Stochastic formulation of incompressible fluid flows in wall-bounded regions

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Abstract

In this paper we establish a mathematical framework which may be used to design Monte-Carlo simulations for a class of time irreversible dynamic systems, such as incompressible fluid flows, including turbulent flows in wall-bounded regions, and some other (non-linear) dynamic systems. Path integral representations for solutions of forward parabolic equations are obtained, and, in combining with the vorticity transport equations, probabilistic formulations for solutions of the Navier-Stokes equations are therefore derived in terms of (forward) McKean-Vlasov stochastic differential equations (SDEs), which provides us with the mathematical framework for Monte-Carlo simulations of wall-bounded turbulent flows.

Keywords: Diffusion process, Feynman-Kac formulas, McKean-Vlasov SDEs, Monte Carlo method, Navier-Stokes equation, parabolic equations, pinned diffusion measures, turbulent flows, velocity field, vorticity, vorticity transport equation, wall-bounded fluid flows

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1 Introduction

It was pointed out that, in a report [32] by von Neumann about 70 years ago, turbulence may be understood with the assistance of powerful electronic computer simulations. With the increasing computational power it is now possible to obtain information about turbulent flows via direct numerical simulations (DNS), large-eddy simulation (LES) and other simulation technologies (cf. [13, 33, 21, 39] for example). One of the common difficulties for implementing turbulent simulations (such as the finite element method, the finite difference method and the finite volume method) lies in the non-linear and non-local nature of the governing fluid dynamic equations, the Navier-Stokes equations. According to Kolmogorov’s small scale theory of isotropic turbulence, in order to obtain meaningful simulations of turbulent flows, the space separation has to be smaller than \( \text{Re}^{-3/4}L \), where Re is the Reynolds number and \( L \) the typical length. This requirement for a small mesh in numerical schemes leads to huge computation hours for simulating turbulence, see [12, Chapter 7] for a very useful analysis about this aspect. It is thus desirable to develop Monte-Carlo schemes for simulations of turbulent flows which may avoid the use of small grids in order to numerically solve non-linear fluid dynamic equations.

Monte-Carlo simulations are based on the law of large numbers, which says the expectation of a random variable can be approximated by the average of independent samplings. Although convergence rates of Monte-Carlo methods are in general slow, but the advantage of Monte-Carlo schemes in most cases lies in their capability of dealing with multivariate dynamic variables. To implement Monte-Carlo schemes for numerically calculating solutions of some linear and non-linear partial differential equations, explicit representations of solutions in terms of some distributions, or in terms of stochastic differential equations, have to be established. An archetypal example is the numerical scheme for computing solutions of Schrödinger type partial differential equations, where the Feynman-Kac formula provides us with a useful representation to the solutions of the Schrödinger equations.

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In this paper we establish a mathematical framework for designing Monte-Carlo schemes for numerically computing solutions of incompressible fluid flows in wall-bounded regions. The basic idea in our approach is to derive an equivalent formulation of the fluid dynamic equations in terms of McKean-Vlasov type (ordinary) stochastic differential equations which can be solved numerically. Let us describe the basic idea and the main technical issues which will be solved in the present paper.

The basic dynamic variable for a fluid flow in a region $D$ is its velocity $u(x,t)$ for $x \in D$ and $t \geq 0$, a time dependent vector field. The main scientific question in turbulence is to give a good description of $u(x,t)$. Mathematically determining the velocity $u(x,t)$ is equivalent to describing its integral curve $X(\xi,t)$ which are solutions to the following differential equation

$$\frac{d}{dt}X(\xi,t) = u(X(\xi,t),t), \quad X(\xi,0) = \xi. \quad (1.1)$$

For a viscous fluid flow with viscosity constant $\nu > 0$, according to Taylor [42], one may consider the Brownian trajectories $X(\xi,t)$ instead which are solutions to the random dynamic system

$$dX(\xi,t) = u(X(\xi,t),t)dt + \sqrt{2\nu}dB(t), \quad X(\xi,0) = \xi, \quad (1.2)$$

where $B(t)$ is a Brownian motion on some probability space. The main idea is to rewrite the velocity $u(x,t)$ in terms of the distribution of the Brownian particles $X(\xi,t)$, substituting it into (1.2), so that (1.2) becomes a stochastic differential equation involving the distribution of its solution. We may call this technique as solving the closure problem of Taylor’s diffusion. The closure problem for incompressible fluid flows freely moving in the whole space (without boundary) has been solved in a recent paper [37] (see also for a similar probabilistic representation in [9]) by using an idea from the vortex methods [4, 5, 11, 29]. The most important case of fluid flows in wall-bounded regions will be solved in the present paper. Let us recall some basic ideas in the study of the vortex dynamics and the vorticity-velocity formulation, for details one may refer to [6, 7, 11, 28, 29].

Consider an incompressible fluid flow with viscosity constant $\nu > 0$, whose velocity $u = (u^1,u^2,u^3)$ and pressure $P$ solve the Navier-Stokes equations

$$\frac{\partial}{\partial t} u^i + (u \cdot \nabla) u^i - \nu \Delta u^i + \frac{\partial}{\partial x^i} P = 0 \quad (1.3)$$

and

$$\sum_{j=1}^{3} \frac{\partial}{\partial x^j} u^j = 0 \quad (1.4)$$

where $i = 1,2,3$. The sum in (1.4) is the divergence of $u$, $\nabla \cdot u$ at any instance, so the second equation means that the velocity $u(x,t)$ is divergence free (which can be formulated in the distribution sense on $\mathbb{R}^3$), $u_0(x) = u(x,0)$ is called the initial data. The boundary condition at infinity has to be supplied such as the velocity is bounded at infinity, but let us ignore this kind of technical issues. It is known that the vorticity $\omega = \nabla \times u$, whose components $\omega^i = \varepsilon^{ijk} \frac{\partial}{\partial x^j} u^k$, plays a dominate rôle in the description of turbulent flows. The vorticity $\omega$ evolves according to the vorticity transport equations

$$\frac{\partial}{\partial t} \omega^i + (u \cdot \nabla) \omega^i - \nu \Delta \omega^i - \sum_{j=1}^{3} S^j_i \omega^j = 0 \quad (1.5)$$

for $i = 1,2,3$, which are obtained by differentiating the Navier-Stokes equations. Here

$$S^j_i = \frac{1}{2} \left( \frac{\partial u^j}{\partial x^i} + \frac{\partial u^i}{\partial x^j} \right) \quad (1.6)$$

is the symmetric tensor of rate-of-strain. Note that the dynamic variables $u$, $\omega$ and $S$ depend on $(x,t)$ and are time irreversible. Our goal is to express the velocity $u(x,t)$ in terms of the distribution of Taylor’s diffusion $X(\xi,t)$ defined by (1.2). To achieve this goal, we observe that $X(\xi,t)$ is a diffusion process
with its infinitesimal generator $L_u = \nabla u + u \cdot \nabla$, whose transition probability density $p_u(\tau, \xi, t, x)$ is the fundamental solution to $L_u + \frac{\partial}{\partial \tau}$, cf. [41], [23] for example. According to a general fact from the parabolic theory (cf. [18]), $p_u(\tau, \xi, t, x)$ coincides with the fundamental solution to the forward heat operator $L_u^* - \frac{\partial}{\partial \tau}$. Since $u$ is divergence-free, so that $L_u^* = L_{-u}$, hence the vorticity transport equations (1.5) may be written as Schrödinger type equations

$$
\left( L_{-u} - \frac{\partial}{\partial \tau} \right) \omega^i + \sum_{j=1}^{3} S^j_i \omega^j = 0,
$$

(1.7)

and therefore it is possible to express $\omega^i$ in terms of the distribution of the Taylor diffusion, the data $S^j_i$ and the initial vorticity $\omega_0$. For fluid flows for which the “potential” term $\sum^3_{j=1} S^j_i \omega^j$ vanish identically (for $i = 1, 2, 3$); in fact it is the case for 2D fluid flows; the vorticity transport equations become $\left( L_{-u} - \frac{\partial}{\partial \tau} \right) \omega^i = 0$, thus $\omega^i$ can be written as an integral of the initial vorticity $\omega_0 = \nabla \wedge u_0$ against the fundamental solution $p_u(\tau, \xi, t, x)$. Therefore the closure problem for this case can be solved by using the Biot-Savart law, cf. [28, 29]. For turbulent flows, it is typical that the non-linear vorticity stretching term $\sum^3_{j=1} S^j_i \omega^j$ do not vanish. For this case one may apply the Feynman-Kac formula to (1.7) in terms of the time reversed diffusion with generator $L_{-u}$ (but not $L_u$). To resolve the closure problem to (1.2), we need to rewrite the Feynman-Kac formula in terms of the law of the Taylor diffusion (1.2), which will be achieved by using the duality between the conditional law of the time reversed $L_{-u}$-diffusion and the conditional law of $L_u$-diffusion, cf. [37] for details.

In this paper we aim to solve the closure problem for incompressible fluid flows constrained in a domain $D$ with a boundary $\partial D$, so that schemes may be devised for numerically calculating the solutions, including the flows within their boundary layers. For a wall-bounded flow, the velocity $u(x, t)$ satisfies the Navier-Stokes equations (1.3, 1.4) only for $x \in D$ and $t > 0$, and has to satisfy the no-slip condition, i.e. $u(x, t)$ vanishes for $x \in \partial D$. We extend the definition of $u(x, t)$ to all $x \in \mathbb{R}^3$ such that $\nabla \cdot u = 0$ on $\mathbb{R}^3$ in the distribution sense. This latter requirement is needed in order to ensure again the duality for the conditional laws of some diffusions involved. Hence we can define the Taylor diffusion again by (1.2) on the whole space $\mathbb{R}^3$.

To solve the closure problem for the boundary problem of the fluid dynamic equations, we develop several technical tools, mainly the duality of conditional laws, and the forward type Feynman-Kac formula for a general class of diffusion processes. We believe that these results have independent interest by their own and may be useful for the study of other problems associated with diffusion-reaction equations (cf. [16, 17]). More precisely we will establish these results for the laws of diffusion processes with infinitesimal generator being an elliptic operator of second order

$$
\mathcal{L} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^i} a^{ij}(x, t) \frac{\partial}{\partial x^j} + \sum_{i=1}^{d} b^i(x, t) \frac{\partial}{\partial x^i},
$$

although some results are established for more general elliptic operators, i.e. for elliptic operators not in divergence form.

This paper is organised as the following. In Section 2, we introduce a class of time dependent elliptic operators of second order which set up the basic data used throughout the paper. We recall in this section a few analytic and probabilistic structures, and recall several well known relations among forward, backward fundamental solutions and transition probability density functions, scattered in literature, stated as Lemma 2.2, Lemma 2.3 and Lemma 2.5. In Section 3, by using the classical Feynman-Kac formula, we establish a general duality relation between the fundamental solution and the transition probability density function (Lemma 3.2 and Lemma 3.3), and under the divergence free condition on the drift vector field $b(x, t)$ establish the duality (Lemma 3.4). In Section 4, we prove the main technical tool, a time reverse theorem (Theorem 4.1) for the conditional laws of the $\mathcal{L}$-diffusion when the vector field $b(x, t)$ is divergence free. Thus far we have established the basic tools for proving our main results. In Section 5, we prove a forward Feynman-Kac formula for solutions of Schrödinger type parabolic systems on the
whole space, which generalises the results in [37]. Section 6 and Section 7 contain the main contributions of this paper. In Section 6, we establish a forward Feynman-Kac formula for solutions of the (non homogeneous) boundary value problems, Theorem 5.3, and in Section 7 we apply the theory to solve the closure problem for solutions of the Navier-Stokes equations satisfying the no-slip boundary condition.

Numerical experiments are not included in the present paper for the reason that numerical simulations based on the formulation of the Navier-Stokes equations in Section 7 must be done case by case, and some of numerical experiments will be published in separate papers.

In the past decades, many excellent works addressing some probabilistic aspects of fluid dynamics have been published by various authors, although not from a viewpoint of numerically calculating solutions, cf. [2], [3], [7, 8], [9], [44, 45] for example and the literature therein. There are still many papers the author may be not aware of, to the best knowledge of the present author however, only the Navier-Stokes equations on \( \mathbb{R}^d \) or \( \mathbb{T}^d \) (\( d = 2, 3 \)) were considered in a view of stochastic analysis in particular via the stochastic flow method, and the boundary value problems are not treated yet in the existing literature. Also in this paper we adopt an approach of weak solutions both for PDEs and SDEs, and therefore stochastic flows do not play a rôle in our study. This makes distinct difference between our approach and the existing methods.

### 2 Assumptions and notations

In this section, we introduce several notions and notations which will be used throughout the paper.

Let \( D \subset \mathbb{R}^d \) be an open subset of the Euclidean space of \( d \) dimensions, with a smooth boundary \( \partial D \), where \( d \geq 2 \). \( \partial D \) is an embedded sub-manifold of dimension \( d - 1 \), though not necessary connected. \( \bar{D} \) denotes the closure of \( D \).

Let \( \nu > 0 \) be a positive constant representing “the viscosity constant”.

Let \( a(x,t) = (a^{ij}(x,t))_{i,j \leq d} \) be a Borel measurable, \( d \times d \) symmetric matrix-valued function defined for \( x \in \mathbb{R}^d \) and \( t \in \mathbb{R} \). It is assumed that \( a(x,t) \) is uniformly elliptic in the sense that there is a constant \( \lambda \geq 1 \) such that

\[
\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^d \xi^i a^{ij}(x,t) \xi^j \leq \lambda |\xi|^2
\]

for every \( \xi = (\xi^1, \cdots, \xi^d) \in \mathbb{R}^d \) and \( (x,t) \in \mathbb{R}^d \times \mathbb{R} \). It is also assumed that there is a symmetric matrix-valued function \( \sigma(x,t) = (\sigma_i^j(x,t)) \) such that \( \sigma_i^j(x,t) = \sigma_j^i(x,t) \) and

\[
a^{ij}(x,t) = \sum_{i=1}^d \sigma_i^i(x,t) \sigma_i^j(x,t)
\]

for all \( i, j \leq d \).

Let \( b(x,t) = (b^1(x,t), \ldots, b^d(x,t)) \) be a Borel measurable, time-dependent vector field on \( \mathbb{R}^d \), and \( c(x,t) \) be a Borel measurable scalar function on \( \mathbb{R}^d \times \mathbb{R} \). \( a(x,t), b(x,t) \) and \( c(x,t) \) may be defined originally only for \( (x,t) \in D \times J \) (where \( J \subset \mathbb{R} \) is an interval), but without further specification, these functions are automatically extended to all \( (x,t) \in \mathbb{R}^d \times \mathbb{R} \) according to the following rules: \( b(x,t) = 0 \), \( c(x,t) = 0 \) and \( a^{ij}(x,t) = \delta_i^j \) for \( (x,t) \notin D \times J \) unless otherwise specified.

Define a time-dependent differential operator of second order

\[
L_{a,b,c} = \nu \sum_{i,j=1}^d a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i \frac{\partial}{\partial x^i} + c.
\]

The first term involving \( a^{ij}(x,t) \) on the right-hand side is called the diffusion part, \( b(x,t) \) is called the drift vector field, and the scalar multiplication part \( c(x,t) \) is called the zero-th order term. If \( c = 0 \), then \( L_{a,b,b} \) is a diffusion operator, which will be denoted by \( L_{a,b} \) for simplicity. If \( a^{ij}(x,t) = \delta_i^j \) for all \( (x,t) \in \mathbb{R}^d \times \mathbb{R} \), then \( L_{a,b,b} \) is denoted by \( L_b \), and similarly, if \( a^{ij}(x,t) = \delta_i^j \) and \( c(x,t) = 0 \) for all \( (x,t) \in \mathbb{R}^d \times \mathbb{R} \), then \( L_{a,b,b} \) is denoted by \( L_b \).
\( L_{a,b,c} \) may be written in a form whose diffusion part is written in divergence form

\[
L_{a,b,c} = v \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j} a_{ij} \frac{\partial}{\partial x^i} + \sum_{i=1}^{d} \left( b^i - v \sum_{j=1}^{d} \frac{\partial a_{ij}^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} + c, \tag{2.4}
\]

as long as \( \frac{\partial a_{ij}^i}{\partial x^j} \) exist. According to integration by parts, the formal adjoint \( L_{a,b,c}^* \) of \( L_{a,b,c} \) is again a second order differentiable operator and \( L_{a,b,c}^* = L_{a;b,c} \), where

\[
b^i_*(x,t) = -b^i(x,t) + 2v \sum_{j=1}^{d} \frac{\partial a_{ij}^i(x,t)}{\partial x^j} \tag{2.5}
\]

and

\[
c^i_*(x,t) = c(x,t) - \nabla \cdot b(x,t) + v \sum_{i,j=1}^{d} \frac{\partial^2 a_{ij}^i(x,t)}{\partial x^j \partial x^i} \tag{2.6}
\]

respectively. The formal adjoint of the (forward) heat operator \( L_{a,b,c} - \frac{\partial}{\partial t} \) is the (backward) heat operator \( L_{a,b,c} + \frac{\partial}{\partial t} \).

**Example 2.1.** If \( a_{ij}^i = \delta_{ij} \), then \( L_{b,c} = v \Delta + b \cdot \nabla + c \) and \( L_{b,c}^* = L_{b,c} - \nabla b \), which is a very important relation in the study of random vortex methods, cf. [4, 11, 29] for details.

Similarly

\[
\mathcal{L}_{a,b,c} = v \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j} a_{ij} \frac{\partial}{\partial x^i} + \sum_{i=1}^{d} b^i \frac{\partial}{\partial x^i} + c \tag{2.7}
\]

is an elliptic operator of second order, and \( \mathcal{L}_{a,b,c} = L_{a;b,c} \), where

\[
\hat{b}^i = b^i + v \sum_{j=1}^{d} \frac{\partial a_{ij}^i}{\partial x^j} \tag{2.8}
\]

as long as \( \frac{\partial a_{ij}^i}{\partial x^j} \) exist. Note that \( \hat{b} \) is independent of \( c \).

In general Einstein’s convention that repeated indices are summed up in their ranges is applied, unless said otherwise. The following convention will also be applied to quantities which rely on \( a, b \) and \( c \). If a quantity depends on \( a, b \) and \( c \) then it may be labelled with a lower subscript \( a;b,c \). If \( a_{ij}^i(x,t) = \delta_{ij} \) for all \( x \) and \( t \), then the part \( a \); will be omitted, so that

\[
L_{b,c} = v \Delta + \sum_{i=1}^{d} b^i \frac{\partial}{\partial x^i} + c. \tag{2.9}
\]

If \( c = 0 \), then the part , \( c \) will be omitted, hence

\[
L_{a;b} = v \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x^j \partial x^i} + \sum_{i=1}^{d} b^i \frac{\partial}{\partial x^i} \tag{2.10}
\]

and therefore

\[
L_{b} = v \Delta + \sum_{i=1}^{d} b^i \frac{\partial}{\partial x^i}. \tag{2.11}
\]

This convention will be applied to other quantities such as the fundamental solutions, transition probability density functions, and so on. In particular \( \mathcal{L}_{a;b} = L_{a;b} \).

We finally make some comments on further regularity assumptions on the data \( a, b, c \) and \( q \) (to be introduced later on) in addition to the assumptions we have already made. For simplicity, we assume that all data \( a, b, c \) and \( q \) are smooth and bounded, although these regularity conditions may be too
demanding for applications. In fact, in additional to the assumptions made, the following assumptions are sufficient for our arguments to be true.

1) The results and arguments about $L_{a,b}$ are valid if $a^{ij}(x,t)$ are uniformly continuous in $(x,t)$, $b^j(x,t)$, $c(x,t)$ and $q(x,t)$ are bounded, Borel measurable and continuous in $t$, although the boundedness conditions may be weaken.

2) The results and arguments about $\mathcal{L}^a_{a,b}$ are valid if $a^{ij}(x,t)$ are Borel measurable and continuous in $t$, $b^j(x,t)$, $c(x,t)$ and $q(x,t)$ are bounded, Borel measurable and continuous in $t$, although the boundedness conditions may be weaken.

2.1 Analytic structures

In this section we recall several analytic structures associated with the basic data $a(x,t)$, $b(x,t)$ and $c(x,t)$.

The fundamental solutions are the basic analytic quantities associated with the elliptic operators of second order, and the reader may refer to [1] and [18, 25] for the most fundamental results in this aspect.

$\Gamma_{a,b,c}(x,t;\xi,\tau)$, defined for $x,\xi \in \mathbb{R}^d$ and $\tau < t$, is the fundamental solution of the forward parabolic equation

$$\left(L_{a,b,c} - \frac{\partial}{\partial t}\right)f(x,t) = 0$$  \hspace{1cm} (2.12)$$

in the sense that for every fixed $\tau \geq 0$ and $\xi \in \mathbb{R}^d$, $f(x,t) = \Gamma_{a,b,c}(x,t;\xi,\tau)$ solves (2.12), and for every bounded continuous function $\varphi$ on $\mathbb{R}^d$

$$\lim_{t \downarrow \tau} \int_{\mathbb{R}^d} \Gamma_{a,b,c}(x,t;\xi,\tau)\varphi(\xi)d\xi = \varphi(x)$$  \hspace{1cm} (2.13)$$

for every $x \in \mathbb{R}^d$.

Similarly $\Gamma^*_{a,b,c}(x,t;\xi,\tau)$ defined for $x,\xi \in \mathbb{R}^d$ and for $t < \tau$ is a fundamental solution of the backward heat operator $L_{a,b,c} + \frac{\partial}{\partial t}$, if for any $\xi \in \mathbb{R}^d$ and $\tau$, $f(x,t) = \Gamma^*_{a,b,c}(x,t;\xi,\tau)$ satisfies the backward parabolic equation

$$\left(L_{a,b,c} + \frac{\partial}{\partial t}\right)f(x,t) = 0$$  \hspace{1cm} (2.14)$$

and

$$\lim_{t \uparrow \tau} \int_{\mathbb{R}^d} \Gamma^*_{a,b,c}(x,t;\xi,\tau)\varphi(\xi)d\xi = \varphi(x)$$  \hspace{1cm} (2.15)$$

for every continuous function $\varphi$ on $\mathbb{R}^d$ and for every $x \in \mathbb{R}^d$.

**Lemma 2.2.** The following relation holds:

$$\Gamma_{a,b,c}(x,t;\xi,\tau) = \Gamma^*_{a,b,c,*}(\xi,\tau;x,t)$$  \hspace{1cm} (2.16)$$

for any $\tau < t$ and $x,\xi \in \mathbb{R}^d$.

For a proof of this basic fact, see [18, Theorem 15, page 28].

2.2 Probabilistic structures

In this section we recall several probabilistic structures associated with the elliptic operator of second order $L_{a,b}$. If $a^{ij}(x,t)$ are uniformly continuous and $b^j(x,t)$ are bounded and Borel measurable, then according to [40] there is a unique family of probability measures $\mathbb{P}_{a,b}^{\xi,\tau}$ (where $\tau \geq 0$ and $\xi \in \mathbb{R}^d$) on the path space $\Omega = C([0,\infty),\mathbb{R}^d)$ of all continuous paths in $\mathbb{R}^d$, equipped with its Borel $\sigma$-algebra, such that

1) the diffusion starts from $\xi$ at time $\tau$ in the following sense that

$$\mathbb{P}_{a,b}^{\xi,\tau}[\psi \in \Omega : \psi(t) = \xi \text{ for all } 0 \leq t \leq \tau] = 1.$$  \hspace{1cm} (2.17)$$
2) (local martingale property) for every \( f \in C^{2,1}(\mathbb{R}^d \times [\tau, \infty)) \),

\[
M_t^{[f]} = f(\psi(t), t) - f(\psi(\tau), \tau) - \int_\tau^t \left( L_{a,b} + \frac{\partial}{\partial s} \right) f(\psi(s), s) \, ds
\]

for \( t \geq \tau \geq 0 \) and \( M_t^{[f]} = 0 \) for \( 0 \leq t \leq \tau \), is a local martingale under the probability measure \( \mathbb{P}_{\tilde{\xi}, \tau}^{\hat{a}, \tilde{b}} \).

The family \( \mathbb{P}_{\tilde{\xi}, \tau}^{\hat{a}, \tilde{b}} \) (where \( \tilde{\xi} \in \mathbb{R}^d, \tau \geq 0 \)) of probability measures is simply called the \( L_{a,b} \)-diffusion.

If \( a^{ij}(x, t) = \sigma_i^j(x, t) \sigma_j^k(x, t) \) and \( b^i(x, t) \) are Lipschitz continuous, then the \( L_{a,b} \)-diffusion may be constructed by solving It\'s stochastic differential equations:

\[
dx^i = b^i(X, t) \, dt + \sqrt{2} \sum_{k=1}^d \sigma_i^k(X, t) \, dB^k, \quad X_s = \xi \text{ for } s \leq \tau
\]

for \( i = 1, \ldots, d \), where \( B = (B^1, \cdots, B^d) \) is the standard Brownian motion in \( \mathbb{R}^d \). The distribution of the strong solution \( X \) is the probability measure \( \mathbb{P}_{\tilde{\xi}, \tau}^{\hat{a}, \tilde{b}} \).

Let \( P_{a,b}(\tau, \xi, t, dx) = \mathbb{P}_{\tilde{\xi}, \tau}^{\hat{a}, \tilde{b}} \{ \psi(t) \in dx \} \) (where \( t > \tau \geq 0 \)), called the transition probability function of the \( L_{a,b} \)-diffusion. Since \( a(x, t) \) is uniformly elliptic, \( P_{a,b}(\tau, \xi, t, dx) \) has a probability density function \( p_{a,b}(\tau, \xi, t, x) \) with respect to the Lebesgue measure, so that

\[
P_{a,b}(\tau, \xi, t, dx) = p_{a,b}(\tau, \xi, t, x) \, dx \quad \text{for } t > \tau \geq 0 \text{ and } \xi \in \mathbb{R}^d.
\]

The transition density function \( p_{a,b}(\tau, \xi, t, x) \) (for \( t > \tau \geq 0 \)) associated with the \( L_{a,b} \)-diffusion is positive and Hölder continuous (cf. [1]).

**Lemma 2.3.** Let \( p_{a,b}(\tau, \xi, t, x) \) be the transition probability density function of the \( L_{a,b} \)-diffusion. Then

\[
p_{a,b}(\tau, \xi, t, x) = \Gamma_{a,b}^t(\xi, \tau; x, t) = \Gamma_{a,b,c}^t(x, t; \xi, \tau)
\]

for all \( 0 \leq \tau < t \) and \( x, \xi \in \mathbb{R}^d \), where

\[
\bar{b}(x, t) = -b^i(x, t) + 2 \nu \sum_{j=1}^d \frac{\partial a^{ij}(x, t)}{\partial x^j}
\]

and

\[
\bar{c}(x, t) = -\nabla \cdot b(x, t) + \nu \sum_{i,j=1}^d \frac{\partial^2 a^{ij}(x, t)}{\partial x^i \partial x^j}.
\]

This follows immediately from Lemma 2.2 and the well known fact that \( p_{a,b}(\tau, \xi, t, x) = \Gamma_{a,b}^t(\xi, \tau; x, t) \) for all \( 0 \leq \tau < t \) and \( x, \xi \in \mathbb{R}^d \) (cf. [41]).

**Remark 2.4.** If \( a^{ij}(x, t) = \delta^{ij} \) and \( b(x, t) \) is divergence free for every \( t \), that is, \( \nabla \cdot b = 0 \) in the distribution sense, then

\[
p_b(\tau, \xi, t, x) = \Gamma_b^t(\xi, \tau; x, t) = \Gamma_{-b}(x, t; \xi, \tau)
\]

for all \( 0 \leq t < \tau \). In particular, if \( \nabla \cdot b = 0 \), then

\[
\Gamma_b(x, t; \xi, t) = p_{-b}(\tau, \xi, t, x)
\]

for all \( t \geq \tau \geq 0 \). The relationship (2.21) plays the crucial rôle in the random vortex method (cf. [28, 29]).

On the other hand, it is known that a forward parabolic equation can be solved by running back the time from a future time, which gives rise to another set of relations between fundamental solutions and the transition probability density functions. If \( T > 0 \) and \( f(x, t) \) is a function, then define \( f_T(x, t) = f(x, (T-t)^+) \).
Lemma 2.5. Let $T > 0$. Then

$$p_{aT,b_T}(T-t,x,T-\tau,\xi) = \Gamma_{a,b}(x,t;\xi,\tau,\tau) = \Gamma_{a,b,\xi}(\xi,\tau,\tau)$$  \hspace{1cm} (2.22)

for all $0 \leq \tau < t \leq T$ and $x, \xi \in \mathbb{R}^n$, where $\hat{b}$ and $\hat{c}$ are given by (2.18) and (2.19) respectively.

\textbf{Proof.} Let $\Theta(x,t;\xi,\tau) = p_{aT,b_T}(T-t,x,T-\tau,\xi)$ for $0 \leq \tau < t \leq T$ and $\xi, x \in \mathbb{R}^d$. As a function of $(x,t)$, $p_{aT,b_T}(t,x,\tau,\xi)$ (for $t < \tau$) solves the backward parabolic equation, so that $\Theta$ solves the forward parabolic equation

$$\left( L_{aT,b_T} + \frac{\partial}{\partial t} \right) \Theta(x,t;\xi,\tau) = 0$$  \hspace{1cm} (2.23)

for $0 \leq \tau < t \leq T$. Since $b_T(x,T-t) = b(x,t)$ and $a_T(x,T-t) = a(x,t)$ for $0 \leq t \leq T$, the previous equality is equivalent to the forward parabolic equation:

$$\left( L_{aT,b_T} + \frac{\partial}{\partial t} \right) \Theta(x,t;\xi,\tau) = 0$$

on $\mathbb{R}^d \times [\tau,T]$ for every $\tau \in (0,T]$. Suppose $\varphi(\xi)$ is continuous and bounded on $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \Theta(x,t;\xi,\tau) \varphi(\xi) d\xi = \int_{\mathbb{R}^d} p_{aT,b_T}(T-t,x,T-\tau,\xi) \varphi(\xi) d\xi \rightarrow \varphi(x)$$

as $T-t \uparrow T - \tau$, i.e. as $t \downarrow \tau$. By the uniqueness of the fundamental solution $\Theta(x,t;\xi,\tau)$ coincides with $\Gamma_{a,b}(x,t;\xi,\tau)$, and therefore the conclusion follows immediately. \hfill \Box

Remark 2.6. For the case where $a^{ij}(x,t) = \delta^{ij}$ and $b(x,t)$ is divergence-free, then $\hat{b} = -b$ and $\hat{c} = 0$, so that

$$p_{aT}(T-t,x,T-\tau,\xi) = \Gamma_b(x,t;\xi,\tau) = \Gamma_{b}(\xi,\tau,x,t) = p_{-b}(\tau,\xi,t,x)$$  \hspace{1cm} (2.24)

for all $0 \leq \tau < t \leq T$ and $x, \xi \in \mathbb{R}^n$. Therefore for this case the fundamental solution $\Gamma_{b}(\xi,\tau,x,t)$ is a probability density in $x$ and $\xi$ respectively.

In general by definition $\Gamma_{a,b,\xi}(\xi,\tau,x,t)$ is a probability density in the variable $\xi$, while in general it is not a probability density function with respect to the variable $x$, thus can not be a transition probability density function of an diffusion. A sufficient condition to ensure that $\Gamma_{a,b,\xi}(\xi,\tau,x,t)$ is a transition probability function (with respect to $x$) of some diffusion is given in the next section.

\section{3 The Feynman-Kac formula}

The Feynman-Kac formula is a functional integration representation to the solutions $f^i(x,t)$ of a backward parabolic equation:

$$\left( L_{aT,b_T} + \frac{\partial}{\partial t} \right) f^i(x,t) + \sum_{j=1}^{n} c_j^i(x,t)f^j(x,t) = 0 \quad \text{in} \quad \mathbb{R}^d \times (0,\infty),$$  \hspace{1cm} (3.1)

where $i = 1, \ldots, n$, and $n$ is some positive integer.

For each $\psi \in C([0,\infty),\mathbb{R}^d)$, let $Q(\tau,\psi,\tau) = (Q^j(\tau,\psi,\tau))_{i,j \leq n}$ be the solution to the ordinary differential equations:

$$\frac{d}{dt}Q_j(t) = Q^j(t)Q^j(t)\psi(t), \quad Q^j(\tau) = \delta^i_j \hspace{1cm} (3.2)$$

for $t \geq \tau$, if the solution exists and is unique.
Lemma 3.1. (The Feynman-Kac formula) Suppose \(a_i^j(x,t)\) are uniformly continuous, \(b_i^j(x,t)\) are bounded and Borel measurable, and \(q_i^j(x,t)\) are bounded and continuous. Suppose \(f^i(x,t)\) are \(C^{2,1}\) solutions to (3.1) with bounded first derivatives. Then

\[
f^i(\eta, \tau) = \int_{\Omega} Q^i_j(\tau, \eta; t) f^i(\eta(t), t) P^\eta_{a,b}(d\psi)
\]

for \(t \geq \tau\) and \(\eta \in \mathbb{R}^d\).

As a consequence we have the following

Lemma 3.2. Under the same regularity assumptions on \(a(x,t)\) and \(b(x,t)\) as in Lemma 3.1, and suppose \(c(x,t)\) is bounded and continuous. Then

\[
\Gamma^*_{a,b,c}(\xi, \tau; x, t) = p_{a,b}(\tau, \xi, t; x) \int_{\Omega} C(\tau, \xi; t) P_{a,b}^\xi_{\tau\rightarrow x}(d\psi)
\]

for \(x, \xi \in \mathbb{R}^d\) and \(t > \tau \geq 0\), where \(C(\tau, \xi; r)\) for each \(\psi \in C([0,\infty); \mathbb{R}^d)\) is the unique solution to the ordinary differential equation:

\[
C(\tau, \psi; t) = 1 + \int_{\tau}^{\tau+t} C(\tau, \psi; s)c(\psi(s), s)ds
\]

for \(t \geq 0\).

Proof. Let \(T > 0\) and \(f\) be a \(C^{2,1}\) solution to the backward parabolic equation:

\[
\left( L_{a,b,c} + \frac{\partial}{\partial t} \right) f = 0
\]

such that \(f(x,t) \rightarrow f_0(x)\) as \(t \uparrow T\). The previous equation may be rewritten as the following:

\[
\left( L_{a,b} + \frac{\partial}{\partial t} \right) f(x,t) + c(x,t) f(x,t) = 0
\]

so that, according to Lemma 3.1

\[
f(\xi, \tau) = \int_{\Omega} C(\tau, \psi; T) f(\psi(T), T) P_{a,b}^\xi_{a,b}(d\psi)
\]

for \(0 \leq \tau < T\), and therefore

\[
f(\xi, \tau) = \int_{\mathbb{R}^d} \left( p_{a,b}(\tau, \xi, T, x) \int_{\Omega} C(\tau, \psi; T) P_{a,b}^\xi_{a,b} [d\psi | \psi(T) = x] \right) f(x, T) dx
\]

which yields that

\[
\Gamma^*_{a,b,c}(\xi, \tau; x, T) = p_{a,b}(\tau, \xi, T, x) \int_{\Omega} C(\tau, \psi; T) P_{a,b}^\xi_{a,b} [d\psi | \psi(T) = x].
\]

The proof is complete.

Lemma 3.3. Let \(T > 0\). Then

\[
p_{at,bT}(T-t, x, T-\tau, \xi) = p_{a,b}(\tau, \xi, T, x) \int_{\Omega} C(\tau, \psi; t) P_{a,b}^\xi_{a,b} (d\psi)
\]

for any \(0 \leq \tau < t \leq T\) and \(x, \xi \in \mathbb{R}^d\), where \(\tilde{b}\) and \(\tilde{c}\) are given by (2.18) and (2.19) respectively, and

\[
\tilde{C}(\tau, \psi; t) = 1 + \int_{\tau}^{\tau+t} \tilde{C}(\tau, \psi; s)\tilde{c}(\psi(s), s) ds
\]

for \(t \geq 0\) and \(\psi \in C([0,\infty); \mathbb{R}^d)\).
Proof. We have
\[ p_{a,T,b}(T-t,x,T-\tau,\xi) = \Gamma^*_{a,b,c}(\xi,\tau;x,t) \]
for \( t > \tau \geq 0 \), so the corollary follows immediately from Lemma 3.2.

We may apply Lemma 3.3 to the \( \mathcal{L}_{a,b} \)-diffusion, whose transition probability density function is denoted by \( h_{a,b}(t,x,\tau,\xi) \) for \( 0 \leq t < \tau \). The formal adjoint operator of \( \mathcal{L}_{a,b} \) is given by
\[
\mathcal{L}^*_a = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^i} a^{ij} \frac{\partial}{\partial x^j} - \sum_{i=1}^{d} b^i \frac{\partial}{\partial x^i} - \nabla \cdot b
\]
and therefore \( \tilde{c} = \nabla \cdot b \). Therefore we have the following consequence.

Lemma 3.4. Suppose \( \nabla \cdot b(\cdot,t) = 0 \) in the distribution sense for every \( t \geq 0 \). Then for every \( T > 0 \)
\[
h_{a,T,b}(T-t,x,T-\tau,\xi) = h_{a,b}(\tau,\xi;T,t,x)
\]  
(3.6)
for \( 0 \leq \tau < t \leq T \) and \( \xi, x \in \mathbb{R}^d \).

Proof. Apply Lemma 3.3 to the differential operator \( \mathcal{L}_{a,b} = L_{a,b} \). Since \( \nabla \cdot b = 0 \) identically for every \( t \), \( \tilde{c} = 0 \), and therefore the gauge process \( \tilde{C} \equiv 1 \), the equality follows immediately.

4 Diffusion bridges

In this section we establish a duality among the conditional laws of the \( \mathcal{L}_{a,b} \)-diffusions. Let \( \mathbb{Q}^\eta_{a,b,T} \) be the distribution of the \( \mathcal{L}_{a,b} \)-diffusion started from \( \eta \) at \( \tau \geq 0 \). If \( T > 0 \) then \( \mathbb{Q}_{a,b}^{\eta,0 \to \zeta,T} \) denotes the conditional law of the \( \mathcal{L}_{a,b} \)-diffusion started from \( \eta \) at instance 0 and arrived at \( \zeta \) at time \( T \).

Theorem 4.1. Suppose \( b(x,t) \) is divergence free and bounded, then for every \( T > 0 \)
\[
\mathbb{Q}_{a,b}^{\eta,0 \to \zeta,T} \circ \tau_T = \mathbb{Q}_{a,T,b}^{\eta,T}
\]  
(4.1)
where \( \tau_T \) is the time reverse which sends \( \psi \) to \( \tau_T \psi \), for \( \psi \in C([0,T]; \mathbb{R}^d) \), that is, \( \tau_T \psi(t) = \psi(T-t) \) for \( t \in [0,T] \).

Proof. It is known that the conditional law of \( \mathbb{Q}_{a,b}^{\eta,0} \) given \( \psi(T) = \zeta \), where \( \eta \) and \( \zeta \) are the initial and final points, according to (14.1) in [14], denoted by \( \mathbb{Q}_{a,b}^{\eta,0 \to \zeta,T} \), is also Markovian (time non-homogeneous) whose transition density function is given by
\[
q(\tau,\xi;T,t,x) = \frac{h_{a,b}(\tau,\xi;T,t,x)h_{a,b}(t,x,T,\xi)}{h_{a,b}(\tau,\xi,T,\zeta)}
\]  
(4.2)
that is
\[
q(\tau,\xi;T,t,x) = \mathbb{Q}_{a,b}^{\eta,0 \to \zeta,T} \left[ \psi \in \Omega : \psi(t) \in dx | \psi(\tau) = \xi \right]
\]
for \( 0 < \tau < t \leq T \). Let \( 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T \). Then
\[
\mathbb{Q}_{a,b}^{\eta,0 \to \zeta,T} \left[ w_{t_0} \in dx_{t_0}, w_{t_1} \in dx_{t_1}, \cdots, w_{t_n} \in dx_{t_n}, w_{t_{n+1}} \in dx_{t_{n+1}} \right]
\]
equals the measure
\[
q(0,\eta;\tau_1,x_1) \cdots q(t_{i-1},x_{i-1},x_i) \cdots q(t_n,x_n,\tau,\zeta) dx_1 \cdots dx_n.
\]
By using (4.2), this measure has a pdf
\[
\frac{h_{a,b}(0,\eta;\tau_1,x_1) \cdots h_{a,b}(t_{i-1},x_{i-1},x_i) \cdots h_{a,b}(t_n,x_n,\tau,\zeta)}{h_{a,b}(0,\eta;T,\zeta)}
\]
with respect to the measure \( \delta_\eta(dx_0)dx_1 \cdots dx_n \delta_\xi(dx_{n+1}) \). According to Lemma 3.4, the previous pdf equals

\[
h_{ar;-br}(0, \zeta, T-t_n, x_n) \cdots h_{ar;-br}(T-t_1, x_1, T-t_{i-1}, x_{i-1}) \cdots p_{ar;-br}(T-t_1, x_1, T, \eta)
\]

and we therefore conclude that

\[
Q_{ar;br}^{\eta,0 \rightarrow \zeta,T} \left[ w_{t_0} \in dx_0, w_{t_1} \in dx_1, \cdots, w_{t_n} \in dx_n, w_{t_{n+1}} \in dx_{n+1} \right]
\]

coincides with

\[
Q_{ar;br}^{\xi,0 \rightarrow \eta,T} \left[ w_{T-t_n} \in dx_n, \cdots, w_{T-t_1} \in dx_1, w_T \in dx_0, w_0 \in dx_{n+1} \right].
\]

\[
\square
\]

5 Feynman-Kac formula for forward parabolic equation

Let us begin with the following general construction of the gauge functional.

Given \( q(x,t) = (q^{ij}(x,t))_{i,j \leq n} \), an \( n \times n \) square-matrix valued function defined for \( x \in \mathbb{R}^d \) for \( t \geq 0 \). For each \( T > 0 \) and each continuous path \( \psi \in C([0,T]; \mathbb{R}^d) \), consider the following ordinary differential equation

\[
\frac{d}{dt} Q_j^I(t) = \sum_{k=1}^n Q_k^I(t) q_j^k(\psi(s), (T-t)^+) \quad Q_j^I(0) = \delta_{ij}
\]

which is linear in \( Q \). The solution depends on \( T \) and on \( \psi \) as well, so it is denoted by \( Q(\psi, T;t) \). By definition

\[
Q_j^I(\psi, T;t) = \delta_{ij} + \int_0^T \sum_{k=1}^n Q_k^I(\psi, T;s) q_j^k(\psi(s), (T-s)^+) ds
\]

for \( t \in [0,T] \), where \( i, j = 1, \ldots, n \).

The time reversal operator \( \tau_T \), we recall, on \( C([0,T]; \mathbb{R}^d) \) maps \( \psi \) to \( \tau_T \psi(t) = \psi(T-t) \). Therefore, if we substitute \( \psi \) by \( \tau_T \psi \) and \( t \) by \( T-t \) we obtain

\[
Q_j^I(\tau_T \psi, T; T-t) = \delta_{ij} + \int_0^{T-t} \sum_{k=1}^n Q_k^I(\tau_T \psi, T; s) q_j^k(\psi(s), (T-s)^+) ds
\]

\[
= \delta_{ij} + \int_t^T \sum_{k=1}^n Q_k^I(\tau_T \psi, T; T-s) q_j^k(\psi(s), s) ds.
\]

Hence we have the following elementary fact.

**Lemma 5.1.** For each \( \psi \in C([0,T]; \mathbb{R}^d) \), \( Q(\psi, T;t) \) denotes the solution to ODE (5.1) and \( G(\psi, T;t) = Q(\tau_T \psi, T; T-t) \) for \( t \in [0,T] \). Then \( G \) is the solution to the ordinary differential equation

\[
\frac{d}{dt} G_j^I(t) = -\sum_{k=1}^n G_k^I(t) q_j^k(\psi(t), t), \quad G_j^I(T) = I.
\]

Moreover \( Q(\psi, T;t) = G(\tau_T \psi, T; T-t) \) for \( t \in [0,T] \).

Now we are in a position to study the initial value problem of the forward parabolic equation:

\[
\left( L - \frac{\partial}{\partial t} \right) w^I(x,t) + \sum_{j=1}^n q_j^I(x,t) w^j(x,t) + f^I(x,t) = 0
\]

subject to the initial value \( w^I(x,0) = w^I_0(x) \), where \( i = 1, \ldots, n \), where \( L = L_{abc} \) (for this case we assume that \( d^{ij}(x,t) \) are uniformly continuous) or \( L = L_{abc} \), for both cases, \( b^I(x,t) \) are bounded and Borel measurable.
Lemma 5.2. Let $P^0$ denote the distribution of the $L_{a,b}$-diffusion or the $\mathcal{L}_{a,-b}$-diffusion, depending on whether $L = L_{a,b}$ or $L = \mathcal{L}_{a,b}$, started from $\eta \in \mathbb{R}^d$ at time 0. Then

$$w^i(x, T) = P^x \left[ \sum_{j=1}^n Q_j^i(\psi, T; T)w_0^j(\psi(T)) + \int_0^T Q_j^i(\psi, T; t)f^j(\psi(t), T-t)dt \right]. \quad (5.5)$$

Proof. This is the classical Feynman-Kac formula applying to $w(x, T-t)$.

Theorem 5.3. Suppose $b$ is divergence free on $\mathbb{R}^d$, i.e. $\nabla \cdot b = 0$ in the distribution sense. Suppose $w(x, t) = (w^i(x, t))_{i \leq n}$ is a solution to the parabolic system

$$\left( \mathcal{L}_{a,b} - \frac{\partial}{\partial t} \right) w^i(x, t) + \sum_{j=1}^n q_j(x,t)w^j(x, t) + f^i(x, t) = 0 \quad (5.6)$$

subject to the initial value $w(x, 0) = w_0(x)$, where $i = 1, \ldots, n$. Then

$$w^i(x, T) = \sum_{j=1}^n \int_{\mathbb{R}^d} \left( \int_0^T G_j^i(\psi, T; t)Q_{a,-b}^{\xi,0 \rightarrow x,T}(d\psi) \right) w_0^j(\xi)h_{a,-b}(0, \xi, T, x)d\xi + \int_0^T \int_{\mathbb{R}^d} G_j^i(\psi, T; t)f^j(\psi(t), t)d\psi Q_{a,-b}^{\xi,0 \rightarrow x,T}(d\psi) h_{a,-b}(0, \xi, T, x)d\xi. \quad (5.7)$$

where $G(\psi, T; t) = G(t)$ (for every $T > 0$ and every continuous path $\psi$) is the unique solution to the ordinary differential equation:

$$\frac{d}{dt}G_j^i(t) = -G_j^i(t)d^j(\psi(t), t), \quad G_j^i(0) = \delta_{ij}. \quad (5.8)$$

Proof. According to the previous Lemma 5.2

$$w^i(x, T) = \sum_{j=1}^n \int_{\mathbb{R}^d} \left[ Q_j^i(\psi, T; T)w_0^j(\psi(T)) \right] + \int_0^T G_j^i(\psi, T; t)f^j(\psi(t), T-t)dt \right] d\xi$$

$$= \sum_{j=1}^n \int_{\mathbb{R}^d} \left[ Q_j^i(\psi, T; T)w_0^j(\psi(T)) \right] \delta(\psi(T) - \xi)Q_{a,0}(\xi)d\xi + \int_0^T \int_{\mathbb{R}^d} G_j^i(\psi, T; t)f^j(\psi(t), T-t)dt \left[ Q_{a,0}(\xi)d\xi \right]$$

$$= \sum_{j=1}^n \int_{\mathbb{R}^d} \left[ Q_j^i(\psi, T; T)w_0^j(\xi)h_{a,b}(0, x, T, \xi)d\xi \right] + \int_0^T \int_{\mathbb{R}^d} G_j^i(\psi, T; t)f^j(\psi(t), T-t)h_{a,b}(0, x, T, \xi)d\xi dt.$$

Since $b(x, t)$ is divergence-free, so that $\mathcal{L}_{a,b} = \mathcal{L}_{a,-b}$. By Theorem 4.1

$$Q_{a,0}^{\xi,0 \rightarrow \eta,T} \circ \tau_T = Q_{a,-b}^{\eta,0 \rightarrow \xi,T},$$

together with the relation that $Q(\psi, T; t) = G(\tau_T \psi, T; T-t)$, we obtain

$$w^i(x, T) = \sum_{j=1}^n \int_{\mathbb{R}^d} \left[ G_j^i(\tau_T \psi, T; 0)w_0^j(\xi)h_{a,b}(0, x, T, \xi)d\xi \right] + \int_0^T \int_{\mathbb{R}^d} G_j^i(\tau_T \psi, T; t)f^j(\tau_T \psi(T-t), T-t)h_{a,b}(0, x, T, \xi)d\xi dt$$

$$= \sum_{j=1}^n \int_{\mathbb{R}^d} \left[ G_j^i(\psi, T; 0)w_0^j(\xi)h_{a,b}(0, x, T, \xi)d\xi \right] + \int_0^T \int_{\mathbb{R}^d} G_j^i(\psi, T; t)f^j(\psi(T-t), T-t)h_{a,b}(0, x, T, \xi)d\xi dt.$$
+ \sum_{j=1}^{n} \int_{a_{d-j}}^{b} \int_{0}^{T} G_{j}^{q}(\psi; T-t) f^{j}(\psi(T-t), T-t) dt \, dh_{d-y}(x, T, \xi) d\xi.

Finally according to Lemma 3.4

\[ h_{d-y}(T-t, x, T-t, \xi) = h_{d-b}(\tau, \xi, t, x), \]

which yields that \( h_{d-y}(0, x, T, \xi) = h_{d-b}(0, \xi, T, x) \) for \( 0 \leq \tau < t \leq T \). Substituting this equation into the representation for \( w^{i} \) we obtain (5.7).

## 6 Boundary value problems

We first recall the Feynman-Kac formula for solutions of the initial and boundary value problem:

\[
\left( L_{a:b} + \frac{\partial}{\partial t} \right) f^{i}(x, t) + q^{i}_{j}(x, t) f^{k}(x, t) = g^{i}(x, t) \quad \text{in } D \times [0, \infty) \tag{6.1}
\]

subject to the Dirichlet boundary condition that

\[
f^{i}(x, t) = \beta^{i}(x) \quad \text{for } x \in \partial D \text{ and } t > 0, \tag{6.2}
\]

where \( j = 1, \ldots, n \).

**Lemma 6.1.** Assume that \( a \) is uniformly continuous and \( b \) is bounded and Borel measurable. For each \( \psi \in C([0, \infty); \mathbb{R}^{d}) \) denote \( Q(\psi, t) \) the solution to the (linear) ordinary differential equation

\[
\frac{d}{dt} Q^{i}_{j}(t) = Q^{i}_{j}(t)1_{D}(\psi(t)) q^{i}_{j}(\psi(t), t), \quad Q^{i}_{j}(0) = \delta^{i}_{j} \tag{6.3}
\]

and

\[
\zeta_{D}(\psi) = \inf\{t \geq 0 : \psi(t) \notin D\}. \tag{6.4}
\]

Then the following integration representation holds:

\[
f^{i}(\eta, 0) = \int_{\Omega} O^{j}_{i}(\psi, t) f^{i}(\psi(t), t) 1_{\{\zeta_{D}(\psi) > t\}} \mathbb{P}^{\eta}(d\psi)
+ \int_{\Omega} O^{j}_{i}(\psi, \zeta_{D}(\psi)) \beta^{i}(\psi(\zeta_{D}(\psi))) 1_{\{\zeta_{D}(\psi) \leq t\}} \mathbb{P}^{\eta}(d\psi)
- \int_{\Omega} \left[ \int_{0}^{\zeta_{D}(\psi)} O^{j}_{i}(\psi, s) g^{i}(\psi(s), s) ds \right] \mathbb{P}^{\eta}(d\psi) \tag{6.5}
\]

for all \( t \geq 0 \) and \( \eta \in \mathbb{R}^{d} \), where \( i = 1, \ldots, n \) and \( \mathbb{P}^{\eta} = \mathbb{P}^{\eta, 0}_{a:b} \) for simplicity.

**Proof.** For a slightly different version of Feynman-Kac formula and its proof, see for example [16, Theorem 2.3. page133]. For completeness we include a proof here. We may assume that \( q^{i}_{j}(x, t) = 0 \) for \( x \notin D \) otherwise we may use \( 1_{D}(x) q^{i}_{j}(x, t) \) instead. Let \( X \) be the weak solution of the stochastic differential equation

\[
dX^{k}(t) = b^{k}(X(t), t) dt + \sqrt{2\nu} \sigma^{k}_{i}(X(t), t) dB^{i}(t).
\]

for \( k = 1, \ldots, d \), with initial \( X(0) = \eta \). Consider the linear ordinary differential equation:

\[
\frac{d}{dt} Q^{j}_{i}(t) = Q^{j}_{i}(t) q^{j}_{i}(X(t), t), \quad Q^{j}_{i}(0) = \delta^{j}_{i}. \tag{6.7}
\]

Suppose \( \rho \) is smooth such that \( \rho(x) = 1 \) for \( x \in \mathcal{D} \), and \( \tilde{f}^{j}(x, t) = \rho(x) f^{j}(x, t) \). Then \( \tilde{f}^{j}(x, t) \) (\( i = 1, \ldots, n \)) are \( C^{2,1} \)-functions on \( \mathbb{R}^{d} \times [0, \infty) \) and

\[
\frac{\partial}{\partial t} f^{j} + L_{a:b} f^{j} + q^{i}_{j} f^{k} = F^{j} \quad \text{in } \mathbb{R}^{d} \times [0, \infty). \tag{6.8}
\]
Consider $M'_t = Q_j(t)f^j(X(t), t)$ for $t \geq 0$. Then, according to Itô’s formula

$$
M'_t = M'_0 + \int_0^t Q_j(s) \sqrt{2\sigma^j(s)}(X(s), s)\frac{\partial \tilde{f}^j}{\partial x^j}(X(s), s)dB_s^j \\
+ \int_0^t Q_j(s) \left( \frac{\partial \tilde{f}^j}{\partial s} + L_{w,b}f^j + q_k^j \tilde{f}^k \right)(X(s), s)ds.
$$

(6.9)

Let

$$
\xi_D = \inf\{t \geq 0 : X(t) \notin D\}
$$

be the first time the diffusion leaves the region $D$. Then

$$
E\left[M'_{t\wedge \xi_D}\right] = E\left[M'_0\right] + E\int_0^{t\wedge \xi_D} Q_j(s) \left( \frac{\partial \tilde{f}^j}{\partial s} + L_{w,b}f^j + q_k^j \tilde{f}^k \right)(X(s), s)ds.
$$

(6.10)

Since $f^j$ solve the differential equations (6.8), so that

$$
E\left[M'_{t\wedge \xi_D}\right] = E\left[M'_0\right] + E\left[\int_0^{t\wedge \xi_D} Q_j(s)F^j(X(s), s)ds\right].
$$

(6.11)

Since $M'_0 = f^j(\eta, 0)$ and

$$
E\left[M'_{t\wedge \xi_D}\right] = E\left[M'_0 : t \geq \xi_D\right] + E\left[M'_0 : t < \xi_D\right] \\
= E\left[Q_j(\xi_D)f^j(X(\xi_D), \xi_D) : t \geq \xi_D\right] \\
+ E\left[Q_j(t)f^j(X(t), t) : t < \xi_D\right] \\
= E\left[Q_j(t)f^j(X(t), t) : t < \xi_D\right] \\
+ E\left[Q_j(\xi_D)\beta^j(X(\xi_D)) : t \geq \xi_D\right]
$$

where the last equality follows from the Dirichlet boundary condition: $X(\xi_D) \in \partial D$ on $\xi_D < \infty$, and $f^j(x, t) = \beta^j(x)$ for $x \in \partial D$. Substituting this equality into (6.11),

$$
E\left[Q_j(t)f^j(X(t), t) : t < \xi_D\right] = f^j(\eta, 0) + E\left[\int_0^{t\wedge \xi_D} Q_j(s)F^j(X(s), s)ds\right] \\
- E\left[Q_j(\xi_D)\beta^j(X(\xi_D)) : t \geq \xi_D\right]
$$

The functional integration representation follows by an approximating procedure. \[\square\]

We next establish a forward Feynman-Kac formula.

For every $\psi \in C([0, \infty); \mathbb{R}^d)$ and $T > 0$, $\hat{Q}(\psi, T; t)$ denotes the solution to the following linear ordinary differential equations

$$
\frac{d}{dt} \hat{Q}_j(\psi, T; t) = -\hat{Q}_j(\psi, T; t)1_D(\psi(t))q_k^j(\psi(t), t), \quad \hat{Q}_j(\psi, T; T) = \delta^j
$$

(6.12)

for $i, j = 1, \cdots, n$.

**Theorem 6.2.** Suppose $b(x, t)$ is bounded, Borel measurable and $\nabla \cdot b = 0$ in the distribution sense on $\mathbb{R}^d$. Let $w(x, t)$ be the solution to Cauchy’s initial and boundary problem of the following parabolic system:

$$
\left(\mathcal{L}_{w,b} - \frac{\partial}{\partial t}\right)w^j(x, t) + \sum_{k=1}^n q_k^j(x, t)w^k(x, t) = g^j(x, t) \quad \text{in } D
$$

(6.13)

subject to the initial and boundary conditions that

$$
w^j(x, 0) = w^j_0(x) \text{ for } x \in D, \text{ and } w^j(x, t) = \beta^j(x) \text{ for } x \in \partial D, t > 0
$$

(6.14)
for $j = 1, \ldots, n$. Then

\[
w^j(\eta, T) = \int_D \left( \int_{\Omega} \tilde{Q}^j_j(\psi, T; \xi) 1_{\{\xi_0(\psi) > T\}} Q_{\eta_{a_r-b_r}}^\eta (d \psi) \right) w^j_0(\xi) h(0, \xi, T, \eta) d\xi \\
+ \int_{\mathbb{R}^d} \left( \int_{\Omega} \tilde{Q}^j_j(\psi, \lambda T, D(\psi)) \beta^j(\psi(\lambda T, D(\psi))) 1_{\{\xi_0(\psi) > T\}} Q_{\eta_{a_r-b_r}}^\eta (d \psi) \right) h(0, \xi, T, \eta) d\xi \\
- \int_0^T \left[ \int_{\mathbb{R}^d} \left( \int_{\Omega} \tilde{Q}^j_j(\psi, T, s) g^j(\psi(s), s) 1_{\{\xi_0(\psi) > T-s\}} Q_{\eta_{a_r-b_r}}^\eta (d \psi) \right) h(0, \xi, T, \eta) d\xi \right] ds
\]

(6.15)

for every $\eta \in D$ and $T > 0$, where $h(\tau, \xi, t, \eta)$ denotes $h_{a_r-b_r}(\tau, \xi, t, \eta)$ for simplicity.

\[
\lambda_{T, D}(\psi) = \sup \{ t \in [0, T] : \psi(t) \in \partial D \}
\]

and

\[
\theta_r : \Omega \to \Omega, \quad \theta_r \psi(t) = \psi(t + s) \text{ for } s, t \geq 0.
\]

**Proof.** Let $T > 0$ and $f^i(x, t) = w^i(x, T-t)$, so that $u^i$ satisfy the following parabolic equations

\[
\left( \mathcal{L}_{a_r-b_r} + \frac{\partial}{\partial t} \right) f^i(x, t) + \sum_{k=1}^n q^i_{k,T}(x, t) f^k(x, t) = g^i_T(x, t) \quad \text{in } D.
\]

Then according to (6.6)

\[
w^i(\eta, T) = \int_{\Omega} \left( Q^i_j(\psi, T) w^j_0(\psi(T)) 1_{\{\xi_0(\psi) > T\}} \right) Q_{a_r-b_r}^\eta (d \psi) \\
+ \int_{\Omega} Q^i_j(\psi, \xi_0(\psi)) \beta^i(\psi(\xi_0(\psi))) 1_{\{\xi_0(\psi) \leq T\}} Q_{a_r-b_r}^\eta (d \psi) \\
- \int_0^T \left[ \int_{\Omega} Q^i_j(\psi, s) g^i(\psi(s), T-s) 1_{\{s < T \land \xi_0(\psi) \}} Q_{a_r-b_r}^\eta (d \psi) \right] ds
\]

(6.17)

where $Q_{a_r-b_r}^\eta$ is the law of the $\mathcal{L}_{a_r-b_r}$-diffusion started from $\eta$ at instance 0, and

\[
Q^i_j(\psi, t) = \delta^i_j + \int_0^t Q^i_k(\psi, s) 1_{D}(\psi(s)) q^k_j(\psi(s), T-s) ds.
\]

Replace $t$ by $T-t$ and $\psi$ by $\tau_T \psi$ one obtains that

\[
Q^i_j(\tau_T \psi, T-t) = \delta^i_j + \int_{T-t}^T Q^i_k(\tau_T \psi, s) 1_{D}(\psi(T-s)) q^k_j(\psi(T-s), T-s) ds \\
= \delta^i_j - \int_T^{T-t} Q^i_k(\tau_T \psi, T-s) 1_{D}(\psi(s)) q^k_j(\psi(s), s) ds.
\]

Hence, by setting $\bar{Q}(\psi, T-t) = Q^i_j(\tau_T \psi, T-t)$, one deduce that

\[
\bar{Q}^i_j(\psi, T-t) = \delta^i_j - \int_T^t Q^i_k(\psi, T-s) 1_{D}(\psi(s)) q^k_j(\psi(s), s) ds.
\]

Moreover $Q^i_j(\tau_T \psi, t) = \bar{Q}(\psi, T-t)$ for every $t \in [0, T]$.

We rewrite (6.18) by conditioning on the values of the diffusion at $T$, to obtain that

\[
w^i(\eta, T) = R^i_j(\eta, T) + R^i_{N}(\eta, T)
\]

(6.20)

where the first term

\[
R^i_j(\eta, T) = \int_D \left( \int_{\Omega} Q^i_j(\psi, T) 1_{\{\xi_0(\psi) > T\}} Q_{a_r-b_r}^\eta (d \psi) \right) w^j_0(\xi) h_{a_r-b_r}(0, \eta, T, \xi) d\xi,
\]
the boundary term

\[ R_B^i(\eta, T) = \int_\Omega Q_j(\psi, \zeta_D(\psi)) \beta^j(\psi(\zeta_D(\psi))) 1_{\{\zeta_D(\psi) \leq T\}} \mathbb{Q}_{ar;br}^\eta(\eta, \psi) \]

and finally the inhomogeneous term

\[ R_N^i(\eta, T) = \int_0^T \left[ \int_\Omega (Q_j(\psi) d\psi) 1_{\{\zeta_D(\psi) \leq T\}} \mathbb{Q}_{ar;br}^\eta(\eta, \psi, d\eta) \right] \eta(0, \eta, T, \xi) d\xi \]

Since \( \mathcal{L}^*_{a;\xi} = \mathcal{L}_{a;-\xi} \) under our assumptions, \( \mathbb{Q}_{a;\eta} \mathcal{L}_{a;\xi} \psi \) is equivalent to \( \mathbb{Q}_{a;\xi} \mathcal{L}_{a;\eta} \psi \) and (cf. Lemma 3.4)

\[ h_{ar;br}(0, \eta, T, \xi) = h(0, \xi, T, \eta). \]

Thanks to these dualities, we are able to rewrite the three terms on the right-hand side of the representation (6.20) for \( w^i \). Indeed the first term

\[ R_N^i(\eta, T) = \int_D \left( \int_\Omega Q_j(\psi, T) 1_{\{\zeta_D(\psi) > T\}} \mathbb{Q}_{ar;br}^\eta(\eta, \psi) \right) w^i(\eta, \psi) h_{ar;br}(0, \eta, T, \xi) d\xi \]

We notice that, if \( \xi, \eta \in D \), then under the conditional law \( \mathbb{Q}_{ar;\eta}^\xi(\psi(T) = \eta) \), \( \zeta_D(\tau_T \psi) > T \) is equivalent to \( \psi(T) \in D \) for all \( t \in [0, T] \), which in turn is equivalent to that \( \psi(t) \in D \). While the last is equivalent to that \( \zeta_D(\psi) > T \). Therefore

\[ R_N^i(\eta, T) = \int_D \left( \int_\Omega Q_j(\psi, T; 0) 1_{\{\zeta_D(\psi) > T\}} \mathbb{Q}_{ar;br}^\eta(\eta, \psi) \right) w^i(\eta, \psi) h(0, \xi, T, \eta) d\xi. \]

To handle the second term which arises from the inhomogeneous boundary data \( \beta(x) \). By using the conditional law we may rewrite

\[ R_B^i(\eta, T) = \int_D \left( \int_\Omega Q_j(\psi, \zeta_D(\psi)) \beta^j(\psi(\zeta_D(\psi))) 1_{\{\zeta_D(\psi) \leq T\}} \mathbb{Q}_{ar;br}^\eta(\eta, \psi, d\eta) \right) h(0, \xi, T, \eta) d\xi \]

where the complication arises due to the integral against the variable \( \xi \) takes place over the whole space \( \mathbb{R}^d \). Observe that \( \zeta_D(\tau_T \psi) \leq T \) if and only if there is \( t_0 \in [0, T] \) such that \( \psi(t_0) \in \partial D \), which is therefore equivalent to that \( \zeta_D(\psi) \leq T \), and

\[ \zeta_D(\tau_T \psi) = \inf \{ t \geq 0 : \psi(T - t) \in \partial D \} = \inf \{ T - s \geq 0 : \psi(s) \in \partial D \} = T - \sup \{ s : 0 \leq s \leq T \ \text{s.t.} \ \psi(s) \in \partial D \}. \]

Therefore \( Q_j(\tau_T \psi, \zeta_D(\tau_T \psi)) = \mathbb{Q}_j(\psi, \lambda_T, \partial D) \) on \( \{ \zeta_D(\psi) \leq T \} \) and

\[ \beta^j(\tau_T \psi(\zeta_D(\tau_T \psi))) 1_{\{\zeta_D(\tau_T \psi) \leq T\}} = \beta^j(\psi(\lambda_T, \partial D)) 1_{\{\zeta_D(\psi) \leq T\}}. \]

By using these relations we can rewrite the boundary term

\[ R_B^i(\eta, T) = \int_\Omega \left( \int_\Omega \mathbb{Q}_j(\psi, \lambda_T, \partial D) \beta^j(\psi(\lambda_T, \partial D)) 1_{\{\zeta_D(\psi) \leq T\}} \mathbb{Q}_{ar;br}^\eta(\eta, \psi) \right) h(0, \xi, T, \eta) d\xi. \]
Finally let us consider the third term arising from the inhomogeneous term in the parabolic system. In fact, by using duality we may rewrite

$$R_N^i(\eta, T) = \int_0^T \int_{\mathbb{R}^d} \int_{\Omega} Q^i(\psi, s)g^i(\psi(s), T - s)1_{(s < \zeta^u(x))} Q^i_{\eta, \psi, s} (d\psi) h_{\eta, \psi, s}(0, \eta, T, \zeta) d\zeta ds$$

$$= \int_0^T \int_{\mathbb{R}^d} \int_{\Omega} Q^i(\tau_T, s)g^i(\tau_T, s)1_{(s < \zeta^u(\tau_T))} Q^i_{\eta, \psi, s} (d\psi) h(0, \eta, T, \zeta) d\zeta ds$$

$$= \int_0^T \int_{\mathbb{R}^d} \int_{\Omega} \tilde{Q}^i(\psi, s)g^i(\psi(s), s)1_{(s < \zeta^u(0, \psi))} Q^i_{\eta, \psi, s} (d\psi) h(0, \xi, T, \eta) d\zeta ds.$$

Putting these three equations together we deduce the functional integration representation.

In particular we have the following forward Feynman-Kac formula.

**Theorem 6.3.** Suppose $\nabla \cdot b = 0$ on $\mathbb{R}^d$ in the distribution sense. Let $w^j(x, t)$ be the solution to Cauchy’s initial problem of the parabolic system:

$$\left(\mathcal{L}_{ab} - \frac{\partial}{\partial t}\right) w^j(x, t) + \sum_{k=1}^n a^j_k(x, t) w^k(x, t) = 0 \quad \text{in } D \times [0, T]$$

(6.24)

subject to the initial and Dirichlet boundary conditions:

$$w^j(x, 0) = w^j_0(x) \text{ for } x \in D, \text{ and } w^j(x, t) = 0 \text{ for } x \in \partial D, t > 0$$

(6.25)

where $j = 1, \cdots, n$. Then

$$w^j(\eta, T) = \int_D \left( \int_{\Omega} \tilde{Q}^i(\psi, T; 0) 1_{(\zeta^u(\psi) > T)} Q^i_{\eta, \psi, s} (d\psi) \right) w^j_0(\xi) h_{\eta, \psi, s}(0, \xi, T, \eta) d\zeta$$

for every $\eta \in D$.

**Remark 6.4.** We would like to emphasize the assumptions on $a^j_i(x, t)$ and $b^j(x, t)$, both are defined for all $x \in \mathbb{R}^d$ and $t \geq 0$. In order to ensure the previous functional integration representation to be valid, we assume that $b^j(x, t)$ is bounded (this condition can be weaken) and Borel measurable, but the most crucial assumption is that $b(x, t)$ is divergence free on $\mathbb{R}^d$ (not only on $D$!) in the distribution sense. The probability measure used in the representation is the distribution associated with the differential operator

$$\mathcal{L}_{u, \psi} = \mathbf{v} \sum_{i,j=1}^d \frac{\partial}{\partial x^j} a^j_i(x, t) \frac{\partial}{\partial x^i} - \sum_{i=1}^d b^j(x, t) \frac{\partial}{\partial x^i}$$

which is the formal adjoint operator of $\mathcal{L}_{ab}$.

## 7 Navier-Stokes equations

In this section we apply the forward Feynman-Kac formula to derive a stochastic representation for solutions of the Navier-Stokes equations in domains via their vortex dynamics, cf. [11, 29]. We begin with the following elementary fact.

**Lemma 7.1.** Suppose $b(x)$ is a $C^1$-vector field in $\overline{D}$, and $\nabla \cdot b(x) = 0$ for all $x \in D$. Suppose $b(x) = 0$ for $x \in \partial D$. Extend $b(x)$ to all $x \in \mathbb{R}^d$ by setting $b(x) = 0$ for $x \notin \overline{D}$. Then $b(x)$ is divergence-free in distribution sense on $\mathbb{R}^d$.

**Proof.** According to assumptions, $\partial D$ is a smooth manifold of $d - 1$ dimensions, so has zero Lebesgue measure. Therefore $\nabla \cdot b = 0$ a.e. on $\mathbb{R}^d$. Suppose $\varphi$ is a smooth function on $\mathbb{R}^d$ with a compact support. Then

$$\nabla \cdot (\varphi b) = \nabla \cdot \varphi b + \varphi \nabla \cdot b = \nabla \varphi \cdot b \quad \text{a.e.}$$
Hence
\[ \int_{\mathbb{R}^d} \nabla \varphi \cdot b = \int_{\mathbb{R}^d} \nabla \cdot (\varphi b) = \int_{\partial D} \varphi b \cdot \nu + \int_{\partial D^c} \varphi b \cdot \nu = 0 \]
where the last equality follows from the assumption that \( b(x) = 0 \) for \( x \in \partial D \). Therefore \( \nabla \cdot b = 0 \) on \( \mathbb{R}^d \) in the distribution sense.

Recall that the velocity \( u(x,t) \) and the pressure \( P(x,t) \) of an incompressible fluid flow constrained in \( D \) are solutions of the Navier-Stokes equations
\[
\frac{\partial}{\partial t} u + (u \cdot \nabla) u - \nu \Delta u - \nabla P = 0 \quad \text{in } D \times [0,\infty) \tag{7.1}
\]
and
\[
\nabla \cdot u = 0 \quad \text{in } D \times [0,\infty), \tag{7.2}
\]
subject to the no-slip condition
\[
u \quad \text{u(x,t) = 0 for x } \in \partial D \text{ and } t \geq 0. \tag{7.3}
\]
Suppose the initial data \( u_0 \in C^\infty(D) \). Then, in dimension two, \( u(x,t) \) remains smooth in \( (x,t) \) up to the boundary, while in dimension three, the regularity of \( u(x,t) \) remains open. Let \( \omega = \nabla \times u \) be the vorticity and \( \omega_0 = \nabla \times u_0 \). It can be verified by a simple calculation that the boundary vorticity can be related to the shearing stress \( \tau_{ij} = \nu \left( \frac{\partial u}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right) \) applied immediately to the boundary surface. In fact it can be demonstrated that the normal part \( \omega^\perp \) of the boundary vorticity \( \omega \big|_{\partial D} \), vanishes identically along the boundary surface, while its tangential vorticity \( \omega^\parallel \) along the boundary coincides with the normal shearing stress \( \tau^\perp \) up to a numerical factor \( \nu^{-1} \). The calculation of the boundary vorticity is an important problem which will not be discussed in this paper in detail. However let us point out that the normal stress applied immediately to the wall is a fluid dynamical quantity to be measured or to be controlled, and can be calculated approximately by using boundary layer equation, cf. [38, Chapter 6].

### 7.1 Two dimensional flows

Let us first establish a representation for 2D flows. In dimension two, the vorticity \( \omega \) can be identified with the scalar function \( \frac{\partial}{\partial x^1} u^2 - \frac{\partial}{\partial x^2} u^1 \) and \( \omega \) is a solution to the vorticity transport equation
\[
\frac{\partial}{\partial t} \omega + (u \cdot \nabla) \omega - \nu \Delta \omega = 0 \quad \text{in } D \times [0,\infty), \tag{7.4}
\]
where the boundary value of \( \omega \) along the wall \( \partial D \) may be identified with the stress of the fluid flow immediately injected to the wall, cf. [38, 27, 43]. Let us denote the stress along the wall by \( \sigma \), whose explicit expression need to be calculated in terms of the geometry of \( \partial D \) as well, so they are must be treated case by case.

Since \( \nabla \cdot u = 0 \), the velocity field may be recovered in terms of \( \omega \) by solving the Poisson equations
\[
\Delta u^1 = -\frac{\partial \omega}{\partial x^1}, \quad \Delta u^2 = \frac{\partial \omega}{\partial x^2} \tag{7.5}
\]
subject to the Dirichlet boundary condition \( u^1(x,t) = u^2(x,t) = 0 \) for \( x \in \partial D \). Hence according to Green formula
\[
u \quad u^i(x,t) = \int_D K^i(x,\eta) \omega(\eta,t) \mathrm{d} \eta = \int_{\mathbb{R}^2} K^i_D(x,\eta) \omega(\eta,t) \mathrm{d} \eta \tag{7.6}
\]
where
\[
u \quad K^i_D(x,\eta) = 1_D(\eta) K^i(x,\eta) \tag{7.7}
\]
and the integral kernel \( K \) depend on the region \( D \) only.
Theorem 7.2. Let $u$ be extended to be a vector field on $\mathbb{R}^2 \times [0, \infty)$ such that $u(\cdot, t)$ is divergence free in the distribution sense on the whole plane $\mathbb{R}^2$. Let $X(\xi, t)$ (for $\xi \in \mathbb{R}^2$ and $t \geq 0$) be the solution to the stochastic differential equation:

$$dX(t) = u(X(t), t)dt + \sqrt{2}\nu dB(t), \quad X(0) = \xi. \quad (7.8)$$

Then

$$u(x, t) = \int_D \omega_0(\xi) \mathbb{E}[K_D(x, X(\xi, t))J_1(\xi, X(\xi, t), t)]d\xi$$

$$+ \int_{\mathbb{R}^2} \mathbb{E}[K_D^r(x, X(\xi, t))J_2(\xi, X(\xi, t), t)]d\xi \quad (7.9)$$

for every $x \in D$ and $t > 0$, where

$$J_1(\xi, \eta, t) = \mathbb{P}[\xi_D(X(\xi, \cdot)) > t | X(\xi, t) = \eta] \quad (7.10)$$

and

$$J_2(\xi, \eta, t) = \mathbb{E}[\sigma(X(\xi, \lambda, \eta, t_D(X(\xi, \cdot))))1_{\{\xi_D(X(\xi, \cdot)) \leq t\}} | X(\xi, t) = \eta] \quad (7.11)$$

for any $\xi, \eta \in \mathbb{R}^2$ and $t > 0$.

Proof. The vorticity transport equation may be formulated in terms of $\mathcal{L}_u$, that is,

$$\left( \frac{\partial}{\partial t} - \mathcal{L}_u \right) \omega = 0 \quad \text{in } D \times [0, \infty). \quad (7.12)$$

Since $u(\cdot, t)$ is divergence free in the distribution sense for every $t$ on $\mathbb{R}^2$, therefore we may apply Theorem 6.3 to $\omega$, to obtain

$$\omega(\eta, t) = \int_D \omega_0(\xi) \mathbb{P}_{\xi, 0 \rightarrow \eta, t}^D[\xi_D(\psi) > t | h(0, \xi, t, \eta)]d\xi$$

$$+ \int_{\Omega} \left( \int_{\Omega} \sigma(\psi(\lambda, \eta, t_D(\psi)))1_{\{\xi_D(\psi) \leq t\}} \mathbb{P}_{\xi, 0 \rightarrow \eta, t}^D(d\psi) \right) h(0, \xi, t, \eta)d\xi \quad (7.13)$$

for $\eta \in D$ and $t > 0$, where $\mathbb{P}_{\xi, 0 \rightarrow \eta, t}^D$ denotes the conditional distribution of the $\mathcal{L}_u$-diffusion started from $\xi$ at time zero given $\psi(t) = \eta$, and $h(\tau, \xi, t, \eta)$ is the transition probability density function of the $\mathcal{L}_u$-diffusion. Since the distribution of $X(\xi, t)$ is exactly $\mathbb{P}_{\xi, 0}^D$, the conclusion therefore follows immediately. \hfill \Box

7.2 Three dimensional flows

Next we consider an incompressible fluid flow with its velocity $u = (u^1, u^2, u^3)$, constrained in a region $D \subset \mathbb{R}^3$. Therefore $u(x, t)$ is a solution to the 3D Navier-Stokes equations (7.1, 7.2, 7.3). The vorticity $\omega^i = \epsilon^{ijk} \frac{\partial}{\partial x^j} u^k$ are solutions to the 3D vorticity transport equations

$$\frac{\partial}{\partial t} \omega^i + (u \cdot \nabla) \omega^i - \nu \Delta \omega^i - \frac{\partial u^i}{\partial x^j} \omega^j = 0 \quad \text{in } D \times [0, \infty), \quad (14.14)$$

where $\nu > 0$ is the viscosity constant as usual, which can be written as

$$\left( \mathcal{L}_u - \frac{\partial}{\partial t} \right) \omega^i + S^i_j \omega^j = 0, \quad (15.15)$$

where

$$S^i_j = \frac{1}{2} \left( \frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right)$$
is the symmetric stress tensor. The boundary value of $\omega$ along the wall $\partial D$ can be again identified with the stress tensor, denoted by $\beta$. Since $\nabla \cdot u = 0$ in $D$ so that $\Delta u = -\nabla \times \omega$ in $D$ and $u$ satisfies the Dirichlet boundary condition along $\partial D$. Therefore

$$u(x,t) = -\int_D H(x,\eta)\nabla \times \omega(\eta,t) d\eta,$$

(7.16)

where $H(x,y)$ is the Green function of $D$. By integration by parts we obtain

$$u(x,t) = \int_D K(x,\eta) \times \omega(\eta,t) d\eta$$

$$= \int_{\mathbb{R}^3} K_D(x,\eta) \times \omega(\eta,t) d\eta$$

(7.17)

where $K_D(x,\eta) = 1_D(\eta)K(x,\eta)$. That is

$$u'(x,t) = \int_D \epsilon^{ijk} K^i(x,\eta) \omega^j(\eta,t) d\eta.$$  

(7.18)

It follows that

$$S_j^i(x,t) = \int_D K_{ij}^k(x,\eta) \omega^k(\eta,t) d\eta$$

$$= \int_{\mathbb{R}^3} K_{ij}^k(x,\eta) \omega^k(\eta,t) d\eta,$$

(7.19)

where

$$K_{ij}^k(x,\eta) = \epsilon^{ijk} \frac{\partial}{\partial x^l} K^l(x,\eta),$$

(7.20)

and

$$K_{ij}^k(x,\eta) = 1_D(\eta) \left( \epsilon^{ijk} \frac{\partial}{\partial x^l} K^l(x,\eta) + \epsilon^{ijk} \frac{\partial}{\partial x^l} K^l(x,\eta) \right).$$

(7.21)

We notice that the Green function $G$, the integral kernels $K$ and $K_D$ are determined solely by the domain $D \subset \mathbb{R}^3$.

Let $u(x,t)$ be extended to be a divergence free (in the distribution sense) vector field on $\mathbb{R}^3$, and $X(\xi;t)$ and $\bar{Q}_j(x,t)$ are the solutions to the stochastic differential equations

$$dX(\xi;t) = u(X(\xi;t),t)dt + \sqrt{2} dB(t), \quad X(\xi;0) = \xi$$

(7.22)

and

$$d\bar{Q}_j(\xi,t;\delta) = -\partial_{\bar{Q}_j'}(\xi,t;\delta) 1_D(X(\xi;\delta)) S_j^i(X(\xi;\delta),\delta), \quad \bar{Q}_j(\xi,t;\delta) = \delta_j$$

(7.23)

for $t > 0$ and $s \geq 0$.

**Theorem 7.3.** Suppose $u(x,t)$ is smooth and bounded on $\overline{D} \times [0,T]$, then

$$u^k(x,t) = \int_D \omega^0(\xi) \mathbb{E} \left[ e^{ijk} K_D^j(x,X(\xi,t)) J_i^l(\xi,X(\xi,t),t) \right] d\xi$$

$$+ \int_{\mathbb{R}^3} \mathbb{E} \left[ e^{ijk} K_D^j(x,X(\xi,t)) B^j(\xi,X(\xi,t),t) \right] d\xi,$$

(7.24)

and

$$S_j^i(x,t) = \int_D \omega^0(\xi) \mathbb{E} \left[ K_{ij}^k(x,\eta) J_i^l(\xi,X(\xi,t),t) \right] d\xi$$

$$+ \int_{\mathbb{R}^3} \mathbb{E} \left[ K_{ij}^k(x,\eta) B^j(\xi,X(\xi,t),t) \right] d\xi,$$

(7.25)

for all $x \in D$ and $t \in (0,T)$, where $k = 1,2,3$,

$$J_i^j(\xi,\eta,t) = \mathbb{E} \left[ Q_j^i(\xi,t;0) 1_{\delta_D(X(\xi,t)) > t} \right] | X(\xi,t) = \eta \right]$$

and

$$B^i(\xi,\eta,t) = \mathbb{E} \left[ Q_j^i(\xi,t;\lambda_T,\epsilon) 1_{\delta_D(X(\xi,t)) \leq t} \right] | X(\xi,t) = \eta \right],$$

(7.27)
Proof. The proof is similar to that of two dimensional case. According to Theorem 5.3

\[ \omega'(\eta, T) = \int_D f_i'(\xi, \eta, t) \omega_i(\xi, t) h_u(0, \xi, T, \eta) d\xi \]

and the representation formula follows from (7.16) and the Fubini theorem immediately. □

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