EULER-MAHONIAN STATISTICS AND DESCENT BASES FOR SEMIGROUP ALGEBRAS

BENJAMIN BRAUN AND MCCABE OLSEN

Abstract. We consider quotients of the unit cube semigroup algebra by particular \(\mathbb{Z}_rI_{S_n}\)-invariant ideals. Using Gröbner basis methods, we show that the resulting graded quotient algebra has a basis where each element is indexed by colored permutations \((\pi, \epsilon) \in \mathbb{Z}_rI_{S_n}\) and each element encodes the negative descent and negative major index statistics on \((\pi, \epsilon)\). This gives an algebraic interpretation of these statistics that was previously unknown. This basis of the \(\mathbb{Z}_rI_{S_n}\)-quotients allows us to recover certain combinatorial identities involving Euler-Mahonian distributions of statistics.

1. Introduction

Let \([0, 1]^n \subset \mathbb{R}^n\) denote the \(n\)-dimensional unit cube. Let \(S_n\) denote the symmetric group on \(n\) elements. Let \([n] := \{1, 2, \ldots, n\}\).

1.1. Polytope semigroup algebras. Let \(\mathcal{P} \subset \mathbb{R}^n\) be an \(n\)-dimensional convex lattice polytope, let \(m \cdot \mathcal{P} = \{m\alpha : \alpha \in \mathcal{P}\}\) denote the \(m\)th dilate of \(\mathcal{P}\), and consider the cone over \(\mathcal{P}\)

\[\text{cone}(\mathcal{P}) := \text{span}_{\mathbb{R} \geq 0}\{(1, p) : p \in \mathcal{P}\} .\]

The affine semigroup algebra of \(\mathcal{P}\) over \(\mathbb{C}\) is

\[\mathbb{C}[\mathcal{P}] := \mathbb{C}[t^m \cdot x^p : (m, p) \in \text{cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}] \subset \mathbb{C}[t, x_1^\pm 1, x_2^\pm 1, \ldots, x_n^\pm 1] ,\]

where \(x^p = x_1^{p_1}x_2^{p_2}\ldots x_n^{p_n}\) when \((m, p) \in \text{cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}\). Given that \(\text{cone}(\mathcal{P})\) is a pointed, rational cone in \(\mathbb{R}^{n+1}\), \(\text{cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}\) has a unique minimal generating set called a Hilbert basis. Subsequently, the algebra \(\mathbb{C}[\mathcal{P}]\) is a finitely generated, graded commutative algebra. If \(\mathcal{P}\) satisfies the integer decomposition property, that is for any \(q \in m \cdot \mathcal{P} \cap \mathbb{Z}^n\), we can express \(q = q_1 + q_2 + \ldots + q_m\) where each \(q_i \in \mathcal{P} \cap \mathbb{Z}^n\), then we can more concisely describe \(\mathbb{C}[\mathcal{P}]\). In particular, if \(\mathcal{P} \cap \mathbb{Z}^n = \{p_1, p_2, \ldots, p_k\}\) and \(\mathcal{P}\) satisfies the integer decomposition property, then

\[\mathbb{C}[\mathcal{P}] := \mathbb{C}[t \cdot x^{p_i} : 1 \leq i \leq k] \subset \mathbb{C}[t, x_1^\pm 1, x_2^\pm 1, \ldots, x_n^\pm 1] .\]

For greater detail and background of semigroup algebras and cones over polytopes, see [19].

Let \(\mathcal{P} = [0, 1]^n\), which is known to satisfy the integer decomposition property. Let \(R_n := \mathbb{C}[[0, 1]^n]\) denote the affine semigroup algebra of \([0, 1]^n\) which has the following description:

\[R_n = \mathbb{C}[t \cdot x_{a_1} \cdots x_{a_i} \mid A = \{a_1, \ldots, a_i\} \subseteq [n]] \subset \mathbb{C}[t, x_1, x_2, \ldots, x_n] .\]

Alternatively, we can define \(R_n\) as the quotient of a polynomial ring by a toric ideal. Let \(T_n\) be a polynomial ring in \(2^n\) variables, where each variable corresponds to a subset of \([n]\), thus

\[T_n := \mathbb{C}[z_A : A \subseteq [n]] .\]
Define the toric ideal
\[ I_n := \langle z_A z_B - z_{A \cap B} z_{A \cup B} | A \not\subseteq B \text{ and } B \not\subseteq A \rangle. \]

It is known that \( R_n \cong T_n/I_n \). For background and details see [19]. This algebra also arises as the Hibi ring for the antichain on \( n \) elements, as the unit cube is the order polytope of the antichain (see e.g. [3, 14, 15] for additional details of Hibi rings). We will use \( R_n \) to denote \( T_n/I_n \) when it is convenient.

Let \( A = \bigoplus_{b \in \mathbb{Z}^n} A_b \) be a finitely generated, \( \mathbb{Z}^n \)-graded commutative \( \mathbb{C} \)-algebra. The Hilbert series of \( A \) is
\[
\text{Hilb}(A; z) = \sum_{b \in \mathbb{Z}^n} \dim_\mathbb{C}(A_b) \cdot z^b.
\]

For a polytope semigroup algebra \( \mathbb{C}[P] \), it is common to consider \( \mathbb{C}[P] \) as an \( \mathbb{N} \)-graded algebra where the grading is given by the \( t \)-degree. In this case, we have
\[
\text{Hilb}(\mathbb{C}[P]; t) = \sum_{m \geq 0} \# (mP \cap \mathbb{Z}^n) \cdot t^m
\]
which coincides with the Ehrhart series of \( P \). The reader is invited to consult [19] and [6] for background on Hilbert series and Ehrhart theory respectively. In the case of \( R_n \), it is well-known that the Hilbert series with respect to the \( t \)-degree is
\[
\sum_{k \geq 0} \frac{(k + 1)^n t^k}{(1 - t)^{n+1}};
\]
first studied by Euler [11]. This identity was generalized to a bivariate identity usually attributed to Carlitz using the major index; see [5] and the references therein for more details on the history of these identities. Given \( \pi \in S_n \), the major index \( \pi \) is defined to be
\[
\text{maj}(\pi) := \sum_{j \in \text{Des}(\pi)} j.
\]

**Theorem 1.1** (Carlitz, [7]). For all \( n \geq 1 \),
\[
\sum_{k \geq 0} [k + 1]_q^n t^k \frac{t^{\text{des}(\pi)} q^{\text{maj}(\pi)}}{\prod_{j=0}^n (1 - tq^j)}
\]
where \([k + 1]_q = 1 + q + q^2 + \cdots + q^k\).

In this form, this identity is due to Carlitz [7], though with some effort one can derive it from the works of MacMahon [18, Volume 2, Chapter IV, §462]. We will call this identity the Euler-Mahonian identity, which has arisen in a variety of contexts in recent years. Some such scenarios include lecture hall partition generating function identities [20], polyhedral-geometric studies of the semigroup algebra for \( \text{cone}([0, 1]^n) \) [3], Hilbert series related to a descent basis for the coinvariant algebra of \( S_n \) [2], 0-Hecke algebra actions on Stanley-Reisner rings [17], and quasisymmetric function identities [21].
Generalizing to colored permutation groups \( \mathbb{Z}_r \wr S_n \), one can consider the flag statistics as well as the negative statistics, the latter of which we define in Section 2. These statistics were originally introduced for the hyperoctahedral group \( B_n \cong \mathbb{Z}_2 \wr S_n \) and generalized for \( r \geq 2 \) to \( \mathbb{Z}_r \wr S_n \) [3, 4]. For these families of statistics, the following Euler-Mahonian identities exist.

**Theorem 1.2** (Bagno, [3]). Given any \( r \geq 2 \), for all \( n \geq 1 \),

\[
\sum_{k \geq 0} [k + 1]_q^n t^k = \frac{\sum_{\rho, \epsilon \in \mathbb{Z}_r \wr S_n} \text{Inder}_\rho(\epsilon) q^{\text{Imaj}_\rho(\epsilon)}}{(1 - t) \prod_{j=1}^n (1 - \epsilon^r q^{j^2})}
\]

**Theorem 1.3** (Bagno-Biagioli, [4]). Given any \( r \geq 2 \), for all \( n \geq 1 \),

\[
\sum_{k \geq 0} [k + 1]_q^n t^k = \frac{\sum_{\rho, \epsilon \in \mathbb{Z}_r \wr S_n} \text{Inder}_\rho(\epsilon) q^{\text{Imaj}_\rho(\epsilon)}}{(1 - t) \prod_{j=1}^n (1 - \epsilon^r q^{j^2})}
\]

1.3. Our Contributions. The goal of this paper is twofold. First, we produce a new algebraic interpretation of negative permutation statistics by considering \( \mathbb{Z}_r \wr S_n \)-quotient algebras of \( R_n \). To do so, we consider an ideal \( \text{invar}(r, n) \subset R_n \) which is generated by certain invariants of \( R_n \) under a \( \mathbb{Z}_r \wr S_n \)-action, defined in detail in Section 2. We obtain the following theorem using Gröbner basis techniques.

**Theorem 1.4** (see Theorem 1.2). There exists a basis of \( R_n/\text{invar}(r, n) \) of the form \( \{ b_{(\sigma, X)} + \text{invar}(r, n) \} \) with elements indexed by pairs \( (\sigma, X) \) that are in bijection with colored permutations \( (\pi, \epsilon) \in \mathbb{Z}_r \wr S_n \). Further, \( b_{(\sigma, X)} \) encodes \( \text{Inder}(\pi, \epsilon) \) and \( \text{Imaj}(\pi, \epsilon) \). The bijective correspondence of \( (\sigma, X) \leftrightarrow (\pi, \epsilon) \) is given in Remark 2.4.

Our second goal is to consider a multigraded Hilbert series of \( R_n \) and the quotient \( R_n/\text{invar}(r, n) \). These computations allow us to recover the identities given by Theorem 1.1 and Theorem 1.2. These new proofs provide a new perspective on identities of this type.

Moreover, the new proof of Theorem 1.1 serves to expand connections between the commutative-algebraic and representation-theoretic methods [2] for the \( S_n \)-coinvariant algebra \( \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}_n \), where \( \mathcal{I}_n := \langle e_1, \ldots, e_n \rangle \) with \( e_i \) denoting the \( i \)-th elementary symmetric function, and polyhedral-geometric methods for \( \text{cone}(0, 1)^n \) [5]. Additionally, we provide a short proof that this quotient algebra is isomorphic as a graded \( S_n \)-module to the \( S_n \)-coinvariant algebra \( \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}_n \). We believe that these results, like those given in [5], support the idea that \( \text{cone}(0, 1)^n \) and its associated semigroup algebra are analogues of the polynomial ring in \( n \) variables that give rise to interesting and different structures and results in similar contexts.

2. Colored permutation groups and decent sets

The wreath product \( \mathbb{Z}_r \wr S_n \cong (\mathbb{Z}_r)^n \rtimes S_n \) of a cyclic group of order \( r \) with \( S_n \) consists of pairs \( (\pi, \epsilon) \) where \( \pi \in S_n \) and \( \epsilon \in \{ \omega^0, \omega^1, \ldots, \omega^{r-1} \}^n \) for \( \omega := e^{2\pi i/r} \) a primitive \( r \)-th root of unity. These groups are often called colored permutation groups and the elements are commonly referred to as colored or indexed permutations. We adopt the usual window notation, denoting the pair \( (\pi, \epsilon) \) by \([\pi(1)^{\epsilon_1} \pi(2)^{\epsilon_2} \cdots \pi(n)^{\epsilon_n}]\) where \( \epsilon_j = \omega^{\epsilon_j} \). Additionally, we will use the notation \( \epsilon^{(j)} \) and \( (\omega^{\epsilon_j}, j) \) to denote elements of \( \{ \omega^0, \omega^1, \ldots, \omega^{r-1} \} \times \{ n \} \).

Elements \( (\pi, \epsilon) \in \mathbb{Z}_r \wr S_n \) can be identified as a permutation matrix for \( \pi \) where the 1 in position \( (\pi(i), i) \) is replaced with \( \epsilon_i \). The algebraic structure of \( \mathbb{Z}_r \wr S_n \) is described by matrix multiplication where entry-by-entry multiplication of the nonzero entries is given by the group operation of \( \mathbb{Z}_r \). This means that given \( (\pi, \epsilon), (\pi', \epsilon') \in \mathbb{Z}_r \wr S_n \)

\[(\pi', \epsilon') \circ (\pi, \epsilon) = (\pi' \circ \pi, (\epsilon_1 \cdot \epsilon_{\pi(1)}, \ldots, \epsilon_n \cdot \epsilon_{\pi(n)})],\]
or represented in window notation we have
\[
[\pi'(1)^{c_1} \cdots \pi'(n)^{c_n}] \circ [\pi(1)^{c_1} \cdots \pi(n)^{c_n}] = [\pi' \circ \pi(1)^{c_1+c_{i}(1)} \cdots \pi' \circ \pi(n)^{c_n+c_{i}(n)}]
\]
where the addition is modulo \(r\). A more explicit understanding of these wreath products may be found in [3 4 5 20].

To review one definition of descents for wreath products, we define a total order as follows. Given \(j^{c_j}, k^{c_k} \in \{\omega^0, \omega^1, \cdots, \omega^{r-1}\} \times [n]\) we say that \(j^{c_j} < k^{c_k}\) if \(c_j > c_k\) or if \(c_j = c_k\) and \(j < k\) hold.

**Definition 2.1.** Let \((\pi, \epsilon) \in \mathbb{Z}_r \wr S_n\). The type-A descent set is defined to be
\[
\text{Des}_A(\pi, \epsilon) := \{i \in [n-1] : \pi^{c_i}_i > \pi^{c_{i+1}}_{i+1}\}
\]
and the type-A descent statistic is
\[
\text{des}_A(\pi, \epsilon) := \#\text{Des}_A(\pi, \epsilon).
\]
The type-A major index is
\[
\text{major}_A(\pi, \epsilon) := \sum_{j \in \text{Des}_A(\pi, \epsilon)} j
\]

**Example 2.2.** Let \((\pi, \epsilon) = [2^1 6^3 4^3 1^0 5^2 3^0] \in \mathbb{Z}_4 \wr S_6\). Then \(\text{Des}_A(\pi, \epsilon) = \{1, 2, 4\}\), \(\text{des}_A(\pi, \epsilon) = 3\), and \(\text{major}_A(\pi, \epsilon) = 7\).

We now review a different notion of descent statistics for \(\mathbb{Z}_r \wr S_n\), namely the negative statistics.

**Definition 2.3.** For an element \((\pi, \epsilon) \in \mathbb{Z}_r \wr S_n\), we define the negative inverse multiset as
\[
\text{NNeg}(\pi, \epsilon) := \{i, i, \ldots, i : i \in [n]\}.\]

The negative descent multiset is
\[
\text{NDes}(\pi, \epsilon) := \text{Des}_A(\pi, \epsilon) \cup \text{NNeg}((\pi, \epsilon)^{-1}).
\]
The negative descent statistic is
\[
\text{ndes}(\pi, \epsilon) := \#\text{NDes}(\pi, \epsilon).
\]
The negative major index is
\[
\text{nmajor}(\pi, \epsilon) := \sum_{i \in \text{NDes}(\pi, \epsilon)} i.
\]

**Example 2.4.** If \((\pi, \epsilon) = [2^1 6^3 4^3 1^0 5^2 3^0] \in \mathbb{Z}_4 \wr S_6\), then \((\pi, \epsilon)^{-1} = [4^0 1^3 6^0 3^1 5^2 2^1]\) and hence \(\text{NNeg}((\pi, \epsilon)^{-1}) = \{2, 2, 2, 4, 5, 5, 6\}\) and hence \(\text{NDes}(\pi, \epsilon) = \{1, 2, 4\} \cup \{2, 2, 2, 4, 5, 5, 6\} = \{1, 2, 2, 2, 4, 4, 5, 5, 6\}\)
and thus \(\text{ndes}(\pi, \epsilon) = 10\) and \(\text{nmajor}(\pi, \epsilon) = 33\).

We will use the following representation for elements of \(\mathbb{Z}_r \wr S_n\).

**Definition 2.5.** The increasing elements of \(\mathbb{Z}_r \wr S_n\), denoted \(I_{r,n}\), is the subset of elements satisfying \(\text{des}_A(\pi, \epsilon) = 0\).

It is a simple exercise to see that any element of \((\pi, \epsilon) \in \mathbb{Z}_r \wr S_n\) can be represented uniquely as
\[
(\pi, \epsilon) = (\rho, \delta) \circ (\sigma, (1, 1, \ldots, 1))
\]
for some \(\sigma \in S_n\) and \((\rho, \delta) \in I_{r,n}\). Subsequently, we have that
\[
\mathbb{Z}_r \wr S_n = \bigcup_{\sigma \in S_n} I_{r,n} \sigma
\]
where we use \(\sigma\) in place of \((\sigma, (1, 1, \ldots, 1))\) for ease.

We also have the following observation.
Proposition 2.6. [5] Proposition 5.11] For $(\rho, \delta) \in \mathcal{I}_{r,n}$ and $\sigma \in S_n$,
\[ \text{NNeg}((\rho, \delta)\sigma^{-1}) = \text{NNeg}((\rho, \delta)^{-1}). \]
Further, each permutation $(\rho, \delta) \in \mathcal{I}_{r,n}$ is uniquely determined by $\text{NNeg}((\rho, \delta)^{-1})$.

Remark 2.7. We will often denote $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$ by the pair $(\sigma, X)$ where $\sigma \in S_n$ satisfies $(\rho, \delta)\sigma = (\pi, \epsilon)$ with $(\rho, \delta) \in \mathcal{I}_{r,n}$ and $X = \text{NNeg}((\pi, \epsilon)^{-1})$. This establishes a bijective correspondence between elements of $\mathbb{Z}_r \wr S_n$ and pairs $(\sigma, X)$ with $\sigma \in S_n$ and $X$ a multiset of elements of $[n]$ in which each element appears with multiplicity strictly less than $r$. For convenience of notation, we will write $(\sigma, X) \in \mathbb{Z}_r \wr S_n$ when this interpretation is preferred.

Example 2.8. Let $(\pi, \epsilon) = [2^1 6^3 4^3 1^0 5^3 3^0] \in \mathbb{Z}_r \wr S_6$ and consider $(\rho, \delta) = [4^3 6^3 5^2 2^1 1^0 3^0] \in \mathcal{I}_{4,6}$ and $\sigma = 421536 \in S_6$. Note that $(\pi, \epsilon) = (\rho, \delta) \circ (\sigma, (1, \ldots, 1))$ as
\[ [2^1 6^3 4^3 1^0 5^3 3^0] = [4^3 6^3 5^2 2^1 1^0 3^0] \circ [4^0 2^0 1^0 5^0 3^0 6^0]. \]
Moreover, $(\rho, \delta)^{-1} = [5^0 4^3 6^0 1^1 3^2 2^1]$ and $\text{NNeg}((\rho, \delta)^{-1}) = \{2, 2, 2, 4, 5, 5, 6\}$. Therefore, $\text{NNeg}((\rho, \delta)^{-1}) = \text{NNeg}(((\rho, \delta)\sigma)^{-1}) = \text{NNeg}((\pi, \epsilon)^{-1})$. Thus, $(\pi, \epsilon)$ corresponds to the pair $(\sigma, X) = (421536, \{2, 2, 2, 4, 5, 5, 6\})$.

3. $\mathbb{Z}_r \wr S_n$-quotient algebras of $R_n$ and descent bases

For convenience, we will view $R_n \cong T_n/I_n$ as the quotient of a polynomial ring by the toric ideal $I_n$. First consider the $S_n$ case. We define an $S_n$ action on $T_n$ given as $S_n \times T_n \to T_n$ defined on the variables by $(\pi, z_A) \mapsto z_{\pi(A)} = z_{(\pi(a_1), \ldots, \pi(a_k))}$ where $A = \{a_1, \ldots, a_k\}$. Note that this action passes to $R_n \cong T_n/I_n$, where it corresponds to the usual action of $S_n$ on $\{x_1, \ldots, x_n\}$ of permutation of the variables because
\[ z_{\{a_1, a_2, \ldots, a_k\}} \mapsto x_{a_1}x_{a_2} \cdots x_{a_k} \]
and
\[ z_{\{\pi(a_1), \ldots, \pi(a_k)\}} \mapsto x_{\pi(a_1)}x_{\pi(a_2)} \cdots x_{\pi(a_k)} \]
which is the usual permutation of variables action of $S_n$. We consider the following ideal of elements which are invariant under this action:
\[ \text{invar}(1, n) := \left\langle \hat{e}_k := \sum_{|A| = k} z_A \mid \text{for all } 0 \leq k \leq n \right\rangle. \]
The elements $\hat{e}_k$ are the $T_n$–analog of the usual elementary symmetric functions $e_k$ in the polynomial ring on $n$ variables. Notice that this ideal cannot be the full ideal of invariants for this action on $T_n$ because there must be $2^n$ algebraically independent invariants [23, Proposition 2.1.1]. However, the generators are indeed invariant and this is the appropriate ideal for our purposes. We say the $S_n$ quotient algebra of $R_n$ is $R_n/\text{invar}(1, n)$, where $\text{invar}(1, n)$ is the image of $\text{invar}(1, n)$ in the quotient $T_n/I_n$. For convenience, we will consider the ring $T_n/J_{1,n}$ where $J_{1,n} := \text{invar}(1, n) + I_n$, as it is a straightforward exercise in algebra to show that $T_n/J_{1,n} \cong R_n/\text{invar}(1, n)$.

Next, we consider $\mathbb{Z}_r \wr S_n$ for $r \geq 2$. Consider the action $\mathbb{Z}_r \wr S_n \times T_n \to T_n$ defined on the variables by $(\pi, \epsilon, z_A) \mapsto (\prod_{i \in A} e_{\epsilon_i}) \cdot z_{\pi(A)} = (\prod_{i \in A} e_{\epsilon_i}) \cdot z_{\{\pi(a_1), \ldots, \pi(a_k)\}}$ where $A = \{a_1, \ldots, a_k\}$. We consider an ideal generated by invariant elements of this action:
\[ \text{invar}(r, n) := \left\langle z_\sigma \hat{e}_{r,k} := \sum_{|A| = k} z_A^r \mid \text{for all } 1 \leq k \leq n \right\rangle. \]
This is consistent with the above in the $r = 1$ case. This ideal also does not contain all of the invariants of $T_n$ under this action, but the ideal is the appropriate choice of invariant generators for our scenario. We say the $\mathbb{Z}_r \wr S_n$ quotient algebra of $R_n$ is $R_n/\text{invar}(r, n)$, and we will consider the ring $T_n/J_{r,n}$ where $J_{r,n} := \text{invar}(r, n) + I_n$, as we have $T_n/J_{r,n} \cong R_n/\text{invar}(r, n)$.
Now, we will define descent bases for our quotients. First consider \( T_n/J_{1,n} \). We wish to construct a basis based on the descent sets of \( S_n \) that is analogous to the Garsia-Stanton descent basis. The Garsia-Stanton descent basis is a basis for the \( S_n \)-coinvariant algebra \( \mathbb{C}[x_1, \ldots, x_n]/\mathcal{J}_n \) with coset representatives

\[
a_{\pi} = \prod_{j \in \text{Des}(\pi)} x_{\pi(1)} \cdots x_{\pi(j)}
\]

for all \( \pi \in S_n \), where the ideal \( \mathcal{J}_n = (e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)) \) is generated by the elementary symmetric functions \( e_i(x_1, \ldots, x_n) = \sum_{a_1 < \cdots < a_i} x_{a_1} \cdots x_{a_i} \). Garsia and Stanton originally showed this was a basis in [12] using the theory of Stanley-Reisner rings. In [2], Adin, Brenti, and Roichman provide another proof of this result and use the basis heavily in their proof of the monomial descent basis is a basis for the coinvariant algebra \( \mathbb{C}[x_1, \ldots, x_n]/\mathcal{J}_n \). We introduce an analogue of the Garsia-Stanton descent basis for \( T_n/J_{1,n} \), which is

\[
\hat{a}_{\pi} := \prod_{j \in \text{Des}(\pi)} z_{\pi(1),\pi(2),\ldots,\pi(j)}
\]

for all \( \pi \in S_n \).

**Example 3.1.** Let \( \pi = 421536 \in S_6 \). Since \( \text{Des}(\pi) = \{1, 2, 4\} \), we have

\[
a_{\pi} = z_{\{4\}} z_{\{2,4\}} z_{\{1,2,4,5\}}
\]

Because of the correspondence given in Theorem 6.1, in this paper we will refer to the set \( \{\hat{a}_{\pi} : \pi \in S_n\} \) as the *Garsia-Stanton basis*. Using Gröbner basis arguments in Section 4 we will show that this is indeed a basis for \( T_n/J_{1,n} \).

We can generalize this to a basis of \( T_n/J_{r,n} \) for \( r \geq 2 \).

**Definition 3.2.** The *negative descent basis* of \( T_n/J_{r,n} \) consists of the elements

\[
b_{(\sigma,X)}^r := \hat{a}_{\sigma} \cdot \prod_{j \in X} z_{\sigma(1),\sigma(2),\ldots,\sigma(j)}
\]

for all \( \sigma \in S_n \) and \( X \) a multiset of \([n]\) where no element has multiplicity greater than \( r - 1 \).

**Example 3.3.** Let \((\sigma,X) = (421536, \{2, 2, 2, 4, 5, 5, 6\})\) corresponding to \((\pi, \epsilon) = [2^1 \ 6^3 \ 4^3 \ 1^0 \ 5^2 \ 3^0] \in \mathbb{Z}_4 \wr S_6 \). Then

\[
b_{(\sigma,X)} = z_{\{4\}} z_{\{2,4\}} z_{\{1,2,4,5\}} \cdot (z_{\{2,4\}})^3 \cdot (z_{\{1,2,4,5\}})^2 \cdot (z_{\{1,2,3,4,5,6\}})^2\]

We will show that this is a basis in Section 4. It follows from Remark 2.7 that if \((\sigma,X)\) corresponds to \((\rho, \epsilon) \in \mathbb{Z}_r \wr S_n\), then \( \text{NNeg}((\rho, \epsilon)^{-1}) = X \) and \( \text{Des}_A(\rho, \epsilon) = \text{Des}(\sigma) \). So, elements of this basis correspond to NDes sets of \( \mathbb{Z}_r \wr S_n \), hence the name “negative descent basis.” It is important to observe that this is distinct from the basis developed by R. Adin, F. Brenti, and Y. Roichman [2] for the hyperoctohedral group \( B_n \cong \mathbb{Z}_2 \wr S_n \), as their basis related to the *flag descent sets*.

4. DESCENT BASES VIA GRÖBNER BASES FOR \( J_{r,n} \)

Our goal in this section is to prove the following theorem by finding a Gröbner basis for the ideal \( J_{r,n} \).

**Theorem 4.1.** For \( r \geq 2 \), \( \{b_{(\sigma,X)}^r : (\sigma,X) \in \mathbb{Z}_r \wr S_n\} \) is a basis of \( T_n/J_{r,n} \). When \( r = 1 \), \( \{\hat{a}_{\pi} : \pi \in S_n\} \) is a basis of \( T_n/J_{1,n} \).
Before proving this theorem, we briefly review Gröbner bases. For a detailed reference on the theory and computation of Gröbner bases, we invite the reader to consult [8, 10]. Consider the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$. Recall that a term order $\prec_{\text{mon}}$ on $S$ is a relation on $\mathbb{Z}_{>0}^n$ which is a total ordering, a well ordering, and satisfies the condition that if $\alpha \prec_{\text{mon}} \beta$ and $\gamma \in \mathbb{Z}_{>0}^n$, then $\alpha + \gamma \prec_{\text{mon}} \beta + \gamma$. Given two monomials $m_1 = \prod_{i=1}^n x_i^{\alpha_i}$ and $m_2 = \prod_{i=1}^n x_i^{\beta_i}$, we say that $m_1 \prec_{\text{mon}} m_2$ if $(\alpha_1, \alpha_2, \ldots, \alpha) \prec_{\text{mon}} (\beta_1, \beta_2, \ldots, \beta_n)$. Given $f \in S$, the leading monomial of $f$, denoted $\text{LM}(f)$, is the largest monomial of $f$ with respect to the term order $\prec_{\text{mon}}$. For notation, we will denote monomials as $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ where $\alpha \in \mathbb{Z}_{>0}^n$. The leading term of $f$, denoted $\text{LT}(f)$, is the leading monomial with its coefficient. Given an ideal $I \subseteq S$, $\text{LT}(I) = \{cx^\alpha : \exists f \in I \text{ s.t. } \text{LT}(f) = cx^\alpha\}$ and $\langle \text{LT}(I) \rangle$ is the ideal generated by elements of $\text{LT}(I)$, which we call the leading term ideal of $I$. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal $I$ is called a Gröbner basis for $I$ if

$$
\langle \text{LT}(g_1), \ldots, \text{LT}(g_t) \rangle = \langle \text{LT}(I) \rangle.
$$

Given a polynomial ideal $I \subset S$ and a fixed term order, we can algorithmically construct a Gröbner basis using a classical result known as Buchberger’s Algorithm. However, one can optimize this classical algorithm to be more efficient. Before stating an optimized version, we must introduce notation. Given two polynomials $f, g$, the $S$-polynomial of $f$ and $g$ is

$$
\text{S}(f, g) = \frac{x^\gamma}{\text{LT}(f)} \cdot f - \frac{x^\gamma}{\text{LT}(g)} \cdot g
$$

where $x^\gamma = \text{LCM}(\text{LM}(f), \text{LM}(g))$. Given a polynomial $f$ and an ordered $s$-tuple of polynomials $F = (f_1, \ldots, f_s)$, let $f^F$ denote the reminder of $f$ after division by each polynomial in $F$ performed in order. The reader should consult [8] for a thorough discussion of multivariate polynomial division.

**Algorithm 4.2** (Optimized Buchberger Algorithm). Let $I = \langle f_1, \ldots, f_s \rangle \subset S$. Then a Gröbner basis for $I$ can be constructed in a finite number of steps as follows:

**Input:** $F = (f_1, \ldots, f_s)$

**Output:** $G$, a Gröbner basis for $I$

**Initial state:** $B := \{(i, j) : 1 \leq i < j \leq s\}$; $G := F$; $t := s$

**WHILE** $B \neq \emptyset$ **DO**

Select $(i, j) \in B$

**IF** $\text{LCM}(\text{LT}(f_i), \text{LT}(f_j)) \neq \text{LT}(f_i) \cdot \text{LT}(f_j)$, **AND** Criterion($f_i, f_j, B$) is false **THEN**

$$
S := \text{S}(f_i, f_j)^G
$$

**IF** $S \neq 0$ **THEN**

$$
t := t + 1 ; \quad f_t := S \ ; \quad G := G \cup \{f_t\} \ ; \quad B := B \cup \{(i, t) : 1 \leq i \leq t - 1\}
$$

Our motivation to compute a Gröbner basis for the ideal $J_{r,n}$ is the following theorem attributed to Macaulay.

**Theorem 4.3** (Macaulay, c.f. [10]). Let $\prec$ be a term order and let $I \subseteq S$ be an ideal. Then the monomials in $S$ which do not belong to $\langle \text{LT}(I) \rangle$ form a $C$-basis for $S/I$.

Determining a Gröbner basis for $J_{r,n}$ yields a useful description of $\langle \text{LT}(J_{r,n}) \rangle$. Thus, Theorem 4.3 is an immediate consequence of Theorem 4.3, Proposition 4.3, and Theorem 4.10 below.

**Proposition 4.4.** Fix $r \geq 1$ and $n \geq 1$. Consider the monomial ideal $N_{r,n}$ in $T_n$ generated by the following elements:

- $z_{\emptyset}$
- $z_{A}$, where $A = [k]$ for all $1 \leq k \leq n$
\[ z_A^{r+1} \text{ where } A \neq [k] \text{ for any } 0 \leq k \leq n \]
\[ z_A z_B \text{ such that } A \not\subseteq B \text{ and } B \not\subseteq A \]
\[ z_A^r z_B^r \text{ where } A \neq [k] \text{ for any } 0 \leq k \leq n, \text{ such that } A \subset B \text{ and } \min(B \setminus A) > \max(A) \]
\[ z_A z_B \text{ where } B \neq [k] \text{ for any } 0 \leq k \leq n, \text{ such that } A \subset B \text{ and there is an } \ell \text{ with } [\ell] \not\subseteq A, [\ell] \subset B, \text{ and } B \setminus A \subset [\ell] \]
\[ z_A z_B z_{A_1} \text{ where } A_2 \neq [k] \text{ for any } 0 \leq k \leq n, \text{ such that } A_1 \subset A_2 \subset A_3 \text{ and } \max(A_2 \setminus A_1) < \min(A_3 \setminus A_2) \]

The monomials outside of this ideal are precisely the elements of the negative descent basis for \( T_n/J_{r,n} \) (for \( r = 1 \), this is the Garsia-Stanton basis described above).

**Proof.** We will first show the argument for \( r = 1 \), the Garsia-Stanton basis, then we will generalize the argument for \( r \geq 2 \). Assume unless otherwise stated that elements of sets are written in ascending order, e.g. \( A = \{a_1, a_2, \ldots, a_\ell\} \) implies \( a_1 < a_2 < \cdots < a_\ell \). First, note that the following observations imply that every monomial \( \hat{a}_\pi \) is not divisible by any of the generators of \( N_{1,n} \).

- \( z_\emptyset \) clearly cannot divide \( \hat{a}_\pi \) by construction.
- \( z_{\{1,2,\ldots,k\}} \) cannot divide \( \hat{a}_\pi \), as this would imply that there is a descent at the position \( k \), but there is no element smaller than \( k \) which has not already appeared.
- \( z^2_A \) cannot divide \( \hat{a}_\pi \), as by definition each set \( A \) which arises from \( \text{Des}(\pi) \) must be unique.
- By definition, if \( z_A z_B \) is a factor of \( \hat{a}_\pi \), it implies that \( A \subset B \) or vice versa.
- If \( z_A z_B \) divides \( \hat{a}_\pi \) with \( A \subset B \) such that \( A = \{a_1, a_2, \ldots, a_\ell\} \) and \( B = A \cup \{b_1, \ldots, b_k\} \), we must have that \( b_1 < a_\ell \) else there is no descent possible at position \( \ell \).
- If \( z_A z_B \) divides \( \hat{a}_\pi \) with \( A \subset B \) such that \( [\ell] \not\subseteq A \) and \( [\ell] \subseteq B \), we must have some element \( x \in B \setminus A \) such that \( x \not\in [\ell] \), else no descent could occur since \( \pi(|B|) \in [\ell] \) and \( [\ell] \subseteq \{\pi(1), \ldots, \pi(|B|)\} \).
- If \( z_A z_B z_{A_1} \) divides \( \hat{a}_\pi \) where \( A_1 \subset A_2 \subset A_3 \) and \( \max(A_2 \setminus A_1) < \min(A_3 \setminus A_2) \), then no descent could occur between set \( A_2 \) and \( A_3 \), i.e. in position \( \pi(|A_2|) \).

Suppose next that we have a monomial in \( m \in T_n \), which is divisible by none of the generators of \( N_{1,n} \). We claim that there exists some \( \pi \in S_n \) such that \( m = \hat{a}_\pi \); to prove this claim, first we write

\[ m = z_{B_1} z_{B_2} \cdots z_{B_s} \]

where \( B_1 \subset B_2 \subset \cdots \subset B_s \). We denote \( B_1 = \{\beta_1, \ldots, \beta_{m_1}\} \) and \( B_i = B_{i-1} \cup \{\beta_{i+1}, \ldots, \beta_{m_i}\} \) for all \( 1 \leq i \leq s \). Note that this union corresponds to the permutation

\[ \pi = \beta_1 \cdots \beta_{m_1} \beta_2 \cdots \beta_{m_2} \cdots \beta_{m_s} \gamma_1 \cdots \gamma_t \]

where \( \gamma_1 < \gamma_2 < \cdots < \gamma_t \) are the elements which do not appear in any \( B_i \) set. Moreover, we have that \( \beta_{m_i} > \beta_{i+1} \) and \( \beta_{m_s} > \gamma_1 \) and these will be the only such descents since \( m \) is not divisible by any of the generators of \( N_{r,n} \). Hence, \( m \) is a Garsia-Stanton descent element \( \hat{a}_\pi \). (This argument is similar to standard \( P \)-partition arguments [22] Lemma 3.15.3.)

Now suppose that \( r \geq 2 \). By a similar argument to that just given, \( b^r_{(\pi,X)} \) is not divisible by a monomial from among the generators of \( N_{r,n} \), since:

- \( z_\emptyset \) clearly cannot divide \( b^r_{(\pi,X)} \) by construction.
- \( z^r_{\{k\}} \) cannot appear in \( b^r_{(\pi,X)} \), as, since the greatest possible multiplicity of any element in \( X \) is \( r - 1 \), this would imply that there is a position \( k \) descent in \( \pi \) when all smaller elements than \( \pi(k) \) have already appeared in \( \pi \).
- \( z_A^{r+1} \) for \( A \neq [k] \) cannot appear in \( b^r_{(\pi,X)} \) as we only obtain a single \( z_A \) from \( \hat{a}_\pi \), and we can obtain at most \( r - 1 \) copies of \( z_A \) from the product over \( X \). Note that if \( z_A \) appears in \( b^r_{(\pi,X)} \), then one of the \( z_A \) terms must have come from the product indexed by \( \text{Des}(\pi) \), and thus \( |A| \in \text{Des}(\pi) \).
- By definition, \( z_A z_B \) a factor of \( b^r_{(\pi,X)} \) implies \( A \subseteq B \) or vice-versa.
• If \( z_A^r z_B \) appears in \( b_{(\pi,X)}^r \) where \( A \subset B \) with \( A = \{a_1, a_2, \ldots, a_\ell \} \) and \( B = A \cup \{b_1, \ldots, b_k \} \), it follows that \( |A| \in \text{Des}(\pi) \), thus we must have that \( b_1 < a_\ell \) else there is no descent occurring in \( \pi \) in position \( |A| \).

• If \( z_A z_B^r \) appears in \( b_{(\pi,X)}^r \) where \( A \subset B \) with \( \ell \not\subset A \) and \( \ell \subset B \), then \( |B| \in \text{Des}(\pi) \). Hence, there must exist an element \( x \in B \setminus A \) such that \( x \not\in \ell \), else no descent can occur.

• If \( z_{A_1} z_{A_2}^r z_{A_3} \) appears in \( b_{(\pi,X)}^r \) such that \( A_1 \subset A_2 \subset A_3 \) and \( \max(A_2 \setminus A_1) < \min(A_3 \setminus A_2) \), then no descent can occur between set \( A_2 \) and \( A_3 \), i.e. in position \( \pi(\{A_2\}) \), but the power of \( r \) on \( z_{A_2}^r \) forces that there is such a descent. Hence this divisibility is not possible.

Suppose next that we have a monomial \( m_r \in T_n \) that is divisible by none of the generators of \( N_{r,n} \). We claim that there exists some \( \pi \in S_n \) and \( X \) a multiset of \( [n] \) with every element having multiplicity strictly less than \( r \) such that \( m_r = b_{(\pi,X)}^r \). An example illustrating the following proof is given in Example 4.5. To prove this claim, first we write

\[
m_r = z_{B_1}^{b_1} z_{B_2}^{b_2} \cdots z_{B_s}^{b_s}
\]

where we have \( B_1 \subset B_2 \subset \cdots \subset B_s \). Note that \( b_1 \leq r \) if \( B_1 \neq [k] \) and \( b_1 \leq r - 1 \) if \( B_1 = [k] \). As in the previous case, inductively define \( B_i = B_{i-1} \cup \{\beta_{i_1}, \ldots, \beta_{i_m} \} \). Construct a new monomial

\[
m'_r = \begin{cases} z_{B_1} z_{B_2} \cdots z_{B_s} & \text{if } B_s \neq [n] \\ z_{B_1} z_{B_2} \cdots z_{B_{s-1}} & \text{if } B_s = [n] \end{cases}
\]

and the set

\[
\tilde{X} = \begin{cases} \{c_1, \ldots, c_1, c_2, \ldots, c_2, \ldots, c_s, \ldots, c_s\} & \text{if } B_s \neq [n] \\ \{c_1, \ldots, c_1, c_2, \ldots, c_2, \ldots, c_{s-1}, c_s, \ldots, c_s, \ldots, c_s\} & \text{if } B_s = [n] \end{cases}
\]

where \( c_i = |B_i| \). We associate to \( m'_r \) the permutation

\[
\pi = \beta_{i_1} \cdots \beta_{i_{m_1}} \beta_{i_2} \cdots \beta_{i_{m_2}} \cdots \beta_{i_1} \cdots \beta_{i_{m_s}} \gamma_1 \cdots \gamma_t
\]

where \( \gamma_1 < \gamma_2 < \cdots < \gamma_t \) are any elements which do not appear in any \( B_i \) set. Since the \( \beta \)-values within each \( B_i \) are increasing, the only possible descents occur between \( \beta_{i_{m_i}} \) and \( \beta_{i_{m_i+1}} \). If we have \( \beta_{i_{m_i}} > \beta_{i_{m_i+1}} \), then we have a descent and we do nothing. (Note that the final three types of generators of \( N_{r,n} \) force a descent to occur if \( b_i \) takes on a maximal value of \( r \) or \( r - 1 \), showing that all seven of the types of generators of \( N_{r,n} \) are required for this argument to hold.) If we have

\[
(3) \quad \beta_{i_{m_i}} < \beta_{i_{m_i+1}}
\]

then there is no descent. Let \( m_{fail} \) be the product of \( z_{B_i} \) over all the \( i \) values such that (3) holds and define

\[
\tilde{m}_r := m'_r / m_{fail}.
\]

We have that \( \hat{\alpha}_r = \tilde{m}_r \) by our argument in the \( r = 1 \) case. Moreover, we set \( X := \tilde{X} \cup \{c_i : \beta_{i_{m_i}} < \beta_{i_{m_i+1}} \} \), where as before \( c_i = |B_i| \). With this choice of permutation and multiset we obtain \( m = b_{(\pi,X)}^r \).

\begin{example}
Let \( r = 4 \) and \( n = 6 \) and use the notation from the preceding proof. Consider the monomial

\[
z_{(4)} z_{(2,4)}^2 z_{\{1,2,3,4,5\}}^2 z_{\{1,2,3,4,5,6\}}.
\]

Thus, \( B_2 = \{2, 4\} \), \( B_5 = \{1, 2, 3, 4, 5, 6\} \), and so on. We have that \( m'_r = z_{\{4\}} z_{\{2,4\}}^2 z_{\{1,2,3,4,5\}}^2 z_{\{1,2,3,4,5,6\}} \) since \( B_4 = \{1, 2, 3, 4, 5, 6\} \). We have that \( \tilde{X} = \{2, 2, 2, 4, 5, 6\} \), where the 6 is included since \( B_5 = \{1, 2, 3, 4, 5, 6\} \). In this case, the permutation \( \pi = 421536 \), and \( m_{fail} = z_{\{1,2,3,4,5\}} \). Thus, we have that

\[
\tilde{m}_r = z_{\{4\}} z_{\{2,4\}} \hat{\alpha}_r = \tilde{m}_r.
\]
\end{example}
and

\[ X = \{2, 2, 2, 4, 5, 5, 6\} \, . \]

It is straightforward to check that

\[ b^4_{(421536, 22224555, 6)} = z^4 \cdot z^2 \cdot z^5 \, . \]

as desired. Note that here we have recovered the correspondence given in Example 3.3.

**Definition 4.6.** Given two sets \( A \) and \( B \) such that \( |A| = |B| = k \), we say that \( A \) is lexicographically before \( B \) if there exists \( i \in A \) such that \( i \notin B \) and given any \( j \in B \) such that \( j < i \) we have \( j \in A \).

For example, the ordering of 3-subsets of the 5-set would be \( 1, 2, 3 < 1, 2, 4 < 1, 2, 5 < 1, 3, 4 < 1, 3, 5 < 1, 4, 5 < 2, 3, 4 < 2, 3, 5 < 2, 4, 5 < 3, 4, 5 \). Our next step is to prove that the monomials listed in Proposition 4.4 arise as leading terms of \( J_{r,n} \) when the following monomial term order is imposed on \( T_n \).

**Definition 4.7.** Give the variables of \( T_n \) the linear order \( z_A > z_B \) if \( |A| < |B| \) or if \( |A| = |B| \) and \( A \) is lexicographically before \( B \). With respect to this ordering of variables, endow \( T_n \) with the graded reverse lexicographic (or grevlex) term order. In this setting, grevlex order is as follows. Let \((\alpha_A)_{A \subseteq \{n\}}\) and \((\beta_A)_{A \subseteq \{n\}}\) be vectors in \( \mathbb{Z}_\geq 0 \) with entries totally ordered by setting the \( A \)-th coordinate to be larger than the \( B \)-th coordinate if and only if \( z_A > z_B \). For two monomials in \( T_n \), we have

\[ \prod_{A \subseteq \{n\}} z^\alpha_A > \text{grevlex} \quad \prod_{A \subseteq \{n\}} z^\beta_A \]

if either (1) \( \sum_{A \subseteq \{n\}} \alpha_A > \sum_{A \subseteq \{n\}} \beta_A \) or (2) \( \sum_{A \subseteq \{n\}} \alpha_A = \sum_{A \subseteq \{n\}} \beta_A \) and in \((\alpha_A - \beta_A)_{A \subseteq \{n\}}\) the right most non-zero entry is negative.

**Example 4.8.** The variables in \( T_3 \) are ordered as follows:

\[ z_\emptyset > z_1 > z_2 > z_3 > z_{\{1,2\}} > z_{\{1,3\}} > z_{\{2,3\}} > z_{\{1,2,3\}} \]

We have that

\[ z^4_{\{2\}} > z^2_\emptyset z^2_{\{1\}} z_{\{1,2\}} \]

since the exponent vectors for these monomials with respect to the linear order of the variables above are \((0, 0, 0, 0, 0, 0, 0, 0)\) and \((1, 2, 0, 0, 1, 0, 0, 0)\), hence we have

\[ (0, 0, 4, 0, 0, 0, 0, 0) - (1, 2, 0, 0, 1, 0, 0, 0) = (-1, -2, 4, 0, -1, 0, 0, 0) \]

with negative right-most non-zero entry.

We will need the following definition for the proof of Theorem 4.10.

**Definition 4.9.** We call a pair of subsets \( A \) and \( B \) such that \( A \not\subseteq B \) and \( B \not\subseteq A \) a Sperner 2-pair.

**Theorem 4.10.** There exists a Gröbner basis \( G_{r,n} \) of \( J_{r,n} \) for which the ideal generated by \( \text{LT}(G_{r,n}) \) is the ideal \( N_{r,n} \) generated by terms of the form listed in Proposition 4.4.

Prior to proving the general Gröbner basis result, it is useful to consider a small example. Take \( J_{3,2} = \langle z_{\{1\}} z_{\{2\}} - z_\emptyset z_{\{2\}}, z_\emptyset, z^3_{\{1\}} + z^3_{\{2\}}, z^3_{\{2\}} \rangle \). From the list of desired leading terms given in Proposition 4.4, the only term not immediately accounted for is \( z^3_{\{2\}} \). The only nontrivial \( S \)-polynomial to consider initially is

\[ S(z_{\{1\}} z_{\{2\}} - z_\emptyset z_{\{2\}}, z^3_{\{1\}} + z^3_{\{2\}}) = \frac{z^3_{\{1\}} z_{\{2\}}}{z^3_{\{2\}} z_{\{1\}}} \cdot (z_{\{1\}} z_{\{2\}} - z_\emptyset z_{\{2\}}) - \frac{z^3_{\{2\}}}{z^3_{\{1\}}} \cdot (z^3_{\{1\}} + z^3_{\{2\}}) \]

\[ = -z^4_{\{2\}} - z_\emptyset z^2_{\{1\}} z_{\{1,2\}} \, . \]
Under our term order, the leading term is $-z^4_{(2)}$, which is as desired. In order to show that no additional polynomials appear in the Gröbner basis, an exhaustive check of all other $S$-polynomials shows they reduce to 0. Alternatively, we can argue that no other terms will appear because we can compute that \( \dim_C(T_2/J_{3,2}) = 3^2 \cdot 2 = 18 \) via a Hilbert series argument that is explicitly given by [4] in the proof below, thus no other leading terms can appear without contradicting this known dimension. In small examples, either argument will suffice. However, for arbitrary $r$ and $n$, the latter argument is more efficient.

**Proof of Theorem 4.10.** Use the term order for $T_n$ described above. Our proof will involve computing $S$-polynomials starting from the generators of $J_{r,n}$. To minimize the number of computations required, we first make a dimension argument showing that the number of monomials outside of the leading term ideal for $J_{r,n}$ is the number of elements of the negative descent basis. We then compute $S$-polynomials to produce elements with all of the leading terms listed in Proposition 4.4, which will complete the proof. We will compute the $S$-polynomials for arbitrary $r$, but we will make two dimension arguments, for $r = 1$ and $r \geq 2$.

Consider $r = 1$. It is a straightforward observation to notice that the number of elements of $p \in R_n$ such that $\deg(p) = t^k$ are precisely the lattice points at height $k$ in the cone([0, 1]^n) and the cardinality of these elements is $(k + 1)^n$. Combining this observation with [22, Proposition 1.4.4], we see that the Hilbert series of $R_n$ is given by

\[
\text{Hilb}(R_n; t) = \sum_{k \geq 0} (k + 1)^n t^k = \frac{A_n(t)}{(1-t)^n+1}
\]

where $A(n) = \sum_{\pi \in S_n} t^{\text{des}(<\pi>)}$ is the Eulerian polynomial. Let $C_{1,n} := \mathbb{C}[\hat{e}_k + I_n | 0 \leq k \leq n]$, and note that the elements $\hat{e}_k + I_n$ are algebraically independent since they specialize in $R_n$ (by setting $t = 1$) to the usual elementary symmetric functions; note that $\text{Hilb}(C_{1,n}; t) = \frac{1}{(1-t)^{n+1}}$. Hochster’s Theorem implies that $R_n$ is Cohen-Macaulay [10], and since $\text{inv}(1,n)$ is an ideal generated by an algebraically independent system of parameters, we have

\[
\text{Hilb}(T_n/J_{1,n}; t) = A_n(t)
\]

by [13, Lemma 17.1]. The $\mathbb{C}$-dimension of $T_n/J_{1,n}$ is

\[
\dim_C(T_n/J_{1,n}) = \text{Hilb}(T_n/J_{1,n}; 1) = A(1) = n!,
\]

which is the number of elements in the Garsia-Stanton descent basis, as desired.

Now, suppose that $r \geq 2$. Let $C_{r,n} = \mathbb{C}[z_{\varnothing} + I_n, e_{r,k} + I_n | 1 \leq k \leq n]$. Given that $R_n$ is Cohen-Macaulay and that $e_{r,k} + I_n$ and $z_{\varnothing} + I_n$ are algebraically independent, hence $\text{Hilb}(C_{r,n}; t) = \frac{1}{(1-t)(1-t^r)^n}$, we have that

\[
\text{Hilb}(R_n; t) = \sum_{k \geq 0} (k + 1)^n t^k = \frac{B_{r,n}(t)}{(1-t)(1-t^r)^n}
\]

where $B_{r,n}(t) = A_n(t) \cdot (1 + t + \cdots + t^{r-1})^n$ by our previous calculation for $r = 1$. Thus,

\[
\text{Hilb}(T_n/J_{r,n}; t) = A_n(t) \cdot (1 + t + \cdots + t^{r-1})^n
\]

from which we can conclude that

\[
\dim_C(T_n/J_{r,n}) = \text{Hilb}(T_n/J_{r,n}; 1) = B_{r,n}(1) = r^n n!,
\]

which is the number of elements in the negative descent basis, as desired.

Next, we move to $S$-polynomial calculations. Our goal is to compute $S$-polynomials until all the elements listed in Proposition 4.4 arise as leading terms; since at that point we will have reached the correct value of $\dim_C(T_n/J_{r,n}) = \text{Hilb}(T_n/\text{LT}(J_{r,n}))$, we must have a Gröbner basis.
We begin by noting that some of our desired leading terms arise from the generators of \( J_{r,n} \). First, \( z_A z_B \) such that \( A \not\subseteq B \) and \( B \not\subseteq A \) where \( A \neq [k] \neq B \) for any \( k \) are leading terms of \( I_n \). The monomials \( z_{\emptyset} \) and \( z_A^r \) where \( A = [k] \) for \( k = 1, \ldots, n \) are the leading terms of \( \text{invar}(r, n) \). These account for the fourth, first, and second items listed in Proposition 4.4, respectively.

To obtain an element with the leading term \( z_A^{r+1} \) as given in the third bullet of Proposition 4.4, suppose that \( |A| = k \) and consider the following \( S \)-polynomial:

\[
S(\hat{e}_{r,k}, z_{[k]} z_A - z_{[k]} \cap A z_{[k] \cup A}) = \frac{z_{[k]}^r z_A}{z_{[k]}^r} \left( z_{[k]}^r + z_{A_1}^r + z_{A_2}^r + \cdots + z_{A}^r + \cdots + z_{A_{(k)}^r} \right) - \frac{z_{[k]}^r z_A}{z_{[k]}^r} \left( z_{[k]} z_A - z_{[k]} \cap A z_{[k] \cup A} \right)
\]

\[
= z_A \left( z_{A_1}^r + z_{A_2}^r + \cdots + z_{A}^r + \cdots + z_{A_{(k)}^r} \right) + z_{A_{(k)}^r-1} z_{[k]} z_A z_{[k] \cup A}
\]

Note that the term order implies that

\[
z_A z_{A_1}^r > z_A z_{A_2}^r > \cdots > z_A z_{A}^r > \cdots > z_A z_{A_{(k)}^r} > z_{A_{(k)}^r-1} z_{[k]} z_A z_{[k] \cup A}
\]

However, for each \( i \) where \( A_i \neq A \), \( A_i z_{A_i} \) is the leading term of a polynomial of \( J_{r,n} \), and we use \( z_A z_{A_i} - z_{A \cap A_i} z_{A_i \cup A_i} \in J_{r,n} \) to rewrite \( z_A z_{A_i} \), yielding

\[
S(\hat{e}_{r,k}, z_{[k]} z_A - z_{[k]} \cap A z_{[k] \cup A}) = z_A^{r+1} + \sum_j z_A \cap A_j z_{A_j}^{r-1} z_{A \cup A_j}
\]

where the sum is over all \( j \) such that \( |A_j| = k \), \( A_j \neq A \), and \( A \cap A_j \neq \emptyset \), since any terms involving \( z_{\emptyset} \) are elements of \( J_{r,n} \). The observation that \( |A| < |A \cup A_j| \) for all such \( j \) implies that \( z_A^{r+1} \) is the leading term of this polynomial, as desired.

Assume that we have added all prior \( S \)-polynomial calculations to the generators of \( J_{r,n} \). To obtain terms of the form \( z_A^r z_B \), where \( A \subseteq B \) with \( \max(A) < \min(B \setminus A) \) as listed in the fifth bullet of Proposition 4.4, let \( |A| = k \). We compute the \( S \)-polynomial of \( \hat{e}_{r,k} \) and the generator of \( I_n \) with leading term \( z_{[k]} z_B \). Note that \( z_{[k]} z_B \) is the leading term of a generator of \( I_n \), since by assumption \( A \neq [k] \) thus if \( [k] \subseteq B \) this would violate the condition \( \max(A) < \min(B \setminus A) \). We compute:

\[
S(\hat{e}_{r,k}, z_{[k]} z_B - z_{[k]} \cap B z_{[k] \cup B}) = \frac{z_{[k]}^r z_B}{z_{[k]}^r} \left( z_{[k]}^r + z_{A_1}^r + z_{A_2}^r + \cdots + z_{A}^r + \cdots + z_{A_{(k)}^r} \right) - \frac{z_{[k]}^r z_B}{z_{[k]}^r} \left( z_{[k]} z_B - z_{[k]} \cap B z_{[k] \cup B} \right)
\]

\[
= z_B \left( z_{A_1}^r + z_{A_2}^r + \cdots + z_{A}^r + \cdots + z_{A_{(k)}^r} \right) + z_{A_{(k)}^r-1} z_{[k]} z_B z_{[k] \cup B}
\]

We have the ordering

\[
z_{A_1}^r z_B > z_{A_2}^r z_B > \cdots > z_{A}^r z_B > \cdots > z_{A_{(k)}^r} z_B > z_{A_{(k)}^r-1} z_{[k]} z_B z_{[k] \cup B}
\]

Moreover, by the condition \( \max(A) < \min(B \setminus A) \) and the use of lexicographic order on subsets, we know that \( A_i \not\subseteq B \) for all \( i \) such that \( z_{A_i}^r z_B > z_{A}^r z_B \). This is true because if \( A_i \subseteq B \), then there exists some \( j \in A_i, j \not\in A \) so that \( \max(A) < j \) and the condition that \( |A| = |A_i| \) implies
that there must exist some \( s \in A \) such that \( s \not\in A_i \) and for all \( t \in A_i \) such that \( t < s \) we have 
\[ t \in A, \text{ which would contradict } z^r_A, z_B > z^r_A, z_B \text{ by the definition of our variable ordering arising from the lexicographic ordering on subsets.} \]
The condition that \( A_i \not\subseteq B \) implies that \( z^r_A, z_B \) is a leading term of a polynomial in \( I_n \). Applying \( z_A, z_B - z_{A \cap B} z_{A \cup B} \in J_{r,n} \) to the term \( z^r_A, z_B \) will produce 
\[ z_A, z_B - z_{A \cap B} z_{A \cup B} < z^r_A, z_B. \]
Therefore, we will have 
\[ S(\hat{e}_{r,k}, z_B - z_{[k]} \cap B z_B) = z^r_A z_B + \sum_j z^r_A, z_B + m z_{A \cap B} z^r_A z_{A \cup B} \]
where the first sum is over all \( j \) so that \( |A_j| = |A|, A_j \neq A, \) and \( A_j \subseteq B \), which implies that \( z_A > z_A \) by condition \( \max(A) < \min(B \setminus A) \). The second sum is over all \( m \) such that \( |A_m| = |A| \) where \( A_m \) and \( B \) are a Sperner 2-pair with \( A_m \cap B \neq \emptyset \), as if the intersection was empty then the resulting term would be a multiple of \( z_B \) and hence an element of \( J_{r,n} \). It follows from a simple cardinality argument that \( z_{A \cap B} < z_A \), and thus \( z^r_A, z_B \) is a leading term in \( J_{r,n} \).

Assume again that we have added all prior \( S \)-polynomial calculations to the generators of \( J_{r,n} \). To obtain terms of the form \( z_A z_B \), there is an \( \ell \) such that \( [\ell] \not\subseteq A, [\ell] \subseteq B \) and \( B \setminus A \subseteq [\ell] \), as listed in the sixth bullet of Proposition 4.4. Let \( |B| = k \). We compute the \( S \)-polynomial of \( \hat{e}_{r,k} \) and the generator of \( I_n \) with leading term \( z_A z_B \), which is a leading term since there exists an element \( x \in [\ell] \subseteq [k] \) such that \( x \not\in A \) and there also exists \( y = \max(A) = \max(B) \not\in [k] \):
\[ S(\hat{e}_{r,k}, z_A z_B) = \frac{z_A z_B}{z_A z_B} \left( z^r_A, z_B + \ldots + z^r_B \right) - \frac{z_A z_B}{z_A z_B} \left( z_A z_B - z_B z_A \right) = z_A \left( z^r_B + \ldots + z^r_B \right) + z^r_B z_A \]
which yields the term order of
\[ z_A z_B > z_A z_B > \ldots > z_A z_B > \ldots z_A z_B \]
Note that \( A \not\subseteq B_i \) for all \( i \) such that \( z_{B_i} > z_B \). This is true because if \( A \subseteq B_i \) for \( B_i \neq B \), then given that \( |B| = |B_i| \) we must have \( z_B > z_{B_i} \) because \( B \setminus A \) contains precisely the smallest elements not contained in \( A \) and thus \( B_i \setminus A \) must contain at least one larger element meaning that \( B_i \) is lexicographically after \( B \). The condition that \( A \not\subseteq B_i \) for all \( i \) such that \( z_{B_i} > z_B \) implies that \( z_A z_{B_i} \) is the leading term of a polynomial in \( I_n \). As in our previous cases, this leads to the calculation 
\[ S(\hat{e}_{r,k}, z_A z_B) = z_A z_B + \sum_j z_A z_B m z_{A \cap B m} z^r_B z_A \]
where the first sum is over all \( j \) such that \( |B_j| = |B|, B \neq B_j, \) and \( A \subseteq B_j \). The second sum is over all \( m \) such that \( A \) and \( B_m \) are a Sperner 2-pair with \( A \cap B_m \neq \emptyset \). Notice that we know that \( |B| = k \) and \( B \neq |k| \) which says that there is at least some subset \( \{j_1, \ldots, j_l\} \subseteq B \) such that \( j_i > k \) for all \( i \) and the defining condition \( [\ell] \not\subseteq A, [\ell] \subseteq B \) and \( B \setminus A \subseteq [\ell] \) implies that \( j_i \in A \) for some \( i \). Thus, \( |A \cap [k]| > k = |B| \). Ergo, we have \( z_A z_B \) as the leading term.

Our final case is to obtain the terms listed in the seventh bullet of Proposition 4.4 i.e. those of type \( z_A z_A z_A \) where \( A_1 \subseteq A_2 \subseteq A_3 \) and \( \max(A_2 \setminus A_1) < \min(A_3 \setminus A_2) \) with \( A_2 \neq |j| \) for all \( j \). Assume that we have added all prior \( S \)-polynomials to the generators of \( J_{r,n} \). We consider the \( S \)-polynomial for the elements \( z_A z_A \) and the generator from [5] given by
$z_{A_2}^{r+1} + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j}$ where $|C_j| = |A_2| = k$, $A_2 \neq C_j$, and $A_2 \cap C_j \neq \emptyset$. Let $B := A_1 \cup (A_3 \setminus A_2)$ for convenience of notation, and compute:

$$S \left( z_{A_2}^{r+1} + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j}, z_{A_2} z_B - z_{A_1} z_{A_3} \right)$$

$$= \frac{z_{A_2}^{r+1} z_B}{z_{A_2}^{r+1}} \left( z_{A_2}^{r+1} + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j} \right)$$

$$= z_B \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j} + z_{A_1} z_{A_2}^{r} z_{A_3}$$

$$= z_{A_1 \cup (A_3 \setminus A_2)} \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j} + z_{A_1} z_{A_2}^{r} z_{A_3}$$

We now wish to show the $z_{A_1} z_{A_2}^{r} z_{A_3}$ is the leading term. Consider the terms involving $C_j$. There are three possible cases

1. $|A_2 \cup C_j| > |A_3|$
2. $|A_2 \cup C_j| < |A_3|$
3. $|A_2 \cup C_j| = |A_3|$

which we consider individually.

**Case 1:** If we have that $|A_2 \cup C_j| > |A_3|$, then we have $z_{A_1} z_{A_2}^{r} z_{A_3} > z_{A_2 \cap C_j} z_{A_1 \cup (A_3 \setminus A_2)} z_{C_j}^{r-1} z_{A_2 \cup C_j}$ immediately by the definition of graded reverse lexicographic order.

**Case 2:** Suppose that we have $|A_2 \cup C_j| < |A_3|$. Note that this implies that there exists $x \in A_3$ such that $x \not\in A_2 \cup C_j$ and hence $x \in A_1 \cup (A_3 \setminus A_2)$. We also have $y \in A_2 \cup C_j$ such that $y \not\in A_1 \cup (A_3 \setminus A_2)$. Hence, we have that $A_1 \cup (A_3 \setminus A_2)$ and $A_2 \cup C_j$ are a Sperner 2-pair. This implies that we can replace the monomial $z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (A_3 \setminus A_2)} z_{A_2 \cup C_j}$ with the monomial

$$z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{(A_1 \cup (A_3 \setminus A_2)) \cap (A_2 \cup C_j)} z_{(A_1 \cup (A_3 \setminus A_2)) \cup (A_2 \cup C_j)}$$

$$= z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (A_3 \setminus A_2) \cap (A_2 \cup C_j)} z_{A_2 \cup C_j}$$

It is clear that $|A_3 \cup C_j| \geq |A_3|$. If the inequality is strict, then we are done. If $A_3 \cup C_j = A_3$, note that $C_j \subset A_3$ and that $C_j \cap (A_3 \setminus A_2) \neq \emptyset$ since $|C_j| = |A_2|$. We will now consider the variable $z_{A_1 \cup (C_j \cap (A_3 \setminus A_2))}$. We note that two subcases arise:

2.1. $A_1 \cup (C_j \cap (A_3 \setminus A_2)) = A_1 \cup C_j$ (equivalently $C_j \cap A_1 = C_j \cap A_2$)

2.2. $A_1 \cup (C_j \cap (A_3 \setminus A_2))$ and $A_2 \cap C_j$ are a Sperner 2-pair.

**Subcase 2.1:** Note that $|A_1 \cup C_j| \geq |A_2|$ with equality occurring if $A_1 \cup C_j = C_j$. If the inequality is strict, we are done. If $A_1 \cup C_j = C_j$, then $|C_j| = |A_2|$, but since $C_j \cap A_1 = C_j \cap A_2$ and $C_j \cap (A_3 \setminus A_2) \neq \emptyset$, the condition $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$ implies that $A_2$ is lexicographically before $C_j$, which is desired.

**Subcase 2.2:** The existence of such a Sperner 2-pair allows us to replace the monomial through division by

$$z_{(A_1 \cup (C_j \cap (A_3 \setminus A_2)) \cap (A_2 \cup C_j))} z_{(A_1 \cup (C_j \cap (A_3 \setminus A_2)) \cup (A_2 \cup C_j)} z_{C_j}^{r-1} z_{A_3}$$

$$= z_{A_1 \cup (C_j \cap (A_3 \setminus A_2))} z_{A_2 \cup C_j} z_{C_j}^{r-1} z_{A_3}$$

Showing the desired outcome is now identical to the argument in Subcase 2.1.
Case 3: Suppose we have \(|A_2 \cup C_j| = |A_3|\). In this case, it is sufficient to consider the following three plausible sub-cases.

3.i. \(A_2 \cap C_j\) and \(A_1 \cup (A_3 \setminus A_2)\) are a Sperner 2-pair.
3.ii. The subcase 3.i. is false, but \(A_2 \cap C_j\) and \(A_1 \cup (A_3 \setminus A_2)\) are a Sperner 2-pair.
3.iii. \(A_2 \cap C_j\), \(A_2 \cup C_j\), and \(A_1 \cup (A_3 \setminus A_2)\) have no Sperner 2-pairs between them.

Subcase 3.i: Suppose we have that the sets \(A_2 \cup C_j\) and \(A_1 \cup (A_3 \setminus A_2)\) are a Sperner 2-pair. This means that via division, we can replace the existing monomial \(z_{A_2 \cap C_j} z_{C_j} \cdot z_{A_1 \cup (A_3 \setminus A_2)} z_{A_2 \cup C_j}\) with the monomial

\[
z_{A_2 \cap C_j} z_{C_j}^{-1} z_{(A_2 \cup C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{(A_2 \cup C_j) \cup (A_1 \cup (A_3 \setminus A_2))} z_{A_2 \cup C_j} = z_{A_2 \cap C_j} z_{C_j}^{-1} z_{(A_2 \cup C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{A_2 \cup C_j}
\]

By virtue of the Sperner 2-pair assumptions, we have that there exists \(x \in C_j\) such that \(x \notin A_3\), which yields \(|A_3 \cup C_j| > |A_3|\) and hence

\[
z_{A_1} z_{A_2} z_{A_3} > z_{A_2 \cap C_j} z_{C_j}^{-1} z_{(A_2 \cup C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{A_2 \cup C_j}
\]

and we are done.

Subcase 3.ii: Suppose that \(A_2 \cap C_j\) and \(A_1 \cup (A_3 \setminus A_2)\) are a Sperner 2-pair, but that \(A_2 \cup C_j\) and \(A_1 \cup (A_3 \setminus A_2)\) are not. Then note that we have \(A_1 \cup (A_3 \setminus A_2) \subset A_2 \cup C_j\), which implies that \(A_3 \setminus A_2 \subset C_j\), and hence \(A_2 \cup C_j = A_3\) by the cardinality assumption. Now, by the existence of the Sperner 2-pair, we can replace via division the existing monomial \(z_{A_2 \cap C_j} z_{C_j}^{-1} z_{A_1 \cup (A_3 \setminus A_2)} z_{A_2 \cup C_j}\) with the monomial

\[
z_{(A_2 \cap C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{(A_2 \cup C_j) \cup (A_1 \cup (A_3 \setminus A_2))} z_{C_j}^{-1} z_{A_3}
\]

Moreover, notice that \(C_j \subseteq ((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))\). If the equality is strict, we have that \(|A_2| < |((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))|\) and we are done. If we have equality, then we know \(|A_2| = |((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))|\). By the assumption that \(\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)\), this implies that \(A_2\) is lexicographically before \(((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))\). Thus we will have

\[
z_{A_1} z_{A_2} z_{A_3} > z_{(A_2 \cap C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{(A_2 \cup C_j) \cup (A_1 \cup (A_3 \setminus A_2))} z_{C_j}^{-1} z_{A_3}
\]

which is as desired.

Subcase 3.iii: Suppose that the sets \(A_2 \cap C_j, A_2 \cup C_j,\) and \(A_1 \cup (A_3 \setminus A_2)\) have no Sperner 2-pairs between them. This implies the following containment

\(A_2 \cap C_j \subseteq A_1 \cup (A_3 \setminus A_2) \subset A_2 \cup C_j = A_3\)

because \(A_2 \cap C_j \subseteq A_1\) and \(A_3 \subseteq A_2 \cup C_j\), which follows from the necessary containment and the fact that these sets have the same cardinality. These observations allow us to conclude that \(C_j \subseteq A_1 \cup (A_3 \setminus A_2)\). If the containment is strict, we have that \(|A_1 \cup (A_3 \setminus A_2)| > |A_2|\) and we are done. If equality holds, we have \(|A_1 \cup (A_3 \setminus A_2)| = |A_2|\). However, the assumed condition that \(\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)\) implies that \(A_2\) is lexicographically before \(A_1 \cup (A_3 \setminus A_2)\). Thus, we have that

\[
z_{A_1} z_{A_2} z_{A_3} > z_{A_2 \cap C_j} z_{A_1 \cup (A_3 \setminus A_2)} z_{C_j}^{-1} z_{A_3}
\]

which is our desired result.

Given all of the above, we can conclude that

\[
S \left( z_{A_2}^r + \sum_j z_{A_2 \cap C_j} z_{C_j}^{-1} z_{A_2 \cup C_j} z_{A_2} z_{A_3} - z_{A_1} z_{A_3} \right) = z_{A_1} z_{A_2} z_{A_3} + p_{A_1 A_2 A_3}
\]

where \(p_{A_1 A_2 A_3}\) is a polynomial with \(\text{LT}(p_{A_1 A_2 A_3}) < z_{A_1} z_{A_2} z_{A_3}\).
We have now shown that all of our desired leading terms appear through the optimized Buchberger Algorithm. Because of our previous dimension calculation for \( \text{deg} (z_A) = t q^{|A|} \), we know that no additional leading terms can result from further computations, thus we have a Gröbner basis. □

We have thus established Theorem 4.1 as it follows immediately from Theorem 4.3, Proposition 4.4, and Theorem 4.10.

5. Combinatorial identities

We will now compute multigraded Hilbert series to prove Theorems 1.1 and 1.2. Recall from Section 1 that we can define a Hilbert series with respect to a \( \mathbb{Z}^m \)-grading for any \( m \geq 1 \) as in (1).

We now define the \( \mathbb{Z}_2 \)-grading which arises from the the defined degree on variable \( s \), \( \text{deg} (z_A) = t q^{|A|} \), where we note that \( \text{deg} (z_\emptyset) = t \). We denote this bivariate Hilbert series as \( \text{Hilb}(R_n; t, q) = \sum_{k \geq 0} [k+1]_q^n t^k \), which we assume for both of the following proofs. We will use the notation \( C_{r,n} \) introduced in the proof of Theorem 4.10.

Proof of Theorem 1.1. Given that \( R_n \) is Cohen-Macaulay and the elements of \( \text{invar}(1,n) \) are an algebraically independent homogeneous system of parameters as argued in the proof of Theorem 4.10, we can express the Hilbert series in the form

\[
\text{Hilb}(R_n; t, q) = \text{Hilb}(T_n/J_{1,n}; t, q) \prod_{j=0}^{n-1} (1 - t q^j).
\]

This follows because it is an elementary exercise to compute that

\[
\text{Hilb}(C_{1,n}; t, q) = \frac{1}{(1-t)(1-tq)(1-tq^2)\cdots(1-tq^n)}.
\]

To compute the numerator, we have

\[
\text{Hilb}(T_n/J_{1,n}; t, q) = \sum_{\pi \in S_n} \text{deg}(\hat{a}_\pi) = \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}
\]

by using the basis for \( T_n/J_{1,n} \) given by Theorem 4.1. This completes the proof.

□

Proof of Theorem 1.2. Given that \( R_n \) is Cohen-Macaulay and \( \text{invar}(r,n) \) is an algebraically independent homogeneous system of parameters, we can express the Hilbert series as

\[
\text{Hilb}(R_n; t, q) = \frac{\text{Hilb}(T_n/J_{r,n}; t, q)}{(1-t) \prod_{j=1}^{r} (1 - t^r q^{j})}.
\]

This follows because, as in the previous proof, it is straightforward to show that

\[
\text{Hilb}(C_{r,n}; t, q) = \frac{1}{(1-t)(1-t^r q^r)(1-t^r q^{2r})\cdots(1-t^r q^{rn})}.
\]

Hence, we compute the numerator by employing the basis given in Theorem 4.1

\[
\text{Hilb}(T_n/J_{r,n}; t, q) = \sum_{(\pi,X) \in Z_{r} \wr S_n} \text{deg}(b_{(\pi,X)}^r) = \sum_{(\pi,X) \in Z_{r} \wr S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} q^{|X|} \prod_{i \in X} i
\]

completing the proof.

□
6. Concluding Remarks

It is worth mentioning that when \( r = 1 \) there is a graded \( S_n \)-module isomorphism between \( T_n/J_{1,n} \) and \( \mathbb{C}[x_1, x_2, \ldots, x_n]/J_{1,n} \).

**Theorem 6.1.** The map \( \varphi : T_n/J_{1,n} \to \mathbb{C}[x_1, x_2, \ldots, x_n]/J_{1,n} \) defined by algebraically extending \( z_A + J_{1,n} \mapsto \prod_{i \in A} x_i + J_{1,n} \) is an \( S_n \)-isomorphism.

**Proof.** Consider \( T_n/J_{1,n} \) under the \( q \)-grading used in the multigrading for Section \( \ref{section:multigrading} \) i.e. \( \deg(z_A) = |A| \). Let \( \mathbb{C}[x_1, x_2, \ldots, x_n]/J_{1,n} \) be graded by total degree. It is clear that \( \varphi \) respects grading, by definition. Moreover, it is clear that \( \varphi \) is an algebra isomorphism, since

\[
\varphi(z_A + J_{1,n}) \cdot \varphi(z_B + J_{1,n}) = (\prod_{i \in A} x_i + J_{1,n}) \cdot (\prod_{j \in B} x_j + J_{1,n})
\]

\[
= (\prod_{i \in A} x_i) \cdot (\prod_{j \in B} x_j) + J_{1,n}
\]

\[
= \varphi(z_A z_B + J_{1,n})
\]

which implies \( \varphi(\hat{a}_\pi + J_{1,n}) = a_\pi + J_{1,n} \) for all \( \pi \in S_n \).

Now we show that the action is preserved. Consider \( z_A + J_{1,n} \) and \( \sigma \in S_n \), and observe that

\[
\sigma \circ \varphi(z_A + J_{1,n}) = \sigma (\prod_{i \in A} x_i) + J_{1,n}
\]

\[
= \prod_{i \in A} x_{\sigma(i)} + J_{1,n}
\]

\[
= \prod_{i \in \sigma(A)} x_i + J_{1,n}
\]

\[
= \varphi(z_{\sigma(A)} + J_{1,n})
\]

\[
= \varphi \circ \sigma(z_A + J_{1,n}).
\]

\( \square \)

It would be interesting to determine if the representation-theoretic results of \( \ref{section:representation-theory} \) are easier to establish in the context of \( T_n/J_{1,n} \) rather than \( \mathbb{C}[x_1, x_2, \ldots, x_n]/J_{1,n} \).

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**Department of Mathematics, University of Kentucky, Lexington, KY 40506–0027**

*E-mail address:* benjamin.braun@uky.edu

**Department of Mathematics, University of Kentucky, Lexington, KY 40506–0027**

*E-mail address:* mccabe.olsen@uky.edu