ON THE BOUNDARY BEHAVIOR OF THE CURVATURE OF $L^2$-METRICS

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Abstract. For one-parameter degenerations of compact Kähler manifolds, we determine the asymptotic behavior of the first Chern form of the direct image of a Nakano semi-positive vector bundle twisted by the relative canonical bundle, when the direct image is equipped with the $L^2$-metric.

1. Introduction

Let $X$ be a connected Kähler manifold of dimension $n + 1$ with Kähler metric $h_X$ and let $S = \{ s \in \mathbb{C}; |s| < 1 \}$ be the unit disc. Set $S^o := S \setminus \{ 0 \}$. Let $\pi: X \to S$ be a proper surjective holomorphic map with connected fibers. Let $\Sigma_s$ be the critical locus of $\pi$. We assume that $\pi^{-1}(\Sigma_s) = \{ 0 \}$. We set $X_s = \pi^{-1}(s)$ for $s \in S$. Then $X_s$ is non-singular for $s \in S^o$. Let $\omega_X = \Omega^{n+1}_X$ be the canonical bundle of $X$ and let $\omega_{X/S} = \Omega^{n+1}_X \otimes (\pi^* \Omega^1_S)^{-1}$ be the relative canonical bundle of $\pi: X \to S$. The Kähler metric $h_X$ induces a Hermitian metric $h_{X/S}$ on $TX/S = \ker \pi_*|_{X \setminus \Sigma_s}$, and $h_{X/S}$ induces a Hermitian metric $h_{\omega_{X/S}}$ on $\omega_{X/S}$.

Let $\xi \to X$ be a holomorphic vector bundle on $X$ equipped with a Hermitian metric $h_\xi$. We write $\omega_{X/S}(\xi) = \omega_{X/S} \otimes \xi$. In this note, we assume that $(\xi, h_\xi)$ is a Nakano semi-positive vector bundle on $X$. Namely, if $R^k$ denotes the curvature form of $(\xi, h_\xi)$ with respect to the holomorphic Hermitian connection, then the Hermitian form $h_\xi(\sqrt{-1} R^k(\cdot, \cdot))$ on the holomorphic vector bundle $TX \otimes \xi$ is semi-positive. Since $\dim S = 1$ and since $(\xi, h_\xi)$ is Nakano semi-positive, all direct image sheaves $R^q \pi_* \omega_{X/S}(\xi)$ are locally free by [12]. By the fiberwise Hodge theory, $R^q \pi_* \omega_{X/S}(\xi)$ is equipped with the $L^2$-metric $h_{L^2}$ with respect to $h_{X/S}$ and $h_{\omega_{X/S}} \otimes h_\xi$. By Berndtsson [2] and Mourougane-Takayama [7], the holomorphic Hermitian vector bundle $(R^q \pi_* \omega_{X/S}(\xi), h_{L^2})$ is again Nakano semi-positive on $S^o$. By Mourougane-Takayama [8], $h_{L^2}$ induces a singular Hermitian metric with semi-positive curvature current on the tautological quotient bundle over the projective-space bundle $\mathbf{P}(R^q \pi_* \omega_{X/S}(\xi))$. (We remark that there is no restrictions of the dimension of the base space $S$ in the works [2], [7], [8].)

After these results, one of the natural problems to be considered is the quantitative estimates for the singularities of the $L^2$-metric and its curvature. In [14], we gave a formula for the singularity of the $L^2$-metric on $R^q \pi_* \omega_{X/S}(\xi)$ (cf. Sect. 2). As a consequence, if $\sigma_q$ is a nowhere vanishing holomorphic section of $\det R^q \pi_* \omega_{X/S}(\xi)$, then there exist a rational number $a_q \in \mathbb{Q}$, an integer $\ell_q \geq 0$ and a real number $c_q$ such that (cf. [14, Th. 6.8])

$$\log \| \sigma_q(s) \|_{L^2}^2 = a_q \log |s|^2 + \ell_q \log(- \log |s|^2) + c_q + O(1 \log |s|) \quad (s \to 0).$$

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In this note, we study the boundary behavior of the curvature of the holomorphic Hermitian vector bundle \((R^q\pi_\ast\omega_{X/S}(\xi), h_{L^2})\) as an application of the description of the singularity of the \(L^2\)-metric \(h_{L^2}\) given in [14]. In this sense, this note is a supplement to the article [14].

Let us state our results. Let \(\mathcal{R}(s)\ ds \wedge d\bar{s}\) be the curvature form of \(R^q\pi_\ast\omega_{X/S}(\xi)\) with respect to the holomorphic Hermitian connection associated to \(h_{L^2}\). By the Nakano semi-positivity [2], \([7]\), \(\sqrt{-1}\mathcal{R}(s)\) is a semi-positive Hermitian endomorphism on the Hermitian bundle \((R^q\pi_\ast\omega_{X/S}(\xi), h_{L^2})\) on \(S^o\).

**Theorem 1.1.** The curvature form \(\mathcal{R}(s)\ ds \wedge d\bar{s}\) has Poincaré growth near \(0 \in S\). Namely, there exists a constant \(C > 0\) such that the following inequality of Hermitian endomorphisms holds for all \(s \in S^o\)

\[
0 \leq \sqrt{-1}\mathcal{R}(s) \leq \frac{C}{|s|^2(\log|s|)^2} \text{Id}_{R^q\pi_\ast\omega_{X/S}(\xi)}.
\]

Moreover, the Chern form \(c_1(R^q\pi_\ast\omega_{X/S}(\xi), h_{L^2})\) has the following asymptotic behavior as \(s \to 0\):

\[
c_1(R^q\pi_\ast\omega_{X/S}(\xi), h_{L^2}) = \left\{ \frac{\ell_q}{|s|^2(\log|s|)^2} + O\left(\frac{1}{|s|^2(\log|s|)^3}\right) \right\} \sqrt{-1}\ ds \wedge d\bar{s}.
\]

Considering the trivial example \(X = M \times S\), \(\xi = \mathcal{O}_X\), \(\pi = pr_2\), where \(M\) is a compact Kähler manifold, we can not expect any lower bound of \(\sqrt{-1}\mathcal{R}(s)\) (resp. \(c_1(R^q\pi_\ast\omega_{X/S}(\xi), h_{L^2})\)) by a non-zero semi-positive Hermitian endomorphism (resp. real \((1,1)\)-form). We remark that, when \(X_0\) is reduced and has only canonical singularities, then we get a better estimate (cf. Sect. 5).

As an application of Theorem 1.1, we get an estimate for the complex Hessian of analytic torsion. Set \(X_s := \pi^{-1}(s)\) and \(\xi_s := \xi|_{X_s}\) for \(s \in S\). Let \(\omega_{X_s}\) be the canonical line bundle of \(X_s\) and let \(h_{\omega_{X_s}}\) be the Hermitian metric on \(\omega_{X_s}\) induced from \(h_X\). For \(s \in S^o\), let \(\tau(X_s, \omega_{X_s}(\xi_s))\) be the analytic torsion [10], [3] of the holomorphic Hermitian vector bundle \((\xi_s \oplus \omega_{X_s}, h_{\xi_s} \oplus h_{\omega_{X_s}})\) on the compact Kähler manifold \((X_s, h_{\omega_X(X_s)})\). Let \(\log \tau(X/S, \omega_{X/S}(\xi))\) be the function defined as

\[
\log \tau(X/S, \omega_{X/S}(\xi))(s) := \log \tau(X_s, \omega_{X_s}(\xi_s)), \quad s \in S^o.
\]

By Bismut-Gillet-Soulé [3], \(\log \tau(X/S, \omega_{X/S}(\xi))\) is a \(C^\infty\) function on \(S^o\). Moreover, under certain algebraicity assumption of the family \(\pi: X \to S\) and the vector bundle \(\xi\), there exist by [14] constants \(\alpha \in \mathbb{Q}, \beta \in \mathbb{Z}, \gamma \in \mathbb{R}\) such that

\[
\log \tau(X/S, \omega_{X/S}(\xi))(s) = \alpha |s|^2 - (\sum_{q \geq 0} (-1)^q \ell_q) \log(-\log|s|^2) + \gamma + O(1/\log|s|)
\]

as \(s \to 0\). By this asymptotic expansion, it is reasonable to expect that the complex Hessian of analytic torsion has a similar behavior to the Poincaré metric on \(S^o\).

**Theorem 1.2.** The complex Hessian \(\partial_{\bar{s}} \log \tau(X/S, \omega_{X/S}(\xi))\) has the following asymptotic behavior as \(s \to 0\):

\[
\partial_{\bar{s}} \log \tau(X/S, \omega_{X/S}(\xi)) = -\sum_{q \geq 0} (-1)^q \ell_q \frac{|s|^2 (\log|s|)^2}{|s|^2(\log|s|)^2} + O\left(\frac{1}{|s|^2(\log|s|)^3}\right).
\]

This note is organized as follows. In Sect. 2, we recall the structure of the singularity of the \(L^2\)-metric \(h_{L^2}\) on \(R^q\pi_\ast\omega_{X/S}(\xi)\). In Sect. 3, we prove some technical lemmas used in the proof of Theorem 1.1. In Sect. 4, we prove Theorems 1.1 and 1.2. In Sect. 5, we study the case where \(X_0\) has only canonical singularities.

Throughout this note, we keep the notation and the assumptions in Sect. 1.
2. The singularity of the $L^2$-metric

2.1. The structure of the singularity of the $L^2$-metric. Let $\kappa_X$ be the Kähler form of $h_X$. In the rest of this note, we assume that $(\xi, h_\xi)$ is Nakano semi-positive on $X$ and that $(S,0) \cong (\Delta,0)$. By [12, Th. 6.5 (i)], $R^q\pi_*\omega_{X/S}(\xi)$ is locally free on $S$. By shrinking $S$ if necessary, we may also assume that $R^q\pi_*\omega_{X/S}(\xi)$ is a free $O_S$-module on $S$. Let $\rho_q \in \mathbb{Z}_{\geq 0}$ be the rank of $R^q\pi_*\omega_{X/S}^{p+1}(\xi)$ as a free $O_S$-module on $S$. Let $\{\psi_1, \ldots, \psi_{\rho_q}\} \subset H^0(S, R^q\pi_*\omega_{X/S}(\xi))$ be a free basis of the locally free sheaf $R^q\pi_*\omega_{X/S}(\xi)$ on $S$.

Let $T$ be another unit disc. By the semistable reduction theorem [9, Chap II], there exists a positive integer $\nu > 0$ such that the family $X \times_S T \to T$ induced from $\pi: X \to S$ by the map $\mu: T \to S$, $\mu(t) = t^\nu$, admits a semistable model. Namely, there is a resolution $r: Y \to X \times_S T$ such that the family $f := pr_2 \circ r: Y \to T$ is semistable, i.e., $Y_0 := f^{-1}(0)$ is a reduced normal crossing divisor of $Y$. We fix such an integer $\nu > 0$. Let $\text{Herm}(r)$ be the set of $r \times r$-Hermitian matrices.

**Theorem 2.1.** By choosing a basis $\{\psi_1, \ldots, \psi_{\rho_q}\}$ of $R^q\pi_*\omega_{X/S}(\xi)$ as a free $O_S$-module appropriately, the $\rho_q \times \rho_q$-Hermitian matrix

$$G(s) := (h_L^2(\psi_\alpha|_X, \psi_\beta|_X))$$

has the following expression

$$G(t^\nu) = D(t) \cdot H(t) \cdot D(t), \quad D(t) = \text{diag}(t^{-e_1}, \ldots, t^{-e_{\rho_q}}).$$

Here $e_1, \ldots, e_{\rho_q} \geq 0$ are integers and the Hermitian matrix $H(t)$ has the following structure: There exist $A_m(t) \in C^\infty(T, \text{Herm}(\rho_q))$, $0 \leq m \leq n$, with

$$H(t) = \sum_{m=0}^n A_m(t) (\log |t|^2)^m.$$ 

Moreover, by defining the real-valued functions $a_m(t) \in C^\infty(T)$, $0 \leq m \leq n\rho_q$ as

$$\det H(t) = \sum_{m=0}^{n\rho_q} a_m(t) (\log |t|^2)^m,$$

one has $a_m(0) \neq 0$ for some $0 \leq m \leq n\rho_q$.

**Proof.** See [14, Th. 6.8 and Lemmas 6.3 and 6.4].

**Remark 2.2.** The meaning of the Hermitian matrix $H(t)$ and the diagonal matrix $D(t)$ is explained as follows. Let $F: Y \to X$ be the map defined as the composition of $r: Y \to X \times_S T$ and $pr_1: X \times_S T \to X$. Then $h_Y := r^* (h_X + dt \otimes dt)$ is a Kähler metric on $Y \setminus Y_0$. There is a basis $\{\theta_1, \ldots, \theta_{\rho_q}\}$ of $R^q f_*\omega_{Y/Y}(F^*\xi)$ such that $H(t) = (H_{\alpha\beta}(t))$, $H_{\alpha\beta}(t) = (\mu^* h_{L^2})(\theta_\alpha, \theta_\beta)$, where $\mu^* h_{L^2}$ is the $L^2$-metric on $R^q f_*\omega_{Y/Y/Y}(F^*\xi)$ with respect to $h_Y$ and $F^*\xi$. By [8, Lemma 3.3], $R^q f_*\omega_{Y/Y/Y}(F^*\xi)$ is regarded as a subsheaf of $\mu^* R^q f_*\omega_{X/S}(\xi)$. Then the relation between the two bases $\{\theta_1, \ldots, \theta_{\rho_q}\}$ and $\{\mu^* \psi_1, \ldots, \mu^* \psi_{\rho_q}\}$ is given by $D(t)$, i.e., $\theta_\alpha = t^{-e_\alpha} \mu^* \psi_\alpha$. Moreover, by [8, Lemma 4.2], $\mu^* h_{L^2}|_{T^0}$ is indeed the pull-back of the $L^2$-metric $h_{L^2}|_{S^0}$ via $\mu$, where $T^0 := T \setminus \{0\}$, which implies the relation $G(\mu(t)) = D(t)H(t)\overline{D(t)}$.

The proof of Theorem 2.1 heavily relies on a theorem of Barlet [1, Th. 4 bis.]. This is the major reason why we need the assumption $\dim S = 1$. 

2.2. A Hodge theoretic proof of Theorem 2.1 for a trivial line bundle.

Assume that \((\xi, h_\xi)\) is a trivial Hermitian line bundle on \(X\), that \(\pi : X \to S\) is a family of polarized projective manifolds with unipotent monodromy and that the Kähler class of \(h_\xi\) is the first Chern class of an ample bundle on \(X\). We see that the expansion in Theorem 2.1 follows from the nilpotent orbit theorem of Schmid [11] in this case.

Let \(\kappa_X\) be the Kähler class of \(h_\xi\). By assumption, there is a very ample line bundle \(L\) on \(X\) with \([\kappa_X] = c_1(L)/N\). Replacing \(\kappa_X\) by \(N\kappa_X\) if necessary, we may assume that \(L\) is very ample. Let \(H_1, \ldots, H_n \in |L|\) be sufficiently generic hyperplane sections such that the following hold for all \(0 \leq k \leq n\) after shrinking \(S\) if necessary:

(i) \(X \cap H_1 \cap \cdots \cap H_k\) is a complex manifold of dimension \(n - k + 1\).

(ii) The restriction of \(\pi\) to \(X \cap H_1 \cap \cdots \cap H_k\) is a flat holomorphic map from \(X \cap H_1 \cap \cdots \cap H_k\) to \(S\).

(iii) \(X_s \cap H_1 \cap \cdots \cap H_k\) is a projective manifold of dimension \(n - k\) for \(s \in S^o\).

We set \(X_s^{(k)} := (\pi^{(k)})^{-1}(s) = X_s \cap H_1 \cap \cdots \cap H_k\) for \(s \in S\).

Let \(\{\psi_1, \ldots, \psi_m\} \subset \Omega^0(X, R^0\pi_*\omega_{X/S})\) be a free basis of the locally free sheaf \(R^0\pi_*\omega_{X/S}\) on \(S\). There exists \(\Psi_1, \ldots, \Psi_m \in \Omega^0(X, \Omega_X^{n+1-q})\) by [12, Th. 5.2] (after shrinking \(S\) if necessary) such that

\[
\psi_\alpha = [(\Psi_\alpha \wedge \kappa_X^q) \otimes (\pi^*ds)^{-1}], \quad \pi^*ds \wedge \Psi_\alpha = 0.
\]

By the condition \(\pi^*ds \wedge \Psi_\alpha = 0\), there exist relative holomorphic differentials \(\psi'_\alpha \in \Omega^0(X \setminus \Sigma, \Omega_X^{n-q})\) such that \(\Psi_\alpha = \psi'_\alpha \wedge \pi^*ds\). Then the harmonic representative of the cohomology class \(\psi_X\), is given by \(\psi'_\alpha \wedge \kappa_X|_{X_s}\). Since \(\kappa_X = c_1(L)\), we get

\[
h_{L^2}(\psi_\alpha, \psi_\beta)(s) = i^{(n-q)^2} \int_{X_s} \psi'_\alpha \wedge \overline{\psi}'_\beta \wedge \kappa_X^{(n-q)}|_{X_s} = i^{(n-q)^2} \int_{X_s^{(q)}} \psi'_\alpha \wedge \overline{\psi}'_\beta |_{X_s^{(q)}}.
\]

Hence Theorem 2.1 is reduced to the case \(q = 0\). In the case \(q = 0\), Theorem 2.1 is a consequence of Fujita’s estimate [4, 1.12] and the following:

**Lemma 2.3.** For \(\varphi, \psi \in \Omega^0(X, \Omega_X^{n+1})\), there exist \(a_m(s) \in C^0(S)\), \(0 \leq m \leq n\) such that

\[
\pi_* (\varphi \wedge \psi)(s) = \sum_{m=0}^n (\log |s|^2)^m a_m(s) ds \wedge d\bar{s}.
\]

**Proof.** Fix \(\varphi \in S^o\). Let \(\gamma \in GL(H^n(X_\delta, \mathbb{C}))\) be the monodromy. By assumption, \(\gamma\) is unipotent. Set \(H := R^n\pi_*\mathcal{O} \otimes_{\mathcal{O}_S} \mathcal{O}_S\), which is equipped with the Gauss-Manin connection. Let \(\{v_1, \ldots, v_m\}\) be a basis of \(H^n(X_\delta, \mathbb{C})\). Since \(\gamma\) is unipotent, there exists a nilpotent \(N \in \text{End}(H^n(X_\delta, \mathbb{C}))\) such that \(\gamma = \exp(N)\). Let \(p : S^o \ni z \to \exp(2\pi iz) \in S^o\) be the universal covering. Since \(H\) is flat, \(v_k\) extend to flat sections \(v_k \in \Gamma(S^o, p^*H)\), which induces an isomorphism \(p^*H \cong \mathcal{O}_{S^o} \otimes_{\mathbb{C}} H^n(X_\delta, \mathbb{C})\). Under this trivialization, we have \(v_k(z + 1) = \gamma \cdot v_k(z)\). We define \(s_k(\exp 2\pi iz) := \exp(-zN) v_k(z)\). Then \(s_1, \ldots, s_m \in \Gamma(S^o, p^*H)\) descend to single-valued holomorphic frame fields of \(H\). The canonical extension of \(H\) is the locally free sheaf on \(S\) defined as \(\mathcal{H} := \mathcal{O}_S s_1 \oplus \cdots \oplus \mathcal{O}_S s_m\). Set \(\mathcal{F}^n := \pi_*\Omega_X^n|_{S^o} \subset \mathcal{H}\).

By [11, p. 235], \(\mathcal{F}^n\) extends to a subbundle \(\mathcal{F}^n \subset \mathcal{H}\).
There exists $\varphi', \psi' \in H^0(X \setminus X_0, \Omega^r_X|_{X \setminus X_0})$ such that $\varphi = \pi^* ds \wedge \varphi'$ and $\psi = \pi^* ds \wedge \psi'$ on $X \setminus X_0$. Then $\varphi'$ and $\psi'$ are identified with $\varphi \otimes (\pi^* ds)^{-1}, \psi \otimes (\pi^* ds)^{-1} \in H^0(X_0, \omega_X/S)$, respectively. Since $F^n \subset H$, there exist $b_k(t), c_k(t) \in \mathcal{O}(S^n)$ such that $[\varphi'|_{X_0}] = \sum_{k=1}^m b_k(s) s_k(s)$ and $[\psi'|_{X_0}] = \sum_{k=1}^m c_k(s) s_k(s)$. Since $\pi_* \omega_X/S = F^n$ by Kawamata [5, Lemma 1], we get $b_k(s), c_k(s) \in \mathcal{O}(S)$. Then
\[
\pi_* (\varphi \wedge \overline{\psi})(s) = \left\{ \int_{X_0} \varphi' \wedge \overline{\psi'} \right\} ds \wedge d\overline{s} = \left\{ \int_{X_0} \sum_{j=1}^m b_j(s) s_j(s) + \sum_{k=1}^m c_k(s) s_k(s) \right\} ds \wedge d\overline{s}.
\]
Substituting $s_k(s) = \exp(-z N) v_k(z) = \sum_{0 \leq m \leq n} \frac{(-z)^m}{m!} N^m v_k(z)$, we get
\[
\pi_* (\varphi \wedge \overline{\psi})(s) = \left\{ \sum_{j,k=1}^m b_j(s) c_k(s) \sum_{0 \leq a, b \leq n} \frac{(-1)^{a+b}}{ab!} s^a z^b C^a_{j,k} \right\} ds \wedge d\overline{s},
\]
where $z = \frac{1}{2\pi i} \log s$ and $C^a_{j,k} = \int_{X_0} (N^a v_j) \wedge (N^b v_k)$. Since $\pi_* (\varphi \wedge \overline{\psi})$ is single-valued, so is the expression $\sum_{a+b=m} \frac{(-1)^{a+b}}{ab!} s^a z^b C^a_{j,k}$. As a result, there exists a constant $C^a_{j,k} \in \mathbb{C}$ such that $\sum_{a+b=m} \frac{(-1)^{a+b}}{ab!} s^a z^b C^a_{j,k} = C^a_{j,k} (\log |s|^2)^m$. Setting $a_m(s) := \sum_{j,k=1}^m C^a_{j,k} b_j(s) c_k(s)$, we get the result. □

Remark 2.4. In the proof of Theorem 2.1, the role of the nilpotent orbit theorem is played by Barlet’s theorem [1, Th. 4bis.] on the asymptotic expansion of fiber integrals associated to the function $f(z) = z_0 \cdots z_n$ near the origin. See [14, Sects. 6.3 and 6.4] for more details.

3. SOME TECHNICAL LEMMATA

We denote by $G^\infty_T(T)$ the set of real-valued $C^\infty$ functions on $T$.

**Lemma 3.1.** Let $\varphi(t) \in C^\infty_T(T)$ and let $r \in \mathbb{Q}$ and $\ell \in \mathbb{Z}$. Set $h(t) := |t|^{2r} (\log |t|^2)^\ell \varphi(t)$. Then the following identities hold:

1. $\partial_t h(t) = \left( \frac{\ell}{\ell + \ell (\log |t|^2)} + \frac{\partial_t \varphi(t)}{\varphi(t)} \right) h(t)$,
2. $\partial_{tt} h(t) = \left( -\frac{2r}{|t|^2 (\log |t|^2)} + \frac{\partial_{tt} \varphi(t)}{\varphi(t)} - \frac{|\partial_t \varphi(t)|^2}{\varphi(t)^2} + \frac{\ell}{\ell + \ell (\log |t|^2)} + \frac{\partial_t \varphi(t)}{\varphi(t)} \right)^2 h(t)$.

Proof. The proof is elementary and is left to the reader. □

**Lemma 3.2.** Let $I$ be a finite set. For $i \in I$, let $r_i \in \mathbb{Q}$, $\ell_i \in \mathbb{Z}$ and $\varphi_i(t) \in C^\infty_T(T)$. Set $g_i(t) := |t|^{2r_i} (\log |t|^2)^{\ell_i} \varphi_i(t)$ for $i \in I$ and $g(t) := \sum_{i \in I} g_i(t)$.

1. If $g(t) > 0$ on $T^0$, then the following equalities of functions on $T^0$ hold:

\[
\partial_t \log g = \sum_{i \in I} \left( \frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_t \varphi_i}{\varphi_i} \right) g_i / g,
\]
\[
\partial_{tt} \log g = -\frac{1}{2} \sum_{i,j} \left( \partial_{tt} \varphi_i \varphi_j + \partial_{tt} \varphi_i \varphi_j \right) g_i g_j / g^2 + \frac{1}{2} \sum_{i,j} \left( \frac{r_i - r_j}{t} + \frac{\ell_i - \ell_j}{t (\log |t|^2)} \right)^2 g_i g_j / g^2,
\]
\[
+ \frac{1}{2} \sum_{i,j} \left( \frac{\partial_{tt} \varphi_i}{\varphi_i} + \frac{\partial_{tt} \varphi_j}{\varphi_j} \right) g_i g_j / g^2 + \frac{1}{2} \sum_{i,j} \left( \partial_t \varphi_i / \varphi_i - \partial_t \varphi_j / \varphi_j \right) g_i g_j / g^2,
\]
\[
+ \sum_{i \neq j} \left( \frac{r_i - r_j}{t} + \frac{\ell_i - \ell_j}{t (\log |t|^2)} \right) \left( \frac{\partial_t \varphi_i}{\varphi_i} - \frac{\partial_t \varphi_j}{\varphi_j} \right) g_i g_j / g^2.
\]
(2) If \( r_i \geq 0 \) and \( 0 \leq \ell_i \leq N \) for all \( i \in I \), then as \( t \to 0 \)

\[
\partial_{t_i} \log g = -\frac{1}{2} \sum_{i,j} \frac{\ell_i + \ell_j}{|t|^2 (|t|^2)^2} \frac{g_i g_j}{g^2} + \frac{1}{2} \sum_{i,j} \left( \frac{r_i - r_j}{t} + \frac{\ell_i - \ell_j}{t (\log |t|^2)} \right)^2 \frac{g_i g_j}{g^2} + O\left( \frac{(- \log |t|)^2 N}{|t|g(t)^2} \right).
\]

**Proof.** (1) The first equality of (1) follows from Lemma 3.1 (1). Since

\[
\partial_{t_i} g(t) = \sum_{i \in I} \left( -\frac{\ell_i}{|t|^2 (\log |t|^2)} + \frac{\partial_{t_i} \varphi_i}{\varphi_i} - \frac{\partial_{x_i} \varphi_i}{\varphi_i^2} + \left( \frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_{x_i} \varphi_i}{\varphi_i} \right)^2 \right) g_i
\]

by Lemma 3.1 (2), we get

\[
(3.1) \quad g \partial_{t_i} g = \sum_{i,j \in I} g_j \left( -\frac{\ell_i + \ell_j}{|t|^2 (\log |t|^2)} + \frac{\partial_{t_i} \varphi_i}{\varphi_i} - \frac{\partial_{x_i} \varphi_i}{\varphi_i^2} - \frac{\partial_{x_j} \varphi_j}{\varphi_j^2} + \left( \frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_{x_i} \varphi_i}{\varphi_i} \right)^2 \right) g_i.
\]

By the first equality of Lemma 3.2 (1), we get

\[
(3.2) \quad |\partial_{t_i} g|^2 = \sum_{i,j \in I} \left( \frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_{x_i} \varphi_i}{\varphi_i} \right) \left( \frac{r_j}{t} + \frac{\ell_j}{t (\log |t|^2)} + \frac{\partial_{x_j} \varphi_j}{\varphi_j} \right) g_i g_j
\]

By (3.1) and (3.2), we get

\[
(3.3) \quad g \partial_{t_i} g - |\partial_{t_i} g|^2 = \frac{1}{2} \sum_{i,j \in I} \left( -\frac{\ell_i + \ell_j}{|t|^2 (\log |t|^2)} + \frac{\partial_{t_i} \varphi_i}{\varphi_i} + \frac{\partial_{t_j} \varphi_j}{\varphi_j} - \frac{\partial_{x_j} \varphi_j}{\varphi_j^2} - \frac{\partial_{x_i} \varphi_i}{\varphi_i^2} \right)
\]

\[
+ \left( \frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_{x_i} \varphi_i}{\varphi_i} \right) - \left( \frac{r_j}{t} + \frac{\ell_j}{t (\log |t|^2)} + \frac{\partial_{x_j} \varphi_j}{\varphi_j} \right)^2 g_i g_j.
\]

Since \( \partial_{t_i} \log g = (g \partial_{t_i} g - |\partial_{t_i} g|^2)/g^2 \), the second equality of Lemma 3.2 (1) follows from (3.3). This proves (1).

(2) By the definition of \( g_i(t) \), we get

\[
(3.4) \quad \left( \frac{\partial_{t_i} \varphi_i}{\varphi_i} + \frac{\partial_{t_i} \varphi_j}{\varphi_j} \right) \frac{g_i g_j}{g^2} = \frac{\varphi_i \partial_{t_i} \varphi_j + \varphi_j \partial_{t_i} \varphi_i}{\varphi_i^2 + \varphi_j^2} \cdot t^{2(r_i + r_j)} (\log |t|^2)^{\ell_i + \ell_j},
\]
Then there exists a constant $C$.

By the first equality of Lemma 3.2 (1) and (3.7), there are constants that $g_i$ are bounded near $t = 0$ and since $|t|^{2(r_i + r_j)}(-\log |t|^2)^{\ell_i + \ell_j} \leq (-\log |t|^2)^{2N}$ by the definition of $N$, we get (2) by the second equality of Lemma 3.2 (1) and (3.4), (3.5), (3.6).

Lemma 3.3. Let $\varphi_i \in C^\infty_T(\mathbb{T})$ for $0 \leq i \leq N$ and set $g(t) = \sum_{i=0}^{N} (\log |t|^2)^i \varphi_i(t)$. Assume that $g(t) > 0$ on $T^O$ and that $\varphi_i(0) \neq 0$ for some $0 \leq i \leq N$. Set

$$\ell := \max_{0 \leq i \leq N, \varphi_i(0) \neq 0} \{i\} \in \mathbb{Z}_{\geq 0}.$$ Then there exists a constant $C > 0$ such that the following inequalities hold

$$|\partial_t \log g(t)| \leq \frac{C}{|t|(-\log |t|)}, \quad \left| \partial_{it} \log g(t) + \frac{\ell}{|t|^2(-\log |t|)^2} \right| \leq \frac{C}{|t|^2(-\log |t|)^2}.$$ Proof. Set $I = \{0, 1, \ldots, N\}$ and $g_i(t) := (-\log |t|)^i \varphi_i(t)$ for $i \in I$. Namely, we set $(r_i, \ell_i) = (0, i)$ in Lemma 3.2. Since $g(t) = \varphi_i(0)(-\log |t|)^i (1 + O(1/\log |t|))$ as $t \to 0$, we get for each $0 \leq i \leq N$ the following asymptotic behavior as $t \to 0$:

(3.7) $\frac{g_i(t)}{g(t)} = \begin{cases} O(|t|(-\log |t|)^i) & (i > \ell), \\ 1 + O(|t|\log |t|)^n) & (i = \ell), \\ O((-\log |t|)^{-(\ell-i)}) & (i < \ell). \end{cases}$

By the first equality of Lemma 3.2 (1) and (3.7), there are constants $C, C' > 0$ such that

$$|\partial_t \log g(t)| \leq \frac{C}{|t|(-\log |t|)} \sum_{i=0}^{N} \left| \frac{g_i}{g} \right| \leq \frac{C'}{|t|(-\log |t|)}.$$ This proves the first inequality. Since $g(t) = \varphi_i(0)(-\log |t|)^i (1 + O(1/\log |t|))$, there exists $c > 0$ such that $g(t) \geq c > 0$ on $T^O$. In particular $O(1/g(t)) = O(1)$.

This, together with Lemma 3.2 (2), yields that

(3.8) $\partial_{it} \log g = -\frac{\ell}{|t|^2(-\log |t|)^2} + \frac{1}{2} \sum_{(i,j) \neq (t,\ell)} \frac{i+j}{|t|^2(-\log |t|)^2} \left( \frac{g_i}{g} \right) \left( \frac{g_j}{g} \right)$

$$+ \frac{1}{2} \sum_{i \neq j} \frac{(i-j)^2}{|t|^2(-\log |t|)^2} \left( \frac{g_i}{g} \right) \left( \frac{g_j}{g} \right) + O \left( \frac{(-\log |t|)^{2N}}{|t|} \right).$$

Since $|g_i(t)g_j(t)/g(t)^2| = O(1/\log |t|)$ when $i \neq j$ by (3.7), the second and the third term in the right hand side of (3.8) is bounded by $|t|^{-2}(-\log |t|)^{-3}$ as $t \to 0$. Similarly, it follows from (3.7) that the second term of the right hand side of (3.8) is bounded by $|t|^{-2}(-\log |t|)^{-3}$. The second inequality follows from (3.8). \[\square\]
4. The boundary behavior of the curvature of the $L^2$-metric

In this section, we define $N, \ell_q \in \mathbb{Z}_{\geq 0}$ as

$$N := n_{\rho_q}, \quad \ell_q := \max_{0 \leq i \leq N, a_i(0) \neq 0} \{i\},$$

where $a_i(t) \in C^\infty(T), 0 \leq i \leq N$, are the functions in Theorem 2.1. Recall that the integer $\nu > 0$ was defined in Sect. 2.

4.1. The singularity of the first Chern form.

**Theorem 4.1.** The following formula holds as $s \to 0$:

$$c_1(R^q \pi_* \omega_{X/S}(\xi), h_{L^2}) = \left\{ \frac{\ell_q}{|s|^2(\log |s|)^2} + O \left( \frac{1}{|s|^2(\log |s|)^3} \right) \right\} \sqrt{-1} ds \wedge d\bar{s}$$

*Proof.* Recall that $T$ is another unit disc and that the map $\mu: T \to S$ is defined as $s = \mu(t) = t^\nu$. By Theorem 2.1, we get

$$\mu^* c_1(R^q \pi_* \omega_{X/S}(\xi), h_{L^2}) = -\frac{\sqrt{-1}}{2\pi} \mu^* \partial \bar{\partial} \log \det G(s) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \det H(t)$$

$$= -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \sum_{m=0}^N a_m(t) (\log |t|^2)^m.$$  

We set $g(t) = \det H(t) = \sum_{i=0}^N (\log |t|^2)^i a_i(t)$ in Lemma 3.3. Since $a_i(0) \neq 0$ for some $0 \leq i \leq N$ by Theorem 2.1, we deduce from Lemma 3.3 that

(4.1) $$\mu^* c_1(R^q \pi_* \omega_{X/S}(\xi), h_{L^2}) = \ell_q \frac{\sqrt{-1}}{|t|^2(\log |t|)^2} \frac{dt \wedge d\bar{t}}{|t|^2(\log |t|)^2} + O \left( \frac{\sqrt{-1}}{|t|^2(\log |t|)^3} \right).$$

Since $\mu^* \sqrt{-1} ds \wedge d\bar{s}/(|s|^2(\log |s|)^2) = \nu^2 \sqrt{-1} dt \wedge d\bar{t}/(|t|^2(\log |t|)^2)$, the desired inequality follows from (4.1).  

**Remark 4.2.** The Hermitian metric $\mu^* \det h_{L^2}$ on the line bundle $R^q f_* \omega_{Y/T}(\xi)$ is good in the sense of Mumford [9]. Namely, the following estimates hold:

1. There exist constants $C, \ell > 0$ such that

$$\det H(t) \leq C(\log |t|^2)^\ell, \quad (\det H(t))^{-1} \leq C(\log |t|^2)^\ell.$$

2. There exists a constant $C > 0$ such that

$$|\partial \bar{t} \log \det H(t)| \leq \frac{C}{|t|(\log |t|)^2}, \quad |\partial t \log \det H(t)| \leq \frac{C}{|t|^2(\log |t|)^2}.$$ 

The inequalities (1) follow from Theorem 2.1. By setting $g(t) = \det H(t)$ in Lemma 3.3, we get (2) because $\det H(t) = g(t) = \sum_{i=0}^N (\log |t|^2)^i a_i(t)$, $a_i(t) \in C^\infty(T)$ with $a_i(0) \neq 0$ for some $0 \leq i \leq N$ by Theorem 2.1.

We do not know if the $L^2$-metric $\mu^* h_{L^2}$ on $R^q f_* \omega_{Y/T}(F^* \xi)$ is good in the sense of Mumford, because the estimates

$$\|\partial \bar{t} H \cdot H^{-1}\| \leq \frac{C}{|t|(\log |t|)^2}, \quad \|\partial t (\partial \bar{t} H \cdot H^{-1})\| \leq \frac{C}{|t|^2(\log |t|)^2}$$

do not necessarily follow from Theorem 2.1; from Theorem 2.1, we have only the estimates $\|\partial \bar{t} H \cdot H^{-1}\| \leq C(\log |t|^2)^\ell/|t|$ and $\|\partial t (\partial \bar{t} H \cdot H^{-1})\| \leq C(\log |t|^2)^\ell/|t|^2$, where $\|A\| = \sum_{i,j} |a_{ij}|$ for a matrix $A = (a_{ij})$. 


4.2. **Proof of Theorem 1.1.** Let $\lambda_1, \ldots, \lambda_{\rho_q}$ be the eigenvalues of the Hermitian endomorphism $\sqrt{-1} \text{Tr}(s)$. By the Nakano semi-positivity of $(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$, we get $\lambda_\alpha \geq 0$ for all $1 \leq \alpha \leq \rho_q$. By Theorem 4.1, we have the following inequality on $S^0$

$$0 \leq \sqrt{-1} \text{Tr}[\mathcal{R}(s)] = \sum_{\alpha} \lambda_\alpha \leq \frac{C}{|s|^2(-\log |s|)^2}.$$ 

In particular, we get $\Lambda := \max_\alpha \{\lambda_\alpha\} \leq C/(|s|^2(-\log |s|)^2)$. We get the desired inequality for $\sqrt{-1} \mathcal{R}(s)$ from the inequality $\sqrt{-1} \mathcal{R}(s) \leq \Lambda \cdot \text{Id}_{R^q\pi_*\omega_{X/S}(\xi)}$. The inequality for $c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$ is already proved in Theorem 4.1. This completes the proof. □

4.3. **Proof of Theorem 1.2.** By the curvature formula for Quillen metrics [3], the following equation of currents on $S$ holds

$$-d^F \log \tau(X/S, \omega_{X/S}) + \sum_{q \geq 0} (-1)^q c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$$

$$= [\pi_*\{\text{Td}(TX/S, h_{X/S})\text{ch}(\omega_{X/S}(\xi))\}]^{(2)},$$

where $[A]^{(p)}$ denotes the component of degree $p$ of a differential form $A$. By [13, Lemma 9.2], there exists $r \in \mathbb{Q}_{>0}$ such that as $s \to 0$

$$[\pi_*\{\text{Td}(TX/S, h_{X/S})\text{ch}(\omega_{X/S}(\xi))\}]^{(2)}(s) = O\left(\frac{\sqrt{-1} |s|^{2r}(-\log |s|)^n ds \wedge d\bar{s}}{|s|^2(-\log |s|)^3}\right).$$

By Theorem 1.1, we get

$$\sum_{q \geq 0} (-1)^q c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2}) = \frac{\sum_{q \geq 0} (-1)^q \ell_q \sqrt{-1} ds \wedge d\bar{s}}{2\pi |s|^2(-\log |s|)^2} + O\left(\frac{\sqrt{-1} ds \wedge d\bar{s}}{|s|^2(-\log |s|)^3}\right).$$

By (4.2), (4.3), (4.4), we get on $S^0$

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \tau(X/S, \omega_{X/S}) = \frac{\sum_{q \geq 0} (-1)^q \ell_q \sqrt{-1} ds \wedge d\bar{s}}{2\pi |s|^2(-\log |s|)^2} + O\left(\frac{\sqrt{-1} ds \wedge d\bar{s}}{|s|^2(-\log |s|)^3}\right).$$

This completes the proof. □

5. **Canonical singularities and the curvature of $L^2$-metric**

In this section, we assume that the central fiber $X_0$ is reduced and irreducible and has only canonical (equivalently rational) singularities. Then $G(s) = (G_{\alpha\beta}(s))$ is expected to have better regularity than usual. To see this, set

$$B(S) := C^\infty(S) \oplus \bigoplus_{r \in \mathbb{Q}^0(0,1)} \bigoplus_{k=0}^n |s|^{2r} (\log |s|)^k C^\infty(S) \subset C^0(S).$$

By [14, Th. 7.2], the $L^2$-metric $h_{L^2}$ on $R^q\pi_*\omega_{X/S}(\xi)$ is a continuous Hermitian metric lying in the class $B(S)$. Namely, $G_{\alpha\beta}(s) \in B(S)$ for all $1 \leq \alpha, \beta \leq \rho_q$.

**Proposition 5.1.** If $X_0$ has only canonical singularities, then there exists $r \in \mathbb{Q}_{>0}$ and $C > 0$ such that the following inequality of real $(1,1)$-forms on $S^0$ holds

$$0 \leq c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2}) \leq C\frac{\sqrt{-1} |s|^{2r} ds \wedge d\bar{s}}{|s|^2(-\log |s|)^2}.$$
In particular, the curvature $iR(s) \, ds \wedge ds$ satisfies the following estimate:

$$0 \leq \sqrt{-1} R(s) \leq \frac{C|s|^{2r}}{|s|^2(- \log |s|)^2} \text{Id}_{R^s \pi \ast \omega_{X/s}(\xi)}.$$

**Proof.** Since $G_{\alpha \beta}(s)$ is continuous on $S$, we may assume by an appropriate choice of basis that $G_{\alpha \beta}(0) = \delta_{\alpha \beta}$. Since $G_{\alpha}(0) \leq 1$, we have $G_{\alpha \beta}(0) = 1$. Let $s = \sum_{i,j} (\log |s|)^{r_i} \varphi_i(s)$. By Lemma 3.2 (2) applied to $G(s)$, we get

$$- \partial_s \log \det G(s) = \frac{1}{2} \sum_{i,j \in I \cup \{0\}} \frac{\ell_i + \ell_j}{|s|^2(\log |s|^2)^2} g_{ij} - \frac{1}{2} \sum_{i,j \in I \cup \{0\}} \left| \frac{r_i - r_j}{s} + \frac{\ell_i - \ell_j}{s(\log |s|^2)} \right|^2 \frac{g_{ij}}{g^2}$$

$$+ O \left( \frac{(- \log |s|)^{2N}}{|s|^2} \right) \leq C \sum_{i,j \in I \cup \{0\}} \frac{(\ell_i + \ell_j)|s|^{2(r_i + r_j)}}{|s|^2(\log |s|^2)^2} + C \sum_{i,j \in I \cup \{0\}} \left| \frac{r_i - r_j}{s} + \frac{\ell_i - \ell_j}{s(\log |s|^2)} \right|^2 |s|^{2(r_i + r_j)}$$

$$+ O \left( \frac{(- \log |s|)^{2N}}{|s|^2} \right).$$

Set $r := \min_{i \in I} r_i > 0$. Since $r_i + r_j > r$ for all $(i, j) \in (I \cup \{0\}) \times (I \cup \{0\}) \setminus \{(0, 0)\}$, we get

$$- \partial_s \log \det G(s) \leq C \frac{2\ell_0}{|s|^2(\log |s|^2)^2} + C \sum_{(i,j) \neq (0,0)} \left| \frac{|s|^{2(r_i + r_j)}}{|s|^2(\log |s|^2)^2} \right|^2 \leq \frac{C|s|^{2r}}{|s|^2}. \quad (1)$$

because $\ell_0 = 0$. Since $- \partial_s \log \det G(s) \geq 0$ by the Nakano semi-positivity of $(R^s \pi \ast \omega_{X/s}(\xi), h_{L^2})$ by [2], [7], we get the first inequality. The proof of the second inequality is the same as that of the corresponding inequality of Theorem 1.1. \hfill $\Box$

**References**

[1] Barlet, D. Développement asymptotique des fonctions obtenues par intégration sur les fibres, Invent. Math. 68 (1982), 129-174.

[2] Berndtsson, B. Curvatures of vector bundles associated to holomorphic fibrations, Ann. of Math. 169 (2009) 531–560.

[3] Bismut, J.-M., Gillet, H., Soulé, C. Analytic torsion and holomorphic determinant bundles I,II,III, Commun. Math. Phys. 115 (1988), 49–78, 79-126, 301-351.

[4] Fujita, T. On Kähler fiber spaces over curves, J. Math. Soc. Japan 30 (1978), 779–794.

[5] Kawamata, Y. Kodaira dimension of algebraic fiber spaces over curves, Invent. Math. 66 (1982), 57–71.

[6] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B. Toroidal Embeddings I, Lecture Notes Math. 339 (1973).

[7] Mourougane, C., Takayama, S. Hodge metrics and the curvature of higher direct images, Ann. Sci. Éc. Norm. Sup. 41 (2008), 905–924.

[8] Extension of twisted Hodge metrics for Kähler morphisms, J. Differential Geom. 83 (2009), 131–161.

[9] Mumford, D. Hirzebruch’s proportionality theorem in the non-compact case, Invent. Math. 42 (1977), 239-272.
[10] Ray, D.B., Singer, I.M. Analytic torsion for complex manifolds, Ann. Math. 98 (1973), 154–177.

[11] Schmid, W. Variation of Hodge structure: The singularities of the period mapping, Invent. Math. 22 (1973), 211–319.

[12] Takegoshi, K. Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms, Math. Ann. 303 (1995), 389–416.

[13] Yoshikawa, K.-I. On the singularity of Quillen metrics, Math. Ann. 337 (2007), 61–89.

[14] Singularities and analytic torsion, preprint, (2010).

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