A new blocks estimator for the extremal index

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ABSTRACT
The occurrence of successive extreme observations can have an impact on society. In extreme value theory there are parameters to evaluate the effect of clustering of high values, such as the extremal index. The estimation of the extremal index is a recurrent theme in the literature and there are several methodologies for this purpose. The majority of existing methods depend on two parameters whose choice affects the performance of the estimators. Here we consider a new estimator depending only on one of the parameters, thus contributing to a decrease in the degree of uncertainty. A simulation study presents motivating results. An application to financial data will also be presented.

ARTICLE HISTORY
Received 22 December 2021
Accepted 2 March 2022

KEYWORDS
Extreme value theory; stationary sequences; tail dependence; extremal index

AMS 2000 SUBJECT CLASSIFICATION
Primary: 60G70;
Secondary: 62G32

1. Introduction
Serial extremal dependence leads to the occurrence of clusters of high values. This is an issue of major concern if associated with damaging phenomena, as for example, heatwaves whose duration in time can cause drought and wildfires. On the other hand, it may indicate a desirable situation, like successive high stock returns attracting possible profits.

The extremal index, often denoted \( \theta \), is a key parameter in assessing the extremal clustering degree. A stationary sequence \( \{X_n\}_n \) is said to have extremal index \( \theta \) if for each \( \tau > 0 \) there is a sequence \( \{u_n\}_n \), such that, as \( n \to \infty \),

\[
nP(X_1 > u_n) \to \tau \quad \text{and} \quad P\left(\bigvee_{i=1}^n X_i \leq u_n\right) \to \exp(-\theta \tau),
\]

where “\( \bigvee \)” stands for the maximum operator and we will also use “\( \bigwedge \)” for the minimum.

The extremal index ranges between 0 and 1, where smaller values mean stronger extremal dependence. Independent sequences have \( \theta = 1 \) and no clustering of extremes takes place. A broad overview about the extremal index and, in particular, its applications in several areas can be seen in Moloney, Faranda, and Sato (2019) and references therein.

One interpretation of \( \theta \) is that it corresponds to the reciprocal limiting mean cluster size (Hsing, Hüsler, and Leadbetter 1988). The blocks and the runs estimators were
developed upon this idea (Smith and Weissman 1994; Weissman and Novak 1998). Both estimators depend on the specification of two unknown parameters: a high threshold above which observations are considered extreme values and a cluster identifier. These are crucial since the methods show sensiveness on their specification. In order to overcome the arbitrariness of these choices, alternative methods were proposed, such as the estimator of Ferro and Segers (2003) which only needs the threshold specification or the estimator of Northrop (2015) solely requiring the choice of the blocks length. Other estimation procedures can also be found in literature, such as the $K$-gaps estimator of Søveges and Davidson (2010) involving the choice of $K$ and of the threshold ($K=0$ leads to the estimator of Ferro and Segers (2003)), estimators holding under a local dependence condition $D^{(c)}$ and thus requiring the indication of $s$ besides the threshold. See Ferreira and Ferreira (2018), Cai (2019), Gomes and Neves (2020) and references therein.

In this work we present a new estimator for the extremal index which only requires a block length parameter. Therefore, it intends to contribute to a decrease in the degree of uncertainty associated to the choice of parameters involving inference on $h$. The direct competitors are Ferro and Segers (2003) and Northrop (2015) estimators, although our methodology based on choosing a block size is closer to the second one. In Section 2 we introduce the new proposal. Section 3 addresses a simulation study in order to evaluate the performance of our estimator. In Section 4 we present an application to a financial time series. Final remarks and future work are pointed in Section 5.

2. Methodology

Let $\{X_n\}_n$ be a stationary sequence with extremal index $\theta$ having, without loss of generality, standard Fréchet marginals, with distribution function (df) $F_X(x) = \exp(-1/x)$, $x > 0$, and $\{\tilde{X}_n\}_n$ an associated iid sequence, i.e., an independent sequence having marginals also standard Fréchet. Consider the bivariate sequence

$$\{(Y_{n,1} = \tilde{X}_n, Y_{n,2} = (1/2)\tilde{X}_n \lor (1/2)X_n)\}_n.$$

We have that

$$\lim_{n \to \infty} P\left( \bigwedge_{i=1}^{n} Y_{i,1} \leq n/\tau_1, \bigwedge_{i=1}^{n} Y_{i,2} \leq n/\tau_2 \right)$$

$$= \lim_{n \to \infty} P\left( \bigwedge_{i=1}^{n} \tilde{X}_i \leq n/(\tau_1 \lor (\tau_2/2)) \right) P\left( \bigwedge_{i=1}^{n} X_i \leq 2n/\tau_2 \right)$$

$$= \exp(-\tau_1 \lor (\tau_2/2)) \exp(-\theta \tau_2/2)$$

$$= (\exp(-\tau_1) \exp(-\theta \tau_2/2)) \land \exp(-(1 + \theta) \tau_2/2),$$

where in the second equality we have applied (1). Thus the limiting bivariate extreme value (BEV) copula is $C(u, v) = uv^{\theta/\kappa} \land v$ (Martins and Ferreira 2014), which has tail dependence coefficient $\lambda_C$, given by

$$\lambda_C = 2 - \lim_{u \to 1} \frac{1 - u^{1+\frac{n}{m}} \land u}{1 - u} = 1 - \frac{\theta}{1 + \theta}. \quad (2)$$
Our estimator is based on relation

\[ \theta = \frac{1}{\lambda_C} - 1 \]  

(3)

derived from (2). Since \( \theta \in [0, 1] \), then we must have \( \lambda_C \in [1/2, 1] \).

Estimators of the tail dependence coefficient of a random pair \((Z_1, Z_2)\) having a BEV df \(G(x_1, x_2) = C_C(G_1(x_1), G_2(x_2))\) are addressed in literature with a threshold-free formulation (see, e.g., Frahm, Junker, and Schmidt 2005; Ferreira 2013 and references there in). The BEV copula \(C_C\) can be stated as 

\[ C_C(G_1(x_1), G_2(x_2)) = \exp\left(-l(-\log x_1, -\log x_2)\right), \]

where \(l\) is the so called stable tail dependence function (Huang 1992).

Consider \((Z_{1,1}, Z_{1,2}, \ldots, Z_{n,1}, Z_{n,2})\) a random sample of \((Z_1, Z_2)\) with BEV copula \(C_C\) and stable tail dependence function \(l\). We are going to use estimator

\[ \tilde{\lambda} = 2 - \tilde{l}(1, 1), \]  

(4)

where 

\[ \tilde{l}(1, 1) = \frac{1}{1 - \frac{1}{n} \sum_{i=1}^{n} (\tilde{G}_1(Z_{i,1}) \lor \tilde{G}_2(Z_{i,2})) - 1} \]  

(5)

and \(\tilde{G}_j(x) = \frac{1}{n+1} \sum_{i=1}^{n} \mathbb{I}(Z_{i,j} \leq x), j = 1, 2,\) is the respective (modified) empirical df. Thus we have

\[ \tilde{\lambda} = 3 - \frac{1}{1 - \frac{1}{n} \sum_{i=1}^{n} (\tilde{G}_1(Z_{i,1}) \lor \tilde{G}_2(Z_{i,2}))}, \]  

(6)

and by (3) we obtain estimator

\[ \tilde{\theta} = \frac{1}{\tilde{\lambda} \lor 1/2} - 1. \]  

(7)

For more details on formulas (4) and (5) see Ferreira and Ferreira (2012a) and references therein. See also Ferreira and Ferreira (2012b, 2018).

The following algorithm describes our estimation proposal of the extremal index of a stationary sequence \(X_1, \ldots, X_n\).

Step 1. In order to have standard Fréchet marginals, consider the marginal transformation \(-\frac{1}{\log \tilde{F}_X(x)}\), where \(\tilde{F}_X\) is the (modified) empirical df of \(X_1, \ldots, X_n\) as defined above.

Step 2. Generate an iid sequence with standard Fréchet marginals, \(\tilde{X}_1, \ldots, \tilde{X}_n\), and consider random pairs, \((\tilde{X}_1, (1/2)\tilde{X}_1 \lor (1/2)X_i), i = 1, \ldots, n\).

Step 3. Since we are going to first estimate \(\lambda\) on the limiting BEV model of the component-wise maximum, we choose the blocks length \(r\) where to take the component-wise maxima, in order to obtain a sample of maximums

\[
(Z_{j,1}, Z_{j,2}) = \left( \bigvee_{i=(j-1)r+1}^{jr} \tilde{X}_i, \bigvee_{i=(j-1)r+1}^{jr} (1/2)\tilde{X}_i \lor (1/2)X_i \right), 1 \leq j \leq n/r.
\]

Step 4. Apply estimator \(\tilde{\lambda}\) given in (6) on the random pairs of the previous step and calculate \(\tilde{\theta}\) in (7).
Step 5. Repeat steps 2-4 a large number $M$ of times, obtain estimates $\tilde{\theta}_1, \ldots, \tilde{\theta}_M$ and take the mean, $\bar{\theta} = \frac{1}{M} \sum_{s=1}^M \tilde{\theta}_s$, in order to achieve robustness given the existence of arbitrariness in the generation of a random sample (step 2) in each estimate. Here we consider $M = 10000$.

3. Simulations

Our simulations are based on 100 replicates of samples of size 1000 and 5000, of each of the following models: a first order autoregressive process with Cauchy marginals and autoregressive parameter $\rho = -0.6$ (Chernick 1978), a negatively correlated uniform AR(1) process with $r = 2$ (Chernick, Hsing, and McCormick 1991), respectively denoted ARCau and ARUnif, a moving maxima (MM) process with coefficients $a_0 = 2/6$, $a_1 = 1/6$, $a_2 = 3/6$ (Deheuvels 1983), a first order MAR process with standard Fréchet marginals and autoregressive parameter $\phi = 0.5$ (Davis and Resnick 1989), a Markov chain (MC) with standard Gumbel marginals and logistic joint distribution with dependence parameter $\alpha = 0.5$ (Smith 1992), an ARCH(1) process with Gaussian innovations, autoregressive parameter $\lambda = 0.5$ and variance parameter $\beta = 1.9 \times 10^{-5}$ (Embrechts, Klüppelberg, and Mikosch 1997). The theoretical extremal index values of the processes ARCau, ARUnif, MM, MAR, MC and ARCH are, respectively, 0.64, 0.75, 0.5, 0.5, 0.328 and 0.835.

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### Table 1. The root mean squared error (rmse) obtained for estimator $\tilde{\theta}$ (with block lengths $r = 10, 20, 30, 40, 50, 70$ and $n = 1000$) and estimator $\tilde{\theta}_{FS}$ (with levels $u$ corresponding to the empirical quantiles 0.80, 0.85, 0.90, 0.925, 0.95, 0.975, and 0.99, respectively denoted, $q_{0.80}$, $q_{0.85}$, $q_{0.90}$, $q_{0.925}$, $q_{0.95}$, $q_{0.975}$ and $q_{0.99}$).

|        | MAR   | MM    | ARUnif | ARCau | ARCH | MC    |
|--------|-------|-------|--------|--------|------|-------|
| $\tilde{\theta}$ (r = 10) | 0.097 | 0.082 | 0.105 | 0.101 | 0.058 | 0.140 |
| $\tilde{\theta}$ (r = 20) | 0.093 | 0.069 | **0.068** | 0.078 | 0.090 | 0.142 |
| $\tilde{\theta}$ (r = 30) | 0.106 | 0.072 | 0.071 | 0.078 | 0.122 | 0.122 |
| $\tilde{\theta}$ (r = 40) | 0.107 | 0.066 | **0.068** | **0.075** | 0.145 | 0.159 |
| $\tilde{\theta}$ (r = 50) | 0.112 | 0.072 | 0.091 | 0.080 | 0.161 | 0.164 |
| $\tilde{\theta}$ (r = 70) | 0.107 | 0.084 | 0.129 | 0.090 | 0.195 | 0.172 |
| $\tilde{\theta}_{N}$ (r = 10) | 0.060 | 0.069 | 0.250 | 0.195 | 0.083 | 0.108 |
| $\tilde{\theta}_{N}$ (r = 20) | **0.054** | **0.048** | 0.219 | 0.111 | 0.079 | 0.081 |
| $\tilde{\theta}_{N}$ (r = 30) | 0.064 | 0.051 | 0.186 | 0.099 | 0.092 | 0.083 |
| $\tilde{\theta}_{N}$ (r = 40) | 0.070 | 0.061 | 0.168 | 0.100 | 0.100 | 0.087 |
| $\tilde{\theta}_{N}$ (r = 50) | 0.074 | 0.075 | 0.157 | 0.107 | 0.108 | 0.091 |
| $\tilde{\theta}_{N}$ (r = 70) | 0.090 | 0.091 | 0.146 | 0.128 | 0.125 | 0.108 |
| $\tilde{\theta}_{FS}$ ($q_{0.80}$) | 0.059 | 0.057 | 0.250 | 0.195 | 0.098 | **0.065** |
| $\tilde{\theta}_{FS}$ ($q_{0.85}$) | 0.067 | 0.059 | 0.245 | 0.153 | 0.095 | **0.076** |
| $\tilde{\theta}_{FS}$ ($q_{0.90}$) | 0.080 | 0.075 | 0.239 | 0.125 | 0.119 | 0.088 |
| $\tilde{\theta}_{FS}$ ($q_{0.925}$) | 0.086 | 0.092 | 0.209 | 0.129 | 0.112 | 0.098 |
| $\tilde{\theta}_{FS}$ ($q_{0.95}$) | 0.100 | 0.111 | 0.201 | 0.151 | 0.122 | 0.165 |
| $\tilde{\theta}_{FS}$ ($q_{0.975}$) | 0.165 | 0.130 | 0.176 | 0.182 | 0.140 | 0.172 |
| $\tilde{\theta}_{FS}$ ($q_{0.99}$) | 0.250 | 0.201 | 0.208 | 0.231 | 0.165 | 0.323 |

The results in bold correspond to the best performance and the italic denotes the second best performance within each model.
The root mean squared errors (rmse) and the absolute mean biases (abias) are given in Tables 1 and 2 for \( n = 1000 \) and Tables 3 and 4 for \( n = 5000 \).

For comparison, we consider two direct competitors of our estimator, as already mentioned in the Introduction, also requiring only one tuning parameter: the sliding blocks estimator of Northrop (2015) based on a block length choice and the estimator of Ferro and Segers (2003) which depends on the choice of the high threshold. The first is denoted \( \tilde{\theta}^{N} \) and is computed for the same block lengths \( r = 10, 20, 30, 40, 50, 70 \) used in our proposal \( \tilde{\theta} \). Ferro and Segers estimator is denoted \( \tilde{\theta}^{FS} \) and obtained for levels \( u_n \) corresponding to the empirical quantiles 0.80, 0.85, 0.90, 0.925, 0.95, 0.975, and 0.99, respectively denoted, \( q_{0.80}, q_{0.85}, q_{0.90}, q_{0.925}, q_{0.95}, q_{0.975} \) and \( q_{0.99} \). We use software R and we compute \( \tilde{\theta}^{FS} \) and \( \tilde{\theta}^{N} \), respectively, within packages evd (Stephenson 2002) and exdex (Northrop and Christodoulides 2019).

Estimator \( \tilde{\theta} \) seems to be competitive, particularly in models ARUnif, ARCau and ARCH.

### Table 2.

The absolute mean bias (abias) obtained for estimator \( \tilde{\theta} \) (with block lengths \( r = 10, 20, 30, 40, 50, 70 \) and \( n = 1000 \)) and estimator \( \tilde{\theta}^{FS} \) (with levels \( u_n \) corresponding to the empirical quantiles 0.80, 0.85, 0.90, 0.925, 0.95, 0.975, and 0.99, respectively denoted, \( q_{0.80}, q_{0.85}, q_{0.90}, q_{0.925}, q_{0.95}, q_{0.975} \) and \( q_{0.99} \)).

| abias | MAR | MM | ARUnif | ARCau | ARCH | MC |
|-------|-----|----|--------|-------|------|----|
| \( \tilde{\theta} (r = 10) \) | 0.066 | 0.070 | 0.097 | 0.085 | 0.024 | 0.119 |
| \( \tilde{\theta} (r = 20) \) | 0.046 | 0.041 | 0.040 | 0.036 | 0.061 | 0.098 |
| \( \tilde{\theta} (r = 30) \) | 0.041 | 0.033 | \textbf{0.006} | 0.013 | 0.096 | 0.096 |
| \( \tilde{\theta} (r = 40) \) | 0.037 | 0.029 | 0.019 | \textbf{0.000} | 0.120 | 0.096 |
| \( \tilde{\theta} (r = 50) \) | 0.033 | 0.025 | 0.049 | 0.009 | 0.140 | 0.103 |
| \( \tilde{\theta} (r = 70) \) | 0.024 | 0.013 | 0.101 | 0.039 | 0.180 | 0.110 |
| \( \tilde{\theta}^{N} (r = 10) \) | 0.052 | 0.062 | 0.250 | 0.191 | 0.068 | 0.102 |
| \( \tilde{\theta}^{N} (r = 20) \) | 0.027 | 0.024 | 0.215 | 0.094 | 0.032 | 0.063 |
| \( \tilde{\theta}^{N} (r = 30) \) | 0.019 | 0.012 | 0.173 | 0.064 | 0.015 | 0.051 |
| \( \tilde{\theta}^{N} (r = 40) \) | 0.013 | 0.008 | 0.143 | 0.048 | 0.006 | 0.045 |
| \( \tilde{\theta}^{N} (r = 50) \) | 0.007 | 0.006 | 0.122 | 0.040 | \textbf{0.003} | 0.040 |
| \( \tilde{\theta}^{N} (r = 70) \) | 0.011 | \textbf{0.001} | 0.093 | 0.035 | 0.018 | 0.039 |
| \( \tilde{\theta}^{FS} (q_{0.80}) \) | 0.019 | 0.012 | 0.250 | 0.177 | 0.069 | \textbf{0.017} |
| \( \tilde{\theta}^{FS} (q_{0.85}) \) | \textbf{0.005} | 0.004 | 0.242 | 0.125 | 0.048 | 0.024 |
| \( \tilde{\theta}^{FS} (q_{0.90}) \) | 0.014 | 0.010 | 0.232 | 0.079 | 0.031 | 0.036 |
| \( \tilde{\theta}^{FS} (q_{0.925}) \) | 0.020 | 0.012 | 0.196 | 0.069 | 0.023 | 0.046 |
| \( \tilde{\theta}^{FS} (q_{0.95}) \) | 0.019 | 0.015 | 0.178 | 0.056 | 0.024 | 0.068 |
| \( \tilde{\theta}^{FS} (q_{0.975}) \) | 0.086 | 0.033 | 0.133 | 0.103 | 0.030 | 0.092 |
| \( \tilde{\theta}^{FS} (q_{0.99}) \) | 0.124 | 0.103 | 0.133 | 0.130 | 0.030 | 0.219 |

The results in bold correspond to the best performance and the italic denotes the second best performance within each model.

The root mean squared errors (rmse) and the absolute mean biases (abias) are given in Tables 1 and 2 for \( n = 1000 \) and Tables 3 and 4 for \( n = 5000 \).

For comparison, we consider two direct competitors of our estimator, as already mentioned in the Introduction, also requiring only one tuning parameter: the sliding blocks estimator of Northrop (2015) based on a block length choice and the estimator of Ferro and Segers (2003) which depends on the choice of the high threshold. The first is denoted \( \tilde{\theta}^{N} \) and is computed for the same block lengths \( r = 10, 20, 30, 40, 50, 70 \) used in our proposal \( \tilde{\theta} \). Ferro and Segers estimator is denoted \( \tilde{\theta}^{FS} \) and obtained for levels \( u_n \) corresponding to the empirical quantiles 0.80, 0.85, 0.90, 0.925, 0.95, 0.975, and 0.99, with respective notation, \( q_{0.80}, q_{0.85}, q_{0.90}, q_{0.925}, q_{0.95}, q_{0.975} \) and \( q_{0.99} \). We use software R and we compute \( \tilde{\theta}^{FS} \) and \( \tilde{\theta}^{N} \), respectively, within packages evd (Stephenson 2002) and exdex (Northrop and Christodoulides 2019).

Estimator \( \tilde{\theta} \) seems to be competitive, particularly in models ARUnif, ARCau and ARCH.

### 4. Application to financial data

The data consists of the log-returns of the exchange rate US dollar versus UK pound, from January 2 of 1980 to May 21 of 1996 (Figure 1). An ARCH(1) fit was performed in Embrechts, Klüppelberg, and Mikosch (1997) leading to \( \theta = 0.835 \).
In Figure 2, we can see the estimates of $\tilde{\theta}$ and $\tilde{\theta}^N$, for block length $r$ ranging from 10 to 100, jointly with estimates of $\tilde{\theta}^{FS}$ for thresholds corresponding to quantiles between 0.9 and 0.99. The 95% confidence bands of estimator $\tilde{\theta}^N$, obtained within R package exdex (Northrop and Christodoulides 2019), are also plotted. We can see a

Table 3. The root mean squared error (rmse) obtained for estimator $\tilde{\theta}$ (with block lengths $r = 10, 20, 30, 40, 50, 70$ and $n = 5000$) and estimator $\tilde{\theta}^{FS}$ (with levels $u$, corresponding to the empirical quantiles 0.80, 0.85, 0.90, 0.925, 0.95, 0.975, and 0.99, respectively denoted, $q_{0.80}$, $q_{0.85}$, $q_{0.90}$, $q_{0.925}$, $q_{0.95}$, $q_{0.975}$ and $q_{0.99}$).

| rmse  | MAR | MM | ARUnif | ARCau | ARCH | MC |
|-------|-----|----|--------|-------|------|----|
| $\tilde{\theta}$ (r = 10) | 0.055 | 0.071 | 0.136 | 0.092 | 0.033 | 0.100 |
| $\tilde{\theta}$ (r = 20) | 0.053 | 0.045 | 0.088 | 0.064 | 0.043 | 0.077 |
| $\tilde{\theta}$ (r = 30) | 0.061 | 0.038 | 0.072 | 0.062 | 0.053 | 0.077 |
| $\tilde{\theta}$ (r = 40) | 0.069 | 0.040 | 0.061 | 0.067 | 0.066 | 0.081 |
| $\tilde{\theta}$ (r = 50) | 0.074 | 0.045 | 0.055 | 0.072 | 0.076 | 0.086 |
| $\tilde{\theta}$ (r = 70) | 0.089 | 0.051 | 0.055 | 0.072 | 0.090 | 0.099 |
| $\tilde{\theta}^N$ (r = 10) | 0.050 | 0.068 | 0.250 | 0.188 | 0.074 | 0.105 |
| $\tilde{\theta}^N$ (r = 20) | 0.031 | 0.038 | 0.223 | 0.093 | 0.045 | 0.063 |
| $\tilde{\theta}^N$ (r = 30) | 0.030 | 0.030 | 0.177 | 0.066 | 0.044 | 0.055 |
| $\tilde{\theta}^N$ (r = 40) | 0.032 | 0.029 | 0.153 | 0.054 | 0.046 | 0.045 |
| $\tilde{\theta}^N$ (r = 50) | 0.034 | 0.031 | 0.136 | 0.050 | 0.050 | 0.043 |
| $\tilde{\theta}^N$ (r = 70) | 0.040 | 0.034 | 0.116 | 0.054 | 0.059 | 0.045 |
| $\tilde{\theta}^{FS}$ (q$_{0.80}$) | 0.034 | 0.026 | 0.250 | 0.174 | 0.074 | 0.024 |
| $\tilde{\theta}^{FS}$ (q$_{0.85}$) | 0.034 | 0.030 | 0.250 | 0.114 | 0.060 | 0.032 |
| $\tilde{\theta}^{FS}$ (q$_{0.90}$) | 0.041 | 0.035 | 0.247 | 0.077 | 0.057 | 0.032 |
| $\tilde{\theta}^{FS}$ (q$_{0.925}$) | 0.044 | 0.041 | 0.194 | 0.064 | 0.063 | 0.042 |
| $\tilde{\theta}^{FS}$ (q$_{0.95}$) | 0.047 | 0.049 | 0.170 | 0.057 | 0.065 | 0.045 |
| $\tilde{\theta}^{FS}$ (q$_{0.975}$) | 0.071 | 0.071 | 0.141 | 0.084 | 0.089 | 0.077 |
| $\tilde{\theta}^{FS}$ (q$_{0.99}$) | 0.119 | 0.083 | 0.145 | 0.131 | 0.115 | 0.118 |

The results in bold correspond to the best performance and the italic denotes the second best performance within each model.

Figure 1. Log-returns of the exchange rate US dollar versus UK pound, from January 2 of 1980 to May 21 of 1996.

In Figure 2, we can see the estimates of $\tilde{\theta}$ and $\tilde{\theta}^N$, for block length $r$ ranging from 10 to 100, jointly with estimates of $\tilde{\theta}^{FS}$ for thresholds corresponding to quantiles between 0.9 and 0.99. The 95% confidence bands of estimator $\tilde{\theta}^N$, obtained within R package exdex (Northrop and Christodoulides 2019), are also plotted. We can see a
Table 4. The absolute mean bias (abias) obtained for estimator $\tilde{\theta}$ (with block lengths $r = 10, 20, 30, 40, 50, 70$ and $n = 5000$) and estimator $\tilde{\theta}^\text{FS}$ (with levels $u$, corresponding to the empirical quantiles $0.80, 0.85, 0.90, 0.925, 0.95, 0.975$, and $0.99$, respectively denoted, $q_{0.80}, q_{0.85}, q_{0.90}, q_{0.925}, q_{0.95}, q_{0.975}$ and $q_{0.99}$).

| abias   | MAR | MM  | ARUnif | ARCau | ARCH | MC   |
|---------|-----|------|--------|-------|------|------|
| $\tilde{\theta}$ ($r = 10$) | 0.047 | 0.068 | 0.133  | 0.088 | 0.017 | 0.045 |
| $\tilde{\theta}$ ($r = 20$) | 0.027 | 0.035 | 0.078  | 0.046 | 0.008 | 0.056 |
| $\tilde{\theta}$ ($r = 30$) | 0.022 | 0.024 | 0.057  | 0.034 | 0.021 | 0.095 |
| $\tilde{\theta}$ ($r = 40$) | 0.023 | 0.020 | 0.042  | 0.031 | 0.037 | 0.041 |
| $\tilde{\theta}$ ($r = 50$) | 0.024 | 0.023 | 0.027  | 0.027 | 0.045 | 0.040 |
| $\tilde{\theta}$ ($r = 70$) | 0.021 | 0.021 | 0.014  | 0.018 | 0.060 | 0.046 |
| $\tilde{\theta}$ ($r = 10$) | 0.048 | 0.067 | 0.250  | 0.187 | 0.071 | 0.018 |
| $\tilde{\theta}$ ($r = 20$) | 0.024 | 0.034 | 0.222  | 0.088 | 0.033 | 0.024 |
| $\tilde{\theta}$ ($r = 30$) | 0.015 | 0.021 | 0.174  | 0.055 | 0.018 | 0.040 |
| $\tilde{\theta}$ ($r = 40$) | 0.010 | 0.016 | 0.148  | 0.038 | 0.010 | 0.049 |
| $\tilde{\theta}$ ($r = 50$) | 0.006 | 0.013 | 0.128  | 0.027 | 0.004 | 0.054 |
| $\tilde{\theta}$ ($r = 70$) | 0.000 | 0.008 | 0.103  | 0.018 | 0.005 | 0.063 |
| $\tilde{\theta}$ ($q_{0.80}$) | 0.023 | 0.007 | 0.250  | 0.169 | 0.063 | 0.010 |
| $\tilde{\theta}$ ($q_{0.85}$) | 0.014 | 0.000 | 0.250  | 0.106 | 0.045 | 0.014 |
| $\tilde{\theta}$ ($q_{0.90}$) | 0.015 | 0.000 | 0.245  | 0.057 | 0.024 | 0.014 |
| $\tilde{\theta}$ ($q_{0.925}$) | 0.010 | 0.002 | 0.188  | 0.036 | 0.014 | 0.013 |
| $\tilde{\theta}$ ($q_{0.95}$) | 0.007 | 0.005 | 0.156  | 0.021 | 0.011 | 0.011 |
| $\tilde{\theta}$ ($q_{0.975}$) | 0.006 | 0.004 | 0.104  | 0.023 | 0.005 | 0.020 |
| $\tilde{\theta}$ ($q_{0.99}$) | 0.044 | 0.005 | 0.087  | 0.053 | 0.016 | 0.048 |

The results in bold correspond to the best performance and the italic denotes the second best performance within each model.

Figure 2. Plot of the estimates: of $\tilde{\theta}^\text{FS}$ for thresholds corresponding to sample quantiles between 0.90 and 0.99, of $\tilde{\theta}^N$ and $\tilde{\theta}$ for block lengths between $r = 10$ and $r = 100$. The dashed line corresponds to the 95% confidence intervals of $\tilde{\theta}^N$. 

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more stable path in the region between block sizes $r = 40$ and $r = 70$, also corresponding to threshold quantiles from 93% to 96%, where the estimates vary among 0.60 and 0.67. The confidence bands include the previous estimate 0.835 for block sizes below $r = 20$.

5. Conclusion

The idea of relating the extremal index with the tail dependence coefficient is not new. For instance, in Ferreira and Ferreira (2012c), $\theta$ was derived as a linear combination of lag-$m$ serial tail dependence coefficients, under some local dependence conditions.

The new proposed estimator of the extremal index is based on a relation between $\theta$ and the tail dependence coefficient $\lambda$ of a BEV copula, without any assumptions on the dependence between the variables of the sequence that has extremal index. This work shows that once we find a relation between the extremal index and the tail dependence coefficient of some BEV copula, we can always explore it to obtain other estimators of $\theta$ from $\lambda$ estimation. Depending on the bivariate sequences we use to explore that relation, we may find estimators that will work better for some class of models than others. This approach opens up new avenues of investigation for the estimation of the extremal index. The estimator of Northrop (2015) uses sliding blocks to increase efficiency of estimation. This can also be considered in Step 3 of our estimation algorithm in Section 2. The study of the asymptotic behavior, in particular, obtaining confidence intervals for the new estimator is also an important aspect to address in a next work.

Acknowledgements

The authors are very grateful to the referees for the comments, suggestions and corrections that contributed to the improvement of this work.

Funding

The first author was partially supported by the research unit Center of Mathematics and Applications of University of Beira Interior UIDB/00212/2020 - FCT (Fundação para a Ciência e a Tecnologia). The second author was financed by Portuguese Funds through FCT - Fundação para a Ciência e a Tecnologia within the Projects UIDB/00013/2020 and UIDP/00013/2020 of Center of Mathematics of the University of Minho, UIDB/00006/2020 and UIDP/00006/2020 of Center of Statistics and its Applications of University of Lisbon, UIDB/04621/2020 and UIDP/04621/2020 of Center for Computational and Stochastic Mathematics and PTDC/MAT-STA/28243/2017.

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References

Cai, J. J. 2019. A nonparametric estimator of the extremal index. arXiv 1911.06674
Chernick, M. R. 1978. Mixing conditions and limit theorems for maxima of some stationary sequences. PhD diss., Stanford University.
Chernick, M. R., T. Hsing, and W. P. McCormick. 1991. Calculating the extremal index for a class of stationary sequences. *Advances in Applied Probability* 23 (4):835–50. doi:10.2307/1427679.

Davis, R., and S. Resnick. 1989. Basic properties and prediction of max-ARMA processes. *Advances in Applied Probability* 21 (4):781–803. doi:10.2307/1427767.

Deheuvels, P. 1983. Point processes and multivariate extreme values. *Journal of Multivariate Analysis* 13 (2):257–72. doi:10.1016/0047-259X(83)90025-8.

Embrechts, P., C. Klüppelberg, and T. Mikosch. 1997. *Modelling extremal events*. Berlin: Springer.

Ferreira, H., and M. Ferreira. 2012a. On extremal dependence of block vectors. *Kybernetika* 48 (5):988–1006.

Ferreira, H., and M. Ferreira. 2012b. Fragility index of block tailed vectors. *Journal of Statistical Planning and Inference* 142 (7):1837–48. doi:10.1016/j.jspi.2012.01.021.

Ferreira, H., and M. Ferreira. 2018. Estimating the extremal index through local dependence. *Annales de l’Institut Henri Poincaré - Probabilités et Statistiques* 54 (2):587–605.

Ferreira, M. 2013. Nonparametric estimation of the tail-dependence coefficient. *REVSTAT - Statistical Journal* 11 (1):1–16.

Ferreira, M., and H. Ferreira. 2012c. On extremal dependence: Some contributions. *TEST* 21 (3):566–83. doi:10.1007/s11749-014-0358-6.

Ferro, C. A. T., and J. Segers. 2003. Inference for clusters of extreme values. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 65 (2):545–56. doi:10.1111/1467-9868.00401.

Frahm, G., M. Junker, and R. Schmidt. 2005. Estimating the tail-dependence coefficient: Properties and pitfalls. *Insurance: Mathematics & Economics* 37 (1):80–100.

Gomes, D. P., and M. M. Neves. 2020. Estimating the extremal index through local dependence. *Extremes* 18 (4):585–603. doi:10.1007/s10687-015-0221-5.

Hsing, T., J. Hüsler, and M. R. Leadbetter. 1988. On the exceedance point process for a stationary sequence. *Probability Theory and Related Fields* 78 (1):97–112. doi:10.1007/BF00718038.

Huang, X. 1992. Statistics of bivariate extreme values. PhD thes., Erasmus University Rotterdam, Tinbergen, Institute Research Series 22.

Martins, A. P., and H. Ferreira. 2014. Extremal properties of M4 processes. *Test* 23 (2):388–408. doi:10.1007/s11749-014-0358-6.

Moloney, N. R., D. Faranda, and Y. Sato. 2019. An overview of the extremal index. *Chaos* 29 (2):022101. doi:10.1063/1.5079656.

Northrop, P. J. 2015. An efficient semiparametric maxima estimator of the extremal index. *Extremes* 18 (4):585–603. doi:10.1007/s10687-015-0221-5.

Northrop, P. J., and C. Christodoulides. 2019. Exdex: Estimation of the Extremal Index. https://CRAN.R-project.org/package=exdex.

Séveges, M., and A. C. Davison. 2010. Model misspecification in peaks over threshold analysis. *Annals of Applied Statistics* 4:203–21.

Weissman, I., and S. Y. Novak. 1998. On blocks and runs estimators of the extremal index. *Journal of Statistical Planning and Inference* 66 (2):281–8. doi:10.1016/S0378-3758(97)00095-5.