BIJECTIONS FOR ENTRINGER FAMILIES

YOANN GELINEAU, HEESUNG SHIN, AND JIANG ZENG

Abstract. André proved that the number of down-up permutations on \{1, 2, ..., n\} is equal to the Euler number \(E_n\). A refinement of André’s result was given by Entringer, who proved that counting down-up permutations according to the first element gives rise to Seidel’s triangle \((E_{n,k})\) for computing the Euler numbers. In a series of papers, using generating function method and induction, Poupard gave several further combinatorial interpretations for \(E_{n,k}\) both in down-up permutations and increasing trees. Kuznetsov, Pak, and Postnikov have given more combinatorial interpretations of \(E_{n,k}\) in the model of trees. The aim of this paper is to provide bijections between the different models for \(E_{n,k}\) as well as some new interpretations. In particular, we give the first explicit one-to-one correspondence between Entringer’s down-up permutation model and Poupard’s increasing tree model.

Contents

1. Introduction 2
2. The left-to-right coding \(\psi\) of down-up permutations 4
3. The left-to-right coding of binary trees 7
  3.1. The bijection \(\varphi : ES_{n,k} \rightarrow BT_{n,k}\) 7
  3.2. Interpretation of Entringer’s formula in \(BT_n\) 10
4. Poupard’s other Entringer families 11
  4.1. Another interpretation in increasing trees 11
  4.2. Another interpretation in down-up permutations 11
4.3. Interpretations in min-max alternating permutations 12
5. New Entringer families 13
  5.1. Interpretations in G-words and R-words 13
  5.2. Interpretations in U-words 15
6. Concluding remarks 16
  6.1. List of bijections for Entringer families 16
  6.2. Illustration for \(n = 4\) 16
  6.3. An open problem 16
Acknowledgement 17
References 17

Date: November 2, 2018.
1. Introduction

The Euler numbers $E_n$ are defined by the generating function
\[ \sum_{n \geq 0} E_n \frac{x^n}{n!} = \tan(x) + \sec(x) \]
\[ = 1 + x + \frac{x^2}{2!} + 2 \frac{x^3}{3!} + 5 \frac{x^4}{4!} + 16 \frac{x^5}{5!} + 61 \frac{x^6}{6!} + 272 \frac{x^7}{7!} + 1385 \frac{x^8}{8!} + \cdots. \]

Let $DU_n$ be the set of down-up permutations of $[n] := \{1, 2, \ldots, n\}$, that is, the permutations $\pi = \pi_1 \pi_2 \ldots \pi_n$ on $[n]$ satisfying $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. For example, the down-up permutations of $[4]$ are:

2 1 4 3, 3 2 4 1, 3 1 4 2, 4 2 3 1, 4 1 3 2.

André [And79] proved that the cardinality of the set $DU_n$ equals the Euler number $E_n$.

Counting the down-up permutations according to the first term leads to the Entringer numbers [Ent66]. More precisely, let $DU_{n,k}$ be the set of permutations $\pi \in DU_n$ such that $\pi_1 = k$ and $E_{n,k}$ the cardinality of $DU_{n,k}$. The first values of $E_{n,k}$ are given in Table 1.

| $n \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------------|---|---|---|---|---|---|---|
| 1               | 1 |   |   |   |   |   |   |
| 2               | 0 | 1 |   |   |   |   |   |
| 3               | 0 | 1 | 1 |   |   |   |   |
| 4               | 0 | 1 | 2 | 2 |   |   |   |
| 5               | 0 | 2 | 4 | 5 |   |   |   |
| 6               | 0 | 5 | 10| 14| 16|   |   |
| 7               | 0 | 16| 32| 46| 56| 61| 61|

Table 1. The first values of Entringer numbers $E_{n,k}$

Theorem 1.1 (Entringer). The numbers $(E_{n,k}) \ (n \geq k \geq 1)$ are defined by
\[ E_{1,1} = 1, \quad E_{n,1} = 0 \ (n \geq 2), \quad E_{n,k} = E_{n,k-1} + E_{n-1,n+1-k}. \] (1)

Iterating the above recurrence, we get $E_{n+1,n+1} = E_{n,n} + E_{n,n-1} + \cdots + E_{n,1}$, which is equal to $E_n$ by André’s result. Hence the Euler numbers $E_n = E_{n+1,n+1}$ are the diagonal entries in Table 1. As an historical remark, Entringer’s recurrence (1) is just a combinatorial interpretation of the Seidel’s scheme [Sei77] to compute Euler numbers, i.e.,

The above scheme was later rediscovered several times in the literature (see [Kem33, MSY96]). A recent survey on down-up permutations and Euler numbers is given by Stanley [Sta09].

A sequence of sets $(X_{n,k})_{1 \leq k \leq n}$ is called an Entringer family if the cardinality of $X_{n,k}$ is equal to $E_{n,k}$ for $1 \leq k \leq n$. 

2 YOANN GELINEAU, HEESUNG SHIN, AND JIANG ZENG
Let \( X = \{x_1, \ldots, x_n\}^< \) be an ordered set such that \( x_1 < \cdots < x_n \). An increasing tree on \( X \) is a spanning tree of the complete graph on \( X \), rooted at \( x_1 \) and oriented from the smallest vertex \( x_1 \), such that the vertices increase along the edges. Let \( BT_n \) be the set of binary increasing trees \( T \) on \([n]\), i.e., the increasing trees such that at most two edges go out from every vertex (see Figure 1).

![Figure 1. The binary increasing trees on [4]](image-url)

Foata and Schützenberger proved in [FS73, §5] that the Euler number \( E_n \) is the cardinality of \( BT_n \). A one-to-one correspondence between \( DU_n \) and \( BT_n \) was then constructed by Donaghey [Don75] (see also [Cal05]). However the tree counterpart of Entringer’s result was found only in 1982 by Poupard [Pou82].

If \( T \) is a binary increasing tree and if \((i, j)\) is an edge in \( T \), \( i < j \), we call \( i \) the parent of \( j \), and \( j \) a child of \( i \). If \( i \) has no child, we say that \( i \) is a leaf of \( T \). A path in \( T \) is a sequence of vertices \((a_i)\) such that \( a_i \) is a child of \( a_{i-1} \) in \( T \), and the minimal path of \( T \) is the path \((a_i)_{1 \leq i \leq \ell} \) such that \( a_1 = 1 \), \( a_\ell \) is the smallest child of \( a_{\ell-1} \), and \( a_k \) is a leaf, denoted by \( p(T) \). Let’s denote by \( BT_{n,k} \) the set of trees \( T \in BT_n \) such that \( p(T) = k \).

**Theorem 1.2** (Poupard). The sequence \((BT_{n,k})_{1 \leq k \leq n}\) is an Entringer family.

Note that contrary to the case of down-up permutations, it is not easy to interpret recurrence (1) in the model of binary increasing increasing trees. Indeed, Donaghey’s bijection doesn’t induce a bijection between \( DU_{n,k} \) and \( BT_{n,k} \) and Poupard’s proof in [Pou82] was analytic in nature. Finding a direct explanation in the model of trees was then raised as an open problem in [KPP94]. The first aim of this paper is to build a bijection between \( DU_{n,k} \) and \( BT_{n,k} \) and answer the above open problem. In other words, we have the following theorem.

**Theorem 1.3.** For all \( n \geq 1 \), there is an explicit bijection \( \Psi : DU_n \rightarrow BT_n \) satisfying

\[
\forall \pi \in DU_n, \quad \text{First}(\pi) = \text{Leaf}(\Psi(\pi)),
\]

where First(\( \pi \)) is the first element of the permutation \( \pi \) and Leaf(\( \Psi(\pi) \)) is the leaf of the minimal path of the tree \( \Psi(\pi) \).

Poupard [Pou82, Pou97] gave also other interpretations for Entringer numbers \( E_{n,k} \) (see Section 1) in binary increasing trees and down-up permutations with induction proofs. Our second aim is to provide simple bijections between the other interpretations of Poupard in down-up permutations and the original interpretation in \( DU_{n,k} \). Note that some other interpretations of Entringer numbers \( E_{n,k} \) in the model of increasing trees were given in [KPP94]. Recently, two new interpretations of Euler numbers were given by Martin and Wagner [MW09] in the model of G-words and R-words. We shall give the corresponding interpretations of the Entringer number \( E_{n,k} \) in the later models.

The rest of this paper is organized as follows. In Section 2 we introduce an intermediate model \( ES_{n,k} \) and present a bijection \( \psi \) between \( DU_{n,k} \) and \( ES_{n,k} \). In Section 3 we describe a bijection \( \varphi \) between \( ES_{n,k} \) and \( BT_{n,k} \) so that \( \Psi = \varphi \circ \psi \) provides the bijection for Theorem 1.3. As an application, in Subsection 3.2 we give a direct interpretation of (1) in the...
model of increasing trees. In Section 3, we recall the other interpretations of \(E_{n,k}\) found by Poupard and establish simple bijections between these models. In Section 4, we give some new interpretations for \(E_{n,k}\), first refining the results of Martin and Wagner [MW09] in their model of G-words and R-words, and secondly introducing the new model of U-words.

2. The left-to-right coding \(\psi\) of down-up permutations

Consider down-up permutations on any finite subset \(I = \{a_1, a_2, \ldots, a_m\} < \mathbb{N}\). Two elements \(a\) and \(b\) in \(I\) are said to be adjacent if there is no \(c \in I\) between \(a\) and \(b\). Let \(\pi\) be a down-up permutation on \(I\), i.e., \(\pi_1 > \pi_2 < \pi_3 > \pi_4 < \ldots\). Suppose \(\pi_1 = a_i\) and \(\pi_2 = a_j\) with \(a_i > a_j\). If \(\pi_1\) and \(\pi_2\) are adjacent, then, deleting \(\pi_1\pi_2\), we obtain again a down-up permutation on \(I \setminus \{\pi_1, \pi_2\}\), otherwise, we can apply successively the adjacent transpositions \((a_i, a_{i-1}), (a_{i-1}, a_{i-2}), \ldots, (a_{j+2}, a_{j+1})\) to \(\pi\) (from left-to-right):

\[
\pi^{(1)} = (a_i, a_{i-1}) \circ \pi, \quad \pi^{(2)} = (a_{i-1}, a_{i-2}) \circ \pi^{(1)}, \quad \ldots, \quad \pi^{(i-j-1)} = (a_{j+2}, a_{j+1}) \circ \pi^{(i-j-2)},
\]

so that all the permutations \(\pi^{(1)}, \ldots, \pi^{(k-j-1)}\) are down-up permutations and the first two elements in \(\pi^{(i-j-1)}\) are adjacent. Deleting the first two elements, we get again a down-up permutation, say \(\pi^{(i-j)}\), on \(I \setminus \{a_{j+1}, a_j\}\). If we register \((a, b)\) for the composition from left with the adjacent involution \((a, b)\), and \((a, b)^*\) for the deletion of the first two letters \(a\) and \(b\), then the operations in the above process can be encoded by the word

\[
(a_i, a_{i-1}) (a_{i-1}, a_{i-2}) \ldots (a_{j+2}, a_{j+1}) (a_{j+1}, a_j)^*.
\]

Since the resulting permutation \(\pi^{(i-j)}\) is still down-up, we can iterate this process until we obtain the empty permutation. Clearly the last deletion is \((n)^*\) if \(n\) is odd. We shall call left-to-right code the resulting sequence of the successive operations in this process and denote it by \(\psi(\pi) = (\Delta_\ell)^\ell\), where each entry \(\Delta_\ell\) is either a transposition \((j, i)\), a deletion \((j, i)^*\), \(1 \leq i < j \leq n\), or the deletion \((n)^*\). Formally, we can write the algorithm as follows:

1. Start with \((\pi, \Delta = \emptyset)\) and support set \(I = \{a_1, a_2, \ldots, a_m\} < \mathbb{N}\).
2. While \(\text{Card}(A) \geq 2\), do:
   a. While there is \(a \in I\) such that \(\pi_1 > a > \pi_2\), do:
      \[\Delta \leftarrow (\Delta, (\pi_1, a'))\], where \(a' = \max\{a \in I | \pi_1 > a > \pi_2\}\),
      \[\pi \leftarrow (\pi_1, a') \circ \pi\].
   b. If there is no \(a \in I\) such that \(\pi_1 > a > \pi_2\), do:
      \[\Delta \leftarrow (\Delta, (\pi_1, \pi_2)^*)\],
      \[\pi \leftarrow \pi_3 \pi_4 \ldots \pi_n\] (eventually \(\pi = \emptyset\),
      \[I \leftarrow I \setminus \{\pi_1, \pi_2\}\].
3. If \(\text{Card}(I) = 1\) with \(I = \{a_m\}\), do:
   \[D_\pi \leftarrow (\Delta, (a_m)^*)\],
   \[\pi \leftarrow \emptyset\],
   \[I \leftarrow \emptyset\].
Example 2.1. If $\pi = 748591623 \in DU_{9,7}$, then the algorithm goes as follows:

| Step | $\pi^{(r)}$ | $\Delta^{(r)}$ |
|------|-------------|---------------|
| 0    | $748591623$ | $\emptyset$  |
| 1    | $648591723$ | (7,6)         |
| 2    | $548691723$ | (6,5)         |
| 3    | $8691723$   | (5,4)*        |
| 4    | $7691823$   | (8,7)         |
| 5    | $91823$     | (7,6)*        |
| 6    | $81923$     | (9,8)         |
| 7    | $31928$     | (8,3)         |
| 8    | $21938$     | (3,2)         |
| 9    | $938$       | (2,1)*        |
| 10   | $839$       | (9,8)         |
| 11   | $9$         | (8,3)*        |
| 12   | $\emptyset$| (9)*          |

Thus, the left-to-right code of $\pi$ is

$$\psi(\pi) = (7,6) (6,5) (5,4)* (8,7) (7,6)* (9,8) (8,3) (3,2) (2,1)* (9,8) (8,3)* (9)*.$$

A domino on $[n]$ is an ordered pairs $(j, i)$ $(1 \leq i < j \leq n)$ and a starred domino on $[n]$ is a starred ordered pairs $(j, i)^*$ $(1 \leq i < j \leq n)$ or $(n)^* = (n, n)^*$. Let $A_n$ be the alphabet consisting of dominos (starred or non) on $[n]$.

Definition 2.2. A word $\Delta = \Delta_1 \ldots \Delta_\ell$ on $A_n$ is an encoding sequence of $[n]$ if the following conditions are verified:

(i) the entries of starred dominos are disjoint and their union equals $[n]$.

(ii) if $\Delta_i = (j, i)^*$, then the next domino (if there is one) starts with an entry $> i$, and no entry of a later domino lies between $i$ and $j$.

(iii) if $\Delta_i = (j, i)$, then both $i$ and $j$ appear in a later domino, with $i$ the first entry of the next domino, and each integer between $i$ and $j$ appears in an earlier starred domino.

Remark 2.3. It is clear from the definition that $(n, i)^*$ $(1 \leq i \leq n)$ can only take the last position in an encoding sequence and an encoding sequence must start with $(k, k - 1)$ or $(k, k - 1)^*$ for $2 \leq k \leq n$.

We denote by $\mathcal{E}S_n$ the set of encoding sequences of $[n]$, and by $\mathcal{E}S_{n,k}$ the subset of $\mathcal{E}S_n$ consisting of encoding sequences starting with $(k, k - 1)$ or $(k, k - 1)^*$, $2 \leq k \leq n$. For example, the set $\mathcal{E}S_4$ is the union of the three subsets:

- $\mathcal{E}S_{4,2} = \{(2,1)^* (4,3)^*\}$
- $\mathcal{E}S_{4,3} = \{(3,2)^* (4,1)^*, (3,2) (2,1)^* (4,3)^*\}$
- $\mathcal{E}S_{4,4} = \{(4,3) (3,2)^* (4,1)^*, (4,3) (3,2) (2,1)^* (4,3)^*\}$

Theorem 2.4. For all $n \geq 1$ and $k \in [n]$, the mapping $\psi : DU_{n,k} \rightarrow \mathcal{E}S_{n,k}$ is a bijection. Therefore, the sequence $(\mathcal{E}S_{n,k})_{1 \leq k \leq n}$ is an Entringer family.

Proof. Let $\pi = \pi_1 \ldots \pi_n$ be an element in $DU_{n,k}$. Then $\pi_1 = k$, so the first letter of $\psi(\pi)$ is $(k, i)$ or $(k, i)^*$ $(1 \leq i < k)$ by definition of $\psi$. It remains to show that the word $\psi(\pi)$ verifies the conditions (i)-(iii) of Definition 2.2. Since the process reduces the permutation $\pi$ to empty permutation, the condition (i) is verified.
• If \( \Delta_\ell = (j, i)\), as \( i \) and \( j \) are adjacent in the support set of \( \pi^{(\ell)} \), the integers between \( i \) and \( j \) have been removed in previous starred dominos, also the first entry of the next domino is greater than \( i \) because \( \pi^{(\ell)} \) is down-up.

• If \( \Delta_\ell = (j, i) \), as \( i \) and \( j \) are adjacent in the support set of \( \pi^{(\ell)} \), the integers between \( i \) and \( j \) have been removed in previous starred dominos, also the next domino must be \((i, m)\) or \((i, m)^*\) with \( i > m \) because \( i \) is the first entry of \( \pi^{(\ell)} \).

It results that \( \psi(\pi) \in \mathcal{ES}_{n,k} \).

Conversely, starting from an encoding sequence \( \Delta = \Delta_1 \ldots \Delta_{\ell} \in \mathcal{ES}_{n,k} \), we construct by induction \( \pi^{(j)} \) such that \( \text{First}(\pi^{(j)}) \) equals the first entry of \( \Delta_j \) for \( j = \ell, \ell-1, \ldots, 1 \).

First, if \( \Delta_\ell = (n)^* \) then define \( \pi^{(\ell)} = n \), if \( \Delta_\ell = (n, i)^* \) with \( i < n \), then define \( \pi^{(\ell)} = n^i \).

Assume that \( \pi^{(j+1)} \) is constructed with \( \text{First}(\pi^{(j+1)}) = k_{j+1} \). By definition of \( \Delta \), there are two cases:

(i) if \( \Delta_j = (k_j, k_{j+1}) \), where \( k_j \) and \( k_{j+1} \) are adjacent in the support set of \( \pi^{(j+1)} \), then define \( \pi^{(j)} := (k_j, k_{j+1}) \circ \pi^{(j+1)} \). This permutation is still down-up and the first element of \( \pi^{(j)} \) is \( k_j \);

(ii) if \( \Delta_j = (a_j, b_j)^* \), where \( a_j > b_j < k_{j+1} \), and \( a_j, b_j \) are not in the support set of \( \pi^{(j+1)} \), then define \( \pi^{(j)} \) as the word \( a_j b_j \pi^{(j+1)} \). Since \( a_j > b_j < k_{j+1} \), the permutation \( \pi^{(j)} \) is down-up with \( a_j \) as the first element.

Let \( \psi^{-1}(\Delta) := \pi^{(1)} \), which is an element in \( \mathcal{DU}_{n,k} \).

Remark 2.5. Denote the largest integer less than \( x \) by \( \lfloor x \rfloor \) and the number of ordered pairs \( (i, j) \in \{1, \ldots, n\} \) such that \( i + 1 < j \) and \( \pi_i > \pi_{i+1} < \pi_j < \pi_i \) by \((31-2)\pi\). Then, one can show that the length of the sequence \( \psi(\pi) \) is equal to

\[
(31-2)\pi + \left\lfloor \frac{n+1}{2} \right\rfloor.
\]

Indeed, \((31-2)\pi\) corresponds to the number of occurrences of terms \((j, i)^*\), \( j > i \), in \( \psi(\pi) \), and there are \( \left\lfloor \frac{n+1}{2} \right\rfloor \) occurrences of terms \((j, i)^*, j > i \), in \( \psi(\pi) \). Note that various formulae for counting \(31-2\)-patterns in down-up permutations are given in [Che08, JV10, SZ10].

Proposition 2.6. Let \( n \geq 2 \) and \( k \geq 2 \). The number of elements starting with \((k, k-1)\) equals \( E_{n,k-1} \), and the number of elements starting with \((k, k-1)^*\) equals \( E_{n-1, n+1-k} \).

Proof. Let \( \Delta \in \mathcal{ES}_{n,k} \). If \( \Delta_1 = (k, k-1) \), the remaining sequence \( (\Delta_2, \Delta_3, \ldots) \) is still an encoding sequence of \([n]\), starting with \( \Delta_2 \in \{(k-1, i), (k-1, i)^*, 1 \leq i \leq k-2\} \). Thus, there are \( E_{n-1,k-1} \) encoding sequences starting by \((k, k-1)\). If \( \Delta_1 = (k, k-1)^* \), the remaining sequence \( (\Delta_2, \Delta_3, \ldots) \) doesn’t contain the elements \( k \) and \( k+1 \) and starts with an element in \( \{(i, j), (i, j)^*, 1 \leq j \leq i-1\} \) with \( i \geq k+1 \). In other words, this is an encoding sequence of \( n-2 \) elements, starting with an index \( i \) that must be greater than the \( k-2 \) first elements. Thus, there are \( E_{n-2,k-1} + E_{n-2,k} + \cdots + E_{n-2,n-2} = E_{n-1, n+1-k} \) encoding sequences starting by \((k, k-1)^*\).

Since any sequence in \( \mathcal{ES}_{n,k} \) begins with either \((k, k-1)\) or \((k, k-1)^*\) (\(2 \leq k \leq n\)), Entringer’s formula [1] results from the above proposition.
3. The left-to-right coding of binary trees

3.1. The bijection $\varphi: \mathcal{ES}_{n,k} \to \mathcal{BT}_{n,k}$. Starting from an encoding sequence $\Delta = \Delta_1 \ldots \Delta_\ell \in \mathcal{ES}_{n,k}$, we construct a tree $T = \varphi(\Delta) \in \mathcal{BT}_{n,k}$ by reading the sequence $\Delta$ in reverse order, i.e., from right to left. More precisely, for $m = \ell, \ell - 1, \ldots, 1$, we shall construct a tree $T_m$ corresponding to the word $\Delta_m \ldots \Delta_{\ell-1} \Delta_\ell$ such that

$$\Delta_m = (j_m, i_m) \text{ or } (j_m, i_m)^* \implies \text{Leaf}(T_m) = j_m,$$

and define $T = T_1 := \varphi(\Delta)$. The algorithm goes as follows:

(i) If $\Delta_\ell = (n)^*$, construct the tree $T_\ell$ with only one vertex $n$; if $\Delta_\ell = (n, i)^*$, construct the increasing tree $T_\ell$ with only one edge $i \rightarrow n$. Clearly (2) is verified.

Assume that we have constructed such a tree $T_{m+1}$ corresponding to the word $\Delta_{m+1} \ldots \Delta_\ell$.

(ii) If $\Delta_m = (j_m, i_m)^*$, we add vertices $i_m$ and $j_m$ to the tree $T_{m+1}$ to obtain $T_m$. Suppose that the minimal path of $T_{m+1}$ is $(a_1, \ldots, a_{p_m})$.

- If $i_m < a_1$, add the edges $(i_m, a_1)$ and $(i_m, j_m)$ to the tree $T_{m+1}$. Then, the tree $T_m$ is an increasing tree rooted at $i_m$ with $(i_m, j_m)$ as the minimal path.

- If $i_m > a_1$, by induction hypothesis and property (ii) of encoding sequences, we see that $a_1 < m$. Hence, there exists $k \in \{1, \ldots, p_m - 1\}$ such that $a_k < i_m < a_{k+1}$. Then, erase the edge $(a_k, a_{k+1})$, create the edges $(a_k, i_m), (i_m, a_{k+1})$ and $(i_m, j_m)$. Clearly, the tree $T_m$ is an increasing tree with $(i_m, j_m)$ as the last edge of the minimal path.

(ii) If $\Delta_m = (j_m, i_m)$, where $i_m$ and $j_m$ are not siblings in $T_{m+1}$, by induction hypothesis and property (iii) of encoding sequences, we derive that $i_m$ is at the end of the minimal path. Then, we transform the tree $T_{m+1}$ as follows: just exchange the places of $i_m$ and $j_m$ in $T_{m+1}$. The tree remains increasing because Then $j_m$ is at the end of the
minimal path in $T_m$.

(iii) If $\Delta_m = (j_m, i_m)$, where $i_m$ and $j_m$ are siblings in $T_{m+1}$, as in the previous case, $i_m$ is at the end of the minimal path. Then, transform $T_{m+1}$ with the following procedure. If $m_1$ denotes the parent of $i_m$ and $j_m$ in $T$, erase the edge $(m_1, j_m)$, create an edge $(i_m, j_m)$, then if $A$ and $B$ are the two subtrees starting from $j_m$ with $\text{min}(A) < \text{min}(B)$ (eventually $B$ is empty), cut the subtree $A$ from $j_m$ and add it as a direct subtree of $m_1$, cut the subtree $B$ from $j_m$ and add it as a direct subtree of $i_m$. The procedure can be illustrated with the following picture:

Let $\varphi(\Delta) := T_1$, which is an element in $B\mathcal{T}_{n,k}$.

**Theorem 3.1.** For all $n \geq 1$ and $k \in [n]$, the mapping $\varphi : \mathcal{E}S_{n,k} \rightarrow B\mathcal{T}_{n,k}$ is a bijection.

**Proof.** It is sufficient to construct the inverse mapping of $\varphi$ to show that this is a bijection. Given $T$ an increasing tree on the ordered set $\{a_1, \ldots, a_n\}$ with $a_1 < \cdots < a_n$, such that $p(T) = a_k$ (that can be interpreted by an element of $B\mathcal{T}_{n,k}$), we construct an encoding sequence $\Delta = \varphi^{-1}(T)$ of $[n]$ recursively as follows:

(a) If $a_{k-1}$ is the parent of $a_k$ in $T$, then let $m$ ($m > a_k$) be the other child of $a_{k-1}$ ($m = \infty$ if $a_k$ is the only child of $a_{k-1}$) and $s$ ($s > k$) be a sibling of $a_{k-1}$ ($s = \infty$ if $a_{k-1}$ has no sibling), and $j$ the parent of $a_{k-1}$ in $T$.

(a1) If $m < \infty$ and $m < s$, then define $\varphi^{-1}(T) = ((a_k, a_{k-1})^*, \varphi^{-1}(T'))$, where $T'$ is the tree obtained from $T$ by deleting the vertices $a_{k-1}, a_k$ and their adjacent
edges in $T$, and adding a new edge between $m$ and $j$.

(a2) In the other cases ($m = \infty$ or $m > s$), then define $\varphi^{-1}(T) = ((a_k, a_{k-1}), \varphi^{-1}(T'))$, where $T'$ is the tree obtained from $T$ by erasing the edges $(a_{k-1}, a_k)$, $(a_{k-1}, m)$ and $(j, s)$ in $T$, and adding the edges $(j, a_k)$, $(a_k, s)$, $(a_k, m)$.

The procedure can be illustrated with the following picture:

(b) If $a_{k-1}$ is not the parent of $a_k$ in $T$, then define $\varphi^{-1}(T) = ((a_k, a_{k-1}), \varphi^{-1}(T'))$, where $T'$ is the tree obtained from $T$ by exchanging the labels $a_{k-1}$ and $a_k$.

Note that cases (a1), (a2) and (b) in the construction of $\varphi^{-1}$ correspond, respectively, to cases (i), (ii) and (iii) of the construction of $\varphi$.

It remains to prove that the obtained sequence $\Delta$ verifies the points (i)-(iii) of Definition 2.2:

- It is easily seen that each integer of $[n]$ is removed once off $T$. So (i) is verified.
- If an element $(j, i)^*$ appears in $\Delta$, that corresponds to the case (a1), when we delete the vertices $i$ and $j$ from the tree $T$. Then the next elements in $\Delta$ don’t contain either $i$ or $j$ since they correspond to $\varphi^{-1}(T')$. Moreover, if we are in the case (a1), the minimal path in the tree $T'$ contains at least one element $m$ with $m > j > i$, so the next element in $\Delta$ must be $(m, k)$ with $m > k$. Thus (ii) is verified.
- If an element $(j, i)$ appears in $\Delta$, in both Case (a2) or Case (b), the tree $T'$ has $i$ as the leaf of the minimal path. Then, the next element in $\Delta$ must be $(i, k)$ with
Moreover, \( i \) and \( j \) must be consecutive elements in the ordered set of labels in \( T \). Then the elements \( \ell \) such that \( i < \ell < j \) don’t appear in \( T \). Thus (iii) is verified.

Let \( \Psi = \varphi \circ \psi \). Then \( \Psi : DU_{n,k} \to BT_{n,k} \) is a bijection satisfying \( \pi_1 = p(\Psi(\pi)) \) for all \( \pi \in DU_{n,k} \). Thus Theorem 1.3 is proved.

**Example 3.2.** Continuing the Example 2.1, we apply \( \Psi \) to \( \pi \) by using the known LR-code of \( \pi = 7 \ 4 \ 8 \ 5 \ 9 \ 1 \ 6 \ 2 \ 3 \). The details are given in Figure 2.

**Figure 2.** The construction of the tree \( \Psi(7 \ 4 \ 8 \ 5 \ 9 \ 1 \ 6 \ 2 \ 3) \)

### 3.2. Interpretation of Entriger’s formula in \( BT_n \). Following the interpretation of (11) in \( ES_n \) (cf Remark 2.6) and the bijection \( \varphi \), we must consider the decomposition of the set \( BT_{n,k} \). The first step in the construction of \( \varphi^{-1} \) would consist in either removing the elements \( k - 1 \) and \( k \), or the first step transforms the tree to obtain another tree of \( BT_n \).

For \( T \) an element of \( BT_{n,k} \), we say that the edge \((k - 1, k)\) is **removable** if \( k - 1 \) is the parent of \( k \) and if \( k - 1 \) has another child \( m \) that is not greater than the sibling of \( k - 1 \) (if such a sibling exists). For a visual representation, a tree \( T \) has its edge \((k - 1, k)\) removable if it corresponds to the case A-1 in the proof of Theorem 3.1.

If the edge \((k - 1, k)\) is not removable, the tree obtained after the first operation in the construction of \( \varphi^{-1} \) will be an increasing tree with \( n \) elements such that \( k - 1 \) is the leaf of the main chain. Then, there are exactly \( E_{n,k-1} \) trees such that the edge \((k - 1, k)\) is not removable.

If the edge is removable, the tree obtained with the first operation in the construction of \( \varphi^{-1} \) will be an increasing tree with \( n - 2 \) elements (without the elements \( k - 1 \) and \( k \)), and the end of the minimal path must be an element \( i \) greater than the \( k - 2 \) first elements. Thus,
there are $E_{n-2,k-1} + E_{n-2,k} + \cdots + E_{n-2,n-2} = E_{n-1,n-k+1}$ increasing trees such that the edge $(k-1,k)$ is removable.

Finally, an interpretation of $\{1\}$ appears in the model of $T_n$. The decomposition according to the removability of the edge $(k-1,k)$ in $T \in \mathcal{BT}_{n,k}$ gives $\{1\}$.

4. **Poupard’s other Entringer families**

4.1. Another interpretation in increasing trees. Let $\mathcal{BT}_{n,k}$ be the set of trees $T \in \mathcal{BT}_n$ such that the parent of $n$ in $T$ is $k-1$. By using recurrence relations Poupard proved that $E_{n,k}$ is also the number of trees in $\mathcal{BT}_{n,k}$. A bijection $\varphi'$ between $\mathcal{BT}_{n,k}$ and $\mathcal{BT}'_{n,k}$ was given in [KPP94] §6 for a more general class of increasing trees that they call geometric.

4.2. Another interpretation in down-up permutations. If $\pi$ is a permutation of $\mathcal{DU}_{n,k}$, define $\theta(\pi)$ as follows:

- if $k < n - k + 1 + \pi_2$, then $\theta(\pi) = (n - k + 1 + \pi_2, n - k + \pi_2, \ldots, k + 1, k) \circ \pi$,
- if $k > n - k + 1 + \pi_2$, then $\theta(\pi) = (n - k + 1 + \pi_2, n - k + 2 + \pi_2, \ldots, k - 1, k) \circ \pi$.

Since $\pi$ is down-up, $\pi_2 < k = 1$. If $k < n - k + 1 + \pi_2$, $\pi_2$ is unchanged by the cycle and then $\sigma(\pi)_2 = \pi_2$. Thus $\sigma(\pi)_2 < k < n - k + 1 + \pi_2 = \sigma(\pi)_1$ and $\theta(\pi)$ is still down-up. If $k > n - k + 1 + \pi_2$, since $k \leq n$, then $n - k + 1 + \pi_2 \geq \pi_2 + 1$, so $\pi_2$ is unchanged by the cycle, $\sigma(\pi)_1 = n - k + 1 + \pi_2 > \pi_2 = \sigma(\pi)_2$ and $\theta(\pi)$ is still down-up.

Let’s denote by $\mathcal{DU}'_{n,k}$ the set of permutations $\pi \in \mathcal{DU}_n$ such that $\pi_1 - \pi_2 = n + 1 - k$.

**Theorem 4.1.** For all $n \geq 1$ and $k \in [n]$, the mapping $\theta$ is a bijection from $\mathcal{DU}_{n,k}$ to $\mathcal{DU}'_{n,k}$. Moreover, for every $\pi \in \mathcal{DU}_{n,k}$, we have $\theta(\pi)_2 = \pi_2$.

**Proof.** By construction, the mapping $\theta$ is clearly invertible. Moreover, for $\sigma \in \mathcal{DU}_n$ with $\sigma_1 - \sigma_2 = n - k + 1$,

- if $k < n - k + 1 + \sigma_2$, then $\theta^{-1}(\sigma) = (k, k + 1, \ldots, n - k + \sigma_2, n - k + 1 + \sigma_2) \circ \sigma$,
- if $k > n - k + 1 + \sigma_2$, then $\theta^{-1}(\sigma) = (k, k - 1, \ldots, n - k + 2 + \sigma_2, n - k + 1 + \sigma_2) \circ \sigma$,

thus $\theta^{-1}(\sigma) \in \mathcal{DU}_{n,k}$.

With Theorem 4.1, the following interpretation of Poupard, proved in [Pou97] by recurrence relations, can be recovered.

**Corollary 4.2.** The sequence $(\mathcal{DU}'_{n,k})_{1 \leq k \leq n}$ is an Entringer family.

Since $\mathcal{DU}_{n,k} \subset \mathcal{DU}_n$, we can define $\theta^i(\pi)$ for $\pi \in \mathcal{DU}_n$. Actually, it is easy to see that the mapping $\theta$ is an involution on $\mathcal{DU}_n$. The result can also be generalized with the following observation. For any $\pi \in \mathcal{DU}_n$, define the complement permutation $\overline{\pi}$ with $\overline{\pi}_i = n + 1 - \pi_i$ for $i \in [n]$. Denote by $\mathcal{DU}_n^*$ the set of permutations $\pi$ such that $\overline{\pi} \in \mathcal{DU}_n$.

**Corollary 4.3.** For $n \geq 1$, we have

$$\sum_{\pi \in \mathcal{DU}_n^*} q^{\pi_1} p^{\pi_2 - \pi_1} = \sum_{\pi \in \mathcal{DU}_n^*} p^{\pi_1} q^{\pi_2 - \pi_1}.$$  

**Proof.** The mapping $\pi \mapsto \theta(\overline{\pi})$ is a bijection between $\{\pi \in \mathcal{DU}_n^* : \pi_1 = k\}$ and $\{\pi \in \mathcal{DU}_n^* : \pi_2 - \pi_1 = k\}$. Thus, the two statistics $\pi_1$ and $\pi_2 - \pi_1$ are equidistributed on $\mathcal{DU}_n^*$. Indeed, with proof of Theorem 4.1, $\pi \mapsto \theta(\overline{\pi})$ is a bijection between $\{\pi \in \mathcal{DU}_n^* : \pi_1 = k, \pi_2 - \pi_1 = \ell\}$ and $\{\pi \in \mathcal{DU}_n^* : \pi_1 = \ell, \pi_2 - \pi_1 = k\}$. Thus, the distribution of the two statistics is symmetric.
4.3. Interpretations in min-max alternating permutations. Recall that a permutation \( \pi \) on \([n]\) is an alternating permutation if \( \pi_1 > \pi_2 < \pi_3 > \cdots \) or \( \pi_1 < \pi_2 > \pi_3 < \cdots \). A min-max alternating permutation of \([n]\) is an alternating permutation in which 1 precedes \( n \).

For example, the min-max alternating permutations of \([4]\) are

\[
1423, \ 1324, \ 3142, \ 2314, \ 2143.
\]

Let \( \mathcal{M}_n \) be the set of \textit{min-max alternating permutations} of \([n]\). Denote by \( \mathcal{M}_{n,k} \) the set of \( \pi \in \mathcal{M}_n \) such that \( \pi_1 - \pi_2 = n + 1 - k \). The set \( \mathcal{D}_n^k \) can be split in two disjoint subsets \( \mathcal{D}_n^k \cap \mathcal{M}_n \) and \( \mathcal{D}_n^k \setminus \mathcal{M}_n \).

Let \( \pi \in \mathcal{D}_n^k \cap \mathcal{M}_n \), define \( \beta(\pi) = \pi \). Then, \( \beta(\pi) \in \mathcal{M}_n^k \) and \( \beta(\pi) = - (n + 1 - k) \).

\[\text{Theorem 4.4. For all } n \geq 1 \text{ and } k \in [n], \text{ the mapping } \beta \text{ is a bijection between } \mathcal{D}_n^k \text{ and } \mathcal{M}_n^k.\]

With the previous theorem, the interpretation of Poupard, proved in \([Pou97]\) by recurrence relations, can be recovered.

\[\text{Corollary 4.5. The sequence } (\mathcal{M}_n^k)_{1 \leq k \leq n} \text{ is an Entringer family.}\]

Denote by \( \mathcal{M}_n^k \) the set of \( \pi \in \mathcal{M}_n \) such that the term immediately before 1 is \( k \), if \( k \leq n - 1 \), and by \( \mathcal{M}_n^k \) the set of \( \pi \in \mathcal{M}_n \) such that \( \pi_1 = 1 \).

We want to construct a bijection \( \rho \) between \( \mathcal{D}_n^k \) and \( \mathcal{M}_n^k \).

If \( k = n \), it suffices to define for \( \pi \in \mathcal{D}_n^k \), \( \rho(\pi) = \pi \). Then, \( \rho(\pi) \in \mathcal{M}_n^k \).

Assume that \( k \leq n - 1 \). The set \( \mathcal{D}_n^k \) can be split in two disjoint subsets \( \mathcal{D}_n^k \cap \mathcal{M}_n \) and \( \mathcal{D}_n^k \setminus \mathcal{M}_n \). For an ordered set \( I = \{a_1, \ldots, a_n\} \) with \( a_1 < \cdots < a_n \), denote by \( \sigma_I \) the permutation:

\[
\sigma_I = \begin{pmatrix}
a_1 & a_2 & \cdots & a_n \\
a_n & a_{n-1} & \cdots & a_1
\end{pmatrix}
\]

Then, for a permutation \( \pi = \pi_1 \cdots \pi_n \) on the ordered set \( I \), denote by \( \pi^R \) the reverse permutation:

\[
\pi^R := \pi_n \pi_{n-1} \cdots \pi_1.
\]

Note that when \( I = [n] \), the definition of the complement permutation coincides with the one in the Remark of Subsection 4.2.

Then, for a permutation \( \pi \in \mathcal{D}_n^k \),

- If \( \pi \in \mathcal{D}_n^k \cap \mathcal{M}_n^k \), we can write \( \pi = \sigma_I \pi_2 \). Then, define \( \rho(\pi) = \sigma_I^R \pi_2 \). Since \( \pi_1 < \pi_2 \), \( \rho(\pi) \) is still down-up, and the term just before 1 in \( \rho(\pi) \) is \( \pi_1 = k \).

- If \( \pi \in \mathcal{D}_n^k \setminus \mathcal{M}_n^k \), we can write \( \pi = \sigma_I \pi_2 \). Then, define \( \rho(\pi) = \sigma_I^R \pi_2 \). Since \( \pi_1 < \pi_2 \) and \( \sigma_I^R \) is down-up, \( \rho(\pi) \) is still down-up, and the term just before 1 in \( \rho(\pi) \) is \( \pi_1 = k \).

\[\text{Theorem 4.6. For all } n \geq 1 \text{ and } k \in [n], \text{ the mapping } \rho \text{ is a bijection between } \mathcal{D}_n^k \text{ and } \mathcal{M}_n^k.\]

\[\text{Proof. In order to prove that } \rho \text{ is a bijection, it suffices to describe the inverse of } \rho. \text{ Let } \pi \text{ be an element in } \mathcal{M}_n \text{ such that the term immediately before 1 is } k. \text{ Following the construction of } \rho, \text{ we have:}\]

- If \( \pi \in \mathcal{D}_n^k \), write \( \pi = \pi_1 \pi_2 \). Then, \( \rho^{-1}(\pi) = \pi_1^R \pi_2 \).
• If \( \pi \notin DU_{n,k} \), write \( \pi = \tau_1 1 \tau_2 \). Then, \( \rho^{-1}(\pi) = \tau_1 R n \tau_2 \).

With the previous theorem, the following interpretation of Poupard, proved in [Pou97] by recurrence relations, can be recovered.

**Corollary 4.7.** The sequence \((MM'_{n,k})_{1 \leq k \leq n}\) is an Entringer family.

Denote by \(MM''_{n,k}\) the set of \( \pi \in MM_n \) such that the term immediately after \( n \) is \( n + 1 - k \), if \( k \leq n - 1 \), and \(MM''_{n,n}\) the set of \( \pi \in MM_n \) such that \( \pi_n = n \).

Denote by \( \rho' \) the mapping defined for \( \pi \in MM''_{n,k} \) by \( \rho'(\pi) = \overline{\pi}R^n \).

**Theorem 4.8.** For all \( n \geq 1 \) and \( k \in [n] \), the mapping \( \rho' \) is a bijection between \( MM'_{n,k} \) and \( MM''_{n,k} \). Therefore, the sequence \((MM''_{n,k})_{1 \leq k \leq n}\) is an Entringer family.

**Proof.** For \( k \leq n - 1 \), \( \pi \in MM_n \) has \( k \) just before 1 if and only if \( \rho'(\pi) \) has \( n + 1 - k \) just after \( n \).

\[\] 5. New Entringer families

5.1. **Interpretations in G-words and R-words.** A permutation \( \pi \) of \( I = \{a_1, \ldots, a_n\} \) with \( a_1 < \cdots < a_n \) is called a **G-word** if

(i) \( \pi_1 = a_n, \pi_n = a_{n-1} \),
(ii) \( \pi_2 > \pi_{n-1} \) (if \( n \geq 4 \)).

Similarly, a permutation \( \pi \) of \( I \) is called an **R-word** if previous conditions are satisfied when \( (ii) \) is replaced by

(ii') \( \pi_2 < \pi_{n-1} \) (if \( n \geq 4 \)).

A G-word (resp. an R-word) is said to be **primitive** if for all \( 1 \leq i < j \leq n \), neither the word \( \pi_i \pi_{i+1} \cdots \pi_j \) nor the word \( \pi_j \pi_{j-1} \cdots \pi_i \) is a G-word (resp. an R-word). Denote respectively by \( GW_n \) and \( RW_n \) the set of primitive G-words on \([n+2]\) and primitive R-words on \([n]\). For examples, the G-words in \( GW_4 \) are:

\[634215, \ 642315, \ 623415, \ 643215, \ 624315,\]

and the R-words in \( RW_4 \) are:

\[621435, \ 623145, \ 614235, \ 631245, \ 624135.\]

These permutations were introduced in [Mar06] with the following problem. Let \( I_n \) be the ideal of all algebraic relations on the slopes of all lines that can be formed by placing \( n \) points in a plane. Then, under two orders, \( I_n \) is generated by monomials corresponding to respectively primitive G-words and primitive R-words.

Martin and Wagner proved [MW99] that \( E_n \) is the number of primitive G-words (resp. the number of primitive R-words) on \([n+2]\). Actually, this result can be refined to Entringer numbers by introducing a statistic on G-words.

Given a primitive G-word or an R-word \( \pi \) on \([n+2]\), the **route** of \( \pi \) is the sequence \((\alpha_i)\) defined by the following procedure:

- \( \alpha_1 = n + 2 (= \pi_1) \), \( \alpha_2 = n + 1 (= \pi_{n+2}) \),
• for $k \geq 2$, if $\alpha_k = \pi_i$, define $A_k = \{\alpha_1, \ldots, \alpha_{k-1}\}$ and

$$\alpha_{k+1} = \begin{cases} 
\alpha_k 
\max \left[ \left\{ \pi_j | j < i \mathrm{\ and\ } \pi_j, \pi_{j+1}, \ldots, \pi_{i-1} \notin A_k \right\} 
\cup \left\{ \pi_j | j > i \mathrm{\ and\ } \pi_{i+1}, \ldots, \pi_j, \pi_i \notin A_k \right\} \right] & \text{if } \{\pi_{i-1}, \pi_{i+1}\} \subseteq A_k, \\
\text{otherwise}. & 
\end{cases}$$

One can represent the route of a G-word or an R-word $\pi$ as a graph with the vertices $\pi_1, \pi_2, \ldots, \pi_n$ ordered in a line, with only one path starting from $n$ drawn upon the line and going successively, if it’s possible, to $n-1$, $n-2$, $\ldots$, $1$ without crossings (see Figure 3 for an example). Denote $GW_{n,k}$ (resp. $RW_{n,k}$) the set of primitive G-words $\pi$ on $[n+2]$ (resp. primitive R-words $\pi$ on $[n+2]$) such that $\alpha_{n+2} = n+1-k$.

![Figure 3. The route of the G-word $\pi = 82546317$](image)

**Theorem 5.1.** The sequences $(GW_{n,k})_{1 \leq k \leq n}$ and $(RW_{n,k})_{1 \leq k \leq n}$ are Entringer families.

**Proof.** Use the bijection $\delta$ between $GW_n$ and $BT_n$ present in [MW09]. For $\pi$ a primitive G-word on $\{a_1, \ldots, a_{n+2}\}$ with $a_1 < \cdots < a_{n+2}$, denote by $\pi'$ the word $\pi_2 \ldots \pi_{n+1}$. If $\pi'$ is a word on $\{a_1, \ldots, a_n\}$, with $a_1 < \cdots < a_n$ and $a_n = \pi'_k$ for $k \in \{1, \ldots, n\}$, define $T = \alpha(\pi')$ as the tree with root $a_1$, from which two subgraphs go out, that are $\alpha(\pi'_1 \pi'_2 \ldots \pi'_{k-1})$ and $\alpha(\pi'_{k+1} \pi'_{k+2} \ldots \pi'_n)$ (eventually one of them or both are empty). The tree $\delta(\pi) = \alpha(\pi')$ is a binary increasing increasing tree and the application $\delta$ is a bijection from $GW_n$ to $BT_n$ (see [MW09] for further details).

Moreover, it is easy to see that the labels upon the minimal path of $T = \delta(\pi)$ are successively $(n+1-a_1), (n+1-a_2), \ldots, (n+1-a_m)$, where $a_1, \ldots, a_m$ ($a_1 > \cdots > a_m$) are the different values that appear in the route of $\pi$. Thus, the leaf of the minimal path is $k$. Then, $\delta$ is a bijection between $GW_{n,k}$ and $T_{n,k}$.

For example, one can construct the tree that corresponds with the G-word $\pi = 82546317$ with this construction:

![Tree example](image)

The analogous result for the R-word can be proved using the same method with the bijection $\delta'$ between $RW_n$ and $BT_n$ present in [MW09].
5.2. Interpretations in U-words. We introduce here two new Entringer families.

**Definition 5.2.** A \( U \)-word of length \( n \) is a sequence \( u = (u_i)_{1 \leq i \leq n} \) such that \( u_1 = 1 \) and \( u_i + u_{i-1} \leq i \) for \( i \in \{2, \ldots, n\} \). We denote by \( UW_n \) the set of U-words of length \( n \).

For example, the U-words of length 4 are:

\[
1111, 1112, 1113, 1121, 1122.
\]

Denote by \( UW_{n,k} \) the set of U-words \( (u_i) \in UW_n \) such that \( u_n = n + 1 - k \).

**Theorem 5.3.** The sequence \( (UW_{n,k})_{1 \leq k \leq n} \) is an Entringer family.

**Proof.** For any finite set \( X \), let \( \#X \) denotes its cardinality. For \( \pi \in DU_{n,k} \), let \( \gamma(\pi) = w^{RW} \), where \( w = w_1 \ldots w_n \) is the word defined by

\[
w_i = \begin{cases} 
\#\{j \geq \pi_i, j \not\in \{\pi_1, \pi_2, \ldots, \pi_{i-1}\}\}, & \text{if } i \text{ is odd}, \\
\#\{j \leq \pi_i, j \not\in \{\pi_1, \pi_2, \ldots, \pi_{i-1}\}\}, & \text{if } i \text{ is even}.
\end{cases}
\]

For example, if \( \pi = 6351724 \in DU_{7,6} \), then the word \( w \) is computed as follows:

- \( \{j \geq 6\} = \{6, 7\} \), so \( w_1 = 2 \),
- \( \{j \leq 3, j \not\in 6\} = \{1, 2, 3\} \), so \( w_2 = 3 \),
- \( \{j \geq 5, j \not\in 3, 6\} = \{5, 7\} \), so \( w_3 = 2 \),
- \( \{j \leq 1, j \not\in 3, 5, 6\} = \{1\} \), so \( w_4 = 1 \),
- \( \{j \geq 7, j \not\in 1, 3, 5, 6\} = \{7\} \), so \( w_5 = 1 \),
- \( \{j \leq 2, j \not\in 1, 3, 5, 6, 7\} = \{2\} \), so \( w_6 = 1 \),
- \( \{j \geq 4, j \not\in 1, 2, 3, 5, 6, 7\} = \{4\} \), so \( w_7 = 1 \).

Then, \( w = 23211111 \) and \( \gamma(\pi) = 1111232 \).

We show that the mapping \( \gamma \) is a bijection between \( DU_{n,k} \) and \( UW_{n,k} \). Following the construction, \( \gamma(\pi)_n = w_1 = n + 1 - \pi_1 = n + 1 - k \). Moreover, when \( \gamma(\pi)_i = w_{n+1-i} \) is written, \( n-i \) elements have been read in \( \pi \) before, thus the number of elements counted by \( \gamma(\pi)_i \) must be less than \( i \). Moreover, the numbers counted by \( \gamma(\pi)_{i-1} \) and \( \gamma(\pi)_i \) are in the \( n-i \) elements that have not been read in \( \pi \) and are two disjoint sets since \( \pi \) is down-up. Thus \( \gamma(\pi)_i + \gamma(\pi)_{i-1} \) must be less than \( i \). Finally, \( \gamma(\pi) \in UW_{n,k} \).

Conversely, if \( u \in UW_{n,k} \), the permutation \( \pi = \gamma^{-1}(u) \in DU_{n,k} \) can be recovered with:

- \( \pi_1 = n + 1 - u_n \)
- \( \forall n \geq 1, \pi_{2i} \) is the \( u_{n-2i+1} \)-st smallest element in \( [n] \setminus \{\pi_1, \ldots, \pi_{2i-1}\} \).
- \( \forall n \geq 1, \pi_{2i+1} \) is the \( u_{n-2i} \)-st greatest element in \( [n] \setminus \{\pi_1, \ldots, \pi_{2i}\} \).

We are done.

Denote by \( UW'_{n,k} \) the set of U-words \( (u_i) \in UW_n \) such that \( u_{n-1} + u_n = k \).

**Theorem 5.4.** The sequence \( (UW'_{n,k})_{1 \leq k \leq n} \) is an Entringer family.

**Proof.** There are two possibilities to prove this result.

Firstly, the mapping \( \gamma \) also induces a bijection between \( DU'_{n,k} \) and \( UW'_{n,k} \). For \( \pi \in DU'_{n,k} \), there exists \( j \in [n] \) such that \( \pi \in DU_{n,j} \), so we can define \( v = \gamma(\pi) \in UW_{n,j} \subset UW_n \). It suffices to show that \( v \in UW'_{n,k} \). In the construction of \( \gamma(\pi) \), \( v_n \) is the number of elements that are greater than \( \pi_1 \), and \( v_{n-1} \) is the number of elements that are less than \( \pi_2 \). Then \( v_n = n + 1 - \pi_1 \) and \( v_{n-1} = \pi_2 \), and \( v_{n-1} + v_n = n + 1 - (\pi_1 - \pi_2) = k \) since \( \pi \in DU'_{n,k} \).

Secondly, it is easy to construct a bijection \( \alpha : UW_{n,k} \rightarrow UW'_{n,k} \). For \( u = (u_1, \ldots, u_n) \in UW_{n,k} \), let \( \alpha(u) = (u_1, \ldots, u_{n-1}, n + 1 - u_{n-1} - u_n) \). Since \( u \in UW_{n,k} \), \( u_n - u_{n-1} \leq n + 1 \).
so we have $\alpha(u) \in UW_n$. Moreover, the last element $\alpha(u)_n = n + 1 - v_{n-1} - (n + 1 - k) = k - v_{n-1} = k - \alpha(u)_{n-1}$, so $\alpha(u) \in UW'_{n,k}$. The mapping $\alpha$ is then clearly a bijection between $UW_{n,k}$ and $UW'_{n,k}$.

It follows immediately from the above theorems that the Euler number $E_n$ is the number of U-words of length $n$ for all integer $n \geq 1$.

6. Concluding remarks

6.1. List of bijections for Entringer families. In what follows, we list all the twelve interpretations for Entringer families along with the bijections discussed in this paper:

1. the permutation $\pi \in DU_{n,k}$ such that $\pi_1 = k$,
2. the encoding sequence $\Delta \in ES_{n,k}$, obtained by $\Delta = \psi(\pi)$, where $\psi$ is the bijection described in Section 2, then $k$ is the first element read in $\Delta$,
3. the binary increasing increasing tree $T \in BT_{n,k}$, obtained by $T = \varphi(\Delta)$, where $\varphi$ is the bijection described in Section 5, then $k$ is the leaf of the minimal path of $T$,
4. the binary increasing increasing tree $T' \in BT'_{n,k}$, obtained by $T' = \varphi'(T)$, where $\varphi'$ is the bijection described in [KPP94, §6], then $k - 1$ is the parent of $n$ in $T'$,
5. the down-up permutation $\sigma \in DU'_{n,k}$, obtained by $\sigma = \theta(\pi)$, where $\theta$ is the bijection described in Subsection 4.2, then $k = n + 1 - \sigma_1 + \sigma_2$,
6. the min-max alternating permutation $\sigma' \in MM_{n,k}$, obtained by $\sigma' = \beta(\sigma)$, where $\beta$ is the bijection described in Subsection 4.3, then $k = n + 1 - |\sigma_1 - \sigma_2|$
7. the min-max alternating permutation $\tau_1 \in MM'_{n,k}$, obtained by $\tau_1 = \rho(\pi)$, where $\rho$ is the bijection described in Subsection 4.3, then $k$ is the term immediately before 1 (or $n$ if $\tau_1$ starts with 1),
8. the min-max alternating permutation $\tau_2 \in MM''_{n,k}$, obtained by $\tau_2 = \rho'(\tau_2)$, where $\rho'$ is the bijection described in Subsection 4.3, then $n + 1 - k$ is the term immediately after $n$ (or 1 if $\tau_2$ ends with $n$),
9. the G-word $\pi' \in GW_{n,k}$, obtained by $\pi' = \delta^{-1}(T)$, where $\delta$ is the bijection described in Subsection 5.1, then $n + 1 - k$ is the end of the route of $\pi'$,
10. the R-word $\pi'' \in RW_{n,k}$, obtained by $\pi'' = (\delta')^{-1}(T)$, where $\delta'$ is the bijection described in Subsection 5.1, then $n + 1 - k$ is the end of the route of $\pi''$, 
11. the sequence $u \in UW_{n,k}$, obtained by $u = \gamma(\pi)$, where $\gamma$ is the bijection described in Subsection 5.2, then $n + 1 - k$ is the last element of $u$,
12. the sequence $v \in UW'_{n,k}$, obtained by $v = \gamma(\sigma) = \alpha(u)$, where $\alpha$ and $\gamma$ are the bijections described in Subsection 5.2, then $k$ is the sum of the two last elements of $v$.

We summarize the bijections of this paper in the diagram of Figure 4 where at the left we gather all the models in down-up permutations, and at the right we gather the models in the increasing trees.

6.2. Illustration for $n = 4$. In Figure 5 we summarize twelve interpretations for $E_{4,k}$, $k \in \{2, 3, 4\}$. In every column, the corresponding elements are described via the different bijections mentioned in the paper. Moreover, in the table, boxes point out the statistic $k = \pi_1$ if $\pi \in DU_{n,k}$ and the corresponding statistics in the other models.

6.3. An open problem. Consider the so-called reduced tangent numbers $t_n = E_{2n+1}/2^n$. Poupart [Pou89] proved that $t_n$ is the number of 0-2 increasing trees (i.e., the trees in $BT_n$ such that every vertex has 0 or 2 children). However, it seems that there is no interpretation
à la André for \( t_n \) in down-up permutations. Furthermore, let \( t_{n,k} \) denote the number of 0-2 increasing trees such that the leaf of the minimal path is \( k \), then the sequence \( (t_{n,k}) \) is obviously a refinement of \( t_n \) as Entringer numbers are for Euler numbers.

Let \( s_n \) (resp. \( s_{n,k} \)) be the number of split-pair arrangements of \([n]\), that are arrangements \( \sigma \) of the multi-set \( \{0, 0, 1, 1, 2, 2, \ldots, n, n\} \) such that \( \sigma(1) = n \) (resp. \( \sigma(1) = \sigma(k + 1) = n \)) and, between the two occurrences of \( i \) in \( \sigma \) (\( 0 \leq i \leq n-1 \)), the number \( i+1 \) appears exactly once.

Recently, Graham and Zang [GZ08] proved that for \( 1 \leq k \leq n \), \( s_{n,k} = t_{n,k} \). In particular, \( s_n = t_n \). There is no bijective proof between Poupard’s model and Graham and Zang’s model.

Acknowledgement

We thank the two referees for their careful readings and helpful comments on a previous version of this paper. This work was partially supported by the French National Research Agency under the grant ANR-08-BLAN-0243-03.

References

[And79] D. André, Développement de sec x et tan x, C. R. Math. Acad. Sci. Paris 88 (1879), 965–979.
[Cal05] D. Callan, A note on downup permutations and increasing 0-1-2 trees, preprint (2009).
[Che08] D. Chebikin, Variations on descents and inversions in permutations, Electron. J. Combin. 15 (2008), no. 1, Research Paper 132, 34.
[Don75] R. Donaghey, Alternating permutations and binary increasing trees, J. Combinatorial Theory Ser. A, 18 (1975), 141–148.
[Ent66] R.C. Entringer, A combinatorial interpretation of the Euler and Bernoulli numbers, Nieuw Arch. Wisk. 14 (1966), 241–246.
[FS73] D. Foata and M.-P. Schützenberger, Nombres d’Euler et permutations alternantes, A survey of combinatorial theory [J.N. Srivastava et al., eds.], Amsterdam, North-Holland (1973), 173–187.
[GZ08] R. Graham and N. Zang, Enumerating split-pair arrangements, J. Combin. Theory Ser. A 115 (2008), no.2, 293–303.
[JV10] M. Josuat-Vergès, A \( q \)-enumeration of alternating permutations, European J. Combin. (2010), no. doi:10.1016/j.ejc.2010.01.008.
[Kem33] A. J. Kempner, On the shape of polynomial curves, Tôhoku Math. Journal 37 (1933) 347–362.
[KPP94] A. G. Kuznetsov, I. M. Pak, and A. E. Postnikov, Increasing trees and alternating permutations, Uspekhi Mat. Nauk 49 (1994), 79–110.
J. L. Martin, *The slopes determined by $n$ points in the plane*, Duke Math. J. 131 (2006), no.1, p.119–165.

J. L. Martin and J. D. Wagner, *Updown numbers and the initial monomials of the slope variety*, Electron. J. Combin. 16 (2009), no.1, Research Paper 82, 8pp.

J. Millar, N. J. A. Sloane and N. E. Young, *A new operation on sequences: the Boustrouphedon transform*, J. Combinatorial Theory, Series A 76(1):44–54 (1996).

C. Poupard, *De nouvelles significations énumératives des nombres d’Entringer*, Discrete Math. 38 (1982), 265–271.

C. Poupard, *Deux propriétés des arbres binaires ordonnés stricts*, European J. Combin. 10 (1989), 369–374.

C. Poupard, *Two other interpretations of the Entringer numbers*, European Journal of Combinatorics 18 (1997), 939–943.

L. Seidel, *Über eine einfache Entstehungsweise der Bernoullischen Zahlen und einiger verwandten Reihen*, Sitzungsber. Münch. Akad. 4 (1877), 157–187.

H. Shin and J. Zeng, The q-tangent and q-secant numbers via continued fractions, [arXiv:0911.4658](http://arxiv.org/abs/0911.4658) to appear in European Journal of Combinatorics, 2010.

R. Stanley, *A Survey of Alternating Permutations*, [arXiv:0912.4240](http://arxiv.org/abs/0912.4240) (2009).
|   | $k$ | 2 | 3 | 4 |
|---|---|---|---|---|
| (1) | $\pi \in DU_{4,k}$ | 1 4 3 | 2 4 1 | 3 1 4 | 4 2 3 1 | 4 1 3 2 |
| (2) | $\Delta \in ES_{4,k}$ | (2 1)* | (3 2)* | (3 2) | (4 3) | (4 3) |
| (3) | $T \in BT_{4,k}$ | 1 2 4 | 2 1 4 | 2 1 4 | 2 1 4 |
| (4) | $T' \in BT'_{4,k}$ | 2 4 3 | 2 4 3 | 2 4 3 | 2 4 3 |
| (5) | $\sigma \in DU'_{4,k}$ | 4 1 3 2 | 4 2 3 1 | 3 1 4 2 | 3 2 1 4 | 3 2 1 4 |
| (6) | $\sigma' \in MM_{4,k}$ | 1 4 2 3 | 1 3 2 4 | 3 1 4 2 | 2 3 1 4 | 2 3 1 4 |
| (7) | $\tau_1 \in MM'_{4,k}$ | 2 1 4 3 | 2 3 1 4 | 3 1 4 2 | (4) 1 3 2 4 | (4) 1 3 2 4 |
| (8) | $\tau_2 \in MM''_{4,k}$ | 2 1 4 3 | 2 1 4 3 | 3 1 4 2 | 3 1 4 2 | 3 1 4 2 |
| (9) | $\pi' \in GW_{4,k}$ | 6 3 4 2 1 5 | 6 4 2 3 1 5 | 6 2 3 4 1 5 | 6 4 3 2 1 5 | 6 2 4 3 1 5 |
| (10) | $\pi'' \in RW_{4,k}$ | 6 2 1 4 3 5 | 6 2 4 1 3 5 | 6 1 4 3 5 | 6 2 4 1 3 5 | 6 2 4 1 3 5 |
| (11) | $u \in UW_{4,k}$ | 1 1 1 | 1 1 2 | 1 1 2 | 1 1 2 | 1 1 2 |
| (12) | $v \in UW'_{4,k}$ | 1 1 1 | 1 1 2 | 1 1 2 | 1 1 2 | 1 1 2 |

Figure 5. Twelve interpretations for $E_{4,k}, 1 \leq k \leq 4$