ABSTRACT

Classical Electrodynamics is not a consistent theory because of its field inadequate behaviour in the vicinity of their sources. Its problems with the electron equation of motion and with non-integrable singularity of the electron self field and of its stress tensor are well known. These inconsistencies are eliminated if the discrete and localized (classical photons) character of the electromagnetic interaction is anticipatively recognized already in a classical context. This is possible, in a manifestly covariant way, with a new model of spacetime structure, shown in a previous paper \(^1\), that invalidates the Lorentz-Dirac equation. For a point classical electron there is no field singularity, no causality violation and no conflict with energy conservation in the electron equation of motion. The electromagnetic field must be re-interpreted in terms of average flux of classical photons. Implications of a singularity-free formalism to field theory are discussed.

I. INTRODUCTION

This paper concern is about 3 old, but still not solved problems of classical field theory, more particularly, of Classical Electrodynamics for a point charge: its field singularity, the non-integrable singularities of its energy-momentum tensor, and its bizarre equation of motion for the electron (the Lorentz-Dirac equation, LDE). They are the reasons for Classical Electrodynamics not being considered an entirely consistent theory; it cannot simultaneously handle its fields and their sources defined at a same point. We show in this paper that these problems are overcome with the anticipated adoption of some notions proper of the quantum theory: the granular, that is, the discrete and localized, character of a quantum interaction (classical photons).

This paper is a sequel of reference [1] of, where a proposed geometric vision of the causality constraints in field theory

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and its possible relevance to the problems of the LDE are discussed. The discussion of this geometry is repeated in an amplified way, in section III. The following 2 sections are applications of these ideas to the Lienard-Wiechert solution (LWS) of Classical Electrodynamics: the natural (in the sense of not ad hoc nor forced) elimination of the stress tensor non-integrable singularities, in section II, and of the LDE problems in section IV. The relevance of these ideas to Quantum Field Theory, as also to the question of the origin and meaning of mass, is developed elsewhere\cite{2,3}, but for the sake of completeness, a synopsis is presented in section V, after a short qualitative discussion on the meaning of the Maxwell-Faraday concept of field.

The LDE\cite{4} is the greatest paradox of classical field theory as it cannot simultaneously preserve both the causality and the energy conservation, although there is nothing in the premisses for its derivation that justify such violations. It is obtained from energy-momentum conservation in the LWS

\[
A = \frac{V}{\rho} \bigg|_{\tau_{\text{ret}}} , \quad \text{for } \rho > 0, \tag{1}
\]

in the limit of $\rho \to 0$. $V$ is the electron 4-velocity and $\rho$ is the invariant distance between a charge and its field. Taking this limit represents an extrapolation of conditions not valid for a region where the electron and its electromagnetic field are simultaneously defined. This limit passage, as it is argued in \cite{1}, is the weak point in all demonstrations of the LDE, because it corresponds to the assumption that the Minkowski model of spacetime is valid also in this limit. But, the Minkowski spacetime does not represent the necessary causality requirements for the description of two interacting fields at a same point. This has been pointed\cite{1} as the source of all problems associated to the LDE.

Many physicists nowadays, as a consequence of the undisputed success of the Einstein’s Theory of Special Relativity, see a Minkowski manifold as the natural or even absolute description of the spacetime structure of the world — in the absence of gravity, it is added. There is nothing special with the Minkowski spacetime. It just represents a geometric implementation of a well established principle of physics: the speed of light is a universal constant. It is a geometrization of this physical principle in the same way that in General Theory of Relativity the Newtonian concept of gravitational force is replaced by the notion of a curved spacetime. For the geometrization of special relativity, time, considered before as just an event-ordering parameter, has to be treated as a fourth dimension of the world. It must transform among inertial observers, in exactly the same way that the other 3 bona fide space dimensions do. These two examples show the importance of geometrization as a tool in theoretical physics: transferring a mandatory
physical requirement to the structure of a background geometry is a way of assuring its automatic implementation. The postulate of a constant speed of light is connected also to the important concept of relativistic causality: no physical object can move faster than light in a vacuum. It is geometrically implemented through the lightcone structure, seen as a restriction of access to regions of spacetime for physical objects. The importance of relativistic causality to modern physics has raised it to the category of a principle. But it is exactly this principle that is violated by solutions of the Lorentz-Dirac equation\(^{(5)}\). This is a paradoxical situation because the Minkowski spacetime is one of the premises always assumed in its derivations. The strategy adopted in reference\(^{(1)}\) was to create a model that represents the geometrization of the causality implementation found in the LWS, which is also one of the premises in the derivation of the LDE. In section II we present a commented review of the content of\(^{(1)}\). The geometrization of the causality constraints that govern the propagation of physical objects, transforms these constraints on restrictions of access to regions of the spacetime, with possible change on its metric structure. In section III the Maxwell formalism for the LWS in this new geometry (with preservation of causality) is presented. It is shown then that, irrespective the spacetime metric structure, the non-integrable singularity of the stress tensor disappears together with the terms that would generate, in the usual treatment, the Schott term and the Teitelboim’s bound-momentum. Previous attempts on taming these singularities have relied on modifications of the Maxwell’s theory with ad hoc addition of extra-terms (see for example the reviews\(^{(9−11)}\)) to the field stress tensor on the electron worldline; it is particularly interesting that, as we will show here, instead of adding anything we should actually not drop out some null terms. Is their contribution (not null, in the limit) that avoid the infinities.

The same problem happens in the derivations of the electron equation of motion: they are done with incomplete field expressions that do not contain these terms that are null only off the particle worldline. The Schott term in the LDE is a consequence of this; it does not appear in the equation when the full field expression is correctly used.

The electron equation of motion,

\[
m \dot{a}^{\mu} - \frac{1}{4} \dot{a}^2 a^{\mu} = F_{\mu}^{\nu} - \left\langle \frac{2}{3} \dot{a}^2 V^{\mu} \right\rangle, \tag{2}
\]

in this geometrical settings, is derived in section IV. The absence of the problematic Schott term assures that there will be no causality violation, and \(\left\langle \frac{2}{3} \dot{a}^2 V^{\mu} \right\rangle\), where the brackets represent a mean or average, in the place of \(\frac{2}{3} \dot{a}^2 V^{\mu}\), guarantees the conservation laws, as explained at the end of section IV. The new term \(\left\langle \frac{1}{4} \dot{a}^2 a^{\mu} \right\rangle\) is associated to
the electron spacetime curvature. The existence or not of this curvature term must be determined by experimental means and evaluated by its theoretical implications.

The recognition that the contribution from the electron self-field to its equation of motion is realized through average quantities requires a re-evaluation of the physical meaning of the Faraday-Maxwell concept of field, as qualitatively discussed in section V. In this new geometrical context, it must be seen as an average flux of “classical photons” around the charge. Then, its singularity loses all of its physical content, because it becomes just a consequence of working with this average field in the place of the fundamental one. This new geometrical structure of spacetime generates a formalism that is free of singularities; it is being more properly discussed elsewhere\(^{(2)}\), but for the sake of completeness we include a synopsis of its main results as well as the fresh and good results of the work\(^{(6−8)}\) with students (their M.Sc. theses, all in Portuguese), at the end of section V. The paper ends with an appendix showing some calculations that have been omitted in section IV.

\section*{II. GEOMETRY OF CAUSALITY}

Causality is implemented in the LWS through the constraint

\[ \Delta x \eta. \Delta x := \Delta x^\mu \eta_{\mu\nu} \Delta x^\nu = 0, \quad \Delta x^0 > 0, \quad (3) \]

where \( \Delta x = x - z(\tau) \), \( z(\tau) \) describes the electron world-line, parameterized by its propertime \( \tau \), and \( x \) is the point where the electromagnetic field (the photon) is observed.

If \( z(\tau_{\text{ret}}) \) is the solution to (3), the geometrical meaning of \( \Delta x \eta. \Delta x = 0 \) is of a light-cone with vertex at \( z(\tau_{\text{ret}}) \), and corresponds to the requirement that the electromagnetic field emitted at \( z(\tau_{\text{ret}}) \) remain on its light-cone. Its extension to the electromagnetic field at any spacetime point requires that \( \Delta x \eta. dx = 0 \) or

\[ d\tau + K. \eta. dx = 0, \quad (4) \]

where

\[ K^\mu := \frac{\Delta x^\mu}{\rho} = \frac{\Delta x^\mu}{\Delta \tau}, \quad (5) \]

is a null 4-vector, \( K^2 = 0 \), and represents a light-cone generator; \( \rho := -V. \eta. \Delta x \). We want to underline here the well known important role of the light-cone generator, \( K \), in the propagation of the electromagnetic field. \( d\tau \)
in (4) is the change in the electron proper-time, which is then connected to the required change in $x$, the point where the electromagnetic wave front is being observed. If $d\tau = 0$ (no change in the electron position), then $K dx = 0$ and $dx dx = 0$; the wave front moved a distance $dx$ along the light-cone generator $K^\mu$. If $d\tau \neq 0$, then $K dx \neq 0$ and $dx dx \neq 0$, which then means that the field emitted at $z(\tau + d\tau)$ must be observed not at $x$ but at $x + dx$, which belongs to the light-cone of $z(\tau_{ret} + d\tau)$, and not to the light-cone of $z(\tau_{ret})$. Then, $d\tau \neq 0$ implies, in (4), that $dx$ represents the difference in the positions of two distinct wave fronts (photons), each one on a different lightcone, with vertices at, respectively $z(\tau)$ and $z(\tau + d\tau)$. A clear vision of this is important for understanding the pictures of the subsequent problems.

The restriction (4) can be absorbed in the definition of a directional derivative

$$\nabla_\mu \equiv \partial_\mu - K_\mu \partial_\tau,\tag{6}$$

which is useful in calculations related to (1), because $\tau$ can be treated then as independent of $x$:

$$\frac{\partial}{\partial x^\mu} A(x, \tau) \bigg|_{\tau_{ret}} = (\partial_\mu - K_\mu \partial_\tau) A(x, \tau).$$

For the electromagnetic field $\tau$ is always $\tau_{ret}$. This is all the causality content in the LWS for the photon, but it is important to remark that it does not apply to the electron. An extension (1) of (3-6) for the electron, requires their replacement, respectively, by

$$- (\Delta \tau)^2 = \Delta z,\eta, \Delta z,\tag{7}$$

$$d\tau + V \cdot \eta, dz = 0,\tag{8}$$

and

$$\nabla_\mu \equiv \partial_\mu - V_\mu \partial_\tau,\tag{9}$$

The differences among (3-6) and (7-9) are an expression of the differences in the causality requirements for massless and for massive fields. Observe that there is a dichotomy in the treatment given to the electron and to its electromagnetic field. In the set (3-6) of causality restrictions imposed on the propagation of the electromagnetic field, $\tau$ is the electron propertime; so it explicitly depends on the electron conditions. But the same does not occur for the propagation of the
electron: (7-9) does not depend on any parameter of the electromagnetic field. This biased description of these two physical objects (the electron and its electromagnetic field) does not correspond to what is directly observed in nature, where both are equally elementary. The production of photons by an electron is so fundamental as the production of electron-positron pairs by a photon. It is just a question of charge, 4-momentum and angular momentum conservation. This seems to be already a clue that this classical formalism cannot produce a correct description of nature, not just for being classical, but for its biased treatment of two equally fundamental physical objects.

Returning to the limit passage of \( \rho \to 0 \), in the usual derivations of the equation of motion (LDE) of a classical electron, which involves its four-momentum conservation, we can see from (3), that the energy-momentum tensor obtained from the LWS has an explicit dependence on \( K \), which dictates the propagation of the photon, not of the electron. This should not be a problem because

\[
\lim_{\rho \to 0} K^\mu = V^\mu \tag{10}
\]

and so, in this limit (3) \( \to \) (7-9). Nonetheless, it can still be a problem because the metric structure of spacetime is wrongly supposed to be kept Minkowskian in this limit. In order to show why this may not be correct, it is necessary first to (geometrize) incorporate (3-9) into a single background geometrical structure of spacetime.

Consider all the physical objects (electrons, electromagnetic fields, etc) immersed in a flat 5- dimensional space, \( R_5 \equiv R_4 \otimes R_1 \), whose line elements are defined by

\[
(\Delta S_5)^2 = \Delta x^M \eta_{MN} \Delta x^N = (\Delta S_4)^2 - (\Delta x^5)^2 = \Delta x.\eta.\Delta x - (\Delta x^5)^2, \tag{11}
\]

where \( M, N = 1 \) to 5. Immersed in this larger space, every physical object is restricted to a 4-dimensional submanifold, its SPACETIME, by

\[
- (\Delta x^5)^2 = \Delta x.\eta.\Delta x. \tag{12}
\]

In other words, the CHANGES of \( x^5 \) of a physical object is identified with the CHANGES of its very propertime, \( \Delta x^5 = \Delta \tau \). For a physical object, \( (\Delta S_5)^2 = -2(\Delta \tau)^2 \), always. This identification of \( \Delta x^5 \) of a physical object with the variation of its proper-time, \( \Delta \tau \), represents the geometrization of causality for massless \( (\Delta \tau = 0) \) and for massive \( (\Delta \tau \neq 0) \) physical objects. So, the constraints on the propagation of physical objects become restrictions on their allowed domain in \( R_5 \), that is, in the definition of their allowed spacetime.
The evolution or propagation of physical objects, in this geometric setting, is restricted by the differential of (12), 
\[ \Delta \tau d\tau + \Delta x \eta . dx = 0, \] 
or
\[ d\tau + f. \eta . dx = 0, \] 
and by its induced directional derivative

\[ \nabla_f := \partial_\mu - f'_\mu \partial\tau, \] 
where \( f^\mu = \frac{\Delta x^\mu}{\Delta \tau} \), \( f'_\mu := \eta_{\mu\nu} f^\nu \), and \( f \) is a timelike 4-vector if \( d\tau \neq 0 \), or (extending (13) to include) a light-like 4-vector if \( d\tau = 0 \). A light-like \( f \) corresponds to \( K \) of (4) while a timelike \( f \) stands for \( V \) of (9).

Let us now discuss the geometrical constructions behind the basic equations (12, 13 and 14). With a more transparent notation, (12) may be written, as

\[ (\Delta t)^2 = (\Delta \tau)^2 + (\Delta \vec{x})^2, \] 
which defines a 4-dimensional hypercone in the local tangent space of \( R_5 \). It is a P-CAUSALITY-CONE, with vertex at a point \( P \), a generalization of the Minkowski light-cone. A light-cone, the domain of massless physical objects, is an intersection of a causality-cone and a 4-dimensional hyperplane defined by \( x^5 = \text{const} \). The interior of a light-cone is the projection of a causality-cone on such a \( (x^5 = \text{const}) \)-hyperplane. Each observer perceives an strictly \((1 + 3)\)-dimensional world and his \( \Delta x^5 \) coincides with the elapsed time measured on his own clock, as required by special relativity; it represents his aging, according to his own clock.

Eq. (13) may be written as \( f_M dx^M = 0 \), with \( f_M := (f'_\mu, f_5 = 1) \). For a fixed point \( P \), it defines a family of 4-dimensional hyperplanes parameterized by \( f'_\mu \), which is both, normal to the hyperplane and tangent to the P-causality-cone (15). On the other hand, \( \bar{f}^M := (\bar{f}^\mu, 1) \) where \( \bar{f}^\mu = (f^0, -f^i) \) is the common (intersection) 5-vector, tangent to the P-causality-cone and to the hyperplane that passes by \( P \) and is orthogonal to \( f_M \). Both, the normal, \( f_M \), and the tangent, \( \bar{f}^M \), to the P-hyperplane, are tangent vectors to the P-causality-cone, at the point \( P \).

At the vertex of the causality-cone, both, the normal to the hyperplane, \( f'_\mu \), and its tangent on the cone, \( f^\mu \), represent two cone-generators, in opposition to each other. See the figure. They represent motions (4-velocities) in opposite space directions but with a same time direction. So, for a given \( f \), there are two solutions for the intersection of (15) and (13): generator \( f^\mu := (f^0, \vec{f}) \) and generator \( \bar{f}^\mu := (f^0, -\vec{f}) \). In section V they will be associated to the
description of creation and annihilation of particles.

Let us introduce the concept of **CAUSALITY-LINE** in contradistinction to the Minkowski worldline, which becomes the projection of the first one on a \((x^5 = \text{const})\)-hypersurface (that is, on its lightcone, or on its interior). The causality-line of a free physical object is a causality-cone generator; an object, under an interaction, may change of causality-line, defining then, a new causality-cone vertex. A causality-cone vertex is defined by the intersection of 3 generators and corresponds to a pictorial description of a Lagrange interaction term, like \(L = e\bar{\Psi}fA_f\Psi f'\), where, in this notation (to be better explained in section V), an incoming field in the generator \(f\), changes to another generator \(f''\), after emitting/absorbing a photon in a generator \(f'\). This picture of cones and generators is appropriate to the description of discrete interactions among point (localized) objects. We will return to this point in section V.

The directional derivative \(\nabla f\) has then a clear and natural meaning: it is the spacetime evolution of a physical object, allowed by its causality restrictions. Let \(A_f\) and \(B_{f'}\) be objects whose causality restrictions on their evolution are given, respectively, by \(f\) and \(f'\), then

\[
\partial_{\mu} \left. \left( \frac{A}{f_{M}.dx^M=0} \right| \frac{B}{f'_{M}.dx^M=0} \right) \equiv (\nabla_{\mu} A_f) B_{f'} + A_f (\nabla'_{\mu} B_{f'}),
\]

(16)

where \(\nabla_{\mu} := \partial_{\mu} - f_{\mu} \partial_{\tau} \) and \(\nabla'_{\mu} := \partial_{\mu} - f'_{\mu} \partial_{\tau}\.\) In other words, the label \(f\) in a directional derivative, like in \(\nabla f\), is defined by the causal restrictions of the object on which it is acting. There is no meaning on an expression like \(\nabla B_{f'}\), unless \(f' = f\).

The equations (3-9) have been replaced in this geometric description by (12-14), but now with a very significative difference: each of these equations involve parameters of just one physical object. For example, the change of \(\tau\) in (12) and in (13) for any given photon is always null. The electron and the electromagnetic field, in this respect, are being treated on a same foot; each one is defined on its appropriate causality-cone generator. This picture has deep physical consequences that are connected to fundamental problems of Quantum Mechanics and of Quantum Field Theory; they will be briefly discussed in section V, as they are outside the scope of this paper.
But it is the metric structure induced by (13) on the spacetime of a physical object that most clearly exposes the
departure from a Minkowski structure. We have from (11) and (13), that 
\[(dS_5)^2 = dx.\eta.dx - (f'.dx)^2 = dx.(\eta - f'f').dx,\]
for massive objects, and that 
\[(dS_5)^2 = dx.\eta.dx\]
for massless objects, as \(d\tau = f'.dx = 0\). Then, the induced
metric is given by

\[g_{\mu\nu} = \begin{cases} 
\eta_{\mu\nu} & \text{if } m = 0; \\
\eta_{\mu\nu} - f'_\mu f'_\nu & \text{if } m \neq 0.
\end{cases} \tag{17}\]

The distinct causality requirements of massive and of massless fields and particles are, therefore, represented by
immersions with distinct metric structures.

This represents a very strong inference: eq. (17) is saying that the change in the propertime of a physical object
contributes to the length of its causality-line. This contribution is materialized in a Riemannian manifold with the
metric (17). Observe that it could be in the other way, since the change in the propertime of a physical object
represents just its aging, and so why should it be taken into account in the computing of its relevant line element?

We could, just as well, assume that the physically relevant line element is given by the worldline, and not by the
causality-line; then, the above \(g_{\mu\nu}\) would be a meaningless concept. This would not be the most esthetics Nature’s
choice since it would spoil the beauty of this geometric picture, but it is indeed a question to be settled by experimental
verifications. We will return to this point in section IV.

While \(T^{\mu\nu}_{,\rho} = 0\) for \(\rho > 0\) remains valid in this new picture, its limit when \(\rho \to 0\) is not as simple as fore-assumed
because it may involve now a local change of manifolds with different metric structure (\(\eta \to \eta - ff\)). The existence of
two distinct metric structures for a massive and a massless field, which has not been considered in the determination
of the LDE is just one of the reasons that invalidates it as the electron equation of motion. As a matter of fact, we
will see in the following sections, that regardless the metric structure of the physical spacetime, be it Minkowskian or
Riemannian, just the understanding of this new geometric picture is enough to clear electrodynamics from some of its
inconsistencies and to show that the Lorentz-Dirac equation is not justified. However, \(g_{\mu\nu}\) and \(\eta_{\mu\nu}\) produce distinct
measurable effects, as shown in (2), that can, in principle, be detected in a laboratory.
III. THE INTEGRABLE STRESS TENSOR

We can have a better understanding of the meaning of this new geometry, studying the behaviour of the retarded Maxwell tensor, $F_{\mu \nu}^{\text{ret}}$, and of its stress-tensor, $\Theta_{\mu \nu}^{\text{ret}}$, both obtained from the retarded Lienard-Wiechert solution ($1$), and both singular at the electron world-line, $\rho = 0$, an immediate consequence of ($1$). In the standard or Minkowskian approach we write $\Theta_{\mu \nu}^{\text{ret}} = \Theta_2 + \Theta_3 + \Theta_4$, where the indices indicate the order of the respective singularities at $\rho = 0$. $\Theta_2$, although singular at $\rho = 0$, is nonetheless integrable. By that it is meant that $\int d^4x \Theta_2$ exists ($9$), while $\Theta_3$ and $\Theta_4$ are not integrable; they generate, respectively, the problematic Schott term in the LDE and a divergent expression, the electron bound 4-momentum ($12$). The most update prescription ($9$, $11$) is to redefine $\Theta_3$ and $\Theta_4$ at the electron worldline in order to make them integrable, but without changing them at $\rho > 0$, so to preserve the standard results of Classical Electrodynamics. This is possible with the use of distribution theory, but it is always an introduction of something strange to the system and in an ad hoc way. The most unsatisfactory aspect of this procedure is that it regularizes the above integral but leaves an unexplained and unphysical discontinuity in the flux of 4-momentum from the charge worldline: $\Theta(\rho = 0) \neq \Theta(\rho \sim 0)$. We show, in this section, that these problems have been solved: no discontinuity and no non-integrable singularity in the electron self field stress tensor.

For a unified treatment of the electron and of its electromagnetic field we introduce a parameter $\chi$:

$$\chi = \begin{cases} 0, & \text{if } \rho > 0; \\ 1, & \text{if } \rho = 0. \end{cases}$$

(18)

so that the tensor metric, after ($10$, $17$), may be written as

$$g_{\mu \nu} = \eta_{\mu \nu} - \chi V'_\mu V'_\nu,$$

(19)

with $g^{\mu \nu} = \eta^{\mu \nu} + \frac{1}{2} V'_\mu V'_\nu$ and $V'_\mu := \eta_{\mu \nu} V^n$, to be distinguished from $V_\mu = g_{\mu \nu} V^n = (1 + \chi) V'_\mu$.

$\chi$ has also the double role of pin pointing the contribution from the non-Minkowskian geometry on the electron world-line to its self-field, stress tensor, and equation of motion, as well as of allowing us to turn off the effects of the inference ($17$) by just taking $\chi = 0$, even for $\rho = 0$.

The metric ($19$) induces a covariant derivative, $D$:

$$D_\alpha g_{\mu \nu} := \nabla_\alpha g_{\mu \nu} - g_{\mu \beta} \Gamma^\beta_\nu_\alpha - g_{\beta \nu} \Gamma^\beta_\mu_\alpha = 0,$$
where $\Gamma^\beta_{\alpha\nu}$ is the Christoffel symbol, which is given by

\[ \Gamma^\beta_{\alpha\nu} = \frac{1}{2} \{ \partial_\alpha V^\beta_{\nu} + V^\beta_{\nu} (x^\alpha - a_{1}\alpha) - f^\beta (V^\nu_{\alpha}, a_{\alpha}) \}, \]

as $g^\mu_{\nu} f^\nu_{\mu} = f^\mu - \frac{\chi}{2} V^\mu$, $g^\mu_{\nu} V^\nu_{\mu} = V^\mu (1 - \frac{\chi}{2})$, $\chi^2 = \chi$, and $a = \frac{dV}{dx}$.

Therefore, from $f^\mu = \frac{(x - z)^\mu}{\Delta x}$ and $\Delta x = -V_\mu (x - z) = -\frac{1}{\chi V g} V g (x - z)$, we find that

\[ D_\alpha \rho = \nabla_\alpha \rho = \{ a f_\rho f_\alpha + f_\alpha - V_{\alpha} \}. \quad (20) \]

\[ D_\alpha A^\mu = -\frac{1}{\rho} \{(a^\mu + a f V^\mu) f_\alpha - (f_\mu - V_{\mu}) V^\mu + \frac{V^\beta}{\rho} \Gamma^\mu_{\alpha\beta} \}. \quad (21) \]

\[ D_\nu A_\mu = g_{\mu\beta} D_\nu A^\beta = -\frac{f_\nu a_{\mu}}{\rho} - \frac{1 + \chi}{\rho} V^\prime_{\nu} (f_\nu a_f + f_\nu - V^\nu) - \frac{\chi}{2 \rho} \{(a_{\mu} f^\nu_{\nu} + (a_{\mu}, V^\nu)\}; \quad (22) \]

as $V^\beta g_{\beta\mu} = (1 + \chi) V^\mu$, $f^\beta g_{\beta\mu} = f^\mu + \chi V^\nu$, $a^\beta g_{\beta\mu} = a_{\mu}$, $g_{\nu\beta} V^\alpha \Gamma^\beta_{\alpha\beta} = -\frac{\chi}{2} \{(a_{\mu} f^\nu_{\nu} + (a_{\mu}, V^\nu)\}$ and where we are using the short notation $\{ A, B \} := AB + BA$, $[A, B] := AB - BA$ and $a_f := a f$.

The retarded Maxwell field, $F_{\mu\nu \text{ret}} := D_\nu A_\mu$, is given by

\[ F_{\mu\nu \text{ret}} = \frac{1 + \chi}{\rho} \{(f_\mu + V^\nu), a_{\nu} + a_f V^\nu \} = (1 + \chi) \{ \frac{1}{\rho} \{ f'_{\mu}, a_{\nu} \} + \frac{a_f}{\rho} \{ f'_{\mu}, V^\nu \} + \frac{[f'_{\mu}, V^\nu]}{\rho^2} \}; \quad (23) \]

\[ F^\mu_{\nu \text{ret}} = \frac{1}{\rho^2} \{ [f^\mu - \frac{\chi}{2} V^\mu, (1 + \chi) a'_{\nu} \rho + V^\nu (1 + \rho a_f) \}. \quad (24) \]

It is more convenient to work with covariant indices. So, in this paper, where not explicitly shown, as in (25-28), below, we will be working with covariant indices.

Using (28) in $4\pi \Theta_{\mu\nu} = F_{\alpha\beta} g^{\alpha\beta} F_{\alpha\nu} - g_{\mu\nu} F^{\alpha\beta} F_{\alpha\nu}$, we find

\[ 4\pi \rho^2 \Theta = (1 + 3 \chi) \{ [f', \rho a + V'(1 + \rho a_f)], g_{\nu}[f', \rho a + V'(1 + \rho a_f)] - \frac{g}{4} [f', \rho a + V'(1 + \rho a_f)]^2 \}, \quad (25) \]

or $\Theta = \Theta_2 + \Theta_3 + \Theta_4$, with

\[ 4\pi \rho^2 \Theta_2 = (1 + 3 \chi) \{ [f', a + V a_f], g_{\nu} [f', a + V a_f] - \frac{g}{4} [f', a + V a_f]^2 \}, \quad (26) \]

\[ 4\pi \rho^2 \Theta_3 = (1 + 3 \chi) \{ [f', V'], g_{\nu} [f', a + V a_f] + [f', a + V a_f], g_{\nu} [f, V'] - \frac{g}{2} T r [f', V'], g_{\nu} [f', a] \} \quad (27) \]

\[ 4\pi \rho^2 \Theta_4 = (1 + 3 \chi) \{ [f', V'], g_{\nu} [f', V'] - \frac{g}{2} [f', V']^2 \}, \quad (28) \]

It is worth to explicitly write (28) and make some comments.

\[ 4\pi \rho^2 \Theta_{2 \mu\nu} = (1 + \chi) \{ -f'_{\mu} f'_{\nu} (1 + \chi) a^2 - a_j^2 \} + (f'_{\mu}, T_{\nu}) a_f \chi - T_{\mu} T_{\nu} \{ \chi + f^2 (1 + \chi) \} - \frac{g_{\mu\nu}}{2} \{ \chi (a_f^2 - a^2) - f^2 (1 + \chi) a_f^2 - a_f \}. \quad (29) \]
\[4\pi \rho^4 \Theta_{3 \mu\nu} = (1 + \chi) \left\{ 2 f'_\mu f'_\nu \mathcal{A}_f - (f'_\mu, T_\nu) + \chi \mathcal{A}_f (V'_\mu, f'_\nu) - (V'_\mu, T_\nu) \left\{ (\chi + (1 + \chi) f^2) - g_{\mu\nu} f^2 \mathcal{A}_f^2 \right\} \right\}, \quad (30)\]

\[4\pi \rho^4 \Theta_{4 \mu\nu} = (1 + \chi) \left\{ f'_\mu f'_\nu - (f'_\mu, V'_\nu) - V'_\mu V'_\nu \left\{ (1 + \chi) f^2 \right\} - \frac{g_{\mu\nu}}{2} (1 + f^2) \right\}, \quad (31)\]

where \( T_\mu \) is a short notation for \( \mathcal{A}_\mu + V'_\mu \mathcal{A}_f \); then \( f.T \equiv 0 \).

For \( \chi = 0 \) (or, equivalently, \( \rho > 0 \)), (29-31), with \( f^2 = 0 \), coincide with their well known expressions\(^{10} \) in the usual formalism. In particular, we have

\[
f_\mu \Theta^{\mu\nu}_2 \bigg|_{\rho > 0} = f_\mu \Theta^{\mu\nu}_3 \bigg|_{\rho > 0} = 0
\]

as it must be for a radiation term. \( \Theta_2 \) is the radiated portion of \( \Theta \).

But the terms with \( f^2 \), even with \( f^2 = 0 \), cannot be dropped from the above equations, since they are necessary for producing the correct limits when \( \rho \to 0 \), or \( x \to z \). As \( f^\mu := \frac{(x-z)^\mu}{\Delta \tau} = \frac{R^\mu}{\rho} \), in this limit we have a \( 0/0 \)-type of indeterminacy, which can be raised with the L’Hospital rule and \( \frac{\partial}{\partial \tau} \) (more specifically, \( \frac{\partial}{\partial \tau_x} \), as \( \Delta \tau = \tau_x - \tau_z \), and so, \( \frac{\partial \Delta \tau}{\partial \tau_x} = -1 \), \( \frac{\partial R}{\partial \tau} = -V \), etc). Therefore, \( \lim_{R \to 0} f = V \), and

\[
\lim_{R \to 0} f'_\mu f^\mu = \lim_{R \to 0} \frac{R.\eta.R}{\rho^2} = -1.
\]

To find the limit of something when \( \rho \to 0 \) will be done so many times in this paper that it better be systematized. We want to find

\[
\lim_{R \to 0} \frac{N(R)}{\rho^n},
\]

where \( N(R) \) is a homogeneous function of \( R \), \( N(R) \bigg|_{R=0} = 0 \). Then, we have to apply the L’Hospital rule consecutively until the indeterminacy is resolved. As \( \frac{\partial \rho}{\partial \tau} = -(1 + \mathcal{A}.R) \), the denominator of \( \frac{\partial}{\partial \tau} \) at \( R = 0 \) will be different of zero only after the \( n^{th} \)-application of the L’Hospital rule, and then, its value will be \( (-1)^n n! \)

If \( p \) is the smallest integer such that \( N(R)_p \bigg|_{R=0} \neq 0 \), where \( N(R)_p := \frac{\partial^p}{\partial \tau^p} N(R) \), then

\[
\lim_{R \to 0} \frac{N(R)}{\rho^n} = \begin{cases} 
\infty, & \text{if } p < n \\
(-1)^n \frac{N(R)_p}{n!}, & \text{if } p = n \\
0, & \text{if } p > n
\end{cases}
\]

(34)
Example 1: \[
\begin{aligned}
f &= \frac{R}{\rho}, \\
f^2 &= \frac{R^2}{\rho^2},
\end{aligned}
\]
\[n = p = 1 \implies \lim_{R \to 0} f = V \]
\[n = p = 2 \implies \lim_{R \to 0} f^2 = -1.\]

Example 2: \[
\begin{aligned}
\frac{[f', \mathcal{A}_\mu]}{\rho} &= \frac{|R, \mathcal{A}_\mu|}{\rho^2} \\
&\implies p = 1 < n = 2 \implies \lim_{R \to 0} \frac{[f', \mathcal{A}_\mu]}{\rho} \text{ diverge}
\end{aligned}
\]

Example 3: \[
\begin{aligned}
\frac{[f, \mathcal{V}]}{\rho} &= \frac{|R, \mathcal{V}|}{\rho^2} \\
&\implies p = 4 > n = 3 \implies \lim_{\rho \to 0} \frac{[f, \mathcal{V}]}{\rho} = 0
\end{aligned}
\]

Example 4: \[
\begin{aligned}
\frac{[f, \mathcal{V}]}{\rho^2} &= \frac{|R, \mathcal{V}|}{\rho^2} \\
&\implies p = 2 < n = 3 \implies \lim_{R \to 0} \frac{[f, \mathcal{V}]}{\rho^2} \text{ diverge}
\end{aligned}
\]

The second term of the RHS of (23) does not contribute to the electron self-field at \(\rho = 0\), but the first and the third terms diverge, as expected, although they produce integrable contributions to the electron self field stress tensor, as we show now. Let us find the integral of the electron self field stress tensor at the electron causality-line: \(\lim_{\rho \to 0} \int dx^4 \Theta\), or \(\lim_{\rho \to 0} \int d\tau d\rho^2 d\rho d\Omega \Theta\), in terms of retarded coordinates\(^9,13,14\), \(x^\mu = z^\mu + \rho f^\mu\), where \(d^2\Omega\) is the element of solid angle in the charge rest-frame.

But, we will first prove a useful result from (34), when the numerator has the form \(N_0 = A_0 g_0 B_0\), where \(A\) and \(B\) represent two possibly distinct homogeneous functions of \(R\), and the subindexes indicate the order of \(\frac{d}{d\tau}\). Then

\[N_1 = (A_1 g_0 B_0 + A_0 g_1 B_1) + A_0 g_1 B_0;\]
\[N_2 = (A_2 g_0 B_0 + 2A_1 g_1 B_1 + A_0 g_2 B_2) + (A_1 g_1 B_0 + A_0 g_1 B_1) + A_0 g_2 B_0;\]

and, generically,

\[N_p = \sum_{a=0}^{p} \binom{p}{a} A_{p-a} g_0 B_a + \sum_{a=0}^{p-1} \binom{p}{a} A_{p-a-1} g_1 B_a + \ldots = \sum_{b=0}^{p-b} \sum_{a=0}^{p-b} \binom{p-b}{a} A_{p-a-b} g_b B_a\]

As, for the cases we are considering in this paper, the metric \(g\) is independent of \(R\), \(b\) in (34) will always be zero. So, for applying (34) in this case, we just have to find the \(\tau\)-derivatives of \(A\) and \(B\) that produce the first non-null term at the limit of \(R \to 0\).

Applying (34) and (35) for finding \(\lim_{\rho \to 0} \int dx^4 \Theta\) we just have to consider the first term of the RHS of (25); the second one, as the trace of the first, will have a similar behaviour.

Example 5

\[
\lim_{\rho \to 0} \frac{\rho^2 [f', \rho \mathcal{A} + V'(1 + \rho \mathcal{A}_f)] g_0 [f', \rho \mathcal{A} + V'(1 + \rho \mathcal{A}_f)]}{\rho^4} = \]

13
\[ \lim_{\rho \to 0} \frac{[R, \rho \dot{a} + V'(1 + \rho \dot{a} R)] \cdot g \cdot [R, \rho \dot{a} + V'(1 + \rho \dot{a} R)]}{\rho^4} \]

As

\[ A_0 = B_0 = [R, \rho \dot{a} + V'(1 + \rho \dot{a} R)] \implies A_2 = B_2 = [\dot{a}, V'] + O(R) \]

Therefore, according to (35), for producing a non null \( N_p \), \( a \) and \( p \) (as \( b = 0 \)) must be given by

\[ p - a = a = 2 \implies p = 4 = n \implies N_4 = 6[\dot{a}, V'].g.[\dot{a}, V'] + O(R). \]

So, from (34),

\[ \lim_{\rho \to 0} \int dx^4 \Theta = \lim_{\rho \to 0} \int dx^4 \Theta = \frac{g}{4}[\dot{a}, V']^2 \]

Therefore, from (27), with \( \chi = 1 \),

\[ \lim_{\rho \to 0} \int dx^4 \Theta = [\dot{a}, V'].g.[\dot{a}, V'] - \frac{g}{4}[\dot{a}, V']^2 \]

It is interesting that (36) comes entirely from the velocity term, \( \frac{f'^2}{\rho^4} \), as we can see from the following example.

\textbullet \ Example 6

\[ \lim_{\rho \to 0} \frac{\rho^2 [f', \rho \dot{a} + V'(1 + \rho \dot{a} R)] \cdot g \cdot [f', \rho \dot{a} + V'(1 + \rho \dot{a} R)]}{\rho^4} = \frac{1}{4}[\dot{a}, V'].g.[\dot{a}, V'] \]

as \( A_2 = B_2 = [\dot{a}, V'] + [R', \dot{a}] \implies p - a = a = 2 \implies N_4 = 6[\dot{a}, V'] + O(R) \) and \( p = n = 4 \). So,

\[ \lim_{\rho \to 0} \int dx^4 \Theta = \lim_{\rho \to 0} \int dx^4 \Theta_4 \]

The other contributions just cancel to zero,

\[ \lim_{\rho \to 0} \int dx^4 \Theta_2 = - \lim_{\rho \to 0} \int dx^4 \Theta_3. \]

IV. THE NEW EQUATION OF MOTION

Let us now derive the electron equation of motion, which can be obtained from

\[ \lim_{\varepsilon \to 0} \int dx^4 D_{\nu} T^{\nu \nu} \theta (\rho - \varepsilon) = 0, \]
where \( T^{\mu\nu} \) is the total electron energy-momentum tensor, which includes the contribution from the electron kinetic energy, from its interaction with external fields and from its self field. Let us move directly to the part that will produce novel results:

\[
m \int a^\mu d\tau = \int F_\text{ext}^\mu d\tau - \lim_{\varepsilon \to 0} \int dx^4 D_\nu \Theta^{\mu\nu} \theta(\rho - \varepsilon), \tag{37}
\]

where \( F_\text{ext}^\mu \) is the external forces acting on the electron, and the last term represents the impulse carried out by the emitted electromagnetic field in the Bhabha tube surrounding the electron worldline, defined by the Heaviside function, \( \theta(\rho - \varepsilon) \). Using \( D_\nu \Theta^{\mu\nu} = \nabla_\nu \Theta^{\mu\nu} + \Theta^{\alpha\nu} \Gamma^{\mu}_{\alpha\nu} \), as \( \Gamma^{\mu}_{\alpha\nu} = 0 \), and the divergence theorem, we have that the last term of the RHS of (37) is transformed into

\[
\lim_{\varepsilon \to 0} \int dx^4 \left\{ \Theta^{\nu\mu} \Gamma^{\mu}_{\nu\rho} \theta(\rho - \varepsilon) - \Theta^{\mu\nu} \nabla_\nu \rho \delta(\rho - \varepsilon) \right\}. \tag{38}
\]

The last term of this equation represents the flux of 4-momentum through the cylindrical hypersurface \( \rho = \varepsilon \). Let us denote it by \( P^\mu \):

\[
P^\mu = \lim_{\varepsilon \to 0} \int dx^4 \Theta^{\mu\nu} \nabla_\nu \rho \delta(\rho - \varepsilon), \tag{39}
\]

As \( \nabla_\nu \rho = \rho \partial f f'_\nu + f'_\nu - V'_\nu \), and \( \Theta = \Theta_2 + \Theta_3 + \Theta_4 \), we can write \( P^\mu := \tilde{P}_0^\mu + \tilde{P}_1^\mu + \tilde{P}_2^\mu \), with

\[
\tilde{P}_2^\mu = \lim_{\varepsilon \to 0} \int dx^4 \Theta^{\mu\nu} (f' - V')_\nu \delta(\rho - \varepsilon), \tag{40}
\]

\[
\tilde{P}_1^\mu = \lim_{\varepsilon \to 0} \int dx^4 \{ \Theta^{\mu\nu} f'_\nu \rho \partial f + \Theta^{\mu\nu} (f' - V')_\nu \} \delta(\varepsilon - \rho), \tag{41}
\]

\[
\tilde{P}_0^\mu = \lim_{\varepsilon \to 0} \int dx^4 \{ \Theta^{\mu\nu} f'_\nu \rho \partial f + \Theta^{\mu\nu} (f' - V')_\nu \} \delta(\rho - \varepsilon), \tag{42}
\]

\( \tilde{P}_1^\mu \) and \( \tilde{P}_2^\mu \) are both null. In order to show this we need to apply (34) for a \( N(R) \) that has a generic form, \( N = A.g.B C \), where \( A, B, \) and \( C \) are functions of \( R \), such that \( N(R = 0) = 0 \). Then it is easy to show, from (34), that

\[
N_p = \sum_{b=0}^{p} \sum_{a=0}^{p-b} \sum_{c=0}^{a} \left( \begin{array}{c} p - b \\ a \\ c \end{array} \right) A_{p-a-b} . g_b . B_{a-c} C_c \tag{43}
\]

• From (28), the integrand of (40), produces (again, we do not need to consider the trace term)

\[
\lim_{\rho \to 0} \rho^2 [f', V'] . g . [f', V'] . (f - V) = \lim_{\rho \to 0} \rho^4 [R, V'] . g . [R, V'] . g . (R - \rho V),
\]

15
or, schematically

\[ \lim_{\rho \to 0} \frac{A \cdot g \cdot A C}{\rho^5} \]

with \( A_0 = B_0 = [R, V] \), and \( C_0 = (R - V \rho) \). Then, \( A_2 = [a, V] + \mathcal{O}(R) \), \( C_2 = a + \mathcal{O}(R) \), and we have, from (43), the following restrictions on \( a, b, c \) for producing a \( N(R = 0) \)_\( \rho \neq 0 \):

- \( c = 2 \); \( b = 0 \) (always, as \( g \) does not depend on \( R \)); \( a - c = 2 \); and \( p - a = 2 \) or \( p = 6 > n = 5 \). Therefore, according to (43)

\[ P_2^{\mu} = 0 \quad (44) \]

- From the second term of the integrand of (41) and from (27) we have

\[ \lim_{\rho \to 0} \frac{\rho^2[f', V'].g.[f', a + V'a f](f - V)}{\rho^3} = \lim_{\rho \to 0} \frac{[R, V'].g.[R, \rho \ a + V'a R](R - \rho V)}{\rho^5} . \]

So, from (43) with

\[ C_0 = R - V \rho \implies C_2 = a + \mathcal{O}(R) \implies c = 2. \]
\[ B_0 = [R, V] \implies B_2 = [a, V] + \mathcal{O}(R) \implies a = 4. \]
\[ A_0 = [R, a] \implies A_1 = [a, V] + \mathcal{O}(R) \implies p = 5 = n. \]

But \( [a, V], g, [a, V] \), \( a = (1 - \chi^2)a^2a \) and it is cancelled by the contribution from the trace term of (27). Therefore,

\[ \lim_{\rho \to 0} \rho^2 \Theta^{\mu \nu}_g (f' - V')_{\nu} = 0 \]

- From the first term of the integrand of (41) and from (28) we have

\[ \lim_{\rho \to 0} \frac{\rho^2[f', V'],g.[f', V']g f' \rho a f}{\rho^4} = \lim_{\rho \to 0} \frac{[R, V'],g.[R, V']g R a R}{\rho^5} . \]

So, from (43), with

\[ C = C_0 = g g R \ a \ R \implies C_3 = -3V a^2 + \mathcal{O}(R) \implies c = 3. \]
\[ A = B = B_0 = [R, V] \implies A_2 = B_2 = [a, V] + \mathcal{O}(R) \implies a = 5 \text{ and } p = 7 > n = 5. \]
Therefore, from (34)

\[
\lim_{\rho \to 0} \rho^2 \Theta_4^{\mu\nu} f'_\nu \rho \, \alpha_f = 0
\]

Consequently

\[
P_1^\mu = 0 \tag{45}
\]

and

\[
P^\mu = P_0^\mu \tag{46}
\]

\(P_0^\mu\) is distinguished from \(P_1^\mu\) and \(P_2^\mu\) for not being \(\rho\)-dependent. This has 3 important implications:

- It is not necessary to use the L’Hospital rule on its determination;
- It is, therefore, not affected by the limit of \(\varepsilon \to 0\);
- The presence of \(\delta(\rho - \varepsilon)\) with \(\varepsilon > 0\) requires \(\chi = 0\),

which is in agreement with (32). The physical meaning of this is that the flux of 4-moment through the cylindrical surface \(\rho = \varepsilon\) comes entirely from the photon field and it requires, therefore, \(\chi = 0\). But we will keep \(\chi\), as \(\chi'\) through the calculation just to see what would be its contribution to \(P_0^\mu\). We just have to make \(\chi' = 0\), at the end. It is convenient now to introduce a spacelike 4-vector \(N\), defined by \(N^\mu := (f - V)^\mu\) and such that \(N.\eta.V = N.g.V = 0\) and \(N.g.N = N.\eta.N = 1\). \(N\) satisfies

\[
\frac{1}{4\pi} \int d\Omega \, N \cdots N = 0 \tag{47}
\]

and

\[
\frac{1}{4\pi} \int d\Omega NN = \frac{\eta + VV}{3} \tag{48}
\]
as can be found, for example, in references [4,13] or in the appendix B of reference [10]. From (27) we have

\[
\int d^4x \Theta_3^{\mu\nu} f'_\nu \rho \alpha_f = \frac{\chi'}{4\pi} \int d\tau d^2\Omega (\alpha N)^2 N^\mu = 0 \tag{49}
\]

From (29), we have

\[
4\pi \rho^2 \Theta_2^{\mu\nu} N_\nu = (1 + \chi') \left((\alpha N)^2 - \Delta^2\right) \left(V^\mu + (1 + \chi') N^\mu\right), \tag{50}
\]
which, with (47) and (48), gives
\[
\lim_{\rho \to 0} \int d^4x \Theta_2^{\mu \nu} N_{\mu} = - \int d\tau \frac{2(1 + \chi')}{3} a^2 V^\mu. \tag{51}
\]
Then, from (51), (49) and (46), we have
\[
P^\mu = - \int d\tau \frac{2(1 + \chi')}{3} a^2 V^\mu, \tag{52}
\]
As we have to make \(\chi' = 0\), it is just the Larmor term. It is interesting that the contribution from a \(\chi \neq 0\) would be just to make it to be twice its usual value.

In order to get to the electron equation of motion we still need to work out the first part of (38). In contradistinction to the second part, this one comes multiplied by \(\Theta(\rho - \varepsilon)\) and not by \(\delta(\rho - \varepsilon)\). Therefore, it has no part that is independent of \(\rho\) (and of \(\chi\)). Its limit when \(\rho \to 0\) can be evaluated with (43) and with \(g^{\rho \alpha}g^{\sigma \beta} \Gamma_{\alpha \beta}^{\mu} = \chi (\alpha^\mu(\nu, V^\sigma) - f^\mu(\nu, V^\sigma) + V^\mu(\alpha^\nu, f^\sigma) - \alpha^\mu V^\nu V^\sigma)\):

\[
\lim_{\rho \to 0} \rho^2 \frac{\rho [f', \rho a + V'(1 + \rho a)]g.[f', a + V'(1 + \rho a)] \left( a(f, V) + V(a, f) - f(a, V) - \alpha V V \right) \chi}{\rho^4} = \lim_{\rho \to 0} \frac{[a V + V(1 + a) \rho].g.\rho [a + V'(1 + a) \rho] \left( a(a, V) + V(a, \rho V) - R(a, V) - \rho \alpha V V \right) \chi}{\rho^5},
\]
with
\[
C_0 = a(R, V) + V(a, R) - R(a, V) - \rho a V V \implies C_1 = -\alpha a V V + O(R) \implies c = 1,
\]
and, like in the example 5,
\[
A_2 = B_2 = [a, V'] + O(R) \implies a - c = 2 \quad \text{and} \quad p - a = 2 \implies a = 3 \quad \text{and} \quad p = 5 = n
\]
Therefore,
\[
N(R = 0) = \frac{1}{5!} \left( \begin{array}{c} 5 \\ 3 \\ 1 \end{array} \right) \chi (a, V')g.(a, V')\chi(VV)a \frac{\chi}{4}.
\]
Then,
\[
\lim_{\varepsilon \to 0} \int dx^4 \Theta^\alpha\Gamma^\mu_{\alpha\nu} \theta(\rho - \varepsilon) = \frac{(1 + 3\chi)}{2} \int d\tau \left( [a_\alpha, V_\rho]g^{\rho\sigma} [a_\sigma, V_\beta] - g_{\alpha\beta} \frac{[a_\alpha, V_\beta]^2}{4} \right) \frac{\chi}{4} V^\alpha V^\beta a^\mu = -\frac{\chi}{4} \int d\tau \dot{a}^2 a^\mu. \tag{53}
\]

This term corresponds to the energy associated to the electron manifold curvature. Observe that it is proportional to \(\chi\).

Finally, from (37), (38), (39), (52) and (53), we can write the electron equation of motion, obtained from the Lienard-Wiechert solution, as

\[
(m - \frac{1}{4} \dot{a}^2)\dot{a}^\mu = F_{\mu}^{\text{ext}} - \frac{2a^2}{3} V^\mu, \tag{54}
\]

The external force provides the work for changing the charge velocity, for the energy dissipated by the radiation, and for the curvature of the electron manifold. The correctness of the presence of the \(a^2 \dot{a}^\mu\) term in the above equation must be decided through experimental means. It can, in principle, be detected in synchrotron accelerators, despite the difficulties caused by the smallness of \(a^2\) face the experimental uncertainties\(^{(15)}\). Indirect evidences would be the analysis of the theoretical implications of this term in (54). This is left for future works.

**Energy conservation**

Eq. (54) is non-linear, like the Lorentz-Dirac equation, but it does not contain the Schott term, the responsible for its spurious behaviour. This is good since it signals that there will be no problem with causality violation. But the Schott term in the LDE has also the role of giving the guaranty of energy conservation, which is obviously missing in (54). Assuming that the external force is of electromagnetic origin, \((F_{\mu}^{\text{ext}} = F_{\mu}^{\text{ext}} V^\nu)\), the contraction of V with eq. (34) would require a contradictory \(a^2 \equiv 0\). But this is just an evidence that (54) cannot be regarded as a fundamental equation. It would be better represented as

\[
m\dot{a}^\mu - \frac{1}{4} a^2 a^\mu = F_{\mu}^{\text{ext}} - \frac{2}{3} a^2 V^\mu, \tag{55}
\]

with

\[
\frac{2}{3} a^2 V^\mu = P^\mu = \lim_{\varepsilon \to 0} \int dx^4 \Theta^{\alpha\nu} \nabla_\nu \theta(\rho - \varepsilon),
\]

\[
\frac{1}{4} a^2 \dot{a}^\mu = \lim_{\varepsilon \to 0} \int dx^4 \Theta^{\alpha\nu} \Gamma^\mu_{\alpha\nu} \theta(\rho - \varepsilon).
\]
It is just an effective or average result, in the sense that the contributions from the electron self field must be calculated, as in (57), by the electromagnetic energy-momentum content of a spacetime volume containing the charge causality-line, in the limit of $\rho \to 0$:

$$m \int \mathbf{a} \cdot \mathbf{V} \, d\tau = \int F_{\text{ext}} \cdot \mathbf{V} \, d\tau - \lim_{\varepsilon \to 0} \int dx^4 f_\mu D_\nu \Theta^{\mu\nu} \theta(\rho - \varepsilon). \quad (56)$$

Observe that in the last term, $\mathbf{V}$, the speed of the electron, is replaced by $f$ the speed of the electromagnetic interaction; only in the limit of $R \to 0$ is that $f \to V$. We have to repeat the same steps from (37) to (54) in order to calculate this last term and to prove (done in the appendix) that it is null:

$$\lim_{\varepsilon \to 0} \int dx^4 f_\mu D_\nu \Theta^{\mu\nu} \theta(\rho - \varepsilon) = 0, \quad (57)$$

So, there is no contradiction anymore. Besides, it throws some light on the meaning of $F_{\mu\nu}$, as we discuss in the following section.

V. THE MEANING OF THE MAXWELL’S FIELDS

With this new spacetime structures we have eliminated the non-integrable singularities and understood the problems with conservation of energy and causality violation that plague the old LDE, but there is still a remaining problem. How to conciliate this with the unboundness of $\Theta^{\mu\nu}$ as $\rho \to 0$? This remaining singularity comes from (1), the solution of the wave equation

$$\square A^\mu = J^\mu, \quad (58)$$

of the Minkowskian formalism. We must realize now, that we are still giving a dichotomic treatment to the electron and to its field. The electron is treated as a classical particle, that is, as a localized object in a well defined trajectory. When we use directional derivatives (14) and the induced metric (17), we are requiring that the electron follows a causality-cone generator of its (instantaneous) causality-cone, which corresponds to a local (point to point) causality implementation. But the LWS (1) describes a wave front propagating in the entire light-cone (not just one of its generator, as for the electron) and this corresponds to a global causality implementation.

By local implementation of causality we refer to the restriction on physical objects to remain in a causality-cone-generator, so that we know its location, point after point; while a global implementation of causality just requires that
a physical object be restricted to a causality cone, or in Minkowski term, on a light-cone if it is massless, or inside a light-cone, if it is massive. One could argue that this global characteristic is something intrinsic to the very nature of a field. But it does not have to be this way. The concept of field, as a distributed object, can be compatible with the concept of photon, a localized object, if the field equations have soliton-like solutions.

As far as the actually fundamental electromagnetic interaction is the one represented by the exchange of a single photon, it cannot be represented by the Maxwell electromagnetic field. It represents rather a kind of effective or average interaction. Consider, for example, the definitions contained in the Coulomb’s law and in the Gauss’s law. While the Coulomb’s law defines the observed electric force between two point charges as a vector physical manifestation acting on each charge, and that exists only in the direction given by the straight line that passes by them, the Gauss’s law describes the inferred electric field as existing around a single charge, independent of the presence of the other charge.

The electric field, as it is well known, is extracted from the Gauss’s law through the integration of its flux across a surface, having the appropriate symmetry, enclosing the charge,

\[ \vec{E}(x) = \hat{e} \int_{V} \rho dV \int_{\partial V} dS. \]  

(59)

The eq. (59) puts in evidence the effective or average character of the Maxwell’s concept of field; it gives also a hint on the meaning and origin of the field singularity. If the electric field can be visualized in terms of exchanged photons, then according to (59), the frequency or the number of these exchanged photons must be proportional to the enclosed net charge. And if we take \( \vec{E} \), as suggested by the Gauss’ law, as a measure of the average number of photons emitted/absorbed by a point charge, we can schematically write, \( E \sim \frac{n}{4\pi r^2} \), where \( n \) is the number of photon per unit of time crossing an spherical surface of radius \( r \) and centred on the charge. Then, the unbounded dependence of \( E \) with \( r \) when \( r \to 0 \) does not represent a physical fact like an increasing number of photons, but just an increasing average number of photons per unit area, as the number of photons remains constant but the area tends to zero. So, a field singularity would have no physical meaning, it would just be a consequence of this average nature of the Maxwell’s field.

In order to give a consistent classical treatment to the electron and to its field in this geometry, we have to consider the classical electromagnetic field as described by a punctiform classical photon propagating along a light-cone generator. Let us represent this classical photon by \( A^{\mu}(x) f \), where \( f \) labels its light-cone generator, its causality-line. This label
f implies that the photon is causally constrained to remain in this f-generator, that is

$$\partial_\mu A_f \bigg|_{d\tau + f.d\mathbf{x} = 0} \equiv \nabla^f_\mu A_f.$$  \hspace{1cm} \text{(60)}$$

Its wave equation is then

$$\Box_f A^f_\mu := \nabla^f_\alpha \eta^{\alpha\beta} \nabla^f_\beta A^f_\mu = 4\pi J_\mu,$$  \hspace{1cm} \text{(61)}$$

which corresponds to the replacement, in the standard equation, of the usual derivative, $\partial_\mu$, by the directional derivative along the f-causality-cone generator, $\nabla^f_\mu$.

The differences between (58) and (61) are very significative. The field $A(x)$ represents a distributed wave propagating along every direction; the operators $\partial_x$, $\partial_y$, $\partial_z$, and $\partial_t$ has, each one, an equal importance and are equally weighted in $\Box$. $A^\mu(x)_f$ represents a localized object propagating along a generator f; only the derivative along f is important and is considered in $\Box_f$. The use of $A(x)$ constitutes an average or statistical approach, considering that the electromagnetic wave consist of a huge number of photons, each one propagating along a null direction f, and described by an $A^\mu(x)_f$.

We can expect then, that an integration of $A^\mu(x)_f$ over all the generators of its light-cone reproduces the Minkowskian $A^\mu(x)$,

$$A(x) = \frac{1}{4\pi} \int d\Omega_f A(x, \tau, f),$$  \hspace{1cm} \text{(62)}$$

and that, the same integration over (61) reproduces (58). This is possible because $A_f$ has an even\(^{(2)}\) functional dependence on f, and so, the f-odd term in (61) is integrated out to zero, by symmetry ($\int d\Omega_f f \equiv 0$).

The equation (61) is solved, in reference\(^{(2)}\), for a fixed and constant f, by the following Green function,

$$G(x, \tau)_f = \frac{1}{2} \theta(-af.\Delta x) \theta(-a\bar{f}.\Delta x) \delta(\Delta \tau + f.\Delta x),$$  \hspace{1cm} \text{(63)}$$

where $a = \pm 1$.

$G(x, \tau)_f$ and its formalism have some remarkable properties to be fully discussed elsewhere\(^{(2)}\) and from which we underline:

1. It is singularity-free. In contradistinction to (64), below, the argument of the delta function in (63) is linear in $\Delta x$, which shows that the “classical photon” field propagates without changing its amplitude and its shape, like a soliton or a classical particle;
2. The Dirac’s delta, $\delta(\Delta \tau + f.\Delta x)$, together with the overall global causality-constraint (3) implies that the photon, after being emitted by a charge at their common causality-cone vertex, freely propagates along a $f$-generator of this causality-cone. Even though the physical description remains in $(3 + 1)$ dimensions, the field dynamics, described through (63), is essentially $(1 + 1)$ dimensional, along the space direction of propagation of the exchanged photon;

3. It is conformally invariant, which is the appropriate symmetry (not shared by the usual Green function, (64) below) for describing the photon propagation. It is free of scale dependent parameters ($f^2 = 0$);

4. The constraint (13) replaces and generalizes (3). Both sides of $(\Delta \tau)^2 = (\Delta t)^2 + (\Delta \vec{x})^2$ are invariant under transformations of the $SO(3,1)$, the invariance group of a Minkowski spacetime $(\vec{x}, t)$, but in $(\Delta t)^2 = (\Delta \tau)^2 + (\Delta \vec{x})^2$, both sides are invariant(1) under transformations of $O_4$, the rotation group of a 4-dimension Euclidean spacetime $(\vec{x}, \tau)$. The change of $(\vec{x}, t)$ to $(\vec{x}, \tau)$ is a Wick rotation(16,17) without the need of an uncomprehensible imaginary time. Physically it means that, for an Euclidean 4-dimensional spacetime, events should be labelled not with the time measured in the observer clock, but with their propertime, measured on their local clocks. $O_4$ is the causality-cone invariance under rotation around its t-axis. Care must be taken with the interpretation of transformations that corresponds to rotation of the $\tau$-axis, as they involve Lorentz and conformal transformations;

5. Following the recipe (62), the integration of $G(x, \tau)f$ over the $f$-directions, for $\Delta \tau = 0$ and for a fixed $x$, reproduces(2) the standard $G(x)$

$$G(x) = \frac{1}{4\pi} \int d\Omega_f \, G(x, \tau, f) = \frac{1}{2} \frac{\theta(at)}{r} \delta(r + at) = \theta(at) \frac{\theta(r^2 - t^2)}{2},$$

(64)

This shows that the singularity at $r = 0$ is not physical; it is merely a consequence of the average meaning of $G(x)$ and of $A(x)$, expressed in (62).

6. The $f$ of $G(x - y)f$, represents a generator of the causality-cone with vertex at $x$, and describes two types of propagating signals (see the figure):

(a) $\Delta x^0 = x^0 - y^0_1 > 0$. $G(x - y)f$ describes the propagation of a photon emitted at $y_1$, along $f$, and being observed at $x$. $J(y_1)$ is its source.
(b) $\Delta x^0 = x^0 - y_2^0 < 0$. $G(x - y) \tilde{f}$ describes the propagation of a photon that is being observed at $x$, propagating along $\tilde{f}$, and that will be absorbed by $J$ at $y_2$. $J(y_2)$ is its sink.

$y_1$ and $y_2$ are the intersections of $J(y)$, the charge causality-line with the $x$- causality-cone. The $f$’s of these two kinds of solutions are related as $f$ and $\tilde{f}$, as defined in section II. This process of creation and annihilation of particles in classical physics replaces the usual scheme of retarded and advanced solutions. All solutions are retarded and there is no causality violation.

7. The set of states, in the Quantum Mechanics language, labelled by $f$ and represented also by $[\mathcal{E}]$, in contradistinction to (64), constitutes a vector space; it obeys the Principle of Linear Superposition and is, therefore, appropriate for a quantum theory representation. This includes the additional hypothesis, required to fit the observational data, of adding the amplitudes of probability (not the probabilities), which is possible only in a vector space;

8. Considering the finite and macroscopic resolution of our measuring apparatus, and that the $f$-causality lines are unidimensional geometric objects, then, any determination of, let’s say, a photon $A_f$, is unavoidably made within an uncertainty window, $\Delta f$, that contains the uncertainties in the photon position and velocity. This lays a bridge to the Uncertainty Principle of Quantum Mechanics\(^{(2,6)}\);

9. A fundamental hypothesis in the derivation of (63) is that of a constant $f$. But $f$ is also the velocity of the propagating fields. Therefore, they freely propagate along a causality-cone generator and their interactions are discrete and localized at points, the causality-cone vertices. As explained in section II, a causality-cone vertex corresponds to an interaction Lagrangian term, like $\mathcal{L}_f = e \bar{\Psi} f A' \Psi'$. The localized and discrete radiation emission at a single point is the most essential and definitive quantum aspect of a field theory. This is a classical formalism with respect to dealing with classical fields instead of with operators, whereas it is already a quantum formalism with respect to interactions and field propagation.

10. It is interesting that (63) prohibits self interactions and vacuum fluctuations: its delta function requires that each field follow and remain strictly on their respective causality-lines, which never cross each other again. For example, an electron and its emitted photon follow different causality-lines, with respectively, $f^2 = -1$ and
\( f^2 = 0 \), and so the probability that this photon be reabsorbed by this electron is null. On the other hand, as the field equation have been changed by the use of directional derivatives \( (14) \), this may compensate for the absence of these renormalization effects. In other words, one may hope, that the exact solutions of the new field equations produce the exact number in effects like the Lamb shift and the electron anomalous magnetic moment, where, as well known, these quantum corrections have so well fitted the data.

11. Although the conformally invariant \( (33) \) cannot have any explicit dependence on any field mass, for \( \Delta \tau \neq 0 \), which is possible only for non-abelian fields, it describes the propagation of a massive field. The crux point is that, in this case, \( f.d\mathbf{x} \neq 0 \), and so, \( f \) does not represent the field velocity anymore. To make a change for the physical velocities corresponds to diagonalize the kinetic lagrangian term. For sets of non-abelian fields, it can be shown, that their masses are eigenvalues of their kinetic Lagrangian term, independently of any interaction, and that they are just determined by their correspondingly enlarged spacetime symmetry. This indicates the intrinsically kinematical character of the mass. The interested reader can find in the reference [3], in a compact communication, an example where, for the \( SU(2) \otimes U(1) \) symmetry the weak boson mass sector of the Lagrangian of the Weinberg-Salam electroweak theory is obtained.

12. The geometrical ideas behind \( (33) \) is entirely compatible and consistent with Lagrangian and variational methods, with a consistent interpretation of the Noether theorem and of the Poincaré Group algebra\(^{(6)}\);

13. From \( (33) \) one can develop a consistent, entirely free of singularity and of ambiguity, Classical Electrodynamics of a point charge, as done in reference [7]. In particular, the Einstein-de Broglie relation \( (E, \mathbf{p}) = h\nu(1, \mathbf{p}) \), that equates the photon 4-momentum, \( p^\mu \), to its 4-wave vector times the Planck constant, is obtained from \( p^\mu = \int d^4x \Theta f^{\mu\nu} dS_\nu \), if the time component of \( f \) is taken as a measure of \( \nu \), the photon frequency. Then, the Planck constant can be identified with a given constant integral over the photon sources and is closely related to the fundamental discrete process of radiation emission described in this \( f \)-formalism;

14. These same ideas can be applied to the General Theory of Relativity\(^{(8)}\). For a spherically symmetric distribution of masses one finds a solution \( g^{\mu\nu}_f \), which is free of singularity and reproduces the Schwarzschild solution, upon following the recipe \( (12) \).
VI. APPENDIX

Let us prove (57). Its RHS corresponds to

$$\lim_{\varepsilon \to 0} \int dx^4 f^\beta g_{\beta \mu} \left\{ \nabla_\nu \Theta^{\mu \nu} + \Gamma^\nu_{\alpha \nu} \Theta^{\alpha \nu} \right\} \theta(\rho - \varepsilon),$$  

(65)

and then,

$$\lim_{\varepsilon \to 0} \int dx^4 \left\{ \left( f^\beta g_{\beta \mu} \Gamma^\mu_{\alpha \nu} - \nabla_\nu (f^\beta g_{\beta \alpha}) \right) \Theta^{\alpha \nu} \theta(\rho - \varepsilon) - f_\mu \Theta^{\mu \nu} \nabla_\nu \delta(\rho - \varepsilon) \right\}.$$  

(66)

But,  

$$f_\mu \Gamma^\mu_{\alpha \nu} = \frac{\chi^2}{2} \left( a f(V'_\alpha, f'_\nu) - (a_\alpha, f'_\nu) - f'(a_\alpha, V'_\nu) \right) = \frac{\chi^2}{2} \left( a R(V'_\alpha, R'_\nu) - a_\alpha (R'_\alpha, R'_\nu) - R(R'(a_\alpha, V'_\nu)) \right)$$

and so, from (27), the behaviour of the first term of the integrand of (66) in the limit of $R \to 0$ is given by

$$\lim_{R \to 0} \left[ R, \rho \frac{a \cdot V}{3} + R(V, R) \right] g_{\alpha \beta} \left( a R(V, R) - \rho (a, R) - R(R'(a_\alpha, V'_\nu)) \right) \equiv 0,$$

(67)

Then, according to the notation used in (43),

$$C_0 = a R(V, R) - \rho (a, R) - R(R'(a, V)) \frac{\chi}{2} \implies C_2 = -2 \chi(a, V) + O(R) \to c = 2$$

$$A_0 = B_0 = [R, \rho \cdot a + V' + (1 + a R)] \implies A_2 = B_2 = [a, V'] + O(R) \implies a - 2 = 2$$

$$\implies p - a = 2 \implies p = 6 = n \implies N_6 = \left( \frac{6}{4} \right) \left( \frac{4}{2} \right) [a, V'] g_{\alpha \beta} [a, V'] \left( -2 \chi(a, V) \right) \equiv 0,$$

according to (35) and (53). Therefore,

$$\lim_{\varepsilon \to 0} \int dx^4 f_\mu \Gamma^\mu_{\alpha \nu} \Theta^{\alpha \nu} \theta(\rho - \varepsilon) = 0$$  

(68)

To find how the second term of the integrand of (66) behaves in the limit of $R \to 0$, we need to find $\nabla_\nu f'_\mu$ from $f^\mu = \frac{R^\mu}{2 \Delta \tau}$ and $\rho = \Delta \tau$. The difference between the derivatives of $\rho$ and of $\Delta \tau$ tends to zero in the limit of $\rho \to 0$. So, it is irrelevant if we use one or the other in the definition of $f$; we use the simplest one, $\Delta \tau$, and $\frac{\partial \Delta \tau}{\partial \tau} = -1$ to find:

$$\nabla_\nu (g_{\mu \alpha} f') = \frac{1}{\Delta \tau} \left\{ \Delta \tau (V'_\mu a_f - a_\mu) + g_{\mu \nu} f'_\nu (V' - f'_{\mu}) \right\} =$$

$$= \frac{1}{(\Delta \tau)^2} \left\{ (\Delta \tau)^2 (g_{\mu \nu} - \chi a_\mu R'_\nu) + \Delta \tau \left( V'_\mu (1 + \chi a \cdot R) \right) R'_\nu - R'_\mu R'_\nu \right\}$$

(69)
We just have to replace, in the previous case, its $C_0$ by

$$C_0 = (\Delta \tau)^2 (g - \chi aR') + \Delta \tau R' V'(1 + \chi a) - R'R'$$

Thus,

$$C_2 = 2g + \mathcal{O}(R) \implies c = 2 \implies p = 6 < n = 7$$

According to (34), this would produce a divergent result if $N_6 \neq 0$, but $\Theta$ and its limit are traceless $\Theta^{\mu\nu} g_{\mu\nu} = 0$, and so, $N_6 = 0$. Therefore, the indeterminacy, $0/0$ remains and a new application of the L'Hôpital rule is demanded.

Then, from (35), for $p = 6$, $a = 4$, $c = 2$, we have

$$N_6 = \binom{6}{2} \binom{2}{2} A_2 . g_0 . B_2 C_2 = 0,$$

and then

$$N_7 = \dot{N}_6 = \binom{6}{2} \binom{2}{2} \left\{ A_3 . g_0 . A_2 C_2 + A_2 . g_0 . A_2 C_2 + A_2 . g_1 . A_2 C_2 + A_2 . g_0 . A_3 C_2 + A_2 . g_0 . A_2 C_3 \right\}.$$  

The terms containing $C_2$ will still give a null contribution ($\Theta$ is traceless). With $C_3 = 3(V, a)(\chi + \frac{1}{2}) + \mathcal{O}(R),$

$$\left\{ A_2 . g_0 . A_2 C_3 \right\} = \left\{ [V, a] . g . [V, a] gg (9V a + 6 aV) \right\} \equiv 0, \quad (69)$$

and then,

$$\lim_{\varepsilon \to 0} \int dx^4 \nabla_\nu f_\alpha \Theta^{\alpha\nu} \theta(\rho - \varepsilon) = 0. \quad (70)$$

The last term in the integrand of (66) is related to (33-12). So, we can write

$$\langle f . P_2 \rangle := \lim_{\varepsilon \to 0} \int dx^4 f^\alpha g_{\alpha\mu} \Theta^{\mu\nu}_4 (f'_\nu - V'_\nu) \delta(\rho - \varepsilon), \quad (71)$$

which is null because

$$\lim_{\nu \to 0} \frac{R . ([R, V'_\nu] . g . [R, V'_\nu]) . g . (R - V \rho)}{\rho^6} = 0;$$

and

$$\langle f . P_1 \rangle := \lim_{\varepsilon \to 0} \int dx^4 f^\alpha g_{\alpha\mu} \left\{ \Theta^{\mu\nu}_4 (f'_\nu - V'_\nu) + \Theta^{\mu\nu}_4 f'_{\nu \rho} a_f \right\} \delta(\rho - \varepsilon), \quad (72)$$
which is also null because

$$\lim_{\rho \to 0} \frac{R.\{[R,V^\prime].g.[R,\rho \alpha + V^\prime \alpha,R].(R - \rho V)\}}{\rho^5} = \lim_{\rho \to 0} \frac{R.g.\{[R,V^\prime].g.[R,V^\prime]\}.R\alpha.R}{\rho^5} = 0,$$

as they can be easily verified. Finally,

$$< f.P_0 > := \int dx^4 f^\alpha g_{\alpha\mu} \left\{ \Theta_2^{\mu\nu} \left( f^\nu - V^\nu \right) + \Theta_3^{\mu\nu} \left( f^\nu \rho \alpha_f \right) \right\} \delta(\rho - \varepsilon) = 0,$$

as a consequence of \((32)\).
Creation ($\bar{f}$) and destruction (f) of particles.

f and $\bar{f}$ represent two distinct types of propagating solutions of $\Box A_f = J$:

1. Along $\bar{f}$ propagates a photon that was emitted at $y_1$ and that has been observed at x. $\Delta x^0 = x^0 - y_1^0 > 0$. 
   $J(y_1)$ is its source.

2. Along f propagates a photon that has been observed at x and that will be absorbed by J at $y_2$. $\Delta x^0 = x^0 - y_2^0 < 0$. 
   $J(y_2)$ is its sink.

There is no advanced field, only retarded fields. $\Delta x^0 > 0$ and $\Delta x^0 <$ describe creation and destruction of fields, respectively.
Figure: creation (f) and annihilation (\(\bar{f}\)) of particles.