Planar Turán Number of the 6-Cycle

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Abstract

Let $\text{ex}_P(n, T, H)$ denote the maximum number of copies of $T$ in an $n$-vertex planar graph which does not contain $H$ as a subgraph. When $T = K_2$, $\text{ex}_P(n, T, H)$ is the well studied function, the planar Turán number of $H$, denoted by $\text{ex}_P(n, H)$. The topic of extremal planar graphs was initiated by Dowden (2016). He obtained sharp upper bound for both $\text{ex}_P(n, C_4)$ and $\text{ex}_P(n, C_5)$. Later on, Y. Lan, et al. continued this topic and proved that $\text{ex}_P(n, C_6) \leq \frac{18(n-2)}{7}$. In this paper, we give a sharp upper bound $\text{ex}_P(n, C_6) \leq \frac{5}{2}n - 7$, for all $n \geq 18$, which improves Lan’s result. We also pose a conjecture on $\text{ex}_P(n, C_k)$, for $k \geq 7$.

Keywords  Planar Turán number, Extremal planar graph

1 Introduction and Main Results

In this paper, all graphs considered are planar, undirected, finite and contain neither loops nor multiple edges. We use $C_k$ to denote the cycle on $k$ vertices and $K_r$ to denote the complete graph on $r$ vertices.

One of the well-known results in extremal graph theory is the Turán Theorem \cite{5}, which gives the maximum number of edges that a graph on $n$ vertices can have without containing
a $K_r$ as a subgraph. The Erdős-Stone-Simonovits Theorem \cite{2 3} then generalized this result and asymptotically determines $\text{ex}(n, H)$ for all non-bipartite graphs $H$: $\text{ex}(n, H) = (1 - \frac{1}{\chi(H)-1})\binom{n}{2} + o(n^2)$, where $\chi(H)$ denotes the chromatic number of $H$. Over the last decade, a considerable amount of research work has been carried out in Turán-type problems, i.e., when host graphs are $K_n$, $k$-uniform hypergraphs or $k$-partite graphs, see \cite{3 6}.

In 2016, Dowden \cite{1} initiated the study of Turán-type problems when host graphs are planar, i.e., how many edges can a planar graph on $n$ vertices have, without containing a given smaller graph? The planar Turán number of a graph $H$, $\text{ex}_P(n, H)$, is the maximum number of edges in a planar graph on $n$ vertices which does not contain $H$ as a subgraph. Dowden \cite{1} obtained the tight bounds $\text{ex}_P(n, C_4) \leq \frac{15(n-2)}{7}$, for all $n \geq 4$ and $\text{ex}_P(n, C_5) \leq \frac{12n-33}{5}$, for all $n \geq 11$. Later on, Y. Lan, et al. \cite{4} obtained bounds $\text{ex}_P(n, \Theta_4) \leq \frac{12(n-2)}{5}$, for all $n \geq 4$, $\text{ex}_P(n, \Theta_5) \leq \frac{5(n-2)}{2}$, for all $n \geq 5$ and $\text{ex}_P(n, \Theta_6) \leq \frac{18(n-2)}{7}$, for all $n \geq 7$, where $\Theta_k$ is obtained from a cycle $C_k$ by adding an additional edge joining any two non-consecutive vertices. They also demonstrated that their bounds for $\Theta_4$ and $\Theta_5$ are tight by showing infinitely many values of $n$ and planar graph on $n$ vertices attaining the stated bounds. As a consequence of the bound for $\Theta_6$ in the same paper, they presented the following corollary.

**Corollary 1** (Y. Lan, et al.\cite{4}).

$$\text{ex}_P(n, C_6) \leq \frac{18(n-2)}{7}$$

for all $n \geq 6$, with equality when $n = 9$.

In this paper we present a tight bound for $\text{ex}_P(n, C_6)$. In particular, we prove the following two theorems to give the tight bound.

We denote the vertex and the edge sets of a graph $G$ by $V(G)$ and $E(G)$ respectively. We also denote the number of vertices and edges of $G$ by $v(G)$ and $e(G)$ respectively. The minimum degree of $G$ is denoted $\delta(G)$. The main ingredient of the result is as follows:

**Theorem 2.** Let $G$ be a 2-connected, $C_6$-free plane graph on $n$ ($n \geq 6$) vertices with $\delta(G) \geq 3$. Then $e(G) \leq \frac{5}{2}n - 7$. 

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We use Theorem 2, which considers only 2-connected graphs with no degree 2 (or 1) vertices and order at least 6, in order to establish our desired result, which bounds gives the desired bound of $\frac{5}{2}n - 7$ for all $C_6$-free plane graphs with at least 18 vertices.

**Theorem 3.** Let $G$ be a $C_6$-free plane graph on $n$ ($n \geq 18$) vertices. Then

$$e(G) \leq \frac{5}{2}n - 7.$$ 

Indeed, there are 17-vertex graphs on 17 vertices with 36 edges, but $\frac{5}{2}(17) - 7 = 35.5 < 36$. One such graph can be seen in Figure 1.

![Figure 1: Example of $G$ on 17 vertices such that $e(G) > (5/2)v(G) - 7$.](image)

We show that, for large graphs, Theorem 3 is tight:

**Theorem 4.** For every $n \equiv 2 \pmod{5}$, there exists a $C_6$-free plane graph $G$ with $v(G) = \frac{18n+14}{5}$ and $e(G) = 9n$, hence $e(G) = \frac{5}{2}v(G) - 7$.

For a vertex $v$ in $G$, the neighborhood of $v$, denoted $N_G(v)$, is the set of all vertices in $G$ which are adjacent to $v$. We denote the degree of $v$ by $d_G(v) = |N_G(v)|$. We may avoid the subscripts if the underlying graph is clear. The minimum degree of $G$ is denoted by $\delta(G)$, the number of components of $G$ is denoted by $c(G)$. For the sake of simplicity, we may use the term $k$-cycle to mean a cycle of length $k$ and $k$-face to mean a face bounded by a $k$-cycle. A $k$-path is a path with $k$ edges.

## 2 Proof of Theorem 4: Extremal Graph Construction

First we show that for a plane graph $G_0$ with $n$ vertices ($n \equiv 7 \pmod{10}$), each face having length 7 and each vertex in $G_0$ having degree either 2 or 3, we can construct $G$, where $G$ is a $C_6$-free plane graph with $v(G) = \frac{18n+14}{5}$ and $e(G) = 9n$. We then give a construction for such a $G_0$ as long as $n \equiv 7 \pmod{10}$. 

Using Euler’s formula, the fact that every face has length 7 and every degree is 2 or 3, we have \( e(G_0) = \frac{7(n-2)}{5} \) and the number of degree 2 and degree 3 vertices in \( G_0 \) are \( \frac{n+28}{5} \) and \( \frac{4n-28}{5} \), respectively.

Given \( G_0 \), we construct first an intermediate graph \( G' \) by step (1):

1. Add halving vertices to each edge of \( G_0 \) and join the pair of halving vertices with distance 2, see an example in Figure 2. Let \( G' \) denote this new graph, then \( v(G') = v(G_0) + e(G_0) = \frac{12n-14}{5} \) and the number of degree 2 and degree 3 vertices in \( G' \) is equal to the number of degree 2 and degree 3 vertices in \( G_0 \), respectively.

![Figure 2: Adding a halving vertex to each edge of \( G_0 \).](image)

To get \( G \), we apply the following steps (2) and (3) on the degree 2 and 3 vertices in \( G' \), respectively.

2. For each degree 2 vertex \( v \) in \( G_0 \), let \( N(v) = \{v_1, v_2\} \), and so \( v_1v_2 \) forms an induced triangle in \( G' \). Fix \( v_1 \) and \( v_2 \), replace \( v_1v_2 \) with a \( K_5^- \) by adding vertices \( v'_1, v'_2 \) to \( V(G') \) and edges \( v'_1v, v'_1v_1, v'_1v_2, v'_2v_1, v'_2v_2 \) to \( E(G') \). See Figure 3

![Figure 3: Replacing a degree-2 vertex of \( G_0 \) with a \( K_5^- \).](image)
For each degree 3 vertex \( v \) in \( G_0 \), such that \( N(v) = \{v_1, v_2, v_3\} \), the set of vertices \( \{v, v_1, v_2, v_3\} \) then forms an induced \( K_4 \) in \( G' \). Fix \( v_1, v_2 \) and \( v_3 \), replace this \( K_4 \) with a \( K_{n-5} \) by adding a new vertex \( v' \) to \( V(G') \) and edges \( v'v, v'v_1, v'v_2 \) to \( E(G') \). See Figure 4.

Figure 4: Replacing a degree-3 vertex of \( G_0 \) with a \( K_{n-5} \).

For each integer \( k \geq 0 \), and \( n = 10k + 7 \) we present a construction for such a \( G_0 \), call it \( G_k^0 \): Let \( v^t_i \) and \( v^b_i \) (\( 1 \leq i \leq k+1 \)) be the top and bottom vertices of the heptagonal grids with 3 layers and \( k \) columns, respectively (see the red vertices in Figure 5) and \( v \) be the extra vertex in \( G_k^0 \) but not in the heptagonal grid. We join \( v^t_1v, vv^b_1 \) and \( v^t_i v^b_i \) (\( 2 \leq i \leq k+1 \)). Clearly, \( G_k^0 \) is a \((10k+7)\)-vertex plane graph and each face of \( G_k^0 \) is a 7-face. Obviously \( e(G_k^0) = 14k + 7 \), and the number of degree 2 and 3 vertices are \( 2k + 7 = \frac{n+28}{5} \) and \( 8k = \frac{4n-28}{5} \) respectively.

After applying steps [1] [2] and [3] on \( G_k^0 \), we get \( G \). It is easy to verify that \( G \) is a \( C_6 \)-free plane graph with

\[
\begin{align*}
v(G) &= v(G_k^0) + e(G_k^0) + 2(2k + 7) + 8k = (10k + 7) + (14k + 7) + 12k + 14 = 36k + 28 \\
e(G) &= 9v(G_k^0) = 90k + 63.
\end{align*}
\]

Thus, \( e(G) = \frac{5}{2}v(G) - 7 \).

**Remark 1.** In fact, for \( k \geq 1 \) and \( n = 10k + 2 \), there exists a graph \( H_k^0 \) which is obtained from \( G_k^0 \) by deleting vertices (colored green in Figure 5) \( x_1, x_2, x_3, x_4, x_5 \) and adding the edge \( v^t_1y \). Clearly, \( H_k^0 \) is an \( 10k + 2 \)-vertex plane graph such that all faces have length 7. Moreover, \( e(H_k^0) = 14k \), the number of degree-2 and degree-3 vertices are \( 2k + 6 = \frac{n+28}{5} \) and
Figure 5: The graph $G_k^k$, $k \geq 1$, in which each face has length 7. The graph $H_0^k$ (see Remark 1) is obtained by deleting $x_1, \ldots, x_5$ and adding the edge $v_1^ty$.

$8k - 4 = \frac{4n - 28}{5}$, respectively. After applying steps (1), (2), and (3) to $H_0^k$, we get a graph $H$ that is a $C_6$-free plane graph with $e(H) = (5/2)v(H) - 7$.

Thus, for any $k \equiv 2 \pmod{5}$, we have the graphs above such that each face is a 7-gon and we get a $C_6$-free plane graph on $n$ vertices with $(5/2)n - 7$ edges for $n \equiv 10 \pmod{18}$ if $n \geq 28$.

3 Definitions and Preliminaries

We give some necessary definitions and preliminary results which are needed in the proof of Theorems 2 and 3.

Definition 5. Let $G$ be a plane graph and $e \in E(G)$. If $e$ is not in a 3-face of $G$, then we call it a trivial triangular-block. Otherwise, we recursively construct a triangular-block in the following way. Start with $H$ as a subgraph of $G$, such that $E(H) = \{e\}$.

(1) Add the other edges of the 3-face containing $e$ to $E(H)$. 

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(2) Take $e' \in E(H)$ and search for a 3-face containing $e'$. Add these other edge(s) in this 3-face to $E(H)$.

(3) Repeat step (2) till we cannot find a 3-face for any edge in $E(H)$.

We denote the triangular-block obtained from $e$ as the starting edge, by $B(e)$.

Let $G$ be a plane graph. We have the following three observations:

(i) If $H$ is a non-trivial triangular-block and $e_1, e_2 \in E(H)$, then $B(e_1) = B(e_2) = H$.

(ii) Any two triangular-blocks of $G$ are edge disjoint.

(iii) If $B$ is a triangular-block with the unbounded region being a 3-face, then $B$ is a triangulation graph.

Let $\mathcal{B}$ be the family of triangular-blocks of $G$. From observation [ii] above, we have

$$e(G) = \sum_{B \in \mathcal{B}} e(B),$$

where $e(G)$ and $e(B)$ are the number of edges of $G$ and $B$ respectively.

Next, we distinguish the types of triangular-blocks that a $C_6$-free plane graph may contain. The following lemma gives us the bound on the number of vertices of triangular-blocks.

**Lemma 6.** Every triangular-block of $G$ contains at most 5 vertices.

**Proof.** We prove it by contradiction. Let $B$ be a triangular-block of $G$ containing at least 6 vertices. We perform the following operations: delete one vertex from the boundary of the unbounded face of $B$ sequentially until the number of vertices of the new triangular block $B'$ is 6. Next, we show that $B'$ is not a triangular-block in $G$. Suppose that it is. We consider the following two cases to complete the proof.

**Case 1.** $B'$ contains a separating triangle.

Let $v_1v_2v_3$ be the separating triangle. Without loss of generality, assume that the inner region of the triangle contains two vertices say, $v_4$ and $v_5$. The outer region of the triangle
contains one vertex, say \( v_6 \). Since the unbounded face is a 3-face, the inner structure is a triangulation. Without loss of generality, let the inner structure be as shown in Figure 6(a). Now consider the vertex \( v_6 \). If \( v_1, v_2 \in N(v_6) \), then \( v_3v_4v_5v_2v_6v_1v_3 \) is a 6-cycle in \( G \), a contradiction. Similarly for the cases when \( v_1, v_3 \in N(v_6) \) and \( v_2, v_3 \in N(v_6) \).

**Case 2.** \( B' \) contains no separating triangle.

Consider a triangular face \( v_1v_2v_3v_1 \). Let \( v_4 \) be a vertex in the triangular-block such that \( v_2v_3v_4v_2 \) is a 3-face. Notice that \( v_1v_4 \notin E(B') \), otherwise we get a separating triangle in \( B' \). Let \( v_5 \) be a vertex in \( B' \) such that \( v_2v_4v_5v_2 \) is a 3-face. Notice that \( v_6 \) cannot be adjacent to both vertices in any of the pairs \( \{v_1, v_2\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_2, v_5\}, \{v_3, v_4\} \), or \( \{v_4, v_5\} \). Otherwise, \( C_6 \subset G \). Also \( v_3v_5 \notin E(B') \), otherwise we have a separating triangle. So, let \( v_1v_5 \in E(B') \) and \( v_1, v_5 \in N(v_6) \) (see Figure 6(b)). In this case \( v_1v_6v_5v_4v_3v_1 \) results in a 6-cycle, a contradiction.

Figure 6: The structure of \( B' \) when it contains a separating triangle or not, respectively.

Now we describe all possible triangular-blocks in \( G \) based on the number of vertices the block contains. For \( k \in \{2, 3, 4, 5\} \), we denote the triangular-blocks on \( k \) vertices as \( B_k \).

**Triangular-blocks on 5 vertices.**

There are four types of triangular-blocks on 5 vertices (see Figure 7). Notice that \( B_{5,a} \) is a \( K_5^- \).
Triangular-blocks on 4, 3, and 3 vertices.

There are two types of triangular-blocks on 4 vertices. See Figure 8. Observe that $B_{4,a}$ is a $K_4$. The 3-vertex and 2-vertex triangular-blocks are simply $K_3$ and $K_2$ (the trivial triangular-block), respectively.

Definition 7. Let $G$ be a plane graph.

(i) A vertex $v$ in $G$ is called a junction vertex if it is in at least two distinct triangular-blocks of $G$.

(ii) Let $B$ be a triangular-block in $G$. An edge of $B$ is called an exterior edge if it is on a boundary of non-triangular face of $G$. Otherwise, we call it an interior edge. An endvertex of an exterior edge is called an exterior vertex. We denote the set of all exterior and interior edges of $B$ by $\text{Ext}(B)$ and $\text{Int}(B)$ respectively. Let $e \in \text{Ext}(B)$, a non-triangular face of $G$ with $e$ on the boundary is called the exterior face of $e$.

Notice that an exterior edge of a non-trivial triangular-block has exactly one exterior face. On the other hand, if $G$ is a 2-connected plane graph, then every trivial triangular-block has two exterior faces. For a non-trivial triangular-block $B$ of a plane graph $G$, we
call a path $P = v_1v_2v_3 \ldots v_k$ an exterior path of $B$, if $v_1$ and $v_k$ are junction vertices and $v_i, v_{i+1}$ are exterior edges of $B$ for $i \in \{1, 2, \ldots, k - 1\}$ and $v_j$ is not junction vertex for all $j \in \{2, 3, \ldots, k - 1\}$. The corresponding face in $G$ where $P$ is on the boundary of the face is called the exterior face of $P$.

Next, we give the definition of the contribution of a vertex and an edge to the number of vertices and faces of $C_6$-free plane graph $G$. All graphs discussed from now on are $C_6$-free plane graph.

**Definition 8.** Let $G$ be a plane graph, $B$ be a triangular-block in $G$ and $v \in V(B)$. The contribution of $v$ to the vertex number of $B$ is denoted by $n_B(v)$, and is defined as

$$n_B(v) = \frac{1}{\# \text{ triangular-blocks in } G \text{ containing } v}.$$  

We define the contribution of $B$ to the number of vertices of $G$ as $n(B) = \sum_{v \in V(B)} n_B(v)$.

Obviously, $v(G) = \sum_{B \in B} n(B)$, where $v(G)$ is the number of vertices in $G$ and $B$ is the family of triangular-blocks of $G$.

Let $B_{K_5}$ be a triangular-block of $G$ isomorphic to a $B_{5,a}$ with exterior vertices $v_1, v_2, v_3$, where $v_1$ and $v_3$ are junction vertices, see Figure 9 for an example. Let $F$ be a face in $G$ such that $V(F)$ contains all exterior vertices $v_{1,1}, \ldots, v_{1,m}, v_{2,1}, \ldots, v_{2,m}, v_{3,1}, \ldots, v_{3,m}$ of $m$ ($m \geq 1$) copies of $B_{K_5}$, such that $v_{1,i}, v_{2,i}, v_{3,i}$ are the exterior vertices of the $i$-th $B_{K_5}$ and $v_{1,i}, v_{3,i}$ ($1 \leq i \leq m$) are junction vertices. Let $C_F$ denote the cycle associated with the face $F$. We alter $E(C_F)$ in the following way:

$$E(C'_F) := E(C_F) - \{v_{1,1}v_{2,1}v_{3,1}\} - \cdots - \{v_{1,m}v_{2,m}v_{3,m}\} \cup \{v_{1,1}v_{3,1}\} \cup \cdots \cup \{v_{1,m}, v_{3,m}\}.$$

Hence, the length of $F$ as $|E(C_F)| = |E(C_F)| - m$. For example, in Figure 9 $|E(C_F)| = 11$ but $|E(C'_F)| = 9$.

Now we are able to define the **contribution** of an “edge” to the number of faces of $C_6$-free plane graph $G$.

**Definition 9.** Let $F$ be a exterior face of $G$ and $C_F := \{e_1, e_2, \ldots, e_k\}$ be the cycle associated with $F$. The contribution of an exterior edge $e$ to the face number of the exterior face $F$, is denoted by $f_F(e)$, and is defined as follows.
Figure 9: An example of a face containing all the exterior vertices of at least one $B_{K_5}$.

(i) If $e_1$ and $e_2$ are adjacent exterior edges of $B_{K_5}$, then $f_F(e_1) + f_F(e_2) = \frac{1}{|C'_F|}$, and
$f_F(e_i) = \frac{1}{|C'_F|}$, where $i \in \{3, 4, \ldots, k\}$.

(ii) Otherwise, $f_F(e) = \frac{1}{|C_F|}$.

Note that $\sum_{e \in E(F)} f_F(e) = 1$. For a triangular-block $B$, the total face contribution of $B$ is denoted by $f_B$ and defined as $f_B = (\# \text{ interior faces of } B) + \sum_{e \in \text{Ext}(B)} f_F(e)$, where $F$ is the exterior face of $B$ with respective to $e$. Obviously, $f(G) = \sum_{B \in \mathcal{B}} f(B)$, where $f(G)$ is the number of faces of $G$.

4 Proof of Theorem 2

We begin by outlining our proof. Let $f$, $n$, and $e$ be the number of faces, vertices, and edges of $G$ respectively. Let $\mathcal{B}$ be the family of all triangular-blocks of $G$.

The main target of the proof is to show that

$$7f + 2n - 5e \leq 0. \tag{1}$$

Once we show (1), then by using Euler’s Formula, $e = f + n - 2$, we can finish the proof of Theorem 2. To prove (1), we show the existence of a partition $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ of $\mathcal{B}$ such that
\[
7 \sum_{B \in \cal P_i} f(B) + 2 \sum_{B \in \cal P_i} n(B) - 5 \sum_{B \in \cal P_i} e(B) \leq 0, \text{ for all } i \in \{1, 2, 3 \ldots, m\}. \text{ Since } f = \sum_{B \in \cal B} f(B),
\]
\[
n = \sum_{B \in \cal B} n(B) \text{ and } e = \sum_{B \in \cal B} e(B) \text{ we have}
\]
\[
7f + 2n - 5e = 7 \sum_{i} \sum_{B \in \cal P_i} f(B) + 2 \sum_{i} \sum_{B \in \cal P_i} n(B) - 5 \sum_{i} \sum_{B \in \cal P_i} e(B)
\]
\[
= \sum_{i} \left(7 \sum_{B \in \cal P_i} f(B) + 2 \sum_{B \in \cal P_i} n(B) - 5 \sum_{B \in \cal P_i} e(B)\right) \leq 0.
\]

The following proposition will be useful in many lemmas.

**Proposition 10.** Let \( G \) be a 2-connected, \( C_6 \)-free plane graph on \( n \) (\( n \geq 6 \)) vertices with \( \delta(G) \geq 3 \).

(i) If \( B \) is a nontrivial triangular-block (that is, not \( B_2 \)), then none of the exterior faces can have length 5.

(ii) If \( B \) is in \( \{B_{5,a}, B_{5,b}, B_{5,c}, B_{4,a}\} \), then none of the exterior faces can have length 4.

(iii) If \( B \) is in \( \{B_{5,d}, B_{4,b}\} \) and an exterior face of \( B \) has length 4, then that 4-face must share a 2-path with \( B \) (shown in blue in Figures 13 and 14) and the other edges of the face must be in trivial triangular-blocks.

(iv) No two 4-faces can be adjacent to each other.

**Proof.**

(i) Observe that any pair of consecutive exterior vertices of a nontrivial triangular-block has a path of length 2 (counted by the number of edges) between them and any pair of nonconsecutive exterior vertices has a path of length 3 between them. So having a face of length 5 incident to this triangular-block would yield a \( C_6 \), a contradiction.

(ii) If \( B \) is in \( \{B_{5,a}, B_{5,b}, B_{5,c}, B_{4,a}\} \), then any pair of consecutive exterior vertices of the listed triangular-blocks has a path of length 3 between them. It remains to consider nonconsecutive vertices for \( \{B_{5,b}, B_{5,c}\} \). For \( B_{5,b} \) each pair of nonconsecutive exterior vertices has a path of length 3 between them. In the case where \( B \) is \( B_{5,c} \), this is true for all pairs without an edge between them. As for the other pairs, if they are in the
same 4-face, then at least one of the degree-2 vertices in $B$ must have degree 2 in $G$, a contradiction.

\[\text{(iii)}\] In both $B_{5,d}$ and $B_{4,b}$, any pair of consecutive exterior vertices has a path of length 3 between them. For $B_{5,d}$, in Figure 13, we see that there is a path of length 4 between $v_2$ and $v_4$ and so the only way a 4-face can be adjacent to $B$ is via a 2-path with endvertices $v_1$ and $v_3$. In fact, because there is no vertex of degree 2, the path must be $v_1v_4v_3$. For $B_{4,b}$, in Figure 13, we see that because $B$ cannot have a vertex of degree 2, the 4-face and $B$ cannot share the path $v_2v_1v_4$ or the path $v_2v_3v_4$. Thus the only paths that can share a boundary with a 4-face are $v_1v_4v_3$ and $v_1v_2v_3$.

As to the other blocks that form edges of such a 4-face. In Figure 10, we see that if, say, $v_1u$ is in a nontrivial triangular-block, then there is a vertex $w$ in that block, in which case $wv_1xv_4uw$ forms a 6-cycle, a contradiction.

\[\text{(iv)}\] If two 4-faces share an edge, then there is a 6-cycle formed by deleting that edge. If two 4-faces share a 2-path, then the midpoint of that path is a vertex of degree 2 in $G$. In both cases, a contradiction.

\begin{proof}
We separate the proof into several cases.
\end{proof}

![Figure 10: Proposition 10(iii)](image)

The blocks defined by blue edges must be trivial.

To show the existence of such a partition we need the following lemmas.

**Lemma 11.** Let $G$ be a 2-connected, $C_6$-free plane graph on $n$ ($n \geq 6$) vertices with $\delta(G) \geq 3$. If $B$ is a triangular-block in $G$ such that $B \notin \{B_{5,d}, B_{4,b}\}$, then $7f(B) + 2n(B) - 5e(B) \leq 0$.

**Proof.** We separate the proof into several cases.
Case 1: \( B \) is \( B_{5,a} \).

Let \( v_1, v_2 \) and \( v_3 \) be the exterior vertices of \( K_5^- \). At least two of them must be junction vertices, otherwise \( G \) contains a cut vertex. We consider 2 possibilities to justify this case.

(a) Let \( B \) be \( B_{5,a} \) with 3 junction vertices (see Figure 11(a)). By Proposition 10, every exterior edge in \( B \) is contained in an exterior face with length at least 7. Thus, 
\[
f(B) = (\text{# interior faces of } B) + \sum_{e \in \text{Ext}(B)} f_F(e) \leq 5 + 3/7.
\]
Moreover, every junction vertex is contained in at least 2 triangular-blocks, so we have \( n(B) \leq 2 + 3/2 \). With \( e(B) = 9 \), we obtain
\[
7f(B) + 2n(B) - 5e(B) \leq 0.
\]

(b) Let \( B \) be \( B_{5,a} \) with 2 junction vertices, say \( v_2 \) and \( v_3 \) (see Figure 11(b)). Let \( F \) and \( F_1 \) are exterior faces of the exterior edge \( v_2v_3 \) and exterior path \( v_2v_1v_3 \) of the triangular-block respectively. Notice that \( v_1v_2 \) and \( v_2v_3 \) are the adjacent exterior edges in the same face \( F_1 \), hence \( |C(F_1)| \geq 8 \). By Definition 9 we have \( f_{F_1}(v_1v_2) + f_{F_1}(v_1v_3) \leq 1/7 \). Because there can be no \( C_6 \), one can see that regardless of the configuration of the \( B_{K_5^-} \), it is the case that \( f_F(v_2v_3) \leq 1/7 \). Thus, \( f(B) \leq 5 + 2/7 \). Moreover, since \( v_1 \) and \( v_3 \) are contained in at least 2 triangular-blocks, we have \( n(B) \leq 3 + 2/2 \). With \( e(B) = 9 \), we obtain
\[
7f(B) + 2n(B) - 5e(B) \leq 0.
\]

Figure 11: A \( B_{5,a} \) triangular-block with 3 and 2 junction vertices, respectively.
Case 2: $B$ is in $\{B_{4,a}, B_{5,b}, B_{5,c}\}$.

(a) Let $B$ be a $B_{4,a}$. By Proposition 10, each face incident to this triangular-block has length at least 7. So, $f(B) \leq 3 + 3/7$. Because there is no cut-vertex, this triangular-block must have at least two junction vertices, hence $n(B) \leq 2 + 2/2$. With $e(B) = 6$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$.

(b) Let $B$ be a $B_{5,b}$. There are 4 faces inside the triangular-block and each face incident to this triangular-block has length at least 7. So, $f(B) \leq 4 + 4/7$. Because there is no cut-vertex, this triangular-block must have at least two junction vertices, hence $n(B) \leq 3 + 2/2$. With $e(B) = 8$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$, as seen in Table 2.

(c) Let $B$ be a $B_{5,c}$. Similarly, $f(B) \leq 3 + 5/7$ and because there are at least two junction vertices, $n(B) \leq 3 + 2/2$. With $e(B) = 7$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -1$.

Case 3: $B$ is $B_3$.

Let $v_1$, $v_2$ and $v_3$ be the exterior vertices of triangular-block $B$. Each of these three must be junction vertices since there is no degree 2 vertex in $G$, which implies that each is contained in at least 2 triangular-blocks. We consider two possibilities:

(a) Let the three exterior vertices be contained in exactly 2 triangular-blocks. By Proposition 10(i), the length of each exterior face is either 4 or at least 7. We want to show that at most one exterior face has length 4.

If not, then let $x_1$ be a vertex that is in two such faces. Consider the triangular-block incident to $B$ at $x_1$, call it $B'$. By Proposition 10, $B'$ is not in $\{B_{5,a}, B_{5,b}, B_{5,c}, B_{4,a}\}$.

If $B'$ is in $\{B_{5,d}, B_{4,b}, B_3\}$, then the triangular-block has vertices $\ell_2, \ell_3$, each adjacent to $x_1$ and the length-4 faces consist of $\{v_1, \ell_2, m_2, v_2\}$ and $\{v_1, \ell_3, m_3, v_3\}$. Either $\ell_2 \sim \ell_3$ (in which case $\ell_2 m_2 v_2 v_3 m_3 \ell_3 \ell_2$ is a 6-cycle, see Figure 12(a)) or there is a $\ell'$ distinct from $v_1$ that is adjacent to both $\ell_2$ and $\ell_3$ (in which case $\ell' \ell_2 m_2 v_2 v_1 \ell_3 \ell_2$ is a 6-cycle, see Figure 12(b)).
If $B'$ is $B_2$, then the trivial triangular-block is $\{v_1, \ell\}$, in which case $\{\ell, m_2, v_1, v_3, m_3\}$ is a $C_6$, see Figure 12(c). Thus, we may conclude that if each of the three exterior vertices are in exactly 2 triangular-blocks, then $f(B) \leq 1 + 2/7 + 1/4$ and $n(B) \leq 3/2$. With $e(B) = 3$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -5/4$.

Figure 12: A $B_3$ triangular-block, $B$ and the various cases of what must occur if $B$ is incident to two 4-faces.

(b) Let at least one exterior vertex be contained in at least 3 triangular-blocks and the others be contained at least 2 triangular-blocks. In this case, we have $f(B) \leq 1 + 3/4$ and $n(B) \leq 2/2 + 2/3$. With $e(B) = 3$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -1/12$.

Case 4: $B$ is $B_2$.

Note that the fact that there is no vertex of degree 2 gives that if an endvertex is in exactly two triangular-blocks, then the other one cannot be a $B_2$. We consider three possibilities:

(a) Let each endvertex be contained in exactly 2 triangular-blocks. Since neither of the triangular-blocks incident to $B$ can be trivial, they cannot be incident to a face of length 5 by Proposition 10(i). Thus, $B$ cannot be incident to a face of length 5. Moreover, the two faces incident to $B$ cannot both be of length 4, again by Proposition 10(iv). Hence, $f(B) \leq 1/4 + 1/7$. Clearly $n(B) \leq 2/2$ and with $e(B) = 1$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -1/4$.

(b) Let one endvertex be contained in exactly 2 triangular-blocks and the other endvertex be contained in at least 3 triangular-blocks. This is similar to case [a] in that neither
face can have length 5 and they cannot both have length 4. The only difference is that $n(B) \leq 1/2 + 1/3$ and so $7f(B) + 2n(B) - 5e(B) \leq -7/12$.

(c) Let each endvertex be contained in at least 3 triangular-blocks. The two faces cannot both be of length 4 by Proposition [10][iv]. Hence, $f(B) \leq 1/4 + 1/5$ and $n(B) \leq 2/3$. With $e(B) = 1$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -31/60$.

\[\square\]

**Lemma 12.** Let $G$ be a 2-connected, $C_6$-free plane graph on $n \geq 6$ vertices with $\delta(G) \geq 3$. If $B$ is $B_{5,d}$, then $7f(B) + 2n(B) - 5e(B) \leq 1/2$. Moreover, $7f(B) + 2n(B) - 5e(B) \leq 0$ unless $B$ shares a 2-path with a 4-face.

---

**Proof.** Let $B$ be $B_{5,d}$ with vertices $v_1, v_2, v_3, v_4$, and $v_5$, as shown in Figure 13(a). By Proposition [10][i], no exterior face of $B$ can have length 5. By Proposition [10][iii], if there is an exterior face of $B$ that has length 4, this 4-face must contain the path $v_1v_4v_3$.

Moreover, since there is no vertex of degree 2, $v_2$ is a junction vertex. Because $G$ has no cut-vertex, there is at least one other junction vertex. We may consider the following cases:

(a) Let $v_4$ be a junction vertex. This prevents an exterior face of length 4. Thus, each exterior face has length at least 7. Hence, $f(B) \leq 4 + 4/7$ and $n(B) \leq 3 + 2/2$. With $e(B) = 8$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$. 

![Figure 13: A $B_{5,d}$ triangular-block and how a 4-face must be incident to it.](image-url)
(b) Let $v_4$ fail to be a junction vertex and exactly one of $v_1, v_3$ be a junction vertex. Without loss of generality let it be $v_3$. In this case, again, each exterior face has length\(^1\) at least 7. Again, $f(B) \leq 4 + 4/7$ and $n(B) \leq 3 + 2/2$. With $e(B) = 8$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$.

(c) Let $v_4$ fail to be a junction vertex and both $v_1$ and $v_3$ be junction vertices. Here either the exterior path $v_1v_4v_3$ is part of an exterior face of length at least 4 or each edge must be in a face of length at least 7. If the exterior face is of length at least 7, then $f(B) \leq 4 + 4/7$, otherwise $f(B) \leq 4 + 2/4 + 2/7$. In both cases, $n(B) \leq 2 + 3/2$ and $e(B) = 8$. Hence we obtain $7f(B) + 2n(B) - 5e(B) \leq -1$ in the first instance and $7f(B) + 2n(B) - 5e(B) \leq 1/2$ in the case where $B$ is incident to a 4-face.

\[\square\]

**Lemma 13.** Let $G$ be a 2-connected, $C_6$-free plane graph on $n \geq 6$ vertices with $\delta(G) \geq 3$. If $B$ is $B_{4,b}$, then $7f(B) + 2n(B) - 5e(B) \leq 4/3$. Moreover, $7f(B) + 2n(B) - 5e(B) \leq 1/6$ if $B$ shares a 2-path with exactly one 4-face and $7f(B) + 2n(B) - 5e(B) \leq 0$ if $B$ fails to share a 2-path with any 4-face.

![Figure 14: A $B_{4,b}$ triangular-block and how a 4-face must be incident to it.](image)

**Proof.** Let $B$ be with vertices $v_1, v_2, v_3, v_4$, as shown in Figure 14(a). By Proposition 10(i), no exterior face of $B$ can have length 5. If there is an exterior face of $B$ that has

\[^1\text{In fact, it can be shown that the length of the exterior face containing the path } v_2v_1v_4v_3 \text{ is at least 9. This yields } f(B) \leq 4 + 1/7 + 3/9 \text{ and } 7f(B) + 2n(B) - 5e(B) \leq -2/3. \text{ However, this precision is unnecessary.}\]
length 4, it is easy to verify that being $C_6$-free and having no vertex of degree 2 means that the junction vertices must be $v_1$ and $v_3$. We may consider the following cases.

(a) Let either $v_2$ or $v_4$ be a junction vertex and, without loss of generality, let it be $v_2$. All the exterior faces have length at least 7 except for the possibility that the path $v_1v_4v_3$ may form two sides of a 4-face. Hence, $f(B) \leq 2 + 2/4 + 2/7$ and $n(B) \leq 1 + 3/2$. With $e(B) = 5$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -1/2$.

(b) Let neither $v_2$ nor $v_4$ be a junction vertex. Because there is no cut-vertex, this requires both $v_1$ and $v_3$ to be junction vertices. Hence, there are two exterior faces: One that shares the exterior path $v_1v_4v_3$ and the other shares the exterior path $v_1v_2v_3$. Each exterior face has length either 4 or at least 7. We consider several subcases:

(i) If both faces are of length at least 7, then $f(B) \leq 2 + 4/7$, and $n(B) \leq 2 + 2/2$. With $e(B) = 5$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -1$.

(ii) If only one of the exterior faces is of length 4, then $f(B) \leq 2 + 2/7 + 2/4$. Moreover, at least one of $v_1$, $v_3$ must be a junction vertex for more than two triangular-blocks, otherwise either $v(G) = 5$ or the vertex incident to two blue edges in Figure 14(b) is a cut-vertex. Hence, $n(B) \leq 2 + 1/3 + 1/2$ and with $e(B) = 5$, we have $7f(B) + 2n(B) - 5e(B) \leq 1/6$.

(iii) Both exterior faces are of length 4. Thus $f(B) \leq 2 + 4/4$. By Proposition 14(iii), the blocks represented by the blue edges in Figure 14(c) are each trivial. Hence $n(B) \leq 2 + 2/3$. With $e(B) = 5$, we get $7f(B) + 2n(B) - 5e(B) \leq 4/3$.

Tables 2 and 3 in Appendix A give a summary of Lemmas 11, 12, and 13.

\noindent**Lemma 14.** Let $G$ be a 2-connected, $C_6$-free plane graph on $n$ ($n \geq 6$) vertices with $\delta(G) \geq 3$. Then the triangular-blocks of $G$ can be partitioned into sets, $\mathcal{P}_1$, $\mathcal{P}_2$, ..., $\mathcal{P}_m$ such that

$$7 \sum_{B \in \mathcal{P}_i} f(B) + 2 \sum_{B \in \mathcal{P}_i} n(B) - 5 \sum_{B \in \mathcal{P}_i} e(B) \leq 0$$

for all $i \in [m]$. 

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Proof. As it can be seen from Tables 2 and 3 in Appendix A there are three possible cases where $7f(B) + 2n(B) - 5e(B)$ assumes a positive value. We deal with each of these blocks as follows.

![Structure of a $B_{5,d}$](image)

Figure 15: Structure of a $B_{5,d}$ if it is incident to a 4-face, as in Lemma 14. The triangular-blocks $B'$ and $B''$ are trivial.

(1) Let $B$ be a $B_{5,d}$ triangular-block as described in the proof of Lemma 12(c). See Figure 15.

By Proposition 10(iii) the edges $v_1u$ and $v_3u$ are trivial triangular-blocks. Denote these triangular-blocks as $B'$ and $B''$. Consider $B'$. One of the exterior faces of $B'$ has length 4 whereas by Proposition 10(iv) the other has length at least 5. It must have length at least 7 because if it had length 5, then the path $v_1v_3u$ would complete it to a 6-cycle. Thus, $f(B') \leq 1/4 + 1/7$. Since the vertex $u$ cannot be of degree 2, then this vertex is shared in at least three triangular-blocks. Thus, $n(B') \leq 1/2 + 1/3$. With $e(B') = 1$, we obtain $7f(B') + 2n(B') - 5e(B') \leq -7/12$ and similarly, $7f(B'') + 2n(B'') - 5e(B'') \leq -7/12$. Define $\mathcal{P}' = \{B, B', B''\}$. Thus, $7 \sum_{B' \in \mathcal{P}'} f(B') + 2 \sum_{B' \in \mathcal{P}'} n(B') - 5 \sum_{B' \in \mathcal{P}'} e(B') \leq 1/2 + 2(-7/12) = -2/3$.

Therefore, for each triangular-block in $G$ as described in Lemma 12(c), it belongs to a set $\mathcal{P}'$ of three triangular-blocks such that $7 \sum_{B' \in \mathcal{P}'} f(B') + 2 \sum_{B' \in \mathcal{P}'} n(B') - 5 \sum_{B' \in \mathcal{P}'} e(B') \leq 0$. Denote such sets as $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{m_1}$ if they exist.

(2) Let $B$ be a $B_{4,b}$ triangular-block as described in the proof of Lemma 13(b)(ii). See Figure 16(a).

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Figure 16: Structure of a $B_{4,b}$ triangular-block if it is incident to a 4-face, as in Lemma 14. The triangular-blocks $B'$, $B''$, $B'''$, and $B''''$ are all trivial.

By Proposition 10(iii), the edges $v_1u_1$ and $v_3u_1$ are trivial triangular-blocks. Denote them as $B'$ and $B''$, respectively. Consider $B'$. One of the exterior faces of $B'$ has length 4 and by Proposition 10(iv), the other has length at least 5. Thus, $f(B') \leq 1/4 + 1/5$. Since the vertex $u_1$ cannot be of degree 2, then this vertex is shared in at least three triangular-blocks. Thus, $n(B') \leq 1/2 + 1/3$. With $e(B') = 1$, we obtain $7f(B') + 2n(B') - 5e(B') \leq -11/60$ and similarly, $7f(B'') + 2n(B'') - 5e(B'') \leq -11/60$. Define $P'' = \{B, B', B''\}$. Thus, $7 \sum_{B^* \in P''} f(B^*) + 2 \sum_{B^* \in P''} n(B^*) - 5 \sum_{B^* \in P''} e(B^*) \leq 1/6 + 2(-11/60) = -1/5$.

Therefore, for each triangular-block in $G$ as described in Lemma 13(b)(ii), it belongs to a set $P''$ of three triangular-blocks such that $7 \sum_{B^* \in P''} f(B^*) + 2 \sum_{B^* \in P''} n(B^*) - 5 \sum_{B^* \in P''} e(B^*) \leq 0$. Denote such sets as $\mathcal{P}_{m_1 + 1}, \mathcal{P}_{m_1 + 2}, \ldots, \mathcal{P}_{m_2}$ if they exist.

(3) Let $B$ be a $B_{4,b}$ triangular-block as described in the proof of Lemma 13(b)(iii). See Figure 16(b).

By Proposition 10(iii), the edges $v_1u_1$, $v_3u_1$, $v_1u_2$, and $v_3u_2$ are trivial triangular-blocks. Denote them as $B'$, $B''$, $B'''$ and $B''''$ respectively. Consider $B'$. One of the exterior faces of $B'$ has length 4 whereas the other has length at least 5. Thus, $f(B') \leq 1/4 + 1/5$. Since the vertex $u_1$ cannot be of degree 2, then this vertex is shared in at least three triangular-blocks. Clearly $v_1$ is in at least three triangular-blocks. Thus, $n(B') \leq 2/3$. With $e(B') = 1$, we obtain $7f(B') + 2n(B') - 5e(B') \leq -31/60$ and the same inequality
holds for \( B'' \), \( B''' \), and \( B'''' \).

Define \( P''' = \{ B, B', B'', B''', B'''', B'''' \} \). Thus, 
\[
7 \sum_{B^* \in P'''} f(B^*) + 2 \sum_{B^* \in P'''} n(B^*) - 5 \sum_{B^* \in P'''} e(B^*) \leq 4/3 + 4(-31/60) = -11/15.
\]

Therefore, for each triangular-block in \( G \) as described in Lemma 13(b)(iii), it belongs to a set \( P' \) of three triangular-blocks such that 
\[
7 \sum_{B^* \in P'''} f(B^*) + 2 \sum_{B^* \in P'''} n(B^*) - 5 \sum_{B^* \in P'''} e(B^*) \leq 0.
\]
Denote such sets as \( P_{m_2+1}, P_{m_2+2}, \ldots, P_{m_3} \) if they exist.

Now define \( P_{m_3+1} = B - \bigcup_{i=1}^{m_3} P_i \), where \( B \) is the set of all blocks of \( G \). Clearly, for each block \( B \in P_{m_3+1} \),
\[
7f(B) + 2n(B) - 5e(B) \leq 0.
\]
Thus, 
\[
7 \sum_{B \in P_{m_3+1}} f(B) + 2 \sum_{B \in P_{m_3+1}} n(B) - 5 \sum_{B \in P_{m_3+1}} e(B) \leq 0.
\]
Putting \( m := m_3 + 1 \) we got the partition \( \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m \) of \( B \) meeting the condition of the lemma.

This completes the proof of Theorem 2.

5 Proof of Theorem 3

Let \( G \) be a \( C_6 \)-free plane graph. We will show that either \( 5v(G) - 2e(G) \geq 14 \) or \( v(G) \leq 17 \).

If we delete a vertex \( x \) from \( G \), then
\[
5v(G - x) - 2e(G - x) = 5(v(G) - 1) - 2(e(G) - \deg(x))
= 5v(G) - 2e(G) - 5 + 2\deg(x)
\geq 5v(G) - 2e(G) - 1.
\]

So, graph \( G \) has an induced subgraph \( G' \) with \( \delta(G) \geq 3 \) with
\[
5v(G) - 2e(G) \geq 5v(G') - 2e(G') + (v(G) - v(G'))
\]

In line with usual graph theoretic terminology, we call a maximal 2-connected subgraph a block. Let \( B' \) denote the set of blocks of \( G' \) with the \( i \)th block having \( n_i \) vertices and \( e_i \) edges. Let \( b \) be the total number of blocks of \( G' \). Specifically, let \( b_2, b_3, b_4, \) and \( b_5 \) denote
the number of blocks of size 2, 3, 4, and 5, respectively. Let \( b_6 \) denote the number of blocks of size at least 6. Then we have \( b = b_6 + b_5 + b_4 + b_3 + b_2 \) and, using Table 1

\[
5v(G') - 2e(G') = 5 \left( \sum_{i=1}^{b} n_i - (b - 1) \right) - 2 \sum_{i=1}^{b} e_i \\
= \sum_{i=1}^{b} (5n_i - 2e_i - 5) + 5 \\
\geq 9b_6 + 2b_5 + 3b_4 + 4b_3 + 3b_2 + 5
\]

Combining (2) and (3), we obtain

\[
5v(G) - 2e(G) \geq 9b_6 + 2b_5 + 3b_4 + 4b_3 + 3b_2 + 5 + (v(G) - v(G'))
\]

If \( b_6 \geq 1 \), then the right-hand side of (4) is at least 14, as desired.

So, let us assume that \( b_6 = 0 \) and \( b = b_5 + b_4 + b_3 + b_2 \). Furthermore,

\[
v(G') = 5b_5 + 4b_4 + 3b_3 + 2b_2 - (b - 1) \\
= 4b_5 + 3b_4 + 2b_3 + b_2 + 1.
\]

So, substituting \( 2b_5 \) from (5) into (4), we have

\[
5v(G) - 2e(G) \geq 2b_5 + 3b_4 + 4b_3 + 3b_2 + 5 + (v(G) - v(G')) \\
= \left( \frac{1}{2}v(G') - \frac{3}{2}b_4 - b_3 - \frac{1}{2}b_2 - \frac{1}{2} \right) + 3b_4 + 4b_3 + 3b_2 + 5 + (v(G) - v(G')) \\
= v(G) - \frac{1}{2}v(G') + \frac{3}{2}b_4 + 3b_3 + \frac{5}{2}v_2 + \frac{9}{2} \\
\geq \frac{1}{2}v(G) + \frac{9}{2},
\]

which is strictly larger than 13 if \( v(G) \geq 18 \). Since \( 5v(G) - 2e(G) \) is an integer, it is at least 14 and this completes the proof of Theorem 3.

Table 1: Estimates of \( 5n - 2e - 5 \) for various block sizes.

| \( n \) | \( 5n - 2e - 5 \) | Theorem | \( n \) | \( 5n - 2e - 5 \) | Figure |
|---|---|---|---|---|---|
| \( n \geq 6 \) | \( 14 - 5 \geq 9 \) | [2] | \( n = 5 \) | \( 5(5) - 2(9) - 5 \geq 2 \) | \( B_{5,a}, \text{Figure 7} \) |
| \( n = 4 \) | \( 5(4) - 2(6) - 5 \geq 3 \) | \( B_{4,a}, \text{Figure 8} \) |
| \( n = 3 \) | \( 5(3) - 2(3) - 5 \geq 4 \) | \( B_3, \text{Figure 8} \) |
| \( n = 2 \) | \( 5(2) - 2(2) - 5 \geq 3 \) | \( B_2, \text{Figure 8} \) |
Remark 2. Observe that for \( n \geq 17 \), the only graphs on \( n \) vertices with \( e \) edges such that \( e > (5/2)n − 7 \) have blocks of order 5 or less and by (1), there are at most 4 such triangular blocks. A bit of analysis shows that the maximum number of edges is achieved when the number of blocks of order 5 is as large as possible.

6 Conclusions

We note that the proof of Theorem 2, particularly Lemma 14, can be rephrased in terms of a discharging argument.

We believe that our construction in Theorem 4 can be generalized to prove \( \text{exp}(n, C_\ell) \) for \( \ell \) sufficiently large. That is, for certain values of \( n \), we try to construct \( G_0 \), a plane graph with all faces of length \( \ell + 1 \) with all vertices having degree 3 or degree 2.

If such a \( G_0 \) exists, then the number of degree-2 and degree-3 vertices are \( \frac{(\ell - 5)n + 4(\ell + 1)}{\ell - 1} \) and \( \frac{4(n - \ell - 1)}{\ell - 1} \), respectively. We could then apply steps similar to (1), (2), and (3) in the proof of Theorem 4 in that we add halving vertices and insert a graph \( B_{\ell - 1} \) (see Figure 17) in place of vertices of degree 2 and 3. For the resulting graph \( G \),

\[
\begin{align*}
\text{v}(G) &= \text{v}(G_0) + \text{e}(G_0) + (\ell - 4)\frac{(\ell - 5)n + 4(\ell + 1)}{\ell - 1} + (\ell - 5)\frac{4(n - \ell - 1)}{\ell - 1} \\
&= n + \frac{\ell + 1}{\ell - 1}(n - 2) + \frac{(\ell^2 - 5\ell)n + 2(\ell + 1)}{\ell - 1} \\
&= \frac{\ell^2 - 3\ell}{\ell - 1}n + \frac{2(\ell + 1)}{\ell} \\
\text{e}(G) &= (3\ell - 9)\text{v}(G_0) = (3\ell - 9)n
\end{align*}
\]

Therefore, \( \text{e}(G) = \frac{3(\ell - 1)}{\ell} \text{v}(G) - \frac{6(\ell + 1)}{\ell} \). We conjecture that this is the maximum number of edges in a \( C_\ell \)-free planar graph.

Conjecture 15. Let \( G \) be an \( n \)-vertex \( C_\ell \)-free plane graph \( (\ell \geq 7) \), then there exists an integer \( N_0 > 0 \), such that when \( n \geq N_0 \), \( \text{e}(G) \leq \frac{3(\ell - 1)}{\ell}n - \frac{6(\ell + 1)}{\ell} \).
Figure 17: $B_{\ell-1}$ is used in the construction of a $C_{\ell}$-free graph.

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References

[1] C. Dowden, Extremal $C_4$-free/$C_5$-free planar graphs, J. Graph Theory 83 (2016), 213–230.

[2] P. Erdős. On the structure of linear graphs. Israel Journal of Mathematics 1 (1963) 156–160.

[3] P. Erdős. On the number of complete subgraphs contained in certain graphs. Publ. Math. Inst. Hung. Acad. Sci. 7 (1962) 459–464.

[4] Y. Lan, Y. Shi, Z. Song. Extremal theta-free planar graphs. Discrete Mathematics 342(12) (2019), Article 111610.

[5] P. Turán. On an extremal problem in Graph Theory. Mat. Fiz. Lapok (in Hungarian). 48 (1941) 436–452.

[6] A. Zykov. On some properties of linear complexes. Mat. Sb. (N.S.) 24(66) (1949) 163–188.
A Tables

The following tables give a summary of the results from Lemmas 11, 12, and 13.

A red edge incident to a vertex of a triangular-block indicates the corresponding vertex is a junction vertex. Moreover, if a vertex has only one red edge, it is to indicate the vertex is shared in at least two triangular-blocks. Whereas if a vertex has two red edges, it means that the vertex is shared in at least three blocks.

A pair of blue edges indicates the boundary of a 4-face.

| Case      | $B$      | Diagram | $f(B) \leq$ | $n(B) \leq$ | $e(B) =$ | $7f + 2n - 5e \leq$ |
|-----------|----------|---------|-------------|-------------|----------|---------------------|
| Lemma 11  | $B_{5,a}$ | ![Diagram](#) | $5 + \frac{3}{7}$ | $2 + \frac{3}{2}$ | 9        | 0                   |
| (a)       |          |         |             |             |          |                     |
| Lemma 11  | $B_{5,a}$ | ![Diagram](#) | $5 + \frac{2}{7}$ | $3 + \frac{2}{2}$ | 9        | 0                   |
| (b)       |          |         |             |             |          |                     |
| Lemma 11  | $B_{5,b}$ | ![Diagram](#) | $4 + \frac{4}{7}$ | $3 + \frac{2}{2}$ | 8        | 0                   |
| (b)       |          |         |             |             |          |                     |
| Lemma 11  | $B_{5,c}$ | ![Diagram](#) | $3 + \frac{5}{7}$ | $3 + \frac{2}{2}$ | 7        | $-1$                |
| (c)       |          |         |             |             |          |                     |
| Lemma 12  | $B_{5,d}$ | ![Diagram](#) | $4 + \frac{4}{7}$ | $3 + \frac{2}{2}$ | 8        | 0                   |
| (a)       |          |         |             |             |          |                     |
| Lemma 12  | $B_{5,d}$ | ![Diagram](#) | $4 + \frac{4}{7}$ | $3 + \frac{2}{2}$ | 8        | 0                   |
| (b)       |          |         |             |             |          |                     |
| Lemma 12  | $B_{5,d}$ | ![Diagram](#) | $4 + \frac{2}{4} + \frac{4}{7}$ | $2 + \frac{3}{2}$ | 8        | $\frac{1}{2} \star$ |
| (c)       |          |         |             |             |          |                     |

Table 2: All possible $B_5$ blocks in $G$ and the estimation of $7f(B) + 2n(B) - 5e(B)$.  

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| Case | $B$ | Diagram | $f(B) \leq$ | $n(B) \leq$ | $e(B) = 7f + 2n - 5e \leq$ |
|------|-----|---------|-------------|-------------|-----------------|
| **Lemma 11 (a)** | $B_{4,a}$ | ![Diagram](image) | $3 + \frac{3}{7}$ | $2 + \frac{2}{2}$ | 6 | 0 |
| **Lemma 13 (a)** | $B_{4,b}$ | ![Diagram](image) | $2 + \frac{2}{4} + \frac{2}{7}$ | $1 + \frac{3}{2}$ | 5 | $-\frac{1}{2}$ |
| **Lemma 13 (b)(i)** | $B_{4,b}$ | ![Diagram](image) | $2 + \frac{4}{7}$ | $2 + \frac{2}{2}$ | 5 | $-1$ |
| **Lemma 13 (b)(ii)** | $B_{4,b}$ | ![Diagram](image) | $2 + \frac{2}{4} + \frac{2}{7}$ | $2 + \frac{1}{3} + \frac{1}{2}$ | 5 | $\frac{1}{6} \star$ |
| **Lemma 13 (b)(iii)** | $B_{4,b}$ | ![Diagram](image) | $2 + \frac{2}{4} + \frac{2}{4}$ | $2 + \frac{2}{3}$ | 5 | $\frac{4}{3} \star$ |
| **Lemma 13 (3)(a)** | $B_{3}$ | ![Diagram](image) | $1 + \frac{2}{7} + \frac{1}{4}$ | $\frac{3}{2}$ | 3 | $-\frac{5}{4}$ |
| **Lemma 13 (3)(b)** | $B_{3}$ | ![Diagram](image) | $1 + \frac{3}{4}$ | $\frac{2}{2} + \frac{1}{3}$ | 3 | $-\frac{1}{12}$ |
| **Lemma 13 (4)(a)** | $B_{2}$ | ![Diagram](image) | $\frac{1}{4} + \frac{1}{7}$ | $\frac{2}{2}$ | 1 | $-\frac{1}{4}$ |
| **Lemma 13 (4)(b)** | $B_{2}$ | ![Diagram](image) | $\frac{1}{4} + \frac{1}{7}$ | $\frac{1}{2} + \frac{1}{3}$ | 1 | $-\frac{7}{12}$ |
| **Lemma 13 (4)(c)** | $B_{2}$ | ![Diagram](image) | $\frac{1}{4} + \frac{1}{5}$ | $\frac{2}{3}$ | 1 | $-\frac{31}{60}$ |

Table 3: All possible $B_{4}, B_{3}$ and $B_{2}$ blocks in $G$ and the estimate of $7f(B) + 2n(B) - 5e(B)$.