Splitting Root-Locus Plot into Algebraic Plane Curves

Francisco Mota

Departamento de Engenharia de Computação e Automação
Universidade Federal do Rio Grande do Norte – Brasil
e-mail:mota@dca.ufrn.br

May 15, 2015

Abstract

In this paper we show how to split the root-locus plot for an irreducible rational transfer function into several individual algebraic plane curves, like lines, circles, conics, etc. To achieve this goal we use results of a previous paper of the author to represent the Root Locus as an algebraic variety generated by an ideal over a polynomial ring, and whose primary decomposition allow us to isolate the planes curves that composes the Root Locus. As a by-product, using the concept of duality in projective algebraic geometry, we show how to obtain the dual curve of each plane curve that composes the Root Locus and unite them to obtain what we denominate the “Algebraic Dual Root Locus”.

Index terms— Root-Locus, Projective Root-Locus, Ideal of Polynomials, Primary Decomposition, Real Projective Plane, Dual Algebraic Curve, Grobner Basis.

1 Introduction

Root-Locus (RL) is a parametric plot of the roots of the polynomial \( p(s) = d(s) + k n(s) \) over the complex plane, equivalently over the affine plane \( \mathbb{R}^2 \), as the parameter \( k \) spans \( \mathbb{R} \); \( d \) and \( n \) are fixed coprime polynomials, and \( d \) is monic with degree, in general, greater than the degree of \( n \). The polynomial \( p \) can represent the denominator of a (proportional) control feedback loop of a linear time invariant plant with transfer function \( G(s) = n(s)/d(s) \) (see Figure 1), and this makes the RL a classical approach to study stability and performance of closed loop feedback systems. The rules for sketching the plot are discussed in most textbooks on feedback control theory of linear systems (see [2]). In a previous paper ([1]) the author showed that the RL plot can be extended to the real projective plane \( (\mathbb{P}^2) \) and be interpreted as a projective algebraic variety, that we denominated projective root-locus (PjRL). In this approach, the RL points, including the ones at infinity, can be calculated as the solution of a set of homogeneous polynomial equations, as well as, we can obtain complementary plots of RL over the different affine planes that make up the projective plane.

In this paper we use the approach presented in [1] to show how to decompose the RL into several planes curves that can be plotted independently to form the final RL plot. In fact, at least in some simple cases, we can easily visualize the RL as a union of several plane curves: for example, the plot presented in Figure 2, that represents the RL for \( G(s) = (s + 1)/s^2 \), is composed by the (parametrized) circle \( (x + 1)^2 + y^2 = 1 \) and by the (parametrized) line \( y = 0 \). In order to deal with this question in a systematic way, however, we need concepts from algebraic geometry, considering the RL as an algebraic variety, and the goal is to find its decomposition into irreducible components (see [3 Chap. 4]). We cannot, in general, obtain the irreducibles components of an algebraic variety “by hand”, but considering the curve as the set of zeros of an ideal in a polynomial ring, the question becomes strongly related to the computation of primary decomposition of ideals (see [2 Chaps. 4,7]), a fundamental topic in abstract algebra, and for which computing algorithms there

\[ u \rightarrow k \rightarrow G(s) \rightarrow y \]

Figure 1: Control Feedback Loop with a Proportional Controller
exists since a long time \((3)\). In particular, Macaulay2 package \((4)\) incorporates a command to compute the primary decomposition of an ideal.

We also present in this paper, mainly as a matter of mathematical curiosity, a new root-locus plot that we denominate “Algebraic Dual Root-Locus” or \(\text{(ADRL)}\), associated to the conventional RL plot, that is obtained by computing the dual curve (in projective geometry sense) for each individual plane curve that makes up the projective root-locus. We leave the analysis of the properties of ADRL for a possible future work.

Bellow we present some concepts used in the paper:

\(\mathbb{R}, \mathbb{C}\) and \(\mathbb{R}[x_1, x_2, \ldots, x_n]\): Represents the field of real numbers, the field of complex numbers and the ring of polynomials with coefficient’s in \(\mathbb{R}\) and with indeterminates \((x_1, x_2, \ldots, x_n)\), respectively.

**Homogeneous polynomial** A polynomial (in several variables) is homogeneous when all of its nonzero terms (monomials) have the same total degree (sum of the degree of each variable). We always can turn a non-homogeneous polynomial \((q)\) into a homogeneous one \((q^h)\) by adding a new variable \((x_{n+1})\), with the following procedure: \(q^h(x_1, \ldots, x_n, x_{n+1}) = x_{n+1}^d q(x_1/x_{n+1}, x_2/x_{n+1}, \ldots, x_n/x_{n+1})\), where \(d\) is the total degree of \(q\); this process is denominated “homogenization” of \(q\). We can always “de-homogenize” \(q^h\) by setting \(x_{n+1} = 1\) and recover back \(q\).

**Ideal of Polynomials:** A set of polynomials \(I \subseteq \mathbb{R}[x_1, x_2, \ldots, x_n]\) is an ideal when it satisfies the following properties \((6, 7)\): (i) \(0 \in I\); (ii) \(p, q \in I\) implies \(p + q \in I\); and (iii) \(p \in I\) and \(q \in \mathbb{R}[x_1, x_2, \ldots, x_n]\) implies \(pq \in I\). One important fact about ideals of the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\) is that they are finitely generated, that is, for every ideal \(I\) always there exists a finite subset of polynomials in \(I\), denoted by \(\{p_1, p_2, \ldots, p_t\}\), such that:

\[
I = \sum_{i=1}^{t} h_i p_i, \quad h_i \in \mathbb{R}[x_1, x_2, \ldots, x_n].
\]

The set \(\{p_1, p_2, \ldots, p_t\}\) is denominated a generating set for \(I\); in this case we write \(I = \langle p_1, p_2, \ldots, p_t \rangle\). A Grobner Basis for the ideal \(I\) is a particular kind of generating set that allows many important properties of the ideal to be deduced easily. Given a generating set \(\{p_1, p_2, \ldots, p_t\}\) for \(I\), we can obtain a Grobner basis \(\{g_1, g_2, \ldots, g_s\}\) for \(I\) algorithmically (see \(6\) Ch. 2). The most basic ideals are \((0)\), the zero ideal, and \((1)\), the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\) itself. In fact, if \(1 \in I\), we immediatelly conclude \(I = \mathbb{R}[x_1, x_2, \ldots, x_n]\). We also have that the intersection of any family of ideals results in an ideal. Another important result related to ideals in Noetherian rings (like \(\mathbb{R}[x_1, x_2, \ldots, x_n]\)) is the “Lask-Noether Theorem” which states that every ideal in a Noetherian ring can be written as an finite intersection of primary ideals.

**Variety generated by an ideal:** An (real) algebraic variety is a subset of \(\mathbb{R}^n\) whose elements are the (real) solutions a system of polynomial equations in \(n\) variables (in \(\mathbb{R}[x_1, x_2, \ldots, x_n]\)). We can see this set of polynomials as a generating set of an ideal \(I\), and so we say that the variety is generated by the ideal \(I\), and represented by \(V(I)\). There exists several important relationships between ideals and varieties, in particular: \(V(\{0\}) = \mathbb{R}^n\), \(V(\{1\}) = 0\) and if \(I\) and \(J\) are ideals we have that \(V(I \cap J) = V(I) \cup V(J)\).

For more details about the concepts above see \((6, 7)\).
2 Decomposing projective root-locus into irreducible components

In a previous paper ([1]) the author showed how to extended the RL plot from the affine plane $\mathbb{R}^2$ to the projective plane $\mathbb{P}^2$ by considering the parametric plot roots of the “modified” polynomial

$$p(s) = k_j \hat{d}(s) + k_n n(s)$$

(1)

over $\mathbb{P}^2$ as $k_n/k_d$ spans the projective line $\mathbb{P}^1$. Considering the ideal $I = \langle u, v \rangle$, generated by the polynomials $u = \text{Re}(p(x + iy))$ and $v = \text{Im}(p(x + iy))$, the projective root-locus (PjRL) is obtained from the Grobner basis $\{g_1, g_2, \ldots, g_s\}$ for the ideal $I$, defined in the ring $\mathbb{R}[x, y, k_d, k_n]$, with respect a graded monomial order. If we define the homogenization of the ideal $I = \langle u, v \rangle$ as the ideal $I^h = \langle g_1^h, g_2^h, \ldots, g_s^h \rangle$, where $g_i^h$ is the (homogeneous) polynomial obtained by the homogenization of $g_i$, we can obtain the projective root-locus from the (projective) variety generated by the ideal $I^h$, denoted by $V(I^h)$, where $I^h$ is defined in the ring $\mathbb{R}[x, y, z, k_d, k_n]$ (see [1] for details).

To decompose the PjRL into irreducible components we need to obtain a primary decomposition of $I^h$, in order to write it as a finite intersection of ideals, that is

$$I^h = J_1 \cap J_2 \cap \cdots \cap J_m,$$

(2)

where each $J_i$ is a primary ideal (see [2, Thm. 7.13]). Based on this, we can write $V(I^h)$, the variety generated by $I^h$, as

$$V(I^h) = V(J_1) \cup V(J_2) \cup \cdots \cup V(J_m),$$

where $V(J_i)$ is the variety generated by the primary ideal $J_i$, and it is an irreducible component of the variety $V(I^h)$. We note that, given a generating set for the ideal $I^h$, namely the Grobner basis $\{g_1^h, g_2^h, \ldots, g_s^h\}$, we can obtain a generating set for each primary ideal $J_i$ in (2) by a computational algorithm, like the command “primaryDecomposition” in Macaulay2 package. In this way, we have the following procedure for finding the irreducible components of the PjRL for an irreducible transfer function $G(s) = n(s)/d(s)$:

1. Define $p(s) = k_d \hat{d}(s) + k_n n(s)$ and taking $s = x + jy$, obtain $p(x + jy) = u(x, y, k_d, k_n) + jv(x, y, k_d, k_n)$;
2. Obtain a Grobner basis for the ideal $I = \langle u, v \rangle$, and denote it by $\{g_1, g_2, \ldots, g_s\}$;
3. Let $I^h = \langle g_1^h, g_2^h, \ldots, g_s^h \rangle$, the homogenization of $I$, and obtain the primary decomposition of the ideal $I^h$, as presented in Equation (2);
4. The zeros of the generating set for each $J_i$ in (2) is an irreducible variety, whose union for $i = 1, 2, \ldots, m$ makes up the PjRL.

2.1 Examples

In all examples below we used the Macaulay2 software ([1]) to make the calculations and all polynomials are defined with coefficients in the field of rationals (that is in $\mathbb{Q}[x_1, x_2, \ldots, x_n]$) so that we can get infinite precision in calculations.

Example 2.1. Let be $G(s) = (s + 1)/s^2$, whose RL plot is shown in Figure 2 and

$$p(s) = k_d x^2 + k_n (s + 1).$$

Defining $u = \text{Re}(p(x + jy))$ and $v = \text{Im}(p(x + jy))$ we have:

$$u(x, y, k_d, k_n) = k_d (x^2 - y^2) + k_n (x + 1), \quad v(x, y, k_d, k_n) = 2k_d xy + k_n y.$$

Now we compute the Grobner basis for the ideal $\langle u, v \rangle$ using the graded reversed lexicographic order with $x > y > k_d > k_n$ and obtain $\{g_1, g_2, g_3, g_4\}$, where:

$$g_1(x, y, k_d, k_n) = 2xy k_d + yk_n \quad (= r)$$
$$g_2(x, y, k_d, k_n) = x^2 k_d - y^2 k_d + xk_n + k_n \quad (= q)$$
$$g_3(x, y, k_d, k_n) = x^2 yk_n + y^2 k_n + 2xy k_n$$
$$g_4(x, y, k_d, k_n) = 2y^3 k_d - xyk_n - 2yk_n$$

Homogenizing of the polynomials $g_i$, using the procedure indicated in the Introduction we obtain:

$$g_1^h = z^3 g_1(x/z, y/z, k_d/z, k_n/z) = 2xy k_d + yzk_n$$
$$g_2^h = z^3 g_2(x/z, y/z, k_d/z, k_n/z) = x^2 k_d - y^2 k_d + xzk_n + z^2 k_n$$
$$g_3^h = z^4 g_3(x/z, y/z, k_d/z, k_n/z) = x^2 yk_n + y^2 k_n + 2xy z k_n$$
$$g_4^h = z^4 g_4(x/z, y/z, k_d/z, k_n/z) = 2y^3 k_d - xy zk_n - 2yz^2 k_n$$

3
Now we compute the primary decomposition for the ideal \( I^h = \langle g_1^h, g_2^h, g_3^h, g_4^h \rangle \) to obtain \( I^h = J_1 \cap J_2 \cap J_3 \), where:

\[
\begin{align*}
J_1 &= \langle y, x^2k_d + xzk_n + z^2k_n \rangle \\
J_2 &= \langle x^2 + y^2 + 2xz, 2xk_d + zk_n \rangle \\
J_3 &= \langle k_d, k_n \rangle
\end{align*}
\]

Using the fact that \( k_d \) and \( k_n \) belong to the set of reals and that they can’t both simultaneously zero, we have that \( 1 \in J_3 \) (suppose \( k_n \neq 0, \) so \( (1/k_n) \times k_n = 1 \in J_3 \)) and then \( J_3 = \mathbb{R}[x, y, z, k_d, k_n] \) can be deleted from the primary decomposition of \( I^h \), that is, \( I^h = J_1 \cap J_2 \). Then we have that \( \mathbb{V}(I^h) = \mathbb{V}(J_1) \cup \mathbb{V}(J_2) \), where \( \mathbb{V}(J_1) \) is defined by \( y = 0 \) and \( \mathbb{V}(J_2) \) is defined by \( x^2 + y^2 + 2xz = 0 \). To analyze these varieties in the affine \( XY \) plane we set \( z = 1 \) and we obtain the components of the plot shown in Figure 2, that is the line \( y = 0 \) and the circle \((x + 1)^2 + y^2 = 1\), as desired. Also, from the ideals \( J_1 \) and \( J_2 \) above we can obtain the parametrization of these curves as well as the initial and terminal points of the PjRL, as was done by the author in [1]:

**Variety \( \mathbb{V}(J_1) \):** Defined by the ideal \( J_1 \), as shown in Equation (3)

- Initial Points: \( k_d = 1 \) and \( k_n = 0 \). From (3), we get \( y = 0 \) and \( x^2 = 0 \) or \( x = 0 \). Therefore the initial point for \( \mathbb{V}(J_1) \) is \((0 : 0 : 1) \) or \((0, 0)\) in affine plane \( XY \).
- Terminal points: \( k_d = 0 \) and \( k_n = 1 \). We get from (3), \( y = 0 \) and \( xz + z^2 = 0 \). Then we have (a) \( z = 0 \) and \( x = 1 \), which is the point at infinity \((1 : 0 : 0)\) (horizontal lines) and (b) \( z = 1 \) which implies \( x = -1 \) and the point is \((-1 : 0 : 1) \) or \((-1, 0)\) in affine plane \( XY \).
- Intermediary Points: \( k_d = 1 \) and \( k_n = \lambda \neq 0 \). Again from (3), we have \( y = 0 \) and \( x^2 + xz \lambda + z^2 \lambda = 0 \), and we must have \( z = 1 \) (\( z = 0 \) would imply \( x = 0 \) what is impossible); so, all intermediary points are at finite position and is given by \( y = 0 \) and \( x^2 + x \lambda + \lambda = 0, \lambda \neq 0 \). Then \( x = -\lambda \pm \sqrt{\lambda^2 - 4\lambda} \), which give us the RL over \( y = 0 \) line.

**Variety \( \mathbb{V}(J_2) \):** Defined by the ideal \( J_2 \), as shown in Equation (4)

- Initial Points: \( k_d = 1 \) and \( k_n = 0 \). From (4), we get \( x^2 + y^2 + 2xz = 0 \) and \( 2x = 0 \) or \( x = 0 \). Then we have \( x^2 + y^2 = 0 \) or \( y = 0 \). Therefore the initial point for \( \mathbb{V}(J_2) \) is \((0 : 0 : 1) \) or \((0, 0)\) in affine plane \( XY \).
- Terminal points: \( k_d = 0 \) and \( k_n = 1 \). We get from (4), \( x^2 + y^2 + 2xz = 0 \) and \( z = 0 \). Then we get \( x^2 + y^2 = 0 \) what implies \( x = y = z = 0 \) which is not allowed, so this variety has no terminal points.
- Intermediary Points: \( k_d = 1 \) and \( k_n = \lambda \neq 0 \). Again from (4), we have \( x^2 + y^2 + 2xz = 0 \) and \( 2x + \lambda z = 0 \), and we must have \( z = 1 \) (\( z = 0 \) would imply \( x = y = 0 \) what is impossible); so, all intermediary points are at finite position and is given by \( x^2 + y^2 + 2x = 0 \) and \( 2x + \lambda = 0, \lambda \neq 0 \), which is the parametrized equation of the circle as shown in RL plot.

Now, since the complete PjRL is \( \mathbb{V}(J_1) \cup \mathbb{V}(J_2) \), we have:

- Initial points: \((0 : 0 : 1)\) (from \( \mathbb{V}(J_1) \)) plus \((0 : 0 : 1)\) (from \( \mathbb{V}(J_2) \)); so we have a duplicate point at \((0 : 0 : 1)\) or at \((0, 0)\) in affine plane \( XY \).
- Terminal Points: \{(1 : 0 : 0), (-1 : 0 : 1)\}, only from \( \mathbb{V}(J_1) \)
- Intermediary points: \{(x : 0 : 1), x = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\lambda}}{2}, \lambda \neq 0\} from \( \mathbb{V}(J_1) \) plus \{(x : y : 1), x = -\lambda/2, \lambda \neq 0 \) and \( x^2 + y^2 + 2x = 0 \)\} from \( \mathbb{V}(J_2) \).

**Example 2.2.** We now consider a modification of Example 2.1 above by defining \( G(s) = \frac{s + 1}{s^2(s + 4)} \), whose RL plot is shown in Figure 3. Also in this case we see that the RL is the union of the line \( y = 0 \) and the “weird” curve shown in red. In this case:

\[
p(s) = k_d(s^3 + 4s^2) + k_n(s + 1)
\]

and we have

\[
\begin{align*}
\text{Re}(p(x + jy)) &= k_d(x^3 - 3xy^2 + 4x^2 - 4y^2) + k_n(x + 1) \\
\text{Im}(p(x + jy)) &= k_d(-y^3 + 3x^2y + 8xy) + k_n y.
\end{align*}
\]
The Grobner basis for the ideal $I$ and computing the primary decomposition for $J$.

Once more, the projective plane

3 Algebraic Dual Root-Locus – ADRL

Duality is a fundamental concept in projective algebraic geometry. In fact, it is a basic property of the real projective plane $\mathbb{RP}^2$ that a “point” with (nonzero) homogeneous coordinate $(u : v : w)$ can be associated to a “line” with equation $ux + by + cz = 0$ in and vice-versa. This kind of duality can be extended from a projective line to a projective plane curve $(W)$ defined by $f(x, y, z) = 0$, where $f(x, y, z) \in \mathbb{R}[x, y, z]$ is a homogeneous polynomial. By the natural duality between lines and points in $\mathbb{RP}^2$, each tangent line to the curve can be associated to a point with homogeneous coordinate, for instance, $(u : v : w)$, and the main result is that this set of points is also the solution to some equation $g(u, v, w) = 0$, where $g(u, v, w) \in \mathbb{R}[u, v, w]$ is a homogeneous polynomial, that represents a curve $(W^*)$ over $\mathbb{RP}^2$ (in fact over the dual of $\mathbb{RP}^2$, which it is itself). Therefore $W^*$ is denominated the dual curve of $W$. Interestingly, we also have that if we take the dual of the dual of a curve we restore back the original curve, that is $(W^*)^* = W$ (see [3]). Mathematically, the dual curve of $f(x, y, z) = 0$ is the set of points $(u : v : w) = (\partial f/\partial x : \partial f/\partial y : \partial f/\partial z)$ in $\mathbb{P}^2$, or $u = \lambda(\partial f/\partial x), v = \lambda(\partial f/\partial y)$ and $w = \lambda(\partial f/\partial z)$, for some $\lambda \neq 0$. To find the curve which these points belongs to, we can restate the problem as the one of eliminating $x, y, z$ and $\lambda$ from the set of equations $f(x, y, z) = 0, u - \lambda(\partial f/\partial x) = 0, v - \lambda(\partial f/\partial y) = 0$ and $w - \lambda(\partial f/\partial z) = 0$, which, in turn, can be solved by finding a Grobner basis for the ideal

$$I = \left\langle f(x, y, z), u - \lambda\frac{\partial f}{\partial x}, v - \lambda\frac{\partial f}{\partial y}, w - \lambda\frac{\partial f}{\partial z} \right\rangle$$
In our context, we are primarily interested in obtaining the dual curve for each irreducible component of the RL, as computed in Section 2 above, and collate them to construct a new RL plot that we denominate “Algebraic Dual Root-Locus” (ADRL). So, we will calculate the ideal defined in (9) for the examples presented in Section 2.

3.1 Examples

Example 3.1. Let be \( G(s) = (s + 1)/s^2 \), whose RL plot is shown in Figure 2 and we found in Example 2.1 that the \( P\)RL can be represented by the set of ideals in Equations (3,4), that is

\[
\begin{align*}
J_1 &= \langle y, x^2k_d + xzk_n + z^2k_n \rangle \\
J_2 &= \langle x^2 + y^2 + 2xz, 2xk_d + zk_n \rangle
\end{align*}
\]

(10) (11)

To find the dual curve of \( V(J_1) \), we have to calculate the dual curve of \( f(x, y, z) = x^2k_d + xzk_n + z^2k_n \), and therefore the ideal in Equation (9) for this curve is:

\[
I = \langle x^2k_d + xzk_n + z^2k_n, u - \lambda(2xk_d + zk_n), v, w - \lambda(zk_n + 2zk_n) \rangle
\]

Finding a Grobner basis for this ideal with “Lex” monomial ordering with \( x > y > z > k_d > k_n > \lambda > u > v > w \) we eliminate \( x, y, z, \lambda \) and obtain the curve:

\[
g(u, v, w) = k_d u^2 + k_n u^2 - k_n uw
\]

So we can define the “dual ideal” for \( J_1 \) as:

\[
J_1^d = \langle v, k_d u^2 + k_n u^2 - k_n uw \rangle
\]

(12)

and the dual curve for \( V(J_1) \) is \( V(J_1^d) \).

To find the dual curve of \( V(J_2) \), we have to calculate the dual curve of \( f(x, y, z) = x^2 + y^2 + 2xz \), and therefore the ideal in Equation (9) for this curve is:

\[
I = \langle x^2 + y^2 + 2xz, u - \lambda(2x + 2z), v - \lambda(2y), w - \lambda(2x) \rangle
\]

Finding a Grobner basis for this ideal, to eliminate \( x, y, z, \lambda \), we obtain

\[
h(u, v, w) = v^2 + 2uw - w^2
\]

To find the parametrization for \( h \) above we use the ideal as shown below

\[
I = \langle 2xk_d + zk_n, u - \lambda(2x + 2z), v - \lambda(2y), w - \lambda(2x) \rangle
\]

and finding a Grobner basis for this ideal, to eliminate \( x, y, z, \lambda \), we obtain the (parametrized) curve

\[
h_1(u, v, w) = k_n u + (2k_d - k_n) w
\]

and then we can define the again “dual ideal” for \( J_2 \) as:

\[
J_2^d = \langle v^2 + 2uw - w^2, k_n u + (2k_d - k_n) w \rangle
\]

(13)

Now we can define the Algebraic Dual Root-Locus for \( G(s) = (s + 1)/s^2 \) as the union of the varieties \( V(J_1^d) \) and \( V(J_2^d) \) where \( J_1^d \) and \( J_2^d \) are defined in (12) and (13), respectively. Below we analyse each variety in order to obtain the complete plot for the ADRL:

**Variety** \( V(J_1^d) \): Defined by the ideal \( J_1^d \), as shown in Equation (12)

- Initial Points: \( k_d = 1 \) and \( k_n = 0 \). From (12), we get \( v = 0 \) and \( w^2 = 0 \) or \( w = 0 \). Therefore the initial point for \( V(J_1) \) is \((1 : 0 : 0)\), a point at infinity, or the intersection of horizontal lines in affine \( UV \) plane.
- Terminal points: \( k_d = 0 \) and \( k_n = 1 \). We get from (12), \( v = 0 \) and \( w^2 - uw = 0 \). Then we have
  - (a) \( u = 0 \) and \( w = 1 \), which is the point \((0 : 0 : 1)\), or the point \((0,0)\) in affine plane \( UV \) and
  - (b) \( u = w = 1 \) which is the point \((1 : 0 : 1)\) or \((1,0)\) in affine \( UV \) plane.
- Intermediary Points: \( k_d = 1 \) and \( k_n = \lambda \neq 0 \). Again from (12), we have \( v = 0 \) and \( \lambda u^2 - \lambda uw + w^2 = 0 \), and we must have \( w = 1 \) \((w = 0 \) would imply \( u = 0 \) what is impossible); so, all intermediary points are at finite position \((w = 1)\) and is given by \( v = 0 \) and \( w^2 - u + 1/\lambda = 0 \), \( \lambda \neq 0 \). Then \( u = \frac{1+\sqrt{1-4\lambda}}{2} \), which give us the ADRL over \( v = 0 \) line in affine \( UV \) plane.

**Variety** \( V(J_2^d) \): Defined by the ideal \( J_2^d \), as shown in Equation (13)
We now consider Example 3.2.

Now, since the complete ADRL is \(\mathcal{V}(J_1^d) \cup \mathcal{V}(J_2^d)\), we have:

- Initial points: \((1 : 0 : 0)\) (from \(\mathcal{V}(J_1^d)\)) plus \((1 : 0 : 0)\) (from \(\mathcal{V}(J_2^d)\)); so we have a duplicate point at \((1 : 0 : 0)\) or at infinity in affine plane \(UV\).
- Terminal Points: \(\{(0 : 1 : 1)\}\) only from \(\mathcal{V}(J_1^d)\); or \((0 : 0)\) and \((1, 0)\) in \(UV\) plane
- Intermediary points: \(\{(u : v : 1), (1 : 0 : 0)\}\) from \(\mathcal{V}(J_1^d)\) plus \(\{(u : v : 1), u = 1 - 2/\lambda, \lambda \neq 0\}\) and \(v^2 + 2u - 1 = 0\) from \(\mathcal{V}(J_2^d)\), in \(UV\) plane.

The ADRL plot for \(G(s) = (s + 1)/s^2\) is shown in Figure 4.

It is important to note that if we take the duals of \(J_1^d\) and \(J_2^d\), defined in Equations (12) and (13), respectively, we get back, the ideals \(J_1\) and \(J_2\), as defined in Equations (10) and (11), respectively. The procedure for doing that is eliminating \(u, v, w\) and \(\lambda\) in the ideals \(I_1, I_2\) and \(I_3\) (defined below), using Lex monomial ordering with \(u > v > w > x > y > z > k_d > k_n > \lambda\).

\[
\begin{align*}
I_1 &= \langle k_d w^2 + k_n u^2 - k_n u w, x - \lambda(2 u k_n - k_n w), y, z - \lambda(2 k_d w - k_n w) \rangle \\
I_2 &= \langle v^2 + 2 u w - w^2, x - \lambda(2 w), y - \lambda(2 v), z - \lambda(2 u - 2 w) \rangle \\
I_3 &= \langle k_n u + (2 k_d - k_n) w, x - \lambda(2 w) y - \lambda(2 v), z - \lambda(2 u - 2 w) \rangle
\end{align*}
\]

**Example 3.2.** We now consider \(G(s) = \frac{s + 1}{s^2(s + 4)}\), whose RL plot is shown in Figure 3 and whose PJRL is represented by the ideals in Equations (10):

\[
\begin{align*}
J_1 &= \langle y, x^3 k_d + 4 x^2 z k_d + x z^2 k_n + z^3 k_n \rangle \\
J_2 &= \langle 2 x^3 + 2 x y^2 + 7 x^2 z + 3 y^2 z + 8 x z^2, 3 x^2 k_d - y^2 k_d + 8 x z k_d + z^2 k_n \rangle
\end{align*}
\]

Repeating the reasoning used in Example 3.1 above, we obtain the “duals” ideals:

\[
\begin{align*}
J_1^d &= \langle v, 4 k_d w u^3 - k_d w^3 + k_n u^3 - k_n u^2 w \rangle \\
J_2^d &= \langle f(u, v, w), g(u, v, w) \rangle
\end{align*}
\]
investigate the properties of the ADRL more deeply in future works. We intend to compute the dual curve (in projective algebraic geometry sense) to each plane curve and join them to compose what we denominated "Algebraic Dual Root Locus" (ADRL). Some examples were worked out in order to show the effectiveness of the procedure. We also showed how to compute the dual curve (ADRL). The plot for an irreducible transfer function. This procedure can be easily implemented in a computational algebra software package. We have presented in this paper a procedure for isolating the planes curves that makes up the root-locus plot for an irreducible transfer function. This procedure can be easily implemented in a computational algebra software package.

where $f$ and $g$ are the “astonishing” polynomials. The plot for the correspondig ADRL, obtained by the Scilab package (\cite{scilab}) is shown in Figure 5.

\begin{align*}
f(u, v, w) &= 216u^5w + 144u^4v^2 - 621u^3v^2w - 912u^2v^2w^2 + 720uv^2w^3 - 352uv^2w^2 + 718u^2v^2w^2 - 442uw^4 - 232uw^4w - 176uw^2w^3 + 128uw^5 - 240u^6 + 107v^4w^2 - 8x^2w^4 - 16uw^6 \\
g(u, v, w) &= 294912k_0^4u^3w - 327680k_0^4v^2 - 798720k_0^4u^2w^2 + 1671168k_0^4uw^2w + 512000k_0^4uw^3 - 1048576k_0^4v^2 - 675840k_0^4v^2w^2 - 96000k_0^4uw^4 + 73728k_0^4u^4 + 448512k_0^4u^4w^2 + 430080k_0^4k_0^4u^2w^2 - 788480k_0^4k_0^4uw^2w - 403200k_0^4k_0^4uw^3 + 131072k_0^4k_0^4v^4 + 300288k_0^4k_0^4v^2w^2 + 75600k_0^4k_0^4uw^4 - 46656k_0^4k_0^4u^4 + 174816k_0^4k_0^4u^2w^2 - 68928k_0^4k_0^4v^2w^2 - 224292k_0^4k_0^4v^2w^2 - 109920k_0^4k_0^4uw^4w + 119040k_0^4k_0^4uw^5w - 6144k_0^4k_0^4v^2 - 41364k_0^4k_0^4v^2w^2 - 22320k_0^4k_0^4uw^4w + 8208k_0^4k_0^4uw^2w - 2617k_0^4k_0^4u^2w^2 + 308k_0^4k_0^4u^2w^2 + 30636k_0^4k_0^4u^2w^2 - 4764k_0^4k_0^4uw^2w - 15616k_0^4k_0^4uw^3w + 128k_0^4k_0^4u^3w - 178k_0^4k_0^4u^3w^2 - 2928k_0^4k_0^4u^3w^2 - 441k_0^4u^4w^2w - 42k_0^4u^2w^2 - 1528k_0^4u^2w^2 + 64k_0^4u^2w^2 + 768k_0^4uw^3w - k_0^4u^2w^2 - 24k_0^4u^2w^2 - 144k_0^4w^4 \\
\end{align*}

Remembering that the dual curve for $2x^3 + 2xy^2 + 7x^2z + 3y^2z + 8xz^2 = 0$ is $f(u, v, w) = 0$ and that $g(u, v, w) = 0$ is used just for obtaining a parametrization. The plot for the correspondig ADRL, obtained by the Scilab package (\cite{scilab}) is shown in Figure 5.

4 Conclusions

We have presented in this paper a procedure for isolating the planes curves that makes up the root-locus plot for an irreducible transfer function. This procedure can be easily implemented in a computational algebra software package. We also showed how to compute the dual curve (in projective algebraic geometry sense) to each plane curve and join them to compose what we denominated “Algebraic Dual Root Locus” (ADRL). Some examples were worked out in order to show the effectiveness of the procedure. We intend to investigate the properties of the ADRL more deeply in future works.

\footnote{In fact, a well known result in projective algebraic geometry states that if a curve has degree $d$ (and no singularities) its dual has degree $d(d - 1)$ \cite{projectiveGeometry} pp. 173; in our case the polynomial in the ideal $J_2$ has degree 3, so its dual has degree 6.}
References

[1] F. Mota, *Projective Root-Locus: An Extension of Root-Locus Plot to the Projective Plane*. arXiv:1409.4476 [cs.SY], 2014.

[2] J. D’Azzo and C. Houpis, *Linear Control System Analysis and Design*. Second Edition. MacGraw-Hill Kogakusha, Ltd., 1981.

[3] Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. 22 July 2004. Web. 29 April, 2015. Available at [http://en.wikipedia.org/wiki/Primary_decomposition](http://en.wikipedia.org/wiki/Primary_decomposition).

[4] D. Grayson and M. Stillman. Macaulay2, A Software System for Research in Algebraic Geometry. Available at [http://www.math.uiuc.edu/Macaulay2](http://www.math.uiuc.edu/Macaulay2).

[5] Scilab Enterprises. Scilab: Free and Open Source Software for Numerical Computation. Orsay, France, 2012. Available at [http://www.scilab.org](http://www.scilab.org).

[6] D. Cox, J. Little and D. O’Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Second Edition. Springer-Verlag New York Inc., 1997.

[7] M. F. Atiyah and I. G. MacDonald. *Introduction to Commutative Algebra*. West View Press, 1969.

[8] J. Gray. *Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century*. Springer-Verlag London Limited, 2007, 2011.