ON NONEXISTENCE AND EXISTENCE OF POSITIVE GLOBAL SOLUTIONS TO HEAT EQUATION WITH A POTENTIAL TERM ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We reinvestigate nonexistence and existence of global positive solutions to heat equation with a potential term on Riemannian manifolds. Especially, we give a very natural sharp condition only in terms of the volume of geodesic ball to obtain nonexistence results.

1. Introduction

In this paper we investigate nonexistence and existence of global positive solutions to the following problem

\[
\begin{aligned}
\partial_t u &= \Delta u - V(x)u + u^p \quad \text{in } M \times (0, \infty), \\
 u(x,0) &= u_0(x) \quad \text{in } M,
\end{aligned}
\]

where \( p > 1 \), and \( M \) is a connected non-compact geodesically complete Riemannian manifold with \( \dim M \geq 3 \), \( \Delta \) is the Laplace-Beltrami operator on \( M \), \( V(x) \) is a smooth function and can be allowed to be negative, and \( u_0 \) is a nonnegative function which is not identically zero.

The main objective of this paper is to illustrate the following questions:

1. What are the influences of potential \( V \) and \( p \) on the nonexistence and existence of global positive solutions to problem (1.1)?
2. Are these influences of \( p \) sharp in some kind of sense for different potential \( V \)?

Before answering these questions, let us firstly recall some history in this area. When \( M = \mathbb{R}^N \), problem (1.1) and its variations have been investigated widely in different respects, see [2, 3, 10, 31, 32], and also a very good survey paper by Levine [21].

Among these literatures, the first celebrated result on problem (1.1) is due to Fujita’s famous paper [9] dealing with the case when \( M = \mathbb{R}^N \) and \( V(x) \equiv 0 \). He proved that

1. If \( 1 < p < 1 + \frac{2}{N} \), and \( u_0 > 0 \), then (1.1) possesses no global positive solution.
2. If \( p > 1 + \frac{2}{N} \), and \( u_0 \) is smaller than a small Gaussian, then (1.1) has global solutions.

Here the number \( 1 + \frac{2}{N} \) is called the Fujita exponent, and usually denoted by \( p^* \). The question of whether \( p^* = 1 + \frac{2}{N} \) belongs to the blow-up case is much more difficult. The case \( p = 1 + \frac{2}{N} \) was decided by Hayakawa [16] for \( N = 1, 2 \) and by Kobayashi, Sirao and Tanaka [18] for general \( N \). One can also see the papers [11, 32] for different methods and further developments.
Zhang investigated problem (1.1) when $V(x)$ has the asymptotic behavior like $\frac{\omega}{1+|x|^p}$ for some $\omega \neq 0$ and $b > 0$. He showed that

**Theorem 1.1.** [35, Zhang] Let $M = \mathbb{R}^N$ with $N \geq 3$.

1. If, for some $b > 2$ and $\omega > 0$, $0 \leq V(x) \leq \frac{\omega}{1+|x|^p}$ holds, then $p^* = 1 + \frac{2}{N}$.
2. If, for some $b \in (0, 2)$ and $\omega > 0$, $V(x) \geq \frac{\omega}{1+|x|^p}$ holds, then $p^* = 1$ and there exists global solutions for all $p > 1$.
3. If, for some $b > 2$ and $\omega < 0$ with $|\omega|$ small enough, $\frac{\omega}{1+|x|^p} \leq V(x) \leq 0$ holds, then $p^* = 1 + \frac{2}{N}$.
4. If, for some $b \in (0, 2)$ and $\omega < 0$, $V(x) \leq \frac{\omega}{1+|x|^p}$ holds, then $p^* = \infty$, which means there exist no global solutions to (1.1) for any $p > 1$.

When $V(x)$ behaves like $\frac{\omega}{|x|^p}$ for some $\omega > 0$ and large $|x|$, Ishige proved that

**Theorem 1.2.** [20, Ishige] Let $M = \mathbb{R}^N$ with $N \geq 3$. Assume that $V(x) \geq 0$. Let $\omega > 0$.

1. If $V(x) \geq \frac{\omega}{|x|^p}$ for large $x$, then for $p > p^*(\omega)$, there exists global positive solution to (1.1);
2. If $V(x) \leq \frac{\omega}{|x|^p}$ for large $x$, then for $1 < p \leq p^*(\omega)$, there exists no global positive solution to (1.1);

where

$$p^*(\omega) = 1 + \frac{2}{N + \alpha(\omega)}, \quad (1.2)$$

and

$$\alpha(\omega) = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\omega}}{2} \quad (1.3)$$

is the larger root of the equation $\alpha(\omega + N-2) = \omega$.

When $V(x)$ behaves like $\frac{\omega}{|x|^p}$, and $\omega$ can be allowed to be negative satisfying $-\frac{(N-2)^2}{4} \leq \omega < 0$, Pinsky obtained that

**Theorem 1.3.** [27, Pinsky] Let $M = \mathbb{R}^N$ with $N \geq 3$.

1. If $V(x) \geq \frac{\omega}{|x|^p}$, then there exists global solution to (1.1) when $p > p^*(\omega)$;
2. If $V(x) \leq \frac{\omega}{|x|^p}$, for large $|x|$, then there are no global solutions to (1.1) when $1 < p \leq p^*(\omega)$.

Now let us transfer our attentions from Euclidean space to manifold. We make a rough assumption on manifold: assume that $M$ is a connected non-compact geodesically complete Riemannian manifold, $d$ is the geodesic distance on $M$, and $\mu_0$ is the Riemannian measure of $M$. Fix a reference point $x_0 \in M$, let $B(x_0, r)$ denote the geodesic ball on $M$ centered at $x_0$ with radii $r > 0$.

The study of nonlinear parabolic equations on manifolds become more and more intriguing, not only because that it has so many applications in geometry and many other areas, but also because usually the approach which is applied for the manifold case is quite different from the Euclidean ones.

In [34], Zhang provided a unified approach to obtain blow-up results for several variations of problem (1.1) when $V(x) = 0$. To cite his result more precisely, let us introduce his assumptions on the manifold

(i). $\mu_0(B(x, r)) \leq Cr^\alpha$, when $r$ is large and for all $x \in M$.

(ii). $\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) \leq C \frac{1}{r}$, where $r = d(x_0, x)$ is smooth. Here $x_0$ is a fixed reference point, and $g^2$ is the volume density of the manifold.
Zhang obtained Fujita exponent of problem (1.1) when $V(x) = 0$.

**Theorem 1.4.** [34, Zhang] Assume conditions (i) and (ii) on manifold are satisfied, and $\alpha \geq 1$. If $1 < p \leq 1 + \frac{2}{\alpha}$, then problem (1.1) possesses no global positive solution to (1.1).

The approach applied by Zhang in [34] is quite powerful, and even very effective to nonlinear homogeneous and inhomogeneous equations, semilinear parabolic equations and porous medium equations with nonlinear source, even to the blow-up problems in exterior domains [36]. Zhang’s approach is by first constructing a suitable integral functional to show that the integral functional in selected fixed domain will blow-up or will be identically equal to zero, then one can derive the blow-up results of nonlinear parabolic equations on manifolds. However, after a very careful examination of Zhang’s paper [34], one can find that the assumptions (i) and (ii) on manifold are essential in his approach, either cannot be relaxed or cannot be dropped, and also, the paper [34] needs to deal with the critical case in a separate way to obtain the blow-up results.

In [23], Mastrolia, Monticelli and Punzo investigated the problem (1.1) with $V(x) \equiv 0$:

$$\begin{cases}
\partial_t u = \Delta u + u^p & \text{in } M \times (0, \infty), \\
u(x, 0) = u_0(x) & \text{in } M,
\end{cases} \quad (1.4)$$

They showed that Zhang’s result can be improved: assumption (ii) can be dropped and assumption (i) can be relaxed to a milder version

$$\mu_0(B(x_0, r)) \leq C r^\alpha \ln \frac{r}{\alpha^2} r, \quad \text{for large enough } r, \quad (1.5)$$

for some reference point $x_0$, the same result still holds. Their technique is to multiply the equation (1.1) by $u^{\alpha} \varphi^p$, and to obtain an integral estimate involving $u$ to show the nonexistence results. This technique is called the nonlinear capacity method, which is systematically studied by Mitidieri and Pohozaev to deal with the elliptic inequality and parabolic differential inequalities. Let us refer to [4, 5, 24, 25] for more details.

Here we point out that their proof relies on a very delicate choice of test function $\varphi$. Moreover, the sharpness of $\frac{\alpha}{2}$ is not shown in their paper [23].

In this paper, the purpose of the paper is threefold: the first one is to provide a sufficient condition for the nonexistence of global solution to problem (1.1) with general $V$; the second one is to attempt to show a unified approach to deal with the parabolic equation with the potential term, moreover, we present a totally different test function $\varphi$ from the one used in [23]; the third one is to show the sharpness of the (general) volume assumption of $\frac{\alpha}{2}$, which has not been shown before.

The idea of using the upper bound of volume of geodesic ball to derive Liouville’s uniqueness type result has already been widely used in literature. It originated from the celebrated work of Cheng and Yau [7]. They proved that if on a geodesically complete Riemannian manifold $M$, for some reference point $x_0 \in M$, the following

$$\mu_0(B(x_0, r)) \leq C r^2,$$

holds for all large enough $r$, then any non-negative superharmonic function on $M$ is identically constant. For other related studies in this area we refer the readers to [11, 13, 23, 30].

Our paper is inspired by the elliptic results in [13, 15] and [29], and parabolic results in [23]. In the paper [14], Grigor’yan and the second author investigated the following differential inequality on $M$

$$\Delta u + u^\sigma \leq 0, \quad (1.6)$$

and proved that if, for some reference point $x_0 \in M$ and $\alpha > 2$, the following

$$\mu_0(B(x_0, r)) \leq C r^\alpha \ln \frac{r}{\alpha^2} r, \quad (1.7)$$
holds for all large enough $r$, then, for any $\sigma \leq \frac{\alpha}{\alpha - 2}$, the only nonnegative solution to (1.6) is identically equal to zero. They also showed the exponents $\alpha$ and $\frac{\alpha - 2}{2}$ in (1.7) are sharp, and can not be relaxed. Otherwise, there exists some model manifold which satisfies (1.7) and admits positive solution to (1.6). The main technique applied in [14] relies on a very delicate choice of test function on manifolds.

Recently in [15], Grigor’yan, the second author and Verbitsky generalized the above results to the integrated form, they obtained the necessary and sufficient condition for the existence of positive solutions in terms of Green function of $\Delta$. Especially, when $M$ has nonnegative Ricci curvature, they showed that problem (1.7) admits a positive $C^2$-solution if and only if

$$\int_{r_0}^{\infty} r^\sigma \frac{1}{\mu_0(B(x_0, r)))} \sigma - 1 \mathrm{d}r < \infty,$$

for some reference point $x_0$ and $r_0 > 0$.

Further in [29], the second author used two different test functions to show that if the volume of geodesic ball satisfies some suitable growth, then the uniqueness result of nonnegative solutions for semi-linear elliptic differential inequalities holds.

Throughout the paper, we require that $V$ admits a smooth positive solution to

$$\Delta h = V h,$$

(1.9)

on $M$. Actually, such a solution $h$ exists widely, for example,

**Lemma 1.5.** [12] Lemmas 10.1 and 10.3] For any smooth non-negative function $\Psi$ on $M$, there exists a smooth positive function $h$ such that

$$\Delta h = \Psi h \quad \text{on } M.$$  

(1.10)

If in addition $\Psi$ is Green bounded, namely,

$$\sup_{x \in M} \int_M G(x, y) \Psi(y) d\mu_0(y) < \infty,$$

(1.11)

then the equation (1.10) has a solution $h \asymp 1$ on $M$. Here, $G(x, y)$ is a finite positive Green function with respect to $\Delta$ on $M$, and the sign $\asymp$ means the ratio of the left-hand and right-hand is bounded from above and below by two positive constants.

We then apply the technique of Doob’s $h$-transform. Consider the weighted manifold $(M, \mu)$, where $\mu$ is a measure on $M$ defined by

$$d\mu := h^2 d\mu_0.$$  

(1.12)

The weighted Laplacian $\tilde{\Delta}$ of $(M, \mu)$ is defined by

$$\tilde{\Delta} := \frac{1}{h^2} \text{div}(h^2 \nabla).$$

In particular, if $h \equiv 1$ then $\tilde{\Delta}$ is the Laplace-Beltrami operator $\Delta$ on $M$.

By using $\Delta h = V h$, for any smooth function $v(x)$, we know

$$\tilde{\Delta} v + V v = (\Delta v + 2 \frac{\nabla h \cdot \nabla v}{h}) + \frac{\Delta h}{h} v = \frac{1}{h} (h \Delta v + 2 \nabla h \cdot \nabla v + \Delta hv) = \frac{\Delta(hv)}{h},$$

Whence

$$\tilde{\Delta} v = \frac{1}{h} \Delta(hv),$$

and

$$\tilde{\Delta} = \frac{1}{h} \circ (\Delta - V) \circ h.$$
Let \( u \) be a smooth positive solution to (1.1) and let \( u = hv \), we know from the above \( v \) is a smooth positive global solution to the following Cauchy problem

\[
\begin{aligned}
\partial_t v &= \Delta v + h^{p-1}v^p & \text{in } M \times (0, \infty), \\
v(x, 0) &= v_0(x) & \text{in } M,
\end{aligned}
\]  

(1.13)

where \( v_0(x) = \frac{u_0}{h}(x) \). Conversely, if \( v \) is a smooth positive solution to problem (1.13), then \( u = hv \) is a solution to (1.1) with \( u_0 = hv_0 \). Hence, the two problems (1.1) and (1.13) are equivalent in the classical sense so that we only need to deal with (1.13) in the following. Actually, problems (1.1) and (1.13) can also be seen equivalent from the weak sense in the below.

Denote by \( W^{1,2}_{loc} (M, d\mu) \) the space of functions \( f \in L^2_{loc} (M, d\mu) \) whose weak gradient \( \nabla f \) is also in \( L^2_{loc} (M, d\mu) \). Denote by \( W^{1,2}_c (M, d\mu) \) the subspace of \( W^{1,2}_{loc} (M, d\mu) \) of functions with compact support. Spaces \( W^{1,2}_{loc} (M \times [0, \infty), d\mu dt), W^{1,2}_c (M \times [0, \infty), d\mu dt) \) are defined similarly.

**Definition 1.6.** \( v \) is called a global weak solution to (1.13) if \( v \) is a nonnegative \( W^{1,2}_{loc} (M \times [0, \infty), d\mu dt) \) function, and for any nonnegative function \( \psi \in W^{1,2}_c (M \times [0, \infty), d\mu dt) \), the following holds

\[
\int_M \psi(x, 0)v_0 d\mu + \int_0^\infty \int_M [v\partial_t \psi - (\nabla v, \nabla \psi) + h^{p-1} v^p \psi] d\mu dt = 0. \tag{1.14}
\]

**Remark 1.7.** From Definition 1.6 we know if \( v \) is a weak solution to (1.1), and \( v_0 \) is nonnegative, we obtain, for any nonnegative function \( \psi \in W^{1,2}_c (M \times [0, \infty), d\mu dt) \)

\[
\int_0^\infty \int_M h^{p-1} v^p \psi d\mu dt \leq \int_0^\infty \int_M (\nabla v, \nabla \psi) d\mu dt - \int_0^\infty \int_M v\partial_t \psi d\mu dt. \tag{1.15}
\]

Before presenting the main results, we introduce some notations. Let us define

\[
P := \frac{2}{p-1}, \quad Q := \frac{1}{p-1}, \tag{1.16}
\]

and a new measure \( \nu \) on \( M \) by

\[
d\nu = h^{-1}d\mu = hd\mu_0. \tag{1.17}
\]

We say that condition (\( H \)) holds: if \( \Delta h = Vh \) admits a smooth positive solution \( h \) and there exist two nonnegative constants \( \delta_1, \delta_2 \), and some reference point \( x_0 \) such that

\[
cr^{-\delta_1} \leq h(x) \leq Cr^{\delta_2}, \quad \text{for large enough } r = d(x, x_0). \tag{H}
\]

Our main result is the following.

**Theorem 1.8.** Assume that condition (\( H \)) is satisfied on \( M \). If the following

\[
\nu(B(x_0, r)) \leq Cr^P \ln^Q r, \tag{1.18}
\]

holds for all large enough \( r \), then problem (1.1) admits no global positive solution. Here \( P \) and \( Q \) are defined as in (1.16).

In particular, when \( V \equiv 0 \), we choose \( h \equiv 1 \), and hence condition (\( H \)) is satisfied. By Theorem 1.8 we have

**Corollary 1.9.** For \( V \equiv 0 \), if, for some reference point \( x_0 \in M \), the following

\[
\mu_0(B(x_0, r)) \leq Cr^P \ln^Q r, \tag{1.19}
\]

holds for all large enough \( r \), then problem (1.1) admits no global positive solution either.
Remark 1.10. Theorem 1.8 and Corollary 1.9 provide us an affirmative answer to the following question: how much could we relax the assumption on the volume growth of geodesic balls to ensure that problem (1.1) admits no global positive solution when the nonlinear term $u^p$ is fixed? In Section 4, we show the sharpness of (1.19), which means that if we relax $P, Q$ a little, there exists a global positive solution to (1.1) on $M$ for small $u_0$.

Our method is to multiply the equation (1.1) by $v^a \varphi^b$ (here $a, b$ are variable parameters). By building suitable integral estimates of $v$ and choosing suitable test function $\varphi$, we can obtain the blow-up results. Actually, the test function $\varphi$ we use here can be considered as a parabolic version used in [29].

Corollary 1.9 can be presented in another equivalent form

Corollary 1.11. For $V \equiv 0$, if, for some reference point $x_0 \in M$ and $\alpha > 0$, the following

$$
\mu_0(B(x_0, r)) \leq C r^\alpha \ln^2 \frac{\alpha}{2r},
$$

(1.20)

holds for all large enough $r$. If $1 < p \leq 1 + \frac{2}{\alpha}$, then problem (1.1) admits no global positive solution.

Remark 1.12. Corollary 1.11 tells us if we know the upper bound of the volume of geodesic ball, then we can determine the range of $p$ to suffice that problem (1.1) admits no global positive solution. Here the volume upper bound condition (1.20) is also sharp, and can not be relaxed either, please see Theorems 1.14 and 1.15.

Corollary 1.11 is a generalization of Zhang’s result, please see Theorem 1.4. Corollary 1.11 was first obtained by Mastrolia, Monticelli, and Punzo in [23].

We then turn to study the existence of global solutions to problem (1.1). For that, we need slightly strengthen our assumptions on $M$. Let $\tilde{P}_t(x, y)$ be the smallest fundamental solution of the heat equation

$$
\partial_t v = \tilde{\Delta} v \quad \text{on } M.
$$

We know $\tilde{P}_t(x, y)$ is called the heat kernel of $\tilde{\Delta}$, and has the following properties

- Symmetry: $\tilde{P}_t(x, y) = \tilde{P}_t(y, x)$, for all $x, y \in M, t > 0$.
- Markovian property: $\tilde{P}_t(x, y) \geq 0$, for all $x, y \in M$ and $t > 0$, and

$$
\int_M \tilde{P}_t(x, y) d\mu(y) \leq 1, \quad \text{for all } x \in M \text{ and } t > 0.
$$

(1.21)

- The semigroup identity: for all $x, y \in M$ and $t, s > 0$,

$$
\tilde{P}_{t+s}(x, y) = \int_M \tilde{P}_t(x, z) \tilde{P}_s(z, y) d\mu(z).
$$

(1.22)

- Approximation of identity: for any $f \in L^2(M, d\mu)$,

$$
\left\| \int_M \tilde{P}_t(x, y) f(y) d\mu(y) - f \right\|_{L^2(M, d\mu)} \rightarrow 0, \quad \text{as } t \rightarrow 0_+.
$$

(1.23)

Let $P^V_t(x, y)$ denote the heat kernel of $-\Delta + V$ on $(M, \mu_0)$. When $V = 0$, we denote by $P_t(x, y) := P^0_t(x, y)$ the heat kernel of $\Delta$. When $M$ has nonnegative Ricci curvature, by famous Li-Yau estimate in [22], we have

$$
P_t(x, y) \asymp \frac{C}{\mu_0(x, \sqrt{t})} \exp \left( - \frac{d^2(x, y)}{ct} \right).
$$

(1.24)
Especially, when $M = \mathbb{R}^N$

$$P_t(x, y) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp \left( -\frac{|x - y|^2}{4t} \right).$$

The questions to obtain the lower bound and upper bound of heat kernels $\tilde{P}_t(x, y)$ and $P_t^V(x, y)$ under different geometric conditions on the underlying manifold have been extensively studied in the past few decades, let us refer to the papers [6, 8, 12, 13, 28].

We say $P_t^V$ satisfies the condition (DUE), if $P_t^V$ has the following upper estimate

$$P_t^V(x, y) \leq \frac{C_1}{\mu_0(x, \sqrt{t})},$$

(DUE)

for some constant $C_1$.

The heat kernels $P_t^V$ and $\tilde{P}_t$ are bridged by the following lemma.

**Lemma 1.13.** [12, Lemma 4.7] The heat kernels $P_t^V$ and $\tilde{P}_t$ have the following relation:

$$P_t^V(x, y) = \tilde{P}_t(x, y) h(x) h(y).$$

(1.25)

If condition (DUE) is satisfied on $M$, and $V$ is Green bounded and nonnegative, by Lemma 1.13 we have

$$\tilde{P}_t(x, y) \leq \frac{C}{\mu_0(x, \sqrt{t})},$$

(1.26)

for some constant $C$.

Our existence result is stated as follows.

**Theorem 1.14.** Assume that $V \geq 0$ is Green bounded and $P_t^V$ satisfies condition (DUE). If, for some $\epsilon > 0$, the following inequality

$$\mu_0(B(x_0, r)) \geq c r^p \ln^{q+\epsilon} r,$$

(1.27)

holds for all large enough $r$, then there exists a global positive solution to (1.1) for some small $u_0$. Here $P, Q$ are defined as in (1.16).

Theorem 1.14 also has an equivalent form.

**Theorem 1.15.** Assume that $V \geq 0$ is Green bounded and $P_t^V$ satisfies condition (DUE). Assume also, for some $\epsilon > 0$, the following inequality

$$\mu_0(B(x_0, r)) \geq c r^q \ln^{q+\epsilon} r,$$

(1.28)

holds for all large enough $r$. If $p > 1 + \frac{2}{\alpha}$, then there exists a global positive solution to (1.1) for some small $u_0$.

The paper is organized as follows: In Section 2 we present some examples to see the applications of our main result. In Section 3 we give the proof of Theorem 1.8. In Section 4 we present the proof of Theorem 1.14.

**Notation.** The letters $C, C', C_0, C_1, c_0, c_1, \ldots$ denote positive constants whose values are unimportant and may vary at different occurrences.

2. Some examples

In this section we present several examples to show the applications of Theorem 1.8 and Corollary 1.11.

First, let us make some preliminary works. Define the Riesz potential on $\mathbb{R}^N$ for $0 < \alpha < N$ by

$$I_\alpha f(x) = c(N, \alpha) \int_{\mathbb{R}^N} \frac{f(y)}{|x - y|^{N-\alpha}} dy,$$

(2.1)
where \( f \in L^1_{\text{loc}}(\mathbb{R}^N) \), and \( \int_{|x| \geq 1} |x|^{N-\alpha} |f(x)| \, dx < \infty \), and
\[
c(N, \alpha) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\pi^{\frac{N}{2} - \alpha} \Gamma\left(\frac{\alpha}{2}\right)}.
\]

Here \( \Gamma(\cdot) \) is the Gamma function.

**Lemma 2.1.** [17 Corollary 2.9] If \( V \leq 0 \), and there exists some constant \( C_2(N) \) such that
\[
I_1[(I_1 V)^2](x) \leq -C_2(N) I_1 V(x),
\]
then there exists a positive solution \( h \) to
\[
\Delta h = V h \quad \text{in} \quad \mathbb{R}^N.
\]
Moreover, if \( I_2(-V) < \infty \), then the solution \( h \) satisfies
\[
\exp(-I_3 V) \leq h \leq \exp(-C_3 I_2 V),
\]
for some constant \( C_3 = C_3(N) > 0 \).

**Proposition 2.2.** [33 Proposition 2.1] Let \( V(x) = \frac{1}{1 + |x|^b} \) for some \( b > 2 \). Then
\[
\sup_{x \in \mathbb{R}^N} I_2 V(x) < \infty.
\]

**Proposition 2.3.** Let \( V(x) = \frac{1}{1 + |x|^b} \) for some \( b > 2 \). Then there exists a constant \( C(N, b) > 0 \) such that for all \( x \in \mathbb{R}^N \),
\[
I_1[(I_1 V)^2](x) \leq C(N, b) I_1 V(x).
\]

**Proof.** We divide the proof into two steps.

**Step 1.** We show the following estimate
\[
I_1 V(x) \asymp \begin{cases} 
1, & |x| \leq 1; \\
|x|^{1-b} + |x|^{1-N} \left(1 + \int_1^{|x|} r^{N-b-1} \, dr\right), & |x| > 1.
\end{cases}
\]
By definition of \( I_1 V \), we have
\[
I_1 V(x) = C(N) \int_{\mathbb{R}^N} \frac{dy}{(1 + |x - y|^b)|y|^{N-1}}.
\]

Firstly, we deal with the case that \( |x| \leq 1 \). The integral of the right hand side of (2.7) can be written as
\[
\int_{\mathbb{R}^N} \frac{dy}{(1 + |x - y|^b)|y|^{N-1}} = \int_{|y| \leq 2|x|} \frac{dy}{(1 + |x - y|^b)|y|^{N-1}} + \int_{|y| > 2|x|} \frac{dy}{(1 + |x - y|^b)|y|^{N-1}}
\]
\[
= J_1 + J_2.
\]

Then for \( |y| \leq 2|x| \), we have \( |y - x| \leq 3|x| \leq 3, \) and \( 1 + |x - y|^b \asymp 1 \). Using polar coordinates, we obtain
\[
J_1 \asymp \int_{|y| \leq 2|x|} \frac{dy}{|y|^{N-1}} \asymp \int_0^{2|x|} \frac{r^{N-1}}{r^N} dr \asymp |x|.
\]
For \( |y| > 2|x| \), we have \( |y|/2 \leq |y - x| \leq 3|y|/2 \), and \( 1 + |x - y|^b \asymp 1 + |y|^b \), then by the fact \( b > 2 \), we obtain
\[
J_2 \asymp \int_{|y| > 2|x|} \frac{dy}{(1 + |y|^b)|y|^{N-1}} = C(N) \int_2^\infty \frac{dr}{1 + r^b} \asymp 1.
\]
By substituting the two estimates to (2.8), we obtain
\[ I_1 V(x) \asymp 1 + |x| \asymp 1, \]
which is the first estimate in (2.10).

Secondly, when \(|x| > 1\), let us write the integral in (2.7) as
\[ \int_{\mathbb{R}^N} \frac{dy}{(1 + |x - y|^{1+b})|y|^{N-1}} = \int_{|y|\leq|x|} + \int_{|x|/2 < |y| \leq 2|x|} + \int_{|y| > 2|x|} =: K_1 + K_2 + K_3. \quad (2.9) \]

Then we estimate \(K_1, K_2, K_3\) respectively.

For \(|y| \leq |x|/2\), we have \(|y - x| \asymp |x|\). Thus
\[ K_1 \asymp \frac{1}{|x|^b} \int_{|y|\leq|x|/2} \frac{dy}{|y|^{N-1}} \asymp |x|^{1-b}. \]

For \(|x|/2 < |y| \leq 2|x|\), we have \(|y| \asymp |x|\). Thus
\[ K_2 \asymp \frac{1}{|x|^{N-1}} \int_{|x|/2 < |y| \leq 2|x|} \frac{dy}{1 + |x - y|^b}. \]

Noting that \(\{y : |y - x| \leq |x|/2\} \subseteq \{y : |x|/2 < |y| \leq 2|x|\} \subseteq \{y : |y - x| \leq 3|x|\}\), we have
\[ \int_{|x|/2 < |y| \leq 2|x|} \frac{dy}{1 + |x - y|^b} \leq \int_{|z| \leq 3|x|} \frac{dz}{1 + |z|^b} \asymp \int_0^3 r^{N-1}dr + \int_3^{3|x|} r^{N-b-1}dr \]
\[ \asymp 1 + \int_1^{|x|} r^{N-b-1}dr, \]
and similarly,
\[ \int_{|x|/2 < |y| \leq 2|x|} \frac{dy}{1 + |x - y|^b} \geq \int_{|z| \leq |x|/2} \frac{dz}{1 + |z|^b} \asymp 1 + \int_1^{|x|} r^{N-b-1}dr. \]

Combining the above estimates, we obtain
\[ K_2 \asymp |x|^{1-N} \left(1 + \int_1^{|x|} r^{N+1-b}dr\right). \]

For \(|y| > 2|x|\), we have \(|y - x| \asymp |y|\). Thus
\[ K_3 \asymp \int_{|y| > 2|x|} \frac{dy}{(1 + |y|^{1+b})|y|^{N-1}} \asymp \int_{|y| > 2|x|} \frac{dy}{|y|^{b+N-1}} \asymp |x|^{1-b}. \]

By substituting the estimates of \(K_1, K_2, K_3\) into (2.9), we obtain the second estimate in (2.10).

**Step 2.** Now we apply (2.6) to show (2.5) with \(V = \frac{1}{1+|x|^r}\). We separate the proof into two cases.

**Case of \(N \geq b\).** We show that there is a constant \(C = C(N,b) > 0\) such that for all \(x \in \mathbb{R}^N\),
\[ (I_1 V(x))^2 \leq CV(x), \quad (2.10) \]
and (2.5) follows immediately by taking \(I_1\) on both sides of (2.10).

When \(|x| \leq 1\), (2.10) is true, since we have
\[ (I_1 V)(x) \asymp V(x) \asymp 1. \]

When \(|x| > 1\), by (2.6), we have
\[ (I_1 V)(x) \asymp \begin{cases} (1 + \log |x|)|x|^{1-b}, & N = b, \\ |x|^{1-b}, & N > b. \end{cases} \]
Noting $b > 2$, we have
\[
(I_V(x))^2 \leq C(1 + \log |x|)^2 |x|^{2-2b} \leq C|x|^{-b} \leq CV(x),
\]
which proves (2.10).

Case of $N < b$. By (2.6), we have
\[
\frac{1}{|x|^{1-N}}, \quad |x| \leq 1,
\]
\[
|I_V(x)| < 1, \quad |x| > 1.
\]

By definition
\[
I_1[(I_1V)^2](x) = C(N) \int_{\mathbb{R}^N} \frac{(I_1V)^2(y)dy}{|x-y|^{N-1}}.
\]

Let us first consider $|x| \leq 1$. Applying (2.11), we obtain
\[
\int_{\mathbb{R}^N} \frac{(I_1V)^2(y)dy}{|x-y|^{N-1}} = \int_{|y| \leq 2} \frac{(I_1V)^2(y)dy}{|x-y|^{N-1}} + \int_{|y| > 2} \frac{(I_1V)^2(y)dy}{|x-y|^{N-1}}
\]
\[
\lesssim \int_{|y| \leq 2} \frac{dy}{|x-y|^{N-1}} + \int_{|y| > 2} \frac{|y|^{2-2N}dy}{|x-y|^{N-1}}
\]
\[
\lesssim \int_{|z| \leq 3} \frac{dz}{|z|^{N-1}} + \int_{|y| > 2} \frac{dy}{|y|^{3N-3}}
\]
\[
\lesssim 1,
\]
which together with $I_1V(x) \asymp 1$, implies for $|x| \leq 1$,
\[
I_1[(I_1V)^2](x) \leq C(I_1V)(x).
\]

Then we consider $|x| > 1$. Rewrite the integral in (2.12) as
\[
\int_{\mathbb{R}^N} \frac{(I_1V)^2(y)dy}{|x-y|^{N-1}} = \left( \int_{|y| \leq 1/2} + \int_{1/2 < |y| \leq |x|/2} + \int_{|x|/2 < |y| \leq 2|x|} + \int_{|y| > 2|x|} \right) \frac{(I_1V)^2(y)dy}{|x-y|^{N-1}}
\]
\[
= L_1 + L_2 + L_3 + L_4.
\]

We estimate $L_i (i = 1, 2, 3, 4)$ as follows.

For $|y| \leq 1/2$, we have by (2.11) that $I_1V(y) \asymp 1$, thus
\[
L_1 \asymp \int_{|y| \leq 1/2} \frac{dy}{|x-y|^{N-1}} \asymp \int_{|y| \leq 1/2} \frac{dy}{|x|^{N-1}} \asymp |x|^{1-N}.
\]

For $1/2 < |y| \leq |x|/2$, we have by (2.11) that $(I_1V(y))^2 \asymp |y|^{2-2N}$, and $|x-y| \asymp |x|$, thus
\[
L_2 \asymp \frac{1}{|x|^{N-1}} \int_{1/2 < |y| \leq |x|/2} |y|^{2-2N}dy \asymp \frac{1-|x|^{2-N}}{|x|^{N-1}}.
\]

For $|x|/2 < |y| \leq 2|x|$, we have $(I_1V(y))^2 \asymp |y|^{2-2N} \asymp |x|^{2-2N}$, thus
\[
L_3 \asymp |x|^{2-2N} \int_{|x|/2 < |y| \leq 2|x|} \frac{dy}{|x-y|^{N-1}} \asymp |x|^{3-2N}.
\]

For $|y| > 2|x|$, we have $(I_1V(y))^2 \asymp |y|^{2-2N}$, and $|x-y| \asymp |y|$, thus
\[
L_4 \asymp \int_{|y| > 2|x|} \frac{|y|^{2-2N}dy}{|y|^{N-1}} \asymp |x|^{3-2N}.
\]

Combining the above estimates, we obtain
\[
I_1[(I_1V)^2](x) \asymp L_1 + L_2 + L_3 + L_4
\]
\[
\asymp |x|^{1-N} + \frac{1-|x|^{2-N}}{|x|^{N-1}} + |x|^{3-2N} + |x|^{3-2N}
\]
Thus applying (2.11), we obtain for \(|x| > 1,\)

\[ I_1[(I_1 V)^2](x) \leq C I_1 V(x). \]

Hence, (2.25) also holds for the case of \(N < b.\) The proof is complete. \(\square\)

**Lemma 2.4.** If \(V(x) = \frac{\omega}{1+|x|^b}\) for some \(\omega < 0\) and \(b > 2,\) then there exists a positive solution \(h\) to

\[ \Delta h = V(x)h. \]

Moreover, \(h \asymp 1.\)

**Proof.** Combining Lemma 2.1 and Proposition 2.3, we obtain there exists a positive solution \(h\) to

\[ \Delta h = Vh, \]

and by Proposition 2.2, we have

\[ \sup_{x \in \mathbb{R}^N} I_2(-V) < \infty. \] (2.13)

Hence, from (2.14), we obtain

\[ h \asymp 1. \] (2.14)

where \(\alpha(\omega)\) is defined as in (1.3).

**Lemma 2.5.** [19, Lemma 2.2] Assume that \(V\) satisfies the following conditions for some \(\omega > 0\) and \(\theta > 0\)

1. \(V = V(|x|) \in C^1(\mathbb{R}^N),\) and \(V(r) \geq 0\) on \([0, \infty),\)
2. \(\sup_{r \geq 1} r^{2+2\theta} |V(r) - \frac{\omega}{r^2}| < \infty,\)
3. \(\sup_{r \geq 1} |r^3 V'(r)| < \infty.\)

Then there exists a unique \(C^2\) solution \(h(r) > 0\) to

\[ \Delta h = hV, \quad \text{in } \mathbb{R}^N, \]

such that

\[ h(r) \asymp r^{\alpha(\omega)}, \quad \text{for large enough } r. \] (2.14)

where \(\alpha(\omega)\) is defined as in (1.3).

**Example 2.6.** Let \(V(x) = 0,\) and \(M = \mathbb{R}^k_g \times S^l\) be endowed with product metric. Here \(\mathbb{R}^k_g = (\mathbb{R}^k, g)\) is a model manifold with induced metric \(g = dr^2 + \psi(r)^2d\theta^2,\) where \((r, \theta)\) is the polar coordinates in \(\mathbb{R}^k,\) and \(\psi(r)\) is a smooth, positive function on \((0, \infty)\) such that

\[ \psi(r) = \begin{cases} r, & \text{for small } r, \\ r^{\alpha-1} \ln^{\frac{\alpha}{2}} r, & \text{for large } r. \end{cases} \]

If \(V(x) = 0,\) we could choose \(h = 1,\) and hence, in (1.17) \(d\nu = d\mu = d\mu_0.\) Then the volume of the ball \(B_r := B_r(0)\) in \(\mathbb{R}^k_g\) can be determined by

\[ \mu_0(B_r) = \int_0^r S(\tau)d\tau, \]

where \(S\) is the surface area defined by

\[ S(r) = \begin{cases} r^{k-1}, & \text{for small } r, \\ r^{\alpha-1} \ln^{\frac{\alpha}{2}} r, & \text{for large } r. \end{cases} \]
Hence, we obtain
\[ \mu_0(B_r) \leq C r^\alpha \ln^2 r, \quad \text{for large enough } r. \]

If follows that the geodesic ball \( B(0,r) \) in \( M \) satisfies
\[ \mu_0(B(0,r)) \leq C r^\alpha \ln^2 r, \quad \text{for large enough } r. \]

Applying Corollary 1.11, we derive that when \( p \leq 1 + \frac{2}{\alpha} \), then (1.1) on \( \mathbb{R}^k \times S^l \) admits no global positive solution. Especially, when \( \mathbb{R}^k = \mathbb{R}^k \), we know that the critical exponent for \( \mathbb{R}^k \times S^l \) is \( 1 + \frac{2}{\alpha} \).

**Example 2.7.** When \( M = \mathbb{R}^N \), we consider the following classes of \( V(x) \).

1. If \( 0 \leq V(x) \leq \frac{\omega}{1 + |x|^b} \) for some \( b > 2 \) and \( \omega > 0 \), we know by Proposition 2.2

\[ \sup_{x \in \mathbb{R}^N} I_2 V < \infty, \]

which means that \( V(x) \) is Green bounded. By Lemma 1.5, we know that
\[ \Delta h = V h, \]

admits a solution \( h \asymp 1 \). Noting that
\[ \nu(B(0,r)) = \int_{B(0,r)} h dx \asymp r^N. \]

By Theorem 1.8, we know if
\[ \nu(B(0,r)) \leq C r^p \ln^Q r, \]

or more precisely, when
\[ p \leq 1 + \frac{2}{N}, \]

there exists no global solution to (1.1). This result also covers the result (1) of Theorem 1.1.

(2) When \( 0 \leq V(x) \leq \frac{\omega}{|x|^2} \), for large \( |x| \), by employing Comparison principle, we can replace \( V(x) \) by \( \frac{\omega}{|x|^2}(1 + |x|^{-\theta}) \) for large \( |x| \) still denoted by \( V(x) \). If we can show that (1.1) admits no global positive solution with \( V(x) = \frac{\omega}{|x|^2}(1 + |x|^{-\theta}) \) for large \( |x| \), then the original problems admits no global positive solution by Comparison principle.

Applying Lemma 2.5, we know the following problem with \( V(x) = \frac{\omega}{|x|^2}(1 + |x|^{-\theta}) \) for large \( |x| \)
\[ \Delta h = V h, \quad \text{in } \mathbb{R}^N, \]

admits a unique solution \( h > 0 \) such that
\[ h(x) \asymp |x|^\alpha(\omega), \quad \text{for large } |x|. \]

where \( \alpha(\omega) \) is defined as in (1.3).

For large \( r \), we obtain that
\[ \nu(B(0,r)) = \int_{B(0,r)} h dx \asymp r^{N + \alpha(\omega)}, \quad \text{for large } r. \]

Applying Theorem 1.8, we obtain that when \( p \leq 1 + \frac{2}{N + \alpha(\omega)} \), there exists no global positive solution to (1.1).

In this case, the result is also in accordance with the (2) in Theorem 1.2.
(3) When \( \frac{\omega}{1+|x|^2} \leq V(x) \leq 0 \) for some \( \omega < 0 \), and \( b > 2 \). By Lemma 2.3, we obtain that \( h \approx 1 \). Applying Theorem 1.8 we obtain that when \( p \leq 1 + \frac{2}{N} \), there exists no global positive solution to \((1.1)\).

In this case, the result is also in accordance with the (3) in Theorem 1.1. Actually, we remove the restriction that \( \omega \) is small enough, and we improve the result obtained in Theorem 1.1.

(4) When \( V = \frac{\alpha(\omega)N + \omega |x|}{(1+|x|^2)^2} \) for \( \omega \in \left[-\frac{(N-2)^2}{4}, 0\right) \), we know \( V(x) \leq \omega (1 + |x|^2) \). By Theorem 1.8, we know if \( p \leq 1 + \frac{2}{N} \), there exists no global positive solution to \((1.1)\).

Remark 2.8. Here we can not cover the case of \( V(x) \leq \frac{\omega}{1+|x|^2} \leq 0 \), the difficulty is that we do not know the asymptotic behavior of \( h \) when \( |x| \to \infty \). However, we conjecture that when \( V(x) \) behaves like \( \frac{\omega}{1+|x|^2} \), \( \Delta h = Vh \) admits a solution \( h(x) \approx |x|^{-a(\omega)} \) for large \( |x| \).

Example 2.9. Assume that \( M \) satisfies
\[
\mu_0(B(x_0, r)) \leq Cr^\alpha, \quad \text{for large enough } r, \tag{2.19}
\]
and
\[
G(x, y) \asymp d(x, y)^{2-\alpha}, \quad \text{for large enough } d(x, y).
\]
Let \( V(x) \asymp \frac{\omega}{1+|x|^2} \) for \( b > 2 \), and \( \omega > 0 \), we know
\[
\sup_x \int_M G(x, y)V(y)dy < \infty, \tag{2.20}
\]
hence \( V(x) \) is Green bounded. Hence by Lemma 1.5, we know there exists a function \( h(x) \approx 1 \) satisfying \( \Delta h = Vh \). Applying Theorem 1.8, we know if
\[
p \leq 1 + \frac{2}{\alpha} \tag{2.21}
\]
then there is no positive global solution to \((1.1)\).

3. Nonexistence of global positive solution

Proof of Theorem 1.8. Let \( \varphi \in W^{1,2}_c(M \times [0, +\infty), d\mu dt) \) be a function satisfying \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \) in a neighborhood of \( D_R := B(x_0, R) \times [0, R^2] \). Define
\[
\psi(x, t) = v(x, t)^{-a} \varphi(x, t)^b, \tag{3.1}
\]
where \( a \) will take arbitrarily small positive value near zero, and \( b \) will be chosen to be a large enough fixed constant.

Without loss of generality, let us assume that \( 1/v \) is locally bounded, otherwise we can replace \( v \) by \( v + \varepsilon \) for \( \varepsilon > 0 \), at last we can let \( \varepsilon \to 0_+ \). From (3.1), we know that \( \psi \) has compact support and is bounded. Note that
\[
\nabla \psi = bv^{-a} \varphi^{b-1} \nabla \varphi - av^{-a-1} \varphi^b \nabla v, \tag{3.2}
\]
and
\[
\partial_t \psi = bv^{-a} \varphi^{b-1} \partial_t \varphi - av^{-a-1} \varphi^b \partial_t v. \tag{3.3}
\]
Thus
\[
\psi \in W^{1,2}_c(M \times [0, +\infty), d\mu dt).
\]
Substituting (3.2) into (1.15), we obtain
\[
\int_0^\infty \int_M h^{p-1}v^p \psi^p d\mu dt + a \int_0^\infty \int_M v^{-a-1} |\nabla v|^2 \varphi^b d\mu dt \\
\leq \int_0^\infty \int_M (\nabla v, bv^{-a} \varphi^{-1} \nabla \varphi) d\mu dt - \int_0^\infty \int_M \varphi \partial_t \psi d\mu dt. \tag{3.4}
\]
Applying the Young’s inequality to the first term in the right-hand side of (3.4), we obtain
\[
\int_0^\infty \int_M (\nabla v, bv^{-a} \varphi^{-1} \nabla \varphi) d\mu dt \\
= \int_0^\infty \int_M \left( \frac{1}{2} v^{-\frac{a}{2}} \varphi^2 \nabla v, ba^{-\frac{1}{2}} v^{\frac{1-a}{2}} \varphi^{-1} \nabla \varphi \right) d\mu dt \\
\leq \frac{a}{2} \int_0^\infty \int_M v^{-a-1} |\nabla v|^2 \varphi^b d\mu dt + \frac{b^2}{2a} \int_0^\infty \int_M v^{1-a} \varphi^{-b-2} |\nabla \varphi|^2 d\mu dt.
\]
Substituting the above into (3.4), we obtain
\[
\int_0^\infty \int_M h^{p-1}v^p \psi^p d\mu dt + \frac{a}{2} \int_0^\infty \int_M v^{-a-1} |\nabla v|^2 \varphi^b d\mu dt \\
\leq \frac{b^2}{2a} \int_0^\infty \int_M v^{1-a} \varphi^{-b-2} |\nabla \varphi|^2 d\mu dt - \int_0^\infty \int_M \varphi \partial_t \psi d\mu dt. \tag{3.5}
\]
Combining (3.5) with (3.3), we obtain
\[
\int_0^\infty \int_M v^{-a-1} |\nabla v|^2 \varphi^b d\mu dt \leq \frac{b^2}{a^2} \int_0^\infty \int_M v^{1-a} \varphi^{-b-2} |\nabla \varphi|^2 d\mu dt \\
- \frac{2}{a} \int_0^\infty \int_M v |bv^{-a} \varphi^{-1} \partial_t \varphi - av^{-a-1} \varphi^{-1} \partial_t v| d\mu dt,
\]
which is
\[
\int_0^\infty \int_M v^{-a-1} |\nabla v|^2 \varphi^b d\mu dt \leq \frac{b^2}{a^2} \int_0^\infty \int_M v^{1-a} \varphi^{-b-2} |\nabla \varphi|^2 d\mu dt \\
- \frac{2}{a} \int_0^\infty \int_M (bv^{-a} \varphi^{-1} \partial_t \varphi - av^{-a} \varphi^{-1} \partial_t v) d\mu dt. \tag{3.6}
\]
Let us use another feasible test function \(\psi(x,t) = \varphi(x,t)^b\). Substituting \(\psi = \varphi^b\) into (1.15), we obtain
\[
\int_0^\infty \int_M h^{p-1}v^p \varphi^b d\mu dt \leq \int_0^\infty \int_M (\nabla v, b \varphi^{-1} \nabla \varphi) d\mu dt - b \int_0^\infty \int_M \varphi^{-b-1} \partial_t \varphi d\mu dt. \tag{3.7}
\]
Let us estimate the first term in the right-hand side of (3.7) via the Young’s inequality
\[
\int_0^\infty \int_M (\nabla v, b \varphi^{-1} \nabla \varphi) d\mu dt \\
= \int_0^\infty \int_M \left( \frac{1}{2} v^{-\frac{a}{2}} \varphi^2 \nabla v, ba^{-\frac{1}{2}} v^{\frac{1+a}{2}} \varphi^{-1} \nabla \varphi \right) d\mu dt \\
\leq \frac{a}{2} \int_0^\infty \int_M v^{-a-1} |\nabla v|^2 \varphi^b d\mu dt + \frac{b^2}{2a} \int_0^\infty \int_M v^{a+1} \varphi^{-b-2} |\nabla \varphi|^2 d\mu dt.
\]
Combining the above with (3.7), we obtain
\[
\int_0^\infty \int_M h^{p-1}v^p \varphi^b d\mu dt \leq \frac{a}{2} \int_0^\infty \int_M v^{-a-1} |\nabla v|^2 \varphi^b d\mu dt \\
+ \frac{b^2}{2a} \int_0^\infty \int_M v^{a+1} \varphi^{-b-2} |\nabla \varphi|^2 d\mu dt - b \int_0^\infty \int_M \varphi^{-b-1} \partial_t \varphi d\mu dt.
\]
Substituting (3.6) into the above, we obtain

\[
\int_0^\infty \int_M h^{p-1} v^p \varphi^b \, d\mu \, dt \leq \frac{b^2}{2a} \int_0^\infty \int_M v^{1-a} \varphi^{b-2} \left| \nabla \varphi \right|^2 \, d\mu \, dt \\
- \int_0^\infty \int_M (bv^{1-a} \varphi^{b-1} \delta \varphi - av^{-a} \varphi^b \delta v) \, d\mu \, dt \\
+ \frac{b^2}{2a} \int_0^\infty \int_M v^{a+1} \varphi^{b-2} \left| \nabla \varphi \right|^2 \, d\mu \, dt \\
- b \int_0^\infty \int_M v \varphi^{b-1} \delta \varphi \, d\mu \, dt.
\]

Then (3.8) can be written as follows

\[
I \leq K_1 + K_2 + K_3 + K_4.
\]

Before estimating (3.9), let us introduce some notations

\[
J(\theta_1, \theta_2) := \int_0^\infty \int_M h^{\theta_1} \left| \nabla \varphi \right|^2 \, dv \, dt, \quad L(\theta_1, \theta_2) := \int_0^\infty \int_M h^{\theta_1} \left| \delta \varphi \right|^2 \, dv \, dt.
\]

Noting \( \varphi \equiv 1 \) in a neighborhood of \( D_R \), and applying the Hölder’s inequality, we obtain

\[
K_1 = \frac{b^2}{2a} \int_{D_R} \int_M v^{1-a} \varphi^{b-2} \left| \nabla \varphi \right|^2 \, d\mu \, dt \\
= \frac{b^2}{2a} \int_{D_R} \int_M \left( \frac{(p-1)(1-a)}{p} v^{1-a} \varphi^{b-2} \right) \left( h \frac{(p-1)(1-a)}{p} \varphi^{b-2} \right) \left| \nabla \varphi \right|^2 \, d\mu \, dt \\
\leq \frac{b^2}{2a} \left( \int_{D_R} \int_M h^{p-1} v^{p} \varphi^{b} \, d\mu \, dt \right)^{\frac{1}{p}} \\
\times \left( \int_0^\infty \int_M h^{\frac{(p-1)(1-a)}{p}} \varphi^{b} \left| \nabla \varphi \right|^2 \, dv \, dt \right)^{\frac{2p}{p+1}} \\
\leq \frac{b^2}{2a} \left( \int_{D_R} \int_M h^{p-1} v^{p} \varphi^{b} \, d\mu \, dt \right)^{\frac{1}{p}} \\
\times \left( \int_0^\infty \int_M h^{\frac{ap}{p+a-1}} \varphi^{b} \left| \nabla \varphi \right|^2 \, dv \, dt \right)^{\frac{2p}{p+1}}
\]

Here we have used that \( D_R = M \times [0, \infty) \setminus D_R \), and \( dv = h^{-1} \, d\mu \).
Applying integration by parts to $K$, we similarly obtain

$$K_1 \leq \frac{C}{a} \left( \int_{D_R} h^{p-1} v^p \varphi^b dµdt \right)^{\frac{1-a}{p}} J \left( \frac{a p}{p+a-1}, \frac{2 p}{p+a-1} \right)^{\frac{p+a-1}{p}}. \quad (3.11)$$

Applying integration by parts to $K_2$, we obtain

$$K_2 = - \int_0^\infty \int_M (b v^{1-a} \varphi^{b-1} \partial_t \varphi - a v^{-a} \varphi^b \partial_t v) dµdt$$

$$= - \int_0^\infty \int_M v^{1-a} \partial_t \varphi \phi^b - \frac{a}{1-a} \partial_t (v^{1-a}) \varphi^b dµdt$$

$$\leq - \frac{1}{1-a} \int_0^\infty \int_M v^{1-a} \partial_t \varphi dµdt$$

$$= - \frac{b}{1-a} \int_{D_R} v^{1-a} \varphi^{b-1} \partial_t \varphi dµdt.$$ 

Using Hölder’s inequality again and by similar arguments as in $K_1$, we obtain

$$K_2 \leq \frac{b}{1-a} \left( \int_{D_R} h^{p-1} v^p \varphi^b dµdt \right)^{\frac{1-a}{p}}$$

$$\times \left( \int_0^\infty \int_M h^{(\frac{p-1)(1-a)}{p+a-1}} \varphi^{[b-1-a]} \partial_t \varphi dµ dt \right)^{\frac{p+a-1}{p}}$$

$$\leq \frac{C}{1-a} \left( \int_{D_R} h^{p-1} v^p \varphi^b dµdt \right)^{\frac{1-a}{p}} L \left( \frac{a p}{p+a-1}, \frac{p}{p+a-1} \right)^{\frac{p+a-1}{p}}. \quad (3.12)$$

Similarly, we obtain

$$K_3 = \frac{b^2}{2a} \int_{D_R} v^{a+1} \varphi^{b-2} |\nabla \varphi|^2 dµdt$$

$$= \frac{b^2}{2a} \left( \int_{D_R} h^{(\frac{p-1)(a+1)}{p}} v^{a+1} \varphi^{\frac{a+1}{p}} \right) \left( \int_{D_R} h^{(\frac{p-1)(a+1)}{p}} \varphi^{b-2-a+1 \frac{a+1}{p}} |\nabla \varphi|^2 dµdt \right)$$

$$\leq \frac{b^2}{2a} \left( \int_{D_R} h^{p-1} v^p \varphi^b dµdt \right)^{\frac{a+1}{p}}$$

$$\times \left( \int_0^\infty \int_M h^{-(\frac{p-1)(a+1)}{p-a-1}} \varphi^{[b-2-a+1 \frac{a+1}{p}] \frac{p}{p-a-1}} |\nabla \varphi|^2 \frac{2p}{p-a-1} dµ dt \right)^{\frac{p-a-1}{p}}$$

$$\leq \frac{C}{a} \left( \int_{D_R} h^{p-1} v^p \varphi^b dµdt \right)^{\frac{a+1}{p}} J \left( \frac{a p}{p-a-1}, \frac{2 p}{p-a-1} \right)^{\frac{p-a-1}{p}}, \quad (3.13)$$
and
\[
K_4 = -b \int \int_{D_R} v\varphi^{b-1} \partial_t \varphi \, d\mu dt
\]
\[
\leq b \int \int_{D_R} (h^{\frac{b-1}{p}} \varphi^\frac{1}{p})(h^{\frac{b-1}{p}} \varphi^{b-1} \partial_t \varphi) \, d\mu dt
\]
\[
\leq b \left( \int \int_{D_R} h^{p-1} v^p \varphi^\frac{b}{p} \, d\mu dt \right) \left( \frac{1}{p} \int_0^\infty \int_M h^{\frac{b-1}{p} - \frac{1}{p} - \frac{1}{p}} \partial_t \varphi \left| \varphi^{\frac{1}{p}} \right| \, d\mu dt \right)^{p-1}
\]
\[
\leq C \left( \int \int_{D_R} h^{p-1} v^p \varphi^\frac{b}{p} \, d\mu dt \right) \frac{1}{p} L \left( 0, \frac{1}{p-1} \right)^{\frac{p-1}{p}}.
\]
(3.14)

Substituting (3.11), (3.12), (3.13) and (3.14) into (3.8), we obtain
\[
I \leq C \left( \int \int_{D_R} h^{p-1} v^p \varphi^\frac{b}{p} \, d\mu dt \right) \frac{1}{p} J \left( \frac{ap}{p + a - 1}, \frac{2p}{p + a - 1} \right)^{\frac{p-1}{p}}
\]
\[
+ \frac{C}{1 - a} \left( \int \int_{D_R} h^{p-1} v^p \varphi^\frac{b}{p} \, d\mu dt \right) \frac{1}{p} L \left( \frac{ap}{p + a - 1}, \frac{p}{p + a - 1} \right)^{\frac{p-1}{p}}
\]
\[
+ \frac{C}{a} \left( \int \int_{D_R} h^{p-1} v^p \varphi^\frac{b}{p} \, d\mu dt \right) \frac{1}{p} J \left( -\frac{ap}{p - a - 1}, \frac{2p}{p - a - 1} \right)^{\frac{p-1}{p}}
\]
\[
+ C \left( \int \int_{D_R} h^{p-1} v^p \varphi^\frac{b}{p} \, d\mu dt \right) \frac{1}{p} L \left( 0, \frac{p}{p - 1} \right)^{\frac{p-1}{p}}.
\]
(3.15)

which is
\[
I \leq \frac{C}{a} \frac{1}{p} J \left( \frac{ap}{p + a - 1}, \frac{2p}{p + a - 1} \right)^{\frac{p-1+a}{p}} + \frac{C}{1 - a} \frac{1}{p} L \left( \frac{ap}{p + a - 1}, \frac{p}{p + a - 1} \right)^{\frac{p-1+a}{p}}
\]
\[
+ \frac{C}{a} \frac{1}{p} J \left( -\frac{ap}{p - a - 1}, \frac{2p}{p - a - 1} \right)^{\frac{p-1-a}{p}} + C \frac{1}{p} L \left( 0, \frac{p}{p - 1} \right)^{\frac{p-1}{p}}.
\]
(3.16)

We claim that there exists a constant $C_0 > 0$ such that
\[
\int_0^\infty \int_M h^{p-1} v^p d\mu dt \leq C_0 < \infty.
\]
(3.17)

We divide the proof into two cases:

**Case 1:** if
\[
\int_0^\infty \int_M h^{p-1} v^p d\mu dt \leq 1,
\]
then we let $C_0 = 1$, and it follow that (3.17) is true.

**Case 2:** If Case 1 is not satisfied, then we obtain
\[
\int_0^\infty \int_M h^{p-1} v^p d\mu dt > 1,
\]
Hence, we can find a large enough $R$ such that
\[
\int \int_{D_R} h^{p-1} v^p d\mu dt > 1.
\]
(3.18)
Recall \( p > 1 \), and choose a positive constant \( \beta \) satisfying
\[
\frac{1 + \beta}{p} < 1.
\]
Let \( a \) satisfy \( 0 < a \ll \min\{1, \beta\} \). Combining (3.15) and (3.18), we obtain
\[
I \leq C I^{1 + \frac{\beta}{p}} \left[ \frac{1}{a} J \left( \frac{ap}{p + a - 1}, \frac{2p}{p + a - 1} \right)^{\frac{p + \alpha - 1}{p}} + L \left( \frac{ap}{p + a - 1}, \frac{p}{p + a - 1} \right)^{\frac{p + \alpha - 1}{p}} \right] + \frac{1}{a} J \left( -\frac{ap}{p - a - 1}, \frac{2p}{p - a - 1} \right)^{\frac{p - \alpha - 1}{p}} + L \left( 0, \frac{p}{p - 1} \right)^{\frac{p - 1}{p}}.
\]
It follows that
\[
I^{1 + \frac{\beta}{p}} \leq C \left[ \frac{1}{a} J \left( \frac{ap}{p + a - 1}, \frac{2p}{p + a - 1} \right)^{\frac{p + \alpha - 1}{p}} + L \left( \frac{ap}{p + a - 1}, \frac{p}{p + a - 1} \right)^{\frac{p + \alpha - 1}{p}} \right] + \frac{1}{a} J \left( -\frac{ap}{p - a - 1}, \frac{2p}{p - a - 1} \right)^{\frac{p - \alpha - 1}{p}} + L \left( 0, \frac{p}{p - 1} \right)^{\frac{p - 1}{p}}.
\]
(3.19)
Let \( g \in C^\infty[0, \infty) \) be a nonnegative function satisfying
\[
g(t) = 1 \text{ on } [0, 1]; \quad g(t) = 0 \text{ on } [2, \infty); \quad |g'| \leq C_1 < \infty.
\]
Let \{\( \eta_k \)\}_{k \in \mathbb{N}}, \{\( \gamma_k \)\}_{k \in \mathbb{N}} \in C^\infty[0, \infty) \) be two sequences of functions defined respectively by
\[
\eta_k(t) = g \left( \frac{t}{2^{2k}} \right),
\]
and
\[
\gamma_k(x) = g \left( \frac{r(x)}{2^k} \right),
\]
where \( r(x) = d(x_0, x) \).

From (3.20) and (3.21), we have
\[
|\partial_t \eta_k| \leq \frac{C}{2^{2k}}, \quad t \in [2^{2k}, 2^{2k+1}],
\]
(3.22)
\[
= 0, \quad \text{otherwise},
\]
and
\[
|\nabla \gamma_k| \leq \frac{C}{2^k}, \quad x \in B(x_0, 2^{k+1}) \setminus B(x_0, 2^k),
\]
(3.23)
\[
= 0, \quad \text{otherwise}.
\]
Let us define a sequence of functions \( \{\varphi_i(x, t)\}_{i \in \mathbb{N}} \) by
\[
\varphi_i(x, t) = \frac{1}{i} \sum_{k=i+1}^{2i} \eta_k(t) \gamma_k(x),
\]
(3.24)
It follows that \( \varphi_i(x, t) = 1 \) when \( (x, t) \in B(x_0, 2^i) \times [0, 2^i] \). Moreover, for distinct \( k \), noting that \( \text{supp}(\partial_t \eta_k) \) and \( \text{supp}(\nabla \gamma_k) \) are disjoint respectively, we obtain for any \( \theta > 0 \)
\[
|\partial_t \varphi_i|^\theta = \frac{1}{i^\theta} \sum_{k=i+1}^{2i} |\gamma_k \partial_t (\eta_k)|^\theta.
\]
(3.25)
\[
|\nabla \varphi_i|^\theta = \frac{1}{i^\theta} \sum_{k=i+1}^{2i} |\eta_k \nabla \gamma_k|^\theta.
\]
(3.26)
Hence
\[ \varphi_i \in W^{1,2}_c(M \times [0, +\infty)). \]

Let
\[ a = \frac{1}{i}. \] (3.27)

Substituting the above with \( \varphi = \varphi_i \) into (3.19), we obtain
\[ P^{1-\frac{1}{p}-i} \leq C \left[ iJ \left( \frac{p}{p + 1/i - 1}, \frac{2p}{p + 1/i - 1} \right) \frac{p + 1/i - 1}{p} + L \left( \frac{p}{p + 1/i - 1}, \frac{p}{p + 1/i - 1} \right) \frac{p + 1/i - 1}{p} \right. \]
\[ + iJ \left( \frac{p}{p - 1/i - 1}, \frac{2p}{p - 1/i - 1} \right) \frac{p + 1/i - 1}{p} + L \left( 0, \frac{p}{p - 1} \right) \frac{p + 1/i - 1}{p} \right]. \] (3.28)

Substituting (3.24) into (3.10), and combining (3.23) and (3.26), noting \( \eta_k \leq 1 \), we obtain
\[ iJ \left( \frac{p}{p - 1/i - 1}, \frac{2p}{p - 1/i - 1} \right) \frac{p + 1/i - 1}{p} \]
\[ = i \left( \int_0^\infty \int_M h(p, 1/i - 1, p - 1/i - 1, p - 1/i - 1) \sum_{k=1}^{2i} \eta_k \nabla \gamma_k \frac{2p}{p + 1/i - 1} \, dv \, dt \right) \]
\[ = i^{-1} \left( \sum_{k=1}^{2i} \int_0^\infty \eta_k \frac{2p}{p + 1/i - 1} \int_{B(x_0, 2k+1) \setminus B(x_0, 2k)} h(p, 1/i - 1, p - 1/i - 1) \nabla \gamma_k \frac{2p}{p + 1/i - 1} \, dv \, dt \right) \]
\[ \leq C_i^{-1} \left( \sum_{k=1}^{2i} 2^{2k+1} (2k+1)^2 \frac{kp/i}{p + 1/i - 1} \left( \nu(B(x_0, 2k+1)) \right)^{p + 1/i - 1} \right), \]
where we have used the condition \((H)\).

Applying volume condition \( \nu(B(x_0, r)) \leq C r^P \ln^Q r \) with \( P = \frac{2}{p - 1} \) and \( Q = \frac{1}{p - 1} \), we obtain
\[ iJ \left( \frac{p}{p + 1/i - 1}, \frac{2p}{p + 1/i - 1} \right) \frac{p + 1/i - 1}{p} \]
\[ \leq C_i^{-1} \left( \sum_{k=1}^{2i} 2^{k+1} \frac{2kp/i}{p + 1/i - 1} \frac{kp/2p}{p + 1/i - 1} \right) \frac{p + 1/i - 1}{p} \]
\[ \leq C_i^{-1+Q} \frac{p + 1/i - 1}{p} \left( \sum_{k=1}^{2i} 2^{k+1} \frac{2kp/i}{p + 1/i - 1} \frac{kp/2p}{p + 1/i - 1} + P \right) \]
\[ \leq C_i^{-1+Q} \frac{p + 1/i - 1}{p} + \frac{p + 1/i - 1}{p}. \] (3.29)

Here we have used that
\[ \left( \sum_{k=1}^{2i} 2^{k+1} \frac{2kp/i}{p + 1/i - 1} \frac{kp/2p}{p + 1/i - 1} + P \right) \frac{p + 1/i - 1}{p} \]
\[ \leq 2 \delta \frac{2p}{p - 1/i - 1} \left( \sum_{k=1}^{2i} 2^{k+1} \frac{2kp/i}{p + 1/i - 1} \frac{kp/2p}{p + 1/i - 1} \right) \]
\[ \leq C_i \frac{p + 1/i - 1}{p}. \] (3.30)
Noting that
\[
\limsup_{i \to \infty} i^{-1+Q}\frac{i^{p+1/i-1} + p^{+1/i-1}}{p^{1/i-1}} = \limsup_{i \to \infty} i^{1/p} = 1,
\]
we obtain
\[
iJ \left( \frac{p/i}{p+1/i-1}, \frac{2p}{p+1/i-1} \right) \frac{p^{+1/i-1}}{p} \leq C.
\]
Substituting (3.24) into (3.10), applying (3.22) and (3.25), noting \( \gamma_k \leq 1 \), we obtain
\[
L \left( \frac{p/i}{p+1/i-1}, \frac{p}{p+1/i-1} \right) \frac{p^{+1/i-1}}{p} \leq C \left( i^{-p/i+1+Q} \binom{p+1/i-1}{p} \right).
\]
where the term \( 2^{\frac{k\rho \rho}{1/i-1}} \) has been absorbed into constant \( C' \).
Using (3.30) again, we have
\[
\left( L \left( \frac{p/i}{p+1/i-1}, \frac{p}{p+1/i-1} \right) \right) \frac{p^{+1/i-1}}{p} \leq C \left( i^{-p/i+1+Q+1} \binom{p+1/i-1}{p} \right).
\]
Since
\[
\limsup_{i \to \infty} \left( i^{-p/i+1+Q+1} \binom{p+1/i-1}{p} \right) = 1,
\]
we obtain
\[
\left( L \left( \frac{p/i}{p+1/i-1}, \frac{p}{p+1/i-1} \right) \right) \frac{p^{+1/i-1}}{p} \leq C.
\]
Similarly,
\[
iJ \left( \frac{-p/i}{p-1/i-1}, \frac{-2p}{p-1/i-1} \right) \frac{p^{-1/i-1}}{p} \leq C^{-1} \left( \sum_{k=i+1}^{2i} \binom{2k+1}{k} \frac{p^p}{p-1/i-1} \nu(B(x_0, 2^k)) \right) \frac{p^{-1/i-1}}{p}.
\]
\[
\leq C' i^{-1+Q \frac{p-1}{p}} \left( \sum_{k=i+1}^{2i} 2^k \left( 2- \frac{2p}{p-1} + P \right) \right)^{\frac{p-1}{p}} \]
\[
\leq C i^{-1+Q \frac{p-1}{p}} \frac{1}{i^{p-1}} \]
\[
\leq C i^{-1+Q \frac{p-1}{p}} < \infty.
\]

(3.33)

and

\[
\left( \begin{array}{c}
\text{L} \left( 0, \frac{p}{p-1} \right) \\
\end{array} \right)^{\frac{p-1}{p}} = \left( \begin{array}{c}
\sum_{k=i+1}^{2i} 2^k \left( 2- \frac{2p}{p-1} + 2k \nu(B(x_0, 2^k+1)) \right) \end{array} \right)^{\frac{p-1}{p}} \]
\[
\leq C \left( \begin{array}{c}
\sum_{k=i+1}^{2i} 2^k \left( 2- \frac{2p}{p-1} + 2k \nu(B(x_0, 2^k+1)) \right) \end{array} \right)^{\frac{p+1}{p}} \]
\[
= \left( \begin{array}{c}
\sum_{k=i+1}^{2i} 2^k \left( 2- \frac{2p}{p-1} + 2k \nu(B(x_0, 2^k+1)) \right) \end{array} \right)^{\frac{p+1}{p}} \]
\[
= C < \infty.
\]

(3.34)

Combining (3.31), (3.32), (3.33), (3.34) with (3.28), we have

\[
\int_0^{2i} \int_{B(x_0, 2^i)} h^{p-1} v^p d\mu dt \leq C < \infty.
\]

(3.35)

It follows by letting \( i \to \infty \) that

\[
\int_0^{\infty} \int_M h^{p-1} v^p d\mu dt \leq C < \infty.
\]

Hence, the claim (3.17) is true.

Substituting \( \varphi = \varphi_i \) and \( R = 2^i \) into (3.15), combining with (3.31), (3.32), (3.33), (3.34) and (3.17), repeating the same procedures in (3.15), we obtain

\[
\int_0^{2i} \int_{B(0, 2^i)} h^{p-1} v^p d\mu dt \leq C \left\{ \left( \int_{D_{2^i}} h^{p-1} v^p d\mu dt \right)^{\frac{1}{p}} \right\}^{\frac{i+1}{p}} + \left( \int_{D_{2^i}} h^{p-1} v^p d\mu dt \right)^{\frac{i}{p}} \}
\]
\[
+ \left( \int_{D_{2^i}} h^{p-1} v^p d\mu dt \right)^{\frac{1}{p}}.
\]

(3.36)

Letting \( i \to \infty \), from (3.17), we have

\[
\int_0^{\infty} \int_M h^{p-1} v^p d\mu dt = 0,
\]

which implies

\[
v \equiv 0.
\]

Noting that \( u = hv \), hence \( u \equiv 0 \). However, the above leads to the contradiction with the positiveness of \( u \). Hence, there exists no global positive solution to problem (1.1). \( \Box \)
4. Global existence of positive solution

In this section, we show the sharpness of \( P, Q \) in Theorem 1.14. It suffices to show that \( Q \) in (1.19) cannot be relaxed.

**Proof of Theorem 1.14.** Define the operator

\[
Tv(x, t) = \int_M \tilde{P}_t(x, y)v_0(y)d\mu(y) + \int_0^t \int_M \tilde{P}_{t-s}(x, y)h^{p-1}v^p(y, s)d\mu(y)ds.
\]

acting on the following space

\[
S_M = \left\{ v \in L^\infty(M \times [0, \infty)) | 0 \leq v(x, t) \leq \lambda \tilde{P}_t(x, x_0) \right\}.
\]

where \( \lambda > 0 \) is a constant to be chosen later, and \( \delta > 1 \) is a large fixed constant. It follows that \( S_M \) is a closed set of \( L^\infty(M \times [0, \infty), d\mu) \).

Let \( v_0 \) satisfy

\[
0 \leq v_0(x) \leq \frac{\lambda}{2} \tilde{P}_t(x, x_0).
\]

Now let us show \( TS_M \subset S_M \).

From (4.3), and applying (1.22), we have

\[
\int_M \tilde{P}_t(x, y)v_0(y)d\mu(y) \leq \frac{\lambda}{2} \int_M \tilde{P}_t(x, y)\tilde{P}_\delta(y, x_0)d\mu(y) = \frac{\lambda}{2} \tilde{P}_t(x, x_0).
\]

From (DUE) and (1.2), we have

\[
\int_0^t \int_M \tilde{P}_{t-s}(x, y)h^{p-1}v^p(y, s)d\mu(y)ds
\leq C_1 \lambda_p \int_0^t \int_M \tilde{P}_{t-s}(x, y)\tilde{P}_s^{p+\delta}(y, x_0)d\mu(y)ds
\leq C_2 \lambda_p \int_0^t \frac{1}{\mu(B(x_0, \sqrt{s+\delta}))^{p-1}} ds \int_M \tilde{P}_{t-s}(x, y)\tilde{P}_s^{p+\delta}(y, x_0)d\mu(y)
\leq C_3 \lambda_p \tilde{P}_t^{p+\delta}(x, x_0) \int_0^t \frac{1}{\mu(B(x_0, \sqrt{s+\delta}))^{p-1}} ds,
\]

where we have used that \( h \asymp 1 \).

Recalling that for large enough \( r \),

\[
\mu_0(B(x_0, r)) \geq c_2 r^p \ln Q + \varepsilon r,
\]

and since \( h \asymp 1 \), and \( d\mu = h^2d\mu_0 \), we have

\[
\mu(B(x_0, r)) \geq c_3 r^p \ln Q + \varepsilon r.
\]

When \( \delta \) is large enough, we obtain

\[
\int_0^t \frac{1}{\mu(B(x_0, \sqrt{s+\delta}))^{p-1}} ds
\leq \int_0^t \frac{1}{C_1(s+\delta)^p (\ln \sqrt{s+\delta})^{Q+\varepsilon}}^{p-1} ds
\leq C_4 2^{1+\varepsilon(p-1)} \int_0^t \frac{1}{(s+\delta)(\ln(s+\delta))^{1+\varepsilon(p-1)}} ds
\]
Hence where we have used that \( P = \frac{2}{p-1}, \ Q = \frac{1}{p-1}. \)

Combining (4.5) with (4.6), we obtain, for small enough \( \lambda \),

\[
\int_0^t \int_M \tilde{P}_{t-s}(x,y)h^{p-1}v^p(y,s)d\mu(y)ds \leq C_5 \lambda^p \tilde{P}_{t+\delta}(x,x_0) \\
\leq \frac{\lambda}{2} \tilde{P}_{t+\delta}(x,x_0). \tag{4.7}
\]

Combining (4.1), (4.4) with (4.7), we obtain

\[
0 \leq Tv \leq \lambda \tilde{P}_{t+\delta}(x,x_0).
\]

Hence

\[
TS_M \subset S_M.
\]

Now we show that \( T \) is a contraction map. For \( v_1, v_2 \in S_M, \) we have

\[
|Tv_1(x,t) - Tv_2(x,t)| \leq \int_0^t \int_M \tilde{P}_{t-s}(x,y)h^{p-1}|v_1^p(y,s) - v_2^p(y,s)|d\mu(y)ds. \tag{4.8}
\]

Noting that

\[
|v_1^p(y,s) - v_2^p(y,s)| \leq p \max\{v_1^{p-1}(y,s), v_2^{p-1}(y,s)\} |v_1(y,s) - v_2(y,s)|,
\]

and combining with (DUE), (4.2) and (4.6), and using that \( h \approx 1, \) we obtain from (4.8) that

\[
|Tv_1(x,t) - Tv_2(x,t)| \\
\leq C_6 p \lambda^{p-1} \|v_1 - v_2\|_{L^\infty} \int_0^t \int_M \tilde{P}_{t-s}(x,y)\tilde{P}_{s+\delta}^{p-1}(y,x_0)d\mu(y)ds \\
\leq C_7 p \lambda^{p-1} \|v_1 - v_2\|_{L^\infty} \int_0^t \int_M \frac{1}{\mu(B(x_0, \sqrt{s+\delta})^{p-1})}ds \\
\leq C_8 p \lambda^{p-1} C_1^p \|v_1 - v_2\|_{L^\infty} \int_0^t \int_M \frac{1}{\mu(B(x_0, \sqrt{s+\delta})^{p-1})}ds \\
\leq C_9 p \lambda^{p-1} \|v_1 - v_2\|_{L^\infty},
\]

where we have used that (1.21) and (4.6).

Choosing \( \lambda \) small enough so that \( C_9 p \lambda^{p-1} < 1, \) we obtain that \( T \) is a contraction map. Applying fixed point theorem, we know there exists a fixed point \( v \in S_M \) satisfying

\[
v(x,t) = \int_M \tilde{P}_{t}(x,y)v_0(y)d\mu(y) + \int_0^t \int_M \tilde{P}_{t-s}(x,y)h^{p-1}(y,v^p(y,s))d\mu(y)ds. \tag{4.9}
\]

Since \( v_0 \geq 0, \) then \( v \) is positive on \( M. \) Since \( v_0, v \in L^2(M, d\mu), \) by [13] Theorem 7.6 and 7.7], we know the integrals in (4.9) are both smooth on \( M \times (0, \infty), \) hence we obtain that \( v \) is a global positive solution of problem (1.13). Furthermore, \( u = hv \) is a global positive solution of problem (1.10).

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