On the 16-rank of class groups of $\mathbb{Q}(\sqrt{-2p})$ for primes $p \equiv 1 \mod 4$

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Abstract
We use Vinogradov’s method to prove equidistribution of a spin symbol governing the 16-rank of class groups of quadratic number fields $\mathbb{Q}(\sqrt{-2p})$, where $p \equiv 1 \mod 4$ is a prime.

1 Introduction
Recently, the authors have used Vinogradov’s method to prove density results about elements of order 16 in class groups in certain thin families of quadratic number fields parametrized by a single prime number, namely the families $\{\mathbb{Q}(\sqrt{-2p})\}_{p \equiv -1 \mod 4}$ and $\{\mathbb{Q}(\sqrt{-p})\}_{p \equiv 1 \mod 4}$ [17, 13]. In this paper, we establish a density result for the family $\{\mathbb{Q}(\sqrt{-2p})\}_{p \equiv 1 \mod 4}$, thereby completing the picture for the 16-rank in thin families of imaginary quadratic fields of even discriminant. Although our overarching methods are similar to those originally developed in the work of Friedlander et al. [10], the technical difficulties in the present case are different and require a more careful study of the spin symbols governing the 16-rank. The main distinguishing feature of the present work is that this careful study allows us to avoid relying on a conjecture about short character sums appearing in [10, 13], thus making our results unconditional.

More generally, given a sequence of complex numbers $\{a_n\}_n$ indexed by natural numbers, a problem of interest in analytic number theory is to prove an asymptotic formula for the sum over primes

$$S(X) := \sum_{\substack{p \text{ prime} \\ p \leq X}} a_p$$

as $X \to \infty$. Many sequences $\{a_n\}_n$ admit asymptotic formulas for $S(X)$ via various generalizations of the Prime Number Theorem, with essentially the best known error terms coming from ideas of de la Valée Poussin already in 1899 [4]. In 1947, Vinogradov [25, 26] invented another method to treat certain sequences which could not be handled with a variant of the Prime Number Theorem. His method has since been clarified and made easier to apply, most notably by Vaughan [24] and, for applications relating to more general number fields, by Friedlander et al. [10]. Nonetheless, there is a relative paucity of interesting sequences $\{a_n\}_n$ that admit an asymptotic formula for $S(X)$ via Vinogradov’s method. The purpose of this paper is to present yet another such sequence, of a similar nature as those appearing in [10, 13]; similarly as in [13], the asymptotics we obtain have implications in the arithmetic statistics of class groups of number fields.
Let \( p \equiv 1 \mod 4 \) be a prime number, and let \( \text{Cl}(-8p) \) denote the class group of the quadratic number field \( \mathbb{Q}(\sqrt{-2p}) \) of discriminant \(-8p\). The finite abelian group \( \text{Cl}(-8p) \) measures the failure of unique factorization in the ring \( \mathbb{Z}[\sqrt{-2p}] \). By Gauss’s genus theory \[12\], the 2-part of \( \text{Cl}(-8p) \) is cyclic and non-trivial, and hence determined by the largest power of 2 dividing the order of \( \text{Cl}(-8p) \). For each integer \( k \geq 1 \), we define a density \( \delta(2^k) \), if it exists, as
\[
\delta(2^k) := \lim_{X \to \infty} \frac{\#\{ p \leq X : p \equiv 1 \mod 4, 2^k | \#\text{Cl}(-8p) \}}{\#\{ p \leq X : p \equiv 1 \mod 4 \}}.
\]

As stated above, the 2-part of \( \text{Cl}(-8p) \) is cyclic and non-trivial, so \( \delta(2) = 1 \). It follows from the Chebotarev Density Theorem (a generalization of the Prime Number Theorem) that \( \delta(4) = \frac{1}{2} \) and \( \delta(8) = \frac{1}{4} \); indeed, Rédei \[19\] proved that \( 4 | \#\text{Cl}(-8p) \) if and only if \( p \) splits completely in \( \mathbb{Q}(\zeta_8) \), and Stevenhagen \[23\] proved that \( 8 | \#\text{Cl}(-8p) \) if and only if \( p \) splits completely in \( \mathbb{Q}(\zeta_8, \sqrt{2}) \), where \( \zeta_8 \) denotes a primitive 8th root of unity. The qualitative behavior of divisibility by 16 departs from that of divisibility by lower 2-powers in that it can no longer be proved by a simple application of the Chebotarev Density Theorem. We instead use Vinogradov’s method to prove

**Theorem 1.** For a prime number \( p \equiv 1 \mod 4 \), let \( e_p = 0 \) if \( \text{Cl}(-8p) \) does not have an element of order 8, let \( e_p = 1 \) if \( \text{Cl}(-8p) \) has an element of order 16, and let \( e_p = -1 \) otherwise. Then for all \( X > 0 \), we have
\[
\sum_{\substack{p \leq X \\mid \\text{p \equiv 1 mod 4}}} e_p \ll X^{1 - \frac{1}{3200}},
\]
where the implied constant is absolute. In particular, \( \delta(16) = \frac{1}{8} \).

In combination with \[17\], we get

**Corollary 2.** For a prime number \( p \), let \( h_2(-2p) \) denote the cardinality of the 2-part of the class group \( \text{Cl}(-8p) \). For an integer \( k \geq 0 \), let \( \delta'(2^k) \) denote the natural density (in the set of all primes) of primes \( p \) such that \( h_2(-2p) = 2^k \), if it exists. Then \( \delta'(1) = 0 \), \( \delta'(2) = \frac{1}{2} \), \( \delta'(4) = \frac{1}{4} \), and \( \delta'(8) = \frac{1}{8} \).

The power-saving bound in Theorem 1 similarly to the main results in \[17\] and \[13\], is another piece of evidence that governing fields for the 16-rank do not exist. For a sampling on previous work about governing fields, see \[2\], \[3\], \[18\], and \[22\].

The strategy to prove Theorem 1 is to construct a sequence \( \{a_n\}_n \) which simultaneously carries arithmetic information about divisibility by 16 when \( n \) is a prime number congruent to 1 modulo 4 and is conducive to Vinogradov’s method. On one hand, the criterion for divisibility by 16 cannot be stated naturally over the rational numbers \( \mathbb{Q} \). For instance, even the criterion for divisibility by 8 is most naturally stated over a field of degree 8 over \( \mathbb{Q} \). On the other hand, proving analytic estimates in a number field generally becomes more difficult as the degree of the number field increases, as exemplified by the reliance on a conjecture on short character sums in \[10\]. We manage to work over \( \mathbb{Q}(\zeta_8) \), a field of degree 4. Although the methods of Friedlander et al. \[10\] narrowly miss the mark of being unconditional for number fields of degree 4, we manage to exploit the arithmetic structure of our sequence to ensure that Theorem 1 is unconditional.
Lastly, for work concerning the average behavior of the 2-parts of class groups of quadratic number fields in families that are not thin, i.e., for which the average number of primes dividing the discriminant grows as the discriminant grows, we point the reader to the extensive work of Fouvry and Klüners [5, 6, 7, 8] on the 4-rank and certain cases of the 8-rank and more recently to the work of Smith on the 8- and higher 2-power-ranks [20, 21]. While Smith’s methods in [21] appear to be very powerful, the authors believe that they are unlikely to be applicable to thin families of the type appearing in this paper.

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2 Encoding the 16-rank of Cl(−8p) into sequences \( \{a_n\}_n \)

Given an integer \( k \geq 1 \), the \( 2^k \)-rank of a finite abelian group \( G \), denoted by \( \text{rk}_{2^k} G \), is defined as the dimension of the \( \mathbb{F}_2 \)-vector space \( 2^k - 1 G / 2^k G \). If the 2-part of \( G \) is cyclic, then \( \text{rk}_{2^k} G \in \{0, 1\} \), and \( \text{rk}_{2^k} G = 1 \) if and only if \( 2^k | \#G \). The order of a class group is called the class number, and we denote the class number of \( \text{Cl}(−8p) \) by \( h(−8p) \).

The criterion for divisibility of \( h(−8p) \) by 16 that we will use is due to Leonard and Williams [15, Theorem 2, p. 204]. Given a prime number \( p \equiv 1 \mod 8 \) (so that \( 4 | h(−8p) \)), there exist integers \( u \) and \( v \) such that

\[
p = u^2 - 2v^2, \quad u > 0.
\]

(2.1)

The criterion for divisibility by 8 can be restated in terms of a quadratic residue symbol; one has

\[
8 | h(−8p) \iff \left( \frac{u}{p} \right)_4 = 1.
\]

Note that \( 1 = (u/p)_2 = (p/u)_2 = (−2/u)_2 \), so that \( 8 | h(−8p) \) if and only if \( u \equiv 1, 3 \mod 8 \). As \( \varepsilon^2(u + v\sqrt{2}) = (3u + 4v) + (2u + 3v)\sqrt{2} \) and \( \varepsilon \) is even, we can always choose \( u \) and \( v \) in (2.1) so that \( u \equiv 1 \mod 8 \). The criterion for divisibility of \( h(−8p) \) by 16 states that if \( u \) and \( v \) are integers satisfying (2.1) and \( u \equiv 1 \mod 8 \), then

\[
16 | h(−8p) \iff \left( \frac{u}{p} \right)_4 = 1,
\]

where \( (u/p)_4 \) is equal to 1 or \(-1\) depending on whether or not \( u \) is a fourth power modulo \( p \). To take advantage of the multiplicative properties of the fourth-power residue symbol, one has to work over a field containing \( i = \sqrt{-1} \), a primitive fourth root of unity. Since \( u \) is naturally defined via the splitting of \( p \) in \( \mathbb{Q}(\sqrt{2}) \), we see that the natural setting for the criterion above is the number field

\[
M := \mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\zeta_8),
\]

of degree 4 over \( \mathbb{Q} \). It is straightforward to check that the class number of \( M \) and each of its subfields is 1, that 2 is totally ramified in \( M \), and that the unit group of its ring of integers
\( \mathcal{O}_M = \mathbb{Z}[\zeta_8] \) is generated by \( \zeta_8 \) and \( \varepsilon = 1 + \sqrt{2} \). Note that \( M/\mathbb{Q} \) is a normal extension with Galois group isomorphic to the Klein four group, say \( \{1, \sigma, \tau, \sigma\tau\} \), where \( \sigma \) fixes \( \mathbb{Q}(i) \) and \( \tau \) fixes \( \mathbb{Q}(\sqrt{2}) \).

\[
\begin{array}{c}
\mathbb{Q}(\zeta_8) \\
\downarrow (\sigma) \\
\mathbb{Q}(i) \\
\downarrow (\sigma\tau) \\
\mathbb{Q}(i\sqrt{2}) \\
\downarrow (\tau) \\
\mathbb{Q}(\sqrt{2}) \\
\downarrow \mathbb{Q} \\
\end{array}
\]

Let \( p \equiv 1 \mod 8 \) be a prime, so that \( p \) splits completely in \( M \). Then there exists \( w \in \mathcal{O}_M \) such that \( N(w) = p \), i.e., such that \( p = w\sigma(w)\tau(w)\sigma\tau(w) \). Note that the inclusion \( \mathbb{Z} \rightarrow \mathcal{O}_M \) induces an isomorphism \( \mathbb{Z}/(p) \cong \mathcal{O}_M/(w) \), so that an integer \( n \) is a fourth power modulo \( p \) exactly when it is a fourth power modulo \( w \). As \( w\tau(w) \in \mathbb{Z}[\sqrt{2}] \), there exist integers \( u \) and \( v \) such that \( w\tau(w) = u + v\sqrt{2} \). Then \( u = (w\tau(w) + \sigma(w)\sigma\tau(w))/2 \). With this in mind, we define, for any \( \alpha \in \mathbb{Z}[\sqrt{2}] \),

\[
\begin{aligned}
\rho(\alpha) &= \frac{1}{2}(\alpha + \sigma(\alpha)) \\
\text{and, for any odd (i.e., coprime to 2) } w \in \mathcal{O}_M, \text{ not necessarily prime,}
\end{aligned}
\]

\[
[w] := \left( \frac{r(w\tau(w))}{w} \right)_4,
\]

where \((\cdot/\cdot)_4\) is the quartic residue symbol in \( M \); we recall the definition of \((\cdot/\cdot)_4\) in the next section. A simple computation shows that \( r(w\tau(w)) > 0 \) for any non-zero \( w \in \mathcal{O}_M \). Hence \( 16|h(-8p) \) if and only if \([w] = 1 \), where \( w \) is any element of \( \mathcal{O}_M \) such that \( N(w) = p \) and \( r(w) \equiv 1 \mod 8 \).

Given a Dirichlet character \( \chi \) modulo 8, we define, for any odd \( w \in \mathcal{O}_M \),

\[
[w]_\chi := [w] \cdot \chi(r(w\tau(w))).
\]

Then

\[
\frac{1}{4} \sum_{\chi \mod 8} [w]_\chi = \begin{cases} [w] & \text{if } r(w\tau(w)) \equiv 1 \mod 8, \\ 0 & \text{otherwise}, \end{cases}
\]

where the sum is over all Dirichlet characters modulo 8. Another simple computation shows that, for all odd \( w \in \mathcal{O}_M \), we have \([\zeta_8 w] = [w] \). We note that \( r(\varepsilon^2\alpha) \equiv 3 \cdot r(\alpha) \mod 8 \) for any \( \alpha \in \mathbb{Z}[\sqrt{2}] \), so that \( \chi(r(\varepsilon^2 w\tau(\varepsilon^2 w))) = \chi(r(w\tau(w))) \) for every Dirichlet character \( \chi \) modulo 8. Finally, we note that

\[
[w] = \left( \frac{16r(w\tau(w))}{w} \right)_4 = \left( \frac{8\sigma(w)\sigma\tau(w)}{w} \right)_4,
\]

so that

\[
[\varepsilon w] = \left( \frac{\sigma(\varepsilon)}{w} \right)_2 [w],
\]

where \( \mathcal{O}_M \) is generated by \( \zeta_8 \) and \( \varepsilon = 1 + \sqrt{2} \).
and hence \([\varepsilon^2 w] = [w]\). Having determined the action of the units \(\mathcal{O}_M^\times\) on \([\cdot]_\chi\), we can define, for each Dirichlet character \(\chi\) modulo 8, a sequence \(\{a(\chi)_n\}_n\) indexed by ideals of \(\mathcal{O}_M\) by setting \(a(\chi)_n := [w]_\chi + [\varepsilon w]_\chi\),

\[
a(\chi)_n := [w]_\chi + [\varepsilon w]_\chi, \tag{2.3}
\]

where \(w\) is any generator of the odd ideal \(n\). Again because \(r(\varepsilon^2 \alpha) \equiv 3 \cdot r(\alpha) \mod 8\) for any \(\alpha \in \mathbb{Z}[\sqrt{2}]\), we see that if \(8|h(−8\nu)|\), then exactly one of \(r(w\tau(w))\) and \(r(\varepsilon w\tau(\varepsilon w))\) is 1 mod 8, and if \(8 \mid h(−8\nu)\), then neither is 1 mod 8. We have proved

**Proposition 2.1.** Let \(p \equiv 1 \mod 8\) be a prime, and let \(\mathfrak{p}\) be a prime ideal of \(\mathcal{O}_M\) lying above \(p\). Then

\[
\frac{1}{4} \sum_{\chi \mod 8} a(\chi)_p = \begin{cases} 
1 & \text{if } 16|h(−8\nu)|, \\
−1 & \text{if } 8|h(−8\nu) \text{ but } 16 \nmid h(−8\nu), \\
0 & \text{otherwise},
\end{cases}
\]

where the sum is over Dirichlet characters modulo 8.

## 3 Prerequisites

We now collect some definitions and facts that we will use in our proof of Theorem 2.1.

### 3.1 Quartic residue symbols and quartic reciprocity

Let \(L\) be a number field with ring of integers \(\mathcal{O}_L\). Let \(\mathfrak{p}\) be an odd prime ideal of \(\mathcal{O}_L\) and let \(\alpha \in \mathcal{O}_L\). One defines the quadratic residue symbol \((\alpha/\mathfrak{p})_{L,2}\) by setting

\[
(\frac{\alpha}{\mathfrak{p}})_{L,2} := \begin{cases} 
0 & \text{if } \alpha \in \mathfrak{p}, \\
1 & \text{if } \alpha \notin \mathfrak{p} \text{ and } \alpha \equiv \beta^2 \mod \mathfrak{p} \text{ for some } \beta \in \mathcal{O}_L, \\
−1 & \text{otherwise}.
\end{cases}
\]

Then we have \((\alpha/\mathfrak{p})_{L,2} \equiv \alpha^{\frac{\mathcal{O}_L(\mathfrak{p})-1}{2}} \mod \mathfrak{p}\). The quadratic residue symbol is then extended multiplicatively to all odd ideals \(\mathfrak{n}\), and then also to all odd elements \(\beta\) in \(\mathcal{O}_L\) by setting \((\alpha/\beta)_{L,2} = (\alpha/\beta_{L,2})_{L,2}\). To define the quartic residue symbol, we assume that \(L\) contains \(\mathbb{Q}(\sqrt{-1})\). Then one can define the quartic residue symbol \((\alpha/\mathfrak{p})_{L,4}\) as the element of \(\{±1, ±i, 0\}\) such that

\[
(\frac{\alpha}{\mathfrak{p}})_{L,4} \equiv \alpha^{\frac{\mathcal{O}_L(\mathfrak{p})-1}{4}} \mod \mathfrak{p},
\]

and extend this to all odd ideals \(\mathfrak{n}\) and odd elements \(\beta\) in the same way as the quadratic residue symbol. A key property of the quartic residue symbol that we will use extensively is the following weak version of quartic reciprocity in \(M := \mathbb{Q}(\zeta_8)\).

**Lemma 3.1.** Let \(\alpha, \beta \in \mathcal{O}_M\) with \(\beta\) odd. Then \((\alpha/\beta)_{M,4}\) depends only on the congruence class of \(\beta\) modulo 16\(\mathcal{O}_M\). Moreover, if \(\alpha\) is also odd, then

\[
(\frac{\alpha}{\beta})_{M,4} = \mu \cdot (\frac{\beta}{\alpha})_{M,4},
\]

where \(\mu \in \{±1, ±i\}\) depends only on the congruence classes of \(\alpha\) and \(\beta\) modulo 16\(\mathcal{O}_M\).

**Proof.** This follows from [14, Proposition 6.11, p. 199].
3.2 Field lowering

A key feature of our proof is the reduction of quartic residue symbols in a quartic number field to quadratic residue symbols in a quadratic field. We do this by using the following three lemmas.

**Lemma 3.2.** Let $K$ be a number field and let $p$ be an odd prime ideal of $K$. Suppose that $L$ is a quadratic extension of $K$ such that $L$ contains $\mathbb{Q}(i)$ and $p$ splits in $L$. Denote by $\psi$ the non-trivial element in $\text{Gal}(L/K)$. Then if $\psi$ fixes $\mathbb{Q}(i)$ we have for all $\alpha \in \mathcal{O}_K$

$$\left( \frac{\alpha}{p\mathcal{O}_L} \right)_{L,4} = \left( \frac{\alpha}{p\mathcal{O}_K} \right)_{K,2},$$

and if $\psi$ does not fix $\mathbb{Q}(i)$ we have for all $\alpha \in \mathcal{O}_K$ with $p \nmid \alpha$

$$\left( \frac{\alpha}{p\mathcal{O}_L} \right)_{L,4} = 1.$$ 

**Proof.** Since $p$ splits in $L$, we can write $p = q \psi(q)$ for some prime ideal $q$ of $L$. Hence we have

$$\left( \frac{\alpha}{p\mathcal{O}_L} \right)_{L,4} = \left( \frac{\alpha}{q} \right)_{L,4} \left( \frac{\alpha}{\psi(q)} \right)_{L,4} = \left( \frac{\alpha}{\psi(q)} \right)_{L,4}.$$ 

If $\psi$ fixes $i$ we find that

$$\left( \frac{\alpha}{q} \right)_{L,4} = \psi \left( \left( \frac{\alpha}{q} \right)_{L,4} \right) = \left( \frac{\psi(\alpha)}{\psi(q)} \right)_{L,4} = \left( \frac{\alpha}{\psi(q)} \right)_{L,4}.$$ 

Combining this with the previous identity gives

$$\left( \frac{\alpha}{p\mathcal{O}_L} \right)_{L,4} = \left( \frac{\alpha}{q} \right)_{L,4} \left( \frac{\alpha}{\psi(q)} \right)_{L,4} = \left( \frac{\alpha}{\psi(q)} \right)_{L,4} = \left( \frac{\alpha}{\psi(q)} \right)_{L,4} = 1,$$

establishing the first part of the lemma. If $\psi$ does not fix $i$ we find that

$$\left( \frac{\alpha}{p\mathcal{O}_L} \right)_{L,4} = \left( \frac{\alpha}{q} \right)_{L,4} \left( \frac{\alpha}{\psi(q)} \right)_{L,4} = \left( \frac{\alpha}{\psi(q)} \right)_{L,4} \left( \frac{\alpha}{\psi(q)} \right)_{L,4} = 1,$$

by checking this for all values of $(\alpha/q)_{L,4} \in \{\pm 1, \pm i\}$. This completes the proof. 

**Lemma 3.3.** Let $K$ be a number field and let $p$ be an odd prime ideal of $K$ of degree 1 lying above $p$. Suppose that $L$ is a quadratic extension of $K$ such that $L$ contains $\mathbb{Q}(i)$ and $p$ stays inert in $L$. Then we have for all $\alpha \in \mathcal{O}_K$

$$\left( \frac{\alpha}{p\mathcal{O}_L} \right)_{L,4} = \left( \frac{\alpha}{p\mathcal{O}_K} \right)_{K,2}^{\frac{p+1}{2}}.$$ 

**Proof.** We have

$$\left( \frac{\alpha}{p\mathcal{O}_L} \right)_{L,4} \equiv \alpha^{\frac{N_L(p)-1}{4}} \equiv \alpha^2 \equiv \left( \frac{\alpha}{p} \right)^{\frac{p+1}{2}} \equiv \left( \frac{\alpha}{p\mathcal{O}_K} \right)_{K,2}^{\frac{p+1}{2}} \mod p,$$

which immediately implies the lemma. 

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Hence \( \beta \) under the maps given as in (3.1). This implies that to \( x \), \( \psi \) one checks that prime ideal \( q \) element of \( O \). \( \equiv \) \( \beta \) \( \psi \) follows that of \( p \). We conclude that \( \psi \) \( L \) inert in \( \text{Gal}(L/K) \) of \( \beta \). \( \equiv \) \( \beta \) \( \psi \). \( L \) is trivial by the assumption that \( \beta \) \( \equiv \) \( \beta \) \( \mod \) \( p \). We conclude that \( \psi \) \( \beta \) comes from \( O_K/p \), which we had to prove.

We still have to prove the lemma if \( p \) splits. In this case we can write \( p = q\psi(q) \) for some prime ideal \( q \) of \( L \). Note that

\[
O_K/p \hookrightarrow O_L/pO_L \cong O_L/q \times O_L/\psi(q). \tag{3.1}
\]

One checks that \( \psi \) is the automorphism of \( O_L/q \times O_L/\psi(q) \) that maps \( (x,y) \) to \( (\psi(y),\psi(x)) \). Hence \( \beta \equiv \psi(\beta) \mod pO_L \) implies that there is some \( x \in O_L/q \) such that \( \beta = (x,\psi(x)) \) as an element of \( O_L/q \times O_L/\psi(q) \). Since \( O_K/p \cong O_L/q \), we can pick \( \beta' \in O_K \) such that \( \beta' \) maps to \( x \) under the natural inclusion \( O_K/p \hookrightarrow O_L/q \). Then it follows that \( \beta \) maps to \( (\beta',\psi(\beta')) \) under the maps given as in (3.1). This implies that \( \beta' \equiv \beta \mod pO_L \) as desired.

3.3 A fundamental domain for the action of \( O_M^\times \)

In defining \( a(\chi)_n \) for odd ideals \( n \) of \( O_M \), we had to choose a generator \( w \) for the ideal \( n \). There are many such choices, since the group of units of \( O_M \) is quite large, i.e.,

\[
O_M^\times = \langle \zeta_8 \rangle \times \langle \varepsilon \rangle,
\]

where \( \varepsilon = 1 + \sqrt{2} \) as before. It will be important to us that we can choose generators that are in some sense as small as possible. The following lemma allows us to do so.

Lemma 3.5. There exists a subset \( \mathcal{D} \) of \( O_M \) such that:

1. \( \mathcal{D} \) is a fundamental domain for the natural action of \( O_M^\times \) on \( O_M \), i.e., for all \( w \in O_M \), there exists a unique \( n \in \mathbb{Z} \) such that \( \varepsilon^n w \in \mathcal{D} \); and

2. every non-zero ideal \( n \) in \( O_M \) has exactly 8 generators in \( \mathcal{D} \); if \( w \) is one such generator, then all such generators are of the form \( \zeta w \), where \( \zeta \in \langle \zeta_8 \rangle \); and

3. there exists a constant \( C > 0 \) such that for all \( w = w_1 + w_2 \zeta_8 + w_3 \zeta_8^2 + w_4 \zeta_8^3 \in \mathcal{D} \) with \( w_i \in \mathbb{Z} \), we have \( |w_1| \leq C \cdot N(\alpha)^{\frac{3}{4}} \).

Proof. For a similar construction see [13, Section 2.6, p. 12], which is based on [16, Chapter 6, p. 158-181].
3.4 The sieve

We will prove Theorem 1 by a sieve of Friedlander et al. [10] that generalizes the ideas of Vinogradov [25, 26] to the setting of number fields. Let \( \chi \) be a Dirichlet character modulo 8, and let \( a(\chi)_n \) be defined as in (2.3). We will prove the following two propositions.

**Proposition 3.6.** There exists a real number \( \theta_1 \in (0, 1) \) such that for every \( \epsilon > 0 \), we have
\[
\sum_{N(n) \leq X, \ m|n} a(\chi)_n \ll \epsilon X^{1-\theta_1+\epsilon}
\]
uniformly for all non-zero ideals \( m \) of \( \mathcal{O}_M \) and all \( X \geq 2 \). One can take \( \theta_1 = \frac{1}{64} \).

**Proposition 3.7.** There exists a real number \( \theta_2 \in (0, 1) \) such that for every \( \epsilon > 0 \), we have
\[
\sum_{N(m) \leq M} \sum_{N(n) \leq N} \alpha_m \beta_n a(\chi)_{mn} \ll \epsilon (M + N)^{\theta_2} (MN)^{1-\theta_2+\epsilon}
\]
uniformly for all \( M, N \geq 2 \) and sequences of complex numbers \( \{\alpha_m\} \) and \( \{\beta_n\} \) satisfying \( |\alpha_m|, |\beta_n| \leq 1 \). One can take \( \theta_2 = \frac{1}{124} \).

Assuming Propositions 3.6 and 3.7, [10, Proposition 5.2, p. 722] implies that there exists \( \theta \in (0, 1) \) such that for every \( \epsilon > 0 \), we have
\[
\sum_{N(n) \leq X} a(\chi)_n A(n) \ll \epsilon X^{1-\theta+\epsilon}
\]
uniformly for all \( X \geq 2 \). Moreover, we can take \( \theta = 1/(49 \cdot 64) = 1/3136 \). By partial summation, it follows that, say,
\[
\sum_{N(p) \leq X} a(\chi)_p \ll X^{1-\frac{1}{3136}}. \quad (3.2)
\]

As
\[
\sum_{N(p) \leq X} 1 \ll X^{1/2},
\]

Theorem 1 follows from (3.2) and Proposition 2.1. It now remains to prove Propositions 3.6 and 3.7.

4 Proof of Proposition 3.6

Let \( \chi \) be a Dirichlet character modulo 8. Let \( m \) be an odd ideal of \( \mathcal{O}_M \). In view of Proposition 2.1 we must bound the following sum
\[
A(x) = A(x; \chi, m) := \sum_{\substack{N(\alpha) \leq x \\ (\alpha, 2) = 1, \ m|\alpha}} ([\alpha]_x + [\varepsilon \alpha]_x),
\]
where \( \alpha \) is chosen to be any generator of \( a \). Our proof is based on the argument in [13, Section 3, p. 12-19], which is in turn based on [10, Section 6, p. 722-733]. Each non-zero ideal \( a \) has
exactly 8 generators \( \alpha \in \mathcal{D} \), where \( \mathcal{D} \) is the fundamental domain from Lemma 3.5. Set \( u_1 = 1 \) and \( u_2 = \varepsilon \). Set \( F = 16 \). Note that \( \chi(r(\alpha \tau(a))) \) depends only on the congruence class of \( \alpha \) modulo 8. After splitting the above sum into congruence classes modulo \( F \), and using (2.2) and Lemma 3.1, we find that

\[
A(x) = \frac{1}{8} \sum_{i=1}^{2} \sum_{\substack{\rho \mod F \mod m \atop (\rho,F) = 1}} \mu(\rho, u_i) A(x; \rho, u_i),
\]

where \( \mu(\rho, u_i) \in \{ \pm 1, \pm i \} \) depends only on \( \rho \) and \( u_i \) and where

\[
A(x; \rho, u_i) := \sum_{\alpha \in u_i \mathcal{D}, N(\alpha) \leq x} \left( \frac{\sigma(\alpha)}{\alpha} \right)_{M_A} \left( \frac{\sigma\tau(\alpha)}{\alpha} \right)_{M_A}.
\]

Our goal is to estimate \( A(x; \rho, u_i) \) separately for each congruence class \( \rho \mod F \), \( (\rho, F) = 1 \) and unit \( u_i \). We view \( \mathcal{O}_M \) as a \( \mathbb{Z} \)-module of rank 4 and decompose it as \( \mathcal{O}_M = \mathbb{Z} \oplus \mathbb{M} \), where \( \mathbb{M} = \mathbb{Z}\zeta_8 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_8^3 \) is a free \( \mathbb{Z} \)-module of rank 3. We can write \( \alpha \) uniquely as

\[
\alpha = a + \beta, \quad \text{with} \quad a \in \mathbb{Z}, \beta \in \mathbb{M},
\]

so that the summation conditions above are equivalent to

\[
a + \beta \in u_i \mathcal{D}, \quad N(a + \beta) \leq x, \quad a + \beta \equiv \rho \mod F, \quad a + \beta \equiv 0 \mod m. \quad (*)
\]

We are now going to rewrite \( (\sigma(\alpha)/\alpha)_{M_A} \) and \( (\sigma\tau(\alpha)/\alpha)_{M_A} \) by using the same trick as in [10, p. 725]. Put

\[
\sigma(\beta) - \beta = \eta^2 c_0 c \quad \text{and} \quad \sigma\tau(\beta) - \beta = \eta^2 c'_0 c'
\]

with \( c_0, c'_0, c, c', \eta, \eta' \in \mathcal{O}_M \), \( c_0, c'_0 \mid F \) squarefree, \( \eta, \eta' \mid F^\infty \) and \( (c, F) = (c', F) = 1 \). By multiplying with an appropriate unit we can even ensure that \( c \in \mathbb{Z}[i] \) and \( c' \in \mathbb{Z}[\sqrt{-2}] \). Indeed, observe that

\[
\alpha' := \frac{\sigma(\alpha) - \alpha}{\zeta_8} = \frac{\sigma(\beta) - \beta}{\zeta_8} \in \mathbb{Z}[i], \quad (4.1)
\]

and we have a similar identity for \( \sigma\tau(\beta) - \beta \). Then we obtain, just as in [13, p. 14], by Lemma 3.1,

\[
\left( \frac{\sigma(\alpha)}{\alpha} \right)_{M_A} = \mu_1 \cdot \left( \frac{a + \beta}{c\mathcal{O}_M} \right)_{M_A} \quad \text{and} \quad \left( \frac{\sigma\tau(\alpha)}{\alpha} \right)_{M_A} = \mu_2 \cdot \left( \frac{a + \beta}{c'\mathcal{O}_M} \right)_{M_A},
\]

where \( \mu_1, \mu_2 \in \{ \pm 1, \pm i \} \) depend only on \( \rho \) and \( \beta \). Hence

\[
A(x; \rho, u_i) \leq \sum_{\beta \in \mathbb{M}} |T(x; \beta, \rho, u_i)|,
\]

where

\[
T(x; \beta, \rho, u_i) := \sum_{a \in \mathbb{Z}, a + \beta \text{ sat. } (*)} \left( \frac{a + \beta}{c\mathcal{O}_M} \right)_{M_A} \left( \frac{a + \beta}{c'\mathcal{O}_M} \right)_{M_A}.
\]
From now on we treat $\beta$ as fixed and estimate $T(x; \beta, \rho, u_i)$. It is here that we deviate from \[10\] and \[13\]. Since we chose $c' \in \mathbb{Z}[\sqrt{-2}]$, we can factor the principal ideal $(c') \subset \mathbb{Z}[\sqrt{-2}]$ into prime ideals in $\mathbb{Z}[\sqrt{-2}]$ that do not ramify in $M$, say, $(c') = \prod_{i=1}^{k} \mathfrak{p}_i^{e_i}$, so that
\[
\left(\frac{a + \beta}{c'\mathcal{O}_M}\right)_{M,4} = \prod_{i=1}^{k} \left(\frac{a + \beta}{\mathfrak{p}_i^{e_i}}\right)_{M,4}.
\]

We claim that \( (a + \beta)/\mathfrak{p}\mathcal{O}_M)_{M,4} = 1 \) if $\mathfrak{p} \nmid a + \beta$. As a first step we can replace $\beta$ by some $\beta' \in \mathbb{Z}[\sqrt{-2}]$ due to Lemma 3.3. Then Lemma 3.2 proves the claim if $\mathfrak{p}$ splits in $M$. Finally suppose that $\mathfrak{p}$ stays inert in $M$. If we define $p := \mathfrak{p} \cap \mathbb{Z}$, we find that $p \equiv 3 \mod 8$. Hence Lemma 3.3 finishes the proof of the claim.

The factor \((a + \beta)/c\mathcal{O}_M)_{M,4}\) is handled more similarly to \[10\] (6.21), p. 727). Since we chose $c \in \mathbb{Z}[\pi]$ we factor $(c) \subset \mathbb{Z}[\pi]$ in $\mathbb{Z}[\pi]$ as $(c) = q\mathfrak{q}$, where $\mathfrak{q}$ is the product of all prime ideal powers $\mathfrak{p}^{e_i} \subset \mathbb{Z}[\pi]$ dividing $(c)$ such that the residue degree of $\mathfrak{p}$ is greater than one or such that $\mathfrak{p}$ is an unramified prime of degree one for which some conjugate of $\mathfrak{p}$ also divides $(c)$. By construction $q := N_{\mathbb{Q}(i)/\mathbb{Q}}(\mathfrak{q})$ is a squarefree odd integer and $g := N_{\mathbb{Q}(i)/\mathbb{Q}}(\mathfrak{q})$ is an odd squarefree number coprime with $g$.

Lemma 3.3 and the Chinese remainder theorem imply that there exists $\beta' \in \mathbb{Z}[\pi]$ such that $\beta \equiv \beta' \mod q\mathcal{O}_M$. Next, Lemma 3.2 and Lemma 3.3 imply that $(a + \beta')/q\mathcal{O}_M)_{M,4} = ((a + \beta')/q)_{\mathbb{Q}(i),2}$. Finally, as $q$ is squarefree, the Chinese remainder theorem guarantees the existence of a rational integer $b$ such that $\beta' \equiv b \mod q$. Combining all of this gives
\[
\left(\frac{a + \beta}{c\mathcal{O}_M}\right)_{M,4} = \left(\frac{a + \beta}{q\mathcal{O}_M}\right)_{M,4} \left(\frac{a + b}{q}\right)_{\mathbb{Q}(i),2}.
\]

Since $c$ depends on $\beta$ and not on $a$, we find that $b$ depends on $\beta$ and not on $a$. Now define $g_0$ as the radical of $g$, i.e., $g_0 := \prod_{\mathfrak{p} | g} \mathfrak{p}$. We observe that the quartic residue symbol $(\alpha/q\mathcal{O}_M)_{M,4}$ is periodic in $\alpha$ modulo $g^* := \prod_{\mathfrak{p} | g} \mathfrak{p}$. But clearly $g^*$ divides $g_0$, and hence we conclude that \((a + \beta)/q\mathcal{O}_M)_{M,4}\) is periodic of period $g_0$ when viewed as a function of $a \in \mathbb{Z}$. So we split $T(x; \beta, \rho, u_i)$ into congruence classes modulo $g_0$, giving
\[
|T(x; \beta, \rho, u_i)| \leq \sum_{a \equiv a_0 \mod g_0} |T(x; \beta, \rho, u_i, a_0)|,
\]
where
\[
T(x; \beta, \rho, u_i, a_0) = \sum_{a + \beta \sat (\pi)} \left(\frac{a + b}{q}\right)_{\mathbb{Q}(i),2} \left(\frac{a + \beta}{c\mathcal{O}_M}\right)_{M,4}.
\]

We have already proven that \((a + \beta)/c\mathcal{O}_M)_{M,4} = 1\) unless $\gcd(a + \beta, c') \neq 1$ and in this case we have \((a + \beta)/c\mathcal{O}_M)_{M,4} = 0\). An application of inclusion-exclusion gives
\[
|T(x; \beta, \rho, u_i, a_0)| \leq \sum_{\mathfrak{d} | c\mathcal{O}_M} |T(x; \beta, \rho, u_i, a_0, \mathfrak{d})|,
\]
where
\[
T(x; \beta, \rho, u_i, a_0, \mathfrak{d}) := \sum_{a + \beta \sat (\pi)} \left(\frac{a + b}{q}\right)_{\mathbb{Q}(i),2}.
\]
We unwrap the summation conditions above similarly as in [10, p. 728]. Certainly \(a + \beta \in u_i \mathcal{D}\) implies that \(a \ll x^{1/4}\), where the implied constant depends only on one of the two fixed units \(u_i\). The condition \(N_{M/Q}(a + \beta) \leq x\) is for fixed \(\beta\) and \(x\) a polynomial inequality of degree 4 in \(a\). Hence the summation variable \(a \in \mathbb{Z}\) runs over at most 4 intervals of length \(\ll x^{1/4}\) with endpoints depending on \(\beta\) and \(x\).

Next, the congruence conditions \(a + \beta \equiv \rho \mod F\), \(a + \beta \equiv 0 \mod m\), \(a \equiv a_0 \mod g_0\) and \(a + \beta \equiv 0 \mod \mathfrak{d}\) imply that \(a\) runs over some arithmetic progression of modulus \(k\) dividing \(g_0mdF\), where we define \(m := N_{M/Q}(m)\) and \(d := N_{M/Q}(\mathfrak{d})\). Moreover, as \(q = N_{Q(i)/Q}(q)\) is squarefree, \((\cdot/q)_{Q(i),2} : \mathbb{Z} \to \{\pm 1, 0\}\) is the real primitive Dirichlet character of modulus \(q\).

All in all, the sum in (4.2) can be rewritten as at most 4 incomplete real character sums of length \(\ll x^{1/4}\) and modulus \(q \ll x^{2}\), each of which runs over an arithmetic progression of modulus \(k\). When the modulus \(q\) of the Dirichlet character divides the modulus \(k\) of the arithmetic progression, one does not get the desired cancellation. So for now we assume that \(q \nmid k\), and we will handle the case \(q \mid k\) later. As has been explained in [11, 7., p. 924-925], Burgess’s bound for short character sums \([4]\) implies that for each integer \(r \geq 2\), we have

\[
T(x; \beta, \rho, u_i, a_0, \mathfrak{d}) \ll_{\epsilon, r} x^{1/2}(1 - \frac{r}{2}) \cdot x^{\frac{1}{r} + \epsilon},
\]

so that on taking \(r = 2\), we obtain

\[
T(x; \beta, \rho, u_i) \ll \epsilon g_0 x^{1/4 - \frac{1}{4} + \epsilon}.
\]

It remains to do the case \(q \mid k\). Certainly, this implies \(q \mid md\). So (4.3) holds if \(q \mid md\). Recall that \((c) = \mathfrak{gq}\), hence we have (4.3) unless

\[
p \mid N_{Q(i)/Q}(\alpha') \implies p^2 \mid mdFN_{Q(i)/Q}(\alpha'),
\]

where \(\alpha'\) is defined as in (4.1). Define \(A_{\square}(x; \rho, u_i)\) as the contribution to \(A(x; \rho, u_i)\) with (4.4). Then we get

\[
A_{\square}(x; \rho, u_i) \leq \{\alpha \in u_i \mathcal{D} : N_{M/Q}(\alpha) \leq x, \ p \mid N_{Q(i)/Q}(\alpha') \implies p^2 \mid mdFN_{Q(i)/Q}(\alpha')\}\].

We decompose \(\mathcal{O}_M\) as \(\mathcal{O}_M = \mathbb{Z}[i] \oplus \mathcal{M}'\), where \(\mathcal{M}' = \mathbb{Z}\zeta_8 \oplus \mathbb{Z}\zeta_8^3 = \mathbb{Z}[i] \cdot \zeta_8\) is a free \(\mathbb{Z}\)-module of rank 2. The linear map \(\mathcal{M}' \to \mathbb{Z}[i]\) given by \(\alpha \mapsto \alpha'\) is injective. Now suppose \(\alpha \in u_i \mathcal{D}\) and \(N_{M/Q}(\alpha) \leq x\). Then by Lemma 3.5, if we write \(\alpha = a_1 + a_2i + (a_3 + a_4i)\zeta_8\), we have \(a_j \ll x^{1/4}\) for \(1 \leq j \leq 4\). Hence the norm \(N_{Q(i)/Q}(\cdot)\) of \(\alpha' = -2(a_3 + a_4i)\) is \(\ll x^{1/2}\), and so

\[
A_{\square}(x; \rho, u_i) \ll x^{1/2}\{\alpha' \in \mathbb{Z}[i] : N_{Q(i)/Q}(\alpha') \ll x^{1/2}, \ p \mid N_{Q(i)/Q}(\alpha') \implies p^2 \mid mdFN_{Q(i)/Q}(\alpha')\}\].

Note that there are at most \(b^r\) elements \(\alpha' \in \mathbb{Z}[i]\) such that \(N_{Q(i)/Q}(\alpha') = b\). This gives

\[
A_{\square}(x; \rho, u_i) \ll x^{1/2 + \epsilon} \sum_{\substack{b \ll x^{1/2} \colon \ p | b \implies p^2 | mdFb}} 1,
\]

where \(b\) runs over the positive rational integers. We assume that \(m \leq x\) because otherwise \(A(x)\) is the empty sum. This shows that \(md \ll x^2\) and we conclude that

\[
A_{\square}(x; \rho, u_i) \ll x^{1/2 + \epsilon}.
\]
Let \( A_0(x; \rho, u_i) \) be the contribution to \( A(x; \rho, u_i) \) of the terms \( \alpha = a + \beta \) not satisfying (4.4). Then we can split \( A(x; \rho, u_i) \) as
\[
A(x; \rho, u_i) = A_{\square}(x; \rho, u_i) + A_0(x; \rho, u_i).
\]

To estimate \( A_0(x; \rho, u_i) \) we can try to use our bound (4.3) for every relevant \( \beta \), but for this we need \( g_0 \) to be small. Hence we make the further partition
\[
A_0(x; \rho, u_i) = A_1(x; \rho, u_i) + A_2(x; \rho, u_i),
\]
where \( \beta \) satisfies the additional constraint
\[
g_0 \leq Z \text{ in the sum } A_1(x; \rho, u_i),
g_0 > Z \text{ in the sum } A_2(x; \rho, u_i).
\]

Here \( Z \) is at our disposal, and we choose it later. We estimate \( A_1(x; \rho, u_i) \) as in [10] by using (4.3) and summing over \( \beta \in \mathbb{M} \) satisfying \( |\beta(1)|, \ldots, |\beta(4)| \ll x^{\frac{1}{2}} \) to obtain
\[
A_1(x; \rho, u_i) \ll \epsilon Z x^{1 - \frac{1}{4} + \epsilon}.
\]

To finish the proof of Proposition 3.6 it remains to estimate \( A_2(x; \rho, u_i) \). Note that \( g_0 \leq \sqrt{g} \) and \( g \leq N_{\mathbb{Q}(i)/\mathbb{Q}}(e) \leq N_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha') \ll x^{\frac{1}{4}} \). Hence, similarly as for \( A_{\square}(x; \rho, u_i) \), with \( b = N_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha') \), we have
\[
A_2(x; \rho, u_i) \ll \epsilon x^{\frac{1}{4} + \epsilon} \sum_{Z < g_0 \ll x^{\frac{1}{4}}} \sum_{b \ll z^{\frac{1}{4}} \atop g_0 | b} 1 \ll \epsilon Z^{-1} x^{1 + \epsilon}.
\]

Picking \( Z = x^{\frac{1}{16}} \) finishes the proof of Proposition 3.6.

5 Proof of Proposition 3.7

Let \( w \) and \( z \) be odd elements in \( \mathcal{O}_M \). All quadratic and quartic residue symbols that follow are over \( M \). By [222], we have
\[
[wz] = \left( \frac{\sigma(wz) \sigma_t(wz)}{wz} \right) = [w][z] \left( \frac{\sigma(w)}{z} \right)_4 \left( \frac{\sigma_t(w)}{z} \right)_4 \left( \frac{\sigma(z)}{w} \right)_4 \left( \frac{\sigma_t(z)}{w} \right)_4.
\]

By Lemma 3.1 we have, for some \( \mu_1 \in \{ \pm 1, \pm i \} \) that depends only on the congruence classes of \( w \) and \( z \) modulo 16,
\[
\left( \frac{\sigma(w)}{z} \right)_4 \left( \frac{\sigma(z)}{w} \right)_4 = \mu_1 \left( \frac{z}{\sigma(w)} \right)_4 \sigma_t \left( \frac{z}{\sigma(w)} \right)_4 = \mu_1 \left( \frac{z}{\sigma(w)} \right)_4 \mu_1 \left( \frac{z}{\sigma(w)} \right)_4,
\]
because \( \sigma(i) = i \). Similarly, for some \( \mu_2 \in \{ \pm 1, \pm i \} \) that depends only on the congruence classes of \( w \) and \( z \) modulo 16,
\[
\left( \frac{\sigma_t(w)}{z} \right)_4 \left( \frac{\sigma_t(z)}{w} \right)_4 = \mu_2 \sigma_t \left( \frac{z}{\sigma(w)} \right)_4 \sigma_t \left( \frac{z}{\sigma(w)} \right)_4 = \mu_2,
\]

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because $\sigma \tau(i) = -i$. Hence we get, for $\mu_3 = \mu_1 \mu_2$,

$$[wz] = \mu_3[w][z] \left( \frac{z}{\sigma(w)} \right)_2. \quad (5.1)$$

This twisted multiplicativity formula for the symbol $[\cdot]$ is what makes the estimate in Proposition 3.7 possible; it is analogous to [9] Lemma 20.1, p. 1021, [10] (3.8), p. 708, [17] Proposition 8, p. 31, and [13] (4.1), p. 19. From here, the proof of Proposition 3.7 is very similar to the proofs of [9] Proposition 21.3, p. 1027, [10] Proposition 7.1, p. 733, [17] Proposition 7, p. 29, and [13] Proposition 2.6, p. 11. For the sake of completeness, we recall the main steps here.

Let $\chi$ be a Dirichlet character modulo 8, and let $\{a(\chi)\}_n$ be the sequence defined in (2.3). Let $\{\alpha_m\}_m$ and $\{\beta_n\}_n$ be any two bounded sequences of complex numbers. Since each ideal of $\mathcal{O}_M$ has 8 different generators in $\mathcal{D}$, we have

$$\sum_{N(m) \leq M} \sum_{N(n) \leq N} \alpha_m \beta_n a(\chi)_{mn} = \frac{1}{8} \sum_{w \in \mathcal{D}} \sum_{N(w) \leq M} \sum_{z \in \mathcal{D}} \sum_{N(z) \leq N} \alpha_w \beta_z ([wz]_\chi + [\varepsilon wz]_\chi).$$

Here $\varepsilon = 1 + \sqrt{2}$, $\alpha_w := a(w)$ and $\beta_z := a(z)$. Note that for any odd element $\alpha \in \mathcal{O}_M$, we have $[\alpha]_\chi = \mu_4 \cdot [\alpha]$ for some $\mu_4 \in \{\pm 1, \pm i\}$ that depends only on the congruence class of $\alpha$ modulo 8 (and so also modulo 16). Also note that (5.1) implies that $[\varepsilon wz] = \mu_5[wz]$ for some $\mu_5 \in \{\pm 1, \pm i\}$ that depends only on the congruence class of $wz$ modulo 16. Hence, by restricting $w$ and $z$ to congruence classes modulo 16, we may break up the sum above into $2 \cdot 16^2$ sums of the shape

$$\mu_6 \sum_{w \in \mathcal{D}} \sum_{N(w) \leq M} \sum_{z \in \mathcal{D}} \sum_{N(z) \leq N} \alpha_w \beta_z [wz],$$

where $\mu_6 \in \{\pm 1, \pm i\}$ depends only on the congruence classes $w$ and $z$ modulo 16. Again by (5.1), we can replace $\alpha_w$ and $\beta_z$ by $\alpha_w[w]$ and $\beta_z[z]$ to arrive at the sum

$$\mu_7 \sum_{w \in \mathcal{D}} \sum_{N(w) \leq M} \sum_{z \in \mathcal{D}} \sum_{N(z) \leq N} \alpha_w \beta_z \left( \frac{z}{\sigma(w)} \right)_2,$$

where $\mu_7 \in \{\pm 1, \pm i\}$ depends only on $w$ and $z$. Hence proving Proposition 3.7 reduces to proving the same estimate for sums of the type

$$B(M, N; \omega, \zeta) := \sum_{w \in \mathcal{D}} \sum_{N(w) \leq M} \sum_{z \in \mathcal{D}} \sum_{N(z) \leq N} \alpha_w \beta_z \left( \frac{z}{\sigma(w)} \right)_2. \quad (5.2)$$

We will prove that

$$B(M, N; \omega, \zeta) \ll_{\varepsilon} M^{2 \ell \frac{d}{d-\ell}} N(M)^{\ell}, \quad (5.3)$$

whenever $N \geq M$; Proposition 3.7 then immediately follows from the symmetry of the sum $B(M, N; \omega, \zeta)$ coming from quadratic reciprocity. So suppose that $N \geq M$. Similarly as in [9] [10], we fix an integer $k \geq 4$, and we apply Hölder’s inequality to the $w$ variable to get

$$|B(M, N; \omega, \zeta)|^{2k-2} \ll M^{(2k-1) \cdot 2} \left( \sum_{w} \left| \sum_{z} \beta_z \left( \frac{z}{\sigma(w)} \right)_2 \right|^{2k} \right)^2,$$
where the summations over \(w\) and \(z\) are as above in (5.2). After expanding the inner sum in the second factor above, we get

\[
|B(M, N; \omega, \zeta)|^{4k} \ll M^{4k-2} \left( \sum_w \sum_z \beta'_z \left( \frac{z}{\sigma(w)} \right)^2 \right),
\]

where

\[
\beta'_z = \sum_{z = z_1 \cdots z_{2k} \in \mathcal{D}_{N(z_1), \ldots, N(z_{2k})} \leq N \atop z_1 \equiv \cdots \equiv z_{2k} \equiv \zeta \mod 16} z_1 z_2 \cdots z_{2k-1} \beta_{z_{2k}},
\]

and again the summation conditions for \(w\) are as in (5.2). Applying the Cauchy-Schwarz inequality to the \(z\)-variable above, we get

\[
|B(M, N; \omega, \zeta)|^{4k} \ll \epsilon, k M^{4k-2} \sum_{w_1} \sum_{w_2} \sum_z \left( \frac{z}{\sigma(w_1 w_2)} \right)^2,
\]

where the summation conditions for \(w_1\) and \(w_2\) are as those for \(w\) in (5.2), while the sum over \(z\) is implicitly over those \(z = z_1 + z_2 \zeta^8 + z_3 \zeta^{16} + z_4 \zeta^{24} \in \mathbb{Z}[\zeta_8]\) with \(|z_j| \ll N^{\frac{2k}{p}}\) for \(1 \leq j \leq 4\), the implied constant being absolute. We break up the sum over \(z\) into congruence classes \(\xi \mod \sigma(w_1 w_2)\) and note that

\[
\sum_{\xi \mod \sigma(w_1 w_2)} \left( \frac{\xi}{\sigma(w_1 w_2)} \right)^2 = 0
\]

unless \((\sigma(w_1 w_2))\) is the square of an ideal in \(\mathcal{O}_M\), and in particular, unless \(N(w_1 w_2)\) is a square. This gives

\[
\sum_z \left( \frac{z}{\sigma(w_1 w_2)} \right)^2 \ll \begin{cases} N^{2k} & \text{if } N(w_1 w_2) = 1 \\ M^2 N^{\frac{2k}{p}} + M^4 N^k + M^6 N^{\frac{2k}{p}} + M^8 & \text{otherwise.} \end{cases}
\]

Since we took \(k \geq 4\) and since \(N \geq M\), we have \(N^\frac{p}{2} \geq M^2\), so the last bound can be simplified to \(M^2 N^{\frac{2k}{p}}\). Hence

\[
|B(M, N; \omega, \zeta)|^{4k} \ll \epsilon, k \left( M^{4k-2} N^{2k} \left( N^2 + M^2 \cdot M^2 N^{\frac{2k}{p}} \right) (MN)^{\epsilon} \right)
\]

\[
\ll \epsilon, k \left( M^{4k-1} N^{4k} + M^{4k+2} N^{\frac{2k}{p}} \right) (MN)^{\epsilon}.
\]

Taking \(k = 6\), we get

\[
|B(M, N; \omega, \zeta)| \ll \epsilon \left( M^{\frac{18}{p}} N + M^{\frac{18}{p}} N^{\frac{2}{p}} \right) (MN)^{\epsilon}.
\]

Since \(N \geq M\), we have \(M^{\frac{12}{p}} N^{\frac{p}{2}} \leq M^{\frac{23}{p}} N\), and this finishes the proof of (5.3) and hence also of Proposition 3.7.
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