Switching transformations for control of opinion patterns in signed networks: application to dynamic task allocation

Anastasia Bizyaeva, Giovanna Amorim, María Santos, Alessio Franci, and Naomi Ehrich Leonard

Abstract—We propose a new design method to control opinion patterns on signed networks of agents making decisions about two options and to switch the network from any opinion pattern to a new desired one. Our method relies on switching transformations, which switch the sign of an agent’s opinion at a stable equilibrium by flipping the sign of its local interactions with its neighbors. The global dynamical behavior of the switched network can be predicted rigorously when the original, and thus the switched, networks are structurally balanced. Structural balance ensures that the network dynamics are monotone, which makes the study of the basin of attraction of the various opinion patterns amenable to rigorous analysis through monotone systems theory. We illustrate the utility of the approach through scenarios motivated by multi-robot coordination and dynamic task allocation.

I. INTRODUCTION

Modern networked technologies require decentralized mechanisms for decision making and allocation of tasks. For example, systems such as smart power grids, cloud computing services, or multi-robot teams, call for strategies that dynamically distribute tasks among individual units so that system performance is optimized even as task requirements change or individual units experience failure.

We use the model of networked nonlinear opinion dynamics of [1], [2] to illustrate how network interconnection topology can be designed so a group of decision makers converges to a desired opinion pattern and how the network can be transformed so the group switches to a desired alternative opinion pattern. When all agents commit to the same option we say the network is in agreement, and for any other configuration of opinions the network is in disagreement. The emergence of agreement and disagreement equilibria on these nonlinear opinion networks has been studied in [1], [3], [4]. However, the analysis in these works has assumed that all of the network interactions are of the same type: purely cooperative or purely competitive. In this paper we add to this body of analysis by relaxing the homogeneity assumption and allowing for mixed-sign interactions.

Decision-making with signed interactions has been studied on linear networks with averaging dynamics [5], [6], as well as with nonlinear consensus models [7], [8] and biased assimilation models [9]. The novelty of our approach is to use signed interactions on a network as a design tool. Our design methodology drives a distributed system to a desired network state and allows any individual agent to respond to local contextual changes and adjust its allocation by dynamically adjusting the sign of interaction with its neighbors.

Our contributions are as follows. First, we prove that a network system can be easily and intuitively controlled to any agreement or disagreement opinion pattern using standard tools from signed graph theory grounded in switching transformations of graphs. Second, we prove a sufficient condition for the networked state to converge to one of two available equilibrium configurations. Third, we show how a pattern of equilibrium opinions can be changed dynamically through local updates of the network weights that follow the structure of a switching transformation. Fourth, we validate the theory by applying both centralized and local switching mechanisms in the context of task allocation for multi-robot systems.

In Section II we introduce notation. Section III describes the opinion dynamics model and summarizes some of its properties. In Section IV we present new analysis of the model on signed graphs and propose a systematic design approach to allocate agents across two tasks. In Section V we describe the asymptotic dynamics of trajectories on structurally balanced graphs. Section VI relates the features of the approach in the context of multi-robot task allocation. Final remarks are included in Section VII.

II. NOTATION AND MATHEMATICAL PRELIMINARIES

For any vectors \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \), \( y = (y_1, \ldots, y_N) \in \mathbb{R}^N \), we define the standard Euclidean inner product \( \langle x, y \rangle = \sum_{i=1}^{N} x_i y_i \). We say \( x \succeq y \) if \( x_i \geq y_i \) for all \( i = 1, \ldots, N \), and we say \( x \succ y \) if \( x_i > y_i \) for all \( i = 1, \ldots, N \). We define the operation \( \odot \) as the element-wise product of vectors, \( x \odot y = (x_1 y_1, \ldots, x_N y_N) \). The symbol \( 0_N \) denotes the vector with all zero entries in \( \mathbb{R}^N \), and \( \mathbb{I}_N \) denotes the identity matrix in \( \mathbb{R}^{N \times N} \).

In this paper we study networks of \( N \) agents with a signed communication graph \( \mathcal{G} := (\mathcal{V}, \mathcal{E}, \sigma) \) where \( \mathcal{V} = \{1, \ldots, N\} \) is the vertex set, \( \mathcal{E} \) is the edge set, and \( \sigma : \mathcal{E} \to \{1, -1\} \) is a sign function or signature of the graph \( \mathcal{G} \). We use the sensing convention such that we let \( e_{ik} \in \mathcal{E} \) denote a directed edge in \( \mathcal{G} \) that points from vertex \( i \) to vertex \( k \), indicating that \( k \) is a neighbor of \( i \). We assume that the unsigned directed graph \( \Gamma = (\mathcal{V}, \mathcal{E}) \) underlying \( \mathcal{G} \) is simple, i.e. contains no self-loops \( e_{ii} \not\in \mathcal{E} \) for all \( i \in \mathcal{V} \), and there is at most one edge \( e_{ik} \) in \( \mathcal{E} \) that begins at vertex \( i \) and ends at vertex \( k \).
for all $i, k \in \mathcal{V}$. We say that $G$ is strongly connected if the edges contained in $E$ form a path between any two nodes.

Define $A = (a_{ik})$ to be the $N \times N$ signed adjacency matrix of the graph $G$ whose entries $a_{ik} \in \{0, 1, -1\}$ satisfy $a_{ik} = 0$ if $e_{ik} \not\in E$ and $a_{ik} = \sigma(e_{ik})$ if $e_{ik} \in E$. We use the symbol $\lambda^*$ to distinguish, when it exists, the real and unique eigenvalue of $A$ that satisfies $\text{Re}(\lambda^*) > \text{Re}(\lambda_i)$ for all eigenvalues $\lambda_i \neq \lambda^*$. We define the right and left eigenvectors of $A$ corresponding to $\lambda^*$ as $v^*$ and $w^*$, respectively. We always assume $v^*, w^*$ are normalized to satisfy $\langle v^*, v^* \rangle = 1$. In the following proposition we adapt the statement of the standard Perron-Frobenius theorem, e.g. as presented in [10], to specialize to adjacency matrices of graphs with an all-positive signature.

**Proposition II.1** (Perron-Frobenius). Suppose $\sigma(e_{ik}) = 1$ for all $e_{ik} \in E$ for some strongly connected graph $G$. Then the following hold: 1) $\lambda^*$ exists; 2) $\lambda^* > 0$; 3) we can choose $v^*, w^*$ to satisfy $v^* > 0_N$ and $w^* > 0_N$.

### III. Nonlinear Opinion Dynamics Model

The evolution of the opinion of $N$ agents on a signed network choosing between two options is modeled in this work according to the continuous-time multi-agent, multi-option nonlinear opinion dynamics model in [1].

Let $x_i \in \mathbb{R}$ denote the opinion of agent $i$, where the magnitude of $x_i$ determines the agent’s commitment to one of the two options such that a stronger (weaker) commitment to an option corresponds to a larger (smaller) $|x_i|$. If $x_i = 0$, the agent is said to be unopinionated, and, if $x_i > 0 (< 0)$, agent $i$ prefers option 1 (option 2). We define the opinion state of the network as $x = (x_1, ..., x_N) \in \mathbb{R}^N$, with $x = 0_N$ being the neutral state of the group. The network is in an agreement state when $\text{sign}(x_i) = \text{sign}(x_k)$ for all $i, k \in \{1, ..., N\}$ (i.e., if all the agents commit to the same option), and in a disagreement state if at least one pair of agents disagrees (i.e. $\text{sign}(x_i) \neq \text{sign}(x_k)$ for some $i, k$).

Using the model in [1], the opinion of agent $i$ evolves according to a linear damping on the agent’s own opinion and a nonlinear saturation on its network interactions,

$$
\dot{x}_i = -dx_i + u_i S \left( \alpha x_i + \gamma \sum_{k=1}^{N} a_{ik} x_k \right),
$$

where the relative influence of both terms is regulated by the attention parameter, $u_i > 0$. The linear term is characterized by a damping coefficient, $d > 0$, while the nonlinear term consists of an odd saturating function $S : \mathbb{R} \to \mathbb{R}$ acting on network interactions. This saturating function satisfies $S(0) = 0, S'(0) = 1$, $\text{sign}(S'(x)) = -\text{sign}(x)$. Network interactions comprise self-reinforcement interactions, weighted by $\alpha \geq 0$, and neighbor interactions, weighted by $\gamma > 0$. Note that neighbor interactions are also weighted by the adjacency term $a_{ik} \in \{0, 1, -1\}$ so that agent $i$ cooperates (competes) with agent $k$ when $a_{ik} = 1$ ($-1$) and is indifferent to agent $k$'s opinion when $a_{ik} = 0$.

### A. Network opinion formation through bifurcation

The following proposition, adapted from [1, Theorem IV.1] and [4, Theorem IV.1] and stated without proof, summarizes several key features of the opinion dynamics $\mathbf{1}$.

**Proposition III.1** (Opinion formation as a pitchfork bifurcation). Consider $\mathbf{1}$ on a graph $G$ with $u_i = u \geq 0$ for all $i = 1, ..., N$, and assume a simple, real largest eigenvalue $\lambda^*$ exists. Suppose $\alpha + \gamma \lambda^* > 0$ and $(w^*, (v^*)^*) > 0$, where $(v^*)^* = v^* \odot v^* \odot v^*$. Then 1) for $0 \leq u < u^* := d/(\alpha + \gamma \lambda^*)$, the neutral equilibrium $x = 0_N$ is locally exponentially stable; 2) for $u > u^*$, $x = 0_N$ is unstable, and two branches of locally exponentially stable equilibria $x = x_1^*, x_2^*$ branch off from $(x, u) = (0_N, u^*)$ along a manifold tangent at $x = 0_N$ to span$(v^*)$. The two nonzero equilibria differ by a sign, i.e. $x_2^* = -x_1^*$.

Proposition III.1 highlights the ability of a group of decision-makers with opinion dynamics $\mathbf{1}$ to break deadlock and commit to an opinionated configuration when their level of attention, $u$, is sufficiently large. Next we state several properties of this system that are used in later sections.

**Corollary III.1.1** (Sufficient condition for agreement). When $G$ is strongly connected with an all-positive signature, conditions of Proposition III.1 are always satisfied. For $u > u^*$, one of the two new stable equilibria satisfies $x^* > 0_N$.

**Proof.** The corollary follows from Proposition III.1 and Proposition II.1 since $\lambda^*$ is the Perron-Frobenius eigenvalue, and its eigenvectors $w^*, v^*$ have all-positive entries. □

In the following corollary we show that the equilibria predicted by Proposition III.1 are the only equilibria admitted by the dynamics $\mathbf{1}$ for a range of values of $u$, following similar arguments as those used for Laplacian-weighted nonlinear consensus networks in [7]. Before proving the corollary we state a necessary lemma.

**Lemma III.2** (Boundedness). Any compact set $\Omega_r \subset \mathbb{R}^N$ of the form $\Omega_r = \{x \in \mathbb{R}^N \text{ s.t. } |x_i| < r \max \{u_j\}, \forall i, j \in \mathcal{V}\}$ with $r > 1$ is forward-invariant for $\mathbf{1}$.

**Proof.** The lemma follows directly from the more general result [1, Theorem A.2]. □

**Corollary III.2.1** (Uniqueness of Equilibria). Suppose conditions of Proposition III.1 are satisfied, and let $\lambda_2$ be an eigenvalue of $A$ satisfying $\text{Re}(\lambda_2) \geq \text{Re}(\lambda_1)$ for all eigenvalues $\lambda_j \neq \lambda^*$ of $A$. 1) $x = 0_N$ is globally asymptotically stable on a forward-invariant compact set $\Omega \subset \mathbb{R}^N$ containing the origin $x = 0_N$, for all $u \in [0, u^*)$; 2) when $\text{Re}(\lambda_2) \geq -\alpha/\gamma$, $u \in (u^*, u_2)$, the only equilibria the system admits are $0_N$, $x_1^*$, and $x_2^*$, where $u_2 = d/\alpha + \gamma \text{Re}(\lambda_2)$; 3) when $\text{Re}(\lambda_2) < -\alpha/\gamma$, the only equilibria the system admits in $\Omega$ for all $u > u^*$ are $0_N$, $x_1^*$, and $x_2^*$.

**Proof.** 1) Existence of $\Omega$ is established in Lemma III.2. Define $\bar{A} = \alpha \mathbb{1}_N + \gamma A$ with components $\bar{a}_{ij}$, and let $f_i(x) = \sum_{j=1}^{N} \bar{a}_{ij} x_j$. Consider the continuously differentiable function $V(x) = \sum_{i=1}^{N} f_i(x) S(\eta)d\eta$. The derivative...
of $V$ along trajectories of $\mathbf{1}$ is
$$
\dot{V}(x) = S(\tilde{A}x)^T \tilde{A}x = S(\tilde{A}x)^T \tilde{A}(-dx + uS(\tilde{A}x)) \\
\leq -d - u(\alpha + \gamma \lambda^*) S(\tilde{A}x)^T S(\tilde{A}x) \leq 0. \quad (2)
$$

The set on which $\mathbf{2}$ is exactly zero is $\mathcal{N}(\tilde{A}) = \{x \in \mathbb{R}^N \text{ s.t. } \tilde{A}x = 0_N\}$. By LaSalle’s invariance principle [11, Theorem 4.4] we conclude that the trajectories $x(t)$ approach the largest invariant set in $\mathcal{N}(\tilde{A})$ as $t \to \infty$. If $\mathcal{N}(\tilde{A}) = \{0_N\}$, the corollary follows trivially. Let $x \in \mathcal{N}(\tilde{A})$ and suppose $x \neq 0$. Then $\dot{x} = -dx$, i.e. all trajectories that start in $\mathcal{N}(\tilde{A})$ decay to the origin exponentially in time, and the corollary follows. Under the assumptions on $u$ stated in 2) and 3), the Jacobian matrix $J(x) = -dI_N + u\text{diag}(S'(\tilde{A}x))\tilde{A}$ is Hurwitz for all $x \in \mathbb{R}^N \setminus \{0\}$. The proof of this statement follows closely the argument presented in [8, Lemma 6] and we omit the details. For values of $u$ in a small neighborhood above $u^*$ exactly three equilibria exist, as described in Proposition III.1. Since the Jacobian is never singular for the considered parameter regime, there are no bifurcation points in $\Omega$ thereby by the implicit function theorem the number of equilibria remains unchanged.

IV. Switching Transformation as a Design Tool for Synthesis of Opinion Patterns

When the communication graph $\mathcal{G}$ contains edges with a negative signature, the opinion-forming bifurcation of Proposition III.1 can result in disagreement network equilibria. In this section we describe a simple synthesis technique for generating a signed adjacency matrix which results in a desired pattern of opinions among the decision-makers following opinion dynamics $\mathbf{1}$. In order to do this we first introduce a few important concepts from the theory of signed graphs. For a more comprehensive exposition to signed graph theory we refer the reader to [12, 13].

A. Signed graphs and switching

Let $\mathcal{W} \subset V$ be a subset of nodes on a signed graph $\mathcal{G}$. Switching a set $\mathcal{W}$ on the graph $\mathcal{G}$ refers to a mapping of the graph $\mathcal{G}$ to $\mathcal{G}'$ on $V \setminus \mathcal{W}$ where the signature of all the edges in $E$ between nodes in $\mathcal{W}$ and nodes in its complement $V \setminus \mathcal{W}$ reverses sign. We introduce the switching function $\theta : V \to \{1, -1\}$, where for any $i \in V$, $\theta(i) = -1$ if $i \in \mathcal{W}$ and $\theta(i) = 1$ otherwise. Then the signature of the switched graph $\mathcal{G}'$ is generated as
$$
\sigma_{\mathcal{W}}(e_{ik}) = \theta(i)\sigma(e_{ik})\theta(k) \quad (3)
$$
for all $e_{ik} \in E$. Importantly, switching a set $\mathcal{W}$ all at once generates the same graph $\mathcal{G}'$ as sequentially switching individual vertices in $\mathcal{W}$. If $\mathcal{G}$ can be transformed into $\mathcal{G}'$ by switching, $\mathcal{G}$ and $\mathcal{G}'$ are switching equivalent graphs.

Suppose $\theta$ is the switching function that defines a switching of the graph $\mathcal{G}$ to $\mathcal{G}'$ whose adjacency matrices are $A$ and $A'$, respectively. Define the switching matrix $\Theta = \text{diag}(\theta(1), \theta(2), \ldots, \theta(N))$. The adjacency matrices of $\mathcal{G}$ and its switching $\mathcal{G}'$ are related as
$$
A' = \Theta^{-1} A \Theta. \quad (4)
$$
Observe that $\Theta^{-1} = \Theta$. We refer to $\mathbf{4}$ as a switching transformation of the adjacency matrix $A$, and we say that $A$ and $A'$ are switching equivalent adjacency matrices.

Proposition IV.1. Suppose $\mathcal{G}$, $\mathcal{G}'$ are switching equivalent with adjacency matrices $A$ and $A'$ and the associated switching matrix $\Theta$. Then 1) $A$ and $A'$ are co-spectral, i.e. have the same set of eigenvalues; 2) $v(\Theta w)$ is a right (left) eigenvector of $A$ corresponding to the eigenvalue $\lambda$ if and only if $\Theta v(\Theta w)$ is a right (left) eigenvector of $A'$ with the same eigenvalue.

Proof. The proposition follows from the standard properties of a matrix similarity transformation, since $A$ and $A'$ are related through a similarity transformation $\mathbf{4}$.

Proposition IV.1 implies that the eigenvectors of the switched adjacency matrix $A'$ are generated from the eigenvectors of the original adjacency matrix $A$ by flipping the sign of each entry that corresponds to a node which is being switched. We will take advantage of this observation in our design of nonlinear opinion patterns on a network.

B. Nonlinear opinion patterns on switch equivalent graphs

In this section we show that a switching transformation of the nonlinear opinion dynamics $\mathbf{1}$ is effectively a coordinate change, and two switching equivalent networks generate topologically equivalent flow and bifurcation diagrams.

Theorem IV.2 (Diffeomorphism between trajectories of switching equivalent systems). Consider switching equivalent graphs $\mathcal{G}$, $\mathcal{G}'$ with adjacency matrices $A$ and $A'$ and with switching matrix $\Theta$. The trajectory $x(t)$ is a solution to $\mathbf{1}$ on $\mathcal{G}$ if and only if $\Theta x(t)$ is a solution to $\mathbf{1}$ on $\mathcal{G}'$.

Proof. Suppose $x(t)$ is a solution of $\mathbf{1}$ on $\mathcal{G}$. Then
$$
\frac{dx}{dt}(t) = -dx(t) + U S(\alpha x(t) + \gamma A x(t))
$$
where $U = \text{diag}(u_1, \ldots, u_N)$ and $S(y) := (S(y_1), \ldots, S(y_N))$ for any $y \in \mathbb{R}^N$. Multiplying both sides of this expression by the switching matrix $\Theta$ yields
$$
\frac{d}{dt}(\Theta x(t)) = \Theta (-dx(t) + U S(\alpha x(t) + \gamma A x(t)))
$$
$$
= -d\Theta x(t) + \Theta U S(\alpha x(t) + \gamma A x(t))
$$
$$
= -d\Theta x(t) + U S(\alpha \Theta x(t) + \gamma A^\mathcal{W} \Theta x(t)),
$$
where the last step follows since $\Theta U = U \Theta$ and $-S(y) = S(-y)$. This shows that $\Theta x(t)$ is a solution of $\mathbf{1}$ on $\mathcal{G}'$. The other direction follows by an identical proof.

Corollary IV.2.1 (Switching a graph “rotates” a pitchfork bifurcation). Consider $\mathbf{1}$ with $u_i = u \geq 0$ for all $i = 1, \ldots, N$ on the graphs described in Theorem IV.2. Suppose $\mathcal{G}$ satisfies the conditions of Proposition III.1. Then $\mathcal{G}'$ also satisfies the conditions of Proposition III.1. Furthermore, $x^*$ is an equilibrium on the bifurcation diagram on $\mathcal{G}$ at some $u$ if and only if $\Theta x^*$ is an equilibrium on the bifurcation diagram of $\mathcal{G}'$ at the same $u$. 


We illustrate the intuition of Corollary [V.2.1] in Fig. 1.

**Theorem IV.3** (Switching complementary vertex sets generate the same flow). Consider two switching equivalent graphs $G^W$, $G^{V \setminus W}$, generated by switching a set of vertices $W$ or its complement $V \setminus W$ on graph $G$. Let the switching matrices in relation to $G$ of these two graphs be $\Theta^W$ and $\Theta^{V \setminus W}$ respectively. The trajectory $x(t)$ is a solution of (1) on $G^W$ if and only if it is also a solution of (1) on $G^{V \setminus W}$.

**Proof.** Suppose $x(t)$ is a solution of (1) on $G^W$. Then by Theorem IV.2 $\Theta^W x(t)$ is an equilibrium of (1) on $G$, and therefore $G^{V \setminus W} \Theta^W x(t) = -x(t)$ is a solution of (1) on $G^{V \setminus W}$. By odd symmetry of the dynamic equations (1), $x(t)$ is also a solution of (1) on $G^{V \setminus W}$. The proof in the other direction follows the same steps in opposite order. \qed

C. Synthesis of nonlinear opinion patterns

Using the theoretical results of Section [V.2.2] we propose a simple design procedure that generates a signed adjacency matrix that realizes a desired allocation of agents across the two options. **Step 1.** Start with a strongly connected $G$ with an all-positive signature. By Corollary III.2.1 (1) on $G$ will result in an all-positive equilibrium $x_1^*$ or an all-negative equilibrium $x_2^*$. **Step 2.** Apply switching transformation with switching matrix $\Theta$ for a desired set of nodes $W \subset V$ to generate signed graph $G^W$. This has the effect of grouping all nodes in $W$ and all nodes in $V \setminus W$ together by sign, i.e. the dynamics (1) on $G^W$ will generate bistable equilibria $\Theta x_1^*$, $\Theta x_2^*$. If $|W| = M$, the equilibrium $\Theta x_1^*$ will have $M$ negative nodes, and $\Theta x_2^*$ will have $N - M$ negative nodes. The designer can choose which nodes are grouped together through the switching matrix $\Theta$. We illustrate in Fig. 2.

V. Dynamic Switching

We next investigate the asymptotic opinion dynamics of (1) when the underlying communication graph $G$ instantaneously changes to a switching equivalent graph $G^W$.

A. Monotonicity and structural balance

First, we introduce some relevant definitions from the study of monotone systems. Let $K$ be an orthant of $R^N$, $K = \{x \in R^N s.t. \ (-1)^{m_i} x_i \geq 0, \ i = 1, \ldots, N \}$ with each $m_i \in \{0, 1\}$. The orthant $K$ generates a partial ordering $\preceq_K$ on $R^N$ where if $x, y \in R^N$, $y \preceq_K x$ if and only if $x - y \in K$. We say a system $\dot{x} = f(x)$ on $U \subseteq R^N$ is type $K$ monotone if its flow preserves the partial ordering $\preceq_K$, i.e. if $x(t) \preceq_K x_2(t)$ implies $x_1(t) \preceq_K x_2(t)$ for all $t > 0$.

**Lemma VI.1.** Consider (1) on a signed graph $G$. It is a type $K$ monotone system if and only if $G$ is switching equivalent to $G^+$, for which $\sigma(e_{ik}) = 1$ for all $e_{ik} \in E$, i.e. $G$ is structurally balanced.

**Proof.** The off-diagonal terms of the Jacobian matrix $J(x)$ are $u_\gamma \text{diag}(S^\prime((\alpha z_N + \gamma A)x))A$. Let $\Theta$ be the switching matrix between $G$ and $G^+$. Since $S^\prime(y) > 0$ for all $y \in R$, the matrix $u_\gamma \Theta \text{diag}(S^\prime((\alpha z_N + \gamma A)x))A\Theta$ has nonnegative components, and the lemma follows by [14, Lemma 2.1]. \qed

B. Instantaneous switching

Suppose $x^*$ is a hyperbolic equilibrium of (1), i.e. the linearization of the system at $x^*$ has $m$ unstable eigenvalues and $N - m$ stable eigenvalues. Then by [15, Theorem 1.3.2], there exist smooth local unstable and stable manifolds $W^u(x^*), W^s(x^*)$ of dimensions $m, N - m$ that are tangent to the unstable and stable eigenspaces of the linearized systems at $x^*$ and invariant under the dynamics. Global stable and unstable manifolds $W^s(x^*), W^u(x^*)$ invariant under the dynamics can be obtained by continuing the trajectories in their local counterparts forwards or backwards in time.

**Assumption 1** (Stable manifold of origin is bounded; Fig. 2). Consider (1) on some structurally balanced graph $G$ with $u_i = u > u^*$ and $u < u_2$ when appropriate, as defined in Corollary III.2.1. Let $U \subset R^N$ be an open neighborhood containing the origin, and let $x \in W^s(0)$, $1 \ |(w^*, x)| < \epsilon |x|^2$ for some $0 < \epsilon < 1$; 2) for equilibria $x^*_k \not= 0$ of Proposition III.1 with $k \in \{1, 2\}$, $|w^*, x^*_k| > \epsilon |x|^2$.

Estimating the $\epsilon$ bound described above requires a lengthy computation of the center manifold approximation which we do not carry out for space considerations, however this assumption should hold at least locally as a consequence of the (Un)Stable Manifold Theorem [15, Theorem 1.3.2].

![Fig. 1: Illustration of Corollary IV.2.1](image1.png) The bifurcation diagram of the switched system is a “rotated” version of the original diagram because the sign of $e_j$ flips.

![Fig. 2: Assigning agents to a 30-70% distribution by switching agents 1, 2 and 3. (a) Time trajectory of the opinion dynamics. (b) Final agent distribution. (c) Network diagram with the opinion of agents 1, 2 and 3.](image2.png)
and monotonicity of the flow. We verified the assumption numerically for several graphs. In our simulations, \( \varepsilon \) was on the order of 0.05 or smaller for all graphs considered when \( w^* \) was normalized to unit norm, and its precise value will likely depend on choice of \( u, d, \alpha, \gamma \) parameters.

**Lemma V.2 (Regions of attraction).** Consider (1) on some structurally balanced graph \( G \) with \( u_i = u > u^* \) for all \( i = 1, \ldots, N, \) on an open and bounded neighborhood \( \Omega_r \) as defined in Lemma [III.2]. Let \( x_1^*, x_2^* \) be the nonzero equilibria described in Proposition [III.1] with \( \langle w^*, x_1^* \rangle > 0 \). Consider an initial condition \( x(0) \) at \( t = 0 \) and let \( W^* \subset V \) be the set of indices \( i \) for which \( x_1(0)w_i^* < 0 \). If \( \langle w^*, x(0) \rangle > \varepsilon \|x(0)\|^2 \) then as \( t \to \infty \), \( x(t) \to x_1^* \) (1).

**Proof.** We established in Corollary [III.2.1] that the only equilibria the system admits are \( 0, x_1^*, x_2^*, \) and \( \Omega_r \) is positively invariant by Lemma [III.2]. Let \( B(x_1^*) \) be the basin of attraction of equilibrium \( x_1 \) in \( \Omega_r \). By monotonicity (Lemma [V.1]) and [14, Theorem 2.6], the set \( \text{Inf}(B(x_1^*)) \cup \text{Inf}(B(x_2^*)) \) is open and dense in \( \Omega_r \), where Inf signifies the interior points. Then following Assumption [I] the stable manifold partitions \( U \) into the basins of attraction of the two locally asymptotically stable equilibria. The sets \( U_+ = \{ x \in \Omega_r \ s.t. (w^*, x) > \varepsilon \|x\|^2 \}, U_- = \{ x \in \Omega_r \ s.t. (w^*, x) < -\varepsilon \|x\|^2 \} \) do not intersect the center manifold and are therefore positively invariant. Then since \( x_1 \in U_+ \) and \( x_2 \in U_- \), \( U_+ \subset B(x_1^*) \) and \( U_- \subset B(x_2^*) \).

**Remark V.1.** In practice, without a precise value for the bound \( \varepsilon \) from Assumption [I] for most points \( x(0) \in \mathbb{R}^N \), it is sufficient to check whether the projection of \( x(0) \) onto \( w^* \) is positive or negative to determine which region of attraction the points belong to, i.e. \( \langle w^*, x(0) \rangle > 0 < 0 \) where \( > \) implies convergence to \( x_1^* \) and \( < \) to \( x_2^* \). This is because the stable manifold that partitions the space of possible opinion configurations occurs near the plane of points normal to \( w^* \) at the origin; see Fig. 3 for illustration. As long as \( x(0) \) is not too close to this plane, the projection is a reliable heuristic for the asymptotic dynamics of the network opinions.

**Theorem V.3.** Consider (1) on some \( G \) and let \( x_1, x_2 \) be the nonzero equilibria described in Proposition [III.1] with \( \langle w^*, x_i \rangle > 0 \). Let \( G^W \) be switch equivalent to \( G \) with the associated switching matrix \( \Theta \). Suppose at \( t = 0 \), \( x(0) \in B_{\mu}(x_i) \) with \( i \in \{1, 2\} \). If \( \|\Theta w^*, x_i\| > \varepsilon \|x_i\|^2 \) and \( \langle \Theta w^*, x_i \rangle > 0 < 0 \) then for (1) on \( G^W \) as \( t \to \infty \), \( x(t) \to \Theta x_i (\to -\Theta x_i) \).

**Proof.** Without loss of generality, let \( x(0) =: x \in B_{\mu}(x_1) \) with \( \mu \) sufficiently small so that \( \langle \Theta w^*, x \rangle > \varepsilon \|x(0)\|^2 \) and \( w^* x_i(0) > 0 \) for all \( i \in V \) (these are true at \( x_1 \) by assumption; sufficiently close nearby points will satisfy the conditions by continuity). By Theorem [V.2] we know that for (1) on \( G^W \), \( \Theta x_1 \) is an equilibrium, and the vector \( \Theta w^* \) is normal to the stable eigenspace at the origin. The theorem follows by Lemma [V.2].

Intuitively, Theorem [V.3] suggests that instantaneously changing a structurally balanced graph \( G \) to its switching equivalent \( G^W \) will result in a predictable transition of the system state. Namely, if the number of nodes in \( W \) is small in comparison with the total cardinality of \( V \) we expect that all nodes in \( W \) will change sign, and all of the nodes in \( V \setminus W \) will not. A simulation example of this behavior is shown in Fig. 3. The precise number of nodes that can be switched simultaneously to generate this behavior depends on the eigenvector \( w^* \) of the graph adjacency matrix, the value of the equilibrium \( x_i \), and the bound \( \varepsilon \); however in practice we observe that it is often sufficient that \( |W| < \frac{1}{2} |V| \).

The analysis in this section suggests that the dynamics (1) should be well-behaved if the transition between \( G \) and \( G^W \) is smooth, e.g. implemented through a dynamic feedback law. We consider an example with such a continuous transition in the following section, with the simulation in Fig. 5.

**VI. APPLICATIONS TO MULTI-ROBOT TASK ALLOCATION**

We illustrate how our method can be applied to regulate the relative number of robots dedicated to a task, how the approach is robust to individual robot failures or robots self-assigning to tasks, and how we can ensure switches are conducted appropriately when triggered locally.

a) Task distributions: Many multi-robot applications need for subteams to be assigned to different tasks in a certain proportion (see e.g. [16] for a review on multi-robot task allocation). Our method can control the distribution of agents among two different tasks by applying switching transformations—coordinated centrally—to a network of agents. These switching transformations influence the agreement/disagreement interactions among neighboring robots, making the robots commit to one of two tasks.

An example illustrating this feature is shown in Fig. 2 where a team of agents are distributed among two tasks in predetermined proportion by applying switching transformations (4). This feature can be considered for homogeneous multi-robot teams, where robots are fully interchangeable, and heterogeneous teams, where robots can contribute to tasks in different capacities (e.g. robots can have different speed, battery life, or task efficiency). For the heterogeneous case, we can consider a scenario where robots can be divided in subteams such that they can indiscriminately contribute to one of the tasks. In this case, our method guarantees the relative proportion among tasks, but does not control which
subteam is assigned to which task. For example, in Fig. 2, the team composed of agents 1, 2, and 3 could execute either task 1 or 2. This affords flexibility from the application point of view, where the initial condition on the opinion (e.g., how close is a particular robot to an area where the demand for a task is abundant) can determine the final distribution of agents, i.e. which subteam is assigned to which task.

b) Local flexibility: Our approach provides flexibility at the local level, letting agents/robots make individual decisions without disconnecting from the network. This feature is essential in long-duration autonomy applications where team performance should not be affected by failures or individual that stop contributing to tasks, e.g. to charge batteries [17]. See Fig. 4 where one agent switches between tasks without affecting the task preference of its neighbors in the graph.

To illustrate suppose that agent $i$ wants to switch options. Agent $i$ alerts its neighbors that $\theta_i = -1$. We define

$$\tau_a \dot{\theta}_i = -a_{ik}(0)\theta_i \theta_k,$$

(5)

where $\tau_a$ is a time-scale parameter. The term $a_{ik}(0) \in \{0, 1, -1\}$ is the initial signature of the edge between agents $i$ and $k$. Fig. 5 depicts a scenario where two neighbors execute a local switch simultaneously. The dynamics (5) allow locally originated switches to take place simultaneously between neighbors, which can be useful in applications involving switching cascades, where a robot switching to a new task can trigger its neighbors to switch. This feature is relevant in dynamic task allocation for multi-robot systems where robots can assign themselves to new tasks as a result of interacting with the environment or with their neighboring robots [18].

VII. Final Remarks

We analyzed the nonlinear networked opinion dynamics [1] on signed graphs and proposed a novel approach for allocating a group of agents across two tasks. In future work, we aim to generalize the results in Section V to graphs that are not structurally balanced, to derive an estimate for the $\varepsilon$ bound from Assumption [1] and to extend this analysis to the more general multi-option opinion dynamics of [1].

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