Mild assumptions for the derivation of Einstein’s effective viscosity formula

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\textbf{ABSTRACT}

We provide a rigorous derivation of Einstein’s formula for the effective viscosity of dilute suspensions of \( n \) rigid balls, \( n \gg 1 \), set in a volume of size 1. So far, most justifications were carried under a strong assumption on the minimal distance between the balls: \( d_{\text{min}} \geq cn^{-\frac{3}{2}}, \) \( c > 0 \). We relax this assumption into a set of two much weaker conditions: one expresses essentially that the balls do not overlap, while the other one gives a control of the number of balls that are close to one another. In particular, our analysis covers the case of suspensions modeled by standard Poisson processes with almost minimal hardcore condition.

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\section{1. Introduction}

Mixtures of particles and fluids, called suspensions, are involved in many natural phenomena and industrial processes. The understanding of their rheology, notably the so-called effective viscosity \( \mu_{\text{eff}} \) induced by the particles, is therefore crucial. Many experiments or simulations have been carried out to determine \( \mu_{\text{eff}} \) [1]. For \( \lambda \) large enough, they seem to exhibit some generic behavior, in terms of the ratio between the solid volume fraction \( \lambda \) and the maximal flowable solid volume fraction \( \lambda_c \), cf. [1]. Still, a theoretical derivation of the relation \( \mu_{\text{eff}} = \mu_{\text{eff}}(\lambda/\lambda_c) \) observed experimentally is missing, due to the complex interactions involved: hydrodynamic interactions, direct contacts, … Mathematical works related to the analysis of suspensions are mostly limited to the dilute regime, that is when \( \lambda \) is small.

In these mathematical works, the typical model under consideration is as follows. One considers \( n \) rigid balls \( B_i = \overline{B(x_i, r_n)}, 1 \leq i \leq n \), in a fixed compact subset of \( \mathbb{R}^3 \), surrounded by a viscous fluid.

The inertia of the fluid is neglected, leading to the Stokes equations

\begin{equation}
\begin{aligned}
-\mu \Delta u_n + \nabla p_n &= f_n, \quad x \in \Omega_n = \mathbb{R}^3 \setminus \bigcup B_i, \\
\text{div} u_n &= 0, \quad x \in \Omega_n, \\
\left. u_n \right|_{B_i} &= u_{n,i} + \omega_{n,i} \times (x - x_i).
\end{aligned}
\end{equation}
The last condition expresses a no-slip condition at the rigid spheres, where the velocity is given by some translation velocities $u_{n,i}$ and some rotation vectors $\omega_{n,i}$. We neglect the inertia of the balls: the $2n$ vectors $u_{n,i}, \omega_{n,i}$ can then be seen as Lagrange multipliers for the $2n$ conditions

$$\int_{\partial B_i} \sigma_\mu(u, p) \nu = -\int_{B_i} f_n, \quad \int_{\partial B_i} \sigma_\mu(u, p) \nu \times (x - x_i) = -\int_{B_i} (x - x_i) \times f_n$$

(2)

where $\sigma_\mu = 2\mu D(u) \nu - p \nu$ is the usual Newtonian tensor, and $\nu$ the normal vector pointing outward $B_i$.

The general belief is that one should be able to replace (1)–(2) by an effective Stokes model, with a modified viscosity taking into account the average effect of the particles:

$$\begin{cases}
-\text{div}(2\mu_{eff} Du_{eff}) + \nabla p_{eff} = f, & x \in \mathbb{R}^3, \\
\text{div} u_{eff} = 0, & x \in \mathbb{R}^3,
\end{cases}$$

(3)

with $D = \frac{1}{2}(\nabla + \nabla^t)$ the symmetric gradient. Of course, such average model can only be obtained asymptotically, namely when the number of particles $n$ gets very large. Moreover, for averaging to hold, it is very natural to impose some averaging on the distribution of the balls itself. Our basic hypothesis will therefore be the existence of a limit density, through

$$\frac{1}{n} \sum_i \delta_{x_i} \rightharpoonup \rho(x) dx \quad \text{weakly in the sense of measures} \quad (A0)$$

where $\rho \in L^\infty(\mathbb{R}^3)$ is assumed to be zero outside a smooth open bounded set $\mathcal{O}$. After playing on the length scale, we can always assume that $|\mathcal{O}| = 1$. Of course, we expect $\mu_{eff}$ to be different from $\mu$ only in this region $\mathcal{O}$ where the particles are present.

The volume fraction of the balls is then given by $\lambda = \frac{4}{3}\pi n r_3^3$. We shall consider the case where $\lambda$ is small (dilute suspension), but independent of $n$ so as to derive a non-trivial effect as $n \to +\infty$. The mathematical questions that follow are:

- Q1: Can we approximate system (1)–(2) by a system of the form (3) for large $n$?
- Q2: If so, can we provide a formula for $\mu_{eff}$ inside $\mathcal{O}$? In particular, for small $\lambda$, can we derive an expansion

$$\mu_{eff} = \mu + \lambda \mu_1 + \ldots ?$$

Regarding Q1, the only work we are aware of is the recent paper [2]. It shows that $u_n$ converges to the solution $u_{eff}$ of an effective model of the type (2), under two natural conditions:

i. the balls satisfy the separation condition $\inf_{i \neq j} |x_i - x_j| \geq Mr_n, M > 2$. Note that this is a slight reinforcement of the natural constraint that the balls do not overlap.
ii. the centers of the balls are obtained from a stationary ergodic point process.

We refer to [2] for all details. Note that in the scalar case, with the Laplacian instead of the Stokes operator, similar results can be found in [3], paragraph 8.6.
Q2, and more broadly quantitative aspects of dilute suspensions, have been studied for long. The pioneering work is due to Einstein [4]. By neglecting the interaction between the particles, he computed a first order approximation of the effective viscosity of homogeneous suspensions:

$$\mu_{\text{eff}} = \left(1 + \frac{5}{2} \lambda\right) \mu$$ \quad in \( \mathcal{O} \).

This celebrated formula was confirmed experimentally afterwards. It was later extended to the inhomogenous case, with formula

$$\mu_{\text{eff}} = \left(1 + \frac{5}{2} \lambda \rho\right) \mu,$$

see [5], page 16]. Further works investigated the \( O(\lambda^2) \) approximation of the effective viscosity, cf. [6] and the recent analysis [7,8].

Our concern in the present paper is the justification of Einstein’s formula. To our knowledge, the first rigorous studies on this topic are [9] and [10]: they rely on homogenization techniques, and are restricted to suspensions that are periodically distributed in a bounded domain. A more complete justification, still in the periodic setting but based on variational principles, can be found in [11]. Recently, the periodicity assumption was relaxed in [12,13], and replaced by an assumption on the minimal distance:

There exists an absolute constant \( c \), such that \( \forall n, \forall 1 \leq i \neq j \leq n, \quad |x_i - x_j| \geq cn^{-\frac{3}{2}}. \) (A1)

For instance, introducing the solution \( u_E \) of the Einstein’s approximate model

$$-\text{div}(2\mu_{E}Du_{E}) + \nabla \rho_{E} = f, \quad \text{div} \ u = 0 \quad \text{in} \ \mathbb{R}^3$$

with \( \mu_{E} = \left(1 + \frac{5}{2} \lambda \rho\right) \mu \), it is shown in [12] that for all \( 1 \leq \rho < \frac{3}{2} \),

$$\limsup_{n \to \infty} \|u_n - u_E\|_{L^p_\rho(\mathbb{R}^3)} = O(\lambda^{1+\theta}), \quad \theta = \frac{1}{P} - \frac{2}{3}.$$ (5)

We refer to [12] for refined statements, including quantitative convergence in \( n \) and treatment of polydisperse suspensions.

Although it is a substantial gain over the periodicity assumption, hypothesis (A1) on the minimal distance is still strong. In particular, it is much more stringent that the condition that the rigid balls can not overlap. Indeed, this latter condition reads: \( \forall i \neq j, |x_i - x_j| \geq 2r_n \), or equivalently \( |x_i - x_j| \geq c \lambda^{1/3} n^{-\frac{4}{3}}, \) with \( c = 2(\frac{2n}{3})^{1/3} \). It follows from (A1) at small \( \lambda \). On the other hand, one could argue that a simple non-overlapping condition is not enough to ensure the validity of Einstein’s formula. Indeed, it is based on neglecting interaction between particles, which is incompatible with too much clustering in the suspension. Still, one can hope that if the balls are not too close from one another on average, the formula still holds.

This is the kind of result that we prove here. Namely, we shall replace (A1) by a set of two relaxed conditions:

There exists \( M > 2 \), such that \( \forall n, \forall 1 \leq i \neq j \leq n, \quad |x_i - x_j| \geq Mr_n. \) (B1)
There exist $C, \alpha > 0$, such that $\forall \eta > 0, \# \{(i, j) \mid |x_i - x_j| \leq \eta n^{-\frac{1}{2}}\} \leq Cn^{\alpha} n$ \quad (B2)

Note that (B1) is slightly stronger than the non-overlapping condition, and was already present in the work [2] to ensure the existence of an effective model. It is possible to relax this condition into a moment bound on the particle separation, see Remark 3 and Section 5. As regards (B2), one can show that it is satisfied almost surely as $n \to \infty$ in the case when the particle positions are generated by a stationary ergodic point process if the process does not favor too much close pairs of points. In particular, it is satisfied by a (hard-core) Poisson point process for $\alpha = 3$. Moreover, (B2) is satisfied for $\alpha = 3$ with probability tending to 1 as $n \to \infty$ for independent and identically distributed particles. We postpone further discussion to Section 6.

Under these general assumptions, we obtain:

**Theorem 1.** Let $\lambda > 0, f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. For all $n$, let $r_n$ such that $\lambda = \frac{4\pi}{3} n m_3$, let $f_n \in L^5(\mathbb{R}^3)$, and $u_n$ in $\dot{H}^1(\mathbb{R}^3)$ the solution of (1)-(2). Assume (A0)-(B1)-(B2), and that $f_n \to f$ in $L^5(\mathbb{R}^3)$. Then, there exists $p_{\min} > 1$ such that for any $p < p_{\min}$, any $q < \frac{3p_{\min}}{3-p_{\min}}$ one can find $\delta > 0$ with the estimate

$$||\nabla(u - u_E)||_{L^p(\mathbb{R}^3)} + \limsup_{n \to +\infty} ||u_n - u_E||_{L^q(K)} = O(\lambda^{1+\delta}), \quad \forall K \subset \mathbb{R}^3, \quad \text{as } \lambda \to 0,$$

where $u$ is any weak accumulation point of $u_n$ in $\dot{H}^1(\mathbb{R}^3)$ and $u_E$ satisfies Einstein’s approximate model (5).

Here, we use the notation $\dot{H}^1(\mathbb{R}^3)$ for the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3) = \{w \in L^6(\mathbb{R}^3) : \nabla w \in L^2(\mathbb{R}^3)\}$ equipped with the $L^2$ norm of the gradient.

**Remark 2.** The following explicit formula for $p_{\min}$ and $\delta$ will be obtained in the proof of the theorem:

$$p_{\min} = 1 + \frac{\alpha}{6 + \alpha}, \quad \delta = \frac{1}{r} - \frac{6}{6 + (2 - r)\alpha}, \quad r = \max\left\{p, \frac{3q}{3 + q}\right\}.$$

**Remark 3.** Since the preprint of our paper, several further results have appeared which we briefly discuss in this remark.

In [14], version 1, an extensive study of the effective viscosity at low volume fraction was performed in the context of stationary ergodic particle configurations, under suitable versions of (B1)-(B2). It includes results on the $O(\lambda^2)$ and higher order corrections, see also the recent paper [15]. As regards the $O(\lambda)$ Einstein’s formula, a result analogous to Theorem 1 was shown with methods of a more probabilistic flavor.

It was subsequently shown in [16] and [14], version 2 that both the existence of an effective viscosity and the Einstein’s formula hold when relaxing condition (B1) into a moment bound on the particle separation. We will argue in Section 5 that our main result still holds under similar milder assumption.

Finally, in [17], results have been obtained concerning the coupling of Einstein’s formula to the time evolution of sedimenting particles.

The rest of the paper is dedicated to the proof of Theorem 1.
2. Main steps of proof

To prove Theorem 1, we shall rely on an enhancement of the general strategy explained in [18], to justify various effective models for conducting and fluid media. Let us point out that one of the examples considered in [18] is the scalar version of (1)-(2). It leads to a proof of a scalar analogue of Einstein’s formula, under assumptions (A0), (B1), plus an abstract assumption intermediate between (A1) and (B2). We refer to the discussion at the end of [18] for more details. Nevertheless, to justify the effective fluid model (5) under the mild assumption (B2) will require several new steps. The main difficulty will be to handle particles that are close to one another, and will involve sharp $L^p$ estimates similar to those of [8].

Concretely, let $\varphi$ be a smooth and compactly supported divergence-free vector field. For each $n$, we introduce the solution $\phi_n \in \dot{H}^1(\mathbb{R}^3)$ of

$$
-\text{div}(2\mu D\phi_n) + \nabla q_n = \text{div}(5\lambda \mu \rho D\varphi) \quad \text{in } \Omega_n, \\
\text{div}\phi_n = 0 \quad \text{in } \Omega_n, \\
\phi_n = \varphi + \phi_{n,i} + w_{n,i} \times (x - x_i) \quad \text{in } B_i, 1 \leq i \leq n
$$

where the constant vectors $\phi_{n,i}, w_{n,i}$ are associated to the constraints

$$
\int_{\partial B} \sigma_\mu(\phi_n, q_n) \nu = -\int_{\partial B} 5\lambda \mu \rho D\varphi \nu, \\
\int_{\partial B} (x - x_i) \times \sigma_\mu(\phi_n, q_n) \nu = -\int_{\partial B} (x - x_i) \times 5\lambda \mu \rho D\varphi \nu.
$$

Testing (1) with $\varphi - \phi_n$, we find after a few integration by parts that

$$
\int_{\mathbb{R}^3} 2\mu E D u_n : D\varphi = \int_{\mathbb{R}^3} f_n \cdot \varphi - \int_{\mathbb{R}^3} f_n \cdot \phi_n.
$$

Testing (5) with $\varphi$, we find

$$
\int_{\mathbb{R}^3} 2\mu E D u_E : D\varphi = \int_{\mathbb{R}^3} f \cdot \varphi.
$$

Combining both, we end up with

$$
\int_{\mathbb{R}^3} 2\mu D(u_n - u_E) : D\varphi = \int_{\mathbb{R}^3} (f_n - f) \cdot \varphi - \int_{\mathbb{R}^3} f_n \cdot \phi_n.
$$

We remind that vector fields $u_n, u_E, \phi_n$ depend implicitly on $\lambda$.

The main point will be to show

**Proposition 4.** There exists $p_{\text{min}} > 1$ such that for all $p < p_{\text{min}}$, there exists $\delta > 0$ and $C > 0$, independent of $\varphi$, such that

$$
\limsup_{n \to \infty} \left| \int_{\mathbb{R}^3} f_n \cdot \phi_n \right| \leq C \lambda^{1+\delta} \| \nabla \varphi \|_{L^p}, \quad p' = \frac{p}{p - 1}.
$$

Let us show how the theorem follows from the proposition. First, by standard energy estimates, we find that $u_n$ is bounded in $\dot{H}^1(\mathbb{R}^3)$ uniformly in $n$. Let $u = \lim u_n$ be a weak accumulation point of $u_n$ in this space. Taking the limit in (8), we get

$$
\int_{\mathbb{R}^3} 2\mu E D(u - u_E) : D\varphi = \langle R, \varphi \rangle
$$
where \( \langle R, \phi \rangle = \lim_{k \to +\infty} \int_{\mathbb{R}^3} f_{n_k} \cdot \phi_{n_k} \). Recall that \( \phi \) is an arbitrary smooth and compactly supported divergence-free vector field and that such functions are dense in the homogeneous Sobolev space of divergence-free functions \( W^{1, p}_\sigma \). Thus, Proposition 4 implies that \( R \) is an element of \( W^{-1, p}_\sigma \) with \( \| R \|_{W^{-1, p}_\sigma} = O(\lambda^{1+\delta}) \). Moreover, the previous identity is the weak formulation of

\[
-\text{div}(2\mu E D(u - u_E)) + \nabla q = R, \quad \text{div}(u - u_E) = 0 \quad \text{in } \mathbb{R}^3.
\]

Writing these Stokes equations with non-constant viscosity as

\[
-\mu \Delta (u - u_E) + \nabla q = R + \text{div}(5\lambda \mu \rho D(u - u_E)), \quad \text{div}(u - u_E) = 0 \quad \text{in } \mathbb{R}^3.
\]

and using standard estimates for this system, we get

\[
\| \nabla (u - u_E) \|_{L^p} \leq C\left( \| R \|_{W^{-1, p}_\sigma} + \lambda \| \nabla (u - u_E) \|_{L^p} \right).
\]

For \( \lambda \) small enough, the last term is absorbed by the left-hand side, and finally

\[
\| \nabla (u - u_E) \|_{L^p(\mathbb{R}^3)} \leq C\lambda^{1+\delta}
\]

which implies the first estimate of the theorem. Then, by Sobolev imbedding, for any \( q \leq \frac{3p}{3-p} \), and any compact \( K \),

\[
\| u - u_E \|_{L^q(K)} \leq C_{K,q} \lambda^{1+\delta}.
\]

We claim that \( \limsup_{n \to \infty} \| u_n - u_E \|_{L^q(K)} \leq C_{K,q} \lambda^{1+\delta} \). Otherwise, there exists a subsequence \( u_{n_k} \) and \( \varepsilon > 0 \) such that \( \| u_{n_k} - u_E \|_{L^q(K)} \geq C_{K,q} \lambda^{1+\delta} + \varepsilon \) for all \( k \). Denoting by \( u \) a (weak) accumulation point of \( u_{n_k} \) in \( H^1 \), Rellich’s theorem implies that, for a subsequence still denoted \( u_{n_k} \), \( \| u_{n_k} - u \|_{L^q(K)} \to 0 \), because \( q < 6 \) (for \( p_{\min} \) taken small enough). Combining this with (10), we reach a contradiction. As \( p \) is arbitrary in \( (1, p_{\min}) \), \( q \leq \frac{3p}{3-p} \) is arbitrary in \( (1, \frac{3p}{3-p}) \). The last estimate of the theorem is proved.

It remains to prove Proposition 4. Therefore, we need a better understanding of the solution \( \phi_n \) of (6)-(7). Neglecting any interaction between the balls, a natural attempt is to approximate \( \phi_n \) by

\[
\phi_n \approx \phi_{\mathbb{R}^3} + \sum_i \phi_{i,n}
\]

where \( \phi_{\mathbb{R}^3} \) is the solution of

\[
-\mu \Delta \phi_{\mathbb{R}^3} + \nabla p_{\mathbb{R}^3} = \text{div}(5\lambda \mu \rho D\phi), \quad \text{div} \phi_{\mathbb{R}^3} = 0 \quad \text{in } \mathbb{R}^3
\]

and \( \phi_{i,n} \) solves

\[
-\mu \Delta \phi_{i,n} + \nabla p_{i,n} = 0, \quad \text{div} \phi_{i,n} = 0 \quad \text{outside } B_i, \quad \phi_{i,n}|_{B_i}(x) = D\phi(x_i) (x - x_i)
\]

Roughly, the idea of approximation (11) is that \( \phi_{\mathbb{R}^3} \) adjusts to the source term in (6), while for all \( i \), \( \phi_{i,n} \) adjusts to the boundary condition at the ball \( B_i \). Indeed, using a Taylor expansion of \( \phi \) at \( x_i \), and splitting \( \nabla \phi(x_i) \) between its symmetric and skew-symmetric part, we find

\[
\phi_n|_{B_i}(x) \approx D\phi(x_i) (x - x_i) + \text{rigid vector field} = \phi_{i,n}|_{B_i}(x) + \text{rigid vector field}.
\]
Moreover, $\phi_{i,n}$ can be shown to generate no force and torque, so that the extra rigid vector fields (whose role is to ensure the no-force and no-torque conditions), should be small.

Still, approximation (11) may be too crude: the vector fields $\phi_{j,n}, j \neq i$, have a non-trivial contribution at $B_i$, and for the balls $B_j$ close to $B_i$, which are not excluded by our relaxed assumption (B1), these contributions may be relatively big. We shall therefore modify the approximation, restricting the sum in (11) to balls far enough from the others.

Therefore, for $\eta > 0$, we introduce a good and a bad set of indices:

$$G_\eta = \{ 1 \leq i \leq n, \forall j \neq i, |x_i - x_j| \geq \eta n^{-1} \}, \quad B_\eta = \{ 1, \ldots, n \} \setminus G_\eta. \quad (14)$$

The good set $G_\eta$ corresponds to balls that are at least $\eta n^{-\frac{1}{4}}$ away from all the others. The parameter $\eta > 0$ will be specified later: we shall consider $\eta = \kappa^\theta$ for some appropriate power $0 < \theta < 1/3$.

We set

$$\phi_{app,n} = \phi_{\mathbb{R}^3} + \sum_{i \in G_\eta} \phi_{i,n} \quad (15)$$

Note that $\phi_{\mathbb{R}^3}$ and $\phi_{i,n}$ are explicit:

$$\phi_{\mathbb{R}^3} = U \ast \text{div}(5\lambda \rho D\phi), \quad U(x) = \frac{1}{8\pi} \left( \frac{I}{|x|} + \frac{x \otimes x}{|x|^3} \right)$$

and

$$\phi_{i,n} = r_n V[D\phi(x_i)] \left( \frac{x - x_i}{r_n} \right) \quad (16)$$

where for all trace-free symmetric matrix $S$, $V[S]$ solves

$$-\Delta V[S] + \nabla P[S] = 0, \text{div} V[S] = 0 \quad \text{outside } B(0,1), \quad V[S](x) = Sx, x \in B(0,1).$$

with expressions

$$V[S] = \frac{5}{2} S : (x \otimes x) \frac{x}{|x|^2} + Sx \frac{1}{|x|^3} - \frac{5}{2} (S : x \otimes x) \frac{x}{|x|^7}, \quad P[S] = 5 S : x \otimes x \frac{x}{|x|^5}.$$ 

Eventually, we denote

$$\psi_n = \phi_n - \phi_{app,n}.$$ 

Tedious but straightforward calculations show that

$$-\text{div}(\sigma_{\mu}(V[S], P[S])) = 5\mu S x^1 = -\text{div}(5\mu S1_{B(0,1)}) \quad \text{in } \mathbb{R}^3$$

where $s^1$ denotes the surface measure at the unit sphere. It follows that

$$-\mu \Delta \phi_{app,n} + \nabla p_{app,n} = \text{div}(5\lambda \mu D\phi - \sum_{i \in G_\eta} 5\mu D\phi(x_i) 1_{B_i}), \quad \text{div}\phi_{app,n} = 0 \quad \text{in } \mathbb{R}^3,$$

(17)
Moreover, for all \(1 \leq i \leq n\),
\[
\int_{\partial B_i} \sigma_\mu (\phi_{\text{app},n}, p_{\text{app},n}) \nu = -\int_{\partial B_i} 5\lambda \mu D\phi \nu, \\
\int_{\partial B_i} (x - x_i) \times \sigma_\mu (\phi_{\text{app},n}, p_{\text{app},n}) \nu = -\int_{\partial B_i} (x - x_i) \times 5\lambda \mu D\phi \nu.
\]
Hence, the remainder \(\psi_n\) satisfies
\[
-\mu \Delta \psi_n + \nabla q_n = 0 \text{ in } \Omega_n, \\
\text{div} \psi_n = 0 \text{ in } \Omega_n,
\]
where the constant vectors \(\psi_{n,i}, w_{n,i}\) are associated to the constraints
\[
\int_{\partial B_i} \sigma_\mu (\psi_n, q_n) \nu = 0, \\
\int_{\partial B_i} (x - x_i) \times \sigma_\mu (\psi_n, q_n) \nu = 0.
\]
Regarding \(\phi_{\text{app},n}\) and \(\psi_n\) will be postponed to sections 3 and 4, respectively.

**Proposition 5.** For all \(p \geq 1\),
\[
\limsup_{n \to \infty} \left| \int_{\mathbb{R}^d} f \cdot \phi_{\text{app},n} \right| \leq C_{p,f} (\lambda \eta^2)^{\frac{1}{p}} \| \nabla \phi \|_{L^p}.
\]

**Proposition 6.** For all \(1 < p < 2\), there exists \(c > 0\) independent of \(\lambda\) such that for all \(1 \geq \eta \geq c\lambda^{1/3}\),
\[
\limsup_{n \to \infty} \left| \int_{\mathbb{R}^d} f \cdot \psi_n \right| \leq C_{p,f} \lambda^2 (\lambda^{1+\frac{2}{p}} \eta^{-\frac{1}{p}} + (\eta^2 \lambda)^{\frac{2}{p}}) \| \nabla \phi \|_{L^p}.
\]

Let us explain how to deduce Proposition 4 from these two propositions. Let \(1 < p < 2\). By standard estimates, we see that \(\phi_n\) is bounded uniformly in \(n\) in \(\dot{H}^1\). It follows that
\[
\limsup_{n \to \infty} \left| \int_{\mathbb{R}^d} f_n \cdot \phi_n \right| = \limsup_{n \to \infty} \left| \int_{\mathbb{R}^d} f \cdot \phi_n \right| \leq \limsup_{n \to \infty} \left| \int_{\mathbb{R}^d} f \cdot \phi_{\text{app},n} \right| + \limsup_{n \to \infty} \left| \int_{\mathbb{R}^d} f \cdot \psi_n \right| \leq C_{p,f} (\lambda \eta^2)^{\frac{1}{p}} + \lambda^2 (\lambda^{1+\frac{2}{p}} \eta^{-\frac{1}{p}} + (\eta^2 \lambda)^{\frac{2}{p}}) \| \nabla \phi \|_{L^p}.
\]
To conclude, we adjust properly the parameters \(p\) and \(\eta\). We look for \(\eta\) in the form \(\eta = \lambda^\theta\), with \(0 < \theta < \frac{1}{2}\), so that the lower bound on \(\eta\) needed in Proposition 6 will be satisfied for small enough \(\lambda\). Then, we choose \(p_{\min} = 1 + \frac{2}{6+\theta}\), and for \(p < p_{\min}\) we choose \(\theta = \frac{2p}{6+2(p-1)}\). It is straightforward to check that this yields a right-hand side \(\lambda^{1+\delta}\) with \(\delta = \frac{1}{p} - \frac{1}{6+(2-p)\theta}\) in accordance with Remark 2.

### 3. Bound on the approximation

This section is devoted to the proof of Proposition 5. We decompose
\[
\phi_{\text{app},n} = \phi_{1,\text{app},n} + \phi_{2,\text{app},n} + \phi_{3,\text{app},n}
\]
where
\[ -\mu \Delta \phi^1_{\text{app},n} + \nabla p^1_{\text{app},n} = \text{div} \left( 5\lambda \mu D\varphi - \sum_{1 \leq i \leq n} 5\mu D\varphi(x_i)1_{B_i} \right), \quad \text{div} \phi^1_{\text{app},n} = 0 \quad \text{in} \ \mathbb{R}^3, \]
\[ -\mu \Delta \phi^2_{\text{app},n} + \nabla p^2_{\text{app},n} = \text{div} \left( \sum_{i \in B_n} 5\mu D\varphi(x)1_{B_i} \right), \quad \text{div} \phi^1_{\text{app},n} = 0 \quad \text{in} \ \mathbb{R}^3, \]
\[ -\mu \Delta \phi^3_{\text{app},n} + \nabla p^3_{\text{app},n} = \text{div} \left( \sum_{i \in B_n} 5\mu(D\varphi(x_i) - D\varphi(x))1_{B_i} \right), \quad \text{div} \phi^1_{\text{app},n} = 0 \quad \text{in} \ \mathbb{R}^3. \]

By standard energy estimates, \( \phi^k_{\text{app},n} \) is seen to be bounded in \( n \) in \( H^1 \), for all \( 1 \leq k \leq 3 \). We shall prove next that \( \phi^k_{\text{app},n} \) and \( \phi^3_{\text{app},n} \) converge in the sense of distributions to zero, while for any \( f \) with \( D(\Delta)^{-1} f \in L^\infty \) (\( \mathbb{P} \) denoting the standard Helmholtz projection), for any \( p \geq 1 \),
\[
\left| \int_{\mathbb{R}^3} f \cdot \phi^2_{\text{app},n} \right| \leq C_{f,p}(\lambda \eta^3)\|\nabla \varphi\|_{L^p}, \quad p' = \frac{p}{p-1}.
\] (21)

Proposition 5 follows easily from those properties.

We start with

**Lemma 7.** Under assumption (A0), \( \sum_{1 \leq i \leq n} D\varphi(x_i)1_{B_i} \to \lambda \rho D\varphi \) weakly* in \( L^\infty \).

**Proof.** As the balls are disjoint, \( \sum_{1 \leq i \leq n} D\varphi(x_i)1_{B_i} \leq \|D\varphi\|_{L^\infty} \). Let \( g \in C_c(\mathbb{R}^3) \), and denote \( \delta_n = \frac{1}{n} \sum \delta_{x_i} \) the empirical measure. We write
\[
\int_{\mathbb{R}^3} \sum_{1 \leq i \leq n} D\varphi(x_i)1_{B_i}(y)g(y)dy = \sum_{1 \leq i \leq n} D\varphi(x_i) \int_{B(0,r_n)} g(x_i + y)dy
\]
\[= n \int_{\mathbb{R}^3} D\varphi(x) \int_{B(0,r_n)} g(x + y)dyd\delta_n(x) \]
\[= nr^3 \int_{\mathbb{R}^3} \int_{B(0,1)} g(x + r_nz)dzd\delta_n(x). \]

The sequence of bounded continuous functions \( x \to \int_{B(0,1)} g(x + r_nz)dz \) converges uniformly to the function \( x \to \frac{4\pi}{3} g(x) \) as \( n \to +\infty \). We deduce:
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} \sum_{1 \leq i \leq n} D\varphi(x_i)1_{B_i}(y)g(y)dy = \lim_{n \to \infty} \lambda \int_{\mathbb{R}^3} D\varphi(x)g(x)d\delta_n(x) = \lambda \int_{\mathbb{R}^3} D\varphi(x)g(x)\rho(x)dx
\]
where the last equality comes from (A0). The lemma follows by density of \( C_c \) in \( L^1 \).

Let now \( h \in C_c(\mathbb{R}^3) \) and \( \nu = (\Delta)^{-1} \mathbb{P} h \). We find
\[
\langle \phi^1_{\text{app},n}, h \rangle = \langle \phi^1_{\text{app},n}, \Delta \nu \rangle = \langle \Delta \phi^1_{\text{app},n}, \nu \rangle = \int_{\mathbb{R}^3} (5\lambda \mu D\varphi - \sum_{1 \leq i \leq n} 5\mu D\varphi(x_i)1_{B_i}) \cdot D\nu \to 0 \quad \text{as} \quad n \to +\infty
\]
where we used the previous lemma and the fact that \( D\nu \) belongs to \( L^1_{\text{loc}} \) and \( \varphi \) has compact support. Hence, \( \phi^1_{\text{app},n} \) converges to zero in the sense of distributions. As regards \( \phi^3_{\text{app},n} \), we notice that
Proposition 3.2. More precisely, we use a duality argument to prove the following proposition, corresponding to [8].

\[ \left\| \sum_{i \in \mathcal{B}_n} 5\mu(D\phi(x) - D\phi(x_i))1_{B_i} \right\|_{L^1} \leq \left\| \nabla^2 \phi \right\|_{L^\infty} \sum_{1 \leq i \leq n} \int_{B_i} |x - x_i| \, dx \]
\[ \leq \left\| \nabla^2 \phi \right\|_{L^\infty} \lambda r_n \to 0 \quad \text{as } n \to +\infty \]

Using the same duality argument as for \( \phi_{app,n}^1 \) (see also below), we get that \( \phi_{app,n}^3 \) converges to zero in the sense of distributions.

It remains to show (21). We use a simple Hölder estimate, and write for all \( p \geq 1 : \)

\[ \left\| \sum_{i \in \mathcal{B}_n} 5\mu D\phi 1_{B_i} \right\|_{L^1} \leq 5\mu \left\| \sum_{i \in \mathcal{B}_n} 1_{B_i} \right\|_{L^p} \left\| D\phi \right\|_{L^{p'}} = 5\mu \left( \text{card}\mathcal{B}_n \frac{4\pi}{3} \right)^\frac{1}{p} \left\| D\phi \right\|_{L^{p'}} \]
\[ \leq C(n^2 \lambda^2)^{\frac{1}{p}} \left\| D\phi \right\|_{L^{p'}} \]

where the last inequality follows from (B2). Denoting \( \nu = (\Delta)^{-1}Pf \), we have this time

\[ \int_{\mathbb{R}^3} f \cdot \phi_{app,n}^2 = \int_{\mathbb{R}^3} D\nu \cdot \sum_{i \in \mathcal{B}_n} 5\mu D\phi 1_{B_i} \leq C \left\| D\nu \right\|_{L^\infty} (\eta^2 \lambda^2)^{\frac{1}{p}} \left\| D\phi \right\|_{L^{p'}} \]

which implies (21).

4. Bound on the remainder

We focus here on estimates for the remainder \( \psi_n = \phi_n - \phi_{app,n} \), which satisfies (18)-(19). The proof of Proposition 6 relies on properties of the solutions of the system

\[ -\mu \Delta \psi + \nabla p = 0, \quad \text{div} \psi = 0 \quad \text{in } \Omega_n, \quad D\psi = D\tilde{\psi} \quad \text{in } B_i, \quad 1 \leq i \leq n \]  

(22)

together with the constraints

\[ \int_{\partial B_i} \sigma_\mu(\psi, p) \nu = \int_{\partial B_i} (x - x_i) \times \sigma_\mu(\psi, p) \nu = 0, \quad 1 \leq i \leq n. \]

(23)

More precisely, we use a duality argument to prove the following proposition, corresponding to [8], Proposition 3.2.

**Proposition 8.** Let \( q > 3 \). Then, under assumption (B1) for all \( g \in L^q(\mathbb{R}^3) \) and all \( \tilde{\psi} \in H^1(\cup_i B_i) \), the weak solution \( \psi \in \dot{H}^1(\mathbb{R}^3) \) to (22)-(23) satisfies

\[ \left\| \int_{\mathbb{R}^3} g \psi \right\| \leq C_g \lambda^2 \left\| D\tilde{\psi} \right\|_{L^2(\cup_i B_i)}. \]

(24)

**Proof.** We introduce the solution \( u_g \) of the Stokes equation

\[ -\Delta u_g + \nabla p_g = g, \quad \text{div} g = 0, \quad \text{in } \mathbb{R}^3. \]

(25)

As \( g \in L^q, \quad q > 3 \), \( u_g \in W^{2, q}_{\text{loc}}, \) so that \( D(u_g) \) is continuous. Integrations by parts yield

\[ \int_{\mathbb{R}^3} g \psi = \int_{\mathbb{R}^3} (-\Delta u_g + \nabla p_g) \psi = 2 \int_{\mathbb{R}^3} D(u_g) : D(\psi) \]
\[ = 2 \int_{\cup_i B_i} D(u_g) : D(\psi) - \sum_i \int_{\partial B_i} u_g \cdot \sigma(\psi, p) \nu \]
\[ = 2 \int_{\cup_i B_i} D(u_g) : D(\psi) - \sum_i \int_{\partial B_i} (u_g + u_g^i + \omega_g^i \times (x - x_i)) \cdot \sigma(\psi, p) \nu \]
for any constant vectors $u_i^j, \omega_{g_i}^j, 1 \leq i \leq n$, by the force-free and torque-free conditions on $\psi$. As $u^j_g + u_i^j + \omega_{g_i}^j \times (x - x_i)$ is divergence-free, one has

$$\int_{\partial B_i} (u^j_g + u_i^j + \omega_{g_i}^j \times (x - x_i)) \cdot \nu = 0.$$ 

We can apply classical considerations on the Bogovskii operator: for any $1 \leq i \leq n$, there exists $U^i_g \in H^1_0(B(x_i, (M/2)r_n))$ such that

$$\text{div } U^i_g = 0 \quad \text{in } B(x_i, \frac{M}{2}r_n), \quad U^i_g = u^j_g + u_i^j + \omega_{g_i}^j \times (x - x_i) \quad \text{in } B_i$$

and with

$$||\nabla U^i_g||_{L^2} \leq C_i,n ||u^j_g + u_i^j + \omega_{g_i}^j \times (x - x_i)||_{W^{1,2}(B_i)}$$

Furthermore, by a proper choice of $u_i^j$ and $\omega_{g_i}^j$, we can ensure the Korn inequality:

$$||u^j_g + u_i^j + \omega_{g_i}^j \times (x - x_i)||_{W^{1,2}(B_i)} \leq c_i,n ||D(u^j_g)||_{L^2(B_i)}$$

resulting in

$$||\nabla U^i_g||_{L^2} \leq C ||D(u^j_g)||_{L^2(B_i)}$$

where the constant $C$ in the last inequality can be taken independent of $i$ and $n$ by translation and scaling arguments. Extending $U^i_g$ by zero, and denoting $U^i_g = \sum U^i_g$, we have

$$||\nabla U^i_g||_{L^2} \leq C ||D(u^j_g)||_{L^2(\cup B_i)} \tag{26}$$

Thus, we find

$$\int_{\mathbb{R}^3} g\psi = 2\int_{\cup B_i} D(U^i_g) : D(\psi) - \sum_i \int_{\partial B_i} U^i_g \cdot \sigma(\psi, q) \nu$$

$$= 2\int_{\mathbb{R}^3} D(U^i_g) : D(\psi)$$

By using (26) and Cauchy-Schwarz inequality, we end up with

$$\left|\int_{\mathbb{R}^3} g\psi\right| \leq C ||D(u^j_g)||_{L^2(\cup B_i)} ||D(\psi)||_{L^2(\mathbb{R}^3)} \leq C ||D(u^j_g)||_{L^\infty} ||\nabla\psi||_{L^2(\mathbb{R}^3)}$$

Now the assertion follows from the somehow standard estimate

$$||\nabla \psi||_{L^2(\mathbb{R}^3)} \leq C ||D\tilde{\psi}||_{L^2(\cup B_i)} \tag{27}$$

for a constant $C$ independent of $n$. Indeed, by a classical variational characterization of $\psi$, we have

$$||\nabla \psi||_{L^2(\mathbb{R}^3)}^2 = 2||D\tilde{\psi}||_{L^2(\mathbb{R}^3)}^2 = \inf \left\{ 2||DU||_{L^2(\mathbb{R}^3)}^2, DU = D\tilde{\psi} \text{ on } \cup B_i \right\}.$$ 

Thus, (27) follows by constructing such a vector field $U$ from $\tilde{\psi}$ in the same manner as we constructed $U^i_g$ from $u^j_g$ above and applying (26). \qed

By (18) we can apply this proposition with $g = f$, $\psi = \psi_n$ and $\tilde{\psi}_n = \varphi - \phi_{\text{app},n}$. Thus, for the proof of Proposition 6, it remains to show
\[
\limsup_{n \to \infty} \|D(\varphi - \phi_{app,n})\|_{L^2(\cup B_i)} \leq C(\lambda^{1/2} \eta^{-1/2} + \eta^{3/2} \lambda^{1/2}) \|\nabla \varphi\|_{L^p}.
\]

(28)

We decompose
\[
\varphi - \phi_{app,n} = \tilde{\psi}_n^1 + \tilde{\psi}_n^2 + \tilde{\psi}_n^3
\]

where
\[
\forall 1 \leq i \leq n, \forall x \in B_i, \quad \tilde{\psi}_n^1(x) = -\varphi(x) - \sum_{j \in \mathcal{G}_i \cap \mathcal{G}_n} \phi_{j,n}(x)
\]

and
\[
\forall i \in \mathcal{G}_n, \forall x \in B_i, \quad \tilde{\psi}_n^2(x) = \varphi(x) - \varphi(x_i) - \nabla \varphi(x_i)(x - x_i) + \left(\varphi(x_i) + \frac{1}{2} \text{curl} \varphi(x_i) \times (x - x_i)\right),
\]

\[
\forall i \in \mathcal{B}_{\eta}, \forall x \in B_i, \quad \tilde{\psi}_n^3(x) = 0,
\]

\[
\forall i \in \mathcal{G}_n, \forall x \in B_i, \quad \tilde{\psi}_n^3(x) = 0,
\]

\[
\forall i \in \mathcal{B}_\eta, \forall x \in B_i, \quad \tilde{\psi}_n^3(x) = \varphi(x).
\]

We remind that the sum in (15) is restricted to indices \(i \in \mathcal{G}_i\) and that \(\phi_{i,n}(x) = D\varphi(x_i)(x - x_i)\) for \(x \in B_i\). This explains the distinction between \(\tilde{\psi}_n^2\) and \(\tilde{\psi}_n^3\).

The control of \(\tilde{\psi}_n^2\) is the simplest:
\[
\|D\tilde{\psi}_n^2\|_{L^2(\cup B_i)} \leq C\|D^2\varphi\|_{L^\infty} \left(\sum_{i \in \mathcal{G}_i} \int_{B_i} |x - x_i|^2 \, dx\right)^{1/2} \leq C' \lambda^{1/2} r_n.
\]

(29)

Hence,
\[
\lim_{n \to +\infty} \|D\tilde{\psi}_n^2\|_{L^2(\cup B_i)} = 0.
\]

(30)

Next, we estimate \(\tilde{\psi}_n^3\). This term expresses the effect of the balls \(B_{\eta}\) that are close to one another. By assumption (B2), \(\text{card}B_{\eta} \leq C\eta^2 n\). Thus,
\[
\|D\tilde{\psi}_n^3\|_{L^2(\cup B_i)} \leq C\|1_{\cup B_i} B_i\|_{L^2(\mathbb{R}^3)} \|D\varphi\|_{L^p(\cup B_i)} \leq C' \eta^2 \lambda^{1/2} \|\nabla \varphi\|_{L^p}.
\]

(31)

The final step in the proof of Proposition 6 is to establish bounds on \(\tilde{\psi}_n^1\). We have
\[
\|D\tilde{\psi}_n^1\|_{L^2(\cup B_i)} \leq C \left(\|D\phi_{\mathbb{R}^3}\|_{L^2(\cup B_i)} + \left(\sum_i \int_{B_i} \sum_{j \in \mathcal{G}_i \cap \mathcal{G}_n} D\phi_{j,n} \right)^2\right)^{1/2}
\]

(32)

For any \(r, s < +\infty\) with \(\frac{1}{r} + \frac{1}{s} = \frac{1}{2}\), we obtain
\[
\|D\phi_{\mathbb{R}^3}\|_{L^2(\cup B_i)} \leq \|1_{\cup B_i} \|_{L^r(\mathbb{R}^3)} \|D\phi_{\mathbb{R}^3}\|_{L^s(\mathbb{R}^3)} \leq C\|1_{\cup B_i}\|_{L^r(\mathbb{R}^3)} \|\nabla \varphi D\varphi\|_{L^2(\mathbb{R}^3)}
\]

using standard \(L^s\) estimate for system (12). Hence,
\[
\|D\phi_{\mathbb{R}^3}\|_{L^2(\cup B_i)} \leq C' \lambda^{1/2} \|\nabla \varphi\|_{L^2(\mathbb{R}^3)}.
\]

(33)
Note that we can choose any \( s > 2 \), this lower bound coming from the requirement \( \frac{1}{r} + \frac{1}{s} = \frac{1}{2} \). Introducing \( p \) such that \( s = p' \), we find that for any \( p < 2 \),

\[
||D\phi_{j,n}(x)||_{L^2(B_j)} \leq C' \lambda^{n-\frac{1}{2}} ||D\phi||_{L^{p'}(\mathbb{R}^d)}.
\]

(34)

The treatment of the second term at the r.h.s. of (32) is more delicate. We write, see (16):

\[
D\phi_{j,n}(x) = DV[D\phi(x_j)] \left( \frac{x - x_j}{r_n} \right) = \mathcal{V}[D\phi(x_j)] \left( \frac{x - x_j}{r_n} \right) + \mathcal{W}[D\phi(x_j)] \left( \frac{x - x_j}{r_n} \right)
\]

(35)

where \( \mathcal{V}[S] = D \left( \frac{S}{2} : (x \otimes x) \frac{x}{|x|} \right) \), \( \mathcal{W}[S] = D \left( \frac{S}{2} - \frac{S}{2} (S : x \otimes x) \frac{x}{|x|} \right) \).

We have:

\[
\sum_i \left[ \sum_{j \in G_{i,j}} \mathcal{W}[D\phi(x_j)] \left( \frac{x - x_j}{r_n} \right) \right]^2 \leq C \sum_i \left[ \sum_{j \in G_{i,j}} |D\phi(x_j)| |x - x_j|^{-5} \right]^2 dx
\]

For all \( i \), for all \( j \in G_{i,j} \) with \( j \neq i \), and all \( (x, y) \in B_i \times B(x_j, \frac{q}{4} n^{-\frac{1}{2}}) \), we have for some absolute constants \( c, c' > 0 \):

\[
|x - x_j| \geq c |x - y| \geq c' \eta n^{-\frac{1}{2}}.
\]

Denoting \( B_{y} = B \left( x_j, \frac{q}{4} n^{-\frac{1}{2}} \right) \) we deduce

\[
\sum_i \left[ \sum_{j \in G_{i,j}} \mathcal{W}[D\phi(x_j)] \left( \frac{x - x_j}{r_n} \right) \right]^2 dx 
\]

\[
\leq C \sum_i \left| \sum_{j \in G_{i,j} \neq i} \mathcal{W}[D\phi(x_j)] \left( \frac{x - x_j}{r_n} \right) \right|^2 dx
\]

\[
\leq C' n^{2} r_n \sum_i \left| \sum_{j \in G_{i,j} \neq i} \mathcal{W}[D\phi(x_j)] \left( \frac{x - x_j}{r_n} \right) \right|^2 dx
\]

Using Hölder and Young's convolution inequalities, we find that for all \( r, s \) with \( \frac{1}{r} + \frac{1}{s} = 1 \),

\[
\int_{\mathbb{R}^d} \mathcal{V}[1_{B_i}(x)] \left( \int_{\mathbb{R}^d} |x - y|^{-5} \cdot 1_{|x - y| > \epsilon n^{-\frac{1}{2}}} (x - y) \sum_{1 \leq j \leq n} |D\phi(x_j)| 1_{B_{y}'}(y) dy \right)^2 dx
\]

\[
\leq ||1_{B_i}||_{L^r} \left( \left( \int_{\mathbb{R}^d} |x|^{-5} \cdot 1_{|x| > \epsilon n^{-\frac{1}{2}}} \right) \right) \sum_{1 \leq j \leq n} ||D\phi(x_j)||_{L^{s'}}^2
\]

\[
\leq ||1_{B_i}||_{L^r} \left( \left( \int_{\mathbb{R}^d} |x|^{-5} \cdot 1_{|x| > \epsilon n^{-\frac{1}{2}}} \right) \right) \sum_{1 \leq j \leq n} ||D\phi(x_j)||_{L^{s'}}^2
\]

\[
\leq C \lambda^{2} (\eta n^{-\frac{1}{2}})^{-4} \sum_{j} ||D\phi(x_j)||_{L^{s'}}^2 \eta^2 n^{-1}
\]

Note that, by (A0), \( \frac{1}{n} \sum_j |D\phi(x_j)|^p \to \int_{\mathbb{R}^d} |D\phi|^p \rho \) as \( n \to +\infty \). We end up with

\[
\limsup_{n \to \infty} \left( \sum_i \left[ \sum_{j \in G_{i,j} \neq i} \mathcal{W}[D\phi(x_j)] \left( \frac{x - x_j}{r_n} \right) \right]^2 dx \right) \leq C \lambda^{\frac{n+1}{2}} \eta^{-10^{1/2}} ||D\phi||_{L^{2c}(C)}^2.
\]
We can take any $s > 1$, which yields by setting $p$ such that $p' = 2s$: for any $p < 2$
\[
\limsup_{n \to \infty} \sum_i \int_{B_i} \left| \sum_{j \in \mathcal{G}_n \setminus \{i\}} \mathcal{W}[D\varphi(x_j)] \left( \frac{x - x_j}{r_n} \right) \right|^2 \, dx \leq C \lambda^{\frac{4 + 2p}{p} - \frac{2}{s}} \eta^{-\frac{2}{s}} \|D\varphi\|^2_{L^2(\mathcal{O})}.
\]  
(36)

To treat the first term in the decomposition (35), we write
\[
\mathcal{V}[D\varphi(x_j)] \left( \frac{x - x_j}{r_n} \right) = r_n^3 \mathcal{M}(x - x_j) \, D\varphi(x_j)
\]
for a matrix-valued Calderon-Zygmund operator.

We use that for all $i$ and all $j \neq i$, $j \in \mathcal{G}_n$ we have for all $(x,y) \in B_i \times B_j^*$
\[
|\mathcal{M}(x - x_j) - \mathcal{M}(x - y)| \leq C\eta^{-1/3} |x - y|^{-4}
\]
Thus, by similar manipulations as before
\[
\sum_i \int_{B_i} \left| \sum_{j \in \mathcal{G}_n \setminus \{i\}} \mathcal{V}[D\varphi(x_j)] \left( \frac{x - x_j}{r_n} \right) \right|^2 \, dx
\]
\[
\leq C \eta^4 \sum_i \int_{B_i} \left( \sum_{j \in \mathcal{G}_n \setminus \{i\}} \left| \mathcal{M}(x - y) \right|^2 \right) \left( \left\{ |x - y| > \eta n^{\frac{1}{3}} \right\} \right) (x-y) |D\varphi(x_j)| \, dy \, dx
\]
\[
+ C \eta^2 \frac{r_n^3}{n^2} \sum_i \int_{B_i} \left( \sum_{j \in \mathcal{G}_n \setminus \{i\}} \frac{1}{|B_j^*|} \right) \left( \left| x - y \right|^4 \right) \left( \left\{ |x - y| > \eta n^{\frac{1}{3}} \right\} \right) (x-y) |D\varphi(x_j)| \, dy \, dx
\]
\[
\leq C \eta^4 \sum_i \int_{B_i} \left| D\varphi(x_j) \right|^2 \, dx \leq C \lambda^{\frac{4 + 2p}{p} - \frac{2}{s}} \eta^{-\frac{2}{s}} \|D\varphi\|^2_{L^2(\mathcal{O})}
\]
As seen in [7, Lemma 2.4], the kernel $\mathcal{M}(x)1_{\{|x| > \eta n^{\frac{1}{3}}\}}$ defines a singular integral that is continuous over $L^t$ for any $1 < t < \infty$, with operator norm bounded independently of the value $\eta n^{-\frac{2}{3}}$ (by scaling considerations). Applying this continuity property with $t = 2s$, writing as before $p' = 2s$, we get for all $p < 2$,
\[
\limsup_{n \to \infty} \sum_i \int_{B_i} \left| \sum_{j \in \mathcal{G}_n \setminus \{i\}} \mathcal{W}[D\varphi(x_j)] \left( \frac{x - x_j}{r_n} \right) \right|^2 \, dx \leq C \lambda^{\frac{4 + 2p}{p} - \frac{2}{s}} \eta^{-\frac{2}{s}} \|D\varphi\|^2_{L^2(\mathcal{O})}
\]
Combining this last inequality with (35) and (36), we finally get: for all $p < 2$,
\[
\limsup_{n \to \infty} \left( \sum_i \int_{B_i} \left| \sum_{j \in \mathcal{G}_n \setminus \{i\}} D\varphi_j \right|^2 \right)^{1/2} \leq C' \lambda^{\frac{4 + 2p}{p} - \frac{2}{s}} \eta^{-\frac{2}{s}} \|D\varphi\|_{L^p(\mathcal{O})}
\]  
(37)

Finally, if we inject (34) and (37) in (32), we obtain that for any $p < 2$,
\[
\limsup_{n \to \infty} \|D\varphi_n\|_{L^2(\cup B_i)} \leq C \lambda^{\frac{4 + 2p}{p} - \frac{2}{s}} \eta^{-\frac{2}{s}} \|D\varphi\|_{L^p(\mathbb{R}^3)}
\]  
(38)

Here we used $\eta \leq 1$.

The desired estimate (28) follows from collecting (30, 38) and (31). This concludes the proof of Proposition 6.
5. Discussion of assumption (B1)

In the light of the recent paper [16], we will show how condition (B1) can be replaced by the following assumption:

\[ \forall i, \quad \rho_i := \sup_{j \neq i} r^{-1}_n |x_i - x_j| - 2 > 0, \quad \exists \alpha > 1, \quad \limsup_{n \to \infty} \frac{1}{n} \sum_{i} \rho_i^{-\alpha} < \infty. \]  

(B1)

We will argue that Theorem 1 remains valid with \( p_{\min} \) depending in addition on the power \( s \) from (B1'). More precisely, \( p_{\min} \) in Remark 2 needs to be replaced by

\[ p_{\min} = \min \left\{ 1 + \frac{\alpha}{6 + \alpha}, 1 + \frac{s - 1}{s + 1}, \frac{3}{2} \right\}. \]  

(39)

There are only two instances where we have used assumption (B1), which are both contained in the proof of Proposition 8: one is to prove the estimate (27) for the solution \( \psi \) to the system (22)–(23), and the other one is to prove the analogue estimate (26). The proof has been based on the construction of suitable functions \( \Psi_i \in H^1_0(B(x_i, M/2r_n)) \) with \( D(\Psi_i) = D(\tilde{\psi}) \) in \( B_r \). If we drop assumption (B1), we can still replace the balls \( B(x_i, M/2r_n) \) by disjoint neighborhoods \( B_i^+ \) satisfying the assumptions of [16], section 3.1 (with \( I_n, I^+_n \) replaced by \( B_i, B_i^+ \)). By [16], Lemma 3.3), it then follows that for all \( r > 2 \) and all \( q \geq \max(2, \frac{6r}{5r - \alpha}) \), there exists \( \Psi_i \in H^1_0(B_i^+) \), such that

\[ ||\nabla \Psi_i||_{L^2(B_i^+)} \leq C_r \rho_i^{3-\frac{s}{2}} r_n^{3-\frac{s}{2}} ||D(\tilde{\psi})||_{L^s(B_i^+)}. \]

Setting \( \Psi = \sum_i \Psi_i \), we find that

\[ ||\nabla \Psi||_{L^2}^2 \leq C_r \sum i \rho_i^{3-\frac{s}{2}} r_n^{3-\frac{s}{2}} ||D(\tilde{\psi})||_{L^s(B_i^+)}^2 \leq C_r \frac{2}{n} \sum \left( \rho_i^{3-\frac{s}{2}} \right) \frac{s}{s+1} \frac{\rho_i^{3-\frac{s}{2}}}{s+1} ||D(\tilde{\psi})||_{L^s(\cup B_i)}^2 \]

(40)

Note that for \( s \) the exponent in (B1'), \( q > 3 \) and \( \frac{q}{q-2} < s \), taking \( r \) close enough to 2, one can ensure that \( q \geq \max(2, \frac{6r}{5r - \alpha}) \) and that the first factor at the right-hand side of (40) is finite. In conclusion this argument shows that Proposition 8 remains valid under assumption (B1') with the estimate (24) replaced by

\[ \left| \int_{\mathbb{R}^3} g \psi \right| \leq C_{G,q} \rho_i^{\frac{q}{2} + \frac{3q}{s+2}} ||D\tilde{\psi}||_{L^q(\cup B_i)}. \]

(41)

It is not difficult to check that this change of the estimate still allows to conclude the argument in Section 4 along the same lines as before. Indeed, whenever we used (24), we also applied Hölder’s estimate to replace \( ||D\tilde{\psi}||_{L^q(\cup B_i)} \) by a higher Lebesgue norm in order to gain powers in \( \lambda \). One could say that the modified estimate (41) has just partly anticipated Hölder’s estimate. The additional restrictions on \( q \) \((q > 3, q < \frac{s}{s-2})\) are the reason for the additional constraints in \( p_{\min} \) in (39). The estimates in Section 4 where we use Proposition 8 concern the terms \( \tilde{\psi}_{i,n}^3, \ i = 1, 2, 3 \). First, in the estimate for \( \tilde{\psi}_{i,n}^3 \) corresponding to (29), we can just use (41) with \( q = \infty \). Second for \( \tilde{\psi}_{i}^3 \), previously estimated in (31), we use (41) with \( q = p' \). Finally, for \( \tilde{\psi}_{i} \), if one carefully follows the estimates in Section 4, one observes that (41) with \( q = p' \) is again sufficient.
6. Discussion of assumption (B2)

6.1. Stationary ergodic processes

Let $\Phi^\delta = \{y_i\}_i \subset \mathbb{R}^3$ be a stationary ergodic point process on $\mathbb{R}^3$ with intensity $\delta$ and hard-core radius $R$, i.e., $|y_i - y_j| \geq R$ for all $i \neq j$. An example of such a process is a hard-core Poisson point process, which is obtained from a Poisson point process upon deleting all points with a neighboring point closer than $R$. We refer to [19], Section 8.1] for the construction and properties of such processes.

Assume that $\mathcal{O}$ is convex and contains the origin. For $\varepsilon > 0$, we consider the set

$$\varepsilon\Phi^\delta \cap \mathcal{O} =: \{x^\varepsilon_i, i = 1, \ldots, n_\varepsilon\}.$$ 

Let $r < R/2$ and denote $r_\varepsilon = \varepsilon r$ and consider $B_i = B(x_i, r_\varepsilon)$. The volume fraction of the particles depends on $\varepsilon$ in this case. However, it is not difficult to generalize our result to the case when the volume fraction converges to $\lambda$ and this holds in the setting under consideration since

$$\frac{4\pi}{3} n_\varepsilon r_\varepsilon^3 \to \frac{4\pi}{3} \delta r^3 =: \lambda(r, \delta) \text{ almost surely as } \varepsilon \to 0.$$ 

Clearly, $\lambda(r, \delta) \to 0$, both if $r \to 0$ and if $\delta \to 0$. However, the process behaves fundamentally differently in those cases. Indeed, if we take $r \to 0$ (for $\delta$ and $R$ fixed), we find that condition (A1), which implies (B2), is satisfied almost surely for $\varepsilon$ sufficiently small as

$$n_\varepsilon^{1/3}|x^\varepsilon_i - x^\varepsilon_j| \geq n_\varepsilon^{1/3}\varepsilon R \to \delta^{1/3}R.$$ 

In the case when we fix $r$ and consider $\delta \to 0$ (e.g. by randomly deleting points from a process), (A1) is in general not satisfied. We want to characterize processes for which (B2) is still fulfilled almost surely as $\varepsilon \to 0$. Indeed, using again the relation between $\varepsilon$ and $n_\varepsilon$, it suffices to show

$$\forall \eta > 0, \#\{i, \exists j, |x_i - x_j| \leq \eta \varepsilon\} \leq C\eta^2 \delta^{1+\frac{2}{3}}\varepsilon^{-3}. \quad (42)$$ 

Let $\Phi^\delta_\eta$ be the process obtained from $\Phi^\delta$ by deleting those points $y$ with $B(y, \eta) \cap \Phi^\delta = \{y\}$. Then, the process $\Phi^\delta_\eta$ is again stationary ergodic (since deleting those points commutes with translations), so that almost surely as $\varepsilon \to 0$

$$\varepsilon^{3} \#\{i, \exists j, |x_i - x_j| \leq \eta \varepsilon\} \to \mathbb{E}\left[\#\Phi^\delta_\eta \cap Q\right],$$

where $Q = [0, 1]^3$. Clearly,

$$\mathbb{E}\left[\#\Phi^\delta_\eta \cap Q\right] \leq \mathbb{E} \sum_{y \in \Phi^\delta_\eta \cap Q} \sum_{y' \notin \Phi^\delta} 1_{B(0, \eta)}(y' - y).$$

We can express this expectation in terms of the 2-point correlation function $\rho^\delta_2(y, y')$ of $\Phi^\delta$ yielding

1In detail: let $\mathcal{E}_\eta$ be the operator that erases all points without a neighboring point closer than $\eta$, and let $T_x$ denote a translation by $x$. Now, let $\mu^\varepsilon$ be the measure for the original process $\Phi^\delta$. Then the measure for $\Phi^\delta_\eta$ is given by $\mu^\varepsilon \mathcal{E}_\eta^{-1}$. Since $\mathcal{E}_\eta T_x = T_x \mathcal{E}_\eta$ (for all $x$, in particular for $T_x = T_x^{-1}$), we have for any measurable set $A$ that $T_x \mathcal{E}_\eta^{-1}A = \mathcal{E}_\eta T_x^{-1}A$. This immediately implies that the new process adopts stationarity and ergodicity.
\[ \mathbb{E}\left[ \# \Phi^\delta_{\eta} \cap Q \right] \leq \int_{\mathbb{R}^d} 1_{Q(y)} 1_{B(0, \eta)}(y' - y) \rho^\delta_2(y, y') \, dy \, dy'. \]

Hence, (42) and therefore also (B2) is in particular satisfied with \( \alpha = 3 \) if \( \rho^\delta_2 \leq C\delta^2 \) which is the case for a (hard-core) Poisson point process.

Moreover, we observe that (B2) with \( \alpha < 3 \) is satisfied even for processes that favor clustering: (42) holds if \( \rho^\delta_2(y, y') \leq C\delta^{3 - \frac{\alpha}{2}} |y - y'|^{\frac{\alpha}{2} - 3} \). This means that \( \rho^\delta_2 \) can be quite singular at the diagonal and of much higher intensity than \( \delta^2 \). Examples for such clustering point processes are Neyman-Scott processes (see e.g. [3, Section 6.3]).

### 6.2. Identically, independently distributed particles

Focusing on assumption (B2), we neglect the non-overlapping condition (B1) in the following, which is not satisfied for i.i.d. particles. As in the case of hard-core Poisson point processes, it is nevertheless possible to construct a process that satisfies (B1), by deleting points which have a too close neighbor. As those points will be few for small volume fractions, this will not affect the discussion of (B2) qualitatively.

We will show the following result: for \( x_1, \ldots, x_n \) i.i.d. with a law \( \rho \in L^\infty (\rho \geq 0, \int \rho = 1) \), for all \( \eta > 0 \) :

\[ n^{-1} \# \left\{ i, \exists j \neq i, |x_i - x_j| \leq \eta n^{-1/3} \right\} \xrightarrow{n \to +\infty} 1 - \int_{\mathbb{R}^3} \rho(x) e^{-\rho(x)\frac{\eta}{18} x^3} \, dx \]  

in probability. This implies (B2) with \( \alpha = 3 \) in probability. We first set

\[ \eta_n := \eta n^{-1/3}, \quad B^n_j := B(x_j, \eta_n), \quad Y^n_i := \prod_{j \neq i} 1_{B^n_j}(x_i). \]

Note that the random variables \( Y^n_i \) are identically distributed, but not independent. Note also that

\[ n^{-1} \# \left\{ i, \exists j \neq i, |x_i - x_j| \leq \eta_n \right\} = \frac{1}{n} \sum_{i=1}^n (1 - Y^n_i). \]

Hence, we need to show that \( \frac{1}{n} \sum_{i=1}^n Y^n_i \) converges to \( I_{\rho, \eta} := \int_{\mathbb{R}^3} \rho(x) e^{-\rho(x)\frac{\eta}{18} x^3} \, dx \) in probability.

**Step 1.** We show that \( \mathbb{E} Y^n_i \xrightarrow{n \to +\infty} I_{\rho, \eta} \). Indeed, by independence,

\[ \mathbb{E} Y^n_i = \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} 1_{B(y, \eta_n)}(x) \rho(y) \, dy \right)^{n^{-1}} \rho(x) \, dx = \int_{\mathbb{R}^3} \left( 1 - \int_{B(x, \eta_n)} \rho(y) \, dy \right)^{n^{-1}} \rho(x) \, dx \]

At each Lebesgue point \( x \) of \( \rho \), one has \( \frac{1}{B(x, \eta_n)} \int_{B(x, \eta_n)} \rho(y) \, dy \to \rho(x) \), so that

\[ \left( 1 - \int_{B(x, \eta_n)} \rho(y) \, dy \right)^{n^{-1}} \to e^{-\rho(x)\frac{\eta}{18} x^3} \quad \text{for a.e.} \, x \]

and the result follows by the dominated convergence theorem.

**Step 2.** We show that

\[ \text{var} \left( \frac{1}{n} \sum_{i=1}^n Y^n_i \right) \to 0 \quad \text{as} \, n \to +\infty \]

By Markov inequality and Step 1, this implies (43).
We have

\[
\text{var} \left( \frac{1}{n} \sum_{i=1}^{n} Y_i^n \right) = \frac{1}{n} \text{var} Y_1^n + \frac{n(n-1)}{n^2} \text{Cov} Y_1^n Y_2^n = \text{Cov} Y_1^n Y_2^n + O \left( \frac{1}{n} \right)
\]

using that \( 0 \leq Y_1^n \leq 1 \). It remains to show that the covariance goes to zero. Using the independence of the \( x_i \)'s, we have the explicit formula

\[
\mathbb{E} Y_1^n Y_2^n = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}} 1_{|x-x_1| \geq \eta_n} 1_{|x-x_2| \geq \eta_n} \rho(x) dx \right)^{n-2} 1_{|x_1-x_2| \geq \eta_n} \rho(x_1) \rho(x_2) dx_1 dx_2.
\]

We have

\[
\left( \int_{\mathbb{R}} 1_{|x-x_1| \geq \eta_n} 1_{|x-x_2| \geq \eta_n} \rho(x) dx \right)^{n-2} = \left( 1 - \int_{B(x_1, \eta_n)} \rho - \int_{B(x_2, \eta_n)} \rho + \left( \int_{B(x_1, \eta_n) \cap B(x_2, \eta_n)} \rho \right) 1_{|x_1-x_2| \leq 2\eta_n} \right)^{n-2}
\]

\[
= e^{-\frac{4\pi}{\sqrt{\pi}} \int_{B(x_1, \eta_n)} \rho} e^{-\frac{4\pi}{\sqrt{\pi}} \int_{B(x_2, \eta_n)} \rho} e^{R_n(x_1, x_2)} \left| R_n(x_1, x_2) \right| \leq C \left| \frac{1}{|x_1-x_2| \leq 2\eta_n} \right| + C n^{-1}.
\]

This quantity converges almost surely to \( e^{-\frac{4\pi}{\sqrt{\pi}} \rho(x_1)} e^{-\frac{4\pi}{\sqrt{\pi}} \rho(x_2)} \) and it follows by the dominated convergence theorem that

\[
\mathbb{E} Y_1^n Y_2^n \rightarrow \left( \int_{\mathbb{R}} e^{-\frac{4\pi}{\sqrt{\pi}} \rho(x_1)} dx_1 \right) \left( \int_{\mathbb{R}} e^{-\frac{4\pi}{\sqrt{\pi}} \rho(x_2)} dx_2 \right) = \lim_{n \to +\infty} (\mathbb{E} Y_1^n)^2
\]

which yields the result.

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