DOMINATING SURFACE GROUP REPRESENTATIONS BY FUCHSIAN ONES

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Abstract. We prove that a representation from the fundamental group of a closed surface of negative Euler characteristic with values in the isometry group of a Riemannian manifold of sectional curvature bounded by \(-1\) can be dominated by a Fuchsian representation. Moreover, we prove that the domination can be made strict, unless the representation is discrete and faithful in restriction to an invariant totally geodesic 2-plane of curvature \(-1\). When applied to representations into PSL(2, \(\mathbb{R}\)) of non-extremal Euler class, our result is a step forward in understanding the space of closed anti-de Sitter 3-manifolds. For representations into PSU(\(n,1\)), it sheds a new light on a famous theorem by Toledo.

Introduction

Let \(\Gamma\) be a group, and \(\rho_i : \Gamma \to \text{Isom}(M_i, g_i)\) be representations of \(\Gamma\) in the groups of isometries of Riemannian manifolds \((M_i, g_i)\), for \(i = 1, 2\). We will say that \(\rho_1\) dominates \(\rho_2\) if there exists a 1-Lipschitz map \(f : M_1 \to M_2\) which is \((\rho_1, \rho_2)\)-equivariant, i.e. which satisfies

\[
f(\rho_1(\gamma) \cdot x) = \rho_2(\gamma) \cdot f(x)
\]

for every \(x \in M_1\) and \(\gamma \in \Gamma\). The domination will be called strict if the map \(f\) can be chosen to be \(\lambda\)-Lipschitz for some constant \(0 < \lambda < 1\).

We will be mainly interested in the case where \(\Gamma\) is the fundamental group of a closed oriented connected surface of negative Euler characteristic. Those surfaces are endowed with metrics of constant negative curvature \(-1\), and any such metric gives rise to an isometric identification of \(S\) with a quotient \(j(\Gamma) \backslash \mathbb{H}^2\), where \(\mathbb{H}^2\) is the Poincaré half-plane, and \(j\) a representation of \(\Gamma\) into \(\text{Isom}^+(\mathbb{H}^2) \cong \text{PSL}(2, \mathbb{R})\). The representations obtained by this procedure are called Fuchsian.

We are interested in the following question: given a representation \(\rho\) of the fundamental group of a closed oriented surface of negative Euler characteristic into the group of isometries of a complete Riemannian manifold, is there a Fuchsian representation of \(\Gamma\) that dominates \(\rho\)? When \(\rho\) takes values into \(\text{PSL}(2, \mathbb{R})\), the question has been raised by Kassel in her work on closed anti-de Sitter 3-manifolds (cf. subsection 0.2).

Here we prove:

**Theorem A.** Let \(S\) be a closed oriented surface of negative Euler characteristic, \(\Gamma\) its fundamental group, \((M, g)\) a smooth, simply connected, complete Riemannian manifold and \(\rho : \Gamma \to \text{Isom}(M, g)\) a representation. Assume that the sectional curvature of \((M, g)\) is bounded above by \(-1\). Then there exists a Fuchsian representation \(j\) of \(\Gamma\) that dominates \(\rho\). Moreover, the domination can be made strict unless \(\rho\) stabilizes a totally geodesic plane \(\mathbb{H} \subset M\) of curvature \(-1\), in restriction to which \(\rho\) is Fuchsian.

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Note that the strict domination cannot occur if $\rho$ stabilizes a totally geodesic copy of $\mathbb{H}^2$ in restriction to which it is Fuchsian. Indeed, composing the equivariant map $f$ with the orthogonal projection on $\mathbb{H}$ would provide a contracting map from the original hyperbolic surface $\rho(\Gamma) \setminus \mathbb{H}$. This would contradict the fact that both surfaces have the same volume $-2\pi \chi(S)$.

One can think of several generalizations of theorem A. In section 3, we will extend it to representations of lattices in $\text{PSL}(2, \mathbb{R})$ with torsion. In a forthcoming paper, we will refine Theorem A by proving that the same statement holds if $(M, g)$ is replaced by any complete geodesic CAT($-1$) space. Hereafter we discuss some applications of our result, and some possible developments.

0.1. Contraction of the length spectrum and universality of the Bers constant.

Recall that the translation length of an isometry $\varphi$ of a Riemannian manifold $(M, g)$ is defined by

$$l(\varphi) := \inf_{x \in M} d(x, \varphi(x)),$$

where $d(\cdot, \cdot)$ is the distance induced by $g$. If $\rho : \Gamma \to \text{Isom}(M, g)$ is a representation, the length spectrum of $\rho$ is the map $l_\rho : \gamma \in \Gamma \mapsto l(\rho(\gamma)) \in \mathbb{R}^+$. The length spectrum of a representation contains a lot of information, and sometimes determines completely the representation. For instance, Zariski dense representations in $\text{PSO}(n, 1)$ are determined by their length spectrum. Another illustration of this phenomenon is a famous result of Otal [17] stating that one can recover a negatively curved metric on a closed surface by the knowledge of the length spectrum of the action of its fundamental group on the universal cover.

Given two representations $\rho_1 : \Gamma \to \text{Isom}(M_1, g_1)$, we say that the length spectrum of $\rho_1$ dominates the one of $\rho_2$ if $l_{\rho_2} \leq l_{\rho_1}$. Similarly, the domination is called strict if there is a positive constant $\lambda < 1$ such that $l_{\rho_2} \leq \lambda l_{\rho_1}$.

It is clear that if $\rho_1$ (strictly) dominates $\rho_2$, then the length spectrum of $\rho_1$ (strictly) dominates the length spectrum of $\rho_2$. The converse has been proven in several cases by Guéritaud–Kassel [9]. They pointed out to us that their method would work in our setting. The following corollary can therefore be seen as a reformulation of theorem A.

**Corollary B.** Let $\Gamma$ be the fundamental group of a closed oriented surface of negative Euler characteristic, and $\rho$ a representation of $\Gamma$ into the isometry group of a smooth simply connected complete Riemannian manifold of curvature bounded above by $-1$. Then there exists a Fuchsian representation of $\Gamma$ whose length spectrum dominates $l_\rho$. Moreover, the domination can be made strict unless there exists a totally geodesic copy of $\mathbb{H}^2$ preserved by $\rho$, in restriction to which $\rho$ is Fuchsian.

Recall that the Bers constant in genus $g$ is the smallest constant $B_g$ such that for any hyperbolic metric on a closed surface $S$ of genus $g$, there is a decomposition of $S$ into pairs of pants such that all the geodesics of this decomposition have length at most $B_g$. The Bers constant in genus 2 has been explicitly computed by Gendulphe [6].

Since the length spectrum of a representation of a surface group into the isometry group of any simply connected Riemannian manifold of sectional curvature $\leq -1$ can be dominated by the length spectrum of a Fuchsian representation, the Bers constant naturally extends to those representations, and we get the following corollary:
Corollary C. Let $\Gamma$ be the fundamental group of a closed oriented surface $S$ of genus $g \geq 2$, $M$ a smooth, simply connected, complete riemannian manifold of curvature bounded above by $-1$, and $\rho$ a representation of $\Gamma$ into $\text{Isom}(M)$. Then there exists a pants decomposition of $S$ for which the image of any curve of the decomposition has translation length at most $B_g$.

Marché and Wolff [16] recently used this extension of the Bers constant to solve a conjecture of Bowditch in genus 2. They proved that, given a closed oriented surface $S$ of genus 2 and a representation $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{R})$ of Euler class 1, there always exists a simple closed curve in $S$ whose image by $\rho$ is not hyperbolic.

0.2. Closed anti-de Sitter manifolds of dimension 3. Anti-de Sitter (AdS) manifolds are Lorentz manifolds of constant negative sectional curvature. In dimension 3, they are locally modelled on $\text{PSL}(2, \mathbb{R})$ equipped with its Killing metric, whose isometry group is (up to finite index) $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ acting on $\text{PSL}(2, \mathbb{R})$ by

$$(\gamma_1, \gamma_2) \cdot x = \gamma_1 x \gamma_2^{-1}.$$ 

Klingler [13], generalizing a result of Carri`ere [2], proved that closed Lorentz manifolds of constant curvature are always geodesically complete. A consequence is that a closed AdS manifold of dimension 3 is a quotient of the universal cover $\tilde{\text{PSL}}(2, \mathbb{R})$ by a subgroup of $\tilde{\text{PSL}}(2, \mathbb{R}) \times \tilde{\text{PSL}}(2, \mathbb{R})$ acting freely, properly discontinuously and cocompactly on $\tilde{\text{PSL}}(2, \mathbb{R})$.

Those quotients have been described by works of Goldman [8], Kulkarni and Raymond [14], Salein [18], and Kassel [11]. Kulkarni and Raymond proved that, up to a finite cover and a finite quotient, closed anti-de Sitter manifolds are isometric to $\Gamma_{j,\rho}\backslash\text{PSL}(2, \mathbb{R})$, where $\Gamma$ is a surface group, $j, \rho$ two representations of $\Gamma$ into $\text{PSL}(2, \mathbb{R})$, $j$ Fuchsian, and $\Gamma_{j,\rho}$ is the image of $\Gamma$ into $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ by the embedding

$$\gamma \mapsto (j(\gamma), \rho(\gamma)).$$

A pair $(j, \rho)$ of representations such that $\Gamma_{j,\rho}$ acts properly discontinuously on $\text{PSL}(2, \mathbb{R})$ is called an admissible pair. Salein noticed that a sufficient condition for $(j, \rho)$ to be admissible is that $j$ strictly dominates $\rho$. As a consequence, he obtains the existence of admissible pairs $(j, \rho)$ with $\rho$ of any non-extremal Euler class. Lastly, Kassel [11, Chapter 5] proved that Salein’s sufficient condition is also necessary.

Describing the space of closed anti-de Sitter 3-manifolds (up to finite coverings) thus reduces to describing the set of triples $(\Gamma, j, \rho)$, where $\Gamma$ is a surface group, $j$ a Fuchsian representation of $\Gamma$ into $\text{PSL}(2, \mathbb{R})$ and $\rho$ another representation that is strictly dominated by $j$. A natural question is whether any non-Fuchsian representation $\rho$ can appear in an admissible pair. It is answered positively by theorem A. We thus get the following corollary:

Corollary D. Let $S$ be a closed oriented surface of negative Euler characteristic, $\Gamma$ its fundamental group, and $\rho: \Gamma \to \text{PSL}(2, \mathbb{R})$ a representation of intermediate Euler class. Then there exists a Fuchsian representation $j$ of $\Gamma$ such that $\Gamma_{j,\rho}$ acts properly discontinuously and cocompactly on $\text{PSL}(2, \mathbb{R})$.

This result has been obtained independantly and with different methods by Guéritaud, Kassel and Wolff, see [10].

Remark 0.1. Theorems of Kulkarni–Raymond and Kassel have been generalized by Kassel [12] and Guéritaud–Kassel [9] to compact quotients of $\text{PSO}(n, 1)$ by discrete subgroups of
PSO(n, 1) \times \text{PSO}(n, 1).

Namely, they proved that those quotients have (up to finite index) the form
\[ \Gamma_{j, \rho} \backslash \text{PSO}(n, 1) \]
with \( \Gamma \) the fundamental group of a closed hyperbolic \( n \)-manifold and \( j, \rho \) two representations of \( \Gamma \) into \( \text{PSO}(n, 1) \), \( j \) discrete and faithful, and \( \rho \) strictly dominated by \( j \). However, when \( n \geq 3 \), the picture is very different. Indeed, by Mostow’s rigidity theorem, there is only one discrete and faithful representation of \( \Gamma \) up to conjugacy. Therefore, one cannot change the translation lengths of the elements \( j(\gamma) \), and when \( H^1(\Gamma, \mathbb{R}) \neq 0 \) it is possible to construct abelian representations \( \rho \) that are not dominated by \( j \).

0.3. Representations in higher dimensional rank 1 Lie groups. Let \( G \) be a simple Lie group of rank 1, and \( K \) a maximal compact subgroup. The homogeneous space \( X = G/K \) carries a unique \( G \)-invariant Riemannian metric whose sectional curvature is either constant equal to \(-1\) or pinched between \(-1\) and \(-4\). Every 2-plane (in the tangent space) of curvature \(-1\) generates (by the exponential map) a totally geodesic copy of \( \mathbb{H}^2 \). We see \( G \) as a group of isometries of \( X \). Then our theorem applies to representations of \( \Gamma \) into \( G \), and we obtain the following corollary:

**Corollary E.** Let \( G \) be a simple Lie group of rank 1, \( S \) a closed oriented surface of negative Euler characteristic, \( \Gamma \) its fundamental group, and \( \rho \) a representation of \( \Gamma \) into \( G \). Then either there exists a Fuchsian representation \( j \) that strictly dominates \( \rho \), or \( \rho(\Gamma) \) preserves a totally geodesic hyperbolic plane of curvature \(-1\) in \( X \), and \( \rho \) is Fuchsian in restriction to this plane.

When \( G \) is \( \text{PSU}(n, 1) \), its symmetric space \( \mathbb{H}^n_{\mathbb{C}} \) carries a \( G \)-invariant Kähler form \( \omega \), which is unique up to rescaling. Once normalized so that the associated hermitian metric has curvature in \([-4, -1]\), this form allows to define an analogous of the Euler class. Consider a closed oriented surface \( S \) of genus \( g \geq 2 \) and \( \Gamma \) its fundamental group. Let \( \rho \) be a representation of \( \Gamma \) into \( \text{PSU}(n, 1) \), and \( f \) be any \( (\Gamma, \rho) \)-equivariant map from \( \tilde{S} \) to \( \mathbb{H}^n_{\mathbb{C}} \). Then the number
\[ \tau(\rho) = \frac{1}{2\pi} \int_S f^* \omega \]
is an integer independent of \( f \), called the **Toledo invariant**.

Toledo proved \([21, 22]\) that this invariant is bounded between \( 2 - 2g \) and \( 2g - 2 \), and is extremal if and only if the representation is Fuchsian in restriction to some stable hyperbolic plane of curvature \(-1\). Here, we can recover Toledo’s theorem as a consequence of corollary E. Indeed, assume there exists a hyperbolic metric \( g_0 \) on \( S \) and a \( (\Gamma, \rho) \)-equivariant contracting map \( f : (\tilde{S}, \tilde{g}_0) \to (\mathbb{H}^n_{\mathbb{C}}, g) \). Then we have
\[ \tau(\rho) = \frac{1}{2\pi} \int_S f^* \omega \leq \frac{1}{2\pi} \int_S |\text{Vol}_{f^* g}| < \int_S \text{Vol}_{g_0} = 2g - 2. \]

0.4. Open questions for higher rank representations. The main reason why we only obtain a result for representations in a Lie group of rank 1 is that if \( G \) is a simple Lie group of rank \( \geq 2 \), and \( X \) its symmetric space, the sectional curvature of \( X \) is not negative everywhere. We shall state some remarks and questions in this context.
First we need to conveniently choose a normalization of the metric on $X$, as in subsection 0.3. To do this, one can note that, though the sectional curvature is not pinched away from 0, there is a negative upper bound on the curvature of totally geodesic hyperbolic planes. We choose to normalize the metric on $X$ so that every totally geodesic hyperbolic plane has curvature $\leq -1$, and one has curvature $-1$.

With this convention, one could ask whether any representation of $\Gamma$ into $G$ can be dominated by a Fuchsian one unless it is Fuchsian in restriction to some stable hyperbolic plane of curvature $-1$. It is not true in general. For instance, the symmetric spaces of the Lie groups $\text{Sp}(2n, \mathbb{R})$ or $\text{SU}(q, q)$ carry a Kähler metric. For surface group representations into those Lie groups, the Toledo invariant can be defined, and for the same reason as before, a representation with maximal Toledo invariant cannot be strictly dominated by a Fuchsian one. In [1, Section 9], Burger, Iozzi and Wienhard proved that some of these representations have Zariski dense image. In particular, they do not preserve a totally geodesic copy of $\mathbb{H}^2$.

For representations with values in $\text{SL}(n, \mathbb{R})$, there is no similar obstruction to domination. Guichard pointed out to us that a bending construction enables to deform a highest weight Fuchsian representation into a Hitchin representation that is dominated by the initial one. The question of which Hitchin representations can be dominated by Fuchsian ones seems non trivial.

From these few remarks, the following questions naturally arise:

**Question 1.** Given $\Gamma$ a surface group, $G$ a simple Lie group of rank $\geq 2$ and $X$ its symmetric space, is there a constant $C$ such that for any representation $\rho : \Gamma \to G$, there exists a Fuchsian representation $j : \Gamma \to \text{PSL}(2, \mathbb{R})$ and a $(j, \rho)$-equivariant map from $\mathbb{H}^2$ to $X$ that is $C$-lipschitz?

**Question 2.** Let $\Gamma$ be a surface group, $G$ a simple Lie group of rank $\geq 2$ and $X$ its symmetric space. Given a representation $\rho : \Gamma \to G$, is there a representation $j : \Gamma \to G$ in the connected component of a highest weight Fuchsian representation whose length spectrum dominates the length spectrum of $\rho$? (For $G = \text{SL}(n, \mathbb{R})$, $j$ would be a Hitchin representation. For $G$ of hermitian type, $j$ would be a maximal representation.)

We believe that the answer to the first question should be no and the answer to the second question may be yes. Anyway, we hope that the point of view of domination will prove interesting in the study of higher Teichmüller theory.

**0.5. Strategy of the proof.** Our approach shares some similarities with the technique used by Toledo in [21, 22]. Starting with any hyperbolic metric on $S$, one can consider a $(\Gamma, \rho)$-equivariant map from $\tilde{S}$ to $M$ that is harmonic. (It almost always exists, according to a theorem of Labourie [15].) Toledo noticed that such a map may not be 1-lipschitz, though if contracts volume in average.

However, it turns out that this harmonic map becomes contracting after suitably changing the hyperbolic metric on $S$. The new hyperbolic metric on $S$ is constructed using a uniformization theorem of Wolf [24] (see subsection 1.3.2). The domination is proved via analytic methods that refine Sampson’s argument, see the proof of [19, Theorem 13].

We introduce in the next section the main results we need about harmonic maps, and we prove theorem A in section 2. In section 3 we note that our proof naturally extends to representations of lattices of $\text{PSL}(2, \mathbb{R})$ that are not necessarily torsion-free.
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1. Harmonic maps between Riemannian manifolds

A harmonic map between two Riemannian manifolds is a critical point of the energy functional, which associates to a map $f$ the mean value of the square norm of its differential. Though most of the fundamental results we state here are true in a general setting, we will restrict ourselves to the theory of harmonic maps with the source being a surface. In this case, the energy of a map only depends on the conformal class of the metric on the source. More details about the theory of harmonic maps and its relation to Teichmüller theory can be found in the survey of Daskalopoulos and Wentworth [3].

1.1. Energy of a map and harmonicity. Let $(S, g^0)$ be a Riemann surface, $(M, g)$ a Riemannian manifold, and $f : S \to M$ a smooth map. One can measure how “stretchy” the map $f$ is by comparing $f^*g$ with $g^0$.

**Definition 1.1.** Let $g_1$ be a non-negative Riemann tensor on $S$. Let $A$ be the field of symmetric endomorphisms of $(TS, g^0)$ such that $g_1 = g^0(\cdot, A \cdot)$. The energy density of $g_1$ (with respect to the Riemann structure $g^0$) is the function

$$x \mapsto e_{g^0}(g_1)(x) = \frac{1}{2} \text{Tr}(A_x).$$

Let $f$ be a map from $(S, g^0)$ to a Riemannian manifold $(M, g)$. The energy density of $f$ is the function

$$x \mapsto e_{g^0}(f)(x) = e_{g^0}(f^*g)(x).$$

When there is no confusion on the ambient Riemann structure $g^0$, we will usually omit to index the energy density on $g^0$ and simply write “$e(f)$”.

**Definition 1.2.** The total energy of $f$ is the integral of the energy density:

$$E(f) = \int_S e(f)(x)\text{Vol}_0(x),$$

where $\text{Vol}_0$ is the volume form induced on $S$ by the metric $g^0$.

Consider now a representation $\rho : \Gamma = \pi_1(S) \to \text{Isom}(M, g)$, and $f : \tilde{S} \to M$ a $(\Gamma, \rho)$-equivariant map. Since $\rho(\Gamma)$ acts on $M$ by isometries, the Riemann tensor $f^*g$ on $\tilde{S}$ is preserved by the action of the fundamental group, and so is the energy density $e_{\tilde{g}^0}(f)$. We will denote $e_{g^0}(f)$ the induced function on $S$. Then we similarly call the integral of $e_{g^0}(f)$ against $\text{Vol}_0$ on $S$ the total energy of $f$ and denote it $E(f)$.

For now on we assume $S$ is closed. Suppose $f$ is a map from $S$ to $M$ that minimizes the energy functional among all smooth maps homotopic to $f$. Then $f$ must verify a certain partial differential equation that can be expressed as the vanishing of a differential operator called the tension field of the map.
Definition 1.3. The second fundamental form of $f$ is the section of $\text{Sym}^2 T^*S \otimes f^*TM$ defined by
\begin{equation}
II^f(X, Y) := \nabla^f_X df(Y) - df(\nabla^g_X Y),
\end{equation}
where $\nabla^f$ is the pull-back on $f^*TM$ of the Levi-Civita connexion $\nabla^g$.

The tension field of $f$ is
\begin{equation}
\tau(f) := \text{Tr}_{g_0} II^f = II^f(e_1, e_1) + II^f(e_2, e_2)
\end{equation}
where $(e_1, e_2)$ is any orthonormal basis of $(TS, g_0)$.

The map $f$ is harmonic if the tension field $\tau(f)$ vanishes everywhere.

Proposition 1.4. If $f : S \to M$ minimizes the energy functional among all maps homotopic to $f$, then $f$ is harmonic.

For the same reason, if a $(\Gamma, \rho)$-equivariant map from $\tilde{S}$ to $M$ minimizes the energy functional among all $(\Gamma, \rho)$-equivariant maps, then it is harmonic. Note that the total energy of $f$ depends on the metric $g$ on $M$, but only on the conformal class of $g_0$ on $S$. Indeed, if we multiply $g_0$ by some positive function $\sigma$, the energy density of $f$ is divided by $\sigma$, but the volume is multiplied by $\sigma$, and therefore the total energy is preserved. From this, one can deduce that harmonicity only depends on the conformal class of $g_0$. This is specific to the case where $S$ has dimension 2.

1.2. Existence results. The fundamental work on harmonic maps is the work of Eells and Sampson. In [4], they study the flow of the equation
\begin{equation}
\frac{\partial}{\partial t} f_t(x) = -\tau(f_t)_x
\end{equation}
and prove that, under some curvature conditions, the Lipschitz constant of $f_t$ can be controlled along the trajectory. As a consequence, they obtain the first existence result:

**Theorem (Eells–Sampson).** Let $S$ be a closed Riemann surface, $(M, g)$ be a closed Riemannian manifold of non-positive sectional curvature and $f : S \to M$ a continuous map. Then there exists a harmonic map $f' : S \to M$ homotopic to $f$, which minimizes the energy among all maps homotopic to $f$. Moreover, if the sectional curvature of $M$ is negative, then this map is unique, unless $f'$ maps $S$ into a geodesic of $M$.

For equivariant maps from $\tilde{S}$ to $M$, their method can be generalized, but one has to be sure that a sequence of equivariant maps with uniformly bounded Lipschitz constant cannot “go to infinity”.

Recall that a simply connected Riemannian manifold $M$ with sectional curvature $\leq -1$ is Gromov hyperbolic. One can thus define its boundary $\partial M$ as the space of geodesic rays, where two such rays are identified when they remain at bounded distance. Any isometry of $M$ induces a transformation of $\partial M$. Therefore, a representation $\rho : \Gamma \to \text{Isom}(M)$ induces an action of $\Gamma$ on $\partial M$.

We can now state the useful refinement of Eells–Sampson theorem, see [15]:

**Theorem (Labourie).** Let $S$ be a closed Riemann surface, $\Gamma$ its fundamental group, $(M, g)$ a complete simply connected Riemannian manifold of sectional curvature $\leq -1$, and $\rho$ a representation of $\Gamma$ into $\text{Isom}(M, g)$. If $\rho$ does not fix a point in the boundary of $M$, or if $\rho$ fixes a geodesic in $M$, then there exists a $\rho$-equivariant harmonic map from $\tilde{S}$ to $M$. If $\rho$ does not
fix a point in the boundary, this map is unique.

1.3. Hopf differential and parametrization of Teichmüller space.

1.3.1. Harmonic maps and quadratic differentials. Consider a surface $S$ with a Riemannian metric $g_0$. This metric induces a conformal structure on $S$ and thus a complex structure. Any complex symmetric 2-form on $S$ splits into a $(1,1)$-part, a $(2,0)$-part and a $(0,2)$-part. In particular, if $f : S \to (M, g)$ is a smooth map, $f^*g$ can be written in the form $\alpha g_0 + \Phi + \bar{\Phi}$, where $\Phi$ is a quadratic differential called the Hopf differential. The following proposition is classical. A proof can be found in [3, Section 2.2.3].

**Proposition 1.5.** If $f : (S, [g_0]) \to (M, g)$ is harmonic, then its Hopf differential is holomorphic. This necessary condition is also sufficient when $M$ is a surface.

Observe that if $f : \tilde{S} \to (M, g)$ is an equivariant harmonic map, with respect to some representation $\rho : \pi_1(S) \to \text{Isom}(M, g)$, then the Hopf differential of $f$ on $\tilde{S}$ is invariant under the action of the fundamental group of $S$, and thus defines a holomorphic quadratic differential on $S$.

1.3.2. Harmonic maps and Teichmüller space. Let $S$ be a closed surface of negative Euler characteristic. We view here the Teichmüller space $T(S)$ as the space of hyperbolic metrics on $S$, where two such metrics are identified when there is an isometry between them which is homotopic to the identity on $S$. Fixing a base point $g_0$ in Teichmüller space, and another point $g_1$, one can look at the unique harmonic map from $(S, g_0)$ to $(S, g_1)$ homotopic to the identity. The Schoen–Yau theorem [20] states that this map is a diffeomorphism. By Proposition 1.5, the Hopf differential of this map is holomorphic with respect to the complex structure given by $g_0$. This constructs a map from the Teichmüller space to the vector space of holomorphic quadratic differentials on $(S, [g_0])$. Sampson [19] proved that this map is injective and Wolf [24] proved that it is surjective. Those results can be summed up in one theorem.

**Theorem** (Schoen–Yau, Sampson, Wolf). Let $(S, [g_0])$ be a closed Riemann surface, and $\Phi$ a holomorphic quadratic differential on $S$. Then there is a unique Riemannian metric $g_1$ on $S$ of curvature $-1$ such that

$$g_1 = \alpha g_0 + \Phi + \bar{\Phi}$$

for some positive function $\alpha$.

2. Proof of theorem A

2.1. Representations fixing a point at infinity. Our strategy for proving that any representation $\rho : \pi_1(S) \to \text{Isom}(M, g)$ can be dominated is based on the existence of a $\rho$-equivariant harmonic map from $\tilde{S}$ to $M$. Hence, we must first say something about representations fixing a point in $\partial M$, for which Labourie’s theorem does not hold. Fortunately, in this case, there is a trick which reduces the problem to the (easy) abelian case.

Assume that $\rho(\pi_1(S))$ fixes a point $a \in \partial M$. Given a geodesic ray $\gamma : [0, \infty) \to M$ which tends to $a$ at infinity, we classically define the Busemann function as

$$\beta_\gamma(x) := \lim_{t \to \infty} (d(\gamma(t), x) - t).$$
Under the assumption that \((M, g)\) is simply connected with curvature bounded away from 0, the following properties hold: for every element \(\varphi \in \text{Isom}(M, g)\) such that \(\varphi(a) = a\), there exists a real number \(m(\varphi)\) such that for every \(x \in M\),

\[
\beta_\gamma(\varphi(x)) = \beta_\gamma(x) + m(\varphi),
\]

and moreover

\[
l(\varphi) = |m(\varphi)|.
\]

We refer to the survey [5] for details about the theory of Gromov hyperbolic spaces.

In the case where \(\rho : \pi_1(S) \to \text{Isom}(M, g)\) fixes the point \(a \in \partial M\), the function \(m \circ \rho : \pi_1(S) \to \mathbb{R}\) is a morphism by (6). Let \(\gamma'\) be an oriented bi-infinite geodesic in the hyperbolic plane \(\mathbb{H}^2\) of constant curvature \(-1\) and let \(\rho'\) be the representation from \(\pi_1(S)\) to \(\mathbb{H}^2\) preserving the geodesic \(\gamma'\) and acting on \(\gamma'\) by translations given by \(m \circ \rho\). Equation (7) reduces the problem of dominating \(\rho\) to the problem of dominating \(\rho'\). But now, \(\rho'\) fixes a geodesic, and Labourie’s theorem can thus be applied to \(\rho'\). We obtain a \((\Gamma, \rho')\)-equivariant harmonic map taking values in the geodesic. This harmonic map can be constructed by integrating the harmonic 1-form having the cohomology class of \(m \circ \rho \in H^1(S, \mathbb{R})\).

2.2. Proof of theorem A. Let \(S\) be a closed oriented surface of negative Euler characteristic, \(\Gamma\) its fundamental group, and \(\rho\) a representation of \(\Gamma\) into \(\text{Isom}(M, g)\) that does not fix a point in \(\partial M\). Fix an arbitrary hyperbolic metric \(g_0\) on \(S\). By Labourie’s theorem, we can consider a \((\Gamma, \rho)\)-equivariant harmonic map \(f : (\tilde{S}, \tilde{g}_0) \to (M, g)\). Let \(\Phi\) be the holomorphic quadratic differential on \(S\) such that

\[
f^*g = \alpha g_0 + \Phi + \bar{\Phi}
\]

for some real function \(\alpha\).

Wolf’s theorem (combined with Schoen–Yau) gives us a hyperbolic metric \(g_1\) on \(S\) such that \(g_1 = \alpha_1 g_0 + \Phi + \bar{\Phi}\) for some positive function \(\alpha_1\). Then our main theorem is the direct consequence of the following lemma:

**Lemma 2.1.** Either \(f\) induces a diffeomorphism from \(\tilde{S}\) to a totally geodesic plane \(\mathbb{H}^2 \subset M\) of curvature \(-1\), or we have

\[
f^*g < g_1
\]

on all \(S\).

Indeed, if we know this inequality holds, and since \(S\) is compact, there is a constant \(\lambda < 1\) such that \(f^*g \leq \lambda g_1\). Let \(j_1\) be a holonomy representation of \(g_1\) (i.e. a representation of \(\Gamma\) into \(\text{PSL}(2, \mathbb{R})\)) such that \((S, g_1)\) is isometric to \(j_1(\Gamma) \setminus \mathbb{H}^2\). Then \(f : (\tilde{S}, g_1) \to \mathbb{H}^2\) is \(\lambda\)-Lipschitz and \((j_1, \rho)\)-equivariant, and therefore \(j_1\) dominates \(\rho\).

2.2.1. The functions \(H_i\) and \(L_i\). Recall that \(g_0\) induces a natural hermitian metric on the line bundle \(K^2_S\). When given a quadratic differential \(\Phi\), the function \(|\Phi|^2_{g_0}\) is thus well-defined on \(S\).

Let us fix a metric \(g_0\) of curvature \(-1\) on \(S\), and let \(g'\) be a non-negative Riemann tensor on \(S\) of the form

\[
\alpha g_0 + \Phi + \bar{\Phi},
\]

with \(\alpha\) a real function and \(\Phi\) a holomorphic quadratic differential.
Proposition 2.2. We have:

\[
\begin{align*}
    e_{g_0}(g') &= \alpha, \\
    \det g' &= e(g')^2 - 4|\Phi|_{g_0}^2.
\end{align*}
\]

Proof. In a local complex coordinate \( z \), we have

\[ g_0 = \sigma dz d\bar{z} \]

for some real positive function \( \sigma \), and

\[ g' = \alpha \sigma dz d\bar{z} + \varphi dz^2 + \bar{\varphi} d\bar{z}^2 \]

for some complex valued (holomorphic) function \( \varphi \). In coordinates \( x = \Re(z), y = \Im(z) \), we thus get

\[ g_0 = \sigma (dx^2 + dy^2) \]

and

\[ g' = (\alpha \sigma + 2\Re(\varphi)) dx^2 + (\alpha \sigma - 2\Re(\varphi)) dy^2 + 4\Im(\varphi) dx dy. \]

From this, we deduce that

\[ e_{g_0}(g') = \alpha \]

and

\[ \det(g') = \alpha^2 - 4|\varphi|^2 = \alpha^2 - |\Phi|_{g_0}^2. \]

Since \( g' \) is non-negative, we obtain that \( e(g')^2 - 4|\Phi|_{g_0}^2 \geq 0 \), from which we can deduce that the system of equations

\[
\begin{align*}
    x + y &= e(g') \\
    xy &= |\Phi|_{g_0}^2
\end{align*}
\]

has two non-negative (eventually identical) solutions. We will use the following lemma:

Lemma 2.3. Let \( H \) and \( L \) be the two functions on \( S \) such that

- \( H \geq L \)
- \( H + L = e_{g_0}(g') \)
- \( HL = |\Phi|_{g_0}^2 \)

Then \( (H - L)^2 = \det_{g_0} g' \), and wherever \( g' \) is non degenerate, \( H \) and \( L \) are solutions of the partial differential equation:

\[ \Delta_0 \log(u) = -2\kappa(g') \left( u - \frac{|\Phi|_{g_0}^2}{u} \right) - 2 \]

where \( \kappa(g') \) denotes the Gauss curvature of the Riemannian metric \( g' \).

This lemma is the direct consequence of several remarks that can be found in Sampson’s paper [19]. The way we present things, it can be proven by a straightforward (but rather tedious) computation in local conformal coordinates.

Let’s go back to our setting, where \( f^* g = e(f)g_0 + \Phi + \bar{\Phi} \) and where \( g_1 = e(g_1)g_0 + \Phi + \bar{\Phi} \). We introduce \( H_1 \) and \( L_1 \) such that

\[
\begin{align*}
    H_1 &\geq L_1, \\
    e(g_1) &= H_1 + L_1
\end{align*}
\]

and

\[ H_1 L_1 = |\Phi|^2 \]
Similarly, let $H_2$ and $L_2$ be such that $H_2 \geq L_2$,

$$e(f) = H_2 + L_2$$

and

$$H_2L_2 = |\Phi|^2.$$ 

Then we have:

- $L_1, H_1, L_2, H_2$ are non-negative, and $H_1 > L_1$ everywhere,
- $\text{Vol}_1 = (H_1 - L_1)\text{Vol}_0$,
- $\det_{g_0} f^*g = (H_2 - L_2)^2$,
- $H_1, L_1$ are both solutions of the following partial differential equation:

$$\Delta_0 \log(u) = 2u - \frac{|\Phi|^2}{u} - 2$$

where $\Delta_0$ is the Laplace operator associated to the metric $g_0$.
- On the open set $U$ of $S$ where $f^*g$ is non degenerate, $H_2, L_2$ are both smooth and solutions of

$$\Delta_0 \log(u) = 2\beta u - \frac{|\Phi|^2}{u} - 2,$$

where $\beta = -\kappa$ and $\kappa$ is the sectional curvature of the Riemannian metric $f^*g$.

**Remark 2.4.** The domain $U$ where $f^*g$ is non-degenerate is either empty or dense, by [19, Corollary of Theorem 3].

We will need the following presumably well-known result (see [19, Theorem 7] for a slightly weaker statement):

**Lemma 2.5.** For all $x \in U$ we have $\kappa(f^*g)(x) \leq -1$. Moreover, the inequality is strict, unless the second fundamental form of $f(S)$ vanishes at $f(\tilde{x})$ ($\tilde{x}$ being any lift of $x$ in $\tilde{S}$). In particular, if $\kappa(f^*g)$ is identically $-1$ on $U$, then the image of $\tilde{U}$ by $f$ is totally geodesic.

**Proof.** By definition of $U$, $f$ restricted to the lift $\tilde{U}$ is an immersion. Let $V \subset \tilde{U}$ be an open subset, small enough so that $N := f(V)$ is an embedded submanifold. Since $f : (V, f^*g) \to (N, g_N)$ is an isometry, the only thing we want to prove is that $(N, g_N)$ has curvature $\leq -1$. Take an orthonormal frame $e_1, e_2$ of $TN$. The curvature of $N$ is related to the sectional curvature of $TN$ in the ambient space $M$ by the following relation:

$$\kappa^N = \kappa^M(TN) + \langle II^N(e_1, e_1), II^N(e_2, e_2) \rangle - \|II^N(e_1, e_2)\|^2,$$

where $II^N(u, v)$ is the second fundamental form of $N$. This formula can be re-expressed as

$$\kappa^N = \kappa^M(TN) + \frac{\text{codim}(N)}{\text{Jac}(f)^2} \cdot \mathbb{E} \left( \det_{g_0}(<II^N(df(\cdot), df(\cdot)), n>) > 0 \right)$$

where the average is taken over all the unitary vectors $n$ normal to $N$ with respect to normalized Haar measure, and $\text{Jac}(f)$ stands for the Jacobian of the map $f : (V, g_0) \to (N, g)$. The second fundamental form of $f$ (see (3)) and of $N$ are related by

$$II^f(X, Y) = II^N(df(X), df(Y)) + df \left( \nabla^f_X Y - \nabla^g_{fX} Y \right).$$

In particular, since both summands on the right hand side are orthogonal, using the harmonicity of $f$, we infer that

$$\text{Tr}_{g_0} II^N(df(\cdot), df(\cdot)) = 0,$$
and in particular for every unitary vector \( n \in TM \) normal to \( TN \), we get
\[
\text{Tr}_{g_0} < II^N(df(\cdot), df(\cdot)), n > = 0.
\]
This shows that the eigenvalues of the quadratic form \( < II^N(df(\cdot), df(\cdot)), n > \) are opposite, hence \( \det_{g_0}(< II^N(df(\cdot), df(\cdot)), n >) \leq 0 \), with equality if only if the quadratic form \( < II^N(df(\cdot), df(\cdot)), n > \) vanishes. From (11), we deduce that \( \kappa^N \leq \kappa^M(TN) \leq -1 \). Equality implies the vanishing of the second fundamental form. This proves the lemma.

**Lemma 2.6.** Either \( f \) induces a diffeomorphism from \( \tilde{S} \) to a totally geodesic plane \( \mathbb{H}^2 \subset M \) of curvature \(-1\), or we have
\[
H_2 < H_1.
\]

**Proof.** Recall that \( U \) is the domain of \( S \) where \( f^* g \) is non-degenerate. First, note that on the complement of \( U \), we have
\[
H_2 = L_2 = \sqrt{\Phi^2_{g_0}} = \sqrt{L_1 H_1 < H_1}.
\]
We shall then only focus on what happens in the domain \( U \).

Recall that \( H_1 \) does not vanish since \( H_1 > L_1 \geq 0 \). Let \( x \in S \) be a point on \( S \) such that \( H_2(x)/H_1(x) \) is maximal. Assume by contradiction that \( H_2(x) > H_1(x) \). Then \( L_2(x) < H_2(x) \), and therefore \( x \) is in \( U \). Equations (8) and (9) and the relation \( H_2 L_2 = |\Phi|^2_{g_0} \) show that at the point \( x \),
\[
(13) \quad \Delta_0 \log \left( \frac{H_2}{H_1} \right) = 2(H_2 - H_1) + 2|\Phi|^2_{g_0} \left( \frac{1}{H_1} - \frac{1}{H_2} \right) + 2(\beta - 1)(H_2 - L_2),
\]
where \( \beta = -\kappa(f^* g) \). By lemma 2.5, we have \( \beta \geq 1 \). Since \( H_2 \geq L_2 \) by hypothesis, the last summand in equation (13) is non-negative. Therefore, the assumption \( H_2 > H_1 \) clearly implies that \( \Delta_0 \log \left( \frac{H_2}{H_1} \right)(x) > 0 \), which contradicts the fact that \( \log \left( \frac{H_2}{H_1} \right) \) admits a maximum at \( x \). At this maximum, we must have \( H_2 \leq H_1 \), and thus \( H_2 \leq H_1 \) everywhere.

To get the strict bound, we will use the following version of the strict maximum principle, which was probably already known by Picard:

**Lemma 2.7** (Picard). Let \( w \) be a real non-positive function on a domain \( U \) of \( \mathbb{C} \), such that \( \Delta w \geq Kw \) for some constant \( K > 0 \). Then either \( w \equiv 0 \) on \( U \), or \( w < 0 \) on \( U \).

In this theorem, \( \Delta \) is a priori the laplace operator associated to a flat metric, but, since it is a local result, the conclusion still holds for any conformal metric. We apply Lemma 2.7 to the function \( w = \log \left( \frac{H_2}{H_1} \right) \). Since \( \beta \geq 1 \), equation (13) shows that
\[
\Delta_0 \log \left( \frac{H_2}{H_1} \right) \geq 2 + \frac{2|\Phi|^2_{g_0}}{H_1 H_2} (H_2 - H_1),
\]
so that we get
\[
\Delta_0 w \geq 2(H_1 + L_2)(e^w - 1) \geq Kw,
\]
where \( K := \max_S(H_1 + L_2) \). Lemma 2.7 shows that either \( H_2 \) is identically equal to \( H_1 \) on \( U \), or \( H_2 < H_1 \) on \( U \). But on the complement of \( U \), we already saw that \( H_2 < H_1 \). In the case \( H_2 = H_1 \), the complement of \( U \) is thus empty and equation (13) shows that necessarily \( \beta = 1 \) on \( S \), hence \( \kappa(f^* g) = -1 \) on \( S \). Lemma 2.5 shows that \( \mathbb{H}^2 = f(S) \) is a totally geodesic plane of curvature \(-1\). Moreover, in that case, we also have \( L_2 = L_1 \) and thus \( f^* g = g_1 \), which means that \( f : (S, g_1) \to M \) is an isometric embedding. Hence Lemma 2.6 is proved. \( \square \)
Let us finish the proof of lemma 2.1. According to lemma 2.6, if \( f \) is not an isometric embedding with totally geodesic image, then we have \( H_2 < H_1 \) everywhere. Since \( H_2 L_2 = H_1 L_1 \) we also have \( L_2 > L_1 \). Therefore, \((H_2 - L_2)^2 < (H_1 - L_1)^2\), and by adding \( 4H_2 L_2 \) to each member, we get that \((H_2 + L_2)^2 < (H_1 + L_1)^2\).

Now, remember that \( f^*g = (H_2 + L_2)g_0 + \Phi + \Phi \) and \( g_1 = (H_1 + L_1)g_0 + \Phi + \Phi \). It is then clear that \( H_2 + L_2 < H_1 + L_1 \) implies \( f^*g < g_1 \).

3. Extension of the theorem to lattices in \( \text{PSL}(2, \mathbb{R}) \) with torsion

Here we extend the theorem when \( \Gamma \) is a lattice in \( \text{PSL}(2, \mathbb{R}) \) with torsion.

**Theorem F.** Let \( \Gamma \) be a lattice in \( \text{PSL}(2, \mathbb{R}) \) and \( \rho \) a representation of \( \Gamma \) into the isometry group of a smooth complete simply connected Riemannian manifold \( M \) of sectional curvature \( \leq -1 \). Assume the action of \( \rho(\Gamma) \) on \( \partial M \) does not have a finite orbit. Then there exists a Fuchsian representation \( j : \Gamma \rightarrow \text{PSL}(2, \mathbb{R}) \) and a \((j, \rho)\)-equivariant map from \( \mathbb{H}^2 \) to \( M \) which is either a contraction or an isometric and totally geodesic embedding.

**Remark 3.1.** Note that this result is relevant to the study of anti-de Sitter lorentz manifolds of dimension 3. Indeed, even when \( \Gamma \) is a lattice in \( \text{PSL}(2, \mathbb{R}) \) may not have any fixed point. If \( j \) strictly dominates \( \rho \), then the action of \( \Gamma_j, \rho \) on \( \text{PSL}(2, \mathbb{R}) \) is properly discontinuous and cocompact. It is free if, furthermore, for any \( \gamma \in \Gamma \) with torsion, \( \rho(\gamma) \) has order strictly smaller than \( \gamma \). In that case, the quotient \( \Gamma_j, \rho \backslash \text{PSL}(2, \mathbb{R}) \) is a smooth anti-de Sitter 3-manifold which is a Seifert bundle over the orbifold \( j(\Gamma) \backslash \mathbb{H}^2 \).

**Proof.** By Selberg’s lemma, we can consider a finite index normal subgroup \( \Gamma_0 \subset \Gamma \) which is torsion-free. Hence the quotient \( S = \Gamma_0 \backslash \mathbb{H}^2 \) is a closed hyperbolic surface. We now mimic the proof of the main theorem.

Since the action of \( \rho(\Gamma) \) on \( \partial M \) does not have any finite orbit, \( \rho(\Gamma_0) \) does not fix a point in \( \partial M \). Let \( f \) be the unique \((\Gamma_0, \rho)\)-equivariant harmonic map from \( \mathbb{H}^2 \) to \( M \). We prove that \( f \) is actually \((\Gamma, \rho)\)-equivariant. Indeed, for some \( \gamma \in \Gamma \), consider the map \( f' : x \rightarrow \rho(\gamma)^{-1}f(\gamma \cdot x) \). It is harmonic and \((\Gamma_0, \rho)\)-equivariant. By uniqueness, \( f' = f \), and we get that \( f(\gamma \cdot x) = \rho(\gamma) \cdot f(x) \). This being true for any \( \gamma \in \Gamma \), we obtain that \( f \) is \((\Gamma, \rho)\)-equivariant.

Let \( g_0 \) denote the hyperbolic metric on \( S = \Gamma_0 \backslash \mathbb{H}^2 \). Then \( g_0 \) is \( \Gamma/\Gamma_0 \)-invariant, and so is the Hopf differential \( \Phi \) of \( f \). Hence, the unique hyperbolic metric \( g_1 \) on \( S \) of the form \( g_1 = \alpha g_0 + \Phi + \Phi \) is \( \Gamma/\Gamma_0 \)-invariant. Therefore, \( g_1 \) induces an orbifold hyperbolic metric on the quotient of \( S \) by \( \Gamma/\Gamma_0 \). The holonomy of this metric gives a representation \( j : \Gamma \rightarrow \text{PSL}(2, \mathbb{R}) \).

By construction, the map \( f \) is \((j, \rho)\)-equivariant and for the same reason as before, it is either a contraction or an isometric and totally geodesic embedding.

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