ANALYSIS OF NOVEL ADAPTIVE TWO-GRID FINITE ELEMENT ALGORITHMS FOR LINEAR AND NONLINEAR PROBLEMS

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Abstract. This paper proposes some novel efficient and accurate adaptive two-grid (ATG) finite element algorithms for linear and nonlinear partial differential equations (PDEs). In these algorithms, they use the information of the solutions on $k$-th level adaptive meshes, instead of on the uniform meshes, to find the solutions on $(k+1)$-th level adaptive meshes. They transform the non-symmetric positive definite (non-SPD) PDEs into symmetric positive definite (SPD) PDEs, and transform the nonlinear PDEs into the linear PDEs. These algorithms have the following advantages:

1. Comparing with adaptive methods, they do not need to solve the nonlinear systems;
2. Comparing with two-grid methods, the degrees of freedom are largely reduced;
3. Comparing with the cases when uniform meshes are used for coarse level approximation, they are easily implemented; they are more efficient and accurate since only the interpolation of the solution on newly refined meshes needs to be computed, and the interpolation error is also reduced; they are especially efficient when many steps of mesh refinements are used since the computational cost of computing solutions on uniform meshes is large then.

Next, this paper constructs a residue-type a posteriori error estimator for general non-SPD linear problems. We prove the upper bound of the oscillation term, and this gets rid of the assumption that the oscillation term is a high order term (h.o.t.), which may not be true generally due to the low regularity of the numerical solution. Based on this result, the reliability and efficiency of the error estimator are established. Finally, the convergence of the error on the adaptive meshes is proved when bisection is used for the mesh refinement.

Key words. adaptive two-grid finite element algorithms, residue-type a posteriori error estimator, symmetric positive definite, convergence

AMS subject classifications. 65N12, 65N15, 65N22, 65N30, 65M50, 65N55

1. Introduction. The two-grid discretization techniques for solving second order non-SPD linear PDEs and nonlinear PDEs were proposed by Xu [18, 19, 20]. In these algorithms, two spaces $V_h$ and $V_H$ are employed for the finite element discretization, where mesh size $h \ll H$. They first solve the original non-SPD linear PDE or nonlinear PDE on the coarser finite element space $V_H$, and then find the solution $u_h$ of a linearized PDE on the finer finite element space $V_h$ based on the coarser level solution $u_H$. The computational cost is saved a lot since $\dim(V_H) \ll \dim(V_h)$, and the optimal accuracy can be maintained by choosing appropriate coarser mesh size $H$.

These two-grid techniques have been applied for solving mixed (Navier-)Stokes/Darcy model [13, 5], time-harmonic Maxwell equations [23], eigenvalue problems [22] and so on.

For adaptive methods, the idea is to homogenize the errors on all mesh elements, and to further improve the accuracy and efficiency of solving PDEs. The upper and lower bounds of the error estimator for general elliptic and parabolic PDEs are derived [15, 16] when the conforming finite element methods are used for the discretization. The convergence of the error and the bound of the convergence rate are considered in [6], but there are some stringent restrictions on the initial mesh, which makes the algorithm not practical. It introduces the concept of data oscillation in [11, 12] to circumvent the requirement on the initial mesh. In [2], it proposes a modification of the algorithm in [12] and proves optimal estimates by incorporating coarsening of the meshes. The convergence rate of the conforming finite element methods is

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also analyzed in [4]. For adaptive discontinuous Galerkin methods, the construction of different types of error estimators are introduced in [9], and the convergence is studied in [10].

Recently in [1], the uniform meshes are used for coarse-grid approximation, and adaptive meshes are used for fine-grid approximation. In this paper, it proposes some novel algorithms which use $k$-th level adaptive meshes as coarse grids, and $(k + 1)$-th level adaptive meshes as fine grids. Based on the solutions on $k$-th coarser level adaptive meshes, we only need to solve linear systems on $(k + 1)$-th finer level adaptive meshes. The following are some main advantages of these novel algorithms in comparison with the algorithms in [1]. First, interpolating the solutions from uniform coarse meshes to adaptive fine meshes will bring in more interpolation errors and increase the computational cost. In the novel algorithms, only the values on newly added mesh points need to be interpolated, so they should have smaller interpolation error and lower computational cost. Second, coarser meshes $T_h^k$ are always contained in finer meshes $T_h^{k+1}$, so the novel algorithms are much easier to implement. Third, in [1], the error estimator and the error are bounded by each other up to a high order term under the condition that $H = O(h^{\mu})$, $\mu \geq \frac{1}{2}$, where $h$ is the smallest size of the adaptive meshes since the inverse inequality is used many times. Therefore, when the number of bisections increases, $h$ and $H$ decrease, then the computational cost can be very large when solving non-SPD linear or nonlinear PDEs on uniform meshes; For the novel algorithms in this paper, only linear systems need to be solved.

Based on these novel algorithms, this paper constructs a residue-type error estimator, and prove its upper and lower bounds with respect to the exact error up to a high order term. It gets rid of the assumption in [1] that the oscillation term is a high order term, which may be wrong due to the low regularity of the solution of the two-grid algorithm, by stating a lemma which proves the oscillation term is bounded by the error and a high order term. Furthermore, a convergence result of the error is established based on the above results. Finally, several algorithms for nonlinear PDEs are proposed in the end, and the analysis of the their convergence can be similarly proved by the ideas in this paper.

The organization of this paper is as follows. In section 2, we introduce some notation, and state some preliminary results. In section 3, we give the adaptive two-grid finite element algorithm for non-SPD linear problems. We construct an error estimator, and then we prove the reliability and efficiency of the proposed error estimator. Furthermore, we prove that the error converges to zero up to a high order term as the number of bisection increases. In section 4, some adaptive two-grid finite element algorithms for nonlinear problems are proposed. In section 5, some numerical tests are given to verify the accuracy and efficiency of the algorithms.

2. Preliminaries. Through this paper, denote $\Omega$ as a convex polygonal domain, and the standard Sobolev notations are used, i.e., for any set $A$,

\[
\|v\|_{L^p(A)} = \left( \int_A |v|^p \, dx \right)^{1/p} \quad 1 \leq p < \infty,
\]

\[
|v|_{W^{m,p}(A)} = \left( \sum_{|\alpha| = m} \|D^\alpha v\|_{L^p(A)}^p \right)^{1/p} \quad 1 \leq p < \infty,
\]

\[
\|v\|_{W^{m,p}(A)} = \left( \sum_{|\alpha| \leq m} |D^\alpha v|_{L^p(A)}^p \right)^{1/p} \quad 1 \leq p < \infty.
\]
When \( m = 0 \), denote \( W_{0,p}(A) = L^p(A) \), and when \( p = 2 \), denote \( W_{m,2}(A) = H^m(A) \). Also, denote \( H^1_0(\Omega) = \{ v \in H^1(\Omega) : v|_{\partial \Omega} = 0 \} \).

Consider the following second order quasi-linear elliptic equation [20]

\[
\begin{align*}
(2.1) & \quad -\text{div}(f(x, u, \nabla u)) + g(x, u, \nabla u) = 0 \quad \text{in} \ \Omega, \\
(2.2) & \quad u = 0 \quad \text{on} \ \partial \Omega,
\end{align*}
\]

where \( f(x, y, z) : \Omega \times \mathbb{R}^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and \( g(x, y, z) : \Omega \times \mathbb{R}^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}^1 \) are smooth functions. Assume the solution of (2.1)–(2.2) satisfies \( u \in H^1_0(\Omega) \cap W^{2,2+\kappa}(\Omega) \) for some \( \kappa > 0 \). For any \( w \in W^{1,\infty}(\Omega) \), we denote

\[
\begin{align*}
a(w) &= D_zf(x, w, \nabla w) \in \mathbb{R}^{2 \times 2}, \quad b(w) = D_yf(x, w, \nabla w) \in \mathbb{R}^2, \\
c(w) &= D_zg(x, w, \nabla w) \in \mathbb{R}^2, \quad d(w) = D_yg(x, w, \nabla w) \in \mathbb{R}^1.
\end{align*}
\]

Introduce \( \delta_1 \) and \( \delta_2 \) below

\[
\delta_1 = \begin{cases} 
0 & \text{if } D_z^2f(x, y, z) = 0, \ D_z^2g(x, y, z) = 0, \\
1 & \text{otherwise},
\end{cases}
\]

and

\[
\delta_2 = \begin{cases} 
0 & \text{if } \delta_1 = 0, \ D_yD_zf(x, y, z) = 0, \ D_yD_zg(x, y, z) = 0, \\
1 & \text{otherwise}.
\end{cases}
\]

If \( \delta_1 = 0 \) and \( \delta_2 = 1 \), then equation (2.1) becomes

\[
(2.3) \quad -\text{div} (\alpha_1(x, u)\nabla u + \alpha_2(x, u)) + \beta(x, u) \cdot \nabla u + \gamma(x, u) = 0.
\]

If \( \delta_1 = \delta_2 = 0 \), then equation (2.1) becomes

\[
(2.4) \quad -\text{div} (\alpha_1(x)\nabla u + \alpha_2(x, u)) + \beta(x) \cdot \nabla u + \gamma(x, u) = 0.
\]

Consider (2.3), we incorporate term \( \text{div}(\alpha_2(x, u)) \) into term \( \beta(x, u) \cdot \nabla u + \gamma(x, u) \), then (2.3) can be written as

\[
(2.5) \quad -\text{div} (\alpha(x, u)\nabla u) + \beta(x, u) \cdot \nabla u + \gamma(x, u) = 0.
\]

Corresponding to (2.5), we consider the non-symmetric positive definite (non-SPD) problem with homogeneous Dirichlet boundary condition below

\[
\begin{align*}
(2.6) & \quad -\text{div}(\alpha(x)\nabla u) + \beta(x) \cdot \nabla u + \gamma(x)u = 0 \quad \text{in} \ \Omega, \\
(2.7) & \quad u = 0 \quad \text{on} \ \partial \Omega.
\end{align*}
\]

Here the coefficients \( \alpha(x) \in \mathbb{R}^{2 \times 2}, \beta(x) \in \mathbb{R}^2 \) and \( \gamma(x) \in \mathbb{R}^1 \) are assumed to be smooth such that the solution \( u \in H^1_0(\Omega) \cap W^{2,2+\kappa}(\Omega) \), where \( \kappa > 0 \).

Define the following notations

\[
\begin{align*}
A(u, v) &= (\alpha(x)\nabla u, \nabla v) + (\beta(x) \cdot \nabla u + \gamma(x)u, v), \\
A_S(u, v) &= (\alpha(x)\nabla u, \nabla v), \\
A_N(u, v) &= (\beta(x) \cdot \nabla u + \gamma(x)u, v).
\end{align*}
\]
The weak form of (2.6)-(2.7) is to seek $u \in H^1_0(\Omega) \cap W^{2,2+\kappa}(\Omega)$ such that
\begin{align}
(2.8) \quad A_S(u, v) + A_N(u, v) &= 0 \quad \text{in } \Omega, \\
(2.9) \quad u &= 0 \quad \text{on } \partial \Omega. 
\end{align}

The bilinear form $A_S$ induces the semi-norm
\begin{equation}
(2.10) \quad ||u|| := A_S(u, u)^{1/2}.
\end{equation}

We also assume $\alpha(x)$ satisfies, for some constant $0 < C_1 < C_2 < \infty$, that
\begin{equation}
(2.11) \quad C_1 |\xi|^2 \leq \xi^T \alpha(x) \xi \leq C_2 |\xi|^2 \quad \forall \xi \in \mathbb{R}^d,
\end{equation}

where $d$ denotes the dimension of the space.

3. A novel adaptive two-grid finite element algorithm for non-SPD problems. In this section, we present an adaptive two-grid finite element algorithm for non-SPD problems. The idea is to utilize the solutions on $k$-th level adaptive meshes to transform the non-SPD problems into an SPD problems, and then to find the solutions of the SPD problems on $(k+1)$-th level adaptive meshes. Denote $T_k$ as the set of meshes in $k$-th bisection, $H_K^k$ as the mesh size of $K$ in $T_k$, $E_k$ as the mesh edges in $k$-th bisection, and $V_{H_K^k}$ as the finite element space on $T_k$. Let us present the two-grid finite element algorithm:

\begin{algorithm}
STEP 1: Find $u_{H_K^0} \in V_{H_K^0}$ such that
\[ A(u_{H_K^0}, v_{H_K^0}) = 0 \quad \forall v_{H_K^0} \in V_{H_K^0}; \]

STEP 2: For $k = 0, 1, 2, \cdots$, find $u_{H_K^{k+1}} \in V_{H_K^{k+1}}$ such that
\[ A_S(u_{H_K^{k+1}}, v_{H_K^{k+1}}) + A_N(u_{H_K^{k}}, v_{H_K^{k+1}}) = 0 \quad \forall v_{H_K^{k+1}} \in V_{H_K^{k+1}}. \]
\end{algorithm}

Notice in STEP 1, $\{H_K^0\}_{K \in T_0}$ are usually chosen to be the uniform meshes, i.e., $H_K^0 = H^0$. In practice, the non-symmetric part $A_N(u_{H_K^k}, v_{H_K^{k+1}})$ can be computed by interpolating $u_{H_K^k}$ from $T_k$ to $T_{k+1}$. This algorithm has many advantages in the aspects of both efficiency and accuracy, as are stated with details in section 1.

Next we construct an error estimator and prove its reliability and efficiency.

3.1. A posteriori error estimates for the novel adaptive two-grid finite element algorithm. Denote $n_{K\pm}$ as the unit outward normal vector of $\partial K^\pm$, and $[u]$ as the jump of $u$ between two triangles. Define the element residue and the edge jump by
\begin{align}
(3.1) \quad R_K^k &= -\text{div}(\alpha(x)\nabla u_{H_K^k}) + \beta(x) \cdot \nabla u_{H_K^k} + \gamma(x) u_{H_K^k}, \\
(3.2) \quad J_{E}^k &= ||\alpha(x)\nabla u_{H_K^k}||_E.
\end{align}

Define the local error estimators $\eta_{R,K}^k$, $\eta_{J,E}^k$, and $\eta_T^k$ on $K$ by
\begin{align}
(3.3) \quad (\eta_{R,K}^k)^2 &= (H_K^k)^2 ||H_K^k||^2_{L^2(K)} , \\
(3.4) \quad (\eta_{J,E}^k)^2 &= H_K^k ||J_{E}^k||^2_{L^2(E)} , \\
(3.5) \quad (\eta_T^k)^2 &= (\eta_{R,K}^k)^2 + (\eta_{J,E}^k)^2.
\end{align}
Define the global error estimators $\eta^k_R$, $\eta^k_J$ and $\eta(u_{H^k}, T_k)$ on mesh $T_k$ by

\begin{align}
\eta^k_R &= \left( \sum_{K \in T_k} (\eta^k_{R,K})^2 \right)^{1/2}, \\
\eta^k_J &= \left( \sum_{E \in \mathcal{E}_k} (\eta^k_{J,E})^2 \right)^{1/2}, \\
\eta(u_{H^k}, T_k) &= ((\eta^k_R)^2 + (\eta^k_J)^2)^{1/2}.
\end{align}

Denote $\bar{R}^k_E$ and $\bar{J}^k_E$ as the average of $R^k_E$ and $J^k_E$ respectively, then define the oscillation terms:

\begin{align}
osc^R(u_{H^k}, T) &= H^k_K \| R^k_K - \bar{R}^k_E \| L^2(K), \\
osc^J(u_{H^k}, T) &= (H^k_K)^{1/2} \| J^k_E - \bar{J}^k_E \| L^2(E), \\
osc(u_{H^k}, T) &= \left( \sum_{K \in T} (osc^R(u_{H^k}, T))^2 + \sum_{E \in \mathcal{E}} (osc^J(u_{H^k}, T))^2 \right)^{1/2}.
\end{align}

### 3.1.1. A reliable upper bound of the error

In this subsection, we will prove that the error of the adaptive two-grid finite element algorithm 3.1 can be bounded by the error estimator.

**Theorem 3.1.** Let $u$ and $u_{H^{k+1}}$ be the solution of (2.8)–(2.9) and the adaptive two-grid finite element algorithm 3.1, then

$$\| u - u_{H^{k+1}} \| \leq C (\eta^k_R + \eta^k_J) + C \| u_{H^k} - u \| L^2(\Omega).$$

**Proof.** Let $v^I$ be the Scott-Zhang interpolation of $v$ on $V_{H^k+1}$. Using equations (2.8)–(2.9) and the two-grid finite element algorithm, then $\forall v \in H^1_0(\Omega)$, we have

\begin{align}
(\alpha(x) \nabla (u - u_{H^{k+1}}), \nabla v) &= (-\beta(x) \cdot \nabla u - \gamma(x) u, v) - (\alpha(x) \nabla u_{H^{k+1}}, \nabla v) \\
&= (-\beta(x) \cdot \nabla u - \gamma(x) u, v) - (\alpha(x) \nabla u_{H^{k+1}}, \nabla (v - v^I)) \\
&\quad + (\beta(x) \cdot \nabla u_{H^k} + \gamma(x) u_{H^k}, v^I) \\
&= (\beta(x) \cdot \nabla (u_{H^k} - u) + \gamma(x) (u_{H^k} - u), v) \\
&\quad - \sum_{E \in \mathcal{E}_{k+1}} (\| \alpha(x) \nabla (u_{H^{k+1}}) \| L^2(v - v^I) + \sum_{K \in T_{k+1}} (\operatorname{div}(\alpha(x) \nabla (u_{H^{k+1}})), v - v^I) \\
&\quad - (\beta(x) \cdot \nabla u_{H^k} + \gamma(x) u_{H^k}, v - v^I) \\
&= - \sum_{K \in T_{k+1}} (R^k_{H^k+1}, v - v^I) - \sum_{K \in T_{k+1}} (J^k_{H^k+1}, v - v^I) \\
&\quad + (\beta(x) \cdot \nabla (u_{H^k} - u) + \gamma(x) (u_{H^k} - u), v).
\end{align}

Then using the Cauchy-Schwarz inequality, trace inequality and properties of the
Scott-Zhang interpolation operator, we have

\[(3.13) \quad (\alpha(x) \nabla (u - u_{H_{K}^{k+1}}), \nabla v) \leq \left( \sum_{K \in T_{k+1}} (H_{K}^{k+1})^2 \| R_{k+1}^2 \|_{L^2(K)} \right)^{\frac{1}{2}} \left( \sum_{K \in T_{k+1}} (H_{K}^{k+1})^{-2} \| v - v^{t} \|_{L^2(K)}^2 \right)^{\frac{1}{2}} + C \left( \sum_{E \in \mathcal{E}_{k+1}} (H_{K}^{k+1})^2 \| J_{E}^{k+1} \|_{L^2(E)}^2 \right)^{\frac{1}{2}} \left( \sum_{E \in \mathcal{E}_{k+1}} (H_{K}^{k+1})^{-1} \| v - v^{t} \|_{L^2(E)}^2 \right)^{\frac{1}{2}} + (\beta(x) \cdot \nabla (u_{H_{K}^{k}} - u) + \gamma(x)(u_{H_{K}^{k}} - u), v) \leq \eta_{R}^{k+1} \left( \sum_{K \in T_{k+1}} (H_{K}^{k+1})^{-2} \| v - v^{t} \|_{L^2(K)}^2 \right)^{\frac{1}{2}} + C \| u_{H_{K}^{k}} - u \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)} \right) \]

Choosing \( v = u - u_{H_{K}^{k+1}} \), then the theorem is proved.

3.1.2. A efficient lower bound of the error. For the a posteriori error estimates of the classical finite element algorithms, the oscillation terms are usually high order terms, but for the proposed adaptive two-grid algorithm 3.1, the oscillation term may not be the high order term because of the low regularity of the solution. One main result in this section is to give an upper bound of the oscillation term, which plays crucial roles in proving the efficiency of the estimator of the adaptive two-grid algorithm. As is seen below, it is bounded by the summation of the error and the high order term under some assumptions (see Remark 3.3).

**Lemma 3.2.** The oscillation term can be bounded by

\[ \text{osc}(u_{H_{K}^{k+1}}, T_{k+1}) \leq e_{1}^{k+1} + e_{2}^{k+1}. \]

where

\[ e_{1}^{k+1} := C \| \nabla (u - u_{H_{K}^{k+1}}) \|_{L^2(\Omega)}, \]

\[ e_{2}^{k+1} := C \| u - u_{H_{K}^{k}} \|_{L^2(\Omega)} + C \left( \sum_{K \in T_{k+1}} (H_{K}^{k+1})^2 \| D^2 u - D^2 u_{H_{K}^{k+1}} \|_{L^2(K)}^2 \right)^{\frac{1}{2}} + C \left( \sum_{K \in T_{k+1}} (H_{K}^{k+1})^2 \| \sigma - \tilde{\sigma} \|_{L^2(K)}^2 \right)^{\frac{1}{2}} + C \left( \sum_{K \in T_{k+1}} \| \nabla u - \nabla u_{H_{K}^{k+1}} \|_{L^2(K)}^2 \right)^{\frac{1}{2}}, \]

and \( \sigma \) and \( \tilde{\sigma} \) are defined in the beginning of the proof.

**Proof.** For the residue estimator, define \( \sigma^k, \tilde{\sigma}^k \) and \( \sigma \) by

\[ \tilde{\sigma}^k := -\alpha(x)^T : D^2 u_{H_{K}^{k+1}} - \text{div}(\alpha(x)^T) \cdot \nabla u_{H_{K}^{k+1}} + \beta(x) \cdot \nabla u_{H_{K}^{k}} + \gamma(x) u_{H_{K}^{k}}, \]

\[ \sigma := -\alpha(x)^T : D^2 u - \text{div}(\alpha(x)^T) \cdot \nabla u + \beta(x) \cdot \nabla u + \gamma(x) u. \]
Denote $\tilde{\delta}^k$ and $\bar{\sigma}$ as the average of $\hat{\delta}^k$ and $\sigma$ on $K$ respectively, then

$$\left(3.14\right) \quad \|\hat{\delta}^k - \tilde{\delta}^k\|_{L^2(K)} \leq \|\hat{\delta}^k - \bar{\delta}\|_{L^2(K)} \leq \|\hat{\delta}^k - \sigma\|_{L^2(K)} + \|\sigma - \bar{\sigma}\|_{L^2(K)} \leq C\|D^2u - D^2u_{H^k_{K+1}}\|_{L^2(K)} + C\|\nabla(u - u_{H^k_{K+1}})\|_{L^2(K)} + C\|\nabla(u - u_{H^k_K})\|_{L^2(K)} + C\|\sigma - \bar{\sigma}\|_{L^2(K)}.$$

For the jump estimator, define $\hat{\delta}^k$ and $\delta$ by

$$\hat{\delta}^k := \|\alpha(x)\nabla u_{H^k_{K+1}}\|_E, \quad \delta := \|\alpha(x)\nabla u\|_E.$$

Denote $\tilde{\delta}^k$ as the average of $\hat{\delta}^k$ on $E$, and $\bar{\nabla u}$ as the average of $\nabla u$ on $K$. Using the trace inequality, we have

$$\left(3.15\right) \quad \|\hat{\delta}^k - \tilde{\delta}^k\|_{L^2(E)} \leq \|\hat{\delta}^k - \delta\|_{L^2(E)} \leq C(H^k_{K+1})^{-\frac{1}{2}}\|\nabla u_{H^k_{K+1}} - \bar{\nabla u}\|_{L^2(K)} \leq C(H^k_{K+1})^{-\frac{1}{2}}\|\nabla(u - u_{H^k_{K+1}})\|_{L^2(K)} + C(H^k_{K+1})^{-\frac{1}{2}}\|\nabla u - \bar{\nabla u}\|_{L^2(K)}.$$

By (3.11), (3.14), (3.15), and the inverse inequality, we have

$$\left(3.16\right) \quad \text{osc}(u_{H^k_{K+1}}, \mathcal{T}_{k+1}) \leq C \left( \sum_{K \in \mathcal{T}_{k+1}} (H^k_{K+1})^2\|D^2u - D^2u_{H^k_{K+1}}\|_{L^2(K)}^2 \right)^{\frac{1}{2}} + C \left( \sum_{K \in \mathcal{T}_{k+1}} (H^k_{K+1})^2\|\nabla(u - u_{H^k_{K+1}})\|_{L^2(K)}^2 \right)^{\frac{1}{2}} + C \left( \sum_{K \in \mathcal{T}_{k+1}} (H^k_{K+1})^2\|\nabla(u - u_{H^k_K})\|_{L^2(K)}^2 \right)^{\frac{1}{2}} + C\|\nabla(u - u_{H^k_K})\|_{L^2(\Omega)} + C \left( \sum_{K \in \mathcal{T}_{k+1}} \|\sigma - \bar{\sigma}\|_{L^2(K)}^2 \right)^{\frac{1}{2}} + C\|\nabla(u - u_{H^k_{K+1}})\|_{L^2(\Omega)} + C \left( \sum_{K \in \mathcal{T}_{k+1}} \|\nabla u - \bar{\nabla u}\|_{L^2(K)}^2 \right)^{\frac{1}{2}}.
By (3.17) and the triangle inequality, we get
\[
\leq C\|\nabla (u - u_{H_{k+1}^K})\|_{L^2(\Omega)} + C\|u - u_{H_{k+1}^K}\|_{L^2(\Omega)}
\]
\[
+ C\left( \sum_{K \in T_{k+1}} (H_{k+1}^K)^2 \|D^2u - D^2u_{H_{k+1}^K}\|_{L^2(K)}^2 \right)^{\frac{1}{2}}
\]
\[
+ C\left( \sum_{K \in T_{k+1}} (H_{k+1}^K)^2 \|\sigma - \bar{\sigma}\|_{L^2(K)}^2 \right)^{\frac{1}{2}}
\]
\[
= \epsilon_1^{k+1} + \epsilon_2^{k+1},
\]
where \(\epsilon_1^{k+1}, \epsilon_2^{k+1}\) denote the error term (the first term) and the high order terms (the last four terms) respectively.

**Remark 3.3.** If the linear element is used, the first terms in the definition of \(\tilde{\sigma}^k\) and \(\sigma\) should be removed, then the second term of \(\epsilon_2^{k+1}\) does not exist. Moreover, if the regularity of \(u\) is high enough, the third and the fourth terms of \(\epsilon_2^{k+1}\) are high order terms. So when the quasi-uniform mesh is used, we have

\[
\text{osc}(u_{H_{k+1}^K}, T_{k+1}) \leq C\|\nabla (u - u_{H_{k+1}^K})\|_{L^2(\Omega)} + h.o.t..
\]

Based on lemma 3.2, we can prove the lower bound of the error estimator below.

**Theorem 3.4.** Let \(u\) and \(u_{H_{k+1}^K}\) be the solution of (2.8)-(2.9) and the two-grid finite element algorithm 3.1, then

\[
\eta_{R_k}^{k+1} + \eta_{J_k}^{k+1} \leq \epsilon_1^{k+1} + \epsilon_2^{k+1}.
\]

**Proof.** We divide the proof into three steps:

Step 1: Using the properties of the element bubble functions \(\varphi_K\), we have

\[
\frac{9}{20} \left\| \tilde{R}_{k+1}^K \right\|_{L^2(K)}^2
\]
\[
= \left( \tilde{R}_{k+1}^K, \varphi_K \tilde{R}_{k+1}^K \right)
\]
\[
= \left( \tilde{R}_{k+1}^K, \varphi_K R_{k+1}^K \right) - \left( R_{k+1}^K - \tilde{R}_{k+1}^K, \varphi_K \tilde{R}_{k+1}^K \right)
\]
\[
= \left( \beta(x) \cdot \nabla (u_{H_{k+1}^K} - u) + (x) (u_{H_{k+1}^K} - u), \varphi_K \tilde{R}_{k+1}^K \right)
\]
\[
+ \left( \alpha(x) \cdot \nabla (u_{H_{k+1}^K} - u), \varphi_K \tilde{R}_{k+1}^K \right)
\]
\[
\leq C(H_{k+1}^K)^{-1}\|u_{H_{k+1}^K} - u\|_{L^2(K)} \|\tilde{R}_{k+1}^K\|_{L^2(K)} + C(H_{k+1}^K)^{-1}\|R_{k+1}^K - \tilde{R}_{k+1}^K\|_{L^2(K)}\|\tilde{R}_{k+1}^K\|_{L^2(K)}.
\]

By (3.17) and the triangle inequality, we get

\[
H_{k+1}^K \|R_{k+1}^K\|_{L^2(K)} \leq C\|u_{H_{k+1}^K} - u\|_{L^2(K)} + C\|u_{H_{k+1}^K} - u\|_{L^2(K)}
\]
\[
+ C H_{k+1}^K \|R_{k+1}^K - \tilde{R}_{k+1}^K\|_{L^2(K)}.
\]
Step 2: Using the properties of the edge bubble functions $\psi_K$, we have

\begin{align}
\tag{3.19}
\frac{2}{3} \| \tilde{J}_{E+1}^{k} \|^2_{L^2(E)} & = (\tilde{J}_{E+1}^{k}, \psi_K \tilde{J}_{E+1}^{k})_E \\
& = (J_{E+1}^{k}, \psi_K J_{E+1}^{k+1})_E - (J_{E+1}^{k}, \psi_K J_{E+1}^{k+1})_E \\
& = (\alpha(x) \nabla u_{H_k}^{k+1}, \nabla (\psi_K J_{E+1}^{k+1})) + (\text{div}(\alpha(x) \nabla u_{H_k}^{k+1}), \psi_K J_{E+1}^{k+1}) \\
& \quad - (J_{E+1}^{k+1}, J_{E+1}^{k+1}, \psi_K J_{E+1}^{k+1})_E \\
& = (\alpha(x) \nabla (u_{H_k}^{k+1} - u), \nabla (\psi_K J_{E+1}^{k+1})) + (\beta(x) \cdot \nabla (u_{H_k}^{k+1} - u), \psi_K J_{E+1}^{k+1}) \\
& \quad + (\gamma(x) |u_{H_k}^{k+1} - u|, \psi_K J_{E+1}^{k+1}) - (R_{k+1}^{k+1}, \psi_K J_{E+1}^{k+1}) \\
& \quad - (J_{E+1}^{k+1} - J_{E+1}^{k+1}, \psi_K J_{E+1}^{k+1})_E \\
\leq C (H_{K+1}^{k+1})^2 \| |u_{H_k}^{k+1} - u| \|_{L^2(E)} \| J_{E+1}^{k+1} \|_{L^2(E)}^2 \\
& \quad + C (H_{K+1}^{k+1}) \| R_{k+1}^{k+1} \|_{L^2(E)} \| J_{E+1}^{k+1} \|_{L^2(E)} + C \| \psi_K J_{E+1}^{k+1} \|_{L^2(E)} \| J_{E+1}^{k+1} \|_{L^2(E)},
\end{align}

where $w_E$ denotes all the triangles contain $E$ as an edge.

By (3.18) and triangle inequality, we have

\begin{align}
\tag{3.20}
(H_{K+1}^{k+1}) \frac{1}{2} \| J_{E+1}^{k+1} \|_{L^2(E)} & \leq C \| u_{H_k}^{k+1} - u \|_{L^2(E)} + C \| u_{H_k}^{k+1} - u \|_{L^2(E)} \\
& \quad + C (H_{K+1}^{k+1}) \frac{1}{2} \| J_{E+1}^{k+1} - J_{E+1}^{k+1} \|_{L^2(E)} + CH_{K+1} \| R_{k+1}^{k+1} - R_{k+1}^{k+1} \|_{L^2(E)}.
\end{align}

Step 3: Summing (3.19) over all $K$, then

\begin{align}
\tag{3.21}
(\eta_{k+1}^R)^2 \leq & C \| u_{H_k}^{k+1} - u \|_{L^2(\Omega)}^2 + C \| u_{H_k}^{k+1} - u \|^2 + \\
& + C \sum_{K \in T_{k+1}} (\text{osc}^R(u_{H_k}^{k+1}, T_{k+1}))^2.
\end{align}

Summing (3.20) over all $E$, then

\begin{align}
\tag{3.22}
(\eta_{E+1}^k)^2 \leq & C \| u_{H_k}^{k+1} - u \|_{L^2(\Omega)}^2 + C \| u_{H_k}^{k+1} - u \|^2 + \\
& + \sum_{E \in T_{k+1}} \sum_{K \in T_{k+1}} (\text{osc}^R(u_{H_k}^{k+1}, T_{k+1}))^2.
\end{align}

Combining (3.21), (3.22) and Lemma 3.2, we get

\begin{align}
\tag{3.23}
\eta_{k+1}^R + \eta_{E+1}^k & \leq c_1^{k+1} + c_2^{k+1}. \qedhere
\end{align}

### 3.2. Convergence of adaptive two-grid algorithm.

Define \( \| u - u_{H_k}^{k+1} \|^2 + (\text{osc}(u_{H_k}^{k+1}, T))^2 \) as the total error, and \( \| u - u_{H_k}^{k+1} \|^2 + C \eta^2(u_{H_k}^{k+1}, T) \) as the quasi error. We consider the convergence of the adaptive two-grid finite element algorithm. The following two lemmas are needed to prove the error reduction property. The first lemma is from [8].

**Lemma 3.5.** Define \( \rho = 1 - \frac{1}{\sqrt{2}} \), then

\[ \eta^2(u_{H_k}^{k+1}, T_{k+1}) \leq \eta^2(u_{H_k}^{k+1}, T_k) - \rho \eta^2(u_{H_k}^{k+1}, T_k \setminus T_{k+1}). \]
Proof. By the definition of the estimator, we have

\[
\eta^2(u_{H_k}^+, T_{k+1}) = \eta^2(u_{H_k}^+, T_{k+1} \cap T_k) + \eta^2(u_{H_k}^+, T_{k+1} \setminus T_k),
\]

Assume \( T \) is subdivided into \( T = T^1 \cup T^2 \) with \( T^1, T^2 \in \mathcal{T}_{k+1} \) and \( |T^1| = |T^2| = \frac{1}{2}|T| \).

It is easy to show that

\[
\sum_{i=1}^{2} \eta^2(u_{H_k}^+, T_{k+1}^i) \leq \frac{1}{\sqrt{2}} \eta^2(u_{H_k}^+, T_k),
\]

Taking the sum over all \( T^i \) \((i = 1, 2, 3, \cdots)\), then we have

\[
\sum_{T \in \mathcal{T}_{k+1} \setminus \mathcal{T}_k} \eta^2(u_{H_k}^+, T_{k+1}^i) \leq \frac{1}{\sqrt{2}} \eta^2(u_{H_k}^+, T_k \setminus \mathcal{T}_{k+1}).
\]

Plug (3.26) into (3.24), we get

\[
\eta^2(u_{H_k}^+, T_{k+1}) = \eta^2(u_{H_k}^+, T_{k+1} \cap T_k) + \frac{1}{\sqrt{2}} \eta^2(u_{H_k}^+, T_k \setminus T_{k+1}),
\]

\[
\leq \eta^2(u_{H_k}^+, T_k) - \rho \eta^2(u_{H_k}^+, T_k \setminus T_{k+1}).
\]

The lemma is proved. \( \square \)

**Lemma 3.6.** For any \( \epsilon > 0 \), we have

\[
\eta^2(u_{H_k}^{k+1}, T_{k+1}) \leq (1 + \epsilon)(1 - \rho \theta) \eta^2(u_{H_k}^+, T_k)
\]

\[
+ C \| \nabla (u_{H_k}^{k+1} - u_{H_k}^+) \|_{L^2(\Omega)}^2 + C \| u_{H_k}^{k+1} - u_{H_k}^{k-1} \|_{L^2(\Omega)}^2.
\]

Proof. Using the definition of the estimator, we have

\[
|\eta(u_{H_k}^{k+1}, T_{k+1}) - \eta(u_{H_k}^+, T_{k+1})|
\]

\[
= \left( \sum_{K \in \mathcal{T}_{k+1}} (H_K^{k+1})^2 \| - \text{div}(\alpha(x) \nabla u_{H_k}^{k+1}) + \beta(x) \cdot \nabla u_{H_k}^+ \right.
\]

\[
\left. + \gamma(x)u_{H_k}^+ \|_{L^2(K)}^2 \right)^{1/2}
\]

\[
- \left( \sum_{K \in \mathcal{T}_{k+1}} (H_K^{k+1})^2 \| - \text{div}(\alpha(x) \nabla u_{H_k}^+) + \beta(x) \cdot \nabla u_{H_k}^{k-1} \right.
\]

\[
\left. + \gamma(x)u_{H_k}^{k-1} \|_{L^2(K)}^2 \right)^{1/2}
\]

\[
+ \left( \sum_{E \in \mathcal{E}_{k+1}} H_K^{k+1} \| [\alpha(x) \nabla u_{H_k}^{k+1} \cdot \mathbf{n}] \|_{L^2(E)}^2 \right)^{1/2}
\]

\[
- \left( \sum_{E \in \mathcal{E}_{k+1}} H_K^{k+1} \| [\alpha(x) \nabla u_{H_k}^+ \cdot \mathbf{n}] \|_{L^2(E)}^2 \right)^{1/2}
\]

\[
:= T^1 + T^2 + T^3 + T^4.
\]
The first and second terms on the right-hand side of (3.28) can be bounded by

\[(3.29) \quad T^1 + T^2 = \left( \sum_{K \in T_{k+1}} (H_{k+1}^{k+1})^2 \right) - \alpha(x)^T : D^2 u_{H_k}^{k+1} - \text{div}(\alpha(x)^T \cdot \nabla u_{H_k}^{k+1} + \beta(x) \cdot \nabla u_{H_k}^{k} + \gamma(x) u_{H_k}^{k} \|L^2(K)_H \right)^{1/2} \]

\[- \left( \sum_{K \in T_{k+1}} (H_{k+1}^{k+1})^2 \right) - \alpha(x)^T : D^2 u_{H_k}^{k} - \text{div}(\alpha(x)^T \cdot \nabla u_{H_k}^{k} + \beta(x) \cdot \nabla u_{H_k}^{k-1} + \gamma(x) u_{H_k}^{k-1} \|L^2(K)_H \right)^{1/2} \]

\[\leq C \|\nabla (u_{H_k}^{k+1} - u_{H_k}^{k})\|_{L^2(\Omega)} + C \|u_{H_k}^{k} - u_{H_k}^{k-1}\|_{L^2(\Omega)} ,\]

where the following Hölder’s inequality has been used in the last step

\[(3.30) \quad \sum_{K \in T_{k+1}} ab \leq \sqrt{\sum_{K \in T_{k+1}} a^2} \sqrt{\sum_{K \in T_{k+1}} b^2} .\]

Similarly, using (3.30) and the trace inequality, the third and fourth terms on the right-hand side of (3.28) can be bounded by

\[(3.31) \quad T^3 + T^4 = \left( \sum_{E \in \mathcal{E}_{k+1}} H_{k+1}^E \|\nabla u_{H_k}^{k+1} \cdot n\|_{L^2(E)}^2 \right)^{1/2} \]

\[- \left( \sum_{E \in \mathcal{E}_{k+1}} H_{k+1}^E \|\nabla u_{H_k}^{k} \cdot n\|_{L^2(E)}^2 \right)^{1/2} \]

\[\leq C \left( \sum_{E \in \mathcal{E}_{k+1}} H_{k+1}^E \|\nabla (u_{H_k}^{k+1} - u_{H_k}^{k}) \cdot n\|_{L^2(E)}^2 \right)^{1/2} \]

\[\leq C \|\nabla (u_{H_k}^{k+1} - u_{H_k}^{k})\|_{L^2(\Omega)} .\]

Using Young’s inequality and Lemma 3.5, we get

\[(3.32) \quad \eta^2 (u_{H_k}^{k+1}, \mathcal{T}_{k+1}) \leq \eta^2 (u_{H_k}^k, \mathcal{T}_k) + (1 + \epsilon) \rho \eta^2 (u_{H_k}^k, \mathcal{T}_k \setminus \mathcal{T}_{k+1}) \]

\[+ C \|\nabla (u_{H_k}^{k+1} - u_{H_k}^k)\|_{L^2(\Omega)}^2 + C \|u_{H_k}^{k} - u_{H_k}^{k-1}\|_{L^2(\Omega)}^2 .\]

Together with the bulk criterion

\[(3.33) \quad \eta^2 (u_{H_k}^k, \mathcal{T}_k \setminus \mathcal{T}_{k+1}) \geq \theta \eta^2 (u_{H_k}^k, \mathcal{T}_k) ,\]

where \(0 < \theta < 1\), then we have

\[(3.34) \quad \eta^2 (u_{H_k}^{k+1}, \mathcal{T}_{k+1}) \leq (1 + \epsilon) (1 - \rho \theta) \eta^2 (u_{H_k}^k, \mathcal{T}_k) \]

\[+ C \|\nabla (u_{H_k}^{k+1} - u_{H_k}^k)\|_{L^2(\Omega)}^2 + C \|u_{H_k}^{k} - u_{H_k}^{k-1}\|_{L^2(\Omega)}^2 .\]

Then we obtain the lemma.

Based on Lemmas 3.5–3.6, next we prove the quasi error decreases with respect to the number of mesh bisections up to some \(L^2\)-norms of the errors, which are high order terms on the uniform meshes.
Theorem 3.7. (Error reduction) We have the following error reduction property
\[ \| u - u_{H_k}^k \|^2 + C\eta^2(u_{H_k}^{k+1}, T_{k+1}) \leq \zeta(\epsilon)(\| u - u_{H_k}^k \|^2 + C\eta^2(u_{H_k}^k, T_k)) + C\| u - u_{H_k}^{k+1} \|^2_{L^2(\Omega)}. \]

Proof. By the Galerkin orthogonality, we have
\[ \| u - u_{H_k}^k \|^2 = \| u - u_{H_k}^k \|^2 - \| u - u_{H_k}^{k+1} \|^2. \]

Using Young’s inequality and Poincaré’s inequality, we get
\[ \| u - u_{H_k}^k \|^2 \leq \| u - u_{H_k}^k \|^2 + \frac{1}{4}\| u_{H_k}^{k+1} - u_{H_k}^k \|^2 + \frac{1}{C}\| u_{H_k}^{k+1} - u_{H_k}^k \|^2_{L^2(\Omega)}. \]

Combining (3.35) and (3.36), we have
\[ \| u - u_{H_k}^k \|^2 \leq 2\| u - u_{H_k}^k \|^2 - 2\| u - u_{H_k}^{k+1} \|^2 + C\| u - u_{H_k}^k \|^2_{L^2(\Omega)}. \]

Using (3.37), lemma 3.6 and the reliability of the estimator, then there exists an \( \epsilon \) and a \( \zeta(\epsilon) \) such that
\[ \| u - u_{H_k}^k \|^2 + C\eta^2(u_{H_k}^{k+1}, T_{k+1}) \leq \| u - u_{H_k}^k \|^2 + \| u - u_{H_k}^{k+1} \|^2_{L^2(\Omega)} + C\| u_{H_k}^{k+1} - u_{H_k}^k \|^2_{L^2(\Omega)} \leq \zeta(\epsilon)(\| u - u_{H_k}^k \|^2 + C\eta^2(u_{H_k}^k, T_k)) + C\| u - u_{H_k}^{k+1} \|^2_{L^2(\Omega)}, \]
where \( 0 < \zeta(\epsilon) < 1. \)

Based on the above Theorem 3.7, we can prove the following convergence theorem:

Theorem 3.8. (Convergence) We have the following convergence result
\[ \| u - u_{H_k}^k \|^2 + C\eta^2(u_{H_k}^{k+1}, T_{k+1}) \leq \zeta(\epsilon)(\| u - u_{H_k}^k \|^2 + C\eta^2(u_{H_k}^k, T_k)) \]
\[ + C\sum_{i=1}^{k} \| u - u_{H_k}^{k-i} \|^2_{L^2(\Omega)}(\zeta(\epsilon))^i + C\sum_{i=1}^{k} \| u - u_{H_k}^{k-i} \|^2_{L^2(\Omega)}(\zeta(\epsilon))^i. \]

Proof. Define \( \| u - u_{H_k}^{k+1} \|^2_{L^2(\Omega)} = \| u - u_{H_k}^k \|^2_{L^2(\Omega)}, \) then by theorem 3.7, we get
\[ \| u - u_{H_k}^k \|^2 + C\eta^2(u_{H_k}^{k+1}, T_{k+1}) \leq \zeta(\epsilon)(\| u - u_{H_k}^k \|^2 + C\eta^2(u_{H_k}^k, T_k)) \]
\[ + C\sum_{i=1}^{k} \| u - u_{H_k}^{k-i} \|^2_{L^2(\Omega)}(\zeta(\epsilon))^i + C\sum_{i=1}^{k} \| u - u_{H_k}^{k-i} \|^2_{L^2(\Omega)}(\zeta(\epsilon))^i. \]
As a by product of this paper, we prove the equivalence between the total error, the quasi error and the error estimator.

**Theorem 3.9.** The total error, quasi error and the error estimator are equivalent up to a term $C\|u_{H_K} - u\|_{L^2(\Omega)} + C\|u_{H_K^{k+1}} - u\|_{L^2(\Omega)} + C\epsilon_2^{k+1}$.

*Proof.* By the reliability and efficiency (equations (3.21) and (3.22)) of the estimator in Section 3.1, we have

\[
\|u - u_{H_K^{k+1}}\| \leq C\eta(u_{H_K^{k+1}}, T_{k+1}) + C\|u_{H_K} - u\|_{L^2(\Omega)},
\]

(3.40)

\[
\eta(u_{H_K^{k+1}}, T_{k+1}) \leq \|u - u_{H_K^{k+1}}\| + \text{osc}(u_{H_K^{k+1}}, T_{k+1}) + C\|u - u_{H_K}\|_{L^2(\Omega)}.
\]

(3.41)

By Lemma 3.2 and (3.40), we have

\[
\|u - u_{H_K^{k+1}}\| + \text{osc}(u_{H_K^{k+1}}, T_{k+1}) \leq C\eta(u_{H_K^{k+1}}, T_{k+1}) + C\|u_{H_K} - u\|_{L^2(\Omega)} + \epsilon_2^{k+1} + C\|u_{H_K^{k+1}} - u\|_{L^2(\Omega)}.
\]

(3.42)

The equivalence of the total error and the error estimator has been proven.

Moreover, by (3.40), we have

\[
\|u - u_{H_K^{k+1}}\|^2 + C\eta^2(u_{H_K^{k+1}}, T_{k+1}) \leq C\eta^2(u_{H_K^{k+1}}, T_{k+1}) + C\|u_{H_K^{k+1}} - u\|_{L^2(\Omega)} + C\|u_{H_K} - u\|^2_{L^2(\Omega)}.
\]

(3.43)

The equivalence of the quasi error and the error estimator has been proven. \qed

### 3.3. Adaptive two-grid finite element algorithm for non-SPD problems on uniform meshes.

The following two-grid finite element algorithm uses uniform meshes, instead of adaptive meshes, as the coarse grid approximation. A similar version of this algorithm was proposed in [1], and they proved the reliability and efficiency of the error estimator, assuming the oscillation term is the high order term. The analysis in this paper removes this assumption.

In the following algorithm 3.2, $H^k$ denotes the uniform mesh size in the $k$-th bisection, and $H_{K}^{k+1}$ denotes the mesh size in $K$ in the $(k+1)$-th bisection.

**Algorithm 3.2** ATG finite element algorithm for non-SPD problems

**STEP 1:** Find $u_{H_K} \in V_{H_K}$ such that

\[
A(u_{H_K}, v_{H_K}) = 0 \quad \forall v_{H_K} \in V_{H_K};
\]

**STEP 2:** Find $u_{H_K^{k+1}} \in V_{H_K^{k+1}}$ such that

\[
A_S(u_{H_K^{k+1}}, v_{H_K^{k+1}}) + A_N(u_{H_K}, v_{H_K^{k+1}}) = 0 \quad \forall v_{H_K^{k+1}} \in V_{H_K^{k+1}}.
\]

The meshes in algorithm 3.2 are uniform on the coarse level, and the theorems below are special cases of the theorems using adaptive meshes as coarse grid approximation, so we skip their proofs.

Corresponding to theorem 3.1, the reliability of the error estimator is given below.

**Theorem 3.10.** Let $u$ and $u_{H_K^{k+1}}$ be the solution of (2.8)-(2.9) and the two-grid finite element algorithm 3.2, then

\[
\|u - u_{H_K^{k+1}}\| \leq C(\eta_R^{k+1} + \eta_j^{k+1}) + C(H^k)^2.
\]
On uniform meshes, the oscillation term in lemma 3.2 is bounded below.

**Lemma 3.11.** Denote $h^k$ as the largest adaptive mesh size in $\mathcal{T}_k$, then the oscillation term can be bounded by

$$\text{osc}(u_{H^k_{T+1}}, \mathcal{T}_{k+1}) \leq C\|\nabla (u - u_{H^k_{T+1}})\|_{L^2(\Omega)} + C h^k + C(H^k)^2.$$ 

Corresponding to theorem 3.4, the efficiency of the error estimator is given below.

**Theorem 3.12.** Let $u$ and $u_{H^k_{T+1}}$ be the solution of (2.8)-(2.9) and the two-grid finite element algorithm 3.2, then

$$\eta_R^k + \eta_J^k \leq C\|\nabla (u - u_{H^k_{T+1}})\|_{L^2(\Omega)} + C h^k + C(H^k)^2.$$ 

Corresponding to the convergence results using adaptive meshes as the coarse grid approximation in lemma 3.6 and theorems 3.7-3.8, we can obtain the following lemma 3.13 and theorems 3.14-3.15.

**Lemma 3.13.** For any $\epsilon > 0$, we have

$$\eta^2(u_{H^k_{T+1}}, \mathcal{T}_{T+1}) \leq (1 + \epsilon)(1 - \rho\theta)\eta^2(u_{H^k}, \mathcal{T}_k) + C\|\nabla (u_{H^k_{T+1}} - u)\|_{L^2(\Omega)} + C(H^{k-1})^4.$$ 

**Theorem 3.14.** (Error reduction) We have the following error reduction property

$$\|u - u_{H^k_{T+1}}\|^2 + C\eta^2(u_{H^k_{T+1}}, \mathcal{T}_{T+1}) \leq \zeta(\epsilon)\left(\|u - u_{H^k_{T}}\|^2 + C\eta^2(u_{H^k_{T}}, \mathcal{T}_k)\right) + C(H^{k-1})^4.$$ 

**Theorem 3.15.** (Convergence) We have the following convergence result

$$\|u - u_{H^k_{T+1}}\|^2 + C\eta^2(u_{H^k_{T+1}}, \mathcal{T}_{T+1}) \leq (\zeta(\epsilon))^{k+1}\left(\|u - u_{H^k_{T}}\|^2 + C\eta^2(u_{H^k_{T}}, \mathcal{T}_T)\right) + C H_0^4 \frac{(1/4)^k - (\zeta(\epsilon))^k}{1/4 - \zeta(\epsilon)}.$$ 

**Proof.** By theorem 3.14, we get

$$\|u - u_{H^k_{T+1}}\|^2 + C\eta^2(u_{H^k_{T+1}}, \mathcal{T}_{T+1}) \leq (\zeta(\epsilon))^{k+1}\left(\|u - u_{H^k_{T}}\|^2 + C\eta^2(u_{H^k_{T}}, \mathcal{T}_T)\right) + C(H^0)^4 \frac{(1/4)^k - (\zeta(\epsilon))^k}{1/4 - \zeta(\epsilon)}.$$ 

4. **Adaptive two-grid finite element algorithm for nonlinear PDEs.** In this section, we state some novel algorithms for solving nonlinear PDEs. The idea is to transform nonlinear PDEs into linear ones using the coarse level solutions.
4.1. Mild nonlinear PDEs. Next we consider a the mild nonlinear PDEs (2.5). It corresponds to the case when $\delta_1 = 0$ and $\delta_2 = 1$. The problem can be written as

\begin{align}
-\text{div} (\alpha(x, u) \nabla u) + \beta(x, u) \cdot \nabla u + \gamma(x, u) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega. 
\end{align}

To state the algorithm, for $u, v, \xi \in W^{1,\infty}(\Omega) \cap H^1_0(\Omega)$, we define

\[ A_1(u, v, \xi) := (\alpha(x, u) \nabla v, \nabla \xi) + (\beta(x, u) \cdot \nabla v + \gamma(x, u), \xi). \]

Also, by testing (2.1) with $\xi \in H^1_0(\Omega)$, we have

\[ (f(x, u, \nabla u), \nabla \xi) + (g(x, u, \nabla u), \xi) = 0. \]

Taking the Fréchet derivative of the above functional at $u$ along direction $v$, and incorporating term $(b(u)v, \nabla \xi)$ into term $(c(u) \cdot \nabla v + d(u)v, \xi)$ by integration by parts, we have

\[ A_2(u, v, \xi) := (a(u) \nabla v, \nabla \xi) + (c(u) \cdot \nabla v + d(u)v, \xi). \]

Similar with the idea of algorithm 3.1, the solutions on the coarse level meshes are used to transform the nonlinear PDEs into linear ones. In the following algorithms, $\{H^0_K\}_{K \in T_0}$ are usually chosen to be the uniform meshes, i.e., $H^0_K = H^0$.

Algorithm 4.1 is proposed by directly substituting the coefficients of the nonlinear PDE with coarse grid solutions.

**Algorithm 4.1** ATG finite element algorithm for mild nonlinear problems

STEP 1: Find $u_{H^0_K} \in \mathcal{V}_{H^0_K}$ such that

\[ A(u_{H^0_K}, v_{H^0_K}) = 0 \quad \forall v_{H^0_K} \in \mathcal{V}_{H^0_K}; \]

STEP 2: Find $u_{H^{k+1}_K} \in \mathcal{V}_{H^{k+1}_K}$ such that

\[ A_1(u_{H^k_K}, u_{H^{k+1}_K}, v_{H^{k+1}_K}) = 0 \quad \forall v_{H^{k+1}_K} \in \mathcal{V}_{H^{k+1}_K}. \]

Algorithm 4.2 is obtained by adding one-step Newton iteration in algorithm 4.1, and the results may become more accurate.

**Algorithm 4.2** ATG finite element algorithm with one-step Newton correction for mild nonlinear problems

STEP 1: Find $u_{H^0_K} \in \mathcal{V}_{H^0_K}$ such that

\[ A(u_{H^0_K}, v_{H^0_K}) = 0 \quad \forall v_{H^0_K} \in \mathcal{V}_{H^0_K}; \]

STEP 2: Find $u_\ast \in \mathcal{V}_{H^{k+1}_K}$ such that

\[ A_1(u_{H^k_K}, u_\ast, v_{H^{k+1}_K}) = 0 \quad \forall v_{H^{k+1}_K} \in \mathcal{V}_{H^{k+1}_K}; \]

STEP 3: Find $u_{H^{k+1}_K} \in \mathcal{V}_{H^{k+1}_K}$ such that

\[ A_2(u_\ast, u_{H^{k+1}_K}, v_{H^{k+1}_K}) = A_2(u_\ast, u_{H^{k+1}_K}) - A(u_\ast, v_{H^{k+1}_K}) \quad \forall v_{H^{k+1}_K} \in \mathcal{V}_{H^{k+1}_K}. \]
4.2. General nonlinear PDEs. The Newton method is employed in this section to solve the general nonlinear PDEs. One step of Newton iteration is used in the Algorithm 4.3.

Algorithm 4.3 Novel ATG finite element algorithm with one-step Newton correction for general nonlinear problems

STEP 1: Find $u_{H_K}^0 \in \mathcal{V}_{H_K}^0$ such that

$$A(u_{H_K}^0, v_{H_K}^0) = 0 \quad \forall v_{H_K}^0 \in \mathcal{V}_{H_K}^0;$$

STEP 2: Find $u_{H_K}^{k+1} \in \mathcal{V}_{H_K}^{k+1}$ such that

$$A_2(u_{H_K}^k, u_{H_K}^{k+1}, v_{H_K}^{k+1}) = A_2(u_{H_K}^k, u_{H_K}^k, v_{H_K}^{k+1}) - A(u_{H_K}^k, v_{H_K}^{k+1}) \quad \forall v_{H_K}^{k+1} \in \mathcal{V}_{H_K}^{k+1}.$$ 

Denote $I_{k+1}^{K+1}$ as an interpolation operator from solutions on $\mathcal{T}_k$ to solutions on $\mathcal{T}_{k+1}$, then we change the STEP 2 of algorithm 4.3 by using interpolation of $u_{H_K}^0$, instead of interpolation of $u_{H_K}^k$, in each iterative step.

Algorithm 4.4 Novel ATG finite element algorithm with one-step Newton correction for general nonlinear problems

STEP 1: Find $u_{H_K}^0 \in \mathcal{V}_{H_K}^0$ such that

$$A(u_{H_K}^0, v_{H_K}^0) = 0 \quad \forall v_{H_K}^0 \in \mathcal{V}_{H_K}^0;$$

STEP 2: Find $u_{H_K}^{k+1} \in \mathcal{V}_{H_K}^{k+1}$ such that

$$A_2(I_{k+1}^{K+1} u_{H_K}^0, u_{H_K}^{k+1}, v_{H_K}^{k+1}) = A_2(I_{k+1}^{K+1} u_{H_K}^0, I_{k+1}^{K+1} u_{H_K}^0, v_{H_K}^{k+1}) - A(I_{k+1}^{K+1} u_{H_K}^0, v_{H_K}^{k+1}) \quad \forall v_{H_K}^{k+1} \in \mathcal{V}_{H_K}^{k+1}.$$ 

Corresponding to algorithm 4.3, two steps of Newton iteration is used in the Algorithm 4.5.

Algorithm 4.5 Novel ATG finite element algorithm with two-step Newton correction for general nonlinear problems

STEP 1: Find $u_{H_K}^0 \in \mathcal{V}_{H_K}^0$ such that

$$A(u_{H_K}^0, v_{H_K}^0) = 0 \quad \forall v_{H_K}^0 \in \mathcal{V}_{H_K}^0;$$

STEP 2: Find $u_* \in \mathcal{V}_{H_K}^{k+1}$ such that

$$A_2(u_{H_K}^k, u_*, v_{H_K}^{k+1}) = A_2(u_{H_K}^k, u_{H_K}^k, v_{H_K}^{k+1}) - A(u_{H_K}^k, v_{H_K}^{k+1}) \quad \forall v_{H_K}^{k+1} \in \mathcal{V}_{H_K}^{k+1};$$

STEP 3: Find $u_{H_K}^{k+1} \in \mathcal{V}_{H_K}^{k+1}$ such that

$$A_2(u_*, u_{H_K}^{k+1}, v_{H_K}^{k+1}) = A_2(u_*, u_*, v_{H_K}^{k+1}) - A(u_*, v_{H_K}^{k+1}) \quad \forall v_{H_K}^{k+1} \in \mathcal{V}_{H_K}^{k+1}.$$
5. **Numerical experiment.** In this section, some numerical experiments will be given to test the proposed algorithms in this paper. Comparing with the non-SPD linear problems, the algorithms in section 4 are much more efficient since solving nonlinear problems is circumvented. Therefore, we will implement algorithms 4.3–4.5 for nonlinear PDEs.

Before the numerical results are shown, we would like to explain the differences between algorithms 4.3–4.5. Consider figure 1 as an example: assume the left graph is the initial meshes, the middle one is the meshes after one step of bisection, and the right one is the meshes after two steps of bisection.

![Fig. 1. Test 1: The initial meshes (left), the meshes after one step of bisection (middle), and the meshes after two steps of bisection (right).](image)

The following steps explain the differences between algorithms 4.3–4.5.

1. Assume the numerical solution on the initial meshes (left graph in figure 1) is obtained. Denote this numerical solution, which can be represented by a $25 \times 1$ column vector, by $[u^0_1, u^0_2, \ldots, u^0_{25}]^T$. Besides, denote the numerical solution in the middle by $[u^1_1, u^1_2, \ldots, u^1_{27}]^T$, and the numerical solution on the right by $[u^2_1, u^2_2, \ldots, u^2_{29}]^T$. The error estimator is computed to determine how to refine or coarsen meshes. In the first step of bisections, mesh points 26 and 27 need to be added;

2. In the first step of bisections, the coefficients of the nonlinear PDEs are all computed by $u^0_{26} = \frac{u^0_9 + u^0_{13}}{2}$ and $u^0_{27} = \frac{u^0_9 + u^0_{12}}{2}$, and then solve linear PDEs. These algorithms are especially efficient in this step since only the newly-added mesh points need to be interpolated, which is different with algorithm 3.2, where all mesh points need to be interpolated in each step of bisections;

3. In the second step of bisections, the coefficients of the nonlinear PDEs in algorithm 4.3 are computed by $u^1_{28} = \frac{u^1_8 + u^1_{12}}{2}$ and $u^1_{29} = \frac{u^1_8 + u^1_{13}}{2}$; the coefficients of the nonlinear PDEs in algorithm 4.4 are computed by $u^1_{28} = \frac{u^0_9 + u^0_{12}}{2}$ and $u^1_{29} = \frac{u^0_9 + u^0_{13}}{2}$; the coefficients of the nonlinear PDEs in STEP 3 in algorithm 4.5 are computed by $u^2_{28}$ and $u^2_{29}$, which were solved in the last step of Newton iteration.

Next we consider the following nonlinear PDE with Neumann boundary condition:

\begin{align}
(5.1) & \quad u - k \Delta u + \frac{k}{\epsilon^2} (u^3 - u) = \tanh(\frac{d(x)}{\sqrt{2}\epsilon}) \quad \text{in } \Omega, \\
(5.2) & \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{align}

where $d(x)$ denotes the distance function from point $x$ to the circle $x^2 + y^2 = 0.3^2$, $\epsilon$ is
chosen to be a small positive constant, i.e., \( \epsilon = 0.02 \), and domain \( \Omega = [-1, 1] \times [-1, 1] \).

Here we choose \( k = \epsilon^2/10 \). We want to remark that when \( k = 1 \), equation (5.1) is similar with the indefinite PDE below

\[
-\Delta u + \frac{1}{\epsilon^2}(u^3 - u) = 0 \quad \text{in} \ \Omega,
\]

and when \( k \) is small, solving equation (5.1) is equivalent to solve one time step Allen-Cahn equation with initial condition \( \tanh\left(\frac{d(x)}{\sqrt{2\epsilon}}\right) \) and time step \( k \) [7, 21].

Algorithm 4.3–4.5 are implemented for this test. The following figures 2 are the meshes at some bisection steps. The meshes of these three algorithms are very similar, so only one set of graphs are placed here to make this paper tight.

Fig. 2. The meshes for algorithms 4.3, 4.4 and 4.5. The initial meshes (left), the meshes after 3-step bisection (middle), and the meshes after 10-step bisection (right).

Next, the \( H^1 \) error is given in the table 1 below. The initial mesh sizes are \( H^0_K = \sqrt{2}/64 \). We can see that the error decays as number of bisections increases.

| N  | ATG 4.3 | ATG 4.4 | ATG 4.5 | N  | ATG 4.3 | ATG 4.4 | ATG 4.5 |
|----|---------|---------|---------|----|---------|---------|---------|
| 1  | 1.392   | 1.392   | 1.392   | 9  | 0.955   | 0.896   | 0.955   |
| 2  | 1.390   | 1.385   | 1.390   | 10 | 0.935   | 0.868   | 0.935   |
| 3  | 1.367   | 1.347   | 1.367   | 11 | 0.901   | 0.811   | 0.901   |
| 4  | 1.271   | 1.218   | 1.271   | 12 | 0.851   | 0.732   | 0.851   |
| 5  | 1.160   | 1.122   | 1.160   | 13 | 0.841   | 0.695   | 0.841   |
| 6  | 1.127   | 1.093   | 1.127   | 14 | 0.844   | 0.673   | 0.844   |
| 7  | 1.065   | 1.029   | 1.065   | 15 | 0.844   | 0.631   | 0.844   |
| 8  | 0.976   | 0.934   | 0.976   | 16 | 0.837   | 0.595   | 0.837   |

Table 1
The relation between \( H^1 \) error and bisection steps.

From the table and our other numerical tests, errors in ATG 4.4 usually decay faster than errors decay in ATG 4.3 and ATG 4.5. For ATG 4.3 and ATG 4.5, the errors are almost the same since the mesh sizes in \( H^k_K \) and \( H^{k+1}_K \) are close and only a small proportion of meshes are refined.

The algorithm 3.2 is designed for solving non-SPD linear problems with uniform meshes as the coarse grid approximation. For the nonlinear problems, we can write the corresponding algorithm similarly, and we assume the smallest mesh sizes on the fine grids of this algorithm and algorithms 4.3–4.5 are the same. Then after 16 bisections,
the uniform mesh size on coarse grid satisfies

$$H^k \leq \sqrt{H^{k+1}_K} = \sqrt{\frac{\sqrt{2}}{64}(\frac{1}{2})^8}.$$

Then the degree of freedom on the uniform meshes is $$(2 \times 2^4 \times 128)^2 > 9 \times 10^4$$. However, the degrees of freedom of algorithms 4.3–4.5 is less than $9 \times 10^3$ as we observed, so the novel algorithms in this manuscript using the adaptive meshes as coarse grid approximation are highly efficient.

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REFERENCES

[1] C. Bi, C. Wang and Y. Lin, A posteriori error estimates of two-grid finite element methods for nonlinear elliptic problems, J. Sci. Comput., 74 (2018), pp. 23–48.
[2] P. Binev, W. Dahmen and R. Devore, Adaptive finite element methods with convergence rates, Numer. Math., 97 (2004), pp. 219–268.
[3] S. Brenner and R. Scott, The mathematical theory of finite element methods, Springer Science & Business Media, 15, 2007.
[4] J. Cascon, C. Kreuzer, R. Nochetto and K. Siebert, Quasi-optimal convergence rate for an adaptive finite element method, SIAM J. Numer. Anal., 46 (2008), pp. 2524–2550.
[5] M. Cai, M. Mu and J. Xu, Numerical solution to a mixed Navier–Stokes/Darcy model by the two-grid approach, SIAM J. Numer. Anal., 47 (2009), pp. 3325–3338.
[6] W. Dörfler, A convergent adaptive algorithm for Poisson’s equation, SIAM J. Numer. Anal., 33 (1996), pp. 1106–1124.
[7] X. Feng and Y. Li, Analysis of symmetric interior penalty discontinuous Galerkin methods for the Allen–Cahn equation and the mean curvature flow, IMA J. Numer. Anal., 35 (2015), pp. 1622-1651.
[8] J. Hu and J. Xu, Convergence and optimality of the adaptive nonconforming linear element method for the Stokes problem, J. Sci. Comput., 55 (2013), pp. 125–148.
[9] O. Karakashian and F. Pascal, A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems, SIAM J. Numer. Anal., 41 (2003), pp. 2374–2399.
[10] O. Karakashian and F. Pascal, Convergence of adaptive discontinuous Galerkin approximations of second-order elliptic problems, SIAM J. Numer. Anal., 45 (2007), pp. 641–665.
[11] P. Morin, R. Nochetto and K. Siebert, Data oscillation and convergence of adaptive FEM, SIAM J. Numer. Anal., 38 (2000), pp. 466–488.
[12] P. Morin, R. Nochetto and K. Siebert, Convergence of adaptive finite element methods, SIAM Rev., 44 (2002), pp. 631–658.
[13] M. Mu and J. Xu, A two-grid method of a mixed Stokes–Darcy model for coupling fluid flow with porous media flow, SIAM J. Numer. Anal., 45 (2007), pp. 1801–1813.
[14] R. Scott and S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, Math. Comp., 54 (1990), pp. 483–493.
[15] R. Verfürth, A review of a posteriori error estimation and adaptive mesh-refinement techniques, John Wiley & Sons Inc, 1996.
[16] R. Verfürth, A posteriori error estimation techniques for finite element methods, OUP Oxford, 2013.
[17] J. Xu, Iterative methods by space decomposition and subspace correction. SIAM Rev., 34 (1992), pp. 581–613.
[18] J. Xu, A novel two-grid method for semilinear elliptic equations. SIAM J. Sci. Comput., 15 (1994), pp. 231–237.
[19] M. Marion and J. Xu, Error estimates on a new nonlinear Galerkin method based on two-grid finite elements. SIAM J. Numer. Anal., 32 (1995), pp. 1170–1184.
(20) J. XU, Two-grid discretization techniques for linear and nonlinear PDEs. SIAM J. Numer. Anal., 33 (1996), pp. 1759–1777.
(21) J. XU, Y. LI, S. WU AND A. BOUSQUET, On the stability and accuracy of partially and fully implicit schemes for phase field modeling, arXiv preprint arXiv:1604.05402 (2016)
(22) J. XU AND A. ZHOU, A two-grid discretization scheme for eigenvalue problems. Math. Comp., 70 (2001), pp. 17–25.
(23) L. ZHONG, S. SHU, J. WANG AND J. XU, Two-grid methods for time-harmonic Maxwell equations. Numerical Linear Algebra with Applications, 20 (2013), pp. 93–111.