In this article, we study the distribution of values of Dirichlet $L$-functions, the distribution of values of the random models for Dirichlet $L$-functions, and the discrepancy between these two kinds of distributions. For each question, we consider the cases of $\frac{1}{2} < \Re s < 1$ and $\Re s = 1$ separately.

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1 INTRODUCTION

The analytic theory of $L$-functions is a central part of modern number theory. The study of distribution of values of $L$-functions is an important topic in the analytic theory of $L$-functions. In [1], Bohr and Jessen introduced a probability treatment to study the distribution of values of the Riemann zeta function. They proved that $\log \zeta(\sigma + it)$ has a continuous limiting distribution for any $\sigma > \frac{1}{2}$. On the critical line, we have Selberg’s central limit theorem. On the 1-line, Granville and Soundararajan [4] studied the distribution of $|\zeta(1 + it)|$, which is asymptotically a double exponentially decreasing function. Their method was also adjusted to apply to the distribution of values on the 1-line of other $L$-functions. In 2003, they [3] studied the distribution of the Dirichlet $L$-functions of quadratic characters $L(1, \chi_d)$, which proves part of Montgomery and Vaughan’s conjecture in [11]. In 2007, Wu [13] improved this result by giving a high-order expansion in the exponent of the distribution function. In 2008, Liu, Royer, and Wu [9] studied the distribution of a kind of symmetric power $L$-functions. In 2010, Lamzouri [6] studied a generalized $L$-function which can cover the results of [3, 9]. In the critical strip $\frac{1}{2} < \Re(s) < 1$, Lamzouri [7] in 2011 studied...
the distribution of $\log|\zeta(\sigma + it)|$ with any fixed $\frac{1}{2} < \sigma < 1$ and also got the asymptotic distribution function. In 2019, Lamzouri, Lester, and Radziwiłł [8] studied the discrepancy between the distribution of $\log|\zeta(\sigma + it)|$ and that of their random models. Later, in 2021, Xiao and Zhai [14] generalized this result to automorphic $L$-functions.

For each prime $p$, $X(p)$ denotes the independent random variable uniformly distributed on the unit circle. Then, the product

$$L(\sigma, X) := \prod_p \left(1 - \frac{X(p)}{p^\sigma}\right)^{-1}$$

converges almost surely for any $\sigma > \frac{1}{2}$. This random $L$-function turns out to be a very good model for the Riemann zeta function $\zeta(\sigma + it)$, where $t \in [T, 2T]$ as $T \to \infty$.

Now we turn our attention to Dirichlet $L$-functions. Let $q$ be a large prime number, and $\chi(\mod q)$ be any character modulo $q$. As $\chi$ varies modulo $q$, the values of $L(\sigma, \chi)$ behavior similarly to $\zeta(\sigma + it)$. So, the random $L$-function $L(\sigma, X)$ is a good model for $L(\sigma, \chi)$ as well. For every real positive number $\tau$ and any fixed $\frac{1}{2} < \sigma < 1$, we define the distribution functions separately by:

$$\Phi_q(\tau) = \Phi_q(\sigma, \tau) := \frac{1}{\phi(q)} \#\{\chi(\mod q) : \log|L(\sigma, \chi)| > \tau\},$$

and

$$\Psi(\tau) = \Psi(\sigma, \tau) := \text{Prob}(\log|L(\sigma, X)| > \tau).$$

In 2011, Lamzouri [7, Theorem 4.5] showed that there is a constant $a_0$ such that

$$\Phi(\tau) = \exp \left\{-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ a_0 + O \left( \frac{1}{\log \tau} + \left( \frac{(\tau \log \tau)^{\frac{1-\sigma}{1-\sigma}}}{\log q} \right)^{\frac{\sigma-1}{2}} \right) \right\} \right\},$$

holds for $1 \ll \tau < b(\sigma)(\log q)^{1-\gamma}(\log_2 q)^{-1}$ with some constant $b(\sigma)$.

For $\sigma = 1$, we define the distribution functions slightly differently:

$$\Phi_{1,q}(\tau) := \frac{1}{\phi(q)} \#\{\chi(\mod q) : |L(1, \chi)| > e^{\gamma}\tau\},$$

and

$$\Psi_1(\tau) := \text{Prob}(|L(1, X)| > \tau),$$

where $\gamma$ is the Euler’s constant. In 2006, Granville and Soundararajan [4] showed that uniformly for $1 \ll \tau < \log_2 q - 20$,

$$\Phi_{1,q}(\tau) = \exp \left\{-\frac{e^\tau - A_0 - 1}{\tau} \left\{ 1 + O \left( \frac{1}{\sqrt{\tau}} + \sqrt{\frac{e^\tau}{\log q}} \right) \right\} \right\}. $$
First, we give asymptotic formula of $\Psi(\tau)$ and $\Psi_1(\tau)$ with a high-order expansion in the exponent, which improve [8, Corollary 7.7].

**Theorem 1.1.** Let $1/2 < \sigma < 1$ be fixed. Then, for any integer $N \geq 1$, there exist computable polynomials $a_0(\cdot), \ldots, a_N(\cdot)$ with $\deg a_i \leq i$ which depend only on $\sigma$ and $N$, such that for any $\tau \geq 2$, we have

$$
\Psi(\tau) = \exp \left( -(\tau \log^\sigma \tau)^{-1/1-\sigma} \left( \sum_{n=0}^{N} \frac{a_n(\log_2 \tau)}{(\log \tau)^n} + O \left( \frac{\log_2 \tau \log \tau}{\log \tau} \right) \right) \right),
$$

with $a_0 > 0$. When $\sigma = 1$, there is a sequence of real numbers $\{b_n\}_{n \geq 1}$ such that for any integer $N \geq 1$, we have

$$
\Psi_1(\tau) = \exp \left( -e^{-A_0-1} \tau \left( 1 + \sum_{n=1}^{N} b_n \frac{\tau^n}{\tau^n} + O \left( \frac{1}{\tau^{N+1}} \right) \right) \right)
$$

uniformly for $\tau \geq 2$, where $A_0 = A_{(0)}$ is defined in (12).

Now we present the relations between $\Psi(\tau)$ and $\Phi_q(\tau)$, and between $\Psi_1(\tau)$ and $\Phi_{1,q}(\tau)$, which are similar to [8, Theorem 1.3]

**Theorem 1.2.** Let $1/2 < \sigma < 1$ be fixed. There exists a positive constant $b(\sigma)$ such that for $3 \leq \tau \leq b(\sigma)(\log q)^{1-\sigma}(\log_2 q)^{-1}$, we have

$$
\Phi_q(\tau) = \Psi(\tau) \left( 1 + O \left( \frac{(\tau \log \tau)^{1-\sigma} \log_2 q}{(\log q)^\sigma} \right) \right).
$$

When $\sigma = 1$, there exists a constant $b$ such that uniformly for $1 \ll \tau \leq \log_2 q - b$, we have

$$
\Phi_{1,q}(\tau) = \Psi_1(\tau) \left( 1 + O \left( \frac{e^\tau \log_2 q}{\tau \log q} \right) \right).
$$

At last, we study the discrepancy between the distribution of Dirichlet $L$-functions and that of their random models. Let $R$ be any rectangle with sides parallel to the coordinate axis. Let

$$
\Phi_q(R) := \frac{1}{q} \# \{ \chi(\mod q) : \log L(\sigma, \chi) \in R \},
$$

and

$$
\Psi(R) := \text{Prob}(\log L(\sigma, X) \in R).
$$

The discrepancy between the above two probabilities is defined by

$$
D_\sigma(q) := \sup_R |\Phi_q(R) - \Psi(R)|,
$$
where \( R \) runs through all the rectangles with sides parallel to the coordinate axis. We have the following theorem.

**Theorem 1.3.** Let \( \frac{1}{2} < \sigma < 1 \) be fixed and \( q \) be a large prime number. Then we have

\[
D_\sigma(q) \ll \frac{1}{(\log q)^\sigma}.
\]

When \( \sigma = 1 \), we have

\[
D_1(q) \ll \frac{(\log_2 q)^2}{\log q}.
\]

This article is organized as follows. In §2, we introduce some preliminary lemmas. In §3, we study the distribution of random models and prove Theorem 1.1. In §4, we study large deviations between the distribution of Dirichlet \( L \)-functions and that of their random models. We prove Theorem 1.2. In §5, we study the discrepancy bound for Dirichlet \( L \)-functions and prove Theorem 1.3.

## 2 | PRELIMINARY LEMMAS

First, we state a lemma which approximates Dirichlet \( L \)-functions by their truncating sums.

**Lemma 2.1.** Let \( s = \sigma + it \) with \( |t| \leq 3q \), and let \( y \geq 2 \) be a real number. Let \( 1/2 \leq \sigma_0 < \sigma \), and suppose that the rectangle \( \{ z : \sigma_0 < \text{Re} z \leq 1, |\text{Im} z - t| \leq y + 3 \} \) contains no zeros of \( L(z, \chi) \). Then

\[
\log |L(s, \chi)| \ll \frac{\log q}{\sigma - \sigma_0}.
\]

Further, now putting \( \sigma_1 = \min(\sigma_0 + \frac{1}{\log y}, \frac{\sigma + \sigma_0}{2}) \), we have

\[
\log L(\sigma + it, \chi) = \sum_{n=2}^{\gamma} \frac{\Lambda(n)\chi(n)}{n^{\sigma+it} \log n} + O\left(\frac{\log q}{(\sigma_1 - \sigma)^2 \sigma_1^{-\sigma}}\right).
\]

**Proof.** See [3, Lemma 2.1]. \( \Box \)

We need the following zero-density estimates for Dirichlet \( L \)-functions.

**Lemma 2.2.** Let \( 1/2 \leq \sigma \leq 1 \) and \( N(\sigma, T, \chi) \) denote the number of zeros of \( L(s, \chi) \) in the region \( \text{Re} s \geq \sigma \) and \( |\text{Im} s| \leq T \). Then we have

\[
\sum_{\chi(\text{mod} q)} N(\sigma, T, \chi) \ll (qT)^{\frac{3-3\sigma}{1-\sigma}} (\log qT)^{14}.
\]

**Proof.** See [10, Theorem 12.1]. \( \Box \)
For any \( y > 1 \) and \( 1/2 < \sigma \leq 1 \), let

\[
R_y(\sigma, \chi) := \sum_{p^n \leq y} \frac{\chi(p)^n}{np^{n\sigma}}.
\]

With the help of Lemmas 2.1 and 2.2, we can show that with very few exceptions, the logarithms of the \( L \)-functions can be approximated by \( R_y(\sigma, \chi) \).

**Lemma 2.3.** Let \( q \) be a large prime number and \( 1/2 < \sigma \leq 1 \). Let \( (\log q)^{A(\sigma)} \leq y \leq q^{a(\sigma)} \) be a real number, where \( 0 < a(\sigma) < \frac{4\sigma-2}{7-2\sigma} < \frac{4}{2\sigma-1} \) and \( A(\sigma) \) are any constants. Then we have

\[
\log L(\sigma, \chi) = R_y(\sigma, \chi) + O\left(\frac{y^{1-2\sigma}}{4 (\log y)^2 \log q}\right)
\]

for all but at most \( q^{\frac{9-6\sigma}{7-2\sigma} y (\log q)^{14}} \) primitive characters \( \chi \pmod{q} \).

**Proof.** This follows from Lemmas 2.1 and 2.2 with the choice of \( \sigma_0 = (\sigma + 1/2)/2 \). \( \square \)

Now we need to give an estimation on the power of \( R_y(\sigma, \chi) \). The idea is to divide \( R_y(\sigma, \chi) \) into three parts. The following lemma gives the bound of the main part.

**Lemma 2.4.** Let \( q \) be a prime number and \( 1/2 < \sigma < 1 \). Suppose that \( y = (\log q)^a \) for some \( a \geq 1 \) and \( k \) is an integer with \( 1 < k < \log q / 3a \log_2 q \). Then there exists a constant \( c_1(\sigma) > 0 \) such that

\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} |R_y(\sigma, \chi)|^{2k} \ll \left( \frac{c_1(\sigma)k^{1-\sigma}}{(\log k)^2} \right)^{2k}.
\]

**Proof.** See [7, Lemma 4.4] \( \square \)

**Lemma 2.5.** Let \( q \) be a prime number and \( 1/2 < \sigma < 1 \). Suppose that \( y = (\log q)^a \) for some \( a \geq 1 \) and \( k \) is an integer with \( 1 < k < \frac{\log q}{3a \log_2 q} \). Then there exists a constant \( c_1(\sigma) > 0 \) such that

\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} |R_y(\sigma, \chi)|^{2k} \ll \left( \frac{c_1(\sigma)k^{1-\sigma}}{(\log k)^2} \right)^{2k}.
\]

**Proof.** The power mean inequality gives

\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} |R_y(\sigma, \chi)|^{2k} \ll \sum_{\chi} \left( \sum_{p \leq k \log k} \left| \frac{1}{p^{\sigma}} \right|^{2k} \right) + \sum_{k \log k \leq p \leq y} \left| \frac{\chi(p)}{p^{\sigma}} \right|^{2k} + \sum_{n \geq 2, p^n \leq y} \left| \frac{1}{np^{n\sigma}} \right|^{2k}.
\]

Note that when \( 1/2 < \sigma < 1 \), we have

\[
\sum_{p \leq x} \frac{1}{p^{\sigma}} \ll \frac{x^{1-\sigma}}{(1 - \sigma) \log x}.
\]
and when $\sigma > 1$, we have
\[ \sum_{x_0 \leq p \leq x_1} \frac{1}{p^\sigma} \ll \int_{x_0}^{\infty} \frac{1}{t^\sigma} d\pi(t) \ll \frac{x_0^{1-\sigma}}{(\sigma - 1) \log x_0}. \tag{1} \]

So, together with Lemma 2.4, we have
\[ \left( \sum_{p \leq k \log k} \frac{1}{p^\sigma} \right)^{2k} \ll \left( (k \log k)^{1-\sigma} \right)^{2k}, \]
and
\[ \frac{1}{\phi(q)} \sum_{\chi} \left| \sum_{k \log k \leq p \leq y} \chi(p) \right|^{2k} \ll \left( \sum_{k \log k \leq p \leq y} \frac{k}{p^{2\sigma}} \chi(p) \right)^{k} + O(q^{-1/2}), \]
\[ \ll \left( \frac{k(k \log k)^{1-2\sigma}}{(2\sigma - 1) \log k} \right)^{k} + O(q^{-1/2}). \]

Finally, the third term is dominated by $\left( \sum_{m \geq 2} m^{-2\sigma} \right)^{2k}$. This completes the proof. \( \square \)

The following lemma shows that there is only a small number of characters $\chi$ such that the values of $|R_y(\sigma, \chi)|$ are large.

**Lemma 2.6.** Let $q$ be a large prime number and $1/2 < \sigma < 1$. Suppose that $y = (\log q)^a$ for some $a \geq 1$. We define
\[ \mathcal{A}_q = \left\{ \chi(\text{mod } q) : |R_y(\sigma, \chi)| \geq \frac{(\log q)^{1-\sigma}}{\log_2 q} \right\}. \]

Then there exists a constant $c_2(\sigma) > 0$ such that
\[ \frac{\# \mathcal{A}_q}{\phi(q)} \ll \exp \left( - \frac{c_2(\sigma) \log q}{\log_2 q} \right). \]

**Proof.** It is easy to see that
\[ \# \mathcal{A}_q \left( \frac{(\log q)^{1-\sigma}}{\log_2 q} \right)^{2k} \ll \sum_{\chi(\text{mod } q)} |R_y(\sigma, \chi)|^{2k}. \]

We choose $k = \left\lfloor \frac{\log q}{c_1(\sigma) \log_2 q} \right\rfloor$ with $c_1'(\sigma) = \max\{1 + c_1(\sigma)\}^{1-\sigma}$, $3a$, then Lemma 2.5 gives
\[ \frac{\# \mathcal{A}_q}{\phi(q)} \ll \left( \frac{c_1(\sigma)}{c_1'(\sigma)^{1-\sigma}} \right)^{2 \log q} \frac{c_2(\sigma)}{c_1'(\sigma) \log_2 q} = \exp \left( - \frac{c_2(\sigma) \log q}{\log_2 q} \right), \]
where \( c_2(\sigma) = \frac{2}{c_1(\sigma)} \log \left( \frac{c_1(\sigma)^{1-\sigma}}{c_1(\sigma)} \right) \).

Similar to \( R_y(\sigma, \chi) \), we let

\[
R_y(\sigma, X) = \sum_{p^n \leq y} \frac{X(p)^n}{n p^\sigma}.
\]

The following lemma gives the relation between \( R_y(\sigma, \chi) \) and \( R_y(\sigma, X) \).

**Lemma 2.7.** Let \( 1/2 < \sigma < 1 \) and \( q \) be a large prime number. Suppose that \( y = (\log q)^a \) for some \( a \geq 1 \). Then, for any integers \( 0 \leq k, \ell \leq \frac{\log q}{a \log_2 q} \), we have

\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} R_y(\sigma, \chi)^k R_y(\sigma, \chi)^\ell = \mathbb{E} \left( R_y(\sigma, X)^k R_y(\sigma, X)^\ell \right).
\]

**Proof.** By expanding the \( R_y(\sigma, \chi) \), we have

\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} R_y(\sigma, \chi)^k R_y(\sigma, \chi)^\ell = \frac{1}{\phi(q)} \sum_{p_{n_1}^{n_{k+\ell}}} \sum_{\chi \pmod{q}} \frac{\chi(p_1^{n_1} \cdots p_k^{n_k}) \chi(p_{k+1}^{n_{k+1}} \cdots p_{k+\ell}^{n_{k+\ell}})}{n_1 p_1^{n_1} \cdots n_{k+\ell} p_{k+\ell}^{n_{k+\ell}}}.\]

The orthogonality of Dirichlet character shows that the inner sum vanishes unless \( p_1^{n_1} \cdots p_k^{n_k} \equiv p_{k+1}^{n_{k+1}} \cdots p_{k+\ell}^{n_{k+\ell}} \pmod{q} \). But we note that

\[
p_1^{n_1} \cdots p_k^{n_k} \leq y^k \leq q.
\]

This implies that the inner sum is nonvanishing if and only if \( p_1^{n_1} \cdots p_k^{n_k} = p_{k+1}^{n_{k+1}} \cdots p_{k+\ell}^{n_{k+\ell}} \). This exactly gives \( \mathbb{E}(R_y(\sigma, X)^k R_y(\sigma, X)^\ell) \). \( \square \)

We give a simple estimation on the partial sum that will be used frequently in the proof.

**Lemma 2.8.** Fix \( D > 0 \) and \( 1/2 \leq \sigma < 1 \). Let \( N = T/(D \log T) \). Then for any \( z \leq \frac{1}{2D e^2 T^\sigma} \), we have

\[
\sum_{n \geq N} \frac{1}{n!} \left( \frac{zT^{1-\sigma}}{\log T} \right)^n \leq e^{-N}.
\]

**Proof.** The Stirling’s approximation shows that \( n! \geq (n/e)^n \). Then, we have

\[
\sum_{n \geq N} \frac{1}{n!} \left( \frac{zT^{1-\sigma}}{\log T} \right)^n \leq \sum_{n \geq N} \left( \frac{e z T^{1-\sigma}}{N \log T} \right)^n = \sum_{n \geq N} (eDz T^{-\sigma})^n.
\]
By the assumption $eDzT^{-\sigma} \leq \frac{1}{2e}$, we have
\[
\sum_{n \geq N} \frac{1}{n!} \left( \frac{zT^{1-\sigma}}{\log T} \right)^n \leq \sum_{n \geq N} \frac{1}{(2e)^n} \leq e^{-N}.
\]
□

Next, we will investigate the discrepancy between the exponential of $R_y(\sigma, \chi)$ and that of the random variable $R_y(\sigma, X)$. The idea is to expand the exponential function, and the main part is given by Lemma 2.7. Then we only need to estimate the error terms.

**Lemma 2.9.** Let $q$ be a large prime number and $1/2 < \sigma < 1$. Suppose that $y = (\log q)^a$ for some $a \geq 1$. Then there exists a constant $c(\sigma, a)$ such that for any complex numbers $z_1, z_2$ with $|z_1|, |z_2| \leq c(\sigma, a)(\log q)^\sigma$, we have
\[
\frac{1}{\phi(q)} \sum_{\chi \in A_q^c} \exp \left( z_1 R_y(\sigma, \chi) + z_2 \overline{R}_y(\sigma, \chi) \right)
= \mathbb{E} \left( \exp \left( z_1 R_y(\sigma, X) + z_2 \overline{R}_y(\sigma, X) \right) \right) + \exp \left( -\frac{c_3(\sigma) \log q}{\log_2 q} \right),
\]
for some constant $c_3(\sigma) > 0$, where $A_q^c$ is the set of all $\chi(\mod q)$ with $\chi \notin A_q$.

**Proof.** Let $N = \frac{\log q}{2(a+\sigma) \log_2 q}$, then we have
\[
\frac{1}{\phi(q)} \sum_{\chi \in A_q^c} \exp \left( z_1 R_y(\sigma, \chi) + z_2 \overline{R}_y(\sigma, \chi) \right)
= \frac{1}{\phi(q)} \sum_{k + \ell \leq N} \sum_{\chi} - \frac{1}{\phi(q)} \sum_{k + \ell \leq N} \sum_{\chi \in A_q} + \frac{1}{\phi(q)} \sum_{k + \ell \geq N} \sum_{\chi \in A_q^c} \frac{z_1^k z_2^\ell R_y(\sigma, \chi)^k R_y(\sigma, \chi)^\ell}{k! \ell!}.
\]

Lemma 2.7 gives the estimation of the first term of (2):
\[
\frac{1}{\phi(q)} \sum_{k + \ell \leq N} \sum_{\chi} \frac{z_1^k z_2^\ell R_y(\sigma, \chi)^k R_y(\sigma, \chi)^\ell}{k! \ell!}
= \sum_{k + \ell \leq N} \left( \frac{z_1^k z_2^\ell}{k! \ell!} \mathbb{E} \left( R_y(\sigma, X)^k \overline{R}_y(\sigma, X)^\ell \right) + O(y^{k+\ell} / q) \right) = \mathbb{E} \left( \exp \left( z_1 R_y(\sigma, X) + z_2 \overline{R}_y(\sigma, X) \right) \right)
- \sum_{k + \ell \geq N} \frac{z_1^k z_2^\ell}{k! \ell!} \mathbb{E} \left( R_y(\sigma, X)^k \overline{R}_y(\sigma, X)^\ell \right) + O \left( \sum_{k + \ell \leq N} \frac{z_1^k z_2^\ell y^{k+\ell}}{k! \ell! q} \right).
\]

By Lemmas 2.5 and 2.7, the second term is bounded by
\[
\sum_{k + \ell \geq N} \frac{(z_1 + z_2)^{k+\ell}}{k! \ell!} \left( \frac{c_1(\sigma)(k + \ell)^{1-\sigma}}{(\log(k + \ell))^{\sigma}} \right) = \sum_{n \geq N} \frac{1}{n!} \left( \frac{c_1(\sigma)(z_1 + z_2)n^{1-\sigma}}{(\log n)^{\sigma}} \right)^n \sum_{k + \ell = n} \frac{n!}{k! \ell!}.
\]
Since $\sum_{k+\ell=n} \frac{n!}{k!\ell!} = 2^n$, so Lemma 2.8 shows that the summation above is bounded by $e^{-N}$ for some suitable constant $c(\sigma, a)$. For the third term, since $(yz_1)^N \ll \sqrt{q}$, we have
\[
\sum_{k+\ell \leq N} \frac{z_1^k z_2^\ell}{k!\ell!} q^{\frac{1}{2}} \ll \frac{(2(z_1 + z_2))^N}{q} \ll q^{-1/2}.
\]
Next, we consider the second term of Equation (2). By the Cauchy–Schwarz inequality, we have
\[
\left| \sum_{\chi \in \mathcal{A}_q} R_y(\sigma, \chi)^k \overline{R_y(\sigma, \chi)}^\ell \right| \leq \left( \# \mathcal{A}_q \sum_{\chi} |R_y(\sigma, \chi)|^{k+\ell} \right)^{1/2}.
\]
Together with Lemmas 2.5 and 2.6, we have
\[
\frac{1}{\phi(q)} \sum_{k+\ell \leq N} \sum_{\chi \in \mathcal{A}_q} \frac{z_1^k z_2^\ell}{k!\ell!} \overline{R_y(\sigma, \chi)^k R_y(\sigma, \chi)^\ell} \leq \exp \left( - \frac{c_2(\sigma) \log q}{2 \log_2 q} \right) \sum_{k+\ell \leq N} \frac{1}{k!\ell!} \left( c_1(\sigma) (k+\ell)^{1-\sigma} \right)^{k+\ell} \leq \exp \left( - \frac{c'_2(\sigma) \log q}{2 \log_2 q} \right).
\]
Finally, we consider the last term of Equation (2). Since $|R(\sigma, \chi)| < (\log q)^{1-\sigma} / \log_2 q$ when $\chi \in \mathcal{A}_q^c$, the third term is bounded by
\[
\sum_{n \geq N} \left( \frac{(z_1 + z_2)(\log q)^{1-\sigma}}{\log_2 q} \right)^n \sum_{k+\ell = n} \frac{1}{k!\ell!} = \sum_{n \geq N} \frac{1}{n!} \left( \frac{2(z_1 + z_2)(\log q)^{1-\sigma}}{\log_2 q} \right)^n.
\]
Then following from Lemma 2.8, it is bounded by $e^{-N}$. This completes the proof. \qed

Now we are able to prove the discrepancy between the exponential of $L(\sigma, \chi)$ and that of the random variable $L(\sigma, X)$ via Lemma 2.9. The following lemma gives the difference between the exponential of $\log L(\sigma, X)$ and the exponential of $R_y(\sigma, X)$.

**Lemma 2.10.** Let $y$ be a large positive real number, then for any real numbers $u, v$ with $|u| + |v| \leq y^{\sigma-1/2}$, we have
\[
\mathbb{E}(\exp (i u \text{ Re } \log L(\sigma, X) + i v \text{ Im } \log L(\sigma, X))) = \mathbb{E}(\exp (i u \text{ Re } R_y(\sigma, X) + i v \text{ Im } R_y(\sigma, X))) + O((|u| + |v|)/y^{\sigma-1/2}).
\]

**Proof.** [8, Lemma 4.1] \qed
Lemma 2.11. Let \( q \) be a prime number, \( A \geq 1 \) and \( 1/2 < \sigma < 1 \). Then there exists a constant \( c \) such that for any \( |u|, |v| \leq c(\log q)^{\sigma} \), we have

\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \exp \left( iu \Re \log L(\sigma, \chi) + iv \Im \log L(\sigma, \chi) \right)
\]

\[=\mathbb{E}(\exp \left( iu \Re \log L(\sigma, X) + iv \Im \log L(\sigma, X) \right)) + O((\log q)^{-A}).\]

**Proof.** Choose \( y = (\log q)^{\frac{4A+8}{2\sigma-1}} \), then Lemma 2.3 shows that there is a set \( B_q \) with \( \#B_q \ll q^{\frac{9-6\sigma}{7-2\sigma}} (\log q)^{\frac{4A+8}{2\sigma-1}+14} \) such that

\[
\log L(\sigma, \chi) = R_y(\sigma, \chi) + O((\log q)^{-A}), \quad \forall \chi \in B_q.
\]

Note that \( \frac{9-6\sigma}{7-2\sigma} < 1 \), so \( \frac{1}{\phi(q)} \#B_q \ll (\log q)^{-A} \). So, we have

\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \exp \left( iu \Re L(\sigma, \chi) + iv \Im L(\sigma, \chi) \right)
\]

\[=\frac{1}{\phi(q)} \sum_{\chi \in B_q^c} \exp \left( iu \Re R_y(\sigma, \chi) + iv \Im R_y(\sigma, \chi) \right) + O((\log q)^{-A})
\]

\[=\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \exp \left( iu \Re R_y(\sigma, \chi) + iv \Im R_y(\sigma, \chi) \right) + O((\log q)^{-A}).
\]

The last equality follows from Lemma 2.6 and \( \exp \left( -\frac{c_3(\sigma) log q}{\log \log q} \right) \ll (\log q)^{-A} \). On the other hand, by taking \( z_1 = (iu + v)/2, z_2 = (iu - v)/2 \) in Lemma 2.9, the quantity above equals to

\[
\mathbb{E}(\exp(\exp \left( iu \Re R_y(\sigma, X) + iv \Im R_y(\sigma, X) \right)) + O((\log q)^{-A}).
\]

Then this theorem just follows from Lemma 2.10. \( \Box \)

Finally, we give the analog of Lemma 2.11 for \( \sigma = 1 \).

Lemma 2.12. Let \( q \) be a large prime number. Then uniformly for all complex numbers \( z_1, z_2 \) in the region \( |z_1|, |z_2| \leq \frac{\log q}{50(\log q)^2} \), we have

\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \exp \left( iu \Re \log L(1, \chi) + iv \Im \log L(1, \chi) \right)
\]

\[=\mathbb{E}(\exp \left( iu \Re \log L(1, X) + iv \Im \log L(1, X) \right)) + O\left( \exp \left( -\frac{\log q}{2\log q} \right) \right).
\]

**Proof.** See [5, Theorem 9.2]. \( \Box \)
3 DISTRIBUTION OF RANDOM MODELS: PROOF OF THEOREM 1.1

3.1 When $\frac{1}{2} < \sigma < 1$

In this subsection, we will prove the first part of Theorem 1.1. We first give some basic asymptotic properties of Bessel function. The modified Bessel function of first kind is defined to be

$$I_0(u) := \sum_{n \geq 0} \frac{(u/2)^{2n}}{n!^2}.$$

Let $f(u) := \log I_0(u)$. We have the following properties for $f$ and its derivatives.

**Lemma 3.1.** For $0 \leq u \leq 1$, we have

$$f^{(m)}(u) \asymp \begin{cases} u^2 & \text{if } m = 0 \\ u & \text{if } m \geq 1 \text{ and } m \text{ odd} \\ 1 & \text{if } m \geq 1 \text{ and } m \text{ even}. \end{cases}$$

And when $u \gg 1$, we have

$$f^{(m)}(u) = \begin{cases} u + O(\log u) & \text{if } m = 0 \\ 1 + O(u^{-1}) & \text{if } m = 1 \\ (-1)^m \frac{(m-1)!}{2} u^{-m} + O(u^{-m-1}) & \text{if } m \geq 2. \end{cases}$$

For $z \in \mathbb{C}$, let us put

$$M(z) := \log \mathbb{E}(|L(\sigma, X)|^2).$$

For any integer $m \geq 0$ and prime number $p$, we define

$$E^{(m)}(z) := \frac{\partial^m}{\partial z^m} \mathbb{E}\left(|1 - \frac{X(p)}{p^{\sigma}}|^{-z}\right),$$

and

$$M^{(m)}_p(z) := \frac{\partial^m}{\partial z^m} \log \mathbb{E}\left(|1 - \frac{X(p)}{p^{\sigma}}|^{-z}\right).$$

To prove Theorem 1.1, we need to know the asymptotic property of $M^{(m)}_p$ for every prime number $p$. When $p$ is much larger than $\kappa$, the asymptotic behavior of $M^{(m)}_p$ relies on the first few terms in $p^{-\sigma}$. To be precise, we have the following.

**Lemma 3.2.** Let $1/2 < \sigma \leq 1$ and $\kappa$ be a real positive number. Suppose that $p$ is a large prime number with $\kappa < p^{(1+\epsilon)\sigma}$ for certain $\epsilon > 0$, then for any integer $m \geq 0$, we have

$$M^{(m)}_p(\kappa) = p^{-m\sigma} f^{(m)}\left(\frac{\kappa}{p^{\sigma}}\right) + O\left(\frac{1}{p^{(m+1-\epsilon)\sigma}}\right).$$
When \( m = 0 \), we have

\[
M_p(\kappa) = f\left(\frac{\kappa}{p^2}\right) + O\left(\frac{\kappa}{p^{2\sigma}}\right),
\]

where all the implicit constants depend only on \( m, \sigma, \) and \( \varepsilon \).

**Proof.** We begin with the estimation of \( \mathbb{E}(m) \). Since \( \kappa < p^{(1+\varepsilon)\sigma} \), we have

\[
\mathbb{E}(m) = \frac{1}{2\pi} \int_0^{2\pi} \left| e^{i\theta} p^{-\sigma} \right|^{-\kappa} |1 - e^{i\theta} p^{-\sigma}|^{-1} d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \exp\left(\frac{\kappa \cos \theta}{p^\sigma} + O\left(\frac{\kappa}{p^{2\sigma}}\right)\right)\left(\cos^m \theta + O\left(\frac{1}{p^{(m+1)\sigma}}\right)\right) d\theta
\]

\[
= p^{-m\sigma} I_0^{(m)}\left(\frac{\kappa}{p^\sigma}\right) + O\left(\frac{1}{p^{(m+1-\varepsilon)\sigma}}\right).
\]

Therefore,

\[
\gamma_m := \frac{1}{m!} \frac{\mathbb{E}(m)}{\mathbb{E}} = \frac{1}{m!} \frac{p^{-m\sigma} I_0^{(m)}(\kappa/p^\sigma)}{I_0(\kappa/p^\sigma)} + O\left(\frac{1}{p^{(m+1-\varepsilon)\sigma}}\right).
\]

By applying Faà di Bruno’s formula, we have at last

\[
M_{p}^{(m)}(\kappa) = \sum_{i_1 \geq 0, \ldots, i_N \geq 0} \frac{(-1)^{i_1 + \cdots + i_N - 1}}{i_1! \cdots i_N!} m!(i_1 + \cdots + i_N - 1)! \gamma_{i_1}^{i_1} \cdots \gamma_{i_N}^{i_N} \frac{1}{I_0(\kappa/p^\sigma)}
\]

\[
= p^{-m\sigma}(\log I_0)^{(m)}\left(\frac{\kappa}{p^\sigma}\right) + O\left(\frac{1}{p^{(m+1-\varepsilon)\sigma}}\right).
\]

In the case \( m = 0 \), we just keep the term \( O(\kappa/p^{2\sigma}) \) to get the desired result. \( \square \)

When \( p \) is smaller, for every single \( M_{p}^{(m)} \), the asymptotic expansion can be deduced from Watson’s lemma. By definition, we have

\[
\mathbb{E}(m) = \frac{1}{2\pi} \int_0^{2\pi} \left| e^{i\theta} p^{-\sigma} \right|^{-\kappa} |1 - e^{i\theta} p^{-\sigma}|^{-1} d\theta
\]

\[
= (-1)^m \int_0^{2\pi} e^{-\frac{\varepsilon}{2} g(\theta)} g^m(\theta) d\theta + O\left(e^{-\kappa^2}\right),
\]
where \( g(\theta) = \log \left( 1 - \frac{2 \cos \theta}{p^2} + \frac{1}{p^{2\sigma}} \right) \). Let \( G(\theta) = g(\theta) - g(0) \). It is easy to see that \( G(\theta) \) is monotonically increasing in the interval \([0, \frac{\pi}{2}]\) and reaches its maximal value \( A = G(\frac{\pi}{2}) \) at \( \theta = \frac{\pi}{2} \). Then for \( t \in [0, A(a)] \), we have

\[
\varphi(t) = \frac{1}{G'(G^{-1}(t))} = \frac{e^{\frac{t}{2} \sinh \frac{a}{2}}}{\sqrt{\cosh a - \cosh a \cosh t + \sinh t}},
\]

where \( a = \sigma \log p \in \left( \frac{1}{2} \log 2, \infty \right) \) and \( A(a) = \log \cosh a - \log(\cosh a - 1) \). The function \( \varphi(t) \) has the following expansion at \( t = 0 \):

\[
\varphi(t) = \sinh \frac{a}{2} t^{-\frac{1}{2}} + \frac{1}{4} (2 + \cosh a) \sinh \frac{a}{2} t^{\frac{1}{2}} + \frac{1}{96} (2 + 3 \cosh a)^2 \sinh \frac{a}{2} t^{\frac{3}{2}}
\]

\[
+ \frac{1}{384} (15 \cosh^3 a + 18 \cosh^2 a - 4 \cosh a - 8) \sinh \frac{a}{2} t^{\frac{5}{2}} + O\left(t^{\frac{7}{2}}\right).
\]

However, the remainder of the above expansion depends on \( p \). To give a uniform estimation for \( \mathbb{E}^{(m)} \) for \( p \), one must seek for a global asymptotic expansion. So, we need an elaborated estimation on the function \( \varphi \).

**Lemma 3.3.** There exists \( M > 0 \) such that the following uniform bound holds:

\[
|\varphi(t) - \sinh \frac{a}{2} t^{-\frac{1}{2}} - \frac{1}{4} (2 + \cosh a) \sinh \frac{a}{2} t^{\frac{1}{2}} - \frac{1}{96} (2 + 3 \cosh a)^2 \sinh \frac{a}{2} t^{\frac{3}{2}}| \leq \frac{M}{384} (15 \cosh^3 a + 18 \cosh^2 a - 4 \cosh a - 8) \sinh \frac{a}{2} t^{\frac{5}{2}},
\]

for all \( a \in \left[ \frac{1}{2} \log 2, \infty \right) \) and \( t \in [0, A(a)] \).

**Proof.** The function \( g(t) = \text{csch} \frac{a}{2} t^{\frac{1}{2}} \varphi(t) \) is analytic in the disk \( \{ t \in \mathbb{C} \mid |t| < R(a) \} \) where

\[
R(a) = \log \left( \frac{\cosh a + 1}{\cosh a - 1} \right) \in \left( 0, \log \left( 17 + 12 \sqrt{2} \right) \right].
\]

The function \( A(a)/R(a) \) is monotone decreasing with respect to \( a \). It attains its maximal value \( \rho = 0.811 \ldots \) when \( a = \frac{1}{2} \log 2 \). The Cauchy bound of \( g(t) \) for \( |t| < A(a) \) gives

\[
|g^{(3)}(t)| \leq \frac{3! 2\pi}{(\rho_0 - \rho)^4} R(a)^{-3} \max_{|t|=\rho_0 R} |g(t)|.
\]

Take \( \rho_0 = \frac{9}{10} \) and we get \( \max_{|t|=\rho_0 R} |g(t)| = |g(\rho_0 R(a))| < 17 \) for all \( a \). This implies

\[
|g^{(3)}(t)| \leq 2 \times 10^7 R(a)^{-3} \asymp e^{3a}
\]

as \( a \to \infty \). On the other hand,

\[
|g^{(3)}(0)| = \frac{1}{384} (15 \cosh^3 a + 18 \cosh^2 a - 4 \cosh a - 8) \asymp e^{3a}.
\]
Thus, there exists certain $M > 0$ such that $|g^{(3)}(t)| \leq M|g^{(3)}(0)|$. □

With the above estimations, we are now able to prove the following.

**Lemma 3.4.** Let $\kappa$ be a large real number. For any prime number $p$ with $\kappa > p^{(1+\varepsilon)\sigma}$ for some certain $\varepsilon > 0$, we have

$$M_p(\kappa) = -\kappa \log(1 - p^{-\sigma}) + O(\log \kappa),$$

$$M'_p(\kappa) = -\log(1 - p^{-\sigma})(1 + O(\kappa^{-\varepsilon})).$$

where the implicit constants depend only on $\sigma$ and $\varepsilon$.

**Proof.** We have

$$E^{(m)}(|1 - X(p)p^{-\sigma}|^{-\kappa}) = \frac{(-1)^m}{2^m\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{\kappa}{2} g(\theta)} g^m(\theta) d\theta + O\left(e^{-\kappa}\right)$$

$$= \frac{(-1)^m}{2^m\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{\kappa}{2}(G(\theta)+g(0))} (G(\theta) + g(0))^m d\theta + O\left(e^{-\kappa}\right)$$

$$= \frac{(-1)^m}{2^m\pi} e^{-\frac{\kappa}{2} g(0)} \int_0^{\frac{1}{2}\kappa A} e^{-t} \left(\frac{2t}{\kappa} + g(0)\right)^m \varphi\left(\frac{2t}{\kappa}\right) dt + O\left(e^{-\kappa}\right).$$

If $m = 0$, since $\cosh a = \frac{p^\sigma + p^{-\sigma}}{2} \ll \kappa^{1-\varepsilon}$, we have

$$E(|1 - X(p)p^{-\sigma}|^{-\kappa}) = \frac{1}{\pi \kappa} e^{-\frac{\kappa}{2} g(0)} \sinh \frac{a}{2} \left(\left(\frac{2}{\kappa}\right)^{-\frac{1}{2}} \gamma\left(\frac{1}{2}, \frac{\kappa A}{2}\right) + O\left(\kappa^{-\frac{1}{2}-\varepsilon}\right)\right),$$

where the implicit constant depends only on $\sigma$ and $\varepsilon$. Since

$$A = \log \frac{1 + p^{-\sigma}}{1 - p^{-\sigma}} = 2p^{-\sigma} + O(p^{-2\sigma}) \gg \kappa^{-1-\varepsilon},$$

so we have $\kappa A \gg \kappa^\varepsilon$ and

$$\gamma\left(\frac{1}{2}, \frac{\kappa A}{2}\right) = \Gamma\left(\frac{1}{2}\right) + O\left(e^{-\kappa}\right).$$

Hence,

$$E(|1 - X(p)p^{-\sigma}|^{-\kappa}) = \frac{1}{\sqrt{2\pi}} \sinh \frac{a}{2} e^{-\kappa \log(1 - p^{-\sigma})\kappa^{-\frac{1}{2}}(1 + O(\kappa^{-\varepsilon}))}.$$

Take logarithm and we get

$$M_p(\kappa) = -\kappa \log(1 - p^{-\sigma}) + O(\log \kappa).$$ (3)
If \( m = 1 \), in the same manner as Equation (3), one has
\[
\mathbb{E}'(1 - X(p)p^{-\sigma} | -\kappa) = -\frac{1}{\sqrt{2\pi}} \sinh \frac{a}{2} e^{-a \log(1-p^{-\sigma})} \log(1 - p^{-\sigma}) \kappa^{-\frac{1}{2}} (1 + O(\kappa^{-\varepsilon})).
\]
This yields
\[
M'_p(\kappa) = \frac{\mathbb{E}'}{\mathbb{E}} = -\log(1 - p^{-\sigma})(1 + O(\kappa^{-\varepsilon})).
\]
Uniform asymptotic behaviors for higher order derivatives can also be obtained for all smaller \( p \) and for all \( z \) in a fixed sector.

**Lemma 3.5.** Let \( z \in \mathbb{C} \). Suppose \( \arg z \in (-\theta, \theta) \) for certain fixed \( 0 < \theta < \frac{\pi}{2} \). For any prime number \( p \) with \( |z| > p^{(2+\varepsilon)\sigma} \) for some certain \( \varepsilon > 0 \), we have
\[
M^{(m)}_p(z) = (-1)^m \frac{(m-1)!}{2} z^{-m} + O(z^{-m-\varepsilon}),
\]
where the implicit constant depends only on \( m, \sigma, \varepsilon \) and \( \theta \).

**Proof.** In the same way as Lemma 3.4 but using \( \cosh a = \frac{p^\sigma + p^{-\sigma}}{2} \ll |z|^{\frac{1}{2}-\varepsilon} \), one gets
\[
\mathbb{E}''(1 - X(p)p^{-\sigma} | -z) = \frac{1}{\sqrt{2\pi}} \sinh \frac{a}{2} e^{-a \log z} \lambda_p^2 \left(z^{-\frac{1}{2}} + \frac{1}{4} (4 + 2 \lambda_p + \lambda_p \cosh a) \lambda_p z^{\frac{3}{2}} + O_{\sigma,\varepsilon,\theta}(z^{-2-\varepsilon}) \right),
\]
where \( \lambda_p = \log(1 - p^{-\sigma}) \). Similar asymptotic expansions can also be obtained for \( \mathbb{E} \) and \( \mathbb{E}' \). At last, one can get
\[
M''_p(z) = \frac{\mathbb{E}''}{\mathbb{E}} = \frac{\mathbb{E}'^2}{\mathbb{E}^2} = \frac{1}{2} z^{-2} + O_{m,\sigma,\varepsilon,\theta}(z^{-2-\varepsilon}).
\]
Taking derivatives with respect to \( z \) and using Cauchy’s estimate, we get the desired result for \( m > 2 \). \( \square \)

With the above preparation, we are able to determine the asymptotic behaviors of \( M(\kappa) \) and its derivatives.

**Proposition 3.6.** As the positive real numbers \( \kappa \to \infty \), we have
\[
M(\kappa) = \frac{\kappa^{1/\sigma}}{\log \kappa} \left( a_0^{(0)} + \frac{a_1^{(0)}}{\log \kappa} + \cdots + \frac{a_N^{(0)}}{(\log \kappa)^N} + O\left( \frac{1}{(\log \kappa)^N+1} \right) \right),
\]
where
\[
a_n^{(0)} := \int_0^\infty f(u)(\log u)^n u^{1/\sigma+1} du,
\]
\( f(u) \) is the density function of the Dirichlet distribution.
and

\[ M'(\kappa) = \frac{\kappa^{1/\sigma-1}}{\log \kappa} \left( a_0^{(1)} + \frac{a_1^{(1)}}{\log \kappa} + \cdots + \frac{a_N^{(1)}}{(\log \kappa)^N} + O\left( \frac{1}{(\log \kappa)^{N+1}} \right) \right), \tag{6} \]

where

\[ a_n^{(1)} := \int_0^\infty \frac{f'(u)(\log u)^n}{u^{1/\sigma}} du. \]

**Proof.** So, by combining Lemma 3.2, Lemma 3.4, and prime number theorem, we have

\[
M(\kappa) + O(\kappa \log \kappa) = \sum_{p^\sigma < \kappa^{1/(1+\epsilon)}} M_p(\kappa) + \sum_{p^\sigma > \kappa^{1/(1+\epsilon)}} M_p(\kappa)
= \sum_{p^\sigma < \kappa^{1/(1+\epsilon)}} -\log(1 - p^{-\sigma}) \kappa + \sum_{p^\sigma > \kappa^{1/(1+\epsilon)}} f(p^{-\sigma} \kappa)
= -\kappa \int_{\kappa^{1/(1+\epsilon)}}^{2-\sigma \kappa} \frac{\log(1 - t)}{\log t} t^{-\frac{1}{\sigma} - 1} dt - \int_0^{\kappa^{1/(1+\epsilon)}} \frac{f(t \kappa)}{\log t} t^{-\frac{1}{\sigma} - 1} dt
= -\frac{\kappa^2}{1} \int_{\kappa^{1/(1+\epsilon)}}^{2-\sigma \kappa} \frac{\kappa \log(1 - t/\kappa)}{\log \kappa} t^{-\frac{1}{\sigma} - 1} dt - \int_0^{\kappa^{1/(1+\epsilon)}} \frac{f(t)}{\log t - \log \kappa} t^{-\frac{1}{\sigma} - 1} dt. \tag{7} \]

When \( 0 \leq t \leq 2/\kappa \), by Lemma 3.1, we have \( f(t) \asymp t^2 \), so

\[
\int_0^{2/\kappa} \frac{f(t)}{\log \kappa - \log t} t^{-\frac{1}{\sigma} - 1} dt \asymp \int_0^{2/\kappa} t^{-\frac{1}{\sigma} + 1} \frac{1}{\log \kappa} dt \asymp \frac{1}{2^{1-\frac{\sigma}{2}}} \log \kappa.
\]

Then, a direct expansion with respect to \( \log t \) gives

\[
M(\kappa) = \frac{\kappa^{1/\sigma}}{\log \kappa} \left( \sum_{n \leq N} \frac{1}{(\log \kappa)^n} \int_0^\infty \frac{h(t)(\log t)^n}{t^{1/\sigma}} dt + O((\log \kappa)^{-N-1}) \right), \tag{8} \]

where

\[ h(t) = \begin{cases} 1 & t \geq \kappa^{2/(1+\epsilon)} \\ t^{-1} f(t) & t \leq \kappa^{2/(1+\epsilon)}. \end{cases} \]

We note that \( t^{-1} f(t) - 1 = o(1) \) as \( t \to \infty \), so

\[
\int_0^\infty \frac{h(t)(\log t)^n}{t^{1/\sigma}} dt = \int_0^{\kappa^{2/(1+\epsilon)}} \frac{f(t)(\log t)^n}{t^{1/\sigma+1}} dt + \int_{\kappa^{2/(1+\epsilon)}}^\infty \frac{(t^{-1} f(t) - 1)(\log t)^n}{t^{1/\sigma}} dt
= \int_0^\infty \frac{f(t)(\log t)^n}{t^{1/\sigma+1}} dt + O(\kappa^{-\sigma(1/\sigma-1)}).
\]

This proves Equation (5).
With the same calculation as Equation (7), we get

\[ M'(\chi) = \frac{\chi^{1/\sigma-1}}{\log \chi} \left( \sum_{n \leq N} \frac{1}{(\log n)^n} \int_0^\infty \frac{h_1(t)(\log t)^n}{t^{1/\sigma}} \, dt + O((\log \chi)^{-N-1}) \right), \]  

(9)

where

\[ h_1(t) = \begin{cases} 1 & t \geq \chi^{\varepsilon/(1+\varepsilon)} \\ f'(t) & t \leq \chi^{\varepsilon/(1+\varepsilon)}. \end{cases} \]

Finally, we note that \( f'(t) - 1 = t^{-1} + O(t^{-2}) \) as \( t \to \infty \), so

\[
\int_0^\infty \frac{h_1(t)(\log t)^n}{t^{1/\sigma}} \, dt = \int_0^\infty \frac{f'(t)(\log t)^n}{t^{1/\sigma}} \, dt + \int_0^\infty \frac{(f'(t) - 1)(\log t)^n}{t^{1/\sigma}} \, dt
\]

\[
= \int_0^\infty \frac{f'(t)(\log t)^n}{t^{1/\sigma}} \, dt + O(\chi^{-\varepsilon/\sigma}).
\]

This completes the proof. \( \square \)

**Proposition 3.7.** As the positive real numbers \( \chi \to \infty \), we have for \( m \geq 2 \)

\[ M^{(m)}(\chi) = \frac{\chi^{1-\sigma-m}}{\log \chi} \left( a^{(m)}_0 + \frac{a^{(m)}_1}{\log \chi} + \cdots + \frac{a^{(m)}_N}{(\log \chi)^N} + O\left( \frac{1}{(\log \chi)^{N+1}} \right) \right), \]  

(10)

where

\[ a^{(m)}_n := \int_0^\infty \frac{f^{(m)}(u)(\log u)^n}{u^{1/\sigma+1-m}} \, du. \]

**Proof.** Put \( \alpha_m = (-1)^m \frac{(m-1)!}{2} \). Using Lemma 3.2 (with \( \varepsilon \) replaced by \( 1 + \varepsilon \)) and Lemma 3.5 again, we have

\[
M^{(m)}(\chi) = \sum_{p^2 \leq \chi^{1/(2+\varepsilon)}} M^{(m)}_p(\chi) + \sum_{p^2 > \chi^{1/(2+\varepsilon)}} M^{(m)}_p(\chi)
\]

\[
= \sum_{p^2 \leq \chi^{1/(2+\varepsilon)}} \alpha_m \chi^{-m} + \sum_{p^2 > \chi^{1/(2+\varepsilon)}} p^{-m\sigma} f^{(m)}(p^{-\sigma} \chi) + O(\chi^{-m+1+\varepsilon})
\]

\[
= \alpha_m \chi^{-m} \int_{\chi^{-1/(2+\varepsilon)}}^{\chi^{-1/(2+\varepsilon)}} \frac{1}{\log t} t^{-\frac{1}{\sigma}-1} \, dt + \int_{\chi^{1/(2+\varepsilon)}}^{\chi^{-1/(2+\varepsilon)}} \frac{f^{(m)}(tx)}{\log t} t^{-\frac{1}{\sigma}+m-1} \, dt + O(\chi^{-m+1+\varepsilon})
\]

\[
= \alpha_m \chi^{-m} \int_{\chi^{1/(2+\varepsilon)}}^{\chi^{-1/(2+\varepsilon)}} \frac{1}{\log t - \log \chi} t^{-\frac{1}{\sigma}-1} \, dt + \chi^{\frac{1}{\sigma}-m} \int_0^{\chi^{\varepsilon/(2+\varepsilon)}} \frac{f^{(m)}(t)}{\log t - \log \chi} t^{-\frac{1}{\sigma}+m-1} \, dt.
\]
When $0 \leq t \leq 2/\kappa$, by Lemma 3.1, we have $f^{(m)}(t) = O(1)$, so

$$
\int_0^{2/\kappa} \frac{f^{(m)}(t)}{\log t - \log \kappa} t^{-\frac{1}{\sigma} + 1} dt \ll \int_0^{2/\kappa} \frac{t^{-\frac{1}{\sigma} + 1}}{\log \kappa} dt \ll \frac{1}{\kappa^{2-1/\sigma} \log \kappa}.
$$

Thus,

$$
M^{(m)}(\kappa) = \frac{\kappa^{1/\sigma - m}}{\log \kappa} \left( \sum_{n \leq N} \frac{1}{(\log \kappa)^n} \int_0^\infty h_m(t)(\log t)^n t^{1/\sigma} dt + O((\log \kappa)^{-N-1}) \right),
$$

where

$$
h_m(t) = \begin{cases} 
\alpha_m t^{-1} & t \geq \kappa^{\varepsilon/(2+\varepsilon)} \\
m^{-1} f^{(m)}(t) & t \leq \kappa^{\varepsilon/(2+\varepsilon)}.
\end{cases}
$$

Again from Lemma 3.1, we see that $t^{m-1}f^{(m)}(t) = \alpha_m t^{-1} + o(t^{-2})$ as $t \to \infty$, so

$$
\int_0^\infty \frac{h_m(t)(\log t)^n}{t^{1/\sigma}} dt = \int_0^\infty \frac{f^{(m)}(t)(\log t)^n}{t^{1/\sigma} + 1} dt + \int_{\kappa^{\varepsilon/(2+\varepsilon)}}^\infty \frac{(t^{m-1}f^{(m)}(t) - \alpha_m t^{-1})(\log t)^n}{t^{1/\sigma}} dt
$$

$$
= \int_0^\infty \frac{f^{(m)}(t)(\log t)^n}{t^{1/\sigma - m + 1}} dt + O(\kappa^{-\varepsilon/(2+\varepsilon)}(1/\sigma + 1)).
$$

This proves Equation (5). □

We then apply saddle point method to get the desired asymptotic expansion of $\Psi$. Here, we give a precise estimation for the saddle point $\kappa$.

**Lemma 3.8.** Let $\tau$ be a large real number and $\kappa$ be the unique solution to $M'(\kappa) = \tau$. Then, there exist computable polynomials $f_0(\cdot), f_1(\cdot), \ldots, f_N(\cdot)$ with $\deg f_n \leq n$ for $0 \leq n \leq N$, which depends only on $\sigma$ and $N$, such that

$$
\kappa = g(\sigma)(\tau \log \tau)^{\sigma/(1-\sigma)} \left( f_0(\log_2 \tau) + \frac{f_1(\log_2 \tau)}{\log \tau} + \cdots + \frac{f_N(\log_2 \tau)}{(\log \tau)^N} + O\left( \left( \frac{\log_2 \tau}{\log \tau} \right)^{N+1} \right) \right),
$$

where

$$
g(\sigma) = \left( \frac{1}{a_0^{(1)}} \right)^{\frac{\sigma}{1-\sigma}},
$$

and $a_0^{(1)}$ is defined in Proposition 3.6. More precisely,

$$
f_0(t) = 1, \ f_1(t) = \frac{\sigma}{1-\sigma} t + \log g(\sigma) - \frac{a_1^{(1)}}{a_0^{(1)}}, \ldots
$$

**Proof.** This follows directly from Proposition 3.6 and Lemma 4.3 in [2]. □
Lemma 3.9. Let $1/2 \leq \sigma < 1$. Then
\[
\Psi(\tau) = \mathbb{E}(|L(\sigma, X)|^\kappa) e^{-\tau \kappa} \left( 1 + O(\kappa^{1-1/\sigma \log \kappa}) \right),
\]
holds uniformly for $\tau \geq 1$.

Proof. See [8, Proposition 7.1]. \qed

Proof of Theorem 1.1. Combining Propositions 3.6 and 3.7 and Lemma 3.9, we have
\[
\Psi(\tau) = \exp \left( \frac{\chi^{1/\sigma}}{\log \kappa} \left( a_0^{(0)} + \frac{a_1^{(0)}}{\log \kappa} + \ldots + \frac{a_N^{(0)}}{(\log \kappa)^N} + O \left( \frac{1}{(\log \kappa)^{N+1}} \right) \right) - \tau \kappa \right).
\]

Then, by applying Lemma 3.8, we get
\[
\Psi(\tau) = \exp \left( - \left( \tau \log^2 \tau \right)^{1/\sigma} \left( \sum_{n=0}^N a_n (\log \tau)^n + O \left( \frac{1}{(\log \tau)^{N+1}} \right) \right) \right),
\]
where
\[
a_0(t) = g(\sigma) \left( 1 - \frac{a_0^{(0)}}{a_1^{(1)}} \right), \quad a_1(t) = g(\sigma) \sigma t + g(\sigma)(1-\sigma) \left( \log g(\sigma) - \frac{a_0^{(0)}}{a_1^{(0)}} \right), \ldots
\]
To show $a_0 > 0$, it is equivalent to $a_0^{(0)} < a_0^{(1)}$. But the definition of $a_0^{(0)}$, $a_0^{(1)}$ gives
\[
a_0^{(1)} = \int_0^\infty u^{-1/\sigma} f(u) du = \frac{1}{\sigma} \int_0^\infty \frac{f(u)}{u^{1+1/\sigma}} du = \frac{a_0^{(0)}}{\sigma} > a_0^{(0)}.
\]
Here the second integral converges absolutely from the facts that $f(u) \asymp u^2$ as $u \to 0$ and $f(u) \asymp u$ as $u \to \infty$. \qed

3.2 | When $\sigma = 1$

In this subsection, we switch to the case $\sigma = 1$. We will see that the computation in this case is similar to the case $1/2 < \sigma < 1$ with a few modification. First, we give asymptotic formulas for the function $M(\kappa)$ and its derivatives, like we did in Propositions 3.6 and 3.7.

Proposition 3.10. As the positive real numbers $\kappa \to \infty$, we have
\[
M(\kappa) = -\kappa \sum_{p < \kappa} \log(1 - p^{-1}) + \frac{\kappa}{\log \kappa} \left( \sum_{n \leq N} \frac{A_n^{(0)}}{\log \kappa} + O \left( \frac{1}{(\log \kappa)^{N+1}} \right) \right),
\]
where

\[ A^{(0)}_n = \int_0^1 \frac{(\log t)^n f(t)}{t^2} dt + \int_1^\infty \frac{(\log t)^n f(t) - t}{t^2} dt, \tag{12} \]

and

\[ M'(\kappa) = - \sum_{p < \kappa} \log(1 - p^{-1}) + \frac{1}{\log \kappa} \left( \sum_{n \leq N} \frac{A^{(1)}_n}{(\log \kappa)^n} + O \left( \frac{1}{(\log \kappa)^{N+1}} \right) \right), \]

where

\[ A^{(1)}_n = \int_0^1 \frac{(\log t)^n f'(t)}{t} dt + \int_1^\infty \frac{(\log t)^n f'(t) - 1}{t} dt. \]

**Proof.** Together with prime number theorem, Lemma 3.4 gives

\[ \sum_{p < \kappa^{1/(1+\varepsilon)}} M_p(\kappa) = -\kappa \sum_{p < \kappa^{1/(1+\varepsilon)}} \log(1 - p^{-1}) + O(\kappa^{1/(1+\varepsilon)}). \]

A similar calculation as Equation (7) gives

\[ \sum_{p > \kappa^{1/(1+\varepsilon)}} M_p(\kappa) = \kappa \int_{\kappa^{1/(1+\varepsilon)}}^\infty \frac{f(t)}{t^2 \log \kappa - \log t} dt + O(\kappa^\varepsilon). \]

Following Mertens' formula [12, Section 1.6, Theorem 11], we have

\[ \log(1 - p^{-1})^{-1} = \log \log \kappa - \log_2 \kappa^{1/(1+\varepsilon)} + O \left( e^{-\sqrt{\log \kappa}} \right) \]

\[ = \log(1 + \varepsilon) + O \left( e^{-\sqrt{\log \kappa}} \right). \]

On the other hand, we have

\[ \int_1^{\kappa^{1/(1+\varepsilon)}} \frac{1}{t (\log \kappa - \log t)} dt = -\log(\log \kappa - \log t) \bigg|_1^{\kappa^{1/(1+\varepsilon)}} = \log(1 + \varepsilon). \]

So, we have

\[ M(\kappa) = \sum_{p < \kappa^{1/(1+\varepsilon)}} M_p(\kappa) + \sum_{p > \kappa^{1/(1+\varepsilon)}} M_p(\kappa) \]

\[ = -\kappa \sum_{p < \kappa} \log(1 - p^{-1}) - \kappa \int_1^{\kappa^{1/(1+\varepsilon)}} \frac{dt}{t (\log \kappa - \log t)} + \kappa \int_0^{\kappa^{1/(1+\varepsilon)}} \frac{f(t)}{t^2 \log \kappa - \log t} dt \]

\[ = -\kappa \sum_{p < \kappa} \log(1 - p^{-1}) + \frac{\kappa}{\log \kappa} \sum_{n=0}^{\infty} (\log \kappa)^{-n} \left( \int_0^1 \frac{(\log t)^n f(t)}{t^2} dt + \int_1^{\kappa^{1/(1+\varepsilon)}} \frac{(\log t)^n f(t) - t}{t^2} dt \right). \]
Meanwhile, from $f(t) - t = O(\log t)$ as $t \to \infty$, we see
\[
\int_{x^{-1+\varepsilon}}^{\infty} (\log t)^n \frac{f(t) - t}{t^2} \, dt = O(\kappa^{-\varepsilon}).
\]

We conclude that
\[
M(\kappa) = -\kappa \sum_{p < \kappa} \log(1 - p^{-1}) + \frac{\kappa}{\log \kappa} \left( \sum_{n \leq N} A_n^{(i)} \log \kappa^n + O\left(1 \left(\log \kappa\right)^{N+1}\right)\right).
\]

Along the same line, we can get the desired asymptotic expansion for $M'(\kappa)$. \qed

**Proposition 3.11.** As the positive real number $\kappa \to \infty$, we have for $m \geq 2$
\[
M^{(m)}(\kappa) = \kappa^{1-m} \log \kappa \left( A_0^{(m)} + \frac{A_1^{(m)}}{\log \kappa} + \cdots + \frac{A_N^{(m)}}{(\log \kappa)^N} + O\left(1 \left(\log \kappa\right)^{N+1}\right)\right),
\]
where
\[
A_n^{(m)} := \int_0^{\infty} f^{(m)}(u) u^{m-2} (\log u)^n \, du.
\]

**Proof.** The proof here is exactly the same as the proof of Proposition 3.7 where we replace $\sigma$ by 1. \qed

**Lemma 3.12.** Let $\lambda > 0$ be a real number and $N$ be a positive integer. For any $c > 0$, we have for $y > 0$
\[
0 \leq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s e^{\lambda s} \frac{-1}{s} \, ds - I_{\geq 1}(y) \leq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s e^{\lambda s} \frac{1 - e^{-\lambda s}}{s} \, ds.
\]

**Proof.** See [8, Lemma 7.2]. \qed

**Lemma 3.13.** Let $s = \kappa + it$ where $\kappa$ is a large positive real number. Then, in the range $|t| \geq \kappa$, we have
\[
\mathbb{E}(|L(\sigma, X)|^\delta) \ll \exp\left(-|t|^\frac{1}{\sigma - 1}\right) \mathbb{E}(|L(\sigma, X)|^\kappa) \quad \text{for } \frac{1}{2} < \sigma < 1,
\]
and
\[
\mathbb{E}(|L(1, X)|^\delta) \ll \exp\left(-\sqrt{|t|}\right) \mathbb{E}(|L(1, X)|^\kappa).
\]

**Proof.** The first assertion is [8, Lemma 7.3]. The proof for the second can be easily get by choosing the value of $y = t$ in that of [8, Lemma 7.3]. \qed

Let us recall that
\[
\Psi_1(\tau) := \text{Prob}(|L(1, X)| > e^\tau).
\]
Now using the saddle point method, we can estimate the function $\Psi_1$ with the previous results on $M(\kappa)$.

**Proposition 3.14.** Uniformly for $\tau \geq 1$, let $\kappa$ be a positive number such that $\log \tau + \gamma = M'(\kappa)$. Then we have

$$
\Psi_1(\tau) = \frac{E(|L(1, X)|^{\kappa}) (e^{\gamma \tau})^{-\kappa}}{\kappa \sqrt{2\pi M''(\kappa)}} \left( 1 + O\left( \frac{\log \kappa}{\sqrt{\kappa}} \right) \right).
$$

**Proof.** Let $0 < \lambda < 1/(2\kappa)$ be a real number to be chosen later. Using Lemma 3.12, we obtain

$$
0 \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} E(|L(1, X)|^{\kappa}) (e^{\gamma \tau})^{-s} \frac{e^{\lambda s} - 1}{\lambda s} \frac{ds}{s} - \Psi_1(\tau) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} E(|L(1, X)|^{\kappa}) (e^{\gamma \tau})^{-s} \frac{e^{\lambda s} - 1 - e^{-\lambda s}}{\lambda s} ds. \tag{14}
$$

Since $\lambda \kappa < 1/2$, we have $|e^{\lambda s} - 1| \leq 3$ and $|e^{-\lambda s} - 1| \leq 2$. Therefore, using Lemma 3.13, we obtain

$$
\left( \int_{\kappa-i\kappa}^{\kappa-i\infty} + \int_{\kappa+i\kappa}^{\kappa+i\infty} \right) E(|L(1, X)|^{\kappa}) (e^{\gamma \tau})^{-s} \frac{e^{\lambda s} - 1 - e^{-\lambda s}}{\lambda s} ds \ll \frac{e^{-\sqrt{\kappa}}}{\lambda \kappa} E(|L(1, X)|^{\kappa}) \tau^{-\kappa}, \tag{15}
$$

and similarly,

$$
\left( \int_{\kappa-i\kappa}^{\kappa-i\infty} + \int_{\kappa+i\kappa}^{\kappa+i\infty} \right) E(|L(1, X)|^{\kappa}) (e^{\gamma \tau})^{-s} \frac{e^{\lambda s} - 1 - e^{-\lambda s}}{\lambda s} ds \ll \frac{e^{-\sqrt{\kappa}}}{\lambda \kappa} E(|L(1, X)|^{\kappa}) \tau^{-\kappa}. \tag{16}
$$

Furthermore, if $|t| \leq \kappa$, then $|(1 - e^{-\lambda s})(e^{\lambda s} - 1)| \ll \lambda^2 |s|^2$. Hence, we derive

$$
\int_{\kappa-i\kappa}^{\kappa+i\kappa} E(|L(1, X)|^{\kappa}) (e^{\gamma \tau})^{-s} \frac{e^{\lambda s} - 1 - e^{-\lambda s}}{\lambda s} ds \ll \lambda \kappa E(|L(1, X)|^{\kappa}) \tau^{-\kappa}.
$$

Therefore, combining this estimate with Equations (14)–(16), we deduce that

$$
\Psi_1(\tau) = \frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} E(|L(1, X)|^{\kappa}) (e^{\gamma \tau})^{-s} \frac{e^{\lambda s} - 1}{\lambda s^2} ds \ll \left( \lambda \kappa + \frac{e^{-\sqrt{\kappa}}}{\lambda \kappa} \right) E(|L(1, X)|^{\kappa}) \tau^{-\kappa}. \tag{17}
$$

On the other hand, in the region $|t| \leq \kappa$, we have

$$
\log E(|L(1, X)|^{\kappa+it}) = \log E(|L(1, X)|^{\kappa}) + it M'(\kappa) - \frac{t^2}{2} M''(k) + O\left( \frac{M'''(\kappa)}{\kappa} |t|^3 \right).
$$

Also, note that

$$
\frac{e^{\lambda s} - 1}{\lambda s^2} = \frac{1}{\kappa} \left( 1 - i \frac{t}{\kappa} + O\left( \frac{\lambda \kappa + t}{\kappa^2} \right) \right).
$$
Hence, using that $M'(\kappa) = \log \tau + \gamma$, we obtain

$$
\mathbb{E}(|L(1,X)^{\kappa}|)(e^{\gamma \tau})^{-\kappa} \frac{e^{i\lambda s} - 1}{\lambda s^2}
= \frac{1}{\kappa} \mathbb{E}(|L(1,X)^{\kappa}|) e^{-\gamma \tau - \kappa} \exp \left(-\frac{t^2}{2} M''(\kappa)\right) \left(1 - i \frac{t}{\kappa} + O \left(\frac{t^2}{\kappa^2} + |M'''(\kappa)||t|^3\right)\right).
$$

Therefore, we obtain

$$
\frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \mathbb{E}(|L(1,X)^{\kappa}|)(e^{\gamma \tau})^{-\kappa} \frac{e^{i\lambda s} - 1}{\lambda s^2} ds
= \frac{1}{\kappa} \mathbb{E}(|L(1,X)^{\kappa}|) e^{-\gamma \tau - \kappa} \frac{1}{2\pi} \int_{-\kappa}^{\kappa} \exp \left(-\frac{t^2}{2} M''(\kappa)\right) \left(1 + O \left(\frac{t^2}{\kappa^2} + |M'''(\kappa)||t|^3\right)\right) dt,
$$

since the integral involving $it/\kappa$ vanishes. Further, we have

$$
\frac{1}{2\pi} \int_{-\kappa}^{\kappa} \exp \left(-\frac{t^2}{2} M''(\kappa)\right) dt = \frac{1}{\sqrt{2\pi M''(\kappa)}} \left(1 + O \left(\exp \left(-\frac{1}{2} \kappa^2 M''(\kappa)\right)\right)\right),
$$

and

$$
\int_{-\kappa}^{\kappa} |t|^n \exp \left(-\frac{t^2}{2} M''(\kappa)\right) dt \ll \frac{1}{M''(\kappa)^{(n+1)/2}}.
$$

Thus, using Proposition 3.11 for the order of $M''(\kappa)$ and $M'''(\kappa)$, we deduce that

$$
\frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \mathbb{E}(|L(1,X)^{\kappa}|)(e^{\gamma \tau})^{-\kappa} \frac{e^{i\lambda s} - 1}{\lambda s^2} ds
= \mathbb{E}(|L(1,X)^{\kappa}|) e^{-\gamma \tau - \kappa} \frac{1}{\kappa \sqrt{2\pi M''(\kappa)}} \left(1 + O \left(\frac{\lambda \kappa + \log \kappa}{\sqrt{\kappa}}\right)\right).
$$

(18)

Thus, combining the estimates (17) and (18) and choosing $\lambda = \kappa^{-3}$ completes the proof. 

As a direct result of Proposition 3.14, we have the following asymptotic formula for the distribution function $\Psi_1(\tau)$, which is the second part of Theorem 1.2.

**Corollary 3.15.** There is a sequence of real numbers $\{b_n\}_{n \geq 1}$ such that for any integer $N \geq 1$, we have

$$
\Psi_1(\tau) = \exp \left(-\frac{e^{\gamma} - A_0 - 1}{\tau} \left\{1 + \sum_{n=1}^{N} \frac{b_n}{\tau^n} + O_N \left(\frac{1}{\tau^{N+1}}\right)\right\}\right)
$$

uniformly for $\tau \geq 2$, where $A_0 = A_0^{(0)}$ is defined in (12).
4  |  LARGE DEVIATIONS: PROOF OF THEOREM 1.2

4.1  |  When $\frac{1}{2} < \sigma < 1$

The following proposition is a key to the proof of the first part of Theorem 1.2. It shows that, after removing a small set of “bad” characters, the moment of Dirichlet $L$-functions can be computed well by the random models.

**Proposition 4.1.** Let $\frac{1}{2} < \sigma < 1$ and $A > 1$ be fixed. There exist positive constants $b_3 = b_3(\sigma, A)$ and $b_4 = b_4(\sigma, A)$ and a set $E(q) \subseteq \{\chi : \chi(\text{mod} \ q)\}$ with $|E(q)| \leq q \exp(-b_3 \log q / \log_2 q)$, such that for all complex numbers $z$ with $|z| \leq b_4(\log q)^\sigma$, we have

$$\frac{1}{\phi(q)} \sum_{\chi(\text{mod} \ q), \chi \notin E(q)} |L(\sigma, \chi)|^z = E(|L(\sigma, X)|^z) + O\left(\frac{E(|L(\sigma, X)|^{\Re z})}{(\log q)^{A-\sigma}}\right).$$

**Proof.** Let $E(q) = A_q \cup B_q$ where $A_q$ is defined in Lemma 2.6 and $B_q$ is defined in Lemma 2.11. So, combining with Lemma 2.6, we have

$$\frac{|E(q)|}{\phi(q)} \ll \exp\left(-\frac{c_2(\sigma) \log q}{\log_2 q}\right).$$

We note that $\log L(\sigma, \chi) = R_y(\sigma, \chi) + O((\log q)^{-A})$, for all $\chi \in B_q$. Then, we get

$$\frac{1}{\phi(q)} \sum_{\chi(\text{mod} \ q), \chi \notin E(q)} |L(\sigma, \chi)|^z = \frac{1}{\phi(q)} \sum_{A_q \cap B_q} \exp\left(z \Re R_y(\sigma, \chi) + O((\log q)^{1-A+\sigma})\right)$$

$$= \frac{1}{\phi(q)} \sum_{A_q \cap B_q} \exp\left(z \Re R_y(\sigma, \chi)\right) + O\left(\frac{1}{(\log q)^{A-\sigma}} \sum_{A_q \cap B_q} \exp\left(z \Re R_y(\sigma, \chi)\right)\right)$$

$$= \frac{1}{\phi(q)} \sum_{A_q \cap B_q} \exp\left(z \Re R_y(\sigma, \chi)\right) + O\left(\frac{1}{(\log q)^{A-\sigma}} E(|L(\sigma, X)|^{\Re z})\right).$$

The last identity follows from Lemma 2.9. On the other hand, since for all $\chi \in A_q^c$, we have

$$|R_y(\sigma, \chi)| \leq \frac{(\log q)^{1-\sigma}}{\log_2 q}.$$ 

So,

$$\frac{1}{\phi(q)} \sum_{\chi \in A_q^c, \chi \notin B_q} \exp\left(z \Re R_y(\sigma, \chi)\right) \ll \frac{1}{\phi(q)} \sum_{\chi \in A_q^c, \chi \notin B_q} \exp\left(\frac{b_4 \log q}{\log_2 q}\right) \ll \frac{b_4 q}{\phi(q)} \exp\left(\frac{b_4 \log q}{\log_2 q}\right).$$

(20)
Since Lemma 2.3 implies that \( \# B_q \ll q^{\frac{g-6\sigma}{7-2\sigma}+1} = q^{\frac{8-4\sigma}{7-2\sigma}} \), so Equation (20) is bounded by

\[
q^{\frac{1-2\sigma}{7-2\sigma}} \exp \left( \frac{\log q}{\log_2 q} \right) \ll q^{\frac{1-2\sigma}{14-4\sigma}}.
\]

Together with Equation (19), we get

\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} |L(\sigma, \chi)|^2 = \frac{1}{\phi(q)} \sum_{\chi \not\in \mathcal{A}_q} \exp \left( \pi \Re R_{\chi}(\sigma, \chi) \right) + O \left( \frac{1}{(\log q)^{A-\sigma}} E(|L(\sigma, X)|^{\Re z}) \right).
\]

Finally, the desired result just follows from Lemmas 2.9 and 2.10.

Now we can prove the first part of Theorem 1.2.

**Proof of Theorem 1.2 for \( \frac{1}{2} < \sigma < 1 \).** Let \( y = b_4 (\log q)^\sigma \) and \( \lambda = \frac{1}{2y} \). Define

\[
I(\sigma, \tau) := \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \mathcal{E}(\pi |L(\sigma, X)|^s) e^{-\tau s} \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N ds
\]

and

\[
J_q(\sigma, \tau) := \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \left( \frac{1}{q} \sum_{\chi \pmod{q}} |L(\sigma, \chi)|^s \right) e^{-\tau s} \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N ds.
\]

Then, by Lemma 3.12, we have

\[
\Psi(\tau) \leq I(\sigma, \tau) \leq \Psi(\tau - \lambda N)
\]

and

\[
\Phi_q(\tau) + O(\delta(q)) \leq J_q(\sigma, \tau) \leq \Phi_q(\tau - \lambda N) + O(\delta(q)),
\]

with

\[
\delta(q) = \exp \left( -c_0(\sigma) \frac{\log q}{\log_2 q} \right),
\]

for some positive constant \( c_0(\sigma) \). Divide each integral of \( I(\sigma, \tau) \) and \( J_q(\sigma, \tau) \) into three parts:

\[
\int_{\kappa - i\infty}^{\kappa + i\infty} = \int_{\kappa - i\infty}^{\kappa - iy} + \int_{\kappa - iy}^{\kappa + iy} + \int_{\kappa + iy}^{\kappa + i\infty}.
\]

Since \( \kappa \) is the solution of \( M'(k) = \tau \), by choosing a suitable constant \( b(\sigma) \) so that \( \kappa \leq y \). Then, \( |\lambda s| < 1 \) for any \( s \) on the line from \( \kappa - iy \) to \( \kappa + iy \). So, we have \( |\frac{e^{\lambda s} - 1}{\lambda s}| \leq 3 \). By Proposition 4.1, we
have for the second part
\[
\frac{1}{2\pi i} \int_{\kappa-iy}^{\kappa+iy} \left( \frac{1}{q} \sum_{\chi \not\equiv \chi' \pmod{q}} |L(\sigma, \chi)|^s - \mathbb{E}(|L(\sigma, X)|^s) \right) e^{-\tau s} \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s}
\]
\[
\ll \frac{3^N y}{(\log q)^{10}} \mathbb{E}(|L(\sigma, X)|^\kappa) e^{-\tau \kappa}.
\]

For the first and third parts, by Proposition 4.1, we have
\[
\left( \int_{\kappa-i\infty}^{\kappa+i\infty} + \int_{\kappa-iy}^{\kappa+iy} \right) \mathbb{E}(|L(\sigma, X)|^\kappa) e^{-\tau s} \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s} \ll \left( \frac{3}{\lambda y} \right)^N \mathbb{E}(|L(\sigma, X)|^\kappa) e^{-\tau \kappa},
\]
and
\[
\left( \int_{\kappa-i\infty}^{\kappa+i\infty} + \int_{\kappa-iy}^{\kappa+iy} \right) \left( \frac{1}{q} \sum_{\chi \equiv \chi' \pmod{q}} |L(\sigma, \chi)|^s \right) e^{-\tau s} \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s}
\]
\[
\ll \left( \frac{3}{\lambda y} \right)^N \mathbb{E}(|L(\sigma, X)|^\kappa) e^{-\tau \kappa}.
\]

Combining the above three inequalities, we have
\[
J_q(\sigma, \tau) - I(\sigma, \tau) \ll \mathbb{E}(|L(\sigma, X)|^\kappa) e^{-\tau \kappa} \left( \frac{3^N y}{(\log q)^{10}} + \left( \frac{3}{\lambda y} \right)^N \right).
\]

Since
\[
\Psi(\tau) \asymp \frac{1}{\tau^{\frac{s}{(1-s)(\log \tau)^{\frac{s}{1-s}}}}} \mathbb{E}(|L(\sigma, X)|^\kappa) e^{-\tau \kappa},
\]
by choosing \( N = \lfloor \log_2 q \rfloor \), we have
\[
J_q(\sigma, \tau) - I(\sigma, \tau) \ll \frac{1}{(\log q)^5} \Psi(\tau).
\]

Theorem 1.1 gives
\[
\Psi(\tau \pm \lambda N) = \Psi(\tau) \left( 1 + O \left( \frac{(\tau \log \tau)^{\frac{s}{1-s}} \log_2 q}{(\log q)^{\sigma}} \right) \right),
\]
so we have
\[
\Phi(\tau) \leq J_q(\sigma, \tau) + O(\delta(q)) \leq I(\sigma, \tau) + O \left( \frac{1}{(\log q)^5} \Psi(\tau) + \delta(q) \right).
\]
\[ \leq \Psi(\tau - \lambda N) + O\left(\frac{1}{(\log q)^5} \Psi(\tau) + \delta(q)\right) \]

\[ \leq \Psi(\tau) \left(1 + O\left(\frac{\tau \log \tau/\log q}{(\log q)^\sigma}\right)\right) + O(\delta(q)), \]

and

\[ \Phi(\tau) \geq J_q(\sigma, \tau + \lambda N) + O(\delta(q)) \geq I(\sigma, \tau + \lambda N) + O\left(\frac{1}{(\log q)^5} \Psi(\tau) + \delta(q)\right) \]

\[ \geq \Psi(\tau + \lambda N) + O\left(\frac{1}{(\log q)^5} \Psi(\tau) + \delta(q)\right) \]

\[ \geq \Psi(\tau) \left(1 + O\left(\frac{\tau \log \tau/\log q}{(\log q)^\sigma}\right)\right) + O(\delta(q)). \]

Thus, we have

\[ \Phi(\tau) = \Psi(\tau) \left(1 + O\left(\frac{\tau \log \tau/\log q}{(\log q)^\sigma}\right)\right) \]

by the trivial bound \( \Psi(\tau) \gg \sqrt{\delta(q)}. \)

4.2 | When \( \sigma = 1 \)

**Lemma 4.2.** Let \( y = (\log q)^a \) for some \( a \geq 1 \). Define \( A_{1,q} = \{\chi(\text{mod } q) : |R_y(1, \chi)| > \log_3 q\} \). Then we have

\[ \frac{\#A_{1,q}}{q} \ll \exp\left( -\frac{c \log q \log_3 q}{\log_2 q} \right), \]

for some constant \( c > 0 \).

**Proof.** Similar to the proof of Lemma 2.5, by taking \( k = [\log q/\log_2 q] \), we have

\[ \frac{1}{\phi(q)} \sum_{\chi(\text{mod } q)} |R_y(1, \chi)|^{2k} \ll \left(\frac{3 \log_2 q}{q}\right)^{2k} + \left(\frac{3}{\log k}\right)^{2k}. \]

So, we get

\[ \#A_{1,q}(\log_3 q)^{2k} \leq |R_y(1, \chi)|^{2k} \ll \left(3 \log_2 q\right)^{2k} + q \left(\frac{3}{\log k}\right)^{2k}. \]

We get the desired result. \( \square \)
Then we can prove the second part of Theorem 1.2 for \( \sigma = 1 \).

**Proof of Theorem 1.2 for \( \sigma = 1 \).** The proof is similar to the case of \( \frac{1}{2} < \sigma < 1 \). Fix \( A > 1 \), let \( \mathcal{E}(q) = A_1(q) \cup B_q \), then Lemma 4.2 shows that \( \# \mathcal{E}(q) \leq q \exp(-\frac{b'_1 \log q \log_3 q}{\log_2 q}) \) for some positive constant \( b'_1 \). With the same argument as in Proposition 4.1, one can show that there exists a positive constant \( b'_4 \) such that for all complex numbers \( z \) with \( |z| \leq b'_4 \log q \), we have

\[
\frac{1}{q} \sum_{\substack{\chi \mod q \atop \chi \notin \mathcal{F}_1(q)}} |L(1, \chi)|^z = E(|L(1, X)|^z) + O\left(\frac{E(|L(1, X)|^{\Re z})}{(\log q)^{A-1}}\right).
\]

(21)

Then we take \( y = \frac{1}{2} b'_4 \log q \). Since \( \kappa \) satisfies \( M'(\kappa) = \gamma + \log \tau \), so we get \( \kappa \leq y \). Let \( s = \kappa + it \) with \( |t| \leq y \). Then Equation (21) holds for \( z = s \). Now we define

\[
I_1(\tau) := \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} E(|L(\sigma, X)|^s)(e^{\gamma \tau})^{-s} \left(\frac{e^{\lambda s} - 1}{\lambda s}\right)^N \frac{ds}{s}
\]

and

\[
J_{1, q}(\tau) := \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{1}{q} \sum_{\chi \mod q \atop \chi \notin \mathcal{F}_1(q)} |L(\sigma, \chi)|^s\right)(e^{\gamma \tau})^{-s} \left(\frac{e^{\lambda s} - 1}{\lambda s}\right)^N \frac{ds}{s}.
\]

By Lemma 3.12, we have

\[
\Psi_1(\tau) \leq I_1(\tau) \leq \Psi_1(\tau - \lambda N)
\]

and

\[
\Phi_{1, q}(\tau) \leq J_{1, q}(\sigma) + O\left(\exp\left(-\frac{c \log q \log_3 q}{\log_2 q}\right)\right) \leq \Phi_{1, q}(\tau - \lambda N).
\]

Along the same line of the proof for the case \( \frac{1}{2} < \sigma < 1 \), we can deduce that

\[
J_{1, q}(\tau) - I_1(\tau) \ll \left(\frac{3N y}{(\log q)^{A-1}} + \left(\frac{3}{\lambda y}\right)^N\right) E(|L(1, X)|^\kappa)(e^{\gamma \tau})^{-\kappa}.
\]

Since \( M'(\kappa) = \log \tau + \gamma \), following Proposition 3.14, we have

\[
\Psi_1(\tau) \asymp \sqrt{\frac{\log \kappa}{\kappa}} E(|L(1, X)|^\kappa)(e^{\gamma \tau})^{-\kappa} \asymp \sqrt{\frac{\tau}{e^{\gamma}}} E(|L(1, X)|^\kappa)(e^{\gamma \tau})^{-\kappa}.
\]

Choosing \( N = \lfloor \log_2 q \rfloor \) and \( \lambda = 1/y \), we have

\[
J_{1, q}(\tau) - I_1(\tau) \ll \frac{1}{(\log q)^4} \Psi_1(\tau).
\]
We also note that Corollary 3.15 gives
\[ \Psi_1(\tau \pm \lambda N) = \Psi_1(\tau) \left( 1 + O\left( \frac{e^{\tau \log_2 q}}{\tau \log q} \right) \right). \]

So, with the same argument, we show that
\[ \Phi_{1,q}(\tau) = \Psi_1(\tau) \left( 1 + O\left( \frac{e^{\tau \log_2 q}}{\tau \log q} \right) \right). \]
\[ \square \]

5 | DISCREPANCY BOUNDS: PROOF OF THEOREM 1.3

In order to prove Theorem 1.3, we need an approximation of character function in [8], which originally comes from Selberg. To present the result, we first give some notations. For \( u \in [0, 1] \), let
\[ G(u) = \frac{2u}{\pi} + 2u(1-u) \cot(\pi u). \]

For any \( a, b \in \mathbb{R} \), let
\[ f_{a,b}(u) = \frac{1}{2}(e^{-2\pi i au} - e^{-2\pi i bu}). \]

Lemma 5.1. Let \( \mathcal{R} = \{ z = x + i y \in \mathbb{C} : a_1 < x < a_2, b_1 < y < b_2 \} \) and \( T > 0 \) be a real number. Then for any \( z = x + iy \), we have
\[ 1_{\mathcal{R}}(z) = W_{T, \mathcal{R}}(z) + O(I(T(\Re \log_2 L(x, X) - s)) + I(T(\Im \log_2 L(x, X) - s)) \leq \frac{1}{T}, \quad \text{and} \quad \mathbb{E}(I(T(\Im \log_2 L(x, X) - s))) \leq \frac{1}{T} \]
hold uniformly in \( s \in \mathbb{R} \).

Proof. See [8, Theorem 1.1].
Proof of Theorem 1.3. Let $R$ be any rectangle with sides parallel to the coordinate axis and $R' = R \cap [-\log_2 q, \log_2 q]^2$. Then by taking $\tau = \log_2 q$ in Theorem 1.1, we have

$$
\Phi_q(R) = \Phi_q(R') + O\left( \frac{1}{(\log q)^A} \right) \quad \text{and} \quad \Psi(R) = \Psi(R') + O\left( \frac{1}{(\log q)^A} \right) \quad \forall A > 0.
$$

So, we can reduce to the case $R \subseteq [-\log_2 q, \log_2 q]^2$. When $1/2\sigma < 1$, we choose $T = c (\log q)^{\sigma}$ where the constant $c$ is the same as in Lemma 2.11. By Lemma 5.1, we see that

$$
\Phi_q(R) = \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} W_{T,R}(\log L(\sigma, \chi)) + O\left( \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} I(T(\Re \log L(\sigma, \chi) - a_1)) + I(T(\Re \log L(\sigma, \chi) - a_2)) + I(T(\Im \log L(\sigma, \chi) - b_1)) + I(T(\Im \log L(\sigma, \chi) - b_2)) \right).
$$

Fix any positive real number $A > 3\sigma$ in Lemma 2.11, we have

$$
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} W_{T,R}(\log L(\sigma, \chi)) = \mathbb{E}(W_{T,R}(\log L(\sigma, X))) + O\left( \frac{1}{(\log q)^A} \int_0^T \int_0^T G\left( \frac{u}{T} \right) G\left( \frac{v}{T} \right) |f_{a_1, a_2}(u)f_{b_1, b_2}(v)| \frac{du}{u} \frac{dv}{v} \right).
$$

We note that $0 \leq G(u) \leq 2/\pi$ when $0 \leq u \leq 1$ and $|f_{a,b}(u)| \leq \pi u |b - a|$. So, the error term is bounded by

$$
\frac{1}{(\log q)^A} \int_0^T \int_0^T G\left( \frac{u}{T} \right) G\left( \frac{v}{T} \right) |f_{a_1, a_2}(u)f_{b_1, b_2}(v)| \frac{du}{u} \frac{dv}{v} \ll \frac{(a_1 - a_2)(b_1 - b_2)T^2}{(\log q)^A}.
$$

By our assumption, $|(a_1 - a_2)(b_1 - b_2)| \leq 4(\log_2 q)^2$, so the error term is bounded by $\frac{1}{(\log q)^A}$. So, we get

$$
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} W_{T,R}(\log L(\sigma, \chi)) = \mathbb{E}(W_{T,R}(\log L(\sigma, X))) + O\left( \frac{1}{(\log q)^A} \right).
$$

(23)

On the other hand, since $\mathbb{E}(1_R(\log L(\sigma, X)))$ is exactly the probability of $\log L(\sigma, X) \in R$, by Lemmas 5.1 and 5.2, we have

$$
\mathbb{E}(W_{T,R}(\log L(\sigma, X))) = \Psi(R) + O\left( \frac{1}{T} \right).
$$

(24)

So, combining Equations (23) and (24), we get

$$
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} W_{T,R}(\log L(\sigma, \chi)) = \Psi(R) + O\left( \frac{1}{T} \right).
$$

(25)
Finally, using again Lemmas 2.11 and 5.2, we get
\[ \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} I(T(\Re \log L(\sigma, \chi) - s)) \ll \frac{1}{T}, \] (26)
and
\[ \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} I(T(\Im \log L(\sigma, \chi) - s)) \ll \frac{1}{T}. \] (27)

By combining Equations (22), (24), (26), and (27), we get
\[ \Phi_q(R) = \Psi_q(R) + O\left(\frac{1}{T}\right). \]

When \( \sigma = 1 \), we choose \( T = \frac{\log q}{50(\log_2 q)^2} \). Following Lemma 2.12, with the same argument as above, we get the desired result. \( \square \)

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