NEW REPRESENTATION FOR LAGRANGIANS OF SELF-DUAL NONLINEAR ELECTRODYNAMICS

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We elaborate on a new representation of Lagrangians of 4D nonlinear electrodynamics including the Born-Infeld theory as a particular case. In this new formulation, in parallel with the standard Maxwell field strength $F_{\alpha\beta}, \tilde{F}_{\dot{\alpha}\dot{\beta}}$, an auxiliary bispinor field $V_{\alpha\beta}, \tilde{V}_{\dot{\alpha}\dot{\beta}}$ is introduced. The gauge field strength appears only in bilinear terms of the full Lagrangian, while the interaction Lagrangian $E$ depends on the auxiliary fields, $E = E(V^2, \tilde{V}^2)$. The generic nonlinear Lagrangian depending on $F, \tilde{F}$ emerges as a result of eliminating the auxiliary fields. Two types of self-duality inherent in the nonlinear electrodynamics models admit a simple characterization in terms of the function $E$. The continuous $SO(2)$ duality symmetry between nonlinear equations of motion and Bianchi identities amounts to requiring $E$ to be a function of the $SO(2)$ invariant quartic combination $V^2\tilde{V}^2$, which explicitly solves the well-known self-duality condition for nonlinear Lagrangians. The discrete self-duality (or self-duality under Legendre transformation) amounts to a weaker condition $E(V^2, \tilde{V}^2) = E(-V^2, -\tilde{V}^2)$. We show how to generalize this approach to a system of $n$ Abelian gauge fields exhibiting $U(n)$ duality. The corresponding interaction Lagrangian should be $U(n)$ invariant function of $n$ bispinor auxiliary fields.

1. Introduction

It is well known that the on-shell $SO(2)$ ($U(1)$) duality invariance of Maxwell equations can be generalized to the whole class of the nonlinear electrodynamics models, including the famous Born-Infeld theory. The condition of $SO(2)$ duality can be formulated as a nonlinear differential equation for the Lagrangian of these theories \[.\] Up to now, the general solution of this equation was analysed only in the framework of proper power expansions. Here, based on our recent paper \[,\] we present details of new representation for the Lagrangians of these theories. Alongside with the electromagnetic field strength $F_{\alpha\beta}, \tilde{F}_{\dot{\alpha}\dot{\beta}}$, it involves an auxiliary tensor (bispinor) field $V_{\alpha\beta}, \tilde{V}_{\dot{\alpha}\dot{\beta}}$. It will be shown that the $SO(2)$ duality condition can be explicitly solved in this formalism, without resorting to any perturbative expansion. The general Lagrangian solving this constraint is a sum of an interaction term $E(V^2, \tilde{V}^2)$ depending only on the $U(1)$-invariant combination of the auxiliary fields, $E = \tilde{E}(V^2\tilde{V}^2)$, and non-invariant terms which are bilinear in $V$ and $F$. The Lagrangian involving only the Maxwell field strengths emerges as a result of eliminating the tensor auxiliary fields by their algebraic equations of motion. More general nonlinear electrodynamics Lagrangians respecting the so-called discrete self-duality (or duality under Legendre transformation) also admit a simple characterization in terms of the function $E$. In this case it should be even, $E(V^2, \tilde{V}^2) = E(-V^2, -\tilde{V}^2)$, and otherwise arbitrary.

In Sect. 2 we give a brief account of the continuous and “discrete” dualities in nonlinear electrodynamics in the conventional approach. A novel representation of the appropriate Lagrangians via bispinor auxiliary fields $V_{\alpha\beta}$ and $\tilde{V}_{\dot{\alpha}\dot{\beta}}$ is discussed in Sect. 3. Two examples of duality-invariant models in the new setting, including the Born-Infeld theory, are presented in Sect. 4. An extension to $U(n)$ duality-invariant systems of $n$ Abelian gauge fields is given in Sect. 5. The corresponding Lagrangian in the $V, F$ representation is fully specified by the interaction term which is an $U(n)$ invariant function of $n$ auxiliary fields $V^k_{\alpha\beta}$. The discrete self-duality also amounts to a
simple restriction on the interaction function.

2. **Self-dualities in nonlinear electrodynamics**

We shall discuss nonlinear 4D electrodynamics models which reveal duality properties and include the free Maxwell theory and Born-Infeld theory as particular cases. Detailed motivations why such models are of interest to study can be found, e.g., in \[3\].

2.1 **Continuous on-shell SO(2) duality**

In the spinor notation, the Maxwell field strengths are defined by

\[
F_{\alpha\beta}(A) = \frac{1}{2}(\partial^\beta A_{\alpha\beta} + \partial^\alpha A_{\beta\alpha}) ,
\]

\[
\bar{F}_{\dot{\alpha}\dot{\beta}}(A) = \frac{1}{2}(\partial^\beta A_{\dot{\alpha}\dot{\beta}} + \partial^\dot{\beta} A_{\dot{\alpha}\dot{\beta}}) ,
\]

where \(\partial_{\alpha\dot{\beta}} = \frac{1}{2}(\sigma^m)_{\alpha\dot{\beta}} \partial_m\) and \(A_{\alpha\dot{\beta}}\) is the corresponding vector gauge potential. Below we shall sometimes treat \(F_{\alpha\beta}\) and \(\bar{F}_{\dot{\alpha}\dot{\beta}}\) as independent variables, without assuming Eqs. (1).

Let us introduce the Lorentz-invariant complex variables

\[
\varphi = F^{\alpha\beta} F_{\alpha\beta} , \quad \bar{\varphi} = \bar{F}^{\dot{\alpha}\dot{\beta}} \bar{F}_{\dot{\alpha}\dot{\beta}} .
\]

In this representation, two independent invariants which one can construct out of the Maxwell field strength in the standard vector notation take the following form:

\[
F^{mn} F_{mn} = 2(\varphi + \bar{\varphi}) , \\
F_{mn} \bar{F}^{m\bar{n}} = -2i(\varphi - \bar{\varphi}) .
\]

Here

\[
F_{mn} = \partial_m A_n - \partial_n A_m , \quad \bar{F}^{m\bar{n}} = \frac{1}{2} \varepsilon^{mnpq} F_{pq} .
\]

It will be convenient to deal with dimensionless \(F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}\) and \(\varphi, \bar{\varphi}\), introducing a coupling constant \(f, |f| = 2\). Then the generic nonlinear Lagrangian \(L(F, \bar{F}) = f^{-2} L(\varphi, \bar{\varphi})\), where

\[
L(\varphi, \bar{\varphi}) = -\frac{1}{2}(\varphi + \bar{\varphi}) + L_{\text{int}}(\varphi, \bar{\varphi})
\]

and \(L_{\text{int}}(\varphi, \bar{\varphi})\) collects all possible self-interaction terms of higher-order in \(\varphi, \bar{\varphi}\).

We shall use the following notation for the derivatives of the Lagrangian \(L(\varphi, \bar{\varphi})\)

\[
P_{\alpha\beta}(F) \equiv i \partial L / \partial F^{\alpha\beta} = 2i F_{\alpha\beta} L_{\varphi} , \]

\[
L_{\varphi} = \partial L / \partial \varphi , \quad L_{\bar{\varphi}} = \partial L / \partial \bar{\varphi} , \]

\[
L_{\varphi\bar{\varphi}} = \partial^2 L / \partial \varphi \partial \bar{\varphi} \ldots ,
\]

and for the bilinear combinations of them

\[
\pi \equiv P^{\alpha\beta} P_{\alpha\beta} = -4(\varphi(L_{\varphi})^2 ,
\]

\[
\bar{\pi} \equiv \bar{P}^{\dot{\alpha}\dot{\beta}} \bar{P}_{\dot{\alpha}\dot{\beta}} = -4\bar{\varphi}(L_{\bar{\varphi}})^2 .
\]

In the vector notation, the same quantities read

\[
\tilde{P}^{mn} = \frac{1}{2} \varepsilon^{mnpq} P_{pq} = 2 \partial L / \partial F_{mn}
\]

\[
i \frac{1}{2} P_{mn} \tilde{F}^{mn} = \pi - \bar{\pi} .
\]

The nonlinear equations of motion have the following form in this representation:

\[
E_{\alpha\dot{\alpha}}(F) \equiv \partial^\alpha \bar{P}_{\dot{\alpha}\dot{\beta}}(F) - \partial^{\dot{\alpha}} P_{\alpha\beta}(F) = 0 .
\]

These equations, together with the Bianchi identities

\[
B_{\alpha\dot{\alpha}}(F) \equiv \partial^\alpha \bar{F}_{\dot{\alpha}\dot{\beta}}(F) - \partial^{\dot{\alpha}} F_{\alpha\beta}(F) = 0 ,
\]

constitute a set of first-order equations in which one can treat \(F_{\alpha\beta}\) and \(\bar{F}_{\dot{\alpha}\dot{\beta}}\) as unconstrained conjugated variables.

This set is said to be duality-invariant if the Lagrangian \(L(\varphi, \bar{\varphi})\) satisfies certain nonlinear condition \[\Box\Box\Box\]. The precise form of this self-duality condition in the spinor notation is

\[
S[F, P[F]] \equiv F^{\alpha\beta} F_{\alpha\beta} + P^{\alpha\beta} P_{\alpha\beta} - c.c. = \varphi + \pi - \bar{\varphi} = 0 .
\]

Let us define the nonlinear transformation

\[
\delta \omega F_{\alpha\beta} = \omega P_{\alpha\beta}(F) \equiv 2i \omega F_{\alpha\beta} L_{\varphi} ,
\]

where \(\omega\) is a real parameter. This transformation is a nonlinear realization of the SO(2) group, provided the condition \[\Box\Box\Box\] is satisfied. Indeed, in this case and some subsequent relations it is assumed that the functional argument \(F\) stands for both \(F_{\alpha\beta}\) and \(\bar{F}_{\dot{\alpha}\dot{\beta}}\); we hope that this short-hand notation will not give rise to any confusion.
case $F_{\alpha\beta}$ and $P_{\alpha\beta}(F)$ form an $SO(2)$ vector

$$\delta_\omega P_{\alpha\beta}(F) = -4\omega \frac{\partial}{\partial \phi}[\varphi(L_\varphi)^2 - \bar{\varphi}(L_{\bar{\varphi}})^2]$$

$$= -\omega F_{\alpha\beta}.$$ (11)

The set of equations (11), (3) is clearly invariant under these transformations

$$\delta_\omega \left( \frac{B_{\alpha\bar{\alpha}}}{\mathcal{E}_{\alpha\bar{\alpha}}} \right) = \left( \begin{array}{cc} 0 & \omega \\ -\omega & 0 \end{array} \right) \left( \begin{array}{c} B_{\alpha\bar{\alpha}} \\ \mathcal{E}_{\alpha\bar{\alpha}} \end{array} \right).$$ (12)

Thus these transformations are an obvious generalization of the $SO(2)$ duality transformation in the Maxwell theory:

$$\delta_\omega F_{\alpha\beta} = -i\omega F_{\alpha\beta}, \quad \delta_\omega F_{\alpha\bar{\beta}} = i\omega F_{\alpha\bar{\beta}},$$ (13)

which is a symmetry of the vacuum Maxwell equation $\partial_\beta F_{\alpha\beta} = 0$.

The authors of [2] analyzed the self-duality condition (1) as an equation for the unknown $L$, using the power expansion in some variable. They have found that in each order of the perturbation theory the general solution for $L$ is specified by some arbitrary function of the single variable $\varphi \bar{\varphi}$. In Sect. 3 we propose the explicit solution to this problem beyond the power expansions, taking advantage of a new representation of self-dual Lagrangians via bispinor auxiliary fields.

It should be pointed out that the $SO(2)$ duality transformations in the standard setting described above cannot be realized on the vector potential $A_m$; they provide a symmetry between the equations of motion and Bianchi identity and as such define on-shell symmetry. The manifestly $SO(2)$ duality-invariant Lagrangians can be constructed in the formalism with additional vector and auxiliary fields [3]. Here we shall not discuss connections with this extended formalism.

Although the Lagrangian $L(\varphi, \bar{\varphi})$ satisfying (3) is not invariant with respect to transformation (13), one can still construct, out of $\varphi$ and $\bar{\varphi}$, the $SO(2)$ invariant function

$$I[F, P(F)] = L + \frac{i}{2}(FP - \bar{F}\bar{P})$$

$$= L - \varphi L_\varphi - \bar{\varphi}L_{\bar{\varphi}} \equiv I(\varphi, \bar{\varphi})$$, (14)

$$\delta_\omega I(\varphi, \bar{\varphi}) = \frac{i}{2} \omega (\varphi + \pi - \bar{\varphi} - \bar{\pi}) = 0,$$

where $FP - \bar{F}\bar{P} = P_{\alpha\beta}F^{\alpha\beta} - \bar{P}_{\alpha\beta}\bar{F}^{\alpha\beta}$. This function cannot be taken as a Lagrangian since it contains no free part.

2.2 Self-duality under Legendre transformation. For what follows we shall need a first-order representation of the action corresponding to the Lagrangian (3), such that the Bianchi identities (3) are implemented in the action with the appropriate Lagrange multipliers and so $F_{\alpha\beta}, F_{\alpha\bar{\beta}}$ are unconstrained complex variables. This form of the action is given by

$$\frac{1}{f^2} \int d^4x I(\varphi, \bar{\varphi})$$

$$+ i[F F^D(\varphi) - \bar{F} F^D(\bar{\varphi})]$$, (15)

where

$$F^D_{\alpha\beta}(B) \equiv \frac{1}{2}(\partial_\alpha B_{\beta\bar{\alpha}} + \partial_\beta \bar{B}_{\alpha\bar{\alpha}}).$$ (16)

Varying with respect to the Lagrange multiplier $B_{\alpha\beta}$, one obtains just the Bianchi identities for $F_{\alpha\beta}, F_{\alpha\bar{\beta}}$. Solving them in terms of the gauge potential $A_{\alpha\beta}$ and substituting the result into (15), we come back to (3). On the other hand, the multiplier $B_{\alpha\beta}$ is defined up to the standard Abelian gauge transformation, which suggests interpreting $B_{\alpha\beta}$ and $F^D_{\alpha\beta}(B)$ as the dual gauge potential and gauge field strength, respectively. Using the algebraic equations of motion for the variables $F_{\alpha\beta}, F_{\alpha\bar{\beta}}$, one can express the action (15) in terms of $F^D_{\alpha\beta}(B), F^D_{\alpha\bar{\beta}}(B)$. If the resulting action has the same form as the original one in terms of $F_{\alpha\beta}(A), F_{\alpha\bar{\beta}}(A)$, the corresponding electrodynamics model is said to enjoy the “discrete” self-duality. This sort of duality should not be confused with the on-shell continuous $SO(2)$ duality discussed earlier. The relevant $SO(2)$ symmetry is realized on the variables $F_{\alpha\beta}$ according to (16), and it is not defined on the dual vector potential $B_{\alpha\beta}$. However, as we shall see soon, any $L(\varphi, \bar{\varphi})$ solving the constraint (3) corresponds to a system revealing the discrete self-duality. The inverse statement is not generally true, so the class of nonlinear electrodynamics actions admitting $SO(2)$ duality of equations of motion forms a subclass in the variety of actions which are self-dual in the “discrete” sense.
Let us elaborate on this in some detail. The dual picture is achieved by varying \( \delta F_{\alpha\beta} \) with respect to the independent variables \( F_{\alpha\beta}, \hat{F}_{\alpha\beta} \), which yields

\[
F_{\alpha\beta}^D(B) = i\partial L/\partial F^{\alpha\beta} \equiv P_{\alpha\beta}(F) = 2iF_{\alpha\beta}L_{\varphi}, \tag{17}
\]
where \( P_{\alpha\beta}(F) \) is the same as in (13). Substituting the solution of this algebraic equation, \( F_{\alpha\beta} = F_{\alpha\beta}(P^D) \), into (13) gives us the dual Lagrangian \( \hat{L}(F^D) \)

\[
\hat{L}(F^D) = \hat{L}(\pi^D, \bar{\pi}^D) \equiv \tilde{L}[F(F^D), F^D], \tag{18}
\]
where \( \pi^D \equiv F^D_{\alpha\beta}F^{\alpha\beta} = \pi(F) \) and \( \pi, \bar{\pi} \) were defined in (15).

Using (17) and its conjugate, as well as the definitions (15), (18), one can explicitly check the property

\[
F_{\alpha\beta} = -i\partial\hat{L}/\partial F^{\alpha\beta} \quad \text{and c.c.} \tag{19}
\]
Due to this relation, and keeping in mind the inverse one (17), one can treat the equation

\[
\hat{L}(\pi, \bar{\pi}) = L(\varphi, \bar{\varphi}) + i(F\bar{P} - \bar{F}P) \tag{20}
\]
as setting the direct and inverse Legendre transforms \( L \leftrightarrow \hat{L} \) between two functions of 6 complex variables

\[
F_{\alpha\beta} \Rightarrow P_{\alpha\beta} = i\partial L/\partial F^{\alpha\beta}, \quad dL = -iP_{\alpha\beta}dF^{\alpha\beta} + i\bar{P}_{\alpha\beta}d\bar{F}^{\alpha\beta}; \tag{21}
\]

\[
F_{\alpha\beta} \Rightarrow P_{\alpha\beta} = -i\partial\hat{L}/\partial P^{\alpha\beta}, \quad d\hat{L} = iF^{\alpha\beta}dP_{\alpha\beta} - i\bar{F}^{\alpha\beta}d\bar{P}_{\alpha\beta}. \tag{22}
\]

Within this interpretation, the “discrete” self-duality defined above and amounting to the condition

\[
\hat{L}(\pi, \bar{\pi}) = L(\varphi, \bar{\varphi}) \tag{23}
\]
can be equivalently called “self-duality under Legendre transformation” (13).

Let us show that the \( SO(2) \) duality condition (13) indeed guarantees the self-duality under Legendre transformation (23). The simplest proof of this statement (see, e.g., (3)) makes use of the finite discrete \( SO(2) \) transformation

\[
F_{\alpha\beta} \rightarrow P_{\alpha\beta}, \quad P_{\alpha\beta} \rightarrow -F_{\alpha\beta} \tag{24}
\]
and the invariance of function (13) under the global version of the \( SO(2) \) transformations (10). Due to the latter property, (13) is invariant with respect to (24) too

\[
L(\varphi, \bar{\varphi}) + \frac{i}{2}PF - \frac{i}{2}\bar{F}P = L(\pi, \bar{\pi}) - \frac{i}{2}FP + \frac{i}{2}\bar{F}P.
\]
Comparing this relation with (20), we arrive at the condition (13).

For the dual Lagrangian \( \hat{L}(\pi, \bar{\pi}) \) one can construct the 1st order action similar to (15)

\[
f^{-2} \int d^4x\hat{L}[F, P(A)] = f^{-2} \int d^4x[\hat{L}(\pi, \bar{\pi}) - iP\hat{F}(A) + i\bar{F}\hat{P}(A)], \tag{25}
\]
where \( F_{\alpha\beta}(A), \hat{F}_{\alpha\beta}(A) \) were defined in (1). The self-dual case (in the “discrete” sense) corresponds to identifying \( \hat{L}(\pi, \bar{\pi}) = L(\pi, \bar{\pi}) \). Varying (22) with respect to the gauge potential \( A_{\alpha\beta} \) produces Bianchi identities for the originally unconstrained variable \( P_{\alpha\beta}, \hat{P}_{\alpha\beta} \) as the corresponding equations of motion and so implies

\[
P_{\alpha\beta} = F_{\alpha\beta}^D(B), \quad \hat{P}_{\alpha\beta} = \hat{F}_{\alpha\beta}^D(B). \tag{26}
\]
On the other hand, the equation of motion for \( P_{\alpha\beta} \) in this representation yields

\[
F_{\alpha\beta} = -i\partial L/\partial P^{\alpha\beta} = -2iP_{\alpha\beta}L_{\varphi}. \tag{26}
\]
Solving this equation for the unknown \( P_{\alpha\beta} \) as a function of \( F_{\alpha\beta}(A), \hat{F}_{\alpha\beta}(A) \), we come back to the action corresponding to the original Lagrangian (1). The on-shell \( SO(2) \) duality formulated earlier in terms of the variables \( F_{\alpha\beta}, P_{\alpha\beta}(F) \) admits an equivalent formulation in terms of the dual variables \( P_{\alpha\beta}, F_{\alpha\beta}(P) \):

\[
\delta_\omega P_{\alpha\beta} = -\omega F_{\alpha\beta}(P) = 2i\omega P_{\alpha\beta}L_{\varphi}. \tag{27}
\]
The condition of \( SO(2) \) self-duality has the following form in this representation:

\[
S[F(P), P] = \pi[1 - 4(L_{\varphi}^2)] - \text{c.c.} = 0. \tag{28}
\]
The function \( S[F(P), P] \) is obtained by substituting \( F_{\alpha\beta}(P) \) from (24) into the universal bilinear form (1).
3. A new form of the actions of nonlinear electrodynamics and self-dualities

3.1 Nonlinear electrodynamics Lagrangians in a new setting. The recently constructed $N = 3$ supersymmetric extension of the Born-Infeld theory [6] suggests a new representation for the actions of nonlinear electrodynamics discussed in the previous Section.

The infinite-dimensional off-shell $N = 3$ vector multiplet contains gauge field strengths [1] and auxiliary fields $H_{\alpha\beta}$ and $\tilde{H}_{\alpha\beta}$ [1]. The gauge field part of the off-shell super $\tilde{N} = 3$ Maxwell component Lagrangian is [1]

$$\frac{1}{16}[h + \tilde{h} - 6(\tilde{H}\tilde{F} + HF) + \varphi + \bar{\varphi}],$$

where $h = H^{\alpha\beta}H_{\alpha\beta}$ and $HF = H^{\alpha\beta}F_{\alpha\beta}(A)$. Eliminating the auxiliary fields $H_{\alpha\beta}, \tilde{H}_{\alpha\beta}$ by their algebraic equations of motion we arrive at the standard Maxwell action

$$L_2(F) = -\frac{1}{2}(\varphi + \bar{\varphi}).$$

The $N = 3$ off-shell superfield strengths contain the following combinations of fields [1]:

$$V_{\alpha\beta} = \frac{1}{4}(H_{\alpha\beta} + F_{\alpha\beta}),$$

$$\tilde{V}_{\alpha\beta} = \frac{1}{4} (\tilde{H}_{\alpha\beta} + \tilde{F}_{\alpha\beta}).$$

The free Maxwell Lagrangian [22], being rewritten through $V_{\alpha\beta}, \tilde{V}_{\alpha\beta}$, reads

$$L_2(V,F) = \nu + \bar{\nu} - 2(VF + \bar{V}\bar{F}) + \frac{1}{2}(\varphi + \bar{\varphi}),$$

where

$$\nu \equiv V^{\alpha\beta}V_{\alpha\beta}, \quad \bar{\nu} \equiv \tilde{V}^{\alpha\beta}\tilde{V}_{\alpha\beta},$$

$$VF \equiv V^{\alpha\beta}F_{\alpha\beta}, \quad \bar{V}\bar{F} \equiv \bar{V}^{\alpha\beta}\bar{F}_{\alpha\beta}.$$ (31)

Eliminating $V^{\alpha\beta}$ by its algebraic equation of motion,

$$V^{\alpha\beta} = F^{\alpha\beta}, \quad \tilde{V}^{\alpha\beta} = \tilde{F}^{\alpha\beta},$$

we arrive at the free Lagrangian [8].

Our aim will be to find a nonlinear extension of the free Maxwell Lagrangian using the $N = 3$ supersymmetry-inspired form $L_2(V,F)$ of it, eq. (32), such that this extension becomes the generic nonlinear Lagrangian $L(F^2, F^2)$, eq. (1), after eliminating the auxiliary fields $V_{\alpha\beta}, \tilde{V}_{\alpha\beta}$ by their algebraic (nonlinear) equations of motion.

By Lorentz covariance, such a nonlinear Lagrangian has the following general form:

$$\mathcal{L}[V,F(A)] = \mathcal{L}_2[V,F(A)] + E(\nu, \bar{\nu}) ,$$

where $E$ is a real function encoding self-interaction. Varying the action with respect to $V_{\alpha\beta}$, we derive the algebraic relation between $V$ and $F(A)$ in this formalism

$$F_{\alpha\beta}(A) = V_{\alpha\beta}(1 + E_\nu) \quad \text{and c.c.} ,$$

where $E_\nu \equiv \partial E(\nu, \bar{\nu})/\partial \nu$. This relation is a generalization of the free equation (13) and it can be used to eliminate the auxiliary variable $V^{\alpha\beta}$ in terms of $F^{\alpha\beta}$ and $\tilde{F}^{\alpha\beta}$, $V_{\alpha\beta} \Rightarrow V_{\alpha\beta}[F(A)]$ (see eq. (40) below). The natural restrictions on the interaction function $E(\nu, \bar{\nu})$ are

$$E(0,0) = 0 , \quad E_\nu(0,0) = E_{\bar{\nu}}(0,0) = 0 ,$$

which mean that its $(\nu, \bar{\nu})$-expansion does not contain constant and linear terms. Clearly, given some non-singular interaction Lagrangian $L_{int}(\varphi, \bar{\varphi})$ in (10), one can pick up the appropriate function $E(\nu, \bar{\nu})$, such that the elimination of $V^{\alpha\beta}, \tilde{V}^{\alpha\beta}$ by (30) yields just this self-interaction. Thus (35) with an arbitrary (non-singular) interaction function $E$ is another form of generic nonlinear electrodynamics Lagrangian (1). The second equation of motion in this representation, obtained by varying (32) with respect to $A_{\alpha\beta}$, has the form

$$\partial_\nu^2 [F_{\alpha\beta}(A) - 2V_{\alpha\beta}] + \text{c.c.} = 0 .$$ (38)

After substituting $V_{\alpha\beta} = V_{\alpha\beta}[F(A)]$ from (30), eq. (38) becomes the dynamical equation for $F_{\alpha\beta}(A), \tilde{F}_{\alpha\beta}(A)$ corresponding to the generic Lagrangian (1). Comparing (38) with (1) yields the relation

$$P_{\alpha\beta}(F) = i [F_{\alpha\beta} - 2V_{\alpha\beta}(F)] ,$$ (39)

where $P_{\alpha\beta}(F)$ was defined in (1).
Let us elaborate in more detail on the relation of the $V, F$ representation of the nonlinear electrodynamics Lagrangians to the original “minimal” one (4) which involves only $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}}$. The general solution of the algebraic equation (36) for $V_{\alpha\beta}$ can be written as

$$V_{\alpha\beta}(F) = F_{\alpha\beta}G(\varphi, \bar{\varphi}) .$$  \hfill (40)

The transition function $G(\varphi, \bar{\varphi})$ can be found from the basic requirement that (35) coincides with the initial nonlinear action after eliminating $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$

$$L[V(F), F] = L(\varphi, \bar{\varphi}) .$$  \hfill (41)

Using eq. (10), one can obtain the relations

$$\nu = \varphi G^2, \quad \bar{\nu} = \bar{\varphi} \bar{G}^2 ,$$

$$V(F)F = \varphi G, \quad \bar{V}(F)\bar{F} = \bar{\varphi} \bar{G} .$$  \hfill (42), (43)

After substituting these expressions in (41) with making use of the explicit expressions (32), (35), eq. (41) can be rewritten as

$$E(\nu, \bar{\nu}) = L(\varphi, \bar{\varphi}) - \frac{1}{2} (\varphi + \bar{\varphi}) + \varphi G(2 - G) + \bar{\varphi} \bar{G}(2 - \bar{G}) .$$  \hfill (44)

One should also add the relations:

$$G^{-1} = 1 + \nu E_{\nu}, \quad \bar{G}^{-1} = 1 + \bar{\nu} E_{\bar{\nu}} ,$$  \hfill (45)

which follow from comparing (10) with (16).

Differentiating (14) with respect to $\varphi$ and using the relations

$$\frac{\partial \nu}{\partial \varphi} = G^2 + 2\varphi G \frac{\partial G}{\partial \varphi}, \quad \frac{\partial \bar{\nu}}{\partial \bar{\varphi}} = 2\bar{\varphi} \bar{G} \frac{\partial \bar{G}}{\partial \bar{\varphi}} ,$$

one obtains the simple expression for transition functions

$$G(\varphi, \bar{\varphi}) = \frac{1}{2} - L_{\varphi} .$$  \hfill (46)

A useful corollary of this formula and eqs. (42), (45) is the relation

$$\nu E_{\nu} = \frac{1}{4} \varphi (1 - 4L^2_{\varphi}) .$$  \hfill (47)

Given a fixed $L(\varphi, \bar{\varphi})$, one can express $\varphi, \bar{\varphi}$ in terms of $\nu, \bar{\nu}$ using eqs. (42), (46) and then restore the explicit form of $E(\nu, \bar{\nu})$ by (44). Conversely, given $E(\nu, \bar{\nu})$, one can restore $L(\varphi, \bar{\varphi})$. In practice, finding out such explicit relations is a rather complicated task, as we shall see on two examples in Sect. 4.

3.2 Self-dualities revisited. Until now we did not touch any issues related to self-dualities. A link with the consideration in the previous Section is established by Eq. (13) which relates the functions $P_{\alpha\beta}(F)$ and $V_{\alpha\beta}(F)$.

Substituting this into the $SO(2)$ duality condition (3) and making use of eq. (15) we find

$$S[F, P(F)] = [F^{\alpha\beta} - V^{\alpha\beta}(F)]V_{\alpha\beta}(F) - c.c.$$

$$= \nu E_{\nu} - \bar{\nu} E_{\bar{\nu}} .$$  \hfill (48)

Thus passing to the $V, F$ representation allows one to reduce the nonlinear differential equation (9) to a linear differential equation for the function $E(\nu, \bar{\nu})$:

$$\text{Eq.}(9) \Leftrightarrow \nu E_{\nu} - \bar{\nu} E_{\bar{\nu}} = 0 .$$  \hfill (49)

The corresponding realization of the $SO(2)$ (or $U(1)$) transformations (10), (11) in terms of $F^{\alpha\beta}$ and $V^{\alpha\beta}(F)$ is given by

$$\delta_{\omega} V_{\alpha\beta}(F) = -i\omega V_{\alpha\beta}(F) ,$$

$$\delta_{\omega} F_{\alpha\beta} = i\omega [F_{\alpha\beta} - 2V_{\alpha\beta}(F)] .$$  \hfill (50), (51)

It is worth recalling that the on-shell $SO(2)$ duality transformation mixes the dynamical equations of motion for $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ with the Bianchi identities (8), and so in this extended set of equations one should treat $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ as unconstrained conjugated complex variables (without explicitly solving (8) in terms of gauge potential $A_{i\alpha}$). Correspondingly, the algebraic relation (50) should be viewed to connect two independent sets of variables, $(F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}})$ and $(V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}})$:

$$F_{\alpha\beta} = V_{\alpha\beta}(1 + E_{\nu}) .$$  \hfill (52)

Due to this relation, one can equivalently formulate the dynamics and duality transformations in terms of either the set $(F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}})$ or the set $(V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}})$. Respectively, one can choose as the basic one either the transformation (51) or (52).

It is important to be sure that (52) is covariant under the transformations (50), (51). It is
The interaction Lagrangian $E$ of the Lagrangian (35) similar to (15).

For this we shall need a first-order representation (49) itself are indeed covariant under these transformations on the surface of eq. (49).

It is important to emphasize that the new form (35) of the self-duality constraint (16) admits a transparent interpretation as the condition of invariance of $E(\nu, \bar{\nu})$ with respect to the $U(1)$ transformations (50)

$$\delta_\omega E = 2i\omega(\bar{\nu}E_\nu - \nu E_\bar{\nu}) = 0 .$$

The general solution of (16) is a function $\bar{E}(a)$ which depends on the single real $U(1)$ invariant variable $a = \nu \bar{\nu}$ quartic in the auxiliary fields $V_{\alpha\beta}$ and $\bar{V}_{\dot{\alpha}\dot{\beta}}$

$$E_{sd}(\nu, \bar{\nu}) = \bar{E}(a) = \bar{E}(\nu \bar{\nu}) , \bar{E}(0) = 0 .$$

Thus we come to the notable result that the whole class of nonlinear extensions of the Maxwell action admitting the on-shell $SO(2)$ duality is parametrized by an arbitrary $SO(2)$ invariant real function of one argument $E_{sd} = \bar{E}(\nu \bar{\nu})$ in the representation (51). The remarkable property of $E_{sd}$ is that only terms $\sim \nu^\alpha \bar{\nu}^\beta$ can appear in its power expansion. Below we shall present this expansion for two examples, including the most interesting case of Born-Infeld theory.

In the $V, F$ representation it is very easy to construct invariants of the $SO(2)$ duality rotations. Besides the function $E(\nu, \bar{\nu})$ itself, one more real invariant combination of $V_{\alpha\beta}$ and $\bar{V}_{\dot{\alpha}\dot{\beta}}$ is as follows

$$I_0(V, F) = \nu + \bar{\nu} - VF - \bar{V} \bar{F}$$

$$= -\nu E_\nu - \bar{\nu} E_{\bar{\nu}} .$$

Finally, let us examine which restrictions on the interaction Lagrangian $E(\nu, \bar{\nu})$ are imposed by the requirement of the “discrete” self-duality with respect to the exchange $F(A) \leftrightarrow F^D(B)$. For this we shall need a first-order representation of the Lagrangian (35) similar to (16).

Let us treat $\mathcal{L}(V, F)$ in eq. (35) as a function of two independent variables and implement the Bianchi identities for $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ (amounting to the expressions (2)) in the Lagrangian via the dual field-strength $F^D_{\alpha\beta}(B)$ (55):

$$\bar{\mathcal{L}}[V, F, F^D(B)] \equiv \mathcal{L}(V, F)$$

$$+ i[F^D(B)F - F^D(B)\bar{F}] .$$

The algebraic $V^{\alpha\beta}$ equation of motion $\partial L/\partial V^{\alpha\beta} = 0$ is just the relation (2). On the other hand, varying (2) with respect to $F_{\alpha\bar{\beta}}$, one obtains the linear relation

$$F_{\alpha\bar{\beta}} - 2V_{\alpha\bar{\beta}} = -iF^D_{\alpha\beta}(B)$$

as the corresponding equation of motion. The Bianchi identities for $F^D_{\alpha\beta}(B)$ following from the definition (16) imply for $F_{\alpha\bar{\beta}} = F_{\alpha\beta}(A)$ just the dynamical equation (35) (with $V_{\alpha\beta}$ expressed in terms of $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ from (52)). This proves the equivalence of the dynamics described by (35) with that associated with (16) or (54).

The function $\bar{\mathcal{L}}[V, F, F^D(B)]$ in (60) contains only quadratic and linear terms in $F$ and $\bar{F}$, so one can explicitly find the dual form of (16) in terms of $F^D_{\alpha\beta}(B), \bar{F}^D_{\dot{\alpha}\dot{\beta}}(B)$ and $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$, expressing $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}}$ from Eq. (57):

$$\bar{\mathcal{L}}[V, F, F^D(B)] = \mathcal{L}(U, F^D)$$

$$= \mathcal{L}_2(U, F^D) + E(-u, -\bar{u}) ,$$

where

$$U_{\alpha\beta} \equiv -iV_{\alpha\bar{\beta}} , \quad u = U^{\alpha\beta}U_{\alpha\beta} .$$

Comparing the dual Lagrangian (58) with the original one (35), we observe that the necessary and sufficient condition of the discrete self-duality is the following simple restriction on the function $E$ (4):

$$E(\nu, \bar{\nu}) = E(-\nu, -\bar{\nu}) .$$

Obviously, an arbitrary $SO(2)$-invariant function $E_{sd} = \bar{E}(\nu \bar{\nu})$ corresponding to a $SO(2)$ self-dual system automatically satisfies the discrete self-duality condition (59). This elementary consideration provides us with a simple proof of the fact (mentioned in Sect. 2) that the $SO(2)$ self-dual systems constitute a subclass in the set of those enjoying the discrete self-duality.

4. Examples of self-dual systems

4.1 Born-Infeld theory. The Lagrangian of the Born-Infeld theory has the following form in terms of complex invariants (4)

$$L_{BI}(\varphi, \bar{\varphi}) = \left[1 - Q(\varphi, \bar{\varphi})\right] ,$$

where

$$Q(\varphi, \bar{\varphi}) = \frac{1}{2} \varphi^\alpha \bar{\varphi}^\beta \bar{\varphi}^\gamma \bar{\varphi}^\delta \left[\begin{array}{cccc}
\delta^\alpha_{\delta} & \delta^\alpha_{\gamma} & \delta^\alpha_{\beta} & 0 \\
\delta^\gamma_{\delta} & \delta^\gamma_{\beta} & 0 & \delta^\gamma_{\alpha} \\
\delta^\beta_{\delta} & 0 & \delta^\beta_{\gamma} & \delta^\beta_{\alpha} \\
0 & \delta^\alpha_{\beta} & \delta^\alpha_{\gamma} & \delta^\alpha_{\delta}
\end{array}\right] .$$

The $\varphi^{\alpha\beta}$ equation of motion $\partial L/\partial \varphi^{\alpha\beta} = 0$ is the identity relation (2). On the other hand, varying (2) with respect to $\varphi_{\alpha\beta}$, one obtains the linear relation

$$\varphi_{\alpha\beta} - 2\varphi_{\alpha\bar{\beta}} = -i\varphi^D_{\alpha\beta}(A)$$

as the corresponding equation of motion. The Bianchi identities for $\varphi^D_{\alpha\beta}(A)$ following from the definition (16) imply for $\varphi_{\alpha\bar{\beta}} = \varphi_{\alpha\beta}(A)$ just the dynamical equation (35) (with $\varphi_{\alpha\beta}$ expressed in terms of $\varphi_{\alpha\beta}, \bar{\varphi}_{\dot{\alpha}\dot{\beta}}$ from (52)). This proves the equivalence of the dynamics described by (35) with that associated with (16) or (54).

The function $\bar{\mathcal{L}}[\varphi, \bar{\varphi}, \varphi^D(A)]$ in (60) contains only quadratic and linear terms in $\varphi$ and $\bar{\varphi}$, so one can explicitly find the dual form of (16) in terms of $\varphi^D_{\alpha\beta}(A), \bar{\varphi}^D_{\dot{\alpha}\dot{\beta}}(A)$ and $\varphi_{\alpha\beta}, \bar{\varphi}_{\dot{\alpha}\dot{\beta}}$, expressing $\varphi_{\alpha\beta}$ and $\bar{\varphi}_{\dot{\alpha}\dot{\beta}}$ from Eq. (57):

$$\mathcal{L}_2(U, \varphi^D) = \mathcal{L}(U, F^D)$$

$$= \mathcal{L}_2(U, F^D) + E(-u, -\bar{u}) ,$$

where

$$U_{\alpha\beta} \equiv -i\varphi_{\alpha\bar{\beta}} , \quad u = U^{\alpha\beta}U_{\alpha\beta} .$$

Comparing the dual Lagrangian (58) with the original one (35), we observe that the necessary and sufficient condition of the discrete self-duality is the following simple restriction on the function $E$ (4):

$$E(\varphi, \bar{\varphi}) = E(-\varphi, -\bar{\varphi}) .$$

Obviously, an arbitrary $SO(2)$-invariant function $E_{sd} = \bar{E}(\nu \bar{\nu})$ corresponding to a $SO(2)$ self-dual system automatically satisfies the discrete self-duality condition (59). This elementary consideration provides us with a simple proof of the fact (mentioned in Sect. 2) that the $SO(2)$ self-dual systems constitute a subclass in the set of those enjoying the discrete self-duality.
where

\[ Q(\varphi, \bar{\varphi}) = \sqrt{1 + X}, \]
\[ X(\varphi, \bar{\varphi}) \equiv (\varphi + \bar{\varphi}) + (1/4)(\varphi - \bar{\varphi})^2. \] (61)

The power expansion of the BI-lagrangian is

\[ L_{BI} = -\frac{1}{2} (\varphi + \bar{\varphi}) + \frac{1}{2} \varphi \bar{\varphi} - \frac{1}{2} \varphi \bar{\varphi} (\varphi + \bar{\varphi}) + \frac{1}{8} \varphi \bar{\varphi} (3 \varphi \bar{\varphi} + \varphi^2 + \bar{\varphi}^2) + O(\varphi^5). \] (62)

In the BI theory the function (5) has the following explicit form

\[ P_{\alpha\beta}(F) = i \frac{\partial L_{BI}}{\partial F^{\alpha\beta}} \]
\[ = -i F_{\alpha\beta} Q^{-1}(\varphi, \bar{\varphi}) \left[ 1 + \frac{1}{2}(\varphi - \bar{\varphi}) \right]. \] (63)

It is easy to check that this function satisfies the SO(2) self-duality condition (6), so BI theory belongs to the class of self-dual models (11).

Let us study the Lagrangian \( V, F \) function \( L_{BI}(V, F) \) (13) for this particular case. Our basic purpose will be to find the corresponding function \( E_{BI}(\nu, \bar{\nu}) \).

The function \( G(\varphi, \bar{\varphi}) \) relating the variables \( V_{\alpha\beta} \) and \( F_{\alpha\beta} \) and defined by eq. (44), is given by the expression

\[ G \equiv g = \frac{1}{2} \left\{ 1 + Q^{-1} \left[ 1 + \frac{1}{2}(\varphi - \bar{\varphi}) \right] \right\}. \] (64)

\[ = 1 - \frac{1}{2} \bar{\varphi} + \frac{1}{2} \varphi \bar{\varphi} + \frac{1}{4} (\bar{\varphi})^2 + \ldots . \]

It is easy to find the inverse relation

\[ \varphi = 2 \bar{\varphi} \bar{g} \frac{1 - \bar{g}}{[1 - (g + \bar{g})]^2}. \] (65)

Our aim is to find \( E_{BI} \) as a function of the variables \( \nu = V^2, \bar{\nu} = \bar{V}^2 \). As the first step, one expresses \( \nu, \bar{\nu} \) in terms of \( g \) and \( \bar{g} \), using (13) and (65)

\[ \nu = \varphi g^2 = 2 \bar{\varphi} \bar{g} \left[ \frac{1 - \bar{g}}{[1 - (g + \bar{g})]^2} \right]. \] (66)

Introducing

\[ t \equiv \frac{g \bar{g}}{1 - (g + \bar{g})}, \] (67)

one finds that \( t \), as a consequence of (13) and the fact that \( g(\varphi = 0) = 1 \), satisfies the following quartic equation:

\[ t^4 + t^3 - \frac{1}{4} \nu \bar{\nu} = 0, \quad t(\nu = \varphi = 0) = -1. \] (68)

It allows one to express \( t \) in terms of \( a = \nu \bar{\nu} \)

\[ t(a) = -1 - \frac{a}{4} + \frac{3 a^2}{16} - \frac{15 a^3}{64} + \ldots . \] (69)

One can write a closed expression for \( t(a) \) as the proper solution of (68), but we do not present it here in view of its complexity.

Now we are ready to find \( E_{BI}(\nu, \bar{\nu}) \). Taking into account the explicit expressions (66) and (68) and substituting all this into (14), one finally finds a simple expression for \( E_{BI}(\nu, \bar{\nu}) \) through the real variable \( t(a) \)

\[ E_{BI}(a) = 2[2t^2(a) + 3t(a) + 1] \]
\[ = \frac{a}{2} - \frac{a^2}{8} + \frac{3a^3}{32} + \ldots . \] (70)

4.2 One more example. Let us now consider the self-dual system corresponding to the simplest choice of the function \( E \) in the action (33)

\[ \hat{E} = \frac{1}{2} \nu \bar{\nu} \quad (\nu \hat{E}_\nu = \hat{E}) \] (71)

which is the lowest order self-dual approximation of \( E_{BI} \). This model is distinguished in that the relation (52) and the corresponding representation of equations of motion via variables \( V \) are polynomial.

Using Eq. (13) one can obtain the relation between variables \( \nu, \bar{\nu} \) and \( \varphi, \bar{\varphi} \) in this case

\[ \varphi = \nu(1 + \frac{1}{2} \bar{\varphi})^2 \quad \text{and c.c.} \] (72)

Like in the previous example, we present here first terms in the power expansion of the solution

\[ \nu = \varphi - \varphi \bar{\varphi} + \varphi^2 \bar{\varphi} + \frac{3}{4} \varphi \bar{\varphi}^2 + \ldots . \] (73)

The corresponding expansion of the transition function is

\[ G = \frac{1}{2} - L_\varphi = (1 + \frac{1}{2} \bar{\varphi})^{-1} \]
\[ = 1 - \frac{1}{2} \bar{\varphi} + \frac{1}{2} \varphi \bar{\varphi} + \frac{1}{4} (\bar{\varphi})^2 + \ldots \] (74)

and one can directly find \( \hat{L}(\varphi, \bar{\varphi}) \) for this case. The Lagrangian \( \hat{L}(\varphi, \bar{\varphi}) \) turns out to be highly
non-polynomial, despite the fact that the interaction in the \( V, F \) representation is specified by the simple monomial \(^{(77)}\).

Note that the first two orders of the solution \(^{(77)}\) coincide with the corresponding nonlinear Lagrangian in the Born-Infeld theory \(^{(63)}\), while the terms starting from the 3-rd order are different. It can be also checked that the first three terms in the non-polynomial Lagrangian of this model \( \mathcal{L}(\varphi, \bar{\varphi}) \) coincide with those in \( L_{BI} \). ![](image.png)

5. \( U(n) \) self-duality

Let us consider \( n \) Abelian field-strengths

\[
F_{\alpha\beta}, \quad \bar{F}_{\dot{\alpha}\dot{\beta}},
\]

where \( i = 1, 2 \ldots n \). As the first step, one can realize the group \( SO(n) \) on these variables

\[
\delta \omega F_{\alpha\beta} = \xi^{ik} F_{\alpha\beta}^k, \quad \xi^{ki} = -\xi^{ik}.
\]

This group is assumed to define an off-shell symmetry of the corresponding nonlinear Lagrangian

\[
\mathcal{L}(F^k, \bar{F}^k) = -\frac{1}{2}(F^k F^i) - \frac{1}{2}(F^i F^k) + L_{int}(F^k, \bar{F}^k).
\]

The \( U(n) \) self-duality conditions for the Lagrangian \( L(F^k, \bar{F}^k) \) generalizing the \( U(1) \) condition \(^{(7)}\) have been analyzed in Refs. \(^{(7)}\) \(^{(8)}\) \(^{(3)}\). In the spinor notation, these conditions read

\[
A^{(kl)} = (F^k p^l) - (F^l p^k) - c.c. = 0, \quad (77)
\]

\[
\mathcal{S}^{(kl)} = (F^k F^l) + (F^l p^k) - c.c. = 0, \quad (78)
\]

where

\[
P_{\alpha\beta}^k(F) \equiv i \frac{\partial L}{\partial F_{\alpha\beta}^k},
\]

\[
(F^k p^l) \equiv F^{\alpha\delta} p_{\alpha\beta}^k, \quad \text{etc}.
\]

The condition \(^{(7)}\) amounts to the \( SO(n) \) invariance of the Lagrangian and holds off shell. The second condition is the true analog of \(^{(7)}\). It guarantees the covariance of the equations of motion for \( F^k_{\alpha\beta}, \bar{F}^k_{\dot{\alpha}\dot{\beta}} \) together with Bianchi identities,

\[
\mathcal{E}_{\alpha\dot{\alpha}}^k(F) \equiv \partial_{\dot{\alpha}} \bar{F}^k_{\dot{\alpha}\dot{\beta}}(F) - \partial_{\dot{\alpha}} F^k_{\dot{\alpha}\beta}(F) = 0, \quad (80)
\]

\[
\mathcal{B}_{\alpha\dot{\alpha}}^k(F) \equiv \partial_{\dot{\alpha}} \bar{F}^k_{\dot{\alpha}\dot{\beta}} - \partial_{\dot{\alpha}} F^k_{\dot{\alpha}\beta} = 0,
\]

under the following nonlinear transformations:

\[
\delta_{\alpha} F^k_{\alpha\beta} = -\eta^{ik} P_{\alpha\beta}^l(F),
\]

\[
\delta_{\alpha} P_{\alpha\beta}^k(F) = \eta^{ik} F_{\alpha\beta}^l,
\]

where \( \eta^{kl} = \eta_{lk} \) are real parameters. On the surface of the condition \(^{(78)}\) these transformations, together with \(^{(77)}\), form the group \( U(n) \). The \( U(n) \) group structure becomes manifest after passing to the new variables:

\[
V_{\alpha\beta}^k(F) \equiv \frac{1}{2} [F_{\alpha\beta}^k + i P_{\alpha\beta}^k(F)] ,
\]

\[
\delta V_{\alpha\beta}^k(F) = \omega^{kl} V_{\alpha\beta}^l(F), \quad (82)
\]

\[
\omega^{kl} = \chi^{kl} + i \eta^{kl}, \quad \bar{\omega}^{kl} = -\omega^{kl}.
\]

The particular solution of the \( U(n) \) self-duality conditions \(^{(77)}\), \(^{(78)}\) constructed so far \(^{(6)}\) is formulated in terms of the algebraic equation for auxiliary scalar variables \( \chi^{kl} = \chi_{kl} \)

\[
L = -\frac{1}{2}(\chi^{kk} + \bar{\chi}^{kk}) ,
\]

\[
\chi^{kl} + \frac{1}{2} \chi^{kj} \bar{\chi}^{lj} = (F^k F^l) .
\]

It generalizes a similar representation for the BI Lagrangian (or its \( N = 1 \) extension) \(^{(9)}\) \(^{(10)}\)

\[
L_{BI} = -\frac{1}{2}(\chi + \bar{\chi}) ,
\]

\[
\chi + \frac{1}{2} \chi \bar{\chi} = \varphi = (FF) ,
\]

where \( \chi \) is an auxiliary scalar. This \( n = 1 \) version of Eq.\(^{(83)}\) can be readily solved

\[
\chi = \frac{1}{2}(\varphi - \bar{\varphi}) - L_{BI} .
\]

The solution for an arbitrary \( n \) has been constructed in \(^{(6)}\) within a perturbative expansion.

Passing to an analog of the \( V, F \) representation in the \( U(n) \) case will allow us to find the general solution to \(^{(77)}\), \(^{(78)}\).

Let us define a new representation for the \( SO(n) \) invariant nonlinear electrodynamics Lagrangians in terms of the Abelian gauge field strengths \( F_{\alpha\beta}^k(A_k) \) and auxiliary fields \( V_{\alpha\beta}^k \)

\[
\mathcal{L}(V^k, F^k) = (V^k V^k) + (V^k \bar{V}^k) - 2(V^k F^k) - 2(\bar{V}^k \bar{F}^k) + \frac{1}{2}(F^k F^k)
\]

\[
+ \frac{1}{2}(\bar{F}^k \bar{F}^k) + E(V^k, \bar{V}^k) .
\]

The real Lagrangian of interaction \( E(V, \bar{V}) \) is \( SO(n) \) invariant by definition. In the case without interaction \( (E = 0) \), the bilinear part of \(^{(85)}\)
gives the standard free Lagrangian of $n$ Abelian fields,

$$L_2(F^k, \bar{F}^k) = -\frac{1}{2}(F^k F^k) - \frac{1}{2}(\bar{F}^k \bar{F}^k),$$  \hspace{1cm} (86)

as the result of eliminating the auxiliary fields.

In the general case of $E \neq 0$ the algebraic equation for $V_{\alpha \beta}$ is

$$F_{\alpha \beta}^k = V_{\alpha \beta}^k + \frac{1}{2} \frac{\partial E}{\partial V_{\alpha \beta}^k}. \hspace{1cm} (87)$$

Using the relation

$$P_{\alpha \beta}^k = i[F_{\alpha \beta}^k - 2V_{\alpha \beta}^k],$$  \hspace{1cm} (88)

one can rewrite the $U(n)$ self-duality conditions \((77)\) and \((78)\) in this representation as follows

$$i(F^k V^l) - i(F^k V^l) - c.c. = 0,$$

$$F^k V^k + (F^k V^l) - 2(V^k V^l) - c.c. = 0. \hspace{1cm} (89)$$

One can readily show that, after making use of the relation \((82)\), these conditions can be brought into the form quite similar to the $U(1)$ self-duality condition \((49)\)

$$V_{\alpha \beta}^k \frac{\partial E}{\partial V_{\alpha \beta}^k} - \bar{V}_{\alpha \beta}^k \frac{\partial E}{\partial \bar{V}_{\alpha \beta}^k} = 0. \hspace{1cm} (90)$$

This constraint is none other than the condition of invariance of $E(V, \bar{V})$ with respect to the $U(n)$ transformations \((82)\).

Using the $U(n)$-invariant function $E$ one can also construct the simple invariant

$$I_0 = -V_{\alpha \beta}^k \frac{\partial E}{\partial V_{\alpha \beta}^k} - \bar{V}_{\alpha \beta}^k \frac{\partial E}{\partial \bar{V}_{\alpha \beta}^k}. \hspace{1cm} (91)$$

It is easy to see that the whole Lagrangian \((83)\) is not invariant with respect to the nonlinear part of the $U(n)$ transformations, being invariant only under the off-shell $SO(n)$ ones (corresponding to $\eta^{kl} = 0$ in \((22)\)). This matches with the fact that these $U(n)/SO(n)$ transformations define an on-shell symmetry of the joint set of equations of motion and Bianchi identities. Just these transformations are true analogs of the standard $U(1)$ ($SO(2)$) duality rotations discussed in previous Sections.

The conclusion is that the $V, F$ representation in the case of $n$ Abelian gauge field strengths allows one to reduce the nonlinear $U(n)$ self-duality conditions to the $U(n)$ invariance condition \((34)\) for the interaction function $E(V, \bar{V})$ and so to obtain a general description of the $U(n)$ self-dual models of nonlinear electrodynamics in terms of an $U(n)$ invariant function of $n$ complex auxiliary variables $V^i_{\alpha \beta}, \bar{V}^i_{\alpha \beta}$.

The passing to the standard $F, \bar{F}$ representation of the corresponding Lagrangians requires solving the nonlinear algebraic equations \((85)\) for $V^i_{\alpha \beta}, \bar{V}^i_{\alpha \beta}$. In general, this can be done only within power expansions. It would be interesting to find an example of self-dual system with a few gauge field strengths, where a solution to such equations can be found in a closed form, like in the BI example of Sect. 4, and to establish the precise connection with the Lagrangian \((83)\).

Finally, let us notice that the condition of “discrete” self-duality in the general case is as follows

$$E(V^k_{\alpha \beta}, \bar{V}^k_{\alpha \beta}) = E(-iV^k_{\alpha \beta}, i\bar{V}^k_{\alpha \beta}). \hspace{1cm} (92)$$

It is an obvious generalization of the $n = 1$ condition \((79)\).

6. Conclusion

This talk is an extended version of Sect. 3 of our work \([6]\) devoted to the construction of $N = 3$ supersymmetric Born-Infeld theory (see also \([11]\)). We have introduced a new $V, F$ representation for the Lagrangians of nonlinear electrodynamics and shown that it allows for a simple description of systems exhibiting the properties of on-shell $U(n)$ self-duality or/and off-shell discrete self-duality in terms of real function of auxiliary bispinor complex fields. In the $V, F$ representation, the nonlinear self-duality conditions are reduced to the simple $U(1)$ or $U(n)$ invariance conditions \((49), (50)\) for this function. The general condition of the discrete self-duality, or self-duality under Legendre transformation, is the invariance of this function under some discrete reflections of its arguments, Eqs. \((52), (53)\).

It is an interesting and quite feasible task to extend this consideration to the case of $N = 1$
and $N = 2$ supersymmetric extensions of nonlinear electrodynamics \cite{3,8} in order to obtain a general characterization of the corresponding self-dual systems. Also, finding out a similar $V, F$ representation for non-Abelian BI theory and its superextensions could shed more light on the structure of these theories.

**Note added.** Our general solution for the $SO(2)$ self-dual Lagrangian $L(F^2, \bar{F}^2)$ can be obtained from Eq. (14) by substituting the $U(1)$-invariant function $E_{sd}(\nu, \bar{\nu}) = \tilde{E}(\nu \bar{\nu})$ and solving the algebraic equations (36) for $V_{\alpha \beta}$. A similar form of the general solution to the $SO(2)$ self-duality condition has been found in Refs. [12] by a different method. To the best of our knowledge, the general solution of the $U(n)$ self-duality equation via a $U(n)$ invariant function $E$, as well as the general parametrization of the Lagrangians with discrete self-dualities via the functions $E$ (59), (92) were not earlier considered in the literature.

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