In this paper, we consider the following p-Kirchhoff problem of Brézis-Nirenberg type with singular terms

\[
\begin{aligned}
-\mathcal{K}(u) \left[ \frac{\Delta u - \lambda |u|^{p-2} u}{|x|^{q}} \right] + \mu |u|^{p-2} u &= \frac{|w|^{p-2} w}{|x|^{q}} + \lambda |u|^{p-2} u, \quad \text{in } \Omega, \\
 u = 0, & \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1)

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( 1 < p < N \), \( 0 \in \Omega \), \( \mathcal{K}(u) = a|u|^q + b, a, b, q > 0, 0 \leq \alpha < (N - p)/p \), \( a \leq \beta < a + 1, 0 \leq \gamma < p \), \( \mu \) is the Caffarelli-Kohn-Nirenberg exponent corresponding to the noncompact embedding of \( \mathcal{D}_a(\Omega) \) into \( L_{p'}(\Omega, |x|^{\beta'}) \), where \( \mathcal{D}_a(\Omega) \) is the closure of \( C_0(\Omega) \) with respect to the norm

\[
\|u\|_{p'} := \left( \int_{\Omega} \frac{|\nabla u|^p}{|x|^{q}} - \mu \frac{|u|^p}{|x|^{|\alpha + \gamma|}} \right)^{1/p'} dx,
\]

(2)

and \( L_{p'}(\Omega, |x|^{\beta'}) \) denotes the usual weighted \( L_{p'}(\Omega) \) space with the weight \( |x|^{\beta'} \).

Kirchhoff-type problems are often referred to as being nonlocal because of the presence of the term \( \mathcal{K}(u) \) which implies that the equation in (1) is no longer a pointwise identity. In the case \( p = 2 \) and \( \alpha = \beta = \gamma = \mu = 0 \), it is analogous to the stationary version of equations that arise in the study of string or membrane vibrations, namely,

\[
u_{tt} - \mathcal{K}(u) \Delta u = g(x, u)
\]

(3)

where \( u \) denotes the displacement and \( g(x, u) \) is the external force. Equations of this type were first proposed by Kirchhoff in 1883 [1] to describe the transversal oscillations of a stretched string.

These problems serve also to model other physical phenomena as biological systems where \( u \) describes a process which depends on the average of itself (for example, population density).

In recent years, Kirchhoff-type problems received much attention, mainly after the famous article of Lions [2]; they have been studied in many papers by using variational methods, see [3–9] and the references therein.

The problem (1) without nonlocal term (\( a = 0 \)) and without singular terms (\( \alpha = \beta = \gamma = \mu = 0 \)) has been treated by Brézis and Nirenberg [10] for \( p = 2 \). Subsequently, an increasing number of researchers have paid attention to
semilinear or quasilinear elliptic equations with critical exponent of Sobolev or Caffarelli-Kohn-Nirenberg, for example, see [11, 12] and the references therein.

In [7], Naimen generalized the results of [10] to the nonlocal problem (1) with \( N = 3 \) and without singular terms. Kang in [1] generalized the results of [10] to a quasilinear problem with singular terms and without the nonlocal term (\( p > 1, a = 0 \) and \( (\alpha, \beta, \gamma, \mu) \neq (0, 0, 0, 0) \)).

Thus, it is natural for us to consider the quasilinear Brézis-Nirenberg problem in [10] with nonlocal term and singular weights, \( (p > 1, a \neq 0 \) and \( (\alpha, \beta, \gamma, \mu) \neq (0, 0, 0, 0) \)). The competing effect of the nonlocal term with the critical nonlinearity and the lack of compactness of the embedding of \( \mathcal{D}_a(\Omega) \) into \( L_{p^*}(\Omega, |x|^{\beta p}) \) prevent us from using the variational methods in a standard way. So, motivated by all the works mentioned above, we prove existence results of our problem (1) with \( N = 3 \) and without singular terms, \( \Omega \) a star-shaped domain with respect to the origin, and we can easily verify that the problem (1) has no nontrivial solution by using a Pohozaev-type identity.

Remark 2. In the case where \( \lambda = 0 \) and \( \Omega \) is a star-shaped domain with respect to the origin, we have

\[
\mu_* = \frac{N - (p^2(\alpha + 1) + y(1 - p))}{p} \left( \frac{N - y}{p} \right)^{p-1}. \tag{9}
\]

Then, the problem (1) has a positive solution in the following cases:

1. \( q = p^* - p \) and \( 0 < a < S^-p^*/p \)
2. \( q < p^* - p \) and \( a > 0 \)

This paper is organized as follows. In Section 2, we study the variational framework and give some preliminary results. In Section 3, we show the existence result and we will prove Theorem 1.

### 2. Variational Framework and Preliminary Results

The starting point of the variational approach to problem (1) is the following Caffarelli-Kohn-Nirenberg inequality in [13] which is also called the Hardy-Sobolev inequality. Assume that \( 1 < p < N, 0 < \alpha < (N - p)/p \) and \( \alpha \leq \beta < \alpha + 1 \), and then,

\[
\left( \int_{\Omega} \frac{|u|^{p^*}}{|x|^{p^*}} dx \right)^{1/p^*} \leq C \left( \frac{\|\nabla u\|_p^p}{\mu} \right)^{1/p} \quad \text{for all } u \in C_0^\infty(\Omega),
\tag{10}
\]

for some positive constant \( C \). In the case where \( \beta = \alpha + 1 \), we have \( p^* = p, \bar{C} = 1/\mu \) and we have the following Hardy inequality:

\[
\int_{\Omega} \frac{|u|^p}{|x|^{p(\alpha + 1)}} dx \leq \frac{1}{\mu} \int_{\Omega} \frac{|\nabla u|^p}{|x|^{p\alpha}} dx \quad \text{for all } u \in C_0^\infty(\Omega). \tag{11}
\]

Definition 3. We say that \( u \in \mathcal{D}_a(\Omega) \setminus \{0\} \) is a weak solution of equation (1) if
\[ \mathcal{H}(u) = \int_{\Omega} \left( \frac{|\nabla u|^2 \nabla u \nabla v}{|x|^\alpha} - \mu \frac{|u|^{p-2} u v}{|x|^{p(a+1)}} \right) \, dx \]

\[ = \int_{\Omega} \left( \frac{|u|^{p^* - 2} u}{|x|^{p^*}} + \lambda \frac{|u|^{q^* - 2} u}{|x|^{q^*}} \right) v \, dx = 0 \]

for any \( v \in \mathcal{D}_a(\Omega) \).

Next, we define the energy functional

\[ I_\lambda(u) = \frac{a}{q + p} \|u\|^{q+p} + \frac{b}{p} \|u\|^p - \frac{1}{p^*} \int_{\Omega} \frac{|u|^{p^*}}{|x|^{p^*}} \, dx - \frac{\lambda}{p} \int_{\Omega} \frac{|u|^p}{|x|^p} \, dx, \]

associated to problem (1), for all \( u \in \mathcal{D}_a(\Omega) \).

Notice that the functional \( I_\lambda \) is well defined in \( \mathcal{D}_a(\Omega) \) and belongs to \( C^1(\mathcal{D}_a(\Omega), \mathbb{R}) \) and a critical point of \( I_\lambda \) is a weak solution of problem (1).

**Definition 4.** Let \( c \in \mathbb{R} \), a sequence \((u_n) \subset \mathcal{D}_a(\Omega)\) is called a \((PS)_c\) sequence (Palais-Smale sequence at level \( c \)) if

\[ I_\lambda(u_n) \to c \text{ and } I'_\lambda(u_n) \to 0 \text{ as } n \to +\infty. \]

Let \( c \in \mathbb{R} \). We say that \( I_\lambda \) satisfies the Palais-Smale condition at level \( c \), if any \((PS)_c\) sequence contains a convergent subsequence in \( \mathcal{D}_a(\Omega) \).

**Lemma 5.** Assume \( 1 < p < N \), \( 0 \leq \alpha < (N - p)/p \), \( \beta < \alpha + 1 \), \( 0 \leq \gamma \leq p \beta / b \), \( \lambda < b \lambda_1 \) and \( q \leq p^* - p \). Let \( c \in \mathbb{R}^+ \) and \((u_n) \subset \mathcal{D}_a(\Omega)\) be a \((PS)_c\) sequence for \( I_\lambda \). Then,

\[ u_n \rightharpoonup u \text{ in } \mathcal{D}_a(\Omega) \]

for some \( u \in \mathcal{D}_a(\Omega) \) with \( I'_\lambda(u) = 0 \).

**Proof.** We have

\[ I_\lambda(u_n) \to c, \]

\[ I'_\lambda(u_n) \to 0. \]

That is,

\[ I_\lambda(u_n) = c + o_n(1) = I_\lambda(u_n), \]

\[ o_n(1) \|v\| = \langle I'_\lambda(u_n), v \rangle, \]

for any \( v \in \mathcal{D}_a(\Omega) \).

Then, as \( n \to +\infty \), it follows that

\[ c + o_n(1) - \frac{1}{p} o_n(1) \|u_n\| \]

\[ = I_\lambda(u_n) - \frac{1}{p} \langle I'_\lambda(u_n), u_n \rangle \]

\[ \geq a \frac{p^* - (q + p)}{(q + p)p^*} \|u_n\|^{q+p} + \left( b - \frac{\lambda}{\lambda_1} \right) \frac{p^* - p}{p^*} \|u_n\|^p. \]

As \( \lambda < b \lambda_1 \) and \( q \leq p^* - p \), we obtain that \((u_n)\) is bounded in \( \mathcal{D}_a(\Omega) \). Up to a subsequence if necessary, there exists a function \( u \in \mathcal{D}_a(\Omega) \) such that \( u_n \rightharpoonup u \) in \( \mathcal{D}_a(\Omega) \), \( u_n \to u \) in \( L^{p^*}(\Omega, |x|^{p^*\beta}) \) and \( u_n \to u \) in \( L^r(\Omega, |x|^{p^*\beta}) \), for all \( r < p^* \) and \( u_n \to u \) a.e. on \( \Omega \). Then,

\[ \langle I'_\lambda(u_n), v \rangle = 0 \text{ for all } v \in C_0^\infty(\Omega), \]

and thus \( I'_\lambda(u) = 0 \). This completes the proof of Lemma 5.

The following lemma is very important for giving the local Palais-Smale condition.

**Lemma 6.** Let \( a, b, q > 0, \sigma \geq 1 \), and \( \bar{y} = ((a/\sigma) S^{\sigma p/p'})^{1/\sigma - 1} \) for \( \sigma > 1 \). For \( y \geq 0 \), we consider the function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \), given by

\[ f(y) = S^{\sigma y^\sigma} - a \bar{y}^q y - b. \]

Then,

\[ (1) \text{ If } \sigma = 1 \text{ and } 0 < a < S^{-q/p^p}, \text{ then the equation } f(y) = 0 \text{ has a unique positive solution} \]

\[ y_1 = \frac{b}{S^{q/p^p}(S^{-q/p^p} - a)} \]

and \( f(y) \geq 0 \) for all \( y \geq y_1 \).

\[ (2) \text{ If } \sigma > 1, \text{ then the equation } f(y) = 0 \text{ has a unique positive solution } y_2 > \bar{y} \text{ and } f(y) \geq 0 \text{ for all } y \geq y_2. \]

**Proof.**

\[ (1) \text{ For } \sigma = 1 \text{ and } 0 < a < S^{-q/p^p}, \text{ we have} \]

\[ f(y) = S^{\sigma y^\sigma - a}y - b \]

that is, the equation \( f(y) = 0 \) has a unique positive solution

\[ y_1 = \frac{b}{(S^{-q/p^p} - a)S^{q/p^p}}, \]

and \( f(y) \geq 0 \) for all \( y \geq y_1 \).
(2) For $\sigma > 1$, we have $f'(y) = \sigma S^{-1}y^{\sigma-1} - aS^\theta y^p$ and

$$f''(y) = \sigma(\sigma - 1)S^{-1}y^{\sigma-2} > 0, \forall y > 0. \tag{24}$$

Then, $f''(\tilde{y}) = 0$ if $f'(y) < 0$ for $y < \tilde{y}$ and $f'(y) > 0$ for $y > \tilde{y}$. Hence, $f$ is concave function and

$$\min_{y \geq 0} f(y) = f(\tilde{y}) = -(\sigma - 1)S^{-1}\left(\frac{a S^\theta}{\sigma}\right)^{\frac{\sigma}{\sigma - 1}} < 0. \tag{25}$$

Moreover, we have $f(\tilde{y}) < 0$ and $\lim_{y \to +\infty} f(y) = +\infty$; thus, from (25) and the concavity of $f$, we can conclude that the equation $f(y) = 0$ has a unique positive solution $y_\gamma > \tilde{y}$ and $f(y) \geq 0$ for all $y \geq y_\gamma$.

Now, we prove an important lemma which ensures the local compactness of the Palais-Smale sequence for $Df$ from (25) and the concavity of $f$. Hence, we can conclude that the functional $I_\lambda$ has a unique positive solution $y_\gamma > \tilde{y}$ and $f(y) \geq 0$ for all $y \geq y_\gamma$.

For $i = 1, 2$, let $y_i$ be defined in Lemma 6 and define

$$y^*_i = \begin{cases} y_1 & \text{if } q = p^* - p \text{ and } 0 < a < S^{\frac{q}{p}} \gamma, \\ y_2 & \text{if } q < p^* - p \text{ and } a > 0. \end{cases} \tag{26}$$

Then, $\theta_i \geq b\eta_i + a\eta_i^{p^*+p}$. Therefore, by (29), we deduce that

$$\theta_i \geq b\eta_i + a\eta_i^{p^*+p} - b \geq 0. \tag{32}$$

Assume by contradiction that there exists $i_0 \in I$ such that $\theta_{i_0} \neq 0$. Set $y = (\theta_{i_0})^{\frac{p^*}{p}}$ and $\sigma = p^* - p/q$, then by (32) we get

$$S^{-1}y^{p^*} - aS^{\theta}y - b \geq 0. \tag{33}$$

It is clear that $\sigma \geq 1$ thanks to $q \leq p^* - p$. So, from (33) and the definition of $f$ in Lemma 6 we get

$$f(y) = S^{-1}y^{p^*} - aS^{\theta}y - b \geq 0. \tag{34}$$

We will discuss it in two cases:

**Case 1.** $q = p^* - p, b > 0$ and $0 < a < S^{-q+p/p}$. According to Lemma 6, we have $f(y_1) = 0$ and $f(y) \geq 0$ if $y \geq y_1$ with

$$y_1 = \frac{b}{(S^{q+p/p} - a)S^\theta}, \tag{35}$$

which implies that

$$S^{\theta_{i_0}} \geq S^{\frac{\gamma}{y_1}} = B_1. \tag{36}$$
Case 2. \( q < p^* - p, b > 0 \), and \( a > 0 \). In this case, from Lemma 6, we get \( f(y_2) = 0 \) and \( f(y) \geq 0 \) if \( y \geq y_2 \) with

\[
y_2 > \left( \frac{aq}{p^* - p} \right) \left( \frac{1}{\lambda_{p^*}} \right),
\]

which implies that

\[
S(\Theta_{\theta}) \geq S\gamma_2 = B_2.
\]

Hence, using (29), we deduce \( \eta_{i_0} \geq S\Theta_{\theta_{i_0}} \geq \)

\[
\begin{aligned}
B_1 & \text{ if } q = p^* - p \text{ and } 0 < a < S^{\frac{q}{q+p}} p,
B_2 & \text{ if } q < p^* - p \text{ and } a > 0.
\end{aligned}
\]

By Young inequality we have

\[
c = \lim_{n \to -\infty} I_i(u_n) - \frac{1}{q + p} \left( I_i^*(u_n), u_n \right)
= \lim_{n \to -\infty} \frac{q}{q + p} b \|u_n\|^p + \frac{p^* - (q + p)}{(q + p)p^*} \|u_n\|^{p^*} + \int_\Omega \frac{|u_n|^p}{|x|^{p^*}}
\geq \frac{q}{q + p} b \left( \|u\|^p + \eta_{i_0} \right) + \frac{p^* - (q + p)}{(q + p)p^*} \left( \int_\Omega \frac{|u|^p}{|x|^{p^*}} + \theta_{i_0} \right)
\geq \frac{q}{q + p} b \lambda_{\Omega} \|u\|^p + \frac{p^* - (q + p)}{(q + p)p^*} \int_\Omega \frac{|u|^p}{|x|^{p^*}}
+ \frac{q}{q + p} b \eta_{i_0} + \frac{p^* - (q + p)}{(q + p)p^*} \theta_{i_0},
\]

we observe that \( \frac{q}{(q + p)p}(b - \lambda_{\Omega}) > 0, p^* - q - p \geq 0 \); thus, for \( j \in \{1, 2\} \) we get

\[
c \geq \frac{q}{(q + p)p} b \eta_{i_0} + \frac{p^* - (q + p)}{(q + p)p^*} \eta_{i_0}
\geq \left( \frac{1}{p} - \frac{1}{q + p} \right) b b_j + \frac{p^* - (q + p)}{(q + p)p^*} \left( B_j^c \right) \left( S_{\gamma_2} \right)
\geq \left( \frac{1}{p} - \frac{1}{q + p} \right) b b_j + \frac{p^* - (q + p)}{(q + p)p^*} \left( B_j^c \right) \left( S_{\gamma_2} \right)
+ \frac{p^* - (q + p)}{(q + p)p^*} a B_j^c + \frac{p^* - (q + p)}{(q + p)p^*} a B_j^c
+ \frac{1}{p^*} b b_j - \frac{1}{p} b b_j \geq \left( \frac{p^* - (q + p)}{(q + p)p^*} a B_j^c + \frac{1}{p - p^*} b B_j \right)
\geq \left( \frac{p^* - (q + p)}{(q + p)p^*} a B_j^c + \frac{1}{p - p^*} b B_j \right)
- \frac{p^* - (q + p)}{(q + p)p^*} a B_j^c + \frac{1}{p - p^*} b B_j
\geq \left( \frac{p^* - (q + p)}{(q + p)p^*} a B_j^c + \frac{1}{p - p^*} b B_j \right)
- \frac{p^* - (q + p)}{(q + p)p^*} a B_j^c + \frac{1}{p - p^*} b B_j
\geq \frac{p^* - (q + p)}{(q + p)p^*} a B_j^c + \frac{1}{p - p^*} b B_j.
\]

Now, set \( I = \lim_{n \to +\infty} \|u_n\| \) as \( n \to +\infty \); then, we have

\[
\left( I_i^*(u_n), u_n \right) = (a\|u_n\|^q + b)\|u_n\| - \int_\Omega \frac{|u_n|^p}{|x|^{p^*}}
- \int_\Omega \frac{|u_n|^p}{|x|^{p^*}} dx = o_n(1)
\]

for any \( v \in D_\Omega(\Omega) \). Let \( n \to +\infty \), and then, from (42) and (43), we deduce that

\[
(a^I + b)\left( \int_\Omega \frac{|u|^p}{|x|^{p^*}} dx - \lambda \int_\Omega \frac{|u|^p}{|x|^p} dx = 0, \right.
\]

\[
(a^I + b)\left( \int_\Omega \frac{|u|^p}{|x|^{p^*}} \nabla u \nabla v - \mu \int_\Omega \frac{|u|^p}{|x|^{p^*}} dx \right) = 0.
\]

Taking the test function \( v = u \) in (45), we get

\[
(a^I + b)\|u\|^p - \int_\Omega \frac{|u|^p}{|x|^{p^*}} dx - \lambda \int_\Omega \frac{|u|^p}{|x|^p} dx = 0.
\]

Therefore, the equalities (44) and (45) imply that \( \|u\| = 1 \). Consequently, \( \{u_n\} \) converges strongly in \( D_\Omega(\Omega) \), which is the desired result.
3. Proof of the Main Result

Let \( R \) be a positive constant and set \( \varphi \in C_0^\infty(\Omega) \) such that \( 0 \leq \varphi(x) \leq 1 \) for \( |x| \leq R \) and \( \varphi(x) \equiv 1 \) for \( |x| \leq R/2 \) and \( B_R(0) \subset \Omega \). Set \( z_\varepsilon(x) = \varphi(x)u_\varepsilon(x) \).

We have the well-known estimates as \( \varepsilon \to 0 \):

\[
\begin{align*}
\left\| z_\varepsilon \right\| &= S_{\varepsilon} + O \left( \varepsilon^{p(p+\alpha+1)-N} \right) \\
\int_{|x| = \varepsilon} \left| \frac{z_\varepsilon'}{\varepsilon^p} - \varepsilon^p f(x) \right| dx &= S_{\varepsilon} + O \left( \varepsilon^{p(p+\alpha+1)-N} \right) \\
\int_{|x| = \varepsilon} \frac{z_\varepsilon'}{\varepsilon^p} dx &\geq C_{\varepsilon(p+\alpha+1)} \quad \text{if } \gamma > N - \delta \omega_\mu \text{ and } 0 \leq \mu < \frac{(N - p - 1)}{1} \\
\int_{|x| = \varepsilon} \frac{z_\varepsilon}{|x|^p} dx &\geq C_{\varepsilon(p-\alpha+1)} \quad \text{if } \gamma = N - \delta \omega_\mu \text{ and } 0 \leq \mu < \frac{(N - p - 1)}{1}.
\end{align*}
\]

(47)

where \( \xi_\mu \) and \( \omega_\mu \) are zeroes of the function

\[
f(t) = (p-1)t^p - (N-p(\alpha+1))t^{p-1} + \mu, \quad t \geq 0, \quad 0 \leq \mu < \mu_\varepsilon,
\]

(48)

that satisfy

\[
0 \leq \xi_\mu < t_\varepsilon < \omega_\mu < \frac{N - p(\alpha + 1)}{p - 1}.
\]

(49)

(see [1]).

On the other hand, the function \( f(t) \) has the unique minimal point

\[
t_\varepsilon = \frac{N - p(\alpha + 1)}{p}
\]

(50)

and is increasing on \([t_\varepsilon, +\infty[\). Thus, if \( \gamma \geq N - \delta \omega_\mu \), i.e., \( \omega_\mu \geq N - \gamma/p \) we have

\[
0 = f(\omega_\mu) \geq f\left( \frac{N - \gamma}{p} \right).
\]

(51)

Therefore, the equalities (48) and (51) imply that

\[
0 \leq \mu < \bar{\mu} = \frac{N - p^2(\alpha + 1) - \gamma(1-p)}{p} \left( \frac{N - \gamma}{p} \right)^{p-1}.
\]

(52)

Consequently,

\[
N - p^2(\alpha + 1) - \gamma(1-p) \geq 0, \quad \frac{p^2(\alpha + 1) - N}{p - 1} \leq \gamma < p \text{ and } a < \frac{N - p}{p^2}
\]

(53)

Next, we show that \( \bar{\mu} \leq \bar{\mu} \). For any \( \gamma \geq 0 \), let

\[
\bar{\mu} = \Psi(\gamma) = \frac{N - p^2(\alpha + 1) - \gamma(1-p)}{p} \left( \frac{N - \gamma}{p} \right)^{p-1}.
\]

(54)

We can show easily that since \( \Psi' \) is increasing on \([0,p(\alpha + 1)]\) and decreasing on \([p(\alpha + 1),+\infty[\) and \( \Psi(p(\alpha + 1)) = \mu_0 \). So, \( \mu \leq \bar{\mu} \).

**Lemma 8.** Let \( 1 < p < N, \quad 0 \leq \alpha < (N - p)/p, \quad \alpha \leq \beta < \alpha + 1, \quad 0 \leq \gamma < p, \quad b > 0, \quad 0 < \lambda < \lambda_1 \), and \( 0 \leq \mu < \mu_0 \). Assume that \( q = p^* - p \) and \( 0 < \alpha < S^{-\gamma/p} \), or \( q < p^* - p \) and \( a > 0 \). Then, \( \sup_{\varepsilon \to 0} I_{\lambda}(t_\varepsilon) < C_\varepsilon \).

**Proof.** We define the following functions

\[
g(t) = I_{\lambda}(t_\varepsilon) = \frac{a}{q + p} t^{q+\gamma} p \| z_\varepsilon \|^{q+\gamma} + \frac{b}{p} \| z_\varepsilon \|^{\gamma} - \frac{1}{p^*} \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx
\]

\[
= \frac{a}{q + p} t^{q+\gamma} p \| z_\varepsilon \|^{q+\gamma} + \frac{b}{p} \| z_\varepsilon \|^{\gamma} - \frac{1}{p^*} \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx
\]

\[
= \frac{a}{q + p} t^{q+\gamma} p \| z_\varepsilon \|^{q+\gamma} + \frac{b}{p} \| z_\varepsilon \|^{\gamma} - \frac{1}{p^*} \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx
\]

\[
h(t) = -\frac{1}{p^*} \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx + a T_{\varepsilon}^q \| z_\varepsilon \|^{q+\gamma} + b \| z_\varepsilon \|^{\gamma} - \lambda \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx = 0.
\]

(55)

Note that \( \lim_{\varepsilon \to 0} g(t) = -\infty \) and \( g(t) > 0 \) when \( t \) is close to \( 0 \), so that \( \sup_{\varepsilon \to 0} g(t) \) is attained for some \( T_\varepsilon > 0 \). Furthermore, from \( g'(T_\varepsilon) = 0 \) it follows that

\[
-T_{\varepsilon}^{-1} \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx + a T_{\varepsilon}^q \| z_\varepsilon \|^{q+\gamma} + b \| z_\varepsilon \|^{\gamma} - \lambda \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx = 0.
\]

(56)

Therefore,

\[
T_{\varepsilon}^{-1} \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx = a T_{\varepsilon}^q \| z_\varepsilon \|^{q+\gamma} + b \| z_\varepsilon \|^{\gamma} - \lambda \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx
\]

(57)

\[
\geq \left( b - \frac{\lambda}{\lambda_1} \right) \| z_\varepsilon \|^p
\]

Choose \( \varepsilon \) small enough so that by (47) we have \( T_\varepsilon \geq t_0 \) for some \( t_0 > 0 \).

\( \square \)

Besides, it holds

\[
T_{\varepsilon}^{-(q+\gamma)} \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx = a \| z_\varepsilon \|^{q+\gamma} + b \frac{\lambda_1}{T_{\varepsilon}} \| z_\varepsilon \|^{q+\gamma} - \lambda \frac{T_{\varepsilon}}{T_{\varepsilon}} \int_{|x|^\gamma} |z_\varepsilon|^{\gamma} dx
\]

\[
\leq a \| z_\varepsilon \|^{q+\gamma} + b \frac{\lambda_1}{T_{\varepsilon}} \| z_\varepsilon \|^{q+\gamma}.
\]

(58)
For \( q < p^* - p \), \( a > 0, b > 0 \) we have by (47)

\[
T^*_t = \frac{\|z_t^*\|^p}{\int_{I_\Omega} \left( \frac{|z_t^*|^p}{|x|^p} dx \right)} \leq a\|z_t^*\|^p + \frac{b\|z_t^*\|^p}{t_0^p}.
\]

(59)

Then, for \( \epsilon \) small enough, the above estimates yield \( T^*_\epsilon \) for some \( t_0' > 0 \) (independently of \( \epsilon \)).

For \( q = p^* - p \), \( 0 < a < S_0 q \), \( b > 0 \) and for \( \epsilon \) small enough we have by (56),

\[
T^*_\epsilon = \left( \frac{\|z^*_\epsilon\|^p - \lambda \int_{I_\Omega} (|z^*_\epsilon|^p/|x|^p) dx}{\int_{I_\Omega} \left( |z^*_\epsilon|^p/|x|^p \right) dx - a\|z^*_\epsilon\|^p} \right).
\]

(60)

which implies that \( T^*_\epsilon \) is bounded above for all \( \epsilon > 0 \), that is, there exists a positive real number \( t_0' > 0 \) (independently of \( \epsilon \)).

Now, we estimate \( g(T^*_\epsilon) \). It follows from \( h'^{(t)}(0) = 0 \)

\[
- \left[ \frac{S^{-1} y^p - aS y - b}{p} \right] = -f(y) = 0.
\]

(61)

Set \( y = tS^{-q/p}\|z^*_\epsilon\|^q \) and \( \sigma = p^* - p/q \). Then, by (61) the definition of \( f \) we get

\[
- \left[ \frac{S^{-1} y^p - aS y - b}{p} \right] = -f(y) = 0.
\]

(62)

which implies from (26) and the proof of Lemma 6 that \( f(y) = 0 \). Therefore, \( h'(t^*_\epsilon) = 0 \), where \( t^*_\epsilon = S_0 q \|z^*_\epsilon\|^{-1} \). As \( f(y) \) is concave, then \( h'^{(t)}(t^*_\epsilon) \) is convex and so,

\[
\max_{t \geq 0} h(t) = h(t^*_\epsilon) = \frac{1}{p}S^{-q/p}\|z^*_\epsilon\|^p t^*_\epsilon + \frac{a}{q+p} \|z^*_\epsilon\|^q t^*_\epsilon + \frac{b}{p} \|z^*_\epsilon\|^p t^*_\epsilon.
\]

(63)

By \( h'(t^*_\epsilon) = 0 \), we have

\[
S^{-q/p}\|z^*_\epsilon\|^p t^*_\epsilon = a\|z^*_\epsilon\|^q t^*_\epsilon + b\|z^*_\epsilon\|^p t^*_\epsilon.
\]

(64)

So, from (64) we deduce that

\[
\frac{a}{q+p} \|z^*_\epsilon\|^q t^*_\epsilon + \frac{b}{p} \|z^*_\epsilon\|^p t^*_\epsilon = a\left( \frac{1}{q+p} - \frac{1}{p} \right) t^*_\epsilon t^*_\epsilon + b\left( \frac{1}{p} - \frac{1}{p} \right) t^*_\epsilon \|z^*_\epsilon\|^p
\]

\[
= a\left( \frac{1}{q+p} - \frac{1}{p} \right) S^p y^p + b\left( \frac{1}{p} - \frac{1}{p} \right) S^p y^p = a \|z^*_\epsilon\|^p C^*_\epsilon.
\]

(65)

Consequently, by (47)

\[
\frac{\sup_{t \geq 0} I^*_\epsilon(t^*_\epsilon)}{t^*_\epsilon} \leq \sup_{t \geq 0} \left( \frac{S^{-q/p} \|z^*_\epsilon\|^q - \int_{I_\Omega} \left( \frac{|z_t^*|^p}{|x|^p} dx \right)}{t^*_\epsilon} \right)
\]

\[
+ \frac{\lambda}{p} \|z^*_\epsilon\|^p \int_{I_\Omega} \left( \frac{|z^*_\epsilon|^p}{|x|^p} dx \right) \leq C^*_\epsilon + O\left( e^{q^*_p\lambda_1 N} \right)
\]

\[
= \begin{cases}
C^*_\epsilon \lambda_1 \epsilon & \text{if } y > N - p\omega^*_\epsilon < C^*_\epsilon, \\
C^*_\epsilon \lambda_1 \epsilon \ln \epsilon & \text{if } y = N - p\omega^*_\epsilon.
\end{cases}
\]

(66)

which is the desired result.

Now, we can prove the existence of a positive solution.

**Proof of Theorem 1.** We verify that the functional \( I^*_\epsilon \) satisfies the mountain pass geometry.

Note that \( I^*_\epsilon(0) = 0 \). Let \( u \in D_0(\Omega) \setminus \{0\} \), by Sobolev and Young inequalities, it holds that

\[
I^*_\epsilon(u) = \frac{a}{q+p} \|u\|^q + \frac{b}{p} \|u\|^p - \frac{1}{p} \int_{\Omega} \left( \frac{|u|^p}{|x|^p} dx - \frac{\lambda}{p} \int_{\Omega} |u|^p dx \right)
\]

\[
\geq - \frac{1}{p} S^{-q/p} \|u\|^q + \frac{a}{q+p} \|u\|^q + \frac{b}{p} \left( b - \frac{\lambda}{\lambda_1} \right) \|u\|^p.
\]

(67)

Let \( \rho = \|u\| \), since \( a > 0, b > 0 \) and from (67), one has

\[
I^*_\epsilon(u) \geq - \frac{1}{p} S^{-q/p} \rho^q + \frac{a}{q+p} \rho^q + \left( b - \frac{\lambda}{\lambda_1} \right) \rho^p.
\]

(68)

Then, we need to consider the following cases.

Case (i) \( q < p^* - p \) and \( a > 0, b > 0, 0 < \lambda < b\lambda_1 \).

As \( q + p < p^* \) there exists a sufficiently small positive numbers \( \rho_1 \) and \( \delta_1 \) such that

\[
I^*_\epsilon(u) \geq \delta_1 > 0, \text{ with } \|u\| = \rho_1.
\]

(69)

Since \( I^*_\epsilon(tu) \to -\infty \) as \( t \to +\infty \) there exists \( t_1 > 0 \) such that \( \|t_1 u\| > \rho_1 \) and \( I^*_\epsilon(t_1 u) < 0 \).

Case (ii) \( p = p^* - p \) and \( 0 < a < S_0 q \), \( b > 0, 0 < \lambda < b\lambda_1 \).

In this case, we have \( q + p = p^* \), and then, from (68), one has

\[
I^*_\epsilon(u) \geq \frac{a}{q+p} S^{-q/p} \rho^q + \left( b - \frac{\lambda}{\lambda_1} \right) \rho^p.
\]

(70)

As \( q < a < S_0 q \) there exist \( \rho_1, \delta_2 > 0 \) such that

\[
I^*_\epsilon(u) \geq \delta_2 > 0, \text{ with } \|u\| = \rho_2.
\]

(71)
On the other hand, using (47) and taking \( \varepsilon_1 > 0 \) small enough, we get
\[
I_A(t_{\varepsilon}) \leq \frac{a^{p+1}}{q+p} \|z_{\varepsilon}\|^{q+p} + \frac{bt^p}{P} \|z_{\varepsilon}\|^p - \frac{b^p}{P} \int_{\Omega} \frac{|z_{\varepsilon}|^p}{|x|^p} \, dx \\
\leq \frac{1}{q+p} \left(a^{\frac{q}{q+p}} - 1\right) S^{\frac{q}{q+p}} t^{q+p} + \frac{b}{P} S^{\frac{q}{q+p}} t^p + O\left(e^{\rho_0(x)p(a+1)-N}\right)
\]
(72)
for all \( \varepsilon \in (0, \varepsilon_1) \). Then, as \( 0 < \varepsilon < S^{q+p}/p \), it follows from the above inequality, \( I_A(t_{\varepsilon}) \to -\infty \) as \( t \to +\infty \). Thus, choosing \( t_2 > 0 \) sufficiently large such that \( \|t_2z_{\varepsilon}\| > \rho_2 \) and \( I_A(t_{2}z_{\varepsilon}) < 0 \).

Set
\[
c = \inf_{y \in \Gamma} \max_{t \in [0,1]} I_A(y(t)),
\]
(73)
where
\[
\Gamma = \{ y \in C([0,1], \mathcal{D}_A(\Omega)), y(0) = 0, y(1) = t_{*}z_{\varepsilon} \},
\]
\[
t_{*} = \begin{cases} t_1 & \text{if } q < p^* - p \text{ and } a > 0 \\ t_2 & \text{if } q = p^* - p \text{ and } 0 < a < S^{q+p}/p \end{cases}
\]
(74)

By the Mountain Pass Theorem, there exists a Palais-Smale sequence \( \{u_n\} \) at level \( c \). Using Lemma 5, we have that \( \{u_n\} \) has a subsequence, still denoted by \( \{u_n\} \), such that \( u_n \rightharpoonup u \) in \( \mathcal{D}_A(\Omega) \) as \( n \to +\infty \). Hence, from Lemmas 7 and 8, we have \( u_n \to u \) in \( \mathcal{D}_A(\Omega) \) as \( n \to +\infty \). Hence, \( I_A'(u) = 0 \) and \( I_A(u) = c > 0 \). So, as \( c = 0 = I_1(0) \), we can conclude that \( u \) is a nonzero solution of (1) with positive energy. Now, we show that \( u > 0 \).

Because
\[
0 = \left(I_A'(u), u^-\right) = (a\|u\|^q + b) \left( \int_\Omega \frac{|u|^{p-2}}{|x|^p} \nabla u \nabla \mu - \mu \frac{|u|^{p-2}}{|x|^{p(a+1)}} u \right) \, dx \\
- \int_\Omega \frac{|u|^{p-2}}{|x|^p} u^+ \, dx + \lambda \int_\Omega \frac{|u|^{p-2}}{|x|^p} \, dx
\geq (a\|u\|^q + b) \left( \int_\Omega \frac{|u|^{p-2}}{|x|^p} - \mu \frac{|u|^{p-2}}{|x|^{p(a+1)}} \, dx \right) + \int_\Omega \frac{|u|^{p-2}}{|x|^p} \, dx
+ \lambda \int_\Omega \frac{|u|^{p-2}}{|x|^p} \, dx \geq b\|u\|^p,
\]
(75)
which implies that \( u^- = 0 \). By the strong maximum principle one has \( u > 0 \). This completes the proof of Theorem 1.

4. Conclusion 1

In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem. Our results and setting were more general and delicate, it difficult to obtain the solution in the degenerate case. Our technique was based on variational methods and concentration compactness argument and we needed to estimate the energy levels. We have shown the existence result for our

problem (1) if \( 1 < p < N, \ 0 \leq \alpha < (N-p)p, \alpha \leq \beta < \alpha + 1, \ \lceil p^2(\alpha + 1) - N \rceil / (p-1) \leq \gamma < p \), \( b > 0 < \lambda < bA_1 \), and \( 0 \leq \mu < \mu_* \) with \( \mu_* = N - (p^2(\alpha + 1) + \gamma(1-p)) \left( N - \gamma \right)^{p-1} / p \),
(76)
and the problem (1) has a positive solution in the following cases: (1) \( q = p^* - p \) and \( 0 < a < S^{q+p}/p \), (2) \( q < p^* - p \) and \( a > 0 \).

Data Availability

The [DATA TYPE] data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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