Current-sheet Evolution near a Hyperbolic Magnetic Neutral Line in Hall Magnetohydrodynamics: An Exact Solution

Bhimsen K. Shivamoggi∗
Los Alamos National Laboratory
Los Alamos, NM 87545

Abstract

Current-sheet evolution near a hyperbolic magnetic neutral line in Hall magnetohydrodynamics (MHD) is investigated. An exact analytical solution describing a self-similar evolution is given. This solution shows a hastening of the current-sheet formation process at intermediate times by the Hall effect but subsequently a quenching by the Hall effect of the finite-time singularity exhibited in ideal MHD and hence a prevention of the current density blow-up at large times. The asymptotic result given by this time-dependent solution is in full quantitative agreement with the recent formulation of steady Hall MHD near a X-type magnetic neutral line (Shivamoggi [20]). The latter formulation showed that this asymptotic result indeed corresponds to a hyperbolic configuration of the magnetic field lines in the steady case.

∗Permanent Address: University of Central Florida, Orlando, FL 32816-1364
1 Introduction

When a plasma collapses near the neutral line of the applied magnetic field the continual accumulation of the magnetic flux in the region of the neutral sheet puts the current sheet in a non-stationary state (Syrovatskii [1]). An exact self-similar solution of the MHD equations for a time-dependent, two-dimensional (2D) flow of an incompressible plasma in a hyperbolic magnetic field was given by Uberoi [2] and Chapman and Kendall [3]. This solution had an initial current-free magnetic field, so it is not appropriate for the reconnection problem. This solution was modified (Shivamoggi [4]) so as to relax this restriction, and hence make it suitable for the reconnection problem. This solution was generalized to incorporate a uniform shear-strain rate in the plasma flow (Shivamoggi [5], Rollins and Shivamoggi [6]), so the magnetic field lines now undergo not only sweeping but also shearing by the plasma flow. The above situation predicted a sequence of events associated with the evolution of a current sheet in a hyperbolic magnetic field, in agreement with laboratory experiments (Frank [7]) on the collapse of a plasma near the hyperbolic neutral line.

In recognition of the numerical results (Brunnel et al. [11]) showing the significant effect plasma density variations near the magnetic neutral point have on the magnetic reconnection processes taking place there the above solution was generalized further to incorporate density variations in the plasma (Rollins and Shivamoggi [12]). The current-sheet formation process was found to speed up in the presence of a plasma density build-up near the current sheet. Fast magnetic reconnection processes in laboratory (ex: sawtooth collapse in tokamak discharges) and space (ex: solar flares and magnetospheric substorms) can be described using collisionless plasma models (Yamada et al. [13], Shibata [14] and Nishida [15]). In a high-β collisionless plasma, on length scales shorter than the ion skin depth $d_i$, the electrons decouple from the ions and the electron dynamics is governed by Hall currents (Sonnerup [16]). The Hall currents generate whistler waves which control the dynamics and lead to a faster magnetic reconnection process (Mandt et al. [17] and Biskamp et al. [18]). In recognition of the important role played by the Hall effect in fast magnetic reconnection processes an investigation of the current-sheet evolution process near a hyperbolic magnetic neutral line in Hall MHD is therefore in order - this is the objective of this paper. The asymptotic result given by the time-dependent solution in question turns out to be in full quantitative agreement with the recent formulation of steady Hall MHD near a X-type magnetic neutral line (Shivamoggi [20]).

2 Governing Equations for Hall MHD

Consider an incompressible, two-fluid, quasi-neutral plasma. The equations governing this plasma dynamics are (in usual notation) -

$$nm_e \frac{\partial v_e}{\partial t} + (v_e \cdot \nabla)v_e = -\nabla p_e - ne(\mathbf{E} + \frac{1}{c}v_e \times \mathbf{B}) + ne\eta\mathbf{J}$$

$^1$This problem has also just been considered by Litvinenko [19] (the author is thankful to a referee for bringing [19] to his attention); however, his solution while being rather different from the present solution appears also to be valid only for small times and the results therefore seem to be dubious for larger times.
\[ nm_i \frac{\partial v_i}{\partial t} + (v_i \cdot \nabla)v_i = -\nabla p_i + ne(E + \frac{1}{c} v_i \times B) - ne\eta J \]  

(2)

\[ \nabla \cdot v_e = 0 \]  

(3)

\[ \nabla \cdot v_i = 0 \]  

(4)

\[ \nabla \cdot B = 0 \]  

(5)

\[ \nabla \times B = \frac{1}{c} J \]  

(6)

\[ \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \]  

(7)

where,

\[ J \equiv ne(v_i - v_e). \]  

(8)

Neglecting electron inertia \((m_e \Rightarrow 0)\), equations (1) and (2) can be combined to give an ion equation of motion -

\[ nm_i \frac{\partial v_i}{\partial t} + (v_i \cdot \nabla)v_i = -\nabla (p_e + p_i) + \frac{1}{c} J \times B \]  

(9)

and a generalized Ohm’s law -

\[ E + \frac{1}{c} v_i \times B = \eta J + \frac{1}{nec} J \times B \]  

(10)

Non-dimensionalize distance with respect to a typical length scale \(a\), magnetic field with respect to a typical magnetic field strength \(B_0\), time with respect to the reference Alfvén time \(\tau_A \equiv \frac{a}{V_{A_0}}\) where \(V_{A_0} \equiv \frac{B_0}{\sqrt{\rho}}\), \(\rho \equiv m_i n\), and introduce the magnetic stream function according to

\[ B = \nabla \psi \times \hat{i}_z + b \hat{i}_z \]  

(11)

and write the ion velocity as

\[ v_i = (\hat{i}_z \times v_e) \times \hat{i}_z + w \hat{i}_z \equiv v + w \hat{i}_z \]  

(12)

and assume the physical quantities of interest have no variation along the \(z\)-direction. Equations (9) and (10) then yield

\[ \left[ \frac{\partial}{\partial t} + (v \cdot \nabla) \right] v = -\nabla P - (\nabla^2 \psi) \nabla \psi \]  

(13)

\[ \left[ \frac{\partial}{\partial t} + (v \cdot \nabla) \right] \psi + \sigma [b, \psi] = \hat{\eta} \nabla^2 \psi \]  

(14)

\[ \left[ \frac{\partial}{\partial t} + (v \cdot \nabla) \right] b + \sigma [\psi, \nabla^2 \psi] + [\psi, w] = \hat{\eta} \nabla^2 b \]  

(15)
\[ \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) w - [b, \psi] = 0 \]  \hspace{1cm} (16)

where,
\[ [A, B] \equiv \nabla A \times \nabla B \cdot \hat{i}_z, \quad P \equiv p_e + p_i + b^2, \quad \sigma \equiv \frac{d_i}{\alpha}, \quad \hat{\eta} \equiv \frac{\eta \tau_a}{a^2}. \]

### 3 Hall MHD Near a Hyperbolic Magnetic Neutral Line

Consider the initial-value problem near a hyperbolic magnetic neutral line in Hall MHD with initial conditions -
\[ t = 0 : v_x = -\gamma_0 x, v_y = \gamma_0 y, w = -1/\sigma(kx^2 - y^2), \psi = kx^2 - y^2, b = Cxy \]  \hspace{1cm} (17)

where \( \gamma_0 \) and \( k \) are externally determined parameters with \( \gamma_0 > 0 \) and \( C > 0 \). This initial condition describes a stagnation point plasma flow impinging transversely onto the \( x = 0 \) plane and incorporates equations (4) and (5). The spatial structure for the out-of-plane magnetic field described by this initial condition is in recognition of the quadrupolar out-of-plane magnetic field \( b \) pattern characterizing the Hall effects (Terasawa [21]). Laboratory experiments (Ren et al. [22]) have also confirmed the latter signature of the Hall effect. The Hall magnetic field \( b \) is believed to be produced by the dragging of the in-plane magnetic field in the out-of-plane direction by the electrons near the \( X \)-type magnetic neutral line ([17]).

The Lorentz force due to the initial magnetic field is
\[ t = 0 : \mathbf{J} \times \mathbf{B} = 4(1 - k)kx\hat{i}_x - 4(1 - k)y\hat{i}_y - 2C(kx^2 + y^2)\hat{i}_z. \]  \hspace{1cm} (18)

We take \( k > 1 \) so that this Lorentz force is directed so as to maintain the prescribed initial stagnation-point flow.

Let us assume that the solution for \( t > 0 \) of equation (4), (5) and (13) - (16) with the above initial conditions is of the self-similar form
\[ v_x(x, y, t) = -\dot{\gamma}(t)x, \quad v_y(x, y, t) = \dot{\gamma}(t)y \]
\[ w(x, y, t) = \frac{1}{\sigma}[\beta(t)y^2 - k\alpha(t)x^2] \]
\[ \psi(x, y, t) = k\alpha(t)x^2 - \beta(t)y^2 \]
\[ b(x, y, t) = Cxy \]
\[ P(x, y, t) = -\frac{1}{2}\nu(t)(x^2 + y^2) + P_0, \quad \nu(t) > 0 \]  \hspace{1cm} (19)

with
\[ t = 0 : \alpha = \beta = 1, \dot{\gamma} = \gamma_0. \]  \hspace{1cm} (20)

For the solution (19), \( \nabla^2 \psi = f(t) \) and \( \nabla^2 b = 0 \), so the effect of resistivity in this case is to add a function of \( t \) to \( \psi \) (which leaves the magnetic field unaltered) and hence to introduce
an electric field along the z-axis. We therefore drop the resistivity in the following. Further, for an incompressible plasma, the pressure does not have a dynamical role. So, it is forced to be an enslaved variable in the sense that its form is chosen so as to be compatible with equations (13) - (16) given the ansaetze for \(v_x, v_y, w, \psi, \) and \(b\).

Substituting (19) into equation (13), we obtain

\[
\ddot{\gamma} = 2(k^2 \alpha^2 - \beta^2)
\]  
(21)

while equation (14) gives

\[
\dot{\alpha} - 2(\dot{\gamma} + \sigma C)\alpha = 0
\]  
(22)

\[
\dot{\beta} + 2(\dot{\gamma} + \sigma C)\beta = 0
\]  
(23)

Equations (15) and (16) are identically satisfied by the solution (19).

We have from equations (21)-(23),

\[
\alpha(t) = e^{2(\gamma+\sigma C t)}
\]  
(24)

\[
\beta(t) = e^{-2(\gamma+\sigma C t)}
\]  
(25)

\[
\dot{\gamma}^2 = [k^2 e^{4(\gamma+\sigma C t)} + e^{-4(\gamma+\sigma C t)}] - A
\]  
(26)

with

\[
t = 0 : \gamma = 0.
\]  
(27)

Equation (26) along with (20) and (27) yields

\[
A = (k^2 + 1) - \gamma^2_0.
\]  
(28)

It may be noted that (24) and (25) are consistent with the ion-fluid incompressibility condition -

\[
\frac{d}{dt} \left( \frac{1}{2} \ln [\alpha(t)\beta(t)] \right) = \nabla \cdot \mathbf{v} = 0
\]  
(29)

which is derivable from equation (14) on substituting (19).

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\(^2\)In the generic situation, it may be mentioned that there are six parameters - two for the magnetic flux function \(\psi\), one for the out-of-plane magnetic field \(b\), and three for the velocity field \((\mathbf{v}, w)\), to be determined by only five scalar equations (13)-(16), so there is non-uniqueness in the solution. This is resolved by specifying the parameters as in (19) to close the system, so (19) is one exact solution. However, this exact solution turns out to have considerable physical significance, as discussed in the following.

\(^3\)It may be noted that Litvinenko [19] sets up the solution and the initial conditions for \(w\) and \(b\) rather different from those prescribed in (19) and (20).
For small $t$, equation (26) gives

$$\gamma(t) \approx \gamma_0 t + \left(k^2 - 1\right)(1 + \frac{\sigma C}{\gamma_0}) t^2$$

(30)

while for large $t$, equation (26) gives

$$\gamma(t) \approx \sigma C t.$$  

(31)

(30) shows that Hall effect ($\sigma \neq 0$)

- materializes for intermediate times $\sim O(t^2)$,
- makes the magnetic separatrices rotate toward each other faster.

(31) shows that the finite-time singularity exhibited in ideal MHD (Shivamoggi [23]) is quenched by the Hall effect. Thus, though the Hall effect hastens the current-sheet formation process for intermediate times, it prevents the current density blow-up at large times. Physically, the suppression of the plasma collapse process near a hyperbolic magnetic neutral point in Hall MHD appears to be caused by the dispersive activity of whistler waves which is known to lead to current-sheet broadening, as confirmed by laboratory experiments (Urrutia et al. [24]) as well as numerical simulations (Shay et al. [25] and [26]). (31), in conjunction with (19), also shows that, for large $t$, the level curves of the out-of-plane magnetic field are also the streamlines of the in-plane ion flow. Further, according to (31),

$$\dot{\gamma} \approx \sigma C > O, \ t \ \text{large}$$

(32)

so the streamlines of the in-plane ion flow continue to maintain their initial orientation even for large times. It is pertinent to note that the asymptotic result (32) is in full quantitative agreement with the recent formulation of steady Hall MHD near a $X$-type magnetic neutral line ([20]). The latter formulation showed that (32) indeed corresponds to a hyperbolic configuration of the magnetic field lines in the steady state.

4 Discussion

In recognition of the important role played by the Hall effect in fast magnetic reconnection processes this paper makes an investigation of the current-sheet evolution process near a hyperbolic magnetic neutral line in Hall MHD. The Hall effect is found to hasten the current-sheet evolution process for intermediate times. However, the Hall effect is found to quench the finite-time singularity exhibited in ideal MHD and hence to prevent the current-density blow-up at large times. The asymptotic result (32) is in full quantitative agreement with the recent formulation of steady Hall MHD near a $X$-type magnetic neutral line ([20]) which showed that (32) indeed corresponds to a hyperbolic configuration of the magnetic field lines in the steady case. Besides, in this range of time, the level curves of the out-of-plane magnetic field are also the streamlines of the in-plane ion flow which continue to maintain their initial orientation.

It may be mentioned that Litvinenko’s [19] solution, which stipulates $\dot{\gamma} = \text{const}$, appears to be valid, according to equation (30), only for small times. Consequently, his results
indicating the existence of a finite-time singularity in Hall MHD as well as the absence of one in ideal MHD (contrary to the results in [23]) seem to be dubious.

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4This anomaly appears to be traceable to the fact that Litvinenko [19] deals with the equation of momentum, namely, equation (13), only in a ”coarse-grained” sense via the vorticity equation derivable from it and does not keep track of the full detail in equation (13).
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