Determining a Slater Winner is Complete for Parallel Access to NP

Michael Lampis
Université Paris-Dauphine, PSL University, CNRS, LAMSADE, 75016, Paris, France

Abstract

We consider the complexity of deciding the winner of an election under the Slater rule. In this setting we are given a tournament $T = (V, A)$, where the vertices of $V$ represent candidates and the direction of each arc indicates which of the two endpoints is preferable for the majority of voters. The Slater score of a vertex $v \in V$ is defined as the minimum number of arcs that need to be reversed so that $T$ becomes acyclic and $v$ becomes the winner. We say that $v$ is a Slater winner in $T$ if $v$ has minimum Slater score in $T$.

Deciding if a vertex is a Slater winner in a tournament has long been known to be NP-hard. However, the best known complexity upper bound for this problem is the class $\Theta^p_2$, which corresponds to polynomial-time Turing machines with parallel access to an NP oracle. In this paper we close this gap by showing that the problem is $\Theta^p_2$-complete, and that this hardness applies to instances constructible by aggregating the preferences of 7 voters.

2012 ACM Subject Classification Theory of computation → Computational complexity and cryptography → Problems, reductions and completeness

Keywords and phrases Slater winner, Feedback Arc Set, Tournaments

Digital Object Identifier 10.4230/LIPIcs.STACS.2022.5

Related Version Full version: https://arxiv.org/abs/2103.16416

Acknowledgements I am grateful to Jérôme Lang for letting me know about this problem and for correctly conjecturing that it is complete for $\Theta^p_2$.

1 Introduction

Voting rules, which are a topic of central interest in computational social choice, are schemes which allow us to aggregate the preferences of a set of voters among a set of candidates, in order to select a single winner who is most compatible with the voters’ wishes. The main challenge of this area is that, even if the preferences of each voter are internally consistent (that is, each voter has a complete ranking of all candidates), it is easy to run into situations such as the famous Condorcet paradox where collective preferences are cyclic and hence no clear winner exists. Many rules have therefore been proposed to deal with this situation and select a winner who is as acceptable as possible to as many voters as possible.

In this paper we investigate the computational complexity of a classical and very natural such voting scheme that is often referred to as the Slater rule. Intuitively, the idea of the Slater rule is the following: we consider every possible pair of candidates in our pool $a, b$ and check whether the majority of voters prefers $a$ or $b$. This allows us to construct a tournament $T$ that depicts the results of each pairwise matchup between candidates. If $T$ is transitive (that is, acyclic), then picking a winner is easy. If not, the Slater rule is that we should select as the winner a candidate who is the winner of a transitive tournament $T'$ that is at minimum edit distance from $T$. In other words, a candidate $c$ is a Slater winner if the number of pairwise matchups that we need to ignore to make $c$ a clear winner is minimized.

More formally, the problem we consider is defined as follows. We are given a set $V$ of $n$ candidates and the preferences of $m$ voters, where each voter’s preferences are given as a total ordering of $V$. We determine a pair-wise relation on $V$ as follows: for $a, b \in V$ we say
that a wins against b if the majority of voters prefers candidate a over candidate b. In this way, assuming that there are no ties (which is guaranteed if the number of voters is odd), we can construct a tournament \( T = (V, A) \), where we have the arc \( b \rightarrow a \) (that is \((b, a) \in A\)) if a wins against b. In this setting, the Slater score of a candidate c is the minimum number of arcs of T that need to be reversed so that T becomes transitive (acyclic) with c being placed last (that is, with c being a sink). The Slater winner of a tournament is a candidate with minimum Slater score. Intuitively, a candidate c is a Slater winner if there exists a linear ordering \( \prec \) of the candidates that ranks c as the winner and is as compatible as possible with the voters’ aggregated preferences, in the sense that the edit distance between \( \prec \) and T is minimum.

The notion of Slater winner is very well-studied and can be seen as a special case of Kemeny voting. Indeed, in Kemeny voting we construct a weighted tournament where the weight of the arc \( b \rightarrow a \) denotes the margin of victory of a over b. In this sense, the Slater system corresponds to a version of Kemeny voting where we only retain as information which of the two candidates would win a head-to-head match-up, but ignore the margin of victory. In other words, Slater voting is the special case of Kemeny voting where the arcs are unweighted. For more information about these and other related voting systems, we refer the reader to [5].

The main question we are interested in in this paper is the computational complexity of determining if a vertex v of a tournament is a Slater winner. It has long been known that this question is at least NP-hard [15]. Indeed, it is not hard to see that if we had an oracle for the Slater problem we would be able to produce in polynomial time an ordering of any tournament in a way that minimizes the number of inverted arcs. This would solve the Feedback Arc Set problem, which is known to be NP-complete on tournaments [1, 2, 7, 9]. On the other hand, membership of this problem in NP is not obvious. The best currently known upper bound on its complexity is the class \( \Theta^P_2 \), shown by Hudry [8, 15].

The class \( \Theta^P_2 \) seems like a natural home for the Slater problem. As a reminder, this class captures as a model of computation Turing machines that run in polynomial time and which are allowed to use an NP oracle either a polynomial number of times non-adaptively (that is, with questions not being allowed to depend on previous answers), or a logarithmic number of times adaptively. Hence, this class is often called “Parallel Access to NP” and written as \( P^{NP} \). Intuitively, solving the Slater problem requires us to calculate exactly a value that is NP-hard to compute (the minimum feedback arc set of a tournament). This can be done either by asking polynomially many non-adaptive NP queries to an oracle (for each \( k = 1, 2, \ldots \) we ask if the feedback arc set has size at most \( k \)), or a logarithmic number of adaptive queries (where we essentially perform binary search). It has therefore been conjectured that determining if a candidate is a Slater winner is not just NP-hard, but \( \Theta^P_2 \)-complete [3, 5, 10]. We recall that \( \Theta^P_2 \) is strongly suspected to be a much larger class than NP – indeed, because \( \Theta^P_2 \) contains all of the so-called Boolean hierarchy of classes, it is known that if it were the case that \( \Theta^P_2 = NP \), then the polynomial hierarchy would collapse [9]. Hence, the difference between the known upper and lower bounds on the complexity of determining a Slater winner is not trivial.

The result we present in this paper settles this problem. We confirm the conjecture that determining the Slater winner of a tournament is indeed \( \Theta^P_2 \)-complete. This places Slater voting in the same class as related voting schemes, such as Kemeny [14], Dodgson [12], and Young [16]. It also places it in the same class as the Slater rule used in [11] for a more general judgment aggregation problem. We prove this result by modifying the reduction of Conitzer [9], which showed that Feedback Arc Set on tournaments is NP-complete. The main
difference is that, rather than reducing from SAT, we need to reduce from a $\Theta_2^p$-complete variant, where we are looking for a maximum weight satisfying assignment that sets a certain variable to True. This forces us to significantly complicate the reduction because we need to encode in the objective function not only the number of satisfied clauses but also the weight of the corresponding assignment.

Having settled the worst-case complexity of the problem in general, we go on to consider a related question: what is the minimum number of voters for which determining a Slater winner is $\Theta_2^p$-complete? The motivation behind this question is that, even though any tournament can be constructed by aggregating the preferences of a large enough number of voters, if the number of voters is limited, some tournaments can never arise. Hence the problem may conceivably be easier if the number of voters is bounded. In the case of the Slater rule, Bachmeier et al.\cite{ati} have shown that determining the Slater winner remains NP-hard for 7 voters. By reusing and slightly adjusting their arguments we improve their complexity lower bound to $\Theta_2^p$-completeness for 7 voters.

## 2 Definitions and Preliminaries

A tournament is a directed graph $G = (V, A)$ such that for all $x, y \in V$, exactly one of the arcs $(x, y), (y, x)$ appears in $A$. A feedback arc set (fas) of a digraph $G = (V, A)$ is a set of arcs $A' \subseteq A$ such that deleting $A'$ from $G$ results in an acyclic digraph. If $G$ is a tournament and $A'$ is a fas of $G$, then the tournament obtained from $G$ by reversing the direction of all arcs of $A'$ is acyclic (or transitive). We will say that a total ordering $\prec$ of the vertices of a digraph $G = (V, A)$ implies the fas $S = \{(x, y) \mid (x, y) \in A, y \prec x\}$ (in the sense that $S$ is the set of arcs that disagree with the ordering). We will say that an ordering of $V$ is optimal if the fas it implies has minimum size.

Given a digraph $G = (V, A)$ and $v \in V$, we say that $v$ is a Slater winner if for some $k \geq 0$ the following hold: (i) there exists a fas $S \subseteq A$ of $G$, such that $v$ is a sink of $G - S$ and $|S| = k$ (ii) every fas of $G$ has size at least $k$. If $v$ is a Slater winner in $G = (V, A)$, then a winning ordering for $v$ is a linear ordering of $V$ that places $v$ last and implies a fas of $G$ of minimum size.

In a digraph $G = (V, E)$, a set $M \subseteq V$ is a module if the following holds: for all $x, y \in M$ and $z \not\in M$ we have $(x, z) \in E \iff (y, z) \in E$ and $(z, x) \in E \iff (z, y) \in E$. In other words, every vertex outside $M$ that has an arc to (respectively from) a vertex of $M$, has arcs to (respectively from) all of $M$. The following lemma, given by Conitzer [9] with slightly different terminology, states that the vertices of a module can, without loss of generality, always be ordered together. We say that the vertices of a set $S$ are contiguous in an ordering $\prec$ if there are no $x, y \in S, z \not\in S$ such that $x \prec z \prec y$.

**Lemma 1.** Let $G = (V, A)$ be a digraph, $v \in V$ a vertex, and suppose we have a partition of $V$ into $k$ non-empty modules $V = M_1 \sqcup M_2 \sqcup \ldots \sqcup M_k$. If $v$ is a Slater winner of $G$, then there exists a winning ordering for $v$ such that for all $i \in [k]$, the vertices of $M_i$ are contiguous.

**Proof.** Suppose $k > 1$ (otherwise the claim is trivial) and consider an ordering $\prec$ that is winning for $v$. We will say that a set of vertices $S \subseteq V$ is a block of $\prec$ if (i) $S \subseteq M_i$ for some $i \in \{1, \ldots, k\}$; (ii) $S$ is contiguous; (iii) $S$ is maximal, that is, adding any vertex to $S$ violates one of the two preceding properties.

---

\footnotetext[1]{This is a classical result known in the literature as McGarvey’s theorem.}
If the number of blocks is equal to \( k \) we are done, as each block is equal to a module so we have an ordering where each module is contiguous. If we have at least \( k + 1 \) blocks, we will explain how to edit the ordering so that it remains winning for \( v \), it implies a fas of the same size, and the number of blocks decreases. Repeating this process until we have \( k \) blocks completes the proof.

Consider two vertices \( x, y \), with \( x \prec y \), which belong to the same module, say \( x, y \in M_1 \), but in distinct blocks. Among all such pairs, select \( x, y \) so that their distance in the ordering, that is, the size of the set \( Z = \{ z \mid x \prec z \prec y \} \) is minimized. Let \( X, Y \) be the blocks that contain \( x, y \) respectively. Note that by the selection of \( x, y \) we have that \( x \) is the last vertex of \( X \), \( y \) is the first vertex of \( Y \), \( X \cup Y \subseteq M_1 \), and \( M_1 \cap Z = \emptyset \).

Let \( d_{\text{out}}^Z(x) \) (respectively \( d_{\text{in}}^Z(x) \)) be the out-degree (respectively in-degree) of \( x \) towards the set \( Z \). Because \( M_1 \) is a module, all vertices of \( M_1 \) have the same in-degree and out-degree towards \( Z \), and in particular, \( d_{\text{out}}^Z(x) = d_{\text{out}}^Z(y) \) and \( d_{\text{in}}^Z(x) = d_{\text{in}}^Z(y) \). Now, if \( d_{\text{in}}^Z(x) > d_{\text{in}}^Z(y) \), we can obtain an ordering that implies a smaller fas by placing \( x \) immediately after the last vertex of \( Z \). This would contradict the optimality of \( \prec \), so it must be impossible. Similarly, if \( d_{\text{out}}^Z(x) < d_{\text{out}}^Z(y) \), we have \( d_{\text{in}}^Z(y) < d_{\text{in}}^Z(x) \), and we can obtain a strictly better ordering by placing \( y \) immediately before the first vertex of \( Z \), contradiction. We conclude that \( d_{\text{in}}^Z(x) = d_{\text{in}}^Z(y) \). Therefore, moving all the vertices of \( X \) so that they appear immediately after the last vertex of \( Z \) produces an ordering which is equally good as the current one, is still winning for \( v \), and has a smaller number of blocks.

\[ \blacktriangleleft \]

### 2.1 Complexity

We recall the class \( \Theta_2^p \) which is known to have several equivalent characterizations, including \( \text{P}^{\text{NP}[\log n]} \) (P with the right to make \( O(\log n) \) queries to an NP oracle), \( \text{L}^{\text{NP}} \) (logarithmic-space Turing machines with access to an NP oracle), and \( \text{P}^{\text{NP}} \) (P with parallel non-adaptive access to an NP oracle). We refer the reader to [13] for more information on this class. In [17] it was shown that the following problem is \( \Theta_2^p \)-complete: given a graph \( G \), is the maximum clique size \( \omega(G) \) odd? In [10] it is mentioned that the following problem, called MAX MODEL, is \( \Theta_2^p \)-complete: given a satisfiable \( \text{CNF} \) formula \( \phi \) containing a special variable \( x \), is there a satisfying assignment of \( \phi \) that sets \( x \) to True and has maximum Hamming weight (among all satisfying assignments), where the Hamming weight of an assignment is the number of variables it sets to True.

We will use as a starting point for our reduction a variant of MAX MODEL which we show is \( \Theta_2^p \)-complete below. The main difference between this variant and the standard version is that we assume that the given formula is satisfied by the assignment that sets all variables to False.

\[ \blacktriangleleft \]

**Lemma 2.** The following problem is \( \Theta_2^p \)-complete. Given a \( \text{3-CNF} \) formula \( \phi \) containing a distinguished variable \( x \), such that \( \phi \) is satisfied by the all-False assignment, decide if there exists a satisfying assignment for \( \phi \) that sets \( x \) to True and has maximum weight among all satisfying assignments.

**Proof.** We start with a graph \( G = (V,E) \) for which the question is if the maximum independent set has odd size (clearly this is equivalent to the question of deciding if the maximum clique has odd size by taking the complement of \( G \), so our starting problem is \( \Theta_2^p \)-complete [17]). Let \( |V| = n \) and suppose \( V = \{v_1, \ldots, v_n\} \). We construct a formula \( \phi \) as follows: for each \( i \in \{1, \ldots, n\} \) we build \( (n+1) \) variables \( x_{i1}^1, \ldots, x_{i1}^{n+1} \) and for each \( j, k \in \{1, \ldots, n+1\} \) we add the clause \( (x_{ij}^j \rightarrow x_{ik}^k) \); for each \( (v_{i1}, v_{i2}) \in E \), for each \( j, k \in \{1, \ldots, n+1\} \) we add the clause \( (\neg x_{i1}^j \lor \neg x_{i2}^k) \); we construct \( n \) variables \( y_1, \ldots, y_n \) and clauses
that represent the constraints \((y_1 = x_1^1)\), and for each \(i \in \{2, \ldots, n\}\), \((y_i = y_{i-1} \oplus x_i^1)\). We set \(y_n\) as the distinguished variable of \(\phi\). The formula construction can clearly be carried out in polynomial time, and no clause has size more than three. Furthermore, setting everything to False satisfies all clauses. Intuitively, for each vertex we have constructed \(n+1\) variables that will be set to True if we take this vertex in the independent set. The first set of clauses ensures that we make a consistent choice among the copies; the second set that we indeed select an independent set; and the third calculates the parity of its size.

We now observe that independent sets \(S\) of \(G\) naturally correspond to satisfying assignments of \(\phi\). In particular, given an independent set \(S \subseteq V\) we can construct an assignment by setting, for all \(i, j\), \(x_i^j\) to True if and only if \(v_i \in S\); we then complete the assignment by giving appropriate values to the \(y_i\) variables so that the parity constraints are satisfied. For the converse direction, we can extract an independent set \(S\) from a satisfying assignment by setting \(v_i \in S\) if and only if the assignment sets \(x_i^1\) to True. We observe the \(y_n\) is set to True in a satisfying assignment if and only if the corresponding independent set has odd size (indeed, for each \(i\), \(y_i\) is set to True if the intersection of the independent set with the first \(i\) vertices has odd size).

Suppose now that there exists an independent set \(S\) of maximum size \(k\) and that \(k\) is odd. Then, there exists a satisfying assignment of maximum weight that sets \(y_n\) to True. Indeed, suppose for contradiction that the maximum satisfying assignment \(\sigma\) sets \(y_n\) to False. Then, the corresponding independent set \(S'\) must have even size \(k'\). Since \(k\) is odd and \(S\) is a maximum independent set, \(k' < k\). But then, the weight of \(\sigma\) is at most \(k'(n+1) + n < k(n+1)\). However, the assignment corresponding to \(S\) has weight at least \(k(n+1)\), contradiction.

For the converse direction, suppose there exists a satisfying assignment \(\sigma\) of maximum weight that sets \(y_n\) to True. The corresponding independent set \(S\) has odd size, say \(|S| = k\). If there exists a maximum independent set \(S'\) that has even size \(k'\), then \(k' > k\). However, the corresponding truth assignment \(\sigma'\) would have weight at least \(k'(n+1) > k(n+1) + n\). Since \(\sigma\) has weight at most \(k(n+1) + n\) we get a contradiction to the optimality of \(\sigma\).

We conclude that there is a satisfying assignment to \(\phi\) of maximum weight that sets \(y_n\) to True if and only if the maximum independent set of \(G\) has odd size.

### 3 Reduction to Slater

This section presents the main result of the paper, stated in Theorem 3. The theorem is based on a reduction from the problem of Lemma 2 to the problem of deciding if a vertex of a tournament is a Slater winner. Before we dive into the proof, let us give some high level intuition (we also invite the reader to take a look at Figure 1).

We will build a tournament to represent a CNF formula \(\phi\) with \(n\) variables and \(m\) clauses by constructing \(n\) groups of “large” modules \((A_i, B_i, C_i, D_i, E_i, F_i, \text{ for } i \in \{1, \ldots, n\}\) and \(m\) “small” modules \(T_j\) for \(j \in \{1, \ldots, m\}\). The internal structure of the modules will be irrelevant and we only care about their ordering, which we may assume to be contiguous thanks to Lemma 1. We will make sure to adjust the sizes of the modules and their connections so that we have the following properties:

1. In any reasonable ordering, all six large modules representing variable \(x_i\) come before the six modules representing \(x_{i+1}\). This will naturally order the large modules into \(n\) sections.
2. Inside a section, any reasonable ordering will place \(A_i, B_i, C_i\) first. Then, if the remaining modules are ordered \(D_i \prec E_i \prec F_i\), this encodes that \(x_i\) is set to True.
Determine a Slater Winner is Complete for Parallel Access to NP

(a) \( x_i \) does not appear in \( c_j \)  
(b) \( x_i \in c_j \)  
(c) \( \neg x_i \in c_j \)

Figure 1 Gadgets of the reduction of Theorem 3. On the left of each figure the six large modules \( A_i, B_i, C_i, D_i, E_i, F_i \) represent the variable \( x_i \). Missing (thick) arcs go downwards, so the ordering is forced except for the last three modules. The depicted ordering \( D_i \prec E_i \prec F_i \) encodes that \( x_i \) is True. In the three figures we depict the connections between the six modules and the small module \( T_j \) representing clause \( c_j \) depending on whether \( x_i \) appears in \( c_j \). In the first case placing \( T_j \) anywhere costs at least three arcs. In the second case, placing \( T_j \) after \( E_i \) costs two arcs, because setting \( x_i \) to True satisfies \( c_j \). In the last case, a similarly advantageous placement could be obtained by using the ordering \( E_i \prec F_i \prec D_i \) (which encodes that \( x_i \) is False) and putting \( T_j \) before \( D_i \).

3. Connections between \( T_j \) and variable modules will be such that if \( T_j \) is placed completely before or completely after the section of a variable \( x_i \), then the cost is the same. However, the cost may be lower if \( T_j \) is placed inside the section of \( x_i \). In that case, we must check if \( x_i \) appears in the clause \( c_j \) in the original formula and the ordering of the section of \( x_i \) encodes an assignment to \( x_i \) that satisfies \( c_j \).

4. Variable modules are so large that the ordering must always encode a satisfying assignment to the formula (which exists by assumption). The ordering of \( T_j \) modules among themselves is irrelevant.

5. In order to encode the weight of a satisfying assignment, we make \( E_i \) modules slightly larger (we add 2 extra vertices). Then, the ordering \( D_i \prec E_i \prec F_i \), which encodes that \( x_i \) is True, is better than other orderings that encode satisfying assignments. Hence, the optimal ordering will represent a satisfying assignment to \( \phi \) with maximum weight.

6. Finally, in order to encode that there is a special variable \( x_n \) which must be set to True, we add one extra vertex to \( E_n \). This makes sure that setting \( x_n \) to True is more advantageous than setting any other variable to True, but not more advantageous than setting two other variables to True.

Armed with the intuition of the previous list, we are now ready to present all the details of our reduction.

Theorem 3. The following problem is \( \Theta^p_2 \)-complete: given a tournament \( T = (V, A) \) and a vertex \( v \in V \), decide if \( v \) is a Slater winner.

Proof. We perform a reduction heavily inspired by the reduction of [9] proving that computing the minimum fas of a tournament is NP-complete, though we include a minor modification proposed by Bachmeier et al. [3] which will later allow us to show that our instances are
realizable using seven voters. The main complication compared to the reductions of [9, 3] is that we now need to encode the CNF formula in a way that satisfying assignments of larger weight correspond to orderings with better objective value.

We start with a formula $\phi$, as given in Lemma 2. Let $x_1, \ldots, x_n$ be the variables of $\phi$ and suppose that the question is whether there exists a satisfying assignment of maximum weight that sets $x_i$ to True. Recall that by assumption the all-False assignment satisfies $\phi$. Let $m$ be the number of clauses of $\phi$.

We define two numbers $s_1, s_2$ which satisfy the following properties:

\begin{align*}
  s_1^2 &> (3n - 1)ms_1s_2 + 3ns_1 + m^2s_2^2 + 9m(n - 1)s_2 \quad (1) \\
  s_1s_2 &> 3ns_1 + m^2s_2^2 + 9m(n - 1)s_2 \quad (2) \\
  s_1 &> m^2s_2^2 + 9m(n - 1)s_2 \quad (3)
\end{align*}

For concreteness, set $s_2 = (n + m)^5$ and $s_1 = s_2^5 = (n + m)^{25}$ and the above inequalities are easily satisfied when $n + m$ is sufficiently large. Importantly, $s_1, s_2$ are polynomially bounded in $n + m$. Intuitively, the idea is that $s_1, s_2$ are two very large numbers, and $s_1$ is significantly larger. We will construct modules of size (roughly) $s_1$ or $s_2$, and the values are chosen so that arcs between large modules will be very important, arcs between large and small modules quite important, and arcs between small modules almost irrelevant.

We now construct our tournament as follows: for each $i \in \{1, \ldots, n\}$ we construct 6 modules, call them $A_i, B_i, C_i, D_i, E_i, F_i$. Modules $A_i, B_i, C_i, D_i, F_i$ have size $s_1$, while modules $E_i$ have size $s_1 + 2$ if $i < n$, and the size of $E_n$ is $s_1 + 3$. Internally, each of these modules induces a transitive tournament. For $i < j$ we add all arcs from $A_i \cup B_i \cup C_i \cup D_i \cup E_i \cup F_i$ to $A_j \cup B_j \cup C_j \cup D_j \cup E_j \cup F_j$. For each $i \in \{1, \ldots, n\}$ we add all possible arcs (i) from $A_i$ to $B_i \cup C_i \cup D_i \cup E_i \cup F_i$ (ii) from $B_i$ to $C_i \cup D_i \cup E_i \cup F_i$ (iii) from $C_i$ to $D_i \cup E_i \cup F_i$ (iv) from $D_i$ to $E_i$ (v) from $E_i$ to $F_i$ (vi) from $F_i$ to $D_i$. The graph we have constructed so far is a tournament with $n$ sections, each made up of 6 modules. Each such section represents a variable $x_i$ and the sections are linearly ordered. The structure inside each section is essentially the transitive closure of $A_i \to B_i \to C_i \to D_i \to E_i \to F_i$ with the exception that arcs between $D_i$ and $F_i$ are heading towards $D_i$.

We now complete the construction by adding to the current tournament some vertices that represent the clauses of $\phi$. In particular, for each $j \in \{1, \ldots, m\}$ we construct a module $T_j$ of size $s_2$ to represent the $j$-th clause of $\phi$. Internally, $T_j$ is a transitive tournament. For $j, j' \in \{1, \ldots, m\}$, the arcs between $T_j$ and $T_{j'}$ are set in an arbitrary direction. What remains is to explain how the arcs between $T_j$ and the modules representing the variables are set so as to encode the incidence of variables with clauses. For each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ we do the following:

1. If $x_i$ does not appear in the $j$-th clause we add all arcs from $T_j$ to $A_i \cup B_i \cup C_i$ and all arcs from $D_i \cup E_i \cup F_i$ to $T_j$.
2. If $x_i$ appears positive in the $j$-th clause we add all arcs from $T_j$ to $A_i \cup B_i \cup F_i$ and all arcs from $C_i \cup D_i \cup E_i$.
3. If $x_i$ appears negative in the $j$-th clause we add all arcs from $T_j$ to $A_i \cup C_i \cup D_i$ and all arcs from $B_i \cup E_i \cup F_i$.

This completes the construction and the question we want to answer is whether the last vertex (that is, the sink) of the transitive tournament induced by $F_n$ is a Slater winner of the whole graph.
We need to prove that the designated vertex is a Slater winner if and only if there is a satisfying assignment for \( \phi \) with maximum weight that sets \( x_n \) to True. We will do this by establishing some properties regarding any optimal ordering of the constructed tournament, showing that such an ordering must always have a structure which implies a satisfying assignment of \( \phi \) with maximum weight. We will rely heavily on Lemma 1, since the tournament we have constructed can be decomposed into \( 6n + m \) modules, namely, \( A_i, B_i, C_i, D_i, E_i, \) and \( F_i \), for \( i \in \{1, \ldots, n\} \), and \( T_j \), for \( j \in \{1, \ldots, m\} \). We therefore assume without loss of generality that these sets are placed contiguously in an optimal ordering.

Let us first argue that any optimal ordering must have some desirable structure which necessarily encodes a satisfying assignment for \( \phi \). To do this it will be helpful to start with a baseline ordering and calculate its implied fas, as then any ordering which implies a larger fas will be necessarily suboptimal. Consider the ordering which is defined as \( A_i \prec B_i \prec C_i \prec E_i \prec F_i \prec D_i \) for each \( i \in \{1, \ldots, n\} \) and which sets \( D_i \prec A_{i+1} \), where each module is internally ordered in the optimal way. We insert into this ordering of the modules that represent variables, the modules \( T_j \) as follows: for each \( j \in \{1, \ldots, m\} \), we find a variable \( x_j \) that appears negative in the \( j \)-th clause (such a variable must exist, since \( \phi \) is satisfied by the all-False assignment), and place all of \( x_j \) set \( \sigma \) contiguously in an optimal ordering. Indeed, if there are \( x_j \) incorrectly ordered, that is, ordered so that all arcs between the modules are included in the structure of an optimal ordering. First, observe that for each \( j \), \( i \prec j \), as other orderings are suboptimal. This allows us to draw some conclusions regarding the fas. Indeed, if there are \( n+1 \) such pairs, the fas will have size at least \( (n+1)s_1^2 > B \). We conclude that regarding the \( 6n \) large modules we must have the following ordering:

1. For each \( i < j \), we have that all vertices of \( A_i \cup B_i \cup C_i \cup D_i \cup E_i \cup F_i \) (the section that represents the variable \( x_i \)) are before all vertices of \( A_j \cup B_j \cup C_j \cup D_j \cup E_j \cup F_j \) (the section that represents the variable \( x_j \)).
2. For each \( i \in \{1, \ldots, n\} \), we have \( A_i \prec B_i \prec C_i \) and all vertices of \( A_i \cup B_i \cup C_i \) are before \( D_i \cup E_i \cup F_i \).
3. For each \( i \in \{1, \ldots, n\} \) we have \( D_i \prec E_i \prec F_i \), or \( E_i \prec F_i \prec D_i \), or \( F_i \prec D_i \prec E_i \).

We would now like to construct a correspondence between assignments to \( \phi \) and orderings of the tournament that respect the above conditions. On the one hand, if we are given an assignment \( \sigma \) we construct an ordering of the variable sections as above and for each \( i \), if \( \sigma \) set \( x_i \) to True we set \( D_i \prec E_i \prec F_i \), otherwise we set \( E_i \prec F_i \prec D_i \). In the converse direction, given an ordering that respects the above conditions (which any optimal ordering must do), we extract an assignment by setting, for each \( i \), \( x_i \) to True if and only if \( D_i \prec E_i \prec F_i \).
We now argue that the assignment corresponding to an optimal ordering must also be satisfying for \( \phi \), as otherwise the fas will have size strictly larger than \( B \), contradicting the optimality of the ordering. For the sake of contradiction, suppose we have an optimal ordering which corresponds to an assignment falsifying a clause. As argued above, there are at least \( ns_1^2 \) arcs in the fas contributed by the ordering of the large modules, so we concentrate on the modules \( T_j \) representing clauses. A module \( T_j \) representing any clause must be incident on at least \((3n - 1)s_1s_2\) arcs of the fas connecting it to large modules. To see this, consider the following: we will say that \( T_j \) is in the interior of section \( i \), if \( A_i \prec T_j \) and \( T_j \) is placed before one of \( D_i, E_i, \) or \( F_i \). \( T_j \) can be in the interior of at most one section \( i \), so for each \( i' \neq i \) we observe that at least \( 3s_1s_2 \) arcs incident on \( T_j \) and modules of the group \( i' \) are in the fas. This gives \( 3(n - 1)s_1s_2 \) arcs. In addition, no matter where we place \( T_j \) in the interior of section \( i \), at least a further \( 2s_1s_2 \) arcs of the fas are obtained: if \( T_j \) is after \( C_i \), then we get the arcs to \( A_i \cup B_i \) or the arcs to \( A_i \cup C_i \); if \( T_j \) is between \( A_i \) and \( C_i \), we get the arcs to \( A_i \) and at least \( 2s_1s_2 \) arcs from \( D_i \cup E_i \cup F_i \). Hence, we get at least \((3n - 1)s_1s_2\) arcs in the fas for each \( T_j \).

Furthermore, suppose that the assignment corresponding to the ordering does not satisfy the \( j \)-th clause. Then, we claim that at least \( 3ns_1s_2 \) arcs connecting \( T_j \) to large modules are included in the fas. Indeed, if \( T_j \) is in the interior of section \( i \), it can either be before or after \( C_i \). If it is before \( C_i \), as we observed in the previous paragraph, we always have at least \( 3s_1s_2 \) arcs in the fas between \( T_j \) and the large modules of section \( i \). If \( T_j \) is placed after \( C_i \), we have the following cases: (i) if \( x_i \) does not appear in the clause, then at least \( 3s_1s_2 \) arcs between \( T_j \) and the large modules of section \( i \) are in the fas (ii) if \( x_i \) appears positive in the \( j \)-th clause, we know that \( F_i \) is not placed last in section \( i \) (otherwise the assignment would satisfy the \( j \)-th clause), so wherever we place \( T_j \), at least \( 3s_1s_2 \) arcs are included in the fas (iii) similarly if \( x_i \) appears negative, since the assignment does not satisfy the clause, \( F_i \) is last, so again at least \( 3s_1s_2 \) arcs are included in the fas.

From the above calculations, if the assignment that corresponds to an ordering falsifies a clause, the fas has size at least \( ns_1^2 + (m - 1)(3n - 1)s_1s_2 + 3ns_1s_2 = ns_1^2 + (3n - 1)ms_1s_2 + s_1s_2 > B \), where we used inequality (2). We conclude that an optimal ordering must correspond to a satisfying assignment.

We now need to argue that the assignment corresponding to an optimal ordering of the tournament must be a satisfying assignment of maximum weight. Suppose for contradiction that the assignment corresponding to an optimal ordering, call it \( \sigma_1 \), sets \( k \) variables to True, but there exists another satisfying assignment, call it \( \sigma_2 \), that sets at least \( k + 1 \) variables to True. We will show that starting from \( \sigma_2 \) we can obtain a better ordering of the tournament, contradicting the optimality of the original ordering.

We claim that the ordering from which we extracted \( \sigma_1 \) includes at least \( ns_1^2 + (3n - 1)ms_1s_2 + 2(n - k)s_1 \) arcs in the fas. This is because in the section corresponding to the \( n - k \) variables that \( \sigma_1 \) sets to False, either the arcs from \( E_i \) to \( F_i \), or the arcs from \( D_i \) to \( E_i \) are in the fas (since \( F_i \) is not placed last in the section), and these are at least \( s_1(s_1 + 2) = s_1^2 + 2s_1 \) arcs.

We construct an ordering from \( \sigma_2 \) as follows: we order the variable section in the normal way and inside each section, if \( \sigma_2(x_i) = \text{True} \) we use the ordering \( D_i \prec E_i \prec F_i \), otherwise we use the ordering \( E_i \prec F_i \prec D_i \). For each \( T_j \), we find a variable \( x_i \) that satisfies the \( j \)-th clause and place \( T_j \) in section \( i \) immediately before the last module of this section. If for \( j, j' \) the order of \( T_j, T_{j'} \) is not implied by the above, we set it arbitrarily. The fas implied by this ordering has size at most...
Determining a Slater Winner is Complete for Parallel Access to NP

\[ \begin{align*}
B' & \leq ns_1^2 + 2(n-k-1)s_1 + s_1 + m(2s_1s_2 + 3(n-1)(s_1 + 3)s_2) + m^2s_2^2 = \\
&= ns_1^2 + 2(n-k)s_1 + (3n-1)ms_1s_2 - s_1 + m^2s_2^2 + 9m(n-1)s_2
\end{align*} \]

Here, the calculations for the terms \( m(2s_1s_2 + 3(n-1)(s_1 + 3)s_2) + m^2s_2^2 \) are the same as in the calculation of \( B \); the term \( 2(n-k-1)s_1 \) takes into account that there are \( n-k-1 \) sections that correspond to variables set to False; and the \( s_1 \) term is due to the fact that \( x_n \) may be one of the variables set to False and \( E_n \) has size \( s_1 + 3 \) and not \( s_1 + 2 \). Using inequality (3) we have that \( -s_1 + m^2s_2^2 + 9m(n-1)s_2 < 0 \), so the ordering we have constructed from \( \sigma_2 \) is better than the one from which we extracted \( \sigma_1 \), contradiction.

At this point we are almost done because we have argued that an optimal ordering of the tournament corresponds to a satisfying assignment of maximum weight and furthermore, since the correspondence sets \( x_n \) to True if and only if \( F_n \) is the last module in the ordering, the sink of \( F_n \) will be last in the ordering if and only if the assignment sets \( x_n \) to True. However, \( \phi \) could have several satisfying assignments of the same weight, and since we have set arcs between \( T_j \) modules arbitrarily, it could be the case that a maximum weight assignment that sets \( x_n \) to False results in a better ordering, making another vertex the Slater winner. This is the reason why we have set \( E_n \) to be slightly larger than all other modules \( E_i \), so that setting \( x_n \) to True is always slightly more advantageous than setting any other variable to True.

Concretely, we argue the following: any optimal ordering of the tournament corresponds to a satisfying assignment of \( \phi \) with maximum weight; and furthermore if a satisfying assignment of \( \phi \) with maximum weight sets \( x_n \) to True, then any optimal ordering places \( F_n \) last. We need to argue the second claim, so suppose for contradiction that an optimal ordering does not place \( F_n \) last and that the assignment that corresponds to this ordering is \( \sigma_1 \). Furthermore, suppose that there exists a satisfying assignment \( \sigma_2 \) of maximum weight that sets \( x_n \) to True. Say that both \( \sigma_1, \sigma_2 \) set \( k \) variables to True.

We first observe that the ordering from which we extracted \( \sigma_1 \) implies a fas of size at least \( ns_1^2 + 2(n-k)s_1 + s_1 + (3n-1)ms_1s_2 \). This is because there are \( n-k-1 \) sections where the fas contains \( s_1(s_1 + 2) \) arcs incident on a module \( E_i \), \( k \) sections where the fas contains \( s_1^2 \) arcs incident from \( F_1 \) to \( D_i \), and in the section corresponding to \( x_n \), the fas contains \( s_1(s_1 + 3) \) arcs, incident on \( E_n \).

On the other hand, if we construct an ordering from \( \sigma_2 \) in the same way as we did previously, the fas obtained will have size at most

\[ \begin{align*}
B'' & \leq ns_1^2 + 2(n-k)s_1 + m(2s_1s_2 + 3(n-1)(s_1 + 3)s_2) + m^2s_2^2 = \\
&= ns_1^2 + 2(n-k)s_1 + (3n-1)ms_1s_2 + m^2s_2^2 + 9m(n-1)s_2
\end{align*} \]

Again, using inequality (3) which states that \( s_1 > m^2s_2^2 + 9m(n-1)s_2 \) we conclude that the new ordering is better, contradicting the optimality of the original ordering.

We now summarize our arguments: we have shown that any optimal ordering of the tournament always corresponds to a maximum weight satisfying assignment of \( \phi \) and furthermore, it corresponds to a maximum weight satisfying assignment that sets \( x_n \) to True if this is possible; furthermore, if an optimal ordering corresponds to an assignment that sets \( x_n \) to True then the last vertex of \( F_n \) is a Slater winner. We therefore have two cases: if the last vertex of \( F_n \) is a Slater winner, then since optimal orderings give rise to satisfying assignments of maximum weight, there is a maximum weight satisfying assignment of \( \phi \)
setting $x_n$ to True; if the last vertex of $F_n$ is not a Slater winner, then the maximum weight satisfying assignment we extract from an optimal ordering sets $x_n$ to False, and there is no satisfying assignment of the same weight setting $x_n$ to True. We conclude that determining if a vertex is a Slater winner is equivalent to deciding if $\phi$ has a maximum weight satisfying assignment setting $x_n$ to True, and is therefore $\Theta_2^p$-complete.

## 4 Hardness for 7 Voters

In this section we show that the tournaments constructed in Theorem 3 correspond to instances that could result from the aggregation of the preferences of 7 voters and as a result the problem of determining a Slater winner remains $\Theta_2^p$-complete even for 7 voters. Our approach follows along the lines of the arguments of Bachmeier et al. [3] who proved that determining the Slater winner is NP-hard for 7 voters. Indeed, the proof of [3] consists of an analysis (and tweak) of the construction of Conitzer [9] which establishes that the instances of the reduction can be built by aggregating 7 voter profiles. Since our reduction is very similar to Conitzer’s, we essentially only need to adjust the arguments of Bachmeier et al. to obtain $\Theta_2^p$-completeness.

Our first step is to slightly restrict the $\Theta_2^p$-complete problem that is the starting point of our reduction. We present the following strengthening of Lemma 2, which is similar to the problem used as a starting point in the reduction of [3].

► **Lemma 4.** The problem given in Lemma 2 remains $\Theta_2^p$-complete under the following additional restrictions: (i) we are given a partition of the clauses of $\phi$ in two sets $L, R$ and each variable appears in at most one clause of $L$ and in at most two clauses of $R$ (ii) each literal appears at most once in a clause of $R$.

**Proof.** Given a formula $\phi$ as in Lemma 2 we construct a new formula $\phi'$ as follows. Let $x_1, x_2, \ldots, x_n$ be the variables of $\phi$ and $m$ be the number of its clauses. For each $x_i$, $i \in \{1, \ldots, n\}$ we construct $m$ variables, call them $y_i^1, y_i^2, \ldots, y_i^m$. For each $i \in \{1, \ldots, n\}$, for $j \in \{1, \ldots, m - 1\}$ we construct the clause $(\neg y_{i}^{j} \vee y_{i}^{j+1})$, as well as the clause $(\neg y_{i}^{m} \vee y_{i}^{1})$. Let $R$ be the set of clauses constructed so far and note that each literal appears at most once and each variable at most twice in these clauses. Intuitively, the clauses of $R$ ensure that for each $i$, all variables in the set $\{y_i^1, y_i^2, \ldots, y_i^m\}$ must receive the same value in a satisfying assignment.

Now, we consider the clauses of $\phi$ one by one. If the $j$-th clause contains the variable $x_i$, we replace it by the variable $y_{i}^{j}$. Doing this for all clauses of $\phi$ we obtain a set of clauses, call it $L$, where each variable appears at most once (assuming without loss of generality that clauses of $\phi$ have no repeated literals).

If $x_n$ was the designated variable of $\phi$ we set $y_n^1$ as the designated variable of $\phi'$. It is now not hard to make a correspondence between satisfying assignments of $\phi$ and $\phi'$ ($x_i$ is set to True if all $y_i^j$ are set to True) in a way that preserves weights (the weight of an assignment to $\phi'$ is $m$ times the weight of the corresponding assignment for $\phi$). Hence, determining if a maximum weight satisfying assignment to $\phi'$ sets $y_n^1$ to True is $\Theta_2^p$-complete. Observe also that $\phi'$ is satisfied by the all-False assignment.

We now obtain the result of this section by starting the reduction of Theorem 3 from the problem of Lemma 4. In the statement of the theorem below, when we say that a tournament $T = (V, A)$ can be obtained from 7 voters, we mean that there exist 7 total orderings of $V$ such that for all $(a, b) \in A$ we have that $a \prec b$ in at least 4 of the orderings.

STACS 2022
Theorem 5. Determining if a vertex of a tournament is a Slater winner remains $\Theta_2^P$-complete even for tournaments that can be obtained from 7 voters.

Proof. We perform the same reduction as in Theorem 3 except we start from the special case given in Lemma 1. What remains is to show that the instance we construct can result from aggregating 7 orderings. Recall that our tournament contains $m$ modules representing the clauses, called $T_j$, for $j \in \{1, \ldots, m\}$. Since modules are internally transitive, we will assume that the 7 voters have preferences which agree with the directions of the arcs inside the modules and hence we focus on the arcs between modules. Recall that in the reduction of Theorem 3 arcs between modules $T_j$ are set arbitrarily. To ease presentation, assume that when $j < j'$ we have the arcs $T_j \rightarrow T_{j'}$.

The first voter has preferences $A_1 \prec B_1 \prec C_1 \prec D_1 \prec E_1 \prec F_1 \prec A_2 \prec B_2 \prec C_2 \prec D_2 \prec E_2 \prec F_2 \prec \cdots \prec T_n \prec T_1 \prec T_2 \prec \cdots \prec T_m$. In other words, the first voter orders all the variable modules before all the clause modules, orders variable groups according to their index, and inside each variable group she has the ordering $A_1 \prec B_1 \prec C_1 \prec D_1 \prec E_1 \prec F_1$.

We now add two voters with the intent of constructing all the arcs of the set

$$X_0 = \left( \bigcup_j T_j \times \bigcup_i (A_i \cup B_i \cup C_i) \right) \cup \bigcup_i (F_i \times D_i)$$

The first of these voters has ordering $(E_1 \prec E_2 \prec \cdots \prec E_n) \prec (F_1 \prec D_1 \prec F_2 \prec D_2 \prec \cdots \prec F_n \prec D_n \prec \cdots \prec T_m \prec T_1 \prec T_2 \prec \cdots \prec T_1 \prec (C_n \prec B_n \prec A_n \prec C_{n-1} \prec B_{n-1} \prec A_{n-1} \prec \cdots \prec C_1 \prec B_1 \prec A_1) \prec (F_n \prec D_n \prec F_{n-1} \prec D_{n-1} \prec \cdots \prec F_1 \prec D_1) \prec (E_n \prec E_{n-1} \prec \cdots \prec E_1)$. Note that these two voters agree that all modules of $\bigcup_j T_j$ come before all modules of $\bigcup_i (A_i \cup B_i \cup C_i)$ and that for each $i$ we have $F_i \prec D_i$, but disagree on every other pair of modules, hence the two voters together induce exactly the set of arcs $X_0$ cited above.

If we now consider the three voters we have so far, we observe that much of our construction is already induced:

1. For $i < i'$ we have arcs from $A_i \cup B_i \cup C_i \cup D_i \cup E_i \cup F_i$ to $A_{i'} \cup B_{i'} \cup C_{i'} \cup D_{i'} \cup E_{i'} \cup F_{i'}$ because of the preferences of the first voter, as the other two voters disagree on these arcs.
2. For each $i \in \{1, \ldots, n\}$, inside the group $A_i \cup B_i \cup C_i \cup D_i \cup E_i \cup F_i$, we have arcs that agree with the ordering $A_i \prec B_i \prec C_i \prec D_i \prec E_i \prec F_i$, except that we have arcs from $F_i$ to $D_i$. This is because the second and third voter agree that $F_i \prec D_i$, but disagree on every other pair (hence the preferences of the first voter prevail for the other pairs).
3. For each $j < j'$ we have arcs from $T_j$ to $T_{j'}$, due to the preferences of the first voter, as the other two disagree.
4. For each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ we have arcs from $T_j$ to $A_i \cup B_i \cup C_i$, because the second and third voter agree that $T_j \prec (A_i \cup B_i \cup C_i)$ (though the first voter disagrees).
5. For each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ we have arcs from $D_i \cup E_i \cup F_i$ to $T_j$, because the second and third voter disagree on these pairs, so the preferences of the first voter break the tie.

We therefore have that the tournament that follows from aggregating the preferences of the first three voters almost corresponds to the one we want to construct, except that for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ the arcs between $T_j$ and $A_i \cup B_i \cup C_i \cup D_i \cup E_i \cup F_i$
correspond to the arcs we would want if \( x_i \) did not appear in the \( j \)-th clause (in other words, the three voters we have so far induce the general structure of the construction, but do not encode which variable appears in which clause). Furthermore, if we look at the relationship between any two modules so far, the margin of victory is always exactly one (that is, there do not exist two modules \( X, Y \) such that all three voters agree that \( X \prec Y \)).

Hence, what remains is to use the four remaining voters to “fix” this, so that if \( x_i \) appears (positive or negative) in the \( j \)-th clause, we have the arcs prescribed in the reduction of Theorem \ref{thm:main}. We will achieve this by giving two pairs of voters. Each pair of voters will disagree on all pairs of modules except a specific set of arcs that we want to fix. Hence, adding the pair of voters to the electorate will repair the arcs in question (since the current margin of victory for all arcs is one), while leaving everything else unchanged.

Recall that the clause set is given to us partitioned into two sets \( R, L \) so that each variable appears in at most one clause of \( L \) and each literal in at most one clause of \( R \). We will use the slightly weaker property that each literal appears at most once in each of \( L, R \). Abusing notation we will write \( j \in R \) if the \( j \)-th clause is in \( R \) (similarly for \( j \in L \)). We will also write \( x_i \in c_j \) (respectively \( \neg x_i \in c_j \)) if \( x_i \) appears positive (respectively negative) in the \( j \)-th clause.

Consider now the following two sets of arcs:

\[
X_1 = \left( \bigcup_{j \in R} \cup_{i: x_i \in c_j} (T_j \times F_i) \right) \cup \left( \bigcup_{j \in L} \cup_{i: x_i \in c_j} (T_j \times D_i) \right)
\]

\[
X_2 = \left( \bigcup_{j \in L} \cup_{i: x_i \in c_j} (T_j \times F_i) \right) \cup \left( \bigcup_{j \in R} \cup_{i: x_i \in c_j} (T_j \times D_i) \right)
\]

Our plan is to give a pair of voters whose preferences induce the arcs of \( X_1 \) and another pair whose preferences induce the arcs of \( X_2 \). Here when we say that two voters induce a set of arcs \( X \) we mean that for each \((a, b) \in X\) both voters have \( a \prec b \) and for each \((a, b) \notin X\) one voter has \( a \prec b \) and the other has \( b \prec a \). Before we proceed we observe that if \( X_1, X_2 \) are inducible by a pair of voters each, then adding these four voters to the three voters we have described so far produces the tournament of Theorem \ref{thm:main}. Indeed, suppose that \( x_i \) appears positive in clause \( c_j \) and \( j \in R \). Then, if we consider the arcs in the tournament induced by the first three voters, we need to inverse the arcs between \( T_j \) and \( F_i \) (which currently point \( F_i \to T_j \)), and the arcs between \( T_j \) and \( C_j \) (which currently point \( T_j \to C_j \)). But the arcs between \( T_j \) and \( F_i \) are inversed thanks to \( X_1 \), while the arcs between \( T_j \) and \( C_j \) are inversed thanks to \( X_2 \), where we use the fact that \( X_1, X_2 \) represent the consensus of two voters, while the margin of victory for any arc induced by the first three voters is one. Similar arguments apply if \( x_i \) appears negative in \( c_j \), or \( j \in L \). Hence, if a pair of voters induces \( X_1 \) and another induces \( X_2 \), taking the union of these four voters with the three voters we have described produces the tournament of Theorem \ref{thm:main} and completes the proof.

We now recall that it was shown in \cite{Lampis2011} that \( X_1, X_2 \) are inducible by two voters each, since these sets of arcs are unions of stars (if we contract each module to a vertex). Let us explain in more detail how to represent \( X_1 \) as the union of the preferences of two voters (the arguments for \( X_2 \) are essentially identical). We will make use of the fact that each literal appears at most once in \( L \) and at most once in \( R \).

We will say that a module from a variable group is “active” if it is incident on an arc of \( X_1 \). In particular, modules \( A_i, E_i \), for \( i \in \{1, \ldots, n\} \) are not active, and neither are modules \( F_i \).
such that \( x_i \) does not appear positive in \( R \) (and similarly for \( B_i, C_i, D_i \)). We will concentrate on the ordering of active modules because if we find two voter profiles that order these modules in a way that induces \( X_1 \), we can add an arbitrary ordering of the inactive modules in the beginning of the preferences of the first voter, and the opposite of that ordering at the end of the preferences of the second voter. This will have as effect that the two voters disagree on any pair that involves an element of an inactive module, as desired.

We now observe that for each active module \( M \) from \( \bigcup_i B_i \cup C_i \cup D_i \cup F_i \), there exists exactly one \( T_j \) such that \( M \) has arcs to \( T_j \) in \( X_1 \). This is because every literal appears in at most one clause of \( R \) and at most one clause of \( L \). Now, we construct two voter profiles as follows: one voter orders the \( T_j \) modules in increasing order of index and the other in decreasing order. For each active module \( D_i \) or \( F_i \), we insert the module immediately after the \( T_j \) from which the module receives arcs in \( X_1 \) in both orderings; for active modules \( B_i \) or \( C_i \), we insert them immediately before the \( T_j \) towards which the module has arcs in both orderings. Note that this does not fully specify the ordering, as if two variables \( x_i, x_i' \) appear in \( c_j \) and \( j \in R \), then we need to place \( F_i \) and \( F_i' \) immediately after \( T_j \). We resolve such conflicts by using an arbitrary ordering of the active modules for the first voter and the opposite of that ordering for the second voter, that is, all modules which are supposed to appear immediately after \( T_j \) are sorted in one way for the first voter and in the opposite way for the second voter. We now observe that with this ordering for every active module the two voters agree about the arcs connecting the module to its neighboring \( T_j \), while we obtain no other arcs between the module and any other \( T_j' \) or any other active module. We therefore have two voters whose preferences induce \( X_1 \).

---

**References**

1. Nir Ailon, Moses Charikar, and Alantha Newman. Aggregating inconsistent information: Ranking and clustering. *J. ACM*, 55(5):23:1–23:27, 2008.
2. Noga Alon. Ranking tournaments. *SIAM J. Discret. Math.*, 20(1):137–142, 2006.
3. Georg Bachmeier, Felix Brandt, Christian Geist, Paul Harrenstein, Keyvan Kardel, Dominik Peters, and Hans Georg Seedig. k-majority digraphs and the hardness of voting with a constant number of voters. *J. Comput. Syst. Sci.*, 105:130–157, 2019.
4. Felix Brandt, Markus Brill, and Paul Harrenstein. Tournament solutions. In *Handbook of Computational Social Choice*, pages 57–84. Cambridge University Press, 2016.
5. Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia. Introduction to computational social choice. In *Handbook of Computational Social Choice*, pages 1–20. Cambridge University Press, 2016.
6. Richard Chang and Jim Kadin. The Boolean hierarchy and the polynomial hierarchy: A closer connection. *SIAM J. Comput.*, 25(2):340–354, 1996. URL: [https://doi.org/10.1137/S0097539790178069](https://doi.org/10.1137/S0097539790178069).
7. Pierre Charbit, Stéphan Thomassé, and Anders Yeo. The minimum feedback arc set problem is NP-hard for tournaments. *Comb. Probab. Comput.*, 16(1):1–4, 2007.
8. Irène Charon and Olivier Hudry. An updated survey on the linear ordering problem for weighted or unweighted tournaments. *Ann. Oper. Res.*, 175(1):107–158, 2010.
9. Vincent Conitzer. Computing Slater rankings using similarities among candidates. In *Proceedings, The Twenty-First National Conference on Artificial Intelligence and the Eighteenth Innovative Applications of Artificial Intelligence Conference, July 16-20, 2006, Boston, Massachusetts, USA*, pages 613–619. AAAI Press, 2006. URL: [http://www.aaai.org/Library/AAAI/2006/aaai06-098.php](http://www.aaai.org/Library/AAAI/2006/aaai06-098.php).
10. Ronald de Haan. Parameterized Complexity in the Polynomial Hierarchy - Extending Parameterized Complexity Theory to Higher Levels of the Hierarchy, volume 11880 of Lecture Notes
11 Ulle Endriss and Ronald de Haan. Complexity of the winner determination problem in judgment aggregation: Kemeny, Slater, Tideman, Young. In AAMAS, pages 117–125. ACM, 2015.

12 Edith Hemaspaandra, Lane A. Hemaspaandra, and Jörg Rothe. Exact analysis of Dodgson elections: Lewis Carroll’s 1876 voting system is complete for parallel access to NP. J. ACM, 44(6):806–825, 1997.

13 Edith Hemaspaandra, Lane A. Hemaspaandra, and Jörg Rothe. Raising NP lower bounds to parallel NP lower bounds. SIGACT News, 28(2):2–13, 1997. URL: https://doi.org/10.1145/261342.261344

14 Edith Hemaspaandra, Holger Spakowski, and Jörg Vogel. The complexity of Kemeny elections. Theor. Comput. Sci., 349(3):382–391, 2005.

15 Olivier Hudry. On the complexity of Slater’s problems. Eur. J. Oper. Res., 203(1):216–221, 2010.

16 Jörg Rothe, Holger Spakowski, and Jörg Vogel. Exact complexity of the winner problem for Young elections. Theory Comput. Syst., 36(4):375–386, 2003.

17 Klaus W. Wagner. More complicated questions about maxima and minima, and some closures of NP. Theor. Comput. Sci., 51:53–80, 1987. URL: https://doi.org/10.1016/0304-3975(87)90049-1