MANIFEST (4,0) SUPERSYMMETRY, SIGMA MODELS AND THE ADHM INSTANTON CONSTRUCTION \(^1\) , \(^2\)

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ABSTRACT

Utilizing (4,0) superfields, we discuss aspects of supersymmetric sigma-models and the ADHM construction of instantons à la Witten.

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I. Introduction

Recently \[1\] an argument has been given that suggests that (4,0) sigma models provide a natural setting in which to discuss the instanton construction of Atiyah, Drinfeld, Hitchin and Manin \[2\]. As shown in reference \[1\], there is an elegant relation between (4,0) supersymmetric theories of scalar and spinor multiplets and the prior work on the construction of instantons. In his presentation, Witten, pointed out the need to study the extent to which it is possible to generalize these new results. This is the main purpose of this presentation. Namely, it is our goal to write out the most general possible model along the following lines. Foremost, the (4,0) action must involve scalar and spinor multiplets. Secondly, the spinor multiplets must have (generalized Yukawa-type) interactions. Thirdly, we demand the presence of manifest (4,0) supersymmetry at all stages of investigation.

Coincidently, just prior to the appearance of the work in \[1\], we had exactly developed the requisite tools \[3\] for this study while investigating aspects of N-extended supersymmetry within the confines of 1D supersymmetric quantum mechanical models. In fact, for a fixed N, there is a one-to-one correspondence between 1D models and 2D “heterotic” models. This should come as no surprise since on-shell “heterotic” models are just 1D models. In fact, as we shall shortly see, (4,0) unidexterous supersymmetry has an unimagined richness in the number of representations from which to build models along the lines described above. Furthermore all of the formulations that we shall use possess complete off-shell representations containing all the necessary auxiliary fields. This latter point is particularly important as it permits us to easily establish the explicit forms of the interactions between the scalar and spinor multiplets.

II. Free (4,0) Scalar Multiplets

One unexpected result found during the investigation of 1D, N = 4 supersymmetric models \[3\] was the surprisingly large number of such representations. There are, to our knowledge, four (4,0) scalar multiplets. This is an example of the phenomenon of variant superfield representations \[4\].

The first scalar multiplet contains spin-0 fields, \( A, B \) and spinor fields \( \psi^{-i} \). The supersymmetry variations of the component

\[
\begin{align*}
\delta_Q A &= 2 \epsilon^+ i C_{ij} \psi^{-j} , \\
\delta_Q B &= i 2 \epsilon^+ i \psi^{-i} , \\
\delta_Q \psi^{-i} &= i \epsilon^+ J^{ij} \partial_4 A - \epsilon^+ i \partial_4 B .
\end{align*}
\]
We refer to this as the (4,0) SM-I (scalar multiplet one) theory. The free kinetic
energy for the fields (written below) form a supersymmetric invariant given by

\[ S_{SM-I} = \int d^2 \sigma \left[ (\partial_\mathbf{A}) (\partial_\mathbf{A}) + (\partial_\mathbf{B}) (\partial_\mathbf{B}) + i2\bar{\psi}^- (\partial_\mathbf{\psi}^-) \right] . \tag{2.2} \]

The second scalar multiplet has the component structure \((\phi, \phi^i_j, \lambda^-_i)\) where the 
first two fields are bosons and the latter a fermion. The supersymmetry variations are

\[
\begin{align*}
\delta Q\phi &= i\epsilon^{+i}\lambda^-_i + i\bar{\epsilon}_{-i}\lambda^-_i , \\
\delta Q\phi^i_j &= 2(\epsilon^{+j}\lambda^-_i - \frac{1}{2}\delta^{j}_i\lambda^-_k) - 2(\epsilon^{+i}\lambda^-_j - \frac{1}{2}\delta^{i}_j\lambda^-_k) , \tag{2.3}
\delta Q\lambda^-_i &= -\bar{\epsilon}^+_i \partial_\mathbf{A}\phi - i\epsilon^{+_k}\partial_\mathbf{\phi}^k .
\end{align*}
\]

We refer to this as the (4,0) SM-II (scalar multiplet two) theory and the following 
action is invariant under these variations

\[ S_{SM-II} = \int d^2 \sigma \left[ (\partial_\mathbf{A}) (\partial_\mathbf{A}) + \frac{1}{2}(\partial_\mathbf{\phi}^i_j) (\partial_\mathbf{\phi}^j_i) + i2(\lambda^-_i\partial_\mathbf{\lambda}^-_i) \right] . \tag{2.4} \]

The third scalar multiplet has the component structure \((\mathbf{A}_i, \rho^-, \pi^-)\) where the 
first field is a boson and the latter two are fermions. The supersymmetry variations are

\[
\begin{align*}
\delta Q\mathbf{A}_i &= C_{ij}\epsilon^{+_j}\pi^- + \bar{\epsilon}^{+_i}\rho^- , \\
\delta Q\rho^- &= -i2\epsilon^{+_i}\partial_\mathbf{A}_i , \tag{5.5}
\delta Q\pi^- &= i2C^{ij}\epsilon^{+_i}\partial_\mathbf{A}_j .
\end{align*}
\]

We refer to this as the (4,0) SM-III (scalar multiplet three) theory and the follow-
ing action is invariant under these variations

\[ S_{SM-III} = \int d^2 \sigma \left[ (\partial_\mathbf{A}_i) (\partial_\mathbf{A}_i) + i\frac{1}{2}\rho^-\partial_\mathbf{\rho}^- + i\frac{1}{2}\pi^-\partial_\mathbf{\pi}^- \right] . \tag{2.6} \]

The final scalar multiplet (known to us) has the component structure \((\mathbf{B}_i, \psi^-, \psi^-_i)\) 
where the first field is a boson and the latter two are fermions. The supersymmetry 
variations are

\[
\begin{align*}
\delta Q\mathbf{B}_i &= \bar{\epsilon}^{+_i}\psi^- - i2\bar{\epsilon}^{+_j}\psi^-_i , \\
\delta Q\psi^- &= -i\epsilon^{+_i}\partial_\mathbf{B}_i + i\bar{\epsilon}^{+_i}\partial_\mathbf{B}_i , \tag{7.7}
\delta Q\psi^-_i &= (\epsilon^{+_j}\partial_\mathbf{B}_i - \frac{1}{2}\delta^{j}_i\epsilon^{+_k}\partial_\mathbf{B}_k) + (\epsilon^{+_i}\partial_\mathbf{B}^j - \frac{1}{2}\delta^{i}_j\epsilon^{+_k}\partial_\mathbf{B}^k) .
\end{align*}
\]
We refer to this as the (4,0) SM-IV (scalar multiplet four) theory and the following action is invariant under these variations

\[ S_{\text{SM-IV}} = \int d^2 \sigma \left[ (\partial_4 \mathcal{B}^i) (\partial_{\pm} \mathcal{B}_i) + i\frac{1}{2} \psi^- \partial_{\pm} \psi^- + i\psi^- i \partial_{\pm} \psi^+ i \right] . \]  

(2.8)

We summarize these results in the table below.

| Multiplet | Spin-0 SU(2) Rep | Spin-$\frac{1}{2}$ SU(2) Rep |
|-----------|------------------|-----------------|
| SM – I    | 4s               | $\frac{1}{2}$  |
| SM – II   | 1s1p             | $\frac{1}{2}$  |
| SM – III  | $\frac{1}{2}$   | 4s              |
| SM – IV   | $\frac{1}{2}$   | 1s1p            |

Table I

III. Free (4,0) Spinor Multiplets

Similarly, for “minus spinor multiplets” (MSM) the same large number of theories make their appearance. There are four such multiplets. The fact that there are precisely four minus spinor multiplets is no accident. Each of these multiplets can be paired with one of the scalar multiplets by a recently recognized type \[3\] of “fermionic duality.”

Fermionic Duality Pairs

| Scalar Multiplet | Fermionic Dual Multiplet |
|------------------|--------------------------|
| SM – I           | MSM – I                  |
| SM – II          | MSM – II                 |
| SM – III         | MSM – III                |
| SM – IV          | MSM – IV                 |

Table II

The component fields of our first minus spinor multiplet are \((\rho^+_i, \mathcal{F}, \mathcal{H})\) and their supersymmetry variations are just:

\[ \delta_Q \rho^+_i = -C_{ij} \epsilon^+_j \mathcal{F} - i \epsilon^+_i \mathcal{H} , \]

\[ \delta_Q \mathcal{F} = -i 2 C^{ij} \epsilon^+_i \partial_4 \rho^+_j , \]  

(3.1)

\[ \delta_Q \mathcal{H} = 2 \epsilon^+_i \partial_4 \rho^+_i . \]
We refer to this as the (4,0) MSM-I (minus spinor multiplet one) theory. It has a supersymmetrically invariant action given by

$$S = \int d^2 \sigma \left[ i\bar{\rho}^+(\partial_+ \rho^+ i) + \frac{1}{2} \mathcal{F} \bar{\mathcal{F}} + \frac{1}{2} \mathcal{H} \bar{\mathcal{H}} \right] . \quad (3.2)$$

The next minus spinor is composed of component fields ($\chi^+ i$, $H$, $H^i j$) together with the supersymmetry variations given by,

$$\delta Q \chi^+ i = -\frac{i}{2} \epsilon^+ i H - i 2 \epsilon^+ k H^k \, ,$$
$$\delta Q H = i 2 \epsilon^+ i \partial_+ \chi^+ i + i 2 \epsilon^+ i \partial_+ \bar{\chi}^+ i \, ,$$
$$\delta Q H^i j = \left( \epsilon^{+j} \partial_+ \chi^+ i - \frac{1}{2} \delta^{i j} \epsilon^+ k \partial_+ \chi^+ k \right) - \left( \bar{\epsilon}^+ i \partial_+ \bar{\chi}^+ j - \frac{1}{2} \delta^{i j} \bar{\epsilon}^+ k \partial_+ \bar{\chi}^+ k \right) . \quad (3.3)$$

We refer to this as the (4,0) MSM-II (minus spinor multiplet two) theory and its invariant free action is,

$$S = \int d^2 \sigma \left[ i \bar{\chi}^+ (\partial_+ \chi^+ i) + \frac{1}{8} H^2 + H^i j H^j i \right] . \quad (3.4)$$

Component fields of the third multiplet consist of ($\alpha^+$, $\beta^+$, $C_i$) which have the following supersymmetry variations

$$\delta Q \alpha^+ = -i 2 \epsilon^+ C_i \, ,$$
$$\delta Q \beta^+ = i 2 C^{ij} \epsilon^+ j C_j \, ,$$
$$\delta Q C_i = C^{ij} \epsilon^+ j \partial_+ \beta^+ + \epsilon^+ i \partial_+ \alpha^+ \, . \quad (3.5)$$

We refer to this as the (4,0) MSM-III (minus spinor multiplet three) theory with invariant action given by,

$$S = \int d^2 \sigma \left[ i \bar{\alpha}^+ (\partial_+ \alpha^+) + i \bar{\beta}^+ (\partial_+ \beta^+) - 2 \bar{\alpha} \beta^+ \right] . \quad (3.6)$$

The final such multiplet has fields ($\chi^+$, $\chi^+ i$, $F_i$) whose supersymmetry variations explicitly take the form

$$\delta Q \chi^+ = -i \epsilon^+ i \bar{F}_i + i \bar{\epsilon}^+ i \bar{F}_i \, ,$$
$$\delta Q \chi^+ j = \left( \epsilon^{+j} \bar{F}_i - \frac{1}{2} \delta^{i j} \epsilon^+ k \bar{F}_k \right) + \left( \bar{\epsilon}^{+j} \bar{F}^i - \frac{1}{2} \delta^{i j} \bar{\epsilon}^+ k \bar{F}^k \right) \, ,$$
$$\delta Q F_i = \bar{\epsilon}^+ i \partial_+ \chi^+ j - i 2 \bar{\epsilon}^+ j \partial_+ \chi^+ i \, . \quad (3.7)$$

We refer to this as the (4,0) MSM-IV (minus spinor multiplet four) theory. The action left invariant under these supersymmetry variations is

$$S = \int d^2 \sigma \left[ i \bar{\chi}^+ (\partial_+ \chi^+) + i \bar{\chi}^+ i (\partial_+ \chi^+ j) + \bar{F}_i F_i \right] . \quad (3.8)$$
In closing, we note that our manifest (4,0) supersymmetric formulation shows one interesting modification to the formulation of reference [1]. In his work, Witten introduced on-shell spinors (in his notation \( \lambda^a_+ \), see his equation (2.8)) where the number of these fields was some arbitrary integer \( a = 1, \ldots, n \). Here we see that the number of real component spinors must always be equal to a multiple of four. In fact, if \( n \) is not a multiple of four, the resulting theory is not even (4,0) supersymmetric!

IV. Generalized ADHM (4,0) Mass & Yukawa Interactions

In the previous two sections, we have explicitly seen the great abundance of (4,0) supersymmetric scalar and spinor multiplets. There are a total of eight different multiplets that we must consider in the class of actions of our interest. The general member of this class has the form,

\[
S = S_{\text{Free}} + S_{\text{Mass}} + S_{\text{Yukawa}},
\]

where \( S_{\text{Free}} \) is any linear combination of the free actions that we have seen in the previous two sections. Before proceeding with our considerations, it is useful to note that the problem of introducing the most general potential in (p,0) supersymmetric models has been studied previously in terms of (1,0) superfields [7]. In the remainder of this section, we focus our attention on finding the most general mass and interaction Lagrangian consistent with the proposal of Witten. It simplifies our discussion in that we need only consider mass and ordinary Yukawa-type couplings. A priori, 2D field theory admits generalized Yukawa-type couplings of the form \( B^n \times F \times F \) where \( B \) denotes a bosonic field while \( F \) denotes a fermionic one. In general \( n \) can be an arbitrary integer. However, due to supersymmetry, the restriction that the scalar potential be of no greater than degree (2,2), restricts us to the cases of \( n = 0, 1 \).

We first consider the \( n = 0 \) mass terms. It is simple to see that there is a unique SM-I mass term given by,

\[
S_{M_1} = M_1 \int d^2 \sigma \left[ \psi^{-i} \rho^+_i + \frac{1}{2} A F - \frac{1}{2} BH \right] + \text{h.c.},
\]

while the SM-II mass term is just

\[
S_{M_2} = M_2 \int d^2 \sigma \left[ i \lambda^{-i} \chi^+_i + i \bar{\lambda}^{-i} \chi^+_i - \frac{1}{2} \phi H - \phi_i^j H^i_j \right].
\]

Continuing, we have the SM-III mass term

\[
S_{M_3} = M_3 \int d^2 \sigma \left[ i \frac{1}{2} \pi^{-} \alpha^+ - i \frac{1}{2} \rho^{-} \beta^+ + C^{ij} A_i C_j \right] + \text{h.c.},
\]
and finally the SM-IV mass term given by
\[ S_{M_4} = M_4 \int d^2 \sigma \left[ i \psi^- \chi^+ + i 2 \psi^- i \chi^+_j i - B_i \bar{F}_i - \bar{B}^i F_i \right] . \] (4.5)

It is of interest to note that one linear combination of these mass terms corresponds precisely to the \( N = 4 \) (i.e. \( (4,4) \)) mass term that was recently discussed [5]. This is in accord with a conjecture of Witten [8] that in the limit of vanishing instanton size there should correspond a full \( N = 4 \) ADHM sigma model. The ADHM sigma model mass term for which this is true is the sum of \( S_{M_1} \) and \( S_{M_2} \). The reason for this is obvious, the twisted-I multiplet in reference [5] is the sum of SM-I and MSM-II and the twisted-II multiplet in reference [5] is the sum of SM-II and MSM-I!

Our manifest \( (4,0) \) supersymmetric formulation of the multiplets may be added together without changing the underlying \( (4,0) \) supersymmetry algebra which always takes the form (in terms of the D-algebra),
\[ \{ D_{+i} , D_{+j} \} = 0 , \quad \{ D_{+i} , \bar{D}_{+j} \} = i 2 \delta_{ij} \partial \bar{\tau} . \] (4.6)

This algebra is realized on all of the component fields without the use of any equations of motion.

The next step in our analysis will take advantage of the fact that we have already found the mass terms. We re-write these as
\[ S_{Y_1} = \int d^2 \sigma \left[ s^{-i} \rho^+_i + \frac{1}{2} r_i A F - \frac{1}{2} t_B H \right] + \text{h.c.} , \] (4.7)
\[ S_{Y_2} = \int d^2 \sigma \left[ iq^{-i} \bar{\chi}^+_i + i \bar{q}^{-i} \chi^+_i - \frac{1}{2} H_i H - t_i^j H_j^i \right] , \] (4.8)
\[ S_{Y_3} = \int d^2 \sigma \left[ i \frac{1}{2} p^{-i} \alpha^+ - i \frac{1}{2} h^{-i} \beta^+ + L^i C_j \right] + \text{h.c.} , \] (4.9)
\[ S_{Y_4} = \int d^2 \sigma \left[ i K^- \chi^+ + i 2 K^- i \chi^+_j i - T_i \bar{F}_i - \bar{T}^i F_i \right] . \] (4.10)

With an appropriate identification of the coefficients these reduce back to the mass terms. However, we can use these expressions in a different way to search for the \( n = 1 \) Yukawa terms! Each of the sets of functions that appear in (4.7 - 4.10) constitute a “section” along the lines defined in reference [7]. For example, a very simple choice of
these sections corresponds to the introduction of the (4,0) cosmological term. Interestingly enough, if we maintain SU(2) covariance, the cosmological term only exists for the sections in (4.7) and (4.8). The cosmological term corresponds to

\[(r_A, t_B, s^{-i}) \equiv (c_1, c_2, 0) ,
(r_H, t^i_j, q^{-i}) \equiv (c_3, 0, 0). \quad (4.10)\]

for arbitrary complex constants \(c_1, c_2\) and real constant \(c_3\). It is a simple matter to show that these choices of the two sections are consistent with the supersymmetry variations of SM-I and SM-II multiplets, respectively.

V. Yukawa Section Selection Rules

This brings us straight away to the actual Yukawa-type \(n = 1\) terms. We begin our analysis by a simple enumeration of all such actions. A convenient notational device for this purpose is provided by the introduction of a “model vector” of the form \((\text{SM}, \text{SM}' | \text{MSM})\). The first entry takes on values I,..., IV labelling which scalar multiplet is used. The second entry takes on the same values for the same purpose. The final entry takes on the same values but indicates which minus spinor multiplet appears. Following the construction given by Witten, the first two entries must be chosen to be different. Finally, the SU(2) symmetry (that ultimately arises from (4,0) supergravity) places some restrictions on which minus spinor can appear coupled to particular pairs of scalar multiplets. When all of this is taken into account, we find that there are only twelve possibilities to consider.

\[
\begin{align*}
(I, \ II| \ I) & \quad (III, \ IV| \ I) \\
(I, \ II| \ II) & \quad (III, \ IV| \ II) \\
(I, \ III| \ III) & \quad (I, \ IV| \ III) \quad (II, \ III| \ III) \quad (II, \ IV| \ III) \\
(I, \ III| \ IV) & \quad (I, \ IV| \ IV) \quad (II, \ III| \ IV) \quad (II, \ IV| \ IV)
\end{align*}
\]

(5.1)

As long as the sections transform as the scalar multiplet that is the fermionic dual of the spinor multiplet in each of the actions, (4,0) supersymmetry will be maintained. This is a critical point! The section must not only provide a representation of (4,0) supersymmetry. It must also be in the fermionic dual representation of the spinor multiplet! Thus, for example, \((r_A, t_B, s^+_i)\) must transform like the components of an SM-I type multiplet. Of course, similar statements must be true about the other corresponding terms in the other actions. Since we are only concerned with the \(n = 1\) Yukawa terms, \(s^+_i, r_A\) and \(t_B\) can only depend on monomials of degree (1,1) and
consistent with the first line of (5.1). We have investigated these conditions and we can find no non-trivial solutions! In other words, using all-known manifest (4,0) formulations implies the impossibility of writing Yukawa terms!

VI. (1, 0) and On-shell Supersymmetry Analysis

Since we have found the surprising and striking result that the use of all known manifest (4,0) supermultiplets leads to the impossibility of constructing a model along the lines outlined in reference [1], it seems as though only allowing a lower manifest supersymmetry representation or even an on-shell representation might permit such a construction. For example, we can attempt a (1,0) superfield formulation. Fortunately, this precise problem has been studied in great detail previously [7]. In fact, the portion of the work of [1] that contains the discussion of including the minus spinor fields is covered as a special case of the more general work of [7]. Witten’s equation (2.10) can be recognized as a special case of (2.5) (or (3.1)) in the first (or second) work of reference [7]. Utilizing these previous analyses (after modifying them to accommodate for two commuting sets of quaternionic complex structure (see appendix B)), we find that only in the case of on-shell supersymmetry can a model as described in reference [1] be consistent.

VII. Summary

One point we have found is a remarkable and long overlooked fact in the area of 2D, (4,0) sigma models. Namely, the existence of variant representations implies that there is a great diversity of representations for multiplets and the actual construction of supersymmetric invariant potentials depends crucially on pairing supersymmetrically dual representations.

In this paper, we have solved a problem that was suggested by the work in reference [1]. We have seen that the “missing” auxiliary fields (absent in reference [1]) have greatly facilitated the analysis of what possible actions may be taken as the starting point in the most general manifestly (4,0) supersymmetric action of (4,0) scalar and spinor multiplets. The most surprising result of this analysis is that the construction of Witten lies outside this category of models!

This raises some very interesting questions with regard to the quantum renormalization behavior of these models. Within the usual manifestly (4,0) supersymmetric models, it is possible to derive non-renormalization theorems based on the fact that manifestly (4,0) supersymmetric models are always equivalent to unconstrained...
superfield formulations of these theories. Superfield perturbation theory lies at the heart of the proofs of the non-renormalization theorems. Quantum supergraphs require unconstrained superfield formulations.

On-shell supersymmetric realizations cannot rely on their equivalence to an unconstrained superfield formulations. In fact, on-shell supersymmetric realizations cannot be written in terms of unconstrained superfields. The quantum mechanical behavior of such theories may be quite different from superfield theories. Thus, it is of interest to investigate further the quantized versions of the models of [1]. In fact, this is a very general question that has not been investigated previously to our knowledge. Stated most succinctly this question reduces to: “Is the quantum mechanical behavior of an on-shell supersymmetric representation always the same as the quantum mechanical behavior of an off-shell supersymmetric representation?” This suggests an avenue requiring future study.

Since we have used a manifest (4,0) supersymmetric formulation, it is straightforward to couple our matter systems to (4,0) supergravity. This may prove to be an interesting exercise. The reason for this is that although all of our models appear equivalent at the level of rigid supersymmetry, there are very great differences in the presence of local (4,0) supersymmetry. For example, the scalar and spinors in the SM-I and SM-II multiplets couple in a very different manner to the (4,0) SU(2) supergravity gauge fields than do those in the SM-III or SM-IV multiplets. This also raises the question of whether the models of [1] can be coupled to supergravity. There are cases in the literature of on-shell representations that cannot be coupled to supergravity. Pursuing this question provides yet another interesting avenue for future study.

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APPENDIX A: Conventions and Definitions

In heterotic theories, we always adhere to “helicity index conventions” that were first established in reference [6] and modified in reference [9]. Thus, a single + (−) sign denotes helicity plus (minus) one-half. We denote spinors by $\psi_+$ as an example. In these conventions, the components of a vector must have either two + or − indices. Rather than writing two such indices, we “double them up” by using the symbols $\dagger$ or $\ast$. Thus vectors are typically denoted by $A_\dagger$ or $A_\ast$. The helicity index conventions on such vectors are perfectly equivalent to writing vectors in terms of their light-cone coordinates. This type of convention also has the added advantage that by simply counting the number of helicity indices on a quantity, we can distinguish whether it transforms as a boson or fermion under the 2D Lorentz group. Typically bosons have even numbers of such indices and fermions have odd numbers. Throughout this paper, we generically use $i, j, ...$ to denote the components of the defining representation of SU(2).

APPENDIX B: Real Formulation of (4,0) Multiplets

It may be useful for future applications to re-write some of our results in terms of only real fields. It is obvious that at the bottom of all of the (4,0) multiplets discussed in this paper, there are four real bosons and four real (Majorana-Weyl) spinors. The supersymmetry variations of these multiplets can thus be expressed in totally real form. For this purpose, we will denote the four scalar fields in any of the scalar multiplets by $\varphi_A$ with $A = 1, 2, 3, 4$. Similarly, we introduce four real spinors denoted by $\Psi^-\hat{A}$ with $\hat{A} = 1, 2, 3, 4$. In order to have a (4,0) supersymmetry, a set of supersymmetry variations can take the form,

$$\delta_Q \varphi_A = i \alpha^+ p (L_p)_A^{\hat{A}} \Psi^-_{\hat{A}}, \quad \delta_Q \Psi^-_{\hat{A}} = \alpha^+ p (R_p)_A^{A} \hat{A} \partial_{\hat{A}} \varphi_A \quad (B.1)$$

written in terms of four real constant Grassmann parameters $\alpha^+$. In order to form a (4,0) supersymmetry algebra, the real quantities $(L_p)_A^{\hat{A}}$ and $(R_p)_A^{A}$ must satisfy

$$(L_p)_A^{\hat{A}} (R_q)_\hat{B}^{\hat{A}} + (L_q)_A^{\hat{A}} (R_p)_\hat{A}^{\hat{B}} = -2 \delta_{pq} (I)_{A}^{B}, \quad (R_p)_A^{\hat{A}} (L_q)_\hat{B}^{\hat{A}} + (R_q)_A^{\hat{A}} (L_p)_\hat{A}^{\hat{B}} = -2 \delta_{pq} (I)_{A}^{\hat{B}}. \quad (B.2)$$

In other words, the L-matrices and R-matrices are generalized 4x4 Pauli matrices. Thus, to express the SM-I theory and the SM-II theory in real notation, it is enough to specify the L-matrices and R-matrices associated with each multiplet. A simple calculation reveals that there exists a basis in which the SM-I multiplet is associated
with the set
\[
\begin{align*}
L_1 &= i\sigma^1 \otimes \sigma^2 \quad ; \quad R_1 = i\sigma^1 \otimes \sigma^2 \quad ; \\
L_2 &= i\sigma^2 \otimes I \quad ; \quad R_2 = i\sigma^2 \otimes I \quad ; \\
L_3 &= -i\sigma^3 \otimes \sigma^2 \quad ; \quad R_3 = -i\sigma^3 \otimes \sigma^2 \quad ; \\
L_4 &= -I \otimes I \quad ; \quad R_4 = I \otimes I ,
\end{align*}
\]
(B.3)
and the SM-II multiplet is associated with
\[
\begin{align*}
L_1 &= i\sigma^2 \otimes \sigma^3 \quad ; \quad R_1 = i\sigma^2 \otimes \sigma^3 \quad ; \\
L_2 &= -iI \otimes \sigma^2 \quad ; \quad R_2 = -iI \otimes \sigma^2 \quad ; \\
L_3 &= i\sigma^2 \otimes \sigma^1 \quad ; \quad R_3 = i\sigma^2 \otimes \sigma^1 \quad ; \\
L_4 &= I \otimes I \quad ; \quad R_4 = -I \otimes I .
\end{align*}
\]
(B.4)
These generalized Pauli matrices resemble complex structures. In fact, there is a relation between complex structures and these matrices. If we define \((f_p)_{AB} \equiv (L_p R_r)_{AB}\) for fixed \(r\) not equal to \(p\), it can be seen that \(f_p\) defines a triplet of complex structures. (The same follows if \((f_p)_{\hat{A} \hat{B}} \equiv (R_p L_r)_{\hat{A} \hat{B}}\).) Equivalently, \((f_{pq})_{AB} \equiv \frac{1}{2}(L_p R_q - L_q R_p)_{AB}\) and \((f_{pq})_{\hat{A} \hat{B}} \equiv \frac{1}{2}(R_p L_q - R_q L_p)_{\hat{A} \hat{B}}\) (for unrestricted \(p\) and \(q\)) also define triplets of complex structures. Finally, we point out that if we use \(f_p\) to denote the complex structures associated with SM-I and use \(\tilde{f}_q\) to denote the complex structures associated with SM-II, then \([f_p , \tilde{f}_q] = 0\) (or equivalently \([f_{pq} , \tilde{f}_{rs}] = 0\)).

For the SM-III and SM-IV multiplets we will denote the four scalar fields by \(\varphi_{\hat{A}}\) and the four real spinors by \(\Psi^{-A}_{\hat{A}}\). In order to have \((4,0)\) supersymmetry, the set of supersymmetry variations take the forms,
\[
\delta_Q \varphi_{\hat{A}} = i\alpha^+ \cdot p(R_p)_{\hat{A}} \cdot \hat{A} \cdot \Psi^{-A}_{\hat{A}} \quad , \quad \delta_Q \Psi^{-A}_{\hat{A}} = \alpha^+ \cdot p(L_p)_{\hat{A}} \cdot \hat{A} \cdot \partial \cdot \varphi_{\hat{A}} .
\]
(B.5)
There exists a 1D, non-local duality transformation by which we can actually derive (B.5) starting from (B.1).

Finally very similar results follow for the spinor multiplets. In real notation MSM-I and MSM-II take the form (below \(F_{\hat{A}}\) denote the auxiliary fields),
\[
\delta_Q \Psi^{+A}_{\hat{A}} = i\alpha^+ \cdot p(L_p)_{\hat{A}} \cdot \hat{A} \cdot F_{\hat{A}} \quad , \quad \delta_Q F_{\hat{A}} = \alpha^+ \cdot p(R_p)_{\hat{A}} \cdot \hat{A} \cdot \partial \cdot \Psi^{+A}_{\hat{A}} \quad ,
\]
(B.6)
with MSM-I and MSM-II associated with (B.3) and (B.4), respectively. For MSM-III and MSM-IV we have
\[
\delta_Q \Psi^{+A}_{\hat{A}} = i\alpha^+ \cdot p(R_p)_{\hat{A}} \cdot \hat{A} \cdot F_A \quad , \quad \delta_Q F_A = \alpha^+ \cdot p(L_p)_{\hat{A}} \cdot \hat{A} \cdot \partial \cdot \Psi^{+A}_{\hat{A}} \quad ,
\]
(B.7)
with MSM-III and MSM-IV associated with (B.3) and (B.4), respectively.
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