Topological dimension of singular-hyperbolic attractors

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Abstract

An attractor is a transitive set of a flow to which all positive orbit close to it converges. An attractor is singular-hyperbolic if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction [16]. The geometric Lorenz attractor [6] is an example of a singular-hyperbolic attractor with topological dimension \( \geq 2 \). We shall prove that all singular-hyperbolic attractors on compact 3-manifolds have topological dimension \( \geq 2 \). The proof uses the methods in [15].

1 Introduction

This paper is concerned with the topological dimension of attractors for flows on compact manifolds. By attractor we mean a transitive set of the flow to which all positive orbit close to it converges. The attractors under consideration will be singular-hyperbolic in the sense that they have singularities (all hyperbolic) and are partially hyperbolic with volume expanding central direction [16]. In particular, the singular-hyperbolic attractors are volume hyperbolic sets as defined in [1]. The geometric Lorenz attractor is an example of a singular-hyperbolic attractor with topological dimension \( \geq 2 \). We shall prove that all singular-hyperbolic attractors on compact 3-manifolds have topological dimension \( \geq 2 \). The proof uses the methods developed in [15]. Let us state our result in a precise way.

Hereafter \( X \) will be a \( C^1 \) vector field on a compact manifold \( M \). The flow of \( X \) is denoted by \( X_t, t \in \mathbb{R} \). Given \( p \in M \) we define \( \omega(p) = \omega_X(p) \), the \( \omega \)-limit set of \( p \), as the accumulation point set of the positive orbit of \( p \). The \( \alpha \)-limit set of \( p \)

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is the set $\alpha(p) = \alpha_X(p) = \omega X(p)$. A compact invariant set $\Lambda$ of $X$ is transitive or attracting depending on whether $\Lambda = \omega(p)$ or $\cap_{t>0} X_t(U)$ for some $p \in \Lambda$ or some compact neighborhood $U$ of $\Lambda$ respectively. An attractor is a transitive attracting sets. A closed orbit of $X$ is either periodic or singular. A singularity of $X$ is hyperbolic if none of its eigenvalues have zero real part.

A compact invariant set $\Lambda$ of $X$ is partially hyperbolic [8] if there are an invariant splitting $T\Lambda = E^s \oplus E^c$ and positive constants $K, \lambda$ such that:

1. $E^s$ is contracting, namely
   $$\| DX_t / E^s_x \| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t > 0.$$

2. $E^s$ dominates $E^c$, namely
   $$\| DX_t / E^s_x \| \cdot \| DX_{-t} / E^c_{X_t(x)} \| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t > 0.$$

The central direction $E^c$ of $\Lambda$ is said to be volume expanding if the additional condition
   $$| \text{Det}(DX_t / E^c_x) | \leq Ke^{-\lambda t}$$
holds $\forall x \in \Lambda, \forall t > 0$ where $\text{Det}(\cdot)$ means the jacobian. The above splitting $E^s \oplus E^c$ will be refered to as a $(K, \lambda)$-splitting in the Appendix.

**Definition 1.1.** ([16]) An attractor is singular-hyperbolic if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction.

**Definition 1.2.** ([9]) The topological dimension of a space $E$ is either $-1$ (if $E = \emptyset$) or the last integer $k$ for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than $k$.

The relation between dynamics and topological dimension was considered for hyperbolic systems [7, 18, 2, 3]; for expansive systems [10, 13]; and for singular-hyperbolic systems [14]. The result below generalizes to singular-hyperbolic attractors a well known property of both hyperbolic strange attractors and geometric Lorenz attractors.

**Theorem A.** Singular-hyperbolic attractors on compact 3-manifolds have topological dimension $\geq 2$.

The idea of the proof is the following. Let $\Lambda$ be a singular-hyperbolic attractor of a flow $X$ on a compact 3-manifold $M$. It follows from [16] that all the singularities $\sigma \in \Lambda$ are Lorenz-like, namely the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $\sigma$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. The flow nearby $\sigma$ can be described using the Grobman-Hartman Theorem [14]. In particular, a Lorenz-like singularity exhibits
two singular cross-sections \( S^t, S^b \) and two singular curves \( l^t, l^b \) ([15]). A singular cross section of \( \Lambda \) is by definition a disjoint collection of singular cross sections \( S^t, S^b \) (as \( \sigma \) runs over all the singularities of \( \Lambda \)) whose horizontal boundaries does not intersect \( \Lambda \). The singular curve of \( S \) is the union \( l \) of the respective singular curves \( l^t, l^b \). A singular partition of \( \Lambda \) will be a compact neighborhood \( O \) of \( \Lambda \cap l \) in \( S \), for some singular cross section \( S \) of \( \Lambda \), such that \( \Lambda \cap l \) does not intersect the boundary of \( O \) and every regular orbit of \( \Lambda \) intersect \( O \). The size of the singular partition \( O \) is the minimal \( \epsilon > 0 \) such that there is an invariant cone field in \( O \) (for the return map \( \Pi : \text{Dom}(\Pi) \subset O \rightarrow O \)) on which the derivative of \( \Pi \) has expansion rate bigger than \( \epsilon^{-1} \). In Proposition 2.7 we prove that one-dimensional singular-hyperbolic attractors on compact 3-manifolds have singular partition with arbitrarily small size. The proof of this proposition uses the Lemmas 7.5 and 7.6 in [15]. These lemmas will be proved in the Appendix for the sake of completeness. In Theorem 2.9 we shall prove that singular-hyperbolic attractors \( \Lambda \) on compact 3-manifolds cannot have singular partitions with arbitrarily small size. Theorem 2.11 will follow from Proposition 2.7 and Theorem 2.9.

## 2 Proof

We start with some definitions. Hereafter \( \Lambda \) is a singular-hyperbolic attractor of a \( C^1 \) flow \( X \) on a compact 3-manifold \( M \). Since Lorenz-like singularities \( \sigma \) are hyperbolic they are equipped with three invariant manifolds \( W_\sigma^s(\sigma), W_\sigma^u(\sigma), W_\sigma^{ss}(\sigma) \) each one tangent at \( \sigma \) to the eigenspace corresponding to \( \{\lambda_2, \lambda_3\}, \{\lambda_1\}, \{\lambda_2\} \) respectively. It follows from [16] that every singularity \( \sigma \) of \( X \) in \( \Lambda \) is Lorenz-like and satisfies \( \Lambda \cap W_\sigma^{ss}(\sigma) = \{\sigma\} \). The classical Grobman-Hartman Theorem [5] gives the description of the flow nearby \( \sigma \). This is done at Figure ???. Note that \( W_\sigma^{ss}(\sigma) \) separates \( W_\sigma^s(\sigma) \) in two connected components denoted the top and the bottom respectively. In one of these components, say the top one, we consider a cross-section \( S^t = S^t_\sigma \) together with a curve \( l^t = l^t_\sigma \) as in Figure ???. Similarly we consider a cross-section \( S^b = S^b_\sigma \) and a curve \( l^b = l^b_\sigma \) located in the bottom component of \( W_\sigma^s(\sigma) \). See Figure ???. Both \( S^* \) (for \( * = 1, 2 \)) are homeomorphic to \([0, 1] \times [0, 1] \). \( S^* \) can be chosen in a way that \( l^* \) is contained in \( W_\sigma^s(\sigma) \setminus W_\sigma^{ss}(\sigma) \). The positive flow lines of \( X \) starting at \( S^t \cup S^b \setminus (l^t \cup l^b) \) exit a small neighborhood of \( \sigma \) passing through the cusp region as indicated in Figure ???. The positive orbits starting at \( l^t \cup l^b \) goes directly to \( \sigma \). We note that the boundary of \( S^* \) is formed by four curves, two of them transverse to \( l^* \) and two of them parallel to \( l^* \). The union of the curves in the boundary of \( S^* \) which are parallel (resp. transverse) to \( l^* \) is denoted by \( \partial^v S^* \) (resp. \( \partial^b S^* \)). The interior (as a submanifold) of \( S^* \) is denoted by \( \text{Int}(S^*) \).
Remark 2.1. An immediate consequence of \( \Lambda \cap W^{ss}_X(\sigma) = \{ \sigma \} \) is the following. Let \( \sigma \) be a singularity of \( X \) in \( \Lambda \). Then there are cross-sections \( S^t, S^b \) as above arbitrarily close to \( \sigma \) such that \( \Lambda \cap \partial^h S^* = \emptyset \) (\( * = t, b \)). Since the two boundary points of \( l^* \) are in \( \partial^h S^* \) we have that \( \Lambda \cap l^* \subset \text{Int}(S^*) \).

Definition 2.2. We shall call the cross sections \( S^t, S^b \) as singular cross sections associated to \( \sigma \). The curves \( l^t, l^b \) are called singular curves of \( S^t, S^b \) respectively. A singular cross section of \( \Lambda \) is a finite disjoint collection \( \{ S^t_\sigma, S^b_\sigma : \sigma \text{ is a singularity of } X \text{ in } \Lambda \} \). It follows that \( \Lambda \cap \partial^h S = \emptyset \). The singular curve of \( S \) is the associated collection of singular curves \( l = \{ l^t_\sigma, l^b_\sigma : \sigma \text{ is a singularity of } X \text{ in } \Lambda \} \).

Hereafter we denote by \( T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda \) the singular-hyperbolic splitting of \( \Lambda \). The contracting direction \( E^s \) is one-dimensional and contracting. So, \( E^s_\Lambda \) can be extended to an invariant contracting splitting \( E^s_{U(\Lambda)} \) on a neighborhood \( U(\Lambda) \) of \( \Lambda \). The standard Invariant Manifold Theory \( \mathbb{S} \) implies that \( E^s_{U(\Lambda)} \) is tangent to a continuous foliation \( \mathcal{F} \) on \( U(\Lambda) \). If \( S \) is a singular cross-section contained in \( U(\Lambda) \), we denote by \( \mathcal{F}^s \) the foliation of \( S \) obtained projecting \( \mathcal{F} \) into \( S \) along \( X \). The space of leaves of \( \mathcal{F}^s \) will be denoted by \( I^S \). We extend \( E^s_\Lambda \) continuously to a subbundle \( E^s_{U(\Lambda)} \) of \( T_{U(\Lambda)} M \). In what follows we fix such a neighborhood \( U(\Lambda) \) of \( \Lambda \).

Remark 2.3. It is possible to choose \( S \) arbitrarily close to the singularities of \( \Lambda \) in a way that \( l \) is a finite union of leaves of \( \mathcal{F}^s \) and \( I^S \) is a finite disjoint union of compact intervals.

The following lemma is a direct consequence of standard argument involving topological dimension. We prove it here for the sake of completeness.

Lemma 2.4. Let \( S \) a singular cross-section and \( l \) be its associated singular curve. If \( \Lambda \) is one-dimensional, then there is a compact neighborhood \( O \) of \( \Lambda \cap l \) in \( S \) whose boundary \( \partial O \) satisfies \( \Lambda \cap \partial O = \emptyset \).

Proof. Note that \( \Lambda \cap \partial^h S = \emptyset \) since \( S \) is a singular cross-section. As noted in Remark 2.1 one has \( \Lambda \cap l \subset \text{Int}(S) \). Fix \( x \in \Lambda \cap l \). Then \( x \in \text{Int}(S) \). Because \( \Lambda \) is one dimensional we have that \( \Lambda \cap S \) is zero dimensional \( \mathbb{S} \). Then, by the definition of the topological dimension, one can find an open set \( S_x \) of \( \Lambda \cap S \) containing \( x \) such that \( \partial S_x = \emptyset \). Note that the topology in \( \Lambda \cap S \) is the one induced by \( S \). In follows that \( S_x = (\Lambda \cap S) \cap O_x \) for some open set \( O_x \) of \( M \). Since \( S \) is transversal to \( X \) we can choose \( O_x \) such that \( \partial S_x = (\Lambda \cap S) \cap \partial O_x \) (for this we can use the Tubular Flow-Box Theorem \( \mathbb{S} \)). It follows that

\[
(\Lambda \cap S) \cap \partial O_x = \emptyset.
\]

On the other hand, \( \Lambda \cap l \) is compact in \( S \) and \( \{ S \cap O_x : x \in \Lambda \cap l \} \) is an open covering of \( \Lambda \cap l \). It follows that there is a finite subcollection of \( \{ S \cap O_x : x \in \Lambda \cap l \} \)
covering $\Lambda \cap l$. Denote by $O$ the union of the closures (in $S$) of the elements of such a subcollection. It follows that $O$ is a compact neighborhood of $\Lambda \cap l$ in $S$. Since $O$ is a finite union of $S \cap O_x$'s satisfying $(\Lambda \cap S) \cap \partial O_x = \emptyset$ we have that $\Lambda \cap \partial O = \emptyset$. This proves the lemma. 

Hereafter $O$ is a set contained in a singular cross section $S$. Clearly $O$ defines a return map

$$\Pi : \text{Dom}(\Pi) \subset O \to O$$

given by

$$\Pi(x) = X_{t(x)}(x),$$

where $\text{Dom}(\cdot)$ denotes the domain and $t(\cdot)$ denotes the return time.

**Remark 2.5.** Note that $\Pi$ may be discontinuous in $\Pi^{-1}(\partial O)$. However if $x \in \Pi^{-1}(\text{Int}(O))$ then $\Pi$ is $C^1$ in an open neighborhood of $x$ contained in $\text{Int}(O)$. This is an immediate consequence of the Tubular Flow-Box Theorem.

We denote by $TO$ the tangent space of $O$ relative $S$. If $x \in M$ we denote by $\angle(v_x, w_x)$ the tangent of the angle between $v_x, w_x \in T_xM$. If $L_x$ is a linear subspace of $T_xM$, we define

$$\angle(v_x, L_x) = \inf_{w_x \in L_x} \angle(v_x, w_x).$$

Given $\alpha > 0$ we define the cone

$$C_\alpha(L_x) = \{v_x \in T_xM : \angle(v_x, L_x) \leq \alpha\}.$$ 

If $L : x \in \text{Dom}(L) \to L_x$ is a map and $\alpha > 0$ we define the cone field

$$C_\alpha(L) = \{C_\alpha(L_x) : x \in \text{Dom}(L)\}.$$ 

The case $L = E^c$ will be interesting. The definition below is a minor modification of the corresponding definition in [15]. If $x \in M$ we denote $X_{\mathbb{R}}(x)$ the full orbit of $x$.

**Definition 2.6.** A singular partition of $\Lambda$ is a set $O$ satisfying the following properties:

1. There is a singular cross-section $S$ such that $O \subset \text{Int}(S)$ is a compact neighborhood of $\Lambda \cap l$.
2. $\Lambda \cap \partial O = \emptyset$.
3. $\text{Sing}_X(\Lambda) = \{q \in \Lambda : X_{\mathbb{R}}(q) \cap O = \emptyset\}$. 

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The size of $O$ is the minimal number $\epsilon > 0$ for which there is $\alpha > 0$ such that the cone field $C_\alpha(E^c)$ satisfies :

4. If $x \in \text{Dom}(\Pi)$, then

$$D\Pi(x) \left( C_\alpha(E^c_x) \cap TO_x \right) \subset \text{int} \left( C_{\frac{\epsilon}{2}\alpha}(E^c_{\Pi(x)}) \cap TO_{\Pi(x)} \right).$$

5. If $x \in \text{Dom}(\Pi)$ and $v_x \in C_\alpha(E^c_x) \cap TO_x$, then

$$\|D\Pi(x)(v_x)\| \geq \epsilon^{-1}\|v_x\|.$$

6. $\inf \{ \angle(v_x, E^s_x) : x \in O, v_x \in C_\alpha(E^c_x) \cap TO_x \} > 0.$

The following proposition studies the existence of singular partition with arbitrarily small size for certain one-dimensional singular-hyperbolic sets. Its proof uses the methods developed in [15]. We let $\text{Sing}_X(\Lambda) = \{\sigma_1, \cdots, \sigma_k\}$ be the set of singularities of $X$ in $\Lambda$.

**Proposition 2.7.** One-dimensional singular-hyperbolic attractors on compact 3-manifolds have singular partitions with arbitrarily small size.

**Proof.** Let $\Lambda$ be a singular-hyperbolic attractor of a $C^1$ flow $X$ on a compact 3-manifold $M$. We shall assume that $\Lambda$ has topological dimension 1. We shall prove that $\Lambda$ has singular partition with arbitrarily small size $\epsilon > 0$. For this we proceed as follows. Since $\Lambda$ has topological dimension 1 we have that $\Lambda$ cannot contain hyperbolic sets (the unstable manifold of a hyperbolic set in $\Lambda$ would be two-dimensional and contained in $\Lambda$). It follows that $\omega(x)$ cannot be hyperbolic for all $x \in \Lambda$. By [17] if $L = \omega(x), \alpha(x)$ then

$$L \cap \text{Sing}_X(\Lambda) \neq \emptyset, \ \forall x \in \Lambda.$$

Choose $\alpha > 0$ such that

$$\inf \{ \angle(v_x, E^s_x) : x \in U(\Lambda), v_x \in C_\alpha(E^c_x) \} > 0.$$

By [15] Lemma 7.5] (see Lemma 3.1) we can find a neighborhood $U_\alpha \subset U(\Lambda)$ of $\Lambda$ and positive constants $T_\alpha, K_\alpha, \lambda_\alpha$ such that the following properties hold:

(P1). If $x \in U_\alpha$ and $t \geq T_\alpha$, then

$$DX_t(x)(C_\alpha(E^c_x)) \subset C_{\alpha/2}(E^c_{X_t(x)}).$$

(P2). If $x \in U_\alpha$ is regular, $X(x) \in C_\alpha(E^c_x)$, $t \geq T_\alpha$ and $v_x \in C_\alpha(E^c_x)$ is orthogonal to $X(x)$, then

$$\|P^t_x(v_x)\| \cdot \|X(X_t(x))\| \geq K_\alpha e^{\lambda_\alpha t}. \|v_x\| \cdot \|X(x)\|,$$
where $P^t_x$ denotes the Poincaré flow associated to $X$ (see [4, 15] or the Appendix).

Once we fix $\alpha$ and $U_{\alpha}$ we apply [15, Lemma 7.6] (see Lemma 3.2) to find, for every $i \in \{1, \ldots, k\}$, a pair of singular cross-sections $S^{*0}_i$ associated to $\sigma_i (\ast = t, b)$ such that

$$X(x) \in C_\alpha(x), \ \forall x \in S^{*0}_i.$$ 

Define

$$S^0 = \bigcup_{i=1}^k (S^{*,0}_i \cup S^{*0}_i).$$

It is clear that $S^0$ is a singular cross-section. We denote by $l_0$ the singular curve of $S^0$. Since $S^0$ is transversal to $X$ one can find a constant $D > 0$ (depending on $S^0$) such that

$$\left\|X(x)\right\| > D,$$

for all $x, y \in S^0$. We choose $T_\epsilon > T_\alpha$ large enough so that

$$K_\alpha e^{\lambda_\alpha t} \cdot D > \epsilon^{-1}$$

(2)

for all $t \geq T_\epsilon$.

For every $\delta > 0$ we consider a singular-cross section $S^\delta \subset S^0 (i = 1, \ldots, k$ and $\ast = t, b)$ formed by small bands $S^{*\delta}_i$ of diameter $2\delta$ around the singular curve $l_i^{*}$ of $S^{*0}_i$. Note that the singular curve of $S^\delta$ is $l^0$ (the one of $S^0$) for all $\delta$. Since $\Lambda$ is one-dimensional Lemma 2.4 implies that $\forall x \in \Lambda \cap l^0$ there is a compact neighborhood $O = O^\delta \subset S^\delta$ of $\Lambda \cap l^0$ such that $\Lambda \cap \partial O = \emptyset$. Note that $O$ is a singular partition of $\Lambda$. In fact, (1) and (2) of Definition 2.6 are obvious. And (3) of Definition 2.6 follows from Eq. (1) since $O$ is a compact neighborhood of $\Lambda \cap l^0$.

Let us prove that if $\delta > 0$ small enough, then $O$ has size $\epsilon$. For this we need to prove that for $\delta$ small the cone field $C_\alpha(E^c)$ satisfies the properties (4)-(6) of Definition 2.6. Let $\Pi : Dom(O) \subset O \to O$ be the return map induced by $X$ in $O$. By definition $\Pi(x) = X_{t(x)}(x)$ where $t(x)$ is the return time of $x \in Dom(O) \subset O$ into $O$. To calculate $D \Pi(x)$ we can assume without loss of generality that $S^0$ is orthogonal to $X$. It follows that

$$D \Pi(x) = P^t_{x}$$

for all $x \in Dom(\Pi)$. Shrinking $\delta$ one has $t(x) > T_\epsilon$ for all $x \in Dom(\Pi) \subset O$. This allows us to apply the properties (P1)-(P2) above. In fact, since $D \Pi(x) = P^t_{x}$ one has

$$D \Pi(x)/C_\alpha(E^c_{x}) \cap TO_x = P^t_{x}/C_\alpha(E^c_{x}) \cap TO_x.$$ 

Then, Definition 2.6-(4) follows from (P1), (P2) and Eq. (2) imply

$$\|D \Pi(x)(v_x)\| = \|P^t_{x}(v_x)\| = \epsilon^{-1}$$

(2)
\[
= \|P_x^{t(x)}(v_x)\| \cdot \|X(X_t(x))\| \cdot \|X(X_t(x))\|^{-1} \geq \\
\geq K_\alpha e^{\lambda_\alpha t(x)} \cdot \frac{\|X(x)\|}{\|X(X_t(x))\|} \cdot \|v_x\| \geq K_\alpha e^{\lambda_\alpha t(x)} \cdot D \cdot \|v_x\| \geq e^{-1}\|v_x\|,
\]

\(\forall x \in \text{Dom}(\Pi), \forall v_x \in C_\alpha(E^u_x) \cap TO_x\) because \(X(x) \in C_\alpha(E^u_x) \forall x \in S^0\) and \(t(x) > T_c\) (\(\forall x \in \text{Dom}(\Pi)\)). This proves Definition 2.6-(5). Definition 2.6-(6) is a direct consequence of the choice of \(\alpha\). The result follows. \(\Box\)

The following lemma will be used to prove Theorem 2.9. Recall that if \(S\) is a singular cross-section then \(S\) is endowed with a singular curve \(l\). If \(O \subseteq S\) then \(\Pi : \text{Dom}(\Pi) \subseteq O \rightarrow O\) denotes the return map associated to \(O\).

**Lemma 2.8.** Let \(\Lambda\) be a singular-hyperbolic attractor of a flow \(X\) on a compact 3-manifold. Let \(O\) be a singular partition of \(\Lambda\). Then, there is an open neighborhood \(O' \subseteq \Lambda \cap O\) such that:

1. \(O' \setminus l \subseteq \text{Dom}(\Pi)\).
2. \(\Pi\) is \(C^1\) in \(O' \setminus l\).
3. \(\Pi(O' \setminus l) \subset O'\).

**Proof.** Because \(\Lambda\) is an attractor we have that the unstable manifold of any of its singularities is contained in \(\Lambda\). In particular, every connected component of \(W^u_\Lambda(\sigma_i) \setminus \{\sigma_i\}\) is contained in \(\Lambda\) \(\forall i\). It follows from Definition 2.6-(3) that all such components intersect \(O\). By Definition 2.6-(2) such intersections can occur only in \(\text{Int}(O)\). This implies that there are small open bands, centered at the singular curves in \(l\), whose union \(V(l)\) satisfies \(V(l) \setminus l \subseteq \Pi^{-1}(\text{Int}(O))\).

As noted in Remark 2.5 we have that \(\Pi\) is \(C^1\) in \(V(l) \setminus l\). Again by Definition 2.6-(2)-(3) one has \((\Lambda \cap O) \setminus V(l) \subseteq \Pi^{-1}(\text{Int}(O))\). So, by Remark 2.5 since \((\Lambda \cap O) \setminus V(l)\) is compact, there is an open neighborhood \(V\) of \((\Lambda \cap O) \setminus V(l)\) contained in \(\text{Dom}(\Pi)\) such that \(\Pi\) is \(C^1\) in \(V\). Observe that \(V \cup V(l)\) is an open neighborhood of \(\Lambda \cap O\) such that \(\Pi\) is \(C^1\) in \((V \cup V(l)) \setminus l\). On the other hand, \(\Lambda\) is an attractor by assumption. Then, there is a neighborhood \(U^*\) such that \(X_t(U^*) \subset U^*\) \(\forall t > 0\). Clearly one can choose \(U^*\) to be arbitrarily close to \(\Lambda\). In particular, \(O' := O \cap U^*\) is contained in \(V \cup V(l)\). It follows that \(O' \setminus l \subseteq \text{Dom}(\Pi)\) because \(V \cup (V(l) \setminus l) \subseteq \text{Dom}(\Pi)\). Because \(X_t(U^*) \subset U^*\) for all \(t > 0\) and the return time for the points in \(\text{Dom}(\Pi)\) is positive we conclude that \(\Pi(O' \setminus l) \subset O'\).

As \(\Pi\) is \(C^1\) in \((V \cup V(l)) \setminus l\) and \(O' \subseteq V \cup V(l)\) we conclude that \(\Pi\) is \(C^1\) in \(O' \setminus l\). This proves the result. \(\Box\)

**Theorem 2.9.** Singular-hyperbolic attractors on compact 3-manifolds cannot have singular partitions with arbitrarily small size.
Proof. Let $\Lambda$ be a singular-hyperbolic attractor of a $C^1$ flow $X$ on a compact 3-manifold $M$. By contradiction we assume that $\Lambda$ has a singular partition $O$ with arbitrarily small size $\epsilon > 0$. We fix $\epsilon \in (0, 1/2)$. We let $O'$ be the open neighborhood obtained in Lemma 2.8 for $O$. Hereafter we say that a $C^1$ connected curve $c$ in $O$ is a $C^u$-curve if its tangent vector belongs to the cone field $C\alpha(E^c)$ at Definition 2.6-(4). Definition 2.6-(4) implies that $\Pi$ carries $C^u$ curves in $\Pi^{-1}(Int(O))$ into $C^u$ curves in $O$ (see also Remark 2.3). Definition 2.6-(6) implies that a $C^u$ curve in $O$ intersects $l$ in at most one point $x_c$. In that case $x_c$ divides $c$ in two connected components the largest one being denoted by $c^+$. Clearly if $L(\cdot)$ denotes the length, then

$$L(c^+) \geq (1/2)L(c).$$

Now, fix a $C^u$ curve $c_1 \subset O' \setminus l$. Define $R = (2\epsilon)^{-1}$. The choice of $\epsilon$ implies $\epsilon^{-1} > R > 1$. Lemma 2.8-(1) implies $c_1 \subset \Dom(\Pi)$. Lemma 2.8-(2) implies that $c_2 = \Pi(c_1)$ is a $C^u$ curve contained in $O'$. Definition 2.6-(5) implies $L(c_2) \geq \epsilon^{-1}L(c_1) \geq R \cdot L(c_1)$. Suppose we have constructed a sequence $c_1, c_2, \cdots, c_i$ of $C^u$ curves of $O$ contained in $O'$ satisfying $L(c_j) \geq R \cdot L(c_{j-1})$ for all $2 \leq j \leq i$. If $c_i \cap l = \emptyset$ we define $c_{i+1} = \Dom(c_i)$ and keep going. If $c_i \cap l \neq \emptyset$ we define $c_{i+1} = \text{Closure}(\Dom(c_i))$. In any case $c_{i+1}$ is a $C^u$ curve of $O$ contained in $O'$. In the first case we have $L(c_{i+1}) \geq \epsilon^{-1}L(c_i) \geq R \cdot L(c_i)$. In the second case we have

$$L(c_{i+1}) = L(\Dom(c_i^+)) \geq \epsilon^{-1}L(c_i^+) \geq (\epsilon^{-1}/2) \cdot L(c_i) = R \cdot L(c_i).$$

In this way we can construct an infinite sequence $c_1, \cdots, c_i, c_{i+1}, \cdots$ of $C^u$ curves of $O$ in $O'$ all of which satisfying $L(c_{i+1}) \geq R \cdot L(c_i)$. It follows that

$$L(c_i) \geq R^i \cdot L(c_1),$$

for all $i$. Since $l(c_1) > 0$ and $R = (2\epsilon)^{-1} > 1$ we conclude that

$$\lim_{i \to \infty} L(c_i) = \infty.$$

On the other hand, let $S$ be the singular cross-section containing $O$ given by Definition 2.6-(1). Let $F^S$ be the projection of the stable manifold in $U(\Lambda)$ over $S$. As noted in Remark 2.3 the leave space $I^S$ of $F^S$ is a finite union of compact intervals. In particular $I^S$ has finite diameter. Since $O' \subset O \subset Int(S)$ we have that all the curves $c_i$ are contained in $S$. Since $c_i$ is a $C^u$ curve we have by Definition 2.6-(6) that $c_i$ have positive angle with the leaves of $F^S$ (note that these leaves are tangent to $E^s$). So, we can project $c_i$ to obtain an infinite sequence of intervals in $I^S$. The lenght of these intervals goes to $\infty$ (as $i \to \infty$) since $L(c_i) \to \infty$ (as $i \to \infty$). This is a contradiction since $I^S$ has finite diameter. This contradiction proves the result. \hfill $\Box$
Proof of Theorem A. Let $\Lambda$ be a singular-hyperbolic attractor on a compact 3-manifold. If $\Lambda$ has topological dimension $< 2$ then $\Lambda$ would be one-dimensional because it has regular orbits [9]. It would follow from Proposition 2.7 that $\Lambda$ has singular partitions with arbitrarily small size contradicting Theorem 2.9. The proof follows.

3 Appendix

In this section we state (and prove) two technical lemmas which were used in the proof of Theorem A. These lemmas were proved in [15] and here we reproduce these proofs for the sake of completeness. Let us state some definitions and notations.

First we define the Linear Poincaré Flow [4]. Let $X$ be a flow on a compact 3-manifold $M$. The Riemannian Metric of $M$ is denoted by $\langle \cdot, \cdot \rangle$. If $x$ is a regular point of $X$ (i.e. $X(x) \neq 0$), we denote by $N_x = \{v_x \in T_x M : \langle v_x, X(x) \rangle = 0\}$ the orthogonal complement of $X(x)$ in $T_x M$. Denote $O_x : T_x M \to N_x$ the orthogonal projection onto $N_x$. For every $t \in \mathbb{R}$ we define $P^t_x : N_x \to N_{X_t(x)}$ by

$$P^t_x = O_{X_t(x)} \circ DX_t(x).$$

It follows that $P = \{P^t_x : t \in \mathbb{R}, X(x) \neq 0\}$ satisfies the cocycle relation

$$P^{s+t}_x = P^s_{X_t(x)} \circ P^s_x,$$

for every $t, s \in \mathbb{R}$. The parametrized family $P$ is called the Linear Poincaré Flow of $X$.

We denote by $\text{vol}(v_x, w_x)$ the area of the parallelogram in $T_x M$ generated by $v_x, w_x \in T_x M$. As $M$ is a compact manifold, there is a constant $V \geq 1$ such that $V^{-1} \leq \text{vol}(v_x, w_x) \leq V$, $\forall x \in M$, $\forall v_x, w_x \in T_x M$ satisfying $\|v_x\| = \|w_x\| = 1$ and $\langle v_x, w_x \rangle = 0$. For simplicity we shall assume that $V = 1$. In other words,

$$\text{vol}(v_x, w_x) = \|v_x\| \cdot \|w_x\|,$$

$\forall x \in M$, $\forall v_x, w_x \in T_x M$ with $\langle v_x, w_x \rangle = 0$.

In addition,

$$\text{vol}(v_x, X(x)) = \|O_x v_x\| \cdot \|X(x)\|,$$

$\forall x \in M$ regular, $\forall v_x \in T_x M$. In particular,

$$\text{vol}(DX_t(x)v_x, X(X_t(x))) = \|P^t_x(v_x)\| \cdot \|X(X_t(x))\|,$$  \hspace{1cm} (3)

$\forall x \in M$ regular, $\forall t \in \mathbb{R}$, $\forall v_x \in N_x$. 

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Recall that if $\Lambda$ is a singular hyperbolic set of $X$ with $(K, \lambda)$-splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$, then
\[ |\text{Det}(DX_t/E^c_x)| \geq Ke^{\lambda t}, \]
$\forall x \in \Lambda$, $\forall t \geq 0$, where $\text{Det}(\cdot)$ denotes the jacobian. So, if $\Lambda$ is a singular hyperbolic set as above one has
\[
\text{vol}(DX_t(x)v^s_x, DX_t(x)w^c_x) \geq Ke^{\lambda t} \text{vol}(v^s_x, w^c_x),
\]
$\forall x \in \Lambda$, $\forall t \geq 0$, $\forall v^s_x, w^c_x \in E^c_x$.

Remember that $U(\Lambda)$ denotes a neighborhood of $\Lambda$ where the splitting $E^s_\Lambda \oplus E^c_\Lambda$ extends to $E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$.

**Lemma 3.1.** Let $\Lambda$ be a singular-hyperbolic attractor of a $C^1$ flow $X$ on a compact 3-manifold $M$. Then for every $\alpha \in (0, 1]$ there are a neighborhood $U_\alpha \subset U(\Lambda)$ of $\Lambda$ and constants $T_\alpha, K_\alpha, \lambda_\alpha > 0$ such that:

1. If $x \in U_\alpha$ and $t \geq T_\alpha$, then
   \[ DX_t(x)(C_\alpha(E^c_x)) \subset C_{\frac{\lambda}{2}}(E^c_{X_t(x)}). \]

2. If $x \in U_\alpha$ is regular, $X(x) \in C_\alpha(E^c_x)$, $t \geq T_\alpha$ and $v_x \in C_\alpha(E_x) \cap N_x$, then
   \[ \|P^t_x(v_x)\| \cdot \|X(X_t(x))\| \geq K_\alpha e^{\lambda_\alpha t} \cdot \|v_x\| \cdot \|X(x)\|. \]

**Proof.** Let $\Lambda$ and $\alpha \in (0, 1]$ be as in the statement. As mentioned above $T_{U(\Lambda)}M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$ denotes the extension of the $(K, \lambda)$-splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of $\Lambda$ to a neighborhood $U(\Lambda)$ of $\Lambda$. Let $\pi^s$ the projection of $T_\Lambda$ on $E^s_\Lambda$, and $\pi^c$ be the projection of $T_\Lambda$ on $E^c_\Lambda$. Denote $v_x = v^s_x + v^c_x \in E^s_x \oplus E^c_x = T_xM \forall x \in U(\Lambda)$, $\forall v_x \in T_xM$. In other words, $v^s_x = \pi^s(v_x)$ and $v^c_x = \pi^c(v_x)$.

As $E^s_\Lambda (K, \lambda)$-dominates $E^c_\Lambda$ we have that
\[
\|DX_t(x)/E^s_x\| \leq K^{-1} e^{-\lambda t} m(DX_t(x)/E^c_x), \tag{4}
\]
$\forall x \in \Lambda$, $\forall t \geq 0$.

Fix $R > 4$ such that
\[
\frac{K}{R} < 1. \tag{5}
\]
Choosing $T^1 = T^1_\alpha > 0$ large enough one has
\[
\|DX_{T^1}(x)/E^s_x\| \leq \frac{K_\alpha}{2R} m(DX_t(x)/E^c_x), \tag{6}
\]
$\forall x \in \Lambda$, $\forall t \geq 0$. 

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Since \( E^s_\Lambda \oplus E^c_\Lambda \) is invariant we have \( \pi^s_{X_t(x)} \circ DX_t(x) = DX_t(x) \circ \pi^s_{X_t(x)} \), and so
\[
\angle(DX_t(x)v_x, E^c_{X_t(x)}) = \frac{\|DX_t(x)v^s_x\|}{\|DX_t(x)v^c_x\|}, \quad \forall x \in \Lambda \quad \forall t \geq 0. \tag{7}
\]

Recall that \( \angle \) denotes the tangent of the angle. The inequality (6) and the last equality imply
\[
\angle(DX_T(x)v_x, E^c_{X_T(x)}) \leq \frac{K\alpha}{2R}, \quad \forall x \in \Lambda, \forall v_x \in C_\alpha(E^c_x).
\]

So,
\[
DX_T(x)(C_\alpha(E^c_x)) \subset C_{\frac{K\alpha}{2R}}(E^c_{X_T(x)}), \quad \forall x \in \Lambda.
\]

Choose a neighborhood \( U^1 = U^1_\alpha \subset U(\Lambda) \) of \( \Lambda \) sufficiently close to \( \Lambda \) such that
\[
DX_T(x)(C_\alpha(E^c_x)) \subset C_{\frac{K\alpha}{R}}(E^c_{X_T(x)}), \quad \forall x \in U^1. \tag{8}
\]

On the other hand, using (4) we get
\[
\frac{\|DX_t(x)v^s_x\|}{\|v^s_x\|} \leq K^{-1}e^{-\lambda t} \frac{\|DX_t(x)v^c_x\|}{\|v^c_x\|}
\]
and so,
\[
\frac{\|DX_t(x)v^s_x\|}{\|DX_t(x)v^c_x\|} \leq K^{-1}e^{-\lambda t} \frac{\|v^s_x\|}{\|v^c_x\|} = K^{-1}e^{-\lambda t} \angle(v_x, E^c_x).
\]

So, by (7), we get
\[
\angle(DX_r(x)v_x, E^c_{X_r(x)}) \leq K^{-1}e^{-\lambda r} \angle(v_x, E^c_x) \leq K^{-1} \angle(v_x, E^c_x),
\]
\( \forall x \in \Lambda, \forall r \in [0, T^1], \forall v_x \in T_xM. \) This implies
\[
DX_r(x)\left(C_\frac{K\alpha}{R}(E^c_x)\right) \subset C_\frac{K\alpha}{R}(E^c_{X_r(x)}), \quad \forall x \in \Lambda \quad \forall r \in [0, T^1].
\]

Choose a neighborhood \( V^2 = V^2_\alpha \subset U^1 \) of \( \Lambda \) sufficiently close to \( \Lambda \) such that
\[
DX_r(x)\left(C_\frac{K\alpha}{R}(E^c_x)\right) \subset C_\frac{K\alpha}{R}(E^c_{X_r(x)}), \quad \forall x \in V^2, \forall r \in [0, T^1]. \tag{9}
\]

As \( \Lambda \) is an attractor there is a neighborhood \( U^2 \subset V^2 \) of \( \Lambda \) such that
\[
X_t(U^2) \subset V^2, \quad \forall t \geq 0.
\]

Now, let \( x \in U^2 \) and \( t \geq T^1 \) be given. Then, \( t = nT^1 + r \) for some integer \( n \geq 1 \) and some \( r \in [0, T^1] \). Thus,
\[
DX_t(x)(C_\alpha(E^c_x)) = DX_{nT^1+r}(x)(C_\alpha(E^c_x)) =
\]
\[ DX_t(X_{nT^1}(x))(DX_{nT^1}(x)(C_\alpha(E_x^c))). \]  

Using (S) and (U) recursively, and that \( \frac{K_\alpha}{R} < \alpha \), and \( n \geq 1 \), we obtain

\[ DX_{nT^1}(x)(C_\alpha(E_x^c)) = DX_{(n-1)T^1}(X_{T^1}(x))(DX_{T^1}(x)C_\alpha(E_x^c)) \subset \]

\[ \subset DX_{(n-1)T^1}(X_{T^1}(x)) \left( C_{\frac{K_\alpha}{R}}(E_{X_{nT^1}(x)}^c(x)) \right) \subset DX_{(n-1)T^1}(X_{T^1}(x))(C_\alpha(E_{X_{nT^1}(x)}^c(x))) \subset \]

\[ \subset \cdots \subset DX_{T^1}(X_{(n-1)T^1}(x))(C_\alpha(E_{X_{(n-1)T^1}(x)}^c(x))) \subset C_{\frac{K_\alpha}{R}}(E_{X_{nT^1}(x)}^c(x)). \]

Henceforth

\[ DX_{nT^1}(x)(C_\alpha(E_x^c)) \subset C_{\frac{2\alpha}{R}}(E_{X_{nT^1}(x)}^c). \]

Applying \( DX_t(X_{nT^1}(x)) \) to both sides of the last expression, replacing in (10) and using (9) we obtain

\[ DX_t(x)(C_\alpha(E_x^c)) \subset DX_t(X_{nT^1}(x)) \left( C_{\frac{K_\alpha}{R}}(E_{X_{nT^1}(x)}^c(x)) \right) \subset C_{\frac{2\alpha}{R}}(E_{X_t(x)}^c). \]

As \( R > 4 \) we have \( \frac{2\alpha}{R} < \frac{\alpha}{2} \) and so

\[ DX_t(x)(C_\alpha(E_x^c)) \subset C_{\frac{\alpha}{2}}(E_{X_t(x)}^c). \]

\[ \forall x \in U^2, \forall t \geq T^1, \text{ proving (1) of Lemma 3.1.} \]

Throughout we fix the neighborhood \( U^2 \) of \( \Lambda \) and the constant \( T^1 > 0 \) obtained above.

As \( E_{\Lambda}^c \) is \((K, \lambda)\)-volume expanding we have

\[ \text{vol}(DX_t(x)v_x^c, DX_t(x)w_x^c) \geq Ke^\lambda \text{vol}(v_x^c, w_x^c), \]

\[ \forall x \in \Lambda, \forall t \geq 0, \forall v_x^c, w_x^c \in E_{\Lambda}^c. \]

Clearly there is \( L > 1 \) such that

\[ L^{-1} \cdot \text{vol}(v_x^c, w_x^c) \leq \text{vol}(v_x, w_x) \leq L \cdot \text{vol}(v_x^c, w_x^c), \]

\[ \forall x \in \Lambda, \forall v_x, w_x \in C_\alpha(E_{\Lambda}^c), \forall \alpha \in (0, 1]. \]

Applying the last two relations and the invariance of \( E_{\Lambda}^s \oplus E_{\Lambda}^c \) we obtain

\[ \text{vol}(DX_t(x)v_x, DX_t(x)w_x) \geq L^{-1} \text{vol}(DX_t(x)v_x^c, DX_t(x)w_x^c) \geq \]

\[ \geq L^{-1}Ke^\lambda \text{vol}(v_x^c, w_x^c) \geq L^{-2}Ke^\lambda \text{vol}(v_x, w_x), \]

\[ \forall x \in \Lambda, \forall t \geq T^1, \forall v_x, w_x \in C_\alpha(E_{\Lambda}^c) \text{ (note that } DX_t(x)v_x, DX_t(x)w_x \in C_\alpha(E_{X_t(x)}^c) \text{ since } t \geq T^1). \]
Choose $S > 0$ large so that
\[
\frac{S}{L^{-2}K} > 1. \quad (11)
\]

It follows that there is $T^2 = T^2_\alpha > T^1$ such that
\[
\text{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) \geq \frac{2S}{L^{-2}K} \text{vol}(v_x, w_x),
\]
$\forall x \in \Lambda, \forall v_x, w_x \in C_\alpha(E^c_x)$. In particular,
\[
\inf\{\text{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) : x \in \Lambda, v_x, w_x \in C_\alpha(E^c_x), \|v_x\| = \|w_x\| = 1, < v_x, w_x > = 0\} \geq \frac{2S}{L^{-2}K}.
\]

Since $\Lambda$ is compact there is a neighborhood $V^3 = V^3_\alpha \subset U^2$ of $\Lambda$ so that
\[
\inf\{\text{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) : x \in U^3, v_x, w_x \in C_\alpha(E^c_x), \|v_x\| = \|w_x\| = 1, < v_x, w_x > = 0\} \geq \frac{S}{L^{-2}K}.
\]

Then,
\[
\text{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) \geq \frac{S}{L^{-2}K} \text{vol}(v_x, w_x), \quad (12)
\]
$\forall x \in U^3, \forall v_x, w_x \in C_\alpha(E^c_x)$ with $< v_x, w_x > = 0$. As $\Lambda$ is an attractor there is a neighborhood $U^3 \subset V^3$ of $\Lambda$ such that
\[
X_t(U^3) \subset V^3, \quad \forall t \geq 0.
\]

We have
\[
\|P^{nT^2}_x v_x\| \|X(X_{nT^2}(x))\| = \|P^{nT^2}_x(X_{(n-1)T^2}(x))P^{(n-1)T^2}_x v_x\| \|X(X_{T^2}(X_{(n-1)T^2}(x)))\|.
\]

Call $z = X_{(n-1)T^2}(x)$, and $v_z = P^{(n-1)T^2}_x v_x$. From the last equality, using that $X(X_{nT^2}(x)) = DX_{nT^2}(x)(X(x))$, $v_z$ is orthogonal to $z$, and combining (11) with (12) we get
\[
\|P^{nT^2}_x v_x\| \|X(X_{nT^2}(x))\| = \|P^{T^2}_z v_z\| \|X(X_{T^2}(z))\| = \text{vol}(DX_{T^2}(z)v_z, X(X_{T^2}(z))) = \text{vol}(DX_{T^2}(z)v_z, DX_{T^2}(z)(X(z)) \geq \frac{S}{L^{-2}K} \text{vol}(v_z, X(z)) = \frac{S}{L^{-2}K} \text{vol}(P^{(n-1)T^2}_x v_x, X(X_{(n-1)T^2}(x))).
\]
Thus,
\[
\|P_x^{nT^2} v_x\| \cdot \|X(X_{nT^2}(x))\| \geq \left\{ \frac{S}{L^{-2}K} \right\}^n \|v_x\| \cdot \|X(x)\|,
\]
(13)
\[\forall x \in U^3 \text{ regular with } X(x) \in C_\alpha(E_x^c), \forall n \in N, \forall v_x \in C_\alpha(E_x^c) \cap N_x \text{ (recall that } N_x \text{ denotes the orthogonal complement of } X(x) \text{ in } T_xM).\]

On the other hand
\[
\text{vol}(DX_r(x)v_x, DX_r(x)w_x) \geq L^{-2}K \cdot \text{vol}(v_x, w_x),
\]
\[\forall x \in \Lambda, \forall r \in [0, T^2], \forall v_x, w_x \in C_\alpha(E_x^c).\]

As before there is a neighborhood \(V^4 = V_\alpha^4 \subset U^3\) of \(\Lambda\) such that
\[
\text{vol}(DX_r(x)v_x, DX_r(x)w_x) \geq \frac{L^{-2}K}{2} \text{vol}(v_x, w_x),
\]
(14)
\[\forall x \in V^4, \forall v_x, w_x \in C_\alpha(E_x^c) \text{ with } <v_x, w_x> = 0, r \in [0, T^2].\] As \(\Lambda\) is an attractor there is a neighborhood \(U^4 \subset V^4\) of \(\Lambda\) such that
\[
X_t(U_4) \subset V^4, \forall t \geq 0.
\]

Now, let \(x \in U^4\) regular with \(X(x) \in C_\alpha(E_x^c), t \geq T^2\) and \(v_x \in C_\alpha(E_x^c) \cap N_x\).

Then, \(t = nT^2 + r\) for some integer \(n \geq 1\) and some \(r \in [0, T^2]\).

Applying (13), (14), and using (3) and (11) we obtain
\[
\|P_x^t v_x\| \cdot \|X(X_t(x))\| = \|P_x^{nT^2} v_x\| \cdot \|DX_r(X_{nT^2}(x))X(X_{nT^2}(x))\| = \text{vol}(DX_r(X_{nT^2}(x))P_x^{nT^2} v_x, DX_r(X_{nT^2}(x))X(X_{nT^2}(x))) \geq \frac{L^{-2}K}{2} \text{vol}(P_x^{nT^2} v_x, X(X_{nT^2}(x))) = \frac{L^{-2}K}{2} \|P_x^{nT^2} v_x\| \cdot \|X(X_{nT^2}(x))\| \geq \left( \frac{L^{-2}K}{2} \right) \cdot \left\{ \frac{SN}{L^{-2}K} \right\}^n \cdot \|v_x\| \cdot \|X(x)\| = \left( \frac{L^{-2}K}{2} \right) \cdot \left( \frac{S}{L^{-2}K} \right)^{-\frac{rT}{2}} \left\{ \left( \frac{S}{L^{-2}K} \right)^{\frac{1}{T}} \right\}^t \cdot \|v_x\| \cdot \|X(x)\| \geq \left( \frac{L^{-4}K^2}{2S} \right) \cdot \left\{ \left( \frac{S}{L^{-2}K} \right)^{\frac{1}{T}} \right\}^t \cdot \|v_x\| \cdot \|X(x)\|.
\]

Thus, choosing \(U_\alpha = U^4, T_\alpha = T^2, K_\alpha = \frac{L^{-4}K^2}{2S}\) and \(\lambda_\alpha = \ln \left( \frac{S}{L^{-2}K} \right)^{\frac{1}{T}} > 0\) we obtain (2) of Lemma 3.1. \(\square\)
Lemma 3.2. Let $\Lambda$ a singular-hyperbolic attractor of $X$, $\alpha > 0$ and $U_\alpha \subset U(\Lambda)$ be a neighborhood of $\Lambda$. Let $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the splitting of $\Lambda$ and $E^c_\Lambda U_\alpha$ be a continuous extension of $E^c_\Lambda$ to $U_\alpha$. Then, for every singularity $\sigma$ of $X$ in $\Lambda$ there are singular cross-sections $S^t, S^b$ associated to $\sigma$ such that $S^t \cup S^b \subset U_\alpha$,

$$ \Lambda \cap (\partial^h S^t \cup \partial^h S^b) = \emptyset, \text{ and } X(x) \in C_\alpha(E^c_S), \forall x \in S^t \cup S^b. $$

Proof. Let $\Lambda, \alpha$ be as in the statement. By Remark 2.1 applied to $U = U_\alpha$ there are singular cross-sections $S^t_0, S^b_0$ associated to $\sigma$ such that $S^t_0, S^b_0 \subset U(\Lambda)$ and $\Lambda \cap (\partial^h S^t_0 \cup \partial^h S^b_0) = \emptyset$. Recall that $U(\Lambda)$ is the neighborhood of $\Lambda$ where the $(K, \lambda)$-splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ has an extension to $T_{U(\Lambda)} M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$, with $E^c_{U(\Lambda)}$ invariant and contracting. We denote by $l^t_0, l^b_0$ the singular curves of $S^t_0, S^b_0$ respectively.

Choose two sequences of singular cross-sections $S^t_n \subset S^t_0, S^b_n \subset S^b_0$ associated to $\sigma$ satisfying

(a) $\Lambda \cap (\partial^h S^t_n \cup \partial^h S^b_n) = \emptyset$;
(b) $\text{diam}(S^t_n), \text{diam}(S^b_n) \to 0$ as $n \to \infty$;
(c) the singular curves $l^t_n, l^b_n$ of $S^t_n, S^b_n$ satisfy $l^t_n = l^t_0, l^b_n = l^b_0, \forall n$.

The properties (b) and (c) imply that $\forall n$ large there is $T = T_n > 0$ such that

$$ \angle(X(X_T(x)), E^c_\sigma) < \frac{\alpha}{2}, \forall x \in S^t_n \cup S^b_n. $$

As $E^c_\Lambda U_\alpha$ is a continuous extension of $E^c_\Lambda$, we have that $E^c_{X(x)}$ is close to $E^c_\sigma$, $\forall x \in S^t_n \cup S^b_n$. Then,

$$ X_T(x) \in U_\alpha, \text{ and } X(X_T(x)) \in C_\alpha(E^c_{X(x)}), \forall n \text{ large}, \forall x \in S^t_n \cup S^b_n. $$

By the Property (a) and the above relation we have that for $n$ large enough $S^t = X_T(S^t_n)$ and $S^b = X_T(S^b_n)$ are singular cross-sections associated to $\sigma$ satisfying the required properties.

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