ABEL AVERAGES AND HOLOMORPHICALLY
PSEUDO-CONTRACTION MAPS IN BANACH SPACES

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ABSTRACT. A class of maps in a complex Banach space is studied, which
includes both unbounded linear operators and nonlinear holomorphic
maps. The defining property, which we call pseudo-contractivity, is in-
trduced by means of the Abel averages of such maps. We show that the
studied maps are dissipative in the spirit of the classical Lumer-Phillips
theorem. For pseudo-contractive holomorphic maps, we establish the
power convergence of the Abel averages to holomorphic retractions.

1. INTRODUCTION AND THE RESULTS

1.1. Preliminary and notations. In this paper, $(X, \| \cdot \|)$ will stand for
a complex Banach space. The best studied class of self-maps of $X$ is that
of bounded linear operators $T : X \to X$, which we denote by $B(X)$. For
$T \in B(X)$, the Abel average is defined as

\begin{equation}
A_\alpha = (1 - \alpha) \sum_{n=1}^{\infty} \alpha^n T^n,
\end{equation}

where $\alpha \in (0, 1)$ is such that the series in (1.1) converges in the operator
norm topology. The study of Abel averages of bounded linear operators goes
back to, at least, E. Hille [13] and W. F. Eberlein [6]. Two natural extensions
of $B(X)$ are the class of unbounded linear operators $T : \mathcal{D}(T) \subset X \to X$,
and the class of nonlinear holomorphic maps $h : V \to X$ with various choices
of the domains $V$. Nonlinear holomorphic maps play an important role in
complex analysis and in theory of dynamical systems, see, e.g., [3, 10, 11, 12,
22, 23]. Note that, for infinite dimensional Banach spaces, these two classes
are disjoint. Separately they are studied extensively. However, in many
applications one encounters maps that are neither linear nor continuous. A
typical example is the map $T + g$ where $T$ is a linear unbounded operator
and $g$ is nonlinear and holomorphic. Such maps appear, in particular, in
evolution equations of reaction-diffusion type, see, e.g., [1] and subsection
3.1 below.

The aim of the present paper is:

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(1) to introduce a class of maps – \( \omega \)-pseudo-contractive maps – which includes both unbounded linear operators and nonlinear holomorphic maps, and to study this class by means of properly defined Abel averages;

(2) to study, cf. [3, 11, 12], the Abel averages of nonlinear holomorphic maps in the spirit of [17], where a number of properties of the Abel averages of unbounded linear operators were obtained.

For a linear operator \( T \), we write \( \text{D}(T) \), \( \text{Ker}(T) \), and \( \text{Im}(T) \) for denoting the domain, kernel, and range of \( T \), respectively. By \( I \) we denote the identity operator \( Ix = x \), \( x \in X \). Given \( r > 0 \) and \( x \in X \), we set \( B_r(x) = \{ y \in X : \| y - x \| < r \} \). For brevity, we denote \( B_r(0) \) by \( B_r \), and \( B_1(0) \) by \( B_1 \). An open and connected subset \( V \subset X \) is called a domain. A map \( h : V \to W \subseteq X \) is called holomorphic if it admits the Fréchet derivative \( h'(x) \) at each \( x \in V \).

For \( W \subseteq X \), by \( \text{Hol}(V, W) \) we denote the set of all holomorphic maps from a domain \( V \) to \( W \), and write \( \text{Hol}(V) \) for \( \text{Hol}(V, V) \).

For \( T \in B(X) \), uniform ergodic theorems for Abel and Cesàro averages were established by M. Lin [19, 20]. The following assertion can be deduced from the corresponding classical results of [19, 20], see [17, Assertion 1.3].

**Assertion 1.1.** Let \( T \in B(X) \) be such that \( \|T^n/n\| \to 0 \) as \( n \to +\infty \). Then, for each \( \alpha \in (0, 1) \), the series in (1.1) converges in the norm topology. Moreover, the following statements are equivalent:

(i) \( \text{Im}(I - T) \) is a closed subset of \( X \).

(ii) For some \( \alpha \in (0, 1) \), the sequence \( \{A^n_\alpha\}_{n \in \mathbb{N}} \) converges in the operator norm topology as \( n \to +\infty \).

(iii) For each \( \alpha \in (0, 1) \), the sequence \( \{A^n_\alpha\}_{n \in \mathbb{N}} \) converges in the operator norm topology as \( n \to +\infty \).

The limit in (ii) and (iii) is given by the projection of \( X \) onto \( \text{Ker}(I - T) \) along \( \text{Im}(I - T) \).

For an unbounded operator \( T : \mathcal{D}(T) \subset X \to X \), the Abel average is defined as

\[
A_\alpha = (1 - \alpha)[I - \alpha T]^{-1},
\]

where \( \alpha \in (0, 1) \) is such that \( A_\alpha \in B(X) \). A further extension of this kind consists in defining Abel averages for nonlinear holomorphic maps in \( X \). The Abel averages of these maps are also nonlinear holomorphic maps. They provide effective tools for studying nonlinear holomorphic maps, in particular nonlinear holomorphic semigroups and their fixed point sets [22].

Of course, the condition \( \|T^n/n\| \to 0 \) in Assertion 1.1 is not applicable to unbounded operators. Moreover, the condition is evidently far from necessary for the corresponding convergence to hold; it can be replaced by, e.g., the dissipativity condition used in the classical Lumer-Phillips theorem, see [2] p. 30]. It turns out that the only essential property of \( T \) required to guarantee the convergence in Assertion 1.1 is that the spectrum \( \sigma(T) \),
except possibly for the point \( \zeta = 1 \), lies in the half-plane
\[
\Pi = \{ \zeta \in \mathbb{C} : \text{Re}\zeta < 1 \}.
\]
Note that the conditions \( \| T^n/n \| \to 0 \) and (i) in Assertion 1.1 imply that
\[
(1.3) \quad \text{Ker}(I - T) \oplus \text{Im}(I - T) = X.
\]
Set \( \rho(T) = \mathbb{C} \setminus \sigma(T) \). In [17, Theorem 2.1], a necessary and sufficient condition for the power convergence of the Abel averages (1.2) was presented in the following form.

**Assertion 1.2.** Let \( T \) be a densely defined linear operator such that \( (1, +\infty) \subset \rho(T) \). Let also \( A_\alpha \) be its Abel average (1.2). Then the following statements are equivalent.

(i) For each \( \alpha \in (0, 1) \), \( A_\alpha \in B(X) \) and the sequence \( \{ A_\alpha^n \}_{n \in \mathbb{N}} \) converges in the operator norm topology.

(ii) \( \sigma(T) \subset \Pi \cup \{ 1 \} \) and (1.3) holds.

For each \( \alpha \in (0, 1) \), the limit in (i) is given by the projection of \( X \) onto \( \text{Ker}(I - T) \) along \( \text{Im}(I - T) \).

Let \( X^* \) be the dual space of \( X \). For \( x^* \in X^* \) and \( x \in X \), by \( \langle x, x^* \rangle \) we denote the value of the functional \( x^* \) at \( x \). Then we set \( J(x) = \{ x^* \in X^* : \| x \|^2 = \langle x, x^* \rangle = \| x^* \|^2 \} \). A linear operator \( T \) is called dissipative, see [8, Definition 3.13 and Proposition 3.13], if for each \( x \in D(T) \), there exists \( x^* \in J(x) \) such that the following holds
\[
(1.4) \quad \text{Re}(Tx, x^*) \leq 0.
\]
Similar definitions were used also for nonlinear holomorphic maps, see [12, Definition 2]. Note that a densely defined closed linear operator \( T \) is dissipative if and only if, for all \( \alpha \geq 0 \),
\[
(1.5) \quad \| A_\alpha \| \leq 1.
\]
In view of their numerous applications, dissipative maps constitute an important class of maps in complex Banach spaces. In this work, we introduce the following notion.

**Definition 1.3.** Given \( \omega \in \mathbb{R} \) and a dense subset \( D \subset B \), a map \( h : D \to X \) is called \( \omega \)-dissipative if, for some \( \varepsilon > 0 \) and \( \varsigma \in \mathbb{R} \), and for each \( x \in D \) such that \( 1 - \varepsilon < \| x \| < 1 \), and all \( x^* \in J(x) \), the following inequality holds
\[
(1.6) \quad \text{Re}(h(x), x^*) \leq \omega \| x \|^2 + \varsigma(1 - \| x \|^2).
\]
Obviously, a 0-dissipative linear operator is dissipative in the sense of (1.4). Below we study the connection between \( \omega \)-dissipativity of maps and the existence and contractiveness of their Abel averages, which we define as follows. For a map \( h : D \subset B \to X \), and for real \( \omega \) and \( \alpha \neq 1/\omega \), we set, cf. (1.2),
\[
(1.7) \quad \Phi_\alpha = (I - \alpha h)^{-1} \circ [(1 - \alpha \omega)I], \quad \Phi_0 = I.
\]
Definition 1.4. A map $h : D \to X$ is called $\omega$-pseudo-contractive if there exists $\delta > 0$ such that for each $\alpha \in (0, \delta)$, the Abel average $\Phi_\alpha$ defined in (1.7) is in $\text{Hol}(B, D)$.

Note that in Definition 1.4 we do not require $D$ be dense in $B$.

Remark 1.5. The use of pseudo-contractive nonlinear maps goes back to papers by F. E. Browder [5] and W. A. Kirk [14]. In those works, $h$ is set to be pseudo-contractive if $I - h$ is accretive, and hence $h - I$ is dissipative. A detailed study of accretive nonlinear maps can be found in S. Reich’s paper [21], cf. Theorem 3.1 on page 32. As we will see below in Theorem 1.8, our term $\omega$-pseudo-contractiveness with $\omega = 1$ is parallel to the pseudo-contractiveness in the works just mentioned.

Definition 1.6. A map $h : D \to X$ is said to be closed in the weak topology if, for each norm-convergent sequence $\{x_n\}_{n \in \mathbb{N}} \subset D$ such that $\lim_{n \to \infty} x_n =: x \in D$ and such that the sequence $\{h(x_n)\}_{n \in \mathbb{N}}$ weakly converges to $y \in X$, it follows that $h(x) = y$.

Let $V$ be a domain containing 0. Following L. Harris [10], we say that the spectrum $\sigma(h)$ of $h \in \text{Hol}(V, X)$ consists of those $\lambda \in \mathbb{C}$, for which there exist no open sets $U, W$ such that $0 \in U \subset V, W \subset X$, and $\lambda I - h$ is a holomorphic diffeomorphism of $U$ onto $W$. It is known, see [10], that $\sigma(h) = \sigma(h'(0))$.

Given $h \in \text{Hol}(B, X)$, suppose that $h$ can continuously be extended the boundary $\partial B$. Then the numerical range of $h$ is, cf. [10] [11],

$$N(h) = \{\langle h_s(x), x^* \rangle : s \in (0, 1), \ x \in \partial B, \ x^* \in J(x)\}.$$ 

For $h \in \text{Hol}(B, X)$ and and $s \in (0, 1)$, we let $h_s(x) = h(sx), x \in B$. Then the numerical radius of $h$ is

$$(1.8) \quad L(h) = \limsup_{s \to 1^-} \text{Re}N(h_s).$$

If $h$ has a uniformly continuous extension to $\bar{B}$ then

$$L(h) = \lim_{t \to 0^+} \frac{\|I + th\| - 1}{t},$$

by [10, Theorem 2] (see also the related discussion in [12, Section 2]).

Each $h \in \text{Hol}(V, X)$ has the following property: for every $x \in V$, there exists $r > 0$ such that $h$ is bounded on $\bar{B}_r(x)$. From this one readily gets that $h$ is bounded on each compact subset of $V$. However, $h$ need not be bounded on each bounded subset of $V$. Following [3] [12] we say that a holomorphic map $h : B \to X$ has unit radius of boundedness if, for each $r \in (0, 1)$,

$$\sup_{x \in \bar{B}_r} \|h(x)\| := C_h(r) < \infty.$$
For nonlinear maps \( g : U \to X, U \subseteq X \), we use the following notations

\[
\text{Null}(g) = \{ x \in U : g(x) = 0 \},
\]
\[
\text{Im}(g) = \{ y \in X : \exists x \in U \ g(x) = y \}.
\]

1.2. The results. Let \( \Delta := \{ \zeta \in \mathbb{C} : |\zeta| < 1 \} \) and \( \bar{\Delta} := \{ \zeta \in \mathbb{C} : |\zeta| \leq 1 \} \). Recall that a set \( C \subseteq X \) is called balanced if \( \zeta \in \bar{\Delta} \) for each \( x \in C \) and \( \zeta x \in C \).

**Theorem 1.7.** Let \( D \subset B \) be dense in \( B \) and balanced. Let also \( h : D \to X \) be an \( \omega \)-pseudo-contractive map, closed in the weak topology. Assume that either \( X \) is reflexive, or \( X \) is weakly sequentially complete and \( h \) is bounded on every \( K(x) := \{ \zeta x : \zeta \in \Delta \} \), \( x \in D \). Then \( h \) is \( \omega \)-dissipative.

Note that densely defined closed linear operators are closed in the weak topology and bounded on the sets \( K(x) \). Indeed, for a linear operator \( T \), the graph \( \Gamma(T) \subset X \times X \) is a convex set, which is weakly closed whenever \( T \) is closed in the usual sense. Moreover, \( TK(x) = K(Tx) \), which yields that \( T \) is bounded on compact subsets of \( K(x) \). Set

\[
Q_\omega = [0, 1/\omega), \quad \omega > 0; \quad Q_0 = [0, +\infty),
\]
\[
Q_\omega = (-\infty, 1/\omega) \cup [0, +\infty), \quad \omega < 0.
\]

**Theorem 1.8.** Given \( h \in \text{Hol}(B, X) \) and \( \omega \in \mathbb{R} \), the following statements are equivalent:

(i) \( h \) is \( \omega \)-dissipative.
(ii) \( L(h) \leq \omega \).
(iii) \( h \) has unit radius of boundedness and it is \( \omega \)-pseudo-contractive. Moreover, for all \( x \in B \),

\[
\lim_{\alpha \to 0^+} \Phi_\alpha(x) = x, \quad \lim_{\alpha \to 0^+} \frac{\Phi_\alpha(x) - x}{\alpha} = h(x) - \omega x,
\]

where the convergence is pointwise in the norm topology of \( X \).
(iv) \( h \) has unit radius of boundedness and \( \Phi_\alpha \in \text{Hol}(B) \), for each \( \alpha \in Q_\omega \).

As an immediate consequence of Theorem 1.7 and Theorem 1.8 we have the following:

**Corollary 1.9.** Let \( X \) be a weakly sequentially complete Banach spaces, \( \omega \in \mathbb{R} \) and \( h \in \text{Hol}(B, X) \). Then \( h \) is \( \omega \)-dissipative if and only if it is \( \omega \)-pseudo-contractive.

Next, we study the convergence of nonlinear Abel averages in the holomorphic category. As usually, for \( h \in \text{Hol}(B) \) and \( n \in \mathbb{N}_0 \), by \( h^n \) we denote the \( n \)-th iterate of \( h \), i.e., \( h^n := h^{n-1} \circ h, h^0 = I \).

**Theorem 1.10.** Let \( h \in \text{Hol}(B, X) \) be such that \( h(0) = 0 \). Suppose that, for some \( \omega \in \mathbb{R} \), the following holds

\[
\text{Ker}(\omega I - h'(0)) \oplus \text{Im}(\omega I - h'(0)) = X.
\]
Assume also that \( \Phi_{\alpha} \in \text{Hol}(B) \) for a certain \( \alpha \in \mathbb{R} \setminus \{0\} \) such that \( \omega \neq 1 \).

Then the sequence \( \{\Phi_{\alpha}^n\}_{n \in \mathbb{N}} \) converges in the norm topology of \( X \), uniformly on closed subsets of \( B \), if and only if the following holds

\[
\sigma(h'(0)) \subset \Omega(\alpha, \omega) := \{\omega\} \cup \left\{ \zeta \in \mathbb{C} : \left| \zeta - \frac{1}{\alpha} \right| > \left| \omega - \frac{1}{\alpha} \right| \right\}. 
\]

Moreover, the limit of \( \{\Phi_{\alpha}^n\}_{n \in \mathbb{N}} \) is a holomorphic retraction

\[
\phi_{\alpha} : B \to \text{Null}(\omega I - h) := \{x \in B : h(x) = \omega x\}.
\]

Contrary to the case of linear operators, the retractions \( \phi_{\alpha} \) may depend on \( \alpha \), see Section 3.2 below.

2. Proof of the theorems

2.1. Dissipative and pseudo-contractive maps.

Lemma 2.1. Let \( h : D \to X \) be \( \omega \)-pseudo-contractive. Then \( D \) is dense in \( B \) if and only if, for each \( x \in B \),

\[
\lim_{\alpha \to 0^+} \Phi_{\alpha}(x) = \Phi_0(x) = x,
\]

where the convergence is in the norm topology of \( X \).

Proof. If (1.1) holds for all \( x \in B \), then \( D \) is dense since \( \Phi_{\alpha}(x) \in D \) for all \( \alpha \in (0, \delta) \) and \( x \in B \).

Let us prove the converse. First we prove that (2.1) holds for all \( x \in D \).

For \( \alpha \in (0, \delta) \), we define

\[
y_{\alpha} = \frac{x - \alpha h(x)}{1 - \alpha \omega}, \quad x \in D.
\]

Clearly, \( y_{\alpha} \in B \) for small enough \( \alpha \), and hence we can compute \( \Phi_{\alpha}(y_{\alpha}) \) and get \( \Phi_{\alpha}(x) = x \). Let \( \rho \) be the hyperbolic metric on \( B \) then

\[
\rho(\Phi_{\alpha}(x), x) = \rho(\Phi_{\alpha}(x), \Phi_{\alpha}(y_{\alpha})) \leq \rho(x, y_{\alpha}),
\]

since \( \Phi_{\alpha} \in \text{Hol}(B) \). It is known [22, Theorem 3.7, page 99] that \( \rho \) is locally equivalent to the norm metric of \( B \). Moreover, for each \( x \in B \) and \( l \in (0, 1) \), there exist positive \( c_1(x, l) \) and \( c_2(x, l) \) such that, for all \( y \in B_l \),

\[
c_1(x, l)\|x - y\| \leq \rho(x, y) \leq c_2(x, l)\|x - y\|.
\]

Now we choose some \( l \in (\|x\|, 1) \) and \( \alpha_l > 0 \) such that \( y_{\alpha_l} \) in (2.2) lies in \( B_{\frac{1}{2}} \) for \( \alpha < \alpha_l \). Then by (2.2) and (2.3),

\[
\rho(\Phi_{\alpha}(x), x) \leq \frac{c_2(x, l)\alpha}{1 - \alpha \omega}\|\omega x - h(x)\|,
\]

which implies (2.1) for this \( x \).

Since \( D \) is dense in \( B \), the general case of \( x \in B \) can be handled by the triangle inequality and the result just proven.  \( \square \)
As a direct consequence of (2.1) we have the following result.

**Corollary 2.2.** Let $D \subset B$ be dense and $h : D \subset B \to X$ be $\omega$-pseudo-contractive. Then, for each $x \in B$ and $r \in (\|x\|, 1)$, there exists $\delta_r < \delta$ such that $\Phi_\alpha(x) \in B_r$ whenever $\alpha \in [0, \delta_r].$

Next, we need the following version of [12, Lemma 4].

**Proposition 2.3.** Given $\vartheta > 0$, let $f : \Delta \times [0, \vartheta) \to \Delta$ be holomorphic in the first variable for each fixed $t \in [0, \vartheta)$, and right-differentiable at 0 in the second variable for each fixed $\zeta \in \Delta$. Suppose also that $f(\zeta, 0) = c\zeta$ for all $\zeta \in \Delta$ and some $c \in (0, 1]$. Then, for each $\zeta \in \Delta$,

$$
(2.6) \quad \Re \zeta \frac{f_t'(\zeta, 0)}{f_t'(0, 0)} \leq (1 - c^2|\zeta|^2)\Re \zeta f_t'(0, 0),
$$

where $f_t'(\zeta, t) = \partial f(\zeta, t)/\partial t$.

In the sequel, by $w$-lim we denote the limit in the weak topology of $X$.

**Lemma 2.4.** Let $D \subset B$ be balanced and dense and let $h : D \to X$ be $\omega$-pseudo-contractive for some $\omega \in \mathbb{R}$. Suppose also that, for each $x \in D$, the following holds

$$
(2.7) \quad w\lim_{\alpha \to 0^+} \frac{\Phi_\alpha(x) - x}{\alpha} = h(x) - \omega x.
$$

Then $h$ is $\omega$-dissipative.

**Proof.** For $t > 0$, we set $\alpha_t = t/(1 + \omega t)$, and let $\vartheta_\delta > 0$ be such that $\alpha_t \in [0, \delta]$ for $t \in [0, \vartheta_\delta)$. Fix $x \in D \setminus \{0\}$ and let $\zeta \in \Delta$. Then set $u_t = (1 + \omega t)\zeta x$ where $t \in [0, \vartheta_\delta)$ is such that $u_t \in B$. By (2.1) and (2.3) and by Corollary 2.2 we obtain

$$
\rho(\Phi_{\alpha_t}(u_t), \zeta x) \leq t\|h(\zeta x)||c(\zeta x),
$$

which holds for some positive $c(\zeta x)$. Note that $\zeta x \in D$ since $D$ is balanced. We conclude that, for each $r \in (\|x\|, 1)$, there exists $\vartheta_r < \vartheta_\delta$ such that

$$
\|\Phi_{\alpha_t}(u_t)\| \leq r, \quad \text{whenever} \quad t \in [0, \vartheta_r].
$$

Since $\Phi_{\alpha_t} \in \text{Hol}(B)$,

$$
(2.8) \quad y_t(\zeta) := \Phi_{\alpha_t}(u_t) = \zeta x + \frac{t}{1 + \omega t}h(y_t(\zeta)), \quad y_0(\zeta) = \zeta x
$$

defines a holomorphic map $\Delta \ni \zeta \mapsto y_t(\zeta) \in B \subset X$ for $t \in [0, \vartheta_r]$. At the same time, by (2.7) the map $t \mapsto y_t(\zeta) \in B \subset X$ has a one-sided weak derivative at $t = 0^+$. For $x^* \in J(x)$, let us consider

$$
(2.9) \quad f(\zeta, t) = \frac{1}{\|x\|(1 + \omega t)}(y_t(\zeta), x^*).
$$

For each $t \in [0, \vartheta_r]$, it is a holomorphic function on $\Delta$. By (2.8), $f(\zeta, 0) = \zeta\|x\|$, and

$$
|f(\zeta, t)| \leq \frac{\|y_t(\zeta)\|}{1 + \omega t} \leq \frac{r}{1 + \omega t}.
$$
Thus, if $\omega \geq 0$, $|f(\zeta, t)| \leq r < 1$ for all $t \in [0, \delta_r]$, whereas if $\omega < 0$, $|f(\zeta, t)| < 1$ for sufficiently small $t$. Hence, for such $t$, $f(\cdot, t)$ maps $\Delta$ into itself. For each fixed $\zeta \in \Delta$, $f(\zeta, t)$ has a one-sided derivative at $t = 0^+$. A direct computation from (2.8) and (2.9) yields
\[ f_t(\zeta, 0) = [-\omega \zeta \|x\|^2 + \langle h(\zeta x), x^* \rangle \|x\|. \]
Applying this in (2.6) with $c = \|x\|$ and $\zeta = \tilde{\zeta} = s \in (0, 1)$, we then obtain
\[ \Re \langle h(sx), x^* \rangle \leq \omega s \|x\|^2 + (1 - s^2 \|x\|^2) \Re \langle h(0), x^* \rangle, \]
which holds for all $s < 1$. Thus, in the limit for $s \to 1^-$ we get (1.6). \hfill \Box

**Proof of Theorem 1.7.** By Lemma 2.4 it is enough to show that (2.7) holds for every $x \in D$. By (2.4), (2.5) and Corollary 2.2, for sufficiently small positive $\alpha$,
\[ \|\Phi_\alpha(x) - x\|/\alpha \leq \frac{c_2(x, r)}{c_1(x, r)(1 - \alpha \omega)} \|\omega x - h(x)\|. \]
If $X$ is reflexive, this estimate implies that the set
\[ \{\Phi_\alpha(x) - x\}/\alpha \subset X : \alpha \in (0, \delta) \}

is relatively weakly compact. For every sequence $\{\alpha_n\}_{n \in \mathbb{N}}$, converging to 0 as $n \to \infty$, $\Phi_{\alpha_n}(x) \to x$ by Lemma 2.1. By (1.7),
\[ \Phi_{\alpha_n}(x) - x \over \alpha_n = h(\Phi_{\alpha_n}(x)) - \omega x. \]
The assumed closedness (in the weak topology) of $h$, the weak compactness just mentioned, and the fact that reflexive Banach spaces are weakly sequentially complete then yield: for each $\alpha_n \to 0^+$, the sequence $\{\Phi_{\alpha_n}(x) - x\}/\alpha_n$ converges weakly to the right-hand side of (2.7).

If $X$ is not reflexive but it is sequentially complete, $D$ is balanced and $h$ is bounded on each $K(x)$, $x \in D$, we use the following arguments. Take arbitrary $y^* \in X^*$ and consider
\[ f_\alpha(\zeta) := \left\langle \frac{1}{\alpha}[\Phi_\alpha(\zeta x) - \zeta x], y^* \right\rangle \]
\[ = -\zeta \omega \langle x, y^* \rangle + \langle h(\Phi_\alpha(\zeta x)), y^* \rangle, \quad \zeta \in \Delta, \quad \alpha \in (0, \delta]. \]
For each $\alpha \in (0, \delta)$, $f_\alpha \in \text{Hol}(\Delta, \mathbb{C})$. For a fixed $\zeta \in \Delta$, the first line in (2.11) can be estimated by means of (2.10), from which it follows that
\[ |f_\alpha(\zeta)| \leq \frac{\|y^*\| c_2(\zeta x, r)}{c_1(\zeta x, r)c_3(\delta \omega)} \|\omega x - h(\zeta x)\|, \]
where $c_3(\delta \omega) = \min\{1, (1 - \delta \omega)\}$. Note that $c_2(\zeta x, r)/c_1(\zeta x, r)$ can be estimated on each $K(x)$. Thus, in view of the assumed boundedness of $h$ on the sets $K(x)$, the family $\{f_\alpha : \alpha \in (0, \delta)\}$ is bounded uniformly on compact subsets of $\Delta$. By Montel's theorem this family contains a sequence $\{f_{\alpha_n}\}_{n \in \mathbb{N}}$, which converges as $n \to \infty$, uniformly on compact subsets of $\Delta$ to some $f \in \text{Hol}(\Delta, \mathbb{C})$. Then by (2.11) we get that the sequence
\(\{\langle h(\Phi_{\alpha n}(\zeta x)), y^*\rangle\}_{n \in \mathbb{N}}\) converges to \(f(\zeta) + \zeta \omega(x, y^*)\). Now we use the convergence in (2.1), the weak sequential completeness of \(X\), and the closedness of \(h\) and conclude that

\[\langle h(\Phi_{\alpha n}(\zeta x)), y^*\rangle \to \langle h(\zeta x), y^*\rangle, \quad n \to \infty,\]

for all \(y^* \in X^*\).

\[\square\]

Remark 2.5. From the previous proof it follows that, for weakly sequentially complete spaces \(X\) and maps \(h\) as in Theorem [1.7], if bounded on each \(K(x)\), the map \(\zeta \mapsto h(\zeta x)\) is in \(\text{Hol}(\Delta, X)\) for each \(x \in D\).

2.2. Dissipative and pseudo-contractive holomorphic maps. The following statement, see [12, Theorem 1], is an extension of the well-known Lumer–Phillips theorem, see e.g., [2, p. 30].

**Proposition 2.6.** Given \(h \in \text{Hol}(B, X)\), it follows that \(L(h) \leq 0\) if and only if, for each \(t > 0\), \((I - th)(B) \supseteq B\) and \((I - th)^{-1} \in \text{Hol}(B)\).

> An immediate corollary of Proposition 2.6 is the following statement.

**Proposition 2.7.** If \(L(h) < 0\), then, for each \(y \in B_r, r = -L(h)\), the equation \(y = h(x)\) has a unique solution \(x \in B\). In particular, \(h\) has a unique null point in \(B\).

**Lemma 2.8.** Given \(\omega \in \mathbb{R}\) and \(h \in \text{Hol}(B, X)\), assume that \(L(h) \leq \omega\). Let also \(r > 0\) and \(\lambda \in \mathbb{C}\) be such that \(\text{Re}\lambda > \omega + r\). Then, for each \(y \in B_r\), the equation \(\lambda x - h(x) = y\) has a unique solution in \(B\).

**Proof.** For a given \(y \in B_r\), the null points of \(g(x) := y - \lambda x + h(x)\) solve the equation in question. For \(x \in \partial B, x^* \in J(x),\) and \(s \in (0,1)\), we obtain

\[\text{Re}(g(sx), x^*) = \text{Re}(y, x^*) - s\text{Re}\lambda + \text{Re}\langle h(sx), x^*\rangle.\]

Thus, by (1.8),

\[\limsup_{s \to 1^-} \text{Re}(g(sx), x^*) \leq \|y\| - (\text{Re}\lambda - \omega) \leq r - (\text{Re}\lambda - \omega) < 0.\]

Hence, the result follows by Proposition 2.7.

The next statement was proven in [3, Theorem 1.5].

**Proposition 2.9.** Let \(h \in \text{Hol}(B, X)\) be \(\omega\)-dissipative. Then, for each \(x \in B\), it follows that

\[\|h(x) - h(0)\| \leq \frac{\zeta_h\|x\|}{1 - \|x\|^2} + 4\|h(0)\|\|x\|^2,\]

where the constant \(\zeta_h > 0\) can be calculated explicitly. In particular, \(h\) has unit radius of boundedness.

**Proof of Theorem 2.8.** (i) \(\Rightarrow\) (ii) is immediate.

(i) \(\Rightarrow\) (iv) \(\Rightarrow\) (iii): The part related to the boundedness of \(h\) follows by Proposition 2.9. For \(\alpha \neq 0, 1/\omega\) and \(z \in B\), consider

(2.12)

\[(I - \alpha h)(x) = (1 - \alpha\omega)z.\]
For \( \lambda = 1/\alpha \), \( (2.12) \) becomes \( \lambda x - h(x) = y \) with \( y = (\lambda - \omega)z \). By Lemma 2.8 whenever \( 1/\alpha > \omega \), the latter has a unique solution \( x \in B \), which holds for each \( \alpha \in Q_\omega \), see (1.9). Thus, \( \Phi_\alpha \) is defined as a self-map of \( B \). For \( \omega = 0 \), \( \Phi_\alpha \in \text{Hol}(B) \) for all \( \alpha \geq 0 \) by Proposition 2.6. Similarly, \( \Phi_\alpha \in \text{Hol}(B) \) for all \( \omega \neq 0 \) and each \( \alpha \in Q_\omega \). The existence of the first limit in (1.10) follows by Lemma 2.4 as \( h \) is clearly \( \omega \)-pseudo-contractive, cf. Definition 1.4. The second limit in (1.10) follows by the continuity of \( h \).

(iii) \( \Rightarrow \) (i) follows directly from Lemma 2.4.

(ii) \( \Rightarrow \) (iii): The boundedness follows by [12, Corollary 9]. To prove that, for some \( \delta > 0 \), \( \Phi_\alpha \in \text{Hol}(B) \) for all \( \alpha \in [0, \delta) \) we let \( g := h - \omega I \), so that \( L(g) \leq 0 \). Hence, by [12, Theorem 1], \( (I - tg)^{-1} \in \text{Hol}(B) \) for all \( t > 0 \), i.e., the map \( B \ni x \mapsto y \in B \) is holomorphic, where \( y \) is defined by

\[
y - tg(y) = (1 + \omega t)y - th(y) = x,
\]

which, for \( t \in [0, 1/|\omega|] \), can be rewritten as

\[
y_{\alpha t} - \alpha t h(y_{\alpha t}) = (1 - \alpha t \omega)x.
\]

Thus, for \( 0 < \delta < 1/|\omega| \), it follows that \( \Phi_\alpha \in \text{Hol}(B) \) for all \( \alpha \in [0, \delta] \). \( \square \)

2.3. **Nonlinear Abel averages.** Let \( h \in \text{Hol}(B, X) \) be such that \( h(0) = 0 \). For every nonzero \( \lambda \notin \sigma(h) = \sigma(h'(0)) \), the set \( U \) in the definition of \( \sigma(h) \) also contains 0, and hence one can choose \( r > 0 \) such that \( (I - h)^{-1}(B_r) \subset B \). Fix these \( \lambda \) and \( r \). Then, for \( \alpha = 1/\lambda \) and all \( \omega \in \mathbb{R} \) such that \( |1/\alpha - \omega| \leq r \), the map

\[
\Phi_\alpha = \left( \frac{1}{\alpha} I - h \right)^{-1} \circ \left( \frac{1}{\alpha} \right) I
\]

is in \( \text{Hol}(B) \). Note that here we do not assume that \( h \) is \( \omega \)-dissipative.

Let \( \text{Fix}(\Phi_\alpha) := \{ x \in B : \Phi_\alpha(x) = x \} \) be the set of fixed points of \( \Phi_\alpha \). Since \( \Phi_\alpha(x) = x \) is equivalent to \( (I - \alpha h)(x) = (1 - \alpha \omega)x \), we then get

\[
\text{Fix}(\Phi_\alpha) = \text{Null}(\omega I - h) = \{ x \in B : \omega x - h(x) = 0 \}.
\]

Recall that a subset \( R \subset B \) is called a holomorphic retract if there exists a holomorphic retraction from \( B \) on \( R \), i.e., a holomorphic self-map \( \phi \) of \( B \) such that \( \phi(B) = R \) and \( \phi(z) = z \) for all \( z \in R \). If \( R \) is a holomorphic retract of \( B \) then, in particular, it is a non-singular closed submanifold of \( B \), and it is also totally geodesic with respect to the hyperbolic metric of \( B \).

Combining classical results of Koliha [16, Theorem 0] and Vesentini’s [23, Theorem 1], one can obtain the following characterization of the power convergence of holomorphic maps.

**Proposition 2.10.** Let \( \Psi \in \text{Hol}(B) \) be such that \( 0 \in \text{Fix}(\Psi) \). Then the following statements are equivalent.

(a) The sequence of iterates \( \{ \Psi^n \}_{n \in \mathbb{N}} \) converges, in the operator norm topology, uniformly on closed subsets of \( B \), to a holomorphic retraction of \( B \) onto \( \text{Fix}(\Psi) \).
(b) The sequence \( \{\Psi'(0)^n\}_{n \in \mathbb{N}} \) is convergent in the operator norm topology.

(c) \( \sigma(\Psi'(0)) \subset \Delta \cup \{1\} \) and \( \zeta = 1 \) is at most a simple pole of the resolvent of \( \Psi'(0) \).

Proof of Theorem 1.10. Note that \( (\alpha I - h'(0)) \) is invertible as \( 1/\alpha \notin \sigma(h) = \sigma(h'(0)) \). Since \( \Phi_\alpha \in \text{Hol}(B) \), we can compute its Fréchet derivative \( \Phi'_\alpha(x) \), \( x \in B \). By the chain rule we get from (1.7)

\[
\Phi'_\alpha(x) - \alpha h'(\Phi_\alpha(x))\Phi'_\alpha(x) = (1 - \alpha \omega)I.
\]

Next, we have \( \Phi_\alpha(0) = 0 \) as \( h(0) = 0 \), and hence

\[
(2.13) \quad \Phi'_\alpha(0) = (1 - \alpha \omega)(I - \alpha h'(0))^{-1}.
\]

Set

\[
\psi(\zeta) = \frac{1 - \alpha \omega}{1 - \alpha \zeta}, \quad \zeta \in \mathbb{C}.
\]

This function is holomorphic on \( \Omega(\alpha, \omega) \), and \( \psi(\Omega(\alpha, \omega)) = \Delta \cup \{1\} \). On the other hand, (2.13) and the spectral mapping theorem imply

\[
(2.14) \quad \sigma(\Phi'_\alpha(0)) = \sigma(\psi(h'(0))) = \psi(\sigma(h'(0))).
\]

Suppose now that \( \sigma(h) \subset \Omega(\alpha, \omega) \). Then, by (2.14) it follows that

\[
\sigma(\Phi'_\alpha(0)) \subset \Delta \cup \{1\}.
\]

Taking into account that \( 1/\alpha \notin \sigma(h'(0)) \), and hence \( \text{Im}(I - \alpha h'(0)) = X \), direct computations yield

\[
(2.15) \quad \text{Ker}(I - \Phi'_\alpha(0)) = \text{Ker}(\omega I - h'(0)),
\]

\[
\text{Im}(I - \Phi'_\alpha(0)) = \text{Im}(\omega I - h'(0)).
\]

By (2.15) and (1.11), we then get

\[
\text{Ker}(I - \Phi'_\alpha(0)) \oplus \text{Im}(I - \Phi'_\alpha(0)) = X,
\]

which means that 1 is at most a simple pole of the resolvent of \( \Phi'_\alpha(0) \). Then by Proposition 2.10 the sequence \( \{\Phi^n_\alpha\}_{n \in \mathbb{N}} \) converges, uniformly on closed subsets of \( B, \) to a holomorphic retraction \( \phi_\alpha : B \rightarrow \text{Fix}\Phi_\alpha = \text{Null}(\omega I - h) \). The converse statement follows directly by Proposition 2.10. □

Remark 2.11. As follows from Theorem 1.10, a necessary condition for the sequence \( \{\Phi^n_\alpha\}_{n \in \mathbb{N}} \) to converge is that \( \text{Null}(\omega I - h) \) is a holomorphic retract of \( B \).

3. Examples

3.1. Dissipative maps. As a typical example of a map \( h : D \subset B \rightarrow X \) described by Theorem 1.7 we consider

\[
(3.1) \quad h = T + g,
\]

where \( T : D(T) \subset X \rightarrow X \) is a closed densely defined linear operator with a nonempty resolvent set and \( g \in \text{Hol}(B, X) \). In this case, \( D = D(T) \cap B \). As
we pointed out in Section 1, \( T \) is also closed in the weak topology. Hence so is \( h \). For each \( x \in B \), \( h \) is bounded on compact subsets of \( K(x) \). Moreover, by Remark 2.5, the map \( \zeta \mapsto h(\zeta x) \) is in \( \text{Hol}(\Delta, X) \).

**Proposition 3.1.** Let \( T : \mathcal{D}(T) \subset X \to X \) be closed and densely defined, and such that \( \text{Re}(Tx, x^*) \leq 0 \) for each \( x \in \mathcal{D}(T) \) and \( x^* \in J(x) \). Assume also that \( g \in \text{Hol}(B, B_{\omega}) \) for some \( \omega > 0 \). Then \( h \) defined in (3.1) is \( \omega \)-dissipative.

**Proof.** The assumed properties of \( T \) imply that its Abel average \( A_\alpha \), \( \alpha \geq 0 \), defined in (1.2) exists and satisfies (1.5). For \( \alpha > 0 \), the Abel average of \( h \) maps \( x \in B \) to \( y \in X \) given by the unique solution of the equation

\[
y - \alpha Ty - \alpha g(y) = (1 - \alpha \omega)x.
\]

This can be rewritten as

\[
y = \alpha(A_\alpha \circ g)(y) + (1 - \alpha \omega)A_\alpha x.
\]

By Proposition 2.7, the map

\[
y \mapsto \alpha(A_\alpha \circ g)(y) + (1 - \alpha \omega)A_\alpha x - y
\]

has a unique null point in \( B \). Thus, the map \( x \mapsto y \) is in \( \text{Hol}(B) \). The proof now follows by Theorem 1.7. \( \Box \)

A concrete example of \( h \) as in Proposition 3.1 is given by the following integro-differential map, which appears in nonlinear and nonlocal evolution equations of the Fisher-KPP type [1]. Here \( X \) is a complex Hilbert space \( L^2(\mathbb{R}) \). Let \( a : \mathbb{R} \times \mathbb{R} \to (0, +\infty) \) be symmetric and such that the operator

\[
L^2(\mathbb{R}) \ni x \mapsto \int_{\mathbb{R}} a(\cdot, s)x(s)ds,
\]

maps \( L^2(\mathbb{R}) \) into \( L^\infty(\mathbb{R}) \), and

\[
\left\| \int_{\mathbb{R}} a(\cdot, s)x(s)ds \right\|_{L^\infty(\mathbb{R})} \leq a \|x\|_{L^2(\mathbb{R})},
\]

for some \( a > 0 \). The integro-differential map \( h = T + g \), where

\[
T = \frac{d^2}{dt^2}, \quad g(x) = bx(\cdot) \left[ 1 - \int_{\mathbb{R}} a(\cdot, s)x(s)ds \right], \quad b > 0,
\]

is 1-dissipative for \( b < 1/(1 + a) \). For \( \mathcal{D}(T) \), one can take the Sobolev space \( W^{2,2}(\mathbb{R}) \), see, e.g., [18, Chapters 6 and 7].

3.2. **Nonlinear Abel averages.** For a linear operator with Abel average \( A_\alpha \), \( \alpha \in (0, 1) \), the limit of the sequence \( \{A_\alpha^n\}_{n \in \mathbb{N}} \), if it exists, is one and the same for all \( \alpha \), see Assertion 1.2. In the nonlinear case, this is no longer true. This is related to the non-uniqueness of holomorphic retractions for holomorphic retracts of \( B \). In [4, Section 3] it was proved that any one-dimensional retract of a bounded convex domain in \( \mathbb{C}^n \) with smooth boundary admits a unique holomorphic retraction whose fibers are affine. However, in general, there can also be non-affine retractions. Using Abel
averages for nonlinear holomorphic maps, one can construct such non-affine holomorphic retractions, as it is shown in the following example shows.

Denote $\mathbb{B}^2 := \{ z \in \mathbb{C}^2 : \| z \| < 1 \}$, where $\| \cdot \|$ is the standard Euclidean norm, and set $h(z) = h(\xi, \eta) = (\lambda \xi + e \eta^2, 0)$, $0 < e < 1$. Note that $\sigma(h) = \sigma(h'(0)) = \{ 0, \lambda \}$. Set $\omega = \lambda$. Then $\sigma(h) \subset \Omega(\alpha, \lambda)$ for $|1 - \alpha \lambda| < 1$. A direct computation shows that

$$\Phi_\alpha(\xi, \eta) = \left( \xi + \alpha e (1 - \alpha \lambda) \eta^2, (1 - \alpha \lambda) \eta \right).$$

It is easy to check that $\Phi_\alpha$ is a holomorphic self-map of $\mathbb{B}^2$ for small enough $\alpha$. Since (1.11) is always satisfied in finite dimensional Banach spaces, Theorem 1.10 applies and $\{ \Phi_\alpha^n \}$ converges to a holomorphic retraction $\phi_\alpha$ of $\mathbb{B}^2$ onto $\text{Null}(\lambda I - h) = \{ (\xi, 0) : \xi \in \Delta \}$. By (3.2), for $n \in \mathbb{N}$, we then get

$$\Phi_\alpha^n(\xi, \eta) = (\xi + \alpha e (1 - \alpha \lambda) \eta^2, (1 - \alpha \lambda)^n \eta),$$

which yields

$$\phi_\alpha(\xi, \eta) = \lim_{n \to \infty} \Phi_\alpha^n(\xi, \eta) = (\xi + \frac{1 - \alpha \lambda}{2 - \alpha \lambda} \eta^2, 0).$$

In particular, $\phi_\alpha$ depends on $\alpha$.

It is also interesting to note that $\phi_\alpha$ can be extended to all $\alpha \in \mathbb{R}$ such that $|1 - \alpha \lambda| < 1$. Although the map $\Phi_\alpha$ may no longer be a self-map of $\mathbb{B}^2$ for some $\alpha$. For instance, take $\lambda = 1$, $e = 1/2$, and $\alpha \in (0, 2)$. But, $\lim_{|\eta| \to 1} \| \Phi_\alpha(0, \eta) \| = (1 + \frac{\alpha^2}{4})(1 - \alpha)^2$ and, for $\alpha$ close to 2, this number is bigger than 1. In this case, however, $\phi_\alpha$ is not a self-map if $\mathbb{B}^2$, so that it is not a holomorphic retraction of $\mathbb{B}^2$ onto $\text{Null}(\lambda I - h)$.

For $\omega = 0$, it follows that $\sigma(h) \subset \Omega(\alpha, 0)$ for all $\alpha \in \mathbb{R}$ such that $|1 - \lambda \alpha| > 1$. In this case,

$$\Phi_\alpha(\xi, \eta) = \left( \frac{\xi + \alpha e \eta^2}{1 - \alpha \lambda}, \eta \right).$$

It is easy to check that, for any $\alpha$ such that $|1 - \lambda \alpha| > 1$, the points $\Phi_\alpha(-\frac{\xi}{2} \eta^2, \eta)$ are not in $\mathbb{B}^2$ for $|\eta| \to 1$. This is not surprising, because otherwise $\{ \Phi_\alpha^n \}$ would converge to a holomorphic retraction of $\mathbb{B}^2$ onto $\text{Null}(h) = \{ (\xi, \eta) \in \mathbb{B}^2 : \lambda \xi = -e \eta^2 \}$, which would imply that $\text{Null}(h)$ be a one-dimensional holomorphic retract of $\mathbb{B}^2$ (so-called complex geodesic of $\mathbb{B}^2$), while one-dimensional holomorphic retracts of $\mathbb{B}^2$ are known to be just the intersection of affine complex lines with $\mathbb{B}^2$.

4. Open question

Let $D$ be a balanced dense subset of $B$, and let $h$ be a closed (weakly closed) map on $D$ with values in $X$. Assume that $h$ is 0-dissipative and for some $\alpha_0 > 0$ its Abel average exists, is holomorphic, and maps $B$ into itself. Do the Abel averages exist for all positive $\alpha$?
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