Optimal stability in the identification of a rigid inclusion in an isotropic Kirchhoff-Love plate

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Abstract

In this paper we consider the inverse problem of determining a rigid inclusion inside a thin plate by applying a couple field at the boundary and by measuring the induced transversal displacement and its normal derivative at the boundary of the plate. The plate is made by non-homogeneous, linearly elastic and isotropic material. Under suitable a priori regularity assumptions on the boundary of the inclusion, we prove a constructive stability estimate of log type. Key mathematical tool is a recently proved optimal three spheres inequality at the boundary for solutions to the Kirchhoff-Love plate’s equation.

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1 Introduction

In this paper we consider the inverse problem of the stable determination of a rigid inclusion embedded in a thin elastic plate by measuring the transverse displacement and its normal derivative at the boundary induced by a couple field applied at the boundary of the plate. We prove that the stability estimate of log-log type found in [M-Ro-Ve2] can be improved to a single logarithm in the case in which the plate is made of isotropic linear elastic material.

From the point of view of applications, modern requirements of structural condition assessment demand the identification of defects using non-destructive methods, and, therefore, the present results can be useful in quality control of plates. We refer, among other contributions, to Bonnet and Constantinescu [Bo-Co] for a general overview of inverse problems arising in diagnostic analysis applied to linear elasticity and, in particular, to plate theory ([Bo-Co, Section 5.3]), and to [K] for the identification of a stiff inclusion in a composite thin plate based on wavelet analysis of the eigenfunctions.

In order to describe our stability result, let us introduce the Kirchhoff-Love model of a thin, elastic isotropic plate under infinitesimal deformation; see, for example, [G]. Let the middle plane of the plate \( \Omega \) be a bounded domain of \( \mathbb{R}^2 \) with regular boundary. The rigid inclusion \( D \) is modelled as a simply connected domain compactly contained in \( \Omega \). Under the assumptions of vanishing transversal forces in \( \Omega \), and for a given couple field \( \hat{M} \) acting on \( \partial \Omega \), the transversal displacement \( w \in H^2(\Omega) \) of the plate satisfies the following mixed boundary value problem

\[
\begin{align*}
(1.1) & \quad \text{div}(\text{div}(P \nabla^2 w)) = 0, \quad \text{in } \Omega \setminus \overline{D}, \\
(1.2) & \quad (P \nabla^2 w)n \cdot n = -\hat{M}n, \quad \text{on } \partial \Omega, \\
(1.3) & \quad \text{div}(P \nabla^2 w) \cdot n + ((P \nabla^2 w)n \cdot \tau)_{,s} = (\hat{M}\tau)_{,s}, \quad \text{on } \partial \Omega, \\
(1.4) & \quad w\big|_{\partial D} \in \mathcal{A}, \quad \text{in } \overline{D}, \\
(1.5) & \quad w^e_{,n} = w^i_{,n}, \quad \text{on } \partial D,
\end{align*}
\]

coupled with the equilibrium conditions for the rigid inclusion \( D \)

\[
(1.6) \quad \int_{\partial D} \left( \text{div}(P \nabla^2 w) \cdot n + ((P \nabla^2 w)n \cdot \tau)_{,s} \right) g - ((P \nabla^2 w)n \cdot n)g_{,n} = 0,
\]

for every \( g \in \mathcal{A} \),

where \( \mathcal{A} \) denotes the space of affine functions. We recall that, from the physical point of view, the boundary conditions (1.4)-(1.5) correspond to ideal
connection between the boundary of the rigid inclusion and the surrounding elastic material, see, for example, [O-Ri, Section 10.10]. The unit vectors $n$ and $\tau$ are the outer normal and the tangent vector to the boundary of $\Omega \setminus \overline{D}$, respectively. We denote by $w_{is}$, $w_{in}$ the derivatives of the function $w$ with respect to the arclength $s$ and to the normal direction, respectively. Moreover, we have defined $w^e \equiv w|_{\Omega \setminus \overline{D}}$ and $w^i \equiv w|_{\overline{D}}$. The functions $\widehat{M}_i$, $\widehat{M}_n$ are the twisting and bending component of the assigned couple field $\widehat{M}$, respectively. The plate tensor $P$ is given by $P = h^3 \mathbb{C}$, where $h$ is the constant thickness of the plate and $\mathbb{C}$ is the non-homogeneous Lamé elasticity tensor describing the response of the material.

The existence of a solution $w \in H^2(\Omega)$ of the problem (1.1)–(1.6) is ensured by general results, provided that $\widehat{M} \in H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)$, with $\int_{\partial \Omega} \widehat{M}_i = 0$, for $i = 1, 2$ (where $\widehat{M} = \widehat{M}_2 e_1 + \widehat{M}_1 e_2$ is the representation of $\widehat{M}$ in cartesian coordinates), and $P$ is bounded and strongly convex. Let us notice that $w$ is uniquely determined up to addition of an affine function.

Let us denote by $w_i$ a solution to (1.1)–(1.6) for $D = D_i$, $i = 1, 2$. In order to deal with the stability issue, we found it convenient to replace each solution $w_i$ with $v_i = w_i - g_i$, where $g_i$ is the affine function which coincides with $w_i$ on $\partial D_i$, $i = 1, 2$. By this approach, maintaining the same letter to denote the solution, the equilibrium problem (1.1)–(1.5) can be rephrased in terms of the following mixed boundary value problem in $\Omega \setminus \overline{D}$ with homogeneous Dirichlet conditions on the boundary of the rigid inclusion

\[
\begin{align*}
\text{(1.7)} \quad & \text{div(div}(P \nabla^2 w)) = 0, & \text{in } \Omega \setminus \overline{D}, \\
\text{(1.8)} \quad & (P \nabla^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial \Omega, \\
\text{(1.9)} \quad & \text{div}(P \nabla^2 w) \cdot n + ((P \nabla^2 w)n \cdot \tau)_s = (\widehat{M}_\tau)_s, & \text{on } \partial \Omega, \\
\text{(1.10)} \quad & w = 0, & \text{on } \partial D, \\
\text{(1.11)} \quad & w_{in} = 0, & \text{on } \partial D,
\end{align*}
\]

for which there exists a unique solution $w \in H^2(\Omega \setminus \overline{D})$. The arbitrariness of this normalization, related to the fact that $g_i$ is unknown, $i = 1, 2$, leads to the following formulation of the stability issue.

Given an open portion $\Sigma$ of $\partial \Omega$, satisfying suitable regularity assumptions, and given two solutions $w_i$ to (1.7)–(1.11) when $D = D_i$, $i = 1, 2$, satisfying, for some $\epsilon > 0$,

\[
\text{(1.12)} \quad \min_{g \in A} \left\{ \|w_1 - w_2 - g\|_{L^2(\Sigma)} + \|(w_1 - w_2 - g)_n\|_{L^2(\Sigma)} \right\} \leq \epsilon,
\]

to evaluate the rate at which the Hausdorff distance $d_H(\overline{D_1}, \overline{D_2})$ between $D_1$ and $D_2$ tends to zero as $\epsilon$ tends to zero.
In this paper we prove the following quantitative stability estimate of log

type for inclusions $D$ of $C^{0,\alpha}$ class:

\begin{equation}
    d_H(D_1, D_2) \leq C |\log \epsilon|^{-\eta},
\end{equation}

where $C$, $\eta$, $C > 0$ and $\eta > 0$, are constants only depending on the a priori
data, see Theorem 3.1 for a precise statement.

The above estimate is an improvement of the log-log type stability esti-

mate found in [M-Ro-Ve2], although it must be said that the latter is not

restricted to isotropic materials and also applies to less regular inclusions

(e.g., $D$ of $C^{3,1}$ class). On the other hand, in the present paper we remove

the hypothesis that the support of the Neumann data $\tilde{M}$ is strictly contained

in $\partial \Omega$, which was assumed in [M-Ro-Ve2, Section 2.1].

It is worth to notice that a single logarithmic rate of convergence for the

fourth order elliptic equation modelling the deflection of a Kirchhoff-Love

plate is expected to be optimal, as it is in fact for the analogous inverse

problem in the scalar elliptic case, which models the detection of perfectly

conducting inclusions in an electric conductor in terms of measurements of

potential and current taken on an accessible portion of the boundary of the

body, as shown by the counterexamples due to Alessandrini ([Al]), Alessan-
drini and Rondi ([Al-R]), see also [Dc-R].

The methods used to prove (1.13) are inspired to the approach presented

in the seminal paper [Al-Be-Ro-Ve] where, for the first time, it was shown

how logarithmic stability estimates for the inverse problem of determining

unknown boundaries can be derived by using quantitative estimates of Strong

Unique Continuation at the Boundary (SUCB), which ensure a polynomial

vanishing rate of the solutions satisfying homogeneous Dirichlet or Neumann

conditions at the boundary. Precisely, in [Al-Be-Ro-Ve] the key tool was a

Doubling Inequality at the boundary established by Adolfsson and Escauri-
aza in [A-E].

Following the direction traced in [Al-Be-Ro-Ve], other kinds of quantita-
tive estimates of the SUCB turned out to be crucial properties to prove op-
timal stability estimates for inverse boundary value problems with unknown
boundaries in different frameworks, see for instance [S] where the case of
Robin boundary condition is investigated. Let us recall, in the context of the

case of thermic conductors involving parabolic equations, the three cylin-
ders inequality and the one-sphere two-cylinders inequality at the boundary

([Ca-Ro-Ve1], [Ca-Ro-Ve2], [E-F-Ve], [E-Ve], [Ve1]), and similar estimate at

the boundary for the case of wave equation with time independent coefficients

([S-Ve], [Ve2], [Ve3]).

In the present paper, the SUCB property used to improve the double

logarithmic estimate found in [M-Ro-Ve2] takes the form of an optimal three
spheres inequality at the boundary. This latter result was recently proved in [Al-Ro-Ve] for isotropic elastic plates under homogeneous Dirichlet boundary conditions, and leads to a Finite Vanishing Rate at the Boundary (Proposition 3.6).

Other main mathematical tools are quantitative estimates of Strong Unique Continuation at the Interior, essentially based on a three spheres inequality at the interior obtained in [M-Ro-Ve1] which allows to derive quantitative estimates of unique continuation from Cauchy data (Proposition 3.3), the Finite Vanishing Rate at the Interior (Proposition 3.5) and a Lipschitz estimate of Propagation of Smallness (Proposition 3.4) for the solutions to the plate equation.

Let us observe that estimate (1.13) is the first stability estimate with optimal rate of convergence in the framework of linear elasticity. Indeed, up to now, the analogous estimate for the determination, within isotropic elastic bodies, of rigid inclusions ([M-Ro2]), cavities ([M-Ro1]) or pressurized cavities ([As-Be-Ro]) show a double logarithmic character, and the same convergence rate has been established by Lin, Nakamura and Wang for star-shaped cavities inside anisotropic elastic bodies ([L-N-W]). Finally, it is worth noticing that our approach could be extended to find a log-type stability estimate analogous to (1.13) for the determination of a cavity inside the plate, provided the SUCB property is available for homogeneous Neumann boundary conditions.

The plan of the paper is as follows. Main notation and a priori information are presented in section 2. In section 3, we first state our main result (Theorem 3.1). In the same section we also state some auxiliary propositions regarding the estimate of continuation from Cauchy data (Proposition 3.3) and from the interior (Proposition 3.4), and the determination of the finite vanishing rate of the solutions to the plate equation at the interior (Proposition 3.5) and at the Dirichlet boundary (Proposition 3.6). Finally, in the second part of section 3 we give a proof of Theorem 3.1.

2 Notation

Let $P = (x_1(P), x_2(P))$ be a point of $\mathbb{R}^2$. We shall denote by $B_r(P)$ the disk in $\mathbb{R}^2$ of radius $r$ and center $P$ and by $R_{a,b}(P)$ the rectangle of center $P$ and sides parallel to the coordinate axes, of length $2a$ and $2b$, namely $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a, |x_2 - x_2(P)| < b\}$. To simplify the notation, we shall denote $B_r = B_r(O), R_{a,b} = R_{a,b}(O)$.

Given a bounded domain $\Omega$ in $\mathbb{R}^2$ we shall denote

\begin{equation}
\Omega_{\rho} = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \rho\}.
\end{equation}
When representing locally a boundary as a graph, we use the following definition.

**Definition 2.1.** \((C^{k,\alpha} \text{ regularity})\) Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\). Given \(k, \alpha\), with \(k \in \mathbb{N}, 0 < \alpha \leq 1\), we say that a portion \(S\) of \(\partial \Omega\) is of class \(C^{k,\alpha}\) with constants \(r_0, M_0 > 0\), if, for any \(P \in S\), there exists a rigid transformation of coordinates under which we have \(P = 0\) and

\[
\Omega \cap R_{r_0, 2M_0r_0} = \{ x \in R_{r_0, 2M_0r_0} \mid x_2 > g(x_1) \},
\]

where \(g\) is a \(C^{k,\alpha}\) function on \([-r_0, r_0]\) satisfying

\[
g(0) = g'(0) = 0,
g\|_{C^{k,\alpha}([-r_0, r_0])} \leq M_0 r_0,
\]

We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous and coincide with the standard definition when the dimensional parameter equals one. For instance,

\[
\|w\|_{H^{-1/2}(\partial \Omega)} = r_0^{-1} \left( \int_{\Omega} w^2 + r_0^2 \int_{\Omega} |\nabla w|^2 \right)^{1/2},
\]

and so on for boundary and trace norms.

Given a bounded domain \(\Omega\) in \(\mathbb{R}^2\) such that \(\partial \Omega\) is of class \(C^{k,\alpha}\), with \(k \geq 1\), we consider as positive the orientation of the boundary induced by the outer unit normal \(n\) in the following sense. Given a point \(P \in \partial \Omega\), let us denote by \(\tau = \tau(P)\) the unit tangent at the boundary in \(P\) obtained by applying to \(n\) a counterclockwise rotation of angle \(\pi/2\), that is

\[
\tau = e_3 \times n,
\]

where \(\times\) denotes the vector product in \(\mathbb{R}^3\) and \(\{e_1, e_2, e_3\}\) is the canonical basis in \(\mathbb{R}^3\).

Throughout the paper, in order to simplify the notation, when we shall write \(\int_{\partial \Omega} uv\) with \(u \in H^{-1/2}(\partial \Omega, \mathbb{R}^2), v \in H^{1/2}(\partial \Omega, \mathbb{R}^2)\), we mean the duality pairing \(<u, v>_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)}\), and similarly for other trace norms.

In the sequel we shall denote by \(C\) constants which may change from line to line.
2.1 A priori information

i) A priori information on the domain.

Let us consider a thin plate $\Omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ with middle surface represented by a bounded domain $\Omega$ in $\mathbb{R}^2$ and having uniform thickness $h$, $h << \text{diam}(\Omega)$.

We shall assume that, given $r_0$, $M_1 > 0$,

$$\text{diam}(\Omega) \leq M_1 r_0.$$  \hspace{1cm} (2.3)

We shall also assume that $\Omega$ contains an open simply connected rigid inclusion $D$ such that

$$\text{dist}(D, \partial \Omega) \geq r_0.$$ \hspace{1cm} (2.4)

Moreover, we denote by $\Sigma$ an open portion within $\partial \Omega$ representing the part of the boundary where measurements are taken.

Concerning the regularity of the boundaries, given $M_0 \geq \frac{1}{2}$ and $\alpha$, $0 < \alpha \leq 1$, we assume that

$$\partial \Omega \text{ is of class } C^{2,1} \text{ with constants } r_0, M_0,$$ \hspace{1cm} (2.5)

$$\Sigma \text{ is of class } C^{3,1} \text{ with constants } r_0, M_0.$$ \hspace{1cm} (2.6)

$$\partial D \text{ is of class } C^{6,\alpha} \text{ with constants } r_0, M_0.$$ \hspace{1cm} (2.7)

Let us notice that, without loss of generality, we have chosen $M_0 \geq \frac{1}{2}$ to ensure that $B_{r_0}(P) \subset R_{r_0,2M_0r_0}(P)$ for every $P \in \partial \Omega$.

Moreover, we shall assume that for some $P_0 \in \Sigma$

$$\partial \Omega \cap R_{r_0,2M_0r_0}(P_0) \subset \Sigma.$$ \hspace{1cm} (2.8)

ii) Assumptions about the boundary data.

On the Neumann data $\hat{M}$, $\hat{M} = \hat{M}_n n + \hat{M}_\tau \tau$, we assume that

$$\hat{M} \in L^2(\partial \Omega, \mathbb{R}^2), \quad (\hat{M}_n, (\hat{M}_\tau)_s) \neq 0,$$ \hspace{1cm} (2.9)

$$\text{supp(}\hat{M}) \subset \subset \Sigma,$$ \hspace{1cm} (2.10)

the (obvious) compatibility condition on the $i$th component of $\hat{M}$ in a given Cartesian coordinate system

$$\int_{\partial \Omega} \hat{M}_i = 0, \quad i = 1, 2,$$ \hspace{1cm} (2.11)
and that, for a given constant $F > 0$,

\[ \frac{\| \hat{M} \|_{L^2(\partial \Omega, \mathbb{R}^2)}}{\| \hat{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}} \leq F, \]  

where we denote

\[ \| \hat{M} \|_{L^2(\partial \Omega, \mathbb{R}^2)} = \| \hat{M}_n \|_{L^2(\partial \Omega)} + r_0 \| (\hat{M}_T), s \|_{H^{-1}(\partial \Omega)} , \]  

\[ \| \hat{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} = \| \hat{M}_n \|_{H^{-\frac{1}{2}}(\partial \Omega)} + r_0 \| (\hat{M}_T), s \|_{H^{-\frac{3}{2}}(\partial \Omega)} , \]  

and similarly for other trace norms.

\[ \text{iii) Assumptions about the elasticity tensor.} \]

Let us assume that the plate is made by elastic isotropic material, the plate tensor $\mathbb{P}$ is defined by

\[ \mathbb{P}A = B [(1 - \nu) \lambda \nabla^2 w + \nu \Delta w I_2] , \]

for every $2 \times 2$ matrix $A$, where $I_2$ is the $2 \times 2$ identity matrix and $\text{tr}(A)$ denotes the trace of the matrix $A$. The bending stiffness (per unit length) of the plate is given by the function

\[ B(x) = \frac{h^3}{12} \left( \frac{E(x)}{1 - \nu^2(x)} \right) , \]

where the Young's modulus $E$ and the Poisson's coefficient $\nu$ can be written in terms of the Lamé moduli as follows

\[ E(x) = \frac{\mu(x)(2\mu(x) + 3\lambda(x))}{\mu(x) + \lambda(x)} , \quad \nu(x) = \frac{\lambda(x)}{2(\mu(x) + \lambda(x))} . \]

Hence, in this case, the displacement equation of equilibrium (1.1) is

\[ \text{div} \left( \text{div} \left( B((1 - \nu) \nabla^2 w + \nu \Delta w I_2) \right) \right) = 0, \quad \text{in } \Omega. \]

We make the following strong convexity assumptions on the Lamé moduli

\[ \mu(x) \geq \alpha_0 > 0, \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0 > 0, \quad \text{in } \overline{\Omega}, \]

where $\alpha_0$, $\gamma_0$ are positive constants.

We assume that the Lamé moduli $\lambda, \mu$ satisfy the following regularity assumptions

\[ \| \lambda \|_{C^4(\overline{\Omega})}, \quad \| \mu \|_{C^4(\overline{\Omega})} \leq \Lambda_0. \]
It should be noted that the regularity hypotheses required on the elastic coefficients and on the boundary of the inclusion are required to apply the arguments and techniques used in [Al-Ro-Ve] to derive the SUCB for isotropic elastic plates.

Under the above assumptions, the weak formulation of the problem (1.7)–(1.11) consists in finding \( w \in H^2(\Omega \setminus \overline{D}) \), with \( w = 0 \) and \( w, n = 0 \) on \( \partial D \), such that

\[
\int_{\Omega \setminus \overline{D}} P \nabla^2 w \cdot \nabla^2 v = \int_{\partial \Omega} M_{\tau,s} v - M_n v, n,
\]
for every \( v \in H^2(\Omega \setminus \overline{D}) \), with \( v = 0 \) and \( v, n = 0 \) on \( \partial D \). By standard variational arguments (see, for example, [Ag]), the above problem has a unique solution \( w \in H^2(\Omega \setminus \overline{D}) \) satisfying the stability estimate

\[
\|w\|_{H^2(\Omega \setminus \overline{D})} \leq C \frac{r^2_0}{\|\hat{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}},
\]
where \( C > 0 \) only depends on \( \alpha_0, \gamma_0, M_0, \) and \( M_1 \).

In the sequel, we shall refer to the set of constants \( \alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1 \) and \( F \) as to the \textit{a priori} data.

### 3 Statement and proof of the main result

Here and in the sequel we shall denote by \( G \) the connected component of \( \Omega \setminus (D_1 \cup D_2) \) such that \( \Sigma \subset \partial G \).

**Theorem 3.1** (Stability result). \textit{Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) satisfying (2.3) and (2.5). Let \( D_i, i = 1, 2 \), be two simply connected open subsets of \( \Omega \) satisfying (2.4) and (2.7). Moreover, let \( \Sigma \) be an open portion of \( \partial \Omega \) satisfying (2.6) and (2.8). Let \( \hat{M} \in L^2(\partial \Omega, \mathbb{R}^2) \) satisfy (2.9)–(2.12) and let the plate tensor \( P \) given by (2.15) with Lamé moduli satisfying the regularity assumptions (2.20) and the strong convexity condition (2.19). Let \( w_i \in H^2(\Omega \setminus \overline{D_i}) \) be the solution to (1.7)–(1.11) when \( D = D_i \), \( i = 1, 2 \). If, given \( \epsilon > 0 \), we have

\[
\min_{g \in A} \left\{ \|w_1 - w_2 - g\|_{L^2(\Sigma)} + r_0 \|(w_1 - w_2 - g), n\|_{L^2(\Sigma)} \right\} \leq \epsilon,
\]
then we have

\[
d_H(D_1, D_2) \leq r_0 \omega \left( \frac{\epsilon}{\frac{1}{r_0^2} \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}} \right),
\]

where \( 0 < \omega < 1 \) is a constant. \textit{Proof.}
where $\omega$ is an increasing continuous function on $[0, \infty)$ which satisfies

$$
(3.3) \quad \omega(t) \leq C(|\log |t||)^{-\eta}, \quad \text{for every } t, \ 0 < t < 1,
$$

and $C, \eta, C > 0, \eta > 0$, are constants only depending on the a priori data.

**Remark 3.2.** Before presenting the proof of the theorem, it is appropriate to underline the optimality of the estimate (3.2) and the improvement it provides with respect to previous results. The presence of a logarithm in the stability estimate is expected and inevitable, since this is a consequence of the ill-posedness of the Cauchy problem (see Proposition 3.3). As it was mentioned in the introduction, a log-log type stability estimate was already derived in [M-Ro-Ve2]. In that paper, the additional logarithm was essentially the consequence of the application of a unique continuation result from the interior expressed in the form of the Lipschitz Propagation of Smallness for solutions to the plate equation (see also Proposition 3.5). In the proof of Theorem 3.1, instead, a different line of reasoning was followed, that is, inspired by the paper [Al-Be-Ro-Ve], we exploited the polynomial vanishing rate of the solution, both inside $\Omega \setminus D$ and close to the boundary of $D$. The former was in fact already available from the results in [M-Ro-Ve2], while the latter was only recently proved in [Al-Ro-Ve] for homogeneous Dirichlet boundary conditions on $\partial D$.

The proof of Theorem 3.1 is obtained from a sequence of propositions. The following proposition can be derived by merging Proposition 3.4 of [M-Ro-Ve2] and geometrical arguments contained in Proposition 3.6 of [Al-Be-Ro-Ve].

**Proposition 3.3** (Stability Estimate of Continuation from Cauchy Data [M-Ro-Ve2, Proposition 3.4]). Let the hypotheses of Theorem 3.1 be satisfied. We have

$$
(3.4) \quad \int_{(\Omega \setminus D) \setminus D_1} |\nabla^2 w_1|^2 \leq r_0^2 \|\tilde{M}\|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \omega \left( \frac{\epsilon}{r_0^2 \|\tilde{M}\|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}} \right),
$$

$$
(3.5) \quad \int_{(\Omega \setminus D) \setminus D_2} |\nabla^2 w_2|^2 \leq r_0^2 \|\tilde{M}\|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \omega \left( \frac{\epsilon}{r_0^2 \|\tilde{M}\|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}} \right),
$$

where $\omega$ is an increasing continuous function on $[0, \infty)$ which satisfies

$$
(3.6) \quad \omega(t) \leq C(\log |\log |t||)^{-\frac{1}{2}}, \quad \text{for every } t < e^{-1},
$$

with $C > 0$ only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0$ and $M_1$. 

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Moreover, there exists $d_0 > 0$, with $\frac{d_0}{r_0}$ only depending on $M_0$, such that if $d_H(\Omega \setminus D_1, \Omega \setminus D_2) \leq d_0$ then (3.4)–(3.5) hold with $\omega$ given by

\begin{equation}
\omega(t) \leq C|\log t|^{-\sigma}, \quad \text{for every } t < 1,
\end{equation}

where $\sigma > 0$ and $C > 0$ only depend on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$, $L$ and $\tilde{r}_0$.

Next two propositions are quantitative versions of the SUCP property at the interior for solutions to the plate equilibrium problem. Precisely, Proposition 3.4 has global character and gives a lower bound of the strain energy density over any small disc compactly contained in $\Omega \setminus \overline{D}$ in terms of the Neumann boundary data. Instead, Proposition 3.5 establishes a polynomial order of vanishing for solutions to the plate problem at interior points of $\Omega \setminus \overline{D}$.

**Proposition 3.4** (Lipschitz Propagation of Smallness). Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ satisfying (2.3) and (2.5). Let $D$ be an open simply connected subset of $\Omega$ satisfying (2.4), (2.7). Let $w \in H^2(\Omega \setminus \overline{D})$ be the solution to (1.7)–(1.11), coupled with the equilibrium condition (1.6), where the plate tensor $\mathbb{P}$ is given by (2.15) with Lamé moduli satisfying the regularity assumptions (2.20) and the strong convexity condition (2.19) and with $\hat{M} \in L^2(\partial \Omega, \mathbb{R}^2)$ satisfying (2.9)–(2.12).

There exists $s > 1$, only depending on $\alpha_0$, $\gamma_0$, $\Lambda_0$ and $M_0$, such that for every $\rho > 0$ and every $\bar{x} \in (\Omega \setminus \overline{D})_{s\rho}$, we have

\begin{equation}
\int_{B_\rho(\bar{x})} |\nabla^2 w|^2 \geq \frac{C\gamma_0^3}{\exp \left[ A \left( \frac{r_0}{\rho} \right) B \right]} \| \hat{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}^2,
\end{equation}

where $A > 0$, $B > 0$ and $C > 0$ only depend on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$ and $F$.

**Proof.** The proof of this proposition, rather technical, is mainly based on the derivation of a lower bound of the strain energy density over the disc $B_\rho(\bar{x})$ in terms of the strain energy density over all the domain $\Omega \setminus \overline{D}$. This estimate requires a geometrical construction involving a number of iterated applications of the three spheres inequality (3.8) which leads to an exponential dependence on the radius $\rho$.

The arguments follow the lines of the proof of Proposition 3.3 in [M-Ro-Ve2], the only difference consisting in estimating from below the strain energy $\int_{\Omega \setminus \overline{D}} |\nabla^2 w|^2$ in terms of the $H^{-\frac{1}{2}}$ norm of $\hat{M}$ as defined in (2.14), that is considering only the tangential derivative of the normal component $\hat{M}_\tau$. This
more natural choice allows to remove the technical hypothesis that the support of $\widetilde{M}$ is strictly contained in $\partial \Omega$, assumed in Lemma 4.6 in [M-Ro-Ve2], see also [M-Ro-Ve1, Lemma 7.1] for details.

Precisely, we only need to prove the following trace-type inequality

\begin{equation}
\| \widetilde{M} \|_{H^{-\frac{1}{2}}(\partial \Omega; \mathbb{R}^2)} \leq C \| \nabla^2 w \|_{L^2(\Omega; \mathbb{R}^3)},
\end{equation}

where $C$ only depends on $M_0$, $M_1$ and $\Lambda_0$.

Let us first estimate the $H^{-\frac{1}{2}}(\partial \Omega)$ norm of $\widetilde{M}$. Given any $g \in H^{\frac{1}{2}}(\partial \Omega)$, let $v \in H^2(\Omega \setminus \overline{D})$ such that $v = 0$, $v_n = g$ on $\partial \Omega$, and $\|v\|_{H^2(\Omega; \mathbb{R}^3)} \leq C r_0 \|g\|_{H^{\frac{1}{2}}(\partial \Omega)}$, where $C$ only depends on $M_0$ and $M_1$. By the weak formulation (2.21) of the equilibrium problem (1.7)–(1.11), we have

\begin{equation}
\int_{\partial \Omega} \widetilde{M}_n g = \int_{\partial \Omega} \widetilde{M}_n v_n + \int_{\partial \Omega} \widetilde{M}_{\tau,s} v = - \int_{\Omega \setminus \overline{D}} P \nabla^2 w \cdot \nabla^2 v \leq \leq \left( \int_{\partial \Omega} P \nabla^2 w \cdot \nabla^2 w \right)^{\frac{1}{2}} \left( \int_{\partial \Omega} P \nabla^2 v \cdot \nabla^2 v \right)^{\frac{1}{2}} \leq C \| \nabla^2 w \|_{L^2(\Omega; \mathbb{R}^3)} \|v\|_{H^2(\Omega; \mathbb{R}^3)} \leq C r_0 \|g\|_{H^{\frac{1}{2}}(\partial \Omega)} \|\nabla^2 w\|_{L^2(\Omega; \mathbb{R}^3)},
\end{equation}

with $C$ only depending on $M_0$, $M_1$ and $\Lambda_0$. Therefore

\begin{equation}
\| \widetilde{M}_n \|_{H^{-\frac{1}{2}}(\partial \Omega)} = \sup_{g \in H^{\frac{1}{2}}(\partial \Omega)} \frac{1}{r_0} \int_{\partial \Omega} \widetilde{M}_n g \leq C \| \nabla^2 w \|_{L^2(\Omega; \mathbb{R}^3)},
\end{equation}

with $C$ only depending on $M_0$, $M_1$ and $\Lambda_0$.

Next, let us estimate the $H^{-\frac{1}{2}}(\partial \Omega)$ norm of $\widetilde{M}_{\tau,s}$. Given any $g \in H^{\frac{1}{2}}(\partial \Omega)$, let $v \in H^2(\Omega \setminus \overline{D})$ such that $v = g$ on $\partial \Omega$, and $\|v\|_{H^2(\Omega; \mathbb{R}^3)} \leq C \|g\|_{H^{\frac{1}{2}}(\partial \Omega)}$, where $C$ only depends on $M_0$ and $M_1$. By recalling (3.11), we can compute

\begin{equation}
- \int_{\partial \Omega} \widetilde{M}_{\tau,s} g = \int_{\partial \Omega} -\widetilde{M}_{\tau,s} v - \widetilde{M}_n v_n + \int_{\partial \Omega} \widetilde{M}_n v_n = \leq C \| \nabla^2 w \|_{L^2(\Omega; \mathbb{R}^3)} \|v\|_{H^2(\Omega; \mathbb{R}^3)} + C r_0 \|\widetilde{M}_n\|_{H^{-\frac{1}{2}}(\partial \Omega)} \|v_n\|_{H^{\frac{1}{2}}(\partial \Omega)} \leq \|v\|_{H^2(\Omega; \mathbb{R}^3)} \left( \| \nabla^2 w \|_{L^2(\Omega; \mathbb{R}^3)} + \| \widetilde{M}_n \|_{H^{-\frac{1}{2}}(\partial \Omega)} \right) \leq C \|g\|_{H^{\frac{1}{2}}(\partial \Omega)} \|\nabla^2 w\|_{L^2(\Omega; \mathbb{R}^3)}.
\end{equation}
with $C$ only depending on $M_0$, $M_1$ and $\Lambda_0$. Therefore

$$\|\tilde{M}_{r,s}\|_{H^{-\frac{1}{2}}(\partial\Omega)} = \sup_{g \in H^{3/2}(\partial\Omega), \|g\|_{H^{3/2}(\partial\Omega)} = 1} \frac{1}{r_0} \int_{\partial\Omega} \tilde{M}_{r,s} g \leq \frac{C}{r_0} \|\nabla^2 w\|_{L^2(\Omega \setminus \overline{D})},$$

with $C$ only depending on $M_0$, $M_1$ and $\Lambda_0$. By (3.11) and (3.13), (3.9) follows.

**Proposition 3.5** (Finite Vanishing Rate at the Interior). Under the hypotheses of Proposition 3.4, there exist $\tilde{c}_0 < \frac{1}{2}$ and $C > 1$, only depending on $\alpha_0$, $\gamma_0$ and $\Lambda_0$, such that, for every $r \in (0, r_0)$ and for every $\overline{x} \in \Omega \setminus \overline{D}$ such that $B_r(\overline{x}) \subset \Omega \setminus D$, and for every $r_1 < \tilde{c}_0 r_3$, $r_3 < s_1 r$ and for any solution $v$ to (2.18), we have

$$\int_{B_{r_1}(\overline{x})} |\nabla^2 v|^2 \geq C \left( \frac{r_1}{r} \right)^{\tau_0} \int_{B_r(\overline{x})} |\nabla^2 v|^2,$$

where $\tau_0 \geq 1$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$, $\frac{\omega}{\overline{r}}$ and $F$.

**Proof.** The above estimate is based on the following three spheres inequality at the interior, which was obtained in [M-Ro-Ve1, Theorem 6.3]: there exist $c_0$, $0 < c_0 < 1$, $s_1$, $0 < s_1 < 1$ and $C_1 > 1$ only depending on $\alpha_0$, $\gamma_0$ and $\Lambda_0$, such that for every $r > 0$, for every $\overline{x} \in (\Omega \setminus \overline{D})$ such that $B_r(\overline{x}) \subset \Omega \setminus D$, for every $r_1 < r_2 < c_0 r_3$, $r_3 < s_1 r$ and for any solution $v$ to (2.18), we have

$$H(v; r_2) \leq \left( \frac{C_1 \overline{r}}{r_2} \right)^{C_1} \left( H \left( v; \frac{r_1}{2} \right) \right)^{\theta_0} \left( H \left( v; \frac{r_3}{2} \right) \right)^{1-\theta_0},$$

where

$$H(v; t) = \sum_{k=0}^{3} t^{2k} \int_{B_t(\overline{x})} \left| \nabla^{2k} v \right|^2, \quad \text{for every } t \in (0, \overline{r}]$$

and

$$\theta_0 = \frac{\log \left( \frac{c_0 r_3}{r_2} \right)}{2 \log \left( \frac{\overline{r}}{r_1} \right)}.$$

Let $w$ be the solution to boundary value problem (1.7)–(1.11) and let us denote

$$v(x) = w(x) - a - \gamma \cdot (x - \overline{x}),$$

13
where

\[ a = \frac{1}{|B_{r_1}(x)|} \int_{B_{r_1}(x)} w \quad \text{and} \quad \gamma = \frac{1}{|B_{r_1}(x)|} \int_{B_{r_1}(x)} \nabla w. \]

By Caccioppoli–type inequality [M-Ro-Ve1, Proposition 6.2] and Poincaré inequality we get

\[ H \left( v; \frac{r_1}{2} \right) \leq C r_1^4 \int_{B_{r_1}(x)} |\nabla^2 w|^2, \]

where \( C \) depends on \( \alpha_0, \gamma_0 \) and \( \Lambda_0 \) only.

Now, we estimate from above \( H \left( v; \frac{r_3}{2} \right) \). By Caccioppoli–type inequality we have

\[ H \left( v; \frac{r_3}{2} \right) \leq C \sum_{k=0}^2 r_3^{2k} \int_{B_{2r_3}(x)} |\nabla^k v|^2, \]

where \( C \) depends on \( \alpha_0, \gamma_0 \) and \( \Lambda_0 \) only. In addition, by (3.19) and Sobolev’s inequality [Ag, Theorem 3.9] we have

\[ |a| \leq \|w\|_{L^\infty(B_{2r_3/3}(x))} \leq C \|w\|_{H^2(B_{2r_3/3}(x))}, \]

and

\[ |\gamma| \leq \|\nabla w\|_{L^\infty(B_{2r_3/3}(x))} \leq C r_3^{-1} \|w\|_{H^3(B_{2r_3/3}(x))}, \]

where \( C \) is an absolute constant. Hence, by (3.22), (3.23) and by Caccioppoli–type inequality we have

\[ \sum_{k=0}^2 r_3^{2k} \int_{B_{2r_3}(x)} |\nabla^k v|^2 \leq C \sum_{k=0}^3 r_3^{2k} \int_{B_{2r_3}(x)} |\nabla^k w|^2 \leq C' \sum_{k=0}^2 r_3^{2k} \int_{B_{r_3}(x)} |\nabla^k w|^2, \]

where \( C \) is an absolute constant and \( C' \) depends on \( \alpha_0, \gamma_0 \) and \( \Lambda_0 \) only. In addition, by (2.22), (3.21) and (3.24) we obtain

\[ H \left( v; \frac{r_3}{2} \right) \leq C_2 r_0^6 \|\tilde{M}\|_{H^{-\frac{1}{2}}(\Omega, \mathbb{R}^2)}, \]

where \( C_2 \) depends on \( \alpha_0, \gamma_0, \Lambda_0, M_0 \) and \( M_1 \) only.
By (3.15), (3.20) and (3.25) we have
\[
\begin{align*}
\int_{B_{r_2}(\tau)} |\nabla^2 w|^2 \leq C_3 \left( \frac{C_1}{r_2} \right)^{C_1} \left( \int_{B_{r_1}(\tau)} |\nabla^2 w|^2 \right)^{\frac{\vartheta_0}{2}} \left( C_2 \vartheta_0 \| M \|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \right)^{1-\vartheta_0},
\end{align*}
\]
where $C_3 > 1$ depends on $\alpha_0$, $\gamma_0$ and $\Lambda_0$ only. Let us introduce the following notation
\[
\begin{align*}
g(r) &= \frac{r^4 \int_{B_r(\tau)} |\nabla^2 w|^2}{C_2 \vartheta_0 \| M \|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}}, \\
\text{for every } r \in \left( 0, \frac{r_3}{2} \right],
\end{align*}
\]
and
\[
\begin{align*}
K &= C_3 \left( \frac{C_1}{r_2} \right)^{C_1},
\end{align*}
\]
so that (3.26) is equivalent to
\[
\begin{align*}
g(r_1) \geq \left( \frac{g(r_2)}{K} \right)^{\vartheta_0^{-1}}.
\end{align*}
\]
We notice that
\[
\vartheta_0^{-1} = \log \left( \frac{r_3}{r_1} \right)^{m},
\]
where $m = \frac{2}{\log \left( \frac{\alpha_0}{\tau_2} \right)}$. Recalling the elementary identity $\kappa^{\log \zeta} = \zeta^{\log \kappa}$, by (3.29) we have
\[
\begin{align*}
g(r_1) \geq \left( \frac{r_1}{r_3} \right)^{m \log \left( \frac{K}{\vartheta_0^{-1}} \right)}.
\end{align*}
\]
Note that, by (3.25), (3.27) and (3.28) we have $\frac{K}{g(r_2)} > 1$, so that
\[
\begin{align*}
g(r_1) \geq \left( \frac{r_1}{r_3} \right)^{m \log \left( \frac{K}{\vartheta_0^{-1}} \right)}.
\end{align*}
\]
Choosing $r_2 = \tau_0 \tau$, where $\tau_0 < \frac{1}{2} \epsilon_0$, by Proposition 3.4 we have
\[
\begin{align*}
\frac{K}{g(r_2)} \leq C \exp \left[ A \left( \frac{r_0}{\tau_0 \tau} \right)^{B} \right],
\end{align*}
\]
where $C$ depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$, $F$ and $\tau_0 \tau$ only, and $A$, $B$ are the same as in Proposition 3.4.

Finally, taking into account (2.22), by (3.30) and (3.32) the thesis follows. $\blacksquare$
As noticed in the Introduction, our key SUCB property is stated in the following proposition, which is the counterpart at the boundary $\partial D$ of Proposition 3.5.

**Proposition 3.6** (Finite Vanishing Rate at the Boundary). *Under the hypotheses of Proposition 3.4, there exist $\overline{c} < \frac{1}{2}$ and $C > 1$, only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, \alpha$, such that, for every $x \in \partial D$ and for every $r_1 < \overline{c}r_0$,

\[(3.33) \int_{B_1(x) \cap (\Omega \setminus D)} w^2 \geq C \left( \frac{r_1}{r_0} \right)^\tau \int_{B_0(x) \cap (\Omega \setminus D)} w^2,\]

where $\tau \geq 1$ only depends on $\alpha_0, \gamma_0, \Lambda_0, M_0, \alpha, M_1$ and $F$.

**Proof.** By Corollary 2.3 in [Al-Ro-Ve], there exist $c < 1$, only depending on $M_0, \alpha$, and $C > 1$ only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, \alpha, M_1$ and $F$, such that, for every $x \in \partial D$ and for every $r_1 < r_2 < cr_0$,

\[(3.34) \int_{B_1(x) \cap (\Omega \setminus D)} w^2 \geq C \left( \frac{r_1}{r_0} \right)^{\frac{\log B}{\log \frac{r_0}{r_2}}} \int_{B_0(x) \cap (\Omega \setminus D)} w^2,\]

where $B > 1$ is given by

\[(3.35) B = C \left( \frac{r_0}{r_2} \right) \frac{\int_{B_0(x) \cap (\Omega \setminus D)} w^2}{\int_{B_2(x) \cap (\Omega \setminus D)} w^2},\]

Let us choose in the above inequalities $r_2 = \overline{c}r_0$, with $\overline{c} = \frac{c}{2}$.

By (2.22) we have

\[(3.36) \int_{B_0(x) \cap (\Omega \setminus D)} w^2 \leq C r_0^2 \| \overline{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)},\]

with $C$ depending on $\alpha_0, \gamma_0, M_0, \alpha, M_1$. By interpolation estimates for solutions to elliptic equations (see, for instance, [Al-Ro-Ve, Lemma 4.7], stated for the case of hemidiscs, but which holds also in the present context), we have that

\[(3.37) \int_{B_2(x) \cap (\Omega \setminus D)} w^2 \geq C r_2^2 \int_{B_2(x) \cap (\Omega \setminus D)} |\nabla^2 w|^2,\]

with $C$ depending on $\alpha_0, \gamma_0$ and $\Lambda_0$. By Proposition 3.4 and recalling the definition of $r_2$, we derive

\[(3.37) \int_{B_2(x) \cap (\Omega \setminus D)} w^2 \geq C r_0^2 \| \overline{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)},\]

with $C$ depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, \alpha, M_1$ and $F$.

By (3.36)–(3.37), we can estimate $B$ from above, obtaining the thesis. \(\square\)
Proof of Theorem 3.1. In order to estimate the Hausdorff distance between the inclusions,

\begin{equation}
\delta = d_H(D_1, D_2),
\end{equation}

it is convenient to introduce the following auxiliary distances:

\begin{equation}
d = d_H(\Omega \setminus D_1, \Omega \setminus D_2),
\end{equation}

\begin{equation}
d_m = \max \left\{ \max_{x \in \partial D_1} \text{dist}(x, \Omega \setminus D_2), \max_{x \in \partial D_2} \text{dist}(x, \Omega \setminus D_1) \right\}.
\end{equation}

Let \( \eta > 0 \) such that

\begin{equation}
\max_{i=1,2} \int_{(\Omega \setminus G) \setminus D_i} |\nabla^2 w_i|^2 \leq \eta.
\end{equation}

Following the arguments presented in \([\text{Al-Be-Ro-Ve}]\), the proof of Theorem 3.1 consists of four main steps. In step 1, we control \( d_m \) in terms of \( \eta \). Then, in step 2 we use this estimate to control \( d \) in terms of \( \eta \). The estimate of \( \delta \) in terms of \( d \) is provided in step 3. Finally, in step 4 we use Proposition 3.3 in previous estimates to obtain the thesis.

Step 1. Let us start by proving the inequality

\begin{equation}
d_m \leq C r_0 \left( \frac{\eta}{r_0^{2} \| M \|_{H^{-1/2}(\partial \Omega)}} \right)^{1/2},
\end{equation}

where \( \tau \) has been introduced in Proposition 3.6 and \( C \) is a positive constant only depending on the a priori data.

Let us assume, without loss of generality, that there exists \( x_0 \in \partial D_1 \) such that

\begin{equation}
\text{dist}(x_0, \Omega \setminus D_2) = d_m > 0.
\end{equation}

Since \( B_{d_m}(x_0 \setminus D_2 \subset \Omega \setminus G \), we have

\begin{equation}
B_{d_m}(x_0) \cap (\Omega \setminus D_1) \subset (\Omega \setminus G) \setminus D_1
\end{equation}

and then, by (3.41),

\begin{equation}
\int_{B_{d_m}(x_0) \cap (\Omega \setminus D_1)} |\nabla^2 w_1|^2 \leq \eta.
\end{equation}
Let us assume that

\[ d_m < \tau r_0, \]  

where \( \tau \) is the positive constant appearing in Proposition 3.6. Since \( w_1 = 0, \nabla w_1 = 0 \) on \( \partial D_1 \), by Poincaré inequality (see, for instance, [Al-M-Ro, Example 4.4]) and noticing that \( d_m \leq \text{diam}(\Omega) \leq M_1 r_0 \), we have

\[ \eta \geq \frac{C}{r_0^q} \int_{B_{d_m(x_0) \cap (\Omega \setminus D_1)}} w_1^2, \]

where \( C > 0 \) is a positive constant only depending on \( \alpha, M_0, M_1 \).

By Proposition 3.6, we have

\[ \eta \geq \frac{C}{r_0^q} \left( \frac{d_m}{r_0} \right)^\tau \int_{B_{r_0(x_0) \cap (\Omega \setminus D_1)}} w_1^2, \]

where \( C > 0 \) is a positive constant only depending on \( \alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1 \) and \( F \).

By Lemma 4.7 in [Al-Ro-Ve], we have

\[ \eta \geq C \left( \frac{d_m}{r_0} \right)^\tau \int_{B_{\frac{r_0}{2} x_0) \cap (\Omega \setminus D_1)}} |\nabla^2 w_1|^2, \]

where \( C > 0 \) is a positive constant only depending on \( \alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1 \).

By Proposition 3.4, we have

\[ \eta \geq C \left( \frac{d_m}{r_0} \right)^\tau r_0^2 \| \hat{M} \|_{H^{1/2}(\partial \Omega)}^2, \]

where \( C > 0 \) is a positive constant only depending on \( \alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1, F \), from which we can estimate \( d_m \)

\[ d_m \leq C r_0 \left( \frac{\eta}{r_0^q \| \hat{M} \|_{H^{1/2}(\partial \Omega)}^2} \right)^{\frac{1}{\tau}}, \]

where \( C > 0 \) is a positive constant only depending on \( \alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1, F \).
Now, let us assume that

\[(3.52) \quad d_m \geq c r_0.\]

By starting again from (3.45), and applying Proposition 3.4 and recalling \(d_m \leq M_1 r_0\), we easily have

\[(3.53) \quad d_m \leq C r_0 \left( \frac{\eta}{r_0^2 \Vert \widehat{M} \Vert^2_{H^{-1/2} (\partial \Omega)}} \right),\]

where \(C > 0\) is a positive constant only depending on \(\alpha_0, \gamma_0, \Lambda_0, M_0, M_1, F\). Assuming \(\eta \leq r_0^2 \Vert \widehat{M} \Vert^2_{H^{-1/2} (\partial \Omega)}\), we obtain (3.42).

\textit{Step 2.} Without loss of generality, let \(y_0 \in \Omega \setminus D_1\) such that

\[(3.54) \quad \text{dist}(y_0, \Omega \setminus D_2) = d.\]

It is significant to assume \(d > 0\), so that \(y_0 \in D_2 \setminus D_1\). Let us define

\[(3.55) \quad h = \text{dist}(y_0, \partial D_1),\]

possibly \(h = 0\).

There are three cases to consider:

i) \(h \leq \frac{d}{2}\);

ii) \(h > \frac{d}{2}, h \leq \frac{d_0}{2}\);

iii) \(h > \frac{d}{2}, h > \frac{d_0}{2}\).

Here the number \(d_0, 0 < d_0 < r_0\), is such that \(\frac{d_0}{r_0}\) only depends on \(M_0\), and it is the same constant appearing in Proposition 3.4. In particular, Proposition 3.6 in [Al-Be-Ro-Ve] shows that there exists an absolute constant \(C > 0\) such that if \(d \leq d_0\), then \(d \leq C d_m\).

\textit{Case i).}

By definition, there exists \(z_0 \in \partial D_1\) such that \(|z_0 - y_0| = h\). By applying the triangle inequality, we get \(\text{dist} \left( z_0, \Omega \setminus D_2 \right) \geq \frac{d}{2} \). Since, by definition, \(\text{dist} \left( z_0, \Omega \setminus D_2 \right) \leq d_m\), we obtain \(d \leq 2d_m\).

\textit{Case ii).}

It turns out that \(d < d_0\) and then, by the above recalled property, again we have that \(d \leq C d_m\), for an absolute constant \(C\).

\textit{Case iii).}
Let \( \tilde{h} = \min\{h, r_0\} \). We obviously have that \( B_{\tilde{h}}(y_0) \subset \Omega \setminus \overline{D_1} \) and \( B_d(y_0) \subset D_2 \). Let us set
\[
d_1 = \min \left\{ \frac{d}{2}, \frac{\tilde{c}_0 d_0}{4} \right\} ,
\]
where \( \tilde{c}_0 \) is the positive constant appearing in Proposition 3.5. Since \( d_1 < d \) and \( d_1 < \tilde{h} \), we have that \( B_{d_1}(y_0) \subset D_2 \setminus \overline{D_1} \) and therefore \( \eta \geq \int_{B_{d_1}(y_0)} |\nabla^2 w_1|^2 \).

Since \( \frac{d_0^2}{2} < \tilde{h} \), \( B_{\frac{d_0^2}{2}}(y_0) \subset \Omega \setminus \overline{D_1} \) so that we can apply Proposition 3.5 with \( r_1 = d_1, \tau = \frac{d_0}{2}, \) obtaining \( \eta \geq C \left( \frac{2d_1}{d_0} \right)^{\tau_0} \int_{B_{\frac{d_0}{2}}(y_0)} |\nabla^2 w_1|^2, \) with \( C > 0 \) only depending on \( \alpha_0, \gamma_0, \Lambda_0, M_0, M_1 \) and \( F \). Next, by Proposition 3.4, recalling that \( \frac{d_0}{r_0} \) only depends on \( M_0 \), we derive that
\[
d_1 \leq Cr_0 \left( \frac{\eta}{r_0^2 \|M\|^2_{H^{-1/2}(\partial \Omega)}} \right)^{\frac{1}{\tau_0}},
\]
where \( C > 0 \) only depends on \( \alpha_0, \gamma_0, \Lambda_0, M_0, M_1 \) and \( F \). For \( \eta \) small enough, \( d_1 < \frac{\tilde{c}_0 d_0}{4} \), so that \( d_1 = \frac{d}{2} \) and
\[
d \leq Cr_0 \left( \frac{\eta}{r_0^2 \|M\|^2_{H^{-1/2}(\partial \Omega)}} \right)^{\frac{1}{\tau_0}},
\]
Collecting the three cases, we have
\[
d \leq Cr_0 \left( \frac{\eta}{r_0^2 \|M\|^2_{H^{-1/2}(\partial \Omega)}} \right)^{\frac{1}{\tau_1}},
\]
with \( \tau_1 = \max\{\tau, \tau_0\} \) and \( C > 0 \) only depends on \( \alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1 \) and \( F \).

Step 3. In this step, we improve the results obtained in [Ca-Ro-Ve2, proof of Theorem 1, step 2]. By (3.56), for \( \eta \) small enough, we have that
\[
d < \frac{r_0}{4 \sqrt{1 + M_0^2}}.
\]
Let us notice that if a point \( y \) belongs to \( \overline{D_1} \setminus D_2 \), then \( \text{dist}(y, \partial D_1) \leq d \).

Without loss of generality let \( \overline{x} \in \overline{D_1} \) such that \( \text{dist}(\overline{x}, \overline{D_2}) = \delta > 0 \). Then \( \overline{x} \in \overline{D_1} \setminus D_2 \) and therefore \( \text{dist}(\overline{x}, \partial D_1) \leq d \).
Let \(w \in \partial D_1\) such that \(|w - \bar{x}| = \text{dist}(\bar{x}, \partial D_1) \leq d\).

Letting \(n\) the outer unit normal to \(D_1\) at \(w\), we may write \(x = w - |w - x|n\).

By our regularity assumptions on \(D_1\), the truncated cone \(C(\bar{x}, -n, 2(\pi/2 - \arctan M_0)) \cap B_{\min(\delta, r_0/4)}(\bar{x})\) is contained in \(D_1 \setminus \overline{D_2}\).

On the other hand, by definition of \(\delta\), \(B_\delta(\bar{x}) \subset \Omega \setminus D_2\), so that the truncated cone \(C(\bar{x}, -n, 2(\pi/2 - \arctan M_0)) \cap B_{\min(\delta, r_0/4)}(\bar{x})\) is contained in \(D_1 \setminus \overline{D_2}\).

Let us see that \(\delta < r_0/4\). In fact if, by contradiction, \(\delta \geq r_0/4\), we can consider the point \(z = \bar{x} - \frac{r_0}{4}n\). Since \(z \in D_1 \setminus D_2\), as noticed above, \(\text{dist}(z, \partial D_1) \leq d\). On the other hand, by using the fact that \(|z - w| \leq \frac{r_0}{2}\) and by the regularity of \(D_1\), it is easy to compute that \(\text{dist}(z, \partial D_1) \geq \frac{r_0}{4\sqrt{1+M_0^2}}\), obtaining a contradiction.

Hence \(\min\{\delta, \frac{r_0}{4}\} = \delta\) and, by defining \(\bar{z} = \bar{x} - \delta n\) and by analogous calculations, we can conclude that \(\delta \leq (\sqrt{1 + M_0^2})d\), which is the desired estimate of \(\delta\) in terms of \(d\).

**Step 4.** By Proposition 3.3,

\[
(3.57) \quad d \leq C r_0 \left( \log \log \left( \frac{\epsilon}{r_0^2 \|M\|_H^{-1/2}(\partial \Omega)} \right) \right)^{-\frac{1}{\tau_1}},
\]

with \(\tau_1 \geq 1\) and \(C > 0\) only depends on \(\alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1\) and \(F\). By this first rough estimate, there exists \(\epsilon_0 > 0\), only depending on \(\alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1\) and \(F\), such that, if \(\epsilon \leq \epsilon_0\), then \(d \leq d_0\). Therefore, the second part of Proposition 3.3 applies and the thesis follows.

\[ \Box \]

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