The Spectrum Of Hypercubes Quotiented By Doubly Even Codewords And The Thermodynamics Of Adinkras

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ABSTRACT

In a previous paper, a solution to the problem of determining isomorphism classes of Lie algebra representations was explored using graphs called adinkras and subgraphs called baobabs [1]. In this paper, I show that adinkras contain Shannon entropy and a latent heat from the information stored in their associated baobabs. In Garden algebra, both properties are closely related to the spectrum of hypercubes quotiented by doubly even codewords, which is introduced in this paper.

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“Make things as simple as possible, but not simpler.”

– Albert Einstein

1 Introduction

Adinkras are graphs with distinct edge colors and edge weights that encode infinitesimal generator representations [3] [10] [1]. Special subgraphs that contain all of the degrees of freedom of an adinkra are called baobabs. Baobabs are used to differentiate isomorphism classes of representations of Super Lie algebras via their encoded degrees of freedom[2]. Many baobabs can create a single adinkra, although if baobabs with the same topology and edge colors (i.e. *chromotopology*) are observed among two adinkras, then their isomorphism class can be rapidly determined [1].

In this paper, we will focus on a type of adinkra that is well known for classifying supersymmetry in 1 dimension, called a Garden adinkra[3]. In [1], it was found that traversing any spanning tree and a few well-chosen cycles of a Garden adinkra will determine all of its properties. In this paper, the author will determine the surjective nature of baobabs in order to better understand the isomorphism class of baobabs themselves, as well as the energy needed to create an adinkra from a baobab. This energy can be considered the latent heat of the phase transition between baobabs and adinkras, introducing a previously unexplored thermodynamic interpretation of adinkras.

It is rather non-obvious to determine the number of baobabs in a given adinkra, because one must know the spectrum first. The topology of a Garden adinkra, however, is that of a quotiented hypercube, therefore, the well known spectrum of a hypercube has been used in part to determine this new spectrum.

The paper will be organized as follows: section 2 will give an overview of baobabs and adinkras in Garden algebra. Section 3 will determine the spectrum of hypercubes quotiented by doubly even codewords, which are used to bound the multiplicity of

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2Previous work has shown that determining the isomorphism class of an adinkra \(\implies\) finding the isomorphism class of the associated baobab [1].

3To better understand Garden adinkras, see [3] or [10]. The former characterizes these graphs from a mathematician’s perspective, while the latter, from the perspective of physicists.
baobabs, ignoring directed edges. Section 4 will introduce some thermodynamic properties of adinkras, and section 5 will conclude the paper. Furthermore, an appendix is provided with the proof of the doubly-even quotiented hypercube spectrum. For the rest of the paper, a “quotiented hypercube” is assumed to be quotiented by doubly even codewords.

2 Preliminaries

The elements of Garden algebra can be represented as a graph (a Garden adinkra), such as Fig. # 1, where the adjacency matrix of a distinct edge color (or adinkra adjacency matrix) corresponds to an element’s representation, with the appropriate non-zero matrix elements.

![Figure 1: Two-color adinkras with labels for bosons (open nodes) and fermions (closed nodes).](image)

A garden adinkra adjacency matrix element, $a_{ij}^k$, picks up a minus sign if the respective edge is dashed, while $a_{ij}^k = \alpha \frac{d}{d\tau}$ and $a^j_i = \alpha'$, if $\bullet_i \leftarrow \bullet_j$. Finally if an edge connects an open (boson) node to a closed (fermion) node the matrix element picks up a phase $\dot{i}$. For example, if bosons are labeled $\Phi_i$ and fermions, $\Psi_j$, the relations between nodes for Fig. # 1(a) can be represented as

$$D_{red} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$$

$$D_{red} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = i \begin{pmatrix} 0 & -\frac{d}{d\tau} \\ \frac{d}{d\tau} & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$
We can create a block matrix that makes the above notation more compact, called an adinkra adjacency matrix. With the supervector $\Phi \oplus \Psi$, the adinkra adjacency matrix for each edge color is of the form:

$$\Gamma_I = \left( \begin{array}{cc} 0 & D_{I_L} \\ D_{I_R} & 0 \end{array} \right)$$ \quad \forall I \in \{1, \ldots, n+k\}$$

(3)

where $D_{I_L}$ permutes the the fermions, and $D_{I_R}$, the bosons, in the above examples. The degrees of freedom of the generators can be symbolized by a subgraph called a baobab $\mathbb{P}$. For example, the above graphs in Fig. # 1 can be represented by their respective baobabs in Fig. # 2.

Lie baobabs so far explored seem to have relatively few cycles (this varies among superalgebras) implying that one can determine the weights of an adinkra in almost linear time compared to naively exploring every edge, which may take up to $O(N[N-1])$ if the graph is $K_N$. 

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Because Lie adinkras always contain some constrain with respect to their edge weights, the Lie baobab is always smaller than the adinkra. It is thus possible that many different Lie baobabs can create the same Lie adinkra. Determining the exact multiplicity of these graphs was an open question for all but the simplest examples.

3 The Spectrum Of Quotiented Hypercubes

There are important exceptions, however. Ignoring edge directions, a Garden adinkra with \( n \) edge colors and \( 2^n \) nodes contains:

\[
\sum B(I^n) = \frac{1}{2^n} \prod_{j=1}^{n} (2j)^{\binom{n}{j}}
\]  

spanning trees, and this an equal number of baobabs \footnote{See [5], [12], [13] or [3] to understand the relation between doubly even codewords and Garden algebra.}, because those adinkras are \( n \)-cubes. This calculation was easily derived from the hypercube spectrum, therefore, one may intuitively expect that knowing the spectrum of adinkra topologies in general will help determine the multiplicity of the baobabs. Garden adinkras are either \( n \)-cubes or quotiented \( n \)-cubes \footnote{See [5], [12], [13] or [3] to understand the relation between doubly even codewords and Garden algebra.}, therefore all we have to do is determine the spectrum of quotiented hypercubes. In this section, I will determine their spectrum, and explain some basic properties of this spectrum, before I use it to determine the number of spanning trees in the graphs.

It has been well studied how a quotiented hypercube can be represented by a set of doubly even codewords, \( \{C_i\} \). If each node is labeled on a unit hypercube by a boolean vector, \( \mathbb{Z}_2^n \), then the quotient modulates the vectors: \( \mathbb{Z}_2^n / C \) where \( C \) is the set of doubly even codewords.
Theorem 1: The Laplace spectrum is:

\[ \{ \prod_{m=0}^{n} \prod_{p=0}^{k} (2(m+p)=\text{const.}) [2(m + p)]^{\mathcal{M}(m,p)} \} \]

where \( \lambda_i^N \) means that the eigenvalue \( \lambda_i \) has degeneracy \( N \). The regular spectrum is \( \lambda_i - (n + k) \) because the graph is strongly regular [9]. \( \mathcal{M}(m, p) \) is the multiplicity for each \( m \) and \( p \), equal to:

\[
\mathcal{M}(m, p) = \sum_{i \leq k, p_i = \sum_j (p_j \mod 2)} \|C_i \| - 1 \] \(\sum_{q \leq k, j \leq q, i_j - 1 < i_j \leq k - (q - j), p_{i_1...i_q}}\)

\[
\prod_{r=1}^{k} \prod_{j \leq r, i_j = 1 + i_{j-1}} \left( \|C_i \| - \sum_{q' \leq k, j' \leq q', \ell_{j' \notin \{i_1,...,i_r\}} (1) \|C_i \| \right)
\]

\( p_{i_1...i_q} + \sum_{q' \leq k, j' \leq q', \ell_{j' \notin \{i_1,...,i_r\}} (1) \|C_i \| \right)

\[
(\sum_{q \leq k, j \leq q} \sum_{r=1}^{k} \sum_{j \leq r, i_j = 1 + i_{j-1}} (1) \|C_i \| \right)
\]

\[
(\sum_{q' \leq k, j' \leq q', \ell_{j' \notin \{i_1,...,i_r\}} (1) \|C_i \| \right)
\]

\[
(\sum_{q' \leq k, j' \leq q', \ell_{j' \notin \{i_1,...,i_r\}} (1) \|C_i \| \right)
\]

Proof: See Appendix.

Here, \( N[i_1...i_r...j] \) represents ordering lexicographically, (e.g. \( p_{132} \rightarrow p_{123} \)) and “\( \|C_i \| \)” symbolized the weight of the code. As an aside, although the above formula may at first appear complicated, it can simplify into a few well known cases. For example, when \( k = 0 \), the spectrum is that of a hypercube, as expected. In addition, when \( N = 4 \), or when \( N = 8 \), the quotiented hypercubes become complete bipartite graphs with 8 and 16 nodes, respectively. In both cases, one can compute by hand that the eigenvalues are \( \{0, 4^6\} \) and \( \{0, 8^{12}, 16\} \). Lastly, Cartesian products of the above graphs are also known.

Thm. # 1 implies an important property relating to the set of co-spectral graphs. Any codewords whose columns can be permuted from one to the other have the exact same spectrum, even when they represent completely distinct topologies. For example, the below codewords have their first and fifth columns switched and represent distinct topologies because they are not isomorphic. Their corresponding spectrums, however, are the same.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
The spectrum is not, however, topologically invariant. As studied in [5], the following set of codewords have distinct topologies and distinct spectra:

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(8)

Trivially $\|C_i\|, \|C_i \wedge C_j\|, \ldots$, are distinct, and thus the spectra are distinct. Therefore a meta-equivalence class exists among codewords which are permutation inequivalent but non-trivially still have the same spectrum.

With these results, the overall appearance of the spectrum can be explored. Fig. #4 shows how the Laplace spectrum changes for an $n + k$-cube quotiented $k$ times. For $k = 3$, 3 distinct codes were found, with a different width, but similar maxima. Interestingly the maximum multiplicity shifts to the right as $k$ increases. This agrees with ones intuition because $\lambda_{\text{mode}} \simeq \frac{n}{2}$ for a hypercube graph, and $2^{n-1}$ for a complete bipartite, thus as $k$ changes, there should be some sort of gradual shift in $\lambda_{\text{mode}}$. When $n + k$ is large, it can be shown that the change in $\lambda_{\text{mode}}$ is linear.
Proposition 1:

\[ \lambda_{\text{mode}} \simeq n + k, \ n + k \gg 1 \quad (9) \]

Proof:

To find the mode of the spectral density,

\[ \frac{\partial \lambda}{\partial p} \sum_p M(\lambda/2 - p, p) = 0 \quad (10) \]

To remove the summation, the maximum multiplicity is explored

\[ \prod_{i_1} \left( \frac{\|C_{i_1}\| - \sum_{j < i_2} \|C_{i_1} \wedge C_{i_2}\| \ldots}{p_{i_1} - \sum_{j < i_2} p_{i_1} p_{i_2} \ldots} \right) \prod_{i_2} \left( \frac{\|C_{i_1} \wedge C_{i_2}\| \ldots}{p_{i_1} p_{i_2} \ldots} \right) \ldots \quad (11) \]

are simultaneously maximum when \( p_{i_1} = (\|C_{i_1}\| - 1)/2, \ p_{i_1 i_2} = \|C_{i_1} \wedge C_{i_2}\|/2, \) etc. as can easily be proved by induction. Changing \( p_1, p_2, \ldots \) by 1 makes a negligible change to the the binomial coefficient, although it significantly changes \( p \). Given \( p_1, \ldots, p_k \), there are \( \binom{k}{p} \) ways of changing \( p_i \) such that \( p \) is constant, hence \( M(\lambda/2 - p, p) \) is maximal when \( p = k/2 \). Therefore,
\[ M_{\text{peak}}(\lambda/2 - p, p) = \binom{k}{k/2} \left( \frac{\|C_1\| - \sum_{i' > i} \|C_i \wedge C_i\| + \ldots}{\|C_1\| - \sum_{i' > i} \|C_i \wedge C_i\| - \ldots} \right)^{\lambda/2} \left( \frac{\|C_1\| - \sum_{i' > i} \|C_i \wedge C_i\| + \ldots}{\|C_1\| - \sum_{i' > i} \|C_i \wedge C_i\| - \ldots} \right) \ldots \]

Therefore, the partial derivative can be put inside the sum. By dropping values that are constant:

\[ \partial_{\lambda} \left( \frac{n + k - \alpha}{\lambda/2 - \alpha/2} \right) = 0 \implies \lambda = n + k \] (13)

where \( \alpha = \sum_{i'} \|C_{i'}\| - \sum_{i' > i} \|C_{i'} \wedge C_{i'}\| + \ldots \).

Lastly, it should be noted that, by using the Gaborit mass formula [6], one can determine \( k_{\text{max}} \) (the number of times in \( N \)-cube can be quotiented). This observation comes directly from research done by the DFGHIL collaboration [5]. For an \( N \)-cube, with \( N = n + k = 8m + s \), one can determine \( k_{\text{max}} \) by the function:

\[
k_{\text{max}}(n+k) = \begin{cases} 
4m & \text{for } (n+k) \equiv 0, 1, 2, \text{ or } 3 \pmod{8} \\
4m + 1 & \text{for } (n+k) \equiv 4 \text{ or } 5 \pmod{8} \\
4m + 2 & \text{for } (n+k) \equiv 6 \pmod{8} \\
4m + 3 & \text{for } (n+k) \equiv 7 \pmod{8}
\end{cases}
\] (18)

Therefore, \( C_{k > k_{\text{max}}} = 00 \ldots \) implying \( M(m, p > k_{\text{max}}) = 0 \) and \( M(m > n + k_{\text{max}}, p) = 0 \).

The Matrix-Tree Theorem states that the number of spanning trees in a graph is \( 1/N \lambda_2 \cdots \lambda_N \). For a quotiented hypercube, this becomes:

\[
\sum T(I^n/C) = \prod_{m}^{n} \prod_{p}^{k} [2(m + p)]^{M(m, p)}
\] (19)

Because the baobab is made from spanning trees, bounds on the baobab multiplicity for a given adinkra can now be determined, ignoring edge directions.

4 The Entropy Of An Adinkra

In this section the multiplicity of the baobabs (ignoring edge directions) will be shown to be
This paper only focuses on the multiplicity of dashed edges and not directed edges in a baobab, due to the complications directed edges present. The multiplicity of directed edges is small for any valid baobab topology, but there are topologies, such as a Hamilton path in a hypercube graph, which are invalid under certain circumstances. The affect of these directed edges on the multiplicity is likely to be minimal, therefore they are safely ignored in this paper.

As explain in [1], a Garden baobab (which we will simply label a “baobab” for the rest of the paper) is a spanning tree with $k$ cycles, each of which contains at least four colors that do not appear an even number of times. Furthermore, there exists an edge color in each cycle doesn’t appear in any other cycle. Because there are $2^n-1$ edges of the same color, there are $2^n-1$ equivalent cycles one can create a baobab from. Ignoring overcounting, this would imply the number of baobabs is $\left( \sum T(I^n/C) \right)^2 (n-1)^k$. 

In truth, however, many spanning trees can create the same cycle (for example, a cycle $C_N$ has $N$ spanning trees [9]). How many baobabs there really are is therefore an open question in general. Simple arguments, however, can set limits on the number of baobabs. If every edge was part of a cycle, then there could be no more than $\left\lceil (2^n + k - 1)/k \right\rceil^k$ trees that make the same baobab. One can arrive at this number by imagining a baobab a collection of cycles $\{C_{N_1}, C_{N_2}, ... , C_{N_k}\}$ with exactly one node in common between any cycle and $N_1 + N_2 + ... + N_k = 2^n + k - 1$. This value is clearly largest when $N_1 = N_2 = ... = N_k$. Unrealistically, however, this assumes that the cycles always span the entire baobab. The number of trees in all baobabs is therefore less than $\left\lceil (2^n + k - 1)/k \right\rceil^k$.

The minimum number of trees is when the $k$ cycles are smallest and have the most edges in common. This happens when all cycles are of size $\|C_{i_1}\|$ (where $\|C_{i_1}\|$ is the weight of the codeword), and have $\|C_{i_1} \land C_{i_2} \land ... \|$ edges in common with one another. The total number of trees is therefore less than or equal to $\prod_{i_q \leq r, i_1 < i_2 < ... < i_q} \|C_{i_1} \land C_{i_2} \land ... \land C_{i_q}\|^{(-1)(q-1)}$. This implies (18).

It is given that when $k = 0$, the upper and lower bounds are the same. In addition, one can see that when $n = 3$ and $k = 1$, the total number of baobabs is equal to $(\sum T(I^3_1))/\|C_1\|$ implying the upper bounds is tight, although simple constructions show that the bound is not tight in general. It seems likely that the value approaches
the lower bounds in the limit of $k \ll n$ while it approaches the upper bounds when $k \sim n$.

The log of the baobab multiplicity is the Shannon entropy of the adinkra. Once the adinkra is fully constructed, information is lost about the original baobab. The second law of thermodynamics can be used to determine the minimum energy needed to create an adinkra [1]:

$$\delta Q = \Delta S T$$  \hspace{1cm} (26)

This value can be regarded as the latent heat needed for a phase transition from the baobab to the adinkra. Although this was derived for Garden algebra, the multiplicity and hence the entropy and heat generation is seen for all Lie adinkras.

## 5 Thermodynamics Of Adinkras

By interpreting baobabs as “microstates”, and the complete adinkra as the “macrostate” the multiplicity of an adinkra can be defined as as the number of baobabs that can create same adinkra. This definition is equivalent to Shannon entropy by noting that information is lost under the surjective transformation from a baobab to the adinkra. Just as logic gate turns 2 bit of information into 1 bit, and therefore looses information about the initial input, an adinkra cannot reconstruct the initial baobab and thus looses information.

Note that the logic gate concept of adinkras only works by assuming the topology of the baobab is known. In other words, to create the logic gate set-up from the baobab, organize all edges as ordered pairs of nodes with a bit to determine dashing $\mathbb{Z}_2^n / \mathbb{C} \times \mathbb{Z}_2^n / \mathbb{C} \times \{0, 1\} = (u, v, x)$. The logic gates for the dashing of a baobab are all functions $\text{NDXOR}$ such that:

$$\text{NDXOR}_{\text{baobab}} :$$

$$\{(u, u \oplus C_i, x), (u, v, y), (v, v \oplus C_i, z)\} \rightarrow (u \oplus C_i, v \oplus C_i, \neg(x \oplus y \oplus z))$$  \hspace{1cm} (27)

Once the logic-gate setup is known, edge dashing can be fed into the gates as input bits. This implies that the Shannon entropy is due to the uncertainty of the baobab’s construction.
Because $\sum T(I^n/C) > \sum T(I^n) > 2^n \gg 2^{(n-1)k} > \frac{\sum B(I^n/C)}{\sum T(I^n/C)}$ for large $n$ and $k$:

$$S = k_b \ln(\sum B) \simeq k_b \ln(\sum T(I^n/C)) + \mathcal{O}(k \ln(k)) \quad (26)$$

to a good approximation, if we assume it is equal probable for any valid baobab to create a given adinkra.

Because the entropy of an adinkra is known, so is the minimal energy necessary to construct an adinkra from a baobab. Recall for small changes in energy for systems that are approximately isothermal, the heat,

$$\delta Q \geq k_b \ln(\sum T(I^n/C)) T \quad (27)$$

where $\Delta S$ is the small change in entropy, and $T$ is temperature. Therefore, we can find the minimal heat lost in order to create the adinkra by the value $\Delta S T$.

### 6 Conclusion

In this paper, I introduced the spectrum of a quotiented hypercube, which to the knowledge of the author, has never appeared previously in literature. Furthermore, using the Laplace spectrum, the number of spanning trees was found, allowing for a strong bound on the number of baobabs in an adinkra, if directed edges are ignored. Lastly, knowing the multiplicity of the baobab, one can determine the Shannon entropy, as well as the minimal energy necessary to create an adinkra. This allows for the interpretation that, by completing 2-color loops, one can create a phase transition between a baobab and an adinkra with some latent heat.

There is much left to study, however. Edge directions were completely ignored, for example. By taking edge directions into account, bounds on the multiplicity of baobabs can be significantly improved. Furthermore, it is an open question whether a similar argument can determine the spectrum and thermodynamics of other Lie adinkras.

### 7 Appendix

Equations (5) and (6) will be proven using a construction provided below.
Construction 1: Any \((n, k)\) set of codewords can be put into a form where \(\forall i, \|\alpha'_i\| = \|C_i - \sum_{j \neq i} C_i \land C_j + \sum_{\ell > j, \ell \neq i} C_i \land C_j \land C_\ell - ...\| > 0\). This means each code will contain at least one column where, \(\forall i\), the \(i\)th row contains the only bit.

- If \(\|\alpha'_j\| = 0\), then choose some bit \(x \in C_i\). \(\forall i \neq j\) where \(\|0...0x...0 \land C_j\| \neq 0\), \(C_j \rightarrow C_j \oplus C_i\). It is easy to see that this forces \(\|\alpha'_i\| > 0\).

- If any other rows contain \(\|\alpha'_j\| = 0\), then repeat the first step (by the construction \(\|\alpha'_j\|\) will stay positive definite).

This construction allows us to create a nice form from which to determine the multiplicity of baobabs (ignoring directed edges). It is easy to see from the act of quotienting that an edge, \(\Gamma_i = \Gamma_a \Gamma_b \Gamma_c\ldots\) where \(\{\Gamma_i, \Gamma_a, \Gamma_b, \Gamma_c, \ldots\} \in C_i\). Therefore, while hypercube edges are usually of weight 1, quotiented hypercubes are of weight 1 or \(\|C_i\| - 1\). By choosing \(\Gamma_a \Gamma_b \Gamma_c\ldots \in C_i - x\), where \(x \in \alpha', u \land C_i \equiv u \land \Gamma_i \pmod{2}\) (without this construction, \(\Gamma_a\) could be equal to \(\Gamma_e \Gamma_f \Gamma_g\ldots\) where \(\{\Gamma_a, \Gamma_e, \Gamma_f, \Gamma_g, \ldots\} \in C_j\), implying \(u \land C_i\) may or may not equal \(u \land \Gamma_i\).

Theorem 1: The Laplace spectrum of a quotiented hypercube is (5) with multiplicity (6) and \(k \leq k_{\text{max}}\).

Proof:

All nodes can be labeled as a quotients of \(n + k\) boolean vectors \(\mathbb{Z}_2^n / C\). All edges, by construction \#1, can be codewords of weight 1 or \(\|C_i\| - 1\), where the weight \(\|C_i\| - 1\) codewords are equal to \(C_i \oplus y\) where \(y \in \alpha'_i\).

Fix two vertices \(u\) and \(v\), and let \(p_i = \|u \land (C_i \oplus y)\|\). When \(p_i = a \pmod{2}\), \((-1)^{u \cdot v} = a(-1)^{u \cdot w}\). Here, \(\cdot\) is the inner product of the boolean vectors (i.e. \(\|u \land v\|\)), \(w \sim v\) and \(a = \{0, 1\}\). \(\|u\| = p_1 + p_2 + \ldots + p_k - \beta\), where \(\beta\) is some number that will be determined later, and \(p\) variables will be \(1 \pmod{2}\) while \(k - p, 0 \pmod{2}\). Because \(k\) weight 1 codewords became weight \(\|C_i\| - 1\) codewords, \(n\) of \(v\)'s neighbors have a weight that differs by \(1\). \((-1)^{u \cdot v} = (-1)^{u \cdot w}\) for \(m = \|u\|\) neighbors of \(v\), and \((-1)^{u \cdot v} = (-1)^{u \cdot w}\) for \(n - m\) neighbors of \(v\). Similarly, from the previous observation, \((-1)^{u \cdot v} = (-1)^{u \cdot w}\) for \(p\) neighbors of \(v\), and \((-1)^{u \cdot v} = (-1)^{u \cdot w}\) for \(k - p\) neighbors.
of \( v \). As an ansatz, let \( f_u = \{(-1)^{u \cdot v}\} \) be the eigenvector. Therefore, the \( v \)th row of \( L(f_u) \) is

\[
L(f_u)\bigg|_v = (n + k)(-1)^{u \cdot v} - \sum_{w} (-1)^{u \cdot w}
\]

\[
= (n + k)(-1)^{u \cdot v} - (n - m - k - p)(-1)^{u \cdot v} + (k + p)(-1)^{u \cdot v}
\]

Simplifying, \( L(f_u) = 2(m + p)f_u \). By construction the eigenvectors are clearly linearly independent, therefore our ansatz is valid. The multiplicity is slightly more complicated to compute. The multiplicity of the portion of \( u \in \alpha_i' \) is simply:

\[
\left(p_{i1} - \sum_{i2'} \neq i1 p_{i1 i2'} \sum_{i3'} \neq i1 i2' p_{i1 i2' i3'} - \ldots\right)
\]

where \( p_{i1i2'} = u \cdot (C_{i1} \land C_{i2'}) \leq \|C_{i1} \land C_{i2'}\| \), and \( p_{i1i2'i3'} = u \cdot (C_{i1} \land C_{i2'} \land C_{i3'}) \leq \|C_{i1} \land C_{i2'} \land C_{i3'}\| \), \ldots The sum of \( p_{i1} - \sum_{i2'} \neq i1 p_{i1 i2'} + \sum_{i3'} \neq i1 i2' p_{i1 i2' i3'} - \ldots \) in the binomial coefficient is simply \( u \cdot \alpha_i' \). By summing \( p_{i1i2...} = 0, 1, 2, \ldots \) such that \( p = \sum_j [p_j (mod 2)] \) one can find the total multiplicity. Similarly, the multiplicity of the bits affected by \( C_{i1} \land C_{i2} \) alone is:

\[
\left(\|C_{i1} \land C_{i2}\| - \sum_{i3'} \neq (i1 i2) \|C_{i1} \land C_{i2} \land C_{i3'}\| + \ldots\right)
\]

which is independent of (23). Continuing in this fashion, one can determine the multiplicity up until \( C_{1} \land C_{2} \land \ldots \land C_{k} \). The total multiplicity of the bits affected by quotienting is the product of all these multiplicities. To determine the multiplicity of the bits unaffected by the quotienting, the simple relation below is used:

\[
\left(m - \sum_{i1} p_{i1} + \sum_{i2'} \neq i1 p_{i1 i2'} - \ldots\right)
\]

Therefore the spectrum is (5) and (6).

\[ \square \]

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