Massless on-shell box integral with arbitrary powers of propagators

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Abstract
The massless one-loop box integral with arbitrary indices in arbitrary space-time dimension \(d\) is shown to reduce to a sum over three generalized hypergeometric functions. This result follows from the solution to the third order differential equation of hypergeometric type. The Gröbner basis technique for integrals with noninteger powers of propagators is used to derive the differential equation. A short description of our algorithm for finding the Gröbner basis is given and a complete set of recurrence relations from the Gröbner basis is presented. The first several terms in the \(\varepsilon = (4 - d)/2\) expansion of the result are given.

Keywords: Feynman integrals, differential equations, Gröbner basis

1. Introduction

Modern methods for evaluating Feynman integrals are mainly used to obtain analytical results for integrals with integer powers (indices) of propagators. There is no regular algorithm for the calculation of integrals with noninteger powers of propagators. However, in some important cases, one needs to evaluate integrals with noninteger indices. For example, such integrals are needed when analytic regularization is exploited. Another example is the evaluation of integrals with massive propagators by the Mellin–Barnes technique. In this approach integrals with massive propagators are expressed in terms of Mellin–Barnes integrals. Integrands of these Mellin–Barnes integrals are in fact massless Feynman integrals with arbitrary indices [1]. Also, the computation of multi-loop integrals with insertions of loop integrals with massless propagators can be reduced to evaluation of integrals with noninteger indices.

The goal of our paper is twofold. First, we describe a regular method for deriving various types of equations for integrals with noninteger indices. Solving these equations, one can obtain analytical results for the required integral. Second, the usefulness of the method will be demonstrated on the one-loop four point massless box integral. This integral has been known for a long time with integer indices. In our paper, we present an analytical result for this integral with arbitrary powers of the propagators.
This article is organized as follows: in section 2, definitions and method for deriving different types of equations for Feynman integrals with noninteger powers of propagators are described. The main steps of the algorithm for deriving the Gröbner basis are presented. In section 3, the differential equation and its solution for the on-shell box integral is given. In section 4, different results for epsilon expansion of the result are presented. In section 5, the most important results of our investigation are shortly described.

2. Definitions and method for deriving equations

The one-loop box type integral $I_4^{(d)}(\{\nu_j\}, \{s_{kr}\})$ with massless propagators to be considered in this paper is defined as

$$I_4^{(d)}(\nu_1, \nu_2, \nu_3, \nu_4; s_{12}, s_{23}, s_{34}, s_{24}, s_{13}) = \int \frac{d^d q}{i \pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4}},$$

where

$$D_j = (q - p_j)^2 + i \epsilon, \quad s_{ij} = p_{ij}^2, \quad p_{ij} = p_i - p_j.$$  \hspace{1cm} (2.2)

The diagram corresponding to the integral is presented in figure 1. In the present paper, the on-shell version of this integral will be considered:

$$I_4^{(d)}(\nu_1, \nu_2, \nu_3, \nu_4; s_{24}, s_{13}) \equiv I_4^{(d)}(\nu_1, \nu_2, \nu_3, \nu_4; 0, 0, 0, 0; s_{24}, s_{13}).$$ \hspace{1cm} (2.3)

In what follows, we will use the traditional kinematic invariants $s$ and $t$, defined as

$$s = (p_{12} + p_{41})^2 = s_{24}, \quad t = (p_{12} + p_{23})^2 = s_{13}.$$ \hspace{1cm} (2.4)

The analytical result for this integral when all $\nu_j = 1$ was derived by diverse methods in [2–5]. Some asymptotic values of the integral with noninteger indices were obtained in [6]. The analytic result for the integral with arbitrary values of $\nu_j$ was not known until now.

In the present paper, the analytic result for the integral will be derived as a solution of an equation. To obtain such an equation, we propose to use the Gröbner basis technique. The idea how to derive the Gröbner basis for integrals with noninteger powers of propagators was outlined for the first time in [7]. Exploiting the Gröbner basis technique, one can set up the differential equation for the integral together with recurrence relations either with respect to the parameter of the space-time dimension $d$ or an index $\nu_j$. We derived all these equations, and discovered that solving the differential equation for the integral $I_4^{(d)}$ is easier than solving both types of the recurrence relations. For this reason, we will stick to the derivation and solution of the differential equation.

To derive the Gröbner basis for $I_4^{(d)}$, we developed a method which is based on ideas formulated in [7]. In [7], a system of generalized recurrence relations was represented as a system of partial differential equations with respect to masses. To reduce this system of partial differential equations to a triangular form and find a set of basis integrals, we introduced ordering of Feynman integrals and applied algorithm standard form [8, 9]. In fact, this algorithm is nothing but a Buchberger-like algorithm [10] applied to systems of partial differential equations.

Our algorithm for finding Gröbner basis can be formulated as follows. First, we derive all possible generalized recurrence relations [12] for a considered integral with arbitrary powers of propagators. In the present paper instead of differential operators we will use operators $\mathbf{j}^\pm$ shifting indices $\nu_j$ by one unit.
In order to eliminate redundant recurrence relations, we perform the following procedure, which we will call in the following normalization. In all recurrence relations, we shift indices \( \nu_j \) in such a way that each equation will contain at least one integral with minimal value of the \( j \)th index equal to \( \nu_j - 1 \). For example, if in an equation there are only integrals with \( \nu_1 + r, \nu_1 + r + 1, \ldots \) i.e. \( I_{N}^{(d)}(\nu_1 + r, \ldots) \), \( I_{N}^{(d)}(\nu_1 + r + 1, \ldots) \), where \( r \) is a positive integer, then in this equation we must shift \( \nu_1 \to \nu_1 - r - 1 \). We treat other indices \( \nu_j \) in this equation in a similar way. If there are integrals with negative shifts of indices \( \nu_k - r \), where \( r \) is an integer and \( r \geq 2 \), then one should make shift \( \nu_k \to \nu_k + r - 1 \). Such a normalization allows one to uniquely arrange the system of recurrence relations in accordance with the ordering of integrals which will be described below. Each recurrence relation in our normalized system of generalized recurrence relations can be written as

\[
\sum_{\sigma, \delta} R_{\sigma \delta}(\{s_k\} \{m_s^2\}, d, \{\nu_k\}) (d^+)^{\sigma} J^d I_{N}^{(d)}(\nu_1 - 1, \ldots, \nu_N - 1) = 0,
\]

where

\[
d^+ I_{N}^{(d)}(\ldots) = I_{N}^{(d+2)}(\ldots), \quad J^d = (1^+)^{\delta_1} \ldots (N^+)^{\delta_N}, \quad \delta = \{\delta_1, \ldots, \delta_N\}, \quad \delta_k \geq 0.
\]

Here, the coefficients \( R_{\sigma \delta}(\{s_k\} \{m_s^2\}, d, \{\nu_k\}) \) are polynomials in kinematic variables \( \{s_k\} \), masses \( \{m_s^2\} \), space-time dimension \( d \) and indices \( \{\nu_k\} \).

The main steps of our algorithm to find Gröbner basis can be described briefly as follows.

1. Make an ordered list of integrals from the system of equations. Integrals in this list in our case were ordered according to their weight as follows: the highest weight have integrals with shifted space time dimension. For integrals with the same shift of dimension, the highest weight will have integrals with the largest sum \( \delta_1 + \ldots + \delta_N \). For integrals with the same dimension and sum of indices, ordering must be done according to the largest shift \( \delta_1 \) of \( \nu_1 \), then shift \( \delta_2 \) of \( \nu_2 \), and so on.

2. For each equation, define the leading integral—i.e. the integral with the highest weight—and split the system of equations into subsystems so that each equation in a subsystem has the same leading integral.

3. Arrange all subsystems according to a position of their leading integrals in the arranged list of integrals. From the subsystems with lower leading integrals, produce all possible subsystems with higher leading integrals by appropriate shifts of \( \nu_j \) and space-time dimension \( d \).
4. Start from the subsystem with lowest leading integral. If the subsystem has only one equation, keep it unchanged and go to the subsystem with the higher leading integral. Again, if the number of equations in this subsystem is one, keep it unchanged and go to the next subsystem until you encounter a subsystem with several equations.

5. For a subsystem with several equations, make all possible pairs of equations and in each pair exclude the leading integral. Check whether the resulting equations are combinations of equations with lower leading integrals. In order to improve the efficiency of our calculations, we did this check by setting all parameters to randomly chosen numbers. If there are equations which are not combinations of the lower ones, add them to the considered system together with the shortest equation in this subsystem. Repeat all steps with the new system, starting from step 1.

6. Calculations are to be finished if after some step new equations do not appear—the resulting system will be in a triangular form. Expressions for lower leading integrals should be substituted into equations with higher leading integrals. The set of equations for lowest leading integrals will form our Gröbner basis, and integrals which are not leading integrals in this set of equations will be basis integrals. The leading integrals in all equations from the Gröbner basis cannot be obtained by shifting indices in some other equation in this basis.

The above algorithm is analogous to that of [7]. Variations of the above algorithm are possible. To improve efficiency of our algorithm, we used the following strategy. First, we did not substitute solutions of equations with lower leading integrals into subsystems with higher leading integrals, thus preventing the explosion in size of the considered system of equations in the intermediate stages of calculations. Second, we performed our calculations in several steps. At the first step, we dropped from our system lengthy equations exceeding some fixed length. We did our calculations with computer algebra system Maple and used the built-in ‘length’ function to determine the length of an expression. Also, newly generated equations exceeding this fixed length were not included in our system of equations. Usually, the size of the fixed length was determined experimentally. It should be large enough to allow the generation of new short equations during the iterative procedure described above. After several iterations, our system of equations restricted in this way was reduced to a triangular form. We then increased the fixed length, and repeated our iteration procedure. At the last step, all equations were included in our consideration. Such a stepwise strategy turns out to be rather efficient. The reason is clear. In many cases, large equations were combinations of some number of shorter equations. The most cumbersome equations were included in our Gröbner basis only in the final steps.

A detailed description of the algorithm with illustrative examples will be given in our future publication.

We found that the Gröbner basis for the integral \( I_4^{(d)} \) consists of 12 relations. The recurrence relations from this basis and additional equation (A.10) given in appendix A allow one to reduce any integral of the type

\[
I_4^{(d+2r)}(\nu_1 + n_1, \nu_2 + n_2, \nu_3 + n_3, \nu_4 + n_4; s, t)
\]

with integer \( r \geq 0, n_k \geq 0 \) to a set of three basis integrals:

\[
B_1 \equiv B_1(\nu_1, \nu_2, \nu_3, \nu_4; s, t) = I_4^{(d)}(\nu_1 - 1, \nu_2 - 1, \nu_3, \nu_4 - 1; s, t),
\]

\[
B_2 \equiv B_2(\nu_1, \nu_2, \nu_3, \nu_4; s, t) = I_4^{(d)}(\nu_1 - 1, \nu_2 - 1, \nu_3 - 1, \nu_4; s, t),
\]

\[
B_3 \equiv B_3(\nu_1, \nu_2, \nu_3, \nu_4; s, t) = I_4^{(d)}(\nu_1 - 1, \nu_2 - 1, \nu_3 - 1, \nu_4 - 1; s, t).
\]
Three relations from the Gröbner basis are rather simple:

\[ 3^+ B_3 = B_1, \quad 4^+ B_3 = B_2, \quad 3^+ B_2 = 4^+ B_1. \]  

(2.8)

Other 9 relations from the Gröbner basis are given in appendix A.

Integrals of the type (2.6) can be obtained by applying the product of shifting operators

\[ (d^+) (1^+) (2^+) (3^+) (4^+) \]  

(2.9)
to the basis integral

\[ I_4^{(d)}(\nu_1 - 1, \nu_2 - 1, \nu_3 - 1, \nu_4 - 1; s, t). \]  

(2.10)

After application of a particular operator \( j^+ \) to the integral (2.10), one should use recurrence relations from the Gröbner basis. As a result any integral of the type (2.6) appeared in expressions for derivatives will be reduced to a combination of three basis integrals (2.7). In the same way, we can reduce the integral \( I_4^{(d)}(\nu_1, \nu_2, \nu_3, \nu_4; s, t) \) to the basis integrals (2.7).

### 3. Differential equation and its solution

As was shown in [11], the derivatives of Feynman integrals can be written in terms of integrals with shifted dimension \( d \) and changed indices of the propagators. To obtain an explicit formula for the derivatives of \( I_4^{(d)} \), we will exploit its \( \alpha \) parametric integral representation:

\[
I_4^{(d)}(\nu_1, \nu_2, \nu_3, \nu_4; s, t) = \int_0^\infty \cdots \int_0^\infty \frac{\alpha_1^{\nu_1-1} \alpha_2^{\nu_2-1} \alpha_3^{\nu_3-1} \alpha_4^{\nu_4-1}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \frac{da_\alpha}{D^{(d)/2}} e^{Q/D} \nonumber \]

(3.11)

where \( \{da\} = d\alpha_1 \cdots d\alpha_4 \) and

\[
Q = \alpha_1 \alpha_3 t + \alpha_2 \alpha_4 s, \quad D = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4. \]

(3.12)

According to the method of [11], explicit expressions for the derivatives of the integral \( I_4^{(d)} \) are determined by the polynomial \( Q \) in equation (3.12) and they read

\[
\frac{\partial}{\partial t} I_4^{(d)}(\nu_1, \nu_2, \nu_3, \nu_4; s, t) = \nu_1 \nu_3 I_4^{(d+2)}(\nu_1 + 1, \nu_2, \nu_3 + 1, \nu_4; s, t), \quad (3.13)
\]

\[
\frac{\partial}{\partial s} I_4^{(d)}(\nu_1, \nu_2, \nu_3, \nu_4; s, t) = \nu_2 \nu_4 I_4^{(d+2)}(\nu_1, \nu_2 + 1, \nu_3, \nu_4 + 1; s, t). \quad (3.14)
\]

Exploiting equations (3.13) and (3.14), one can also express higher derivatives of the integral \( I_4^{(d)} \) in terms of integrals with shifted dimension and indices. Integrals with shifted space time dimension must be expressed in terms of integrals without shift of dimension \( d \).

In order to obtain the differential equation, we express the basis integrals in terms of the integral itself and its first and second derivatives. Substituting these expressions into the result for the third derivative leads to the following differential equation for the integral \( I_4^{(d)}(\nu_1, \nu_2, \nu_3, \nu_4; s, t) \equiv I_4^{(d)}(t) \):

\[
\frac{\partial^3 I_4^{(d)}(t)}{\partial t^3} = \frac{\partial^3 I_4^{(d)}(t)}{\partial s^3} = \left\{ \begin{array}{ll}
\nu_1 \nu_3 \frac{\partial I_4^{(d+2)}(t)}{\partial t} & \nu_1 \nu_3 \frac{\partial I_4^{(d+2)}(t)}{\partial s}, \\
\nu_2 \nu_4 \frac{\partial I_4^{(d+2)}(t)}{\partial t} & \nu_2 \nu_4 \frac{\partial I_4^{(d+2)}(t)}{\partial s}.
\end{array} \right. \]
\[4t^2(s + t) \theta^2 I_4^{(d)}(t) - 2t[2(d - 2\nu_1 - \nu_2 - 2\nu_3 - \nu_4 - 3)s + (d - 6 - 4\nu_1 - 2\nu_2 - 4\nu_3 - 2\nu_4)t] \theta^2 I_4^{(d)}(t) + [2(2\nu_1 \nu_2 + 2\nu_1 \nu_4 + 2\nu_2 \nu_4 + 2\nu_2 \nu_3 + 2 + 4\nu_3 - d\nu_1 - \nu_3 d + 2\nu_2 + 6\nu_1 \nu_3 + 2\nu_3^2 - d + 2\nu_1^2 + 4\nu_1)t + s(d - 2\nu + 2\nu_2 - 2)(d - 2\nu + 2\nu_4 - 2)] \theta^4 I_4^{(d)}(t) - 2\nu_1 \nu_3 (d - 2\nu) I_4^{(d)}(t) = 0, \tag{3.15}\]

where
\[\theta = \frac{d}{dt}\]

and
\[\nu = \nu_1 + \nu_2 + \nu_3 + \nu_4. \tag{3.17}\]

A similar prescription can be used to obtain recurrence relations with respect to an index \(\nu_j\) or \(d\).

A solution of the differential equation (3.15) which is valid at small \(t\) reads
\[I_4^{(d)}(t) = \frac{C_1(\nu_1, \nu_2, \nu_3, \nu_4)}{s^{\frac{d}{2} - \nu}} \text{}_3 F_2 \left[ \begin{array}{c} \nu_1, \nu_2, \nu_3, \nu_4 \end{array} ; \begin{array}{c} \nu - \nu_4 + 1, \nu - \nu_2 + 1, \nu - \frac{d}{2} \end{array} ; - \frac{t}{s} \right] + \frac{C_2(\nu_1, \nu_2, \nu_3, \nu_4)}{t^{\frac{d}{2} - \nu} \nu_4} \text{}_3 F_2 \left[ \begin{array}{c} \nu_4, \frac{d}{2} - \nu_1 - \nu_2, \frac{d}{2} - \nu_3 - \nu_2 \end{array} ; \begin{array}{c} 1 + \nu_4 - \nu_2, \frac{d}{2} - \nu + \nu_3 + 1 \end{array} ; - \frac{t}{s} \right] + \frac{C_3(\nu_1, \nu_2, \nu_3, \nu_4)}{t^{\frac{d}{2} - \nu} \nu_4} \text{}_3 F_2 \left[ \begin{array}{c} \nu_2, \frac{d}{2} - \nu_4 - \nu_1, \frac{d}{2} - \nu_3 - \nu_2 \end{array} ; \begin{array}{c} 1 - \nu_4 + \nu_2, \frac{d}{2} - \nu + \nu_3 + 1 \end{array} ; - \frac{t}{s} \right], \tag{3.18}\]

where
\[\text{}_3 F_2 \left[ \begin{array}{c} a_1, a_2, a_3 \end{array} ; \begin{array}{c} b_1, b_2 \end{array} ; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k}{(b_1)_k(b_2)_k} \frac{z^k}{k!}, \tag{3.19}\]

and \((a)_k = \Gamma(a + k)/\Gamma(a)\). The values of the constants \(C_j\) can be determined from boundary conditions and symmetry relations of the integral with respect to indices. The coefficient \(C_1(\nu_1, \nu_2, \nu_3, \nu_4)\) can be found from the value of the integral at \(t = 0\):
\[I_4^{(d)}(t = 0) = t^d \text{d}^{d/2 - \nu} \Gamma \left( \frac{d}{2} - \nu + \nu_2 \right) \Gamma \left( \frac{d}{2} - \nu + \nu_4 \right) \Gamma \left( \frac{d}{2} - \nu + \nu_3 \right) \Gamma \left( \nu - \frac{d}{2} \right) \Gamma(\nu_4) \Gamma(\nu_2) \Gamma(d - \nu). \tag{3.20}\]

Setting \(t = 0\) in equation (3.18) leads to the relation
\[C_1(\nu_1, \nu_2, \nu_3, \nu_4) = s^{d - d/2} I_4^{(d)}(t = 0) = t^d \text{d}^{d/2 - \nu} \Gamma \left( \frac{d}{2} - \nu + \nu_2 \right) \Gamma \left( \frac{d}{2} - \nu + \nu_4 \right) \Gamma \left( \frac{d}{2} - \nu + \nu_3 \right) \Gamma \left( \nu - \frac{d}{2} \right) \Gamma(\nu_4) \Gamma(\nu_2) \Gamma(d - \nu) \tag{3.21}\]

In order to find \(C_2\) and \(C_3\), we first transformed the result (3.18) into the region valid for large \(t\) by using the formula for analytic continuation of \(\text{d}_2 F_2\) functions:
Taking into account the symmetries of the integral with respect to \( \nu \):

\[
I_4^{(d)}(\nu_1, \nu_2, \nu_3, \nu_4; s, t) = I_4^{(d)}(\nu_3, \nu_2, \nu_1, \nu_4; s, t) = I_4^{(d)}(\nu_1, \nu_4, \nu_3, \nu_2; s, t),
\]

and the value of the integral at \( s = 0 \):

\[
I_4^{(d)}(s = 0) = \frac{\Gamma(\nu - \frac{d}{2}) \Gamma\left(\frac{d}{2} - \nu + \nu_1\right) \Gamma\left(\frac{d}{2} - \nu + \nu_3\right)}{\Gamma(\nu_1) \Gamma(\nu_3) \Gamma(d - \nu)},
\]

the following formulae for \( C_2 \) and \( C_3 \) were derived:

\[
C_2(\nu_1, \nu_2, \nu_3, \nu_4) = \frac{\Gamma(\nu - \nu_4 - \frac{d}{2}) \Gamma\left(\frac{d}{2} - \nu + \nu_2\right) \Gamma(\nu_2 - \nu_4) \Gamma\left(\frac{d}{2} - \nu_1\right)}{\Gamma(\nu_1) \Gamma(\nu_3) \Gamma(\nu_2) \Gamma(d - \nu)}.
\]

\[
C_3(\nu_1, \nu_2, \nu_3, \nu_4) = C_2(\nu_1, \nu_4, \nu_3, \nu_2).
\]

Symmetry relations (3.23) can be easily seen from the \( \alpha \) parametric representation of \( I_4^{(d)} \).

From equation (3.23) and hypergeometric representation (3.18), the following relations also follow:

\[
C_2(\nu_1, \nu_2, \nu_3, \nu_4) = C_2(\nu_3, \nu_2, \nu_1, \nu_4),
\]

\[
C_2(\nu_1, \nu_2, \nu_3, \nu_4) = C_2(\nu_1, \nu_4, \nu_3, \nu_2),
\]

\[
C_3(\nu_1, \nu_2, \nu_3, \nu_4) = C_2(\nu_1, \nu_4, \nu_3, \nu_2).
\]

\[
4. \text{Epsilon expansion of } I_4^{(d)}
\]

In this section, we present several terms in the \( \varepsilon = (4 - d)/2 \) expansion of the on-shell integral \( I_4^{(d)} \) for the case when

\[
\nu_1 = 1 + a_1 \varepsilon, \nu_2 = 1 + a_2 \varepsilon, \nu_3 = 1 + a_3 \varepsilon, \nu_4 = 1 + a_4 \varepsilon.
\]

Epsilon expansion of hypergeometric functions was described in [13–18]. The result up to \( O(\varepsilon^2) \) terms reads
\[ -\frac{z}{a}(1 + a_1 + a_4)(1 + a_3 + a_4)(1 + a_2 + a_3)(1 + a_1 + a_2)\epsilon^{(\nu-d/2)}
\Gamma(1 - \epsilon)I_4^{(4-2\epsilon)}(t) \]
\[ = \frac{a}{\epsilon^2} - \left[ (1 + a_1 + a_3)(a - 1) + a_2a_4 - a_1a_3 \right] \ln(-z) \]
\[ + \frac{1}{2} \left[ (1 + a_2 + a_4 + a_2a_4 - a_1a_3)(a_1 + a_3) + a(a_1^2 + a_1^2 + a_1a_3) \right] \ln^2(-z) \]
\[ - \frac{\pi^2}{6} \left[ (a_2 + 1)(1 + a_4)(2a_2 + 3 + 2a_4) + (3a_2 + 7 + 3a_4)a_1a_3 \right] \]
\[ + (5 + 5a_2 + 5a_4 + 2a_1a_3 + 3a_2a_4 + a_1^2 + a_1^2)(a_1 + a_3) \]
\[ + (2 + a_2 + a_4)(a_1^2 + a_2^2) \]
\[ + \epsilon \left\{ (1 + a_1 + a_2)(1 + a_3 + a_4)(1 + a_2 + a_3)(1 + a_1 + a_2) \left[ - \frac{1}{2} \ln(1 - z) \ln(-z)^2 \right] \right. \]
\[ - \ln(-z) \operatorname{Li}_2(z) - \frac{\pi^2}{2} \ln(1 - z) \ln(-z) + \frac{1}{6} S_1 \ln(-z)^3 + \frac{\pi^2}{6} S_2 \ln(-z) + S_3 \zeta_3 \right\} \]
\[ + O(\epsilon^2). \]  
(4.28)

where \( \gamma \) is the Euler–Mascheroni constant,
\[ a = 2 + a_1 + a_3 + a_4, \]  
(4.29)

and \( S_1, S_2, S_3 \) are as given in appendix B. When all \( a_i = 0 \) for the first three terms in the expansion, we find complete agreement with the result given in [3].

Just for completeness, we present here the formula for \( I_4^{(d)} \) which can be used in the frame of analytical regularization. In this case, it is assumed that \( d = 4 \), and indices are defined as in (4.27). The result of \( \epsilon \) expansion including the constant term reads
\[ -\frac{z}{b}(a_1 + a_4)(a_3 + a_4)(a_2 + a_3)(a_1 + a_2)\epsilon^{(\nu-d/2)}\epsilon^{-\epsilon}I_4^{(4)}(t) \]
\[ = \frac{b}{\epsilon^2} - \left[ a_2a_4 - a_1a_3 + b(a_1 + a_3) \right] \ln(-z) \]
\[ - \frac{\pi^2}{6} \left[ (a_2 + a_4)(2a_2a_4 + a_1^2 + a_1^2 + 3a_1a_3) + (a_1 + a_3)(a_1^2 + 3a_2a_4 + a_1^2 + 2a_1a_3) \right] \]
\[ + \frac{1}{2} \left[ a_1^2 + a_1^2 + (a_1a_3 + a_2a_4)b + (a_1 + a_2)(a_1^2 + a_1^2 - a_2a_4) \right] \ln(-z)^2 + O(\epsilon). \]  
(4.30)

where
\[ b = a - 2. \]  
(4.31)

5. Conclusions

In conclusion, we briefly summarize the most important results of the paper. First of all, we outline a regular algorithm for deriving analytical results for Feynman integrals with arbitrary noninteger powers of propagators. We demonstrate that the algorithm works perfectly for an integral which is rather complicated due to arbitrary indices. The Gröbner basis for the integral with general powers of propagators is presented here for the first time. It can be used not only for deriving differential equations but also for deriving recurrence relations with respect to space-time dimension \( d \) or an index \( \nu \) for integrals. From these equations, one can choose the easiest equation by which to obtain an analytical result. The method can be applied to massless integrals as well as to integrals with massive propagators.
In our opinion, application of the set of equations from the Gröbner basis can be more efficient in reducing Feynman diagrams to basis integrals than solving huge numbers of equations for integrals with integer indices obtained from integration-by-parts relations. However, this question needs further investigation.

A detailed description of our method and its application to other integrals will be presented in our future publications.

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Appendix A

Recurrence relations from the Gröbner basis for the integral \( I^{(d)}_{\mu}(\nu_1, \nu_2, \nu_3, \nu_4; s, t) \):

\[
1^+ B_1 = I^{(d)}_{\mu}(\nu_1, \nu_2 - 1, \nu_3, \nu_4 - 1; s, t) = \frac{(d + 2 - \nu)(d - 2 + \nu)(d - 2 + 2\nu_4)}{2\nu_4 (2 - 2\nu_1 - 2\nu_2 + d)} B_1 + \frac{2(\nu_4 - 1)(d + 2 - \nu)}{t(\nu_1 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_2,
\]

(A.1)

\[
2^+ B_1 = I^{(d)}_{\mu}(\nu_1 - 1, \nu_2, \nu_3, \nu_4 - 1; s, t) = \frac{2(\nu_2 - \nu_4)(d + 2\nu_4 - 2\nu_1 - 2\nu_2 + d)}{\nu_1 (2 - 2\nu_1 - 2\nu_2 + d)} B_1 + \frac{(d - 2\nu_1 - 2\nu_2)(d - 2\nu_1 - 2\nu_2 + d)(d - 2\nu_1 - 2\nu_2 + d) + 2(\nu_2 - \nu_4)(d + 2\nu_4 - 2\nu_1 - 2\nu_2 + d)}{(d - 2\nu_1 - 2\nu_2)(\nu_1 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_2 + \frac{(d - 2\nu_1 - 2\nu_2)(d - 2\nu_1 - 2\nu_2 + d) + 2(\nu_2 - \nu_4)(d + 2\nu_4 - 2\nu_1 - 2\nu_2 + d)}{(d - 2\nu_1 - 2\nu_2)(\nu_1 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_3,
\]

(A.2)

\[
3^+ B_1 = I^{(d)}_{\mu}(\nu_1 - 1, \nu_2 - 1, \nu_3, \nu_4 - 1; s, t) = \frac{2(\nu_2 - \nu_4)(d + 2\nu_4 - 2\nu_1 - 2\nu_2 + d)}{\nu_1 (2 - 2\nu_1 - 2\nu_2 + d)} B_1 + \frac{2(\nu_2 - \nu_4)(d - 2\nu_1 - 2\nu_2)(d - 2\nu_1 - 2\nu_2)(d - 2\nu_1 - 2\nu_2)(d - 2\nu_1 - 2\nu_2 + d)}{(d - 2\nu_1 - 2\nu_2)(\nu_1 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_2 + \frac{2(\nu_2 - \nu_4)(d + 2\nu_4 - 2\nu_1 - 2\nu_2 + d)}{\nu_1 (2 - 2\nu_1 - 2\nu_2 + d)(\nu_1 - 1)(d - 2\nu_1 - 2\nu_2)(d - 2\nu_1 - 2\nu_2)(d - 2\nu_1 - 2\nu_2 + d)} B_3,
\]

(A.3)

\[
4^+ B_1 = I^{(d)}_{\mu}(\nu_1 - 1, \nu_2 - 1, \nu_3, \nu_4; s, t) = \frac{(d - 2\nu_1 - 2\nu_2 + d)(d + 2\nu_4 - 2\nu_1 - 2\nu_2 + d)}{t(\nu_1 - 1)(d - 2\nu_1 - 2\nu_2)(\nu_1 - 1)} B_1 + \frac{(4d - 2\nu_2 - 2\nu_4)(d + 2\nu_4 - 2\nu_1 - 2\nu_2 + d)}{t(\nu_1 - 1)(d - 2\nu_1 - 2\nu_2)(\nu_1 - 1)} B_2 + \frac{(d - 2\nu_1 - 2\nu_2 + d)(d + 2\nu_4 - 2\nu_1 - 2\nu_2 + d)}{(d - 2\nu_1 - 2\nu_2)(\nu_1 - 1)(d - 2\nu_1 - 2\nu_2)(\nu_1 - 1)} B_3,
\]

(A.4)
\[1^+ B_2 = \frac{I_4^{(d)}(\nu_1, \nu_2 - 1, \nu_3 - 1, \nu_4; s, t)}{(\nu_4 - 1)(2 - 2\nu_1 - 2\nu_2 - 2\nu_3)} B_1 \]
\[= \frac{(d - 2\nu + 2\nu_3 + 4)(\nu_3 - 1)(d + 2 - \nu)(d + 2 - 2\nu_3 - 2\nu_2)}{(\nu_4 - 1)(d - 2\nu_1 - 2\nu_4)(2 - 2\nu_1 - 2\nu_2 + d)s(\nu_1 - 1)} B_1 \]
\[= \frac{(2(\nu_1 - \nu_3)(d + 2\nu_3 - 2\nu + 4)(4 + 2\nu_2 - 2\nu + d)(2 - 2\nu_1 - 2\nu_2 + d)s)(d + 2 - \nu)}{(2 - 2\nu_1 - 2\nu_2 + d)s(d - 2\nu_1 - 2\nu_4)(\nu_1 - 1)} B_2 \]
\[= \frac{(d - 2\nu + 8)(d + 3 - \nu)(d + 2 - \nu)}{s(\nu_1 - 1)(\nu_4 - 1)(d - 2\nu_1 - 2\nu_4)} B_1. \quad (A.5) \]

\[2^+ B_2 = \frac{I_4^{(d)}(\nu_1 - 1, \nu_2, \nu_3 - 1, \nu_4; s, t)}{(\nu_4 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_1 \]
\[= \frac{2(d + 2 - \nu)(\nu_3 - 1)}{s(\nu_2 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_1 = \frac{(d + 2 - \nu)(d + 2\nu_3 - 2\nu + 4)}{s(\nu_2 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_2, \quad (A.6) \]

\[4^+ B_2 = I_4(\nu_1 - 1, \nu_2 - 1, \nu_3 - 1, \nu_4 + 1; s, t) \]
\[= \frac{2(\nu_1 - 1)(\nu_2 - 2)(d + 2\nu_3 - 2\nu_2)(d + 2 - \nu)}{(d - 2\nu_1 - 2\nu_2)(d - 2\nu_4 - 2\nu_3)s(\nu_4 - 1)} B_1 \]
\[= \frac{(d - \nu + 2)(t(d - 2\nu_4 - 2\nu_3)(d + 2\nu_3 - 2\nu + 4) - 2(\nu_2 - \nu_4 - 1)(d + 2\nu_2 - 2\nu + d)s)}{(d - 2\nu_1 - 2\nu_4)(d - 2\nu_4 - 2\nu_3)s(\nu_4 - 1)} B_2 \]
\[+ \frac{2(\nu_2 - \nu_4 - 1)(d - 2\nu + 8)(d - \nu + 3)(d - \nu)}{(d - 2\nu_4 - 2\nu_3)s(\nu_4 - 1)(d - 2\nu_1 - 2\nu_4)} B_3. \quad (A.7) \]

\[1^+ B_3 = \frac{I_4^{(d)}(\nu_1, \nu_2 - 1, \nu_3 - 1, \nu_4 - 1; s, t)}{(\nu_2 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_1 \]
\[= \frac{(d + 2 - \nu)(\nu_2 - 4)}{(\nu_2 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_1 + \frac{2(\nu_1 - \nu_3)(\nu_4 - 1)}{(\nu_1 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_2, \quad (A.8) \]

\[2^+ B_3 = \frac{I_4^{(d)}(\nu_1 - 1, \nu_2, \nu_3 - 1, \nu_4 - 1; s, t)}{(\nu_2 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_1 \]
\[= \frac{(\nu_2 - 1)(\nu_2 - \nu_4)}{(\nu_2 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_1 + \frac{2(\nu_1 - \nu_3)(\nu_4 - 1)}{(\nu_1 - 1)(2 - 2\nu_1 - 2\nu_2 + d)} B_2. \quad (A.9) \]

\[I_4^{(d+2)}(\nu_1 - 1, \nu_2 - 1, \nu_3 - 1, \nu_4 - 1; s, t) \]
\[= \frac{(2\nu_2 - 4t + (d + 2\nu_2 - 2\nu + 6)s)(\nu_3 - 1)s(d + 2 - 2\nu_3 - 2\nu_2)}{(2d + 3 - \nu)(d - 2\nu + 10)(d + 5 - \nu)(d - \nu + 4)(t + s)} B_1 \]
\[+ \frac{(d - 2\nu_1 - 2\nu_4 + 2)(t(d + 2\nu_1 - 2\nu + 6) + 2\nu_3 - 4\nu)(\nu_1 - 1)}{(2d + 3 - \nu)(d + 10 - 2\nu)(d + 5 - \nu)(d - \nu + 4)(t + s)} B_2 \]
\[+ \frac{2(d - \nu + 4)(d + 10 - 2\nu)(d - \nu + 5)(t + s)}{1} \times \left[ t^2(d + 2\nu_1 - 2\nu + 6)(d + 2\nu_1 - 2\nu + 6) + t(d^2 - 2d\nu + 6d + 4\nu_1\nu_3 - 8 + 4\nu_2\nu_4)s \right. \]
\[+ (d + 2\nu_2 - 2\nu + 6)(d + 2\nu_2 - 2\nu + 6)s^2 ] B_3. \quad (A.10) \]
Appendix B

Some coefficients in the $\varepsilon$ expansion of $I_4^{(d)}$ given in equation (4.28):

\[
S_1 = 1 + 2(a_1 + a_2 + a_3 + a_4) + a_2^2 + a_3^2 + 4a_2a_4 + 3a_3(a_2 + a_4) + 3(a_1 + a_2 + a_4)a_1
- a_1(2a_2^2 - a_1^2) + a_4(a_1 + 3a_3)a_1 + 2a_4(2a_3 + a_4)a_2 + (a_1 + 2a_3 + a_4)a_1^2
+ (4a_4 + 5a_1)a_1a_2 - 2a_1^2 + (a_1 + a_3)(a_2 + a_3)(a_2a_3 - a_1^2)
+ (a_2 - a_1 + a_3)a_1 - (a_2 + a_3 + a_4)a_1^2 + a_1^4.
\]  

(B.1)

\[
S_2 = 3 + 8(a_1 + a_3) + 6(a_2 + a_4) + 7(a_1^2 + a_3^2) + (17a_3 + 12a_2 + 12a_4)a_1
+ 3a_2^2 + 12(a_1 + a_3)a_2 + 3a_3^2 + 12a_2a_3 + (6a_4 + 4a_3 + 4a_1)a_1^2
+ (6a_1^2 + 16(a_1 + a_3)a_4 + 16a_1a_4 + 7a_3^2 + 7a_1^2)a_2
+ (16a_1a_3 + 7a_3^2 + 7a_1^2)a_4 + (a_1 + a_3)(2a_1^2 + 9a_1a_3 + 2a_3^2 + 4a_3^2)
+ (a_1 + a_3)(a_2a_3 + 3a_2a_4 + a_3a_4) + (2a_1^3 + 9a_2a_1a_4 + 3a_2a_3^2 + 3a_3a_2^2
+ 4a_2a_3^2 + 4a_2a_4 + 5a_2a_3 + 5a_2a_3) + (a_2 + a_3 + a_4)a_1
+ (a_2^2 + a_1^2 + 4a_3 + 5a_2a_2 + 5a_4a_2 + 5a_4a_1)a_1^2.
\]  

(B.2)

\[
S_3 = -5 - 11(a_1 + a_2 + a_3 + a_4) - 8(a_1^2 + a_2^2 + a_3^2 + a_4^2) - 17(a_1 + a_3)(a_2 + a_4)
- 19(a_2a_4 + a_1a_3) - 2(a_1^2 + a_3^2 + a_4^2) - (8a_2 - 10a_3 - 8a_4)a_1^2
+ (-8a_2^2 - 19(a_1 + a_3)a_2 - 10a_3^2 - 19a_2a_3 - 8a_3a_2 - 8a_1a_3 + 10a_4)a_1
+ (8a_1^2 + 19a_2a_1 + 10a_3^2 - 19a_2a_3 - 8a_2a_1 - 8a_3a_1 + 10a_4)a_1^2
- (a_2 + 2a_3 + a_4)a_1(a_1^2 + (a_1 + a_2 + 2a_3)a_1 + a_1^2 + 2a_3a_1 + 4a_3a_2 + a_2^2 + 2a_2a_3 + a_3^2)
- a_2a_3(a_2 + a_3)^2 - (a_2 + a_3)(2a_2 + 5a_3)a_2a_3 - 4a_2a_3a_1^2 - 2a_1a_1^2.
\]  

(B.3)

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