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A robust numerical two-level second-order explicit approach to predicting the spread of Covid-2019 pandemic with undetected infectious cases

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1. Introduction and motivation

Deterministic models are important decision tools that can be useful to forecasting different scenarios. Usually, the initial study of such models is based on the use of the theory of ordinary/partial differential equations and a low computational complexity which can permit a better calibration of the model characteristics. Furthermore, deterministic approaches are the only suitable methods that can be used when modeling a new problem with few data. For more details, we refer the readers to [1–4] and references therein. The use of the mathematical models as a predictive tool in the simulation of complex problems arising in a broad range of practical applications in biology, environmental fluid mechanic, chemistry and applied mathematics.

Early in an epidemic, the quality of the data on infections, deaths, tests and other factors often are limited by undetection or inconsistent detection of cases, reporting delays, and poor documentation, all of which affect the quality of any model output. Simpler models may provide less valid predictions since they cannot capture complex and unobserved human mixing patterns and other time-varying parameters of infectious disease spread. Also, complex models may be no more reliable than simpler ones if they miss key biological aspects [5]. At a time when numbers of cases and deaths from coronavirus 2019 (Covid-19) pandemic continue to increase with alarming speed, accurate forecasts from mathematical
models are increasingly important for physicians, politicians, epidemiologists, the public and most importantly, for authorities responsible of organizing care for the populations they serve. Given the unpredictable behavior of severe acute respiratory syndrome Covid-19, it is worth mentioning that efficient numerical approaches are the best tools that can be used to predict the spread of the disease with reasonable accuracy. These predictions have crucial consequences regarding how quickly and strongly the government of a country moves to curb a pandemic. However, assuming the worst-case scenario at state and national levels will lead to inefficiencies (such as: the competition for beds and supplies) and may compromise the quality of care, whereas supposing the best-case scenario can lead to a disastrous lack of preparation.

In this paper, we develop an efficient numerical scheme for solving a mathematical model well adapted to Covid-19 pandemic subjected to special characteristics (effect of undetected infected cases, effect of different sanitary and infectiousness conditions of hospitalized people and estimation of the needs of beds in hospitals) and considering different scenarios [6]. Specifically, the proposed technique should provide the numbers of detected infected and undetected infected cases, numbers of deaths and needs of beds in hospitals in countries (for example, in Cameroon) where Covid-19 is a very serious health problem. It is worth noticing that the model of Covid-19 considered in this work has been obtained under the assumption of "only within-country disease spread" for territories with relevant number of people infected by the virus SARS-Cov-2, where local transmission is the major cause of the virus spread (for instance: case of Cameroon). Furthermore, the parameters of the model used in this note are taken from the literature [6–9]. The study also considers the disease fatality rate together with the proportion of detected cases compared to the total number of infectious (detected or undetected) which allow to analyze the impact of Covid-19 pandemic. In addition, to demonstrate the efficiency and validity of the new approach when applied to the mathematical model for the Covid-19 epidemic, we consider the case of Cameroon, the country of the central Africa where one can observe the highest number of people infected by the new virus SARS-Cov-2. We compare the results produced by the numerical method to the data obtained from this country and those provided by the World Health Organization in its reports [10]. Finally, it is important to mention that the considered area (Cameroon) in the numerical experiments can be replaced by any territory worldwide.

This paper is organized as follows: Section 2 considers some preliminaries together with the mathematical model for the spreading of Covid-19. In Section 3, we provide a full description of the two-level explicit scheme for solving the problem indicated in Section 2. Section 4 analyzes the stability and the convergence rate of the new procedure while a large set of numerical experiments are presented and critically discussed in Section 5. We draw in Section 6 the general conclusion and we describe our future work.

2. Preliminaries and the mathematical model of Covid-19

We use a mathematical formalism [6] that describes how the disease changes as a function of time since infections for a representative cohort of infected persons. We assume that transmission of the Covid-19 is contagious from person to person and not point source. Furthermore, it is also assumed that, at the initial phase of Covid-19 disease, the proportion of the population with immunity to the virus SARS-CoV-2 is negligible [11–14]. At the beginning of an epidemic, a small number of infected individuals spread the disease to the whole population. The model assumes that each individual can go through nine stages: Starting as susceptible ($X_1$: not infected); exposed ($X_2$: the person is in incubation period after being infected, but has no clinical symptoms); infective ($X_3$: the person completed the incubation period, may infect others and develop clinical signs. Here, a person may test positive to the disease or not be tested and continue as infectious); infectious but undetected ($X_4$: the person can infect others, have clinical symptoms, but is not detected and reported (will not die)), hospitalized or in quarantine at home ($X_5$: the person can infect others, but will recover); hospitalized but will die ($X_6$: the person is hospitalized and can infect other people, but will die); recovered after being previously detected as infectious ($X_7$: the person survived the disease, is no longer infectious and has developed a natural immunity to the virus); recovered after being previously infectious but undetected ($X_8$: the person was not previously detected as infectious, survived the disease, is no longer infectious and has developed a natural immunity to the disease), dead because of Covid-19 ($X_9$).

The proposed model is based on thirteen parameters.

- $R_0$ denotes basis reproductive number, that is, the expected number of new infectious cases per infectious case,
- $N$ is the population before the starting of the pandemic,
- $\mu_1 \in [0, 1]$, designates the birth rate ($1/day$) (the number of births per day and per capita),
- $\mu_2 \in [0, 1]$, represents the death rate ($1/day$) (the number of deaths per day and per capita),
- $w(t) \in [\underline{w}, \overline{w}] \subset [0, 1]$, denotes the case fatality rate in the considered territory at time $t$ (the proportion of deaths compared to the total number of infectious people (detected or undetected). Here, $\underline{w}$ and $\overline{w}$ are the minimum and maximum case fatality rates in the country, respectively),
- $\theta(t) \in [\underline{w}, \overline{w}]$, means the fraction of infected people that are detected and reported by the authorities in the country at time $t$. For the convenience of writing, we assume that all the deaths due to Covid-19 disease are detected and reported, so $\theta(t) \geq \overline{w}$,
- $\beta_j \in \mathbb{R}^+$, for $j = 2, 3, \ldots, 6$, are the disease contact rates ($day^{-1}$) of a person in the corresponding compartment $X_j$ in the country (without taking into account the control measures),
- $\beta_d(\theta) \in \mathbb{R}^+$, represents the disease contact rates ($day^{-1}$) of a person in compartment $X_4$, in the country (without taking into account the control measures), where the fraction of infected individuals that are detected is $\theta(t)$,
• \( \gamma_X \in (0, \infty) \), designates the transition rate \((\text{day}^{-1})\) from compartment \( X_2 \) to compartment \( X_3 \). It is the same in all the countries,
• \( \gamma_X(t) \in (0, \infty) \), is the transition rate \((\text{day}^{-1})\) from compartment \( X_3 \) to compartments \( X_4, X_5 \) or \( X_6 \) at time \( t \). It can change from a country to another,
• \( \gamma_{X_4}(t), \gamma_{X_5}(t) \) and \( \gamma_{X_6}(t) \in [0, \infty) \), denote the transition rate \((\text{day}^{-1})\) from compartments \( X_4, X_5 \) or \( X_6 \) to compartments \( X_7, X_8 \) and \( X_9 \), respectively, in the considered country at time \( t \),
• \( m_{X_j}(t) \in [0, 1] \), for \( j = 2, 3, \ldots, 6 \), are functions representing the efficiency of the control measures applied to the corresponding compartment in the considered country at time \( t \),
• \( \tau_1 \) is the person infected that arrives in the territory from other countries per day. \( \tau_2 \) is the person infected that leaves the territory from other countries per day. Both can be modeled following the between-country spread part of the Be-CoDis model, see [15].

The control measures applied by the government to curb the virus SARS-CoV-2 spread are those provided by the WHO in [16,17]:

• isolation: infected people are isolated from contact with other persons. Only health professional is in contact with them. Isolated patients receive an adequate medical treatment that reduces the Covid-19 fatality rate,
• quarantine: movement of people in the area of origin of an infected person is restricted and controlled (for instance: quick sanitary check-points at the airports) to avoid that possible infected people spread the disease,
• tracing: the aim of tracing is to identify potential infectious contacts which may have infected an individual or spread the virus SARS-CoV-2 to other people. Increase the number of tests in order to increase the percentage of detected infected persons,
• increase of sanitary resources: number of operational beds and sanitary personal available to detect and treat affected people is increased, producing a decrease in the infectious period for the compartment \( X_3 \).

Furthermore, the mathematical model of coronavirus 2019 epidemic considers the following assumptions:

\( (a_1) \) the population at risk is large enough and time period of concern is short enough that over the time period of interest, very close to 100% of the population is susceptible,
\( (a_2) \) the pandemic is at the early stage and has not reached the point where the susceptible population decreases so much due to death or post-infection immunity that the average number of secondary cases falls,
\( (a_3) \) unprotected contact results in infection,
\( (a_4) \) the epidemic in the population of interest begins with a single host (note that the equations used in computing cases and deaths are easily modified if this is not the case),
\( (a_5) \) infectivity occurs during the incubation period only (that is, a person is infected by the virus SARS-CoV-2 but has no symptom. This corresponds to individuals in compartment \( X_2 \)),
\( (a_6) \) the models are deterministic, that is, the thirteen parameters of Covid-19 spread cited above are constant values.

Under these assumptions, the mathematical formulation of Covid-19 disease is given by the following system of nonlinear ordinary differential equations:

\[
\begin{align*}
\frac{dX_1}{dt} &= -\frac{X_1}{N} \left[ m_{X_2}(t)\beta_{X_2}(t)X_2 + m_{X_3}(t)\beta_{X_3}(t)X_3 + m_{X_4}(t)\beta_{X_4}(t)X_4 + m_{X_5}(t)\beta_{X_5}(t)X_5 + m_{X_6}(t)\beta_{X_6}(t)X_6 \right] \\
&\quad - \mu_2X_1 + \mu_1[X_1 + X_2 + X_3 + X_4 + X_7 + X_8], \\
\frac{dX_2}{dt} &= \frac{X_1}{N} \left[ m_{X_2}(t)\beta_{X_2}(t)X_2 + m_{X_3}(t)\beta_{X_3}(t)X_3 + m_{X_4}(t)\beta_{X_4}(t)X_4 + m_{X_5}(t)\beta_{X_5}(t)X_5 + m_{X_6}(t)\beta_{X_6}(t)X_6 \right] \\
&\quad - \mu_2X_2 - \gamma_{X_2}(t)X_2 - \tau_1(t) - \tau_2(t), \\
\frac{dX_3}{dt} &= \gamma_{X_2}(t)X_2 - (\mu_2 + \gamma_{X_3}(t))X_3, \quad \frac{dX_4}{dt} = (1 - \theta(t))\gamma_{X_3}(t)X_3 - (\mu_2 + \gamma_{X_4}(t))X_4, \\
\frac{dX_5}{dt} &= (\theta(t) - w(t))\gamma_{X_3}(t)X_3 - \gamma_{X_5}(t)X_5, \quad \frac{dX_6}{dt} = w(t)\gamma_{X_3}(t)X_3 - \gamma_{X_6}(t)X_6, \\
\frac{dX_7}{dt} &= \gamma_{X_4}(t)X_4 - \mu_2X_7, \quad \frac{dX_8}{dt} = \gamma_{X_5}(t)X_5 - \mu_2X_8 \quad \text{and} \quad \frac{dX_9}{dt} = \gamma_{X_6}(t)X_6,
\end{align*}
\]

with the initial conditions
\[
X_j(t_0) = X^0_j \in (0, \infty), \quad \text{for } j = 1, 2, \ldots, 9,
\]
where all the unknowns $X_j$ dependent on the time $t \in [t_0, T_{\text{max}}]$. Setting $X(t) = (X_1, X_2, \ldots, X_9)^T$ and $F(t, X(t)) = (F_1(t, X(t)), F_2(t, X(t)), \ldots, F_9(t, X(t)))^T$, where

\[
F_1(t, X(t)) = -\frac{X_1}{N} \left[ m_{X_2}(t)\beta_{X_2}(t)X_2 + m_{X_3}(t)\beta_{X_3}(t)X_3 + m_{X_4}(t)\beta_{X_4}(t)X_4 + m_{X_5}(t)\beta_{X_5}(t)X_5 \right]
+ m_{X_6}(t)\beta_{X_6}(t)X_6 - \mu_2 X_1 + \mu_1 [X_1 + X_2 + X_3 + X_4 + X_7 + X_8],
\]
\[F_2(t, X(t)) = \frac{X_1}{N} \left[ m_{X_2}(t)\beta_{X_2}(t)X_2 + m_{X_3}(t)\beta_{X_3}(t)X_3 + m_{X_4}(t)\beta_{X_4}(t)X_4 + m_{X_5}(t)\beta_{X_5}(t)X_5 \right]
+ m_{X_6}(t)\beta_{X_6}(t)X_6 - \mu_2 X_2 - \gamma_{X_2}(t)X_2 + \tau(t),
\]
\[F_3(t, X(t)) = \gamma_{X_2}(t)X_2 - (\mu_2 + \gamma_{X_2}(t))X_3,
F_4(t, X(t)) = (1 - \theta(t))\gamma_{X_3}(t)X_3 - (\mu_2 + \gamma_{X_3}(t))X_4,
\]
\[F_5(t, X(t)) = \gamma_{X_3}(t)X_3 - \gamma_{X_5}(t)X_5,
F_6(t, X(t)) = \gamma_{X_5}(t)X_5 - \gamma_{X_6}(t)X_6,
\]
\[F_7(t, X(t)) = \gamma_{X_6}(t)X_6 - \mu_2 X_7,
F_8(t, X(t)) = \gamma_{X_6}(t)X_4 - \mu_2 X_8
\text{ and } F_9(t, X(t)) = \gamma_{X_6}(t)X_6,
\]
the system of nonlinear equations (1)–(5) is equivalent to

\[
\frac{dX}{dt} = F(t, X),
\]

where $\tau(t) = \tau_1(t) - \tau_2(t)$.

**Remark.** In the modeling point of view, the term $\frac{w(t)}{n(t)}$ corresponds to the apparent fatality rate of the disease (obtained by considering only the detected cases) in the considered area at time $t$, whereas $w(t)$ is the real fatality rate of coronavirus 2019 disease.

Since the mathematical model of Covid-19 provided by the system of Eqs. (1)–(5) is too complex and because both natality and mortality (not from the virus SARS-Cov-2) do not seem to be useful factors for this pandemic (at least for relatively short periods of time), we assume in the following that

\[
\mu_m = \mu_n = 0.
\]

It is worth mentioning that the aim of this paper is to compute the following Covid-19 characteristics:

1. the model cumulative of coronavirus 2019 cases at day $t$ given by

\[
c_m(t) = X_5(t) + X_6(t) + X_8(t) + X_9(t).
\]

2. the model cumulative number of deaths (due to Covid-19 disease) at day $t$, which is given by $X_9(t)$.

3. $R_e(t)$ which is the effective reproductive number of Covid-19 epidemic,

4. the number of people in hospital is estimated by the following equation

\[
Host(t) = X_6(t) + p(t)[X_5(t) + (X_7(t) - X_7(t - d_0))],
\]

where $p(t)$ represents the fraction at time $t$, of people in compartment $X_5$ that are hospitalized and $d_0$ days is the period of convalescence (i.e., the time a person is still hospitalized after recovering from Covid-19 disease). This function can help to estimate and plan the number of clinical beds needed to treat all the virus SARS-Cov-2 cases at time $t$.

5. the maximum number of hospitalized persons at the same time in the territory during the time interval $[t_0, T_{\text{max}}]$, which is defined as

\[
\text{MaxHost} = \max_{t_0 \leq t \leq T_{\text{max}}} Host(t).
\]

$\text{MaxHost}$ can help to estimate and plan the number of clinical beds needed to treat all the coronavirus 2019 disease cases over the interval $[t_0, T_{\text{max}}]$.

6. the number of people infected during the time interval $[t_0, T_{\text{max}}]$, by contact with people in compartments $X_2, X_4$ and $X_{10} = X_5 + X_6$, respectively. They are defined as

\[
\Gamma_{X_2}(t) = \frac{1}{N} \int_{t_0}^{t} m_{X_2}(s)\beta_{X_2}(s)X_2(s)X_1(s)ds.
\]
Focusing on the application on the control strategies, the efficiency of these measures indicated in [21] satisfies

\[ m_X(t) := m(t) = \left\{ \begin{array}{ll}
(m_l - m_{l+1}) \exp[-k_{l+1}(t - \lambda_l)], & t \in [\lambda_l, \lambda_{l+1}), l = 0, 1, \ldots, q - 1, \\
(m_q - m_q) \exp[-k_q(t - \lambda_{q-1})], & t \in [\lambda_{q-1}, \infty),
\end{array} \right. \tag{22} \]

for \( j = 1, 2, \ldots, 6 \), where \( m_l \in [0, 1] \) measures the intensity of the control measures (greater value implies lower value of disease contact rates), \( k_l \in [0, \infty) \) (in day\(^{-1}\)) simulates the efficiency of the control strategies (greater value implies lower value of disease contact rates) and \( \lambda_l \in [0, \infty) \), \( l = 1, 2, \ldots, q \), denotes the first day of application of each control strategy. \( \lambda_0 \in [0, \infty) \) is the first day of application of a control measure that was being used before \( t_0 \), if any. In this work, \( q \in \mathbb{N} \) represents the number of changes of control strategy. In general, the values of \( \lambda_l \) are typically taken in the literature (using dates when the countries implement special control measures). It is important to remind that some of the values of \( m_l \) can be also sometimes known. The rest of the parameters needed to be calibrated.

In the following, we assume that the case fatality rate \( w(t) \), depends on the considered country, time \( t \), and it can be affected by the application of the control measures (such as, earlier detection, better sanitary condition, etc.) [15]. Thus, it satisfies equation

\[ w(t) = m(t)w + (1 - m(t))\overline{w}, \tag{23} \]

where \( \overline{w} \in [0, 1] \) is the case fatality rate when no control measures are applied (i.e., \( m(t) = 1 \)) and \( w \in [0, 1] \) is the case fatality rate when implemented control measures are fully applied (\( m(t) = m_q \)).

Denoting by \( d_{X_1}, d_{X_2}, d_{X_3}, \) and \( d_{X_4} \), be the “average” duration in days of a person in compartment \( X_1, X_2, X_3 \), and \( X_4 \), respectively, without the application of control strategies, we assume as in [7,9] that

- the transition rate from \( X_j \) to \( X_k \) depends on the virus and, therefore is considered constant, that is \( \gamma_j(t) = \alpha = \alpha_l \)
- the value of \( \gamma_j(t) := \gamma(t) \) can be increased due to the application of control measures (that is, people with symptoms are detected earlier). As a consequence, the values of \( \gamma_{X_1}(t) \), \( \gamma_{X_2}(t) \) and \( \gamma_{X_3}(t) \) can be decreased (i.e., persons with symptoms stay under observation during more time).
- \( d_{X_1} = d_{X_2} + \delta, \delta > 0 \),
• for the sake of readability, the infectious period for undetected individuals is the same than that of hospitalized people that survive the disease. So \( d_{x_3} = d_{x_4} \). Furthermore, we suppose that the additional time a person is in the compartments \( X_3 \) and \( X_4 \) is constant, so it comes from [15] that

\[
\gamma(t) := \gamma_{x_3}(t) = \frac{1}{d_{x_3} - g(t)}, \quad (\text{day}^{-1})
\]

(24)

\[
\rho(t) := \gamma_{x_3}(t) = \frac{1}{d_{x_3} + g(t)}, \quad (\text{day}^{-1})
\]

(25)

\[
\psi(t) := \gamma_{x_3}(t) = \frac{1}{d_{x_3} + g(t) + \delta}, \quad (\text{day}^{-1})
\]

(26)

where \( g(t) = d_{x_3}(1 - m(t)) \) represents the decrease of the duration of the function \( d_{x_3} \) due to the application of the control measures at time \( t \). \( d_{x_3} \) is the maximum number of days that \( d_{x_3} \) can be decreased due to the control measures.

Finally, the disease contact rate \( \beta_{x_3}(\theta) \) is defined by

\[
\beta_{x_3}(\theta) = \begin{cases} 
\bar{\beta}_{x_3}, & \text{if } \theta = \bar{w}, \\
\beta_{x_3}, & \text{nonincrease}, \\
\underline{\beta}_{x_3}, & \text{if } \theta = \underline{w},
\end{cases}
\]

(27)

where \( \bar{\beta}_{x_3} \) and \( \underline{\beta}_{x_3} \) are suitable lower and upper bounds, respectively. For the convenience of writing, we assume that \( \bar{\beta}_{x_3} = \beta_{x_3} := \beta \). In addition, the people in compartments \( X_2, X_4, X_5 \) and \( X_6 \) are less infectious than people in compartment \( X_3 \) (due to their lower virus load or isolation measures). This fact results in

\[
\beta_{x_2} = c_{x_2}\beta_{x_3}, \quad \beta_{x_4} = c_{x_4}(t)\beta_{x_3}, \quad \beta_{x_5} = c_u\beta_{x_3},
\]

(28)

where \( c_{x_2}, c_{x_4}, c_u \in [0, 1] \).

3. Construction of the two-level explicit numerical scheme

In this section, we develop the robust numerical two-level explicit scheme for solving the mathematical problem (1)–(5) modeling the spread of Covid-19 with undetected cases.

Let \( h := \Delta t = \frac{t_{\text{max}} - t_0}{M} \) be the step size, \( M \) is a positive integer. Set \( t_n = t_0 + nh, t_{n-\frac{1}{2}} = \frac{t_n + t_{n-1}}{2} \) for \( n = 1, 2, \ldots, M \) and \( \Omega_h = \{ t_n, 0 \leq n \leq M \} \) be a regular partition of \([t_0, t_{\text{max}}]\). Let \( \mathcal{F}_h = \{ X_i^n, n = 0, 1, \ldots, M; 1 \leq i \leq 9 \} \) be the grid functions space defined on \( \Omega_h \times \mathbb{C}^9 \subset I \times \mathbb{C}^9 := [t_0, t_{\text{max}}] \times \mathbb{C}^9 \).

Define the following norms

\[
\| X^n \|_\infty = \max_{1 \leq i \leq 9} \| X^n_i \|_\infty \quad \text{and} \quad \| X \|_{l^2(I)} = \left( \frac{1}{h} \sum_{n=0}^{M} \| X^n \|_\infty^2 \right)^{\frac{1}{2}}
\]

(29)

where \( |\cdot| \) designates the norm defined on the field of complex numbers \( \mathbb{C} \). Furthermore, denote

\[
P_j(t, X(t)) = \sum_{l} F_l(t, X(t)) L_l(t),
\]

(30)

where the function \( L_l(t) \) is given by

\[
L_l(t) = \prod_{q \neq l} \frac{t - t_q}{t_l - t_q},
\]

(31)

be a polynomial of degree \( j \) interpolating the function \( F_l(t, X(t)) \) at the node points \( (t_l, F_l(t_l, X(t_l))) \).

Now, integrating both sides of Eq. (12) at the node points \( t_n \) and \( t_{n+\frac{1}{2}} \), this yields

\[
X(t_{n+\frac{1}{2}}) = X(t_n) + \int_{t_n}^{t_{n+\frac{1}{2}}} F(t, X) dt.
\]

(32)

For \( j = 1 \), \( P_j(t, X(t)) \) is a linear polynomial approximating the function \( F_l(t, X(t)) \) at the points \( (t_n, F_l(t_n, X(t_n))) \) and \( (t_{n+\frac{1}{2}}, F_l(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}))) \). Using Eqs. (30) and (31), it is easy to observe that

\[
P_j(t, X(t)) = F_l(t_n, X(t_n)) t - t_{n+\frac{1}{2}} \bigg[ F_l(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) - \frac{t - t_n}{t_{n+\frac{1}{2}} - t_n} \bigg] + \frac{t - t_n}{t_{n+\frac{1}{2}} - t_n} \bigg[ F_l(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) - F_l(t_n, X(t_n)) t + t_{n+\frac{1}{2}} F_l(t_n, X(t_n)) - t_n F_l(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) \bigg].
\]

(33)
where the error term is given by

\[
F_i(t, X(t)) - P_i(t, X(t)) = \frac{1}{2}(t - t_n)(t - t_{n+1}) \frac{d^2 F_i}{dt^2}(t_i, X(t_i)) := O(h^3),
\]  
(34)

where \( t_i \) (respectively, each component \( X_i(t_i) \) of \( X(t_i) \)) is an unknown function which is between the maximum and the minimum of the numbers \( t_n, t_{n+1} \) and \( t \) (respectively: \( X_i(t_n), X_i(t_{n+1}) \) and \( X_i(t) \)). But Eq. (34) can be rewritten as

\[
F_i(t, X(t)) = P_i(t, X(t)) + O(h^3), \quad \text{for } i = 1, 2, \ldots, 9.
\]
(35)

Substituting approximation (35) into the \( i \)th equation of the system (32), we obtain

\[
X_i(t_{n+\frac{1}{2}}) = X_i(t_n) + \int_{t_n}^{t_{n+\frac{1}{2}}} P_i(t, X(t)) dt + O(h^3), \quad \text{for } i = 1, 2, \ldots, 9,
\]
(36)

which is equivalent to the following system

\[
X(t_{n+\frac{1}{2}}) = X(t_n) + \int_{t_n}^{t_{n+\frac{1}{2}}} Q_i(t, X(t)) dt + O(h^3),
\]
(37)

where \( Q_i(t, X(t)) = \left( P_i^{(1)}(t, X(t)), P_i^{(2)}(t, X(t)), \ldots, P_i^{(9)}(t, X(t)) \right)^T \) and \( O(h^3) = (O(h^3), O(h^3), \ldots, O(h^3))^T \).

The integration of both sides of Eq. (33) provides

\[
\int_{t_n}^{t_{n+\frac{1}{2}}} P_i(t, X(t)) dt = \frac{1}{h} \left[ F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) - F_i(t_n, X(t_n)) \right] \left[ t_{n+\frac{1}{2}}^2 - t_n^2 \right] + 2 \left[ t_{n+\frac{1}{2}} F_i(t_n, X(t_n)) - t_n F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) \right] \left[ t_{n+\frac{1}{2}} - t_n \right] - t_n F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) \left[ t_{n+\frac{1}{2}} - t_n \right] = \frac{h}{4} \left[ F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) + F_i(t_n, X(t_n)) \right],
\]
(38)

where the last two equalities come from the identities \( t_{n+\frac{1}{2}}^2 - t_n^2 = (t_{n+\frac{1}{2}} - t_n)(t_{n+\frac{1}{2}} + t_n) \), \( t_{n+\frac{1}{2}} - t_n = \frac{h}{2} \) and \( t_{n+\frac{1}{2}} + t_n = 2t_n + \frac{1}{2} \). To get the desired first-level of the new algorithm, we should approximate the sum \( F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) + F_i(t_n, X(t_n)) \) by the term \( a_1 F_i(t_n, X(t_n)) + a_2 F_i(t_n + p_1 h, X(t_n) + p_2 h F_i(t_n, X(t_n))) \), in which the coefficients \( a_1, a_2, p_1 \) and \( p_2 \), are real numbers and are chosen so that the Taylor expansion

\[
\frac{X_i(t_{n+\frac{1}{2}}) - X_i(t_n)}{h/2} = \frac{1}{2} \left[ F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) + F_i(t_n, X(t_n)) \right] + O(h^2).
\]

Applying the Taylor series expansion for \( X_i \) and \( F_i \) about \( t_n \) and \( (t_n, X(t_n)) \), respectively, with step size \( h/2 \) using forward difference representations, direct calculations yield

\[
\frac{X_i(t_{n+\frac{1}{2}}) - X_i(t_n)}{h/2} = \frac{1}{2} \left[ F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) + F_i(t_n, X(t_n)) \right] = \left( 1 - \frac{a_1 + a_2}{2} \right) F_i(t_n, X(t_n)) +
\]

\[
\frac{h}{2} \left[ \left( \frac{1}{2} - a_2 p_1 \right) \partial t F_i(t_n, X(t_n)) + \left( a_2 - \frac{1}{2} \right) \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial k F_i(t_n, X(t_n)) \right] + O(h^2),
\]

which equals \( O(h^3) \) if and only if \( a_1 + a_2 = 2, a_2 p_1 = \frac{1}{2} \) and \( a_2 p_2 = \frac{1}{2} \). But the last two equations require \( a_2 \neq 0, p_1 \neq 0 \) and \( p_2 \neq 0 \). For instance, take \( p_1 = p_2 = \frac{1}{2} \), so \( a_1 = a_2 = 1 \). Using this, it is not hard to observe that the sum \( F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) + F_i(t_n, X(t_n)) \) is approximated as

\[
F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) + F_i(t_n, X(t_n)) = 2F_i(t_n, X(t_n)) + \frac{h}{2} \left[ \partial t F_i(t_n, X(t_n)) \right] +
\]

\[
\sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial k F_i(t_n, X(t_n)) \right] + O(h^2).
\]
(39)

A combination of Eqs. (38) and (39) results in

\[
\int_{t_n}^{t_{n+\frac{1}{2}}} p_i(t, X(t)) dt = \frac{h}{2} F_i(t_n, X(t_n)) + \frac{h^2}{8} \left[ \partial t F_i(t_n, X(t_n)) + \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial k F_i(t_n, X(t_n)) \right] + O(h^3).
\]
(40)
Substituting Eq. (40) into (36), this provides
\[ X_i(t_{n+1}) = X_i(t_n) + \frac{h}{2} F_i(t_n, X(t_n)) + \frac{h^2}{8} \left[ \partial_t F_i(t_n, X(t_n)) + \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_k F(t_n, X(t_n)) \right] + O(h^3). \] (41)

Tracking the infinitesimal term \( O(h^3) \), Eq. (41) can be approximated as
\[ X_i^{n+\frac{1}{2}} = X_i^n + \frac{h}{2} F_i(t_n, X^n) + \frac{h^2}{8} \left[ \partial_t F_i(t_n, X^n) + \sum_{k=1}^{9} F_k(t_n, X^n) \partial_k F_i(t_n, X^n) \right], \quad \text{for } i = 1, 2, \ldots, 9. \] (42)

The difference equations provided by relation (42) represent the first-level of the new approach.

To develop the second-level of the desired algorithm, we should integrate both sides of system (12) at the node points \( t_{n+\frac{1}{2}} \) and \( t_{n+1} \). This provides
\[ X_i(t_{n+1}) = X_i(t_{n+\frac{1}{2}}) + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} F_i(t, X)dt, \quad \text{for } i = 1, 2, \ldots, 9. \] (43)

Replacing \( F_i(t, X) \) by the linear interpolation polynomial \( P_i^{(1)}(t, X) \) at the node points \( (t_n, F_i(t_n, X^n)) \) and \( (t_{n+\frac{1}{2}}, F_i(t_{n+\frac{1}{2}}, X^n)) \), and using (34) and (43), simple computations give
\[ X_i(t_{n+1}) = X_i(t_{n+\frac{1}{2}}) + \frac{2}{h} \left[ \frac{1}{2} F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) - F_i(t_n, X(t_n)) \right] \left( t_{n+1} - t_{n+\frac{1}{2}} \right) + \left[ t_{n+1} F_i(t_n, X(t_n)) - t_{n+\frac{1}{2}} F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) \right] \left( t_{n+1} - t_{n+\frac{1}{2}} \right) + O(h^3) = X_i(t_{n+\frac{1}{2}}) +
\]
\[ \frac{h}{2} \left[ 3F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) - F_i(t_n, X(t_n)) \right] + O(h^3). \] (44)

Omitting the error term \( O(h^3) \), we obtain the second-level of the new method which is defined as
\[ X_i^{n+1} = X_i^{n+\frac{1}{2}} + \frac{h}{2} \left[ 3F_i(t_{n+\frac{1}{2}}, X^{n+\frac{1}{2}}) - F_i(t_n, X^n) \right], \quad \text{for } i = 1, 2, \ldots, 9, \] (45)
where the functions \( F_i(t, X), (i = 1, 2, \ldots, 9) \) are defined by relations (7)-(11). It is important to recall that assumptions given by Eqs. (13), (22), (23), (25) and (28) allow to get a simplified expression of \( F_i \).

To provide a full description of the two-level explicit formulation for solving the mathematical problem (1)-(5) to predicting the spread of the virus SARS-CoV-2, we should put together relations (42), (45) and the initial condition given by Eq. (6). Specifically, for \( n = 1, 2, \ldots, M - 1 \),
\[ X_i^{n+\frac{1}{2}} = X_i^n + \frac{h}{2} F_i(t_n, X^n) + \frac{h^2}{8} \left[ \partial_t F_i(t_n, X^n) + \sum_{k=1}^{9} F_k(t_n, X^n) \partial_k F_i(t_n, X^n) \right], \quad \text{for } i = 1, 2, \ldots, 9; \] (46)
\[ X_i^{n+1} = X_i^{n+\frac{1}{2}} + \frac{h}{2} \left[ 3F_i(t_{n+\frac{1}{2}}, X^{n+\frac{1}{2}}) - F_i(t_n, X^n) \right], \quad \text{for } i = 1, 2, \ldots, 9; \] (47)
subject to initial condition
\[ X_i(t_0) = X_i^0, \quad \text{for } i = 1, 2, \ldots, 9; \] (48)
where the functions \( F_i (i = 1, 2, \ldots, 9) \) are given by Eqs. (7)-(11).

With the tools provided in Section 3, we are ready to analyze the stability and the convergence rate of the two-level explicit procedure (46)-(48) for predicting the spread of the virus SARS-CoV-2 modeled by Eqs. (1)-(6).

4. Stability analysis and convergence rate of the new algorithm

In this section we wish to examine the stability and convergence rate of the new technique (46)-(48) applied to the initial-value problem (1)-(6).

Firstly, we define the strip \( S = \{(t, X), t_0 \leq t \leq T_{\text{max}}, X \in \mathbb{R}^9 \} \) in which both exact and computed solutions of problem (1)-(6) should lie. Performing direct calculations, it comes from Eqs. (7)-(11) that the functions \( F_i \) and their partial derivatives are continuous on the strip \( S \), while the partial derivatives are unbounded on this set. Thus it follows from the Henrici result [22] that the system of Eqs. (1)-(6) admits a unique solution \( X(t) \) defined in a certain neighborhood \( U(t_0) \subset [t_0, T_{\text{max}}] \) of the initial point \( t_0 \). Without loss of this constraint, we assume in the following that
$U(t_0) = [t_0, T_{\text{max}}]$ (indeed, we are dealing with a real world problem which, in reality should have a unique solution defined over the whole interval $[t_0, T_{\text{max}}]$). This shows the existence and uniqueness of the solution for the initial-value problem (1)–(6).

Let us introduce the functions $\Delta_l(t_i, X(t_i))$ and $\delta_l(t_i, X(t_i))$ (for $l \equiv n, n + \frac{1}{2}$) be the difference quotient of the exact solution $X(t_i)$ of Eqs. (1)–(6) at time $t_i$ and the difference quotient of the approximate solution $X_l^i$ of problem (46)–(48) obtained at time $t_i$, respectively. Moreover, $\Delta_l(t_i, X(t_i))$ and $\delta_l(t_i, X(t_i))$ are given by

\[
\Delta_l(t_i, X(t_i)) = \begin{cases} \frac{X(t_i + \frac{1}{2}) - X(t_i)}{h/2}, & \text{if } h \neq 0, \\ F_l(t_i, X(t_i)), & \text{if } h = 0, \end{cases}
\]

and

\[
\delta_l(t_i, X(t_i)) = \begin{cases} \frac{X_l^{i+\frac{1}{2}} - X_l^i}{h/2}, & \text{if } h \neq 0, \\ F_l(t_i, X_l^i), & \text{if } h = 0. \end{cases}
\]

It is worth mentioning that $\delta_l(t_i, X(t_i)) = \delta_l(t_i, X_l^i)$. The local discretization error at the point $(t_i, X(t_i))$ of the considered scheme is defined as

\[
\sigma_l(t_i, X(t_i)) = \Delta_l(t_i, X(t_i)) - \delta_l(t_i, X(t_i)), \quad l \equiv n, n + \frac{1}{2}, \quad \text{for } i = 1, 2, \ldots, 9,
\]

indicates how well the exact solution of the differential Eqs. (1)–(6) obeys by Eqs. (46)–(48) provided by the two-level explicit formulation.

The following result (Theorem 4.1) analyzes the stability and gives the convergence rate of the proposed approach (46)–(48).

**Theorem 4.1 (Stability Analysis and Convergence Rate).**

Let $e^n = X(t_n) - X^n$ be the global discretization error provided by algorithm (46)–(48), where $X(t_n)$ is the solution of system (1)–(6) obtained at time $t_n$ and $X^n$ is the one provided by (46)–(48) at time $t_n$. Thus, it holds

\[
\|X^n\| = \max_{1 \leq l \leq 9} |X^n_l| \leq C_1,
\]

which implies $\|X\|_{L^2(I)} = \left(\sum_{n=0}^{M} \|X^n\|_{\infty}^2 \right)^{1/2} \leq C_1$

where $C_1$ is a positive constant independent of the step size $h$ and $X$ denotes the approximate solution. Furthermore

\[
\|e^n\| = \max_{1 \leq l \leq 9} |e^n_l| \leq C_2 h^2,
\]

which implies $\|e\|_{L^2(I)} = \left(\sum_{n=0}^{M} \|e^n\|_{\infty}^2 \right)^{1/2} \leq C_2 h^2$

where $C_2$ is a positive constant that does not depend on the step size $h$. In the following we represent the analytical solution by $X(\cdot)$.

**Proof.** For a detailed proof of this Theorem, we refer the readers to Appendix. □

However, the proof of Theorem 4.1 requires the following intermediate result (namely Lemma 4.1).

**Lemma 4.1.** Suppose the vectors $q_l \in \mathbb{C}^l$ satisfy the estimates of the form

\[
\|q_{l+1}\|_{L^\infty(C^l)} \leq (1 + \epsilon)\|q_l\|_{L^\infty(C^l)} + \xi, \quad \epsilon > 0 \text{ and } \xi > 0,
\]

for $j = 0, 1, \ldots, m$. Then

\[
\|q_m\|_{L^\infty(C^l)} \leq \exp(m \epsilon)\|q_0\|_{L^\infty(C^l)} + \frac{\exp(m \epsilon) - 1}{\epsilon} \xi.
\]

**Proof.** We should prove inequality (53) by mathematical induction. Since $\exp(\epsilon) \geq 1 + \epsilon$ and $1 \leq \frac{\exp(\epsilon) - 1}{\epsilon}$ for any $\epsilon \geq 0$, using the assumption of Lemma 4.1, it is not difficult to see that

\[
\|q_1\|_{L^\infty(C^l)} \leq (1 + \epsilon)\|q_0\|_{L^\infty(C^l)} + \xi \leq \exp(\epsilon)\|q_0\|_{L^\infty(C^l)} + \frac{\exp(\epsilon) - 1}{\epsilon} \xi.
\]

Now, we assume that

\[
\|q_{m-1}\|_{L^\infty(C^l)} \leq \exp((m - 1)\epsilon)\|q_0\|_{L^\infty(C^l)} + \frac{\exp((m - 1)\epsilon) - 1}{\epsilon} \xi.
\]
Combining estimates: \( 1 + \epsilon \leq \exp(\epsilon) \) and \( 1 \leq \frac{\exp(\epsilon - 1)}{\epsilon} \) together with inequality (52) provided by the assumption of Lemma 4.1 and (54), direct calculations give
\[
\|q_m\|_{\infty(C^t)} \leq (1 + \epsilon)\|q_{m-1}\|_{\infty(C^t)} + \xi \leq (1 + \epsilon)\left[\exp(\epsilon(m - 1))\|q_0\|_{\infty(C^t)} + \frac{\exp((m - 1)\epsilon - 1)}{\epsilon}\right] + \xi \\
= \exp(\epsilon)\left[\exp((m - 1)\epsilon)\|q_0\|_{\infty(C^t)} + \frac{\exp((m - 1)\epsilon - 1)}{\epsilon}\right] + \xi \leq \exp(me)\|q_0\|_{\infty(C^t)} + \frac{\exp(me) - 1}{\epsilon}\xi.
\]

The last inequality comes the estimate \( \epsilon - \exp(\epsilon) \leq -1 \). This completes the proof of Lemma 4.1. \( \Box \)

5. Numerical experiments

In this section, we use MATLAB R2007b and we present a broad range of numerical evidences to illustrate and demonstrate the efficiency of the proposed approach applied to the mathematical model of the Covid-19 spreading. We stress that in this situation, we obtain satisfactory results, so our algorithm performances are not worse for multidimensional problems. Furthermore, the new approach is more efficient than a wide set of numerical methods analyzed in the literature ([22] and references therein). We consider the particular case of the Covid-19 epidemic in Cameroon where the data are available in [23] and we analyze and discuss the obtained results. The other real data used in this study are taken from the literature [7,9,23]. More precisely, the number of people in this country is approximately \( N = 25.000.000 \), the fraction at time \( t \), of people in compartment \( X_0 \) that are hospitalized is assumed equals 1 \( (p(t) = 1) \), because of the decision made by WHO it was decided to hospitalize all detected cases to reduce the transmission of the SARS-Cov-2 virus [24]), \( d_{X_2} = 5.5 \) days, \( d_5 = 6 \) days, \( d_{X_4} = 7.3 \) days (that is, \( y_{X_4} = 1/7.3 \) \( (day^{-1}) \)), \( C_0 = 0.3 \) days, \( T_{\max} = 150 \) days, \( t_0 = 0 \) (which corresponds to 06 March 2020), \( h = 1 \) (the step size), \( w = 0.00133 \), \( w = 0.0667 \), \( \beta_{X_2} = 0.1125 \), \( \beta_{X_3} = 0.375 \), \( d_0 = 14 \) (the period of convalescence, we recall that this corresponds to the time a person is still hospitalized after recovering from Covid-19 disease), \( y_{X_2} = 0.1818 \), \( y_{X_3} = 0.78895 \), \( C_0 = 0.3 \), \( k = 0.13 \) (efficiency of control measures), \( \lambda = 12 \) (first day of the application of the control measures which corresponds to 17 March 2020 in Cameroon), \( \beta_{X_3} = C_0 \times \beta_{X_2} = 0.1125 \) and \( \beta_{X_3} = 0.375 \).

To prevent the transmission of the Coronavirus 2019 in the other cities of the country, the authorities decided to suspend all fights in the country and to restrict the number of passengers in all public transportation and outbound trains in the cities of Yaounde, Douala, Bafoussam and municipalities on 17 March 2020 [23]. Hence, the following implementation of the control measures is considered in order to indicate the real situation of the measured imposed
\[
m(t) = \begin{cases} 
1, & \text{if } t \in [05 \text{ March } 2020, \lambda]; \\
\exp[-k(t-\lambda)], & \text{if } t \in [\lambda, T_{\max}], 
\end{cases}
\]
where \( \lambda \) corresponds to 17 March 2020. Furthermore, we assume that the fraction of detected infected people \( (\theta(t)) \) is a linear function, the disease contact rate of a person in compartment \( X_2 \) denoted \( w(t) \) (without taking into account the control measures) in this territory is a continuous function and both functions are given by
\[
\theta(t) = \frac{1 - m}{T_{\max}}t \quad \text{and} \quad \beta_{X_4}(\theta(t)) = \beta_{X_3} + (\beta_{X_3} - \beta_{X_2})\frac{1 - \theta(t)}{1 - w(t)}.
\]

We recall that the initial data used in these experiments are taken in [23]. Specifically, on 05 March 2020, there were two cases in which the first case was imported on 24 February 2020 and the second one was occurring by contact with the first case reported. By devoting resources, on 08 March 2020, 108 people from 176 identified individuals of local transmission were traced back to their presumed exposure, either to a known case or to a location linked to spread [23]. From this observation, we set \( X_1(t_0) = N - 176, X_2(t_0) = 172, X_3(t_0) = X_4(t_0) = 1, X_5(t_0) = 2, X_j(t_0) = 0, \) for \( j = 6, 7, 8, 9 \). We perform a wide set of numerical evidences to demonstrate the robustness of the new approach. More specifically:

In Table 1 we show the evolution of the model cumulative number of cases in Cameroon from 06 March 2020 till 30 April 2020 subject to various values of the fraction of detected infected people that are documented \( \theta \), whereas Fig. 1(1) deals with the evolution of the predicted number of cases during the given time \( T_{\max} = 55 \) days, taking into account different values of \( \theta \). The model cumulative number of reported deaths and computed ones corresponding to different values of \( \theta \) are presented in both Table 2 and Fig. 1(2). Table 3 and Fig. 1(3) indicate the model cumulative number of hospitalized people using various values of \( \theta \) while Table 4 and Fig. 1(4) suggest the number of infected individuals, who are not expected to be detected yet, may infect other persons and start to developing clinical signs. In both Table 5 and Fig. 2(5), we present the expected number of people that will recover, but can still infect other persons. Table 6 and Fig. 2(6) indicate the evolution of the cumulative number of people who recovered after being previously infected but were undetected and documented by the government and are no longer infectious. The model cumulative number of individuals exposed to the Covid-19 disease is indicated in both Table 7 and Fig. 2(7). Both Table 8 and Fig. 2(8) present the evolution of the number of persons infected by contact with people in compartment \( X_2 \), whereas Table 9 together with Fig. 3(9) show the model cumulative number of people infected by contact with individuals in compartment \( X_4 \). In Table 10 and Fig. 3(10), we discuss the number of persons infected by contact with people in compartment \( X_{10} = X_3 + X_6 \). Each analysis described above considers various values of \( \theta \). Finally, we draw in Table 11 and sketch in Fig. 3(11) the effective reproduction number of Covid-19 disease in Cameroon.
Model cumulative number of cases, deaths, hospitalized people and infected persons who are not detected yet.

Now, focusing on the values taken by $\theta$, we observe from the tables and figures that the predicted results reproduce quite accurately the evolution of the number of cases, number of deaths and number of hospitalized people for greater values of $\theta$. In particular, for $\theta = 0.3212$ the predicted values on 30 April 2020 (cases, deaths and people in hospital) are approximately 2205.6, 111.2786 and 19.6962, respectively, whereas they become overestimated: 3260.6 cases, 166.1457 deaths and 90.1832 hospitalized people when $\theta = 0.0667$ (the smallest value taken by $\theta$ in $[w, 1]$). Furthermore, for different values of $\theta=0.0667, 0.1515, 0.3212, 0.7455$, we observe the maximum number of undetected cases: 116.0502, 85.1727, 74.3269 and 56.5865, respectively, estimated by the new approach represent: 5.5%, 4.2257%, 4.0439% and 3.6912% of the number of total cases obtained on 30 March 2020. Interestingly, the results provided by our method suggest that, despite the relative control of Covid-19 pandemic in Cameroon, they may still exist an undetected source of infected persons that could cause the increase of the disease in a near future if the implementation of the control measures is significantly relaxed like the government decided at the beginning of May 2020. In addition, Fig. 1(3) indicates that the peak of the persons hospitalized in this country at the same time should be reached on 05 April 2020, with approximately 800 hospitalized patients. This number is associated with the smallest value of $\theta = 0.0667$. However, the obtained results slightly overestimate the observed values by around 693. Focusing on the recovered people, the considered technique suggests a maximum number of 2976.2, 2258.5, 2043.1 and 1677.4 people corresponding to various values of $\theta: 0.0667, 0.1515, 0.3212$ and 0.7455, respectively. For $\theta = 0.1515$, the total number (241.9064 people) is very close to the real observation (around 244 hospitalized individuals on 15 April 2020). This shows that the proposed method is a reasonable decision tool to estimate the number of beds in hospital during an epidemic. Also, it is worth mentioning that our approach is able to detect early a reasonable expected date of this peak.

Finally, we observe from both Table 11 and Fig. 3(11) that the value of the effective reproduction number ($R_e$) decreases since the application of the control measures and it becomes less than 1 after the peak is attained (30 March 2020). These numbers should help to expect the number of beds in hospitals during an epidemic.
Cumulative number of recovered infected, undetected infected who recovered, people exposed to Covid-19, infected by contact with $X_2$.

**Fig. 2.** Number of recovered infected, recovered undetected infected, exposed to Covid-19, infected by contact with $X_2$.

**Table 1**
Model cumulative number of cases with various values of $\theta$.

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| NC (real data) | 2             | 15            | 142           | 848           | 2014          |
| CM(0.0667)     | 2             | 696.3508      | 2149.8        | 3245.3        | 3322.0        |
| CM(0.1515)     | 2             | 681.7603      | 2048.7        | 2467.4        | 2481.5        |
| CM(0.3212)     | 2             | 647.5659      | 1861.4        | 2223.8        | 2236.0        |
| CM(0.7455)     | 2             | 583.5342      | 1538.2        | 1808.4        | 1817.5        |

**Table 2**
The evolution of number of deaths with different values of $\theta$.

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| ND (real data) | 0             | 1             | 6             | 14            | 61            |
| $X_3 (0.0667)$ | 0             | 17.8300       | 79.6390       | 141.2279      | 166.1457      |
| $X_3 (0.1515)$ | 0             | 17.5511       | 73.8069       | 108.6332      | 121.8038      |
| $X_3 (0.3212)$ | 0             | 16.8994       | 68.3353       | 99.5074       | 111.2786      |
| $X_3 (0.7455)$ | 0             | 15.6638       | 58.6885       | 83.6826       | 93.0994       |
Model cumulative number of people infected by contact with $X_4$, $X_{10}$ and effective reproduction number.

![Figure 9](image9.png)

![Figure 10](image10.png)

![Figure 11](image11.png)

**Figure 9**
Number of infected by contact with $X_4$, $X_{10}$ and effectivereproduction number $Re$.

**Figure 10**
Cumulative number of people infected by contact with $X_4$ with various values of $\theta$.

**Figure 11**
Effective reproduction number of Covid-19 with various values of $\theta$.

**Table 3**
The cumulative number of hospitalized people.

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| NH (real obs.) | 2             | 8             | 61            | 244           | 328           |
| Host(0.0667)   | 2             | 335.5282      | 668.8886      | 476.0395      | 86.9377       |
| Host(0.1515)   | 2             | 325.1239      | 673.6724      | 241.0999      | 44.0635       |
| Host(0.3212)   | 2             | 300.8178      | 583.1121      | 209.3954      | 47.4915       |
| Host(0.7455)   | 2             | 255.0717      | 432.0405      | 158.2955      | 54.7413       |

**Table 4**
Cumulative number of infected individuals who are not expected to be detected yet, but start developing clinical signs.

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| $X_3(0.0667)$  | 1             | 70.3135       | 116.0502      | 15.4822       | 0.2363        |
| $X_3(0.1515)$  | 1             | 67.8561       | 85.1727       | 2.6167        | 0.0608        |
| $X_3(0.3212)$  | 1             | 62.0955       | 74.3269       | 2.2831        | 0.0530        |
| $X_3(0.7455)$  | 1             | 51.8569       | 56.5865       | 1.7375        | 0.0404        |

6. General conclusion and future works

We have developed an efficient two-level explicit method for estimating the propagation of Covid-19 disease with undetected infectious cases. The analysis has shown that the new algorithm is stable, at least second-order accuracy and
Table 5  
Model cumulative number of infected people that will recover but can still infect other persons.

| Date            | X0(0.0667) | X0(0.1515) | X0(0.3212) | X0(0.7455) |
|-----------------|------------|------------|------------|------------|
| 06 March 2020   | 2          | 2          | 2          | 2          |
| 17 March 2020   | 307.1123   | 297.3910   | 274.6855   | 231.8949   |
| 30 March 2020   | 608.9473   | 626.9163   | 547.2620   | 413.7484   |
| 15 April 2020   | 486.4522   | 268.6669   | 232.7084   | 173.1906   |
| 30 April 2020   | 106.7015   | 52.8153    | 45.7222    | 33.9913    |

Table 6  
Evolution of the number of undetected infected people who recovered but are no longer infectious.

| Date            | X0(0.0667) | X0(0.1515) | X0(0.3212) | X0(0.7455) |
|-----------------|------------|------------|------------|------------|
| 06 March 2020   | 0          | 0          | 0          | 0          |
| 17 March 2020   | 342.9927   | 339.0853   | 329.8487   | 312.7987   |
| 30 March 2020   | 1402.0     | 1299.3     | 1202.1     | 1030.5     |
| 15 April 2020   | 2582.4     | 2070.9     | 1874.4     | 1537.8     |
| 30 April 2020   | 3037.7     | 2300.8     | 2073.5     | 1686.0     |

Table 7  
Cumulative number of people exposed to Covid-19 disease.

| Date            | X2(0.0667) | X2(0.1515) | X2(0.3212) | X2(0.7455) |
|-----------------|------------|------------|------------|------------|
| 06 March 2020   | 172        | 172        | 172        | 172        |
| 17 March 2020   | 317.5650   | 306.0850   | 279.0561   | 231.3889   |
| 30 March 2020   | 495.4672   | 289.9048   | 252.9417   | 192.4997   |
| 15 April 2020   | 52.1206    | 8.6553     | 7.5517     | 5.7472     |
| 30 April 2020   | 0.7125     | 0.2011     | 0.1754     | 0.1335     |

Table 8  
Model cumulative number of people infected by contact with persons in compartment X2.

| Date            | X2(0.0667) | X2(0.1515) | X2(0.3212) | X2(0.7455) |
|-----------------|------------|------------|------------|------------|
| 06 March 2020   | 19.3499    | 19.3499    | 19.3499    | 19.3499    |
| 17 March 2020   | 35.7245    | 34.4331    | 31.3926    | 26.0303    |
| 30 March 2020   | 37.7351    | 0.0000     | 0.0000     | 0.0000     |
| 15 April 2020   | 0.6432     | 0.0000     | 0.0000     | 0.0000     |
| 30 April 2020   | 0.0013     | 0.0000     | 0.0000     | 0.0000     |

Table 9  
Model cumulative number of people infected by contact with persons in compartment X4.

| Date            | X4(0.0667) | X4(0.1515) | X4(0.3212) | X4(0.7455) |
|-----------------|------------|------------|------------|------------|
| 06 March 2020   | 0.3750     | 0.3641     | 0.3368     | 0.2686     |
| 17 March 2020   | −9.1174    | −8.0160    | −5.6330    | −1.5867    |
| 30 March 2020   | −12.2623   | 0.0000     | 0.0000     | 0.0000     |
| 15 April 2020   | −1.5011    | 0.0000     | 0.0000     | 0.0000     |
| 30 April 2020   | −0.0477    | 0.0000     | 0.0000     | 0.0000     |

Table 10  
Model cumulative number of people infected by contact with persons in compartment X10 = X5 + X6.

| Date            | X10(0.0667) | X10(0.1515) | X10(0.3212) | X10(0.7455) |
|-----------------|------------|------------|------------|------------|
| 06 March 2020   | 0.0044     | 0.0058     | 0.0069     | 0.0076     |
| 17 March 2020   | 0.7379     | 0.9354     | 1.0414     | 0.9745     |
| 30 March 2020   | 0.9748     | 0.0000     | 0.0000     | 0.0000     |
| 15 April 2020   | 0.1191     | 0.0000     | 0.0000     | 0.0000     |
| 30 April 2020   | 0.0038     | 0.0000     | 0.0000     | 0.0000     |

Table 11  
Effective reproduction number of Covid-19 disease in Cameroon.

| Date            | Re(0.0667) | Re(0.1515) | Re(0.3212) | Re(0.7455) |
|-----------------|------------|------------|------------|------------|
| 06 March 2020   | 1.1111     | 1.1164     | 1.1209     | 1.1236     |
| 17 March 2020   | 1.1111     | 1.1163     | 1.1208     | 1.1236     |
| 30 March 2020   | 0.7546     | 0.0000     | 0.0000     | 0.0000     |
| 15 April 2020   | 0.1229     | 0.0000     | 0.0000     | 0.0000     |
| 30 April 2020   | 0.0175     | 0.0000     | 0.0000     | 0.0000     |

can serve as a fast and robust tool for integrating general systems of ordinary differential equations. Numerical results based on the case of Cameroon reproduce quite accurately the evolution of the number of cases (detected or undetected), number of deaths, number of people in hospitals, number of infected detected persons who recovered and number of infected undetected individuals who recovered by natural immunity from 06 March 2020 to 30 April 2020. The approach
presented in this work can help to estimate the number of beds in hospitals during a pandemic. Furthermore, the proposed technique can be considered as a fundamental tool for detecting early a reasonable expected date of the peak during an epidemic. Our future works will consider the numerical solution of a more complex system of ordinary differential equations using the new two-level explicit approach.

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Appendix. Proof of Theorem 4.1

This appendix considers the proof of Theorem 4.1.

Proof. Firstly, we introduce the domains

\[ D_i = \{ (t, x) : t \in [t_0, T_{max}], x \in \mathbb{R}, |X_i(t) - x| \leq \nu \}, \quad S_0 = \{ (t, x) : t_0 \leq t \leq T_{max}, x \in \mathbb{R} \}, \]

and

\[ D = \{ (t, x) : t \in [t_0, T_{max}], x \in \mathbb{R}^9, \|X(t) - X\| \leq \nu \}, \]  \hspace{1cm} (55)

where \( \nu \) is a positive constant independent of the step size \( h \) and \( X_i(t) \) denotes the \( i \)-th component of the exact solution \( X(t) \) of the initial-value problem (1)–(6).

Plugging approximations (46)–(47) and Eq. (50), it is not hard to observe that

\[ \delta_i(t_n, X_i(t_n)) = F_i(t_n, X^n) + \frac{h}{4} \left( \partial_i F_i(t_n, X^n) + \sum_{k=1}^{9} F_k(t_n, X^n) \partial_k F_i(t_n, X^n) \right), \]  \hspace{1cm} (56)

and

\[ \delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) = 3F_i(t_{n+\frac{1}{2}}, X^{n+\frac{1}{2}}) - F_i(t_n, X^n). \]  \hspace{1cm} (57)

So, the functions \( \delta_i \) given by Eq. (50) and their partial derivatives are continuous on the domain \( D_i \). Since \( D_i \) is a compact subset of the strip \( S_0 \), their partial derivatives are bounded on \( D_i \). Applying the Mean-Value Theorem, there exists a positive constant \( L_i \) which does not depend on the step size \( h \) so that

\[ |\delta_i(t, X_i) - \delta_i(t, Y_i)| \leq L_i |X_i - Y_i|, \]  \hspace{1cm} (58)

for every \( (t, X_i) \) and \( (t, Y_i) \) in \( D_i \). Furthermore, a simple manipulation of (41) results in

\[ \frac{X_i(t_{n+\frac{1}{2}}) - X_i(t_n)}{h/2} = F_i(t_n, X(t_n)) + \frac{h}{4} \left( \partial_i F_i(t_n, X(t_n)) + \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_k F_i(t_n, X(t_n)) \right) + O(h^2), \]

which is equivalent to

\[ \frac{X_i(t_{n+\frac{1}{2}}) - X_i(t_n)}{h/2} = \left[ F_i(t_n, X(t_n)) + \frac{h}{4} \left( \partial_i F_i(t_n, X(t_n)) + \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_k F_i(t_n, X(t_n)) \right) \right] = O(h^2). \]

Utilizing Eqs. (49), (51) and (56), this becomes

\[ \sigma_i(t_n, X_i(t_n)) = \Delta_i(t_n, X_i(t_n)) - \delta_i(t_n, X_i(t_n)) = O(h^2), \]

which implies

\[ |\sigma_i(t_n, X_i(t_n))| \leq C_{4i} h^2, \quad \text{for} \quad i = 1, 2, \ldots, 9, \]  \hspace{1cm} (59)

where \( C_{4i} \) are positive constants that do not depend on the step size \( h \). Setting \( C_4 = \max_{1 \leq i \leq 9} C_{4i}, \sigma(t_n, X(t_n)) = (\sigma_1(t_n, X_1(t_n)), \ldots, \sigma_9(t_n, X_9(t_n)))^T \) and taking the maximum over \( i \) of both sides of estimate (59), to get

\[ \|\sigma(t_n, X(t_n))\|_{\mathbb{R}^9} \leq C_4 h^2. \]  \hspace{1cm} (60)

In a similar manner, combining approximation (45), Eqs. (49), (50) and (51), straightforward calculations provide

\[ \frac{X_i(t_{n+1}) - X_i(t_{n+\frac{1}{2}})}{h/2} = \left[ 3F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) - F_i(t_n, X(t_n)) \right] = O(h^2). \]
which can be rewritten as
\[ \Delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) - \delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) = O(h^2). \]

Utilizing the definition of \( \sigma_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) \), this implies
\[ |\sigma_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}}))| \leq C_5 h^2, \]
where \( C_5 \) (1 \( \leq \) i \( \leq \) 9) are positive constants independent of \( h \). Taking the maximum over \( i \), this yields
\[ \|\sigma(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}))\|_{L^\infty} \leq C_5 h^2, \quad (61) \]
where \( C_5 = \max_{1 \leq i \leq 9} \sigma_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) = (\sigma_1(t_{n+\frac{1}{2}}, X_1(t_{n+\frac{1}{2}})), \ldots, \sigma_9(t_{n+\frac{1}{2}}, X_9(t_{n+\frac{1}{2}})))^T. \]

We consider the functions \( \widehat{\delta}_i(t, x) \) defined on the strip \( S_0 = \{(t, x): t_0 \leq t \leq T_{\text{max}}, x \in \mathbb{R} \} \), as
\[ \widehat{\delta}_i(t, x) = \begin{cases} \delta_i(t, X_i(t) + \nu), & \text{for } t \in [t_0, T_{\text{max}}] \text{ and } x > X_i(t) + \nu, \\ \delta_i(t, X_i(t) - \nu), & \text{for } t \in [t_0, T_{\text{max}}] \text{ and } x < X_i(t) - \nu. \end{cases} \quad (62) \]

We remind that \( X_i(t) \) is the \( i \)th component of the exact solution \( X(t) \) of the initial-value problem (1)–(6). Now, using relation (62), it is not hard to observe that each function \( \widehat{\delta}_i \) satisfies the “Lipschitz requirement” in the strip \( S_0 \), that is,
\[ |\widehat{\delta}_i(t, x) - \widehat{\delta}_i(t, y)| \leq L_i|x - y|, \quad (63) \]
for all \((t, x)\) and \((t, y)\) in \( S_0 \), where the positive constant is given by relation (58). Thus, each \( \widehat{\delta}_i \) is continuous on \( S_0 \). Indeed.

Let \((t, x)\) and \((t, y)\) be two elements of \( S_0 \). If \((t, x)\) and \((t, y)\) lie in \( D_i \), then estimate (63) holds thanks to inequality (58), so the function \( \delta_i \) is continuous and satisfies the “Lipschitz requirement” on \( D_i \). Otherwise, either \((t, x)\) or \((t, y)\) does not lie in \( D_i \). This corresponds to three cases: (a) \((t, x)\) lies in \( D_i \) and \((t, y)\) does not lie in \( D_i \), (b) \((t, x)\) does not lie in \( D_i \) and \((t, y)\) does lie in \( D_i \) and (c) \((t, x)\) and \((t, y)\) do not lie in \( D_i \). Here we should prove only one case, for instance \((t, x)\) does not lie in \( D_i \) and \((t, y)\) does lie in \( D_i \), the proof of the other cases is similar.

\((t, x) \notin D_i \Leftrightarrow |x - X_i(t)| > \nu \Leftrightarrow x - X_i(t) > \nu \text{ or } -(x - X_i(t)) > \nu \Leftrightarrow x > X_i(t) + \nu \text{ or } x < X_i(t) - \nu. \)

So,
\[ |\widehat{\delta}_i(t, x) - \widehat{\delta}_i(t, y)| = \begin{cases} |\delta_i(t, X_i(t) + \nu) - \delta_i(t, y)|, & \text{for } t_0 \leq t \leq T_{\text{max}} \text{ and } x > X_i(t) + \nu, \\ |\delta_i(t, X_i(t) - \nu) - \delta_i(t, y)|, & \text{for } t \in [t_0, T_{\text{max}}] \text{ and } x < X_i(t) - \nu, \end{cases} \]
\[ \leq \begin{cases} L_i|x(t) + \nu - y|, & \text{for } t \in [t_0, T_{\text{max}}] \text{ and } x > X_i(t) + \nu, \\ L_i|x(t) - \nu - y|, & \text{for } t_0 \leq t \leq T_{\text{max}} \text{ and } x < X_i(t) - \nu. \end{cases} \quad (64) \]

But
\[(t, y) \in D_i \Leftrightarrow |X_i(t) - y| \leq \nu \Leftrightarrow X_i(t) - y \leq \nu \text{ and } -X_i(t) + y \leq \nu. \]

Using this, it is easy to see that
\[ x > X_i(t) + \nu \Leftrightarrow x - y > X_i(t) + \nu - y \geq 0 \Leftrightarrow |X_i(t) - y| > |X_i(t) + \nu - y|, \]
and
\[ x < X_i(t) - \nu \Leftrightarrow x - y < X_i(t) - \nu - y \leq 0 \Leftrightarrow |X_i(t) - y| > |X_i(t) - \nu - y|. \]

This fact together with estimate (64) results in
\[ |\widehat{\delta}_i(t, x) - \widehat{\delta}_i(t, y)| \leq L_i|x - y|. \]

This ends the proof of the first case (a). Thus, \( \widehat{\delta}_i \) satisfies the “Lipschitz condition” on the strip \( S_0 = \{(t, x): t_0 \leq t \leq T_{\text{max}}, x \in \mathbb{R}\} \). In addition, \( \widehat{\delta}_i \) is also continuous on \( S_0 \).

Since, \((t, X_i(t)) \in D_i\), so \( \widehat{\delta}_i(t, X_i(t)) = \delta_i(t, X_i(t)). \) A combination of \((60)-(61)\) provides
\[ |\Delta_i(t_n, X_i(t_n)) - \widehat{\delta}_i(t_n, X_i(t_n))| = |\Delta_i(t_n, X_i(t_n)) - \delta_i(t_n, X_i(t_n))| \leq C_4 h^2, \quad (65) \]
and
\[ |\Delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) - \widehat{\delta}_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}}))| = |\Delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) - \delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}}))| \leq C_5 h^2. \quad (66) \]
We recall that the two-level explicit method (46)–(48) is generated by the function \( \delta_i \). In fact, plugging Eqs. (46), (47) and (50), \( \delta_i \) is defined as

\[
\delta_i(t_n, X_i(t_n)) = \begin{cases} 
F_i(t_n, X^n) + \frac{h}{4} \left( \delta_i F_i(t_n, X^\alpha) + \sum_{k=1}^9 F_k(t_n, X^\beta) \delta_i F_k(t_n, X^\beta) \right), & \text{if } h \neq 0 \\
F_i(t_n, X^n), & \text{if } h = 0,
\end{cases}
\]

and

\[
\delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) = 3F_i(t_{n+\frac{1}{2}}, X^{n+\frac{1}{2}}) - F_i(t_n, X^n).
\]

Thus, approximations (46) and (47) become

\[
X_i^{n+\frac{1}{2}} = X_i^n + \frac{h}{2} \delta_i(t_n, X_i(t_n)) \quad \text{and} \quad X_i^{n+1} = X_i^{n+\frac{1}{2}} + \frac{h}{2} \delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})).
\]

Analogously, the two-level numerical scheme generated by \( \delta_i \) should provide the approximate solutions which satisfy

\[
\tilde{X}_i^{n+\frac{1}{2}} = \tilde{X}_i^n + \frac{h}{2} \tilde{\delta}_i(t_n, \tilde{X}_i(t_n))
\]

and

\[
\tilde{X}_i^{n+1} = \tilde{X}_i^{n+\frac{1}{2}} + \frac{h}{2} \tilde{\delta}_i(t_{n+\frac{1}{2}}, \tilde{X}_i(t_{n+\frac{1}{2}})).
\]

In view of (49), we have

\[
X_i(t_{n+\frac{1}{2}}) = X_i(t_l) + \frac{h}{2} \Delta_i(t_l, X_i(t_l)), \quad \text{for } l = n, n + \frac{1}{2}.
\]

Plugging relations (67)–(69), and because \( \tilde{e}_i^n = X_i(t_l) - \tilde{X}_i^n \), direct calculations result in

\[
\tilde{e}_i^{n+\frac{1}{2}} = \tilde{e}_i^n + \frac{h}{2} \left( \Delta_i(t_n, X_i(t_n)) - \tilde{\delta}_i(t_n, \tilde{X}_i(t_n)) \right) = \tilde{e}_i^n + \frac{h}{2} \left[ \tilde{\delta}_i(t_n, X_i(t_n)) - \tilde{\delta}_i(t_n, X_i(t_n)) \right] + \Delta_i(t_n, X_i(t_n))
\]

Taking the absolute value, it is easy to see that

\[
|\tilde{e}_i^{n+\frac{1}{2}}| \leq |\tilde{e}_i^n| + \frac{h}{2} \left[ |\tilde{\delta}_i(t_n, X_i(t_n)) - \tilde{\delta}_i(t_n, X_i(t_n))| + |\Delta_i(t_n, X_i(t_n)) - \tilde{\delta}_i(t_n, X_i(t_n))| \right] \leq
\]

\[
|\tilde{e}_i^n| + \frac{h}{2} \left[ L_i |X_i(t_n) - \tilde{X}_i^n| + C_4 h^2 \right] \leq \left( 1 + \frac{L h}{2} \right) |\tilde{e}_i^n| + \frac{C_4}{2} h^3,
\]

where \( L = \max L_i \). The last two estimates follow from inequalities (63) and (65). Taking the maximum over \( i \), estimate (70) gives

\[
|\tilde{e}_i^{n+\frac{1}{2}}| \leq \max_{1 \leq i \leq s} \left\{ \left( 1 + \frac{L h}{2} \right) |\tilde{e}_i^n| + \frac{C_4}{2} h^3 \right\} \leq \left( 1 + \frac{L h}{2} \right) |\tilde{e}_i^n| + \frac{C_4}{2} h^3.
\]

In a similar way, utilizing relations (68), (69), (63) and (66), one easily shows that

\[
|\tilde{e}_i^{n+1}| \leq \left( 1 + \frac{L h}{2} \right) |\tilde{e}_i^{n+\frac{1}{2}}| + \frac{C_2}{2} h^3.
\]

Substitute estimate (71) into (72) to obtain

\[
|\tilde{e}_i^{n+1}| \leq \left( 1 + \frac{L h}{2} \right)^2 |\tilde{e}_i^n| + \left[ \left( 1 + \frac{L h}{2} \right)^2 + \frac{C_4}{2} + \frac{C_5}{2} \right] h^3 = (1 + \alpha_{1h}) |\tilde{e}_i^n| + \alpha_{2h} h^3,
\]

where \( \alpha_{1h} = L \left( 1 + \frac{L h}{2} \right) \) and \( \alpha_{2h} = (1 + \frac{L h}{2}) \frac{C_4}{2} + \frac{C_5}{2} \). To guarantee the convergence of the algorithm, the step size should satisfy \( 0 < h \leq 1 \). This restriction allows to write: \( \alpha_{2h} \leq (1 + \frac{1}{2}) \frac{C_4}{2} + \frac{C_5}{2} \) := \( \alpha_2 \). This fact, together with inequality (73) yield

\[
|\tilde{e}_i^{n+1}| \leq \left( 1 + \frac{L h}{2} \right)^2 |\tilde{e}_i^n| + \alpha_2 h^3.
\]

Applying Lemma 4.1, it holds

\[
|\tilde{e}_i^n| \leq \exp(n\alpha_{1h}) |\tilde{e}_i^{0}| + \exp(n\alpha_{1h}) - 1 \alpha_2 h^3.
\]

(74)
But it comes from the initial condition that $\hat{e}^0 = 0$. Using this, relation (74) becomes
\[
\|\hat{e}^n\| \leq \frac{\exp[nLh(1 + \frac{l}{4}h)] - 1}{L(1 + \frac{l}{4}h)} \alpha \max |\hat{X}^n|.
\]
Since $h = \frac{T_{\text{max}} - t_0}{n}$ and $t_n = t_0 + nh$, then $nh = t_n - t_0 \leq T_{\text{max}}$. Furthermore, $1 < 1 + \frac{l}{4}h \leq 1 + \frac{l}{4}$ (indeed, $0 < h \leq 1$).

Utilizing this fact, we have
\[
\|\hat{e}^n\| \leq \frac{\exp[Lt_{\text{max}}(1 + \frac{l}{4})] - 1}{L} \alpha \max |\hat{X}^n|,
\]
which can be rewritten as
\[
|X(t_n) - \hat{X}^n| \leq \frac{\alpha}{L} \left[\exp[Lt_{\text{max}}(1 + \frac{l}{4})] - 1\right] h^2 \leq \frac{\alpha}{L} \left[\exp[Lt_{\text{max}}(1 + \frac{l}{4})] - 1\right], \text{ for } i = 1, 2, \ldots, 9.
\]

\begin{align*}
&\text{since } 0 < h \leq 1. \\
&\text{Setting } \nu = \frac{\alpha}{L} \left[\exp[Lt_{\text{max}}(1 + \frac{l}{4})] - 1\right] > 0, \text{ estimate (76) indicates that } (t_n, \hat{X}^n) \in D, \text{ for } i = 1, 2, \ldots, 9.
\end{align*}

Now, according to the definition of $\delta_i$, we should have $X^i_0 = \hat{X}_0, e^i_0 = \hat{e}_0$ and $\delta(t_n, X(t_n)) = \delta(t_n, \hat{X}(t_n))$. This fact together with estimates (75)–(76) provide
\[
|X(t_n) - X^i(t_n)| \leq \nu \text{ and } \|\hat{e}^i\| \leq \nu h^2.
\]

It comes from the inequality $\|u\|_{\infty} - \|v\|_{\infty} \leq \|u - v\|_{\infty}$ for any $u, v \in \mathbb{R}^9$ and estimate (77) that
\[
\|X^n\|_{\infty} \leq \|X(t_n)\|_{\infty} + \nu.
\]

The analytical solution $X(\cdot)$ is bounded on the interval $[t_0, T_{\text{max}}]$ because $(t, X(t)) \in D$, where the domain $D$ is given by relation (55). It follows from relation (78) that the approximate solution $X$ is also bounded over the interval $[t_0, T_{\text{max}}]$. Thus, the proposed approach (46)–(48) for solving the initial-value problem (1)–(6) is stable. Finally, the second estimate in relation (77) suggests that the new method is at least second-order convergent. This completes the proof of Theorem 4.1. □

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Further reading

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