Limits on the Measurability of Space-time Distances in
(the Semi-classical Approximation of) Quantum Gravity

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ABSTRACT

By taking into account both quantum mechanical and general
relativistic effects, I derive an equation that describes some limita-
tions on the measurability of space-time distances. I then discuss
possible features of quantum gravity which are suggested by this
equation.

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I Introduction

The study of problems in which both quantum mechanical (QMal) and general relativistic (GRic) effects are important is strongly motivated by the possibility of discovering some features of Quantum Gravity (QG), a yet to be found “more fundamental” theory in which the questions raised by the apparent incompatibilities of Quantum Mechanics (QM) and General Relativity (GR) are solved. One such problem that has recently attracted a lot of attention is the (semi-classical) quantum analysis of black holes, which has lead to the famous black hole information paradox. In this Letter I shall be concerned with another extensively investigated problem that, in most treatments, involves the interplay between QMal and GRic effects: the search for limitations on the quantum measurement of space-time distances (see, for example, Refs.[2-8]; in particular, Ref.5 gives a good review of the results in this area and refers to additional related material). Specifically, I shall analyze from a different point of view the interesting measurement procedure for space-time distances discussed in Refs.[2, 7], and derive a simple equation that indicates some possible features of QG.

II Measurement Procedure

In ordinary QM the distance between two space-time points is given in terms of their coordinates, which are assumed to have a physical meaning independent of observations (no prescription for the measurement of these coordinates is to be given). In GR, instead, space-time points can only be meaningfully identified by events. Therefore, a basic measurement to be considered in (at least the semiclassical limit of) QG is the quantum measurement of the distance between events in space-time. In particular, it is important to establish whether there are limitations on the accuracy achievable in these measurements, because such limitations would render impossible the definition of a traditional coordinate system. A simple way to start analyzing this issue is given by the study of the measurement of the length $L$ of an object, which was considered in Ref.[7]. Clearly, this does not lead to the most general discussion of the quantum measurement of the distance between events in space-time (in particular, the very idea of the “length of an object” can only be introduced in approximately flat regions of space-time); however, its simplicity allows a very intuitive analysis, which is useful in deriving expectations for the features of QG. As discussed in Refs.[2, 4], in the spirit of GR such a length measurement can be carried out by putting a clock, a “light-gun” (i.e. a device capable of sending a light signal when triggered), and a detector at one end of the object and putting a mirror at the other end. [In the discussions of Refs.[2, 4] the light-gun and the detector are not explicitly mentioned; however, their “clock” is capable of sending the signal at $t=0$ (i.e. it is connected to a light-gun), and stops when hit by the signal coming back from the mirror (i.e. it can detect the signal).] This apparatus should be set to send a light signal toward the mirror when the clock reads zero, and to record the time $T$ shown by the clock when the light signal is detected by the detector after being reflected by the mirror. Clearly the time $T$ is related to the length $L$; for example, in

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1Here and in the title I point out that my discussion might only apply to the semiclassical limit (GRic geometry and quantized matter) of QG. In fact, it is possible that, at scales smaller than the Plank length, the very concept of distance may be somewhat foreign to the full QG because geometry might look very different from what we are accustomed to. \[8\].
Minkovski space and neglecting quantum effects one finds that $T$ and $L$ are simply related by

$$L = c \frac{T}{2}, \quad (1)$$

where, as customary, $c$ denotes the speed of light.

I am interested in analyzing this measurement procedure including QM and GRic effects, and therefore the relation between $T$ and $L$ is more complex than Eq. (1). Specifically, the outcome $T$ (time read by the clock) of a measurement procedure and $L$ (the “actual” length of the object) are formally related as follows

$$L = c \frac{T}{2} \pm \delta L \pm \Delta L \pm \delta_g L, \quad (2)$$

where

- $\delta L$ is the total QM uncertainty due to the QM uncertainties in the position and velocity of the various agents in the measurement procedure. As it was already discussed in Refs. [2, 7], contributions to $\delta L$ come, for example, from the spread in the position of the various devices (clock, light-gun, detector, and mirror) during the time interval $T$.

- $\Delta L$ is the total classical correction (to the relation $L = cT/2$) due to the gravitational forces among the agents in the measurement procedure. Examples of such corrections are the ones resulting from the gravitational attraction between the light signal and the devices in the apparatus.

- $\delta_g L$ is the total QM uncertainty which results from the uncertainties in the gravitational forces among the agents in the measurement procedure. For example, as a result of the QM spread in the mass of the clock, there is an uncertainty in the strength of the gravitational attraction exerted by the clock on the light signal.

In this Letter I intend to investigate, as intuitively as possible, the possibility of limitations on the accuracy of quantum measurements of $L$ resulting from the fact that it is not possible to prepare the apparatus so that the uncertainty introduced in the measurement of $L$ by the presence of the $\delta$’s in Eq. (1) be arbitrarily small. I shall not attempt to derive the absolute lower bound (whose rigorous derivation surely involves an extremely complex analysis) on the uncertainty of quantum measurements of $L$, instead I shall look for a lower bound (which may well be lower than the absolute lower bound) for this uncertainty. Consistently with this objective, the only contribution to $\delta L$ that I will consider is $\delta x_{com}$, defined as the spread in the position of the center of mass of the system composed by clock, light-gun, and detector during the time interval $T$. The other (many!) contributions to $\delta L$ (given, among others, by the spread in the position of the clock, light-gun, and detector with respect to the center of mass of the clock+light-gun+detector system, and by the spread in the position of the mirror) could obviously only increase the lower bound on the uncertainty that I will present, but, like the authors of Ref. [7], I prefer to give an intuitive and simple discussion rather than attempting to find a higher lower bound.
\(\delta g L\) has already been considered in Refs.\[3, 8\], where it has been found that \(\delta g L \geq L_p\), \(L_p\) being the Plank length. Therefore, in the following, I shall look for a lower bound \(\delta x_{\text{com}}^{(\text{min})}\) for \(\delta x_{\text{com}}\), which will lead to a lower bound for the uncertainty \(\delta L_{\text{tot}}\) in our length measurement, as indicated by the relation

\[
\delta L_{\text{tot}} \equiv \delta L + \delta g L \geq \delta x_{\text{com}} + L_p \geq \delta x_{\text{com}}^{(\text{min})} + L_p .
\] (3)

In the following, I shall also assume (these simplifying assumptions will allow me to use spherical symmetry) that the clock+light-gun+detector system has (homogeneously distributed) mass \(M\) and occupies a spherically symmetric region of space of radius (“size”) \(s\), and neglect the effects of all the other masses in the problem.

Finally let me observe that the \(\Delta L\) contribution to Eq.(2) does not play any role in my study because, as I indicated, in a context in which both GRic and QMal effects are taken into account, \(\Delta L\) represents a correction, not an uncertainty. In fact, knowing all the masses in the system, \(\Delta L\) can be calculated and accounted for in the analysis of the outcome of the measurement, therefore leading to no additional uncertainty.

In Ref.\[7\] \(\Delta L\) was essentially treated on the same footing as the \(\delta\)’s. This makes the results of Ref.\[7\] relevant to contexts different from the one that I am here considering; they apply to measurements in which the masses in the apparatus are not known, or to measurements whose outcome is analyzed ignoring GRic effects (in which case \(\Delta L\) is to be treated as an experimental error).

### III Analysis

#### III.1 Quantum Mechanics

The evaluation of the spread in the position of (the center of mass of) a body during a time interval \(T\) is an elementary QMal problem. Following Ref.\[2\], one finds that

\[
\delta x_{\text{com}} \equiv \delta x_{\text{com}}(t \epsilon [0, T]) \geq \delta x_{\text{com},i} + \frac{\hbar}{c M \delta x_{\text{com},i}} ,
\] (4)

where \(\delta x_{\text{com},i} \equiv \delta x_{\text{com}}(t = 0)\) is the initial (i.e. at the time \(t=0\) when the light signal is emitted) spread in the position of the center of mass of the clock+light-gun+detector system.

Eq.(4) can be understood as follows\[2, 7\]. Initially, the system is described by a wave packet with position-spread \(\delta x_{\text{com},i}\) and velocity-spread \(\delta v_{\text{com},i}\). During the time interval \(T\) following \(t=0\) the uncertainty in the position of the center of mass of the clock+light-gun+detector system is given by

\[
\delta x_{\text{com}} \sim \delta x_{\text{com},i} + \delta v_{\text{com},i} T \sim \delta x_{\text{com},i} + \delta v_{\text{com},i} \frac{2L}{c} ,
\] (5)

where on the right-hand-side I used the fact that in first approximation \(T \sim 2L/c\).

\^\text{As discussed in Ref.\[7\], accounting for the effects of the other masses (and accounting for the the QMal spread in the mass \(M\)) could only increase the lower limit on \(\delta L_{\text{tot}}\) found in this type of analysis.}
Eq. (5) takes the form of Eq. (4) once the uncertainty principle, which states that

$$\delta x_{\text{com},i} \delta v_{\text{com},i} \geq \frac{\hbar}{M}, \quad (6)$$

is taken into account.

The most important feature of Eq. (4) is that it indicates that for a clock+light-gun+detector system of a given mass $M$ there is no way to prepare the $t = 0$-wave-packet so that $\delta x_{\text{com}} = 0$. In fact, Eq. (4) indicates that, for given $M$, QM leads to the following minimum value of $\delta x_{\text{com}}$

$$[\delta x_{\text{com}}^{(\text{min})}]_{QM} \sim \sqrt{\frac{\hbar L}{c M}}, \quad (7)$$

where, as I shall do in the following, I neglected numerical factors of $O(1)$, which are essentially irrelevant for the discussion presented in this Letter.

### III.2 General Relativity

Up to this point I have only used QM and therefore it is not surprising to discover that, as shown by Eq. (7), $[\delta x_{\text{com}}^{(\text{min})}]_{QM} \to 0$ as $M \to \infty$. Indeed, the uncertainty that I am considering originates from the uncertainty in the kinematics of the bodies involved in the measurement procedure, and clearly this uncertainty vanishes in the limit of infinite masses (the classical limit) because in this limit Eq. (6) is consistent with $\delta x_{\text{com},i} = \delta v_{\text{com},i} = 0$.

However, the central observation of the present paper is that this scenario is significantly modified, as a result of the dramatic consequence on the geometry of large “localized” (what I mean here with localized will become clear later) masses, when the GRic effects relevant to our experimental set up are taken into account.

For the moment, let us consider a fixed size $s$ for the clock+light-gun+detector system. Large values of the mass of the clock+light-gun+detector system necessarily lead to great distortions of the geometry, and well before the $M \to \infty$ limit (which, as I just indicated, is desirable for reducing the uncertainty given by Eq. (7)) our measurement procedure can no longer be followed. In particular, if

$$M \geq \frac{c^2 s}{G} = \frac{\hbar s}{c L_p^2}, \quad (8)$$

where $G$ is Newton’s constant of gravitation, an horizon forms around the center of mass of the clock+light-gun+detector system (here I am using the spherical symmetry) and it is not possible to send a light signal from the clock+light-gun+detector system.

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§This observation plays for example a crucial role in Bhor and Rosenfeld’s measurement analysis concerning quantum electrodynamics. Specifically, it motivates the choice, as an agent in their gedanken experiment, of a continuous charge distribution whose ratio of electric charge versus mass can be taken to zero. Limitations, like the ones that I discuss in the following, on the measurability of spacetime distances in QG are just due to the fact that the QG-charge and the mass are the same thing (equivalence of inertial and gravitational mass), their ratio is fixed to 1, and therefore the type of measurement strategy adopted by Bhor and Rosenfeld is not viable in QG.
system to the mirror positioned on the other side of the object whose length is being measured. This observation, combined with Eq. (7), allows me to derive the following relation giving a $L$-dependent lower bound on the uncertainty

$$\delta L^{tot} \geq \delta x_{com}^{(min)} + L_p \sim \sqrt{\frac{LL^2_p}{s}} + L_p .$$

(9)

It must be noted that $\delta x_{com}^{(min)} \rightarrow 0$ in the $s \rightarrow \infty$ limit; however, it is rather unclear whether it makes any sense to allow that in our experimental set-up the clock+light-gun+detector system has infinite size (or even that $s > L$).

**IV Conclusion**

In this last section, I want to discuss some possible features of QG which might be indicated by Eq. (9).

**IV.1 Classical Device**

My first observation is that Eqs. (7) and (9) can be interpreted as indications of the fact that whereas in QM a classical device is "only" required to be infinitely massive, in (at least the semiclassical approximation of) QG a classical device is required to be infinitely massive and infinitely extended.

It would be interesting to investigate whether this observation is compatible with our present measurement theory, in which the presence of classical devices is a crucial ingredient.

*If (at a given moment) there is a device inside the horizon of a black hole and close to the center, and another device further away from the center (it does not matter whether this second device is inside or outside the horizon as long as it is at a greater distance from the center than the other device), signals emitted by the device which is closer to the center (the clock+light-gun+detector system) cannot reach the device which is further away (the mirror). I thank Mario Bergeron and John Stachel for discussing with me about this.*

∥Moreover, it is easy to realize that (even if we allow the size of the clock+light-gun+detector system to be extremely large) the lower bound on $\delta L^{tot}$ might still be rather strongly $L$-dependent because the size $s$ which is in fact relevant in the determination of a significant lower bound is the size of the region of the clock+light-gun+detector system which is actually interacting with the photon.

**With classical device I intend a device (an example of which would be given in my measurement analysis by the clock+light-gun+detector system if it was infinitely massive and extended) which can perform observations (i.e. can probe with signals the system under measurement), and whose position and velocity are both completely determined, so that the accuracy of its observations is only limited by the application of the uncertainty principle to the quantities observed. In particular, such a device should be able to measure distances (or lengths) with unlimited accuracy (obviously at the price of renouncing to any information concerning momenta), at least up to scales of order $L_p$.**
IV.2 Decoherence

An interesting possibility suggested by Eq. (9) is the one of decoherence. In fact, Eq. (9) indicates that in every real-world \( s \neq \infty \) experiment, unless \( s \sim L \), the uncertainty \( \delta L_{\text{tot}}^\text{ } \) on the measurement of \( L \) depends on \( L \) itself, and, as discussed for example in Refs. [3, 7], this should lead to quantum (de)coherence phenomena. It would not come as a surprise if indeed QG would host a mechanism for decoherence; in particular, this has already been speculated in Refs. [3, 7] and in some approaches to the black hole information paradox. Moreover, decoherence has also been advocated in discussions of the QMal “measurement problem”, and if QG (besides easing the tension between QM and GR) must also give us a satisfactory measurement theory, one might expect decoherence to be present\(^{††} \).

IV.3 Material Reference Systems

In Ref. [12], Rovelli has identified well-defined local observables of QG, by explicitly taking into account the physical nature of the bodies that form the reference system, which is understood as a material reference system. A rather typical material reference system is given by a collection of bodies of size \( \tilde{s} \), and one of the local observables available in such a material reference system is the distance between two of the bodies. Eq. (9) appears to be immediately relevant to this framework. In particular, it indicates that the distance between two of the bodies of a material reference system cannot be measured with infinite accuracy, the uncertainty being bounded from below by \( L_p + \sqrt{L L_p^2 / \tilde{s}} \); moreover, decoherence phenomena should characterize the physics as described by one such material reference system.

Also notice that in this case the \( \tilde{s} \to \infty \) limit is meaningless\(^{‡‡} \), and actually, in order to have as fine as possible a network of bodies, one would like \( \tilde{s} \) to be small, but

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\(^{††}\) On this issue of the relation between Eq. (9), decoherence, and the QMal “measurement problem” let me make parenthetically a speculative but perhaps intriguing comment. In the QG measurement analysis presented in Ref. [11] it has been argued that in any such measurement procedure the information must be first transferred from the system being measured to some intermediate microscopic system (also to be considered part of the apparatus) before being recorded by the appropriate macroscopic devices. In such a scenario one can see [10] that the size \( s \) relevant to the lower bound on the uncertainty of length measurements that I have discussed is the size of such microscopic systems, and therefore the “amount of decoherence” would be rather large. If QG is to give us a consistent measurement theory, one can also envision that QG might prescribe a typical size \( s^* \) (which could be related to or be given by the Plank length \( L_p \), which is the natural length scale in the problem) for the kind of microscopic systems which can appear in the very first stage of the measurement procedure. In this scenario, once substituted \( s^* \) to \( s \), Eq. (9) would give an absolute and general lower bound on the accuracy of measurements of space-time distances in QG.

\(^{‡‡}\) It is not possible to set up the network of bodies necessary to form the material reference system if each one of the bodies occupies all of space. Moreover, Rovelli’s observables are defined only with respect to a given material reference system, i.e. they are characterized by a specific value of \( \tilde{s} \), and therefore increasing \( \tilde{s} \) one would not reach better and better accuracy in the measurement of the same Rovelli’s observable, one would instead span a class of different Rovelli’s observables which are measurable with different accuracy depending on their specific value of \( \tilde{s} \).
this, following Eq.(9), leads to large uncertainty in length measurements and large decoherence.

**IV.4 Outlook**

In conclusion, “probing” (even if only conceptually) an area of physics where both QMαl and GRic effects can be relevant has once again lead to an intruiging result. It appears that the analysis of procedures of measurement of space-time distances holds promises of being instructive not only on the contradictions between QM and GR, but also on the unsolved issues of the QMal measurement theory, and therefore there are reasons of interest in pursuing further the type of investigation preliminarily presented, in their respective limits of validity, in this Letter and in Ref.[1]. In particular, it would be important to verify whether the results are relevantly modified by removing the simplifying assumptions made here and in Ref.[7], which allow a spherically symmetric treatment. It would also be interesting to estimate whether the decoherence phenomena to be expected, as a result of Eq.(9), in real-world laboratories (with their typical sizes) are large enough to be observed.

In the preliminary stages of this work I greatly benefitted from very stimulating discussions with John Stachel, which I am very happy to acknowledge. I am also happy to acknowledge conversations with Jack Ng, with whom I discussed issues related to Ref.[7], and several members of the Center for Theoretical Physics, especially Dongsu Bak, Mario Bergeron, Domenico Seminara, and Philippe Zaugg.
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