COUPLING AND HARNACK INEQUALITIES
FOR SIERPINSKI CARPETS

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Abstract. Uniform Harnack inequalities for harmonic functions on the pre- and
graphical Sierpinski carpets are proved using a probabilistic coupling argument. Var-
ious results follow from this, including the construction of Brownian motion on Sier-
pinski carpets embedded in \( \mathbb{R}^d, d \geq 3 \), estimates on the fundamental solution of the
heat equation, and Sobolev and Poincaré inequalities.

The Sierpinski carpets (SCs) we will study are generalizations of the Cantor set. Let
\( F_0 = [0, 1]^d \) be the unit cube in \( \mathbb{R}^d, d \geq 2 \), centered at \( z_0 = (1/2, \ldots, 1/2) \).
Let \( k, a \) be integers with \( 1 \leq a < k \) and \( a + k \) even. Divide \( F_0 \) into \( k^d \) equal
subcubes, remove a central block of \( a^d \) subcubes, and let \( F_1 \) be what remains:
thus \( F_1 = F_0 - ((k - a)/2k, (k + a)/2k)^d \). Now repeat this operation on each of
the \( k^d - a^d \) remaining subcubes to obtain \( F_2 \). Iterating, we obtain a decreasing
sequence of closed sets \( F_n \); then \( \bigcap_{n=0}^{\infty} F_n \) is a Sierpinski carpet and has
Hausdorff dimension \( d_f = d_f(F) = \log(k^d - a^d)/\log(k) \). (When \( d = 2, k = 3 \), and
\( a = 1 \), we get the usual Sierpinski carpet.) Let \( \hat{F}_n = k^n F_n \subset [0, \infty)^d \), and define
the pre-Sierpinski carpet by \( \hat{F} = \bigcup_{n=1}^{\infty} \hat{F}_n \) (see [10]). The graphical Sierpinski
carpet is the graph \( G = (V, E) \) with vertex set \( V = (z_0 + \mathbb{Z}^d) \cap \hat{F} \) and edge set
\( E = \{ \{x, y\} \in V : |x - y| = 1 \} \).

Thus \( \text{int}(\hat{F}) \) is a domain in \( \mathbb{R}^d \) with a large-scale structure which mimics the
small-scale structure of \( F \). We are interested in the behavior of solutions of the
Laplace and heat equations on \( F, \hat{F}, \) and \( G \). One reason for this is applications to
“transport phenomena” in disordered media (see [6]); another is the new type of
behavior of the heat kernel on these spaces. Let \( W \) be Brownian motion on \( \hat{F} \)
with normal reflection on \( \partial \hat{F} \), and let \( q(t, x, y) \) be the transition density of \( W \), so that \( q \)
solves the heat equation on \( \hat{F} \) with Neumann boundary conditions on \( \partial \hat{F} \).
Theorem 1. There exist $c_1, \ldots, c_6 \in (0, \infty)$ and $d_s = d_s(F) \in (1, d_f)$ such that if $x, y \in \hat{F}$, $t \in (1, \infty)$, $|x - y| \leq t$, then

$$c_1 t^{-d_s/2} \exp \left( -c_2 \frac{|x - y|^{d_w}}{t} \right)^{1/(d_w-1)} \leq q(t, x, y) \leq c_3 t^{-d_s/2} \exp \left( -c_4 \frac{|x - y|^{d_w}}{t} \right)^{1/(d_w-1)},$$

(1)

where $d_w = 2d_f/d_s$; while if $x, y \in \hat{F}$, $t \in (1, \infty)$, $|x - y| > t$, then

$$\exp \left( -c_5 \frac{|x - y|^2}{t} \right) \leq q(t, x, y) \leq \exp \left( -c_6 \frac{|x - y|^2}{t} \right).$$

(2)

The index $d_s$ is called the spectral dimension of $F$ and turns out to be much more significant than the Hausdorff dimension $d_f$ as far as analytic properties of these spaces are concerned. Since $d_s < d$, (1) confirms the physical intuition that the presence of increasingly large reflecting barriers causes heat to dissipate to infinity more slowly. It seems unlikely that there is any simple relationship between $d_s, k, a$, and $d$.

While there is a well-developed approach to the heat equation using analytic tools such as Sobolev or log-Sobolev inequalities (see [7]), these methods do not appear to give the best-possible results on spaces such as $\hat{F}$—compare the upper bound on $q(t, x, y)$ given in Theorem 1 with the results of [10].

The proof of Theorem 1 rests on the following Harnack inequality. Let $D \subset \mathbb{R}^d$ be open: we will say that $h$ is harmonic on $D \cap \hat{F}$ if (i) $\Delta h = 0$ in $\text{int}(D \cap \hat{F})$ and (ii) $h$ has 0 normal derivative a.e. on $D \cap \partial \hat{F}$. Equivalently, $h$ is harmonic with respect to $W(t \wedge T_D)$, where $T_D = \inf\{t : W(t) \notin D\}$. Let $D_n = (-1, k^n)^d$.

Theorem 2. There exists $c_1 \in (0, \infty)$ (depending only on $d, k, a$), such that if $h$ is positive harmonic in $D_n \cap \hat{F}$ and $x, y \in D_{n-1} \cap \hat{F}$, then $h(x)/h(y) \leq c_1$.

Remarks. 1. Note that $c_1$ is independent of $n$; otherwise the result is trivial.

2. The case $d = 2$ was proved in [1]; the proof there relies on the fact that a closed curve in the plane separates the plane into two pieces. Just as in the case of elliptic operators, the results for two dimensions are considerably easier to prove. The result of [1] was extended in [8] to SCs with $d_s(F) < 2$.

3. Using the symmetry of $\hat{F}$, Theorem 2 extends to other domains in $\hat{F}$.

4. Theorems 1 and 2 actually hold for a much wider class of SCs, those satisfying a higher-dimensional generalization of (2.1) of [4].

5. A similar result holds for the graphical Sierpinski carpet $G$.

6. Most existing proofs of Harnack inequalities for selfadjoint operators depend on Sobolev inequalities, which in turn depend on the underlying geometry of the space. Here the appropriate Sobolev inequality involves the spectral dimension $d_s(F)$; however, no geometric definition of $d_s$ is known. Thus we were led to abandon analytic approaches in favor of the probabilistic coupling argument described at the end of this paper.

We now describe some other consequences of Theorem 2. Let $\hat{F} = \bigcup_{n=0}^\infty k^n F$, the SC extended to $[0, \infty)^d$, and write $\mu$ for Hausdorff $x^{d_f}$-measure on $\hat{F}$.
**Theorem 3.** There exists a strong Markov process $X_t$ with state space $\tilde{F}$ such that $X$ has a strong Feller transition semigroup $P_t$ which is $\mu$-symmetric, $X_t$ has continuous paths, $X_t$ is self-similar with respect to dilations of size $k^n$, and the process $X$ is locally invariant with respect to the local isometries of $\tilde{F}$.

Let $p(t, x, y)$ be the transition density of $X_t$ with respect to $\mu$. Then $p(t, x, y)$ is the fundamental solution to the heat equation on $\tilde{F}$: $\partial u/\partial t = \Delta_{\tilde{F}}u$, where $\Delta_{\tilde{F}}$ is the infinitesimal generator of $X_t$. Then we have

**Theorem 4.** There exist $c_1, c_2, c_3, c_4 \in (0, \infty)$ and $d_s = d_s(F) \in (1, d_f)$ such that for all $x, y \in \tilde{F}$, $t \in (0, \infty)$,

$$c_1 t^{-d_s/2} \exp \left( -c_2 \left( \frac{|x-y|^{d_w}}{t} \right)^{1/(d_w - 1)} \right) \leq p(t, x, y) \leq c_3 t^{-d_s/2} \exp \left( -c_4 \left( \frac{|x-y|^{d_w}}{t} \right)^{1/(d_w - 1)} \right).$$

where $d_w = 2d_f/d_s$. Moreover, $p(t, x, y)$ is $C^\infty$ in $t$, and $p(t, x, y)$ and all its partial derivatives with respect to $t$ are jointly Hölder continuous in $x$ and $y$.

Many properties of the process $X$, such as its transience or recurrence, the existence of local times, the existence of self-intersections, and the asymptotic frequency of eigenvalues follow easily from Theorem 4. For example, note that $X$ is point recurrent if and only if $d_s(F) < 2$.

The next set of consequences include Sobolev inequalities, Poincaré inequalities, and electrical resistance inequalities for $\tilde{F}$, $\tilde{F}$, and $G$—nine theorems in total. Since the electrical resistance inequalities are probably the least well-known type, we give the one for $G$ as a representative sample. If $B$ is any subset of $G$, let $|B|$ denote the cardinality of $B$. Then $R(B)$, the resistance from $B$ to infinity, is defined by

$$R(B)^{-1} = \inf \left\{ \sum_{(x, y) \in E(G)} (f(x) - f(y))^2 : f \equiv 1 \text{ on } B, f(x) \to 0 \text{ as } |x| \to \infty \right\}.$$ 

The inverse of $R(B)$ is the conductance from $B$ to infinity and equals the capacity of $B$.

**Theorem 5.** Suppose $d_s = d_s(F) > 2$, and let $\zeta = d_s/(d_s - 2)$. Then there exists $c_1$ such that if $A \subset G$, $|A| \leq c_1 R(A)^{-\zeta}$.

Theorem 5 follows fairly straightforwardly from Theorem 4 by applying ideas of [11] and [12]. As Theorems 1, 3, and 4 follow from Theorem 2 by generalizations and modifications of methods of [1–4, 8, 9], we discuss only Theorem 2.

Let $W_t$ be the Brownian motion on $\tilde{F}$ described above, and let

$$\tau(x, r) = \inf\{t : |W_t - x| \geq r\}, \quad T(x, r) = \inf\{t : |W_t - x| \leq r\}.$$ 

The following lemma is proved in a similar fashion to Lemma 3.2 of [1].

**Lemma 6.** There exist $c_2 > c_1 > 1$, $\delta > 0$ independent of $r$ such that if $x, y \in \tilde{F}$ and $|y - x| \leq c_1 r$, then

$$\mathbb{P}^y(T(x, r) < \tau(x, c_2 r)) > \delta.$$ 

It is known (see Theorem 3.9 of [5], for example) that the Harnack inequality Theorem 2 follows from (3) and an oscillation inequality of the following form.
Lemma 7. There exists $\rho < 1$ such that if $n \geq 1$ and $h$ is positive harmonic on $D_n \cap \tilde{F}$, then
\begin{equation}
|h(x) - h(y)| \leq \rho \sup_{z \in D_n \cap \tilde{F}} |h(z)|, \quad x, y \in D_{n-1} \cap \tilde{F}.
\end{equation}

To show (4), it suffices to construct two $\tilde{F}$-valued Brownian motions $W^x$ and $W^y$, starting from $x$ and $y$ respectively, which couple (i.e., meet) with probability at least $1 - \rho$ before either exits $D_n$.

Fix $n$. Let $S_m$ be the collection of cubes of side length $k^m$ with vertices in $k^m\mathbb{Z}^d$. Say that $x, y \in \tilde{F}$ are $m$-associated if there is an isometry of the cube in $S_m$ containing $x$ onto the cube in $S_m$ containing $y$ that maps $x$ onto $y$. Note that if two points are $m$-associated, then they will also be $\ell$-associated for all $\ell \leq m$.

Suppose first that $x$ and $y$ are $m$-associated. We start a Brownian motion $W^x(t)$ on $\tilde{F}$ at $x$. Let $U_0 = 0$, and $U_{i+1} = \inf \{ t : |W^x(t) - W^x(U_i)| \geq k^m \}$. The key step is to exploit the local symmetry of $\tilde{F}$ to construct, using suitable reflections, another Brownian motion $W^y(t)$ on $\tilde{F}$, starting at $y$, such that (a) $W^x(t)$ and $W^y(t)$ are $m$-associated for all $t \geq 0$, and (b) there exist $j$ and $c_1 > 0$ such that
\begin{equation}
\mathbb{P} \left( W^x(U_j(\omega)) \text{ and } W^y(U_j(\omega)) \text{ are } (m+1)\text{-associated} \right) > c_1.
\end{equation}

A renewal argument and then an induction show that if $x$ and $y$ are $0$-associated, then $W^x$ and $W^y$ couple with probability $c_2 > 0$ before either process leaves $D_n$. Lemma 6 then follows easily.

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