Chiral Fermions from Manifolds Of $G_2$ Holonomy

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$M$-theory compactification on a manifold $X$ of $G_2$ holonomy can give chiral fermions in four dimensions only if $X$ is singular. A number of examples of conical singularity that give chiral fermions are known; the present paper is devoted to describing some additional examples. In some of them, the physics can be determined but the metric is not known explicitly, while in others the metric can be described explicitly but the physics is more challenging to understand.

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1. Introduction

Compactification of $M$-theory on a seven-manifold $X$ of $G_2$ holonomy is a natural way to obtain a four-dimensional theory with $N = 1$ supersymmetry. However, in the case of a smooth manifold $X$, one obtains in this way abelian gauge groups only without chiral fermions.

Singular manifolds of $G_2$ holonomy offer additional possibilities. A variety of different kinds of singularity are possible in $M$-theory. The $A - D - E$ singularities appear in codimension four, so they arise in compactification on a K3 surface. They can be embedded in a manifold of $G_2$ holonomy and give a mechanism to generate gauge symmetry. In codimension six, there are many $M$-theory singularities that can appear in Calabi-Yau threefolds; again, they can be embedded in a manifold of $G_2$ holonomy.

When a codimension four or codimension six singularity is embedded in a manifold of $G_2$ holonomy, the resulting physics largely is determined by what happens on a Calabi-Yau two-fold or three-fold. What is really new for a $G_2$-manifold is the occurrence of singularities of codimension seven, that is, isolated singularities (or singularities that are isolated modulo orbifold singularities). Since chiral fermions are a phenomenon that is special to compactification to four dimensions, it is perhaps not surprising that the isolated singularities turn out to be important for getting chiral fermions.

The known isolated singularities of $G_2$-manifolds are conical singularities. A number of examples of isolated conical singularities of $G_2$-manifolds have been studied recently from different points of view [1-4]. Some of them give chiral fermions. Anomaly cancellation in models with singularities of this kind has been investigated in [5].

The present paper is devoted to constructing and investigating two additional classes of examples of conical singularities of $G_2$-manifolds. Both classes of example are constructed by taking the quotient of a conical hyper-Kahler eight-manifold $\hat{X}$ by a $U(1)$ symmetry group. The two classes differ by what kind of $U(1)$ symmetry group is chosen.

In section two, we used duality with the heterotic string to motivate one construction, which is quite similar to one described for Type II strings on a Calabi-Yau threefold in [6]. In this construction, the $U(1)$ group preserves the hyper-Kahler structure of $\hat{X}$. For these examples, though we predict the existence of a conical $G_2$ metric, we cannot describe it explicitly. This is an interesting open problem. On the other hand, we can describe explicitly the physics that emerges from these examples since they are constructed to "geometrically engineer" a desired result. For example, suitably chosen examples of this
type give the usual chiral matter representations for the familiar grand unified groups such as $SU(5)$, $SO(10)$, and $E_6$.

In section three, we consider examples of a different kind, constructed with the same $\hat{X}$’s that appear in section two, but dividing by a different kind of $U(1)$ symmetry (a subgroup of an $SU(2)$ symmetry that rotates the complex structures of $\hat{X}$). Surprisingly, this also leads to conical metrics of $G_2$ holonomy, metrics that can be described explicitly using known results on self-dual Einstein metrics from quaternionic reduction \cite{4,5} together with the construction of $G_2$ metrics on cones over twistor spaces of self-dual Einstein manifolds \cite{1,2}. As the metric in these examples is known, the problem is to understand the physics. This can be done in some cases, but in general is not as straightforward as in section 2.

2. Singularities From Duality With The Heterotic String

We start by considering duality with the heterotic string. The heterotic string compactified on a Calabi-Yau three-fold $W$ can readily give chiral fermions. On the other hand, most Calabi-Yau manifolds participate in mirror symmetry. For $W$ to participate in mirror symmetry means \cite{11} that, in a suitable limit of its moduli space, it is a $T^3$ fibration (with singularities and monodromies) over a base $Q$. Then, taking the $T^3$’s to be small and using on each fiber the equivalence of the heterotic string on $T^3$ with $M$-theory on K3, it follows that the heterotic string on $W$ is dual to $M$-theory on a seven-manifold $X$ that is K3-fibered over $Q$ (again with singularities and monodromies). $X$ depends on the gauge bundle on $W$. Since the heterotic string on $W$ is supersymmetric, $M$-theory on $X$ is likewise supersymmetric, and hence $X$ is a manifold of $G_2$ holonomy.

Since there are many $W$’s that could be used in this construction (with many possible classes of gauge bundles) it follows that there are many manifolds of $G_2$ holonomy with suitable singularities to give nonabelian gauge symmetry with chiral fermions. The same conclusion can be reached using duality with Type IIA, as many six-dimensional Type IIA orientifolds that give chiral fermions are dual to $M$-theory on a $G_2$ manifold \cite{4}.

Let us try to use this construction to determine what kind of singularity $X$ must have. (The reasoning and the result are very similar to that given in \cite{3} for engineering matter from Type II singularities. In \cite{3}, the Dirac equation is derived directly rather than being motivated – as we will – by using duality with the heterotic string.) Suppose that the heterotic string on $W$ has an unbroken gauge symmetry $G$, which we will suppose to be
simply-laced (in other words, an $A$, $D$, or $E$ group) and at level one. This means that each K3 fiber of $X$ will have a singularity of type $G$. As one moves around in $X$ one will get a family of $G$-singularities parameterized by $Q$. If $Q$ is smooth and the normal space to $Q$ is a smoothly varying family of $G$-singularities, the low energy theory will be $G$ gauge theory on $\mathbf{R}^4 \times Q$ without chiral multiplets. So chiral fermions will have to come from singularities of $Q$ or points where $Q$ passes through a worse-than-orbifold singularity of $X$.

We can use the duality with the heterotic string to determine what kind of singularities are required. The argument will probably be easier to follow if we begin with a specific example, so we will consider the case of the $E_8 \times E_8$ heterotic string with $G = SU(5)$ a subgroup of one of the $E_8$'s. Such a model can very easily get chiral 5's and 10's of $SU(5)$; we want to see how this comes about, in the region of moduli space in which $W$ is $\mathbf{T}^3$-fibered over $Q$ with small fibers, and then we will translate this description to $M$-theory on $X$.

2.1. Computation In Heterotic String Language

Let us consider, for example, the 5's. The commutant of $SU(5)$ in $E_8$ is a second copy of $SU(5)$, which we will denote as $SU(5)'$. Since $SU(5)$ is unbroken, the structure group of the gauge bundle $E$ of $W$ reduces from $E_8$ to $SU(5)'$. Massless fermions in the heterotic string transform in the adjoint representation of $E_8$. The part of the adjoint representation of $E_8$ that transforms as 5 under $SU(5)$ transforms as 10 under $SU(5)'$. So to get massless chiral 5's of $SU(5)$, we must look at the Dirac equation $D$ on $W$ with values in the 10 of $SU(5)'$; the zero modes of that Dirac equation will give us the massless 5's of the unbroken $SU(5)$.

We denote the generic radius of the $\mathbf{T}^3$ fibers as $\alpha$, and we suppose that $\alpha$ is much less than the characteristic radius of $Q$. This is the regime of validity of the argument for duality with $M$-theory on $X$ (and the analysis of mirror symmetry [11]). For small $\alpha$, we can solve the Dirac equation on $X$ by first solving it along the fiber, and then along the base. In other words, we write $D = D' + D''$, where $D''$ is the Dirac operator along the fiber and $D'$ is the Dirac operator along the base. The eigenvalue of $D''$ gives an effective "mass" term in the Dirac equation on $Q$. For generic fibers of $W \to Q$, as we explain momentarily, the eigenvalues of $D''$ are all nonzero and of order $1/\alpha$. This is much too large to be canceled by the behavior of $D'$. So zero modes of $D$ are localized near points in $Q$ above which $D''$ has a zero mode.
When restricted to a $T^3$ fiber, the $SU(5)'$ bundle $E$ can be described as a flat bundle with monodromies around the three directions in $T^3$. For generic monodromies, every vector in the $10$ of $SU(5)'$ undergoes non-trivial “twists” in going around some (or all) of the three directions in $T^3$. When this is the case, the minimum eigenvalue of $D''$ is of order $1/\alpha$. A zero mode of $D''$ above some point $P \in Q$ arises precisely if for some vector in the $10$, the monodromies in the fiber are all trivial.

This means that the monodromies lie in the subgroup of $SU(5)'$ that leaves fixed that vector. If we represent the $10$ by an antisymmetric $5 \times 5$ matrix $B^{ij}$, $i, j = 1, \ldots, 5$, then the monodromy-invariant vector corresponds to an antisymmetric matrix $B$ that has some nonzero matrix element, say $B^{12}$; the subgroup of $SU(5)'$ that leaves $B$ invariant is clearly then a subgroup of $SU(2) \times SU(3)$ (where in these coordinates, $SU(2)$ acts on $i, j = 1, 2$ and $SU(3)$ on $i, j = 3, 4, 5$). Let us consider the case (which we will soon show to be generic) that $B^{12}$ is the only nonzero matrix element of $B$. If so, the subgroup of $SU(5)'$ that leaves $B$ fixed is precisely $SU(2) \times SU(3)$. There is actually a distinguished basis in this problem – the one that diagonalizes the monodromies near $P$ – and it is in this basis that $B$ has only one nonzero matrix element.

The commutant of $SU(2) \times SU(3)$ in $E_8$ is $SU(6)$. So over the point $P$, the monodromies commute not just with $SU(5)$ but with $SU(6)$. Everything of interest will happen inside this $SU(6)$. The reason for this is that the monodromies at $P$ give large masses to all $E_8$ modes except those in the adjoint of $SU(6)$. So we will formulate the rest of the discussion as if the heterotic string gauge group were just $SU(6)$, rather than $E_8$. Away from $P$, the monodromies break $SU(6)$ to $SU(5) \times U(1)$ (the global structure is actually $U(5)$). Restricting the discussion from $E_8$ to $SU(6)$ will mean treating the vacuum gauge bundle as a $U(1)$ bundle (the $U(1)$ being the second factor in $SU(5) \times U(1) \subset SU(6)$) rather than an $SU(5)'$ bundle.

The fact that, over $P$, the heterotic string has unbroken $SU(6)$ means that, in the $M$-theory description, the fiber over $P$ has an $SU(6)$ singularity. Likewise, the fact that away from $P$, the heterotic string has only $SU(5) \times U(1)$ unbroken means that the generic fiber, in the $M$-theory description, must contain an $SU(5)$ singularity only, rather than an $SU(6)$ singularity. As for the unbroken $U(1)$, in the $M$-theory description it must be carried by the $C$-field.

If we move away from the point $P$ in the base, the vector $B$ in the $10$ of $SU(5)'$ is no longer invariant under the monodromies. Under parallel transport around the three directions in $T^3$, it is transformed by phases $e^{2\pi i \theta_j}$, $j = 1, 2, 3$. Thus, the three $\theta_j$ must all
vanish to make $B$ invariant. As $Q$ is three-dimensional, we should expect generically that the point $P$ above which the monodromies are trivial is isolated. (Now we can see why it is natural to consider the case that, in the basis given by the monodromies near $P$, only one matrix element of $B$ is nonzero. Otherwise, the monodromies could act separately on the different matrix elements, and it would be necessary to adjust more than three parameters to make $B$ invariant. This would be a less generic situation.) We will only consider the (presumably generic) case that $P$ is disjoint from the singularities of the fibration $W \to Q$. Thus, the $T^3$ fiber over $P$ is smooth (as we have implicitly assumed in introducing the monodromies on $T^3$).

Since the fermion zero mode we are looking for is localized near $P$, the global behavior of $Q$ does not matter, and we can replace $Q$ by $\mathbb{R}^3$ and take $P$ to be the origin in $\mathbb{R}^3$. Also, since we are working in a small neighborhood of $P$, a variation in the fibers of $W$ will not be important and we can take the fibration to be a simple product. So we can replace $W$ by $\mathbb{R}^3 \times T^3$.

The generic behavior near $P$ is that the monodromy angles $\theta_j$ have a nondegenerate common zero, in which case we can use the $\theta_j$ as local coordinates on $\mathbb{R}^3$ near $P$. The chirality that we will get by solving the Dirac equation depends only on the local behavior of the $\theta_j$ near $P$. We could determine the answer from a general index theorem, but instead we will solve the Dirac equation in a special case. In $W = \mathbb{R}^3 \times T^3$, we take $\mathbb{R}^3$ to have linear coordinates $x_j$, $j = 1, 2, 3$, and $T^3$ to have angular coordinates $\phi_j$, $j = 1, 2, 3$, with the complex structure on $W$ such that $z_j = x_j + i\phi_j$ is holomorphic. Thus, holomorphically, $W = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ with $z_j$ a holomorphic function on the $j^{th}$ copy of $\mathbb{C}^*$. We assume that the Kahler form on $W$ is $\omega = \alpha^2 \sum_j dx_j \wedge d\phi_j$; this determines also the Kahler metric. We suppose that the monodromy angles are $\psi_j = \sigma_j f(x_j)$, where $\sigma_j$ are small nonzero real constants (with a further restriction discussed momentarily) and $f(x)$ is a monotonic function that is $-1$ for $x \to -\infty$, $0$ at $x = 0$, and $1$ for $x \to \infty$. We can get these monodromies with an abelian gauge field $A = \sum_j \sigma_j f(x_j) d\phi_j$. The curvature of the bundle is then $F = dA = \sum_j \sigma_j f'(x_j) dx_j d\phi_j$.

Next, we would like to impose the condition that the bundle is stable, and the connection $A$ obeys the hermitian Yang-Mills equation $\omega \wedge \omega \wedge F = 0$ ($\omega$ being the Kahler form). With our assumed Kahler form, $\omega \wedge \omega \wedge F$ is a multiple of $\sum_j \sigma_j f'(x_j)$. To obey the hermitian Yang-Mills equation, we need to make a complex gauge transformation to set $\omega \wedge \omega \wedge F$ to zero. This can be done in a unique way if and only if $\int \omega \wedge \omega \wedge F = 0$, which in our case (as $f'(x_j) > 0$) is so precisely if two of the $\sigma_j$ are positive and one is negative, or
vice-versa. The Dirac equation that we consider next is invariant under a complex gauge transformation, so we need not consider the details of modifying $A$ to obey the hermitian Yang-Mills equation.

It is easy to solve the Dirac equation in this situation. We have $\mathcal{D} = \sum_j \mathcal{D}_j$, where $\mathcal{D}_j$ is a Dirac equation on the $j^{th}$ copy of $\mathbb{C}^*$. The $\mathcal{D}_j$ anticommute, and a zero mode of $\mathcal{D}$ must be annihilated by each of the $\mathcal{D}_j$. We can explicitly write $\alpha \mathcal{D}_j$ as

$$
\begin{pmatrix}
0 \\
\frac{\partial}{\partial x_j} - i \left( \frac{\partial}{\partial \phi_j} + i \sigma_j f(x_j) \right)
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{\partial}{\partial x_j} + i \left( \frac{\partial}{\partial \phi_j} + i \sigma_j f(x_j) \right)
\end{pmatrix}.
$$

(2.1)

In this basis, the upper and lower components correspond to positive and negative chirality, respectively, or – upon making the standard relation of spinors on a Calabi-Yau manifold to $(0,q)$-forms – to $(0,0)$ and $(0,1)$-forms, respectively. If the $\sigma_j$ are small, zero modes are independent of $\phi_j$. The possible zero modes are

$$
\begin{pmatrix}
\exp(-\sigma_j \int_0^{x_j} dt f(t)) \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
\exp(\sigma_j \int_0^{x_j} dt f(t))
\end{pmatrix}.
$$

(2.2)

The first of these is normalizable if $\sigma_j > 0$ and the second is normalizable if $\sigma_j < 0$.

After tensoring together the zero modes of the $\mathcal{D}_j$ for $j = 1, 2, 3$, we learn that there is always precisely one zero mode in the 5 of $SU(5)$. It is a $(0,1)$-form or $(0,2)$-form depending on whether the number of negative $\sigma_j$ is 1 or 2. Thus, the chirality of this zero mode depends on the sign of the product $\sigma_1 \sigma_2 \sigma_3$. Since $F \wedge F \wedge F$, whose integral gives the third Chern class of the gauge bundle, is proportional in our example to $\sigma_1 \sigma_2 \sigma_3$, this is a local version of the familiar fact that the chiral asymmetry in Calabi-Yau compactification (the number of left-handed 5’s minus the number of right-handed 5’s) is equal to the integral of the third Chern class. Of course, if we want fermion zero modes transforming as the 5, we get them by complex conjugation, which reverses the chirality; so if there is a left-handed 5, there is a right-handed 5, and vice-versa.

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1 Since $W = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ is noncompact in our model calculation, an index theorem governing this situation would involve not only the integral of $F \wedge F \wedge F$ but also the behavior of the gauge field at infinity.
2.2. Description In M-Theory Language

So we have found a local structure in the heterotic string that gives a net chirality – the number of massless left-handed $5$’s minus right-handed $5$’s – of one. Let us see in more detail what it corresponds to in terms of $M$-theory on a manifold of $G_2$ holonomy.

Here it may help to review the case considered in [6], where the goal was geometric engineering of charged matter on a Calabi-Yau threefold in Type IIA. What was considered there was a Calabi-Yau three-fold $R$, fibered by K3’s with a base $Q'$, such that over a distinguished point $P \in Q'$ there is a singularity of type $\hat{G}$, and over the generic point in $Q'$ this singularity is replaced by one of type $G$ – the rank of $\hat{G}$ being one greater than that of $G$. In our example, $\hat{G} = SU(6)$ and $G = SU(5)$. In the application to Type IIA, although $R$ also has a Kahler metric, the focus is on the complex structure. For $\hat{G} = SU(6)$, $G = SU(5)$, let us describe the complex structure of $R$ near the singularities. The $SU(6)$ singularity is described by an equation $xy = z^6$. Its “unfolding” depends on five complex parameters and can be written $zy = z^6 + P_4(z)$, where $P_4(z)$ is a quartic polynomial in $z$. If – as in the present problem – we want to deform the $SU(6)$ singularity while maintaining an $SU(5)$ singularity, then we must pick $P_4$ so that the polynomial $z^6 + P_4$ has a fifth order root. This determines the deformation to be

$$xy = (z + 5\epsilon)(z - \epsilon)^5,$$

where we interpret $\epsilon$ as a complex parameter on the base $Q'$. Thus, (2.3) gives the complex structure of the total space $R$.

What is described in (2.3) is the partial unfolding of the $SU(6)$ singularity, keeping an $SU(5)$ singularity. In our $G_2$ problem, we need a similar construction, but we must view the $SU(6)$ singularity as a hyper-Kahler manifold, not just a complex manifold. In unfolding the $SU(6)$ singularity as a hyper-Kahler manifold, each complex parameter in $P_4$ is accompanied by a real parameter that controls the area of an exceptional divisor in the resolution/deformation of the singularity. The parameters are thus not five complex parameters but five triplets of real parameters. (There is an $SU(2)$ symmetry that rotates each triplet. In perturbative string theory, each triplet combines with a theta angle to make up a four-component hypermultiplet.)

To get a $G_2$-manifold, we must combine the complex parameter seen in (2.3) with a corresponding real parameter. Altogether, this will give a three-parameter family of
deformations of the $SU(6)$ singularity (understood as a hyper-Kahler manifold) to a hyper-
Kahler manifold with an $SU(5)$ singularity. The parameter space of this deformation is
what we have called $Q$, and the total space is a seven-manifold that is our desired singular
$G_2$-manifold $X$, with a singularity that produces the chiral fermions that we analyzed
above in the heterotic string language.

To find the hyper-Kahler unfolding of the $SU(6)$ singularity that preserves an $SU(5)$
singularity is not difficult, using Kronheimer’s description of the general unfolding via
a hyper-Kahler quotient \[12\]. At this stage, we might as well generalize to $SU(N)$, so
we consider a hyper-Kahler unfolding of the $SU(N + 1)$ singularity to give an $SU(N)$
singularity. The unfolding of the $SU(N + 1)$ singularity is obtained by taking a system
of $N + 1$ hypermultiplets $\Phi_0, \Phi_1, \ldots, \Phi_N$ with an action of $K = U(1)^N$. Under the $i$th $U(1)$
for $i = 1, \ldots, N$, $\Phi_i$ has charge 1, $\Phi_{i-1}$ has charge $-1$, and the others are neutral.
This configuration of hypermultiplets and gauge fields is known as the quiver diagram of
$SU(N + 1)$ and appears in studying $D$-branes near the $SU(N + 1)$ singularity \[13\]. We
let $H$ denote $\mathbb{R}^4$, so the hypermultiplets parameterize $H^{N+1}$, the product of $N + 1$ copies
of $\mathbb{R}^4$. The hyper-Kahler quotient of $H^{N+1}$ by $K$ is obtained by setting the $\vec{D}$-field (or components of the hyper-Kahler moment map) to zero and dividing by $K$. It is denoted $H^{N+1} // K$, and is isomorphic to the $SU(N + 1)$ singularity $\mathbb{R}^4 / \mathbb{Z}_{N+1}$. Its unfolding is
described by setting the $\vec{D}$-fields equal to arbitrary constants, not necessarily zero. In all,
there are $3N$ parameters in this unfolding – three times the dimension of $K$ – since for
each $U(1)$, $\vec{D}$ has three components, rotated by an $SU(2)$ group of $R$-symmetries.

We want a partial unfolding keeping an $SU(N)$ singularity. To describe this, we keep
$3(N - 1)$ of the parameters equal to zero and let only the remaining three vary; these three
will be simply the values of $\vec{D}$ for one of the $U(1)$’s. To carry out this procedure, we first
write $K = K' \times U(1)'$ (where $U(1)'$ denotes a chosen $U(1)$ factor of $K = U(1)^N$). Then
we take the hyper-Kahler quotient of $H^{N+1}$ by $K'$ to get a hyper-Kahler eight-manifold
$\tilde{X} = H^{N+1} // K'$, after which we take the ordinary quotient, not the hyper-Kahler quotient,
by $U(1)'$ to get a seven-manifold $X = \tilde{X} / U(1)'$ that should admit a metric of $G_2$-holonomy.
$X$ has a natural map to $Q = \mathbb{R}^3$ given by the value of the $\vec{D}$-field of $U(1)'$ – which was
not set to zero – and this map gives the fibration of $X$ by hyper-Kahler manifolds.

In the present example, we can easily make this explicit. We take $U(1)'$ to be the
“last” $U(1)$ in $K = U(1)^N$, so $U(1)'$ only acts on $\Phi_{N-1}$ and $\Phi_N$. $K'$ is therefore the product
of the first $N - 1$ $U(1)$’s; it acts trivially on $\Phi_N$, and acts on $\Phi_0, \ldots, \Phi_{N-1}$ according to
the standard quiver diagram of $SU(N)$. So the hyper-Kahler quotient $H^{N+1} // K'$ is just
\((H^N//K') \times H',\) where \(H^N//K'\) is the \(SU(N)\) singularity, isomorphic to \(H/Z_N\), and \(H'\) is parameterized by \(\Phi_N\). So finally, \(X\) will be \((H/Z_N \times H')/U(1)'\). To make this completely explicit, we just need to identify the group actions on \(H\) and \(H'\). If we parameterize \(H\) and \(H'\) respectively by pairs of complex variables \(\left(\begin{array}{c} a \\ b \end{array}\right)\) and \(\left(\begin{array}{c} a' \\ b' \end{array}\right)\), then the \(Z_N\) action on \(H\), such that the quotient \(H/Z_N\) is the \(SU(N)\) singularity, is given by

\[
\left(\begin{array}{c} a \\ b \end{array}\right) \to \left(\begin{array}{c} e^{2\pi i k/N} a \\ e^{-2\pi i k/N} b \end{array}\right). \tag{2.4}
\]

while the \(U(1)'\) action that commutes with this (and preserves the hyper-Kähler structure) is

\[
\left(\begin{array}{c} a \\ b \end{array}\right) \to \left(\begin{array}{c} e^{i\psi/N} a \\ e^{-i\psi/N} b \end{array}\right). \tag{2.5}
\]

The \(U(1)'\) action on \(H'\) is similarly

\[
\left(\begin{array}{c} a' \\ b' \end{array}\right) \to \left(\begin{array}{c} e^{-i\psi} a' \\ e^{i\psi} b' \end{array}\right). \tag{2.6}
\]

In all, if we set \(\lambda = e^{i\psi/N}\), \(w_1 = \overline{a'}\), \(w_2 = b'\), \(w_3 = a\), \(w_4 = \overline{b}\), then the quotient \((H/Z_N \times H')/U(1)\) can be described with four complex variables \(w_1, \ldots, w_4\) modulo the equivalence

\[(w_1, w_2, w_3, w_4) \to (\lambda^N w_1, \lambda^N w_2, \lambda w_3, \lambda w_4), \quad |\lambda| = 1. \tag{2.7}\]

This quotient is a cone on a weighted projective space \(\mathbf{WP}^3_{N,N,1,1}\). In fact, if we impose the equivalence relation (2.7) for all nonzero complex \(\lambda\), we would get the weighted projective space itself; by imposing this relation only for \(|\lambda| = 1\), we get a cone on the weighted projective space.

This example was analyzed in [5]. Away from the apex of the cone, it has a family of \(SU(N)\) singularities, giving unbroken \(SU(N)\) gauge symmetry, and an unbroken \(U(1)\) carried by the \(C\)-field. The present arguments show that \(M\)-theory on this cone has a massless chiral multiplet in the fundamental representation of \(U(N)\), a claim that is entirely compatible with the anomaly results [5] and with the claims (based on a quite different duality) in section 3.7 of [3].

Some extensions of this can be worked out in a similar fashion. Consider the case that away from \(P\), the monodromies break \(SU(N+1)\) to \(SU(p) \times SU(q) \times U(1)\), where \(p + q = N + 1\). Analysis of the Dirac equation along the above lines shows that such a model will give chiral fermions transforming as \((p, \overline{q})\) under \(SU(p) \times SU(q)\) (and charged
under the $U(1)$). To describe a dual in $M$-theory on a manifold of $G_2$ holonomy, we let $K = K' \times U(1)'$, where now $K' = K_1 \times K_2$, $K_1$ being the product of the first $p - 1$ $U(1)$’s in $K$ and $K_2$ the product of the last $q - 1$, while $U(1)'$ is the $p^{th}$ $U(1)$. Now we must define $\hat{X} = \mathbb{H}^{N+1}/K'$, and the manifold admitting a metric of $G_2$ holonomy should be $\hat{X}/U(1)'$.

We can compute $\hat{X}$ easily, since $K_1$ acts only on $\Phi_1, \ldots, \Phi_p$ and $K_2$ only on $\Phi_{p+1}, \ldots, \Phi_{N+1}$. The hyper-Kahler quotients by $K_1$ and $K_2$ thus simply construct the $SU(p)$ and $SU(q)$ singularities, and hence $\hat{X} = \mathbb{H}/\mathbb{Z}_p \times \mathbb{H}/\mathbb{Z}_q$. $\hat{X}$ has planes of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ singularities, which will persist in $X = \hat{X}/U(1)'$, which will also have a more severe singularity at the origin. So the model describes a theory with $SU(p) \times SU(q)$ gauge theory and chiral fermions supported at the origin. $U(1)'$ acts on $\mathbb{H}/\mathbb{Z}_p$ and $\mathbb{H}/\mathbb{Z}_q$ as the familiar global symmetry that preserves the hyper-Kahler structure of the $SU(p)$ and $SU(q)$ singularities. Representing those singularities by pairs $(a\,\,b)$ and $(a'\,\,b')$ modulo the usual action of $\mathbb{Z}_p$ and $\mathbb{Z}_q$, $U(1)'$ acts by

\[
\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} e^{i\psi/p}a \\ e^{-i\psi/p}b \end{pmatrix}, \quad \begin{pmatrix} a' \\ b' \end{pmatrix} \mapsto \begin{pmatrix} e^{-i\psi/q}a' \\ e^{i\psi/q}b' \end{pmatrix}.
\]

(2.8)

Now if $p$ and $q$ are relatively prime, we set $\lambda = e^{i\psi/pq}$, and we find that the $U(1)'$ action on the complex coordinates $w_1, \ldots, w_4$ (which are defined in terms of $a, b, a', b'$ by the same formulas as before) is

\[
(w_1, w_2, w_3, w_4) \rightarrow (\lambda^p w_1, \lambda^p w_2, \lambda^q w_3, \lambda^q w_4).
\]

(2.9)

If $p$ and $q$ are relatively prime, then the $U(1)'$ action, upon taking $\lambda$ to be a $p^{th}$ or $q^{th}$ root of 1, generates the $\mathbb{Z}_p \times \mathbb{Z}_q$ orbifolding that is part of the original definition of $\hat{X}$. Hence in forming the quotient $\hat{X}/U(1)'$, we need only to act on the $w$'s by the equivalence relation (2.9). The quotient is therefore a cone on a weighted projective space $\text{WCP}^3_{p,p,q,q}$. If $p$ and $q$ are not relatively prime, we let $(p, q) = r(n, m)$ where $r$ is the greatest common divisor and $n$ and $m$ are relatively prime. Then we let $\lambda = \exp(ir\psi/pq)$, so (2.9) is replaced with

\[
(w_1, w_2, w_3, w_4) \rightarrow (\lambda^n w_1, \lambda^n w_2, \lambda^m w_3, \lambda^m w_4).
\]

(2.10)

Also, in addition to imposing the relation (2.10), to reproduce $\hat{X}/U(1)$ we must divide by $\mathbb{Z}_r$, acting by $(w_1, w_2, w_3, w_4) \rightarrow (\zeta w_1, \zeta w_2, w_3, w_4)$, where $\zeta^r = 1$. So $X$ is a cone on $\text{WCP}^3_{n,n,m,m}/\mathbb{Z}_r$. These results were found in section 3.7 of [3] using duality with Type IIA brane configurations.
2.3. Generalization

It does not take much more effort to state the generalization. Suppose that we want to get chiral fermions in the representation $R$ of a simply-laced group $G$. This can be achieved for certain representations. We find a simply-laced group $\hat{G}$ of rank one more than the rank of $G$, such that $\hat{G}$ contains $G \times U(1)$ and the Lie algebra of $\hat{G}$ decomposes as $g \oplus o \oplus r \oplus \bar{r}$, where $g$ and $o$ are the Lie algebras of $G$ and $U(1)$, $r$ transforms as $R$ under $G$ and of charge 1 under $U(1)$, and $\bar{r}$ transforms as the complex conjugate. Such a $\hat{G}$ exists only for special $R$’s, and these are the $R$’s that we will generate from $G_2$ singularities.

Given $\hat{G}$, we proceed as above on the heterotic string side. We consider a family of $T^3$’s, parameterized by $Q$, with monodromy that at a special point $P \in Q$ leaves unbroken $\hat{G}$, and at a generic point breaks $\hat{G}$ to $G \times U(1)$. We moreover assume that near $P$, the monodromies have the same sort of generic behavior assumed above. Then the same computation as above will show that the heterotic string has, in this situation, a single multiplet of fermion zero modes (of positive or negative chirality depending on the sign of the analog of $\sigma_1\sigma_2\sigma_3$) in the representation $R$, with $U(1)$ charge 1.

Dualizing this to an $M$-theory description, over $P$ we want a $\hat{G}$ singularity, while over a generic point in $Q$, we should have a $G$ singularity. Thus, we want to consider the unfolding of the $\hat{G}$ singularity (as a hyper-Kahler manifold) that preserves a $G$ singularity. To do this is quite simple. We start with the Dynkin diagram of $\hat{G}$. The vertices are labeled with integers $n_i$, the Dynkin indices. In Kronheimer’s construction, the $\hat{G}$ singularity is obtained as the hyper-Kahler quotient of $\mathbf{H}^k$ (for some $k$) by the action of a group $K = \prod_i U(n_i)$. Its unfolding is obtained by allowing the $\vec{D}$-fields of the $U(1)$ factors (the centers of the $U(n_i)$) to vary.

The $G$ Dynkin diagram is obtained from that of $\hat{G}$ by omitting one node, corresponding to one of the $U(n_i)$ groups; we write the center of this group as $U(1)'$. Then we write $K$ (locally) as $K = K' \times U(1)'$, where $K'$ is defined by replacing the relevant $U(n_i)$ by $SU(n_i)$. We get a hyper-Kahler eight-manifold as the hyper-Kahler quotient $\check{X} = \mathbf{H}^k//K'$, and then we get a seven-manifold $X$ by taking the ordinary quotient $X = \check{X}/U(1)'$. This maps to $Q = \mathbb{R}^3$ by taking the value of the $U(1)'$ $\vec{D}$-field, which was not set to zero. The fiber over the origin is obtained by setting this $\vec{D}$-field to zero after all, and gives the original $\hat{G}$ singularity, while the generic fiber has a singularity of type $G$.

Just as in [6], one can readily work out examples of pairs $G, \hat{G}$. We will just consider the cases most relevant for grand unification. For $G = SU(N)$, to get chiral fields in
the antisymmetric tensor representation, $\hat{G}$ should be $SO(2N)$. For $G = SO(10)$, to get chiral fields in the 16, $\hat{G}$ should be $E_6$. For $G = SO(2k)$, to get chiral fields in the 2k, $\hat{G}$ should be $SO(2k + 2)$. (Note in this case that 2k is a real representation. However, the monodromies in the above construction break $SO(2k + 2)$ to $SO(2k) \times U(1)$, and the massless 2k’s obtained from the construction are charged under the $U(1)$; under $SO(2k) \times U(1)$ the representation is complex.) For $2k = 10$, this example might be used in constructing $SO(10)$ GUT’s. For $G = E_6$, to get 27’s, $\hat{G}$ should be $E_7$. A useful way to describe the topology of $X$ in these examples is not clear.

In this construction, we emphasized, on the heterotic string side, the most generic special monodromies that give enhanced gauge symmetry, which corresponds on the $M$-theory side to omitting from the hyper-Kähler quotient a rather special $U(1)$ that is related to a single node of the Dynkin diagram. We could also consider more general heterotic string monodromies; this would correspond in $M$-theory to omitting a more general linear combination of the $U(1)$’s.

2.4. Generation Of A Superpotential

Finally, let us briefly indicate how to map the superpotential as computed on the heterotic string side to an $M$-theory computation.

In heterotic string compactification on a Calabi-Yau threefold $W$, the chiral fermion wavefunctions are spread out over $W$. The superpotential has a classical contribution, coming from the integral of a product of chiral wavefunctions, and corrections from worldsheet instantons of genus zero.

In the present example, $W$ has a map to a three-manifold $Q$. Worldsheet instantons cannot project to a point in $Q$ (the fiber is Lagrangian and does not contain any holomorphic curves), so they must project to one-surfaces or two-surfaces in $Q$. A worldsheet instanton that projects to a two-surface in $Q$ would come in $M$-theory from a fivebrane instanton (since macroscopic heterotic strings in the seven-dimensional theory obtained by compactification on $\mathbf{T}^3$ are dual to strings made from fivebranes wrapped on K3). There are no supersymmetric fivebranes on a $G_2$-manifold $X$ (in fact $b_1(X) = 0$) so we conclude, as could presumably be argued directly, that the relevant heterotic string instantons project to one-manifolds (possibly singular one-manifolds, i.e. graphs) in $Q$.

A worldsheet instanton that projects to a one-manifold describes the propagation of a string wrapped on a one-cycle in the $\mathbf{T}^3$ fiber. Such a string lifts in $M$-theory to a
membrane wrapped on a two-cycle in K3. So worldsheet instantons in the heterotic string correspond to membrane instantons in $M$-theory.

What about the classical contribution from the heterotic string? At first sight, one might expect that this would come from a classical contribution on the $M$-theory side. But, with each $G$-multiplet of chiral fields supported at a distinct singularity, it is very hard to construct anything in $M$-theory that one might regard as a “classical” contribution. (There is no superpotential term that can be constructed locally involving only chiral fields supported at just one singularity, because they are always in an irreducible representation of the gauge group with a fixed and nonzero charge under a $U(1)$ that is unbroken locally.) We will argue, instead, that also the classical contribution on the heterotic string side corresponds in $M$-theory to a membrane instanton sum.

The idea is that on the heterotic string side, the chiral multiplets $\Phi_\alpha$ are supported at points $P_\alpha \in Q$ that we assume to be disjoint. The wavefunction for each $\Phi_\alpha$ decays exponentially in the distance from $P_\alpha$, so in the limit that the fibers of $W \to Q$ have small volume, the classical contributions are exponentially suppressed (and hence can come from membrane instantons on the $M$-theory side). To compute a classical contribution to a term in the superpotential of the form, say, $\Phi_\alpha \Phi_\beta \Phi_\gamma$ for distinct $\alpha, \beta, \gamma$, one would find a graph $\Gamma$ in $Q$ of “steepest descent” connecting $P_\alpha$, $P_\beta$, and $P_\gamma$, in which the exponential decay of the wavefunctions is minimized. Each line in the graph represents propagation of a mode of one of the $\Phi$’s. The components of $\Phi$ are charged strings; they can be regarded as strings wrapped, not on the geometrical fiber $T^3$ of $W$, but on the “internal torus” $T^{16} = T^8 \oplus T^8$ that carries the charged degrees of freedom in the bosonic construction of the heterotic string. Such internally wrapped strings also correspond to wrapped membranes in $M$-theory, so their propagation is again described in $M$-theory by a membrane instanton effect.

One detail should be checked here, namely that the wrapped membranes that correspond to the $\Phi$’s are not collapsed at singularities. The relevant chiral modes of the $\Phi$’s are in the adjoint representation of $\hat{G}$, and not in the adjoint representation of $G$. Once one leaves the points $P_\alpha$, the $M$-theory singularities are of type $G$, and only membranes that represent charges in the adjoint representation of $G$ are collapsed at singularities.

The fact that the propagating strings are charged under $G$ means that the corresponding wrapped membranes pass through the orbifold singularity. Hence, the intersection of the membrane instanton with $Q$ is a graph connecting the points $P_\alpha$, just as we observed in the heterotic string description.
One could consider a more special situation in which on the heterotic string side several of the $P_\alpha$ coincide. Then in $M$-theory there would be some more complicated singularities giving several gauge multiplets of chiral fields; in this situation some of the membrane instantons might collapse and some terms in the superpotential might not have a membrane instanton interpretation. An example is the cone on $SU(3)/U(1)^2$ considered in $\Box$; it gives three gauge multiplets of chiral fermions, with a superpotential that presumably cannot usefully be derived from membrane instantons.

3. Singularities from Twistor Spaces

The conical $G_2$ manifolds whose existence was argued in the previous section were of the form $\hat{X}/U(1)$, where $\hat{X}$ is a conical hyperkahler eight-manifold that is obtained from a hyper-Kahler quotient, and $U(1)$ preserves the hyper-Kahler structure of $\hat{X}$. These examples had the advantage that we know (via duality with the heterotic string) exactly what the physics is, but the disadvantage that we do not know how to actually construct a $G_2$ metric on $\hat{X}/U(1)$.

Here we will consider a different quotient of the form $\hat{X}/U(1)$, with the same $\hat{X}$ but a different $U(1)$ action. The properties will be opposite: the physics is in general harder to understand, but a conical $G_2$-metric on $\hat{X}/U(1)$ can be constructed explicitly. In this case, the $U(1)$ action on $\hat{X}$ does not preserve the hyper-Kahler structure of $\hat{X}$; it is part of an $SU(2)$ that rotates the three complex structures.

Let us first recall why $\hat{X}$ has such an $SU(2)$ action. (It also entered briefly in section 2.) Existence of this $SU(2)$ is inherited from the fact that $\hat{X}$ is a hyper-Kahler quotient of a system of $n$ free hypermultiplets, for some $n$. The bosons in these hypermultiplets parameterize $R^{4n}$ for some $n$. $R^{4n}$ admits the action of $SU(2) \times Sp(n)$, where $SU(2)$ rotates the three complex structures and $Sp(n)$ preserves them. To obtain $\hat{X}$, we pick a subgroup $H$ of $Sp(n)$ (the dimension of $H$ being $n-2$), and form the hyper-Kahler quotient – that is, we divide by $H$ and set to zero the $D$-fields (or hyper-Kahler moment map) of $H$. These operations are $SU(2)$-invariant, so the hyper-Kahler quotient $\hat{X} = R^{4n}/H$ inherits an $SU(2)$ action that rotates the three complex structures.

Now we simply pick a $U(1)$ subgroup of $SU(2)$, and we claim that the quotient $X = \hat{X}/U(1)$ has a conical $G_2$ metric which moreover can be described explicitly.

To show this requires several steps. $\hat{X}$ is a cone on a seven-manifold $V$, and $V$ also admits the action of $SU(2)$. ($V$ has a tri-Sasakian structure $\Box$ with $SU(2)$ rotating
the three vector fields.) $SU(2)$ acts on $V$ with three-dimensional orbits, so the quotient $V/SU(2)$ is, away from some possible singularities, a four-manifold $M$. The metric that $M$ inherits as the quotient of $V$ is actually a self-dual Einstein metric of positive scalar curvature. (The only smooth manifolds of this type are $S^4$ and $\mathbb{CP}^2$, so $M$ is almost always singular; the singularities arise because $SU(2)$ does not act freely on $V$. It can be shown that the singularities of $M$ are orbifold singularities.)

One way to explain that the metric on $M$ is a self-dual Einstein metric is as follows. $\mathbb{R}^{4n}$ to begin with is a cone on a sphere $S^{4n-1}$, and the quotient $S^{4n-1}/SU(2)$ is a copy of quaternionic projective space $\mathbb{HP}^{n-1}$, with a metric of $Sp(1) \cdot Sp(n-1)$ holonomy. For manifolds of such holonomy there is a quaternionic-Kähler quotient construction \cite{14} analogous to the hyper-Kähler quotient. $M$ can be characterized as the quaternionic-Kähler quotient of $\mathbb{HP}^{n-1}$ by $H$, and hence inherits a quaternionic-Kähler structure, which for a four-manifold is the same as a self-dual Einstein metric of positive scalar curvature.

Now let us describe $\hat{X}/U(1)$. For any conical manifold such as $\hat{X}$ or $\mathbb{R}^4$, we let $\hat{X}_0$ or $\mathbb{R}^4_0$ denote the same space with the origin deleted. For example, $\hat{X}_0 = \mathbb{R}^+ \times V$, where $\mathbb{R}^+$ is the positive real numbers. $V$ is an $SU(2)$ bundle over $M$, so $\hat{X}_0$ is an $\mathbb{R}^+ \times SU(2)$ bundle over $M$, or equivalently an $\mathbb{R}^4_0$ bundle over $M$ (since $\mathbb{R}^4_0 = \mathbb{R}^+ \times SU(2)$). $SU(2)$ acts on the $\mathbb{R}^4_0$ fibers via its standard action on $\mathbb{R}^4$, and the $U(1)$ we want to divide by is a subgroup of this $SU(2)$. With this information it is easy to describe $X_0 = \hat{X}_0/U(1)$. Since $\mathbb{R}^4/U(1) = \mathbb{R}^3$ (this being the basis for the standard lift of the $D6$-brane to $M$-theory \cite{15}), we have $\mathbb{R}^4_0/U(1) = \mathbb{R}^3_0$, and hence $X_0$ is an $\mathbb{R}^3_0$ bundle over $M$.

Next we want to include the origin in $\hat{X}$, and there are two natural-looking ways to do this. The origin is a single point in $\hat{X}$, so it descends to a single point in $\hat{X}/U(1)$. So, to describe $\hat{X}/U(1)$, we should add just one point to $X_0$. To do so, we note that $\mathbb{R}^3_0 = \mathbb{R}^+ \times S^2$, so an $\mathbb{R}^3_0$ bundle over $M$, such as $X_0$, is asymptotic at infinity to a cone on an $S^2$ bundle over $M$; we will let $Y$ denote this $S^2$ bundle. Then, if we let $X$ be a cone on $Y$, $X$ differs from $X_0$ precisely by adding a single point at the origin, and so $X = \hat{X}/U(1)$.

Alternatively, we could compactify the interior of $\hat{X}_0$ by replacing the $\mathbb{R}^3_0$ bundle over $M$ by an $\mathbb{R}^3$ bundle over $M$, simply by filling in the “origin” in each fiber. This gives another seven-manifold, which we will call $X'$.

In fact, it can be shown that $X'$ is the bundle of anti-self-dual two-forms over $M$, sometimes denoted $\wedge^{2,-}(M)$. $Y$ is the bundle of unit anti-self-dual two-forms, also known as the twistor space of $M$. 15
Since $M$ is a self-dual Einstein manifold, a general construction \cite{9-10} gives $G_2$ holonomy metrics on the bundles $X$ and $X'$ over $M$. The $G_2$ metric on $X'$ can be written explicitly in terms of the metric $d\Sigma^2$ on $M$, fiber coordinates $u_i$ for the bundle $X' \to M$, together with the Riemannian connection $A$ on the bundle of anti-self-dual two-forms:

\[
d s^2 = \frac{dr^2}{1 - (r_0/r)^4} + \frac{r^2}{4} (1 - (r_0/r)^4) |d_A u|^2 + \frac{r^2}{2} d\Sigma^2 \tag{3.1}
\]

Here $d_A u$ is the covariant derivative $d_A u_i = du_i + \epsilon_{ijk} A_j u_k$. $r_0$ is the modulus of the solution; it determines the volume of the copy of $M$ at the center of $X'$. As $r_0 \to 0$, $X'$ degenerates to $X$, and (3.1) degenerates to a conical $G_2$ metric on $X$.

The asymptotically conical metric (3.1) has also been described in \cite{16} starting from the canonical metric on the twistor space.

3.1. Determination Of $M$ In Some Examples

This construction can be used to construct many possible examples. To make it concrete, one would like an effective way to describe $M$. This has been done in \cite{8} for a simple class of examples; the classical singularities of these examples were also investigated in \cite{8}. In the rest of this paper, we will explain these results and then attempt to analyze the behavior of these examples in $M$-theory.

For these examples, $H = U(1)$ and there are three hypermultiplets $\Phi_i$ of charges $p_1, p_2, p_3$. We write the $\Phi_i$ in terms of complex fields $a_i, \bar{a}_i$. By possibly exchanging some $a$’s and $b$’s, we can assume that the $p_i$ are all positive. Also, by possibly rescaling the generator of $U(1)$, we can assume that the $p_i$ have no common factor.

The hyper-Kahler moment map can be written in terms of a real part

\[
\mu_R(p_i) = \sum_i p_i (|a_i|^2 - |b_i|^2) \tag{3.2}
\]

and a complex part

\[
\mu_C(p_i) = \sum_i p_i a_i \bar{b}_i. \tag{3.3}
\]

If we introduce $x_i = \sqrt{p_i} a_i$ and $y_i = \sqrt{p_i} b_i$, then in terms of the new variables (3.2) and (3.3) become

\[
\mu_R(1) = \sum_i (|x_i|^2 - |y_i|^2) \tag{3.4}
\]

16
\[ \mu_C(1) = \sum_i x_i y_i. \]  

This rescaling commutes with both \( SU(2) \) and \( H \). The submanifold \( N \) of \( H^3 \) on which the hyper-Kahler moment map vanishes coincides with the zero set of \( (3.4) \) and \( (3.5) \). \( N \) is in fact a cone on \( SU(3) \).

To see this, think of an element \( U \in SU(3) \) as a \( 3 \times 3 \) complex matrix whose columns are elements \( u, v, \) and \( w \) of \( \mathbb{C}^3 \). The fact that \( U \) is unitary means that \( u, v, \) and \( w \) are orthonormal. The fact that \( U \) has determinant one means that \( u \times v \cdot w = 1 \), where here \( \times \) is the usual cross product of three-vectors, and \( \cdot \) is the usual dot product of three-vectors. Given these conditions, \( w \) is uniquely determined in terms of \( u \) and \( v \); in fact, \( w = \overline{u} \times \overline{v} \). Hence elements of \( SU(3) \) are in one-to-one correspondence with pairs \( u, v \in \mathbb{C}^3 \) with \( |u|^2 = |v|^2 = 1, u \cdot \overline{v} = 0 \). Modulo scaling of \( x \) so that \( |x|^2 = 1 \), these are precisely the conditions \( (3.4) \) and \( (3.5) \) obeyed by \( x \) and \( y \). So the space of zeroes of \( (3.4) \) and \( (3.5) \) is a cone on \( SU(3) \).

The manifold \( SU(3) \) also admits an action of \( SU(2) \) rotating the first two columns \( u \) and \( v \) among themselves and leaving the third column \( w \) fixed. This is the \( SU(2) \) that rotates the complex structures of \( \hat{X} \). It is easy to take the quotient \( SU(3)/SU(2) \) in this language. Given three orthonormal vectors \( u, v, w \) in \( \mathbb{C}^3 \) with \( u \cdot v \times w = 1 \), the choice of \( w \) uniquely determines \( u \) and \( v \) up to an \( SU(2) \) transformation. So we can divide by \( SU(2) \) by just forgetting \( u \) and \( v \). So the quotient \( SU(3)/SU(2) \) is a copy of \( S^5 \), parameterized by a complex three-vector \( w \) with \( |w|^2 = 1 \). Such a \( w \) determines an element of \( S^5 \); \( N/SU(2) \) is a cone on this \( S^5 \).

To get the desired four-manifold \( M \), we must divide this \( S^5 \) by the original \( U(1) \) that acts on the hypermultiplets with charges \( p_1, p_2, p_3 \). Since \( H \) acts on the components of \( x \) and \( y \) with eigenvalues \( (p_1, p_2, p_3) \), it acts on the components of \( w = \overline{x} \times \overline{y} \) with eigenvalues \( -(p_2 + p_3, p_1 + p_3, p_1 + p_2) \). The overall minus sign can be removed by reversing the sign of the generator of \( H \). After doing this, the \( U(1) \) action on the \( w_i \) is

\[ w_i \to \lambda^{k_i} w_i \]  

where \( (k_1, k_2, k_3) = (p_2 + p_3, p_1 + p_3, p_1 + p_2) \). Since we assumed that the \( p_i \) have no common factor, the \( k_i \) either have no common factor or have a greatest common divisor of 2; the latter possibility arises precisely if the \( p_i \) are all odd.

\( M \) is the quotient \( S^5/H \) and is therefore a copy of \( \mathbb{WCP}^2_{q_1, q_2, q_3} \), where the weights \( q_i \) are \( k_i \) if the \( p_i \) are not all odd, and are \( k_i/2 \) if the \( p_i \) are all odd.
3.2. The Singularities

We want to understand the physics of \( M \) theory on the singular \( G_2 \)-holonomy spaces derived from the examples just considered. We must first understand the singularities of these spaces. We begin by describing the singularities of \( M = WCP^2_{q_1, q_2, q_3} \). We continue to denote the homogeneous coordinates of this weighted projective space as \( w_i \).

Although the \( q_i \) have no common factor, they are not necessarily pairwise relatively prime. Let \( n_1 \) be the greatest common divisor of \( q_2 \) and \( q_3 \), and similarly for \( n_2 \) and \( n_3 \). If we set \( w_i = 0 \), then \( \lambda \) acts trivially in (3.6) if \( \lambda^{n_i} = 1 \), so we get a \( \mathbb{Z}_{n_i} \) orbifold singularity. Setting \( w_i = 0 \) leaves us with a copy of \( \mathbb{C}P^1 \) (note that a weighted \( \mathbb{C}P^1 \) is equivalent topologically to an ordinary one). So in general, if the \( n_i \) are all greater than one, we have three \( \mathbb{C}P^1 \)'s of orbifold singularities in \( M \). The \( i^{th} \) and \( j^{th} \) such \( \mathbb{C}P^1 \)'s meet at the point with \( w_i = w_j = 0 \). The configuration is thus a “triangle” of \( \mathbb{C}P^1 \)'s. The vertices of the triangle – where \( w_i \) and \( w_j \) vanish for some \( i \) and \( j \) – are orbifold singularities of order \( q_k \) where \( k \) is the “third” label.

Now let us analyze the singularities in the bundle \( X' \) of anti-self-dual two-forms over \( M \) and in the twistor space \( Y \) of unit anti-self-dual two-forms. To find a fixed point in \( X' \) or \( Y \), we have to find a point in \( M \) that is invariant under

\[
 w_i \rightarrow \lambda^{n_i} w_i
\]

(for some value of \( \lambda \neq 1 \)) and an anti-self-dual two-form that is also invariant. So the key issue is to determine how (3.7) acts on the anti-self-dual two-forms over the orbifold singularities in \( M \).

Suppose, for example, we consider the fixed \( \mathbb{C}P^1 \) with \( w_1 = 0 \). Near this locus the anti-self-dual two-forms can be written explicitly

\[
 dw_1 \wedge d\overline{w}_1 - dt \wedge d\overline{t}, \quad dw_1 \wedge d\overline{t}, \quad dt \wedge d\overline{w}_1,
\]

where \( t \) is a linear combination of \( w_2 \) and \( w_3 \). For \( \lambda^{n_1} = 1 \), these anti-self-dual two-forms transform by \( 1, \lambda^{n_1}, \lambda^{-n_1} \), respectively. Since \( \lambda^{q_i} \neq 1 \) if \( \lambda \neq 1 \) (given that the \( q_i \) have no common factor) these eigenvalues are those of a nontrivial rotation of \( \mathbb{R}^3 \). The fixed point set in \( \mathbb{R}^3 \) consists of a single line – the multiples of \( \theta = dw_1 \wedge d\overline{w}_1 - dt \wedge d\overline{t} \). In \( S^2 \), the fixed point set consists of two points – the points where the fixed line in \( \mathbb{R}^3 \) meets \( S^2 \).

So in \( X' \), the singular set consists of three copies of \( \mathbb{C}P^1 \times \mathbb{R} \), the \( i^{th} \) copy being an orbifold singularity of order \( n_i \). They meet pairwise at a copy of \( \mathbb{R} \) at which two of the \( w_i \)
vanish. On these sets, we again get an orbifold singularity, of higher order. For example, consider the locus with $w_1 = w_2 = 0$, $w_3 = 1$, so to get a fixed point we need $\lambda^{q_3} = 1$. The anti-self-dual two forms in this locus are $|dw_1|^2 - |dw_2|^2$, $dw_1 \wedge d\overline{w_2}$, $d\overline{w_1} \wedge dw_2$ and transform as $1$ and $\lambda^{\pm(q_1-q_2)}$. The number of values of $\lambda$ for which $\lambda^{q_3} = \lambda^{q_1-q_2} = 1$ is $r_3$, the greatest common divisor of $q_3$ and $q_1-q_2$. So if $r_3 > 1$, the point with $w_1 = w_2 = 0$ lifts in $X'$ to an $\mathbb{R}^3$ of $\mathbb{Z}_{r_3}$ orbifold singularities. If $r_3 = 1$, then to get a fixed point, we must take the anti-self-dual two-form to be a multiple of $|dw_1|^2 - |dw_2|^2$, so in this case the point with $w_1 = w_2 = 0$ lifts to a fixed $\mathbb{R}$ in $X'$. This copy of $\mathbb{R}$ is a locus of $\mathbb{Z}_{q_3}$ orbifold singularities; the action of $\mathbb{Z}_{q_3}$ on the normal bundle can be described by saying that the normal bundle is a copy of $\mathbb{C}^3$ and the eigenvalues are $\lambda^{q_1}$, $\lambda^{q_2}$, and $\lambda^{q_1-q_2}$ (acting on $\overline{w_1}$, $w_2$, and an anti-self-dual two-form). In the twistor space $Y$, the story is similar, except that we want unit anti-self-dual two-forms, so a fixed $\mathbb{R}^3$ is replaced by $\mathbb{S}^2$ and a fixed $\mathbb{R}$ by two points. Of course, there is a similar story to the above with $q_3$ replaced by any of the $q_i$ and $r_i$ defined as the greatest common divisor of $q_i$ and $q_{i+1} - q_{i-1}$.

For $\mathbb{Z}_n$ fixed points of codimension four (of the form $\mathbb{C}\mathbb{P}^1 \times \mathbb{R}$ or $\mathbb{R}^3$), the action on the normal bundle is always that of a standard $A_{n-1}$ singularity (the eigenvalues being $q^{n\pm 1}$ where $q^n = 1$). This is required for $G_2$ holonomy (or supersymmetry) and can be verified from the above formulas.

As $X$ is a cone on $Y$, the fixed points in $X$ are just a cone on the fixed point set in $Y$, so each fixed $\mathbb{S}^2$ (or fixed point) is replaced by a fixed $\mathbb{R}^+ \times \mathbb{S}^2$ (or $\mathbb{R}^+$). Of course $\mathbb{R}^+$ is topologically the same as $\mathbb{R}$, but it is natural to here think of a half-line. What happens to the fixed point set when the conical manifold $X$ is deformed to the orbifold $X'$, the bundle of anti-self-dual two-forms? The two $\mathbb{R}^+ \times \mathbb{S}^2$’s over a fixed $\mathbb{C}\mathbb{P}^1$ in $M$ join together in $X'$ to a fixed $\mathbb{R} \times \mathbb{S}^2$, by gluing two half-lines $\mathbb{R}^+$ into a copy of $\mathbb{R}$. As for fixed $\mathbb{R}^+ \times \mathbb{S}^2$’s that lie over vertices of the triangle (when some $r_i$ are greater than one), here we note that upon adding an “origin,” $\mathbb{R}^+ \times \mathbb{S}^2$ is the same as $\mathbb{R}^3$; these give the fixed $\mathbb{R}^3$’s in $X'$ that were found above.

### 3.3. $M$-Theory Physics on $X$ and $X'$

We are now in a position to say something about the $M$-theory physics near these $G_2$-holonomy singularities. We begin by discussing the abelian symmetries associated with the $C$-field. We then explain why there must be chiral fermions charged under these symmetries supported at the conical singularity in the cone on $Y$. Following this we go on to discuss the physics associated with the additional orbifold singularities.
C-field symmetries and Chiral Fermions.

In $M$-theory on $X$ with $X$ a manifold with $G_2$-holonomy, the Lie algebra of the symmetries which originate from the $C$-field is isomorphic to $H^2(X; \mathbb{R})$, the second de Rham cohomology group. If $Y$ is the twistor space of $\mathbb{WCP}^2_{q_1,q_2,q_3}$, then the second Betti number of $Y$ is two. This is also the second Betti number of the cone on $Y$. Thus in $M$-theory on the cone on $Y$ we have, at least locally, a $U(1)^2$ symmetry group originating from the $C$-field.

On the other hand, the smoothed out cone $X'$ is contractible to $\mathbb{WCP}^2_{q_1,q_2,q_3}$, and therefore its second Betti number is one. Thus, when we deform the cone, the $U(1)^2$ symmetry gets broken to $U(1)$. This is exactly as in the case studied in [3] when all the weights are one. As will become apparent below, other symmetries of the physics also get broken when we deform the cone.

In [3], it was shown that in $M$-theory on the cone on $Y$ we will find chiral fermions charged under $U(1)^2$ if

$$\int_Y \omega_i \wedge \omega_j \wedge \omega_k \neq 0 \quad (3.9)$$

for any $\omega_i \in H^2(Y; \mathbb{Z})$. By taking two $\omega$'s to be pullbacks from the base and the third to have a nonzero integral over the fiber, we can make (3.9) nonzero. This works for any $M$ of positive second Betti number (such as the examples considered here) and implies that there are chiral fermions, charged under the abelian gauge symmetries from the $C$-field, and supported at the conical singularity.

We now go on to consider the physics associated with the singularities in the cone on $Y$ and its smoothed out version. The general discussion is rather complicated, as one might guess from the discussion of the singularities above, and we will focus on certain classes of examples. For simplicity, we will mainly consider cases in which the $r_i$ are all 1.

The Generic Case

We will first consider the case, which one might loosely consider generic, in which the $q_i$ are pairwise relatively prime as well as $r_i = 1$. In this case, $M$ has three isolated orbifold points (where two of the $w_i$ vanish), and $Y$ contains six isolated orbifold points. In the neighbourhood of the $i^{th}$ such point, $Y$ is locally $\mathbb{C}^3/\mathbb{Z}_{q_i}$ and as found above, $\mathbb{Z}_{q_i}$ acts with eigenvalues $\lambda^{-q_i+1}, \lambda^{q_i-1}, \lambda^{q_i+1-q_i-1}$.

There is not a known useful description of the behavior of $M$-theory at a codimension six orbifold singularity of this type. It is believed that $M$-theory on $(\mathbb{C}^3/\mathbb{Z}_{q_i}) \times \mathbb{R}^5$
with generic action of $Z_{q_i}$ generates at low energies a non-trivial conformal theory in five dimensions. (This assertion generalizes special cases that have been studied in [17-19].)

Assuming that this is correct, the low energy behavior of $M$-theory on $R^4 \times X'$ is determined by those conformal field theories. The three fixed lines in $X'$ correspond to copies of $R^4 \times R = R^5$ on which nontrivial conformal field theories are realized. There are no further complications because the fixed $R$'s in $X'$ do not intersect.

If we degenerate the orbifold $X'$ to the cone $X$, things are more complicated as the $R$'s intersect at the conical singularity at the origin. Additional phenomena might be expected at this point, especially since a chiral anomaly is supported there, as we have seen above.

A Special Case

Having discussed a general case where we can say very little, we will now go on to discuss a case which we can more or less understand fully. This is when the $q_i$ take the form $(p, p, q)$ with $p$ and $q$ relatively prime. In this case, $\Sigma$, the singular set of $Y$, consists of a pair of two-spheres of $A_{p-1}$ singularities and a single two-sphere of $A_{q-1}$ singularities. In the cone on $Y$ each of these becomes a copy of $R^3$ and all three meet at the origin. The three $R^3$'s support gauge groups that are respectively $SU(p)$, $SU(p)$, and $SU(q)$.

Locally the topology of this singularity is very similar to a case discussed in [3] in which three $R^3$ families of $A_{n-1}$ singularities met at the conical singularity. In addition to the $SU(n)^3$ gauge symmetry it was found that chiral fermions were residing at the origin in the representation $(n, n, 1) + (1, n, \bar{n}) + (\bar{n}, 1, n)$. In the example we are discussing here, the only difference is that one of the $R^3$'s of singularities is of a different rank from the other two. We therefore propose that in this example there are chiral fermions at the origin in the representations $(p, p, 1) + (1, p, \bar{q}) + (\bar{p}, 1, q)$.

This proposal is compatible with the anomaly calculations in [3]. We can argue for it more precisely by relating the problem to a Type IIA configuration on $R^6$, with three sets of intersecting D6-branes (each filling out a copy of $R^3$ passing through the origin) of multiplicities $(p, p, q)$.

With $WCP_{p, p, q}^2$ defined via the $U(1)$ action on $C^3$ that multiplies the homogeneous coordinates by $(\lambda^p, \lambda^p, \lambda^q)$, consider the $U(1)$ action $\alpha$ that acts by by $(\tau, \tau, 1)$, $|\tau| = 1$. The fixed points of $\alpha$ in the cone on $Y$ are precisely the orbifold singularities $\Sigma$. Proceeding exactly as in section 3.4 of [3] one can show that $Y/U(1) = S^5$ and that the corresponding Type IIA description is in terms of D6-branes in flat space.

For more general weights the reasoning given in [3] does not hold and it is unlikely that one can relate the corresponding $M$-theory physics to $D$-branes in flat space.
The description in terms of branes makes it apparent that there is a cubic super-

potential between the three chiral multiplets. Consider now Higgsing the $SU(p)^2$ gauge symmetry down to its diagonal $SU(p)$ by giving a vev to the $(p, \overline{p}, 1)$ superfield. The superpotential implies that the remaining two fields become massive so that the only remaining massless degrees of freedom are the $SU(p)$ and $SU(q)$ vector multiplets. This is exactly in accord with our expectation of the physics in the smoothed out cone. There the singularities consist of an $\mathbb{R} \times S^2$ of $A_{p-1}$-singularities and an $\mathbb{R}^3$ of $A_{q-1}$-singularities which do not intersect. Therefore, again we see that deforming the cone corresponds to symmetry breaking.

Massless Hypermultiplets

Now we want to consider the situation in which the $q_i$ are not relatively prime. In general, one can meet a mixture of all the phenomena explained above and below, but if the $q_i$ are of the form $(ab, ac, bc)$ for some relatively prime integers $a, b, c$, there is an additional phenomenon that can be described in a simple fashion. So we will focus on this case.

With our choice, the greatest common divisors of pairs of $q_i$ are $a, b,$ and $c$, respectively. So the fixed $\mathbb{CP}^1$’s in $W_{ab,ac,bc}$ are $Z_a, Z_b,$ and $Z_c$ singularities while their intersections are $Z_{ab}, Z_{bc},$ and $Z_{ca}$ singularities. In $Y$, everything is duplicated. In a neighbourhood of one of the orbifold points in $Y$ labeled by $ab$, the orbifold singularity looks like the origin in $\mathbb{C}^3/Z_{a} \times Z_b$. Specifically, $Z_a$ acts on local coordinates of $\mathbb{C}^3$ by multiplication by $(q, q^{-1}, 1), q^a = 1,$ and $Z_b$ acts by $(1, q, q^{-1}), q^b = 1$. This is a special case of what was explained in section 3.2.

The physics of $M$-theory on $\mathbb{R}^5 \times (\mathbb{C}^3/Z_a \times Z_b)$ is believed to be described by a five dimensional supersymmetric gauge theory with $SU(a) \times SU(b)$ gauge group and a single bi-fundamental hypermultiplet \[20,6\] that is supported at the origin. Therefore, in $M$-theory on $\mathbb{R}^4 \times \wedge^2 (W_{ab,ac,bc})$, along with the gauge groups supported on copies of $\mathbb{R}^4 \times S^2 \times \mathbb{R}$, there are bi-fundamental hypermultiplets supported on the intersections $\mathbb{R}^4 \times \mathbb{R}$, transforming as $(a, \overline{b}, 1) + (1, b, \overline{c}) + (\overline{a}, 1, c).$

A cone $X$ on $Y$ has more complicated singularities, so we do not immediately learn how to describe the spectrum in the case of the cone. If we think of $X' = \wedge^2 (W_{ab,ac,bc})$ as a deformation of the cone, can we guess the physics on $X$ by requiring that it reduce

\[22\] Here we have used a terminology that is strictly appropriate if the singularity is embedded in a compact manifold and four-dimensional effective field theory can be used.
to the physics on \( X' \) after symmetry breaking? We will not be able to offer a satisfactory answer to this question but we will offer a suggestion.

When \( X' \) degenerates to \( X \), each fixed \( \mathbb{CP}^1 \times \mathbb{R} \) splits into two copies of \( \mathbb{CP}^1 \times \mathbb{R}^+ \), so the gauge group is \((SU(a) \times SU(b) \times SU(c))^2\), with each factor supported on a separate fixed point component. One set of matter fields we know about are bi-fundamentals \( \Phi_i \) supported on the six half-lines. The \( \Phi \)'s are in the representations

\[
(a, b, 1, 1, 1) + (1, b, c, 1, 1, 1) + (1, 1, c, 1, 1) + (1, 1, 1, a, 1, 1) + (1, 1, 1, b, 1, c) + (1, 1, 1, 1, a, 1) + (1, 1, 1, 1, b, c) + (1, 1, 1, 1, 1, 1) + (1, 1, 1, 1, 1, 1).
\]  

We conjecture that there are also three chiral multiplets \( \psi_m \) which live only at the conical singularity and are in the representations \((a, 1, 1, 1, 1, 1)\), \((1, b, 1, 1, 1, 1)\), and \((1, 1, c, 1, 1, 1)\). Since the \( \Phi \)'s live on half-lines, we need to specify boundary conditions on them at the origin. To make a simple proposal, we write a hypermultiplet \( \Phi \) as a pair of \( \mathcal{N} = 1 \) chiral superfields \((u, \overline{v})\). A reasonable conjecture is that the boundary conditions are such that in the absence of Higgsing, the only non-zero components at the origin are \( u_1, u_2, u_3, v_4, v_5, v_6 \). The idea is then that when the \( \psi \)'s are nonzero (which is supposed to correspond to Higgsing to the orbifold \( X' \)) the boundary conditions are deformed to \( u_4 \sim \psi_1 u_1 \) and similarly for the others. It is plausible that the resulting spectrum could agree with \( M \)-theory on the smoothed out cone. However, in order for us to make a precise proposal, we really have to understand the superpotential of the model and the boundary conditions after Higgsing.

### 3.4. More General Singularities

Obviously, these examples have many generalizations, obtained as \( U(1) \) quotients of more general conical hyperkahler manifolds.

A simple class of examples are “toric” hyper-Kahler manifolds, with \( H = U(1)^k \) acting on \( k+2 \) hypermultiplets. (Of course, one would also like to study examples with nonabelian \( H \).) The charges are now given by a \( k \times (k + 2) \) matrix. The orbifold singularities in this case are encoded in a \((k + 2)\)-gon, generalizing the triangle. In these examples the basic nature of the physics is qualitatively similar to the cases with \( k = 1 \). The main difference is that one finds more and more gauge groups and matter fields present. For instance, the symmetries coming from the \( C \)-field in the cone on \( Y \) are \( U(1)^{k+1} \) which gets broken to \( U(1)^k \) when we smooth out the conical singularity.
It is also interesting to consider the toric hyper-Kahler manifolds as inputs for the construction considered in section two. Let \( \hat{X} = H^{k+2}/H \). Since a set of \( k+2 \) hypermultiplets admits an action of \( U(1)^{k+2} \) which contains and commutes with \( H \), \( \hat{X} \) admits a symmetry group \( F \cong U(1) \times U(1) \) that preserves the hyper-Kahler structure. The moment map of \( F \) has six components (three for each \( U(1) \)) so it gives a natural map \( \hat{X} \to \mathbb{R}^6 \) which commutes with the action of \( F \).

Now suppose we pick a \( U(1) \) subgroup of \( F \) and let \( X = \hat{X}/U(1) \). In the spirit of section 2, we may hope that \( X \) admits a conical \( G_2 \) metric, though this remains to be established. In any event, \( X \) admits the action of \( U(1)' = F/U(1) \) and the moment map on \( \hat{X} \) induces a \( U(1)' \)-invariant map \( X \to \mathbb{R}^6 \). Using the \( U(1)' \) orbits to induce a Type IIA description of \( M \)-theory on \( \mathbb{R}^4 \times X \), we find that this model is equivalent to Type IIA on \( \mathbb{R}^4 \times \mathbb{R}^6 = \mathbb{R}^{10} \), with \( D6 \)-branes coming from \( U(1)' \) fixed points. Using the properties of the hyper-Kahler moment map, one can show that the \( D \)-brane worldvolumes are all copies of \( \mathbb{R}^4 \times \mathbb{R}^3 \), linearly embedded in \( \mathbb{R}^4 \times \mathbb{R}^6 \). This conjecturally generalizes the situation considered in section 3 of [3].

Since the examples we have studied here are much more explicit than those in section 2, one might wonder if the examples in section 2 can be obtained as twistor spaces. This appears to be untrue, since for example the weighted projective space \( \mathbb{WCP}^3_{n,n,m,m} \) discussed in [3] and in section 2 appears not be a twistor space.

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