Existence and Stability of Circular Orbits in Time–Dependent Spherically Symmetric Spacetimes

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Abstract

For a general spherically four–dimensional metric the notion of “circularity” of a family of equatorial geodesic trajectories is defined in geometrical terms. The main object turns out to be the angular–momentum function $J$ obeying a consistency condition involving the mean extrinsic curvature of the submanifold containing the geodesics. The analysis of linear stability is reduced to a simple dynamical system formally describing a damped harmonic oscillator. For static metrics the existence of such geodesics is given when $J^2 > 0$, and $(J^2)' > 0$ for stability. The formalism is then applied to the Schwarzschild–de Sitter solution, both in its static and in its time–dependent cosmological version, as well to the Kerr–de Sitter solution. In addition we present an approximate solution to a cosmological metric containing a massive source and solving the Einstein field equation for a massless scalar.

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## 1 Introduction

The analysis of circular orbits (both time– and lightlike) around an isolated massive source had always been an important tool to determine its physical parameters. For example in Newtonian gravity, the Kepler–law $v^2 = m/r$ for circular orbits in the field of a central attracting body allows to determine its mass–parameter $m$. In the slightly generalized form $v^2 = m(r)/r$ with $v^2 \to \text{const} \neq 0$ it led to the assumption of dark matter in the halo of galaxies. The same argument holds also in Einstein’s gravity for spherically–symmetric and static spacetimes. Its description is slightly more involved, being based on an effective potential. However at least in a cosmological context it is necessary to include the effects of expansion (possibly accelerating), thus leading to time–dependent metrics and preventing the applicability of the effective potential method. For a possible explanation of the anomalies in the trajectories of the Pioneer probes (at approx. 20 – 70 AU), such effects have been studied from diverse perspectives. However they are far too small to be relevant in the context of the standard theory\textsuperscript{3}. In fact, a careful analysis\textsuperscript{2} of the probe geometry revealed that the acceleration anomaly can be explained away by a thermal effect. But data concerning “wide binaries” ($\gg 7000$ AU) seems to conform to $v^2 \to \text{const} \neq 0$ and so could indicate a possible cosmological effect.

Moreover, in the last years interest has been shifted towards galactic systems with their asymptotic constant tangential velocities obeying the baryonic Tully–Fisher relation $v \to (MH)^{1/4} = \text{const.}$\textsuperscript{4} This relation does not fit well into the conventional $\Lambda$CDM model but is the main result of Milgrom’s phenomenological MOND theory\textsuperscript{5}. In all these attempts the velocity of the bound orbits is of paramount importance.

Nevertheless, these studies also revealed some lack of understanding of what is meant by a “circular orbit” in the field of a time–dependent, but spherically–symmetric metric. It is the aim of the present paper to propose a geometric definition of the notion of circular orbit valid in this more general setting. This definition is substantiated by the possibility to formulate a corresponding stability–criterium.

The paper is organized as follows. In section 2 the basic notation and conventions are established. In section 3 the equatorial metrics are defined. In section 4 the notion of circular orbit is introduced geometrically. In section 5 the fundamental results for the existence and stability for circular orbits are formulated and proved. In section 6 the static case is analyzed in some detail. In section 7 some examples are analyzed (including time–dependent ones) before concluding with a discussion of our results in section 8.

The emphasis is on timelike orbits with nonvanishing angular–momentum, $J \neq 0$. However, for completeness also the case of radial timelike trajectories, $J = 0$ is briefly considered, as well as lightlike trajectories. It should be stressed that we deal exclusively with one–parameter families of trajectories sweeping the whole equatorial plane and not with isolated trajectories.

\textsuperscript{1}for a review, see Carrera and Giulini, 2010\textsuperscript{23}
\textsuperscript{2}Turyshev et al., 2012\textsuperscript{27}
\textsuperscript{3}Hernandez, Jimenez, Allen, 2012\textsuperscript{25}
\textsuperscript{4}see McGaugh, 2005\textsuperscript{21}
\textsuperscript{5}for an introduction, see Milgrom, 2008\textsuperscript{24}
2 Conventions and Notations

Although a generalization to spherically–symmetric spacetimes with dimension \(d = 3\) and \(d > 4\) would be straightforward, we will deal exclusively with \(d = 4\) and Lorentz–signature \((-1, 1, 1, 1)\). Later on we will restrict to a corresponding “equatorial” spacetime with \(d = 3\) and the induced metric. As our problem is fundamentally geometric, we well use extensively the conventional geometric index–free notation. As far as possible we will follow the conventions and notations of O’Neill, 2006 \[22\], with the following major exceptions. Perhaps less well–known are the useful “musical isomorphisms” \(\flat\) and \(\sharp\) between vectors \(V\) and one–forms \(F\), where \(F^\flat := g^{-1}(F)\), corresponding to the frequent “lowering” and “raising” of a simple index with the metric tensor. This notation keeps visible the geometric origin either as a vector or as a 1–form. In addition we will denote the contraction between a vector \(V\) and a 1–form \(W\) with the dot–operator to a scalar \(V \cdot W\). This spares us from the sometimes clumsy notation of O’Neill for the metric scalar–product \(\langle U, V \rangle\), which now would be written either as \(U \cdot V^\flat\) or as \(V \cdot U^\flat\). Also instead of O’Neill’s notation \(D_X\) for the covariant–derivative in the direction of \(X\) we will use the more conventional \(\nabla_X\). However in accordance with O’Neill we will also assume a torsion–free and metric–compatible connection \(\nabla_X\):

\[
\nabla_X Y - \nabla_Y X - [X, Y] = 0, \quad (1) \\
\nabla_X \langle Y, Z \rangle - \langle \nabla_X Y, Z \rangle - \langle Y, \nabla_X Z \rangle = 0, \quad (2)
\]
valid for any vectors \(X, Y, Z\). This defines uniquely the standard Levi–Civita connection expressed by the usual metric–based Christoffel–symbols \(\Gamma\).

We use a unit system based on powers of the light–year (ly). Both gravitational coupling constant \(\kappa := 8\pi G\) and velocity of light \(c\) will be set to 1.

3 Equatorial Metrics

The spherical symmetry of the metric allows the line–element to be expressed in the following canonical form (e.g. Carroll 2004, ch. 42 \[15\])

\[
ds^2 = -e^{2a} dt^2 + e^{2b} dr^2 + r^2 d\Omega^2,
\]
valid for any vectors \(X, Y, Z\). This defines uniquely the standard Levi–Civita connection expressed by the usual metric–based Christoffel–symbols \(\Gamma\).

Evidently there is the discrete reflection isometry \(\vartheta \rightarrow \pi - \vartheta\). This isometry is frequently used to motivate the restriction to the equatorial plane when considering geodesics (e.g. Frolov and Zelnikov 2011, ch. 8 \[26\]). A more refined consideration is to base it directly on the separability of the geodesic equation

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6from now on, we cite this work only by its author–name
7more generally, contraction of \(V\) with respect to the first (from the left) free vector slot of the tensorial object \(W\)
8the only exception to this rule is the divergence of a vector \(V\), when expressed as \(\nabla \cdot V\)
9the only instance where we use the \(\langle \cdot, \cdot \rangle\)–notation
Here we want to indicate briefly its geometric origin by showing that the equatorial submanifold is totally geodesic.

By Lemma 4.1 there is the identity \( K(X, Y) = -\varepsilon B(X, Y) \), where \( K \) denotes one–half of the Lie–derivative of the induced metric \( h \) along the normalized normal vector \( n := N\# \), \( B \) denotes the extrinsic curvature, and \( \varepsilon := N\# \cdot N = \pm 1 \).

Using local coordinates, with \( e^c = r \), \( N = r d\vartheta \), \( \varepsilon = 1 \), we have

\[
K_{\mu\nu} := \frac{1}{2} \mathcal{L}_{n} h_{\mu\nu} = \frac{1}{2} \left( \mathcal{L}_{e^{-c}} \partial_{\vartheta} \text{diag} (-e^{2a}, e^{2b}, 0, e^{2c}) \right) \bigg|_{\vartheta = \pi/2} = 0
\]

and so the extrinsic curvature also vanishes, \( B(X, Y) = 0 \). Therefore the equatorial submanifold is **totally geodesic**, implying in particular, that any geodesic in it is also a geodesic in the original manifold. As we deal exclusively with geodesics in the equatorial plane, we could restrict all the following considerations to the three–dimensional canonical equatorial metric (i.e. the induced metric) with line element

\[
ds^2 = -e^{2a} dt^2 + e^{2b} dr^2 + r^2 d\varphi^2,
\]

where \( a, b \) are functions of \((t, r)\) only.

More generally, if the dependency of a spherically symmetric metric on \( \vartheta \) is only through \( \sin^2 \vartheta \) and \( \cos^2 \vartheta \), there is reflection symmetry, and again \( K_{\mu\nu} = 0 \). This is the case for the axially–symmetric metrics with mirror–symmetry, like the Kerr–metric. Also it will be convenient to allow some extra redundancy by introducing a more general radial coordinate than the areal radius, like in most cosmological metrics. Therefore our analysis will deal more generally with the **generalized equatorial metric**, with line–element

\[
ds^2 = -e^{2a} dt^2 + e^{2b} dr^2 + e^{2c} (d\varphi - w dt)^2,
\]

where besides \( a \) and \( b \) also \( c \) and \( w \) are functions only of \((t, r)\). Evidently there is a remaining angular symmetry given by the Killing–vector \( C := \partial_{\varphi} \).

### 4 Circular Orbits

A systematic analysis of circular orbits for general spherically–symmetric time–dependent spacetimes seems not to have been done yet save for very particular cases. Sometimes (e.g. Sultana and Dyer, 2005 [19]), the notion of “circular orbit” is bound to the constancy of some “radius” along the trajectory. For a time–dependent spacetime this can at most be achieved by some isolated orbits.

This is of course not enough to establish Kepler–like relations e.g. in the form \( v = v(r) \) (\( v \): tangential velocity). Another approach is to consider “quasi–circular orbits” as defined by circular orbits in a static setting perturbed by cosmological dynamics (Faraoni and Jacques, 2007 [23]). At the other extreme is the approach of Nolan, 2014 [30] establishing the boundedness of the orbits, which asymptotically for large times approach circular orbits. However this has been shown only for some particular McVittie spacetimes. Also this would not be sufficient to obtain Kepler–like relations.

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10 for a more detailed formulation, see Proposition 13 of ch. 4 in O’Neill
11 \( w \) is usually denoted by \( \omega \)
12 in fact, assuming \( w = 0 \), the canonical equatorial metric is hard to achieve — if at all
13 see also Carrera and Giulini, 2010 [25]
Here we propose a characterization of circular orbits in between these extremes, making essential use of the remaining angular symmetry provided by the Killing–vector $C := \partial_\phi$. Consider a congruence of non–radial trajectories (not necessarily geodesic) with tangents $T$. The symmetry is imposed by the condition $\mathcal{L}_C T = 0$. These trajectories sweep out a congruence of hypersurfaces $\Sigma$, locally given by $\sigma = \text{const}$ with $\mathcal{L}_C \sigma = 0$ by symmetry. Then also $\mathcal{L}_C N = 0$, where $N \sim d\sigma$ denotes the one–form normal to $\Sigma$. The hypersurface–property can be also expressed as $N \wedge dN = 0$.

Our notion of “circularity” will be therefore defined by the following differential conditions encoding the symmetry both for $T$ as well as for $N$,

$$\text{circdiff} \left\{ \begin{array}{l} a) \ [C, T] = 0, \\ b) \ C \cdot dN + d(C \cdot N) = 0, \\ c) \ N \wedge dN = 0. \end{array} \right.$$  

These differential conditions must be supplemented by the following algebraic conditions,

$$\text{circalg} \left\{ \begin{array}{l} a) \ T \cdot N = 0, \\ b) \ C \cdot N = 0, \\ c) \ T \cdot T_\flat = -1, \\ d) \ N^5 \cdot N = +1. \end{array} \right.$$  

Here the condition a) expresses that $T$ is contained in the corresponding hypersurface defined by $N$ — analogously condition b).

The angular symmetry of the equatorial metric can be expressed as

$$\text{circkill} \left\{ \begin{array}{l} a) \ \forall X, Y: (\mathcal{L}_C g)(X, Y) \equiv C \cdot d(X \cdot Y_\flat) + [X, C] \cdot Y_\flat + [Y, C] \cdot X_\flat = 0, \\ b) \ C \cdot C_\flat \succ 0, \end{array} \right.$$  

Condition b) serves only to prevent $C$ to be a timelike Killing–vector.

These conditions are still independent of any geodesy of $T$, here expressed by $\nabla_T T = 0$. To take it into account, it will be convenient to introduce also the notion of the extrinsic curvature $B(X, Y)$ of a (nowhere lightlike) hypersurface $\Sigma$. This object encodes the deviation between the induced connection in the hypersurface $\Sigma$ and the connection in the ambient manifold. In the literature several different but closely related notions are used for this — e.g. in O’Neill, “shape tensor”, “shape operator” and “second fundamental form”. The “second fundamental form” frequently also goes under the name of extrinsic curvature. Following Frankel 1998, ch. 11.4 [12] we define it by means of Gauss’ equation,

$$\nabla_X Y = \nabla_X Y + B(X, Y) N^\sharp,$$  

where $X, Y$ are any vectors in the submanifold $\Sigma$, $\nabla$ denotes the induced connection and $N^\sharp$ the vector corresponding to the normalized normal one–form $N$.

There is a useful relation to the Lie–derivative along $N^\sharp$ of the induced metric $h$ which we express in the following

\[ \text{here the corresponding coordinate–dependent expression } C_{i;k} + C_{k;i} \text{ looks much simpler, but lacks an immediate geometrical meaning as the Lie–derivative of the metric } g_{ik} \text{ along } C^i \]

\[ \text{this case was not considered by O’Neill — there would be some technical difficulties} \]
Lemma 4.1
For vectors $X, Y$ orthogonal to the normalized one-form $N$ normal to $\Sigma$ with induced metric $h$, there is the identity

$$X \cdot Y \cdot (\mathcal{L}_N h) = -\varepsilon B(X, Y).$$  \hspace{1cm} (8)

Proof
The proof will be based on the fundamental identity $(\nabla_X X) \cdot N = \varepsilon B(X, X)$, which is seen to hold due to

$$X \cdot \nabla_X N \equiv \nabla_X (X \cdot N) - \nabla_X X \cdot N = \varepsilon B(X, X)$$  \hspace{1cm} (9)

by Gauss and orthogonality. By polarization of this quadratic relation we immediately get

$$(X \cdot \nabla_Y + Y \cdot \nabla_X) N = \varepsilon B(X, Y).$$  \hspace{1cm} (10)

Again by orthogonality this can also be written as

$$(X \cdot Y + Y \cdot X) \cdot \nabla N = -\varepsilon B(X, Y).$$  \hspace{1cm} (11)

Here $\nabla N$ can be replaced by $\mathcal{L}_N g$. Taking instead of $g$ the induced metric

$$h := g - \varepsilon N \times N$$

the identity still holds and so again by orthogonality we get the claimed result. □

Note that for a spacelike hypersurface $\varepsilon = -1$ and so this definition agrees in this case with the one given e.g. in Wald 1984, ch. 7.2 [9] and in Carroll 2004, app. D [15].

5 Main Results

The above nine conditions are still independent of the geodesy of $T$, and could refer to any congruence of vectors.\footnote{\textit{it is not claimed that they are all independent}} Note that the geodesy, $\nabla_X X = 0$ (a vectorial equation), of any trajectory with tangent $X$ in $\Sigma$ with normal $N$, implies $B(X, X) = 0$ (a scalar equation). But the converse is not true — evidently from $B(X, X) = 0$ follows only $(\nabla_X X) \cdot N = 0$. However assuming circularity, there is a converse when taking into account angular–momentum conservation. This is expressed in proposition 5.1 and its corollary.

5.1 Timelike Orbits

Let us first consider timelike orbits. Then the following proposition holds,

5.1.1 Existence

Proposition 5.1
Assuming the conditions \textit{circdiff}, \textit{circalg} and \textit{circkill}, then the necessary and sufficient conditions for geodesy, $\nabla_T T = 0$, are $B(T, T) = 0$ together with angular–momentum conservation $T \cdot d(C \cdot T_\nu) = 0$.

Proof

1. necessity. \textit{Here we assume geodesy} $\nabla_T T = 0$.\footnote{\textit{it is not claimed that they are all independent}}
a) Differentiating assumption \textit{circalg a)} with $\nabla_T$ and using geodesy gives $\frac{1}{2} \left( \mathcal{L}_{\nabla_T} g \right)(T, T) = 0$. By Lemma 4.1 this is equivalent to $B(T, T) = 0$, as claimed.

b) Contracting assumption \textit{cirediff a)} with $T_b$ and using assumption \textit{circalg c)} as well as the geodesy, results in $T \cdot d(T \cdot C_b) = 0$, as claimed.

2. sufficiency. Here we assume $B(T, T) = 0$ and $T \cdot d(T \cdot C_b) = 0$.

a) Contracting \textit{cirediff a)} with $T_b$ and using $T \cdot d(T \cdot C_b) = 0$ gives $\nabla_T \cdot C_b = 0$.

b) Applying $\nabla_T$ on \textit{circalg a)} and using $B(T, T) = 0$ in its equivalent form $K(T, T) = 0$ gives $\nabla_T \cdot N = 0$.

c) Applying $\nabla_T$ on \textit{circalg c)} results in $\nabla_T \cdot T_b = 0$.

As $(C_b, N, T_b)$ constitutes a complete system of linearly independent 1–forms, there follows geodesy $\nabla_T T = 0$, as claimed. ■

Disregarding the algebraic conditions, the conditions of the previous proposition constitute a simultaneous differential system both for $T$ and implicitly for $N$. Defining the angular–momentum of $T$ by $J := C \cdot T_b$ and assuming $J \neq 0$, this can be made more explicit by the following

\textbf{Corollary 5.1}

For the generalized equatorial metric of equation (6) define the following vectors $C$, $T$, one–form $N$ and scalar $\sigma$

\begin{align*}
C &:= \partial_{\varphi}, \\
N &:= \frac{1}{\sigma} \left( \dot{J} \, dt + J' \, dr \right), \\
T &:= \frac{P}{Q} \left[ J' \partial_t - \dot{J} \partial_r + w \, J' \partial_{\varphi} \right] + J \, e^{-2c} \partial_{\varphi}, \\
\sigma &:= Q \, e^{-(a+b)},
\end{align*}

where $P^2 := 1 + J^2 \, e^{-2c}$, $Q^2 := e^{2a} \, J' - e^{2b} \, \dot{J}^2$.

Then all the assumptions \textit{cirediff}, \textit{circalg} and \textit{cirkill} of Proposition \textit{5.1} are satisfied, including angular–momentum–conservation. And the necessary and sufficient condition for geodesy can be expressed as the single $J$–equation

\begin{equation}
K = - \left( \frac{1 + 2 J^2 / e^{2c}}{P^2 Q^2} \left( e^{2a} \, J' \, c' - e^{2b} \, \dot{J} \, c \right) - \frac{J \, w'}{P Q} \right) \sigma,
\end{equation}

where $K := -\nabla \cdot N^2$ is the mean curvature of $\Sigma$ \footnote{we fix some arbitrary signs by requiring $P > 0$, $Q > 0$} explicitly given by

\begin{equation}
K = \frac{1}{\sigma Q^2} \left[ \left( J^2 \, \ddot{J} + J^2 \, J'' - 2 \, J' \, \dot{J} \, \dot{J}' \right) - e^{2(a-b)} (a' + c') \, J^a - e^{2(b-a)} (b + c) \, J^b \\
+ (2a' - b' + c') \, J^2 \, J' + (2b - a + c) \, J^2 \, \dot{J} \right].
\end{equation}
Proof
All the assumptions of Proposition 5.1 are evidently satisfied. Also angular–momentum conservation is seen to hold due to \( \nabla T \cdot J = T \cdot (\sigma N) = 0 \).

By Lemma 4.1 and orthogonality \( T \cdot N = 0 \),

\[
B(T, T) = 0 \iff T \cdot (\mathcal{L}_N g) = 0. \tag{18}
\]

The inverse metric \( g^{-1} = \sum_{\alpha} \sigma_{\alpha} \left( \partial_t + w \partial_{\phi} \right) \partial_t + e^{-2a} \partial_r^2 + e^{-2b} \partial_{\phi}^2 \) \( \tag{19} \)

can also be expressed by the quasi–orthogonal decomposition

\[
g^{-1} = P^{-2}(T \times + C \times C e^{-2c} - J(C \times 3 \times C) e^{-2c}) + N^2 \times N^2. \tag{20}
\]

Therefore we can write

\[
P^{-2}T \cdot (\mathcal{L}_N g) \equiv - \nabla \cdot N^2 + N^2 \cdot (\nabla N) \]

\[
+ P^{-2} e^{-2c} (C \cdot C \cdot (\nabla N) - J(C \cdot T \cdot + T \cdot C)(\nabla N)). \tag{21}
\]

Evaluating the terms on the r.h.s., then \( N^2 \cdot (\nabla N) = 0 \) by normalization, \( C \cdot C \cdot (\nabla N) = N^2 \cdot d(e^{-2c}) \) by orthogonality and symmetry, and \( (C \cdot T \cdot + T \cdot C)(\nabla N) = N^2 \cdot T \cdot d((d\phi - w dt)e^{2c}) \), again by orthogonality and symmetry, so that finally,

\[
P^{-2}B(T, T) \equiv K + \left( \frac{1 + 2 J^2 / e^{2c}}{P^2 Q^2} \left( e^{2a} J'c' - e^{2b} \dot{J}/c \right) - \frac{J w'}{PQ} \right) \sigma = 0, \tag{22}
\]
as was to be demonstrated. ■

The case of vanishing angular–momentum, \( J = 0 \), can be dealt with similarly.

For a hypersurface \( \Sigma \), locally defined by \( S = \text{const} \), define the vector \( T \), one–form \( N \) and scalar \( \sigma \) now as

\[
N := \frac{1}{\sigma} (\dot{S} \partial_t + S' \partial_r), \tag{23}
\]

\[
T := \frac{1}{Q} \left[ S' \partial_t - \dot{S} \partial_r + w S' \partial_{\phi} \right], \tag{24}
\]

\[
\sigma := Q e^{-(a+b)}, \tag{25}
\]

where \( Q^2 := e^{2a} S'^2 - e^{2b} \dot{S}^2 \).

Then again all the assumptions circdiff, circalg and circkill of Proposition 5.1 are satisfied, including angular–momentum–conservation in the form \( J = 0 \).

The necessary and sufficient condition for geodesy can then be expressed as the single \( S \)–equation

\[
K = - \left( e^{2a} S' c' - e^{2b} \dot{S} c \right) \frac{\sigma}{Q^2}, \tag{26}
\]

where \( K := - \nabla \cdot N^2 \) again denotes the mean curvature of \( \Sigma \).

So now we are left with only one nonlinear second–order partial differential equation for one unknown \( J(t, r) \) (resp. \( S(t, r) \)), albeit with such a high complexity (comparable to the closely related and notoriously complex equation for
minimal surfaces), that in a non–stationary setting only in very special cases it can be hoped to get an exact solution. A corresponding circular orbit is said to exist in a certain \((t, r)\)–region, if the solution \(J\) (resp. \(S\)) is nonvanishing and real there, so that \(J^2 > 0\) (resp. \(S^2 > 0\)). Once we have such a solution, any other relevant quantity can be derived from it — in particular, the tangents \(T\) to the geodesics from equation (14) (resp. equation (24)). For the canonical equatorial metric given by equation (5), if \(J' \neq 0\) the equation \(J(t, r) = j_0 = \text{const}\) can be solved implicitly to give the time–development of the areal radius, \(r = f(t, j_0)\) (similarly for \(S\)).

5.1.2 Stability

Here we will analyze the stability of the solution of the \(J\)–equation for timelike orbits under linear perturbations. For this purpose, consider the perturbed geodesic with normalized tangent \(\tilde{T} := U - T\) and perturbation \(U\). Then from geodesy and normalization of \(T\) in first–order we must have

\[
\nabla_T U = -\nabla U T.
\]

(27)

Note that in order to have a more conventional form for the resulting dynamical system, we will now use the thick–dot notation \(\cdot\) instead of \(\nabla_T \lambda\),

\[
\text{ Proposition 5.2 }
\]

Define \(\mu := U \cdot C\) and \(\nu := U \cdot N\) and assume nonradial trajectories, \(J \neq 0\). Then there results the system

\[
\begin{align*}
\dot{\mu} &= -\sigma \nu, \\
\dot{\nu} &= \kappa \mu + \delta \nu,
\end{align*}
\]

(28, 29)

where \(\kappa := 2 \frac{\sigma J}{P^2 Q^2} (e^{2a} c' J' - e^{2b} c J) e^{-2c} + \frac{\sigma}{P Q} w', \quad \delta := \frac{\dot{\sigma}}{\sigma}.\)

Proof

Condition \(U \cdot T = 0\) with \(U \cdot C = \mu\) and \(U \cdot N = \nu\) is solved with

\[
U := \mu e^{-2c} (C + JT) + \nu N^2.
\]

(30)

Decomposing \(\nabla_T U = -\nabla U T\) into \(\dot{\mu} = -\nabla U J\) and \(\dot{\nu} = 2 \text{ symm}(\nabla N)(U, T)\); inserting \(U\) and using the angular symmetry as well as the \(J\)–equation gives the above system of equations. \(\blacksquare\)

The special case \(J = 0\) can be solved similarly by setting

\[
U := \mu e^{-2c} C + \nu N^2,
\]

(31)

with

\[
\begin{align*}
N &= \frac{1}{\sigma} (\dot{S} dt + S dr), \\
\sigma &= Q e^{-(a+b)}, \\
\text{where } \quad Q^2 &= e^{2a} S^2 - e^{2b} \dot{S}^2,
\end{align*}
\]

19 not to confound with the thin–dot notation \(\lambda \equiv \partial \lambda / \partial t\)
resulting in the following coefficients for the system of equations (28, 29),
\[ \kappa := \frac{\sigma}{Q} w', \quad \delta := \frac{\sigma}{\sigma}. \]

In the case purely spherically–symmetric case \( w = 0 \), the system can be immediately integrated to \( \nu = c \sigma, \mu = -c \sigma^2 + k \), with constants \( c, k \). For such radial timelike geodesics a boundedness criterium does not make much sense. Here we propose a criterium based on the expansion of the trajectory: the additional expansion \( c \nabla \cdot (\sigma N^3) \) should remain smaller than the unperturbed expansion \( \Theta := \nabla \cdot T \). This amounts to the condition
\[ |c| |\Delta S| \ll |\Theta|, \quad (34) \]
where \( \Delta \) denotes the d’Alembertian operator.

For the canonical equatorial metric (i.e. \( c = 1/2 \ln r^2, w = 0 \)) and \( J \neq 0 \) the coefficients of dynamical system simplify somewhat. Formally an angular frequency \( \omega \) for the perturbation can be introduced,
\[ \omega^2 := \kappa \sigma \equiv \frac{1}{r} \frac{(J^2)'}{r^2 + J^2}, \quad (35) \]
as well as a formal damping–term
\[ \delta := \frac{\sigma}{\sigma}, \quad (36) \]
These expressions are valid even in the time–dependent case. Whereas \( \kappa \), \( \sigma \), and \( \delta \) do also depend on the metric, \( \omega \) does not.

The above system of equations equations (28, 29) constitutes a system of first–order linear–homogeneous ordinary differential equations
\[ \dot{x} = A(\tau) x \]
for \( x := (\mu, \nu)^T \) with time–dependent coefficient–matrix \( A \). It has the form of the equation for a damped harmonic oscillator. Assuming the trajectories to be timelike geodesically complete, it can be considered as a nonautonomous dynamical system. The time–dependency comes from functions explicitly dependent on the affine parameter through the metric. The analysis of the stability of our circular orbits is thus reduced to the analysis of the stability of a linear nonautonomous dynamical system. Unfortunately the results of the stability theory for autonomous dynamical systems do not carry over. The appropriate mathematical notions are not as easy to apply and are outside the scope of this paper.

Only in the static case it could be considered an autonomous dynamical system, with associated two–dimensional phase–space, where the stability–behaviour is well–known.

### 5.2 Lightlike Orbits

The lightlike case is somewhat special and cannot immediately be dealt with by adapting the assumptions leading to proposition (5.1). Instead, we follow a more direct way.

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20 note the relation \( \Delta S = -\epsilon \sigma K + N^1 \nabla \sigma \)
21 note that for \( \delta \) to be well–defined, we must have \( J' \neq 0 \)
22 there is already an extensive literature on nonautonomous dynamical systems— see e.g. Kloeden and Pötzche, 2013 [29] for an introduction
23 see section 5 on static spacetimes
5.2.1 Existence

It is well-known (e.g. Wald 1984, ch. 4.2 [9]) that the normal vector to a null-hypersurface is automatically geodesic. That is, assuming the null-hypersurface \( \Sigma \) locally defined by \( S = 0 \) with \( (dS)^\ast \cdot dS = 0 \), then \( T := N^\ast \) with \( N^\ast := dS \) is an affinely parametrized null-geodesic, \( \nabla_T T = 0, T \cdot T = 0 \). This holds a fortiori also for a congruence \( \Sigma_\lambda \) of such hypersurfaces, locally defined by \( S = \lambda = \text{const} \).

Now, consider a symmetry of the metric given by the Killing-vector \( C \) and impose it also on the one-form \( N \), by requiring
\[
\mathcal{L}_C N \equiv d(C \cdot N) + C \cdot (dN) = 0. 
\]
(37)
The last term vanishes due to \( N = dS \), and we are left with the condition \( C \cdot N = \text{const} \). This is somewhat stronger than the usual conservation along \( N^\ast \). Its “equatorial” form is obtained again by setting \( \vartheta = \pi/2 \) in the \( S^2 \)-part of the metric. Without loss of generality we can set \( N := du \). Evidently \( C \cdot N \equiv 0 \), so that the only null geodesics compatible with spherical symmetry are the radial null geodesics, with \( J = 0 \). For the existence, this is the only constraint.

5.2.2 Stability

Again, for linear stability for \( \nabla N^\ast \cdot N = 0 \) with \( N^2 \cdot N = 0 \) we must require \( U \cdot N = 0 \) and \( \nabla N \cdot U = -\nabla U \cdot N^4 \). Introducing the auxiliary null vector \( M \), \( M \cdot M_\nu = 0 \), with \( M \cdot N = -1 \) and \( M \cdot C_\nu = 0 \), the algebraic condition can be satisfied with
\[
U = \mu e^{-2c} C - \nu N^4, 
\]
(39)
so that \( \mu = U \cdot C_\nu \) and \( \nu = U \cdot M_\nu \). Using geodesy and \( C \)-symmetry, we arrive at the trivial system
\[
\dot{\mu} = 0, \quad \text{(40)}
\]
\[
\dot{\nu} = 0. \quad \text{ (41)}
\]
Therefore linear stability does not require any constraint in addition to \( J = 0 \). The condition \( \nu = \text{const} \) amounts to a constant rescaling of \( N \), whereas \( \mu = \text{const} \) leads to a constant rotation of the trajectories and so in this sense they are rigid.

\footnote{E.g. the Schwarzschild line-element can be written \( ds^2 = X (dt + 1/X \, dr) (dt - 1/X \, dr) + r^2 d\Omega^2 \), where \( X := 1 - 2m/r \). With \( N_\pm := dt \pm 1/X \, dr \) we have \( N^2_\pm = 0 \) and \( N_\pm = dS \) with \( S := t \pm \sqrt{1/X \, dr} \). For the timelike Killing-vector \( C := \partial_t \), in fact \( C \cdot N = 1 = \text{const} \).}
6 Static Spacetimes

Here we consider explicitly static metrics in the sense of \( \dot{a} = \dot{b} = \dot{c} = 0 \) and \( w = 0 \) in the region of interest. Although in this case our algorithm is (as of course it should) completely equivalent to the very well-known algorithm based on the effective potential, it is more directly applicable. For example, in this case the fundamental \( J \)-equation of equation (15) is immediately solved by the simple algebraic relation

\[
J^2 = \frac{a'}{c'} e^{2c}.
\]

Assuming \( e^{2c} > 0 \) the necessary and sufficient condition for \( J^2 > 0 \) is either

\[
2a' > c' > a' > 0 \quad \text{or} \quad 2a' < c' < a' < 0.
\]

The local tangential velocity with respect to the static observers

\[
u_{\text{stat}}^2 := -g_{rr} \cdot r^2 + g_{\phi\phi} \cdot \varphi^2 
\equiv \frac{1 + g_{tt} \cdot t^2}{g_{tt} \cdot t^2} = \frac{J^2 e^{-2c}}{1 + J^2 e^{-2c}} = \frac{a'}{c'}.
\]

Note that as long as \( U \) remains timelike and \( J^2 > 0 \), then \( 0 < \nu^2 < 1 \).

In particular, for the canonical equatorial metric, this reduces to the extremely simple relation \( v_{\text{stat}}^2 = r a' \). Let us also note that in view of the asymptotically constant galactic rotation curves, requiring constant velocity \( \nu \) immediately results in the well-known relation\(^{25}\) \( a = \nu^2 \ln (r/r_0) \). This is not very satisfying, as the corresponding metric is not asymptotically Lorentzian.

The stability analysis also becomes particularly simple, as the dynamical system becomes undamped autonomous, with angular frequency\(^{26}\)

\[
\omega^2 := \kappa \sigma = (1 + J^2 e^{-2c})^{-1} (J^2)' c' e^{-2(b+c)}.
\]

Assuming \( c' > 0 \) in addition to \( J^2 > 0 \), the condition for stability reduces to the positivity of the squared frequency,

\[
\omega^2 \sim (J^2)' \sim a''/a' - 2a'/(c''/c' - 2c') > 0.
\]

For the canonical equatorial metric, the existence criterium \( J^2 > 0 \) then reduces to \( 1 > r a' > 0 \), whereas the stability \( (J^2)' > 0 \) criterium reduces to

\[
r^2 a'' + 3r a' - 2 (r a')^2 > 0.
\]

Both these criteria for circular orbits in a static spherically-symmetric metric can be found e.g. in Lake, 2004 \[^{17}\].

Recapitulating, the conditions for existence and stability of circular orbits in a static spherically symmetric metric are just the two conditions \( J^2 > 0 \) and \( (J^2)' > 0 \), where \( J^2 \) is defined by equation (42). Energy conservation is not invoked at all. This sets our algorithm apart from the conventional one where an effective potential \( V(r; L, E) \) has first to be set up, and where the constant parameters \( L, E \) have to be chosen so that for a particular \( r \), the three
conditions \( V = 0 \), \( V' = 0 \) and \( V'' > 0 \) are satisfied. Also the direct physical interpretability of \( J \) vs. \( V \) has the advantage of a better contextuality — after all we are dealing primarily with a problem having angular symmetry. But the most salient advantage is its applicability to a time–dependent setting, where the conventional approach inextricably based on energy conservation cannot be generalized.

7 Examples

In the following we will first apply our algorithm to some spacetimes which are better known under the generic name Schwarzschild–de Sitter spacetimes. They correspond to metrics solving the Einstein–equation including a de Sitter–term, \( \text{Ein} = -\Lambda g \) (\( g \) denotes the Einstein–tensor) and reducing to the Schwarzschild–solution for \( \Lambda = 0 \). Although they are (for \( \Lambda \geq 0 \)) locally isometric, their metric tensors differ markedly. To keep these particular solutions a part, we will here refer them by the name of their discoverers appended to the name of Schwarzschild:

i) Schwarzschild–Kottler metric (Kottler, 1918 [1]) and

ii) Schwarzschild–Robertson metric (Robertson, 1927 [2]).

We will apply our algorithm also to the Kerr–de Sitter metric, which generalizes the Schwarzschild–Kottler metric to a stationary rotating spacetime.

As an additional explicitly time–dependent example we will consider a particular cosmological solution with a scalar field and massive source, where the \( J \)–equation can be solved approximatively.

7.1 Schwarzschild–Kottler Metric

Here

\[
e^{2a} = e^{-2b} := 1 - 2 \frac{m}{r} - \frac{1}{3} \Lambda r^2, \quad e^{2c} = r^2,
\]

so that \( a + b = 0 \). In view of its cosmological reformulation in the next section, we assume \( K^2 := \frac{1}{3} \Lambda m^2 \geq 0 \). In addition assuming \( K^2 < 1/27 \), the equation \( e^{2b} = 0 \) has two positive real solutions \( r_{\pm} \), with \( e^{2a} > 0 \) in the “static” range \( r_- < r < r_+ \), \( r_0 := r_- > 2m \) denotes the black hole horizon, whereas \( r_s := r_+ < 1/H \), where \( H^2 := \frac{1}{3} \Lambda \), denotes the static limit. Despite the cosmological \( \Lambda \)–term this metric is evidently static in the above range, with Killing–vector \( \partial_t \).

Stable circular orbits in such metrics have already been studied by Stuchlík and Hledík, 1999 [13] and more recently by Nolan, 2014 [30] in the context of (non–static) McVittie metrics.

The equatorial circular trajectories satisfy the \( J \)–equation in the purely algebraic form of equation [12]. Defining the reduced areal radius \( \varrho := r/m \),

\[
J_{\varrho K}^2 = \frac{m^2 \varrho^2}{3} \frac{1 - K^2 \varrho^3}{\varrho - 3},
\]

which agrees with the conventional calculation and is positive in the range \( 3 < \varrho < K^{-2/3} \). This existence range slightly overlaps the above static range,
the higher bound exceeding the static limit. Without solving any differential equation, the corresponding trajectories can now be calculated directly from this $J$ using equation (53),

$$ T_{sk} = \left(1 + \frac{J^2}{r^2}\right)^{1/2} e^{-\alpha} \partial_r + \frac{J}{r^2} \partial_\varphi. \quad (50) $$

Then the corresponding local tangential velocity with respect to static observers defined by $U := e^{-\alpha} \partial_\varphi$ is simply

$$ v_{sk}^2 := \frac{e^{2b} r^2 + r^2 \varphi^2}{e^{2a} \ell^2} \equiv \frac{e^{2a} \ell^2 - 1}{e^{2a} \ell^2} = \frac{1 - K^2 g^3}{\ell - K^2 g^3 - 2}, \quad (51) $$

where $J^2/\ell^2$ is always smaller than the "bare" velocity $v_0^2 := 1/(\ell - 2)$ — a manifestation of the repulsive character of the $\Lambda$–term. More precisely,

$$ v_{sk}^2 = \frac{1 - K^2 g^3}{\ell - 2 - K^2 g^3} \equiv \frac{1}{\ell - 2} - \frac{\ell - 3}{(\ell - 2)^2} \left(1 - \frac{K^2 g^3}{\ell - 2}\right)^{-1} K^2 g^3 $$

$$ \approx \frac{1}{\ell - 2} \left(1 - \frac{\ell - 3}{(\ell - 2) K^2 g^3}\right) \quad (52) $$

up to $O(K^2 g^3)^2$, correcting the Kepler–relation $v^2 = 1/\ell$ for $\gg 1$ as long as $K^2 g^3 \ll 1$.

Let us make some rough order of magnitude estimates. Assuming a current Hubble–parameter $H = 75$ km/s Kpc $\approx 7.6 \times 10^{-11}$ ly$^{-1}$ then for a mass of the solar–system ($m \approx 1.6 \times 10^{-15}$ ly) and the orbit of Neptune ($r \approx 30$ AU $\approx 4.8 \times 10^{-4}$ ly), we have $K \approx 1.2 \times 10^{-23}$ and $g \approx 3.1 \times 10^8$. The correction–factor $\Delta$ (where $v_0 := v_0 (1 + \Delta)$) of the bare Kepler–relation results as $\Delta = -1/2 K^2 g^3 \approx -2.0 \times 10^{-18}$ and so would be negligible.

However for galaxies like the Andromeda–galaxy, with $m_G \approx 1.5 \times 10^{11} m_\odot \approx 2.3 \times 10^{-12} ly$ the situation is more favourable. For a circular orbit at the visible rim with $r \approx 1.1 \times 10^5$ ly, $K \approx 1.8 \times 10^{-12}$ and $g \approx 4.7 \times 10^6$, giving $\Delta \approx -1.6 \times 10^{-4}$, which is still below current empirical verification. However, extending beyond the luminous region into the "dark matter halo" — e.g. taking 18 $\ell$ would already give vanishing velocity. So, if the asymptotically flat galactic rotation curves admit an explanation within the current ΛCDM–paradigm, this negative effect of the repulsive $\Lambda$–term should be taken into account.

The squared (unperturbed) angular frequency can be defined as usual by $\Omega^2 := J^2/r^4$, giving

$$ \Omega^2 = \frac{1}{m^2} \frac{1 - K^2 g^3}{(\ell - 3) g^2}. \quad (54) $$

The equations for the dynamical system (28, 29) now have constant coefficients and no damping, $\delta = 0$, so their solutions are periodic in proper time, with

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30 see Abramowicz, 2016
31 we will use throughout units of lightyears (ly)
32 without the mass of the "dark matter halo"
angular frequency by equation (45)

\[ \omega^2 = \frac{1}{m^2} \frac{\varrho - 6 + K^2 \varrho^3 (15 - 4 \varrho)}{(\varrho - 3) \varrho^3}. \] (55)

Therefore the perturbation is bounded and must be considered as stable. The assumption of geodesic completeness for the system to be considered a proper DS is trivially satisfied in this case.

The perturbed angular frequency allows to calculate the perihelion precession as e.g. Wald 1984, ch. 63 [9], giving this time a correction-factor \( \Delta = \frac{1}{2} K^2 \varrho^4 \). For the Andromeda galaxy and a mass point at the rim, this would result in enhancing the standard precession-rate of \( 4.2 \times 10^{-15} \text{ rad/y} \) of the corresponding basic angular velocity of \( \Omega \approx 4.7 \times 10^{-9} \text{ rad/y} \) by a factor of 250 — far too small to be empirically relevant.

In the following fig. 1 the relevant quantities for \( K = 1.0 \times 10^{-6} \) as well as some of their reference quantities for \( K = 0 \) are displayed. The plot of

![Figure 1: SdS–LogLog–curves for \( J^2, J_0^2, \varrho^2, \varphi_0^2, \sigma^2, \Omega^2, \omega^2 (K = 1.0 \times 10^{-6}) \)](image)

\( \sigma^2 \sim (J^2)' \) nicely illustrates (better than \( \omega^2 \)) the stability-range \( (J^2)' > 0 \), properly contained in the existence-range \( J^2 > 0 \). This range is given by 6 < \( \varrho_{\min} \) < \( \varrho \) < \( \varrho_{\max} < (2 K)^{-2/3} \), where the exact limits could be obtained from a quartic equation.

### 7.2 Schwarzschild–Robertson Metric

A closely related metric is given by the line-element

\[ ds^2 = - \left( 1 - \frac{X}{1 + X} \right)^2 dt^2 + e^{2 H t} (1 + X)^4 \left( dr^2 + r^2 d\Omega^2 \right), \] (56)

\[ X := \frac{m}{2} e^{-H t}, \quad H := K/m \equiv \sqrt{\Lambda/3} = \text{const}, \]

which is regular for \( X \neq \pm 1 \) and \( r \neq 0 \). In fact, as already shown by Robertson, 1928 [4], for \( \Lambda > 0 \) this explicitly time-dependent metric is locally isometric to the Schwarzschild–Kottler metric defined by equation (48), transforming it

\( ^{33} \) neglecting eccentricity, this is in accordance with Kerr, Hauck and Mashhoon, 2003 [13]

\( ^{34} \) for galaxies \( K \approx 10^{-12} \) would be more realistic, but the plots would be less compelling

\( ^{35} \) all the second–order invariants are regular for \( X \neq -1 \)
by means of the coordinate–transformation
\[ t \to t + F \left( \frac{1 - X}{1 + X} \right), \tag{57} \]
\[ r \to r (1 + X)^2 e^{H t}, \tag{58} \]
\[ F(x) := \int \left( (1 - x^2) \left( (1 - x^2)^2 x^2 - 4 K^2 \right) \right)^{-1} dx. \]

This metric is the most simple nontrivial metric of the much studied class of spherically–symmetric metrics introduced by McVittie, 1933 [4]. As already noted by Robertson, 1927 [2], for \( \Lambda = 0 \) it goes over to the Schwarzschild–metric in isotropic coordinates, whereas for \( m = 0 \) the expanding de Sitter–metric results. So we have effectively a cosmological model with Dark Energy containing a Schwarzschild Black Hole.

For such an explicitly time–dependent metric apparently (generalized) circular orbits seem not to have been analyzed yet. In Carrera and Giulini, 2010 [25] an approximation is made for the McVittie Ansatz assuming small velocities, leading to the time–development of the areal radius \( R := r (1 + X)^2 e^{H t} \). In the case of the Schwarzschild–Robertson metric it effectively reduces to
\[ \ddot{R} = J^2 / R^3 - M / R^2 + H^2 R. \tag{59} \]

In this case we can even define an effective potential
\[ V = M / R - \frac{1}{2} J^2 / R^2 + \frac{1}{3} H^2 R^2. \tag{60} \]

This admits circular orbits in the proper sense (among time–dependent solutions) by requiring \( V = E = \text{const} \), \( \dot{R} = 0 \). Solving for \( J^2 \) this gives
\[ J^2_{\text{are}} = M R - H^2 R^4, \tag{61} \]
which makes sense in the range \( 0 < R < (M / H^2)^{1/3} = K^{-2/3} M \). The upper existence bound \( R_x := K^{-2/3} M \) thus agrees exactly with the one obtained for the corresponding Schwarzschild–Kottler metric. For the upper stability bound we get however \( R_s := (2 K)^{-2/3} M \), which is somewhat higher than the one for the Schwarzschild–Kottler metric.

In the case of the Schwarzschild–Robertson metric we could dispense with the above approximation and directly use the somewhat unwieldy coordinate–transformation of equations (57, 58) to get all quantities of interest based on the proper circular orbits of the Schwarzschild–Kottler metric.

The more conventional possibility would be to first find the timelike Killing–vector and then apply the standard approach based on the effective potential. However, the \( J \)–based approach allows a much more direct derivation. Recalling that the areal radius \( R \) is an invariant of any spherically–symmetric metric, and that \( J^2 \) is a scalar function, then from any solution \( J^2(r) \) in terms of the standard radial coordinate \( r := e^x \), we can immediately read–off
\[ J^2(R) := J^2(r), \tag{62} \]
using the same function \( J^2 \). Also, from \( \dot{r} = 0 \) then follows \( \dot{R} = 0 \) (with respect to their corresponding affine parametrizations). For the Schwarzschild–Robertson
metric, and using the reduced areal radius \( \rho := R/m \) with \( R = r (1 + X)^2 e^{Ht} \),
\[
J_{\text{SK}}^2 = m^2 \rho^2 \frac{1 - K^2 \rho^3}{\rho - 3}, \quad \text{where} \quad \rho = y \left(1 + \frac{1}{2y}\right), \quad y := \frac{r}{m} e^{Ht}, \quad (63)
\]
with inverse function \( \Upsilon \) in the branch \( \rho \geq 2 \),
\[
y = \Upsilon(\rho) := \frac{1}{2} (\rho - 1) + \frac{1}{2} \rho (1 - 2/\rho)^{1/2}. \quad (64)
\]
Observe that for \( y \to \infty \), \( J_{\text{SK}}^2 = J^2_{\text{Ker}} \), giving additional support to the above approximation, which was based on the assumption of small velocities.

A big advantage of our approach is that now from \( J_{\text{SK}}^2 \) we can calculate any other relevant quantity, like the tangent to the geodesics, \( T \). Calculating it conventionally by means of a coordinate transformation from the SK–solution would have been relatively cumbersome. Evidently, \( T \) will now also get a \( \partial_t \)–component. As an example, we will show how the local tangential velocity gets modified when choosing co–expanding observers \( U = \frac{1 + \bar{w}}{\bar{w}} \partial_t \) (with expansion \( \Theta = 3H \)). Analogously as in the previous derivation of equation (51),
\[
v_{2s}^2 := \frac{e^{2a}}{e^{2a}} \left( 1 - \frac{1}{\Upsilon(\rho)} \right) = \frac{1 - K^2 \rho^3 - Z^2}{\rho - K^2 \rho^3 - 2}, \quad (65)
\]
where \( J := J_{\text{SK}}^2 \), \( Z := \frac{e^{b-a} \bar{w}}{e^{K\rho}} \frac{\mathcal{J}}{\mathcal{J}} = K \frac{1}{\Upsilon(\rho)} \frac{\Upsilon'(\rho)}{\Upsilon(\rho)} - 2 e^{K\rho}, \quad \tau := t/m. \)
Evidently, \( v_{2s}^2 < v_{2s}^2 \). And if \( v_{2s}^2 > 0 \) at some time \( \tau_0 \), then for a fixed radius \( \rho \) there is a time \( \bar{\tau} \), such that \( v_{2s}^2 \geq 0 \) for \( \tau_0 \leq \tau \leq \bar{\tau} \) with \( v_{2s}^2 = 0 \) at \( \tau = \bar{\tau} \) — in this case the trajectory of the co–expanding observers has moved outside the existence region of the angular–momentum. For small \( K \) and \( 1 \ll \rho \), as well as \( \tau \ll 1/K \) (corresponding to \( t \ll 1/H \), the Hubble–time),
\[
v_{2s}^2 \approx v_{2s}^2 - \frac{K^2}{\rho^3} e^{2K\rho} + o(K^4, \rho^{-4}, \tau^2). \quad (66)
\]
Thus the correction–factor to the Schwarzschild–Kottler velocity \( v_{2s} \) of equation (53) for \( \tau = 0 \) is \( \Delta = -K^2/\rho^2 \). For example, for a circular orbit at the rim of the Andromeda galaxy and \( \tau = \tau_0 \), the correction–factor is \( |\Delta| \approx 7.9 \times 10^{-39} \), and so would be completely negligible.

### 7.3 Kerr–de Sitter Metric

Here we will briefly apply our formalism to the equatorial orbits in Kerr–de Sitter spacetime. Assuming time–independence of the metric\(^{35}\), the \( J \)-equation\(^{36}\) reduces to the quadratic equation for \( J^2/e^{2c} \),
\[
(a' + J^2/e^{2c}(\alpha' - c')) e^{2(a-c)} - w^2 J^2/e^{2c} \left(1 + J^2/e^{2c}\right) = 0, \quad (67)
\]
with solutions
\[
J^2/e^{2c} = \frac{\alpha'(\alpha' - \alpha) e^{2(a-c)} + \frac{1}{4} w^2 \pm \sqrt{\frac{1}{4} w^4 (4 \alpha' \alpha' e^{2(a-c)} + w^2)^{1/2}}}{(\alpha' - \alpha)^2 e^{2(a-c)} - w^2}. \quad (68)
\]
\(^{35}\)too complex to be fully displayed — however the quotients \( \mathcal{J}/\mathcal{J}' \) simplify significantly
\(^{36}\)if \( w' \neq 0 \), the Killing–vector \( \partial_t \) is not anymore hypersurface–orthogonal, i.e. the metric is stationary only
We base our derivations on the Kerr–de Sitter metric in the Boyer–Lindquist form as given by Stuchlík and Slaný, 2004 [18] With the auxiliary quantities

\[ \alpha := \frac{a}{m}, \quad x := \frac{m}{r}, \quad K^2 := \frac{1}{3} \Lambda m^2, \quad L^2 := \frac{K^2}{x^2}, \quad M^2 := 1 + K^2 \alpha^2, \quad N^2 := 1 + \alpha^2 x^2, \]

the equatorial form of the metric derived from the full metric is defined by

\[ e^{2a} = \frac{1}{M^2} \frac{(1 - L^2) N^2 - 2 x}{M^2 N^2 + 2 \alpha^2 x^2}, \quad (69) \]
\[ e^{2b} = \left( (1 - L^2) N^2 - 2 x \right)^{-1}, \quad (70) \]
\[ e^{2c} = \frac{m^2}{M^2} x^2 \left( M^2 N^2 + 2 \alpha^2 x^3 \right), \quad (71) \]
\[ w = -\alpha \frac{x^2}{m} \frac{L^2 N^2 + 2 x}{M^2 N^2 + 2 \alpha^2 x^3}. \quad (72) \]

The expression for \( J^2 \) resulting from equation (68) can be simplified to

\[ J^2_{\text{Kerr}} = \frac{m^2}{(1 + K^2 \alpha^2)^2} \frac{\left( 2 \alpha + \alpha x (x^2 + \alpha^2) K^2 \mp x^{-1/2}(x^2 + \alpha^2) \Delta_K^{1/2} \right)^2}{x^2 \left( 1 - \frac{3}{x} - \alpha^2 K^2 \pm 2 \alpha x^{-3/2} \Delta_K^{1/2} \right)}, \quad (73) \]

where \( \Delta_K := 1 - K^2 x^3 \).

Up to the factor \( m^2(1 + K^2 \alpha^2)^{-2} \) this agrees with Stuchlík and Slaný [18] And for \( \alpha = 0 \) it agrees with the expression equation (49) already derived for the Kottler–de Sitter metric, whereas for \( K = 0 \) it agrees with the expression derived by Bardeen, Press and Teukolsky, 1972 [6].

\[ J^2_{\text{Kerr}} = m^2 \frac{(x^2 + \alpha^2 \pm 2 \alpha x^{1/2})^2}{x (x^2 - 3 x \pm 2 \alpha x^{1/2})}. \quad (74) \]

### 7.4 Husain–Martínez–Núñez Metric

Although in general an exact solution to the J–equation seems to be hopeless, certain metrics admit a straightforward first–order approximate solution.

In the context of scalar field collapse some exact solutions to the Einstein field equation with a massless scalar have been found by Husain–Martínez–Núñez, 1994 [10] Assuming without loss of generality \( r > 0 \) and \( t > -1/2H \)

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38 there \( y \) instead of \( K^2 \) is used and \( a, r, t \) are in units of the mass \( m \)
39 they use an angular momentum \( J \) rescaled by \( (1 + K^2 \alpha^2)/m \)
40 see also Chandrasekhar 1983, ch. 7 [3] for a more detailed derivation
41 see also Faraoni 2015, ch. 4.8 [31] for a more detailed exposition
with $H \geq 0$, the metric of their case I. can be more conveniently written as

$$ds^2 = (1 + 2 H t) \left(-F^\alpha dt^2 + F^{-\alpha} dr^2 + r^2 F^{1-\alpha} d\Omega^2\right),$$

$$\phi = \pm \frac{1}{6} \sqrt{6} \ln (F^\alpha (1 + 2 H t)^3), \quad F := 1 - 2 \frac{m}{r}, \quad \alpha := \pm \frac{1}{2} \sqrt{3}. $$  

The constant $H$ can be interpreted as the Hubble–Parameter of the corresponding cosmological FLRW–metric (with scale–factor $a(t) = (1 + 3 H t)^{1/3}$) when setting $m = 0$. On the other hand, for $H = 0$, we get a particular solution with parameter $\alpha = \pm \sqrt{3}/2$ and effective mass $M := \alpha m$ of the two–parameter family of massless scalar solutions with $m \neq 0$ and $\alpha^2 \leq 1$, well–known under the diverse names (in chronological order of their rediscoverers) of Fisher ('48), Buchdahl ('59), Newman–Janis–Winicour ('69) and Wyman ('81).

As already shown by Buchdahl, 1959 [5], this family of solutions not only contains the Schwarzschild–solution as the special case $\alpha = 1$ with $m > 0$, but moreover is PPN–compatible with it, as long as the effective mass $M$ is positive. So the standard gravitational tests cannot fix the value of $\alpha$, except when going to the near–field, where $(m/r)^2$–terms become appreciable.

Let us first calculate the local circular velocity (with respect to static observers) for the Buchdahl–metric. Our static J–algorithm immediately gives

$$v_{\text{max}}^2 := \frac{a'}{c'} = \frac{\alpha m}{r - (1 + \alpha) m} = \alpha \frac{m}{r} \left(1 + (1 + \alpha) \frac{m}{r} + o \left(\frac{m}{r}\right)^2\right),$$

which in fact up to $o(m/r)^2$–terms seems to coincide with the circular velocity corresponding to an effective Schwarzschild–mass $M = \alpha m$. However, for the proper existence of the velocity (i.e. bound circular orbits), we must in addition assume $M > 0$. This restricts the general HMN–solution to solutions where the “naked mass” $m$ has the same sign as the parameter $\alpha = \pm \sqrt{3}/2$. In fact, as shown by Buchdahl, there is a discrete isometry $m \rightarrow -m$, $\alpha \rightarrow -\alpha$ between the apparently different metrics. Therefore, we will restrict in what follows to the “physical metric” with parameters $\alpha = +\sqrt{3}/2$ and $m > 0$.

Now to the approximate solution for the HMN–metric. Noting that this metric is conformally related with factor $1 + 2 H t$ to the above static Buchdahl–metric, this suggests the following Ansatz

$$J_{\text{max}}^2 := \left(\frac{a'}{c'} - \frac{\varphi}{a'}\right)^2 \approx \left(1 + 2 H t\right),$$

where the functions $a$ and $c$ are taken from the corresponding static Buchdahl–metric with $M > 0$. Inserting this Ansatz into the J–equation shows that it is satisfied up to $o(H^2)$–terms — more precisely up to $1/m o(K^2 x)$, where again $K := H m$ and $x := r/m$. In terms of the areal radius $\varphi := r (1+X)^2(1+2 H t)^{1/2}$,

$$J_{\text{max}}^2 \approx M \varphi (1 + H t), \quad M := \alpha m,$$

42in fact they use a slightly different normalization, setting $m = 1$ (their $c$) and at instead of our “cosmological” scale factor $1 + 2 H t$, with $a = \pm 1$ and $t$ with $t \geq 0$
43the sign of $\phi$ is independent of that of $\alpha$
44also Bronnikov, 1973 [7] should be mentioned
45i.e., up to isometries
46in the following this seminal paper will be cited only by name
47he uses the opposite signature of the metric and our parameter $\alpha$ is denoted by $\beta$
for \( q \rightarrow \infty \) and up to \( o(H^2) \).

Taking again the Andromeda galaxy as an example, the approximate distance where our approximation begins to break down is \( d \approx 7.5 \times 10^{21} \) ly. This is many orders of magnitude higher than the extent of our observable universe estimated to be \( r \approx 9.3 \times 10^{29} \) ly (slightly below the Hubble–radius \( r_H := H^{-1} \approx 1.3 \times 10^{10} \) ly). And one of the biggest known cosmological structures, the local Laniakea supercluster, has an radial extent of \( r \approx 2.6 \times 10^{8} \) ly, with about \( 10^5 \) galaxies \( (m \approx 3.9 \times 10^9 \) ly). The breakdown radius is reduced to \( d \approx 4.5 \times 10^{15} \) ly, which is still significantly higher than the extent of our observable universe (even when allowing a mass–factor of 10 for interstellar gas and radiation). Therefore for all practical purposes, even for the biggest known structures, for this particular cosmological model based on the HMN–solution, we can safely ignore the \( H^2 \)– and any higher–order terms.

Also the velocity \( v^2 \) with respect to a coexpanding observer gets the same Hubble–factor \((1+Ht)\). This can be seen as follows. First we get, up to ignorable \( K^2x \)–terms, using equation (14),

\[
-g_{tt} t^2 = P^2 = 1 + J^2_{\text{HMN}} e^{-2c} f(1 + 2Ht) \equiv \frac{\alpha'}{e'} - \frac{e'}{a'} e^{2c},
\]  

as in the static case of equation (42). Then, up to ignorable \( K^2x \)–terms,

\[
v^2_{\text{HMN}} = \frac{\alpha m}{r - (1 + \alpha) m}.
\]  

This is exactly the expression of the velocity of equation (77) for the corresponding static Buchdahl–metric and apparently time–independent. However, when again expressed in terms of the areal radius,

\[
v^2_{\text{HMN}} \approx \frac{M}{\varrho} (1 + Ht).
\]  

Of course, this HMN–metric is very special in that it is not only conformally static, but also in that the squared cosmological scale–factor is linear in time. A slight generalization where our approximation still works would consist in taking an arbitrary power of the above scale–factor — however the corresponding field equations would not anymore be satisfied. Therefore the validity of our approximation referring to the HMN–metric must be considered as fortuitous.\footnote{In particular this approximation fails for the solution of Sultana–Dyer (in the conformally Schwarzschild form of the metric as given by Faraoni 2015, ch. 47 \[31\]).}

Trying to conventionally solve the relatively complex geodesic equation, this approximate solution would hardly have been found. This again shows the superiority of the \( J \)–based approach.

### 8 Discussion

Let us emphasize again that we could of course have done the static analysis in section 6 using the conventional approach based on an effective potential. However here we calculated the relevant quantities in a more transparent and shorter way using only \( J^2 \) and its first derivative. In fact, the usefulness of the
angular–momentum function seems not to have been recognized so far. However, the full potential of our approach would only be revealed when analyzing effectively time–dependent metrics, like inhomogeneous isotropic cosmologies. In particular, our approach would allow to find out if the asymptotically flat galactic rotation curves could be understood as a manifestation of some dynamic gravitational effects — without invoking either dark matter or Milgrom’s phenomenological MOND–theory. Unfortunately such cosmological models cannot be analyzed analytically, except when exact solutions to the J–equation can be found, which is highly improbable. Nevertheless, appropriate approximate solutions to the J–equation could be constructed.

However we were able to analyze two particular time–dependent spacetimes: i) a metric with Λ–term (the Schwarzschild–Robertson solution) and ii) a metric with Hubble–expansion (the Husain–Martinez–Núñez solution). Whereas in the first example cosmological effects first appear in $o(\Lambda) = o(H^2)$ and turn out to be negligible for galaxies, in the second example both $J^2$ and $v^2$ are in $o(H)$ subject to the local Hubble–flow with scale–factor $a = 1 + Ht$.

Perhaps more realistic such cosmological models will be needed, as well as generally applicable approximation methods in order to better understand and perhaps resolve the as yet empirical baryonic Tully–Fisher relation. The present work can be seen as a step in this direction.

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