Corrigendum: Cubic polynomials on Lie groups: reduction of the Hamiltonian system

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The purpose of this corrigendum is to replace lemma 6 on page 13 of the paper to guarantee the accuracy of other results derived from it, in particular, the discussion after remark 4 on page 15. In the original version, the result we prove does not allow us to conclude, as we claim, that the set of constants of the motion we identify can be used with the Lie–Cartan theorem.

The formulation of the lemma is misleading. Besides, we need the additional hypothesis that \( G \) is semisimple to be able to prove the correct statement. Therefore, both the statement and the proof should be replaced by the following.

Lemma 6. If the Lie group \( G \) is semisimple, then \( \{ l_j : j = 1, \ldots, n+1 \} \) is a set of functionally independent functions on an open dense subset of \( O_\eta \times g \times g^* \).

Proof. In the proof and for the sake of simplicity, we identify \( \eta \in g^* \) with an element of \( g \) via the Riemannian metric. We shall also consider \( O_\eta \) to be the adjoint orbit defined by a regular element \( \eta \) in a Cartan subalgebra \( t \) of \( g \) and \( r \) be the rank of \( g \).

Consider the coordinate expression for the invariants, with respect to the natural basis taken from the orthonormal basis \( \{ A_i \}_{i=1,\ldots,n} \) of the Lie algebra \( g \):

\[
\begin{align*}
l_1 &= \sum_{j=1}^{n} y^j \theta_j(v_1, \ldots, v_{2m}) + \frac{1}{2} \sum_{j=1}^{n} (\xi_j)^2 \\
l_{i+1} &= \theta_i(v_1, \ldots, v_{2m}) + \sum_{j,k=1}^{n} C_{jk}^i y^j \xi_k,
\end{align*}
\]

where \( v_1, \ldots, v_{2m} \) are the variables in the orbit \( O_\eta \). The differentials of the invariants can be written as

\[
dl_1 = \sum_{\alpha=1}^{2m} \sum_{j=1}^{n} y^j \frac{\partial \theta_j}{\partial v_{\alpha}} \, dv_{\alpha} + \sum_{j=1}^{n} \theta_j \, dy^j + \sum_{j=1}^{n} \xi_j \, d\xi_j
\]
\[ dl_{i+1} = \sum_{\alpha=1}^{2m} \frac{\partial \theta}{\partial u_{\alpha}} \, dv_{\alpha} + \sum_{j,k=1}^{n} C_{j}^{k} \xi_{j} \, dy^{j} + \sum_{j,k=1}^{n} C_{j}^{k} y^{j} \, d\xi_{j}, \quad i = 1, \ldots, n. \]

We shall prove that \( dl_{1} \wedge dl_{2} \wedge \cdots \wedge dl_{n+1} \neq 0 \) in an open dense subset of \( O_{\eta} \times g \times g^{*} \). The coefficients of the above exterior product corresponding to the elements \( dv_{1} \wedge dv_{2} \wedge \cdots \wedge dv_{2m} \wedge d\xi_{1} \wedge \cdots \wedge d\xi_{n+1} \) are sums containing \( 2m \) terms, not depending on the variables \( \xi_{i} \), and \( r+1 \) terms, each one depending linearly on a different variable \( \xi_{i} \). The \( r+1 \) terms are given by the minors of order \( n \) of the matrix representing the linear map \( F \) from \( T_{\theta}O_{\eta} \times g \) into \( g \) that applies \( (Z, W) \) to \( i_{\epsilon_{\eta}}(Z) - adYW \), where \( i \) is the inclusion of \( O_{\eta} \) into \( g \). If we prove that the map \( F \) has full rank in an open dense subset of \( O_{\eta} \times g \), then the corresponding minor of order \( n \) of the matrix representation gives the non-vanishing term we are looking for.

In order to do so, let us recall the standard root space decomposition (see for instance [1]) for the complexified algebra \( g^{C} \):

\[ g^{C} = g^{C}_0 \oplus \left( \bigoplus_{\alpha \in \Delta} g^{C}_{\alpha} \right) \]

with respect to a Cartan subalgebra \( t^{C} \) (i.e., \( g^{C}_0 \) corresponds to the centralizer of \( t^{C} \) in \( g^{C} \) which is equal to \( t^{C} \) if the algebra is semisimple). The related vectors \( X_{\theta}, Y_{\theta} \in g \) such that \( [T, X_{\theta}] = \alpha(T)Y_{\theta} \) and \( [T, Y_{\theta}] = -\alpha(T)X_{\theta} \), for all \( T \in t \) and for each root \( \alpha \in \Delta \), induce the decomposition

\[ g = t \oplus \left( \sum_{\alpha \in \Delta_{+}} \mathbb{R}X_{\alpha} \oplus \mathbb{R}Y_{\alpha} \right) \]

and give a basis \( B^{l}_{\theta} \) of \( g \). Let us consider the tangent space \( T_{\theta}O_{\eta} = \{[\theta, A], A \in g\} \), for each \( \theta \in O_{\eta} \). Using the basis \( B^{l}_{\theta} \), it is possible to check that there exists an open dense subset of \( O_{\eta} \) defined by elements \( \theta \) such that \( T_{\theta}O_{\eta} \cap t = \{0\} \). Under this condition, it is possible to extend a basis \( B^{l}_{T_{\theta}O_{\eta}} \) of \( T_{\theta}O_{\eta} \), using a basis of \( t \), in order to obtain a basis \( B^{l}_{\theta} \) of \( g \). Now, we consider the basis \( B^{l}_{T_{\theta}O_{\eta}} \times B^{l}_{\theta} \) of \( T_{\theta}O_{\eta} \times g \) and the basis \( B^{l}_{\theta} \) of \( g \). It is clear that the matrix of the map \( F \) relatively to these bases has full rank for all \( \theta \in O_{\eta} \) such that \( T_{\theta}O_{\eta} \cap t = \{0\} \) and for all \( Y = T + \sum_{\alpha \in \Delta_{+}} (b_{\alpha}X_{\alpha} + c_{\alpha}Y_{\alpha}) \) with no null coefficients \( b_{\alpha} \) and \( c_{\alpha} \), for each \( \alpha \in \Delta_{+} \). Therefore, we proved that the map \( F \) has full rank in an open dense subset of \( O_{\eta} \times g \). This implies that there is an open dense subset of \( O_{\eta} \times g \times g^{*} \), where the functions \( \{l_{1}, \cdots, l_{n+1}\} \) are functionally independent.

This new version of the lemma guarantees the accuracy of the results contained in the last part of the original paper. \( \square \)

**Reference**

[1] Helgason S 1978 *Differential Geometry, Lie Groups, and Symmetric Spaces* (New York: Academic)
Cubic polynomials on Lie groups: reduction of the Hamiltonian system

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Abstract

This paper analyzes the optimal control problem of cubic polynomials on compact Lie groups from a Hamiltonian point of view and its symmetries. The dynamics of the problem is described by a presymplectic formalism associated with the canonical symplectic form on the cotangent bundle of the semidirect product of the Lie group and its Lie algebra. Using these control geometric tools, the relation between the Hamiltonian approach developed here and the known variational one is analyzed. After making explicit the left trivialized system, we use the technique of Marsden–Weinstein reduction to remove the symmetries of the Hamiltonian system. In view of the reduced dynamics, we are able to guarantee, by means of the Lie–Cartan theorem, the existence of a considerable number of independent integrals of motion in involution.

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1. Introduction

Riemannian cubic polynomials (RCP), also called Riemannian cubics, can be seen as a generalization of cubic polynomials in Euclidean spaces to Riemannian manifolds. The cubic polynomials on a Riemannian manifold are the smooth solutions of the fourth-order differential equation

\[ \frac{D^4x}{dt^4} + R \left( \frac{D^2x}{dt^2}, \frac{dx}{dt} \right) \frac{dx}{dt} = 0, \]

(1)

where \( D/dt \) denotes the covariant differentiation and \( R \) is the curvature tensor. Equation (1) is the Euler–Lagrange equation of a second-order variational problem with the Lagrangian
given by $\frac{1}{2}(D^2x/dt^2, D^2x/dt^2)$, where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric. This variational problem was first introduced in 1989 (see [31]) and explored from a dynamical interpolation perspective in 1995 (see [17]). Interesting points related to this subject have been developed in the last few years, namely a geometric theory surprisingly close to the Riemannian theory of geodesics (see [2–4, 12, 14–16, 19, 29, 30, 33, 34]). We recall, in particular, a result which says that if $V$ denotes the velocity vector field of a cubic polynomial $x$, then

$$I_1 = \frac{1}{2} \left\{ \frac{DV}{dt}, \frac{DV}{dt} \right\} - \left\{ \frac{D^2V}{dt^2}, V \right\}$$

is invariant along $x$. In Riemannian context, $I_1$ plays a role similar to the one played by the length of the velocity vector field in the theory of geodesics (see, for example, [16]). Recently, in [3, 29, 30, 33], the analysis of RCP from a variational point of view was carried out for locally symmetric manifolds and a second invariant was obtained:

$$I_2 = \left\{ \frac{D^2V}{dt^2}, \frac{D^2V}{dt^2} \right\} - \left\{ \frac{D^3V}{dt^3}, \frac{DV}{dt} \right\}$$

The analysis of RCP given in [3, 29] is qualitative, with special attention to the case of the Lie group $SO(3)$, where RCP correspond to Lie quadratics on the Lie algebra. The article [3] introduces a reduction of the RCP equation for this Lie group of rotations. In [29], some results on asymptotics and symmetries of cubics are proved for the particular case of the so-called null cubics on $SO(3)$. In [30], the author solves by quadratures the linking equation on $SO(3)$ and $SO(1, 2)$ of the Riemannian cubics. Finally, in [33] the author studies $n$th-order generalizations of RCP introduced in [14].

To our knowledge, the first Hamiltonian description of the RCP problem has been considered in [15] (made in collaboration with one of the authors). The present paper deals, for the case of arbitrary compact and connected Lie groups, with a different Hamiltonian description of the problem. Here we use a presymplectic approach to Pontryagin’s maximum principle inspired by some ideas of [9, 11, 18, 25]. Namely, we consider the intrinsic geometric approach used in [18, 25] for a first-order general optimal control problem, and similarly considered in [9] for time-dependent optimal control problems by using the jet bundle framework. In a similar way, reference [11] gives the geometric treatment of the Lagrangian dynamics with higher order constraints. The description of RCP (on an arbitrary manifold) using these geometric ideas was first presented in [4] by the authors of the present paper. Recently, in [5, 6], the authors have treated the particular situation of the dynamic control of the spherical free rigid body, a mechanical system with configuration manifold given by the Lie group $SO(3)$. The new contribution of this presymplectic formalism is to use the Lie group structure of the semidirect product of the Lie group and its Lie algebra, $G \times g$, which is the state space of the optimal control problem. This allows us to use classical results from [1, 10], adapted to this Lie group structure. Namely, we present the left trivialization of the Hamiltonian system, a set of equations which lives in the manifold given by the Cartesian product of the semidirect product mentioned above and the dual of its Lie algebra, $G \times g \times g^* \times g^*$

The main goal of this work is to reduce the degrees of freedom of the left trivialized Hamiltonian system. We first apply the symplectic point reduction theorem [26, 32] and then explore the reduced dynamics using a suitable symplectomorphism. The reduced Hamiltonian vector field lives in the manifold given by the Cartesian product of a co-adjoint orbit, the Lie algebra and its dual, $O_n \times g \times g^*$. Furthermore, some invariants along the extremal trajectories are characterized as a crucial point to develop, in a future work, a study of the integrability of the Hamiltonian system. In fact, using the Lie–Cartan theorem [8], we obtain an interesting result on the number of independent integrals of motion in involution.
The plan of the paper is as follows. Section 2 recalls some notes on compact Lie groups and fixes the notation used in the rest of the paper. Section 3 begins with the introduction of the optimal control problem of cubic polynomials and presents the corresponding presymplectic approach. After that, we provide the left trivialized description equivalent to the variational one [17] in a similar way to what happens in [15]. However, it is important to remark that our approach. After that, we provide the left trivialized description equivalent to the variational optimal control problem of cubic polynomials and presents the corresponding presymplectic frame and co-frame fields on $G$. Furthermore, the elements of $G$ are denoted by $x$ or $g$ and the maps $G \times G \rightarrow G$, $(x, g) \mapsto xg$ and $G \rightarrow G, x \mapsto x^{-1}$ are the multiplication and inversion operations for the Lie group $G$, respectively. Given $x, g \in G$, let $L_x : G \rightarrow G$ and $R_x : G \rightarrow G$ be, respectively, the left and right translations by $x$. The tangent of $L_x$ at $g$ is denoted by $T_gL_x$ and $T^*_gL_x$ represents its transpose. Recall the following definitions.

- The adjoint representation of the Lie group $G$ is denoted by $Ad$. It gives for each $x \in G$ an algebra automorphism defined by $Ad_x = T_e(R_{x^{-1}} \circ L_x)$.
- The adjoint representation of the Lie algebra $g$ is the tangent of $Ad$ at the identity $e$ and it is denoted by $ad$. For each $Y, Z \in g$, we have $ad_Y Z = [Y, Z]$.
- The map $Ad^* : G \rightarrow Aut(g^*)$ defined, for each $X \in g$, by $[Ad^*(x)](\xi) := Ad_x^* \circ \xi = \xi \circ Ad_{x^{-1}}$, for each $x \in G$ and $\xi \in g^*.$, is called the co-adjoint representation of $G$.
- The co-adjoint representation of $g$ is the map $ad^* : g \rightarrow Aut(g^*)$ defined, for each $Y \in g$ and $\xi \in g^*$, by $[ad^*(Y)](\xi) := -ad^*_Y \xi = -\xi \circ ad_Y$.

Since the Lie group is assumed to be connected and compact, we can guarantee the existence of a bi-invariant metric on $G$, which we shall denote by $\langle ., . \rangle$. This statement and the following result can be found for instance in [23].

**Theorem 1** [23]. If $G$ is a Lie group equipped with a bi-invariant metric, the metric connection $\nabla$ and the curvature tensor $R$ associated with that metric are given, respectively, by $\nabla_Y Z = \frac{1}{2}[Y, Z]$ and $R(Y, Z)W = -\frac{1}{4}\{[Y, Z], W\}$, where $Y, Z$ and $W$ are left invariant vector fields. Furthermore, the first equality above implies that $\langle [Y, Z], W \rangle = \langle Y, [Z, W] \rangle$.

In the course of this paper, we shall fix an orthonormal basis in the Lie algebra $g$. The corresponding dual basis is a basis of the dual space $g^*$. These two bases generate left invariant frame and co-frame fields on $G$, respectively. We assume the following notations.

- Let $Y$ be a curve in $g$ and $\xi$ a curve in $g^*$. We represent by $\dot{Y}$ (respectively, $\dot{\xi}$) the element of $g$ (respectively, $g^*$) which has components with respect to the basis of $g$ (respectively, $g^*$) mentioned above, given by the derivative of the components of $Y$ (respectively, $\xi$).
Given $\xi \in \mathfrak{g}^*$, the tangent vector identified with this co-vector by the Riemannian metric will be denoted by $X_\xi \in \mathfrak{g}$. That is, $\xi(Y) = \langle X_\xi, Y \rangle$, $\forall Y \in \mathfrak{g}$.

With the above notation, it is simple to verify that $\dot{X}_\xi = X_{\dot{\xi}}$ and $X_{\text{ad}^* Y} \xi = -\text{ad}_Y X_\xi$.

2.2. The tangent bundle Lie group and its left trivialization

**Lemma 1** [24]. The tangent bundle $T G$ is a Lie group with a group operation defined as the tangent prolongation of the original one on $G$. That is, the multiplication operation for $T G$ is defined by

$$(v_x, v_g) \in T_x G \times T_g G \mapsto -\rightarrow v_x v_g = T_x R_{x^{-1}} v_x + T_g L_{x^{-1}} v_g \in T_{xg} G$$

and the inversion is defined by

$$v_x \in T_x G \mapsto v_x^{-1} = -(T_x L_{x^{-1}} \circ T_x R_{x^{-1}}) (v_x) \in T_{x^{-1}} G.$$ 

Consider the semidirect product $G \times \mathfrak{g}$ of the Lie group $G$ and the Lie algebra $\mathfrak{g}$ regarded as an Abelian group, under the right action of $G$ on $\mathfrak{g}$, $(x, Y) \in G \times \mathfrak{g} \mapsto \text{Ad}^*_Y$.

**Lemma 2.** The semidirect product $G \times \mathfrak{g}$ is a Lie group whose underlying manifold is the Cartesian product $G \times \mathfrak{g}$ and group multiplication law,

$$(x, Y)(g, Z) = (xg, \text{Ad}^*_x Y + Z),$$

for $(x, Y), (g, Z) \in G \times \mathfrak{g}$. The inversion is defined as $(x, Y)^{-1} = (x^{-1}, -\text{Ad}^*_x Y)$.

The semidirect product structure considered here is a special case of the general one defined by a right representation of a Lie group on a vector space that may be found in works on semidirect products, particularly the ones on models of continuum mechanics and plasmas where it is convenient to work with right instead of left representations (see, for example, [22]).

**Proposition 1** [24]. The left trivialization of $T G$ determined by the map

$$\lambda : T G \rightarrow T_x G \times \mathfrak{g} \mapsto (x, T_x L_{x^{-1}} v_x)$$

allows us to write the Lie group diffeomorphism $T G \simeq G \times \mathfrak{g}$.

We introduce now some important notations used in the rest of the paper.

- The elements of the tangent bundle $T(G \times \mathfrak{g})$ are denoted by

$$(v, Y, U) \in T_{(x, Y)}(G \times \mathfrak{g}) = T_x G \times \{Y\} \times \mathfrak{g}.$$ 

- The second tangent bundle of $G$, $T^2 G$, can also be trivialized by using the map $\lambda$ and then realized as a bundle over $G \times \mathfrak{g}$, which is a subbundle of $T(G \times \mathfrak{g})$. We represent this bundle by $T^2 G$ and denote its elements as

$$(v_x, U) \in T_{(x, T_x L_{x^{-1}} v_x)}(G \times \mathfrak{g}) \simeq T_x G \times \mathfrak{g}.$$ 

- The elements of the cotangent bundle $T^* (G \times \mathfrak{g})$ are represented by

$$(\alpha_x, Y, \xi) \in T_{(x, Y)}^*(G \times \mathfrak{g}) = T_x^* G \times \{Y\} \times \mathfrak{g}^*.$$ 

4
In the previous statements, we consider \((x, Y) \in G \times g\). Throughout this paper, we will, for the sake of simplicity, occasionnally assume the identification between the elements of \(T_{(x, Y)}(G \times g)\) (respectively, \(T^*_{(x, Y)}(G \times g)\)) and elements of \(T_x G \times g\) (respectively, \(T^*_x G \times g^*\)).

According to the Lie group structure chosen in lemma 2, we easily compute

\[
T_{(e, 0)} L_{(x, Y)}(Z, U) = (T_x L_x Z, U + \text{ad}_Y Z) \quad (5)
\]

and

\[
\text{ad}_{Y, Z}(Y', Z') = (\text{ad}_Y Y', \text{ad}_Y Z' + \text{ad}_Z Y'), \quad (6)
\]

where \(x \in G\) and \(Y, Z, U, Y', Z' \in g\). Obviously, these formulas can be derived from the general known ones from the theory of semidirect products.

3. Hamiltonian system

The aim of this section is to give a Hamiltonian description of the optimal control problem of cubic polynomials on \(G\) based on some material published in [4], where we used a geometric formulation similar to the one developed in [11] for higher order constrained variational problems. The section begins with the introduction of the optimal control problem, where by means of the left translation on \(G\), the state space has been left trivialized to be \(G \times g\) instead of \(TG\). After that, we apply a presymplectic constraint algorithm and the result is a Hamiltonian system on a space symplectomorphic to \(T^*(G \times g)\). Using again a left trivialization, but now determined by left translation on the group \(G \times g\), we pass to a Hamiltonian description on \(G \times g \times g^* \times g^*\).

3.1. Optimal control problem

Considering the left trivialization (4) of \(TG\), the state space for our problem may be taken to be the semidirect product \(G \times g\) and the bundle of controls as the second tangent bundle \(T^2G\).

The optimal control problem of cubic polynomials on \(G\) consists in finding the \(C^2\) piecewise smooth curve \(\gamma : [0, T] \to T^2G\) with fixed endpoints in state space, minimizing the functional

\[
\int_0^T L(\gamma(t)) \, dt, \quad \text{with} \quad T \in \mathbb{R}^+ \text{ fixed},
\]

for \(L : T^2G \to \mathbb{R}\), the cost functional defined by

\[
L(v_x, U) = \frac{1}{2}(U, U) \quad (7)
\]

and satisfying the control system

\[
\frac{d}{dt} (\tau^1_2(\gamma(t))) = F(\gamma(t)), \quad (8)
\]

where \(\tau^1_2 : T^2G \to G \times g\) is the natural projection and \(F : T^2G \to T(G \times g)\) is the vector field along this projection defined by

\[
F(v_x, U) = (v_x, T_x L_x v_x, U), \quad (9)
\]

Note that, according to the notation set in subsection 2.2, a curve \(\gamma\) in \(T^2G\) is defined by means of three elements: a curve \(x\) in \(G\), a vector field \(Y_x\) along \(x\) (which can be seen as a curve
in $TG$ satisfying $\pi_2 \circ Y = x$, where $\pi_2 : TG \to G$ is the canonical projection) and a curve $U \in g$. So $y(t) = (Y_x(t), U(t)) \in T_{y(t)}G \times g$ and we have $\tau_1^2(y(t)) = (x(t), T_{y(t)}L_{\omega_1}Y_x(t))$. Consequently, using the appropriated basis of left invariant vector field on $G$ to develop the calculus, it is simple to prove that the control system (8) can be written as

$$\dot{x}(t) = Y_x(t), \quad \frac{dY_x}{dt}(t) = T_xL_{\omega_1}U(t),$$

(10)

which is a version of the control system presented in [13, 15].

### 3.2. Dynamics of the optimal control problem

The co-state space of our system is the cotangent bundle $T^*(G \times g)$. The dynamics of the control problem is described by a presymplectic system $(T, \Omega, H)$ whose total space is the bundle over $G \times g$ given by

$$T = T^*(G \times g) \times_{\alpha,g} \overline{T^*G}.$$  

(11)

The elements of this space are points in $T^*G \times \{Y\} \times g^* \times g$ denoted by $(\alpha_x, Y, \xi, U)$, where $(x, Y) \in G \times g$. Consider the canonical projections $pr_1 : T \to T^*(G \times g)$, $(\alpha_x, Y, \xi, U) \mapsto (\alpha_x, Y, \xi)$ and $pr_2 : T \to \overline{T^*G}$, $(\alpha_x, Y, \xi, U) \mapsto (T_xL_gY, U)$. The closed two-form is defined by the pull-back

$$\overline{\Omega} = (pr_1)^*\Omega_1,$$

(12)

with $\Omega_1$ denoting the canonical symplectic two-form on the space $T^*(G \times g)$. The Hamiltonian is defined by $H = \langle \langle pr_1, F \circ pr_2 \rangle \rangle - L \circ pr_2$, where $F$ and $L$ are defined by (7) and (9) and $\langle \langle \cdot, \cdot \rangle \rangle$ stands for the canonical duality product of vectors and co-vectors on $G \times g$. Then,

$$H(\alpha_x, Y, \xi, U) = (T_xL_g\alpha_x(Y) + \xi(U) - \frac{1}{2}L(U, U)).$$

(13)

The dynamical vector field of the system is the vector field $X_{\overline{\Omega}} : T \to TT$ solution of the dynamical system $i_{x_{\overline{\Omega}}} \overline{\Omega} = dH$.

Note that the optimal control problem is obviously regular and thus applying the geometric algorithm of presymplectic systems (see [20, 21]) to $(T, \overline{\Omega}, H)$, we obtain a symplectic system on the manifold $W_1 = \{(\alpha_x, Y, \xi, U) \in T : U = X_\xi\}$, where $X_\xi \in g$ is the tangent vector identified with the co-vector $\xi \in g^*$ by the Riemannian metric of $G$. Hence, $(W_1, \overline{\Omega}_{W_1}, \overline{\Omega}_{W_1})$ is a symplectic system, with $\overline{\Omega}_{W_1}$ and $\overline{H}_{W_1}$ being the restrictions to $W_1$ of (12) and (13), respectively. The map $f$ defined below gives us a diffeomorphism between the symplectic manifolds $(T^*(G \times g), \Omega_1)$ and $(W_1, \overline{\Omega}_{W_1})$:

$$f : (T^*(G \times g), \Omega_1) \longrightarrow (W_1, \overline{\Omega}_{W_1})$$

(14)

$$(\alpha_x, Y, \xi) \longmapsto (\alpha_x, Y, \xi, X_\xi).$$

So we have a symplectomorphism between the two manifolds (see [1], p 177). In this sense, we construct the Hamiltonian $H_1 := \overline{H}_{W_1} \circ f : T^*(G \times g) \to \mathbb{R}$. We obtain

$$H_1(\alpha_x, Y, \xi) = (T_xL_g\alpha_x(Y) + \frac{1}{2}\xi(X_\xi),$$

(15)

for each $(\alpha_x, Y, \xi) \in T^*_xY(G \times g)$, where $(x, Y) \in G \times g$. Furthermore, the existence of the symplectomorphism (14) allows us to conclude that (see [1], p 194) the study of the dynamical system defining the Hamiltonian vector field $X_{\overline{H}_{W_1}}$ associated with $\overline{H}_{W_1}$ is reduced to the study of the system

$$i_{x_{\overline{H}_{W_1}}} \Omega_1 = dH_1,$$

(16)

where the vector field $X_{\overline{H}_{W_1}} : T^*(G \times g) \to T(T^*(G \times g))$ is the push-forward of $X_{\overline{H}_{W_1}}$ by $f^{-1}$, $X_{\overline{H}_{W_1}} = (f^{-1})_*X_{\overline{H}_{W_1}}$. The integral curves of this vector field determine the solutions of the optimal control problem (see [9, 18]).
3.3. Left trivialization of the dynamics

Consider now the left trivialization of the cotangent bundle $T^*G$, determined by the diffeomorphism $\rho$ defined from $T^*(G \times g)$ to the space $G \times g^* \times g^*$ as

$$\rho(\alpha, Y, \xi) = (x, Y, T^*_G L_{\alpha, Y} (\alpha, Y, \xi)),$$

which using (5) gives $\rho(\alpha, Y, \xi) = (x, Y, T^*_G L_{\alpha} + \text{ad}^*_g \xi, \xi)$, for each $(\alpha, Y, \xi)$ in $T^*_G(G \times g)$, where $(x, Y) \in G \times g$. Observe that if $(x, Y, \mu, \xi) \in G \times g \times g^* \times g^*$, then $\rho^{-1}(x, Y, \mu, \xi) = (T^*_G L_{\xi}(\mu - \text{ad}^*_g \xi), Y, \xi) \in T^*_G(G \times g)$.

The left trivialization of Hamiltonian (15) is given by $H := H_1 \circ \rho^{-1}$. We easily conclude that

$$H(x, Y, \mu, \xi) = \mu(Y) + \frac{1}{2} \xi(X_\xi).$$

Since $\rho$ is a diffeomorphism, we can endow (see [1], p 177) $G \times g \times g^* \times g^*$ with a symplectic structure, as $\Omega = (\rho^{-1})^* \Omega_1$. Furthermore, $\rho$ is a symplectomorphism and (see [1], p 194) the Hamiltonian vector field $X_H$ defined by (16) may be left trivialized to $G \times g \times g^* \times g^*$ by considering the push-forward by $\rho$ of the Hamiltonian vector field associated with $H_1$, $X_H := \rho_* X_{H_1}$.

The proposition below leads us to the expression of $X_H$:

$$X_H(x, Y, \mu, \xi) = (T_L Y, X_\xi, 0, -\mu + \text{ad}^*_g \xi),$$

for each $(x, Y, \mu, \xi) \in G \times g \times g^* \times g^*$.

**Proposition 2.** The following set of differential equations describes the motions of the Hamiltonian system $(G \times g \times g^* \times g^*, \Omega, H)$:

$$\begin{align*}
\dot{x} &= T_L Y \\
\dot{Y} &= X_\xi \\
\dot{\mu} &= 0 \\
\dot{\xi} &= -\mu + \text{ad}^*_g \xi.
\end{align*}$$

**Proof.** Let $z = (x, Y, \mu, \xi)$ be an integral curve of $X_H$. In the following ([10], section A.3, example 3) the Hamiltonian equations on $G \times g \times g^* \times g^*$ (the left trivialization of the cotangent bundle of the Lie group $G \times g$) are called the Euler–Arnold equations and are given by

$$\begin{align*}
\left\{ \begin{array}{c}
\dot{x} = T_L y \\
\dot{Y} = \frac{\partial H}{\partial \mu}(z), \\
\dot{\mu} = -T_L^* L_{\mu}(Y), \\
\dot{\xi} = -\mu + \text{ad}^*_g \xi.
\end{array} \right.
\end{align*}$$

In this notation, $\partial H(z)/\partial \mu$ is regarded as an element of $T^*_G G$, $\partial H(z)/\partial Y$ as an element of $g^*$, and $\partial H(z)/\partial \mu$ and $\partial H(z)/\partial \xi$ as elements of $g$. Use (5) and (2.2) to rewrite the previous system as

$$\begin{align*}
\dot{x} &= T_L y \\
\dot{Y} &= \frac{\partial H}{\partial \xi}(z) + \text{ad}^*_g \frac{\partial H}{\partial \mu}(z) \\
\dot{\mu} &= -T^*_L L_{\mu}(Y) - \text{ad}^*_g \frac{\partial H}{\partial Y}(z) + \text{ad}^*_g \frac{\partial H}{\partial Y}(z) + \text{ad}^*_g \frac{\partial H}{\partial Y}(z) + \text{ad}^*_g \frac{\partial H}{\partial Y}(z)
\end{align*}$$
From the expression of the Hamiltonian function (17), we obtain
\[ \frac{\partial H}{\partial x} = 0, \quad \frac{\partial H}{\partial Y} = \mu, \quad \frac{\partial H}{\partial \mu} = Y \quad \text{and} \quad \frac{\partial H}{\partial \xi} = X_\xi. \]
Now, substitute these expressions into the above system, use the fact that \( \text{ad}^*_\xi \xi = 0 \) and the result follows. □

Remark 1. It will now be interesting to see how the dynamics described by (18) is related to the known variational approach of cubic polynomials. To proceed, we begin with the following remarks.

- First, write the last equation of (19) as an equation on the Lie algebra, using the identification of co-vectors and tangent vectors given by the Riemannian metric of \( G \) (see the end of subsection 2.1 for details on notation). We obtain
\[ \dot{X}_\xi = -X_\mu - \text{ad}^*_Y X_\xi. \]
- Differentiate the above equation and use the third equation of (19) to obtain
\[ \ddot{X}_\xi + \text{ad}^*_\dot{Y} X_\xi + \text{ad}^*_Y \dot{X}_\xi = 0. \]
Use the second equation of (19) to obtain...

We have just shown that each solution of the equations of Hamilton (19) gives rise to a solution of the equations
\[ Y = T_{\xi L x}^{-1} x, \quad \dot{Y} + [Y, \dot{Y}] = 0. \]
Conversely, the solutions of (20) satisfying \( \dot{Y} = X_\xi \) and \( \dot{X}_\xi + X_\mu + \text{ad}^*_Y X_\xi = 0 \) correspond to the solutions of (19).

Equations (20) are the Euler–Lagrange equations (1) that define the cubic polynomials on a Lie group, which were proved in [17] as an extension of the proof that had already been given in [31] for \( SO(3) \). (The proof uses some facts derived from theorem 1.)

4. Reduction of the Hamiltonian system

The purpose of this section is to study the symmetries of the Hamiltonian system \( (G \times g \times g^* \times g^*, \Omega, H) \) described in the previous section and use that to reduce the corresponding dynamics, eliminating degrees of freedom in the system. The idea is to apply the symplectic point reduction theorem (see [26] for the original references and [32] for full details in this subject) and carry out the appropriate interpretation of the reduced Hamiltonian system for the study of important questions as the integrability of the system. Namely, we shall focus our attention on the integrals of motion of the reduced Hamiltonian system.

4.1. Symplectic point reduced space

Let \( \phi \) be the smooth left action of the Lie group \( G \) on \( G \times g \times g^* \times g^* \) defined by
\[ \phi(g, (x, Y, \mu, \xi)) = (gx, Y, \mu, \xi), \]
for each \( g \in G \) and \((x, Y, \mu, \xi) \in G \times g \times g^* \times g^* \). The moment map of \( \phi \) is the map \( J : G \times g \times g^* \times g^* \rightarrow g^* \) defined, for each \((x, Y, \mu, \xi) \in G \times g \times g^* \times g^* \), by
\[ J(x, Y, \mu, \xi) = \text{Ad}^*_\mu (\mu - \text{ad}^*_Y \xi). \]

The action \( \phi \) can be seen as the left trivialization to \( G \times g \times g^* \times g^* \) of the cotangent lift of the action of \( G \) on \( G \times g \) given, for each \( g \in G \) and \((x, Y) \in G \times g \), by \((g, (x, Y)) \mapsto L_{g,0}(x, Y) = (gx, Y) \). Recall that every cotangent lift action is symplectic and has momentum map \( \text{Ad}^* \)-equivariant (see [1], p 283). So it is easy to verify the following statement:
(A) $\phi$ is a symplectic action with the momentum map $\text{Ad}^\ast$-equivariant.

Observe now that the action $\phi$ is proper since it is an action of a compact Lie group. Moreover, $\phi$ is obviously free and hence the symmetry algebra of every point in $G \times g \times g^\ast \times g^\ast$ is zero, which is equivalent to saying that every $\eta \in g^\ast$ is a regular value of the momentum map $J$.

Let $\eta \in g^\ast$. Consider the co-adjoint isotropy subgroup of $\eta$, defined by

$$G_\eta := \{ g \in G : \text{Ad}^\ast_g \eta = \eta \}$$

(23)

and also the level set $J^{-1}(\eta)$ of the momentum map $J$. Note that

$$J^{-1}(\eta) = \{ (x, Y, \mu, \xi) \in G \times g \times g^\ast \times g^\ast : \mu = \text{Ad}^\ast_x \eta + \text{ad}^\ast_x \xi \}.$$  

(24)

Because $\phi$ is a symplectic $G$-action on the symplectic manifold $G \times g \times g^\ast \times g^\ast$ and $\eta \in g^\ast$ is a regular value of $J$, we see that $J^{-1}(\eta)$ is a submanifold of $G \times g \times g^\ast \times g^\ast$. Furthermore, as a consequence of $J$ being $\text{Ad}^\ast$-equivariant, we easily prove that $J^{-1}(\eta)$ is $G_\eta$-invariant. The comments now exposed allow us to conclude that $G_\eta$ acts on $J^{-1}(\eta)$ and that the orbit space

$$(G \times g \times g^\ast \times g^\ast)_\eta := J^{-1}(\eta)/G_\eta$$

(25)

is well defined. The action of $G_\eta$ on $J^{-1}(\eta)$ is obtained by restriction of $\phi$ to subgroup (23) and to the $G_\eta$-invariant submanifold (24). It turns out that the action $\phi$ is proper and free and that by definition $G_\eta$ is a closed subgroup of $G$; thus (see [32], p 60).

(B) the action of $G_\eta$ on $J^{-1}(\eta)$ is proper and free.

This result guarantees that the orbit space (25) is a smooth manifold and that the corresponding projection map is a surjective submersion.

Since conditions (A) and (B) are satisfied, we are able to apply the symplectic point reduction theorem. The theorem states the following:

The reduced space $(G \times g \times g^\ast \times g^\ast)_\eta$ has a unique symplectic structure $\Omega_\eta$ characterized by the identity $\pi_\eta^\ast \Omega_\eta = i^\ast \Omega$, where $i_\eta$ is the canonical inclusion from $J^{-1}(\eta)$ to $G \times g \times g^\ast \times g^\ast$ and $\pi_\eta$ is the projection of $J^{-1}(\eta)$ onto $(G \times g \times g^\ast \times g^\ast)_\eta$.

The symplectic manifold $((G \times g \times g^\ast \times g^\ast)_\eta, \Omega_\eta)$ is called the symplectic point reduced space at $\eta$.

Let us now explore in more detail the reduction obtained. More specifically, we will interpret the symplectic point reduced space in a strategic way to conduct further studies. In what follows, we shall adopt the notation $\phi$ for the above $G_\eta$-action on $J^{-1}(\eta)$. First, we note that from (24), the submanifold $J^{-1}(\eta)$ is diffeomorphic to the semidirect product $G \times g \times g^\ast$ (of the Lie group $G \times g$ and the vectorial space $g^\ast$) through the diffeomorphism

$$\Upsilon_\eta : \quad G \times g \times g^\ast \longrightarrow J^{-1}(\eta)$$

$$(x, Y, \xi) \quad \longmapsto \quad (x, Y, \text{Ad}^\ast_x \eta + \text{ad}^\ast_x \xi, \xi).$$

(26)

Consider the $G_\eta$-action on $G \times g \times g^\ast$ given, for each $g \in G_\eta$ and $(x, Y, \xi) \in G \times g \times g^\ast$, by $g \cdot (x, Y, \xi) = (gx, Y, \xi)$ and consider also the corresponding orbit space $G \times g \times g^\ast/G_\eta$.

Lemma 3. Diffeomorphism (26) induces a new diffeomorphism

$$\bar{\Upsilon}_\eta : (G \times g \times g^\ast)/G_\eta \longrightarrow (G \times g \times g^\ast \times g^\ast)_\eta,$$

(27)

which maps the $G_\eta$-orbit of the element $(x, Y, \xi) \in G \times g \times g^\ast$ onto the $G_\eta$-orbit of the element $(x, Y, \text{Ad}^\ast_x \eta + \text{ad}_x \xi, \xi) \in J^{-1}(\eta)$.
Proof. We have just to prove that $\mathcal{T}_\eta$ is equivariant for the $G_f$-action $\phi$ on $J^{-1}(\eta)$ and the $G_h$-action on $G \times g \times g^*$ described above. Indeed, if $g \in G_h$ and $(x, Y, \xi) \in G \times g \times g^*$, then $\mathcal{T}_\eta(g \cdot (x, Y, \xi)) = (gx, Y, Ad^*_\eta \eta + \text{ad}^*_\xi \xi) = (gx, Y, Ad^*_\eta \eta + \text{ad}^*_\xi \xi, \xi) = \phi_\eta(\mathcal{T}_\eta(x, Y, \xi))$. \hfill\Box

Proposition 3. The point reduced space $(G \times g \times g^*)_\eta$ is diffeomorphic to the space $O_\eta \times g \times g^*$, where $O_\eta$ denotes the co-adjoint orbit of the element $\eta$.

Proof. It is clear that the map $\tilde{\psi}_\eta : (G \times g \times g^*) / G_h \rightarrow O_\eta \times g \times g^*$, which takes a $G_h$-orbit of an element $(x, Y, \xi) \in G \times g \times g^*$ to a current point $(Ad^*_\eta \eta, Y, \xi) \in O_\eta \times g \times g^*$, is a diffeomorphism. Hence, one constructs the map $\tilde{\psi}_\eta := \tilde{\psi}_\eta \circ \tilde{\psi}_\eta^{-1}$, that is,

$$
\tilde{\psi}_\eta : (G \times g \times g^*) \rightarrow O_\eta \times g \times g^*
\end{equation}

$$

which gives us the diffeomorphism. \hfill\Box

The result from the previous proposition allows us to conclude now that $\tilde{\psi}_\eta$ is a symplectomorphism, with $\tilde{\Omega}_\eta = (\tilde{\psi}_\eta^{-1})^* \Omega_\eta$ being the symplectic structure on $O_\eta \times g \times g^*$.

To conclude, it is useful to note that the map $\psi_\eta := \tilde{\psi}_\eta \circ \pi_\eta : J^{-1}(\eta) \rightarrow O_\eta \times g \times g^*$ is such that

$$
\psi_\eta(x, Y, Ad^*_\eta \eta + \text{ad}^*_\xi \xi, \xi) = (Ad^*_\eta \eta, Y, \xi).
$$

(29)

Furthermore, $\psi_\eta$ is surjective since $\tilde{\psi}_\eta$ is bijective and the projection $\pi_\eta$ is surjective.

Besides the reduction of the phase space, the symplectic point reduction theorem has a dynamic counterpart, which will be addressed in the following subsection.

4.2. Reduction of the dynamics

We proceed with the analysis of the reduction of the dynamics of the Hamiltonian system of cubic polynomials $(G \times g \times g^*, \Omega, H)$ described in subsection 3.3. We will first present the natural reduction of dynamics that comes from the symplectic point reduction theorem. Then we shall perform this reduction as a dynamics on a Hamiltonian system on $(O_\eta \times g \times g^*, \tilde{\Omega}_\eta)$ in the context of the previous subsection, that is, using diffeomorphism (28).

Consider the Hamiltonian $H$ given by (17) and the associated Hamiltonian vector field $X_H$ defined by (18). Note that $H$ is invariant under the $G$-action defined by (21). The symplectic point reduction theorem allows us to conclude the following:

The flow $f_t$ of the Hamiltonian vector field $X_H$ induces a flow $f^H_t$ on the reduced space $(G \times g \times g^*)_\eta$ defined by $\pi_\eta \circ f_t \circ \eta = f^H_t \circ \pi_\eta$. The vector field generated by the flow $f^H_t$ is Hamiltonian with the associated reduced Hamiltonian function $H_\eta$ defined uniquely by $H_\eta \circ \pi_\eta = H \circ \eta$. Furthermore, the vector fields $X_H$ and $X_{H_\eta}$ are $\pi_\eta$-related.

The triple $((G \times g \times g^*)_\eta, \tilde{\Omega}_\eta, H_\eta)$ is called the reduced Hamiltonian system. We are interested now in characterizing the corresponding system on $(O_\eta \times g \times g^*, \tilde{\Omega}_\eta)$. Namely, we shall determine the expression of the reduced Hamiltonian vector field when interpreted as a vector field on $O_\eta \times g \times g^*$ regarding the description given at the end of the previous subsection. To effect this, one follows the steps below.

From now on, wherever there is no confusion, we will denote an element $Ad^*_\eta \eta \in O_\eta$ by $\theta, \theta := Ad^*_\eta \eta$. Introduce the Hamiltonian function $h := H_\eta \circ \tilde{\psi}_\eta^{-1} : O_\eta \times g \times g^* \rightarrow \mathbb{R}$, with $\tilde{\psi}_\eta$ defined by (28).
Lemma 4. For each \((\theta, Y, \xi) \in \mathcal{O}_\eta \times \mathfrak{g} \times \mathfrak{g}^*\), we have

\[ h(\theta, Y, \xi) = \theta(Y) + \frac{1}{2} \xi(X_\xi). \] (30)

Proof. Since the function \(\varphi\) defined by (29) is surjective, we know that the element \((x, Y, \theta + \text{ad}^0_\eta \xi, \xi) \in J^{-1}(\eta)\) is such that \(\varphi_\eta(x, Y, \theta + \text{ad}^0_\eta \xi, \xi) = (\theta, Y, \xi)\). Thus, \(h(\theta, Y, \xi) = (H_\eta \circ \pi_\eta) \circ (x, Y, \theta + \text{ad}^0_\eta \xi, \xi)\). But from the definition of \(H_\eta\), we have \(H_\eta \circ \pi_\eta = H \circ \iota_\eta\), so use (17) and the result follows.

Since \(\bar{\varphi}_\eta\) is a symplectomorphism, we see that the Hamiltonian vector field \(X_\eta\) associated with \(H_\eta\) is such that \(X_\eta \circ \bar{\varphi}_\eta = T \bar{\varphi}_\eta \circ X_{H_\eta}\). Then, \(X_\eta \circ \bar{\varphi}_\eta \circ \pi_\eta = T \bar{\varphi}_\eta \circ X_{H_\eta} \circ \pi_\eta\), that is, \(X_\eta \circ \varphi_\eta = T \bar{\varphi}_\eta \circ X_{H_\eta} \circ \pi_\eta\), where \(\varphi_\eta\) is the function defined by (29). Now, use the fact that \(X_{H_\eta}\) and \(X_{H_\eta} \circ \pi_\eta\) are \(\pi_\eta\)-related, that is, \(T \pi_\eta \circ X_{H_\eta} \circ \iota_\eta = X_{H_\eta} \circ \pi_\eta\), to obtain

\[ X_\eta \circ \varphi_\eta = T \bar{\varphi}_\eta \circ X_{H_\eta} \circ \iota_\eta. \] (31)

We shall develop the expression of \(X_\eta\) for each point in \(\mathcal{O}_\eta \times \mathfrak{g} \times \mathfrak{g}^*\) using the relation now obtained and then we will present a useful remark.

Lemma 5. If \((a, b, c, d) \in T(x, Y, \theta + \text{ad}^0_\eta \xi, \xi)J^{-1}(\eta)\), then

\[ T(x, Y, \theta + \text{ad}^0_\eta \xi, \xi) \varphi_\eta(a, b, c, d) = (\text{ad}^*_\eta_{\theta+\text{ad}^0_\eta \xi, \xi} \beta(a, b, c, d) \in T(x, Y, \theta + \text{ad}^0_\eta \xi, \xi). \] (32)

Proof. First we would like to clarify that the choice of the element \((a, b, c, d)\) is related to the fact that \(T(x, Y, \theta + \text{ad}^0_\eta \xi, \xi)J^{-1}(\eta)\) is a subset of \(T(x, Y, \theta + \text{ad}^0_\eta \xi, \xi)(\mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^*) = T\mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*\). Now, let \(\beta = (\beta_1, \beta_2, \text{Ad}^*_\eta \eta + \text{ad}^*_\eta \beta_3, \beta_4)\) be a curve in \(J^{-1}(\eta)\) satisfying the initial conditions \(\beta(0) = (x, Y, \theta + \text{ad}^0_\eta \xi, \xi)\) and \(\beta(0) = (a, b, c, d)\). Then, we know that \(T(x, Y, \theta + \text{ad}^0_\eta \xi, \xi) \varphi_\eta(a, b, c, d) = d(\varphi_\eta \circ \beta)(0)/dt\), which is equal to \(\text{ad}^*_\eta_{\theta+\text{ad}^0_\eta \xi, \xi} \beta(a, b, c, d)\), that is, \(\text{ad}^*_\eta_{\theta+\text{ad}^0_\eta \xi, \xi} \beta(a, b, c, d)\) as we wanted to prove.

We are now able to prove the following result.

Proposition 4. For each point \((\theta, Y, \xi) \in \mathcal{O}_\eta \times \mathfrak{g} \times \mathfrak{g}^*\), the dynamical vector field \(X_\eta\) of the Hamiltonian system \((\mathcal{O}_\eta \times \mathfrak{g} \times \mathfrak{g}^*, \tilde{\Omega}_\eta, h)\) is given by

\[ X_\eta(\theta, Y, \xi) = (\text{ad}^*_\eta \theta, X_\xi, -\theta). \] (33)

Proof. We begin by noting that due to relations (29) and (31), we obtain \(X_\eta(\theta, Y, \xi) = (T \varphi_\eta \circ X_{H_\eta})(x, Y, \theta + \text{ad}^0_\eta \xi, \xi)\). Now according to (18), we have \(X_{H_\eta}(x, Y, \theta + \text{ad}^0_\eta \xi, \xi) = (T \eta_{\theta+\text{ad}^0_\eta \xi, \xi} X_\xi, 0, -\theta)\). It remains only to show that \(T(x, Y, \theta + \text{ad}^0_\eta \xi, \xi) \varphi_\eta(T(x, Y, X_\xi, 0, -\theta)) = (\text{ad}^*_\eta \theta, X_\xi, -\theta)\), but this follows from (32) taking \((a, b, c, d) = (T \eta_{\theta+\text{ad}^0_\eta \xi, \xi} X_\xi, 0, -\theta)\).

Thus, the equations of Hamilton on the reduced manifold \(\mathcal{O}_\eta \times \mathfrak{g} \times \mathfrak{g}^*\) are given by

\[ \begin{cases} \dot{\theta} = \text{ad}^*_\eta \theta \\ \dot{Y} = X_\xi \\ \dot{\xi} = -\theta. \end{cases} \] (34)

Remark 2. In remark 1, we have proved the equivalence between the solutions of the equations of Hamilton (19) and the Euler–Lagrange equations (20). It is obvious that the reduced dynamics described by (33) is also related to the variational approach of cubic polynomials. In fact, an integral curve of the reduced Hamiltonian vector field (33) gives rise to a curve that satisfies the second equation of the Euler–Lagrange system (20). Indeed, writing the first equation of (34) as an equation on the Lie algebra (see the end of subsection 2.1 for details on notation), we obtain \(\dot{X}_\eta = [Y, \dot{X}_\eta] = 0\). But, by the other two equations of (34), we know that \(\dot{X}_\eta = -\dot{Y}\). We conclude that a solution of the reduced system gives a solution of

\[ \dot{Y} + [Y, \dot{Y}] = 0. \] (35)
4.3. Invariants along the extremal trajectories

Integrals of motion of a dynamical system are quantities that are conserved along the flow of that system and can sometimes be associated with symmetries of the system. A classical result due to Liouville, exposed in [7] by Arnold (who has also contributed to a more complete version of this result), says that a dynamical system on a phase space of dimension $2N$ is completely integrable if it admits $N$ functionally independent first integrals in involution (i.e. their Poisson brackets all vanish). However, these situations are rather rare. In practice, one often deals with Hamiltonian systems which admit a non-Abelian group of symmetries or an Abelian group of symmetries less in number than that required to have complete integrability. If some special conditions are satisfied, the non-Abelian set of independent integrals can lead us to the integrability of the system, as explained by Fomenko and Mishchenko in [27], the authors of the theorem of the non-commutative integrability. But, in most cases, one naturally expects to find only a number of independent Poisson commuting invariants less than $N$, which can allow us to partially reduce the original system (Poincaré–Lyapunov–Liouville–Arnol’d theorem [28]).

The problem we are concerned with in this subsection is the preliminary analysis of the symmetries of the Hamiltonian system $(\mathcal{O}_q \times g \times g^*, \bar{\Omega}_q, h)$, so that we can study the integrability of the system in a forthcoming work. The problem of reduction of the order of a Hamiltonian system is an old subject of study, with emphasis on several works of Poincaré and Cartan, namely the Lie–Cartan theorem (for more details on this subject, see [8]). We shall find, by using this classical theorem of Lie–Cartan, a maximum number of functionally independent first integrals in involution.

In the rest of this paper, for the sake of simplicity, we shall use the following notation:

$$\dim g = n \quad \text{and} \quad \dim \mathcal{O}_q = 2m.$$  \hfill (36)

(recall that the dimension of the co-adjoint orbit is always even), where obviously $2m \leq n$. So the dimension of the phase space of our Hamiltonian system shall be

$$\dim (\mathcal{O}_q \times g \times g^*) = 2(n + m).$$  \hfill (37)

A function $f : \mathcal{O}_q \times g \times g^* \to \mathbb{R}$ is an integral of motion of our dynamical system (with the associated vector field $X_h$) if $[X_h(w)](f) = 0$, that is, $[(df)(w)](X_h(w)) = 0$, for all $w \in \mathcal{O}_q \times g \times g^*$. It is important to note that $df : \mathcal{O}_q \times g \times g^* \to T^* (\mathcal{O}_q \times g \times g^*)$ is such that $df(w) \in T^*_w \mathcal{O}_q \times g^* \times g \subseteq g \times g^* \times g$. In that sense, we shall assume the notation $df(w) = (\partial f(w)/\partial \theta, \partial f(w)/\partial Y, \partial f(w)/\partial \xi)$. The Hamiltonian function is naturally an integral of motion of the Hamiltonian system. So function (30), that is,

$$l_1 \equiv h = \theta(Y) + \frac{1}{2} \xi(X_\xi),$$

is an integral of motion. But besides that, we are able to prove the following result.

\textbf{Proposition 5.} The functions $l_{i+1} : \mathcal{O}_q \times g \times g^* \to \mathbb{R}$ defined by

$$l_{i+1} = (\theta + \text{ad}^*_\eta)(A_i), \quad \text{with } A_i \text{ a fixed basis element of } g.$$  \hfill (38)

are integrals of motion of the Hamiltonian system $(\mathcal{O}_q \times g \times g^*, \bar{\Omega}_q, h)$.

\textbf{Proof.} Consider $w = (\theta, Y, \xi) \in \mathcal{O}_q \times g \times g^* \subseteq g^* \times g \times g^*$, with $\theta = \text{Ad}^*_x \eta$, for some $x \in G$. An elementary computation gives

$$dl_{i+1}(w) = (A_i, -\text{ad}^*_x \xi, \text{ad}_y A_i).$$  \hfill (39)
Then, by (33), we obtain $[X_0(I_{i+1})](w) = \theta(\text{ad}_Y A_i) - \xi(\text{ad}_Y X_0) - \theta(\text{ad}_Y A_i) = [X_0, X_0], A_i = 0$, which shows that $[\{dI_{i+1}(w)\} X_0(w)] = 0$, for each $i = 1, \ldots, n$, proving that the given functions are invariant. \hfill \square

**Remark 3.** Recall that in the context of the variational approach two invariants are known, (2) and (3). These invariants are related to the invariants now obtained. Indeed, it is simple to prove (using theorem 1) that $l_1 \equiv l_1$ and $2I_2 - \sum_{i=1}^n l_{i+1}^2 \equiv \theta(X_0)$.

It is immediate to see that any linear combination of the $n+1$ integrals of motion described above is also an integral of motion of the system. A natural question arises: how to extract, from the set of invariant functions, a maximal set of independent commuting invariant functions? In order to answer to this problem, we begin with some comments.

– The independence between the invariants $l_{i+1}$, $i = 1, \ldots, n$, defined by (38) is obvious, but these invariants are not in general in involution with each other.

– On the other hand, it is clear that $l_1$ commutes with each of these functions $l_{i+1}$, since $l_1$ coincides with the Hamiltonian function.

Furthermore, we have the following two results.

**Lemma 6.** Consider the invariants $l_{i+1}$, for $i = 1, \ldots, n$, defined by (38). The Hamiltonian $l_1$ is functionally independent of all the invariants $l_{i+1}$.

**Proof.** Consider the coordinate expression for the invariants, with respect to the natural basis taken from the orthonormal basis $\{A_i\}_{i=1}^{n}$ of the Lie algebra $\mathfrak{g}$:

$$l_1 = \sum_{j=1}^n \theta_j y_j^j + \frac{1}{2} \sum_{j=1}^n (\xi_j)^2$$

$$l_2 = \theta_1 + \sum_{j,k=1}^n C_{jk}^i y_j y_k$$

$$\vdots$$

$$l_{n+1} = \theta_n + \sum_{j,k=1}^n C_{jk}^i y_j y_k.$$  

Note that for $i > m$, $\theta_i$ is a function of $\theta_1, \ldots, \theta_m$. We shall prove that $dl_1 \wedge dl_{i+1} \neq 0$, for a dense domain of $\mathcal{O}_n \times \mathfrak{g} \times \mathfrak{g}^*$ ($i = 1, \ldots, n$). But it is immediate to isolate the coefficients corresponding to $\text{d}y_i^j \wedge \text{d}\theta_j$, which are equal to $\text{d}\theta_i \wedge \text{d}y_i^j$. Thus, the condition of the wedge product being equal to zero implies that $\theta_j = 0$, $\forall j$, which is impossible for any regular value of $\eta$ different from zero.

If $\eta = 0$, it is immediate to check that the invariants are also independent, since in this situation the Hamiltonian does not depend on $Y$, while the other invariants do. \hfill \square

**Lemma 7.** Considering the Poisson structure on $\mathcal{O}_n \times \mathfrak{g} \times \mathfrak{g}^*$, the set $\{l_{i+1}\}_{i=1}^{n}$ is endowed with a Lie algebra structure that makes it isomorphic to the Lie algebra $\mathfrak{g}$.

**Proof.** Consider the orthonormal basis $\{A_1, \ldots, A_n\}$ of the Lie algebra $\mathfrak{g}$ and represent by $C_{ji}^k$ the structure constants of this Lie algebra for this basis. If $w = (\theta, Y, \xi)\in \mathcal{O}_n \times \mathfrak{g} \times \mathfrak{g}^*$, we know that $[l_{i+1}, l_{j+1}](w) = [dI_{i+1}(w)] (X_{j+1}(w))$, with $X_{j+1}$ denoting the Hamiltonian vector field associated with $l_{j+1}$. 

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In order to proceed with the proof, let us find now the expression of the Hamiltonian vector field $X_{l_{j+1}}$ of $l_{j+1}$ in a similar way to the one used in proposition 4 for $X_l$. To do this, first consider the function $L_{j+1} : G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^* \to \mathbb{R}$ uniquely characterized by the identity $L_{j+1} \circ l_l = l_{j+1} \circ \psi_l$, where $\psi_l$ is the surjective function defined by (29). More precisely, we have $L_{j+1}(z) = l_{j+1}(\theta, Y, \xi)_{\mathfrak{c}, \mathfrak{c}^*_{\mathfrak{a}_l}} = \mu(A_l)$ where the element $\xi$ is such that $w = \psi_l(z)$, that is, $z = (x, Y, \theta + \text{ad}_Y \xi, \xi) \in J^{-1}(\eta)$. Then, the Hamiltonian vector field $X_{l_{j+1}}$ associated with the function $L_{j+1}$ is related to $X_{l_{j+1}}$, as follows:

$$X_{l_{j+1}}(w) = T \psi_l(X_{l_{j+1}}(z)) = \{\text{ad}_Y \theta, \text{ad}_Y A_l, \text{ad}_Y \xi\},$$

where we use lemma 5 choosing $(a, b, c, d) = X_{l_{j+1}}(z) = (X_{l_{j+1}}^1, X_{l_{j+1}}^2, X_{l_{j+1}}^3, X_{l_{j+1}}^4)$. To completely determine $X_{l_{j+1}}(w)$, we must find the expression of the components $X_{l_{j+1}}^1, X_{l_{j+1}}^2, X_{l_{j+1}}^3, X_{l_{j+1}}^4$. A computation analogous to the one performed for the components of the vector field $X_l$ in proposition 2 shows that $X_{l_{j+1}}^1 = l_k A_l, X_{l_{j+1}}^2 = \text{ad}_Y A_l$ and $X_{l_{j+1}}^3 = \text{ad}_Y^2 \xi$, and hence

$$X_{l_{j+1}}(w) = \{\text{ad}_Y \theta, \text{ad}_Y A_l, \text{ad}_Y^2 \xi\}. \quad (40)$$

Now, recalling expression (39) of $d_l(w)$, we obtain

$$[l_{i+1}, l_{j+1}](w) = \theta ([A_j, A_i]) + \xi ([A_j, [A_i, Y]] + [A_j, [Y, A_i]])$$
$$= \theta ([A_j, A_i]) - \xi ([Y, [A_j, A_i]]) = (\theta + \text{ad}_Y \xi)([A_j, A_i])$$
$$= \sum_{k=1}^n C_{jk}^i \xi \theta_{k+1}.$$

The structure constants of the Lie algebra generated by the functions $l_{i+1}, i = 1, \ldots, n$, and the structure constants of $\mathfrak{g}$ coincide, so the algebras are isomorphic.

Let us summarize the situation.

- We have $n+1$ smooth functions, the integrals of motion $l_1, l_2, \ldots, l_{n+1}$, whose differentials are linearly independent of $\mathfrak{c}_l \times \mathfrak{g} \times \mathfrak{g}^*$ (see lemma 6).
- The linear span of these functions is closed with respect to the Poisson bracket (see lemma 7 and recall that $l_1$ is in involution with all the other functions).

Thus, the linear span $\mathfrak{c}$ of these $n+1$ functions has a structure of a finite-dimensional real Lie algebra, with dim $\mathfrak{c} = n + 1$. This algebra is called the algebra of integrals.

We now present the Lie–Cartan theorem formulated according to [8].

**Theorem 2** (Lie–Cartan). Consider a Hamiltonian system $(M, \omega, H)$ with first integrals $F_1, \ldots, F_k$ such that $[F_i, F_j] = a_{ij}(F_1, \ldots, F_k)$. Let $F : M \to \mathbb{R}^k$ be the natural mapping generated by these sets of integrals.

Suppose that the point $c \in \mathbb{R}^k$ is not a critical value of the mapping $F$ and that in its neighborhood the rank of matrix $(a_{ij})$ is constant. Then in a small neighborhood $U \subset \mathbb{R}^k$ of $c$, one can find $k$ independent functions $\varphi_j : U \to \mathbb{R}$ such that the functions $\phi_j = \varphi_j \circ F : N \to \mathbb{R}$, where $N = F^{-1}(U)$, satisfy the relations

$$[\phi_1, \phi_2] = \cdots = [\phi_{2q-1}, \phi_{2q}] = 1,$$

whereas the remaining brackets $[\phi_i, \phi_j]$ vanish. The number $2q$ equals the rank of the matrix $(a_{ij})$.

We are interested in the following consequence of the above theorem (see [8]).
Remark 4. Under the hypotheses of the Lie–Cartan theorem and using the notation above, there are \( k - q \) independent integrals in involution: \( \phi_2, \phi_4, \ldots, \phi_{2q-2}, \phi_{2q}, \phi_{2q+1}, \ldots, \phi_{2n} \). As a consequence, the original Hamiltonian system can be reduced, by the method of Poincaré, to a system with minus \( k - q \) degrees of freedom than the original one.

We consider now an open dense subset \( D \) in \( O_q \times g \times g^* \) where the functional independence of the \( n+1 \) integrals is satisfied and where the skew-symmetric Poisson bracket matrix \((\{l_i, l_j\})\), \( 1 \leq i, j \leq n+1 \), has maximal rank. Note that, by lemma 7, the maximum rank of this Poisson bracket matrix coincides with the maximum rank of the matrix \( M_g(a) = (M_{ij}(a)) \), with \( M_{ij}(a) = \sum_{k=0}^{n} C_{ij}^k a_k \), for \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) and where \( C_{ij}^k \) are structure constants of the Lie algebra \( g \). We fix the notation (note that this rank is always even)

\[
    r_g := \frac{1}{2} \max_{a \in \mathbb{R}^n} \text{rank} M_g(a). \tag{41}
\]

In consideration of remark 4, the \( 2(m+n) \)-dimensional Hamiltonian system \((O_q \times g \times g^*, \Omega_n, H)\) admits \( n + 1 - r_g \) functions defined on an open subset of \( D \), which form a set of independent integrals of the motion in involution. Thus, we can expect the system to be reduced to a system of dimension equal to \( 2(m + n - r_g) \).

Example 1. Consider the problem of cubic polynomials on the Lie group \( SO(3) \), which can be illustrated by the dynamic optimal control problem of the spherical free rigid body by the authors of this paper in [5, 6]. In this case, it is well known that the co-adjoint orbit \( O_q \) corresponds to a two-dimensional sphere with radius \( \|\eta\| \). (For the singular case \( \eta = 0 \), the orbit reduces to one point.) So, considering the non-singular case, the symplectic reduced manifold \( O_q \times so(3) \times se^*(3) \) has dimension equal to 8 and \( r_g = 1 \). Applying the above theorem, the corresponding reduced Hamiltonian system has three independent invariants in involution and it can be reduced to a two-dimensional system.

As mentioned at the beginning of this section, these results on integrals of the motion may have important implications from the point of view of the integrability of the corresponding dynamical systems. Thus, they are relevant at the level of determining the cubic polynomials. In conclusion, by using the Lie–Cartan theorem, we are able to reduce the Hamiltonian system to a system with at least \( n + 1 - r_g \) degrees of freedom less. We note that if \( m + r_g = 1 \), the system will be completely integrable. But, this condition occurs only in trivial cases, when the algebra \( g \) is Abelian or the co-adjoint orbit \( O_q \) is reduced to a point.

In a future paper, we hope to address this integrability problem in detail.

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