First-Order Rewritability of Frontier-Guarded Ontology-Mediated Queries

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Abstract

We focus on ontology-mediated queries (OMQs) based on (frontier-)guarded existential rules and (unions of) conjunctive queries, and we investigate the problem of FO-rewritability, i.e., whether an OMQ can be rewritten as a first-order query. We adopt two different approaches. The first approach employs standard two-way alternating parity tree automata. Although it does not lead to a tight complexity bound, it provides a transparent solution based on widely known tools. The second approach relies on a sophisticated automata model, known as cost automata. This allows us to show that our problem is 2EXPTIME-complete. In both approaches, we provide semantic characterizations of FO-rewritability that are of independent interest.

1 Introduction

Ontology-based data access (OBDA) is a successful application of KRR technologies in information management systems [Poggi et al., 2008]. One premier goal is to facilitate access to data that is heterogeneous and incomplete. This is achieved via an ontology that enriches the user query, typically a union of conjunctive queries, with domain knowledge. It turned out that the ontology and the user query can be seen as two components of one composite query, called ontology-mediated query (OMQ) [Bienvenu et al., 2014]. The problem of answering OMQs is thus central to OBDA.

Building ontology-aware database systems from scratch, with sophisticated optimization techniques, is a non-trivial task that requires a great effort. An important route towards practical implementation of OMQ answering is thus to use conventional database management systems. The problem that such systems are unaware of ontologies can be addressed by query rewriting: the ontology $O$ and the database query $q$ are combined into a new query $q_O$, the so-called rewriting, which gives the same answer as the OMQ consisting of $O$ and $q$ over all input databases. It is of course essential that the rewriting $q_O$ is expressed in a language that can be handled by standard database systems. The typical language that is considered in this setting is first-order (FO) queries.

Although in the OMQ setting description logics (DLs) are often used for modeling ontologies, it is widely accepted that for handling arbitrary arity relations in relational databases it is convenient to use tuple-generating dependencies (TGDs), a.k.a. existential rules or Datalog² rules. It is known, however, that evaluation of rule-based OMQs is undecidable [Cali et al., 2013]. This has led to a flurry of activity for identifying restrictions on TGDs that lead to decidability. The main decidable classes are (i) (frontier-)guarded TGDs [Baget et al., 2011], [Cali et al., 2013], which includes linear TGDs [Cali et al., 2012a], (ii) acyclic sets of TGDs [Fagin et al., 2005], and (iii) sticky sets of TGDs [Cali et al., 2012b]. There are also extensions that capture Datalog; see the same references.

For OMQs based on linearity, acyclicity, and stickiness, FO-rewritings are always guaranteed to exist [Gottlob et al., 2014]. In contrast, there are (frontier-)guarded OMQs that are inherently recursive, and thus not expressible as a first-order query. This brings us to our main question: Can we check whether a (frontier-)guarded OMQ is FO-rewritable? Notice that for OMQs based on more expressive classes of TGDs that capture Datalog, the answer to the above question is negative, since checking whether a Datalog query is FO-rewritable is an undecidable problem. Actually, we know that a Datalog query is FO-rewritable iff it is bounded [Ajtai and Gurevich, 1994], while the boundedness problem for Datalog is undecidable [Gaifman et al., 1993].

The above question has been studied for OMQ languages based on Horn DLs, including $\mathcal{EL}$ and $\mathcal{EL}^\tau$, which (up to a certain normal form) are a special case of guarded TGDs [Bienvenu et al., 2013]. [Bienvenu et al., 2016] [Lutz and Sabellek, 2017]. More precisely, FO-rewritability is semantically characterized in terms of the existence of certain tree-shaped ABoxes, which in turn allows the authors to pinpoint the complexity of the problem by employing automata-based procedures. As usual in the DL context, schemas consist only of unary and binary relations. However, in our setting we have to deal with relations of higher arity. This indicates that the techniques devised for checking the FO-rewritability of DL-based OMQs cannot be directly applied to rule-based OMQs; this is further explained in Section 3. Therefore, we develop new semantic characterizations...
and procedures that are significantly different from those for OMQs based on description logics.

Our analysis aims to develop specially tailored techniques that allow us to understand the problem of checking whether a (frontier-)guarded OMQ is FO-rewritable, and also to pinpoint its computational complexity. Our plan of attack and results can be summarized as follows:

- We first focus on the simpler OMQ language based on guarded TGDs and atomic queries, and, in Section 3, we provide a characterization of FO-rewritability that forms the basis for applying tree automata techniques.

- We then exploit, in Section 4, standard two-way alternating parity tree automata. In particular, we reduce our problem to the problem of checking the finiteness of the language of an automaton. The reduction relies on a refined version of the characterization of FO-rewritability established in Section 3. This provides a transparent solution to our problem based on standard tools, but it does not lead to an optimal result.

- Towards an optimal result, we use, in Section 5, a more sophisticated automata model, known as cost automata. This allows us to show that FO-rewritability for OMQs based on guarded TGDs and atomic queries is in 2EXPTIME, and in EXPTIME for predicates of bounded arity. Our application of cost automata is quite transparent, which, as above, relies on a refined version of the characterization of FO-rewritability established in Section 3. However, the complexity analysis relies on an intricate result on the boundedness problem for a certain class of cost automata from [Benedikt et al., 2013].

- Finally, in Section 6, by using the results of Section 5, we obtain our main results. We show that FO-rewritability is 2EXPTIME-complete for OMQs based on guarded TGDs and on frontier-guarded TGDs, no matter whether the actual queries are conjunctive queries, unions thereof, or the simple atomic queries. This remains true when the arity of the predicates is bounded by a constant, with the exception of guarded TGDs and atomic queries, for which the complexity then drops to EXPTIME-complete.

In principle, the procedure based on tree automata also provides concrete FO-rewritings when they exist, but it is not tailored towards doing this in an efficient way. Efficiently constructing rewritings is beyond the scope of this work.

2 Preliminaries

Basics. Let C, N, and V be disjoint, countably infinite sets of constants, (labeled) nulls, and (regular) variables, respectively. A schema S is a finite set of relation symbols. The width of S, denoted wd(S), is the maximum arity among all relation symbols of S. We write R/n to denote that the relation symbol R has arity n ≥ 0. A term is either a constant, null, or variable. An atom over S is an expression of the form R(¯v), where R ∈ S is of arity n ≥ 0 and ¯v is an n-tuple of terms. A fact is an atom whose arguments are constants.

Databases. An S-instance is a (possibly infinite) set of atoms over the schema S that contain only constants and nulls, while an S-database is a finite set of facts over S. The active domain of an instance ¯J, denoted adom(¯J), consists of all terms occurring in ¯J. For X ⊆ adom(¯J), we denote by J[X] the subinstance of ¯J induced by X, i.e., the set of all facts R(¯a) with ¯a ⊆ X. A tree decomposition of an instance ¯J is a tuple δ = (T, (X,)(t)∈γT), where T = (T, E) is a (directed) tree with nodes T and edges E, and (X,)(t)∈γT is a collection of subsets of adom(¯J), called bags, such that (i) if R(¯a) ∈ γ, then there is v ∈ T such that ¯a ⊆ Xv, and (ii) for all a ∈ adom(γ), the set {v ∈ T | a ∈ Xv} induces a connected subtree of T. The width of δ is the maximum size among all bags Xv (v ∈ T) minus one. The tree-width of ¯J, denoted tw(¯J), is min{n | there is a tree decomposition of width n of ¯J}.

Conjunctive queries. A conjunctive query (CQ) over S is a first-order formula of the form q(¯x) = Φ(¯y, ϕ(¯x, ¯y)), where ¯x and ¯y are tuples of variables, and ϕ is a conjunction of atoms R1(¯v1) ∧ ⋯ ∧ Rm(¯vm) over S that mention variables from ¯x ⊇ U only. The variables ¯x are the answer variables of q(¯x). If ¯x is empty then q is a Boolean CQ. Let var(q) be the set of variables occurring in q. As usual, the evaluation of CQs over instances is defined in terms of homomorphisms. A homomorphism from q to ¯J is a mapping h: var(q) → adom(¯J) such that Rψ(¯h(i)) ∈ ¯J for each 1 ≤ i ≤ m. We write ¯J |= Φ(¯y, ϕ(¯x, ¯y)) to indicate that there is such a homomorphism h such that h(ϕ(¯x, ¯y)) = ϕ(¯h(1), ¯h(2)). The evaluation of q(¯x) over ¯J, denoted q(¯J), is the set of all tuples a such that ¯J |= q(a). A union of conjunctive queries (UCQ) q(¯x) over S is a disjunction ∪m i=1 qi(¯x) of CQs over S. The evaluation of q(¯x) over ¯J, denoted q(¯J), is the set of tuples ∪1≤i≤m q(a)i. We write ¯J |= q(a) to indicate that ¯J |= qi(a) for some 1 ≤ i ≤ m. Let CQ be the class of conjunctive queries, and UCQ the class of UCQs. We also write AQ0 for the class of atomic queries of the form P(ψ), where P is a 0-ary predicate.

Tuple-generating dependencies. A tuple-generating dependency (TGD) (a.k.a. existential rule) is a first-order sentence of the form τ : ∀ ¯x, ¯y (ϕ(¯x, ¯y) → ∃ ¯z ψ(¯x, ¯z)), where ϕ and ψ are conjunctions of atoms that mention only variables. For brevity, we write ϕ(¯x, ¯y) → ∃ ¯z ψ(¯x, ¯z), and use comma instead of ∧ for conjointing atoms. We assume that each variable of ¯x is mentioned in ψ. We call ϕ and ψ the body and head of the TGD, respectively. The TGD τ is logically equivalent to the sentence ∀ ¯x (qψ(¯x) → qϕ(¯x)), where qψ(¯x) and qϕ(¯x) are the CQs Φ(ϕ(¯x, ¯y)) and Φ(ψ(¯x, ¯z)), respectively. Thus, an instance ¯J satisfies τ if qψ(¯x) ⊆ qϕ(¯x). Also, ¯J satisfies a set of TGDs O, denoted ¯J |= O, if ¯J satisfies every τ ∈ O. Let TGDB be the class of finite sets of TGDs.

Ontology-mediated queries. An ontology-mediated query (OMQ) is a triple Q = (S, O, q(¯x)), where S is a (non-empty) schema (the data schema), O is a set of TGDs (the ontology), and q(¯x) is a UCQ over S ∪ sig(O), where sig(O) is the set of relation symbols in O. Notice that the ontology O can introduce relations that are not in S; this allows us to enrich the schema of q(¯x). We include S in the specification of Q to emphasize that Q will be evaluated over S-databases, even though O and q(¯x) may use additional relation symbols.

The semantics of Q is given in terms of certain answers. The certain answers to a UCQ q(¯x) w.r.t. an S-database D, and a set O of TGDs, is the set of all tuples ¯a of constants, where |a| = |x|, such that (D, O) |= q(a), i.e., ¯J |= q(a) for every instance ¯J ⊇ D that satisfies O. We write D |= Q(¯a) if ¯a is a certain answer to q w.r.t. D and O. Moreover, we set
Ontology-mediated query languages. We write \((C, Q)\) for the class of OMQs \((S, O, q)\), where \(O\) falls in the class of TGDs \(C\), and \(q\) in the query language \(Q\). The evaluation problem for \((TGD, UCQ)\), i.e., given a query \(Q \in (TGD, UCQ)\) with data schema \(S\), an \(S\)-database \(D\), and \(a \in \text{dom}(D)^{|\bar{x}|}\), to decide whether \(D \models Q(a)\), is undecidable; this holds even for \((TGD, AQ\)\) [Cali et al., 2013]. Here we deal with one of the most paradigmatic decidable restrictions, i.e., guardedness. A TGD is guarded if it has a body atom, called guard, that contains the frontier of \(\tau\), i.e., the body variables that appear also in the head. We write \(FG\) for the class of all finite sets of guarded TGDs. A TGD \(\tau\) is called frontier-guarded if its body contains an atom, called frontier-guard, that contains the frontier of \(\tau\), i.e., the body variables that appear also in the head. We write \(FG\) for the class of all finite sets of guarded TGDs. Roughly, the evaluation problem for \((G, UCQ)\) and \((FG, UCQ)\) is decidable since \(G\) and \(FG\) admit tree-like universal models [Cali et al., 2013].

First-order rewritability. A first-order \((FO)\) query over a schema \(S\) is a \((function-free)\) FO formula \(\varphi(\bar{x})\), with \(\bar{x}\) being its free variables, that uses only relations from \(S\). The evaluation of \(\varphi\) over an \(S\)-database \(D\), denoted \(\varphi(D)\), is the set of tuples \(\{\bar{a} \in \text{dom}(D)^{|\bar{x}|} | D \models \varphi(\bar{a})\}\); \(\models\) denotes the standard notion of satisfaction for FO. An OMQ \(Q = (S, O, q(x))\) is FO-rewritable if there exists a (finite) FO query \(\varphi_q(x)\) over \(S\) that is equivalent to \(Q\), i.e., for every \(S\)-database \(D\) it is the case that \(Q(D) = \varphi_D(D)\). We call \(\varphi_q(x)\) an FO-rewriting of \(Q\). A fundamental task for an OMQ language \((C, Q)\), where \(C\) is a class of TGDs and \(Q\) is a class of queries, is deciding first-order rewritability:

\[
\begin{align*}
\text{PROBLEM:} & \quad \text{FORew}(C, Q) \\
\text{INPUT:} & \quad \text{An OMQ } Q \in (C, Q). \\
\text{QUESTION:} & \quad \text{Is it the case that } Q \text{ is FO-rewritable?}
\end{align*}
\]

First-order rewritability of \((FG, UCQ)\)-queries. As shown by the following example, there exist \((G, CQ)\) queries (and thus, \((FG, UCQ)\) queries) that are not FO-rewritable.

Example 1. Consider the OMQ \(Q = (S, O, q) \in (G, CQ)\), where \(S = \{S/3, A/1, B/1\}\), \(O\) consists of

\[
\begin{align*}
S(x, y, z), A(z) & \rightarrow R(x, z), \\
S(x, y, z), R(x, z) & \rightarrow R(x, y),
\end{align*}
\]

and \(q = \exists x, y, z \ (S(x, y, z) \land R(x, z) \land B(y))\). Intuitively, an FO-rewriting of \(Q\) should check for the existence of a set of atoms \(\{S(c, a_i, a_{i-1})\}_{1 \leq i \leq k}\), among others, for \(k \geq 0\). However, since there is no upper bound for \(k\), this cannot be done via a finite FO-query, and thus, \(Q\) is not FO-rewritable. A proof that \(Q\) is not FO-rewritable is given below.

On the other hand, there are (frontier-)guarded OMQs that are FO-rewritable; e.g., the OMQ obtained from the query \(Q\) in Example 1 by adding \(A(z)\) to \(q\) is FO-rewritable with \(\exists x, y, z \ (S(x, y, z) \land B(y) \land A(z))\) being an FO-rewriting.

3 Semantic Characterization

We proceed to give a characterization of FO-rewritability of OMQs from \((G, AQ)\) in terms of the existence of certain tree-like databases. Our characterization is related to, but different from characterizations used for OMQs based on DLS such as \(\mathcal{EL}\) and \(\mathcal{EL}\) [Bienvenu et al., 2013, Bienvenu et al., 2016].

The characterizations in [Bienvenu et al., 2013, Bienvenu et al., 2016] essentially state that a unary OMQ \(Q\) is FO-rewritable if there is a bound \(k\) such that, whenever the root of a tree-shaped database \(D\) is returned as an answer to \(Q\), then this is already true for the restriction of \(D\) up to depth \(k\). The proof of the (contrapositive of the) “only if” direction uses a locality argument: if there is no such bound \(k\), then this is witnessed by an infinite sequence of deeper and deeper tree databases that establish non-locality of \(Q\). For guarded TGDs, we would have to replace tree-shaped databases with databases of bounded tree-width. However, increasing depth of tree decompositions does not correspond to increasing distance in the Gaifman graph, and thus, does not establish non-locality. We therefore depart from imposing a bound on the depth, and instead impose a bound on the number of facts, as detailed below.

It is also interesting to note that, while it is implicit in [Bienvenu et al., 2016] that an OMQ based on \(\mathcal{EL}\) and CQs is FO-rewritable iff it is Gaifman local, there exists an OMQ from \((G, CQ)\) that is Gaifman local, but not FO-rewritable. Such an OMQ is the one obtained from the query \(Q\) given in Example 1 by removing the existential quantification on the variable \(x\) in the CQ \(q\), i.e., converting \(q\) into a unary CQ.

Theorem 1. Consider an OMQ \(Q \in (G, AQ)\) with data schema \(S\). The following are equivalent:

1. \(Q\) is FO-rewritable.
2. There is a \(k \geq 0\) such that, for every \(S\)-database \(D\) of tree-width at most \(\text{wd}(S) - 1\), if \(D \models Q\), then there is a \(D' \subseteq D\) with at most \(k\) facts such that \(D' \models Q\).

For \((1) \Rightarrow (2)\) we exploit the fact that, if \(Q \in (G, AQ)\) is FO-rewritable, then it can be expressed as a UCQ \(q_D\). This follows from the fact that OMQs from \((G, AQ)\) are preserved under homomorphisms [Bienvenu et al., 2014], and Rossman’s Theorem stating that an FO query is preserved under homomorphisms over finite instances iff it is equivalent to a UCQ [Rossman, 2008]. It is then easy to show that \((2)\) holds with \(k\) being the size of the largest disjunct of the UCQ \(q_D\). For \((2) \Rightarrow (1)\), we use the fact that, if there is an \(S\)-database \(D\) that entails \(Q\), then there exists one of tree-width at most \(\text{wd}(S) - 1\) that entails \(Q\), and can be mapped to \(D\). The next example illustrates Theorem 1.

Example 2. Consider the OMQ \(Q = (S, O, P) \in (G, AQ)\), where \(S = \{S/3, A/1, B/1\}\), and \(O\) consists of the TGDs given in Example 1 plus the guarded TGD

\[
S(x, y, z), R(x, z), B(y) \rightarrow P,
\]

which is essentially the CQ \(q\) from Example 1. It is easy to verify that, for an arbitrary \(k \geq 0\), the \(S\)-database

\[
D_k = \{A(a_0), S(c, a_1, a_0), \ldots, S(c, a_{k-1}, a_{k-2}), B(a_{k-1})\}
\]
of tree-width \( \text{wd}(S) - 1 = 2 \) is such that \( \mathcal{D}_k \models Q \), but for every \( \mathcal{D}' \subset \mathcal{D}_k \) with at most \( k \) facts, \( \mathcal{D}' \not\models Q \). Thus, by Theorem 1, \( Q \) is not FO-rewritable.

4 Alternating Tree Automata Approach

In this section, we exploit the well-known algorithmic tool of two-way alternating parity tree automata (2ATA) over finite trees of bounded degree (see, e.g., [Cosmadakis et al., 1988]), and prove that FORew(G, AQ_0) can be solved in elementary time. Although this result is not optimal, our construction provides a transparent solution to FORew(G, AQ_0) based on standard tools. This is in contrast with previous studies on closely related problems for guarded logics, in which all elementary bounds heavily rely on the use of intricate results on cost automata [Blumensath et al., 2014; Benedikt et al., 2015]. We also apply such results later, but only in order to pinpoint the exact complexity of FORew(G, AQ_0).

The idea behind our solution to FORew(G, AQ_0) is, given a query \( Q \in (G, AQ_0) \), to devise a 2ATA \( B_Q \) such that \( Q \) is FO-rewritable iff the language accepted by \( B_Q \) is finite. This is a standard idea with roots in the study of the boundedness problem for monadic Datalog (see, e.g., [Vardi, 1992]). In particular, our main result establishes the following:

**Theorem 2.** Let \( Q \in (G, AQ_0) \) with data schema \( S \). There is a 2ATA \( B_Q \) on trees of degree at most \( 2^{\text{wd}(S)} \) such that \( Q \) is FO-rewritable iff the language of \( B_Q \) is finite. The state set of \( B_Q \) is of double exponential size in \( \text{wd}(S) \), and of exponential size in \( |S \cup \text{sig}(O)| \). Furthermore, \( B_Q \) can be constructed in double exponential time in the size of \( Q \).

As a corollary to Theorem 2, we obtain the following result:

**Corollary 3.** FORew(G, AQ_0) is in \( 3\text{EXPTIME} \), and in \( 2\text{EXPTIME} \) for predicates of bounded arity.

From Theorem 2 to check whether a query \( Q \in (G, AQ_0) \) is FO-rewritable, it suffices to check that the language of \( B_Q \) is finite. The latter is done by first converting \( B_Q \) into a non-deterministic bottom-up tree automaton \( B_Q' \); see, e.g., [Vardi, 1998]. This incurs an exponential blowup, and thus, \( B_Q' \) has triple exponentially many states. We then check the finiteness of the language of \( B_Q' \) in polynomial time in the size of \( B_Q' \) by applying a standard reachability analysis; see [Vardi, 1992]. For predicates of bounded arity, a similar argument as above provides a double exponential time upper bound.

In the rest of Section 4, we explain the proof of Theorem 2. The intuitive idea is to construct a 2ATA \( B_Q' \) whose language corresponds to suitable encodings of databases \( \mathcal{D} \) of bounded tree-width that “minimally” satisfy \( Q \), i.e., \( \mathcal{D} \models Q \), but if we remove any atom from \( \mathcal{D} \), then \( Q \) is no longer satisfied.

A refined semantic characterization. In order to apply an approach based on 2ATA, it is essential to revisit the semantic characterization provided by Theorem 1. To this end, we need to introduce some auxiliary terminology.

Let \( \mathcal{D} \) be a database, and \( \delta = (T, (X_v)_{v \in T}) \), where \( T = (T, E) \), a tree decomposition of \( \mathcal{D} \). An adornment of the pair \(( \mathcal{D}, \delta )\) is a function \( \eta : T \to 2^\mathcal{D} \) such that \( \eta(v) \subseteq \mathcal{D}[X_v] \) for all \( v \in T \), and \( \bigcup_{v \in T} \eta(v) = \mathcal{D} \). Therefore, the pair \(( \delta, \eta )\) can be viewed as a representation of the database \( \mathcal{D} \) along with a tree decomposition of it. For the intended characterization, it is important that this representation is free of redundancies, formalized as follows. We say that \( \delta \) is \( \eta \)-simple if \( |\eta(v)| \leq 1 \) for all \( v \in T \), and non-empty \( \eta \)-labels are unique, that is, \( \eta(v) \neq \eta(w) \) for all distinct \( v, w \in T \) with \( \eta(v) \) and \( \eta(w) \) non-empty. Nodes \( v \in T \) with \( \eta(v) \) empty, called white from now on, are required since we might not have a (unique!) fact available to label them. Note, though, that white nodes \( v \) are still associated with a non-empty set of constants from \( \mathcal{D} \) via \( X_v \). All other nodes are called black. While \( \delta \) being \( \eta \)-simple avoids redundancies that are due to a fact occurring in the inflationary use of white nodes. We say that a node \( v \in T \) is \( \eta \)-well-colored if it is black, or it has at least two successors and all its successors are \( \eta \)-well-colored. We say that \( \delta \) is \( \eta \)-well-colored if every node in \( T \) is \( \eta \)-well-colored. For example, \( \delta \) is not \( \eta \)-well-colored if it has a white leaf, or if it has a white node and its single successor is also white. Informally, requiring \( \delta \) to be \( \eta \)-well-colored makes it impossible to blow up the tree by introducing white nodes without introducing black nodes. For \( i \in \{1, 2\} \), let \( \mathcal{D}_i \) be a database, \( \delta_i \) a tree decomposition of \( \mathcal{D}_i \), and \( \eta_i \) an adornment of \( (\mathcal{D}_i, \delta_i) \). We say that \( (\mathcal{D}_1, \delta_1, \eta_1) \) and \( (\mathcal{D}_2, \delta_2, \eta_2) \) are isomorphic if the latter can be obtained from the former by consistently renaming constants in \( \mathcal{D}_1 \) and tree nodes in \( \delta_1 \).

We are now ready to revisit the characterization of FO-rewritability for OMQs from \((G, AQ_0) \) given in Theorem 1.

**Theorem 4.** Consider an OMQ \( Q \in (G, AQ_0) \) with data schema \( S \). The following are equivalent:

1. Condition 2 from Theorem 1 is satisfied.
2. There are finitely many non-isomorphic triples \((\mathcal{D}, \delta, \eta)\), where \( \mathcal{D} \) is an \( S \)-database, \( \delta \) a tree decomposition of \( \mathcal{D} \) of width at most \( \text{wd}(S) - 1 \), and \( \eta \) an adornment of \((\mathcal{D}, \delta)\), such that
   (a) \( \delta \) is \( \eta \)-simple and \( \eta \)-well-colored,
   (b) \( \mathcal{D} \models Q \), and
   (c) for every \( \alpha \in \mathcal{D} \), it is the case that \( \mathcal{D} \setminus \{\alpha\} \not\models Q \).

**Devising automata.** We proceed to discuss how the 2ATA announced in Theorem 2 is constructed. Consider an OMQ \( Q = (S, \mathcal{O}, P) \) from \((G, AQ_0) \). Our goal is to devise an automaton \( B_Q \) whose language is finite iff Condition 2 from Theorem 4 is satisfied. By Theorems 1 and 4, \( Q \) is then FO-rewritable iff the language of \( B_Q \) is finite.

The 2ATA \( B_Q \) will be the intersection of several automata that check the properties stated in item 2 of Theorem 2. But first we need to say a few words about tree encodings. Let \( \Gamma \) be a finite alphabet, and let \( \{ \mathbb{N} \setminus \{0\} \}^* \) denote the set of all finite words of positive integers, including the empty word. A finite \( \Gamma \)-labeled tree is a partial function \( t : \{ \mathbb{N} \setminus \{0\} \}^* \to \Gamma \) such that the domain of \( t \) is finite and prefix-closed. Moreover, if \( v \cdot i \) belongs to the domain of \( t \), then \( v \cdot (i - 1) \) also
belongs to the domain of $t$. In fact, the elements in the domain of $t$ identify the nodes of the tree. It can be shown that an $S$-database $\mathcal{D}$, a tree decomposition $\delta$ of $\mathcal{D}$ of width $w - 1$, and an adornment $\eta$ of $(\mathcal{D}, \delta)$, can be encoded as a $\Gamma_{S, w}$-labeled tree $t$ of degree at most $2^w$, where $\Gamma_{S, w}$ is an alphabet of size double exponential in $w$ and exponential in $S$, such that each node of $\delta$ corresponds to exactly one node of $t$ and vice versa. Although every $\mathcal{D}$ can be encoded into a $\Gamma_{S, w}$-labeled tree $t$, the converse is not true in general. However, it is possible to define certain syntactic consistency conditions such that every consistent $\Gamma_{S, w}$-labeled tree can be decoded into an $S$-database, denoted $\llbracket t \rrbracket$, whose tree-width is at most $w$. We are going to abbreviate the alphabet $\Gamma_{S, \text{wd}(S)}$ by $\Gamma_S$.

**Lemma 5.** There is a 2ATA $C_S$ that accepts a $\Gamma_S$-labeled tree $t$ iff $t$ is consistent. The number of states of $C_S$ is constant. $C_S$ can be constructed in polynomial time in the size of $\Gamma_S$.

Since a $\Gamma_S$-labeled tree incorporates the information about an adornment, the notions of being well-colored and simple can be naturally defined for $\Gamma_S$-labeled trees. Then:

**Lemma 6.** There is a 2ATA $R_S$ that accepts a consistent $\Gamma_S$-labeled tree iff it is well-colored and simple. The number of states of $R_S$ is exponential in $\text{wd}(S)$ and linear in $|S|$. $R_S$ can be constructed in double exponential time in the size of $\Gamma_S$.

Concerning property 2(b) of Theorem 4, we can devise a 2ATA that accepts those trees whose decoding satisfies $Q$.

**Lemma 7.** There is a 2ATA $A_Q$ that accepts a consistent $\Gamma_S$-labeled tree iff $\llbracket t \rrbracket \models Q$. The number of states of $A_Q$ is exponential in $\text{wd}(S)$ and linear in $|S \cup \text{sig}(O)|$. $A_Q$ can be constructed in double exponential time in the size of $\Gamma_S$.

The crucial task is to check condition 2(c) of Theorem 4, which states the key minimality criterion. Unfortunately, this involves an extra exponential blowup:

**Lemma 8.** There is a 2ATA $M_Q$ that accepts a consistent $\Gamma_S$-labeled tree $t$ iff $\llbracket t \rrbracket \models Q$ for all $\alpha \in \llbracket t \rrbracket$. The state set of $M_Q$ is of double exponential size in $\text{wd}(S)$, and of exponential size in $|S \cup \text{sig}(O)|$. Furthermore, $M_Q$ can be constructed in double exponential time in the size of $\Gamma_S$.

Let us briefly explain how $M_Q$ is constructed. This will expose the source of the extra exponential blowup, which prevents us from obtaining an optimal complexity upper bound for $\text{FORew}(G, AQ_0)$. We first construct a 2ATA $D_Q$ that runs on $\Lambda_S$-labeled trees, where $\Lambda_S$ is an alphabet that extends $\Gamma_S$ with auxiliary symbols that allow us to tag some facts in the input tree. In particular, $D_Q$ accepts a tree $t$ iff $t$ is consistent, there is at least one tagged fact, and $\llbracket t \rrbracket \models Q$ where $\llbracket t \rrbracket$ is obtained from $\llbracket t \rrbracket$ by removing the tagged facts. Having $D_Q$ in place, we can then construct a 2ATA $\exists D_Q$ that accepts a $\Gamma_S$-labeled tree $t$ if there is a way to tag some of its facts so as to obtain a $\Lambda_S$-labeled tree $t'$ with $\llbracket t' \rrbracket \models Q$. This is achieved by applying the projection operator on $D_Q$. Since for 2ATAs projection involves an exponential blowup and $D_Q$ already has exponentially many states, $\exists D_Q$ has double exponentially many. It should be clear now that $M_Q$ is the complement of $\exists D_Q$, and we recall that complementation of 2ATAs can be done in polynomial time.

The desired automaton $B_Q$ is obtained by intersecting the 2ATAs in Lemmas 5 to 8. Since the intersection of 2ATA is feasible in polynomial time, $B_Q$ can be constructed in double exponential time in the size of $Q$.

## 5 Cost Automata Approach

We proceed to study $\text{FORew}(G, AQ_0)$ using the more sophisticated model of cost automata. This allows us to improve the complexity of the problem obtained in Corollary 3 as follows:

**Theorem 9.** $\text{FORew}(G, AQ_0)$ is in 2ExpTime, and in ExpTime for predicates of bounded arity.

As in the previous approach, we develop a semantic characterization that relies on a minimality criterion for trees accepted by cost automata. The extra features provided by cost automata allow us to deal with such a minimality criterion in a more efficient way than standard 2ATA. While our application of cost automata is transparent, the complexity analysis relies on an intricate result on the boundedness problem for a certain class of cost automata from [Benedikt et al., 2015].

Before we proceed further, let us provide a brief overview of the cost automata model that we are going to use.

**Cost automata models.** Cost automata extend traditional automata (on words, trees, etc.) by providing counters that can be manipulated at each transition. Instead of assigning a Boolean value to each input symbol (indicating whether the input is accepted or not), these automata assign a value from $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ to each input.

Here, we focus on cost automata that work on finite trees of unbounded degree, and allow for two-way movements; in fact, the automata that we need are those that extend 2ATA over finite trees with a single counter. The operation of such an automaton $A$ on each input tree $t$ may be viewed as a two-player cost game $G(A, t)$ between players Eve and Adam. Recall that the acceptance of an input tree for a conventional 2ATA can be formalized via a two-player game as well.

### Cost games

In case of a minimizing (resp., maximizing) objective, a strategy $\xi$ of Eve in the cost game $G(A, t)$ is $n$-winning if any play of Adam consistent with $\xi$ has cost at most $n$ (resp., at least $n$).

Given an input tree $t$, one then defines the value of $t$ in $A$ as $[A](t) = \inf \{n \mid \text{Eve has an } n\text{-winning strategy in } G(A, t)\}$, where $\inf = \sup$ (resp., $\inf = \sup$) in case Eve’s objective is to minimize (resp., maximize). Therefore, $[A]$ defines a function from the domain of input trees to $\mathbb{N}_\infty$. We call functions of that type cost functions. A key property of such functions is boundedness. We say that $[A]$ is bounded if there exists an $n \in \mathbb{N}$ such that $[A](t) \leq n$ for every input tree $t$.

We employ automata with a single counter, where Eve’s objective is to minimize the cost, while satisfying the parity condition. Such an automaton is known in the literature as dist $\wedge$ parity-automaton [Benedikt et al., 2015].

To navigate in the tree, it may use the directions $\{0, \uparrow, \downarrow, \leftarrow, \rightarrow\}$.
\[\text{Theorem 10.} \text{ There is a polynomial } f \text{ such that, for every dist } \land \text{ parity-automaton } A \text{ using priorities } \{0, 1\} \text{ for the parity acceptance condition, the boundedness for } |A| \text{ is decidable in time } |A|^{f(m)}, \text{ where } m \text{ is the number of states of } A.\]

Our goal is to reduce FORew(G, AQ_0) to the boundedness problem for dist \land parity-automata.

A refined semantic characterization. We first need to revisit the semantic characterization provided by Theorem[1]

Consider an \textit{S}-database \textit{D}, and a query \(Q = (S, \mathcal{O}, P) \in (G, AQ_0)\). Let \(k_Q = |S \cup \text{sig}(O)| + w^c\), where \(w = \text{wd}(S \cup \text{sig}(O))\). A derivation tree for \(D\) and \(Q\) is a labeled \(k_Q\)-ary tree \(T\), with \(\eta\) being a node labeling function that assigns facts \(R(\alpha)\), where \(R \in S \cup \text{sig}(O)\) and \(\alpha \subseteq \text{dom}(D)\), to its nodes, that satisfies the following conditions:

1. For the root node \(v\) of \(T\), \(\eta(v) = P\).
2. For each leaf node \(v\) of \(T\), \(\eta(v) \in D\).
3. For each non-leaf node \(v\) of \(T\), with \(u_1, \ldots, u_k\) being its children, \(\{\eta(u_1), \ldots, \eta(u_k)\}, O) \Rightarrow \eta(v)\).

Roughly, \(T\) describes how the \(0\)-ary predicate \(P\) can be entailed from \(D\) and \(O\). In fact, it is easy to show that \(D \models Q\) iff there is a derivation tree for \(D\) and \(Q\). The height \(hgt(T)\), denoted \(hgt(T)\), is the maximum length of a branch in \(T\), i.e., of a path from the root to a leaf node. Assuming that \(D \models Q\), the cost of \(D\) w.r.t. \(Q\), denoted \(\text{cost}(D, Q)\), is defined as

\[
\min \{hgt(T) \mid T \text{ is a derivation tree for } D \text{ and } Q\},
\]

while the cost of \(Q\), denoted \(\text{cost}(Q)\), is defined as

\[
\sup \{\text{cost}(D, Q) \mid D \models Q, D \text{ is an } S\text{-database with } \text{tw}(D) \leq \max\{0, \text{wd}(S) - 1\}\}.
\]

In other words, the cost of \(Q\) is the \textit{least upper bound} of the height over all derivation trees for all \(S\)-databases \(D\) of width at most \(\max\{0, \text{wd}(S) - 1\}\) such that \(D \models Q\). If there is no such a database, then the cost of \(Q\) is zero since \(\sup \emptyset = 0\). Actually, \(\text{cost}(Q) = 0\) indicates that \(Q\) is unsatisfiable, which in turn means that \(Q\) is trivially FO-rewritable.

Having the notion of the cost of an OMQ from \((G, AQ_0)\) in place, it should not be difficult to see how we can refine the semantic characterization provided by Theorem[1].

\[\text{Theorem 11.} \text{ Consider an OMQ } Q \in (G, AQ_0) \text{ with data schema } S. \text{ The following are equivalent:}\]

1. Condition 2 from Theorem[1] is satisfied.
2. \(\text{cost}(Q)\) is finite.

\[\text{Devising automata. We briefly describe how we can use cost automata in order to devise an algorithm for FORew}(G, AQ_0)\] that runs in double exponential time.

\[\text{Consider an OMQ } Q = (S, \mathcal{O}, P) \in (G, AQ_0). \text{ Our goal is to devise a dist } \land \text{ parity-automaton } B_Q \text{ such that the cost function } |B_Q| \text{ is bounded iff } \text{cost}(Q)\text{ is finite. Therefore, by Theorems[1] and[11] to check whether } Q \text{ is FO-rewritable we simply need to check if } |B_Q| \text{ is bounded, which, by Theorem[10] can be done in exponential time in the size of } B_Q. \text{ The input trees to our automata will be over the same alphabet } \Gamma_S \text{ that is used to encode tree-like } S\text{-databases in Section 4.}\]

Recall that for a dist \land parity-automaton \(A\), the cost function \(|A|\) is bounded over a certain class \(C\) of trees if there is an \(n \in \mathbb{N}\) such that \(|A|(t) \leq n\) for every input tree \(t \in C\). Then:

\[\text{Lemma 12.} \text{ There is a dist } \land \text{ parity-automaton } H_Q \text{ such that } |H_Q| \text{ is bounded over consistent } \Gamma_S\text{-labeled trees iff } \text{cost}(Q)\text{ is finite. The number of states of } H_Q \text{ is exponential in } \text{wd}(S), \text{ and polynomial in } |S \cup \text{sig}(O)|. \text{ Moreover, } H_Q \text{ can be constructed in double exponential time in the size of } Q.\]

The automaton \(H_Q\) is built in such a way that, on an input tree \(t\), Eve has an \(n\)-winning strategy in \(G(H_Q, t)\) iff there is a derivation tree for \([t]\) and \(Q\) of height at most \(n\). Thus, Eve tries to construct derivation trees of minimal height. The counter is used to count the height of the derivation tree.

Having this automaton in place, we can now complete the proof of Theorem[9]. The desired dist \land parity-automaton \(B_Q\) is defined as \(C_Q \cap H_Q\), where \(C_Q\) is similar to the 2ATA CS (in Lemma[5]) that checks for consistency of \(\Gamma_S\)-labeled trees of bounded degree. Notice that \(C_Q\) is essentially a dist \land parity-automaton that assigns zero (resp., \(\infty\)) to input trees that are consistent (resp., inconsistent), and thus, \(C_Q \cap H_Q\) is well-defined. Since the intersection of dist \land parity-automata is feasible in polynomial time, Lemma[5] and Lemma[12] imply that \(B_Q\) has exponentially many states, and it can be constructed in double exponential time. Lemma[12] implies also that \(|B_Q|\) is bounded iff \(\text{cost}(Q)\) is finite. It remains to show that the boundedness of \(|B_Q|\) can be checked in double exponential time. By Theorem[10] there is a polynomial \(f\) such that the latter task can be carried out in time \(|B_Q|^{f(m)}\), where \(m\) is the number of states of \(B_Q\), and the claim follows. For predicates of bounded arity, a similar complexity analysis as above shows a single exponential time upper bound.

6 Frontier-Guarded OMQs

The goal of this section is to show the following result:

\[\text{Theorem 13.} \text{ It holds that:}\]

- FORew(FG, \(Q\)) is 2EXPTIME-complete, for each \(Q \in \{UCQ, CQ, AQ_0\}\), even for predicates of bounded arity.
- FORew(G, \(Q\)) is 2EXPTIME-complete, for each \(Q \in \{UCQ, CQ\}\), even for predicates of bounded arity.
- FORew(G, AQ_0) is 2EXPTIME-complete. Moreover, for predicates of bounded arity it is EXPTIME-complete.

\[\text{Lower bounds.} \text{ The 2EXPTIME-hardness in the first and the second items is inherited from [Bienvenu et al., 2016], where it is shown that deciding FO-rewritability for OMQs based on ELI and CQs is 2EXPTIME-hard. For the 2EXPTIME-hardness in the third item we exploit the fact that containment for OMQs from \((G, AQ_0)\) is 2EXPTIME-hard, even if}\]
the right-hand side query is FO-rewritable; this is implicit in \cite{Barcelo et al., 2014}. Finally, the \textsc{ExpTime}-hardness in the third item is inherited from \cite{Bienvenu et al., 2013}, where it is shown that deciding FO-rewritability for OMQs based on \c{E}L and atomic queries is \textsc{ExpTime}-hard.

\textbf{Upper bounds.} The fact that for predicates of bounded arity \textsc{FORew}(G, AQ_0) is in \textsc{ExpTime} is obtained from Theorem \ref{thm:upper}. It remains to show that \textsc{FORew}(FG, UCQ) is in 2\textsc{ExpTime}. We reduce \textsc{FORew}(FG, UCQ) to \textsc{FORew}(FG, AQ_0), and then show that the latter is in 2\textsc{ExpTime}. This reduction relies on a construction from \cite{Bienvenu et al., 2016}, which allows us to reduce \textsc{FORew}(FG, UCQ) to \textsc{FORew}(FG, UBCQ) with UBCQ being the class of union of Boolean CQs, and the fact that a Boolean CQ can be seen as a frontier-guarded TGD. To show that \textsc{FORew}(FG, AQ_0) is in 2\textsc{ExpTime}, we reduce it to \textsc{FORew}(G, AQ_0), and then apply Theorem \ref{thm:upper}. This relies on \textit{treeification}, and is inspired by a translation of guarded negation fixed-point sentences into guarded fixed-point sentences \cite{Barany et al., 2015}. Our reduction may give rise to exponentially many guarded TGDs, but the arity is increased only polynomially. Since the procedure for \textsc{FORew}(G, AQ_0) provided by Theorem \ref{thm:upper} is double exponential only in the arity of the schema the claim follows.

\section{Future Work}

The procedure based on 2ATA provides an FO-rewriting in case the input OMQ admits one, but it is not tailored towards doing this in an efficient way. Our next step is to exploit the techniques developed in this work for devising practically efficient algorithms for constructing FO-rewritings.

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Appendix

A Proofs for Section 3

A.1 Proof of Theorem 1

Let us first cite an important lemma that will be used in the proof of Theorem 1 below:

Lemma 14. Let Q be an OMQ from (G, Aquery) with data schema S and consider an S-database D. If D \models Q then there is an S-database D* of tree-width at most max\{0, wd(S) − 1\} such that

1. D* \models Q and
2. there is a homomorphism from D* to D.

Lemma 4 can be proved using the notion of guarded unraveling and applying the compactness theorem (an almost verbatim result can be found in [2]).

Proof of Theorem 2. Assume first that Q is FO-rewritable. Then there is a first-order sentence \( \varphi(Q) \) which is equivalent over all S-databases to Q. Notice that Q is closed under homomorphisms, hence so is \( \varphi(Q) \). By Rossman’s theorem [Rossman, 2008], we thus know that \( \varphi(Q) \) must be equivalent to a (Boolean) UCQ \( q_{D} \). Now we let \( k = \max\{p_{i}: i = 1, \ldots, n\} \), where \( p_{i} \) denotes the number of atoms in \( p_{i} \). We claim that \( k \) is the bound we are looking for in condition 2. Indeed, if D \models Q, for a database of tree-width at most \( \max\{0, \text{wd}(S) − 1\} \), then also D \models q_{D} and so there is a homomorphism \( h \) that maps some \( p_{i} \) to D. The image of \( p_{i} \), under \( h \) is a database of size at most \( k \) that satisfies \( q_{D} \), i.e., \( h(p_{i}) \models q_{D} \). Since Q is equivalent to \( q_{D} \), we infer \( h(p_{i}) \models Q \), as required.

Suppose now that there is a \( k \geq 0 \) such that, for every S-database D of tree-width at most \( \max\{0, \text{wd}(S) − 1\} \), if D \models Q, then there is a D’ \subseteq D with at most k facts such that D’ \models Q. Let S be the class of all S-databases D such that (i) D contains at most k facts and (ii) D \models Q. Consider S factorized modulo isomorphisms. Notice that S is thus finite. We claim that q_{D} = \sqrt{S} (here we consider the databases in S as CQs) is a UCQ equivalent to Q (and thus an FO-rewriting of Q).

To see this, suppose first that D \models Q for some S-database D. By Lemma 14 there is an S-database D* of tree-width at most max\{0, wd(S) − 1\} such that D* \models Q. By assumption, there is a D_0 \subseteq D* of at most k facts such that D_0 \models Q. It follows that some isomorphic representative of D_0 is contained in S. Therefore, D_0 \models q_{D} and, since D_0 \subseteq D, also D \models q_{D}.

Suppose now that D \models q_{D}. Then there is some p \in S such that D \models p. Hence there is a homomorphism \( h \) that maps p to D. Recall that p (viewed as an S-database) also satisfies Q by construction of S. Since Q is closed under homomorphisms, we must also have D \models Q, and the claim follows. □

B Proofs for Section 4

B.1 Proof of Theorem 4

Let \( w = \max\{0, \text{wd}(S) − 1\} \). We are first going to prove some auxiliary statements.

Notation. For any tree T = (T, E), we denote by \( \preceq_{T} \) the natural ancestor relation induced by T, i.e., for v, w \in T, \( v \preceq_{T} w \) if v is an ancestor of w.

Lemma 15. If D has tree-width \( w \), then there is a tree decomposition \( \delta D \) of width \( w \) and an adornment \( \eta D \) for (D, \( \delta D \)) such that \( \delta \) is \( \eta \)-simple.

Proof. Let \( D = (T, (X_v)_{v \in T}) \) be a tree decomposition of D of width \( w \). Let \( \delta D \) be a tree decomposition of D defined as follows. Initially, we define that \( \delta \) equals \( \delta D \). In a second step, we add additional nodes to \( \delta D \). For any v \in T, if \|D[X_v]\| = n then we add to n − 1 copies of v to \( \delta D \) that become children of v in \( \delta D \). Let v \in T and suppose \( D[X_v] = \{\alpha_1, \ldots, \alpha_n\} \). Let v_1, \ldots, v_{n−1} be the copies of v in \( \delta D \). We can set \( \eta(v) = \{\alpha_1\} \) and \( \eta(v_i) = \{\alpha_{i+1}\} \) for i = 1, ..., n − 1. It is easy to check that \( \eta \) is an adornment for (D, \( \delta D \)).

Now \( \eta \) satisfies that \( |\eta(v)| = 1 \), for all v \in T’. We can easily modify \( \delta D \) so that \( \delta \) becomes a \( \eta \)-simple. We simply do so by successively removing from \( \delta D \) all nodes v \in T’ such that there is some v \neq w such that \( \eta(v) = \eta(w) \).

Now we show how we can modify \( \delta D \) in order to become \( \eta \)-well-colored. Let B \subseteq T’ denote the set of black nodes of T. Let \( T^* \) be the smallest set such that (1) B \subseteq T^* and (2) if v is the greatest common ancestor of some \( T_0 \subseteq T^* \), then also v \in T^*. Hence, T^* is B closed off under greatest common ancestors. Let \( \delta^* = (T^*, (Y_v)_{v \in T^*}) \), where, for v, w \in T^*, \( v \preceq_{T^*} w \) if v \preceq w \preceq w. Notice that \( \delta^* \) is a tree decomposition of D that has width \( w \), since it contains all black nodes of T’ w.r.t. \( \eta \). It is now easy to check that \( \delta D \) is \( \eta \)-well-colored. □

Lemma 16. Suppose \( \eta \) is an adornment for (D, \( \delta \)) and that \( \delta \) is \( \eta \)-simple. Then D contains at least as many facts as \( \delta \) contains black nodes w.r.t. \( \eta \).

Proof. By induction on the number n of black nodes of \( \delta = (T, (X_v)_{v \in T}) \) w.r.t. \( \eta \). If n = 1 the claim is trivial. Suppose \( \delta \) has n + 1 black nodes w.r.t. \( \eta \). There is a black node v \in T such that v has no descendant that is also black. Let \( \delta’ = (T’, (X_v)_{v \in T’}) \) be the tree decomposition that arises from \( \delta \) by removing the subtree rooted at v. Let \( D’ = \bigcup_{v \in T’} \eta(v) \). Then \( \delta’ \) is a tree decomposition of D’ and \( \delta’ \) has n black nodes w.r.t. \( \eta \). Now if D’ = D this means that \( \eta(v) = \eta(w) \) for some w \neq v. Hence, \( \delta \) cannot be simple. Therefore, D’ \subseteq D.
By the induction hypothesis, $|\mathcal{D}'| \geq n$ and we thus obtain $|\mathcal{D}| \geq n + 1$. □

**Lemma 17.** If $\delta$ is an $\eta$-well-colored tree decomposition of $\mathcal{D}$, then the number of white nodes of $\delta$ w.r.t. $\eta$ is strictly less than the number of black nodes of $\delta$ w.r.t. $\eta$.

**Proof.** Let $b_\delta (w_\delta$, respectively) denote the number of black (white, respectively) nodes of $\delta = (T, (X_v)_{v \in T})$ w.r.t. $\eta$. We proceed by induction on the depth of $T$, i.e., the maximum length of a branch leading from the root node to a leaf node. If $T$ consists only of a single node and if $\delta$ is $\eta$-well-colored, this single node must be a black node, and so the claim holds trivially. (Recall that we can restrict ourselves to non-empty databases, since we assume non-empty schemas.) Assume that $T$ is of depth $n + 1$ and assume that $\delta$ is $\eta$-well-colored. Let $T_1, \ldots, T_k$ enumerate the subtrees of $T$ rooted at the child nodes of the root of $T$, and let $\delta_i (i = 1, \ldots, k)$ be the tree decomposition that arises from $\delta$ if we restrict $T$ to $T_i$. If the root of $T$ is black, the claim is again trivial. Otherwise, if it is white, we see that $k \geq 2$ since $\delta$ is $\eta$-well-colored. For $i = 1, \ldots, k$, let $b_{\delta_i} (w_{\delta_i}$, respectively) denote the number of black (white, respectively) nodes of $T_i$ w.r.t. $\eta$. Using the induction hypothesis, we conclude that $w_{\delta_i} = w_{\delta_1} + \cdots + w_{\delta_k} + 1 < b_{\delta_i} + \cdots + b_{\delta_k} = b_\delta$. □

**Proof of Theorem 4.** Assume that condition 2 does not hold. That is, there are infinitely many non-isomorphic triples $(\mathcal{D}, \delta, \eta)$ that satisfy conditions (a)–(c). Let $S$ be the set of all these triples and let $S'$ be $S$ factorized modulo our notion of isomorphism, i.e., $S'$ contains a representative for every isomorphism type of $S$. Let $\Phi$ be the set $\{ T \mid (\mathcal{D}, \delta = (T, (X_v)_{v \in T}), \eta) \in S' \}$ of trees factorized modulo usual tree isomorphism. Notice that $\Phi$ must be infinite as well. Hence, $\Phi$ must contain trees of arbitrary size. Thus, by Lemma 17 for every $k \geq 0$, we can find a $(\mathcal{D}_k, \delta_k, \eta_k) \in S'$ such that $\delta_k = (T_k, (X_v)_{v \in T_k})$ and $T_k$ has at least $k$ black nodes w.r.t. $\eta_k$ and thus, by Lemma 16 $D_k$ has at least $k$ facts. Now $D_k \models Q$ by assumption, but $D_0 \not\models Q$ for every $D_0 \subset D_k$. Thus, for every $k$, we can find a database $D$ of tree-width at most $w$ (namely $D_k$) such that $D \models Q$, but for every $D_0 \subset D$ of at most $k$ atoms we have $D_0 \not\models Q$. Hence, condition 1 does not hold.

Suppose now that condition 1 does not hold. That is, for every $k \geq 0$ there is a database $D_k$ of tree-width at most $w$ such that $D \models Q$, yet for every $D_0 \subset D$ of at most $k$ facts we have $D_0 \not\models Q$. Let $S$ be the set of all $S$-databases $D$ of tree-width at most $w$ such that $D \models Q$, yet for any $D_0 \subset D$, we have $D_0 \not\models Q$. Let $S'$ be $S$ factorized modulo database isomorphism. $S'$ must be, by assumption, infinite as well. By Lemma 15 for every $D \in S'$, there is a tree decomposition $\delta_D$ and an adnomption $\eta_D$ of $(\mathcal{D} \delta_D)$ such that $\delta_D$ is $\eta_D$-well-colored and $\eta_D$-simple. Now for two distinct $D \in S'$ it must be the case that $(\mathcal{D}, \delta_D, \eta_D)$ and $(\mathcal{D}, \delta_D', \eta_D')$ are non-isomorphic, for otherwise $D$ and $D'$ would be isomorphic as well. For $D \in S'$ we know that, by construction of $S'$, $D \not\models Q$ for all $\alpha \in D$. Hence, the class $\{ (\mathcal{D}, \delta_D, \eta_D) \mid D \in S' \}$ is a class of infinitely many, pairwise non-isomorphic triples such that properties (a)–(c) of condition 2 are satisfied. Thus, condition 2 does not hold as well. □

**B.2 Preliminaries: Tree Encodings**

One can naturally encode instances of bounded tree-width into trees over a finite alphabet. Our goal here is to appropriately encode databases of bounded tree-width in order to make them accessible to tree automata techniques. Similar encoding techniques are well-known in the context of guarded logics, see e.g. [7] for similar encodings.

**Labeled trees.** Let $\Gamma$ be an alphabet and $(\mathbb{N} \setminus \{0\})^*$ be the set of infinite sequences of positive integers, including the empty sequence $\epsilon$. Let us recall that a $\Gamma$-labeled tree is a partial function $t: (\mathbb{N} \setminus \{0\}) \rightarrow \Gamma$, where $\text{dom}(t)$ is closed under prefixes, i.e., $x \cdot i \in \text{dom}(t)$ implies $x \in \text{dom}(t)$, for all $x \in (\mathbb{N} \setminus \{0\})^*$ and $i \in \mathbb{N} \setminus \{0\}$. The elements contained in $\text{dom}(t)$ identify the nodes of $t$. For $i \in \mathbb{N} \setminus \{0\}$, nodes of the form $\cdot i \in \text{dom}(t)$ are the children of $x$. A leaf node is a node without children. The number of children of a node $x$ is its branching degree. If every node of $t$ has branching degree at most $m$, then we say that $t$ is $m$-ary. A path of length $n$ in $t$ from $x$ to $y$ is a sequence of nodes $x = x_1, \ldots, x_n = y$ such that $x_{i+1}$ is a child of $x_i$. A branch of $t$ is a path that starts from the root node and ends in a leaf node. The height of $t$ is the maximum length of all branches. For $x \in \text{dom}(t)$, we set $x \cdot i = x$, for all $i \in \mathbb{N}$, and $x \cdot 0 = x$. Notice that $\cdot -1$ is not defined.

**Encoding.** Fix a schema $S$ and let $w \geq 1$. Let $U_{S,w}$ be a set containing $2w$ distinct constants. The elements from $U_{S,w}$ will be called names. Names are used to encode constants of an $S$-database of tree-width at most $w - 1$. Neighborhoods may describe overlapping pieces of the encoded database. In particular, if one name is used in neighboring nodes, this means that the name at hand refers to the same element—this is why we use $2w$ elements for bags. Let $K_{S,w}$ be the finite schema capturing the following information:

- For all $a \in U_{S,w}$, there is a unary relation $D_a \in K_{S,w}$.
- For each $R \in S$ and every $n$-tuple $\bar{a} \in U^n_{S,w}$, there is a unary relation $R_{\bar{a}} \in K_{S,w}$.

Let $\Gamma_{S,w} = 2^{U_{S,w}}$ be an alphabet and suppose that $\mathcal{D}$ is an $S$-database of tree-width at most $w - 1$. Consider a tree decomposition $\delta = (T, (X_v)_{v \in T})$ of $\mathcal{D}$ that has width at most $w - 1$. Moreover, consider an adnomption $\eta$ of $(\mathcal{D}, \delta)$. Fix a function $f: \text{dom}(\mathcal{D}) \rightarrow U_{S,w}$ such that different elements that occur in neighboring bags of $\delta$ are always assigned different names from $U_{S,w}$. Using $f$, we can encode $\mathcal{D}$ together with $\delta$ and $\eta$ into a $\Gamma_{S,w}$-labeled tree $t_{\mathcal{D},\delta,\eta}$ such that each node from $T$ corresponds to exactly one node in $t_{\mathcal{D},\delta,\eta}$ and vice versa. For a node $v$ from $T$, we denote the corresponding node of $t_{\mathcal{D},\delta,\eta}$ by $\hat{v}$ in the following and vice versa. In this light, the symbols from $K_{S,w}$ have the following intended meaning:

- $D_a \in t(\hat{v})$ means that $a$ is used as a name for some element of the bag $X_v$.\footnote{We specify that 0 is included in $\mathbb{N}$ as well.}
- $R_{\bar{a}} \in t(\hat{v})$ indicates that $R$ holds in $\mathcal{D}$ for the elements named by $\bar{a}$ in bag $X_v$ and this fact appears in $\eta(v)$.}
Decoding trees. Under certain assumptions, we can decode a \( \Gamma_{S,w} \)-labeled tree \( t \) into an \( S \)-database whose tree-width is bounded by \( w - 1 \). Let \( \text{names}(v) = \{ a \mid D_a \in t(v) \} \). We say that \( t \) is consistent, if it satisfies the following properties:

1. For all nodes \( v \) it holds that \( |\text{names}(v)| \leq w \).
2. For all \( R_a \in K_{S,w} \) and all \( v \in \text{dom}(t) \) it holds that \( R_a \in t(v) \) implies that \( a \subseteq \text{names}(v) \).

Suppose now that \( t \) is consistent. We show how we can decode \( t \) into a database \( [t] \) whose tree-width is at most \( w - 1 \). Let \( a \) be a name used in \( t \). We say that two nodes \( v, w \) of \( t \) are \( a \)-equivalent if \( D_a \in t(u) \) for all nodes \( u \) on the unique shortest path between \( v \) and \( w \). Clearly, \( a \)-equivalence defines an equivalence relation and we let \([v]_a = \{ w \mid (w, a) \in [v]_a \} \). The domain of \([t] \) is the set \( \{ [v]_a \mid v \in \text{dom}(t), a \in \text{names}(v) \} \) and, for \( R/n \in S \), we define
\[
[t] = R([v_1]_{a_1}, \ldots, [v_n]_{a_n}) \iff \text{there is some } v \in [v_1]_{a_1} \cap \cdots \cap [v_n]_{a_n} \text{ s.t. } R_{a_1, \ldots, a_n} \in t(v).
\]

It is not hard to show that, if \( t \) is consistent, \( [t] \) is well-defined and is an \( S \)-database of tree-width at most \( w - 1 \). We refer the reader to [?] for proofs of similar results.

Given a consistent \( t \), we let \( \delta_t^* = (T, (X_v)_{v \in T}) \) be a tree decomposition of \([t] \), where \( T \) is the same tree in structure as \( t \) and \( X_v = \text{names}(v) \), for all \( v \in T \). Moreover, we define the adornment \( \eta_t \) for \([t], \delta_t^* \) by
\[
\eta_t : v \mapsto (R([v]_{a_1}, \ldots, [v]_{a_k}), R_{a_1, \ldots, a_k} \in t(v)).
\]

We say that \( t \) is simple (well-colored, respectively) if \( \delta_t \) is \( \eta_t \)-simple (\( \eta_t \)-well-colored, respectively). Moreover, a node \( v \in \text{dom}(t) \) is black (white, respectively), if it is black (white, respectively) w.r.t. \( \eta_t \). We call \( \delta_t \) (\( \eta_t \), respectively) the standard tree decomposition (standard adornment, respectively) of \( t \).

Bounding the branching degree. For our automata constructions that follow, it will be convenient to work on \( \Gamma_{S} \)-labeled trees whose branching degree can be bounded by the constant \( m_S = 2^{\text{wd}(S)} \) so that we can work automatata that run on \( m_S \)-ary trees. The following statement shows that we can always assume this without loss of generality:

**Lemma 18.** Suppose \( \mathcal{D} \) is an \( S \)-database and \( \delta \) a tree decomposition of \( \mathcal{D} \). Then there exists a tree decomposition \( \delta' \) of \( \mathcal{D} \) such that \( \delta' \) has branching degree at most \( m_S = 2^{\text{wd}(S)} \).

**Proof.** Let \( \delta = (T, (X_v)_{v \in T}) \) be a tree decomposition of \( \mathcal{D} \) of width at most \( w = \max\{0, \text{wd}(S) - 1\} \). For \( v \in T \), let \( d_v \) be the branching degree of \( v \) in \( T \). Moreover, let
\[
d_\delta = \sum_{v \in T} \{ d_v - m_S \mid d_v > m_S, v \in T \}.
\]

We are going to prove the following statement by induction on \( d_\delta \): if \( \delta \) is a tree decomposition of \( \mathcal{D} \) then there is a tree decomposition \( \delta' \) of \( \mathcal{D} \) such that every node of \( \delta' \) has branching degree at most \( m_S = 2^{\text{wd}(S)} \). Moreover, \( \delta' \) results from “reorganizing” nodes of \( \delta \) and we can view any adornment \( \eta \) of \( (\mathcal{D}, \delta) \) is also an adornment of \( (\mathcal{D}, \delta') \).

If \( d_\delta = 0 \) then the claim is trivial, since \( \delta \) has no nodes of branching degree greater than \( m_S \). Suppose now \( d_\delta = n + 1 \). Let \( v \in T \) be a node such that \( d_v > m_S \). Assume \( v \) is chosen such that it has, among all nodes of branching degree greater than \( m_S \), maximal distance to the root. Let \( v_1, \ldots, v_k \) enumerate all children of \( v \). For \( i = 1, \ldots, k \), let \( Y_i = X_v \cap X_{v_i} \). Hence, \( Y_i \subseteq X_v \) and since there are at most \( m_S = 2^{\text{wd}(S)} \) subsets of \( X_v \), it must be the case that \( Y_i = Y_j \) for some \( i \neq j \). Let \( \delta' \) be the tree decomposition that arises from \( \delta \) by removing subtree rooted at \( v_i \) from \( T \), inserting it below \( v_i \) so that \( v_i \) becomes a child node of \( v \). Notice that \( v_i \) still has branching degree at most \( m_S \) by the choice of \( v \). Moreover, \( \delta' \) is still a tree decomposition of \( \mathcal{D} \), since connectedness is clearly ensured. Now \( d_{\delta'} = n \) and an application of the induction hypothesis yields the claim.

By Lemma 18 we can thus always assume that the encoding of an \( S \)-database has branching degree at most \( m_S = 2^{\text{wd}(S)} \) which we will assume for the remainder of this section.

### B.3 Preliminaries: Two-way alternating automata (2ATA)

For a finite set of symbols \( X \), let \( \mathbb{B}^+(X) \) be the set of positive Boolean formulas that can be formed using propositional variables from \( X \), i.e., formulas using \( \wedge, \vee \) and propositional variables from \( X \).

A two-way alternating (parity) automaton (2ATA) on (finite) \( m \)-ary trees is a tuple \( A = (S, \Gamma, \delta, s_0, \Omega, \text{Dir}) \), where

- \( S \) is a finite set of states,
- \( \Gamma \) is the input alphabet,
- \( \delta : S \times \Gamma \to \mathbb{B}^+(\text{tran}(A)) \) is the transition function, where \( \text{tran}(A) = \{(d)s, [d]s \mid d \in \text{Dir}\} \),
- \( s_0 \) is the initial state,
- \( \Omega : S \to \mathbb{N} \) is the parity condition that assigns to each \( s \in S \) a priority \( \Omega(s) \),
- \( \text{Dir} \) is a set of directions and, in our case, always equals \( \{-1, 0, 1, \ldots, m\} \).

Notice that we make explicit the set of directions the automaton may use. Formally, a direction is just a function that maps a node to other nodes. For \( d \in \{-1, 0, 1, \ldots, m\} \), we set
\[
d : \varepsilon \mapsto \{d\}, \quad v \mapsto \{v \cdot d\}, \quad \text{for } d \neq -1,
\]
\[
d : \varepsilon \mapsto \{d\}, \quad v \mapsto \{v \cdot d\}, \quad \text{for } v \neq \varepsilon.
\]
Notice that \(-1(\varepsilon)\) is thus undefined, since the root has no parent. The direction 0 maps every node to itself and thus indicates that the automaton should stay in the current node, the direction \(-1\) indicates the automaton should proceed to \(\varepsilon\). Notice that we always use the same set of directions here, i.e., we work on trees of a fixed branching degree. We nevertheless make this set of directions explicit to avoid confusion, since our cost automatata are going to work with amorphous automata that work on trees of arbitrary branching degree.
the parent node, and a direction \( k \in \{1, \ldots, n\} \) indicates that the automaton should move to the \( k \)-th child of the current node. Transitions of the form \( (d)s \) mean that a copy of the automaton must accept in state \( s \) for at least one node in direction \( d \), while the dual connective, \( [d]s \), means that, for every neighbor in direction \( d \), if a copy of the automaton is sent to that neighbor in state \( s \), it must accept. Notice that our automaton is two-way, since it can proceed to the in both directions—to the parent and to the children.

Remark. We will consider our 2ATA to run on finite trees. The parity condition nevertheless makes sense, since our automata are two-way and two-way movements can give rise to infinite runs.

Given an \( A \) as above and a \( \Gamma \)-labeled input tree \( t \), the notion of acceptance of \( t \) is defined via a game played between two players, Eve and Adam. The goal of Eve is to satisfy the parity condition and prove that \( t \) is accepted by \( A \), while to goal of Adam is to disprove this. We shall make this more precise in the following.

Let \( \chi \in \mathbb{B}^+ (\text{tran}(A)) \) be a positive formula. We assign \( \chi \) to an owner according to its form:

- If \( \chi = \chi_1 \land \chi_2 \) (respectively, \( \chi = \chi_1 \lor \chi_2 \)) then \( \chi \) is owned by Adam (respectively, Eve).
- If \( \chi = [d]s \) (respectively, \( \langle d \rangle s \)) then \( \chi \) is owned by Adam (respectively, Eve).

The acceptance game \( \mathcal{G}(A, t) \) for \( A \) and \( t \) is played in the arena \( \mathbb{B}^+ (\text{tran}(A)) \times \text{dom}(t) \). For each position \( (\chi, v) \) of the arena, we define the set of possible choices:

- If \( \chi = \chi_1 \land \chi_2 \) or \( \chi = \chi_1 \lor \chi_2 \) then the possible choices are \( \{(\chi_1, v), (\chi_2, v)\} \).
- If \( \chi = [d]s \) or \( \chi = \langle d \rangle s \) then the possible choices are \( \{ (\delta(s, t(w)), w) \mid w \in d(v) \} \).

Let \( \chi_0 = \delta(s_0, t(\varepsilon)) \). The initial position of the game \( \mathcal{G}(A, t) \) is \( (\chi_0, v) \) and from any position \( (\chi, v) \):

- The player that owns \( \chi \) selects a \( (\chi', v) \) among the possible choices of \( (\chi, v) \), and
- the game continues from position \( (\chi', v) \).

The transition from \( (\chi, v) \) to \( (\chi', v) \) is called a move. By \( \chi \) in \( \mathcal{G}(A, t) \) we mean a sequence of moves \( (\chi_0, v), (\chi_1, v_1), (\chi_2, v_2), \ldots \) (recall that \( \chi_0 = \delta(s_0, t(\varepsilon)) \)). A strategy for one of the players is a function that returns the next choice for that player given the history of the play. Fixing a strategy for both players thus uniquely determines a play in \( \mathcal{G}(A, t) \). A play \( \pi \) is consistent with a strategy \( \xi \) if there is a strategy \( \xi' \) for the other player such that \( \xi \) and \( \xi' \) yield \( \pi \).

We say that a strategy is winning for Eve, if every play consistent with it satisfies the parity acceptance condition, that is, if every play \( (\chi_0, v), (\chi_1, v_1), (\chi_2, v_2), \ldots \) consistent with that strategy, the maximum priority among \( \Omega(\chi_0), \Omega(\chi_1), \Omega(\chi_2), \ldots \) that occurs infinitely often is even. Here, we set

\[
\Omega(\chi) = \begin{cases} 
\Omega(s), & \text{if } \chi = [d]s \text{ or } \chi = [d]s, \\
\min \Omega(S), & \text{otherwise.}
\end{cases}
\]

The language of \( A \), denoted \( \mathcal{L}(A) \), is the set of all \( \Gamma \)-labeled trees \( t \) such that Eve has a winning strategy in \( \mathcal{G}(A, t) \).

### B.4 Proof of Lemma 5

The construction of this automaton is fairly standard and we only make a few comments on it (cf. [2] for a similar construction). Notice first of all that each of the two conditions for consistency can be checked separately, and taking the intersection of the respective automata yields the desired automaton. Each of the two consistency conditions involves a top-down pass through the tree, while checking the respective condition locally.

### B.5 Proof of Lemma 6

We can devise \( R_S \) as the intersection of two separate 2ATA, \( R_{1,S} \) and \( R_{2,S} \), where the former checks whether the input tree \( t \) is simple and the latter checks whether \( t \) is well-colored. It is well-known that building the intersection of two 2ATA is feasible in polynomial time. Recall that \( \delta_t \) denotes the standard tree decomposition of \( \langle \| \rangle, \delta_t \).

**The automaton \( R_{1,S} \).** In order to check whether \( t \) is simple, we have to check two conditions: (i) whether \( |\eta_t(v)| \leq 1 \) for all \( v \in \text{dom}(t) \) and (ii) whether \( \eta_t(v) \neq \eta_t(w) \) for all \( v \neq w \). The first condition is easy to check (respecting the stated size bounds) and we leave this as an exercise for the reader. We describe how the check the second one and assume that the input tree satisfies the first condition.

We shall describe the game \( \mathcal{G}(R_{1,S}, t) \). Adam will have a winning strategy in \( \mathcal{G}(R_{1,S}, t) \) iff \( \eta_t(v) = \eta_t(w) \) for some nodes \( v \neq w \). Adam first navigates to an arbitrary node \( v \) for which he wants to prove that there is some other \( w \neq v \) such that \( \eta_t(v) = \eta_t(w) \). He then selects the one and only atom \( R_a \in t(v) \) and guesses the path to the node \( w \) for which he thinks that \( \eta_t(v) = \eta_t(w) \). If he finds that node, he wins. By navigating to \( w \), he must remember the atom \( R_a \) in the states and also the direction he came from. He must remember the direction due to the fact that we require \( v \neq w \). Due to that, the number of states of \( R_{1,S} \) also depends linearly on the branching degree \( m_S \) which still allows us the respect the stated size bounds since \( m_S = 2^{w_d(S)} \). Now while navigating to \( w \), Adam is not allowed to traverse the tree backwards in the direction he came from. For storing this information, we need exponentially many states in \( w_d(S) \) and linearly in \(|S| \) and \( m_S \).

**The automaton \( R_{2,S} \).** Recall that a node \( v \in \text{dom}(t) \) is well-colored if it is either black or it has at least two successor nodes which are both well-colored. Having this definition in place, devising \( R_{2,S} \) becomes quite easy. In \( \mathcal{G}(R_{2,S}, t) \), Adam guesses the node \( v \) with a maximum distance from the root of which he wants to prove that this node is not good. Since \( v \) is not good, \( v \) is white and it has less than two successors that are good. Moreover, since \( v \) has maximum distance from the root, \( v \) must have either no successors or it has a single successor that is black. (Two black successors would turn \( v \) to a good node, while one black and a white successor of which both are not good would turn the white successor to a non-good node which has higher distance to the root.) Therefore, all Adam has to do is to challenge Eve to show the existence of the (non-existent) second successor. Adam will win if Eve cannot point to such a second successor. Notice
that the size of the state set of this automaton is independent from $S$. In fact, $R_{2,S}$ has constantly many states.

### B.6 Proof of Lemma 7

The construction of this automaton appears in [?]. Notice that in [?], this automaton is devised for input trees whose decodings are acyclic rather than of tree-width at most $\max\{0, wd(S) - 1\}$. However, the construction works also with our encodings. Alternatively, one can view $A_Q$ as a version of the cost automaton $H_Q$ from Lemma 12 that has no counters at all.

### B.7 Proof of Lemma 8

We shall informally describe the construction of $M_Q$ and describe its size bounds. Then we are going to prove that $M_Q$ indeed can be used to check whether $Q$ is FO-rewritable.

Firstly, we define an auxiliary alphabet $\Lambda_S$ that is a copy of (some parts of) the alphabet $\Gamma_S$. More specifically, for every $\rho \in \Gamma_S$ such that

$$\rho \cap \{R_a | R_i/n \in S, a \in U_{S, wd(S)} \} = \{a_1, \ldots, a_k\},$$

we stipulate that $\Lambda_S$ contains the symbol $\rho^a = \{a^1, \ldots, a^k\}$.

That is, the alphabet $\Lambda_S$ carries information on the facts named in a $\Gamma_S$-labeled tree (we call facts of the form $\alpha^i$ tagged). Intuitively, a fact of the form $R^2_a$ specifies that the minimization procedure (that is yet to be implemented in $M_Q$) should aim to satisfy $Q$ without the need of $R_a$.

For a $\Lambda_S$-labeled tree $t$, we define $t \upharpoonright \Gamma_S$ as the $\Gamma_S$-labeled $t'$ that arises from $t$ by setting $t'(v) = t(v) \cap \bigcup \Gamma_S$ for all $v \in \text{dom}(t')$. We say that $t$ is an extension of $t' = t \upharpoonright \Gamma_S$. We call $t$ consistent if $t \upharpoonright \Gamma_S$ is consistent and at least one node $v \in \text{dom}(t)$ is labeled with a fact of the form $R^2_a$ and, moreover, for all $v \in \text{dom}(t)$, if $R^2_a \in t(v)$ then also $R_a \in t(v)$. We define $[t]^- \upharpoonright \Gamma_S$ to be

$$[t]^- \upharpoonright \Gamma_S = \{R_a[v]_{a_1}, \ldots, [v]_{a_k} | R^2_{a_1}, \ldots, a_k \in t(v)\}.$$

That is, in $[t]^- \upharpoonright \Gamma_S$ we remove the facts that are tagged.

**Lemma 19.** There is a 2ATA $D_Q$ that runs on $m$-ary $\Lambda_S$-labeled trees and accepts a $\Lambda_S$-labeled tree if and only if:

1. $t$ is consistent,
2. $[t]^- = Q$, i.e., $t$ without the tagged facts satisfies $Q$.

The number of states of $D_Q$ is exponential in $wd(S)$ and linear in $|S \cup \text{sig}(O)|$. Moreover, $D_Q$ can be constructed in double exponential time in the size of $Q$.

$D_Q$ can be constructed as the intersection of the 2ATA, where one checks consistency and the other ensures that $[t]^- \models Q$. The former can be constructed in a similar spirit as $A_Q$ from Lemma 7 so that the construction of $D_Q$ respects the same size bounds. Moreover, consistency of $t$ can be checked in a similar fashion as consistency for $\Gamma_S$-labeled trees, with an additional check that the input tree contains at least one tagged fact.

Having $D_Q$ from Lemma 19 in place, we are now going to construct $M_Q$. $M_Q$ will accept a consistent $\Gamma_S$-labeled input tree $t$ if and only if there is no $\Lambda_S$-labeled extension $t'$ of $t$ such that (i) $t'$ is consistent and (ii) $[t']^- = Q$. Equivalently, $M_Q$ will accept a consistent $t$ if there are no facts $a_1, \ldots, a_k \in [t] \ (k \geq 1)$ such that $[t] \setminus \{a_1, \ldots, a_k\} = Q$.

We can, according to [Vardi, 1998], convert $D_Q$ into a nondeterministic parity tree automaton on $m_S$-ary trees $D'_Q$ which is simply a 2ATA on $m_S$-ary trees where all transitions are of the form $\delta(q, a) = \bigvee_{i \in I} q_{i,1} \land \cdots \land (m_S)_{q_{m,1}}$. This conversion causes an exponential blowup on the size of the state set. We can view $\Lambda$ as an alphabet extending $\Gamma$. Hence, we can perform the operation of projection on $D'_Q$ in such a way that the resulting automaton, call it $\exists D_Q$, accepts $\Gamma_S$-labeled trees only. Notice that projection is easy to perform in the case of nondeterministic parity automata. Indeed, in order to construct $\exists D_Q$, the only thing we have to do is to guess symbols from $\Lambda_S$ in the transition function of $\exists D_Q$. This does not involve a blowup on the state set, and $\exists D_Q$ can be constructed in polynomial time in the size of $D_Q$. Notice that $\exists D_Q$ accepts a consistent $\Gamma_S$-labeled tree $t$ iff there is a $\Lambda_S$-labeled extension $t'$ of $t$ such that $t'$ is consistent and $[t']^- = Q$. We thus obtain $M_Q$ from $\exists D_Q$ by building the complement of $\exists D_Q$. Building the complement of a 2ATA is easy—we simply swap the formulas owned by Adam and Eve.

### B.8 Proof of Theorem 2

Let $Q = (S, O, G)$. As said in the main body of the paper, we can obtain $B_Q$ by intersecting the respective 2ATA from Lemmas 5 to 8.

It is clear $B_Q$ has double-exponentially many states in $wd(S)$ and, moreover, $B_Q$ can be constructed in double-exponential time. Thus, $B_Q$ accepts a $\Gamma_S$-labeled tree $t$ if and only if

- $t$ is consistent,
- $t$ is well-colored and simple,
- $[t] = Q$, and
- $\{t\} \not\models Q$ for all $\alpha \in [t]$.

For a proof of Theorem 2 it thus remains to be shown that the language of $B_Q$ is infinite iff $Q$ is not FO-rewritable.

Suppose first that $L(B_Q)$ is infinite. Since the the branching degree of the input trees is bounded (recall that we run on $m_S$-ary trees), $B_Q$ accepts trees of arbitrary height. Hence, there are infinitely many trees $t_0, t_1, \ldots, t_k, \ldots$ and natural numbers $h_0, h_1, \ldots, h_k, \ldots$ such that, for $i \geq 0$,

- $t_i$ has height $h_i$,
- $[t_i] \models Q$,
- $t_i$ is well-colored and simple, and
- $[t_i] \not\models Q$, while $[t_i] \setminus \{\alpha\} \not\models Q$ for all $\alpha \in [t_i]$.

Moreover, we can assume that $i \neq j$ implies $h_i \neq h_j$ (otherwise we simply drop $t_j$). For $i \geq 0$, consider the standard tree decomposition $\delta_i$ and the standard adornment $\eta_i$ of $t_i$. It is clear that $\delta_i$ is $\eta_i$-well-colored and $\eta_i$-simple as well. Moreover, for $i \neq j$, the triples $([t_i], \delta_i, \eta_i)$ must be non-isomorphic, since the height of $t_i$ and $t_j$ differ, i.e., $h_i \neq h_j$. We thus obtain by Theorem 4 that $Q$ cannot be FO-rewritable.

\[4\] A definition of acyclicity will be given in Section 6.
Suppose now that \( Q \) is not FO-rewritable. By \( 4 \) there is an infinite class \( \mathcal{D} \) of pairwise non-isomorphic triples \((\mathcal{D}, \delta, \eta)\) (where \( \mathcal{D} \) is an S-database, \( \delta \) a tree decomposition of \( \mathcal{D} \) of width at most \( \max\{0, \text{wd}(\mathcal{S}) - 1\} \), and \( \eta \) an adornment of \((\mathcal{D}, \delta)\)) such that

- \( \delta \) is \( \eta \)-well-colored and \( \eta \)-simple,
- \( \mathcal{D} \models Q \), and
- for every \( \alpha \in \mathcal{D} \) it holds that \( \mathcal{D} \setminus \{\alpha\} \neq Q \).

Recall that we can encode every such triple \( \gamma = (\mathcal{D}, \delta, \eta) \) as a \( \Gamma_\delta \)-labeled tree \( t_\gamma \). Considering the encoding, for two non-isomorphic triples \( \gamma, \gamma' \in \mathcal{S} \) we must have \( t_\gamma \neq t_{\gamma'} \). By construction we then have \( \{t_\gamma \mid \gamma \in \mathcal{S}\} \subseteq L(B_Q) \). Hence, \( L(B_Q) \) must be infinite since \( \mathcal{S} \) is. This completes the proof of \( 2 \)

\[ \text{B.9 Proof of Corollary} \]

As described in the main body of this paper, in order to decide \( \text{FOW}(\mathcal{G}, \mathcal{A}\mathcal{Q}_0) \), it suffices to decide whether the language of the \( \lambda \text{ATA} \) \( B_Q \) from \( 2 \) is finite. To this end, we first convert \( B_Q \) into a non-deterministic parity tree automaton \( B'_Q \) according to the procedure presented in \( \text{Vardi, 1998} \). The number of states of \( B'_Q \) is exponential in the number of states of \( B_Q \). Since \( B'_Q \) accepts only finite trees, we can view \( B'_Q \) as a conventional bottom-up tree automaton that works on finite trees. We can then check whether \( L(B'_Q) \) is finite in polynomial time in the size of \( B'_Q \) \( \text{Vardi, 1992} \). Since \( B'_Q \) has triple exponentially many states, the 3\text{EXPTIME} upper bound of \( \text{FO}(\mathcal{G}, \mathcal{A}\mathcal{Q}_0) \) follows. In case of bounded arity, \( B'_Q \) has double exponentially many states, which yields the 2\text{EXPTIME} upper bound as stated.

\[ \text{C Proofs for Section} \]

\[ \text{C.1 Proof of Theorem} \]

Let \( Q = (\mathcal{S}, \mathcal{O}, \mathcal{G}) \) be an OMQ from \( (\mathcal{G}, \mathcal{A}\mathcal{Q}_0) \).

**Lemma 20.** Let \( \mathcal{D} \) be an S-database. Then \( \mathcal{D} \models Q \) iff there is a derivation tree for \( \mathcal{D} \) and \( Q \).

**Proof (sketch).** The direction from right to left is straightforward and left to the reader. For the other direction, we remark that a proof of a similar statement appears in \( \text{[2]} \). We only sketch the idea. Basically, since \( \mathcal{O} \) consists of guarded rules only, for any fact \( R(\bar{a}) \) such that \( (\mathcal{D}, \mathcal{O}) \models R(\bar{a}) \), one can find a guarded sequence of facts \( \alpha_1, \ldots, \alpha_k \) such that \( \{(\alpha_1, \ldots, \alpha_k), \mathcal{O}\} \models R(\bar{a}) \). (We say that \( \alpha_1, \ldots, \alpha_k \) is guarded, if there is an \( i = 1, \ldots, k \) such that \( \text{adom}(\{\alpha_i\}) \supseteq \text{adom}(\{\alpha_1, \ldots, \alpha_k\}) \).) One then builds an appropriate derivation tree \( T \) by starting with the root node, labeled by \( \mathcal{G} \), and successively searching for such guarded sequences of facts. The facts contained in the guarded sequence then become labels of leaf nodes of the current node. Continuing this process recursively then gives rise to a derivation tree for \( \mathcal{D} \) and \( Q \). (It is pretty easy to check that, if \( (\mathcal{D}, \mathcal{O}) \models R(\bar{a}) \), the labels of leaf nodes of \( \mathcal{T} \) must be contained in \( \mathcal{D} \).)

It remains therefore to be argued why the branching degree of \( T \) can be bounded by \( k_Q = |\mathcal{S} \cup \text{sig}(\mathcal{O})| \cdot w^w \), where \( w = \text{wd}(\mathcal{S} \cup \text{sig}(\mathcal{O})) \). This is simply the case, since any guarded sequence \( \alpha_1, \ldots, \alpha_k \) with more than \( k_Q \) facts must contain repetitions: for a fixed guard \( \alpha_i \) \( (i = 1, \ldots, k) \), there are at most \( w^w \) different sequences of constants that use elements from \( \text{adom}(\{\alpha_i\}) \). Moreover, there are \( |\mathcal{S} \cup \text{sig}(\mathcal{O})| \) different symbols that we can attach to such sequences. Therefore, \( \alpha_1, \ldots, \alpha_k \) contains at most \( k_Q \) distinct facts. \( \square \)

**Remark.** Notice that from this proof it becomes evident that we can restrict ourselves to derivation trees where the set of children of a non-leaf node is guarded in the above sense. We will assume this of all derivation trees in the following.

**Proof of Theorem** Throughout the proof, we let \( w = \max\{0, \text{wd}(\mathcal{S}) - 1\} \).

Suppose first that there is a \( k \geq 0 \) such that, for every S-database \( \mathcal{D} \) of tree-width at most \( w \), if \( \mathcal{D} \models Q \), then there is a \( \mathcal{D}' \subseteq \mathcal{D} \) with at most \( k \) facts such that \( \mathcal{D}' \models Q \). We show that \( \text{cost}(Q) \) is finite. Let \( \mathcal{S} \) be the set of all S-databases \( \mathcal{D} \) of at most \( k \) facts such that \( \mathcal{D} \models Q \), and consider \( \mathcal{S} \) to be factorized modulo database isomorphism. Clearly, \( \mathcal{S} \) must be finite. For each \( \mathcal{D} \in \mathcal{S} \), let \( \mathcal{T}_\mathcal{D} \) be a derivation tree for \( \mathcal{D} \) and \( Q \). Let \( n = \max\{|\text{ht}(\mathcal{T}_\mathcal{D})| \mid \mathcal{D} \in \mathcal{S}\} \). We claim that \( \text{cost}(Q) \leq n \). Indeed, suppose that \( \mathcal{D} \) is an S-database of tree-width at most \( w \) such that \( \mathcal{D} \models Q \). By assumption, we can find a \( \mathcal{D}' \subseteq \mathcal{D} \) of at most \( k \) atoms such that \( \mathcal{D}' \models Q \). Hence, (some isomorphic copy of) \( \mathcal{D}' \) is contained in \( \mathcal{S} \). Consider an arbitrary derivation tree \( T \) for \( \mathcal{D} \) and \( Q \). In case \( \text{ht}(T) > n \), we know that \( \mathcal{T}_\mathcal{D} \) is also a derivation tree for \( \mathcal{D} \) and \( Q \), and so \( T \) is not the minimal one. Therefore, \( \text{ht}(T) \leq n \) and so \( \text{cost}(Q) \leq n \), as required.

Suppose now that \( \text{cost}(Q) = n \) for some \( n \in \mathbb{N} \). For a derivation tree \( T \), let \( \ell_T \) denote the number of leaf nodes of \( T \). Let

\[ k = \sup \{\ell_T \mid T \text{ is a derivation tree for } \mathcal{D} \text{ and } Q \text{ of minimum height, where } \mathcal{D} \text{ is an S-database with } \text{tw}(\mathcal{D}) \leq w\}. \]

Notice that \( k \) exists (i.e., \( k \in \mathbb{N} \)), since we can bound the height of the derivation trees used in the definition of \( k \) by \( n \), and since the number of leaf nodes of a derivation tree of finite height cannot be arbitrarily large (recall that the branching degree of a derivation tree is bounded). We claim that \( k \) is the bound we are looking for in condition 1 of \( 11 \). Let \( \mathcal{D} \) be an S-database of tree-width at most \( w \) such that \( \mathcal{D} \models Q \). Consider a derivation tree \( T \) for \( \mathcal{D} \) and \( Q \) of minimum height. Let \( \alpha_1, \ldots, \alpha_m \) be the leaf nodes of \( T \). By construction, the number of leaf nodes of \( T \) is surely bounded by \( k \), i.e., \( m \leq k \). Moreover, \( T \) is by definition also a derivation tree for \( \mathcal{D}' = \{\alpha_1, \ldots, \alpha_m\} \) and \( Q \). By Lemma 20 we then have \( \mathcal{D}' \models Q \). Now \( \mathcal{D}' \) is a subset of \( \mathcal{D} \) of at most \( k \) atoms. Therefore, condition 1 of the statement of \( 11 \) holds as well. \( \square \)

\[ \text{C.2 Preliminaries: Cost automata} \]

We remark that in this section we are going to work on labeled trees that are amorphous, i.e., that have an arbitrary branching degree. This is not necessary for a technical reason, but simplifies presentation of the cost automaton from Lemma 12.
Objects. An objective is a triple Obj = (Act, f, goal), where Act is a finite set of actions, f a function (the objective function) that assigns values from $\mathbb{N}_\infty$ to sequences of actions, and goal $\in \{\min, \max\}$. We shall consider a run of a (two-way) alternating cost automaton as a two-player game with players Eve and Adam, where goal specifies whether Eve’s aim is to minimize or maximize the objective function.

An example for an objective can be given by the well-known parity acceptance condition which we also used for plain 2ATA. This condition can be recast into a parity objective parity $= (P, \text{cost}_{\text{parity}}, \text{goal})$, where P is a finite set of priorities and cost_{\text{parity}} is specified as follows: if goal $= \min$ (goal $= \max$, respectively) then cost_{\text{parity}} maps a sequence of priorities to 0 ($\infty$, respectively) if the maximum priority that occurs infinitely often is even, and to $\infty$ (0, respectively) otherwise.

Cost automata model. Let $\Gamma$ be an alphabet. A two-way alternating cost automaton $\mathcal{A}$ on $\Gamma$-labeled trees is a tuple $(S, \Gamma, s_0, \text{Dir}, \text{Obj}, \delta)$, where

- $S$ is a finite set of states,
- $s_0$ is the initial state,
- Dir, as in the case of 2ATA, describes the possible directions; in our case, we always have Dir $= \{0, \downarrow, \uparrow\}$, where
  \[
  \downarrow: \varepsilon \mapsto \{i \mid i \in \{1, \ldots, n_v\}\},
  \quad v \mapsto \{v \cdot i \mid i \in \{-1, 1, \ldots, n_v\}\},
  \quad \text{for } v \neq \varepsilon,
  \]
  and $n_v$ denotes the number of successors of $v$. Hence, the direction $\uparrow$ denotes all possible neighbors of a node, including the parent (the root $\varepsilon$ has no parent and therefore $\uparrow(\varepsilon)$ only includes all children of $\varepsilon$).
- Obj is an objective;
- $\delta: S \times \Gamma \rightarrow \mathbb{B}^+(\text{tran}(\mathcal{A}))$ the transition function, where
  \[
  \text{tran}(\mathcal{A}) = \{\langle d \rangle(s, c), \{d\}(s, c) \mid s \in S, c \in \text{Act}, d \in \text{Dir}\}.
  \]

To emphasize the objective that is used, we often call an automaton in the form of $\mathcal{A}$ an Obj-automaton.

Notice that each transition also carries information on the action that is to be performed when switching to a new state. We will present the concrete actions available to our automata model below. Also note that our cost automata work on trees of arbitrary branching degree.

As in the case of 2ATA, we assign owners to each formula from $\mathbb{B}^+(\text{tran}(\mathcal{A}))$ in the expected manner. That is, conjunctions are owned by Adam, disjunctions are owned by Eve, atomic formulas of the form $\{d\}(s, c)$ are owned by Adam, while those of the form $\langle d \rangle(s, c)$ are owned by Eve.

Let $t$ be a $\Gamma$-labeled tree. As in the case of 2ATA, we define a two-player (cost) acceptance game $\mathcal{G}(\mathcal{A}, t)$ for $\mathcal{A}$ and $t$. The arena of the game is again $\mathbb{B}^+(\text{tran}(\mathcal{A})) \times \text{dom}(t)$, and the notion of possible choices of a position $(\chi, v)$ in the game is defined as in the case of 2ATA:

- If $\chi = \chi_1 \land \chi_2$ or $\chi = \chi_1 \lor \chi_2$ then the possible choices are $\{(\chi_1, v), (\chi_2, v)\}$.
- If $\chi = [d](s, a)$ or $\chi = \langle d \rangle(s, a)$ then the possible choices are $\{(\chi, w) \mid w \in d(v)\}$.

Let $\chi_0 = \delta(s_0, t(\varepsilon))$. The initial position of the game is $(\chi_0, \varepsilon)$ and from any position $(\chi, v)$:

- The player that owns $\chi$ selects a $(\chi', w)$ from the possible choices of $(\chi, v)$, and
- the game continues from position $(\chi', w)$.

The transition from $(\chi, v)$ to $(\chi', w)$ is a move. If $\chi$ is of the form $\langle d \rangle(s, c)$ or $[d](s, c)$, then we say the output of that move is $c$. Otherwise, we simply say that that move has no output. A play in $\mathcal{G}(\mathcal{A}, t)$ is a sequence of moves $(\chi_0, \varepsilon), (\chi_1, v_1), (\chi_2, v_2), \ldots$ and a strategy for one of the players is a function that, given the history of the play, returns the next choice for that player. Again, fixing a strategy for both players uniquely determines a play in $\mathcal{G}(\mathcal{A}, t)$. A play $\pi$ is consistent with a strategy $\xi$ if there is a strategy $\xi'$ for the other player such that $\xi$ and $\xi'$ yield $\pi$. The output of a play $\pi = (\chi_0, v_0), (\chi_1, v_1), (\chi_2, v_2), \ldots$ is the sequence of actions $c_{i_0}, c_{i_1}, c_{i_2}, \ldots$ such that, for all $j \geq 0$, $c_{i_j}$ is the output of $(\chi_j, v_j)$.

Suppose $\text{Obj} = (\text{Act}, f, \text{goal})$. The cost of a play $\pi$ consistent with a winning strategy of Eve is the value of $f$ on the output of that play. If goal $= \min$ (goal $= \max$, respectively) then an $n$-winning strategy for Eve is a strategy such that the cost of any play (also called the cost of that play) consistent with that strategy is at most $n$ (at least $n$, respectively). We define

$[\mathcal{A}](t) = \text{op}(\{n \mid \text{Eve has an n-winning strategy in } \mathcal{G}(\mathcal{A}, t)\})$,

where $\text{op} = \inf$ (sup, respectively) if goal $= \min$ (goal $= \max$, respectively). We say that $[\mathcal{A}]$ is the cost function defined by $\mathcal{A}$.

Counter actions. As in [Benedikt et al., 2015], we are interested in objectives that are based on counters. We use the elementary actions increment & check $ic$, reset $r$, and no change $\varepsilon$. Let $\gamma$ be a counter. Its initial value is 0 and afterwards it can take values from $\mathbb{N}$ according to a sequence $\bar{u}$ of actions from $\{ic, r, \varepsilon\}$. The meanings of $r$ and $\varepsilon$ are clear. The operation $ic$ increments the counter value (the increment) and, at the same time, indicates that we are interested in the current value of the counter (the check). Let $C_{\gamma}(\bar{u})$ denote the set of values at the moment(s) in the sequence $\bar{u}$ when $\gamma$ is checked, i.e., when the operation $ic$ occurs. For example, $C_{\gamma}(iciriciricicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicicic
is minimizing for Eve and may contain multiple counters instead of a single one. Moreover, in this objective, the counter action $\not\in c$ is separated into $\in$ and $c$, i.e., the counters may be incremented but not checked. We refer the reader to [Benedikt et al., 2015] for more details.

### C.3 Proof of Lemma 12

Let $Q = (S, \mathcal{O}, G)$ be an OMQ from $(G, A_Q)$. Consider a consistent $\Gamma_S$-labeled tree $T$. We are going to devise a dist $\land$ parity-automaton $H_Q$ over $\Gamma_S$ such that Eve has an $n$-winning strategy in $G[H_Q, t]$ if and only if there is a derivation tree $T$ for $\langle t \rangle$ and $Q$ of height at most $n$.

Let $H_Q = (S, \Gamma_S, s_0, \{0, \top\}, \text{dist} \land \text{parity}, \delta)$. For the parity condition, we shall only use the priorities $\{0, 1\}$. The remaining components of $H_Q$ are specified in the following.

The state set $S$. Let $U_S$ be the finite set of constants that is used for arguments in $\Gamma_S$. The state set $S$ consists of all atomic formulas $R(a_1, \ldots, a_k)$, where $R/n \in \mathcal{O}$ and $a_1, \ldots, a_k \in U_S$. We set the initial state $s_0$ to equal $G$. For technical reasons, we include an additional sink state denoted sink.

The transition function $\delta$. We define $\delta$ as follows. Consider a symbol $\rho \in \Gamma_S$. Firstly, we set

$$\delta(\text{sink}, \rho) = (\langle 0 \rangle)(\text{sink}, \varepsilon, 0).$$

Secondly, let $R(a_1, \ldots, a_k)$ be a state different from sink. We set $\bar{a} = a_1, \ldots, a_k$ and distinguish cases:

- **(C1)** If $\{a_1, \ldots, a_k\} \not\subseteq \text{names}(\rho)$ then

  $$\delta(R(\bar{a}), \rho) = (\langle 0 \rangle)(R(\bar{a}), \not\in c, 1).$$

  In this case, Eve will lose the game as she loops in this state $R(\bar{a})$ while incrementing the counter and producing an infinite run whose maximum priority that occurs infinitely often (i.e., the priority 1) is odd.

- **(C2)** Otherwise, if $\rho \models R(\bar{a})$ then

  $$\delta(R(\bar{a}), \rho) = (\langle 0 \rangle)(\varepsilon, 1).$$

  In this case, Eve will win the game as she first changes to the sink state and then she loops in this state while not increasing the counter. In the sink state, she produces an infinite play whose maximum priority that occurs infinitely often (the priority 0) is even.

- **(C3)** Otherwise, let

  $$\tau_1 : \alpha_{1,1} \land \cdots \land \alpha_{1,m_1} \ldots \tau_l : \alpha_{l,1} \land \cdots \land \alpha_{l,m_l}$$

  enumerate all guarded conjunctions of atomic facts from $S$ such that $(\{\alpha_{i,1}, \ldots, \alpha_{i,m_i}\}, \mathcal{O}) \models R(\bar{a})$, for all $i = 1, \ldots, l$. We let

  $$\delta(R(\bar{a}), \rho) = \bigvee_{i=1}^l \bigwedge_{j=1}^{m_i} (\langle 0 \rangle)(\alpha_{i,j}, \not\in c, 1) \lor (\langle \top \rangle)(R(\bar{a}), \varepsilon, 1).$$

  Eve may choose between two possibilities here. Either she moves to some neighboring node in the tree while remaining in state $R(\bar{a})$, or she may decide to pick a guarded conjunction $\tau_i : \alpha_{i,1} \land \cdots \land \alpha_{i,m_i}$. In the latter case, Adam challenges Eve’s choice by changing the state to one of the $\alpha_{i,j}$ while incrementing the counter. Notice that this case corresponds to the unfolding of a (series of) rules and thus to the built-up of a derivation tree.

This completes the construction of $H_Q$. We briefly comment on the size of $H_Q$ and the time required to construct the same.

It is clear that the number of states of $H_Q$ is exponential in $\omega(d(S))$ and linear in $|\mathcal{O} \cup \text{sig}(\mathcal{O})|$. Moreover, the overall construction of $H_Q$ takes double exponential time in the size of $Q$. The determining factor for this upper bound is the construction of $\delta(\cdot, \cdot)$; more specifically, the case of condition (C3). Up to logical equivalence, there are at most double exponentially many conjunctions of the form $\tau_1 : \alpha_{1,1} \land \cdots \land \alpha_{1,m_1}$ that imply a given atomic fact under $O$ and $\tau_i$ is of at most exponential size. Moreover, checking whether an atomic fact is implied by a database and a set of guarded rules is feasible in 2EXPSPACE in combined complexity, and in PTIME in data complexity. Therefore, the transition function can in total be constructed in 2EXPSPACE.

It remains to be shown that $H_Q$ is correct, that is:

**Lemma 21.** Suppose $t$ is a consistent $\Gamma_S$-labeled tree. Eve has an $n$-winning strategy in $G[H_Q, t]$ if there is a derivation tree $T$ for $\langle t \rangle$ and $Q$ of height at most $n$.

**Proof (sketch).** Suppose first that there is such a $T$ of height $n_0 \leq n$. We can assume without loss of generality that only the leaf nodes of $T$ of the form $\beta(\bar{a})$ satisfy $\langle t \rangle \models \beta(\bar{a})$; otherwise, we can simply truncate $T$.

Our strategy for Eve will be chosen such that any atomic formula $R(a_1, \ldots, a_k)$ that appears as a state in a play consistent with that strategy occurs in the label of some node of $T$. This is trivially satisfied for the initial state, since the root of $T$ is labeled with $G$. Suppose now that the game is at position $(\chi, v)$, where $v \in \text{dom}(t)$ is a node of the input tree $t$ and $\chi$ is of the form $\langle 0 \rangle(\beta(\bar{a}), \not\in c, 1)$, with $\beta(\bar{a})$ a state of $H_Q$. Eve’s task is to show that she can either match the atom $\beta(\bar{a})$ to the input tree, or she proceeds according to (C3) by finding a guarded conjunction of facts that imply $\beta(\bar{a})$ under $O$. Eve thus proceeds as follows:

- If $t(v) \models \beta(\bar{a})$ then Eve proceeds according to (C2). In fact, in this case she has no other choice to do so and, as explained in the definition of (C2), such a play will be winning for Eve.

- Otherwise, it must be the case that there is a non-leaf node in $T$ with a label $\beta(\bar{a})$. Suppose the children of that node are labeled with $\alpha_1(\bar{a}_1), \ldots, \alpha_k(\bar{a}_k)$. As said, we can restrict ourselves to the case where the conjunction $\alpha_1(\bar{a}_1) \land \cdots \land \alpha_k(\bar{a}_k)$ is guarded and thus, say, $\alpha_1(\bar{a}_1)$ is its guard. Eve first navigates to a node $w \in \text{dom}(t)$ of the input tree whose names comprise all of $\bar{a}_1$ while remaining in state $\beta(\bar{a})$. Notice that this is possible, since all of $\bar{a}$ are contained as names in the nodes on the unique path between $v$ and $w$. When Eve arrives at node $w$, she chooses to challenge Adam by selecting the conjunction $\langle 0 \rangle(\alpha_1(\bar{a}_1), \not\in c, 1) \land \cdots \land \langle 0 \rangle(\alpha_k(\bar{a}_k), \not\in c, 1)$. Adam then selects an arbitrary con-
junct $\langle 0 \rangle (\alpha_i(\bar{a}_i), 1, c, 1)$ and the game continues from the accordin
g position (it is then again Eve’s turn).

It is easy to see that the choices of Eve that are dictated by $T$ lead to an infinite play that satis
the parity condition. Concerning the counter, it is only increased when either Eve chooses to “unf
a rule according to (C3). Moreover, the counter is not increased when she navigates between no
d in the input tree. Therefore, any play consistent with Eve’s strategy has cost at most $n_0 \leq n$.

Conversely, suppose now that Eve has an $n$-winning strategy $\xi$ in $G(H_Q, t)$. Let $\pi$ be a play of maximum cost that is c
consistent with Eve’s strategy. We shall construct a derivation t

We construct a derivation tree $T$ for $\overline{\xi}$ and $Q$ whose height is bounded by $n$. Let $\pi = (x_0, v_0), (x_1, v_1), \ldots, (x_k, v_k), \ldots$ be a play consistent with $\xi$. Consider the sequence $\bar{\chi} = x_0, x_1, \ldots$ and let $\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots$ enumerate the states that appear in atomic formulas in $\bar{\chi}$ such that $\alpha_k$ appears in $\bar{\chi}$ before $\alpha_{k+1}$. That is, $\alpha_0 = G$ and each $\alpha_i (i \geq 1)$ is ei
the sink state or results from a challenge by Adam according to (C3). Notice that, by construction of $H_Q$, if $\alpha_i = \text{sink}$ then $\alpha_j =\text{sink}$ for all $j \geq i$.

We inductively construct a sequence of trees $T_0, T_1, T_2, \ldots$ such that $T_{k+1}$ extends $T_k$ and there is an $m \geq 0$ such that $T_m = T_k$ for all $k \geq m$. $T_m$ will be a tree that can be extended to derivation tree for $\overline{\xi}$ and $Q$. Let $T_0$ be the tree with only the root node that is labeled with $G$. Assume that $T_k$ has been constructed. $T_{k+1}$ is defined according to $\pi$:

- If $\alpha_{k+1} = \text{sink}$ or $\alpha_{k+1} = \alpha_k$ then we set $T_{k+1} = T_k$.
- If $\alpha_{k+1}$ is an atomic fact chosen by Adam according to (C3) in response to Eve’s choice of a guarded conjunct $\beta_1 \land \cdots \land \beta_l$ that imply an atom $\alpha_i (i \leq k)$, then $T_{k+1}$ is obtained from $T_k$ by adding children $v_1, \ldots, v_l$ with the respective labels $\beta_1, \ldots, \beta_l$ to a leaf node $v$ of $T_k$ whose label is an atomic fact of the form $\alpha_i$. Notice that such a leaf node exists, since the state $\alpha_{k+1}$ can only be assumed via the existence of such an $\alpha_i$.

Observe that $\pi$ must assume the sink state at some point, since $\pi$ is winning for Eve and, by construction, it loops in this state (otherwise the parity condition would not be satisfied). Thus, there is an $m \geq 0$ such that $T_m = T_k$ for all $k \geq m$. It remains to be shown that $T_m$ can be extended to a derivation tree $T$ for $\overline{\xi}$ and $Q$. Recall that we chose $\pi$ to be a play consistent to Eve’s strategy that is of maximal cost. Clearly, the cost of $T$ equals the height of $T_m$. Roughly, $T_m$ consists of a finite branch starting at the root (which is labeled with $G$) whose leaf node is labeled by an atom from $\overline{\xi}$. Moreover, each branch may have children that may not be database atoms of $\overline{\xi}$. We can, however, easily check that we can attach subtrees to these “incomplete” nodes such that the resulting tree $T$ becomes a derivation tree for $\overline{\xi}$ and $Q$. If this was not possible, Adam could find a play which forces Eve to lose. Moreover, the height of $T_m$ must equal the height of $T_m$, since otherwise Adam could find a play that has higher cost than $\pi$, which is impossible due to our choice of $\pi$. Notice that, by construction, the cost of $\pi$ equals the height of $T$ and $\text{hgt}(T) \leq n$.

Hence, by Lemma [21] we know that, for any consistent $\Gamma_S$-
labeled tree $t$, $[H_Q](t) = n$ if and only if $n$ is the minimal $n_0$ such that there is a derivation tree of height $n_0$ for $\overline{\xi}$ and $Q$. Therefore, for all $n \in \mathbb{N}$, $[H_Q](t) = n$ iff $\text{cost}(\overline{\xi}) = n$. We thus obtain that $[H_Q]$ is bounded iff $\text{cost}(Q)$ is finite. This concludes the proof of Lemma [22].

D Proofs for Section 6

In this section, we also consider CQs that contain equality atoms of the form $x = y$ in their bodies. Notice that, in non-empty CQs, these can always be removed by appropriately identifying variables. We allow such atoms, since the results we rely upon explicitly make use of such atoms.

We say that a CQ $q(\bar{x})$ is answer-guarded if it contains an atom in its body that has all answer variables of $q(\bar{x})$ as arguments. Notice that every (non-empty) Boolean CQ is trivially answer-guarded. Also notice that the body of any frontier-guarded rule can be seen as an answer-guarded CQ.

D.1 Preliminaries: Treeification

Strictly acyclic queries. Let $q(\bar{x})$ be an answer-guarded CQ over a schema $S$. We say that $q(\bar{x})$ is cyclic, if there is a tree decomposition $\delta = (T, (X_v)_{v \in T})$ of $q(\bar{x})$ such that, for all $v \in T$, there is an atom $\alpha$ of $q(\bar{x})$ such that $X_v \subseteq \text{var}(\alpha)$. If there is such a $\delta$ that, in addition, has a bag containing all answer variables of $q(\bar{x})$, then we say that $q(\bar{x})$ is strictly acyclic.

A guarded formula is a first-order formula where each occurrence of a quantifier is of either forms

$$\forall \bar{y}(\alpha(\bar{x}, \bar{y}) \rightarrow \psi) \quad \text{or} \quad \exists \bar{y}(\alpha(\bar{x}, \bar{y}) \land \psi),$$

where $\alpha$ is an atomic formula, called guard, and all the free variables of $\psi$ (denoted $\text{free}(\psi)$) are contained in $\bar{x} \cup \bar{y}$. We also permit that the guard is an equality atom of the form $x = y$. In the following, we are interested in guarded formulas that contain only existential quantification and conjunction, and we restrict ourselves to those in the remainder of this paper.

We say that a guarded formula $\varphi(\bar{x})$ is strictly guarded, if it is of the form $\exists \bar{y}(\alpha(\bar{x}, \bar{y}) \land \psi)$, i.e., all free variables in a strictly guarded formula are covered by an atom as well. It is well-known that every strictly acyclic formula is equivalent to a strictly guarded formula and vice versa (see [21]).

Treeifying CQs. Given an answer-guarded CQ $q(\bar{x})$ over $S$ and a schema $T \supseteq S$, the $T$-treeification of $q(\bar{x})$ is the set $\Lambda^T_q$ of all strictly acyclic CQs $q'(\bar{x})$ over $T$ such that (i) $q'$ is contained in $q$, in symbols $q' \subseteq q$, that is, for any $T$-database $D$, if $D \models q'$ then also $D \models q$, and (ii) $q'$ is minimal in the sense that removing one atom from $q'$ turns $q'$ into a CQ that is either not strictly acyclic or that is not contained in $q$ anymore.

It can be shown that all the CQs contained in $\Lambda^T_q$ can be restricted as to contain only CQs of size at most $3|q|$, where $|q|$ denotes the number of atoms in $q$. Hence, $\Lambda^T_q$ can be seen as a UCQ that is of exponential size in the size of $q$. Notice that $q(\bar{x})$ is in general not equivalent to its treeification. However, $q(\bar{x})$ and $\Lambda^T_q$ are equivalent over acyclic $T$-databases (acyclicity for databases is defined as for CQs) [Bárany et al., 2015].

\footnote{Note that we can view the empty CQ, denoted $\top$, also as answer-guarded since it is equivalent to $\exists x \ x = x$.}
Treeifying OMQs from (FG, AQ0). Let $Q = \langle S, \mathcal{O}, G \rangle$ from (FG, AQ0). The width of $\mathcal{O}$, denoted wd($\mathcal{O}$), is the maximum number of variables that appear in any body of a rule from $\mathcal{O}$. Fix a new relation symbol $C$ of arity wd($\mathcal{O}$).

We are now going to describe a translation $\eta_C(Q)$ in full detail that takes $Q$ and transforms it into an OMQ $\eta_C(Q)$ from (G, AQ0) with data schema $S \cup \{C\}$. Firstly, we set

$$\eta_C(Q) = \left( S \cup \{C\}, \bigcup_{\tau \in \mathcal{O}} \eta^{\mathcal{O} \setminus \{C\}}(\tau), G \right),$$

where the definition of $\eta^{\mathcal{O} \setminus \{C\}}(\tau)$, for $\tau \in \mathcal{O}$ and a schema $T$, is as follows. Suppose $\tau$ is of the form $\varphi(x, \bar{z}) \rightarrow \exists \bar{y} \beta(x, \bar{y})$. Then we set

$$\eta^{\mathcal{O} \setminus \{C\}}(\tau) = \{ q(x) \rightarrow \exists \bar{y} \beta(x, \bar{y}) | q(x) \in \varphi(x, \bar{z}) \} \cup \bigcup_{\tau \in \mathcal{O}} \eta^{\mathcal{O} \setminus \{C\}}(\tau).$$

Notice that here we somehow did not pedantically follow the detail that takes $Q$ and transforms it into an OMQ $\eta_C(Q)$ from (G, AQ0) with data schema $S \cup \{C\}$. We shall treat this predicates modulo logical equivalence, i.e., we set $T_{\eta_1} = T_{\eta_2}$ iff $\eta_1 \equiv \eta_2$. We unfold the query $q(x)$ inductively according to the construction of $\eta_C(Q)$.

Suppose first that $\chi(x) \equiv \beta(x)$ for some relational atom $\beta(x)$. We translate $q(x)$ to the rule

$$\beta(x) \rightarrow T_{\chi(x)}(x).$$

Suppose now that $\chi(x) \equiv \exists \bar{y} \gamma(x, \bar{y}) \wedge \eta$, where free($\eta$) $\subseteq x \cup \bar{y}$ and $\gamma(x, \bar{y})$ is a relational atom. Let $\eta \equiv \eta(x_1), \ldots, \eta_k$. We translate $q(x)$ to the rules $q(x)$ to the rule

$$\gamma(x, \bar{y}) \rightarrow T_{\eta} (x_1), \ldots, T_{\eta_k}(x_k) \rightarrow T_{\chi(x)}(x),$$

and, in addition, add the according translations for the formulas $\eta(x_1), \ldots, \eta_k(x_k)$.

The unfolding of the rule $q(x) \rightarrow \exists \bar{y} \beta(x, \bar{y})$ is then the set of rules resulting from translating $\chi(x)$ plus the rule

$$T_{\chi(x)}(x) \rightarrow \exists \bar{y} \beta(x, \bar{y}).$$

As mentioned above, we set

$$\eta^{\mathcal{O} \setminus \{C\}}(\tau) = \{ \sigma | \sigma \text{ is a rule contained in the unfolding of } \tau \}.$$
Suppose first that $D \models q$. Then also $D \models Q$ and, by Lemma 23, there is an $S$-database $D^*$ of tree-width at most $w$ such that $D^* \models Q$ and $D^*$ maps to $D$. Thus, also $D^* \models q$. Since $D^*$ has tree-width at most $w$, there is a disjunct $p$ in $q'$ and a $D^* \subseteq D^*$ such that $p$ is homomorphically equivalent to $D'$. Thus $D' \models p$ and so $D^* \models p$. Since $D^*$ maps homomorphically to $D$, we have $D \models p$ and so $D \models q'$. \square

Suppose now that $D \models q'$, i.e., $D \models p$ for some disjunct $p$ of $q'$. Since $p \subseteq L_i$ for some $i = 1, \ldots, n$, there is a homomorphism from $p_i$ to $p$. Hence, it follows that $D \models p_i$ and thus $D \models q$. \square

Suppose now that each $p_i (i = 1, \ldots, n)$ has tree-width at most $w$. Let $\delta_i = (T_i, (X_i,v)_{v \in T_i})$ be a tree decomposition of $p_i$ of width at most $w$. A variant of $p_i$ is a CQ $p$ over $(S \cup \{C\})$ that (i) results from $p_i$ by adding a set of atoms of the form $C(x_0, \ldots, x_w)$ with $\{x_0, \ldots, x_w\} \subseteq X_i, v$ for some $v \in T_i$, and (ii) is acyclic. Let $p'_i$ be the UCQ over $(S \cup \{C\})$ that contains a disjunct for each variant of $p_i$. Moreover, let $q' = \bigvee_{i=1}^n p'_i$ and assume again that $q'$ contains no two distinct disjuncts that are homomorphically equivalent. Obviously, $q'$ is a finite UCQ, and we claim that $q'$ is a UCQ-rewriting of $\eta_{\text{CQ}}(Q)$.

Indeed, suppose $D$ is an $(S \cup \{C\})$-database such that $D \models \eta_{\text{CQ}}(Q)$. According to Lemma 24, there is an acyclic $(S \cup \{C\})$-database $D^*$ such that $D^* \models \eta_{\text{CQ}}(Q)$ and such that $D^*$ maps homomorphically to $D$. By Lemma 22, we have $D^* \models Q$ as well. Let $D^*[S]$ denote the database $D$ restricted to $S$. Since $C$ does not appear in $Q$, we must have $D^*[S] \models Q$ as well and so $D^*[S] \models p_i$ for some $i = 1, \ldots, n$. It is now easy to check that there is a variant $p'_i$ of $p_i$ such that $D^* \models p'_i$. Hence, $D^* \models q'$ and so $D \models q'$ as required.

Conversely, suppose that $D \models q'$, i.e., $D \models p$ for some CQ $p$ that is a variant of some $p_i$. By Lemma 24, there is an acyclic $(S \cup \{C\})$-database $D^*$ such that $D^* \models p$ and $D^*$ homomorphically maps to $D$. Obviously, there is a homomorphism from $p_i$ to $p$, since $p$ is a variant and results from $p_i$ just by adding atoms. Hence, also $D^* \models p_i$ and thus $D^* \models q$ follows. Then we obtain $D^* \models Q$ and by Lemma 22 also $D^* \models \eta_{\text{CQ}}(Q)$. Since $\eta_{\text{CQ}}(Q)$ is closed under homomorphisms, $D \models \eta_{\text{CQ}}(Q)$ follows.

Suppose now that $\eta_{\text{CQ}}(Q)$ is FO-rewrutable and let $q = \bigvee_{i=1}^n p_i$ be a UCQ equivalent to $\eta_{\text{CQ}}(Q)$. We show that $Q$ is FO-rewrutable as well. In this case, we can assume that $q$ is actually a disjunction of acyclic CQs, a proof of this fact can be obtained similarly to the claim above and is left to the reader. Let $p'_i$ be the CQ that results from $p_i$ by dropping all atoms of the form $C(x_0, \ldots, x_w)$. Moreover, let $q' = \bigvee_{i=1}^n p'_i$. We claim that $q'$ is a UCQ equivalent to $Q$.

Suppose first that $D \models Q$. By Lemma 22, there is an $S$-database $D^*$ of tree-width at most $w$ such that $D^* \models Q$ and $D^*$ homomorphically maps to $D$. Fix a tree decomposition $\delta^* = (T, (X,v)_{v \in T})$ of $D^*$. We can turn $D^*$ into an acyclic $(S \cup \{C\})$-database by adding to $D^*$ all facts of the form $C(a_0, \ldots, a_w)$ such that $\{a_0, \ldots, a_w\} \subseteq X_v$ for some $v \in T$. Call the resulting database $D'$. Obviously, $D^* \models Q$ and since $D^*$ is acyclic, we obtain $D \models \eta_{\text{CQ}}(Q)$ by Lemma 22. Therefore, $D' \models p_i$ for some $i = 1, \ldots, n$. Since $p'_i$ contains no atoms of the form $C(x_0, \ldots, x_w)$, it follows that $D^* \models p'_i$ as well and so $D^* \models p'_i$. Thus $D \models p'_i$ and so $D \models q'$ as required.

Conversely, suppose now that $D \models q'$, i.e., $D \models p'_i$ for some $i = 1, \ldots, n$. Notice that $p'_i$ has tree-width at most $w$ by construction. Using Lemma 23, we infer that there is an $S$-database $D^*$ such that $D^* \models p'_i$ and such that there is a homomorphism $h$ from $D^*$ to $D$. Now fix a tree decomposition $\delta_i = (T_i, (X_i,v)_{v \in T_i})$ of $p'_i$ of width at most $w$. We can see $\delta_i$ also as a tree decomposition of $p_i$ that witnesses that $p_i$ is acyclic. Now we extend $D^*$ to an $(S \cup \{C\})$-database as follows. Suppose $C(x_0, \ldots, x_w)$ occurs in $p_i$ but has been deleted from $p'_i$. Then $\{x_0, \ldots, x_w\} \subseteq X_v$ for some $v \in T$. We can assume w.l.o.g. that $\text{dom}(h) \cap \{x_0, \ldots, x_w\} \neq \emptyset$; otherwise we can drop that atom from $p_i$. Pick a $y \in \text{dom}(h) \cap \{x_0, \ldots, x_w\}$. Now we add to $D^*$ the atom $C(a_0, \ldots, a_w)$, where $a_i = h(x_i)$ if $x_i \in \text{dom}(h)$ and $a_i = h(y)$ otherwise. We repeat this construction for all occurrences of an atom of the form $C(x_0, \ldots, x_w)$ in $p_i$. Call the resulting database $D'$. It is clear that $h$ is a homomorphism from $p_i$ to $D'$. Notice also that $D'[S] = D^*$. Now since $D'_i \models p_i$, we must have $D'_i \models \eta_{\text{CQ}}(Q)$ and so by Lemma 24, there is an acyclic $(S \cup \{C\})$-database $D''$ such that $D'' \models \eta_{\text{CQ}}(Q)$ and $D''$ homomorphically maps to $D'$. By Lemma 22, we have $D'' \models Q$ as well and hence also $D \models Q$. But $D[S] = D^*$, whence $D \models Q$ follows since $C$ does not occur in $Q$. Since $D^*$ homomorphically maps to $D$, we obtain $D \models Q$ as required. \square

D.2 Proof of Theorem 13

Lower bounds. Since, according to [Bienvenu et al., 2016], FO-rewrutability for the class $(\mathcal{EL}, \mathcal{B}CQ)$ is already hard for 2EXPTIME according to [Bienvenu et al., 2016], the following hardness results follow immediately:

- 2EXPTIME-hardness for $\text{FORew}(C, Q)$ with $C \in \{G, FG\}$ and $Q \in \{\text{CQ}, \text{UCQ}\}$.
- 2EXPTIME-hardness for $\text{FORew}(FG, \text{AQO}_Q)$.

Moreover, in [Bienvenu et al., 2013], it is shown that $\text{FORew}(\mathcal{EL}, \text{AQO})$ is EXPTIME-hard. Therefore, for OMQs of bounded arity, EXPTIME-hardness for $\text{FORew}(G, \text{AQO}_Q)$ follows.

The only missing lower bound is therefore the 2EXPTIME lower bound for $\text{FORew}(G, \text{AQO}_Q)$.

Let $Q_1$ and $Q_2$ be Boolean OMQs with data schema $S$. We say that $Q_1$ is contained in $Q_2$, if $D \models Q_1$ implies $D \models Q_2$ for every $S$-database $D$. We are going to use the following result which is implicit in [Barceló et al., 2014]:

Theorem 26. The problem of deciding whether a OMQ $Q_1 = (S, O, G_1)$ from $(G, \text{AQO}_Q)$ is contained in an OMQ $Q_2 = (S, O, G_2)$ is hard for 2EXPTIME. This is true even for the case where $Q_2$ is FO-rewrutable.

Remark. In [Barceló et al., 2014], a slightly different statement is proved. The authors prove in fact that deciding whether a guarded Datalog program is contained in a Boolean acyclic UCQ is hard for 2EXPTIME. Guarded Datalog can easily be seen as a fragment of $(G, \text{AQO}_Q)$. Moreover, a Boolean acyclic UCQ can easily be written as an OMQ from
(G, AQ0) (cf. the discussion of “unfolding” strictly acyclic queries in the definition of treeifications).

To prove that FORew(G, AQ0) is hard for 2ExpTIME, we are going to reduce the problem mentioned in Theorem 26 to FORew(G, AQ0).

Let Q1 = (S, O1, G1) and Q2 = (S, O2, G2) be as in the hypothesis of Theorem 26. Without loss of generality, we may assume that the predicates Q1 and Q2 use and that do not appear in S are distinct. We are going to construct an OMQ Q′ that falls in (G, AQ0) such that Q′ is FO-rewritable if Q1 is contained in Q2.

Let Q′ = (S, O′, G2), where

- S′ = S ∪ {R/2, A/1, B/1};
- O′ is the union of O1 and O2 plus the rules

\[ R(x, y), A(y) \rightarrow A(x), \]
\[ A(x), B(x), G1 \rightarrow G2. \]

Notice that G2 is also the query component of Q′.

**Lemma 27.** Q1 is contained in Q2 iff Q′ is FO-rewritable.

**Proof.** Assume first that Q1 is not contained in Q2. Then there is an S′-database D such that D ⊨ Q1 and D ⊭ Q2. By Lemma 14 there is a D* of tree-width at most max{0, \( wd(S) - 1 \)} such that D* ⊨ Q1. Moreover, there also is a homomorphism from D* to D. Since Q2 is closed under homomorphisms, we must also have D* ⊭ Q2. For each k > 0, let Dk be the S′-database extending D* with the facts

\[ B(a_0), R(a_0, a_1), \ldots, R(a_{k-1}, a_k), A(a_k), \]

where a0, ..., ak do not occur in \( \text{adm}(D^*) \). It is easy to check that Dk ⊨ Q′ for all k > 0. Moreover, no proper subset of Dk satisfies Q′. By virtue of Theorem 11 Q′ is thus not FO-rewritable.

Conversely, suppose that Q1 is contained in Q2. Recall that Q2 is FO-rewritable and, therefore, there is a UCQ \( \varphi \) over S that is equivalent to Q2. We claim that q is a UCQ-rewriting for Q′ as well.

Indeed, suppose first that D \models q for some S′-database S. Since q uses only symbols from S, we obtain that D[S] \models q as well. Since q is equivalent to Q2, we get D[S] \models Q2 and, by construction of Q′, so D \models Q′.

Suppose now that D \models Q′ for some S′-database D. By construction of Q′, we must then have D \models Q2 or D \models Q1. In the former case, we are done since Q2 and q are equivalent. In the latter case, we get D[S] \models Q1 whence D[S] \models Q2 since Q1 is contained in Q2. Therefore also D[S] \models q and thus D \models q. This proves the claim.

It is clear that Q′ can be constructed from Q1 and Q2 in polynomial time. Therefore, 2ExpTIME-hardness for FORew(G, AQ0) follows by Lemma 27.

**Upper bounds.** We shall now prove that FORew(FG, UBCQ) is in 2ExpTIME. Following a similar result for description logics in [Bienvenu et al., 2016], we first show that we can focus on Boolean UCQs:

**Lemma 28.** Let C ∈ {FG, G}. Then FORew(C, UCQ) can be reduced in polynomial time to FORew(C, UBCQ).

**Proof (sketch).** Let Q = (S, O, q(\( \bar{x} \))) be an OMQ from (C, UCQ) with \( \bar{x} = x_1, \ldots, x_n \). We let S′ = S ∪ \( \{ A_1, \ldots, A_n \} \), where A1, ..., An are new unary predicates. Let q′(\( \bar{x} \)) be the UCQ that results from q(\( \bar{x} \)) by adding the conjunction \( A_1(x_1) \land \cdots \land A_n(x_n) \) to every disjunct of q(\( \bar{x} \)). Let Q′ = (S′, O, \( \exists \bar{x} q'(\( \bar{x} \)) \)). It is not hard to check that Q is FO-rewritable iff Q′ is.

Indeed, if \( \varphi_Q(x_1, \ldots, x_n) \) is an FO-rewriting of Q, then \( \exists x_1, \ldots, x_n (\varphi_Q(x_1, \ldots, x_n) \land A_1(x_1) \land \cdots \land A_n(x_n)) \) is one of Q′.

Conversely, if Q′ is FO-rewritable then there is a Boolean UCQ p′ that is equivalent to Q′. Now, for any S′-database D, D \models q′ iff there are a1, ..., an ∈ \( \text{adm}(D) \) such that \( A_1(a_1), \ldots, A_n(a_n) \in D \) and D \models (\( A_1(a_1) \land \cdots \land A_n(a_n) \)). Let p be the UCQ that results from p′ by removing all occurrences of \( A_i(x_i) \) and the associated existential quantifier \( \exists x_1 \). It is easy to see that p is a UCQ-rewriting of Q. \( \square \)

Now consider an OMQ Q = (S, O, q) from (FG, UBCQ). In a first step, we transform Q into an equivalent OMQ Q′ that falls in (FG, AQ0). This is easy: we simply choose a fresh predicate C of arity zero and add to O the rules p → G for every disjunct p of q. Notice that O is still frontier-guarded, since q is Boolean. Call the resulting ontology O′, i.e., Q′ = (S, O′, G).

Now we choose a fresh predicate C of arity \( wd(O′) \). We then construct the OMQ QC(Q′) that has data schema \( S \cup \{ C \} \). This translation takes exponential time, and the ontology of QC(Q′) may be of exponential size. However, as already mentioned in the main body of the paper, the arity of each predicate occurring in QC(Q′) is at most \( wd(O′) \).

The OMQ QC(Q′) falls in (G, AQ0). We can, according to Theorem 9, decide FO-rewritability for that class in 2ExpTIME, with a double exponential dependence only on the width of the data schema. Since \( wd(S \cup \{ C \}) = wd(O′) \), it follows that FO-rewritability of QC(Q′) can be decided in 2ExpTIME, where the second exponent of the run-time depends on \( wd(O′) \) only. Hence, we can decide whether QC(Q′) if FO-rewritable in 2ExpTIME. Given that the construction of QC(Q′) (starting with Q) is, of course, also feasibly in 2ExpTIME, the fact that FORew(FG, UBCQ) is in 2ExpTIME follows by Lemma 25. Using Lemma 28 we obtain that FORew(FG, UCQ) is in 2ExpTIME as well.