Some bounds for quantum copying

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We propose new optimality criterion for the estimation of state-dependent cloning. We call this measure the relative error because the one compares the errors in the copies with contiguous size taking into account the similarity of states to be copied. A copying transformation and dimension of state space are not specified. Only the unitarity of quantum mechanical transformations is used. The presented approach is based on the notion of the angle between two states. Firstly, several useful statements simply expressed in terms of angles are proved. Among them there are the spherical triangle inequality and the inequality establishing the upper bound on the modulus of difference between probability distributions generated by two any states for an arbitrary measurement. The tightest lower bound on the relative error is then obtained. Hillery and Bužek originally examined an approximate state-dependent copying and obtained the lower bound on the absolute error. We consider relationship between the size of error and the corresponding probability distributions and obtain the tightest lower bound on the absolute error. Thus, the proposed approach supplements and reinforces the results obtained by Hillery and Bužek. Finally, the basic findings of investigation for the relative error are discussed.

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1. INTRODUCTION

One of the fundamental distinctions of the quantum world from the classical world is expressed by the no-cloning theorem \([1]\). This result was generalized and extended in paper \([2]\). There are many applications of this statement, for example, to the quantum cryptography. In some protocols Alice and Bob encode the bits 1 and 0 into two non-orthogonal pure states \([3]\). Eve, a paradigmatical eavesdropper, then has the following problem. Particle 1 (original mode) is secretly prepared in some state from a set \(\mathfrak{A} = \{\ket{\phi}, \ket{\psi}\}\) of two states. Particle 2 (copy mode) is in some standard state \(\ket{0}\). Eve’s action is to allow subsystem ”1+2” to interact unitarily with auxiliary system CM (copying machine) in some standard state \(\ket{m}\). Eve tries to implement a unitary process

\[
\forall |s\rangle \in \mathfrak{A}: \quad U |s\rangle \otimes |0\rangle \otimes |m\rangle = |s\rangle \otimes |s\rangle \otimes |\Phi^{(s)}\rangle.
\]

But the no-cloning theorem forbids such a ”ideal copying” (except when states \(\ket{\phi}\) and \(\ket{\psi}\) are orthogonal or identical). It is clear, Eve will want to make the process output as near to the ideal output as possible. How well Eve can do?

This question was originally considered by Hillery and Bužek in \([4]\). They examined approximate cloning machines destined for copying of prescribed two non-orthogonal states. In paper \([5]\) such devices were called ’state-dependent cloners’. Writers of \([4]\) introduced the notion of ’global fidelity’ and constructed the optimal symmetric state-dependent cloner which optimizes the global fidelity. Of course, the evaluation of quality for one or another of potential cloners is dependent on the used measure of ’closeness’ to the ideality. As is readily seen from the text, papers \([4]\) and \([5]\) are based on the distinct optimality criteria. Measure used by Hillery and Bužek can be named ’absolute error’ of copying of two-state set. In work \([4]\) the global fidelity was maximized.

However, a degree of similarity for states those are subject to cloning is taken into account neither by the absolute error nor by the global fidelity. It is small wonder, because these measures are simply related. Meanwhile, the consideration of similarity for states to be copied may have practical sense. In general, it would be useful to have some set of optimality criteria every of those elucidates one or another of facets of the state-dependent cloning. In this work we propose the new criterion for estimation of quality for the state-dependent cloning. We call this criterion the relative error because the one compares the errors in the copies with contiguous size that is not independent of degree of similarity for states to be copied. The lower bound on the relative error is obtained. In our analysis we do not specify the dimension of particle state space, although with applied viewpoint the two-dimensional case is most interesting. We also supplement and reinforce the results obtained by Hillery and Bužek \([4]\). In particular, the relationship between the size of error and the absolute value of the deviation of the resulting probability distribution from the desired probability distribution is shown, as well as stronger bound is obtained.

In the examination all the state vectors are normalized to unity. In natura non-unit vectors will sometimes occur, and this cases will be expressly stated. As is customary, the norm of the vector \(\ket{\Phi}\) is defined as \(\|\Phi\| = (\langle \Phi | \Phi \rangle)^{1/2}\).
II. PRELIMINARY LEMMAS

For comparing states $|\Phi\rangle$ and $|\Psi\rangle$ we shall use the angle between these vectors. Many relations can be naturally expressed in terms of angles. In addition, the calculations are simplified by the use of angles.

**Definition 1.** Angle $\delta(\Phi, \Psi) \in [0; \pi/2]$ between two unit vectors $|\Phi\rangle$ and $|\Psi\rangle$ is defined as

$$\delta(\Phi, \Psi) \overset{\text{def}}{=} \arccos\left(|\langle \Phi | \Psi \rangle|\right).$$

This definition is naturally extended on the case of non-unit vectors. It is obvious that function $\delta(\varnothing, \varnothing)$ is symmetric. For brevity we will also often write $\delta_{\Phi, \Psi}$. Expression $\delta_{\Phi, \Psi} = 0$ is equivalent to the identity of states $|\Phi\rangle$ and $|\Psi\rangle$. Let us now prove two statements connected with each other. These relations can be useful in various contexts.

**Lemma 1.** For any triplet $\{|\Phi\rangle, |\Upsilon\rangle, |\Psi\rangle\}$ of unit vectors,

$$\cos \delta_{\Phi, \Psi} \leq \cos (\delta_{\Phi \Upsilon} - \delta_{\Upsilon \Psi}),$$

with equality only if the triplet is coplanar.

**Proof.** It is sufficient to consider the case, in which the triplet has no identical states. Let vectors $|\Theta\rangle$ and $|\Omega\rangle$ be such that $|\Theta\rangle \perp |\Upsilon\rangle$ and $|\Theta\rangle \in \text{span}\{|\Phi\rangle, |\Upsilon\rangle\}$, $|\Omega\rangle \perp |\Upsilon\rangle$ and $|\Omega\rangle \in \text{span}\{|\Psi\rangle, |\Upsilon\rangle\}$. These vectors can be always constructed from non-collinear pairs $\{|\Phi\rangle, |\Upsilon\rangle\}$ and $\{|\Psi\rangle, |\Upsilon\rangle\}$ by the Gram–Schmidt orthogonalization:

$$|\Theta\rangle = \left\{|\Phi\rangle - |\Upsilon\rangle \langle \Phi | \Upsilon \rangle\right\}/\sqrt{1 - |\langle \Upsilon | \Phi \rangle|^2},$$

$$|\Omega\rangle = \left\{|\Psi\rangle - |\Upsilon\rangle \langle \Psi | \Upsilon \rangle\right\}/\sqrt{1 - |\langle \Upsilon | \Psi \rangle|^2}.\tag{2.2}$$

Last equalities can be rewritten as

$$|\Phi\rangle = |\Upsilon\rangle \langle \Phi | \Upsilon \rangle + \sin \delta_{\Phi \Upsilon} |\Theta\rangle,$$

$$|\Psi\rangle = |\Upsilon\rangle \langle \Psi | \Upsilon \rangle + \sin \delta_{\Upsilon \Psi} |\Omega\rangle,$$

Applying the triangle inequality for complex numbers to equality

$$\langle \Phi | \Psi \rangle = \langle \Phi | \Upsilon \rangle \langle \Upsilon | \Psi \rangle + \sin \delta_{\Phi \Upsilon} \sin \delta_{\Upsilon \Psi} \langle \Theta | \Omega \rangle$$

and taking into account that in line with the Schwarz inequality $|\langle \Theta | \Omega \rangle| \leq 1$, we then get

$$|\langle \Phi | \Psi \rangle| \leq \cos \delta_{\Phi \Upsilon} \cos \delta_{\Upsilon \Psi} + \sin \delta_{\Phi \Upsilon} \sin \delta_{\Upsilon \Psi} |\Theta |\Omega \rangle |$$

$$\leq \cos (\delta_{\Phi \Upsilon} - \delta_{\Upsilon \Psi}).$$

The maximal value $\cos(\delta_{\Phi \Upsilon} - \delta_{\Upsilon \Psi})$ is reached only if unit vectors $|\Theta\rangle$ and $|\Omega\rangle$ are collinear. The latter together with (2.2) implies that there is linear dependence for the triplet and the triplet is coplanar.

**Lemma 2.** For any triplet $\{|\Phi\rangle, |\Upsilon\rangle, |\Psi\rangle\}$ of unit vectors,

$$\delta(\Phi, \Upsilon) \leq \delta(\Phi, \Psi) + \delta(\Upsilon, \Psi),$$

with equality only if the triplet is coplanar.

**Proof.** *Ex adverso,* let $\delta_{\Phi \Upsilon} > \delta_{\Phi \Psi} + \delta_{\Upsilon \Psi}$. Taking into account the angle range of values, we get $0 \leq \delta_{\Phi \Psi} < \delta_{\Phi \Upsilon} - \delta_{\Upsilon \Psi} \leq \pi/2$ and $\cos \delta_{\Phi \Psi} > \cos(\delta_{\Phi \Upsilon} - \delta_{\Upsilon \Psi})$. But this contradicts to lemma 1, so that (2.3) is true. Equality in (2.3) gives equality in (2.1), that is possible only if the triplet is coplanar.

The last statement is the base of our approach to obtaining of bounds for state-dependent cloning. Inequality (2.3) can be called ‘spherical triangle inequality’. This becomes obvious if to represent unit vectors $|\Phi\rangle$, $|\Upsilon\rangle$ and $|\Psi\rangle$ by three points on the sphere with unit radius. Then quantities $\delta(\Phi, \Upsilon)$, $\delta(\Phi, \Psi)$ and $\delta(\Upsilon, \Psi)$ are the sides of the spherical triangle formed by these points. Equality $\delta_{\Phi \Upsilon} = \delta_{\Phi \Psi} + \delta_{\Upsilon \Psi}$ holds when the spherical triangle is degenerated into arc of great circle and point $\Psi$ is found between points $\Phi$ and $\Upsilon$. In this case triplet $\{|\Phi\rangle, |\Upsilon\rangle, |\Psi\rangle\}$ is coplanar and vector $|\Psi\rangle$ lies between $|\Phi\rangle$ and $|\Upsilon\rangle$.
Below we shall show that if the angle between two states is small, then the probability distributions generated by them for an arbitrary measurement are close to each other. The measurement over the system in state $|S\rangle$ produces result $R$ with probability (p.e., see (2.4))

$$P(R|S) = \langle S|\Pi|S\rangle,$$  

where $\Pi$ is the operator of the orthogonal projection onto the corresponding subspace.

**Lemma 3.** For arbitrary triplet $\{|\Theta\rangle, |\Phi\rangle, |\Psi\rangle\}$ of unit vectors,

$$\left|\left|\langle\Theta|\Phi\rangle\right|^2 - \left|\langle\Theta|\Psi\rangle\right|^2\right| \leq \sin \delta_{\Phi\Psi}. \quad (2.5)$$

**Proof.** Using lemma 1 and standard trigonometric formula (see (2.1))

$$\cos^2 \alpha - \cos^2 \beta = -\sin(\alpha + \beta)\sin(\alpha - \beta),$$

we have

$$\left|\left|\langle\Theta|\Phi\rangle\right|^2 - \left|\langle\Theta|\Psi\rangle\right|^2\right| \leq \cos^2(\delta_{\Phi\Psi} - \delta_{\Phi\Theta}) - \cos^2\delta_{\Phi\Theta} =$$

$$= \sin \delta_{\Phi\Psi} \sin(2\delta_{\Phi\Theta} - \delta_{\Phi\Psi}) \leq \sin \delta_{\Phi\Psi}.$$

We further get by a parallel argument

$$\left|\left|\langle\Theta|\Psi\rangle\right|^2 - \left|\langle\Theta|\Phi\rangle\right|^2\right| \leq \sin \delta_{\Phi\Psi},$$

and the two last inequalities give (2.5). \hfill \Box

**Lemma 4.** For an arbitrary projector $\Pi$ and two any states $|\Phi\rangle$ and $|\Psi\rangle$,

$$\left|\langle\Phi|\Pi|\Phi\rangle - \langle\Psi|\Pi|\Psi\rangle\right| \leq \sin \delta_{\Phi\Psi}. \quad (2.6)$$

**Proof.** It is sufficient to consider the case, in which vectors $\Pi|\Phi\rangle$ and $\Pi|\Psi\rangle$ are non-zero (other cases are reduced to lemma 3). Let us introduce two unit vectors $|\Theta\rangle = \Pi|\Phi\rangle/\|\Pi|\Phi\rangle\|$ and $|\Omega\rangle = \Pi|\Psi\rangle/\|\Pi|\Psi\rangle\|$, which lie in the subspace generated by $\Pi$. We then have

$$\langle\Phi|\Pi|\Phi\rangle = \left|\langle\Theta|\Phi\rangle\right|^2, \quad \langle\Psi|\Pi|\Psi\rangle = \left|\langle\Omega|\Psi\rangle\right|^2. \quad (2.7)$$

We now express $|\Phi\rangle$ and $|\Psi\rangle$ as

$$|\Phi\rangle = |\Theta\rangle\langle\Theta|\Phi\rangle + |\Phi_\perp\rangle, \quad |\Psi\rangle = |\Omega\rangle\langle\Omega|\Psi\rangle + |\Psi_\perp\rangle,$$

where (generally, non-unit) vectors $|\Phi_\perp\rangle$ and $|\Psi_\perp\rangle$ are orthogonal to span$\{|\Theta\rangle, |\Omega\rangle\}$. Because

$$\left|\langle\Theta|\Phi\rangle\right| = \left|\langle\Theta|\Omega\rangle\right| = \left|\langle\Omega|\Psi\rangle\right| \leq \left|\langle\Omega|\Psi\rangle\right|,$$

we get by the use of lemma 3 the following inequality:

$$\left|\left|\langle\Theta|\Phi\rangle\right|^2 - \left|\langle\Omega|\Psi\rangle\right|^2\right| \leq \left|\langle\Theta|\Phi\rangle\right|^2 - \left|\langle\Theta|\Psi\rangle\right|^2 \leq \sin \delta_{\Phi\Psi}.$$

We next have by a parallel argument

$$\left|\left|\langle\Omega|\Psi\rangle\right|^2 - \left|\langle\Theta|\Phi\rangle\right|^2\right| \leq \sin \delta_{\Phi\Psi}.$$

The two last inequalities and relations (2.7) give (2.6) right away. \hfill \Box

Applying (2.4) and lemma 4, we then have the relation

$$\left|\left|P(R|\Phi) - P(R|\Psi)\right|\right| \leq \sin \delta_{\Phi\Psi} \quad (2.8)$$

for an arbitrary measurement and two any states $|\Phi\rangle$ and $|\Psi\rangle$. In accordance with (2.8) small angle between two states implies the closeness of probability distributions generated by these states for any measurement. Thus, the angle between two states gives a reasonable measure of closeness for two pure states.
Addendum we shall apply (3.6) to the quantum circuit model, in which any unitary transformation is approximated by a network formed from universal gates (see, for example, [8]). Let two unitary transformations U and V obey

$$\|U - V\| \leq \varepsilon,$$

where the matrix norm is induced by the Euclidean norm of vectors. Acting by U or V on some initial state |\Sigma\rangle or |\Upsilon\rangle = V|\Sigma\rangle respectively. Relation (2.9) means that $$\|\|\Gamma\| - |\Upsilon\|\| \leq \varepsilon,$$ whence

$$2(1 - \cos \delta_{\Gamma,\Upsilon}) \leq 2(1 - \Re\langle \Gamma|\Upsilon\rangle) = \|\|\Gamma\| - |\Upsilon\|\| \leq \varepsilon^2.$$

Therefore, $$1 - \cos^2 \delta_{\Gamma,\Upsilon} \leq \varepsilon^2(1 - \varepsilon^2/4),$$ and for any measurement we then have

$$|P(R|\Gamma) - P(R|\Upsilon)| \leq \varepsilon \sqrt{1 - \varepsilon^2/4}.$$ 

Therefore, if V is substituted for U in a quantum network, then the probability of arbitrary measurement outcome on the final state is affected by at most \varepsilon.

### III. BASIC DEFINITIONS

Let \( \mathfrak{A} = \{ |\phi\rangle, |\psi\rangle \} \) be the set of two pure states which we would like to copy. The action of the copying machine can be expressed as

$$\forall |s\rangle \in \mathfrak{A} : \ U |s\rangle \otimes |0\rangle \otimes |m\rangle = |V^{(s)}\rangle,$$

where \( |V^{(s)}\rangle \) is in the state space of composite system "1+2+CM". The unitarity of transformation U implies that

$$\langle \phi|\psi\rangle = \langle V^{(\phi)}|V^{(\psi)}\rangle, \quad \delta_{\phi\psi} = \delta(V^{(\phi)}, V^{(\psi)}).$$

In paper [4] output \( |V^{(s)}\rangle \) was expressed as

\[
\begin{align*}
|V^{(s)}\rangle &= |s\rangle \otimes |s\rangle \otimes |q^{(s)}\rangle + |\perp^{(s)}\rangle, \\
|s\rangle \otimes |s\rangle \otimes |q^{(s)}\rangle &= \{ |s\rangle\langle s| \otimes |s\rangle\langle s| \otimes 1 \}|V^{(s)}\rangle,
\end{align*}
\]

where 1 is the identity operator. The unitarity of copying transformation imposes constraint

$$\|\|q^{(s)}\|\|^2 + \|\|\perp^{(s)}\|\|^2 = 1,$$

because the idempotency of projector \( |s\rangle\langle s| \otimes 1 \) gives

$$\{ |s\rangle\langle s| \otimes 1 \}|\perp^{(s)}\rangle = 0.$$

Hillery and Bužek introduced quantity \( X^{(s)} = \|\|\perp^{(s)}\|\| \) as the size of error of state \( |s\rangle \) copying. However, the relationship between \( X^{(s)} \) and the deviation of the resulting probability distribution from the desired probability distribution was not discussed in paper [4]. We will below study this relationship, and we will define in passing some important objects. Let us introduce magnitude

$$\delta^{(s)} = \inf \{ \delta(V^{(s)}, s \otimes s \otimes k) \mid \langle k|k\rangle = 1 \}.$$

Using relation (3.3), we see that the inner product of unit vectors \( \langle V^{(s)}| \) and \( |s\rangle \otimes |s\rangle \otimes |k\rangle \) is equal to \( \langle q^{(s)}|k\rangle \). Because \( \|\|k\|\| = 1 \), the Schwartz inequality gives

$$\|\langle q^{(s)}|k\rangle \| \leq \|\|q^{(s)}\|\|,$$

where the equality takes place if and only if \( |q^{(s)}\rangle = c |k\rangle \) for some complex number c. The maximal value of the modulus of the inner product of two unit vectors \( \langle V^{(s)}| \) and \( |s\rangle \otimes |s\rangle \otimes |k\rangle \) corresponds to the minimal value of angle \( \delta(V^{(s)}, s \otimes s \otimes k) \) between these vectors, so that if for some vector \( |k\rangle \) infimum (3.6) is reached, then unit vector \( |k\rangle \) and vector \( |q^{(s)}\rangle \) are collinear. For \( \|\|q^{(s)}\|\| \neq 0 \) let us define vectors

$$|k^{(s)}\rangle \overset{\text{def}}{=} \frac{|q^{(s)}\rangle}{\|\|q^{(s)}\|\|} \quad \text{and} \quad |Id^{(s)}\rangle \overset{\text{def}}{=} |s\rangle \otimes |s\rangle \otimes |k^{(s)}\rangle.$$

\( \frac{1}{3} \)
Infimum (3.6) is reached for any vector \( |k⟩ = u |k^{(s)}⟩ \) with unit complex number \( u \), and \( δ^{(s)} \) is angle between unit vectors \( |V^{(s)}⟩ \) and \( |Id^{(s)}⟩ \). Let Hermitian operator \( A \) describes some observable for particle 1. Its measurement over particle in state \( |s⟩ \) produces result \( a \) with probability
\[
p(a|s) = ⟨s|Π_a|s⟩,
\]
where \( Π_a \) is the corresponding projector. If we shall now consider this observable for composite system "1+2+CM", then the measurement of such an observable over system "1+2+CM" in pure state \( |V⟩ \) gives result \( a \) with probability
\[
P(a \text{ for } 1 | V) = ⟨V|Π_a ⊗ 1 ⊗ 1 |V⟩,
\]
where \( Π_a ⊗ 1 ⊗ 1 \) is the projector on the corresponding subspace of the composite system state space. Such a representation for probability is equivalent to expression \( \text{Tr}_1(Π_aρ) \), where density operator \( ρ \) of particle 1 is the partial trace of operator \( |V⟩⟨V| \) over subsystem "2+CM". For particle 2 we analogously have
\[
P(a \text{ for } 2 | V) = ⟨V| 1 ⊗ Π_a ⊗ 1 |V⟩.
\]
For state \( |Id^{(s)}⟩ \) the probability of outcome \( a \) is
\[
P(a \text{ for } j | Id^{(s)}) = ⟨s|Π_a|s⟩ = p(a|s), \tag{3.8}
\]
where \( j = 1, 2 \) and \( s = φ, ψ \). Thus, \( Id^{(s)} \) corresponds to the ideal output. We can now relate angle \( δ^{(s)} \) with objects introduced by Hillery and Bužek \[9\]. In line with definitions (3.6) and (3.7) we have
\[
\cos δ^{(s)} = |⟨V^{(s)}|Id^{(s)}⟩| = |⟨q^{(s)}|k^{(s)}⟩| = \|q^{(s)}\|.
\]
Then relation (3.4) gives \( \|q^{(s)}\| = \sin δ^{(s)} \). As inequality (2.8), equality (3.8) and definition of \( X^{(s)} \) show,
\[
P(a \text{ for } j | V^{(s)}) - p(a|s) ≤ X^{(s)}, \tag{3.9}
\]
i.e. magnitude \( X^{(s)} \) characterizes upon the whole the deviation of the resulting probability distribution from the desired probability distribution. Sum \( X^{(φ)} + X^{(ψ)} \) evaluates the total error of copying of set \( A \).

**Definition 2.** Magnitude \( AE(A) = X^{(φ)} + X^{(ψ)} \) is the absolute error of copying of set \( A = \{|φ⟩, |ψ⟩\} \).

However, the absolute error does not take into account a degree of similarity for states \( |φ⟩ \) and \( |ψ⟩ \). Therefore, we need a new measure for estimation of state-dependent cloning. The notion of the relative error can be motivated in the following way. Let us take that we want distinguishing the input state by measurement made on the copying output. If the modulus \( |⟨φ|ψ⟩| \) is close to 1 then the lower bound on the absolute error is close to 0 and the upper bound on the global fidelity is close to 1. In this case both criteria assert that each copying output can be made near to which would be at the ideality. We know both the ideal output \( |Id^{(φ)}⟩ \) and the ideal output \( |Id^{(ψ)}⟩ \). At first sight it seems that, comparing given output \( |V^{(φ)}⟩ \) to this ideal output and to that one, we can recognize the input state not uneasily. It would be a rashness to think so. Indeed, the closeness for states \( |φ⟩ \) and \( |ψ⟩ \) implies certain closeness for the corresponding ideal outputs. But if so, are we able to decide that given output should be related to this ideal output and not to that one? To express this in quantitative form we should use some measure of closeness for states \( |Id^{(φ)}⟩ \) and \( |Id^{(ψ)}⟩ \). Since according to (2.8)
\[
|P(R | Id^{(φ)}) - P(R | Id^{(ψ)})| ≤ \sin δ(Id^{(φ)}, Id^{(ψ)}),
\]
the quantity \( \sin δ(Id^{(φ)}, Id^{(ψ)}) \) provides such a measure. It stands to reason, this quantity is not independent of similarity for states \( |φ⟩ \) and \( |ψ⟩ \). Are we willing to decide that, for example, given output \( |V^{(φ)}⟩ \) should be related to ideal output \( |Id^{(φ)}⟩ \) and not to \( |Id^{(ψ)}⟩ \), when \( \sin δ(Id^{(φ)}, Id^{(ψ)}) \) is small? The closeness of \( |V^{(φ)}⟩ \) to \( |Id^{(φ)}⟩ \) and the closeness of \( |V^{(ψ)}⟩ \) to \( |Id^{(ψ)}⟩ \) are described by the sizes \( X^{(φ)} = \sin δ^{(φ)} \) and \( X^{(ψ)} = \sin δ^{(ψ)} \) respectively. It is not without signification that \( \sin δ(Id^{(φ)}, Id^{(ψ)}) \) is size of the same kind. Therefore, it is advisable to compare the absolute error with pointed quantity. So, we have in a reasonable way arrived at the definition of 'relative error'.

**Definition 3.** The ratio
\[
RE(A) = (X^{(φ)} + X^{(ψ)}) / \sin δ(Id^{(φ)}, Id^{(ψ)})
\]
is the relative error of copying of set \( A = \{|φ⟩, |ψ⟩\} \).
We shall now derive the angle relations, from which bounds on the errors are simply obtained. Using lemma 2 twice, we have

\[ \delta(Id^{(\phi)}, Id^{(\psi)}) \leq \delta^{(\phi)} + \delta^{(\psi)} + \delta(V^{(\phi)}, V^{(\psi)}). \]  \hspace{1cm} (3.10)

In accordance with the Schwarz inequality, there is

\[ |\langle Id^{(\phi)}| Id^{(\psi)}\rangle| = \left| |\langle \phi|\psi\rangle|^2 \right| (|k^{(\phi)}|k^{(\psi)}) \leq |\langle \phi|\psi\rangle|^2, \]

whence we obtain \( \delta(Id^{(\phi)}, Id^{(\psi)}) \geq \delta(\phi \otimes \phi, \psi \otimes \psi). \) Therefore, \( \delta(\phi \otimes \phi, \psi \otimes \psi) \leq \delta^{(\phi)} + \delta^{(\psi)} + \delta(V^{(\phi)}, V^{(\psi)}), \) or simply

\[ \delta^{(\phi)} + \delta^{(\psi)} \geq \delta(\phi \otimes \phi, \psi \otimes \psi) - \delta(\phi, \psi) \]

in line with Eq. (3.2). Since \( 0 \leq |\langle \phi|\psi\rangle| \leq 1, \) there is \( \delta(\phi \otimes \phi, \psi \otimes \psi) \geq \delta(\phi, \psi). \) Eqs. (3.10) and (3.11) contain the restrictions imposed by the laws of the quantum theory. In particular, the ones allow to derive the lower bounds on both the relative error and the absolute error.

IV. TIGHTEST LOWER BOUNDS

In this section we establish the lower bounds on both the relative error and the absolute error. It should be pointed out that these lower bounds are tightest. Indeed, we shall below describe a cloner that reaches them. At first, let us formulate the result that will be proved.

**Theorem 1.** The relative error \( RE(\mathfrak{A}) \) of cloning for set \( \mathfrak{A} = \{|\phi\rangle, |\psi\rangle\} \) must be at least as large as the quantity

\[ F(z) = z - z^2 / \sqrt{1 + z^2}, \]  \hspace{1cm} (4.1)

where \( z = |\langle \phi|\psi\rangle|. \)

**Proof.** In order to minimize \( RE(\mathfrak{A}) \) the quantity \( \sin \delta(Id^{(\phi)}, Id^{(\psi)}) \) must be as increased as possible. We shall individually consider two cases, to wit (i) \( \delta^{(\phi)} + \delta^{(\psi)} + \delta_{\phi\psi} \leq \pi/2 \) and (ii) \( \delta^{(\phi)} + \delta^{(\psi)} + \delta_{\phi\psi} > \pi/2 \). Using Eqs. (3.2) and (3.10), for the case (i) we have

\[ \sin \delta(Id^{(\phi)}, Id^{(\psi)}) \leq \sin(\delta^{(\phi)} + \delta^{(\psi)} + \delta_{\phi\psi}). \]

The last relation, \( \sin \delta^{(\phi)} + \sin \delta^{(\psi)} \geq \sin(\delta^{(\phi)} + \delta^{(\psi)}) \) and the trigonometric formula for sine of difference give

\[ RE(\mathfrak{A}) \geq \cos \delta_{\phi\psi} - \sin \delta_{\phi\psi} \cot(\delta^{(\phi)} + \delta^{(\psi)} + \delta_{\phi\psi}). \]  \hspace{1cm} (4.2)

It must be stressed that the equality

\[ \sin \delta^{(\phi)} + \sin \delta^{(\psi)} = \sin(\delta^{(\phi)} + \delta^{(\psi)}) \]

is necessary for the equality in Eq. (4.2). We want minimizing the right-hand side of Eq. (4.2) in the interval \( \delta(\phi^{(\phi)}, \psi^{(\phi)}) \leq \delta^{(\phi)} + \delta^{(\psi)} + \delta_{\phi\psi} \leq \pi/2 \) established by Eq. (3.11) and the case (i) condition. Since the right-hand side of Eq. (4.2) increases as the cotangent decreases and the cotangent is a decreasing function of one’s argument, the required minimum is reached at the left boundary point of the interval stated above. Therefore, in the case (i)

\[ RE(\mathfrak{A}) \geq \sin(\delta(\phi^{(\phi)}, \psi^{(\phi)}) - \delta_{\phi\psi}) / \sin \delta(\phi^{(\phi)}, \psi^{(\phi)}). \]  \hspace{1cm} (4.4)

Moreover, we can see that in the case (i) the inequality \( AE(\mathfrak{A}) \geq \sin(\delta(\phi^{(\phi)}, \psi^{(\phi)}) - \delta_{\phi\psi}) \) holds. In the case (ii) we have \( RE(\mathfrak{A}) \geq \sin \delta^{(\phi)} + \sin \delta^{(\psi)}, \) because \( \sin \delta(Id^{(\phi)}, Id^{(\psi)}) \leq 1 \) according to the definition of the angle. Next, the case (ii) condition can be separated into two alternatives, \( \pi/2 - \delta_{\phi\psi} < \delta^{(\phi)} + \delta^{(\psi)} \leq \pi/2 \) and \( \pi/2 < \delta^{(\phi)} + \delta^{(\psi)} \leq \pi. \)

The first alternative contains

\[ RE(\mathfrak{A}) \geq \sin(\delta^{(\phi)} + \delta^{(\psi)}) \geq \cos \delta_{\phi\psi}. \]

In the second alternative the conditions \( \delta^{(\phi)} \leq \pi/2 \) and \( \pi/2 < \delta^{(\phi)} + \delta^{(\psi)} \leq \pi \) provide \( \sin \delta^{(\phi)} + \sin \delta^{(\psi)} \geq 1, \) that can be easily verified by elementary methods. Thus, in the case (ii) \( RE(\mathfrak{A}) \geq \cos \delta_{\phi\psi} \) and \( AE(\mathfrak{A}) \geq \cos \delta_{\phi\psi}. \) To sum up, we see that the lower bound on the relative error is given by the right-hand side of Eq. (4.4). Designating \( z = \cos \delta_{\phi\psi}, \) hence \( \cos \delta(\phi^{(\phi)}, \psi^{(\phi)}) = z^2, \) the right-hand side of (4.4) can be rewritten as \( F(z). \)
Thus, the relative error of copying of two-state set $\mathcal{A}$ must be at least as large as the right-hand side of (4.5) that is plotted in Fig. 1. We can see that in the greater part of interval $z \in [0, 1]$ the function increases and only in the vicinity of the right boundary point the one becomes decreasing. In general, the obtained result is clear. The more states $|\phi\rangle$ and $|\psi\rangle$ are close to each other, the less chances Eve has for the information extraction. Thus, the lower bound on the relative error advises that Alice and Bob should make encoding states $|\phi\rangle$ and $|\psi\rangle$ as close as possible up to the vicinity in which the one slightly decreases. However, the characteristics of a communication system can rather limit a closeness for encoding states. In addition, if two values $z_1$ and $z_2$ (where $z_1 < z_2$) give the same lower bound on the absolute error, value $z_2$ is more preferred, since for $z_2$ the lower bound on the relative error is larger than for $z_1$. As it is shown, in this case the distinction between the bounds on the relative error can be significant.

Next, the reasons used for proof of theorem 1 have established the inequality $AE(\mathcal{A}) \geq \sin(\delta(\phi^{(2)}, \psi^{(2)}) - \delta_{\phi\psi})$ that can be rewritten in the following way.

**Theorem 2** The absolute error of cloning for set $\mathcal{A} = \{|\phi\rangle, |\psi\rangle\}$ has the lower bound:

$$AE(\mathcal{A}) \geq z\sqrt{1 - z^4 - z^2\sqrt{1 - z^2}}. \quad (4.5)$$

Thus, the absolute error of copying of two-state set $\mathcal{A}$ must be at least as large as the right-hand side of (4.5). For small $z$ this function behaves as $z$, for small positive $\xi = 1 - z$ one behaves as $(2 - \sqrt{2})\sqrt{\xi}$. The maximal value of the lower bound (4.1) is equal to $\sqrt{2/27} \approx 0.272$ and occurs for $z = 1/\sqrt{3} \approx 0.577$. The general bound

$$X^{(\phi)} + X^{(\psi)} \geq 2\left(\sqrt{1 + z(1 - z)} - 1\right) \quad (4.6)$$

was obtained in paper [5]. The right-hand side of (4.4) takes its maximal value $\sqrt{5} - 2 \approx 0.236$, when $z = 1/2$. For $z = 1/2$ our bound — the right-hand side of (4.1) — is equal to $\sqrt{3}(\sqrt{5} - 1)/8 \approx 0.268$, and this value is not a maximum. For small $z$ the right-hand side of (4.4) behaves as $z$, for small positive $\xi = 1 - z$ one behaves as $\xi$. Thus, we see that the lower bound given by (4.1) is stronger than the lower bound (4.6). The distinction is perceptable in the intermediate range of values of $z$ and for values of $z$ close to 1. For example, at $z = 4/5$ our bound is approximately 1.5 of the bound given by (4.6). The right-hand side of (4.4) and the right-hand side of (4.5) are plotted as functions of $z$ in Fig. 2 by the solid and dashed lines, respectively. As Fig. 2 shows, the bound (4.6) is symmetric with respect to point $z = 1/2$, whereas the bound (4.5) is asymmetric. Thus, the presented approach has allowed to reinforce the lower bound derived by Hillery and Bužek [4].

In principle, both lower bounds given by theorems 1 and 2 can be reached without auxiliary device. Then a unitary operator $U$ acts on the Hilbert space of 2 qubits:

\[
|V^{(\phi)}\rangle = U\{|\phi\rangle \otimes |0\rangle\}, \quad |V^{(\psi)}\rangle = U\{|\psi\rangle \otimes |0\rangle\}.
\]

The ideal output is $|Id^{(\phi)}\rangle = |s \otimes s\rangle$ for $s = \phi, \psi$, and there is $\sin \delta(Id^{(\phi)}Id^{(\psi)}) = \sqrt{1 - z^4}$. It is clear that the necessary condition for minimization of the errors is the equality in Eq. (3.11). According to lemma 2 the equality in Eq. (3.11) holds only if both final states $|V^{(\phi)}\rangle$ and $|V^{(\psi)}\rangle$ lie in plane span$\{|\phi \otimes \phi\rangle, |\psi \otimes \psi\rangle\}$. This is the necessary condition also for that the global fidelity is maximized [3]. Because unitary operations preserve angles, we have

$$\delta(V^{(\phi)}, V^{(\psi)}) = \delta(\psi \otimes 0, \phi \otimes 0). \quad (4.7)$$

If states $|\phi\rangle$ and $|\phi\rangle$ are non-orthogonal and non-identical then angle $\delta(\phi^{(2)}, \psi^{(2)})$ is larger than the right-hand side of Eq. (4.7) and the ideal copying is impossible. In fact, it is impracticable that angle between $|\phi \otimes 0\rangle$ and $|\psi \otimes 0\rangle$ should be properly increased. To superpose the plane span$\{|\phi \otimes 0\rangle, |\psi \otimes 0\rangle\}$ onto the plane span$\{|\phi \otimes \phi\rangle, |\psi \otimes \psi\rangle\}$ by rigid rotation $U$ is at most that we can achieve. The transformation with characteristics

$$\text{span}\{|V^{(\phi)}\rangle, |V^{(\psi)}\rangle\} = \text{span}\{|\phi \otimes \phi\rangle, |\psi \otimes \psi\rangle\}, \quad \delta^{(\phi)} = \delta^{(\psi)} = \left(\delta(\phi^{(2)}, \psi^{(2)}) - \delta_{\phi\psi}\right)/2$$

is the optimal symmetric state-dependent cloner constructed in papers [3]. This cloner produces equal errors for both states $|\phi\rangle$ and $|\psi\rangle$. The absolute error $AE_S(\mathcal{A})$ is equal to the doubled sine of the angle stated in Eq. (4.4). Using simple trigonometric formulae, we find that the relative error for the cloner defined by Eqs. (4.8) and (4.9) is equal to

$$RE_S(\mathcal{A}) = \sqrt{2} \left[\frac{1 + z + z^2}{1 + z + z^2 + z^3} - \frac{1}{\sqrt{1 + z^2}}\right]^{1/2}. \quad (4.9)$$
The defined by Eqs. (4.8) and (4.9) cloner does not reach the equality in Eq. (4.3) (except when states $|\phi\rangle$ and $|\psi\rangle$ are orthogonal or identical, and, hence, the angle stated in Eq. (4.9) is equal to zero). Therefore, this symmetrical cloner minimizes neither the relative error nor the absolute error. Note that for any symmetric state-dependent cloner the relative error must be at least as large as $RE_S(\mathfrak{A})$.

We shall now propose an asymmetric cloner for which the relative error of copying for set $\mathfrak{A}$ is rigorously equal to the right-hand side of Eq. (4.1). Such a optimal asymmetric state-dependent cloner is defined by

$$\text{span}\{|V(\phi), |V(\psi)\} = \text{span}\{|\phi^{\otimes 2}\}, |\psi^{\otimes 2}\} ,$$

$$\delta(\phi) = 0 \land \delta(\psi) = \delta(\phi^{\otimes 2}, \psi^{\otimes 2}) - \delta_{\phi\psi} .$$

This cloner makes the ideal copying for one from prescribed pair $\mathfrak{A}$ of states, i.e. the one is entirely asymmetric. It is obvious that both the equality in Eq. (3.11) and the equality in Eq. (4.3) are reached. Therefore, for the cloner defined by Eqs. (4.10) and (4.11) the relative error $RE_A(\mathfrak{A}) = F(z)$ and the absolute error $AE_A(\mathfrak{A})$ is equal to the right-hand side of Eq. (1.3). In other words, the optimal asymmetric state-dependent cloner minimizes both the relative and absolute errors. So, without the symmetry requirement it is possible to build the cloner with relative error that is smaller than relative error for state-dependent cloner constructed in $\mathfrak{A}$.

Note that our asymmetric cloner, which makes the ideal copying for one from prescribed pair of states, is not a special example of the Wootters–Zurek cloner. The Wootters–Zurek copying machine implements the ideal copying for the orthogonal basis vectors $|s\rangle$. For pair $\mathfrak{A} = \{|\phi\rangle, |\psi\rangle\}$ of non-orthogonal states, we shall take the unit vector $|\omega\rangle = \text{span}\{|\phi\rangle, |\psi\rangle\}$ and $\langle\phi|\omega\rangle = 0$. Then $|\phi\rangle$ and $|\omega\rangle$ are basis elements, and it may be reasonable to consider the WZ–cloner such that $|s\rangle \otimes |x\rangle \rightarrow |s\rangle \otimes |s\rangle \otimes |k(s)\rangle$ for $s = \phi, \omega$. Using simple calculations we find that $RE_{WZ}(\mathfrak{A}) = \sqrt{3} z / \sqrt{1 + z^2}$. Therefore, the optimal asymmetric cloner defined by Eqs. (4.10) and (4.11) differs from the Wootters–Zurek copying machine.

V. CONCLUSION

We have proposed the notion of the relative error which provides new optimality criterion for the state-dependent cloning. We have beforehand proved several useful statements those maintain our approach. Among them there are the spherical triangle inequality and the inequality establishing the upper bound on the modulus of difference between probability distributions generated by two any states for an arbitrary measurement. These relations can be useful in various questions. Using physical reasons, the notion of the relative error has been then introduced. The tightest lower bounds on the absolute and relative errors of copying of the two-state set were obtained. These bounds succeed the unitarity of quantum mechanical transformations.

The lower bound on the relative error increases as function of $z$ in the greater part of interval $z \in [0; 1]$. Returning to cryptographic example, it is possible to say roughly that the more states $|\phi\rangle$ and $|\psi\rangle$ are close to each other the less chances Eve has for the information extraction. It is not inconceivable that in the vicinity of the right boundary point $z = 1$ the lower bound on the relative error becomes decreasing. Thus, this lower bound advises that Alice and Bob should make encoding states $|\phi\rangle$ and $|\psi\rangle$ as close as possible up to the vicinity in which the one slightly decreases. However, a characteristics of a communication system can rather limit a closeness of encoding states.

As it is shown, the optimal symmetric state-dependent cloner, which maximizes the global fidelity, reaches neither the lower bound on the relative error nor the lower bound on the absolute error. We have described the optimal asymmetric state-dependent cloner that minimizes minimizes both the relative error and the absolute error. It is worth noting that the global fidelity is optimized only if a cloner is symmetric, whereas both the absolute and relative error are optimized only if a cloner is entirely asymmetric.

It should be pointed out that the obtained results have only a partial application to quantum communication problems because in the reality a communication channel will inevitably suffer from noise that will have caused the bits to evolve to mixed states. Authors of paper [2] showed that noncommuting mixed states cannot be ideally broadcast. It would be interesting to consider possible limits on error in the case, where Eve’s copying machine has as input the original mode secretly prepared in one state from a set of two mixed states.

Finally, we would like to point out that the above stated approach can be applied to machines, which make multiple copies. Author intends to examine bounds on errors for such a case in next paper.

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FIG. 1. The function $F(z)$ defined by the right-hand side of (4.1).
FIG. 2. The right-hand side of (4.5) (solid line) and the right-hand side of (4.6) (dashed line) as functions of $z$. 