Formal Equivalences in $C^4$

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Abstract. It is studied the local Equivalence Problem in Complex Analysis. It is proven that the Formal (Holomorphic Segre Preserving) Equivalences, of Real-Formal Hypersurfaces in $C^2$, are determined by their 1-jets. These Equivalences are also reversible and their inverses are also smooth. Then, it is concluded that any Formal Holomorphic Segre Preserving Mapping, between the complexifications of two Real-Analytic Hypersurfaces or Real-Algebraic Hypersurfaces in $C^2$, is convergent or algebraic. In particular, there are derived formal constructions of normal form type, which are convergent if the source manifold is analytic, and respectively algebraic if the source manifold is algebraic.

1. Introduction and Main Results

Let $M, N \subset C^2$ two Real-Formal Hypersurfaces defined near $0 \in C^2$ by

$$ M : \rho_1 (Z, \overline{Z}) = 0, \quad N : \rho_2 (Z, \overline{Z}) = 0, $$

where $Z = (w, z)$ are coordinates in $C^2$.

We recall from Angle\[2\] that any (Formal) Holomorphic Segre Preserving Mapping, between $M$ and $N$, is defined by

$$ \mathcal{H} : C^2 \rightarrow C^2 \quad \text{such that} \quad \mathcal{H}(Z, \zeta) = \left( \mathcal{H}(Z), \mathcal{H}(\zeta) \right), $$

where $\mathcal{H}, \hat{\mathcal{H}} : C^2 \rightarrow C^2$ are formal holomorphic mappings such that

$$ \rho_2 (H(Z), \hat{\mathcal{H}}(\zeta)) = 0, \quad \text{for all } (Z, \zeta) \in C^4 \text{ such that } \rho_1 (Z, \zeta) = 0, $$

where $\zeta = (\nu, \xi)$ replaces $Z = (w, z)$.

Two (Formal) Holomorphic Segre Preserving Mappings $\mathcal{H}_1, \mathcal{H}_2$ are called to be determined by $k$-jets if

$$ J^k(\mathcal{H}_1) = J^k(\mathcal{H}_2) \rightarrow \mathcal{H}_1 = \mathcal{H}_2, $$

where $k \in N^*$ and $J^k$ defines the $k$-jet in the corresponding formal expansion.

Throughout paper, there are studied standard problems\[31, 8\] considered for Formal Holomorphic Segre Preserving Equivalences\[2, 40\] of a class of Real-Formal Hypersurfaces in Complex Space, called non-flat Hypersurfaces. Such mappings are called shortly Formal (Holomorphic) Segre Equivalences. In particular, there are considered formal constructions of normal form type using the methods learned by the author\[9, 10\] from Zaitsev\[35, 59\], recalling that the normal form\[16, 17, 23\] convergence may not occur. We obtain:

**Theorem 1.1.** Let $M \subset C^2$ be a Real-Formal Hypersurface defined near $p = 0$ by

$$ \text{Im} w = (\text{Re} w)^{\nu} \mathcal{L}(z, \overline{z}) + \sum_{m+n+p \geq k_0+1} \varphi_{mnp} z^m \overline{z}^n (\text{Re} w)^p, $$

where $\mathcal{L}(z, \overline{z})$ is a homogeneous polynomial of degree $k_0 - s \geq 3$, for given $k_0, s \in N$, such that

$$ \varphi_{mnp} = 0 \text{ if } s \neq 0, \text{ for all } m, n \in N \text{ with } m + n \geq k_0 + 1. $$

Then, the Formal Holomorphic Segre Equivalences of $M$ are determined by their 1-jets.

In this case \[12\], which is perhaps less difficult, we consider points of infinite type\[22, 23\] for $s \neq 0$, and respectively points of finite type\[22, 23\] for $s = 0$. Then normalizations, derived from Weighted and Pseudo-Weighted Versions of the (Generalized) Fischer Decomposition\[35\], are formally imposed in complexified (formal) local defining equations. Such Fischer Decompositions are applied with respect to a natural System of Weights when $s = 0$. Otherwise, we introduce by \[10\] a System of Pseudo-Weights, defined on individual classes of terms, which verifies that the (pseudo-)weight of the sum of two terms is actually the sum of their (pseudo-)weights, but the (pseudo-)weight of a product of two terms may not the sum of their (pseudo-)weights as usual. Then, it is possible to impose normalizations on sums of weighted and (pseudo-)weighted terms by applying the strategy from \[10\], after a preliminary normalization of the 1-jets of the Formal Segre Preserving Change of Coordinates. Regardless of such non-trivialities in both cases $s = 0$ and $s \neq 0$, the 1−jet determination follows as in the standard cases from Ebenfelt-Lamel-Zaitsev\[12, 13\]. More generally, identifying pseudo-weighted homogeneous terms, or weighted homogeneous terms, in the local defining equation, we obtain:

**Corollary 1.2.** Any Formal Segre-Preserving Mapping, between two non-flat Real-Formal Hypersurfaces in $C^2$, posses the finite jet determination property.

Keywords. Finite jet Determination, Normal Form, Cauchy-Riemann Geometry, CR Equivalence, Equivalence Problem, Algebraicity. Special Thanks (in regard to this large becoming paper) to Science Foundation Ireland grant 10/RFP/MT H2878. I make clear that the reference \[10\] was fully supported by Science Foundation Ireland Grant 06/RFP/MAT 018.
The finite jet determination problem defines a specific property occurring in the theory of Real Submanifolds in Complex Spaces [6], which may not generally hold. The example of the group of biholomorphisms of Levi-flat hypersurfaces shows the nontriviality of the finite jet determination problem [11] in \( \mathbb{C}^2 \) (see also [21]). It is indicated [6] for an extended introduction to this topic [12], [13], [31], and Angle [2], [3] and Zhang [40] for progresses concerning the Segre Holomorphic Preserving Mappings [2] and related to standard problems in CR Geometry [6].

Going forward, it is studied the problem of convergence of a Formal Equivalence. We obtain

**Theorem 1.3.** Any Formal Segre-Equivalence, of two non-flat Real-Analytic Hypersurfaces in \( \mathbb{C}^2 \), is Convergent.

It may be an evidence of the inflexibility of such Formal Equivalences, regardless of the geometrical context. Such conclusion contrasts when \( s \neq 0 \) in the standard case in \( \mathbb{C}^2 \), where Kossovskiy-Shafikov [19] showed that it may not exist Holomorphic Equivalences between two non-minimal Real-Analytic Formally Equivalent Submanifolds in Complex Spaces. In particular, we obtain

**Corollary 1.4.** Any Formal Segre-Preserving Mapping, of two non-flat Real-Analytic Hypersurfaces in \( \mathbb{C}^2 \), is Convergent.

The proofs are based on constructions of analytic systems, which are actually derived from the local defining equations, taking ingredients from Min [27], [28]. Then, it is solved the convergence problem [5], [7] using the Approximation Theorem of Artin [1]. Furthermore, the formal constructions of normal form type are convergent, if the source manifold is analytic, which is surprising, because Kolár [23] constructed an example of divergent normal form for a class of real-analytic hypersurfaces of finite type in \( \mathbb{C}^2 \). Similarly, it turns out that the formal constructions of normal form type are algebraic, if the source manifold is algebraic.

A holomorphic function is called algebraic if it satisfies a non-trivial polynomial equation with polynomial coefficients according to the standard terminology taken from [9], [33]. Then, a real-surface is called algebraic if its local defining equation is smooth and satisfies a non-trivial polynomial equation with polynomial coefficients.

Going forward, it is studied the problem of algebraicity of a Formal Equivalence. We obtain

**Corollary 1.5.** Any Formal Segre-Equivalence, of two non-flat Real-Algebraic Hypersurfaces in \( \mathbb{C}^2 \), is Algebraic.

The algebraicity problem has a long history in Complex Analysis [6]. It has been studied by Baouendi-Rothschild [4], Baouendi-Ebenfelt-Rothschild [5], Zaitsev [37], Min [29] and Mir-Lamel [33]. The proof is based on constructions of analytic systems, which are actually derived from the local defining equations, taking ingredients from Min [29]. In particular, we obtain:

**Corollary 1.6.** Any Formal Segre-Preserving Mapping, of two non-flat Real-Algebraic Hypersurfaces in \( \mathbb{C}^2 \), is Algebraic.

It is a strong consequence, because the considered formal mapping is not assumed to an equivalence. It is a strong characteristic of our case, because it is not required for more non-degeneracy conditions like finite type condition. It contrasts the standard case in \( \mathbb{C}^2 \), where Baouendi-Rothschild [4] showed that it may not exist Algebraic Holomorphic Equivalences between two Real-Algebraic Formally Equivalent Submanifolds in Complex Spaces.

Few words about the organisation of this paper. The case \( s \neq 0 \) has priority, because it generalizes the computations of the case \( s = 0 \), forming the first part, which mainly defines formal constructions of normal form type. Secondly, there are constructed analytic (or algebraic) systems in order to prove the convergence (or algebraicity) results.

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### 2. Preliminaries

#### 2.1. Local Coordinates

It is applied the standard (formal) procedure [9], [10] applied for constructing (formal) normal forms [22], [38], [39] in Complex Analysis, in respect to (1.3) using the following local coordinates:

Let \( M \subset \mathbb{C}^2 \) be the Real-Formal Hypersurface [6] defined by

\[
M : \quad \text{Im } w = (\text{Re } w)^s \mathcal{L}(z, \bar{z}) + \sum_{m+n+p \geq k_0+1} \varphi_{mnp} z^m \bar{z}^n (\text{Re } w)^p,
\]

where \((z, w)\) are coordinates in \( \mathbb{C}^2 \).

Let \( M' \subset \mathbb{C}^2 \) be another Real-Formal Hypersurface [6] defined by

\[
M' : \quad \text{Im } w' = (\text{Re } w')^s \mathcal{L}(z', \bar{z}') + \sum_{m+n+p \geq k_0+1} \varphi'_{mnp} z^m \bar{z}^n (\text{Re } w')^p,
\]

where \((z', w')\) are coordinates in \( \mathbb{C}^2 \).

In order to study the formal (holomorphic) equivalence like (1.2), we complexify (2.1) and (2.2). We obtain

\[
M : \quad \frac{w - \bar{w}}{2\sqrt{-1}} = (\frac{w + \bar{w}}{2})^s \mathcal{L}(z, \bar{z}) + \sum_{m+n+p \geq k_0+1} \varphi_{mnp} z^m \bar{z}^n (\frac{w + \bar{w}}{2})^p,
\]

and respectively, we obtain

\[
M' : \quad \frac{w' - \bar{w}'}{2\sqrt{-1}} = (\frac{w' + \bar{w}'}{2})^s \mathcal{L}(z', \bar{z}') + \sum_{m+n+p \geq k_0+1} \varphi'_{mnp} z^m \bar{z}^n (\frac{w' + \bar{w}'}{2})^p,
\]

where \( \mathcal{L} \) has been replaced with \( \xi' \), and \( \mathcal{L} \) has been replaced with \( \nu' \). From The Theorem of Implicit Functions applied in (2.3), we obtain

\[
Q(z, \xi, \nu) = \frac{w - Q(z, \xi, \nu)}{Q_{k_0}(z, \xi, \nu)},
\]

where \( Q \) is (formal) holomorphic in \((z, \xi, \nu)\), which can be written as follows

\[
Q(z, \xi, \nu) = \nu + Q_{k_0}(z, \xi, \nu) + Q_{k_0+1}(z, \xi, \nu),
\]
where we have used a homogeneous polynomial of degree $k_0$ in $(z, \xi, \nu)$, denoted as
\begin{equation}
Q_{k_0} (z, \xi, \nu),
\end{equation}
which is obtained from $Q (z, \xi, \nu)$, and a (possibly infinite) sum of homogeneous polynomials of degree at least $k_0 + 1$ in $(z, \xi, \nu)$, denoted as
\begin{equation}
Q_{k_0+1} (z, \xi, \nu).
\end{equation}
which is obtained as well from $Q (z, \xi, \nu)$.

Next, we replace (2.6) in (2.5). We obtain
\begin{equation}
\frac{Q_{k_0} (z, \xi, \nu) + Q_{k_0+1} (z, \xi, \nu)}{2\sqrt{-1}} = \left( \frac{2\nu + Q_{k_0} (z, \xi, \nu) + Q_{k_0+1} (z, \xi, \nu)}{2} \right)^{\nu} \mathcal{L} (z, \xi)
\end{equation}
(2.8)
\begin{equation}
\sum_{m+n+p \geq k_0+1} \varphi_{mnp} z^m \xi^n \left( \frac{2\nu + Q_{k_0} (z, \xi, \nu) + Q_{k_0+1} (z, \xi, \nu)}{2} \right)^{\nu} \mathcal{L} (z, \xi) = \mathcal{O} (k_0 + 1),
\end{equation}
Then, we effectuate formal expansions in (2.8). We obtain the following evaluations
\begin{equation}
\sum_{m+n+p \geq k_0+1} \varphi_{mnp} z^m \xi^n \left( \frac{2\nu + Q_{k_0} (z, \xi, \nu) + Q_{k_0+1} (z, \xi, \nu)}{2} \right)^{\nu} \mathcal{L} (z, \xi) = \nu^s \mathcal{L} (z, \xi) + \mathcal{O} (k_0 + 1).
\end{equation}
Then, because of (2.10), according to a simple analysis of terms from the both sides in (2.8), we obtain
\begin{equation}
Q_{k_0} (z, \xi, \nu) = 2\sqrt{-1} \nu^s \mathcal{L} (z, \xi).
\end{equation}
Next, in order to assume that the linear part of the formal (holomorphic) equivalence $\mathcal{H}$ is standard, or equivalently by (1.2) that the formal power series $\mathcal{H}$ and $\tilde{\mathcal{H}}$ have standard linear parts, we consider:

\section*{2.2. Linear (holomorphic Segre preserving) Changes of Coordinates.}

We write the formal expansions of $\mathcal{H}$ and $\tilde{\mathcal{H}}$:
\begin{equation}
\mathcal{H} (z, w) = \left( \sum_{k+l \geq 1} f_{kl} z^k w^l, \sum_{k+l \geq 1} g_{kl} z^k w^l \right), \quad \tilde{\mathcal{H}} (\xi, \nu) = \left( \sum_{k+l \geq 1} \tilde{f}_{kl} \xi^k \nu^l, \sum_{k+l \geq 1} \tilde{g}_{kl} \xi^k \nu^l \right),
\end{equation}
using the hypothesis that $\mathcal{H}$ is an equivalence, which implies
\begin{equation}
f_{10} \neq 0, g_{10} \neq 0 \text{ and } \tilde{f}_{10} \neq 0, \tilde{g}_{10} \neq 0.
\end{equation}
Next, we replace (2.11) in (2.4) using (2.5). Then, there do not exist multiplications of $\nu$, $z$ and $\xi$ with complex numbers in right-hand side from (2.4). We obtain
\begin{equation}
g_{10} = \tilde{g}_{10}, \quad g_{01} = \tilde{g}_{01} = 0.
\end{equation}
Now, in order to normalize $\mathcal{H}$, we consider the following linear change of coordinates
\begin{equation}
(w', z'; \nu', \xi') = \left( \frac{w}{g_{10}}, \frac{z}{(g_{10})^{\nu_{10}}}, \frac{\nu}{g_{10}}, \frac{\xi}{(g_{10})^{\nu_{10}}-\nu_{10}} \right),
\end{equation}
which is well-defined because of (2.12), and which preserves by (2.13) the Model
\begin{equation}
w = \nu + 2\sqrt{-1} \nu^s \mathcal{L} (z, \xi).
\end{equation}
Changing the coordinates according to (2.14), we can assume
\begin{equation}
g_{10} = \tilde{g}_{10} = 1.
\end{equation}
Now, we identify the terms of degree 2 in $(z, \xi)$ in (2.4). We obtain
\begin{equation}
\mathcal{L} (z, \xi) = \mathcal{L} \left( f_{10} z, \tilde{f}_{10} \xi \right), \quad \text{and then } \mathcal{L} \left( \frac{z}{f_{10}}, \frac{\xi}{f_{10}} \right) = \mathcal{L} (z, \xi).
\end{equation}
Then, the Model (2.15) is preserved by the following linear (holomorphic Segre preserving) change of coordinates
\begin{equation}
(w', z'; \nu', \xi') = \left( \frac{w}{f_{10}}, \frac{z}{f_{10}}, \nu, \frac{\xi}{f_{10}} \right).
\end{equation}
Changing the coordinates using (2.17), we can assume
\begin{equation}
f_{10} = \tilde{f}_{10} = 1.
\end{equation}
Next, replacing (2.5) and (2.11) in (2.4) using the assumptions (2.16) and (2.18), we move forward in order to derive formal constructions of normal form type. In particular, we study the following:
2.3. General Equation. We have
\[
\nu + 2\sqrt{-1} \nu^s L(z, \xi) + Q_{k_0 + 1}(z, \xi, \nu) - \nu + \sum_{k+l \geq 2} \left( g_{kl} z^k \left( \nu + 2\sqrt{-1} \nu^s L(z, \xi) + Q_{k_0 + 1}(z, \xi, \nu) \right)^l - \tilde{g}_{kl} \xi^l \right)
\]
\[= \frac{2\sqrt{-1}}{2} \left( \frac{1}{2} \left( \nu^s L(z, \xi) + Q_{k_0 + 1}(z, \xi, \nu) + 2\nu + \sum_{k+l \geq 2} \left( g_{kl} z^k \left( \nu + Q_{k_0}(z, \xi, \nu) + Q_{k_0 + 1}(z, \xi, \nu) \right)^l + \tilde{g}_{kl} \xi^l \right) \right)^s \right).
\]

(2.19)

In order to determine uniquely (2.11) from (2.19), we use:

2.4. Fischer Decompositions\[35\]. Any formal (holomorphic) power series, denoted as \(F(z, \xi, \nu)\) as a sum of the following type
\[
F(z, \xi, \nu) = G(z, \xi, \nu) P(z, \xi, \nu) + R(z, \xi, \nu), \quad \text{where } P^* (R(z, \xi, \nu)) = 0,
\]
where \(G(z, \xi, \nu)\) and \(R(z, \xi, \nu)\) are formal power series in \((z, \xi, \nu)\), and
\[
P^* := \sum_{m+n+p=r} \frac{\partial^r}{\partial z^m \partial \xi^n \partial \nu^p}, \text{ if } P(z, \xi, \nu) = \sum_{m+n+p=r} p_{mnp} z^m \xi^n \nu^p, \text{ where } r \in \mathbb{N}^*.
\]

Furthermore, the Fischer Decomposition (2.20) can be extended, for any homogeneous polynomials \((P_1, P_2, P_3, P_4)\) \((z, \xi, \nu)\) of degree \(r\), in order to write \(F(z, \xi, \nu)\) as a sum of the following type:
\[
F(z, \xi, \nu) = \sum_{i=1}^{4} G_i(z, \xi, \nu) P_i(z, \xi, \nu) + R(z, \xi, \nu), \quad \text{where } R(z, \xi, \nu) \in \bigcap_{i=1}^{4} \ker P_i^*.
\]

where \((G_1, G_2, G_3, G_4)\) \((z, \xi, \nu)\) are formal power series in \((z, \xi, \nu)\).

The uniqueness of the formal power series \(R(z, \xi, \nu)\) is known in (2.20) and (2.22). Also, the uniqueness of the formal power series \(G(z, \xi, \nu)\) is known in (2.20), but it is not generally clear the uniqueness of the formal power series \((G_1, G_2, G_3, G_4)\) \((z, \xi, \nu)\) in (2.22).

We move forward in order to analyse the first occurring case:

3. Case \(s = 0\)

It is required to make exact evaluations of the terms from (2.19) when \(s = 0\). In particular, the Model (2.11) becomes
\[
w = \nu + 2\sqrt{-1} L(z, \xi),
\]
(3.1)

Procedures from Kolik\[22\], Zaitsev\[35\] and Huang-Yin\[18\] are implemented in order to consider suitable normalization conditions in the local defining equation (2.19) when \(s = 0\). It is required to identify convenient interactions of terms from (2.19), which contain undetermined terms of the formal equivalence defined by (1.2) and (2.11). In particular, we linearise in (2.19), but there are not yet clear which normalizations are the best in (2.19), because there exist many terms available in order to impose further normalizations like the vanishing in (2.19) of the coefficient of \(\nu^s\), which uniquely determines
\[
g_{kl}, \text{ for all } l \in \mathbb{N}^* \text{ and } k \in \mathbb{N} \text{ with } N = k + l \geq 2,
\]
(3.2)

and respectively, the difference
\[
g_{N,0} - \tilde{g}_{N,0}, \text{ for all } N \geq 2.
\]
(3.3)

Such normalizations are not satisfactory, because the formal equivalence defined by (1.2) and (2.11) can not be computed entirely, since there are left undetermined an infinite number of parameters. Moreover, there exist better options which may be considered in (2.19) in order to discover new normalizations. In particular, the Model (3.1) provides new homogeneous terms, but the Model (3.1) is not homogeneous, and therefore it can not be used in order to consider classical Fischer Decompositions\[35\].

On the other hand, we make homogeneous the Model (3.1) using:

3.1. System of Weights. We define
\[
\text{wt } \{ \nu \} = k_0, \quad \text{wt } \{ z \} = \text{wt } \{ \xi \} = 1.
\]
(3.4)

Now, the Model (3.1) becomes weighted-homogeneous in respect to the system of weights from (3.3). Then, it is desired to study the interactions of terms from (2.19) using weighted-evaluations in respect to (3.3). It remains to study the eventual overlappings of following terms
\[
g_{kl} z^k \left( \nu + 2\sqrt{-1} L(z, \xi) + Q_{k_0 + 1}(z, \xi, \nu) \right)^l, \quad g_{kl} z^k \left( \nu + 2\sqrt{-1} L(z, \xi) + Q_{k_0 + 1}(z, \xi, \nu) \right)^l, \quad g_{kl} z^k \left( \nu + 2\sqrt{-1} L(z, \xi) + Q_{k_0 + 1}(z, \xi, \nu) \right)^l,
\]
(3.5)

where \(2 \leq N = k_0 + l < N' = k' + l_0'\), for \(k, l, k', l' \in \mathbb{N}\).
Then, these terms from (3.5) do not overlap, because such terms have different weights, according to the following weighted-evaluations

\[
\binom{\nu + 2\sqrt{-1}L}{2\nu - 1} (z, \xi, \nu) = (\nu + 2\sqrt{-1}L (z, \xi), \nu) + \Omega_{0} > k_{0} + 1 (z, \xi, \nu),
\]

Similarly, it does not exist any overlapping among the following terms

\[
f_{kl} z^{L} (z, \xi) (\nu + 2\sqrt{-1}L (z, \xi))^{l}, \quad f_{kl} z^{L} (z, \xi) (\nu + 2\sqrt{-1}L (z, \xi))^{l},
\]

where \( N = k + k_{l} - k_{0} + 1 \leq N' = k' + k_{l}' - k_{0} + 1 \), for \( k, l, k', l' \in N \).

Next, we are ready to make computations in (2.19) by extracting the terms of weight \( N \geq k_{0} + 1 \):

3.2. General Equation. We have

\[
\frac{\sum_{k + k_{0} = N} f_{kl} z^{L} (z, \xi) (\nu + 2\sqrt{-1}L (z, \xi))^{l}}{2\nu - 1} = \sum_{k + l = N} \tilde{g}_{kl} z^{L} (z, \xi) \nu^{l}.
\]

Next, we move forward in order to study the next occurring case:

It is required to make exact evaluations of the terms from (2.19) when \( s \neq 0 \). In particular, we recall the Model (2.18)

\[
w = \nu + 2\sqrt{-1}l_{s} L (z, \xi).
\]

3.3. Weighted Fischer-Decompositions. We consider the Fischer Decompositions (2.20) and (2.22) using the system of weights (3.4). More precisely, the Differential Operator (2.21) is defined in respect to (3.4). In particular, we have

\[
\frac{\partial}{\partial \nu} (\nu) := \frac{\partial^{0}}{\partial \nu^{0}} (\nu) = (k_{0})!, \quad \frac{\partial}{\partial \nu} (\nu^{2}) = 2k_{0} (k_{0})! \nu,
\]

because it is differentiated towards to the weight of \( \nu \), which is \( k_{0} \).

Such Fischer Decompositions hold according to classical Fischer Decompositions recalled in (2.20) and (2.22). In particular, we use Weighted Fischer Decompositions considering \( F (z, \xi, \nu) \) just a weighted-homogeneous polynomial and

\[
\begin{align*}
Q_{1} (z, \xi, \nu) &= z^{L} (z, \xi) (\nu + 2\sqrt{-1}L (z, \xi))^{l}, \\
Q_{2} (z, \xi, \nu) &= \xi^{L} (z, \xi) (\nu + 2\sqrt{-1}L (z, \xi))^{l}, \\
Q_{3} (z, \xi, \nu) &= z^{L} (z, \xi) (\nu + 2\sqrt{-1}L (z, \xi))^{l}, \\
Q_{4} (z, \xi, \nu) &= \xi^{L} (z, \xi) (\nu + 2\sqrt{-1}L (z, \xi))^{l},
\end{align*}
\]

using the following hypothesis

\[
k + k_{0} = k + \tilde{i} = N \geq k_{0} + 1, \quad k' + k'' = \tilde{k} + \tilde{\mu} = N - k_{0} + 1,
\]

where \( k, l, k', l', \tilde{k}, \tilde{\mu} \in N \).

It follows that the formal power series \((G_{1}, G_{2}, G_{3}, G_{4}, R) (z, \xi, \nu)\) are weighted-homogeneous polynomials in \((z, \xi, \nu)\), and also uniquely determined when (3.10) and (3.11) hold. Then, we impose normalizations in (2.19) using the weighted-homogeneous polynomials from (3.10). Now, we move forward in order to study the next occurring case:

4. Case \( s \neq 0 \)

It is required to make exact evaluations of the terms from (2.19) when \( s \neq 0 \). In particular, we recall the Model (2.18)

\[
w = \nu + 2\sqrt{-1}l_{s} L (z, \xi).
\]

Procedures from Kolli [22], Zaitsev [35], [39] and Huang-Yin [18] are implemented in order to consider suitable normalization conditions in the local defining equation (2.19) when \( s \neq 0 \). It is required to identify convenient interactions of terms from (2.19), which contain underdetermined terms of the formal equivalence defined by (2.2) and (2.11). In particular, we linearise in (2.19) like when in the case \( s = 0 \), but it is difficult to understand the interactions of terms in (2.19). There exist many terms available in order to impose normalizations like the terms derived from formal expansions of the Model (4.1). On the other hand, it is difficult to understand their actions in (2.19) when \( s \neq 0 \). Then, in order to make exact evaluations (2.19) when \( s \neq 0 \), it is introduced a system of pseudo-weights in order to make homogeneous the Model (4.1), because it is also not possible to define a system of weights like when \( s = 0 \). Regardless of its non-triviality, we want to make homogeneous the Model (4.1). In particular, it is implemented the strategy from (10), but it is not clear how the system of pseudo-weights should be defined.

It is clear that we should have

\[
\text{wt} \{L(z, \xi)\} = k_{0}.
\]
because, we should have

\[(4.3) \quad \text{wt} \{z\} = \text{wt} \{\xi\} = 1.\]

Then, we should have

\[(4.4) \quad \text{wt} \{\nu\} = k_0, \quad \text{wt} \{\nu^s \xi^L (z, \xi)\} = k_0.\]

More generally, we should have

\[(4.5) \quad \text{wt} \left\{ \nu^{\alpha + \beta} \left( \xi^L (z, \xi) \right)^\beta \right\} = nk_0, \quad \text{for all } \alpha, \beta \in \mathbb{N} \text{ with } \alpha + \beta = n \in \mathbb{N},\]

according to the following expansion

\[(\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi))^n = \sum_{\alpha + \beta = n} C_n^\alpha \nu^\alpha \left( 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^\beta,\]

and because it is desired to have

\[(4.6) \quad \text{wt} \left\{ (\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi))^n \right\} = n \cdot \text{wt} \left\{ \nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right\},\]

in order to be satisfied as much of the axioms of the pseudo-weight.

Then, we should have

\[(4.7) \quad \text{wt} \left\{ z^n \nu^{\alpha + \beta} \left( \xi^L (z, \xi) \right)^\beta \right\} = m + nk_0, \quad \text{for all } \alpha, \beta, m \in \mathbb{N} \text{ with } \alpha + \beta = n \in \mathbb{N}.\]

Also, we should have

\[(4.8) \quad \text{wt} \left\{ \xi^{\nu + s} \right\} = s + \tilde{l} + \tilde{k}, \quad \text{for all } \tilde{l}, \tilde{k} \in \mathbb{N} \text{ with } \tilde{k} \neq 0.\]

Next, we should have

\[(4.9) \quad \text{wt} \left\{ \xi^L (z, \xi) \xi^{\nu + s} \right\} = \tilde{l} + \tilde{k}' + k_0 - 1, \quad \text{for all } \tilde{l}', \tilde{k}' \in \mathbb{N} \text{ with } \tilde{k}' \neq 0,\]

because (4.3) should hold, and because we should have

\[(4.10) \quad \text{wt} \left\{ \xi^L (z, \xi) \nu^{\nu + s} \right\} = \tilde{l} + s + k_0 - 1, \quad \text{for all } \tilde{l} \in \mathbb{N}.\]

Finally, we should have

\[(4.11) \quad \text{wt} \left\{ z^k \xi^L (z, \xi) \nu^{\nu + s} \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = sk_0 + l k_0 + k + k_0 - 1, \quad \text{for all } k, l \in \mathbb{N},\]

because we may have

\[(4.12) \quad \text{wt} \left\{ z^k \xi^L (z, \xi) \nu^{\nu + s} \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = \text{wt} \left\{ z^k \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s + \text{wt} \left\{ \xi^L (z, \xi) \nu^{\nu + s} \right\}, \quad \text{for all } k, l \in \mathbb{N},\]

and as well we may have

\[(4.13) \quad \text{wt} \left\{ z^k \xi^L (z, \xi) \nu^{\nu + s} \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = \text{wt} \left\{ \nu^{\nu + s} \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s + \text{wt} \left\{ z^k \xi^L (z, \xi) \nu^{\nu + s} \right\}, \quad \text{for all } k, l \in \mathbb{N},\]

and as well we may have

\[(4.14) \quad \text{wt} \left\{ z^k \xi^L (z, \xi) \nu^{\nu + s} \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = \text{wt} \left\{ \xi^L (z, \xi) \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s + \text{wt} \left\{ z^k \nu^{\nu + s} \right\}, \quad \text{for all } k, l \in \mathbb{N},\]

using (4.12), (4.13) and (4.11), and it becomes clear that we should have

\[(4.15) \quad \text{wt} \left\{ \xi^L (z, \xi) \nu^{\nu + s} \right\} = s + k_0 - 1,\]

and that we should have

\[(4.16) \quad \text{wt} \left\{ z^k \xi^L (z, \xi) \right\} = k + k_0 - 1, \quad \text{for all } k \in \mathbb{N},\]

and, of course, that we should have

\[(4.17) \quad \text{wt} \left\{ z^k \nu^{\nu + s} \right\} = k + s, \quad \text{for all } k \in \mathbb{N}^*,\]

but it remains to observe by (4.10) that we should have

\[(4.18) \quad \text{wt} \left\{ z^k \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = \text{wt} \left\{ z^k \right\} + \text{wt} \left\{ \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = k + l k_0, \quad \text{for all } k, l \in \mathbb{N},\]

and by (4.14) and (4.15) that we should have

\[(4.19) \quad \text{wt} \left\{ \nu^{\nu + s} \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = \text{wt} \left\{ \nu^{\nu + s} \right\} + \text{wt} \left\{ \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = s + k_0, \quad \text{for all } l \in \mathbb{N},\]

and by (4.11) that we should have

\[(4.20) \quad \text{wt} \left\{ \xi^L (z, \xi) \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = \text{wt} \left\{ \xi^L (z, \xi) \right\} + \text{wt} \left\{ \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} = k_0 - 1 + l k_0, \quad \text{for all } l \in \mathbb{N}.\]

On the other hand, (4.20) may not hold, because (4.11) should hold. Then, (4.11) may not hold, because we could have

\[(4.21) \quad \text{wt} \left\{ z^k \xi^L (z, \xi) \nu^{\nu + s} \right\} = k - 1 + k_0 + l k_0, \quad \text{for all } k, l \in \mathbb{N} \text{ with } k \neq 0,\]

because we could have

\[(4.22) \quad \text{wt} \left\{ z^k \xi^L (z, \xi) \nu^{\nu + s} \right\} = \text{wt} \left\{ \left( \nu^\nu + 2\sqrt{-1} \nu^s \xi^L (z, \xi) \right)^s \right\} + \text{wt} \left\{ z^k \xi^L (z, \xi) \nu^{\nu + s} \right\}, \quad \text{for all } k, l \in \mathbb{N} \text{ with } k \neq 0,\]

because (4.11) should hold, or we could have

\[(4.23) \quad \text{wt} \left\{ z^k \xi^L (z, \xi) \nu^{\nu + s} \right\} = l k_0 + s k_0 + k_0 - 1 + k, \quad \text{for all } k, l \in \mathbb{N} \text{ with } k \neq 0,
because (4.16) should hold, or we could have

\[ \text{wt} \left\{ \mathcal{L}_\nu (z, \xi) \nu^a (\nu + 2 \sqrt{-1} \nu' \mathcal{L}(z, \xi))' \right\} = ik_0 + s + k_0 - 1, \quad \text{for all } k \in \mathbb{N}, \]

because (4.16) should hold.

Now, because we can define the weight of a polynomial expression in different ways (see (4.11), (4.21), (4.25), (4.24)), it is not clear yet which is the most suitable definition. Then, we need to move forward by [11] to:

### 4.1. System of Pseudo-Weights

We define

\[ \text{wt} \left\{ z^{\alpha \xi^\beta} \right\} = \alpha + \beta, \quad \text{for all } \alpha, \beta \in \mathbb{N}, \]

making clear (4.2), (4.3) and (4.16).

Similarly, we define

\[ \text{wt} \left\{ \xi^\alpha \nu^\beta \right\} = \alpha + \beta, \quad \text{for all } \alpha, \beta \in \mathbb{N} \text{ with } \alpha \neq 0. \]

making (4.8) clear.

Similarly, we define

\[ \text{wt} \left\{ z^{\alpha \nu^\beta} \right\} = \alpha + \beta, \quad \text{for all } \alpha, \beta \in \mathbb{N} \text{ with } \alpha \neq 0. \]

making (4.17) clear.

Next, we define

\[ \text{wt} \left\{ \nu^N z^{\alpha \xi^\beta} \right\} = \begin{cases} N + \alpha + \beta, & \text{for all } N, \alpha, \beta \in \mathbb{N} \text{ with } \alpha + \beta < k_0 \text{ and } \alpha \neq 0 \text{ or } \beta \neq 0, \\ N - s + \alpha + \beta, & \text{for all } N, \alpha, \beta \in \mathbb{N} \text{ with } \alpha + \beta = k_0 \text{ and } \alpha \neq 0 \text{ and } \beta \neq 0, \\ \end{cases} \]

making (4.10) and (4.11) clear, and extending (4.25), (4.26) and (4.27).

Now, in order to provide further definitions, we observe that the homogeneous polynomial \( \mathcal{L} \) of degree \( k_0 \), is defined by monomials

\[ z^{a \nu^b}, \quad \text{with } a + b = k_0 \text{ with } a, b \in \mathbb{N}^*. \]

Then, we define

\[ \text{wt} \left\{ \nu^N z^{a \nu^b} \right\} = (N - (s - 1)\beta) k_0, \quad \text{for all } N, \beta, a, b \in \mathbb{N} \text{ with } a + b = k_0 \text{ and } (N - (s - 1)\beta) k_0 \geq 0, \]

making (4.5) clear.

Therefore, it is required to define

\[ \text{wt} \left\{ \nu^N z^{a \nu^b} \right\} = (N - (s - 1)\beta) k_0 + c, \quad \text{for all } c, \beta, a, b \in \mathbb{N} \text{ with } a + b = k_0 \text{ and } (N - (s - 1)\beta) k_0 \geq 0, \]

\[ \text{wt} \left\{ z^{a \nu^b} \right\} = (N - (s - 1)\beta) k_0 + c, \quad \text{for all } c, \beta, a, b \in \mathbb{N} \text{ with } a + b = k_0 \text{ and } (N - (s - 1)\beta) k_0 \geq 0. \]

Otherwise, when \((N - (s - 1)\beta) k_0 \leq 0\), we define

\[ \text{wt} \left\{ \nu^N z^{a \nu^b} \right\} = (N - (s - 1)\beta) k_0 + (a + b) (\beta - \beta'), \]

for all \( N, a, b \in \mathbb{N} \) with \( a + b = k_0 \) and \( \beta', \beta \in \mathbb{N} \) maximal such that \((N - (s - 1)\beta') k_0 \geq 0\).

Clearly, the best definition is attained when the right-hand side of (4.31) is minimal, more precisely when \( \beta' \in \mathbb{N} \) is maximal satisfying the above property, because there exist more evaluations available. We denote them by

\[ \text{wt}_{N,a,b} \left\{ \nu^N z^{a \nu^b} \right\}, \quad \text{where } N, a, b \in \mathbb{N} \text{ and } a, b \neq 0. \]

Therefore, the best definition is clearly the following

\[ \text{wt} \left\{ \nu^N z^{a \nu^b} \right\} = \min \left( \text{wt}_{N,a,b} \left\{ \nu^N z^{a \nu^b} \right\} \right), \quad \text{where } N, a, b \in \mathbb{N} \text{ and } a, b \neq 0. \]

Now, the Model (4.1) becomes pseudo-weighted-homogeneous in respect to the system of pseudo-weights from (4.25), (4.26), (4.27), (4.28), (4.29), (4.30), (4.31). Then, it remains to study the eventual overlappings of following terms

\[ g_k z^k \left( \nu + 2 \sqrt{-1} \nu' \mathcal{L}(z, \xi) + Q_{k_0+1}(z, \xi, \nu) \right), \quad g_{k'} z^{k'} \left( \nu + 2 \sqrt{-1} \nu' \mathcal{L}(z, \xi) + Q_{k_0+1}(z, \xi, \nu) \right), \]

where \( 2 \leq N = k + k_0 < N' = k' + k_0, \) for \( k, k', k'' \in \mathbb{N} \).

These terms do not overlap in (4.33), because they have different pseudo-weights, according to the following pseudo-weighted-evaluations

\[ \left( \nu + 2 \sqrt{-1} \nu' \mathcal{L}(z, \xi) + Q_{k_0+1}(z, \xi, \nu) \right)' = \left( \nu + 2 \sqrt{-1} \nu' \mathcal{L}(z, \xi) \right)' + \text{wt}_{k_0+1} (\nu, z, \xi, \nu), \]

\[ \left( \nu + 2 \sqrt{-1} \nu' \mathcal{L}(z, \xi) + Q_{k_0+1}(z, \xi, \nu) \right)' = \left( \nu + 2 \sqrt{-1} \nu' \mathcal{L}(z, \xi) \right)' + \text{wt}_{2k_0+1} (\nu, z, \xi, \nu). \]

Similarly, it does not exist any overlapping among the following terms

\[ f_k z^k \mathcal{L}(z, \xi) \nu^a (\nu + 2 \sqrt{-1} \nu' \mathcal{L}(z, \xi))', \quad f_{k'} z^{k'} \mathcal{L}(z, \xi) \nu^a (\nu + 2 \sqrt{-1} \mathcal{L}(z, \xi))', \]

where \( N = k + k_0 + 1 < N' = k' + k_0, \) for \( k, k', k'' \in \mathbb{N}, \) because (4.21) and (4.24) hold according to (4.33).
4.2. General Equation. We have

\[
\sum_{k+k_0l=N} \left( g_{kl} z^k \left( \nu + 2\sqrt{-1}\nu^s \mathcal{L}(z, \xi) \right)^l \right) - \sum_{k+l=N} \tilde{g}_{kl} \xi^k \mu^l \frac{2\sqrt{-1}}{1} + \mathcal{L}_z (z, \xi) \nu^s \sum_{k \neq 0} f_{kl} z^k \left( \nu + 2\sqrt{-1}\nu^s \mathcal{L}(z, \xi) \right)^l + \mathcal{L}_\xi (z, \xi) \nu^s \sum_{l \neq 0} f_{kl} \xi^k \mu^l
\]

(4.37)

where we have used the following expression

\[
E'_N (z, \xi, \nu) = E'_N \left( z, \xi, \nu; (f_{kl})_{k+k_0l<N-k_0+1}, \left( \tilde{f}_{kl} \right)_{k+\tilde{l}<N-k_0+1}, (g_{kl})_{k+k_0l<N}, (\tilde{g}_{kl})_{k+l<N} \right),
\]

\[
E''_N (z, \xi, \nu) = E''_N \left( z, \xi, \nu; (f_{kl})_{k+k_0l<N-k_0+1}, \left( \tilde{f}_{kl} \right)_{k+\tilde{l}<N-k_0+1} \right),
\]

which shows that in order to compute (4.37) from (4.38), it suffices to consider induction depending on \( N \geq k_0 + 1 \).

Then, in order to determine \( \tilde{H} \) defined in (2.19), it suffices to focus on the coefficients of the following terms

(4.38)

\( \mathcal{L}_\xi (z, \xi) \xi^k \mu^l \nu^p \), for all \( k', \tilde{l} \in \mathbb{N} \) with \( k' + \tilde{l} = N - k_0 + 1 \),

and respectively, on the following terms

(4.39)

\( \xi^k \mu^l \nu^p \), for all \( k, \tilde{l} \in \mathbb{N} \) with \( k + \tilde{l} = N \).

Like when \( s = 0 \), the Model (2.11) can not be used in order to consider classical Fischer Decompositions. On the other hand, we make computations in the both sides from (4.37) in order to apply:

4.3. Pseudo-Weighted Fischer Decompositions. We consider Decompositions like in (2.20) and (2.22) according to the previous system of pseudo-weights, and then the Operator (2.21). Such Fischer Decompositions hold according to classical Fischer Decompositions already recalled in (2.20) and (2.22). In particular, we use Pseudo-Weighted Fischer Decompositions considering \( F(z, \xi, \nu) \) just a homogeneous polynomial and

\[
P_1 (z, \xi, \nu) = z^k \left( \nu + 2\sqrt{-1}\nu^s \mathcal{L}(z, \xi) \right)^l,
\]

\[
P_2 (z, \xi, \nu) = \xi^k \mu^l \nu^p,
\]

\[
P_3 (z, \xi, \nu) = z^k \mathcal{L}_z (z, \xi) \nu^s \left( \nu + 2\sqrt{-1}\nu^s \mathcal{L}(z, \xi) \right)^l,
\]

\[
P_4 (z, \xi, \nu) = \xi^k \mathcal{L}_\xi (z, \xi) \xi^p \nu^p + \nu^s,
\]

\[
P_5 (z, \xi, \nu) = \mathcal{L}_z (z, \xi) \nu^s \left( \nu + 2\sqrt{-1}\nu^s \mathcal{L}(z, \xi) \right)^l,
\]

\[
P_6 (z, \xi, \nu) = \mathcal{L}_\xi (z, \xi) \nu^p \nu^p + \nu^s,
\]

using the following hypothesis

\[
k + k_0l = k + l + s = s + k_0 - 1 + l''k_0 = k_0 - 1 + \tilde{l} + s = N \geq k_0 + 1,
\]

\[
k' + k_0l' = k' + \tilde{l}' = N - k_0 + 1,
\]

for \( k, l, k', \tilde{l}, k'', \tilde{l}' ; k', \tilde{l}', \tilde{l}' \in \mathbb{N} \) with \( k, k'', \tilde{k}' \neq 0 \).

Moreover, the formal power series \((G_1, G_2, G_3, G_4, R)(z, \xi, \nu)\) are just homogeneous polynomials in \((z, \xi, \nu)\) in (4.41), which furthermore are uniquely determined in (2.22) when (4.37) holds. Then, the Formal Equivalence (2.11) is determined from (4.37) as follows:

5. Computation of the Formal Equivalence (2.11) for \( s \neq 0 \)

We impose the following normalization condition

\[
\sum_{\text{pseudo-weight} \geq N} \phi_{mnp} z^m \xi^n \nu^p \in \tilde{S}_N, \quad \text{for all } N \geq k_0 + 1,
\]

(5.1)
by defining via (4.41) the Space of Fischer Normalizations 

$$S_N = \left( \bigcap_{k + k_0 = N} \ker \left( z^k (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{N-k_0} \right)^* \right) \cap \left( \bigcap_{k + k_0 = N} \ker \left( \xi_z^k \nu^l + s \right)^* \right)$$

$$\bigcap_{k' + k_0 = N - k_0 + 1} \ker \left( z^{k'} \nu^l L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^l \right)^* \cap \left( \bigcap_{k' + k_0 = N - k_0 + 1} \ker \left( L_\xi (z, \xi) \xi_z^k \nu^l + s \right)^* \right),$$

$$\bigcap_{k' + k_0 = N - k_0 + 1} \ker \left( \nu^k L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{N-k_0+1} \right)^* \cap \ker \left( L_\xi (z, \xi) \nu^{N-k_0+1} \right)^* \bigcup \text{for all } N \geq k_0 + 1.$$

Then, (5.1) is well-defined, because

$$\begin{cases} \xi_z^k \nu^l + s, \xi_z^k \nu^l L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^l, L_\xi (z, \xi) \xi_z^k \nu^l + s, \\ z^{k_1} (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{l_1}, \xi_z^{k_1} \nu^l + s, z^{k_1} \nu^l L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^l, L_\xi (z, \xi) \xi_z^{k_1} \nu^l + s, \\ \nu^k L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{N-k_0+1}, L_\xi (z, \xi) \nu^{N-k_0+1} \end{cases} \subseteq \ker \left( z^k (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{l_1} \right)^*,$$

and

$$\begin{cases} \xi_z^k \nu^l + s, \xi_z^k \nu^l L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^l, L_\xi (z, \xi) \xi_z^k \nu^l + s, \\ z^{k_1} (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{l_1}, \xi_z^{k_1} \nu^l + s, z^{k_1} \nu^l L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^l, L_\xi (z, \xi) \xi_z^{k_1} \nu^l + s, \\ \nu^k L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{N-k_0+1}, L_\xi (z, \xi) \nu^{N-k_0+1} \end{cases} \subseteq \ker \left( \xi_z^k \nu^l + s \right)^*,$$

and

$$\begin{cases} \xi_z^k \nu^l + s, \xi_z^k \nu^l L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^l, L_\xi (z, \xi) \xi_z^k \nu^l + s, \\ z^{k_1} (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{l_1}, \xi_z^{k_1} \nu^l + s, z^{k_1} \nu^l L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^l, L_\xi (z, \xi) \xi_z^{k_1} \nu^l + s, \\ \nu^k L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{N-k_0+1}, L_\xi (z, \xi) \nu^{N-k_0+1} \end{cases} \subseteq \ker \left( \xi_z^k \nu^l + s \right)^*,$$

and

$$\begin{cases} \xi_z^k \nu^l + s, \xi_z^k \nu^l L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^l, L_\xi (z, \xi) \xi_z^k \nu^l + s, \\ z^{k_1} (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{l_1}, \xi_z^{k_1} \nu^l + s, z^{k_1} \nu^l L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^l, L_\xi (z, \xi) \xi_z^{k_1} \nu^l + s, \\ \nu^k L_z (z, \xi) (\nu + 2\sqrt{1 - \nu^2} L(z, \xi))^{N-k_0+1}, L_\xi (z, \xi) \nu^{N-k_0+1} \end{cases} \subseteq \ker \left( \xi_z^k \nu^l + s \right)^*,$$
and
\[
\begin{align*}
\left\{ z^k (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi)) \right\}^{l+1}, z^{k+1} (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi)) \right\}^{l+1},
\end{align*}
\]
(5.9)
\[
\begin{align*}
\nu^a \Lambda (z, \xi) (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi))^{N-k+1},
\end{align*}
\]
\[
\begin{align*}
\nu^a \Lambda (z, \xi) (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi))^{N-k+1} \subseteq \ker \left( \nu^a \Lambda (z, \xi) (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi))^{N-k+1} \right)^*,
\end{align*}
\]
and
\[
\begin{align*}
\left\{ z^k (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi)) \right\}^{l+1}, z^{k+1} (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi)) \right\}^{l+1},
\end{align*}
\]
(5.10)
\[
\begin{align*}
\nu^a \Lambda (z, \xi) (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi))^{N-k+1} \subseteq \ker \left( \nu^a \Lambda (z, \xi) (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi))^{N-k+1} \right)^*,
\end{align*}
\]
and
\[
\begin{align*}
\left\{ z^k (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi)) \right\}^{l+1}, z^{k+1} (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi)) \right\}^{l+1},
\end{align*}
\]
(5.11)
\[
\begin{align*}
\nu^a \Lambda (z, \xi) (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi))^{N-k+1} \subseteq \ker \left( \nu^a \Lambda (z, \xi) (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi))^{N-k+1} \right)^*,
\end{align*}
\]
where we have
\[
\begin{align*}
k + k_0 l = k + l + s = k_0 + 1 - 1 = l_0 k_0 l_0 = k_0 - 1 + l' + s = N \geq k_0 + 1,
k_1 + k_0 l_1 = k_1 + l_1 + s = k_0 - 1 + l_1' + k_0 = k_0 - 1 + l_1' + s = N \geq k_0 + 1,
k' + k_0 l' = k' + l = N - k_0 + 1,
k'' + k_0 l'' = k'' + l = N - k_0 + 1,
k_0 = k = k', k_0 l' = k_0 l'' = k'' = \text{and:}
k_0 \neq k_1, l \neq l_1, k \neq k_1, l \neq l_1, k' \neq k_0, l' \neq l_1.
\end{align*}
\]
Then, (2.11) is determined by induction with respect to $N \geq k_0 + 1$. We obtain:

**Proposition 5.1.** Let $\mathcal{M}$ and $\mathcal{M}'$ be defined as in (2.3) and (2.4) such that $s > 0$. Then, there exists a unique formal mapping defined in (2.11), (2.15) and (2.18), which transforms $\mathcal{M}$ into $\mathcal{M}'$, such that the normalizations [5.1] are satisfied.

### 6. Computation of the Formal Equivalence (2.11) for $s = 0$

We impose by induction the following normalization condition
\[
\begin{align*}
\sum_{m+n+p=k_0+N} \varphi_m \varphi_n \varphi_p \tau^m \xi^n \nu^p \in \mathcal{S}_N,
\end{align*}
\]
(6.1)
by defining the Space of Fischer Normalizations
\[
\begin{align*}
\mathcal{S}_N = \bigcap_{k+l=N} \left( \ker \left( z^k (\nu + \sqrt{-1} \nu^a \Lambda (z, \xi)) \right)^l \right) \bigcap \ker \left( \xi^k \nu^l \right)^*,
\end{align*}
\]
\[
\begin{align*}
\bigcap_{k_l+k_0 l_1=N-k_0+1} \left( \ker \left( z^{k_l} \Lambda (z, \xi) \right)^{l_1} \right) \bigcap \ker \left( \xi^{k_l} \nu^{l_1} \right)^*,
\end{align*}
\]
(5.12)
Then, (6.1) is well-defined, because
\[
\begin{align*}
\left\{ \xi^k \nu^l \right\}^{l_1}, \left\{ \xi^{k_l} \nu^{l_1} \right\}^{l_1},
\end{align*}
\]
\[
\begin{align*}
 z^k (\nu + 2\sqrt{-1} L (z, \xi))^{l'}, \xi^k \nu^{l'}, z^{k'} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l'} , \\
z^{k1} (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, \xi^{k1} \nu^{l1}, z^{k1} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, L_z (z, \xi) \xi^{k1} \nu^{l1} \subseteq \ker \left( L_z (z, \xi) \xi^{k1} \nu^{l1} \right) ,
\end{align*}
\]

Then we move forward:

6.2. Proof of Theorem 1.1.

Before explaining the convergence, it is introduced the following (holomorphic) mapping

\[
\begin{align*}
 z^k (\nu + 2\sqrt{-1} L (z, \xi))^{l'}, \xi^k \nu^{l'}, z^{k'} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l'} , \\
z^{k1} (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, \xi^{k1} \nu^{l1}, z^{k1} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, L_z (z, \xi) \xi^{k1} \nu^{l1} \subseteq \ker \left( z^{k1} (\nu + 2\sqrt{-1} L (z, \xi))^{l1} \right) ,
\end{align*}
\]

\[
\begin{align*}
 z^k (\nu + 2\sqrt{-1} L (z, \xi))^{l'}, \xi^k \nu^{l'}, z^{k'} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l'} , \\
z^{k1} (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, \xi^{k1} \nu^{l1}, z^{k1} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, L_z (z, \xi) \xi^{k1} \nu^{l1} \subseteq \ker \left( z^{k1} (\nu + 2\sqrt{-1} L (z, \xi))^{l1} \right) ,
\end{align*}
\]

where we have

\[
\begin{align*}
 k, \bar{k}, \bar{l}, k', \bar{l}', \bar{k}', \bar{l}', k_1, l_1, \bar{k}_1, \bar{l}_1, k'_1, \bar{l}'_1, \bar{k}'_1, \bar{l}'_1 \in \mathbb{N} \text{ such that:} \\
k + k_0 \bar{l} = \bar{k} + \bar{l}' = k_1 + k_0 \bar{l}_1 = \bar{k}_1 + \bar{l}_1 = N, \\
k' + k_0 \bar{l}' = \bar{k}' + \bar{l}'_1 = k'_1 + k_0 \bar{l}'_1 = \bar{k}'_1 + \bar{l}'_1 = N - k_0 + 1, \\
k \neq k_1, \quad \bar{k} \neq \bar{k}_1, \quad k' \neq k'_1, \quad \bar{k}' \neq \bar{k}'_1.
\end{align*}
\]

Clearly, (6.1) is a well-defined Fischer Decomposition, because we restrict (6.10), (6.12) becomes (6.2). Other computations are similar like above.

We move forward:

6.1. Essential Ingredient. Finally, we move forward using the following diagram:

\[
\begin{align*}
 H_1, H_2 : \mathcal{M} & \to \mathcal{M} \\
 \frac{\mathcal{H}_1}{\mathcal{H}_2} & \cong \frac{\mathcal{M}}{\mathcal{M}}.
\end{align*}
\]

where \( \mathcal{H} \) denotes the construction of formal normal form type.

6.2. Proof of Theorem 1.1. Clearly, any two Formal Equivalences, which have the same 1-jet, are actually identical because of the uniqueness of the Formal Segre Equivalence sending \( \mathcal{M} \) into \( \mathcal{M} \). Then, the determination by 1-jets follows from (6.3).

6.3. Proof of Corollary 1.2. We move forward:

7. Proof of Theorem 1.3

We move forward:

8. Proofs of Corollaries 1.4, 1.5 and 1.6

Before explaining the convergence, it is introduced the following (holomorphic) mapping

\[
\begin{align*}
 H (\nu, z; H (\nu, \xi)) = 0,
\end{align*}
\]

which rewrites \( 2.4 \) as

\[
\begin{align*}
 z^k (\nu + 2\sqrt{-1} L (z, \xi))^{l'}, \xi^k \nu^{l'}, z^{k'} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l'} , \\
z^{k1} (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, \xi^{k1} \nu^{l1}, z^{k1} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, L_z (z, \xi) \xi^{k1} \nu^{l1} \subseteq \ker \left( z^{k1} (\nu + 2\sqrt{-1} L (z, \xi))^{l1} \right) ,
\end{align*}
\]

\[
\begin{align*}
 z^k (\nu + 2\sqrt{-1} L (z, \xi))^{l'}, \xi^k \nu^{l'}, z^{k'} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l'} , \\
z^{k1} (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, \xi^{k1} \nu^{l1}, z^{k1} L_z (z, \xi) (\nu + 2\sqrt{-1} L (z, \xi))^{l1}, L_z (z, \xi) \xi^{k1} \nu^{l1} \subseteq \ker \left( z^{k1} (\nu + 2\sqrt{-1} L (z, \xi))^{l1} \right) ,
\end{align*}
\]

which is obviously of maximum generic rank.
where we have used the following notations

\[
\begin{aligned}
&\frac{1}{2}\left(\sum_{k+l \geq 1} g_{kl}z^k w^l - \sum_{k+l \geq 1} \tilde{g}_{kl} \xi^k \nu^l\right) = \frac{1}{2}\left(\sum_{k+l \geq 1} g_{kl}z^k w^l + \sum_{k+l \geq 2} \tilde{g}_{kl} \xi^k \nu^l\right) \\
&+ \mathcal{L}\left(\sum_{k+l \geq 1} f_{kl} z^k w^l, \sum_{k+l \geq 1} \tilde{f}_{kl} \xi^k \nu^l\right)
\end{aligned}
\]

(8.3)

because such number \(H\) is not trivial, resulting \(\sum_{k+l \geq 1} g_{kl}z^k w^l\).

In particular for \(z = w = 0\), (8.3) implies the convergence of the following formal power series

(8.5)

\[
\sum_{k+l \geq 1} \tilde{g}_{kl} \xi^k \nu^l.
\]

Then (8.4) and (8.5) imply by (8.3) the convergence of the following formal power series

\[
S(w, z; \nu, \xi) := \left(\sum_{k+l \geq 1} g_{kl}z^k w^l + \sum_{k+l \geq 1} \tilde{g}_{kl} \xi^k \nu^l\right)^s \cdot \mathcal{L}\left(\sum_{k+l \geq 1} f_{kl} z^k w^l, \sum_{k+l \geq 1} \tilde{f}_{kl} \xi^k \nu^l\right) + \sum_{m+n+p \geq k_0+1} \varphi_{mnp}^\prime \left(\sum_{k+l \geq 1} f_{kl} z^k w^l\right)^m \cdot \left(\sum_{k+l \geq 1} \tilde{f}_{kl} \xi^k \nu^l\right)^n \cdot \left(\sum_{k+l \geq 1} g_{kl}z^k w^l + \sum_{k+l \geq 1} \tilde{g}_{kl} \xi^k \nu^l\right)^p
\]

(8.6)

In order to move forward, let \(l_0 \in \mathbb{N}\) such that

\[
\frac{\partial^2}{\partial \xi^l_0} \left(\mathcal{L}(z, \xi)\right)|_{\xi=0} \neq 0,
\]

because such number \(l_0\) exists, since the homogeneous polynomial \(\mathcal{L}(z, \xi)\) is not trivial, resulting

\[
\left(\frac{\partial^2}{\partial \xi^l_0} \left(\frac{S(w, z; \nu, \xi)}{H_0(w, z)}\right) - \frac{\partial^0}{\partial \xi^l_0} \left(\mathcal{L}\left(\sum_{k+l \geq 1} f_{kl} z^k w^l, \xi\right)\right)\right)_{\xi=0} = \sum_{m+n+p \geq k_0+1} \varphi_{mnp}^\prime \left(\sum_{k+l \geq 1} f_{kl} z^k w^l\right)^m \cdot H_{n,p}(w, z),
\]

where we have used the following notations

\[
H_0(w, z) := \left(\sum_{k+l \geq 1} g_{kl}z^k w^l\right)^s,
\]

\[
H_{n,p}(w, z) := \left(\sum_{k+l \geq 1} \tilde{f}_{kl} \xi^k \nu^l\right)^n \cdot \left(\sum_{k+l \geq 1} g_{kl}z^k w^l + \sum_{k+l \geq 1} \tilde{g}_{kl} \xi^k \nu^l\right)^p
\]

\[
\left.\frac{\partial^0}{\partial \xi^l_0} \left(\frac{S(w, z; \nu, \xi)}{H_0(w, z)}\right)\right|_{\xi=0}\]

which are holomorphic functions near \(0 \in \mathbb{C}^2\).

Now, in order to have a well-defined fraction in the left-hand side of (8.6) and (8.7), it is required essentially to have

\[
H_0(w, z) \neq 0.
\]

Clearly, \(H_0(w, z)\) is a non-constant holomorphic function near \(0 \in \mathbb{C}^2\). In particular, there exists \((w_0, z_0) \in \mathbb{C}^2\) near \(0 \in \mathbb{C}^2\) such that

\[
H_0(w_0, z_0) \neq 0.
\]

By continuity, we can assume

\[
H_0(w, z) \neq 0,
\]

for all \((w, z) \in \mathbb{C}^2\) in a small open connected vicinity of \((w_0, z_0)\).
In order to obtain the convergence of the following formal power series
\[(8.7) \sum_{k+l \geq 1} f_{kl}z^kw^l, \]
we use the translation \((w, z) \mapsto \tilde{(w, z)} = (w_0, z_0)\).

defined near \((w_0, z_0) \in \mathbb{C}^2\), and then we apply Proposition 4.2 from [Mir27] in the equation (8.5), because the function \((?7)\) is real-analytic. It follows that the formal solution \((8.7)\) is unique and also convergent near \((w_0, z_0) \in \mathbb{C}^2\), and therefore holomorphic near \(w \in \mathbb{C}^2\) according to the Phenomenon of Hartogs, because the formal solution \((8.7)\) is holomorphic on a circular connected domain around \(0 \in \mathbb{C}^2\).

Analogously as previously, we obtain the local convergence of the following formal power series
\[(8.8) \sum_{k+l \geq 1} \tilde{f}_{kl}z^kw^l. \]

Now, (8.4), (8.5), (8.7) and (8.8) imply by (2.11) the convergence of \(\tilde{H}(\xi, \nu)\) and \(H(z, w)\) near \(0 \in \mathbb{C}^2\).

8.2. Proof of Corollary 1.5.

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