Effective Action in $\mathcal{N} = 2,4$ Supersymmetric Yang-Mills Theories

A.T. Banin$^a$, I.L. Buchbinder$^{b}$, N.G. Pletnev$^a$

$^a$Institute of Mathematics, Novosibirsk, 630090, Russia,

$^b$Department of Theoretical Physics
Tomsk State Pedagogical University
Tomsk 634041, Russia

Abstract

We review the approach to calculation of one-loop effective action in $\mathcal{N} = 2,4$ SYM theories. We compute the non-holomorphic corrections to low-energy effective action (higher derivative terms) in $\mathcal{N} = 2$, SU(2) SYM theory coupled to hypermultiplets on a non-abelian background for $R_2$-gauge fixing conditions. A general procedure for calculating the gauge parameters depending contributions to one-loop superfield effective action is developed. The one-loop non-holomorphic effective potential is exactly found in terms of the Euler dilogarithm function for a specific choice of gauge parameters. We also discuss the calculations of hypermultiplet dependence of $\mathcal{N} = 4$ SYM effective action.
1 Introduction

It is well-known that low-energy effective action of $\mathcal{N} = 2$ supersymmetric Yang-Mills theories is determined, in purely gauge superfield sector, by two effective potentials. The leading correction is given by holomorphic potential $\mathcal{F}(\mathcal{W})$, and the next-to-leading correction is written in terms of non-holomorphic potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ where $\mathcal{W}$ and $\bar{\mathcal{W}}$ are $\mathcal{N} = 2$ superfield strengths (see e.g. the review [1]). $\mathcal{N} = 2$ supersymmetry strongly restricts the form of holomorphic potential what was demonstrated by Seiberg and Witten for $SU(2)$ SYM model in the Coulomb branch of inequivalent vacua in which the low energy theory has unbroken $U(1)$ gauge factors [2]. An extension of this result for various gauge groups and coupling to matter was given in Ref. [3] (see also the review [4]). General form of holomorphic potential for an arbitrary $\mathcal{N} = 2$ model is now well established.

Computation of the non-holomorphic potential is more delicate, and a general form of $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ is still unknown although some contributions to $\mathcal{H}$ were obtained for special cases. In $\mathcal{N} = 2$ superconformal invariant models and the $\mathcal{N} = 4$ SYM theory the non-holomorphic potential has been found in the Coulomb phase [5] - [9]. Here all beta functions vanish, and the evolutions under the renormalization group is trivial. This effective potential is turned out to be an exact solution of the $\mathcal{N} = 4$ SYM theory, its explicit form is given only by one-loop contribution, any higher-loop or instanton corrections are absent [6], [9] - [11]. However all above results correspond to Abelian background $\mathcal{W}$ and $\bar{\mathcal{W}}$ for the theory, living on a point of general position of the moduli space, where one has the symmetry-breaking pattern: $SU(N) \rightarrow U(1)^{N-1}$ and all physical quantities vary smoothly over the moduli spaces. As to a non-abelian background, the non-holomorphic potential was found only for very special cases in Refs. [5] - [12].

One of the basic approaches to evaluating the effective action is the derivative expansion [15]. This approach allows one to get the effective action in the form of a series in derivatives of its functional arguments. Within $\mathcal{N} = 1$ supersymmetric derivative expansion, the leading contributions to the effective action are formed by the so-called Kählerian and chiral superfield effective potentials [13], [14]. We point out that the Kählerian effective potential naturally arises in $\mathcal{N} = 2$ SYM models if ones formulate these models in terms of $\mathcal{N} = 1$ superfields [16] and, as a result, it allows to construct the potentials $\mathcal{F}(\mathcal{W})$ and $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ on its base.

Another line of current study of the effective action in extended SUSY theories is associated with a realization of these theories on the world volume of branes. Such a realization provides a dual description of low-energy field dynamics in terms of D-brane theory. Webs of intersecting branes as a tool for studying the gauge theories with reduced number of supersymmetries have been introduced in Ref. [17]. The five-brane construction has been successfully applied to a computation of holomorphic (or rather BPS) quantities of the four-dimensional supersymmetric gauge theory (see Refs. [18], [19]). The five-brane configurations corresponding to these $\mathcal{N} = 1$ supersymmetric gauge theories encode the information about the $\mathcal{N} = 1$ moduli spaces of vacua. The non-holomorphic quantities such as higher derivative terms in $\mathcal{N} = 2$ theories and the Kählerian potential of $\mathcal{N} = 1$ supersymmetric gauge theories are of special interest since they are not protected by supersymmetry. It was shown that the Kählerian potential on the Coulomb branch of $\mathcal{N} = 2$ theories is correctly reproduced from the classical dynamics of M-theory five-brane. As to the non-holomorphic contributions to low-energy effective action, such as the higher derivative terms, a correspondence between string/brane approach and four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theories beyond two-derivative level, is not completely established (see e.g. Refs. [18, 19, 20, 21]).

In this paper we discuss some aspects of structure of the non-holomorphic effective potential for non-abelian background in order to pay attention on a problem of its gauge dependence. This fact is related to the parameterization non-invariance of the conventional effective action (see e.g. Ref. [22]) and leads to a number of different effective actions corresponding to one classical action. But any gauge-fixing condition is equal to a redefinition of fields in each order of the effective action.
loop expansion [30]. The gauge dependence of an effective action for Yang-Mills theories is well-known problem for many years [22,30]. But for \( \mathcal{N} = 1,2,4 \) super-YM theories such a problem has not been considered in detail. A supersymmetric generalization of \( R_\xi \)-gauge seems to be a good tool for studying gauge-dependence in SYM theories. Such a generalization was first suggested in [28]. We present an extended supersymmetric \( R_\xi \)-gauge for SYM models within background field method. The choice of a gauge fixing term in spontaneous broken non-abelian gauge theories is of basic technical importance. It is known that the use of the \( R_\xi \)-gauge became a major step in the proof that Yang-Mills models are unitary, on-shell gauge-independent and renormalizable quantum field theories. Our consideration is mainly based on Ref. [23]. Finally, we study the dependence of the low-energy effective action in \( \mathcal{N} = 4 \) SYM theory on hypermultiplet fields.

## 2 \( \mathcal{N} = 2 \) SYM Theory in \( \mathcal{N} = 1 \) Superspace

The simplest and well developed description of four-dimensional supersymmetric field theories is formulation in terms of \( \mathcal{N} = 1 \) superspace. From the point of view of \( \mathcal{N} = 1 \) supersymmetry, a field content of the pure \( \mathcal{N} = 2 \) SYM model is given by the vector multiplet superfield \( V \) and chiral superfield \( \Phi \), and the field content of the hypermultiplet is given by two chiral superfields \( Q_+, Q_- \). This allows one to write the action \( S \) of the \( \mathcal{N} = 2 \) SYM model coupled to hypermultiplet matter in \( \mathcal{N} = 1 \) superspace as follows:

\[
S = S_{\text{SYM}} + S_{\text{Hyper}}
\]

\[
S_{\text{SYM}} = \frac{1}{2T(R)g^2} \text{tr} \int d^6 z \left[ \frac{1}{2} W^\alpha W_\alpha + \int d^6 \phi e^V \Phi e^{-V} \right] \]

\[
S_{\text{Hyper}} = \int d^6 z (\bar{Q}_+ e^V Q_+ + Q_- e^{-V} \bar{Q}_-) + i \int d^6 z \bar{Q}_+ \Phi Q_+ + i \int d^6 z \bar{Q}_- \Phi Q_-)
\]

where the superfields \( V = V^A T^A \) and \( \Phi = \Phi^A T^A \) form the \( \mathcal{N} = 2 \) gauge multiplet with the component fields \((A_\mu, \lambda_+, \phi)\) belonging to the adjoint representation of the gauge group \( G \), and the superfields \( Q_+ \) form a hypermultiplet with the component fields \((\psi_+, H_+, \psi_-)\) belonging to some representation \( \mathcal{R} \) of \( G \). We use the conventions of Ref. [24].

The classical actions \( S_{\text{SYM}} \) and \( S_{\text{Hyper}} \) are gauge invariant and manifestly \( \mathcal{N} = 1 \) supersymmetric by construction. However, the full action \( S \) is also invariant under the hidden \( \mathcal{N} = 2 \) supersymmetry transformations, which can be written in terms of the covariant chiral superfields \( \Phi_c = e^{\Omega} \Phi e^{-\bar{\Omega}} \), \( Q_+ = e^{\Omega} Q_+ \) etc.:

\[
\delta \Phi_c = e^{\alpha} W_\alpha, \\
\delta W_\alpha = -e_\alpha \nabla^2 \Phi_c + ie^\chi \nabla_{\alpha c} \Phi_c, \\
\delta \bar{W}_{\dot{c}} = -\bar{e}_{\dot{c}} \nabla^2 \bar{\Phi}_c + i e^\chi \nabla_{\dot{c} c} \bar{\Phi}_c, \\
\delta Q_{+c} = \bar{Q}_{+c} (\Delta_1 \Omega) - \nabla^2 (\bar{Q}_- \chi c), \\
\delta \bar{Q}_{-c} = -\bar{Q}_{-c} (\Delta_1 \Omega) \bar{Q}_- + \nabla^2 (\chi Q_+), \\
\delta Q_{+c} = -(\Delta_2 \Omega) Q_{+c} + \nabla^2 (\bar{Q}_- \chi), \\
\delta Q_{-c} = Q_{-c} (\Delta_2 \Omega) - \nabla^2 (\bar{Q}_- \chi c), \\
\Delta_1 \Omega = e^{-\Omega} e^{\Omega} = i \chi \Phi_c, \\
\Delta_2 \Omega = e^{\Omega} e^{-\Omega} = i \bar{\Phi}_c \chi, \\
\chi = \lambda(\theta) + \bar{\lambda}(\bar{\theta}).
\]

Here \( \Omega \) is a complex superfield determining the gauge superfield \( V \) in the form \( e^V = e^{\bar{\Omega}} e^\Omega \), and \( \bar{\lambda} \) are chiral and antichiral space-time-independent superfield parameters with the expansion \( \lambda = \gamma + \frac{1}{2} \theta^\alpha \epsilon_\alpha + \theta^2 (\beta_1 + i \beta_2) \), where the \( \beta_1 \) and \( \beta_2 \) parameterize the \( SU(2)/U(1) \) group, \( \epsilon_\alpha \) are the anticommutative parameters present in the Eqs. (4), and \( \gamma \) parameterizes the central charge transformations. The hypermultiplet action and corresponding \( \mathcal{N} = 2 \) supersymmetry transformations in terms of \( \mathcal{N} = 1 \) superspace were considered in Refs. [24] and [25]. Invariance of the actions \( S_{\text{SYM}} \) and \( S_{\text{Hyper}} \) under the transformations (4, 5) can be checked directly. One also points out that both the \( \mathcal{N} = 2 \) super Yang-Mills model and the hypermultiplet model are the superconformal invariants [26]. Further we will use only the covariant chiral superfields, and subscript \( c \) will be omitted.

The low-energy effective action of the model under consideration is described by the holomor-
phic scale-dependent effective potential \( F(W) \) and the non-holomorphic scale-independent real effective potential \( H(W,W) \) where \( W \) is \( \mathcal{N} = 2 \) superfield strength. The corresponding contributions to the effective action can be expressed in terms of \( \mathcal{N} = 1 \) superfields. The holomorphic part \( \Gamma_F \) of low-energy effective action is written in \( \mathcal{N} = 1 \) form as follows [16]

\[
\Gamma_F = \int d^4x d^2\theta \frac{1}{2} F_{AB}(\Phi) W^A \, W_B^\alpha + \frac{1}{2} F_A(\Phi) \Phi^A + h.c.
\]  

(6)

The non-holomorphic contribution \( \Gamma_H \) can be given in an \( \mathcal{N} = 1 \) form using the metric, connection and curvature of natural Kähler geometry since the \( \mathcal{H} \) is associated with a Kähler potential on a complex manifold defined modulo the real part of a holomorphic function

\[
\Gamma_H = \int d^4x d^2\theta (g_{AB}[-\frac{1}{2} \nabla^\alpha \Phi A \nabla_\alpha \Phi B + i \bar{W}^{B\dot{a}}(\nabla_\dot{a} W^A + \Gamma^A_{CD} \nabla_\dot{a} \Phi C W^D) - i \Gamma^{AB} W^E \bar{\Phi} E (W^F + \Gamma^{BCD} W^D \bar{\Phi}^C \nabla_\dot{a} \Phi B) + \frac{1}{2} \frac{\Gamma}{4} R_{ABCD}(W^A \nabla^C W^B \bar{W}^D \bar{\Phi}^B))
\]  

(7)

where \( g_{AB} = \mathcal{H}_{AB}, \Gamma^A_{BC} = g^{AD} \mathcal{H}_{BCD}, R_{ABCD} = \mathcal{H}_{ACBD} - g_{EF} \mathcal{H}^{EF} \mathcal{H}^{BCD}. \) Being expressed in terms of component fields, the contribution to effective action \( \Gamma_H \) contains at most four space-time derivatives.

We will analyze a general form of the one-loop functionals \( \Gamma_F \) and \( \Gamma_H \) in the model under consideration using functional methods in the \( \mathcal{N} = 1 \) superspace and revise the contributions to the effective action which determine a functional dependence of \( \mathcal{F} \) and \( \mathcal{H} \) on the \( \mathcal{N} = 2 \) vector multiplet. Eqs. (6, 7) play a very important role in such an approach since they ensure a bridge between the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) descriptions and allow one to restore manifestly \( \mathcal{N} = 2 \) supersymmetric functionals on the basis of their \( \mathcal{N} = 1 \) projections.

### 3 Background Field Quantization

The background field method is a powerful and convenient tool for studying the structure of a quantum gauge theory (see Refs. [24, 27]). After fields splitting, the action in the \( \mathcal{N} = 1 \) SYM theory with matter multiplets will be written as a functionals of the background superfields \( \Omega, \bar{\Omega}, \Phi, \bar{\Phi} \) and quantum ones \( V, \phi, \bar{\phi} \). To quantize the theory, we impose the gauge-fixing conditions only on the quantum fields, introduce the corresponding ghosts and consider the background fields as functional arguments of the effective action.

We choose the proper gauge-fixing conditions for the quantum superfields \( V \) and \( \phi \) in the form

\[
\bar{F} = \nabla^2 V^A + i \frac{1}{\lambda} \bar{\nabla}^2 \phi^B \bar{\Phi}^C f^{ABC},
\]

\[
F^A = \nabla^2 V^A - i \frac{1}{\bar{\lambda}} \bar{\nabla}^2 \phi^B \Phi^C f^{ABC},
\]

(8)

where \( \lambda, \bar{\lambda} \) are the arbitrary numerical parameters and \( \square \pm \) are standard notations for Laplace-like operators in the superspace. It is evident that the gauge fixing functions (8) are covariant under background superfield transformations. The gauge fixing functions (8) can be considered as a superfield form of so-called \( R_\xi \)-gauge fixing which are ordinarily used in spontaneously broken gauge theories. An extension of \( R_\xi \)-gauge fixing conditions to \( \mathcal{N} = 1 \) superfield theories has been given in Ref. [28].

The gauge-fixing action corresponding to the functions (8) is constructed in the standard form

\[
S_{GF} = -\frac{1}{|\alpha|^2} \int d^8z (F^A F^A + b^A b^A)
\]

(9)

and depends on the extra parameter \( \alpha \). The quadratic part of the Faddeev-Popov action which is relevant for one loop is

\[
S_{ghost} = \int d^8z (\bar{c}^A c^A - c^A \bar{c}^A + \bar{c} B \bar{\lambda} \bar{X}^{BE} X^{EA} c^A + c^B \bar{\lambda} \bar{X}^{BE} X^{EA} c^A + b^A b^A),
\]

(10)

where \( X^{AB} = f^{ABC} \Phi^C, \ X^{AB} = f^{ABC} \bar{\Phi}^C \).

All one-loop contributions to the effective action are given in terms of the functional trace...
\( \text{Tr} \ln(\hat{H}), \) where the operator \( \hat{H} \) is the matrix of the second variational derivatives of the action \( S_2 \) in all quantum fields. The one-loop effective action in the model under consideration reads

\[
\Gamma[V, \Phi] = \frac{i}{2} \text{Tr} \ln \hat{H}_{\text{SYM}} + i \text{Tr} \ln \hat{H}_{\text{Hyper}} - \frac{i}{2} \text{Tr} \ln \hat{H}_{\text{ghost}}. \tag{11}
\]

It should be noted that operator \( \hat{H}_{\text{SYM}} \) contains the contributions from the \( \mathcal{N} = 1 \) vector and chiral multiplets forming \( \mathcal{N} = 2 \) gauge multiplet. The choice \( \lambda = \bar{\lambda} = \alpha \) in (8, 9) greatly simplifies all calculation because it diagonalizes the matrix \( \hat{H}_{\text{SYM}} \) and decoupled the contributions from the \( \mathcal{N} = 1 \) vector and chiral multiplets. However, we will keep the gauge parameters \( \lambda \) and \( \alpha \) arbitrary and investigate the dependence of the effective action on these parameters.

4 \( \mathcal{N} = 1 \) Kähler and Non-holomorphic \( \mathcal{N} = 2 \) Potentials

4.1 \( \mathcal{N} = 1 \) Kähler potential

In this section we study the form of the non-Abelian low-energy effective action \( \Gamma = \int d^8z K \) and its gauge dependence. It is known that in the non-Abelian case the Kähler potential cannot be written in the form \( \text{Im}(\Phi \mathcal{F}'(\Phi)) \) consistent with the rigid version of special geometry (see e.g. Refs. [5, 14]). The additional terms originate from a real function \( \mathcal{H}_0(\mathcal{W}, \bar{\mathcal{W}}) \) of the \( \mathcal{N} = 2 \) YM superfield strength \( \mathcal{W} \). The results obtained in the present paper are more general as compared with the ones obtained in Refs. [5, 12, 14] since here we have used here the more general and complicated gauges. To calculate these potentials, we consider the diagrams with external \( \Phi, \bar{\Phi} \) lines corresponding only to the constant field background. Such a choice of background superfields leads to a number of technical simplifications due to the absence of the background gauge field, which allows us to replace all background covariant derivatives with flat ones (i.e. \( \nabla \rightarrow \nabla \rightarrow \hat{D} \)). This provides a possibility of using the superspace projectors \( P_1 = \frac{1}{2} \hat{D}^2 \hat{D}^2, \ P_2 = \frac{1}{2} \hat{D}^2 \hat{D}^2 \), \( P_T = -\frac{1}{2} \hat{D} \hat{D} \hat{D} \) and \( \Pi_0 = P_1 + P_2 \) and simplifying the evaluations of the functional determinants (11).

The final result is a sum of three terms:

1) The hypermultiplet contribution to effective action

\[
K^{\text{Hyp}}_{\text{Hyper}} = \frac{-1}{(8\pi)^2} (\Phi \bar{\Phi}) \left( \ln \frac{\Phi^2 \bar{\Phi}^2}{16e^2 \Lambda^4} + s \ln \frac{1 + s}{1 - s} \right), \tag{12}
\]

where \( \Phi \bar{\Phi} \) denotes the scalar product in isospin space, and we have used the notation \( s = 1 - \frac{\Phi \bar{\Phi}}{\lambda^2 < 0} \).

2) The effective action \( \Gamma_{\text{SYM}} = \Gamma_V + \Gamma_{\text{GD}} \) induced by the \( \mathcal{N} = 2 \) vector multiplet contains the vector loop contribution

\[
K_V = \frac{1}{(4\pi)^2} \left( \Phi \bar{\Phi} \ln \frac{\Phi^2 \bar{\Phi}^2}{e^2 \Lambda^4} + (\Phi \bar{\Phi}) \ln t \right) \tag{13}
\]

\[
+ \sqrt{\Phi^2 \bar{\Phi}^2} \left[ \frac{t + 1}{2} \ln \frac{t + 1}{2} + \frac{t - 1}{2} \ln \frac{t - 1}{2} \right],
\]

where the notation \( t = \frac{\Phi \bar{\Phi}}{\sqrt{\Phi^2 \bar{\Phi}^2}} \) was introduced, plus

3) the gauge dependent contribution

\[
K_{\text{GD}} = \int \frac{dk^2}{(4\pi)^2} \ln \left( 1 + \frac{((\Phi \bar{\Phi})^2 - \Phi^2 \bar{\Phi}^2)}{(k^2 + \lambda \Phi \bar{\Phi}) (k^2 + \lambda \Phi \bar{\Phi})} \right)
\]

\[
\left[ - \frac{\lambda \bar{\lambda}}{2} + \alpha \left( \frac{\lambda \bar{\lambda}}{4k^2} (\Phi \bar{\Phi}) + \frac{\lambda + \bar{\lambda}}{2} - \frac{\alpha}{4} \right) \right] =
\]

\[
\int \frac{dk^2}{(4\pi)^2} \ln \left( \frac{(k^2 - e_1)(k^2 - e_2)(k^2 - e_3)}{k^2(k^2 + 1)^2} \right), \tag{14}
\]

which automatically vanishes for Abelian background fields \( \Phi \). This is the main result of the subsection. The dependence of the one-loop effective action on all gauge parameters is given by this expression. When \( \lambda = \bar{\lambda} = 0 \) the result (12, 13, 14) coincides with one given in Ref. [12]. The case \( \lambda = 0, \alpha = 1 \) is known as the Fermi gauge. The corresponding form of the Kählerian potential (12, 13, 14) was found in Ref. [14]. Result for Landau-DeWitt gauge is obtained with \( \alpha = 0, \lambda = \bar{\lambda} = 1 \). Note that (14) in the gauge \( \alpha = \lambda = \bar{\lambda} = 1 \), which can be naturally called the Fermi-DeWitt, two last terms in the first line in (13) are exactly cancelled by (14) while the first term (13) is doubled.
Using the simple integral in (14) we obtain
\[ K_{GD} = e_1 \ln(-e_1) + e_2 \ln(-e_2) + e_2 \ln(-e_2), \] (15)
where \( e \)'s are the roots of the polynomial of \( k^2 \) under the logarithm in (14).

4.2 \( \mathcal{N} = 2 \) non-holomorphic potential

In previous subsection we have found the one-loop Kähler effective potential \( K(\Phi, \bar{\Phi}) \) induced by both \( \mathcal{N} = 2 \) vector multiplets and hypermultiplets. As has been mentioned in Refs. [5, 14], the Kähler potential in the non-Abelian case determines not only by the holomorphic function \( \mathcal{F} \). Additional terms originate from a real function \( \mathcal{H}(\mathcal{W}, \mathcal{W}) \) of the \( \mathcal{N} = 2 \) Yang-Mills superfield strength \( \mathcal{W} \), which is integrated over full \( \mathcal{N} = 2 \) superspace. Comparison the last term in decomposition (7) with the Kähler potential leads to
\[ K(\Phi, \bar{\Phi}) = \Phi^A \mathcal{F}_A + \Phi^2 (\Phi^A \mathcal{H}_A) - (\Phi \bar{\Phi})(\Phi^A \mathcal{H}_A), \] (16)
where
\[ \Phi^A \mathcal{H}_A = 0, \quad \Phi^A \mathcal{H}_A = -\frac{2 \Phi^2}{(\Phi \bar{\Phi})} s^2 \partial \mathcal{H} / \partial s^2. \]

It is well known that the \( \beta \)-function and the axial anomaly arise exactly from the holomorphic potential \( \mathcal{F} \). This fact gives us a unique recipe for extracting the contributions from the Kähler potential, which can be associated with holomorphic and non-holomorphic potentials respectively. Using the expressions (12) and (13) and the reconstruction formula (16), ones find, in accordance with Ref. [5], the contributions to the holomorphic potential \( \mathcal{F}(\mathcal{W}) \) and to the non-holomorphic potential \( \mathcal{H}(\mathcal{W}, \mathcal{W}) \) depending on the \( \mathcal{N} = 2 \) superfield strength \( \mathcal{W} \):
\[ \mathcal{F}^{\text{fund}}_{\text{Hyper}} = -\frac{1}{(8\pi i)^2} \mathcal{W}^2 \ln \frac{\mathcal{W}^2}{e^2 A^2}, \] (17)
\[ \mathcal{F}_{\text{Vector}} = \frac{1}{(4\pi i)^2} \mathcal{W}^2 \ln \frac{\mathcal{W}^2}{e^2 A^2}, \] (18)
\[ \mathcal{H}_{\text{Hyper}} = \frac{1}{(16\pi i)^2} \ln^2 \frac{1 + s}{1 - s}, \] (19)

\[ \mathcal{H}_{\text{Vector}} = \frac{1}{(8\pi i)^2} \left( \text{Li}_2(1 - t^2) - 2 \ln t + \frac{1}{2} \ln t - \frac{1}{2} \right) \] (20)

Our further aim is to obtain the off-shell gauge-dependent contribution to \( \mathcal{H} \) from the gauge-dependent part of the full Kähler potential. In this case Eq. (16) is written in the form
\[ -2s^2(1 - s^2) \frac{d\mathcal{K}_{GD}}{ds^2} = \mathcal{K}_{GD}(s^2), \] (21)
where \( \mathcal{K}_{GD} \) was introduced in Eq.(14), \( s^2 = 1 - 1/t^2 \) and \( t = \frac{W}{\sqrt{2|\mathcal{F}|}} \), \( t \in [0, 1] \). It has already been noticed that \( \mathcal{K}_{GD} = 0 \) at \( s^2 \to 0 \) and therefore \( \mathcal{H}_{GD} \) vanishes on-shell.

We see the holomorphic potential \( \mathcal{F} \) is gauge independent. The whole dependence on the gauge-fixing parameters is concentrated in the term \( \mathcal{H}_{GD} \) of the non-holomorphic potential \( \mathcal{H} \). Let us present (14) as a formal power series. Eq. (15) is nothing but a determination of a symmetrical function via the polynomial roots. According to the fundamental theorem in theory of symmetrical functions (see e.g. Ref. [29]) “any entire rational symmetrical function can be uniquely rewritten as an entire rational function of elementary symmetrical functions” (i.e. coefficients of the polynomial).

To represent (15) as an entire rational function, we expand the logarithms into a formal power series
\[ \mathcal{K}_{GD} = -\sum_{n=1}^{\infty} \frac{1}{n} S_n, \] (22)
where the power symmetrical functions of the roots \( e_1, e_2, e_3 \) of the form
\[ S_n = e_1(1 + e_1)^n + e_2(1 + e_2)^n + e_3(1 + e_3)^n \] (23)
has been used. Using Newton’s classical recursion formulae, we can uniquely express \( S_n \) in terms of elementary symmetrical functions. It is well known that the roots \( e_i \) of an algebraic equation are always satisfy the Vieta relations. For the roots of the polynomial, which appears from the numerator in the logarithm of (14) we have
\[ -e_1 e_2 e_3 = g_3, \quad e_1 e_2 + e_2 e_3 + e_1 e_3 = g_2, \] (24)
\[ e_1 + e_2 + e_3 = -2, \]
where elementary symmetrical functions are given from (14, 24) as 
\[ g_2 = 1 + s^2(-\frac{1}{2} + \gamma(1 - \frac{1}{4})), \quad g_3 = s^2 \frac{\gamma}{4}. \]
Multiplying Eq. (23) by \( e_1 + e_2 + e_3 \) and using identities (24) we obtain the recursion relation
\[ S_{n+1} - S_n - (1 - g_2)S_{n-1} + (1 - g_2 + g_3)S_{n-2} = 0. \]  
(25)

Using this relation, one can evaluate any \( S_n \) step by step. One can check that \( S_n \sim s^2 \) for any \( n \), and each \( S_n \) includes \( g_3 \) linearly. It allows one to simplify integration in Eq. (21).

Moreover, Waring’s well-known formulae (see, e.g. [29]) allow to express \( S_n \) for any \( n \) directly in terms of \( g_2, g_3 \). In order to get all \( S_n \), it is very useful to introduce a generating function defined from (14, 24) as
\[ G(\tau) = \sum_{k=1}^{\infty} \tau^{k-1} S_k, \]  
(26)
then any \( S_n \) can be found with help of differentiations of the generating function \( G \) with respect to \( \tau \). It also allows us to express the general term of the sequence \( S_n \) in terms of symmetrical functions \( g_2 \) and \( g_3 \) instead of the roots \( e_i \). Since the functions \( g_2, g_3 \) are known from the integral (14), we can avoid finding the roots \( e_i \) for analysis \( H_{GD} \) at all. The generating function \( G \) satisfies an algebraic equation which can be derived by multiplying the recursion relation by \( \tau^k \) and summing over powers \( k \). The solution to this equation is
\[ G(\tau) = \frac{2(1 - g_2)(1 - 2\tau + \tau^2) - 3g_3\tau + 2g_3\tau^2}{1 - \tau - (1 - g_2)\tau^2 + (1 - g_2 + g_3)\tau^3}. \]  
(27)
As a result we obtain an expansion of \( K_{GD} \) in terms of elementary symmetrical functions \( g_2, g_3 \):
\[ K_{GD} = -\sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{d\tau} \right)^{n-1} G(\tau) |_{\tau=0}. \]  
(28)

Now, it is useful to introduce the new parameters \( g = -1/2 + (\gamma/2 - 1)^2, \quad g_3 = \gamma/4, \quad p = g + g_3, \quad u = 1 - s^2 \). Using the binomial formula for derivatives of the generating function (27) in (28), we rewrite the equation (21) in the following form
\[ -2u \frac{dH_{GD}}{du} = \sum_{k=0}^{\infty} \frac{1}{(k+3)!} \times \]  
(29)
\[ \times (4g - g_3(k + 1)(k + 5)) \left( \frac{d}{d\tau} \right)^k Y |_{\tau=0}, \]  
where \( Y^{-1} = 1 - \tau - g\tau^2 + pr^3 + u(g\tau^2 - pr^3) \). It is useful to extract, in the right hand side of Eq. (29), the powers of \( u \) and to rewrite this relation in form of double sum. This expression allows one to find \( H_{GD} \) as a series with a coefficients of each given power of \( u \) depending on elementary symmetrical functions. Hence, we finally can rewrite (21) in terms of elementary symmetrical functions. We see that the right hand side (29) can be written via rational functions for any given choice of gauge parameters. For some partial choice of gauge parameters, arbitrary term of series can be found exactly.

Let’s consider the Landau-DeWitt gauge in more detail. At such a choice, \( Y^{(k)} \) in (29) becomes simple enough
\[ Y^{(k)} = k! \left( \frac{1}{1 - a^2} - \frac{(-a)^{k+1}}{2(1 + a)} - \frac{a^{k+1}}{2(1 - a)} \right), \]  
(30)
\[ a^2 = \frac{s^2}{2}, \]  
and the general term in right side (29) can be exactly found. Finally, Eq. (29) becomes
\[ (1 - 2a^2) \frac{dH_{GD}}{da} = \frac{1 - a}{a} \ln(1 - a) + \frac{1 + a}{a} \ln(1 + a) \]  
(31)
and we obtain \( H_{GD} \) by integration
\[ 2(4\pi)^2 H_{GD} = \ln(2) \ln(1 - s^2) + \]  
\[ + \frac{1}{\sqrt{2}} \ln \left( \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) \ln(1 - s^2) - \text{Li}_2 \left( \frac{s^2}{2} \right) + \]  
\[ + \sqrt{2} - 1 \left[ \text{Li}_2 \left( \frac{s - 1}{\sqrt{2} - 1} \right) + \text{Li}_2 \left( -\frac{s + 1}{\sqrt{2} - 1} \right) \right] + \]  
\[ + \sqrt{2} + 1 \left[ \text{Li}_2 \left( \frac{s + 1}{\sqrt{2} + 1} \right) + \text{Li}_2 \left( \frac{1 - s}{\sqrt{2} + 1} \right) \right]. \]  
(32)
We emphasize that expressions (19), (20) and (32) are exact results within the one-loop approximation. Of course, they can be expanded in series in two limit cases: \( t \to 1 \) and \( t \to 0 \). Such a behavior is not unusual and it looks like quite similar to the well-known exactly solvable model in an effective field theory, namely the Euler-Heisenberg effective action. One can point out some more property of the Euler-Heisenberg effective action at small mass (strong external field): it possesses
by logarithmic branch point as well as \( \mathcal{H}_{\text{GD}}, \mathcal{H}_{\text{Vector}} \), while at large mass (weak external field) there exists an asymptotic series expansion in inverse powers of mass.

We have shown that the gauge-dependent part of the off-shell effective action can be found with an arbitrary level of accuracy and at any choice of the gauge fixing parameters. The form of the non-holomorphic effective potential has an essential arbitrariness due to its explicit gauge dependence. In particular, this fact leads to the ambiguous definition of

\[
R_{ABCD}(W^{A\alpha} W^{C\beta} \bar{W}^{B\dot{\alpha}} \bar{W}^{D\dot{\beta}})
\]

term from Eq. (7), which should reproduce the leading term in the expansion of the non-Abelian analog of the Born-Infeld action (see, e.g. [21]). The structure of the tensor \( R_{ABCD} \) is cumbersome enough. In addition, we point out that the symmetrized trace \((F^+)^2(F^-)^2/\phi^2 \bar{\phi}^2\), determining the full set of \( F^4 \)-terms in the effective action, also contains the various contractions \( \phi^A, \bar{\phi}^A \) with \( F^4 \). The existence a large class of gauge theory operators, which correspond to supergravity modes and contain nontrivial extra factors (depending on \( \phi^A, \bar{\phi}^A \), in the non-Abelian Born-Infeld action was discussed in Ref. [21].

To conclude this subsection, we note that, unlike the Abelian case, \( \mathcal{N} = 2 \) supersymmetry itself can not uniquely fix a form of next-to-leading term in the effective action because of its explicit gauge dependence.

5 Complete \( \mathcal{N} = 4 \) structure of the low-energy effective action in \( \mathcal{N} = 4 \) SYM theories

The \( \mathcal{N} = 4 \) SYM theory, being maximally extended rigid supersymmetric model, possesses the remarkable properties on classical and quantum levels. The corresponding quantum theory is finite, scale independent and superconformally invariant. The exact low-energy quantum dynamics of this model is described by a non-holomorphic effective potential [6-9]. The explicit form of the non-holomorphic potential for the \( SU(N) \) gauge group spontaneously broken down to \( U(1)^{N-1} \) is given by the expression

\[
\mathcal{H}(W, \bar{W}) = c \sum_{i<j} \ln \left( \frac{W^i - W^j}{\Lambda} \right) \times \quad (33)
\]

\[
\times \ln \left( \frac{\bar{W}^i - \bar{W}^j}{\Lambda} \right),
\]

where \( \Lambda \) is a scale and \( c = 1/(4\pi)^2 \) (see e.g. [8]). Expression (33) determines exact low-energy effective action in the \( \mathcal{N} = 2 \) gauge superfield sector.

We point out that the result (33) is so general that it can be obtained entirely on the symmetry grounds, from the requirements of scale independence and \( R \)-invariance only up to a numerical factor [6]. Moreover, the potential (33) gets neither quantum corrections beyond one loop nor instanton corrections [6]. These properties are very important for understanding low-energy quantum dynamics in \( \mathcal{N} = 4 \) SYM theory in the Coulomb phase. In particular, this effective potential provides description of sub-leading terms in the interaction between parallel D3-branes in superstring theory [21].

The complete exact low-energy effective action containing the dependence on both the \( \mathcal{N} = 2 \) gauge superfields and the hypermultiplets has been discovered [31] using a techniques of harmonic superspace [33]. It was shown that the algebraic restrictions on the full \( \mathcal{N} = 4 \) supersymmetric structure of the low-energy effective action are so strong that they allows us to restore the dependence of effective action on the hypermultiplets on the basis of the known non-holomorphic effective potential (33). As a result, the additional to (33) hypermultiplet dependent contribution, containing the on-shell \( W, \bar{W} \) and hypermultiplet \( q^a [32] \) superfields, has been obtained in the form

\[
\mathcal{L}_q = c \left\{ (X-1) \frac{\ln(1-X)}{X} + [\text{Li}_2(X) - 1] \right\}, \quad (34)
\]

\[
X = -\frac{q^a q_{ia}}{WW}. 
\]

The effective Lagrangian (34) together with the effective potential (33) define the exact \( \mathcal{N} = 4 \)
supersymmetric low-energy effective action in the
theory under consideration.

The effective Lagrangian (34) has been found in
Ref. [31] on the purely algebraic ground. It
would be extremely interesting to derive this La-
grangian in the framework of quantum field the-
ory. Here we just present such a derivation. To
be more precise, we discuss the calculations of the
one-loop effective action depending on both the
N = 2 gauge and the hypermultiplet background
fields using the formulation of the N = 4 SYM
theory in terms of N = 1 superfields [24, 27] and
exploring the derivative expansion technique in
N = 1 superspace [15]. It allows us to obtain the
exact coefficients by various powers of covariant
derivatives on a constant space-time background
belonging to the Cartan subalgebra of the gauge
group SU(n)

\[ W| = \Phi = \text{Const}, \quad D^i_\alpha |W| = \lambda^i_\alpha = \text{Const}, \quad (35) \]

\[ D^i_{(\alpha} D_{\beta)}|W| = f_{\alpha\beta} = \text{Const}, \quad D^{(i} D^{j)}|W| = 0, \]

where \( \Phi^I = \text{diag}(\Phi^1, \Phi^2, \ldots, \Phi^n) \), \( \sum \Phi^I = 0 \). An-
other approach to derivations of the Lagrangian
(34) can be developed in N = 2 harmonic superspace [32]. The structure of the one- and two-loop
low-energy effective actions in N = 2 supersym-
metric models is also studied in Ref. [35].

The main technical feature used in the given
paper consists in the background covariant gauge-
fixing multi-parametrical conditions (8). Since
the Abelian background is a solution of the equa-
tions of motion, we won’t worry about the choice
of the gauge-fixing parameters. It is therefore
convenient to choose the gauge-fixing which ear-
lier was named the Fermi-DeWitt gauge: \( \alpha = \lambda = 1 \). The choice of the gauge parameters allows us
to avoid the calculation problems with mixed
loops containing vector-chiral superfield propaga-
tors circulating along the loops.

The N = 4 “on-shell” multiplet can be ob-
tained by combining three N = 1 chiral super-
fields and one N = 1 vector superfield [24]. In
this description an SU(3) \( \otimes U(1) \) subgroup of
the SU(4) R-symmetry group is manifest. The form
of the N = 1 supersymmetric action in the chiral
representation is given by

\[ S = \frac{1}{g^2} \text{tr} \{ \int d^4x d^2\theta W^2 + \]

\[ + \int d^4x d^2\bar{\theta} \bar{\Phi} e^V \Phi e^{-V} + \]

\[ + \frac{1}{3!} \int d^4x d^2\bar{\theta} i c_{ijk} \Phi^i [\Phi^j, \Phi^k] + \]

\[ + \frac{1}{3!} \int d^4x d^2\bar{\theta} i c^{ijk} \bar{\Phi}^i [\bar{\Phi}^j, \bar{\Phi}^k] \} \quad (36) \]

It is convenient to introduce the new notations
\( \Phi^1 = \Phi, \Phi^2 = Q, \Phi^3 = \bar{Q} \) and to rewrite the two
last terms in (36) as follows

\[ i \int d^4x d^2\theta Q[\Phi, \bar{Q}] + i \int d^4x d^2\bar{\theta} Q[\Phi, \bar{Q}]. \]

After splitting each field into the background and
quantum parts (i.e. \( e^{\Omega_{\text{tot}}} = e^{\Omega_{\text{g}} V e^{\Omega}} \Phi \to \Phi +
\varphi, \Phi \to \bar{\Phi} + \bar{\varphi}, Q \to Q + q, \bar{Q} \to \bar{Q} + \bar{q}, \bar{Q} \to 
\bar{Q} + \bar{q}, \bar{Q} \to \bar{Q} + \bar{q} \)) we can rewrite the quadratic
part of the classical action (36), and (9) in form
which does not contain any \( V \Phi \) terms

\[ S_{(2)} = -\frac{1}{2} \sum_{I<J} \int d^4x d^2\theta (\mathcal{F} \mathcal{H} \mathcal{F}^\dagger + \]

\[ + \Omega (O_V - M) V) , \]

where \( \mathcal{F} = (\varphi, \varphi, q, \bar{q}, \bar{q}, q) \), \( \mathcal{F}^\dagger = (\varphi, \bar{\varphi}, q, \bar{q}, q, \bar{q})^T \),

\[ M_{IJ} = (\Phi_{\bar{I}J} \Phi_{\bar{J}I} + \bar{Q}_{\bar{I}J} Q_{IJ} + \bar{Q}_{\bar{I}J} \bar{Q}_{IJ}) , \quad (38) \]

\[ O_V = \Box - i W_{I}^\alpha \nabla_\alpha - i \bar{W}_{I}^\alpha \bar{\nabla}_\alpha, \]

where \( W_{I}^\alpha = W_{I}^\alpha - W_{-I}^\alpha, \bar{W}_{I}^\alpha = \bar{W}_{I}^\alpha - \bar{W}_{-I}^\alpha \)
are the background field strengths and \( \Phi_{\bar{I}J} = \Phi_{\bar{I}J} - 
\Phi_{I\bar{J}} \). Here \( \mathcal{H} \) denotes some \( 6 \times 6 \) matrix de-
pending on covariant derivatives and background
fields. The explicit form of this matrix and the
details of the calculation are given in Ref. [34].

According to the Faddev-Popov procedure, we also
need to introduce a gauge-compensating term
(10) in the action. The final step is integration
in the functional integral over all quantum super-
fields. It allows to write the standard represent-
tion for the one-loop effective action

\[ e^{\Omega} = \prod_{I<J} \text{Det}^{-1}(O_V - M) \text{Det}^{-1}(\mathcal{H}) \text{Det}^2(\mathcal{H}_{FP}) \]

Calculation of the functional trace leads to

\[ \Gamma_{\text{SYM}} = i \text{Tr} \ln(\Omega_V - M) + \]

\[ + 2i \text{Tr} \left( \ln(1 - \frac{M}{\square_+}) \nabla_+^2 \nabla_+^2 \right) + \]

\[ + 2i \text{Tr} \left( \ln(1 - \frac{M}{\square_-}) \nabla_-^2 \nabla_-^2 \right) , \quad (39) \]
where $\square_{\pm}$ are standard notation for $\nabla^2\nabla^2$ and $\nabla^2\nabla^2$. In the space of chiral and antichiral superfields these operators act as follows

$$\nabla^2\nabla^2 = \square_{\pm} = \square - i\bar{W}^\alpha\nabla_\alpha - \frac{i}{2}(\nabla \bar{W}),$$

$$\nabla^2\nabla^2 = \square_{\mp} = \square - i\bar{W}^\alpha\nabla_\alpha - \frac{i}{2}(\nabla \bar{W}).$$

Evaluation of $\text{Tr} \ln(H_{FP})$ leads to the following ghost contribution to the effective action

$$\Gamma_{FP} = -2i \left( \text{Tr} \ln(1 - \frac{M}{\square_{\pm}}) \frac{1}{\square_{\pm}} \nabla^2\nabla^2 \right) - 2i \left( \text{Tr} \ln(1 - \frac{M}{\square_{\mp}}) \frac{1}{\square_{\mp}} \nabla^2\nabla^2 \right), \quad (41)$$

which exactly cancels the second and third terms in (39). This surprising cancellation between the ghost and chiral fields contributions in the $\mathcal{N} = 4$ SYM theory effective action was firstly noted in [8].

After the functional trace calculation, the first term in (39) gives known result [15,26]

$$\Gamma = \frac{1}{8\pi^2} \int d^8z \int_0^\infty dt t e^{-t} \frac{W^2\bar{W}^2}{M^2} \omega(t\Psi, t\bar{\Psi}), \quad (42)$$

$$\omega(t\Psi, t\bar{\Psi}) = \frac{\cosh(t\Psi) - 1}{t^2\Psi^2} \times \frac{t^2(\Psi^2 - \bar{\Psi}^2)}{\cosh(t\Psi) - \cosh(t\bar{\Psi})},$$

$\Psi$ and $\bar{\Psi}$ are given by

$$\bar{\Psi}^2 = \frac{1}{M^2} \nabla^2W^2, \quad \Psi^2 = \frac{1}{M^2} \nabla^2\bar{W}^2. \quad (43)$$

One can show that the quantity $M = (\bar{\Phi}\Phi + Q\bar{Q} + \bar{Q}\bar{Q})$ denominators (42, 43) is invariant under the $R$-symmetry group of $\mathcal{N} = 4$ supersymmetry. The function $\omega$ introduced in (42) has the following expansion

$$\omega(x, y) = \frac{1}{2} + \frac{x^2y^2}{4 \cdot 5!} - \frac{5}{12 \cdot 7!} (x^4y^2 + x^2y^4) + \frac{1}{34500} (x^6y^6 + x^6y^2) + \frac{1}{86400} x^4y^4 + \ldots \quad (44)$$

Eq. (44) allows one to expand the effective action (42) in series in powers $\Psi^2, \bar{\Psi}^2$ as follows

$$\Gamma = \Gamma_{(0)} + \Gamma_{(2)} + \Gamma_{(3)} + \cdots. \quad (45)$$

where the term $\Gamma_{(n)}$ contains terms $c_{m,l}\Psi^{2m}\bar{\Psi}^{2l}$ with $m + l = n$.

The effective action (42) and its expansion (45) are given in terms of the $\mathcal{N} = 1$ superfields. Our next aim is to find a manifest $\mathcal{N} = 2$ form of each term in the expansion (45). For this purpose we extract from $M = 1$ form of the $X = -\Phi\bar{Q} + \bar{Q}\Phi$, which was defined in Eq. (34) by $M = \Phi\Phi(1 - X)$, and then expand the denominator $(1/M)^k$ from (42) in a power series over $X$. This expansion of $(1/M)^k$ and the reconstruction expressions

$$\nabla^4\ln W = \nabla^2 \left( \frac{W^\alpha W_\alpha}{\Phi^2} \right) + \ldots,$$

$$\left(\nabla^2\right)^2 \frac{1}{\sqrt{2M}} = \frac{2m(2m+1)W^\alpha W_\alpha}{\Phi^{2m}} + \ldots \quad (46)$$

allow us to obtain the first term in (45)(which is $\sim F^4$)

$$\Gamma_{(0)} = \frac{1}{(4\pi)^2} \int d^4z (\ln W \ln \bar{W} + \sum_{k=1}^\infty \frac{1}{k^2(k+1)} X^k), \quad (47)$$

where $X = \left(\frac{\Phi\bar{Q} + \bar{Q}\Phi}{W\bar{W}}\right)^n$ was defined in (34). The second term in (47) can be transformed to the form (34) using the power series for Euler’s dilogarithm function and we see that this term is just the effective Lagrangian (34) found in [31, 32].

The following terms in the expansion (45) can be calculated using expansion $(1/M)^k$ in $X$. Their $\mathcal{N} = 2$ form is reconstructed by taking into account (46). Using the same analysis, ones get the term $\Gamma_{(2)}$ in (45) in the form

$$\Gamma_{(2)} = \frac{1}{2(4\pi)^2} \int d^2z \Psi^2 \bar{\Psi}^2 (1 + \frac{1}{5!} \sum_{k=1}^\infty \frac{(k+5)(k+4)(k+1)}{(k+3)(k+2)} X^k). \quad (48)$$

Its $X$-independent part was given in [26]. Here the $\mathcal{N} = 2$ chiral combinations $\Psi^2 = \bar{W}^2\nabla^2\ln W$, $\Psi^2 = W^2\nabla^4\ln W$ are the scalars under $\mathcal{N} = 2$ superconformal group transformations. The sum in (48) can be transformed as follows

$$\sum_{k=1}^\infty \frac{(k+5)(k+4)(k+1)}{(k+3)(k+2)} X^k = \frac{1}{(1 - X)^2} + \frac{4}{(1 - X)^3}.$$
\begin{equation}
+6X - \frac{4X^2}{X^3} \ln(1-X) - 4X - 10\frac{X}{X^2} - 3.
\end{equation}

Applying the same procedure for the third term in (45) one obtains
\begin{equation}
\Gamma_3 = -\frac{5}{6(4\pi)^2} \int d^{12}z (\Psi^4 \bar{\Psi}^2 + \Psi^2 \bar{\Psi}^4) \times (49)
\times \frac{1}{7!} \sum_{k=1}^{\infty} (k + 7)(k + 6)(k + 1)X^k,
\end{equation}

where the sum in right hand side is
\begin{equation}
\sum_{k=1}^{\infty} (k + 7)(k + 6)(k + 1)X^k = 2X
\end{equation}

\begin{equation}
(56 - 116X + 84X^2 - 21X^3).
\end{equation}

Thus, we have found the hypermultiplet dependence of the contributions \(\Gamma_0, \Gamma_2\) and \(\Gamma_3\) to the known effective action [26] which depend on \(\mathcal{N} = 2\) vector multiplet. As result we obtained the complete \(\mathcal{N} = 4\) supersymmetric forms for the three first terms of expansion of the effective action (42) in power series in Abelian strength. It is evident, that such a reconstruction procedure can be applied to any term in the expansion (45).

The fourth term in (45) contains two parts. The first one is
\begin{equation}
\Gamma_{(4_1)} = \frac{1}{(4\pi)^2} \int d^{12}z (\Psi^2 \bar{\Psi}^6 + \Psi^6 \bar{\Psi}^2) \times
\end{equation}

\begin{equation}
\frac{12X}{(1-X)^6} \left(450 - 1545X + 2284X^2 - 1779X^3 + 720X^4 - 120X^5\right)
\end{equation}

and the second part is given as follows
\begin{equation}
\Gamma_{(4_2)} = \frac{1}{5 \cdot 6! (4\pi)^2} \int d^{12}z \Psi^4 \bar{\Psi}^4 \times
\end{equation}

\begin{equation}
\left(\frac{12(5X - 4)}{X^5}\right) \ln(1-X) -
\end{equation}

\begin{equation}
\frac{1}{5X^4(1-X)^6} \left(240 - 1620X + 4610X^2 - 7120X^3 + 6363X^4 - 4878X^5 + 6135X^6 - 7560X^7 + 5670X^8 - 2268X^9 + 378X^{10}\right)
\end{equation}

As a result, we see that the reconstruction procedure for the effective action of \(\mathcal{N} = 4\) SYM theory can be realized completely for any terms in the expansion (45), completing them by the corresponding terms containing the hypermultiplet superfields.

6 Summary

We have presented the general approach to evaluation of the one-loop effective action in \(\mathcal{N} = 2\) supersymmetric field theories formulated in terms of \(\mathcal{N} = 1\) superfields. The approach provides a calculation of the effective action by a series in supercovariant derivatives with the coefficients depending on the background superfields. The approach allows one to reproduce the known results on the one-loop holomorphic and non-holomorphic effective potentials depending on Abelian background strengths in \(\mathcal{N} = 2\) SYM theories. We have studied the structure of the low-energy effective action on non-Abelian background superfields using a parametrically dependent family of appropriate superfield \(R_t\)-gauges. For some values of the gauge parameters, the non-Abelian non-holomorphic effective potential is presented in an explicit form in terms of the the Euler dilogarithm function. We have applied this general approach to evaluation of the \(\mathcal{N} = 4\) supersymmetric low-energy effective action in \(\mathcal{N} = 4\) SYM theory. We have found an integral representation of the effective action for the constant Abelian background strength, including the dependence on both the \(\mathcal{N} = 2\) gauge multiplet and the hypermultiplet superfields. The four lowest terms of the effective action power expansion in the Abelian strength are given in an explicit form.

Acknowledgements

The work was supported in part by INTAS grant, INTAS-00-00254 and RFBR grant, project No 03-02-16193. I.L.Buchbinder is also grateful to RFBR-DFG grant, project No 02-02-04002 and to DFG grant, project No 436 RUS 113/669 and grant for Leading Russian Scientific Schools, project No 1252.2003.2 for partial support. The work of N.G.Pletnev and A.T.Banin was supported in part by RFBR grant, project No 02-02-17884. A.T.B and N.G.P are very grateful to the organizers of GRG11 for warm hospitality in Tomsk.

References

[1] E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov, S.M. Kuzenko, B.A. Ovrut, “Low-energy effective action in N=2 supersymmetric field theo-
[2] N. Seiberg, E. Witten, *Nucl. Phys.* **B426** (1994) 19; *Nucl. Phys.* **B431** (1994) 484.

[3] A. Klemm, W. Lerche, S. Theisen and S. Yankielowich, *Phys. Lett.* **B344** (1995); P. Argyres, A. Farraggi, *Phys. Rev. Lett.* **73** (1995) 3931.

[4] E. D’Hoker, D.H. Phong, “Lectures on supersymmetric Yang-Mills theory and integrable systems”, hep-th/9912271.

[5] M. Dine, N. Seiberg, *Phys. Lett.* **B409** (1997) 239.

[6] M. Dine, J. Gray, *Phys. Lett.* **B481** (2000) 427.

[7] I.L. Buchbinder, A. Yu. Petrov, *Phys. Lett.* **B482** (2000) 429.

[8] I.L. Buchbinder, M.T. Grisaru, M. Roček, *Mod. Phys. Lett.* **A13** (1998) 1623; I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, *Phys. Lett.* **B417** (1998) 61; E.L. Buchbinder, I.L. Buchbinder and S.M. Kuzenko, *Phys. Lett.* **B446** (1999) 216.

[9] D.A. Lowe, R. von Unge, *JHEP* **9811** (1998) 014.

[10] M. Dine, J. Gray, *Phys. Lett.* **B383** (1996) 415.

[11] I.L. Buchbinder, S.M. Kuzenko, J.V. Yarevskaya, *Nucl. Phys.* **B411** (1994) 665.

[12] A. Pickering, P. West, *Phys. Lett.* **B383** (1996) 53.

[13] I.L. Buchbinder, S.M. Kuzenko, J.V. Yarevskaya, *Nucl. Phys.* **B411** (1994) 665.

[14] N.G. Pletnev, A.T. Banin, *Phys. Rev. D60* (1999) 105017; A.T. Banin, I.L. Buchbinder, N.G. Pletnev, *Nucl. Phys.* **B598** (2001) 371.

[15] S.J. Gates, Jr., *Nucl. Phys.* **B238** (1984) 349.
[32] I.L. Buchbinder, E.A. Ivanov, A.Yu. Petrov, Nucl.Phys. B 653, 64 (2003).

[33] A.S. Galperin, E.A. Ivanov, S. Kalitzin, V.I. Ogievetsky, E.S. Sokatchev, Class.Quant.Grav. 1, 469 (1984); A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, E.S. Sokatchev, Class.Quant.Grav. 2, 601 (1985); ibid 613; A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev “Harmonic Superspace”, Cambridge Univ. Press, (2001).

[34] A.T. Banin, I.L. Buchbinder and N.G. Pletnev, “One-loop effective action for $\mathcal{N} = 4$ SYM theory in the hypermultiplet sector: leading low-energy approximation and beyond”, hep-th/0304046, to be appeared in Phys. Rev. D.

[35] S.M. Kuzenko, I.M. McArthur, Phys.Lett. B506, 140 (2001); B513, 213 (2002); JHEP 0305 015 (2003); “Low-energy dynamics in N=2 super QED: Two-loop approximation”, hep-th/0308136