Nonlinear dynamics investigation of contact force in a cam–follower system using the Lyapunov exponent parameter, power spectrum analysis, and Poincaré maps

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Abstract
The aim of this article is to detect the detachment between the cam and the follower using the largest Lyapunov exponent parameter, the power density function of fast Fourier transform, and Poincaré maps due to the nonlinear dynamics phenomenon of the follower. The detachment between the cam and the follower was investigated for different cam speeds and different internal distances of the follower guide from inside. This study focuses on the use of the cam–follower system with a bionic quadruped robot through a linkage mechanism. Multishock absorber (spring–damper–mass) systems at the very end of the follower were used to improve the dynamic performance and to reduce the detachment between the cam and the follower. The SolidWorks program was used in the numerical solution, while a high-speed camera at the foreground of the OPTOTRAK 30/20 equipment was used to identify the follower position. The friction and impact were considered between the cam and the follower and between the follower and its guide. In general, when the cam and the follower are in permanent contact, there is no loss in potential energy, and no impact or restitution. The detachment between the cam and the follower increases with increasing coefficient of restitution in the presence of the impact.

KEYWORDS
nonlinear dynamics, Lyapunov exponent, non-periodic motion, contact force

1 | INTRODUCTION

The value of the contact force between the cam and the follower is crucial in determining whether the follower detaches from the cam or remains in permanent contact. Sundar et al.1 used the Hertzian contact theory to calculate the contact stiffness under both line and point contacts. They analyzed the coefficient of a restitution model by using a single degree of freedom when the cam was rotating about a fixed pivot. Under varied cam rotational speeds, Alzate et al.2 observed that the follower is detached from the cam and begins to show the emergence of periodic impacting behavior characterized by many impacts and chattering. Ciulli et al.3 measured the contact force experimentally during the rotation of a circular eccentric at different rotational speeds and preloads. Yousuf4 calculated the contact force against time for a greasy and dry profile of the cam based on Hertzian contact pressure. He applied the theory of a circular plate to calculate the bending deflection on the cam profile. The influence of the maximum contact pressure of the bending deflection of the cam profile is studied for a globoidal cam...
and roller follower system. Moreover, Yousuf used the photo-elastic technique to calculate the value of contact stress between the pear cam and the roller follower mechanism. Four different positions of the follower with respect to the cam such as 0°, 90°, 180°, and 270° at different values of the compression load are considered. Pugliese et al. designed an apparatus to measure the contact force using an optical interferometry sensor and a high-speed camera. They showed different conditions of the contact at low and high magnifications to detect the angular positions of the cam. Yang et al. extended a transient impact model to calculate the value of contact stress between the pear cam and the follower such as 0°, 90°, 180°, and 270° at different motions for the follower such as constant velocity and rotational motions was applied to describe motion of the three degrees of freedom of a flat-faced follower. Figure 1 shows the three degrees of freedom system based on the motion of the follower stem. Jamali et al. discussed the parameters that affect the contact problem between the cam and the flat-faced follower such as the radius of curvature, surface velocities, and applied load under a thin film of lubricants. They improved numerically the level of film thickness lubricant by taking into consideration the parabolic shape of the pressure distribution through cam depth, which reduced the detachment between the cam and the follower. Desai and Patel determined the critical angular speed when the follower jumped off the cam. They determined the kinematic parameters of the dynamic force analysis at different motions for the follower such as constant velocity motion, cycloidal motion, parabolic motion, and simple harmonic motion. Lassaad et al. studied the effect of the error in the cam profile on the nonlinear dynamics of the oscillated roller follower by solving nonlinear second-order differential equations. DasGupta and Ghosh used a constant pressure angle to assess jamming of the follower in its guide based on the follower guide friction.

This study focuses on the use of a cam–follower system with a bionic quadruped robot through a linkage mechanism. Multishock absorber (spring–damper–mass) systems at the very end of the follower were used to improve the dynamic performance and to reduce the detachment between the cam and the follower. This paper is organized as follows:

(a) In Section 2, the analytic displacement of the follower was derived using the Newton–Euler approach in the presence of three degrees of freedom for the follower. (b) In Section 3, the experiment setup that is used in this study with a high-speed camera at the foreground of OPTOTRAK 30/20 equipment to identify the follower position and to calculate the contact force is described. (c) Section 4 describes the method of numerical simulation of the detachment between the cam and the follower through follower displacement and contact force using the SolidWorks program. (d) In Sections 5–7, the detachment between the cam and the follower is detected using the largest Lyapunov exponent parameter, the power density function of FFT, and Poincaré map. The highlights of this paper are as follows:

- The detachment between the cam and the follower was detected using the largest Lyapunov exponent parameter, the power density function using FFT, and Poincaré maps due to the nonlinear dynamics phenomenon of the follower.
- Multishock absorber (spring–damper–mass) systems at the very end of the follower were used to improve the dynamic performance and to reduce the detachment height between the cam and the follower.
- Friction and impact were considered between the cam and the follower and between the follower and its guides.
- The experimental test was carried out using a high-speed camera at the foreground of the OPTOTRAK 30/20 equipment.

2 | EQUATIONS OF MOTION OF A THREE-DEGREE-OF-FREEDOM SYSTEM

In this paper, nonlinear dynamics occurred because of the clearance between the follower stem and its guide and due to motion of the follower with three degrees of freedom (right-left, up-down, and rotation about the z-axis). The nonlinear dynamics is tested over four values of the internal distance (ID) of the follower guide from inside (ID = 16, 17, 18, and 19). The more the ID, the more the clearance between the follower stem and its guide since the follower stem geometry and dimensions are constants. The nonlinear dynamics phenomenon is tested over high values of ID, which yields large translation and rotation (the motion of the follower stem is nonperiodic motion and chaos). Periodic motion of the follower occurs when there is no clearance between the follower and its guide or at a small value of ID The Newton–Euler approach for translational and rotational motions was applied to describe motion of the three-degree-of-freedom follower. Figure 1 shows the three degrees of freedom of a flat-faced follower stem. All springs have the same stiffness $K_1 = K_2 = K_3 = K_4 = K_5 = 73.56$ N/mm, while the damping coefficient has the values $C_1 = C_2 = C_3 = C_4 = C_5 = 9.19$ N/s/mm. The mass of the follower is assumed to be $m = 0.2759$ kg.

The cam–follower mechanism can be treated as a single- or a multi-degree-of-freedom system based on the motion of the follower. Applying the Newton equation for rigid body dynamics $\Sigma \text{Forces} = m \ddot{x}$ to the translational motion in the vertical direction where center of gravity point 2 is the reference point, one obtains

$$P_c - K_6x_2 - C_6 \dot{x}_2 = m\ddot{x}_2.$$ (1)
point 2 is the center of gravity point of the follower stem. Point 1 is the point of the center of the follower stem length.

After simplification, Equation (1) yields

\[ m\ddot{x}_2 + C_3\dot{x}_2 + K_3x_2 = P_c. \] (2)

Applying the Newton equation \( \Sigma \) Forces = \( m\ddot{x}_1 \) to the translational motion in the horizontal direction on the follower stem where center of gravity point 2 is the reference point, one obtains

\[
\begin{align*}
-K_3(x_1 - l_2\dot{\theta}) - C_1(x_1 - l_2\dot{\theta}) - K_2(x_1 - l_2\dot{\theta}) - C_3(x_1 - l_2\dot{\theta}) \\
- K_4(x_1 + l_2\dot{\theta}) - C_2(x_1 + l_2\dot{\theta}) - K_4(x_1 + l_2\dot{\theta}) \\
- C_4(x_1 + l_2\dot{\theta}) \end{align*}
\] (3)

After simplification, Equation (3) leads to

\[
m\ddot{x}_1 + (C_1 + C_2 + C_3 + C_4)x_1 - (C_1l_2 - C_2l_2 + C_3l_2 - C_4l_2)x_1 \\
+ (K_1 + K_2 + K_3)x_1 - (K_1l_2 - K_2l_2 + K_3l_2) \\
- K_4l_2x_1 = 0.
\] (4)

Taking the moment about the center of gravity point 2 in the follower stem structure, the Euler equation \( \Sigma \) Moments = \( l\ddot{\theta} \) leads to

\[
\begin{align*}
-K_3[l_2\dot{\theta} - x_1]l_4 - C_1[l_2\dot{\theta} - x_1]l_4 - K_2[l_2\dot{\theta} - x_1]l_4 \\
- C_3[l_2\dot{\theta} - x_1]l_4 - K_4[l_2\dot{\theta} + x_1]l_4 - C_2[l_2\dot{\theta} + x_1]l_4 \\
- K_4[l_2\dot{\theta} + x_1]l_4 - C_4[l_2\dot{\theta} + x_1]l_4 + P_c \times \sin(\theta) \\
\times (l_3 - l_4 - l_4) = l\ddot{\theta}.
\end{align*}
\] (5)

After simplification, Equation (5) can be written as

\[
l\ddot{\theta} + (C_1 + C_3)l_2^2\dot{\theta} + (C_2 + C_4)l_2^2\dot{\theta} + (K_1 + K_3)l_2^2\dot{\theta} + (K_2 + K_4)l_2^2\dot{\theta} \\
- (C_1 + C_3)l_2^2\dot{\theta} + (C_2 + C_4)l_2^2\dot{\theta} - (K_1 + K_3)l_2^2\dot{\theta} \\
+ (K_2 + K_4)l_2^2\dot{\theta} = P_c \times \sin(\theta) \times (l_3 - l_4 - l_4). \] (6)

The geometry of the follower stem is treated as a beam. Also, the summation of area moments of inertia is obtained around the center of gravity (black dot point 2). After applying the theory of the parallel axis theorem to transfer the area moment of inertia from point 1 to point 2, we obtain

\[ I = \frac{m_2^2}{12} + m \left( \frac{l_3}{l_2} - \left( l_4 + l_4 \right) \right)^2. \]

Equations (2), (4), and (6) can be written in matrix form as

\[
\begin{bmatrix}
\mathbf{m}_{11} & \mathbf{m}_{12} & \mathbf{m}_{13} \\
\mathbf{m}_{21} & \mathbf{m}_{22} & \mathbf{m}_{23} \\
\mathbf{m}_{31} & \mathbf{m}_{32} & \mathbf{m}_{33}
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddot{\theta}
\end{bmatrix}
= \begin{bmatrix}
\mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} \\
\mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} \\
\mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33}
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{\theta}
\end{bmatrix}
\begin{bmatrix}
\mathbf{X}_1 \\
\mathbf{X}_2 \\
\mathbf{X}_3
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}.
\] (7)

where

\[
\begin{align*}
m_{11} = m, & \quad m_{12} = 0, \quad m_{13} = 0, \\
m_{21} = 0, & \quad m_{22} = m, \quad m_{23} = 0, \\
m_{31} = 0, & \quad m_{32} = 0, \quad m_{33} = l, \\
C_{11} = C_1 & + C_2 + C_3 + C_4, \\
C_{12} = 0, & \quad C_{13} = -(C_1l_2 - C_2l_2 + C_3l_2 - C_4l_2), \\
C_{21} = 0, & \quad C_{22} = C_5, \quad C_{23} = 0, \\
C_{31} = -(C_1 + C_3l_2) & - (C_2 - C_4)l_2, \\
C_{32} = 0, & \quad C_{33} = (C_1 + C_3)l_2^2 + (C_2 + C_4)l_2^2, \\
K_{11} = (K_1 & + K_2 + K_3), \\
K_{12} = 0, & \quad K_{13} = -(K_1l_2 - K_2l_2 + K_3l_2), \\
K_{21} = 0, & \quad K_{22} = K_4, \quad K_{23} = 0, \\
K_{31} = -(K_1 + K_3l_2) & + (K_2 + K_4)l_2, \\
K_{32} = 0, & \quad K_{33} = (K_1 + K_3)l_2^2 + (K_2 + K_4)l_2^2, \\
F_1 = 0, & \quad F_2 = P_c, \quad F_3 = P_c \times \sin(\theta) \times (l_3 - l_4 - l_4).
\end{align*}
\]

Equation (7) is applied for a small value of the clearance in which the translation and rotation are very small. The nonlinear dynamics phenomenon arises from the contact force. Equation (7) can be used to define the frequency–response matrix, called the impedance matrix, as follows\textsuperscript{18}:

\[
\begin{bmatrix}
Z_{11}(\Omega) & Z_{12}(\Omega) & Z_{13}(\Omega) \\
Z_{21}(\Omega) & Z_{22}(\Omega) & Z_{23}(\Omega) \\
Z_{31}(\Omega) & Z_{32}(\Omega) & Z_{33}(\Omega)
\end{bmatrix}
\begin{bmatrix}
\mathbf{X}_1 \\
\mathbf{X}_2 \\
\mathbf{X}_3
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}.
\] (8)

where

\[
\begin{align*}
Z_{11}(\Omega) = & -\Omega^2m_{11} + i\Omega C_{11} + K_{11}, \\
Z_{12}(\Omega) = & i\Omega C_{13} + K_{13}, \\
Z_{13}(\Omega) = & -\Omega^2m_{12} + i\Omega C_{22} + K_{22}, \\
Z_{21}(\Omega) = & i\Omega C_{31} + K_{31}, \\
Z_{22}(\Omega) = & -\Omega^2m_{22} + i\Omega C_{33} + K_{33}. \\
\end{align*}
\]

\( \Omega \) is the frequency of external sinusoidal excitations, and \( \ast \) is the complex amplitude of \( \ast \) for a steady–state response with both amplitude and phase information.

The eigenvalue \( \lambda \in \mathbb{C} \) of a damped system is complex and can be solved by substitution of \( i\Omega \rightarrow \lambda \) into Equation (8), with the determinant of the impedance matrix being zero, that is
FIGURE 2 Experimental setup of cam-follower mechanism

FIGURE 3 Applied load as a function of spring deflection

FIGURE 4 Contact force against time

FIGURE 5 Computer-aided design model
The complex amplitude of a steady-state response is calculated by taking the inverse of the impedance matrix and multiplying by the force vector as

$$\begin{pmatrix} Z_{11}(\Omega) & 0 & Z_{13}(\Omega) \\ 0 & Z_{22}(\Omega) & 0 \\ Z_{31}(\Omega) & 0 & Z_{33}(\Omega) \end{pmatrix}^{-1} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = 0. \quad (9)$$

The analytical solution of the steady-state response of the follower is

$$\begin{pmatrix} X_1(t) \\ X_2(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} Z_{11}(\Omega) & Z_{12}(\Omega) & Z_{13}(\Omega) \\ 0 & Z_{22}(\Omega) & 0 \\ Z_{31}(\Omega) & 0 & Z_{33}(\Omega) \end{pmatrix}^{-1} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \sin(\Omega t - \arg \lambda). \quad (10)$$

The analytical solution of the follower displacement is given as

$$X = \sqrt{X_1^2 + X_2^2} \quad (13)$$

since the follower displacement varies in horizontal and vertical directions.

### 3 | EXPERIMENTAL TEST

An internal distance (ID = 16 mm) of the follower guide with different cam speeds (N) was used in the experimental test. The internal dimension of the follower guide is the distance between the two halves of the cylindrical shape of the follower guide from inside. There was a link between the two halves of the cylindrical shape all the way from the back. The internal distance of the follower guide from inside was measured to be 16 mm. The experimental setup included a high-speed camera at the foreground of OPTOTRAK 30/20 equipment to identify the follower position and to calculate the contact force. A multishock absorber (spring-damper-mass) system was used at the very end of the follower stem to maintain both the cam and the follower in contact.
permanent contact and to reduce the detachment heights of the follower. All the springs have the stiffness $K_1 = K_2 = K_3 = K_4 = K_5 = 73.56 \text{ N/mm}$, while the damping coefficients have the values $C_1 = C_2 = C_3 = C_4 = C_5 = 9.19 \text{ N·s/mm}$. The masses of the four boxes are assumed to be $M_1 = M_2 = M_3 = M_4 = 0.2625 \text{ kg}$, with a total mass of 1.05 kg. These masses have the effect of reducing the peak of follower lift displacement. Four cylindrical rollers are moved up and down through a vertical groove from both sides of the rectangular boxes. The follower has three degrees of freedom, and the contact point was considered in the calculation of follower displacement. The signal of follower movement was derived twice with respect to time to determine the follower acceleration. Figure 2 shows the experimental test setup used in this study.

As mentioned earlier in Section 2, the more the clearance between the follower and its guide, the more the translation and rotation. The follower stem goes up and down based on the profile

**FIGURE 8** Follower linear displacement versus time for different values of coefficient of restitution: (A) without using the coefficient of restitution. (B) Coefficient of restitution = 0.2. (C) Coefficient of restitution = 0.3. (D) Coefficient of restitution = 0.4.

**FIGURE 9** Comparison of follower linear displacement
geometry of the cam. A multishock absorber is used in Figure 2 to suppress the translational motion of the follower stem.

To ensure contact between the cam and the follower, a spring with the following specification is used between the vertical table and follower stem:

\[ C = 3, \quad n = 6, \quad G = 80 \text{ GPa}, \quad d = 2.5 \text{ mm}, \quad \text{OD} = 10 \text{ mm}, \quad \Delta = 15 \text{ mm}. \]

The spring was suspended from a fixed point and the load was applied at the very end point of the spring. The spring deflection was recorded against the applied load, as shown in Figure 3, in which the slope represents the spring stiffness \( K = 400 \text{ N/m} \).

The proposed experimental setup is used with bionic quadruped robots since the cam–follower system is connected to robots through a linkage mechanism. The use of a multishock absorber (spring–damper–mass) system at the very end of the follower stem keeps the robot walking at a horizontal level. Figure 4 shows the comparison of the contact force against time at \( ID = 16 \text{ mm} \) and \( N = 200 \text{ rpm} \). An analytic set of data of the contact force was calculated after applying Equation (12), while the numerical simulation of the contact force was determined using a computer-aided design (CAD) program. The experimental set of the data of the contact force was calculated after tracking the follower position using a high-speed camera at the foreground of the OPTOTRAK 30/20 equipment. Newton’s law of translational motion was applied twice in the presence of follower mass and follower acceleration to determine the contact force.

**FIGURE 10** Comparison of follower displacement with other publications

**FIGURE 11** Contact force versus time for different cam speeds for \( ID = 16 \text{ mm} \): (A) \( N = 400 \text{ rpm} \), (B) \( N = 600 \text{ rpm} \), (C) \( N = 1000 \text{ rpm} \), and (D) \( N = 1200 \text{ rpm} \). ID, internal distance.
4 | NUMERICAL SIMULATION

4.1 | Numerical simulation of follower displacement

A CAD program is used to simulate the numerical model of the follower displacement. In the numerical simulation, the follower has three degrees of freedom (up–down, right–left, and rotation about the z-axis). The two rollers in both sides between the wall and the mass of the spring–damper system allowed the multishock absorber systems to move up and down as shown in Figure 5. The GSTIFF integrator solves complex nonlinear dynamics systems and can determine the solution over a range of cam speeds. The integrator GSTIFF is designed with default values of a maximum iteration of 50, an initial integrator step size of 0.0001, a minimum integrator step size of 0.000 000 1, and a maximum integrator step size of 0.001. The elastic constants of the spring and damper used in the numerical simulation are obtained from Section 3.

Figure 6 shows multishock absorber (spring–damper–mass) systems. Four spring–damper–mass systems are used at the very end of the follower stem to ensure that the detachment between the cam and the follower is as low as possible and to maintain both the mechanical parts in continuous contact.

Multishock absorber systems have been proven to be effective in reducing the detachment between the cam and the follower and in improving the dynamic performance through reduction in the peak of the nonlinear response of the follower as

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**FIGURE 12** Contact force versus time for different cam speeds for ID = 17 mm: (A) $N = 200$ rpm, (B) $N = 400$ rpm, (C) $N = 800$ rpm, and (D) $N = 1200$ rpm. ID, internal distance.

**FIGURE 13** Local Lyapunov exponent versus number of samples for ID = 16 mm and $N = 400$ rpm. ID, internal distance.
shown in Figure 7. The peak of the nonlinear response of the follower displacement was reduced to 15%, 32%, 45%, and 62% after using multishock absorber systems. The system with ID = 17 mm and N = 300 rpm was used in the simulation. The amplitude of the nonlinear response of the follower is damped using multi spring-damper-mass systems. In other words, the more the spring-damper-mass systems, the lower the amplitude of vibrations.
Figure 8 shows the follower displacement versus time for different coefficient of restitution values for ID = 19 mm and N = 200 rpm. The coefficient of restitution characterizes the nature of impact and defines loss in potential energy due to the reduction in the peak of the nonlinear response of the follower. There were no losses in potential energy when the cam and the follower were in continuous contact as shown in Figure 8A. The potential energy loss increases with increasing coefficient of restitution and with increasing peak of the nonlinear response of the follower displacement as shown in Figure 8B–D. In general, when the cam and the follower are in continuous contact, there is no loss in potential energy and there is no impact either. The detachment between the cam and the follower increases with increasing coefficient of restitution in the case of impact.

In this paper, the cam profile with return–dwell–rise–dwell–return–dwell–rise–dwell–rise–dwell was selected. The system with ID = 16 mm and N = 200 rpm was used in the verification of the follower displacement as shown in Figure 9 for one cycle of cam rotation.

The analytical follower displacement was obtained using Equation (13). Follower linear displacement was tracked experimentally using a high-speed camera at the foreground of the OPTOTRAK/3020 equipment. In the analytical solution of the follower displacement, the dwell stroke period of time was a straight line, which means that there was no detachment between the cam and the follower. In the numerical simulation, the dwell stroke was divided between the rise and return strokes, and the detachment was intangible between the cam and the follower. In the experimental setup, the dwell stroke was tangible and it varied as a sinusoidal function of time. In terms of the comparison of the present work with the other publications, Figure 10 shows a comparison of nonlinear follower response. In the present work, the system with ID = 17 mm and N = 200 rpm is considered in the
simulation without a linkage mechanism. Ref. 26 assumes that the follower is connected to a linkage mechanism.

4.2 Numerical analysis of contact force

The weight of the follower was found to be sufficient to maintain the contact when the follower moved with simple harmonic motion. Also, the preload spring between the installation table and the follower stem must be properly selected to maintain the contact. Figures 11 and 12 show contact force versus time for ID = 16 and 17 mm at different cam speeds $N$. The detachment height varied with the variation of the contact load value. The detachment occurs between the cam and the follower when the value of the contact force is minimum as shown in Figures 11B–D and 12B–D. There was no noticeable detachment when the follower slipped off the cam as shown in Figures 11A and 12A for $N = 400$ and 200 rpm, respectively. In general, the detachment between the cam and the follower increases when the contact force has a minimum value.

5 | LARGEST LYAPUNOV EXPONENT PARAMETER

The largest Lyapunov exponent is a numeric value that can have either a positive or negative sign. In the design, the largest Lyapunov exponent parameter is one, used to detect separation between the cam and the follower through the contact force. A positive value of

![Figure 19](image19.png)  
**Figure 19** Follower–displacement power spectrum analysis from the experimental test

![Figure 20](image20.png)  
**Figure 20** Follower–displacement power spectrum analysis from numerical simulation
the Lyapunov exponent of the contact force indicates that there is a detachment between the cam and the follower (nonperiodic motion). A negative largest Lyapunov exponent value of the contact force implies periodic motion (the cam and the follower are in continuous contact), and the contact force has a maximum value. The Wolf algorithm based on Matlab software was used to calculate the values of the largest Lyapunov exponent by monitoring the orbital divergence for the contact force.\textsuperscript{27} Equations (14) and (15) were used to develop a Wolf algorithm code of the dynamic tool.\textsuperscript{28}

\[
d(t) = De^{\lambda t},
\]  

\text{(14)}

**FIGURE 21** Poincaré maps for ID = 16 mm for different cam angular speeds: (A) \(N = 400 \text{ rpm}\), (B) \(N = 600 \text{ rpm}\), (C) \(N = 800 \text{ rpm}\), (D) \(N = 1000 \text{ rpm}\), (E) \(N = 1200 \text{ rpm}\), and (F) \(N = 1500 \text{ rpm}\). ID, internal distance.

**FIGURE 22** Poincaré maps for ID = 18 mm for different cam angular speeds: (A) \(N = 200 \text{ rpm}\), (B) \(N = 400 \text{ rpm}\), (C) \(N = 600 \text{ rpm}\), (D) \(N = 800 \text{ rpm}\), (E) \(N = 1000 \text{ rpm}\), and (F) \(N = 1500 \text{ rpm}\). ID, internal distance.
where $D$ is the average displacement between trajectories at $t = 0$, $d(t)$ is the rate of change in the distance between nearest neighbors, $d_j(i)$ is the distance between the $j$th pair at $(i)$ nearest neighbors (in mm), $t$ is the single time series (in s), $y(i)$ is the curve fitting of the least square method for the follower displacement data (in mm), $\lambda$ is the largest Lyapunov exponent parameter, and $\Delta t$ is the time interval (in s).

Figure 13 shows the numerical value of the largest Lyapunov exponent as a function of number of samples for ID = 16 mm and $N = 400$ rpm. The value of the Lyapunov exponent parameter was taken at its respective equilibrium point. The data of the contact force against time are set up as one column in the Wolf algorithm alongside the numeric value of time delay and embedding dimensions to calculate the numerical value of the largest Lyapunov exponent parameter. The contact force against time has been taken from the SolidWorks program and is used in the dynamic tool of the Wolf algorithm code.

Time delay and embedding dimensions are numerical values required in the Wolf algorithm code for the estimation of the local Lyapunov exponent parameter. The algorithm of the dynamic code of global false nearest neighbors of the contact force was used to obtain the value of embedding dimensions estimated when the trend of global false nearest neighbors approaches zero. In this paper, a local and optimal time delay was estimated based on the recommendation that the local time delay should be chosen dependent on the embedding dimensions. The algorithm code of average mutual information was used to obtain the value of time delay, and the first minimum time in the average mutual information trend represents the value of the time delay. The optimal time delay was calculated from the following equation:

$$\tau_w = (P - 1)\tau^*,$$

where $\tau^*$ is the local time delay, $\tau_w$ is the optimal for independence of time series, and $P$ represents the embedding dimensions. Figures 14

![Figure 23](image-url)

**FIGURE 23** Contact force versus time for ID = 18 mm: (A) $N = 400$ rpm, (B) $N = 800$ rpm, (C) $N = 1200$ rpm, and (D) $N = 1500$ rpm. ID, internal distance.
and 15 show the comparison of time delay, while Figures 16 and 17 show the comparison of embedding dimensions.

There is another method, called average logarithmic divergence, for estimating the largest Lyapunov exponent of the contact force. The straight line represents the slope of average logarithmic divergence, which reflects the value of the Lyapunov exponent parameter. The curve represents the logarithm function against time of the contact force. As is known, the logarithm function treats any set of data by a straight line that gives the value of the largest Lyapunov exponent. Figure 18 shows a comparison of the average logarithmic divergence of contact force versus time for ID = 16 mm and N = 200 rpm. The largest analytical Lyapunov exponent of the contact force was calculated after applying Equation (13) on one column of the contact force using the average logarithmic divergence approach, while the numerical simulation of the largest Lyapunov exponent of the contact force was determined using the SolidWorks program. The experimental value of the largest Lyapunov exponent was obtained after tracking the follower position using the OPTOTRAK 30/20 equipment using a high-speed camera and by applying Newton's second law to the follower mass and follower acceleration as previously mentioned.

### 6 | FAST FOURIER TRANSFORM

The power density function was used to detect the detachment between the cam and the follower since it gives six frequencies peaks along with the fundamental frequency. The signal power is the most important factor for signal quality in which the noise can be measured using the signal-to-noise ratio (SNR). A maximum value of the SNR of the follower displacement signal implies no error or noise in the follower displacement. The SNR power of the follower displacement signal is measured using the dB scale since it can be either positive or negative. A negative SNR value means that the signal power is lower than the noise power. The power of the amplitude peak of the fundamental frequency and the other frequencies decreases with
increasing number of samples, which means that separation is about to occur between the cam and the follower (the contact force starts approaching zero). When the frequency peaks start disappearing from the FFT diagram, the motion is quasiperiodic (contact force is approaching zero). Figures 19 and 20 show a comparison of the power spectrum analysis of FFT of the follower displacement for ID = 17 mm and N = 200 rpm, respectively. The SNR value of the follower displacement signal is positive and equal to 17.24 dB.

7 | POINCARÉ MAP

Poincaré maps are used to check for the status of the follower motion based on the value of the contact force against time.32 The contact force in the Poincaré map detects if the motion of the follower is nonperiodic or periodic.32 Figures 21 and 22 show Poincaré maps for ID = 16 and 18 mm, respectively, for different cam speeds N. The algorithm code of the Poincaré map is written using the Matlab program, while the SolidWorks program is used to calculate the contact force against time that is being used in the Poincaré map code. The more the black dots in Poincaré maps, the greater the contact between the cam and the follower since the detachment is noticeable when the follower either slips off or slides off the cam, as shown in Figures 21A,B and 22A,B,D,E. As mentioned previously, the disappearance of black dots from Poincaré maps is indicative of follower–cam detachment within the cycle of cam rotation, as shown in Figures 21E,F and 22F.

8 | RESULTS AND DISCUSSION

Figures 23 and 24 show the contact forces for different cam speeds N and different internal distances of the follower guide ID. The follower–cam detachment occurs when the value of the contact force is minimum as shown in Figures 23C,D and 24B,C. The value of the contact force decreases as the cam speed and the internal distance of the follower guide increase. The system with ID = 18 mm and N = 1500 rpm has a minimum value of contact force.

Figure 25 shows the largest Lyapunov exponent as a function of cam angular velocities. The values of the largest Lyapunov exponent of the contact force increase with increasing cam speeds. The LLE values for all the systems were the same at N = 200–400 rpm, while the LLE value
varied sinusoidally as a function of cam speeds at $N = 400$–1000 rpm. The value of the Lyapunov exponent parameter was determined at its respective equilibrium points. The value of the largest Lyapunov exponent was above zero and positive, indicating follower–cam detachment heights. A positive Lyapunov exponent indicates nonperiodic and chaotic follower motion.

Figures 26 and 27 show the Poincaré maps of the contact force for different internal distances of the follower guide ID and different cam speeds $N$. More black dots indicate more follower–cam contact and fewer detachments. The follower slips off the cam, as shown in Figures 26A,B and 27A,B,F. When the black dots disappear from Poincaré maps, the follower becomes detached from the cam profile within one cycle of the cam rotation, as shown in Figures 26C,E and 27C,D. When the black dots start disappearing from Poincaré maps, the follower experiences nonperiodic and chaotic motion.

**CONCLUSIONS**

In this study, cam–follower detachment heights at different internal distances of the follower guide from inside and at different cam speeds $N$ analyzed and discussed. The values of the largest Lyapunov exponent were above zero and positive, indicating cam–follower detachment. The peak of the follower displacement decreases with increasing coefficients of restitution since the follower potential energy decreases and energy is dissipated due to impact. It can be concluded from the FFT diagram that there is a detachment between the cam and the follower when the frequency peaks start disappearing. When the black dots start disappearing from the Poincaré maps, the follower is detached from the cam profile under the effect of multiple impacts within one cycle of the cam rotation. Moreover, the black dots start disappearing from the Poincaré maps, indicating nonperiodic and chaotic follower motion.

**CONFLICT OF INTEREST**

The author declares no conflict of interest.

**DATA AVAILABILITY STATEMENT**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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