Noisy Linear Convergence of Stochastic Gradient Descent for CV@R Statistical Learning under Polyak-Łojasiewicz Conditions

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Abstract

Conditional Value-at-Risk (CV@R) is one of the most popular measures of risk, which has been recently considered as a performance criterion in supervised statistical learning, as it is related to desirable operational features in modern applications, such as safety, fairness, distributional robustness, and prediction error stability. However, due to its variational definition, CV@R is commonly believed to result in difficult optimization problems, even for smooth and strongly convex loss functions. We disprove this statement by establishing noisy (i.e., fixed-accuracy) linear convergence of stochastic gradient descent for sequential CV@R learning, for a large class of not necessarily strongly-convex (or even convex) loss functions satisfying a set-restricted Polyak-Łojasiewicz inequality. This class contains all smooth and strongly convex losses, confirming that classical problems, such as linear least squares regression, can be solved efficiently under the CV@R criterion, just as their risk-neutral versions. Our results are illustrated numerically on such a risk-aware ridge regression task, also verifying their validity in practice.

Keywords. Statistical Learning, Risk-Aware Learning, Conditional Value-at-Risk, Stochastic Gradient Descent, Stochastic Approximation, Polyak-Łojasiewicz Inequality.

1 Introduction

Risk-awareness is becoming an increasingly important issue in modern statistical learning theory and practice, especially due to the need to meet strict reliability requirements in high-stakes, critical applications. In such settings, risk-aware learning formulations are particularly appealing, since they can explicitly balance the performance of optimal predictors between average-case and “difficult” to learn, infrequent, or worst-case examples, inducing a form of statistical robustness in the learning outcome. The foundational idea of risk-aware statistical learning is to replace the standard, expected loss learning objective by more general loss functionals, called risk measures, whose purpose is to effectively quantify the statistical variability of the random loss function considered, in addition to average performance. Popular examples of risk measures include mean-variance functionals, mean-semideviations, and Conditional Value-at-Risk (CV@R).
CV@R, in particular, plays a significant role in supervised statistical learning, as it is naturally connected not only to prediction error stability (see Section 7), but also to distributional robustness [Shapiro et al., 2014, Curi et al., 2019], fairness [Williamson and Menon, 2019], as well as the formulation of classical learning problems, such as the celebrated (ν-)SVM [Vapnik, 2000, Schölkopf et al., 2000, Takeda and Sugiyama, 2008, Gotoh and Takeda, 2016]. Relevant generalization bounds were recently reported in [Mhammedi et al., 2020] and [Lee et al., 2020], establishing asymptotic consistency for CV@R learning, as well.

But except for operational effectiveness and generalization performance, computational methods for actually obtaining optimal solutions to CV@R learning problems are of paramount importance, especially for practical considerations. The design of such methods is facilitated by the variational definition of CV@R ([Rockafellar and Uryasev, 2000], also see Section 2), allowing the reduction of any CV@R learning problem to a standard stochastic optimization problem with a special loss function. This approach was followed in [Soma and Yoshida, 2020], where several averaged Stochastic Gradient Descent (SGD)-type algorithms were analyzed under a batch setting (i.e., given a dataset available a priori). Almost concurrently, and under the same setting, [Curi et al., 2019] proposed an adaptive sampling algorithm for CV@R learning, by exploiting the distributionally robust representation of CV@R [Shapiro et al., 2014]. In both works, convergence rates reported are at best of the order of $1/\sqrt{T}$, where $T$ denotes the total runtime of the respective algorithm (iterations).

Such rates might seem to be nearly all we can get: Due to its construction, CV@R is commonly conjectured to result in potentially difficult or badly behaved stochastic problems, mainly because standard properties which enable fast convergence of gradient methods, such as strong convexity, are not preserved when transitioning from (data-driven) risk-neutral to CV@R learning, even for smooth and strongly convex losses. In this work, we disprove this argument by showing that SGD attains noisy (i.e., fixed-tunable-accuracy) linear global convergence for sequential CV@R learning (i.e., provided a datastream), for a large class of not necessarily strongly-convex (or even convex) loss functions satisfying a set-restricted Polyak-Łojasiewicz inequality [Polyak, 1963, Karimi et al., 2016]. As a byproduct of this result, we also obtain noisy linear convergence of SGD for smooth and strongly convex losses, since those belong to the aforementioned class. Essentially, our results confirm that at least from an optimization perspective, CV@R learning is almost as easy as risk-neutral learning. This implies that CV@R learning can have widespread use in applications, since risk-aware versions of ubiquitous problems, such as linear least squares estimation, can be solved as efficiently as their risk-neutral counterparts, and with provable and equivalent rate guarantees.

Numerical simulations on such a basic ridge regression task confirm the validity of our results in a practical setting.

2 CV@R Statistical Learning

Let $\mathcal{P}_D$ be an unknown probability measure over an example space $D \triangleq \mathbb{R}^d \times \mathbb{R}$, and consider a known parametric family of functions $\mathcal{F} \triangleq \{ \phi : \mathbb{R}^m \to \mathbb{R} | \phi(\cdot) \equiv f(\cdot, \theta), \theta \in \mathbb{R}^m \}$, called a hypothesis class. We are interested in the problem of discovering or learning a function $f(\cdot, \theta^*) \in \mathcal{F}$ that best approximates $y$ when presented with the input $x$, where the pair $(x, y)$ follows the example distribution $\mathcal{P}_D$. The instantaneous quality of every admissible predictor $f(\cdot, \theta)$ is expressed by a loss function $\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ taking, for each example $(x, y)$, the quantities $f(x, \theta)$ and $y$ and mapping them to an integrable random variable, $\ell(f(x, \theta), y)$. Due to randomness on the example space, it is generally not possible to minimize losses for all possible examples simultaneously. Instead, it is
standard to consider minimizing an expected loss functional of the form

\[
\inf_{\theta \in \mathbb{R}^m} \left[ \mathbb{E}_{\mathcal{P}_D} \{ \ell(f(\mathbf{x}, \theta), y) \} \equiv \int_{\mathcal{D}} \ell(f(\mathbf{x}, \theta), y) d\mathcal{P}_D(\mathbf{x}, y) \right],
\]

which is at the heart of modern machine learning theory and practice and beyond, such as signal processing, statistics, and control.

Despite its wide popularity, though, a fundamental issue with the gold standard expected loss learning formulation is its very nature: It is risk-neutral, i.e., it minimizes losses only on average. Because of this, it lacks robustness and essentially ignores relatively infrequent but statistically significant example instances, treating them as inconsequential. This is important from a practical point of view, since such “difficult” or “extreme” examples will incur high and/or unacceptably significant \(\ell\) values, e.g., due to \(\ell\)’s defining properties, the optimal prediction error has minimal expected value.

As briefly explained in Section 1, the need for a systematic treatment of the shortcomings of the risk-neutral approach motivates and sets the premise of risk-aware statistical learning, in which expectation is replaced by more general loss functionals, called risk measures \([\text{Shapiro et al., 2014}]\). Their purpose is to induce risk-averse characteristics into the learning outcome by explicitly controlling the statistical variability of the random loss \(\ell(f(\mathbf{x}, \cdot), y)\), or, equivalently, its tail behavior. By far one of the most popular risk measures in theory and practice is \(\text{CV@R}\), which for an integrable random loss \(Z\) is defined as \([\text{Rockafellar and Uryasev, 2000}]\)

\[
\text{CV@R}^\alpha(Z) \triangleq \inf_{t \in \mathbb{R}} \left\{ t + \frac{1}{\alpha} \mathbb{E}\{ (Z - t)_+ \} \right\},
\]

at confidence level \(\alpha \in (0, 1]\). Intuitively, \(\text{CV@R}^\alpha(Z)\) is the mean of the worst \(\alpha\)% of the values of \(Z\), and is a strict generalization of expectation; in particular, it is true that

\[
\text{CV@R}^1(Z) \equiv \mathbb{E}\{ Z \} \leq \text{CV@R}^\alpha(Z), \forall \alpha \in (0, 1], \quad \text{and}
\]

\[
\text{CV@R}^0(Z) \triangleq \lim_{\alpha \downarrow 0} \text{CV@R}^\alpha(Z) \equiv \text{esssup} Z.
\]

One of the most important properties of \(\text{CV@R}\) is that it constitutes a coherent risk measure, meaning that it is a convex, monotone, translation equivariant and positively homogeneous functional of its argument; see (\text{Shapiro et al., 2014}, Section 6.3).

By setting \(Z \equiv \ell(f(\mathbf{x}, \theta), y), \theta \in \mathbb{R}^m\), we may now formulate the \(\text{CV@R}\) statistical learning problem as

\[
\inf_{\theta \in \mathbb{R}^m} \text{CV@R}^\alpha_{\mathcal{P}_D}[\ell(f(\mathbf{x}, \theta), y)].
\]

Observe that due to its defining properties, the \(\text{CV@R}\) problem is most intuitive, and allows for an excellent tunable tradeoff between risk neutrality (for \(\alpha \equiv 1\), and minimax robustness (as \(\alpha \downarrow 0\)). Additionally, because \(\text{CV@R}\) is a coherent risk measure, it follows that problem (5) is convex whenever \(\ell(f(\mathbf{x}, \cdot), y)\) is convex for each \((\mathbf{x}, y)\), and strongly convex whenever \(\ell(f(\mathbf{x}, \cdot), y)\) is strongly convex for each \((\mathbf{x}, y)\) \([\text{Kalogerias and Powell, 2018}]\). Thus, problem (5) is favorably structured.

However, because \(\text{CV@R}\) is itself defined as the optimal value of a stochastic program, it is difficult to evaluate analytically, especially in a data-driven setting. Still, we may leverage the
Assumption 1. Unless the function \( \ell(f(x, \cdot), y) \) is convex on \( \mathbb{R}^m \) for \( \mathcal{P}_D \)-almost all \((x, y)\), then for each \( \theta \in \mathbb{R}^m \):

1) \( \ell(f(x, \cdot), y) \) is \( C_\theta(x, y) \)-Lipschitz on a neighborhood \( \theta \) for \( \mathcal{P}_D \)-almost all \((x, y)\), and it is true that \( \mathbb{E}_{\mathcal{P}_D}\{C_\theta(x, y)\} < \infty \).

2) \( \ell(f(x, \cdot), y) \) is differentiable at \( \theta \) for \( \mathcal{P}_D \)-almost all \((x, y)\), and \( \mathcal{P}_D(\ell(f(x, \theta), y) = t) \equiv 0 \) for all \((\theta, t) \in \mathbb{R}^m \times \mathbb{R} \).

For convenience, let us define, for \((\theta, t) \in \mathbb{R}^m \times \mathbb{R} \),

\[
G_\alpha(\theta, t) \triangleq \mathbb{E}_{\mathcal{P}_D}\left\{ t + \frac{1}{\alpha}(\ell(f(x, \theta), y) - t)_+ \right\}.
\]
Then it may be shown that, under Assumption 1, differentiation may be interchanged with expectation for $G_\alpha$ ([Shapiro et al., 2014], Section 7.2.4), yielding, for every $(\theta, t)$, the (sub)gradient representation

$$
\nabla G_\alpha(\theta, t) = \left[ \frac{1}{\alpha} \mathbb{E}_{P^\alpha} \{ 1_{A(\theta,t)}(x,y) \nabla f(x,\theta) \} \right],
$$

where for brevity and for later use we have defined the *event-valued* multifunction $A : \mathbb{R}^m \times \mathbb{R} \Rightarrow \mathcal{D}$ as

$$
A(\theta, t) \triangleq \{ (x, y) \in \mathcal{D} | \ell(f(x, \theta), y) - t > 0 \},
$$

for $(\theta, t) \in \mathbb{R}^m \times \mathbb{R}$. We note that, for each $(\theta, t)$, the set $A(\theta, t)$ contains all examples corresponding to the *positive section* of the function $\ell(f(\bullet, \theta), \cdot) - t$.

Leveraging (8), and given an independent and identically distributed datastream $\{(x^n, y^n)\}_{n=0}^\infty$, we can now outline the simplest and most obvious scheme for possibly tackling the CV@R problem (6), i.e., the standard SGD rule, described via the recursive updates

$$
t^{n+1} = t^n - \gamma \left[ 1 - \frac{1}{\alpha} 1_{A(\theta^n, t^n)}(x^{n+1}, y^{n+1}) \right] \quad \text{and} \quad \theta^{n+1} = \theta^n - \beta \frac{1}{\alpha} 1_{A(\theta^n, t^n)}(x^{n+1}, y^{n+1}) \nabla \ell(f(x^{n+1}, \theta^n), y^{n+1}),
$$

where $n \in \mathbb{N}$ is an iteration index, $\beta > 0$ and $\gamma > 0$ are constant stepsizes, and where $(\theta^0, t^0)$ are appropriately chosen initial values.

We observe that the SGD updates (10) and (11) can be regarded as a modification of the standard risk-neutral SGD (solving (1)), but where learning happens *if and only if* $\ell(f(x^{n+1}, \theta^n), y^{n+1}) - t^n \geq 0$, for each $n$. The update in $t$ controls the frequency of learning, as well as the proportion of examples that participate in learning. Also note that if $\alpha \equiv 1$, then $t^n$ is nonincreasing, and therefore $\theta^n$ should approach a risk-neutral solution. In the following, we suggestively refer to the algorithm comprised by (10) and (11) as CV@R-SGD.

### 4 Polyak-Łojasiewicz Conditions

We next present the standard Polyak-Łojasiewicz (PL) inequality, first appeared in [Polyak, 1963].

**Definition 1. (PL Polyak [1963])** We say that a function $\varphi : \mathbb{R}^L \rightarrow \mathbb{R}$ satisfies the *Polyak-Łojasiewicz (PL) inequality* with parameter $\mu > 0$ on $\Sigma \subseteq \mathbb{R}^L$, if and only if $\varphi$ is differentiable on $\Sigma$ and, for every $x \in \Sigma$,

$$
\frac{1}{2} \| \nabla \varphi(x) \|^2 \geq \mu (\varphi(x) - \varphi^*),
$$

where $\varphi^* \triangleq \inf_{x \in \Sigma} \varphi(x)$.

In a recent seminal article [Karimi et al., 2016], the PL inequality was exploited to show linear convergence of gradient methods under multiple interesting and useful setups. Further, [Karimi et al., 2016] shows that strong convexity implies the PL inequality, but also that there are lots of nonconvex functions obeying the PL inequality. This indeed implies that S(GD) converges *globally and linearly* for such functions.

For our purposes, unfortunately, the standard PL inequality (Definition 1) will not suffice. Instead, we introduce and rely on a generalization, which we call the *set-restricted PL inequality*, as follows.
**Definition 2. (Set-Restricted PL)** Consider a measurable function $\varphi : \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}$, a Borel-valued multifunction $B : \mathbb{R}^L \rightrightarrows \mathbb{R}^M$, and a probability measure $\mathcal{M}$ on $\mathfrak{B}(\mathbb{R}^M)$. We say that $\varphi$ satisfies the (diagonal) $B$-restricted Polyak-Lojasiewicz (PL) inequality with parameter $\mu > 0$, relative to $\mathcal{M}$ and on a subset $\Sigma \subseteq \mathbb{R}^L$, if and only if $\varphi(\cdot, w)$ is subdifferentiable on $\Sigma$ for $\mathcal{M}$-almost every $w \in \mathbb{R}^M$, and it is true that, for every $z \in \Sigma$,

$$
\frac{1}{2} \| E_M(\nabla z \varphi(z, w)|B(z)) \|_2^2 \geq \mu E_M\{ \varphi(z, w) - \varphi^*(z)|B(z)\},
$$

where $\varphi^*(\cdot) \triangleq \inf_{z \in \Sigma} E_M\{ \varphi(z, w)|B(\cdot)\}$.

Although admittedly somewhat mysterious at first sight, the set-restricted PL inequality is essentially the same as the classical PL inequality as considered for standard stochastic optimization [Karimi et al., 2016], with the important difference that expectation is replaced by conditional expectation relative to an event varying in the argument of the function involved (i.e., an event-valued multifunction). From a learning perspective, the set-restricted PL inequality quantifies the curvature of the loss surface by restricting attention on sets of learning examples that matter (in Definition 2, $B$ plays this role).

One fact revealing the importance of the set-restricted PL inequality of Definition 2 is that it is satisfied by all smooth and strongly convex losses. In particular, we have the following result.

**Proposition 1. (Strong Convexity $\implies$ Set-Restricted PL)** Suppose that the loss $\ell(f(x, \cdot), y)$ is $L$-smooth and $\mu$-strongly convex for $\mathcal{P}_\mathcal{D}$-almost all $(x, y)$. Then, for every pair $(\theta, B) \in \mathbb{R}^m \times \mathfrak{B}(\mathcal{D})$ such that $\mathcal{P}_\mathcal{D}(B) > 0$, it is true that

$$
\frac{1}{2} \| E_\theta(\nabla \ell(f(x, \theta), y)|B) \|_2^2 \geq \mu E_\theta\{ \ell(f(x, \theta), y) - \ell^*(B)|B\},
$$

where $\ell^*(B) \equiv \inf_{\tilde{\theta}} E\{ \ell(f(x, \tilde{\theta}), y)|B\}$.

**Proof of Proposition 1.** Taking conditional (rescaled) expectations relative to $B$, we get that, for every qualifying pair $(\theta, \theta')$,

$$
E\{ \ell(f(x, \theta), y)|B\} \geq E\{ \ell(f(x, \theta'), y)|B\} + \langle E_\theta(\nabla \ell(f(x, \theta'), y)|B), \theta - \theta' \rangle + \frac{\mu}{2} \| \theta - \theta' \|_2^2.
$$

By Assumption 1, we may interchange expectation with differentiation, further obtaining

$$
L_B(\theta) \geq L_B(\theta') + \langle \nabla L_B(\theta), \theta - \theta' \rangle + \frac{\mu}{2} \| \theta - \theta' \|_2^2, \quad \forall (\theta, \theta'),
$$

where $L_B(\cdot) \triangleq E\{ \ell(f(x, \cdot), y)|B\}$. This shows that the restricted expected loss $L_B$ is $\mu$-strongly convex. In exactly the same fashion, it follows that $L_B$ is $L$-smooth, as well. Consequently, $L_B$ satisfies the PL inequality with parameter $\mu$ [Karimi et al., 2016], i.e., it is true that, for every qualifying $\theta$,

$$
\frac{1}{2} \| \nabla L_B(\theta) \|_2^2 \geq \mu (L_B(\theta) - \inf_\theta L_B(\theta)).
$$

But $\nabla L_B(\cdot) \equiv E\{ \nabla_\theta \ell(f(x, \cdot), y)|B\}$. Enough said.

From Proposition 1, it follows that every smooth strongly convex loss satisfies the set-restricted PL inequality relative to any qualifying event-valued multifunction of choice. For instance, in the notation of Proposition 1, one may set $B \equiv \mathcal{A}(\theta, t)$, for every fixed pair $(\theta, t)$. This choice is particularly important, as we will see in the next section.
5 Linear Convergence of CV@R-SGD

In this section, we present the main results of the paper. We start by showing that, quite interestingly, if the loss satisfies the set-restricted PL inequality relative to the multifunction $G$, then the objective function $G_{\alpha}$ satisfies the ordinary PL inequality. The relevant result follows.

Lemma 1. ($G$ is Polyak-Łojasiewicz) Consider a subset $\Delta \triangleq \Delta_m \times \Delta_1 \subseteq \mathbb{R}^m \times \mathbb{R}$, and suppose that the following are in effect:

- $0 < \delta \leq \inf_{(\theta,t) \in \Delta} P_D(A(\theta,t))$,
- the random loss $\ell(f(x,\cdot),y)$ satisfies the $A$-restricted PL inequality on with parameter $\mu > 0$, relative to $D$ and on $\Delta_m \times \Delta$, i.e.,
  \[ \frac{1}{2} \|\mathbb{E}\{\nabla_{\theta}\ell(f(x,\theta),y)|A(\theta,t)\}\|^2 \geq \mu \mathbb{E}\{\ell(f(x,\theta),y) - \ell^*(\theta,t)|A(\theta,t)\}, \]
  \[ \text{for all } (\theta,t) \in \Delta, \text{ where } \ell^*(\cdot,\cdot) \equiv \inf_{\tilde{\theta} \in \Delta_m} \mathbb{E}\{\ell(f(x,\tilde{\theta}),y)|A(\cdot,\cdot)\}. \]
- $\arg\min_{(\theta,t) \in \Delta} G_{\alpha}(\theta,t) \neq \emptyset$, with $(\theta^*,t^*)$ being an arbitrary member of this set.

Then, for $\alpha \leq 2\mu \delta$ and as long as
\[ \nabla_t G_{\alpha}(\theta,t)((t^*-t) + \nabla_t G_{\alpha}(\theta,t)) \geq 0, \quad \forall (\theta,t) \in \Delta, \]
the CV@R objective $G_{\alpha}$ obeys the ordinary Polyak-Łojasiewicz inequality with parameter $1/2$, everywhere on $\Delta$.

Proof of Lemma 1. For every $(x,y)$, we have
\[ g_{(x,y)}^\alpha(\theta,t) - g_{(x,y)}^\alpha(\theta^*,t^*) \]
\[ \equiv t - t^* + \frac{1}{\alpha}(\ell(f(x,\theta),y) - t) - \frac{1}{\alpha}(\ell(f(x,\theta^*),y) - t^*) \]
\[ \leq t - t^* + \frac{1}{\alpha}(\ell(f(x,\theta),y) - t) - \frac{1}{\alpha}1_{A(\theta,t)}(x,y)(\ell(f(x,\theta^*),y) - t^*) \]
\[ \equiv t - t^* + \frac{1}{\alpha}1_{A(\theta,t)}(x,y)\left[\ell(f(x,\theta),y) - \ell(f(x,\theta^*),y) + t^* - t\right] \]
\[ = (t^* - t)\left(\frac{1}{\alpha}1_{A(\theta,t)}(x,y) - 1\right) + \frac{1}{\alpha}1_{A(\theta,t)}(x,y)\left[\ell(f(x,\theta),y) - \ell(f(x,\theta^*),y)\right]. \]

By taking expectation on both sides, it follows that
\[ G_{\alpha}(\theta,t) - G_{\alpha}(\theta^*,t^*) \]
\[ \leq (t^* - t)\left(\frac{1}{\alpha}P_D(A(\theta,t)) - 1\right) + \frac{1}{\alpha}\mathbb{E}\{1_{A(\theta,t)}(x,y)[\ell(f(x,\theta),y) - \ell(f(x,\theta^*),y)]\} \]
\[ \equiv (t^* - t)\left(\frac{1}{\alpha}P_D(A(\theta,t)) - 1\right) + \frac{1}{\alpha}\mathbb{E}\{\ell(f(x,\theta),y) - \ell(f(x,\theta^*),y)|A(\theta,t)\}P_D(A(\theta,t)) \]
\[ \equiv (t^* - t)\left(\frac{1}{\alpha}P_D(A(\theta,t)) - 1\right) + \frac{1}{\alpha}(\mathbb{E}\{\ell(f(x,\theta),y)|A(\theta,t)\} - \mathbb{E}\{\ell(f(x,\theta^*),y)|A(\theta,t)\})P_D(A(\theta,t)) \]
\[ \leq (t^* - t)\left(\frac{1}{\alpha}P_D(A(\theta,t)) - 1\right) + \frac{1}{\alpha}(\mathbb{E}\{\ell(f(x,\theta),y)|A(\theta,t)\} - \ell^*(\theta,t))P_D(A(\theta,t)) \]
\[
\equiv (t^* - t) \left( \frac{1}{\alpha} \mathcal{P}_D(\mathcal{A}(\theta, t)) - 1 \right) + \frac{1}{\alpha} \left( \mathbb{E}\{\ell(f(x, \theta), y) - \ell^*(\theta, t)|\mathcal{A}(\theta, t)\} \right) \mathcal{P}_D(\mathcal{A}(\theta, t))
\]

Therefore, from the set-restricted PL inequality we get
\[
G_{\alpha}(\theta, t) - G_{\alpha}(\theta^*, t^*) \leq (t^* - t) \left( \frac{1}{\alpha} \mathcal{P}_D(\mathcal{A}(\theta, t)) - 1 \right) + \frac{1}{2\alpha} \|\mathbb{E}\{\nabla_\theta \ell(f(x, \theta), y)|\mathcal{A}(\theta, t)\}\|_2^2 \mathcal{P}_D(\mathcal{A}(\theta, t)).
\]

Next, assuming that
\[
\nabla_t G_{\alpha}(\theta, t)((t^* - t) + \nabla_t G_{\alpha}(\theta, t)) \geq 0
\]
\[
\iff \left( 1 - \frac{1}{\alpha} \mathcal{P}_D(\mathcal{A}(\theta, t)) \right) (t^* - t) + \left( 1 - \frac{1}{\alpha} \mathcal{P}_D(\mathcal{A}(\theta, t)) \right)^2 \geq 0
\]
and noting that \(\delta \leq \mathcal{P}(\mathcal{A}(\theta, t))\) for all \((\theta, t) \in \Delta\), we may further write
\[
G_{\alpha}(\theta, t) - G_{\alpha}(\theta^*, t^*) \leq \left( \frac{1}{\alpha} \mathcal{P}_D(\mathcal{A}(\theta, t)) - 1 \right)^2 + \frac{1}{2\alpha} \|\mathbb{E}\{\nabla_\theta \ell(f(x, \theta), y)|\mathcal{A}(\theta, t)\}\|_2^2 \mathcal{P}_D(\mathcal{A}(\theta, t))_.
\]

Therefore, with \(\alpha \leq 2\mu\delta\), we obtain
\[
G_{\alpha}(\theta, t) - G_{\alpha}(\theta^*, t^*) \leq \left( \frac{1}{\alpha} \mathcal{P}_D(\mathcal{A}(\theta, t)) - 1 \right)^2 + \frac{1}{2\alpha} \|\mathbb{E}\{\nabla_\theta \ell(f(x, \theta), y)|\mathcal{A}(\theta, t)\}\|_2^2 \mathcal{P}_D(\mathcal{A}(\theta, t)).
\]

Now, observe that
\[
\nabla G_{\alpha}(\theta, t) = \left[ \frac{1}{\alpha} \mathbb{E}\{1_{\mathcal{A}(\theta, t)}(x, y) \nabla_\theta \ell(f(x, \theta), y)\}, 1 - \frac{1}{\alpha} \mathcal{P}_D(\mathcal{A}(\theta, t)) \right],
\]
from where we immediately deduce that, for every \((\theta, t) \in \Delta\),
\[
\frac{1}{2}(G(\theta, t) - G(\theta^*, t^*)) \leq \frac{1}{2} \|\nabla G(\theta, t)\|_2^2,
\]
This shows that \(G\) satisfies the PL inequality with parameter \(\mu' \equiv 1/2\) on \(\Delta\), under the above stated assumptions.

For brevity, let CV@R^\alpha be the optimal value of (5) (given \(\alpha\)). Our main result follows, showing linear convergence of CV@R-SGD under the set-restricted PL inequality.

**Theorem 1. (Linear Convergence of CV@R-SGD)** Let Assumption 1 be in effect and suppose that, for a subset \(\Delta \equiv \Delta_m \times [-\infty, T]\), with \(\Delta_m \subseteq \mathbb{R}^m\), it holds that \(\delta \equiv \inf_{\theta \in \Delta_m} \mathcal{P}_D(\mathcal{A}(\theta, 1)) > 0\), and that the loss \(\ell(f(x, \cdot), y)\) obeys the \(A\)-restricted PL inequality with parameter \(\mu > 0\) relative to \(\mathcal{P}_D\) on \(\Delta\). Choose \(\alpha \in (0, 2\mu\delta] \cap (0, \delta)\) and, provided an optimal solution to (5) exists, say \((\theta^*, t^*)\), suppose further that there is another subset \(\Delta' \equiv \Delta_m' \times \Delta_1' \subseteq \Delta\), where \(\theta^* \in \Delta_m' \subseteq \Delta_m\) and
$\Delta'_1 \triangleq [t^* - (\delta - \alpha)/\alpha, \overline{t}]$. Then, for fixed $T \in \mathbb{N} \cup \{\infty\}$ and $\min\{\beta, \gamma\} < 1$, as long as $(\theta^n, t^n) \in \Delta'$, $n \in \mathbb{N}_T$ and $G_{\alpha}$ is $L$-smooth on $\Delta'$, it is true that

$$
\begin{align*}
\mathbb{E}\{G_{\alpha}(\theta^{T+1}, t^{T+1}) - CV@R^\alpha_T\} \leq (1 - \min\{\beta, \gamma\})^T (G_{\alpha}(\theta^0, t^0) - CV@R^\alpha_T) \\
+ \frac{(\max\{\beta, \gamma\})^2 L (1 + C_T^2)}{2 \alpha^2},
\end{align*}
$$

where $\sup_{n \in \mathbb{N}_T} \mathbb{E}\{\|\nabla_{\theta} \ell(f(x^{n+1}, \theta^n), y^{n+1})\|^2\} \leq C_T^2$.

**Proof of Theorem 1.** First, by the assumptions of the theorem, we observe that

$$
\alpha < \delta \equiv \inf_{\theta \in \Delta_m} \mathcal{P}_{\Delta}(A(\theta, t)) \equiv \inf_{(\theta, t) \in \Delta} \mathcal{P}_{\Delta}(A(\theta, t)) \leq \mathcal{P}_{\Delta}(A(\theta, t)).
$$

Therefore, to invoke Lemma 1 on the set $\Delta'$, we also need to verify that

$$
\nabla_t G_{\alpha}(\theta, t)(t^* - t) + \nabla_i G_{\alpha}(\theta, t) \geq 0, \quad \forall (\theta, t) \in \Delta'.
$$

The case $\nabla_i G_{\alpha}(\theta, t) \equiv 0$ is trivial, and thus it suffices to examine the cases where $\nabla_i G_{\alpha}(\theta, t) \leq 0$. If $\nabla_i G_{\alpha}(\theta, t) > 0$, it must equivalently be true that $\alpha > \mathcal{P}_{\Delta}(A(\theta, t))$, and thus this case never happens either. If, however, $\nabla_i G_{\alpha}(\theta, t) < 0$, it must be the case that

$$
t^* - t + \nabla_i G_{\alpha}(\theta, t) \leq 0 \iff \mathcal{P}(A(\theta, t)) \geq \alpha + \alpha(t^* - t).
$$

But since the assumption that $t \in \Delta'_1$ implies that

$$
t^* - \frac{\delta - \alpha}{\alpha} \leq t \iff t^* - t \leq \frac{\delta - \alpha}{\alpha},
$$

(31) will be true because

$$
\mathcal{P}_{\Delta}(A(\theta, t)) \geq \delta \equiv \alpha + \alpha \frac{\delta - \alpha}{\alpha} \geq \alpha + \alpha(t^* - t).
$$

Therefore, it follows by Lemma 1 that $G_{\alpha}$ obeys the PL inequality on $\Delta$ with parameter $1/2$.

Now, since, also by assumption,

$$
(\theta^n, t^n) \in \Delta', \quad \forall n \in \mathbb{N}_T,
$$

for some $T \in \mathbb{N} \cup \{\infty\}$, we may exploit $L$-smoothness of $G_{\alpha}$ together with the SGD updates to write

$$
G_{\alpha}(\theta^{n+1}, t^{n+1}) \leq G_{\alpha}(\theta^n, t^n) - \langle \nabla G_{\alpha}(\theta^n, t^n), [\beta 1_m \gamma] T \circ \nabla g_{(x^{n+1}, y^{n+1})}^\alpha(\theta^n, t^n) \rangle \\
+ \frac{L}{2}\|\beta 1_m \gamma T \circ \nabla g_{(x^{n+1}, y^{n+1})}^\alpha(\theta^n, t^n)\|_2^2
$$

for each $n \in \mathbb{N}_T$, where “$\circ$” denotes the Hadamard product. Taking expectations relative to $\mathcal{D}_n$, we obtain

$$
\mathbb{E}\{G_{\alpha}(\theta^{n+1}, t^{n+1})|\mathcal{D}_n\} \leq G_{\alpha}(\theta^n, t^n) - \langle \nabla G_{\alpha}(\theta^n, t^n), [\beta 1_m \gamma] T \circ \nabla G_{\alpha}(\theta^n, t^n) \rangle
$$

for each $n \in \mathbb{N}_T$. Then, for fixed $T \in \mathbb{N} \cup \{\infty\}$ and $\min\{\beta, \gamma\} < 1$, as long as $(\theta^n, t^n) \in \Delta'$, $n \in \mathbb{N}_T$ and $G_{\alpha}$ is $L$-smooth on $\Delta'$, it is true that

$$
\mathbb{E}\{G_{\alpha}(\theta^{T+1}, t^{T+1}) - CV@R^\alpha_T\} \leq (1 - \min\{\beta, \gamma\})^T (G_{\alpha}(\theta^0, t^0) - CV@R^\alpha_T) \\
+ \frac{(\max\{\beta, \gamma\})^2 L (1 + C_T^2)}{2 \alpha^2},
$$

where $\sup_{n \in \mathbb{N}_T} \mathbb{E}\{\|\nabla_{\theta} \ell(f(x^{n+1}, \theta^n), y^{n+1})\|^2\} \leq C_T^2$. 

\[
+ \frac{L}{2} \mathbb{E}\left\{ \| \beta \mathbf{1}_m \gamma \|_2 \right\} \circ \nabla g_{(x_{n+1}, y_{n+1})}^\alpha(\theta^n, t^n) \|_2^2 | \mathcal{D}_n \} \\
\leq G_\alpha(\theta^n, t^n) - \min\{\beta, \gamma\}\| \nabla G_\alpha(\theta^n, t^n) \|_2^2 \\
+ \frac{L}{2} (\max\{\beta, \gamma\})^2 \mathbb{E}\{\| \nabla g_{(x_{n+1}, y_{n+1})}^\alpha(\theta^n, t^n) \|_2^2 \mathcal{D}_n \},
\]

By applying the (standard) PL inequality for \( G_\alpha \), and using the fact that

\[
\| \nabla g_{(x_{n+1}, y_{n+1})}^\alpha(\theta^n, t^n) \|_2^2 \equiv \left( 1 - \frac{1}{\alpha} \mathbf{1}_\mathcal{A}(\theta^n, t^n)(x_{n+1}, y_{n+1}) \right)^2 \\
+ \frac{1}{\alpha^2} \mathbf{1}_\mathcal{A}(\theta^n, t^n) \| \nabla \ell(f(x_{n+1}, \theta^n), y_{n+1}) \|_2^2 \\
\leq \max\left\{ 1, \left( \frac{1 - \alpha}{\alpha} \right)^2 \right\} \\
+ \frac{1}{\alpha^2} \| \nabla \ell(f(x_{n+1}, \theta^n), y_{n+1}) \|_2^2 \\
\leq \frac{1}{\alpha^2} + \frac{1}{\alpha^2} \| \nabla \ell(f(x_{n+1}, \theta^n), y_{n+1}) \|_2^2,
\]

we further get

\[
\mathbb{E}\{G_\alpha(\theta^{n+1}, t^{n+1}) | \mathcal{D}_n \} \leq G_\alpha(\theta^n, t^n) - \min\{\beta, \gamma\}(G_\alpha(\theta^n, t^n) - G_\alpha(\theta^*, t^*)) \\
+ \frac{L}{2} (\max\{\beta, \gamma\})^2 1 + \mathbb{E}\{\| \nabla \ell(f(x_{n+1}, \theta^n), y_{n+1}) \|_2^2 \mathcal{D}_n \},
\]

Rearranging and taking expectation one more time, it follows that

\[
\mathbb{E}\{G_\alpha(\theta^{n+1}, t^{n+1}) - G_\alpha(\theta^n, t^n) \} \leq (1 - \min\{\beta, \gamma\})(G_\alpha(\theta^n, t^n) - G_\alpha(\theta^*, t^*)) \\
+ \frac{L}{2} (\max\{\beta, \gamma\})^2 1 + \frac{C_\ell^2}{\alpha^2},
\]

where we have used that \( \sup_{n \in \mathbb{N}} \mathbb{E}\{\| \nabla \ell(f(x_{n+1}, \theta^n), y_{n+1}) \|_2^2 \} \leq C_\ell^2 \). Using that \( \min\{\beta, \gamma\} < 1 \) and applying this inequality recursively, we may easily see that

\[
\mathbb{E}\{G_\alpha(\theta^{T+1}, t^{T+1}) - G_\alpha(\theta^n, t^n) \} \leq (1 - \min\{\beta, \gamma\})^T (G_\alpha(\theta^n, t^n) - G_\alpha(\theta^*, t^*)) \\
+ \frac{(\max\{\beta, \gamma\})^2 L(1 + C_\ell^2)}{\min\{\beta, \gamma\} 2\alpha^2}.
\]

The proof is complete. \( \blacksquare \)

A couple of remarks regarding the assumptions and conclusions of Theorem 1 are essential at this point. First, for a subset \( \Delta^t \) to exist, it must be true that \( t^* \leq \bar{t} + (\delta - \alpha)/\alpha \). From (Shapiro et al., 2014, Section 6.2.4), we know that, given the level \( \alpha \), the smallest optimal \( t^* \) may be chosen equal to the quantile \( F_{\ell(f(x, \theta^*), y)}^{-1}(1 - \alpha) \), where \( F_Z \) denotes the cumulative distribution function of the random variable \( Z \). Then, it must be the case that

\[
t^* \leq \bar{t} + (\delta - \alpha)/\alpha \iff \alpha \geq \mathcal{P}(\mathcal{A}(\theta^*, \bar{t} + (\delta - \alpha)/\alpha)),
\]

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for the particular choice of $t$, and further this is only possible if
\[ \mu \geq (2\delta)^{-1} \mathcal{P}_D(A(\theta^*, t + (\delta - \alpha)/\alpha)), \] (42)
since $\alpha \leq 2\mu\delta$. We see that, on the one hand, $t$ must be chosen small enough such that $\delta$ is large enough, also placing a lower restriction on $\alpha$ (cf. (41)), while, on the other hand, $\mu$ has to be large enough such that the particular choice of $\alpha$ is feasible (cf. (42)).

Although these dependencies might seem fairly restrictive, they are very reasonable, since in order for CV@R-SGD to converge fast, the condition $\ell(f(x^{n+1}, \theta^n), y^{n+1}) - t^n \geq 0$ needs to be satisfied sufficiently often. But all this is reasonable from a practical perspective as well: If $\alpha$ is closer to 1 (risk-neutral setting), risky events are effectively smoothened, whereas, if $\alpha$ approaches zero, only rare events matter, and an essentially robust solution is sought, which does not really exhibit the dynamic character of a risk-aware solution. Therefore, depending on the problem, $\alpha$ should be chosen modestly, providing both non-trivial results and fast linear convergence; from a conceptual point of view, there is a certain logical balance to be respected between moderatism and conservatism.

Second, the set-restricted PL inequality involved in Theorem 1 may still look mysterious, but is indeed useful. In fact, by Proposition 1, a byproduct of Theorem 1 is that CV@R-SGD converges linearly to fixed, user-tunable accuracy whenever $\ell(f(x, \cdot), y)$ is strongly convex and smooth for every $(x, y)$, even though $G_\alpha$ might not be strongly convex. This is especially important, because it shows that classical problems, such as linear least squares regression, can provably be solved most efficiently using SGD under risk-aware performance criteria, i.e., the CV@R, just as their risk-neutral counterparts (for instance, via the celebrated Least-Mean-Squares (LMS) algorithm for linear least squares problems).

Third, a reasonable question is: How one actually ensures that $t^n \in \Delta_1$ in practice? Well, for an appropriate choice of $\alpha$ (see discussion above), this can be achieved by setting $t^n$ small enough. Since $\{t^n\}_n$ is merely a scalar sequence, this is easy to do in practice; also see our numerical results in Section 7.

6 Enforcing Smoothness

There are two potential issues associated with the CV@R problem (6) and the assumptions ensuring linear convergence of CV@R-SGD, as suggested in Theorem 1. The first is that there are useful cases where the demand that $\mathcal{P}_D(\ell(f(x, \cdot), y) = (\cdot)) \equiv 0$ on $\mathbb{R}^m \times \mathbb{R}$ (see Assumption 1.2) might not be satisfied; this happens, e.g., in classification problems where the hypothesis class $\mathcal{F}$ contains hard classifiers, i.e., functions with binary or discrete range. The second issue is that the smoothness assumption on $G_\alpha$, essential to obtain the rate promised by Theorem 1, might not be easy to verify or even hold by merely assuming that the loss $\ell(f(x, \cdot), y)$ is smooth; this is due to the presence of the indicator $1_{A(\bullet, \cdot)}(x, y)$ next to $\nabla \ell(f(x, \bullet), y)$ in (8). It turns out that these two issues are related, and both may be mitigated by a rather simple strategy, which we now discuss.

Consider an augmented example $(x, y, w)$, where $w \sim \mathcal{N}(0, \sigma^2)$, $\sigma^2 > 0$, is a fictitious target, independent of $(x, y)$, which we choose to use adversarially during the training process. In particular, we do that by defining the surrogate loss $\bar{\ell} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ as
\[ \bar{\ell}(f(x, \theta), y, w) \triangleq \ell(f(x, \theta), y) - w, \] (43)
Although such a surrogate loss is meaningless in the risk-neutral setting (since $\mathbb{E}\{w\} \equiv 0$), it provides regularization in risk-aware and, in particular, CV@R statistical learning. In fact, it can be easily
shown that, by choosing \( \tilde{\ell} \) as the loss, Assumption 1.2 is always satisfied, and the resulting objective function in problem (6) is \( L' \)-smooth whenever \( \ell(f(x, \cdot), y) \) is \( G \)-Lipschitz and \( L \)-smooth, with

\[
L' = \frac{L \sigma \sqrt{2\pi} + G^2}{\alpha \sigma \sqrt{2\pi}}. 
\]

(44)

To see those facts, observe that because \( w \) is independent of \((x, y)\), we may write

\[
P_D(\tilde{\ell}(f(x, \theta), y, w) = t) \equiv P_D(\ell(f(x, \theta), y) - w = t) \\
\equiv \mathbb{E}_{P_D}\{P(w(\ell(f(x, \theta), y) - t = w| x, y)) \equiv 0, 
\]

(45)
since \( w \) is a continuous random variable. This shows that Assumption 1.2 is satisfied. Further, recall the expression for the gradient \( \nabla G_\alpha \) which, for the loss \( \tilde{\ell} \) considered here, becomes

\[
\nabla G_\alpha(\theta, t) = \left[ \frac{1}{\alpha} \mathbb{E}_{P_D}\{1_A(\theta, t)(x, y, w)\nabla_\theta \tilde{\ell}(f(x, \theta), y, w)\} - \frac{1}{\alpha} \mathbb{E}_{P_D}\{1_A(\theta, t)(x, y, w)\} + 1 \right],
\]

(46)

where we additionally identify \( \tilde{D} \triangleq \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \). We first readily see that

\[
\mathbb{E}_{P_D}\{1_A(\theta, t)(x, y, w)\} \equiv \mathbb{E}_{P_D}\{\mathbb{E}_{P_w}\{1_A(\theta, t)(x, y, w)|x, y)\} \\
\equiv \mathbb{E}_{P_D}\{P(w(\ell(f(x, \theta), y) - t > w|x, y)) \\
= \mathbb{E}_{P_D}\{\Phi\left(\frac{\ell(f(x, \theta), y) - t}{\sigma}\right)\},
\]

(47)

where \( \Phi : \mathbb{R} \rightarrow [0, 1] \) denotes the standard Gaussian cumulative distribution function. In similar fashion, we also obtain

\[
\mathbb{E}_{P_D}\{1_A(\theta, t)(x, y, w)\nabla_\theta \tilde{\ell}(f(x, \theta), y, w)\} \equiv \mathbb{E}_{P_D}\{1_A(\theta, t)(x, y, w)\nabla_\theta \ell(f(x, \theta), y)\} \\
\equiv \mathbb{E}_{P_D}\{\mathbb{E}_{P_w}\{1_A(\theta, t)(x, y, w)|x, y)\nabla_\theta \ell(f(x, \theta), y)\} \\
\equiv \mathbb{E}_{P_D}\{P(w(\ell(f(x, \theta), y) - t > w|x, y))\nabla_\theta \ell(f(x, \theta), y)\} \\
\equiv \mathbb{E}_{P_D}\{\Phi\left(\frac{\ell(f(x, \theta), y) - t}{\sigma}\right)\nabla_\theta \ell(f(x, \theta), y)\}.
\]

(48)

Therefore, the gradient \( \nabla G_\alpha \) may be equivalently represented as

\[
\nabla G_\alpha(\theta, t) = \left[ \frac{1}{\alpha} \mathbb{E}_{P_D}\{\Phi\left(\frac{\ell(f(x, \theta), y) - t}{\sigma}\right)\nabla_\theta \ell(f(x, \theta), y)\} \\
- \frac{1}{\alpha} \mathbb{E}_{P_D}\{\Phi\left(\frac{\ell(f(x, \theta), y) - t}{\sigma}\right)\} + 1 \right].
\]

(49)

Our claims above readily follow by exploiting this gradient representation.

Further, because it is true that [Kalogerias and Powell, 2018]

\[
\mathbb{E}_{P_w}\{(z - w)_+\} = \sigma \left(\frac{z}{\sigma} \Phi\left(\frac{z}{\sigma}\right) + \phi\left(\frac{z}{\sigma}\right)\right) \equiv \mathcal{R}_\sigma(z), \quad \forall z \in \mathbb{R},
\]

(50)
Figure 7.1: Comparison between risk-neutral (LMS) and risk-aware (CV@R-SGD) ridge regression: Evolution of iterates \( \{\theta^n\}_n \).

where \( \phi : \mathbb{R} \to \mathbb{R}_+ \) denotes the standard Gaussian density, and due to the fact that

\[
(z)_+ \leq \mathcal{R}_\sigma(z) \leq \mathcal{R}_\sigma(0) + (z)_+ \equiv \frac{\sigma}{\sqrt{2\pi}} + (z)_+, \quad \forall z \in \mathbb{R},
\]

we may readily derive uniform estimates in \((\theta, t)\)

\[
\text{CV@R}^\alpha_{\mathcal{P}_D}[\ell(f(x, \theta), y)] \leq \text{CV@R}^\alpha_{\mathcal{P}_D}[\tilde{\ell}(f(x, \theta), y, w)] \leq \text{CV@R}^\alpha_{\mathcal{P}_D}[\ell(f(x, \theta), y)] + \frac{\sigma}{\alpha \sqrt{2\pi}}.
\]

Then, similarly to Theorem 1, we obtain linear convergence up to fixed accuracy

\[
\frac{(\max\{\beta, \gamma\})^2}{\min\{\beta, \gamma\}} \left(1 + C^2_f\right) L\sigma \sqrt{2\pi} + G^2 + \frac{\sigma}{\alpha \sqrt{2\pi}}
\]

which by proper choice of \( \sigma \) results in a quantity of the order of

\[
\left(\sqrt{(\max\{\beta, \gamma\})^2 / \min\{\beta, \gamma\}}\right) / \alpha^2.
\]

We observe that this result is slightly worse than that of Theorem 1.

7 A Simple Numerical Example

In this section, we numerically demonstrate the behavior of CV@R-SGD, confirming the validity of Theorem 1. To this end, we consider the \( \lambda \)-strongly convex, risk-aware ridge regression problem

\[
\inf_{\theta \in \mathbb{R}^m} \text{CV@R}^\alpha_{\mathcal{P}_D}\left[(y - \langle \theta, x \rangle)^2 + \lambda \|\theta\|_2^2\right],
\]

where \( y \equiv \langle \theta_o, x \rangle \in \mathbb{R} \) for a constant \( \theta_o \in \mathbb{R}^7 \) and with the elements of \( x \in \mathbb{R}^7 \) being independent uniform in \([0, 2] \), \( \lambda \equiv 0.1 \) and \( \alpha \equiv 0.2 \). Therefore, our goal is to find a \( \theta^* \) which minimizes the mean of the worst 80% of all possible values of the random error \( (y - \langle \cdot, x \rangle)^2 + \lambda \|\cdot\|^2_2 \). Note that, for \( \alpha \equiv 1 \), problem (55) reduces to ordinary ridge regression, and may be solved via the LMS algorithm.

Figs. 7.1 and 7.2 show the iterate evolution as well as the behavior of the optimal prediction (test) error for both CV@R-SGD (with stepsizes \( \beta \equiv \alpha \times 0.01 \) and \( \gamma \equiv 0.001 \)) and the LMS scheme (with
Figure 7.2: Comparison between risk-neutral (LMS) and risk-aware (CV@R-SGD) ridge regression: Histogram (left) and actual values (right) of the test error.

stepsize \( \beta \equiv 0.01 \), respectively. We observe that both algorithms converge at an essentially identical noisy linear rate, in line with Theorem 1. However, the solutions are radically different. In fact, the risk-aware solution discovered by CV@R-SGD dramatically reduces the volatility of prediction error, and provides prediction stability. Although this apparently comes at the cost sacrificing mean performance, such sacrifice is fully user-customizable by varying the CV@R level \( \alpha \).

8 Conclusion

In this work, we established noisy linear convergence of SGD for sequential CV@R learning, for a large class of possibly nonconvex loss functions satisfying a set-restricted PL inequality, also including all smooth and strongly convex losses as special cases. This result disproves the belief that CV@R learning is fundamentally difficult, and shows that classical learning problems can be solved efficiently under CV@R criteria, just as their risk-neutral versions. Our theory was also illustrated via an indicative numerical example. Future work includes the consideration of special learning settings such as linear least squares, as well as other risk measures beyond CV@R.

References

Mehdi Bennis, Merouane Debbah, and H. Vincent Poor. Ultrareliable and Low-Latency Wireless Communication: Tail, Risk, and Scale. Proceedings of the IEEE, 106(10):1834–1853, October 2018. ISSN 15582256. doi: 10.1109/JPROC.2018.2867029.

Adrian Rivera Cardoso and Huan Xu. Risk-Averse Stochastic Convex Bandit. In International Conference on Artificial Intelligence and Statistics, volume 89, pages 39–47, April 2019.

Christina Chaccour, Mehdi Naderi Soorki, Walid Saad, Mehdi Bennis, and Petar Popovski. Risk-Based Optimization of Virtual Reality over Terahertz Reconfigurable Intelligent Surfaces. In IEEE International Conference on Communications, volume 2020-June. Institute of Electrical and Electronics Engineers Inc., June 2020. ISBN 9781728150895. doi: 10.1109/ICC40277.2020.9149411.
Sebastian Curi, Kfir. Y. Levy, Stefanie Jegelka, and Andreas Krause. Adaptive Sampling for Stochastic Risk-Averse Learning. *arXiv preprint, arXiv:1910.12511*, October 2019.

Jun Ya Gotoh and Akiko Takeda. CVaR Minimizations in Support Vector Machines. In *Financial Signal Processing and Machine Learning*, pages 233–265. John Wiley & Sons, Ltd, Chichester, UK, April 2016. ISBN 9781118745540. doi: 10.1002/9781118745540.ch10.

Mert Gürbüzbalaban, Andrzej Ruszczyński, and Landi Zhu. A Stochastic Subgradient Method for Distributionally Robust Non-Convex Learning. *arXiv preprint, arXiv:2006.04873*, June 2020.

Wenjie Huang and William B. Haskell. Risk-Aware Q-learning for Markov Decision Processes. In *2017 IEEE 56th Annual Conference on Decision and Control, CDC 2017*, volume 2018-Janua, pages 4928–4933. IEEE, December 2018. ISBN 9781509028733. doi: 10.1109/CDC.2017.8264388.

Dionysios S. Kalogerias and Warren B. Powell. Recursive Optimization of Convex Risk Measures: Mean-Semideviation Models. *arXiv preprint, arXiv:1804.00636*, April 2018.

Hamed Karimi, Julie Nutini, and Mark Schmidt. Linear Convergence of Gradient and Proximal-Gradient Methods under the Polyak-Łojasiewicz Condition. In *Machine Learning and Knowledge Discovery in Databases, ECML PKDD 2016, Lecture Notes in Computer Science*, volume 9851 LNAI, pages 795–811. Springer Verlag, 2016. ISBN 9783319461274. doi: 10.1007/978-3-319-46128-1_50.

Sung-Kyun Kim, Rohan Thakker, and Ali-akbar Agha-mohammadi. Bi-directional Value Learning for Risk-aware Planning Under Uncertainty. *IEEE Robotics and Automation Letters*, 4(3):2493–2500, July 2019. ISSN 2377-3766. doi: 10.1109/LRA.2019.2903259.

Alec Koppel, Amrit S. Bedi, and Ketan Rajawat. Controlling the Bias-Variance Tradeoff via Coherent Risk for Robust Learning with Kernels. In *Proceedings of the American Control Conference*, volume 2019-July, pages 3519–3525. Institute of Electrical and Electronics Engineers Inc., July 2019. ISBN 9781538679265. doi: 10.23919/acc.2019.8814879.

Harold J. (Harold Joseph) Kushner and George Yin. *Stochastic Approximation and Recursive Algorithms and Applications*. Springer, 2003. ISBN 9780387008943.

Jaeho Lee, Sejun Park, and Jinwoo Shin. Learning Bounds for Risk-sensitive Learning. *arXiv preprint, arXiv:2006.08138*, June 2020.

Yan Li, Deke Guo, Yayei Zhao, Xiaofeng Cao, and Honghui Chen. Efficient Risk-Averse Request Allocation for Multi-Access Edge Computing. *IEEE Communications Letters*, pages 1–1, September 2020. ISSN 1089-7798. doi: 10.1109/lcomm.2020.3027562.

Wann-Jiun Ma, Chanwook Oh, Yang Liu, Darinka Dentcheva, and Michael M. Zavlanos. Risk-Averse Access Point Selection in Wireless Communication Networks. *IEEE Transactions on Control of Network Systems*, 5870(c):1–1, 2018. ISSN 2325-5870. doi: 10.1109/TCNS.2018.2792309.

Harry Markowitz. Portfolio Selection. *The Journal of Finance*, 7(1):77–91, March 1952. ISSN 15406261. doi: 10.1111/j.1540-6261.1952.tb01525.x.

Zakaria Mhammedi, Benjamin Guedj, and Robert C. Williamson. PAC-Bayesian Bound for the Conditional Value at Risk. *arXiv preprint, arXiv:2006.14763*, June 2020.
B. T. Polyak. Gradient Methods for Minimizing Functionals. *USSR Computational Mathematics and Mathematical Physics*, 3(4):864–878, January 1963. ISSN 00415553. doi: 10.1016/0041-5553(63)90382-3.

R. Tyrrell Rockafellar and Stanislav Uryasev. Optimization of Conditional Value-at-Risk. *Journal of Risk*, 2:21–41, 2000. ISSN 10769986. doi: 10.2307/1165345.

Bernhard Schölkopf, Alex J. Smola, Robert C. Williamson, and Peter L. Bartlett. New Support Vector Algorithms. *Neural Computation*, 12(5):1207–1245, March 2000. ISSN 08997667. doi: 10.1162/089976600300015565.

Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. *Lectures on Stochastic Programming: Modeling and Theory*. Society for Industrial and Applied Mathematics, 2nd edition, 2014. ISBN 089871687X. doi: http://dx.doi.org/10.1137/1.9780898718751.

Tasuku Soma and Yuichi Yoshida. Statistical Learning with Conditional Value at Risk. *arXiv preprint, arXiv:2002.05826*, February 2020.

Akiko Takeda and Takanori Kanamori. A Robust Approach Based on Conditional Value-at-Risk Measure to Statistical Learning Problems. *European Journal of Operational Research*, 198(1):287–296, October 2009. ISSN 03772217. doi: 10.1016/j.ejor.2008.07.027.

Akiko Takeda and Masashi Sugiyama. ν-Support Vector Machine as Conditional Value-at-Risk Minimization. In *Proceedings of the 25th International Conference on Machine Learning*, pages 1056–1063, New York, New York, USA, 2008. Association for Computing Machinery (ACM). ISBN 9781605582054. doi: 10.1145/1390156.1390289.

Vladimir N. Vapnik. *The Nature of Statistical Learning Theory*. Springer, 2000. ISBN 0387987800.

Constantine Alexander Vitt, Darinka Dentcheva, and Hui Xiong. Risk-Averse Classification. *Annals of Operations Research*, August 2019. ISSN 0254-5330. doi: 10.1007/s10479-019-03344-6.

Robert C Williamson and Aditya Krishna Menon. Fairness Risk Measures. In *36th International Conference on Machine Learning, ICML 2019*, volume 2019-June, pages 11763–11774, 2019. ISBN 9781510886988.

Lifeng Zhou and Pratap Tokekar. An Approximation Algorithm for Risk-Averse Submodular Optimization. In *Springer Proceedings in Advanced Robotics, vol. 14*, pages 144–159. Springer, Cham, December 2020. doi: 10.1007/978-3-030-44051-0_9.