METRISABILITY OF PROJECTIVE SURFACES AND PSEUDO-HOLOMORPHIC CURVES

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ABSTRACT. We show that the metrisability of an oriented projective surface is equivalent to the existence of pseudo-holomorphic curves. A projective structure $p$ and a volume form $\sigma$ on an oriented surface $M$ equip the total space of a certain disk bundle $Z \to M$ with a pair $(J_p, J_{p,\sigma})$ of almost complex structures. A conformal structure on $M$ corresponds to a section of $Z \to M$ and $p$ is metrisable by the metric $g$ if and only if $[g] : M \to Z$ is a pseudo-holomorphic curve with respect to $J_p$ and $J_{p,\sigma}$.

1. Introduction

A projective structure on a smooth manifold consists of an equivalence class $p$ of torsion-free connections on its tangent bundle, where two such connections are called equivalent if they have the same geodesics up to parametrisation. A projective structure $p$ is called metrisable if it contains the Levi-Civita connection of some Riemannian metric. The problem of (locally) characterizing the projective structures that are metrisable was first studied in the work of R. Liouville [17] in 1889, but was solved only relatively recently by Bryant, Dunajski and Eastwood for the case of two dimensions [2]. Since then, there has been renewed interest in the problem, see [5, 6, 8, 10, 11, 13, 14, 25, 27] for related recent work.

The purpose of this short note is to show that in the case of an oriented projective surface $(M, p)$, the metrisability of $p$ is equivalent to the existence of certain pseudo-holomorphic curves.

An orientation compatible complex structure on $M$ corresponds to a section of the bundle $\pi : Z \to M$ whose fibre at $x \in M$ consists of the orientation compatible linear complex structures on $T_x M$. The choice of a torsion-free connection $\nabla$ on $TM$ equips $Z$ with an almost complex structure $J$ [7, 26]. Namely, at $j \in Z$ we lift $j$ horizontally and take a natural complex structure on each fibre vertically. It turns out that $J$ is always integrable and does only depend on the projective equivalence class $p$ of $\nabla$, we thus denote it by $J_p$. Reversing the orientation on each fibre yields another almost complex structure $\tilde{J}$ which is however never integrable and is not projectively invariant. Fixing a volume form $\sigma$ on the projective surface $(M, p)$ determines a unique representative connection $\sigma\nabla \in p$ which preserves $\sigma$. We will write $\tilde{J}_{p,\sigma}$ for the non-integrable almost complex structure arising from $\sigma\nabla \in p$.

The choice of a conformal structure $[g]$ on an oriented surface $M$ defines an orientation compatible complex structure by rotating a tangent vector.

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counterclockwise by $\pi/2$ with respect to $[g]$. Thus, we may think of a conformal structure as a section $[g] : M \to Z$. Denoting the area form of a Riemannian metric $g$ by $dA_g$, we show:

**Theorem 1.1.** An oriented projective surface $(M, \mathfrak{p})$ is metrisable by the metric $g$ on $M$ if and only if $[g] : M \to (Z, J_p)$ is a holomorphic curve and $[g] : M \to (Z, \mathfrak{J}_p, dA_g)$ is a pseudo-holomorphic curve.

Applying a general existence result for pseudo-holomorphic curves [24, Theorem III] it follows that locally we can always find a Riemannian metric $g$ so that $[g] : M \to (Z, J_p)$ is a holomorphic curve or so that $[g] : M \to (Z, \mathfrak{J}_p, dA_g)$ is a pseudo-holomorphic curve. The geometric significance of the existence of such (pseudo-)holomorphic curves is given in Proposition 2.8 below.

The construction of the (integrable) almost complex structure $J_p$ on $Z$ given in [7, 26] is adapted from the construction of an almost complex structure $J$ on the twistor space $Y \to N$ of an oriented Riemannian 4-manifold $(N, g)$, see [1]. In the Riemannian setting the almost complex structure $J$ is integrable if and only if $g$ is self-dual. In [12], Eells–Salamon observe that reversing the orientation on each fibre of $Y \to N$ associates another almost complex structure $\mathfrak{J}$ on $Y$ to $(N, g)$ which is never integrable. Thus, the non-integrable almost complex structure $\mathfrak{J}$ used here may be thought of as the affine analogue of the non-integrable almost complex structure in oriented Riemannian 4-manifold geometry.

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2. Pseudo-Holomorphic Curves and Metrisability

Recall that the set of torsion-free connections on a surface $M$ is an affine space modelled on the smooth sections of the vector bundle $V = S^2(T^*M) \otimes TM$. We have a natural trace mapping $\text{tr} : V \to T^*M$, given in abstract index notation by $A^i_{jk} \mapsto A^i_{ik}$, as well as an inclusion $\text{Sym} : T^*M \to V$, given by $b_i \mapsto \delta^i_j b_k + \delta^i_k b_j$. The bundle $V$ thus decomposes as $V = V_0 \oplus T^*M$, where $V_0$ denotes the trace-free part of $V$. We have (Cartan, Eisenhart, Weyl) – the reader may also consult [9] for a modern reference:

**Lemma 2.1.** Two torsion-free connections $\nabla$ and $\nabla'$ on $TM$ are projectively equivalent if and only if there exists a 1-form $\xi$ on $M$ so that $\nabla - \nabla' = \text{Sym}(\xi)$.

This gives immediately:

**Lemma 2.2.** Let $(M, \mathfrak{p})$ be an oriented projective surface and $\sigma$ a volume form on $M$. Then there exists a unique representative connection $\sigma \nabla \in \mathfrak{p}$ preserving $\sigma$. 
Proof. Let $\nabla \in \mathfrak{p}$ be a representative connection. Since $\sigma$ is a volume form there exists a unique 1-form $\alpha$ on $M$ such that $\nabla \sigma = \alpha \otimes \sigma$. An elementary computation shows that the connection $\nabla + \text{Sym}(\xi)$ satisfies

$$(\nabla + \text{Sym}(\xi)) \sigma = \nabla \sigma - 3\xi \otimes \sigma,$$

for all $\xi \in \Omega^1(M)$. Thus the connection $\sigma \nabla = \nabla + \frac{1}{3}\text{Sym}(\alpha)$ preserves $\sigma$ and clearly is the only connection in $\mathfrak{p}$ doing so. \hfill $\square$

We also have:

**Lemma 2.3.** Let $\varphi \in \Gamma(V_0)$ and $\nabla$ be a torsion-free connection on $TM$. Then $\nabla + \varphi$ preserves a volume form $\sigma$ on $M$ if and only if $\nabla$ preserves the volume form $\sigma$.

Proof. Since $\varphi \in \Gamma(V_0)$, an elementary computation shows that the connections $\nabla$ and $\nabla + \varphi$ induce the same connection on the bundle $\Lambda^2(T^*M)$ whose non-vanishing sections are the volume forms. \hfill $\square$

For our purposes it is convenient to construct the almost complex structures $(J, J)$ associated to $\nabla$ in terms of the connection form $\theta$ on the oriented frame bundle of $M$. The oriented frame bundle $F$ of the oriented surface $M$ is the bundle $\nu : F \to M$ whose fibre at $x \in M$ consists of the linear isomorphisms $u : \mathbb{R}^2 \to T_x M$ that are orientation preserving with respect to the standard orientation on $\mathbb{R}^2$ and the given orientation on $T_x M$. The group $\text{GL}^+(2, \mathbb{R})$ acts transitively from the right on each fibre by the rule $R_u(a) = u \circ a$ for all $a \in \text{GL}^+(2, \mathbb{R})$, $u \in F$ and this action turns $\nu : F \to M$ into a principal right $\text{GL}^+(2, \mathbb{R})$-bundle. The total space $F$ carries a tautological $\mathbb{R}^2$-valued 1-form $\omega$ defined by $\omega_u = u^{-1} \circ \nu'_u$ and $\omega$ satisfies the equivariance property

$$R^*_a \omega = a^{-1} \omega$$

for all $a \in \text{GL}^+(2, \mathbb{R})$. We may embed $\text{GL}(1, \mathbb{C})$ as the subgroup of $\text{GL}^+(2, \mathbb{R})$ consisting of matrices that commute with the standard linear complex structure on $\mathbb{R}^2$. Note that may think of the oriented frame bundle $\nu : F \to M$ as a principal $\text{GL}(1, \mathbb{C})$-bundle over $Z = F/\text{GL}(1, \mathbb{C})$. We may describe an almost complex structure on $Z$ by describing the pullback of its $(1,0)$-forms to $F$. The pullback of a 1-form on $Z$ to $F$ is semi-basic for the projection $\nu : F \to Z$, that is, it vanishes when evaluated on vector fields that are tangent to the fibres of $\nu$. For $y \in \mathfrak{gl}(2, \mathbb{R})$ we denote by $Y_y$ the vector field on $F$ that is generated by the flow $R_{\exp(ty)}$. Clearly, the vector fields $Y_y$ for $y \in \mathfrak{gl}(1, \mathbb{C})$ span the vector fields on $F$ that are tangent to the fibres of $\nu$.

Let $\nabla$ be a torsion-free connection on $TM$ with connection form $\theta = (\theta^i_j)$ on $F$. Recall that $\theta$ satisfies the equivariance property

$$R^*_a \theta = a^{-1} \theta a$$

for all $a \in \text{GL}^+(2, \mathbb{R})$ and the structure equations

$$\begin{align*}
d\omega^i &= -\theta^i_j \wedge \omega^j, \\
d\theta^i_j &= -\theta^i_k \wedge \theta^k_j + \Theta^i_j,
\end{align*}$$

(2.3)
where $\Theta = (\Theta^i_j)$ denotes the curvature form of $\theta$. Since $\theta$ is a principal connection on $F$ it also satisfies $\theta(Y_y) = y$ for all $y \in \mathfrak{gl}(2,\mathbb{R})$. Since the Lie algebra of $\text{GL}(1,\mathbb{C})$ is spanned by the matrices of the form

$$
\begin{pmatrix}
z & -w \\
w & z
\end{pmatrix}
$$

for $(z, w) \in \mathbb{R}^2$, the complex-valued 1-forms on $F$ that are semi-basic for the projection $\nu : F \to Z$ are spanned by the forms $\omega = \omega^1 + i\omega^2$ and $\zeta = (\theta^1_1 - \theta^2_2) + i(\theta^1_2 + \theta^2_1)$

and their complex conjugates. We now have:

**Proposition 2.4.** Let $\nabla$ be a torsion-free connection on $TM$ with connection form $\theta = (\theta^i_j)$ on $F$. Then there exists a unique pair $(J, \tilde{J})$ of almost complex structures on $Z$ whose $(1,0)$-forms pull back to become linear combinations of the forms $(\omega, \zeta)$ in the case of $J$ and to $(\omega, \overline{\zeta})$ in the case of $\tilde{J}$. Moreover, the almost complex structure $J$ is always integrable, whereas $\tilde{J}$ is never integrable.

**Proof.** Writing

$$r e^{i\phi} \simeq \begin{pmatrix} r \cos \phi & -r \sin \phi \\
r \sin \phi & r \cos \phi \end{pmatrix}$$

for the elements of $\text{GL}(1,\mathbb{C})$, the equivariance property (2.1) of $\omega$ and (2.2) of $\theta$ implies

$$(R_{re^{i\phi}})^* \omega = \frac{1}{r} e^{i\phi} \omega \quad \text{and} \quad (R_{re^{i\phi}})^* \zeta = e^{-2i\phi} \zeta.$$ 

It follows that there exists a unique almost complex structure $J$ on $Z$ whose $(1,0)$-forms pull back to $F$ to become linear combinations of the forms $\omega, \zeta$. Likewise there exists a unique almost complex structure $\tilde{J}$ on $Z$ whose $(1,0)$-forms pull back to $F$ to become linear combinations of the forms $\omega, \overline{\zeta}$. Furthermore, simple computations using the structure equations (2.3) imply that

$$0 = d\zeta \wedge \omega \wedge \zeta = d\omega \wedge \omega \wedge \zeta.$$

Consequently, the Newlander-Nirenberg theorem [23] implies that $J$ is integrable. On the other hand, we get

$$d\omega \wedge \omega \wedge \overline{\zeta} = \frac{1}{2} \omega \wedge \overline{\omega} \wedge \zeta \wedge \overline{\zeta}$$

so that $\tilde{J}$ is never integrable. \qed

**Remark 2.5.** The equivariance properties (2.4) imply that the bundles

$$H = \nu'\{\text{Re}(\zeta) = 0, \text{Im}(\zeta) = 0\} \quad \text{and} \quad V = \nu'\{\text{Re}(\omega) = 0, \text{Im}(\omega) = 0\}$$

are well-defined distributions on $Z$ that are invariant with respect to $J$ (and $\tilde{J}$). Hence we have $TZ = H \oplus V$.

For the convenience of the reader, we also show [7, 26]:

**Proposition 2.6.** Suppose the torsion-free connections $\nabla$ and $\nabla'$ on $TM$ are projectively equivalent, then they induce the same integrable almost complex structure $J$ on $Z$. 

Proof. The connections $\nabla$ and $\nabla'$ are projectively equivalent if and only if there exists a 1-form $\xi$ on $M$ such that $\nabla' = \nabla + \text{Sym}(\xi)$. Writing $\theta = (\theta^i_j)$ for the connection form of $\nabla$ on $F$ and $\nu^*\xi = x_i\omega^i$ for real-valued functions $x_i$ on $F$, the connection form $\theta'$ of $\nabla'$ becomes

$$\theta' = \theta + \left(\begin{array}{cc} 2x_1\omega^1 + x_2\omega^2 & x_2\omega^1 \\ x_1\omega^2 & x_1\omega^1 + 2x_2\omega^2 \end{array}\right).$$

Consequently, we obtain

$$\zeta' = \zeta + (x_1\omega^1 - x_2\omega^2) + i(x_2\omega^1 + x_1\omega^2) = \zeta + (x_1 + ix_2)\omega$$

which shows that the complex span of $\omega, \zeta$ is the same as the one of $\omega, \zeta'$ and hence the two integrable almost complex structures are the same. \qed

Remark 2.7. For a projective structure $p$ on $M$ we will write $J_p$ for the integrable almost complex structure defined by any representative connection $\nabla \in p$. For a projective structure $p$ and a volume form $\sigma$ on $M$ we will write $\mathfrak{F}_{p,\sigma}$ for the non-integrable almost complex structure defined by the representative connection $\sigma\nabla \in p$. Note that the non-integrable almost complex structure is not projectively invariant.

Recall that a Weyl connection for a conformal structure $[g]$ is a torsion-free connection $[g]\nabla$ on $TM$ which preserves $[g]$. Fixing a Riemannian metric $g \in [g]$, the Weyl connections for $[g]$ can be written as $[g]\nabla = g\nabla + g \otimes B - \text{Sym}(\beta)$ for some 1-form $\beta$ on $M$ and where $B$ denotes the $g$-dual vector field to $\beta$. In [20] and in the language of thermostats in [22], it was observed that for every choice of a conformal structure $[g]$ on a projective surface $(M, p)$, there exists a unique Weyl connection $[g]\nabla$ for $[g]$ and a unique 1-form $\varphi \in \Gamma(V_0)$ so that $[g]\nabla + \varphi$ is a representative connection of $p$. Moreover the endomorphism $\varphi(X)$ is symmetric with respect to $[g]$ for every vector field $X$ on $M$. We call $[g]\nabla$ the Weyl connection determined by $[g]$. Explicitly, if $\nabla$ is any representative connection of $p$, $g \in [g]$ and if we define a vector field $B = \frac{1}{2}\text{tr} (g^2 \otimes (\nabla - g\nabla)\nabla_0)$, then

$$\varphi = (\nabla - g\nabla - g \otimes B)_0 \quad \text{and} \quad [g]\nabla = g\nabla + g \otimes B - \text{Sym}(\beta),$$

where $A_0$ denotes the trace-free part of a tensor field $A \in \Gamma(S^2(T^*M) \otimes TM)$. We refer the reader to [20, 22] for a proof that $[g]\nabla$ and $\varphi$ do satisfy the claimed properties.

Proposition 2.8. Let $(M,p)$ be an oriented projective surface and $g$ a Riemannian metric on $M$. Then we have:

(i) $p$ contains a Weyl connection for $[g]$ if and only if $[g] : M \rightarrow (Z, J_p)$ is a holomorphic curve;

(ii) the Weyl connection determined by $[g]$ is the Levi-Civita connection of $g$ if and only if $[g] : M \rightarrow (Z, \mathfrak{F}_{p,dA_0})$ is a pseudo-holomorphic curve.

Remark 2.9. Here we say $[g] : M \rightarrow (Z, \mathfrak{F})$ is a (pseudo-)holomorphic curve if the image $\Sigma = [g](M) \subset Z$ admits the structure of a (pseudo-)holomorphic curve. By admitting the structure of (pseudo-)holomorphic curve, we mean that $\Sigma$ can be equipped with a complex structure $J$, so that the inclusion $\iota : \Sigma \rightarrow Z$ is $(J, \mathfrak{F})$-linear, that is, satisfies $\mathfrak{F} \circ \iota' = \iota' \circ J$. 


As an immediate consequence, we obtain the Theorem 1.1:

Proof of Theorem 1.1. The projective structure $\mathfrak{p}$ is metrisable by $g$ if and only if the Weyl connection determined by $[g]$ is the Levi-Civita connection of $g$ and the 1-form $\varphi$ vanishes identically. The claim follows by applying Proposition 2.8. \qed

For the proof of Proposition 2.8 we also need the following Lemma:

Lemma 2.10. Let $(Z, \mathcal{J})$ be an almost complex four-manifold and $\omega, \chi \in \Omega^1(Z, \mathbb{C})$ a basis for the $(1,0)$-forms of $Z$. Suppose $\iota : \Sigma \to Z$ is an immersed surface so that $\iota^* (\omega \wedge \bar{\omega})$ is non-vanishing on $\Sigma$. Then $\Sigma$ admits the structure of a pseudo-holomorphic curve if and only if $\iota^*(\omega \wedge \chi)$ vanishes identically on $\Sigma$.

Proof. Since $\iota^*(\omega \wedge \bar{\omega})$ is non-vanishing on $\Sigma$, the forms $\iota^*\omega$ and $\iota^*\bar{\omega}$ span the complex-valued 1-forms on $\Sigma$. Recall that $\iota : \Sigma \to Z$ is $(j, \mathcal{J})$-linear if and only if the pullback of every $(1,0)$-form on $Z$ is a $(1,0)$-form on $\Sigma$, the claim follows. \qed

Proof of Proposition 2.8. Let $g$ be a Riemannian metric on the oriented projective surface $(M, \mathfrak{p})$. Without losing generality we can assume that the projective structure $\mathfrak{p}$ arises from a connection of the form $[g]\nabla + \varphi$. The Weyl connection $[g]\nabla$ satisfies

$$[g]\nabla dA_g = 2\beta \otimes dA_g$$

for some 1-form $\beta$ on $M$ and hence can be written as $[g]\nabla = g\nabla + g \otimes \beta^\sharp - \text{Sym}(\beta)$.

Now suppose $\nabla \in \mathfrak{p}$ preserves the volume form $dA_g$ of $g$. Then, by Lemma 2.3 it must be of the form

$$(2.5) \quad \nabla = [g]\nabla + \varphi + \frac{2}{3}\text{Sym}(\beta) = g\nabla + g \otimes \beta^\sharp - \frac{1}{3}\text{Sym}(\beta) + \varphi.$$ 

Proposition 2.4 and Lemma 2.10 imply that the condition that $[g] : M \to Z$ defines a pseudo-holomorphic curve with respect to $J_p$ respectively $3_p,dA_g$ is equivalent to the condition that on the pullback bundle $[g]^*F \to M$ the form $\omega \wedge \zeta$, respectively $\omega \wedge \bar{\zeta}$ vanishes identically, where $\zeta$ is computed from the connection form of $\nabla$ and where we think of $F$ as fibering over $Z$. Keeping this in mind we now compute the pullback of the forms $\zeta$ and $\bar{\zeta}$ to $[g]^*F$. Recall that the semi-basic 1-forms on $F$ are spanned by the components of $\omega$, hence there exist unique real-valued functions $g_{ij} = g_{ji}$ on $F$ so that $\nu^*g = g_{ij} \omega^i \otimes \omega^j$. Likewise, there exist unique real-valued functions $b_i$ on $F$ so that $\nu^*\beta = b_i \omega^i$ and unique real-valued function $A_{jk}^i = A_{kj}^i$ on $F$ so that $(\nu^*\varphi)^i_j = A_{jk}^i \omega^k$. The functions $A_{jk}^i$ satisfy furthermore $A_{ki}^k = 0$ and $g_{ik}A_{jk}^k = g_{jk}A_{ki}^k$ since $\varphi$ takes values in the endomorphisms of $TM$ that are trace-free and symmetric with respect to $g$. The Levi-Civita connection $(\psi_j^i)$ of $g$ is the unique principal $\text{GL}^+(2, \mathbb{R})$-connection on $F$ that satisfies

$$d\omega^i = -\psi_j^i \wedge \omega^j,$$

$$dg_{ij} = g_{ik}\psi_j^k + g_{kj}\psi_i^k.$$
The pullback bundle \( P := [g]^*F \) is cut out by the equations \( g_{11} = g_{22} \) and \( g_{12} = 0 \). On \( P \) we have
\[
0 = dg_{12} = g_{11}\psi_1^1 + g_{22}\psi_2^2 = g_{11}(\psi_1^1 + \psi_2^2),
0 = dg_{11} - dg_{22} = 2g_{11}\psi_1^1 - 2g_{22}\psi_2^2 = g_{11}(\psi_1^1 - \psi_2^2)
\]
On \( P \) the condition \( g_{ik}A^k_{jl} = g_{jk}A^k_{il} \) implies \( A^2_{11} = -A^2_{22} \) and \( A^2_{22} = -A^1_{11} \). Writing \( A^1_{11} = a_1 \) and \( A^2_{22} = a_2 \) and using (2.5), the connection form \( \theta \) of \( \nabla \) thus becomes
\[
\theta = \begin{pmatrix} \psi_1^1 - \psi_2^2 \\ \psi_2^1 \end{pmatrix} + \begin{pmatrix} b_1\omega^1 & b_1\omega^2 \\ b_2\omega^1 & b_2\omega^2 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 2b_1\omega^1 + b_2\omega^2 & b_2\omega^1 \\ b_1\omega^1 + 2b_2\omega^2 & -b_2\omega^1 \end{pmatrix} + \begin{pmatrix} a_1\omega^1 - a_2\omega^2 & -a_2\omega^1 - a_1\omega^2 \\ -a_2\omega^1 - a_1\omega^2 & a_1\omega^1 + a_2\omega^2 \end{pmatrix}
\]
Introducing the complex notation \( a = a_1 + ia_2 \) and \( b = \frac{1}{2}(b_1 - ib_2) \), we obtain from a simple calculation
\[
\zeta = (\theta_1^1 - \theta_2^2) + i(\theta_1^2 + \theta_2^1) = \frac{4}{3}T\omega + 2\omega^2,
\]
where we write \( \omega = \omega_1 + i\omega_2 \).

Finally, since \( [g] : M \to (Z, J_p) \) is a holomorphic curve if and only if \( \omega \wedge \zeta \) vanishes identically on \( P \), it follows that \( [g] : M \to (Z, J_p) \) is a holomorphic curve if and only if
\[
0 = \omega \wedge \zeta = 2\pi\omega \wedge \overline{\omega}
\]
which is equivalent to \( \varphi \) vanishing identically. This shows (i).

Likewise \( [g] : M \to (Z, \mathfrak{J}_p, dA_0) \) is a pseudo-holomorphic curve if and only if
\[
0 = \omega \wedge \zeta = \frac{4}{3}i\omega \wedge \overline{\omega}
\]
on \( P \). This is equivalent to \( \beta \) vanishing identically. This shows (ii).

As a corollary we obtain:

**Corollary 2.11.** Let \( (M, p) \) be a projective surface. Then locally \( p \) contains

(i) a Weyl connection \( [g] \nabla \) for some conformal structure \( [g] \);
(ii) a connection of the form \( \tilde{g} \nabla + \varphi \) for some Riemannian metric \( \tilde{g} \) and some \( \varphi \in \Gamma(V_0) \) with \( \varphi \) taking values in the endomorphisms that are \( \tilde{g} \)-symmetric.

**Remark 2.12.** The first statement of Proposition 2.8 and Corollary 2.11 was previously obtained in [19].

**Proof of Corollary 2.11.** We first consider the case (ii). We fix a volume form \( \sigma \) on \( M \). We need to show that in a neighbourhood \( U_x \) of every point \( x \in M \) there exists a conformal structure \( [g] \) which is a pseudo-holomorphic curve into the total space of the bundle \( \pi : Z \to M \), where we equip \( Z \) with the almost complex structure \( \mathfrak{J}_{p,\sigma} \). Choose \( j \in Z \) with \( \pi(j) = x \). Recall from Remark 2.5 that the subspace \( H_j \subset T_jZ \) is invariant under \( \mathfrak{J}_{p,\sigma} \). Now [24, Theorem III] implies that there exists a pseudo-holomorphic curve \( \Sigma \subset (Z, \mathfrak{J}_{p,\sigma}) \) which contains \( j \) and has \( H_j \) as its tangent space at \( j \). Since \( H_j \subset T_jZ \) is horizontal, the restriction \( \pi_j^*|_{H_j} : H_j \to T_xM \) is an
isomorphism. Therefore, the restriction of $\pi$ to $\Sigma$ is a local diffeomorphism in some neighbourhood of $j$. Hence there exists a neighbourhood $U_x$ of $x \in M$ and a section $[g] : U_x \to Z$ so that $[g](U_x) \subset \Sigma$. Thus, $[g] : U_x \to (Z, J_p)$ is a pseudo-holomorphic curve in the sense of Remark 2.9. Taking $\hat{g}$ to be the unique metric in $[g]$ with volume form $\sigma$ and applying Proposition 2.8 shows the claim. The case (i) follows in the same fashion, except that [24] is not needed, as $J_p$ is integrable and hence the construction of a holomorphic curve realising a prescribed $J_p$-invariant tangent plane is an elementary exercise.

Remark 2.13. Locally we can always find a holomorphic curve $[g] : M \to (Z, J_p)$, but globally this is not always possible. A properly convex projective structure $p$ on a closed surface $M$ with $\chi(M) < 0$ admits a holomorphic curve $[g] : M \to (Z, J_p)$ if and only if $p$ is hyperbolic [22]. One would expect that a corresponding global non-existence result should also hold in the pseudo-holomorphic setting for a suitable class of projective surfaces.

Remark 2.14. If $(M, p)$ is a closed oriented projective surface of with $\chi(M) < 0$, then there exists at most one holomorphic curve $[g] : M \to (Z, J_p)$, see [21].

Remark 2.15. Hitchin [15] gave a twistorial construction of (complex) two-dimensional holomorphic projective structures. In the holomorphic category such a projective structure corresponds to a complex surface $Z$ having a family of rational curves with self-intersection number one. Denoting the canonical bundle of $Z$ by $K_Z$, such a holomorphic projective surface is metrizable if and only if $K_Z^{-2/3}$ admits a holomorphic section which intersects each rational curve in $Z$ at two points [2, 3, 16].

Remark 2.16. The notion of a projective structure also makes sense in the complex setting and such structures are referred to as c-projective, see [4]. Correspondingly, there is a Kähler metrizability problem of c-projective structures. Some obstructions to Kähler metrizability of a (complex) two-dimensional c-projective structure have been obtained in [18].

We conclude by describing the holomorphic curves for the standard projective structure $p_0$ on the 2-sphere whose geodesics are the great circles.

Example 2.17. Let $S^2$ denote the sphere of radius 1 centered at the origin in $\mathbb{R}^3$ and $g$ its induced round metric of constant Gauss curvature 1 whose geodesics are the great circles. We equip $S^2$ with its standard orientation.

Recall that the unit tangent bundle $\lambda : T_1S^2 \to S^2$ of $(S^2, g)$ carries a canonical coframing $(\omega_1, \omega_2, \psi)$, where $\omega_1, \omega_2$ span the 1-forms on $T_1S^2$ that are semi-basic for the projection $\lambda$ and $\psi$ denotes the Levi-Civita connection form of $g$. The 1-forms $(\omega_1, \omega_2, \psi)$ satisfy the structure equations

\[
\begin{align*}
\omega_1 &= -\omega_2 \wedge \psi \\
\omega_2 &= -\psi \wedge \omega_1 \\
\psi &= -\omega_1 \wedge \omega_2.
\end{align*}
\]

Let $\hat{g}$ be a Riemannian metric on $S^2$ and write $\lambda^* \hat{g} = \hat{g}_{ij} \omega_i \otimes \omega_j$ for unique real-valued functions $\hat{g}_{ij} = \hat{g}_{ji}$ on $T_1S^2$. Phrased in modern language (c.f. [2]) and applied to the case of the 2-sphere, R. Liouville’s result [17] implies that if the metrics $\hat{g}$ and $g$ have the same unparametrised geodesics then the
functions \( h_{ij} := \hat{g}_{ij}(\hat{g}_{11}\hat{g}_{22} - \hat{g}_{12}^2)^{-2/3} \) satisfy the linear differential equations

\[
\begin{align*}
\text{d}h_{11} &= -2h_1\omega_2 + 2h_{12}\psi, \\
\text{d}h_{12} &= h_1\omega_1 - h_2\omega_2 - (h_{11} - h_{22})\psi, \\
\text{d}h_{22} &= 2h_2\omega_1 - 2h_{12}\psi,
\end{align*}
\tag{2.7}
\]

for some smooth real-valued functions \( h_i \) on \( T_1S^2 \). Conversely, a solution to (2.7) on \( T_1S^2 \) satisfying \( h_{11}h_{22} - h_{12}^2 \neq 0 \) gives a Riemannian metric \( \hat{g} \) on \( S^2 \) with \( \lambda^*\hat{g} = (h_{ij}(h_{11}h_{22} - h_{12}^2)^{-2})\omega_i \otimes \omega_j \) and that has the same unparametrised geodesics as \( g \).

Applying the exterior derivative to the above system of equations implies the existence of a unique real-valued function \( h \) on \( T_1S^2 \) such that

\[
\begin{align*}
\text{d}h_1 &= -h_{12}\omega_1 + (h_{11} + h)\omega_2 + 2h_2\psi, \\
\text{d}h_2 &= -(h_{22} + h)\omega_1 + h_1\omega_2 - h_1\psi.
\end{align*}
\]

Taking yet another exterior derivative gives that

\[
\text{d}h = -2h_1\omega_1 + 2h_2\omega_2.
\]

Writing

\[
\vartheta = \begin{pmatrix} 0 & -\omega_1 & -\omega_2 \\ \omega_1 & 0 & -\psi \\ \omega_2 & \psi & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} h & h_2 & -h_1 \\ h_2 & -h_{22} & h_{12} \\ -h_1 & h_{12} & -h_{11} \end{pmatrix}
\]

the above system of differential equations can be expressed as

\[
\text{d}H + \vartheta H + H\vartheta^t = 0.
\]

The structure equations (2.6) imply that \( \text{d}\vartheta + \vartheta \wedge \vartheta = 0 \), hence we may write \( \vartheta = \Xi^{-1}\text{d}\Xi \) for some diffeomorphism \( \Xi : T_1S^2 \rightarrow SO(3) \). It follows that the solutions are of the form \( H = \Xi^{-1}C(\Xi^{-1})^t \) for some constant symmetric 3-by-3 matrix \( C \). In particular, taking \( C = AA^t \) for some \( A \in \text{SL}(3, \mathbb{R}) \), we obtain a solution \( H_A \) providing a metric \( \hat{g}_A \) on \( S^2 \) having the great circles as its geodesics.

Finally, in order to construct the holomorphic curve \( [\hat{g}_A] : S^2 \rightarrow Z \) from \( H_A \), we interpret \( Z \) as an associated bundle to \( T_1S^2 \). We will only give a sketch of the construction and refer the reader to [22, §4] for additional details. The orientation and metric turn \( S^2 \) into a Riemann surface and hence a conformal structure on \( S^2 \) is given in terms of a Beltrami differential. Denoting the canonical bundle of \( S^2 \) by \( K_{S^2} \), a Beltrami differential is a section \( \mu \) of \( K_{S^2} \otimes K_{S^2}^{-1} \) satisfying \( |\mu(x)| < 1 \) for all \( x \in S^2 \), where \( | \cdot | \) denotes the norm induced by the natural Hermitian bundle metric on \( K_{S^2} \otimes K_{S^2}^{-1} \). The Riemannian metric \( g \) gives an isomorphism \( K_{S^2} \otimes K_{S^2}^{-1} \simeq K_{S^2}^{-1} \) and thus \( Z \) may be identified with \( T_1S^2 \times_{S^1} \mathbb{D} \), where \( S^1 \) acts by usual rotation on \( T_1S^2 \) and by \( z \cdot e^{i\phi} = ze^{-2i\phi} \) on the open unit disk \( \mathbb{D} \subset \mathbb{C} \). A holomorphic curve \( [\hat{g}] : S^2 \rightarrow Z \) is therefore represented by a map \( \mu : T_1S^2 \rightarrow \mathbb{D} \). Explicitly, the conformal structure arising from a Riemannian metric \( \hat{g} \) on \( S^2 \) is represented by the map

\[
\mu = \frac{p - q + 2ir}{p + q + 2\sqrt{pq - r^2}},
\]
where we write $\lambda^*\hat{g} = p\omega_1 \otimes \omega_1 + 2r\omega_1 \circ \omega_2 + q\omega_2 \otimes \omega_2$ for unique real-valued functions $p, q, r$ on $T_1S^2$. In our case, the holomorphic curve $[\hat{g}_A] : S^2 \to Z$ is thus represented by $\mu$ with

$$p = \frac{h_{11}}{(h_{11}h_{22} - h_{12}^2)^2}, \quad r = \frac{h_{12}}{(h_{11}h_{22} - h_{12}^2)^2}, \quad q = \frac{h_{22}}{(h_{11}h_{22} - h_{12}^2)^2}$$

and where the functions $h_{ij}$ arise from $H_A$ as above.

**Remark 2.18.** In the case of the standard projective structure on $S^2$ the complex surface $(Z,J_{p0})$ is biholomorphic to $\mathbb{CP}^2 \setminus \mathbb{RP}^2$ and moreover, the image of a holomorphic curve $[g] : S^2 \to Z$ is a smooth quadric, see [19]. Trying to explicitly relate the holomorphic curve $[\hat{g}_A]$ to its image quadric does in general however not seem to give manageable expressions.

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