Effects of turbulent mixing on critical behaviour: renormalization-group analysis of the Potts model

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Abstract
The critical behaviour of a system, subjected to strongly anisotropic turbulent mixing, is studied by means of the field-theoretic renormalization group. Specifically, the relaxational stochastic dynamics of a non-conserved multicomponent order parameter of the Ashkin–Teller–Potts model, coupled to a random velocity field with prescribed statistics, is considered. The velocity is taken to be Gaussian, white in time, with a correlation function of the form \( \propto \delta(t - t')/|k_\perp|^{d-1+\xi} \), where \( k_\perp \) is the component of the wave vector, perpendicular to the distinguished direction (‘direction of the flow’)—the \( d \)-dimensional generalization of the ensemble was introduced by Avellaneda and Majda (1990 Commun. Math. Phys. 131 381) within the context of passive scalar advection. This model can describe a rich class of physical situations. It is shown that, depending on the values of the parameters that define the self-interaction of the order parameter and the relation between the exponent \( \xi \) and the space dimension \( d \), the system exhibits various types of large-scale scaling behaviour, associated with different infrared attractive fixed points of the renormalization-group equations. In addition to known asymptotic regimes (critical dynamics of the Potts model and passively advected field without self-interaction), the existence of a new, non-equilibrium and strongly anisotropic, type of critical behaviour (universality class) is established, and the corresponding critical dimensions are calculated to the leading order of the double expansion in \( \xi \) and \( \varepsilon = 6 - d \) (one-loop approximation). The scaling appears to be strongly anisotropic in the sense that the critical dimensions related to the directions parallel and perpendicular to the flow are essentially different.

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1. Introduction

Numerous systems of very different physical natures reveal interesting singular behaviour in the vicinity of their critical points. Their correlation functions exhibit self-similar (scaling) behaviour with universal critical dimensions: they depend only on few global characteristics of the system (such as symmetry or space dimension). A consistent qualitative and quantitative description of the critical behaviour is provided by the field-theoretic renormalization group (RG). In the RG approach, possible types of critical regimes (universality classes) are associated with infrared (IR) attractive fixed points of renormalizable field-theoretic models.

Most typical equilibrium phase transitions belong to the universality class of the $O_n$-symmetric $\varphi^4$ model of an $n$-component scalar order parameter $\varphi$. Universal characteristics of the critical behaviour depend only on $n$ and the space dimension $d$, and can be calculated within systematic perturbation schemes, in particular, in the form of an expansion in $\varepsilon = 4 - d$, the deviation of the space dimension from its upper critical value $d = 4$; see [1, 2] and the literature cited therein.

Another important example is provided by the Ashkin–Teller–Potts (ATP) class of models [3–9]. In the continuous formulation, they are described by the effective Hamiltonian for the $n$-component order parameter with a trilinear interaction term, invariant under the hypertetrahedron symmetry group [5–9]. Such models have numerous physical applications: magnetic materials and solids with nontrivial symmetry, Edwards–Anderson spin-glass models within the replica formalism [10] and so on. In general, the ATP models describe systems which locally have $n$ states, but the energy of any given configuration depends on whether pairs of neighbouring sites are in the same state or not [3]. The case $n = 2$ corresponds to nematic-to-isotropic transitions in the liquid crystals [11], while the formal limits $n = 0$ and $n = -1$ correspond to the percolation problem and the random resistor network, respectively [12–14]. Recently, models with trilinear interactions have attracted new interest due to their interesting formal properties [15, 16] and applications to the dynamics of first-order phase transitions [17]. The application of the cubic model to the Yang–Lee edge singularity has long been known [18].

The problem of the nature of the phase transition in the ATP model has a long and rather entangled history; see e.g. [4–8] and references therein. According to Landau’s phenomenological theory, the existence of a trilinear term excludes the possibility of the second-order transition. In contrast, exact two-dimensional results, numerical simulations and RG analysis suggest that for small $n$, the phase transition in the ATP model is of second order, while for $n$ large enough ($n > 3$ in two dimensions [4] and $n > 7/3$ in the vicinity of the upper critical dimension $d = 6$ [7, 8]), the transition becomes a first-order one. In this paper, we accept the point of view that the existence of an IR attractive fixed point of the RG equations implies the existence of a self-similar (scaling) asymptotic regime and thus the existence of a kind of critical state.

It is well known that dynamical critical behaviour (critical singularities of relaxation and correlation times, various kinetic and transport coefficients, etc) appears much richer, less universal and is comparatively less understood. The different nature of the order parameter (conserved or non-conserved), inclusion of additional slow modes (densities of entropy or energy) and interaction with hydrodynamical degrees of freedom produce different types of critical dynamics for the same static model [2, 19].

The behaviour of a real system near its critical point is extremely sensitive to external disturbances, gravity, geometry of the experimental setup, presence of impurities and so on; see [20] for the general discussion and references. The ‘ideal’ equilibrium critical behaviour
of an infinite system can be obscured by the limited accuracy of measuring the temperature, finite-size effects, finite time of evolution (ageing) and so on. In the presence of a distinguished direction, the scaling behaviour of such systems can become strongly anisotropic, with different critical dimensions corresponding to different spatial directions. In addition, some disturbances (randomly distributed impurities in magnets and turbulent mixing of fluid systems) can change the type of phase transition (first-order to second-order, and vice versa) and produce completely new types of critical behaviour (universality classes) with rich and rather exotic properties.

Investigation of the effects of various kinds of deterministic or chaotic flows (laminar shear flows, turbulent convection and so on) on the behaviour of critical fluids (like liquid crystals or binary mixtures near their consolodation points) has shown that the flow can destroy the usual critical behaviour: it can change to the mean-field behaviour or, under some conditions, to a more complex behaviour described by new non-equilibrium universality classes [21–32].

In this paper, we apply the field-theoretic RG to study the effects of turbulent mixing on the dynamical critical behaviour of systems described by the generalized ATP model. Special attention is paid to the anisotropy of the flow. Bearing in mind the application to liquid crystals or percolation in liquid media, we consider a purely relaxational stochastic dynamics of a non-conserved order parameter of the ATP model, coupled to a random velocity field with prescribed Gaussian statistics.

In most studies of fully developed turbulence, based on the standard RG techniques, a stochastic Navier–Stokes (NS) equation with a random stirring force is employed. The force provides the energy input to the system; its correlation function is chosen in the power-law form \( \propto k^{4-d-y} \), where \( k \) is the wave number and the exponent \( y \) plays the part analogous to that played by \( \varepsilon \) in the RG theory of critical behaviour; see [33] for a detailed review and the references therein. The limit \( y \to 4 \) corresponds to the energy input to the largest scales, \( \propto \delta(k) \), when the standard picture of the fluid turbulence with the direct energy cascade is recovered.

Results of the RG analysis of that model are reliable and internally consistent for asymptotically small \( y \), while the possibility of their extrapolation to the physical value \( y = 4 \) is far from obvious. Of course, the physical value of \( \varepsilon \) in models of critical behaviour is also not small (especially for models with trilinear interactions). But there, no qualitative changeover is expected when \( \varepsilon \) increases from asymptotically small values to the real ones \( \varepsilon \sim 1 \), so that the possibility of such an extrapolation is usually not disputed. The situation with the stirred NS equation appears rather different. New physical effects (self-sweeping by large-scale vortices and multiscale with an infinite set of anomalous exponents) are encountered as \( y \) grows, and they can be lost or misrepresented if the expansion in \( y \) is too naively applied; see e.g. [34] for a discussion of those issues.

However, in this paper we do not study the fluid turbulence itself, rather we study its effects on the dynamics of the order parameter. Thus, we can use a simplified model for the turbulent velocity field: a Gaussian statistics with prescribed correlation function. Recently, the models involving passive scalar fields advected by such ‘synthetic’ velocity ensembles attracted a great deal of attention because of the insight they offer into the origin of intermittency and anomalous scaling in the real fluid turbulence; see the review paper [35] and references therein. The most popular is Kraichnan’s rapid-change model with a power-like spectrum and vanishing correlation time [36]. For the passive scalar field, that model reproduces the standard phenomenological picture with the direct energy cascade, intermittency and dissipative anomaly; see the discussion in [37]. The RG approach to the problem of passive advection is reviewed in [38].
In spite of their relative simplicity, such models render many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. In the context of our study, it is especially important that they allow us to easily model anisotropy of the flow, which in more realistic models would be introduced by the initial and/or boundary conditions. More specifically, we employ the $d$-dimensional generalization of a strongly anisotropic ensemble introduced in [39] and further discussed in a number of papers, e.g. [40–43], in connection with the passive scalar problem. A detailed description of that model is given below in section 2, and here we stress only the two main points.

(i) The velocity field is oriented along a chosen direction $n$, while its correlation function depends only on the coordinates perpendicular to $n$. Thus, the model provides a simplified description of a turbulent shear flow. Strong anisotropy leads to drastic distinctions with the isotropic case already for the linear advection–diffusion equation; see e.g. [43] for a recent discussion. In this connection, the model of the magnetohydrodynamic turbulence in the solar crown is also worth mentioning [44], where the large-scale magnetic field introduces the distinguished direction $n$, while the small-scale turbulent activity is concentrated in the perpendicular plane.

(ii) In the spirit of standard turbulent phenomenology, the inertial-range spectrum of the velocity is taken in the power-like form, \( \propto \delta(t-t')/|k_{\perp}|^{d-1+\xi} \), where $k_{\perp}$ is the component of the wave vector $k$, perpendicular to $n$. The most realistic ‘Kolmogorov’ value $\xi = 4/3$ is not small, but here it is not exceptional: for the linear case (passive scalar advection), no crossover in the behaviour of the model is expected at $\xi = 4/3$; see e.g. [43]. We also note that for the isotropic linear case, the validity of the RG approach, based on the small-$\xi$ expansion, is demonstrated by the agreement with the non-perturbative zero-mode approach and numerical simulations; see the discussion in [38].

Although simplified, the model appears rather nontrivial and captures the main property of the problem: the existence of a new, non-equilibrium and strongly anisotropic, universality class of scaling behaviour.

The plan of the paper is as follows. In section 2, we present a detailed description of the model and its field-theoretic formulation. In section 3, we analyse the ultraviolet (UV) divergences, relying upon the power counting and additional symmetry considerations. We show that the model, after proper extension, appears multiplicatively renormalizable. Thus, we can derive the RG equations and introduce the RG functions ($\beta$ functions and anomalous dimensions $\gamma$) in the standard manner; see section 4.

In section 5, we show that, depending on the relation between the spatial dimension $d$ and the exponent $\xi$ in the velocity correlator, the model reveals four different types of critical behaviour, associated with four fixed points of the corresponding RG equations. Three fixed points correspond to known regimes: Gaussian or free field theory, non-interacting scalar field passively advected by the flow (the ATP nonlinearity in the original dynamical equations appears irrelevant) and the original critical behaviour of the model without mixing. The most interesting fourth point corresponds to a new full-scale non-equilibrium universality class, in which both the nonlinearity and turbulent mixing are relevant.

The corresponding critical dimensions can be calculated as double expansions in two parameters: $\xi$ and $\varepsilon = 6 - d$. The scaling behaviour appears strongly anisotropic in the sense that the critical dimensions related to the directions parallel and perpendicular to the flow are essentially different. The practical calculation of the renormalization constants, RG functions, regions of stability and critical dimensions was performed in the leading order (one-loop approximation); some of the results, however, are exact (valid to all orders of the double $\varepsilon-\xi$ expansion). These issues are discussed in section 6, while section 7 is reserved for the conclusion.
2. Description of the model and the field-theoretic formulation

The relaxational dynamics of a non-conserved $n$-component order parameter $\varphi(x)$ with
$x \equiv [t, \mathbf{x}]$ is described by a stochastic differential equation

$$\partial_t \varphi(x) = -\lambda_0 \frac{\delta H(\varphi)}{\delta \varphi(x)} \bigg|_{\varphi(x) \rightarrow \varphi(x)} + \eta_x(x), \quad (2.1)$$

where $\partial_t = \partial / \partial t$, $\lambda_0$ is the (constant) kinetic coefficient and $\eta_x(x)$ is a Gaussian random noise
with zero mean and the pair correlation function

$$\langle \eta_x(x) \eta_x(x') \rangle = \delta_{x x} D_\eta (x - x'), \quad (2.2)$$

with $d$ being the dimension of the $\mathbf{x}$ space. Near the critical point, the static Hamiltonian $H(\varphi)$ of the ATP model is taken in the form [5–7]

$$H(\varphi) = \int \mathrm{d}x \left\{ -\frac{1}{2} \varphi_\alpha(x) \partial^2 \varphi_\alpha(x) + \frac{\tau_0}{2} \varphi_\alpha(x) \varphi_\alpha(x) + \frac{g_0}{3!} R_{abc} \varphi_\alpha(x) \varphi_\alpha(x) \varphi_\alpha(x) \right\}, \quad (2.3)$$

where $\partial_i = \partial / \partial x_i$ is the spatial derivative, $\partial^2 = \partial_i \partial_i$ is the Laplacian, $\tau_0 \propto (T - T_c)$
measures deviation of the temperature (or its analogue) from the critical value and $g_0$ is the
 coupling constant. Summations over repeated indices are always implied ($a, b, c = 1, \ldots, n$
and $i = 1, \ldots, d$). After taking the functional derivative $\delta H(\varphi)/\varphi(x)$, one has to replace
$\varphi(x) \rightarrow \varphi(x)$ in (2.1).

Following [9], we consider the generalized case of a certain symmetry group $G$, which has
the only irreducible invariant third-rank tensor $R_{abc}$; with no loss of generality, it is assumed
to be symmetric. In the original ATP model, $G$ is the symmetry group of the hypertetrahedron
in $n$ dimensions. Then the tensor $R_{abc}$ is conveniently expressed in terms of the set of $n + 1$
vectors $e^\alpha$ which define its vertices [5, 6]:

$$R_{abc} = \sum_\alpha e^\alpha_a e^\alpha_b e^\alpha_c,$$

where the $e^\alpha_a$ satisfy

$$\sum_{a=1}^{n+1} e^\alpha_a = 0, \quad \sum_{a=1}^{n+1} e^\alpha_a e^\beta_a = (n + 1) \delta_{ab}, \quad \sum_{a=1}^n e^\alpha_a e^\alpha_a = (n + 1) \delta^{\alpha \beta} - 1. \quad (2.4)$$

Using equations (2.4), all the contractions with the tensor $R_{abc}$ can be calculated. For example,
the following contractions of two and three tensors have the form

$$R_{abc} R_{abc} = R_1 \delta_{cd}, \quad R_{abc} R_{cde} R_{def} = R_2 R_{eef}, \quad (2.5)$$

where

$$R_1 = (n + 1)^2 (n - 1), \quad R_2 = (n + 1)^2 (n - 2). \quad (2.6)$$

Coupling with the velocity field $\mathbf{v} = \{v_i(x)\}$ is introduced by the replacement

$$\partial_t \rightarrow \nabla_t = \partial_t + v_i \partial_i, \quad (2.7)$$

where $\nabla_t$ is the Lagrangian (Galilean covariant) derivative. For an incompressible fluid, the
velocity field $\mathbf{v}$ is transverse due to the continuity relation $\partial_t v_i = 0$. The velocity ensemble
is defined as follows [39]. Let $\mathbf{n}$ be a unit constant vector that determines some distinguished
direction (‘direction of the flow’). Then any vector can be decomposed into the components
perpendicular and parallel to the flow, for example, \( \mathbf{x} = \mathbf{x}_\perp + \mathbf{n} x_1 \) with \( \mathbf{x}_\perp \cdot \mathbf{n} = 0 \). The velocity field will be taken in the form

\[
\mathbf{v} = \mathbf{n} v(t, \mathbf{x}_\perp),
\]

where \( v(t, \mathbf{x}_\perp) \) is a scalar function independent of \( x_1 \). Then the incompressibility condition is automatically satisfied:

\[
\partial_t v_i = \partial_{i}v(t, \mathbf{x}_\perp) = 0.
\]

For \( v(t, \mathbf{x}_\perp) \), we assume a Gaussian distribution with zero mean and the pair correlation function of the form

\[
\langle v(t, \mathbf{x}_\perp) v(t', \mathbf{x}_\perp') \rangle = \delta(t - t') \int \frac{dk}{(2\pi)^d} \exp \{ik \cdot (\mathbf{x} - \mathbf{x}') \} D_v(k) = \delta(t - t') \int \frac{dk}{(2\pi)^d} \exp \{ik \cdot (\mathbf{x}_\perp - \mathbf{x}_\perp') \} \tilde{D}_v(k_\perp), \quad k_\perp = |\mathbf{k}_\perp|,
\]

with the scalar coefficient functions

\[
D_v(k) = 2\pi \delta(k_0) \tilde{D}_v(k_\perp), \quad \tilde{D}_v(k_\perp) = D_0 k^{-d+1-\xi}.
\]

Here and below, \( d \) is the dimension of the \( \mathbf{x} \) space, \( D_0 > 0 \) is a constant amplitude factor and \( \xi \) is an arbitrary exponent. The IR regularization in (2.8) is provided by the cutoff \( k_\perp > m \) (by dimension, \( \tau_0 \propto m^2 \)). The precise form of the IR regularization is inessential; sharp cutoff is the most convenient choice from the calculational viewpoints. The natural interval for the exponent is \( 0 < \xi < 2 \), when the so-called effective eddy diffusivity has a finite limit for \( m \to 0 \); it includes the most realistic Kolmogorov value \( \xi = 4/3 \).

In order to ensure the multiplicative renormalizability of the model, it is necessary to split the Laplacian in (2.1) into the parallel and perpendicular parts \( \partial^2 \to \partial^2_\perp + f_0 \partial^2_{\|} \) by introducing a new parameter \( f_0 > 0 \). Here \( \partial^2_\perp \) is the Laplacian in the subspace orthogonal to the vector \( \mathbf{n} \) and \( \partial_{\|} = \partial/\partial x_1 \). In the anisotropic case, these two terms will be renormalized in a different way. Thus, equation (2.1) becomes

\[
\nabla \varphi = \lambda_0 (\partial^2_\perp + f_0 \partial^2_{\|} - \tau_0) \varphi - \frac{g_0 \lambda_0}{2} R_{abc} \varphi_b \varphi_c + \eta_0; \quad (2.10)
\]

this completes the formulation of the model.

Interpretation of the splitting can be twofold; cf the discussion in [31, 32]. On the one hand, stochastic models of type (2.1) are phenomenological and, by construction, they should include all the relevant terms allowed by symmetry. The fact that the splitting is required by the renormalization procedure means that it is not forbidden by dimensionality or symmetry considerations and, therefore, it is reasonable to include the general value \( f_0 = 1 \) in the model from the very beginning. On the other hand, one can insist on discussing the original model with \( f_0 = 1 \) and \( O_{\perp} \) covariant Laplacian term, although that symmetry is broken to \( O_{\perp-1} \otimes Z_2 \) by the interaction with the anisotropic velocity ensemble (here \( Z_2 \) is the residual reflection symmetry \( x_1 \to -x_1 \)). Then the extension of the model to the case \( f_0 = 1 \) can be viewed as a purely technical trick which is only needed to ensure the multiplicative renormalizability and to derive the RG equations. The latter should then be solved with the special initial data corresponding to \( f_0 = 1 \). However, in renormalized variables, this would correspond to general initial data with \( f \neq 1 \) anyway. Thus, the value \( f_0 = 1 \) is not exceptional, and the resulting IR behaviour (governed by the same IR attractive fixed points) would be the same as for the general case with \( f_0 \neq 1 \).
According to the general theorem (see e.g. chapter 5 of [2]), our stochastic problem is equivalent to the field-theoretic model of the extended set of fields \( \Phi = \{ \varphi_0, \varphi_a, \psi \} \) with the action functional

\[
S(\Phi) = \frac{\varphi_0 D_0 \varphi_0}{2} + \int - \nabla_i + \lambda_0 (\partial_\perp^2 + f_0 \partial_\parallel^2) - \lambda_0 \tau_0 \varphi_a - \frac{\lambda_0 \varphi_0 f_0^{1/4}}{2} R_{abc} \varphi_a \partial_\perp^a \partial_\perp^c \varphi_c + S_0(\psi),
\]

(2.11)

where we segregated the factor \( f_0^{1/4} \) from the charge \( g_0 \). The first few terms represent the De Dominics–Janssen action functional for the stochastic problem (2.1), (2.2) at fixed \( \psi \); it involves the auxiliary scalar response field \( \varphi_a(x) \). All the required integrations over \( x = [t, \mathbf{x}] \) and summations over the vector indices are implied, for example,

\[
\varphi_a^\prime \partial_\perp^2 \varphi_a = \sum_{a=1}^n \int dt \int d\mathbf{x} \varphi_a^\prime(x) \partial_\perp^2 \varphi_a(x).
\]

The last term in (2.11) corresponds to the Gaussian averaging over \( \psi \) with correlator (2.8) and has the form

\[
S_0(\psi) = - \frac{1}{2} \int dt \int d\mathbf{x} \partial_\perp^2 \partial_\perp^2 v(t, \mathbf{x}) - v'(t, \mathbf{x}') \psi(t', \mathbf{x}'),
\]

where

\[
\tilde{D}_t^{-1}(\mathbf{r}_\perp) \propto D_0^{-1} r_\perp^{2(1-d) - \xi}
\]

is the kernel of the inverse linear operation \( D_0^{-1} \) for the correlation function \( D_0 \) in (2.9).

This formulation means that statistical averages of random quantities in the original stochastic problem coincide with the Green functions of the field-theoretic model with action (2.11), given by functional averages with the weight exp \( S(\Phi) \). This allows one to apply the standard Feynman diagrammatic technique, the field-theoretic renormalization theory and RG to our stochastic problem. The role of the two coupling constants (expansion parameters of the perturbation theory) is played by the parameter \( g_0 \) from the static Hamiltonian (2.3) and the additional parameter \( u_0 = D_0 / (f_0 \lambda_0) \) that describes the interaction with the velocity field. By dimension, \( g_0 \sim \ell^{1-\xi} \) and \( u_0 \sim \ell^{-\xi} \), where \( \ell \) has the order of the smallest length scale of our problem.

3. Canonical dimensions, UV divergences and renormalization

It is well known that the analysis of UV divergences is based on the analysis of canonical dimensions; see e.g. [1, 2]. In general, dynamic models have two scales: the canonical dimension of some quantity \( F \) (a field or a parameter in the action functional) is completely characterized by two numbers, the frequency dimension \( d_f^0 \) and the momentum dimension \( d_k^0 \), see e.g. chapter 5 in [2]. They are determined such that \( [F] \sim [T]^{-d_f} [L]^{-d_k} \), where \( L \) is some length scale and \( T \) is the time scale.

Our strongly anisotropic model, however, has two independent momentum scales, related to the directions perpendicular and parallel to the vector \( \mathbf{n} \), and requires a more detailed specification of the canonical dimensions. Namely, one has to introduce two independent canonical dimensions \( d_f^\perp \) and \( d_k^\perp \) so that

\[
[F] \sim [T]^{-d_f^\perp} [L_\perp]^{-d_k^\perp} [L_\parallel]^{-d_k^\parallel},
\]

where \( L_\perp \) and \( L_\parallel \) are (independent) length scales in the corresponding subspaces. The dimensions are found from the obvious normalization conditions \( d_k^\perp = -d_k^\parallel = 1 \),

\[
d_k^\perp = -d_k^\parallel = 0, d_f^\perp = d_f^\parallel = 0, d_k^\perp = -d_k^\parallel = 1 \text{ and so on, and from the requirement}
\]

\[
[d_f^\perp] \sim [T]^{-d_k^\perp} [L_\perp]^{-d_k^\perp} [L_\parallel]^{-d_k^\parallel},
\]

where \( d_f^\perp \) and \( d_k^\perp \) are (independent) length scales in the corresponding subspaces. The dimensions are found from the obvious normalization conditions \( d_k^\perp = -d_k^\parallel = 1 \),

\[
d_k^\perp = -d_k^\parallel = 0, d_f^\perp = d_f^\parallel = 0, d_k^\perp = -d_k^\parallel = 1 \text{ and so on, and from the requirement}
\]

\[
[d_f^\perp] \sim [T]^{-d_k^\perp} [L_\perp]^{-d_k^\perp} [L_\parallel]^{-d_k^\parallel},
\]
that each term of the action functional (2.11) be dimensionless (with respect to all three independent dimensions separately).

The canonical dimensions of model (2.11) are given in Table 1, including renormalized parameters, which will be introduced a bit later. From Table 1 it follows that the model is logarithmic (the coupling constants parameters, which will be introduced a bit later. From Table 1 it follows that the model is independent dimensions separately).

Here \( \lambda \) enters the renormalized action only in the form of the Lagrangian derivative (in the free theory, \( \tilde{\lambda} \propto \tilde{\delta}^2 \)). In the renormalization theory, it plays the same part as the conventional (momentum) dimension does in one-scale static problems: superficial UV divergences, whose removal requires counterterms, can be present only in those 1-irreducible Green functions for which the total canonical dimension at the logarithmic values \( \varepsilon = \xi = 0 \) (formal index of divergence) is a non-negative integer.

The careful analysis of Table 1, augmented by symmetry considerations, shows that all the counterterms needed to cancel the UV divergences in our model are present in action (2.11).

Thus, our model appears multiplicatively renormalizable with the renormalized action of the form

\[
S_R(\Phi) = Z_1 \left( \frac{\psi'_a D_0 \psi'_a}{2} + \psi'_a \left\{ -Z_2 \nabla_i + \lambda (Z_3 \partial^2_i + Z_4 f \partial^0_i) - Z_5 \lambda \tau \right\} \psi_a \right.
\]

\[
- Z_6 \frac{\lambda_R u^{1/2} f^{1/2}}{2} R_{abc} \psi'_a \psi'_b \psi'_c + S_0(\nu).
\]

(3.1)

Here \( \lambda, \tau, f, g \) and \( w \) are renormalized analogues of the bare parameters (with the subscripts ‘0’) and \( u \) is the reference mass scale (additional arbitrary parameter of the renormalized theory). The renormalization constants capture all the divergences at \( \varepsilon, \xi = 0 \) so that the correlation functions of the renormalized model (3.1) have finite limits for \( \varepsilon, \xi \to 0 \), when expressed in the renormalized parameters \( \lambda, \mu \) and so on.

A simple analysis shows that the Feynman diagrams needed for the calculation of the renormalization constants in (3.1) are the same as in the one-component \( \psi^3 \) theory (model 2 from [32]), multiplied with the appropriate tensor contractions. Moreover, in the

| \( \xi \) |
|---|
| \( \frac{d}{d-1} )/2 \) |
| \( \frac{d}{d-3} )/2 \) |
| \( \frac{d}{2} \) |

Table 1. Canonical dimensions of the fields and parameters in model (2.11).
one-loop approximation, only contractions (2.5) appear. Therefore, one can easily generalize
the results of [32] to the case at hand with the aid of (2.5).

In [32], the minimal subtraction (MS) renormalization scheme was employed. In the MS
scheme, the renormalization constants have the forms ‘Z = unity plus only singularities in
ε and ξ’, with the coefficients depending on the two completely dimensionless parameters—
renormalized coupling constants g and w. The one-loop results for the renormalization
constants in (3.1) are as follows:

\[ Z_1 = 1 - \frac{uR_1}{2\varepsilon}, \quad Z_2 = 1 - \frac{uR_1}{3\varepsilon}, \quad Z_3 = 1 - \frac{uR_1 - \frac{w}{\xi}}{3\varepsilon}, \quad Z_4 = 1 - \frac{uR_1}{3\varepsilon} - \frac{w}{\xi}. \]  

(3.2)

Here we have passed to the more convenient coupling constants u → \(g^2/128\pi^3\) and
w → w/24\(\pi^3\).

The parameters R_1 and R_2 are related to the dimension n of the order parameter by expression (2.6). Although we are especially interested in the cases n = 0 and 2, for
completeness the coefficients R_1 and R_2 in what follows are assumed to be arbitrary.

Expression (3.1) is equivalent to the multiplicative renormalization of the fields \(\varphi_a \rightarrow \phi_a Z_{\varphi_a}\), \(\varphi_a \rightarrow \phi_a Z_{\varphi_a}\) and the parameters

\[ \lambda_0 = \lambda Z_{\lambda_0}, \quad \tau_0 = \tau Z_{\tau}, \quad f_0 = f Z_f, \]

\[ g_0 = g u^{1/2} Z_{g_0}(u_0 = u/\mu^2 Z_{\mu}), \quad w_0 = w/\mu^3 Z_w \]  

(no renormalization of the velocity field is needed; \(Z_v = 1\)). The constants in equations (3.1)
and (3.3) are related as follows:

\[ Z_1 = Z_1 Z_{\varphi}, \quad Z_2 = Z_{\varphi} Z_{\varphi}, \quad Z_3 = Z_{\varphi} Z_{\varphi} Z_{\varphi}, \]

\[ Z_4 = Z_{\varphi} Z_{\varphi} Z_{\varphi} Z_{\varphi}, \quad Z_5 = Z_{\varphi} Z_{\varphi} Z_{\varphi} Z_{\varphi}, \quad Z_6 = Z_{\varphi} Z_{\varphi} Z_{\varphi} Z_{\varphi}. \]  

(3.4)

Since the last term \(S_\varphi(v)\) is not renormalized, the amplitude \(D_0\) is expressed in renormalized
parameters as

\[ D_0 = w_0 f_0 \lambda_0 = w_0 f_0 \mu^3 \lambda, \]

which leads to the relation

\[ Z_0 Z_1 Z_3 = 1 \]  

(3.5)

between the renormalization constants.

4. RG equations and RG functions

Let us recall an elementary derivation of the RG equations; a more detailed discussion can be found in [1, 2]. The RG equations are written for the renormalized correlation functions
\(G_R = \langle \Phi \cdots \Phi \rangle_R\), which differ from the original (unrenormalized) ones \(G = \langle \Phi \cdots \Phi \rangle\) only
by normalization and the choice of parameters, and therefore can equally be used for analysing
the critical behaviour. The relation \(S_{\Phi}(Z_{\Phi} \Phi, \varepsilon, \mu) = \tilde{S}(\Phi, \epsilon_0)\) between functionals (2.11) and
(3.1) results in the relations

\[ G(\epsilon_0, \ldots) = Z_{\Phi}^N \Phi Z_{\Phi}^N G_R(\varepsilon, \mu, \ldots) \]  

(4.1)

between the correlation functions. Here, \(N_{\varphi}\) and \(N_{\varphi}\) are the full numbers of corresponding
fields entering into \(\Gamma\) (we recall that in our model \(Z_{\varphi} = 1\)); \(\epsilon_0 = \{\lambda_0, \tau_0, f_0, u_0, w_0\}\) is the full
set of bare parameters and \(\varepsilon = \{\lambda, \tau, f, u, w\}\) are their renormalized counterparts; the ellipsis
stands for the other arguments (times, coordinates, momenta, etc).
We use \( \bar{D}_\mu \) to denote the differential operation \( \mu \partial_\mu \) for fixed \( \epsilon_0 \) and operate on both sides of equation (4.1) with it. This gives the basic RG differential equation

\[
\{ \bar{D}_\mu + N_\epsilon \gamma_\psi + N_\mu \gamma_\psi \} G^\mu(\epsilon, \mu, \ldots) = 0,
\]

where \( \bar{D}_\mu \) is the operation \( \bar{D}_\mu \) expressed in the renormalized variables:

\[
\bar{D}_\mu \equiv D_\mu + \beta_u \partial_u + \beta_w \partial_w - \gamma_f D_f - \gamma_\psi D_\psi - \gamma_\sigma D_\sigma.
\]

Here we have written \( D_\mu \equiv x \partial_\mu \) for any variable \( x \); the anomalous dimensions \( \gamma \) are defined as

\[
\gamma_F \equiv \bar{D}_\mu \ln Z_F \quad \text{for any quantity } F,
\]

and the \( \beta \) functions for the two dimensionless couplings \( u \) and \( w \) are

\[
\beta_u \equiv \bar{D}_\mu u = u [-\epsilon - \gamma_\epsilon], \quad \beta_w \equiv \bar{D}_\mu w = w [-\xi - \gamma_\xi],
\]

where the second equalities come from the definitions and relations (3.3).

Equations (3.4) result in the following relations between the anomalous dimensions:

\[
\begin{align*}
\gamma_1 &= \gamma_\epsilon + 2\gamma_\psi, \\
\gamma_2 &= \gamma_\epsilon + \gamma_\psi, \\
\gamma_3 &= \gamma_\epsilon + \gamma_\psi + \gamma_\psi, \\
\gamma_4 &= \gamma_f + \gamma_\psi, \\
\gamma_5 &= \gamma_f + \gamma_\psi, \\
\gamma_6 &= \gamma_\epsilon + \gamma_\psi/2 + \gamma_f/4 + \gamma_\psi/4 + 2\gamma_\psi,
\end{align*}
\]

while from (3.5), one obtains

\[
\gamma_\epsilon + \gamma_\psi + \gamma_\psi = 0.
\]

The dimensions \( \gamma_1 - \gamma_6 \) are calculated from the corresponding renormalization constants using definition (4.4):

\[
\begin{align*}
\gamma_1 &= \frac{uR_1}{2}, \\
\gamma_2 &= \frac{uR_1}{3}, \\
\gamma_3 &= \frac{uR_1}{3} + w, \\
\gamma_4 &= 2uR_1, \\
\gamma_5 &= 2uR_1, \\
\gamma_6 &= 2uR_2,
\end{align*}
\]

with the corrections of the order \( u^2, w^2, uw \) and higher.

The RG functions entering equation (4.3) are easily found from relations (3.2), (4.6) and (4.7):

\[
\begin{align*}
\gamma_\psi &= \gamma_2 - \gamma_3/2 = uR_1/3, \\
\gamma_\epsilon &= \gamma_3/2 = \frac{uR_1}{6}, \\
\gamma_f &= \gamma_4 - \gamma_3 = w, \\
\gamma_\sigma &= \gamma_3 - \gamma_3 = 5uR_1/3, \\
\gamma_w &= \gamma_2 - \gamma_4 = uR_1/6 - w, \\
\gamma_u &= 2\gamma_6 - 5\gamma_3/2 - \gamma_4/2 = (4R_2 - R_1)u - w/2,
\end{align*}
\]

with higher order corrections.

From definitions (4.5), relations (4.9) and explicit expressions (4.8) for the anomalous dimensions, we derive the following leading-order expressions for the \( \beta \) functions of our model:

\[
\begin{align*}
\beta_u &= u [-\epsilon + Ru + u/2], \\
\beta_w &= w [-\xi - uR_1/6 + w],
\end{align*}
\]

where we have introduced a new convenient parameter \( R = R_1 - 4R_2 \).
5. Fixed points and scaling regimes

It is well known that possible large-scale scaling regimes of a renormalizable model are associated with IR attractive fixed points of the corresponding RG equations. In our model, the coordinates $u_*, w_*$ of the fixed points are found from the equations

$$\beta_u(u_*, w_*) = 0, \quad \beta_w(u_*, w_*) = 0,$$

with the $\beta$ functions given in (4.5). The type of a fixed point is determined by the matrix

$$\Omega = \{\Omega_{ij} = \partial \beta_i/\partial u_j\},$$

where $\beta_i$ denotes the full set of the $\beta$ functions and $u_j = \{u, w\}$ is the full set of couplings. For IR stable fixed points, the matrix $\Omega$ is positive, i.e. the real parts of all its eigenvalues are positive. This condition defines the regions of IR stability for the corresponding scaling regimes.

The couplings $u$ and $w$ should be non-negative (by definition, $u \propto g^2 \geq 0$ and $w \propto D_0/f_{\lambda} \geq 0$), and in the following we will be interested only in the ‘good’ (admissible from the physics viewpoints) fixed points, which satisfy the conditions

$$u_* \geq 0, \quad w_* \geq 0$$

and can be IR attractive for some values of the model parameters.

In order to give the complete picture of possible scaling regimes, it is instructive to discuss at first a more general case, specified by the $\beta$ functions of the form

$$\beta_u = u[-\varepsilon + au + bw], \quad \beta_w = w[-\xi + cu + dw],$$

(5.4)

with arbitrary real coefficients $a$–$d$.

From equations (5.1) and (5.4), we can identify four different fixed points. For the first three points, the matrix $\Omega$ appears to be triangular, so that its eigenvalues (and hence the regions of IR stability of the corresponding scaling regimes) are simply determined by the diagonal elements: $\Omega_u = \partial \beta_u/\partial u > 0$ and $\Omega_w = \partial \beta_w/\partial w > 0$.

1. Gaussian (free) fixed point: $u_* = w_* = 0$; $\Omega_u = -\varepsilon$, $\Omega_w = -\xi$.
2. $u_* = 0$ (exact result to all orders), $w_* = \xi/d$; $\Omega_u = -\varepsilon + b\xi/d$, $\Omega_w = \xi$. This point can be ‘good’ only if $d > 0$; otherwise the conditions $\Omega_w > 0$ and $w_* > 0$ cannot be simultaneously satisfied.
3. $w_* = 0$ (exact result to all orders), $u_* = \varepsilon/a$; $\Omega_u = \varepsilon$, $\Omega_w = -\xi + c\varepsilon/a$. Similarly to case (2), this point can be ‘good’ only if $a > 0$.

The last, fully nontrivial, fixed point requires a more detailed discussion.

4. The coordinates of this point are

$$u_* = (de - b\xi)/\Delta, \quad w_* = (a\xi - c\varepsilon)/\Delta, \quad \Delta = ad - bc,$$

(5.5)

while the matrix $\Omega$ can be written in the form

$$\Omega = \begin{pmatrix} au_* & bu_* \\ cw_* & dw_* \end{pmatrix}$$

(5.6)

(it is useful not to substitute explicit expressions (5.5) into (5.6) for a while).

The necessary and sufficient condition for the IR stability of this point can be restated as the fulfilment of two inequalities:

$$\det \Omega > 0, \quad \text{tr} \Omega > 0.$$

(5.7)
From (5.6), one obtains
\[
\det \Omega = u_* w_* \Delta > 0,
\]
which along with (5.3) shows that this point can be ‘good’ only if $\Delta > 0$.
For the trace of $\Omega$, we obtain
\[
\text{tr} \Omega = au_* + dw_* > 0.
\]
There are three possibilities.

1. $a > 0, d > 0$. In this case, inequality (5.9) is an automatic consequence of (5.3). Four regions of stability of the fixed points (1)–(4) divide the $\varepsilon$–$\xi$ plane without ‘gaps’ or overlaps. This is the most typical situation, realized for the $\varphi^4$ model or the Gribov process in various kinds of random flows [30–32].

2. $a < 0, d < 0$. Then (5.9) contradicts (5.3) and this point can never be ‘good.’ Such a situation has not yet been encountered.

3. The parameters $a$ and $d$ are opposite in sign: $ad < 0$. For definiteness, we assume that $a < 0, d > 0$. We will see in short that this situation can be realized for the ATP model. In this case, one obtains from (5.9)
\[
w_* > -au_*/d > 0,
\]
where the last inequality follows from $a < 0, u_* > 0$. The second inequality in (5.3) is implied by (5.10) and thus becomes superfluous. The region where the fixed point is IR attractive and positive is given by the two inequalities
\[
u_*>0, \quad au_* + dw_* > 0.
\]
Summing up, we conclude that the scaling regime corresponding to the fixed point (4) exists if $\Delta > 0$ and at least one of the two parameters $a$ and $d$ is positive. The region where the fourth fixed point is ‘good’ is determined by inequalities (5.3) if $a$ and $d$ are simultaneously positive, and by the conditions of type (5.11) if $a$ and $d$ are opposite in sign.

Let us turn to our specific model with the $\beta$ functions (4.10). Identifying them with (5.4) gives
\[
a = R_1 - 4R_2 \equiv R, \quad b = \frac{1}{2}, \quad c = -\frac{R_1}{6}, \quad d = 1;
\]
\[
\Delta = R + R_1/12 \equiv \Delta^R.
\]
The coordinates of the four possible fixed points are obtained by substituting expressions (5.12) into the general results.

In the scaling regime corresponding to the fixed point (2), the nonlinearity $\varphi^2$ in the stochastic equation (2.1) becomes irrelevant in the sense of Wilson due to the exact relation $u_* = 0$. Thus, we arrive at the linear advection–diffusion equation for a passive scalar field $\varphi$. In turn, the effects of the velocity field become irrelevant in the third regime (fixed point (3)) due to the exact relation $w_* = 0$. The isotropy violated by the velocity ensemble is restored and the leading terms of the IR behaviour coincide with those of the equilibrium dynamic model ATP. Finally, the fixed point (4) corresponds to a new nontrivial IR scaling regime, in which both the nonlinearities in the stochastic equation for $\varphi$ are important; the corresponding critical dimensions reveal strong anisotropy, depend essentially on both the RG expansion parameters $\varepsilon$ and $\xi$, and are calculated as double series in those parameters; see section 6.

The regions of IR stability for all possible fixed points in the $\varepsilon$–$\xi$ plane for different values of the parameters $R_1$ and $R_2$ are shown in figures 1–5. In the one-loop approximation, all the boundaries of the regions are given by straight lines.
Figures 3 and 5 show that there are overlaps between the IR stability regions of fixed points (2) and (3) and points (1) and (4), respectively. This means that for the values of \( \varepsilon \) and \( \xi \), corresponding to the region of an overlap, the system has two variants of the IR scaling behaviour. Which one of them is realized depends on the initial data for the parameters \( u \) and \( w \) in the RG equations.
On the other hand, from figure 4 one can see that if the parameters $R_1$ and $R_2$ are such that $\Delta R < 0$ and $R < 0$, there is a gap. The system does not exhibit scaling behaviour for the corresponding values of $\varepsilon$ and $\xi$, which can be interpreted as the existence of a first-order phase transition.

It is interesting to note that for $a < 0$, the original static model has no ‘good’ fixed point (which is usually interpreted as a first-order transition), but a ‘good’ point of type (5.5) can appear in the full dynamic model with two couplings, as illustrated by figure 5. One can say that the phase transition changes its type and becomes a second-order one owing to the turbulent mixing.

Let us briefly discuss the pattern of the RG flows in the plane of couplings $u$-$w$ for two special cases. Consider first the simplest (and the most typical) situation, illustrated by figure 2. The schematic picture of the fixed points and RG flows is sketched in figure 6 for the case when the values of the parameters $\varepsilon$ and $\xi$ lie in region 4 in figure 2. Then all the fixed points lie in the physical domain $u \geq 0, w \geq 0$: point (4) is IR attractive, point (1) is repulsive and points (2) and (3) are saddle points. If the parameters $\varepsilon$ and $\xi$ change such that the corresponding point in the $\varepsilon$-$\xi$ plane crosses the boundary $\xi = 2\varepsilon$ and moves into region 2, the fixed point (4) in figure 6 crosses the ray $u = 0, w \geq 0$, going through point (2), and moves into the unphysical domain $u < 0$. Point (2) becomes IR attractive. If the point in figure 2 crosses the boundary $\xi = -\varepsilon R_1/6R$, moving from region 4 to 3, the fixed point (4) crosses the ray $u \geq 0, w = 0$, going through point (3), and moves into the unphysical domain $w < 0$; point (3) becomes IR attractive. Although this picture is based on the explicit one-loop expressions.
for the $β$ functions and regions of stability, it appears robust with respect to the higher order corrections. It is crucial here that the functions $β_u$ at $w = 0$ and $β_w$ at $u = 0$ coincide with the $β$ functions of the corresponding single-charge problems: the ATP model and the passive scalar case, respectively. The latter are supposed to have a unique nontrivial fixed point each: for the passive scalar, the $β$ function is known exactly, while for the ATP model (with $R > 0$) this is true at least within the $ε$ expansion. Thus, we may conclude that the coincidence of the fixed points (2) and (4) or (3) and (4) takes place simultaneously with the changeover in their type of stability. In turn, this means that, although the boundaries between regions 2 and 4 or 3 and 4 in the $ε−ξ$ plane can become curved beyond the one-loop approximation due to the higher order corrections to the $β$ functions in (5.4), no gaps nor overlaps can appear between those regions. This is equally true for the boundaries between regions 1 and 2 or 1 and 3 in figure 2, but there the boundaries are not affected by the higher order corrections due to the simple exact expressions for the eigenvalues $Ω_{u,w}$ at the Gaussian fixed point.

The RG flows for another interesting situation, illustrated by figure 3, are depicted in figure 7 for the case when the values of the parameters $ε$ and $ξ$ lie in the overlap of regions 2 and 3. Then both the fixed points (2) and (3) are ‘good’ and the asymptotic behaviour of a flow depends on the initial data for $u$ and $w$. Point (4) is unphysical: although it lies in the domain $u ≥ 0$, $w ≥ 0$, inspection of the explicit expressions (5.5) and (5.8) shows that it is a saddle point for this case. It is also worth noting that, for all possible situations, the RG flow with the
initial data in the physical domain $u \geq 0, w \geq 0$ can never leave it, because the function $\beta_u$ vanishes for $u = 0$ and arbitrary $w$, while $\beta_w$ vanishes for $w = 0$ and arbitrary $u$.

Let us conclude this section with a brief discussion of the most interesting physical cases of the original ATP model. Then from (2.6) and (5.12), we obtain $R = (n + 1)^2 (7 - 3n)$ and $\Delta^R = (n + 1)^2 (83 - 35n)/12$.

The case $n = 2$ corresponds to the nematic-to-isotropic transition in a liquid crystal. Then

$$R = R_1 = 9, \quad R_2 = 0, \quad \Delta^R = 39/4,$$  \hspace{1cm} (5.13)

and the case represented by figure 1 is realized. One can see that the interval of the most realistic values of the model parameters, $d = 3$ and $0 < \xi < 2$, belongs completely to the region of stability of the most nontrivial fixed point (4).

The second case of interest is the critical behaviour of bond percolation, obtained in the limit $n = 0$. Strictly speaking, a more realistic description of dynamical percolation is given by a special model with a nonlocal in-time interaction [45], but it is interesting to look at the dynamics of the ATP model with $n = 0$ as some kind of approximation. For $n = 0$, relations (2.6) give

$$R_1 = -1, \quad R_2 = -2, \quad R = 7, \quad \Delta^R = 83/12.$$ \hspace{1cm} (5.14)

Thus, we arrive at the case shown in figure 2. Again, the most realistic values of the model parameters ($d = 3$ and $\xi = 4/3$) belong to the stability region of the new anisotropic scaling regime, corresponding to the fixed point (4).

For $n \geq 3$, the case shown in figure 4 is realized. Now the values $d = 3$ and $\xi = 4/3$ lie in the ‘empty’ region where none of the fixed points are ‘good.’ This fact is usually interpreted as a first-order transition; the turbulent mixing does not change its type. The interesting situation illustrated by figure 5, where the mixing gives rise to the changeover in the type of the phase transition to the second-order one, is realized for the interval $7/3 < n < 83/35$, which does not contain integer values and has no documented physical interpretation.

### 6. Critical scaling and critical dimensions

We recall the definition of generalized homogeneity. Let $F$ be a function of $n$ independent arguments $\{x_1, \ldots, x_n\}$ that satisfies the dimensional relation

$$F(\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n) = \lambda^{a_F} F(x_1, \ldots, x_n)$$ \hspace{1cm} (6.1)

with a certain set of constant coefficients (scaling dimensions) $\{\alpha_1, \ldots, \alpha_n, \alpha_F\}$ and an arbitrary positive parameter $\lambda > 0$. Differentiating relation (6.1) with respect to $\lambda$ and then setting $\lambda = 1$, we obtain a first-order differential equation with constant coefficients

$$\frac{\partial}{\partial \lambda} F(x_1, \ldots, x_n) = \alpha_F F(x_1, \ldots, x_n), \quad \frac{\partial}{\partial \lambda} = x_i \partial / \partial x_i.$$ \hspace{1cm} (6.2)

Its general solution has the form

$$F(x_1, x_2, \ldots, x_n) = x_1^{\alpha_F/\alpha_1} \tilde{F} \left( \frac{x_2}{\alpha_2/\alpha_1}, \ldots, \frac{x_n}{\alpha_n/\alpha_1} \right),$$

where $\tilde{F}$ is an arbitrary function of $(n - 1)$ arguments. Obviously, the dimensions are defined up to a common factor (this can be seen by replacing $\lambda \rightarrow \lambda^d$ in (6.1) or multiplying equation (6.2) by $\alpha$); this arbitrariness can be eliminated, for example, if we set $\alpha_1 = 1$. If $\alpha_i = 0$ for some $x_i$, this variable is not dilated in (6.1), and the corresponding derivative is absent from (6.2).
It is well known that the leading terms, determining the asymptotic behaviour of (renormalized) correlation functions at large distances, satisfy the RG equation (4.2), in which the renormalized coupling constants are replaced with their values at the fixed points. In our case, this leads to the equation

\[ \{D_\mu - \gamma^*_\mu D_\mu - \gamma^*_\nu D_\nu + \gamma^*_\phi D_\phi + \gamma^*_\psi D_\psi\} G^* = 0, \]  

(6.3)

where \( \gamma^* = \gamma,(\mu = u_\mu, w = w_\psi) \) for all the anomalous dimensions.

We are interested in the critical scaling behaviour, that is, behaviour of type (6.1) in which all the IR relevant parameters (momenta/coordinates, frequencies/times, deviation of the temperature from its critical value \( T \propto (T - T_c) \)) are dilated, while the IR irrelevant parameters (those which remain finite at the fixed point: \( \lambda, \mu \) and \( f \)) are fixed [1, 2]. Thus, we combine equation (6.3) with the analogous equations, corresponding to the canonical scale invariance (see section 3), so that the derivatives with respect to the IR irrelevant parameters are eliminated; this gives the desired equation which describes the critical scaling behaviour (for more details, see e.g. [31, 32]):

\[ \{D_\perp + \Delta_\perp D_1 + \Delta_\phi D_\phi + \Delta_\tau D_\tau - N_0 \Delta_\phi\} G_{N_0} = 0, \]

where \( D_\perp = k_\perp \partial / \partial k_\perp \) and \( D_\phi = k_\phi \partial / \partial k_\phi \). Here, \( \Delta_\perp = 1 \) is the normalization condition, and the critical dimension of any IR-relevant parameter \( F \) is given by the general expression

\[ \Delta_F = d_F^+ + \Delta_\phi d_F^\phi + \Delta_\tau d_F^\tau + \gamma_F^*, \]

(6.4)

with canonical dimensions from table 1 and the relations

\[ \Delta_\omega = 2 = \gamma_\phi^*, \quad \Delta_\parallel = (2 + \gamma_\phi^*) / 2. \]  

(6.5)

We are in a position to write the final one-loop results for the critical dimensions. Substituting (4.9) into the general formulae (6.4) and (6.5) gives

\[ \Delta_\omega = 2 + \frac{u_\phi R_1}{6}, \quad \Delta_\parallel = 1 + \frac{u_\phi}{2}, \quad \Delta_\tau = 2 + \frac{5u_\phi R_1}{3}, \]

\[ \Delta_\varphi = \frac{d}{2} + 1 + \frac{u_\phi R_1}{3} + \frac{w^*}{4}, \quad \Delta_\psi = \frac{d}{2} - 1 + \frac{u_\phi R_1}{6} + \frac{w^*}{4}. \]

By inserting the explicit expressions for the fixed point coordinates and taking the equality \( d = 6 - \epsilon \) into account, one obtains the leading-order expressions for the critical dimensions. The results for all scaling regimes are summarized in table 2.

The expressions for the first and second fixed points are exact. Other dimensions have corrections, given by higher powers of \( \epsilon \) for the third fixed point and higher powers of \( \epsilon \) and \( \xi \) for the fourth one. The critical dimensions for the models of a liquid crystal and bond
percolation are derived from the general results by substituting expressions (5.13) and (5.14), respectively.

Let us discuss the consequences of the general scaling relations for the most interesting special case of the pair correlation function. They result in the scaling expression

$$\langle \varphi_a(x + r, t + t') \varphi_b(x, t') \rangle = \delta_{ab} r_\perp^{-2\Delta_\perp} \mathcal{F}\left(\tau_0, r_\perp^\Delta_\perp, r_\parallel^\Delta_\parallel, \right),$$

(6.6)

where \(r_\perp = |r_\perp|, r_\parallel = |r_\parallel|\) and \(\mathcal{F}\) is some scaling function. This representation is valid in the symmetric phase \((\tau_0 \geq 0)\), where the tensor structure is simply given by the \(\perp\) symbol. It is natural to assume that \(\mathcal{F}\) has a finite limit for \(\tau_0 = 0\) (that is, exactly at the critical point) and/or for \(t = 0\) (equal-time correlation function). Then from (6.6), one obtains

$$\langle \varphi_a(x + r, t) \varphi_b(x, t) \rangle = \delta_{ab} r_\perp^{-2\Delta_\perp} \overline{\mathcal{F}}(r_\parallel / r_\perp^\Delta_\perp)$$

with another nontrivial function \(\overline{\mathcal{F}}(x) = \mathcal{F}(0, 0, x)\).

The two last arguments in the scaling representation (6.6) can also be chosen in the form \(r_\perp / L_\perp(t)\) and \(r_\parallel / L_\parallel(t)\) with two different characteristic length scales

$$L_\perp(t) \sim t^{\alpha_\perp}, \quad L_\parallel(t) \sim t^{\alpha_\parallel}, \quad \alpha_\perp = 1/\Delta_\perp, \quad \alpha_\parallel = \Delta_\parallel / \Delta_\perp. \quad (6.7)$$

For the most realistic values \(\varepsilon = 3\) (\(d = 3\)) and \(\xi = 4/3\) (Kolmogorov spectrum of the velocity) and for the \(n = 2\) case of our model (liquid crystals), the explicit results from table 2 and expressions (5.13) give

$$\Delta_\perp \approx 2.359, \quad \Delta_\parallel \approx 1.846, \quad \alpha_\perp \approx 0.424, \quad \alpha_\parallel \approx 0.783, \quad \Delta_\varphi \approx 1.487,$$

while for the percolation limit \(n = 0\) from (5.14), one obtains

$$\Delta_\perp \approx 1.944, \quad \Delta_\parallel \approx 1.639, \quad \alpha_\perp \approx 0.514, \quad \alpha_\parallel \approx 0.843, \quad \Delta_\varphi \approx 0.731.$$

The existence of two different length scales (6.7) with a power-law dependence on the time was established earlier in a number of studies within numerical simulations [26], approximate analytical solutions [27], RG analysis [31, 32] and exactly soluble simplified models [28]. It is interesting to note that the inequality \(\alpha_\parallel > \alpha_\perp\) also holds for all those cases.

7. Conclusion

We studied effects of turbulent mixing on the critical behaviour of the system, described by the relaxational dynamics of a non-conserved order parameter of the ATP model. The mixing was modelled by a Gaussian statistics with vanishing correlation time and strongly anisotropic correlation function \(\propto \delta(t - t') / k_\perp^{2 \Delta_\perp} k_\parallel^{1 + \varepsilon}\); see equations (2.8) and (2.9). Such ensembles were employed earlier in [39–43] in the analysis of the two-dimensional passive turbulent advection (a linear equation for the scalar field).

The model, originally described by stochastic differential equations (2.1)–(2.3), (2.7), can be reformulated as a multiplicatively renormalizable field theory with action (2.11), which allows one to employ the field-theoretic RG to study its critical behaviour. The model reveals four different IR scaling regimes, related to the four different fixed points of the RG equations. Their regions of stability in the \(\varepsilon - \xi\) plane were identified in the leading order of the double expansion in \(\varepsilon\) and \(\xi\) and are shown in figures 1–5. These regimes correspond to

1. Gaussian (free) model;
2. linear passive scalar advection (the self-action term in the ATP Hamiltonian (2.3) is irrelevant in the sense of Wilson);
3. equilibrium critical dynamics of the ATP model (interaction with the velocity field is irrelevant) and
the full-fledged strongly anisotropic scaling regime in which both interactions are important; it corresponds to a new non-equilibrium universality class.

It was shown that the equilibrium critical regimes for both physically interesting cases (liquid crystals and percolation process) become unstable for the realistic range of parameters $d = 3$ and $0 < \xi \leq 2$, which includes the Kolmogorov spectrum ($\xi = 4/3$) and the Batchelor limit ($\xi \to 2$) and is replaced with the new non-equilibrium regime. The corresponding critical dimensions were calculated to first order of the corresponding RG expansion, which in this case takes on the form of the double expansion in $\varepsilon$ and $\xi$; explicit expressions are given in table 2.

Those results were derived within the leading (one-loop) approximation, that is, in the leading order of the double expansion in $\varepsilon$ and $\xi$, and their validity for finite physical values of these parameters can be called into question (especially because of large physical values $\varepsilon = 6 - d = 3$ and $\xi = 4/3$). A careful analysis of this problem requires the calculation of the higher order corrections and applying some kind of summation procedure to the results obtained, as was done e.g. in [9] for the scalar static $\phi^3$ model. Such an analysis goes far beyond the scope of this paper, and we hope to address it in the future. Nevertheless, the discussion of the RG flows, given in section 5, suggests that the pattern of fixed points (and thus of critical regimes), obtained within the one-loop approximation, appears robust with respect to higher order corrections and is preserved for finite values of $\varepsilon$ and $\xi$.

Thus, we hope that our simplified model of a non-conserved order parameter and Gaussian velocity ensemble captures the most important features of the full-fledged problem: emergence of a new non-equilibrium universality class with a new set of critical exponents, completely different from those of the classical ATP model; the existence (for a strongly anisotropic velocity ensemble) of two different length scales (with a power-law time dependence) and so on. Further investigation should take into account conservation of the order parameter, compressibility of the fluid, non-Gaussian character and finite correlation time of the velocity field and so on. This work is partly in progress and partly remains for the future.

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