HAMiltonians for the zeros of a general family of zeta functions

Su Hu and Min-Soo Kim

Abstract. Towards the Hilbert-Pólya conjecture, in this paper, we present a general construction of Hamiltonian $\hat{H}_f$, which leads a general family of Hurwitz zeta functions $(-1)^{n-1}L(f, z_n, x - 1)$ defined by Mellin transform becomes their eigenstates under a suitable boundary condition, and the eigenvalues $E_n$ have the property that $z_n = \frac{1}{2}(1 - iE_n)$ are the zeros of a general family of zeta functions $L(f, z)$.

1. Introduction

The Riemann zeta function $\zeta(z)$ is defined as

\begin{equation}
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \text{Re}(z) > 1
\end{equation}

and it has the integral representation

\begin{equation}
\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt
\end{equation}

([1, Theorem 12.2]).

It is well-known that $\zeta(z)$ can be analytically continued to the whole complex plane except for a single pole at $z = 1$ with residue 1 and $\zeta(z)$ satisfies the following functional equation

\begin{equation}
\zeta(z) = 2(2\pi)^{1-z}\Gamma(1 - z) \sin\left(\frac{\pi z}{2}\right) \zeta(1 - z)
\end{equation}

([1, Theorem 12.7]). From this, we see that $\zeta(z)$ is symmetric with the axis $\text{Re}(z) = \frac{1}{2}$.

By (1.3), from the vanishing of the function $\sin\left(\frac{\pi z}{2}\right)$, the negative even integers are also zeros of $\zeta(z)$, which are called the trivial zeros. In 1859, Riemann [5] conjectured that the nontrivial zeros of $\zeta(z)$ are all lie on the line $\text{Re}(z) = \frac{1}{2}$. This becomes one of most remarkable conjectures in mathematics, known as the Riemann hypothesis. To the purpose of solving Riemann hypothesis, Hilbert-Pólya [6] conjectured that the imaginary parts of the zeros of $\zeta(z)$ might correspond to the eigenvalues of a Hermitian, self-adjoint operator. Berry and Keating [2] conjectured that the classical counterpart of such a Hamiltonian may have the form $\hat{H} = \hat{x}\hat{p}$, but a Hamiltonian possessing this property has not yet been found.
Let \( \hat{p} = -i \partial_x \) be the momentum operator. Recently, Bender, Brody and Müller [3] constructed a non-Hermitian Hamiltonian
\[
\hat{H} = \frac{1}{1 - e^{-i\hat{p}}}(\hat{x} \hat{p} + \hat{p} \hat{x})(1 - e^{-i\hat{p}}),
\]
which satisfies the conditions of the Hilbert-Pólya conjecture, that is, if the eigenfunctions of \( \hat{H} \) satisfy the boundary condition \( \psi_n(0) = 0 \) for all \( n \), then the eigenvalues \( E_n \) have the property that \( \frac{1}{2}(1 - iE_n) \) are the nontrivial zeros of the Riemann zeta function, and the Hurwitz zeta functions \( \psi_n(x) = -\zeta(z_n, x + 1) \) are the corresponding eigenstates. They also constructed the metric operator to define an inner-product space, on which the Hamiltonian is Hermitian. They remarked that if the analysis may be made rigorous to show that \( \hat{H} \) is manifestly self-adjoint, then this implies that the Riemann hypothesis holds true.

In this paper, we present a general construction of Hamiltonian \( \hat{H}_f \), which leads a general family of Hurwitz zeta functions \((-1)^{z_n - 1}L(f, z_n, x - 1)\) becomes their eigenstates under a suitable boundary condition, and the eigenvalues \( E_n \) have the property that \( z_n = \frac{1}{2}(1 - iE_n) \) are the zeros of a general family of zeta functions \( L(f, z) \). The definitions of \( L(f, z, x) \) and \( L(f, z) \) will be given in the next section.

2. Mellin transform and zeta functions

Recall that a function \( f \) tends rapidly to 0 at infinity if for any \( k \geq 0 \), as \( t \to +\infty \) the functions \( t^k f(t) \) tends to 0. Let \( tf(-t) \) be a \( C^\infty \) function on \([0, \infty)\) tending rapidly to 0 at infinity and for \( \text{Re}(z) > 1 \), we define a general family of zeta functions \( L(f, z, x) \) from the following integral representation
\[
L(f, z) = \frac{1}{\Gamma(z)} \int_0^\infty t^z f(-t) \frac{dt}{t}.
\]

As pointed out by Cohen [4, Proposition 10.2.2 (2)], if \( f(-t) \) itself is a \( C^\infty \) function on \([0, \infty)\) and it tends rapidly to 0 at infinity, then we have \( L(f, z) \) can be analytically continued to the whole of \( \mathbb{C} \) into a holomorphic function.

The Hurwitz zeta function \( \zeta(s, x) \) is defined by the series
\[
\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^z}, \quad \text{Re}(z) > 1
\]
and it has the integral representation
\[
\zeta(s, x) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-(x-1)t} t^{z-1} \frac{dt}{e^t - 1}.
\]

We also define a general family of Hurwitz zeta function \( L(f, z, x) \) as the following integral representation
\[
L(f, z, x) = \frac{1}{\Gamma(z)} \int_0^\infty t^z e^{-(x-1)t} f(-t) \frac{dt}{t}, \quad \text{Re}(z) > 1.
\]
It is to see that \( L(f, z, 1) = L(f, z) \), and letting \( f(t) = \frac{1}{e^{-t} - 1} \) in (2.1) and (2.4), we recover the Riemann zeta function \( \zeta(z) \) and the Hurwitz zeta function \( \zeta(z, x) \), respectively.

Assume the function \( tf(t) \) is analytic on an open disc of radius \( r \) centered at 0 in the complex plane \( \mathbb{C} \), we have the following result.

**Proposition 2.1.** Let \( C \) be a loop around the negative real axis. If \( 0 < x \leq 1 \) the function defined by the contour integral

\[
I(f, z, x) = \frac{1}{2\pi i} \int_{C} t^{z-1} e^{(1-x)t} f(t) \frac{dt}{t}
\]

is an entire function of \( z \). Moreover, we have

\[
L(f, z, x) = \Gamma(1 - z) I(f, z, x), \quad \text{Re}(z) > 1.
\]

**Remark 2.2.** If \( \text{Re}(z) \leq 1 \), we define \( L(f, z, x) \) by the equation

\[
L(f, z, x) = \Gamma(1 - z) I(f, z, x).
\]

This equation provides the analytic continuation of \( L(f, z, x) \) in the entire complex plane.

**Proof.** The proof follows from similar arguments as [1, Theorem 12.3].

We regard the interval \([0, \infty)\) of integration as a path of a complex integral, and then expanding this a little. Here we consider the following contour \( C \). For \( \epsilon > 0 \), we define \( C \) by a curve \( \varphi : (-\infty, \infty) \to \mathbb{C} \) given by

\[
C : \quad \varphi(u) = \begin{cases} 
  u & u < -\varepsilon, \\
  \varepsilon \exp \left( \pi i \frac{u+\varepsilon}{\varepsilon} \right) & -\varepsilon \leq u \leq \varepsilon, \\
  -u & u > \varepsilon.
\end{cases}
\]

In the definition of \( C \), the parts for \( u < -\varepsilon \) and \( u > \varepsilon \) overlap, but we interpret it that for \( u < -\varepsilon \) we take the path above the real axis and for \( u > \varepsilon \) below the real axis. This path is illustrated in Figure 1.

![Figure 1. Path of C](image)

Now consider the complex contour integral

\[
\int_{C} t^{z-1} e^{(1-x)t} f(t) \frac{dt}{t}.
\]

Since we have to treat \( t^{z-1} \) on \( C \), we shall choose a complex power \( t^z \) for \( z \in \mathbb{C} \). Denoting the argument of \( t \) by \( \arg t \), we can define a single-valued function \( \log t \) on \( \mathbb{C} \setminus \{ s = a + ib \mid b = 0, a \leq 0 \} \) by

\[
\log t = \log |t| + i \arg t \quad (-\pi < \arg t < \pi).
\]
Using this, a single-valued analytic function $t^z$ on $\mathbb{C} \setminus \{s = a + ib \mid b = 0, a \leq 0\}$ is defined by

$$t^z = e^{z \log t} = e^{z(\log |t| + i \arg t)},$$

where $-\pi < \arg t < \pi$. We divide the contour $C$ into three pieces as follows.

- $C_1$: the part of the real axis from $-\infty$ to $-\varepsilon$,
- $C_2$: the positively oriented circle of radius $\varepsilon$ with center at the origin,
- $C_3$: the part of the real axis from $-\varepsilon$ to $-\infty$.

We have $C = C_1 + C_2 + C_3$. For $t$ on $C_1$, we put $\arg t = -\pi$, and for $t$ on $C_3$, we put $\arg t = \pi$. Then the integral on $C_1$ is given by

$$\int_{C_1} t^z e^{(1-x)t} f(t) \frac{dt}{t} = \int_{\infty}^{\varepsilon} t^{z-1} e^{-\pi iz} e^{-(1-x)t} f(-t) dt$$

and on $C_3$ by

$$\int_{C_3} t^z e^{(1-x)t} f(t) \frac{dt}{t} = \int_{\varepsilon}^{\infty} t^{z-1} e^{\pi iz} e^{-(1-x)t} f(-t) dt$$

Thus, together we get

$$\int_C t^z e^{(1-x)t} f(t) \frac{dt}{t} = (e^{\pi iz} - e^{-\pi iz}) \int_{\varepsilon}^{\infty} t^{z-1} e^{-(1-x)t} f(-t) dt + \int_{C_2} t^z e^{(1-x)t} f(t) \frac{dt}{t}.$$

The circle $C_2$ is parameterized as $t = \varepsilon e^{i\theta} (-\pi \leq \theta \leq \pi)$, so on $C_2$ we have $t^{z-1} = \varepsilon^{z-1} e^{i(z-1)\theta}$, and the absolute value of the integral is estimated from above as

$$\left| \int_{C_2} t^z e^{(1-x)t} f(t) \frac{dt}{t} \right| \leq \int_{-\pi}^{\pi} \left| \varepsilon^{z-1} e^{i(z-1)\theta} e^{(1-x)\varepsilon e^{i\theta}} f(\varepsilon e^{i\theta}) \varepsilon e^{i\theta} \right| d\theta$$

$$\leq \varepsilon^{|\text{Re}(z)|} \int_{-\pi}^{\pi} \left| e^{i(z-1)\theta} e^{(1-x)\varepsilon e^{i\theta}} f(\varepsilon e^{i\theta}) \right| d\theta.$$

Since $tf(t)$ is analytic on an open disc of radius $r$ centered at 0, we have $\varepsilon f(\varepsilon e^{i\theta})$ is bounded as a function of $\varepsilon$ and $\theta$ if $\varepsilon \leq r/2$ and $-\pi \leq \theta \leq \pi$. So if we take the limit $\varepsilon \to 0$, then, for $\text{Re}(z) > 1$, by (2.10) we get

$$\lim_{\varepsilon \to 0} \int_{C_2} t^z e^{(1-x)t} f(t) \frac{dt}{t} = 0.$$

Therefore, using (2.11), we get

$$\int_C t^z e^{(1-x)t} f(t) \frac{dt}{t} = (e^{\pi iz} - e^{-\pi iz}) \int_{0}^{\infty} t^{z-1} e^{-(1-x)t} f(-t) dt.$$

Let

$$I(f, z, x) = \frac{1}{2\pi i} \int_C t^z e^{(1-x)t} f(t) \frac{dt}{t}.$$
By (2.4) and (2.12), we have

\[ I(f, z, x) = \frac{1}{2\pi i} \int_C t^z e^{(1-x)t} f(t) \frac{dt}{t} \]

\[ = \frac{e^{\pi iz} - e^{-\pi iz}}{2\pi i} \int_0^\infty t^{z-1} e^{-(1-x)t} f(-t) dt \]

\[ = \frac{\sin \pi z}{\pi} \frac{\Gamma(z)}{\Gamma(1-z)} L(f, z, x) \]

which is the desired result.

\[ \square \]

3. Main results

Let \( H \) be a separable Hilbert space. Assume the following Taylor expansion at \( t = 0 \)

\[ (-t)f(t) = \sum_{n=0}^\infty a_n t^n \]

with \( a_0 \neq 0 \). Define an operator on \( H \) by

\[ \hat{\Delta} f \Psi(x) = \frac{1}{f(-i\hat{p})} \Psi(x), \]

for any \( \Psi(x) \in H \), then by (3.1) we have

\[ \hat{\Delta}^{-1} f \Psi(x) = f(-i\hat{p}) \Psi(x) = \frac{1}{i\hat{p}} \sum_{n=0}^\infty a_n (-i\hat{p})^n \Psi(x) \]

Interpreting \((i\hat{p})^{-1}\) as an integral operator with a boundary at infinity

\[ \frac{1}{i\hat{p}} g(x) = \int_x^\infty g(t) dt, \]

we have

\[ \hat{\Delta}^{-1} f^{-z} = \sum_{n=0}^\infty a_n (-i\hat{p})^n x^{1-z} \]

\[ = \frac{1}{1-z} \sum_{n=0}^\infty a_n (-i\hat{p})^n x^{1-z}. \]

Since \( i\hat{p} = \partial_x \) and \( \partial_x^n x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)} x^{\mu-n} \), from (3.4), we have

\[ \hat{\Delta}^{-1} f^{-z} = \frac{\Gamma(2-z)}{1-z} \sum_{n=0}^\infty a_n (-1)^n \frac{x^{1-z-n}}{\Gamma(2-z-n)}. \]

Since \( \Gamma(2-z-n) \) has the following integral representation

\[ \frac{1}{\Gamma(2-z-n)} = \frac{1}{2\pi i} \int_C e^u u^{n+z-2} du, \]
where \( C \) denotes a Hankel contour that encircles the negative-\( u \) axis in the positive orientation. From (3.5), we have

\[
\hat{\Delta}^{-1} x^{-z} = \frac{\Gamma(1 - z)}{2\pi i} x^{1-z} \int_{C} e^{u x^{-z}} du \sum_{n=0}^{\infty} a_{n} \left( \frac{-u}{x} \right)^{n}
\]

(3.6)

Letting \( u = t \) in (3.6), we have

\[
\hat{\Delta}^{-1} x^{-z} = \frac{\Gamma(1 - z)}{2\pi i} \int_{C} t^{z} e^{xt} f(-t) \frac{dt}{t}
\]

(3.7)

By (2.6) and (2.7),

\[
\hat{\Delta}^{-1} x^{-z} = (-1)^{z-1} L(f, z, x - 1).
\]

(3.8)

Let \( \hat{H}_{f} = \hat{\Delta}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta} \). Denote by \( \Psi(f, z, x) = (-1)^{z-1} L(f, z, x - 1) \), by (3.8), we have

\[
\hat{H}_{f} \Psi(f, z, x) = \hat{\Delta}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta} \hat{\Delta}^{-1} x^{-z}
\]

(3.9)

\[
= \hat{\Delta}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})x^{-z}
\]

\[
= i(2z - 1) \Psi(f, z, x).
\]

Thus if we propose the boundary condition \( \Psi(f, z, 2) = 0 \) to the differential equation (3.9), then we have the \( n \)th eigenstates of the Hamiltonian \( \hat{H}_{f} \) are \( \Psi(f, z_{n}, x) = (-1)^{z_{n}-1} L(f, z_{n}, x - 1) \), the \( n \)th eigenvalues are \( E_{n} = i(2z_{n}-1) \), and from the boundary condition, \( z_{n} = \frac{1}{2}(1-iE_{n}) \) are the zeros of the general family zeta functions \( L(f, z) \) (since \( L(f, z, 1) = L(f, z) \)).

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Department of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China
E-mail address: mahusu@scut.edu.cn

Division of Mathematics, Science, and Computers, Kyungnam University, 7(Woryeong-dong) kyungnamdaehak-ro, Masanhappo-gu, Changwon-si, Gyeongsangnam-do 51767, Republic of Korea
E-mail address: mskim@kyungnam.ac.kr