1. Introduction

Given a pair of Hilbert spaces $H_1, H_2$, unitary operators $U_1 : H_1 \to H_1$, $U_2 : H_2 \to H_2$, and a bounded operator $A : H_1 \to H_2$, define the future and past wave operators $W^+, W^-$ as the limits of the sequence $\{U_2^{-n}AU_1^n\}_{n \in \mathbb{Z}}$ as $n \to +\infty$, $n \to -\infty$, respectively.

There are a number of different meanings of the above term “limit”. In the classical scattering theory it is proved that the limits $\lim_{n \to \pm \infty} U_2^{-n}AU_1^n$ exist in the strong operator topology, under the assumptions that the spectral measures of $U_1, U_2$ are absolutely continuous with respect to the Lebesgue measure and the commutator $AU_1 - U_2A$ is of trace class [5].

Much less is known about similar results for unitary operators $U_1, U_2$ having singular spectrum. Simple examples in a one-dimensional space show that the limits $\lim_{n \to \pm \infty} U_2^{-n}AU_1^n$ may not exist. A natural way to define the wave operators in this situation is to consider the limits of averages of the sequence $U_2^{-n}AU_1^n$ by using some summation (averaging) method. Important examples of summation methods which will be used here are the Cesaro summation method taking a sequence to the sequence of its arithmetical means and the Abel-Poisson summation method.

Now we give a precise definition of summation methods and averaged wave operators. Let $(E, \prec)$ be a totally ordered set, $\mathbb{Z}_+$ the set of nonnegative integers. A family of nonnegative numbers $\{p_{\alpha,n}\}_{\alpha \in E, n \in \mathbb{Z}_+}$ determines an s-regular summation (averaging) method if the following conditions are fulfilled:

1) $\sum_n p_{\alpha,n} = 1$ for every $\alpha \in E$,
2) $\lim_{\alpha} \sum_n |p_{\alpha,n} - p_{\alpha,n+1}| = 0$,
3) $\lim_{\alpha} p_{\alpha,n} = 0$ for every $n \in \mathbb{Z}_+$.

For a sequence $\{x_n\}$ of elements of a Banach space $X$, which is bounded in norm, define its averages by $x(\alpha) = \sum_n p_{\alpha,n}x_n$, $\alpha \in E$. If the limit $x = \lim_\alpha x(\alpha)$ exists in some topology $\sigma$ on $X$, we will say that the averages of $\{x_n\}$ converge to $x$ in $\sigma$. Usually we will work with the space $X = B(H)$ of
all bounded operators on a Hilbert space $H$ equipped by the weak operator topology $\sigma$. All s-regular methods possess the following natural properties:

a) Regularity: averages of every convergent sequence converge to its limit.

b) Linearity: if averages of sequences $\{x_n\}, \{y_n\}$ converge to $x, y$, then averages of $\{x_n + cy_n\}$ converge to $x + cy$ for any complex number $c$;

c) Stability: averages of sequences $\{x_n\}$ and $\{x_{n+1}\}$ do or do not converge simultaneously, and if they converge, then the corresponding limits coincide.

Moreover, every s-regular summation method averages to zero any nonconstant unimodular geometric progression of complex numbers:

d) $\lim_{\alpha} \sum_{n=0}^{\infty} p_{\alpha,n} z^n = 0$ if $z \in \mathbb{C} \setminus \{1\}, |z| = 1$.

The well-known examples of s-regular methods are the Cesaro summation method (with $E = (\mathbb{N}, \leq)$ and $p_{\alpha,n} = \frac{1}{\alpha}$ whenever $n < \alpha$, $p_{\alpha,n} = 0$ otherwise), and the Abel-Poisson summation method (with $E = ((0, 1), \leq)$ and $p_{\alpha,n} = (1 - \alpha)^n\alpha^n$).

Fix some s-regular summation method. In what follows the term “averaging” will always mean the use of the fixed summation method.

**Definition.** Let $H_{1,2}, U_{1,2}, A$ be as above, denote by $W_+ (\alpha), W_- (\alpha)$ the averages of the sequences $\{U_{1}^{-n}AU_{1}^{n}\}_{n \in \mathbb{Z}_+}$ and $\{U_{2}^{-n}AU_{1}^{-n}\}_{n \in \mathbb{Z}_+}$:

$$W_+ (\alpha) = \sum_{n=0}^{+\infty} p_{\alpha,n} U_{1}^{-n}AU_{1}^{n} \quad \text{and} \quad W_- (\alpha) = \sum_{n=0}^{+\infty} p_{\alpha,n} U_{2}^{n}AU_{1}^{-n}.$$  

The future weak averaged wave operator $W_+$ is the limit $\lim_{\alpha} W_+ (\alpha)$ in the weak operator topology if the limit exists. Similarly, the past weak averaged wave operator $W_-$ is the limit $\lim_{\alpha} W_- (\alpha)$.

We borrow the term “wave operator” from the classical scattering theory, where the case of absolutely continuous spectrum is studied.

If $AU_1 = U_2A$ we have $U_{1}^{-n}AU_{1}^{n} = A$, therefore operators $W_\pm$ obviously exist and equal $A$. It seems natural to examine the problem of existence of $W_\pm$ for operators $U_1, U_2, A$ with a “small” commutator $AU_1 - U_2A$. A general conjecture can be formulated as follows: If $AU_1 - U_2A$ is a finite-rank operator, then the future and past averaged wave operators exist for every s-regular summation method. For the case of rank-one commutators this conjecture was essentially proven in [1], see also [2]. We study the case of rank-two commutators. Since the problem is solved for operators with absolutely continuous spectrum, one can restrict the consideration to singular unitary operators $U_1, U_2$. Our main result is the following:

**Theorem 1.** Let $U_1 : H_1 \to H_1, U_2 : H_2 \to H_2$ be singular unitary operators, and let $A : H_1 \to H_2$ be a bounded operator such that
rank(\(AU_1 - U_2A\) \(\leq 2\)). Then the weak limit \(\lim_\alpha (W_+(\alpha) - W_-(\alpha))\) exists and equals zero. In particular, the weak averaged wave operators \(W_\pm\) do or do not exist simultaneously. If \(W_\pm\) exist, they coincide.

In this paper we follow the approach developed in [2]. The general problem of existence of the limits \(\lim_\alpha W_\pm(\alpha)\) and \(\lim_\alpha (W_+(\alpha) - W_-(\alpha))\) for rank-two commutators reduces (see [2, Theorem 7.2]) to the following particular case:

1) \(H_1 = H_2 = L^2(\mu)\), where \(\mu\) is a Borel singular measure on the unit circle \(\mathbb{T}\) of the complex plane \(\mathbb{C}\). The measure \(\mu\) is free of point masses.
2) \(U_1 = U_2 = U\) is the operator of multiplication by the independent variable on \(L^2(\mu)\),
3) \(AU - UA = (\cdot, \varphi)1 - (\cdot, 1)\varphi\) for some real-valued function \(\varphi \in L^2(\mu)\).

In this situation the problem of existence of wave operators \(W_\pm\) can be restated in terms of functions \(\varphi\) from item 3). In [3] one can find a sufficient condition on the function \(\varphi\) guaranteeing the existence of \(W_\pm\). Namely, if \(\mu\) is the Clark measure \(\sigma_1\) for an inner function \(\theta\) and \(\varphi\) coincides \(\mu\)-almost everywhere with a trace of some function \(h \in \mathcal{K}_\theta\) having a continuous trace on \(\sigma_\alpha\), \(\alpha \neq 1\), then the operator \(W_-\) exists. The precise definitions will be given in Section 5, where we discuss the related results.

We show that it suffices to consider only functions \(\varphi\) that coincide \(\sigma_1\)-almost everywhere with a continuous function on \(\mathbb{T}\).

**Theorem 2.** Suppose that for every triple \(\mu, U, A\) satisfying conditions 1) – 3), where, moreover,

4) \(\varphi\) coincides \(\mu\)-almost everywhere with a continuous function on \(\mathbb{T}\),

one of the limits \(\lim_\alpha W_+(\alpha)\), \(\lim_\alpha W_-(\alpha)\), \(\lim_\alpha (W_+(\alpha) - W_-(\alpha))\) exists in the weak operator topology. Then the same limit exists for any operators \(A, U_1, U_2\) from Theorem 7 in the general case of \(\text{rank}(AU_1 - U_2A) = 2\).

However, not all continuous functions arise from commutators of type 1) – 3). Denote by \(C(\mathbb{T})\) the set of all continuous functions on the unit circle \(\mathbb{T}\).

**Theorem 3.** There exist a function \(\varphi \in C(\mathbb{T})\) and a singular measure \(\mu\) without atomic masses such that the operator \((\cdot, \varphi)1 - (\cdot, 1)\varphi\) on \(L^2(\mu)\) cannot be represented as a commutator \(AU - UA\) for a bounded operator \(A\).

From the proof of Theorem 3 it follows some consequences concerning to a boundary behaviour of pseudocontinuable functions. We discuss them in Section 5.
2. Proof of Theorem 1

For the proof of Theorem 1 we need some preliminary technical results. In what follows we will assume that for the operators $A, U$ conditions 1) and 2) from Section 1 are fulfilled. Define operators $W_{\pm}(\alpha)$ by formula (1).

**Lemma 4.** Let $K$ be the commutator $AU - UA$. We have

$$W_+ (\alpha) U - UW_- (\alpha) = \sum_{n=0}^{\infty} p_{\alpha,n} \sum_{l=-n}^{n} U^l K U^{-l}.$$  

**Proof.** We have

$$W_+ (\alpha) U - UW_- (\alpha) = \sum_{n=0}^{\infty} p_{\alpha,n} U^{-n} AU^{n+1} - \sum_{n=0}^{\infty} p_{\alpha,n} U^{n+1} AU^{-n}$$

$$= \sum_{n=0}^{\infty} p_{\alpha,n} (U^{-n} AU^{n+1} - U^{n+1} AU^{-n})$$

$$= \sum_{n=0}^{\infty} p_{\alpha,n} \sum_{l=-n}^{n} (U^l AU^{-l+1} - U^{l+1} AU^{-l})$$

$$= \sum_{n=0}^{\infty} p_{\alpha,n} \sum_{l=-n}^{n} U^l K U^{-l}.$$

**Lemma 5.** Assume that the commutator $K$ is of trace class, $Kh = \sum_{m=0}^{\infty} (h, u_m)v_m$, $h \in L^2(\mu)$ where the sum $\sum_{m=0}^{\infty} ||u_m|| \cdot ||v_m||$ is finite. Then

$$(W_+ (\alpha) U - UW_- (\alpha))h = \sum_{m=0}^{\infty} v_m \cdot \left[ (\overline{u_m}h) \ast \sum_{n=0}^{\infty} p_{\alpha,n} D_n \right],$$

where $D_n(\xi) = \sum_{l=-n}^{n} \xi^l$ is the Dirichlet kernel of order $n$ and the symbol $\ast$ denotes the convolution.

**Proof.** Assume at first that $K = (\cdot, u)v$ and consider the sum $\sum_{l=-n}^{n} U^l K U^{-l}$ applied to a vector $h \in L^2(\mu)$:

$$\sum_{l=-n}^{n} U^l K U^{-l}h = \sum_{l=-n}^{n} U^l K z^{-l}h = \sum_{l=-n}^{n} U^l (z^{-l}h, u)v = \sum_{l=-n}^{n} (z^{-l}h, u)z^l v =$$

$$= \sum_{l=-n}^{n} z^l v \int \xi^l h(\xi) \overline{u(\xi)} \, d\mu(\xi) = v \int h(\xi) \overline{u(\xi)} \sum_{l=-n}^{n} (z \xi)^l \, d\mu(\xi) = v \cdot [(\overline{uh}) \ast D_n].$$

By linearity arguments, we obtain the conclusion of the lemma.  \[\square\]
**Lemma 6.** Let $k(\xi, z)$ be a real-valued function from $L^\infty(\mu \times \mu)$ such that $k(\xi, z) = k(z, \xi)$. Then the operator $L$ given by the bilinear form

$$ (Lf, g) = \iint (f(\xi) - f(z))\overline{g(z)}k(\xi, z)\, d\mu(\xi)\, d\mu(z) $$

is a selfadjoint bounded operator.

**Proof.** The quadratic form of the operator $L$ is

$$ (Lf, f) = \iint (f(\xi) - f(z))\overline{f(z)}k(\xi, z)\, d\mu(\xi)\, d\mu(z), $$

and, symmetrically,

$$ (Lf, f) = \iint (f(z) - f(\xi))\overline{f(\xi)}k(z, \xi)\, d\mu(z)\, d\mu(\xi). $$

Therefore the sum $2(Lf, f) = -\iint |f(\xi) - f(z)|^2k(\xi, z)\, d\mu(z)\, d\mu(\xi)$ is real, which proves the statement. \( \square \)

**Proof of Theorem 1.** Assume that a triple $\mu, U, A$ satisfies conditions 1) – 3) from Section 1. This assumption makes no loss of generality, see \[2, Theorem 7.2\]. The theorem will be proved if we check the formula

$$ (W_+(\alpha) - W_-(\alpha))h_1, h_2 = 0 \text{ for every pair of vectors } h_1, h_2 \in L^2(\mu). $$

A simple computation shows that

$$ U^{-1}W_+(\alpha)U = W_+(\alpha) - p_{\alpha,0}A + \sum_{n=0}^\infty (p_{\alpha,n} - p_{\alpha,n+1})U^{-n-1}AU^{n+1}. $$

By this formula and properties 2), 3) from the definition of the s-regular summation method, the weak limits of $W_+(\alpha) - W_-(\alpha)$ and $W_+(\alpha)U - UW_-(\alpha)$ do or do not exist simultaneously. Moreover, if they exist, they are or are not equal to zero simultaneously. By Lemma 6 we have

$$ (W_+(\alpha)U - UW_-(\alpha))h_1 = (\varphi h_1) * \sum_{n=0}^\infty p_{\alpha,n}D_n - \varphi \cdot [h_1 * \sum_{n=0}^\infty p_{\alpha,n}D_n] $$

on every element $h_1 \in L^2(\mu)$. Since the set of all vectors $h_1 \in L^2(\mu)$, that satisfy the condition $\lim_{n}(W_+(\alpha)U - UW_+(\alpha))h_1, h_2 = 0$ for every element $h_2 \in L^2(\mu)$, forms a reducing subspace of $U$, without loss of generality one can consider $h_1 = 1$. Set $k_\alpha(z, \xi) = \sum_{n=0}^\infty p_{\alpha,n}D_n(z\xi)$. We have

$$ (2) \quad p_{\alpha}(z) = (W_+(\alpha)U - UW_-(\alpha))1 = \int (\varphi(\xi) - \varphi(z))k_\alpha(z, \xi)\, d\mu(\xi). $$

The norms of $p_{\alpha}(z)$ in $L^2(\mu)$ are uniformly bounded by $2\|A\|$. Hence one can check the required condition $\lim_{\alpha}(p_{\alpha}(z), h_2) = 0, h \in L^2(\mu)$, only on a dense subset of $L^2(\mu)$. Take the set of all smooth functions on $\mathbb{T}$, which is dense in $L^2(\mu)$. Consider

$$ (p_{\alpha}(z), h_2) = \iint (\varphi(\xi) - \varphi(z))h_2(z)k_\alpha(z, \xi)\, d\mu(\xi)\, d\mu(z), $$
Proof. In the proof we consider only the case of operators $\sup$. Moreover, if $W_\alpha$ tends to zero in the weak operator topology without assuming that the wave kernel and $\xi \neq z$ we have

$$\lim_{\alpha} k_{\alpha}(z, \xi) = 0.$$  

Therefore, the limit $\lim_{\alpha} k_{\alpha}(z, \xi)$ equals zero if $\xi \neq z$. By the dominated convergence theorem we obtain $\lim_{\alpha} (p_{\alpha}(z), h_2) = 0$.

**Remark.** We have proved the fact that the operators $W_+^{\alpha} - W_-^{\alpha}$ tend to zero in the weak operator topology without assuming that the wave operators $W_\pm$ exist.

3. **Proof of Theorem 2**

Let $\mu$, $U$, $A$ be triple satisfying conditions 1) – 3), and let the operators $W_\pm^{\alpha}$ be defined by formula \(11\). For the proof of Theorem 2 we need the following lemma:

**Lemma 7.** Let $\{h_n\}_{n \in \mathbb{N}}$ be a sequence of elements from $L^2(\mu)$ such that $h_n(z) \to h(z) \in L^2(\mu)$ for $\mu$-almost all points $z \in \mathbb{T}$, and such that the limit $\lim_{\alpha}(W_-^{\alpha} h_n, h_n)$ exists for all $n \in \mathbb{N}$. Then the limit $\lim_{\alpha}(W_-^{\alpha} h, h)$ exists. Moreover, if $\sup_n \|h_n\| < \infty$, then the double limit $\lim_{\alpha,n}(W_-^{\alpha} h_n, h_n)$ exists. The same holds for the families $W_+^{\alpha}$ and $W_+^{\alpha} - W_-^{\alpha}$.

**Proof.** In the proof we consider only the case of operators $W_-^{\alpha}$, other cases can be proven in a similar way.

Fix an arbitrary $\varepsilon > 0$. By the Egorov theorem, one can choose a set $F_\varepsilon \subset \mathbb{T}$ such that $\mu(\mathbb{T} \setminus F_\varepsilon) \leq \varepsilon$ and $h_n$ tend to $h$ uniformly on $F_\varepsilon$. Set $h_n^\varepsilon = \chi_{F_\varepsilon} h_n$ and $h^\varepsilon = \chi_{F_\varepsilon} h$, where $\chi_{F_\varepsilon}$ is the characteristic function of the set $F_\varepsilon$. Since $h_n^\varepsilon$ belongs to the reducing subspace of $U$ generated by $h_n$, the limits $\lim_{\alpha}(W_-^{\alpha} h_n^\varepsilon, h_n^\varepsilon)$ exist for every $n \in \mathbb{N}$. Next, let $n(\varepsilon)$ be the number for which $\|h^\varepsilon - h_{n(\varepsilon)}^\varepsilon\| < \varepsilon$. Choose the element $\alpha(\varepsilon) \in E$ such that $\|(W_+^{\alpha(\varepsilon)} - W_-^{\alpha(\varepsilon)}) h_{n(\varepsilon)}, h_{n(\varepsilon)}\| < \varepsilon$ for every $\alpha_1, \alpha_2 > \alpha(\varepsilon)$. We have

$$\|(W_-^{\alpha(1)} - W_-^{\alpha(2)}) h, h\| \leq 4\|A\|\|h\|\varepsilon + ||(W_-^{\alpha(1)} - W_-^{\alpha(2)}) h^\varepsilon, h^\varepsilon\| \leq 8\|A\|\|h\|\varepsilon + ||(W_-^{\alpha(1)} - W_-^{\alpha(2)}) h_{n(\varepsilon)}^\varepsilon, h_{n(\varepsilon)}^\varepsilon\| \leq (1 + 8\|A\|\|h\|)\varepsilon,$$

which implies the existence of $\lim_{\alpha}(W_+^{\alpha} h, h)$. 

where $h_2 \in C^1(\mathbb{T})$. By Lemma 6 we have

$$(p_{\alpha}(z), h_2) = \int \int \varphi(z)(\bar{h}_2(\xi) - h_2(\xi))k_{\alpha}(z, \xi) d\mu(\xi) d\mu(z).$$

Since $|D_n(\xi z)| \leq \frac{2}{|\xi - z|}$ for every $n \in \mathbb{N}$ (see formula (3) below), and $\sum_{n} p_{\alpha,n} = 1$, the integrand is bounded by $2|\varphi(z)|\sup_{\xi \in \mathbb{T}} |h_2(\xi)|$ for every $\alpha \in E$ and $z \in \mathbb{T}$.

By property $d)$ of s-regular summation method, the averages of every non-constant unimodular geometric progression tend to zero. For the Dirichlet number for which $\lim_{\alpha} p_{\alpha,\alpha} = 1$, the integrand is bounded by $2\Re((\xi z)^n - (\xi z)^{n+1})$.

Therefore, the limit $\lim_{\alpha} k_{\alpha}(z, \xi)$ equals zero if $\xi \neq z$. By the dominated convergence theorem we obtain $\lim_{\alpha} (p_{\alpha}(z), h_2) = 0$. 

$$D_n(\xi z) = \sum_{-n}^{n} (\xi z)^l = \frac{2 \Re((\xi z)^n - (\xi z)^{n+1})}{|1 - \xi z|^2}. $$

Remark. We have proved the fact that the operators $W_+^{\alpha} - W_-^{\alpha}$ tend to zero in the weak operator topology without assuming that the wave operators $W_\pm$ exist.
Assume now that norms of $h_n$ in $L^2(\mu)$ are uniformly bounded by $C$. For every elements $\alpha_1, \alpha_2 \succ \alpha(\varepsilon)$ and natural numbers $n_1, n_2 > n(\varepsilon)$ we have

\[
|(W_-(\alpha_1)h_{n_1}, h_{n_1}) - (W_-(\alpha_2)h_{n_2}, h_{n_2})| \leq 4\|A\|C\varepsilon + |(W_-(\alpha_1)h_{n_1}^\varepsilon, h_{n_1}^\varepsilon) - (W_-(\alpha_2)h_{n_2}^\varepsilon, h_{n_2}^\varepsilon)| \leq 8\|A\|C\varepsilon + |(W_-(\alpha_1) - W_-(\alpha_2))h_{n_1}^\varepsilon, h_{n_2}^\varepsilon)| \leq (1 + 8\|A\|C)\varepsilon,
\]

which implies the existence of the double limit $\lim_{\alpha, n}(W_-(\alpha)h_n, h_n).$ \hfill $\square$

**Proof of Theorem 2.** We give the proof for the operators $W_{\alpha}$ which implies the existence of the double limit $\lim_{\alpha, n}(W_-(\alpha)h_n, h_n).$

It is shown in [2, Theorem 7.2] that Theorem 2 holds if we omit item 4) in the statement. Hence, our aim is to prove the following fact: If the limit $\lim_n W_-(\alpha)$ exists for every triple $\mu$, $U$, $A$ satisfying conditions 1) – 4), then this limit exists for every triple satisfying conditions 1) – 3).

By the Luzin theorem, we can choose compacts $K_n$ such that $\mu(T \setminus K_n) \leq \frac{1}{n}$ and $\varphi$ coincides with $\varphi_n \in C(T)$ on $K_n$. Set $h_n = \chi_{K_n}$ and $\mu_n = \chi_{K_n} \, d\mu$ (as usual, $\chi_{K_n}$ denotes the characteristic function of the set $K_n$). For every number $n$ we have

\[(W_-(\alpha)h_n, h_n) = (M_{h_n} W_-(\alpha)M_{h_n}1, 1)\]

and the operator $M_{h_n} W_-(\alpha)M_{h_n}$ is exactly the past wave operator constructed for the operator $A_n = M_{h_n}AM_{h_n}$. Every triple $\mu_n$, $U$, $A_n$ satisfies properties 1) – 4), because the commutator $A_nU - UA_n$ equals

\[(\cdot, h_n\varphi)h_n - (\cdot, h_n)h_n\varphi_n = (\cdot, \varphi_n)1 - (\cdot, 1)\varphi_n.\]

An application of Lemma 7 ends the proof, due to the fact of pointwise convergence $h_n \to 1$ on the set $\cup_n K_n$ of full measure. \hfill $\square$

4. PROOF OF THEOREM 3

Define operators $P_r$, $r \in (0, 1)$, on $L^2(\mu)$ by the formula

\[P_r : \varphi \mapsto \int_T (\varphi(\xi) - \varphi(z)) \frac{1 - r^2}{|1 - r\xi z|^2} \, d\mu(\xi).\]

**Lemma 8.** There exist a Borel singular measure $\mu$ on $T$ without atomic masses and a function $\varphi \in C(T)$ such that the norms of $P_r\varphi$ in $L^2(\mu)$ are unbounded.

**Proof.** Define a function $\varphi$ on $T$ by

\[
\varphi(z) = \begin{cases} 
\sqrt[4]{\text{arg}(z)}, & \text{if } \text{arg}(z) \in [0, \frac{\pi}{4}]; \\
0, & \text{if } \text{arg}(z) \in (0, -\frac{\pi}{4}); \\
g(z), & \text{if } \text{arg}(z) \in (\frac{\pi}{4}, \frac{3\pi}{4}).
\end{cases}
\]
Here \( g(z) \) is an arbitrary nonnegative function such that \( \varphi \in C(\mathbb{T}) \). Let \( \psi \in C(\mathbb{T}) \) be determined by \( \psi(z) = \varphi(z) \). For every measure \( \mu \) supported on \( I = \{ z : \arg(z) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \} \), we have

\[
(P_r \varphi, \psi) = \int \int \varphi(\xi)\psi(z) \frac{1 - r^2}{|1 - r\xi z|^2} d\mu(\xi) d\mu(z),
\]
due to the fact that \( \varphi\psi = 0 \) on \( I \). Find a family of arcs \( I_k \subset \mathbb{T}, k \in \mathbb{Z} \setminus \{0\} \), such that

1) \( a_k = \exp(\text{sgn}(k)2^{-|k|} i) \) is the center of \( I_k \), \( \text{sgn}(k) = \frac{k}{|k|} \),

2) \( \varphi(z) \geq \frac{1}{2} \varphi(a_k) \) and \( \psi(z) \geq \frac{1}{2} \psi(a_k) \) for every \( z \in I_k \),

3) \( \sup_{z \in I_k, \xi \in I_k} |z - \xi| \leq 2^{-|k|+2} \).

Now define the measure \( \mu \) on \( I \) by the formula \( \mu = \sum_{k \in \mathbb{Z} \setminus \{0\}} k^{-2} \mu_k \), where probability measures \( \mu_k \) are supported on the arcs \( I_k \). As follows from the construction, the measure \( \mu \) can be chosen to be singular and free of point masses. For the measure \( \mu \) and the number \( r_m = 1 - 2^{-m} \), \( m \geq 1 \), we have

\[
(P_r \varphi, \psi) \geq \frac{\varphi(a_m)\psi(a_m)}{4m^4} \int_{I_m \times I_m} \frac{1 - r_m^2}{|1 - r_m\xi z|^2} d\mu(\xi) d\mu(z)
\]

\[
\geq \frac{\varphi(a_m)\psi(a_m)}{4m^4} \inf_{\xi \in I_m} \frac{1 - r_m}{|z - r_m \xi|^2 + |r_m z - r_m \xi|^2}
\]

\[
\geq \frac{1}{100m^4} 2^m,
\]

which tends to infinity when \( m \) increases. Thus, we have \( \sup_r (P_r \varphi, \psi) = \infty \), therefore \( P_r \varphi \) are cannot be bounded in \( L^2(\mu) \).

\[\square\]

Remark. In fact, we have proved that \( P_r \varphi \) are unbounded even in the norm of \( L^1(\mu) \). This makes the convergence of \( P_r \varphi \) impossible in any reasonable topology.

Proof of Theorem 3. Let \( \varphi \) be the function constructed in Lemma 8. Assume that \( AU - UA = (\cdot, \varphi)I - (\cdot, 1)\varphi \) and consider the averaged wave operators \( W_\pm(r) \) with respect to the Abel-Poisson summation method. In our notations this means that \( E = ((0,1), \leq) \), \( p_{r,n} = (1 - r)r^n \), and the operators \( W_\pm(r) \) are defined by formula (1) from Section 1 (now we use the letter \( r \) in place of \( \alpha \)). The Abel-Poisson means of the Dirichlet kernels \( D_n(\xi z) \) are equal to the Poisson kernel \( \frac{1 - r^2}{|1 - r\xi z|^2} \). Formula (2) gives us

\[
(W_+(r)U - UW_-(r))I = P_r \varphi.
\]

In particular, we have \( \sup_r \|P_r \varphi\|_{L^2(\mu)} \leq 2 \|A\| \), which contradicts the conclusion of Lemma 8. \[\square\]
5. Boundary behaviour of Cauchy-type integrals.

Lemma 8 is of special interest because of recent result by V.V. Kapustin concerning the boundary behaviour of Cauchy-type integrals. To formulate this result, we introduce several standard objects from the theory of pseudocontinuable functions. The reader can find a more detailed discussion in [3].

Given a singular probability measure $\mu$ on $\mathbb{T}$, define an inner function $\theta$ by the formula

$$\text{Re} \left( \frac{1 + \theta(z)}{1 - \theta(z)} \right) = \int \frac{1 - |z|^2}{|1 - \xi z|^2} d\mu(\xi), \quad |z| < 1.$$ 

The function $\theta$ generates a family of singular probability measures $\{\sigma_\alpha\}_{\alpha \in \mathbb{T}}$ by

$$\text{Re} \left( \frac{\alpha + \theta(z)}{\alpha - \theta(z)} \right) = \int \frac{1 - |z|^2}{|1 - \xi z|^2} d\sigma_\alpha(\xi), \quad |z| < 1.$$ 

By the definition, we have $\mu = \sigma_1$. With the inner function $\theta$ we associate the subspace $K_\theta = H^2 \ominus \theta H^2$ of the Hardy space $H^2$ in the unit disk of the complex plane. Functions from $K_\theta$ (also referred to as $\theta$-pseudocontinuable functions) have boundary values $\sigma_\alpha$-almost everywhere for every measure $\sigma_\alpha$, see [4].

Define operators $C_r$, $r \in (0,1)$ on $L^2(\mu)$ by the formula

$$C_r : \varphi \mapsto \int_\mathbb{T} (\varphi(\xi) - \varphi(z)) \frac{1}{1 - r\xi z} d\mu(\xi).$$

Theorem 9 (Theorem 1.3 of [3]). Let a function $h \in K_\theta$ coincide $\sigma_\alpha$-almost everywhere with a continuous function $\varphi_\alpha$, where $\alpha \neq 1$. Take the function $\varphi \in L^2(\mu)$ such that $h = \varphi \sigma_1$-almost everywhere. If $\sigma_1$ has no atomic masses, the family $\{C_r \varphi\}$ converges in norm of $L^2(\mu)$ to $\frac{\varphi_1 - \varphi}{\alpha - 1}$.

As is shown in [2], for the Abel-Poisson summation method the general problem of existence of $W_\pm$ in the case of rank-two commutator is equivalent to the convergence of $C_r \varphi$ for every function $\varphi$ corresponding to a commutator 1)–3) from Section 1. Theorem 9 establishes the convergence of $C_r \varphi$ for functions $\varphi$ with a continuous “transplantation” $\varphi_\alpha$. On the other hand, it follows from Theorem 2 that we can check the convergence of $\{C_r \varphi\}$ only for continuous functions $\varphi$ without loss of generality. By the definition we have $\varphi_1 = \varphi$. Unfortunately, the convergence in Theorem 9 fails if we remove the assumption $\alpha \neq 1$.

Proposition 10. There exists a Borel singular measure $\mu$ on $\mathbb{T}$ without atomic masses and a continuous function $\varphi \in C(\mathbb{T})$ such that the norms of $C_r \varphi$ in $L^2(\mu)$ are unbounded.
Proof. Using the identities
\[
\frac{2}{1 - r\overline{\xi}z} - 1 = \frac{1 + r\overline{\xi}z}{1 - r\overline{\xi}z} \quad \text{and} \quad \text{Re} \left( \frac{1 + r\overline{\xi}z}{1 - r\overline{\xi}z} \right) = \frac{1 - r^2}{|1 - r\overline{\xi}z|^2},
\]
one can to show that \( \|P_r(\varphi)\|_{L^2(\mu)} \leq 2\|C_r(\varphi)\|_{L^2(\mu)} + \|\varphi\|_{L^2(\mu)} \) for every real-valued function \( \varphi \in L^2(\mu) \). It remains to apply Lemma 8.

As is easily seen from Proposition 10, the operator \( C : f \mapsto \lim_r C_r f \) is not a bounded operator on \( L^2(\mu) \). On the other hand, the operator \( C \) is well defined on smooth functions. One of natural ways to define it on a wider subset of \( L^2(\mu) \) could be the following: at first we define the operator \( C \) as the limit \( \lim_r C_r f \) on all functions \( f \in L^2(\mu) \) for which the limit exists in \( L^2(\mu) \), and then take its closure of the graph of \( C \). Unfortunately, this way does not work, as the following proposition shows.

Proposition 11. The operator \( C \), defined on all functions \( f \) for which the limit \( \lim_r C_r f \) exists in \( L^2(\mu) \), is not a closable operator on \( L^2(\mu) \).

Proof. By Theorem 9 we have \( C \varphi = 0 \) if \( \varphi \) is the trace on \( \mathbb{T} \) of a continuous function \( h \in K_\theta \) (in this case \( \varphi_\alpha = \varphi \) almost everywhere with respect to \( \sigma_1 \)). A well-known result by A.B.Aleksandrov says that continuous functions from \( K_\theta \) form a dense subset in \( K_\theta \). Since the operator taking a function from \( K_\theta \) to its boundary values \( \sigma_\alpha \)—almost everywhere is a unitary operator from \( K_\theta \) to \( L^2(\sigma_\alpha) \), see [4, 6], the traces of continuous functions from \( K_\theta \) are dense in \( L^2(\mu) \). Hence the fact that \( C \) a closable operator would imply \( C = 0 \). But, obviously, \( Cf = -\overline{z} \neq 0 \) for the function \( f(z) \equiv \overline{z} \).

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