THE CONSTRAINED KP HIERARCHY AND THE GENERALISED MIURA TRANSFORMATION

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ABSTRACT

Recently much attention has been paid to the restriction of KP to the submanifold of operators which can be represented as a ratio of two purely differential operators $L = AB^{-1}$. Whereas most of the aspects concerning this reduced hierarchy, like the Lax flows and the Hamiltonians, are by now well understood, there still lacks a clear and conclusive statement about the associated Poisson structure. We fill this gap by placing the problem in a more general framework and then showing how the required result follows from an interesting property of the second Gelfand-Dickey brackets under multiplication and inversion of Lax operators. As a byproduct we give an elegant and simple proof of the generalised Kupershmidt-Wilson theorem.

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\( \mathcal{W} \) algebras provide a link between two dimensional conformal field theories and integrable systems. At the heart of their relation with the former lies the so called “free field realization”, which in the context of the latter receives the name of (quantized) Miura transformation \([1]\). On the other hand, the approach based on integrable systems has provided a unifying framework for those algebras via the formalism of pseudo-differential operators \((\Psi\text{DO}); \text{ i.e. there exists a basic common ingredient, the so called Adler map, from which the } \mathcal{W} \text{ algebras can be readily constructed as Poisson bracket algebras. The data that specify a particular } \mathcal{W} \text{ algebra is encoded in the particular form of the associated Lax operator. This scheme has raised the hope of establishing a full classification or atlas. Whereas only partial results have been achieved, the increasing size of this atlas, with new examples being constructed everyday, calls for a deeper understanding of the way in which different } \mathcal{W} \text{ algebras can be related.}

This letter is a modest step towards this direction. Although many of the results presented here were already known, our approach makes emphasis on the astonishing simplicity that lies behind a number of important and, up to now, disperse results, and whose individual proof required a considerable amount of insight and calculational thrust.

Our original motivation stemmed from the construction of free (multi) boson realization for non-linear \( \mathcal{W}_\infty \) type of algebras\([2]\). From the point of view of integrable systems, the associated piece of data is a KP-Lax operator of the form \( L = AB^{-1} \), where \( A = (\partial + \varphi_1) \cdots (\partial + \varphi_m) \) and \( B = (\partial + \phi_1) \cdots (\partial + \phi_n) \). This kind of Lax operator arises naturally in the context of matrix models \([3]\) and the associated hierarchy and Hamiltonian structure has been the subject of recent intense research \([4]\). The possibility of inducing a \( \mathcal{W}_\infty \) type of algebra from the free boson Poisson brackets for \( \varphi_i \) and \( \phi_j \) is the content of what we call the generalized Kupershmidt-Wilson theorem. As the reader will see, this theorem follows as a trivial corollary from the property of “self-similarity” of the second Gelfand-Dickey brackets against product and inversion of Lax operators.

The natural geometrical arena for KP is the space of pseudodifferential operators \((\Psi\text{DO})\) of the form

\[
L = \partial^n + \sum_{i \geq 1} u_i \partial^{n-i}.
\]  

(1)

Each \( L \) parameterizes a point on the infinite dimensional manifold \( \mathcal{M}_L \) whose coordinates are the functions in the unbounded set \( \{u_1, u_2, \ldots\} \), which in addition may be considered as generators of a differential polynomial algebra \( \mathbb{A}_L \). On this manifold we may consider functions or rather functionals \( F(L) \),

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$G(L)$ and Poisson brackets thereof $\{F, G\}$. The interesting Poisson brackets, from the point of view of CFT, are the ones of the associated integrable hierarchy, i.e. the KP-hierarchy. They are called second Gelfand-Dickey brackets [5][6], and are explicitly given by

$$\left\{ F, G \right\} = \text{Tr} \left( \frac{\delta F}{\delta L} (L \frac{\delta G}{\delta L}) + L - \frac{\delta F}{\delta L} L \frac{\delta G}{\delta L} \right).$$  \hspace{1cm} (2)

As usual, $\text{Tr} L$ stands for $\int \text{res} L$, where $\text{res} L$ is the coefficient of $\partial^{-1}$ in $L$, the subscript $+$ stands for the projection onto the differential part, and $\frac{\delta F}{\delta L}$ and $\frac{\delta G}{\delta L}$ are the gradients of $F$ and $G$ evaluated at the point $L$:

$$\frac{\delta F}{\delta L} = \sum_{i \in \mathbb{N}} \partial^{-n+i-1} \frac{\delta F}{\delta u_i}.$$  \hspace{1cm} (3)

It is a nontrivial fact that these formulas actually define consistent Poisson brackets. In fact, an essential point behind this property is the form of the Lax operator $L$. If $L$ did not have an infinite tail or, in general, if some constraints appeared among the fields $u_i$ the bracket as defined above would not enjoy the Poisson properties, unless the definition of the gradients would be modified accordingly, an extremely difficult task in general. However there exist some peculiar constrained forms for $L$ for which the formula (2) still defines consistent Poisson brackets as it stands without further modification. The first well-known case is when $L$ is a purely differential operator, i.e. $u_{n+1} = u_{n+2} = \ldots = 0$. This reduction leads to the Poisson structures associated with the famous n-KdV hierarchy.

We will call Lax operators for which (2) defines a consistent set of Poisson brackets, consistent Lax operators, of which we have already given two examples: the so called unreduced n-KP, where $n \in \mathbb{Z}$, ($n \neq 0$), and n-KdV Lax operators with $n \in \mathbb{N}$. The expert in constrained dynamical systems will readily recognise that, in this context, consistent is the same as either unconstrained, or else, first class constrained.

From (2), a set of fundamental Poisson brackets among the generators $\{u_i\}$ of the polynomial algebra $\mathcal{A}_L$ may be derived in a straightforward manner by considering linear functionals. For $L$ the n-KP operator in (1), the infinite set $u_1, u_2, \ldots$ spans a nonlinear algebra called $\mathcal{W}^{(n)}_{KP}$ [7][8] (also named $\hat{\mathcal{W}}^{(n)}_{1+\infty}$). When $L$ is the n-KdV operator, after setting $u_{n+1} = u_{n+2} = \ldots = 0$ in $\mathcal{W}^{(n)}_{KP}$ the remaining fields close another nonlinear algebra named $\mathcal{GD}_n$ after I.M. Gelfand and L.A. Dickey.
All the information about the Poisson brackets is neatly encoded in the essential “geometrical” structure behind it: the Adler map. For any one-form \( X = \sum_i \partial^{-i+n-1} x_i \), dual to the deformations of \( L \) under the pairing defined by the trace, the Adler map is then defined as

\[
J_L(X) = (LX)_+ L - L(XL)_+ \tag{4}
\]

Therefore, the Poisson brackets in (2) can be equivalently written as

\[
\{ F, G \} = \text{Tr} \frac{\delta F}{\delta L} J_L \left( \frac{\delta G}{\delta L} \right). \tag{5}
\]

The question of the consistency of \( L \) can be traced back to the fact that \( J_L(X) \) should be understood as a deformation of \( L \). Hence the Adler map, in order not take us out of the (constrained) manifold, should yield an operator of the same form as \( L - \partial^n \).

**Theorem I:** Let \( A = \sum_i a_i \partial^i \) and \( B = \sum_j b_j \partial^j \) be two consistent Lax \( \Psi DO \), and let the Poisson brackets among the generators \( \{ a_i \} \) of \( A_A \) and \( \{ b_j \} \) of \( A_B \) be given by the second Gelfand-Dickey bracket (2); moreover let \( \{ a_i, b_j \} = 0 \ \forall \ i, j \). Then we can form the product \( AB = L = \sum_k u_k \partial^k \) and the Poisson brackets induced on the \( u_k \)'s via the embedding \( u_k = u_k(a_i, b_j) \) are nothing but the second Gelfand-Dickey brackets computed on \( A_L \); \( L \) is again a consistent Lax operator.

**Proof:** Considered as functions of \( A \) and \( B \) the functional \( F \) has infinitesimal increment of the form

\[
\delta F(A, B) = \text{Tr} \left( \frac{\delta F}{\delta A} \delta A + \frac{\delta F}{\delta B} \delta B \right) \tag{6}
\]

whereas considered as a function of \( L \) the same increment looks like

\[
\delta F(L) = \text{Tr} \frac{\delta F}{\delta L} \delta L = \text{Tr} \frac{\delta F}{\delta L} (\delta A B + A \delta B)
\]

using the cyclic property of the trace and comparing with (6) we arrive at the following identification

\[
\frac{\delta F}{\delta A} = B \frac{\delta F}{\delta L} ; \quad \frac{\delta F}{\delta B} = \frac{\delta F}{\delta L} A. \tag{7}
\]
The computation is now straightforward

\[
\{F, G\} = \text{Tr} \left( \left( \frac{\delta F}{\delta A} A \frac{\delta G}{\delta A} \right)_+ + \left( \frac{\delta F}{\delta B} B \frac{\delta G}{\delta B} \right)_+ \right)
\]

\[
= \text{Tr} \left( \frac{\delta F}{\delta L} L \frac{\delta G}{\delta L} \right)_+ - \frac{\delta F}{\delta L} L \left( \frac{\delta G}{\delta L} \right)_+ \right).
\]

(8)

The careful reader may suspect that we have sloppily omitted projectors onto differential or integral parts. Namely, suppose that A were purely differential, then the right hand of both equalities in (7) should be enclosed in ( _ ) _. However this is not necessary because, as usual, the Adler map in the expression for the Poisson brackets takes care of all projections automatically.

We could have phrased Theorem I in a different way which emphasizes the fundamental Poisson-bracket algebra. For example, if A and B are KdV (differential) operators of order n and m, the associated non-linear algebras spanned by the \{a_i\} and the \{b_j\} are GD_n and GD_m. As a result of this theorem, we find that the set of \(n+m\) differential polynomials \(\{u_k = u_k(a_i, b_j)\}\) defined by the relation \(L = AB\) span the algebra GD\(_{n+m}\). Hence GD\(_{n+m}\) is a Poisson subalgebra of GD\(_n \times GD_m\).

**Corollary I:** The Kupershmidt-Wilson theorem [9] follows; namely, the Miura transformation defines a Poisson algebra homomorphism.

**Proof:** First, notice that the content of theorem I admits, by repeated application, a straightforward extension to the case of multiple factorization \(L = ABC \cdots D\). Next choose \(A = \partial + a\) and \(B = \partial + b\). The specialization of the first expression in (8) to these cases leads to

\[
\{F, G\} = \int \left( \frac{\delta F}{\delta a} \right) \left( \frac{\delta G'}{\delta a} \right) + \left( \frac{\delta F}{\delta b} \right) \left( \frac{\delta G'}{\delta b} \right)
\]

i.e., the basic building blocks \(a\) and \(b\) are mutually commuting free fields:

\[
\{a(x), a(y)\} = \delta'(x - y) \quad \{b(x), b(y)\} = \delta'(x - y) \quad \{a(x), b(y)\} = 0
\]

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Theorem II: Let \( A = \sum_i a_i \partial^i \) be an invertible consistent Lax operator and let the Poisson brackets among the \( a_i \) be given by (2). Take \( L = A^{-1} = \sum_j u_j \partial^j \). The Poisson brackets among the \( u_j \) induced by the mapping \( u_j \rightarrow u_j(a_i) \) are given again by the second Gelfand-Dickey bracket on \( A_L \) except for a relative minus sign; \( L \) is also a consistent Lax operator.

Proof: Following mutatis mutandi, identical steps as for Theorem I, we have
\[
\frac{\delta F}{\delta A} = -L \frac{\delta F}{\delta L} L,
\]
and, by direct substitution we find
\[
\{ F, G \} (A) = \text{Tr} \left( \frac{\delta F}{\delta A} \left( A \frac{\delta G}{\delta A} A - \frac{\delta F}{\delta A} A \left( \frac{\delta G}{\delta A} A \right) \right) \right) = -\text{Tr} \left( \frac{\delta F}{\delta L} \left( L \frac{\delta G}{\delta L} L - \frac{\delta F}{\delta L} L \left( \frac{\delta G}{\delta L} L \right) \right) \right) = -\{ F, G \} (L)
\]

We consider both theorems as a kind of generating rules for consistent Lax operators under the Adler map. Of special importance for KP is the second one since it constitutes a kind of Miura transform for the KP operator. Namely writing \( L = A^{-1} \) where \( A = \partial + a \) the potentials in \( L = \partial^{-1} + u_1 \partial^{-2} + u_2 \partial^{-3} + ... \) are expressed as differential polynomials in \( a \). The algebra of the former ones is named \( W_{KP}^{(-1)} \), and is induced by the embedding \( u_i \rightarrow u_i(a) \) where \( a \) satisfies the free field Poisson brackets. More generally, if \( A \) is of order \( n \) and its fields \( a_i \) span the algebra \( \text{GD}_n \), \( L \) is of order \( -n \) and its fields span the algebra \( W_{KP}^{(-n)} \)

The recent literature on the topic is concerned with the so called multi-boson reduction of KP. In particular the juxtaposition of theorems I and II helps to clarify the analysis of ref. [10] in what concerns the Poisson structure induced by the mapping \( A, B \rightarrow AB^{-1} \). In particular it explains the fact that the Lax flows are hamiltonian with respect to the direct “difference” of hamiltonian structures for \( A \) and \( B \).

The two comments above can be more precisely stated in the form of two corollaries as follows:

Corollary II: Let \( A = \sum_i a_i \partial^i \) and \( B = \sum_j b_j \partial^j \) be two consistent Lax PDO, and let the Poisson brackets among the generators \( \{ a_i \} \) of \( A \) and \( \{ b_j \} \) of \( A_B \) be given respectively by the Gelfand-Dickey bracket (2) with opposite signs; moreover let \( \{ a_i, b_j \} = 0 \ \forall \ i, j \). Then we can form the
product $AB^{-1} = L = \sum_k u_k \partial^k$ and the Poisson brackets induced on the $u_k$’s via the embedding $u_k = u_k(a_i, b_j)$ are equal to the second Gelfand-Dickey brackets computed on $A_L$; $L$ is again a consistent Lax operator.

**Corollary III:** Kupershmidt-Wilson-Yu theorem[9][11]. Let $L$ be a $\Psi$DO of order $n - m$. If $L$ admits a factorization (generalised Miura transformation) $L = (\partial + \varphi_1) \cdots (\partial + \varphi_n)(\partial + \phi_1) \cdots (\partial + \phi_m)^{-1} = \sum_k u_k \partial^k$, and the only nonzero Poisson brackets among the generalised Miura fields are given by \{\varphi_i(y), \varphi_j(x)\} = \delta_{ij} \delta'(y - x) and \{\phi_i(y), \phi_j(x)\} = -\delta_{ij} \delta'(y - x), then the Poisson algebra induced on the $u_k$’s via the embedding $u_k = u_k(\varphi_i, \phi_j)$ is $W_{KP}^{(n-m)}$.

From the point of view of the Lax equations, the first field $u_1$ in $L = \partial^n + u_1 \partial^{n-1} + u_2 \partial^{n-2} + \ldots$ is a kind of spectator that can be set to zero. However, from the point of view of the symplectic structure, this constraint is second class, and the reduction changes the Poisson bracket algebra. For example, performing this reduction in GD$_n$ yields the algebra $W_n$. Correspondingly, in the case of a KP type of Lax operator, $W_{KP}^{(n)}$ reduces to the non-linear $\hat{W}_{\infty}^{(n)}$. The interest in these reduced algebras resides in the fact that they contain a Virasoro subalgebra spanned by the field $u_2$, and that the higher generators $u_3, u_4, \ldots$ can be redefined to transform like tensors of $Diff(S^1)$ under Poisson brackets with $u_2$.

The reduced Lax operator $\tilde{L} \equiv L|_{u_1=0}$ is in general not consistent. That is, the Adler map (4) yields for generic $X$, another $J_{\tilde{L}}(X)$ where the coefficient of $\partial^{n-1}$ is different from zero, and hence “sticks out” of the constrained manifold. This fact can be avoided if $X$ is required to satisfy the equation $\text{res} \left[ X, \tilde{L} \right] = 0$ [6]. Translated to the language of Poisson brackets, the reduced manifold $\mathcal{M}_{\tilde{L}} = \{u_1 = 0, u_2, u_3, \ldots\}$ admits the same Poisson structure (2) with the proviso that now the gradients are no more given by (3) since in particular $\delta F/\delta u_1$ is ill defined. Instead they should be defined by

$$\frac{\delta F}{\delta L} = \partial^{-n} \frac{\delta F}{\delta u_1} + \sum_{i \geq 2} \partial^{-n+i-1} \frac{\delta F}{\delta u_i}$$

with $\delta F/\delta u_1$ solved in terms of $\delta F/\delta u_i, \ (i = 2, 3, \ldots)$ such that

$$\text{res} \left[ \frac{\delta F}{\delta L}, \tilde{L} \right] = 0.$$ 

holds identically.
It is now easy to check that the above property is preserved under the operation of taking the inverse of the Lax operator. Explicitly, using the relation

\[ \frac{\delta F}{\delta \tilde{A}} = -\tilde{L} \frac{\delta F}{\delta \tilde{L}} \]

it follows that

\[ \text{res} \left[ \frac{\delta F}{\delta \tilde{A}}, \tilde{A} \right] = 0 \iff \text{res} \left[ \frac{\delta F}{\delta \tilde{L}}, \tilde{L} \right] = 0. \]

In the language of \( \mathcal{W} \) algebras, it means that if the fields in \( a_i \) (\( i \geq 2 \)) in \( A = \partial^n + a_2 \partial^{n-2} + \ldots \) span \( \mathcal{W}_n \), the fields \( u_i \) in \( L = A^{-1} = \partial^{-n} + u_2 \partial^{-n-2} + \ldots \) span \( \hat{\mathcal{W}}^{-n}_\infty \).

Unfortunately, we have not yet been able to reach a similar result in the context of theorem I, but we hope to report on this point in the near future.

Summarizing, we have found that the second Gelfand-Dickey brackets exhibit a particularly simple behavior under the product and inversion of \( \Psi \)DO’s, and this property is crucial for inducing associated Poisson structures in the so-called constrained KP-hierarchies. To point other directions of further study, it would be interesting to know to what extent the Adler map is completely determined by this set of properties.

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Appendix A The Classical Limit

This appendix is concerned with the classical (also called dispersionless or commutative) limit of KP. This dynamical system can also be endowed with a Poisson structure that arises as a classical limit of the second Gelfand-Dickey brackets [12]. The formal procedure requires first the mapping of any “normal ordered” \( \Psi \)DO (all \( \partial \)’s to the right of the \( u_i(x) \)’s) to its associated pseudo-differential symbol.

\[ A = \sum_i a_i \partial^{n-i} \to \hat{A} = \sum_i a_i \xi^{n-i}. \]

The Poisson bracket of two functionals \( F(\hat{L}) \) and \( G(\hat{L}) \) is given by the same
expression (5) with $J_L(X)$ replaced by

$$J_L^{cl}(X) = [[\hat{L}, \hat{X}]] + \hat{L} - [[\hat{L}, (\hat{X}\hat{L})_+]]$$

(1)

where for any two $\hat{P}$ and $\hat{Q}$, $\hat{P}\hat{Q} = \hat{Q}\hat{P}$ and $[[\hat{P}, \hat{Q}]]$ stands for the ordinary 2-dimensional Poisson bracket, i.e.

$$[[\hat{P}, \hat{Q}]] = \frac{\partial \hat{P}}{\partial x} \frac{\partial \hat{Q}}{\partial \xi} - \frac{\partial \hat{Q}}{\partial x} \frac{\partial \hat{P}}{\partial \xi}.$$ 

The following theorem is proven by a straightforward calculation of the same type as before.

**Theorem III:** Both theorem I and II hold in the classical limit, i.e. with $J_L^{cl}$ in place of $J_L$.

In the classical case even a stronger result holds: the mapping $L \rightarrow L^p$ where $p \in \mathbb{N}$ is a Poisson isomorphism. Namely let $u_i$ be the fields in $L$, and $w_j$ those of $L^p$, and compute the Poisson brackets among the $u_i$’s with the help of (1). The brackets induced on the $w_j(u_i)$ by the above mapping coincide with those computed with $J_L^{cl}$, up to a global factor $p^2$ [12].

It is worth noting that even for the higher dimensional generalization of the classical Gelfand-Dickey brackets that were constructed in [13] the two previous theorems hold. This is not at all obvious since, for example, this construction uses not one but two different splittings. Again this fact calls for an understanding of the uniqueness of the bracket under the requirement that it “replicates” upon multiplication of Lax operators.

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