ON THE NUMBER OF SUBRINGS OF $\mathbb{Z}^n$ OF PRIME POWER INDEX

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Abstract. Let $n$ and $k$ be positive integers, and $f_n(k)$ (resp. $g_n(k)$) be the number of unital subrings (resp. unital irreducible subrings) of $\mathbb{Z}^n$ of index $k$. The numbers $f_n(k)$ are coefficients of certain zeta functions of natural interest. The function $k \mapsto f_n(k)$ is multiplicative, and the study of the numbers $f_n(k)$ reduces to computing the values at prime powers $k = p^e$. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ of $e$ into $n - 1$ positive integers, let $g_{n, \alpha}(p)$ denote the number of irreducible subrings of $\mathbb{Z}^n$ for which the associated upper triangular matrix in Hermite normal form has diagonal $(p^{\alpha_1}, \ldots, p^{\alpha_{n-1}}, 1)$. Via combinatorial analysis, the computation of $f_n(p^e)$ reduces to the computation of $g_{n, \alpha}(p)$ for all compositions of $\alpha$ into $e$ parts, where $i \leq e$ and $j \leq n - 1$. We extend results of Liu and Atanasov-Kaplan-Krakoff-Menzel, who explicitly compute $f_n(p^e)$ for $e \leq 8$. The case $e = 9$ proves to be significantly more involved. We evaluate $f_n(e^3)$ explicitly in terms of a polynomial in $n$ and $p$ up to a single term which is conjecturally a polynomial. Our results provide further evidence for a conjecture of Bhargava on the asymptotics for $f_n(k)$ as a function of $k$ motivates the study of the asymptotics for $g_{n, \alpha}(p)$ for certain infinite families of compositions $\alpha$, for which we are able to obtain general estimates using techniques from the geometry of numbers.

1. Introduction

Let $G$ be an infinite group. Given a finite index subgroup $H$, we set $[G : H]$ to denote the index of $H$ in $G$. For $k \in \mathbb{Z}_{\geq 1}$, let $a_k(G)$ be the number of finite-index subgroups $H$ of $G$ such that $[G : H] = k$. In [GSS88], Grunewald, Segal and Smith introduced the zeta function of $G$, defined as follows

$$\zeta_G(s) := \sum_{H} [G : H]^{-s} = \sum_{k=1}^{\infty} a_k(G) k^{-s},$$

where the above sum runs over all finite index subgroups $H$ of $G$. The properties of such zeta functions are described in [DSWP08]. In this context, zeta functions measure subgroup growth. For a comprehensive account of this theme, we refer to [LS03]. Throughout, $n > 1$ is an integer and $\mathbb{Z}^n$ is the ring consisting of integer $n$-tuples with componentwise addition and multiplication. It is well known that the zeta function $\zeta_{\mathbb{Z}^n}(s)$ is given by

$$\zeta_{\mathbb{Z}^n}(s) = \zeta(s)\zeta(s-1)\ldots\zeta(s-(n-1))$$

(cf. loc. cit. for five different proofs of the above result).

We study distribution questions for the number of finite-index unital subrings of $\mathbb{Z}^n$ of prescribed index. This problem is closely related to the distributions of orders in a fixed number ring $O_K$, a problem studied by Brackenhoff [Bra09], Kaplan Marcinek and Takloo-Bighash (cf. [KMTB15]). Given $k \in \mathbb{Z}_{\geq 1}$, following [Liu07], let $f_n(k)$ be the number of commutative unital subrings $S$ of $\mathbb{Z}^n$ such that $[\mathbb{Z}^n : S] = k$. The subring zeta function is given by

$$\zeta^R_{\mathbb{Z}^n}(s) = \sum_{k=1}^{\infty} f_n(k) k^{-s},$$

which decomposes into an Euler product

$$\zeta^R_{\mathbb{Z}^n}(s) = \prod_{p} \zeta^R_{\mathbb{Z}^n,p}(s),$$

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where the product is over all primes $p$. The local Euler factor at $p$ is given by

$$
\zeta_{\mathbb{Z}_p}(s) := \sum_{c=0}^{\infty} f_n(p^c)p^{-cs}.
$$

It is not hard to show that $\zeta_{\mathbb{Z}_p}(s) = \zeta(s)$, and that

$$
\zeta_{\mathbb{Z}_p}(s) = \frac{\zeta(3s-1)(s)^3}{\zeta(2s)^2},
$$

cf. [Liu07, Proposition 4.1]. The formula for $n = 3$ was first obtained by Datskovsky and Wright [DW88]. A formula for $n = 4$ was obtained by Nakagawa cf. [Nak96]. Recently, there has been interest in the following question (cf. [Ish22b, Question 1.2]).

**Question 1.1.** Let $n,e$ be a pair of natural numbers. What can be said about $f_n(p^e)$ as a function of $p$?

For $e \leq 5$, Liu [Liu07] gives explicit formulae for $f_n(p^e)$. Subsequently, these formulae were generalized by Atanasov, Kaplan, Krakoff and Menzel [AKKM21] for $e = 6,7,8$. These computations indicate that for a fixed pair $(n,e)$, the function $p \mapsto f_n(p^e)$ is a polynomial in $p$ (cf. [AKKM21, Question 1.13]), and this has been further studied by Isham in [Ish22a]. The zeta function $\zeta_{\mathbb{Z}_p}(s)$ is said to be uniform if there is a rational function $W(X,Y) \in \mathbb{Q}(X,Y)$ such that for every prime $p$, $\zeta_{\mathbb{Z}_p}(s) = W(p,p^{-s})$. It is not hard to show that the zeta function $\zeta_{\mathbb{Z}_p}(s)$ is uniform if for all $e$, $f_n(p^e)$ is a polynomial in $p$. For further details, we refer to [AKKM21, section 5.1]. Below we summarize the known values of $f_n(p^e)$ for $e \leq 5$.

**Theorem 1.2** (Liu [Liu07]). With respect to notation above,

$$
\begin{align*}
 f_n(1) &= 1, f_n(p) = \binom{n}{2}, f_n(p^2) = \binom{n}{2} + \binom{n}{3} + 3\binom{n}{4}, \\
 f_n(p^3) &= \binom{n}{2} + (p + 1)\binom{n}{3} + 7\binom{n}{4} + 10\binom{n}{5} + 15\binom{n}{6}, \\
 f_n(p^4) &= \binom{n}{2} + (3p + 1)\binom{n}{3} + (p^2 + p + 10)\binom{n}{4} + (10p + 21)\binom{n}{5} + 70\binom{n}{6} \\
 &\quad + 105\binom{n}{7} + 105\binom{n}{8}, \\
 f_n(p^5) &= \binom{n}{2} + (4p + 1)\binom{n}{3} + (7p^2 + p + 13)\binom{n}{4} + (p^3 + p^2 + 41p + 31)\binom{n}{5} \\
 &\quad + (15p^2 + 35p + 141)\binom{n}{6}.
\end{align*}
$$

For the values for $e$ in the range $6 \leq e \leq 8$, we refer to [KMTB15].

In this manuscript, we extend the above mentioned results to $e = 9$. First, we introduce some further notation. A subring of prime power index decomposes into a direct product of irreducible subrings, which we now proceed to describe.

**Definition 1.3.** A subring $L$ of index $p^e$ is said to be irreducible if for each vector $x = (x_1, \ldots, x_n)^t \in L$, we have that

$$
x_1 \equiv x_2 \equiv \cdots \equiv x_n \pmod{p}.
$$

An irreducible subring $A$ is a matrix $A$ in Hermite normal form

$$
A = \begin{pmatrix}
p^{a_1} & p^{a_1,1} & \cdots & p^{a_1,n-1} & 1 \\
p^{a_2} & p^{a_2,1} & \cdots & p^{a_2,n-1} & 1 \\
p^{a_3} & p^{a_3,1} & \cdots & p^{a_3,n-1} & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
$$

(1.1)
with \( \alpha_i \geq 1 \) for all \( i \). The subring \( L \) of index \( p^e \) is irreducible if and only if \( A_L \) is an irreducible subring matrix. Let \( g_n(p^e) \) be the number of irreducible subrings of \( \mathbb{Z}^n \) of index \( p^e \). It is shown that every subring of \( p \)-power index is a direct sum of irreducible subrings, and as a consequence, we have the following recursive formula (cf. Proposition 4.4 of \[AKKM21\])

\[
(1.2) \quad f_n(p^e) = \sum_{i=0}^{n} \sum_{j=1}^{e} \binom{n-1}{j-1} f_{n-j}(p^{e-1}) g_{j}(p^i).
\]

In order to compute \( f_n(p^e) \) it thus suffices to compute \( g_j(p^e) \) for all values \( e \leq 9, 1 \leq j \leq n \). We remark that the convention for the definition of \( g_n(p^e) \) used here is that of \[AKKM21\], and differs from that in \[Liu07\].

A composition of \( m \) into \( r \) parts consists of a tuple \( \alpha = (\alpha_1, \ldots, \alpha_r) \) of positive integers such that \( \sum_{i=1}^{r} \alpha_i = m \). Let \( C_{n,e} \) be set of all compositions of \( e \) into \( (n-1) \) parts. Given \( \alpha \in C_{n,e} \), let \( g_{\alpha}(p) \) be the number of irreducible subring matrices (cf. Definition 2.3) of the form (1.1). We find that \( g_n(p^e) = \sum_{\alpha \in C_{n,e}} g_{\alpha}(p) \), and thus to compute \( g_n(p^e) \), we need to compute \( g_{\alpha}(p) \) for all \( \alpha \in C_{n,e} \). Therefore, the numbers \( g_{\alpha}(p) \) can be thought of as the basic building blocks for computing the numbers \( f_n(k) \). In practice, it is possible to compute \( g_{\alpha}(p) \) in many cases via combinatorial case by case analysis. For larger values of \( e \), the computation of \( f_n(p^e) \) becomes significantly more involved. The total number of new compositions that one must consider does grow exponentially, and even though one may compute the value of \( g_{\alpha}(p) \) for various types of compositions that fit into a general framework, there are many exceptional compositions that do not fit into such a framework. Furthermore, the combinatorial (case by case) analysis does get increasing more challenging for exceptional compositions of longer length. With this in mind, we state the main result below.

**Theorem 1.4.** With respect to notation above, we have that

\[
f_n(p^g) - \gamma(n, p) = \sum_{i=0}^{n} \sum_{j=1}^{g} \binom{n-1}{j-1} f_{n-j}(p^{g-1}) g_{j}(p^i).
\]

where \( \gamma(n, p) := \left( \sum_{k=2}^{n} \binom{k-1}{5} \right) g_{(3,2,2,1,1)}(p) \).

It proves to be difficult to compute the exact value of \( g_{(3,2,2,1,1)}(p) \) since it reduces to the explicit computation of the number of solutions to the following system of polynomial equations
over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

\[
\begin{align*}
(x_2^2 - x_3) - x_2(x_2^2 - x_7) - x_1(x_2^2 - x_5) &= 0 \\
(x_2^2 - x_4) - x_2(x_2^2 - x_8) - x_1(x_2^2 - x_6) &= 0 \\
x_3x_4 - x_2x_7x_8 - x_1x_3x_6 &= 0.
\end{align*}
\]

There are 3 equations in 8 variables. Computations done for primes $p \leq 19$ suggest that the number of solutions for the above equations is given by $N_p = p^5 + 12p^4 - 20p^3 + 30p^2 - 10p$.

Our computations then suggest that

$$g_{(3,2,2,1,1,1)}(p) = p^7 + 24p^6 - 29p^5 + 21p^4 - 4p^3.$$  

We are unable to verify this claim for primes $p > 19$, since it seems to be a difficult problem in general to compute the number of points on varieties over finite fields in a large number of variables. We note that in very specific cases, exact formulae are known. For instance, in the sense of $\zeta$-functions, one may however bound the size of $g_{(3,2,2,1,1,1)}(p)$, and it follows from Proposition 7.5 that $g_{(3,2,2,1,1,1)}(p) \geq p^4$.

We describe the method of proof in more detail. It follows from results in [Liu07, AKKM21] that $g_n(p)$ can be computed for compositions $\alpha$ of one of the following types

- $\alpha = (\beta, 1, \ldots, 1)$ and $n \geq 2$,
- $\alpha = (2, 1, \ldots, 1, \beta, 1, \ldots, 1)$ and $n \geq 3$.

We refer to results in section 2 for further details. In section 3, we prove some general results which allow us to compute $g_n(p)$ for compositions $\alpha$ of one of the following types

- $\alpha = (\beta, 1, \ldots, 1, \gamma)$ is a composition of length $(n - 1) \geq 3$ such that $\beta > 2$ and $\gamma \geq \beta - 1$,
- $\alpha$ is of the form $\alpha = (2, 1, \ldots, 1, 2, 1, \ldots, 1, \beta)$, where $\beta > 1$,
- $\alpha$ is of the form $\alpha = (2, 1, \ldots, 1, 2, 1, \ldots, 1, \beta, 1)$, where $\beta > 1$,
- $\alpha$ is of the form $\alpha = (2, 1, \ldots, 1, 3, 1, \ldots, 1, 2)$.

We refer to a composition that does not fit into any of the above families as an exceptional composition. In section 4 (resp. section 5), we compute $g_n(p)$ for all relevant compositions beginning with 2 (resp. 3). In section 6, we compute $g_n(p)$ for all relevant compositions beginning with 4, 5 or 6. The proof of Theorem 1.4 is provided in section 7.1. All known computations indicate that for any composition $\alpha$, the function $g_n(p)$ is a polynomial in $p$. The conditions for a matrix $A$ give rise to polynomial conditions on the entries $a_{i,j}$, and $g_n(p)$ in many cases is the number of $\mathbb{F}_p$-points on a scheme defined by integral polynomial equations. It is certainly of interest to note that in all the examples considered, these schemes have polynomial point count in the sense of [HRV08, p.616, ll.-6 to -2].

We now come to discussing the general asymptotic results proved in this manuscript. The analysis of the numbers $f_n(p^e)$ can be translated into properties of the associated subring zeta functions. For instance, Isham in [Ish22b] proves lower bounds for $g_n(p^e)$ and deduces that the subring zeta function $\zeta_{\mathbb{F}_p}(s)$ diverges for all $s$ such that $\text{Re} s \leq c_7(n)$, where $c_7(n) := \max_{0 \leq d \leq n-1} \frac{d(n-1-d)}{(n-1+d)}$. This comes as a consequence of proving lower bounds for $f_n(p^e)$. Some related results are also proven by Brackenhoff in [Bra09]. Liu attributes the following conjecture regarding the asymptotics for the numbers $f_n(k)$ to Bhargava (cf. [Liu07, p.298]). The conjecture was communicated to Liu via personal communication, cf. loc. cit.

**Conjecture 1.5.** For $n$ odd, $f_n(k) = O(k^{\frac{n-1}{2}+\epsilon})$, and for $n$ even, $f_n(k) = O(k^{\frac{n^2-2n}{6n-8}+\epsilon})$.

In particular, for a fixed prime number $p$ and a fixed value of $n$, the conjecture predicts that

$$f_n(p^e) = \begin{cases} O(p^{\frac{2n(n-1)}{n+1}}) & \text{if } n \text{ is odd,} \\ O(p^{\frac{2n^2-2n}{6n-8}}) & \text{if } n \text{ is odd.} \end{cases}$$

Conjecture 1.5 motivates our results in section 7.2, where we investigate the asymptotics for $g_n(p)$ for certain natural families of compositions $\alpha$. Let $n > 1$ be a natural number and $t$ be in the range $1 \leq t \leq n - 1$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ be a composition of length $(n-1)$ and for $k \in \mathbb{Z}_{\geq 1}$, set $\alpha_{k,t} := (\alpha_1, \alpha_2, \ldots, k\alpha_t, \ldots, \alpha_{n-2}, \alpha_{n-1})$, i.e., the composition obtained
upon multiplying the \( t \)-th coordinate of \( \alpha \) by \( k \). Note that \( g_{\alpha/k,t}(p) \) contributes to \( g_n(p^{s_k}) \), where \( e_k := \sum_{i=1}^{n-1} \alpha_i + (k-1)\alpha_t \). Therefore, the asymptotic growth of \( g_{\alpha/k,t}(p) \) as a function of \( k \) provides insight into that of \( g_n(p^{s_k}) \). The following result is proven via a combination of techniques from the geometry of numbers and the combinatorial analysis of large subring matrices.

**Theorem 1.6.** Let \( n \geq 1 \) be a natural number and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \) a composition of length \((n-1)\) such that \( \alpha_i > 1 \) are integers for each \( i \in \{1, 2, \ldots, n-1\} \) and \( 1 \leq t \leq n-1 \). Then,

\[
ge_{\alpha/k,t}(p) = O(p^{\gamma k(n-2)}) \quad \text{as} \quad k \to \infty,
\]

where \( \gamma = \max\{\alpha_1, \ldots, \alpha_t\} \). The implied constant depends on \( p \) and \( n \), and not on \( k \).

Although the above result is weaker than the prediction of Conjecture 1.5, the authors expect that such arguments have the potential to lead to stronger results in the future.

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## 2. Preliminaries

Let \( n \geq 1 \) be an integer and \( M_n(Z) \) denote the ring of \( n \times n \)-matrices with integer entries. By definition, a **lattice** in \( Z^n \) is a subgroup \( L \) of finite index in \( Z^n \). This index is denoted \([Z^n : L]\). Set \( u_1, \ldots, u_n \) to be the standard basis of \( Z^n \), with \( u_i = (0, \ldots, 0, 1, 0, \ldots) \), with 1 in the \( i \)-th position and 0s in all other positions. Let \( v_1, \ldots, v_n \) be a basis of \( L \), i.e., a set of vectors such that every element \( v \in L \) is uniquely represented as an integral linear combination \( v = \sum_{i=1}^{n} b_i v_i \). Expressing \( v_i = \sum_{j=1}^{n} a_{i,j} u_j \), consider the integer matrix \( A = (a_{i,j}) \in M_n(Z) \). We may choose \( v_1, \ldots, v_n \) such that the associated matrix \( A \) is in Hermite normal form, i.e.,

\[
A = \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
av_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
v_{n,1} & \cdots & \cdots & \cdots & a_{n,n} \\
\end{pmatrix}
\]

with \( 0 \leq a_{i,j} < a_{i,i} \) for all tuples \((i, j)\) such that \( 1 \leq i < j \leq n \). The following result is used to reinterpret counting problems for subrings of \( Z^n \) in terms of integral matrices satisfying prescribed conditions.

**Lemma 2.1.** There is a bijection between lattices \( L \subset Z^n \) of index \( k > 0 \) and integer \( n \times n \) matrices \( A \) in Hermite normal form with determinant \( k \).

**Proof.** The reader is referred to the proof of [Liu07, Proposition 2.1]. \( \square \)

Thus, to a lattice \( L \) we associate the matrix \( A_L \) in Hermite normal form, and to a matrix \( A \) in Hermite normal form, we associate a unique lattice \( L_A \). Given two integral vectors \( u = (u_1, \ldots, u_n)^t \) and \( v = (v_1, \ldots, v_n)^t \), denote the **composite** by \( u \circ v := (u_1 v_1, \ldots, u_n v_n)^t \). A lattice \( L \) is multiplicatively closed if \( u \circ v \in L \) for all elements \( u, v \in L \). A subring of \( Z^n \) shall in this paper be taken to mean a multiplicatively closed lattice which contains the identity element \( 1 := (1, 1, \ldots, 1)^t \). In particular, the index of a subring is finite. Given a positive integer \( k \), let \( f_n(k) \) be the number of subrings of \( Z^n \) with index equal to \( k \). Define the subring zeta function as follows

\[
\zeta_{Z^n}^R(s) := \sum_{k=1}^{\infty} f_n(k) k^{-s}.
\]

The function \( f_n(k) \) is multiplicative (cf. [Liu07, Proposition 2.7]), i.e., given two coprime integers \( k_1 > 0 \) and \( k_2 > 0 \), we have that \( f_n(k_1 k_2) = f_n(k_1) f_n(k_2) \). The zeta function exhibits an Euler product

\[
\zeta_{Z^n}^R(s) = \prod_p \zeta_{Z^n,p}^R(s),
\]
where \( \zeta_{Z^n,p}(s) = \sum_{n=0}^{\infty} f_n(p^n)s^{-n} \). Thus, the study of the function \( f_n(k) \) and the subring zeta function, comes down to the determination of \( f_n(p^n) \), where \( n > 0, c \geq 0 \) and \( p \) is a prime number. More precisely, given \( n \) and \( e \), we would like to determine \( f_n(p^e) \) as a function of \( p \).

We recall Liu’s bijection between subrings of \( Z^n \) and integral matrices in Hermite normal form.

**Proposition 2.2.** Let \( n,k > 0 \) be integers. The association \( L \mapsto A_L \) gives a bijection between subrings \( L \) of \( Z^n \) of index \( k \) and matrices \( A \in M_n(Z) \) in Hermite normal form with \( \det(A) = k \) and columns \( v_1, \ldots, v_n \) such that

1. \( 1 \) is in the column span of \( A \).
2. for all \( i,j \) in the range \( 1 \leq i,j \leq n \), \( v_i \circ v_j \) is in the column span of \( A \).

**Proof.** The reader is referred to [Liu07, Proposition 2.1, 2.2]. \( \square \)

**Definition 2.3.** A matrix \( A \) satisfying the conditions of Proposition 2.2 is referred to as a subring matrix. The column span of \( A \) is denoted \( \text{Col}(A) \).

Let \( \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \) be a composition of length \((n-1)\). Recall from the introduction that \( g_\alpha(p) \) is the number of irreducible subring matrices with diagonal \((p^{\alpha_1}, \ldots, p^{\alpha_{n-1}}, 1)\). We recall some useful results from [AKKM21, Liu07] which shall allow us to deduce the values of \( g_\alpha(p) \) for specific choices of \( \alpha \).

**Lemma 2.4.** Let \( n \geq 2 \) and \( \alpha = (\beta,1,\ldots,1) \) be a composition of length \((n-1)\). Then, the following assertions hold.

1. If \( \beta = 2 \), then \( g_\alpha(p) = p^{n-2} \).
2. If \( \beta \geq 3 \), then \( g_\alpha(p) = (n-1)p^{n-2} \).

**Proof.** The above result is [AKKM21, Lemma 3.5]. \( \square \)

**Lemma 2.5.** Let \( n \geq 3 \) and let \( \alpha = (2,1,\ldots,1,\beta,1,\ldots,1) \) be a composition of length \((n-1)\) where \( \beta \) is in the \( k \)-th position. Let \( r = n-1-k \), i.e., the number of \( 1 \)s after \( \beta \). Then, the following assertions hold.

1. If \( \beta = 2 \), then, \( g_\alpha(p) = p^{n-3+r} + (r+1)p^{n-3}(p-1) \).
2. If \( \beta \geq 3 \), then \( g_\alpha(p) = (r+1)\left(p^{n-3+r} + p^{n-3}(p-1)\right) \).

**Proof.** The above result is [AKKM21, Lemma 3.6]. \( \square \)

3. General results for computing the value of \( g_\alpha(p) \)

In this section, we prove a number of general results computing \( g_\alpha(p) \) for various choices of \( \alpha \). Given an irreducible subring with associated matrix \( A \in M_n(Z) \), recall that \( v_1, \ldots, v_n \) shall denote the columns of \( A \), where \( v_i \) is the \( i \)-th column.

**Lemma 3.1.** Let \( n \geq 4 \), \( \alpha = (\beta,1,\ldots,1,\gamma) \) be a composition of length \((n-1)\) such that \( \beta > 2 \) and \( \gamma \geq \beta - 1 \). Then, we have \( g_\alpha(p) = p^{n-3+\lceil \frac{\beta}{2} \rceil} + (n-3)p^{n-2} \).

**Proof.** Consider a matrix \( A \in M_n(Z) \) in Hermite normal form

\[
A = \begin{pmatrix}
p^\beta & p a_1 & p a_2 & \cdots & p a_{n-2} & 1 \\
p & 0 & 0 & \cdots & 0 & 1 \\
p & \cdots & 0 & \cdots & \cdots & \cdots \\
p & \cdots & \cdots & \cdots & \cdots & \cdots \\
p^\gamma & 1 \\
1,
\end{pmatrix}
\]

where the entries \( a_1, \ldots, a_{n-2} \) satisfy \( 0 \leq a_i \leq p^{\beta-1} - 1 \). First we write down conditions for \( v_j^2 \in \text{Col}(A) \). Clearly, this condition is automatically satisfied for \( j = 1 \) and \( j = n \). Consider the
values of $j$ that lie in the range $2 \leq j \leq n - 1$. We may write $j = i + 1$, where $i$ lies in the range $1 \leq i \leq n - 2$. First, we consider the case when $i \leq n - 3$. Observe that

$$v_{i+1}^2 = (a_i^2, p^2, 0, \ldots, 0, p^2, 0, \ldots, 0)^t.$$  

Note that $v_{i+1}^2$ is contained in $\text{Col}(A)$ if and only if $v_{i+1}^2 - pv_{i+1}$ is contained in $\text{Col}(A)$. We find that

$$v_{i+1}^2 - pv_{i+1} = ((a_i^2 - a_i)p^2, 0, \ldots, 0)^t.$$  

Therefore, $v_{i+1}^2$ is contained in $\text{Col}(A)$ if and only if

$$a_i^2 - a_i \equiv 0 \pmod{p^3 - 1}.$$  

We find that $v_{n-1}^2 = (a_{n-2}^2, 0, \ldots, 0, p^{2\gamma}, 0)^t$ is in $\text{Col}(A)$ if and only if

$$v_{n-1}^2 - p^\gamma v_{n-1} = (a_{n-2}^2 - a_{n-2} p^{\gamma + 1}, 0, \ldots, 0)^t$$  

is contained in $\text{Col}(A)$. Therefore, we deduce that $v_{n-1}^2$ is in $\text{Col}(A)$ if and only if

$$a_{n-2}^2 - a_{n-2} p^{\gamma + 1} \equiv 0 \pmod{p^3}.$$  

It is assumed that $\gamma \geq \beta - 1$ and hence the above condition is equivalent to

$$a_{n-2}^2 \equiv 0 \pmod{p^{3-2}},$$  

i.e.,

$$a_{n-2} \equiv 0 \pmod{p^{\lceil \frac{3}{2} \rceil - 1}}.$$  

For $1 \leq i < j \leq n - 2$, we find that $v_{i+1} \circ v_{j+1}$ is contained in $\text{Col}(A)$ if and only if

$$a_i a_j \equiv 0 \pmod{p^{3-2}}.$$  

Putting it all together, we find that $A$ is a subring matrix if and only if the following conditions are satisfied

1. $a_i(a_i - 1) \equiv 0 \pmod{p^{3-2}}$ for all $i$ in the range $1 \leq i \leq n - 3$,
2. $a_{n-2} \equiv 0 \pmod{p^{\lceil \beta/2 \rceil - 1}}$,
3. $a_i a_j \equiv 0 \pmod{p^{3-2}}$ for all $i, j$ in the range $1 \leq i < j \leq n - 2$.

From condition (1) above, we find that at most one of $a_i \equiv 1 \pmod{p^{3-2}}$ for $i = 1, \ldots, n - 3$. We thus are led to the following cases.

**Case 1:** First, we consider the case when $a_i \equiv 0 \pmod{p^{3-2}}$ for all $i$ in the range $1 \leq i \leq n - 3$. Since $a_i$ is in the range $0 \leq a_i < p^{\beta - 1}$, we find that there are $p$ choices for each $a_i$ and $p^{\beta - \lceil \beta/2 \rceil} = p^{\lceil \beta/2 \rceil}$ choices for $a_{n-2}$. In total, we find that there are $p^{n-3+\lceil \beta/2 \rceil}$ matrices in this case.

**Case 2:** Consider the case when one of the $a_i$ satisfies $a_i \equiv 0 \pmod{p^{3-2}}$. Then, from (3) we find that

$$a_{n-2} \equiv a_i a_{n-2} \equiv 0 \pmod{p^{3-2}}.$$  

Thus, for each index $i$ in the range $1 \leq i \leq n - 2$, we have $p^{n-2}$ choices for which $a_i \equiv 1 \pmod{p^{3-2}}$. There are $(n - 3)$ values taken by $i$ for which $a_i \equiv 1 \pmod{p^{3-2}}$. Thus, in this case, there are $(n - 3)p^{n-2}$ choices.

Putting together the calculations from cases 1 and 2, we find that $g_\alpha(p) = p^{n-3+\lceil \beta/2 \rceil} + (n - 3)p^{n-2}$.

**Lemma 3.2.** Let $\alpha$ be of the form $\alpha = (2, 1, \ldots, 1, 2, 1, \ldots, 1, \beta)$, where $\beta > 1$ and the second 2 occurs at position $k \geq 2$, then

$$g_\alpha(p) = p^{n-3+r} + p^{n-3}(p - 1)r,$$

where $r = n - 1 - k$. \qed
Proof. Let $A$ be an integral matrix in Hermite normal form

$$
A = \begin{pmatrix}
p^2 & pa_1 & pa_2 & \cdots & pa_{k-1} & \cdots & \cdots & pa_{n-2} & 1 \\
p & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 1 \\
p & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \vdots & 1 \\
p^2 & pb_1 & \cdots & pb_r & 1 \\
p & \cdots & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & 1 \\
p^3 & 1 \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1.
\end{pmatrix}
$$

Since the first entry of $\alpha$ is equal to 2, it is easy to see that $v_1^2$ is in $\text{Col}(A)$ for $i = 1, \ldots, k-1$. We find that $v_r^2$ is in $\text{Col}(A)$ if and only if $v_k^2 - p^2 v_k$ is in $\text{Col}(A)$. Since $p^2$ divides $a_{k-1}^2 p^2 - p^3 a_{k-1}$, this condition is seen to be satisfied. For $i = 1, \ldots, r$, we observe that

$$v_{k+i}^2 = (a_{k-1+i}^2 p^2, \cdots, b_i^2 p^2, \cdots, p^2)^t$$

is in $\text{Col}(A)$ if and only if

$$v_{k+i}^2 = (a_{k-1+i}^2 - a_{k-1+i+1}) p^2, \cdots, (b_i^2 - b_i) p^2, 0, \ldots, 0)^t$$

is in $\text{Col}(A)$. Subtracting $(b_i^2 - b_i) v_k$ from the above, we get

$$v_{k+i}^2 - pv_{k+i} - (b_i^2 - b_i) v_k = \left(\left(a_{k-1+i}^2 - a_{k-1+i+1}\right) p^2 - (b_i^2 - b_i) a_{k-1} p, 0, \ldots, 0\right)^t,$$

which is in $\text{Col}(A)$ if and only if first entry is divisible by $p^2$. We have thus shown that $v_{k+i}^2$ is in $\text{Col}(A)$ if and only if

$$\left(b_i^2 - b_i\right) a_{k-1} \equiv 0 \mod p.$$ (3.5)

Now, $v_{n-1}^2 = (a_{n-2}^2 p^2, \cdots, b_i^2 p^2, 0, \ldots, 0, p^2 a, 0, \ldots, 0)^t$ is in $\text{Col}(A)$ if and only if $v_{n-1}^2 - p^2 v_{n-1}$ is in $\text{Col}(A)$. Note that therefore, $v_{n-1}^2$ is in $\text{Col}(A)$ if and only if $b_i a_{k-1} \equiv 0 \mod p$.

It is clear that for all $1 \leq i \leq k$ and all values of $j$, $v_i v_j \in \text{Col}(A)$. Now consider $v_{k+i} v_{k+j}$ for $1 \leq i < j \leq r$, it is equal to

$$\left(a_{k-1+i} a_{k-1+j} p^2, 0, \ldots, 0, b_i b_j p^2, 0, \ldots, 0\right)^t.$$ We find that $v_{k+i} v_{k+j}$ is contained in $\text{Col}(A)$ if and only if

$$v_{k+i} v_{k+j} = b_i b_j v_{k-1} \in \text{Col}(A).$$

Therefore, $v_{k+i} v_{k+j}$ is contained in $\text{Col}(A)$ if and only if $b_i b_j a_{k-1} \equiv 0 \mod p$. Therefore, $A$ is a subring matrix if and only if

1. $(b_i^2 - b_i) a_{k-1} \equiv 0 \mod p$ for $i = 1, \ldots, r - 1$,
2. $b_i a_{k-1} \equiv 0 \mod p$,
3. $b_i b_j a_{k-1} \equiv 0 \mod p$, for all $i, j$ such that $1 \leq i < j \leq r$.

We consider two cases.

Case 1: Consider the case when $a_{k-1} = 0$. In this case, the total number of choices is the total number of choices of $(a_1, \ldots, a_{k-2}, a_{k+1}, \ldots, a_{n-2})$ and $(b_1, \ldots, b_r)$. Thus, the total number of choices are $p^{n-3+k}$.

Case 2: Consider the other case, i.e., when $a_{k-1} \neq 0$. Since $a_{k-1} < p$, we find that $p \nmid a_{k-1}$.

Therefore, the conditions are as follows

1. $b_i^2 - b_i \equiv 0 \mod p$,
2. $b_i \equiv 0 \mod p$,
3. $b_i b_j \equiv 0 \mod p$, $1 \leq i < j \leq r$.

Note that all elements $b_i$ satisfy the bounds $0 \leq b_i < p$. Therefore, at most one of the $b_i$ satisfies $b_i = 1$. Subdivide into two cases, first consider the case when all the $b_i$ are equal to 0 and then consider the case when at most one of the $b_i$ is equal to 1. The total number of choices is
Lemma 3.3. Let \( \alpha \) be of the form \( \alpha = (2, 1, \ldots, 1, \beta, 1) \), where \( \beta > 1 \), the second 2 occurs at position \( k \geq 2 \) and set \( r := n - k \). Then, the following assertions hold.

1. If \( \beta = 2 \), then we find that
   \[
   g_\alpha(p) = 2p^{n-3+r} + p^{n-5+r}(p-2) + 2rp^{n-3}(p-1) + p^{n-4}(p-1)(p+r-2).
   \]

2. If \( \beta > 3 \), then,
   \[
   g_\alpha(p) = 2p^{n-3+r} + 2p^{n-5+r}(p-1) + 2rp^{n-3}(p-1) + 2p^{n-4}(p-1)^2(p+r-2).
   \]

Proof. Let \( A \) be an integral matrix in Hermite normal form

\[
A = \begin{pmatrix}
  p^2 & pa_1 & pa_2 & \cdots & pa_{k-1} & \cdots & \cdots & pa_{n-2} & 1 \\
p & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 1 \\
p & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \n\vdots & \vdots & \vdots & \vdots & p^2 & pb_1 & \cdots & \cdots & pb_r & 1 \\
p & \cdots & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & 1 \\
p^3 & pc_1 & 1 \\
p & 1 \\
1
\end{pmatrix}
\]

Since the first entry is \( p^2 \), it is clear that \( v_i^2 \) is contained in \( \text{Col}(A) \) for \( i = 1, 2, \ldots, k \). For \( i = 1, 2, \ldots, r - 2 \), we find that \( v_i^2 \) is contained in \( \text{Col}(A) \) if and only if \( v_{k+i}^2 \) is contained in \( \text{Col}(A) \). So,

\[
(a_{k-1+i}^2p^2 - a_{k-1}p^2, \ldots, b_j^2p^2 - b_jp^2, \ldots)^t
\]

should be in \( \text{Col}(A) \). It is clear that this is true if and only if

\[
(3.6) \quad a_{k-1}(b_i^2 - b_i) \equiv 0 \mod p
\]

Similarly we see that \( v_{n-2}^2 \) is contained in \( \text{Col}(A) \) if and only if \( v_{n-2}^2 - p^3v_{n-2} \) is contained in \( \text{Col}(A) \). Now,

\[
(a_{n-3}^2b^2 - a_{n-3}p\beta^3 + 1, \ldots, b_{r-1}^2p^2 - b_{r-1}p\beta^3 + 1, \ldots)^t
\]

should be in \( \text{Col}(A) \). It is again clear that this is true if and only if

\[
(3.7) \quad a_{k-1}b_{r-1} \equiv 0 \mod p
\]

as \( \beta > 1 \). Again, \( v_{n-1}^2 \) is contained in \( \text{Col}(A) \) if and only if \( v_{n-1}^2 - pv_{n-1} \) is contained in \( \text{Col}(A) \), so

\[
(a_{n-2}^2p^2 - a_{n-2}p^2, \ldots, (b_r^2 - b_r)p^2, \ldots, (c_i^2 - c_i)^2, 0, 0)^t
\]

should be in \( \text{Col}(A) \). Then we have \( c_i^2 - c_i \equiv 0 \mod p^{\beta-2} \), and

\[
(3.8) \quad b_{r-1}(c_i^2 - c_i) \equiv 0 \mod p^{\beta-1}
\]

\[
(3.9) \quad a_{k-1} \left( b_i^2 - b_i - \frac{b_{r-1}(c_i^2 - c_i)}{p^{\beta-1}} \right) + a_{n-3} \frac{c_i^2 - c_i}{p^{\beta-2}} \equiv 0 \mod p
\]

The requirement that \( v_i v_j \) belongs to \( \text{Col}(A) \) for \( i \neq j \) translates the following conditions on entries of the matrix (for all \( 1 \leq i < j \leq r \) and \( \{i, j\} \neq \{r - 1, r\} \))

\[
(3.10) \quad a_{k-1}b_i b_j \equiv 0 \mod p
\]

First, we suppose that \( \beta \geq 3 \), then we have conditions,

1. \( a_{k-1}(b_i^2 - b_i) \equiv 0 \mod p, i = 1, \ldots, r - 2 \),
2. \( b_{r-1}a_{k-1} \equiv 0 \mod p \),
3. \( c_i^2 - c_i \equiv 0 \mod p^{\beta-2} \),
(4) \( b_{r-1}(c_i^2 - c_1) \equiv 0 \mod p^{3-1} \),
(5) \( a_{k-1} \left( b_i^2 - b_r - \frac{b_{r-1}(c_i^2 - c_1)}{p^{3-1}} \right) + a_{n-3} \frac{c_i^2 - c_1}{p^{3-1}} \equiv 0 \mod p, \)
(6) \( a_{k-1} b_i b_j \equiv 0 \mod p, 1 \leq i < j \leq r, \) and \( \{i, j\} \neq \{r-1, r\} \).

We consider two cases.

Case 1: First, we consider the case when \( a_{k-1} = 0 \).
Then, the above equations reduce to the following:
(1) \( c_i^2 - c_1 \equiv 0 \mod p^{3-2} \),
(2) \( b_{r-1}(c_i^2 - c_1) \equiv 0 \mod p^{3-1} \),
(3) \( a_{n-3}(c_i^2 - c_1) \equiv 0 \mod p \).

If \( c_1 \in \{0, 1\} \), then the number of such matrices is \( 2p^{n-3+r} \). On the other hand, if \( c_1 \notin \{0, 1\} \),
then, there are \( 2p - 2 \) choices for \( c_1 \). Since \( b_{r-1} = 0, a_{n-3} = 0 \), we deduce that there are
\( 2(p-1)p^{n+r-5} \) more matrices.

Case 2: Suppose \( a_{k-1} \neq 0 \), then equations reduce to
(1) \( b_i^2 - b_1 \equiv 0 \mod p, i = 1, \ldots, r - 2, \)
(2) \( b_{r-1} = 0, \)
(3) \( c_i^2 - c_1 \equiv 0 \mod p^{3-2} \),
(4) \( a_{k-1} \left( b_i^2 - b_r \right) + a_{n-3} \frac{c_i^2 - c_1}{p^{3-2}} \equiv 0 \mod p, \)
(5) \( b_i b_j \equiv 0 \mod p, 1 \leq i < j \leq r, \) and \( \{i, j\} \neq \{r-1, r\} \)

We divide into two sub-cases.

- Case 2A: Suppose that \( c_1 \in \{0, 1\} \), then we get that \( b_r \in \{0, 1\} \) and as in Lemma 3.2,
we get that at most one of the \( b_i \) satisfies \( b_i = 1 \). Subdivide into two cases, first consider
the case when all the \( b_i \) are equal to 0 and then consider the case when at most one of
the \( b_i \) is equal to 1. Therefore, the total number of choices is \( 2r p^{n-3}(p-1) \).

- Case 2B: Consider the other subcase, i.e., \( c_1 \notin \{0, 1\} \). Then, there is a unique value
of \( a_{n-3} \) for each value of \( b_r, a_{k-1}, c_1 \), so we get \( 2p^{n-3}(p-1)^2(p + r - 2) \) matrices.

Putting it all together, the result is proven. The case \( \beta = 2 \) is similar, and the number of
matrices change only in Case 1, with \( c_1 \notin \{0, 1\} \) and Case 2B.

Lemma 3.4. Let \( \alpha \) be of the form \( \alpha = (2, 1, \ldots, 1, 3, 1, \ldots, 1, 2) \), where 3 occurs at position
\( k \geq 2 \). Then, we have that
\[
g_\alpha(p) = rp^{n-3+r} + p^{n-3}(p-1)(p + r - 1),
\]
where \( r := n - 1 - k \).

Proof. Let \( A \) be an integral matrix in Hermite normal form
\[
A = \begin{pmatrix}
p^2 & pa_1 & pa_2 & \cdots & pa_{k-1} & \cdots & pa_{n-2} & 1 \\
p & 0 & \cdots & 0 & \cdots & 0 & 1 \\
p & 0 & \cdots & 0 & \cdots & 0 & 1 \\
p & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
p^2 & pb_1 & \cdots & pb_r & 1 \\
p & 0 & \cdots & 0 & \cdots & 0 & 1 \\
p & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
p^2 & 1 & \cdots & 0 & \cdots & 0 & 1 \\
p^2 & 1 & \cdots & 0 & \cdots & 0 & 1 \\
\end{pmatrix}
\]
Since the first diagonal entry is \( p^2 \), we see that \( v_i^2 \) is contained in \( \text{Col}(A) \) for all \( i = 1, \ldots, k \).
Note that \( v_{k+i}^2 \in \text{Col}(A) \) if and only if \( v_{k+i}^2 - pv_{k+i} \in \text{Col}(A) \) for \( i = 1, 2, \ldots, r - 1 \). Hence, we find that
\[
\left( (a_{k+i}^2 - a_{k-1+i})p^2, 0, \ldots, 0, (b_i^2 - b_1)p^2, 0, \ldots, 0 \right)^t
\]
should be in \( \text{Col}(A) \). We deduce that that this is the case if and only if the following conditions are satisfied

\[
\begin{align*}
& b_i^2 - b_i \equiv 0 \pmod{p}, \\
& a_{k-1}(b_i^2 - b_i) \equiv 0 \pmod{p^2}.
\end{align*}
\]

It is also required that \( v_{n-1}^2 \) is contained in \( \text{Col}(A) \), which is the case if and only if \( v_{n-1}^2 - p^2v_{n-1} \) is contained in \( \text{Col}(A) \). Therefore, we find that

\[
\left((a_n^2 - a_{n-2}p)^2, 0, \ldots, 0, (b_i^2 - brp)p^2, 0, \ldots, 0\right)^t
\]

should be in \( \text{Col}(A) \). We get that \( b_i^2 - brp \equiv 0 \pmod{p} \) and \( a_{k-1}(b_i^2 - brp) \equiv 0 \pmod{p^2} \), therefore, we find that \( p \mid brp \). Using similar arguments we deduce that \( v_{k+i}v_{k+j} \in \text{Col}(A) \) for \( 1 \leq i < j \leq r \) if and only if the following conditions are satisfied

\[
\begin{align*}
& b_i b_j \equiv 0 \pmod{p}, \\
& a_{k-1}b_i b_j \equiv 0 \pmod{p^2}.
\end{align*}
\]

It is clear that \( v_i v_j \) is contained in \( \text{Col}(A) \) for \( 1 \leq i < j \leq k \). Therefore, we have following conditions on the entries of \( A \)

1. \( b_i^2 - b_i \equiv 0 \pmod{p} \) and \( a_{k-1}(b - i^2 - b_i) \equiv 0 \pmod{p^2} \) for \( i = 1, \ldots, r-1 \), 
2. \( b_r = b_r'p \), where \( 0 \leq b_r' \leq p - 1 \), 
3. \( b_i b_j \equiv 0 \pmod{p} \) and \( a_{k-1}b_i b_j \equiv 0 \pmod{p^2} \) for all values \( 1 \leq i < j \leq r \).

We consider two cases as follows.

**Case 1**: First, we consider the case when \( a_{k-1} = 0 \). In this case, the conditions on \( A \) reduce to the following

1. \( b_i^2 - b_i \equiv 0 \pmod{p} \) for all \( i = 1, \ldots, r-1 \), 
2. \( b_r = b_r'p \), where \( 0 \leq b_r' \leq p - 1 \), 
3. \( b_i b_j \equiv 0 \pmod{p} \) for \( 1 \leq i < j \leq r \).

As in Lemma 3.3, either all the values of \( b_i \) for \( 1 \leq i \leq r-1 \) are \( 0 \pmod{p} \), or at most one of these values is \( 1 \pmod{p} \). Further dividing into cases, it is easy to see that the number of matrices is \( r^{p^{n-3+r}} \).

**Case 2**: Next, we consider the other case, namely assume that \( a_{k-1} \neq 0 \). The conditions on \( A \) then reduce to the following

1. \( b_i^2 - b_i \equiv 0 \pmod{p^2} \) for all \( i = 1, \ldots, r-1 \), 
2. \( b_r = b_r'p \), where \( 0 \leq b_r' \leq p - 1 \), 
3. \( b_i b_j \equiv 0 \pmod{p^2} \) for all \((i, j)\) satisfying \( 1 \leq i < j \leq r \).

Consequently, it follows that either \( b_i = 0 \) for all \( i = 1, \ldots, r-1 \), or at most one of them is 1. Therefore, we find that the number of matrices in this case is equal to \( p^{n-3}(p-1)(p+r-1) \) matrices. Adding up our conclusions, we prove the assertion of the lemma.

4. **Calculating the values of \( g_n(p) \) for compositions beginning with 2**

4.1. **Compositions of length 4**. We consider the compositions \( \alpha \) of length 4. In all, there are 15 of them, listed below

\[
\begin{array}{ll}
(2, 5, 1, 1) & (2, 1, 2, 4) \\
(2, 1, 5, 1) & (2, 3, 3, 1) \\
(2, 1, 1, 5) & (2, 3, 1, 3) \\
(2, 4, 2, 1) & (2, 1, 3, 3) \\
(2, 4, 1, 2) & (2, 3, 2, 2) \\
(2, 2, 4, 1) & (2, 2, 3, 2) \\
(2, 2, 1, 4) & (2, 2, 2, 3) \\
(2, 1, 4, 2) & \\
\end{array}
\]

It follows directly from Lemmas 2.5, 3.2 and 3.3 that
Proof. Let \( g \) be the matrix,
\[
\begin{pmatrix}
 p^2 & a_1p & a_2p & a_3p & 1 \\
 0 & p & 0 & 0 & 1 \\
 0 & 0 & p^4 & b_1p & 1 \\
 0 & 0 & 0 & p^2 & 1 \\
 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
where \( 0 \leq a_1 \leq p - 1 \) and \( 0 \leq b_1 \leq p^3 - 1 \). We arrive at conditions for \( A \) to be a subring matrix. It is easy to see that
- \( v_3^2, v_2v_3, v_2v_4, v_3v_4 \) are in \( \text{Col}(A) \).
- We find that \( v_1^3 \) is in \( \text{Col}(A) \) if and only if \( v_1^3 - p^2v_4 \) is contained in \( \text{Col}(A) \). In other words,
\[
(a_3^2p^2, 0, b_1^2p^2, p^4, 0)^t - p^2(a_3p, 0, b_1p, p^2, 0)^t
\]
\[
= \left( (a_3^2p^2 - a_3p)^2, 0, (b_1^2p^2 - b_1p^2), 0, 0 \right)^t
\]
is in \( \text{Col}(A) \). The expression \( (b_1^2p^2 - b_1p^2) = b_1(b_1 - p)p^2 \) must be divisible by \( p^4 \) and moreover, we find that \( v_3^2 - p^2v_4 \) is contained in \( \text{Col}(A) \) if and only if \( v_3^2 - p^2v_4 = \frac{b_1(b_1 - p)p^2}{p}v_4 \) is in \( \text{Col}(A) \). This holds if and only if in addition, we have that \( b_1(b_1 - p)a_2 \equiv 0 \mod p^3 \).

We deduce from above that the necessary conditions are as follows
\begin{enumerate}
  \item \( b_1(b_1 - p) \equiv 0 \mod p^2 \),
  \item \( b_1(b_1 - p)a_2 \equiv 0 \mod p^3 \).
\end{enumerate}
From equation (1), we get that \( b_1 = b_1'p \), where \( 0 \leq b_1' \leq p^2 - 1 \). Equation (2) asserts that \( b_1'(b_1' - 1)a_2 \equiv 0 \mod p \). Consider the following case decomposition.
- **Case 1**: Assume that \( a_2 = 0 \). In this case, the number of such matrices is \( p^4 \).
- **Case 2**: Consider the other case, i.e., \( a_2 \neq 0 \). Then, we have \( b_1'(b_1' - 1) \equiv 0 \mod p \).

Hence, the total number of such matrices is easily seen to be \( 2p^3(p - 1) \).

We conclude from the above that \( g_{(2,1,4,2)}(p) = p^4 + 2p^3(p - 1) = 3p^4 - 2p^3 \). \( \square \)

**Lemma 4.2.** With respect to notation above, we find that \( g_{(2,1,4,2)}(p) = p^3 + 3p^4 - p^3 - p^2 \).

Proof. Let \( A \) be the matrix
\[
\begin{pmatrix}
 p^2 & a_1p & a_2p & a_3p & 1 \\
 0 & p^4 & b_1p & b_2p & 1 \\
 0 & 0 & p & 0 & 1 \\
 0 & 0 & 0 & p^2 & 1 \\
 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
where \( 0 \leq a_i \leq p - 1 \) and \( 0 \leq b_j \leq p^3 - 1 \). We obtain conditions for \( A \) to be a subring matrix.

By the same arguments as in Lemma 4.1, we find that
- \( v_3^2, v_2v_3, v_2v_4 \) are in \( \text{Col}(A) \).
- We find that \( v_3^2 \) is in \( \text{Col}(A) \) if and only if \( b_1(b_1 - 1) \equiv 0 \mod p^2 \) and \( b_1(b_1 - 1)a_1 \equiv 0 \mod p^3 \).
- We find that \( v_4^2 \) is in \( \text{Col}(A) \) if and only if \( b_2 = b_2'p, 0 \leq b_2' \leq p^2 - 1 \) and \( b_2'(b_2' - 1)a_1 \equiv 0 \mod p \).

\( g_{(2,1,4,2)}(p) \) for the rest of the compositions. We note that many arguments are similar, and we summarize the arguments that tend to repeat.

**Lemma 4.1.** With respect to notation above, we have that \( g_{(2,1,4,2)}(p) = 3p^4 - 2p^3 \).

Proof. Let \( A \) be the matrix,
\[
\begin{pmatrix}
 p^2 & a_1p & a_2p & a_3p & 1 \\
 0 & p & 0 & 0 & 1 \\
 0 & 0 & p^4 & b_1p & 1 \\
 0 & 0 & 0 & p^2 & 1 \\
 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
We find that $v_3v_4$ is in $\text{Col}(A)$ if and only if $b_1b'_2 \equiv 0 \mod p$ and $a_1b_1b'_2 \equiv 0 \mod p^2$.

Hence, $A$ is a subring matrix if and only if

1. $b_1(b_1 - 1) \equiv 0 \mod p^2$,
2. $a_1b_1(b_1 - 1) \equiv 0 \mod p^3$,
3. $b_2 = b'_2p$ for $0 \leq b'_2 \leq p^2 - 1$,
4. $a_1b'_2(b'_2 - 1) \equiv 0 \mod p$,
5. $b_1b'_2 \equiv 0 \mod p$,
6. $a_1b_1b'_2 \equiv 0 \mod p^2$.

In order to compute the total number of matrices satisfying all of the above conditions, we consider the following cases.

- **Case 1**: Assume that $a_1 = 0$. If $b_1 \equiv 0 \mod p^2$ then number of such matrices is $p^6$ otherwise $b_1 \equiv 1 \mod p^2$, number of such matrices is $p^4$.
- **Case 2**: Consider the case when $a_1 \neq 0$. We get $b_1 = 0$ or $b_1 = 1$. If $b_1 = 0$, number of such matrices is $2p^3(p - 1)$. Otherwise $b_1 = 1$ and number of such matrices is $p^2(p - 1)$ (as $b_2 = 0$ in this case).

Therefore, we find that $g_{(2,4,1,2)}(p) = p^5 + p^4 + 2p^3(p - 1) + p^2(p - 1) = p^5 + 3p^4 - p^3 - p^2$. □

**Lemma 4.3.** We have that

\[
g_{(2,3,1,3)}(p) = 3p^4 - p^2,
\]

\[
g_{(2,1,3,3)}(p) = p^4,
\]

\[
g_{(2,2,3,2)} = p^5 + p^4 - p^3.
\]

**Proof.** The proof is similar to Lemma 4.2, and we omit it. □

**Lemma 4.4.** We have that $g_{(2,3,2,2)}(p) = p^5 + 4p^4 - 9p^3 + 4p$.

**Proof.** Let $A$ be the matrix,

\[
\begin{pmatrix}
p^2 & a_1p & a_2p & a_3p & 1 \\
0 & p^3 & b_1p & b_2p & 1 \\
0 & 0 & p^2 & c_1p & 1 \\
0 & 0 & 0 & p^3 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

where $0 \leq a_i \leq p - 1$, $0 \leq b_j \leq p^2 - 1$ and $0 \leq c_1 \leq p - 1$. Below are the conditions for $A$ to be a subring matrix.

- It is easy to see that $v_3^2$ is in $\text{Col}(A)$.
- We find that $v_3^2$ is contained in $\text{Col}(A)$ if and only if $v_3^2 - p^2v_3$ is contained in $\text{Col}(A)$. Hence, we find that $v_3^2 \in \text{Col}(A)$ if and only if $b_1 = b'_1p$, where $0 \leq b'_1 \leq p - 1$.
- Similar reasoning shows that $v_3^2 \in \text{Col}(A)$ if and only if $b_2^2 - (c_1^2 - c_1)b'_1 \equiv 0 \mod p$ and $(\frac{b_2^2 - c_1^2}{p}b'_1 - b_2) a_1 + c_1(c_1 - 1)a_2 \equiv 0 \mod p$.
- We next consider the requirement that $v_i v_j$ is in $\text{Col}(A)$ for distinct values of $i$ and $j$.
- It is easy to see that $v_1v_3, v_1v_4, v_2v_3, v_2v_4$ and $v_2v_4$ are in $\text{Col}(A)$ and that $v_3v_4$ is in $\text{Col}(A)$ if and only if $b'_1a_1(b_2 - c_1) \equiv 0 \mod p$.

Summarizing the above, the conditions we arrive at are as follows

1. $b_1 = b'_1p$, where $0 \leq b'_1 \leq p - 1$,
2. $b_2^2 - (c_1^2 - c_1)b'_1 \equiv 0 \mod p$,
3. $(\frac{b_2^2 - c_1^2}{p}b'_1 - b_2) a_1 + c_1(c_1 - 1)a_2 \equiv 0 \mod p$,
4. $b'_1a_1(b_2 - c_1) \equiv 0 \mod p$.

Consider the following cases.

- **Case 1**: First consider the case when $a_1 = 0$. We further subdivide our argument into the following subcases.
  - **Case 1A**: Assume that $c_1 \in \{0, 1\}$. Then, we find that $b_2 \equiv 0 \mod p$. There are $2p^4$ such matrices.
○ **Case 1B**: Assume that $c_1 \notin \{0, 1\}$. In this case, we get $a_2 = 0$, and thus, there are $p^3(p-2)$ such matrices.

- **Case 2**: Next consider the case when $a_1 \neq 0$. Then $b_1(b_2 - c_1) \equiv 0 \mod p$. We have the following subcases.
  - **Case 2A**: Assume that $b_1' = 0$, then we find that $b_2 \equiv 0 \mod p$. Now if $c_1 \in \{0, 1\}$ then we get $2p^3(p-1)$ such matrices. If $c_1 \notin \{0, 1\}$ we get $p^5(p-1)(p-2)$ such matrices.
  - **Case 2B**: Assume that $b_1' \neq 0$, then we get $b_2 \equiv c_1 \mod p$. If $c_1 = 0$ we get $p^3(p-1)^2$ such matrices. If $c_1 = 1$ we get no subring matrices as we get $b_2 \equiv 1 \mod p$ and $b_2 \equiv 0 \mod p$, but no such integer exists. If $c_1 \notin \{0, 1\}$, we get $p^5(p-1)(p-2)$ subring matrices.

Adding up the numbers obtained in the above case decomposition, we find that $g_{(2,3,2,2)}(p) = p^5 + 4p^4 - 9p^3 + 4p$.

**Lemma 4.5.** We have that

\[
g_{(2,4,2,1)}(p) = 2p^5 + 8p^4 - 15p^3 + 6p^2, \\
g_{(2,3,3,1)}(p) = 12p^4 - 10p^3 + 2p^2.
\]

**Proof.** The proof is similar to that of Lemma 4.4, and is omitted.

4.2. **Compositions of length 5.** In this subsection we consider the compositions of length 5 that begin with 2. We need to compute $g_5(p)$ for 20 compositions of this form. They are listed below,

| (2, 4, 1, 1, 1) | (2, 1, 4, 1, 1) |
| (2, 1, 1, 4, 1) | (2, 1, 1, 1, 4) |
| (2, 3, 2, 1, 1) | (2, 3, 1, 2, 1) |
| (2, 3, 1, 1, 2) | (2, 2, 3, 1, 1) |
| (2, 2, 1, 3, 1) | (2, 2, 1, 1, 3) |
| (2, 1, 3, 1, 2) | (2, 1, 3, 2, 1) |
| (2, 1, 2, 3, 1) | (2, 1, 2, 1, 3) |
| (2, 1, 1, 3, 2) | (2, 1, 1, 2, 3) |
| (2, 2, 2, 2, 1) | (2, 2, 2, 1, 2) |
| (2, 2, 1, 2, 1) | (2, 2, 1, 2, 1) |
| (2, 1, 2, 2, 2) | (2, 1, 2, 2, 2) |

**Table 1.** Compositions of length 5 that begin with 2

In the next two lemmas we obtain values of $g_5(p)$ for all the compositions listed above.

**Lemma 4.6.** We have following values

| $g_{(2,4,1,1,1)}(p)$ | $p^5 + 4p^4 - 4p^3$ |
| $g_{(2,1,4,1,1)}(p)$ | $p^5 + 3p^4 - 3p^3$ |
| $g_{(2,1,1,4,1)}(p)$ | $3p^5 - 2p^3$ |
| $g_{(2,1,1,1,4)}(p)$ | $p^4$ |
| $g_{(2,3,1,1,2)}(p)$ | $3p^6 + p^5 + p^4 - 2p^3$ |
| $g_{(2,1,3,1,2)}(p)$ | $3p^5 - p^3$ |
| $g_{(2,1,3,2,1)}(p)$ | $p^5$ |
| $g_{(2,2,1,1,3)}(p)$ | $p^6 + 3p^4 - 3p^3$ |
| $g_{(2,1,2,1,3)}(p)$ | $p^5 + 2p^4 - 2p^3$ |
| $g_{(2,1,1,2,3)}(p)$ | $2p^4 - p^3$ |
| $g_{(2,2,1,3,1)}(p)$ | $2p^6 + 4p^5 + 2p^4 - 8p^3 + 2p^2$ |
| $g_{(2,1,2,3,1)}(p)$ | $4p^5 + 4p^4 - 4p^3$ |

**Proof.** The result follows immediately from Lemmas 2.5, 3.2, 3.3 and 3.4.
Next, we obtain values of \( g_{\alpha}(p) \) for remaining 8 compositions.

**Lemma 4.7.** We have following values,

\[
\begin{align*}
g_{(2,3,2,1,1)}(p) &= 3p^7 + 9p^6 + 7p^5 - 18p^4 - 3p^2 \\
g_{(2,2,3,1,1)}(p) &= p^5 + 5p^3 + 8p^2 - 11p^4 \\
g_{(2,1,3,2,1)}(p) &= 10p^4 - 11p^4 + 4p^2 \\
g_{(3,3,1,2,1)}(p) &= 7p^6 + 2p^5 - 2p^4 - 5p^3 + 2p^2 \\
g_{(2,2,2,2,1)}(p) &= 9p^6 + 2p^5 - 18p^4 + 12p^3 - 4p^2 \\
g_{(2,2,2,1,2)}(p) &= 2p^6 + 9p^5 - 16p^4 + 8p^3 - 2p^2 \\
g_{(2,2,1,2,2)}(p) &= p^6 + 3p^5 - 3p^4 \\
g_{(2,1,2,2,2)}(p) &= 2p^5 - p^3
\end{align*}
\]

**Proof.** We will prove this for the composition \((2,1,2,2,2)\), expression for other compositions can be obtained in a similar way. Let \( A \) be a matrix in Hermite normal form of following type,

\[
\begin{pmatrix}
p^2 & a_1p & a_2p & a_3p & a_4p & 1 \\
p & 0 & 0 & 0 & 1 \\
p & b_1p & b_2p & 1 \\
p & c_1p & 1 \\
p & c_2p & 1
\end{pmatrix}
\]

First, we determine the conditions on entries of the matrix which make \( A \) a subring matrix.

1. First, we note that \( v_2^1 \in \text{Col}(A) \) and \( v_2^3 \in \text{Col}(A) \).
2. Next, we know that \( v_2^1 \in \text{Col}(A) \) if and only if \( v_2^1 - p^2v_4 \in \text{Col}(A) \). Now, this is the case if and only if \( b_1a_2 \equiv 0 \pmod{p} \).
3. Similarly, \( v_2^3 \in \text{Col}(A) \) if and only if \( v_2^1 - p^2v_5 \in \text{Col}(A) \) and this is true if and only if \( b_1c_1 \equiv 0 \pmod{p} \).
4. We also note that if above conditions are satisfied then \( v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5 \) and \( v_4v_5 \) are in \( \text{Col}(A) \).

The conditions we get are,

- \( b_1a_2 = 0 \),
- \( b_1c_1 = 0 \),
- \( b_2^2a_2 - c_1^2a_3 \equiv 0 \pmod{p} \).

First we assume that \( b_1 = 0 \). Now further if \( c_1 = 0 \) then we get \( p^3(2p - 1) \) matrices. If \( c_1 \neq 0 \) then we get \( p^4(p - 1) \) matrices. Next, we assume that \( b_1 \neq 0 \) then \( a_2 = 0 \) and \( c_1 = 0 \), therefore we get \( p^4(p - 1) \) matrices. Therefore, we get that \( g_{(2,1,2,2,2)}(p) = 2p^5 - p^3 \).

5. **Calculating the values of \( g_{\alpha}(p) \) for compositions beginning with 3**

5.1. **Compositions of length 4.** In this section, we compute \( g_{\alpha}(p) \) for 10 compositions \( \alpha \) beginning with 3 and of length 4. They are listed as follows

\[
\begin{align*}
(3, 4, 1, 1) & \quad (3, 3, 1, 2) \\
(3, 1, 4, 1) & \quad (3, 2, 3, 1) \\
(3, 1, 1, 4) & \quad (3, 1, 2, 3) \\
(3, 2, 1, 3) & \quad (3, 3, 2, 1) \\
(3, 1, 3, 2) & \quad (3, 2, 2, 2).
\end{align*}
\]

We make note of some known computations.

**Lemma 5.1.** The following values of \( g_{\alpha}(p) \) are known

\[
\begin{align*}
g_{(3,4,1,1)}(p) &= 12p^3 - 9p^3 + p^2 \\
g_{(3,1,4,1)}(p) &= 2p^3 + 6p^3 - 2p^2 \\
g_{(3,1,1,4)}(p) &= 3p^3.
\end{align*}
\]
Proof. The above result follows from [Ish22a, Lemma 4.6]. □

Lemma 5.2. We have that \( g_{(3,2,1,3)}(p) = 5p^4 - 4p^3 + 2p^2 \).

Proof. Let \( A \) be any integer matrix in Hermite normal form of the type

\[
\begin{pmatrix}
p^3 & a_1p & a_2p & a_3p \\
p^2 & b_1p & b_2p & 1 \\
p & 0 & 1 \\
p^3 & 1 \\
1 & \\
\end{pmatrix}
\]

We determine the conditions so that \( A \) is a subring matrix.

1. It is required that \( v_2^3 \in \text{Col}(A) \) and this is the case if and only if
   \[
   (a_1^2p^2, p^3, 0, 0, 0)^t - p^3(a_1p, p^2, 0, 0, 0)^t
   \]
   is in \( \text{Col}(A) \). We deduce that \( a_1 \equiv 0 \mod p \), hence \( a_1 = a'_1p \) where \( 0 \leq a'_1 \leq p - 1 \).

2. It is required that \( v_2 \in \text{Col}(A) \); clearly this is equivalent to the statement that \( v_2^3 - pv_3 \in \text{Col}(A) \). Now, it is easy to see that latter condition is true if and only if
   \[
   (5.1)
   \]
   \[
   (a_2^2 - a_3) - a_1'(b_1^2 - b_1) \equiv 0 \mod p.
   \]

3. Similarly, it is clear that \( v_2^3 \in \text{Col}(A) \) if and only if \( v_2^3 - p^3v_4 \in \text{Col}(A) \) and latter condition is equivalent to
   \[
   (5.2)
   \]
   \[
   a_3^2 - b_3a_1' \equiv 0 \mod p.
   \]

4. One can easily check that \( v_2v_3 \) and \( v_2v_4 \) are both in \( \text{Col}(A) \) and \( v_3v_4 \in \text{Col}(A) \) if and only
   \[
   (5.3)
   \]
   \[
   a_2a_3 - a_1'b_1b_2 \equiv 0 \mod p.
   \]

Thus, to summarize, we arrive at the following conditions

1. \( a_1 = a_1'p \) where \( 0 \leq a_1' \leq p - 1 \),
2. \( (a_2^2 - a_3) - a_1'(b_1^2 - b_1) \equiv 0 \mod p \),
3. \( a_3^2 - b_3a_1' \equiv 0 \mod p \),
4. \( a_2a_3 - a_1'b_1b_2 \equiv 0 \mod p \).

We count matrices \( A \) satisfying these conditions by dividing into cases.

Case 1 : First consider the case when \( a_3 \equiv 0 \mod p \). From equation (5.2) we deduce that \( a_1'b_2 \equiv 0 \mod p \). We consider two subcases below.

- Case 1A : We consider the case when \( b_2 = 0 \). If \( b_1 \in \{0, 1\} \), then, there are \( 4p^3 \) matrices. If \( b_1 \notin \{0, 1\} \), then, there are \( p^3(p - 2) \) matrices. Thus, there are \( p^3(p + 2) \) in this case in total.

- Case 1B : Next consider the case when \( b_2 \neq 0 \). From equation (5.2) we find that \( a_1' = 0 \). From (5.1), we deduce \( a_2^2 - a_3 \equiv 0 \mod p \). Hence, there are \( 2p^2(p - 1) \) matrices in this case.

Case 2 : In this case we count matrices such that \( a_3 \neq 0 \mod p \). From (5.2), we deduce that \( b_2 \neq 0 \). Consider following subcases.

- Case 2A : Suppose \( b_1 = 0 \), then from equation (5.3) we get that \( a_2 \equiv 0 \mod p \). Hence, there are \( p^2(p - 1)^2 \) matrices in this case.

- Case 2B : Assume that \( b_1 = 1 \), then from equation (5.1) we find that \( a_2(a_2 - 1) \equiv 0 \mod p \). However, from (5.2) we cannot have \( a_2 \equiv 0 \mod p \), therefore \( a_2 \equiv 1 \mod p \). Hence, we find that \( a_3 \equiv b_2 \mod p \). As a result, there are \( p^2(p - 1) \) matrices in this case.

- Case 2C : In the last case we consider the case when \( b_1 \notin \{0, 1\} \) then from equations (5.1), (5.2), (5.3) we get that \( a_2 \neq 0 \mod p \). Now, we get that from the three equations (5.1), (5.2), (5.3) that,

\[
a_1' \equiv \frac{a_2(a_2 - 1)}{b_1(b_1 - 1)} \mod p,
\]
To summarize, the conditions are as follows:

\[ a'_1 \equiv \frac{a_2}{b_2} \mod p, \]
\[ a'_1 \equiv \frac{a_2a_3}{b_1b_2} \mod p. \]

From these we find that \( a_2 \equiv b_1 \mod p \) and \( a_3 \equiv b_2 \mod p \). Hence, there are \((p - 2)(p - 1)p^2\) matrices in this case.

Adding up the values in each case, we prove the assertion of the lemma. \(\square\)

**Lemma 5.3.** We have that \( g_\alpha(p) = 2p^3 \) where \( \alpha = (3, 1, 3, 2) \).

**Proof.** Let \( A \) be any integer matrix in Hermite normal form of the type

\[
\begin{pmatrix}
 p^3 & a_1p & a_2p & a_3p & 1 \\
 p & 0 & 0 & 1 \\
 p^3 & c_1p & 1 \\
 p^2 & 1 & 1
\end{pmatrix}
\]

We derive the conditions so that \( A \) is a subring matrix and use them to evaluate \( g_{(3,1,3,2)}(p) \).

1. It is clear that \( v^2_1 \in \text{Col}(A) \) if and only if \( v^2_1 - pv_2 \in \text{Col}(A) \), the second condition translates to

\[ a'_1 \equiv 0 \mod p. \]  
(5.4)

2. Similarly, \( v^2_3 \in \text{Col}(A) \) if and only if \( v^2_3 - p^3v_3 \in \text{Col}(A) \), and therefore, \( a_2 \equiv 0 \mod p \).

In other words, \( a_2 = a'_2p \), where \( 0 \leq a'_2 \leq p - 1 \).

3. Arguing as in previous two cases, we see that \( v^2_4 \in \text{Col}(A) \) if and only if \( c_1 = c'_1p \), where \( 0 \leq c'_1 \leq p - 1 \) and \( a_3 = a'_3p \) with \( 0 \leq a'_3 \leq p - 1 \).

4. It is easy to see that if the above conditions are satisfied, then \( v_2v_3, v_2v_4 \) and \( v_3v_4 \) are in \( \text{Col}(A) \).

To summarize, the conditions are as follows:

1. \( a'_1 - a_1 \equiv 0 \mod p \),
2. \( a_2 = a'_2p \) where \( 0 \leq a'_2 \leq p - 1 \),
3. \( c_1 = c'_1p \) where \( 0 \leq c'_1 \leq p - 1 \),
4. \( a_3 = a'_3p \) with \( 0 \leq a'_3 \leq p - 1 \).

From the above, it is clear that \( g_{(3,1,3,2)}(p) = 2p^4 \). \(\square\)

**Lemma 5.4.** We have that

\[ g_{(3,3,1,2)}(p) = 8p^4 - 6p^3 + 2p^2. \]  
(5.5)

**Proof.** Let \( A \) be an integer matrix in Hermite normal form

\[
\begin{pmatrix}
 p^3 & a_1p & a_2p & a_3p & 1 \\
 p^3 & b_1p & b_2p & 1 \\
 p & 0 & 1 \\
 p^2 & 1 & 1
\end{pmatrix}
\]

We determine the conditions so that \( A \) is a subring matrix.

1. Arguing as in Lemma 5.3, we find that the condition \( v^2_1 \in \text{Col}(A) \) translates to the condition that \( a_1 = a'_1p \), with \( 0 \leq a'_1 \leq p - 1 \).

2. Next, we see that \( v^2_3 \in \text{Col}(A) \) if and only if \( v^2_3 - pv_3 \in \text{Col}(A) \). Therefore we deduce that \( b^2_1 - b_1 \equiv 0 \mod p \) and

\[ (a_2^2 - a_2)p - a'_1(b^2_1 - b_1) \equiv 0 \mod p. \]  
(5.6)

3. It is again clear that \( v^2_4 \in \text{Col}(A) \) if and only if \( v^2_4 - p^2v_4 \in \text{Col}(A) \). We deduce that this is true if and only if \( b_2 = b'_2p \), where \( 0 \leq b'_2 \leq p - 1 \), and \( a_3 = a'_3p \), where \( 0 \leq a'_3 \leq p - 1 \).

4. It is clear that \( v_2v_3 \) and \( v_2v_4 \) are in \( \text{Col}(A) \) whenever above conditions are satisfied. By similar arguments as above we see that \( v_3v_4 \in \text{Col}(A) \) if and only if \( a'_1b_1b'_2 \equiv 0 \mod p \).

We have the following conditions on \( A \).
(1) \(a_1 = a_1'p\) with \(0 \leq a_1' \leq p - 1\),
(2) \(b_1^2 - b_1 \equiv 0 \pmod{p}\) and \((a_2^2 - a_2)p - a_1'(b_1^2 - b_1) \equiv 0 \pmod{p}\),
(3) \(b_2 = b_2'p\) where \(0 \leq b_2' \leq p - 1\),
(4) \(a_3 = a_3'p\) where \(0 \leq a_3' \leq p - 1\),
(5) \(a_1'b_1b_2' \equiv 0 \pmod{p}\).

We consider three cases depending upon whether \(b_1 = 0\) or not in \(\{0, 1\}\), and count number of matrices in each case.

**Case 1** : First consider the case when \(b_1 = 0\). Clearly there are \(2p^4\) matrices in this case.

**Case 2** : Consider the case in which \(b_1 = 1\). It is easy to see that there are \(2p^4(2p - 1)\) matrices in this case.

**Case 3** : Assume that \(b_1 \notin \{0, 1\}\). Then if \(b_2' = 0\) we get \(2p^3(p - 1)\) matrices. On the other hand, if \(b_2' \neq 0\) then \(a_1' = 0\) follows from \(a_1'b_2' \equiv 0 \pmod{p}\). Therefore, we find that \(a_2^2 - a_2 \equiv 0 \pmod{p}\), and there are \(4p^3(p - 1)^2\) matrices. Adding up the number of matrices from each case we arrive at the assertion. 

**Lemma 5.5.** The following equality holds

\[
g_{(3,2,3,1)}(p) = 12p^4 - 2p^3 + 4p^2.
\]

**Proof.** Let \(A\) be any integer matrix in Hermite normal form of the type

\[
\begin{pmatrix}
p^3 & a_1p & a_2p & a_3p & 1 \\
p^2 & b_1p & b_2p & 1 \\
p^1 & c_1p & 1 \\
p^0 & 1 
\end{pmatrix}
\]

We determine the conditions so that \(A\) is a subring matrix.

(1) It is easy to see that \(v_3^2 \in \text{Col}(A)\) if and only if \(a_1 = a_1'p\) where \(0 \leq a_1' \leq p - 1\).

(2) We note that \(v_3^2\) is in \(\text{Col}(A)\) if and only if \(v_3^3 - p^3v_3 \in \text{Col}(A)\). Therefore, we require that

\[
(a_2^2p^2, b_2^2p^2, p^6, 0, 0, 0)^t - p^3(a_2p, b_1p, p^3, 0, 0, 0)^t
\]

is in \(\text{Col}(A)\). Clearly, this is equivalent to the congruence

\[
a_2^2 - a_1'b_2^2 \equiv 0 \pmod{p}.
\]

(3) It is required that \(v_2^3 \in \text{Col}(A)\). It is easy to see that this condition is equivalent to the condition that \(v_2^3 - pv_4 \in \text{Col}(A)\). The last entry in the column \(v_2^3 - pv_4\) is \((c_1^2 - c_1)p^2\) therefore, first condition is \(c_1^2 - c_1 \equiv 0 \pmod{p}\). As in previous lemmas we obtain following second condition, \(b_1(c_1^2 - c_1) \equiv 0 \pmod{p^2}\) and from this we deduce that \(b_1(c_1^2 - c_1) = 0\) and the third condition is,

\[
(a_3^2 - a_3)p^2 - a_2(c_1^2 - c_1) - a_1'(b_2^2 - b_2)p^2 \equiv 0 \pmod{p^3}.
\]

(4) If entries of matrix \(A\) satisfy above conditions then it is easy to see that \(v_2v_3, v_2v_4\) are in \(\text{Col}(A)\) and \(v_3v_4 \in \text{Col}(A)\) if and only if the following congruence holds

\[
a_2(a_3 - c_1) - a_1'b_1(b_2 - c_1) \equiv 0 \pmod{p}.
\]

Thus, the conditions so that \(A\) is a subring matrix are as follows

(1) \(a_1 = a_1'p\) with \(0 \leq a_1' \leq p - 1\),

(2) \(c_1^2 - c_1 \equiv 0 \pmod{p}\) and \(b_1(c_1^2 - c_1) = 0\),

(3) \((a_2^2 - a_3)p^2 - a_2(c_1^2 - c_1) - a_1'(b_2^2 - b_2)p^2 \equiv 0 \pmod{p^3}\),

(4) \(a_2(a_3 - c_1) - a_1'b_1(b_2 - c_1) \equiv 0 \pmod{p}.

There are three cases to consider, depending upon the value of \(c_1\).

**Case 1** : First suppose that \(c_1 = 0\). Then by symmetry, the set of matrices is in bijection with the set of irreducible subring matrices with diagonal \((3,2,1,3)\). Therefore, we deduce that

\[
g_{(3,2,1,3)}(p) = 5p^2 - 4p^3 + 2p^2
\]

(4) (from the argument in Lemma 5.2).

**Case 2** : Consider the case when \(c_1 = 1\). The equation (5.9) reduces to the following

\[
(a_3^2 - a_3) - a_1'(b_2^2 - b_2) \equiv 0 \pmod{p}.
\]
First we suppose that \( b_1 = 0 \). Then we find that \( a_2 \equiv 0 \mod p \). Again considering two cases when \( b_2 \in \{0, 1\} \) or \( \not\in \{0, 1\} \) it is clear that there are \( p^3(p + 2) \) matrices in this case with \( b_1 = 0 \). If, \( b_1 \neq 0 \), then we further divide into cases such that \( b_2 = 0, 1 \) or not in \( \{0, 1\} \). It is easy to see that \( 2p^2(p - 1)(2p - 1) \) in this case with \( b_1 \neq 0 \).

Case 3 : Finally, we consider the case when \( c_1 \not\in \{0, 1\} \). From conditions above we find that \( b_1 = 0 \); from equation 5.9 we deduce that \( a_2 = 0 \) and equation 5.9 gets reduced to the equation 5.11. We consider sub-cases depending upon whether \( b_2 \in \{0, 1\} \) or not. There are \( 8p^3 \) matrices in the case when \( b_2 \in \{0, 1\} \). On the other hand, there are \( 2p^3(p - 2) \) matrices for which \( b_2 \not\in \{0, 1\} \).

Adding up the number of matrices in each case we prove the result. \( \square \)

**Lemma 5.6.** The following equality holds

\[
g_{(3,1,2,3)}(p) = p^4 + 2p^3 - p^2, \]

\[
g_{(3,3,2,1)}(p) = p^6 + 2p^5 - 13p^4 + 9p^3, \]

\[
g_{(3,2,2,2)}(p) = p^6 - 2p^5 + 6p^4 - 4p^3 + 3p^2 - 5p + 2. \]

**Proof.** The proof is very similar to previous results, and we omit it. \( \square \)

### 5.2. Compositions of length 5

We consider the composition of length 5 that begin with 3 we need to evaluate \( g_3(p) \) for 10 compositions of this form. We list all compositions of this form in the table below.

| (3, 3, 1, 1, 1) | (3, 2, 1, 1, 2) |
|----------------|-----------------|
| (3, 1, 3, 1, 1) | (3, 1, 2, 1, 2) |
| (3, 1, 1, 3, 1) | (3, 1, 2, 2, 1) |
| (3, 1, 1, 1, 3) | (3, 1, 1, 2, 2) |
| (3, 2, 1, 2, 1) | (3, 2, 2, 1, 1) |

*Table 2.* Compositions of length 5 that begin with 3.

For some of these compositions, we can use results from [Ish22a] to deduce value of \( g_3(p) \). The next result follows from the results in loc. cit.

**Lemma 5.7.** We have following values

| \( g_{(3,3,1,1,1)}(p) \) | \( 16p^6 + 12p^5 - 20p^4 + 8p^3 \) |
|------------------------|----------------------------------|
| \( g_{(3,1,3,1,1)}(p) \) | \( 18p^6 - 6p^4 \) |
| \( g_{(3,1,1,3,1)}(p) \) | \( 2p^5 + 10p^4 - 4p^3 \) |
| \( g_{(3,1,1,1,3)}(p) \) | \( 4p^4 \). |

**Proof.** The above results are direct consequence of [Ish22a, Lemma 4.6]. \( \square \)

Next we evaluate the value of \( g_3(p) \) for the remaining compositions.

**Lemma 5.8.** The following table of relations holds

| \( g_{(3,1,2,2,1)}(p) \) | \( 13p^6 - 8p^4 + 8p^3 - 2p^2 \) |
|------------------------|----------------------------------|
| \( g_{(3,1,2,1,2)}(p) \) | \( 5p^6 - 4p^4 + 2p^3 \) |
| \( g_{(3,1,1,2,2)}(p) \) | \( p^5 + 3p^3 - 2p^3 \) |
| \( g_{(3,2,1,1,2)}(p) \) | \( 4p^6 - 8p^5 + 35p^4 - 31p^3 + 4p^2 \) |
| \( g_{(3,2,1,2,1)}(p) \) | \( 9p^6 - 4p^5 + 28p^4 - 39p^3 + 20p^2 - 10p \). |
Proof. We will prove this result only for the composition \((3,1,1,2,2)\), and the proof is similar for the other compositions listed above. Therefore, we will count the number of integer subring matrices \(A\) in Hermite normal form of the type

\[
\begin{pmatrix}
p^3 & a_1p & a_2p & a_3p & a_4p & 1 \\
p & 0 & 0 & 0 & 1 \\
p & 0 & 0 & 1 \\
p^2 & b_1p & 1 \\
p^2 & 1 \\
p^2 & 1 \\
\end{pmatrix}
\]

We determine the conditions on entries of the matrix which make \(A\) a subring matrix.

1. We note that \(v_1^2\) and \(v_2^2\) are in \(\text{Col}(A)\) if and only if \(a_1^2 - a_1 \equiv 0 \mod p\) and \(a_2^2 - a_2 \equiv 0 \mod p\).

2. We also want \(v_3^2\) to be in \(\text{Col}(A)\) and this is true if and only if

\[
\begin{pmatrix}
p^3 & a_1p & a_2p & a_3p & a_4p & 1 \\
p & 0 & 0 & 0 & 1 \\
p & 0 & 0 & 1 \\
p^2 & b_1p & 1 \\
p^2 & 1 \\
p^2 & 1 \\
\end{pmatrix}
\]

is in \(\text{Col}(A)\). Therefore, \(v_3^2\) is in \(\text{Col}(A)\) if and only if \(a_3^2 = a_4p\) with \(0 \leq a_3^2 \leq p - 1\).

3. Similarly, \(v_4^2\) is in \(\text{Col}(A)\) if and only if \(v_4^2 \equiv p\) \(\mod p\) and \(v_5^2 \equiv p\) \(\mod p\).

4. Similarly, \(v_5^2\) is in \(\text{Col}(A)\) if and only if \(v_5^2 = \alpha^2\) \(\mod p\). On the other hand, either \(a_3\) or \(a_4\) is divisible by \(p\). Thus, the total number of matrices for this case is \(3p^5(p+2)(p-1)\).

Putting together the number of matrices from both cases we deduce that \(g(3,1,1,2,2)(p) = p^5 + 3p^6 - 2p^3\).

There is one composition \(\alpha\) for which we are not able to compute \(g_\alpha(p)\) using the above arguments. This is the composition \(\alpha = (3,2,2,1,1)\). Let us consider this case in some further detail. Let \(A\) be an integer subring matrix of the form

\[
A = \begin{pmatrix}
p^3 & a_1p & a_2p & a_3p & a_4p & 1 \\
p & 0 & 0 & 0 & 1 \\
p & 0 & 0 & 1 \\
p^2 & b_1p & 1 \\
p & 0 & 1 \\
p & 1 \\
\end{pmatrix}
\]

Since the above matrix is in Hermite normal form, \(1 \leq a_i \leq p^2 - 1\), and \(1 \leq b_i, c_j \leq p - 1\).

Below, we list the conditions for \(A\) to be a subring matrix

\[
\begin{align*}
a_1^2 - a_1 b_i^2 &\equiv 0 \mod p, \\
b_1 c_1 (c_1 - 1) &\equiv 0, b_1 c_2 (c_2 - 1) = 0, b_1 c_1 c_2 = 0, \\
(a_2^2 - a_3) - a_2 c_1 (c_1 - 1)/p &\equiv 0 \mod p, \\
(a_3^2 - a_4) - a_2 c_2 (c_2 - 1)/p &\equiv 0 \mod p, \\
a_2 (a_3 - a_1) - a_1 b_1 (b_2 - c_1) &\equiv 0 \mod p, \\
(a_3 - a_4) - a_1 b_1 (b_2 - c_1) &\equiv 0 \mod p, \\
&\text{and } a_2 (a_4 - c_2) - a_1 b_1 (b_3 - c_2) \equiv 0 \mod p, \\
a_3 a_4 - a_2 c_1 c_2/p &\equiv 0 \mod p.
\end{align*}
\]
In the subcase when \( b_1 = 0 \) counting solutions to the equations is reduced to counting number of solutions in \( \mathbb{F}_p^6 \) to following system of polynomial equations:
\[
\begin{align*}
(x_3^2 - x_3) - x_2(x_2^2 - x_7) - x_1(x_5^2 - x_5) &= 0 \\
(x_4^2 - x_4) - x_2(x_5^2 - x_8) - x_1(x_6^2 - x_6) &= 0 \\
x_3x_4 - x_2x_7x_8 - x_1x_5x_6 &= 0
\end{align*}
\]
Let \( N_p \) denote the number of solutions to this system in \( \mathbb{F}_p^6 \). Using SageMath we calculated \( N_p \) for \( p = 2, 3, 5, 7, 11, 13, 17, 19 \) and these computations suggest that \( N_p = p^5 + 12p^4 - 20p^3 + 30p^2 - 10p \). Using this and arguments similar to those used in proving previous results we deduce that \( g_{(3,2,2,1,1)}(p) = p^7 + 24p^6 - 29p^5 + 21p^4 - 4p^3 \). In summary, our computations lead us to make the following conjecture.

**Conjecture 5.9.** We have that \( g_{(3,2,2,1,1)}(p) = p^7 + 24p^6 - 29p^5 + 21p^4 - 4p^3 \).

6. Calculating the values of \( g_\alpha(p) \) for compositions beginning with 4, 5 or 6

6.1. Values of \( g_\alpha(p) \) for compositions beginning with 4.

6.1.1. Compositions of length 4. We will evaluate \( g_\alpha(p) \) for 6 compositions of length 4 that begin with 4. They are listed below.

| (4, 2, 2, 1) | (4, 2, 1, 2) |
| (4, 1, 2, 2) | (4, 3, 1, 1) |
| (4, 1, 3, 1) | (4, 1, 1, 3) |

**Table 3.** Compositions of length 4 that begin with 4.

**Lemma 6.1.** We have following values,
\[
\begin{align*}
g_{(4,2,2,1)}(p) &= 6p^6 + 12p^5 - 10p^4 - 7p^3 + 4p^2 \\
g_{(4,2,1,2)}(p) &= p^5 + 9p^4 - 5p^3 - 4p^2 \\
g_{(4,1,2,2)}(p) &= p^6 + 2p^5 - p^2 \\
g_{(4,3,1,1)}(p) &= 2p^6 - 2p^5 + 17p^4 - 6p^3 - 2p^2 \\
g_{(4,1,3,1)}(p) &= 2p^6 + 6p^3 \\
g_{(4,1,1,3)}(p) &= p^3 + 2p^2 
\end{align*}
\]

**Proof.** The proof is omitted. \( \square \)

6.1.2. Compositions of length 5. We will evaluate \( g_\alpha(p) \) for 4 compositions of length 5 that begin with 4. We list them below.

| (4, 2, 1, 1, 1) | (4, 1, 2, 1, 1) |
| (4, 1, 1, 2, 1) | (4, 1, 1, 1, 2) |

**Table 4.** Compositions of length 5 that begin with 4.

**Lemma 6.2.** We have following values,
\[
\begin{align*}
g_{(4,2,1,1,1)}(p) &= 5p^8 + 16p^7 - 17p^6 - 27p^5 + 12p^4 \\
g_{(4,1,2,1,1)}(p) &= p^6 + 13p^5 - 8p^4 + 3p^3 + 2p^2 \\
g_{(4,1,1,2,1)}(p) &= p^6 + 8p^5 - 4p^3 \\
g_{(4,1,1,1,2)}(p) &= p^5 + 3p^4 
\end{align*}
\]
Proof. We will explicitly obtain the expression for $g_{(4,1,1,1,2)}(p)$. The other computations are similar, and thus omitted. Let $A$ be a matrix in Hermite normal form of following type

$$
\begin{pmatrix}
  p^4 & a_1 p & a_2 p & a_3 p & a_4 p & 1 \\
  p^4 & 0 & 0 & 0 & 1 \\
  p^4 & 0 & 0 & 1 \\
  p^4 & 0 & 1 \\
  p^4 & 1 \\
\end{pmatrix}
$$

First, we determine the conditions on entries of the matrix which make $A$ a subring matrix. Note that for $i = 2, 3, 4$, $v_i^2 \in \text{Col}(A)$ if and only if $a_{i-1}(a_{i-1} - 1) \equiv 0 \mod p^2$ and $v_i^2 \in \text{Col}(A)$ if and only $a_4 \equiv 0 \mod p$. We also note that for $1 < i < j$, $v_i v_j \in \text{Col}(A)$ if and only if $a_{i-1}a_{j-1} \equiv 0 \mod p^2$. First we suppose that for $i = 2, 3, 4$, we have that $a_{i-1} \equiv 0 \mod p^2$. There are $p^5$ matrices of this form. Next, suppose for exactly one of $a_1, a_2, a_3$ is congruent to 1 modulo $p^2$ and others are divisible by $p^2$ then we get $3p^4$ matrices of this form. Therefore, $g_{(4,1,1,1,2)}(p) = p^5 + 3p^4$.

6.2. Values of $g_a(p)$ for compositions beginning with 5.

6.2.1. Compositions of length 4. We calculate $g_a(p)$ for 3 compositions of length 4 that begin with 5. The compositions are listed below

| Composition |
|-------------|
| (5, 2, 1, 1) |
| (5, 1, 2, 1) |
| (5, 1, 1, 2) |

Table 5. Compositions of length 4 that begin with 5.

Lemma 6.3. The following table of relations hold

| Relation | Expression |
|----------|------------|
| $g_{(5,2,1,1)}(p)$ | $p^5 - 3p^3 + 17p - 8p^4 - 5p^2 + 2p$ |
| $g_{(5,1,2,1)}(p)$ | $7p^5 - p^3 - 2p^2$ |
| $g_{(5,1,1,2)}(p)$ | $p^4 + 2p^3$ |

Proof. We will prove the result only for the composition $(5, 1, 2, 1)$. The other cases are omitted since the arguments are similar to this case. Let $A$ be a matrix in Hermite normal form of following type

$$
\begin{pmatrix}
  p^5 & a_1 p & a_2 p & a_3 p & 1 \\
  p^5 & 0 & 0 & 0 & 1 \\
  p^5 & 0 & 0 & 1 \\
  p^5 & 0 & 1 \\
  p^5 & 1 \\
\end{pmatrix}
$$

We obtain the conditions on the entries of $A$ below.

1. First, we must have that $v_2^2 \in \text{Col}(A)$ and this is true if and only if $v_2^2 - pv_2 \in \text{Col}(A)$.

   The latter condition is equivalent to the condition $a_1(a_1 - 1) \equiv 0 \mod p$.

2. We also want that $v_3^2 \in \text{Col}(A)$ and we see that this is equivalent to the condition $v_3^2 - p^3v_3 \in \text{Col}(A)$, which is true if and only if $a_2(a_2 - p) \equiv 0 \mod p$. Therefore we get that $a_2 = a_2'p$ with $0 \leq a_2' \leq p^3 - 1$ and $a_2'(a_2' - 1) \equiv 0 \mod p$.

3. Using similar arguments we obtain that $v_i^2 \in \text{Col}(A)$ if and only if $a_3(a_3 - 1) - b_i(b_i - 1)a_2' \equiv 0 \mod p^3$. We also deduce that $v_2v_3, v_2v_4$ and $v_3v_4$ are in $\text{Col}(A)$ if and only if $a_1a_2 \equiv 0 \mod p$, $a_1a_3 \equiv 0 \mod p$ and $a_2(a_2 - b_i) \equiv 0 \mod p$.

The conditions we get are as follows

- $a_1(a_1 - 1) \equiv 0 \mod p$,
- $a_2 = a_2'p$ with $0 \leq a_2' \leq p^3 - 1$ and $a_2'(a_2' - 1) \equiv 0 \mod p$,
- $a_3(a_3 - 1) - b_i(b_i - 1)a_2' \equiv 0 \mod p^3$. 


\begin{itemize}
\item $a_1 a_2 \equiv 0 \mod p$, $a_1 a_3 \equiv 0 \mod p$ and $a_2 (a_3 - b_1) \equiv 0 \mod p$.
\end{itemize}

Next, we count the number of matrices by dividing in two cases depending upon if $a_1 \equiv 0 \mod p$ or $a_1 \equiv 0 \mod p^3$.

\textbf{Case 1 :} First consider the case when $a_1 \equiv 1 \mod p^3$, as we must have $a_1 a_2 \mod p^3$ we deduce that $a_2 = a_2'' p^2$ and $a_3 \equiv 0 \mod p$. Therefore, conditions reduce to $b_1 (b_1 - 1) a_2'' \equiv 0 \mod p$. Counting we see that there are $3p^3 - 2p^2$ matrices in this case.

\textbf{Case 2 :} Now, we consider that case when $a_1 \equiv 0 \mod p^3$. We further consider two separate cases depending upon whether $a_2''$ is divisible by $p$ or not.

\begin{itemize}
\item \textbf{Case 2A :} Consider the case when $a_2''$ is divisible by $p$. we have two possibilities, either $p^3 \nmid a_2$ or $p^3 \mid a_2$, if $p^3 \nmid a_2$ then we deduce that $a_3 \equiv b_1 \mod p$, therefore $b_1 = 0$ or $1$. We get $2p^3 (p - 1)$ matrices. if $p^3 \mid a_2$, again considering two cases depending upon whether $b_1 \in \{0, 1\}$ or not and counting we see that we get $2p^4$ matrices.

\item \textbf{Case 2B :} We consider the case when $a_2'' \equiv 1 \mod p$, we deduce that $a_3 \equiv b_1 \mod p^3$. Conditions reduce to $b_1 (b_1 - 1) (1 - a_2'') \equiv 0 \mod p^3$. Now, suppose $b_1 \in \{0, 1\}$, then we get $2p^4$ matrices. Otherwise, $a_2'' = 1$ and we get $p^2 (p - 2)$ matrices.
\end{itemize}

It follows from the above observations that the number of matrices in each case we deduce that $g(5, 1, 2, 1)(p) = 7p^4 - p^3 - 2p^2$. \hfill \qed

6.2.2. \textit{Compositions of length 5.} In this section we evaluate $g_5(p)$ for the composition $\alpha = (5, 1, 1, 1, 1)$. This is the only composition of length 5 which starts with 5. We have the following result.

\textbf{Lemma 6.4.} We have that $g_5(p) = 5p^4$, where $\alpha = (5, 1, 1, 1, 1)$.

\textit{Proof.} The result follows directly from [AKKM21, Lemma 3.5]. \hfill \qed

6.3. \textit{Values of }$g_5(p)$\textit{ for compositions beginning with 6.} We evaluate $g_5(p)$ for the only composition $\alpha = (6, 1, 1, 1, 1)$ of length 4 which begins with 6.

\textbf{Lemma 6.5.} We have that $g(6, 1, 1, 1, 1)(p) = 4p^3$.

\textit{Proof.} The above result is an immediate consequence [AKKM21, Lemma 3.5]. \hfill \qed

7. \textbf{Main results}

7.1. \textbf{Proof of the main result.} In this section we prove Theorem 1.4. We note that for $n > 1$ and $e \geq n - 1$ the following relation holds

\[(7.1)\quad g_n(p^e) = g_{n-1}(p^{e-1}) + \sum_{\alpha \in \mathcal{C}^n_{n,e}} g_\alpha(p),\]

where $\mathcal{C}^n_{n,e}$ denotes the set of compositions in $\mathcal{C}_{n,e}$ whose first coordinate is greater than 1.

\textbf{Proposition 7.1.} We have that,

\[g_5(p^9) = 11p^6 + 14p^5 + 137p^4 - 16p^3 + p^2 + 2p + 3,\]

and

\[g_6(p^9) = 4p^7 + 76p^6 + 128p^5 + 56p^4 - 111p^3 + 43p^2 - 9p + 1 + g(3, 2, 2, 1, 1)(p).\]

\textit{Proof.} The result follows immediately from (7.1) and computations in previous sections. \hfill \qed

Liu computes the values of $g_n(p^e)$ for $n = 3, 4$ (cf. [Liu07, Proposition 6.1 and 6.2]). We record these values for $e = 9$ and refer to [AKKM21, p. 231] the values of $e$ which lie in the range $4 \leq e \leq 8$. We find that

\[g_3(p^9) = p^3 + 4p^2 + 4p + 1,\]

and that

\[g_4(p^9) = 11p^4 + 30p^3 + 9p^2 + p + 1.\]
Proposition 7.2. For $n > 1$, the following relation holds
\[ f_n(p^9) - f_{n-1}(p^9) = Q(n)P_{10}(n)p^{10} + 276480P_3(n)p^9 + 138240P_5(n)p^8 + 34560P_7(n)p^7 + 34560P_9(n)p^6 + 11520P_5(n)p^5 + 11520P_2(n)p^4 + 1440P_3(n)p^3 + 1440P_2(n)p^2 + 240P_1(n)p + P_0(n) + \binom{n-1}{5}g_{(3,2,2,1,1)}(p). \]

where,
\[
Q(n) = \frac{n-1}{1393459200},
\]
\[
P_{10}(n) = (n-2)(n-3)(n-4)(n-5)(n-6)(n-7),
\]
\[
P_9(n) = n(n-2)(n-3)(n-4)(n-5)(n-6),
\]
\[
P_8(n) = (n-2)(n-3)(n-4)(n-5)(9n^2 - 131n + 784),
\]
\[
P_7(n) = (n-2)(n-3)(n-4)(n-5)(37n^3 - 761n^2 + 6482n - 18144),
\]
\[
P_6(n) = (n-2)(n-3)(n-4)(75n^5 - 2468n^4 + 36349n^3 - 279672n^2 + 1101372n - 1732080),
\]
\[
P_5(n) = (n-2)(n-3)(n-4)(33n^5 - 1220n^4 + 21757n^3 - 208732n^2 + 1053542n^2 - 2414076n + 1592640),
\]
\[
P_4(n) = (n-2)(n-3)(42n^7 - 2102n^6 + 51106n^5 - 749135n^4 + 7010050n^4 - 41819711n^3 + 152633830n^2 - 307862664n + 260890560),
\]
\[
P_3(n) = (n-2)(273n^{10} - 18123n^9 + 571282n^8 - 1099986n^7 + 140978185n^6 - 1241293667n^5 + 7531038196n^4 - 30849468932n^3 + 81115302432n^2 - 123180801984n + 81613163520),
\]
\[
P_2(n) = (n-2)(147n^{11} - 10843n^{10} + 377832n^9 - 8095642n^8 + 117343063n^7 - 1198590955n^6 + 8745674590n^5 - 45324367680n^4 + 162663439888n^3 - 383226514464n^2 + 531138427776n - 326776343040),
\]
\[
P_1(n) = (n-2)(315n^{12} - 25368n^{11} + 956025n^{10} - 22147762n^9 + 349611873n^8 - 3946086924n^7 + 32542876455n^6 - 196948068290n^5 + 865325849028n^4 - 2683887407672n^3 + 556047149216n^2 - 6887555482752n + 3844971970560),
\]
\[
P_0(n) = 135n^{16} - 14400n^{15} + 730980n^{14} - 23334360n^{13} + 522513250n^{12} - 8678453720n^{11} + 110319301164n^{10} - 109320229312n^9 + 849128605343n^8 - 51930963880392n^7 + 248190356069720n^6 - 915171074718208n^5 + 2545385435757472n^4 - 5146605000021888n^3 + 7110255039457536n^2 - 5073926003435520n + 2289569122713600.
\]

Proof. Recall the recurrence relation (1.2)
\[ f_n(p^9) = \sum_{i=0}^{9} \sum_{j=1}^{n} \binom{n-1}{j-1} f_{n-j}(p^{9-i})g_j(p). \]

Note that since $i \leq 9$, we find that $j \leq 10$ in the above recurrence relation. We compute the values of $g_j(p^i)$ and then recursively, use the above to compute the value of $f_n(p^9)$. In greater
detail, we use values from [AKKM21, p. 231], the Theorem 1.2, and the values computed in previous sections, to obtain the above recurrence relation. 

Using the recurrence relation above and noting that

$$f_n(p^9) = \sum_{k=2}^{n} \left( f_k(p^9) - f_{k-1}(p^9) \right)$$

we obtain Theorem 1.4. We performed all the computations in SageMath.

7.2. Some bounds on $g_\alpha(p)$. In this section we obtain upper bounds on $g_\alpha(p)$ for certain compositions $\alpha$. Let $u_1, \ldots, u_n$ be the standard basis of $\mathbb{Z}^n$. The vector $u_i = (0, 0, \ldots, 1, 0, \ldots, 0)$ consists of 0 in all entries except for 1 in the $i$-th entry. Let $L$ be an irreducible subring of $\mathbb{Z}^n$ and let $m_L$ be the ideal in $L$ consisting of vectors all of whose coordinates are divisible by $p$. Setting $\rho_L := \dim_{\mathbb{F}_p} m_L/m_L^2$. Following [Liu07, p. 292, 1.-1], an irreducible subring is full if $\rho_L = n - 1$. Assume that the matrix for $L$ is in Hermite normal form with respect to the basis $u_1, \ldots, u_n$. Let $\pi : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n-1}$ be the projection onto the last $(n - 1)$ coordinates $u_2, \ldots, u_n$. Suppose that the basis is chosen so that with respect to the ordered basis $(u_1, \ldots, u_n)$, the matrix $A$ associated with $L$ is in the Hermite normal form. Let $v_1, \ldots, v_n$ be the columns of $A$, and let $\tau(L)$ be the lattice generated by $L$ and by $v_i' = \frac{1}{p} v_i$. It is easy to see that $\tau(L)$ is a subring (cf. [Liu07, p.291, 1.-2]).

**Proposition 7.3.** Let $L$ be an irreducible subring of index $p^{e+n}$ such that $\pi(L)$ is full and $pu_1 \notin L$. Then the map $L \mapsto \frac{1}{p} m_{\pi(L)}$ is a $p^{n-2}$-to-one surjection onto subrings of index $p^e$.

**Proof.** The above result is [Liu07, Proposition 5.6].

Let $\beta := (\beta_1, \beta_2, \ldots, \beta_{n-1})$ be a composition of length $n - 1$ such that $\beta_1 > 1$, $\beta_i > 0$ for $i \in \{2, \ldots, n - 1\}$. By abuse of notation, we say that an irreducible subring $L$ has diagonal $\beta$ if the entries on the diagonal of the matrix $A_L$ are $p^{\beta_1}, \ldots, p^{\beta_{n-1}}$. We set $\mathcal{R}_\beta$ to denote the set of irreducible subrings with diagonal $\beta$. Let $\beta' := (\beta_1 - 2, \beta_2 - 1, \ldots, \beta_{n-1} - 1)$, let $S_{\beta'}$ denote the set of subrings with diagonal $\beta'$. Let $\mathcal{R}'_{\beta}$ be the subset of $\mathcal{R}_\beta$ consisting of irreducible subrings with diagonal $\beta$ such that $\pi(L)$ is full and $pu_1 \notin L$, with respect to the basis $u_1, \ldots, u_n$ for which $A_L$ is in Hermite normal form. Given a finite set $S$, set $\# S$ to denote the cardinality of $S$.

**Lemma 7.4.** Let $\beta := (\beta_1, \beta_2, \ldots, \beta_{n-1})$ be a composition of length $n - 1$ such that the following conditions are satisfied

1. $\beta_1 > 1$,
2. $\beta_i > 0$ for $i \in \{2, \ldots, n - 1\}$,
3. $\sum_{i=1}^{n-1} \beta_i > n$.

Then, there is a $p^{n-2}$-to-one surjection from $\mathcal{R}'_{\beta}$ to $S_{\beta'}$. In particular, we have that

$$\# \mathcal{R}_\beta \geq \# \mathcal{R}'_{\beta} \geq p^{n-2} \# S_{\beta'}.$$

**Proof.** It follows from Proposition 7.3 that the map $L \mapsto \frac{1}{p} m_{\pi(L)}$ defines a $p^{n-2}$ to one surjective map $\Phi : \mathcal{R}'_{\beta} \rightarrow S_{\beta'}$. The inequality follows immediately from this.

The following result comes as a consequence of the above Lemma.

**Proposition 7.5.** Let $\beta := (\beta_1, \beta_2, \ldots, \beta_{n-1})$ be a composition of length $n - 1$ such that $\beta_1 > 1$, $\beta_i > 0$ for $i \in \{2, \ldots, n - 1\}$ and $\sum_{i=1}^{n-1} \beta_i > n$. Setting $m_{\beta} := \min\{\lfloor \frac{1}{2} \rfloor, \beta_2, \ldots, \beta_{n-1}\}$, we have the following lower bound for $g_{\beta}(p)$

$$g_{\beta}(p) \geq p^{(n-2)m_{\beta}}.$$

**Proof.** The result follows upon repeatedly applying the Lemma 7.4. 

In particular we see that if $\alpha := (k, \ell, \ldots, \ell)$ of length $n - 1$, then $g_\alpha(p) \geq p^{(\frac{1}{2}) (n-2)}$. We note that this is a special case of [Ish22b, Corollary 4.6].
Theorem 7.6. We have the following polynomial lower bound

$$g_n(p^e) \geq \sum_{j=1}^{\lfloor e/n \rfloor} p^{(n-2)j} \left( \frac{e - 1 - nj}{n - 2} - \frac{e - 1 - n(j + 1)}{n - 2} \right).$$

Proof. Let $C_{n,e}^j$ be the subset of $C_{n,e}$ consisting of all compositions $\beta = (\beta_1, \ldots, \beta_{n-1})$ such that $m_\beta = j$. It is easy to see that

$$\#C_{n,e}^j = \#C_{n,e-nj} - \#C_{n,e-n(j+1)} = \left( \frac{e - 1 - nj}{n - 2} \right) - \left( \frac{e - 1 - n(j + 1)}{n - 2} \right).$$

It follows from Proposition 7.5 that

$$g_n(p^e) \geq \sum_{j=1}^{\lfloor e/n \rfloor} p^{(n-2)j} \#C_{n,e}^j,$$

and the result follows. \(\square\)

We now prove Theorem 1.6.

Proof of Theorem 1.6. Let us assume for simplicity that $t = n - 1$. The proof in generality is identical to this case. Given an $n \times n$ matrix $A$, let $A_{i,j}$ be the $i \times j$ matrix obtained upon deleting the last $(n - i)$ rows and $(n - j)$ columns. Thus, $A_{i,j}$ is the upper left $i \times j$ submatrix of $A$. Let $S$ denote the set of subring matrices of the form

$$A = \begin{pmatrix}
p^{a_1} & \cdots & \cdots & \cdots & \cdots & a_1p^1 \\
p^{a_2} & \cdots & \cdots & \cdots & \cdots & a_2p^1 \\
& \ddots & \cdots & \cdots & \cdots & \cdots \\
& & \cdots & \cdots & \cdots & \cdots \\
& & & \cdots & \cdots & \cdots \\
p^{k_{\alpha_{n-1}}} & \cdots & \cdots & \cdots & \cdots & 1 \\
& & & \cdots & \cdots & 1 \\
& & & & \cdots & 1
\end{pmatrix}\begin{pmatrix}
1 \\
1 \\
\vdots \\
\vdots \\
\vdots \\
1
\end{pmatrix}$$

that are in Hermite normal form. We note that the following conditions hold

1. $0 \leq a_i \leq p^{\gamma - 1} - 1$,
2. setting $A := A_{[n-2,n-2]}$, we note that

$$(a_1^2p^2 - a_1p^{k_{\alpha_{n-1}}}, \ldots, a_{n-2}^2p^2 - a_{n-2}p^{k_{\alpha_{n-1}}} + 1) \in \text{Col } A = \hat{A}(\mathbb{Z}^{n-2}).$$

We note in passing that $g_{a_{k,i}}(p) = \#S$. We denote the set of all irreducible subring matrices of size $(n - 1)$ and with diagonal $(p^{a_1}, p^{a_2}, \ldots, p^{a_{n-2}}, 1)$ by $A$.

We note that for each index $i$ in the range $1 \leq i \leq n - 2$ we have that $|a_i^2p^2 - a_ip^{k_{\alpha_{n-1}}} + 1| \leq |a_i^2p^2| + |a_ip^{k_{\alpha_{n-1}}} + 1| \leq 2p^{(k+1)\gamma}$. In order to simplify notation, for $B \in A$, set $B' := B_{[n-2,n-2]}$. Setting $C := [-2p^{(k+1)\gamma}, 2p^{(k+1)\gamma}]^{n-2}$, we find that

$$\#S \leq \sum_{B \in A} \# \left( B' \mathbb{Z}^{n-2} \cap C \right).$$

For $B \in A$, the number of points in $v \in \mathbb{Z}^{n-2}$ such that $B'v \in C$ is same as the number of points $w \in C$ such that $(B')^{-1}(w) \in \mathbb{Z}^{n-2}$. Therefore using inequality 7.2 we deduce that

$$\#S \leq \sum_{B \in A} \#((B')^{-1}(C) \cap \mathbb{Z}^{n-2}).$$

For $w = (w_1, \ldots, w_{n-2}) \in \mathbb{Z}^{n-2}$, set $\|w\|$ to be the Euclidean norm $\sqrt{\sum w_i^2}$. For any $w \in C$ we have that $\|(B')^{-1}w\| \leq \|(B')^{-1}\|\|w\|$. It is easy to see that $\|w\| \leq 2p^{(k+1)\gamma} \sqrt{n - 2}$. On the other hand, since $A$ is a finite set which is defined independent of $k$, we find that for any $B \in A$ and $w \in C$, $\|(B')^{-1}w\| \leq M\sqrt{n - 2}p^{(k+1)\gamma}$ where $M \in \mathbb{R}_{>0}$ is a suitably large constant (not depending on $k$). Using the inequality (7.3), we deduce that

$$\#S \leq \sum_{L \in \mathcal{A}} \#([-M\sqrt{n - 2}p^{(k+1)\gamma}, M\sqrt{n - 2}p^{(k+1)\gamma}]^{n-2} \cap \mathbb{Z}^{n-2})$$

26.

H. MISHRA AND A. RAY
Fixing an integer $N > M\sqrt{n-2}$, we deduce that
\begin{equation}
#S \leq \sum_{L \in A} \# \left( \left[ -Np^{(k+1)\gamma}, Np^{(k+1)\gamma} \right] \cap \mathbb{Z}^{n-2} \right)
\end{equation}
Therefore, $#S \leq \#(A)(2N)^{n-2}p^{(k+1)\gamma(n-2)}$. We have proved that
\[g_{\alpha/k,t}(p) = #S \leq \#(A)(2N)^{n-2}p^{\gamma(n-2)p^{\gamma k(n-2)}}
\]
Therefore, $g_{\alpha/k,t}(p) = O(p^{\gamma k(n-2)})$ as $k \to \infty$. \hfill \Box

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