Multi-focal spherical media and geodesic lenses in geometrical optics

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Abstract

This paper presents a general approach to designing isotropic spherical media with complex spatial structure that provide different types of imaging for different light rays. It is based on equivalence of the spherical medium and the corresponding geodesic lens. We use this approach to design multi-focal gradient-index lenses embedded in an optically homogeneous region and multi-focal absolute instruments that provide stigmatic imaging of three-dimensional domains.

Keywords: multi-focal spherical medium, geodesic lens, gradient-index lens, absolute instrument, inverse scattering, geometrical optics

(Some figures may appear in colour only in the online journal)

1. Introduction

An interesting class of isotropic optical media is represented by the so-called spherical media, i.e. media with a spherically symmetric distribution of refractive index. They have found application in proposals for many optical devices that provide remarkable properties within geometrical optics. We mention spherical gradient-index lenses embedded in an optically homogeneous region that provide stigmatic imaging of two concentric spherical surfaces. Representatives of these are Luneburg and Eaton lenses [1, 2]. Next, we mention absolute instruments that provide stigmatic imaging of three-dimensional domains [3]. A famous representative of these is Maxwell’s fish-eye [4, 5].

The general approach to designing a spherical medium with specific properties lies in solving an inverse scattering problem. This is considerably simplified due to the symmetry of the spherical medium which enables one to treat it as two-dimensional, because each light ray lies in a plane through the center of symmetry. Classical methods for solving the inverse problem have been derived for both spherical gradient-index lenses [1] and absolute instruments [6]. Recently, we have described an alternative method [7] based on optical conformal mapping [8]. It utilizes an equivalence of the central section of the spherical medium and a corresponding geodesic lens [9–12]. The main advantage of this method is that the shape of a geodesic lens can easily be deduced from the desired ray trajectories for both focusing spherical gradient-index lenses and spherical absolute instruments. Once the shape of a geodesic lens is known, the refractive index of the spherical medium is calculated by a straightforward procedure.

In this paper we use this alternative approach to design new optical devices referred to as multi-focal gradient-index lenses and multi-focal absolute instruments. These devices formed of isotropic spherical media with complex spatial structure provide different imaging properties for different light rays and, therefore, enable one to achieve remarkable optical effects. In fact, we have already presented examples of such devices before [6]. Here, we show that both these devices can easily be designed using the concept of geodesic lenses and that there is a close connection between them. As in [7], we reduce the description to two dimensions and we keep it within geometrical optics.

The paper is organized as follows. Section 2 is devoted to the derivation of multi-focal gradient-index lenses embedded in an optically homogeneous region that provide stigmatic imaging of multiple pairs of concentric spherical surfaces.
Section 3 is aimed to the derivation of multi-focal absolute instruments that provide different imaging properties for different spatial domains. Section 4 concludes the paper.

2. Gradient-index lenses

In this section we briefly summarize the inverse scattering problem for a spherical gradient-index lens embedded in a homogeneous region (for details see [7]). Then we use the general results to design multi-focal lenses that provide stigmatic imaging of multiple pairs of concentric spherical surfaces.

2.1. Gradient-index lens embedded in an optically homogeneous region

We consider a gradient-index lens of unit radius that is specified in polar coordinates $(r, \theta)$ by the refractive index $n(r)$, where $n(1) = 1$. It is embedded in a homogeneous region with unit refractive index (see figure 1(a)). In terms of optical conformal mapping, the gradient-index lens is equivalent to a geodesic lens which is a curved rotationally symmetric surface with the refractive index set to unity (see figure 1(b)). We describe its points by a radial coordinate $\rho$, a polar angle $\theta$ and a value $s(\rho)$ that represents the length of the surface measured along the meridian from the axis of symmetry to a given point. Assuming $\theta = \varphi$, the coordinate transform is

$$ \rho = nr, \quad dx = n \, dr. \quad (1) $$

The shape of a geodesic lens is qualitatively given by the form of the function $\rho(r)$. In this section we assume that for $r \in [0, 1]$ the function $\rho(r)$ is increasing with a single maximum at $r = 1$, so the geodesic lens has the shape sketched in figure 1(b). Although $s(\rho)$ is unambiguous, it will be referred to as $s_1(\rho)$ to keep consistency with section 3.

Each light ray propagating in a spherical medium or on a geodesic lens is specified by a constant quantity $L$ referred to as the angular momentum. It is given by

$$ L = nr \sin \alpha = \rho \sin \alpha \quad (2) $$

where $\alpha$ is the angle between the tangent to the ray trajectory and the radius vector (in a spherical medium) or the meridian (on a geodesic lens). The turning point where $\alpha = \pi/2$ is given by the radial coordinate $r_+(L)$. Denoting $s_1'(\rho) = ds_1(\rho)/d\rho$, the ray propagation is governed by the differential equations

$$ d\varphi = \pm \frac{L \, dr}{r \sqrt{n^2 r^2 - L^2}} = \pm \frac{L s_1'(\rho) \, d\rho}{\rho \sqrt{\rho^2 - L^2}}. \quad (3) $$

2.2. Inverse scattering problem

The classical formulation of the inverse scattering problem for a gradient-index lens embedded in a homogeneous region lies in the derivation of the refractive index $n(r)$ from a given dependence of the scattering angle $\chi(L)$ on the angular momentum $L$ (see figure 1). Here, we specify the bending of the light rays in another equivalent way. For a ray with angular momentum $L$, we introduce the radial coordinates $r_s(L)$ and $r_i(L)$ of a ‘source’ point $P_s$ and an ‘image’ point $P_i$ which are located on the straight lines along which it propagates before and after passing through the lens, respectively. Next, we introduce an angle $\Delta \varphi(L)$ swept by the ray between the points $P_s$ and $P_i$. The notation suggests that $r_s(L)$, $r_i(L)$ and $\Delta \varphi(L)$ are, in general, functions of $L$; this is necessary for considering the multi-focal lenses discussed later. Therefore, the inverse problem lies in the derivation of the refractive index $n(r)$ from the given functions $r_s(L)$, $r_i(L)$ and $\Delta \varphi(L)$.

The mathematical formulation of the inverse problem lies in expressing the angle $\Delta \varphi(L)$ using (3). Under the assumptions given above, we get the integral equations

$$ \int_{r_s(L)}^{1} \frac{L \, dr}{r \sqrt{n^2 r^2 - L^2}} = \int_{L}^{1} \frac{L s_1'(\rho) \, d\rho}{\rho \sqrt{\rho^2 - L^2}} = g(L), \quad (4) $$

where

$$ g(L) = \frac{1}{2} \left( \Delta \varphi(L) + \arcsin \frac{L}{r_s(L)} + \arcsin \frac{L}{r_i(L)} - 2 \arcsin L \right). \quad (5) $$

The standard procedure for directly solving (4) with respect to $n(r)$ is given in [1]. Here, we use an alternative approach that lies in solving (4) with respect to $s_1'(\rho)$ and subsequent calculation of the refractive index $n(r)$.

Equation (4) for the function $s_1'(\rho)$ is a kind of Abel integral equation. Assuming that $g(L)$ is an absolutely continuous function on the interval $[0, 1]$, it has a unique solution in the class of Lebesgue-integrable functions [13] given by

$$ s_1'(\rho) = -\frac{2 \rho}{\pi} \frac{d}{d\rho} \left( \int_{\rho}^{1} \frac{g(L) \, dL}{\sqrt{L^2 - \rho^2}} \right). \quad (6) $$

Figure 1. (a) A gradient-index lens embedded in an optically homogeneous region. (b) An equivalent geodesic lens.
Once we have $s'_1(\rho)$, we derive the refractive index $n(r)$. Using the formulas (1) we get $dr/r = s'_1(\rho)\,d\rho/\rho$. Integrating this equation from the boundary of the lens to the turning point, given in the spherical medium by the radial coordinate $r$ equal to $r_+(L)$ and on the geodesic lens by $\rho$ equal to $L$, we obtain
\[
r_{+}(L) = \exp \left( \int_{r_{-}(L)}^{r_{+}(L)} s'_1(\rho)\,d\rho/\rho \right).\tag{7}
\]
Since we have assumed that the function $r(r)$ is monotonic, the function $r_{+}(L)$ is also monotonic and invertible. Therefore, in principle we can find the inverse function $L(r_{+})$ and use it for calculation of the refractive index by means of the relation
\[
n(r_{+}) = \frac{L(r_{+})}{r_{+}}\tag{8}
\]
which holds for the turning point since there $\rho = L$. Omitting the lower index of $r_{+}$, we get the final result for the refractive index $n(r)$ of the gradient-index lens.

2.3. Multi-focal gradient-index lenses

Let us deal with gradient-index lenses that provide stigmatic imaging of pairs of concentric spherical surfaces.

In our previous paper [7], we have discussed the simplest case when there is only one pair of these surfaces. Then all the light rays with $L \in [0, 1]$ that emerge from a point source located at a given surface and that pass through the lens are focused into an image point located at the second given surface. Such lenses can be found by solving the Luneburg inverse problem [1] and well-known representatives are Maxwell’s fish-eye lens, the Luneburg lens and the Eaton lens.

Here, we focus on multi-focal gradient-index lenses that provide stigmatic imaging of multiple pairs of concentric spherical surfaces. For simplicity, we start with the case when there are two pairs of these surfaces, i.e. the light rays passing through the lens are divided into two bundles specified by subintervals of angular momentum $[0, L_1]$ and $[L_1, 1]$ which, in general, have the source and image points located at spherical surfaces of different radii. Then the functions $r_{s}(L)$ and $r_{i}(L)$ are given by
\[
r_{s}(L) = \begin{cases} r_{s1} & \text{for } L \in [0, L_1], \\ r_{s2} & \text{for } L \in (L_1, 1], \end{cases}
\]
\[
r_{i}(L) = \begin{cases} r_{i1} & \text{for } L \in [0, L_1], \\ r_{i2} & \text{for } L \in (L_1, 1], \end{cases}
\]
where $r_{s1}, r_{s2}, r_{i1}$ and $r_{i2}$ are given constants greater than or equal to unity. Similarly, we write the angle $\Delta \psi(L)$ in the form
\[
\Delta \psi(L) = \begin{cases} M_1\pi & \text{for } L \in [0, L_1], \\ M_2\pi & \text{for } L \in (L_1, 1], \end{cases}
\]
where $M_1 \geq M_2$ are positive constants (a more detailed analysis shows that for $M_1 < M_2$ the integral equation (4) has no solution). The different imaging properties for the two bundles of rays imply that the refractive index must have different functional dependences in two different regions separated by a boundary of radius $r_{+}(L_1)$. Therefore, we denote
\[
n(r) = \begin{cases} n_{11}(r) & \text{for } r \in [0, r_{+}(L_1)], \\ n_{12}(r) & \text{for } r \in [r_{+}(L_1), 1]. \end{cases}
\]
Similarly, an equivalent geodesic lens must be parameterized gradually from the top downwards by two different parts of the function $s_1(\rho)$ formally written as
\[
s_1(\rho) = \begin{cases} s_{11}(\rho) & \text{for } \rho \in [0, L_1], \\ s_{12}(\rho) & \text{for } \rho \in [L_1, 1]. \end{cases}
\]

Equipped with the notation, we proceed to solving the inverse scattering problem. After a lengthy calculation using (6), we get the results in a compact form,
\[
s'_1(\rho) = A_1 + A_2 + \frac{B_1}{\sqrt{1 - (\rho/L_1)^2}} + \frac{B_2}{\sqrt{1 - \rho^2}}\tag{13}
\]
where
\[
A_1 = \frac{1}{\pi} \left( \arcsin \frac{L_1^2 - \rho^2}{r_{s1}^2 - \rho^2} - \arcsin \frac{L_1^2 - \rho^2}{r_{s1}^2} \right)
\]
\[
+ \arcsin \frac{L_1^2 - \rho^2}{r_{s1}^2 - \rho^2} - \arcsin \frac{L_1^2 - \rho^2}{r_{s1}^2},
\]
\[
B_1 = (M_1 - M_2) + \frac{1}{\pi} \left( \arcsin \frac{L_1}{r_{s1}} - \arcsin \frac{L_1}{r_{s2}} \right)
\]
\[
+ \arcsin \frac{L_1}{r_{s1}} - \arcsin \frac{L_1}{r_{s2}},
\]
\[
A_2 = 1 - \frac{1}{\pi} \left( \arcsin \frac{1 - \rho^2}{r_{i1}^2 - \rho^2} + \arcsin \frac{1 - \rho^2}{r_{i2}^2} \right)
\]
\[
B_2 = (M_2 - 1) + \frac{1}{\pi} \left( \arcsin \frac{1}{r_{i1}^2} + \arcsin \frac{1}{r_{i2}^2} \right).
\]

Clearly, the results for the function $s'_1(\rho)$ and the parameters $A_1$ and $A_2$ are formally identical to the results presented in [7] as a solution of the Luneburg inverse problem.

The refractive index $n(r)$ can be calculated by the procedure given in section 2.2. In general, the parameters $A_1$ and $A_2$ are functions of $\rho$, so $n(r)$ can be calculated only numerically. An analytical result can be obtained when the source and image points are located on a unit circle or at infinity and also assuming $r_{s1} = r_{s2}$ and $r_{i1} = r_{i2}$. Then $A_1 = 0$, the parameter $A_2$ becomes independent of $\rho$ and we get an implicit formula
\[
r^{2/B_2} - 2L_1^{1/B_2}(n_{12}(r))^{B_2/2} + (n_{12}(r))^{2A_2/B_2} = 0\tag{15}
\]
for the function $n_{12}(r)$ and a similar formula for the function $n_{11}(r) = \rho(r)/r$.
\[
r = \rho^{A_2 - B_1 - B_2 \left( L_1 - \sqrt{L_1^2 - \rho^2} \right)^{B_1} \left( 1 - \sqrt{1 - \rho^2} \right)^{B_2}}.\tag{16}
\]
However, in most cases the last equation can be solved only numerically.
The described solution of the inverse scattering problem for a gradient-index lens that provides stigmatic imaging of two pairs of concentric spherical surfaces can easily be generalized for cases with multiple pairs of these surfaces. Then the light rays passing through the lens are divided into $N$ bundles specified by $N$ subintervals of angular momentum that are separated by $N-1$ real constants $0 < L_1 < L_2 < \cdots < L_{N-1} < 1$. In the $j$th subinterval of angular momentum, the functions $r_j(L)$, $r_j(L)$ and $\phi(L)$ are given by the quantities $r_{ij}$, $r_{ij}$ and $M_j \pi$, respectively, which represent generalization of the above notation. The refractive index is given by the function $n_1(r)$ in an annular region $r \in [r_+ (L_{j-1}), r_+ (L_j)]$ and the corresponding part of an equivalent geodesic lens is described by the function $s_{ij}(\rho)$ or its derivative $s'_{ij}(\rho)$. Using (6), we get

$$s'_{ij}(\rho) = \sum_{k=0}^{N} \left( A_k + \frac{B_k}{\sqrt{1 - (\rho/L_k)^2}} \right)$$

while we define $L_N = 1$ and the parameters $A_k$ and $B_k$ are given as follows. For $k \in \{1, 2, \ldots, N - 1\}$, we get the parameters $A_k$ and $B_k$ from $A_1$ and $B_1$ listed in (14) by substitution of all indices $1$ with $k$ and $2$ with $k + 1$. Similarly, for $k = N$ we get $A_N$ and $B_N$ from $A_2$ and $B_2$ listed in (14) by substitution of all indices $2$ with $N$. Finally, we calculate the refractive index $n(r)$ by the procedure given in section 2.2.

The general formula (17) for the function $s'_{ij}(\rho)$ together with the formulas for the parameters $A_k$ and $B_k$ obtained by generalization of (14) represent the main result of this section. It proves that the concept of geodesic lenses is a very efficient tool for the design of multi-focal gradient-index lenses.

2.4. Examples

In figure 2 we show examples of multi-focal gradient-index lenses that provide stigmatic imaging of two pairs of concentric spherical surfaces. The bundles of rays specified by two subintervals of angular momentum $[0, L_1]$ and $(L_1, 1]$ are colored in blue and red, respectively, and we refer to the lenses according to the effective behavior of the rays.

The first example is a lens with $A_1 = A_2 = 0, B_1 = B_2 = 1$ and $L_1 = 0.5$ which acts as the combination of a classical and a generalized Maxwell’s fish-eye lens. The second example with $A_1 = 0, A_2 = B_2 = \frac{1}{2}, B_1 = 1$ and $L_1 = 0.5$ is the combination of a classical and a generalized Luneburg lens. The third example with $A_1 = 0, A_2 = B_1 = B_2 = 1, L_1 = 0.65$ acts as the combination of an Eaton lens and an invisible sphere.

The last example with $L_1 = 0.5$ is probably the most interesting one. It effectively acts as the combination of a Luneburg lens and an invisible sphere, so it provides an interesting optical effect. When looking through the lens, the central region shows an undisturbed image of the scene behind while the peripheral region shows a deformed image of the surrounding scene. This effect is apparent from the visualization shown in figure 3 which corresponds to the situation when the lens is placed between an observer and a surrounding scene (for the lens shown in figure 2(d), the observer is on the left-hand side). The image transformation was calculated using the software Mathematica.

3. Absolute instruments

In this section we deal with absolute instruments formed of spherical media. They provide stigmatic imaging of three-dimensional domains in the sense of geometrical optics which is achieved when all the light rays propagate along closed trajectories. We briefly review the inverse scattering problem for a spherical medium with spatially confined light rays (for details see [7]). Then we design so-called multi-focal
Figure 3. A visualization of the optical effect provided by the lens shown in figure 2(d) which effectively acts as the combination of a Luneburg lens and an invisible sphere.

absolute instruments that provide different imaging properties for different light rays.

3.1. Spherical medium with spatially confined light rays

We consider a spherical medium of radius $R > 1$ that is specified in polar coordinates $(r, \varphi)$ by the refractive index $n(r)$. To ensure that all the light rays are spatially confined in the medium (see figure 4(a)) we assume that the function $\rho(r) = nr$ has a single global maximum $\rho(1) = 1$, the value $\rho(0) = 0$ and $\lim_{r\to R} \rho(r) = 0$. Then the inverse function $r(\rho)$ is multivalued; we denote the corresponding branches by $r^{\pm}(\rho)$. The equivalent geodesic lens shown in figure 4(b) is parameterized by the functions $s_1'(\rho)$ and $s_2'(\rho)$ above and below the circle $\rho = 1$ referred to as the equator. For definiteness, we assume that the upper part of the geodesic lens corresponds in the spherical medium to the region inside the unit circle.

A light ray with angular momentum $L$ is in a spherical medium or on a geodesic lens spatially confined to a region bounded by the circles $r_+(L)$ and $r_-(L)$ or $\rho = L$ (one above and one below the equator), respectively, on which the turning points are located. As justified in [7], the angle between two consecutive turning points $\Delta \varphi_{tp}(L)$ is the same for infinitely many geodesic lenses that can be mutually reshaped into each other and that are specified by the functions

$$
\begin{align*}
\dot{s}_1'(\rho) &= s_2'(\rho) + \dot{s}_2'(\rho) \\
\dot{s}_2'(\rho) &= s_1'(\rho) - \dot{s}_2'(\rho)
\end{align*}
$$

(18)

where the functions $\dot{s}_1'(\rho)$ and $\dot{s}_2'(\rho)$ describe the symmetric and antisymmetric parts of a geodesic lens, respectively.

3.2. Inverse scattering problem

The inverse scattering problem for spatially confined light rays lies in the derivation of the refractive index $n(r)$ from the given function $\Delta \varphi_{tp}(L)$. Using (3) and (18), we get the integral equations

$$
\int_{r_-(L)}^{r_+(L)} \frac{L\,dr}{r\sqrt{n^2r^2 - L^2}} = 2 \int_{L}^{1} \frac{L'\,d\rho}{\rho\sqrt{\rho^2 - L^2}} = \Delta \varphi_{tp}(L).
$$

(19)

The procedure for solving (19) with respect to $n(r)$ is given in [6]. Here, we use an alternative approach that lies in solving (19) with respect to $s_1'(\rho)$, choosing $s_2'(\rho)$ and subsequently calculating the refractive index $n(r)$.

Equation (19) for the function $s_1'(\rho)$ is again a kind of Abel integral equation. Assuming that $\Delta \varphi_{tp}(L)$ is an absolutely continuous function on the interval $[0, 1]$, it has a unique solution in the class of Lebesgue-integrable functions

$$
\dot{s}_1'(\rho) = -\frac{\rho}{\pi} \frac{d}{d\rho} \left( \int_{\rho}^{1} \frac{\Delta \varphi_{tp}(L) \,dL}{\sqrt{L^2 - \rho^2}} \right).
$$

(20)

Once we know the shape of a symmetrical geodesic lens, we can choose the function $s_2'(\rho)$ and design a variety of asymmetrical geodesic lenses with a common function $\Delta \varphi_{tp}(L)$.

Figure 4. (a) A spherical medium with spatially confined light rays. (b) An equivalent geodesic lens.
The refractive index \( n(r) \) can be calculated by the method given in section 2.2. For \( r \in [0, 1] \) the method is directly applicable; for \( r \in [1, R] \) it is necessary to use formally identical relations obtained by a simple substitution \( s'_1(\rho) \rightarrow s'_2(\rho) \) and \( r_+(L) \rightarrow r_-(L) \).

3.3. Multi-focal absolute instruments

Let us utilize the general results for discussion of absolute instruments that provide stigmatic imaging of three-dimensional domains.

In our previous paper [7], we have discussed the simplest case when the angle \( \Delta \varphi_p(L) \) is independent of \( L \) and equals a rational fraction of \( \pi \). Then the ray trajectories are closed and the imaging properties are common for all light rays.

Here, we deal with so-called multi-focal absolute instruments that provide different imaging properties for light rays with different angular momenta. Analogously to section 2, we start with the case when the angle \( \Delta \varphi_p(L) \) is piecewise constant on two subintervals of \( L \). We denote

\[
\Delta \varphi_p(L) = \begin{cases} B_1 \pi & \text{for } L \in [0, L_1], \\ B_2 \pi & \text{for } L \in (L_1, 1]. 
\end{cases}
\] (21)

where \( B_1 \geq B_2 \) are positive rational numbers and \( 0 < L_1 < 1 \) (the first condition is analogous to the relation \( M_1 \geq M_2 \) given in section 2.3). Then \( n(r) \) has different functional dependences in four annular regions separated by the circles \( r_+(L_1), 1 \) and \( r_-(L_1) \), respectively. Inside the unit circle we use the notation \( (11) \). Next we denote

\[
n(r) = \begin{cases} n_{22}(r) & \text{for } r \in [1, r_-(L_1)] \\ n_{21}(r) & \text{for } r \in [r_-(L_1), 1]. 
\end{cases}
\] (22)

An equivalent geodesic lens is parameterized by two different parts of the functions \( s_1(\rho) \) and \( s_2(\rho) \). The former is already specified by \( (12) \); the latter is given by

\[
s_2(\rho) = \begin{cases} s_{22}(\rho) & \text{for } \rho \in [L_1, 1] \\ s_{21}(\rho) & \text{for } \rho \in [0, L_1]. 
\end{cases}
\] (23)

where the functions \( s_{22}(\rho) \) and \( s_{21}(\rho) \) parameterize the lower part of a geodesic lens gradually from the equator downwards. Similarly, we split the functions

\[
s_2(\rho) = \begin{cases} s_{2}(\rho) & \text{for } \rho \in [0, L_1] \\ s_{1}(\rho) & \text{for } \rho \in [L_1, 1]. 
\end{cases}
\] (24)

Equipped with the notation, we proceed to solving the inverse scattering problem. Using the general formula \( (20) \) we get the functions

\[
s'_1(\rho) = \frac{B_1 - B_2}{\sqrt{1 - (\rho/L_1)^2}} + \frac{B_2}{\sqrt{1 - \rho^2}}
\]

\[
s'_2(\rho) = \frac{B_2}{\sqrt{1 - \rho^2}}
\] (25)

that describe a symmetrical geodesic lens. Choosing \( s'_1(\rho) \) and \( s'_2(\rho) \) we get a variety of asymmetrical geodesic lenses that are equivalent to multi-focal absolute instruments.

In analogy with the results of section 2.3, we focus on a class of geodesic lenses given by the choice \( s'_1(\rho) = A_1 \) and \( s'_2(\rho) = A_2 \), where \( A_1 \) and \( A_2 \) are non-negative real numbers. Then the geodesic lens is described by a set of functions

\[
s'_1(\rho) = A_1 + \frac{B_1 - B_2}{\sqrt{1 - (\rho/L_1)^2}} + \frac{B_2}{\sqrt{1 - \rho^2}}, \\

s'_2(\rho) = \begin{cases} A_2 + \frac{B_2}{\sqrt{1 - \rho^2}} & \text{for } 0 < \rho < 1 \\
A_2 - \frac{B_2}{\sqrt{1 - \rho^2}} & \text{for } 0 < \rho < 1 \\
A_2 - \frac{B_2}{\sqrt{1 - \rho^2}} & \text{for } 0 < \rho < 1. 
\end{cases}
\] (26)

The refractive index \( n(r) \) can be calculated using equations derived in section 2.3. The functions \( n_{22}(r) \) and \( n_{21}(r) \) are given by a formula obtained from \( (15) \) by a substitution \( A_2 \rightarrow A_2 \) and \( B_2 \rightarrow B_2 \). Similarly, the functions \( n_{11}(r) \) and \( n_{21}(r) \) are given by an equation obtained from \( (16) \) by replacing \( A_1 \rightarrow A_1, B_1 \rightarrow B_1 - B_2 \) and \( B_2 \rightarrow B_2 \).

Similarly to section 2, the solution of the inverse scattering problem for the case of two subintervals of \( L \) can easily be generalized for the cases with \( N \) subintervals separated by \( N - 1 \) real constants \( 0 < L_1 < L_2 < \cdots < L_{N-1} < 1 \). Then \( \Delta \varphi_p(L) \) is given in the \( j \)th subinterval of angular momentum by the quantity \( B_j \pi \). The refractive index is given by the functions \( n_{1j}(r) \) and \( n_{2j}(r) \) inside and outside the unit circle, respectively. An equivalent geodesic lens is specified by the functions \( s_{1j}(\rho) \) and \( s_{2j}(\rho) \) above and below the equator, and its symmetric and antisymmetric parts are described by the functions \( s_{1j}(\rho) \) and \( s_{2j}(\rho) \), respectively. Using \( (20) \), we get

\[
s'_j(\rho) = \sum_{k=j}^{N} \frac{B_k - B_{k+1}}{\sqrt{1 - (\rho/L_k)^2}}
\] (27)

while we define \( L_1 = 1 \) and \( B_{N+1} = 0 \). Choosing the functions \( s'_j(\rho) \), we get a variety of asymmetrical geodesic lenses and, finally, we calculate the refractive indices of the corresponding multi-focal absolute instruments by the procedure given in section 2.2.

The general formula \( (27) \) represents the main result of this section. It proves that the concept of geodesic lenses is an efficient tool for the design of multi-focal absolute instruments. Moreover, the similarity of the formulas \( (17) \) and \( (27) \) reveals that there is a close connection between gradient-index lenses and absolute instruments.

3.4. Examples

In figure 5 we present examples of multi-focal absolute instruments that provide different imaging for the light rays specified by two subintervals of angular momentum \( [0, L_1] \)
Figure 5. Ray tracing and equivalent geodesic lenses for the multi-focal absolute instruments given by the following parameters:
(a) $A_1 = A_2 = 0$, $B_1 = 2$, $B_2 = 1$, (b) $A_1 = 0$, $A_2 = B_2 = 1$, $B_1 = 1$, (c) $A_1 = A_2 = B_2 = 1$, $B_1 = 2$. The black circle corresponds to the circular trajectory at $r = 1$ with maximum possible angular momentum $L = 1$; the dashed circles have the radii $r_{\pm}(L_1)$.

and $(L_1, 1)$. As in section 2.4, the rays are colored in blue and red, respectively.

The first example is given by the parameters $A_1 = A_2 = 0$, $B_1 = 2$, $B_2 = 1$ and $L_1 = 0.8$, so the geodesic lens is symmetrical and its part for $\rho \geq L_1$ is formed of a spherical surface on which the rays propagate along great circles. The refractive index $n(r)$ is given by the Maxwell’s fish-eye profile in an annular region bounded by the dashed circles $r_{\pm}(L_1)$ and the numerically calculated profiles outside this region. The red rays emerging from the point P propagate along the circular trajectories in the dashed bounded annulus and they meet at the image point Q before returning to P. On the other hand, the blue rays also propagate outside the dashed bounded annulus and they meet at the image point T before returning to P. Moreover, the angle between the turning points for the blue rays is twice that of the red rays.

The second example is specified by the parameters $A_1 = 0$, $A_2 = B_2 = 1$, $B_1 = 1$ and $L_1 = 0.95$. The geodesic lens is asymmetrical and the refractive index is given by a Luneburg profile in the dashed bounded annulus. The red rays emerging from the point P propagate along the concentric ellipses and meet at the image point Q before returning to P. The blue rays emerging from P meet at the image point T which is different from Q and the angle between the turning points is twice that of the red rays.

Finally, the third example is given by the parameters $A_1 = A_2 = B_2 = 1$, $B_1 = 2$ and $L_1 = 0.87$. The refractive index is given by an Eaton profile in the dashed bounded annulus, so the light rays propagate in this region along confocal ellipses. The red rays emerging from the point P return back after making one loop around the origin, but the blue rays need to make two loops to close their trajectory.

4. Conclusion

In this paper we have presented a general approach to the design of isotropic spherical media with complex spatial structure that provide different types of imaging for different light rays. This approach based on equivalence of the spherical medium and the corresponding geodesic lens proved to be very efficient for the design of multi-focal gradient-index lenses embedded in an optically homogeneous region as well as multi-focal absolute instruments that provide stigmatic imaging of three-dimensional domains.

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