Magnetic Permeability in Constrained Fermionic Vacuum

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Abstract

We obtain using Schwinger’s proper time approach the Casimir-Euler-Heisenberg effective action of fermion fluctuations for the case of an applied magnetic field. We implement here the compactification of one space dimension into a circle through anti-periodic boundary condition. Aside of higher order non-linear field effects we identify a novel contribution to the vacuum permeability. These contributions are exceedingly small for normal electromagnetism due to the smallness of the electron Compton wavelength compared to the size of the compactified dimension, if we take the latter as the typical size of laboratory cavities, but their presence is thought provoking, also considering the context of strong interactions.

There is a major ongoing effort [1, 2, 3] to measure the Euler-Heisenberg (EH) effect [4, 5]. As it is well understood, the EH-effective Lagrangian arises from deformation of fermion-antifermion pair fluctuations caused by an applied strong (classical = absence of virtual photon diagrams) electromagnetic field. Since renormalization defines the electric charge in an Abelian theory in the long wavelength limit, the field dependent terms in the effective action after renormalization must be of higher order than quadratic and thus introduce non-linear effects in the electromagnetic field. A (coherent) light wave within a field-filled volume experiences matter-like scattering effects from the field-polarized vacuum [6]. The hope and expectation is that the birefringence effect [7] can be experimentally observed in near future.

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There are several reasons that make the search for such a macroscopic confirmation of vacuum fluctuation effects worthwhile:

i) the experiment constitutes a test of quantum electrodynamics in the far infrared domain \( q \to 0 \) in which limit the usually dominant lowest order vacuum polarization effect vanishes exactly in consequence of charge renormalization. Thus one probes the interaction of a light wave with the external fields at the level \( \mathcal{O}(\alpha^2) \);

ii) there is hope that when the experimental techniques are refined, certain otherwise invisible higher order effects, such as is the interference with quantum chromodynamic vacuum structure could become observable. An effect is in principle possible since quarks are carriers of both Maxwell and strong charge [8].

The properties of the (relativistic) quantum vacuum are also influenced by boundary conditions constraints, an effect generally known as the Casimir effect [9] first studied for the case of two uncharged conductive parallel plates restraining the photon fluctuations, causing an observable attractive force. The effect of constrains on the quantum vacuum of a massive field is usually significant when the dimension of the support region is comparable to the Compton wavelength \( \lambda = \hbar c/m \) of the field excitations. The Casimir effect is thus negligible for electrons in nano-cavities. On the other hand, in QCD the hadronic confinement region and thus presumably also the quark field fluctuation region is smaller than \( \lambda_q \) and we should expect significant interference between Euler-Heisenberg and Casimir effects for quark fluctuations.

In view of this observation it is necessary to study what new physical phenomena could arise when the effective action is evaluated for a space-time region subject to the combined effect of an applied field and boundary conditions for the charged fermion fluctuations, amalgamating the Euler-Heisenberg and Casimir effects in a Euler-Heisenberg-Casimir (HEC) effective action. We are not interested here in the enhancement of the Casimir effect caused by the external field [10]. Our objective is to understand the modifications of the vacuum properties, specifically here magnetic permeability induced by the Casimir effect.

In other words, we would like to derive the vacuum polarization effect in the infrared limit \( q \to 0 \) for finite \( ma \), where \( m \) is the mass of the fermion and \( a \) is a length related to the boundary condition for the Casimir effect. In several aspects our interests parallels the study of the cavity Casimir effect [11] as well as the consideration of radiative corrections to the Casimir effect recently obtained by Kong and Ravndal [12]. Unlike these efforts to understand fluctuations of a confined quantum electromagnetic vacuum our study deals with a confined quantum fermionic vacuum in the presence of an (external) classical electromagnetic field. While our work is carried out for the Abelian Maxwell external fields, it is a study case for the physics applicable to the non-Abelian strong interactions, where the magnitude of the expected effects is significant.

To obtain the interference of the Casimir effect with the external field we impose anti-periodic boundary condition on the fermionic field in the \( z \)-direction from \( z = -a \) to \( z = a \). This boundary condition corresponds topologically to a confinement of twisted spinors into a circle \( S^1 \) (of radius \( a/\pi \)) resulting from a compactification of the \( z \)-axis dimension [13]; this choice of twisted spinors avoids the problem of non-causal propagation that occurs with the untwisted spinors of the periodic boundary condition [13]. The anti-periodic boundary condition gives rise to a Casimir effect as well as the boundary condition of confinement between impermeable plates (see the reviews in [9] and for the bosonic case also [14]) and avoids the mathematical complexity of the latter (the \( \gamma \) matrices dependent MIT boundary condition [15]). The xy-planes we take as large squares with side \( \ell \) and the limit \( \ell \to \infty \) can be taken at the end of the calculations.

We will consider the case of a constant uniform magnetic field \( \mathbf{B} \) perpendicular to the xy-plane applied in the vacuum of the anti-periodic fermion field. These choices for the geometry, the boundary condition and the external field are intended to simplify the formalism, in order for us to concentrate on the physical effects. Once their basic features are understood the path is open
to consider more complicated situations, in particular more complicated geometries and boundary conditions \[10\].

We note that a simple dimensional and symmetry consideration leads to the effective action in lowest order in field strength in the form

\[
\mathcal{L}_{\text{eff}} = \Pi(ma) \frac{\vec{E}^2 - \vec{B}^2}{2} + Q(ma) \vec{E} \cdot \vec{B} + F_1(ma) \frac{1}{2} (\vec{E} \cdot \vec{n})^2 - F_2(ma) \frac{1}{2} (\vec{B} \cdot \vec{n})^2 + F_3(ma) \vec{B} \cdot \vec{n} \vec{E} \cdot \vec{n} + \mathcal{O}(E^4, B^4, E^2 B^2),
\]

where \( \vec{n} \) is the normal vector of the xy-plane. We note that the terms odd in \( \vec{n} \) cannot occur, since there is no sense of orientation introduced by the boundary condition: there are as many fluctuations moving to the 'right', as there are moving to the 'left'. In consequence terms odd in \( \vec{n} \) cannot occur, which along with particle-antiparticle symmetry (Furry theorem) assures that the effective action is an even function in the electromagnetic fields.

Given the breaking of the symmetry by the boundary condition we expect the vacuum state to be birefringent \[1\]. For the term \( F_3(am) \) to induce the same magnitude effect as the original Euler-Heisenberg birefringence \[1\] we must have:

\[
F_3^{\text{equiv}} \approx |B|^2 7 A, \quad A = \frac{2 \alpha^2 (\hbar/mc)^3}{45 \frac{mc^2}{me^2}} = 0.265 \text{ fm}^3/\text{MeV}.
\]

A is the coefficient of the first non vanishing EH term, see also Eq. \(14\) below. The energy density \( B^2/2 \) of a 1 Tesla field is \( 0.321 \times 10^{-18} \text{keV/fm}^3 \). For a 5 Tesla field we thus find \( F_3^{\text{equiv}}(ma) \approx 3 \times 10^{-17} \), a small number indeed. This clearly illustrates the difficulty of the experimental effort which must reduce birefringence due to matter to below this contribution. A different view at the smallness of the Maxwell theory effects arises recalling that the fields are measured in units of the ‘critical’ field, here \( B_{\text{cr}} = m^2c^3/\hbar e = 4.414 \times 10^9 \text{T} \).

The evaluation of all the five factor functions in Eq. \(1\), allowing for the strong interaction structure of the vacuum, is a formidable task which one should undertake only upon confirmation that there are interesting physical properties waiting to be discovered. We can explore one interesting aspect in a relatively easy way and thus motivate further study of this complex subject matter. It turns out that the QED-case \( \vec{n} \parallel \vec{B} \) with \( \vec{E} = 0 \) is easily analytically soluble using Schwinger’s proper time technique \[3\] \[11\] and we shall thus address this case in detail here. It amounts to the evaluation of the vacuum permeability:

\[
\mathcal{L}_{\text{eff}}(\vec{E} = 0, \vec{n} \parallel \vec{B}) = -[\Pi(ma) + F_2(ma)] \frac{|B|^2}{2} = \frac{-1}{\mu(\text{am})} \frac{|B|^2}{2}.
\]

The magnetic permeability introduced here has the vacuum limit \( \mu \to 1 \) for \( ma \to \infty \), assured by the renormalization process carried out below.

We now turn to determine the effective Casimir-Euler-Heisenberg action and \( \mu(\text{am}) \) in particular. So let us consider the quantum vacuum of a Dirac field of mass \( m \) and charge \( e \) in the non-trivial topology of \( \mathbb{R} \times \mathbb{R}^2(\text{xy-plane}) \times S^1 \) in the presence of the constant uniform magnetic field \( \mathbf{B} \). We use Schwinger’s proper-time method \[9\] in order to calculate the effective Lagrangian. We start our calculations with the proper-time representation for the effective action with regularization provided by a cutoff \( s_o \) in the proper-time \( s \):

\[
\mathcal{W}^{(1)} = \frac{i}{2} \int_{s_o}^{\infty} ds \int Tr e^{-isH},
\]

where \( Tr \) stands for the total trace and \( H \) is the proper-time Hamiltonian, which for the case at hand is given by \( H = (p - eA)^2 - (e/2)\sigma_{\mu\nu}F^{\mu\nu} + m^2 \), where \( p \) has components \( p_\mu = -i\partial_\mu \), \( A \) is
the electromagnetic potential and \( F \) is the electromagnetic field, which is being contracted with the combination of gamma matrices \( \sigma_{\mu \nu} = i[\gamma_\mu, \gamma_\nu] / 2 \). Using the anti-periodic boundary condition we find for \( p_z \) the eigenvalues \( \pm \pi n / 2a \) \( (n \in 2N - 1) \), where \( N \) represents the set of positive integers.

The components which are parallel to the plates are constrained into the Landau levels created by the magnetic field \( B \); we call \( B \) the component of \( B \) perpendicular to the xy-plane and consider \( B \) oriented in such a way that \( eB \) is positive. A straightforward calculation yields for the trace in Eq. (5) the following expression:

\[
\text{Tr} \ e^{-isH} = e^{-ism^2} \sum_{\alpha=\pm} 2 \sum_{n \in 2N - 1} 2 e^{-i(n\pi / 2a)^2} \sum_{n' \in N - 1} \frac{eB \ell^2}{2\pi} e^{-iseB(2n'+1-\alpha)} \int \frac{dt \ d\omega}{2\pi} e^{is\omega^2},
\]

(5)

where the first sum is due to the four components of the Dirac spinor, the second sum is over the eigenvalues obtained from the anti-periodic boundary condition, the third sum comes from the Landau levels with the corresponding multiplicity factor due to degeneracy, and the integration is done over the observation time \( T \) and the continuum of eigenvalues \( \omega \) of the operator \( p^\alpha \). Following Schwinger \[17\] we now use Poisson’s formula \[18\] to rewrite the second sum in Eq. (5) in the following form:

\[
\sum_{n \in 2N - 1} 2 e^{-i(n\pi / 2a)^2} = \frac{2a}{2\sqrt{i\pi s}} \left[ 1 + 2 \sum_{n \in N} (-1)^n e^{i(an)^2 / s} \right],
\]

(6)

The sum over the Landau levels is trivially obtained in terms of hyperbolic functions in such a way that the trace in (5) takes the form:

\[
\text{Tr} \ e^{-isH} = \frac{T \ 2a \ell^2}{4\pi^2 i} e^{-ism^2} \left[ 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{i(an)^2 / s} \right] [1 + iseB L(isB)],
\]

(7)

where \( L(\xi) = \coth \xi - \xi^{-1} \) is the Langevin function. Substituting this expression into equation Eq. (2) we obtain the effective action as

\[
\mathcal{W}^{(1)} = [\mathcal{L}^{(1)}_{HE}(B) + \mathcal{L}^{(1)}_{HEC}(B, a)] T2a \ell^2,
\]

(8)

where

\[
\mathcal{L}^{(1)}_{HE}(B) = \frac{1}{8\pi^2} \int_{s_0}^\infty \frac{ds}{s^3} e^{-ism^2} \left[ 1 + iseB L(isB) \right]
\]

(9)

is the Euler-Heisenberg contribution to the effective Lagrangian and

\[
\mathcal{L}^{(1)}_{HEC}(B, a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4\pi^2} \int_{s_0}^\infty \frac{ds}{s^3} e^{-ism^2 + i(an)^2 / s} \left[ 1 + iseB L(isB) \right],
\]

(10)

is the Casimir-Euler-Heisenberg contribution to the effective Lagrangian. Both contributions are in unrenormalized form and renormalization is required before we can remove the cutoff \( s_0 \).

The usual renormalization condition requiring that the observable Maxwell charge is seen at large distances requires that the compactification size \( a \) is taken to infinity first. However, since the HEC contribution is finite for \( s_0 \to 0 \) for any \( a \), a careful study of these limits in the Abelian theory is not necessary, and we can proceed as usual: an expansion of \( 1 + iseBL(isB) \) in powers of \( eB \) yields the one subtraction and one renomalization constant. The first, constant, expansion term can be subtracted from the HE-part of the effective Lagrangian, as it is not field dependent. In the limit \( s_0 \to 0 \) this constant vacuum action due to free fermi fluctuations tends to \(-m^4\Gamma(-2)/8\pi^2\), where \( \Gamma \) is the Euler gamma function. It is generally believed that a complete quantum field theory will have equal number of fermion and boson degrees of freedom, in which case the divergent component in the zero point contributions to the World action can cancel out.
The second term in the expansion is proportional to the Maxwell Lagrangian,

$$\mathcal{L}^{(0)}(B) = -\frac{1}{2} B^2,$$

with a constant of proportionality $Z_3^{-1} - 1 = e^2 \Gamma(0)/12\pi^2$ in the limit $s_o \to 0$. We absorb this constant into Maxwell Lagrangian by a renormalization of $B$: define the renormalized field as $B_R = BZ_3^{-1/2}$ and the renormalized charge as $e_R = e Z_3^{1/2}$. After subtractions and conversions to these renormalized quantities, $\mathcal{L}^{(1)}_{HE}$ in Eq. (4) becomes free of spurious terms and is well-defined in the limit $s_o \to 0$ while in the Maxwell Lagrangian Eq. (11), the bare field is replaced by $B_R$. The HEC-component $\mathcal{L}^{(1)}_{HEC}$ in Eq. (10) depends on $e$ and $B$ only through the product $eB = eRB_R$ and thus it does not change in form. Since it is finite and vanishes for $a \to \infty$ no further renormalization is required to render $\mathcal{L}^{(1)}_{HEC}$ in Eq. (10) well-defined.

We immediately drop the subindex $R$ indicating renormalization, and assemble the renormalized contributions of Eq. (11), Eq. (9) and Eq. (10) to write the complete renormalized effective Lagrangian as:

$$\mathcal{L}(B,a) = -\frac{1}{2} B^2 - \frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-sm^2}}{s^3} \left[ seB \mathcal{L}(seB) - \frac{1}{3}(seB)^2 \right] +$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4\pi^2} \int_0^\infty ds \frac{e^{-sm^2-(an)^2/s}}{s^3} \left[ 1 + seB \mathcal{L}(seB) \right],$$

where the integration axis $s$ has been rotated to $-is$ and the cutoff $s_o$ has been removed. In the remainder of this paper we explore interesting features of this effective action.

We notice three contributions in Eq. (12):

i) the (renormalized) Maxwell Lagrangian $\mathcal{L}^{(0)}(B)$, given by the quadratic term $-B^2/2$;

ii) the renormalized Euler-Heisenberg Lagrangian Eq. (9):

$$\mathcal{L}^{(1)}_{HE}(B) = -\frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-sm^2}}{s^3} \left[ seB \mathcal{L}(seB) - \frac{1}{3}(seB)^2 \right].$$

(13)

For $B$ small compared to the critical field $B_{cr} = m^2/e$ we can expand this Lagrangian in powers of $B^2$ to obtain

$$\mathcal{L}^{(1)}_{HE} = \sum_{k=2}^{\infty} \frac{(-1)^k}{8\pi^2} \frac{2^{2k}|B_{2k}| m^4}{2k(2k-1)(2k-2)} \frac{B^{2k}}{B_{cr}^{2k}},$$

(14)

where $B_{2k}$ is the $2k$-th Bernoulli number. This expression shows that the lowest order contribution from the (renormalized) Euler-Heisenberg Lagrangian to Maxwell Lagrangian is a term in $B^4$.

iii) The third contribution in Eq. (12) is given by the Casimir-Euler-Heisenberg Lagrangian $\mathcal{L}^{(1)}_{HEC}$ in Eq. (10) with renormalized charge and field.

We are in particular interested here in any modification of the magnetic permeability of the vacuum, $\mu(B,am)$ as defined by the derivative of the complete effective Lagrangian Eq. (12) with respect to $-B^2/2$:

$$H \equiv -\frac{\partial \mathcal{L}}{\partial B} \equiv \frac{1}{\mu(B,am)} B.$$

(15)

By using Eq. (14) and employing the formula 3.471.9 in Ref. [19] we obtain:

$$\frac{1}{\mu(B,am)} = \frac{1}{\mu(am)} -$$

$$- \sum_{k=2}^{\infty} \left[ 1 - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^2(ammn)^{2k-2}}{(2k-3)!} K_{2k-2}(2ammn) \right] \frac{(-1)^k}{8\pi^2} \frac{2^{2k}|B_{2k}|e^2}{(2k-1)(2k-2)} \frac{B^{2k-2}}{B_{cr}^{2k-2}},$$

(16)
where we introduced
\[
\frac{1}{\mu(B \to 0, am)} \equiv \frac{1}{\mu(am)} = 1 - \frac{e^2}{3\pi^2} \sum_{n=1}^{\infty} (-1)^{n-1} K_0(2amn).
\] (17)

In the weak field regime the vacuum permeability Eq. (17) is the dominant term in the expansion given by Eq. (16).

The corresponding separation of the Casimir-Euler-Heisenberg term in Eq. (17) is:
\[
\mathcal{L}_{HEC}^{(1)}(B) = \frac{(am)^2}{2\pi^2a^4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} K_2(2amn) + \frac{1}{2} \left[ 1 - \frac{1}{\mu(am)} \right] B^2 + \mathcal{L}_{HEC}^{(1)\gamma}(a, B),
\] (18)
with
\[
\mathcal{L}_{HEC}^{(1)\gamma}(a, B) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-s\frac{(am)^2}{s}} [s\epsilon B L(s\epsilon B) - \frac{1}{3}(s\epsilon B)^2].
\] (19)

The first term in Eq. (18) is the negative of the Casimir energy density of the fermionic field with anti-periodic boundary condition; it has no importance in the present discussion because it is independent of \(B\).

We illustrate the behavior of the vacuum permeability \(\mu(am)\), Eq. (17), as a function of \(am\) in figure 1. We see that it reduces to 1 when the compactification disappears \((a \to \infty)\). In the general case \(a < \infty\) the permeability of the Fermionic vacuum confined in \(\mathbb{R}^2 \times S^1\) is determined by the series of Bessel functions in Eq. (17). From the properties of Bessel functions \(\mathcal{L}_{HEC}^{(1)}\) it is easy to see that the behavior obtained numerically in figure 1 taking \(\alpha = 3\pi/4\) for illustrative purposes, is correct. There is a value \(\gamma_{cr} \equiv ma_{cr}\) at which a transition between paramagnetic and diamagnetic permeability occurs. For \(am < \gamma_{cr}\) we have a diamagnetic permeability and for \(am > \gamma_{cr}\) we have a paramagnetic permeability. From Eq. (17) we obtain this critical value \(\gamma_{cr}\) as defined by the equality
\[
\sum_{n=1}^{\infty} (-1)^{n-1} K_0(2ma_{cr}n) = \frac{3\pi}{4\alpha},
\] (20)
where it appears the fine structure constant of the Dirac particle, \(\alpha = e^2/4\pi\), which is the magnitude that determines \(a_{cr}\). Typical values of the Bessel function \(K_0\) show that \(a_{cr}\) is extremely small for the usual values of \(\alpha\) in the case of electrons and quarks. In this case it is easy to find the estimation (cf. formula 8.526.2 in [13])
\[
ma_{cr} \simeq \frac{\pi}{2} e^{\frac{3\pi/2\alpha}{2}} C,
\] (21)
where \(C\) is the Euler constant. This estimation shows that \(ma_{cr}\) is exceedingly small for the cases under consideration (for the electron its order of magnitude is \(10^{-282}\)); in those cases the vertical asymptote in Figure 1 becomes the vertical axis of the graph, which is reduced to the curve at the right of this asymptote. Therefore, we are more than justified in attributing paramagnetic properties to the Fermionic vacuum under consideration. On the other hand it is reassuring to obtain from Eq. (17) a sensible result even in the limiting situation of \(a \ll a_{cr}\). In fact, for extremely dense packed vacuum fluctuations, intuition favors diamagnetism as the dominant effect.

Clearly the fermionic vacuum permeability arising in typical physical sizes involving electromagnetism is very small. If we take for the electron vacuum field \(a\) in the range of nanometer the change in permeability given by Eq. (17) can be taken as zero (it is less than \(10^{-200}\)). On the other hand if we consider the vacuum field of \(u, d\) and \(s\) quarks inside a hadron, assuming that our expression derived for confinement in \(\mathbb{R}^2 \times S^1\) provides a magnitude estimate for quarks confined in the region inside the spherical surface \(S^2\) (and assuming the radius of \(S^1\) and \(S^2\) are roughly the same). Numerically obtained changes in permeability are of magnitude \(10^{-3}, 10^{-3}\) and \(10^{-4}\), respectively,
for \( u, d \) and \( s \) quarks and \( a \approx 1 \) fm. This effect falls in the range of typical material constants: for example aluminum, for which \( \mu - 1 \approx 2.3 \times 10^{-4} \) at 20°C. These changes in vacuum permeability show that equation Eq. (17) yields no contribution to QED at realistic domain scale. However, these formal results are of potential importance to the understanding of the QED vacuum inside a hadron. Moreover, the transition point from paramagnetic to diamagnetic vacuum deserves special attention in the studies of dimensional compactification. We further observe that our study can be adapted to the more complicated gauge groups and be applied to the study of QCD in particular, where the coupling constant is large, and thus color magnetic permeability (and permittivity as well) is expected to be quite different from the perturbative vacuum value within the confinement volume.

We have obtained by an explicit calculation the magnetic permeability of the QED vacuum in the infrared limit, including the dependence on the size of confining regions. The effect is found to be exceedingly small and should not influence the birefringence experimental tests of the Euler-Heisenberg effective Lagrangian. On the other hand both QED and QCD effects within the hadron de-confinement region promise to be of interest and we hope to return to study these effects in near future.

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