Finite temperature topological order in 2D topological color codes

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In this work the topological order at finite temperature in two-dimensional color code is studied. The topological entropy is used to measure the behavior of the topological order. Topological order in color code arises from the colored string-net structures. By imposing the hard constrained limit the exact solution of the entanglement entropy becomes possible. For finite size systems, by rising the temperature one type of string-net structures is thermalized and the associative topological entropy vanishes. In the thermodynamic limit the underlying topological order is fragile even at very low temperatures. Taking first thermodynamic limit and then zero-temperature limit and vice versa don’t commute, and their difference is related only to the topology of regions. The contribution of the colors and symmetry of the model in the topological entropy is also discussed. It is shown how the gauge symmetry of the color code underlies the topological entropy.

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I. INTRODUCTION

Exotic sates of matter are those that defy the usual description in terms of well-known Landau-Ginzburg-Wilson paradigm, where an appropriate local order parameter characterizes different behaviors of two phases on either side of the critical point. These are new phases of matter carrying a kind of quantum order called topological order, and the transition between various phases does not depend on the symmetry breaking mechanism. From the experimental point of view, the fractional quantum Hall liquid exhibits topological order. Different phases of electron liquids carry the same symmetry and the phase transition between them is possible. Therefore, the Landau theory of classical phase transitions fails in order to describe these phases. The most remarkable properties of such new states are the dependency of the ground state degeneracy on genus or handles of the space, gapped excitations and non-trivial braiding of excitations. This latter property introduces new class of emerging particles with the statistical properties that are neither fermions nor bosons. In fact, they are anyons. They interact topologically independent of their distances much like that of Aharonov-Bohm interaction. If by winding of an anyon around another one the wavefunction picks up an overall phase, the anyons are called abelian anyons. But, if evolution of the wavefunction is captured by a unitary matrix, the anyons are called non-abelian anyons, i.e. they obey non-abelian braiding statistics.

The questions such as what the objects underlying the topological phases are and how different phases can be classified are still under debate. However, some physical mechanisms such as string/membrane-net condensation, branyons that are analogous of the particle condensation in the symmetry-breaking phases and a description in terms of quantum groups have been introduced. The string-nets are extended nonlocal objects and the ground state is a coherent superposition of all possible string’s configurations appearing at all length scale. This physical picture clarifies the topological order based on the microscopic degrees of freedom. Emerging particles such as fermions or anyons are collective excitations of strings.

The fact that in topological ordered phases the ground state subspace has a robust degeneracy and the excitations above the ground state are gapped give them the ability of being rigorous quantum memory in the sense of error correction. However, these are not sufficient conditions for being self-correcting code. Perhaps thermal noises spoil the self-stability of the code. The construction of fault-tolerant topological quantum computation exploits the emerging properties of topological ordered phases. The quantum information can be stored in the topologically protected subspace being free from the decoherence. The robust manipulation of quantum information is done by braiding of anyons, where the unitary gate operations are carried out by braiding of anyons.

The ground state of topological ordered phases is highly entangled state and an indicator for topological order which is not based on symmetry has been emerged, notably the topological entanglement entropy. This topological quantity appears as subleading correction to the entanglement entropy of a convex region. It is a general feature of all gapped phases that the entanglement entropy scales with the boundary of the region, the so-called area law. However, the appearance of constant subleading term is a new feature related to topological order. The topological phase can be described as a phase of matter for which the low-energy effective theory is a topological field theory (TQFT). Topological entanglement entropy is related to one of the basic parameter of the TQFT, the total quantum dimension of emerging quasiparticles in the theory.

The best known model for studying topological order, emerging abelian or non-abelian anyons and examining its capabilities for topological quantum computations is the famous Kitaev’s model. In the abelian phase, the model becomes the well-known toric code with a stabi-
lizer structure\textsuperscript{23}. The stabilizer structure of the code is given by a set of star and plaquette operators. Stabilizer operators fix a subspace in the Hilbert space of the model in which different states are distinguished by means of topological numbers. Topological color code is another relevant example of stabilizer codes, with enhanced computational capabilities\textsuperscript{24,25}. The stabilizer operators are only colored plaquette operators. However, both codes are topological because the stabilizer operators are local and non-detectable errors have non-trivial supports on the manifold. Non-trivial string operators stand for encoded logical operators. Topological order in toric code and color code are related to different gauge symmetry. Indeed, the topological order in the toric code is related to the $\mathbb{Z}_2$ gauge symmetry, whereas the topological order in the color code is related to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge symmetry. This latter symmetry arises from the contribution of the colors in the construction of the code and is responsible of the string-net structure of the code. However, both models have almost the same error threshold when one considers the error syndrome measurement\textsuperscript{26}.

Entanglement properties of both models at zero-temperature have been extensively studied. The entanglement entropy of a region with its complement depends on the degrees of freedom living on the boundary of the region supplemented with a topological term. But, the topological order survives even at non-zero temperatures\textsuperscript{30,31}. Further increasing of temperature destroys the loop structures of the model implying the fragility of the toric code\textsuperscript{32,33}.

In this work we address the above problem, i.e the fate of topological order at finite temperature, in the color codes. The loop structures of the color code is different from that of toric code since they are related to different gauge symmetries. We discuss how gauge symmetry affects the finite temperature properties of the code. We attach to each set of plaquettes with the same color an energy scale. In the lattice gauge theory these energy scales are translated into the chemical potential for creating the respective charges. Following the derivation of C. Castelnovo, et al\textsuperscript{33}, we first impose the hard constrained limit on the string-net structure in $\sigma^z$ bases and allow for the thermalization of the string-net structure in $\sigma^x$ bases. Then, in order to identify the contribution of colored strings in the topological entanglement entropy, we fix other loop structures and examine the residual topological order. Also in the high temperature limit the description in terms of classical topological order is recovered.

The organization of the paper is as follows. In the next section the color codes is briefly reviewed. Then in Sec(III) the thermal density matrix that is needed for subsequent arguments is derived. In Sec(V) the entanglement entropy is derived from the density matrix. Then, limiting behavior of the entanglement is discussed in Sec(V). Topological entanglement entropy and its behavior in terms of temperature is the subject of Sec(VI). The case of open boundary conditions is discussed in Sec(VII). Sec(VIII) is devoted to conclusions.

II. 2-COLEX AND COLOR CODE: FIXING NOTATIONS

Let us start by a brief introductory on the color code model. Consider a two-dimensional trivalent lattice composed of plaquettes, vertices and links. Such structure is shown in Fig(1). To keep track of vertices, plaquettes and links, we use color as a bookkeeping tool. We will use three colors: red, green and blue. Then we color the plaquettes of the lattice in which two neighboring plaquettes don’t share in the same color. In this way we can also color links so that a c-link (the letter c stand for color throughout the paper unless stated another meaning) connects two c-plaquettes. We call this two-dimensional lattice a $2$-colex. The lattice can also be embedded in higher spatial dimensions, the so-called $D$-
of the space. In fact for each homology class only two colors are independent related to the symmetry of the code. As long as we consider closed spaces such as torus, the closed non-contractible loops are enough to form bases for the coding space. The coding space is called color code. From now on, unless it is stated else, we suppose the 2-colex spanned by 3N plaquettes, N of each color. Notice that on a compact surface like a torus all plaquette operators are not independent. In fact they are subject to the constraint $\prod_{g \in G} \Omega_g = \prod_{b \in B} \Omega_b = \prod_{r \in R} \Omega_r$, where $\Omega = X, Z$, and $G, B, R$ are sets of green, blue and red plaquettes, respectively.

The protected subspace is ground state of a many-body Hamiltonian that is minus sum of all plaquette operators equipped with some coupling constants as follows:

$$H = -\lambda_x^g \sum_{g \in G} X_g - \lambda_z^b \sum_{b \in B} X_b - \lambda_x^z \sum_{r \in R} X_r - \lambda_x^g \sum_{g \in G} Z_g - \lambda_z^b \sum_{b \in B} Z_b - \lambda_x^z \sum_{r \in R} Z_r, \quad (1)$$

where $\lambda_x^g, \lambda_z^b, \lambda_x^z$’s are coupling constants and each sum runs over all corresponding c-plaquettes. The ground states spanning the coding space are all vectors in which $X_n |\psi\rangle = Z_n |\psi\rangle = |\psi\rangle$ for all plaquette operators. Any violation of this condition amounts to an excited state or alternatively as an error. Therefore, the ground state is said to be vortex free in the sense of being closed string-net condensate. In fact, the ground state is an equal weighted superposition of all string-net configurations. Such configurations can be visualized if we consider the product of plaquette operators. For example consider the product of two neighboring red and blue plaquettes. In the product, the Pauli operators of qubits shared between two plaquettes square identity and we are left with a green string being boundary of two plaquettes. This closed green string has been shown in Fig.1(a), which connects green plaquettes. Such closed strings of either size and color commute with Hamiltonian. Another interesting extended object underlying the ground state structure is string-net that is a collection of colored strings. A typical string-net has been depicted in Fig.1(b). The associated string-net operator can be the product of either $\sigma^x$ or $\sigma^y$ acting on the qubits it contains (filled colored circles on the string-net). This string operator commutes with all plaquettes since they share either nothing or even number of qubits. The appearance of such string-net is crucial for the full implementation of the Clifford group.

Excitations appear as end points of open colored strings. An open string anticommutes with the plaquettes lying at its ends as they share odd number of qubits. To have a simple picture of excitations, let us focus on some simple cases. Consider a rotation $\sigma^x$ applied to a certain qubit as in Fig.1(c). As $\sigma^x$ anticommutes with $X_x$, $X_b$ and $X_y$ plaquette operators adjacent to the qubit, it will increase the energy of plaquettes by $2\lambda_x^x$, $2\lambda_x^b$ and $2\lambda_x^y$, respectively. In this case the excitation on
a c-plaquette is revealed by an c-star. Similarly, if we perform a $\sigma^y$ rotation on a qubit as in Fig.1(d), the anticommutation of $\sigma^y$ with $Z_c$ of neighboring plaquettes leads the energy of red, blue and green plaquettes increase by $2\lambda_c^z$, $2\lambda_b^y$ and $2\lambda_r^z$, respectively. In this case the excitation on a c-plaquette is revealed by a c-circle. If a rotation $\sigma^y$ is performed, as in Fig.1(e), each plaquette contains both above excitations, i.e. the excitations on the c-plaquette increase the energy by $2\lambda_c^z + 2\lambda_r^z$. All quasiparticles are by themselves boson. However, they may have mutual semionic statistics. For instance, if a red particle of Fig.1(f) go around a green particle of Fig.1(f) , the wavefunction picks up a minus sign implying they are anyons. There are also composite particles of two excitations as shown by white dashed ellipses. If two excitations differ in both color and type, they form a boson as in Fig.1(f). Otherwise, they form a fermion as in Fig.1(e).

All above emerging particles that are the low energy excitations of the model can be classified in terms of underlying gauge group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Before that, let us make a convention for colors which will be useful for subsequent discussions. We refer to colors by a bar operation over colors which will be useful for subsequent discussions. We refer to colors by a bar operation over colors which will be useful for subsequent discussions.

**III. THERMAL DENSITY MATRIX OF COLOR CODE**

Zero-temperature entanglement of the color code is given by writing the ground state of the Hamiltonian in Eq.1. Let $G$ denotes the group constructed by a set of generators of the spin-flip plaquette operators, i.e $X_c$. By starting from a polarized state, the ground state will be an equal weighted superposition of all elements of the stabilizer group as follows.

$$|\psi\rangle = |G\rangle^{-1/2} \sum_{h \in G} h|0\rangle \bigotimes |V\rangle,$$

where $h$ is an element of the stabilizer group, $|G\rangle$ is the cardinality of the group, $|V\rangle$ is total number of vertices on the lattice and $\sigma^y|0\rangle = |0\rangle$. Other ground states can be constructed through the action of the non-local spin-flip operators with non-trivial supports winding the handles of the space. They are distinguished by different topological numbers. However, all of them have the same entanglement properties and we consider only one of them as above.

Topological order in the ground state of the color code Hamiltonian arises from the condensed string-net structures, which are loops without open ends. The ground state $|\psi\rangle$ has been written in the $\sigma^z$ bases. In each term of the superposition a collection of spins related to the group element $h$ have been flipped. Spins on the corresponding string-net have negative $\sigma^x$ component. In this string-net structure $Z_c = 1$ for all plaquettes. There are also same string-net structures if we worked in the $\sigma^z$ bases preserving the $X_c = 1$. This is because the model is symmetric upon the exchange of the $\sigma^z$ and $\sigma^x$ operators on the plaquettes. Unlike the toric code, both loop structures in the color code exist on the direct lattice since for each plaquette we attach two operators. The thermal fluctuations can break the string-nets and limit them with open ends. That how much thermal effects are able to destroy the topological order associated to a string-net structure depends on the coupling constants in the Hamiltonian as well as size of the system. Definitely, we calculate the thermal density matrix and the topological entanglement entropy is used as a measure of topological order. Since the ground state is fixed point of all plaquette operators, its underlying topological order arises from coherent superposition of two string-net structures, each related to one of the bases. We impose a constraint in which the string-net structures related to the $\sigma^z$ bases are preserved. The so called hard constrained limit $\beta \to \infty$ can be recasted into the limit $\lambda_c^z \to \infty$. One may expect that in this limit the thermal fluctuations can only create excitations with topological charges $(e, \chi_c)$. We can describe all excited states as $|\gamma\rangle = 2^{|\gamma|} |\gamma\rangle |\psi\rangle$, where $\gamma$ is any arbitrary product of $\sigma^z$ Pauli operators. The density matrix $\rho = \exp(-\beta H)$ in the basis spanned by all states $|\psi\rangle |\gamma\rangle$ is diagonal. However, things get more simplified if we work in the original bases $h|0\rangle \bigotimes |V\rangle$. Let us for simplicity drop the superscript $\bigotimes |V\rangle$. In these bases the density matrix reads as follows

$$\rho = \frac{\sum_{h, h' \in G} |0\rangle \langle 0| \langle \text{exp}(\beta H)|0\rangle}{\sum_{h' \in G} \langle \text{exp}(\beta H)|0\rangle}.$$  

Without loss of generality let $\lambda_c^z = \lambda_b^y$ for all plaquettes and $G_z = \sum_x X_y, B_x = \sum_y X_y, R_x = \sum_x X_x$ and $Z = \sum_x X_z$. Then the Hamiltonian proposed in Eq.1 can be written as

$$H = -\lambda_c^z G_x - \lambda_b^y B_x - \lambda_r^y R_x - \lambda_z Z.$$  

Since $[Z, h] = 0$ for all $h$, then $Z |\psi\rangle = 3Nh|0\rangle$ and the numerator of Eq.1 reads as

$$\langle 0| \langle \text{exp}(\beta H)|0\rangle = e^{3\beta\lambda_r^y N (0)e^{3\beta\lambda_e^z G_x + 3\beta\lambda_b^y B_x + 3\beta\lambda_c^z R_x} Z |0\rangle.$$  

Since each plaquette operator squares identity, i.e. $X_c^2 = 1$, each exponential term can be expressed in terms of
plaquette operators as follows.

\[ e^{\beta \lambda^y G_x} = \prod_{g \in G} \left[ \cosh(\beta \lambda^y_x) + \sinh(\beta \lambda^y_x) X_g \right]. \]

(6)

The two remaining exponential terms \( e^{\beta \lambda^s B_x} \) and \( e^{\beta \lambda^r R_x} \) have the same expressions and obtained by replacing the color index \( g \) by \( b \) and \( r \), respectively. Every element of the stabilizer group \( \hat{h} \) has a decomposition in terms of plaquette operators that it involves as \( \hat{h} = \prod_{g \in \hat{h}} X_g \prod_{b \in \hat{b}} X_b \prod_{r \in \hat{r}} X_r \), where \( c \in \hat{h} \) we imply if the \( c \)-plaquette operator exists in the decomposition of \( \hat{h} \). Let \( n_c \) be the number of \( c \)-plaquettes operators that exist in the decomposition of \( \hat{h} \). Hence, the only nonzero terms in Eq. (6) are as follows.

\[
(0)|0\rangle = (\cosh(\beta \lambda^y_x))^N (\cosh(\beta \lambda^b_x))^N (\cosh(\beta \lambda^r_x))^N \times \\
\left[ \sum_c e^{-(k_c+k_e)N-k_c n_c + k_e n_e + e^{-k_g n_g - k_h n_h - k_r n_r}} \right],
\]

(7)

where \( k_c = -\ln(\tanh(\beta \lambda^y_x)) \) and the sum runs over three colors \( c \in \{r, b, g\} \). Notice that the operator \( \hat{h} \) not only given by the product of green, blue and red plaquettes belonging to \( \hat{h} \) but also by other combinations of plaquettes due the periodic boundary conditions. In fact, the operator \( \hat{h} \) has the following expressions in terms of plaquettes operators.

\[
\hat{h} = \prod_{g \in \hat{h}} X_g \prod_{b \in \hat{b}} X_b \prod_{r \in \hat{r}} X_r = \prod_{g \in \hat{h}} X_g \prod_{b \in \hat{b}} X_b \prod_{r \in \hat{r}} X_r = \prod_{g \notin \hat{h}} X_g \prod_{b \notin \hat{b}} X_b \prod_{r \notin \hat{r}} X_r.
\]

(8)

Note that in the above expressions by \( c \in \mathcal{C} \setminus \hat{h} \) we mean a \( c \)-plaquette in \( \mathcal{C} \) (\( \mathcal{C} = \mathcal{R}, \mathcal{G}, \mathcal{B} \) that is not in \( \hat{h} \). Now we can see that the ambiguity in Eq. (7) amounts to the above expressions for \( \hat{h} \). The partition function can be evaluated in a similar way as follows:

\[
\mathcal{Z} = e^{3 \beta \lambda_z N} (\cosh(\beta \lambda^y_x))^N (\cosh(\beta \lambda^b_x))^N \times \\
(\cosh(\beta \lambda^r_x))^N \left( 1 + \sum_c e^{-(k_c+k_e)N} \right).
\]

(9)

Thus, the density matrix can be recasted into the following form:

\[
\rho = \frac{1}{|G|} \sum_{\hat{h}, \hat{h} \in G} \eta_T(\hat{h}) \times \hat{h} |0\rangle \langle 0| \hat{h},
\]

(10)

where

\[
\eta_T(\hat{h}) = \frac{\sum_c e^{-(k_c+k_e)N-k_c n_c + k_e n_e + e^{-k_g n_g - k_h n_h - k_r n_r}}}{\sum_c e^{-(k_c+k_e)N + 1}}.
\]

The limiting behavior of the density matrix at zero and very high temperatures is compatible with the respective known results. As temperature tends to zero \( (k_c \to 0) \), the pure density matrix \( \rho = \frac{1}{|G|} \sum_{\hat{h}} |0\rangle \langle 0| \hat{h} \) is recovered. At high-temperature limit, the totally mixed state \( \rho = \frac{1}{|G|} \sum_{\hat{h}} |0\rangle \langle 0| \hat{h} \) yields the classical limit of the model upon the hard constrained limit.

### IV. Entanglement Entropy

In order to calculate the entanglement entropy we consider a generic bipartition of the system into subsystems \( A \) and \( B \). Notice that each bipartition may be composed of several disconnected regions. Suppose \( m_A \) and \( m_B \) stand for the number of respective bipartitions. Let \( \Sigma_A \) and \( \Sigma_B \) be the number of plaquette operators acting solely on \( A \) and \( B \), respectively, and let \( \Sigma_{AB} \) stands for the number of plaquette operators acting simultaneously on \( A \) and \( B \), i.e. boundary operators. We focus on the entanglement entropy between two partitions \( A \) and \( B \) of the system. To this end, first the reduced density operator of the one subsystem is evaluated and then the entanglement entropy is measured using von Neumann entropy. The reduced density matrix of a region, say \( A \), is obtained by tracing out \( \rho \) with respect to degrees of freedom of the subsystem \( B \). Using the properties of the group, the reduced density matrix reads

\[
\rho_A = \frac{1}{|G|} \sum_{\hat{h}, \hat{h} \in G_A} \eta_T(\hat{h}) \hat{h}_A |0_A\rangle \langle A_0| \hat{h}_A,
\]

(11)

where \( G_A \) and \( G_B \) are subgroups of \( G \) which act trivially on subsystems \( B \) and \( A \). The complete description of the subgroup \( G_A \) is given by a set of plaquette operators acting solely on \( A \) as well as collective operators. The latter is a collection of operators acting solely on \( A \), but they are not the product of plaquettes in \( G_A \). Suppose \( A \) is a simply connected region and the set \( \{B_1, B_2, ..., B_{m_B}\} \) presents the disconnected components of \( B \). Let \( AB_i \) be the collection of plaquette operators acting simultaneously on \( A \) and \( B_i \). The product of \( c \)- and \( \bar{c} \)-plaquettes of component \( B_i \) with the \( c \)- and \( \bar{c} \)-plaquettes of the boundary \( AB_i \) produces a \( \bar{c} \)-string acting solely on \( A \). Denoting this string by \( \gamma_{\bar{c}}^e \), its expression will be

\[
\gamma_{\bar{c}}^e = \prod_{c \in B_i \cup AB_i} X_c X_{\bar{c}}.
\]

(12)

A schematic representation of these collective strings is shown in Fig. (1). Therefore, for each disconnected
region we can realize three collective operators $\gamma_i^c$, $\gamma_i^b$ and $\gamma_i^g$. For each disconnected region however only two of them are independent due to the local constraint $\gamma_i^c\gamma_i^b\gamma_i^g = 1$. There is also another constraint on the total number of collective operators. It is a global constraint: for a given color, say $c$, the product $\prod_{i} \gamma_i^c$ can be produced by product of all $c$-and $\bar{c}$-plaquettes acting solely on $A$, namely $\prod_{i} \gamma_i^c = \prod_{c,\bar{c}\in A} X_c X_{\bar{c}}$. Taking into account all plaquettes, collective operators and constraints on them, the cardinality of the subgroups $G_A$ and $G_B$ will be $d_A = 2^2n+2m_A-2$ and $d_B = 2^2n+2m_A-2$, respectively.

The von Neumann entropy as a measure of entanglement between two bipartitions is given by $S_A = -\text{Tr}(\rho_A \ln \rho_A)$. We don’t use this relation to measure the entanglement. Here, we find it useful to instead compute it using the replica $S_A = \lim_{n \to 1} \partial_n \text{Tr}[\rho_A^n]$. Hence, the trace of $n$-th power of the reduced density matrix will be

$$
\text{Tr}[\rho_A^n] = \left(\frac{d_B}{|G|}\right)^{n-1} \sum_{l=1}^{n} \prod_{\bar{h}_i \in G_A} \eta_T(\bar{h}_i)\langle 0|\hat{h}_{1, A}\hat{h}_{2, A}...\hat{h}_{n, A}|0\rangle.
$$

In order to bring the above relation to a manageable form, we resort to a map between the group elements $\bar{h}$ and the Ising variables $\theta_i$. Via this map the group elements are labeled by Ising variables based on which plaquettes and/or collective operators are involved in its expression. Let $\theta_c = -1(+1)$ be a Ising variables related to appearing (not appearing) a $c$-plaquette in $\bar{h}$. Similarly, $\theta_{\bar{c}} = -1(+1)$ related to appearing (not appearing) a collective $c$-string $\gamma_i^c$ in $\bar{h}$. Notice that this map is not a one-to-one map. In fact, it is a four-to-one map. This latter point arises from the fact that by considering three colors, if we reverse the sign of Ising variables related to two colors, they represent the same element in the group. This point is consistent with Eq.\(^\text{[13]}\) and will be clear in following. The number of $c$-plaquettes, $n_c$, that the element $\bar{h}$ involved can be expressed in terms of Ising variables as

$$
n_c = \sum_{c \in C} \frac{1 - \theta_c}{2} + \sum_{i} \left(\Sigma_{B_i}^c + \Sigma_{\bar{B}_i}^c\right) \frac{1 - \Theta_i^c \Theta_i^\bar{c}}{2}.
$$

Here, Once the color index in $n_c$ was fixed, the first sum runs over all $c$-plaquettes in $C(G, B, \mathcal{R})$ and $\theta_i$’s take values of $-1(+1)$ as defined above accordingly, and the $\Sigma_{B_i}^c$ is the number of all $c$-plaquettes in $B_i$, let $\Sigma_{B_i}^c + \Sigma_{\bar{B}_i}^c = \Sigma_i$ and since $\Sigma_i^c + \Sigma_i^\bar{c} = N$, Eq.\(^\text{[12]}\) can be written as

$$
n_c = \frac{N}{2} - \sum_{c \in C} \theta_c - \frac{1}{2} \sum_{i} \Sigma_i^\bar{c} \Theta_i^c \Theta_i^\bar{c}.
$$

As we stated above, for a disconnected region $B_i$, the strings $\gamma_i^c$, $\gamma_i^b$ and $\gamma_i^g$ are not independent. With such dependency, the appearance of $\Theta_i^c \Theta_i^\bar{c}$ in above relation is meaningful. To see this, consider the case in which both $c$- and $\bar{c}$-strings are present in the group element. This, in terms of Ising variables, yields $\Theta_i^c = \Theta_i^\bar{c} = -1$. But, the local constraint on strings implies that the product of two strings yield the $c$-string. Since there is not any $c$-plaquette in the decomposition of a $c$-string, the number of $c$-plaquettes arising from the collective operators will be zero.

In terms of Ising variables, the Eq.\(^\text{[12]}\) will become

$$
\text{Tr}[\rho_A^n] = \left(\frac{d_B}{|G|}\right)^{n-1} \prod_{l=1}^{n} \sum_{\{\theta_i\}} \eta_T\{\theta_i\} \Theta_i^c \Theta_i^\bar{c} (\text{const}) (16)
$$

The coefficient $\frac{1}{4}$ comes form the $4 - 1$ mapping between Ising variables and plaquettes. The sum runs over all possible Ising variables subject to a constraint. This constraint arises by requiring the non-zero value for $\langle 0|\hat{h}_{1, A}\hat{h}_{2, A}...\hat{h}_{n, A}|0\rangle$ in Eq.\(^\text{[13]}\), which implies that the product of $\hat{h}$’s must be trivial to give a non-zero value for the expectation value. The constraint can be applied by the following expression:

$$
\sum_{I_c, I_b, I_g} \left\{ \prod_{i} \left[ \delta \left( \prod_{l=1}^{n} \Theta_i^{(l)b} \Theta_i^{(l)g} - I_c \right) \delta \left( \prod_{l=1}^{n} \Theta_i^{(l)r} \Theta_i^{(l)g} - I_b \right) \delta \left( \prod_{l=1}^{n} \Theta_i^{(l)r} \Theta_i^{(l)b} - I_g \right) \right] \times \prod_r \delta \left( \prod_{l=1}^{n} \Theta_i^{(l)} - I_r \right) \prod_b \delta \left( \prod_{l=1}^{n} \Theta_i^{(l)} - I_b \right) \prod_g \delta \left( \prod_{l=1}^{n} \Theta_i^{(l)} - I_g \right) \delta (I_c I_b I_g - 1),
$$

where $I_c = \pm 1$ subject to $I_c I_b I_g = 1$ which is imposed by the delta function in the second line of above expression. Notice that three delta functions in the first line are not independent and we could already drop one of them. However, we keep them to make the expression more symmetric. In fact, once two of them are fixed, the third one satisfied. This is just a reinterpretation of the local constraint that for a disconnected region only two strings are independent.

Regarding the map between Ising variables and pla-
quartets, we can fully bring the $\eta_T$ in Eq. (10) into an expression in terms of Ising variables. Using Eq. (15), the $\eta_T$ function reads as follows:

$$\eta_T = \frac{1}{4Z_0} \sum_{J_r, J_b, J_g} \left\{ \prod_r e^{\frac{\epsilon}{2} J_r \theta_r} \prod_b e^{\frac{\epsilon}{2} J_b \theta_b} \prod_g e^{\frac{\epsilon}{2} J_g \theta_g} \prod_i e^{\frac{\epsilon}{2} J_i \sum_r \Theta_i^{(r)b} \Theta_i^{(r)b} + \frac{\epsilon}{2} J_i \sum_r \Theta_i^{(r)r} \Theta_i^{(r)r} + \frac{\epsilon}{2} J_g \sum_r \Theta_i^{(r)b} \Theta_i^{(r)b}} \delta(J_r J_b J_g - 1) \right\} \delta(J_r J_b J_g - 1),$$

where

$$Z_0 = \frac{1}{4} \left( e^{\frac{\epsilon}{2} (k_r + k_b + k_g)} + e^{\frac{\epsilon}{2} (k_r - k_b - k_g)} + e^{\frac{\epsilon}{2} (k_r + k_b - k_g)} + e^{\frac{\epsilon}{2} (k_r - k_b + k_g)} \right),$$

and $J_c = \pm$ subject to $J_r J_b J_g = 1$ imposed by the boundary conditions, respectively. Thus,

$$\prod_r e^{\frac{\epsilon}{2} \sum \delta(J_r J_b J_g - 1)},$$

The upper limits of sums are just implying the constraint in Eq. (17). All expressions in the above parentheses can be restated as a partition function of a classical Ising model. Before that, notice that the index $r$ in the typical expression $\sum_{\{\theta_i\}_{r=1}^n} e^{\frac{\epsilon}{2} \sum \delta(J_r J_b J_g - 1)}$ is mute since the sum over configuration of $\theta_i$'s is done first. Other indices $b, g$ as well as $i$ are mute. Since the $\theta_i$ is an Ising variable, it is possible to write it as $\delta^{\epsilon} = \tau^r \tau^{r+1}$, which $\tau^r$'s are classical spins. So mapping to the classical Ising model provides a tool to have an analytical expression for entropy. The constraints $\Delta_r = +1$ and $\Delta_r = -1$ are also satisfied by considering the periodic or antiperiodic boundary conditions, respectively. Thus,

$$\prod_r e^{\frac{\epsilon}{2} \sum \delta(J_r J_b J_g - 1)}.$$
Thus, regarding the constraints \( \Delta_p \) that are indicated by the superscript expression, the partition function reads as follows:

\[
\mathcal{Z} \left( \kappa, \Sigma_r^b, k_b \Sigma^b_i, k_g \Sigma^g_i; \{ J^r_i \}, \{ J^b_i \}, \{ J^g_i \} \right) = \sum_{\{ s', s'' \}} \exp \left( \sum_i \left( \frac{1}{4} \kappa J^r_i s'^{r+1}_i s'^{r+1}_i s'^{r+1}_i s'^{r+1}_i + \frac{1}{4} k_b J^b_i s''_i s''_i + \frac{1}{4} k_g J^g_i s''_i s''_i + \frac{1}{4} J^c_i \right) \right),
\]

(22)

First, consider the case with constraints \( \Delta^c = 1 \) for \( c = r, b, g \). The transfer matrix will be a 4 \( \times \) 4 matrix as follows:

\[
T_i = \begin{pmatrix}
  e^{-J^b_i b + J^g_i g - J^r_i r} & e^{-J^b_i b - J^g_i g + J^r_i r} & e^{-J^b_i b + J^g_i g - J^r_i r} & e^{-J^b_i b - J^g_i g + J^r_i r} \\
  e^{-J^b_i b - J^g_i g + J^r_i r} & e^{-J^b_i b + J^g_i g - J^r_i r} & e^{-J^b_i b - J^g_i g + J^r_i r} & e^{-J^b_i b + J^g_i g - J^r_i r} \\
  e^{-J^b_i b + J^g_i g - J^r_i r} & e^{-J^b_i b - J^g_i g + J^r_i r} & e^{-J^b_i b - J^g_i g + J^r_i r} & e^{-J^b_i b + J^g_i g - J^r_i r} \\
  e^{-J^b_i b - J^g_i g + J^r_i r} & e^{-J^b_i b + J^g_i g - J^r_i r} & e^{-J^b_i b - J^g_i g + J^r_i r} & e^{-J^b_i b + J^g_i g - J^r_i r}
\end{pmatrix}
\]

(23)

where \( b = \frac{k_b}{4} \Sigma^b_i, b = \frac{k_b}{4} \Sigma^b_i \) and \( r = \frac{k_r}{4} \Sigma^r_i \). The eigenvalues of the transfer matrix can be easily calculated. Let us denote them by \( \xi^b_{11} = 4 J^b_i \xi^b_{11}, \xi^b_{21} = 4 J^b_i \xi^b_{21}, \xi^r_3 = 4 J^r_i \xi^r_3 \) and \( 4 \xi_{4i} \), where

\[
\begin{align*}
\xi^b_{11} &= \frac{1}{2} e^b \cosh (g + r) - \frac{1}{2} e^{-b} \cosh (g - r) \\
\xi^b_{21} &= \frac{1}{2} e^b \cosh (b + r) - \frac{1}{2} e^{-b} \cosh (b - r) \\
\xi^r_{3i} &= \frac{1}{2} e^r \cosh (b + g) - \frac{1}{2} e^{-r} \cosh (b - g) \\
\xi^g_{4i} &= \frac{1}{2} e^g \cosh (b + g) + \frac{1}{2} e^{-g} \cosh (b - g).
\end{align*}
\]

(24)

Thus, regarding the constraints \( \Delta^c = 1 \) for \( c = r, b, g \) that are indicated by the superscript ppp in the following expression, the partition function reads as follows:

\[
\mathcal{Z}_{ppp}^n = 4^n \left[ J^b_0 \xi^b_{11} + J^b_1 \xi^b_{21} + J^r_3 \xi^r_{3i} + \xi^g_{4i} \right],
\]

(25)

where the simplification \( J_c = \prod_{i=1}^n J^c_i \) has been used. Considering other constraints that amount to applying the antiperiodic boundary conditions on Ising spins, the sings of coefficients \( J^b_i \) and \( J^r_i \) become minus. Therefore, other partition functions are as follows:

\[
\begin{align*}
\mathcal{Z}_{n}^{ppa} &= 4^n \left[ -J^b_0 \xi^b_{11} + J^b_1 \xi^b_{21} - J^r_3 \xi^r_{3i} + \xi^g_{4i} \right], \\
\mathcal{Z}_{n}^{paa} &= 4^n \left[ -J^b_0 \xi^b_{11} - J^b_1 \xi^b_{21} + J^r_3 \xi^r_{3i} + \xi^g_{4i} \right], \\
\mathcal{Z}_{n}^{app} &= 4^n \left[ J^b_0 \xi^b_{11} - J^b_1 \xi^b_{21} - J^r_3 \xi^r_{3i} + \xi^g_{4i} \right].
\end{align*}
\]

(26)

Thus, the expression in Eq. (19) is entirely given in terms of partition functions studied here as follows:
\[ \text{Tr}[\rho_A^n] = \left( \frac{d_B}{|G|} \right)^{n-1} \frac{1}{4^{2n}Z_0^n} \sum_{\{J_i\}_{i=1}^n} \sum_{\{J'_i\}_{i=1}^n} \sum_{\{J''_i\}_{i=1}^n} \left[ \prod_{i=1}^n \delta(J_i'J_iJ_i' - 1) \right] \times \]

\[ \left\{ \left( Z_n^{(p)}(k_r, \{J_i\}) \right)^{\Sigma_A} \left( Z_n^{(p)}(k_b, \{J_i\}) \right)^{\Sigma_b} \left( Z_n^{(p)}(k_g, \{J_i\}) \right)^{\Sigma_g} \prod_i Z_{n,pp}(k_r, k_b, k_g, \{J_i\}, \{J'_i\}, \{J''_i\}) \right\} \]

\[ + \left( Z_n^{(a)}(k_r, \{J_i\}) \right)^{\Sigma_A} \left( Z_n^{(a)}(k_b, \{J_i\}) \right)^{\Sigma_b} \left( Z_n^{(a)}(k_g, \{J_i\}) \right)^{\Sigma_g} \prod_i Z_{n,ap}(k_r, k_b, k_g, \{J_i\}, \{J'_i\}, \{J''_i\}) \]

\[ + \left( Z_n^{(p)}(k_r, \{J_i\}) \right)^{\Sigma_A} \left( Z_n^{(a)}(k_b, \{J_i\}) \right)^{\Sigma_b} \left( Z_n^{(a)}(k_g, \{J_i\}) \right)^{\Sigma_g} \prod_i Z_{n,ap}(k_r, k_b, k_g, \{J_i\}, \{J'_i\}, \{J''_i\}) \]

\[ + \left( Z_n^{(a)}(k_r, \{J_i\}) \right)^{\Sigma_A} \left( Z_n^{(p)}(k_b, \{J_i\}) \right)^{\Sigma_b} \left( Z_n^{(a)}(k_g, \{J_i\}) \right)^{\Sigma_g} \prod_i Z_{n,pa}(k_r, k_b, k_g, \{J_i\}, \{J'_i\}, \{J''_i\}) \right\} \]

(27)

where \(\Sigma_A = \Sigma_A^r + \Sigma_A^b + \Sigma_A^g\). The sum over \(J_i'\)'s can be easily taken. Notice \(J = \pm 1\). Due to the delta function, this leads to \(J_i J_i J_i = 1\). So, in summation the \(Z_2 \times Z_2\) symmetry is automatically held. Therefore, the factors \(J = \prod_{i=1}^n J_i\) in the above expression can be safely dropped since the mentioned symmetry get simply exchanged the terms between brackets. The sums give a multiplicative factor \((\frac{1}{4} \times 8)^n\), where the coefficient \(\frac{1}{4}\) arises because of the delta function. It is convenient to introduce the notations \(x_c = \cosh(\frac{c}{2})\) and \(y_c = \sinh(\frac{c}{2})\). By inserting Eq. (21), Eq. (23) and Eq. (26) into Eq. (27), we eventually arrive at the following expression.

\[ \text{Tr}[\rho_A^n] = \frac{1}{4Z_0} \left( \frac{d_A d_B}{|G|} \right)^{n-1} \times \left( F_1^{(n)} + F_2^{(n)} + F_3^{(n)} + F_4^{(n)} \right), \]

(28)

where \(F_i\)'s are functions of \(x_c, y_c\) and \(\xi_i\) (see Appendix [X]). In particular, the replica trick gives the entanglement entropy as follows

\[ S_A(T) = -\ln \left( \frac{d_A d_B}{|G|} \right) + \ln(Z_0) \]

\[ -\frac{1}{4Z_0} (\partial F_1 + \partial F_2 + \partial F_3 + \partial F_4), \]

(29)

where \(\partial F_i\)'s are given in Appendix [X]. This relation is at the heart of our subsequent discussions.

V. LIMITING BEHAVIOR OF ENTANGLEMENT ENTROPY AND MUTUAL INFORMATION

Eq. (20) gives all we need to explore the dependency of entanglement entropy on temperature since apart from the first term the remaining ones depend on temperature. First, suppose the size of the system is finite.

As the temperature goes to zero, all terms that depend on temperature vanish and the entanglement entropy of ground state is recovered. The zero-temperature entropy is \(S_A(T = 0) = -\ln(\frac{d_A d_B}{|G|})\). High-temperature limit of the model corresponds to a classical model captured through the "hard constrained limit". As the temperature tends to infinity, the couplings \(k_c \to \infty\) leading to \(x_c \sim y_c \sim \frac{1}{4} e^{k_c/2}\) and \(\xi_i \sim \xi_i \sim \xi_i \sim \xi_i e^{b r + r} \). At this limit the entanglement entropy acquires a contribution from the bulk degrees of freedom of region \(A\) as follows

\[ S_A(T \to \infty) = (\Sigma_{AB} + \Sigma_A - 2m_A) \ln 2. \]

(30)

This is expected in the sense that at high-temperature limit the thermal entropy that scales with the volume of the region must be reached. This also verifies that the extension of von Neumann entropy to finite temperatures makes sense, since it contains a bulk contribution (scaling with the volume of subsystem \(A\)) that corresponds to the ordinary classical entropy. However, the above entropy, despite being at high temperature limit, carries a constant term. This term \(2m_A\) depends only on the topology of the region \(A\). This exhibits even at high temperature limit the underlying system may carry topological order. Here, the classical system is constructed by thermalization of a pure density matrix via the high constrained limit. In this way precisely half of the original topological order is preserved at the classical limit. We will refer to this point in next section where the topological entanglement entropy as a measure of the topological order is calculated.

Now, we turn on to take first the thermodynamic limit. In this limit, the entanglement entropy behaves as

\[ S_A(T) = -\ln \left( \frac{d_A d_B}{|G|} \right) + \frac{N}{2} (k_r + k_b + k_g) - \ln(4) \]

\[ -e^{-\frac{1}{2} (k_r + k_b + k_g)} (\partial F_1 + \partial F_2 + \partial F_3 + \partial F_4). \]

(31)
expression for the entanglement entropy to \( \Sigma \). Let us consider the limit of large bipartitions which amount to have a more intuitive picture of above expression, let

\[
\partial F \rightarrow \infty, \quad N \rightarrow \infty.
\]

Both quantities scale with the size of the boundary, i.e. the area law holds no matter which limit is taken first. However, the subleading terms are different. The difference between two quantities is \( 2(m_B - 1) \), which implies that the difference depends only on the topological properties of the regions. Like the toric code model\textsuperscript{34}, the topological contribution to the entanglement entropy in the color code can be extracted by a single bipartition of the model, provided the region \( B \) be a multicomponent region, i.e. \( m_B > 1 \).

At zero-temperature the entanglement entropy is symmetric upon the exchange of two subsystems \( A \) and \( B \), namely \( S_A(0) = S_B(0) \). But, this is no longer true at finite temperature. This is because the entanglement entropy acquires an extensive contribution from the bulk degrees of freedom of the region. At finite temperature the entanglement entropy is not a measure of quantum correlations between subsystems. The relevant quantity that drops the bulk dependency and is symmetric between two subsystems as well is the so called "mutual information". The mutual information grasps the total correlations (quantum and classical) between two subsystems. In particular, it is a linear combination of entanglement entropy as follows

\[
\mathcal{I}_{AB} = \frac{1}{2} (S_A(T) + S_B(T) - S_{A\cup B}(T)).
\]

Again, the von Neumann entropies for subsystems \( A \) and \( B \) as well as whole system \( A\cup B \) are captured by Eq. (29). Notice that \( \Sigma_{A\cup B} = 3N \). The derivation of the mutual information is straightforward. However, its full expression will be lengthy. Instead of given the full expression, we only give the limiting behavior as for the entanglement entropy. For finite size systems, the zero-temperature behavior coincides with the entanglement entropy since at zero-temperature the system will be in a pure state and the entropy \( S_{A\cup B}(T) \) vanishes. However, at the high-temperature limit, where the classical description holds, the mutual information is symmetric between two subsystems and scales with boundary supplemented with topological terms as follows.

\[
\mathcal{I}_{AB} = \frac{1}{2} (S_A(T) - 2m_A - 2m_B + 2) \ln 2.
\]

This quantity reveals that in the classical model precisely half of the mutual information at zero-temperature survives. Apart from the boundary term, the remaining terms are topological depending on the topology of the subsystems and underlying topological order manifests...
itself through them. Once again, classical system supports half of the topological order of the quantum system at zero-temperature. Taking first thermodynamic limit and then zero-temperature limit don’t commute with the opposite case. While both limits scale with the boundary, the contribution of topological terms are different. The topological contribution to the mutual information can be filtered out by taking the difference between two following limits:

\[
\lim_{T \to 0, N \to \infty} I_{AB} = (\Sigma_{AB} - m_A - m_B + 1) \ln 2,
\]

\[
\lim_{N \to \infty, T \to 0} I_{AB} = (\Sigma_{AB} - 2m_A - 2m_B + 2) \ln 2.
\]

The difference

\[
\Delta I_{AB} = \lim_{T \to 0, N \to \infty} I_{AB} - \lim_{N \to \infty, T \to 0} I_{AB}
\]

depends only on the topology of the regions as \(\Delta I_{AB} = (m_A + m_B - 1) \ln 2\). The advantage of this latter relation, in contrast to Eq. (34), is that even a simply connected bipartition of both subsystems give rises to a nonvanishing value. So, the mutual information paves the way in which we can define a purely topological order quantity using a simple connected region, in contrast to linear combination over different bipartitions.19,20

VI. TOPOLOGICAL ENTANGLEMENT ENTROPY

A. finite values of \(\lambda^c\)

Topological entanglement entropy appears as a subleading term of the entanglement entropy of a region with its complement in the topological phases. It is proposed that this subleading term can be used as a measure of topological order. In this section we use this measure to evaluate the behavior of topological order versus temperature in the topological color code. Ground state of the color code is appeared as a superposition of two underlying string-net structures. Both of them can be realized on the direct 2-colex lattice (unlike the toric code where one of the loop structure is appeared on the dual lattice). String-net structures are identified by considering the \(\sigma^+\) or \(\sigma^-\) bases. The string-net structure related to the latter bases is preserved by imposing the hard constrained limit. But, the string-net related to the former bases can be evaporated against the temperature. The dependency of the topological entanglement entropy on temperature clarifies how much robust the topological order related to the string-net structure is against the thermal fluctuations. In this subsection we assume that all three couplings \(\lambda^c\) are finite.

Thermal fluctuations may be so strong that are able to break the closed strings and leave them with end points. The broken strings as we explained in Sec carries excitations on their end points. Imposing the hard constrained limit on the couplings \(\lambda^c \to \infty\) in the Hamiltonian in Eq. (1), the appearance of respective excitations is restricted. So, part of topological order related to such strings is preserved. As other couplings, \(\lambda^c\), are finite, one may expect that the thermal fluctuations create the star excitations (See Fig. 11 at plaquettes). This means that only excitations with charge \((e, \chi_c)\) are created. As we will see the accumulation of these excitations in the model destroys the topological order partially.

As Eq. (29) suggests, the entanglement entropy scales with the boundary as well as bulk degrees of freedom. To get rid of boundary and bulk contribution, it is convenient to use a set of bipartitions, and then a linear combination of their entanglement entropies unveils the topological contribution. This set of bipartitions is shown in Fig. 2(b). The following linear combination of four bipartitions gives a purely topological contribution to the entanglement entropy.

\[
S_{\text{topo}} = \lim_{R,r \to \infty} (-S_{1A} + S_{2A} + S_{3A} - S_{4A}),
\]

where each term is given by the von Neumann entanglement entropy of the corresponding bipartition, \(R\) and \(r\) are the linear size of the subsystems as shown in Fig. 2(b). Large sizes of the regions are taken to make the size of the regions much larger than the correlation length in the topological phase. Notice that with the above bipartitions we have the following identifications for the number of connected and disconnected regions.

\[
m_{1A} = m_{2A} = m_{3A} = m_{2B} = m_{3B} = m_{4B} = 1,
\]

\[
m_{1B} = m_{4B} = 2.
\]

The following relations between number of plaquettes related to different bipartitions also hold.

\[
\Sigma_{1A} + \Sigma_{4A} = \Sigma_{2A} + \Sigma_{3A},
\]

\[
\Sigma^c_{1A} + \Sigma^c_{1,1} + \Sigma^c_{1,2} = N,
\]

\[
\Sigma^c_{2A} + \Sigma^c_{2,1} = N,
\]

\[
\Sigma^c_{3A} + \Sigma^c_{3,1} = N,
\]

\[
\Sigma^c_{4A} + \Sigma^c_{4,1} = N.
\]

Notice the ambiguity to the definition of low indices in \(\Sigma^c_{j,i}\), where \(j\) stands for one of the bipartitions in Fig. 2(b) and \(i\) has the same meaning as in Eq. (13). The full expression of \(S_{\text{topo}}\) is very lengthy (see Appendix X B). It is rather hard to see the behavior of the topological entanglement entropy versus temperature from this equation. To make this quantity more clear, let us consider some limiting cases. First, we consider the finite size systems. Zero-temperature and high-temperature limits are as follows:

\[
T \to 0 \quad (k_c \to 0) : \quad S_{\text{topo}} - S_{cc} = 0,
\]

\[
T \to \infty \quad (k_c \to \infty) : \quad S_{\text{topo}} - S_{cc} = -2 \ln 2,
\]
where $S_{cc} = 4 \ln 2$ is the topological entanglement entropy of color code at zero-temperature. In the zero-temperature limit the model coincides with the pure ground state. However, at high-temperature limit, as it is clear from the above relation, precisely half of the topological entropy is removed. This result is obtained along the hard constrained limit that has already been taken. As we discussed in preceding section, one could expect such result. In fact, while in the zero-temperature limit the system is fully topological order, at the high-temperature limit only the string-net structures related to $Z_c$ plaquettes survive. Since both string-net structures related to $X_c$ and $Z_c$ plaquettes have the same contribution to the topological order of the ground state, destroying one of them gives rise to reduction of the topological entanglement entropy by half.

The ground state has topological entropy $S_{cc} = \ln D^2$, where $D = 4$ is the so-called quantum dimension of the system. By rising the temperature, the populations of open $\gamma^z$ string-nets would be favorable since the open strings anticommute with plaquette operators $X_c$ that they share at odd qubits. At the very high temperature limit, $S_{topo}(T \to \infty) = \ln D$ implies that each underlying string-net structure contributes $\ln D$ to the topological entropy of the ground state.

Now, let us comment on whether the thermalization process and taking the classical limit affect the topological sectors of the color code model. The ground state of the color code is $4^{2g}$-fold degenerate for the systems that live on a manifold with genus $g$. The $4^{2g}$ topological sectors are identified by the eigenvalues of the non-local operators winding the handles of the manifold. These non-local operators are product of $\sigma^z$ operators along the winding closed string as $S^{\sigma^z}_\mu = \prod_{i \in \Gamma^c_\mu} \sigma^z_i$, where $\mu$ and $c$ stand for homology and color of the respective string and $\Gamma^c_\mu$ is the support of qubits winding the handle. By a closed $c$-string we mean a set of links that connect $c$-plaquettes. So, it commutes with all plaquettes. Notice that for each homology class there are three winding strings each of one color. However, because of the interplay between color and homology only two of them are independent. Besides, any other closed strings belonging to the same homology class are equivalent up to a deformation.

Within each sector, the ground state is the equal superposition of all bases obtained by any given state in the sector and applying the plaquette operators as in Eq.(2). For any ground state, the respective totally mixed state that corresponds to the density matrix at high-temperature limit is obtained by removing all non-diagonal elements of pure density matrix. The value of non-local string operators will be preserved by taking the high-temperature limit. This latter point implies that topological sectors are not get mixed through the thermalization of the code. In fact, for any mixed density matrix, the expectation value of the any closed non-winding loops that are product of plaquette operators will be +1 in the hard constrained system. Classically changing the topological sectors by flipping the spins along winding loops are exponentially suppressed leading to the so-called broken ergodicity, where the phase space is divided into topological sectors.

What happens if we take the thermodynamic limit first? As the size of the system goes to infinity, all $\Sigma_{j,i}^c$ tends to infinity except one $\Sigma_{1,1}^c$ related to the inner region of Fig.2(b(1)). Thus, the expression in Ap-
To get more intuitive picture of above relation, we plot the variation of $S_{\text{topo}}(T) - S_{cc}(0)$ in terms of $K \Sigma$ and temperature in Fig.3(b) and Fig.4, respectively. For simplicity we have set $K = k_c$ and $\Sigma = \Sigma_{1,1}'$. For finite size partition, the topological entropy drops as long as the temperature increases. However, even at the high-temperature limit half of topological entropy survives due to the hard constrained limit. As it is clear in Fig.3 at $K \Sigma = \frac{\pi}{4}$, a drop occurs at topological entropy. We use this to estimate a characteristic temperature above which the topological order disappears. The approximate temperature reads as follows:

$$T_{\text{drop}} \simeq \frac{\lambda_c}{\ln(\sqrt{2\Sigma'})},$$

where $\Sigma' = 3\Sigma$ is the total number of plaquettes in the inner region of Fig.3(b(1)). Thus, we can conclude that the dropping temperature depends on both coupling $\lambda_c$ and size of the partition. When the size of the partition becomes large, the dropping temperature tends to smaller values. This is transparent from Fig.3 where the dropping of the topological entropy versus temperature for different sizes of partition has been plotted. Since we are considering the large sizes of the partition in the very definition of the entanglement entropy, it is natural to consider the limit of $\Sigma \gg 1$ in Eq.(42) that amounts to the following relation:

$$S_{\text{topo}}(T) - S_{cc}(0) \rightarrow N \rightarrow \infty - 2\ln 2,$$

which explicitly implies that in the thermodynamic limit the topological entropy is fragile at any non-zero temperature. This is also expected from the dropping temperature if the bipartitions grow proportional to each other. By this we mean that each bipartition scales by a coefficient proportional to the size of the system. For instance, for the bipartitions in Fig.3(b(1)), we can write $\Sigma_A^c = \gamma_A N$ and $\Sigma_1^c = \gamma_1 N$, where $0 < \gamma < 1$ and $\gamma_1 + \gamma_1 + \gamma_2 = 1$. With this identification, now, we can rewrite the dropping temperature in terms of size of the system as follows:

$$T_{\text{drop}} \simeq \frac{\lambda_c}{\ln(\sqrt{2\Sigma'})},$$

which clearly shows that as the thermodynamic limit is reached, the dropping temperature tends to zero. In the thermodynamic limit only at zero temperature the topological order entirely exists.

### B. finite values of $\lambda_c^x$ but $\lambda_c^{x}, \lambda_c^{x} \rightarrow \infty$

Thus far, we have considered all $\lambda_c^{x}$’s are finite implying that excitations are allowed in all plaquettes by rising the temperature. As we derived in preceding subsection, at the high temperature limit precisely half of the topological entanglement entropy vanishes. In fact, the string-net structures related to the three colored strings are evaporated and excitations in all plaquettes are favorable. In the ground state, the string-net structures contribute $2\ln D$ to the topological entropy. Now, a question arises. How does an individual colored string impact on the topological entropy? We give an answer to this question in what follows. To do so, we allow for excitations to occur only on $c$-plaquettes while excitations on other $\bar{c}$- and $\bar{c}$-plaquettes are restricted due to the energy cost. This can be done by applying a similar hard constrained limit on the couplings $\lambda_c^{x}$ and $\lambda_c^{x}$ in the Hamiltonian of Eq.1. Let $c = g$, $\bar{c} = r$ and $\bar{c} = b$. Thus, thermal fluctuations produce only open green strings carrying topological charges $(e, \chi_g)$. The low lying states of the system are $|\psi[\gamma_g^g]\rangle = \gamma_g^g |\psi\rangle$, where $\gamma_g^g$ is an arbitrary open green string. Within these states, red and blue plaquettes are vortex free since $|X_r, \gamma_g^g\rangle = 0$ and $|X_b, \gamma_g^g\rangle = 0$.

Taking the limits $\lambda_c^{b} \rightarrow \infty (k_b \rightarrow 0)$, the topological entanglement in Eq.(42) behaves as

$$S_{\text{topo}}(T) - S_{cc}(0) \rightarrow N \rightarrow \infty$$

$$\left\{ e^{-\frac{k_b}{2} \Sigma_{1,1}^q} \left[ (\cosh \frac{k_g}{2} \Sigma_1^g) \ln(\cosh \frac{k_g}{2} \Sigma_1^g) + (\sinh \frac{k_g}{2} \Sigma_1^g) \ln(\sinh \frac{k_g}{2} \Sigma_1^g) \right] - \frac{k_g}{2} \Sigma_{1,1}^g \right\}$$

As we are eventually interested in large sizes of the bipartitions, i.e. $\Sigma_{1,1}^q \gg 1$, any non-zero temperature subsides the topological entanglement as follows

$$S_{\text{topo}}(T) - S_{cc}(0) \rightarrow N \rightarrow \infty - \ln 2.$$
the ground state level, the expectation values of non-local strings are nonzero and independent of any deformation of the strings, i.e. \( \Upsilon^c(0) = \langle \psi | S_{g,x}^c | \psi \rangle = +1(-1) \), depending on the topological sector we are analyzing it. But, at finite temperature the expectation value is replaced by thermal average as \( \Upsilon^c(T) = \frac{1}{N_T} \sum_{|T|} \langle S_{g,x}^c | \rangle_T \).

The excitations carrying by open green strings are deconfined and the expectation value of two non-local red (blue) strings on opposite sides of excitations are different. For instance consider an open green string carrying star excitations on its ends as shown in Fig.5(a). The winding blue and red strings cross the green string either odd times (solid wavy lines) or nothing (dashed wavy lines). The solid lines anticommute with green string while dashed lines commute. In the expectation value we must take average over all possible cases. So at finite temperature, the emerging excitations destroy the non-local order parameter \( \Upsilon^c(T) \simeq 0 \). However, the expectation value of a non-local green string remains finite since it commutes with defects. These lead to destroying the string-net structure of the ground state through the thermalization. The thermal states still contain closed green strings since they commute with defects, i.e. \([\lambda_g^x, \bar{\gamma}_g^x] = 0\), where \( \lambda_g^x \) is an arbitrary product of \( \sigma^x \) operators living on a closed green string. Notice that closed green strings are product of red and blue plaquettes \( X_r \) and \( X_b \) for which the expectation values with respect to thermal states are +1, i.e. they are stabilized by red and blue plaquettes. Thus, one may expect that the topological order in \( \sigma^x \) bases is partially preserved. This is just the message of the Eq.(47) that topological entanglement entropy can be reexpressed into \( S_{\text{topo}} = \ln D + \frac{1}{2} \ln D \), where \( \ln D \) is due to the string-net structure related to the \( \sigma^z \) bases and \( \frac{1}{2} \ln D \) is ascribed to one type of closed colored strings (here green) in \( \sigma^x \) bases survived even at finite temperature.

It is tempting to infer that each colored string contribute \( \frac{1}{2} \ln D \) to the topological entropy. However, the string structure of the topological order in the ground state of the color code is subject to the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) gauge symmetry. This is a property of the color code that each colored string is a \( \mathbb{Z}_2 \) gauge degree of freedom, but all strings form a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) gauge structure that is rooted in the string-net structure of the model. Nature of the topological order in the ground state comes from the fact that the ground state is invariant not only under deformation of strings but also the splitting of a \( c \)-string into \( \bar{c} \)- and \( \bar{c} \)-strings as shown in Fig.5(b). The latter point about splitting corresponds to the structure of the gauge group of the color code as we explained in Sec.4.

The relation between topological entropy and colored strings can be further understood if we soften more couplings. For example let \( \lambda_g^x, \lambda_r^x \) be finite while \( \lambda_b^x/T \gg 1 \). Now, the creation of defects in green and red plaquettes becomes favorable. Although the blue plaquettes remain immune against the thermal fluctuations, the average value of all colored winding strings vanishes, since they anticommute with green or red defects. Thus, all topological order vanishes. This can also be seen from the Eq.(42) by letting \( \lambda_g^x/T \gg 1(k_b \rightarrow 0) \). In the limit of large bipartitions, all topological order vanishes as follows

\[
S_{\text{topo}}(T) - S_{cc}(0) \rightarrow N \rightarrow -2 \ln 2. \tag{48}
\]

Once again when two colored strings are allowed to evaporate, the topological order in the system subsides. Putting all things together, we can conclude that the topological entanglement entropy related to the string-net structure in the \( \sigma^x \) bases receives contributions from the colored strings. In fact, each colored string contributes \( \frac{1}{2} \ln D \) to the topological entropy, but the symmetry of the color code model gives rise to the total contribution \( 2 \times \frac{1}{2} \ln D \).

VII. OPEN BOUNDARY CONDITIONS:

PLANAR COLORED CODES

Up to this point, we have only considered the system embedded in a closed surface such as torus, i.e. with periodic boundary conditions. However, the physical system must have boundary in which we can confine to a certain
piece of space. It is simple to obtain an open surface from a closed one by removing some plaquettes of the 2-colex. Such lattices are called planar color code. For example, consider a 2-colex embedded on a sphere. In this case the lattice encodes zero qubits, i.e. the ground state subspace is spanned by a single vector. If we remove a single qubit and its three neighboring links and plaquettes, the obtained 2-colex will encode a single qubit. In fact, three missed plaquettes form three colored borders for the lattice, and only strings with the same color of border will have end points on that border. An example of such bordered lattice is shown in Fig.5(c).

The most important class of the planar color code is the so called triangular color code. The triangular code has three colored borders each of one color (see Fig.5(c)). The logical operators are given by a string-net and its deformation in which make a two-dimensional algebra, since the code encodes only a single qubit. Such string-net is crucial in implementation of the Clifford group. The stabilizer group of the code is again given by a set of plaquettes. However, all plaquettes are independent because of boundary, which affect the number of collective operators needed for evaluating of entanglement entropy. Although a 2-colex on closed and open surfaces may represent different properties for the encoding and implementation of quantum information processing, they represent the same entanglement properties as we will see what follows. This is due to the fact that both structures have same symmetry and topological order comes from the string-net structure of the model.

The group element \( \hat{h} \) can be produced only in one way that is the product of some plaquettes. This amounts to consider only one of the terms in Eq.(47) and Eq.(49). This leads to the following simple relation for \( \eta_T(\hat{h}) \).

\[
\eta_T(\hat{h}) = e^{-k_g n_g - k_b n_b - k_r n_r}.
\] (49)

We can follow the method in Sec(IV) to map the contribution of plaquettes and collective strings in the group element into Ising variables. However, we should take care about the collective strings. Consider two disjoint regions \( B_1 \) and \( B_2 \) and connected region \( A \) as in Fig.2(a), but, on an open surface. Each disjoint region is surrounded by three colored strings in which only two of them are independent. Observe that two colored strings, say red, surrounding regions \( B_1 \) and \( B_2 \) are independent, in contrast to the closed surface. By this we mean that if \( \gamma^i \) be a collective string around the disjoint region \( B_i \), the product \( \prod_i \gamma^i \) is not the product of blue and green plaquettes of region \( A \). Implication of this point arises in the cardinalities of subgroups \( G_A \) and \( G_B \).

Since the open boundary conditions establish only one construction for an group element \( \hat{h} \) in terms of plaquette operators, the preceding arguments leading to the expression in Eq.(27) give only the first term as follows:

\[
\text{Tr} \left[ \rho_A^n \right] = \left( \frac{d_B}{|G|} \right)^{n-1} \frac{1}{Z_1} 2^{2\Sigma_A + 2m_B} \sum_{(J^g)} \sum_{(J^b)} \sum_{(J^r)} \left[ \prod_{i=1}^n \delta (J^g_i) \right] \times
\]

\[
\left\{ \left( Z_n^p(k_r, \{J^r_i\}) \right)^\Sigma^g_A \left( Z_n^p(k_b, \{J^b_i\}) \right)^\Sigma^b_A \left( Z_n^p(k_g, \{J^g_i\}) \right)^\Sigma^g_A \right\}
\]

\[
\prod_i Z_n^{ppp}(k_r, \Sigma^g_i, k_b, \Sigma^b_i, k_g, \Sigma^g_i, \{J^r_i\}, \{J^b_i\}, \{J^g_i\}).
\] (50)

where \( Z_1 = e^{\Sigma^g} \). In particular, the replica gives the entanglement entropy as

\[
S_A(T) = - \ln \left( \frac{d_A d_B}{|G|} \right) + \ln(Z_1) - \frac{1}{Z_1} \partial F_1.
\] (51)

The expression \( \partial F_1 \) is given in Appendix X.A.

For finite size systems, taking the zero-temperature limit gives the entanglement entropy of a single bipartition, \( m_A = m_B = 1 \), in the ground state of the triangular color code, i.e. \( S^t_A(0) = (\Sigma + 2) \ln 2 \), where the upper index \( t \) denotes the triangular code. If we were to take first the thermodynamic limit and then the zero-temperature limit, the limits don’t commute with each other. The difference between two limits depends only on the topology of the regions as \( \Delta S^t_A = \lim_{T \to 0, N \to \infty} S_A - \lim_{N \to \infty, T \to 0} S_A = 2m_B \ln 2 \), which implies that the topological contribution to the entanglement entropy can be extracted even by a single bipartition.

Topological entanglement entropy is given by Eq.(38) and the respective bipartitions in Fig.2(b). We should take into account the restriction imposed by boundary. The cardinalities of bipartitions in Fig.2(b(1)) are \( d_A = 2^{2\Sigma + 2} \) and \( d_B = 2^{2\Sigma + 2} \). The corresponding quantities for bipartitions in Fig.2(b(2)) and Fig.2(b(3)) are \( d_A = 2^{2\Sigma + 4} \) and \( d_B = 2^{2\Sigma + 4} \). Last bipartition in Fig.2(b(4)) yields the cardinalities \( d_A = 2^{2\Sigma_A} \) and \( d_B = 2^{2\Sigma_B + 4} \). This leads to the topological entropy of triangular color code at zero temperature \( \infty \) as \( S^t_{tr} = 4 \ln 2 \), which is similar with that of the periodic boundary conditions.

Finite temperature topological entropy behaves as
what we derived for closed surface. For finite size systems and at high-temperature limit precisely half of the topological entanglement is preserved as \( S_{\text{topo}}(T \to \infty) - S_{\text{topo}}^L = -2\ln 2 \), which is a consequence of destroying of string-net structure in the \( \sigma^x \) or \( \sigma^z \) bases. If we take first the thermodynamic limit, any non-zero temperature subsides the topological entropy to half of its value at zero temperature.

### VIII. CONCLUSIONS

In this work the topological order in a class of two-dimensional topological stabilizer codes, the so called color code, at finite temperature has been addressed. Both closed and open surfaces, where the lattice embedded on, were considered. This is because quasiparticle excitations are different for each embedding. The stabilizer structure of the color code comes from the plaquette operators which can be of \( X \) and \( Z \) Pauli operators. The plaquettes are labeled by their color and type. The stabilizer group is adjusted into a many-body Hamiltonian in which the coding space is its ground state subspace. The Hamiltonian is supplemented with energy scales such as \( \lambda_x \)’s and \( \lambda_z \)’s. The ground state is a string-net condensate. Closed string-nets are collections of colored strings in which no end points are left. Topological order in the ground state of the model arises from the coherent superposition of two string-net structures that can be visualized by adopting the string-nets in \( \sigma^x \) and \( \sigma^z \) bases. This topological order can be characterized by the topological entanglement entropy that is \( 2 \ln D \) where \( D = 4 \).

Considering the limit \( \lambda_z \to \infty \), the exact solution of the model at finite temperature becomes possible. As the temperature is increased, for finite size systems, the topological entropy reduced to \( \ln D \). This implies that by thermalization of one of the string-net structures the respective topological order is destroyed. Both string-net structures have equal contribution to topological entropy.

The temperature at which the topological entropy is dropped is a function of both coupling \( \lambda_x \) and size of system as in Eq. (45). This relation can be used to distinguish a length scale for separated defects. It can be recast into \( N e^{-2\lambda_x/T_{\text{drop}}} \simeq 1 \). This means that topological order is destroyed when density of defects in the model becomes of order unity as the temperature is increased. Similar behavior observed in the toric code. This interpretation for density of defects allows us to define the respective length scale as \( \zeta^x = e^{\lambda_x/T} \). For temperatures well below the dropping temperature \( T < T_{\text{drop}} \), the characteristic length scale is large implying the density of defects \( N e^{-2\lambda_x/T} \) is much less than unity. In this case the thermal defects are not able to shave out the string-net structure of the model and topological order is preserved. As the temperature increases, the density of defects becomes of order unity that destroy the topological order in the system. In the thermodynamic limit and at finite temperature the distance between defects is much less than the system size. Thus at this limit any nonzero temperature can destroy topological order as in Eq. (44). The robustness of the string-net structure in the hard constrained limit can be understood via such identification for distance between defects. In the limit \( \lambda_x \to \infty \) the distance between respective defects \( \zeta^x = e^{\lambda_x/T} \) is infinity. So, the respective topological order is immune even in the high temperature limit. This is the reason why in the high constrained system half of the topological order is preserved.

Topological order in the color code arises from coherent superposition of string-net structures in \( \sigma^x \) and \( \sigma^z \) bases. Both structures are similar having the same contribution in the topological entropy. By imposing extra conditions on the couplings \( \lambda_x \), we can examine the contribution of colors and underlying symmetry on the topological entropy. If we let for defects to occur only in \( c \)-plaquettes, i.e. \( \lambda^c_x, \lambda^c_z \to \infty \), the thermal states still carry topological order in the \( \sigma^c \) bases even at thermodynamic limit as in Eq. (47). However, if we soften the latter condition and let for defects to occur in both \( c \)- and \( \bar{c} \)-plaquettes, i.e. only let \( \lambda^c_x \to \infty \), as Eq. (48) suggests topological order is entirely destroyed. These observations reveal that both colors and symmetry determine the topological entropy. In each loop structure, either in \( \sigma^x \) or \( \sigma^z \) bases, string-nets are collections of closed colored strings. Each colored closed string contributes \( \frac{1}{2} \ln D \) to the topological entropy. Considering the gauge symmetry of the model, the total contributions of closed colored strings in the topological entropy will be \( \ln D \). In the ground state both string-net structures in \( \sigma^x \) and \( \sigma^z \) bases contribute equally yielding the topological entropy as \( 2 \ln D \).

All above results also hold for the case of open boundary conditions. For this case we considered the triangular color code with borders. Although lattices embedded in closed or open surfaces present different properties from the quantum information perspective, they have similar topological order. This is because they have similar gauge symmetry group and consequently same underlying string structure that is reflected in topological entropy.

The topological order in both toric code and color code models in two-dimensional space is fragile against thermal fluctuations in the sense that their underlying structures are one-dimensional objects, i.e. the strings. This fragility limits their capabilities as self-stability codes. The deconfinement of open strings carrying excitations can be restricted by coupling the defects to bosonic fields. Toric code model in three-dimension have a membrane structure that are robust against thermal noises. The color code model can also be generalized to \( D \)-dimensional space with branyon and brane-net structures. It is instructive to generalize the approach presented here to color codes based on the D-colexes in the sense that they have different underlying objects as physical mechanism for topological order.
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X. APPENDIX

A. F’s expressions and their derivatives

$F_j^{(n)}$ for $j = 1, 2, 3, 4$ stand for four terms between the two brackets in Eq. (27). By using the partition functions related to plaquettes and strings that are given in Eq. (21), Eq. (24) and Eq. (26), the following relations for $F_j^{(n)}$’s are obtained.

\[
F_1^{(n)} = (x_r^n + y_r^n)\sum_{b}(x_b^n + y_b^n)\sum_{g}(x_g^n + y_g^n)\prod_i(\xi_{1i}^n + \xi_{2i}^n + \xi_{3i}^n + \xi_{4i}^n)
\]
\[
F_2^{(n)} = (x_r^n - y_r^n)\sum_{b}(x_b^n - y_b^n)\sum_{g}(x_g^n + y_g^n)\prod_i(-\xi_{1i}^n + \xi_{2i}^n - \xi_{3i}^n + \xi_{4i}^n)
\]
\[
F_3^{(n)} = (x_r^n + y_r^n)\sum_{b}(x_b^n - y_b^n)\sum_{g}(x_g^n - y_g^n)\prod_i(-\xi_{1i}^n - \xi_{2i}^n + \xi_{3i}^n + \xi_{4i}^n)
\]
\[
F_4^{(n)} = (x_r^n - y_r^n)\sum_{b}(x_b^n + y_b^n)\sum_{g}(x_g^n - y_g^n)\prod_i(\xi_{1i}^n - \xi_{2i}^n - \xi_{3i}^n + \xi_{4i}^n)
\]

The case $n = 1$ and their derivatives are needed for evaluation of entanglement entropy. So, the above expressions get more simplified as follows:

\[
F_1^{(1)} = e^\frac{\Phi}{4}(k_r + k_b + k_g), \quad F_2^{(1)} = e^\frac{\Phi}{4}(-k_r + k_b + k_g),
\]
\[
F_3^{(1)} = e^\frac{\Phi}{4}(-k_r - k_b + k_g), \quad F_4^{(1)} = e^\frac{\Phi}{4}(-k_r + k_b - k_g)
\]

Trivially $Z_0 = \frac{1}{4}(F_1^{(1)} + F_2^{(1)} + F_3^{(1)} + F_4^{(1)})$, and their derivatives at $n = 1$ are
\[ \partial F_1 = (\partial F_1^{(n)}/\partial n)_{n \to 1} = F_1^{(1)} \left[ \Sigma_A e^{x_A} (x_r \ln x_r + y_r \ln y_r) + \Sigma_A e^{-x_A} (x_b \ln x_b + y_b \ln y_b) + \Sigma_A e^{-x_A} (x_g \ln x_g + y_g \ln y_g) \\
+ \sum_i e^{-\frac{k_i}{2} \Sigma_i + \frac{k}{2} \Sigma_i^0 - \frac{k}{2} \Sigma^g} (\xi_{1i} \ln \xi_{1i} + \xi_{2i} \ln \xi_{2i} + \xi_{3i} \ln \xi_{3i} + \xi_{4i} \ln \xi_{4i}) \right], \]

\[ \partial F_2 = (\partial F_2^{(n)}/\partial n)_{n \to 1} = F_2^{(2)} \left[ \Sigma_A e^{y_A} (x_r \ln x_r - y_r \ln y_r) + \Sigma_A e^{y_A} (x_b \ln x_b - y_b \ln y_b) + \Sigma_A e^{y_A} (x_g \ln x_g + y_g \ln y_g) \\
+ \sum_i e^{\frac{k_i}{2} \Sigma_i + \frac{k}{2} \Sigma_i^0 - \frac{k}{2} \Sigma^g} (\xi_{1i} \ln \xi_{1i} + \xi_{2i} \ln \xi_{2i} - \xi_{3i} \ln \xi_{3i} + \xi_{4i} \ln \xi_{4i}) \right], \]

\[ \partial F_3 = (\partial F_3^{(n)}/\partial n)_{n \to 1} = F_3^{(3)} \left[ \Sigma_A e^{-y_A} (x_r \ln x_r + y_r \ln y_r) + \Sigma_A e^{-y_A} (x_b \ln x_b - y_b \ln y_b) + \Sigma_A e^{-y_A} (x_g \ln x_g - y_g \ln y_g) \\
+ \sum_i e^{\frac{k_i}{2} \Sigma_i + \frac{k}{2} \Sigma_i^0 + \frac{k}{2} \Sigma^g} (\xi_{1i} \ln \xi_{1i} - \xi_{2i} \ln \xi_{2i} + \xi_{3i} \ln \xi_{3i} + \xi_{4i} \ln \xi_{4i}) \right], \]

\[ \partial F_4 = (\partial F_4^{(n)}/\partial n)_{n \to 1} = F_4^{(4)} \left[ \Sigma_A e^{y_A} (x_r \ln x_r - y_r \ln y_r) + \Sigma_A e^{y_A} (x_b \ln x_b + y_b \ln y_b) + \Sigma_A e^{y_A} (x_g \ln x_g - y_g \ln y_g) \\
+ \sum_i e^{\frac{k_i}{2} \Sigma_i - \frac{k}{2} \Sigma_i^0 + \frac{k}{2} \Sigma^g} (\xi_{1i} \ln \xi_{1i} - \xi_{2i} \ln \xi_{2i} - \xi_{3i} \ln \xi_{3i} + \xi_{4i} \ln \xi_{4i}) \right], \]

(53)

**B. Topological Entropy**

entropy then reads as follows:

Applying Eq. (29) about each bipartition of Fig. (2b), and inserting into Eq. (38), the topological entanglement
\[ S_{\text{topo}}(T) - S_{\text{cc}}(0) = \]
\[
\frac{1}{4Z_0} \sum_{i=1}^{2} \left\{ e^{k_1(N-S_{1i})} \frac{k_1}{k_2}(N-S_{1i}) \right\}(\xi_1^{(1)} \ln \xi_1^{(1)} + \xi_2^{(1)} \ln \xi_2^{(1)} + \xi_3^{(1)} \ln \xi_3^{(1)} + \xi_4^{(1)} \ln \xi_4^{(1)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(1)} \ln \xi_1^{(1)} + \xi_2^{(1)} \ln \xi_2^{(1)} - \xi_3^{(1)} \ln \xi_3^{(1)} + \xi_4^{(1)} \ln \xi_4^{(1)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(1)} \ln \xi_1^{(1)} - \xi_2^{(1)} \ln \xi_2^{(1)} + \xi_3^{(1)} \ln \xi_3^{(1)} + \xi_4^{(1)} \ln \xi_4^{(1)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(1)} \ln \xi_1^{(1)} + \xi_2^{(1)} \ln \xi_2^{(1)} + \xi_3^{(1)} \ln \xi_3^{(1)} + \xi_4^{(1)} \ln \xi_4^{(1)})
\]
\[
- \frac{1}{4Z_0} \left\{ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \right\}(\xi_1^{(2)} \ln \xi_1^{(2)} + \xi_2^{(2)} \ln \xi_2^{(2)} + \xi_3^{(2)} \ln \xi_3^{(2)} + \xi_4^{(2)} \ln \xi_4^{(2)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(2)} \ln \xi_1^{(2)} + \xi_2^{(2)} \ln \xi_2^{(2)} - \xi_3^{(2)} \ln \xi_3^{(2)} + \xi_4^{(2)} \ln \xi_4^{(2)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(2)} \ln \xi_1^{(2)} - \xi_2^{(2)} \ln \xi_2^{(2)} + \xi_3^{(2)} \ln \xi_3^{(2)} + \xi_4^{(2)} \ln \xi_4^{(2)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(2)} \ln \xi_1^{(2)} + \xi_2^{(2)} \ln \xi_2^{(2)} + \xi_3^{(2)} \ln \xi_3^{(2)} + \xi_4^{(2)} \ln \xi_4^{(2)})
\]
\[
- \frac{1}{4Z_0} \left\{ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \right\}(\xi_1^{(3)} \ln \xi_1^{(3)} + \xi_2^{(3)} \ln \xi_2^{(3)} + \xi_3^{(3)} \ln \xi_3^{(3)} + \xi_4^{(3)} \ln \xi_4^{(3)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(3)} \ln \xi_1^{(3)} + \xi_2^{(3)} \ln \xi_2^{(3)} - \xi_3^{(3)} \ln \xi_3^{(3)} + \xi_4^{(3)} \ln \xi_4^{(3)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(3)} \ln \xi_1^{(3)} - \xi_2^{(3)} \ln \xi_2^{(3)} + \xi_3^{(3)} \ln \xi_3^{(3)} + \xi_4^{(3)} \ln \xi_4^{(3)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(3)} \ln \xi_1^{(3)} + \xi_2^{(3)} \ln \xi_2^{(3)} + \xi_3^{(3)} \ln \xi_3^{(3)} + \xi_4^{(3)} \ln \xi_4^{(3)})
\]
\[
+ \frac{1}{4Z_0} \left\{ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \right\}(\xi_1^{(4)} \ln \xi_1^{(4)} + \xi_2^{(4)} \ln \xi_2^{(4)} + \xi_3^{(4)} \ln \xi_3^{(4)} + \xi_4^{(4)} \ln \xi_4^{(4)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(4)} \ln \xi_1^{(4)} + \xi_2^{(4)} \ln \xi_2^{(4)} - \xi_3^{(4)} \ln \xi_3^{(4)} + \xi_4^{(4)} \ln \xi_4^{(4)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(4)} \ln \xi_1^{(4)} - \xi_2^{(4)} \ln \xi_2^{(4)} + \xi_3^{(4)} \ln \xi_3^{(4)} + \xi_4^{(4)} \ln \xi_4^{(4)})
\]
\[
+ e^{k_1(N-S_{1i})} \cdot \frac{k_1}{k_2}(N-S_{1i}) \left\{ (-\xi_1^{(4)} \ln \xi_1^{(4)} + \xi_2^{(4)} \ln \xi_2^{(4)} + \xi_3^{(4)} \ln \xi_3^{(4)} + \xi_4^{(4)} \ln \xi_4^{(4)})
\]
\[
\right\},
\]

(54)

where \( S_{\text{cc}}(0) \) stands for the topological entanglement of the color code at zero-temperature, i.e. \( S_{\text{cc}}(0) = 4 \ln 2 \).

Notice that the upper indices of \( \xi \)'s are related to the respective bipartitions in Fig 2(b).

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