Patience Sorting and Its Generalizations

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What you see and hear depends a great deal on where you are standing; it also depends on what sort of person you are.

— C. S. Lewis, *The Magician’s Nephew*
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Patience Sorting and Its Generalizations

Abstract

Despite having been introduced in the 1960s, the card game and combinatorial algorithm *Patience Sorting* is only now beginning to receive significant attention. This is due in part to recent results like the Baik-Deift-Johansson Theorem, which suggest connections with Probabilistic Combinatorics and Random Matrix Theory.

Patience Sorting (a.k.a. *Floyd’s Game*) can be viewed as an idealized model for the immensely popular single-person card game Klondike Solitaire. Klondike is interesting from a mathematical perspective as it has long resisted the analysis of optimality for its strategies. While there is a well-known optimal greedy strategy for Floyd’s Game, we provide a detailed analysis of this strategy as well as a suitable adaption for studying more Klondike-like generalizations of Patience Sorting.

At the same time, Patience Sorting can also be viewed as an iterated, non-recursive form of the Schensted Insertion Algorithm. We study the combinatorial objects that result from this viewpoint and then extend Patience Sorting to a full bijection between the symmetric group and certain pairs of these combinatorial objects. This *Extended Patience Sorting Algorithm* is similar to the Robinson-Schensted-Knuth (or RSK) Correspondence, which is itself built from repeated application of the Schensted Insertion Algorithm.

This analysis of Patience Sorting and its generalizations naturally encounters the language of barred pattern avoidance. We also introduce a geometric form for the Extended Patience Sorting Algorithm that is naturally dual to X. G. Viennot’s celebrated Geometric RSK Algorithm. Unlike Geometric RSK, though, the lattice paths coming from Patience Sorting are allowed to intersect. We thus also give a characterization for the intersections of these lattice paths.
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Chapter 1

Introduction

1.1 Informal Overview and Motivation

Given a positive integer \( n \in \mathbb{Z}_+ \), we use \( \mathfrak{S}_n \) to denote the *symmetric group* on the set \([n] = \{1, 2, \ldots, n\}\). In other words, \( \mathfrak{S}_n \) is the set of all bijective functions on \([n]\). Each element \( \sigma \in \mathfrak{S}_n \) is called a *permutation*, and \( \sigma_i = \sigma(i) \) denotes the \( i^{th} \) function value for each \( i \in [n] \). However, even though \( \sigma \) is defined as a *function* on the set \([n]\), it is often convenient to instead regard \( \sigma \) as a *rearrangement* of the sequence of numbers \(1, 2, \ldots, n\). This, in particular, motives the so-called *two-line notation*

\[
\sigma = \begin{pmatrix}
1 & 2 & \cdots & n \\
\sigma_1 & \sigma_2 & \cdots & \sigma_n
\end{pmatrix}
\]

and its associated *one-line notation* \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \).

In keeping with this emphasis on the values \( \sigma_i \), a *subsequence* (a.k.a. *subpermutation*) of a permutation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \) is any sequence of the form \( \pi = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \), where \( k \in [n] \) and \( i_1 < i_2 < \cdots < i_k \). We denote the *length* of the subsequence \( \pi \) by \( |\pi| = k \). It is also common to call \( \pi \) a *partial permutation on* \([n]\) since it is the restriction of the bijective function \( \sigma \) to the subset \( \{i_1, i_2, \ldots, i_k\} \) of \([n]\). Note that
the components of $\sigma_1 \sigma_2 \cdots \sigma_k$ are not required to be contiguous in $\sigma$. As such, subsequences are sometimes called *scattered subsequences* in order to distinguish them from so-called (contiguous) *substrings*.

### 1.1.1 Longest Increasing Subsequences and Row Bumping

Given two permutations (thought of as rearrangement of the numbers $1, 2, \ldots, n$), it is natural to ask whether one permutation is more “out of order” than the other when compared with the strictly increasing arrangement $12\cdots n$. For example, one would probably consider $\sigma^{(1)} = 53241$ to be more “out of order” than something like $\sigma^{(2)} = 21354$. While there are various metrics for justifying such intuitive notions of “disorder”, one of the most well-studied involves the examination of a permutation’s subsequences that are themselves in strictly increasing order.

An *increasing subsequence* of a permutation is any subsequence that increases when read from left to right. For example, $234$ is an increasing subsequence of $2154$. One can also see that $235$ is an increasing subsequence of $21354$. This illustrates the nonuniqueness of *longest increasing subsequences*, and such subsequences can even be disjoint as in $456123$. The *length* of every longest increasing subsequence is nonetheless a well-defined property for a given permutation $\sigma \in \mathfrak{S}_n$, and we denote this statistic by $\ell_n(\sigma)$. For example, with notation as above, $\ell_5(\sigma^{(1)}) = \ell_5(53241) = 2$ and $\ell_5(\sigma^{(2)}) = \ell_5(21354) = 3$, which provides one possible heuristic justification for regarding $\sigma^{(1)}$ as more “out of order” than $\sigma^{(2)}$.

Given a permutation $\sigma \in \mathfrak{S}_n$, there are various methods for calculating the length of the longest increasing subsequence $\ell_n(\sigma)$. The most obvious algorithm involves directly examining every subsequence of $\sigma$, but such an approach is far from being computationally efficient as there are $2^n$ total subsequences to examine. Since a given
increasing subsequence is essentially built up from shorter increasing subsequences, a fairly routine application of so-called dynamic programming methodologies allows us to calculate $\ell_n(\sigma)$ using, in the worst possible case, $O(n^2)$ operations.

**Algorithm 1.1.1** (Calculating $\ell_n(\sigma)$ via Dynamic Programming).

**Input:** a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathcal{S}_n$

**Output:** the sequence of positive integers $L_1(\sigma), \ldots, L_n(\sigma)$

1. First set $L_1(\sigma) = 1$.

2. Then, for each $i = 2, \ldots, n$, determine the value $L_i(\sigma)$ as follows:

   (a) If $\sigma_i > \min_{1 \leq j < i} \{\sigma_j\}$, then set

   $$L_i(\sigma) = 1 + \max_{1 \leq j < i} \{L_j(\sigma) \mid \sigma_i > \sigma_j\}.$$

   (b) Otherwise, set $L_i(\sigma) = 1$.

Each value $L_i(\sigma)$ computed in Algorithm 1.1.1 is the length of the longest increasing subsequence (when reading from left to right) that is terminated by the entry $\sigma_i$ in $\sigma$. Given this data, it is then clear that

$$\ell_n(\sigma) = \max_{1 \leq i \leq n} \{L_i(\sigma)\}.$$

We illustrate Algorithm 1.1.1 in the following example.

**Example 1.1.2.** Given $\sigma = \sigma_1 \sigma_2 \cdots \sigma_9 = 364827159 \in \mathcal{S}_9$, we use Algorithm 1.1.1 to compute the sequence $L_1(\sigma), L_2(\sigma), \ldots, L_9(\sigma)$ by

- first setting $L_1(\sigma) = 1$, and then,
• for each $i = 2, \ldots, 9$, computing the values $L_i(\sigma)$ as follows:
  
  - Since $6 = \sigma_2 > \min\{\sigma_1\} = 3$, set $L_2(\sigma) = 1 + L_1(\sigma) = 2$.
  - Since $4 = \sigma_3 > \min\{\sigma_1, \sigma_2\} = 3$, set $L_3(\sigma) = 1 + L_1(\sigma) = 2$.
  - Since $8 = \sigma_4 > \min\{\sigma_1, \sigma_2, \sigma_3\} = 3$, set $L_4(\sigma) = 1 + L_3(\sigma) = 3$.
  - Since $2 = \sigma_5 < \min\{\sigma_1, \ldots, \sigma_4\} = 2$, set $L_5(\sigma) = 1$.
  - Since $7 = \sigma_6 > \min\{\sigma_1, \ldots, \sigma_5\} = 2$, set $L_6(\sigma) = 1 + L_3(\sigma) = 3$.
  - Since $1 = \sigma_7 < \min\{\sigma_1, \ldots, \sigma_6\} = 2$, set $L_7(\sigma) = 1$.
  - Since $5 = \sigma_8 > \min\{\sigma_1, \ldots, \sigma_7\} = 2$, set $L_8(\sigma) = 1 + L_3(\sigma) = 3$.
  - Since $9 = \sigma_9 > \min\{\sigma_1, \ldots, \sigma_8\} = 2$, set $L_9(\sigma) = 1 + L_8(\sigma) = 4$.

It follows that $\ell_9(\sigma) = \ell_9(364827159) = \max\{L_i(\sigma) \mid i = 1, 2, \ldots, 9\} = 4$, which can be checked by direct inspection. E.g., 3679 and 3459 are two longest increasing subsequences in 364827159.

While each term in the sequence $L_1(\sigma), L_2(\sigma), \ldots, L_n(\sigma)$ has significance in describing various combinatorial properties of the permutation $\sigma \in \mathfrak{S}_n$, there is no need to explicitly calculate every value if one is only interested in finding the length $\ell_n(\sigma)$ of the longest increasing subsequence in $\sigma$. To see this, suppose that the value $v \in \mathbb{Z}_+$ occurs in the sequence $L_1(\sigma), L_2(\sigma), \ldots, L_n(\sigma)$ at positions $i_1, i_2, \ldots, i_k \in [n]$, where $i_1 < i_2 < \cdots < i_k$. Then, from the definitions of $L_{i_1}(\sigma), L_{i_2}(\sigma), \ldots, L_{i_k}(\sigma)$, we must have that $\sigma_{i_1} > \sigma_{i_2} > \cdots > \sigma_{i_k}$. (In other words, $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$ is a decreasing subsequence of $\sigma$, which we call the $v$th left-to-right minima subsequences of $\sigma$. Such subsequences will play an important role in the analysis of Patience Sorting throughout this Dissertation. See Section 3.1) Moreover, given a particular element $\sigma_{i_j}$ in the subsequence $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$, $L_m(\sigma) > v$ for some $m > i_j$ if and only if $\sigma_m > \sigma_{i_j}$.
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Consequently, it suffices to solely keep track of the element $\sigma_{ij}$ in the subsequence $\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_k}$ in order to determine the value of $L_{i_{j+1}}(\sigma)$. This observation allows us to significantly reduce the number of steps required for computing $\ell_n(\sigma)$. The resulting *Single Row Bumping Algorithm* was first implicitly introduced by Craige Schensted [32] in 1961 and then made explicit by Knuth [21] in 1970.

**Algorithm 1.1.3** (Single Row Bumping).

**Input:** a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$

**Output:** the partial permutation $w$

1. First set $w = \sigma_1$.

2. Then, for each $i = 2, \ldots, n$, insert $\sigma_i$ into $w = w_1w_2 \cdots w_k$ using the following rule:

   (a) If $\sigma_i > w_k$, then append $\sigma_i$ to $w$ to obtain $w = w_1w_2 \cdots w_k\sigma_i$.

   (b) Otherwise, use $\sigma_i$ to "bump" the left-most element $w_j$ of $w$ that is larger than $\sigma_i$. In other words, set

   $$w = w_1w_2 \cdots w_{j-1}\sigma_iw_{j+1} \cdots w_k \quad \text{where} \quad j = \min_{1 \leq m \leq k} \{m \mid \sigma_i < \sigma_m\}.$$  

In particular, one can show that $L_i(\sigma) \geq r$ in Algorithm [1.1.1] if and only if $\sigma_i$ was inserted into $w$ at position $r$ during some iteration of Algorithm [1.1.3]. It follows that $\ell_n(\sigma)$ is equal to the length $|w|$ of $w$. (The interplay between the sequence $L_1(\sigma), \ldots, L_n(\sigma)$ and the formation of $w$ can most easily be seen by comparing Example [1.1.2] with Example [1.1.4] below.)

In terms of improved computational efficiency over Algorithm [1.1.1], note that the elements in $w$ must necessarily increase when read from left to right. Consequently,
Step 2(b) of Algorithm 1.1.3 can be accomplished with a Binary Search (see [13]), under which Single Row Bumping requires, in the worst possible case, $O(n \log(n))$ operations in order to calculate $\ell_n(\sigma)$ for a given permutation $\sigma \in S_n$. This can actually be even further reduced to $O(n \log(\log(n)))$ operations if $w$ is formed as a special type of associative array known as a van Emde Boas tree [5]; see [39] for the appropriate definitions. (Hunt and Szymanski [20] also independently gave an $O(n \log(\log(n)))$ algorithm in 1977. Their algorithm, however, computes $\ell_n(\sigma)$ as a special case of the length of the longest common subsequence in two permutations, where one of the two permutations is taken to be $12\cdots n$.)

In the following example, we illustrate Single Row Bumping (Algorithm 1.1.3) using a row of boxes for reasons that will become clear in Section 1.1.2 below. We also explicitly indicate each “bumped” value, using the null symbol “$\emptyset$” to denote the empty partial permutation. (E.g., “$[3 \, 6] \leftarrow 4 = [3 \, 4] \leadsto 6$” means that 4 has been inserted into $[3 \, 6]$ and has “bumped” 6, whereas “$[3 \, 4] \leftarrow 8 = [3 \, 4 \, 8] \leadsto \emptyset$” means that 8 has been inserted into $[3 \, 4]$ and nothing has been “bumped”.)

Example 1.1.4. Given $\sigma = 364827159 \in S_9$, we use Single Row Bumping (Algorithm 1.1.3) to form the partial permutation $w$ as follows:

- Start with $w = \emptyset$, and insert 3 to obtain $w = \emptyset \leftarrow 3 = [3]$.
- Append 6 so that $w = [3] \leftarrow 6 = [3 \, 6] \leadsto \emptyset$.
- Use 4 to bump 6: $w = [3 \, 6] \leftarrow 4 = [3 \, 4] \leadsto 6$.
- Append 8 so that $w = [3 \, 4] \leftarrow 8 = [3 \, 4 \, 8] \leadsto \emptyset$.
- Use 2 to bump 3: $w = [3 \, 4 \, 8] \leftarrow 2 = [2 \, 4 \, 8] \leadsto 3$.
- Use 7 to bump 8: $w = [2 \, 4 \, 8] \leftarrow 7 = [2 \, 4 \, 7] \leadsto 8$. 


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Even though \( w \) is not a longest increasing subsequence of \( \sigma \), one can nonetheless check that \( \ell_9(\sigma) = 4 = |w| \). (Cf. Example 1.1.2)

Algorithm 1.1.3 provides an efficient method for computing the length of the longest increasing subsequence statistic, but it is also combinatorially wasteful. In particular, it is reasonable to anticipate that even more combinatorial information would be obtained by placing additional structure upon the “bumped” values. The most classical and well-studied generalization involves recursively reapplying Algorithm 1.1.3 in order to create additional partial permutations from the “bumped” values. The resulting construction is called the Schensted Insertion Algorithm. Historically, this was the original framework within which Schensted \([32]\) invented Row Bumping while studying the length of the longest increasing subsequence statistic. We review Schensted Insertion and some of its more fundamental properties (including the well-studied and widely generalized RSK Correspondence based upon it) in Section 1.1.2 below.

The remainder of this Dissertation then describes various parallels and differences between Schensted Insertion and another natural extension of Single Row Bumping called Patience Sorting. We first describe Patience Sorting in Section 1.1.3. Further background material on permutation patterns is then given in Section 1.1.4. We then provide a summary of the main results of this Dissertation in Section 1.2.
1.1.2 Schensted Insertion and the RSK Correspondence

As discussed in Section 1.1.1 above, Single Row Bumping (Algorithm 1.1.3) can be viewed as combinatorially wasteful since nothing is done with the values as they are “bumped”. The most classical extension repeatedly employs Single Row Bumping in order to construct a collection of partial permutations \( P = P(\sigma) = (w_1, w_2, \ldots, w_r) \). This results in the following algorithm, in which we use the same “⇝” notation for “bumping” as in Example 1.1.4.

Algorithm 1.1.5 (Schensted Insertion).

Input: a permutation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathfrak{S}_n \)

Output: the sequence of partial permutations \( w_1, w_2, \ldots, w_r \)

1. First set \( w_1 = \sigma_1 \).

2. Then, for each \( i = 2, \ldots, n \), insert \( \sigma_i \) into the partial permutations \( w_1, w_2, \ldots, w_m \) as follows:

   (a) Insert \( \sigma_i \) into \( w_1 \) using Single Row Bumping (Algorithm 1.1.3).
      If a value is "bumped", then denote it by \( \sigma_1^* \). Otherwise, set \( \sigma_1^* = \emptyset \). We denote this redefinition of \( w_1 \) as
      \[
      w_1 \leftarrow \sigma_i = (w_1 \leftarrow \sigma_i) \rightsquigarrow \sigma_1^*.
      \]

   (b) For \( j = 2, \ldots, m \), redefine each \( w_j \) as
      \[
      w_j \leftarrow \sigma_j^* = (w_j \leftarrow \sigma_j^*) \rightsquigarrow \sigma_j^*.
      \]

      using the convention that \( w_j = w_j \leftarrow \emptyset = (w_j \leftarrow \emptyset) \rightsquigarrow \emptyset = w_j \rightsquigarrow \emptyset \).
In other words, one forms $w = w_1$ as usual under Single Row Bumping (Algorithm 1.1.3) while simultaneously forming a new partial permutation $w_2$ (also using Single Row Bumping) from each value as it is “bumped” from $w_1$. The values “bumped” from $w_2$ are furthermore employed to form a new partial permutation $w_3$ (again using Single Row Bumping), and so forth until $w_1, \ldots, w_r$ have been formed.

We illustrate Schensted Insertion in the following example, where we form the object $P = (w_1, w_2, \ldots, w_r)$ as a collection of left- and top-justified boxes. This extends the notation used in Example 1.1.4 so that $P$ becomes a so-called standard Young tableau written in English notation.

**Example 1.1.6.** Given $\sigma = 364827159 \in \mathfrak{S}_9$, we form $P(\sigma) = (w_1, w_2, w_3, w_4)$ under Schensted Insertion (Algorithm 1.1.5) as follows:

- Start with $P = \emptyset$ and insert 3 into $P$ to obtain
  \[
P = \emptyset \leftarrow 3 = \begin{array}{c}
  3
  \end{array}
  \]

- Append 6 to obtain
  \[
P = \begin{array}{c}
  3
  \end{array} \leftarrow 6 = \begin{array}{c|c}
  3 & 6 \\
  \hline
  & 
  \end{array} \sim \emptyset.
  \]

- Use 4 to bump 6:
  \[
P = \begin{array}{c|c}
  3 & 6 \\
  \hline
  & 
  \end{array} \leftarrow 4 = \begin{array}{c|c}
  3 & 4 \\
  \hline
  & \emptyset
  \end{array} \sim 6 = \begin{array}{c|c}
  3 & 4 \\
  \hline
  \emptyset & 6
  \end{array}.
  \]

- Append 8 to obtain
  \[
P = \begin{array}{c|c}
  3 & 4 \\
  \hline
  \emptyset & 6
  \end{array} \leftarrow 8 = \begin{array}{c|c|c}
  3 & 4 & 8 \\
  \hline
  \emptyset & 6
  \end{array}.
  \]

- Use 2 to bump 3:
  \[
P = \begin{array}{c|c|c}
  3 & 4 & 8 \\
  \hline
  \emptyset & 6
  \end{array} \leftarrow 2 = \begin{array}{c|c|c}
  2 & 4 & 8 \\
  \hline
  3 & 6
  \end{array} \sim 3 = \begin{array}{c|c|c}
  2 & 4 & 8 \\
  \hline
  3 & 6
  \end{array} \sim 6 = \begin{array}{c|c|c}
  2 & 4 & 8 \\
  \hline
  3 & 6
  \end{array}.
  \]
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- Use 7 to bump 8:

\[
P = \begin{array}{c}
\begin{array}{ccc}
2 & 4 & 8 \\
3 & 6 \\
\end{array}
\end{array} \leftarrow 7 = \begin{array}{c}
\begin{array}{ccc}
2 & 4 & 7 \\
3 & 6 \\
\end{array}
\end{array} \Rightarrow 8 = \begin{array}{c}
\begin{array}{ccc}
2 & 4 & 7 \\
3 & 8 & 6 \\
\end{array}
\end{array}
\]

- Use 1 to bump 2:

\[
P = \begin{array}{c}
\begin{array}{ccc}
2 & 4 & 7 \\
3 & 8 & 6 \\
\end{array}
\end{array} \leftarrow 1 = \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 7 \\
3 & 8 & 6 \\
\end{array}
\end{array} \Rightarrow 2 = \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 7 \\
2 & 8 & 6 \\
\end{array}
\end{array} \Rightarrow 3
\]

\[
= \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 7 \\
2 & 8 & 6 \\
\end{array}
\end{array} \Rightarrow 6 = \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 7 \\
2 & 8 & 6 \\
\end{array}
\end{array}
\]

- Use 5 to bump 7:

\[
P = \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 7 \\
2 & 8 & 6 \\
\end{array}
\end{array} \leftarrow 5 = \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 5 \\
2 & 8 & 6 \\
\end{array}
\end{array} \Rightarrow 7 = \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 5 \\
2 & 7 & 6 \\
\end{array}
\end{array} \Rightarrow 8 = \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 5 \\
2 & 7 & 8 & 6 \\
\end{array}
\end{array}
\]

- Append 9 to obtain

\[
P = \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 5 \\
2 & 7 & 8 & 6 \\
\end{array}
\end{array} \leftarrow 9 = \begin{array}{c}
\begin{array}{ccc}
1 & 4 & 5 & 9 \\
2 & 7 & 8 & 6 \\
\end{array}
\end{array}
\]

It follows that \(w_1 = 1459\), \(w_2 = 27\), \(w_3 = 38\), and \(w_4 = 6\). As in Example 1.1.4, the length of the longest increasing subsequence of \(\sigma\) is given by \(\ell_9(\sigma) = 4 = |w_1|\). This is the exact result due to Schensted [32] in 1961, with a sweeping generalization involving more than just \(w_1\) also given by Greene [19] in 1974.

The standard Young tableau \(P(\sigma) = (w_1, w_2, \ldots, w_r)\) contains a wealth of combinatorial information about the permutation \(\sigma \in \mathfrak{S}_n\). To get a sense of this, we first define the shape \(\lambda\) of \(P\) to be the sequence of lengths \(\lambda = (|w_1|, |w_2|, \ldots, |w_r|)\). One
can prove (see [18]) that the partial permutations constituting $P$ much satisfy the conditions

$$|w_1| \geq |w_2| \geq \cdots \geq |w_r| \quad \text{and} \quad |w_1| + |w_2| + \cdots + |w_r| = n$$

so that $\lambda$ is an example of a so-called partition of $n$. (This is denoted $\lambda \vdash n$.) It follows that $P$ can always be represented by a top- and left-justified shape as in Example 1.1.6 above. One can furthermore prove (again, see [18]) that the entries in $P$ must increase when read down each column and when read (from left to right) across each row.

In general, one defines a standard Young tableau to be any “filling” of a partition shape $\lambda \vdash n$ using each element from the set $[n]$ exactly once and such that both this row and column condition are satisfied.

When presented with a combinatorial algorithm like Schensted Insertion, it is natural to explore the invertibility of the steps involved. Such considerations lead not only to a better understanding of the combinatorial objects output from the algorithm but also to potentially useful inverse constructions. With this in mind, the following is a natural first question to ask:

**Question 1.1.7.** Given a standard Young tableau $P$, is there some “canonical” permutation $\sigma$ such that Schensted Insertion applied to $\sigma$ yields $P = P(\sigma)$?

To answer this questions, we first exploit the column-filling condition for standard Young tableaux. In particular, since elements must increase when read down each column, it follows that Schensted Insertion applied to a decreasing sequence will yield a tableau having exactly one column:
Now, suppose that we have two decreasing sequences

\[ d_1 > d_2 > \cdots > d_k \quad \text{and} \quad e_1 > e_2 > \cdots > e_\ell \]

with \( k \geq \ell \) and each \( d_i < e_i \) for \( i = 1, \ldots, \ell \). Then it is easy to see that

\[
\begin{pmatrix}
  d_1 e_1 \\
  d_2 e_2 \\
  \vdots \\
  d_k e_\ell \\
  \vdots \\
  d_1
\end{pmatrix}
\]

This motivates the definition of the *column word* \( \text{col}(P) \) of a standard Young tableau \( P \), which is formed by reading up each of the columns of \( P \) from left to right.

**Example 1.1.8.** We have that

\[
\begin{pmatrix}
  1 & 4 & 5 & 9 \\
  2 & 7 \\
  3 & 8 \\
  6
\end{pmatrix}
\]

\[ \text{col} = 632187459. \]

One can also check that

\[
P \left( \begin{pmatrix}
  1 & 4 & 5 & 9 \\
  2 & 7 \\
  3 & 8 \\
  6
\end{pmatrix}
\right) = P(632187459) = \begin{pmatrix}
  1 & 4 & 5 & 9 \\
  2 & 7 \\
  3 & 8 \\
  6
\end{pmatrix}.
\]
This illustrates the following important property of column words (which is proven by induction and repeated use of Equation (1.1.1)):

**Lemma 1.1.9.** Given a permutation $\sigma \in S_n$, $P(\text{col}(P(\sigma))) = P(\sigma)$.

For a more algebraic view of this result, denote by $\mathfrak{T}_n$ the set of all standard Young tableaux with some partition shape $\lambda \vdash n$. Schensted Insertion and the above column word operation can then be viewed as maps $P : S_n \rightarrow \mathfrak{T}_n$ and $\text{col} : \mathfrak{T}_n \rightarrow S_n$, respectively. With this notation, Lemma 1.1.9 becomes

**Lemma 1.1.9.** The composition $P \circ \text{col}$ is the identity map on the set $\mathfrak{T}_n$.

In particular, even though $\text{col}(P(\mathfrak{S}_n)) = \text{col}(\mathfrak{T}_n)$ is a proper subset of the symmetric group $\mathfrak{S}_n$, we nonetheless have that $P(\text{col}(P(\mathfrak{S}_n))) = P(\mathfrak{S}_n) = \mathfrak{T}_n$. As such, it makes sense to define the following non-trivial equivalence relation on $\mathfrak{S}_n$, with each element of $\text{col}(\mathfrak{T}_n)$ being the most natural choice of representative for the distinct equivalence class to which it belongs.

**Definition 1.1.10.** Two permutation $\sigma, \tau \in \mathfrak{S}_n$ are Knuth equivalent, written $\sigma \overset{K}{\sim} \tau$, if they yield the same standard Young tableau $P(\sigma) = P(\tau)$ under Schensted Insertion (Algorithm 1.1.5).

**Example 1.1.11.** One can check (using the so-called Hook Length Formula; see [18]) that there are 216 permutations $\sigma \in \mathfrak{S}_9$ satisfying $\sigma \overset{K}{\sim} 632187459$. E.g., as illustrated in Example 1.1.8, $364827159 \overset{K}{\sim} 632187459$.

In order to characterize the equivalence classes formed under $\overset{K}{\sim}$, it turns out (see [18]) that the following two examples are generic enough to characterize Knuth
equivalence up to transitivity and order-isomorphism:

\[ P(213) = P(231) = \begin{array}{ccc} 1 & 3 \\ 2 \end{array} \quad \text{and} \quad P(312) = P(132) = \begin{array}{ccc} 1 & 2 \\ 3 \end{array}. \]

In particular, these motivate the so-called Knuth relations:

**Definition 1.1.12.** Given two permutations \( \sigma, \tau \in \mathfrak{S}_n \), we define the Knuth relations \( \sim^{K_1} \) and \( \sim^{K_2} \) on \( \mathfrak{S}_n \) as

(K1) \( \sigma \sim^{K_1} \tau \) if \( \sigma \) can be obtained from \( \tau \) either by

(K1-1) changing a substring order-isomorphic to 213 in \( \sigma \) into a substring order-isomorphic to 231 in \( \tau \)

(K1-2) or by changing a substring order-isomorphic to 231 in \( \sigma \) into a substring order-isomorphic to 213 in \( \tau \).

(K2) \( \sigma \sim^{K_2} \tau \) if \( \sigma \) can be obtained from \( \tau \) either by

(K2-1) changing a substring order-isomorphic to 312 in \( \sigma \) into a substring order-isomorphic to 132 in \( \tau \)

(K2-2) or by changing a substring order-isomorphic to 132 in \( \sigma \) into a substring order-isomorphic to 312 in \( \tau \).

One can show (again, see [18]) that Knuth equivalence \( \sim^K \) on the symmetric group \( \mathfrak{S}_n \) is the equivalence relation generated by \( \sim^{K_1} \) and \( \sim^{K_2} \).

**Example 1.1.13.** From Example [1.1.11], we see that 364827159 \( \sim^K \) 632187459. This equivalence under Schensted Insertion can be obtained by the following sequence of Knuth relations, where we have underlined the appropriate order-isomorphic substring being permuted in order to move from one step to the next. (E.g., the substring
364 in 364827159 is order-isomorphic to 132 because 3, 6, and 4 occur in the same order with the same relative magnitudes as do 1, 3, and 2 in 132.

\[ 364827159 \sim K^2 \quad 634287159 \sim K^1 \quad 632487159 \sim K^2 \quad 632481759 \sim K^1 \quad 632418759 \sim K^2 \quad 632148759 \sim K^1 \quad 632187459 \sim K^2 \]

Given how many intermediate permutations are needed in order to realizing the Knuth equivalence of 364827159 and 632187459 via the two Knuth relations, one might expect Knuth equivalence classes to be fairly large in general. This suggests the next question, which is a natural follow-up to Question 1.1.7.

**Question 1.1.14.** Given a standard Young tableau \( P \), how much extra “bookkeeping” is necessary in order to uniquely specify a permutation \( \sigma \) such that \( P = P(\sigma) \)?

One can show (see [18] for a detailed account) that, due to the row- and column-filling conditions on \( P \), it suffices to keep track of the order in which new boxes are added to the shape of \( P \). In particular, each entry in \( P \) has a unique “bumping path” by which it reached its final position, and so this path can be inverted in order to “unbump” (a.k.a. reverse row bump) the entry. This motivates a bijective extension of Schensted Insertion called the RSK Correspondence, which originated from the study of representations of the symmetric group by Robinson [30] in 1938. The modern “bumping” version that we present, though, resulted from work by Schensted [32] in 1961 that extended Robinson’s algorithm to combinatorial words. Further generalization was also done by Knuth [21] for so-called \( N \)-matrices (or, equivalently, so-called two-line arrays) in 1970. (See Fulton [18] for the appropriate definitions and for a detailed account of the differences between these algorithms; Knuth also includes a description in [22].)
Algorithm 1.1.15 (RSK Correspondence).

Input: a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$

Output: the pair of standard Young tableaux $P = P(\sigma)$ and $Q = Q(\sigma)$

- Use Schensted Insertion (Algorithm 1.1.5) to build $P$.

- For each $i = 1, \ldots, n$, when the $i^{th}$ box is added to the shape of $P$, add the box $i$ to $Q$ so that $P$ and $Q$ maintain the same shape.

We call $P(\sigma)$ the insertion tableau and $Q(\sigma)$ the recording tableau corresponding to $\sigma \in \mathfrak{S}_n$ under the RSK Correspondence. Since the construction of the recording tableau suffices to invert Schensted Insertion, this correspondence yields a bijection between the symmetric group $\mathfrak{S}_n$ and the set of all ordered pairs of standard Young tableaux such that the two tableaux have the same partition shape. We denote this bijection at the element level by $\sigma \overset{\text{RSK}}{\longleftrightarrow} (P(\sigma), Q(\sigma))$.

Example 1.1.16. Given the permutation $\sigma = 364827159 \in \mathfrak{S}_9$, we form the insertion tableau $P = P(\sigma)$ (cf. Example 1.1.6 above) and the recording tableau $Q = Q(\sigma)$ under the RSK Correspondence (Algorithm 1.1.15) as follows:

- Start with $P = \emptyset$ and insert 3 into $P$ to obtain $P = \begin{array}{c} 3 \end{array}$ and $Q = \begin{array}{c} 1 \end{array}$.

- Append 6 to $P$ so that

$$P = \begin{array}{c} 3 \end{array} 6 \quad \text{and} \quad Q = \begin{array}{c} 1 \end{array} 2.$$

- Use 4 to bump 6 in $P$ so that

$$P = \begin{array}{c} 3 \end{array} 4 \begin{array}{c} 6 \end{array} \quad \text{and} \quad Q = \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 3 \end{array}.$$

- Append 8 to the top row of $P$ so that

$$P = \begin{array}{c} 3 \end{array} 4 \begin{array}{c} 8 \end{array} \begin{array}{c} 6 \end{array} \quad \text{and} \quad Q = \begin{array}{c} 1 \end{array} 2 \begin{array}{c} 4 \end{array} \begin{array}{c} 3 \end{array}.$$
• Use 2 to bump 3 in $P$ so that

$$P = \begin{pmatrix} 2 & 4 & 8 \\ 3 & \_ & \_ \\ 6 & \_ & \_ \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 2 & 4 \\ 3 & \_ & \_ \\ 5 & \_ & \_ \end{pmatrix}.$$ 

• Use 7 to bump 8 in $P$ so that

$$P = \begin{pmatrix} 2 & 4 & 7 \\ 3 & 8 & \_ \\ 6 & \_ & \_ \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & \_ \\ 5 & \_ & \_ \end{pmatrix}.$$ 

• Use 1 to bump 2 in $P$ so that

$$P = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 8 & \_ \\ 3 & 8 & \_ \\ 6 & \_ & \_ \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & \_ \\ 5 & 8 & \_ \\ 7 & \_ & \_ \end{pmatrix}.$$ 

• Use 5 to bump 7 in $P$ so that

$$P = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 7 & \_ \\ 3 & 8 & \_ \\ 6 & \_ & \_ \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & \_ \\ 5 & 8 & \_ \\ 7 & \_ & \_ \end{pmatrix}.$$ 

• Append 9 to the top row of $P$ so that

$$P = \begin{pmatrix} 1 & 4 & 5 & 9 \\ 2 & 7 & \_ & \_ \\ 3 & 8 & \_ & \_ \\ 6 & \_ & \_ & \_ \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 2 & 4 & 9 \\ 3 & 6 & \_ & \_ \\ 5 & 8 & \_ & \_ \\ 7 & \_ & \_ & \_ \end{pmatrix}.$$ 

It follows that $364827159 \overset{\text{RSK}}{\longleftrightarrow} \begin{pmatrix} 1 & 4 & 5 & 9 \\ 2 & 7 & 3 & 6 \\ 3 & 8 & 5 & 8 \\ 6 & \_ & 7 & \_ \end{pmatrix}.$

One can also check that the inverse of $\sigma$ (when thought of a function) satisfies

$$\sigma^{-1} = 751382649 \overset{\text{RSK}}{\longleftrightarrow} \begin{pmatrix} 1 & 2 & 4 & 9 \\ 3 & 6 & 2 & 7 \\ 5 & 8 & 3 & 8 \\ 7 & \_ & 6 & \_ \end{pmatrix}.$$
1.1. Informal Overview and Motivation

This illustrates the following amazing fact about the RSK Correspondence:

**Theorem 1.1.17** (Schützenberger Symmetry for the RSK Correspondence). Given a permutation \( \sigma \in \mathfrak{S}_n \), \( \sigma \overset{\text{RSK}}{\longleftrightarrow} (P(\sigma), Q(\sigma)) \) if and only if \( \sigma^{-1} \overset{\text{RSK}}{\longleftrightarrow} (Q(\sigma), P(\sigma)) \).

Schützenberger Symmetry, which was first proven by a direct combinatorial argument in [33], is only one of many remarkable properties of the RSK Correspondence. A particularly good account of the numerous consequences of the RSK Correspondence in such fields as Representation Theory can be found in Sagan [31]. One can also read about the RSK Correspondence on words and on \( N \)-matrices in Fulton [18].

Another remarkable fact about the RSK Correspondence is that it can be realized without explicit involvement of Single Row Bumping (Algorithm 1.1.3). One of the more striking alternatives involves the so-called shadow diagram of a permutation, as introduced by Viennot [40] in the context of further explicating Schützenberger Symmetry. As we review in Section 4.3 below, Theorem 1.1.17 follows trivially from Viennot’s use of shadow diagrams.

At the same time, it is also interesting to look at when the full RSK Correspondence is essentially unnecessary. This motivates the following question:

**Question 1.1.18.** Given a standard Young tableau \( P \), under what conditions is no extra “bookkeeping” necessary in order to uniquely recover a permutation \( \sigma \) such that \( P = P(\sigma) \)?

As a direct consequence of the Schützenberger Symmetry property for the RSK Correspondence, there is a bijection between the set \( \mathcal{I}_n \) of involutions in \( \mathfrak{S}_n \) and the set \( \mathfrak{T}_n \) of all standard Young tableaux. (An involution is any permutation that is equal to its own inverse.) However, this bijection doesn’t yield any information about the size of the equivalence classes containing each involution. Thus, in order
to answer Question 1.1.18 one must instead consider permutations that “avoid” the order-isomorphic substrings (a.k.a. block patterns as defined in Section 1.1.4 below) used to define the Knuth relations. A Knuth equivalence class cannot be a singleton set unless the only permutation it contains cannot be transformed into a Knuth equivalent permutation via a Knuth relation. Given how restrictive a condition this is, a tableau $P$ satisfies Question 1.1.18 if and only if $P$ is a single row or column.

Chapters 3 and 4 of this Dissertation are focused upon addressing Questions 1.1.7, 1.1.14, and 1.1.18 when adapted to the combinatorial algorithm Patience Sorting. As we will see, the notion of “pattern avoidance”, which is made explicit in Section 1.1.4, provides the unifying language for characterizing our responses.

1.1.3 Patience Sorting as “Non-recursive” Schensted Insertion

The term Patience Sorting was introduced in 1962 by Mallows [25, 26] as the name of a two-part card sorting algorithm invented by A. S. C. Ross. The first part of this algorithm, which Mallows referred to as a “patience sorting procedure”, involves partitioning a shuffled deck of cards into a collection of sorted subsequences called piles. Unless otherwise noted, we take our “(shuffled) deck of cards” to be a permutation.

Algorithm 1.1.19 (Mallows’ Patience Sorting Procedure).

Input: a shuffled deck of cards $\sigma = c_1c_2\cdots c_n \in \mathcal{S}_n$

Output: the sequence of partial permutations $r_1, r_2, \ldots, r_m$

1. First form a new pile $r_1$ using the card $c_1$.

2. Then, for each $i = 2, \ldots, n$, consider the cards $d_1, d_2, \ldots, d_k$ atop the piles $r_1, r_2, \ldots, r_k$ that have already been formed.
(a) If \( c_i > \max_{1 \leq j \leq k} \{d_j\} \), then form a new pile \( r_{k+1} \) using \( c_i \).

(b) Otherwise, find the left-most card \( d_j \) that is larger than \( c_i \) and place the card \( c_i \) atop pile \( r_j \). In other words, set

\[
d_j = c_i \quad \text{where} \quad j = \min_{1 \leq m \leq k} \{m \mid c_i < d_m\}.
\]

We call the collection of subsequences \( R = R(\sigma) = \{r_1, r_2, \ldots, r_m\} \) the pile configuration associated to the deck of cards \( \sigma \in \mathfrak{S}_n \) and illustrate their formation in Example 1.1.21 below. In keeping with the language of “piles”, we will often write the constituent subsequences \( r_1, r_2, \ldots, r_m \) vertically, in order, and bottom-justified with respect to the largest value in the pile. This is illustrated in the following example.

**Example 1.1.20.** The pile configuration \( R(64518723) = \{641, 52, 873\} \) (formed in Example 1.1.21 below) has piles \( r_1 = 641 \), \( r_2 = 52 \), and \( r_3 = 873 \). We represent this visually as

\[
R(64518723) = \begin{array}{cccc}
1 & 3 \\
4 & 2 & 7 \\
6 & 5 & 8 \\
\end{array}
\]

**Example 1.1.21.** Given the deck of cards \( \sigma = 64518723 \in \mathfrak{S}_8 \), we form \( R(\sigma) \) under Patience Sorting (Algorithm 1.1.19) as follows:

First, use 6 to form a new pile:

\[
\begin{array}{c}
6
\end{array}
\]

Then place 4 atop this new pile:

\[
\begin{array}{c}
6 \\
4
\end{array}
\]

Use 5 to form a new pile:

\[
\begin{array}{cc}
4 & 5
\end{array}
\]

Then place 1 atop the left-most pile:

\[
\begin{array}{ccc}
1 \\
4 & 5
\end{array}
\]
Use 8 to form a new pile:

\[
\begin{array}{cccc}
1 & 4 & 6 & 5 \\
\end{array}
\]

Then place 7 atop this new pile:

\[
\begin{array}{cccc}
1 & 4 & 7 & 6 & 5 & 8 \\
\end{array}
\]

Place 2 atop the middle pile:

\[
\begin{array}{cccc}
1 & 2 & 7 & 4 & 6 & 5 & 8 \\
\end{array}
\]

Finally, place 3 atop the right-most pile:

\[
\begin{array}{cccc}
1 & 3 & 4 & 2 & 7 & 6 & 5 & 8 \\
\end{array}
\]

Now, in order to affect sorting, cards can be removed one at a time from these piles in the order 1, 2, \ldots, 8. Note that, by construction, the cards in each pile decrease when read from bottom to top. Consequently, the appropriate card will always be atop a pile at each step of this removal process:

- After removing 1:
  \[
  \begin{array}{cccc}
  3 & 4 & 2 & 7 & 6 & 5 & 8 \\
  \end{array}
  \]

- After removing 2:
  \[
  \begin{array}{cccc}
  3 & 4 & 7 & 6 & 5 & 8 \\
  \end{array}
  \]

- After removing 3:
  \[
  \begin{array}{cccc}
  7 & 6 & 5 & 8 \\
  \end{array}
  \]

- After removing 5:
  \[
  \begin{array}{cccc}
  7 & 6 & 8 \\
  \end{array}
  \]

And so on.

When applying Algorithm 1.1.19 to a permutation \(\sigma = c_1c_2\cdots c_n \in S_n\), each card \(c_i\) is either larger than the top card of every pile or is placed atop the left-most top card \(d_j\) larger than it. As such, the cards \(d_1, d_2, \ldots, d_k\) atop the piles will always be in increasing order (from left to right) at each step of the algorithm, and it is in this sense that Algorithm 1.1.19 resembles Schensted Insertion (Algorithm 1.1.5). The distinction is that cards remain in place and have other cards placed on top of them when “bumped” rather than endure insertion into successive rows of a standard Young
tableau through recursive application of Single Row Bumping (Algorithm 1.1.3).

Note, in particular, that the cards $d_1, d_2, \ldots, d_k$ correspond exactly to the top row of the standard Young tableau formed at each stage of Schensted Insertion. Consequently, the number of piles formed under Patience Sorting is exactly equal to the length of the longest increasing subsequence in the deck of cards.

Given the algorithmic simplicity of Patience Sorting, the pile configuration $R(\sigma)$ is commonly offered as a tangible realization for the length $\ell_n(\sigma)$ of the longest increasing subsequence in a permutation $\sigma \in \mathfrak{S}_n$. This is particularly true in the context of Probabilistic Combinatorics since the asymptotic number of piles formed must follow the highly celebrated Baik-Deift-Johansson Theorem from [4]. In other words, even though it is easy to construct $R(\sigma)$, there is no simple way to completely describe the number of piles formed $\ell_n(\sigma)$ without borrowing from Random Matrix Theory. (A good historical survey of attempts previous to [4] can be found in both [2] and [37].)

According to the Baik-Deift-Johansson Theorem, the distribution for the number of piles formed under Patience Sorting converges asymptotically to the Tracy-Widom $F_2$ distribution (up to an appropriate rescaling). Remarkably, though, $F_2$ originated in work by Tracy and Widom [38] as the asymptotic distribution for the largest eigenvalue of a random Hermitian matrix (again, up to rescaling). Because of this deep connection between Patience Sorting and Probabilistic Combinatorics, it has been suggested (see, e.g., [23], [24] and [28]; cf. [14]) that studying generalizations of Patience Sorting might be the key to tackling certain open problems that can be viewed from the standpoint of Random Matrix Theory — the most notable being the Riemann Hypothesis.

Another natural direction of study involves characterizing the objects output from
both Algorithm 1.1.19 and an appropriate bijective extension. Chapter 3 is largely devoted to the combinatorics that arises from various characterizations for pile configurations. Then, in Chapter 4, we study the combinatorics that arises from a full, non-recursive analog of the RSK Correspondence. In particular, we mimic the RSK recording tableau construction so that “recording piles” $S(\sigma)$ are assembled along with the usual pile configuration $R(\sigma)$ under Patience Sorting (which by analogy to RSK we will similarly now call “insertion piles”). We refer to the resulting (ordered) pair of pile configurations as a stable pair and denote $\sigma \overset{XPS}{\leftrightarrow} (R(\sigma), S(\sigma))$.

**Algorithm 1.1.22** (Extended Patience Sorting).

Input: a shuffled deck of cards $\sigma = c_1c_2\cdots c_n \in \mathcal{S}_n$

Output: the ordered pair $(R(\sigma), S(\sigma))$ of pile configurations, where we denote $R(\sigma) = \{r_1, r_2, \ldots, r_m\}$ and $S(\sigma) = \{s_1, s_2, \ldots, s_m\}$

1. First form a new pile $r_1$ using the card $c_1$ and set $s_1 = 1$.

2. Then, for each $i = 2, \ldots, n$, consider the cards $d_1, d_2, \ldots, d_k$ atop the piles $r_1, r_2, \ldots, r_k$ that have already been formed.

   (a) If $c_i > \max_{1 \leq j \leq k} \{d_j\}$, then form a new pile $r_{k+1}$ using $c_i$ and set $s_{k+1} = i$.

   (b) Otherwise, find the left-most card $d_j$ that is larger than $c_i$ and place the card $c_i$ atop pile $r_j$ while simultaneously placing $i$ at the bottom of pile $s_j$. In other words, set

   $$d_j = c_i \quad \text{where } j = \min_{1 \leq m \leq k} \{m \mid c_i < d_m\}.$$ 

   and insert $i$ at the bottom of pile $s_j$. 
By construction, the pile configurations in the resulting stable pair must have the same notion of “shape”, which we define as follows.

**Definition 1.1.23.** Given a pile configuration \( R = \{ r_1, r_2, \ldots, r_m \} \) that has been formed from \( n \) cards, we call the \( m \)-tuple \( \text{sh}(R) \) the shape of \( R \), where

\[
\text{sh}(R) = (|r_1|, |r_2|, \ldots, |r_m|).
\]

Note, in particular, that \( \text{sh}(R) \) satisfies

\[
|r_1|, |r_2|, \ldots, |r_m| \in [n] \quad \text{and} \quad |r_1| + |r_2| + \cdots + |r_m| = n
\]

and thus is an example of a so-called composition of \( n \). By partial analogy to the notation for a partition of \( n \) (and since partitions are a special case of compositions), this is denoted by \( \text{sh}(R) \models n \).

Having established this shape convention, we now illustrate Extended Patience Sorting (Algorithm 1.1.22) in the following example.

**Example 1.1.24.** Given \( \sigma = 64518723 \in \mathcal{S}_8 \), we form the following stable pair \((R(\sigma), S(\sigma))\) under Algorithm [1.1.22] with shape \( \text{sh}(R(\sigma)) = \text{sh}(S(\sigma)) = (3, 2, 3) \models 8 \).

| insertion piles | recording piles | insertion piles | recording piles |
|----------------|----------------|----------------|----------------|
| After inserting 6: | 6 | 1 | After inserting 4: | 4 | 1 |
| | | | | 6 | 2 |
| After inserting 5: | 4 | 1 | After inserting 1: | 1 | 1 |
| | 6 5 | 2 | | 4 | 2 |
| | | | | 6 5 | 4 | 3 |
After inserting 8:

\[
\begin{array}{cccc}
1 & 2 & 4 & 8 \\
6 & 5 & 3 & \end{array}
\]

After inserting 7:

\[
\begin{array}{cccc}
1 & 7 & 2 & 5 \\
6 & 5 & 8 & 4 \end{array}
\]

After inserting 2:

\[
\begin{array}{cccc}
1 & 3 & 4 & 2 \\
6 & 5 & 8 & 4 \end{array}
\]

After inserting 3:

\[
\begin{array}{cccc}
1 & 5 & 3 & 6 \\
6 & 5 & 8 & 4 \end{array}
\]

Given a permutation \( \sigma \in \mathfrak{S}_n \), the recording piles \( S(\sigma) \) indirectly label the order in which cards are added to the insertion piles \( R(\sigma) \) under Patience Sorting (Algorithm 1.1.19). With this information, \( \sigma \) can be reconstructed by removing cards from \( R(\sigma) \) in the opposite order that they were added to \( R(\sigma) \). (This removal process, though, should not be confused with the process of sorting that was illustrated in Example 1.1.21.) Consequently, reversing Extended Patience Sorting (Algorithm 1.1.22) is significantly less involved than reversing the RSK Correspondence (Algorithm 1.1.15) through recursive “reverse row bumping”. The trade-off is that entries in the pile configurations resulting from the former are not independent (see Section 4.1) whereas the tableaux generated under the RSK Correspondence are completely independent (up to shape).

To see that \( S(\sigma) = \{s_1, s_2, \ldots, s_m\} \) records the order in which cards are added to the insertion piles, instead add cards atop new piles \( s'_j \) in Algorithm 1.1.22 rather than to the bottoms of the piles \( s_j \). This yields modified recording piles \( S'(\sigma) \) from which each original recording pile \( s_j \in S(\sigma) \) can be recovered by simply reflecting the corresponding pile \( s'_j \) vertically. Moreover, it is easy to see that the entries in \( S'(\sigma) \) directly label the order in which cards are added to \( R(\sigma) \) under Patience Sorting.

**Example 1.1.25.** As in Example 1.1.24 above, let \( \sigma = 64518723 \in \mathfrak{S}_8 \). Then \( R(\sigma) \) is formed as before and
1.1. Informal Overview and Motivation

As Example 1.1.25 illustrates, Extended Patience Sorting (Algorithm 1.1.22) is only one possible way to bijectively extend Patience Sorting (Algorithm 1.1.19). What suggests Extended Patience Sorting as the right extension is a symmetric property (see Section 1.2) directly analogous to Schützenberger Symmetry for the RSK Correspondence (Theorem 1.1.17): \( \sigma \in \mathfrak{S}_n \) yields \((R(\sigma), S(\sigma))\) under Extended Patience Sorting if and only if \(\sigma^{-1}\) yields \((S(\sigma), R(\sigma))\). Moreover, this symmetry result follows trivially from the geometric realization of Extended Patience Sorting given in Section 4.3.2, which is also a direct analogy to the geometric realization for the RSK Correspondence reviewed in Section 4.3.1.

While such interesting combinatorics result from questions suggested by the resemblance between Patience Sorting and Schensted Insertion, this is not the only possible direction of study. In particular, after applying Algorithm 1.1.19 to a deck of cards, it is easy to recollect each card in ascending order from amongst the current top cards of the piles (and thus complete A. S. C. Ross’ original card sorting algorithm as in Example 1.1.21). While this is not necessarily the fastest sorting algorithm that one might apply to a deck of cards, the patience in Patience Sorting is not intended to describe a prerequisite for its use. Instead, it refers to how pile formation in Algorithm 1.1.19 resembles the placement of cards into piles when playing the popular single-person card game Klondike Solitaire, and Klondike Solitaire is often called Patience in the UK. This is more than a coincidence, though, as Algorithm 1.1.19 also happens to be an optimal strategy (in the sense of forming as few piles as possible; see Section 2.2) when playing an idealized model of Klondike Solitaire known as Floyd’s Game:
1.1. Informal Overview and Motivation

**Card Game 1.1.26 (Floyd’s Game).** Given a deck of cards $c_1 c_2 \cdots c_n \in S_n$,

- place the first card $c_1$ from the deck into a pile by itself.

- Then, for each card $c_i$ ($i = 2, \ldots, n$), either
  
  - put $c_i$ into a new pile by itself
  
  - or play $c_i$ on top of any pile whose current top card is larger than $c_i$.

- The object of the game is to end with as few piles as possible.

In other words, cards are played one at a time according to the order that they appear in the deck, and piles are created in much the same way that they are formed under Patience Sorting. According to [2], Floyd’s Game was developed independently of Mallows’ work during the 1960s as an idealized model for Klondike Solitaire in unpublished correspondence between computer scientists Bob Floyd and Donald Knuth.

Note that, unlike Klondike Solitaire, there is a known strategy (Algorithm 1.1.19) for Floyd’s Game under which one will always win. In fact, Klondike Solitaire — though so popular that it has come pre-installed on the vast majority of personal computers shipped since 1989 — is still poorly understood mathematically. (Recent progress, however, has been made in developing an optimal strategy for a version called *thoughtful solitaire* [43].) As such, Persi Diaconis ([2] and private communication with the author) has suggested that a deeper understanding of Patience Sorting and its generalization would undoubtedly help in developing a better mathematical model for analyzing Klondike Solitaire.

Chapter 2 is largely dedicated to the careful study of strategies for Floyd’s Game as well as a generalization of Patience Sorting called *Two-color Patience Sorting*. 
1.1.4 The Language of Pattern Avoidance

Given two positive integers $m, n \in \mathbb{Z}_+$, with $1 < m < n$, we begin this section with the following definition.

**Definition 1.1.27.** Let $\pi = \pi_1 \pi_2 \cdots \pi_m \in S_m$. Then we say that $\pi$ is a *(classical permutation)* pattern contained in $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ if $\sigma$ contains a subsequence $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_m}$ that is order-isomorphic to $\pi$. I.e., for each $j, k \in [m]$,

$$\sigma_{i_j} < \sigma_{i_k} \text{ if and only if } \pi_j < \pi_k.$$

Despite its ostensive complexity, the importance of Definition 1.1.27 cannot be overstated. In particular, the containment (and, conversely, avoidance) of patterns provides a remarkably flexible language for characterizing collections of permutations that share a common combinatorial property. Such a point of view is sufficiently general that it provides the foundation for an entire field of study commonly called *Pattern Avoidance*. There are also many natural ways to extend the definition of a classical pattern, as we will discuss after the following examples.

**Example 1.1.28.**

1. Given $k \in \mathbb{Z}_+$, a pattern of the form $\imath_k = 12 \cdots k \in S_k$ is called a (classical) *monotone increasing* pattern, and $\sigma \in S_n$ contains $\imath_k$ if and only if the length of the longest increasing subsequence in $\sigma$ satisfies $\ell_n(\sigma) \geq k$.

E.g., $\sigma = 364827159 \in S_9$ has several increasing subsequences of length four, and so $\sigma$ contains quite a few occurrences of the patterns 12, 123, and 1234. Moreover, as we saw in Example 1.1.4, $\ell_9(364827159) = 4$.

2. The permutation $\sigma = 364827159 \in S_9$ also contains occurrences of such pat-
terns as $231 \in \mathfrak{S}_3$ (e.g., via the subsequence 271), $2431 \in \mathfrak{S}_4$ (e.g., via the subsequence 3871), $23541 \in \mathfrak{S}_5$ (e.g., via the subsequence 34871), and $235416 \in \mathfrak{S}_6$ (e.g., via the subsequence 368719).

Given a permutation $\sigma \in \mathfrak{S}_n$ containing a pattern $\pi \in \mathfrak{S}_m$, note that the components of the order-isomorphic subsequence $\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_m}$ are not required to occur contiguously within $\sigma$. In other words, the difference between consecutive indices in the subsequence is allowed to be more than one. It is sometimes convenient, though, to consider patterns formed using only consecutive subsequences. This gives rise to the definition of a block pattern (a.k.a. consecutive pattern or segment pattern).

**Definition 1.1.29.** Let $\pi \in \mathfrak{S}_m$ be a permutation. Then we say that $\pi$ is a block (permutation) pattern contained in $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$ if $\sigma$ contains a subsequence $\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_m-1}$ that is order-isomorphic to $\pi$. I.e., $\sigma$ contains $\pi$ as a classical pattern but with the subsequence $\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_m-1}$ necessarily comprised of consecutive entries in $\sigma$.

In effect, block patterns can be seen as the most restrictive special case of a classical pattern, and both notions of pattern have numerous combinatorial applications. E.g., an inversion in a permutation is an occurrence of a (classical) 21 pattern, while a descent is an occurrence of a (block) 21 pattern. At the same time, though, there is no reason to insist on either extreme. The following definition, which originated from the classification of so-called Mahonian Statistics by Babson and Steingrímsson [3], naturally generalizes both Definition 1.1.27 and Definition 1.1.29.

**Definition 1.1.30.** Let $\pi \in \mathfrak{S}_m$ be a permutation, and let $D$ be a distinguished subset $D = \{d_1, d_2, \ldots, d_k\} \subset [m-1]$ of zero or more of its subscripts. Then we say that $\pi$ is a generalized (permutation) pattern contained in $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$ if $\sigma$
contains a subsequence $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_m}$ that is order-isomorphic to $\pi$ such that

$$i_{j+1} - i_j = 1 \text{ if } j \in [m - 1] \setminus D.$$ 

I.e., $\sigma$ contains $\pi$ as a classical pattern but with $\sigma_{i_j}$ and $\sigma_{i_{j+1}}$ necessarily consecutive entries in $\sigma$ unless $j \in D$ is a distinguished subscript of $\pi$.

When denoting a generalized pattern $\pi \in \mathfrak{S}_m$ with distinguished subscript set $D = \{d_1, d_2, \ldots, d_k\}$, we will often insert a dash between the entries $\pi_{d_j}$ and $\pi_{d_{j+1}}$ in $\pi$ for each $j \in [k]$. We illustrate this convention in the following examples.

**Example 1.1.31.**

1. Let $\pi \in \mathfrak{S}_m$ be a generalized pattern. Then $\pi$ is a classical pattern if the subscript set $D = [m - 1]$. I.e.,

$$\pi = \pi_1 - \pi_2 - \cdots - \pi_{m-1} - \pi_m.$$ 

Similarly, $\pi$ is a block pattern if the subscript set $D = \emptyset$, in which case it is written with no dashes: $\pi = \pi_1\pi_2\cdots\pi_m$.

2. The permutation $\sigma = 364827159 \in \mathfrak{S}_9$ from Example [11.28] contains many examples of generalized patterns. The subsequence 271 is an occurrence of (the block pattern) 231, while 371 is not. However, both of the subsequences 271 and 371 are occurrences of 2–31, while 381 is not. In a similar way, both 361 and 481 are occurrences of 23–1, while 381 is again not.

Finally, we note that each of the subsequences 362, 361, 342, 341, 382, 381, 371, 682, 681, 685, 671, 675, 482, 481, 471, and 271 form an occurrence of the generalized pattern 2–3–1 (a.k.a. the classical pattern 231) in 364827159.
1.1. Informal Overview and Motivation

An important further generalization of Definition 1.1.30 requires that the context in which the occurrence of a generalized pattern occurs also be taken into account. The resulting concept of barred pattern first arose within the study of so-called stack-sortability of permutations by West [42] (though West’s barred patterns were based upon the definition of a classical pattern and not upon the definition of a generalized pattern as below). As we will illustrated in Section 1.2 these barred patterns arise as a natural combinatorial tool in the study of Patience Sorting.

**Definition 1.1.32.** Let \( \pi = \pi_1\pi_2\cdots\pi_m \in S_m \) be a generalized pattern with an additional distinguished subset \( B = \{b_1, b_2, \ldots, b_\ell\} \subset [m] \) of zero or more of its subscripts. Then we say that \( \pi \) is a barred (generalized permutation) pattern contained in \( \sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n \) if \( \sigma \) contains a subsequence \( \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m} \) such that

- the (index restricted) subsequence \( \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}[m]\setminus B = \sigma_{b_1}\sigma_{b_2}\cdots\sigma_{b_\ell} \) is order-isomorphic to the subsequence \( \pi_{[m]\setminus B} = \pi_{b_1}\pi_{b_2}\cdots\pi_{b_\ell} \) of \( \pi \)

- and the (unrestricted) subsequence \( \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m} \) is not order-isomorphic to \( \pi \).

I.e., \( \sigma \) contains the subsequence of \( \pi \) indexed by \( [m]\setminus B \) as a generalized pattern unless this occurrence of \( \pi_{[m]\setminus B} \) is part of an occurrence of \( \pi \) as a generalized pattern.

When denoting a barred (generalized) pattern \( \pi \in S_m \) with distinguished subscript set \( B = \{b_1, b_2, \ldots, b_\ell\} \), we will often place overbars atop the entries \( \pi_{b_j} \) in \( \pi \) for each \( j \in [\ell] \). This convention is illustrated in the following examples.

**Example 1.1.33.**

1. Let \( \pi \in S_m \) be a barred pattern. Then \( \pi \) is an ordinary generalized pattern (as in Definition 1.1.31) if the subscript set \( B = \emptyset \).
2. A permutation contains an occurrence of the barred pattern 3–T–42 (where $\pi = 3142 \in \mathcal{S}_4$, $D = \{1, 2\}$, and $B = \{2\}$) if it contains an occurrence of the generalized pattern 2–31 that is not part of an occurrence of the generalized pattern 3–1–42.

E.g., the permutation $\sigma = 364827159 \in \mathcal{S}_9$ from Example 1.1.28 contains the generalized pattern 2–31 (and hence 3–1–42) via each of the subsequences 382, 371, 682, 671, 482, 471, and 271. This is because none of these subsequences occur within a larger occurrence of the generalized pattern 3–1–42. In fact, one can check that no subsequence of 364827159 constitutes an occurrence of the pattern 3–1–42.

When a permutation $\sigma \in \mathcal{S}_n$ fails to contain a given pattern $\pi \in \mathcal{S}_m$, then we say that $\sigma$ avoids $\pi$. Questions of pattern avoidance have tended to dominate the study of patterns in general. This is at least partially due to the initial choice of terminology forbidden subsequence, which was only later refined to distinguish between pattern containment and pattern avoidance. Nonetheless, such bias continues to be reflected in the following (now standard) definitions.

**Definition 1.1.34.** Given any collection $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}$ of patterns, we denote by

$$S_n(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}) = \bigcap_{i=1}^{k} S_n(\pi^{(i)}) = \bigcap_{i=1}^{k} \{ \sigma \in \mathcal{S}_n \mid \sigma \text{ avoids } \pi^{(i)} \}$$

the avoidance set consisting of all permutations $\sigma \in \mathcal{S}_n$ such that $\sigma$ simultaneously avoids each of the patterns $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}$. Furthermore, the set

$$Av(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}) = \bigcup_{n \geq 1} S_n(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)})$$
is called the \textit{(pattern) avoidance class} with \textit{basis} \{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}\}. We will also refer to the sequence of cardinalities

\[|S_1(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)})|, |S_2(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)})|, \ldots, |S_n(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)})|, \ldots\]

as the \textit{avoidance sequence} for the basis \{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}\}.

In effect, avoidance sets provide a language for characterizing permutations that share common combinatorial properties, and the associated avoidance sequences are commonly used to motivate the formulation of (often otherwise unmotivated) bijections with equinumerous combinatorial sets. Section 3.3 contains a discussion along these lines for the avoidance set \(S_n(3\rightarrow 1\rightarrow 42)\), which turns out to be especially important in the study of Patience Sorting. (See also Section 1.2 below.)

More information about permutation patterns in general can be found in [6].

\subsection*{1.2 Summary of Main Results}

In this section, we summarize the main results in this Dissertation concerning Patience Sorting as an algorithm. Many of these results first appeared in [7, 8, 9]. (Chapter 2 is largely concerned with comparing Patience Sorting to other strategies for Floyd’s Game, so we do not summarize it here.)

In Section 3.2 we define a “column word” operation on pile configurations (Definition 3.2.1) called the \textit{reverse patience word \textit{(RPW)}}. This is by direct analogy with the column word of a standard Young tableau (see Example 1.1.8). In Section 3.3 we then provide the following characterization of reverse patience words in terms of pattern avoidance.
1.2. Summary of Main Results

Theorem 3.3.1. The set of permutations $S_n(3-\overline{1}-42)$ avoiding the barred pattern $3-\overline{1}-42$ is exactly the set $RPW(R(\mathfrak{S}_n))$ of reverse patience words obtainable from the symmetric group $\mathfrak{S}_n$. Moreover, the number of elements in $S_n(3-\overline{1}-42)$ is given by the $n^{th}$ Bell number

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$ 

This theorem is actually a special case of a more general construction. In Section 3.2 we define the equivalence relation $\mathcal{PS}$ (Definition 3.2.4) on the symmetric group $\mathfrak{S}_n$ by analogy to Knuth equivalence (Definition 1.1.10). Specifically, two permutations $\sigma, \tau \in \mathfrak{S}_n$ are called patience sorting equivalent (denoted $\sigma \mathcal{PS} \tau$) if they result in the same pile configurations $R(\sigma) = R(\tau)$ under Patience Sorting (Algorithm 1.1.19). We then provide the following characterization of this equivalence relation:

Theorem 3.2.6. Let $\sigma, \tau \in \mathfrak{S}_n$. Then $\sigma \mathcal{PS} \tau$ if and only if $\sigma$ and $\tau$ can be transformed into the same permutation by changing one or more occurrences of the pattern $2-31$ into occurrences of the pattern $2-13$ such that none of these $2-31$ patterns are contained within an occurrence of a $3-1-42$ pattern.

In other words, $\mathcal{PS}$ is the equivalence relation generated by changing $3-\overline{1}-42$ patterns into $3-\overline{1}-24$ patterns.

In Section 3.4 we then use this result to additionally characterize those permutations within singleton equivalence classes under $\mathcal{PS}$. (We also enumerate the resulting avoidance set, which involves convolved Fibonacci numbers as defined in [34].)

Theorem 3.4.1. A pile configuration pile $R$ has a unique preimage $\sigma \in \mathfrak{S}_n$ under Patience Sorting if and only if $\sigma \in S_n(3-\overline{1}-42, 3-\overline{1}-24)$. 
1.2. Summary of Main Results

In Chapter 4 we next turn our attention to Extended Patience Sorting (Algorithm 1.1.22). Specifically, we first characterize the resulting stable pairs in Section 4.1 using further pattern avoidance (where $S'$ is the same notation as in Example 1.1.25):

**Theorem 4.1.4.** Extended Patience Sorting gives a bijection between the symmetric group $\mathfrak{S}_n$ and the set of all ordered pairs of pile configurations $(R, S)$ such that both $\text{sh}(R) = \text{sh}(S)$ and $(\text{RPW}(R), \text{RPW}(S'))$ avoids simultaneous occurrences of the pairs of patterns $(31-2, 13-2)$, $(31-2, 32-1)$ and $(32-1, 13-2)$ at the same positions in $\text{RPW}(R)$ and $\text{RPW}(S')$.

We also give a geometric realization for Extended Patience Sorting that is, in the sense described in Section 4.3 naturally dual to the Geometric RSK Correspondence reviewed in Section 4.3.1.

**Theorem 4.3.5.** The Geometric Patience Sorting process described in Section 4.3.2 yields the same pair of pile configurations as Extended Patience Sorting.

Unlike the Geometric RSK Correspondence, though, Geometric Patience Sorting can result in intersecting lattice paths. In Section 4.3.4 we provide the following characterization of those permutations that do no result in intersecting lattice paths.

**Theorem 4.3.10.** Geometric Patience Sorting applied to $\sigma \in \mathfrak{S}_n$ results in non-crossing lattice paths at each iteration of the algorithm if and only if every row in both $R(\sigma)$ and $S(\sigma)$ is monotone increasing from left to right.
Chapter 2

Patience Sorting as a Card Game: Floyd’s Game

As discussed in Section 1.1.3, Patience Sorting (Algorithm 1.1.19) can be simultaneously viewed as a card sorting algorithm, as a tangible realization of the length of the longest increasing subsequence in a permutation, and as an optimal strategy for Floyd’s Game (Card Game 1.1.26). From this last standpoint, there are two natural directions of study: comparing Patience Sorting to other strategies for Floyd’s Game and appropriately modifying Patience Sorting so that it becomes a strategy for generalizations of Floyd’s Game. Note that the most important property of any such strategy is the number of piles formed since this is the statistic that determines whether or not the strategy is optimal.

In Section 2.1 we explicitly describe how Floyd’s Game should be viewed as an idealized model for Klondike Solitaire. Then, in Section 2.2 we motive two particular optimal strategies. These are the Greedy Strategy, as defined in Section 2.3 and the Look-ahead from Right Strategy, which we introduce in Section 2.5. Even though the
Greedy Strategy in Section 2.3 is conceptually more general than Algorithm 1.1.19 (which is also called the “Greedy Strategy” in [2] but which we will call the Simplified Greedy Strategy), we will see in Section 2.4 that both strategies are fundamentally the same when applied to Floyd’s Game.

The distinction between Mallows’ “patience sorting procedure” and the Greedy Strategy of Section 2.3 becomes apparent when both strategies are extended to generalizations of Floyd’s Game. In particular, for the game Two-color Patience Sorting introduced in Section 2.6 the Greedy Strategy remains optimal, mutatis mutandis, while the Simplified Greedy Strategy does not.

2.1 Floyd’s Game as an Idealized Model for Klondike Solitaire

One of the most popular games in the world is what many people commonly refer to as either Solitaire (in the U.S.) or Patience (in the U.K.). Properly called Klondike Solitaire (and also sometimes called Demon Patience or Fascination), this is the game that many card players in the English-speaking world use to while away the hours if left alone with a deck of playing cards; and yet, no one knows the odds of winning or if there is even an optimal strategy for maximizing one’s chance of winning.

Klondike Solitaire is played with a standard deck of 52 cards, with each card uniquely labeled by a combination of suit and rank. There are four suits (♣, ♥, ♠, ♦) and thirteen ranks, which are labeled in increasing order as

A (for Ace), 2, 3, . . . , 10, J (for Jack), Q (for Queen), and K (for King).

The majority of actual gameplay involves placing cards into a so-called tableau that consists of (at most) seven piles. Each pile starts with exactly one face-up card in it,
and additional cards are then placed atop these piles according to the rule that the ranks must decrease in order (from $K$ to $A$) and that cards must alternate between red suits ($\heartsuit, \diamondsuit$) and black suits ($\spadesuit, \clubsuit$). A player is also allowed to move multiple cards between piles, and it has been frequently suggested (see, e.g., [2], [23], [28]) that this form of player choice is responsible for the difficulties encountered in analyzing Klondike Solitaire.

Floyd’s Game can be viewed as a particularly simplistic idealized model for Klondike Solitaire. In particular, Floyd’s Game abstracts the formation of piles with descending value (yet allows gaps in the sequence $n, \ldots, 1$), and all notions of card color and repeated rank values are eliminated. Furthermore, the objective in Floyd’s Game is to end with as few piles as possible, while Klondike Solitaire is concerned with forming four so-called foundation piles. One also expects to end up with many piles under Floyd’s Game since cards cannot be moved once they are placed in a pile.

When viewed as an abstraction of Klondike Solitaire, there are two natural directions for generalizing Floyd’s Game so that it to be more like Klondike Solitaire. Namely, one can introduce either repeated card ranks or card colors (or both). In the former case, analysis of pile formation is relatively straightforward, though care must be taken to specify whether or not identical ranks can be played atop each other. (Both “repeated rank” variants are briefly considered in [2] and [17] under the names “ties allowed” and “ties forbidden”.) In the “card colors” case, though, analyzing pile formation becomes significantly more subtle. We introduce a two-color generalization of Floyd’s Game called Two-color Patience Sorting in Section 2.6 below.
2.2 Strategies for Floyd’s Game and their Optimality

There are essentially two broad classes of strategies with which one can play Floyd’s Game: look-ahead and non-look-ahead strategies. In the former class, one is allowed to take into account the structure of the entire deck of cards when deciding the formation of card piles. When actually playing a game, though, such strategies are often undesirable in comparison to non-look-ahead strategies. By taking a more restricted view of the deck, non-look-ahead strategies require that each card be played in order without detailed consideration of the cards remaining to be played. In this sense, look-ahead strategies are global algorithms applied to the deck of cards, while non-look-ahead strategies are local algorithms.

One might intuitively guess that look-ahead strategies are superior to non-look-ahead strategies since “looking ahead” eliminates any surprises about how the remainder of the deck of cards is ordered. Thus, it may come as a surprise that look-ahead strategies can do no better than non-look-ahead strategies when playing Floyd’s Game. (We also prove an analogous result for Two-color Patience Sorting in Section 2.6). Specifically, the prototypical non-look-ahead Greedy Strategy defined in Section 2.3 is easily shown to be optimal using a so-called “strategy stealing” argument. In other words, the Greedy Strategy always results in the fewest number of piles possible.

Even though one cannot form fewer piles under a look-ahead strategy for Floyd’s Game, it can still be useful to compare optimal look-ahead strategies to the Greedy Strategy. In particular, the Look-ahead from Right Strategy introduced in Section 2.6 yields exactly the same piles as the Greedy Strategy, but it also brings additional
2.3. The Greedy Strategy for Floyd’s Game

Under the Greedy Strategy for Floyd’s Game, one plays as follows.

**Strategy 2.3.1** (Greedy Strategy for Floyd’s Game). Given \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathfrak{S}_n \), construct the set of piles \( \{p_i\} \) by

1. first forming a new pile \( p_1 \) with top card \( \sigma_1 \).

2. Then, for \( l = 2, \ldots, n \), suppose that \( \sigma_1, \sigma_2, \ldots, \sigma_{l-1} \) have been used to form the piles

   \[
   p_1 = \begin{cases} 
   \sigma_{1s_1} \\ 
   \vdots \\ 
   \sigma_{11} 
   \end{cases}, \quad
   p_2 = \begin{cases} 
   \sigma_{2s_2} \\ 
   \vdots \\ 
   \sigma_{21} 
   \end{cases}, \quad
   p_k = \begin{cases} 
   \sigma_{ks_k} \\ 
   \vdots \\ 
   \sigma_{k1} 
   \end{cases}
   \]

   (a) If \( \sigma_l > \sigma_{js_j} \) for each \( j = 1, \ldots, k \), form a new pile \( p_{k+1} \) with top card \( \sigma_l \).

   (b) Otherwise redefine pile \( p_m \) to be

   \[
   p_m = \begin{cases} 
   \sigma_l \\ 
   \sigma_{ms_m} \\ 
   \vdots \\ 
   \sigma_{m1} 
   \end{cases}
   \]

   where \( \sigma_{ms_m} = \min_{1 \leq j \leq k} \{ \sigma_{js_j} \mid \sigma_l < \sigma_{js_j} \} \).

In other words, Strategy [2.3.1] initially creates a single pile using the first card from the deck. Then, if possible, each remaining card is played atop the pre-existing pile having smallest top card that is larger than the given card. Otherwise, if no such pile exists, one forms a new pile.
2.3. The Greedy Strategy for Floyd’s Game

The objective of Floyd’s Game is to form as few piles as possible, so, intuitively, an optimal strategy is one that forms a new pile only when absolutely necessary. The Greedy Strategy fulfills this intuition by “eliminating” as few possible future plays. We make this explicit in the proof of the following theorem.

**Theorem 2.3.2.** The Greedy Strategy (Strategy \(2.3.1\)) is an optimal strategy for Floyd’s Game (Game \(1.1.26\)) in the sense that it forms the fewest number of piles possible under any strategy.

**Proof.** We use an inductive strategy stealing argument to show that the position in which each card is played under the Greedy Strategy cannot be improved upon so that fewer piles are formed: Suppose that, at a given moment in playing according to the Greedy Strategy, card \(c\) will be played atop pile \(p\); suppose further that, according to some optimal strategy \(S\), card \(c\) is played atop pile \(q\). We will show that it is optimal to play \(c\) atop \(p\) by “stealing” the latter strategy.

Denote by \(c_p\) the card currently atop pile \(p\) and by \(c_q\) the card currently atop pile \(q\). Since the Greedy Strategy places each card atop the pile having top card that is both larger than \(c\) and smaller than all other top cards that are larger than \(c\), we have that \(c < p \leq q\). Thus, if we were to play \(c\) on top of pile \(q\), then, from that point forward, any card playable atop the resulting pile could also be played atop pile \(p\). As such, we can construct a new optimal strategy \(T\) that mimics \(S\) verbatim but with the roles of piles \(p\) and \(q\) interchanged from the moment that \(c\) is played. Playing card \(c\) atop pile \(p\) is therefore optimal.

Since the above argument applied equally well to each card in the deck, it follows that no strategy can form fewer piles than are formed under the Greedy Strategy. \(\square\)

**Remark 2.3.3.** Even though we call Strategy \(2.3.1\) the “Greedy Strategy” for Floyd’s Game, it is subtly different from what Aldous and Diaconis call the “Greedy Strategy”
2.4. Simplifying the Greedy Strategy

The difference lies in how much work is performed when choosing where to play each card atop a pre-existing pile. However, as we will see in Section 2.4 below, the distinction is actually somewhat artificial.

2.4 Simplifying the Greedy Strategy

Under the Simplified Greedy Strategy for Floyd’s Game, one plays as follows.

Strategy 2.4.1 (Simplified Greedy Strategy for Floyd’s Game). Given a permutation \( \sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathcal{S}_n \), construct the set of piles \( \{p_l\} \) by

1. first forming a new pile \( p_1 \) with top card \( \sigma_1 \).

2. Then, for \( l = 2, \ldots, n \), suppose that \( \sigma_1, \sigma_2, \ldots, \sigma_{l-1} \) have been used to form the piles

\[
 p_1 = \begin{cases} 
 \sigma_{1s_1} \\
 \vdots \\
 \sigma_{11} 
\end{cases}, \quad 
 p_2 = \begin{cases} 
 \sigma_{2s_2} \\
 \vdots \\
 \sigma_{21} 
\end{cases}, \quad 
 p_k = \begin{cases} 
 \sigma_{ks_k} \\
 \vdots \\
 \sigma_{k1} 
\end{cases}
\]

(a) If \( \sigma_l > \sigma_{js_j} \) for each \( j = 1, \ldots, k \), form a new pile \( p_{k+1} \) with top card \( \sigma_l \).

(b) Otherwise redefine pile \( p_m \) to be

\[
 p_m = \begin{cases} 
 \sigma_l \\
 \sigma_{ms_m} \\
 \vdots \\
 \sigma_{m1} 
\end{cases} \quad \text{where} \quad m = \min_{1 \leq j \leq k} \left\{ j \mid \sigma_l < \sigma_{js_j} \right\}.
\]

It is not difficult to see that the Simplified Greedy Strategy (Strategy 2.4.1) produces the same piles as the Greedy Strategy (Strategy 2.3.1). However, before
2.4. Simplifying the Greedy Strategy

providing a proof of this fact, we first give an independent proof of the optimality of the Simplified Greedy Strategy. In particular, we use the following two-step approach due to Aldous and Diaconis in [2]: First, we prove a lower bound on the number of piles formed under any strategy. Then we show that Strategy 2.4.1 achieves this bound.

Given a positive integer \( n \in \mathbb{Z}_+ \), recall that an increasing subsequence of a permutation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n \) is any subsequence \( s = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \) (meaning that \( i_1 < i_2 < \cdots < i_k \)) for which \( \sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k} \). Note that, while a permutation may have many longest increasing subsequences, the length of the longest increasing subsequence of \( \sigma \) (which we denote by \( \ell_n(\sigma) \)) is nonetheless well-defined.

**Lemma 2.4.2.** The number of piles that result from applying any strategy for Floyd’s Game to a permutation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n \) is bounded from below by the length \( \ell_n(\sigma) \) of the longest increasing subsequence of \( \sigma \).

**Proof.** Let \( s = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \) be a longest increasing subsequence of \( \sigma \). Since the cards played atop each pile must decrease in value while the components of \( s \) increase in value, it follows that each component of \( s \) must be played into a different pile. Consequently, there must be at least \( k \) piles, regardless of one’s strategy. \( \square \)

**Proposition 2.4.3.** The number of piles that result from applying the Simplified Greedy Strategy (Strategy 2.4.1) to a permutation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n \) is bounded from above by the length \( \ell_n(\sigma) \) of the longest increasing subsequence of \( \sigma \).

**Proof.** While applying the Simplified Greedy Strategy to \( \sigma \), we impose the following bookkeeping device:

- If a card is placed on top of the left most pile, then proceed as normal.

- If a card is placed on top of any other pile, then draw an arrow from it to the current top card of the pile immediately to the left.
When the Simplified Greedy Strategy terminates, denote by $\sigma_{i_k} (= \sigma_{k_{s_k}})$ the top card of the right most pile. Then there must exist a sequence of arrows from right to left connecting the cards

$$\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k},$$

where $\sigma_{i_j}$ is contained in the $j^{th}$ pile and $k$ denotes the number of resulting piles. Moreover, these cards must appear in increasing order from left to right in $\sigma$ by construction of the arrows.

**Corollary 2.4.4.** The Simplified Greedy Strategy (Strategy 2.4.1) is an optimal strategy for Floyd’s Game (Game 1.1.26) in the sense that it forms the fewest number of piles possible under any strategy.

**Proof.** This follow from the combination of Proposition 2.4.3 and Lemma 2.4.2. □

Finally, to see that the Greedy Strategy and the Simplified Greedy Strategy actually do produce the same piles, note first that the strategies differ only in their respective Step 2(b). In particular, under the Greedy Strategy, one plays each card atop the pile whose top card is both larger than the current card and, at the same time, is the smallest top card among all top cards larger than it. However, with a bit of thought, it is easy to see that the cards atop each pile naturally form an increasing sequence from left to right since new piles are always created to the right of all pre-existing piles. (Cf. the formation of piles in Example 1.1.21.) Thus, the Greedy Strategy reduces to playing each card as far to the left as possible, but this is exactly how cards are played under the Simplified Greedy Strategy.

We have both proven the following Proposition and provided an independent proof of Theorem 2.3.2.
2.5 A Look-Ahead Strategy for Floyd’s Game

Proposition 2.4.5. Strategies $[2.3.1]$ and $[2.4.1]$ produce the same piles when applied to a permutation $\sigma \in \mathcal{S}_n$.

2.5 A Look-Ahead Strategy for Floyd’s Game

In this section, we outline yet another proof of Theorem $[2.3.2]$ by again constructing a strategy that yields the same piles as the Greedy Strategy (Strategy $[2.3.1]$). Unlike the Greedy Strategy, though, this new strategy is actually a look-ahead strategy in the sense that one takes into account the entire structure of the deck of cards when forming piles. Aside from providing a proof of Theorem $[2.3.2]$ that does not rely upon Lemma $[2.4.2]$ or “strategy stealing”, this approach also has the advantage of resembling how one actually plays Klondike Solitaire. Specifically, it consists of moving sub-piles around (in this case, strictly from right to left) in a manner that mimics the arbitrary rearrangement of piles in Klondike Solitaire.

The following strategy essentially builds the left-to-right minima subsequences (see Definition $[3.1.3]$) of the deck of cards by reading through the permutation repeatedly from right to left. The example that follows should make the usage clear.

Definition 2.5.1. Given a subset $E \subset \mathbb{R}$, we define the minimum (positive) excluded integer of $E$ to be

$$\text{mex}^+(E) = \min(\mathbb{Z}_+ \setminus E),$$

where $\mathbb{Z}_+$ denotes the set of positive integers.

Strategy 2.5.2 (Look-ahead from Right Strategy for Floyd’s Game). Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$, set $E = \{\}$ and inductively refine the set of piles $\{p_i\}$ as follows:
2.5. A Look-Ahead Strategy for Floyd’s Game

- (Base Step) Form initial piles $p_1, \ldots, p_n$, where the top card of each $p_i$ is $\sigma_i$.

- (Inductive Step) Suppose that we currently have the $k$ piles

\[
\begin{align*}
p_1 &= \begin{cases} 
\sigma_{1s_1} \\
\vdots \\
\sigma_{11}
\end{cases}, & p_2 &= \begin{cases} 
\sigma_{2s_2} \\
\vdots \\
\sigma_{21}
\end{cases}, & \ldots, & p_k &= \begin{cases} 
\sigma_{ks_k} \\
\vdots \\
\sigma_{k1}
\end{cases},
\end{align*}
\]

and set $m = \text{mex}^+(E)$.

(i) If $m \geq n + 1$, then cease playing Floyd’s Game.

(ii) Otherwise, by construction, there exists a pile $p_l$ having top card $m$. Thus, we iteratively combine certain piles, starting at pile $p_l$, as follows:

1. If $\sigma_{i1} > \sigma_{is_i}$ for $i = 1, 2, \ldots, l - 1$ (i.e., the bottom card of the $i^{th}$ pile is larger than the top card of each pile to its left), then we redefine $E := E \cup \{\sigma_{1\mu}\}$ and return to the start of the inductive step (since pile $p_l$ cannot legally be placed on top of any pile to its left).

2. Otherwise, take $p_t$ to be the right-most pile that is both to the left of $p_l$ and for which $\sigma_{i1}$ is larger than the bottom card of $p_t$. I.e.,

\[
t = \max_{1 \leq i < l} \{i \mid \sigma_{i1} < \sigma_{i1}\}.
\]

   (a) If $|t - l| \leq 1$, then we redefine $E := E \cup \{\sigma_{1\mu}\}$ and return to the start of the induction step to avoid moving pile $p_t$ past pile $p_l$.

   (b) Otherwise, we place pile $p_t$ atop the pile $p_{\mu}$ between piles $p_t$ and $p_l$ such that the current top card of $p_{\mu}$ is the smallest card greater
2.5. A Look-Ahead Strategy for Floyd’s Game

than \( \sigma_{11} \). I.e., we redefine the pile \( p_\mu \) to be

\[
p_\mu = \begin{cases} 
p_l & \text{where } \mu = \min_{t < i < k} \{i \mid m < \sigma_{is} \}, 
p_\mu & \end{cases}
\]

and then we redefine \( E := E \cup \{\sigma_{1\mu}\} \).

**Example 2.5.3.** Let \( \sigma = 64518723 \in \mathcal{S}_8 \). Then one plays Floyd’s Game under the Look-ahead from Right Strategy (Strategy [2.5.2]) as follows:

- Start with \( E = \{\} \) and the initial piles

\[
6 \ 4 \ 5 \ 1 \ 8 \ 7 \ 2 \ 3
\]

- \( \text{mex}^+(E) = 1 \), so move the pile “1” onto the pile “4” and then the resulting pile onto the pile “6”. This results in \( E = \{1, 4, 6\} \) and the piles

\[
1 \\
4 \\
6 \ 5 \ 8 \ 7 \ 2 \ 3
\]

- \( \text{mex}^+(E) = 2 \), so move the pile “2” onto the pile “5”. This results in \( E = \{1, 2, 4, 5, 6\} \) and the piles

\[
1 \\
4 \ 2 \\
6 \ 5 \ 8 \ 7 \ 3
\]

- \( \text{mex}^+(E) = 3 \), so move the pile “3” onto the pile “7” and then the resulting pile onto pile the “8”. This results in \( E = \{1, 2, 3, 4, 5, 6, 7, 8\} \) and the piles

\[
1 \ 3 \\
4 \ 2 \ 7 \\
6 \ 5 \ 8
\]
2.6. The Greedy Strategy for Two-color Patience Sorting

- Finally, $\text{mex}^+(E) = 9 > 8$, so we cease playing Floyd’s Game.

Remark 2.5.4. In view of how Strategy 2.5.2 was played out in the above example, it should be fairly clear that it is an optimal strategy for Floyd’s Game. Though we do not include a proof, we nonetheless note that Geometric Patience Sorting (see Section 4.3) can be used to show that Strategies 2.3.1 and 2.5.2 always result in the same piles.

2.6 The Greedy Strategy for Two-color Patience Sorting

The following is a natural generalization of Floyd’s Game:

**Card Game 2.6.1** (Two-color Patience Sorting). Given a deck of (uniquely labeled) bi-colored cards $c_1, c_2, \ldots, c_{2n}$,

- place the first card $c_1$ from the deck into a pile by itself.
- Then, for each card $c_i$ ($i = 2, \ldots, 2n$), either
  - put $c_i$ into a new pile by itself
  - or play $c_i$ on top of any pile whose current top card is larger than $c_i$ and of opposite color.
- The object of the game is to end with as few piles as possible.

In other words, Two-color Patience Sorting (TCPS) is played exactly like Floyd’s Game except that each card is one of two colors and the colors of the cards in each pile must alternate. As discussed in Section 2.1 this introduction of card color
results in piles that are significantly more like those produced while playing Klondike Solitaire.

In order to model a bi-colored deck of cards, we introduce the following conventions. Given a positive integer \( n \in \mathbb{Z}_+ \), we denote \( [n] = \{1, 2, \ldots, n\} \) and by \( \mathfrak{S}_n \) the set of all permutations on \([n]\). Then, as a natural extension of these conventions, we define the overbarred integers \( [\bar{n}] = \{\bar{1}, \bar{2}, \ldots, \bar{n}\} \) by analogy and use \( \mathfrak{S}_{\bar{n}} \) to denote the set of all permutations on \([\bar{n}]\). Letting \( \mathcal{N} = [n] \cup [\bar{n}] \), we will use \( i^\pm \) to denote \( i \in \mathcal{N} \) when it is unimportant whether or not \( i \) has an overbar.

With this notation, we can now define the elements in our bi-colored deck of cards as follows.

**Definition 2.6.2.** A 2-permutation \( w = \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ w_1 & w_2 & w_3 & w_4 \\ \vdots & \vdots & \vdots & \vdots \\ 2n-1 & 2n \\ w_{2n-1} & w_{2n} \end{array} \right) \) is any permutation on the set \( \mathcal{N} \).

As with normal permutations, we will often also denote the 2-permutation \( w \) by the bottom row of the two-line array. That is, by abuse of notation, we write \( w = w_1 w_2 \cdots w_{2n} \). Furthermore, we denote by \( \mathfrak{S}_\mathcal{N} \) the set of all 2-permutations on \( \mathcal{N} \). Note that \( \mathfrak{S}_\mathcal{N} \) is isomorphic to \( \mathfrak{S}_{2n} \) as a set, but we distinguish \( \mathfrak{S}_\mathcal{N} \) from \( \mathfrak{S}_{2n} \) in order to emphasize the difference in how we will treat the former combinatorially. In particular, there is no transitive order relation between barred and unbarred numbers.

Finally, we introduce a useful function \( \mathcal{N} \times \mathcal{N} \to \{0, 1\} \) that makes dealing with a mixture of barred and unbarred numbers more convenient:

\[
bar(i, j) = \begin{cases} 
1, & \text{if exactly one of } i \text{ and } j \text{ is overbarred} \\
0, & \text{otherwise}
\end{cases}
\]

Now that we have established notation for our bi-colored deck of cards, we show
that the Greedy Strategy for Floyd’s Game can be naturally extended in order to form an optimal strategy for TCPS while the Simplified Greedy Strategy cannot. More precisely, we begin by defining the following strategy.

**Strategy 2.6.3** (Naïve Greedy Strategy for Two-color Patience Sorting). Given a random 2-permutation \( w = w_1 w_2 \cdots w_{2n} \in S_N \), build the set of piles \( \{ p_i \} \) by

1. first forming a new pile \( p_1 \) with top card \( w_1 \).
2. Then, for each \( l = 2, \ldots, 2n \), suppose that \( w_1, w_2, \ldots, w_{l-1} \), have been used to form piles

\[
\begin{align*}
p_1 &= \left\{ \begin{array}{c}
w_{1s_1} \\
\vdots \\
w_{11} \\
\end{array} \right\}, &
p_2 &= \left\{ \begin{array}{c}
w_{2s_2} \\
\vdots \\
w_{21} \\
\end{array} \right\}, &
p_k &= \left\{ \begin{array}{c}
w_{ks_k} \\
\vdots \\
w_{k1} \\
\end{array} \right\},
\end{align*}
\]

(a) If \( w^+_l \geq w^+_j \) for each \( j = 1, \ldots, k \) such that \( \overline{w_l, w_j s_j} = 1 \), then form a new pile \( p_{k+1} \) with top card \( w_l \).

(b) Otherwise, redefine pile \( p_m \) to be

\[
p_m = \left\{ \begin{array}{c}
w_l \\
w_{ms_m} \\
\vdots \\
w_{m1} \\
\end{array} \right\}
\]

where \( m = \min \left\{ j \mid w^+_l < w^+_j, \overline{w_l, w_j s_j} = 1 \right\} \).

In other words, we play each card as far to the left as possible (up to card color). We both illustrate this algorithm and show that it is not an optimal strategy for TCPS in the following example.
Example 2.6.4. Let \( w = \overline{323112} \in \mathcal{S}_{[3] \cup [\overline{3}]} \). Then one applies Strategy 2.6.3 to \( w \) as follows:

After playing \( \overline{3} \):
\[
\overline{3}
\]

After playing \( \overline{2} \):
\[
\overline{3} \quad \overline{2}
\]

After playing \( 3 \):
\[
\overline{3} \quad \overline{2} \quad 3
\]

After playing \( \overline{1} \):
\[
\overline{1} \quad \overline{3} \quad \overline{2} \quad 3
\]

After playing \( 1 \):
\[
1 \quad \overline{1} \quad \overline{3} \quad \overline{2} \quad 3
\]

After playing \( 2 \):
\[
\overline{3} \quad \overline{2} \quad 3 \quad \overline{2}
\]

However, note that one could also play to obtain the following piles (as given by Strategy 2.6.5 below):

After playing the \( \overline{3} \):
\[
\overline{3}
\]

After playing the \( \overline{2} \):
\[
\overline{3} \quad \overline{2}
\]

After playing the \( 3 \):
\[
\overline{3} \quad \overline{2} \quad 3
\]

After playing the \( \overline{1} \):
\[
\overline{1} \quad \overline{3} \quad \overline{2} \quad 3
\]

After playing the \( 1 \):
\[
1 \quad \overline{1} \quad \overline{3} \quad \overline{2} \quad 3
\]

After playing the \( 2 \):
\[
\overline{3} \quad \overline{2} \quad 3 \quad \overline{2}
\]

Strategy 2.6.5 (Greedy Strategy for Two-color Patience Sorting). Given a random 2-permutation \( w = w_1 w_2 \cdots w_{2n} \in \mathcal{S}_N \), build the set of piles \( \{p_i\} \) by

- first forming a new pile \( p_1 \) with top card \( w_1 \).

- Then, for \( l = 2, \ldots, 2n \), suppose that \( w_1, w_2, \ldots, w_{l-1} \), have been used to form piles

\[
p_1 = \begin{cases} 
    w_{1s_1} \\
    \vdots \\
    w_{11} 
\end{cases},
\quad
p_2 = \begin{cases} 
    w_{2s_2} \\
    \vdots \\
    w_{21} 
\end{cases},
\quad
p_k = \begin{cases} 
    w_{ks_k} \\
    \vdots \\
    w_{k1} 
\end{cases}
\]
2.6. The Greedy Strategy for Two-color Patience Sorting

1. If \( w_i^\pm \geq w_{js_j}^\pm \) for each \( j = 1, \ldots, k \) such that \( \bar{\text{bar}}(w_l, w_{js_j}) = 1 \), then form a new pile \( p_{k+1} \) with top card \( w_l \).

2. Otherwise, redefine pile \( p_m \) to be

\[
p_m = \begin{cases} 
  w_l \\
  w_{ms_m} \\
  \vdots \\
  w_{m1} 
\end{cases}
\]

where \( w_{ms_m} = \min_{1 \leq j \leq k} \{ w_{js_j} \mid w_l^\pm < w_{js_j}^\pm, \ \bar{\text{bar}}(w_l, w_{js_j}) = 1 \} \).

In other words, just as for Floyd’s Game, the above Greedy Strategy (Strategy 2.6.5) starts with a single pile consisting of the first card from the deck. Then, if possible, one plays each remaining card on top of the pre-existing pile having smallest top card that is both larger than the given card and of opposite color. Otherwise, if no such pile exists, a new pile is formed. Moreover, just as with Floyd’s Game, this strategy will again be optimal for TCPS since it forms new piles only when absolutely necessary.

**Theorem 2.6.6.** The Greedy Strategy (Strategy 2.6.5) is an optimal strategy for Two-color Patience Sorting in the sense that it forms the fewest number of piles possible under any strategy.

**Proof.** We use an inductive strategy stealing argument to show that the position in which each card is played under the Greedy Strategy cannot be improved upon so that fewer piles are formed: Suppose that, at a given moment in playing according to the Greedy Strategy, card \( c \) will be played atop pile \( p \); suppose further that, according to some optimal strategy \( S \), card \( c \) is played atop pile \( q \). We will show that it is optimal to play \( c \) onto \( p \) by “stealing” the latter strategy.
Denote by $c_p$ the card atop pile $p$ and by $c_q$ the card atop pile $q$. Since the Greedy Strategy plays each card atop the pile having top card that is larger than $c$, of opposite color, and smaller than all other top cards that are both larger than $c$ and of opposite color, we have that $c < p \leq q$ with $\text{bar}(c, c_p) = \text{bar}(c, c_q) = 0$ and $\text{bar}(c_p, c_q) = 1$. Thus, if we were to play $c$ atop $c_q$, then any card playable atop the modified pile $q$ from that point onward can also be played atop $c_p$. As such, we can construct a new optimal strategy $T$ that mimics $S$ verbatim but with the roles of piles $p$ and $q$ interchanged from the moment that card $c$ is played. It is therefore optimal to play card $c$ atop pile $p$.

Since we can apply the above argument to each card in the deck, it follows that no other strategy can form fewer piles than are formed under the Greedy Strategy.

We conclude by making the following useful observation, which, unlike Floyd's Game, does not have an affirmative converse (as illustrated in the example that follows).

**Proposition 2.6.7.** The number of piles that results from playing Two-color Patience Sorting under any strategy on the 2-permutation $w = w_1w_2 \cdots w_{2n} \in S_N$ is bounded from below by the length of the longest weakly increasing subsequence in the word $w^\pm$.

**Proof.** The proof is identically to that of Lemma 2.4.2. \qed

**Example 2.6.8.** Let $w = 23143241 \in S_{[4] \cap [4]}$. Then, upon applying the Greedy Strategy (Strategy 2.6.5) to $w$, one obtains the following five piles:

$$
\begin{array}{cccc}
1 & 1 & 2 \\
2 & 3 & 4 & 3 & 4
\end{array}
$$

However, the length of the longest weakly increasing subsequence of $w^\pm = 23143241$ is four, which corresponds to the unique longest weakly increasing subsequence 2334.
2.6. The Greedy Strategy for Two-color Patience Sorting

In fact, one can easily construct pathological examples in which the “gap” between the length of the longest weakly increasing subsequence and the number of piles formed under the Greedy Strategy grows arbitrarily large. E.g., consider

\[ w = n, n - 1, \ldots, 2, 1, \overline{n}, \overline{n - 1}, \ldots, \overline{2}, 1 \in \mathcal{S}_N. \]

Unlike Floyd’s Game, no known combinatorial statistic on 2-permutations is equidistributed with the number of piles formed when TCPS is played under the Greedy Strategy. Since Proposition 2.6.7 yields a lower bound for the number of piles formed, though, one could conceivably study the asymptotic “gap” between the length of the longest weakly increasing subsequence and the number of piles formed in order to gain insight into the expected number of piles formed.
Chapter 3

Patience as an Algorithm:
Mallows’ Patience Sorting
Procedure

3.1 Pile Configurations and
Northeast Shadow Diagrams

Given a positive integer $n \in \mathbb{Z}_+$, we begin by explicitly characterizing the objects that result when Patience Sorting (Algorithm 1.1.19) is applied to a permutation $\sigma \in \mathfrak{S}_n$:

**Lemma 3.1.1.** Let $\sigma \in \mathfrak{S}_n$ be a permutation and $R(\sigma) = \{r_1, r_2, \ldots, r_k\}$ be the pile configuration associated to $\sigma$ under Algorithm 1.1.19. Then $R(\sigma)$ is a partition of the set $[n] = \{1, 2, \ldots, n\}$ such that, denoting $r_j = \{r_{j_1} > r_{j_2} > \cdots > r_{j_{s_j}}\}$,

$$r_{j_{s_j}} < r_{i_{s_i}} \quad \text{if} \quad j < i.$$  \hfill (3.1.1)
Moreover, for every set partition \( T = \{t_1, t_2, \ldots, t_k\} \) satisfying Equation (3.1.1), there is a permutation \( \tau \in \mathfrak{S}_n \) such that \( R(\tau) = T \).

**Proof.** Given the pile configuration \( R(\sigma) = \{r_1, r_2, \ldots, r_k\} \) associated to \( \sigma \in \mathfrak{S}_n \), suppose that, for some pair of indices \( i, j \in [k] \), we have that \( j < i \) but \( r_{js_j} > r_{is_i} \). Then the card \( r_{is_i} \) was put atop pile \( r_i \) when pile \( r_j \) had top card \( d_j \geq r_{js_j} \) so that \( d_j > r_{is_i} \) as well. However, it then follows that the card \( r_{is_i} \) would actually have been placed atop either pile \( r_j \) or atop some pile to the left of \( r_j \) instead of atop pile \( r_i \). This resulting contradiction implies that \( r_{js_j} < r_{is_i} \) for each \( j < i \).

Conversely, let \( T = \{t_1, t_2, \ldots, t_k\} \) be any set partition of \([n]\), with the block \( t_j = \{t_{j1} > t_{j2} > \cdots > t_{js_j}\} \) for each \( j \in [k] \) and with \( t_{js_j} < t_{is_i} \) for every pair of indices \( i, j \in [k] \) such that \( j < i \). Then, setting

\[
\tau = t_{11}t_{12}\cdots t_{is_i}t_{21}t_{22}\cdots t_{js_j}t_{ks_k},
\]

it is easy to see that \( \tau \in \mathfrak{S}_n \) and that \( R(\tau) = T \). \( \square \)

According to Lemma 3.1.1, pile configurations formed from \([n]\) are set partitions of \([n]\) in which the constituent blocks have been ordered by their minimal element. (Cf. Example 1.1.20.) We devote the remainder of this section to an alternate characterization involving the (northeast) shadow diagram of a permutation. The construction of shadow diagrams was first used by Viennot \[40\] to study properties of the RSK Correspondence (Algorithm 1.1.15) for permutations. (See Section 4.3.)

**Definition 3.1.2.** Given a lattice point \((m, n) \in \mathbb{Z}^2\), we define the northeast shadow of \((m, n)\) to be the quarter space

\[
S_{NE}(m, n) = \{(x, y) \in \mathbb{R}^2 \mid x \geq m, \ y \geq n\}.
\]
3.1. Pile Configurations and Northeast Shadow Diagrams

Figure 3.1: Examples of Northeast Shadow and Shadowline Construction

See Figure 3.1(a) for an example of a point’s shadow.

By itself, the notion of shadow doesn’t come across as particularly exciting. However, one can use these shadows in order to associate a lattice path to any (finite) collection of lattice points.

**Definition 3.1.3.** Given lattice points \((m_1, n_1), (m_2, n_2), \ldots, (m_k, n_k) \in \mathbb{Z}^2\), we define their **northeast shadowline** to be the boundary of the union of the northeast shadows \(S_{NE}(m_1, n_1), S_{NE}(m_2, n_2), \ldots, S_{NE}(m_k, n_k)\).

In particular, we wish to associate to every permutation a certain collection of shadowlines (as illustrated in Figure 3.1(b)–(d));

**Definition 3.1.4.** Given a permutation \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathfrak{S}_n\), the **northeast shadow diagram** \(D_{NE}(\sigma) = D_{NE}^{(0)}(\sigma)\) of \(\sigma\) consists of the shadowlines \(L_1(\sigma), L_2(\sigma), \ldots, L_k(\sigma)\) formed as follows:


- $L_1(\sigma)$ is the northeast shadowline for the set of lattice points

\[ \{(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)\}. \]

- Then, while at least one of the points $(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)$ is not contained in the shadowlines $L_1(\sigma), L_2(\sigma), \ldots, L_j(\sigma)$, define $L_{j+1}(\sigma)$ to be the northeast shadowline for the points

\[ \{(i, \sigma_i) \mid i \in [n], (i, \sigma_i) \notin \bigcup_{k=1}^{j} L_k(\sigma)\}. \]

In other words, the shadow diagram $D_{NE}(\sigma) = \{L_1(\sigma), L_2(\sigma), \ldots, L_k(\sigma)\}$ of the permutation $\sigma \in S_n$ is defined inductively by first taking $L_1(\sigma)$ to be the shadowline for the so-called diagram $\{(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)\}$ of the permutation $\sigma \in S_n$. Then we ignore the points whose shadows were actually used in building $L_1(\sigma)$ and define $L_2(\sigma)$ to be the shadowline of the resulting subset of the permutation diagram. We then build $L_3(\sigma)$ as the shadowline for the points not yet used in constructing either $L_1(\sigma)$ or $L_2(\sigma)$, and this process continues until each of the points in the permutation’s diagram has been exhausted.

One can show that the shadow diagram for a permutation $\sigma \in S_n$ encodes the top row of the RSK Correspondence insertion tableau $P(\sigma)$ (resp. recording tableau $Q(\sigma)$) as the smallest ordinates (resp. smallest abscissae) of all points along the constituent shadowlines $L_1(\sigma), L_2(\sigma), \ldots, L_k(\sigma)$. (See Sagan [31] for a proof.) In particular, if $\sigma$ has pile configuration $R(\sigma) = \{r_1, r_2, \ldots, r_m\}$, then $m = k$ since the number of piles is equal to the length of the top row of $P(\sigma)$ (as both are the length of the longest increasing subsequence of $\sigma$; see Sections 1.1.2 and 1.1.3). We can say
even more about the relationship between $D_{NE}(\sigma)$ and $R(\sigma)$ when both are viewed in terms of *left-to-right minima subsequences* (a.k.a. *basic subsequences*).

**Definition 3.1.5.** Let $\pi = \pi_1 \pi_2 \cdots \pi_l$ be a partial permutation on $[n] = \{1, 2, \ldots, n\}$. Then the *left-to-right minima subsequence* of $\pi$ consists of those components $\pi_j$ of $\pi$ such that

$$\pi_j = \min_{1 \leq i \leq j} \{\pi_i\}.$$ 

We then inductively define the left-to-right minima subsequences $s_1, s_2, \ldots, s_k$ of a permutation $\sigma \in \mathfrak{S}_n$ by first taking $s_1$ to be the left-to-right minima subsequence for $\sigma$ itself. Then each subsequent $i^{th}$ *left-to-right minima subsequence* $s_i$ is defined to be the left-to-right minima subsequence for the partial permutation obtained by removing the elements of $s_1, s_2, \ldots, s_{i-1}$ from $\sigma$.

We are now in a position to give a particularly nice correspondence between the piles formed under Patience Sorting and the shadowlines that constitute the shadow diagram of a permutation via these left-to-right minima subsequences. We will rely heavily upon this correspondence in the sections that follow.

**Lemma 3.1.6.** Suppose $\sigma \in \mathfrak{S}_n$ has shadow diagram $D_{NE}(\sigma) = \{L_1(\sigma), \ldots, L_k(\sigma)\}$. Then the ordinates of the southwest corners of each $L_i$ are exactly the cards in the $i^{th}$ pile $r_i \in R(\sigma)$ formed by applying Patience Sorting (Algorithm [1.1.19]) to $\sigma$. In other words, the $i^{th}$ pile $r_i$ is exactly the $i^{th}$ left-to-right minima subsequence of $\sigma$.

**Proof.** The $i^{th}$ left-to-right minima subsequence $s_i$ of $\sigma$ consists of the entries in $\sigma$ that appear at the end of an increasing subsequence of length $i$ but not at the end of an increasing subsequence of length $i + 1$. Thus, since each element added to a pile must be smaller than all other elements already in the pile, $s_1 = r_1$. It then follows similarly by induction that $s_i = r_i$ for each $i = 2, \ldots, k$. 

The proof that the ordinates of the southwest corners of the shadowlines $L_i$ are also exactly the elements of the left-to-right minima subsequences $s_i$ is similar.

We conclude this section with an example.

**Example 3.1.7.** Consider $\sigma = 64518723 \in S_8$. From Figure 3.1, we see that $D_{NE}(\sigma) = \{L_1(\sigma), L_2(\sigma), L_3(\sigma)\}$,

where $L_1(\sigma)$ has southwest corners $\{(1,6), (2,4), (4,1)\}$, $L_2(\sigma)$ has southwest corners $\{(3,5), (7,2)\}$, and $L_3(\sigma)$ has southwest corners $\{(5,8), (6,7), (8,3)\}$. Moreover, one can check that $\sigma$ has left-to-right minima subsequence $s_1 = 641$ (corresponding to the ordinates of the southwest corners for $L_1(\sigma)$), $s_2 = 52$ (corresponding to the ordinates of the southwest corners for $L_2(\sigma)$), and $s_3 = 873$ (corresponding to the ordinates of the southwest corners for $L_3(\sigma)$).

Similarly, $R(\sigma) = \{s_1, s_2, s_3\}$ as in Example 1.1.20.

### 3.2 Reverse Patience Words and Patience Sorting Equivalence

In the proof of Lemma 3.1.1, we gave the construction of a special permutation that can be used to generate any set partition under Patience Sorting (Algorithm 1.1.19). At the same time, though, it should be clear that there are in general many permutations resulting in a given set partition. In this section, we characterize the corresponding equivalence relation on the symmetric group $S_n$. We also characterize the most natural choice of generators for the resulting equivalence classes.
Definition 3.2.1. Given a pile configuration \( R = \{r_1, \ldots, r_k\} \), the reverse patience word \( \text{RPW}(R) \) of \( R \) is the permutation formed by concatenating the piles \( r_1, r_2, \ldots, r_k \) together, with each pile \( r_j = \{r_{j1} > r_{j2} > \cdots > r_{js_j}\} \) written in decreasing order. Using the notation of Lemma 3.1.1,

\[
\text{RPW}(R) = r_{11}r_{12}\cdots r_{1s_1} r_{21}r_{22}\cdots r_{2s_2} \cdots r_{k1}r_{k2}\cdots r_{ks_k}.
\]

Example 3.2.2. The pile configuration \( R = \{\{6 > 4 > 1\}, \{5 > 2\}, \{8 > 7 > 3\}\} \) from Example 1.1.20 is represented by the piles

\[
\begin{array}{ccc}
1 & 3 \\
4 & 2 & 7 \\
6 & 5 & 8 \\
\end{array}
\]

and has reverse patience word \( \text{RPW}(R) = 64152873 \). Moreover, as in the proof of Lemma 3.1.1

\[
R(\text{RPW}(R)) = R(64152873) = R.
\]

This illustrates the following Lemma.

Lemma 3.2.3. Given a permutation \( \sigma \in \mathfrak{S}_n \), \( R(\text{RPW}(R(\sigma))) = R(\sigma) \).

Proof. Suppose that \( T \) is a set partition of \([n]\) satisfying Equation (3.1.1), and let \( \tau = \text{RPW}(T) \) as in the proof of Lemma 3.1.1. Then \( T = R(\tau) = R(\text{RPW}(T)) \). In particular, given \( \sigma \in \mathfrak{S}_n \), \( R(\sigma) = T = R(\text{RPW}(R(\sigma))) \).

As with the column word operation on standard Young tableaux (from Example 1.1.8), Lemma 3.2.3 can also be recast from an algebraic point of view. Denote by \( \mathfrak{P}_n \) the set of all pile configurations with some composition shape \( \gamma \models n \). Patience Sorting and the reverse patience word operation can then be viewed as maps
3.2. Reverse Patience Words and Patience Sorting Equivalence

\( R : \mathfrak{S}_n \to \mathfrak{P}_n \) and \( RPW : \mathfrak{P}_n \to \mathfrak{S}_n \), respectively. With this notation, Lemma 3.2.3 becomes

**Lemma 3.2.3.** The composition \( R \circ RPW \) is the identity map on the set \( \mathfrak{P}_n \).

In particular, even though \( RPW(R(\mathfrak{S}_n)) = RPW(\mathfrak{P}_n) \) is a proper subset of the symmetric group \( \mathfrak{S}_n \), we nonetheless have that \( R(RPW(R(\mathfrak{S}_n))) = R(\mathfrak{S}_n) = \mathfrak{P}_n \).

As such, it makes sense to define the following non-trivial equivalence relation on \( \mathfrak{S}_n \), with each element of \( RPW(\mathfrak{P}_n) \) being the most natural choice of representative for the distinct equivalence class to which it belongs.

**Definition 3.2.4.** Two permutations \( \sigma, \tau \in \mathfrak{S}_n \) are said to be patience sorting equivalent, written \( \sigma \overset{PS}{\sim} \tau \), if they yield the same pile configuration \( R(\sigma) = R(\tau) \) under Patience Sorting (Algorithm 1.1.19). We denote the equivalence class generated by \( \sigma \) as \( \tilde{\sigma} \).

From Lemma 3.1.6 we know that the pile configurations \( R(\sigma) \) and \( R(\tau) \) correspond to the shadow diagrams of \( \sigma, \tau \in \mathfrak{S}_n \), respectively. Thus, it should be intuitively clear that preserving a given pile configuration is equivalent to preserving the ordinates for the southwest corners of the shadowlines. In particular, this means that we are limited to horizontally “stretching” shadowlines up to the point of not allowing them to cross. This is illustrated in Figure 3.2 and the following examples.

**Example 3.2.5.**

1. The only non-singleton patience sorting equivalence class for \( \mathfrak{S}_3 \) consists of \( \widetilde{231} = \{231, 213\} \). We illustrate \( 231 \overset{PS}{\sim} 213 \) in Figure 3.2(a).

2. One can similarly check that \( 645187236 \overset{PS}{\sim} 64158723 \).
3.2. Reverse Patience Words and Patience Sorting Equivalence

Note, in particular, that the actual values of the elements interchanged in Example 3.2.5 are immaterial so long as they have the same relative magnitudes as the literal values in $231 \in S_3$. (I.e., they have to be order-isomorphic to the interchange $231 \sim 213$ as in $451 \sim 415$ from Example 3.2.5(2).) Moreover, it should also be clear that any value greater than the element playing the role of “1” can be inserted between the elements playing the roles of “2” and “3” in “231” without affecting the ability to interchange the “1” and “3” elements. Problems with this interchange only start to arise when a value smaller than the element playing the role of “1” is inserted between the elements playing the roles of “2” and “3”. More formally, one describes this idea using the language of generalized permutation patterns (Definition 1.1.30):

**Theorem 3.2.6.** Let $\sigma, \tau \in S_n$. Then $\sigma \overset{PS}{\sim} \tau$ if and only if $\sigma$ and $\tau$ can be transformed into the same permutation by changing one or more occurrences of the pattern $2\leftarrow 31$ into occurrences of the pattern $2\leftarrow 13$ such that none of these $2\leftarrow 31$ patterns are contained within an occurrence of a $3\leftarrow 1\leftarrow 42$ pattern.

In other words, $\overset{PS}{\sim}$ is the equivalence relation generated by changing $3\leftarrow 1\leftarrow 42$ patterns into $3\leftarrow 1\leftarrow 24$ patterns.

**Proof.** By Lemma 3.2.3, it suffices to show that $\sigma$ can be transformed into the reverse
3.2. Reverse Patience Words and Patience Sorting Equivalence

patience word $RPW(R(\sigma))$ via a sequence of pattern interchanges

$$\sigma = \sigma^{(0)} \leadsto \sigma^{(1)} \leadsto \sigma^{(2)} \leadsto \cdots \leadsto \sigma^{(\ell)} = RPW(R(\sigma)),$$

where each “$\leadsto$” denotes a pattern interchange and each $\sigma^{(i)} \overset{PS}{\sim} \sigma^{(i+1)}$. However, this should be clear by the interpretation of pile configurations in terms shadowlines as given by Lemma 3.1.6.

**Example 3.2.7.**

1. Notice that 2431 contains exactly one instance of a 2–31 pattern as the bold underlined subsequence 2431. (Conversely, 2431 is an instance of 23–1 but not of 2–31.) Moreover, it is clear that 2431 $\overset{PS}{\sim}$ 2413.

2. Even though 3142 contains a 2–31 pattern (as the subsequence 3142), we cannot interchange “4” and “2”, and so $R(3142) \neq R(3124)$. As illustrated in Figure 3.2(b), this is because “4” and “2” are on the same shadowline.

**Remark 3.2.8.** It follows from Theorem 3.2.6 that Examples 3.2.5(1) and 3.2.7(2) sufficiently characterize $\overset{PS}{\sim}$. It is worth pointing out, though, that these examples also begin to illustrate an infinite sequence of generalized permutation patterns (all of them containing either 2–13 or 2–31) with the following property: an interchange of the pattern 2–13 with the pattern 2–31 is allowed within an odd-length pattern in this sequence unless the elements used to form the odd-length pattern can also be used as part of a longer even-length pattern in this sequence.

**Example 3.2.9.** Even though the permutation 34152 contains a 3–1–42 pattern in the suffix “4152”, one can still directly interchange the “5” and the “2” because of the “3” prefix (or via the sequence of interchanges 34152 $\leadsto$ 31452 $\leadsto$ 31425 $\leadsto$ 34125).
3.3 Enumerating $S_n(3−\overline{1}−42)$ and Related Avoidance Sets

In this section, we use results from Sections 3.1 and 3.2 to enumerate and characterize the permutations that avoid the generalized permutation pattern 2–31 unless it occurs as part of an occurrence of the generalized pattern 3–1–42. As in Section 1.1.4, we call this restricted form of the generalized pattern 2–31 a barred (generalized) permutation pattern and denote it by 3−\overline{1}−42.

Theorem 3.3.1.

1. $S_n(3−\overline{1}−42) = \{RPW(R(\sigma)) \mid \sigma \in S_n\}$.

   In particular, given $\sigma \in S_n(3−\overline{1}−42)$, the entries in each column of $R(\sigma)$ (when read from bottom to top) occupy successive positions in the permutation $\sigma$.

2. $S_n(\overline{2}−41−3) = \{RPW(R(\sigma))^{-1} \mid \sigma \in S_n\}$.

   In particular, given $\sigma \in S_n(\overline{2}−41−3)$, the columns of $R(\sigma)$ (when read from top to bottom) contain successive values.

3. The size of $S_n(3−\overline{1}−42)$ is given by the $n^{th}$ Bell number $B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$.

Proof.

1. Let $\sigma \in S_n(3−\overline{1}−42)$. Then, for each $i = 1, 2, \ldots, n − 1$, define

   $$\sigma_{m_i} = \min_{i \leq j \leq n} \{\sigma_j\}.$$  

   Since $\sigma$ avoids $3−\overline{1}−42$, the subpermutation $\sigma_i\sigma_{i+1} \cdots \sigma_{m_i}$ is a decreasing subsequence of $\sigma$. (Otherwise, $\sigma$ would necessarily contain an occurrence of a 2–31...
3.3. Enumerating $S_n(3\bar{1}\bar{42})$ and Related Avoidance Sets

pattern that is not part of an occurrence of a $3\bar{1}\bar{42}$ pattern.) It follows that
the left-to-right minima subsequences $s_1, s_2, \ldots, s_k$ of $\sigma$ must be disjoint and
satisfy Equation (3.1.1). The result then follows by Lemmas 3.1.1 and 3.1.6.

2. This follows immediate by taking inverses in Part (1).

3. Recall that the Bell number $B_n$ enumerates all set partitions $\mathcal{P} = \{1, 2, \ldots, n\}.
(See [35].) From Part (1), the elements of $S_n(3\bar{1}\bar{42})$ are in bijection with pile
configurations. Thus, since pile configurations are themselves set partitions by
Lemma 3.1.1 we need only show that every set partition is also a pile configu-
ration. But this follows by ordering the components of a given set partition by
their smallest element so that Equation (3.1.1) is satisfied.

Even though the set $S_n(3\bar{1}\bar{42})$ is enumerated by such a well-known sequence
as the Bell numbers, it cannot be described in a simpler way using classical pattern
avoidance. This means that there is no countable set of non-generalized (a.k.a. clas-
sical) permutation patterns $\pi_1, \pi_2, \ldots$ such that

$$ S_n(3\bar{1}\bar{42}) = S_n(\pi_1, \pi_2, \ldots) = \bigcap_{i \geq 1} S_n(\pi_i). $$

There are two very important reasons that this cannot happen.

First of all, the Bell numbers satisfy $\log B_n = n(\log n - \log \log n + O(1))$ and so
exhibit superexponential growth. However, in light of the Stanley-Wilf ex-Conjecture
(proven by Marcus and Tardos [27] in 2004), the set of permutations $S_n(\pi)$ avoiding
any classical pattern $\pi$ can only grow at most exponentially in $n$. 
Second, the so-called avoidance class with basis \(\{3-\bar{1}-42\}\),

\[
Av(3-\bar{1}-42) = \bigcup_{n \geq 1} S_n(3-\bar{1}-42),
\]

is not closed under taking order-isomophic subpermutations, whereas it is easy to see that classes of permutations defined by classical pattern avoidance must be closed. (See Chapter 5 of [6].) In particular, \(3142 \in Av(3-\bar{1}-42)\), but \(231 \not\in Av(3-\bar{1}-42)\).

At the same time, Theorem 3.3.1 implies that \(3-\bar{1}-42\) belongs to the so-called Wilf Equivalence class for the generalized pattern \(1-23\). That is, if

\[
\pi \in \{1-23, 3-21, 12-3, 32-1, 1-32, 3-12, 21-3, 23-1\},
\]

then \(|S_n(\pi)| = B_n\). In particular, Claesson [10] showed that \(|S_n(23-1)| = B_n\) using a direct bijection between permutations avoiding 23–1 and set partitions. Furthermore, given \(\sigma \in S_n(3-\bar{1}-42)\), each segment between consecutive right-to-left minima must form a decreasing subsequence (when read from left to right), so it is easy to see that \(S_n(3-\bar{1}-42) = S_n(23-1)\). Thus, the barred pattern \(3-\bar{1}-42\) and the generalized pattern \(23-1\) are not just in the same Wilf equivalence class. They also have identical avoidance classes.

We collect this and similar results together in the following Theorem.

**Theorem 3.3.2.** Let \(B_n\) denote the \(n^{th}\) Bell number. Then

1. \(S_n(3-\bar{1}-42) = S_n(3-\bar{1}-4-2) = S_n(23-1)\).
2. \(S_n(31-\bar{4}-2) = S_n(3-1-\bar{4}-2) = S_n(3-12)\).
3. \(S_n(\bar{2}-\bar{4}1-3) = S_n(\bar{2}-4-1-3) = S_n(2-4-1-\bar{3}) = S_n(2-41-\bar{3})\).
3.3. Enumerating \( S_n(3-\bar{1}-42) \) and Related Avoidance Sets

4. \( |S_n(2-41-3)| = |S_n(31-\bar{4}-2)| = |S_n(3-\bar{1}-42)| = B_n \).

Proof.

1. This is proven above.

2. This follows from Part (1) by taking the reverse complement (as defined in \([6]\)) of each element in \( S_n(3-\bar{1}-42) \).

3. The proof is similar to that in Part (2). (This part is also proven in \([1]\).)

4. This follows from the fact that the patterns \( 3-1-\bar{4}-2 \) and \( 2-4-1-3 \) are inverses of each other (as classical permutation patterns).

\[ \square \]

Remark 3.3.3. Even though \( S_n(3-\bar{1}-42) = S_n(23-1) \), it is more natural to use avoidance of \( 3-\bar{1}-42 \) when studying Patience Sorting. Fundamentally, this lets us look at \( S_n(3-\bar{1}-42) \) as the set of equivalence classes in \( S_n \) modulo \( 3-\bar{1}-42 \) \( \overset{PS}{\sim} \) \( 3-\bar{1}-24 \), where each equivalence class corresponds to a unique pile configuration. The same equivalence relation is not easy to describe when starting with an occurrence of \( 23-1 \).

In other words, both

\[ 23-1 \sim 2-13 \quad \text{and} \quad 23-1 \sim 21-3 \]

are wrong since they incorrectly suggest

\[ 2431 \sim 2314 \quad \text{and} \quad 2431 \sim 2134, \]

respectively, instead of the correct \( 2431 \overset{PS}{\sim} 2413 \).
3.4. Invertibility of Patience Sorting

We conclude this section with an immediate corollary of Theorem 3.3.2 that characterizes an important category of classical permutation patterns.

**Definition 3.3.4.** Given a composition \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \models n \), the (classical) layered permutation pattern \( \pi_\gamma \in \mathfrak{S}_n \) is the permutation

\[
\gamma_1(\gamma_1-1) \cdots 1(\gamma_1+\gamma_2)(\gamma_1+\gamma_2-1) \cdots (\gamma_1+1) \cdots n(n-1) \cdots (\gamma_1+\gamma_2+\cdots+\gamma_m-1+1).
\]

**Example 3.3.5.** Given \( \gamma = (3, 2, 3) \models 8 \), the corresponding layered pattern (following the notation in [29]) is \( \pi_{(3,2,3)} = \hat{3}21\hat{5}4\hat{8}76 \in \mathfrak{S}_8 \).

**Corollary 3.3.6.** \( S_n(3-\bar{1}-42, 2-41-3) \) is the set of layered patterns in \( \mathfrak{S}_n \).

**Proof.** By Theorem 3.3.2, we have that \( S_n(3-\bar{1}-42, 2-41-3) = S_n(23-1, 31-2) \), the latter being a characterization for layered patterns given in [11].

3.4 Invertibility of Patience Sorting

As discussed in Section 3.2, many different permutations can correspond to the same pile configuration under Patience Sorting (Algorithm 1.1.19). E.g., \( R(3142) = R(3412) \). In this section, we use barred permutation patterns to characterize permutations for which this does not hold. We then establish a non-trivial enumeration for the resulting avoidance sets.

**Theorem 3.4.1.** A pile configuration pile \( R \) has a unique preimage \( \sigma \in \mathfrak{S}_n \) under Patience Sorting if and only if \( \sigma \in S_n(3-\bar{1}-42, 3-\bar{1}-24) \).

**Proof.** From the proof of Lemma 3.1.1, we know that every pile configuration \( R \) has at least its reverse patience word \( RPW(R) \) as a preimage under Patience Sorting, and,
3.4. Invertibility of Patience Sorting

by Theorem 3.3.1, \( RPW(R) \in S_n(3-\bar{1}-42) \). Furthermore, by Theorem 3.2.6, two permutations yield the same pile configurations under Patience Sorting if and only if one can be obtained from the other by a sequence of order-isomorphic exchanges of the form

\[
3-\bar{1}-24 \leftrightarrow 3-\bar{1}-42 \text{ or } 3-\bar{1}-42 \leftrightarrow 3-\bar{1}-24.
\]

(I.e., the occurrence of one pattern is reordered to form an occurrence of the other pattern.) Thus, it is easy to see that \( R \) has the unique preimage \( RPW(R) \) if and only if \( RPW(R) \) avoids both \( 3-\bar{1}-42 \) and \( 3-\bar{1}-24 \).

Given this pattern avoidance characterization of invertibility of Patience Sorting, we have the following recurrence relation for the size of the avoidance sets in Theorem 3.4.1.

**Lemma 3.4.2.** Set \( f(n) = |S_n(3-\bar{1}-42, 3-\bar{1}-24)| \) and, for \( k \leq n \), denote by \( f(n, k) \) the cardinality

\[
f(n, k) = \#\{\sigma \in S_n(3-\bar{1}-42, 3-\bar{1}-24) \mid \sigma(1) = k\}.
\]

Then \( f(n) = \sum_{k=1}^{n} f(n, k) \), and \( f(n, k) \) satisfies the four part recurrence relation

\[
f(n, 0) = 0 \quad \text{for} \quad n \geq 1 \quad (3.4.1)
\]
\[
f(n, 1) = f(n, n) = f(n - 1) \quad \text{for} \quad n \geq 1 \quad (3.4.2)
\]
\[
f(n, 2) = 0 \quad \text{for} \quad n \geq 3 \quad (3.4.3)
\]
\[
f(n, k) = f(n, k - 1) + f(n - 1, k - 1) + f(n - 2, k - 2) \quad \text{for} \quad n \geq 3 \quad (3.4.4)
\]

subject to the initial conditions \( f(0, 0) = f(0) = 1 \).
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Proof. Note first that Equation (3.4.1) is the obvious boundary condition for $k = 0$.

Now, suppose that the first component of $\sigma \in S_n(3-\bar{1}-42,3-\bar{1}-24)$ is either $\sigma(1) = 1$ or $\sigma(1) = n$. Then $\sigma(1)$ cannot be part of any occurrence of $3-\bar{1}-42$ or $3-\bar{1}-24$ in $\sigma$. Thus, upon removing $\sigma(1)$ from $\sigma$, and subtracting one from each component if $\sigma(1) = 1$, a bijection is formed with $S_{n-1}(3-\bar{1}-42,3-\bar{1}-24)$. Therefore, Equation (3.4.2) follows.

Next, suppose that the first component of $\sigma \in S_n(3-\bar{1}-42,3-\bar{1}-24)$ is $\sigma(1) = 2$. Then the first column of $R(\sigma)$ must be $r_1 = 21$ regardless of where 1 occurs in $\sigma$. Therefore, $R(\sigma)$ has the unique preimage $\sigma$ if and only if $\sigma = 21 \in \mathcal{S}_2$, and from this Equation (3.4.3) follows.

Finally, suppose that $\sigma \in S_n(3-\bar{1}-42,3-\bar{1}-24)$ with $3 \leq k \leq n$. Since $\sigma$ avoids $3-\bar{1}-42$, $\sigma$ is a RPW by Theorem 3.3.1 and hence the left prefix of $\sigma$ from $k$ to 1 is a decreasing subsequence. Let $\sigma'$ be the permutation obtained by interchanging the values $k$ and $k-1$ in $\sigma$. Then the only instances of the patterns $3-\bar{1}-42$ and $3-\bar{1}-24$ in $\sigma'$ must involve both $k$ and $k-1$. Note that the number of $\sigma$ for which no instances of these patterns are created by interchanging $k$ and $k-1$ is $f(n,k-1)$.

There are now two cases in which an instance of the barred pattern $3-\bar{1}-42$ or $3-\bar{1}-24$ will be created in $\sigma'$ by this interchange:

Case 1. If $k-1$ occurs between $\sigma(1) = k$ and 1 in $\sigma$, then $\sigma(2) = k - 1$, so interchanging $k$ and $k-1$ will create an instance of the pattern 23-1 via the subsequence $(k-1,k,1)$ in $\sigma'$. Thus, by Theorem 3.3.2, $\sigma'$ contains $3-\bar{1}-42$ from which $\sigma' \in S_n(3-\bar{1}-42)$ if and only if $k-1$ occurs after 1 in $\sigma$. Note also that if $\sigma(2) = k - 1$, then removing $k$ from $\sigma$ yields a bijection with permutations in $S_{n-1}(3-\bar{1}-42,3-\bar{1}-24)$ that start with $k-1$. Therefore, the number of permutations counted in Case 1 is $f(n-1,k-1)$. 
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Case 2. If \( k - 1 \) occurs to the right of 1 in \( \sigma \), then \( \sigma' \) both contains the subsequence \((k - 1, 1, k)\) and avoids the pattern \(3-\bar{1}-42\), so it must also contain the pattern \(3-\bar{1}-24\). If an instance of \(3-\bar{1}-24\) in \( \sigma' \) involves both \( k - 1 \) and \( k \), then \( k - 1 \) and \( k \) must play the roles of “3” and “4”, respectively. Moreover, if the value \( \ell \) preceding \( k \) is not 1, then the subsequence \((k - 1, \ell, k)\) is an instance of \(3-1-24\), so \((k - 1, \ell, k)\) is not an instance of \(3-\bar{1}-24\). Therefore, for \( \sigma' \) to contain \(3-\bar{1}-24\), \( k \) must follow 1 in \( \sigma \), and so \( k - 1 \) follows 1 in \( \sigma \). Similarly, if the letter preceding 1 is some \( m < k \), then the subsequence \((m, 1, k - 1)\) is an instance of \(3-\bar{1}-24\) in \( \sigma \), which is impossible. Therefore, \( k \) must precede 1 in \( \sigma \), from which \( \sigma \) must start with the initial segment \((k, 1, k - 1)\). It follows that removing the values \( k \) and 1 from \( \sigma \) and then subtracting 1 from each component yields a bijection with permutations in \(S_{n-2}(3-\bar{1}-42, 3-\bar{1}-24)\) that start with \( k - 2 \). Thus, the number of permutations counted in Case 2 is then exactly \( f(n - 2, k - 2) \), which yields Equation (3.4.4). \( \square \)

If we denote by

\[
\Phi(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f(n, k) x^n y^k
\]

the bivariate generating function for the sequence \( \{f(n, k)\}_{n,k \geq 0} \); then Equation (3.4.4) implies that

\[
(1 - y - xy - x^2 y^2) \Phi(x, y) = 1 - y - xy + xy^2 - xy^2 \Phi(xy, 1) + xy(1 - y - xy) \Phi(x, 1).
\]

We conclude this section with the following enumerative result.

Theorem 3.4.3. Denote by \( F_n \) the \( n \)th Fibonacci number (with \( F_0 = F_1 = 1 \)) and by

\[
a(n, k) = \sum_{n_1, \ldots, n_{k+1} \geq 0}^{n_1 + \cdots + n_{k+1} = (n-2)-(k+1)=n-k-3} F_{n_1} F_{n_2} \cdots F_{n_{k+1}}
\]
the convolved Fibonacci numbers for \( n \geq k + 2 \) (with \( a(n, k) := 0 \) otherwise). Then, defining

\[
X = \begin{bmatrix}
f(0) \\
f(1) \\
f(2) \\
f(3) \\
f(4) \\
\vdots
\end{bmatrix}, \quad F = \begin{bmatrix}1 \\ F_0 \\ F_1 \\ F_2 \\ F_3 \\ \vdots\end{bmatrix}, \quad \text{and} \quad A = (a(n, k))_{n,k \geq 0},
\]

we have that \( X = (I - A)^{-1} F \), where \( I \) is the infinite identity matrix and \( A \) is, by definition, lower triangular with zero main diagonal.

**Proof.** From Equations (3.4.1)–(3.4.4), one can conjecture an equivalent recurrence in which Equations (3.4.3) and (3.4.4) are replaced by the following equation (and where \( \delta_{nk} \) denotes the Kronecker delta function):

\[
f(n, k) = \sum_{m=0}^{k-3} c(k, m) f(n - k + m) + \delta_{nk} F_{k-2}, \quad n \geq k \geq 2. \tag{3.4.5}
\]

For this relation to hold, the coefficients \( c(k, m) \) must satisfy the recurrence relation

\[
c(k, m) = c(k - 1, m - 1) + c(k - 1, m) + c(k - 2, m), \quad k \geq 2,
\]

or, equivalently,

\[
c(k - 1, m - 1) = c(k, m) - c(k - 1, m) - c(k - 2, m), \quad k \geq 2,
\]

with \( c(2, 0) = 1 \) and \( c(k, m) = 0 \) in the case that \( k < 2, m < 0, \) or \( m > k - 2 \). This implies that the generating function for the sequence \( \{c(k, m)\}_{k \geq 0} \) (for each \( m \geq 0 \)
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is

\[ \sum_{n \geq 0} c(k, m)x^k = \frac{x^{m+2}}{(1 - x - x^2)^{m+1}}. \]

Thus, the coefficients \( c(k, m) = a(k, m) \) in Equation (3.4.5) are the convolved Fibonacci numbers \(^{34}\) and form the so-called skew Fibonacci-Pascal triangle in the matrix \( A = (a(k, m))_{k, m \geq 0} \). In particular, the sequence of nonzero entries in column \( m \geq 0 \) of \( A \) is the \( m \)th convolution of the sequence \( \{F_n\}_{n \geq 0} \).

Finally, upon combining the expansion of \( f(n, n) \) from Equation (3.4.5) with Equation (3.4.2),

\[ f(n) = \sum_{m=0}^{n-2} a(n, m)f(m) + F_{n-1}, \]

which is equivalent to the matrix equation \( X = AX + F \). Therefore, since \( I - A \) is clearly invertible, the result follows.

\[ \square \]

Remark 3.4.4. Since \( A \) is strictly lower triangular with zero main diagonal and zero sub-diagonal, it follows that multiplication of a matrix \( B \) by \( A \) shifts the position of the highest nonzero diagonal in \( B \) down by two rows. Thus, \( (I - A)^{-1} = \sum_{n \geq 0} A^n \) as a Neumann series, and all nonzero entries of \( (I - A)^{-1} \) are positive integers.

In particular, one can explicitly compute

\[ A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 \\
3 & 5 & 3 & 1 & 0 & 0 \\
5 & 10 & 9 & 4 & 1 & 0 \\
8 & 20 & 22 & 14 & 5 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} \]

from which
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\[(I - A)^{-1} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \\
66 & 66 & 66 & 66 & 66 & 66 & 66 & 66 & 66 & 66 & 66 & 66 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}.
\]

It follows that the first few values of the sequence \( \{f(n)\}_{n \geq 0} \) are

1, 1, 2, 4, 9, 23, 66, 209, 718, 2645, 10373, 43090, 188803, 869191, 4189511, \ldots
Chapter 4

Bijectively Extending Patience Sorting

4.1 Characterizing “Stable Pairs”

According to Theorem 3.3.1, the number of pile configurations that can be formed from $[n]$ is given by the Bell number $B_n$. Comparing this to the number of standard Young tableau $|\Sigma_n|$ (see, e.g., [22]), it is clear that there are significantly more possible pile configurations than standard Young tableau. Consequently, not every ordered pair of pile configurations with the same shape can result from Extended Patience Sorting (Algorithm 1.1.22). In this section, we characterize the “stable pairs” of pile configurations that result from applying Extended Patience Sorting to a permutation.

The following example, though very small, illustrates the most generic behavior that must be avoided in constructing these “stable pairs”. As in Example 1.1.25, we denote by $S'$ the “reversed pile configuration” corresponding to $S$ (which has all piles listed in reverse order).
4.1. Characterizing “Stable Pairs”

**Example 4.1.1.** Even though the pile configuration \( R = \{\{3 > 1\}, \{2\}\} \) cannot result as the insertion piles of an involution under Extended Patience Sorting, we can nonetheless look at the shadow diagram for the pre-image of the pair \((R, R)\) under Algorithm 1.1.22:

\[
R = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad S' = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix} \implies \quad 0 \quad 1 \quad 2 \quad 3 \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

Note that there are two competing constructions here. On the one hand, we have the diagram \(\{(1,3), (2,2), (3,1)\}\) of the permutation \(321 \in \mathcal{S}_3\) given by the entries in the pile configurations. (In particular, the values in \(R\) specify the ordinates, and the values in the corresponding boxes of \(S'\) specify the abscissae.) On the other hand, the piles in \(R\) also specify shadowlines with respect to this permutation diagram. Here, the pair \((R, S) = (R, R)\) of pile configurations is “unstable” because their combination yields crossing shadowlines — which is clearly not allowed.

Similar considerations lead to crossings of the form

\[
\begin{array}{c}
\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \\
\bullet & \bullet & \bullet & \bullet
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \\
\bullet & \bullet & \bullet & \bullet
\end{array}
\]

Note also that these latter two crossings can also be used together to build something like the first crossing but with “extra” elements on the boundary of the polygon formed:
4.1. Characterizing “Stable Pairs”

We are now in a position to make the following fundamental definitions:

**Definition 4.1.2.** Given a composition $\gamma$ of $n$ (denoted $\gamma \models n$), we define $\mathcal{P}_n(\gamma)$ to be the set of all pile configurations $R$ having shape $\text{sh}(R) = \gamma$ and put

$$\mathcal{P}_n = \bigcup_{\gamma \models n} \mathcal{P}_n(\gamma).$$

**Definition 4.1.3.** Define the set $\Sigma(n) \subset \mathcal{P}_n \times \mathcal{P}_n$ to consist of all ordered pairs $(R, S)$ with $\text{sh}(R) = \text{sh}(S)$ such that the ordered pair $(\text{RPW}(R), \text{RPW}(S'))$ avoids simultaneous occurrences of the pairs of patterns $(31-2, 13-2)$, $(31-2, 32-1)$ and $(32-1, 13-2)$ at the same positions in $\text{RPW}(R)$ and $\text{RPW}(S')$.

In other words, if $\text{RPW}(R)$ contains an occurrence of the first pattern in any of the above pairs, then $\text{RPW}(S')$ cannot contain an occurrence at the same positions of the second pattern in the same pair, and vice versa. In effect, Definition 4.1.3 characterizes “stable pairs” of pile configurations $(R, S)$ by forcing $R$ and $S$ to avoid certain sub-pile pattern pairs. As in Example 4.1.1 we are characterizing when the induced shadowlines cross.

**Theorem 4.1.4.** Extended Patience Sorting (Algorithm 1.1.22) gives a bijection between the symmetric group $\mathfrak{S}_n$ and the “stable pairs” set $\Sigma(n)$.

**Proof.** We show that, for any “stable pair” $(R, S) \in \Sigma(n)$ and any permutation
4.1. Characterizing “Stable Pairs”

\[ \sigma \in \mathfrak{S}_n, \ (R, S) = (R(\sigma), S(\sigma)) \text{ if and only if } \]

\[
\sigma = \begin{pmatrix} \text{RPW}(S') \\ \text{RPW}(R) \end{pmatrix} \quad \text{(in the two-line notation)}.
\]

Clearly, if \((R, S) = (R(\sigma), S(\sigma))\) for some \(\sigma \in \mathfrak{S}_n\), then \(\sigma = \begin{pmatrix} \text{RPW}(S') \\ \text{RPW}(R) \end{pmatrix}\). Thus, we need only to prove that \((R, S) \in \Sigma(n)\). In particular, if \((R, S) \notin \Sigma(n)\), then \(\text{RPW}(R)\) and \(\text{RPW}(S')\) contain simultaneous occurrences of (at least) one of the three forbidden pattern pairs given in Definition 4.1.3.

Suppose that \(\text{RPW}(R)\) contains an occurrence \((r_3, r_1, r_2)\) of 31–2, and suppose also that \(\text{RPW}(S')\) contains an occurrence \((s_1', s_3', s_2')\) of 13–2, with both occurrences at the same positions. Since \(r_3 > r_1\) and since \(r_3\) and \(r_1\) are consecutive entries in \(\text{RPW}(R)\), it follows that \(r_3\) and \(r_1\) must be in the same column \(c_i(R)\) of \(R\) (in fact, \(r_1\) is immediately on top of \(r_3\)). Furthermore, since \(r_1 < r_2\) and since \(r_2\) is to the right of \(r_1\) in \(R\), it follows that the column \(c_j(R)\) of \(R\) containing \(r_2\) must be to the right of the column containing \(r_1\) atop \(r_3\). Therefore, \(s_2'\) must also be in a column \(c_i(S')\) of \(S'\) to the right of the column \(c_j(S')\) containing \(s_3'\) atop \(s_1'\).

Now, consider the subpermutation \(\tau\) of \(\sigma\) formed by removing all components of \(\text{RPW}(R)\) and \(\text{RPW}(S')\) that are not in these two columns. Alternatively, let \(R_s\) and \(S_s'\) consist only of the columns \((c_i(R), c_j(R))\) of \(R\) and \((c_i(S'), c_j(S'))\) of \(S'\), respectively. Then

\[
\tau = \begin{pmatrix} \text{RPW}(S_s') \\ \text{RPW}(R_s) \end{pmatrix}.
\]

Note that the values \(r_3\) and \(r_1\) in \(c_i(S')\) are consecutive left-to-right minima of \(\tau\), whereas \(r_2\) is not a left-to-right minimum of \(\tau\). Since \(r_1 < r_2 < r_3\), it follows
4.1. Characterizing “Stable Pairs”

that \( r_2 \) cannot occur between \( r_1 \) and \( r_3 \) in \( \tau \). However, since

\[
\begin{pmatrix}
    s'_1 & s'_3 & s'_2 \\
    r_3 & r_1 & r_2
\end{pmatrix}
\]

is a subpermutation of \( \tau \) and since \( s'_1 < s'_2 < s'_3 \), it follows that \( r_2 \) does occur between \( r_1 \)
and \( r_3 \), which is a contradiction.

A similar argument applies to both \((31-2, 32-1)\) and \((32-1, 13-2)\), which then
implying \((R, S) \in \Sigma(n)\).

Conversely, given \((R, S) \in \Sigma(n)\), set \( \sigma = \begin{pmatrix} \text{RPW}(S') \\ \text{RPW}(R) \end{pmatrix} \). Then, since the pattern avoidance conditions defining \( \Sigma(n) \) forbid intersections in the northeast shadow di-
gram corresponding to \( \sigma \) (as illustrated in Example 4.1.1), it follows by Lemma 3.1.6
that \((R, S) = (R(\sigma), S(\sigma))\).

**Example 4.1.5.** The pair of piles

\[
(R, S) = \begin{pmatrix}
    1 & 3 & 1 & 5 \\
    4 & 2 & 7 & 2 & 3 & 6 \\
    6 & 5 & 8 & 4 & 7 & 8
\end{pmatrix} \in \Sigma(8)
\]

corresponds to the permutation

\[
\sigma = \begin{pmatrix} \text{RPW}(S') \\ \text{RPW}(R) \end{pmatrix} = \begin{pmatrix}
    1 & 2 & 4 & 3 & 7 & 5 & 6 & 8 \\
    6 & 4 & 1 & 5 & 2 & 8 & 7 & 3
\end{pmatrix} = 64518723 \in \mathcal{S}_8.
\]

The similarities between Extended Patience Sorting (Algorithm 1.1.22) and the
RSK Correspondence (Algorithm 1.1.15) are perhaps most observable in the following
simple Proposition.

**Proposition 4.1.6.** Let \( \downarrow_k = 1-2-\cdots-k \) and \( \uparrow_k = k-\cdots-2-1 \) be the classical
monotone permutation patterns. Then there is a bijection...
4.1. Characterizing “Stable Pairs”

1. between $S_n(\mathcal{I}_{k+1})$ and “stable pairs” of pile configurations having the same composition shape $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \models n$ but with at most $k$ piles (i.e., $m \leq k$),

2. as well as a bijection between $S_n(\mathcal{I}_{k+1})$ and “stable pairs” of pile configurations having the same composition shape $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \models n$ but with no pile having more than $k$ cards in it (i.e., $\gamma_i \leq k$ for each $i = 1, 2, \ldots, m$).

Proof.

1. Given $\sigma \in S_n$, the proof of Proposition 2.4.3 yields a bijection between the set of piles $R(\sigma) = \{r_1, r_2, \ldots, r_k\}$ formed under Patience Sorting and the components of a particular longest increasing subsequence in $\sigma$. Since avoiding the monotone pattern $\mathcal{I}_{k+1}$ is equivalent to restricting the length of the longest increasing subsequence in a permutation, the result then follows.

2. Follows from Part (1) by reversing the order of the components in each of the permutations in $S_n(\mathcal{I}_{k+1})$ in order to form $S_n(\mathcal{J}_{k+1})$.

Proposition 4.1.6 states that Patience Sorting can be used to efficiently compute the length of both the longest increasing and longest decreasing subsequences in a given permutation. In particular, one can compute these lengths without examining every subsequence of a permutation, just as with the RSK Correspondence. However, while both the RSK Correspondence and Patience Sorting can be used to implement this computation in $O(n \log(n))$ time, an extension is given in [5] that also simultaneously tabulates all of the longest increasing or decreasing subsequences without incurring any additional asymptotic computational cost.
4.2 Schützenberger-type Symmetry and a Bijection with Involutions

We are now in a position to prove that Extended Patience Sorting (Algorithm 1.1.22) has the same form of symmetry as does the RSK Correspondence (Algorithm 1.1.15).

**Theorem 4.2.1.** Let \((R(\sigma), S(\sigma))\) be the insertion and recording piles, respectively, formed by applying Algorithm 1.1.22 to \(\sigma \in \mathfrak{S}_n\). Then, applying Algorithm 1.1.22 to the inverse permutation \(\sigma^{-1}\), one obtains the pair \((S(\sigma), R(\sigma))\).

**Proof.** Construct \(S'(\sigma)\) from \(S(\sigma)\) as discussed in Example 1.1.25 and form the \(n\) ordered pairs \((r_{ij}, s'_{ij})\), with \(i\) indexing the individual piles and \(j\) indexing the cards in the \(i^{th}\) piles. Then these \(n\) points correspond to the diagram of a permutation \(\tau \in \mathfrak{S}_n\). However, since reflecting these points through the line \(y = x\) yields the diagram for \(\sigma\), it follows that \(\tau = \sigma^{-1}\). \(\square\)

Proposition 4.2.1 suggests that Extended Patience Sorting is the right generalization of Patience Sorting (Algorithm 1.1.19) since we obtain the same symmetry property as for the RSK Correspondence (Theorem 1.1.17). Moreover, Proposition 4.2.1 also implies that there is a bijection between involutions and pile configurations that avoid simultaneously containing the symmetric sub-pile patterns corresponding to the patterns given in Definition 4.1.3. This corresponds to the reverse patience word for a pile configuration simultaneously avoiding a symmetric pair of the generalized patterns \(31-2\) and \(32-1\), etc. As such, it is interesting to compare this construction to the following results obtained by Claesson and Mansour [12]:

1. The size of \(S_n(3-12, 3-21)\) is equal to the number of involutions \(|\mathfrak{I}_n|\) in \(\mathfrak{S}_n\).

2. The size of \(S_n(31-2, 32-1)\) is \(2^{n-1}\).
The first result suggests that there should be a way to relate the result in Theorem 4.1.4 to simultaneous avoidance of the similar looking patterns $3\!-\!12$ and $3\!-\!21$. The second result suggests that restricting to complete avoidance of all simultaneous occurrences of $31\!-\!2$ and $32\!-\!1$ will yield a natural bijection between $S_n(31\!-\!2, 32\!-\!1)$ and a subset $\mathfrak{N} \subset \mathfrak{P}_n$ such that $\mathfrak{N} \cap \mathfrak{P}_n(\gamma)$ contains exactly one pile configuration of each shape $\gamma$. A natural family for this collection of pile configurations consists of what we call non-crossing pile configurations; namely, for the composition 

$$
\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \models n,
$$

$$
\mathfrak{N} \cap \mathfrak{P}_n(\gamma) = \{\{\gamma_1 > \cdots > 1\}, \{\gamma_1 + \gamma_2 > \cdots > \gamma_1 + 1\}, \ldots, \{n > \cdots > n - \gamma_k - 1\}\}
$$

so that there are exactly $2^{n-1}$ such pile configurations. One can also show that $\mathfrak{N}$ is the image $R(S_n(3\!-\!1\!-\!2))$ of all permutations avoiding the classical pattern $3\!-\!1\!-\!2$ under the Patience Sorting map $R : \mathfrak{S}_n \rightarrow \mathfrak{P}_n$.

4.3 A Geometric Form for the Extended Patience Sorting Algorithm

Viennot introduced the shadow diagram of a permutation while studying Schützenberger Symmetry for the RSK Correspondence (Theorem 1.1.17 which was first proven using a direct combinatorial argument in [33]). Specifically, using a particular labelling of the constituent “shadow lines” in recursively defined shadow diagrams, one recovers successive rows in the usual RSK insertion and recording tableaux. Schützenberger Symmetry for the RSK Correspondence then immediately follows since reflecting these shadow diagrams through the line “$y = x$” both inverts the
4.3. Geometric Form for Extended Patience Sorting

permutation and exactly interchanges these labellings.

We review Viennot’s Geometric RSK Algorithm in Section 4.3.1 below. Then, in Section 4.3.2 we define a natural dual to Viennot’s construction that similarly produces a geometric characterization for Extended Patience Sorting. As with the RSK Correspondence, the analog of Schützenberger Symmetry follows as an immediate consequence. Unlike Geometric RSK, though, the lattice paths formed under Geometric Patience Sorting are allowed to intersect. Thus, having defined these two algorithms, we classify in Section 4.3.3 the types of intersections that can occur under Geometric Patience Sorting and then characterize them in Section 4.3.4.

4.3.1 Northeast Shadowlines and Geometric RSK

In this section, we briefly review Viennot’s geometric form for the RSK Correspondence in order to motivate the geometric form for Extended Patience Sorting that is given in Section 4.3.2 (Viennot’s Geometric RSK Algorithm was first introduced in [40]; an English version can also be found in [41] and in [31].)

In Section 3.1, the northeast shadow diagram for a collection of lattice points was defined by inductively taking northeast shadowlines for those lattice points not employed in forming the previous shadowlines. In particular, given a permutation $\sigma \in S_n$, we form the northeast shadow diagram $D_{NE}(\sigma) = \{L_1(\sigma), \ldots, L_k(\sigma)\}$ for $\sigma$ by first forming the northeast shadowline $L_1(\sigma)$ for $\{(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)\}$. Then we ignore the lattice points whose northeast shadows were used in building $L_1(\sigma)$ and define $L_2(\sigma)$ to be the northeast shadowline of the resulting subset of the permutation diagram. We then take $L_3(\sigma)$ to be the northeast shadowline for the points not yet used in constructing either $L_1(\sigma)$ or $L_2(\sigma)$, and this process continues until all points in the permutation diagram are exhausted.
We can characterize the points whose shadows define the shadowlines at each stage of this process as follows: they are the smallest collection of unused points whose shadows collectively contain all other remaining unused points (and hence also contain the shadows of those points). As a consequence of this shadow containment property, the shadowlines in a northeast shadow diagram will never cross. However, as we will see in Section 4.3.2 below, the dual construction to the definition of northeast shadow diagrams will allow for crossing shadowlines, which are then classified and characterized in Section 4.3.3 and 4.3.4, respectively. This distinction results from the reversal of the above shadow containment property.

As simple as northeast shadowlines were to define in Section 3.1, a great deal of information can still be gotten from them. One of the most basic properties of the northeast shadow diagram \( D_{NE}^{(0)}(\sigma) \) for a permutation \( \sigma \in \mathfrak{S}_n \) is that it encodes the top row of the RSK insertion tableau \( P(\sigma) \) (resp. recording tableau \( Q(\sigma) \)) as the smallest ordinates (resp. smallest abscissae) of all points belonging to the constituent shadowlines \( L_1(\sigma), L_2(\sigma), \ldots, L_k(\sigma) \). One proves this by comparing the use of Schensted Insertion on the top row of the insertion tableau with the intersection of vertical lines having the form \( x = a \). In particular, as \( a \) increases from 0 to \( n \), the line \( x = a \) intersects the lattice points in the permutation diagram in the order that they are inserted into the top row, and so shadowlines connect elements of \( \sigma \) to those smaller elements that will eventually bump them. (See Sagan [31] for more details.)

Remarkably, one can then use the southwest corners (called the salient points) of \( D_{NE}^{(0)}(\sigma) \) to form a new shadow diagram \( D_{NE}^{(1)}(\sigma) \) that similarly gives the second rows of \( P(\sigma) \) and \( Q(\sigma) \). Then, inductively, the salient points of \( D_{NE}^{(1)}(\sigma) \) can be used to give the third rows of \( P(\sigma) \) and \( Q(\sigma) \), and so on. As such, one can view this recursive formation of shadow diagrams as a geometric form for the RSK correspondence. We
illustrate this process in Figure 4.1 for the following permutation:

\[ \sigma = 64518723 \xrightarrow{RSK} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 7 \\ 6 & 8 \\ 2 & 6 & 8 \\ 1 & 3 & 5 \\ 4 & 7 \end{pmatrix} \]

4.3.2 Southwest Shadowlines and Geometric Patience Sorting

In this section, we introduce a natural dual to Viennot’s Geometric RSK construction as given in Section 4.3.1. We begin with the following fundamental definition.

**Definition 4.3.1.** Given a lattice point \((m, n) \in \mathbb{Z}^2\), we define the southwest shadow of \((m, n)\) to be the quarter space

\[ S_{SW}(m, n) = \{(x, y) \in \mathbb{R}^2 \mid x \leq m, \ y \leq n\}. \]
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Figure 4.2: Examples of Southwest Shadow and Shadowline Constructions

See Figure 4.2(a) for an example of a point’s southwest shadow.

As with their northeast counterparts, the most important use of these shadows is in building southwest shadowlines.

**Definition 4.3.2.** Given lattice points \((m_1, n_1), (m_2, n_2), \ldots, (m_k, n_k) \in \mathbb{Z}^2\), we define their southwest shadowline to be the boundary of the union of the shadows \(S_{SW}(m_1, n_1), S_{SW}(m_2, n_2), \ldots, S_{SW}(m_k, n_k)\).

In particular, we wish to associate to each permutation a specific collection of southwest shadowlines. However, unlike the northeast case, these shadowlines are allowed to cross (as illustrated in Figures 4.2(b)–(d) and Figures 4.3(a)–(b)).

**Definition 4.3.3.** Given \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathcal{S}_n\), the southwest shadow diagram \(D_{SW}^{(0)}(\sigma)\) of \(\sigma\) consists of the southwest shadowlines \(L_1^{(0)}(\sigma), L_2^{(0)}(\sigma), \ldots, L_k^{(0)}(\sigma)\) formed as follows:
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- $L_1^{(0)}(\sigma)$ is the shadowline for those points $(x, y) \in \{(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)\}$ such that $S_{SW}(x, y)$ does not contain any other lattice points.

- Then, while at least one of the points $(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)$ is not contained in the shadowlines $L_1^{(0)}(\sigma), L_2^{(0)}(\sigma), \ldots, L_j^{(0)}(\sigma)$, define $L_{j+1}^{(0)}(\sigma)$ to be the shadowline for the points

  $$(x, y) \in \{(i, \sigma_i) \mid i \in [n], (i, \sigma_i) \notin \bigcup_{k=1}^j L_k^{(0)}(\sigma)\}$$

such that $S_{SW}(x, y)$ does not contain any other lattice points in the same set.

In other words, we again define a shadow diagram by recursively eliminating certain points in the permutation diagram until every point has been used to define a shadowline. Here, however, we are reversing both the direction of the shadows and the shadow containment property used in the northeast case. It is in this sense that the geometric form for Extended Patience Sorting given below can be viewed as “dual” to Viennot’s geometric form for the RSK Correspondence.

As with northeast shadow diagrams, one can also produce a sequence

$$D_{SW}(\sigma) = (D_{SW}^{(0)}(\sigma), D_{SW}^{(1)}(\sigma), D_{SW}^{(2)}(\sigma), \ldots)$$

of southwest shadow diagrams for a given permutation $\sigma \in \mathfrak{S}_n$ by recursively applying Definition 4.3.3 to salient points, with the restriction that new shadowlines can only connect points that were on the same shadowline in the previous iteration. (The reason for this important distinction from Geometric RSK is discussed further in Section 4.3.3 below.) The salient points in this case are naturally defined to be the northeast corner points of a given set of shadowlines. See Figure 4.3 for an example.
Definition 4.3.4. We call $D_{SW}^{(k)}(\sigma)$ the $k^{th}$ iterate of the exhaustive shadow diagram $D_{SW}(\sigma)$ for $\sigma \in \mathfrak{S}_n$.

As mentioned above, the resulting sequence of shadow diagrams can be used to reconstruct the pair of pile configurations given by Extended Patience Sorting (Algorithm 1.1.22). To accomplish this, index the cards in a pile configuration using the French convention for tableaux (see [18]) so that the row index increases from bottom to top and the column index from left to right. (In other words, we are labelling boxes as we would lattice points in the first quadrant of $\mathbb{R}^2$). Then, for a given permutation $\sigma \in \mathfrak{S}_n$, the elements of the $i^{th}$ row of the insertion piles $R(\sigma)$ (resp. recording piles $S(\sigma)$) are given by the largest ordinates (resp. abscissae) of the shadowlines that comprise $D_{SW}^{(i)}$.

The main difference between this process and Viennot’s Geometric RSK is that
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Care must be taken to assemble each row in its proper order. Unlike the entries of a standard Young tableau, the elements in the rows of a pile configuration do not necessarily increase from left to right, and they do not have to be contiguous. As such, the components of each row should be recorded in the order that the shadowlines are formed. The rows can then uniquely be assembled into a legal pile configuration since the elements in the columns of a pile configuration must both decrease (when read from bottom to top) and appear in the leftmost pile possible.

To prove that this process works, one argues along the same lines as with Viennot’s Geometric RSK. In other words, one thinks of the shadowlines produced by Definition 4.3.3 as a visual record for how cards are played atop each other under Algorithm 1.1.22. In particular, it should be clear that, given a permutation $\sigma \in \mathcal{S}_n$, the shadowlines in both of the shadow diagrams $D_{SW}^{(0)}(\sigma)$ and $D_{NE}^{(0)}(\sigma)$ are defined by the same lattice points from the permutation diagram for $\sigma$. By Lemma 3.1.6, the points along a given northeast shadowline correspond exactly to the elements in some column of $R(\sigma)$ (as both correspond to one of the left-to-right minima subsequences of $\sigma$). Thus, by reading the lattice points in the permutation diagram in increasing order of their abscissae, one can uniquely reconstruct both the piles in $R(\sigma)$ and the exact order in which cards are added to these piles (which implicitly yields $S(\sigma)$). In this sense, both $D_{SW}^{(0)}(\sigma)$ and $D_{NE}^{(0)}(\sigma)$ encode the bottom rows of $R(\sigma)$ and $S(\sigma)$.

It is then easy to see by induction that the salient points of $D_{SW}^{(k-1)}(\sigma)$ yield the $k^{th}$ rows of $R(\sigma)$ and $S(\sigma)$, and so this gives the following Theorem.

**Theorem 4.3.5.** The process described above for creating a pair of pile configurations $(R, S)$ from the Geometric Patience Sorting construction yields the same pair of pile configurations $(R(\sigma), S(\sigma))$ as Extended Patience Sorting (Algorithm 1.1.22) applied to $\sigma \in \mathcal{S}_n$. 
4.3.3 Types of Crossings in Geometric Patience Sorting

As discussed in Section 1.1.3, Extended Patience Sorting (Algorithm 1.1.22) can be viewed as a “non-bumping” version of the RSK Correspondence (Algorithm 1.1.15) in that cards are permanently placed into piles and are covered by other cards rather than being displaced by them. In this sense, one of the main differences between their geometric realizations lies in how and in what order (when read from left to right) the salient points of their respective shadow diagrams are determined. In particular, as playing a card atop a pre-existing pile under Patience Sorting is essentially like non-recursive Schensted Insertion, certain particularly egregious “multiple bumps” that occur under Schensted Insertion prove to be too complicated to be properly modeled by the “static insertions” of Patience Sorting.

At the same time, it is also easy to see that, for a given $\sigma \in \mathfrak{S}_n$, the cards atop the piles in the pile configurations $R(\sigma)$ and $S(\sigma)$ (as given by Algorithm 1.1.22) are exactly the cards in the top rows of the RSK insertion tableau $P(\sigma)$ and recording tableau $Q(\sigma)$, respectively. Thus, this raises the question of when the remaining rows of $P(\sigma)$ and $Q(\sigma)$ can likewise be recovered from $R(\sigma)$ and $S(\sigma)$. While this appears to be directly related to the order in which salient points are read (as illustrated in Example 4.3.6 below), one would ultimately hope to characterize the answer in terms of generalized pattern avoidance similar to the description of reverse patience words for pile configurations (as given in Section 4.1).

**Example 4.3.6.** Consider the northeast and southwest shadow diagrams

$$D_{NE}^{(0)}(2431) = \begin{array}{c}
0 & 1 & 2 & 3 & 4 \\
\bullet & \circ & \circ & \circ & \bullet
\end{array}$$

vs.

$$D_{SW}^{(0)}(2431) = \begin{array}{c}
0 & 1 & 2 & 3 & 4 \\
\bullet & \circ & \circ & \bullet & \circ
\end{array}.$$
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In particular, note that the order in which the salient points are formed (when read from left to right) is reversed. Such reversals serve to illustrate one of the inherent philosophical differences between the RSK Correspondence and the Extended Patience Sorting.

As previously mentioned, another fundamental difference between Geometric RSK and Geometric Patience Sorting is that the latter allows certain crossings to occur in the lattice paths formed during the same iteration of the algorithm. We classify these crossings below and then characterize those permutations that yield entirely non-intersecting lattice paths in Section 4.3.4.

Given \( \sigma \in S_n \), we can classify the basic types of crossings in \( D_{SW}^{(0)}(\sigma) \) as follows: First note that each southwest shadowline in \( D_{SW}^{(0)}(\sigma) \) corresponds to a pair of decreasing sequences of the same length, namely a column from the insertion piles \( R(\sigma) \) and its corresponding column from the recording piles \( S(\sigma) \). Then, given two different pairs of such columns in \( R(\sigma) \) and \( S(\sigma) \), the shadowline corresponding to the rightmost (resp. leftmost) pair — under the convention that new columns are always added to the right of all other columns in Algorithm 1.122 — is called the upper (resp. lower) shadowline. More formally:

**Definition 4.3.7.** Given shadowlines \( L_i^{(m)}(\sigma), L_j^{(m)}(\sigma) \in D_{SW}^{(m)}(\sigma) \) with \( i < j \), we call \( L_i^{(m)}(\sigma) \) the lower shadowline and \( L_j^{(m)}(\sigma) \) the upper shadowline. Moreover, should \( L_i^{(m)}(\sigma) \) and \( L_j^{(m)}(\sigma) \) intersect, then we call this a vertical crossing (resp. horizontal crossing) if it involves a vertical (resp. horizontal) segment of \( L_j^{(m)}(\sigma) \).

We illustrate these crossings in the following example. In particular, note that the only permutations \( \sigma \in S_3 \) of length three having intersections in their 0th iterate shadow diagram \( D_{SW}^{(0)}(\sigma) \) are 312, 231 \( \in S_3 \).
Example 4.3.8.

1. The smallest permutation for which $D_{SW}^{(0)}(\sigma)$ contains a horizontal crossing is $\sigma = 312$ as illustrated in Figure 4.4(a). The upper shadowline involved in this crossing is the one with only two segments.

2. The smallest permutation for which $D_{SW}^{(0)}(\sigma)$ has a vertical crossing is $\sigma = 231$ as illustrated in Figure 4.4(b). As in part (1), the upper shadowline involved in this crossing is again the one with only two segments.

3. Consider $\sigma = 4231 \in \mathfrak{S}_4$. From Figure 4.4(c), $D_{SW}^{(0)}(\sigma)$ contains exactly two southwest shadowlines, and these shadowlines form a horizontal crossing followed by a vertical crossing. We call a configuration like this a “polygonal crossing.” Note, in particular, that $D_{SW}^{(1)}(\sigma)$ (trivially) has no crossings.
4. Consider $\sigma = 45312 \in \mathfrak{S}_5$. From Figure 4.4(d), $D_{SW}^{(0)}(\sigma)$ not only has a “polygonal crossing” (as two shadowlines with a vertical crossing followed by a horizontal one) but $D_{SW}^{(1)}(\sigma)$ does as well.

Polygonal crossings are what make it necessary to read only the salient points along the same shadowline in the order in which shadowlines are formed (as opposed to constructing the subsequent shadowlines using the entire partial permutation of salient points as in Viennot’s Geometric RSK).

**Example 4.3.9.** Consider the shadow diagram of $\sigma = 45312 \in \mathfrak{S}_5$ as illustrated in Figure 4.4(d). The 0\textsuperscript{th} iterate shadow diagram $D_{SW}^{(0)}$ contain a polygonal crossing, and so the 1\textsuperscript{st} iterate shadow diagram $D_{SW}^{(1)}$ needs to be formed as indicated in order to properly describe the pile configurations $R(\sigma)$ and $S(\sigma)$ since

$$\sigma = 45312 \xrightarrow{XPS} \begin{pmatrix} 1 & 1 \\ 3 & 2 & 3 & 2 \\ 4 & 5 & 4 & 5 \end{pmatrix}$$

under Extended Patience Sorting.

### 4.3.4 Characterizing Crossings in Geometric Patience Sorting

Unlike the rows of standard Young tableaux, the values in the rows of a pile configuration need not increase when read from left to right. As we show below, descents in the rows of pile configurations are closely related to the crossings given by Geometric Patience Sorting.

As noted in Section 4.3.2 above, Geometric Patience Sorting is ostensibly simpler than Geometric RSK in that one can essentially recover both the insertion piles
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$R(\sigma)$ and the recording piles $S(\sigma)$ from the $0^{th}$ iterate shadow diagram $D_{SW}^{(0)}$. The fundamental use, then, of the iterates $D_{SW}^{(i+1)}, D_{SW}^{(i+2)}, \ldots$ is in understanding the intersections in the $i^{th}$ iterate shadow diagram $D_{SW}^{(i)}$. In particular, each shadowline $L_i^{(m)}(\sigma) \in D_{SW}^{(m)}(\sigma)$ corresponds to the pair of segments of the $i^{th}$ columns of $R(\sigma)$ and $S(\sigma)$ that are above the $m^{th}$ row (or are the $i^{th}$ columns if $m = 0$), where rows are numbered from bottom to top.

**Theorem 4.3.10.** Each iterate $D_{SW}^{(m)}(\sigma)$ ($m \geq 0$) of $\sigma \in \mathfrak{S}_n$ is free from crossings if and only if every row in both $R(\sigma)$ and $S(\sigma)$ is monotone increasing from left to right.

**Proof.** Since each shadowline $L_i^{(m)} = L_i^{(m)}(\sigma)$ in the shadow diagram $D_{SW}^{(m)}(\sigma)$ depends only on the $i^{th}$ columns of $R = R(\sigma)$ and $S = S(\sigma)$ above row $m$, we may assume, without loss of generality, that $R$ and $S$ have the same shape with exactly two columns.

Let $m + 1$ be the highest row where a descent occurs in either $R$ or $S$. If this descent occurs in $R$, then $L_2^{(m)}$ is the upper shadowline in a horizontal crossing since $L_2^{(m)}$ has ordinate below that of $L_1^{(m)}$, which is the lower shadowline in this crossing (as in 312). If this descent occurs in $S$, then $L_2^{(m)}$ is the upper shadowline in a vertical crossing since $L_2^{(m)}$ has abscissa to the left of $L_1^{(m)}$, which is the lower shadowline in this crossing (as in 231). Note that both types of descents may occur simultaneously (as in 4231 or 45312).

Conversely, suppose $m$ is the last iterate at which a crossing occurs in $D_{SW}^{(m)}(\sigma)$ (i.e., $D_{SW}^{(\ell)}(\sigma)$ has no crossings for $\ell > m$). We prove that the shadowline $L_2^{(m)}$ can only form a crossing using its first or last segment. This, in turn, implies that row $m$ in $R$ or $S$ is decreasing. Note that a crossing occurs when there is a vertex of $L_1^{(m)}$ that is not in the shadow of any point of $L_2^{(m)}$. Thus, we need only show that this can only involve the
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first or last vertex. Let \( \{(s_1, r_1), (s_2, r_2), \ldots \} \) and \( \{(u_1, t_1), (u_2, t_2), \ldots \} \) be the vertices that define \( L_1^{(m)} \) and \( L_2^{(m)} \), respectively. Then \( \{r_i\}_{i \geq 1} \) and \( \{t_i\}_{i \geq 1} \) are decreasing while \( \{s_i\}_{i \geq 1} \) and \( \{u_i\}_{i \geq 1} \) are increasing. Write \((a, b) \leq (c, d)\) if \((a, b)\) is in the shadow of \((c, d)\) (i.e., if \(a \leq b\) and \(c \leq d\)), and consider \( L_1^{(m+1)} \) and \( L_2^{(m+1)} \). By hypothesis, these are noncrossing shadowlines defined by the salient points \( \{(s_1, r_2), (s_2, r_3), \ldots \} \) and \( \{(u_1, t_2), (u_2, t_3), \ldots \} \), respectively. Moreover, given any index \( i \), there is an index \( j \) such that \((s_i, r_{i+1}) \leq (u_j, t_{j+1})\). Suppose, in particular, that \((s_i, r_{i+1}) \leq (u_j, t_{j+1})\) and \((s_{i+1}, r_{i+2}) \leq (u_k, t_{k+1})\) for some \( j < k \). Each upper shadowline vertex must contain some lower shadowline vertex in its shadow, so, for all indices \( \ell \) satisfying \( j \leq \ell \leq k \), either \((s_i, r_{i+1}) \leq (u_{\ell}, t_{\ell+1})\) or \((s_{i+1}, r_{i+2}) \leq (u_{\ell}, t_{\ell+1})\). Let \( \ell \) be the smallest such index such that \((s_{i+1}, r_{i+2}) \leq (u_{\ell}, t_{\ell+1})\). If \((s_i, r_{i+1}) \leq (u_{\ell}, t_{\ell+1})\), then \((s_{i+1}, r_{i+1}) \leq (u_{\ell}, t_{\ell+1}) \leq (u_{\ell}, t_{\ell})\). Similarly, if \((s_i, r_{i+1}) \nleq (u_{\ell}, t_{\ell+1})\), then \((s_i, r_{i+1}) \leq (u_{\ell-1}, t_{\ell})\), from which \((s_{i+1}, r_{i+1}) \leq (u_{\ell}, t_{\ell})\). Thus, in both cases, \((s_{i+1}, r_{i+1}) \leq (u_{\ell}, t_{\ell})\), and so the desired conclusion follows.

An immediate corollary of the above proof is that each row \( i \) \((i \geq m)\) in both \( R(\sigma) \) and \( S(\sigma) \) is monotone increasing (from left to right) if and only if every iterate \( D_{SW}^{(i)}(\sigma) \) \((i \geq m)\) is free from crossings.

We conclude this section by noting that Theorem 4.3.10 only characterizes the output of the Extended Patience Sorting Algorithm. At the time of this writing, a full description of the permutations themselves remains elusive. We nonetheless provide the following theorem as a first step toward characterizing those permutations that result in non-crossing lattice paths under Geometric Patience Sorting.

**Theorem 4.3.11.** The set \( S_n(3-1-42, 31-4-2) \) consists of all reverse patience words having non-intersecting shadow diagrams (i.e., no shadowlines cross in the 0th iterate shadow diagram). Moreover, given a permutation \( \sigma \in S_n(3-1-42, 31-4-2)\),
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the values in the bottom rows of $R(\sigma)$ and $S(\sigma)$ increase from left to right.

Proof. From Theorem 3.3.2, $R(S_n(3-1-42, 31-4-2)) = R(S_n(23-1, 3-12))$ consists exactly of set partitions of $[n] = \{1, 2, \ldots, n\}$ whose components can be ordered so that both the minimal and maximal elements of the components simultaneously increase. (These are called strongly monotone partitions in [12].)

Let $\sigma \in S_n(3-1-42, 31-4-2)$. Since $\sigma$ avoids $3-1-42$, we must have that $\sigma = RPW(R(\sigma))$ by Theorem 3.3.2. Thus, the $i$th shadowline $L^{(0)}_i(\sigma)$ is the boundary of the union of shadows generated by the $i$th left-to-right minima subsequence $s_i$ of $\sigma$. In particular, we can write $s_i = \varsigma_i a_i$ where $a_k > \cdots > a_2 > a_1$ form the right-to-left minima subsequence of $\sigma$. Let $b_i$ be the $i$th left-to-right maximum of $\sigma$. Then $b_i$ is the left-most (i.e., maximal) entry of $\varsigma_i a_i$, so $\varsigma_i a_i = b_i \varsigma'_i a_i$ for some decreasing subsequence $\varsigma'_i$. Note that $\varsigma'_i$ may be empty so that $b_i = a_i$.

Since $b_i$ is the $i$th left-to-right maximum of $\sigma$, it must be at the bottom of the $i$th column of $R(\sigma)$. (Similarly, $a_i$ is at the top of the $i$th column.) So the bottom rows of both $R(\sigma)$ and $S(\sigma)$ must be in increasing order.

Now consider the $i$th and $j$th shadowlines $L^{(0)}_i(\sigma)$ and $L^{(0)}_j(\sigma)$ of $\sigma$, respectively, where $i < j$. We have that $b_i < b_j$ from which the initial horizontal segment of the $i$th shadowline is lower than that of the $j$th shadowline. Moreover, $a_i$ is to the left of $b_j$, so the remaining segment of the $i$th shadowline is completely to the left of the remaining segment of the $j$th shadowline. Thus, $L^{(0)}_i(\sigma)$ and $L^{(0)}_j(\sigma)$ do not intersect. \qed
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