Hubble constants and luminosity distance in the renormalized cosmological models due to general-relativistic second-order perturbations

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1. Introduction
In order to discuss the cosmological tension on the difference between the Hubble constant derived from the Planck measurements[1, 2] and that from the direct measurements of the Hubble constant[3–8], we studied cosmological models[9–11] which were derived using the general-relativistic second-order perturbation theory ([12] for non-zero Λ and [13–15] for zero Λ). It was found in these models that the cosmological random adiabatic density fluctuations[16] play an important role as the first-order perturbations for producing the gap of Hubble constants due to the non-linear process.

After the publication of our above papers, we found a necessity of the revision for the derivation of averages of second-order metric perturbations in the first paper[9], which changed the derived value of the Hubble constant slightly.

In the present paper we first derive our correct averages of metric perturbations, and use them to derive the Hubble constants and optical quantities such as redshift and luminosity distance necessary for the observations.

In Sect. 2, we show our background model and the outline of our perturbation theory. In Sect. 3, we show the revised second-order metric perturbations, and in Sect. 4, derive two kinds of the Hubble parameter, which are found to be comparable, and larger than the background value. In Sect. 5, we derive the optical quantities, and the observational relation using the revised metric perturbations. In Sect. 6, concluding remarks are given. In Appendix A, the basic formulation is compactly reviewed. In Appendix B, the deceleration parameter $q$ is derived, and in Appendix C, the luminosity distance $d_L$ is derived, based on the revised metric perturbations. In Appendix D, dependence of renormalized model parameters on background model parameters is shown.
2. Background and the perturbation theory

The space-time of our spatially flat background universe is expressed by the line element

\[ ds^2 = g_{\mu\nu}dx^\mu dy^\nu = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j], \]  

where the Greek and Roman letters denote 0, 1, 2, 3 and 1, 2, 3, respectively. The conformal time \( \eta(=x^0) \) is related to the cosmic time \( t \) by \( dt = a(\eta)d\eta \). The background Hubble parameter \( H \equiv a'/a^2 = \dot{a}/a \) satisfies

\[ H = [(\rho + \Lambda)/3]^{1/2} = H_0 (\Omega_M a^{-3} + \Omega_\Lambda)^{1/2}, \]  

where a prime and a dot denote \( d/d\eta \) and \( d/dt \), respectively. We use a background model with model parameters given by

\[ H_0 = 67.3 \text{ km s}^{-1}\text{Mpc}^{-1} \quad \text{and} \quad (\Omega_M, \Omega_\Lambda) = (0.22, 0.78), \]  

where

\[ \Omega_M = \frac{8\pi G \rho_0}{3H_0^2} = \frac{\rho_0}{3H_0^2} \quad \text{and} \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} = \frac{\Lambda}{3H_0^2}. \]  

\( H_0 \) and \( \rho_0 \) are the present Hubble constant and matter density, and the units \( 8\pi G = c = 1 \) are used. In Appendix D, cases of \( (\Omega_M, \Omega_\Lambda) \) = (0.24, 0.76) and (0.28, 0.72) are treated for comparison.

For perturbations on large scales with \( x \equiv k/k_{eq} \leq x_{max} \), the perturbed metric, velocity and density perturbations are expressed as

\[ \delta g_{\mu\nu} = h_{\mu\nu} + \ell_{\mu\nu}, \]  

\[ \delta u^\mu = \delta_1 u^\mu + \frac{1}{2} \delta_2 u^\mu, \]  

\[ \delta \rho/\rho = \delta_1 \rho/\rho + \frac{1}{2} \delta_2 \rho/\rho, \]  

where the definition of \( k_{eq} \) and \( x_{max} \) are shown in Appendix A. Here we assume the synchronous and comoving coordinates, that is

\[ h_{00} = 0, \quad h_{0i} = 0 \quad \text{and} \quad \delta_1 u^0 = 0, \quad \delta_1 u^i = 0, \quad \ell_{00} = 0, \quad \ell_{0i} = 0 \quad \text{and} \quad \delta_2 u^0 = 0, \quad \delta_2 u^i = 0 \]  

in the same way as the previous paper[9], cited as [I]. In the previous paper[12], the expressions of metric in the Poisson coordinates also were shown, but here our treatments are confined only to the synchronous and comoving coordinates.

The first-order perturbations in the growing mode are expressed in Eq.(14) of [I] by use of an arbitrary potential function \( F(x) \), where \( x \) is the spatial coordinates. The amplitude of \( F(x) \) is related to the cosmological adiabatic density fluctuations.[16] The second-order perturbations corresponding to the first-order perturbations are expressed in Eqs. (20) - (23) of [I].

The average values of second-order density perturbations are shown in Appendix A with some small corrections. For those of second-order metric perturbations, the revised version is shown in the next section.

The scale with \( x > x_{max} \) represents the scale which is always sub-horizon at the matter-dominant stage after the epoch such as \( 1 + z = 1500 \) (cf [10]). Perturbations on small scales
with \( x \geq x_{\text{max}} \) were separately treated in the Newtonian approximation and their effect to the large-scale quantities was found to be negligible. So the following analyses are confined to the above perturbations on large scales with \( x \leq x_{\text{max}} \).

3. Revised second-order metric perturbations

The average of second-order perturbations of the scale-factor (\( \delta_2 a \)) is expressed using the second-order metric perturbations \( l_{ij} \) as

\[
\delta_2(a^2)/a^2 = \frac{1}{3} \langle l_{m}^m \rangle, \tag{8}
\]

where the average process is shown in Appendix A and \( \langle l_{m}^m \rangle = \langle l_1^1 + l_2^2 + l_3^3 \rangle \). We have this relation (8), because \( a^2 l_{ij} \) represents the perturbations corresponding to the background space-time Eq.(1). So, Eq.(40) of [I] is wrong with respect to the factor \( a^2 \), and the following Eqs. (42) - (51) of [I] should be replaced by the correct ones, which are shown in the following:

\[
\langle l_{ij} \rangle = P(\eta)\langle L_{ij} \rangle + P^2(\eta)\langle M_{ij} \rangle + Q(\eta)\langle N_{ij} \rangle + \langle C_{ij} \rangle. \tag{9}
\]

Here \( L_{ij}, M_{ij} \) and \( N_{ij} \) are metric components being functions of spatial variable \( x \), and \( C_{ij} \) represents the components of gravitational waves, which were used in Eqs. (20), (21), and (A1) - (A4) of [I]. Since \( L_j^j = L_{ij}, M_j^j = M_{ij} \) and

\[
L_i^i = -\frac{1}{4} [2F\Delta F + \frac{3}{2} F_{ij} F_{ji}],
\]
\[
M_i^i = \frac{1}{28} [10F_{,jl} F_{,jl} - 3(\Delta F)^2],
\]
\[
\Delta N = \frac{1}{28} [\langle \Delta F \rangle^2 - F_{,kl} F_{,kl}],
\]
\[
\Box C_i^i = 0, \tag{10}
\]

we obtain

\[
\langle L_i^i \rangle = -\frac{1}{4} \langle F\Delta F \rangle,
\]
\[
\langle M_i^i \rangle = \frac{1}{4} \langle (\Delta F)^2 \rangle, \tag{11}
\]
\[
\langle \Delta N \rangle = \langle C_i^i \rangle = 0.
\]

Then we get using Eqs.(A5) and (A7)

\[
\langle l_{ii} \rangle = \langle l_i^i \rangle = \frac{1}{4} (2\pi)^{-2} P(\eta) \left[ \int dk k^2 P_F(k) + P(\eta) \int dk k^4 P_F(k) \right]. \tag{12}
\]

Here \( P_F \) is replaced by \( P_R \) with \( P_{R0} \) in Eq.(A10), and we obtain

\[
\langle l_{ii} \rangle = 2\pi \ 32.4^4 \ P_{R0} \ Z(a) [32.4^{-2} A + Z(a) B], \tag{13}
\]

where \( P(\eta) \) is expressed using \( Z(a) \) (defined in Eqs. (A14) and (A15)), and the constants \( A \) and \( B \) (defined by Eq. (A13)) reflect the amplitude of the BBKS adiabatic fluctuations. [16].
The average metric perturbations $\langle l_{ii}\rangle$ are spatially constant and isotropic, and so we can consider the renormalized scale-factor $a_{rem}$ defined by

$$a_{rem} = a \left(1 + \frac{1}{3} \langle l_{ii}\rangle\right)^{1/2} = a \left(1 + \frac{1}{6} \langle l_{ii}\rangle\right),$$  \hspace{1cm} (14)

where we neglect the terms of higher-orders than second-order. The renormalized Hubble parameter ($H_{kin}$) is defined as

$$H_{kin} = \frac{a_{rem}}{a_{rem}} = \frac{\dot{a}}{a} + \frac{1}{6} \langle l_{ii}\rangle,$$  \hspace{1cm} (15)

or

$$H_{kin}/H = 1 + \frac{1}{6} \langle l_{ii}\rangle/H,$$  \hspace{1cm} (16)

where $H (\equiv \dot{a}/a)$ is the background Hubble parameter. Here $H_{kin}$ denotes the kinematic definition of the Hubble parameter. Differentiating Eq.(13), we obtain

$$\langle l_{ii}\rangle = 2\pi \frac{32.4^4}{3} \mathcal{P}_R \left[32.4^{-2} A + 2Z(a)B\right] \frac{dZ(a)}{da} \dot{a},$$  \hspace{1cm} (17)

where

$$\frac{dZ(a)}{da} = (H_0)^2 \frac{P^a}{a^3} = (H_0/H)^2 \frac{Y(a)}{a^3} = \frac{Y(a)}{\Omega_M + \Omega_{\Lambda} a^3},$$  \hspace{1cm} (18)

using Eqs. (2) and (A15). Therefore, we get using Eq. (18)

$$\langle l_{ii}\rangle/H = 2\pi \frac{32.4^4}{3} \mathcal{P}_R \left[32.4^{-2} A + 2Z(a)B\right] \frac{Y(a)a}{\Omega_M + \Omega_{\Lambda} a^3},$$  \hspace{1cm} (19)

and then the kinematic second-order Hubble parameter is given by Eq.(15). At present epoch, it is expressed as

$$(H_{kin})_0/H_0 = 1 + \frac{1}{6} \langle l_{ii}\rangle/H \big|_0 = 1 + \frac{2\pi}{3} \frac{32.4^4}{3} \mathcal{P}_R \left[\frac{1}{2} \times 32.4^{-2} A + Z(1)B\right] Y(1).$$  \hspace{1cm} (20)

Here, $Y(1)$ and $Z(1)$ are expressed as

$$Y(1) = I(1), \quad Z(1) = \frac{2}{3\Omega_M} [1 - I(1)], \quad \text{and} \quad I(1) = \int_0^1 \frac{d\dot{b}^3}{(\Omega_M + \Omega_{\Lambda} b^3)^{1/2}}.$$  \hspace{1cm} (21)

For the background model parameters (3), we get

$$I(1) = Y(1) = 0.566, \quad Z(1) = 1.316.$$  \hspace{1cm} (22)

So, we obtain using $A$ and $B$ in Eq.(A18)

$$(H_{kin})_0 = 72.6 \text{ km s}^{-1}\text{Mpc}^{-1}.$$  \hspace{1cm} (23)

In order to explain the histories of $H_{kin}$ and $a_{rem}$, the behaviors of $H_{kin}$ ($H_0/H$) and $\xi (\equiv a_{rem}(t)/a(t) - 1)$ are shown as functions of $a$ in Figs. 1 and 2, respectively. Corresponding previous figures (Figs. 2 and 4 in [I]) must be replaced by these figures.
Fig. 1: History of $H_{\text{kin}}$. The ordinate denotes $H_{\text{kin}}(H_0/H)$ is expressed as a function of $a$. The scale factor $a$ has 1 at the present epoch.

4. Renormalization of model parameters

For the second-order density perturbations, we have no revision, and so using Eqs.(A12) and (A19) for the density perturbations, we can consider the renormalization of model parameters, similarly to that in [I]. Since $\langle \delta_2 \rho \rangle$ is spatially constant and isotropic, we assume that it is a part of the renormalized matter density $\rho_{\text{rem}}$. So we have

$$\rho_{\text{rem}} = \rho + \langle \delta_2 \rho \rangle. \quad (24)$$

Then the renormalized ones corresponding to $\Omega_M$ and $\Omega_\Lambda$ are given by

$$[(\Omega_M)_{\text{rem}}, (\Omega_\Lambda)_{\text{rem}}] = [(\Omega_M)(1 + \langle \delta_2 \rho / \rho \rangle), \quad \Omega_\Lambda]/(1 + \langle \delta_2 \rho / \tilde{\rho} \rangle), \quad (25)$$

where $\tilde{\rho} \equiv \rho + \Lambda$.

Next, we define a renormalized Hubble parameter $H_{\text{dyn}}$ corresponding to Eq.(24) as

$$H_{\text{dyn}} = \left[ \frac{1}{3}(\rho_{\text{rem}} + \Lambda) \right]^{1/2} = \left[ \frac{1}{3}(\tilde{\rho} + \langle \delta_2 \rho \rangle) \right]^{1/2}. \quad (26)$$
Fig. 2: The relative scale-factors $\xi (\equiv a_{rem}/a - 1)$ is expressed as a function of $a$. The scale factor $a$ has 1 at the present epoch.

This Hubble parameter appears when we describe the dynamical evolution of the perturbed model, as used in the previous paper[10], and so we call it the dynamical Hubble parameter. Its present value is expressed as

$$\left(\frac{H_{dyn}}{H_0}\right)_0 = 1 + \frac{2\pi}{3} \times 32.4^4 \mathcal{P}_{R_0} \left[ \frac{-5}{2} \times 32.4^{-2} A + Z(1)B \right] [1 - Y(1)],$$  \hspace{1cm} (27)

where $A$ and $B$ are given in Eqs.(A13) and (A18).

Corresponding to the background model parameter (3), the present value of renormalized model parameters are found to be

$$\left(\Omega_M\right)_{rem} = 0.305, \quad \left(\Omega_A\right)_{rem} = 0.695,$$  \hspace{1cm} (28)

and

$$\left(\frac{H_{dyn}}{H_0}\right)_0 = 71.4 \text{ km s}^{-1}\text{Mpc}^{-1}.$$

(29)

As for the histories of $\langle \delta_2 \rho \rangle$ and $\langle \Omega_M \rangle_{rem}$, Fig. 1 and Fig. 3 in [I] are useful also in the present paper.
On the other hand, we have the kinematic Hubble parameter $H_{\text{kin}}$, which was derived in the previous section. $H_{\text{kin}}$ and $H_{\text{dyn}}$ are different with respect to the factor $Y(a)$, and $H_{\text{kin}}$ is a little larger than $H_{\text{dyn}}$. So we discriminate these two Hubble parameters. On the other hand, both Hubble parameters are found to be larger than the background Hubble parameter $H$.

These Hubble parameters depend on the value of $B$ which is sensitively related to the upper limit $x_{\text{max}}$ in Eq.(A13). The terms with $A$ are negligibly small. We have used $x_{\text{max}} = 5.7$ here and in previous papers. In Table 1, we show the dependence of $(H_{\text{kin}})_0$ and $(H_{\text{dyn}})_0$ on the value of $x_{\text{max}}$ and $L_{\text{max}} (=2\pi/k_{\text{max}})$ given in Eq.(53) of [I], which may represent the boundary for whether the general-relativistic non-linearity is effective for the evolution of perturbations, as was discussed in a previous paper[11]. For larger $x_{\text{max}}$ (or smaller $L_{\text{max}}$), we have larger Hubble constant.

Table 1: Dependence of $(H_{\text{kin}})_0$ and $(H_{\text{dyn}})_0$ on $x_{\text{max}}$ and $L_{\text{max}}$ in the case of background model parameters (3).

| $x_{\text{max}}$ | $L_{\text{max}}$ | $(H_{\text{kin}})_0$ | $(H_{\text{dyn}})_0$ |
|------------------|------------------|----------------------|----------------------|
| 5.7              | 102/h            | 72.6                 | 71.4                 |
| 6.0              | 97/h             | 73.3                 | 71.7                 |
| 6.3              | 92/h             | 74.0                 | 72.4                 |

5. Optics and observations

The renormalized line-element can be expressed as

$$ds^2 = -dt^2 + a_{\text{rem}}(t)^2 \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right], \tag{30}$$

where the renormalized scale factor $a_{\text{rem}}(t)$ is given by Eq. (14), and $x^1, x^2$ and $x^3$ are comoving coordinates.

5.1. Redshift

The light path is given by the null condition

$$dt = \pm a_{\text{rem}}(t) dr, \tag{31}$$

where $r \equiv [(dx^1)^2 + (dx^2)^2 + (dx^3)^2]^{1/2}$.

If a light ray starts from a distant source with $r$ at epoch $t_1$ and reaches an observer at epoch $t_0$, we have

$$\int_{t_1}^{t_0} dt/a_{\text{rem}} = r. \tag{32}$$

If we receive two subsequent signals with intervals $\delta t_0$ and $\delta t_1$ from a comoving source, we obtain

$$\frac{\delta t_1}{a_{\text{rem}}(t_1)} = \frac{\delta t_0}{a_{\text{rem}}(t_0)}. \tag{33}$$

For the frequencies $\nu_0$ and $\nu_1$ (given by $\nu_1/\nu_0 = \delta t_0/\delta t_1$), we have

$$1 + z_{\text{rem}} = \nu_1/\nu_0 = a_{\text{rem}}(t_0)/a_{\text{rem}}(t_1), \tag{34}$$

where $z_{\text{rem}}$ is the redshift.
For nearby sources, we can expand $a_{\text{rem}}(t)$ as
\[ a_{\text{rem}}(t) = a_{\text{rem}}(t_0)[1 + (t - t_0)(H_{\text{kin}}_0) + \cdots], \tag{35} \]
where $(H_{\text{kin}})_0$ gives the relation
\[ (H_{\text{kin}})_0 = \dot{a}_{\text{rem}}(t_0)/a_{\text{rem}}(t_0). \tag{36} \]
Moreover, we have
\[ z_{\text{rem}} = (H_{\text{kin}})_0 \Delta t, \tag{37} \]
where $\Delta t = t_0 - t_1$.

### 5.2. Luminosity distance

The relation between the apparent luminosity $l$ and the absolute luminosity $L$ is expressed by
\[ l = L/(4\pi d_L^2), \tag{38} \]
where $d_L$ is the luminosity distance between a source and an observer. In the expanding universe with the metric (30), the time intervals $\delta t_1$ and $\delta t_0$ in the source and the observer are not equal and given by Eq.(33). Moreover, the received and emitted energies of a photon are different and given by the redshift factor. As a result, the above relation is expressed as
\[ l = L/[4\pi(d_L)^2_{\text{rem}}], \tag{39} \]
using the renormalized luminosity distance $(d_L)_{\text{rem}}$, which is given by
\[ (d_L)_{\text{rem}} = a_{\text{rem}}(t_0) r(z_{\text{rem}}) (1 + z_{\text{rem}}). \tag{40} \]
Here $r(z_{\text{rem}})$ is the coordinate distance between the observer and the source with $z_{\text{rem}}$, which is derived eliminating $t_1$ from Eqs. (32) and (34).

For $z_{\text{rem}} \ll 1$, we have
\[ z_{\text{rem}} = (H_{\text{kin}})_0(t_0 - t_1) + \frac{1}{2}[(q_{\text{kin}})_0 + 2] (H_{\text{kin}})_0^2 (t_0 - t_1)^2 + \cdots, \tag{41} \]
where the kinematic deceleration parameter $(q_{\text{kin}})$ is defined as
\[ q_{\text{kin}} = -\left(\frac{d^2a_{\text{rem}}(t)}{dt^2}\right)/[(H_{\text{kin}})^2 a_{\text{rem}}]. \tag{42} \]

From Eq.(41), we obtain inversely
\[ (H_{\text{kin}})_0(t_0 - t_1) = z_{\text{rem}} - \frac{1}{2}[(q_{\text{kin}})_0 + 2] z_{\text{rem}}^2 + \cdots \tag{43} \]
Therefore, we obtain from Eq.(40)
\[ (d_L)_{\text{rem}} = (H_{\text{kin}})_0^{-1}\left\{z_{\text{rem}} + \frac{1}{2}[(1 - (q_{\text{kin}})_0)] z_{\text{rem}}^2 + \cdots \right\}, \tag{44} \]
where the value of $q_{\text{kin}}$ is derived in Appendix B. Its present value $(q_{\text{kin}})_0$ is a little larger than $q_0$ in the background:
\[ q_0 = -0.67 \quad \text{and} \quad (q_{\text{kin}})_0 = -0.659 \tag{45} \]
for the background model parameter (3).
The background equation corresponding to Eq.(44) is [17]

\[ d_L = (H_0)^{-1} \left\{ z + \frac{1}{2} [1 - q_0] z^2 + \cdots \right\}. \] (46)

Here the ratio \([(q_{\text{kin}})_0/q_0]^{-1} (= 1.016)\) is smaller than the ratio \((H_{\text{kin}})_0/H_0 (= 1.079)\). As for coefficients of the second terms \(\left(\frac{1}{2} [1 - (q_{\text{kin}})_0]\right)\) and \(\frac{1}{2} [1 - q_0]\) for \(d_L\) also, the ratio is \((0.993)^{-1} (= 1.007)\), and so smaller than \((H_{\text{kin}})_0/H_0\).

Now let us show the \(z_{\text{rem}}\)-dependence of \((d_L)_{\text{rem}}\) from the definition (Eq.(40)) for arbitrary \(z_{\text{rem}}\):

\[(H_{\text{kin}})_0 (d_L)_{\text{rem}} = z_{\text{rem}} (1 + z_{\text{rem}}) \Phi(z_{\text{rem}}) \times \frac{1}{z} \int_{1/(1+z)}^{1} \frac{da}{a^2} (\Omega_M a^{-3} + \Omega_\Lambda)^{-1/2} \left\{ 1 + \frac{1}{6} \zeta [(Z(1))^2 - (Z(a))^2] \right\},\] (47)

where \(\zeta \equiv 2\pi \times 32.4^d \mathcal{P}_{R_0} B (= 0.319)\), \(Z(a)\) is given by Eq.(A15), and

\[ \Phi(z_{\text{rem}}) = \frac{(H_{\text{kin}})_0}{H_0} \frac{z_{\text{rem}}}{z_{\text{rem}}}. \] (48)

The derivation of Eq.(47) is shown in Appendix C. On the other hand, we have the relation between \(z_{\text{rem}}\) and \(z (= 1/a - 1)\):

\[ z_{\text{rem}} = z + \frac{1}{6} \zeta \{ (Z(1))^2 - (Z(a))^2 \}, \] (49)

which is derived from Eq.(C6). So we can get \((H_{\text{kin}})_0 (d_L)_{\text{rem}}\) as a function of given \(z_{\text{rem}}\). The expanded form of this \((d_L)_{\text{rem}}\) for small \(z_{\text{rem}}\) is found to be consistent with Eq.(44), in which the approximate forms of \(W\) and \(\Phi\) in Eqs.(C11) and (C12) are used.

On the other hand, the background equation corresponding to Eq.(47) is [17]

\[ H_0 d_L = (1 + z) \times \int_{1/(1+z)}^{1} \frac{da}{a^2} (\Omega_M a^{-3} + \Omega_\Lambda)^{-1/2}. \] (50)

The difference between \((H_{\text{kin}})_0 (d_L)_{\text{rem}}\) and \(H_0 d_L\) for equal \(z_{\text{obs}}\) is found to be \(\sim 0.2\%\), for \(z_{\text{obs}} = 0.5\). But, \((d_L)_{\text{rem}}\) and \(d_L\) have a larger difference \(\sim 8\%\) due to the ratio \((H_{\text{kin}})_0/H_0\). Here \(z_{\text{obs}}\) is defined to be equal to \(z_{\text{rem}}\) and \(z\) for the renormalized case and the background case, respectively.

The relation between \(\log_{10} (d_L)_{\text{rem}}\) and the observed redshift \(z_{\text{obs}}\) is shown in Fig. 3 for the cases of \(z_{\text{obs}} < 1\), in a comparison with the background counterpart \(\log_{10} d_L\). The absolute magnitude \(M\) of an object is defined in terms of an apparent magnitude \(m\) and the luminosity distance \((d_L)_{\text{rem}}\) as

\[ m - M = 5 \log_{10} \left[ (d_L)_{\text{rem}}/10\text{pc} \right], \] (51)

which is applicable to the observational redshift-magnitude relation for SNIa[3].
Fig. 3: Luminosity distances. Solid and dotted curves show \((d_L)_{rem}\) and \(d_L\), respectively, relative to the observed redshift \(z_{obs}\), where \(z_{obs}\) is equal to \(z_{rem}\) and \(z\), respectively.

5.3. Angular diameter distance

The angular diameter distance \(d_A\) is related to \(d_L\) as\(^{[17, 18]}\)

\[
d_A = (1 + z)^{-2} d_L. \tag{52}
\]

On the other hand, the angular diameter distance \((d_A)_{rem}\) is related to \((d_L)_{rem}\) in the line-element (30) as

\[
(d_A)_{rem} = (1 + z_{rem})^{-2} (d_L)_{rem}, \tag{53}
\]

where \(z_{rem}\) is related to \(z\) by Eq.(49). So, the difference of \((H_{kin})_0(d_A)_{rem}\) and \(H_0d_A\) is very small, for equal \(z_{obs}\), similarly to that of the luminosity distance.

6. Concluding remarks

It was shown that there are two kinds of renormalized Hubble parameters: the dynamical parameter \((H_{dyn})\) and the kinematic parameter \((H_{kin})\). As for their present values, the ratio
is

\[ H_0 : (H_{dyn})_0 : (H_{kin})_0 = 1 : 1.061 : 1.079, \]  

so that \((H_{dyn})_0\) and \((H_{kin})_0\) are larger than the background constant \(H_0\) by the factors \(6 \sim 8\%\), respectively.

The roles of \(H_{dyn}\) and \(H_{kin}\) are for dynamical motions (including phenomena treated in the second paper\([10]\)) and the optical phenomena (which were treated in the present paper), respectively. In the latter, we found that \(H_0d_L(z_{obs})\) and \((H_{kin})_0(d_L)_{rem}(z_{obs})\) are almost equal, so that \(d_L(z_{obs})\) and \((d_L)_{rem}(z_{obs})\) are different by the factor \((H_{kin}/H_0)\). This situation is quite same also in that of the angular diameter distances \(d_A\) and \((d_A)_{rem}\) in Eqs.(52) and (53).

In the determination of the Hubble constant due to the gravitational-wave measurements\([19]\) also, we have the kinematic constant \((H_{kin})_0\) in the same way as the optical observation.

### A. Average second-order perturbations

The arbitrary potential function is given the following expression

\[
F(x) = \int dk \alpha(k) e^{ikx},
\]

(A1)

where \(\alpha(k)\) is a random variable and the average of \(F\) expressed as \(\langle F \rangle\) vanishes, and the average of their products is given by

\[
\langle \alpha(k)\alpha(k') \rangle = (2\pi)^{-2}P_F(k)\delta(k+k').
\]

(A2)

Here we have

\[
\langle \delta_1 \rho / \rho \rangle = 0
\]

(A3)

for the first-order density perturbation. For the second-order perturbations, we have

\[
\langle F, i F, i \rangle = -\int \int dk d k' \langle \alpha(k)\alpha(k') \rangle kk' e^{i(k+k')x},
\]

\[
\langle F \Delta F \rangle = -\int \int dk d k' \langle \alpha(k)\alpha(k') \rangle k^2 e^{i(k+k')x},
\]

(A4)

so that we obtain

\[
\langle F, i F, i \rangle = -\langle F \Delta F \rangle = (2\pi)^{-2} \int dk k^2 P_F(k),
\]

(A5)

where we corrected some careless misprints in \([I]\).
Similarly, we have

$$\langle F_{ij} F_{ij} \rangle = \int \int dk d k' \langle \alpha(k) \alpha(k') \rangle (k k')^2 e^{i(k+k')x}, \quad (A6)$$

$$\langle (\Delta F)^2 \rangle = \int \int dk d k' \langle \alpha(k) \alpha(k') \rangle k^2 (k')^2 e^{i(k+k')x},$$

so that we obtain

$$\langle F_{ij} F_{ij} \rangle = \langle (\Delta F)^2 \rangle = (2\pi)^{-2} \int dk \ k^4 P_F(k). \quad (A7)$$

The second-order density perturbations are expressed by Eq.(23) of [I] using $F(k)$, and so average second-order density perturbations are shown as follows:

$$\langle \delta \rho \rangle = \frac{1}{2 \rho a^2} (2\pi)^{-2} \left[ - \frac{5}{2} \int dk k^2 P_F(k) + P \int dk k^4 P_F(k) \right]. \quad (A8)$$

Here $F$ is related to the curvature fluctuation $\mathcal{R}$ by $F = 2 \mathcal{R}$, and so we have the relation

$$P_F(k) = 4 P_R(k), \quad (A9)$$

where $P_R$ is expressed using the power spectrum[17, 18] as

$$P_R = 2\pi^2 P_{R0} \ k^{-3}(k/k_{eq})^{n-1} T_s^2(k/k_{eq}) \quad (A10)$$

and $P_{R0} = 2.2 \times 10^{-9}$ according to the result of Planck measurements.[1, 2] The transfer function $T_s(x)$ is expressed as a function of $x = k/k_{eq}$, where

$$k_{eq} (\equiv a_{eq} H_{eq}) = 219 (\Omega_M h) H_0 = 32.4 H_0. \quad (A11)$$

Here $H_0 \ (\equiv 100h)$ is the present background Hubble constant, $(a_{eq}, H_{eq})$ is $(a, H)$ at the epoch of equal energy density, and $(\Omega_M, h) = (0.22, 0.673)$ (given in Eq. (3)).

Moreover, we assume $n = 1$ here and in the following. Then we obtain for arbitrary $a$

$$\langle \delta \rho \rangle = \frac{4\pi}{3} 32A^4 \frac{P_{R0}}{(\Omega_M/a + \Omega A^2)} \left[ - \frac{5}{2} 32A^{-2} A + Z(a)B \right], \quad (A12)$$

where $\tilde{\rho} \equiv \rho + \Lambda$, and $A$ and $B$ are expressed as

$$A \equiv \int_{x_{min}}^{x_{max}} dx \ x \ T_s^2(x), \quad B \equiv \int_{x_{min}}^{x_{max}} dx \ x^3 \ T_s^2(x) \quad (A13)$$

using the transfer function $T_s(x)$ for the interval $(x_{max}, x_{min})$. Here we have

$$Y(a) \equiv \frac{a'}{a} P', \quad Z(a) \equiv (H_0)^2 P. \quad (A14)$$

These functions are reduced to

$$Y(a) = a^{-5/2}(\Omega_M + \Omega A^3)^{1/2} I(a), \quad Z(a) = \frac{2}{3\Omega_M a^{2}}[1 - Y(a)], \quad (A15)$$

where

$$I(a) \equiv \int_0^a db \ [b^3/(\Omega_M + \Omega A^3)]^{1/2}. \quad (A16)$$

For $T_s$, we assume the simplest transfer function (BBKS) for cold matter, adiabatic
fluctuations, given by \[ T_s(x) = \frac{\ln(1 + 0.171x)}{0.171x} [1 + 0.284x + (1.18x)^2 + (0.399x)^3 + (0.490x)^4]^{-1/4}. \] (A17)

For \( x_{\text{max}} \) and \( x_{\text{min}} \), we take \( x_{\text{max}} = 5.7 \) and \( x_{\text{min}} = 0.01 \), which were used in [I]. This value of \( x_{\text{max}} \) corresponds to the lower limit of linear scales of super-horizon perturbations at the matter-dominant stage.[11]

Then we obtain

\[ A = 2.22, \quad B = 20.95, \] (A18)

\( Y(1) \) and \( Z(1) \) are shown in Eqs. (21) and (22) for the background parameter (3), and \( \langle \delta \rho / \bar{\rho} \rangle = 0.121 \), (A19) at the present epoch \((a = 1)\), where \( \bar{\rho} \equiv \rho + \Lambda \).

**B. Derivation of the deceleration parameter \( q_{\text{kin}} \)**

The background deceleration parameter \( q \) is defined by

\[ q \equiv -\ddot{a} / (\dot{a})^2. \] (B1)

The corresponding parameter \( q_{\text{kin}} \) is expressed as

\[ q_{\text{kin}} \equiv -\ddot{a}_{\text{rem}} a_{\text{rem}} / (\dot{a}_{\text{rem}})^2, \] (B2)

where \( a_{\text{rem}} \) is given using \( a \) and \( \langle l_{ii} \rangle \) in Eq.(14). Differentiating \( a_{\text{rem}} \), we obtain

\[ \dot{a}_{\text{rem}} = \dot{a} \left( 1 + \frac{1}{6} \langle l_{ii} \rangle \right) + \frac{1}{6} a \langle l_{ii} \rangle, \]

\[ \ddot{a}_{\text{rem}} = \ddot{a} \left( 1 + \frac{1}{6} \langle l_{ii} \rangle \right) + \frac{1}{3} \langle l_{ii} \rangle \dot{a} + \frac{1}{6} a \langle l_{ii} \rangle. \] (B3)

Using them, we obtain the Hubble parameter \( H_{\text{kin}} \) (\( \equiv \dot{a}_{\text{rem}} / a_{\text{rem}} \)) in Eq. (15), and

\[ q_{\text{kin}} = q - \frac{1}{6H^2} \langle l_{ii} \rangle \dot{a} - \frac{1}{3} \frac{1 + q}{H} \langle l_{ii} \rangle. \] (B4)

Here we have the expression of \( \langle l_{ii} \rangle \) in Eq.(17) with \( dZ(a)/da \).

Now we neglect the small terms with \( A \), i.e.

\[ \langle l_{ii} \rangle = \zeta [Z(a)]^2, \] (B5)

and

\[ \langle l_{ii} \rangle \dot{a} = 2\zeta \frac{Z(a) Y(a) \dot{a}}{\Omega_M + \Omega_A a^3}, \] (B6)

where

\[ \zeta \equiv 2 \pi \times 32 A^4 P_{R0} B = 0.01523 \times 20.95 = 0.319. \] (B7)

First, differentiating Eq.(B5), we get

\[ \langle l_{ii} \rangle \dot{a} = 2\zeta (\Omega_M + \Omega_A a^3)^{-1} Z(a) Y(a) \]

\[ \times \left\{ \ddot{a} + (\dot{a})^2 \left[ \frac{dZ(a)/da}{Z(a)} + \frac{dY(a)/da}{Y(a)} - \frac{3\Omega_A a^2}{\Omega_M + \Omega_A a^3} \right] \right\}. \] (B8)
Using Eqs.(18), (A14) and (A15), we can derive

\[-dY(a)/da = \frac{1}{a} \left\{ \left[ 1 + \frac{3}{2} \Omega_M/(\Omega_M + \Omega_\Lambda a^3) \right] Y - 1 \right\}. \quad (B9)\]

From Eqs.(B8) and (B9), we obtain

\[\langle l_{ii} \rangle = 2z \frac{aH^2}{\Omega_M + \Omega_\Lambda a^3} \left\{ -q + \frac{5}{2} \frac{3 \Omega_\Lambda a^3}{\Omega_M + \Omega_\Lambda a^3} \right\} Z Y + Z + \frac{aY^2}{\Omega_M + \Omega_\Lambda a^3}. \quad (B10)\]

Using Eqs.(B4) and (B10), we obtain

\[q_{\text{kin}} = q - \frac{4z}{3z} \frac{aH^2}{\Omega_M + \Omega_\Lambda a^3} \left\{ -q + \frac{1}{2} \frac{4 \Omega_\Lambda a^3}{\Omega_M + \Omega_\Lambda a^3} \right\} Z Y + Z + \frac{aY^2}{\Omega_M + \Omega_\Lambda a^3}. \quad (B11)\]

On the other hand, the background model gives

\[q \equiv \left( \frac{1}{2} \Omega_M - \Omega_\Lambda a^3 \right)/(\Omega_M + \Omega_\Lambda a^3), \quad (B12)\]

where the present value is \(q_0 = -0.67\).

The present value of \(q_{\text{kin}}\) is

\[(q_{\text{kin}})_0 - q_0 = -\frac{1}{3} \zeta \left[ -3 \Omega_\Lambda Z Y(1) + Z(1) + Y(1)^2 \right]. \quad (B13)\]

Using the values of \(Y(1)\) and \(Z(1)\) in Eqs.(21) and (22) for the background model parameters, we obtain

\[(q_{\text{kin}})_0 = -0.670 + 0.011 = -0.659. \quad (B14)\]

C. Derivation of \((d_L)_{\text{rem}}\)

Using the background equations for a light path, we obtain

\[a_{\text{rem}}(t) = \frac{da}{H_0} \left( \frac{\Omega Ma^{-3} + \Omega_\Lambda}{aH_0} \right)^{-1/2} [1 + z_{\text{rem}}(a)], \quad (C1)\]

where \(z_{\text{rem}}(a)\) is expressed using Eq.(14) as

\[1 + z_{\text{rem}}(a) = (1 + z) \left\{ 1 + \frac{1}{6} \langle l_{ii} \rangle_0 - \langle l_{ii} \rangle \right\} \quad (C2)\]

and \(1 + z = 1/a(t)\). Using \((H_{\text{kin}})_0\) and Eq.(40), therefore, \((d_L)_{\text{rem}}\) is expressed as

\[(H_{\text{kin}})_0 \ (d_L)_{\text{rem}} = (H_{\text{kin}})_0 \frac{a}{H_0} (1 + z_{\text{rem}}) \times W(a), \quad (C3)\]

where

\[W(a) \equiv \frac{1}{z} \int_a^1 \frac{da}{a^2} (\Omega Ma^{-3} + \Omega_\Lambda)^{-1/2} \left[ 1 + \frac{1}{6} \langle l_{ii} \rangle_0 - \langle l_{ii} \rangle \right]. \quad (C4)\]

Using Eq.(13), moreover, we can express \(W(a)\) as

\[W(a) = \frac{1}{z} \int_a^1 \frac{da}{a^2} (\Omega Ma^{-3} + \Omega_\Lambda)^{-1/2} \left[ 1 + \frac{1}{6} \zeta \left[ Z(1)^2 - Z(a)^2 \right] \right]. \quad (C5)\]

Here we have the relation for \(z\) as

\[z (= 1/a - 1) = z_{\text{rem}} - \frac{1}{6} \zeta \left[ Z(1)^2 - Z(a)^2 \right]. \quad (C6)\]

For \(z \ll 1\), we get

\[z = z_{\text{rem}} - \frac{1}{3} \zeta \left[ Z(a)Z(a) \right] a \ (1 - a) = z_{\text{rem}} \left[ 1 - \frac{1}{3} \zeta \left[ Z(1)Y(1) \right] + O(z^2), \quad (C7)\]

where we used the relation \(dZ(a)/da = Y(a)/(\Omega_M + \Omega_\Lambda a^3)\).
On the other hand, we have a relation
\[
\frac{(H_{\text{kin}})}{H_0} = 1 + \frac{1}{3} \zeta Z(1)Y(1),
\]  
(C8)
neglecting small terms with $A$. So we obtain finally
\[
(H_{\text{kin}})_0 (d_L)_{\text{rem}} = z_{\text{rem}}(1 + z_{\text{rem}}) \Phi(z_{\text{rem}}) W(a),
\]  
(C9)
where $W(a)$ is given by Eq. (C5), $a = 1/(1 + z)$, and $z$ is related to $z_{\text{rem}}$ by Eq. (C6). The auxiliary function $\Phi(z_{\text{rem}})$ is expressed as
\[
\Phi(z_{\text{rem}}) \equiv \left( \frac{(H_{\text{kin}})_0}{H_0} \right)_{\text{rem}} z_{\text{rem}} = 1 + \frac{1}{3} \zeta Z(1)Y(1) - \frac{1}{6} \zeta Z(1)^2 - Z(a)^2) / z_{\text{rem}}.
\]  
(C10)
For $z_{\text{rem}} \ll 1$, $W$ and $\Phi$ are expanded as
\[
(1 + z_{\text{rem}}) W = 1 + \left[ 1 - \frac{3}{4} \Omega_M - \frac{1}{6} \zeta Z(1)Y(1) \right] z_{\text{rem}} + \cdots,
\]  
(C11)
and
\[
\Phi = 1 + \frac{1}{6} \zeta \left[ \frac{1}{2} (1 - 3\Omega_\Lambda) Z(1)Y(1) + Y(1)^2 + \frac{3}{2} \Omega_M Z(1)^2 \right] z_{\text{rem}} + \cdots.
\]  
(C12)

D. Renormalized model parameters for other background model parameters

Let us show the renormalized model parameters for other background model parameters such as
\[
H_0 = 67.3 \text{ km s}^{-1} \text{Mpc}^{-1}, \quad \Omega_\Lambda = 1 - \Omega_M, \quad \text{and } \Omega_M > 0.22.
\]  
(D1)
In these cases, we have
\[
k_{eq} \equiv a_{eq}H_{eq} = 219 \ (\Omega_M h) = 147.4 \ \Omega_M,
\]  
(D2)
From Eqs. (20), (25), (27) and (A12), we obtain
\[
\frac{(H_{\text{kin}})}{H_0} = 1 + \frac{2\pi}{3} (k_{eq})^4 P_{\text{R0}} Y(1)Z(1)B,
\]  
(D3)
\[
\frac{(H_{\text{dyn}})}{H_0} = 1 + \frac{2\pi}{3} (k_{eq})^4 P_{\text{R0}} [1 - Y(1)]Z(1)B,
\]  
(D4)
\[
(\Omega_\Lambda)_{\text{rem}} = \Omega_\Lambda [H_0/(H_{\text{dyn}})^2],
\]  
(D5)
where we neglected small terms with $A$. Here $Y(a)$ and $Z(a)$ depend on $\Omega_M$ and $\Omega_\Lambda$, while $B$ does not depend on them, but on $x_{\text{max}}$ and $x_{\text{min}}$.

For $(\Omega_M, \Omega_\Lambda) = (0.24, 0.76)$, we have
\[
k_{eq} = 35.4, \quad Y(1) = 0.557, \quad Z(1) = 1.232,
\]  
(D6)
so that
\[
\frac{(H_{\text{kin}})}{H_0} = 1 + 0.1043 (B/20.95),
\]  
(D7)
\[
\frac{(H_{\text{dyn}})}{H_0} = 1 + 0.0827 (B/20.95).
\]  
(D7)
Then we obtain
\[
(H_{\text{kin}})_0 = 74.3, \quad (H_{\text{dyn}})_0 = 72.8, \quad (\Omega_\Lambda)_{\text{rem}} = 0.65
\]  
(D8)
for $B = 20.95$ (with $x_{\text{max}} = 5.7$). From Eqs. (B12) and (B13), moreover, we have
\[
[q_0, \ (q_{\text{kin}})_0] = (-0.640, -0.638).
\]  
(D9)
For \((\Omega_M, \Omega_\Lambda) = (0.28, 0.72)\), we have

\[ k_{eq} = 41.3, \quad Y(1) = 0.540, \quad Z(1) = 1.095, \quad (\Omega_\Lambda)_{rem} = 0.65 \]

so that

\[
(H_{kin})_0 / H_0 = 1 + 0.166 \ (B/20.95),
\]

\[
(H_{dyn})_0 / H_0 = 1 + 0.141 \ (B/20.95). \tag{D11}
\]

Then we obtain

\[
(H_{kin})_0 = 71.5, \quad (H_{dyn})_0 = 70.8, \quad (\Omega_\Lambda)_{rem} = 0.65 \tag{D12}
\]

for \(B = 7.8\) (with \(x_{max} = 3.77\)). Moreover, we have

\[
[q_0, \ (q_{kin})_0] = (-0.580, \ -0.568). \tag{D13}
\]

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