Conformal field theory and mapping class groups

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Abstract. Rational conformal field theories produce a tower of finite-dimensional representations of surface mapping class groups, acting on the conformal blocks of the theory. We review this formalism. We show that many recent mathematical developments can be fit into the first 2 floors of this tower. We also review what is known in higher genus.

1. Introduction.

Conformal field theory (CFT) is an exceptionally symmetric quantum field theory. It takes classical mathematical structures, and 'loops' or 'complexifies' them, to produce infinite-dimensional structures such as affine algebras or vertex operator algebras. Its importance to math – which is considerable – is that the resulting structures tend to straddle several areas, such as geometry, algebra, number theory, functional analysis, ....

One of its most beautiful but least appreciated accomplishments is the reorganisation of several recent mathematical developments, such as Monstrous Moonshine, Jones’ knot invariant, the modularity of affine Kac-Moody characters, and braid group monodromy of the KZ equation, into the first two floors of an infinite tower. This paper describes the resulting picture. For more details see the book [18] and references therein.

2. Mapping class groups.

Up to homeomorphism, a (connected oriented real) surface is completely characterised by its genus $g$ and number $n$ of boundary components (punctures). In fact, there’s only one way (up to equivalence) to give it a real-differential structure. But a real surface can also be a complex curve – it can usually be given a complex-differential (equivalently, a conformal) structure in infinitely many different ways.

For example, the torus $\mathbb{R}^2/\mathbb{Z}^2$ can be given a complex structure by replacing $\mathbb{R}^2$ with $\mathbb{C}$ and $\mathbb{Z}^2$ by $\mathbb{Z} + \tau\mathbb{Z}$ for any $\tau \in \mathbb{C}$ with nonzero imaginary part. In fact, any torus is conformally equivalent to one of the form $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) =: T_\tau$, where $\tau$ lies in the upper-half plane $\mathbb{H} := \{ x + iy \in \mathbb{C} | y > 0 \}$. Moreover, the tori $T_\tau$ and $T_{\tau'}$ are themselves conformally equivalent, iff $\tau' = \frac{a\tau+b}{c\tau+d}$ for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$.

The set of possible complex structures on the torus forms the moduli space $\mathcal{M}_{1,0}$, so labelled because the torus has genus 1 and 0 punctures. This moduli space can be identified with the orbifold $\mathbb{H}/\text{SL}_2(\mathbb{Z})$: we call $\text{SL}_2(\mathbb{Z}) =: \Gamma_{1,0}$ its mapping class group, and $\mathbb{H} =: \mathcal{T}_{1,0}$ its Teichmüller space. Similarly, $\Gamma_{g,n}$, $\mathcal{T}_{g,n}$, and $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\Gamma_{g,n}$ can be defined for any other
In particular, the Teichmüller space $T_{g,n}$ (a simply connected complex manifold) accounts for ‘continuous’ conformal equivalences, while the mapping class group $\Gamma_{g,n}$ (almost always an infinite discrete group) contains the left-over ‘discontinuous’ ones. The moduli spaces usually have conical singularities, corresponding to surfaces with extra symmetries; taking into account these stabilisers, $\Gamma_{g,n}$ will be the (orbifold) fundamental group of $M_{g,n}$.

For example, $\mathcal{M}_{1,0}$ is a sphere with a puncture (corresponding to the cusp $Q \cup i\infty$), and conical singularities at $\tau = i$ and $e^{2\pi i/3}$. Because $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ can also be interpreted as a torus with a special point, namely the additive identity $0$, we also have $T_{1,1} = \mathbb{H}$ and $\Gamma_{1,1} = \text{SL}_2(\mathbb{Z})$.

The surfaces relevant to our story possess additional structure. Let $\Sigma$ be a compact genus-$g$ surface with $n$ marked points $p_i \in \Sigma$. About each point $p_i$ choose a local coordinate $z_i$, vanishing at $p_i$ – this identifies a neighbourhood of $p_i$ with a neighbourhood of $0 \in \mathbb{C}$. We call $(\Sigma, \{p_i\}, \{z_i\})$ an enhanced surface of type $(g, n)$. The resulting moduli space $\hat{\mathcal{M}}_{g,n}$ is infinite-dimensional, but its mapping class group $\hat{\Gamma}_{g,n}$ is an extension of $\Gamma_{g,n}$ by $\mathbb{Z}^n$. For example, $\hat{\Gamma}_{1,1}$ is the braid group $B_3$.

As we will see below, a rational conformal field theory gives finite-dimensional representations of each $\hat{\Gamma}_{g,n}$ – merely projective for $n = 0$, but it seems a true one for $n \geq 1$ (though a proof of trueness is to my knowledge only available for $g \leq 1$). Enhanced surfaces are important because they have canonical sewings. Nevertheless it is common to restrict instead to the projective representations of $\Gamma_{g,n}$, and pay at most lip service to the coordinates $z_i$.

### 3. Conformal field theory.

This section introduces the correlation functions and chiral blocks of conformal field theory.

A conformal field theory (CFT) is a quantum field theory, usually on a two-dimensional space-time $\Sigma$, whose symmetries include the conformal transformations (so conformally equivalent space-times are identified). We restrict to compact orientable $\Sigma$. The same CFT lives simultaneously on all such $\Sigma$. See e.g. [9,16,19,27], and Chapter 4 of [18] for reviews.

Two dimensions are special for CFT because the local conformal maps, which form the Lie algebra $\mathfrak{so}_{n+1,1}(\mathbb{R})$ in $\mathbb{R}^n$ for $n > 2$, becomes infinite-dimensional in $\mathbb{R}^2$ (thanks to their identification with (anti-)holomorphic maps). The conformal algebra in two dimensions consists of two commuting copies of the Witt algebra $\mathfrak{Witt}$ (one for the holomorphic maps, and the other for anti-holomorphic ones). $\mathfrak{Witt}$ is the infinite-dimensional Lie algebra of vector fields on $S^1$, and has a basis $\ell_n, n \in \mathbb{Z}$, satisfying

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n}. \quad (3.1)$$
Its unique nontrivial central extension is the Virasoro algebra \( \mathfrak{vir} \), with basis \( L_n, C \) satisfying
\[
[L_m, L_n] = (m - n)L_{m+n} + \delta_{n,-m}\frac{m(m^2 - 1)}{12}C.
\] (3.2)

Basic data in the CFT are the quantum fields \( \varphi(z) \), called vertex operators, living on space-time \( \Sigma = S^2 = \mathbb{C} \cup \{\infty\} \) and centred at \( z = 0 \). Being quantum fields, these \( \varphi \) are ‘operator-valued distributions’ on \( \Sigma \), acting on the space \( \mathcal{H} \) of states for \( \Sigma \). The most important vertex operators are the stress-energy tensors \( T(z), \overline{T}(z) \), which are the conserved currents of the conformal symmetry, as promised by Noether’s Theorem; the corresponding conserved charges are operators \( L_n, \overline{L}_m \) defining a \( \mathfrak{vir} \)-representation, with central term \( C \) given by scalars \( c_I, \overline{c}_I \) called the central charges.

In a typical quantum field theory, a theoretical physicist makes contact with experiment by computing transition amplitudes \( \langle \text{out}|\text{in} \rangle \) between incoming and out-going states, given mathematically by a Hermitian product \( (|\text{out}\rangle, S|\text{in}\rangle) \) in the Hilbert space \( \mathcal{H} \) of states, for some operator \( S \) called the scattering matrix. In practise these can only be calculated in infinite time \( (t \to \pm \infty) \) limits. The typical way (‘LSZ reduction formulae’) to express these asymptotic amplitudes is via artifacts sometimes called correlation functions. The theory is regarded as solved if all correlation functions can be computed. We are interested in the correlation functions
\[
\langle \varphi_1(z_1) \varphi_2(z_2) \cdots \varphi_n(z_n) \rangle_{\Sigma; p_1, \ldots, p_n}
\] (3.3)
of CFT, for any vertex operators \( \varphi_i \) and any enhanced surface \( (\Sigma, \{p_i\}, z_i) \) (so \( \varphi_i(z_i) \) is ‘centred’ at \( p_i \in \Sigma \)). The remainder of this section explains how physicists think of these CFT correlation functions. Their intuition is provided by string theory.

In a typical quantum theory, correlation functions are calculated perturbatively by Taylor-expanding in some coupling constant. For this purpose, Feynman’s path integral formulation – the quantisation of Hamilton’s action principle in classical mechanics – is convenient. Each term in this perturbation series is computed separately using Feynman diagrams and rules. The Feynman diagrams of quantum field theory are graphs, with a different kind of edge for each species of particle, and a different kind of vertex for every term in the interaction part of the Lagrangian; Feynman’s rules describe how to go from these diagrams to certain integral expressions and hence to the individual terms in the Taylor series expansion of the given correlation function. Feynman diagrams are combinatorial artifacts describing ‘virtual’ (non-real) processes; topologically equivalent ones are identified, and in practise only the simplest are ever considered.

Applying this perturbation formalism to string theory recovers CFT. Consider for convenience closed strings. Then CFT lives on the worldsheet \( \Sigma \) (string theory’s Feynman diagrams) traced by the strings as they
virtually evolve, colliding and separating, through time: string amplitudes (in e.g. 26-dimensional space-time) can be expressed as correlation functions of a (point-particle) CFT in two dimensions. The boundaries of these world-sheets are the in-coming and out-going strings; the world-sheets for asymptotic amplitudes have semi-infinite end-tubes and can be conformally mapped to compact surfaces with punctures $p_i$ (one for every external string). The corresponding Feynman integral is over moduli space $\hat{\mathcal{M}}_{g,n}$. The data of those external strings are stored in the appropriate vertex operator attached to that point $p_i$. The Witt algebra arises here as infinitesimal reparametrisations of the string (a circle).

Everything in CFT comes in a combination of strictly holomorphic, and strictly anti-holomorphic, quantities. Here, ‘holomorphic’ is in terms of space-time $\Sigma$ (which locally looks like $\mathbb{C}$), or the appropriate moduli space (which usually locally looks like $\mathbb{C}^\infty$). These holomorphic and anti-holomorphic building blocks are called chiral. A CFT is studied by first analysing its chiral parts, and then determining explicitly how they piece together to form the physical (=bi-chiral) quantities. Almost all attention by mathematicians has focused on the chiral (as opposed to physical) data. In string theory, this holomorphic/anti-holomorphic alternative corresponds to classical ripples travelling clockwise/anti-clockwise around the string.

A typical vertex operator $\varphi(z)$ depends neither holomorphically nor anti-holomorphically on $z$. Let $\mathcal{V}$ consist of all the holomorphic, and $\overline{\mathcal{V}}$ the anti-holomorphic, ones. For example, the stress-energy tensor $T(z)$ and all of its derivatives lie in $\mathcal{V}$. In the very simplest CFTs, called the minimal models, $\mathcal{V}$ consists only of $T(z)$ and its derivatives.

These chiral algebras $\mathcal{V}, \overline{\mathcal{V}}$ have a rich mathematical structure, with a ‘multiplication’ coming from the so-called operator product expansion, and are examples of vertex operator algebras (see [28], or Chapter 5 of [18]). $\mathcal{V}$ and $\overline{\mathcal{V}}$ mutually commute and the full symmetry ‘algebra’ of the CFT can be identified with $\mathcal{V} \oplus \overline{\mathcal{V}}$. Since quantum fields act on state-space $\mathcal{H}$, it carries a representation of $\mathcal{V} \oplus \overline{\mathcal{V}}$ and decomposes into a direct integral of irreducible $\mathcal{V} \oplus \overline{\mathcal{V}}$-modules. A rational conformal field theory (RCFT) is one whose state-space $\mathcal{H}$ decomposes in fact into a finite direct sum

$$\mathcal{H} = \bigoplus_{M \in \Phi, N \in \overline{\Phi}} \mathcal{Z}_{M, N} M \otimes \overline{N}, \quad (3.4)$$

where $\Phi$ and $\overline{\Phi}$ denote the (finite) sets of irreducible $\mathcal{V}$- and $\overline{\mathcal{V}}$-modules, and the $\mathcal{Z}_{M, N} \geq 0$ are multiplicities. The RCFT are especially symmetric and well-defined quantum field theories and are the CFTs we’re interested in. The name ‘rational’ arises because their central charges $c, \overline{c}$ lie in $\mathbb{Q}$.

The correlation functions (3.3) can be expressed in terms of purely chiral quantities called conformal or chiral blocks, denoted

$$\langle \mathcal{I}_1(v_1, z_1) \mathcal{I}_2(v_2, z_2) \cdots \mathcal{I}_n(v_n, z_n) \rangle_{(\Sigma; p_1, \ldots, p_n; M^1, \ldots, M^n)} . \quad (3.5)$$
As usual, \((\Sigma, \{p_i\}, \{z_i\})\) is an enhanced surface, and to each \(p_i\) we assign a \(L\)-module \(M^i \in \Phi\) and a state \(v_i \in M^i\). The holomorphic field \(\mathcal{I}_i(v_i, z_i)\) centred at \(p_i\) is an operator-valued distribution called an \textit{intertwiner sending} (‘intertwining’) one \(L\)-module (say \(M \in \Phi\)) to another (say \(N \in \Phi\)). For fixed \(M, N, M^i\), the dimension of the space of intertwiners is called the \textit{fusion coefficient} \(N^N_{M^i, M}\), and is given by Verlinde’s formula

\[
N^N_{M^i, M} = \sum_{P \in \Phi} \frac{S_{M^i, P} S_{M, P} S_{N, P}}{S_{V, P}} \tag{3.6}
\]

where the matrix \(S\) (no relation to the scattering matrix) is defined in (5.1a) below. For example, \(V\) is always a module for itself and \(N^V_V = 1\); the unique (up to scaling) intertwiner \(\mathcal{I}(v, z)\) bijectively associates states \(v\) with vertex operators \(\varphi(z) = \mathcal{I}(v, z)\) (the so-called ‘state-field correspondence’). Thus intertwiners generalise vertex operators \(\varphi \in V\).

To solve a given RCFT, the strategy then is to:
(a) construct all possible chiral blocks (3.5); and
(b) construct the correlation functions (3.3) from those chiral blocks.

4. The chiral blocks of RCFT.

For a fixed \((g, n; M^1, \ldots, M^n)\), an RCFT assigns a finite-dimensional space \(\mathcal{F}^{(g, n)}\) of chiral blocks (3.5). Chiral blocks are important to RCFT because finite combinations of them are the correlation functions, and knowing the latter is equivalent to solving the theory.

Each chiral block depends multi-linearly on the states \(v_i \in M^i\), and holomorphically on the \(z_i\), provided branch-cuts in \(\Sigma\) between \(p_i\) are made; locally, it can be regarded as a holomorphic function on \(\hat{M}_{g,n}\). The dimension of this space is given by Verlinde’s formula

\[
\dim \mathcal{F}^{(g, n)} = \sum_{P \in \Phi} \frac{S_{M^1, P} S_{M, P} S_{N, P}}{S_{V, P}} S^{2(1-g)}_{V, P} \tag{4.1}
\]

a generalisation of (3.6).

Moore and Seiberg [30] – see also [2] – isolated the data (finite-dimensional vector spaces and linear transformations) defining each chiral half of RCFT, and provided a complete set of relations they satisfy. Huang is pursuing the explicit construction for all sufficiently nice chiral algebras \(V\) (see e.g. [21]).

A basis for \(\mathcal{F}^{(g, n)}\) is found by performing the following Feynman rules (called ‘conformal bootstrap’). Fix a surface \(\Sigma\) of type \((g, n)\). The space \(\mathcal{F}^{(0,3)}\) consists of intertwiners; arbitrarily fix bases for all those spaces. Now, dissect \(\Sigma\) into pairs-of-pants, as in Figure 1; assign a dummy label \(N_j \in \Phi\) to each internal cut in the dissection; to each vertex in your dissection, choose an intertwinning operator from the basis of the appropriate
space \( \mathfrak{F}^{(0,3)} \); ‘evaluate’ the corresponding chiral block – e.g. for each cut, a trace is taken of the product of intertwiners. Repeating, by running through all possible values of the dummy labels, the result is a basis of chiral blocks.

Figure 1. Dissecting a surface into pairs-of-pants

For an important example, let \( \Sigma \) be the torus \( \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) with one puncture \( p \) (say at 0), assigned the module \( M^1 = \mathcal{V} \) and state \( v_1 = |0\rangle \) (the vacuum, the state of lowest energy). One cut suffices to unfold it into a sphere with 3 punctures, assigned \( \mathcal{V} \)-modules \( \mathcal{V}, M, M \in \Phi \) (\( M \) is the dummy label). The fusion coefficient \( N_{\mathcal{V}, M}^M \) always equals 1, and so for each \( M \in \Phi \) there is a unique intertwiner, say \( \mathcal{I}_1^{(M)} \). Hence \( \dim \mathfrak{F}^{(1,1)} = \| \Phi \| \).

These Feynman rules yield the chiral block

\[
\chi_M(\tau) := \text{tr}_M e^{2\pi i \tau (L_0 - c/24)},
\]

where \( c \) is the central charge and the Virasoro element \( L_0 \) corresponds to energy (the trace comes from the dissection). These span \( \mathfrak{F}^{(1,0)} \).

One of the simplest RCFT is the Ising model, a minimal model. It has central charge \( c = \tau = 0.5 \), and its chiral algebra has 3 irreducible modules, which we’ll label \( \Phi = \{ \mathcal{V}, \epsilon, \sigma \} \). Its toroidal chiral blocks (4.2) are

\[
\begin{align*}
\chi_\mathcal{V}(\tau) &= q^{-1/48} (1 + q^2 + q^3 + 2q^4 + 3q^6 + 3q^7 + \cdots), \\
\chi_\epsilon(\tau) &= q^{23/48} (1 + q + q^2 + q^3 + 2q^4 + 3q^5 + 3q^6 + 3q^7 + \cdots), \\
\chi_\sigma(\tau) &= q^{1/24} (1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + 5q^6 + 6q^7 + \cdots),
\end{align*}
\]

where \( q = e^{2\pi i \tau} \).

Other important RCFT are the Wess–Zumino–Witten (WZW) models. These correspond to strings living on a compact Lie group \( G \). The chiral algebra \( \mathcal{V} \) is closely related to the affine Kac–Moody algebra \( \hat{g}^{(1)} \) (see [23]), where \( \hat{g} \) is the Lie algebra of \( G \) (\( \hat{g}^{(1)} \) is the nontrivial central extension of the loop algebra \( g \otimes \mathbb{C}[z^{\pm 1}] \)). Its modules \( M \in \Phi \) can be identified with the integrable highest-weight modules \( L(\lambda) \) at a level \( k \) determined by the central charge \( c \). The chiral blocks \( \chi_M(\tau) \) for the WZW models are a specialisation of the corresponding affine algebra \( \hat{g}^{(1)} \)-character \( \chi_\lambda(h) \), and for this reason \( \chi_M(\tau) \) in any RCFT is called the character of the \( \mathcal{V} \)-module \( M \). We’ll return to the WZW and Ising models shortly.

Each dissection produces a basis for the space \( \mathfrak{F}^{(g,n)} \). However, any \( \Sigma \) can be dissected in different ways. The over-used term duality means here
for the invertible matrices relating the chiral blocks of different dissections. For example, the left dissection in Figure 2 of the \((g, n) = (0, 4)\) surface corresponds to a matrix \(F = \begin{bmatrix} M & N \\ L & P \end{bmatrix}\) of size \(n \times n\) for \(n = \dim \mathcal{F}_{(L,M,N,P)}^{(0,4)}\) (for an appropriate orientation of surfaces and punctures – a minor technicality we’ve been ignoring), called the fusing matrix. Likewise, the right dissection defines the braiding matrix \(B\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The fusing and braiding matrices \(F, B\)}
\end{figure}

All duality transformations are built up from a few elementary ones, like \(B\) and \(F\). By decomposing surfaces in different ways, we get relations between these elementary dualities. For example, the \(B\)-matrices obey an equation of the form \(BBB = BBB\) called the Yang–Baxter equation, and this is the source of its name ‘braiding’.

Consider four marked points \(w_i\) on the sphere \(\mathbb{C} \cup \{\infty\}\). Using the Möbius (=conformal) symmetry of the sphere, move \(w_i\) to 0, \(w, 1, \infty\), respectively, where \(w\) is the cross-ratio \(\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_3)(w_2-w_4)}\). If we label all four marked points with the Ising module \(\sigma \in \Phi\), then the space \(\mathcal{F}_{(0,4)}^{(\sigma\sigma\sigma\sigma)}\) of chiral blocks is two-dimensional; choosing the smallest energy state in \(\sigma\), the chiral blocks are spanned by

\[
\mathcal{F}_1(w) = \sqrt{1+\sqrt{1-w}} \quad \text{and} \quad \mathcal{F}_2(w) = \sqrt{1-\sqrt{1-w}} .
\]

Fusing interchanges \(w_1 = 0\) and \(w_3 = 1\), hence involves the Möbius transformation \(w \mapsto (1-w)/(1-0) = 1-w\). Likewise, braiding interchanges \(w_2\) with \(w_3\), and so involves \(w \mapsto (0-1)/(0-w) = 1/w\). The braiding and fusing matrices here become

\[
B = \begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = \frac{e^{-\pi i/8}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},
\]

\[
F = \begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

5. Monodromy in RCFT.

The fractional powers in the Ising blocks \((4.4)\) tell us they have branch-point singularities – we must make cuts in the \(w\)-plane to get holomorphic
functions there. If instead we analytically continue these functions along a closed curve, the value of the block need not return to the same value. For example, consider the circle \( w(t) = r e^{2\pi i t} \) for \( r \) small: the value of \( \mathcal{F}_i(w) \) at \( t = 1 \) is \( e^{-\pi i/4} \) times its value at \( t = 0 \). This factor \( e^{-\pi i/4} \) is the monodromy about \( w = 0 \). Likewise, their monodromy about \( w = 1 \) is
\[
\begin{pmatrix}
\mathcal{F}_1(w) \\
\mathcal{F}_2(w)
\end{pmatrix} \mapsto
\begin{pmatrix}
0 & e^{-2\pi i/8} \\
e^{-2\pi i/8} & 0
\end{pmatrix}
\begin{pmatrix}
\mathcal{F}_1(w) \\
\mathcal{F}_2(w)
\end{pmatrix}.
\]

Reintroducing the four coordinates \( w_i \), the chiral blocks \( \mathcal{F}_i \) will be holomorphic on the universal cover of the configuration space \( \mathcal{E}_4(S^2) = \{(w_1, w_2, w_3, w_4) \in (S^2)^4 \mid w_i \neq w_j \} \) of the Riemann sphere. Analytically continuing along any closed path \( \gamma \) in \( \mathcal{E}_4(S^2) \) defines an action of the fundamental group \( \pi_1(\mathcal{E}_4(S^2)) \) – the pure braid group of the sphere with four strands – on the space \( \mathfrak{F}^{(0,4)}_{(\sigma\sigma\sigma\sigma)} \) of chiral blocks. For example, the monodromy about \( w = 1 \) found above corresponds to the pure braid \( \sigma_1^2 \), where \( \sigma_1 \) is the twist of the 1-strand with the \( w \)-strand (each strand corresponds to one of the points \( 0, 1, w, \infty \)). Actually, the full spherical braid group \( \mathfrak{B}_4(S^2) \) acts: \( \beta \in \mathfrak{B}_4(S^2) \) maps the space \( \mathfrak{F}^{(0,4)}_{(M^1, M^2, M^3, M^4)} \) to \( \mathfrak{F}^{(0,4)}_{(M^{\beta_1}, M^{\beta_2}, M^{\beta_3}, M^{\beta_4})} \), where \( \beta_i \) is the associated permutation. For example, the twist \( \sigma_1 w \in \mathfrak{B}_4(S^2) \) is the braiding matrix \( B \begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} \).

Equivalently, as a ‘function’ on the configuration space, the chiral blocks form holomorphic horizontal sections of a projectively flat vector bundle. What this means is that each chiral block satisfies a system of partial differential equations (the Knizhnik–Zamolodchikov or KZ equations) describing how to parallel-transport it around configuration space, and flatness says it locally depends only on the moduli space parameters (and not on the path chosen). Globally, however, there will be monodromy.

More generally, a chiral block \( \mathcal{F} \) for an enhanced surface \( \Sigma \) is a multivalued function on the appropriate moduli space. To make it well-defined, \( \mathcal{F} \) can be lifted to the corresponding Teichmüller space. There will be an action of the corresponding mapping class group \( \hat{\Gamma}_{g,n} \), coming from monodromy. In other words, the space \( \mathfrak{F}^{(g,n)}_{(M^i)} \) of chiral blocks carries a representation \( \rho^{(g,n)}_{(M^i)} \) of \( \hat{\Gamma}_{g,n} \). This \( \hat{\Gamma}_{g,n} \)-representation is built up from the duality matrices, such as the braiding and fusing matrices. As we shall see below, this picture unifies the Jones knot polynomial, the modularity of Monstrous Moonshine, and many other phenomena.

For example, we can dissect the torus using a single vertical cross-sectional cut, or using a horizontal equatorial cut; one basis is \( \chi_M(\tau) \) and the other is \( \chi_M(-1/\tau) \). Duality says that they both span the same space:
\[
\chi_M(-1/\tau) = \sum_{N \in \Phi} S_{MN} \chi_N(\tau).
\] (5.1a)
Likewise, performing a Dehn twist about the vertical cut, we obtain

$$\chi_M(\tau + 1) = \sum_{N \in \Phi} T_{MN} \chi_N(\tau). \quad (5.1a)$$

Here, $S, T$ are complex matrices. Together, $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + 1$ generate the modular group $\text{PSL}_2(\mathbb{Z})$, and $S, T$ generate a true representation $\rho^{(1,0)}$ of the central extension $\text{SL}_2(\mathbb{Z}) = \Gamma_{1,0}$. Hence RCFT characters $\chi_M$ form a weight-0 vector-valued modular form for $\text{SL}_2(\mathbb{Z})$, with multiplier $\rho^{(1,0)}$.

For example, the matrix $T$ for WZW models involves the quadratic Casimir of $G$, while the matrix $S$ involves characters of $G$ evaluated at elements of finite order. For the Ising model, these matrices are

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} e^{-\pi i/24} & 0 & 0 \\ 0 & -e^{-\pi i/24} & 0 \\ 0 & 0 & e^{\pi i/12} \end{pmatrix} \quad (5.2)$$

Perhaps the most elegant treatment of the finite-dimensional representations of a compact Lie group $G$ is Borel–Weil theory, which constructs them via the $G$-action on line bundles over the flag manifold $G_{\mathbb{C}}/B$. Something similar happens to the Virasoro algebra $\mathfrak{Vir}$, with now the moduli spaces of curves playing the role of the flag manifold and mapping class groups taking the place of the Weyl group. A copy of $\mathfrak{Witt} = \text{Vect}(S^1)$ attached to the $i$th puncture on an enhanced surface of type $(g,n)$ acts naturally on the moduli space $\mathfrak{M}_{g,n}$: the vector field $z_i^\ell \partial/\partial z_i$ for $\ell \geq 1$ changes the local coordinate $z_i$; $\partial/\partial z_i$ moves the puncture; and $z_i^\ell \partial/\partial z_i$ for $\ell \leq -1$ can change the conformal structure of the surface. This infinitesimal action fills out the tangent space to any point on $\mathfrak{M}_{g,n}$. In this picture, the central extension of $\mathfrak{Witt}$ to $\mathfrak{Vir}$ arises geometrically as a curvature effect. The KZ equations say roughly that the desired sections of $\mathfrak{M}_{g,n}$-vector bundles should respect this $\mathfrak{Vir}$ action.

6. Correlation functions.

Our primary interest is the $\Gamma_{g,n}$-action $\rho^{(g,n)}(M^i)$ on the spaces $\mathcal{F}^{(g,n)}(M^i)$. Correlation functions – the quantities of physical interest – are sesquilinear combinations of chiral blocks which have trivial monodromy. In a typical quantum field theory the correlation functions are computed perturbatively, but in RCFT they can be found exactly.

For example, the toroidal correlation function is

$$\mathcal{Z}(\tau) := \sum_{M \in \Phi, N \in \Phi} \mathcal{Z}_{M,N} \chi_M(\tau) \chi_N(\tau). \quad (6.1)$$
It is required to be invariant under the $\text{SL}_2(\mathbb{Z})$ action (5.1) on the chiral blocks $\chi_M(\tau)$ of the torus. For the special case of the Ising model (recall (5.2)), the unique solution to the various constraints is

$$Z(\tau) = \chi_V(\tau) \chi_V(\overline{\tau}) + \chi_{\epsilon}(\tau) \chi_{\epsilon}(\overline{\tau}) + \chi_{\sigma}(\tau) \chi_{\sigma}(\overline{\tau}) \cdot$$

(6.2)

Its $\text{SL}_2(\mathbb{Z})$-invariance follows from the unitarity of the matrices (5.2). The classification of possible toroidal correlation functions for WZW models involves quite interesting Lie theory and number theory – see e.g. [17] for a short proof of Cappelli–Itzykson–Zuber’s A-D-E classification of $\text{SU}_2(\mathbb{C})$.

The most elegant and general construction of correlation functions from chiral blocks uses topological field theories and the language of category theory [14].

7. Genus 0: braids and knots.

In the 1980s, Jones studied the combinatorial characterisation of embedding one von Neumann algebra (a factor) in another, and as an unexpected by-product obtained new representations of braid groups $B_n$. We can obtain a knotted link from a braid by gluing the $n$ top endpoints of the braid to the corresponding bottom ones. It is possible to characterise completely (using the ‘Markov moves’) the different braids which yield the same knot, and remarkably, Jones’ $B_n$-representations respect this redundancy, in the sense that Jones could obtain from his representations (using the trace in the underlying von Neumann algebra), a polynomial knot invariant [22].

Witten [32] reinterpreted Jones’ braid group representations as due to the (projective) representation of the genus-0 mapping class groups $\Gamma_{0,n}$, coming from the $\text{SU}_2(\mathbb{C})$ WZW model (as we shall see, $\Gamma_{0,n}$ is essentially a braid group). Witten showed how the $\Gamma_{0,n}$-representation of any other RCFT similarly gives rise to other knot invariants, thus generalising Jones’ invariant considerably by embedding it naturally in a much broader context.

von Neumann algebras arise in quantum field theory (hence RCFT) through the assignment to each region of space-time of the observables measurable in that region; when one region is a subset of another, then its algebra of observables is embedded in the other.

$\Gamma_{0,n}$ has presentation

$$\Gamma_{0,n} = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i - j| > 1), \ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 = 1 = (\sigma_1 \cdots \sigma_{n-1})^4 \rangle$$

$$\cong B_n(S^2)/\mathbb{Z}_2,$$

(7.1)

the quotient of the spherical braid group by its centre. RCFT obtains the $\Gamma_{0,n}$-representation $\rho_{(M)}^{(0,n)}$ by assigning each generator $\sigma_i$ to a matrix in block-diagonal form, whose blocks are braiding matrices. We get a different representation, of dimension given by (4.1), for every choice $M^1, \ldots, M^n$...
of \( V \)-modules. An element \( \beta \in \Gamma_{0,n} \) sends the space \( \mathfrak{F}_{(M^i)}^{(0,n)} \) to \( \mathfrak{F}_{(M^\beta i)}^{(0,n)} \), so to get a representation of the full \( \Gamma_{0,n} \) we should sum \( \mathfrak{F}_{(M^i)}^{(0,n)} \) over all reorderings of \( (M^i) \). It is common to lift this \( \Gamma_{0,n} \)-action to the braid group \( B_n \) in the manner clear from (7.1). In all known examples it seems, these representations are always defined over some cyclotomic field.

The groups \( \Gamma_{0,0} \cong \Gamma_{0,1} \cong 1 \), \( \Gamma_{0,2} \cong \mathbb{Z}_2 \), \( \Gamma_{0,3} \cong S_3 \) are all finite and so aren’t very interesting. This is because the Möbius symmetry on \( S^2 \) is triply transitive. However, the image of \( \Gamma_{0,n} \) for \( n \geq 4 \) will usually be infinite – e.g. in the special case of two-dimensions, the question of \( \Gamma \) image reduces to Schwarz’ classical analysis of the finite monodromy of the hypergeometric equation, and as such is very rare.

\( \Gamma_{0,4} \) is an extension of \( \text{PSL}_2(\mathbb{Z}) \) by \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), and the part of \( \Gamma_{0,4} \) corresponding to a trivial permutation \( \beta i = i \) is isomorphic to the principal congruence subgroup \( \Gamma(2) / \pm 1 \). Using the \( \Gamma(2) \)-Hauptmodul \( \theta_2(\tau)^4 / \theta_3(\tau)^4 \), we can lift the chiral blocks \( \mathcal{F}(w) \) to the upper-half plane \( \mathbb{H} \), and in this way interpret these chiral blocks as vector-valued modular forms for \( \text{PSL}_2(\mathbb{Z}) \). For example, the Ising blocks (4.4) would now become

\[
\mathcal{F}_1(\tau) = q^{-\frac{1}{12}} (1 + q^{\frac{1}{2}} + 3q + 4q^{\frac{3}{2}} + 5q^2 + 8q^{\frac{5}{2}} + 11q^3 + \cdots), \\
\mathcal{F}_2(\tau) = q^{\frac{1}{12}} (2 + 2q^{\frac{1}{2}} + 2q + 4q^{\frac{3}{2}} + 8q^2 + 10q^{\frac{5}{2}} + 12q^3 + \cdots). \tag{7.2}
\]

These lifts \( \mathcal{F}(\tau) \) will always be holomorphic in \( \mathbb{H} \), but can have poles at the cusps. The weight can be rational because the \( \Gamma(2) \)-representation will typically be projective; in fact arbitrary rational weight is possible. The theory of these vector-valued modular forms of arbitrary rational weight for \( \text{PSL}_2(\mathbb{Z}) \), and no restriction on the kernel of the multiplier, has been developed recently [4], [26] and is quite rich. It is interesting that RCFT produces plenty of examples of these.

The \( \Gamma_{0,n} \)-action coming from WZW models is especially interesting. Consider \( \text{SU}_2(\mathbb{C}) \) for concreteness. Choose \( n \) distinct points \( z_1, \ldots, z_n \in \mathbb{C} \) and \( n \) \( \mathfrak{sl}_2(\mathbb{C}) \)-modules \( \mathcal{V}^i \); write \( \hat{\mathcal{V}}^i \) for the corresponding \( \mathfrak{sl}_2^{(1)} \)-modules. Then the conformal blocks \( \mathcal{F} \in \mathfrak{F}_{(\hat{\mathcal{V}}^i)}^{(0,n)} \) are precisely the functions \( \mathcal{F} : \mathfrak{C}_n(S^2) \rightarrow V_1 \otimes \cdots \otimes V_n \) satisfying the KZ equations [27]

\[
\frac{\partial \mathcal{F}}{\partial z_i} = \frac{1}{k + 2} \sum_{j \neq i} \Omega_{ij} (z_i - z_j) \mathcal{F}, \tag{7.3}
\]

where \( \Omega_{ij} / (z_i - z_j) \) is the classical Yang–Baxter \( r \)-matrix for \( \text{SU}_2(\mathbb{C}) \). As mentioned earlier, any solution to (7.3) can be parallel-transported through \( \mathfrak{C}_n(S^2) \); projective flatness means that this parallel-transport along a closed loop depends (up to a projective factor) only on the homotopy-class of the loop. In other words, the space \( \mathfrak{F}_{(\hat{\mathcal{V}}^i)}^{(0,n)} \) of solutions to (7.3) carries a projective representation of the pure spherical braid group \( \pi_1(\mathfrak{C}_n(S^2)) \). The
Drinfel’d–Kohno monodromy theorem expresses this monodromy in terms of the 6j-symbols of the quantum group \( U_q(\mathfrak{sl}_2(\mathbb{C})) \), for \( q = e^{\pi i/(k+2)} \), which are straightforward to compute \([25]\). Something similar happens for any \( G \).

The infinitely many irreducible finite-dimensional modules of a simple Lie algebra \( \mathfrak{g} \) naturally span a symmetric monoidal category (see \([31]\) for definitions); its representation ring is isomorphic to a polynomial ring in \( r \) variables, where \( r \) is the rank of the algebra. On the other hand, the finitely many level \( k \) irreducible integrable modules of the affine algebra \( \mathfrak{g}^{(1)} \) span (among other things) a braided monoidal category; the corresponding representation ring is called a fusion ring and has structure constants equal to the fusion coefficients (3.6). The key ingredient in this category – the braiding – comes from the braid group monodromy of (7.3). Something similar happens for any RCFT.

8. Genus 1: modularity.

The characters \( \chi_\lambda \) of the affine algebra \( \mathfrak{g}^{(1)} \) are defined exactly as for semi-simple \( \mathfrak{g} \), as a sum of exponentials of the Cartan subalgebra, though the sum will now be infinite. In fact a miracle happens: the character \( \chi_\lambda \) will be a modular function for some subgroup of \( \text{SL}_2(\mathbb{Z}) \)! One of the coordinates of the Cartan subalgebra of \( \mathfrak{g}^{(1)} \) plays the role of \( \tau \in \mathbb{H} \), and the others come along for the ride. The algebraic proof of this modularity makes it look accidental: the character \( \chi_\lambda \) is expressed as a fraction; the denominator is automatically a modular form for \( \text{SL}_2(\mathbb{Z}) \), by the simple combinatorics of affine algebras; the numerator is a modular form (in fact a lattice theta function) for some congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \), because the Weyl group of \( \mathfrak{g}^{(1)} \) contains translations in a lattice; their quotient yields a modular function.

RCFT provides a much more satisfying explanation for this unexpected modularity. The characters \( \chi_\lambda(\tau) \) of these affine algebra modules equal the chiral blocks (4.2) of the corresponding WZW model, and the action (5.1) of \( \Gamma_{1,0} = \text{SL}_2(\mathbb{Z}) \) coming from RCFT explains their unexpected modularity. This relation with RCFT also tells us the \( \mathfrak{g}^{(1)} \)-modules are simultaneously \( \mathfrak{Vir} \)-modules. All of this was known to algebraists before the relation of affine algebras to RCFT was developed, but this relation emphasises that these properties of affine algebra modules are not accidental but naturally fit into a much broader perspective.

There is more to being a modular form or function than transforming nicely with respect to \( \text{SL}_2(\mathbb{Z}) \). Good behaviour at the cusps of \( \mathbb{H} \) is also crucial, as they compactify the domain. These cusps correspond to a pinched torus; their analogue for the other moduli spaces are surfaces with nodes (this is the Deligne–Mumford compactification). RCFT requires nice behaviour (‘factorisation’) of chiral blocks as we move in moduli space toward these degenerate surfaces. This connects the moduli spaces of different topologies, and tells us RCFT is naturally defined on a ‘universal tower’ of moduli spaces.
Interesting modularity certainly isn’t restricted to affine algebras. Indeed, the Monstrous Moonshine conjectures (see e.g. [18]) relate character values of the monster finite simple group to various Hauptmoduls. For example, the first nontrivial coefficient (196884) in the $j$-function nearly equals the dimension 196883 of the first nontrivial representation of the monster. Conjecturally, Hauptmoduls are associated to pairs of commuting elements in the Monster – e.g. the $j$-function is assigned to $(e, e)$. The starting point to our (still incomplete) understanding of these conjectures is the construction [13] of an RCFT (with central charge $c = 24$, $∥\Phi∥ = 1$, and anti-holomorphic chiral algebra $V = \mathbb{C}$) whose symmetry group equals the Monster and whose single character (4.2) equals the $j$-function.

To some crude extent, Moonshine can then be interpreted as the conjunction of two different pictures of quantum field theory, applied to that very special RCFT: the Hamiltonian picture, which provides us a Hilbert space (state-space) carrying an action of the Monster, and an energy operator $L_0$ such that (4.2) is defined; and the Feynman picture, which lives in moduli space and which makes modularity manifest. As explained at the end of Section 5, the Virasoro algebra, through its action on the moduli spaces $\mathcal{M}_{g,n}$, lies at the heart of Moonshine.

Can we see more directly why the RCFT characters $\chi_M(\tau)$ of (4.2) should have anything to do with modularity? The chiral blocks on the torus can be obtained from those of the plane $\mathbb{C}$, by first considering the map $z \mapsto w := e^{2\pi i z}$. Though holomorphic, it changes the global topology, sending the plane $\mathbb{C}$ to the annulus $\mathbb{C}\setminus\{0\}$, and this topology change is responsible for the $-c/24$ in (4.2). To obtain our torus, we now identify $w$ and $qw$, where as always $q = e^{2\pi i \tau}$. This is equivalent to taking the finite annulus $\{w \in \mathbb{C} | |q| < |w| < 1\}$ and sewing together its two boundary circles appropriately. The resulting torus is conformally equivalent to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau)$. Applying this construction to chiral blocks, we find that those for the torus are indeed given by (4.2) (e.g. the trace comes from sewing). The proof [33] of modularity of vertex operator algebra characters follows this outline.

The $\text{SL}_2(\mathbb{Z})$-representation (5.1) is defined over a cyclotomic field for any RCFT [8], and its kernel contains a congruence subgroup [3]. Perhaps the latter isn’t so surprising, considering that $\tilde{\mathcal{F}}^{(1,0)}$ has a basis (4.2) with integer $q$-expansions. Intimately connected with this congruence subgroup property, the matrices $S, T$ have nice properties with respect to the cyclotomic Galois group [8,3].

$\Gamma_{1,1}$ is also $\text{SL}_2(\mathbb{Z})$, and so the chiral blocks in $\tilde{\mathcal{F}}^{(1,1)}_M$ also form vector-valued modular forms for $\text{SL}_2(\mathbb{Z})$. The weight though can be arbitrary rational numbers and the kernel need not be of finite index. So together with the $\mathcal{F} \in \tilde{\mathcal{F}}^{(0,4)}_{(M)}$ (as explained last section), RCFT is a rich source of vector-valued modular forms. These $\Gamma_{1,1}$-representations are explicitly known in terms of the duality matrices (see e.g. [30]), and so the machinery of [4] allows these chiral blocks to be explicitly found [5].
But specialising to the WZW models, we should expect a nice Lie theoretic answer, and indeed the complete answer is known for SU\(_2(\mathbb{C})\) [24,11]. The matrix representing \(\tau \mapsto \tau + 1\) will again be given by quadratic Casimirs, but the matrix representing \(\tau \mapsto -1/\tau\) involves the continuous \(q\)-ultraspherical polynomials, also known as the Macdonald polynomials for \(\mathfrak{sl}_2(\mathbb{C})\) (and as such are a natural generalisation of SU\(_2(\mathbb{C})\)-characters, which describe \(\tau \mapsto -1/\tau\) for \(\rho^{(1,0)}\)). Similarly, the chiral blocks in \(\mathfrak{F}_\lambda^{(1,1)}\) can be expressed using Macdonald ‘polynomials’ for \(\mathfrak{sl}_2(\mathbb{C})\) (a natural generalisation of the \(\mathfrak{sl}_2(\mathbb{C})\)-characters which are the chiral blocks in \(\mathfrak{F}^{(1,0)}\)). Something similar can be expected for the other WZW models on the punctured torus. Likewise, evaluating these chiral blocks for the Moonshine RCFT should extend Moonshine to noncommuting pairs \(g, h\) in the monster [5].

The space \(\mathfrak{F}_\mathcal{V}^{(1,1)}\), with the \(\mathcal{V}\)-module \(\mathcal{V}\), can be identified with \(\mathfrak{F}^{(1,0)}\) except that in the former we have the freedom to evaluate the block at any state \(v \in \mathcal{V}\). In particular, taking \(v\) to have the minimum energy – the vacuum – recovers the characters (4.2), but taking other \(v\) will give a vector-modular form of even weight, with the same SL\(_2(\mathbb{Z})\)-multiplier \(\rho^{(1,0)}\) as in (5.1). [10] evaluated these for the Moonshine RCFT and found that all modular forms (of the right shape) arise. What is interesting is that the coefficients of these modular forms will have Moonshine-like interpretations involving characters of the stabiliser of the state \(v\).

9. Higher genus.

For a fixed RCFT, chiral blocks \(\mathcal{F} \in \mathfrak{F}^{(g,n)}_{(M^1,\ldots,M^n)}\) yield vector-valued automorphic functions for the infinite discrete groups \(\Gamma_{g,n}\), each realising a finite-dimensional representation \(\rho^{(g,n)}_{(M^i)}\) of \(\Gamma_{g,n}\). This tower of automorphic functions is coherent in the sense that it respects basic operations like sewing or pinching the surfaces. As mentioned earlier, \((g,n) = (1,0), (1,1), (0,4)\) all give a vector-valued modular form for SL\(_2(\mathbb{Z})\); for \((1,0)\) this is a classical object, being weight-0 and invariant under some congruence subgroup, but for \((1,1)\) and \((0,4)\) the weight is rational and the image of the multiplier will usually be infinite. Relatively little is known in higher genus \(g\), and surely it is a direction for important future research. The main open challenge is to identify the special features and structures occurring here. In this sense most of the work done has been negative. In this section we sample a few of the highlights.

Most of the work has focused on the kernel and image of these representations \(\rho^{(g,n)}_{(M^i)}\). The groups \(\Gamma_{0,n}\), \(n \leq 3\), are finite; all other \(\ker \rho^{(g,n)}_{(M^i)}\) will be infinite, since \(\Gamma_{g,n}\) is generated by infinite-order Dehn twists but \(\rho^{(g,n)}_{(M^i)}\) maps each of these to a finite-order matrix. However, for fixed \(n\), the intersection over all \(k\) of the kernel of \(\rho^{(g,0)}\) for the SU\(_n(\mathbb{C})\) WZW model at level \(k\), is trivial in any genus \(g > 2\) [1].

The image of \(\rho^{(1,0)}\) is always finite [3], but this is atypical: it is expected
that a generic RCFT will have all other images infinite. For example, Funar [15] found that all \( \text{im} \rho^{(g,0)} \) will be infinite for \( SU_2(\mathbb{C}) \) WZW models at all levels \( k > 8 \), and all genus \( g > 1 \). Moreover, Masbaum [29] found an infinite-order matrix in \( \text{im} \rho^{(0,4)} \) for those RCFT.

In the RCFT associated to even self-dual lattices \( L \) (where the strings live on the torus \( \mathbb{R}^n/L \) for \( n = \dim L \)), the conformal blocks in \( \mathfrak{F}^{(g,0)} \) can be expressed in terms of Siegel theta functions, and the Torelli subgroup of \( \Gamma_{g,0} \) is in the kernel of \( \rho^{(g,0)} \). This is very atypical for RCFT, e.g. it is known to fail for \( SU_2(\mathbb{C}) \) WZW at most levels.

On the other hand, these representations for all known RCFT seem to be always definable over a cyclotomic field. A notion of integrality for these representations is being developed [20].

A class of RCFT very conducive to study are the so-called holomorphic orbifolds, associated to the Drinfel’d double of a finite group \( G \). In this case, the chiral blocks in \( \mathfrak{F}^{(g,0)} \) are parametrised by \( \text{Hom}(\pi_1(\Sigma_g), G)/G \), i.e. group homomorphisms \( \varphi \) from \( \pi_1 \) of a genus-\( g \) surface \( \Sigma_g \) into \( G \), where we identify \( \varphi(\sigma) \) and \( g^{-1}\varphi(\sigma)g \). \( \Gamma_{g,0} \) acts naturally on \( \pi_1(\Sigma_g) \) and hence \( \rho^{(g,0)} \) here becomes a permutation representation. This means \( \text{im} \rho^{(g,0)} \) is manifestly finite. However, we can see from this explicitly that \( \ker \rho^{(g,0)} \) won’t contain the Torelli generators listed by [6], at least for generic groups \( G \), and so even in this extremely well-behaved theory, \( \rho^{(g,0)} \) doesn’t factor through to a representation of Siegel’s modular group \( \text{Sp}_{2g}(\mathbb{Z}) \). [12] show that \( \text{im} \rho^{(0,n)} \) will always be finite here, and it is tempting to guess that all \( \text{im} \rho^{(g,n)} \) is finite here.

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