{k-i} \right)^{(s)}.
\end{align*}
\] (18)

Proof. Using the explicit formula (15) and the Vandermonde formula, we get the result. □

The special case \( s = 1 \) gives the identity given by the authors [8, eq 6] when \( r = 1 \).

The \( s \)-associated Lah numbers satisfy the following vertical recurrence relation.

\[
\begin{align*}
\binom{n}{k}^{(s)} &= n! \sum_{i=0}^{p} \frac{(k-i)!}{(n-p-(s-1)i)!} \left( \binom{n-(s-1)i-p}{k-i} \right)^{(s)}.
\end{align*}
\] (18)

Proof. Using the explicit formula (15) and the Vandermonde formula, we get the result. □

The special case \( s = 1 \) gives the identity given by the authors [8, eq 6] when \( r = 1 \).

The \( s \)-associated Lah numbers satisfy the following vertical recurrence relation.

\[
\begin{align*}
\binom{n}{k}^{(s)} &= n! \sum_{i=0}^{p} \frac{(k-i)!}{(n-p-(s-1)i)!} \left( \binom{n-(s-1)i-p}{k-i} \right)^{(s)}.
\end{align*}
\] (18)

Proof. Using the explicit formula (15) and the Vandermonde formula, we get the result. □

The special case \( s = 1 \) gives the identity given by the authors [8, eq 6] when \( r = 1 \).
Let us consider the \((k - 1)\) first lists, they contain \(i\) \((s(k - 1) \leq i \leq n - s)\) elements. So, we choose the \(i\) elements and we have \(\binom{n - i}{i}\) ways to do it, and constitute the \(k - 1\) lists such that each list have at least \(s\) elements, which can be done by \(\binom{i}{s}\) ways, then order the \((n - i)\) remaining elements in a list with \((n - i)!\) possibilities. We conclude by summing over \(i\). \(\square\)

The exponential generating function of the \(s\)-associated Lah numbers is given by the following. It is a complement list to (9) and (10).

**Theorem 8** Let \(n, k\) and \(s\) be integers, we have

\[
\sum_{n \geq k} \binom{n}{k} \frac{x^n}{n!} = \frac{1}{k!} \left( \frac{x^s}{1-x} \right)^k.
\]  

**Proof.** Using the explicit formula (15), with the following identity for \(x \in \mathbb{N}\), see for instance [18],

\[
\sum_{n \geq 0} \frac{(n + x)}{x} t^n = \left( \frac{1}{1-t} \right)^{x+1},
\]

we get the result. \(\square\)

According to identity (13), the double generating function is given by

**Theorem 9** We have

\[
\sum_{n \geq k} \sum_{k=0}^{\lfloor n/s \rfloor} \binom{n}{k} y^k \frac{x^n}{n!} = \exp \left\{ \frac{x^s}{1-x} \right\}.
\]  

**Proof.** Interchanging the order of summation and using equation (20), we get the result. \(\square\)

3 Nested sums related to binomial coefficients

In this section, we evaluate some symmetric functions. We start by the following result.

**Lemma 10** Let \(\alpha\) and \(\beta\) be integers such that \(\beta \geq \alpha\). We have

\[
\sum_{n \geq 0} \binom{n+\alpha}{\beta} z^n = \frac{z^{\beta-\alpha}}{(1-z)^{\beta+1}}.
\]  

**Proof.** \(\sum_{n \geq 0} \binom{n+\alpha}{\beta} z^n = \left( \sum_{n \geq 0} \binom{n+\beta}{\beta} z^n \right) z^{\beta-\alpha} = \frac{z^{\beta-\alpha}}{(1-z)^{\beta+1}}.\) \(\square\)

The following result seems to be nice as an independent one.

**Theorem 11** Let \(\alpha_1, \ldots, \alpha_r, \alpha, \beta_1, \ldots, \beta_r, \beta, k_1, \ldots, k_r\) and \(k\) be integers such that \(\alpha_1 + \cdots + \alpha_r = \alpha, \beta_1 + \cdots + \beta_r = \beta\) and \(k_1 + \cdots + k_r = k\) with \(\beta_i \geq \alpha_i\). The following identity holds

\[
\sum_{k_1+\cdots+k_r=k} \binom{k_1+\alpha_1}{\beta_1} \cdots \binom{k_r+\alpha_r}{\beta_r} = \binom{k+\alpha+r-1}{\beta+r-1}.
\]  

**Proof.** By induction over \(r\), we get the result. So It suffices to do the proof for \(r = 2\). Thus, we have to establish

\[
\sum_{k_1+k_2=k} \binom{k_1+\alpha_1}{\beta_1} \binom{k_2+\alpha_2}{\beta_2} = \binom{k+\alpha+1}{\beta+1}.
\]  

(24)
We consider the following product \( \sum_{n \geq 0} \sum_{k_1 + k_2 + \cdots + k_r = n} (k_1 + \alpha_1)(k_2 + \alpha_2) \cdot \cdots \cdot (k_r + \alpha_r) t^n = \left( \sum_{n \geq 0} \binom{k_1 + \alpha_1}{\beta_1} t^n \right) \left( \sum_{n \geq 0} \binom{k_2 + \alpha_2}{\beta_2} t^n \right) \) using (22), we get \( \frac{e^{t^1} - e^{-1}}{(1-t)^{\beta_1}} \cdot \frac{e^{t^{2}} - e^{-1}}{(1-t)^{\beta_2}} = \sum_{k} \binom{k + \alpha + 1}{\beta + 1} t^k \). □

As a consequence, we evaluate the sum of all possible integer products having the same summation.

**Corollary 12** For \( \alpha_i = 0 \) and \( \beta_i = 1 \) we get

\[
\sum_{k_1 + \cdots + k_r = n} k_1 k_2 \cdots k_r = \binom{n + r - 1}{2r - 1}.
\] (25)

The above identity can be interpreted as the number of ways to choose \( r \) leaders of \( r \) groups constituted from \( n \) persons: we choose one person of each group and we have \( \binom{k_1}{1} \cdots \binom{k_r}{r} \) ways to do it. This is equivalent to choose \( r \) persons and \((r-1)\) separators from the \( n \) persons and the \( r - 1 \) separators and we have \( \binom{n + r - 1}{2r - 1} \) ways to do it.

Now, we are able to produce a general result. Also, it will be used to establish the next theorem.

**Corollary 13** Let \( r, p \) and \( k \) be integers such that \( r \geq p \), we have

\[
\sum_{k_1 + \cdots + k_p + \cdots + k_r = n} k_1 k_2 \cdots k_p = \binom{n + r - 1}{r + p - 1}.
\] (26)

**Proof.**

\[
\sum_{k_1 + \cdots + k_p + \cdots + k_r = k} k_1 k_2 \cdots k_p = \sum_{m=0}^{k} \binom{k - m + \alpha_1}{\beta_1} \cdot \sum_{k_1 + \cdots + k_p + \cdots + k_r = m} 1
\]

using identity (25) and \( \sum_{i_1 + i_2 + \cdots + i_r = m} 1 = \binom{m + r - 1}{r - 1} \) we get

\[
\sum_{k_1 + \cdots + k_p + \cdots + k_r = k} k_1 k_2 \cdots k_p = \sum_{m=0}^{k} \binom{m + p - 1}{2p - 1} \binom{k - m + r - p - 1}{r - p - 1},
\]

applying relation (24), we get the result. □

Now, we are able to evaluate the sum of all products of \( k \) terms, all translated by \( \alpha \), and having a fixed summation.

**Theorem 14** We have

\[
\sum_{i_1 + \cdots + i_k = n} (i_1 + \alpha)(i_2 + \alpha) \cdots (i_k + \alpha) = \sum_{j=0}^{k} \binom{k}{j} \binom{n + k - 1}{n - j} \alpha^{k - j}.
\] (27)

**Proof.** We have

\[
\sum_{i_1 + \cdots + i_k = n} (i_1 + \alpha) \cdots (i_k + \alpha) = \sum_{j=0}^{k} \binom{k}{j} I_{k, k-j} \alpha^j,
\]

where \( I_{k,j} := \sum_{i_1 + i_2 + \cdots + i_k = n} i_1 i_2 \cdots i_j \) and from (20), we get the result. □

This nice result will be used to evaluate the explicit formula of the \( s \)-associated \( r \)-Stirling numbers which are introduced in the following section.
4 The \(s\)-associated \(r\)-Stirling numbers of the both kinds and the \(s\)-associated \(r\)-Lah numbers

Now, we introduce the \(s\)-associated \(r\)-Stirling numbers of the both kinds and the \(s\)-associated \(r\)-Lah numbers.

**Definition 15** The \(s\)-associated \(r\)-Stirling numbers of the first kind count the number of permutations of the set \(Z_n\) with \(k\) cycles such that the \(r\) first elements are in distinct cycles and each cycle contains at least \(s\) elements.

The \(s\)-associated \(r\)-Stirling numbers of the second kind count the number of partitions of the set \(Z_n\) into \(k\) subsets such that the \(r\) first elements are in distinct subsets and each subset contains at least \(s\) elements.

The \(s\)-associated \(r\)-Lah numbers, called also the \(s\)-associated \(r\)-Stirling numbers of the third kind, count the number of partitions of the set \(Z_n\) into \(k\) ordered lists such that the \(r\) first elements are in distinct lists and each list contains at least \(s\) elements.

Here is given, for each kind, the table for \(r = s = 2\).

| \(n\) | \(2\) | \(3\) | \(4\) | \(5\) | \(6\) | \(7\) |
|------|------|------|------|------|------|------|
| 4    | 2    |      |      |      |      |      |
| 5    | 12   |      |      |      |      |      |
| 6    | 72   | 12   |      |      |      |      |
| 7    | 480  | 160  |      |      |      |      |
| 8    | 3600 | 1740 | 90   |      |      |      |
| 9    | 3024 | 18648| 2100 |      |      |      |
| 10   | 282240| 207648| 35840| 840  |      |      |
| 11   | 2903040| 2446848| 560448| 3024 |      |      |
| 12   | 32659200| 30702240| 8641080| 743400| 9450 |      |
| 13   | 399168000| 410731200| 135519120| 15935920| 485100|      |
| 14   | 5269017600| 5852755280| 2194121952| 324416400| 16216200| 124740 |
| 15   | 74724249600| 88663610880| 36941553792| 6522721920| 455975520| 8648640 |
| 16   | 1133317785600| 1424644865280| 649046990592| 132205465392| 11835944120| 377116740 |

**Table 1:** Some values for the 2-associated 2-Stirling numbers of the first kind

| \(n\) | \(2\) | \(3\) | \(4\) | \(5\) | \(6\) | \(7\) | \(8\) |
|------|------|------|------|------|------|------|------|
| 4    | 2    |      |      |      |      |      |      |
| 5    | 6    |      |      |      |      |      |      |
| 6    | 14   | 12   |      |      |      |      |      |
| 7    | 30   | 80   |      |      |      |      |      |
| 8    | 62   | 360  | 90   |      |      |      |      |
| 9    | 126  | 1372 | 1050 |      |      |      |      |
| 10   | 254  | 4788 | 7700 | 840  |      |      |      |
| 11   | 510  | 15864| 45612| 15120|      |      |      |
| 12   | 1022 | 50880| 239190| 163800| 9450 |      |      |
| 13   | 2046 | 159764| 1161270| 1389080| 242550|      |      |
| 14   | 4094 | 494604| 5353392| 10182480| 3638250| 124740|      |
| 15   | 8190 | 1516528| 23809200| 67822040| 41771730| 4324320|      |
| 16   | 16382| 4619160| 103099994| 422534112| 407246840| 85765680| 1891890 |
| 17   | 32766| 14004876| 438124050| 2507785280| 3555852300| 1280178900| 85135050 |      |

**Table 2:** Some values of the 2-associated 2-Stirling numbers of the second kind
The s-associated r-Stirling numbers of the three kinds have the following explicit formulas.

**Theorem 16** For \( n \geq sk \) and \( k \geq r \), we have

\[
\begin{align*}
\left[ \begin{array}{c} n \\ k \end{array} \right]_{r}^{(s)} &= \frac{(n-r)!}{(k-r)!} \sum_{m=0}^{n-sk} \frac{1}{r!} \sum_{i_{1}+\ldots+i_{r}=n-sk-m} (m+r-1) \cdots (i_{r}+s) \cdots (i_{k}+s), \\
\left\{ \begin{array}{c} n \\ k \end{array} \right\}_{r}^{(s)} &= \frac{(n-r)!}{(k-r)!} \sum_{i_{1}+\ldots+i_{k}=n-sk} \frac{1}{r!} (i_{1}+s-1)! \cdots (i_{r}+s-1)! (i_{r+1}+s)! \cdots (i_{k}+s)!, \\
\left[ \begin{array}{c} n \\ k \end{array} \right]_{r}^{(s)} &= \frac{(n-r)!}{(k-r)!} \sum_{j=0}^{r} \binom{r}{j} (n-1)^{j} \sum_{j+1}^{k} (n-r-1) \cdots k_{j-1} (k-j+1)^{k-j},
\end{align*}
\]

**Proof.** We first prove the identity (28). To constitute a partition of \( Z_{n} \) into \( k \) parts such that each part has at least \( s \) elements and the \( r \) first elements are in distinct parts, we proceed as follows: we put the \( r \) first elements in \( r \) parts (one by part). Now we partition the \( n-r \) remaining elements into \( k \) parts such that \( r \) parts have at least \( s-1 \) elements and \( k-r \) parts have at least \( s \) elements, and we have \( \frac{(n-r)!}{(k-r)!} \binom{n-r}{i_{1}, i_{2}, \ldots, i_{k}} \) ways to do it, with \( i_{j} \geq s-1 \) for \( j = 1, \ldots, r \) and \( i_{j} \geq s \) for \( j = r+1, \ldots, k \), which gives identity (28).

With the same specifications used to establish relation (29), to count the number of permutations of \( Z_{n} \) into \( k \) cycles it suffices to constitute the cycles by considering all the possible arrangements in the parts and we have \( \frac{(n-r)!}{(k-r)!} \binom{n-r}{i_{1}, i_{2}, \ldots, i_{k}} \) ways. So we can write:

\[
\begin{align*}
\left[ \begin{array}{c} n \\ k \end{array} \right]_{r}^{(s)} &= \frac{1}{(k-r)!} \sum_{i_{1}+\ldots+i_{k}=n-r} \frac{1}{i_{1}! \cdots i_{k}!} (n-r-1)! \cdots (i_{r}+1)! \cdots (i_{k}-1)!, \\
&= \frac{(n-r)!}{(k-r)!} \sum_{m=r(s-1)}^{n-r} \frac{1}{i_{r+1} \cdots i_{k}!} \sum_{i_{j} \geq s} \frac{1}{i_{1} \cdots i_{r}!} \sum_{i_{r+1} \cdots i_{k}!} 1, \\
&= \frac{(n-r)!}{(k-r)!} \sum_{m=0}^{n-sk} \frac{1}{(i_{r+1}+\cdots+i_{k}+n-sk-m)!} \sum_{i_{1}+\cdots+i_{r}=m} 1,
\end{align*}
\]

finally using \( \sum_{i_{1}+\cdots+i_{r}=m} 1 = \binom{m+r-1}{r-1} \), we get identity (29).
The same approach works, to constitute partitions of \( Z_n \) into \( k \) ordered lists we have to consider the arrangement in the parts and we have \( (i_1 + 1)! (i_2 + 1)! \cdots (i_r + 1)! i_{r+1}! \cdots i_k! \) ways to do it. Thus we get

\[
\binom{n}{k}_r^{(s)} = \frac{1}{(k-r)!} \sum_{i_1 + \cdots + i_k = n-r} \binom{n-r}{i_1, \ldots, i_k} (i_1 + 1)! \cdots (i_r + 1)! i_{r+1}! \cdots i_k!
\]

using relation (27), we get

\[
= \frac{(n-r)!}{(k-r)!} \sum_{m=s(k-r)}^{n-r} \sum_{i_j \geq s} (i_1 + 1) \cdots (i_r + 1) \sum_{i_{r+1} + \cdots + i_k = m} 1
\]

using relation (24), we get identity (30).

From (31), we can write a second kind explicit formula according to (27) and generalizing relation (15).

\[
\binom{n}{k}_r^{(s)} = \frac{(n-r)!}{(k-r)!} \sum_{j=0}^{r} \binom{r}{j} \sum_{m=0}^{n-sk-m} \binom{m + k - r - 1}{k - r - 1} \binom{n - sk - m - r - 1}{r - 1 + j} s^{r-j},
\]

using relation (27), we get

\[
\binom{n}{k}_r^{(s)} = \frac{(n-r)!}{(k-r)!} \sum_{i_1 + i_2 + \cdots + i_k = n-sk} (i_1 + s) (i_2 + s) \cdots (i_r + s).
\]

From (32), we can write a second kind explicit formula according to (27) and generalizing relation (15).

\[
\binom{n}{k}_r^{(s)} = \frac{1}{(k-r)!} \sum_{i_1 + i_2 + \cdots + i_k = n-sk} (i_1 + s) (i_2 + s) \cdots (i_r + s).
\]

The precedent theorem works for \( k = r \). Furthermore, the identities are more explicit.

**Remark 17** For \( k = r \), we get respectively

\[
\binom{n}{r}_r^{(s)} = \frac{(n-r)! (n-r(s-1) - 1)}{r - 1},
\]

\[
\binom{n}{r}_r^{(s)} = \sum_{i_1 + i_2 + \cdots + i_k = n-r} \binom{n-r}{i_1, i_2, \ldots, i_r},
\]

\[
\binom{n}{r}_r^{(s)} = \frac{(n-r)!}{i_0 + 1} \binom{n-(s-1)r - 1}{r + i - 1} s^{r-i}.
\]

The following special values can be easily computed, extending relations given by relations (11) and (16)

\[
\binom{sk}{k}_r^{(s)} = \frac{(n-r)!}{(k-r)!} s^k, \quad (36)
\]

\[
\binom{sk}{k}_r^{(s)} = \frac{(n-r)!}{(k-r)!} s^{(s-1)! r s^{k-r}}, \quad (37)
\]

\[
\binom{sk}{k}_r^{(s)} = \frac{(n-r)!}{(k-r)!} s^r. \quad (38)
\]

Here is given an other explicit formula of the \( s \)-associated \( r \)-Lah numbers. This one is more interesting than relation (32). It is evaluated using one summation

**Theorem 18** Let \( n, k, r \) and \( s \) be nonnegative integers such that \( k \geq r \) and \( n \geq sk \), we have

\[
\binom{n}{k}_r^{(s)} = \frac{(n-r)!}{(k-r)!} \sum_{j=0}^{r} \binom{r}{j} \binom{n + j - (s-1)k - 1}{k + j - 1} (s-1)^{r-j}.
\]
Proof. To constitute the \( k \) lists we use the \( r \) first elements which are supposed in different lists to identify the \( r \) first lists and we choose \( k - r \) elements form the remaining elements, with \( \binom{n-r}{k-r} \) possibilities, as head list of the \( k - r \) remaining lists. Now to retch the condition that in each list we have at least \( s \) elements, we constitute \( k \) groups of \( (s - 1) \) elements form the \( n - k \) remaining elements and we have \( \binom{n-k}{s-1,n-s} \) ways to do it, and consider all the permutations of each group so we get \( (s - 1)! \) possibilities. Now, for the \( r \) first elements we suppose that \( j \) of them are head lists so we choose them with \( \binom{k}{j} \) ways and order the \( r - j \) elements after an element of each group and we have \( (s - 1)r-j \) possibilities. It remains to affect the \( n - sk \) remaining elements, so the first one has \( (k + j) \) possibilities (\( k \): at the end of each lists or before the \( j \) suppose head lists), the second one have \( (k + j + 1) \) possibilities (one possibilities added by the previews element) and so one . . . the last element have \( (k + j) + (n - sk - 1) = n + j - (s - 1)k - 1 \) possibilities. This gives \( \frac{(n+j-(s-1)k-1)!}{(k+j-1)!} = (k+j)\cdots(n+j-(s-1)k-1) \) possibilities. Summing over all possible values of \( j \) we get \( \frac{\binom{n-k}{(s-1)n-s}}{\binom{n-k}{n-s}} \sum_{j=0}^{r} \binom{r}{j} \frac{(n+j-(s-1)k-1)!}{(k+j-1)!} (s-1)^{r-j} \) which, after simplification, gives the result. \( \square \)

Note that the explicit formula of the \( s \)-associated Lah numbers [15] is obtained form [39] for \( r = 0 \) and \( r = 1 \). Also, for \( s = 1 \), we get the explicit formula of the \( r \)-Lah numbers [3] Eq 3]

From [39] and [30] we can state the following, which is very nice in terms of identities related to binomial coefficients.

**Proposition 19** We have

\[
\sum_{j=0}^{r} \binom{r}{j} \binom{n+k+j-1}{n} (s-1)^{r-j} = \sum_{j=0}^{r} \binom{r}{j} \binom{n-(s-1)k-1}{k+j-1} (s-1)^{r-j},
\]

(40)

From [41] and [30] we get a second expression, dual to relation [27].

**Proposition 20** we have

\[
\sum_{i_1+i_2+\ldots+i_k=n} (i_1+s) (i_2+s) \cdots (i_r+s) = \sum_{j=0}^{r} \binom{r}{j} \binom{n+k+j-1}{n} (s-1)^{r-j}
\]

5 Recurrence relations

The \( s \)-associated \( r \)-Stirling numbers satisfy recurrence relations as the regular \( s \)-associated Stirling numbers, using three terms of two triangles: the \((r-1)\)-Stirling triangle and the \( r \)-Stirling triangle.

![Triangular recurrence relation](image)

Figure 3: Triangular recurrence relation given the value of the black element as linear combination of the values of the three others, for the \( s \)-associate \( r \)-Stirling numbers of the three kinds

The recurrence relation of the \( s \)-associated \( r \)-Stirling numbers of the first kind is given as follows.
Theorem 21 Let \( r, k, s \), and \( n \) be nonnegative integers such that \( n \geq sk \) and \( k \geq r \), we have

\[
\begin{align*}
\binom{n}{k}_r^{(s)} &= \binom{n-r-1}{s-1}(s-1)!\binom{n-s-1}{k-1}_r^{(s)} + r\binom{n-r-1}{s-2}(s-2)!\binom{n-s-2}{k-1}_{r-1}^{(s)} + (n-1)\binom{n-1}{k}_r^{(s)}.
\end{align*}
\]

Proof. Let us consider the \( n \)-th element, if it belongs to a cycle containing exactly \( s \) elements not from the \( r \) elements, we have \( \binom{n-r-1}{s-1} \) ways to choose the \( (s-1) \) remaining elements and \((s-1)!\) ways to constitute the cycle, then distribute the \((n-s)\) remaining elements on the \((k-1)\) remaining cycles such that each cycle has at least \( s \) element and the \( r \) first elements are in distinct cycles, so we have \( \binom{n-s}{k-1}_r^{(s)} \) ways to do it. Thus we get \( \binom{n-r-1}{s-1}(s-1)!\binom{n-s}{k-1}_r^{(s)} \) possibilities. Else, if one of the \( r \) first elements belongs to the cycle, we have \( r \) ways to choose one of the \( r \) first elements, \( \binom{n-r-1}{s-2} \) ways to choose the remaining \( (s-2) \) ones and \((s-1)!\) ways to constitute the cycle, then distribute the \((n-s)\) remaining elements on the \((k-1)\) remaining cycles such that, in each cycle, there is at least \( s \) elements and the \( r-1 \) first elements are in distinct cycles, so we have \( \binom{n-s}{k-1}_{r-1}^{(s)} \) possibilities to do it. Thus we get \( r\binom{n-r-1}{s-2}(s-2)!\binom{n-s}{k-1}_{r-1}^{(s)} \) possibilities. Else, we consider all the permutations of \((n-1)\) elements with \( k \) cycles under the usual conditions which can be done by \( \binom{n-1}{k}_r^{(s)} \) ways, then add the \( n \)-th element to the \( k \) cycles and we have \((n-1)\) possibilities. \( \square \)

For \( s = 1 \) we get relation (41), and for \( r = 1 \) using Pascal’s formula we get the recurrence relation of the \( s \)-associated Stirling numbers of first kind [20, eq 4.8].

The \( s \)-associated \( r \)-Stirling numbers of the second kind satisfy the following recurrence relation.

Theorem 22 Let \( r, k, s \), and \( n \) be nonnegative integers such that \( n \geq sk \) and \( k \geq r \), we have

\[
\binom{n}{k}_r^{(s)} = \binom{n-r-1}{s-1}\binom{n-s-1}{k-1}_r^{(s)} + r\binom{n-r-1}{s-2}\binom{n-s-2}{k-1}_{r-1}^{(s)} + k\binom{n-1}{k}_r^{(s)}.
\]

Proof. Let us consider the \( n \)-th elements, if it belongs to a part containing exactly \( s \) elements not from the \( r \) first ones, so we have \( \binom{n-r-1}{s-1} \) ways to choose the remaining \( (s-1) \) elements and \( \binom{n-s}{k-1}_r^{(s)} \) ways to distribute the \((n-s)\) remaining elements on the \((k-1)\) remaining parts such that, the \( r \) first elements are in distinct parts, and each part, have at least \( s \) elements which gives \( \binom{n-s}{k-1}_r^{(s)} \) possibilities. Else, if one of the \( r \) first elements belongs to that part, we have \( r \) ways to choose it, and \( \binom{n-r-1}{s-2} \) ways to choose the remaining \( (s-2) \), then distribute the \((n-s)\) remaining elements on the \((k-1)\) remaining parts such that, the \( r-1 \) first elements are in distinct parts, and each part, have at least \( s \) elements which can be done by \( \binom{n-s}{k-1}_{r-1}^{(s)} \) ways. So we have \( r\binom{n-r-1}{s-2}\binom{n-s}{k-1}_{r-1}^{(s)} \) possibilities. Else, we consider all the partitions of \((n-1)\) elements on \( k \) blocks under the usual conditions which can be done by \( \binom{n-1}{k}_r^{(s)} \) ways, then add the \( n \)-th elements to the \( k \) cycles with \((n-1)\) possibilities. \( \square \)

For \( s = 1 \) we get relation (45), and for \( r = 1 \) using Pascal’s formula we get the recurrence relation of the \( s \)-associated Stirling numbers of the second kind [20, eq 4.1].

The \( s \)-associated \( r \)-Lah numbers satisfy the following recurrence relation.

Theorem 23 Let \( r, k, s \), and \( n \) be nonnegative integers such that \( n \geq sk \) and \( k \geq r \) we have

\[
\binom{n}{k}_r^{(s)} = \binom{n-r-1}{s-1}s!\binom{n-s-1}{k-1}_r^{(s)} + r\binom{n-r-1}{s-2}s!\binom{n-s}{k-1}_{r-1}^{(s)} + (n+k-1)\binom{n-1}{k}_r^{(s)}.
\]
least $s$ elements and the $r$ first elements are in distinct lists with $|_{k-1}^{n-s}(s)$ ways. Thus we get $\binom{n - s}{k - 1} s! |_{k-1}^{n-s}(s)$ possibilities. Else, if one of the $r$ first elements belongs to the list, we have $\binom{s}{1} = r$ ways to choose one of the $r$ first elements and $\binom{n-r-1}{s-2}$ ways to choose the remaining $(s-2)$ elements and $s!$ ways to constitute the list, then distribute the $(n-s)$ remaining elements into the $(k-1)$ remaining lists such that each list has at least $s$ elements and the $r-1$ first elements are in distinct lists and we have $|_{k-1}^{n-s}(s)$ possibilities. Else, if one of the $r$ first elements belongs to the $k$ lists and we have $(n-s)$ possibilities as a head list, which gives $(n+k-1)|_{k-1}^{n-s}(s)$ possibilities. \hfill $\square$

For $s = 1$ we get relation (6), and for $r = 1$ and using Pascal’s formula we get the recurrence relation (14).

6 Combinatorial identities or convolution relations

In this section, we establish some combinatorial identities for the $s$-associated $r$-Stirling numbers using a combinatorial approach. We can also consider them as convolution relations.

The next identity is an expression of $s$-associated $r$-Stirling numbers in terms of the $s$-associated $r'$-Stirling numbers with $r' \leq r$.

Theorem 24 Let $p, r, k$ and $n$ be nonnegative integers such that $p \leq r \leq k$ and $n \geq sk$, we have

$$\binom{n}{k}_r^{(s)} = \sum_{i=(s-1)p}^{n-p-s(k-p)} \frac{(n-r)!}{(n-r-i)!} \binom{i-p(s-2)-1}{r-1} \left[ \binom{n-p-i}{k-p} \right]_{r-p}^{(s)}.$$ (44)

Proof. Let us consider the $i$ ($(s-1)p \leq i \leq n-p-s(k-p)$) elements which belongs to the $p$ cycles containing the elements $1, \ldots, p$. We have $\binom{n-r}{i}$ possibilities to choose the $i$ elements and $i+p \binom{(s)}{p}_p$ ways to construct the corresponding cycles. The remaining $n-p-i$ elements must form the $k-p$ remaining cycles; this can be done in $\binom{n-p-i}{k-p}_{r-p}^{(s)}$ ways. Using equation (33) and summing for all $i$, we get the proof. \hfill $\square$

For $p = r$, we obtain an expression of the $s$-associated $r$-Stirling numbers of the first kind in terms of the regular $s$-associated Stirling numbers of the first kind

$$\binom{n}{k}_r^{(s)} = \sum_{i=(s-1)r}^{n-r-s(k-r)} \frac{(n-r)!}{(n-r-i)!} \binom{i-r(s-2)-1}{r-1} \left[ \binom{n-r-i}{k-r} \right]^{(s)}.$$ (45)

For $s = 1$, we obtain the equation given by Broder [12, eq 26] and for $r = 1$, we get a vertical recurrence relation for the classical $s$-associated Stirling numbers of the first kind

$$\binom{n}{k}_1^{(s)} = \sum_{i=s-1}^{n-s(k-1)-1} \frac{(n-1)!}{(n-i-1)!} \left[ \binom{n-i-1}{k-1} \right]^{(s)}.$$ (46)

Theorem 25 Let $p, r, k$ and $n$ be nonnegative integers such that $p \leq r \leq k$ and $n \geq sk$, we have

$$\binom{n}{k}_r^{(s)} = \sum_{i=p-r+s(k-p)}^{n-r-(s-1)p} \frac{(n-r)!}{(s-1)!} \binom{n-p(s-1)-r}{i} \binom{i+r-p}{k-p}_{r-p}^{(s)} p^{n-p(s-1)-r-i}.$$ (47)
Proof. Let us consider $p$ first elements ($p \leq r$), they constitute $p$ parts with $p(s - 1)$ elements so we choose those elements by \( \binom{n-r}{s-1,n-r,\ldots,n-r(s-1)-r} = \frac{(n-r)!}{((s-1)!)^p(n-p(s-1)-r)!} \) ways. Then we choose the $i$ elements \((s-1)(r-p) + s(k-r) \leq i \leq n-r - (s-1)p\) which belongs to the remaining $k - p$ parts and we have \(\binom{n-p(s-1)-r}{i} \) ways to do it. Then, distribute them on $k - p$ parts such that the $r - p$ fixed elements are in distinct parts and each part have at least $s$ elements, which can be done by \(\binom{k-p}{r_p}^{(s)}\) possibilities. It remains now to distribute the remaining $n - p(s-1) - r - i$ elements on the $p$ first parts and we have \(p^{n-p(s-1)-r-i} \) possibilities. We conclude by summing over all possible values of $i$. \(\square\)

Figure 4: The value of an element in the $s$-associated $r$-Stirling table in terms of the consecutive vertical elements in the $s$-associated $r - p$-Stirling table as an inner product result

For $p = r$ we get an expression of the $s$-associated $r$-Stirling numbers of the second kind in terms of the regular $s$-associated Stirling numbers of the second kind

\[
\binom{n}{k}_r^{(s)} = \sum_{i=s(k-r)}^{n-r} \frac{(n-r)!}{((s-1)!)^i(n-s)!} \binom{n-sr}{i} \binom{i}{k-r}_r^{(s)} \; r^{n-sr-i},
\]

also, for $s = 1$, we obtain the equation given by Broder \[12\] eq 31] and for $r = 1$, we get a vertical recurrence relation for the classical $s$-associated Stirling numbers of the second kind

\[
\binom{n}{k}_1^{(s)} = \sum_{i=s(k-1)}^{n-s} \binom{n-1}{i} \binom{n-s}{i} \binom{i}{k-1}_1^{(s)}.
\]

Theorem 26 Let $p, r, k$ and $n$ be nonnegative integers such that $p \leq r \leq k$ and $n \geq sk$, we have

\[
\binom{n}{k}_r^{(s)} = \sum_{i=0}^{p} \frac{(k-r+p-i)!}{(k-r)!} \binom{p}{i} \binom{n-r}{i(s-1)} \frac{(i(s-1))!}{((s-1)!)^i} \binom{n-p-i(s-1)}{k-i}_r^{(s)}.
\]

Proof. Let us consider the $p$ first elements, and focus on the $i$ ($0 \leq i \leq p$) parts containing exactly $s$ elements, we have \(\binom{p}{i}\) ways to choose the $i$ elements from the $p$ first ones, and \(\binom{n-r}{i(s-1)}\) ways to choose the $i(s-1)$ remaining elements to have $s$ elements by part, and \(\frac{(i(s-1))!}{((s-1)!)^i}\) ways to partition the $i(s-1)$ elements on $i$ groups such that each group have at least $(s-1)$ elements, then affect each group to the $i$ elements and we have $i!$. Then, we partition the $n-p-i(s-1)$ remaining elements into $(k-i)$ parts such that each group has at least $s$ elements and the remaining $r-p$ elements are in distinct subsets, and we have \(\binom{n-p-i(s-1)}{k-i}_r^{(s)}\) ways to do it. Now, it reminds $(p-i)$ elements not yet affected.
Thus we have \((k - r + p - i)\) choice for the first one, \((k - r + p - i - 1)\) choice for the second one and so on until the last one have \((k - r + 1)\) which gives \((k - r + p - i)(k - r + p - i - 1) \cdots (k - r + 1) = \frac{(k - r + p - i)!}{(k - r)!}\) possibilities. We conclude by summing.

For \(s = 1\) we get the relation given by the authors [5 eq 5].

An expression of the \(s\)-associated \(r\)-Stirling numbers of the second kind in terms of the \(s\)-associated Stirling numbers can be deduced from equation (50), for \(p = r\), as follows

\[
\left\{ \frac{n}{k} \right\}_r^{(s)} = \sum_{i=0}^{r} \frac{(k-i)!}{(k-r)!} \binom{n-r}{i} \binom{i(s-1)!}{(s-1)!} \left\{ n - r - i(s-1) \right\}_r^{(s)} - (k-i).
\]

(51)

Also, for \(r = 1\), we obtain the recurrence relation of the \(s\)-associated Stirling numbers [20 eq 4.1].

![Diagram](image)

Figure 5: Value of \(s\)-associated \(r\)-Stirling element (in black) as an inner product of a periodic sequence of elements of the \(s\)-associated \(r-p\)-Stirling table (in white) with a sequence deriving from binomial coefficient.

**Theorem 27** Let \(p, r, k\) and \(n\) be nonnegative integers such that \(p \leq r \leq k\) and \(n \geq sk\), we have

\[
\left[ \frac{n}{k} \right]_r^{(s)} = \sum_{i=0}^{n-sk} \frac{(n-r)!}{(n-r-j+(s-1)p)!} \binom{p}{i} \binom{p+j-1}{j-i} \left[ \frac{n-sp-j}{k-p} \right]_r^{(s)} s^{r-i}.
\]

(52)

**Proof.** Let us consider the \(p\) first elements, they are in \(p\) distinct lists with \(i_j\) \((i_j \geq s-1; j = 1..p)\) other elements, such that \(i_1 + i_2 + \cdots + i_p = j \((s-1)p \leq j \leq n - p - s (k-p)\)\). Then there are \(\binom{n-r}{i_1} \binom{n-r-i_1}{i_2} \cdots \binom{n-r-i_1-i_2-\cdots-i_p-1}{i_p}\) ways to choose the \(i_1, i_2, \ldots, i_p\) elements and \((i_1 + 1)! (i_2 + 1)! \cdots (i_p + 1)!\) ways to constitute the \(p\) lists. Now, it remains to distribute the \(n - p - j\) remaining elements into \(k - p\) lists such that each list have at least \(s\) elements and the \(r - p\) elements are in distinct lists, which gives \(\left[ \frac{n-p-j}{k-p} \right]_r^{(s)} \) possibilities. We sum over all value of \(j\) we get

\[
\left[ \frac{n}{k} \right]_r^{(s)} = \sum_{j=(s-1)p}^{n-p-s(k-p)} \sum_{\substack{i_1 + i_2 + \cdots + i_p = j \\text{and } i_j \geq s-1}} \binom{i_1 + 1}{i_1} \binom{i_2 + 1}{i_2} \cdots \binom{i_p + 1}{i_p} \left[ \frac{n-r}{k-p} \right]_r^{(s)}.
\]

the inner summations can be evaluated using (27). This gives the result. □

For \(p = r\), we get an expression of the \(s\)-associated \(r\)-Lah numbers in terms of the \(s\)-associated Lah numbers

\[
\left[ \frac{n}{k} \right]_r^{(s)} = \sum_{i=0}^{n-sk} \sum_{j=i}^{r} \binom{r}{j} \binom{r+j-1}{j-i} \frac{(n-r)!}{(n-j+(s-2)r)!} s^{r-i} \left[ \frac{n-sr-j}{k-r} \right]_r^{(s)}.
\]

(53)
Also, for \( r = 1 \), we get relation \([19]\), and for \( s = 1 \) we get the identity \([3\ \text{eq}\ 7]\).

7 Cross recurrence relations

From equations \([44]\) and \([52]\), for \( p = 1 \), we get some vertical cross recurrence relations.

**Corollary 28** We have

\[
\begin{align*}
\binom{n}{k}_r &= \binom{n-s(k-1)-1}{(n-r)!} \binom{n-i-1}{k-1}_{r-1}, \\
\binom{n}{k}_r &= \binom{n-s(k-1)-1}{(i+1)(n-r)!} \binom{n-i-1}{k-1}_{r-1}.
\end{align*}
\]

(54) (55)

For \( r = 1 \), we get relation \([19]\) and for \( s = 1 \) we get the identity given by the authors in \([8]\).

**Theorem 29** Let \( r, k, n \) be nonnegative integers such that \( n \geq sk \), we have

\[
\binom{n}{k}_r = \binom{n-r}{s-1} \binom{n-s}{k-1}_{r-1} + (k-r+1) \binom{n-1}{k}_{r-1}.
\]

(56)

**Proof.** Let us consider the \( r \)th elements. If it belongs to a group containing exactly \( s \) elements, we have \( \binom{n-r}{s-1} \binom{n-s}{k-1}_{r-1} \) ways to choose the remaining \((s-1)\) elements and \( \binom{n-r}{s-1} \binom{n-s}{k-1}_{r-1} \) ways to partition the remaining \((n-s)\) elements on \((k-1)\) parts such that the \((r-1)\) first elements are in distinct parts, and each part, have at least \( s \) elements. Thus we get \( \binom{n-r}{s-1} \binom{n-s}{k-1}_{r-1} \) possibilities. Else, we have \( \binom{n-r}{s-1} \binom{n-s}{k-1}_{r-1} \) possibilities to partition the remaining \((n-1)\) elements into \( k \) parts such that the \((r-1)\) first elements are in distinct parts, and each part, have at least \( s \) elements, then add the \( r \)th elements to on of the \((k-(r-1))\) parts and we have \( (k-r+1) \binom{n-1}{k}_{r-1} \) possibilities. It gives \( (k-r+1) \binom{n-1}{k}_{r-1} \).

For \( s = 1 \) we get the cross recurrence \([5\ \text{eq}\ 3]\) and for \( r = 1 \) we get the recurrence relation of the \( s \) associated Stirling numbers of the second kind \([20\ \text{eq}\ 4.1]\).

8 Convolution identities (revisited)

The \( s \)-associated \( r \)-Stirling numbers of the three kinds can be expressed as a convolution using the binomial coefficients.

**Theorem 30** Let \( r, k \) and \( n \) be nonnegative integers such that \( n \geq sk \) with \( k_1 + \cdots + k_p = k \) and \( r_1 + \cdots + r_p = r \), we have

\[
\binom{k}{k_1, \ldots, k_p}_r \binom{n}{k+r}_r = \sum_{l_i \geq s k_i + (s-1) r_i, l_i + \cdots + l_p = n} \binom{n}{l_1, \ldots, l_p} \binom{l_1 + r_1}{k_1 + r_1}_r \cdots \binom{l_p + r_p}{k_p + r_p}_r.
\]

(57)

**Proof.** We consider permutations of \( Z_{n+r} \) with \( k+r \) cycles such that the \( r \) first elements are in distinct cycles and each cycle has at least \( s \) elements and we have \( \binom{n+r}{k+r}_r \) possibilities. We color the cycles with \( p \) colors such that each \( r_i \) cycles containing the \( r_i \) elements with \( k_i \) other cycles have the same color, thus we choose the \( k_i \) cycles and we have \( \binom{k_i}{k_i}_r \) possibilities this is to choose the \( l_i \) elements that have the
same color of the \( r_i \) first and we have \( \binom{n}{ l_1, \ldots, l_p} \) possibilities, then consider all the permutations of the \( l_i + r_i \) elements with \( k_i + r_i \) cycles such that the \( r_i \) elements are in distinct cycles and each cycle has at least \( s \) element and we have \( \binom{[l_i + r_i]^{(s)}}{k_i + r_i} \) ways to do it. Summing over all possible values of \( l_i \) gives the result. \( \square \)

**Theorem 31** Let \( k, r, k \) and \( n \) be nonnegative integers such that \( n \geq sk \) with \( k_1 + \cdots + k_p = k \) and \( r_1 + \cdots + r_p = r \). The \( s \)-associated \( r \)-Stirling numbers of the second kind satisfy

\[
\left( \begin{array}{c} k \\ k_1, \ldots, k_p \end{array} \right) \binom{n + r}{k + r} = \sum_{l_1 + \cdots + l_p = n \atop l_i \geq sk_i + (s-1)r_i} \binom{n}{l_1, \ldots, l_p} \binom{l_1 + r_1}{k_1 + r_1} \cdots \binom{l_p + r_p}{k_p + r_p} \cdot (s) . \tag{58}
\]

**Proof.** We use an adapted analogous bijective proof as for the identity \( \text{(57)} \). \( \square \)

Relations \( \text{(57)} \) and \( \text{(58)} \) extend those given by the others [5, Eq 8, Eq 12] to the \( s \)-associated situation.

**Theorem 32** Let \( r, k \) and \( n \) be nonnegative integers such that \( n \geq sk \) with \( k_1 + \cdots + k_p = k \) and \( r_1 + \cdots + r_p = r \). The \( s \)-associated \( r \)-Lah numbers satisfy

\[
\left( \begin{array}{c} k \\ k_1, \ldots, k_p \end{array} \right) \binom{n + r}{k + r} = \sum_{l_1 + \cdots + l_p = n \atop l_i \geq sk_i + (s-1)r_i} \binom{n}{l_1, \ldots, l_p} \binom{l_1 + r_1}{k_1 + r_1} \cdots \binom{l_p + r_p}{k_p + r_p} \cdot (s) . \tag{59}
\]

**Proof.** We use an adapted analogous bijective proof as for the identity \( \text{(57)} \). \( \square \)

For \( s = 1 \), we get

\[
\left( \begin{array}{c} k \\ k_1, \ldots, k_p \end{array} \right) \binom{n + r}{k + r} = \sum_{l_1 + \cdots + l_p = n \atop l_i \geq sk_i + (s-1)r_i} \binom{n}{l_1, \ldots, l_p} \binom{l_1 + r_1}{k_1 + r_1} \cdots \binom{l_p + r_p}{k_p + r_p} . \tag{60}
\]

### 9 Generating functions

The \( s \)-associated \( r \)-Stirling numbers of the first kind have the following exponential generating function.

**Theorem 33** We have

\[
\sum_{n \geq sk + (s-1)r} \binom{n + r}{k + r} \binom{x^n}{n!} = \frac{(-1)^k}{k!} \left( \ln \left( \frac{1}{1 - x} \right) + \sum_{i=1}^{s-1} \frac{x^i}{i} \right)^k \left( \frac{x^{n-1}}{1 - x} \right)^r . \tag{61}
\]

**Proof.** Using the identity \( \text{(13)} \), we get

\[
\sum_{n \geq sk + (s-1)r} \binom{n + r}{k + r} \binom{x^n}{n!} = \sum_i \binom{i - r(s - 2) - 1}{r - 1} x^i \sum_{n \geq sk + (s-1)r} \binom{n - i}{k} \binom{x^{n-i}}{(n-i)!} ,
\]

from \( \text{(9)} \), we obtain

\[
\sum_{n \geq sk + (s-1)r} \binom{n + r}{k + r} \binom{x^n}{n!} = \frac{1}{k!} \left( \ln \left( \frac{1}{1 - x} \right) - \sum_{i=1}^{s-1} \frac{x^i}{i} \right) \sum_i \binom{i - r(s - 2) - 1}{r - 1} x^i ,
\]

using relation \( \text{(22)} \) we get the result. \( \square \)

The above theorem implies the double generating function.
Theorem 34 The $s$-associated $r$-Stirling numbers of the first kind satisfy
\[
\sum_{n,k} \left[ \frac{n+r}{k+r} \right]^{(s)} \frac{x^n}{n!} y^k = \frac{\exp \left( y \ln \left( \frac{1}{1-x} \right) - y \sum_{i=0}^{s-1} \frac{x^i}{i!} \right) \left( x^{s-1} \right)^r}{1-x}.
\] (62)

Proof. Interchanging the order of summation and using equation (61) we get the result. \qed

The $s$-associated $r$-Stirling numbers of the second kind have the following exponential generating function
\[
\sum_{n \geq sk+(s-1)r} \left\{ \frac{n+r}{k+r} \right\}^{(s)} \frac{x^n}{n!} = \frac{\exp \left( y - \sum_{i=0}^{s-1} \frac{x^i}{i!} \right) \left( x^{s-1} \right)^r}{1-x}.
\] (63)

Proof. Using the identity (51), we get
\[
\sum_{n \geq sk+(s-1)r} \left\{ \frac{n+r}{k+r} \right\}^{(s)} \frac{x^n}{n!} = \frac{(k+r-i)!}{k!} \sum_{i=0}^{r} \left( \frac{x^{i-1}}{(s-1)!} \right)^i \sum_{n \geq sk+(s-1)r} \left\{ \frac{n-i}{k+r+i} \right\}^{(s)} \frac{x^n}{n!}.
\]
the second summation can be evaluated using (10) and gives
\[
\sum_{n \geq sk+(s-1)r} \left\{ \frac{n+r}{k+r} \right\}^{(s)} \frac{x^n}{n!} = \frac{1}{k!} \left( \exp \left( y - \sum_{i=0}^{s-1} \frac{x^i}{i!} \right) \right)^r,
\]
using the binomial theorem we get the result. \qed

The double generating function for $s$-associated $r$-Stirling numbers of the second kind is
\[
\sum_{n,k} \left\{ \frac{n+r}{k+r} \right\}^{(s)} \frac{x^n}{n!} y^k = \exp \left( y \exp \left( x \right) - y \sum_{i=0}^{s-1} \frac{x^i}{i!} \right) \left( x^{s-1} \right)^r.
\] (64)

The $s$-associated $r$-Lah numbers have the following exponential generating function
\[
\sum_{n,k} \left\{ \frac{n+r}{k+r} \right\}^{(s)} \frac{x^n}{n!} y^k = \exp \left( y \exp \left( x \right) - y \sum_{i=0}^{s-1} \frac{x^i}{i!} \right) \left( x^{s-1} \right)^r.
\] (65)

Proof. Using the explicit formula (39) in the left hand side we get
\[
\sum_{n \geq sk+(s-1)r} \left\{ \frac{n+r}{k+r} \right\}^{(s)} \frac{x^n}{n!} = \frac{1}{k!} \sum_{j=0}^{r} \left( \frac{x^j}{(s-1)!} \right)^r \sum_{n \geq sk+(s-1)r} \left( \frac{n+r+j-(s-1)(k+r)-1}{k+r+j} \right) x^n,
\]
the second summation in the right side, due to relation (22), gives
\[
\sum_{n \geq sk+(s-1)r} \left\{ \frac{n+r}{k+r} \right\}^{(s)} \frac{x^n}{n!} = \frac{1}{k!} \sum_{j=0}^{r} \left( \frac{x^j}{(s-1)!} \right)^r \sum_{j=0}^{r} \left( \frac{x^j}{(s-1)!} \right)^r \left( \frac{1}{1-x} \right)^{r-j}
\]
using the binomial theorem we get the result. \qed

The double generating function of the $s$-associated $r$-Lah numbers is given by
Theorem 38

\[
\sum_{n \geq s k + (s-1)r} \sum_{k \geq 0} \left[\begin{array}{c} n+r \\ k+r \end{array}\right]_{r, \lambda, n!} \frac{x^n}{n!} y^k = \left[\exp \left\{ \frac{y}{1-x} \right\} \right] \left[\frac{x^{s-1}}{(1-x)^2} (s-(s-1)x) \right]^r.
\] (66)

Proof. Interchanging the order of summation and using equation (65) we get the result. \(\square\)

10 Conclusion and perspectives

Roughly speaking, there are many recurrence and congruence relations known about the \(r\)-Stirling numbers and the associated Stirling numbers which can be generalized to \([n]\alpha, \beta)\(r\), \([n]\alpha, \beta)\(k\), \([n]\alpha, \beta)\(r\), \([n]\alpha, \beta)\(k\). We have treated a few of these. In this section, we propose some problems:

- Howard \([21]\) gave, as perspectives, an extension of the weighted associated Stirling numbers to the Weighted \(s\)-associated Stirling numbers without specifying the expressions. In this perspective, as continuity to our work, we propose the Weighted \(s\)-associated Stirling numbers of the first and the second kind, denoted \([n]\alpha, \beta)\(k\) and \([n]\alpha, \beta)\(r\), respectively, by the following

\[
\sum_{n \geq s} \left[\begin{array}{c} n \\ k \end{array}\right]_{\lambda}^\alpha \frac{x^n}{n!} = \frac{(-1)^k}{k!} \left( \frac{1}{1-x} \right)^k \left( -\sum_{i=1}^{s-1} \frac{(-\lambda) x^i}{i} \right) \left( \ln (1-x) + \sum_{i=1}^{s-1} \frac{x^i}{i} \right)^k,
\] (67)

where \(\alpha, \beta\) are parameters.

\[
\sum_{n \geq s} \left(\begin{array}{c} n \\ k \end{array}\right)_{r, \lambda, n!} \frac{x^n}{n!} = \frac{1}{k!} \left( \exp (\lambda x) - \sum_{i=0}^{s-1} \frac{(\lambda x)^i}{i!} \right) \left( \exp (x) - \sum_{i=0}^{s-1} \frac{x^i}{i!} \right)^k.
\] (68)

Note that for \(s = 2\), we get weighted associated Stirling numbers. It seems possible to derive analog relations of the weighted associated Stirling numbers, and establish other identities.

- By the same reasoning, it is interesting to extend these generalization to the \(r\)-Stirling numbers. We define the weighted \(s\)-associated \(r\)-Stirling numbers of the first and the second kind respectively

\[
\sum_{n} \left[\begin{array}{c} n+r \\ k+r \end{array}\right]_{r, \lambda, n!} \frac{x^n}{n!} = \frac{(-1)^k}{k!} \left( \frac{1}{1-x} \right)^k \left( -\sum_{i=1}^{s-1} \frac{(-\lambda) x^i}{i} \right) \left( \ln (1-x) + \sum_{i=1}^{s-1} \frac{x^i}{i} \right)^k \left( \frac{x^{s-1}}{1-x} \right),
\] (69)

where \(\alpha, \beta\) are parameters.

\[
\sum_{n} \left(\begin{array}{c} n+r \\ k+r \end{array}\right)_{r, \lambda, n!} \frac{x^n}{n!} = \frac{1}{k!} \left( \exp (\lambda x) - \sum_{i=0}^{s-1} \frac{(\lambda x)^i}{i!} \right) \left( \exp (x) - \sum_{i=0}^{s-1} \frac{x^i}{i!} \right)^k \left( \exp (x) - \sum_{i=0}^{s-2} \frac{x^i}{i!} \right)^r.
\] (70)

It will be nice to investigate the combinatorial meaning and drive all the combinatorial identities. Also, for \(s = 1\), we get the definition of the weighted \(r\)-Stirling numbers as follows

\[
\sum_{n \geq s k + (s-1)r} \left[\begin{array}{c} n+r \\ k+r \end{array}\right]_{r, \lambda, n!} \frac{x^n}{n!} = \frac{(-1)^k}{k!} \left( \frac{1}{1-x} \right)^k \left( \ln (1-x) \right)^k,
\] (71)

\[
\sum_{n \geq s k + (s-1)r} \left(\begin{array}{c} n+r \\ k+r \end{array}\right)_{r, \lambda, n!} \frac{x^n}{n!} = \frac{1}{k!} \left( \exp (\lambda x) - 1 \right) \left( \exp (x) - 1 \right)^k \exp (rx).
\] (72)

- It will be nice to investigate the different generalization (weighted, degenerated) of the Lah numbers and \(r\)-Lah numbers.

- The authors and Belkhir \([4]\) define the \([n]_{k}^{\alpha, \beta}\) as the weight of a partition of \(n\) elements into \(k\) lists such that the element inserted as head list has weight \(\beta\) except the first inserted one which has weight 1 and the element inserted after an other one has weight \(\alpha\). This interpretation allow an extension to the \(s\)-associated aspect by adding the known restriction (at least \(s\) elements by list).
• An other perspective of this work is to consider the Whitney numbers (see [9, 10, 11, 2]) and \( r \)-Whitney numbers (see [16]) and to introduce the \( s \)-associated situation by two approaches: the first one via the generating function and the second one using the combinatorial interpretation (see [6]).

11 Tables of the \( s \)-associated \( r \)-Stirling numbers of the three kinds

| \( n \) \( k \) | 3  | 4  | 5  | 6  | 7  | 8  |
|----------------|----|----|----|----|----|----|
| 6              | 6  |    |    |    |    |    |
| 7              | 72 |    |    |    |    |    |
| 8              | 720| 60 |    |    |    |    |
| 9              | 7200| 1320||| | |
| 10             | 75600| 21420| 630| | | |
| 11             | 8467200| 3205440| 21840| | | |
| 12             | 101606400| 47537280| 519120| 7560| | |
| 13             | 1306368000| 720057600| 10795680| 378000| | |
| 14             | 17962560000| 11297880000| 213804360| 12335400| 103950| |
| 15             | 263450880000| 184823040000| 4191881760| 339255840| 7068600| |
| 16             | 41098337280000| 31640678784000| 83018048256| 8627739120| 302702400| 1621620| |
| 17             | 6799906713600000| 5670985582080000| 1679434428672| 212106454560| 10621490880| 143783640| |

Some values of the 2-associated 3-Stirling numbers of the first kind

| \( n \) \( k \) | 2  | 3  | 4  | 5  | 6  |
|----------------|----|----|----|----|----|
| 6              | 24 |    |    |    |    |
| 7              | 240|    |    |    |    |
| 8              | 2160|    |    |    |    |
| 9              | 20160| 1680|    |    |    |
| 10             | 201600| 36960|    |    |    |
| 11             | 2177280| 616896|    |    |    |
| 12             | 25401600| 9616320| 201600|    |    |
| 13             | 319334400| 145774080| 7761600|    |    |
| 14             | 4311014400| 2329015680| 206569440|    |    |
| 15             | 622702080000| 391659840000| 4817292480| 38438400|    |
| 16             | 9589612032000| 6728987865600| 106815893184| 2287084800|    |
| 17             | 156920924160000| 120809864448000| 23376236083200| 88691803200|    |
| 18             | 2719962685440000| 2268394232832000| 514852847308800| 28861664832000| 10762752000| |
| 19             | 49796239933440000| 44538726503520000| 11537634473164800| 863628051686400| 914833920000| |

Some values of the 3-associated 2-Stirling numbers of the first kind

| \( n \) \( k \) | 3  | 4  | 5  | 6  |
|----------------|----|----|----|----|
| 9              | 720|    |    |    |
| 10             | 15120|    |    |    |
| 11             | 241920|    |    |    |
| 12             | 3628800| 120960|    |    |
| 13             | 54432000| 4536000|    |    |
| 14             | 838252800| 117754560|    |    |
| 15             | 134120448000| 268240896000| 266112000|    |
| 16             | 224172748000| 579168921600| 1556755200|    |
| 17             | 39230231040000| 12391009344000| 593902108800|    |
| 18             | 719220902400000| 2654453628288000| 19024845840000| 8072064000| |
| 19             | 13809041326080000| 592364034662400000| 5607556708360000| 678053376000| |
| 20             | 2774361939148800000| 13356216902246400000| 158911827250176000| 356510778624000| |
### Some values of the 3-associated 3-Stirling numbers of the first kind

| $n \setminus k$ | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------------|----|----|----|----|----|----|
| 6               | 6  |    |    |    |    |    |
| 7               | 36 |    |    |    |    |    |
| 8               | 150| 60 |    |    |    |    |
| 9               | 540| 660|    |    |    |    |
| 10              | 1806|4620|630|    |    |    |
| 11              | 5796|26376|10920|    |    |    |
| 12              | 18150|134316|114660|7560|    |    |
| 13              | 55980|637020|947520|189000|    |    |
| 14              | 171006|2882220|6798330|2772000|103950|    |
| 15              | 519156|12623952|44482680|31221960|3534300|    |
| 16              | 1569750|54031692|273060216|299459160|68918850|1621620|
| 17              | 4733820|227425380|1690815216|2578495920|1013632620|71891820|
| 18              | 14250606|945535500|9069810750|20561420880|12509597100|1797295500|
| 19              | 42850116|3895163928|50074806600|154904109360|136912175400|33423390000|

### Some values of the 2-associated 3-Stirling numbers of the second kind

| $n \setminus k$ | 2  | 3  | 4  | 5  | 6  | 7  |
|-----------------|----|----|----|----|----|----|
| 6               | 6  |    |    |    |    |    |
| 7               | 20 |    |    |    |    |    |
| 8               | 50 |    |    |    |    |    |
| 9               | 112| 210|    |    |    |    |
| 10              | 238| 1540|    |    |    |    |
| 11              | 492| 7476|    |    |    |    |
| 12              | 1002|30240|12600|    |    |    |
| 13              | 2024|110550|161700|    |    |    |
| 14              | 4070|379764|1286670|    |    |    |
| 15              | 8164|1252680|8168160|1201200|    |    |
| 16              | 16354|4020016|45411366|23823800|    |    |
| 17              | 32736|12656826|231591360|281331050|    |    |
| 18              | 65502|39315588|1112731620|2574371800|168168000|    |
| 19              | 131036|120953436|5122253136|20176035880|4764760000|    |
| 20              | 262106|369535392|22845529356|142501719360|78189711600|    |
| 21              | 524248|1123340382|99494683548|934588410756|973654882200|32590958400|

### Some values of the 3-associated 2-Stirling numbers of the second kind
\[
\begin{array}{|c|cccccc|}
\hline
n \backslash k & 3 & 4 & 5 & 6 & 7 \\
\hline
9 & 90 \\
10 & 630 \\
11 & 2940 \\
12 & 11508 & 7560 \\
13 & 40950 & 94500 \\
14 & 137610 & 734580 \\
15 & 445896 & 4569180 & 831600 \\
16 & 1410552 & 24959220 & 16216200 \\
17 & 4390386 & 125381256 & 188558370 \\
18 & 13514046 & 594714120 & 1701649950 & 126126000 \\
19 & 41278068 & 2707865160 & 13172479320 & 3531528000 \\
20 & 125405532 & 11965834608 & 92024532600 & 57320062800 \\
21 & 379557198 & 51706343676 & 597753095940 & 706637731800 & 25729704000 \\
22 & 1145747538 & 219672404652 & 3679670518524 & 7344721664280 & 977728752000 \\
23 & 3452182656 & 921197481924 & 21746705483880 & 67927123063800 & 2110264564000 \\
24 & 10388002848 & 3824306218236 & 124527413730720 & 5772111312567600 & 340398980922000 \\
\hline
\end{array}
\]

Some values of the 3-associated 3-Stirling numbers of the second kind

\[
\begin{array}{|c|cccccc|}
\hline
n \backslash k & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
6 & 48 \\
7 & 864 \\
8 & 12240 & 960 \\
9 & 166320 & 31680 \\
10 & 2298240 & 735840 & 20160 \\
11 & 33022080 & 15200640 & 1048320 \\
12 & 497871360 & 302279040 & 35925120 & 483840 \\
13 & 7903526400 & 5994777600 & 104328000 & 36288000 \\
14 & 13220441600 & 120708403200 & 28101427200 & 1716422400 \\
15 & 2328905779200 & 2491766323200 & 732872448000 & 66501388000 \\
16 & 43153254144000 & 53016855091200 & 18942597273600 & 2325792268800 \\
17 & 839788479129600 & 1166096823091200 & 491947097241600 & 77022028752000 \\
\hline
\end{array}
\]

Some values for the 2-associated 3-Lah numbers

\[
\begin{array}{|c|cccc|}
\hline
n \backslash k & 2 & 3 & 4 & 5 \\
\hline
6 & 216 \\
7 & 2880 \\
8 & 33120 \\
9 & 383040 & 45360 \\
10 & 4636800 & 1330560 \\
11 & 59512320 & 28667520 \\
12 & 812851200 & 562464000 & 163296000 \\
13 & 11815372800 & 10777536000 & 838252800 \\
14 & 182499609600 & 207886694400 & 28979596800 \\
15 & 2988969984000 & 4097379686400 & 859328870400 & 93405312000 \\
16 & 5178309472800 & 8316809804800 & 23799673497600 & 7410154752000 \\
17 & 946756242432000 & 1745745281280000 & 640760440320000 & 374866652160000 \\
\hline
\end{array}
\]

Some values for the 3-associated 2-Lah numbers
### Some values for the 3-associated 3-Lah numbers

| \(n\backslash k\) | 3            | 4            | 5            | 6            |
|------------------|--------------|--------------|--------------|--------------|
| 9                | 19440        |              |              |              |
| 10               | 544320       |              |              |              |
| 11               | 11249280     |              |              |              |
| 12               | 212647680    | 9797760      |              |              |
| 13               | 3940876800   | 489888000    |              |              |
| 14               | 73766246400  | 16525555200  |              |              |
| 15               | 1414970726400| 479001600000 | 6466521600   |              |
| 16               | 2802159360000| 12989565388800| 504388684800 |              |
| 17               | 575115187046400| 342959397580800| 25107347865600|              |
| 18               | 1225552417689600| 9007261046784000| 1030447401984000| 5884534656000 |
| 19               | 271347662057472000| 2382687312445440000| 383096282849280000| 659067881472000 |
| 20               | 62423143630848000000| 639703839430656000000| 135090085190860800000| 453501470822400000 |

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