ON ROOT-CLASS RESIDUALITY OF
GENERALIZED FREE PRODUCTS

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Abstract. Root-class residuality of free product of root-class residual groups is demonstrated. A sufficient condition for root-class residuality of generalized free product $G$ of groups $A$ and $B$ amalgamating subgroups $H$ and $K$ through the isomorphism $\varphi$ is derived. For the particular case when $A = B$, $H = K$ and $\varphi$ is the identity mapping, it is shown that group $G$ is root-class residual if and only if $A$ is root-class residual and subgroup $H$ of $A$ is root-class closed. These results are extended to generalized free product of infinitely many groups amalgamating a common subgroup.

1. Introduction

Let $\mathcal{K}$ be an abstract class of groups containing at least one non-identity group. Then $\mathcal{K}$ is called a root-class if the following conditions are satisfied:

1. If $A \in \mathcal{K}$ and $B \leq A$, then $B \in \mathcal{K}$.
2. If $A \in \mathcal{K}$ and $B \in \mathcal{K}$, then $A \times B \in \mathcal{K}$.
3. If $1 \leq C \leq B \leq A$ is a subnormal sequence and $A/B$, $B/C \in \mathcal{K}$, then there exists a normal subgroup $D$ in group $A$ such that $D \leq C$ and $A/D \in \mathcal{K}$. See for example [6], p. 428 for details about this definition.

In this paper, we study root-class residuality of generalized free products.

We recall that a group $G$ is root-class residual (or $\mathcal{K}$-residual, for a root-class $\mathcal{K}$) if, for every non-identity element $g \in G$, there exists an homomorphism $\varphi$ from $G$ to some group $G'$ of root-class $\mathcal{K}$ such that $g\varphi \neq 1$. Equivalently, $G$ is $\mathcal{K}$-residual if, for every non-identity element $g \in G$, there exists a normal subgroup $N$ of $G$ such that $G/N \in \mathcal{K}$ and $g \notin N$. The most investigated residual properties of groups are residual finiteness and finite $p$-groups residuality, (i.e. residuality by the classes of all finite groups and all finite $p$-groups respectively), and also residuality by the class of all soluble groups. All these three classes are root-classes. Therefore results about root-class residuality have sufficiently enough general character.

In [6] (p. 429) the following result obtained by Gruenberg is given:

Free product of root-class residual groups is root-class residual if and only if every free group is root-class residual.

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The following theorem shown in item 2 asserts that the given above necessary and sufficient condition is satisfied for any root-class:

**Theorem 1.** Every free group is root-class residual.

So, Gruenberg’s result can be reformulated as follows:

**Theorem 2.** Free product of root-class residual groups is root-class residual.

From theorem 2 and H. Neumann’s theorem ([5], p. 122), the following result is easily established:

**Theorem 3.** The generalized free product $G$ of groups $A$ and $B$ amalgamating subgroup $H$ is root-class residual if groups $A$ and $B$ are root-class residual and there exists an homomorphism $\varphi$ from $G$ to a group $G'$ of a root-class such that $\varphi$ is injective on $H$.

Let’s remark that theorem 2 can be considered as a particular case of theorem 3.

We also see that, if the amalgamated subgroup $H$ is finite, then the formulated above sufficient condition of root-class residuality of group $G$ will be as well necessary.

Another restriction permitting to obtain simple criteria of root-class residuality of generalized free product of groups $A$ and $B$ amalgamating subgroup $H$ is the equality of the free factors $A$ and $B$.

More precisely, let $G$ be the generalized free product of groups $A$ and $B$ amalgamating subgroups $H$ and $K$ through the isomorphism $\varphi$. If $A = B$, $H = K$ and $\varphi$ is the identity map, we denote group $G$ by $Q = A \star^H A$. We call $Q$ the generalized free square of group $A$ over subgroup $H$. Then for such group $Q$ we prove the following criterium:

**Theorem 4.** Group $Q = A \star^H A$ is root-class residual if and only if group $A$ is root-class residual and subgroup $H$ of $A$ is root-class closed.

In [3] the above result is obtained for the particular case of the class of all finite $p$-groups. We recall that subgroup $H$ of a group $A$ is root-class closed (or $K$-closed, for a root-class $K$) if, for any element $a$ of $A$ and $a \notin H$, there exists an homomorphism $\varphi$ from $A$ to a group of root-class $K$ such that $a\varphi \notin H\varphi$. This means that, for each $a \in A \setminus H$, there exists a normal subgroup $N$ of $A$ such that $A/N \in K$ and $a \notin NH$. A $p$-closeness analogue of this definition is given in [4].

Further, the generalized free product of infinitely many groups amalgamating subgroup is introduced in [7]. Some results on residual properties of this construction are shown in [1]. We extend theorems 3 and 4 above to generalized free products of every family $(G_\lambda)_{\lambda \in \Lambda}$ of groups $G_\lambda$ amalgamating a common subgroup $H$ (theorems 5 and 6). The set $\Lambda$ can be infinite. Theorems 5 and 6 generalize some of the results obtained in [1].
2. Proofs of theorems 1-4

Let $\mathcal{K}$ be a root-class of groups.

**Lemma.** Then
1. If a group $G$ has a subnormal sequence with factors belonging to class $\mathcal{K}$, then $G \in \mathcal{K}$.
2. If $F \trianglelefteq G$, $G/F \in \mathcal{K}$ and $F$ is $\mathcal{K}$-residual, then group $G$ is also $\mathcal{K}$-residual.
3. If $A \trianglelefteq G$, $B \trianglelefteq G$, $G/A \in \mathcal{K}$ and $G/B \in \mathcal{K}$, then $G/(A \cap B) \in \mathcal{K}$.

In fact, from the definition of root-class, it follows that root-class is closed for extensions. So the first property of lemma is satisfied. The second and third properties are also easily verified by the definition of root-class.

For the proof of theorem 1, let’s remark that every root-class $\mathcal{K}$ contains a non-identity cyclic group (property 1 of the definition of root-class). If $\mathcal{K}$ contains an infinite cyclic group then, by lemma, $\mathcal{K}$ contains any group possessing subnormal sequence with infinite cyclic factors; and thus all finitely generated nilpotent torsion-free groups belong to class $\mathcal{K}$. If $\mathcal{K}$ contains a finite non-identity cyclic group, then $\mathcal{K}$ contains group of prime order $p$ and consequently, by lemma, $\mathcal{K}$ contains all groups possessing subnormal sequence with factors of order $p$; hence all finite $p$-groups belong to $\mathcal{K}$. So any root-class contains all finitely generated nilpotent torsion-free groups or all finite $p$-groups for some prime $p$. Therefore, to end the proof of theorem 1, let’s remind that free groups are residually finitely generated nilpotent torsion-free groups and also residually finite $p$-groups (see [2], p. 121 and [6], p. 347).

From the proof of theorem 1 and the Grunberg’s theorem formulated above theorem 2 directly follows.

Let’s prove theorem 3.

Let $G$ be the generalized free product of groups $A$ and $B$ amalgamating subgroup $H$ and let groups $A$ and $B$ be $\mathcal{K}$-residual. Suppose there exists an homomorphism $\sigma$ of $G$ to a group of class $\mathcal{K}$, which is one-to-one on $H$. Let’s denote by $N$ the kernel of the homomorphism $\sigma$. Then $G/N \in \mathcal{K}$ and $N \cap H = 1$. By H. Neumann’s theorem ([5], p. 122) $N$ is the the free product of a free group $F$ and some subgroups of group $G$ of the form

$$g^{-1}Ag \cap N, \quad g^{-1}Bg \cap N,$$

where $g \in G$. The subgroups of the form (*) are root-class residual since are groups $A$ and $B$. By theorem 1, free group $F$ is also root-class residual. Thus $N$ is a free product of root-class residual groups. Therefore, by theorem 2, $N$ is root-class residual. Moreover, since $G/N \in \mathcal{K}$, by property 2 of lemma, it follows that group $G$ is root-class residual. Theorem 3 is proven.

Let’s now prove theorem 4.
Let $Q = A \ast_H A$. For any normal subgroup $N$ of group $A$ one can define the generalized free square

$$Q_N = A/N \ast_{HN/N} A/N$$

of group $A/N$ over subgroup $HN/N$ and the homomorphism $\varepsilon_N : Q \rightarrow Q_N$, extending the canonical homomorphism $A \rightarrow A/N$. It is evident that group $Q_N$ is an extension of free group with group $A/N$. So, if $A/N$ belongs to root-class $\mathcal{K}$ then, by lemma and theorem 1, $Q_N$ is $\mathcal{K}$-residual. Thus, to end the proof, it is enough to show that $Q$ is residually a group of the form $Q_N$ such that $A/N \in \mathcal{K}$.

Suppose group $A$ is $\mathcal{K}$-residual and subgroup $H$ of $A$ is $\mathcal{K}$-closed. Let $g \in Q$ such that $g \neq 1$. And let $g = a_1 \cdots a_s$ be the irreducible form of element $g$. Two cases arise:

1. $s > 1$. In this case $a_i \in A \setminus H$ for all $i = 1, \ldots, s$. From $\mathcal{K}$-closeness of $H$, it follows that, for every $i = 1, \ldots, s$, there exits a normal subgroup $N_i$ of group $A$ such that $A/N_i \in \mathcal{K}$ and $a_i \notin HN_i$. Let $N = N_1 \cap \cdots \cap N_s$. By lemma, $A/N \in \mathcal{K}$ and, it is clear that, for all $i = 1, \ldots, s$, $a_i \notin HN$ i.e. $a_i N \notin HN/N$. So, for all $i = 1, \ldots, s$, $a_i \notin H\varepsilon_N$. Therefore the form

$$g\varepsilon_N = a_1 \varepsilon_N \cdots a_s \varepsilon_N$$

is irreducible and has length $s > 1$.

Consequently $g\varepsilon_N \neq 1$.

2. $s = 1$ i.e. $g \in A$. As group $A$ is $\mathcal{K}$-residual, there exists a normal subgroup $N$ of $A$ such that $A/N \in \mathcal{K}$ and $g \notin N$, i.e. $gN \neq N$. Hence $g\varepsilon_N \neq 1$.

Thus, in any case, for an element $g \neq 1$ in group $A$, there exists a normal subgroup $N$ such that $A/N \in \mathcal{K}$ and the homomorphism $\varepsilon_N : Q \rightarrow Q_N$ transforms $g$ to a non identity element. Hence group $Q$ is residually a group $Q_N$ where $A/N \in \mathcal{K}$. Therefore $Q$ is $\mathcal{K}$-residual.

Conversely, suppose group $Q$ is $\mathcal{K}$-residual. Evidently, his subgroup $A$ has the same property. Let’s prove that $H$ is a $\mathcal{K}$-closed subgroup of group $A$. Let $\gamma$ be an automorphism of group $Q$ canonically permuting the free factor. Let $a \in A \setminus H$. Then $a\gamma \neq a$. As $Q$ is $\mathcal{K}$-residual, there exists a normal subgroup $N$ of $Q$ such that $Q/N \in \mathcal{K}$ and $aN \neq a\gamma N$. Let $M = N \cap N\gamma$. Then

$$M\gamma = N\gamma \cap N\gamma^2 = N\gamma \cap N = M.$$
3. Generalization

Let \((G_\lambda)_{\lambda \in \Lambda}\) be a family of groups, where the set \(\Lambda\) can be infinite. Let \(H_\lambda \leq G_\lambda\), for every \(\lambda \in \Lambda\). Suppose also that, for every \(\lambda, \mu \in \Lambda\), there exists an isomorphism \(\varphi_{\lambda\mu} : H_\lambda \rightarrow H_\mu\) such that, for all \(\lambda, \mu, \nu \in \Lambda\), the following conditions are satisfied:

\[\varphi_{\lambda\lambda} = \text{id}_{H_\lambda}, \quad \varphi_{\lambda\mu}^{-1} = \varphi_{\mu\lambda}, \quad \varphi_{\lambda\mu}\varphi_{\mu\nu} = \varphi_{\lambda\nu}.\]

Let now \(G = (G_\lambda (\lambda \in \Lambda); \ h\varphi_{\lambda\mu} = h \ (h \in H_\lambda, \ \lambda, \mu \in \Lambda))\) be the group generated by groups \(G_\lambda (\lambda \in \Lambda)\) and defined by all the relators of these groups and moreover by all possible relations: \(h\varphi_{\lambda\mu} = h, \ h \in H_\lambda, \ \lambda, \mu \in \Lambda\). It is evident every \(G_\lambda\) can be canonically embedded in group \(G\) and if we consider \(G_\lambda \leq G\) then, for all different \(\lambda, \mu \in \Lambda\),

\[G_\lambda \cap G_\mu = H_\lambda = H_\mu.\]

Let’s denote by \(H\) the subgroup of group \(G\), equal to the common subgroups \(H_\lambda\). Then \(G\) is the generalized free product of the family \((G_\lambda)_{\lambda \in \Lambda}\) of groups \(G_\lambda (\lambda \in \Lambda)\) amalgamating subgroup \(H\). We will consider, as well, that \(G_\lambda \leq G\), for all \(\lambda \in \Lambda\). See [1] or [7] for details about generalized free product of a family of groups.

**Theorem 5.** The generalized free product \(G\) of the family \((G_\lambda)_{\lambda \in \Lambda}\) of groups \(G_\lambda\) amalgamating subgroup \(H\) is root-class residual if every group \(G_\lambda\) is root-class residual and there exists an homomorphism \(\varphi\) from \(G\) to a group \(G'\) of a root-class such that \(\varphi\) is injective on \(H\).

**Proof.** The proof is the same as that of theorem 3.

In fact, let groups \(G_\lambda\) be \(K\)-residual, for all \(\lambda \in \Lambda\). Suppose there exists an homomorphism \(\sigma\) of \(G\) to a group of class \(K\), which is one-to-one on \(H\) and let \(N = \ker\sigma\). Then \(G/N \in K\) and \(N \cap H = 1\). But \(N\) is the free product of a free group \(F\) and some subgroups of group \(G\) of the form

\[g^{-1}G_\lambda g \cap N,\]

(\(g \in G\) and \(\lambda \in \Lambda\)) which are root-class residual. Since \(F\) is also root-class residual by theorem 1, \(N\) is a free product of root-class residual groups. Thus, by theorem 2, \(N\) is root-class residual. Moreover, since \(G/N \in K\), by property 2 of lemma, it follows that group \(G\) is root-class residual and the theorem is proven.

Suppose now that, for all \(\lambda \in \Lambda\), \(G_\lambda = A\) and denote \(P\), the generalized free power of group \(A\) over subgroup \(H\), by \(A_{H} \ast \cdots \ast A_{H}\). For such group \(P\) we have the following criterium:

**Theorem 6.** Group \(P = A_{H} \ast \cdots \ast A_{H}\) is root-class residual if and only if group \(A\) is root-class residual and subgroup \(H\) of \(A\) is root-class closed.

The proof is similar to that of theorem 4.
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