Lattices of Matrices*

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ABSTRACT: We study a new class of matrix models, formulated on a lattice. On each site are $N$ states with random energies governed by a Gaussian random matrix Hamiltonian. The states on different sites are coupled randomly. We calculate the density of and correlation between the eigenvalues of the total Hamiltonian in the large $N$ limit. We find that this correlation exhibits the same type of universal behavior we discovered recently. Several derivations of this result are given. This class of random matrices allows us to model the transition between the “localized” and “extended” regimes within the limited context of random matrix theory.

We formulate and study a class of matrix models defined in the simplest version by the random Hamiltonian

$$H = \begin{pmatrix} H_1 & V \\ V^\dagger & H_2 \end{pmatrix}$$

(1)

taken from the Gaussian distribution

$$P(H) = \frac{1}{Z} e^{-\text{Tr}(\frac{1}{2}(m_1^2 H_1^2 + m_2^2 H_2^2) + M^2 V^\dagger V)}.$$  

(2)

Here $H_{1,2}$, and $V$ denote $N$ by $N$ matrices, and $H$ a $2N$ by $2N$ matrix, with $H$ and $H_{1,2}$ hermitean. The parameters $m_{1,2}^2$ and $M^2$ are both treated as order $N^0$ in the large $N$ limit.
A Hamiltonian of the form in (1) with $V$ equal to zero would naturally arise when a conserved symmetry prevents two sectors in the Hilbert space of states to mix. The off-diagonal coupling $V$ would then represent symmetry-breaking effects.

It is natural to generalize our model immediately to contain $C$ sectors which mix due to some symmetry-breaking effects. The Hamiltonian $H$ in (1) is generalized from a 2 by 2 block matrix to a $C$ by $C$ block matrix. Each block is an $N$ by $N$ matrix, which we will denote by $H_{\alpha\beta}$ where $\alpha$, $\beta$, ... run from 1 to $C$. The distribution in (2) is generalized to

$$P(H) = \frac{1}{Z} e^{-N \sum_{\alpha,\beta} \frac{1}{2} \mathcal{M}_{\alpha\beta}^2 \text{tr}(H_{\alpha\beta}H_{\beta\alpha})}$$

(3)

Here $\mathcal{M}^2$ represents a $C$ by $C$ real symmetric matrix. The distribution considered in (2) corresponds to the case $C = 2$ with $\mathcal{M}^2_{11} = m_1^2$, $\mathcal{M}^2_{22} = m_2^2$, and $\mathcal{M}^2_{12} = \mathcal{M}^2_{21} = M^2$.

It is natural also to think of the $C$ sectors as $C$ sites on a lattice. On site $\alpha$ live $N$ states with energies determined by $H_{\alpha\alpha}$. The states on sites $\alpha$ and $\beta$ are coupled by $H_{\alpha\beta}$. The matrix $\mathcal{M}^2$ determines the “connectivity” of the lattice. The standard matrix model corresponds to the case of a lattice with one site. With this lattice interpretation, this model is essentially the same as the model proposed and studied by Wegner\textsuperscript{1} some years ago. Our results overlap those of Wegner; nevertheless it may be fruitful to approach the model from a somewhat different point of view and within the context of our recent study of universal correlation in matrix models.\textsuperscript{2,3,4}

This class of models does not appear to be analyzable by the standard method of orthogonal polynomials. On the other hands, the density and correlation in this class of models can be readily obtained using the diagrammatic approach recently developed by us.\textsuperscript{4} The counting and summing of diagrams provides a somewhat amusing exercise in perturbative field theory.

We note in passing how we are led to consider such a model. In localization theory it is well known that the distribution of spacing between neighboring energy levels changes character as one varies the energy from the localized to the extended regime. The physics behind this change is clear. Extended wave functions overlap in space, and any perturbation would have a significant matrix element between two wave functions close together in energy. The familiar phenomenon of level repulsion as described by second order perturbation theory tells us that the likelihood of having neighboring energy levels separated by an energy $s$ becomes vanishingly small as $s$ goes to zero. This simple argument, essentially due to Wigner\textsuperscript{5} and to Landau and Smorodinsky,\textsuperscript{6} is known as the Wigner surmise. We expect
the probability $p(s)$ of a spacing $s$ between neighboring energy levels to rise from zero for $s = 0$, reach a peak, and then to decrease rapidly\textsuperscript{7}. The curve traced out by $p(s)$ is sometimes referred\textsuperscript{8} to informally as “Wigner’s head.” On the other hand, localized wave functions have little overlap with each other, and thus do not repel each other significantly. In the localized regime $p(s)$ simply decreases from some finite value as $s$ increases from zero, following a Poisson distribution. The curve traced out by $p(s)$ is known informally as “Poisson’s tail.”

Recently, there have been some attempts\textsuperscript{9,10} to incorporate this transition from the “Wigner regime” to the “Poisson regime” in the context of the theory of random matrices.\textsuperscript{11,7,12} In matrix models the concept of space does not appear of course; nevertheless one can ask whether the distribution of eigenvalues, as manifested in the density of, and the correlation between, eigenvalues may exhibit the type of transition described above for localization theory. In particular, Moshe, Neuberger, and Shapiro\textsuperscript{10} proposed a new class of matrix models with a parameter they called $b$ such that as $b$ varies the distribution of eigenvalues goes from a Wigner to a quasi-Poisson distribution. We found their model somewhat unsatisfactory in that the parameter $b$ has to be order $N^2$ larger than the other parameters in the model, where $N$ as usual in discussions of matrix models denotes the size of the matrices. This actually reflects the universality of the Wigner distribution, that is, roughly speaking, its tendency to resist change. We feel that the model studied here does not suffer from this difficulty.

Take the $C = 2$ case as an example. We note that at the special value $M^2 = m_1^2 = m_2^2 = 2m_{eff}^2$ (which we will refer as the Wigner point), the distribution (2) collapses to

$$P(H) = \frac{1}{Z} e^{-2NtrH^2}$$

the standard Gaussian distribution for $2N$ by $2N$ matrices and we should recover the results of Ref. 4 as a check. For $M^2 = \infty$, the sectors 1 and 2 clearly decouples, and the spacing distribution is trivially Poissonian in the sense described above. We will refer to this as the Poisson point. For $M^2 < \infty$, the off-diagonal perturbation $V$ generates level repulsion, and the spacing distribution should be Wignerian.

We now calculate the density of eigenvalues and the correlation between eigenvalues using a diagrammatic method we developed recently.\textsuperscript{4} As usual, these quantities can be obtained from the Green’s functions

$$G(z) \equiv \left\langle \frac{1}{CN} tr \frac{1}{z - H} \right\rangle_c$$

$$G(z, w)_c \equiv \left\langle \frac{1}{CN} tr \frac{1}{z - H} \frac{1}{CN} tr \frac{1}{w - H} \right\rangle_c$$

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where \( \langle 0(H) \rangle \equiv \int dH \ 0(H)P(H) \) and the subscript \( c \) indicates the connected Green’s function. The density of eigenvalues is then given by

\[
\rho(\mu) = \left\langle \frac{1}{N} \text{tr} \delta(\mu - H) \right\rangle = -\frac{1}{\pi} \text{Im} G(\mu + i\epsilon)
\]

(7)

and the correlation between eigenvalues, by

\[
\rho_c(\mu, \nu) = \left\langle \frac{1}{N} \text{tr} \delta(\mu - H) \frac{1}{N} \text{tr} \delta(\nu - H) \right\rangle_c
\]

\[
= \left( -\frac{1}{4\pi^2} (G_c(++) + G_c(--) - G_c(+-) - G_c(-+)) \right)
\]

(8)

with the obvious notation \( G_c(\pm, \pm) \equiv G_c(\mu \pm i\epsilon, \nu \pm i\delta) \) (signs uncorrelated).

The Feynman diagram expansion corresponds to an expansion in powers of \( 1/z \) and \( 1/w \). Let us illustrate with the correlation function and expand

\[
(CN)^2 G_c(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{z^{m+1}w^{n+1}} \left\langle \text{tr} H^m \text{tr} H^n \right\rangle_c
\]

(9)

The calculation proceeds along the same line as in Ref. 4 suitably generalized to include the block indices \( \alpha, \beta, \ldots \). Diagrammatically, we may borrow the terminology of large N QCD and describe the expression for \( G_c(z, w) \) as two separate quark loops, of type \( z \) and type \( w \) respectively, interacting by emitting and absorbing gluons. Now each gluon line also carries two block indices \( \alpha\beta \). Some readers may find it natural to think of the \( C \) block indices as representing a second “color,” or “technicolor” index. We take the limit \( N \) large for fixed \( C \). With the Gaussian distribution in (3) we can readily “Wick-contract” (9). The gluon propagator is given by

\[
\left\langle H_{\alpha\beta}^{ij} H_{\gamma}^{kl} \right\rangle = \frac{1}{N} \delta_{il} \delta_{jk} \delta_{\alpha\epsilon} \delta_{\gamma\beta} \sigma_{\alpha\beta}
\]

(10)

where we have defined the \( C \) by \( C \) real symmetric matrix \( \sigma \) by \( \sigma_{\alpha\beta} = 1/\mathcal{M}^2_{\alpha\beta} \). (Note that the matrix \( \sigma \) is not the inverse of the matrix \( \mathcal{M}^2 \).) Thus, the gluon is represented by a double line while a quark is represented by a single line. This convention greatly facilitates counting the powers of \( N \) as explained in Ref. 4.

Let us begin by calculating the propagator, which is easily seen diagrammatically to be diagonal in block indices. Define \( g_\alpha \) by

\[
G_{\alpha\beta,ij} = \left\langle \frac{1}{z-H} \right\rangle_{\alpha\beta,ij} \equiv \delta_{\alpha\beta} \delta_{ij} g_\alpha
\]

(11)

In the large \( N \) limit, the sum of the leading planar diagrams (“generalized rainbow” diagrams) is determined by the self-consistent equation

\[
g_\alpha = \frac{1}{z - \sum_\beta \sigma_{\alpha\beta} g_\beta}
\]

(12)
For general $M^2$ this represents a system of coupled non-linear equations to be solved for $g_\alpha$. It may be interesting to study the solution numerically for arbitrary $M^2$.

We are now ready to calculate $G_c(z, w)$. As in Ref. 4 we begin by ignoring in (9) contractions within the same trace (in which case $m$ and $n$ are required to be equal). In the large $N$ limit, the dominant graphs are given essentially by “ladder graphs” (with one crossing). We now have a combinatorial problem which is most easily solved by distorting the ladder graph in Fig. (1a) to the “wheel” graph in Fig. (1b). (Here we admit distortions as long as they preserve the combinatorial factor.) We see that each “spoke” of the wheel is associated with a factor of $\sigma_{\alpha\beta}$. Thus, the graphs with $n$ loops should be multiplied by the factor $tr\sigma^n$. We obtain

$$(CN)^2G_c(z, w) = \sum_{n=0}^{\infty} \frac{ntr\sigma^n}{(zw)^{n+1}}$$

We next include Wick-contractions within the same trace in $\langle trH^mtrH^n \rangle$. Graphically these contractions correspond to decorating the ladder graphs by vertex and self energy corrections. First, we have to correct the upper part of the graph in Fig. (1a) with vertex corrections, as shown in Fig. (2), and similarly for the lower part of the graph. These vertex corrections lead to a rather complicated expression. For example, the factor $tr\sigma^n$ in (13) for $n = 3$ is to be replaced by the sum of terms, one of which has the form

$$\sum_{\alpha\rho\omega\beta} \sigma_{\alpha\rho}\sigma_{\rho\omega}\sigma_{\omega\beta} \sum_\beta \left( \frac{1}{1 - \sigma/z^2} \right)_{\alpha\beta} \sum_\gamma \left( \frac{1}{1 - \sigma/w^2} \right)_{\alpha\gamma}$$

and another with the factor $\sum_\gamma \left( \frac{1}{1 - \sigma/w^2} \right)_{\alpha\gamma}$ in the expression above replaced by $\sum_\omega \left( \frac{1}{1 - \sigma/w^2} \right)_{\omega\gamma}$. This clearly leads to a rather unwieldy expression into which we still have to put in the self energy corrections.

Clearly we should not expect to obtain a simple and closed expression for arbitrary $\sigma$: every sector or every site would be unique in its local properties. Instead, we made the rather mild and physically reasonable assumption of homogeneity by supposing that

$$\sum_\beta \sigma_{\alpha\beta} = \text{independent of } \alpha.$$ 

Every site (or sector, if the reader prefers) is treated on the same footing. In particular, this assumption holds if the lattice is translation invariant.

Quite remarkably, we now notice that this model may in fact be solved on any lattice in any dimension as long as the condition (15) holds. Consider the eigenvalue decomposition
the real symmetric matrix $\sigma$, which we may identify as a “hopping matrix”,

$$\sigma = \sum_k |k > \epsilon_k < k|$$  \hspace{1cm} (16)

where $k$ takes on $C$ values. Denote the eigenvalues by $\epsilon_k$. The condition (15) merely states that of the eigenvectors of $\sigma$ there is one (which we labeled as $|k = 0 >$) with components all equal to $1/\sqrt{C}$.

We find immediately that in (12) $g_\alpha$ is independent of $\alpha$ and equal to

$$G(z) = \frac{z - \sqrt{z^2 - 4\epsilon_0}}{2\epsilon_0}$$ \hspace{1cm} (17)

This has the same form as the propagator in the simple one matrix model and gives immediately Wigner's semi-circle law for the density of eigenvalues

$$\rho(\mu) = \frac{1}{2\pi\epsilon_0} \sqrt{4\epsilon_0 - \mu^2}$$ \hspace{1cm} (18)

We now proceed to the two-point Green's function. The vertex correction factor we encountered before now simplifies drastically

$$\sum_\beta (\frac{1}{1 - \sigma/\epsilon_0^2})_{\alpha \beta} = (\frac{1}{1 - \epsilon_0/\epsilon_0^2})$$ \hspace{1cm} (19)

to a factor independent of $\alpha$. Putting in vertex corrections just amounts to multiplying by this factor and a corresponding one with $z$ replaced by $w$. The trace and sum in (13) can be performed immediately to give

$$(CN)^2 G_c(z, w) = \sum_k \frac{\epsilon_k}{(zw - \epsilon_k)^2}.$$ \hspace{1cm} (20)

After putting in vertex and self-energy corrections, we find

$$(CN)^2 G_c(z, w) = \left( \frac{G^2(z)}{1 - \epsilon_0 G^2(z)} \right) \left( \frac{G^2(w)}{1 - \epsilon_0 G^2(w)} \right) \left( \sum_k \frac{\epsilon_k}{(1 - \epsilon_k G(z) G(w))^2} \right).$$ \hspace{1cm} (21)

Taking the appropriate absorptive parts we can calculate the connected correlation function $\rho_c(\mu, \nu)$. As in our earlier work we find it natural to introduce angular variables defined by

$$\sin \theta = \frac{\mu}{2\epsilon_0}$$ \hspace{1cm} (22)
and

\[ \sin \varphi = \frac{\nu}{2\epsilon_0}. \]

(23)
The angles \( \theta \) and \( \varphi \) vary from \(-\pi/2\) to \(+\pi/2\) over the range of the eigenvalues. The connected correlation function is given by

\[ \rho_c(\mu, \nu) = -\frac{1}{16\pi^2 N^2 C^2 \epsilon_0 \cos \theta \cos \varphi} \times \sum_k \tau_k \left\{ \frac{\tau_k + \cos(\theta + \varphi)}{(1 + \tau_k \cos(\theta + \varphi))^2} + \frac{\tau_k - \cos(\theta - \varphi)}{(1 - \tau_k \cos(\theta - \varphi))^2} \right\} \]

(24)
where we have defined

\[ \tau_k = \frac{2\epsilon_0 \epsilon_k}{\epsilon_0^2 + \epsilon_k^2}. \]

(25)
The expressions (21) and (24) represent the central results of this paper. It is worth emphasizing that our results hold for any lattice with any connectivity as long as the homogeneity condition (15) is satisfied.

As a simple special case, consider the simple case in which the matrix \( M^2 \) is such that all its diagonal elements are equal to \( m^2 \) and all its off-diagonal elements are equal to \( M^2 \). Every site on the lattice couples to every other site. The matrix \( \sigma \) has one eigenvalue \( \epsilon_0 = 1/m_s^2 \) and \( C - 1 \) degenerate eigenvalues equal to \( 1/m_d^2 \). Thus we have

\[ \rho_c(\mu, \nu) = -\frac{m_s^2}{16\pi^2 N^2 C^2 \cos \theta \cos \varphi} \left\{ \frac{1}{1 + \cos(\theta + \varphi)} + \frac{1}{1 - \cos(\theta - \varphi)} \right\} \]

\[ + (C - 1)\tau \frac{\tau + \cos(\theta + \varphi)}{(1 + \tau \cos(\theta + \varphi))^2} + (C - 1)\tau \frac{\tau - \cos(\theta - \varphi)}{(1 - \tau \cos(\theta - \varphi))^2} \]

(26)
where

\[ \tau = \frac{2m_s^2 m_d^2}{m_s^4 + m_d^4}. \]

(27)
Note that \( |\tau| < 1 \) and thus only the second term in \( \rho_c(\mu, \nu) \) is singular when \( \mu = \nu \). As we vary \( M^2 \), the correlation moves from a Poissonian ("localized") regime to a Wignerian ("extended") regime. In particular, at the Wigner point, \( \tau = 0 \) and we recover trivially the result for the one matrix model first obtained in Ref. 2. At the Poisson point, \( \tau = 1 \) we have a short distance (i.e., when \( \theta \) and \( \varphi \) approach each other) singularity with a residue \( (C - 1 + 1)/C^2 \). (Recall from Ref. 2 that this short distance singularity occurs because we are dealing with the "smoothed" correlation. The exact correlation does not have this singularity. Nevertheless, this singularity tells us about the short distance correlation between the density of eigenvalues.) Thus, the short distance singularity is reduced from the standard \( C = 1 \)
case and is completely suppressed in the large $C$ limit, namely in the limit of many sectors forbidden to mix by many conservation laws. More generally, for any $\tau$, it is sufficient to take $C$ large for this short distance singularity to be suppressed.\(^{14}\)

For an arbitrary lattice, we simply find the eigenvalues of the hopping matrix $\sigma$. For example, for a hypercubic lattice we have

$$\epsilon_k = \frac{1}{m^2} + \frac{2}{M^2} \sum_a \cos k_a$$

(28)

where the sum over $a$ runs over the dimension $D$ of the lattice. In the expressions (21) and (24) the sum over $k$ runs as usual over the Brillouin zone $k_a = 2\pi j_a/L_a, j_a = 0, 1, 2, \ldots L_a - 1$ with $L_1L_2\ldots L_D = C$. Note that to have nearest neighbor coupling the appropriate entries in the matrix $\mathcal{M}^2$ have to be taken to infinity. This simply means that some of the matrices $H_{\alpha\beta}$ in (3) are to be set equal to zero. It may be interesting to study our result for a variety of lattices. Also, one may consider the limit $C \to \infty$ in which case the sum over $k$ is replaced by an integral. While our results are obtained in the context of random matrix theory, they may conceivably be of relevance to some condensed matter systems.

This correlation function, while it is not quite the same as the correlation functions we\(^2,3,4\) and others\(^{15,16}\) have found previously in a number of different situations, has the same general structure as these other correlation functions.\(^{17}\) This in itself is a statement of universality. Next, we may make a remark about going beyond Gaussian distributions. In our previous work\(^3,4\) we identified two classes of matrix models: the Wigner class and the trace class. Briefly, in the Wigner class the individual matrix element obeys a probability distribution. (In the present context, the distributions for different $H_{\alpha\beta}$ may in general be different.) As was explained in Ref. 3, universality, that is, the independence of the density and the correlation on these probability distributions, can be proved almost immediately using our diagrammatic method. In contrast, in the trace class, each of the $H_{\alpha\beta}$ obeys some probability distribution of the form

$$P(H_{\alpha\beta}) \propto e^{-N\text{tr}V^{\alpha\beta}(H_{\alpha\beta})}$$

(29)

(no sum over repeated indices) where $V^{\alpha\beta}$ denotes a polynomial, different for different $\alpha\beta$. In this case, we do not know how to prove universality, that is, the independence of the correlation function $\rho_c$ (when expressed in terms of scaling variables) on $V^{\alpha\beta}$. It would be interesting to see numerically whether universality indeed holds.

We would now like to present two other derivations of our central result. Let us for notational simplicity consider the case studied in Ref. 4, that of one Gaussian-random matrix,
which we will denote by $H$. In other words, we first consider $H$ for $C = 1$, a lattice with one point. We will first recover the results of Ref. 4 and then generalize these results to an arbitrary lattice. Consider the “scattering amplitude”

$$\Gamma_{ln}^{im} \equiv N \left< \left( \frac{1}{z - H} \right)_l^i \left( \frac{1}{w - H} \right)_n^m \right>_c \equiv \delta_n^i \delta_l^m A + \delta_l^i \delta_n^m B$$

The two “scalar” functions $A$ and $B$ depend on $z$ and $w$ of course. The Green’s function is defined as usual by

$$G(z) \equiv \left< \frac{1}{N} \text{tr} \left( \frac{1}{z - H} \right) \right>$$

we now calculate (repeated indices summed)

$$\Gamma_{li}^l = N^2 A + NB = N^2 \left[ \frac{G(z) - G(w)}{w - z} - G(z)G(w) \right]$$

This equation tells us that $A$ is of order $N^0$ while $B$ is of order $1/N$. Furthermore, the connected two-point Green’s function is given by

$$N^2 G_c(z, w) \equiv \left< \text{tr} \left( \frac{1}{z - H} \text{tr} \frac{1}{w - H} \right) \right>_c = A + NB$$

Thus, if we know $G$ and $G_c$ we know $A$ and $B$ and hence the full scattering amplitude $\Gamma_{ln}^{im}$.

For our lattice problem we have to generalize the definition above to

$$\Gamma_{\beta\epsilon n}^{\alpha i \gamma m} \equiv N \left< \left( \frac{1}{z - H} \right)^{\alpha i}_{\beta l} \left( \frac{1}{w - H} \right)^{\gamma m}_{\epsilon n} \right>_c \equiv \delta_n^\epsilon \delta_n^\gamma \delta_l^\alpha \delta_l^\beta A + \delta_l^\alpha \delta_l^\beta \delta_n^\gamma \delta_n^\epsilon B$$

All is as before except that now the scattering functions $A$ and $B$ can depend on the site labels $\alpha$, $\beta$, $\gamma$ and $\epsilon$ or technicolor indices because the probability distribution is not unitary invariant in these indices. Note that nevertheless the kronecker delta structure in (34) holds. This can be seen graphically since a line carrying color and technicolor going in must come out carrying the same color and technicolor.

Now we note the remarkable fact that if we are interested in questions of localization, namely what happens when the site $\alpha$ is equal to the site $\epsilon$, and far away from the site $\beta = \gamma$, we don’t need to know anything about $B$. But $A$ is determined by the Green’s function $G(z)$ completely!
Recalling that for the Gaussian and single matrix case (see (12) and (17)) that \( G(z) \) is the solution of the quadratic equation

\[
z = G + \frac{1}{G}
\]  

we obtain immediately from (32) that

\[
A(z, w) = \frac{G^2(z)G^2(w)}{1 - G(z)G(w)}
\]  

One can see readily that for the case of a lattice of matrices considered here we only have to modify this result to

\[
A_{\alpha\beta}(z, w) = G^2(z)G^2(w) \left( \frac{\sigma}{1 - \sigma G(z)G(w)} \right)_{\alpha\beta}^\sigma
\]

\[
= G^2(z)G^2(w) \int \frac{d^dk}{(2\pi)^d} \frac{\epsilon_k e^{i\vec{k} \cdot (\vec{\alpha} - \vec{\beta})}}{1 - \epsilon_k G(z)G(w)}
\]  

Note that as anticipated \( A \) depends on \( \alpha \) and \( \beta \).

This result appears to agree with Wegner’s result\(^1\). We would like to emphasize the simplicity and ease with which this result can be obtained. For the sake of completeness, let us recall that the standard localization formalism instructs us to set \( z = E + i\omega \) and \( w = E - i\omega \) and then let the distance \( r \equiv |\vec{\alpha} - \vec{\beta}| \) tend to infinity. Since \( G(E + i\omega)G(E - i\omega) \to 1 \) as \( \omega \to 0 \) and \( \epsilon_k \to 1 - O(k^2) \) as \( k \to 0 \) for a simple rectilinear lattice, we see that \( A \to 1/r^{d-2} \), as is well known from the work of Wegner.

As a check, we may also easily calculate \( A \) directly. From the structure of the indices we see that \( A \) is given by summing the ladder graphs in Fig (3). Note that vertex corrections are suppressed. We merely have to dress the quark propagators. We thus have

\[
A(z, w) = \frac{1}{zw} \sum_{n=1}^{\infty} \frac{1}{(zw)^n} \frac{\sigma^n}{\alpha} \beta
\]

After summing and dressing the quark propagators by replacing \( 1/z \) by \( G(z) \) and \( 1/w \) by \( G(w) \) we obtain (37) immediately.

In this approach, in order to obtain our correlation function, we must calculate \( B \). Thus, there is conservation of labor after all. While it is somewhat more involved to calculate \( B \) than to calculate \( A \), it is still easy enough. The relevant graphs are given in Fig. (4). Curiously, because of the structure of indices only graphs with vertex corrections contribute
to $B$. In particular, the simple one-gluon exchange graph does not contribute. We obtain (again for notational simplicity we do the $C = 1$ case here)

$$NB(z, w) = \left( \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{z^{2p} w^{2q}} - 1 \right) \sum_{n=0}^{\infty} \frac{1}{(zw)^{n+2}}$$

$$\rightarrow \frac{G^2(z) + G^2(w) - G^2(z)G^2(w)}{(1 - G^2(z))(1 - G^2(w))} \frac{G^2(z)G^2(w)}{1 - G(z)G(w)}$$

after dressing the quark propagators. Note the $(-1)$ inside the parenthesis. We leave it to the reader to show that when this somewhat strange looking expression is combined with $A$ according to (33) we recover our result\textsuperscript{2,4} for the correlation function $G_c(z, w)$. It is a straightforward exercise to put in the $\sigma$ matrix to recover (21).

We conclude by giving yet another derivation of our central result. Note that (6) may be written as

$$G_c(z, w) \equiv \left\langle \frac{1}{CN} \text{tr} \frac{1}{z - H} \frac{1}{w - H} \right\rangle_c$$

$$= \frac{\partial}{\partial z} \frac{\partial}{\partial w} \left\langle \frac{1}{CN} \text{tr} \log(z - H) \frac{1}{CN} \text{tr} \log(w - H) \right\rangle$$

Again, for notational simplicity, let us consider the $C = 1$ case first. Expanding the logarithms, we find

$$G_c(z, w) = \frac{\partial}{\partial z} \frac{\partial}{\partial w} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{z^n w^m} \left\langle \frac{1}{Nn} \text{tr} H^n \frac{1}{Nm} \text{tr} H^m \right\rangle$$

This is represented by the RwheelS graph of Fig (1b). We see immediately that vertex corrections are suppressed in the large $N$ limit. Thus, we may set $n = m$ in (41) and since $\langle \text{tr} H^n \text{tr} H^n \rangle = n$ we can immediately evaluate the sum to obtain

$$N^2G(z, w)_c = -\frac{\partial}{\partial z} \frac{\partial}{\partial w} \log(1 - \frac{1}{zw})$$

Dressing the quark propagators, we obtain immediately

$$N^2G(z, w)_c = -\frac{\partial}{\partial z} \frac{\partial}{\partial w} \log(1 - G(z)G(w))$$

This derivation is simpler than that given in Ref. 4. We also obtain a more compact form as given in Ref. 18. It is now simple to go to the case of an arbitrary lattice. We now have $\langle \text{tr} H^n \text{tr} H^n \rangle = ntr \sigma^n$ and thus we find

$$N^2G(z, w)_c = -\frac{\partial}{\partial z} \frac{\partial}{\partial w} \text{tr} \log(1 - \sigma G(z)G(w)) = -\sum_k \frac{\partial}{\partial z} \frac{\partial}{\partial w} \log(1 - \epsilon_k G(z)G(w))$$
We leave it to the reader to check that we have indeed recovered our previous result (21).

Finally, we are tempted to indulge in a speculation. It has long been known that matrix models are intimately related to the Calogero-Sutherland model. Specifically, the Calogero-Sutherland model for a certain coupling constant $\lambda$ equal to 1/2, 1, and 2 correspond respectively to matrix models with real symmetric, hermitean, and quaternionic matrices. Recently, Ha$^{19}$ was able to solve the Calogero-Sutherland model for $\lambda$ equal to an arbitrary rational number. The question naturally arises whether or not for these values of the coupling constant the Calogero-Sutherland model corresponds to matrix models. Is it possible that the class of matrix models considered here may be the correspondents? And if not, it would be interesting to ask whether the matrix models considered here corresponds to some generalized Calogero-Sutherland models.

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FIGURE CAPTIONS

Fig (1a) A crossed ladder graph that contributes to leading order in the large $N$ limit; (1b) As far as the combinatorics is concerned, graphs of the type drawn in (1a) are equivalent to the graphs of the type drawn in (1b).

Fig (2) Vertex corrections to the upper part of the graph in (1a)

Fig (3) Graphs contributing to $A$ in the large $N$ limit.

Fig (4) Some graphs contributing to $B$; note that only graphs with vertex corrections are to be included.
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