On the phase and group velocities of optical solitons

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Abstract. The method of analytical continuation of the dispersion parameters on the complex plane is analyzed. On this basis the new general formulae for phase and group velocities of the soliton-like pulses are obtained.

1. Introduction
One of the trends of modern nonlinear optics is the creation in the laboratory conditions of pulses of shorter durations $\tau_p$. In the sixties of the last century, after the appearance of lasers with Q-switching and mode synchronization, pulses of nanosecond duration were generated. At the turn of the 70's and 80's in many laboratories in the world was overcome by a picosecond barrier. By now, we can confidently talk about femtosecond and even attosecond pulses [1]. The characteristic frequency for red color is \( \omega \approx 3 \cdot 10^{15} \text{s}^{-1} \), which corresponds to the period \( T \approx 2 \cdot 10^{-15} \text{s} \). The number of electromagnetic oscillations contained within the pulse can be estimated as \( N \approx \tau_p / T \sim \omega \tau_p \). Then nano- and picosecond pulses contain, respectively, about one million and thousands of periods of light oscillations. Thus, here we can determine a small parameter \( \mu \approx 1 / N \sim (\omega \tau_p)^{-1} \). The presence of this small parameter makes it possible to use the approximation of a slowly varying envelope (SVE) [2].

A pulse lasting several femtoseconds contains about one period of electromagnetic oscillations. There is already a parameter \( \mu \) of the order of unity and it is impossible to introduce the concept of an envelope. In this case, it is necessary to derive equations not for the envelope, but for the electric field itself of the light pulse [3 - 5].

With the shortening of the pulse duration with an unchanged period of electromagnetic oscillations, its spectral width increases. To a few-cycle pulse (FCP), the notion of the carrier frequency is not applicable. Therefore, well-known expressions for the phase and group velocities of a quasimonochromatic signal are also subject to revision. The generalization of these expressions to the case of short pulses is the goal of the work. To achieve this, a method of analytical continuation of dispersion parameters (ACDP) onto the complex plane will be used below.

2. Analytical continuation of dispersion parameters
The method presented in this section makes it possible to determine the dependence of the phase and group velocities of soliton-like pulses from their duration and the central frequency of the spectrum from the linear dispersion relation. It is sufficient to know that the investigated nonlinear equation has solutions in the form of localized traveling waves. As a rule, this localization has an exponential
character, i.e. on the "tails" of the pulse, its field falls off as follows (assuming that the pulse propagates along the $z$-axis)

$$E \sim \exp\left[-\frac{t-z/v_g}{\tau_p}\right] \cos\left[\omega(t-z/v_{ph})\right] = \Re\left\{\exp\left[i(\omega+i\gamma)t-(k+iq)z\right]\right\},$$  \hspace{1cm} (1)$$

where $\gamma = 1/\tau_p$, phase $v_{ph}$ and group $v_g$ velocities are determining by the expressions [6]

$$\frac{1}{v_g} = \frac{q}{\gamma}, \hspace{1cm} \frac{1}{v_{ph}} = \frac{k}{\omega}.$$  \hspace{1cm} (2)$$

Let us describe the ACDP procedure. On the "tails" of a localized pulse, the field is weak. Therefore, the linear approximation is applicable here. Let on the basis of the linearized version of the nonlinear equation under investigation, the following dispersion equation is derived:

$$k = k(\omega)$$  \hspace{1cm} (3)$$

Next, the operation of the ACDP is carried out:

$$\omega \rightarrow \omega + i\gamma, \hspace{1cm} k \rightarrow k + iq.$$  \hspace{1cm} (4)$$

Substituting (4) into (3), we obtain $k + iq = k(\omega + i\gamma)$. Separating the real and imaginary parts, we arrive at two equations

$$k = k(\omega, \gamma), \hspace{1cm} q = q(\omega, \gamma).$$  \hspace{1cm} (5)$$

Substitution of these expressions into (2) leads to the dependences $v_g(\omega, \tau_p)$ and $v_{ph}(\omega, \tau_p)$.

In the case of quasimonochromatic pulses, the expression for the group velocity is well known. A natural question arises as to how this expression will change as the pulse duration becomes shorter. In the case of FCP, the notion of the carrier frequency is missing, but the notion of the center frequency of the spectrum remains. It is clear that the group velocity and phase velocity can no longer be calculated from the formula $v_g = \partial \omega / \partial k$ and $v_{ph} = \omega / k$, respectively. The ACDP method is able to give an answer to this question. Let us note that $k = \Re\left[k(\omega+i\gamma)\right] = [k(\omega+i\gamma)+k(\omega-i\gamma)]/2$, $q = \Im\left[k(\omega+i\gamma)\right] = [k(\omega+i\gamma)-k(\omega-i\gamma)]/2i$. Then the last two formulae (2) are written in the form

$$\frac{1}{v_{ph}} = \frac{k(\omega + i/\tau_p) + k(\omega - i/\tau_p)}{2\omega}, \hspace{1cm} \frac{1}{v_g} = \frac{\tau_p}{2i} \frac{k(\omega + i/\tau_p) - k(\omega - i/\tau_p)}{2i}.$$  \hspace{1cm} (6)$$
Introducing the translation operator in the frequency space by the formula

\[ k(\omega \pm i / \tau_p) = \exp \left( \pm i \frac{1}{\tau_p} \frac{\partial}{\partial \omega} \right) k(\omega) , \]

we rewrite (6) in the form

\[ \frac{1}{v_g} = \tau_p \sin \left( \frac{1}{\tau_p} \frac{\partial}{\partial \omega} \right) k , \quad \frac{1}{v_{ph}} = \frac{1}{\omega} \cos \left( \frac{1}{\tau_p} \frac{\partial}{\partial \omega} \right) k . \] (7)

These operator expressions generalize well-known formulae for the phase and group velocities of quasimonochromatic pulses. Here the parameter \( \omega \tau_p \) has the meaning of the central frequency of the pulse spectrum. In the limit \( \omega \tau_p >> 1 \), this parameter goes into the carrier frequency of a quasimonochromatic signal.

The operator expressions (7) have the meaning of the corresponding Taylor series expansions:

\[ \frac{1}{v_g} = \partial k - \frac{1}{3!} \frac{\partial^3 k}{\partial \omega^3} + \frac{1}{5!} \frac{\partial^5 k}{\partial \omega^5} + \ldots , \] (8)

\[ \frac{1}{v_{ph}} = \frac{k}{\omega} - \frac{1}{2! \omega \tau_p^2} \frac{\partial^2 k}{\partial \omega^2} + \frac{1}{4! \omega \tau_p^4} \frac{\partial^4 k}{\partial \omega^4} + \ldots \] (9)

Let us illustrate the application of the expressions obtained here to the example of solitons of some equations. The envelope of a nonresonant quasimonochromatic pulse in an isotropic medium with a weak nonlinearity and dispersion is described by a nonlinear Schrödinger equation

\[ i \frac{\partial \psi}{\partial z} = - \frac{k_z}{\tau_p} \frac{\partial^2 \psi}{\partial \omega^2} + \alpha |\psi|^2 \psi , \] (10)

where \( k_z = \frac{\partial^2 k}{\partial \omega^2} \) is the parameter of dispersion of the group velocity (DGV), \( \alpha \) is the parameter of the Kerr nonlinearity, \( \tau = t - z / v_{gl} \), \( v_{gl} \) is the linear group velocity determined by the first term in expansions of the right-hand side of expression (8).

Soliton solution of equation (10) looks like

\[ \psi = \frac{1}{\tau_p} \sqrt{-k_z / \alpha} \exp \left( i \frac{k_z z}{2 \tau_p^2} \right) \text{sech} \left( \frac{t - z / v_{gl}}{\tau_p} \right) . \] (11)

The phase factor in (11) corresponds to a nonlinear addition to the inverse phase velocity (or to the refractive index). In fact, taking (11) into account, we represent the phase \( \varphi \) of the light pulse in the form \( \varphi = \omega t - k_z z / (2 \tau_p^2) = \omega \left( t - z / v_{ph} \right) \). Hence we have

\[ \frac{1}{v_{ph}} = \frac{k}{\omega} - \frac{k_z}{2 \omega \tau_p^2} . \] (12)
It is easy to see that expression (12) corresponds to the first two terms of the expansion on the right-hand side of expression (9). The group velocity of a soliton is equal to the linear group velocity and is determined only by the first term of the expansion on the right-hand side of expression (8).

To obtain the dependence of the group velocity on the duration, one can take into account the terms of a higher order in the equation for the envelope. This means that we must generalize equation (10). Taking into account the dispersion of nonlinearity and the linear dispersion of the group velocity of the third order, we obtain the Hirota equation [7]

\[ i \frac{\partial \psi}{\partial z} = -\frac{k_3}{2} \frac{\partial^3 \psi}{\partial \tau^3} + i \frac{k_3}{6} \frac{\partial^2 \psi}{\partial \tau^2} + a \left| \psi \right|^2 \psi + i \beta \left| \psi \right|^2 \frac{\partial \psi}{\partial \tau}, \]  

(13)

where \( k_3 = \partial^3 k / \partial \omega^3 \), \( \beta \) is the coefficient characterizing the dispersion of the Kerr nonlinearity.

Under the condition \( k_3 \beta = -k_\omega \alpha \), equation (13) has a soliton solution of the form (11). In this case, the group velocity is given by the expression

\[ \frac{1}{v} = \frac{1}{v_g} - \frac{k_3}{6\tau_p^2}. \]  

(14)

It is easy to see that the right-hand side of this expression corresponds to the first two terms of the expansion on the right-hand side of (8). In this case, the phase velocity is still determined by the expression (12).

The procedure for further complicating the equation for the envelope can be continued. Then the terms of higher order with respect to the parameter \( \mu \) will be consistently taken into account. A shorter and more rational procedure is that it is necessary to abandon the approximation of a slowly varying envelope. I.e., it is necessary to derive an equation for the electric field \( E \) of the pulse. In the case of a nonresonant nonlinear interaction with a dielectric, the dynamics of the pulse field is described by the equation [4]

\[ \frac{\partial E}{\partial z} + aE^2 \frac{\partial E}{\partial \tau} - b \frac{\partial^3 E}{\partial \tau^3} + g \int \limits_{-\infty}^{\tau} E d\tau' = 0. \]  

(15)

Here \( a \) is a coefficient that takes into account the Kerr nonlinearity, parameters \( b \) and \( g \) characterize the dispersion of the electron and ion responses, respectively, \( \tau = t - n_0 z / c \), \( c \) is the speed of light, \( n_0 \) is the dispersionless refractive index.

Equation (15) does not have exact analytic solutions in the form of soliton-like pulses. However, in this case we can also find the dependences of the group and phase velocities on the central frequency \( \omega \) of the pulse spectrum and its duration \( \tau_p \). The linearized version of equation (15) corresponds to the dispersion relation

\[ k = \frac{n_0 \omega}{c} + b \omega^3 - \frac{g}{\omega}. \]  

(16)

Substituting this expression into (8) and (9) we find
\[
\frac{1}{v_g} = \frac{n_0}{c} + b \left( 3\omega^2 \frac{1}{\tau_p^2} \right) + \frac{g}{\omega^2} \left[ 1 - \left( \frac{1}{\omega \tau_p} \right)^2 + \left( \frac{1}{\omega \tau_p} \right)^4 \right] \ldots ,
\]
\[
\frac{1}{v_{ph}} = \frac{n_0}{c} + b \left( \omega^2 \frac{3}{\tau_p^2} \right) - \frac{g}{\omega^2} \left[ 1 - \left( \frac{1}{\omega \tau_p} \right)^2 + \left( \frac{1}{\omega \tau_p} \right)^4 \right] \ldots .
\]

Summing up here infinite geometric progressions, we have [8]
\[
\frac{1}{v_g} = \frac{n_0}{c} + b \left( 3\omega^2 - \frac{1}{\tau_p^2} \right) + \frac{g}{\omega^2 + \tau_p^2} ,
\]
\[
\frac{1}{v_{ph}} = \frac{n_0}{c} + b \left( \omega^2 - 3\tau_p^2 \right) - \frac{g}{\omega^2 + \tau_p^2} .
\]  

Setting here first, and then, we arrive at the known expressions for the breather solutions of the modified Korteweg–de Vries equation [9] and the Schäfer–Wayne equation [10, 11], respectively.

The above geometric progressions converge, if \( \omega \tau_p > 1 \). On the other hand, when (16) is substituted into expressions (6), we arrive at the expressions (17) without using this restriction. Hence we come to the conclusion that formulae (6) are more general than (7) - (9). However, expressions (7), (8) and (9) are more intuitive, since it allows us to compare the general expressions for the group and phase velocities of solitons with the corresponding linear expressions. Here the situation is analogous to the hypothesis of "elimination of unpleasant divergences" in quantum field theory. This hypothesis allows us to summarize the Feynman diagrams of all orders in describing the systems of interacting quasiparticles [12].

3. Concluding remarks

Thus, in this paper a detailed analysis of the ACDP method is carried out. It is not always possible to find analytical solutions to these nonlinear wave equations or systems. It is in these cases that this method comes to the rescue.

The expressions (6) - (9) for the group and phase velocities can be used for both envelope pulses and FCP. In the first case, the terms containing the pulse duration should be considered as small corrections to the first terms on the right-hand sides of expansions (8) and (9) containing the carrier frequency. In the second case, the parameter has the meaning of the central frequency of the pulse spectrum and all terms of the series (8), (9) can make a significant contribution.

The procedure used in this article allows a quasiparticle interpretation of solitons. When \( \gamma = q = 0 \), we have a superposition of quasiparticles characterized by a linear dispersion equation. Accounting for nonlinearity leads to interaction between quasiparticles. As a result, the wave field is localized. The imaginary additions to \( \omega \) and \( k \) in (4) should be treated as the inverse lifetime and the inverse mean free path of the quasiparticle. Thus, the duration of the soliton \( \tau_p = 1/\gamma \) denotes the mean lifetime of quasiparticles in a state with energy \( \hbar \omega \) and momentum \( \hbar k \) (\( \hbar \) is the Planck constant). When \( \gamma / \omega \ll 1 \), the quasiparticles interact weakly with one another, we have which corresponds to the envelope soliton. If \( \gamma / \omega \sim 1 \), we have a few-cycle pulse, consisting of quasiparticles with strong interaction.

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