On the properties of Poisson random measures associated with a $G$-Lévy process

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Abstract

In this paper we study the properties of the Poisson random measure and the Poisson integral associated with a $G$-Lévy process. We prove that a Poisson integral is a $G$-Lévy process and give the conditions which ensure that a Poisson integral belongs to a good space of random variables. In particular, we study the relation between the quasi-continuity of an integrand and the quasi-continuity of the integral. Lastly, we apply the results to establish the pathwise decomposition of a $G$-Lévy process into a generalized $G$-Brownian motion and a pure-jump $G$-Lévy process and prove that both processes belong to a good space of random variables.

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1 Introduction

In the last years much effort has been made to investigate the problem of model uncertainty. The problem has roots in the real life: participants of the financial markets are interested in measuring the risk of losses connected with the financial positions which they take. It is worth to note that besides the model-implicit risks resulting from the movement of prices, there is also a risk (or rather the uncertainty) that one has misspecified the model by either assuming the wrong parameters or taking a wrong class of models. The uncertainty connected with the model misspecification is very interesting from the mathematical point of view as it is on the one hand difficult to quantify, and on the other hand, it can have serious consequences to the conclusions drawn from the misspecified model. It is also very challenging as often it leads to the family of the probability measures which cannot be dominated by a single reference measure.

Such undominated families of models were studied first by Denis and Martini in [1] where they proposed the framework for investigating the volatility uncertainty via quasi-sure approach. At the same time Shige Peng introduced his $G$-Brownian motion, a process on a canonical space equipped with a sublinear expectation called a $G$-expectation (see [13]). Peng constructed the $G$-expectation using the viscosity solutions of a non-linear heat equation reflecting the unknown level of volatility. Both approaches are closely connected, as

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it was shown in \cite{3}, and the interest spurred by both papers resulted in many interesting papers: \cite{15}, \cite{18}, \cite{21}, \cite{17}, \cite{16}, \cite{6}, \cite{5}, \cite{8}, \cite{10} and many others.

The natural generalization of both frameworks is to consider a jump processes and the uncertainty associated with the drift, the volatility and the jump component. Such jump process was first considered in \cite{7} in which a process called a $G$-Lévy process was introduced. A $G$-Lévy process is a càdlàg process defined on a sublinear expectation space which has increments stationary and independent of the past and which might be decomposed into a pure-jump part and a continuous part. A sublinear expectation associated with a $G$-Lévy process may be defined by some non-linear IPDE describing all three sources of uncertainty. Ren in \cite{17} showed that such a sublinear expectation might be represented as supremum of ordinary expectations over a relatively compact family of probability measures (which again cannot be dominated by a single reference probability measure). Nutz and Neufeld in \cite{9} and we in gave a characterization of the laws used in this representation. The difference between our approach and the one by Nutz and Neufeld is that we consider a strong formulation via some Itô-Lévy integrals, whereas Nutz et al. characterizes the laws using much weaker conditions on characteristics of a semimartingale. More information on the $G$-Lévy processes can be found in Section 2.

In this paper we investigate the properties of the Poisson random measure and a Poisson integral for $G$-Lévy processes. We investigate both the distributional properties of these objects and the regularity w.r.t. omega. Among other results, we show that a Poisson integral is itself a $G$-Lévy process and its regularity w.r.t. omega (i.e. quasi-continuity) is strictly connected with the regularity of an integrand. The ultimate goal of the paper is to show that the decomposition of a $G$-Lévy process into a generalized $G$-Brownian motion and a process of finite variation might be done pathwise on the same sublinear expectation space. Moreover, the may require the finite variation part to be a $G$-martingale. At last we show that we cannot require the finite variation part to be a symmetric $G$-martingale, unless we extend the sublinear expectation space. Then, however, the decomposition is only meant in the distributional sense.

The structure of the paper is as follows. In Section 2 we give an introduction to the framework and present the most important results used throughout the paper. In Section 3 we consider two different ways of compensating a pure-jump $G$-Lévy process on an auxiliary sublinear expectation space. Section 4 is devoted to studying the continuity of an Poisson integral as an operator and investigating its distributional properties. We also establish the characterization of the spaces of integrands in terms of their tightness, uniform integrability and regularity w.r.t. $z$. In Section 5 we investigate the regularity of the Poisson integral w.r.t. $\omega$. We give sufficient condition for the quasi-continuity of the integral in terms of the quasi-continuity of the integrand. We also establish the conditions for the quasi-continuity of the jump times of a $G$-Lévy process and use them to give the necessary conditions for the quasi-continuity of the Poisson integral. Lastly, in Section 6 we apply the previous results to establish the decomposition of a $G$-Lévy process into a generalized $G$-Brownian motion and a pure-jump $G$-Lévy process, which are defined without introducing the auxiliary sublinear expectation space. We also investigate in that section the possibility of compensating a pure jump process into a $G$-Lévy martingale.

2 Preliminaries

Let $\Omega$ be a given space and $\mathcal{H}$ be a vector lattice of real functions defined on $\Omega$, i.e. a linear space containing 1 such that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. We will treat elements of $\mathcal{H}$ as random variables.
Definition 1. A sublinear expectation $\mathbb{E}$ is a functional $\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties:

1. **Monotonicity:** If $X, Y \in \mathcal{H}$ and $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

2. **Constant preserving:** For all $c \in \mathbb{R}$ we have $\mathbb{E}[c] = c$.

3. **Sub-additivity:** For all $X, Y \in \mathcal{H}$ we have $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$.

4. **Positive homogeneity:** For all $X \in \mathcal{H}$ we have $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space.

We will consider a space $\mathcal{H}$ of random variables having the following property: if $X_i \in \mathcal{H}$, $i = 1, \ldots, n$ then

$$\phi(X_1, \ldots, X_n) \in \mathcal{H}, \quad \forall \phi \in C_{b,Lip}(\mathbb{R}^n),$$

where $C_{b,Lip}(\mathbb{R}^n)$ is the space of all bounded Lipschitz continuous functions on $\mathbb{R}^n$. We will express the notions of a distribution and an independence of the random vectors using test functions in $C_{b,Lip}(\mathbb{R}^n)$.

Definition 2. An $n$-dimensional random vector $Y = (Y_1, \ldots, Y_m)$ is said to be independent of an $n$-dimensional random vector $X = (X_1, \ldots, X_n)$ if for every $\phi \in C_{b,Lip}(\mathbb{R}^n \times \mathbb{R}^m)$

$$\mathbb{E}[\phi(X, Y)] = \mathbb{E}[\mathbb{E}[\phi(x, Y)]_{x=X}].$$

Let $X_1$ and $X_2$ be $n$-dimensional random vectors defined on sublinear random spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$ respectively. We say that $X_1$ and $X_2$ are identically distributed and denote it by $X_1 \sim X_2$, if for each $\phi \in C_{b,Lip}(\mathbb{R}^n)$ one has

$$\mathbb{E}_1[\phi(X_1)] = \mathbb{E}_2[\phi(X_2)].$$

Now we give the definition of G-Lévy process (after [7]).

Definition 3. Let $X = (X_t)_{t \geq 0}$ be a $d$-dimensional càdlàg process on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We say that $X$ is a Lévy process if:

1. $X_0 = 0$,

2. for each $t, s \geq 0$ the increment $X_{t+s} - X_t$ is independent of $(X_t, \ldots, X_{t_n})$ for every $n \in \mathbb{N}$ and every partition $0 \leq t_1 \leq \ldots \leq t_n \leq t$,

3. the distribution of the increment $X_{t+s} - X_t$, $t, s \geq 0$ is stationary, i.e. does not depend on $t$.

Moreover, we say that a Lévy process $X$ is a G-Lévy process, if satisfies additionally following conditions:

4. there a 2$d$-dimensional Lévy process $(X^c_t, X^d_t)_{t \geq 0}$ such for each $t \geq 0$ $X_t = X^c_t + X^d_t$, where the equality is meant in the distributional sense,

5. processes $X^c$ and $X^d$ satisfy the following growth conditions

$$\lim_{t \downarrow 0} \mathbb{E}[||X^c_t||^2]^{1/2} = 0; \quad \mathbb{E}[||X^d_t||] < Ct \text{ for all } t \geq 0.$$
**Remark 4.** The condition $\delta$ implies that $X^c$ is a $d$-dimensional generalized $G$-Brownian motion (in particular, it has continuous paths), whereas the jump part $X^d$ is of finite variation.

Peng and Hu noticed in their paper that each $G$-Lévy process $X$ might be characterized by a non-local operator $G$.

**Theorem 5** (Lévy-Khintchine representation, Theorem 35 in [7]). Let $X$ be a $G$-Lévy process in $\mathbb{R}^d$. For every $f \in C_b^d(\mathbb{R}^d)$ such that $f(0) = 0$ we put
\[ G[f(.):= \lim_{\delta \downarrow 0} \mathbb{E}[f(X_\delta)]\delta^{-1}. \]

The above limit exists. Moreover, $G$ has the following Lévy-Khintchine representation
\[ \mathbb{E}[f(.)] = \sup_{(v,p,Q) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d} f(z)v(dz) + \langle Df(0),q \rangle + \frac{1}{2} \text{tr}[D^2f(0)QQ^T] \right\}, \]

where $\mathbb{R}^d := \mathbb{R}^d \setminus \{0\}$, $\mathcal{U}$ is a subset $\mathcal{U} \subset \mathcal{M}([0,1]) \times \mathbb{R}^d \times \mathbb{R}^{d \times d}$ and $\mathcal{M}(\mathbb{R}^d)$ is a set of all Borel measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. We know additionally that $\mathcal{U}$ has the property
\[ \sup_{(v,p,Q) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d} |z|v(dz) + |q| + \text{tr}[QQ^T] \right\} < \infty. \quad (1) \]

**Theorem 6** (Theorem 36 in [7]). Let $X$ be a $d$-dimensional $G$-Lévy process. For each $\phi \in C_{b,Lip}(\mathbb{R}^d)$, define $u(t,x) := \mathbb{E}[\phi(x + X_t)]$. Then $u$ is the unique viscosity solution of the following integro-PDE
\[ 0 = \partial_t u(t,x) - G[u(t,x + .) - u(t,x)] \]
\[ = \partial_t u(t,x) - \sup_{(v,p,Q) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d} [u(t,x+z) - u(t,x)]v(dz) \right\} \]
\[ + \langle Du(t,x),q \rangle + \frac{1}{2} \text{tr}[D^2u(t,x)QQ^T] \quad (2) \]

with initial condition $u(0,x) = \phi(x)$.

It turns out that the set $\mathcal{U}$ used to represent the non-local operator $G$ fully characterize $X$, namely having $X$ we can define $\mathcal{U}$ satisfying eq. (1) and vice versa.

**Theorem 7.** Let $\mathcal{U}$ satisfy (1). Consider the canonical space $\Omega := \mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)$ of all càdlàg functions taking values in $\mathbb{R}^d$ equipped with the Skorohod topology. Then there exists a sublinear expectation $\mathbb{E}$ on $\mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that the canonical process $(X_t)_{t \geq 0}$ is a $G$-Lévy process satisfying Lévy-Khintchine representation with the same set $\mathcal{U}$.

The proof might be found in [7] (Theorem 38 and 40). We will give however the construction of $\mathbb{E}$, as it is important to understand it.

Begin with defining the sets of random variables. Put
\[ Lip(\Omega_T) := \{ \xi \in L^0(\Omega) : \xi = \phi(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}), \]
\[ \phi \in C_{b,Lip}(\mathbb{R}^{d \times n}), 0 \leq t_1 < \ldots < t_n < T \}, \]

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where $X_t(\omega) = \omega_t$ is the canonical process on the space $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and $L^1(\Omega)$ is the space of all random variables, which are measurable to the filtration generated by the canonical process. We also set

$$Lip(\Omega) := \bigcup_{T=1}^{\infty} Lip(\Omega_T).$$

Firstly, consider the random variable $\xi = \phi(X_{t+s} - X_t)$, $\phi \in C_b, Lip(\mathbb{R}^d)$. We define

$$\hat{\mathbb{E}}[\xi] := u(s, 0),$$

where $u$ is a unique viscosity solution of integro-PDE (2) with the initial condition $u(0, x) = \phi(x)$. For general

$$\xi = \phi(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}), \quad \phi \in C_b, Lip(\mathbb{R}^{d \times n})$$

we set $\hat{\mathbb{E}}[\xi] := \phi_n$, where $\phi_n$ is obtained via the following iterated procedure

$$\phi_1(x_1, \ldots, x_{n-1}) = \hat{\mathbb{E}}[\phi(x_1, \ldots, X_t - X_{t_{n-1}})],$$

$$\phi_2(x_1, \ldots, x_{n-2}) = \hat{\mathbb{E}}[\phi_1(x_1, \ldots, X_t - X_{t_{n-1}})],$$

$$\vdots$$

$$\phi_{n-1}(x_1) = \hat{\mathbb{E}}[\phi_{n-1}(x_1, X_{t_2} - X_{t_1})],$$

$$\phi_n = \hat{\mathbb{E}}[\phi_{n-1}(X_{t_1})].$$

Lastly, we extend definition of $\hat{\mathbb{E}}[\cdot]$ on the completion of $Lip(\Omega_T)$ (respectively $Lip(\Omega)$) under the norm $\|x\|_p := \hat{\mathbb{E}}[|x|^p]^\frac{1}{p}$, $p \geq 1$. We denote such a completion by $L_G^p(\Omega_T)$ (or resp. $L_G^p(\Omega)$).

Note that we can equip the Skorohod space $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ with the canonical filtration $\mathcal{F}_t := \mathcal{B}(\Omega_t)$, where $\Omega_t := \{\omega_{\mathcal{F}_t} : \omega \in \Omega\}$. Then using the procedure above we may in fact define the time-consistent conditional sublinear expectation $\hat{\mathbb{E}}[\xi|\mathcal{F}_t]$. Namely, w.l.o.g. we may assume that $t = t_i$ for some $i$ and then

$$\hat{\mathbb{E}}[\xi|\mathcal{F}_{t_i}] := \phi_{n-1}(X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_i} - X_{t_{i-1}}).$$

One can easily prove that such an operator is continuous w.r.t. the norm $\|\cdot\|_1$ and might be extended to the whole space $L_G^1(\Omega)$. By construction above, it is clear that the conditional expectation satisfies the tower property, i.e. is dynamically consistent.

**Definition 8.** A stochastic process $(M_t)_{t \in [0,T]}$ is called a G-martingale if $M_t \in L_G^1(\Omega_t)$ for every $t \in [0,T]$ and for each $0 \leq s \leq t \leq T$ one has

$$M_s = \hat{\mathbb{E}}[M_t|\mathcal{F}_s].$$

Moreover, a G-martingale $M$ is called symmetric, if $-M$ is also a G-martingale.

### 2.1 Representation of $\hat{\mathbb{E}}[\cdot]$ as an upper-expectation

$\hat{\mathbb{E}}[\cdot]$ satisfies the definition of a coherent risk measure and, as it is well known, coherent risk measures exhibit representation as a supremum of some expectations over a family of some probabilities. In [17] it has been proved that the sublinear expectation associated with a G-Lévy process can be represented as such an upper-expectation. Moreover, in [12] we characterized that family of probability measures as laws of some Itô-Lévy integrals under some conditions on the family of Lévy measures (see Section 3 in [12]). We will use this characterization and take the following assumption throughout this paper.
Moreover, we can choose the functions \( g \) explicitly in Appendix, Subsection 7.1.

It is elementary that the equation (2) is equivalent to the following equation

\[
\sup_{v \in \mathcal{V}} \int_{|z| < 1} |z|^q v(dz) < \infty.
\]

**Remark 9.** Let \( \mathcal{G}_B \) denote the set of all Borel function \( g : \mathbb{R}^d \to \mathbb{R}^d \) such that \( g(0) = 0 \). It might be checked that for all Lévy measures \( \mu \in \mathcal{M}(\mathbb{R}^d) \) which are absolutely continuous w.r.t. Lebesgue measure there exists a function \( g_v \in \mathcal{G}_B \) such that

\[
v(B) = \mu(g_v^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R}^d).
\]

Moreover, we can choose the functions \( g_v \) in such a way that for all \( \epsilon > 0 \) there exists \( \eta > 0 \) such that for all \( v \in \mathcal{V} \) we have \( g_v^{-1}(B(0, \epsilon^c)) \subset B(0, \eta^c) \). We construct such a function explicitly in Appendix, Subsection 7.1.

We fix a measure \( \mu \) and assume additionally that \( \int_{\mathbb{R}^d} |z| \mu(dz) < \infty \). We may consider a different parametrizing set in the Lévy-Khintchine formula. Namely, using

\[
\hat{\mathcal{U}} := \{(v, p, q) \in \mathcal{G}_B \times \mathbb{R}^d \times \mathbb{R}^{d \times d} : (v, p, q) \in \mathcal{U}\}
\]

it is elementary that the equation (2) is equivalent to the following equation

\[
0 = \partial_t u(t, x) - \sup_{(p, p, Q) \in \hat{\mathcal{U}}} \left\{ \int_{\mathbb{R}^d} [u(t, x + g(z)) - u(t, x)] \mu(dz) \right\} + \langle Du(t, x), p \rangle + \frac{1}{2} \text{tr}[D^2 u(t, x)QQ^T].
\]

Let \((\hat{\Omega}, \hat{\mathcal{G}}, \hat{\mathbb{P}}_0)\) be a probability space carrying a Brownian motion \( W \) and a Lévy process with a Lévy triplet \((0, 0, \mu)\), which is independent of \( W \). Let \( N(dt, dz) \) be a Poisson random measure associated with that Lévy process. Define \( N_t = \int_0^t N(t, dz) \), which is finite \( \mathbb{P}_0 \)-a.s. as we assume that \( \mu \) integrates \(|z|\). We also define the filtration generated by \( W \) and \( N \):

\[
\mathcal{G}_t := \sigma(W_s, N_s : 0 \leq s \leq t) \vee \mathcal{N}; \quad \mathcal{N} := \{A \in \hat{\mathcal{G}} : \mathbb{P}_0(A) = 0\}; \quad \mathcal{G} := (\mathcal{G}_t)_{t \geq 0}.
\]

**Theorem 10** (Theorem 11-13 and Corollary 14 in [12]). *Introduce a set of integrands \( \mathcal{A}_{t,T} \), \( 0 \leq t < T \), associated with \( \mathcal{U} \) as a set of all processes \( \theta = (\theta^1, \theta^2, \theta^3) \) defined on \([t, T]\) satisfying the following properties:

1. \((\theta^1, \theta^2, \theta^3)\) is \( \mathcal{G} \)-adapted process and \( \theta^d \) is \( \mathcal{G} \)-predictable random field on \([t, T] \times \mathbb{R}^d \).
2. For \( \mathbb{P}_0 \)-a.a. \( \omega \in \hat{\Omega} \) and a.e. \( s \in [t, T] \) we have that \((\theta^d(s, \cdot)(\omega), \theta^1_s(\omega), \theta^2_s(\omega)) \in \hat{\mathcal{U}}\).
3. \( \theta \) satisfies the following integrability condition

\[
\mathbb{E}^{\mathbb{P}_0}\left[ \int_t^T \left[ |\theta^1_s| + |\theta^2_s|^2 + \int_{\mathbb{R}^d} |\theta^d(s, z)| \mu(dz) \right] ds \right] < \infty.
\]
For \( \theta \in \mathcal{A}_{0,\infty}^d \) denote the following Lévy-Itô integral as

\[
B_t^\theta = \int_t^T \theta_s^1 \, ds + \int_t^T \theta_s^2 \, dW_s + \int_{[t,T]} \int_{\mathbb{R}^d} \theta(s,z) N(ds,dz).
\]

Lastly, for a fixed \( \phi \in C_{b,Lip}(\mathbb{R}^d) \) and fixed \( T > 0 \) define for each \( (t,x) \in [0,T] \times \mathbb{R}^d \)

\[
u(t,x) := \sup_{\theta \in \mathcal{A}^d_{0,\infty}} \mathbb{E}_\theta^0[\phi(x + B_T^\theta)].
\]

Then under Assumption \( u \) is the viscosity solution of the following integro-PDE

\[
\partial_t u(t,x) + G[u(t,x + .) - u(t,x)] = 0 \quad (5)
\]

with the terminal condition \( u(T,x) = \phi(x) \). Moreover, for every \( \xi \in L^1(\Omega) \) we can represent the sublinear expectation in the following way

\[
\hat{E}[\xi] = \sup_{\theta \in \mathcal{A}^d_{0,\infty}} \mathbb{E}^{\theta}[\xi],
\]

where \( \mathbb{E}^{\theta} := \mathbb{P}_0 \circ (B^d,0,\theta)^{-1}, \theta \in \mathcal{A}^d_{0,\infty} \). We will introduce also the following notation \( \mathcal{P} := \{\mathbb{P}^{\theta} : \theta \in \mathcal{A}^d_{0,\infty}\} \).

We can define the capacity \( c \) associated with \( \hat{E} \).

**Definition 11.** Let \( \hat{E}[\cdot] \) has a following representation \( \hat{E}[\cdot] = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\theta}[\cdot] \). Then capacity \( c \) associated with \( \hat{E} \) is defined as

\[
c(A) := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(A), \quad A \in \mathcal{B}(\Omega).
\]

We will say that a set \( A \in \mathcal{B}(\Omega) \) is polar if \( c(A) = 0 \). We say that a property holds quasi-surely (q.s.) if it holds outside a polar set.

**Remark 12.** We can also extend our sublinear expectation to all random variables \( Y \) on \( \Omega_T \) (or \( \Omega \)) for which the following expression has sense

\[
\hat{E}[Y] := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\theta}[Y].
\]

We can thus can also extend the definition of the norm \( \|\cdot\|_p \) and define following spaces

1. Let \( L^0(\Omega_T) \) be the space of all random variables on \( \Omega_T \). Let \( L^p(\Omega_T) \), \( p \geq 1 \) be a space of all equivalence classes of functions in \( L^0(\Omega_T) \) s.t. \( \|\cdot\|_p \) norm is finite.
2. Let \( B_b(\Omega_T) \) be the space of all bounded random variables in \( L^0(\Omega_T) \). The completion of \( B_b(\Omega_T) \) in the norm \( \|\cdot\|_p \) will be denoted as \( L^p_{c,b}(\Omega_T) \).
3. Let \( C_b(\Omega_T) \) be the space of all continuous and bounded random variables in \( L^0(\Omega_T) \). The completion of \( C_b(\Omega_T) \) in the norm \( \|\cdot\|_p \) will be denoted as \( L^p_{c,b}(\Omega_T) \).
4. Let \( C_{b,Lip}(\Omega_T) \) be the space of all Lipschitz continuous random variables in \( C_b(\Omega_T) \). The completion of \( C_{b,Lip}(\Omega_T) \) in the norm \( \|\cdot\|_p \) will be denoted as \( L^p_{c,Lip}(\Omega_T) \).
Definition 13. We will say that the random variable $Y \in L^0(\Omega)$ is quasi-continuous, if for all $\epsilon > 0$ there exists an open set $O$ such that $c(O) < \epsilon$ and $Y|_{O^c}$ is continuous. For convenience, we will often use the abbreviation q.c.

It is well known that the following characterization of $\mathbb{L}^p_{c}(\Omega)$ and $\mathbb{L}^p_{c}(\Omega)$ holds (see Theorem 25 in [3]).

Proposition 14. For each $p \geq 1$ one has

$$\mathbb{L}^p_{c}(\Omega) = \{ Y \in L^p(\Omega) : \lim_{n \to \infty} \hat{E}[|Y|^p \mathbb{I}_{\{|Y| > n\}}] = 0 \}$$

and

$$\mathbb{L}^p(\Omega) = \{ Y \in L^p(\Omega) : \lim_{n \to \infty} \hat{E}[|Y|^p \mathbb{I}_{\{|Y| > n\}}] = 0, \ Y \ has \ a \ q.c. \ version. \}$$

Ren proved in [17] that under Assumption [1] we have the following inclusion $L^p_{c}(\Omega T) \subset \mathbb{L}^p_{c}(\Omega T)$. In [12] we proved that for a $G$-Lévy process with finite activity we have that $L^p_{c}(\Omega T) = \mathbb{L}^p_{c}(\Omega T)$. Modifying the same proof we can obtain this equality also in our more general framework, what is done in the following proposition and theorem.

Proposition 15. We have the following inclusion

$C_{b,lip}(\Omega T) \subset L^1_{c}(\Omega T)$.

As a consequence

$$\mathbb{L}^p_{c,lip}(\Omega T) \subset L^p_{c}(\Omega T).$$

Theorem 16. The space $C_{b, lip}(\Omega T)$ is dense in $C_{lip}(\Omega T)$ under the norm $\hat{E}[|.|]$. Thus $L^1_{c}(\Omega T) = \mathbb{L}^1_{c}(\Omega T)$.

The proof of Proposition [15] is given in Appendix as it is similar to Proposition 18 in [12], whereas the proof of Theorem [16] is skipped as it is identical to the proof of Theorem 21 in the same paper.

3 Compensating a jump part of a $G$-Lévy process

Let $X$ be a $d$-dimensional $G$-Lévy process on some sublinear expectation space $(\Omega, L^1_{c}(\Omega), \hat{E}[.|])$ associated with a set $\mathcal{U}$ via Lévy-Khintchine decomposition. We know by definition that there exists a sublinear expectation space $(\Omega, L^1_{c}(\Omega), \hat{E}[.|])$ and a $2d$-dimensional $G$-Lévy process $(X^c, X^d)$ such that the distribution of $X^c + X^d$ is the same as $X$ and $\lim_{t \downarrow 0} \hat{E}[(X_t^c)^2] t^{-1} = 0$ and $\hat{E}[(X_t^d)^2] \leq Ct$ for some constant $C$.

Hu and Peng showed in [7] that $\hat{\Omega}$ might be chosen to be $D_0(\mathbb{R}^+, \mathbb{R}^{2d})$. The jump part $X^d$ satisfies then the Lévy-Khintchine formula with the set $\mathcal{V} \times \{0\} \times \{0\}$, where $\mathcal{V}$ has the definition as in eq. [3]. One may ask the question how to compensate the $X^d$ to obtain a $G$-martingale. It turns out that it might be done in two different manners.

The first alternative is just to substract the expectation of $X^d_t$. It is easy to check directly by the construction of $\hat{E}$ via the viscosity solution of an IPDE that $\hat{E}[X^d_t] = t \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} vz(dz)$. Consider then the process

$$Y_t := X^d_t - t \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} vz(dz).$$
It is easy to see that $Y$ is a $G$-Lévy process associated with the Lévy triple $\mathcal{V} \times \{0\} \times \{-\sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d_+} zv(dz)\}$ and hence it has stationary increments independent of the past and with 0 expectation. Hence it must be a $G$-martingale. In [11] we showed much stronger result for $G$-Itô-Lévy integral compensated by its expectation under the assumption that the $G$-Lévy process driving the integral is of finite activity.

However, it is also trivial to note that $-Y$ is a $G$-martingale iff $\mathcal{V} = \{ v \}$, i.e. $Y$ is a classical Lévy process without jump-measure uncertainty. Hence, if we want to have a compensating factor which would lead to a $G$-Lévy process which is a symmetric $G$-martingale, we need to be slightly cleverer. The easiest way to do that is to introduce the drift uncertainty which would exactly compensate the jump measure uncertainty. We mainly consider a $G$-Lévy process $\tilde{Z}$ associated with the set

$$\{(v, 0, -\int_{\mathbb{R}^d_+} zv(dz)) : v \in \mathcal{V}\}.$$ 

We stress that usually such a process is defined on a different sublinear expectation space than $X^d$. It is not a problem as long as we are interested only in the distributional properties of a $G$-Lévy process. We will return to that problem in Section 5.

It is easy to see that both $Z_t$ and $-Z_t$ have 0 expectation. To see it note that $u(t, x) = \pm x$ is a viscosity solution of the following IPDE

$$0 = \partial_t u(t, x) - \sup_{v \in \mathcal{V}} \left[ \int_{\mathbb{R}^d_+} [u(t, x + z) - u(t, x)] v(dz) + \left(-\int_{\mathbb{R}^d_+} zv(dz), Du(t, x)\right)\right],$$

$$u(0, x) = \pm x.$$ 

Hence both $Z$ and $-Z$ are $G$-martingales as processes with independent stationary increments and 0 expectation. We summarize this section with the following Proposition.

**Proposition 17.** Let $X$ be a $G$-Lévy process associated with a set $\mathcal{U}$. We may define a sublinear expectation $\tilde{E}$ on $\mathbb{D}_0(\mathbb{R}_+, \mathbb{R}^{2d})$ such that the canonical process $Y_t(\omega^d, \omega^\mathcal{V}) := (X_t^d(\omega^d), X_t^\mathcal{V}(\omega^\mathcal{V})) := (\omega^d_t, \omega^\mathcal{V}_t)$ is a $G$-Lévy process (under $\tilde{E}[\cdot]$) associated with a set $\tilde{U}$ defined as

$$\tilde{U} := \{((v \otimes 0), (-\int_{\mathbb{R}^d_+} zv(dz), p + \int_{\mathbb{R}^d_+} zv(dz)), (0, q)) : (v, p, q) \in \mathcal{U}\}.$$ 

Then $X^d_t + X^\mathcal{V}_t$ has the same distribution as $X_t$, $\lim_{t \to 0} \tilde{E}[(X^\mathcal{V}_t)^3]t^{-1} = 0$ and $\tilde{E}[|X^d_t|] \leq C t$ for some constant $C > 0$ and $X^d$ is a symmetric $\tilde{G}$-martingale.

4 Poisson random measure and a Poisson integral associated with a $G$-Lévy process

In [12] we introduced a Poisson jump measure associated with a $G$-Lévy process with finite activity. In the same paper we defined a pathwise integrals $\int_t^T \int_{\mathbb{R}^d} K(s, z) N(ds, dz)$ w.r.t. that measure for regularly enough random fields $K$. We proved that the integral defined in such a way has good properties: it is continuous as an operator into $L^p_{G}(\Omega_T)$, $p = 1, 2$, it satisfies the Itô formula etc. The class of integrands considered in that paper was large enough to consider the martingale representation, as it was shown in [11] (see Theorem 25). However, the analysis carried out in the two papers was done under the assumption of some continuity of the integrand w.r.t. $z$, i.e. jump size. That assumption prevented us from
considering a much simpler Poisson integrals \( \int_A \phi(z)N([0,t],dz) \), \( A \in B(\mathbb{R}_0^d) \), \( 0 \notin \bar{A} \) and integrals w.r.t. \( G \)-Lévy processes which might not have finite activity. We also didn’t say anything about the properties of a Poisson random measure. In this section we will deal with both questions.

Some of the results in the subsequent section will be given under the following assumption:

**Assumption 2.** Let the family of Lévy measures have the following property: there exists \( p > 1 \) such that 
\[
\sup_{v \in \mathcal{V}} \int_{\{|z| \geq 1\}} |z|^p v(dz) < \infty.
\]

First, let us introduce a Poisson random measure for a Lévy process \( X \) on a canonical space \( D([0,\infty),\mathbb{R}^d) \) associated with a set \( \mathcal{U} \) via Lévy-Khintchine formula. Let also \( \mathcal{V} \) be a set of Lévy measures considered in \( \mathcal{U} \). For such a Lévy process we introduce a Poisson random measure \( L \) defined by

\[
L([s,t],A) := \sum_{0 < u \leq t} 1_A(\Delta X_u), \quad 0 \leq s < t, \; A \in B(\mathbb{R}_0^d), \; 0 \notin \bar{A}.
\]

Note that \( L([s,t],A) \) is well-defined as all paths of \( X \) are càdlàg functions. Moreover, it is obvious that \( L([s,t],A) \in L^0(\Omega_T) \). If we consider only interval \([0,t]\) we will often shorten the notation and write \( L(t,A) \) instead of \( L([0,t],A) \).

In the same way we define the Poisson integral. Let \( \phi \) be a deterministic Borel function on \( \mathbb{R}^d \) which is finite on \( A \in B(\mathbb{R}_0^d) \), \( 0 \notin \bar{A} \). We introduce

\[
\int_A \phi(z)L(t,dz) := \sum_{0 < u \leq t} \phi(\Delta X_u)1_A(\Delta X_u).
\]

The integral is well defined and it belongs to \( L^0(\Omega_T) \). We remind that we have extended the notion of the sublinear expectation \( \hat{E}[] \) using its representation as an upper-expectation (compare with Remark 12). This enables us to examine the continuity w.r.t. \( \hat{E}[\|\cdot\|] \)-norm of the integral as an operator or to study later the distributional properties of a process \( t \mapsto \int_A \phi(z)L(t,dz) \). To study those problems we will introduce and characterize the function spaces on \( \mathbb{R}_0^d \).

### 4.1 Function spaces on \( \mathbb{R}_0^d \) and their characterization

We introduce the following capacity related to \( \mathcal{V} \).

**Definition 18.** For a set of Lévy measures \( \mathcal{V} \) define a set function \( c^\mathcal{V} \) on \( B(\mathbb{R}_0^d) \) by

\[
c^\mathcal{V}(A) := \sup_{v \in \mathcal{V}} v(A), \quad A \in B(\mathbb{R}_0^d).
\]

We will call this function \( \mathcal{V} \)-capacity. We will say that a set \( A \) is \( \mathcal{V} \)-polar if \( c^\mathcal{V}(A) = 0 \). Similarly we will say that a property holds \( \mathcal{V} \)-quasi-surely (abbr. \( \mathcal{V} \)-q.s.) if it holds outside a \( \mathcal{V} \)-polar set. Finally, we will say that a function \( f : \mathbb{R}_0^d \to \mathbb{R} \) is \( \mathcal{V} \)-quasi-continuous (abbr. \( \mathcal{V} \)-q.c.), if for every \( \epsilon > 0 \) there exists an open subset \( O \) of \( \mathbb{R}_0^d \) such that \( c^\mathcal{V}(O) < \epsilon \) and \( f|_O \) is a continuous function.

We introduce the following function spaces connected with \( c^\mathcal{V} \). Let \( A \in B(\mathbb{R}_0^d) \) and \( p \geq 1 \).

1. \( L^0(A) \) is the space of all Borel functions \( f \) on \( \mathbb{R}^d \) such that \( f \equiv 0 \) outside \( A \).
2. $\mathbb{L}^p(A, \mathcal{V})$ is the space of all equivalent classes of functions $f \in L^0(A)$ such that $\|f\|_{p,A,\mathcal{V}}^p := \sup_{v \in \mathcal{V}} \int_A |f(z)|^p v(dz) < \infty$. Equivalent classes are taken w.r.t. the $\mathcal{V}$-quasi-sure equivalence.

3. $B_b(A)$ is the space of all bounded Borel functions in $L^0(A)$. $C_b(A)$ is the space of all continuous functions in $B_b(A)$. Note that $C_b(\text{Int } A) = C_b(\bar{A})$.

4. If $0 \notin \bar{A}$ then $\mathbb{L}^p_0(A, \mathcal{V})$ and $\mathbb{L}^p(A, \mathcal{V})$ are defined as a completion of a $B_b(A)$ (respectively $C_b(A)$) under the norm $\|\cdot\|_{p,A,\mathcal{V}}$.

5. If $0 \in \bar{A}$ then we define $\mathbb{L}^p_0(A, \mathcal{V})$ and $\mathbb{L}^p(A, \mathcal{V})$ spaces as a completion under $\|\cdot\|_{p,A,\mathcal{V}}$ norm of the following sets (respectively)

$$\bigcup_{B \subset A \atop 0 \notin B} B_b(B) \quad \text{and} \quad \bigcup_{B \subset A \atop 0 \notin B} C_b(B).$$

Note that $\mathbb{L}^p(\text{Int } A, \mathcal{V}) = \mathbb{L}^p(\bar{A}, \mathcal{V})$.

**Remark 19.** Note that $\mathcal{V}$-capacity is a Choquet capacity, however it is usually unnormed (or even infinite). One needs to mention that the general results from [3] have been proven for normed capacities, so a priori they might not hold for our $\mathcal{V}$-capacity. Fortunately, many results do not require this condition. In particular, we have exactly the same characterization of spaces $\mathbb{L}^p_0(A, \mathcal{V})$ and $\mathbb{L}^p(A, \mathcal{V})$ as in Proposition [13] if $0 \notin \bar{A}$.

In order to generalize this characterization, we introduce the following natural definitions of tightness and the uniform integrability of functions w.r.t. the $\mathcal{V}$-capacity.

**Definition 20.** Fix $A \in \mathcal{B}(\mathbb{R}_0^d)$ and $f \in L^p(A, \mathcal{V})$

We will say that $f$ is $\mathcal{V}$-tight if for all $\epsilon > 0$ there exists a compact set $F \subset \bar{A} \cap \mathbb{R}_0^d$ such that and $\sup_{v \in \mathcal{V}} \int_{F^c} |f(z)| dz < \epsilon$.

We will say that $f$ is $\mathcal{V}$-uniformly integrable if

$$\lim_{n \to \infty} \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_0^d} |f(z)| 1_{\{|f(z)| \geq n\}} v(dz) = 0.$$ 

**Proposition 21.** Let $A \in \mathcal{B}(\mathbb{R}_0^d)$. Then we have the following characterizations of spaces $\mathbb{L}^p_0(A, \mathcal{V})$ and $\mathbb{L}^p(A, \mathcal{V})$

$$\mathbb{L}^p_0(A, \mathcal{V}) = \{ f \in L^p(A, \mathcal{V}) : |f|^p \text{ is } \mathcal{V}\text{-tight and } \mathcal{V}\text{-uniformly integrable} \}$$

and

$$\mathbb{L}^p(A, \mathcal{V}) = \{ f \in \mathbb{L}^p_0(A, \mathcal{V}) : f \text{ has a } \mathcal{V}\text{-q.c. version} \}.$$ 

*Proof.* We will follow the ideas Proposition 18, Proposition 24 and Theorem 25 in [3]. We will proceed in three steps.

**Step 1.** $\mathbb{L}^p_0(A, \mathcal{V}) = \{ f \in \mathbb{L}^p(A, \mathcal{V}) : |f|^p \text{ is } \mathcal{V}\text{-tight and } \mathcal{V}\text{-uniformly integrable} \}$.

Define $J_p := \{ f \in \mathbb{L}^p(A, \mathcal{V}) : |f|^p \text{ is } \mathcal{V}\text{-tight and } \mathcal{V}\text{-uniformly integrable} \}$. Fix $f \in J_p$ and $\epsilon > 0$. By the tightness of $f$ we may define the compact and bounded away from 0 subset $F_\epsilon \subset A$ s.t.

$$\sup_{v \in \mathcal{V}} \int_{F_\epsilon^c} |f(z)|^p v(dz) < \epsilon/2.$$
Define $f_{n,\epsilon} = [(f \land n^{1/p}) \lor (-n^{1/p})]\mathbb{1}_{F_c \cap A}$. Then $f_{n} \in B_0(F_c \cap A)$ and we have by definition of $J_p$ that

$$\sup_{v \in V} \int_{F_n} |f(z) - f_{n,\epsilon}(z)|^p v(dz) \leq \sup_{v \in V} \left\{ \int_{F_n \cap A} |f(z)|^p v(dz) + \int_{F_n \cap A} (|f(z)|^p - n) \mathbb{1}_{(|f(z)|^p > n)} v(dz) \right\}$$

$$\leq \sup_{v \in V} \int_{F_n} |f(z)|^p v(dz) + \sup_{v \in V} \int_{\mathbb{R}^d} |f(z)|^p \mathbb{1}_{(|f(z)|^p > n)} v(dz) < \epsilon$$

for $n$ large enough. Hence $J_\epsilon \subseteq L^p_b(A, \mathcal{V})$.

On the other hand, for each $f \in L^p_b(A, \mathcal{V})$ we may find a sequence $\{g_n\}_{n=1}^\infty$ such that $g_n \in B_0(A_n)$ $n = 1, 2, \ldots$, where $A_n \subset A$ satisfying $0 \notin A_n$ and $A_n \uparrow A$, and $\|f - g_n\|_{p,A,\mathcal{V}} \to 0$.

First, we claim that each $|g_n|^p$ is $\mathcal{V}$-tight. The proof is straightforward. We fix $\epsilon > 0$ Since $\mathcal{V}$ restricted to $A_n$ is tight (see [7], p. 14), we may choose a compact set $K_n$ such that $\epsilon^p (K_n^c \cap A_n) \leq \epsilon/(M_n)^p$, where $M_n$ is a bound of $g_n$. We put $F_n := K_n \cap A_n$. Then $F_n$ is compact and bounded away from 0. We also have

$$\sup_{v \in V} \int_{F_n} |g_n(z)|^p v(dz) = \sup_{v \in V} \int_{F_n \cap A} |g_n(z)|^p v(dz) \leq M_n \sup_{v \in V} \int_{F_n \cap A} v(dz) < \epsilon.$$

To prove that $f$ is also $\mathcal{V}$-tight, fix $\epsilon > 0$ again and $N$ such that $\|f(z) - g_1(\mathcal{V})\|_{p,A,\mathcal{V}} < \epsilon/3$. We also fix a compact set $F \subset A \subset A$ and bounded away from 0 set such that

$$\left( \sup_{v \in V} \int_{F} |g_1(\mathcal{V})|^p v(dz) \right)^{1/p} < \epsilon/3.$$

We have then the estimate

$$\left( \sup_{v \in V} \int_{F} |f(z)|^p v(dz) \right)^{1/p} = \|f(z) - f(z)\mathbb{1}_F(z)\|_{p,\mathbb{R}^d,\mathcal{V}}$$

$$\leq \|f(z) - g_1(\mathcal{V})\|_{p,\mathbb{R}^d,\mathcal{V}} + \|g_1(\mathcal{V}) - f(z)\mathbb{1}_F(z)\|_{p,\mathbb{R}^d,\mathcal{V}}$$

$$\leq \|f(z) - g_1(\mathcal{V})\|_{p,A,\mathcal{V}} + \left( \sup_{v \in V} \int_{F} |g_1(\mathcal{V})|^p v(dz) \right)^{1/p}$$

$$+ \left( \sup_{v \in V} \int_{F} |g_1(\mathcal{V}) - f(z)|^p v(dz) \right)^{1/p} < \epsilon/3.$$

and consequently $|f|^p$ is $\mathcal{V}$-tight.

To prove $\mathcal{V}$-uniform integrability of $|f|^p$, we fix $\epsilon > 0$ and a compact set $F_\epsilon \subset \mathbb{R}^d$ such that $\sup_{\nu \in V} \int_{F_\epsilon} |f(z)|^p v(dz) < \epsilon/2$. Note that

$$\sup_{v \in V} \int_{\mathbb{R}^d} |f(z)|^p \mathbb{1}_{(|f(z)|^p \geq n)} v(dz) < \sup_{v \in V} \int_{F_\epsilon} |f(z)|^p \mathbb{1}_{(|f(z)|^p \geq n)} v(dz) + \epsilon/2 \quad (6)$$

Let $y_\eta := \sup_{z \in A} |g_n(z)|$ and $f_{n,\epsilon} := [(f \land y_\eta) \lor (-y_\eta)]\mathbb{1}_{F_c}$. Then we have $|f - f_{n,\epsilon}| \leq |f - g_n|$ on $F_\epsilon$ and consequently $\|f - f_{n,\epsilon}\|_{p,F_\epsilon,\mathcal{V}} \to 0$. Hence it is not difficult to prove that for any sequence $(\eta_n)_{n=1}^\infty$ tending to $\infty$ one has

$$\sup_{v \in V} \int_{F_\epsilon} |f(z) - (f(z) \land \eta_n) \lor (-\eta_n)|^p v(dz) \to 0.$$
and using exactly the same arguments as in Proposition 18 in \[3\] we get that for \(n\) large enough
\[
\sup_{v \in \mathcal{V}} \int_{F_n} |f(z)|^p L_{\{z \in (f(z))^\geq n\}} v(\mathrm{d}z) \leq \epsilon/2
\]
This together with (6) guarantees that \(|f|^p\) is \(\mathcal{V}\)-uniformly integrable.

**Step 2.** Each element in \(L^p_{\mathcal{V}}(A,\mathcal{V})\) has a q.c. version.

The argument here is exactly the same as in Proposition 24 in \[3\]: we choose the appropriate sequence of the elements in \(C_b(A_n)\) converging to a fixed function \(f \in L^p(A,\mathcal{V})\) and define open sets of small \(\mathcal{V}\)-capacity, such that the convergence is uniform on their complement. For details, see the original paper.

**Step 3.** \(L^p_{\mathcal{V}}(A,\mathcal{V}) = \{f \in L^p_{\mathcal{V}}(A,\mathcal{V}): f\text{ has a }\mathcal{V}\text{-q.c. version}\}\). First note that we may just prove the characterization for set \(A\) open as \(L^p_{\mathcal{V}}(\text{Int} A,\mathcal{V}) = L^p_{\mathcal{V}}(\bar{A},\mathcal{V})\).

By step 1 and 2 we have \(L^p_{\mathcal{V}}(A,\mathcal{V}) \subset \{f \in L^p_{\mathcal{V}}(A,\mathcal{V}): f\text{ has a }\mathcal{V}\text{-q.c. version}\} =: J_p\).

To prove the inclusion in the other direction we fix \(f \in J_p\). Define \(h_n := (f \wedge n^{1/p}) \vee (-n^{1/p})\). Note that \(h_n\) is \(\mathcal{V}\)-q.c. Hence we may find an open set \(O_n\) with small \(\mathcal{V}\)-capacity s.t. \(h_n\) is continuous outside \(O_n\). Moreover we may assume that \(O_n \subset A\), as \(h_n^\circ \equiv 0\) on \(A^c\) (which is a closed set). By Tietze’s theorem, we extend \(h_n\) to a continuous functions \(g_n \in C_b(A)\) (in particular, \(g_n \equiv 0\) outside \(A\)). Moreover, we if we choose sets \(O_n\) small enough, we can get that \(\|f - g_n\|_{p,A,F} \to 0\). See the proof of Theorem 25 in \[3\] for details. Hence, without loss of generality we may assume that
\[
\|f - g_n\|_{p,A,\mathcal{V}} \leq \frac{1}{5n}\tag{7}
\]
We will use now \(\mathcal{V}\)-tightness to construct functions with supports bounded away from 0 which approximate \(f\). For every \(n = 1,2,\ldots\) we choose a compact set \(F_n \subset \bar{A}\) \((0 \not\in F_n)\) such that \(\sup_{v \in \mathcal{V}} \int_{F_n} |f(z)|^p v(\mathrm{d}z) < \frac{1}{n}\). Hence, by (7) we get
\[
\left(\sup_{v \in \mathcal{V}} \int_{F_n} |g_n(z)|^p v(\mathrm{d}z)\right)^{1/p} < \frac{2}{5n}.
\]
It is now easy to construct a function \(f_n\) satisfying the following conditions:

1. \(f_n\) is a continuous bounded function with support \(A_n \subset A\) bounded away from 0.
2. \(f_n \equiv g_n\) on \(F_n\).
3. \(|f_n| \leq |g_n|\) on \(F_n^c\).

Then we have the following
\[
\|f_n - f\|_{p,A,\mathcal{V}} \leq \|f_n - g_n\|_{p,A,\mathcal{V}} + \|g_n - f\|_{p,A,\mathcal{V}}
\]
\[
= \left(\sup_{v \in \mathcal{V}} \int_{F_n} |f_n(z) - g_n(z)|^p v(\mathrm{d}z)\right)^{1/p} + \frac{1}{5n}
\]
\[
= \left(\sup_{v \in \mathcal{V}} \int_{F_n} |f_n(z)|^p v(\mathrm{d}z)\right)^{1/p} + \left(\sup_{v \in \mathcal{V}} \int_{F_n} |g_n(z)|^p v(\mathrm{d}z)\right)^{1/p} + \frac{1}{5n}
\]
\[
= 2 \left(\sup_{v \in \mathcal{V}} \int_{F_n} |g_n(z)|^p v(\mathrm{d}z)\right)^{1/p} + \frac{1}{5n} < \frac{1}{n}.
\]
4.2 Continuity of the Poisson integral as an operator

**Proposition 22.** For each \( A \in \mathcal{B}(\mathbb{R}^d_0) \), \( 0 \notin \hat{A} \) we have that the integral \( \int_A L(t, dz) \) is a continuous operator from the space \( \mathbb{L}^1_0(A, \mathcal{V}) \) to \( \mathbb{L}^1_0(\Omega_t) \).

**Proof.** The proof is similar to the proofs of Theorem 27 and 28 in [12]. For the sake of completeness, we will sketch it. First we take a \( \phi \in \mathcal{B}_b(A) \).

We prove that the integral of \( \phi \) is \( \mathbb{L}^1_0(\Omega_t) \). We use the extended notion of a sublinear expectation as an upper-expectation and the characterization of \( \mathbb{L}^1_0(\Omega_t) \) from Proposition [13].

First, we prove the continuity of the integral w.r.t. the appropriate norms.

\[
\mathbb{E} \left[ \left| \int_A \phi(z)L(t, dz) \right| \right] = \sup_{\theta \in \mathcal{A}_{0,t}} \mathbb{E}^{P_0} \left[ \sum_{0<u \leq t} \phi(\Delta B_u^0, \theta) \mathbb{1}_A(\Delta B_u^0, \theta) \right]
\]

\[
= \sup_{\theta \in \mathcal{A}_{0,t}} \mathbb{E}^{P_0} \left[ \sum_{0<u \leq t} \phi(\theta^d(u, \Delta N_u)) \mathbb{1}_A(\theta^d(u, \Delta N_u)) \right]
\]

\[
= \sup_{\theta \in \mathcal{A}_{0,t}} \mathbb{E}^{P_0} \left[ \int_0^t \int_{\mathbb{R}^d_0} \phi^\theta(u, z) N(du, dz) \right], \quad (8)
\]

where \( \phi^\theta \) is a predictable random field defined as

\[
\phi^\theta(u, z) := \phi(\theta^d(u, z)) \mathbb{1}_A(\theta^d(u, z)).
\]

Define now a random measure \( \pi^\theta(u)(\omega) := \mu \circ (\theta^d(u, .)(\omega))^{-1} \). By the definition of a set \( \mathcal{A}_{0,\infty} \) we know that for a.a. \( \omega \) and \( u \) we have that \( \pi^\theta(t, \omega, .) \in \mathcal{V} \). Hence it is now easy to see that

\[
\mathbb{E} \left[ \left| \int_A \phi(z)L(t, dz) \right| \right] \leq \sup_{\theta \in \mathcal{A}_{0,t}} \mathbb{E}^{P_0} \left[ \int_0^t \int_{\mathbb{R}^d_0} |\phi^\theta(u, z)| N(du, dz) \right]
\]

\[
= \sup_{\theta \in \mathcal{A}_{0,t}} \mathbb{E}^{P_0} \left[ \int_0^t \int_{\mathbb{R}^d_0} |\phi^\theta(u, z)| \mu(dz) du \right]
\]

\[
= \sup_{\theta \in \mathcal{A}_{0,t}} \mathbb{E}^{P_0} \left[ \int_0^t \int_{\mathbb{R}^d_0} |\phi(z)| \pi^\theta_n(dz) du \right] = t \sup_{\phi \in \mathcal{V}} \int_A |\phi(z)| v(dz). \quad (9)
\]

The last equality is true as we may always take process \( \theta^d \) which is deterministic.

Hence, if \( \phi \in \mathcal{B}_b(A) \), then \( \int_A \phi(z)L(t, dz) \in L^1(\Omega_t) \).

Now we prove that the integral is in fact in \( \mathbb{L}^1_0(\Omega_t) \). Let \( K \) be a bound of \( \phi \). Without the loss of generality we may assume that \( K = 1 \). Take a set \( B \in \mathcal{B}(\mathbb{R}^d_0) \) such that \( 0 \notin \hat{B} \) and \( B \supset \bigcup_{v \in \mathcal{V}} g_v^{-1}(A) \) (it is possible by Remark [9]). Then for any \( \theta \in \mathcal{A}_{0,\infty} \) we have the following inclusion

\[
\left\{ \left| \sum_{0<u \leq t} \phi(\theta^d(u, \Delta N_u)) \mathbb{1}_A(\theta^d(u, \Delta N_u)) \right| > n \right\} \subset \left\{ \sum_{0<u \leq t} |\phi(\theta^d(u, \Delta N_u))| \mathbb{1}_B(\Delta N_u) | > n \right\}
\]

\[
\subset \{ u \mapsto N(u, B) \text{ has at least } n \text{ jumps in } [s, t] \} =: B_n,
\]

as the sum of jumps grows only at jump times and only by a value bounded by 1. Introduce

\[
C_n := B_n \setminus B_{n+1} = \{ u \mapsto N(u, B) \text{ has } n \text{ jumps in the interval } [s, t] \}.
\]
Hence we have the estimate

\[
\mathbb{E}\left[ \int_A \phi(z) L(t, dz) \right] I_{\{ \int_A \phi(z) L(t, dz) > n \}} \\
= \sup_{\theta \in A_{\Omega, t}} \mathbb{E}^{P_0}_{\theta} \left[ \sum_{u \leq t} \phi(\theta^d(u, \Delta N_u)) I_A(\theta^d(u, \Delta N_u)) \right] I\{ |\sum_{u \leq t} \psi(\theta^d(u, \Delta N_u)) I_A(\theta^d(u, \Delta N_u))| > n \}
\]

\[
\leq \sup_{\theta \in A_{\Omega, t}} \mathbb{E}^{P_0}_{\theta} \left[ \sum_{u \leq t} |\phi(\theta^d(u, \Delta N_u)) I_B(\Delta N_u)| I_{B_n} \right] \\
\leq \sum_{m=n}^{\infty} \sup_{\theta \in A_{\Omega, t}} \mathbb{E}^{P_0}_{\theta} \left[ \sum_{u \leq t} |\phi(\theta^d(u, \Delta N_u)) I_B(\Delta N_u)| I_{C_m} \right]
\]

because the Poisson random variable has first moment finite and the number of jumps of the Poisson process \( u \mapsto N(u, B) \) in the fixed interval is Poisson-distributed. Consequently, the integral belongs to \( L^1(\Omega) \) by the characterization from Proposition 23. For details see the proof of Theorem 28 in [12].

Now it is easy to extend this result to the whole space \( L^1(\Omega) \) by using [14].

**Remark 23.**
1. Using Proposition 22 one can prove that for each \( A \in \mathcal{B}(\mathbb{R}^d_0) \) and \( \phi \in L^1(\Omega, \mathcal{B}, \mathbb{P}) \) the following integral is well defined

\[
\int_A \phi(z) L(t, dz).
\]

In particular, we note that under Assumption 2 (and of course Assumption 1), for \( A = \mathbb{R}^d_0 \) since \( (z \mapsto z) \in L^1(\mathbb{R}^d_0, \mathcal{V}) \) (compare with Proposition 28 in [3]). Hence, we have that we can define the integral \( \int_{\mathbb{R}^d_0} z L(t, dz) \) and it is an element of \( L^1(\Omega) \).

2. The extended integral is a continuous operator from \( L^1(\mathbb{R}^d_0, \mathcal{V}) \) to \( L^1(\Omega) \).

3. By using the same argument as in eq. (8) and (9) we may conclude that \( \mathbb{E}[\int_A \phi(z) L(t, dz)] = t \sup_{v \in \mathcal{V}} \int_A \phi(z) v(dz) \).

### 4.3 Distributional properties of Poisson integrals

The second important property, which we will investigate in this section, is the distribution of a Poisson integral. Thanks to the proof of the representation of \( \mathbb{E}[\cdot] \) we easily get the following Proposition.

**Proposition 24.** For each \( A \in \mathcal{B}(\mathbb{R}^d_0) \) and \( \phi \in L^1(\mathcal{A}, \mathcal{V}) \) the stochastic process \( t \mapsto \int_A \phi(z) L(t, dz) \) is a \( G^{\phi, A} \)-Lévy process, where \( G^{\phi, A} \) is a non-local operator associated with set \( \mathcal{U}^{\phi, A} = \mathcal{V}^{\phi, A} \times \{0\} \times \{0\} \) via Lévy-Khintchine formula in Theorem 5 with

\[
\mathcal{V}^{\phi, A} := \{ v \circ \phi^{-1}(\cdot \cap A) : v \in \mathcal{V} \}.
\]

Before we go to the proof, note that by this proposition it is trivial that the Poisson random measure \( t \mapsto L(t, A) \) is a \( G \)-Poisson process for any \( A \in \mathcal{B}(\mathbb{R}^d_0), \ 0 \notin \overline{A} \).
Proof. We need to check the definition of a G-\textit{Lévy} process for a process \( t \mapsto Y_t := \int_A \phi(z)L(t, dz). \) It is obvious that \( Y_0 = 0. \) Fix \( t, s \geq 0, \) take a partition \( 0 = t_0 \leq \ldots \leq t_n \leq t \) and a function \( \psi \in C_{b, L\psi}(\mathbb{R}^{d \times (n + 1)}). \) Then we have via representation of \( \mathbb{E}[\cdot] \) that
\[
\mathbb{E}[\psi(Y_{t_1}, \ldots, Y_{t_n} - Y_{t_{n-1}}, Y_{t+s} - Y_t)] = \sup_{\theta \in \mathcal{A}_{t+s}^{s+t}} \mathbb{E}^\theta_0 \left[ \psi(\tilde{B}_{t_1}^{0, \theta}, \ldots, \tilde{B}_{t_n}^{t, \theta}, \tilde{B}_{t+s}^{t, \theta}) \right],
\]
where
\[
\tilde{B}_b^{t, \theta} := \int_a^b \int_{\mathbb{R}^d} (\phi \cdot 1_A) \circ \theta^d(s, z)N(ds, dz).
\]
The Poisson random measure \( N \) and the probability measure \( \mathbb{P}_0 \) are taken from the representation of \( \mathbb{E}. \) It is now easy to see that \( \tilde{B}_b^{t, \theta} \) is an Itô-Lévy integral which is used to prove the representation of some sublinear expectation \( \mathbb{E} \) associated with a \( G^{\phi, \Lambda} \)-\textit{Lévy} process. By the proof of representation of \( \mathbb{E} \) (see Section 3 in [12]) it is easy to see that the following property holds (compare with Theorem 13 in [12]):
\[
\sup_{\theta \in \mathcal{A}_{t+s}^{s+t}} \mathbb{E}^\theta_0 \left[ \psi(\tilde{B}_{t_1}^{0, \theta}, \ldots, \tilde{B}_{t_n}^{t, \theta}, \tilde{B}_{t+s}^{t, \theta}) \right] = \sup_{\theta \in \mathcal{A}_{t+s}^{s+t}} \mathbb{E}^\theta_0 \left[ \sup_{\theta \in \mathcal{A}_{t+s}^{s+t}} \mathbb{E}^\theta_0 \left[ \psi(x, \tilde{B}_{t_n}^{t, \theta}) \right] \right],
\]
where \( B = (\tilde{B}_{t_1}^{0, \theta}, \tilde{B}_{t_n}^{t, \theta}). \) Hence, again by the representation of \( \mathbb{E}[\cdot] \) we get that
\[
\mathbb{E}[\psi(Y_{t_1}, \ldots, Y_{t_n} - Y_{t_{n-1}}, Y_{t+s} - Y_t)] = \mathbb{E} \left[ \mathbb{E} \left[ \psi(x, Y_{t+s} - Y_t) \right] | x = (Y_{t_1}, \ldots, Y_{t_n} - Y_{t_{n-1}}) \right],
\]
therefore the increment of \( Y \) is independent of the past. We prove that the increment has stationary distribution in the same way. It is now obvious that \( Y \) is a \( G^{\phi, \Lambda} \)-\textit{Lévy} process. \( \square \)

Just as in the classical case, one can consider a very simple function \( \phi(z) = z1_A(z) \) for some Borel set \( A \) bounded away from 0. Using a technique similar to the proof above, one can prove the following result.

**Proposition 25.** Under Assumption 4 for any \( A \in \mathcal{B}^{\mathbb{R}^d}_c \) stochastic process \( t \mapsto (X_t - \int_A zL(t, dz), \int_A zL(t, dz)) \) is a \( \tilde{G}^A \)-\textit{Lévy} process, where \( \tilde{G}^A \) is a non-local operator associated with set \( U^A \) via Lévy-Khintchine formula in Theorem 5 with
\[
\tilde{U}^A := \{(v \cdot A) \otimes v | A, (p, 0), (q, 0)) : (v, p, q) \in U^A \}.
\]
The direct consequence of this proposition is the following corollary.

**Corollary 26.** Let \( X \) be a \( G \)-\textit{Lévy} process defined on the canonical sublinear expectation space \( (\Omega, \mathbb{L}^1_1(\Omega), \mathbb{E}[\cdot]) \), \( \Omega = \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) satisfying Assumption 4. Then the decomposition \( X_t = X^c_t + X^d_t \) as in Point 4 of Definition 4 might be taken on the same sublinear expectation space with \( X^c_t := X_t - \int_{\mathbb{R}^d} zL(t, dz) \) and \( X^d_t := \int_{\mathbb{R}^d} zL(t, dz) \), i.e. \( X^c_t \) and \( X^d_t \) belong to \( \mathbb{L}^1_1(\Omega) \).

5 Quasi-continuity of Poisson integrals

In this section we concentrate on the quasi-continuity of the Poisson integrals. In particular, we introduce the tools which enables us to investigate when processes \( X^c \) and \( X^d \) used in the decomposition established in the Corollary 26 belong to the space \( L^1_1(\Omega) \). Such a particular form of the decomposition is useful, as many objects (as for example the conditional expectation) are defined only on the space \( L^1_1(\Omega) \).

First, let us deal with the simple situation of a continuous function \( \phi \).
Proposition 27. Let φ ∈ C_b(A), A ∈ B(\mathbb{R}_0^n) and 0 ∉ \bar{A}. Then for all t > 0 we have that the function \( \omega^t \mapsto \int_{\mathbb{R}_0^n} φ(z)L(t,dz)(\omega^t) \) is continuous. Moreover, we also have that \( \omega \mapsto \int_{\mathbb{R}_0^n} φ(z)L(t,dz)(\omega) \) has a.q. version. Hence, \( \int_{\mathbb{R}_0^n} φ(z)L(t,dz) \in L^1_ω(Ω_t) \).

The proof of this proposition is very similar to the proof of Theorem 28 in [12], so we omit it. The remarkable thing about this proposition is that even though the integral \( t \mapsto \int_{\mathbb{R}_0^n} φ(z)L(t,dz) \) is adapted, its continuity w.r.t. \( ω \) depends on what happens just after time \( t \). We need to underline that the function \( ω \mapsto \int_{\mathbb{R}_0^n} φ(z)L(t,dz)(ω) \) usually has some discontinuities even for very regular φ’s.

Corollary 28. Let φ ∈ L^1_ω(\mathbb{R}_0^n, V). Then for every t > 0 the integral \( \int_{\mathbb{R}_0^n} φ(z)L(t,dz) \in L^1_ω(Ω_t) \).

Proof. By the definition of L^1_ω(\mathbb{R}_0^n, V) there exists a sequence \( (φ_n)_n \), \( φ_n \in C_b(A_n) (0 ∉ \bar{A}_n) \) under the norm \( ||·||_{1,\mathbb{R}_0^n, V} \). We know that the integral is a continuous operator from L^1_ω(\mathbb{R}_0^n, V) to L^1_ω(Ω_t) (see Remark 23), hence \( X_n := \int_{\mathbb{R}_0^n} φ_n(z)L(t,dz) \) converges to \( \int_{\mathbb{R}_0^n} φ(z)L(t,dz) \) in \( \mathbb{E}[||·||] \) norm. But we also know by Proposition 27 that \( X_n \in L^1_ω(Ω_t) \), therefore so does its limit.

Hence, we have found a sufficient condition for the Poisson integral to be quasi-continuous. However, what about the necessary conditions? In the following subsections we will establish that the regularity of the integral w.r.t. \( ω \) also implies the regularity of the integrand w.r.t. \( z \).

5.1 The regularity of the jumps times and related quasi-continuity theorems

Let \( A \) be an open set in \( \mathbb{R}_0^n \) such that \( 0 ∉ \bar{A} \). We will define the following stopping times

\[ τ_A^0 = \tau_\bar{A}^0 := 0, \quad τ_A^k := \inf\{t > τ_A^{k-1} : ΔX_t ∈ A\}, \quad τ_\bar{A}^k := \inf\{t > τ_\bar{A}^{k-1} : ΔX_t ∈ \bar{A}\} \quad k = 1, 2 \ldots \]

The stopping times are well-defined and we have that \( \tau_\bar{A}^k \leq \tau_A^k \). We have also the lemma.

Lemma 29. Let \( A \) be an open subset of \( \mathbb{R}_0^n \) such that \( 0 ∉ \bar{A} \) and \( C(V(∂A) = 0 \). Then for any \( t > 0 \) we have that

\[ c(∃ u ∈ [0,t] s.t. ΔX_u ∈ ∂A) = c(τ_A^k ∧ t < τ_\bar{A}^k ∧ t) = 0. \]

Consequently,

\[ c(∃ u > 0 s.t. ΔX_u ∈ ∂A) = c(τ_A^k < τ_\bar{A}^k) = 0. \]

Proof. First note that

\[ c(τ_A^k ∧ t < τ_\bar{A}^k ∧ t) = c(∃ u s.t. τ_A^{k-1} ∧ t < u ≤ τ_A^k ∧ t \quad and \quad ΔX_u ∈ ∂A) \]

\[ ≤ c(∃ u ∈ [0,t] s.t. ΔX_u ∈ ∂A). \]

So to prove the first assertion it is sufficient to check that \( c(∃ u ∈ [0,t] s.t. ΔX_u ∈ ∂A) = 0 \). Assume the opposite. Then \( L(t,∂A) ≥ 1 \) on a set with positive capacity. Then by the definition of c there exists a measure \( Q ∈ \Psi \) s.t. \( L(t,∂A) ≥ 1 \) with positive \( Q \)-probability. Since the
Poisson random measure is non-negative, we know then that \( \mathbb{E}[L(t, \partial A)] \geq \mathbb{E}^\mathbb{Q}[L(t, \partial A)] > 0 \). But by Remark 29, Point 3, we know that \( \mathbb{E}[L(t, \partial A)] = t \cdot c^V(\partial A) = 0 \). Hence

\[
0 = c(\exists \ u \in [0, t] \ s.t. \ \Delta X_u \in \partial A) = c(\bar{\tau}_A \wedge t < \tau_A \wedge t).
\]

Now, note that the second assertion is a simple consequence of the first one:

\[
c(\bar{\tau}_A < \tau_A) = c(\exists \ q \in \mathbb{Q}_+ \ s.t. \ \bar{\tau}_A < q < \tau_A) \leq \sum_{q \in \mathbb{Q}_+} c(\bar{\tau}_A < q < \tau_A)
\]

\[
\leq \sum_{q \in \mathbb{Q}_+} c(\bar{\tau}_A < \tau_A \wedge q) = 0,
\]

using the first assertion of the lemma. Similarly

\[
c(\exists \ u > 0 \ s.t. \ \Delta X_u \in \partial A) \leq \sum_{n=1}^\infty c(\exists \ u \in [0, n] \ s.t. \ \Delta X_u \in \partial A) = 0.
\]

Using this lemma it is easy to prove the quasi-continuity of the stopping times \( \tau_A^k \).

**Proposition 30.** Let \( A \) be an open subset of \( \mathbb{R}^d_0 \) such that \( 0 \notin \bar{A} \) and \( c^V(\partial A) = 0 \). Then for any \( t > 0 \) the random variables \( \tau_A^k, \mathbf{1}_{\{\tau_A^k \leq t}\} \) and \( \Delta X_{\tau_A^k} \) are q.c.

**Proof.** Consider a set \( G = \{\tau_A^k = \bar{x}_A^k\} \). It is easy to see that \( \omega \mapsto \tau_A^k(\omega) \) is a continuous function on \( G \). On the other hand \( G^c \subset \{\omega \in \Omega: \exists \ u > 0 \ s.t. \ \Delta w_u \in \partial A\} = F \). \( F \) is a closed set and by Lemma 29 we know that \( c(F) = 0 \). Hence, by Lemma 3.4. in [20] for each \( \epsilon > 0 \) there exists an open set \( O' \) with \( c(O') < \epsilon \ s.t. \ F \subset O' \). But \( (O')^c \subset G \) and therefore \( \tau_A^k \) is continuous on \( O' \). As a consequence, \( \tau_A^k \) is q.c.

To prove that \( \mathbf{1}_{\{\tau_A^k \leq t}\} \) and \( \Delta X_{\tau_A^k} \) are q.c. we fix \( \epsilon > 0 \) and define the set \( O'^{1/2} \) as above. Consider also a set \( H := \{\omega \in \Omega: \Delta w_\omega \in \bar{A}\} \). It is easy to check by the definition of \( \mathbb{P}^\theta \) used in the definition of \( c \) that this set is polar. It is also a closed set, hence there exists an open set \( O'^{1/2} \) containing \( H \) and \( c(O'^{1/2}) < \epsilon/2 \). Now define \( O := O'^{1/2} \cup O'^{1/2} \) and note that \( c(O) < \epsilon \).

Take any sequence \( (\omega^n)_n \subset O^c \) which converges in Skorohod topology to \( \omega \) (also in \( O^c \) by its closedness). Then by the choice of \( O'^{1/2} \) we have \( \tau_A^k(\omega^n) \to \tau_A^k(\omega) \). We also know that \( \tau_A^k(\omega) \not\in t \) by the choice of \( O'^{1/2} \), hence either \( \tau_A^k(\omega^n) < t \) for large \( n \) if \( \tau_A^k(\omega) < t \) or \( \tau_A^k(\omega^n) > t \) for large \( n \) if \( \tau_A^k(\omega) > t \). In any case we have the convergence \( \mathbf{1}_{\{\tau_A^k \leq t\}}(\omega^n) \to \mathbf{1}_{\{\tau_A^k \leq t\}}(\omega) \). Hence \( \mathbf{1}_{\{\tau_A^k \leq t\}} \) is q.c. Similarly, by the convergence \( \omega^n \to \omega \) we get that \( \Delta X_{\tau_A^k}(\omega^n) \to \Delta X_{\tau_A^k}(\omega) \). Hence \( \Delta X_{\tau_A^k} \) is also q.c.

We may also include another jump times in the following manner without losing the quasi-continuity.

**Proposition 31.** Let \( A \) be an open subset of \( \mathbb{R}^d_0 \) such that \( 0 \notin \bar{A} \) and \( c^V(\partial A) = 0 \). Then for any \( k > l > 0 \) and \( C > 0 \) the random variable \( \mathbf{1}_{\{\tau_A^k \leq \tau_A^l - C\}} \) is q.c.

**Proof.** Fix \( \epsilon > 0 \). By the quasi-continuity of \( \tau_A^k \) and \( \tau_A^l \) we may find an open set \( O_1 \) s.t. \( c(O_1) < \epsilon/2 \) and both stopping times are continuous w.r.t. \( \omega \).

Let \( F := \{\tau_A^k = \tau_A^l - C\} \). We claim that \( F \) is polar, too. We use again the representation theorem for sublinear expectations.

\[
c(F) = \sup_{\theta \in \mathcal{A}^0_{0,\infty}} \mathbb{P}^\theta(F) \leq \sup_{\theta \in \mathcal{A}^0_{0,\infty}} \mathbb{P}^\theta(\exists u > 0 \ \Delta X_{u-C} \in A, \ \Delta X_u \in A)
\]

\[
= \sup_{\theta \in \mathcal{A}^0_{0,\infty}} \mathbb{P}_0\left(\exists u > 0 \ \Delta B_{u-C}^{0,\theta} \in A, \ \Delta B_u^{0,\theta} \in A\right).
\]
Let \( B = \bigcup_{v \in V} g^{-1}_v(A) \). By the construction of \( g_v \) we know that \( 0 \notin B \) (see Remark \( 9 \) and Appendix, Subsection \( 7.1 \)). Then

\[
c(F) \leq P_0(\exists u \geq 0 \Delta N_{u-C} \in B, \Delta N_u \in B) = 0
\]

by standard properties of Lévy processes.

By the continuity of both stopping times on \( O^c \) it is obvious that \( F \cap O^c \) is a closed set. Then, just as in Proposition \( 30 \) we use Lemma \( 3.4 \). from \( [20] \) to find an open set \( O \) containing \( F \cap O^c \) with capacity \( c(O) < \epsilon/2 \). Setting \( O := O_1 \cup O_2 \) we can easily check via argument identical to the one at the end of the proof of Proposition \( 30 \) that \( 1_{\{\tau_A^k \leq \tau_A^k - C\}} \) is continuous on \( O^c \). Of course \( c(O) < \epsilon \), hence the indicator is q.c. \( \square \)

The following result will also be useful. It is also interesting by itself.

**Proposition 32.** Let \( A, B \in B(\mathbb{R}^d) \), s.t. \( 0 \notin A \) and \( C \in B(\mathbb{R}^d) \). Assume that \( v(A) > 0 \) for all \( v \in V \). Then for all \( k \geq 1 \) we have

\[
c(\Delta X^{\tau_A^k}_A \in B) \geq \sup_{v \in V} \frac{v(B \cap A)}{v(A)} \geq \frac{\mathbb{d}(B \cap A)}{\mathbb{d}(A)}
\]

and

\[
c(\Delta X^{\tau_A^k}_A \in B, \tau_A^k \in C) \geq \sup_{v \in V} \frac{v(B \cap A) \mu^{v,A,k}(C)}{v(A)} \geq \frac{\mathbb{d}(B \cap A) \inf_{v \in V} \mu^{v,A,k}(C)}{\mathbb{d}(A)}
\]

where \( \mu^{v,A,k} \) is Erlang distribution with the shape parameter \( k \) and the mean \( \frac{k}{v(A)} \).

**Proof.** Let \( \mathcal{A}^{d,\infty}_0 \) denote the subset of \( \mathcal{A}^{d,\infty}_0 \) consisting of all deterministic and constant \( \theta \)'s. Hence for an arbitrary \( \theta \in \mathcal{A}^{d,\infty}_0 \) we have that \( \theta \equiv (g_v(\cdot), a, \sigma) \) for some \( (v, a, \sigma) \in U \). It is easy to see that the canonical process \( X \) under \( \mathbb{P}_\theta \) for \( \theta \in \mathcal{A}^{d,\infty}_0 \) is a classical Lévy process with the Lévy triplet \( (a, \sigma^2, v) \). By the standard theory of Lévy processes and the properties of the capacity \( c \) we get then that

\[
c(\Delta X^{\tau_A^k}_A \in B) \geq \sup_{\theta \in \mathcal{A}^{d,\infty}_0} \mathbb{P}_\theta(\Delta X^{\tau_A^k}_A \in B) = \sup_{(v, a, \sigma) \in U} \frac{v(B \cap A)}{v(A)} = \sup_{v \in V} \frac{v(B \cap A)}{v(A)} \geq \frac{\mathbb{d}(B \cap A)}{\mathbb{d}(A)}.
\]

Similarly we have that

\[
c(\Delta X^{\tau_A^k}_A \in B, \tau_A^k \in C) \geq \sup_{\theta \in \mathcal{A}^{d,\infty}_0} \mathbb{P}_\theta(\Delta X^{\tau_A^k}_A \in B, \tau_A^k \in C) = \sup_{(v, a, \sigma) \in U} \mathbb{P}_\theta(\Delta X^{\tau_A^k}_A \in B) \mathbb{P}_\theta(\tau_A^k \in C)
\]

\[
\geq \inf_{v \in V} \frac{\mu^{v,A,k}(C)}{\mathbb{d}(A)} \frac{\mathbb{d}(B \cap A)}{\mathbb{d}(A)}.
\]

We have used here the standard properties of a classical Lévy process such as the independence a jump magnitude from the time it occurs, the distribution of the k-th jump time of size in a set \( A \) etc. \( \square \)

Using Proposition \( 30 \) and \( 32 \) we are able to prove the following theorem.

**Theorem 33.** Let \( \phi \in L^0(\mathbb{R}^d)^d \). Assume that there exists an open set \( A \) containing the support of \( \phi \) such that \( 0 \notin A \) and \( \mathbb{d}(\partial A) = 0 \). Assume moreover that \( \inf_{v \in V} v(A) > 0 \).
1. If \( \phi(\Delta X_{\tau^k} I_{\tau^k \leq t}) \) is q.c. for some \( t > 0 \) and \( k \geq 1 \) then \( \phi \) is \( \mathcal{V} \)-q.c.

2. If \( \phi \) is \( \mathcal{V} \)-q.c. then \( \phi(\Delta X_{\tau^k} I_{\tau^k \leq t}) \) is q.c. for all \( t > 0 \) and \( k \geq 1 \).

Proof. Point 1.

First note that by assumption we have that \( 0 < \inf_{v \in \mathcal{V}} v(A) \) and hence \( \sup_{v \in \mathcal{V}} \frac{1}{v(A)} < \infty \).

Consequently, \( \inf_{v \in \mathcal{V}} \mu^{v,A,k}([0, t]) > 0 \) as \( \{\mu^{v,A,k} : v \in \mathcal{V}\} \) is a family of Erlang distributions with shape parameter \( k \) fixed and a bounded mean (hence the mass of distribution does not accumulate in the neighbourhood of the infinity).

Fix \( \epsilon > 0 \) and take \( \eta = \epsilon \inf_{v \in \mathcal{V}} \mu^{v,A,k}([0, t])/(4e^\Psi(A)) \). Note that \( c^\Psi(A) < \infty \) as \( 0 \notin \tilde{A} \) (compare with [7], p. 14).

By the quasi-continuity of \( \Delta X_{\tau^k}, I_{\tau^k \leq t} \) and \( \phi(\Delta X_{\tau^k} I_{\tau^k \leq t}) \) there exist open sets \( O_1, O_2 \) and \( O_3 \) such that \( c(O_1) < \eta, i = 1, 2, 3 \) and \( \Delta X_{\tau^k}, I_{\tau^k \leq t} \) and \( \phi(\Delta X_{\tau^k} I_{\tau^k \leq t}) \) are continuous respectively on \( O_1 \), \( O_2 \) and \( O_3 \).

Ren proved in [17] that the family \( \mathfrak{F} \) is relatively compact hence by Prohorov’s theorem there exists a compact set \( K \) such that \( c(K^c) < \eta \). Take then a set \( F = O_1 \cap O_2 \cap O_3 \cap K \).

Note that \( F \) is a compact set as a closed subset of a compact set \( K \) and that \( c(F^c) < 4\eta \). By the choice of \( F \) we know that both \( \Delta X_{\tau^k} I_{\tau^k \leq t} \) and \( \phi(\Delta X_{\tau^k} I_{\tau^k \leq t}) \) are continuous on \( F \), therefore \( \phi \) is continuous on the set \( J := \{\Delta X_{\tau^k} \omega \} I_{\tau^k \leq t} ; \omega \in F \}. \) Then \( J \subset A \cup \{0\} \) by the choice of \( O^c \). \( J \) is also a closed set (or even compact) as an image of a compact set \( F \) under a continuous function.

Note also that \( \phi \) is continuous on \( A^c \) as its support lies in \( A \). As both \( J \) and \( A^c \) are closed sets, we deduce that \( \phi \) is a continuous function on \( J \cup A^c \), also a closed set.

Our target is to show that \( \mathcal{V} \)-capacity of \( (J \cup A^c)^c = J^c \cap A \) is small. We will do that by investigating the capacity of the following event: \( \{\Delta X_{\tau^k} \in J^c, \tau^k \leq t\} \). By Proposition [22] we have that

\[
e c(\Delta X_{\tau^k} \in J^c, \tau^k \leq t) \geq \frac{c^\Psi(J^c \cap A)}{c^\Psi(A)} \inf_{v \in \mathcal{V}} \mu^{v,A,k}([0, t])\]

hence

\[
c^\Psi(J^c \cap A) \leq \frac{c^\Psi(A)}{\inf_{v \in \mathcal{V}} \mu^{v,A,k}([0, t])} c(\Delta X_{\tau^k} \in J^c, \tau^k \leq t)
\]

Note also that we have the following set inclusion \( F \subset \{\Delta X_{\tau^k} \in J\} \cup \{\tau^k > t\} \). Since, of course, \( \{\Delta X_{\tau^k} \in J^c, \tau^k \leq t\} = \{\Delta X_{\tau^k} \in J\} \cup \{\tau^k > t\} \), we easily get \( \{\Delta X_{\tau^k} \in J^c, \tau^k \leq t\} \subset F^c \) and

\[
c^\Psi(J^c \cap A) \leq 4\eta \frac{c^\Psi(A)}{\inf_{v \in \mathcal{V}} \mu^{v,A,k}([0, t])} = \epsilon.
\]

Hence, we have proved the \( \mathcal{V} \)-quasi-continuity of \( \phi \).

Point 2.

Fix \( \epsilon > 0 \), \( k \geq 1 \) and \( t > 0 \) and choose an open subset \( \hat{O}_1 \subset \mathbb{R}^d_+ \) such that \( c^\Psi(\hat{O}_t) < \epsilon/(2t) \) and \( \phi \) is continuous on \( \hat{O}_1 \). Note that \( \phi \) is also continuous on \( A^c \) as the support of \( \phi \) lies in \( A \). Define the set

\[
O_t := \{\omega \in \Omega : \exists u \leq t \Delta u \in \hat{O}_1 \cap A\}.
\]

Hence \( O_t = \{\omega \in \Omega : \forall u \leq t \Delta u \in \hat{O}_1 \cup A^c\} \) and it is a closed set. Note also that we can
express \( O_t \) in the following manner: \( O_t = \{ L(t, \bar{O}_t \cap A) \geq 1 \} \). Hence

\[
\frac{\varepsilon}{2} \geq t \cdot c^V(\bar{O}_t \cap A) = \mathbb{E}[L(t, \bar{O}_t \cap A)] = \sup_{P \in \mathcal{P}} \mathbb{E}^P[L(t, \bar{O}_t \cap A)] = \sup_{P \in \mathcal{P}} \sum_{k=1}^{\infty} k \mathbb{P}(\{ L(t, \bar{O}_t \cap A) = k \}) \geq \sup_{P \in \mathcal{P}} \sum_{k=1}^{\infty} \mathbb{P}(\{ L(t, \bar{O}_t \cap A) = k \}) = \sup_{P \in \mathcal{P}} \mathbb{P}(\{ L(t, \bar{O}_t \cap A) \geq 1 \}) = c(O_t).
\]

Take also a closed set \( F \) with \( c(F^c) < \varepsilon/2 \) such that \( \Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^- \leq t \}} \) is continuous on \( F \). Put \( O := O_t \cup F^c \). Then \( \Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^+ \leq t \}} \) is continuous on \( O^c \) and takes values only in \( \bar{O}_t \cap \{ 0 \} \). Hence, \( \phi(\Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^+ \leq t \}}) \) is continuous on \( O^c \).

**Remark 34.** We might have also prove Theorem 33 Point 1 assuming the quasi-continuity of \( \phi(\Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^+ \leq t \}}) \). The proof would be a nearly identical (and we need to require only \( c^V(A) > 0 \) instead of \( \inf_{v \in \mathcal{V}} v(A) > 0 \)). However by Proposition 30 it is trivial that if \( \phi(\Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^+ \leq t \}}) \) is q.c. then so is \( \Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^+ \leq t \}} \). On the other hand, one can also prove that if \( \inf_{v \in \mathcal{V}} v(A) > 0 \) and \( \phi \in \mathcal{L}_d(A, \mathcal{V}) \) then the quasi-continuity of \( \phi(\Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^+ \leq t \}}) \) for all \( t > 0 \) implies the quasi-continuity of \( \phi(\Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^+ \leq t \}}) \). Hence Point 2 in the theorem above might have slightly been stronger.

Theorem 33 is very interesting, but it doesn’t give us the answer to the problem, if the quasi-continuity of the Poisson integral implies the \( \mathcal{V} \)-quasi-continuity of the integrand. We have of course that

\[
\int_A \phi(z) L(t, dz) = \sum_{k=1}^{\infty} \phi(\Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^+ \leq t \}})
\]

and it is easy to see that if \( \phi(\Delta X_{\tau^-} A \mathbb{1}_{\{ \tau_k^+ \leq t \}}) \) is q.c. for all \( k \geq 1 \) then the integral is also q.c. However, we are interested in proving the opposite relation. Even if we assumed that the integral is quasi-continuous for all \( t > 0 \) (which seems to be a reasonable assumption), it is not trivial how to proceed with such a proof. Hence we will take slightly stronger assumptions. First, we remind the notion of a quasi-continuity of a stochastic process (as it was introduced in [21]).

**Definition 35.** Let \( Y \) be a stochastic process. We say that \( Y \) is quasi-continuous on \( I \) (\( I = [0, T] \) or \( \mathbb{R}_+ \)) if for all \( \varepsilon > 0 \) there exists an open set \( O \) such that \( c(O) < \varepsilon \) and \((t, \omega) \mapsto Y_t(\omega)\) is a continuous function on \( I \times O \).

**Remark 36.** Assume for a moment that the integral \( \int_{\mathbb{R}_+} \phi(z) L(., dz) \) is quasi-continuous on \( \mathbb{R}_+ \) and that the support of \( \phi \) lies in an open set \( A \), \( 0 \notin \bar{A} \) with \( c^V(\partial A) = 0 \). Then of course the stopping times \( \tau_k^- \) are q.c. and it wouldn’t be difficult to prove by stopping the integral that \( \phi(\Delta X) \) would also be q.c. By Theorem 33 and Remark 34 we would get the \( \mathcal{V} \)-q.c. of \( \phi \). However, the assumption of q.c. of the integral is too strong, as it is show in the next proposition.

**Proposition 37.** Let \( \phi \in C_b(\mathbb{R}_+^d) \) with a support bounded away from 0 and of positive \( \mathcal{V} \)-capacity. Then the process \( t \mapsto \int_{\mathbb{R}_+} \phi(z) L(t, dz) \) is not quasi-continuous neither on \( \mathbb{R}_+ \) nor on any \( [0, T] \), \( T > 0 \).
Proof. The proof will be given only for $J = \mathbb{R}_+$, as the other case follows exactly the same argument. We assume the contrary. First note that by Proposition 19 in [8] and Corollary 25 we have that for each $\epsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{B}(\Omega)$ with $c(A) \leq \delta$ then

$$E[\int_{\mathbb{R}^d} |\phi(z)| L(t, dz) I_A] \leq \epsilon$$

(10)

Hence we fix $t, \epsilon > 0$ and take $\delta$ as above. By assumed quasi-continuity we can choose an open set $O$ such that $c(O) < \delta$ and $(t, \omega) \mapsto \int_{\mathbb{R}^d} |\phi(z)| L(t, dz)(\omega)$ is continuous on $[0, \infty] \times O^c$. Let $A$ be the support of $\phi$. Fix any $r > 0$ and $\omega \in O^c$ and take a sequence $r_n \uparrow r$. By the assumption we have that

$$\sum_{0 < u \leq r_n} \phi(\Delta \omega_u) \rightarrow \sum_{0 < u \leq r} \phi(\Delta \omega_u),$$

hence $\phi(\Delta \omega_r) = 0$. But $r$ and $\omega$ were arbitrary, hence

$$\phi(\Delta X(\omega)) \equiv 0$$

on $[0, \infty] \times O^c$ and we conclude that $\int_{\mathbb{R}^d} |\phi(z)| L(\cdot, dz) \equiv 0$ on $[0, \infty] \times O^c$.

Now we know that $E[\int_{\mathbb{R}^d} |\phi(z)| L(t, dz)] = t \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} |\phi(z)| v(dz)$. Moreover, by assumption that the support of $\phi$ has positive $\mathcal{V}$-capacity we know that this expectation must be also positive.

However, by (10) we have that

$$E[\int_{\mathbb{R}^d} |\phi(z)| L(t, dz)] \leq E[\int_{\mathbb{R}^d} |\phi(z)| L(t, dz) I_O] + E[\int_{\mathbb{R}^d} |\phi(z)| L(t, dz) I_{O^c}] \leq \epsilon.$$

We get the contradiction hence the integral cannot be quasi-continuous on $\mathbb{R}_+$. \qed

It is trivial to see why there is a problem with non-quasi-continuity of a Poisson integral as a stochastic process: for every path it has "large" discontinuities at jump times. However smoothing it a bit by making "new" jumps "small", helps to obtain quasi-continuity without distorting the process to much, hence the stopping time technique might be still successfully applied to get the desired result.

**Theorem 38.** Fix a bounded function $\phi$ such that the support of $\phi$ lies in an open set $A$ with $c^V(\partial A) = 0$, $c^V(A) > 0$ and $0 \notin \bar{A}$. Assume for each $t > 0$ we can find a sequence of functions $f^t_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

1. $f^t_n$ are continuous, non-increasing and $f^t_n \downarrow 1_{[0,t]}$,
2. $f^t_n(u) = 0$ for $u \geq t$, $f^t_n(u \vee 0) = 1$ for $u \leq t - 1/n$.
3. the stochastic process $Y^n$ defined as

$$Y^n_t := \int_0^t \int_{\mathbb{R}^d} \phi(z) f^t_n(s) L(ds, dz) := \sum_{0 < u \leq t} \phi(\Delta X_u) f^t_n(u)$$

is quasi-continuous on $\mathbb{R}_+$ for all $n$.

Then $\phi(\Delta X_{t+k})$ is q.c. for all $k$. Consequently, $\phi$ is $\mathcal{V}$-q.c.
Before we prove the theorem we note that it is easy to check that for any \( \phi \in C_b(\mathbb{R}_+^d) \) with the support bounded away from 0 we can easily find functions \( f_n^{\phi} \) such that \( Y^n \) is continuous on \([0, \infty[ \times \Omega \). If \( \phi \) is \( \mathcal{V} \)-q.c. and bounded with support bounded away from 0 one can also check that the assumptions above are satisfied.

**Proposition 39.** Let \( \phi \) be a bounded, \( \mathcal{V} \)-q.c. function such that the support of \( \phi \) lies in an open set \( A \) with \( c^\mathcal{V}(\partial A) = 0 \), \( c^\mathcal{V}(A) > 0 \) and \( 0 \notin A \). Then the process \( Y^n \) defined in Theorem 38 is quasi-continuous on \( \mathbb{R}_+ \) for any family of functions \( f_n^{\phi} \) satisfying properties 1-2.

**Proof.** First we fix \( T > 0 \) and we prove the quasi-continuity on \([0, T] \). Fix \( \epsilon > 0 \) and an open subset \( \tilde{O}_T \subset \mathbb{R}_+^d \) such that \( c^\mathcal{V}(\tilde{O}_T) < \epsilon / T \) and \( \phi \) is continuous on \( \tilde{O}_T \). Define \( O_T \) as follows

\[
O_T := \{ \omega \in \Omega : \exists u \leq T \Delta \omega_u \in \tilde{O}_T \cap A \}.
\]

Hence \( O_T^c = \{ \omega \in \Omega : \forall u \leq T \Delta \omega_u \in \tilde{O}_T \cup A^c \} \) and it is a closed set. Moreover it is easy to see that \( Y^n \) is continuous on \([0, T] \times O_T \) if \( f_n^{\phi} \) satisfies the properties 1 and 2. Note also that \( c(O_T) < \epsilon \) by exactly the same argument as in the proof of Theorem 33 Point 2. Therefore, \( Y^n \) is q.c. on \([0, T] \). But \( T \) was arbitrary. So fix \( \epsilon > 0 \) again and for each \( T = n, n = 1, 2 \ldots \) we can choose an open set \( O_n \) with capacity \( c(O_n) \leq \epsilon / 2^n \) such that \( Y^n \) is continuous on \([0, T] \times O_n \). By putting \( O = \bigcup_{n=1}^{\infty} \) we immediately see that \( Y^n \) is also continuous on \( \mathbb{R}_+ \times O^c \).

Therefore \( Y^n \) is q.c. on \( \mathbb{R}_+ \).

**Proof of Theorem 38.** We follow the idea presented in Remark 36. We fix \( k, n \in \mathbb{N} \) and \( \epsilon > 0 \). By the quasi-continuity of \( Y^n \) we may find an open set \( O_1 \) with capacity \( c(O_1) < \epsilon / 2 \) such that \( \tau^k_A, \tau^{k+1}_A, \mathbb{1}_{\{\tau^{k-1}_A \leq \tau^k_A \leq 1/n \}} \) are continuous on \( O_1^c \).

Take \( O = O_1 \cup O_2 \). Then we have that by the choice of \( O \) and the definition of \( Y^n \) that

\[
Z^n := \left( Y^n_{\tau^k_A + 1} - Y^n_{\tau^k_A} \right) \mathbb{1}_{\{\tau^k_A \leq \tau^{k+1}_A \leq 1/n \} \cap \{\tau^{k-1}_A \leq \tau^k_A \leq 1/n \}}
\]

is continuous on \( O \). Consequently, by boundedness of \( \phi \) we get that \( Z^n \in L^1_G(\Omega) \).

We will show now that \( Z^n \) converges to \( \phi(\Delta X^k_A) \) in \( L^1_G(\Omega) \), hence \( \phi(\Delta X^k_A) \in L^1_G(\Omega) \) and, in particular, is quasi-continuous.

\[\hat{E} [Z_n - \phi(\Delta X^k_A)] = \hat{E} [\phi(\Delta X^k_A)] \left| 1 - \mathbb{1}_{\{\tau^k_A \leq \tau^{k+1}_A \leq 1/n \} \cap \{\tau^{k-1}_A \leq \tau^k_A \leq 1/n \}} \right| \leq B c(\tau^k_A > \tau^{k+1}_A - 1/n) + B c(\tau^{k-1}_A > 1/n),\]

where \( B \) is a bound of \( \phi \). We will deal with the first summand, as the second has exactly the same structure.

Take again the set \( O_2 \). Note that

\[c(\tau^k_A > \tau^{k+1}_A - 1/n) \leq c(\{\tau^k_A - \tau^{k+1}_A \geq -1/n \} \cap O^c_2) + c(O_2) < c(\{\tau^k_A - \tau^{k+1}_A \geq -1/n \} \cap O^c_2) + \frac{\epsilon}{2} \]

Both \( \tau^k_A \) and \( \tau^{k+1}_A \) are continuous on \( O^c_2 \) so \( F_n := \{\{\tau^k_A - \tau^{k+1}_A \geq -1/n \} \cap O^c_2 \) is a sequence of closed sets decreasing to \( \emptyset \) (as \( \tau^k_A < \tau^{k+1}_A \) by its definition). Hence by the relative compactness of \( \mathcal{P} \) (see [17]) and Theorem 12 in [3] we may find \( N > 0 \) s.t. for all \( n > N \) \( c(F_n) < \epsilon / 2 \). Consequently \( Z_n \rightarrow \phi(\Delta X^k_A) \) in \( L^1_G(\Omega) \).
5.2 Characterization of the $\mathcal{V}$-quasi continuity

As it was seen in the Subsection 5.1, the $\mathcal{V}$-quasi continuity (under some mild conditions) determine the quasi-continuity of the integral. In this subsection we will give the quick criterion for $\mathcal{V}$-quasi-continuity.

**Proposition 40.** Let $\phi \in L^0(\mathbb{R}^d)$ and let $A$ be the set of all discontinuity points of $\phi$. Then if $\mathcal{V}$ is relatively compact and $c^\mathcal{V}(\bar{A}) = 0$, then $\phi$ is $\mathcal{V}$-q.c. On the other hand, if $\phi$ is $\mathcal{V}$-q.c. then

$$\inf\{c^\mathcal{V}(O): O - an \ open \ subset \ of \ \mathbb{R}^d, \ A \subset \bar{O}\} = 0.$$

**Proof.** Assume that $c^\mathcal{V}(\bar{A}) = 0$. Then by Lemma 3.4 in [20] we have that

$$0 = \inf\{c^\mathcal{V}(O): \bar{A} \subset O, O \ is \ open\},$$

so for each $\epsilon > 0$ we can choose an open set containing $\bar{A}$ with $\mathcal{V}$-capacity less than $\epsilon$. Of course $\phi$ is continuous on $O^c$, hence it’s $\mathcal{V}$-q.c. Note that Lemma 3.4. is formulated for the case of a capacity normed by 1. However the proof does not depend on this property and only the relative compactness is crucial for it.

To prove the second assertion of the proposition, take $\epsilon > 0$. By the $\mathcal{V}$-quasi-continuity there exists an open set $O$ such that $c^\mathcal{V}(O) < \epsilon$ and $\phi$ is continuous on $O^c$.

Take any $x \in A$. By the definition of $A$ there exists a sequence $x_n \to x$ s.t. $\phi(x_n)$ doesn’t converge to $\phi(x)$. We have two cases:

1. Infinitely many $x_n$ belong to $O^c$. W.l.o.g. we can assume that all $x_n \in O^c$. By the closedness of $O^c$ we deduce that also $x \in O^c$. But this contradicts the continuity of $\phi$ on $O^c$.

2. Infinitely many $x_n$ belong to $O$. Again w.l.o.g. we can assume that all $x_n \in O$. Hence $x \in \bar{O}$. But $x \in A$ was an arbitrary point, hence $A \subset \bar{O}$ and we have

$$\inf\{c^\mathcal{V}(O): \bar{A} \subset O, O \ is \ open\} < \epsilon.$$  

\[ \square \]

5.3 Example of a Poisson integral which is not quasi-continuous

In this subsection we consider a $G$-Lévy process $X$ associated with the following set $\mathcal{U} := \{\delta_x: x \in [1,2]\} \times \{0\} \times \{0\}$. We show directly from the definition that a very simple integral $Y_T := \int_{\mathbb{R}^d} \mathbf{1}_{\{1\}}(z) N(T, dz)$ is not quasi-continuous. Note that it is easy to check using Proposition 10 that $\phi = \mathbf{1}_{\{1\}}$ is not $\mathcal{V}$-q.c., just as it is predicted by the theory in Subsection 5.1.

**Proposition 41.** The random variable $Y_T$ is not q.c. for any $T > 0$.

**Sketch of the proof.** We fix $T > 0$. For simplicity we take $\Omega = \mathbb{D}_0([0, T], \mathbb{R}^d)$ and we will drop the subscript $T$ in $Y_T$. Assume the contrary, i.e. that $Y$ is q.c. Fix $0 < t_1 < t_2 < T$ and take $0 < \epsilon < Q(M([t_1, t_2], 1) = 1)$, where $M$ is a Poisson random measure associated with a $G$-Poisson process with parameter 1. By the assumed quasi-continuity of $Y$ we can choose a open set $O$ with $c(O) < \epsilon$ s.t. $Y|_{O^c}$ is continuous.

Introduce the following family of closed sets for $x \in [1,2]$

$$\Omega^x := \{\omega \in \Omega: \Delta \omega_s \in \{0, x\} \ \forall \ s \in [0, T]\}.$$

Of course we have that $c(\Omega^x) = 1$ for all $x \in [1,2]$. We can have two cases:
1. \( O^c \cap \Omega^1 = \emptyset \). Then \( O \supset \Omega^1 \) and consequently \( 1 > \epsilon > c(O) \geq 1 \). Contradiction.

2. \( O^c \cap \Omega^1 \neq \emptyset \). Then w.l.o.g. we may assume that \( O^c \supset \Omega^1 \) as \( Y \) is continuous on \( \Omega^1 \), which is a closed set. For \( t \in [0,T] \) and \( x \in [1,2] \) fine the following paths:

\[
\omega^{i,x} := 0 \cdot 1_{[0,t]} + x \cdot 1_{[t,T]}
\]

Note that if \( x_n \downarrow 1 \) and \( t_n \to t \) then \( \omega^{i,x_n} \to \omega^{i,x} \), but \( Y(\omega^{i,x_n}) = 0 \) for all \( n \) and \( Y(\omega^{i,x}) = 1 \). Hence for each \( t \in [0,T] \) there exist constants \( 0 \leq s^t < t < S^t \leq T \) and \( 1 < A^t \leq 2 \) s.t. for all \( u \in [s^t,S^t] \) and \( x \in [1,A^t] \) we have that \( \omega^{u,x} \notin O^c \) (as \( Y \) is continuous on \( O^c \)). Define \( I^t := [s^t,S^t] \). Then it is obvious that the family \( \{I^t: t \in [0,T]\} \) constitutes an open covering of a compact interval \([t^1,t^2]\) which was introduced earlier.

Hence, we can choose the finite subcovering \( \{I^t: t \in \{u_1, \ldots, u_n\}\} \). We also can define then \( A = \min_{0 \leq i \leq n} A^{u_i} \). Note that \( A > 1 \). Consequently we have

\[
\{\omega^{u,x}: u \in [t^1,t^2], x \in [1,A]\} \subset O.
\]

But we also have the following

\[
\epsilon > c(O) \geq c(\{\omega^{u,x}: u \in [t^1,t^2], x \in [1,A]\}) \geq Q(M([t^1,t^2], 1) = 1 > \epsilon.
\]

The third inequality is the consequence of the fact that for each \( x \in [1,A] \) there is a \( \mathbb{P}^x \in \mathcal{Q} \) such that the canonical process under \( \mathbb{P}^x \) is a standard Poisson process multiplied by \( x \) and \( \mathbb{P}^x(\{\omega^{u,x}: u \in [t^1,t^2], x \in [1,A]\}) = Q(M([t^1,t^2], 1) = 1) \).

In both cases we obtained the contradiction, hence \( Y \) cannot be q.c. \( \square \)

6 Applications

In this section we will apply the results of the previous sections to the problem of the decomposition of \( G \)-Lévy processes. We will prove that we can require the jump part to be a \( G \)-martingale in \( L^1_G(\Omega) \), but we cannot make the jump part a symmetric \( G \)-martingale (unless we extend the canonical space).

First, we have the following easy corollary of Corollary 28 and Proposition 41.

**Corollary 42.** Let \( A \subset \mathbb{R}^d \) be such that \( 0 \notin \overline{A} \) and \( c^V(\partial A) = 0 \). Let \( \phi \) be a continuous function with linear growth. Then under Assumption 2 we have that \( \int_A \phi(z) L(t,dz) \) is in \( L^1_G(\Omega) \) for all \( t > 0 \). In particular, the integral \( \int_A \phi(z) L(t,dz) \in L^1_G(\Omega) \).

**Proof.** It is sufficient to prove that \( L_A \phi \in L^1_G(A,V) \). We use the characterization of functions in \( L^1_G(A,V) \) (compare with Remark 19 and Proposition 21). By the assumption on \( A \) we know that there exists \( \epsilon > 0 \) such that \( A \subset B(0,\epsilon)^c \). We also know that \( V \) restricted to \( B(0,\epsilon)^c \) is relatively compact (compare with [7], p. 14), hence the \( c^{\|v\|_{B(0,\epsilon)^c}}(\partial A) = 0 \) and we can prove the \( V|_{B(0,\epsilon)^c} \)-q.c. of \( L_A \phi \), which then we easily extend to \( V \)-quasi-continuity. The "uniform integrability condition" might be easily obtained via the the linear growth and the fact that \( \sup_{v \in V} \int_{\{|z| \geq 1\}} |z|^p v(dz) < \infty \) for some \( p > 1 \).

Very similarly we get the following theorem.

**Theorem 43.** Let \( X \) be a \( G \)-Lévy process defined on the canonical sublinear expectation space \( (\Omega,L^1_G(\Omega),\mathbb{E}[\cdot]), \Omega = D(\mathbb{R}^+,\mathbb{R}^d) \), satisfying Assumption 2.
1. Then the decomposition $X_t = X^c_t + X^d_t$ as in Point 4 of Definition 3 might be taken on the same sublinear expectation space with $X^c_t := X_t - \int_{\mathbb{R}^d_+} zL(t,dz)$ and $X^d_t := \int_{\mathbb{R}^d_+} zL(t,dz)$ and both processes belong to $L^1_G(\Omega)$ for each $t$.

2. We may also require the discontinuous part to be a $G$-martingale without losing the property that it belongs to $L^1_G(\Omega)$. We simply take $X^d_t := \int_{\mathbb{R}^d_+} zL(t,dz) - t \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d_+} zv(dz)$. However, such defined discontinuous part is not a symmetric $G$-martingale, unless $\mathcal{V} = \{v\}$.

3. If there exist disjoint sets $A_1, \ldots, A_n \subset \mathbb{R}^d_0$ such that $0 \notin A_i$ and $\mathcal{C}^\mathcal{U}(\partial A_i) = 0$ for each $i = 1, \ldots, n$ then the following process

$$t \mapsto (X_t - \sum_{i=1}^n \int_{A_i} zL(t,dz), \int_{A_1} zL(t,dz), \ldots, \int_{A_n} zL(t,dz))$$

is a $G^A_1, \ldots, A_n$-Lévy process on sublinear expectation space $(\Omega, L^1_G(\Omega), \hat{\mathcal{E}}[\cdot])$, with $\Omega = \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, where $G^A_1, \ldots, A_n$ is a non-local operator associated with set $U$ via Lévy-Khintchine formula in Theorem 2 where

$$\hat{U} := \{(v|_{\mathbb{R}^d_0} \supset A_i \otimes \cdots \otimes v|_{A_i}, (p,0,\ldots,0), (q,0,\ldots,0)) : (v,p,q) \in \mathcal{U}\}.$$ 

4. In a similar manner we may compensate each of the discontinuous components of the $G^A_1, \ldots, A_n$-Lévy process defined in Point 3.

Lastly, we show that we cannot compensate the discontinuous part of $X$ with a factor which would make it a symmetric $G$-martingale without extending the space.

**Theorem 44.** Let $X$ be a $G$-Lévy process with finite activity defined on the canonical sublinear expectation space $(\Omega, L^1_G(\Omega), \hat{\mathcal{E}}[\cdot])$, $\Omega = \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. Assume that set $\mathcal{U}$ is of the following form

$$\mathcal{U} = \mathcal{V} \times \{0\} \times \mathcal{Q}.$$ 

and that there exists a measure $\pi$ on $\mathcal{B}(\mathbb{R}^d_0)$ such that each $v \in \mathcal{V}$ is equivalent to $\pi$ and we have the following bounds $0 < \underline{c} \leq \bar{c} < \infty$ for all $B \in \mathcal{B}(\mathbb{R}^d_0)$

$$\underline{c}\pi(B) \leq v(B) \leq \bar{c}\pi(B).$$

Assume also that there exists $p > 2$ s.t.

$$\sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d_+} |z|^p v(dz) < \infty. \quad(11)$$

Define $X^d_t := \int_{\mathbb{R}^d_+} zL(t,dz)$. If there exists a process with finite variation $Y$ such that $Y_t \in L^p_G(\Omega_t)$ some $q > 2$ and $X^d_t := X^d_t - Y_t$ is a symmetric $G$-martingale then $\mathcal{V} = \{v\}$.

**Proof.** Assume that there exists such a process. Fix $T > 0$. Note that by the Kunita’s inequality (see Corollary 4.4.24 in [1]) and the moment assumption in eq. (11), we have that $X^d_t \in L^q_G(\Omega_T)$ for $2 < s < p$. Then $X^d_t \in L^r_G(\Omega_T)$ for some $r = \min\{s,q\} > 2$. Note that and by Theorem 25 and Proposition 26 in [11] we get that $X^d_t$ must be written as a sum of a stochastic integral w.r.t. a $G$-Brownian motion, a non-increasing continuous $G$-martingale $K^c$ and a Itô-Lévy integral compensated by its mean. Since $X^d_t$ has a finite variation we deduce that the first summand is equal to 0, hence $X^d_t = \int_0^t \int_{\mathbb{R}^d_+} K(s,z)L(ds,dz) - \int_0^t \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d_+} K(s,z)v(dz)ds + K^c$. But the latter is a symmetric $G$-martingale if $\mathcal{V} = \{v\}$ and $K^c \equiv 0.$ \qed
7 Appendix

7.1 Construction of $g_v$

Take any measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ s.t. $\mu$ is a Lévy measure absolutely continuous w.r.t. Lebesgue measure and $\mu(\mathbb{R}^d_+) = \infty$.

Fix a sequence $\{\epsilon_n\}_{n \geq 0}$ s.t. $\epsilon_0 = \infty$ and $\epsilon_n \downarrow 0$ as $n \to \infty$.

Let $r_n := \sup_{v \in \mathcal{V}} \nu(\{\epsilon_n < |z| \leq \epsilon_{n-1}\})$, $n = 1, 2, \ldots$. We know that $r_n < \infty$ (see p. 14 in [7]). By the assumptions on $\mu$ we may find a decreasing sequence $\{\eta_n\}_{n \geq 0}$ s.t. $\eta_0 = \infty$ and $\mu(\{\eta_n < |z| \leq \eta_{n-1}\}) = r_n$, $n = 1, 2, \ldots$. We introduce the notation

$$O_n := \{z \in \mathbb{R}^d_0 : \epsilon_n < |z| \leq \epsilon_{n-1}\} \quad \text{and} \quad U_n := \{z \in \mathbb{R}^d_0 : \eta_n < |z| \leq \eta_{n-1}\}, \quad n = 1, 2, \ldots.$$

Note that $\nu(O_n) \leq \mu(U_n)$, $n = 1, 2, \ldots$ for every $v \in \mathcal{V}$. Again by the properties of $\mu$ we may find a subset $U_n^v \subset U_n$ such that $\nu(O_n) = \mu(U_n^v) = r_n^v$ for all $v \in \mathcal{V}$ and $n = 1, 2, \ldots$.

Since $\nu(O_n/r_n^v)$ and $\mu(U_n/r_n^v)$ are two probability measures and the second one is absolutely continuous w.r.t. Lebesgue measure, we may use the Knothe-Rosenblatt rearrangement (see for example [22], p.8-9) to find a function $g_{v,n}: U_n^v \to O_n$ such that

$$\frac{\nu(O_n(A))}{r_n^v} = \frac{\mu(U_n^v(g_{v,n}(A)))}{r_n^v} \quad \forall A \in \mathcal{B}(O_n).$$

It is now trivial that by putting $g_v := g_{v,n}$ on every $U_n^v$ and $g_v \equiv 0$ outside $\bigcup_n U_n^v$ we get a function which transports measure $\mu$ onto $v$. Of course by the construction for each $\epsilon > 0$ there exists an $\eta > 0$ s.t.

$$\bigcup_{v \in \mathcal{V}} g_v^{-1}(B(0,\epsilon^c)) \subset B(0,\eta)^c.$$

We may also take $\mu$ which integrates $|z|$.

7.2 Proof of characterization of $L^1_G(\Omega)$

Before we will go with the proof let us remind the properties of càdlàg modulus

**Lemma 45.** For any $\delta > 0$ and a càdlàg function $x: [0,T] \to \mathbb{R}^d$ define the following càdlàg modulus

$$\omega_x(\delta) := \inf_\pi \max_{0 < i \leq r} \sup_{s,t \in [t_{i-1},t_i]} |x(s) - x(t)|,$$

where infimum runs over all partitions $\pi = \{t_0, \ldots, t_r\}$ of the interval $[0,T]$ satisfying $0 = t_0 < t_1 < \ldots < t_r = T$ and $t_i - t_{i-1} > \delta$ for all $i = 1, 2, \ldots, r$. Define also

$$w'_x(\delta) := \sup_{t_1 \leq t_2 \leq \delta} \min \{|x(s) - x(t_1)|, |x(t_2) - x(s)|\}.$$

Then

1. $w'_x(\delta) \leq w'_x(\delta)$ for all $\delta > 0$ and $x \in \mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)$.

2. For every $\epsilon > 0$ and a subinterval $[\alpha, \beta] \subset [0,T]$ if $x$ does not have any jumps of magnitude $> \epsilon$ in the interval $[\alpha, \beta]$ then

$$\sup_{t_1, t_2 \in [\alpha, \beta], |t_2 - t_1| \leq \delta} |x(t_1) - x(t_2)| \leq 2w'_x(\delta) + \epsilon \leq 2w'_x(\delta) + \epsilon.$$
3. The function \( x \mapsto w'_x(\delta) \) is upper semicontinuous for all \( \delta > 0 \).
4. \( \lim_{\delta \downarrow 0} w'_x(\delta) = 0 \) for all \( x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \).

These properties are standard and might be found in \cite{2} for properties 1, 3 and 4 (see Chapter 3, Lemma 1, eq. (14.39) and (14.46)) and \cite{13} for property 2 (see Lemma 6.4 in Chapter VII).

Proof of Proposition \cite{12} Fix a random variable \( Y \in \mathcal{C}_{b, lip}(\Omega_T) \). For any \( n \in \mathbb{N} \) define the operator \( T^n : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \rightarrow \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) as

\[
T^n(\omega)(t) = \begin{cases} 
\omega_{\lfloor \frac{kT}{n} \rfloor}, & \text{if } t \in (kT/n, (k+1)T/n], \ k = 0, 1, \ldots, n-1.
\omega_T, & \text{if } t = T.
\end{cases}
\]

Define \( Y^n := Y \circ T^n \). Then \( Y^n \) depend only on \( \{\omega_{kT/n}\}_{k=0}^n \) thus there exists a function \( \phi^n : \mathbb{R}^{(n+1)\times d} \rightarrow \mathbb{R} \) such that

\[
Y^n(\omega) = \phi^n(\omega_0, \omega_{\frac{T}{n}}, \ldots, \omega_T).
\]

By the boundedness and Lipschitz continuity of \( Y \) we can easily prove that also \( \phi^n \) must be bounded and Lipschitz continuous (all we have to do is to consider the paths, which are constant on the intervals \([kT/n, (k+1)T/n])\). Note however that

\[
\tilde{E}[|Y - Y^n||] = \tilde{E}[|Y - Y \circ T^n||] \leq L \tilde{E}[d(X^T, X^T \circ T^n) \wedge 2K],
\]

where \( L > 0 \) and \( K \cdot L \) are respectively a Lipschitz constant and bound of \( Y, X^T \) is a canonical process, i.e. our \( G \)-Lévy process, stopped at time \( T \) and \( d \) is the Skorohod metric.

Fix now \( \epsilon > 0 \). Then for any \( \omega \in \Omega_T \) we have that a number of jumps with the magnitude > \( \epsilon \) is finite. Fix \( \omega \in \Omega_T \) and let \( 0 < r_1 < \ldots < r_{m-1} < T \) be the times of jumps with magnitude > \( \epsilon \). We can possibly have such a jump also at \( r_m := T \). We can choose \( n \) big enough such that \( r_{i+1} - r_i \geq T/n \) for \( i = 0, \ldots, m-1 \). Define \( A_{T, \epsilon}^n \) as a set of all \( \omega \in \Omega_T \) for which the minimal distance between jumps of magnitude > \( \epsilon \) is larger or equal to \( T/n \). We want to have an estimate of the Skorohod metric for \( \omega \in A_{T, \epsilon}^n \). To obtain it we construct the piecewise linear function \( \lambda^n \) as follows \( \lambda^n(0) = 0, \lambda^n(T) = T \), for each \( k = 1, \ldots, n-1 \) define

\[
\lambda^n \left( \frac{kT}{n} \right) := \begin{cases} 
\frac{kT}{n}, & \text{if } r_i \notin \left( \frac{(k-1)T}{n}, \frac{kT}{n} \right], \ i = 1, \ldots, m,
\frac{(k-1)T}{n}, & \text{if } r_i \in \left( \frac{(k-1)T}{n}, \frac{kT}{n} \right], \ i = 1, \ldots, m.
\end{cases}
\]

Moreover, let \( \lambda^n \) be linear between these nods. By the construction \( \|\lambda^n - Id\|_\infty \leq 2T/n \). Define \( t_k := \lambda^n(kT/n) \) for \( k = 0, \ldots, n \). Note that \( \omega \) does not have any jump of magnitude > \( \epsilon \) on \([t_k, t_{k+1}]\). Then by definition of the Skorohod metric and property 2 in Lemma \cite{12} we have

\[
d(\omega, T^n(\omega)) \wedge 2K = \left( \inf_{\lambda \in \Lambda} \max_{\lambda \in \Lambda} \left\{ \|\lambda - Id\|_\infty, \|T^n(\omega) - \omega \circ \lambda\|_\infty \right\} \right) \wedge 2K
\leq (\|\lambda^n - Id\|_\infty + \|T^n(\omega) - \omega \circ \lambda^n\|_\infty) \wedge 2K
\leq \left( \frac{2T}{n} + \max_{k=0,\ldots,n-1} \sup_{s,t \in [t_k, t_{k+1}]} |\omega(s) - \omega(t)| \right) \wedge 2K
\leq \left( \frac{2T}{n} + 2w'_x \left( \frac{2T}{n} \right) + \epsilon \right) \wedge 2K.
\]

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Thus we can define yet another bound $K^{n,\epsilon}$ as

$$K^{n,\epsilon}(\omega) := \left\{ \frac{2T}{n} + 2w_\omega \left( \frac{2T}{n} \right) + \epsilon \right\} \wedge 2K,$$

if $\omega \in A_{T}^{m,\epsilon}$, and $\omega \in \Omega_T \setminus A_{T}^{m,\epsilon}$.

Then $d(X^T, X^T \circ T^n) \wedge 2K \leq K^{n,\epsilon}$ and thus $\hat{E}[d(X^T, X^T \circ T^n) \wedge 2K] \leq \hat{E}[K^{n,\epsilon}]$. We also have $K^{n,\epsilon} \downarrow \epsilon$ as $n \to \infty$ on every $A_{T}^{m,\epsilon}$, $m$ and $\epsilon$ are fixed. This follows from property 4 in Lemma 45. Moreover we claim that $K^{n,\epsilon}$ is upper semi-continuous on every set $A_{T}^{m,\epsilon}$ for $m \leq n$ and a fixed $\epsilon$. Firstly, note that the set $A_{T}^{m,\epsilon}$ is closed under the Skorohod topology. This is clear from the definition of the set: if $\{\omega^k\}_k \subset A_{T}^{m,\epsilon}$ then the distance between the jumps of magnitude $> \epsilon$ is $\geq T/m$ for each $k$. But if $\omega^k \to \omega$ then also $\omega$ must satisfy this property and hence it belong to $A_{T}^{m,\epsilon}$ in Lemma 45. Moreover we claim than $K^{n,\epsilon}$ is upper semi-continuous on each closed set $A_{T}^{m,\epsilon}$, $m \leq n$. Then we have that $\omega \mapsto (2T/n + 2w_\omega(2T/n + \epsilon)) \wedge 2K$ is upper semi-continuous as a minimum of two upper semi-continuous functions and thus

$$\limsup_{k \to \infty} K^{n,\epsilon}(\omega^k) = \limsup_{k \to \infty} \left( \frac{2T}{n} + 2w_\omega \left( \frac{2T}{n} \right) + \epsilon \right) \wedge 2K \leq \left( \frac{2T}{n} + 2w_\omega \left( \frac{2T}{n} \right) + \epsilon \right) \wedge 2K = K^{n,\epsilon}(\omega).$$

Thus $K^{n,\epsilon}$ is upper semi-continuous on each closed set $A_{T}^{m,\epsilon}$, $m \leq n$. Then we have that $\omega \mapsto (2T/n + 2w_\omega(2T/n + \epsilon)) \wedge 2K$ is upper semi-continuous as a minimum of two upper semi-continuous functions and thus

$$\limsup_{k \to \infty} K^{n,\epsilon}(\omega^k) = \limsup_{k \to \infty} \left( \frac{2T}{n} + 2w_\omega \left( \frac{2T}{n} \right) + \epsilon \right) \wedge 2K \leq \left( \frac{2T}{n} + 2w_\omega \left( \frac{2T}{n} \right) + \epsilon \right) \wedge 2K = K^{n,\epsilon}(\omega).$$

Thus $K^{n,\epsilon}$ is upper semi-continuous on each closed set $A_{T}^{m,\epsilon}$, $m \leq n$. Then we have that $\omega \mapsto (2T/n + 2w_\omega(2T/n + \epsilon)) \wedge 2K$ is upper semi-continuous as a minimum of two upper semi-continuous functions and thus

$$\limsup_{k \to \infty} K^{n,\epsilon}(\omega^k) = \limsup_{k \to \infty} \left( \frac{2T}{n} + 2w_\omega \left( \frac{2T}{n} \right) + \epsilon \right) \wedge 2K \leq \left( \frac{2T}{n} + 2w_\omega \left( \frac{2T}{n} \right) + \epsilon \right) \wedge 2K = K^{n,\epsilon}(\omega).$$

We also claim that the sets $A_{T}^{m,\epsilon}$ are 'big' in the sense, that the capacity of the complement is decreasing to 0. Note that

$$(A_{T}^{m,\epsilon})^c = \{ \omega \in \Omega_T : \exists t, s \leq T, |t-s| < \frac{T}{m} \text{ and } |\Delta \omega_t| > \epsilon, |\Delta \omega_s| > \epsilon \}.$$

For any $\theta \in \mathcal{A}_{0,T}^m$ define the set

$$(A_{T}^{m,\epsilon,\theta})^c = \{ \omega \in \Omega_T : \exists t, s \leq T, |t-s| < \frac{T}{m} \text{ and } |\Delta B_{t}^{0,\theta}(\omega)| > \epsilon, |\Delta B_{s}^{0,\theta}(\omega)| > \epsilon \}.$$

We want to use the definition of $c$ and the fact that $\mathbb{P}^0$ is the law of $B^{0,\theta}$. We remind that $N_t = \int_0^t zN([0,t], dz)$. If $B^{0,\theta}$ has a jump of magnitude greater than $\epsilon$ then $N$ must have also a jump at $t$ of magnitude greater then $\eta' > 0$ (compare with Remark 6 and Subsection 7.1).

$$c[(A_{T}^{m,\epsilon})^c] = \sup_{\theta \in \mathcal{A}_{0,T}^m} \mathbb{P}^0[(A_{T}^{m,\epsilon})^c] = \sup_{\theta \in \mathcal{A}_{0,T}^m} \mathbb{P}_0[(A_{T}^{m,\epsilon,\theta})^c] \leq \mathbb{P}_0(\exists t, s \leq T, |t-s| < \frac{T}{m} \text{ and } |\Delta N_t| > \eta \epsilon' \text{ and } |\Delta N_s| > \eta \epsilon') = \mathbb{P}(B^{m,\epsilon}).$$

$N$ has $(\mathbb{P}_0 = a.a.)$ paths càdlàg and hence $B^{m,\epsilon} \supset B^{m+1,\epsilon}$ and $\mathbb{P}_0(\bigcap_{m=1}^\infty B^{m,\epsilon}) = 0$. Hence, by continuity of probability we have $\mathbb{P}_0(B^{m,\epsilon}) \to 0$ and consequently for every $\epsilon > 0$ one has $c[(A_{T}^{m,\epsilon})^c] \to 0$ as $m \to \infty$.

Note that we will prove the assertion of our proposition if we use the following lemma (proof below).

**Lemma 46.** For every $\epsilon > 0$ let $\{X_n,\epsilon\}_n$ be a sequence of non-negative uniformly bounded random variables on $\Omega_T$ such that there exists a sequence of closed sets $(F_m,\epsilon)_m$ having the following properties

4Note that it is not a problem for us that a jump of magnitude $> \epsilon$ for $\omega^k$ may have jump of magnitude exactly $\epsilon$ in the limit, as then the distance between "large" jumps would only increase.
1. \( c(F_{m,ε}^c) \to 0 \) as \( m \to \infty \).
2. \( X_{n,ε} \downarrow ε \) on every \( F_{m,ε} \).
3. \( X_{n,ε} \) is upper semi-continuous on every \( F_{m,ε} \ m ≤ n \).

Then for all \( ε > 0 \) there exists \( N(ε) \) such that for all \( n > N(ε) \) we have \( \hat{E}[X_{n,ε}] < 2ε \).

Applying this lemma to our sequence \( \{K^{n,ε}\}_n \) together with the closed sets \( (A_T^{m,ε})_m \) we get that for all \( n \) big enough we have

\[
\hat{E}[Y^n - Y] \leq L\hat{E}[d(X^T, X^T \circ T^n) \wedge 2K] \leq L\hat{E}[K^{n,ε}] < 2Lε.
\]

\( \square \)

Proof of Lemma \([46]\) Fix \( ε > 0 \). Let \( M \) be the bound of all \( X_{n,ε} \). By the representation of the sublinear expectation we have

\[
\hat{E}[X_{n,ε}] = \sup_{θ \in A_{θ,T}} \mathbb{E}^θ [X_{n,ε}] = \sup_{θ \in A_{θ,T}} \int_0^M \mathbb{P}^θ (X_{n,ε} ≥ t) dt ≤ \epsilon + \sup_{θ \in A_{θ,T}} \int_0^M \mathbb{P}^θ (X_{n,ε} ≥ t) dt
\]

\[
≤ \epsilon + \sup_{θ \in A_{θ,T}} \int_0^M \mathbb{P}^θ (\{X_{n,ε} ≥ t\} \cap F_{m,ε}) dt ≤ \epsilon + \sup_{θ \in A_{θ,T}} \int_0^M c(X_{n,ε}|F_{m,ε} ≥ t) dt + Mc(F_{m,ε}).
\]

By the first property of sets \( F_{m,ε} \) we can choose \( m \) big enough so that \( c(F_{m,ε}) \leq \frac{ε}{M} \). Choose \( n ≥ m \). By the upper semi-continuity of \( X_{n,ε} \) on \( F_{m,ε} \) we get that each \( \{X_{n,ε}|F_{m,ε} ≥ t\} \) is closed in the subspace topology on \( F_{m,ε} \). But \( F_{m,ε} \) is also a closed set in the Skorohod topology, thus \( \{X_{n,ε}|F_{m,ε} ≥ t\} \) is also closed in it. Moreover, due to monotone convergence to \( ε \) on \( F_{m,ε} \) we have that \( \{X_{n,ε}|F_{m,ε} ≥ t\} \downarrow \emptyset \) for every \( t > ε \) as \( n \uparrow \infty \). Thus by Lemma 7 in \([3]\) we get that \( c(X_{n,ε}|F_{m,ε} ≤ t) \downarrow 0 \) for every \( t > ε \) as \( n \uparrow \infty \) and we get the assertion of the lemma by applying monotone convergence theorem for the Lebesgue integral and choosing \( n ≥ m \) big enough, so that the integral is less then \( \frac{ε}{2} \). Thus

\[
0 ≤ \hat{E}[X_{n,ε}] ≤ 2ε \quad \text{for } n \text{ big enough.} \]

\( \square \)

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