Beyond Value Iteration for Parity Games: Strategy Iteration with Universal Trees

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Abstract

Parity games have witnessed several new quasi-polynomial algorithms since the breakthrough result of Calude et al. (2017). The central combinatorial object underlying these approaches is a universal tree, as identified by Czerwiński et al. (2019). By providing a quasi-polynomial lower bound on the size of universal trees, they have highlighted a barrier that must be overcome by all existing approaches to attain polynomial runtime. This is due to the existence of worst case instances which force these algorithms to explore a large portion of the tree.

As an attempt to overcome this barrier, we propose a strategy iteration framework which can be applied on any universal tree. It is at least as fast as its value iteration counterparts, while allowing one to take bigger leaps in the universal tree. Value iteration — asymptotically the fastest known algorithm for parity games — is a repeated application of operators associated with arcs in the game graph to obtain the least fixed point. Our main technical contribution is an efficient method for computing the least fixed point of operators associated with arcs in a strategy subgraph. This is achieved via a careful adaptation of shortest path algorithms to the setting of ordered trees. By plugging in the universal tree of Jurdziński and Lazić (2017), or the Strahler universal tree of Daviaud et al. (2020), we obtain instantiations of the general framework that take time $O(mn^2 \log n \log d)$ and $O(mn^2 \log^3 n \log d)$ respectively per iteration.

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1 Introduction

Parity games A parity game is an infinite duration game between two players Even and Odd. It takes place on a sinkless directed graph $G = (V, E)$ equipped with a priority function $\pi: V \rightarrow [d]$, where $d \in \mathbb{N}$ is even. The node set $V$ is partitioned into $V_0 \sqcup V_1$ such that nodes in $V_0$ and $V_1$ are owned by Even and Odd respectively. The game starts when a token is placed on a node. In each turn, the owner of the current node moves the token along an outgoing arc to the next node, resulting in an infinite walk. If the highest priority occurring infinitely often in this walk is even, then Even wins. Otherwise, Odd wins.

Due to the positional determinacy of parity games [12], there exists a partition of $V$ into two subsets from which Even and Odd can force a win respectively. The main algorithmic problem of parity games is to determine this partition, or equivalently, decide the winner given a starting node. While its containment in $\text{NP} \cap \text{co-NP}$ has been known for quite some time [13], only recently a breakthrough result by Calude et al. [6] gave the first quasi-polynomial algorithm. The problem is also known to be in $\text{UP} \cap \text{co-UP}$ [23]. Therefore, it is unlikely to be NP-complete and the quest for a polynomial time algorithm is still ongoing.

Relevance Apart from its intriguing complexity status, parity games have several important applications. They play a fundamental role in formal verification and synthesis, the model checking problem for modal $\mu$-calculus [22, 33, 13], and emptiness checking of non-deterministic parity tree automata [28]. Furthermore, parity games form a relatively well-understood subclass of mean payoff games and stochastic games, which allows for a fruitful interplay between the advances on these classes [16, 38, 7, 33]. Finally, parity games played a crucial role in understanding the complexity of strategy iteration and the simplex method for linear programming. A family of parity games [18], on which strategy iteration takes exponentially many steps, led to an exponential lower bound for policy iteration on Markov decision processes [14]. A similar idea was then used to construct examples for several pivot rules for the simplex method on which they take at least (sub)exponentially many steps [20, 19].

Quasi-polynomial algorithms and universal trees After the first quasi-polynomial algorithm by Calude et al. [6], many others followed soon after. The central combinatorial object underlying these approaches is a universal tree, as identified and introduced by Czerwiński et al. [8]. Roughly speaking, a universal tree is an ordered tree into which every ordered tree of a certain size can be embedded. We elaborate on two main classes of quasi-polynomial algorithms, namely attractor decomposition and progress measure lifting.

The first class, going back to work of McNaughton [31] and Zielonka [37], recursively decomposes the game graph based on attractors to identify winning regions. The original version, which is exponential in the worst case, was improved to subexponential time by Jurdziński et al. [27]. More recently, a quasi-polynomial version was devised by Parys [32] through an ingenious pruning of the recursion tree. This was later refined by Lehtinen et al. [30], who observed that the tree of recursive calls follows a particular construction of universal trees. Subsequently, Jurdziński and Morvan [26] generalized it to a generic algorithm parameterized by a universal tree.

The second line of attack started with the work of Jurdzinski [24], who designed small progress measures for solving parity games. Following the breakthrough result of Calude et al. [6], this was improved by Jurdziński and Lazić [25] to succinct progress measures, which is asymptotically the fastest quasi-polynomial algorithm to date. As observed by Fijalkow [15] and Czerwiński et al. [8], they are essentially value iteration with different constructions of universal trees. In this approach, nodes in the game graph are labeled with leaves of a universal tree. Starting from the smallest
leaf, the labels are monotonically lifted until convergence, at which point they certify the winning region for one of the players.

By proving a quasi-polynomial lower bound on the size of universal trees, Czerwiński et al. [8] have highlighted a barrier that must be overcome by all existing approaches to attain polynomial runtime. This is due to the existence of worst-case instances which force these algorithms to explore a large portion of the tree. Inspired by the register index of Lehtinen [29], Daviaud et al. [9] recently introduced Strahler universal trees, whose size depends on structural properties of the winning regions.

**Strategy iteration** A different class of algorithms for solving parity games is strategy iteration. A (positional) strategy τ of a player is a choice of an outgoing arc from each node owned by the player. Removing the unchosen outgoing arcs from each node owned by the player results in a strategy subgraph $G_τ$. Starting with a strategy $τ_1$ of say Odd, strategy iteration evaluates the current strategy and modifies it according to a pivot rule. This procedure is repeated until an optimal strategy is reached. The strategy evaluation relies on certain labels on the nodes, which can be real numbers or more combinatorial quantities. This technique, introduced by Hoffman and Karp for stochastic games [21], was adapted to parity games by Puri [33]. While the latter uses real-numbered labels, the variants of Vöge and Jurdzinski [36], Björklund et al. [5], and Schewe [35] use discrete labels. These strategy iteration algorithms usually perform well in practice, but tedious constructions of their worst case (sub)exponential complexity are known [18].

**Our contribution** It is instructive to compare parity game algorithms to algorithms for Markov Decision Processes (MDPs). Value iteration and strategy iteration (also known as policy iteration) are two key algorithmic frameworks for MDPs, see, e.g., [34]. Value iteration proceeds through a sequence of values (real numbers) associated with each state. If an action at a state violates feasibility, then we increase the value of the state accordingly – this is a simple ‘lifting’ operation. Strategy iteration, on the other hand, selects an action at each state (a strategy) and computes the associated values by solving a system of linear equations. If there are some violating actions, then we modify the strategy by swapping in some of these actions. The particular choice of these swaps is governed by a pivot rule. In the number of iterations, strategy iteration is more efficient than value iteration because it ‘jumps’ over a long sequence of lifting steps; however, each iteration of strategy iteration is computationally more expensive.

For parity games, the values/labels are usually sourced from an ordered discrete set, such as the leaves of an ordered tree. The aforementioned papers on strategy iteration develop careful implementations of strategy evaluation for particular label schemes, but they do not seem easily extendable to universal trees. While the basic idea of adapting shortest path algorithms is still useful, a significantly deeper understanding of the structure of universal trees is necessary.

Motivated by the construction of small universal trees [25, 9], a natural question is whether the (sub)exponential worst-case complexity of strategy iteration algorithms can be improved using universal trees. We develop a robust and general framework for strategy iteration that can be applied with any choice of a universal tree. We coin the name Cramer computation for the strategy evaluation in each iteration, as it is analogous to the computation of reduced costs in the simplex method for linear programming, following Cramer’s rule for solving linear systems. Abstractly, this is the problem of finding the least simultaneous fixed point of a set of inflationary and monotone operators associated with arcs in a strategy subgraph $G_τ$. We remark that the superset of operators associated with arcs in $G$ are the lifting steps used in value iteration. One can obtain a finite algorithm for Cramer computation by running value iteration on $G_τ$, but its running time depends
on the tree size.

Our main contribution is a more efficient combinatorial method for Cramer computation. It relies on adapting the classical techniques of label-correcting and label-setting from combinatorial optimization to the setting of ordered trees. Applying it to specific universal trees constructed in the literature, we obtain the following runtimes for Cramer computation:

- The universal tree of Jurdziński and Lazić \([25]\) takes \(O(mn^2 \log n \log d)\).
- The Strahler universal tree of Daviaud et al. \([9]\) takes \(O(mn^2 \log^3 n \log d)\).
- The perfect \(n\)-ary tree of height \(d/2\) takes \(O(d(m + n \log n))\).

As usual, \(n := |V|\) and \(m := |E|\). The details of these techniques are elaborated in Section 1.1.

We remark that the total runtime of our strategy iteration framework is never worse than its value iteration counterparts \([24, 25, 9]\). To evaluate a current strategy \(\tau\), one could always run our combinatorial method in parallel with the aforementioned naive approach of value iteration on \(G_\tau\). In this way, our framework can be used to improve upon any value iteration algorithm based on a universal tree.

Another way to understand our main contribution is via a small detour to tropical geometry: Akian et al. \([2]\) have shown that tropical linear programming is equivalent to determining the winning regions for mean payoff games. A closer look at the projection operators on tropical polyhedra considered in Akian et al. \([3]\) reveals that the algorithm of Björklund et al. \([4]\) for mean payoff games can be seen as a careful combination of these operators to obtain a strategy iteration algorithm. In a similar spirit, we combine the lifting operators used for value iteration with universal trees to obtain a strategy iteration algorithm. The connection between parity games and mean payoff games is sketched in Appendix A.

### 1.1 Main Techniques for Cramer Computation

Let \(\tau\) be a strategy of Odd. The restriction to a strategy subgraph \(G_\tau\) gives rise to a one-player parity game for Even. Iterating the lifting operators in \(G_\tau\) to compute the least simultaneous fixed point is not polynomial in general. Therefore, we need a more efficient method — we approach the least fixed point from above instead of from below. This is reminiscent of shortest path algorithms, where node labels form upper bounds on the shortest path distances throughout the algorithm. In a label-correcting method, to compute shortest paths to a target node \(t\), the label at \(t\) is initialized to 0 while the label at other nodes are initialized to \(+\infty\). By iteratively checking if an arc violates feasibility, the node labels are monotonically decreased. We refer to Ahuja et al. \([1]\) for an overview on label-correcting and label-setting techniques for computing shortest paths.

There are several obstacles in adapting this framework to node labels given by the leaves of an ordered tree \(T\). The first difficulty is that there are no pre-specified target nodes in \(G_\tau\). From the definition of lifting operators, one sees directly that their fixed points stabilize on even cycles in \(G_\tau\). These are cycles whose maximum priority is even. However, it is not clear a priori which even cycles are the crucial ones for the desired least fixed point. To address this issue, we consider base nodes as candidate target nodes. A node \(v\) is a base node if it dominates an even cycle in \(G_\tau\), i.e. it is a node with the highest priority in the cycle.

The second difficulty is initialization. The presumably obvious choice is to assign the smallest leaf in \(T\) to each base node, and a top element \(\top\) to all other nodes, where \(\top\) is bigger than all the leaves in \(T\). However, this fails because the label on a base node may already be too small. On the other hand, if the label on a base node is too large, we may converge to a fixed point that is above the least fixed point.
In order to correctly initialize the label on a base node \( w \), we consider the set \( \mathcal{T}_j \) of distinct subtrees of \( T \) rooted at depth \( j = \pi(w)/2 \). This set of trees forms a poset with respect to the partial order determined by whether an ordered tree can be embedded into another ordered tree. Let \( \mathcal{C}_j \) be a chain cover of \( \mathcal{T}_j \), and let \( \mathcal{C}^*_j \) be a chain in \( \mathcal{C}_j \). Observe that the trees in \( \mathcal{C}^*_j \) are ordered from ‘narrowest’ to ‘widest’. It turns out that the task of initialization at \( w \) boils down to finding the narrowest tree in \( \mathcal{C}^*_j \) whose leaves can encode a fixed point on a cycle dominated by \( w \). We refer to the width of this tree as the width of base node \( w \).

Instead of determining the width of \( w \) exactly, we compute an under-estimate which is sufficient for our purposes. To accomplish this, we define an arc-weighted auxiliary digraph on the set of base nodes. Each arc \( uv \) represents the ‘narrowest’ paths from base node \( u \) to base node \( v \) in \( G_\tau \). We show how to estimate the width of base nodes by computing minimum bottleneck cycles in the auxiliary graph. Equipped with such an approximate width for \( w \), we then locate the corresponding subtree in the main ordered tree \( T \). Finally, the desired label at \( w \) is obtained by ‘raising’ the current label at \( w \) to the smallest leaf in that subtree. This is a significantly bigger step than what lifting would achieve.

The running time of this label-correcting framework is proportional to the size of the chain cover \( \mathcal{C}_j \). We prove that the quasi-polynomial universal trees constructed in the literature \([25, 9]\) admit small chain covers. Using this result, we then give efficient implementations of this framework for these trees.

We give a brief overview of the label-setting framework, which is useful for Cramer computation with perfect \( n \)-ary trees. Unlike label-correcting approaches, in a label-setting algorithm such as Dijkstra’s algorithm, the label of a node is fixed in each iteration. In the shortest path problem, Dijkstra’s algorithm selects a node with the smallest label to be fixed in every iteration. When working with labels given by the leaves of an ordered tree \( T \), this criterion does not work anymore. It turns out that the right potential function is obtained by interlacing each label with a tuple that encodes topological orders in \( G_\tau \) which are induced by the even priorities. In every iteration, a node with the smallest potential is selected and its label is fixed.

1.2 Paper Organization

In Section 2, we introduce notation and provide the necessary preliminaries on parity games and ordered trees. In Section 3, we give a strategy iteration framework based on universal trees and introduce Cramer computation — the key computational task in each iteration. Section 3.1 is a prelude to the development of shortest path algorithms for ordered trees. In Section 4, we develop a label-correcting framework for Cramer computation, and apply it to the quasi-polynomial universal trees constructed in the literature. In Section 5, we develop a label-setting framework for Cramer computation, and illustrate its utility on perfect \( n \)-ary trees.

2 Preliminaries on Parity Games and Ordered Trees

Let \( G = (V, E) \) be a sinkless directed graph with \( V = V_0 \sqcup V_1 \), and let \( \pi : V \to [d] \) be a priority function where \( d \in \mathbb{N} \) is even. Denote \( m := |E| \) and \( n := |V| \). In this paper, we are only concerned with positional strategies. A strategy for Odd is a function \( \tau : V_1 \to V \) such that \( v\tau(v) \in E \) for all \( v \in V_1 \). The associated strategy subgraph is denoted by \( G_\tau = (V, E_\tau) \), where \( E_\tau := \{vw \in E : v \in V_0\} \cup \{v\tau(v) : v \in V_1\} \). A strategy for Even and its associated strategy subgraph are defined analogously. We generally denote a strategy for Even as \( \sigma \), and a strategy for Odd as \( \tau \). For a fixed strategy \( \tau \) for Odd, the resulting instance \((G_\tau, \pi)\) is a one-player parity game for Even.
For the sake of brevity, we overload the priority function $\pi$ as follows. Given a subgraph $H$ of $G$, denote $\pi(H)$ as the highest priority in $H$. The subgraph $H$ is said to be even if $\pi(H)$ is even, and odd otherwise. We use $\Pi(H)$ to denote the set of nodes in $H$ with priority $\pi(H)$. If $v \in \Pi(H)$, we say that $v$ dominates $H$. For a fixed $p \in [d]$, $H_p$ refers to the subgraph of $H$ induced by nodes with priority at most $p$. For a node $v$, let $\delta^-_H(v)$ and $\delta^+_H(v)$ be the incoming and outgoing arcs of $v$ in $H$ respectively. Similarly, let $N^-_H(v)$ and $N^+_H(v)$ be the in-neighbors and out-neighbors of $v$ in $H$ respectively. When the subgraph $H$ is clear from context, we will omit $H$ from the subscripts.

2.1 Ordered Trees and Universal Trees

An ordered tree $T$ is a prefix-closed set of tuples, whose elements are drawn from a linearly ordered set $M$. The linear order of $M$ lexicographically extends to $T$. Note that a prefix of a tuple is smaller than the tuple itself, unless the prefix is the whole tuple. Equivalently, $T$ can be thought of as a rooted tree, whose root we denote by $r$. Under this interpretation, elements in $M$ correspond to the branching directions at each vertex of $T$ (see Figure 1 and 2 for examples). Every tuple then corresponds to a vertex $v \in V(T)$. This is because the tuple can be read by traversing the unique $r$-$v$ path in $T$. Observe that $v$ is an $h$-tuple if and only if $v$ is at depth $h$ in $T$. In particular, $r$ is the empty tuple.

In this paper, we always use the terms ‘vertex’ and ‘edge’ when referring to an ordered tree $T$. The terms ‘node’ and ‘arc’ are reserved for subgraphs of the game graph $G$.

Given an ordered tree $T$ of height $h$, let $L(T)$ denote the set of leaves in $T$. For convenience, we assume that every leaf in $T$ is at depth $h$. As they encode information about odd priorities, the tuple representing a leaf $\xi \in L(T)$ is denoted as $\xi = (\xi_{2h-1}, \xi_{2h-3}, \ldots, \xi_1)$, where $\xi_i \in M$ for all $i$. We refer to $\xi_{2h-1}$ as the first component of $\xi$, even though it has index $2h - 1$. For a fixed $p \in [2h]$, the $p$-truncation of $\xi$ is defined as

$$
\xi|_p := \begin{cases} 
(\xi_{2h-1}, \xi_{2h-3}, \ldots, \xi_{p+1}), & \text{if } p \text{ is even} \\
(\xi_{2h-1}, \xi_{2h-3}, \ldots, \xi_p), & \text{if } p \text{ is odd.}
\end{cases}
$$

In other words, the $p$-truncation of a tuple is obtained by deleting the components with index less than $p$. Note that a truncated tuple corresponds to an ancestor of the untruncated tuple in $T$.

**Definition 2.1.** Given ordered trees $T$ and $T'$, we say that $T$ embeds into $T'$ if there exists an injective and order-preserving homomorphism from $T$ to $T'$. Formally, this is an injective function $f : V(T) \to V(T')$ which satisfies the following properties:

1. For all $u, v \in V(T)$, $uv \in E(T)$ implies $f(u)f(v) \in E(T')$;
2. For all $u, v \in V(T)$, $u \leq v$ implies $f(u) \leq f(v)$.

Denote $T \subseteq T'$ if $T$ embeds into $T'$. As usual, $T = T'$ if $T \subseteq T'$ and $T' \subseteq T$. We also write $T \varsubsetneq T'$ if $T \subseteq T'$ and $T \neq T'$.

In the definition above, since $f$ is order-preserving, the children of every vertex in $T$ are mapped to the children of its image injectively such that their order is preserved. It is easy to verify that $\subseteq$ is a partial order on the set of all ordered trees.

**Definition 2.2.** An $(\ell, h)$-universal tree is an ordered tree $T'$ such that $T \subseteq T'$ for every ordered tree $T$ of height at most $h$ and with at most $\ell$ leaves.
The simplest example of an \((\ell, h)\)-universal tree is the perfect \(\ell\)-ary tree of height \(h\), which we call a perfect universal tree. The linearly ordered set \(M\) for this tree can be chosen as \(\{0, 1, \ldots, \ell - 1\}\) (see Figure 1 for an example). It has \(\ell^h\) leaves, which grows exponentially with \(h\). Jurdziński and Lazić [25] constructed an \((\ell, h)\)-universal tree with at most \(\ell \log h + O(1)\) leaves, which we call a succinct universal tree. In this tree, every leaf \(\xi\) corresponds to an \(h\)-tuple of binary strings with at most \(\lfloor \log(\ell) \rfloor\) bits in total\(^1\). We use \(|\xi|\) and \(|\xi_i|\) to denote the total number of bits in \(\xi\) and \(\xi_i\) respectively. The linearly ordered set \(M\) for this tree consists of finite binary strings, where \(\varepsilon \in M\) denotes the empty string (see Figure 2 for an example). For any pair of binary strings \(s, s' \in M\) and a bit \(b\), the linear order on \(M\) is defined as

\[
0s < \varepsilon < 1s' \quad \text{for all} \quad s, s' \in M.
\]

Note that the succinct \((3, 2)\)-universal tree embeds into the perfect \((3, 2)\)-universal tree. Recently, Daviaud et al. [9] introduced Strahler universal trees, which refine the concept of universal trees. They also gave a quasi-polynomial construction of these trees, which we refer to as succinct Strahler universal trees. As the construction is more involved, we defer it to Section 4.4.2.

2.2 Node Labels from Ordered Trees

Let \((G, \pi)\) be a parity game instance and \(T\) be an ordered tree of height \(d/2\). Let us augment the set of leaves with an extra top element \(\top\), denoted \(\bar{L}(T) := L(T) \cup \{\top\}\), such that \(\top > v\) for all \(v \in V(T)\). We also set \(\top|_p := \top\) for all \(p \in [d]\). A function \(\mu : V \to \bar{L}(T)\) which maps nodes in the game graph \(G\) to \(\bar{L}(T)\) is called node labels.

For a subgraph \(H\) of \(G\), we say that \(\mu\) is feasible in \(H\) if there exists a strategy \(\sigma : V_0 \to V\) for Even such that the following condition holds for every arc \(vw\) in \(H \cap G_\sigma\):

- If \(\pi(v)\) is even, then \(\mu(v)|_{\pi(v)} \geq \mu(w)|_{\pi(v)}\).
- If \(\pi(v)\) is odd, then \(\mu(v)|_{\pi(v)} > \mu(w)|_{\pi(v)}\) or \(\mu(v) = \mu(w) = \top\).

An arc which does not satisfy this condition is said to be violated with respect to \(\mu\). In the literature, such node labels are usually formulated as a condition for \(H = G\) and called progress measures. Note that the node labels defined by \(\mu(v) = \top\) for all \(v \in V\) are trivially feasible in \(G\). However, we are primarily interested in feasible node labels with minimal top support, i.e. the set of nodes having label \(\top\) is inclusion-wise minimal.

\(^1\)A slightly looser bound of \(\lceil \log \ell \rceil\) was derived in [25, Lemma 1]. It can be strengthened to \(\lfloor \log \ell \rfloor\) with virtually no change in the proof.
Theorem 2.3 ([24, Corollaries 7–8]). Given an \((n,d/2)\)-universal tree \(T\), let \(\mu^* : V \rightarrow \bar{L}(T)\) be node labels which are feasible in \(G\) and have minimal top support. Then, Even wins from \(v \in V\) if and only if \(\mu^*(v) \neq \top\).

The following definition gives a finer distinction of non-violated arcs.

Definition 2.4. Given arbitrary node labels \(\mu : V \rightarrow \bar{L}(T)\), a non-violated arc \(vw\) is tight if \(\mu(v)\) is the smallest element in \(\bar{L}(T)\) such that \(vw\) is non-violated. Otherwise, we say that the non-violated arc is loose.

We say that a subgraph is tight if it consists of tight arcs. The next observation is well-known (see, e.g., [25, Lemma 2]) and follows directly from the definition of feasibility.

Lemma 2.5 (Cycle Lemma). Let \(\mu\) be node labels on a cycle \(C\) such that \(\mu(v) \neq \top\) for all \(v \in V(C)\). If there are no violated arcs in \(C\), then \(C\) is even. If \(C\) is also tight, then \(\mu(v) = \mu(w)\) for all \(v, w \in \Pi(C)\).

Lastly, we assume we have access to the following algorithmic primitive, whose running time we denote by \(\theta(T)\). Its implementation depends on the ordered tree \(T\). For example, \(\theta(T) = O(d)\) if \(T\) is a perfect \((n,d/2)\)-universal tree. If \(T\) is a succinct \((n,d/2)\)-universal tree, Jurdziński and Lazic [25, Theorem 7] showed that \(\theta(T) = O(\log n \log d)\). The latter running time also applies for succinct Strahler universal trees [9, Lemma 22].

\[
\text{Tighten}(\mu, vw)
\]

Given arbitrary node labels \(\mu : V \rightarrow \bar{L}(T)\) and an arc \(vw \in E\), return the unique element \(\xi \in \bar{L}(T)\) such that \(vw\) is tight if \(\mu(v) = \xi\).

3 Strategy Iteration with Tree Labels

In this section, we present a strategy iteration algorithm (Algorithm 1) whose pivots are guided by a universal tree \(T\). At the start of every iteration, the algorithm maintains a strategy \(\tau\) for Odd and node labels \(\mu : V \rightarrow \bar{L}(T)\) which are feasible in \(G_\tau\). Furthermore, there are no loose arcs in \(G_\tau\) with respect to \(\mu\). For \(v \in V_1\), a violated arc \(vw \in E\) with respect to \(\mu\) is called admissible (as it admits Odd to perform an improvement). If there are no admissible arcs in \(G\), then the algorithm terminates. In this case, observe that \(\mu\) is feasible in \(G\). By the Cycle Lemma, Even wins from every node \(v\) where \(\mu(v) \neq \top\), and a winning strategy can be recovered from the feasibility of \(\mu\). On the other hand, if there is at least one admissible arc in \(G\), Odd modifies its current strategy \(\tau\) by switching to admissible arc(s) at a subset of nodes in \(V_1\). The choice of which admissible arc(s) to pick is governed by a pivot rule.

Let \(\tau'\) denote the resulting strategy for Odd. Note that there are no loose arcs in \(G_{\tau'}\). To complete the iteration, it remains to update the node labels \(\mu\) such that they become feasible in the new strategy subgraph \(G_{\tau'}\). In particular, our goal is to compute the pointwise minimal node labels \(\mu' \geq \mu\) which are feasible in \(G_{\tau'}\). Note that it is not clear whether \(\mu'\) exists at this point; its existence will be evident in the proof of Theorem 3.2. Observe that by the minimality of \(\mu'\), there are no loose arcs in \(G_{\tau'}\) with respect to \(\mu'\), so this invariant continues to hold. We coin the term Cramer computation for this goal, which is the focus in the rest of the paper.
Cramer computation\((G_\tau, \pi), \mu\)

Given a one-player parity game \((G_\tau, \pi)\) for Even and node labels \(\mu : V \to \bar{L}(T)\) such that there are no loose arcs in \(G_\tau\), return the pointwise minimal node labels \(\mu^* : V \to \bar{L}(T)\) which are feasible in \(G_\tau\) and satisfy \(\mu^* \geq \mu\).

At the start of the algorithm, the node labels \(\mu\) are initialized as the smallest leaf of \(T\) at every node \(v \in V\). Then, we use Cramer computation to update \(\mu\) to the pointwise minimal node labels which are feasible in \(G_\tau\) and satisfy \(\mu^* \geq \mu\).

Algorithm 1 Strategy iteration with tree labels: \((G, \pi)\) parity game instance, \(T\) universal tree, \(\tau_1\) initial strategy for Odd

1: procedure StrategyIteration\(((G, \pi), T, \tau_1)\)
2: \(\mu(v) \leftarrow \min \bar{L}(T) \ \forall v \in V\)
3: \(\tau \leftarrow \tau_1\)
4: \(\mu \leftarrow \text{CramerComputation}((G_\tau, \pi), \mu)\)
5: while \(\exists\) an admissible arc in \(G\) with respect to \(\mu\) do
6: Pivot to a strategy \(\tau'\) by selecting admissible arc(s) \(\triangleright \text{requires a pivot rule}\)
7: \(\tau \leftarrow \tau'\)
8: \(\mu \leftarrow \text{CramerComputation}((G_\tau, \pi), \mu)\)
9: return \(\tau, \mu\)

Lifting operators To prove the correctness of Algorithm 1, it is instructive to discuss about the lifting operators associated with arcs in \(G\). Given arbitrary node labels \(\mu : V \to \bar{L}(T)\) and an arc \(vw \in E\), let \(\text{lift}(\mu, vw)\) be the smallest element \(\xi \in \bar{L}(T)\) such that \(\xi \geq \mu(v)\) and \(vw\) is non-violated if \(\mu(v) = \xi\). Observe that if \(vw\) is violated, then \(\text{lift}(\mu, vw)\) is given by \(\text{Tighten}(\mu, vw)\). Otherwise, it is equal to \(\mu(v)\). Hence, it can be computed in running time \(\theta(T)\) as \(\text{Tighten}\).

Let \(\mathcal{L}\) be the finite lattice of node labels mapping \(V\) to \(\bar{L}(T)\). For every node \(v \in V_0\), define the operator \(\text{Lift}_v : \mathcal{L} \times V \to \bar{L}(T)\) as

\[
\text{Lift}_v(\mu, u) := \begin{cases} 
\min_{vw \in E} \text{lift}(\mu, vw) & \text{if } u = v, \\
\mu(u) & \text{otherwise}.
\end{cases}
\]

For every arc \(vw \in E\) where \(v \in V_1\), define the operator \(\text{Lift}_{vw} : \mathcal{L} \times V \to \bar{L}(T)\) as

\[
\text{Lift}_{vw}(\mu, u) := \begin{cases} 
\text{lift}(\mu, vw) & \text{if } u = v, \\
\mu(u) & \text{otherwise}.
\end{cases}
\]

Equipped with this terminology, we recall the generic value iteration algorithm (Algorithm 2) as stated, e.g., in [15, Algorithm 3]. We remark that the operators \(2\) are defined slightly differently than usual to provide more flexibility in the way we apply them. We also allow for an additional argument \(\mu_0\) of initial node labels, which is usually set as \(\mu_0(v) := \min \bar{L}(T)\) for all \(v \in V\).

Fixed points in lattices We recall a fundamental result on the existence of fixed points. Let \(\mathcal{L}\) be a non-empty finite lattice with least element \(\min \mathcal{L}\). Let \(\mu, \nu \in \mathcal{L}\). An operator \(\phi : \mathcal{L} \to \mathcal{L}\) with

\[
\phi(\mu) := \begin{cases} 
\min_{vw \in E} \text{lift}(\mu, vw) & \text{if } \mu \geq \nu, \\
\nu & \text{otherwise}.
\end{cases}
\]

is a fixed point of \(\phi\) if and only if \(\mu = \nu\) and it is a minimal fixed point if it is the least element of \(\mathcal{L}\).
Algorithm 2 Value iteration with tree labels: \((G, \pi)\) parity game instance, \(\mu_0 : V \to L(T)\) node labels from a universal tree \(T\)

1: procedure \textsc{ValueIteration}((\(G, \pi\), \(\mu_0\))
2: \(\mu \leftarrow \mu_0\)
3: repeat
4: \(\mu(v) \leftarrow \text{Lift}_v(\mu, v)\) for any \(v \in V_0\) where \(\mu(v) < \text{Lift}_v(\mu, v)\) or
5: \(\mu(v) \leftarrow \text{Lift}_{vw}(\mu, v)\) for any \(vw \in E\) where \(v \in V_1\) and \(\mu(v) < \text{Lift}_{vw}(\mu, v)\)
6: until \(\mu\) is feasible in \(G\)
7: return \(\mu\)

is monotone if \(\mu \leq \nu \Rightarrow \phi(\mu) \leq \phi(\nu)\), and it is inflationary if \(\mu \leq \phi(\mu)\). Let \(\mathcal{G}\) be a family of inflationary, monotone operators on \(\mathcal{L}\) and \(H \subseteq \mathcal{G}\) a subfamily. By \(\mu^H\) we denote the least simultaneous fixed point of \(H\) which is at least \(\mu\).

**Proposition 3.1** (Folklore). (i) The least simultaneous fixed point \(\mu^H\) exists.

(ii) The least fixed point is non-decreasing with the set of operators: \(\mu^H \leq \mu^G\).

(iii) The least fixed point is monotone: if \(\mu \leq \nu\) then \(\mu^H \leq \nu^H\).

An operator \(\psi : \mathcal{L} \to \mathcal{L}\) is deflationary if \(\mu \geq \psi(\mu)\). Considering the lattice through an order-reversing poset isomorphism implies that the analogous statements of Proposition 3.1 hold for deflationary monotone operators. Note that if \(\mathcal{G}\) is a family of deflationary monotone operators and \(H \subseteq \mathcal{G}\), then \(\mu^H\) denotes the greatest simultaneous fixed point of \(H\) which is at most \(\mu\).

For the purpose of this paper, \(\mathcal{L}\) is the lattice of node labels mapping \(V\) to \(L(T)\) for an ordered tree \(T\). The set of operators is \(\mathcal{G} := \{\text{Lift}_v : v \in V_0\} \cup \{\text{Lift}_{vw} : vw \in E, v \in V_1\}\). From the definition of \(\text{lift}(\cdot, \cdot)\), they are inflationary and monotone. If we choose \(H = \{g\}\) for some \(g \in \mathcal{G}\), then \(\mu^H\) is the result of applying a single Lift operator to \(\mu\), i.e. one iteration of Algorithm 2. On the other hand, if we choose \(H = \mathcal{G}\), then \(\mu^H\) is the pointwise minimal node labels feasible in \(G\) which is at least \(\mu\). The former is easy to compute, while the latter is much harder as it enables us to solve parity games. The main idea behind Algorithm 1 is to select a large \(H \subseteq \mathcal{G}\) while ensuring that \(\mu^H\) remains efficiently computable. The operators in strategy subgraphs turn out to be the right choice.

We are ready to prove the correctness of Algorithm 1.

**Theorem 3.2.** Algorithm 1 returns the pointwise minimal node labels \(\mu^\ast : V \to \bar{L}(T)\) which are feasible in \(G\).

**Proof.** Since \(\mathcal{L}\) is finite, the algorithm terminates. Let \(\mu \in \mathcal{L}\). From the definition of feasibility, \(\mu\) is feasible in \(G\) if and only if \(\mu\) is a simultaneous fixed point of \(\mathcal{G}\). Moreover, the output of \textsc{Cramer Computation}((\(G_\tau, \pi\), \(\mu\)) is precisely the least simultaneous fixed point \(\mu^H\) of the subfamily \(H := \{\text{Lift}_v : v \in V_0\} \cup \{\text{Lift}_e : e \in \tau\}\). Correctness then follows from Proposition 3.1 as the operators \((\mu \mapsto \mu^H)\) themselves are inflationary and monotone.

Thus, by Theorem 2.3, the algorithm correctly determines the winning positions for Even.

**Runtime** We can ensure that the total runtime of Algorithm 1 is not asymptotically worse than running value iteration on \(G\), i.e. \textsc{ValueIteration}((\(G, \pi\), \(\mu_0\)) where \(\mu_0(v) = \min L(T)\) for all \(v \in V\). A primitive way of solving Cramer computation is to run value iteration on the strategy subgraph \(G_\tau\) with node labels \(\mu\), i.e. \textsc{ValueIteration}((\(G_\tau, \pi\), \(\mu\)). Since the label at every node can be
lifted at most $|L(T)|$ times, this yields a total runtime of $O(\theta(T)m|L(T)|)$, matching value iteration algorithms \cite{24,25,9}. However, this method is not preferable as it may take superpolynomial time to perform a single Cramer computation. Our goal is to ensure that every iteration of Algorithm 1 runs in polynomial time. Nevertheless, running VALUE\_ITERATION($\langle G_\tau, \pi \rangle, \mu$) in parallel with a more efficient method of Cramer computation in every iteration of Algorithm 1 is still useful in ensuring that we are not slower than VALUE\_ITERATION($\langle G, \pi \rangle, \mu_0$) overall.

### 3.1 Prelude to Shortest Path Algorithms with Ordered Trees

In order to efficiently perform Cramer computation, we adapt classical shortest path algorithms to the setting of ordered trees. Given an ordered tree $T$ and a one-player parity game $G_\tau$ for Even, let $\mu : V \to \bar{L}(T)$ be node labels such that there are no loose arcs in $G_\tau$. Recall that the goal of Cramer computation is to compute the pointwise minimal node labels $\mu^* : V \to \bar{L}(T)$ which are feasible in $G_\tau$ and satisfy $\mu^* \geq \mu$.

From the definition of feasibility, Even has a strategy $\sigma$ such that $G_{\sigma \tau} := (V, E_\sigma \cap E_\tau)$ does not contain violated arcs with respect to $\mu^*$. The subgraph $G_{\sigma \tau}$ arises as the intersection of the strategy subgraphs for $\sigma$ and $\tau$. Consider the subgraph of $G_{\sigma \tau}$ induced by the nodes $v \in V$ with $\mu^*(v) \neq \top$. Since every node has outdegree 1, each component is a disjoint union of in-trees whose roots lie in a common directed cycle. Note that the cycle is even by the Cycle Lemma. These cycles are analogous to the destination node in the shortest path problem. However, unlike for shortest paths, we do not know them a priori.

To address this issue, we use nodes which dominate an even cycle as destination nodes.

**Definition 3.3.** Given a one-player parity game $(G_\tau, \pi)$ for Even, we call $v \in V$ a base node if $v \in \Pi(C)$ for some even cycle $C$ in $G_\tau$. Denote $B(G_\tau)$ as the set of base nodes in $G_\tau$.

The base nodes can be found by recursively decomposing $G_\tau$ into strongly connected components. Initially, for each strongly connected component $K$ of $G_\tau$, we delete $\Pi(K)$. If $\pi(K)$ is even and $|V(K)| > 1$, then $\Pi(K)$ are base nodes and we collect them. Otherwise, we ignore them. Then, we are left with a smaller subgraph of $G_\tau$, so we repeat the process. Using Tarjan’s strongly connected components algorithm, this procedure takes $O(dm)$ time.

To make Cramer computation efficient, we approach the target labels $\mu^*$ from above instead of from below. First, we define the deflationary operators that complement the inflationary Lift operators. For arbitrary node labels $\nu : V \to \bar{L}(T)$ and an arc $vw$, let $\text{drop}(\nu, vw)$ be the largest element $\xi \in \bar{L}(T)$ such that $\xi \leq \nu(v)$ and $vw$ is not loose if $\nu(v) = \xi$. Its computation will serve as the main algorithmic primitive for the shortest path algorithms in this paper. Observe that if $vw$ is loose, then $\text{drop}(\nu, vw)$ can be obtained by calling the subroutine TIGHTEN($\nu, vw$). Otherwise, it is equal to $\nu(v)$. Hence, it can be computed in $\theta(T)$ time.

Let $\mathcal{L}$ be the finite lattice of node labels mapping $V$ to $\bar{L}(T)$. For every arc $vw \in E_\tau$, define the operator $\text{Drop}_{vw} : \mathcal{L} \times V \to \bar{L}(T)$ as

$$\text{Drop}_{vw}(\nu, u) := \begin{cases} \text{drop}(\nu, vw) & \text{if } u = v, \\
\mu(u) & \text{otherwise}.\end{cases}$$

Denote the family of these operators by $\mathcal{G}_\tau := \{\text{Drop}_{vw} : vw \in E_\tau\}$. From the definition of $\text{drop}(\cdot, \cdot)$, they are deflationary and monotone. Hence, for any $\nu \in \mathcal{L}$, the greatest simultaneous fixed point of $\mathcal{G}_\tau$ in $\mathcal{L}$ which is at most $\nu$ exists. We denote this point as $\nu^{\mathcal{G}_\tau}$. One way to obtain it is by applying the operators in $\mathcal{G}_\tau$ to $\nu$ in any order until convergence.
Note that for any \( \nu \in \mathcal{L} \), there are no loose arcs in \( G_\tau \) with respect to \( \nu \) if and only if \( \nu \) is a simultaneous fixed point of \( G_\tau \).

The most prevalent methods for computing shortest paths are label-setting and label-correcting algorithms. Label-setting algorithms are asymptotically faster than label-correcting algorithms, but the former requires the underlying graph to have nonnegative (reduced) arc costs. We first focus on the label-correcting framework, applicable for Cramer computation with arbitrary ordered trees, in particular the quasipolynomial-sized universal trees constructed for parity games. We defer the development of the label-setting framework to Section 5, which is more suited for Cramer computation with perfect universal trees.

## 4 Label-Correcting Framework

The Bellman–Ford algorithm for the shortest path problem is a well-known implementation of the generic label-correcting method. We start by giving its analogue for ordered trees.

Algorithm 3 takes as input a one-player parity game \((G_\tau, \pi)\) for Even and node labels \( \nu_0 : V \to \bar{L}(T) \) from some ordered tree \( T \). During its execution, it maintains node labels \( \nu : V \to \bar{L}(T) \), which are initialized to \( \nu_0 \). Like its classical version for shortest paths, the algorithm runs for \( n-1 \) iterations. In each iteration, it updates \( \nu \) by dropping the tail label of every arc in \( G_\tau \). Clearly, the running time is \( O(mn\theta(T)) \). Moreover, the returned node labels \( \nu \) satisfy \( \nu \geq \nu_0^{\bar{G}_\tau} \), where \( \bar{G}_\tau \) is the set of Drop operators in \( G_\tau \).

### Algorithm 3 Bellman–Ford: \((G_\tau, \pi)\) one-player parity game for Even, \( \nu_0 : V \to \bar{L}(T) \) node labels from an ordered tree \( T \)

```algorithm
1: \textbf{procedure} \textsc{BellmanFord}((\(G_\tau, \pi\)), \( \nu_0 \))
2: \quad \nu \leftarrow \nu_0
3: \quad \textbf{for} \ i = 1 \ \textbf{to} \ n-1 \ \textbf{do}
4: \quad \quad \textbf{for} \ \text{all} \ \textit{vw} \in E_\tau \ \textbf{do} \quad \triangleright \text{In any order}
5: \quad \quad \nu(v) \leftarrow \text{drop}(\nu, \textit{vw})
6: \quad \textbf{return} \ \nu
```

Now, fix a one-player parity game \((G_\tau, \pi)\) for Even and node labels \( \mu : V \to \bar{L}(T) \) with no loose arc in \( G_\tau \). Let \( \mu^* \) denote the desired output of \textsc{Cramer Computation}((\(G_\tau, \pi\)), \( \mu \)). We state a sufficient condition on the input node labels \( \nu_0 \) which guarantees that Algorithm 3 returns \( \mu^* \). Recall that in the shortest path problem, the input labels are set to \( \infty \) everywhere except at the destination node. Differently, for ordered trees, one has to ensure that the algorithm does not terminate with labels bigger than \( \mu^* \), motivating the following definition.

**Definition 4.1.** For a base node \( v \in B(G_\tau) \), define \( \hat{\mu}(v) \) as the smallest element in \( \bar{L}(T) \) such that \( \hat{\mu}(v) = \tilde{\mu}(v) \) for some node labels \( \tilde{\mu} : V \to \bar{L}(T) \) which satisfy the following two properties: (1) \( \tilde{\mu} \) is feasible in a cycle dominated by \( v \) in \( G_\tau \); and (2) \( \tilde{\mu}(v) \geq \mu(v) \).

We call \( \hat{\mu}(v) \) the threshold label for \( v \), as it is an upper bound on \( \nu_0(v) \) which ensures that Algorithm 3 returns \( \mu^* \). The next lemma guarantees that the range between \( \mu^*(v) \) and \( \hat{\mu}(v) \) is not empty for all base nodes \( v \in B(G_\tau) \).

**Lemma 4.2.** For every base node \( v \in B(G_\tau) \), we have \( \hat{\mu}(v) \geq \mu^*(v) \).

**Proof.** Fix a base node \( v \). Let \( \tilde{\mu} \) be node labels which are feasible on a cycle \( C \) dominated by \( v \) and satisfy \( \tilde{\mu}(v) \geq \mu(v) \). We can extend \( \tilde{\mu} \) to feasible node labels on the whole strategy subgraph.
$G_\tau$ by setting it to $\top$ on all nodes outside $C$. Then, $\tilde{\mu}$ is clearly feasible in $G_\tau$. Since there are no loose arcs in $C$ with respect to $\mu$ and $\tilde{\mu}(v) \geq \mu(v)$, we obtain $\tilde{\mu} \geq \mu$. Due to the minimality of $\mu^*$ as feasible node labels in $G_\tau$, this implies that $\tilde{\mu} \geq \mu^*$. As $\tilde{\mu}$ was chosen arbitrarily (with the desired properties from Definition 4.1), the claim also holds for the minimum $\tilde{\mu}(v)$ of the potential values $\tilde{\mu}(v)$.

**Theorem 4.3.** Let $(G_\tau, \pi)$ be a one-player parity game for Even and let $\mu : V \rightarrow \bar{L}(T)$ be node labels such that there are no loose arcs in $G_\tau$. If $v_0 : V \rightarrow \bar{L}(T)$ satisfies $v_0 \geq \mu^*$ and $v_0(v) \leq \tilde{\mu}(v)$ for all $v \in B(G_\tau)$, then Algorithm 3 returns $\mu^*$.

**Proof.** Let $\nu$ be the node labels returned by Algorithm 3. Since there are no loose arcs in $G_\tau$ with respect to $\mu$, the node labels $\mu$ is a simultaneous fixed point of $G_\tau$. By the minimality of $\mu^*$, there are also no loose arcs in $G_\tau$ with respect to $\mu^*$. Hence, $\mu^*$ is a simultaneous fixed point of $G_\tau$. Now, $\mu \leq \mu^* \leq \nu_0$ implies $\mu \leq \mu^* \leq \nu_0^{G_\tau} \leq \nu$ as $\nu$ arises from a repeated application of operators in $G_\tau$.

First, we show that there are no loose arcs in $G_\tau$ with respect to $\nu$, i.e., $\nu = \nu_0^{G_\tau}$. For the purpose of contradiction, let $v_1v_2$ be a loose arc with respect to $\nu$. Let $R$ be the sequence of arcs encountered chronologically in Algorithm 3 on which drop$(\cdot, \cdot)$ decreases the tail label. Note that $|R| \leq (n-1)|E_\tau|$. Starting from the end of $R$, we construct a walk $P$ from $v_2$ as follows. Add the last arc in $R$ with tail $v_2$. Let $\xi_2 \in L(T)$ be the corresponding label assigned to $v_2$ by drop$(\cdot, \cdot)$ at that point of Algorithm 3. Note that $\nu(v_2) = \xi_2 \neq \top$. Let $v_3$ be the head of this arc, and continue the construction in the same way with $v_3$. We finish with a walk $P = (v_2, v_3, \ldots, v_\ell)$ when there are no more arcs in $R$ before $(v_{\ell-1}, v_\ell)$ whose tail is $v_\ell$. Let $\xi_\ell = \nu_0(v_\ell)$. Observe that $\xi_i \neq \top$ for all $2 \leq i \leq \ell$.

Since the arc $v_1v_2$ is loose with respect to $\nu$, we have $\ell > n$ because Algorithm 3 ran for $n-1$ iterations. Hence, the walk $v_1v_2 + P$ contains a cycle $C = (v_j, v_{j+1}, \ldots, v_k)$, where $v_j = v_k$. Let $v' : V(C) \rightarrow L(T)$ be the node labels defined by $v'(v_i) := \xi_i$ for all $j < i \leq k$. Then, $v'$ is feasible in $C$. In particular, every arc except $v_jv_{j+1} = v_kv_{j+1}$ is tight with respect to $v'$. As $v'(v) \neq \top$ for all $v \in V(C)$, $C$ is even by the Cycle Lemma. Pick $v_i \in \Pi(C) \subseteq B(G_\tau)$. Since $v'(v_i) \geq \nu(v_i) \geq \mu(v_i)$, we obtain $v'(v_i) \geq \tilde{\mu}(v_i)$ from the definition of threshold label. We also have $v'(v_i) \leq v_0(v_i)$ as node labels are non-increasing throughout the algorithm. Combining these two inequalities yield $\tilde{\mu}(v_i) \geq v'(v_i) \leq v_0(v_i)$. It follows that $\tilde{\mu}(v_i) = v'(v_i) = v_0(v_i)$ due to our choice of $v_0$. So, if $i = j = k$, then this contradicts the minimality of $\tilde{\mu}$ because $v_jv_{j+1}$ is loose with respect to $v'$. If $i \notin \{j, k\}$, then this contradicts $v'(v_i) = v_0(v_i)$, as the label at $v_i$ was decreased.

Next, we show that $\nu = \mu^*$. Let $S := \{v \in V : \mu^*(v) < \nu(v)\}$, and suppose that $S \neq \emptyset$ for the sake of contradiction. Since there are no loose arcs in $G_\tau$ with respect to $\nu$, every arc in $\delta^+_\tau(S)$ is violated with respect to $\mu^*$. From the feasibility of $\mu^*$ in $G_\tau$, there exists a strategy $\sigma$ for Even such that $G_{\sigma\tau}$ does not contain any violated arc with respect to $\mu^*$. As every node in $G_{\sigma\tau}$ has outdegree 1, there exists a cycle $C$ in $G_{\sigma\tau}[S]$. By the Cycle Lemma, $C$ is even because $\mu^*(v) \neq \top$ for all $v \in S$. Pick any $u \in \Pi(C) \subseteq B(G_\tau)$. Then, $\tilde{\mu}(u) \leq \mu^*(u)$ by the minimality of $\tilde{\mu}$. However, we arrive at the contradiction $\tilde{\mu}(u) \leq \mu^*(u) < \nu(u) \leq v_0(u) \leq \tilde{\mu}(u)$ due to our choice of $v_0$.

According to Theorem 4.3, it suffices to compute node labels $v_0 : V \rightarrow \bar{L}(T)$ which satisfy $\mu^*(v) \leq v_0(v) \leq \tilde{\mu}(v)$ for all $v \in B(G_\tau)$ for the purpose of Cramer computation. This is because we can simply set $v_0(v) := \top \geq \mu^*$ for all $v \notin B(G_\tau)$. To tackle this problem, we need to understand the subtrees of $T$ via the embedding partial order $\subseteq$. 

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4.1 Covering an Ordered Tree with Chains

We fix an ordered tree $T$ of height $h$. Two ordered trees $T'$ and $T''$ are said to be distinct if $T' \neq T''$. For $0 \leq j \leq h$, denote $\mathcal{T}_j$ as the set of distinct subtrees of $T$ rooted at vertices of depth $h - j$. For example, $\mathcal{T}_h = \{T\}$ and $\mathcal{T}_0$ contains a trivial tree with a single vertex. Since we assumed that all leaves in $T$ are at the same depth, every tree in $\mathcal{T}_j$ has height $j$. We denote $\mathcal{T} = \cup_{j=0}^{h} \mathcal{T}_j$, the union of all these subtrees of $T$. The sets $\mathcal{T}$ and $\mathcal{T}_j$ of trees form posets where the partial order is given by embedding $\sqsubseteq$ of trees.

**Definition 4.4.** For $0 \leq j \leq h$, let $\mathcal{C}_j = (\mathcal{C}^0_j, \mathcal{C}^1_j, \ldots, \mathcal{C}^{|\mathcal{C}_j| - 1}_j)$ be a tuple of chains in the poset $((\mathcal{T}_j, \sqsubseteq))$. We say that $\mathcal{C}_j$ is a cover of $\mathcal{T}_j$ if $\cup_k \mathcal{C}^k_j = \mathcal{T}_j$. A cover of $\mathcal{T}$ is a tuple $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_h)$ such that $\mathcal{C}_j$ is a cover of $\mathcal{T}_j$ for all $0 \leq j \leq h$. We call $\mathcal{C}_j$ the $j$th-subcover of $\mathcal{C}$. For $0 \leq j \leq h$ and $0 \leq k < |\mathcal{C}_j|$, denote the subtrees in the chain $\mathcal{C}^k_j$ as $\mathcal{C}^0_{0,j} \sqsubseteq \mathcal{C}^1_{1,j} \sqsubseteq \cdots \sqsubseteq \mathcal{C}^{|\mathcal{C}_j| - 1,j}_k$.

We are ready to introduce the key concept of this subsection. It is important that we defined a subcover as an indexed tuple and not just as a set, as can be seen in the next definition.

**Definition 4.5.** Let $\mathcal{C}$ be a cover of $\mathcal{T}$. Given a subgraph $H$ of $G_r$, let $j = \lceil (\pi(H)/2) \rceil$. For a fixed chain $\mathcal{C}^k_j$ with $0 \leq k < |\mathcal{C}_j|$, the $k$th-width of $H$, denoted $\alpha^k(H)$, is the smallest integer $i \geq 0$ such that there exist node labels $\nu : V(H) \rightarrow L(\mathcal{T}^k_j)$ from the tree $\mathcal{T}^k_j$ in the chain $\mathcal{C}^k_j$ which are feasible in $H$. If $i$ does not exist, then $\alpha^k(H) = \infty$.

When the cover $\mathcal{C}$ is clear from context, we will omit it from the subscript and write $\alpha^k(H)$. We would like to emphasize that $\nu(v) \neq \top$ is required for all $v \in V(H)$ in the definition above. Therefore, an odd cycle has infinite $k$th-width by the Cycle Lemma. Since $(\mathcal{C}^k_j, \sqsubseteq)$ is a chain, for all finite $i \geq \alpha^k(H)$, there exist node labels $\nu : V(H) \rightarrow L(\mathcal{T}^k_{i,j})$ which are feasible in $H$. The next lemma illustrates the connection between the $k$th-width of an even cycle and its path decomposition.

**Lemma 4.6.** Let $\mathcal{C}$ be a cover of $\mathcal{T}$. Given an even cycle $C$, let $\Pi(C) = \{v_1, v_2, \ldots, v_\ell\}$. Decompose $C$ into node-disjoint paths $P_1, P_2, \ldots, P_\ell$ such that each $P_i$ ends at $v_i$ and $\cup_{i \in [\ell]} V(P_i) = V(C)$. For any $0 \leq k < \lceil |\mathcal{C}_{\pi(C)}|/2 \rceil$, we have

$$\alpha^k(C) = \max_{i \in [\ell]} \alpha^k(P_i).$$

**Proof.** Fix a $k$ and let $\alpha^* = \max_{i \in [\ell]} \alpha^k(P_i)$. Since each $P_i$ is a subgraph of $C$, node labels which are feasible in $C$ are also feasible in $P_i$. So, $\alpha^k(C) \geq \alpha^*$. For the reverse inequality, we know that there exist node labels $\nu_i : V(P_i) \rightarrow L(\mathcal{T}^k_{\alpha^*, \pi(C)/2})$ which are feasible in $P_i$ for all $i \in [\ell]$. Let $\nu : V(C) \rightarrow L(\mathcal{T}^k_{\alpha^*, \pi(C)/2})$ be the node labels obtained by aggregating the $\nu_i$’s, i.e. $\nu(v) := \nu_i(v)$ if $v \in V(P_i)$. Then, $\nu$ is feasible in $C$ because $\pi(\nu_i) = \pi(C)$ for all $i \in [\ell]$ and $\nu(v) \neq \top$ for all $v \in V(C)$. Hence, $\alpha^k(C) \leq \alpha^*$. \hfill $\square$

Next, we extend the concept of $k$th-width to base nodes.

**Definition 4.7.** Let $\mathcal{C}$ be a cover of $\mathcal{T}$. Given a base node $v \in B(G_r)$, let $j = \pi(v)/2$. For a fixed $0 \leq k < |\mathcal{C}_j|$, the $k$th-width of $v$ is

$$\alpha^k_v := \min \left\{ \alpha^k(C) : C \text{ is a cycle dominated by } v \text{ in } G_r \right\}.$$
Again, we omit $C$ from the subscript when the cover is clear from context. The $k$-th-width of $v$ gives rise to the subtree $T_{\alpha^k(v),\pi(v)/2}$. It is the smallest tree in the chain $C_{\pi(v)/2}$ that is capable of serving as the codomain to feasible node labels in some cycle dominated by $v$. This is the crucial quantity that enables one to compute threshold labels (Definition 4.1) using the following subroutine.

**RAISE($\mu, v, k, i$)**

Given node labels $\mu : V \rightarrow \bar{L}(T)$, a node $v \in V$, $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, return the smallest element $\xi \in \bar{L}(T)$ such that: (1) $\xi \geq \mu(v)$; and (2) $\xi$ is the smallest leaf in $T_{i',\pi(v)/2}^k$ for some $i' \geq i$. If $\xi$ does not exist, return $\top$.

For a base node $v \in B(G_r)$, using the $k$-th-width $\alpha^k(v)$ as input $i$ in RAISE will produce the threshold label $\hat{\mu}(v)$. In particular, $\hat{\mu}(v)$ is equal to the smallest label returned by RAISE over all the chains in the subcover $C_{\pi(v)/2}$, as the next lemma shows.

**Lemma 4.8.** Given a base node $v \in B(G_r)$, let $\xi^k$ be the label returned by RAISE($\mu, v, k, \alpha^k(v)$) for all $0 \leq k < |C_{\pi(v)/2}|$. If there are no loose arcs in $G_r$ with respect to $\mu$, then $\hat{\mu}(v) = \min_k \xi^k$.

**Proof.** To simplify notation, let $j = \pi(v)/2$. First, we prove that $\hat{\mu}(v) \leq \xi^k$ for all $0 \leq k < |C_j|$. Fix a $k$ and assume that $\xi^k \neq \top$. Since $\xi^k$ is returned by RAISE($\mu, v, k, \alpha^k(v)$), it is the smallest leaf in $T_{i,j}^k$ for some $i \geq \alpha^k(v)$. According to the definition of the $k$-th-width $\alpha^k(v)$, there exists a cycle $C$ dominated by $v$ in $G_r$ and node labels $\nu : V(C) \rightarrow L(T_{i,j}^k)$ which are feasible in $C$. Note that $\nu(w) \neq \top$ for all $w \in V(C)$. Since $\nu(v) = 2j$, we may assume that $\nu(v) = \min L(T_{i,j}^k)$ without losing feasibility of $\nu$ in $C$. Now, consider the node labels $\bar{\mu} : V \rightarrow \bar{L}(T)$ defined as follows. If $w \in V(C)$, set $\bar{\mu}(w)$ as the concatenation of $\xi^k|_{i,j}$ and $\nu(v)$, which is an $h$-tuple like $\xi^k$. Otherwise, $\bar{\mu}(v) := \top$. Then, $\bar{\mu}$ is feasible in $C$ and $\bar{\mu}(v) = \xi^k \geq \mu(v)$, where the equality is due to our assumption $\nu(v) = \min L(T_{i,j}^k)$. Therefore, $\xi^k \geq \bar{\mu}(v)$ by the minimality of $\bar{\mu}(v)$ in Definition 4.1.

Next, we prove that $\xi^k \leq \hat{\mu}(v)$ for some $0 \leq k < |C_j|$. We may assume that $\bar{\mu}(v) \neq \top$. From Definition 4.1, there exist node labels $\hat{\mu} : V \rightarrow \bar{L}(T)$ and a cycle $C$ dominated by $v$ in $G_r$ such that $\hat{\mu}$ is feasible in $C$ and $\hat{\mu}(v) = \bar{\mu}(v) \geq \mu(v)$. Since $C$ is a cover of $T$, the subtree of $T$ rooted at $\bar{\mu}(v)\mid_{i,j}$ is a copy of $T_{i,j}^k$ for some $0 \leq k < |C_j|$ and $0 \leq i < |C_j|$. The parameter $i$ of this tree fulfills $i \geq \alpha^k(C)$ because $\bar{\mu}(v) \neq \top$ and $\bar{\mu}(w)\mid_{i,j} = \bar{\mu}(w')\mid_{i,j}$ for all $w, w' \in V(C)$. In other words, all the labels in $C$ lie in the same subtree, a copy of $T_{i,j}^k$. As $v \in \Pi(C)$, we also have $\alpha^k(C) \geq \alpha^k(v)$, yielding $i \geq \alpha^k(v)$.

We will prove that $\xi^k \leq \hat{\mu}(v)$ for this choice of $k$. Since $\xi^k$ is given by RAISE($\mu, v, k, \alpha^k(v)$), it is the smallest element in $\bar{L}(T)$ such that (1) $\xi^k \geq \mu(v)$ and (2) $\xi^k$ is the smallest leaf in $T_{i',j}^k$ for some $i' \geq \alpha^k(v)$. As $i \geq \alpha^k(v)$, it suffices to show that $\hat{\mu}(v)$ is the smallest leaf in the subtree of $T$ rooted at $\bar{\mu}(v)\mid_{i,j}$, which is a copy of $T_{i,j}^k$. Suppose otherwise for a contradiction. Then, $\hat{\mu}(v) = \mu(v)$, as otherwise $\hat{\mu}(v)$ can be decreased, contradicting its minimality in Definition 4.1. Starting from the incoming arc of $v$, where we have the equality $\hat{\mu}(v) = \bar{\mu}(v) = \mu(v)$, we trace the cycle $C$ backwards. As there are no loose arcs in $G_r$ with respect to $\mu$, we get $\hat{\mu}(w) \geq \mu(v)$ for all $w \in V(C)$. However, this implies that the outgoing arc of $v$ in $C$ is loose with respect to $\mu$ because $\mu(v)$ is not the smallest leaf in the subtree of $T$ rooted at $\mu(v)\mid_{i,j}$. We have reached a contradiction.

The necessary number of chains $|C_{\pi(v)/2}|$ can be very large if $T$ is an arbitrary ordered tree. Fortunately, the universal trees constructed for parity games so far are ‘structured’. Formally, this means that there exists a cover such that the number of chains in each subcover is small. For instance, a succinct $(n, h)$-universal tree has a cover with only 1 chain per subcover. On the other hand, a succinct Strahler $(n, h)$-universal tree has a cover with at most $\log n$ chains per subcover.
To efficiently navigate $T$, we assume to have access to RAISE, whose running time we denote by $\rho(T, C)$. Its implementation depends on the cover $C$. After we elaborate on how to approximate threshold labels and conclude Cramer computation, we provide efficient implementations of RAISE for succinct universal trees and succinct Strahler universal trees in Subsections 4.4.1 and 4.4.2.

4.2 Raising the Labels on Base Nodes

In light of the previous discussion, our attention now shifts to computing the $k$-width of a base node $w \in B(G_\tau)$ for all $0 \leq k < |\mathcal{C}(w)\pi(w)/2|$. Fix a $k$ in this range. Since we ultimately need a label $\nu(w)$ that lies between $\mu(w)$ and $\hat{\mu}(w)$ to initialize Algorithm 3, it suffices to compute a ‘good’ under-estimate of $\alpha^k(w)$. We approach this problem by computing minimum bottleneck cycles in an auxiliary digraph $D_\tau$ with nonnegative arc costs $c^k \geq 0$. For a base node $w \in B(G_\tau)$, let $K_w$ be the maximal strongly connected subgraph containing $w$ in the subgraph of $G_\tau$ induced by nodes with priority at most $\pi(w)$. Let $J_w$ be the subgraph of $K_w$ induced by nodes which can reach $w$ after deleting the incoming arcs $\delta^-(v)$ for all $v \in \Pi(K_w) \setminus w$.

![Figure 3: An example of a strategy subgraph $G_\tau$ is shown on the left, with its auxiliary graph $D_\tau$ on the right. In $G_\tau$, nodes in $V_0$ and $V_1$ are drawn as squares and circles respectively. The base nodes are labeled as $w_1$, $w_2$, $w_3$ and $w_4$. The light gray region is $K_{w_4}$, while the dark gray region is $J_{w_4}$.](image)

**Auxiliary digraph construction** The node set of $D_\tau$ is the set of base nodes $B(G_\tau)$. For every ordered pair $(v, w)$ of base nodes where $\pi(v) = \pi(w)$, add the arc $vw$ if $v$ has an outgoing arc in $J_w$. As ordered pairs of the form $(v, v)$ are also considered, $D_\tau$ may contain self-loops. Observe that $D_\tau$ is a disjoint union of strongly connected components, each of which corresponds to base nodes of the same priority (see Figure 3 for an example). For each $w \in B(G_\tau)$, we use $D_\tau(w)$ to denote the component in $D_\tau$ that contains $w$. Note that the graph structure of $D_\tau$ is independent of $k$.

To finish the description of $D_\tau$, it is left to assign the arc costs $c^k$, which does depend on $k$. We give a range in which the cost of each arc should lie. Fix a base node $w \in B(G_\tau)$ and let $j = \pi(w)/2$. Denote $J_w$ as the family of Drop operators induced by the subgraph $J_w$, i.e. $J_w := \{\text{Drop}_e : e \in E(J_w)\}$. For each $0 \leq i < |c^k|$, let $\lambda^k_{i,w} : V(J_w) \to \hat{L}(T_{i,j}^k)$ be the greatest simultaneous fixed point of $J_w$ subject to $\lambda^k_{i,w}(w) = \min L(T_{i,j}^k)$. Then, for each arc $vw \in E(D_\tau)$, the lower and
upper bounds of $c^k(vw)$ are given by

\[
\mathcal{L}^k(vw) := \min_{u \in N_{\alpha^k}(v)} \left\{ i : \lambda_{i,u}^k(u) \neq \top \right\}
\]

\[
\mathcal{E}^k(vw) := \min_{u \in N_{\alpha^k}(v)} \left\{ \alpha^k(P) : P \text{ is a } u-w \text{ path in } J_w \right\}
\]

respectively. The lower bound $\mathcal{L}^k(vw)$ is the smallest integer $i \geq 0$ such that the greatest simultaneous fixed point $\lambda_{i,u}^k$ assigns a non-top label to an out-neighbor of $v$ in $J_w$. On the other hand, the upper bound $\mathcal{E}^k(vw)$ is the minimum $k$th-width of a path from an out-neighbor of $v$ to $w$ in $J_w$. Note that these quantities could be equal to $+\infty$.

**Lemma 4.9.** For every arc $vw \in E(D_\tau)$, we have $\mathcal{L}^k(vw) \leq \mathcal{E}^k(vw)$.

**Proof.** We may assume that $\mathcal{E}^k(vw) < \infty$. Let $P$ be a $u-w$ path in $J_w$ for some $u \in N_{\alpha^k}(v)$ such that $\alpha^k(P) = \mathcal{E}^k(vw)$. From the definition of $k$th-width, there exist node labels $\nu : V(P) \to L(T_{\alpha^k}(P), j)$ which are feasible in $P$. Recall that every node in $P$ receives a non-top label from $\nu$. Without loss of generality, we may assume that $\nu(w) = \min L(T_{\alpha^k}(P), j)$. Extend $\nu$ to the subgraph $J_w$ by assigning $\top$ to the nodes outside $P$. Then, $\nu \geq \lambda_{i,u}^k(v,w)$ because $\nu(w) = \lambda_{i,u}^k(v,w)(w)$ and there are no loose arcs in $P$ with respect to $\lambda_{i,u}^k(v,w)$. Since $\nu(u) \neq \top$, it follows that $\alpha^k(P) \geq \mathcal{L}^k(vw)$.

The next lemma relates the finiteness of greatest simultaneous fixed points of $J_w$ if comparable trees are used as codomains.

**Lemma 4.10.** For every node $u \in V(J_w)$, if $\lambda_{i,u}^k(u) \neq \top$, then $\lambda_{i',u}^k(u) \neq \top$ for all $i' \geq i$.

**Proof.** For convenience of notation, let $j = \pi(w)/2$. Fix $i' \geq i$. Define node labels $\nu_i^{(0)} : V(J_w) \to \bar{L}(T_{i,j})$ and $\nu_i^{(0)} : V(J_w) \to \bar{L}(T_{i,j})$ by $\nu_i^{(0)}(w) := \min L(T_{i,j}), \nu_i^{(0)}(u) := \nu_i^{(0)}(u) = \top$ for all $u \neq w$. We know that $\lambda_{i,u}^k$ can be obtained by applying a sequence $R$ of Drop operators from $J_w$ to $\nu_i^{(0)}$. For each $0 \leq \ell \leq |R|$, let $R_\ell$ be the subsequence of $R$ which consists of the first $\ell$ elements. Let $\nu_i^{(\ell)}$ and $\nu_i^{(\ell)}$ be the node labels obtained by applying $R_\ell$ to $\nu_i^{(0)}$ and $\nu_i^{(0)}$ respectively. Since $T_{i,j} \subseteq T_{i,j}$, there exists an injective and order-preserving homomorphism $f : V(T_{i,j}) \to V(T_{i,j})$. Extend $f$ by setting $f(\top) := \top$. As $\lambda_{i,u}^k \leq \nu_i^{(|R|)}$, it suffices to show that $\nu_i^{(\ell)}(u) \leq f(\nu_i^{(\ell)}(u))$ for all $u \in V(J_w)$ and $0 \leq \ell \leq |R|$. We proceed by induction on $\ell$. The base case $\ell = 0$ is true by construction. Suppose that the inductive hypothesis is true for some $\ell \geq 0$, and let $\text{Drop}_{uw}(\cdot, \cdot)$ be the $(\ell + 1)$-th element in $R$. It suffices to compare the label at node $u$. Since $f$ is order-preserving and Drop is monotone, we obtain $\nu_i^{(\ell+1)}(u) \leq f(\nu_i^{(\ell+1)}(u))$ as required.

For a cycle $C$ in $D_\tau$, its (bottleneck) $c^k$-cost is defined as $c^k(C) := \max_{e \in E(C)} c^k(e)$. We remark that self-loops in $D_\tau$ are considered cycles. The next theorem allows us to obtain the desired node labels $\nu_i^k$ by computing minimum bottleneck cycles in $D_\tau$.

**Theorem 4.11.** Let $(G, \pi)$ be a one-player parity game for Even and $D_\tau$ be its auxiliary graph. Given node labels $\mu : V \to \bar{L}(T)$ with no loose arc in $G_\tau$, let $C$ be a cover of $T$. Fix a base node $w \in B(G_\tau)$, and let $c^k$ be arc costs in $D_\tau(w)$ such that $\mathcal{L}^k \leq c^k \leq \mathcal{E}^k$ for all $0 \leq k < |C_{\pi(w)}/2|$. For each $k$, let $i^k$ be the minimum $c^k$-cost of a cycle containing $w$ in $D_\tau$, and let $\xi_k$ be the label returned by $\text{RAISE}((\mu, w, k, i^k))$. Then,

$$\mu^*(w) \leq \min_k \xi_k \leq \hat{\mu}(w).$$
Proof. For ease of notation, let \( j = \pi(w)/2 \). We first prove the lower bound \( \mu^*(w) \leq \xi^k \) for all \( 0 \leq k < |C_j| \). Fix a \( k \) and assume that \( \xi^k \neq T \). Let \( C \) be a minimum \( c^k \)-cost cycle in \( D_T \) containing \( w \). Denote \( C = (w_1, w_2, \ldots, w_\ell) \) where \( w = w_1 = w_\ell \). For every \( s \in [\ell] \), let \( \nu_s : V \to \tilde{L}(T) \) be the node labels defined by \( \nu_s(w_s) := \xi^k \) and \( \nu_s(v) := T \) for all \( v \neq w_s \). Then, consider the greatest simultaneous fixed point \( \nu_s^{\mathcal{J}_w} \), which can be obtained by applying the Drop operators in \( \mathcal{J}_w \) to \( \nu_s \) until convergence. Since \( \pi(J_{w_s}) = \pi(w_s) = 2j \), we have \( \nu_s^{\mathcal{J}_w}(w_s) = \nu_s(w_s) = \xi^k \) because \( \xi^k \) is the smallest leaf in the subtree of \( T \) rooted at \( \xi^k \). This also implies that \( \nu_s^{\mathcal{J}_w}(v) \geq \xi^k \) for all \( v \in V \).

Furthermore, \( \nu_s^{\mathcal{J}_w} \) is feasible in \( G_\tau \setminus \delta^+(w_s), \) as \( \nu_s \) is feasible in \( G_\tau \setminus \delta^+(w_s) \).

**Claim 1.** For each \( 1 < s \leq \ell \), there exists a node \( u_s \in N^+_j(w_{s-1}) \) such that \( \nu_s^{\mathcal{J}_w}(u_s)|_{2j} = \xi^k|_{2j} \).

Proof. Fix an \( s \) and let \( i = \xi^k(w_{s-1}w_s). \) From the definition of \( \xi^k \) in (3), there exists a node \( u_s \in N^+_j(w_{s-1}) \) such that \( \lambda^k_{i,w_s}(u_s) \neq T \). Recall that \( \lambda^k_{i,w_s}(u_s) : V(J_{w_s}) \to \tilde{L}(T_{i,j}) \) is the greatest simultaneous fixed point of \( \mathcal{J}_{w_s} \) subject to \( \lambda^k_{i,w_s}(w_s) = \min(\tilde{L}(T_{i,j})) \). We will show that \( \nu_s^{\mathcal{J}_w}(u_s)|_{2j} = \xi^k|_{2j} \). First, observe that \( \nu_s^{\mathcal{J}_w}_s : V \to \tilde{L}(T) \) is the greatest simultaneous fixed point of \( \mathcal{J}_{w_s} \) subject to \( \lambda^k_{i,w_s}(u_s) = \xi^k \). Since \( \xi^k \) is returned by \( RAISE(\cdot, k, \ell) \), it is the smallest leaf in a copy of \( T^k_{i,j} \) in the main tree \( T \) for some \( i' \geq i^k \). As \( \xi^k(C) = i^k \) and \( w_{s-1}w_s \in E(C) \), we also have \( i \geq i^k \). Observing that \( w_{s-1}w_{s-1} = i^k \) and \( w_{s-1}w_s = i^k \), we know that the last inequality is due to our choice of the arc costs \( c^k \). It follows that \( u' \geq i \) and \( T_{i,j} \subseteq T_{i',j} \). Thus, we obtain \( \nu_s^{\mathcal{J}_w}(u_s)|_{2j} = \xi^k|_{2j} \) by Lemma 4.10 because \( \lambda^k_{i,w_s}(u_s) \neq T \).

Now, consider the node labels \( \nu \) defined by \( \nu(v) := \min_{s \in [\ell]} \nu_s^{\mathcal{J}_w}(v) \) for all \( v \in V \). Note that \( \nu(w_s) = \xi^k \) for all \( s \in [\ell] \). It suffices to show that \( \nu \) is feasible in \( G_\tau \) and \( \nu \geq \mu \). This is because it would then imply \( \nu \geq \mu^* \) by the pointwise minimality of \( \mu^* \), in particular, \( \xi^k = \nu(w) \geq \mu^*(w) \).

We first prove that \( \nu \) is feasible in \( G_\tau \). Notice that \( \nu \) is feasible in \( G_\tau \setminus \bigcup_{s=1}^{\ell} \delta^+(w_s) \) because \( \nu_s^{\mathcal{J}_w} \) is feasible in \( G_\tau \setminus \delta^+(w_s) \) for all \( s \in [\ell] \). So, it suffices to show that every \( w_s \) has a non-violated outgoing arc in \( G_\tau \) with respect to \( \nu \). Fix an \( 1 < s \leq \ell \). By Claim 1, there exists a node \( u_s \in N^+_j(w_{s-1}) \) such that \( \nu_s^{\mathcal{J}_w}(u_s)|_{2j} = \xi^k|_{2j} \). Hence, \( \nu(u_s)|_{2j} = \xi^k|_{2j} \). As \( \nu(w_{s-1}) = \xi^k \) and \( \pi(w_{s-1}) = 2j \), the arc \( w_{s-1}u_s \) is non-violated with respect to \( \nu \).

Next, we prove that \( \nu_s^{\mathcal{J}_w} \geq \mu \) for all \( s \in [\ell] \), which will then imply \( \nu \geq \mu \) as desired. We proceed by induction on \( s \). For the base case \( s = \ell \), we know that \( \nu^{\mathcal{J}_w}_\ell \) is the greatest simultaneous fixed point of \( \mathcal{J}_{w_\ell} \) subject to \( \nu^{\mathcal{J}_w}_\ell(w_\ell) = \xi^k \). Observe that \( \mu \) is also a simultaneous fixed point of \( \mathcal{J}_{w_\ell} \) because there are no loose arcs in \( G_\tau \) with respect to \( \mu \). As \( \nu^{\mathcal{J}_w}_\ell(w_\ell) = \xi^k \geq \mu(w_\ell) \) due to \( w_\ell = w \), we obtain \( \nu^{\mathcal{J}_w}_\ell \geq \mu \). For the inductive step, suppose that \( \nu_s^{\mathcal{J}_w} \geq \mu \) for some \( 1 < s \leq \ell \). By Claim 1, there exists a node \( u_s \in N^+_j(w_{s-1}) \) such that \( \nu_s^{\mathcal{J}_w}(u_s)|_{2j} = \xi^k|_{2j} \). Then, \( \mu(w_{s-1}) \leq \nu_s^{\mathcal{J}_w}(w_{s-1}) = \xi^k \), where equality follows from the tightness of \( w_{s-1}u_s \) with respect to \( \nu_s^{\mathcal{J}_w} \). Since \( \nu_s^{\mathcal{J}_w}(w_{s-1}) \) is the greatest simultaneous fixed point of \( \mathcal{J}_{w_{s-1}} \) subject to \( \nu_s^{\mathcal{J}_w}(w_{s-1}) = \xi^k \geq \mu(w_{s-1}) \), we get \( \nu_s^{\mathcal{J}_w}(w_{s-1}) \geq \mu \) because \( \mu \) is also a simultaneous fixed point of \( \mathcal{J}_{w_{s-1}} \).

It is left to show that \( \xi^k \leq \mu(w) \) for some \( 0 \leq k < |C_j| \). We may assume that \( \widehat{\mu}(w) \neq T \). By Lemma 4.8, \( \widehat{\mu}(w) \) is returned by \( RAISE(\mu, w, k, \alpha^k(w)) \) for some \( 0 \leq k < |C_j| \). Let \( H \) be a cycle in \( G_\tau \) such that \( w \in \Pi(H) \) and \( \alpha^k(H) = \alpha^k(w) \). We denote \( \Pi(H) = \{w_1, w_2, \ldots, w_\ell\} \) and decompose \( H \) into arc-disjoint paths \( P_1, P_2, \ldots, P_\ell \) such that \( P_s \) is a \( w_{s-1}w_s \) path for all \( s \in [\ell] \), where \( w_0 := w_\tau \). Since each \( P_s \) lies in the subgraph \( J_{w_s} \), we have \( w_{s-1}w_s \in E(D_T) \) for all \( s \in [\ell] \),
and their union induces a cycle \( H' \) containing \( w \) in \( D_\tau \). Then,

\[
i^k \leq c^k(H') = \max_{s \in [r]} c^k(w_{s-1}w_s) \leq \max_{s \in [r]} \bar{c}^k(w_{s-1}w_s) \leq \max_{s \in [r]} \alpha^k(P_s) = \alpha^k(H) = \alpha^k(w).
\]

The third inequality follows from the definition of \( \bar{c}^k \), while the second equality is due to Lemma 4.6.

It follows that \( c^k \leq \hat{\mu}(w) \) because \( \text{RAISE} \) is monotone with respect to its fourth argument. \( \square \)

### 4.3 Cramer Computation

The overall implementation of Cramer Computation is given in Algorithm 4, which we call CramerLabels as it is a rather generic label-correcting algorithm with labels from an ordered tree. Throughout the algorithm, we maintain node labels \( \nu : V \rightarrow \bar{L}(T) \), which are initialized to \( \nu(v) := \top \) for all \( v \in V \). First, construct the auxiliary digraph \( D_\tau \). Next, for every component \( H \) in \( D_\tau \), pick a node \( w \in V(H) \) and let \( j = \pi(w)/2 \). For each \( 0 \leq k < |C_j| \), assign arc costs \( c^k \) to \( H \) which satisfy \( \underline{c}^k \leq c^k \leq \bar{c}^k \). Compute the minimum \( c^k \)-cost of a cycle containing \( v \) in \( H \) for all \( v \in V(H) \). Then, these cycle costs are used to raise \( \mu(v) \) and update \( \nu(v) \) for all \( v \in \Pi(K_w) = V(H) \). Note that only the smallest \( \nu \) labels are kept. Finally, Algorithm 3 is run on \( G_\tau \) with input node labels \( \nu \).

#### Algorithm 4 Cramer computation: \( (G_\tau, \pi) \) one-player parity game for Even, \( \mu : V \rightarrow \bar{L}(T) \) node labels from an ordered tree \( T \) with no loose arc in \( G_\tau \), \( L \) cover of \( T \).

```plaintext
1: procedure CramerLabels\(((G_\tau, \pi), \mu)\)
2: \( \nu(v) \leftarrow \top \) for all \( v \in V \)
3: Construct auxiliary graph \( D_\tau \)
4: for all components \( H \) in \( D_\tau \) do
5: \( j \leftarrow \pi(w)/2 \) for any \( w \in V(H) \)
6: for \( k = 0 \) to \( |C_j| - 1 \) do
7: Assign arc costs \( c^k \) to \( H \) where \( \underline{c}^k \leq c^k \leq \bar{c}^k \)
8: for all \( v \in V(H) \) do
9: \( i^k \leftarrow \) minimum \( c^k \)-cost of a cycle containing \( v \) in \( H \)
10: \( \nu(v) \leftarrow \min(\nu(v), \text{RAISE}(\mu, v, i^k)) \)
11: \( \nu \leftarrow \text{BELLMANFORD}((G_\tau, \pi), \nu) \)
12: return \( \nu \)
```

In the next two paragraphs, we elaborate on how the arc costs \( c^k \) and the minimum \( c^k \)-cost cycles are computed.

#### Computing arc costs

Pick a node \( w \in V(D_\tau) \) and let \( j = \pi(w)/2 \). If the chain \( C_j^k \) is short, a straightforward way to compute arc costs \( c^k \) for the incoming arcs \( \delta^-_{D_\tau}(w) \) is by running Algorithm 3 on the subgraph \( J_w \) for \( \lceil C_j^k \rceil \) times. In each run \( 0 \leq i < |C_j^k| \), we supply the algorithm with input node labels \( \nu_i : V(J_w) \rightarrow \bar{L}(T)^{i_j^k} \) defined by \( \nu_i(v) := \min L(T_{i_j^k}) \) and \( \nu_i(u) := \top \) for all \( u \neq w \). Let \( \nu'_i \) be the returned node labels. Clearly, \( \nu'_i \geq \lambda^k_{i,w} \). Moreover, for each node \( u \in V(J_w) \), if \( \nu'_i(u) = \top \), then \( \alpha^k(P) > i \) for every \( u-w \) path \( P \) in \( J_w \). This is due to our choice of \( \nu_i \), and the fact that Algorithm 3 ran for \( n - 1 \) iterations. Hence, the cost of each arc \( vw \in \delta^-_{D_\tau}(w) \) can be set as \( c^k(vw) := \min_{u \in N^-_{J_w}(v)} \{ i : \nu'_i(u) \neq \top \} \). Since \( \lambda^k_{c^k(vw),u}(u) \leq \nu'_i(vw)(u) \neq \top \) for some \( u \in N^-_{J_w}(v) \), we have \( c^k(vw) \geq \underline{c}^k(vw) \). We also have \( c^k(vw) \leq \bar{c}^k(vw) \) because \( \nu'_i(u) = \top \) for all \( i < c^k(vw) \) and \( u \in N^+_J(v) \).
If the chain \( C_j^k \) is long, then we incorporate the above approach in a binary search framework. Start by selecting the middle tree \( T_{i,j}^k \) in the chain, i.e. \( i = \lfloor |C_j^k|/2 \rfloor \). Run Algorithm 3 on the subgraph \( J_w \) with input node labels \( \nu_i : V(J_w) \to L(T_{i,j}^k) \) as defined above, and let \( \nu'_i \) be the returned node labels. For each node \( u \in V(J_w) \), if \( \nu'_i(u) = \top \), then we restrict ourselves to the trees in the chain which are bigger than \( T_{i,j}^k \). Otherwise, we disregard these trees. Then, the process is repeated until we locate the optimal tree. Hence, the time taken to compute \( c^k \) for the whole graph \( D_\tau \) is \( O(mn^2\theta(T) \cdot \max_j \min(|C_j^k|, n \log |C_j^k|)) \).

Computing minimum bottleneck cycles It is well-known how to compute minimum bottleneck directed cycles in a digraph with arbitrary arc costs. For the sake of completeness, we give a brief description. Let \( H \) be a component in \( D_\tau \) and \( c^k < c^k_2 < \cdots < c^k_j \) be the distinct arc costs in \( H \). Consider the subgraph of \( H \) induced by arcs with cost at most \( c^k_{\lceil \ell/2 \rceil} \), and let \( K_1, K_2, \ldots, K_t \) be its strongly connected components. For each \( i \in [t] \), if \( |E(K_i)| > 0 \), then every node in \( K_i \) has a cycle going through it with cost at most \( c^k_{\lceil \ell/2 \rceil} \). Otherwise, \( K_i \) is a singleton with no self-loops. Hence, every cycle going through it has cost greater than \( c^k_{\lceil \ell/2 \rceil} \). This observation allows us to split the instance into two. The first instance is given by the disjoint union of \( K_1, K_2, \ldots, K_t \). The second instance is obtained by contracting each \( K_i \) into a node in \( H \), destroying any self-loops on the resulting node. Then, the procedure is repeated on these two smaller instances. Since \( D_\tau \) contains \( O(n^2) \) arcs, and each arc appears in at most \( O(\log \ell) = O(\log n) \) instances, the total running time over all the components in \( D_\tau \) is \( O(n^2 \log n) \).

We are ready to prove a generic bound on the running time of Algorithm 4 for an arbitrary ordered tree \( T \) with an arbitrary cover \( C \).

**Theorem 4.12.** Algorithm 4 returns \( \mu^* \) in \( O(\max_j k |C_j^k| + mn^2 \theta(T) \cdot \min(|C_j^k|, n \log |C_j^k|)) \) time.

**Proof.** Correctness follows immediately from Theorem 4.11 and Theorem 4.3. In terms of running time, identifying the base nodes \( B(G_\tau) \) takes \( O(dm) \) time, while constructing the auxiliary digraph \( D_\tau \) takes \( O(mn) \) time. Next, for each \( 0 \leq k < \max_j |C_j^k| \), computing the arc costs \( c^k \) for \( D_\tau \) takes \( O(mn^2 \theta(T) \cdot \max_j \min(|C_j^k|, n \log |C_j^k|)) \). Then, finding minimum \( c^k \)-cost cycles takes \( O(n^2 \log n) \) time, while raising the labels on \( B(G_\tau) \) takes \( O(n \rho(T, C)) \) time. Finally, Bellman–Ford runs in \( O(mn \theta(T)) \) time. \( \Box \)

### 4.4 Quasi-polynomial Universal Trees

In this section, we apply Algorithm 4 to the quasi-polynomial universal trees constructed in the literature, namely succinct universal trees and succinct Strahler universal trees. The vertices in these trees are encoded using tuples of binary strings.

**Definition 4.13.** Given a tuple \( \xi = (\xi_{2h-1}, \xi_{2h-3}, \ldots, \xi_1) \) of binary strings, denote \( \zeta(\xi) \) as the number of leading zeroes in \( \xi_{2h-1} \). We also define \( \zeta(\top) := -1 \).

For a pair of tuples \( \xi, \xi' \) of binary strings, note that if \( \xi \geq \xi' \), then \( \zeta(\xi) \leq \zeta(\xi') \) by the lexicographic order on tuples. For convenience, we introduce the following operation on these tuples.

**Definition 4.14.** Given a tuple \( \xi = (\xi_{2h-1}, \xi_{2h-3}, \ldots, \xi_1) \) of binary strings and an integer \( \kappa \geq 0 \), let \( \xi^\kappa \) be the tuple obtained by deleting \( \kappa \) leading zeroes from \( \xi_{2h-1} \). If \( \kappa > \zeta(\xi) \), then \( \xi^\kappa := \top \). We also define \( \top^\kappa := \top \).
4.4.1 Succinct Universal Trees

Let \( T \) be a succinct \((n, h)\)-universal tree. Recall that every leaf \( \xi \in L(T) \) corresponds to an \( h \)-tuple of binary strings where \(|\xi| \leq \lfloor \log n \rfloor\). First, we show that each \( T_j \) has a cover of size 1. Equivalently, each \((T_j, \subseteq)\) is a chain.

**Lemma 4.15.** There exists a cover \( \mathcal{C} \) of \( T \) such that \(|\mathcal{C}_j| = 1\) for all \( 0 \leq j \leq h \).

**Proof.** Fix \( 0 \leq j \leq h \) and pick two vertices \( r_1, r_2 \in V(T) \) at depth \( h - j \). Let \( T_1 \) and \( T_2 \) be the subtrees of \( T \) rooted at \( r_1 \) and \( r_2 \) respectively. Every leaf \( \xi_1 \in L(T_1) \) and \( \xi_2 \in L(T_2) \) corresponds to a \( j \)-tuple of binary strings where \(|\xi_1| \leq |\log n| - |r_1|\) and \(|\xi_2| \leq |\log n| - |r_2|\). Without loss of generality, assume that \(|r_1| \geq |r_2|\). Then, the identity map from \( V(T_1) \) to \( V(T_2) \) is an order-preserving and injective homomorphism. Hence, \( T_1 \subseteq T_2 \).

Since each subcover \( \mathcal{C}_j \) of \( \mathcal{C} \) consists of a single chain, we write \( \mathcal{C}_j = T_j \) and omit the superscript \( k \). The subtrees in \( T_j \) are \( T_{0,j} \subseteq T_{1,j} \subseteq \cdots \subseteq T_{\lfloor \log n \rfloor,j} \). Observe that every leaf \( \xi \in L(T_{i,j}) \) corresponds to a \( j \)-tuple of binary strings where \(|\xi| \leq i\).

Next, we give an efficient implementation of the \textsc{Raise} subroutine.

**Lemma 4.16.** For a succinct \((n, d/2)\)-universal tree \( T \) with cover \( \mathcal{C} = (T_0, T_1, \ldots, T_{d/2}) \), the \textsc{Raise}(\( \mu, v, k, i \)) subroutine runs in \( O(\log n \log d) \) time.

**Proof.** Without loss of generality, we may assume that \( \pi(v) \) is even. We may also assume that \( \mu(v) \) is the smallest leaf in the subtree rooted at \( \mu(v)|_{\pi(v)} \). Otherwise, we can set it to the smallest leaf of the next subtree rooted at that depth using \textsc{Tighten}. Recall that the \textsc{Tighten} subroutine for \( T \) also runs in \( O(\log n \log d) \) time. It follows that \( \mu(v)|_{\pi(v)-1} \in \{0 \cdots 0, \varepsilon\} \) and \( \mu(v)_j = \varepsilon \) for all \( j < \pi(v) - 1 \). If \( |\mu(v)|_{\pi(v)-1} | \geq i \), then we simply return \( \mu(v) \). Otherwise, let \( p \in [d] \) be the smallest even integer such that \( \mu(v)_p \) has a child bigger than \( \mu(v)|_{p-1} \) with at most \( |\log n| - i \) bits. If \( p \) does not exist, then we return \( T \). Otherwise, \( p > \pi(v) \). Let \( r = |\log n| - i - |\mu(v)|_{p-1} \). There are two cases.

- **Case 1:** \( r > 0 \). Return \[
(\mu(v)_{d-1}, \ldots, \mu(v)_{p-1}10\cdots0, \varepsilon, \ldots, \varepsilon, 0\cdots0, \varepsilon, \ldots, \varepsilon)
\]

where the string of \( i \) zeroes is at index \( \pi(v) - 1 \).

- **Case 2:** \( r \leq 0 \). Denote \( \mu(v)_p = b_1b_2\cdots b_\ell \) where \( b_j \in \{0, 1\} \) for all \( j \in [\ell] \). Note that \( \ell \geq 1 \) by our choice of \( p \). Furthermore, there exists a largest \( k \in [\ell] \) such that \( b_k = 0 \) and \( r' := r + \ell - k + 1 \geq 0 \). Then,
  - If \( p = \pi(v) + 2 \), return \[
(\mu(v)_{d-1}, \ldots, \mu(v)_{p+1}, b_1 \cdots b_{k-1}, 0\cdots0, \varepsilon, \ldots, \varepsilon)
\]
  - If \( p > \pi(v) + 2 \), return \[
(\mu(v)_{d-1}, \ldots, \mu(v)_{p+1}, b_1 \cdots b_{k-1}, 0\cdots0, \varepsilon, \ldots, \varepsilon, 0\cdots0, \varepsilon, \ldots, \varepsilon)
\]

where the string of \( i \) zeroes is at index \( \pi(v) - 1 \).
Due to the structure of succinct universal trees, the running time of Algorithm 4 given in Theorem 4.12 can be improved. The following lemma yields a faster method for computing arc costs for the auxiliary digraph $D_T$. The key observation is that for any pair of trees in a chain $T_j$, the smaller tree can be obtained from the larger tree by deleting vertices in decreasing lexicographic order. For example, a succinct $(3,2)$-universal tree can be obtained from a succinct $(7,2)$-universal tree by deleting vertices whose first component does not contain a leading zero (compare Figures 2 and 4).

![Diagram of a succinct (7,2)-universal tree]

**Lemma 4.17.** Given integers $0 \leq i_1 \leq i_2$ and $j \geq 0$, let $\nu_1 : V \rightarrow \bar{L}(T_{i_1,j})$ and $\nu_2 : V \rightarrow \bar{L}(T_{i_2,j})$ be node labels such that $\nu_1(u) = \nu_2(u)^{i_2-i_1}$ for all $u \in V$. For any arc $vw \in E_T$ where $\pi(v) < 2^j$, we have $\text{drop}(\nu_1, vw) = \text{drop}(\nu_2, vw)^{i_2-i_1}$.

**Proof.** Let $\xi_1 = \text{drop}(\nu_1, vw)$ and $\xi_2 = \text{drop}(\nu_2, vw)$. First, assume that $vw$ is violated with respect to $\nu_1$. Then, $\nu_1(v) \neq \top$, which implies that $\nu_2(v) \neq \top$. In particular, $\zeta(\nu_2(v)) \geq i_2 - i_1$. We claim that $vw$ is also violated with respect to $\nu_2$. This is clear if $\nu_1(v) = \top$. If $\nu_1(v) = \top$, then $\zeta(\nu_2(v)) < i_2 - i_1$. As $\nu_2(v)[2j-1] < \nu_2(w)[2j-1]$ and $\pi(v) < 2^j$, the arc $vw$ is indeed violated with respect to $\nu_2$. Hence, $\xi_1 = \nu_1(v)$ and $\xi_2 = \nu_2(v)$.

Next, assume that $vw$ is not violated with respect to $\nu_1$. If $\nu_1(v) \neq \top$, then $vw$ is also not violated with respect to $\nu_2$. It is easy to verify that $\xi_1 = \xi_2^{i_2-i_1}$. On the other hand, if $\nu_1(v) = \top$, then $\nu_1(v) = \top$. So, we have $\zeta(\nu_2(v)) < i_2 - i_1$ and $\zeta(\nu_2(w)) < i_2 - i_1$. Since $\pi(v) < 2^j$, we also have $\zeta(\xi_2) < i_2 - i_1$. Thus, $\xi_1 = \top = \xi_2^{i_2-i_1}$ as required.

Pick a node $v \in V(D_T)$ and let $j = \pi(v)/2$. For each $0 \leq i \leq \lceil \log n \rceil$, let $\nu_i : V(J_w) \rightarrow \bar{L}(T_{i,j})$ be the node labels defined by $\nu_i(u) := \min L(T_{i,j})$ and $\nu_i(u) := \top$ for all $u \neq w$. To compute the cost of incoming arcs $\delta_{D_{\nu}}(w)$, we only need to run Algorithm 3 on $J_w$ once, with input node labels $\nu_{\lceil \log n \rceil}$. Let $\nu$ be the returned node labels. Note that $\nu(u) \neq \top$ for all $u \in V(J_w)$ because $T_{\lceil \log n \rceil,j}$ is an $(n,j)$-universal tree. Without loss of generality, we may assume that Algorithm 3 is run on the subgraph $J_w \setminus \delta^{\top}(w)$ instead, as $\nu(w) = \nu_{\lceil \log n \rceil}(w)$. Then, by Lemma 4.17, for any $0 \leq i \leq \lceil \log n \rceil$, $\nu_{\lceil \log n \rceil - i}$ are the node labels returned by Bellman–Ford on $J_w$ with input labels $\nu_i$. Hence, the cost of each arc $vw \in \delta_{D_{\nu}}(w)$ is set as $c(vw) := \lfloor \log n \rfloor - \max_{u \in N_j^w(v)} \zeta(\nu(u))$.

We are ready to prove the running time of Algorithm 4 for succinct universal trees.

**Theorem 4.18.** For a succinct $(n, d/2)$-universal tree $T$ with cover $C = (T_0, T_1, \ldots, T_{d/2})$, Algorithm 4 runs in $O(mn^2 \log n \log d)$ time.
4.15, we have \(|C_j| = 1\) for all \(0 \leq j \leq h\). Computing arc costs for the auxiliary digraph \(D_r\) takes \(O(mn^2\theta(T))\) time. Hence, the running time of Algorithm 4 becomes \(O(n\rho(T,C) + mn^2\theta(T))\). The result then follows from \(\theta(T) = O(\log n \log d) = \rho(T,C)\), where the latter equality is due to Lemma 4.16.

\[\square\]

4.4.2 Succinct Strahler Universal Trees

In this subsection, we demonstrate the Cramer computation for succinct Strahler universal trees. Let us start by introducing the necessary definitions. The Strahler number of a rooted tree \(T\) is the largest height of a perfect binary tree that is a minor of \(T\). For example, a perfect \((\ell,h)\)-universal tree has Strahler number 0 if \(\ell = 1\), and \(h\) otherwise.

**Definition 4.19.** A \(g\)-Strahler \((\ell,h)\)-universal tree is an ordered tree \(T'\) such that \(T \subseteq T'\) for every ordered tree \(T\) of Strahler number at most \(g\), height at most \(h\), and with at most \(\ell\) leaves.

In the definition above, we may assume that \(g \leq \min(h, [\log \ell])\). Daviaud et al. \[9\] constructed a \(g\)-Strahler \((\ell,h)\)-universal tree with \(f^{O(1)}(h/g)^g\) leaves. Note that this is quasipolynomial in \(\ell\) and \(h\) by our previous remark. In this paper, we refer to it as a succinct \(g\)-Strahler \((\ell,h)\)-universal tree.

Every leaf \(\xi = (\xi_{2h-1}, \xi_{2h-2}, \ldots, \xi_1)\) in this tree corresponds to an \(h\)-tuple of binary strings which satisfies the following three properties:

1. There are \(g\) nonempty bit strings, i.e., \(\{|i : \xi_i \neq \varepsilon|\} = g\);
2. The total number of bits \(|\xi|\) is at most \(g + [\log \ell]\);
3. For each odd \(i \in [2h],\)
   
   (a) If there are \(f < g\) nonempty bit strings in \(\xi_i\) and \(|\xi_i| = f + [\log \ell]\), then \(\xi_i = 0\).
   
   (b) If \(x_j \neq \varepsilon\) for all odd \(j \in [i]\), then \(x_j\) starts with 0 for all odd \(j \in [i]\).

This is the construction of \(B^g_{\ell,h}\) in \[9, Definition 19\]. In each string \(\xi_i\), the first bit is called the leading bit, while the remaining bits are the non-leading bits. Properties 1 and 2 imply that \(\xi\) contains exactly \(g\) leading bits and at most \([\log \ell]\) non-leading bits. Observe that if \(g = h\), then the tree is identical to a succinct \((\ell,h)\)-universal tree. Indeed, one can arrive at the encoding of Jurdziński and Lazić \[25\] by removing the leading zero in every string.

Let \(T\) be a succinct \(g\)-Strahler \((\ell,h)\)-universal tree. Let \(v \in V(T)\) be a vertex at depth \(h - j\) for some \(0 \leq j \leq h\). If \(v\) has \(k\) nonempty strings and \(i\) non-leading bits, then the subtree rooted at \(v\) is a succinct \((g-k)\)-Strahler \((2^{|\log \ell|-i}, j)\)-universal tree. Note that if \(i = [\log \ell]\), then the subtree is a path. This fact is actually independent of \(k\). Indeed, varying \(k\) only yields different encodings of the same path.

The crucial fact for obtaining a small cover is that the subtrees of \(T\) with fixed height \(j\) and fixed Strahler number \(k\) form a chain. This leads to the following statement.

**Lemma 4.20.** There exists a cover \(C\) of \(T\) such that \(|C_j| \leq g\) for all \(0 \leq j \leq h\).

**Proof.** Fix a \(0 \leq j \leq h\). For each \(0 \leq k \leq g\), let \(C^k_j\) be the set of \(k\)-Strahler \((.,j)\)-universal trees in \(T_j\). Then, \(\cup_k C^k_j = T_j\). Note that \(C^k_j \neq \emptyset\) if and only if \(\max(0, g-j) \leq k \leq \min(j,g)\). It is left to show that \((C^k_j, \subseteq)\) is a chain for all \(k\). Fix a \(k\) and pick two vertices \(r_1, r_2 \in V(T)\) at depth \(h - j\) such that they each have \(g-k\) nonempty bit strings. Let \(T_1\) and \(T_2\) be the subtrees of \(T\) rooted at \(r_1\) and \(r_2\) respectively. Observe that \(T_1\) is a succinct \(k\)-Strahler \((|\log n| - |r_1| + g-k, j)\)-universal tree. Similarly, \(T_2\) is a succinct \(k\)-Strahler \((|\log n| - |r_2| + g-k, j)\)-universal tree. Without loss of generality, assume that \(|r_1| \geq |r_2|\). Then, the identity map from \(V(T_1)\) to \(V(T_2)\) is an order-preserving and injective homomorphism. Hence, \(T_1 \subseteq T_2\). \(\square\)
Let $C$ be the cover given in the proof of Lemma 4.20, i.e. $C_j = (C_j^0, C_j^1, \ldots, C_j^h)$ for all $0 \leq j \leq h$. Note that $T_{0,j}^k = T_{0,j}^{k'}$ for all $k, k'$ where $C_j^k, C_j^{k'} \neq \emptyset$. The next lemma shows that the RAISE subroutine can be implemented efficiently using this cover.

**Lemma 4.21.** For a succinct $g$-Strahler $(n, d/2)$-universal tree $T$ with cover $C$, the RAISE($\mu, v, k, i$) subroutine runs in $O(\log n \log d)$ time.

**Proof.** Without loss of generality, we may assume that $\pi(v)$ is even. We may also assume that $\mu(v)$ is the smallest leaf in the subtree of $T$ rooted at $\mu(v)|_{\pi(v)}$. Otherwise, we can set it to the smallest leaf of the next subtree rooted at that depth using TIGHTEN. Recall that the TIGHTEN subroutine for $T$ also runs in $O(\log n \log d)$ time. It follows that $\mu(v)|_{\pi(v)} \in \{0 \cdots 0, \varepsilon\}$ and $\mu(v)_{j} \in \{0, \varepsilon\}$ for all $j < \pi(v) - 1$. If $i = 0$, then we output $\mu(v)$ because $T_{0,\pi(v)/2}^k$ is identical for all $0 \leq k < |C_{\pi(w)/2}|$. So, let us assume that $i > 0$ from now on.

If there are at least $i$ non-leading bits in $\mu(v)|_{\pi(v) - 1}$ and exactly $k$ non-empty strings among $\mu(v)|_{\pi(v) - 1}, \ldots, \mu(v)_1$, then we simply return $\mu(v)$. Otherwise, let $p \in [d]$ be the smallest even integer such that $\mu(v)|_p$ has a child bigger than $\mu(v)|_{p - 1}$ with at most $\lceil \log n \rceil - i$ non-leading bits and exactly $x$ non-empty strings for some $g - k - p - \pi(v)/2 + 1 \leq x \leq g - k$. If $p$ does not exist, then we return $T$. Otherwise, $p > \pi(v)$.

Our goal is to increase $\mu(v)$ minimally such that it can accommodate a label from $T_{v', \pi(v)/2}^k$ for some $i' \geq i$, using only the components after $\mu(v)|_p$. Let $z$ be the number of non-empty strings in $\mu(v)|_{p - 1}$ and let $y$ be the number of non-leading bits in $\mu(v)|_{p - 1}$. We set $s = g - k - z$, representing the discrepancy on the number of non-empty bit strings (Property 1). We also set $r = \lceil \log n \rceil - i - y$, representing the discrepancy on the number of non-leading bits (Property 2). We split the remaining analysis into cases based on the emptiness of $\mu(v)|_{p - 1}$ and the signs of $r, s$.

**Case 1: $\mu(v)|_{p - 1} = \varepsilon$.** Note that $y = \lceil \log n \rceil - i$ and $z < g - k$. So, $r \geq 0$ and $s > 0$. Return

$$(\mu(v)|_{d - 1}, \ldots, \mu(v)|_{p + 1}, 10 \cdots 0, 0, \ldots, \varepsilon, \ldots, \varepsilon, 0 \cdots 0, 0, \ldots, 0, \varepsilon, \ldots, \varepsilon)$$

where the string of $i + 1$ zeroes is at index $\pi(v) - 1$. Property 3b holds because there is at least one empty string among the last $p/2$ components. As all the other properties are also fulfilled by construction, this is a valid tuple encoding a leaf.

**Case 2: $\mu(v)|_{p - 1} \neq \varepsilon$ and $s < 0$.** Note that $s = -1$ and $\mu(v)|_{p - 1}$ has a leading zero due to our choice of $p$. Let $t$ be the number of non-leading bits in $\mu(v)|_{p - 1}$. Then, $y - t \leq \lceil \log n \rceil - i$, which implies that $r + t \geq 0$. Return

$$(\mu(v)|_{d - 1}, \ldots, \mu(v)|_{p + 1}, \varepsilon, \ldots, \varepsilon, 0 \cdots 0, 0, \ldots, 0, \varepsilon, \ldots, \varepsilon)$$

where the string of $i + 1 + r + t$ zeroes is at index $\pi(v) - 1$.

**Case 3: $\mu(v)|_{p - 1} \neq \varepsilon$, $r > 0$ and $s \geq 0$.** Return

$$(\mu(v)|_{d - 1}, \ldots, \mu(v)|_{p - 1}, 10 \cdots 0, 0, \ldots, 0, \varepsilon, \ldots, \varepsilon, 0 \cdots 0, 0, \ldots, 0, \varepsilon, \ldots, \varepsilon)$$

where the string of $i + 1$ zeroes is at index $\pi(v) - 1$.

**Case 4: $\mu(v)|_{p - 1} \neq \varepsilon$, $r \leq 0$ and $s \geq 0$.** Denote $\mu(v)|_{p - 1} = b_1b_2 \cdots b_t$ where $b_j \in \{0, 1\}$ for all $j \in [t]$. By our choice of $p$, there exists a largest $t \in [t]$ such that $b_t = 0$ and $r' := r + t - \max(t - 1, 1) \geq 0$. Then,
• If \( t = 1 \), return

\[
(\mu(v)_{d-1}, \ldots, \mu(v)_{p+1}, \varepsilon, 0 \cdots 0, 0, \ldots, 0, \varepsilon, \ldots, 0, \varepsilon, \ldots, 0, \varepsilon)
\]

where the string of \( i + 1 \) zeroes is at index \( \pi(v) - 1 \).

• If \( s = 0 \) and \( t > 1 \), return

\[
(\mu(v)_{d-1}, \ldots, \mu(v)_{p+1}, b_1 \cdot b_{t-1}, \varepsilon, \ldots, 0 \cdots 0, 0, \ldots, 0, \varepsilon, \ldots, 0, \varepsilon)
\]

where the string of \( i + 1 + r' \) zeroes is at index \( \pi(v) - 1 \).

• If \( s > 0 \) and \( t > 1 \), return

\[
(\mu(v)_{d-1}, \ldots, \mu(v)_{p+1}, b_1 \cdot b_{t-1}, 0 \cdots 0, 0, \ldots, 0, \varepsilon, \ldots, 0, \varepsilon, \ldots, 0, \varepsilon)
\]

where the string of \( i + 1 \) zeroes is at index \( \pi(v) - 1 \).

\[\square\]

We are ready to prove the running time of Algorithm 4 for succinct Strahler universal trees. Recall that \( g \leq \log n \).

**Theorem 4.22.** For a succinct \( g \)-Strahler \((n, d/2)\)-universal tree \( T \) with cover \( C \), Algorithm 4 runs in \( O(mn^2 g \log^2 n \log d) \) time.

**Proof.** By Lemma 4.20, we have \( |C_j| \leq g \) for all \( 0 \leq j \leq h \). Furthermore, \( |C_j^p| \leq \log n \) for all \( 0 \leq j \leq h \) and \( 0 \leq k < |C_j| \). As \( \theta(T) = O(\log n \log d) = r(T, C) \), where the latter equality is due to Lemma 4.21, the result follows from Theorem 4.12. \[\square\]

## 5 Label-Setting Framework

The epitome of a label-setting algorithm is none other than Dijkstra’s algorithm [10]. In this subsection, we develop its analogue for ordered trees. Despite having a fast running time, it requires prior knowledge of \( \mu^*(v) \) for all \( v \in B(G_\tau) \) in order to compute \( \mu^* \) (Corollary 5.4). Nevertheless, we demonstrate its applicability to Cramer computation with perfect universal trees.

The algorithm takes as input a one-player parity game \((G_\tau, \pi)\) for Even and node labels \( \nu_0 : V \to \tilde{L}(T) \) from some ordered tree \( T \). During its execution, the algorithm maintains node labels \( \nu : V \to \tilde{L}(T) \) such that \( \nu(v) = \nu_0(v) \) for all \( v \in B(G_\tau) \). It also maintains a node set \( S \subseteq V \) such that \( \nu(v) \) remains fixed for all \( v \in S \). For the sake of brevity, denote \( H := G_\tau \setminus B(G_\tau) \).

In every iteration, a new node is added to \( S \), whose label is set. To determine this node, we introduce a label function \( \Phi \), which remains fixed throughout the algorithm. It encodes a family of topological orders in \( H \) induced by the even priorities. The node is then selected using a potential function \( \Phi^\nu \), defined based on the labels \( \Phi \) and \( \nu \). This selection criteria accounts for the fact that the representation of a parity game as a mean payoff game could have negative arc weights.

To describe \( \Phi \), we define a family of functions \( \Phi_p \), parametrized by the even priorities in \( H \). For an even \( p \in [d] \), \( \Phi_p \) encodes the topological order of nodes in the subgraph \( H_p \). Recall that \( H_p \) is the subgraph of \( H \) induced by the nodes with priority at most \( p \). Formally, \( \Phi_p : V(H) \to \mathbb{Z}_+ \) is any function which satisfies the following three properties:
The label function $\Phi : V(H) \to \mathbb{Z}_+^{d/2}$ is then defined as

$$\Phi(v) := (\Phi_d(v), \Phi_{d-2}(v), \ldots, \Phi_2(v)).$$

A linear order on $\Phi(v)$ is obtained by extending the linear order of its components lexicographically.

**Remark 5.1.** Given a pair of nodes $v$ and $w$, comparing $\Phi(v)$ and $\Phi(w)$ amounts to finding the largest $p \in [d]$ such that $\Phi_p(v) \neq \Phi_p(w)$. Observe that if $\Phi_q(v) = \Phi_q(w) > 0$ for some $q \in [d]$, then $\Phi_r(v) = \Phi_r(w)$ for all $r \geq q$. On the other hand, if $\Phi_q(v) = \Phi_q(w) = 0$, then $\Phi_r(v) = \Phi_r(w) = 0$ for all $r \leq q$. Hence, such a $p$ can be computed in $O(\log d)$ time via binary search.

Finally, given node labels $\nu : V \to \bar{L}(T)$, the potential function $\Phi^\nu : V(H) \to (\mathbb{Z}_+^{d/2} \times L(T)) \cup \{\infty\}$ is obtained by interlacing the components of $\Phi$ and $\nu$ as follows

$$\Phi^\nu(v) := \begin{cases} 
(\Phi_d(v), \nu(v)_{d-1}, \ldots, \Phi_2(v), \nu(v)_1), & \text{if } \nu(v) \neq \top. \\
\infty, & \text{otherwise.}
\end{cases}$$

A linear order on $\Phi^\nu(v)$ is acquired by extending the linear order of its components lexicographically. For any $p \in [d]$, the $p$-truncation of $\Phi(v)$ and $\Phi^\nu(v)$, denoted $\Phi(v)|_p$ and $\Phi^\nu(v)|_p$ respectively, are obtained by deleting the components with index less than $p$, with the convention $\infty|_p := \infty$.

We are ready to state Dijkstra’s algorithm for ordered trees (Algorithm 5). First, it initializes the node set $S$ to $B(G_\tau)$, and the node labels $\nu$ as $\nu(v) := \nu_0(v)$ for all $v \in S$ and $\nu(v) := \top$ for all $v \in V \setminus S$. Then, for each even $p \in [d]$, it computes the topological order $\Phi_p$ by running Tarjan’s strongly connected component algorithm on $H_p$. Next, $\nu$ is updated by dropping the tail labels of the incoming arcs $\delta^-(S)$. At the start of every iteration, the algorithm selects a node $u$ with minimum potential $\Phi^\nu(u)$ among all the nodes in $V \setminus S$ (ties are broken arbitrarily). Then, it adds $u$ to $S$ and updates $\nu$ by dropping the tail labels of the incoming arcs $\delta^-(u) \cap \delta^-(S)$. The algorithm terminates when $S = V$.

**Algorithm 5** Dijkstra: $(G_\tau, \pi)$ one-player parity game for Even, $\nu_0 : V \to \bar{L}(T)$ node labels from an ordered tree $T$

1: procedure DIJKSTRA($(G_\tau, \pi), \nu_0$)
2: $S \leftarrow B(G_\tau)$
3: $\nu(v) \leftarrow \nu_0(v)$ for all $v \in S$
4: $\nu(v) \leftarrow \top$ for all $v \in V \setminus S$
5: Compute topological order $\Phi_p$ for all even $p \in [d]$ ▷ Using Tarjan’s SCC algorithm
6: for all $vw \in \delta^-(S)$ do ▷ Break ties arbitrarily
7: $\nu(v) \leftarrow \text{drop}(\nu, vw)$
8: while $S \subseteq V$ do
9: $u \in \arg\min_{v \in V \setminus S} \Phi^\nu(v)$
10: $S \leftarrow S \cup \{u\}$
11: for all $vw \in \delta^-(u)$ where $v \notin S$ do
12: $\nu(v) \leftarrow \text{drop}(\nu, vu)$
13: return $\nu$

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An efficient implementation of Dijkstra’s algorithm using Fibonacci heaps was given by Fredman and Tarjan [17]. Its running time is $O(m + n \log n)$ when the keys in the heap are real numbers, assuming that each elementary operation on the reals takes constant time. In our setting, the keys are the node potentials $\Phi^\nu$. Since computing $d_p(v, e)$ takes $\Theta(T)$ time while comparing the potential of two nodes takes $\Theta(T) + \log d$ time, their result translates to $O(\Theta(T)m + (\Theta(T) + \log d)n \log n)$ time here. We also need to compute the base nodes $B(G_T)$ and the topological orders $\Phi_p$, which take $O(dm)$ time. Hence, the total running time of Algorithm 5 is $O((\Theta(T) + d)m + (\Theta(T) + \log d)n \log n)$.

Before proving the algorithm’s correctness, we illustrate an important connection between the potential function $\Phi^\nu$ and even cycles in $H$.

**Lemma 5.2.** Let $vw \in E(H)$ be a tight arc with respect to some node labels $\nu$. If $\Phi^\nu(v) < \Phi^\nu(w)$, then there exists an even cycle $C$ in $H$ such that $vw \in E(C)$ and $\pi(C) = \pi(v)$.

**Proof.** First, note that $\nu(v) \neq \top$ because $\Phi^\nu(v) < \Phi^\nu(w)$. This in turn implies that $\nu(w) \neq \top$ as $vw$ is tight with respect to $\nu$. Next, since $v$ can reach $w$ in $H$, we have $\Phi(v)_{\pi(v)} \geq \Phi(w)_{\pi(v)}$. We also have $\nu(v)_{\pi(v)} \geq \nu(w)_{\pi(v)}$ due to the tightness of $vw$. Then, combining these two inequalities yield $\Phi^\nu(v)_{\pi(v)} \geq \Phi^\nu(w)_{\pi(v)}$. In fact, we get $\Phi^\nu(v)_{\pi(v)} = \Phi^\nu(w)_{\pi(v)}$ because $\Phi^\nu(v) < \Phi^\nu(w)$. Since $vw$ is tight, we conclude that $\pi(v)$ is even. It follows that $0 < \Phi_{\pi(v)}(v) = \Phi_{\pi(w)}(w)$, which implies that $v$ and $w$ are strongly connected in $H_{\pi(v)}$. Thus, there exists a cycle $C$ in $H_{\pi(v)}$ such that $vw \in E(C)$. Clearly, $\pi(C) = \pi(v)$.

The next theorem shows that Algorithm 5 returns the pointwise minimal node labels which are ‘almost’ feasible in $G_T$. The key observation is that the sequence of node potentials admitted to $S$ during the algorithm is monotonically nondecreasing.

**Theorem 5.3.** Algorithm 5 returns the pointwise minimal node labels $\nu$ which are feasible in $G_T \setminus \cup_{v \in B(G_T)} \delta^+(v)$ and satisfy $\nu(v) = \nu_0(v)$ for all $v \in B(G_T)$.

**Proof.** For every $i \geq 1$, let $\nu_i$ be the node labels at the start of iteration $i$. Note that $\nu_n = \nu$. To show the feasibility of $\nu$, we prove that $\nu_i$ is feasible in $G_T \setminus \cup_{v \in B(G_T)} \delta^+(v)$ by induction on $i \geq 1$. The base case $i = 1$ is clearly true, and the inductive step is straightforward. Furthermore, it is easy to see that $\nu(v) = \nu_0(v)$ for all $v \in B(G_T)$ due to our initialization.

For every $i \geq 1$, let $u_i$ be the node added to $S$ in iteration $i$. We show that $\Phi^\nu(u_i) \geq \Phi^\nu(u_{i-1})$ for all $i > 1$. For the purpose of contradiction, suppose that $\Phi^\nu(u_i) < \Phi^\nu(u_{i-1})$ for some $i > 1$. We claim that $u_iu_{i-1} \in E(H)$ and it is tight with respect to $\nu_i$. Suppose otherwise for a contradiction. Then, $\nu_i(u_i) = \nu_{i-1}(u_i)$. As $\nu(u_i) = \nu_i(u_i)$ and $\nu(u_{i-1}) = \nu_{i-1}(u_{i-1})$, it follows that $\Phi^\nu(u_i) < \Phi^\nu(u_{i-1})$. This is a contradiction because $u_i$ would have been added to $S$ in iteration $i - 1$ instead of $u_{i-1}$. By the claim above and Lemma 5.2, there exists an even cycle $C$ in $H$. However, $\Pi(C) \subseteq B(G_T)$, which is a contradiction.

Next, we show that there are no loose arcs in $G_T \setminus \cup_{v \in B(G_T)} \delta^+(v)$ with respect to $\nu$. It suffices to prove that $u_iu_j$ is not loose with respect to $\nu$ for all $1 \leq i < j$ where $u_iu_j \in E(H)$. For the purpose of contradiction, let $u_iu_j \in E(H)$ be a loose arc with respect to $\nu$. Note that $\nu(u_i) \neq \top$, as otherwise it would imply $\nu(u_j) = \top$ because $\Phi^\nu(u_i) \leq \Phi^\nu(u_j)$. Since the label $\nu(u_i)$ was given by $\text{Tighten}$, it is the smallest leaf in the subtree of $T$ rooted at $\nu(u_i)_{\pi(u_i)}$. Therefore, $\nu(u_i)_{\pi(u_i)} > \nu(u_j)_{\pi(u_j)}$. Let $p$ be the smallest even integer such that $p \geq \pi(u_i)$. Then, $\Phi_{\pi(p)}(u_i) \geq \Phi(u_j)_{\pi(u_j)}$, because either $\pi(u_j) > p$ or $u_iu_j \in E(H_p)$. However, these two inequalities yield $\Phi^\nu(u_i) \geq \Phi^\nu(u_j)$, which is a contradiction.

It is left to show the pointwise minimality of $\nu$. For the purpose of contradiction, let $\nu' : V \to \hat{L}(T)$ be node labels feasible in $G_T \setminus \cup_{v \in B(G_T)} \delta^+(v)$ such that $\nu'(v) = \nu_0(v)$ for all $v \in B(G_T)$ and
\[ \nu'(u) < \nu(u) \] for some \( u \in V(H) \). Note that \( \nu'(u) \neq \top \). From the definition of feasibility, there exists a strategy \( \sigma \) for Even such that \( G_{\sigma} \rightarrow \cup v \in B(G_{\sigma}) \delta^+(v) \) does not contain violated arcs with respect to \( \nu' \). In this subgraph, \( u \) can reach a node in \( B(G_{\sigma}) \). Indeed, if it reaches a cycle \( C \), then \( C \) is even by the Cycle Lemma because \( \nu'(v) \neq \top \) for all \( v \in V(C) \). So, \( \Pi(C) \subseteq B(G_{\sigma}) \). Let \( P \) be a \( u-w \) path in this subgraph for some \( w \in B(G_{\sigma}) \). Since \( \nu(u) > \nu'(u) \) and \( \nu(w) = \nu'(w) \), there exists a loose arc in \( P \) with respect to \( \nu \). We have reached a contradiction. \( \Box \)

Consequently, if we can determine \( \mu^*(v) \) for all \( v \in B(G_{\sigma}) \) beforehand, then we can use Algorithm 5 to compute \( \mu^* \).

**Corollary 5.4.** If \( \nu_0(v) = \mu^*(v) \) for all \( v \in B(G_{\sigma}) \), then Algorithm 5 returns \( \mu^* \).

**Proof.** Let \( \nu \) be the node labels returned by Algorithm 5. By Theorem 5.3, we have \( \nu \leq \mu^* \). Since \( \nu(v) = \mu^*(v) \) for all \( v \in B(G_{\sigma}) \), this implies that \( \nu \) is feasible in \( G_{\sigma} \). To show that \( \nu \geq \mu^* \), it suffices to prove that \( \nu \geq \mu \) due to the pointwise minimality of \( \mu^* \). For the purpose of contradiction, suppose that \( \nu(u) < \mu(u) \) for some \( u \in V(H) \). Let \( \sigma \) be a strategy for Even such that \( G_{\sigma} \) does not contain violated arcs with respect to \( \nu \). Since \( \nu(u) \neq \top \), there exists a loose arc in \( G_{\sigma} \) for some \( w \in B(G_{\sigma}) \). As \( \mu(u) > \nu(u) \) and \( \mu(w) \leq \mu^*(w) = \nu(w) \), there exists a loose arc in \( P \) with respect to \( \mu \). This contradicts our assumption on \( \mu \). \( \Box \)

### 5.1 Perfect Universal Trees

In this subsection, we show that Algorithm 5 yields a faster method for Cramer computation with perfect universal trees.

**Theorem 5.5.** For a perfect \((n, d/2)\)-universal tree \( T \), Cramer computation can be performed in \( O(dn + n \log n) \) time.

**Proof.** Given a one-player parity game \( G_{\sigma} \) for Even, let \( \mu : V \rightarrow \mathbb{L}(T) \) be node labels such that there are no loose arcs in \( G_{\sigma} \). Let \( \mu^* : V \rightarrow \mathbb{L}(T) \) be the pointwise minimal node labels which are feasible in \( G_{\sigma} \) and satisfy \( \mu^* \geq \mu \).

First, we show that for every \( v \in V \), we may assume that \( \mu(v) \) is either \( \top \) or the smallest leaf in the subtree of \( T \) rooted at \( \mu(v) \) without loss of generality. If a node \( w \) violates this condition, then we know that \( \mu^*(w) > \mu(w) \) because there are no loose arcs in \( G_{\sigma} \) with respect to \( \mu \). Hence, we can set \( \mu(w) \) as the smallest leaf of the next subtree rooted at that depth using Tighten. Recall that the Tighten subroutine for \( T \) takes \( O(d) \) time. In the worst case, we incur \( O(dn) \) extra time. It suffices to prove that \( \mu^*(v) = \mu(v) \) for all \( v \in B(G_{\sigma}) \). Then, the result follows from Corollary 5.4. Pick a base node \( w \in B(G_{\sigma}) \). We may assume that \( \mu(w) \neq \top \). Let \( C \) be a cycle dominated by \( w \) in \( G_{\sigma} \), and consider the path \( P := C \setminus wu \) where \( wu \in E(C) \). Let \( \bar{\mu} : V \rightarrow \mathbb{L}(T) \) be node labels such that \( \bar{\mu}(v) = \top \) for all \( v \notin V(P) \), \( \bar{\mu}(w) = \mu(w) \) and \( P \) is tight with respect to \( \bar{\mu} \). Then, \( \bar{\mu} \) is feasible in \( G_{\sigma} \setminus \delta^+(w) \). Moreover, \( \bar{\mu} \geq \mu \) because there are no loose arcs in \( P \) with respect to \( \mu \). Now, recall that \( \mu(w) \) is the smallest leaf in the subtree of \( T \) rooted at \( \mu(w) \). Since \( \pi(w) = \pi(P) \) is even, we have \( \bar{\mu}(u) |_{\pi(w)} = \bar{\mu}(w) |_{\pi(w)} \) because \( |V(P)| \leq n \). Hence, the arc \( wu \) is tight with respect to \( \bar{\mu} \). It follows that \( \bar{\mu} \) is feasible in \( G_{\sigma} \). Thus, \( \mu^* \leq \bar{\mu} \). In particular, \( \mu^*(w) = \mu(w) \). \( \Box \)

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A Connection to tropical linear programming

In this section, we describe the well-known connection between parity games and tropical linear programming. To this end, let us assume that $G$ is bipartite with bipartition $V_0 \sqcup V_1$ without loss of generality. Consider the tropical (min-plus) semiring $(\mathbb{T}, \oplus, \odot)$, where the set $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ is equipped with binary operations $a \oplus b = \min\{a, b\}$ and $a \odot b = a + b$. Let $n_1 = |V_1|$ and $y \in \mathbb{T}^{n_1}$ be a vector of variables indexed over the nodes in $V_1$. A parity game can be formulated as the following system of tropical linear inequalities

$$y_u + (-n)^{\pi(u)} \geq \min_{vw \in E} \left\{ y_w - (-n)^{\pi(v)} \right\} \quad \forall uv \in \delta^+(V_1). \quad \text{(P)}$$

This is precisely the reduction from parity games to mean payoff games [11, 33]. In particular, for each node $v \in V$, we assign a cost of $-(-n)^{\pi(v)}$ to every arc in $\delta^+(v)$. A trivial feasible solution to (P) is given by $\infty \cdot 1$. However, we are interested in feasible solutions of maximal finite support due to the following statement.

**Theorem A.1.** Let $y^* \in \mathbb{T}^{n_1}$ be a feasible solution to (P) of maximal finite support. Then, Even wins from $u \in V_1$ if and only if $y^*_u < \infty$.

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