PAPER

Some algebraic structures for the bosonic three-level systems

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Abstract

A simple systematic procedure to construct the three-level unitary and orthogonal algebras which appear in the bosonic three-level pairing models and for arbitrary choice of level degeneracies is presented. We draw attention to the existence of the three types of dynamical symmetry. The new analytical expressions of the energies are derived. The results presented show very clearly the duality between orthogonal algebras and the quasi-spin algebras for the three-level system with pairing interactions. The seniority selection rules of the transition operators in three level boson system are given. Illustrative calculations are carried out for $U_{21}$ algebra.

1. Introduction

Since the birth of quantum mechanics, symmetry has acquired a central role in all branches of physics and group theory provides the mathematical tool to formulate symmetry principles. Modern developments in symmetry are putting more emphasis on the concept of dynamical symmetry (DS) \cite{1–3}. The DS is a type of symmetry in which the Hamiltonian is expanded in elements of a Lie algebra, (G\textsubscript{0}), called the spectrum generating algebra (SGA). The DS occurs if the Hamiltonian can be written in terms of the Casimir operators (COs) of a chain of nested algebras, G\textsubscript{0} ⊃ G\textsubscript{1} ⊃ … ⊃ G\textsubscript{n}. The DS has been widely exploited in many different fields ranging from hadronic \cite{4, 5} and molecular physics \cite{6, 7} to nuclear physics \cite{8–11}. On the other hand, in some applications \cite{12, 13}, the predictions of an exact DS are not fulfilled and one is compelled to break it. The required symmetry breaking is achieved by including terms associated with (two or more) different subalgebra chains of the parent SGA in the Hamiltonian.

The notable application of DS and SGA in nuclear physics is the study of collective states for even-even nuclei using the interacting boson model (IBM) \cite{14, 15}. In the original version of the IBM, the identical bosons with angular momentum $l$ is referred to as the $l$-bosons. Hence, the nucleus is described in terms of interacting $s = (l = 0)$ and $d = (l = 2)$ bosons. The SGA of the IBM is the unitary algebra $U_2$. One obvious extension is gIBM \cite{16–21} in which the next even angular momentum is considered hence to include a $g = (l = 4)$ boson. The $U_{15}$ is the SGA of the gIBM. In addition to those mentioned above, more important versions of the IBM \cite{22–24} consists in the introduction of additional angular momenta ($p$, $f$ with $l = 1$, 3, respectively) together with the $s$ and $d$ boson. In the extensions of IBM, the bosons with $p$, $f$ and $g$ angular momentum are useful mostly as supplements of dipole and octupole degrees of freedom, which are generated by the system of the $s$ and $d$ boson.

Over the last few years, more attention is paid to the development of the IBM in a more general form \cite{25–27}. The two-level boson model, or $s = a$ boson model, is described in terms of interacting $s = a$–$a$–bose where $a$ is the positive integer angular momentum. The SGA of the $s = a$ IBM is $U_{n_a+1}$, where $n_a = 2a + 1$.

Within this framework, the purpose of this paper is to extend the work of the two-level pairing model to the generic three-level pairing model. Consequently, we have a more explicit insight into the algebraic structure of the IBM. Clearly, such an algebraic approach is preferable not only because it gives the opportunity to expand the model to other cases without having to develop a completely new model, but also because it also provides the analytical expressions of physical observations.

The generic three-level boson model or $s = ab$ boson model can be defined in terms of the $s$, $a$ and $b$-bosons. The SGA of the $s = ab$ IBM is $U_{n_a+n_b+1}$, where $n_a = 2a + 1$ and $n_b = 2b + 1$. Two versions of IBM in...
this class include the $U_{15}^{sdg}$ IBM and the $U_{8}^{spd}$ IBM. The $U_{n_\alpha+n_\beta+1}^{k}$ three-level boson model is a special case of the three-level pairing model. We focus on a significant class of general symmetries which exist in the three-level pairing model. The expressions for the COs of the generic three level unitary algebra and three level orthogonal algebra are formulated. In addition, the analytical expressions of the energies for three types of the DS are presented.

Quasi-spin (QS) algebra is SGA of the standard pairing model [28–32]. A close relation between three level unitary algebra and three level QS subalgebra arises. The relation between pairing operator and the multipole operator for the generic three level boson system is given in a closed form. Furthermore, we emphasize that the duality relations hold in the case of three level algebras.

The layout of the paper is as follows. In section 2, the general symmetries in three-level boson model were presented. In section 3, the generic structure of the three-level unitary and orthogonal algebras was described in detail. In section 4, the relation between pairing operator and the multipole operator for the three-level boson system was given in a closed form. In section 5, the selection rules of the transition operators were derived. In section 6, the theoretical analysis of the three types of the DS was provided, with emphasis on the energy eigenvalues. Finally, in section 7, our results were summarized.

2. General symmetries in IBM

By studying the algebraic structure of the $spd$IBM, $sdg$IBM, $spdf$IBM and $sdgp$IBM, it becomes clear that there is a general algebra structure. It is as follows

\[
\text{Chain 1.} \quad U_{n_\alpha} \supset \left\{ O_{n_\alpha} \otimes O_{n_\beta}, \quad O_{n_\beta} \otimes O_{n_\gamma}, \quad O_{n_\gamma} \otimes O_{n_\alpha} \right\} \supset K, \quad (1)
\]

\[
\text{Chain 2.} \quad U_{n_\alpha} \otimes U_{n_\beta} \supset \left\{ O_{n_\alpha} \otimes O_{n_\beta}, \quad O_{n_\beta} \otimes O_{n_\gamma}, \quad O_{n_\gamma} \otimes O_{n_\alpha} \right\} \supset K, \quad (2)
\]

\[
\text{Chain 3.} \quad K \equiv O_{n_\alpha} \otimes O_{n_\beta} \otimes O_{n_\gamma} \supset O_{n_\alpha} \otimes O_{n_\beta}, \quad O_{n_\beta} \otimes O_{n_\gamma}, \quad O_{n_\gamma} \otimes O_{n_\alpha} \supset O_{3,abc}, \quad (2)
\]

where the indices $\alpha$, $\beta$ and $\gamma$ take the values $a$, $b$ and $c$ in cyclic order, $n_\alpha = 2\alpha + 1$, $n_\beta = 2\beta + 1$ and $n_\gamma = 2\gamma + 1$, and $n_{abc} = n_a + n_b + n_c$. For instance, figure 1 shows lattice of algebras of $U_{21}^{pf/fg}$ using (1). It is important to note that if $a$, $b$ or $c$ equals zero then the corresponding unitary and orthogonal algebra will be $U_n \equiv U_1 \supset O_1$. Therefore, $O_3,abc$ will be neglected in (2). Later, figure 2 illustrates this case for the algebra $U_{21}$.

In the subalgebra chains, $SU(1, 1)$, seniority quantum number is the most important. Seniority refers to the number of particles that are not in pairs coupled to angular momentum $J = 0$. Seniority can be given a group-theoretical definition starting from the Lie subalgebra chain $U_n \supset O_n \supset O_{3,k}$ with $k = a$, $b$ or $c$. It arises as a label $\nu_k$ of the orthogonal algebra $O_n$. Alternatively, seniority can be introduced via the QS formalism [30–32] where it arises as a label of the algebras $SU(1, 1)$.

For $U_n$ and $O_n$, the irreducible representations (IRs) are $[N] \equiv [N0 \ldots 0]$ and $[\nu] \equiv [\nu0 \ldots 0]$, respectively. The IRs for all algebras in (1) are given in table 1. The basis states are denoted by

\[
\text{Chain 1.} \quad [N], \quad [\nu_{abc}], \quad [\nu_k], \quad [\nu_{c}], \quad [\nu_{l}], \quad a, \quad b, \quad c, \quad l, \quad (3a)
\]

\[
\text{Chain 2.} \quad [N], \quad [N_0], \quad [N_1], \quad [\nu_k], \quad [\nu_{c}], \quad [\nu_{l}], \quad a, \quad b, \quad c, \quad l, \quad (3b)
\]

\[
\text{Chain 3.} \quad [N], \quad [N_0], \quad [N_1], \quad [\nu_k], \quad [\nu_{c}], \quad [\nu_{l}], \quad a, \quad b, \quad c, \quad l. \quad (3c)
\]
In general, this classification is not complete as indicated by the dots in the equations (3a)–(3c). The reduction $O_{n_j} \supset O_{n,k}$ is not multiplicity free and generally requires a multiplicity label $\alpha$. The multiplicity of the angular momentum $n$ in the decomposition of the IR $[\nu, \lambda]$ may be calculated by means of Littlewood rules [33].

The reduction $U_{n_1 + n_j} \supset U_{n_1} \otimes U_{n_j}$ is given by $N_j = 0, 1, …, N$ and $N = N - N_j$. However, for the reduction $O_{n_1 + n_j} \supset O_{n_1} \otimes O_{n_j}$ one finds $\nu = 2q_j + \nu_i + \nu_j$ and $q_j = 0, 1, 2, …, \left\lfloor \frac{N}{2} \right\rfloor$, where $\left\lfloor \frac{N}{2} \right\rfloor$ denotes the integer part of $\frac{N}{2}$. Finally, the reduction $U_n \supset O_n$ is given by $N = 2q + \nu$ and $q = 0, 1, 2, …, \left\lfloor \frac{N}{2} \right\rfloor$.

### 3.3. Three level unitary and orthogonal algebras

Beginning, in order to simplify the notations which will appear later, $A$, $B$, and $C$ are used to designate spherical tensors (STs) with integer angular momenta $a$, $b$, and $c$ respectively (i.e., $A_{a} \equiv A_{a}$ and $A_{abc} \equiv A_{abc}$). An irreducible tensor product $[A \otimes B]$, of two STs $A$ and $B$ is defined as the tensor of rank $e$ whose components $[A \otimes B]_{\alpha, \beta}$ can be expressed in terms of $A_{\alpha}$ and $B_{\beta}$ according to, $[A \otimes B]_{\alpha, \beta} = \sum_{\alpha, \beta} C_{\alpha, \beta}^{\alpha', \beta'} A_{\alpha', \beta'}$, where $C_{\alpha, \beta}^{\alpha', \beta'}$ are the Clebsch-Gordan coefficients of the tensor commutator, [34, 35], with definite angular momentum $e$ and $z$ component $\epsilon$ is defined as $[A, B]_{e, \epsilon} = \sum_{\alpha, \beta} C_{\alpha, \beta}^{\alpha', \beta'} [A_{\alpha'}, B_{\beta'}]$.

The creation operators transform by definition as STs under rotation. However, the annihilation operators do not. Whereas, the modified annihilation operators $\tilde{A}_n = (-1)^{n-a} A_{-a}$, transform as ST. The boson creation $A^\dagger_{a, \beta}$ and modified annihilation $\tilde{A}^\dagger_{a, \beta}$ operators satisfy commutation relations

$$[\tilde{A}, B^\dagger] = \sqrt{2a + 1} \delta_{ab} \delta_{\alpha\beta}, \quad [\tilde{A}, B] = [A^\dagger, B^\dagger] = 0.$$  

(4)

The tensor commutator [26] of two coupled operators is

$$[A^\dagger \otimes B^\dagger]_k, [C \otimes D]_k = (-1)^{\epsilon + d + f} \Delta_{df} \begin{pmatrix} e & f \\ a & b \end{pmatrix} [A^\dagger \otimes D]_k \delta_{bc}$$

$$- (-1)^{\epsilon + d + f} \Delta_{df} \begin{pmatrix} e & f \\ c & b \end{pmatrix} [C \otimes B]_k \delta_{ad}$$  

(5)

where $\Delta_{ab…d} = \sqrt{(2a + 1)(2b + 1)…(2d + 1)}$.

Consider, the unitary algebra $U_{nabc}$, which appears in the problem of many bosons placed in a three-level system, of possibly unequal degeneracies, with integer angular momenta $a$, $b$, and $c$ and we call it three level unitary algebra. The algebra $U_{nabc}$ is spanned by the generators

$$T_t(A, A) = [A^\dagger \otimes A^\dagger]_t, \quad t = 0, 1, …, 2a,$$

$$T_t(B, B) = [B^\dagger \otimes B^\dagger]_t, \quad t = 0, 1, …, 2b,$$

$$T_t(C, C) = [C^\dagger \otimes C^\dagger]_t, \quad t = 0, 1, …, 2c,$$

$$T_t(A, B) = [A^\dagger \otimes B^\dagger]_t, \quad t = |a - b|, …, a + b,$$

$$T_t(A, C) = [A^\dagger \otimes C^\dagger]_t, \quad t = |a - c|, …, a + c,$$

$$T_t(B, C) = [B^\dagger \otimes C^\dagger]_t, \quad t = |b - c|, …, b + c.$$  

(6)

The two level subalgebra $U_{nab}$ is spanned by the generators $T_t(B, B), T_t(C, C), T_t(B, C)$ and $T_t(C, B)$ which are obtained by omitting all terms with $A$ in (6).

CO plays a central role in the formulation of the laws of DS. By definition COs commute with all the generators of the $U_{nabc}$. The number of COs of the $U_{nabc}$ is identical with that of the algebra. So, there are $n_{abc}$ COs of the $U_{nabc}$. Since the Hamiltonian of IBM includes one and two body interactions, in most applications, there will be no need for COs beyond quadratic. The following expressions for the linear and quadratic COs of three level unitary algebra $U_{nabc}$ are derived. We obtain

| Algebra | IRs | Algebra | IRs |
|---------|-----|---------|-----|
| $U_{nabc}$ | $[N_{abc}]$ | $O_{abc}$ | $[\nu_{abc}]$ |
| $U_{nbc}$ | $[N_{abc}]$ | $O_{abc}$ | $[\nu_{abc}]$ |
| $U_{nac}$ | $[N_{abc}]$ | $O_{abc}$ | $[\nu_{abc}]$ |
| $U_{nba}$ | $[N_{abc}]$ | $O_{abc}$ | $[\nu_{abc}]$ |
| $U_{nca}$ | $[N_{abc}]$ | $O_{abc}$ | $[\nu_{abc}]$ |
| $U_{nab}$ | $[N_{abc}]$ | $O_{abc}$ | $[\nu_{abc}]$ |

### Table 1. The IRs of all subalgebras in the chains (1).
The quadratic CO of single level unitary algebra \( U_n \) is given by
\[
C_2[U_{n_a}] = \Delta_a[A^\dagger \otimes \bar{A}]_b + \Delta_b[B^\dagger \otimes \bar{B}]_c + \Delta_c[C^\dagger \otimes \bar{C}]_d,
\]
and
\[
C_2[U_{n_b}] = \sum_{k=a,b,c} C_2[U_{n_a}] + \sum_{(i,j)} (2N_iN_j + n_iN_j + n_jN_i),
\]
where the quadratic CO of single level unitary algebra \( U_n \) is given by
\[
C_2[U_{n_a}] = \sum_{t=0}^a \Delta_t (-1)^t [T_t(A, A) \otimes T_t(A, A)]_0,
\]
and \( N_a \) is the number operator,
\[
N_a = \Delta_a[A^\dagger \otimes \bar{A}]_0.
\]

From now on we assume, unless stated explicitly otherwise, that the summation over the pair of indices \((i,j)\) and \((I,J)\) ranges over the elements of the sets \{(a, b), (a, c), (b, c)\} and \{(A, B), (A, C), (B, C)\}, respectively.

In general, the quadratic COs of any multi-level unitary algebra have many interesting properties. For instance, they will be reduced to combination of linear and quadratic COs of single level unitary algebras. For example, the COs of four level unitary algebra can be represented as a combination of linear and quadratic COs of single level unitary algebras
\[
C_2[U_{n_a}] = C_2[U_{n_b}] + C_2[U_{n_c}] + n_{ab}N_{ab} + n_{ac}N_{ac} + 2N_{ab}N_{ac},
\]
where
\[
C_2[U_{n_b}] = C_2[U_{n_c}] + n_iN_j + n_jN_i + 2N_{ij}.
\]

Now, let us consider a single level orthogonal algebra. The generators \(T_t(A, A)\) with \(t = 1, 3, \ldots, 2a - 1\) form a subalgebra of the \(U_n\) which is referred to as the \(O_{n_a}\). Moreover, for the single-level algebra, \(O_{n_a}\), the operator \(T(A, A) \circ T(A, A)\) defined by
\[
T(A, A) \circ T(A, A) = \sum_{t \text{ odd}} \Delta_t [T_t(A, A) \otimes T_t(A, A)]_0,
\]
commutes with all the generators of the \(O_{n_a}\). As a result, the normalized CO of the \(O_{n_a}\) is given by
\[
C_2[O_{n_a}] = 4T(A, A) \circ T(A, A).
\]
The \(O_{n_{ab}}\) subalgebra is spanned by the generators
\[
T_t(A, A) = [A^\dagger \otimes \bar{A}]_b, \quad t = 1, 3, \ldots, 2a - 1, \quad (15a)
\]
\[
T_t(B, B) = [B^\dagger \otimes \bar{B}]_c, \quad t = 1, 3, \ldots, 2b - 1, \quad (15b)
\]
\[
T_t(C, C) = [C^\dagger \otimes \bar{C}]_d, \quad t = 1, 3, \ldots, 2c - 1, \quad (15c)
\]
and three linear combinations of the form
\[
K_t(A, B) = \rho_{ab} T_t(A, B) + \alpha_{bc} T_t(B, A), \quad t = |a - b|, \ldots, a + b, \quad (16a)
\]
\[
K_t(A, C) = \rho_{ac} T_t(A, C) + \alpha_{bc} T_t(C, A), \quad t = |a - c|, \ldots, a + c, \quad (16b)
\]
\[
K_t(B, C) = \rho_{bc} T_t(B, C) + \alpha_{bc} T_t(C, B), \quad t = |b - c|, \ldots, b + c, \quad (16c)
\]
the coefficients of which can be determined by the requirement of closure of the commutation relation. Using (5), the closure is obtained if
\[
\frac{\rho_{ab}'}{\rho_{ab}} = (-1)^i \omega_{ab}, \quad \frac{\rho_{ac}'}{\rho_{ac}} = (-1)^i \omega_{ac}, \quad \frac{\rho_{bc}'}{\rho_{bc}} = (-1)^i \omega_{bc},
\]
and
\[
\omega_{ab} \omega_{bc} = -\omega_{ac},
\]
where
\[
\omega_{ij} = \pm 1, \quad i, j = a, b, c \quad \text{and} \quad i \neq j.
\]
Combining (17) with (16) leads to the relations
\[
K_t(A, B) = \rho_{ab} [T_t(A, B) + (-1)^i \omega_{ab} T_t(B, A)], \quad (20a)
\]
\[
K_t(A, C) = \rho_{ac} [T_t(A, C) + (-1)^i \omega_{ac} T_t(C, A)], \quad (20b)
\]
\[
K_t(B, C) = \rho_{bc} [T_t(B, C) + (-1)^i \omega_{bc} T_t(C, B)]. \quad (20c)
\]
Let $\omega_{ij} = (-1)^{j}$ with $\eta = 0$ or $1$ for $i,j = a, b, c$ and $i \neq j$. Then, we choose overall arbitrary phases
\[
\varrho'_{ab} = \varrho'_{ac} = \varrho'_{bc} = 0 \quad \text{if} \quad i + j + \eta = odd, \quad i, j = a, b, c, \quad i \neq j.
\] (21)

Accordingly, the $O_{n_{ab}}$ subalgebra is spanned by the generators in (15) and the set of generators
\[
K_{i}(A, B) = T_{i}(A, B) + (-1)^{i}\omega_{ab}T_{i}(B, A), \quad (22a)
\]
\[
K_{i}(A, C) = T_{i}(A, C) + (-1)^{i}\omega_{ac}T_{i}(C, A), \quad (22b)
\]
\[
K_{i}(B, C) = T_{i}(B, C) + (-1)^{i}\omega_{bc}T_{i}(C, B). \quad (22c)
\]

The two level subalgebra $O_{n_{bc}}$ is spanned by the generators $T_{d}(B, B), T_{d}(C, C)$ and $K_{i}(B, C)$ which are obtained by omitting all terms with $A$ in (15) and (22).

In order to write CO of the three-level orthogonal algebra in terms of multipole operators, the following operators are introduced
\[
K(A, B) \circ K(B, A) = \sum_{i}(-1)^{i}K_{i}(A, B) \cdot K_{i}(B, A), \quad (23a)
\]
\[
K(A, C) \circ K(C, A) = \sum_{i}(-1)^{i}K_{i}(A, C) \cdot K_{i}(C, A), \quad (23b)
\]
\[
K(B, C) \circ K(C, B) = \sum_{i}(-1)^{i}K_{i}(B, C) \cdot K_{i}(C, B). \quad (23c)
\]

The operators $K(A, B) \circ K(B, A), K(A, C) \circ K(C, A)$ and $K(B, C) \circ K(C, B)$ generalize the multipole operator to the generic three level model. The quadratic CO of $O_{n_{bc}}$ is given by,
\[
C_{2}[O_{n_{bc}}] = \sum_{(j, j)} K(j, j) + \sum_{k = a, b, c} C_{2}[O_{n_{k}}]. \quad (24)
\]

The result follows by explicitly evaluating the commutator of $C_{2}[O_{n_{bc}}]$ with the generators (15) and (22), using (5). The quadratic CO of two level orthogonal algebra $O_{n_{bc}}$ is a special case of the (24).
\[
C_{2}[O_{n_{bc}}] = 2K(A, B) \circ K(B, A) + C_{2}[O_{n_{a}}] + C_{2}[O_{n_{c}}]. \quad (25)
\]

The generators and COs of the three-level unitary and orthogonal algebras in all subalgebra chains of the sdIBM are established in [17] and they are consistent with our results (6), (8), (15), (22) and (24), with $a = 0$, $b = 2$ and $c = 4$. The main result of this section is that the COs of three level unitary algebras can be expressed in terms of the COs of single level algebras. However, the COs of $O_{n_{ab}}$ can be written in terms of the COs of single level algebras and multipole operators.

4. The multipole operator and quasi-spin algebra

First, let us consider the QS algebra $SU(1, 1)$ which appears in the problem of many bosons placed in a single level of angular momentum $a$ (an integer) which we denote by $SU_{a}(1, 1)$ and call it single level QS algebra. The $SU_{a}(1, 1)$ algebra is spanned by $S_{a}^{+}$, $S_{a}^{-}$ and $S_{a}^{0}$ which obeying the following commutation relations
\[
[S_{a}^{0}, S_{a}^{+}] = +S_{a}^{+}, \quad [S_{a}^{0}, S_{a}^{-}] = -S_{a}^{-}, \quad [S_{a}^{+}, S_{a}^{-}] = -2S_{a}^{0}, \quad (26)
\]
and satisfy unitary conditions
\[
S_{a}^{+}\dagger = S_{a}^{-} \quad \text{and} \quad S_{a}^{0}\dagger = S_{a}^{0}. \quad (27)
\]

Next, we shall introduce the pair creation operator ($S_{ab}$) and pair annihilation operator ($S_{ab}$) as,
\[
S_{ab} = \frac{1}{2}\Delta_{a}[A^{\dagger} \otimes A^{\dagger}]_{00}, \quad S_{a}^{-} = \frac{1}{2}\Delta_{a}[\bar{A} \otimes \bar{A}]_{00}. \quad (28)
\]

Denoting $S_{a0}$ by
\[
S_{a0} = \frac{1}{2}\Delta_{a}([A^{\dagger} \otimes \bar{A}]_{00} + [\bar{A} \otimes A^{\dagger}]_{00}), \quad (29)
\]
we get the $SU_{a}(1, 1)$ commutation relations.

A QS algebra for the three-level system which we denote by $SU_{abc}(1, 1)$ is spanned by the sum generators
\[
S_{+} = S_{a0} + S_{ab}S_{b0} + S_{ac}S_{c0}, \quad (30a)
\]
\[
S_{-} = S_{a0} + S_{ab}S_{b0} + S_{ac}S_{c0} \quad (30b)
\]
\[
S_{0} = S_{a0} + S_{b0} + S_{c0}. \quad (30c)
\]
A QS algebra is obtained with either choice of sign $\varpi_{ab} = \pm$, $\varpi_{ac} = \pm$ in the ladder operators.
For a single \( a \) level, recoupling and commutation of creation operators yields
\[
4S_{a+}S_{a-} = \Delta_a^4 [A^\dagger \otimes A^\dagger]_{00} [\tilde{A} \otimes \tilde{A}]_{00} + \sum_\varepsilon \psi_\varepsilon [A^\dagger \otimes \tilde{A}]_\varepsilon [A^\dagger \otimes \tilde{A}]_\varepsilon
\]
(31)

Thus, the relations between QS operators and COs for the three single level algebras are
\[
4S_{k+}S_{k-} = -N_k + \frac{1}{2} C_2 [O_{n_k}], \quad k = a, b, c.
\]
(32)

For the three-level system, the product \( S_+ S_- \) is given by
\[
S_+ S_- = \sum_{k=a,b,c} 4S_{k+}S_{k-} + \sum_{(i,j)} 4\omega_{ij} (S_{i+}S_{j-} + S_{j+}S_{i-}).
\]
(33)

Using the recoupling and commutation relations, the mixed term of QS operators, \( 4(S_{a+}S_{b-} + S_{b+}S_{a-}) \), is given by
\[
4(S_{a+}S_{b-} + S_{b+}S_{a-}) = \psi_{ab} (A^\dagger \otimes A^\dagger + B^\dagger \otimes B^\dagger) + \psi_{ac} (A^\dagger \otimes A^\dagger + B^\dagger \otimes B^\dagger)
\]
(34)

In the same way, we get
\[
4(S_{a+}S_{c-} + S_{c+}S_{a-}) = -N_{ac} + \frac{1}{2} C_2 [O_{n_{ac}}], \quad 4(S_{b+}S_{c-} + S_{c+}S_{b-}) = -N_{bc} + \frac{1}{2} C_2 [O_{n_{bc}}].
\]
(35a, 35b)

The one level terms and mixed terms of \( S_+ S_- \) may thus be combined to give an expression involving the \( O_{n_{ab}}, O_{n_{bc}}, O_{n_{ac}} \) COs if and only if the sign \( \omega_{ij} \) arising in the definition of the sum QS algebra and the sign \( \omega_{ij} \) entering into the definition of \( \psi_{ij} \) are related by
\[
\omega_{ab} = \omega_{ba}, \quad \omega_{ac} = \omega_{ca}, \quad \omega_{bc} = \omega_{cb}.
\]
(36a, 36b, 36c)

We again have an expression of the same form as (32):
\[
4S_{a+}S_{a-} = -N_{abc} + \frac{1}{2} C_2 [O_{n_{abc}}].
\]
(37)

Using (8), (9) and (14), the \( K \circ K \) operators and the \( S_+ S_- \) operators are related by
\[
\sum_{(i,j)} K(I, J) \circ K(J, I) = -\frac{1}{2} \sum_{k=a,b,c} C_2 [O_{n_k}] + C_2 [O_{n_{abc}}] - N_{abc} - 4S_+ S_-
\]
(38)

The three-level pairing Hamiltonian in QS notation can be written as
\[
H = \sum_{k=a,b,c} \epsilon_k (2S_{k+} - \Omega_k) + G_{k,0} S_{k+} S_{k-} + \sum_{(i,j)} G_{ij} (S_{i+}S_{j-} + S_{j+}S_{i-}),
\]
(39)

where we used equations (42) and (44) in [26]. Substituting from (32), (34) and (35) into (39) one obtains
\[
H = \sum_{k=a,b,c} \epsilon_k - \frac{1}{4} G_{k,0} + \frac{1}{8} \sum_{(i,j)} (-1)^{i+j} G_{ij} (2C_2 [U_{n_i}] + C_2 [O_{n_j}]) + \frac{1}{8} (G_{a,b} + (-1)^{a+b} G_{a,b} + (-1)^{a+c} G_{a,c}) (2C_2 [U_{n_b}] + C_2 [O_{n_a}])
\]
\[
+ \frac{1}{8} (G_{b,c} + (-1)^{b+c} G_{b,c}) (2C_2 [U_{n_c}] + C_2 [O_{n_b}]) + \frac{1}{8} (G_{c,a} + (-1)^{a+c} G_{c,a}) (2C_2 [U_{n_a}] + C_2 [O_{n_c}]).
\]
(40)

The interesting result is that the three-level pairing Hamiltonian can be expressed in terms of COs of algebras appearing in the subalgebra chains of \( U_{n_{abc}} \) (not just at the DS limit). Furthermore, by using results of section 3, the three-level pairing Hamiltonian can be written in terms of COs of single level algebras \( (N_k, C_2 [U_{n_k}], C_2 [O_{n_k}], k = a, b, c) \) and the multipole operators.

5. Three level transition operators

Usually, in the three level systems, operators inducing electromagnetic transitions of multipolarity \( L \) can be written in terms of one-body operators as
where the following properties are then observed. The DS plays an essential role in IBM. The main advantage of DS is that, whenever one such symmetry occurs, the three-level system with pairing interactions. In the language of group theory, the scalar operators tensorial nature, what part is diagonal in seniority. In other words, we can derive selection rules for allowed transitions. The operators behave like scalars in the quasispin space under quasispin transformations. Consequently, we can see what part of the operators changes seniority and what part is diagonal in seniority. In other words, we can derive selection rules for allowed transitions. The tensorial nature, of the operators $Q^b_{Lq}(A, B), Q^c_{Lq}(A, C)$ and $Q^d_{Lq}(B, C)$ depend on the $R_{(i,j)}^{(X)L}$ choice. The operators behave like scalars in the quasispin space (i.e., they do not change seniority) in the case

$$R_{(A,B)}^{(X)L} = (-1)^{L+1} \mathbf{1}_{ab}, \quad R_{(A,C)}^{(X)L} = (-1)^{L+1} \mathbf{1}_{ac}, \quad R_{(B,C)}^{(X)L} = (-1)^{L+1} \mathbf{1}_{ab} \mathbf{1}_{ac}. \quad (43)$$

However, the operators become quasispin vectors in the choice

$$R_{(A,B)}^{(X)L} = (-1)^i \mathbf{1}_{ab}, \quad R_{(A,C)}^{(X)L} = (-1)^j \mathbf{1}_{ac}, \quad R_{(B,C)}^{(X)L} = (-1)^k \mathbf{1}_{ab} \mathbf{1}_{ac}. \quad (44)$$

The results of the section 4 show very clearly the duality between orthogonal algebras and the QS algebras for the three-level system with pairing interactions. In the language of group theory, the scalar operators $Q^a_{Lq}(A, B), Q^b_{Lq}(A, C)$ and $Q^c_{Lq}(B, C)$ are the generators of the $O_{n_0}$ algebra, (22). Again, (36) plays the central role in the duality between orthogonal algebras and the QS algebras. Similarly, the operators $[A^t \otimes \tilde{A}]_{Lq}, [B^t \otimes \tilde{B}]_{Lq}$ and $[C^t \otimes \tilde{C}]_{Lq}$ with $L$ odd and even are scalars ($T^0_L$) and quasispin vectors ($T^0_L$) in the quasispin space respectively.

If we consider the parity invariance $T^{E,L}_{L-even}$ and $T^{M,L}_{L=odd}$ operators to be $T^0_L$ and $T^0_L$ in the quasispin space, then

$$T_q^{(X)L} = \sum_{K=A,B,C} C_{(K,K)}^{(X)L} [K^\dagger \otimes \tilde{K}]_{Lq} + \sum_{(i,j)} C_{(i,j)}^{(X)L} Q_{Lq}(I, J). \quad (45)$$

Similarly, if the parity changing $T^{E,L}_{L=odd}$ and $T^{M,L}_{L=even}$ operators are $T^0_L$ and $T^0_L$ in the quasispin space respectively, we have

$$T_q^{(X)L} = \sum_{(i,j)} C_{(i,j)}^{(E,L)} Q_{Lq}(I, J). \quad (46)$$

On the other hand, if we impose the condition that $T^{E,L}_{L=1}$ is $T^0_L$. The parity invariance operator is

$$T_q^{(E,L=even)} = \sum_{(i,j)} C_{(i,j)}^{(E,L)1} Q_{Lq}(I, J). \quad (47)$$

However, the parity changing operator is

$$T_q^{(E,L=odd)} = \sum_{K=A,B,C} C_{(K,K)}^{(E,L)} [K^\dagger \otimes \tilde{K}]_{Lq} + \sum_{(i,j)} C_{(i,j)}^{(E,L)} Q_{Lq}^+(I, J). \quad (48)$$

Similarly, $T^{M,L}$ can be chosen to quasispin vectors.

6. The dynamical symmetries

The DS plays an essential role in IBM. The main advantage of DS is that, whenever one such symmetry occurs, the following properties are then observed. (i) Analytic expressions are available for physical observables. (ii) All states are classified by quantum numbers $\lambda_0, \lambda_1, \ldots, \lambda_n$ which are the labels of the IRs of the subalgebras in SGA.

More generally, when considering phase transitions, it is helpful to construct the Hamiltonian in terms of COs from parallel subalgebra chains, here the algebras in (1), as

$$H = \sum_{i=a,b,c,ab,ac,bc,abc} \theta_i C_i [U_n] + \phi_i C_2 [U_n] + \psi_i C_3 [O_{n_i}] + H_{abcd}, \quad (49)$$

where

$$H_{abcd} = \sum_{k=a,b,c} \rho_k C_2 [O_{3,k}] + \chi C_2 [O_{3,abc}]. \quad (50)$$

Considering only the linear COs of unitary algebra, the Hamiltonian can be written as

$$H = \sum_{i=a,b,c,ab,ac,bc,abc} \theta_i N_i + \psi_i C_3 [O_{n_i}] + H_{abcd}. \quad (51)$$
There are three types of DS which correspond to the subalgebra chains in (1). The first DS, 
\( O_{n_{abc}} \supset O_{n_b} \otimes O_{n_c} \), is obtained by taking \( \theta_{abc} = \theta_{ac}, \theta_{bc} = \theta_a = \theta_b = \psi_{abc} = \psi_{ac} = 0 \) in the Hamiltonian (51). The Hamiltonian reduces to

\[
H_1 = \sum_{i=a,b,c} \psi_i C_2[O_{n_i}] + \theta_{abc} N_{abc} + H_{abc}. \tag{52}
\]

This Hamiltonian is diagonal in the basis (3a). Its eigenvalues can be found by taking the expectation value of \( H_1 \) in (3a). The eigenvalues are given by

\[
E_1 = \sum_{i=a,b,c} 2\psi_i v_i (v_i + n_i - 2) + \theta_{abc} N_{abc} + E_{abc}, \tag{53}
\]

where

\[
E_{abc} = \sum_{k=a,b,c} \rho_{kk} (k + 1) + \chi l (l + 1). \tag{54}
\]

The second DS, \( U_{n_a} \otimes U_{n_b} \supset O_{n_b} \otimes O_{n_a} \), corresponds to taking \( \theta_{abc} = \theta_{ac} = \theta_0 = \theta_c = \psi_{abc} = \psi_{ac} = \psi_{bc} = 0 \) in the Hamiltonian (51). The Hamiltonian and its eigenvalues are given by

\[
H_2 = \sum_{i=a,b,c} \psi_i C_2[O_{n_i}] + \sum_{j=a,b,c} \theta_{ij} N_{ij} + H_{abc}. \tag{55a}
\]

\[
E_2 = \sum_{i=a,b,c} 2\psi_i v_i (v_i + n_i - 2) + \sum_{j=a,b,c} \theta_{ij} N_{ij} + E_{abc}. \tag{55b}
\]

The third DS, \( U_{n_b} \otimes U_{n_a} \supset U_{n_b} \otimes U_{n_a} \otimes U_{n_b} \supset O_{n_b} \otimes O_{n_a} \otimes O_{n_b} \), corresponds to taking \( \theta_{abc} = \theta_{ac} = \psi_{abc} = \psi_{ac} = \psi_{bc} = 0 \) in the Hamiltonian (51). The Hamiltonian and its eigenvalues are given by

\[
H_3 = \sum_{i=a,b,c} \psi_i C_2[O_{n_i}] + \sum_{j=a,b,c} \theta_{ij} N_{ij} + H_{abc}. \tag{56a}
\]

\[
E_3 = \sum_{i=a,b,c} 2\psi_i v_i (v_i + n_i - 2) + \sum_{j=a,b,c} \theta_{ij} N_{ij} + E_{abc}. \tag{56b}
\]

Interestingly, some of our considered DSs and energy formulae have particular realizations in IBM. Kota et al [16–21], categorized the DSs of sdgIBM into seven types. Some of them are unraveled in our general three DSs. In the second DS \( U_{n_{abc}} \supset U_{n_b} \otimes U_{n_c} \supset O_{n_b} \otimes O_{n_c} \), when (\(a, b, c\)) are substituted with (0, 2, 4), (2, 0, 4) and (4, 0, 2), we obtain the weak coupling limits \( U_{1} \otimes U_{14}, U_{5} \otimes U_{110} \) and \( U_{9} \otimes U_{90} \), respectively. Moreover, in the first DS \( U_{n_{abc}} \supset O_{n_{abc}} \supset O_{n_b} \otimes O_{n_c} \), when (\(a, b, c\)) takes the value (0, 2, 4), we obtain the strong coupling limits \( U_{15} \supset O_{15} \supset O_{14} \). Similarly, the three general DSs are in agreement with the three level algebras and its DSs in the spdIBM, [22–24].

Recently, using proxy-\( SU_3 \) scheme [38, 39] simple predictions of shape observables for deformed nuclei can be made. In the proxy-\( SU_3 \) scheme, the neutrons of the 82–126 shell live in a proxy\( pfh \) shell having an approximate\( U_{21} \) symmetry. In addition to figure 1 that shows the lattice of algebras of\( U_{21,pfh} \) using (1), we present one of the possible reduction of\( U_{21,pfh} \) to\( SU_{3,ph} \)

\[
U_{21,pfh} \supset \begin{cases}
U_{3,p} \otimes U_{18,p} \\
U_{7,f} \otimes U_{4,pfh} \\
U_{11,h} \otimes U_{10,pf}
\end{cases} \supset U_{3,p} \otimes U_{7,f} \otimes U_{11,h} \supset K, \tag{57}
\]

\[
K \equiv U_{3,p} \otimes U_{3,f} \otimes U_{3,h} \supset U_{3,pfh} \supset SU_{3,ph}. \tag{58}
\]

In the following part, we briefly discuss the spectra of eigenvalues of the pairing Hamiltonian in the DS \( O_{n_{abc}} \supset O_{n_b} \otimes O_{n_c} \) for the \( U_{21} \) algebra. There are seven three element sets of unequal odd integers that have a sum of members equal to 21. The \( U_{21} \) algebra can be constructed using seven different level systems. Figure 2 represents these systems. The \( O_{n_{abc}} \) is used as the label of orthogonal algebra generated by the three levels \( a, b \) and \( c \). Figure 2 shows also the lattice of algebras in the \( U_{21} \) symmetry generated by \( a = b = c = levels \), with the DS \( O_{n_{abc}} \supset O_{n_b} \otimes O_{n_c} \). In fact, there are 21 subalgebra chains for the algebra \( U_{21} \) with the DS \( O_{n_{abc}} \supset O_{n_b} \otimes O_{n_c} \). If the Hamiltonian is chosen as

\[
H = -K(A, B) \otimes K(B, A) = K(A, C) \otimes K(C, A), \tag{59}
\]

by taking \( \psi_{abc} = -\frac{1}{2} \), \( \psi_{bc} = \psi_{c} = \frac{1}{2} \), \( \psi_{b} = \psi_{a} = 0 \), \( \theta_{abc} = \rho_{a} = \rho_{b} = \rho_{c} = \chi = 0 \) in the Hamiltonian (52) then its eigenvalues are given by

\[
E(\nu_{abc}, \nu_{bc}, \nu_{b}) = n_{abc}(\nu_{bc} - \nu_{abc}) + n_{a}(\nu_{a} - \nu_{bc}) + (2n_{abc} - 2\nu_{a} - 2\nu_{bc} - \nu_{abc}^{2} + \nu_{bc}^{2}). \tag{60}
\]
The equation (60) enables us to compare the spectra of eigenvalues of the Hamiltonian operator (59) that produced by the different three level systems. From (60), the different three level systems with the same DS $O_n \supset O_{n1} \otimes O_{n2}$ have the identical spectra of the eigenvalues, $E(\nu_{abc}, \nu_{bc}, \nu_a)$. There are 9 distinct subalgebra chains, namely

\begin{align*}
\text{Chain 1} & \equiv O_{21} \supset O_{20} \otimes O_{18} \\
\text{Chain 2} & \equiv O_{21} \supset O_{19} \otimes O_{16} \\
\text{Chain 3} & \equiv O_{21} \supset O_{18} \otimes O_{15} \otimes O_6 \\
\text{Chain 4} & \equiv O_{21} \supset O_{17} \otimes O_{14} \otimes O_6 \\
\text{Chain 5} & \equiv O_{21} \supset O_{16} \otimes O_{13} \otimes O_6 \\
\text{Chain 6} & \equiv O_{21} \supset O_{15} \otimes O_{12} \otimes O_6 \\
\text{Chain 7} & \equiv O_{21} \supset O_{14} \otimes O_{11} \otimes O_6 \\
\text{Chain 8} & \equiv O_{21} \supset O_{13} \otimes O_{10} \otimes O_6 \\
\text{Chain 9} & \equiv O_{21} \supset O_{12} \otimes O_{10} \otimes O_9 \\
\end{align*}

As a result, the number of the spectra of the eigenvalues will be reduced from 21 spectra to 9 distinct spectra. For example, the spectra of the eigenvalues, $E(\nu_{abc}, \nu_{bc}, \nu_a)$, of the DSs $O_{21}^{[18]} \supset O_{10}^{[18]}$, $O_{21}^{[27]} \supset O_{10}^{[27]}$, $O_{21}^{[36]} \supset O_{10}^{[36]}$ and $O_{21}^{[45]} \supset O_{10}^{[45]}$ are identical. We called these subalgebras chain 1. Figure 3 shows the energy diagram for DS of the chain 1. The total occupation number is $N = 4$. The pairing Hamiltonian is chosen as $-K(A, B) \circ K(B, A) = K(A, C) \circ K(C, A)$. With $N = 4$, the values of $\nu_{abc}, \nu_{bc}$ and $\nu_a$ are generated using the reduction rules of the subalgebra $O_n \supset O_{n1} \otimes O_{n2}$ in the end of section 1. The energy diagram is divided into three parts. These parts represent energy levels with $\nu_{abc} = 4$, $\nu_{abc} = 2$ and $\nu_{abc} = 0$, respectively. The degeneracy within IRs of the three-level algebra (labeled by $\nu_{abc}$) is split according to the IRs of the single and two-level algebras (labeled by $\nu_{bc}$, $\nu_a$). Figure 4 compares the energy levels at each value $(\nu_{abc}, \nu_{bc}, \nu_a)$ for the 9 distinct chains of the DSs $O_{21} \supset O_{n1} \otimes O_{n2}$. Interestingly, a closer look at the figure reveals that the eigenvalues of the Hamiltonian (59) showed invariant values at certain points. For instance, $E(2, 1, 1)$ equals $-23$, in all spectra produced by the different three level systems.

7. Conclusions

We shed light on the class of general subalgebra chains that exist in the three-level pairing models. The expressions for the generators and COs of the generic three level unitary algebra and three level orthogonal
algebra are formulated. Thus, we draw attention to the existence of three types of DS. For the three general symmetry limits discussed in this paper, $O_{21}^{[1]} \supset O_{21}^{[2]} \supset O_{21}^{[3]}$, $O_{21}^{[4]} \supset O_{21}^{[5]}$, and $O_{21}^{[6]} \supset O_{21}^{[7]}$, the results for electromagnetic transition strengths will be presented elsewhere—beyond the special cases conventionally considered. The group theoretical problems needed for these are being solved.

In addition, an important finding that emerges from this study is that the same unitary algebra and consequently its orthogonal subalgebra can be constructed using many different three level systems. Therefore, we can compare the spectra of eigenvalues of the pairing operator that produced by the different three level systems.

The relation between the orthogonal algebras and the QS algebras was discussed. The relation between pairing operators and the multipole operators for the three-level boson system is given in a closed form. The results presented here show very clearly the duality between orthogonal algebras and the QS algebras for the three-level system with pairing interactions.

Finally, using $F$ spin and double spherical tensors, the extended pairing algebras with two-species boson will be the topic of future work, for which the quantum phase transitions have not yet been completely studied.

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