LOCAL UNSTABLE ENTROPY AND LOCAL UNSTABLE PRESSURE FOR RANDOM PARTIALLY HYPERBOLIC DYNAMICAL SYSTEMS

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Abstract. Let \( F \) be a random partially hyperbolic dynamical system generated by random compositions of a set of \( C^2 \)-diffeomorphisms. For the unstable foliation, the corresponding local unstable measure-theoretic entropy, local unstable topological entropy and local unstable pressure via the dynamics of \( F \) along the unstable foliation are introduced and investigated. And variational principles for local unstable entropy and local unstable pressure are obtained respectively.

1. Introduction. It is well known that among the concepts in differentiable dynamical systems, entropy including measure-theoretic entropy and topological entropy, pressure, and Lyapunov exponents play important roles for both deterministic and random cases, which describe the complexity of a given dynamical system from different points of view.

Plentiful results on entropies have been obtained, since measure-theoretic entropy was introduced by Kolmogorov and topological entropy was introduced by Adler, Konheim and McAndrew respectively. One important result is the so-called variational principle relating measure-theoretic entropy and topological entropy, which says that for a given dynamical system, its topological entropy is equal to

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the supremum of measure-theoretic entropies over all invariant measures. Another momentous result is the famous Pesin’s entropy formula relating measure-theoretic entropy and Lyapunov exponents for a $C^2$ diffeomorphism with an SRB measure, which was first introduced by Pesin and was explored in depth by Ledrappier and Young in [4]. Moreover, as a generalization of Pesin’s entropy formula, for a $C^2$ diffeomorphism, Ledrappier and Young gave the dimension formula for all invariant measures in [5]. All above results are also generalized to random cases under different settings, and the reader can refer to [7] for a review.

In order to describe the complexity of a dynamical system more finely, various forms of entropies were introduced. One useful entropy is the so-called local entropy including local measure-theoretic entropy and local topological entropy, and a local version of variational principle was also obtained. For a topological dynamical system $(X,T)$ and an open cover $U$ of $X$, Romagnoli [12] introduced two notions of local measure-theoretic entropies with respect to $U$, $h_\mu(T,U)$ and $h^+_\mu(T,U)$ with $h_\mu(T,U) \leq h^+_\mu(T,U)$, then a variational principle was obtained as follows:

$$h_{\text{top}}(T,U) = \sup_{\mu \text{ is } T\text{-invariant}} \{h_\mu(T,U)\},$$

where $h_{\text{top}}(T,U)$ is the local topological entropy with respect to $U$.

For random case, Ma and Chen [9] introduced random entropies $h^{(r)-}_\mu(T,U)$, $h^{(r)+}_\mu(T,U)$ and $h_{\text{top}}(T,U)$ for random bundle transformation $T$ with an invariant measure $\mu$ and an open cover $U$, then they gave a random version of local variational principle as follows:

$$\max\{h^{(r)-}_\mu(T,U): \mu \text{ is } T\text{-invariant}\} = h_{\text{top}}(T,U).$$

An interesting question is that can we introduce unstable entropies including both topological and measure-theoretic versions such that a similar variational principle can be obtained? Further more, can we obtain similar results for local entropies? Recently, Hu, Hua and Wu in [2] and Wu in [15] gave positive answers to the two questions for deterministic case respectively. In [2], for a $C^1$-partially hyperbolic diffeomorphism on a closed Riemannian manifold, Hu, Hua and Wu introduced the so-called unstable measure-theoretic entropy and unstable topological entropy, and a variational principle was obtained relating them; in [15], in the same settings, the so-called local unstable measure-theoretic entropy and local unstable topological entropy were introduced by Wu, and a variational principle relating them was also established. For random case, a positive answer to the first question mentioned above under the condition of partial hyperbolicity is also obtained in [14] by Wang, Wu and Zhu. For a random dynamical system $\mathcal{F}$ with an invariant measure $\mu$, they introduce the definitions of $h^{u}_{\text{top}}(\mathcal{F})$ and $h^{u}_{\mu}(\mathcal{F})$ and following the line in [2] they establish the corresponding variational principle as follows

$$\sup\{h^{u}_{\mu}(\mathcal{F}): \mu \text{ is } \mathcal{F}\text{-invariant}\} = h^{u}_{\text{top}}(\mathcal{F}).$$

In this paper, we want to give a positive answer to the second question mentioned above for random case. Given a $C^2$ random partially hyperbolic dynamical system $\mathcal{F}$ (refer to Section 2 for its definition), an invariant measure $\mu$ respect to $\mathcal{F}$, and an open cover $U$ (refer to Section 2 for its precise definition), we introduce the definitions of $h^{u}_{\mu}(\mathcal{F},U)$, $h^{u,+}_{\mu}(\mathcal{F},U)$, and $h^{u}_{\text{top}}(\mathcal{F},U)$. And as a generalization of the topological entropy, a local version of pressure is also introduced. In order to establish a variational principle relating $h^{u}_{\mu}(\mathcal{F},U)$, $h^{u,+}_{\mu}(\mathcal{F},U)$, and $h^{u}_{\text{top}}(\mathcal{F},U)$, some relations are obtained between unstable entropies and their corresponding local
versions; furthermore, the unstable topological conditional entropy and unstable tail entropy are introduced. In fact, both of them vanish in our settings, which implies us that the principle obtained in [14] can be applied in our proofs. In the end, as a generalization of the entropy, we establish the variational principle for local pressure.

This paper is organised as follows. In Section 2, we give some basic knowledge necessary for our goal and state our main results. In Section 3, we give the definitions of two kinds of local unstable measure-theoretic entropies, and some properties of these two local entropies and relations between them are also obtained. In Section 4, we give the definition of local unstable topological entropy with some important properties of them. In Section 5, we give the definitions of unstable topological conditional entropy and unstable tail entropy and their relations with local unstable entropies and unstable entropies, which are crucial to the proof of our variational principles. In the last section, i.e. Section 6, we give the proofs of the variational principles for both local entropies and local pressure.

2. Preliminaries and main results. Throughout this paper, let $M$ be a $C^\infty$ compact Riemannian manifold without boundary. Denote by $\mathcal{B}(M)$ the Borel $\sigma$-algebra of $M$. Let $(\Omega, \mathcal{F}, P)$ be a Polish probability space and $\theta$ be an invertible and ergodic measure-preserving transformation on $\Omega$.

Definition 2.1. A $C^2$ random dynamical system $F$ on $M$ over $(\Omega, \mathcal{F}, P, \theta)$ is defined as a map

$$F: \mathbb{Z} \times \Omega \times M \to M$$

$$(n, \omega, x) \mapsto F(n, \omega)x,$$

which has the following properties:

(i) $F$ is measurable;

(ii) the maps $F(n, \omega): M \to M$ form a cocycle over $\theta$, i.e. they satisfy

$$F(0, \omega) = \text{id},$$

$$F(n + m, \omega) = F(m, \theta^n \omega) \circ F(n, \omega),$$

for all $n, m \in \mathbb{Z}$ and $\omega \in \Omega$;

(iii) the maps $F(n, \omega): M \to M$ are $C^2$ for all $n \in \mathbb{Z}$ and $\omega \in \Omega$.

For each $\omega \in \Omega$, we define

$$f^\omega_n := \begin{cases} F(1, \theta^{n-1} \omega) \circ \cdots \circ F(1, \omega) & \text{if } n > 0, \\ \text{id} & \text{if } n = 0, \\ F(1, \theta^n \omega)^{-1} \circ \cdots \circ F(1, \theta^{-1} \omega)^{-1} & \text{if } n < 0. \end{cases}$$

Associated with $\Omega \times M$, there is a skew product $\Theta$ induced by $F$, i.e.,

$$\Theta: \Omega \times M \to \Omega \times M$$

$$(\omega, x) \mapsto (\theta \omega, F(1, \omega)x).$$

Definition 2.2 (Invariant measure). A measure $\mu$ on $\Omega \times M$ is said to be an $F$-invariant measure, if it is $\Theta$-invariant and has marginal measure $P$ on $\Omega$. In particular, an $F$-invariant measure $\mu$ is said to be ergodic, if it is ergodic with respect to $\Theta$. 
We denote by $\mathcal{M}_p(\mathcal{F})$ the set of all $\mathcal{F}$-invariant measures.

In order to apply Rokhlin’s results for Lebesgue space, in the following part of this paper, for $\mu \in \mathcal{M}_p(\mathcal{F})$ we always consider the $\mu$-completion of $\mathcal{F} \times \mathcal{B}(M)$, which is still denoted by $\mathcal{F} \times \mathcal{B}(M)$ for simplicity.

According to Rokhlin’s paper [11] and Liu and Qian’s monograph [8], for each $\mu \in \mathcal{M}_p(\mathcal{F})$, there exists a family of sample measures $\mu(\cdot) : \Omega \times \mathcal{B}(M) \to [0,1]$ of $\mu$ satisfying the following properties:

(i) for all $B \in \mathcal{B}(M)$, $\omega \mapsto \mu_\omega(B)$ is $\mathcal{F}$-measurable;
(ii) for $\mathcal{P}$-a.e. $\omega \in \Omega$, $\mu_\omega : \mathcal{B}(M) \to [0,1]$ is a probability measure on $M$;
(iii) for $A \in \mathcal{F} \times \mathcal{B}(M)$,

$$
\mu(A) = \int_{\Omega} \int_{M} 1_A(\omega, x) d\mu_\omega(x) d\mathcal{P}(\omega),
$$

where $1_A$ is the characteristic function of $A \subset \Omega \times M$.

**Remark 1.** For $\mu \in \mathcal{M}_p(\mathcal{F})$, it is clear that $f_n^\omega(\omega) = \mu_{\theta^n \omega}$ for all $n \in \mathbb{Z}$ and $\mathcal{P}$-a.e. $\omega \in \Omega$.

Throughout this paper, we always assume that the Probability $\mathcal{P}$ on $\Omega$ satisfies

$$
\int_\Omega \left( \log^+ |f_1(1, \omega)|_{C^2} + \log^+ |f(-1, \omega)|_{C^2} \right) d\mathcal{P}(\omega) < \infty,
$$

where $|f|_{C^2}$ denotes the usual $C^2$ norm of $f \in \text{Diff}^2(M)$, and $\log^+ a = \max\{\log a, 0\}$.

Similar to the deterministic case, we can define the Lyapunov exponents for $\mathcal{F}$. Let $\Lambda$ be the set of all regular points $(\omega, x) \in \Omega \times M$ in the sense of Oseledec. For $(\omega, x) \in \Lambda$, Let $\lambda_1(\omega, x) > \cdots > \lambda_r(\omega, x)$ be its distinct Lyapunov exponents of $\mathcal{F}$ with multiplicities $m_j(\omega, x)$ ($1 \leq j \leq r(\omega, x)$).

Let $\mu \in \mathcal{M}_p(\mathcal{F})$, and denote by $\|\cdot\|$ the norm of vectors in the tangent space of $M$. By the Oseledec Multiplicative Ergodic Theorem, we know that $\Lambda$ is $\Theta$-invariant and $\mu(\Lambda) = 1$. For each $(\omega, x) \in \Lambda$, there is a splitting of $T_x M$ as follows

$$
T_x M = E_1(\omega, x) \oplus \cdots \oplus E_r(\omega, x)(\omega, x)
$$

such that for $i = 1, \ldots, r(\omega, x)$, $\dim E_i(\omega, x) = m_i(\omega, x)$ and

$$
\lim_{n \to \pm \infty} \frac{1}{n} \log \|D_x f_n^\omega v\| = \lambda_i(\omega, x), \quad \text{for all } v \in E_i(\omega, x) \setminus \{0\}.
$$

(By (1) we can choose $\Lambda$ such that all the above Lyapunov exponents are finite.)

Let

$$
u(\omega, x) = \max\{j : \lambda_j(\omega, x) > 0\}.
$$

For $(\omega, x) \in \Lambda$, we define the set

$$
W^u(\omega, x) = \{y \in M : \limsup_{n \to +\infty} \frac{1}{n} \log d(f_{\omega}^{-n} y, f_{\omega}^{-n} x) \leq -\nu(\omega, x)(\omega, x)\},
$$

where $d(\cdot, \cdot)$ is the metric on $M$ induced by its Riemannian structure. Let

$$
E^u(\omega, x) = \bigoplus_{j=1}^{\nu(\omega, x)} E_j(\omega, x), \quad F^u(\omega, x) = \bigoplus_{j=\nu(\omega, x)+1}^{r(\omega, x)} E_j(\omega, x).
$$

It is clear that both $E^u(\omega, x)$ and $F^u(\omega, x)$ are invariant under the tangent map, i.e. for $n \in \mathbb{Z}$,

$$
D_x f_n^\omega E^u(\omega, x) = E^u(\theta^n \omega, f_n^\omega x) \quad \text{and} \quad D_x f_n^\omega F^u(\omega, x) = F^u(\theta^n \omega, f_n^\omega x).
$$

The following proposition from [1] ensures that $W^u(\omega, x)$ is an immersed submanifold of $M$. 
Proposition 1. For $(\omega, x) \in \Lambda$, the set $W^n(\omega, x)$ is a $C^{1,1}$ immersed submanifold of $M$ tangent at $x$ to $E^n(\omega, x)$.

When $(\omega, x) \in \Omega \times M \setminus \Lambda$, we let $W^n(\omega, x) = \{x\}$, making the definitions of unstable topological entropy in subsequent sections more transparent. We call the collection $\{W^n(\omega, x) : (\omega, x) \in \Omega \times M\}$ the $W^n$-foliation.

A family of sets $\mathcal{U} = \{U_i\}_{i \in I}$ is called a cover of $\Omega \times M$, if it satisfies

$$\bigcup_{i \in I} U_i \supset \Omega \times M,$$

where $I$ is an index set. Denote by $\mathcal{U}(\omega, x)$ the element of $\mathcal{U}$ containing $(\omega, x)$. It is clear that $\mathcal{U}_\omega := \{U_i(\omega)\}_{i \in I}$ is a cover of $M$, where $U_i(\omega) := \{x \in M : (\omega, x) \in \mathcal{U}(\omega, x)\}$. $\mathcal{U}$ is called a Borel cover, if $\mathcal{U}_\omega$ is a Borel cover of $M$, for all $\omega \in \Omega$. Specially, if $\mathcal{U}_\omega$ is an open cover of $M$, for all $\omega \in \Omega$, $\mathcal{U}$ is called an open cover. A cover is said to be finite, if $I$ is a finite set. Denote by $C_{\Omega \times M}$ and $C^0_{\Omega \times M}$ the set of finite Borel covers and the set of finite open covers respectively.

For a Borel cover $\mathcal{U}$ of $\Omega \times M$ and $\omega \in \Omega$, define $\text{diam}(\mathcal{U}_\omega)$ as follows

$$\text{diam}(\mathcal{U}_\omega) := \max_{U \in \mathcal{U}_\omega} \text{diam}(U),$$

where $\text{diam}(U) = \sup_{x,y \in U} d(x, y)$. Then the diameter of $\mathcal{U}$ is defined as a variable $\text{diam}(\mathcal{U}) : \Omega \to \mathbb{R}$ such that

$$\text{diam}(\mathcal{U})(\omega) = \text{diam}(\mathcal{U}_\omega).$$

It is clear that a measurable partition $\alpha$ of $\Omega \times M$ can be regarded as a Borel cover of $\Omega \times M$, and $\alpha_\omega$ is a measurable partition of $M$, for all $\omega \in \Omega$.

For two covers $\alpha$ and $\beta$ of $\Omega \times M$, $\alpha \geq \beta$ means for any element $A \in \alpha$, there is an element $B \in \beta$ such that $A \subset B$. Let $\mu \in \mathcal{M}_p(\mathcal{F})$. A measurable partition $\xi$ of $\Omega \times M$ with $\xi \geq \sigma_0$, where $\sigma_0$ is the partition $\{\{\omega\} \times M : \omega \in \Omega\}$, is said to be subordinate to the $W^u$-foliation, if for $\mu$-a.e. $(\omega, x) \in \Omega \times M$, $\xi_\omega(x) \subset W^u(\omega, x)$ and it contains a neighborhood of $x$ in $W^u(\omega, x)$.

For each measurable partition $\eta$ subordinate to $W^u$-foliation, there is a canonical system of conditional measures $\{\mu_{\omega, x}^{\eta}\}_{(\omega, x) \in \Omega \times M}$ of $\mu$ by a classical result of Rokhlin [11]. And $\mu_{\omega, x}^{\eta}$ can be regarded as a measure on $\eta_\omega(x)$, if we identify $\{\omega\} \times \eta_\omega(x)$ with $\eta_\omega(x)$.

We call a partition $\alpha$ of $\Omega \times M$ fiberwise finite if for $\mathcal{P}$-a.e. $\omega \in \Omega$, $\alpha_\omega$ is finite and the cardinality of $\alpha_\omega K(\omega)$ is integrable with respect to $\omega$, i.e.

$$\int_{\Omega} K(\omega) d\mathcal{P} < \infty.$$
A partition satisfying Proposition 2 is called an increasing partition subordinate to unstable manifolds, and denote by $Q^u(\Omega \times M)$ the set of all such partitions.

Now we give the definition of random partially hyperbolic dynamical systems. A random variable $s: \Omega \rightarrow \mathbb{R}^+$ is called $\theta$-tempered, if $\lim_{n \rightarrow \pm \infty} \frac{1}{n} \log s(\theta^n) = 0$ for $P$-a.e. $\omega \in \Omega$, and $t$ is called $\theta$-invariant if $t(\theta^\omega) = t(\omega)$ for $P$-a.e. $\omega \in \Omega$.

**Definition 2.3.** An RDS $\mathcal{F}$ is called partially hyperbolic, if

(i) there exist a $\theta$-tempered random variable $C(\omega) > 1$ and a $\theta$-invariant random variable $0 < \lambda(\omega) < 1$ such that for all $(x, \omega) \in \Lambda$ and $n \in \mathbb{N}$, $u \in F^n(\omega, x) \setminus \{0\}$ and $v \in E^n(\omega, x) \setminus \{0\}$,

$$\frac{\|Df^n u\|}{\|u\|} \leq C(\omega)(\lambda(\omega))^n \frac{\|Df^n v\|}{\|v\|};$$

(ii) $\mathcal{F}$ is uniformly expanding in $x$ along $W^u$-foliation, i.e., there exists a random variable $\tilde{\lambda}(\omega) > 1$ such that $\|Df^n|E^n(\omega, x)\| > \tilde{\lambda}(\omega)$.

**Remark 2.** (i) In fact, $\lambda(\omega)$ is a.e. constant, since $\theta$ is $P$-ergodic.
(ii) Property (i) in Definition 2.3 is called $u$-domination (cf. [13] for deterministic case). Notice that the uniform expansion in (ii) is a crucial property for our proofs of the main results. We think that the similar results should hold for $C^2$ RDSs with $u$-domination, but more complicated techniques involving Pesin theory must be applied.

**Basic assumption.** In the remaining of this paper, we always assume that $\mathcal{F}$ is a random partially hyperbolic dynamical system.

**Example.** Let $f$ be a $C^2$ partially hyperbolic diffeomorphism. Combining the techniques in [6] and [3], we can obtain a random partially hyperbolic dynamical system satisfying the above assumption via small $C^2$ random perturbations of $f$.

Define $W^u(\omega, x, \delta) = \{y \in W^u(\omega, x): d^u_\omega(y, x) \leq \delta\}$, where $d^u_\omega(\cdot, \cdot)$ is the distance along $W^u(\omega, x)$. For two variables $s, t: \Omega \rightarrow \mathbb{R}$, $s = (\text{resp.} s >, \geq)t$ means $s(\omega) = (\text{resp.} s >, \geq)t(\omega)$ for $P$-a.e. $\omega \in \Omega$. We can choose a variable $\epsilon_1 > 0$ small enough and a $\theta$-tempered variable $C_0 > 1$ such that $d(\cdot, \cdot) \leq d^u_\omega(\cdot, \cdot) \leq C_0(\omega)d(\cdot, \cdot)$ on any local unstable manifold $W^u(\omega, x, \epsilon_1(\omega))$.

Choose and fix a variable $\lambda_0$ such that $\lambda_0(\omega) > \|Dx f^1|E^u(\omega, x)\| > 1$ for any $x \in M$. Choose $L: \Omega \rightarrow \mathbb{R}$ with $L > 0$ and $0 < \epsilon_0 \ll \min\{\epsilon_1, L\}$ such that

(i) $\epsilon_0$ is $\theta$-tempered;
(ii) for any $(\omega, x) \in \Lambda$, $W^u(\omega, x, L(\omega)) \cap B(x, \epsilon_0(\omega))$ has only one connected component.
(iii) $\lambda_0(\omega)C_0(\omega)\epsilon_0(\omega) \ll L(\theta)\omega$.

Let $\mathcal{P}(\Omega \times M)$ denote the set of all fiberwise finite Borel partitions $\alpha$ of $\Omega \times M$ with $\text{diam}(\alpha) < \epsilon_0$. For a partition $\alpha \in \mathcal{P}(\Omega \times M)$, adapting method in [2] we can construct $\alpha^u \geq \alpha$ satisfying $\alpha^u(\omega) = \alpha_{\omega}(\omega) \cap W^u(\omega, x, \epsilon_0(\omega))$ for any $(\omega, x) \in \Lambda$, where $C_0(\omega)\epsilon_0(\omega) < \epsilon_0(\omega) < L(\omega)$ and $C_0(\omega)\epsilon_0(\omega) < \epsilon_0(\theta)\omega$.

Denote by $\mathcal{P}^n(\Omega \times M)$ the set of all partitions constructed by the above method.

**Remark 3.** It is easy to check that if for $P$-a.e. $\omega$, $\mu_\omega(\cup_{A \in \alpha_{\omega}} \partial A) = 0$, then a measurable partition described as above is a partition subordinate to the $W^u$-foliation.

Given $U \in C^0_{\Omega \times M}$, as generalizations of local unstable measure-theoretical entropies and local unstable topological entropy in [15], we introduce corresponding
notions for random case, denote them by $h^u_\mu(F, U|\zeta)$, $h^{u,+}_\mu(F, U|\zeta)$ (see Section 3) and $h^u_{\text{top}}(F, U)$ (see Section 4) respectively, where $\zeta \in P^u(\Omega \times M) \cup Q^u(\Omega \times M)$.

Now we can give our main results as follows.

Theorem A. Let $F$ be a random partially hyperbolic dynamical system, and $U \in C^0_{\Omega \times M}$ with small enough diameter. Then for any $\mu \in M(F)$ and $\zeta \in P^u(\Omega \times M) \cup Q^u(\Omega \times M)$, we have

$$h^u_\mu(F, U|\zeta) = h^{u,+}_\mu(F, U|\zeta) = h^u_\mu(F|\zeta) = h^u_\mu(F),$$

and

$$h^u_{\text{top}}(F, U) = h^u_{\text{top}}(F),$$

where for the definition of $h^u_\mu(F|\zeta)$ and $h^u_\mu(F)$, see Definition 3.2; $h^u_{\text{top}}(F)$ is the unstable topological entropy (for more details, we refer the reader to the paper [14]).

As a corollary of Theorem A and Theorem D in [14], we have the following theorem.

Theorem B. Let $F$ be a random partially hyperbolic dynamical system, and $U \in C^0_{\Omega \times M}$ with small enough diameter. Then for any $\zeta \in P^u(\Omega \times M) \cup Q^u(\Omega \times M)$, we have

$$h^u_{\text{top}}(F, U) = \sup_{\mu \in M(F)} h^u_\mu(F, U|\zeta) = \sup_{\mu \in M(F)} h^{u,+}_\mu(F, U|\zeta).$$

We can generalize above results to the unstable pressure for random case. Firstly, we have the following theorem. Denote the set

$$\{ \phi \in L^1(\Omega \times M) : \phi \text{ is measurable in } \omega, \text{ continuous in } x \}$$

by $L^1(\Omega, C(M))$.

Theorem C. Let $F$ be a random partially hyperbolic dynamical system, and $U \in C^0_{\Omega \times M}$ with small enough diameter. Then for any $\phi \in L^1(\Omega, C(M))$, we have

$$P^u(F, \phi, U) = P^u(F, \phi).$$

Applying Theorem C in [14], we can obtain a variational principle as follows.

Corollary D. Let $F$ be a random partially hyperbolic dynamical system, and $U \in C^0_{\Omega \times M}$ with small enough diameter. Then for any $\phi \in L^1(\Omega, C(M))$, we have

$$P^u(F, \phi, U) = \sup_{\mu \in M(F)} \left\{ h^u_\mu(F, U|\zeta) + \int_{\Omega \times M} \phi d\mu \right\} = \sup_{\mu \in M(F)} \left\{ h^{u,+}_\mu(F, U|\zeta) + \int_{\Omega \times M} \phi d\mu \right\}.$$
Definition 3.1. Let $\mu$ be an invariant measure of $(\Omega \times M, \Theta)$, $\alpha$ and $\eta$ be two measurable partitions of $\Omega \times M$. The information function of $\alpha$ with respect to $\mu$ is defined as

$$I_\mu(\alpha)(\omega, x) := -\log \mu(\alpha(\omega, x)),$$

and the entropy of $\alpha$ with respect to $\mu$ is defined as

$$H_\mu(\alpha) := \int_{\Omega \times M} I_\mu(\alpha)(\omega, x) d\mu(\omega, x) = -\int_{\Omega \times M} \log \mu(\alpha(\omega, x)) d\mu(\omega, x).$$

The conditional information function of $\alpha$ with respect to $\eta$ is defined as

$$I_\mu(\alpha|\eta)(\omega, x) := -\log \mu_{(\omega, x)}(\alpha(\omega, x)),$$

where $\{\mu_{(\omega, x)}^\eta\}_{(\omega, x) \in \Omega \times M}$ is the canonical system of conditional measures of $\mu$ with respect to $\eta$. Then the conditional entropy of $\alpha$ with respect to $\eta$ is defined as

$$H_\mu(\alpha|\eta) := \int_{\Omega \times M} I_\mu(\alpha|\eta)(\omega, x) d\mu(\omega, x) = -\int_{\Omega \times M} \log \mu_{(\omega, x)}^\eta(\alpha(\omega, x)) d\mu(\omega, x).$$

For simplicity, sometimes we will use the following notations. For $\omega \in \Omega$, denote

$$H_{\mu_\omega}(\alpha) := \int_M I_\mu(\alpha)(\omega, x) d\mu_\omega,$$

and

$$H_{\mu_\omega}(\alpha|\eta) := \int_M I_\mu(\alpha|\eta)(\omega, x) d\mu_\omega.$$

It is clear that

$$H_\mu(\alpha) = \int_\Omega H_{\mu_\omega}(\alpha) dP,$$

and

$$H_\mu(\alpha|\eta) = \int_\Omega H_{\mu_\omega}(\alpha|\eta) dP.$$

The following definition comes from [14].

Definition 3.2. Let $\mu \in \mathcal{MP}(\mathcal{F})$. The conditional entropy of $\mathcal{F}$ for a fiberwise finite measurable partition $\alpha$ with respect to $\zeta \in \mathcal{P}^u(\Omega \times M) \cup \mathcal{Q}^u(\Omega \times M)$ is defined as

$$h^u_\mu(\mathcal{F}, \alpha|\zeta) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha^{n-1}|\zeta).$$

The conditional entropy of $\mathcal{F}$ with respect to $\zeta$ is defined as

$$h^u_\mu(\mathcal{F}|\zeta) = \sup_{\alpha \in \mathcal{P}(\Omega \times M)} h^u_\mu(\mathcal{F}, \alpha|\zeta),$$

and the conditional entropy of $\mathcal{F}$ along $W^u$-foliation is defined as

$$h^u_\mu(\mathcal{F}) = \sup_{\zeta \in \mathcal{P}^u(\Omega \times M) \cup \mathcal{Q}^u(\Omega \times M)} h^u_\mu(\mathcal{F}|\zeta).$$

In the following, we give some useful conclusions from [14]. For deterministic case, see [2, 16].

Lemma 3.3. For any $\alpha \in \mathcal{P}(\Omega \times M)$ and $\eta \in \mathcal{P}^u(\Omega \times M)$, the map $\mu \mapsto H_\mu(\alpha|\eta)$ from $\mathcal{MP}(\mathcal{F})$ to $\mathbb{R}$ is concave. Moreover, the map $\mu \mapsto h^u_\mu(\mathcal{F})$ from $\mathcal{MP}(\mathcal{F})$ to $\mathbb{R}$ is affine.
Moreover, the function $\mu \mapsto H_{\mu}(\alpha|\eta)$ is upper semi-continuous at $\mu$, i.e.,

$$\limsup_{\mu' \to \mu} H_{\mu'}(\alpha|\eta) \leq H_\mu(\alpha|\eta).$$

Moreover, the function $\mu' \mapsto h_{\mu'}^u(F)$ is upper semi-continuous at $\mu$, i.e.,

$$\limsup_{\mu' \to \mu} h_{\mu'}^u(F) \leq h_{\mu}^u(F).$$

Lemma 3.5. Suppose $\mu$ is ergodic, then for any $\alpha \in \mathcal{P}(\Omega \times M)$ and $\eta \in \mathcal{P}^u(\Omega \times M)$, we have

$$h_{\mu}^u(F) = h_{\mu}^u(F,\alpha|\eta) = \lim_{n \to \infty} \frac{1}{n} H_{\mu}(\alpha_{n-1}^n|\eta).$$

We denote $\bigvee_{j=m}^n \Theta^{-j}\alpha$ by $\alpha^n_m$, for any $m \leq n$, $m, n \in \mathbb{Z} \cup \{\pm \infty\}$. Given $\mu \in \mathcal{M}_P(F)$, now, we give the definition of $h_{\mu}^{u,+}(F,\mathcal{U}(\zeta))$.

Definition 3.6. For any $\mathcal{U} \subset \mathcal{C}_{\Omega \times M}$, and $\zeta \in \mathcal{P}^u(\Omega \times M) \cup \mathcal{Q}^u(\Omega \times M)$, define

$$h_{\mu}^{u,+}(F,\mathcal{U}(\zeta)) := \inf_{\alpha \in \mathcal{P}(\Omega \times M), \alpha \geq \mathcal{U}} h_{\mu}^u(F,\alpha|\zeta).$$

In order to study properties of $h_{\mu}^{u,+}(F,\mathcal{U}(\zeta))$, which are crucial to the proof of our main results, we give some lemmas as follows.

Lemma 3.7. Let $\alpha \in \mathcal{P}(\Omega \times M)$ with diameter smaller than $\epsilon_0/\lambda_0$, and $\eta \in \mathcal{P}^u(\Omega \times M)$, then for any $n \in \mathbb{N}$, we have

$$\alpha^{-1}_n \vee \Theta^n \eta \geq (\Theta^\alpha)^n.$$

Proof. Given $(\omega, x) \in \Omega \times M$, let $y \in (\alpha^{-1}_n \vee \Theta^n \eta)_{\theta^n \omega}(x)$. Thus, we have $f_{\theta^n \omega}^{-n}(y) \in \eta_\omega(f_{\theta^n \omega}^{-n}(x))$, which implies that

$$d_\omega^n(f_{\theta^n \omega}^{-n}(x), f_{\theta^n \omega}^{-n}(y)) < C_0(\omega) \epsilon_0(\omega).$$

Thus, we get

$$d_{\theta^n \omega}^n(f_{\theta^n \omega}^{-(n-1)}(x), f_{\theta^n \omega}^{-(n-1)}(y)) < C_0(\omega) \epsilon_0(\omega) \lambda(\omega) \ll L(\theta \omega).$$

For the choice of $y$, we have $f_{\theta^n \omega}^{-(n-1)}(y) \in \alpha_{\theta^n \omega}(f_{\theta^n \omega}^{-(n-1)}(x))$, which implies that

$$d(f_{\theta^n \omega}^{-(n-1)}(x), f_{\theta^n \omega}^{-(n-1)}(y)) < \frac{\epsilon_0(\theta \omega)}{\lambda_0(\theta \omega)}.$$

As $\epsilon_0$ is small enough, we have

$$d_{\theta^n \omega}^n(f_{\theta^n \omega}^{-(n-1)}(x), f_{\theta^n \omega}^{-(n-1)}(y)) < \frac{C_0(\theta \omega) \epsilon_0(\theta \omega)}{\lambda_0(\theta \omega)}.$$

Then, we can obtain

$$d_{\theta^n-1 \omega}^n(f_{\theta^n-1 \omega}^{-1}(x), f_{\theta^n-1 \omega}^{-1}(y)) < \frac{C_0(\theta^{n-1} \omega) \epsilon_0(\theta^{n-1} \omega)}{\lambda_0(\theta^{n-1} \omega)}$$

by induction. Recall that $\mathcal{F}$ is uniformly expanding on $W^u$, so we have

$$d_{\theta^n \omega}^n(x, y) < C_0(\theta^{n-1} \omega) \epsilon_0(\theta^{n-1} \omega) < \epsilon_0(\theta^n \omega).$$
which implies that \( y \in W^u(\theta^n\omega, x, \bar{\epsilon}_0(\theta^n\omega)) \). Noticing that \( y \in (\Theta^n\alpha)_{\bar{\theta}^n}\omega(x) \), we have \( y \in (\Theta^n\alpha)^{u\bar{\theta}^n}\omega(x) \), hence we have

\[
(\alpha_n^{-1} \lor (\Theta^n\eta))_{\bar{\theta}^n}\omega \geq (\Theta\alpha)^{u\bar{\theta}^n}\omega.
\]

Let \( \omega = \theta^{-n}\omega \), then we have

\[
(\alpha_n^{-1} \lor (\Theta^n\eta))_{\omega} \geq (\Theta\alpha)^u_{\omega}.
\]

Because of the arbitrariness of \( \omega \), we obtain the result we need. \( \square \)

**Lemma 3.8.** For any \( \eta \in \mathcal{P}^u(\Omega \times M) \), \( \zeta \in \mathcal{P}^u(\Omega \times M) \cup \mathcal{Q}^u(\Omega \times M) \), \( H_\mu(\eta|\zeta) \) is finite.

**Proof.** Let \( \beta \in \mathcal{P}(\Omega \times M) \) such that \( \eta = \beta^n \), then

\[
H_\mu(\eta|\zeta) = -\int_{\Omega \times M} \log \mu^\zeta(\omega, x) \mu(\omega, x) d\mu(\omega, x)
\]

\[
= -\int_{\Omega \times M} \log \mu^\zeta(\omega, x) (\beta(\omega, x) \cap W^u(\omega, x, \bar{\epsilon}_0)) d\mu(\omega, x)
\]

\[
= -\int_{\Omega \times M} \log \mu^\zeta(\omega, x) (\beta(\omega, x)) d\mu(\omega, x)
\]

\[
= H_\mu(\beta|\zeta) \leq H_\mu(\beta) < \infty,
\]

where the last inequality is due to the definition of fiberwise finite partition. \( \square \)

**Lemma 3.9.** \( h^u_\mu(\mathcal{F}, \alpha|\eta) := \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) \) exists for any \( \eta \in \mathcal{P}^u(\Omega \times M) \) and any \( \alpha \in \mathcal{P}(\Omega \times M) \) with \( \text{diam}(\alpha) < \epsilon_0/\lambda_0 \).

**Proof.** Let \( \beta \in \mathcal{P} \) such that \( \eta = \beta^n \). Then we have

\[
H_\mu(\alpha_0^{n+m-1}|\eta) = H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\Theta^{-n} \alpha_0^{m-1}|\alpha_0^{n-1} \lor \eta)
\]

\[
= H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\alpha_0^{m-1}|\alpha_0^{-1} \lor \Theta^n\eta)
\]

\[
\leq H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\alpha_0^{m-1}|(\Theta\alpha)^u)
\]

\[
\leq H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\alpha_0^{m-1}|\eta) + H_\mu(\eta|(\Theta\alpha)^u),
\]

where in the third inequality, Lemma 3.7 is used. The diameter of \( (\Theta\alpha)_0^u(x) \) with respect to \( d^u_\alpha \) is no more than \( C_0(\theta^{-1}\omega)\epsilon_0(\theta^{-1}\omega) \), which implies that \( (\Theta\alpha)_0^u(x) \subset W^u(\omega, x, \bar{\epsilon}_0(\omega)) \). Hence by Lemma 3.8, we have

\[
H_\mu(\eta|(\Theta\alpha)^u) \leq H_\mu(\beta).
\]

Then we have

\[
H_\mu(\alpha_0^{n+m-1}|\eta) \leq H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\alpha_0^{m-1}|\eta) + H_\mu(\beta),
\]

which means that the sequence \( \{H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\beta)\} \) is a subadditive sequence. So we have

\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) = \lim_{n \to \infty} \frac{1}{n} (H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\beta)) = \inf_{n \in \mathbb{N}} \frac{1}{n} (H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\beta)).
\]

\( \square \)

**Lemma 3.10.** \( h^u_\mu(\mathcal{F}) = h^u_\mu(\mathcal{F}|\eta) = h^u_\mu(\mathcal{F}, \alpha|\eta) \) for any \( \eta \in \mathcal{P}^u(\Omega \times M) \) and \( \alpha \in \mathcal{P}(\Omega \times M) \) with \( \text{diam}(\alpha) < \epsilon_0/\lambda_0 \).
Proof. First we show that \( \alpha^\infty \cap \eta = \varepsilon \).

Fix \( \omega \in \Omega \). If \( x \neq y \) and \( y \in (\alpha^\infty \cap \eta)(\omega, x) \), then we have \( y \in \eta(\omega, x) \) and \( f^j_\omega(y) \in \alpha_{\theta^\omega}(f^j_\omega(x)) \) for any \( j \in \mathbb{N} \). Let \( k \) be the first number such that \( d^{\mu}_{\theta^k \omega}(f^k_\omega(x), f^k_\omega(y)) > C_0(\theta^k \omega)\varepsilon_0(\theta^k \omega) \). We have \( d^{\mu}_{\theta^k \omega}(f^k_\omega(x), f^k_\omega(y)) \leq \lambda_0(\theta^{k-1} \omega)d^{\mu}_{\theta^{k-1} \omega}(f^{k-1}_{\omega}(x), f^{k-1}_{\omega}(y)) \)
\[ \leq C_0(\theta^{k-1} \omega)\lambda_0(\theta^{k-1} \omega)\varepsilon_0(\theta^{k-1} \omega) \]
by the uniform expansion of \( W^u \). Meanwhile, \( f^k_\omega(y) \in \alpha_{\theta^k \omega}(f^k_\omega(x)) \). A contradiction is obtained.

Pick up \( \beta \in \mathcal{P}^u(\Omega \times \mathbb{M}) \). Since \( H_{\mu}(\beta|\alpha^\infty \cap \eta) = 0 \), for any \( \rho > 0 \), we can choose \( k \in \mathbb{N} \) such that \( H_{\mu}(\beta|\alpha^\infty \cap \eta) < \rho \). Then we have \( H_{\mu}(\beta|\alpha^\infty \cap \eta) \leq H_{\mu}(\beta|\alpha^\infty \cap \eta) + H_{\mu}(\beta|\alpha^\infty \cap \eta) \)
\[ \leq nH_{\mu}(\beta|\alpha^\infty \cap \eta) + H_{\mu}(\beta|\alpha^\infty \cap \eta) \]
\[ \leq n\rho + H_{\mu}(\beta|\alpha^\infty \cap \eta) \]

On the other hand, we have
\[ H_{\mu}((\alpha^\infty \cap \eta)^n \cap \eta) = H_{\mu}(\alpha^{n+k-2} \cap \Theta^{-(n-1)} \eta \cap \eta) \]
\[ \leq H_{\mu}(\alpha^{n+k-2} \cap \eta) + H_{\mu}(\Theta^{-(n-1)} \eta \cap \alpha^{n+k-2} \cap \eta) \]
\[ \leq H_{\mu}(\alpha^{n+k-2} \cap \eta) + H_{\mu}(\eta \cap \Theta^{-(n-1)} \eta) \]
\[ \leq H_{\mu}(\alpha^{n+k-2} \cap \eta) + H_{\mu}(\eta) \]

where in the last inequality, Lemma 3.7 is applied. Then by Lemma 3.9 we have
\[ h_{\mu}^u(\mathcal{F}, \beta|\eta) = \lim_{n \to \infty} \frac{1}{n} H_{\mu}(\beta^{n-1} \cap \eta) \]
\[ \leq \rho + \lim_{n \to \infty} \frac{1}{n} H_{\mu}(\alpha^{n+k-2} \cap \eta) + \lim_{n \to \infty} \frac{1}{n} H_{\mu}(\eta) \]
\[ = \rho + h_{\mu}^u(\mathcal{F}, \alpha|\eta) \]

Since \( \rho > 0 \) is arbitrary, we have
\[ h_{\mu}^u(\mathcal{F}, \beta|\eta) \leq h_{\mu}^u(\mathcal{F}, \alpha|\eta) \]
then by the arbitrariness of \( \beta \), we complete the proof.

The following lemma gives the relationship between \( h_{\mu}^u(\mathcal{F}, \beta|\eta) \) and \( h_{\mu}^u(\mathcal{F}, \beta|\xi) \), whose proof is similar to that in [15], so we omit its proof.

**Lemma 3.11.** Let \( \alpha \in \mathcal{P}(\Omega \times \mathbb{M}) \), \( \eta \in \mathcal{P}^u(\Omega \times \mathbb{M}) \) and \( \xi \in \mathcal{Q}^u(\Omega \times \mathbb{M}) \). Then for \( \mu \)-a.e. \((\omega, x) \in \Omega \times \mathbb{M} \), we have
\[ \liminf_{n \to \infty} \frac{1}{n} I_{\mu}(\alpha^{n-1} \cap \xi)(\omega, x) = \liminf_{n \to \infty} \frac{1}{n} I_{\mu}(\alpha^{n-1} \cap \eta)(\omega, x) \]
and
\[ \limsup_{n \to \infty} \frac{1}{n} I_{\mu}(\alpha^{n-1} \cap \xi)(\omega, x) = \limsup_{n \to \infty} \frac{1}{n} I_{\mu}(\alpha^{n-1} \cap \eta)(\omega, x) \]

We also need the following theorem from [14].

**Theorem 3.12** (Theorem B in [14]). Suppose \( \mu \) is an ergodic measure of \( \mathcal{F} \), and \( \eta \in \mathcal{P}^u(\Omega \times \mathbb{M}) \). Then for any measurable partition \( \alpha \) of \( \Omega \times \mathbb{M} \) with \( H_{\mu}(\alpha|\eta) < \infty \), we have
\[ \lim_{n \to \infty} \frac{1}{n} I_{\mu}(\alpha^{n-1} \cap \eta)(\omega, x) = h_{\mu}^u(\mathcal{F}, \alpha|\eta) \]
Now we give the following proposition, which plays an important role in this paper.

**Proposition 3.** For any \( \zeta \in \mathcal{P}^u(\Omega \times M) \cup \mathcal{Q}^u(\Omega \times M) \), and \( \alpha \in \mathcal{P}(\Omega \times M) \) with \( \text{diam}(\alpha) < \epsilon_0/\lambda_0 \),

\[
    h^u_\mu(\mathcal{F}) = h^u_\mu(\mathcal{F}|\zeta) = h^u_\mu(\mathcal{F}, \alpha|\zeta) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\zeta).
\]

**Proof.** We prove Proposition 3 in two cases.

**Case 1** For \( \zeta \in \mathcal{P}^u(\Omega \times M) \).

This is the results of Lemma 3.9 and 3.10.

**Case 2** For \( \zeta \in \mathcal{Q}^u(\Omega \times M) \).

Let \( \eta \in \mathcal{P}^u(\Omega \times M) \), and denote \( \zeta \) by \( \xi \). By Lemma 3.3 and Lemma 3.4, we know that \( \mu \to h_\mu(\mathcal{F}, \alpha|\eta) \) is affine and upper semi-continuous, for any \( \mu \in \mathcal{M}_p(\mathcal{F}) \), by the Ergodic Decomposition Theorem, let \( \mu = \int_{\mathcal{M}_p(\mathcal{F})} \nu d\tau(\nu) \), then by Theorem 3.12, we have

\[
    h^u_\mu(\mathcal{F}, \alpha|\eta) = \int_{\mathcal{M}_p(\mathcal{F})} h^u_\nu(\mathcal{F}, \alpha|\eta) d\tau(\nu) = \int_{\Omega \times M} \lim_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(\omega, x) d\mu.
\]

Then by Fatou’s Lemma and Lemma 3.11, we have

\[
    h^u_\mu(\mathcal{F}, \alpha|\xi) \geq \liminf_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\xi)
\]

\[
    \geq \int_{\Omega \times M} \liminf_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(\omega, x) d\mu
\]

\[
    = \int_{\Omega \times M} \liminf_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(\omega, x) d\mu
\]

\[
    = h^u_\mu(\mathcal{F}, \alpha|\eta).
\]

On the other hand, we have

\[
    H_\mu(\alpha_0^{n-1}|\xi) \leq H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\eta|\xi),
\]

by Lemma 3.8 we know that \( H_\mu(\eta|\xi) < \infty \), thus we have

\[
    h^u_\mu(\mathcal{F}, \alpha|\xi) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\xi) \leq \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) = h^u_\mu(\mathcal{F}, \alpha|\eta).
\]

Combining Lemma 3.5 and Lemma 3.10, we complete the proof of 3.

\[\square\]

The following corollary can be obtained easily from Proposition 3.

**Corollary 1.** If \( \mathcal{U} \in \mathcal{C}_{\Omega \times M} \), has diameter smaller than \( \epsilon_0/\lambda_0 \), then

\[
    h^u_\mu(\mathcal{F}, \mathcal{U}|\zeta) = h^u_\mu(\mathcal{F}|\zeta) = h^u_\mu(\mathcal{F})
\]

for any \( \zeta \in \mathcal{P}^u(\Omega \times M) \cup \mathcal{Q}^u(\Omega \times M) \).

Now we begin to define another notation of local unstable measure-theoretic entropy \( h^u_\mu(f, \mathcal{U}|\zeta) \).

**Definition 3.13.** For \( \zeta \in \mathcal{P}^u(\Omega \times M) \cup \mathcal{Q}^u(\Omega \times M) \), define

\[
    h^u_\mu(\mathcal{F}, \mathcal{U}|\zeta) := \limsup_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\zeta),
\]

where \( H_\mu(\mathcal{U}|\zeta) = \inf_{\alpha \in \mathcal{P}(\Omega \times M), \alpha \geq \mathcal{U}} H_\mu(\alpha|\zeta) \).
The following proposition gives the relation between \( h_\mu^u(\mathcal{F}, \mathcal{U}|\zeta) \) and \( h_\mu^{u,+}(\mathcal{F}, \mathcal{U}|\zeta) \), whose proof is similar to that of Proposition 3.16 in [15] and omitted here.

**Proposition 4.** \( h_\mu(\mathcal{F}, \mathcal{U}|\zeta) \leq h_\mu^{u,+}(\mathcal{F}, \mathcal{U}|\zeta) \) for any \( \zeta \in \mathcal{P}^u(\Omega \times M) \cup \mathcal{Q}^u(\Omega \times M) \).

In the definition of \( h_\mu^u(\mathcal{F}, \mathcal{U}|\zeta) \), we use “\( \limsup \)”; in fact, we can show that for any \( \eta \in \mathcal{P}^u(\Omega \times M) \), it can be replaced by “\( \lim \)”. To prove this, we need some lemmas.

**Lemma 3.14.** Let \( \alpha \in \mathcal{P}(\Omega \times M) \) with \( \alpha \geq \mathcal{U}_{0}^{n-1} \) and \( \eta \in \mathcal{P}^u(\Omega \times M) \). Then \( \Theta^n_\alpha \lor \Theta^n_\eta \geq (\Theta^n_\alpha)^u \) for any \( n \in \mathbb{N} \).

**Proof.** Given \( (\omega, x) \in \Omega \times M \), let \( y \in (\Theta^n_\alpha \lor \Theta^n_\eta)_{\bar{\theta}^n_\omega}(x) \). Thus, we have \( f_{\bar{\theta}^n_\omega}(y) \in \eta_{\omega}(f_{\bar{\theta}^n_\omega}(x)) \) and \( f_{\bar{\theta}^n_\omega}(y) \in U_j(\theta^{n-j}_\omega)(f_{\bar{\theta}^n_\omega}(x)) \) for \( 1 \leq j \leq n \), where \( U_j \) is the element of \( \mathcal{U} \) that contains \( (\theta^{n-j}_\omega, f_{\bar{\theta}^n_\omega}(x)) \), which implies that
\[
d^n_\omega(f_{\bar{\theta}^n_\omega}(x), f_{\bar{\theta}^n_\omega}(y)) < C_0(\omega)\epsilon_0(\omega).
\]
Thus, we get
\[
d^n_{\theta^j_\omega}(f_{\bar{\theta}^n_\omega}^{(n-1)}(x), f_{\bar{\theta}^n_\omega}^{(n-1)}(y)) < C_0(\omega)\epsilon_0(\omega) \lambda_0(\omega) \ll L(\theta_\omega).
\]
For the choice of \( y \), we have \( f_{\bar{\theta}^n_\omega}^{(n-1)}(y) \in U_{n-1}(\theta_\omega)(f_{\bar{\theta}^n_\omega}^{(n-1)}(x)) \), which implies that
\[
d^n_{\theta^j_\omega}(f_{\bar{\theta}^n_\omega}^{(n-1)}(x), f_{\bar{\theta}^n_\omega}^{(n-1)}(y)) < \frac{\epsilon_0(\theta_\omega)}{\lambda_0(\theta_\omega)}.
\]
As \( \epsilon_0 \) is small enough, we have
\[
d^n_{\theta^j_\omega}(f_{\bar{\theta}^n_\omega}^{(n-1)}(x), f_{\bar{\theta}^n_\omega}^{(n-1)}(y)) < \frac{C_0(\theta_\omega)\epsilon_0(\theta_\omega)}{\lambda_0(\theta_\omega)}.
\]
Then, we can obtain
\[
d^n_{\theta^{n-1}_\omega}(f_{\bar{\theta}^n_\omega}^{(n-1)}(x), f_{\bar{\theta}^n_\omega}^{(n-1)}(y)) < \frac{C_0(\theta^{n-1}_\omega)\epsilon_0(\theta^{n-1}_\omega)}{\lambda_0(\theta^{n-1}_\omega)}
\]
by induction. By uniform expansion of \( W^u \), we have
\[
d^n_{\theta^{n}_\omega}(x, y) < C_0(\theta^{n-1}_\omega)\epsilon_0(\theta^{n-1}_\omega) < \bar{\epsilon}_0(\theta^{n}_\omega),
\]
which implies \( y \in W^u(\theta^{n}_\omega, x, \bar{\epsilon}_0(\theta^{n}_\omega)) \). Noticing that \( y \in (\Theta^n_\alpha)^u(\omega, x) \), we have
\[
(\Theta^n_\alpha \lor (\Theta^n_\eta)_{\theta^{n}_\omega} \geq (\Theta^n_\alpha)^u_{\bar{\theta}^n_\omega}.
\]
Let \( \omega = \theta^{-n}_\omega \), then we have
\[
(\Theta^n_\alpha \lor (\Theta^n_\eta)_{\omega} \geq (\Theta^n_\alpha)^u_{\omega}.
\]
Because of the arbitrariness of \( \omega \), we obtain the result we need.

**Lemma 3.15.** For any \( \eta \in \mathcal{P}^u(\Omega \times M) \), \( h_\mu^u(\mathcal{F}, \mathcal{U}|\eta) := \lim_{n \to \infty} \frac{1}{n} H_\mu(U_0^{n-1}|\eta) \) exists.
Let $\beta \in \mathcal{P}(\Omega \times M)$ with $\eta = \beta^u$. Choose any $\alpha, \gamma \in \mathcal{P}(\Omega \times M)$ such that $\alpha \geq U_0^{n-1}$ and $\gamma \geq U_0^{m-1}$, then we have

$$H_\mu(U_0^{m+n-1}|\eta) \leq H_\mu(\alpha \vee \Theta^{-n}\gamma|\eta)$$

$$\leq H_\mu(\alpha|\eta) + H_\mu(\Theta^{-n}\gamma|\alpha \vee \eta)$$

$$= H_\mu(\alpha|\eta) + H_\mu(\gamma|\Theta^\alpha \vee \Theta^n\eta)$$

$$\leq H_\mu(\alpha|\eta) + H_\mu(\gamma|(\Theta^\alpha)^\eta)$$

$$\leq H_\mu(\alpha|\eta) + H_\mu(\gamma|\eta) + H_\mu(\beta).$$

In the third and last inequality Lemma 3.14 and Lemma 3.8 are applied respectively. Because of the arbitrariness of $\alpha$ and $\gamma$, we have

$$H_\mu(U_0^{m+n-1}|\eta) \leq H_\mu(U_0^{n-1}|\eta) + H_\mu(U_0^{m-1}|\eta) + H_\mu(\beta).$$

As in the proof of Lemma 3.9, we have showed that $\{H_\mu(U_0^{n-1}|\eta) + H_\mu(\beta)\}$ is a subadditive sequence, which implies what we need. \hfill \square

**Lemma 3.16.** $h_\mu^u(\mathcal{F}, U|\eta)$ is independent of $\eta \in \mathcal{P}^u(\Omega \times M)$.

**Proof.** For the proof, we refer the reader to Proposition 3.19 in [15]. \hfill \square

The following lemmas are useful for the proof of Theorem A.

**Lemma 3.17.** Fix $N \in \mathbb{N}$, for any $k \geq 1$ and $\alpha \geq U_0^{N-1}$ we have

$$\Theta^{N_k}\alpha \vee \cdots \vee \Theta^N\alpha \vee \Theta^{N_k}\eta \geq (\Theta^N\alpha)^u.$$

**Proof.** The proof is similar to those of Lemma 3.7 and Lemma 3.17, and we refer the reader to the proof of Lemma 3.20 in [15]. \hfill \square

**Lemma 3.18.** For $\eta \in \mathcal{P}^u(\Omega \times M)$, we have

1. $h_\mu^u(\mathcal{F}, U|\eta) = \frac{1}{n} h_\mu^u(\mathcal{F}^n, U_0^{n-1}|\eta)$ for any $n \in \mathbb{N}$,
2. $h_\mu^u(\mathcal{F}, U|\eta) = \lim_{n \to \infty} \frac{1}{n} h_\mu^u(\mathcal{F}^n, U_0^{n-1}|\eta)$.

where $\mathcal{F}^n$ is induced by $\Theta^n$.

**Proof.** In the proof, Lemma 3.17, Lemma 3.8 and Proposition 4 are used, which is completely parallel to that of Lemma 3.21 in [15], so we omit it here. \hfill \square

**Lemma 3.19.** $h_\mu^{u,+}(\mathcal{F}^n, U_0^{n-1}|\eta) = nh_\mu^u(\mathcal{F}|\eta)$ for any $\eta \in \mathcal{P}^u(\Omega \times M)$ and $n \in \mathbb{N}$.

**Proof.** Choose arbitrary $\alpha \geq U_0^{n-1}$, as in Lemma 3.10, we can show that

$$\eta \vee \bigvee_{i=0}^{\infty} \Theta^{-ni}\alpha = \varepsilon.$$

Then follow the line of proof of Lemma 3.22 in [15], for any $\beta \in \mathcal{P}(\Omega \times M)$ and $\rho > 0$ we can show that

$$h_\mu^u(\mathcal{F}^n, \beta|\eta) \leq \rho + h_\mu^u(\mathcal{F}^n, \alpha|\eta).$$

Then by the arbitrariness of $\beta$, $\rho$ and $\alpha$, we have

$$nh_\mu^u(\mathcal{F}|\eta) = h_\mu^u(\mathcal{F}^n|\eta) \leq h_\mu^{u,+}(\mathcal{F}^n, U_0^{n-1}|\eta).$$

And it is clear that $nh_\mu^u(\mathcal{F}|\eta) \geq h_\mu^{u,+}(\mathcal{F}^n, U_0^{n-1}|\eta)$. \hfill \square
4. Local unstable topological entropy and pressure. In this section, we give the definition of local unstable topological entropy of $F$ with respect to a Borel cover $U \in \mathcal{C}_{\Omega \times M}$.

Let $K \subset \Omega \times M$. For any $U \in \mathcal{C}_{\Omega \times M}$, denote $\min\{\text{the cardinality of } V : V \subset U, \bigcup_{V \in V} V \supseteq K\}$ by $N(K, U)$, and denote $\log N(K, U)$ by $H(U|K)$.

**Definition 4.1.** For any $U \in \mathcal{C}_{\Omega \times M}$, we define

$$h_{\text{top}}^u(F, U) := \lim_{\delta \to 0} h_{\text{top}}^u(F, U, \delta),$$

where

$$h_{\text{top}}^u(F, U, \delta) := \int_{\Omega} \sup_{x \in M} h_{\text{top}}(F, U|W^u(\omega, x, \delta)) d\mathcal{P},$$

and

$$h_{\text{top}}(F, U|W^u(\omega, x, \delta)) := \lim_{n \to \infty} \frac{1}{n} H(U_0^{n-1}|W^u(\omega, x, \delta)),$$

where we identify $W^u(\omega, x, \delta)$ with $\{\omega\} \times W^u(\omega, x, \delta)$.

**Lemma 4.2.**

$$h_{\text{top}}^u(F, U) = h_{\text{top}}^u(F, U, \delta), \text{ for any } \delta > 0.$$

**Proof.** It is clear that $h_{\text{top}}^u(F, U) \leq h_{\text{top}}^u(F, U, \delta)$ for any $\delta > 0$, since the function $\delta \mapsto h_{\text{top}}^u(F, U, \delta)$ is increasing.

Now, let $\delta > 0$. For any $\rho > 0$ and each $\omega \in \Omega$, there exists $y_\omega$ such that

$$\sup_{x \in M} h_{\text{top}}(F, U|W^u(\omega, x, \delta)) \leq h_{\text{top}}(F, U|W^u(\omega, y_\omega, \delta)) + \frac{\rho}{3}.$$

Then choose $0 < \delta_1 < \delta$ small enough such that

$$h_{\text{top}}^u(F, U) + \frac{\rho}{3} \geq \int_{\Omega} \sup_{x \in M} h_{\text{top}}(F, U|W^u(\omega, x, \delta_1)) d\mathcal{P} = h_{\text{top}}^u(F, U, \delta_1). \quad (2)$$

There exists a positive number $N = N(\omega)$ which depends on $\delta, \delta_1$ and the Riemannian structure on $W^u(\omega, y_\omega, \delta)$ such that

$$W^u(\omega, y_\omega, \delta) \subset \bigcup_{j=1}^N W^u(\omega, y_j, \delta_1)$$

for some $y_j \in W^u(\omega, y_\omega, \delta)$, $j = 1, 2, \ldots, N$. Then we have

$$\sup_{x \in M} h_{\text{top}}(F, U|W^u(\omega, x, \delta)) \leq h_{\text{top}}(F, U|W^u(\omega, y_\omega, \delta)) + \frac{\rho}{3}$$

$$= \limsup_{n \to \infty} \frac{1}{n} H(U_0^{n-1}|W^u(\omega, y_\omega, \delta)) + \frac{\rho}{3}$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \log \left( \sum_{j=1}^N N(W^u(\omega, y_j, \delta_1), U_0^{n-1}) \right) + \frac{\rho}{3}$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \log N(W^u(\omega, y_l, \delta_1), U_0^{n-1}) + \frac{\rho}{3} \text{ for some } 1 \leq l \leq N$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log N(W^u(\omega, y_l, \delta_1), U_0^{n-1}) + \frac{\rho}{3}$$

$$\leq \sup_{x \in M} h_{\text{top}}(F, U|W^u(\omega, x, \delta_1)) + \frac{\rho}{3}.$$
Integrating both sides of the above inequality, we get
\[ h_{\text{top}}^u(F, \mathcal{U}, \delta) \leq h_{\text{top}}^u(F, \mathcal{U}, \delta_1) + \frac{\rho}{3}. \]
Thus, by (2) we have
\[ h_{\text{top}}^u(F, \mathcal{U}, \delta) \leq h_{\text{top}}^u(F, \mathcal{U}) + \frac{2\rho}{3}. \]
Since \( \rho \) is arbitrary, we have
\[ h_{\text{top}}^u(F, \mathcal{U}, \delta) \leq h_{\text{top}}^u(F, \mathcal{U}), \]
which completes the proof of Lemma 4.2.

As a generalization of local unstable topological entropy, we can give the definition of local unstable pressure of \( F \).

**Definition 4.3.** Let \( \phi \in L^1(\Omega, C(M)) \). Define
\[ P^u(F, \phi, \mathcal{U}|W^u(\omega, x, \delta)) = \inf_{V \in P(\Omega \times M)} \{ \sum_{V \in V} \sup_{y \in W^u(\omega, x, \delta)} \exp(S_n \phi)(y) : V \geq U^n \} \].
Then \( P^u(F, \phi, \mathcal{U}|W^u(\omega, x, \delta)) \) is defined as
\[ P^u(F, \phi, \mathcal{U}|W^u(\omega, x, \delta)) = \limsup_{n \to \infty} \frac{1}{n} \log P^u(F, \phi, \mathcal{U}, \delta, n, \mathcal{U}). \]
Next, define
\[ P^u(F, \phi, \mathcal{U}, \delta) = \sup_{x \in M} P^u(F, \phi, \mathcal{U}|W^u(\omega, x, \delta)) \]
and
\[ P^u(F, \phi, \mathcal{U}, \delta) = \int_{\Omega} P^u(F, \phi, \mathcal{U}, \delta) d\mathcal{P}(\omega). \]
Then the local unstable pressure of \( F \) with respect to \( \phi \) is defined as
\[ P^u(F, \phi, \mathcal{U}) = \lim_{\delta \to 0} P^u(F, \phi, \mathcal{U}, \delta). \]

Now we give the relation between local unstable measure-theoretic entropy and local unstable topological entropy.

**Proposition 5.** Let \( \mu \in \mathcal{M}_p(F) \) and \( \mathcal{U} \in C_{\Omega \times M} \). Then for any \( \zeta \in P^u(\Omega \times M) \cup P^u(\Omega \times M) \), we have
\[ h^u_\mu(F, \mathcal{U}|\zeta) \leq h_{\text{top}}^u(F, \mathcal{U}). \]

**Proof.** Fix \( \delta > 0 \) such that \( \zeta(\omega, x) \subseteq W^u(\omega, x, \delta) \). In the next, for any \( V = \{ V_j \}_{j=1}^k \in C_{\Omega \times M} \), we will construct a fiberwise finite partition \( \alpha \in P(\Omega \times M) \) with \( \alpha \geq V \) such that
\[ H_\mu(\alpha|\zeta) \leq \int_{\Omega \times M} \log N(W^u(\omega, x, \delta), \mathcal{V}) d\mu(\omega, x). \]
Fix \( \omega \in \Omega \), for any \( y \in M \), we can find a subset \( I_\omega(y) \) of \( \{1, 2, \cdots, k\} \) with minimal cardinality no more than \( N(W^u(\omega, y, \delta), \mathcal{V}) \) such that
\[ \bigcup_{j \in I_\omega(y)} V_j \supseteq \zeta_\omega(y). \]
Then follow the way in [15] we can construct a partition \( \tilde{\alpha}_\omega = \{ A_{\omega,j} \}_{j=1}^{N(\omega)} \) of \( M \) with \( \tilde{\alpha}_\omega \geq \mathcal{V}_\omega \) such that
\[
H_{\mu_\omega}(\tilde{\alpha}_\omega|\xi_\omega) \leq \int_M \log N(\overline{W}^u(\omega,x,\delta),\mathcal{V})\mu_\omega(x). \tag{3}
\]
Then we can construct a fiberwise partition \( \alpha \) of \( \Omega \times M \) such that \( \alpha_\omega = \tilde{\alpha}_\omega \), it is clear that \( \alpha \geq \mathcal{V} \). Then by (3) we have
\[
H_{\mu}(\alpha|\xi) = \int_\Omega H_{\mu_\omega}(\alpha|\xi_\omega)\mu(x) \leq \int_\Omega \int_M \log N(\overline{W}^u(\omega,x,\delta),\mathcal{V})\mu_\omega(x)\mu(x) = \int_{\Omega \times M} \log N(\overline{W}^u(\omega,x,\delta),\mathcal{V})d\mu(x).
\]
Hence \( H_{\mu}(\mathcal{V}|\xi) \leq \int_{\Omega \times M} \log N(\overline{W}^u(\omega,x,\delta),\mathcal{V})d\mu(x) \).

By Fatou’s Lemma, we have
\[
h^u_{\mu}(F,\mathcal{U}|\xi) = \lim sup_{n \to \infty} \frac{1}{n} H_{\mu}(U^{n-1}|\xi) \leq \lim sup_{n \to \infty} \int_{\Omega \times M} \frac{1}{n} \log N(\overline{W}^u(\omega,x,\delta),U^{n-1}_0)d\mu(\omega,x) \\
\leq \int_{\Omega \times M} \lim sup_{n \to \infty} \frac{1}{n} \log N(\overline{W}^u(\omega,x,\delta),U^{n-1}_0)d\mu(\omega,x) \\
= \int_{\Omega \times M} h_{\text{top}}(F,\mathcal{U}|\overline{W}^u(\omega,x,\delta))d\mu(\omega,x) \\
\leq \int_{\Omega \times M} \max_{x \in M} h_{\text{top}}(F,\mathcal{U}|\overline{W}^u(\omega,x,\delta))d\mu(\omega,x) \\
= \int_{\Omega \times M} \max_{x \in M} h^u_{\mu}(F,\mathcal{U}|\overline{W}^u(\omega,x,\delta))d\mu(\omega,x) \\
= h^u_{\text{top}}(F,\mathcal{U},\delta) \\
= h^u_{\text{top}}(F,\mathcal{U}),
\]
which completes the proof of Proposition 5.

5. Unstable topological conditional entropy and unstable tail entropy. In this section, we give the definitions of unstable topological conditional entropy and unstable tail entropy, which are useful in the proof of Theorem A, for the case when \( \xi \in Q^u(\Omega \times M) \). For \( Y \subset \Omega \times M \) and \( \omega \in \Omega \), denote \( Y_\omega = \{ x \in M : (\omega,x) \in Y \} \).

**Definition 5.1.** For \( Y \in \Omega \times M \) and any two covers \( \mathcal{U}, \mathcal{V} \in \mathcal{C}_{\Omega \times M} \), define
\[
N^u_\omega(Y,\mathcal{U}) = \sup_{y \in Y_\omega} N(Y_\omega \cap \overline{W}^u(\omega,y,\delta),\mathcal{U}),
\]
and
\[
H^u(Y,\mathcal{U}) = \int_\Omega \log N^u_\omega(Y,\mathcal{U})\mu(x) \mu(x) = \max_{V \in \mathcal{V}} N^u_\omega(V_\omega,\mathcal{U}),
\]
and if \( Y = \emptyset \), we set \( H^u(\emptyset,\mathcal{U}) = 0 \). Define
\[
N^u_\omega(\mathcal{U}|\mathcal{V}) = \max_{V \in \mathcal{V}} N^u_\omega(V_\omega,\mathcal{U}),
\]
and
\[ H^n(\mathcal{U}|V) = \int_{\Omega} \log N^n(\mathcal{U}|V)d\mathbf{P}. \]

The following proposition is a collection of properties for \( H^n \), whose proof is simple.

**Proposition 6.**
1. \( H^n(Y, \mathcal{U}) \leq H^n(Z, V) \) if \( Y \subset Z \) and \( V \supseteq \mathcal{U} \).
2. \( H^n(\mathcal{U}_1|V_1) \leq H^n(\mathcal{U}_2|V_2) \) if \( \mathcal{U}_2 \supseteq \mathcal{U}_1 \) and \( V_1 \supseteq V_2 \).
3. \( H^n(Y, \mathcal{U}) = \int_{\Omega} \sup_{\mathcal{V} \subseteq Y} \log \left( (f^n_\mathcal{U}^{-1}(Y_\omega) \cap (f_\mathcal{V}^{-1}(V_\omega, y, \delta))^n(\mathcal{U}_\omega))d\mathbf{P}. \right) \)
4. \( H^n(\mathcal{U} \vee \mathcal{V}|W) \leq H^n(\mathcal{U}|W) + H^n(\mathcal{V}|W) \).
5. \( H^n(\mathcal{U}_1 \vee \mathcal{V}_1 \vee \mathcal{V}_2) \leq H^n(\mathcal{U}_1|\mathcal{V}_1) + H^n(\mathcal{V}_1|\mathcal{V}_2) \).
6. \( H^n(\mathcal{Y}, \mathcal{U}) \leq H^n(\mathcal{Y}|\mathcal{U}) + H^n(\mathcal{U}) \).
7. \( H^n(\Omega \times M, \mathcal{U}) \leq H^n(\Omega \times M, V) + H^n(\mathcal{U}|V) \).
8. \( H^n(\mathcal{U}|V) \leq H^n(\mathcal{U}|\mathcal{V}) + H^n(\mathcal{V}) \).

**Lemma 5.2.** If \( \text{diam}(V) < \epsilon_0 \ll \delta \), then \( \lim_{n \to \infty} \frac{1}{n} H^n(U_0^{-1}|V_0^{-1}) \) exists.

**Proof.** Firstly, we show that \( V_{\theta^n \omega} \cap f_{\omega}^n \overline{W}^u(\omega, y, \delta) = V_{\theta^n \omega} \cap \overline{W}^u(\theta^n \omega, f_{\omega}^n y, \delta) \) for any \( V \in \mathcal{Y}_m^{-1} \) and \( y \in f_{\omega}^n V_{\theta^n \omega} \). Because \( V \in \mathcal{Y}_m^{-1} \), we know that if \( z \in V_{\theta^n \omega} \cap f_{\omega}^n \overline{W}^u(\omega, y, \delta) \), then \( d^n_{\theta^n \omega}(z, f_{\omega}^n y) \leq C_0(\theta^n \omega) \epsilon_0(\theta^n \omega) \) for \( 0 \leq j \leq n \), so we have \( d^n_{\theta^n \omega}(z, f_{\omega}^n y) \leq C_0(\theta^n \omega) \epsilon_0(\theta^n \omega) \) which implies that \( z \in V_{\theta^n \omega} \cap \overline{W}^u(\theta^n \omega, f_{\omega}^n y, \delta) \). Then by Proposition 6, we have
\[
H^n(U_0^{-1}|V_0^{-1}) - H^n(U_0^{-1}|V_0^{-1}) \leq H^n(U_0^{-1}|V_0^{-1}) + H^n(\Theta^{-n} U_0^{-1}|\Theta^{-n} V_0^{-1})
\]
\[
= \int_{\Omega} \log \max_{V \in \mathcal{Y}_m^{-1}} \sup_{y \in f_{\omega}^n V_{\theta^n \omega}} N(f_{\omega}^{-n} V_{\theta^n \omega} \cap (\overline{W}^u(\omega, y, \delta)), \Theta^{-n} U_0^{-1})d\mathbf{P}
\]
\[
= H^n(U_0^{-1}|V_0^{-1}) \leq H^n(U_0^{-1}|V_0^{-1}) + H^n(U_0^{-1}|V_0^{-1})
\]
which means that \( \{H^n(U_0^{-1}|V_0^{-1})\} \) is subadditive, hence we complete the proof.

Because of Lemma 5.2, we have the following definition.

**Definition 5.3.** The unstable conditional entropy of \( \mathcal{F} \) on the cover \( \mathcal{U} \) with respect to the cover \( V \) is defined as
\[
h^u(\mathcal{F}, \mathcal{U}|V) = \lim_{n \to \infty} \frac{1}{n} H^n(U_0^{-1}|V_0^{-1}).
\]

And define \( h^u(\mathcal{F}, \mathcal{Y}, \mathcal{U}) = \limsup_{n \to \infty} \frac{1}{n} H^n(Y, U_0^{-1}). \)

The unstable conditional entropy with respect to \( V \) is defined as
\[
h^u(\mathcal{F}|V) = \sup_{\mathcal{U} \in \mathcal{C}_0^{-1}} h^u(\mathcal{F}, \mathcal{U}|V).
\]
Proposition 8.

And define $h^u(\mathcal{F}, Y) = \sup_{\mathcal{U} \in C_{\text{top}}^u M} h^u(\mathcal{F}, Y, \mathcal{U})$.

The unstable topological conditional entropy of $\mathcal{F}$ in the sense of Misiurewicz is defined as

$$h^{u\text{top}}(\mathcal{F}) := \inf_{\mathcal{V} \in C_{\text{top}}^u M} h^u(\mathcal{F}, \mathcal{V}).$$

The following proposition is a collection of properties of unstable conditional entropy.

**Proposition 7.**

(i) $h^u(\mathcal{F}, Y, \mathcal{U}) \leq h^u(\mathcal{F}, Z, \mathcal{V})$ if $Y \subset Z$ and $\mathcal{V} \supset \mathcal{U}$.

(ii) $h^u(\mathcal{F}, \mathcal{U}_1 | \mathcal{V}_1) \leq h^u(\mathcal{F}, \mathcal{U}_2 | \mathcal{V}_2)$ if $\mathcal{U}_2 \supset \mathcal{U}_1$ and $\mathcal{V}_1 \supset \mathcal{V}_2$.

(iii) $h^u(\mathcal{F}, \Omega \times M, \mathcal{U}) \leq h^u(\mathcal{F}, \Omega \times M, \mathcal{V}) + h^u(\mathcal{F}, \mathcal{U} | \mathcal{V})$.

(iv) $h^u(\mathcal{F}, \mathcal{U} | \mathcal{V}) \leq h^u(\mathcal{F}, \mathcal{V}) + h^u(\mathcal{F}, \mathcal{U} | \mathcal{V})$.

(v) $h^{u\text{top}}(\mathcal{F}, \mathcal{U}) \leq h^{u\text{top}}(\mathcal{F}, \mathcal{V}) + h^u(\mathcal{F}, \mathcal{U} | \mathcal{V})$.

**Proof.** (i)–(iv) are simple. For (v), by Lebesgue Dominated Convergence Theorem, we have

$$h^u(\mathcal{F}, \Omega \times M, \mathcal{U}) = h^{u\text{top}}(\mathcal{F}, \mathcal{U}),$$

then by Proposition 6, we obtained what we need.

The following proposition is important.

**Proposition 8.**

$$h^u(\mathcal{F}, \Omega \times M) \leq h^u(\mathcal{F}, \Omega \times M, \mathcal{U}) + h^u(\mathcal{F} | \mathcal{U}),$$

and

$$h^{u\text{top}}(\mathcal{F}) \leq h^{u\text{top}}(\mathcal{F}, \mathcal{U}) + h^u(\mathcal{F} | \mathcal{U}).$$

**Proof.** The two inequalities can be obtained by Proposition 7 (3) and (5) respectively.

In the next, we begin to define the unstable tail entropy of $\mathcal{F}$ in the sense of Bowen.

Fix $\delta > 0$, for a $\theta$-tempered variable $\epsilon > 0$, $(\omega, x) \in \Lambda$ and $Y \in \Omega \times M$, a subset $E_n$ of $W^u(\omega, x, \delta)$ is called an $(\omega, n, \epsilon)$-spanning set of $Y \cap W^u(\omega, x, \delta)$ if for any $y_1, y_2 \in E_n$, we have $d^u_n(y_1, y_2) \leq \epsilon$, which means

$$d^u_n(f^j y_1, f^j y_2) \leq \epsilon(\theta^j \omega), \quad \text{for } 0 \leq j \leq n - 1.$$  

Denote the smallest cardinality of an $(\omega, n, \epsilon)$-spanning set of $Y \cap W^u(\omega, x, \delta)$ by $R^u_n(W^u(\omega, x, \delta), \epsilon)$. Then define

$$r^u_n(Y, \omega, \epsilon) = \sup_{y \in Y \omega} R^u_n(W^u(\omega, y, \delta), \epsilon),$$

$$\bar{r}^u(Y, \omega, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r^u_n(Y, \omega, \epsilon),$$

and

$$h^u(\mathcal{F}, \omega, Y) := \lim_{\epsilon \to 0} \bar{r}^u(Y, \omega, \epsilon).$$

For a $\theta$-tempered variable $\epsilon$, denote $\bigcup_{\omega \in \Omega} \bigcap_{n=1}^{\infty} \{\omega\} \times B_n(\omega, x, \epsilon)$ by $\Phi(x, \epsilon)$, where $B_n(\omega, x, \epsilon) = \{y \in M : d^u(x, y) < \epsilon\}$, and $d^u(x, y) < \epsilon$ means $d(f^k_x, f^k_y) < \epsilon(\theta^k \omega)$ for $0 \leq k \leq n - 1$. New we can give the following definition.
Definition 5.4.

\[ h^u(F, \epsilon) = \int_{\Omega} \sup_{x \in M} \tilde{h}^u(F, \omega, \Phi(x, \epsilon)) dP. \]

The following proposition gives the relation between unstable conditional entropy and unstable tail entropy, whose is completely similar to that of Proposition 4.9 in [15], so we omit it here. For \( U \in C_{\Omega \times M}^0 \), define a variable \( \text{Leb}(U) : \Omega \to \mathbb{R} \) such that \( \text{Leb}(U)(\omega) = \text{Leb}(U_\omega) \), where \( \text{Leb}(U_\omega) \) is the Lebesgue number of \( U_\omega \).

Proposition 9. For \( Y \in \Omega \times M \) and \( U, V \in C_{\Omega \times M}^0 \) with \( \text{diam}(U) < \epsilon < \frac{\text{Leb}(V)}{2} \),

\[ N(Y_\omega \cap \overline{W^u(\omega, y, \delta)}, \mathcal{V}_0^{n-1}) \leq r_n^{u}(Y_\omega, \epsilon) \leq N(Y_\omega \cap \overline{W^u(\omega, y, \delta)}, \mathcal{U}_0^{n-1}). \]

In fact, in our setting, both unstable conditional entropy and unstable tail entropy vanish. So we have the following theorem.

Theorem 5.5.

\[ h^u(F) = 0, \]
and

\[ h^u(F, \epsilon) = 0 \]
for any \( \epsilon > 0 \) small enough.

Proof. For \( U \in C_{\Omega \times M}^0 \) with \( \text{diam}(U) \ll \epsilon_0 \), we show that \( h^u(F|U) = 0 \). For any \( \theta \)-tempered variable \( \epsilon > 0 \), choose \( W \in C_{\Omega \times M}^0 \) with \( \text{Leb}(W) = 3\epsilon \). Then by Proposition 9, we have

\[
\begin{align*}
\max_{U \in \mathcal{C}_n^{u-1}} \sup_{y \in U_\omega} \log N(U_\omega \cap \overline{W^u(\omega, y, \delta)}, \mathcal{W}_0^{n-1}) & \leq \max_{U \in \mathcal{C}_n^{u-1}} \log r_n^{u}(U_\omega, \omega, \epsilon) \\
& \leq \log r_n^{u}(B_n(\omega, x, \epsilon_0), \omega, \epsilon). \tag{4}
\end{align*}
\]

Fix \( y \in B_n(\omega, x, \epsilon_0) \). Let \( z \in B_{n}(\omega, x, \epsilon_0) \cap \overline{W^u(\omega, y, \delta)} \), which implies that

\[ d(f_{k}^{z}(z), f_{k}^{y}(y)) \leq 2\epsilon_0(\theta^k \omega) \]

for any \( 0 \leq k \leq n - 1 \). Thus we have

\[ d_{W^u}(f_{k}^{z}(z), f_{k}^{y}(y)) \leq 2C_0(\theta^k \omega)\epsilon_0(\theta^k \omega), \]

for any \( 0 \leq k \leq n - 1 \), which means

\[ B_{n}(\omega, x, \epsilon_0) \cap \overline{W^u(\omega, y, \delta)} \subset B_{n}(\omega, y, 2C_0\epsilon_0). \]

Noticing that

\[ B_{n}(\omega, y, 2C_0\epsilon_0) \subset (f_{n}^{z})^{-1}B_{n}^{u}(\theta^n \omega, f_{n}^{z}(y), 2C_0\epsilon_0), \]

where \( B_{n}^{u}(\omega, x, \rho) \) is the ball in \( W^u(\omega, x) \) with radius \( \rho \), we can find \( z_j, 1 \leq j \leq N \) such that

\[ B_{n}(\theta^n \omega, f_{n}^{z}(y), 2C_0(\theta^n \omega)\epsilon_0(\theta^n \omega)) \subset \bigcup_{j=1}^{N} B_{n}(\theta^n \omega, f_{n}^{z}(z_j), \epsilon(\theta^n \omega)), \]

where \( N = D(\theta^n \omega)(2C_0(\theta^n \omega)\epsilon_0(\theta^n \omega)/\epsilon(\theta^n \omega))^{\dim E_n} \) for some \( D(\theta^n \omega) > 1 \). So we have

\[ B_{n}(\omega, x, \epsilon_0) \cap \overline{W^u(\omega, y, \delta)} \subset \bigcup_{j=1}^{N} B_{n}(z_j, \epsilon). \]
Thus

\[ r_n^u(B_n(\omega, x, \epsilon_0), \omega, \epsilon) \leq N. \]

By (4) and Dominated Convergence Theorem we have

\[
\lim_{n \to \infty} \frac{1}{n} H_n^u(\omega_0^{n-1} | \mathcal{U}_0^{n-1}) \\
= \lim_{n \to \infty} \frac{1}{n} \int_{\Omega} \max_{u \in \mathcal{U}_0^{n-1}} \sup_{y \in U} \log N(U_\omega \cap \overline{W^u(\omega, y, \delta)}, \omega_0^{n-1}) d\mathcal{P} \\
= \int_{\Omega} \lim_{n \to \infty} \frac{1}{n} \max_{u \in \mathcal{U}_0^{n-1}} \sup_{y \in U} \log N(U_\omega \cap \overline{W^u(\omega, y, \delta)}, \omega_0^{n-1}) d\mathcal{P} \\
= \int_{\Omega} \lim_{n \to \infty} \frac{1}{n} r_n^u(B_n(\omega, x, \epsilon_0), \omega, \epsilon) d\mathcal{P} \\
\leq \int_{\Omega} \lim_{n \to \infty} \frac{1}{n} \log D(\theta^n \omega)(2C_0(\theta^n \omega) \epsilon_0(\theta^n \omega)/\epsilon(\theta^n \omega))^{\dim E^n} d\mathcal{P} \\
= 0,
\]

which implies that \( h^u(\mathcal{W} | \mathcal{U}) = 0 \), because of the arbitrariness of \( \mathcal{W} \), we know that \( h^u(\mathcal{F} | \mathcal{U}) = 0 \).

As in the proof of Theorem 1.4 in [15], we have \( \overline{W^u(\omega, y, \delta)} \cap \Phi(\omega, x, \epsilon) \) contains at most one point due to the expansion of \( \mathcal{F} \) along \( W^u \). The following the line of the proof of Theorem 1.4 in [15], we have

\[ h^{u^u}(\mathcal{F}, \epsilon) = 0 \]

for \( \epsilon > 0 \) small enough. Now we complete the proof of Theorem 5.5. \( \square \)

6. Variational principles for local unstable entropy and unstable pressure.

In this section, we prove Theorem A, then variational principles for local unstable entropy and unstable pressure are obtained. First of all, we give two propositions as follows.

**Proposition 10.** For any \( \mu \in \mathcal{M}_p(\mathcal{F}) \), \( \zeta \in P^u(\Omega \times M) \cup Q^u(\Omega \times M) \), and \( \mathcal{U}, \mathcal{V} \in \mathcal{C}_{1 \times M} \) with small enough diameter, we have

\[
H^u_\mu(\mathcal{V}|\zeta) \leq H^u_\mu(\mathcal{U}|\zeta) + H^u(\mathcal{V}|\mathcal{U}).
\]

**Proof.** Let \( \mathcal{V} = \{V_1, V_2, \ldots, V_m\} \) and \( \beta \in \mathcal{P}(\Omega \times M) \) such that \( \beta \geq \mathcal{U} \). Fix \( \omega \in \Omega \).

Let \( B \in \beta \). For each \( y \in B_\omega \), there exists \( I_y(\omega) \subset \{1, 2, \ldots, m\} \) with minimal cardinality no more than \( N^\omega_\mu(\mathcal{V}|\beta) \) such that \( \bigcup_{i \in I_y(\omega)} V_i \supset B_\omega \cap \zeta_\omega(y) \). Thus we can choose \( y_1, y_2, \ldots, y_s \in B_\omega \) such that for each \( y \in B_\omega \), \( I_y(\omega) = I_{y_i}(\omega) \) for some \( 1 \leq i \leq s \). Then as in the proof of Proposition 5, we can construct a partition \( \tilde{\gamma}_{\omega,B} \) of \( B_\omega \), then a partition \( \tilde{\gamma}_\omega = \bigcup_{B \in \beta_\omega} \tilde{\gamma}_{\omega,B} \) of \( M \). By the same way we can construct a partition \( \gamma \) of \( \Omega \times M \) with \( \gamma_\omega = \tilde{\gamma}_\omega \). According to the construct of \( \gamma \), we know that

\[
N_\omega(\gamma|\beta \lor \zeta) \leq N^\mu_\omega(\mathcal{V}|\beta),
\]

where

\[
N_\omega(\gamma|\beta \lor \zeta) = \max_{\mathcal{U} \in \beta} \sup_{y \in B_\omega \cap \zeta_\omega(y)} N(B_\omega \cap \zeta_\omega(y), \gamma).
\]
Then we have

\[ H_\mu(\gamma|\zeta) \leq H_\mu(\beta|\zeta) + H_\mu(\gamma|\beta \vee \zeta) \]

\[ = H_\mu(\beta|\zeta) + \int_\Omega H_{\mu^\beta \vee \zeta}(\gamma) d\mathbb{P} \]

\[ \leq H_\mu(\beta|\zeta) + \int_\Omega \log N_\mu^u(\gamma|\beta \vee \zeta) d\mathbb{P} \]

\[ \leq H_\mu(\beta|\zeta) + \int_\Omega \log N_\mu^u(\mathcal{U}|\zeta) d\mathbb{P}. \]

Thus

\[ H_\mu^u(\mathcal{U}|\zeta) \leq H_\mu(\gamma|\zeta) \leq H_\mu(\beta|\zeta) + H_\mu^u(\mathcal{U}|\zeta). \]

Since \( \beta \geq \mathcal{U} \) is arbitrary, we complete the proof. \( \square \)

**Proposition 11.** For any \( \mu \in \mathcal{M}(\mathcal{F}), \zeta \in \mathcal{P}^u(\Omega \times M) \cup \mathcal{Q}^u(\Omega \times M), \mathcal{U} \in \mathcal{C}_{\Omega \times M} \) with sufficiently small diameter, we have

\[ h_\mu^u(\mathcal{U}|\zeta) \leq h_\mu^u(\mathcal{F}, \mathcal{U}|\zeta) + h_\mu^u(\mathcal{F}|\mathcal{U}). \]

**Proof.** Proposition 11 can be obtained from 10 easily, so we omit the proof. \( \square \)

**Proof of Theorem A.** We divide the proof into two cases.

**Case 1** for \( \eta \in \mathcal{P}^u(\mathcal{P} \times M) \).

Let \( \mathcal{U} \in \mathcal{C}_{\Omega \times M} \) with \( \text{diam}(\mathcal{U}) \ll \epsilon_0 \). By Corollary 1, we know that

\[ h_\mu^u(\mathcal{F}, \mathcal{U}|\zeta) = h_\mu^u(\mathcal{F}|\zeta) = h_\mu^u(\mathcal{F}) \tag{5} \]

By Lemma 3.18 and Lemma 3.19 we have

\[ h_\mu^u(\mathcal{U}|\eta) = \lim_{n \to \infty} \frac{1}{n} h_\mu^u(\mathcal{F}^n, \mathcal{U}_0^{n-1}|\eta) = \lim_{n \to \infty} \frac{1}{n} n h_\mu^u(\mathcal{F}|\eta) = h_\mu^u(\mathcal{F}|\eta). \tag{6} \]

Then by (5), (6) and Proposition 5, we know that

\[ h_\mu^u(\mathcal{F}) = h_\mu^u(\mathcal{F}, \mathcal{U}|\zeta) \leq h_{\text{top}}^u(\mathcal{F}, \mathcal{U}). \]

By the variational principle for unstable entropy of \( \mathcal{F} \) we can obtain

\[ h_{\text{top}}^u(\mathcal{F}) = \sup_{\mu \in \mathcal{M}(\mathcal{F})} h_\mu^u(\mathcal{F}) \leq h_{\text{top}}^u(\mathcal{F}, \mathcal{U}). \]

And it is easy to see that \( h_{\text{top}}^u(\mathcal{F}, \mathcal{U}) \leq h_{\text{top}}^u(\mathcal{F}), \) then we have

\[ h_{\text{top}}^u(\mathcal{F}, \mathcal{U}) = h_{\text{top}}^u(\mathcal{F}). \]

This ends the proof of Theorem A for \( \eta \in \mathcal{P}^u(\mathcal{P} \times M) \).

**Case 2** for \( \xi \in \mathcal{Q}^u(\mathcal{P} \times M) \).

Let \( \mathcal{U} \in \mathcal{C}_{\Omega \times M} \). By Theorem 5.5 and Proposition 11 we have

\[ h_\mu^u(\mathcal{F}|\zeta) = h_\mu^u(\mathcal{F}, \mathcal{U}|\zeta). \]

By Corollary 1, we know that

\[ h_\mu^u(\mathcal{F}|\zeta) = h_\mu^{u,\pm}(\mathcal{F}, \mathcal{U}|\zeta). \]

By Proposition 8 and Theorem 5.5 we have

\[ h_{\text{top}}^u(\mathcal{F}) \leq h_{\text{top}}^u(\mathcal{F}, \mathcal{U}), \]
and it is clear that for $\mathcal{U} \in C_{\Omega \times M}^0$, we have

$$h^u_{top}(\mathcal{F}) \geq h^u_{top}(\mathcal{F}, \mathcal{U}),$$

which completes the proof.

As an application of Theorem A, following the line of Proposition 3.25 in [15], we have the following proposition.

**Proposition 12.** For $\mathcal{U} \in C_{\Omega \times M}^0$, the local unstable entropy map $\mu \mapsto h^u_\mu(\mathcal{F}, \mathcal{U}|\eta)$ is upper semi-continuous for $\eta \in P^u(\Omega \times M)$.

Now we begin to discuss the variational principle for local pressure. Firstly, we need the following lemma from [10], which is adapted in our context. For $\mathcal{V} \in C_{\Omega \times M}$, let $\alpha$ be the Borel partition generated by $\mathcal{V}$; given $\omega \in \Omega$, let

$$P^\ast \omega(\mathcal{V}) = \{ \beta \in P(M) : \beta \geq \mathcal{V}_\omega \text{ and each atom of } \beta \text{ is the union of some atoms of } \alpha \omega \}.$$

**Lemma 6.1 (Lemma 2.1 in [10]).** Let $\mathcal{U} \in C_{\Omega \times M}$, and $\phi \in L^1(\Omega, C(M))$, then we have

$$\inf_{\mathcal{V} \in C_{\Omega \times M}, \mathcal{V} \geq \mathcal{U}} \{ \sum_{B \in \mathcal{V}} \sup_{y \in B \cap W^u(\omega, x)} \phi(\omega, y) \} = \min_{\beta \in P^\ast(\mathcal{U})} \{ \sum_{B \in \beta} \sup_{y \in B \cap W^u(\omega, x)} \phi(\omega, y) \}.$$

Now we begin to prove Theorem C. Firstly, we prove the following proposition.

**Proposition 13.** For any $\zeta \in P^u(\Omega \times M) \cap Q^u(\Omega \times M)$, we have

$$h^u_\mu(\mathcal{F}, \mathcal{U}|\zeta) + \int_{\Omega \times M} \phi \, d\mu \leq P^u(\mathcal{F}, \phi, \mathcal{U}).$$

**Proof.** Let $\zeta \in P^u(\Omega \times M) \cap Q^u(\Omega \times M)$, we have

$$h^u_\mu(\mathcal{F}, \mathcal{U}|\zeta) + \int_{\Omega \times M} \phi \, d\mu = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\zeta) + \int_{\Omega \times M} \phi \, d\mu(\omega, x)$$

$$= \limsup_{n \to \infty} \frac{1}{n} \int_{\Omega \times M} \left( H_{\mu_{(\omega, x)}}(\mathcal{U}_0^{n-1}) + \sum_{n} S_n \phi \, d\mu_{(\omega, x)} \right) \, d\mu(\omega, x). \quad (7)$$

Choose $\delta > 0$ such that $\zeta(x) \subset W^u(\omega, x, \delta)$ for every $(\omega, x) \in \Lambda'$, where $\mu(\Lambda') = 1$. For any $\beta \in \mathcal{P}(\Omega \times M)$, for any $(\omega, x) \in \Lambda'$, denote $\{ C : C = B \cap \zeta(x) \text{ for some } B \in \beta \}$ by $\beta(x)$. Then by Lemma 6.1, we know that there exists a $\gamma \in P^\ast(\mathcal{U}_0^{n-1})$ such that

$$\log P^u(\mathcal{F}, \omega, \phi, x, \delta, n, \mathcal{U}) = \log \left( \sum_{B \in \beta} \sup_{y \in B \cap W^u(\omega, x, \delta)} \exp((S_n \phi)(\omega, y)) \right)$$

$$\geq \log \left( \sum_{C \in \beta(x)} \sup_{y \in C} \exp((S_n \phi)(\omega, y)) \right).
\[
\geq \sum_{C \in \beta(\omega, x)} \mu_{\omega,x}^\uparrow(C) \left( \sup_{y \in C} (S_n \phi)(\omega, y) - \log \mu_{\omega,x}^\uparrow(C) \right) \\
= H_{\mu_{\omega,x}^\uparrow}(\beta_{\omega,x}) + \sum_{C \in \beta(\omega, x)} \mu_{\omega,x}^\uparrow(C) \sup_{y \in C} (S_n \phi)(\omega, y) \\
\geq H_{\mu_{\omega,x}^\uparrow}(\beta_{\omega,x}) + \int_M S_n \phi d\mu_{\omega,x}^\uparrow \\
\geq H_{\mu_{\omega,x}^\uparrow}(U_0^{n-1}) + \int_M S_n \phi d\mu_{\omega,x}^\uparrow. \\
\tag{8}
\]

By (7) and (8) we know that
\[
\begin{align*}
\left. h^u_{\mu}(\mathcal{F}, \Omega_M) \right|_{\mathcal{M} \mathcal{P}(\mathcal{F})} + \int_{\Omega \times M} \phi d\mu &= \int_{\Omega \times M} \left( H_{\mu_{\omega,x}^\uparrow}(U_0^{n-1}) + \int_M S_n \phi d\mu_{\omega,x}^\uparrow \right) d\mu(\omega, x) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \int_{\Omega \times M} \log P^n(\mathcal{F}, \omega, x, \delta, n, \mathcal{U}) d\mu(\omega, x) \\
&\leq \int_{\Omega \times M} \limsup_{n \to \infty} \frac{1}{n} \log P^n(\mathcal{F}, \omega, x, \delta, n, \mathcal{U}) d\mu(\omega, x) \\
&= \int_{\Omega \times M} \log P^n(\mathcal{F}, \omega, x, \mathcal{U}) d\mu(\omega, x) \\
&\leq P^n(\mathcal{F}, \omega, \mathcal{U}),
\end{align*}
\]
in the third inequality, Fatou’s Lemma is applied. This completes the proof. \(\square\)

**Proof of Theorem C.** By Proposition 13, Theorem A, and the principle for unstable pressure for RDSs obtained in [14], we have
\[
P^n(\mathcal{F}, \phi) = \sup \left\{ h^u_{\mu}(\mathcal{F} \mid \zeta) + \int_{\Omega \times M} \phi d\mu : \mu \in \mathcal{M} \mathcal{P}(\mathcal{F}) \right\} \\
= \sup \left\{ h^u_{\mu}(\mathcal{F}, \mathcal{U}) + \int_{\Omega \times M} \phi d\mu : \mu \in \mathcal{M} \mathcal{P}(\mathcal{F}) \right\} \\
\leq P^n(\mathcal{F}, \phi, \mathcal{U}).
\]

On the other hand, it is clear that when \( \mathcal{U} \in \mathcal{C}^0_{\Omega \times M}, P^n(\mathcal{F}, \phi, \mathcal{U}) \leq P^n(\mathcal{F}, \phi) \), which completes the proof. \(\square\)

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