Exact correlations in the one-dimensional coagulation–diffusion process investigated by the empty-interval method

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Abstract. The long-time dynamics of reaction–diffusion processes in low dimensions is dominated by fluctuation effects. The one-dimensional coagulation–diffusion process is used to describe the kinetics of particles which freely hop between the sites of a chain and where, upon two particles meeting, one of them disappears with probability 1. The empty-interval method has, for a long time, proved a convenient tool for use in the exact calculation of time-dependent particle densities using this model. We generalize the empty-interval method by considering the probability distributions of two simultaneously empty intervals at a given distance. While the equations of motion of these probabilities reduce for the coagulation–diffusion process to a simple diffusion equation in the continuum limit, consistency with the single-interval distribution introduces several non-trivial boundary conditions which are solved for the first time for arbitrary initial
configurations. In this way, exact space–time-dependent correlation functions can be directly obtained and their dynamic scaling behaviour is analysed for large classes of initial conditions.

**Keywords:** correlation functions, exact results, phase transitions into absorbing states (theory), diffusion
1. Introduction

The precise description of cooperative effects in strongly interacting many-body systems continues to pose many challenges. Paradigmatic examples are systems which may be described in terms of diffusion-limited reaction–diffusion processes. Applications of these systems and their non-equilibrium phase transitions have arisen in fields as different as solid-state physics, physical chemistry, physical and chemical ageing, cosmology, biology, financial markets and population evolution in social sciences. If the spatial dimension of these systems is low enough, that is \( d \leq d^* \) where \( d^* \) is the upper critical dimension, fluctuation effects dominate the long-time kinetics of these systems and their behaviour is different from the one expected from the solutions of (mean-field) reaction–diffusion equations, which attempt to describe the interactions of the elementary constituents in terms of the macroscopic law of mass action; see e.g. [37, 26, 42, 6, 34, 18].

One of the motivations for this work is the continuing practical interest in systems with reduced dimensionality, and such that homogenization through stirring is not possible. Specifically, we shall consider the one-dimensional coagulation–diffusion process, which is defined as follows. Consider a single species \( A \) of indistinguishable particles, such that each site of an infinitely long chain can be either empty or occupied by a single particle. The dynamics of the system is described in terms of a Markov process, where allowed two-site microscopic reactions \( A + \emptyset \leftrightarrow \emptyset + A \) and \( A + A \rightarrow A + \emptyset \) or \( \emptyset + A \) are implemented as follows: at each microscopic time step, a randomly selected single particle hops to a nearest-neighbour site, with a rate \( \Gamma := Da^2 \), where \( a \) is the lattice constant. If that site was empty, the particle is placed there. On the other hand, if the site was already occupied, one of the two particles is removed from the system with probability 1.

This model is one of the best-studied examples of a diffusion-limited process and at least since the work of Toussaint and Wilczek [46] it has been known that the mean particle concentration \( c(t) \sim t^{-1/2} \) for large times and with an amplitude which is thought to be universal as confirmed by the field-theoretical renormalization group [24, 7]; in contrast a mean-field treatment would have predicted \( c(t) \sim t^{-1} \). These theoretically predicted fluctuation effects have been confirmed experimentally, for example using the kinetics of excitons on long chains of the polymer TMMC = (CH\( _3 \))\( _4 \)N(MnCl\( _3 \)) [23], but also in other polymers confined to quasi-one-dimensional geometries [36, 21]; see also the reviews in [37]. Another recent application of diffusion-limited reactions concerns carbon nanotubes, for example the relaxation of photoexcitations [41] or the photoluminescence saturation [44]. On the other hand, the 1D coagulation–diffusion process has also received attention from mathematicians [9, 31] and is simple enough that it can be related to integrable quantum chains; see [3, 42]. Hence, by a consideration of the quantum chain Hamiltonians, which can be derived from the master equation, the time dependence of its observables could in principle be found via a Bethe ansatz; see [40]. In practice, however, it has turned out to be easier to find the time-dependent densities from the empty-interval method, which considers the time-dependent probabilities \( E_n(t) \) that \( n \geq 1 \) consecutive sites of the
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chain are empty [5, 11, 4, 6, 27]; see also [43]. The $E_n(t)$ satisfy a closed set of differential–difference equations, subject to the boundary condition $E_0(t) = 1$, and the average particle concentration is obtained as $c(t) = (1 - E_1(t))/a$. The scaling behaviour of the averages can be directly studied in the continuum limit $a \to 0$, when $E_n(t) \to E(x, t)$ which in turn satisfies the diffusion equation $(\partial_t - 2D\partial_x^2)E(x, t) = 0$ with the boundary condition $E(0, t) = 1$ such that the concentration now becomes $c(t) = -\partial_x E(x, t)_{|x=0}$. Still, the direct solution of the problem is usually considered to be complicated enough that one prefers to consider instead $\rho(x, t) := \partial_x^2 E(x, t)$ where the boundary condition becomes $\rho(0, t) = 0$ such that standard Green’s functions of the diffusion equation can be used; see [6] and references therein.

Remarkably, the empty-interval method can be applied to a large class of coagulation–diffusion models, where several additional reactions can be added, see e.g. [5, 11, 4, 6, 27, 29, 17, 20, 2, 31, 32, 28]. Furthermore, the quantum Hamiltonian/Liouvillian of coagulation–diffusion models with the (reversible) reaction $2A \leftrightarrow A$ can, by a stochastic similarity transformation [22, 15, 10], be transformed to that of a pair annihilation/creation $2A \leftrightarrow \emptyset$ which in one dimension can be solved by free-fermion methods; see e.g. [9, 45, 39, 25, 3, 14, 43, 30]. Those one-dimensional reaction–diffusion systems which can be treated with free-fermion methods have been classified [16, 42], but the empty-interval technique has the advantage that further reactions can be treated, such as $\emptyset \rightarrow A$ or $A \emptyset A \rightarrow A A A$, which have no known analogue in a free-fermion description. In particular, Peschel et al. [35] suggested a systematic way to identify observables for which closed systems of equations of motion can be derived from the reformulation of the master equation in terms of a Hamiltonian matrix in a controllable way. Their approach includes the method of empty intervals as the simplest special case. In principle, their method can be extended to include the probabilities of having several empty intervals of sizes $n_1, n_2, \ldots$ at certain distances which allows one to find correlation functions as well. Their study is the main subject of this paper.

In particular, our approach allows to consider arbitrary initial configurations of particles and hence our results will include many of the existing results in the literature as special cases. As we shall see, there exists a natural decomposition of the time-dependent observables which may be arranged in terms of the information required on the initial state. This can be formulated through single-interval or two-interval probabilities for those quantities which we consider explicitly. We shall give examples which suggest a clear order of relevance in the long-time limit. On the other hand, we shall assume spatial translation invariance from the outset, which simplifies the equations to be analysed. However, if one were to investigate the effects of disorder, one would have to revert to a formalism [11, 4] where translation invariance is not required.

The study of correlation functions of reaction–diffusion systems is also motivated by the recent interest in ageing phenomena: following the initial study of slow relaxation in glassy systems brought out of equilibrium after a rapid change in the thermodynamic parameters, it was later realized that the three main characteristics of physical ageing, namely (i) slow, non-exponential relaxation, (ii) breaking of time translation invariance and (iii) dynamical scaling, also occur in many-body systems, which is in contrast to glasses being neither disordered nor frustrated; see [13] for a brief review and a forthcoming book [19]. Furthermore, these characteristics have also been found in several many-particle systems with absorbing stationary states, such as the contact process [12, 38, 8],

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the non-equilibrium kinetic Ising model [33] and kinetically constrained systems such as the Fredrickson–Andersen model [28]. One particular point of interest in these ageing systems is the relation between two-time correlations and responses, and a study of the coagulation–diffusion process (along with its exactly solved extensions) should be useful, since exact results can be expected, at least in one dimension. However, while such an analysis is readily formulated in terms of the empty-interval method at two different times, the explicit calculation requires the knowledge of the exact equal-time two-interval probabilities. In this paper, we shall provide this information, which will become an initial condition for the two-time correlator, and is going to be used in a following paper where the ageing behaviour in exactly solvable reaction–diffusion processes will be addressed.

This paper is organized as follows. In order to make the presentation more self-contained, we recall in section 2 the derivation of the equation of motion for the empty-single-interval probability \( E_n(t) \) before we proceed to show that the boundary condition \( E_0(t) = 1 \) can be fixed through an analytic continuation to negative values of \( n \). The techniques thereby developed are to be generalized to the two-interval probability in the remainder of the paper. The passage from the initial state towards the scaling long-time regime as a function of the initial distribution is analysed and we also compare between the discrete model and its continuum limit. In section 3, the equations of motion and the formal solution are given, followed by the derivation of the consistency conditions with the single-interval probabilities. The general two-interval probability for arbitrary initial conditions is derived in section 4, and in section 5 we use the results for the derivation of the equal-time correlators. We conclude in section 6. Several appendices A–G contain technical details of the calculations.

2. The single-interval probability

2.1. Equations of motion

Using the definition of the coagulation–diffusion process as given in section 1, we begin by recalling the derivation of the equations for the empty-interval probabilities [5]. The same equations can also be found within a quantum Hamiltonian formalism [35], but this will not be repeated here. We denote by \( E_n(t) \) the time-dependent probability of having an interval of \( n \) consecutive empty sites at time \( t \). Since the system is assumed to be homogeneous, \( E_n(t) \) is site independent and will depend only on the interval size \( n \) and time \( t \). The time evolution of this quantity is governed by the rate at which particles move on adjacent intervals of size \( n \) or \( n - 1 \). In an interval of length \( n \), which will be denoted by \([n]\), a particle (•) can enter from the left or the right between the time period \( t \) and \( t + dt \), and \( E_n(t) \) decreases during this period of time by the amount

\[ -[\Pr(• \rightarrow [n]) + \Pr([n] \rightarrow •)]. \]

The probability \( \Pr(• \rightarrow [n]) \) is proportional to the probability that a particle lies on the left of the interval, or \( \Pr(• \rightarrow [n]) = \Pr(•[n]) \Gamma dt \), which can be evaluated using the relation

\[ \Pr(•[n]) + \Pr(◦[n]) = \Pr([n]) = E_n(t) \]

where the symbol (◦) refers to an empty site. Since by definition \( \Pr(◦[n]) = E_{n+1}(t) \), we obtain directly

\[ \Pr(•[n]) = \Pr([n]•) = E_n(t) - E_{n+1}(t). \]

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$E_n(t)$ may also increase, if we consider the possibility that a particle sitting next to an interval of size $n - 1$ moves away from this interval,

$$+ [\Pr(\bigcirc \bullet [n - 1]) + \Pr([n - 1] \bigcirc)].$$

This is possible because the process $A + A \to A$ constrains each site to contain at most one particle. Hence there is no need to consider the case where the particle encounters another particle when it moves away from the interval. As before, we have $\Pr(\bigcirc \bullet [n - 1]) = \Pr([n - 1] \bigcirc) \Gamma dt = (E_{n-1}(t) - E_n(t)) \Gamma dt$. Summing the contributions, the rate of change for $E_n(t)$ is given by

$$\partial_t E_n(t) = 2\Gamma[-\{E_n(t) - E_{n+1}(t)\} + \{E_{n-1}(t) - E_n(t)\}] = 2\Gamma(E_{n-1} - 2E_n + E_{n+1}). \quad (2)$$

This equation is valid only for a positive index $n > 1$. For $n = 1$ the rate of change for $E_1(t)$ is given as previously by the equation

$$\partial_t E_1(t) dt = [\Pr(\bigcirc \bullet) + \Pr(\bullet \bigcirc) - \Pr(\bigcirc \bigcirc) - \Pr(\bullet \bullet)].$$

We also have $\Pr(\bigcirc \bullet) = \Pr(\bullet \bigcirc) \Gamma dt$ and $\Pr(\bullet \bigcirc) = \Pr(\bullet \bullet) \Gamma dt$. The solutions for each of these quantities can be found by considering the probability conditions

$$\Pr(\bullet) + \Pr(\circ) = 1 \Rightarrow \Pr(\bullet) = 1 - E_1(t)$$

$$\Pr(\bullet \circ) + \Pr(\circ \circ) = \Pr(\circ) \Rightarrow \Pr(\bullet \circ) = E_1(t) - E_2(t). \quad (3)$$

Therefore, the equation for $n = 1$ is given by

$$\partial_t E_1(t) = 2\Gamma[1 - 2E_1(t) + E_2(t)]. \quad (4)$$

In order to be able to write this as the extension of equation (2) for $n = 1$, it appears convenient and is, indeed, common, to introduce the constraint $E_0(t) = 1$. We shall do the same, but return to this condition below. However, the boundary conditions, including $E(0, t) = 1$, were considered to be so complicated so that an explicit solution of (5) is usually avoided. Ingenious ways have been developed for extracting physically interesting information, such as the particle density $c(t)$. We shall require the explicit form of $E(x, t)$ below when looking for correlation functions and shall now give it. In the continuum limit, when $a$ is small, we set $x = na$ and $E(x, t) = E_n(t)$. The previous relation (2) can be expanded with respect to $a$ and a rescaled hopping rate $D = \Gamma/a^2$, which leads to a simple diffusion equation, together with a boundary condition

$$\partial_t E(x, t) = 2D\partial_{xx} E(x, t), \quad \text{and} \quad E(0, t) = 1. \quad (5)$$

If we could use a spatially infinite Fourier transform $E(x, t) = \int_{-\infty}^{+\infty} dk/2\pi \exp(ikx) \tilde{E}(k, t)$ to solve the previous equation, we would obtain in the standard fashion

$$E(x, t) = \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{\pi} \ell_0} \exp \left[-\frac{1}{\ell_0^2} (x - x')^2 \right] E(x', 0), \quad (6)$$

where the integrals over the real axis are unrestricted. In the above expression, a diffusion length

$$\ell_0 := \sqrt{8Dt} \quad (7)$$

acts as the scaling length of the function $E(x, t) = E(x/\ell_0)$. 

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2.2. The effect of the boundary condition: the continuum limit

The simplistic approach outlined at the end of section 2.1 must evidently be modified in order to take the boundary condition $E(0, t) = 1$ into account. This amounts to defining in equation (6) the meaning of the probability $E(x', 0)$ for negative $x'$ and this is achieved through the following result.

**Lemma 2.1.** If one extends the validity of equation (2) to all $n \in \mathbb{Z}$, together with the boundary condition $E_0(t) = 1$, one has

$$E_{-n}(t) = 2 - E_n(t). \quad (8)$$

**Proof.** This is proven by induction. First, we consider the case $n = 0$. Using equation (2) and $E_0(t) = 1$, we obtain

$$\partial_t E_0(t) = 2\Gamma(E - 1 - 2E_0 + E_1),$$

which implies $E_1(t) = 2E_0(t) - E_1(t) = 2 - E_1(t)$. In the general case, let us consider the equation of motion for the index $-n - 1$ and use the assumption (8) for the indices $-n$ and $-n + 1$:

$$E_{-n-1} = 2E_{-n} - E_{-n+1} + \frac{1}{2\Gamma} \partial_t E_{-n}$$

$$= 2(2 - E_n) - (2 - E_{n-1}) + \frac{1}{2\Gamma} \partial_t (2 - E_n)$$

$$= 2 - E_{n+1}$$

where the equations of motion (2) were used again. This completes the proof. \hfill \Box

In the continuum limit, this relation allows us to rewrite the integral (6) over the positive axis only:

$$E(x, t) = \text{erfc}(x/\ell_0) + \int_0^{\infty} \frac{dx'}{\sqrt{\pi\ell_0}} E(x', 0)\left[e^{-\left(x-x'\right)^2/\ell_0^2} - e^{-\left(x+x'\right)^2/\ell_0^2}\right] \quad (9)$$

where erfc is the complementary error function [1].

Equation (9) is the general solution for the probability $E(x, t)$ of having an empty interval, at least of length $x$ and at time $t$, where the initial state is described using the function $E(x, 0)$. The particle concentration $c(t) = Pr(\bullet)/a$ can be obtained in the continuum limit from the relation (3):

$$\text{Pr}(\bullet) + \text{Pr}(\circ) = 1 \Rightarrow \text{Pr}(\bullet) = ac(t) = 1 - E_1(t),$$

where $ac(t) = 1 - E_1(t) \simeq 1 - E(0, t) - a\partial_x E(x = 0, t)$, and therefore

$$c(t) = -\partial_x E(x, t)|_{x=0}. \quad (10)$$

The function $E(x, t)$ can by definition be written as a cumulative sum of the probabilities for being bounded on the left, of size at least equal to $x'$, or $P(x', t) = Pr(\leftarrow x')$:

$$E(x, t) = \int_x^\infty dx' P(x', t). \quad (11)$$
This imposes two boundary conditions: first, we have $E(0, t) = \int_0^\infty dx P(x, t) = 1$ by normalization; then, in the limit $x \to \infty$, one must have $E(x, t) \to 0$.

We can express $E(x, t)$ as function of $P(x, 0)$ by performing an integration by parts of (9):

$$E(x, t) = 1 - \frac{1}{2} \int_0^\infty dx' P(x', 0) \left[ \text{erf} \left( \frac{x' + x}{\ell_0} \right) - \text{erf} \left( \frac{x' - x}{\ell_0} \right) \right].$$

(12)

By differentiation with respect to $x$, we obtain the expression for the concentration:

$$c(t) = \frac{2}{\sqrt{\pi \ell_0}} \int_0^\infty dx' P(x', 0) \exp \left( -\frac{x'^2}{\ell_0^2} \right).$$

(13)

From this, all initial conditions, characterized by $P(x, 0)$, lead to the long-time behaviour of the concentration:

**Lemma 2.2.** For sufficiently long times and any initial distribution $P(x, 0)$, the concentration decreases as

$$c(t) \simeq \frac{2}{\sqrt{\pi \ell_0}} + o(\ell_0^{-1}).$$

(14)

**Proof.** Since $P(x, 0)$ is a normalized probability distribution, $\int_0^\infty dx P(x, 0) = 1$, we must have $P(x, 0) = o(1/x)$ for $x \to \infty$. We rewrite equation (13) as follows:

$$c(t) = \frac{2}{\sqrt{\pi \ell_0}} - \frac{2}{\sqrt{\pi}} \int_0^\infty dx P(x \ell_0, 0) (1 - e^{-x^2})_{o(1/\ell_0)}$$

(15)

where the last estimate follows from the large-$x$ behaviour of $P(x, 0)$. $\square$

In a more explicit way, this may be obtained from equation (13) by a formal expansion of the exponential. If the second moment of $P$ is well defined, this leads to the long-time behaviour

$$c(t) \simeq \frac{2}{\sqrt{\pi \ell_0}} \left( 1 - \frac{\langle x^2 \rangle}{\ell_0^2} \right) + o(\ell_0^{-2}).$$

(16)

On the other hand, the case of a diverging second moment is illustrated by the example

$$P(x, 0) = \frac{a_0}{1 + x^{1+\alpha}},$$

(17)

where $0 < \alpha < 1$ and $a_0 = (1 + \alpha) \sin[\pi/(1 + \alpha)]/\pi$ is the normalization factor. In this case, a calculation analogous to the proof of lemma 2.2 gives

$$c(t) = \frac{2}{\sqrt{\pi \ell_0}} \int_0^\infty dx a_0 \exp \left( -\frac{x'^2}{\ell_0^2} \right) \frac{1}{1 + x^{1+\alpha}}$$

$$= \frac{2}{\sqrt{\pi \ell_0}} \left[ 1 - \frac{a_0}{\ell_0^\alpha} \int_0^\infty dx \frac{1 - e^{-x^2}}{x^{1+\alpha}} + \cdots \right],$$

(18)

which gives the leading correction in the long-time limit as a function of the exponent $\alpha$. We also notice that if particles occupy each site with probability $p$, the concentration is

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defined by \( c_0 = p/a \). When the system is filled with a concentration \( c_0 \) of particles, the function \( E_n(0) \) is proportional to

\[
E_n(0) \sim (1 - p)^n = (1 - ac_0)^{x/a} \quad E(x, 0) = \exp(-c_0x).
\] (19)

However, if the system is entirely filled with particles, \( p = 1 \) and \( E_n(0) = 0 \) for \( n \neq 0 \), then, from (9), \( E(x, t) \) is simply given by \( \text{erfc}(x/\ell_0) \), and \( c(t) = 2/\sqrt{\pi} \ell_0 \). In the general case of a given concentration \( c_0 \) where \( E_0(x) = e^{-c_0x} \), we simply have \( P(x, 0) = c_0e^{-c_0x} \) and from (13)

\[
c(t) = c_0 \exp \left( \frac{1}{4}c_0^2\ell_0^2 \right) \text{erfc} \left( \frac{c_0\ell_0}{2} \right).
\] (20)

In figure 1 we illustrate the effect of several initial empty-interval distributions \( E(x, 0) \). Clearly, as expected from the above discussion, all initial distributions lead to the same long-time asymptotics \( c(t) \sim t^{-1/2} \), but the way that this asymptotic regime is reached depends on the initial state. This can be better understood when plotting the interparticle distribution function (IPDF)

\[
p(x, t) := \frac{1}{c(t)} \frac{\partial^2 E(x, t)}{\partial x^2},
\] (21)

which gives the probability density for the next neighbour of a particle being at distance \( x \) at time \( t \) [5, 6]. This function is shown for \( t = 0 \) in the inset of figure 1. One observes that in those cases where \( p(x, 0) \) decays monotonically with \( x \), the transition to the asymptotic regime is more gradual. On the other hand, in the third case there is an initial non-vanishing distance the particles must overcome before they can react. This leads to a very sharp transition between the initial and the asymptotic regimes.

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2.3. The discrete case

We now give the solution for the discrete case, without taking a continuum limit.

First we recall the solution of the differential equation (2). The generating function $F(z, t)$ satisfies as usual the differential equation $\partial_t F(z, t) = 2D(z + 1/z - 2)F(z, t)$, with the solution

$$F(z, t) = F(z, 0)e^{2D(z + 1/z - 2)t} = e^{-4Dt} \sum_{n=-\infty}^{+\infty} z^n \sum_{m=-\infty}^{+\infty} E_{n-m}(0)I_m(4Dt). \quad (22)$$

Identifying $E_n(t)$ in the previous expression, we write

$$E_n(t) = e^{-4Dt} \sum_{m=-\infty}^{+\infty} E_m(0)I_{n-m}(4Dt). \quad (23)$$

We now must take the boundary condition $E_0(t) = 1$ into account. As in the continuum case, we replace the summation over negative values of the index $m$ by using the discrete relation $E_{-n}(t) = 2 - E_n(t)$ and find

$$E_n(t) = e^{-4Dt} \left[ \sum_{m=0}^{+\infty} E_m(0)\left( I_{n-m}(4Dt) - I_{n+m}(4Dt) \right) + \sum_{m=1}^{+\infty} 2I_{n+m}(4Dt) + I_1(4Dt) \right]. \quad (24)$$

In the discrete case, the particle concentration is given by $c(t) = 1 - E_1(t)$. Using summation and recurrence relations for modified Bessel functions, we obtain

$$c(t) = e^{-4Dt} \left( I_0(4Dt) + I_1(4Dt) - \sum_{m=1}^{+\infty} \frac{m}{2Dt} E_m(0)I_m(4Dt) \right), \quad (25)$$

which generalizes earlier results of Spouge [43]. In figure 2, we compare the particle concentration according to the discrete case equation (25) with the previously obtained solution equation (20) in the continuum limit, for three values of the initial concentration $c_0$. We used, respectively, the initial distributions $E_n(0) = (1-c_0/a)^n$ and $E(x, 0) = e^{-c_0x}$. As expected, the same asymptotics is found in all cases, independently of the initial concentration. We observe that the passage between the initial and the asymptotic regimes is more gradual in the continuum limit. As above, we interpret this as coming from the fact that in the discrete case the particles must first overcome a finite distance before they can react.

3. The two-interval probability

In this section, we generalize the previous result and evaluate the probability $E_{n_1,n_2}(d, t)$ for having two empty intervals, at least of sizes $n_1$ and $n_2$ and separated by the distance $d$: we denote it by $E_{n_1,n_2}(d, t) = \Pr(\begin{bmatrix} n_1 \\ d \\ n_2 \end{bmatrix})$. This function is expected to have the following symmetries:

$$E_{n_1,n_2}(d, t) = E_{n_2,n_1}(d, t) \quad E_{n_1,0}(d, t) = E_{n_1}(t) \quad \text{and} \quad E_{0,n_2}(d, t) = E_{n_2}(t) \quad (26)$$
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Figure 2. Time evolution of the concentration in the discrete case (full curves) and the continuous limit (dashed lines). The initial concentration $c_0$ is $[1,0.5,0.1]$ from top to bottom.

3.1. Equations of motion

As before and using the notation of section 2, we consider the different possibilities for the variation of $E_{n_1,n_2}(d,t)$ between the time $t$ and $t+dt$:

$$\partial_t E_{n_1,n_2}(d,t)dt = -[\Pr(\bullet \cdot n_1 d n_2) + \Pr(n_1 \cdot d n_2 \cdot \bullet)]$$
$$+ [\Pr(\bullet n_1 d -1 n_2) + \Pr(n_1 d -1 \bullet \cdot n_2)]$$
$$+ \Pr(n_1 -1 \cdot d n_2) + \Pr(n_1 d -1 \cdot \bullet)$$
$$+ \Pr(n_1 -1 \cdot d n_2) + \Pr(n_1 d -1 \cdot \bullet)\cdot n_2].$$

(27)

The probability rates are given by considering the sum rules for static probabilities. First, we consider the negative contributions for which we obtain the relations

$$\Pr(\bullet n_1 d n_2) = \Pr(\cdot n_1 d n_2) \Gamma dt,$$

$$\Pr(\bullet n_1 d n_2) + \Pr(\cdot n_1 d n_2) = \Pr(n_1 d n_2)$$
$$\Rightarrow \Pr(\bullet n_1 d n_2) = E_{n_1,n_2}(d,t) - E_{n_1+1,n_2}(d,t)$$

and

$$\Pr(n_1 -1 \cdot d -1 n_2) = \Pr(n_1 \cdot d -1 n_2) \Gamma dt,$$

$$\Pr(n_1 -1 \cdot d -1 n_2) + \Pr(n_1 \cdot d -1 n_2) = \Pr(n_1 d n_2)$$
$$\Rightarrow \Pr(n_1 \cdot d -1 n_2) = E_{n_1,n_2}(d,t) - E_{n_1+1,n_2}(d-1,t).$$

For the positive contributions, we have

$$\Pr(\cdot n_1 -1 d n_2) = \Pr(\bullet n_1 -1 d n_2) \Gamma dt,$$

$$\Pr(\cdot n_1 -1 d n_2) + \Pr(\bullet n_1 -1 d n_2) = \Pr(n_1 -1 d n_2)$$
$$\Rightarrow \Pr(\bullet n_1 -1 d n_2) = E_{n_1-1,n_2}(d,t) - E_{n_1,n_2}(d,t).$$

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and
\[
\Pr(n_1 - 1 \bullet d n_2) = \Pr(n_1 - 1 \bullet d n_2) \Gamma dt,
\]
\[
\Pr(n_1 - 1 \bullet d n_2) + \Pr(n_1 - 1 \circ d n_2) = \Pr(n_1 - 1 \circ d n_2)
\]
\[
\Rightarrow \Pr(n_1 - 1 \bullet d n_2) = E_{n_1-1,n_2}(d + 1, t) - E_{n_1,n_2}(d, t)
\]
(and similarly for the other terms which are symmetric). After gathering together all the contributions, we finally find (the time variable is from now on suppressed)
\[
\partial_t E_{n_1,n_2}(d) = \Gamma[-8E_{n_1,n_2}(d) + E_{n_1+1,n_2}(d) + E_{n_1,n_2+1}(d) + E_{n_1-1,n_2}(d) + E_{n_1,n_2-1}(d)]
\]
\[
+ E_{n_1+1,n_2}(d-1) + E_{n_1,n_2+1}(d-1) + E_{n_1-1,n_2}(d+1) + E_{n_1,n_2-1}(d+1)].
\]
(28)

We have checked that the same closed system of equations of motion is also obtained when the master equation is rewritten in terms of a quantum Hamiltonian [35].

The continuum limit of this diffusion equation is obtained by expanding the terms up to the second order in the lattice step \(a\) (the \(a^2\) will be absorbed in \(\Gamma\)). Setting \(x = n_1 a\), \(y = n_2 a\) and \(z = d a\) we obtain the following linear differential equation:
\[
\partial_t E(x, y, z) = 2D[\partial_x^2 + \partial_y^2 + \partial_z^2 - (\partial_x \partial_z + \partial_y \partial_z)]E(x, y, z).
\]
(29)

### 3.2. The general solution

The general solution for (29) is obtained by diagonalizing the quadratic form associated with the differential operator \(\hat{P} := \partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_x \partial_z - \partial_y \partial_z \). We find that the following change of variables \((x, y, z) \rightarrow (X, Y, Z)\) diagonalizes \(\hat{P}\):
\[
X = \alpha(x + y + \sqrt{2}z), \quad Y = \beta(x + y - \sqrt{2}z), \quad Z = \gamma(x - y),
\]
(30)

with the positive constants
\[
\alpha = [2(2 - \sqrt{2})]^{-1/2}, \quad \beta = [2(2 + \sqrt{2})]^{-1/2}, \quad \gamma = 1/\sqrt{2},
\]
such that \(\alpha^2 + \beta^2 = 1\) and \(\alpha^2 - \beta^2 = 1/\sqrt{2}\).

In the new variables, the operator \(\hat{P} = \partial_X^2 + \partial_Y^2 + \partial_Z^2\) is diagonal. As in the previous section (see (6)) for the one-interval problem, and in the continuum limit, if the boundary conditions are ignored for a moment, the function \(E(x, y, z)\) can be found via a Fourier transformation, and explicitly expressed as a kernel integral depending on the initial conditions \(E_0(x, y, z) := E(x, y, z, 0)\):
\[
E(x, y, z) = \frac{\sqrt{2}}{(\sqrt{\pi} \ell_0)^3} \int_{-\infty}^{\infty} dx' dy' dz' \mathcal{W}(x - x', y - y', z - z') E_0(x', y', z')
\]
(31)

where the Gaussian kernel \(\mathcal{W}(u, v, w)\) is given by
\[
\mathcal{W}(u, v, w) = \exp \frac{1}{\ell_0^2} [-\alpha^2(u + v + \sqrt{2}w)^2 - \beta^2(u + v - \sqrt{2}w)^2 - \gamma^2(u - v)^2]
\]
\[
= \exp \frac{1}{\ell_0^2} \left[-(u + v + w)^2 - w^2 - \frac{1}{2}(u - v)^2\right],
\]
(32)
the Jacobian of the transformation (30) being equal to \(4\sqrt{2}\alpha\beta\gamma = \sqrt{2}\).
3.3. Compatibility conditions

As indicated before by equation (8), it is important to make a correspondence between intervals of formally negative and positive lengths. This will be needed in the formal solution (31) which requires real variables, whereas the probability \( E(x, y, z) \) has an obvious physical meaning for positive distances only. We have to consider three cases, depending on whether \( x, y \) and \( z \) are negative or positive. By symmetry considerations (26), it is only necessary to consider the case where \( x \) or \( z \) are individually negative (the case \( y < 0 \) being deduced by means of the first equation (26)), the other variables being positive. The explicit evaluation of equation (31) in the following sections requires several identities, stated as lemmata for clarity and proven in appendices A and B, respectively. The first one treats the case of formally negative interval lengths.

**Lemma 3.1.** The probability of two empty intervals of negative lengths is related to the probability of positive lengths as follows:

\[
\begin{align*}
E_{-n_1,n_2}(d) &= 2E_{n_2} - E_{n_1,n_2}(d - n_1), \\
E_{n_1,-n_2}(d) &= 2E_{n_1} - E_{n_1,n_2}(d - n_2), \\
E_{-n_1,-n_2}(d) &= 4 - 2E_{n_1} - 2E_{n_2} + E_{n_1,n_2}(d - n_1 - n_2).
\end{align*}
\]

A further relation connects the negative separations \(-d\) between two intervals to the positive ones.

**Lemma 3.2.** We have

\[
E_{n_1,n_2}(-d) = 2E_{n_1+n_2-d} - E_{n_1-d,n_2-d}(d).
\]

Later on, we shall require these results in the continuum limit, where the expressions (8), (33)–(35) take the following form:

\[
\begin{align*}
E(-x) &= 2 - E(x), \\
E(-x, y, z) &= 2E(y) - E(x, y, z - x), \\
E(x, -y, z) &= 2E(x) - E(x, y, z - y), \\
E(-x, -y, z) &= 4 - 2E(x) - 2E(y) + E(x, y, z - x - y), \\
E(x, y, -z) &= 2E(x + y - z) - E(x - z, y - z, z).
\end{align*}
\]

This allows us to rewrite (31) in the restricted domain where \((x', y', z')\) are all positive, and where \(E_0(x', y', z')\) is physically well defined.

4. The general solution for \( E(x, y, z, t) \)

From the general equation (31), we separate the eight different domains of integration around the origin, for example \((x' > 0, y' > 0, z' > 0)\), \((x' < 0, y' > 0, z' > 0)\) etc, and use relations (8), (33), (34) and (35) to map all domains into the single domain \((x' > 0, y' > 0, z' > 0)\). This calculation is done in appendix C and, here, we just summarize the results. The general solution can be decomposed as follows:

\[
E(x, y, z, t) = E^{(0)}(x, y, z, t) + E^{(1)}(x, y, z, t) + E^{(2)}(x, y, z, t),
\]

where \(E^{(0)}(x, y, z, t)\) is obtained from the terms independent of the initial conditions, \(E^{(1)}(x, y, z, t)\) from the initial one-interval probability \(E_0(x')\) and \(E^{(2)}(x, y, z, t)\) from the
initial two-interval probability $E_0(x, y, z)$. Note that $E^{(1)}(x, y, z, t)$ and $E^{(2)}(x, y, z, t)$ depend on $E_0(x^{'}, y^{'}, z^{'})$ with positive arguments; hence this gives us the physical answer for the diffusion process in the coagulation problem starting from arbitrary initial conditions and constraints on the differential equation. We now analyse these three terms one by one.

4.1. The special case of a system initially entirely filled with particles

We notice that (C.6)–(C.8) contain initial conditions for the single-interval distribution $E_0(x^{'})$, some constants independent of the initial conditions, and initial conditions for the two-interval distribution $E_0(x^{'}, y^{'}, z^{'})$. To simplify the notation, we shall rescale all lengths by $\ell_0$ such that $E_0(x, y, z) = E(x\ell_0, y\ell_0, z\ell_0, t)$. In (C.8), we can isolate from $E_0(-x^{'}, -y^{'}, z^{'})$ two terms independent of the initial conditions,

$$4 - 4\theta(y^{'}, -z^{'})\theta(x^{'}, -z^{'})\theta(x^{'}, y^{'}, -z^{'}) = \theta(y^{'}, z^{'})\theta(x^{'}, -z^{'})$$

It is obvious that (C.6)–(C.8) contain initial conditions for the single-interval distribution $E_0(x^{'})$, some constants independent of the initial conditions, and initial conditions for the two-interval distribution $E_0(x^{'}, y^{'}, z^{'})$. To simplify the notation, we shall rescale all lengths by $\ell_0$ such that $E_0(x, y, z) = E(x\ell_0, y\ell_0, z\ell_0, t)$. In (C.8), we can isolate from $E_0(-x^{'}, -y^{'}, z^{'})$ two terms independent of the initial conditions,

$$4 - 4\theta(y^{'}, -z^{'})\theta(x^{'}, -z^{'})\theta(x^{'}, y^{'}, -z^{'}) = \theta(y^{'}, z^{'})\theta(x^{'}, -z^{'})$$

These two terms, plus the first term in (C.3), give a contribution to the general function equal to

$$E_0^{(0)}(x, y, z) = \text{erfc}(z)\text{erfc}(x + y + z)$$

$$+ \sqrt{\frac{2}{\pi^3}} \int_{\mathbb{R}^3_+} dx' dy' dz' \tilde{W}_0(-x^{'}, -y^{'}, z^{'}) \{4 - 4\theta(y^{'}, -z^{'})\theta(x^{'}, -z^{'})\}.$$ 

Performing the translations $x^{'}, z^{'}, y^{'}, z^{'} \to x, y, z$ in the last contribution, we obtain

$$E_0^{(0)}(x, y, z) = \text{erfc}(z)\text{erfc}(x + y + z)$$

$$+ \sqrt{\frac{2}{\pi^3}} \int_{\mathbb{R}^3_+} dx' dy' dz' \{\tilde{W}_0(-x^{'}, -y^{'}, z^{'}) - \tilde{W}_0(-x^{'}, -y^{'}, z^{'})\}.$$ 

Using the relation (C.4), and the identity

$$W_0(x - x^{'}, y - y^{'}, z - z^{'}, z + z^{'}) = W_0(x - x^{'}, y - y^{'}, z - z^{'}, z^{'})e^{-4zz^{'}}$$

$$\Rightarrow W_0(x^{'}, z^{'}, y^{'}, z^{'}, -z^{'}) = -W_0(x^{'}, y^{'}, z^{'})$$

we can rewrite $E$ as

$$E_0^{(0)}(x, y, z) = \text{erfc}(z)\text{erfc}(x + y + z)$$

$$+ \sqrt{\frac{32}{\pi^3}} \int_{\mathbb{R}^3_+} dx' dy' dz' \{\tilde{W}_0(-x^{'}, -y^{'}, z^{'}) + \tilde{W}_0(-x^{'}, -y^{'}, -z^{'})\}$$

$$= \text{erfc}(z)\text{erfc}(x + y + z) + \sqrt{\frac{32}{\pi^3}} \int_{\mathbb{R}^3_+} dx' dy' \int_{\mathbb{R}} dz' \tilde{W}_0(-x^{'}, -y^{'}, z^{'}).$$

The integral over $z^{'},$ now gives a Gaussian exponential. Then the two remaining integrations can also be carried out explicitly (see appendix G). Introducing again the diffusion length $\ell_0,$ we find

$$E^{(0)}(x, y, z) = \text{erfc}\left(\frac{x}{\ell_0}\right)\text{erfc}\left(\frac{y}{\ell_0}\right) + \text{erfc}\left(\frac{z}{\ell_0}\right)\text{erfc}\left(\frac{x + y + z}{\ell_0}\right)$$

$$- \text{erfc}\left(\frac{x + z}{\ell_0}\right)\text{erfc}\left(\frac{y + z}{\ell_0}\right).$$

(39)

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This is the exact two-interval probability in the case of an initially fully filled lattice (where both \( E_0(x) \) and \( E_0(x, y, z) \) vanish). In particular, the solution (39) satisfies the symmetry conditions (26). In the limit of \( z \) large, one has the factorization \( E(x, y, z, t) \approx E(x, t)E(y, t) \).

The remaining terms of the full solution depend on the initial conditions. They are of two kinds, and they involve either the single-interval or else the two-interval initial probabilities. We turn to them now.

### 4.2. Contributions to \( E(x, y, z, t) \) from terms with a single-interval initial distribution

The contributions to \( E(x, y, z, t) \) of single-interval distributions come from the previous relations (C.6)–(C.8), where we can isolate the following individual terms:

- the second term of equation (C.3),
- the first three terms in (C.6) and (C.7),
- terms 2, 3, 4 in (C.8).

On the whole, there are 10 terms contributing to the initial conditions given for a given choice of \( E_0(x') \). Gathering these terms and performing successive translations in \( x', y' \) or \( z' \) when necessary, we obtain

\[
E_{0}^{(1)}(x, y, z) = \text{erfc}(z) \int_{0}^{\infty} \frac{dx'}{\sqrt{\pi}} E_{0, \ell_0}(x')(e^{-x+y+z-x'} - e^{-x+y+z+x'})
+ \sqrt{\frac{8}{\pi^3}} \int_{\mathbb{R}_+^3} dx' dy' dz' \ E_{0}(x') \left\{ \hat{\mathcal{W}}_{\ell_0}(x', -y', z') - \hat{\mathcal{W}}_{\ell_0}(x' - z', -y' - z', z') \right\}_{I_1}
+ \hat{\mathcal{W}}_{\ell_0}(-y', x', z') - \hat{\mathcal{W}}_{\ell_0}(-y' - z', x' - z', z') \right\}_{I_2}
+ \hat{\mathcal{W}}_{\ell_0}(-x' - z', -y' - z', z') - \hat{\mathcal{W}}_{\ell_0}(-x', -y', z') \right\}_{I_3}
+ \hat{\mathcal{W}}_{\ell_0}(-y' - z', -x' - z', z') - \hat{\mathcal{W}}_{\ell_0}(-y', -x', z') \right\}_{I_4}
+ \theta(-x' - y' + z') \left\{ \hat{\mathcal{W}}_{\ell_0}(y', -z', x') + \hat{\mathcal{W}}_{\ell_0}(-z', y', x') \right\}_{I_5}
+ \theta(-x' + y' + z') \hat{\mathcal{W}}_{\ell_0}(-z', -y', x') \right\}_{I_6}.
\]

By performing partial translations and integrations and simplifying all terms from \( I_1 \) to \( I_6 \) (see appendix D for details), we obtain

\[
E_{0}^{(1)}(x, y, z) = \int_{0}^{\infty} \frac{dx'}{\sqrt{\pi}} E_{0, \ell_0}(x') \left\{ \text{erfc}(z)[e^{-(x'-x-y-z)^2} - e^{-(x'+x+y+z)^2}] + \text{erfc}(x)[e^{-(x'-y)^2} - e^{-(x'+y)^2}] - \text{erfc}(x + z)[e^{-(x'-y-z)^2} - e^{-(x'+y+z)^2}] + \text{erfc}(y)[e^{-(x'-x)^2} - e^{-(x'+x)^2}] - \text{erfc}(y + z)[e^{-(x'-x-z)^2} - e^{-(x'+x+z)^2}] \right\}.
\]

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The coagulation–diffusion process in 1D investigated by the empty-interval method

\[ E^{(1)}(x, y, z) = \text{erfc} \left( \frac{z}{\ell_0} \right) F_{c_0}(x + y + z) + \text{erfc} \left( \frac{x}{\ell_0} \right) F_{c_0}(y) - \text{erfc} \left( \frac{x + z}{\ell_0} \right) F_{c_0}(y + z) + \text{erfc} \left( \frac{y + z}{\ell_0} \right) F_{c_0}(x) - \text{erfc} \left( \frac{y + z}{\ell_0} \right) F_{c_0}(x + z) + \text{erfc} \left( \frac{x + y + z}{\ell_0} \right) F_{c_0}(z), \]

with the following abbreviation:

\[ F_{c_0}(x) := \int_{0}^{\infty} \frac{dx'}{\ell_0 \sqrt{\pi}} \ e^{-c_0x'} \left[ e^{-(x'-x)^2/\ell_0^2} - e^{-(x'+x)^2/\ell_0^2} \right] = \frac{1}{2} e^{c_0^2 \ell_0^2/4} \left\{ e^{-c_0x} \text{erf} \left( \frac{x}{\ell_0} \right) - e^{c_0x} \text{erf} \left( \frac{x + \ell_0 c_0}{2} \right) - 2 \sinh(c_0x) \right\}. \]

The limiting values of this function read

\[ F_{c_0}(x) \xrightarrow{x=\infty} e^{c_0^2 \ell_0^2/4-c_0x}, \quad F_{c_0}(x) \xrightarrow{x=0} \left( \frac{2}{\ell_0 \sqrt{\pi}} - c_0 e^{c_0^2 \ell_0^2/4} \text{erfc}(\ell_0 c_0/2) \right) x, \]

and \( F_0(x) = \text{erf}(x/\ell_0). \)

In the long-time limit, the above expression goes to zero like \( 1/\ell_0^3 \):

\[ F_{c_0}(x) \approx \frac{4x}{\sqrt{\pi c_0^2 \ell_0^3}} \left( 1 - \frac{6}{c_0^2 \ell_0^2} \frac{x^2}{\ell_0^2} + \cdots \right) \]

and the dominant part of contribution \( E^{(1)}(x, y, z, t) \) in the same limit behaves like

\[ E^{(1)}(x, y, z) \approx \frac{4(x + y)}{\sqrt{\pi c_0^2 \ell_0^3}}. \]

It is interesting to compare this expansion with the long-time limit expansion of \( E^{(0)}(x, y, z) \), which is independent of \( c_0 \):

\[ E^{(0)}(x, y, z) \approx 1 - \frac{2(x + y)}{\sqrt{\pi \ell_0^2}} + \frac{2[(x + y)^3 + 6xyz]}{3\sqrt{\pi \ell_0^3}}. \]

The first term (46) tends to increase the two-interval probability by a factor independent of the distance, since there are fewer particles in the system for a finite concentration of particles.

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4.3. Contributions to $E(x, y, z)$ for two-interval initial distributions

As noticed previously, we can isolate from equations (C.6)–(C.8) terms involving $E(x', y', z', t = 0) = E_0(x', y', z')$. In particular:

- 1 term $E_0(x', y', z')$ when all variables are positive, in combination with $\tilde{W}_{t_0}(x', y', z')$;
- $2 \times 3$ terms in (C.6) and (C.7);
- five terms in (C.8).

On combining and simplifying these terms (see appendix E for details), we obtain a reduced integral form for $E_{t_0}^{(2)}(x, y, z)$, as a function of a kernel $K_{2,t_0}(x', y', z'; x, y, z)$:

$$E_{t_0}^{(2)}(x, y, z) = \sqrt{\frac{2}{\pi^3}} \int_{\mathbb{R}^3_+} dx' dy' dz' E_{0,t_0}(x', y', z') \tilde{W}_{t_0}(x - x', y - y', z - z')$$

$$\times K_{2,t_0}(x', y', z'; x, y, z)$$

with

$$K_{2,t_0}(x', y', z'; x, y, z) = [1 - e^{-4(x' + y' + z')(x + y + z)}](1 - e^{-4z'})$$

$$+ e^{-4x'x - 4y'y}[1 - e^{-4(x' + y')(x + y + z)}][1 - e^{-4(x' + y' + z')z}]$$

$$- e^{-4x'x}[1 - e^{-4(y' + z')(x + y + z)}][1 - e^{-4(x' + z')z}]$$

$$- e^{-4y'y}[1 - e^{-4(x' + z')(x + y + z)}][1 - e^{-4(y' + z')z}]$$

$$+ e^{-4x'x - 4z'(x+z)}[1 - e^{-4(y' + z')(x + y + z)}](1 - e^{-4z'})$$

$$+ e^{-4y'y - 4z'(y+z)}[1 - e^{-4x'(x+y+z)}](1 - e^{-4y'z}).$$

(48)

It is important to notice for the following sections that $K_{2}(x', y', z', 0, 0, z) = \partial_x K_{2}(x', y', z', x, y, z)|_{x=y=0} = \partial_y K_{2}(x', y', z', x, y, z)|_{x=y=0} = 0$.

4.4. Sum rules and the large distance limit

From the results of section 4.3, we can write the total two-interval distribution as a sum of three terms:

$$E(x, y, z, t) = E^{(0)}(x, y, z, t) + \int_0^\infty \frac{dx'}{\ell_0 \sqrt{\pi}} E_0(x') K_1(x'; x, y, z)$$

$$+ \frac{\sqrt{2}}{\ell_0 \sqrt{\pi}^{3/2}} \int_{\mathbb{R}^3_+} dx' dy' dz' E_0(x', y', z') W(x - x', y - y', z - z')$$

$$\times K_2(x', y', z'; x, y, z).$$

(49)

As a check on our calculations, we consider the particular case of initial conditions given by a configuration where there is no particle in the system (or $c_0 = 0$). In this case, $E(x)$ and $E(x, y, z)$ are always equal to unity for any time. This means that if we put the conditions $E_0(x') = 1$ and $E_0(x', y', z') = 1$ into equation (49) we should recover the result $E(x, y, z, t) = 1$. The first contribution comes from (39) and is independent of the
initial conditions. The second contribution comes from (41) or (42) with $c_0 = 0$: 

$$E(x, y, z, t) = \text{erfc}\left(\frac{x}{\ell_0}\right) \text{erfc}\left(\frac{y}{\ell_0}\right) + \text{erfc}\left(\frac{z}{\ell_0}\right) \text{erfc}\left(\frac{x+y+z}{\ell_0}\right)$$

$$- \text{erfc}\left(\frac{x+z}{\ell_0}\right) \text{erfc}\left(\frac{y+z}{\ell_0}\right) + \text{erfc}\left(\frac{y}{\ell_0}\right) \text{erf}\left(\frac{x}{\ell_0}\right) \text{erfc}\left(\frac{y}{\ell_0}\right)$$

$$+ \text{erfc}\left(\frac{z}{\ell_0}\right) \text{erf}\left(\frac{x+y+z}{\ell_0}\right) - \text{erf}\left(\frac{x+z}{\ell_0}\right) \text{erfc}\left(\frac{y+z}{\ell_0}\right)$$

$$+ \int_{\mathbb{R}_+^3} \frac{\sqrt{2}\, dx'\, dy'\, dz'}{\ell_0^3 \pi^{3/2}} \mathcal{W}(x-x', y-y', z-z') K_2(x', y', z'; x, y, z).$$

The terms can be rearranged and we obtain the following equality, since $E(x, y, z, t) = 1$ for all times:

$$\frac{\sqrt{2}}{\ell_0^3 \pi^{3/2}} \int_{\mathbb{R}_+^3} \text{d}x'\, dy'\, dz' \mathcal{W}(x-x', y-y', z-z') K_2(x', y', z'; x, y, z)$$

$$= \text{erf}\left(\frac{x}{\ell_0}\right) \text{erf}\left(\frac{y}{\ell_0}\right) + \text{erf}\left(\frac{z}{\ell_0}\right) \text{erf}\left(\frac{x+y+z}{\ell_0}\right)$$

$$- \text{erf}\left(\frac{x+z}{\ell_0}\right) \text{erf}\left(\frac{y+z}{\ell_0}\right).$$

The last expression gives an identity for the complicated integral involving only the different weights when the initial functions are set to unity. On comparing to expression (39), we see that this is the same expression except for the erfc functions being replaced by erf. We also see that all correlators vanish on the empty lattice, as expected. From the general expression (49) we can compute the limit when the distance $z$ is large. As seen previously with (39) when the system is entirely filled with particles, the quantity $E^{(0)}(x, y, z, t)$ is easily factorized as

$$\lim_{z \to \infty} E^{(0)}(x, y, z, t) = \text{erfc}(x/\ell_0)\text{erfc}(y/\ell_0) = E(x, t) E(y, t).$$

In the general case, for a finite concentration at initial time for example, we can show that the limit is still valid from (49). Indeed the two other parts $E^{(1)}$ and $E^{(2)}$ have simpler behaviour in this limit. For $E^{(1)}$, the kernel $K_1$ given in (41) has two dominant terms:

$$K_{1,\ell_0}(x'; x, y, z) \simeq \text{erfc}(x)[e^{-(x'-y)^2} - e^{-(x'+y)^2}] + \text{erfc}(y)[e^{-(x'-x)^2} - e^{-(x'+x)^2}],$$

and then $E^{(1)}$ can be approximated by

$$E^{(1)}(x, y, z, t) \simeq \frac{\text{erfc}(x/\ell_0)}{\sqrt{\pi}\ell_0} \int_0^\infty \frac{dx'}{\sqrt{\pi}\ell_0} E(x', 0)[e^{-(x'-y)^2/\ell_0^2} - e^{-(x'+y)^2/\ell_0^2}]$$

$$+ \frac{\text{erfc}(y/\ell_0)}{\sqrt{\pi}\ell_0} \int_0^\infty \frac{dx'}{\sqrt{\pi}\ell_0} E(x', 0)[e^{-(x'-x)^2/\ell_0^2} - e^{-(x'+x)^2/\ell_0^2}].$$

The other kernel $K_2$ behaves like

$$K_{2,\ell_0}(x', y', z'; x, y, z) \simeq (1 - e^{-4xx'}) (1 - e^{-4yy'}).$$

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Then
\[ E^{(2)}(x, y, z, t) \simeq \int_{\mathbb{R}_+^3} \sqrt{2} \frac{dz'}{\ell_0^3} \frac{dy'}{\ell_0} \frac{dz'}{\ell_0^3} \sqrt{\pi} E_0(x', y', z') W_{\ell_0}(x - x', y - y', z - z') \times (1 - e^{-4yy'})(1 - e^{-4yy'}). \]

In the last expression, we set \( z'' = z' - z \), so \( E_0(x', y', z') = E_0(x', y', z'' + z) \). We also assume that \( E_0(x', y', z'' + z) \simeq E_0(x') E_0(y') \), and that the integral over \( z'' \) can be extended over the real axis since the lower bound \(-z\) is large and negative. Then there remains an integration over the weight \( W_{\ell_0}(x - x', y - y', -z'') \), which is independent of \( z \), and which can be performed exactly. We find in particular that
\[
\int_{-\infty}^{\infty} \sqrt{2} \frac{dz''}{\ell_0 \sqrt{\pi}} W_{\ell_0}(x - x', y - y', -z'') = e^{-(x-x')^2/\ell_0^2-(y-y')^2/\ell_0^2}.
\]

If we multiply this exponential with the kernel \( K_2 \), we obtain that
\[
E^{(2)}(x, y, z, t) \simeq \int_{0}^{\infty} \frac{dz'}{\sqrt{\pi} \ell_0} E(x', 0)[e^{-(x'-x)^2/\ell_0^2} - e^{-(x'+x)^2/\ell_0^2}] \\
\times \int_{0}^{\infty} \frac{dy'}{\sqrt{\pi} \ell_0} E(y', 0)[e^{-(y'-y)^2/\ell_0^2} - e^{-(y'+y)^2/\ell_0^2}].
\]

The sum of the expressions (51), (52), combined with the limit of (39), gives exactly, after factorization, the expected limit, which is the generalization of the result seen with (39) in the particular case of an initially filled system:
\[
\lim_{z \to 1} E(x, y, z, t) = E(x, t) E(y, t)
\]

This limit is exact for any given initial condition at any time.

Summarizing the contents of this section, we have the decomposition (37) of the two-interval probability, where the individual terms are given by equations (39), (41) and (48).

5. The two-point correlation function

5.1. The definition

The two-point connected correlation function can be defined in the discrete case as the probability of having two particles separated by the distance \( d \):
\[
C_d(t) := \Pr(\bullet \ d \bullet) - \Pr(\bullet) \Pr(\bullet).
\]

We can express the previous function in terms of the discrete two-interval functions. Indeed
\[
\Pr(\bullet \ d \bullet) + \Pr(\circ \ d \bullet) = \Pr(\bullet) = 1 - \Pr(\circ) = 1 - E_{0,1}(d).
\]

Since \( \Pr(\circ \ d \bullet) + \Pr(\circ \ d \circ) = \Pr(\circ) = E_{1,0}(d) \), we have \( \Pr(\circ \ d \bullet) = E_{1,0}(d) - E_{1,1}(d) \) and the correlator becomes
\[
C_d(t) = 1 - E_{0,1}(d) - E_{1,0}(d) + E_{1,1}(d) - (1 - E_{1,0}(d))^2.
\]

Using the fact that \( E_{0,0}(d) = 1 \), we can expand \( C_d(t) \) up to the second order in the lattice step \( a \), setting \( z = da \), and \( C(z, t) = C_d(t)/a^2 \), so
\[
C(z, t) = \partial_x^2 E(x, y, z)|_{x=0, y=0} - \partial_x E(x)|_{x=0} \partial_y E(y)|_{y=0}.
\]

This is the general expression for the correlation functions, which depends on the one- and two-interval probability functions.

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5.2. Decomposition: the general formalism

There are three contributions to $C(z, t)$ which come from the second derivatives with respect to $x$ and $y$ of $E^{(0)}$, $E^{(1)}$ and $E^{(2)}$. From the two-interval probability equation (37), we have the decomposition

$$ C(z, t) = C^{(0)}(z, t) + C^{(1)}(z, t) + C^{(2)}(z, t), \quad (56) $$

where the different contributions are

$$ C^{(0)}(z, t) = \partial^2_{xy} E^{(0)}|_{x=0, y=0} \quad \text{and} \quad C^{(1)}(z, t) = \partial^2_{xy} E^{(1)}|_{x=0, y=0} - 2\partial_x I(x)|_{x=0} \partial_y \text{erfc}(y/\ell_0)|_{y=0} \quad (57) $$

with

$$ I(x) = \int_0^\infty \frac{dx'}{\sqrt{\pi \ell_0}} E(x', 0)(e^{-x'^2/\ell_0^2} - e^{-(x'+x)^2/\ell_0^2}). \quad (58) $$

From (39), we obtain the first non-connected contribution of $C^{(0)}(z, t)$, which does not depend on the initial conditions:

$$ \partial^2_{xy} E^{(0)}|_{x=0, y=0} = \frac{4e^{-z^2/\ell_0^2}}{\pi \ell_0^2} \left[ 2 \sinh(z^2/\ell_0^2) + \sqrt{\pi \ell_0^2} \text{erfc} \left( \frac{z}{\ell_0} \right) \right], $$

and so

$$ C^{(0)}(z, t) = \frac{4e^{-z^2/\ell_0^2}}{\pi \ell_0^2} \left[ -e^{-z^2/\ell_0^2} + \sqrt{\pi \ell_0^2} \text{erfc} \left( \frac{z}{\ell_0} \right) \right] =: \frac{1}{\ell_0^2} f_0(z/\ell_0) \quad (59) $$

where the scaling function $f_0$ has the following limit behaviour:

$$ f_0(\ell) \sim \begin{cases} 
-\frac{4}{\pi} & \text{for } \ell \to 0 \\
\frac{4}{\pi} e^{-\ell^2} & \text{for } \ell \to \infty.
\end{cases} \quad (60) $$

Equation (59) is in exact agreement with the result announced in the literature [3,4]. The asymptotic forms (60) have also been obtained several times, either for the coagulation–diffusion process [28,31,32] or for the equivalent [22,15] pair annihilation–diffusion process; see [3,14,30,42] and references therein. However, the present discussion is not restricted to the rather special case of an initially fully occupied lattice. As we shall see, the second and third contributions in (56) are corrections to the leading behaviour in the long-time limit (see appendix F for the details of the proof). We obtain a hierarchy in the inverse powers $1/\ell_0$, where $|C^{(0)}(z, t)| \gg |C^{(1)}(z, t)| \gg |C^{(2)}(z, t)|$, with dominant contributions of order $1/\ell_0^2$, $1/\ell_0^4$ and $1/\ell_0^6$, respectively.
5.3. Application

In what follows, we consider the special case of initial conditions $E_0(x') = \exp(-c_0 x')$ and $E_0(x', y', z') = \exp(-c_0 (x' + y'))$, where $c_0$ is the concentration of uncorrelated particles at time $t = 0$. We assume that initially the two-interval distribution is independent of the distance between the intervals. It satisfies nevertheless the condition $E_0(x', y', z' = 0) = E_0(x' + y')$. The solutions for the correlation function given in an appendix by (F.1) and (F.2) can be computed except for the triple integral, where the successive Gaussian integrals cannot be expressed, at least to our knowledge, in terms of known special functions. Since we are merely interested in the long-time limit and in the influence of the initial conditions in this limit, we can nevertheless perform an expansion in $x'/\ell_0$ and $y'/\ell_0$ inside the kernel $K_{2, \ell_0}$ derivatives and the weight function $W_0(-x', -y', z - z')$ in (F.2). This is so because only small values of $x'$ and $y'$ are relevant when $c_0 \ell_0$ is large when these integrands are combined with the exponential factor $\exp(-c_0 (x' + y'))$. The latter function renders the integral finite after expansion of the weight and kernel. Indeed the natural small parameter of the expansion is $1/(c_0 \ell_0)$, relatively to the other dimensionless parameter $z/\ell_0$, and the series expansion is assumed to break down only in the limit of low concentration or short times. But the previous result (F.2) provides a general form suitable for series expansion of the correlated function with generic initial distributions. The other connected part of the correlation function $C^{(1)}(z, t)$, which is defined in equation (F.1), can be computed exactly with the chosen exponential initial condition $E_0(x') = \exp(-c_0 x')$, or by differentiating twice, with respect to $x$ and $y$, the previous result given in (42). In the asymptotic limit and for $c_0 \ell_0$ large, we obtain

$$C^{(1)}(z, t) = \frac{8 z/\ell_0}{\sqrt{\pi \ell_0^2 (c_0 \ell_0)^2}} \left( -3 + 2 \frac{z^2}{\ell_0^2} \right) \text{erf}(z/\ell_0) e^{-z^2/\ell_0^2} + \frac{16}{\pi \ell_0^2 (c_0 \ell_0)^2} \left( 1 - \frac{z^2}{\ell_0^2} \right) e^{-2z^2/\ell_0^2}$$

$$+ \frac{16 z/\ell_0}{\sqrt{\pi \ell_0^2 (c_0 \ell_0)^4}} \left( 15 - 20 \frac{z^2}{\ell_0^2} + 4 \frac{z^4}{\ell_0^4} \right) \text{erf}(z/\ell_0) e^{-z^2/\ell_0^2}$$

$$+ \frac{32}{\pi \ell_0^2 (c_0 \ell_0)^4} \left( -3 + 9 \frac{z^2}{\ell_0^2} - 2 \frac{z^4}{\ell_0^4} \right) e^{-2z^2/\ell_0^2} + o(1/(c_0 \ell_0)^4). \tag{61}$$

We notice that these first terms contribute to the correlation function in the large time limit at least like $1/\ell_0^4$ times a scaling function of $z/\ell_0$. This is a correction to $C^{(0)}(z, t)$ which behaves like $1/\ell_0^2$ instead.

The last term $C^{(2)}(z, t)$ can also be expanded as a power series in $1/(c_0 \ell_0)$ whose coefficients are scaling functions of $z/\ell_0$:

$$C^{(2)}(z, t) = \frac{16}{\pi \ell_0^2} \exp(-2z^2/\ell_0^2) \sum_{k=2}^{\infty} \frac{P_{2k}(z/\ell_0)}{(c_0 \ell_0)^{2k}} \tag{62}$$

where the first few polynomials $P_{2k}(z)$ read

$$P_4(z) = -1 - 2z^2, \quad P_6(z) = 12 + 24z^2 - 16z^4,$$

$$P_8(z) = -156 - 312z^2 + 528z^4 - 96z^6, \tag{63}$$

$$P_{10}(z) = 2400 + 4800z^2 - 15360z^4 + 5888z^6 - 512z^8.$$
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Figure 3. Time evolution of the connected two-point correlation function $C(z,t)$ for $c_0 = 1$ and $z = 1/2$. The full solid line shows the leading contribution $C^{(0)}(1/2, t)$, the dashed line also includes the effect of the leading correction $C^{(0)}(1/2, t) + C^{(1)}(1/2, t)$ while the dashed–dotted line includes all contributions $C^{(0)}(1/2, t) + C^{(1)}(1/2, t) + C^{(2)}(1/2, t)$.

In general, $C(z,t)$ can be expanded as a series in $1/(c_0 \ell_0)$, with coefficients being scaling functions of $z/\ell_0$. Only the contribution $C^{(0)}(z,t)$ is dominant at zeroth order in $1/(c_0 \ell_0)$, while $C^{(1)}(z,t)$ contributes at order $1/(c_0 \ell_0)^2$. Otherwise the dominant part of $C^{(2)}(z,t)$, as seen in equation (62), is of order $1/(c_0 \ell_0)^4$, and therefore smaller than the previous ones. The correlations are negative over the whole range of values of $z$ considered.

In figure 3, the time evolution of the correlator $C(1/2,t)$ is illustrated, for initially uncorrelated particles with a concentration $c_0 = 1$ and $8D = 1$, so $\ell_0 = \sqrt{t}$. In particular, we plot the leading contribution for $t \to \infty$, $C^{(0)}(1/2, t)$, along with the curve resulting when the first correction $C^{(1)}(1/2, t)$ is included, as well as when both corrective terms $C^{(1)}(1/2, t) + C^{(2)}(1/2, t)$ are added. In agreement with our asymptotic estimates, we clearly see that even for relatively small times, a clear hierarchy $|C^{(0)}(1/2, t)| \gg |C^{(1)}(1/2, t)| \gg |C^{(2)}(1/2, t)|$ emerges.

5.4. Extensions

Having found the single- and two-hole probabilities, we can immediately derive further quantities of physical interest. For example, the effective reaction rate is controlled by the pair probability $Pr(\bullet\bullet)$ for finding two particles on neighbouring sites. Similarly, one may define the triplet probability $Pr(\bullet\bullet\bullet)$. Simple enumeration leads to

\[
Pr(\bullet\bullet) = E_0 - 2E_1 + E_2 \equiv a^0 \frac{\partial^2 E(x,t)}{\partial x^2} |_{x=0}
\]

\[
Pr(\bullet\bullet\bullet) = E_0 - 3E_1 + 2E_2 - E_3 + E_{1,1,1} \equiv a^0 \alpha^3 [\partial_{xyz} E(x,y,z)|_{x,y,z=0} - \partial_{xxx} E(x,y,z)|_{x,y,z=0}].
\]

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While the pair probability can be expressed in terms of the single-hole probability alone\(^4\), the triplet probability already depends on a two-hole probability as well. In the continuum limit, the pair probability reads, for the three examples of initial distributions already considered in figure 1,

\[
\begin{align*}
\text{Pr}(\bullet \bullet) &= \begin{cases} 
  c_0^2 \exp \left( \frac{c_0 \ell_0}{2} \right) \text{erfc} \left( \frac{c_0 \ell_0}{2} \right) ; & \text{if } E_0(x) = e^{-c_0x} \\
  \frac{4c_0^2}{\sqrt{\pi}} \int_0^\infty dx \ e^{-x^2} \frac{1}{(1 + c_0 \ell_0 x)^3} ; & \text{if } E_0(x) = (1 + c_0 x)^{-1} \\
  0; & \text{if } E_0(x) = \text{erfc} \left( \frac{\sqrt{\pi} c_0 x}{2} \right)
\end{cases}
\end{align*}
\]

where \(c_0\) characterizes the width in the initial state. In the first two cases, the long-time behaviour is given by \(\text{Pr}(\bullet \bullet) \sim (2/\sqrt{\pi}) (c_0/\ell_0) \sim t^{-1/2}\) when \(c_0\) is kept fixed. Although the long-time behaviour is algebraic, the associated amplitude depends explicitly on the initial distribution, in contrast to what we had seen for equation (14) for the particle concentration \(c(t)\).

Similarly, from the results derived in this paper further correlators can be directly obtained, for example

\[
\begin{align*}
\text{Pr}(\bullet \bullet \bullet) &= E_0 - 3E_1 + E_2 + E_{1,1}(d) + E_{1,1}(d + 1) - E_{2,1}(d) \\
&\quad + a \left[ \partial_{x,y,z} E(x, y, z) \right]_{x,y=0} - \partial_{x,y,z} E(x, y, z) \big|_{x,y=0}
\end{align*}
\]

\[
\begin{align*}
\text{Pr}(\bullet \bullet \bullet \bullet) &= \text{Pr}(\bullet \bullet \bullet) - E_1 + E_2 + E_{1,1}(d + 1) + E_{1,1}(d + 2) \\
&\quad - E_{2,1}(d + 1) - E_{1,2}(d + 1) - E_{1,2}(d) + E_{2,2}(d) \\
&\quad + a \left[ \partial_{x,y,z} E(x, y, z) \right]_{x,y=0} - \partial_{x,y,z} E(x, y, z) \big|_{x,y=0} \\
&\quad + \partial_{x,y,z} E(x, y, z) \big|_{x,y=0} + \partial_{x,y,z} E(x, y, z) \big|_{x,y=0}.
\end{align*}
\]

6. Conclusions

We have analysed a method for the computation of time-dependent correlators in the 1D coagulation–diffusion process. The relevant quantities are the probabilities of finding either a single empty interval of a given size or else two empty intervals of given sizes and at a given distance from each other. These probabilities satisfy simple linear differential or difference equations, but working out the explicit solution is rendered difficult through boundary conditions, whose symmetry properties are not the same as those of the differential equations. Rather than circumventing this difficulty, we have shown how, by analytic continuation to negative sizes and distances, the full solution may be found for an arbitrary initial distribution which in turn can be characterized in terms of single- and double-interval probabilities.

Specifically, we have seen the following:

(i) The leading long-time behaviour of the particle density is explicitly confirmed to be independent of the initial conditions, as expected from the field-theoretical

\(^4\) The dependence on \(x\) of \(\partial_x^2 E(x,t)/c(t)\) describes the interparticle distribution function and has been analysed in detail in the past; see [5,6] and references therein.
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renormalization group [24, 7] and in agreement with earlier calculations using a formulation of the empty-interval method where spatial translation invariance is not immediately built in [5, 6]. Our results also allow us to compare directly the continuum limit with the result found for the discrete lattice and we can derive systematically the finite-time corrections to the leading dynamical scaling behaviour.

(ii) The leading long-time behaviour equation (59) for the density–density correlator not only agrees with the earlier asymptotic estimate [4] but is furthermore explicitly shown to be independent of the initial conditions as well. The corrections to the leading scaling behaviour have also been analysed and the relative importance of the initial single- or double-interval probabilities can be quantitatively studied.

(iii) By considering general initial conditions, we have identified a natural decomposition (see equations (37) and (56)) into terms which develop a clear hierarchy in the long-time limit.

This information will become important in a following article where we plan to analyse the ageing behaviour in the coagulation–diffusion process or generalizations thereof, where methods analogous to the ones developed here should apply, and the single-time correlators studied here will serve as initial values for the two-time correlators whose dynamical scaling behaviour will be searched for.

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Appendix A. Proof of lemma 3.1

Starting from the discrete case (28), we consider the time differential equation where $n_1 = 0$, which leads to

\[ \partial_t E_{0,n_2}(d) = \partial_t E_{n_2}. \]  

(A.1)

The first term is evaluated from (28) and is equal, after some algebra, to

\[ \partial_t E_{0,n_2}(d) = \Gamma[2E_{n_2-1} - 4E_{n_2} + 2E_{n_2+1} + E^{-1}_{-1,n_2}(d) + E_{-1,n_2}(d+1) - 4E_{n_2} + E_{1,n_2}(d) + E_{1,n_2}(d-1)]. \]

Using (2), the right-hand side in (A.1) is equal to

\[ \partial_t E_{n_2} = 2\Gamma[E_{n_2-1} - 2E_{n_2} + E_{n_2+1}]. \]

5 As discussed in section 5, the various asymptotic estimates of either the diffusion–coagulation process or the equivalent pair annihilation–diffusion process are fully reproduced.
By comparing the last two expressions, we obtain the equality between $n_1 = -1$ and 1:

$$E_{-1,n_1}(d) + E_{-1,n_2}(d + 1) + E_{1,n_2}(d) + E_{1,n_2}(d - 1) = 4E_{n_2}. $$

We then differentiate this relation with respect to time, and use (28) for each of the four terms on the lhs (also (2) for the evaluation of $\partial_t E_{n_2}$) in order to obtain a new relation between index $n_1 = -2$ and index $n_1 = 2$:

$$E_{-2,n_2}(d) + 2E_{-2,n_2}(d + 1) + E_{-2,n_2}(d + 2) + E_{2,n_2}(d) + 2E_{2,n_2}(d - 1) + E_{2,n_2}(d - 2) = 8E_{n_2}. $$

By recursion, we can extend this result to any positive index $n_1$:

$$\sum_{k=0}^{n_1} \binom{k}{n_1} E_{-n_1,n_2}(d + k) + \sum_{k=0}^{n_1} \binom{k}{n_1} E_{1,n_2}(d - k) = 2^{n_1+1}E_{n_2}. \quad \text{(A.2)}$$

In particular, for $n_2 = 0$ we recover the result (8): $E_{-n_1} + E_{n_1} = 2$, by using the relation $\sum_{k=0}^{n_1} \binom{k}{n_1} = 2^{n_1}$. Equation (A.2) can be inverted in order to obtain a relation between $E_{-n_1,n_2}(d)$ and functions of positive indices. We use a discrete Fourier transform

$$E_{n_1,n_2}(d) = \int_0^1 dz \tilde{E}_{n_1,n_2}(z)e^{2i\pi zd}, \quad \tilde{E}_{n_1,n_2}(z) = \sum_{d=-\infty}^{\infty} E_{n_1,n_2}(d)e^{-2i\pi zd}$$

to solve the previous Green’s function (A.2):

$$\tilde{E}_{-n_1,n_2}(z) \left(1 + e^{2i\pi z}\right)^{n_1} = -\tilde{E}_{n_1,n_2}(z) \left(1 + e^{-2i\pi z}\right)^{n_1} + 2^{n_1+1}E_{n_2} \sum_{d'} e^{2i\pi (d-d')} ,$$

where we used the Dirac relation

$$\sum_{d'=-\infty}^{\infty} e^{2i\pi zd'} = \sum_{d'=-\infty}^{\infty} \delta(z - d') \quad \text{(A.3)}$$

Inverting this relation, we obtain

$$E_{-n_1,n_2}(d) = -\sum_{d'} E_{n_1,n_2}(d') \int_0^1 dz e^{2i\pi (d-d'-n_1)} + 2^{n_1+1}E_{n_2} \int_0^1 dz \sum_{d'} \frac{e^{2i\pi zd'}}{(1+e^{2i\pi z})^{n_1}}. $$

The first integral over the variable $z$ gives simply a Kronecker function $\delta_{d-d'-n_1,0}$, whereas, for the second one, we use the Dirac sum to select the value $z = d' = 0$ or $z = d' = 1$ when performing the integration over $z$. We obtain the simple result

$$\int_0^1 dz \sum_{d'} \frac{e^{2i\pi zd'}}{(1+e^{2i\pi z})^{n_1}} = \int_0^1 dz \sum_{d'} \delta(z - d') \frac{1}{(1+e^{2i\pi z})^{n_1}} = 2^{-n_1}.$$ 

Therefore, we obtain the symmetry relations between the negative and positive indices $n_1$ (and $n_2$). Relation (34) is seen to be directly connected to the first two parts of the lemma, by considering the case where the interval lengths are both negative. \hfill $\Box$
Appendix B. Proof of lemma 3.2

We consider the case of $d = 0$ in the discrete equation (28). Since $E_{n_1,n_2}(0) = E_{n_1+n_2}$, we have

$$\partial_tE_{n_1,n_2}(0) = \partial_tE_{n_1+n_2}.$$  

Using (28) the first term gives

$$\partial_tE_{n_1,n_2}(0) = \Gamma[2E_{n_1+n_2+1} - 8E_{n_1+n_2} + E_{n_1+1,n_2}(-1) + E_{n_1,n_2+1}(-1) + 2E_{n_1+n_2-1} + E_{n_1-1,n_2}(1) + E_{n_1,n_2-1}(1)],$$

whereas the second term gives

$$\partial_tE_{n_1+n_2} = 2\Gamma(E_{n_1+n_2-1} - 2E_{n_1+n_2} + E_{n_1+n_2+1}).$$

By comparing the last two expressions, we obtain

$$E_{n_1+1,n_2}(-1) + E_{n_1,n_2+1}(-1) + E_{n_1-1,n_2}(1) + E_{n_1,n_2-1}(1) = 4E_{n_1+n_2}.$$  

By successively differentiating the last expression with respect to time, we obtain a general Green’s equation for the terms involving a negative and a positive distance between intervals:

$$\sum_{k=0}^{d} \binom{k}{d} E_{n_1+k,n_2+d-k}(-d) + \sum_{k=0}^{d} \binom{k}{d} E_{n_1-k,n_2-d+k}(d) = 2^{d+1}E_{n_1+n_2}.$$  

Notice that the sum of all three indices is always equal to $n_1 + n_2$. As before, we introduce the general Fourier transform

$$E_{n_1,n_2}(d) = \int_{0}^{1} dx \int_{0}^{1} dy E(x,y,d)e^{2i\pi(n_1x+n_2y)},$$

$$\tilde{E}(x,y,d) = \sum_{n_1,n_2=-\infty}^{\infty} E_{n_1,n_2}(d)e^{-2i\pi(n_1x+n_2y)}$$

to obtain the relation

$$\tilde{E}(x,y,-d) + \tilde{E}(x,y,d) \left( \frac{e^{-2i\pi x} + e^{2i\pi y}}{e^{2i\pi x} + e^{2i\pi y}} \right)^{d} = \tilde{E}(x,y,0) \frac{2^{d+1}}{(e^{2i\pi x} + e^{2i\pi y})^{d}}.$$  

The Fourier inverse then reads explicitly

$$E_{n_1,n_2}(-d) = \sum_{n_1',n_2'} E_{n_1',n_2'}(0) \int_{0}^{1} dx \int_{0}^{1} dy \frac{2^{d+1}\exp(2i\pi x(n_1-n_1') + 2i\pi y(n_2-n_2'))}{(e^{2i\pi x} + e^{2i\pi y})^{d}} - \sum_{n_1',n_2'} E_{n_1',n_2'}(d) \int_{0}^{1} dx \int_{0}^{1} dy \left( \frac{e^{-2i\pi x} + e^{-2i\pi y}}{e^{2i\pi x} + e^{2i\pi y}} \right)^{d} e^{2i\pi x(n_1-n_1') + 2i\pi y(n_2-n_2')}.$$
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The second integral over $x$ and $y$ gives simply a double Kronecker function $\delta_{n_1-n'_1-d,0}\delta_{n_2-n'_2-d,0}$, whereas the first integral can be transformed as follows:

$$\sum_{n_1',n_2'} E_{n_1',n_2'}(0) \int_0^1 dx \int_0^1 dy \frac{e^{2d+1}}{(e^{2\pi x}+e^{2\pi y})^d} e^{2\pi x(n_1-n'_1)+2\pi y(n_2-n'_2)}$$

$$= \sum_{n_1',n_2'} E_{n_1'+n_2'} \int_0^1 dx \int_0^1 dy \frac{2^{d+1}}{(e^{2\pi x}+e^{2\pi y})^d} e^{2\pi x(n_1-n'_1)+2\pi y(n_2-n'_2)}$$

$$= \sum_{n_1'} E_{n_1} \int_0^1 dx \int_0^1 dy \frac{2^{d+1}}{(e^{2\pi x}+e^{2\pi y})^d} e^{2\pi x(n_1-n'_1)+2\pi y(n_2-n'_2)} \sum_{n_2'} e^{2\pi n_2' (x-y)}.$$

The last sum over $n_2'$ can be performed using equality (A.3), selecting $x = y$ over the interval of integration over $y$. Then, the integration over variable $x$ gives directly a Kronecker function $\delta_{n_1+n_2-d,n_1'0}$. Finally we obtain the relation between the indices $-d$ and $d$. □

### Appendix C. Decomposition of the two-interval probability in the three-dimensional space

We give the detailed calculation for the decomposition of the general solution for the two-hole probability. We begin by separating the regions with $z' > 0$ from those with $z' < 0$:

$$E_{0}(x, y, z, t) = \sqrt{\frac{2}{\pi^3}} \int_{\mathbb{R}^2} \int_{0}^{\infty} dx' \int_{0}^{\infty} \frac{dy'}{2\pi} \frac{dz'}{2\pi} \left[ W_{0}(x-x', y-y', z-z') E_{0,0}(x', y', z') + W_{0}(x-x', y-y', z+z') E_{0,0}(x', y', -z') \right].$$

(C.1)

In the last term, we map the domain of negative values of $z'$ to the positive values by using (35) in the continuum limit $E_{0}(x', y', -z') = -E_{0}(x'-z', y'-z', z') + 2E_{0}(x'+y'-z')$, where $E_{0}(x) = E(x, 0)$. It is also useful to perform two translations $x'-z' \rightarrow x'$, $y'-z' \rightarrow y'$ in the first term $E_{0}(x'-z', y'-z', z')$, and a translation on the variable $x' \rightarrow x'-y'+z'$ in the second term $E_{0}(x'+y'-z')$ so that we have

$$E_{0}(x, y, z) = \sqrt{\frac{2}{\pi^3}} \int_{\mathbb{R}^2} \int_{0}^{\infty} dx' \int_{0}^{\infty} dy' \int_{0}^{\infty} dz' W_{0}(x-x', y-y', z-z') E_{0,0}(x', y', z')$$

$$- W_{0}(x-x'-z', y-y'-z', z+z') E_{0,0}(x', y', z')$$

$$+ 2W_{0}(x-x'+y'-z', y-y', z+z') E_{0}(x').$$

(C.2)

In the third term containing $E_{0}(x')$, we can now perform the integration on $y'$ and $z'$, since the function $W_{0}(0)$ is a Gaussian kernel. When $x'$ is negative, we use also $E_{0}(x') = 2 - E_{0}(-x')$ in order to perform the mapping onto the positive axis. After rearranging the different terms we obtain

$$E_{0}(x, y, z) = \text{erf}(z) \text{erf}(x+y+z)$$

$$+ \text{erf}(z) \int_{0}^{\infty} \frac{dx'}{\sqrt{\pi}} [e^{-(x+y+z-x')^2} - e^{-(x+y+z+x')^2}] E_{0,0}(x').$$

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\[ + \sqrt{\frac{2}{\pi^3}} \int_{\mathbb{R}^2} \int_0^\infty dx' \, dy' \, dz' \left[ W_{10}(x - x', y - y', z - z') \right. \]
\[ - W_{10}(x - x' - z', y - y' - z', z + z') \left. \right] E_{0,0}(x', y', z'). \] (C.3)

In the last term, the difference between the two weight functions \( W_{10}() \) can be simplified by noticing that

\[ W_{10}(x - x', y - y', z - z') = W_{10}(x - x' - z', y - y' - z', z + z') \]
\[ = W_{10}(x - x', y - y', z - z')(1 - e^{-4\pi z}) := \tilde{W}_{10}(x', y', z'). \] (C.4)

From the last multiple integral (C.3), we can divide the integration domain over \( x' \) and \( y' \) into four parts: \( E_0(x', y', z') \), \( E_0(-x', y', z') \), \( E_0(x', -y', z') \), and \( E_0(-x', -y', z') \), with \( (x', y', z') \) all positive variables. \( E_0(-x', y', z') \) can be transformed using the following steps and (33) and (35):

\[ E_0(-x', y', z') = 2E_0(y') - E_0(x', y', z' - x') = 2E_0(y') - \theta(z' - x')E_0(x', y', z' - x') \]
\[ - \theta(x' - z')[2E_0(y' + z') - E_0(z', -x' + y' + z', x' - z')]; \] (C.5)

then

\[ E_0(-x', y', z') = 2E_0(y') - \theta(x' - z')E_0(x', y', z') = 2E_0(y') - \theta(x' - z')\theta(x' - y' - z')2E_0(z') \]
\[ - \theta(z' - x')E_0(x', y', z' - x') + \theta(x' - z')\theta(-x' + y' + z')E_0(z', -x' + y' + z', x' - z') \]
\[ - \theta(x' - z')\theta(x' - y' - z')E_0(z', x' - y' - z', y'). \] (C.6)

All arguments of the functions appearing in the last expression are now positive. An analogous analysis is done for \( E_0(x', -y', z') \), which is symmetric on inverting \( x' \) and \( y' \):

\[ E_0(x', -y', z') = 2E_0(x') - \theta(y' - z')2E_0(x' + z') + \theta(y' - z')\theta(-x' + y' - z')2E_0(z') \]
\[ - \theta(z' - y')E_0(x', y', z' - y') + \theta(y' - z')\theta(-x' + y' + z')E_0(x' - y' + z', z', y' - z') \]
\[ - \theta(y' - z')\theta(-x' + y' + z')E_0(-x' + y' + z', z', x'). \] (C.7)

Finally, using (34) and (35) the last expression \( E_0(-x', -y', z') \) is transformed into

\[ E_0(-x', -y', z') = 4 - 2E_0(x') - 2E_0(y') + E_0(x', y', z' - x' - y') \]
\[ = 4 - 2E_0(x') - 2E_0(y') + \theta(z' - x' - y')E_0(x', y', z' - x' - y') \]
\[ + \theta(x' + y' - z')[2E_0(z') - E_0(z' - y', z' - x', x' + y' - z')]. \]

In the last term, \( z' - y' \) and \( z' - x' \) can be either positive or negative, which leads us to analyse the other possibilities

\[ E_0(-x', -y', z') = 4 - 2E_0(x') - 2E_0(y') + \theta(x' + y' - z')2E_0(z') \]
\[ + \theta(z' - x' - y')E_0(x', y', z' - x' - y') - \theta(x' + y' - z')\theta(z' - y')\theta(z' - x')E_0(z' - y', z' - x', x' + y' - z') \]
\[ + \theta(y' - z')\theta(x' - z')\theta(-E_0(y' - z', z' - x', x') + 2E_0(z' - x')) \]
\[ + \theta(z' - y')\theta(x' - z')\{-E_0(z' - y', x' - z', y') + 2E_0(z' - y')\} \]
\[ + \theta(y' - z')\theta(x' - z')\{4 - 2E_0(y' - z') - 2E_0(x' - z') + E_0(y' - z', x' - z', z')\}. \] (C.8)

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Here, we have (i) terms without any empty-interval probability, (ii) terms which refer to the single-interval probabilities only and (iii) terms which contain also the two-interval probabilities. The first group of terms is needed for an initially fully occupied lattice (see section 4).

Appendix D. Simplification of some integrals for the initial one-interval contribution

We show how to simplify the integrals $I_1, \ldots, I_6$ of section 4.2. In order to arrive at the previous partial result, it is useful to note that a function $\theta(x' - z')$ (in the case where it involves an integral over the one-interval distribution $E_0$ which does not depend on $x'$) with $x'$ and $z'$ positive can be simplified and replaced by unity if we perform the translation $x'' = x' - z'$. The integration over $x''$ extends then from $-z' < 0$ to $\infty$; however the function $\theta$ restricts the integration to the interval $x'' > 0$. Therefore the limits of the integration are unchanged under this procedure, and the function $\theta$ can be set to unity. In the opposite case, where we have a contribution such as $E_0(x')\theta(x' - z')$ for example, a translation on $x'$ will change the argument of $E_0$, which does not conserve the form (40). Instead, we write $\theta(x' - z') = 1 - \theta(z' - x')$ and apply the previous translation to the positive variable $z' = z'' + x'$. This changes the arguments of the $\mathcal{W}_0(\cdot \cdot \cdot)$ functions only and not $E_0$. Using (38), the different groups of terms $I_1, \ldots, I_4$ can be simplified:

\begin{align}
I_1 &= \tilde{\mathcal{W}}_0(x', -y', z') - \mathcal{W}_0(x' - z', -y' - z', z') \\
&= \mathcal{W}_0(x', -y', z') + \mathcal{W}_0(x', -y', -z') \\
I_2 &= \tilde{\mathcal{W}}_0(-y', x', z') - \mathcal{W}_0(-y' - z', x' - z', z') \\
&= \mathcal{W}_0(-y', x', z') + \mathcal{W}_0(-y', x', -z') \\
I_3 &= \tilde{\mathcal{W}}_0(-x' - z', -y' - z', z') - \mathcal{W}_0(-x', -y', z') \\
&= -\tilde{\mathcal{W}}_0(-x', -y', -z') - \mathcal{W}_0(-x', -y', z') \\
I_4 &= \tilde{\mathcal{W}}_0(-y' - z', -x' - z', z') - \mathcal{W}_0(-y', -x', z') \\
&= -\tilde{\mathcal{W}}_0(-y', -x', -z') - \mathcal{W}_0(-y', -x', z').
\end{align}

(D.1)

These relations allow us to extend the integration over $z'$ from $-\infty$ to $+\infty$ since they combine each time two terms depending on $z'$ and $-z'$. Since the variable $z'$ appears only in the Gaussian weights, the integration over $z'$ gives new Gaussian exponentials. Then integration over $y'$ gives erf or erfc functions, which are combined with $E_0(x')$. Terms $I_5$ and $I_6$ can be combined together. The first term of $I_5$ can be transformed as

\begin{align*}
\int_{\mathbb{R}_+^3} dx' \, dy' \, dz' \, E_0(x')\theta(-x' - y' + z') \tilde{\mathcal{W}}_0(y', -z', x') \\
&= \int_{\mathbb{R}_+^3} dx' \, dy' \, dz' \, E_0(x')\theta(-x' + y' - z') \tilde{\mathcal{W}}_0(z', -y', x') \\
&= \int_{0}^{\infty} dx' \, E_0(x') \int_{0}^{\infty} dy' \, \int_{-\infty}^{0} dz' \, \theta(-x' + y' + z') \tilde{\mathcal{W}}_0(-z', -y', x').
\end{align*}
The second term gives
\[
\int_{\mathbb{R}^3_+} dx' dy' dz' \, E_0(x') \theta(-x' - y' + z') \tilde{W}_{t_0}(-z', y', x') = \int_0^\infty \int_0^\infty \int_0^\infty dx' dy' dz' \, \theta(-x' + y' + z') \tilde{W}_{t_0}(-z', -y', x').
\]

Finally, consider \( I_6 \), which is equal to
\[
\int_0^\infty \int_0^\infty \int_0^\infty dx' \, E_0(x') \int_0^\infty dy' \, \int_0^\infty dz' \, \theta(-x' + y' + z') \tilde{W}_{t_0}(-z', -y', x')
\]
and can be combined with the previous two terms. Noticing that \( \theta(-x' + y' + z') = 0 \) in the domain where \( y' \) and \( z' \) are both negative, we can write the integral over \( I_5 \) and \( I_6 \) as follows:
\[
\sqrt{\frac{8}{\pi^3}} \int_{\mathbb{R}^3_+} dx' \, dy' \, dz' \, E_{0, t_0}(x') \left( I_5 + I_6 \right)
\]
\[
= 2\sqrt{2} \int_0^\infty \frac{dx'}{\sqrt{\pi}} \int_0^\infty \frac{dy'}{\pi} \int_{-\infty}^\infty \frac{dz'}{\pi} \, \theta(-x' + y' + z') \tilde{W}_{t_0}(-z', -y', x')
\]
\[
= 2\sqrt{2} \int_0^\infty \frac{dx'}{\sqrt{\pi}} \int_0^\infty \frac{dy'}{\pi} \int_{-\infty}^\infty \frac{dz'}{\pi} \, \theta(-x' + z') \tilde{W}_{t_0}(y' - z', -y', x').
\]

Then, it is possible to perform the integrations over \( y' \) and \( z' \) successively.

**Appendix E. Simplification of some integrals for the initial two-interval contribution**

In this appendix, we show how to simplify the expression for the contribution which depends on the initial two-interval probability.

There are therefore 12 terms involving the weights \( \tilde{W}_{t_0}(\cdots) \). After performing different translation operations, we arrive at the expression
\[
E_{t_0}^{(2)}(x, y, z) = \sqrt{\frac{2}{\pi^3}} \int_{\mathbb{R}^3_+} dx' \, dy' \, dz' \, E_{0, t_0}(x', y', z')
\]
\[
\times \left\{ \tilde{W}_{t_0}(x', y', z') - \tilde{W}_{t_0}(-y' - z', -x' - z', z') \right. \\
+ \tilde{W}_{t_0}(-x', -y', x' + y' + z') - \tilde{W}_{t_0}(-x' - z', -y' - z', x' + y' + z') \\
+ \tilde{W}_{t_0}(-x' - y' - z', -z', x' + z') - \tilde{W}_{t_0}(-x', y', x' + z') \\
+ \tilde{W}_{t_0}(-z', -x' - y' - z', y' + z') - \tilde{W}_{t_0}(x', -y', y' + z') \\
+ \tilde{W}_{t_0}(-x' - z', y' + z', x') - \tilde{W}_{t_0}(-x' - y' - z', z', x') \\
+ \tilde{W}_{t_0}(x' + z', -y' - z', y') - \tilde{W}_{t_0}(z', -x' - y' - z', y'). \right\} \quad (E.1)
\]

We can pair the terms which appear on each line of the previous relation, noticing that
\[
\tilde{W}_{t_0}(u - w, v - w, w - (u + v)) = \tilde{W}_{t_0}(u, v, w - (u + v))e^{-4w(x+y+z)}. \quad (E.2)
\]

This yields the following relations:
\[
\tilde{W}_{t_0}(-y' - z', -x' - z', z') = \tilde{W}_{t_0}(x', y', z')e^{-4(x+y'+z')(x+y+z)},
\]
\[
\tilde{W}_{t_0}(-x' - z', -y' - z', x' + y' + z') = \tilde{W}_{t_0}(-x', -y', x' + y' + z')e^{-4z'(x+y+z)},
\]
\[
\tilde{W}_{t_0}(-x' - z', z', x' + y' + z') = \tilde{W}_{t_0}(-x', -y', x' + y' + z')e^{-4z'(x+y+z)},
\]
\[
\tilde{W}_{t_0}(-z', -x' - y' - z', y' + z') = \tilde{W}_{t_0}(x', -y', y' + z')e^{-4z'(x+y+z)},
\]
\[
\tilde{W}_{t_0}(-x' - y' - z', z', x') = \tilde{W}_{t_0}(-x', y', x' + z')e^{-4z'(x+y+z)},
\]
\[
\tilde{W}_{t_0}(x' + z', -y' - z', y') = \tilde{W}_{t_0}(z', -x' - y' - z', y').
\]
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\begin{align*}
\dot{W}_0(x' - y' - z', -z', x' + z') &= \dot{W}_0(-x', y', x' + z')e^{-4(y' + z')(x + y + z)}, \\
\dot{W}_0(-z', -x' - y' - z', y' + z') &= \dot{W}_0(x', -y', y' + z')e^{-4(x' + z')(x + y + z)}, \\
\dot{W}_0(-x' - y' - z', z', x') &= \dot{W}_0(-x' - z', y' + z', x')e^{-4y'(x + y + z)}, \\
\dot{W}_0(z', -x' - y' - z', y') &= \dot{W}_0(x' + z', -y' - z', y')e^{-4z'(x + y + z)}.
\end{align*}

Moreover, we have

\begin{align*}
\tilde{W}_0(x', y', z') &= \dot{W}_0(x - x', y - y', z - z')(1 - e^{-4z'}), \\
\tilde{W}_0(-x', -y', x' + y' + z') &= \dot{W}_0(x - x', y - y', z - z')e^{-4x'x - 4y'y[1 - e^{-4(x' + y' + z')z}]}, \\
\tilde{W}_0(-x', y', x' + z') &= \dot{W}_0(x - x', y - y', z - z')e^{-4x'x}[1 - e^{-4(x' + z')z}], \\
\tilde{W}_0(x', -y', y' + z') &= \dot{W}_0(x - x', y - y', z - z')e^{-4y'y[1 - e^{-4(y' + z')z}]}, \\
\tilde{W}_0(-x' - z', y' + z', x') &= \dot{W}_0(x - x', y - y', z - z')e^{-4x'x - 4x'(z + x)}(1 - e^{-4x'x}) \\
\tilde{W}_0(x' + z', -y' - z', y') &= \dot{W}_0(x - x', y - y', z - z')e^{-4y'y - 4x'(y + z)}(1 - e^{-4y'y}).
\end{align*}

Appendix F. The two-point correlation function

In this appendix, we show how the terms depending on initial conditions are corrections to the leading behaviour in the long-time limit.

We first consider the contribution \( C^{(1)}(z, t) \) which can be evaluated by deriving the kernel in (41):

\[
C^{(1)}(z, t) = \int_0^\infty \frac{dx'}{\sqrt{\pi \ell_0^3}} E(x', 0) \left( \frac{2 - 4(x' + z)^2}{\ell_0^2} \right) e^{-(x' + z)^2/\ell_0^2} \\
- \frac{4}{\sqrt{\pi \ell_0^3}} \left( \frac{(2x' - z)}{\ell_0^2} e^{-(x' - z)^2/\ell_0^2} + \frac{2x' + z}{\ell_0^2} e^{-(x' + z)^2/\ell_0^2} \right) e^{-x^2/\ell_0^2} =: \int_0^\infty \frac{dx'}{\sqrt{\pi \ell_0^3}} E(x', 0) L_1(x', z). \tag{F.1}
\]

In order to obtain the term \( C^{(2)}(z, t) \), we use the important property previously seen that \( K_2(x', y', z', 0, 0, z) = \partial_x K_2(x', y', z', x, y, z) |_{x=y=0} = \partial_y K_2(x', y', z', x, y, z) |_{x=y=0} = 0 \), which can be evaluated from (48), and we obtain

\[
C^{(2)}(z, t) = \frac{\sqrt{2}}{\pi^{3/2} \ell_0^3} \int_{\mathbb{R}^3} dx' dy' dz' E_0(x', y', z') \dot{W}_0(-x', -y', z - z') \\
\times \left\{ 16 e^{-4z'z/\ell_0^2} [x' y'(e^{4z'/\ell_0^2} + e^{-4(x' + y' + z') z'/\ell_0^2})] \\
- (x' + z')(y' + z')(1 + e^{-4(x' + y')z'/\ell_0^2}) \right\} \frac{1}{\ell_0^4} \\
- \left( \int_0^\infty \frac{dx}{\sqrt{\pi \ell_0^3}} 4x E(x, 0) e^{-x^2/\ell_0^2} \right)^2.
\]
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\[
C^{(0)}(z, t) = \frac{1}{\ell_0} f_0(z/\ell_0), \quad \text{with } f_0(z/\ell_0) \approx -\frac{4}{\pi}. \tag{F.3}
\]

For \(C^{(1)}\), when \(\ell_0\) is large, we can perform a Taylor expansion of \(L_1(x', z)\) around \(x' = 0\), and since \(L_1(0, z) = 0\) we obtain

\[
C^{(1)}(z, t) \approx \frac{f_1(z/\ell_0)}{\ell_0^4} \int_0^\infty dx E(x, 0)x \tag{F.4}
\]

where the scaling function \(f_1\) is given by the expression

\[
f_1 \left( \frac{z}{\ell_0} \right) = \frac{8}{\pi} e^{-z^2/\ell_0^2} \left[ 2 \left( 1 - \frac{z^2}{\ell_0^2} \right) e^{-z^2/\ell_0^2} + \sqrt{\frac{\pi}{\ell_0}} \left( 2 \frac{z^2}{\ell_0^2} - 3 \right) \text{erf} \left( \frac{z}{\ell_0} \right) \right] \approx \frac{16}{\pi} \left( 1 - \frac{6 z^2}{\ell_0^2} \right). \tag{F.5}
\]

The quantity \(E_1 := \int_0^\infty dx E(x, 0)x\) is related to the second moment of the interval distribution \(P(x, 0)\) (11) after performing an integration by parts:

\[
E_1 = \int_0^\infty dx E(x, 0)x = \frac{1}{2} \int_0^\infty dx P(x, 0) x^2. \tag{F.6}
\]

As regards the dominant behaviour of \(C^{(2)}\), we notice that the Gaussian weight \(W(-x', -y', z - z')\) is peaked around the value \((x', y', z') = (0, 0, z)\). We can therefore begin to perform a Taylor expansion of \(E_0(x', y', z')\) around the \(z' = z\) and integrate the variable \(z'\) on the real axis if \(z\) is far enough from the value zero, which is also satisfied if \(z/\ell_0\) is small:

\[
C^{(2)}(z, t) \approx \int_{\mathbb{R}_+^3} dx' dy' E_0(x', y', z) \left\{ \int_{M_0(x', y', z)} \frac{\sqrt{2} dz'}{\sqrt{\pi} \ell_0^3} W(-x', -y', z - z') L_2(x', y', z', z) \right\} M_0(x', y', z)
\]

\[
+ \int_{\mathbb{R}_+^3} dx' dy' \partial_z E_0(x', y', z) \left\{ \int_{M_1(x', y', z)} \frac{\sqrt{2} dz'}{\sqrt{\pi} \ell_0^3} (z' - z) W(-x', -y', z - z') L_2(x', y', z', z) \right\} M_1(x', y', z)
\]

\[
- \left( \int_0^\infty dx \frac{4x}{\sqrt{\pi \ell_0^3}} E(x, 0) e^{-x^2/\ell_0^2} \right)^2. \tag{F.7}
\]
The last term in (F.7) can be approximated by

\[ M_0(x', y', z) \simeq \frac{x'y'}{\ell_0^6} f_2^{(0)} \left( \frac{z}{\ell_0} \right), \quad f_2^{(0)}(u) = -\frac{32}{\pi} e^{-2u^2} (1 + 2u^2 - 6u^4). \]

The next function \( M_1 \) in the development can be expanded as

\[ M_1(x', y', z) \simeq \frac{x'y'}{\ell_0^6} f_2^{(1)} \left( \frac{z}{\ell_0} \right), \quad f_2^{(1)}(u) = -uf_2^{(0)}(u). \]

The last term in (F.7) can be approximated by

\[ \left( \int_0^\infty dx \frac{4x}{\sqrt{\pi} \ell_0} E(x, 0, e^{-x^2/\ell_0^2}) \right)^2 \simeq \frac{16}{\pi \ell_0^6} E_1^2 \]

where \( E_1 \) is the average quantity of the single-interval distribution, given in (F.6).

Hence, \( C^{(2)} \) can be expanded as a series in the inverse diffusion length involving integrals of the two-interval distribution and related moments, as long as these integrals are not diverging:

\[ C^{(2)}(z, t) \simeq \frac{f_2^{(0)}(z/\ell_0)}{\ell_0^6} \int_{\mathbb{R}_+^2} dx' dy' x' y' E_0(x', y', z) + \frac{f_2^{(1)}(z/\ell_0)}{\ell_0^6} \int_{\mathbb{R}_+^2} dx' dy' x' y' \partial_z E_0(x', y', z) - \frac{16}{\pi \ell_0^6} E_1^2. \] (F.8)

The expansion (F.8) has the advantage that if the initial condition \( E_0(x, y, z) \) does not depend on the distance \( z \) between the two intervals \( x \) and \( y \), then the second term of the development involving \( \partial_z E_0(x, y, z) \) vanishes. However, from the previous results (F.3), (F.4) and (F.8), we obtain a hierarchy in the inverse powers \( 1/\ell_0 \), where \( |C^{(0)}(z, t)| \gg |C^{(1)}(z, t)| \gg |C^{(2)}(z, t)| \), with dominant contributions of order \( 1/\ell_0^2 \), \( 1/\ell_0^4 \) and \( 1/\ell_0^6 \), successively. From the previous asymptotic results, it is also easy to check that in every case, the signs of the correlation contributions are respectively \( C^{(0)} < 0 \), \( C^{(1)} > 0 \) and \( C^{(2)} < 0 \). In particular, if \( z \ll \ell_0 \), the dominant contribution to \( C_2 \) comes from the last term of (F.8), since the scaling functions \( f_2^{(0)} \) and \( f_2^{(1)} \) are rapidly decreasing functions of \( z/\ell_0 \).

Hence

\[ C^{(2)}(z, t) \bigg|_{\ell_0 \gg 1} \simeq \frac{16}{\pi \ell_0^6} E_1^2. \] (F.9)

**Appendix G. Some integral identities**

We list some identities involving the kernel \( W \) and the error function \( \text{erf} \), which are used in the main text.

\[ \sqrt{2} \int \int \int_{-\infty}^{\infty} \frac{dx' dy' dz'}{\sqrt{\pi}^3} W_{\ell_0}(x', y', z') = 0 \] (G.1)

\[ \sqrt{2} \int_0^{\infty} \frac{dx'}{\sqrt{\pi}} \int \int_{-\infty}^{\infty} \frac{dy' dz'}{\pi} W_{\ell_0}(x', y', z') = \frac{1}{2} \left[ \text{erf}(x) - \text{erf}(x + z) \right] \] (G.2)
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\[
\sqrt{2} \int_0^\infty \frac{dy'}{\sqrt{\pi}} \int_0^\infty \frac{dx' dz'}{\pi} \tilde{W}_0(x', y', z') = \frac{1}{2} \left[ \text{erf}(y) - \text{erf}(y + z) \right]
\] (G.3)

\[
\sqrt{2} \int \int_{-\infty}^\infty \frac{dx'}{\pi} \frac{dy'}{\pi} \int_{-\infty}^\infty \frac{dz'}{\pi} \tilde{W}_0(x', y', z') = \text{erf}(z)
\] (G.4)

\[
4\sqrt{2} \int_0^\infty \frac{dy'}{\sqrt{\pi}} \int_0^\infty \frac{dx' dz'}{\pi} \tilde{W}_0(x', y', z') = \text{erf}(x) - \text{erf}(x + z) + \text{erf}(y) - \text{erf}(y + z) + \text{erf}(x)\text{erf}(y) - \text{erf}(x + z)\text{erf}(y + z)
\] (G.5)

\[
4\sqrt{2} \int_{-\infty}^\infty \frac{dy'}{\sqrt{\pi}} \int_0^\infty \frac{dx' dz'}{\pi} \tilde{W}_0(x + x' + y' - z', y - y', z + z')
= \text{erfc}(z)\text{erfc}(x + y + z)
\] (G.6)

\[
2\sqrt{2} \int_{-\infty}^\infty \frac{dy'}{\sqrt{\pi}} \int_0^\infty \frac{dx' dz'}{\pi} \tilde{W}_0(-x' - y' + z', y', -z') = -\text{erf}(z)\text{erfc}(x + y + z).
\] (G.7)

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