A Fundamental QCD Axion Model

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Abstract

We construct and study a fundamental field theory of the QCD axion: all couplings flow to zero in the infinite-energy limit realizing the totally asymptotically free (TAF) scenario. Some of the observable quantities (such as the masses of new quarks and scalars) are predicted at low energies by the TAF requirement. Here the minimal model of this sort is explored; the axion sector is charged under an SU(2) gauge group and a dark photon appears at low energies. This model can be TAF and feature an absolutely stable vacuum at the same time.

1 Introduction

QCD is perhaps the most satisfying building block of the Standard Model (SM). Not only it provides us with an accurate description of strong interactions, but is also a non-trivial fundamental theory: asymptotic freedom [1] tells us that QCD remains interacting in the continuum limit.

It is surprising that, while the Yukawa interactions of the SM violate CP, the QCD Lagrangian respects it: the possible CP-violating $\theta$-term $\theta G_{\mu\nu}\tilde{G}^{\mu\nu}$, built with the gluon field strength $G_{\mu\nu}$ and its dual $\tilde{G}_{\mu\nu}$, is strongly constrained by the experiments (for a recent review see [2]).

A possible explanation was proposed by Peccei and Quinn (PQ) [3]: they introduced a global chiral U(1) symmetry (called PQ symmetry and denoted here $U(1)_{PQ}$), under which some colored particles transform. These can be the quarks of the SM and/or some extra (still unobserved) quarks. Since $U(1)_{PQ}$ is anomalous, $\theta$ can be reabsorbed by a quark field redefinition and, therefore, cannot affect any physical observable.

Moreover, since the quarks must be massive, $U(1)_{PQ}$ has to be spontaneously broken too. The corresponding pseudo-Goldstone boson [4], called the axion, is a good dark matter candidate. To realize such breaking in concrete models (e.g. [5, 6]) one typically introduces new scalars and thus new quartic couplings. However, mainly because of the difficulty in having asymptotically free (AF) quartic couplings, all the field-theoretic axion models proposed so far suffers from a Landau pole (LP) and spoil the asymptotic freedom of QCD.

The purpose of this paper is to construct and study the first fundamental and realistic field theory of axions. We focus on a minimal model that can implement $U(1)_{PQ}$ and its breaking and be TAF at the same time.
The TAF requirement has been used in the literature to obtain UV-complete extensions of particle physics models [7–10]. One common feature of these constructions is the presence of several extra fields and potentially further sources of CP violation, which, in the absence of U(1)$_{\text{PQ}}$, may induce a too large radiative contribution to $\theta$ unless a tremendous fine tuning is performed. This is another independent motivation to construct a TAF axion model. Yet another motivation is the fact that TAF models can predict the low-energy values of some observables: this can happen because some couplings must have precise low energy values in order for all couplings to be AF.

Here we assume that gravitational interactions, unlike what happens in Einstein gravity, become very weak at high energy such that their impact on the renormalization group (RG) flow is negligible, but still all successes of Einstein’s theory at accessible energies are reproduced. This scenario, called softened gravity [8] may be realized, for example, in UV modifications of gravity featuring quadratic curvature terms in the action [11] or in non-local extensions of general relativity [12]. We, therefore, neglect gravity in the present study.

2 Building the model

As well-known, the scalars of a TAF model should be charged under some gauge interaction and the gauge group must not have any U(1) factor to avoid LPs.

The minimal possibility (which we consider here) is having the axion sector gauge invariant under an SU(2) group (henceforth SU(2)$_a$). Then the full gauge group contains the factor SU(3)$_c \times$SU(2)$_a$, where SU(3)$_c$ is the ordinary SU(3) of strong interactions. The gauge group should also include extra factors to account for a TAF extension of the SM (explicit realizations were provided in [8–10]). We will refer to such an extension as the SM sector. This sector has to be present, in addition to the axion sector we describe here, for obvious phenomenological reasons. The SM and axion sectors talk to each other through the SU(3)$_c$ gauge interactions. Here we will take as typical example of TAF SM extensions those based on the trinification gauge group SU(3)$_L \times$SU(3)$_c \times$SU(3)$_R$ [10] because SU(3)$_c$ is not embedded in a larger gauge group factor, in contrast to other known TAF models like, for instance, those based on the Pati-Salam group SU(3)$_L \times$SU(4)$_{\text{PS}} \times$SU(3)$_R$ [8, 9]. However, we will not commit ourselves to any specific TAF SM extension here.

In order to have U(1)$_{\text{PQ}}$ invariance we introduce two extra Weyl fermions $q$ and $\bar{q}$ in the fundamental and antifundamental of SU(3)$_c \times$SU(2)$_a$, respectively, and give them the same PQ charge: $\{q, \bar{q}\} \rightarrow e^{i\alpha/2}\{q, \bar{q}\}$, where $\alpha$ is a constant. For the sake of minimality we require the PQ charges of all particles in the SM sector to vanish; from this point of view the model we are constructing is similar to the KSVZ-like axion models [5]. Since the extra-fermion representation of the gauge group is vector like, there are no gauge anomalies as long as the SM sector is free from gauge anomalies; this is clearly the case for the trinification SM sectors, whose fermions can form a representation of the anomaly free $E_6$ group containing SU(3)$_L \times$SU(3)$_c \times$SU(3)$_R$. As usual U(1)$_{\text{PQ}}$ forbids an explicit mass term $\bar{q}q$ and so, in order to give mass to these extra quarks (as required by the experiments), we introduce a scalar field $A$, which spontaneously breaks U(1)$_{\text{PQ}}$. Therefore, $A$ has to be complex and have Yukawa interactions with $q$ and $\bar{q}$,

$$\mathcal{L}_y = -y\bar{q}AQ + \text{H.c.}. \quad (2.1)$$

The PQ symmetry of $\mathcal{L}_y$ requires $A$ to transform under U(1)$_{\text{PQ}}$ as follows: $A \rightarrow e^{-i\alpha}A$. Gauge
invariance, on the other hand, tells us that $A$ has to be invariant under SU(3)$_c$ and belong to the adjoint of SU(2)$_a$. The scalar $A$, being complex, contains two Hermitian adjoint representations $A_R$ and $A_I$ and we can decompose $A = A_R + iA_I$. Note that further Yukawa interactions besides (2.1) and those present in the SM sector are forbidden by the gauge symmetries and U(1)$_{PQ}$.

The potential of $A$ is given by

$$V_A = -m^2 \text{Tr}(A^\dagger A) + \lambda_1 \text{Tr}^2(A^\dagger A) + \lambda_2 |\text{Tr}(AA)|^2,$$

(2.2)

where $m^2$ is taken to be positive to trigger the spontaneous breaking of U(1)$_{PQ}$. Both $\text{Tr}(A^\dagger A)$ and $|\text{Tr}(AA)|^2$ are real and non-negative. Therefore, the couplings $\lambda_i$ (with $i = 1, 2$) are real and vacuum stability at high-field values (henceforth “high-field stability”) is guaranteed for $\lambda_i > 0$. However, these conditions are sufficient but not necessary for high-field stability. Indeed, since $\text{Tr}^2(A^\dagger A) \geq |\text{Tr}(AA)|^2$ the coupling $\lambda_2$ can be negative and the necessary and sufficient conditions for high-field stability are

$$\lambda_1 > 0, \quad \lambda_1 + \lambda_2 > 0.$$

(2.3)

Later on we will show that this model is TAF and stable at high fields for some values of the parameters and for the same values absolute vacuum stability (not only high-field stability) is guaranteed. Here we set to zero the couplings with the scalars of the SM sector; this is consistent at the one-loop level because those couplings are not generated and so they remain zero at the one-loop level if their initial condition in the RG flow is set to zero. The one-loop approximation, on the other hand, is enough for our purposes because total asymptotic freedom implies that all couplings are tiny at high enough energies.

3 The RG flow

The one-loop $\beta$-function of the gauge coupling $g$ of a generic gauge group $G$ is

$$\frac{dg^2}{dt} = -bg^4, \quad b \equiv \frac{11}{3} C_2(G) - \frac{4}{3} S_2(F) - \frac{1}{6} S_2(S),$$

(3.1)

where $t \equiv \ln(\mu^2/\mu_0^2)/(4\pi)^2$, the energy scale $\mu_0$ is arbitrary, $\mu$ is the usual RG scale and $C_2(G)$, $S_2(F)$ and $S_2(S)$ are the Dynkin indices of the adjoint representation ($C_2(G) = N$ for $G = \text{SU}(N)$), the Dirac-spinor representation and the scalar representation, respectively. The general solution to Eq. (3.1) is $g^2(t) = g_0^2/(1 + g_0^2 bt)$, where $g_0 \equiv g(0)$. Then in order to have an AF gauge coupling and avoid a LP we must have $b > 0$. The corresponding Gaussian fixed point is UV attractive: whatever value of $g_0$ is chosen, it is always true that $g \to 0$ as $t \to \infty$. For SU(2)$_a$ we have $S_2(F) = 3/2$, $S_2(S) = 4$ (we have $4 = 2 + 2$ instead of 2 because $A$ is complex) and so the constant $b$ for the corresponding gauge coupling $g_a$ is

$$b_a = \frac{14}{3},$$

(3.2)

which, being positive, gives an AF $g_a$. With a similar computation one finds that the constant $b$ corresponding to SU(3)$_c$ is instead

$$b_s = \frac{29}{3} - \Delta,$$

(3.3)
where \( \Delta \) is the positive extra contribution due to the fermions and scalars in the SM sector. Using, for example, the results of [10] we find that it is possible to have \( g_s \) AF keeping the SM sector TAF. Moreover, since \( q \) and \( \bar{q} \) do not have Yukawa couplings with the SM sector (they transform under \( SU(2)_L \) and \( U(1)_{PQ} \), but the fields in the SM sector do not), these extra quarks favor the total asymptotic freedom in the SM sector: this is because the smaller \( b_s \) is (keeping \( b_s > 0 \)) the bigger \( g_s \) at a fixed energy favoring AF for the Yukawa and scalar quartic in the SM sector [10].

The renormalization group equation (RGE) of \( y \) is instead

\[
\frac{dy^2}{dt} = y^2 \left( \frac{9y^2}{2} - 8g_s^2 - \frac{9g_a^2}{2} \right).
\]

Equations of this type have been studied in [8]. In our case the general solution to (3.4) for any \( b_a \) and \( b_s \) is

\[
y^2(t) = \frac{y_0^2 \left( 1 - \frac{9g_a^2I(t)}{2} \right)^{-1}}{(1 + g_{a0}b_s t)^{8/b_s} (1 + g_{d0}b_a t)^{g_s/(2b_a)}},
\]

where \( y_0 \equiv y(0), g_{a0} \equiv g_s(0), g_{a0} \equiv g_a(0) \) and

\[
I(t) \equiv \int_0^t \frac{dt'}{(1 + g_{a0}b_s t')^{8/b_s} (1 + g_{d0}b_a t')^{9/(2b_a)}}.
\]

We find that \( I(t) \) admits a closed form expression\(^1\). Looking at (3.5) and (3.6) and using the AF conditions for the gauge couplings (\( b_s > 0 \) and \( b_a > 0 \)) we see that \( y \) is AF if and only if \( y_0 \) satisfies

\[
y_0^2 \leq \frac{2}{9I_\infty}, \quad I_\infty \equiv \lim_{t \to \infty} I(t),
\]

otherwise \( y \) has a LP. Note that if \( b_s > 0 \) and \( b_a > 0 \) the integral \( I_\infty \) is positive and convergent whenever \( 8/b_s + 9/(2b_a) > 1 \), which, from (3.2) and (3.3), is satisfied for any value of \( \Delta \) such that \( b_s > 0 \), namely \( \Delta < 29/3 \). This bound is compatible with the values in existing TAF SM sectors discussed in the literature. It follows that by taking \( y_0 \) small enough (compatible with the inequality in (3.9)) one can indeed have an AF \( y \). When the condition in (3.9) is satisfied as a strict inequality \( y \) decreases faster than the gauge couplings at large \( t \). This class of solutions are UV attractive because if we perturb the initial condition \( y_0 \) by a small enough amount (keeping the inequality in (3.9) satisfied) the solution remains AF. When instead \( y_0^2 = 2/(9I_\infty) \) the Yukawa coupling decreases just like the gauge coupling at large \( t \) (see Fig. 1). Such solution is not UV attractive, but IR attractive. This provides us with an interesting prediction of \( y \) at low energy and, therefore, of the mass of the new quarks as discussed below in Sec. 4.

\(^1\)Indeed, the integral in (3.6) is a particular case of

\[
\int_0^t \frac{dt'}{(1 + a_1 t')^{e_1} (1 + a_2 t')^{e_2}} = \mathcal{I}(t) - \mathcal{I}(0), \tag{3.7}
\]

where

\[
\mathcal{I}(t) = \frac{\left(\frac{a_1 + a_1 a_2 t}{a_1 - a_2}\right)^{e_2}}{(1 + a_1 t)^{e_1 - 1} (1 + a_2 t)^{e_2 - 1} (a_1 - a_1 e_1)} 2F_1 \left(1 - e_1, e_2; 2 - e_1; \frac{-a_2 - a_1 a_2 t}{a_1 - a_2}\right), \tag{3.8}
\]

and \( 2F_1 \) is Gauss's hypergeometric function.
The RGEs of $\lambda_1$ and $\lambda_2$ are $\frac{d\lambda_1}{dt} = \beta_1$, and $\frac{d\lambda_2}{dt} = \beta_2$, where

$$\beta_1(g_a, y, \lambda) = \frac{9}{2} g_a^4 + \lambda_1 \left(8 \lambda_2 + 6 y^2 - 12 g_a^2\right) + 14 \lambda_1^2 + 8 \lambda_2^2 - 3 y^4$$

(3.10)

and

$$\beta_2(g_a, y, \lambda) = \frac{3}{2} g_a^4 + \lambda_2 \left(12 \lambda_1 + 6 y^2 - 12 g_a^2\right) + 6 \lambda_2^2 + \frac{3}{2} y^4.$$  

(3.11)

The $\beta$-functions above have been obtained by applying the general formalism of [13–15] to the present model. The RGEs of the $\lambda_i$ are too complicated for us to determine the general solution at any $t$. However, we can understand if all couplings are AF by considering the ansatz

$$\tilde{g}_s^2(t) = \frac{\bar{g}_s^2}{t}, \quad \tilde{g}_a^2(t) = \frac{\bar{g}_a^2}{t}, \quad y^2(t) = \frac{\bar{y}^2}{t}, \quad \lambda_i(t) = \frac{\bar{\lambda}_i}{t},$$

(3.12)

where $\bar{g}_s^2, \bar{g}_a^2, \bar{y}$ and $\bar{\lambda}_i$ are constants. The ansatz above is manifestly TAF and is a fixed flow: although the couplings individually run, their ratios do not. A solution of the form in (3.12) exists if and only if the corresponding algebraic system of equations obtained by plugging (3.12) into the RGEs admits solutions with $\bar{g}_s^2, \bar{g}_a^2, \bar{y}$ and $\bar{\lambda}_i$ real and also $\bar{g}_s^2, \bar{g}_a^2$ and $\bar{y}^2$ positive. Note that, when this condition is satisfied, (3.12) not only is a solution of the RGEs, but also describes the $t \gg 1$ asymptotic behavior of any solution.

Let us first consider the RGEs of the gauge couplings with the fixed-flow ansatz. Here we are interested in the case $\bar{g}_s^2 \neq 0$ (as we want to match the non-trivial low energy QCD running) and $\bar{g}_a^2 \neq 0$ because we want a TAF model. Then from (3.1) $\bar{g}_a^2 = 1/b_a$ and $\bar{g}_s^2 = 1/b_s$. Turning to the Yukawa coupling, we have either $\bar{y}^2 = 0$ or

$$\bar{y}^2 = \frac{2}{9} \left(\frac{9}{2b_a} + \frac{8}{b_s} - 1\right).$$

(3.13)
| $\Delta$ | unstable vacuum | stable vacuum |
|---------|-----------------|--------------|
| 28/3    | (0.183, -3.23)  | (1.68, -0.951)|
| 26/3    | (0.149, -1.05)  | (0.575, -0.343)|
| 8       | (0.145, -0.598) | (0.349, -0.231)|

Table 1: Real solutions $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ to (3.14) (corresponding to TAF solutions) obtained by varying the contribution $\Delta$ to the RGE of the strong coupling (compatibly with a TAF SM sector, see e.g. [10]). The Yukawa coupling is at the fixed-flow, Eq. (3.13). The values of $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ are approximated with three digits.

The latter case corresponds to saturating the bound in (3.9) and is, therefore, an IR attractive solution as discussed above. Finally the corresponding system of algebraic equations for the quartic couplings reads

$$\tilde{\lambda}_i = -\beta_i(\tilde{g}_a, \tilde{y}, \tilde{\lambda}).$$

(3.14)

In Table 1 we show the real solutions $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ to Eq. (3.14) obtained by varying $\Delta$ (considering as an example the values corresponding to the TAF SM sector reported in [10]). In that Table $\tilde{y}$ is at the fixed-flow in (3.13). Taking instead the Yukawa coupling outside the fixed flow, that is setting $\tilde{y} = 0$, produces no TAF solutions. Note that the last column in Table 1 satisfies the vacuum stability condition in (2.3), while the second column does not and the corresponding solutions are then ruled out. By using the general formalism in [8], we find that the solutions in the last column are IR attractive, which results in a prediction for the $\lambda_i$ at low energies and for the scalar spectrum, as discussed below in Sec. 4.

We observe that varying the strong coupling RGE (varying $\Delta$) there is always a solution with an unstable vacuum, which should be excluded, and a stable one. Furthermore, $\lambda_2$ is always negative. These features are quite robust and persist even if we vary $b_a$ in addition to $b_s$. This can be done, for example, by adding a certain number $n_e$ of extra vector-like Dirac fermions, which are neutral under SU(3)$_c$ and U(1)$_{\text{PQ}}$, but in the fundamental of SU(2)$_a$. The values of $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ for all TAF solutions are then shown in Table 2. Again the stable solutions in the last column turn out to be IR attractive.

## 4 Stationary points and the mass spectrum

The two Hermitian adjoint representations $A_R$ and $A_I$ can be expressed in terms of the Pauli matrices $\sigma^k$ as follows: $A_R = A_{Rk}\sigma^k/2$, $A_I = A_{Ik}\sigma^k/2$, where a sum over $k = 1, 2, 3$ is understood. Since SU(2)$_a$ transforms the $A_{Rk}$ and $A_{Ik}$ as ordinary rotations transform the coordinates in three dimensions, it is possible to set $A_{R3} = A_{I2} = A_{I3} = 0$ through an SU(2)$_a$ transformation. In the following we, therefore, do so without loss of generality.

For general values of $A_{R1}$, $A_{I1}$ and $A_{R2}$ the three SU(2)$_a$ gauge fields acquire the following masses:

$$M_V = g_a \sqrt{A_{R1}^2 + A_{R2}^2 + A_{I1}^2}, \quad M_{V\pm} = \sqrt{\frac{M_V^2}{2} \pm \sqrt{\frac{M_V^4}{4} - g_a^4 A_{I1}^2 A_{R2}^2}}. \quad (4.1)$$

The Weyl quarks $q$ and $\bar{q}$ (that are doublets under SU(2)$_a$) form instead two Dirac quarks $Q_\pm$
The scalar spectrum corresponding to (4.3) includes three massive scalars with squared masses
\[ M_{S_1}^2 = 2m^2, \quad M_{S_2}^2 = M_{S_3}^2 = -2\lambda_2 m^2/(\lambda_1 + \lambda_2). \]
Note that also the second squared mass is positive for \( \lambda_2 < 0 \) and \( \lambda_1 + \lambda_2 > 0 \), which is the case for the TAF solutions with stable vacuum reported in the last column of Tables 1 and 2. The vacuum in (4.3) is, therefore, a minimum of the potential (and actually, as we will see, the absolute minimum) using the TAF and high-field stability requirements. The scalar spectrum also includes three massless modes, two of them are

| \( \Delta \) | \( n_e \) | unstable vacuum | stable vacuum |
|-------|-------|----------------|--------------|
| 28/3  | 1     | (0.219, -3.25) | (1.70, -0.965) |
| //    | 2     | (0.268, -3.27) | (1.73, -0.986) |
| //    | 3     | (0.344, -3.30) | (1.77, -1.02) |
| //    | 4     | (0.469, -3.34) | (1.84, -1.08) |
| //    | 5     | (0.722, -3.42) | (1.97, -1.20) |
| //    | 6     | (1.50, -3.49)  | (2.34, -1.70) |
| 26/3  | 1     | (0.185, -1.06) | (0.593, -0.362) |
| //    | 2     | (0.237, -1.07) | (0.619, -0.389) |
| //    | 3     | (0.314, -1.08) | (0.656, -0.435) |
| //    | 4     | (0.447, -1.08) | (0.712, -0.528) |
| 8     | 1     | (0.182, -0.601) | (0.365, -0.255) |
| //    | 2     | (0.236, -0.599) | (0.387, -0.294) |
| //    | 3     | (0.324, -0.570) | (0.411, -0.376) |

Table 2: Real solutions \( (\tilde{\lambda}_1, \tilde{\lambda}_2) \) as in Table 1 except that \( n_e \) vector-like Dirac fermions (in the fundamental of \( SU(2)_a \), but neutral under \( SU(3)_c \) and \( U(1)_{\text{PQ}} \)) are added. The number \( n_e \) is varied until total asymptotic freedom is possible.

(both triplets under \( SU(3)_c \)) with masses

\[
M_{Q^\pm} = \frac{y}{2} \sqrt{A_{R1}^2 + (A_{I1} \pm A_{R2})^2}. \tag{4.2}
\]

There are only three physically inequivalent stationary points. An obvious one is the origin \( (A_{R1} = A_{I1} = A_{R2} = 0) \), which leaves \( SU(2)_a \) unbroken and corresponds to a maximum of \( V_A \). Next, there all the equivalent configurations obtained from

\[
A_{R1} = \frac{m}{\sqrt{\lambda_1 + \lambda_2}}, \quad A_{I1} = 0, \quad A_{R2} = 0, \tag{4.3}
\]

through a \( U(1)_{\text{PQ}} \) and/or an \( SU(2)_a \) transformation. Note that \( A_{R1} \) in (4.3) is guaranteed to be real from the high-field stability condition in (2.3). These stationary points break \( SU(2)_a \) down to a residual Abelian group \( U(1)_a \). Indeed, from (4.1) one has \( M_{V_-} = 0 \), and \( M_{V_+} = M_V = g_a m / \sqrt{\lambda_1 + \lambda_2} \). The value of \( A_{R1} \) given by (4.3) is interpreted as the axion decay constant \( f_a \). The corresponding value of \( V_A \) is

\[
V_A = -\frac{m^4}{4(\lambda_1 + \lambda_2)} \quad \text{(potential at (4.3)).} \tag{4.4}
\]

The quarks \( Q^\pm \) acquire equal non-vanishing masses for \( y \neq 0 \): \( M_{Q^+} = M_{Q^-} = y m / (2\sqrt{\lambda_1 + \lambda_2}) \). The scalar spectrum corresponding to (4.3) includes three massive scalars with squared masses \( M_{S_1}^2 = 2m^2 \), and \( M_{S_2}^2 = M_{S_3}^2 = -2\lambda_2 m^2 / (\lambda_1 + \lambda_2) \). Note that also the second squared mass is positive for \( \lambda_2 < 0 \) and \( \lambda_1 + \lambda_2 > 0 \), which is the case for the TAF solutions with stable vacuum reported in the last column of Tables 1 and 2. The vacuum in (4.3) is, therefore, a minimum of the potential (and actually, as we will see, the absolute minimum) using the TAF and high-field stability requirements. The scalar spectrum also includes three massless modes, two of them are
eaten by the two massive vector bosons. The third one is the axion. Since $U(1)_{PQ}$ is broken by anomalies the axion receives as usual a mass at quantum level.

The last class of stationary points consists of all the equivalent configurations obtained from

$$A_{R1} = 0, \quad A_{I1} = \pm A_{R2} = \pm \frac{m}{\sqrt{2\lambda_1}} \quad (4.5)$$

through a $U(1)_{PQ}$ and/or an $SU(2)_a$ transformation. These configurations break $SU(2)_a$ completely; indeed, from (4.1) one finds $M_V = \sqrt{2}g_a|A_{I1}|$ and $M_{V\pm} = g_a|A_{R1}|$, but are not phenomenologically acceptable because they lead to a massless extra colored fermion (see (4.2)). Inserting (4.5) into $V_A$ one obtains $V_A = -m^4/(4\lambda_1)$. Since $\lambda_1$ must be positive from high-field stability this value of $V_A$ is higher than the one in (4.4) if and only if $\lambda_2 < 0$ and $\lambda_1 + \lambda_2 > 0$, which is the case for the TAF solutions reported in the last column of Table 1 (those with a stable vacuum). Moreover, the scalar squared mass matrix in case (4.5) has eigenvalues $2m^2\lambda_2/\lambda_1$ and $2m^2$. So the TAF requirement and high-field stability automatically allow (and actually force) us to exclude the phenomenologically unacceptable stationary points in (4.5) because they guarantee that the vacuum in (4.3) is the absolute minimum of the potential and the stationary points in (4.5) are only saddle points.

Having a vacuum with the residual $U(1)_a$ is not phenomenologically ruled out. Indeed, one can perform a linear homogeneous transformation on the ordinary hypercharge and the $U(1)_a$ gauge bosons in a way that the extra massless boson (which appears at low energies as a dark photon) does not interact at the renormalizable level with the SM particles; its effective interactions can be generated only via loop contributions involving the extra quarks $q$ and $\bar{q}$. As long as the masses of these fermions, $M_{Q\pm}$, are large enough these interactions appear at low energies as non-renormalizable terms in the Lagrangian suppressed by appropriate powers of the large masses. The dark photon is compatible with the observations given that $M_{Q\pm}$ are around the PQ symmetry breaking scale $f_a$. Indeed, $f_a$ is at least of order $10^8$ GeV and even higher to account for the whole dark matter through the axion (see [2] for a recent review on axion bounds), which is more than enough to satisfy the observational bounds [16].

Finally, we note that the requirement of TAF couplings and vacuum stability leads to a prediction for the scalar mass $M_{S2} = M_{S3}$ (because the $\lambda_i$ are predicted) and for the masses of the new quarks, $M_{Q\pm}$, because the Yukawa coupling has to satisfy the fixed-flow condition (3.13) to obtain TAF solutions. For example, for the setup in the first line of Table 1 we have $M_{S2} = M_{S3} \approx 1.62m$ and $M_{Q\pm} = 1.35m$, where $m = M_{S1}/\sqrt{2} = f_a\sqrt{\lambda_1 + \lambda_2} \approx 0.853 f_a$. This is opposed to known (non-TAF) axion models, where the masses and couplings of the new particles are freely adjustable parameters.

## 5 Conclusions

A fundamental field theory of the QCD axion has to have certain features. In particular, the axion sector should be invariant under a non-Abelian gauge group to ensure total asymptotic freedom. Here, the minimal realistic model of this sort has been explicitly built and studied: it features an $SU(2)_a$ gauge symmetry, a complex scalar $A$ in the adjoint representation of $SU(2)_a$ and one extra Dirac field \{\(q, \bar{q}\)\} in the fundamental representation of $SU(3)_c \times SU(2)_a$ to implement the $U(1)_{PQ}$ symmetry. All PQ charges of the SM particles have been set to zero for simplicity. We have
shown that there are initial conditions for the RG flow such that the model is TAF and features an absolutely stable vacuum at the same time. An interesting feature of this model is the presence of a dark photon in the low-energy spectrum.

Besides the presence of extra non-Abelian gauge symmetries a generic TAF theory can predict a number of observable quantities given that the RG flow typically involves IR attractive fixed points. In the minimal model proposed, indeed, we have seen that some of the masses of the extra particles are predicted. This is the case, for example, for the extra quarks, because the corresponding Yukawa coupling should lie on the fixed-flow (which is IR attractive) to realize the TAF requirement.

Let us conclude by giving some examples of possible outlook. It would be interesting to construct TAF models of the QCD axion where the quarks carrying the PQ charges are those already present in the SM. For example, one could construct a DFSZ-like [6] TAF model. This could have interesting implications for the Higgs physics given that the DFSZ model features an extra Higgs doublet. Also, it would be valuable to know whether the dark photon present in the low energy spectrum of the minimal model generically appears in other TAF axion models. Another example of possible outlook is the construction of fundamental QCD axion models where some couplings flow to an interacting UV fixed point.

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