On the Polya permanent problem over finite fields

Gregor Dolinar
Faculty of Electrical Engineering, University of Ljubljana, Tržaška 25, SI-1000 Ljubljana, Slovenia.

Alexander E. Guterman
Faculty of Algebra, Department of Mathematics and Mechanics, Moscow State University, GSP-1, 119991 Moscow, Russia.

Bojan Kuzma
1University of Primorska, Glagoljaška 8, SI-6000 Koper, Slovenia, and 2IMFM, Jadranska 19, SI-1000 Ljubljana, Slovenia.

Marko Orel
IMFM, Jadranska 19, SI-1000 Ljubljana, Slovenia.
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Corresponding author:
Bojan Kuzma IMFM, Jadranska 19, SI-1000 Ljubljana, Slovenia.
e-mail: bojan.kuzma@pef.upr.si
ON THE POLYA PERMANENT PROBLEM OVER FINITE FIELDS

GREGOR DOLINAR, ALEXANDER E. GUTERMAN, BOJAN KUZMA, AND MARKO OREL

Abstract. Let $F$ be a finite field of characteristics different from two. We show that no bijective map transforms permanent into determinant when the cardinality of $F$ is sufficiently large. We also give an example of non-bijective map when $F$ is arbitrary and an example of a bijective map when $F$ is infinite which do transform permanent into determinant. The developed technique allows us to estimate the probability of the permanent and the determinant of matrices over finite fields to have a given value. Our results are also true over finite rings without zero divisors.

1. Introduction

Let $A = (a_{ij}) \in M_n(F)$ be an $n \times n$ matrix over a field $F$. The permanent function

$$\text{per} A = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

is defined in a very similar way to the definition of the determinant function

$$\det A = \sum_{\sigma \in S_n} sgn(\sigma)a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$ 

In both cases the sum is considered over all permutations $\sigma \in S_n$, where $S_n$ denotes the set of all permutations of the set $\{1, 2, \ldots, n\}$. The value $sgn(\sigma) \in \{-1, 1\}$ is the signum of the permutation $\sigma$, i.e., $sgn(\sigma) = 1$ if $\sigma$ is an even permutation, and $sgn(\sigma) = -1$ if $\sigma$ is an odd permutation.

The determinant is certainly one of the most well-studied functions in mathematics. Geometrically, it is the volume together with orientation of the parallelepiped defined by rows (or columns) and algebraically, it

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is the product of all eigenvalues, counted with their multiplicities. The permanent function is also well-studied, especially in combinatorics, see [20]. For example, if $A$ is a $(0,1)$-matrix, then the value $\text{per} A$ is equal to the number of perfect matchings in a bipartite graph with adjacency matrix $A$. However, no nice geometric or algebraic interpretation is known for permanent. Moreover the permanent does not enjoy the same properties as the determinant, in particular it is neither multiplicative nor invariant under linear combinations of rows or columns.

Computing permanent of a matrix seems to have different computational complexity than computing the determinant. The determinant can be calculated by a polynomial time algorithm. For example, Gauss elimination method requires $O(n^3)$ operations. At the same time no efficient algorithm for computing the permanent function is known, and, in fact, none is believed to exist. When using its definition, the computation of the permanent requires $(n-1)n!$ multiplications and one of the best known algorithms to compute permanent, so-called Ryser’s formula [23], has an exponential complexity and requires $(n-1)\cdot(2^n-1)$ multiplications. Moreover, Valiant [27] has shown that even computing the permanent of a $(0,1)$-matrix is a $\#P$-complete problem, i.e., this problem is an arithmetic analogue of Cook’s hypothesis $P \neq NP$, see [7, 15, 12] for details.

Starting from 1913 researchers are trying to find a way to calculate permanents using determinants. More precisely, the following problems which dates back to the work of Pólya [21] are under intensive investigations for almost a century.

**Problem 1.1.** Does there exist a uniform way of affixing $\pm$ signs to the entries of a matrix $A = (a_{ij}) \in M_n(\mathbb{F})$ such that $\text{per}(a_{ij}) = \text{det}(\pm a_{ij})$?

**Problem 1.2.** Given a $(0,1)$-matrix $A \in M_n(\mathbb{F})$, does there exists a transformed matrix $B$, obtained by changing some of the $+1$ entries of $A$ into $-1$, so that $\text{per} A = \text{det} B$?

**Problem 1.3.** Under what conditions does there exist a transformation $\Phi : M_n(\mathbb{F}) \to M_m(\mathbb{F})$ satisfying

\begin{equation}
\text{per} A = \text{det} \Phi(A).
\end{equation}

In this case the image $\Phi(A)$ is usually called a *Pólya matrix* for $A$. For example if $n = 2$, one can consider the Pólya matrix

\begin{equation}
B = B(A) = \begin{pmatrix}
a_{11} & -a_{12} \\
a_{21} & a_{22}
\end{pmatrix}.
\end{equation}
Problem 1.1 was solved negatively by Szegö in [26], namely he proved that for \( n \geq 3 \) there is no generalization of the formula (2).

Problem 1.2 has been intensively studied since it belongs to the famous class of equivalent problems, containing the following ones: When does a real square matrix have the property that every real matrix with the same sign pattern is non-singular? When does a bipartite graph have a “Pfaffian orientation”? Given a digraph, does it have no direct circuit of even length? See [4, 22, 28] for the detailed and self-contained information.

Problem 1.3 is a natural generalization of Problem 1.1. Namely, affixing \((\pm 1)\) signs to the entries of a matrix is an example of a certain linear transformation with easy structure. One may ask, if there exists some more sophisticated linear transformation \( \Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F}) \) satisfying (1)? In 1961 Marcus and Minc [19], see also Botta [3], proved that if \( n \geq 3 \) there is no linear transformations \( \Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F}) \) satisfying equality (1).

Von zur Gathen [13] investigated the linear transformations \( \Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F}) \) satisfying the equality

\[
\det A = \text{per} \Phi(A)
\]

and proved that if there exists such \( \Phi \), then \( m > \sqrt{2n} - 6\sqrt{n} \). These results were later improved, see for example Cai [5] and references therein.

After that several attempts to further reduce the linearity assumption were made, see for example Coelho, Duffner [6], Kuzma [18], and references therein. In these works no bijectivity or linearity is assumed, but the authors consider transformations \( \Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) satisfying the equality

\[
d_\chi(\Phi(A) + \lambda \Phi(B)) = d_{\chi'}(A + \lambda B), \quad (\lambda \in \mathbb{C})
\]

where \( d_\chi, d_{\chi'} \) are arbitrary immanants. In particular, this also covers the possibility \( d_\chi = \det \) and \( d_{\chi'} = \text{per} \) or vice versa.

In the present paper we show that without any regularity assumptions imposed on \( \Phi \) there do exist transformations (possibly bijective if the underlying field is infinite) that even exchange the permanent and the determinant, i.e. transformations that satisfy both equalities (1) and (3) simultaneously, see Examples 8.2 and 8.5 from the present paper.

The main aim of the present paper, however, is to show that if \( \mathbb{F} \) is a finite field of sufficiently large cardinality, depending on \( n \), then there are no bijective transformations \( \Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F}) \) satisfying (1),
i.e., we obtain a negative solution of Problem 1.3 for bijective maps, defined on matrices over finite fields.

These results are heavily based on the detailed analysis of the cardinality of the set of matrices over finite fields with zero permanent.

As an application of our results we also estimate the probability of the permanent and the determinant of matrices over a finite field to have a given value. This problem dates back to the works by Erdős and Rényi [9, 10], where they estimated the probability for a (0,1)-matrix with a given number of ones to have a zero permanent. For the detailed and self-contained account of the results one may study monograph [21, 25].

Our paper is organized as follows: Section 2 contains basic definitions and notations used in the paper and the statements of the main results. In Section 3 we calculate the number of matrices with the zero permanent in $M_3(\mathbb{F})$. In Section 4 we compute the cardinality of a set of pairs of vectors, which are orthogonal to each other with respect to a matrix of fixed rank $r$ and split the set of all matrices with zero permanent into several subsets by means of the Laplace decomposition. In Section 5 we introduce inductively the lower and upper bounds for the number of matrices with zero permanent and prove these bounds. Section 6 is devoted to the proof of the main result, i.e., that the introduced upper bound for matrices with zero permanent is strictly less than the number of matrices with zero determinant. In Section 7 the probability of the permanent and the determinant to have given values is estimated. In Section 8 we provide some examples, in particular the examples of non-bijective transformations on matrices over any fields and bijective transformations of matrices over infinite field that satisfy both equalities (1) and (3) simultaneously. We also show that our results are valid over finite rings without zero divisors.

2. Preliminaries and statement of the main result

In our paper $\mathbb{F}$ is a finite field of characteristics different from 2 and of the cardinality $|\mathbb{F}| = q$, except in Section 8 where $\mathbb{F}$ is arbitrary.

We denote the identity matrix from $M_n(\mathbb{F})$ by $I_n$ and zero matrix by $O_n$. If the size $n$ is clear from the context, we omit the corresponding index. By $A^t$ we denote the transposed matrix to $A$.

In this paper we use the term per-minor of order $k$ (or $k$-by-$k$ per-minor) of $A \in M_n(\mathbb{F})$ to denote the permanent of a $k \times k$-submatrix of $A$. Principal per-minor of $A \in M_n(\mathbb{F})$ is a per-minor of order $n - 1$. Let $A_{ij}$ denote the matrix obtained from $A$ by deleting the $i$-th row and $j$-th column; $A_{(i\ldots j)(k\ldots l)}$ denote the matrix obtained from $A$ by deleting
the rows from $i$ to $k$ and the columns from $j$ to $l$. Let

$$
\hat{A} = \begin{pmatrix}
\per A_{11} & \ldots & \per A_{1n} \\
\vdots & \ddots & \vdots \\
\per A_{n1} & \ldots & \per A_{nn}
\end{pmatrix}
$$

be a permanental compound of $A$. In this paper we investigate the sets

$$
P_n(F) = \{ A \in M_n(F) : \per A = 0 \}
$$

and

$$
D_n(F) = \{ A \in M_n(F) : \det A = 0 \}
$$
of all matrices with zero permanent and zero determinant, respectively.

It is straightforward to see that as is the case with a classical determinant, the permanent also obeys the Laplace decomposition, see for example [20, Chapter 2, Theorem 1, 2],

$$
\per A = \per (a_{ij}) = a_{i1} \per A_{i1} + a_{i2} \per A_{i2} + \ldots + a_{in} \per A_{in}.
$$

Our main result can be formulated as follows.

**Theorem 2.1.** Suppose $n \geq 3$. Then there exists $q_0$, depending on $n$, such that for any finite field $F$ with at least $q_0$ elements and $\text{ch} F \neq 2$ no bijective map $\Phi : M_n(F) \to M_n(F)$ satisfies

\begin{equation}
(4) \quad \per A = \det \Phi(A).
\end{equation}

When $n = 3$ the conclusion holds for any finite field with $\text{ch} F \neq 2$.

Note that any finite ring without zero divisors is a field (in Section 8 we provide a short proof of this fact for the sake of completeness). Therefore the above result is valid also for matrices over finite rings without zero divisors.

**Corollary 2.2.** Let $n \geq 3$ and let $R$ be a finite ring without zero divisors of sufficiently large cardinality, $\text{ch} R \neq 2$. Then no bijective map $\Phi : M_n(R) \to M_n(R)$ satisfies $\per(A) = \det \Phi(A)$.

**Remark 2.3.** By considering $\Psi = \Phi^{-1}$ the above Corollary shows that $\per \Psi(A) = \det(A)$ is impossible for bijective $\Psi$ acting on matrices over a finite ring without zero divisors of sufficiently large cardinality and characteristic different from 2.

The proof of Theorem 2.1 will be given in Section 6. Here we outline the main idea. Any bijective $\Phi$ satisfying (4) would induce a bijection from the set $P_n(F)$ of $n$–by–$n$ matrices with zero permanent onto the set $D_n(F)$ of $n$–by–$n$ matrices with zero determinant. Consequently, to prove the theorem it suffices to show that the number $|P_n(F)|$ does
not equal \(|D_n(\mathbb{F})|\) for all sufficiently large finite fields of characteristic different from two. We remark that the latter number is well-known. Actually, there exists precisely

\[(5) \quad |D_n(\mathbb{F})| = q^{n^2} - \prod_{k=1}^{n}(q^n - q^{k-1}) = q^{n^2} - q^{\frac{n(n-1)}{2}}(q^n - 1) \ldots (q - 1)\]

\(n\)-by-\(n\) matrices with determinant zero [1, Prop. 2, p. 41], where \(q = |\mathbb{F}|\).

For \(n = 3\) we will exactly calculate the number \(|P_3(\mathbb{F})|\) of matrices with permanent zero, however for \(n \geq 4\) we will not give an exact formula for \(|P_n(\mathbb{F})|\), but we will give its upper bound \(\mathcal{U}_n(\mathbb{F})\) and show that

\(|P_n(\mathbb{F})| \leq \mathcal{U}_n(\mathbb{F}) \leq |D_n(\mathbb{F})|\)

if \(\mathbb{F}\) is a finite field with sufficiently many elements and \(\text{ch} \mathbb{F} \neq 2\).

3. Zero permanents in \(M_3(\mathbb{F})\)

**Lemma 3.1.** Let \(\mathbb{F}\) be a finite field with \(\text{ch} \mathbb{F} \neq 2\). Then

\(|P_3(\mathbb{F})| = |D_3(\mathbb{F})| - q^2(q - 1)^5\).

**Proof.** We decompose \(D_3(\mathbb{F})\) and \(P_3(\mathbb{F})\) into pairwise disjoint union of three sets

\[D_3(\mathbb{F}) = D'_{3}(\mathbb{F}) \cup D''_{3}(\mathbb{F}) \cup D'''_{3}(\mathbb{F})\]

and

\[P_3(\mathbb{F}) = P'_{3}(\mathbb{F}) \cup P''_{3}(\mathbb{F}) \cup P'''_{3}(\mathbb{F})\],

where

\[D'_{3}(\mathbb{F}) = \{A \in D_3(\mathbb{F}) : a_{33} \neq 0, \det A_{11} = 0\}\]

\[D''_{3}(\mathbb{F}) = \{A \in D_3(\mathbb{F}) : a_{33} \neq 0, \det A_{11} \neq 0\}\]

\[D'''_{3}(\mathbb{F}) = \{A \in D_3(\mathbb{F}) : a_{33} = 0\}\]

\[P'_{3}(\mathbb{F}) = \{A \in P_3(\mathbb{F}) : a_{33} \neq 0, \per A_{11} = 0\}\]

\[P''_{3}(\mathbb{F}) = \{A \in P_3(\mathbb{F}) : a_{33} \neq 0, \per A_{11} \neq 0\}\]

\[P'''_{3}(\mathbb{F}) = \{A \in P_3(\mathbb{F}) : a_{33} = 0\}\]

We claim that the cardinality of \((6)\) and \((6')\) are the same; and also the cardinality of \((7)\) and \((7')\) are the same, while \(|P'_{3}(\mathbb{F})| = |D'_{3}(\mathbb{F})| - q^2(q - 1)^5\).

Start with \((6)\). There are \(q^2(q - 1)^2\) many ways of prescribing the values to ‘variables’ \(a_{22}, a_{23}, a_{32}, a_{33}\) to achieve \(\det A_{11} \neq 0 \neq a_{33}\). We
can further arbitrarily prescribe the values of \( a_{12}, a_{13}, a_{21}, a_{31} \), while \( a_{11} \) is then completely determined by \( \det A = 0 \), i.e., by
\[
    a_{11} = \frac{a_{13}(a_{22}a_{31} - a_{21}a_{32}) + a_{12}(a_{21}a_{33} - a_{23}a_{31})}{a_{22}a_{33} - a_{23}a_{32}}.
\]

In total, \( |D'_3(\mathbb{F})| = q^2(q - 1)^2 \cdot q^4 \). A similar computation also gives \( |P'_3(\mathbb{F})| = q^2(q - 1)^2 \cdot q^4 \), as claimed.

We next show that the cardinalities of (7) and (7') are the same. To do this, just notice that
\[
    \Psi_3 : \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & 0 \end{pmatrix} \mapsto \begin{pmatrix} -x_{11} & x_{12} & x_{13} \\ x_{21} & -x_{22} & x_{23} \\ x_{31} & x_{32} & 0 \end{pmatrix}
\]
is a linear bijection with the property \( \det \Psi_3(A) = \det A = 0 \) for every \( A \in M_3(\mathbb{F}) \) with \( a_{33} = 0 \). Whence it also maps the set \( P'_3(\mathbb{F}) \) bijectively onto the set \( D'_3(\mathbb{F}) \), as claimed.

Finally, we compute the cardinalities \( |P'_3(\mathbb{F})| \) and \( |D'_3(\mathbb{F})| \). Consider first the set
\[
    D'_3(\mathbb{F}) = \{ A \in D_3(\mathbb{F}) : a_{33} \neq 0, \det A_{11} = 0 \}
\]
\[
= \left\{ (a_{ij}) \in M_3(\mathbb{F}) : \frac{(a_{13}a_{32} - a_{12}a_{33})(a_{21}a_{33} - a_{23}a_{31})}{a_{33}} = 0, \frac{a_{22} - a_{12}a_{33}}{a_{33}} = 0, a_{33} \neq 0 \right\}.
\]
The number of matrices inside \( D'_3(\mathbb{F}) \) can be computed as follows. We can choose a total of \( q - 1 \) distinct nonzero values for \( a_{33} \), and a total of \( q \) distinct values for each ‘variable’ \( a_{23} \) and \( a_{32} \). Once these are chosen, \( a_{22} \) is uniquely determined from them, by the second equation. All together, we can prescribe \( (q - 1)q^2 \) different values for ‘variables’ \( a_{33}, a_{32}, a_{23}, a_{22} \).

Once we choose the values of these four ‘variables’, we have additional equation \( (a_{13}a_{32} - a_{12}a_{33})(a_{21}a_{33} - a_{23}a_{31}) = 0 \) with four new ‘variables’ \( a_{13}, a_{12}, a_{21}, a_{31} \). There are \( q^4 \) ways of prescribing their values, but only \( q(q - 1) \cdot q(q - 1) \) ways of prescribing their values so that both factors are nonzero — in fact, we can prescribe, say, \( a_{13} \) arbitrarily, and then \( a_{12} \) is determined by \( a_{12} \neq a_{13}a_{33}^{2a_{32}} \), likewise for the second factor. From this we deduce that there are precisely \( q^4 - q^2(q - 1)^2 = q^2(2q - 1) \) ways of prescribing the values for \( a_{13}, a_{12}, a_{21}, a_{31} \) so that the product of the two factors is zero. Finally, due to \( \det A_{11} = 0 \) we may arbitrarily prescribe \( a_{11} \) without affecting \( \det A = 0 \). In total,
\[
    |D'_3(\mathbb{F})| = (q - 1)q^2 \cdot q^2(2q - 1) \cdot q = q^5(q - 1)(2q - 1).
\]
In the set

\[ P'_3(\mathbb{F}) = \{ A \in P_3(\mathbb{F}) : a_{33} \neq 0, \text{ per } A_{11} = 0 \} \]

\[ = \left\{ (a_{ij}) \in M_3(\mathbb{F}) : \frac{a_{11}a_{12}(a_{23}a_{33} - a_{23}a_{33})}{a_{33}} + a_{12}(a_{23}a_{31} + a_{21}a_{33}) = 0, a_{22} + \frac{a_{23}a_{32}}{a_{33}} = 0, a_{33} \neq 0 \right\} \]

the first equation does not split, so we need a different approach to compute \(|P'_3(\mathbb{F})|\). First we count those matrices inside \(P'_3(\mathbb{F})\) which satisfy \((a_{23}a_{31} + a_{21}a_{33}) \neq 0\). As in the determinant case, \(a_{11}\) is arbitrary while the values for ‘variables’ \(a_{33}, a_{32}, a_{23}, a_{22}\) can be prescribed in \((q - 1)q^2\) different ways. Once these values are chosen, we have \(q(q - 1)\) possibilities for ‘variables’ \(a_{31}, a_{21}\) to achieve that \((a_{23}a_{31} + a_{21}a_{33}) \neq 0\).

We may further prescribe \(a_{13}\) arbitrarily, and then \(a_{12}\) is completely determined by

\[ a_{12} = \frac{a_{13}a_{32}(a_{23}a_{31} - a_{21}a_{33})}{a_{33}(a_{23}a_{31} + a_{21}a_{33})}. \]

All together, there are \(q \cdot (q - 1)q^2 \cdot q(q - 1) \cdot q = q^5(q - 1)^2\) matrices in \(P'_3(\mathbb{F})\) which satisfy \((a_{23}a_{31} + a_{21}a_{33}) \neq 0\).

To count the matrices inside \(P'_3(\mathbb{F})\) which satisfy \((a_{23}a_{31} + a_{21}a_{33}) = 0\), note that in this case \(a_{21} = -\frac{a_{23}a_{31}}{a_{33}}\), so the first equation inside [8] reduces to \(\frac{2a_{13}a_{23}a_{31}a_{32}}{a_{33}} = 0\). Hence, we need to count those 3–by–3 matrices which satisfy:

\[ a_{13}a_{23}a_{31}a_{32} = 0, \quad a_{33} \neq 0, \quad a_{21} = -\frac{a_{23}a_{31}}{a_{33}}, \quad a_{22} = -\frac{a_{23}a_{32}}{a_{33}}. \]

We may choose \((q^4 - (q - 1)^4)\) possible values for \(a_{13}, a_{23}, a_{31}, a_{32}\) to have \(a_{13}a_{23}a_{31}a_{32} = 0\), we may choose \((q - 1)\) values for \(a_{33}\), the ‘variables’ \(a_{21}, a_{22}\) are uniquely determined, while \(a_{11}\) and \(a_{12}\) are arbitrary. All together, there are \((q^4 - (q - 1)^4) \cdot (q - 1) \cdot 1 \cdot 1 \cdot q = q^2(q - 1)(q^4 - (q - 1)^4)\) such matrices.

In summary we get \(|P'_3(\mathbb{F})| = q^5(q - 1)^2 + q^2(q - 1)(q^4 - (q - 1)^4)\). Consequently, a simple calculation gives

\[ |D_3(\mathbb{F})| - |P_3(\mathbb{F})| = |D'_3(\mathbb{F})| - |P'_3(\mathbb{F})| \]

\[ = \left( q^5(q - 1)(2q - 1) \right) - \left( q^5(q - 1)^2 + q^2(q - 1)(q^4 - (q - 1)^4) \right) \]

\[ = q^2(q - 1)^5. \]

\[ \square \]
4. Zero permanents in $M_n(\mathbb{F})$ for $n \geq 4$

The following lemma should be known, but unfortunately we were unable to find it in the literature. We include its proof for the sake of convenience.

**Lemma 4.1.** Let $k \geq 2$ be an integer and let $A \in M_k(\mathbb{F})$ be of rank $r$. Then the set $V_k(r)(\mathbb{F}) = \{(x, y) \in \mathbb{F}^k \times \mathbb{F}^k : x^\tr A y = 0\}$ has cardinality

$$|V_k(r)(\mathbb{F})| = q^{2(k-r)}((q^r - 1)q^{r-1} + q^r).$$

This number is a strictly decreasing function of $r$.

**Proof.** When $r = 0$, every pair satisfies $x^\tr A y = 0$, so $|V_k(r)(\mathbb{F})| = q^{2k}$, which agrees with (9). Suppose $r > 0$. There exist invertible matrices $P$ and $Q$ such that $A = P(Id_r \oplus 0_{k-r})Q$. Therefore,

$$x^\tr A y = x^\tr P(Id_r \oplus 0_{k-r})Qy = (x')^\tr (Id_r \oplus 0_{k-r})y'$$

where $x' = P^\tr x$ and $y' = Qy$. Since $P$ and $Q$ are invertible, the map $(x, y) \mapsto (P^\tr x, Qy)$ bijectively maps the zeros of $x^\tr A y$ onto the zeros of $(x')^\tr (Id_r \oplus 0_{k-r})y'$. So we may assume that $A = (Id_r \oplus 0_{k-r})$.

Writing $x = (x_1, \ldots, x_k)^\tr$ and $y = (y_1, \ldots, y_k)^\tr$ we clearly have

$$x^\tr A y = \sum_{i=1}^k x_i y_i.$$ 

Now, given any fixed nonzero $r$-tuple $x_r = (x_1, \ldots, x_r)$ we have that $x^\tr A y = 0$ precisely when $y_r = (y_1, \ldots, y_r)$ lies in the kernel of the functional $F_{x_r} : \mathbb{F}^r \rightarrow \mathbb{F}$ defined by $y_r \mapsto x^\tr_{r} y_r$. Hence, $y_r$ must lie in a hyperplane inside $\mathbb{F}^r$ of codimension 1. Any such hyperplane is isomorphic to $\mathbb{F}^{r-1}$ and contains $q^{r-1}$ vectors $y_r$. Since there are precisely $q^r - 1$ possible nonzero vectors $x_r$, we get $(q^r - 1)q^{r-1}$ tuples $(x_r, y_r) \in \mathbb{F}^r \times \mathbb{F}^r$ which satisfy $\sum_{i=1}^r x_i y_i = 0$, and such that $x_r \neq 0$. If, however, $x_r = 0$ then $y_r$ can be arbitrary, which adds additional $1 \cdot q^r$ tuples. Finally, we may arbitrarily prescribe the values for ‘variables’ $x_{r+1}, \ldots, x_k$, $y_{r+1}, \ldots, y_k$ giving a total of

$$((q^r - 1)q^{r-1} + 1 \cdot q^r) \cdot q^{k-r} = ((q^r - 1)q^{r-1} + q^r) \cdot q^{2(k-r)}$$

tuples $(x, y)$ which solve $x^\tr A y = 0$.

To prove the last statement in the lemma, we simply notice that the derivative $d/dr$ of the above result equals $-(q-1)q^{2k-r-1} \ln q < 0$. □

We now recursively calculate the cardinality of the set $P_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \text{per } A = 0\}$. Recall that the permanent can be computed with
a Laplace decomposition as

\[(10) \quad \text{per } A = a_{11} \text{ per } A_{11} + a_{12} \text{ per } A_{12} + \cdots + a_{1n} \text{ per } A_{1n}.\]

This suggests splitting \( P_n(\mathbb{F}) \) into two disjoint subsets

\[ P_n(\mathbb{F}) = \hat{P}_n(\mathbb{F}) \cup \tilde{P}_n(\mathbb{F}) \]

\[ = \{ A \in P_n(\mathbb{F}) : \text{ per } A_{11} \neq 0 \} \cup \{ A \in P_n(\mathbb{F}) : \text{ per } A_{11} = 0 \}. \]

In \( \hat{P}_n(\mathbb{F}) \) we can choose \((q^{(n-1)^2} - |P_{n-1}(\mathbb{F})|)\) blocks \( A_{11} \) with nonzero permanent. For each fixed block \( A_{11} \) we can arbitrarily prescribe the values for \( 2(n - 1) \) 'variables' \( a_{12}, a_{13}, \ldots, a_{1n}, a_{21}, a_{31}, \ldots, a_{n1} \). However, the value of \( a_{11} \) is then completely determined by \( a_{11} = -(a_{12} \text{ per } A_{12} + \cdots + a_{1n} \text{ per } A_{1n})/\text{ per } A_{11} \). All together, the first subset has cardinality

\[(11) \quad |\hat{P}_n(\mathbb{F})| = (q^{(n-1)^2} - |P_{n-1}(\mathbb{F})|)q^{2(n-1)}.\]

Consider next the second set. Here, we further decompose \((10)\) into

\[ \text{per } A = a_{11} \text{ per } A_{11} + \sum_{i,j=2}^{n} a_{1i}a_{j1} \text{ per } A_{(1i)(j1)}, \]

where \( A_{(1i)(j1)} \) is an \((n - 2)\)-by-\((n - 2)\) submatrix, obtained from \( A \) by deleting the 1-st and the \( j \)-th row and the \( i \)-th and the 1-st column. The second term is a bilinear form. Actually, by introducing column vectors \( \mathbf{x} = (a_{12}, \ldots, a_{1n})^\text{tr}, \mathbf{y} = (a_{21}, \ldots, a_{n1})^\text{tr} \in \mathbb{F}^{n-1} \) we can write

\[ \sum_{i,j=2}^{n} a_{1i}a_{j1} \text{ per } A_{(1i)(j1)} = \mathbf{x}^\text{tr} \overline{A_{11}} \mathbf{y}, \]

where \( \overline{A_{11}} = (\text{per } A_{(1i)(j1)})_{2 \leq i,j \leq n} \) is the \((n - 1)\)-by-\((n - 1)\) matrix of principal per-minors of the matrix \( A_{11} \). By Lemma \ref{lem:principal_minors} applied at \( k = n - 1 \), the number of pairs \((\mathbf{x}, \mathbf{y})\), for which the above equation is zero, equals \( |V_{n-1}^{(r)}(\mathbb{F})| = q^{2(n-r-1)} ((q^r - 1)q^{r-1} + q^r) \), where \( r = \text{rk } \overline{A_{11}}. \) To count the cardinality of \( \tilde{P}_n(\mathbb{F}) \) we have to do the following. First, we multiply \( |V_{n-1}^{(r)}(\mathbb{F})| \) with the number of all lower-right \((n - 1)\)-by-\((n - 1)\) blocks \( A_{11} \) which have permanent equal to zero and \( \text{rk } \overline{A_{11}} = r \). Then we make a sum of these products over all ranks \( r \). Finally, we multiply this sum with \( q \) since \( \text{per } A_{11} = 0 \) and therefore \( a_{11} \) can be arbitrary. So, given an integer \( r \geq 0 \) we define

\[(12) \quad N_{n-1}^{(r)}(\mathbb{F}) = \{ X \in M_{n-1}(\mathbb{F}) : \text{ per } X = 0, \text{ rk } \overline{X} = r \},\]
and then
\[ |\tilde{P}_n(\mathbb{F})| = q \sum_{r=0}^{n-1} |N_{n-1}^{(r)}(\mathbb{F})| \cdot |V_{n-1}^{(r)}(\mathbb{F})|. \]

Combined with equality (11) for \( |\tilde{P}_n(\mathbb{F})| \), we derive the following recursive formula for number of \( n \)-by-\( n \) matrices with permanent zero:

\[ |P_n(\mathbb{F})| = (q^{n-1})^2 - |P_{n-1}(\mathbb{F})|)q^{2(n-1)} + q \sum_{r=0}^{n-1} |N_{n-1}^{(r)}(\mathbb{F})| \cdot |V_{n-1}^{(r)}(\mathbb{F})|. \]

Unfortunately, we were unable to calculate \( |P_n(\mathbb{F})| \) since we could not determine the values for \( |N_{n-1}^{(r)}(\mathbb{F})| \). However, in the next section we obtain an upper bound \( \Upsilon_n(\mathbb{F}) \) for \( |P_n(\mathbb{F})| \) which is sufficient to prove the theorem. Here is a brief sketch of our procedure. We will introduce the functions \( \mathcal{L}_n(\mathbb{F}) \) and \( \Upsilon_n(\mathbb{F}) \), defined inductively by the equalities (16) and (17), correspondingly, and show by the simultaneous induction that \( \mathcal{L}_n(\mathbb{F}) \) is a lower bound for \( |P_n(\mathbb{F})| \) (Step 1 of the proof of Lemma 5.1). Then we estimate the summand \( |\tilde{P}_n(\mathbb{F})| \) using the inductively proved lower bound \( \mathcal{L}_{n-1}(\mathbb{F}) \) (Step 2 of the proof of Lemma 5.1). We split the summand \( |\tilde{P}_n(\mathbb{F})| \) into three summands: for \( r = 0, r = 1, \) and \( r \geq 2 \). In order to estimate the last summand (i.e., with \( r \geq 2 \)) we roughly use the monotonicity proved in Lemma 4.1 for \( |V_{n-1}^{(r)}(\mathbb{F})| \) and argue that a part is less than the whole, i.e., use inductive bound \( \sum_{r=2}^{n-1} |N_{n-1}^{(r)}(\mathbb{F})| \leq |P_{n-1}(\mathbb{F})| \leq \Upsilon_{n-1}(\mathbb{F}) \) (Step 3 of the proof of Lemma 5.1). Then we estimate separately the first two summands (Steps 4 and 5 of the proof of Lemma 5.1).

5. UPPER AND LOWER BOUNDS

In this section we determine the upper bound \( \Upsilon_n(\mathbb{F}) \) for \( |P_n(\mathbb{F})| \). To do this we need a lower bound \( \mathcal{L}_n(\mathbb{F}) \) for \( |P_n(\mathbb{F})| \) as well. To simplify the writing we will also define auxiliary quantities \( \mathfrak{V}_{n-1}^{(0)}(\mathbb{F}) \) and \( \mathfrak{V}_{n-1}^{(1)}(\mathbb{F}) \). It will be shown that they are upper bounds for \( |V_{n-1}^{(0)}(\mathbb{F})| \) and \( |V_{n-1}^{(1)}(\mathbb{F})| \) respectively.

First, we lower-estimate the number of 1--by--1 and 2--by--2 matrices with zero permanent by \( \mathcal{L}_1(\mathbb{F}) = 0, \mathcal{L}_2(\mathbb{F}) = 0 \). It is easy to see that \( |P_1(\mathbb{F})| = 1 \) and \( |P_2(\mathbb{F})| = q^2 + q^2 - q \), so we define \( \Upsilon_1(\mathbb{F}) = |P_1(\mathbb{F})| = 1 \) and \( \Upsilon_2(\mathbb{F}) = |P_2(\mathbb{F})| = q^2 + q^2 - q \). Note that we have already calculated the exact value for \( |P_3(\mathbb{F})| \), see Lemma 3.1 and Formula 5. Hence, we put \( \mathcal{L}_3(\mathbb{F}) = \Upsilon_3(\mathbb{F}) = |P_3(\mathbb{F})| \). We also put \( \mathfrak{V}_{2}^{(0)}(\mathbb{F}) = 1 \). Finally, we define \( \mathfrak{V}_{n-1}^{(0)}(\mathbb{F}), \mathfrak{V}_{n-1}^{(1)}(\mathbb{F}), \mathcal{L}_n(\mathbb{F}), \) and \( \Upsilon_n(\mathbb{F}) \) for \( n \geq 4 \). We do it
recursively as follows

\[
\mathcal{N}_{n-1}(F) = 1 + \sum_{k=1}^{n-3} \left( \frac{n-1}{n-k-2} \right)^2 q^{2(n-k-2)(k+1)} \cdot (q^{(n-k-2)^2} - \mathcal{L}_{n-k-2}(F)),
\]

\[
\mathcal{N}_{n-1}(F) = (q^{(n-1)^2-1} - (q-3)(n-1)^2-1) + \mathcal{N}_{n-2}(F) \cdot q^{2(n-2)+1} + q \cdot U_{n-2}(F) \cdot |V_{n-2}(F)|,
\]

\[
\mathcal{L}_n(F) = (q^{(n-1)^2} - \mathcal{L}_{n-1}(F))q^{2(n-1)},
\]

\[
\mathcal{U}_n(F) = (q^{(n-1)^2} - \mathcal{L}_{n-1}(F))q^{2(n-1)} + q \cdot \mathcal{N}_{n-1}(F) \cdot |V_{n-1}(F)| + q \cdot U_{n-1}(F) \cdot |V_{n-2}(F)|.
\]

**Lemma 5.1.** Suppose $|F| > 3$. Then $\mathcal{L}_n(F) \leq |P_n(F)| \leq \mathcal{U}_n(F)$ for all $n$.

**Proof.** We use induction on $n$. For $n = 1, 2, 3$ this is clear. Now, let $n \geq 4$ and assume that we have already proven that $\mathcal{L}_k(F) \leq |P_k(F)| \leq \mathcal{U}_k(F)$ holds for all $1 \leq k \leq n-1$. Let us show that it holds also for $k = n$.

**Step 1.** To start with, we infer from (13) and from induction hypothesis that

\[
|P_n(F)| = |\hat{P}_n(F)| + |\tilde{P}_n(F)|
\geq |\hat{P}_n(F)|
\geq (q^{(n-1)^2} - |P_{n-1}(F)|)q^{2(n-1)}
\geq (q^{(n-1)^2} - \mathcal{U}_{n-1}(F))q^{2(n-1)} = \mathcal{L}_n(F),
\]

which proves the inductive argument for the lower bound.

We now proceed with the upper bound.

**Step 2.** By the inductive hypothesis

\[
|\hat{P}_n(F)| = (q^{(n-1)^2} - |P_{n-1}(F)|)q^{2(n-1)} \leq (q^{(n-1)^2} - \mathcal{L}_{n-1}(F))q^{2(n-1)}.
\]

**Step 3.** We are using now the boundary obtained at Step 2 and split the second summand into three parts for $r = 0$, $r = 1$, and $r \geq 2$ as
follows

\[
|P_n(F)| \leq (q^{n-1})^2 - \mathcal{L}_{n-1}(F))q^{2(n-1)} + q \cdot |N_{n-1}^{(0)}(F)| \cdot |V_{n-1}^{(0)}(F)| + \\
+ q \cdot |N_{n-1}^{(1)}(F)| \cdot |V_{n-1}^{(1)}(F)| + q \left( \sum_{r=2}^{n-1} |N_{n-1}^{(r)}(F)| \cdot |V_{n-1}^{(r)}(F)| \right)
\]

Since by Lemma 11, \(|V_{n-1}^{(r)}(F)| \) is a decreasing function of \(r\), we estimate its value by \(|V_{n-1}^{(2)}(F)|\). Since the sets \(N_{n-1}^{(r)}(F)\) are obviously disjoint, we have

\[
\sum_{r=2}^{n-1} |N_{n-1}^{(r)}(F)| \leq |P_{n-1}(F)|
\]

and using the inductive bound \(|P_{n-1}(F)| \leq \mathcal{U}_{n-1}(F)\) we obtain

(18)

\[
|P_n(F)| \leq (q^{(n-1)^2} - \mathcal{L}_{n-1}(F))q^{2(n-1)} + q \cdot |N_{n-1}^{(0)}(F)| \cdot |V_{n-1}^{(0)}(F)| + \\
+ q \cdot |N_{n-1}^{(1)}(F)| \cdot |V_{n-1}^{(1)}(F)| + q \cdot \mathcal{U}_{n-1}(F) \cdot |V_{n-1}^{(2)}(F)|.
\]

To show \(|P_{n}(F)| \leq \mathcal{U}_{n}(F)\) it now suffices to demonstrate that \(\mathcal{U}_{n}(F)\), defined by (17), is even greater than the last quantity in (18). And to verify this claim, it is sufficient to prove \(|N_{n-1}^{(0)}(F)| \leq \mathcal{V}_{n-1}^{(0)}(F)\) and \(|N_{n-1}^{(1)}(F)| \leq \mathcal{V}_{n-1}^{(1)}(F)\).

**Step 4.** Let us prove that \(|N_{k}^{(0)}(F)| \leq \mathcal{V}_{k}^{(0)}(F)\) for all \(2 \leq k \leq n - 1\).

To see this, recall that \(\mathcal{L}_{k}(F) \leq |P_{k}(F)|\) for \(1 \leq k \leq n - 1\) by the inductive hypothesis. Note that \(|N_{k}^{(0)}(F)|\) equals the number of all \(k\)-by–\(k\) matrices \(X = (x_{ij})\) in which every principal per-minor vanishes. Then it is easy to see that, when \(k = 2\) all four per-minors of the \(2\)-by–\(2\) matrix \(X\) vanish precisely when \(X = 0\). So, \(|N_{2}^{(0)}(F)| = 1\). By definition we also have \(\mathcal{V}_{2}^{(0)}(F) = 1\). Hence, it remains to prove the claim for \(3 \leq k \leq n - 1\).

To do this we split the set \(N_{k}^{(0)}(F)\) into the union of the following sets of matrices: for any \(j\), \(1 \leq j \leq k - 2\) we consider the set of matrices with all \((k - i)\)-by–\((k - i)\) per-minors equal to 0 for any \(i\), \(1 \leq i \leq j\) and possessing a nonzero \((k - j - 1)\)-by–\((k - j - 1)\) per-minor, and the set consisting just of the zero matrix. Then we estimate the number of matrices in each of these sets.

We first over-estimate the number of matrices from \(N_{k}^{(0)}(F)\) with the additional property that they have a nonzero \((k - 2)\)-by–\((k - 2)\) per-minor. For simplicity assume that this \((k - 2)\)-by–\((k - 2)\) submatrix is in the lower-right corner, i.e., \(X_{(11)(22)} \neq 0\); for other positions the
calculations yield the same results. Note that such \((k - 2)\)–by–\((k - 2)\)
lower-right block can be chosen in \((q^{(k-2)^2} - \mid P_{k-2}(\mathbb{F})\mid)\) ways. But by
the inductive hypothesis, this number is smaller or equal to \((q^{(k-2)^2} - \mathcal{L}_{k-2}(\mathbb{F}))\). So, such \((k - 2)\)–by–\((k - 2)\)
lower-right block can be chosen in not more than \((q^{(k-2)^2} - \mid \mathcal{L}_{k-2}(\mathbb{F})\mid)\) ways. By the assumption every \((k - 1)\)–by–\((k - 1)\) per-minor vanishes. In particular, per \(X_{11} = \per X_{12} = \per X_{21} = \per X_{22} = 0, \) from where all the \(2^2 = 4\) ‘variables’ \(x_{11}, x_{12}, x_{21}, x_{22}\) from the upper-left \(2\)–by–\(2\) corner are uniquely determined by
the block \(X_{(11)(22)}\) and the other ‘variables’ in the first or second row
or column. For example, \(x_{22} = -\sum_{i>2} x_{2i} \per X_{(11)(2i)} / \per X_{(11)(22)}.\)
Now, if we prescribe the values for the \(4(k - 2)\) ‘variables’ \(x_{i3}, \ldots, x_{ik}\)
and \(x_{3i}, \ldots, x_{ki}, i = 1, 2,\) arbitrarily we will obtain the estimate which is
larger or equal to the precise number. Finally, we multiply this estimate
with \(\left(\begin{array}{c} k \\ k - 2 \end{array}\right)^2 q^{4(k-2)} (q^{(k-2)^2} - \mathcal{L}_{k-2}(\mathbb{F}))\).

Among those still remaining in our class of \(k\)–by–\(k\) matrices with all
principal per-minors zero, we next over-estimate the number of those
matrices which have all \((k - 2)\)–by–\((k - 2)\) per-minors zero, but such
that at least one \((k - 3)\)–by–\((k - 3)\) per-minor is nonzero. Proceeding as
above, there are at most \(\left(\begin{array}{c} k \\ k - 2 \end{array}\right)^2 q^{2k^2 - (k-3)^2 - 9} (q^{(k-3)^2} - \mathcal{L}_{k-3}(\mathbb{F}))\) possible such per-minors at
a given position. Having prescribed any one, there are \(3^2 = 9\) ‘variables’
which are completely determined by the demand that every \((k - 2)\)–by–\((k - 2)\) principal per-minor vanishes. We may arbitrarily prescribe the
values for the rest of \(k^2 - (k-3)^2 - 9 = 6(k-3)\) ‘variables.’ Since there
are \(\left(\begin{array}{c} k \\ k - 3 \end{array}\right)^2\) possible positions for a given nonzero \((k - 3)\)–by–\((k - 3)\)
per-minor, there are at most
\(\left(\begin{array}{c} k \\ k - 2 \end{array}\right)^2 q^{k^2 - (k-3)^2 - 9} (q^{(k-3)^2} - \mathcal{L}_{k-3}(\mathbb{F}))\)
matrices inside the present subclass. We now proceed inductively. At
the \(j\)-th stage we over-estimate those \(k\)–by–\(k\) matrices such that every
\((k - i)\)–by–\((k - i)\) per-minor vanishes, for \(i = 1, \ldots, j,\) but there exists
a nonzero per-minor of dimension \((k - j - 1)\)–by–\((k - j - 1)\). Arguing
as above, there are at most
\(\left(\begin{array}{c} k \\ k - j - 1 \end{array}\right)^2 q^{k^2 - (k-j-1)^2 - (j+1)^2} (q^{(k-j-1)^2} - \mathcal{L}_{k-j-1}(\mathbb{F}))\).
of them. This process stops at \( j = k - 1 \), when every 1–by–1 per-minor vanishes, i.e., when \( X = 0 \). Then we do not use the above formula because we clearly have only 1 possibility for \( X = 0 \). Summing up, we over-estimate \( |N_k^{(0)}(\mathbb{F})| \) as

\[
|N_k^{(0)}(\mathbb{F})| \leq 1 + \sum_{j=1}^{k-2} \binom{k}{k-j-1} q^{2(k-j-1)(j+1)} \left( q^{(k-j-1)^2} - \mathcal{L}_{k-j-1}(\mathbb{F}) \right).
\]

By (14) the right side equals \( N_k^{(0)}(\mathbb{F}) \).

**Step 5.** Let us prove that \( |N_{n-1}^{(1)}(\mathbb{F})| \leq N_{n-1}^{(1)}(\mathbb{F}) \).

To see this, we divide the set \( N_{n-1}^{(1)}(\mathbb{F}) \) of all \((n-1)\times(n-1)\) matrices \( X \) with per \( X = 0 \) and \( \text{rk} \hat{X} = 1 \) in three disjoint subsets

\[
\begin{align*}
\hat{N}_{n-1}^{(1)}(\mathbb{F}) &= \{ X \in N_{n-1}^{(1)}(\mathbb{F}) : \text{per} X_{11} \neq 0 \}, \\
\check{N}_{n-1}^{(1)}(\mathbb{F}) &= \{ X \in N_{n-1}^{(1)}(\mathbb{F}) : \text{per} X_{11} = 0 \text{ and } \text{rk} \hat{X}_{11} = 0 \}, \\
\tilde{N}_{n-1}^{(1)}(\mathbb{F}) &= \{ X \in N_{n-1}^{(1)}(\mathbb{F}) : \text{per} X_{11} = 0 \text{ and } \text{rk} \hat{X}_{11} \neq 0 \}
\end{align*}
\]

and then over-estimate the cardinality of each of them. Start with \( \hat{N}_{n-1}^{(1)}(\mathbb{F}) \) and recall that \( \text{rk} \hat{X} \leq 1 \) if and only if all 2–by–2 determinant-minors of \( \hat{X} \) vanish. In particular, the complement of \( \hat{N}_{n-1}^{(1)}(\mathbb{F}) \) inside the set \( \mathcal{W}_{n-1} = \{ X \in M_{n-1}(\mathbb{F}) : \text{per} X = 0, \text{per} X_{11} \neq 0 \} \) contains the subset \( \mathcal{V}_{n-1} \) of all \((n-1)\times(n-1)\) matrices with the following properties

\[
\begin{align*}
(19) & \quad 0 = \text{per} X = x_{11} \text{per} X_{11} + \sum_{i \geq 2} x_{i1} \text{per} X_{i1}, \\
(20) & \quad 0 \neq \text{per} X_{11}, \\
(21) & \quad 0 \neq \text{per} X_{11} \text{ per} X_{22} - \text{per} X_{12} \text{ per} X_{21}.
\end{align*}
\]

From (19)–(20) we express the ‘variable’ \( x_{11} \) and put it into (21). Note that the only factor in (21) which contains \( x_{11} \) is \( \text{per} X_{22} \). Therefore, after elimination of \( x_{11} \) in (21), the set \( \mathcal{V}_{n-1} \) is determined by simultaneously non-vanishing of two polynomials in \((n-1)^2 - 1\) ‘variables’ \( x_{12}, \ldots, x_{1(n-1)}, x_{21}, \ldots, x_{2(n-1)}, \ldots, x_{(n-1)(n-1)} \):

\[
\begin{align*}
(22) & \quad p_1(X) = \text{per} X_{11} \neq 0, \\
(23) & \quad p_2(X) = \text{per} X_{11} (\text{per} X_{22} |_{x_{11} = \frac{\sum_{i \geq 2} x_{i1} \text{per} X_{i1}}{\text{per} X_{11}}} ) - \text{per} X_{12} \text{ per} X_{21} \neq 0,
\end{align*}
\]
By the definition of the permanent $p_1$ is a multilinear polynomial, i.e.,
every ‘variable’ of $p_1$ is linear, and it is also easy to see that every
‘variable’ of $p_2$ is either linear or quadratic. Now, there exists at least
one tuple of ‘variables’ which fulfills both inequalities. To see this, just
notice that
$$X = \left( \begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix} \right) \oplus \text{Id}_{n-3}$$
is a matrix with $\text{per} X = 0$, $\text{per} X_{11} \neq 0$, and with $\text{per} X_{11} \text{ per } X_{22}$ —
$\text{per} X_{12} \text{ per } X_{21} = -2 \neq 0$.

We now claim that at least $(q-3)^{(n-1)^2-1}$ tuples simultaneously sat-
isfy both inequalities (22)–(23). Namely, start with a given tuple that
does satisfy them. Keep all ‘variables’ but one, say $x_{i_0j_0}$ for simplicity,
fixed. Recall that in the first polynomial $x_{i_0j_0}$ is at most linear, while in
the second $x_{i_0j_0}$ is at most quadratic (it may also happen that for some
tuple, the polynomials are constant). So, to satisfy the second inequal-
ity, the ‘variable’ $x_{i_0j_0}$ can take all but perhaps two values — this is
because a quadratic polynomial has at most two zeros. Since the first
polynomial is linear, at most one of the allowed values of $x_{i_0j_0}$ can be
its zero. So, to simultaneously satisfy also the first inequality, we ca
choose at least $q-3$ values for ‘variable’ $x_{i_0j_0}$. In this way we obtained
$(q-3)^2$ tuples which simultaneously satisfy inequalities (22)–(23).

We proceed by choosing another ‘variable’ while keeping all the oth-
ers fixed. In the same way as before we obtain for each of the above
$(q-3)^2$ tuples additional $(q-3)^2$ tuples, hence together $(q-3)^2$ tuples
which simultaneously satisfy the inequalities (22)–(23).

By continuing in the same manner we finally end up with at least
$(q-3)^{(n-1)^2-1}$ matrices inside $V_{n-1} \subseteq W_{n-1} \setminus \tilde{N}^{(1)}_{n-1}(\mathbb{F})$. Recall that $W_{n-1}$
is the set of $(n-1)$–by–$(n-1)$ matrices with $\text{per} X = 0$, $\text{per} X_{11} \neq 0$. Clearly, $x_{11}$ is uniquely
determined with the other elements of a matrix $X$, so there are at most $q^{(n-1)^2-1}$ matrices inside $W_{n-1}$. Therefore,

$$|\tilde{N}^{(1)}_{n-1}(\mathbb{F})| = |W_{n-1}| - |W_{n-1} \setminus \tilde{N}^{(1)}_{n-1}(\mathbb{F})|$$
$$\quad \leq |W_{n-1}| - |V_{n-1}| \leq q^{(n-1)^2-1} - (q-3)^{(n-1)^2-1}.$$

We next over-estimate the cardinality of $\tilde{N}^{(1)}_{n-1}(\mathbb{F})$. Firstly, the num-
ber of $(n-2)$–by–$(n-2)$ matrices $X_{11}$ with $\text{per} X_{11} = 0$ and $\text{per} X_{11} = 0$
equals $|N^{(0)}_{n-2}(\mathbb{F})|$. If we enlarge such block $X_{11}$ to an $(n-1)$–by–
$(n-1)$ matrix by arbitrarily prescribing the values of $2(n-2)+1$ ‘variables’ from the first row and column we always obtain a matrix
with permanent zero. Note that not every extension necessarily satu-
sifies $\text{rk} \tilde{X} = 1$, however we still obtain an upper bound $|\tilde{N}^{(1)}_{n-1}(\mathbb{F})| \leq$
|N_{n-2}(F)| \cdot q^{2(n-2)+1}. By Step 1, |N_{n-2}(F)| \leq \Omega_{n-2}(F), so

|\tilde{N}_{n-1}(F)| \leq \Omega_{n-2}(F) \cdot q^{2(n-2)+1}.

It remains to over-estimate the cardinality of $\tilde{N}_{n-1}(F)$. We will make a rough estimate. By the induction hypothesis there are at most $\mathcal{U}_{n-2}(F)$ blocks $X_{11}$ with per $X_{11} = 0$ and $\tilde{X}_{11} \neq 0$. Every such block can be enlarged to $(n - 1)$--by--$(n - 1)$ matrix $X$ with 0 = per $X = x_{11}$ per $X_{11} + y_{21}^{tr} \tilde{X}_{11} x_{12}$, by prescribing the values for ‘variables’ in the first row and column. Here, $y_{21}$ is the first column of $X$ with the first entry removed, and $x_{12}$ is the first row of $X$ with the first entry removed. Clearly then, the ‘variable’ $x_{11}$ is arbitrary, while the $2(n - 2)$ ‘variables’ inside $y_{21}$, $x_{12}$ must fulfill $y_{21}^{tr} \tilde{X}_{11} x_{12} = 0$. By the assumptions on $\tilde{N}_{n-1}(F)$, we have $\text{rk} \tilde{X}_{11} = r \geq 1$. So, by Lemma 4.1 there are precisely $q \cdot |V^{(r)}_{n-2}(F)| \leq q \cdot |V^{(1)}_{n-2}(F)|$ extensions. All together,

$$|\tilde{N}_{n-1}(F)| \leq q \cdot \mathcal{U}_{n-2}(F) \cdot |V^{(1)}_{n-2}(F)|,$$

wherefrom we further deduce

$$|N_{n-1}(F)| = |\tilde{N}_{n-1}(F)| + |\tilde{N}_{n-1}(F)| + |\tilde{N}_{n-1}(F)| \leq \Omega_{n-1}(F),$$

which ends the proof of Step 2 and consequently also the proof of the lemma.

\[\square\]

6. PROOF OF THE MAIN RESULT

Proof of Theorem 2.1. By Lemma 3.1, $|P_3(F)|$ is strictly smaller than the number $|D_3(F)|$ of 3--by--3 matrices with zero determinant for arbitrary finite field with ch $F \neq 2$. This proves the theorem in the case $n = 3$. Suppose now $n \geq 4$. Recall that $|D_n(F)|$ equals $q^{n^2} - \prod_{k=1}^n(q^n - q^{k-1})$. So to prove the theorem it remains to verify that, given a fixed $n$, then for all sufficiently large $q$ one has

$$\mathcal{U}_n(F) \leq q^{n^2} - \prod_{k=1}^n(q^n - q^{k-1}).$$

Note that each quantity in this expression is a polynomial in $q$. It is easy to see that

\[(24) \quad q^{n^2} - \prod_{k=1}^n(q^n - q^{k-1}) = q^{n^2-1} + q^{n^2-2} + O(q^{n^2-5}),\]
where $O(q^k)$ is a standard notation for a quantity which satisfies 
\[
\limsup_{q \to \infty} |O(q^k)/q^k| < \infty.
\]
Let us prove inductively that 
\[
\mathcal{L}_n(F) = q^{n^2-1} - q^{n^2-2} + O(q^{n^2-3}) \quad (n \geq 4),
\]
\[
\mathcal{U}_n(F) = q^{n^2-1} + O(q^{n^2-3}) \quad (n \geq 4).
\]
To start with, one directly computes from (16) that 
\[
\mathcal{L}_4(F) = q^{15} - q^{14} - 5q^{12} + 11q^{11} - 9q^{10} + 4q^9 - q^8 = q^{15} - q^{14} + O(q^{13})
\]
and from (17), (14), (15), and (9) that 
\[
\mathcal{U}_4(F) = q^{15} + 53q^{13} - 520q^{12} + 3276q^{11} - 12864q^{10} +
\]
\[
+ 32905q^9 - 54445q^8 + 55410q^7 - 30619q^6 + 6561q^5
\]
\[= q^{15} + O(q^{13}).\]
Now, assume $n \geq 5$ and the claim holds for all $\mathcal{L}_k(F)$ and $\mathcal{U}_k(F)$, where $4 \leq k \leq n - 1$. Then, 
\[
\mathcal{L}_n(F) = (q^{(n-1)^2} - \mathcal{L}_{n-1}(F))q^{2(n-1)} = (q^{(n-1)^2} - q^{(n-1)^2-1} - O(q^{(n-1)^2-3}))q^{2(n-1)} = q^{n^2-1} - q^{n^2-2} + O(q^{n^2-4}),
\]
proving the inductive step for the lower bound.
Consider lastly $\mathcal{U}_n(F)$. According to its definition (17), we split it as 
\[
\mathcal{U}_n(F) = I_n + II_n + III_n + IV_n,
\]
where 
\[
I_n = (q^{(n-1)^2} - \mathcal{L}_{n-1}(F))q^{2(n-1)}, \quad II_n = q \cdot \mathcal{U}_{n-1}^{(0)}(F) \cdot |V_{n-1}^{(0)}(F)|,
\]
\[
III_n(F) = q \cdot \mathcal{U}_{n-1}^{(1)}(F) \cdot |V_{n-1}^{(1)}(F)|, \quad IV_n = q \cdot \mathcal{U}_{n-1}(F) \cdot |V_{n-1}^{(2)}(F)|.
\]
The first summand is done as for $\mathcal{L}_n(F)$ and equals 
\[
I_n = q^{n^2-1} - q^{n^2-2} + O(q^{n^2-3}).
\]
In the last summand we use (9) to deduce 
\[
IV_n = q \cdot \mathcal{U}_{n-1}(F) \cdot |V_{n-1}^{(2)}(F)|
\]
\[= q(q^{(n-1)^2-1} + O(q^{(n-1)^2-3})) \cdot (q^{2(n-3)}(q^3 + q^2 - q))
\]
\[= q^{n^2-2} + O(q^{n^2-3}).
\]
To estimate $II_n$, we infer from (14) that 
\[
\mathcal{U}_{n-1}^{(0)}(F) = \sum_{k=1}^{n-3} O(q^{2(n-k-2)(k+1)}) \cdot (q^{(n-k-2)^2} - O(q^{(n-k-2)^2-1}))
\]
\[= \sum_{k=1}^{n-3} O(q^{(n-1)^2-(k+1)^2}) = O(q^{(n-1)^2-4}),
\]
(25)
while (9) implies that $|V_{n-1}^{(0)}(\mathbb{F})| = O(q^{2n-2})$. Consequently, $II_n = q \cdot \mathfrak{M}_{n-1}^{(0)}(\mathbb{F}) \cdot |V_{n-1}^{(0)}(\mathbb{F})| = O(q^{n^2-4})$, which is below the required $O(q^{n^2-3})$.

Consider lastly the third summand. To estimate (15) we note that $(q^{(n-1)^2-1} - (q - 3)^{(n-1)^2-1}) = O(q^{(n-1)^2-2})$. By (25), $\mathfrak{N}_{n-2}^{(0)}(\mathbb{F}) = O(q^{(n-2)^2-4})$, while (9) implies $|V_{n-2}^{(1)}(\mathbb{F})| = O(q^{2n-5})$ and $|V_{n-1}^{(1)}(\mathbb{F})| = O(q^{2n-3})$. Hence,

$$
\mathfrak{N}_{n-1}^{(1)}(\mathbb{F}) = O(q^{(n-1)^2-2}) + O(q^{(n-2)^2-4}) \cdot q^{2(n-2)+1} + q \cdot O(q^{(n-2)^2-1}) \cdot O(q^{2n-5}) = O(q^{(n-1)^2-2}),
$$

and

$$
III_n = q \cdot \mathfrak{N}_{n-1}^{(1)}(\mathbb{F}) \cdot |V_{n-1}^{(1)}(\mathbb{F})| = q \cdot O(q^{(n-1)^2-2}) \cdot O(q^{n^2-3}) = O(q^{n^2-3}).
$$

In total we have $\mathfrak{U}_n(\mathbb{F}) = I_n + II_n + III_n + IV_n = q^{n^2-1} + O(q^{n^2-3})$ which proves the inductive step. Note that this number is strictly smaller than (24) for all sufficiently large $q$, so for such $q$ we have $|P_n(\mathbb{F})| \leq \mathfrak{U}_n(\mathbb{F}) \ll |D_n(\mathbb{F})|$, which proves the theorem.

\[ \square \]

7. Applications

In this section we apply the developed technique and results to estimate the probability of the determinant and permanent functions to have a given value in a finite field. This problem goes back to the works of Erdös and Rényi [9, 10], where they estimated the probability for a (0,1)-matrix with a given number of ones to have a zero permanent. Later many authors investigated this topic for determinant and permanent functions of (0,1)-matrices, see monographs [2, 14] for details. In particular, Sachkov [27] proved that if a uniform distribution is given on the set of all (0,1)-matrices of size $m \times n$, where $m \leq n$, then the probability $P\{\per A \neq 0\} \to 1$ if $n \to \infty$, where $A$ is an arbitrary (0,1)-matrix of size $m \times n$. An asymptotics for cardinality of (0,1)-matrices with zero permanent was given by Everett and Stein in [11], corresponding results for the determinant are due to Komlós, see [16, 17].

Here we investigate the situation over arbitrary finite fields. The application of our technique over a finite field $\mathbb{F}$ of cardinality $q$ shows that for $0 \neq \alpha \in \mathbb{F}$ the probability function $P$ behaves as follows

$$
P(\det A = \alpha) = \frac{1}{q} - \frac{1}{q^3} + O\left(\frac{1}{q^4}\right), \quad P(\det A = 0) = \frac{1}{q} + \frac{1}{q^2} + O\left(\frac{1}{q^3}\right),
$$

$$
\frac{1}{q} - \frac{1}{q^2} + O\left(\frac{1}{q^3}\right) \leq P(\per A = 0) \leq \frac{1}{q} + O\left(\frac{1}{q^3}\right), \text{ and}
$$
\[
\frac{1}{q} + O\left(\frac{1}{q^4}\right) \leq P(\text{per } A = \alpha) \leq \frac{1}{q} + \frac{1}{q^3} + O\left(\frac{1}{q^4}\right),
\]
so, roughly speaking, each of these probabilities approximately equals to \(\frac{1}{q} + O\left(\frac{1}{q^4}\right)\).

In order to prove our result we need the following lemma:

**Lemma 7.1.** Let \(\mathbb{F}\) be a finite field, \(\text{ch } \mathbb{F} \neq 2\). Then for any nonzero \(\alpha, \beta \in \mathbb{F}\) the cardinality of the set of matrices of a given size with the determinant (permanent) \(\alpha\) is equal to the cardinality of the set of matrices of a given size with the determinant (permanent) \(\beta\), i.e.,

\[|\{A \in M_n(\mathbb{F}) : \det A = \alpha\}| = |\{A \in M_n(\mathbb{F}) : \det A = \beta\}| \]

and

\[|\{A \in M_n(\mathbb{F}) : \per A = \alpha\}| = |\{A \in M_n(\mathbb{F}) : \per A = \beta\}|.\]

**Proof.** We denote \(D_n^\alpha(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \det A = \alpha\}\). For any \(A = (a_{ij}) \in D_n^\alpha(\mathbb{F})\) we consider the matrix \(B = (b_{ij})\) defined by \(b_{ij} = a_{ij}\) for \(i = 1, \ldots, n, j = 2, \ldots, n\), \(b_{i1} = \beta \alpha a_{i1}\) for \(i = 1, \ldots, n\). Then \(\det B = \frac{\beta}{\alpha} \det A = \beta\), i.e., \(B \in D_n^\beta(\mathbb{F})\). Since \(\alpha \beta \neq 0\), the mapping from \(A\) to \(B\) is well-defined and injective, hence, \(|D_n^\beta(\mathbb{F})| \geq |D_n^\alpha(\mathbb{F})|\). Similarly, \(|D_n^\beta(\mathbb{F})| \leq |D_n^\alpha(\mathbb{F})|\).

Since permanent is also a linear function of a row or a column, the result for the permanent can be obtained in the same way. \(\square\)

**Theorem 7.2.** Let \(\mathbb{F}\) be a finite field, \(|\mathbb{F}| = q\), \(\text{ch } \mathbb{F} \neq 2\). For any \(\alpha \in \mathbb{F}\) the probability that \(\det A = \alpha\), \(A \in M_n(\mathbb{F})\), is equal to \(\frac{1}{q} + O\left(\frac{1}{q^2}\right)\) and the probability that \(\per A = \alpha\) is also equal to \(\frac{1}{q} + O\left(\frac{1}{q^2}\right)\).

**Proof.** We consider at first \(\alpha = 0\). Then by the proof of Theorem 2.1 it follows that the quantity of matrices with zero determinant \(|D_n(\mathbb{F})| = q^{n^2-1} + q^{n^2-2} + O(q^{n^2-5})\). Hence, the probability

\[P(\det A = 0) = \frac{q^{n^2-1} + q^{n^2-2} + O(q^{n^2-5})}{q^{n^2}} = \frac{1}{q} + \frac{1}{q^2} + O\left(\frac{1}{q^3}\right) = \frac{1}{q} + O\left(\frac{1}{q^2}\right).\]
Similarly, using the proof of Theorem \[2.1\] we have

\[ P(\text{per } A = 0) \leq \frac{\sum_n(F)}{q^n} = \frac{q^{n^2-1} + O(q^{n^2-3})}{q^n} = \frac{1}{q} + O\left(\frac{1}{q^2}\right) \]

and

\[ P(\text{per } A = 0) \geq \frac{\sum_n(F)}{q^n} = \frac{q^{n^2-1} - q^{n^2-2} + O(q^{n^2-3})}{q^n} = \frac{1}{q} - \frac{1}{q^2} + O\left(\frac{1}{q^3}\right) \]

So, \( P(\text{per } A = 0) = \frac{1}{q} + O\left(\frac{1}{q^2}\right) \).

If \( \alpha \neq 0 \) then by Lemma 7.1

\[ |D_n^{\alpha}(F)| = \prod_{k=1}^{n}(q^n - q^{k-1}) = q^{-\frac{n(n-1)}{2}}(q^n - 1)(q^2 - 1) \]

Thus the probability

\[ P(\text{det } A = \alpha) = \frac{q^{n^2-1} - q^{n^2-3} + O(q^{n^2-4})}{q^n} = \frac{1}{q} - \frac{1}{q^2} + O\left(\frac{1}{q^4}\right) \]

Finally,

\[ |P_n^{\alpha}(F)| \leq \frac{q^{n^2} - q^{n^2-1} + q^{n^2-2} + O(q^{n^2-3})}{q - 1} = q^{n^2-1} + q^{n^2-3} + O(q^{n^2-4}) \]

and

\[ |P_n^{\alpha}(F)| \geq \frac{q^{n^2} - q^{n^2-1} + O(q^{n^2-3})}{q - 1} = q^{n^2-1} + O(q^{n^2-4}). \]

Thus the probability

\[ P(\text{per } A = \alpha) = \frac{1}{q} + O\left(\frac{1}{q^2}\right). \]
8. Examples and Remarks

Remark 8.1. In the table below, for a given \( n \) we compute the first integer \( i \) such that for any \( j > i \) the value of the polynomial \( U_n(F) \) at \( q = j \) is strictly less than the value of the polynomial \( |D_n(F)| \) at \( q = j \). In the third row we give the minimal number of elements in a field with this property, i.e., the minimal power of a prime \( q = |F| \) such that \( U_n(F) \preceq |D_n(F)| \). We used Wolfram’s Mathematica 5.1 for the calculations. For example, when \( n = 5 \) we have \( U_5(F) \preceq |D_5(F)| \) whenever the finite field \( F \) has at least 76 elements and its characteristic differs from 2. The smallest such field with at least 76 elements is \( GF(79) \). So, \( q = 79 \).

| \( n \) | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11   |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|------|
| \( i \) | 2   | 43  | 76  | 116 | 164 | 221 | 287 | 362 | 446  |
| \( q \) | 3   | 43  | 79  | 121 | 167 | 223 | 289 | 367 | 449  |
| \( n \) | 358 | 640 | 750 | 869 | 996 | 1133| 1278| 1433| 1596 |
| \( q \) | 541 | 641 | 751 | 877 | 997 | 1151| 1279| 1433| 1597 |

If \( F \) is an infinite field then there do exist bijective converters of permanent into determinant. In the Example 8.2 we give such bijective maps \( \Phi : M_n(F) \to M_n(F), \ n \geq 2 \), that even satisfy \( \text{per}(A) = \text{det}(A) \) and \( \text{det}(A) = \text{per}(A) \) simultaneously for all \( A \in M_n(F) \).

Example 8.2. If \( \text{ch} F = 2 \) then \( \text{per}(A) = \text{det}(A) \) for any \( A \in M_n(F) \) so we take \( \Phi(X) = X \) to achieve \( \text{per}(A) = \text{det}(A) \) and \( \text{det}(A) = \text{per}(A) \). Assume \( \text{ch} F \neq 2 \). Note that the cardinality of infinite sets satisfies \( |F| = |F \times F| \), so \( |M_n(F)| = |F^{n^2}| = |F| \).

We are going to prove now that for any given \( \lambda, \mu \in F \) the cardinality of the set of matrices with permanent \( \lambda \) and determinant \( \mu \) is equal to \( |F| \), so for any given pair of such sets there is a bijection between them. Let us denote

\[
\Omega_n(\lambda, \mu) = \{ A \in M_n(F) : \text{per}(A) = \lambda \ \text{and} \ \text{det}(A) = \mu \}.
\]

1. For given fixed \( \lambda, \mu \in F \) consider the set

\[
\Delta_n(\lambda, \mu) = \left\{ \left( \begin{array}{c} \alpha \\
\lambda + \mu \\
\text{Id}_{n-2} \end{array} \right) : \alpha \in F \setminus \{0\} \right\} \subseteq M_n(F).
\]

2. The cardinality of this set is \( |F| - 1 = |F| \) and every matrix from this set has permanent and determinant equal to \( \lambda \) and \( \mu \) respectively.

3. Moreover,

\[
\Delta_n(\lambda, \mu) \subseteq \Omega_n(\lambda, \mu) \subseteq M_n(F),
\]

and comparing cardinalities, we obtain \( |F| = |\Delta_n(\lambda, \mu)| \preceq |\Omega_n(\lambda, \mu)| \preceq |M_n(F)| = |F| \). By the classic Bernstein-Schroeder’s theorem [8, Cor. II.7.7] we have \( |\Omega_n(\lambda, \mu)| = |F| \).

4. So, there is a bijection \( \Phi_{\lambda, \mu} : \Omega_n(\lambda, \mu) \to \Omega_n(\mu, \lambda) \).
5. However, due to partition
\[ M_n(\mathbb{F}) = \bigcup_{\lambda, \mu \in \mathbb{F}} \Omega_n(\lambda, \mu), \]
the maps \( \Phi_{\lambda, \mu} \) constitute a well-defined bijection \( \Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F}) \) with \( \text{per} A = \text{det} \Phi(A) \) and \( \text{det} A = \text{per} \Phi(A) \). It is given by \( A \mapsto \Phi_{\lambda, \mu}(A) \) if \( A \) satisfies \( \text{per}(A) = \lambda \) and \( \text{det}(A) = \mu \).

**Remark 8.3.** By adopting the above arguments it can be shown that there exists a bijection \( \Phi : M_n(\mathbb{F}) \to M_m(\mathbb{F}^\prime) \) with similar properties as in the previous example, provided that \( \mathbb{F} \) and \( \mathbb{F}^\prime \) are infinite fields of the same cardinality and \( m, n \geq 2 \).

Note that for any field \( \mathbb{F} \) there exist nonbijective converters of permanent into determinant.

**Example 8.4.** As an example, \( \Phi : A \mapsto (\text{Id}_{n-1} \oplus \text{per} A) \) satisfies \( \text{per} A = \text{det} \Phi(A) \). Note that such transformations cannot be linear.

Moreover, there exist also nonbijective transformations \( \Phi : M_n(\mathbb{F}) \to M_m(\mathbb{F}) \) which exchange permanent and determinant of a matrix.

**Example 8.5.** If \( \text{ch} \mathbb{F} = 2 \) then \( \text{per} A = \text{det} A \) for any \( A \in M_n(\mathbb{F}) \) and if \( m \geq n \) the map \( \Phi : A \mapsto A \oplus I_{m-n} \) has the required property.

If \( \text{ch} \mathbb{F} \neq 2 \) then for any field \( \mathbb{F} \) and for all \( m \geq 2 \) we consider
\[ \Phi : A \mapsto \begin{pmatrix} 1 & \frac{1}{2}(\text{det} A - \text{per} A) \\ 1 & \frac{1}{2}(\text{det} A + \text{per} A) \end{pmatrix} \oplus \text{Id}_{m-2}. \]
Hence, \( \Phi \) satisfies \( \text{per} A = \text{det} \Phi(A) \) and \( \text{det} A = \text{per} \Phi(A) \). Note that such transformations cannot be linear.

In order to extend our results to finite rings we need the following lemma, which we include here with its proof for the sake of completeness.

**Lemma 8.6.** Let \( R \) be a finite ring without zero divisors. Then \( R \) is a field.

**Proof.** Since \( R \) has no zero divisors, then for any \( a \in R, a \neq 0 \), the transformations \( r_a : x \to ax \) and \( l_a : x \to xa \) are injective. Thus both these transformations are bijective since they are surjective by the finiteness of \( R \).

Let us check that the neutral element is automatically in \( R \). Since \( r_a \) is surjective, there exists \( x \in R \) such that \( ax = a \). Now, for any \( b \in R \) there exists \( y \in R \) such that \( b = ya \). Thus \( bx = yax = ya = b \), i.e., \( x \) is a right unity. Similarly, there is \( x' \in R \) which is a left unity. Then \( x = x'x = x', \) i.e., \( x \) is a unity. Let us denote it by \( e \).
Now for any \( a \in \mathbb{R}, a \neq 0 \), there exist \( a', a'' \in \mathbb{R} \), such that \( aa' = e \) and \( a''a = e \) by the surjectivity of \( r_a \) and \( l_a \), correspondingly. Considering \( a'' = a''(aa') = (a''a)a' = a' \), we get that \( a \) is invertible. Thus \( R \) is a division ring. By Wedderburn theorem any finite division ring is a field and the result follows. \( \square \)

**Proof of Corollary 2.2.** It follows directly by the application of Theorem 2.1 to the result of Lemma 8.6. \( \square \)

**Remark 8.7.** By Lemma 8.6 the results of Section 7 are valid for finite rings without zero divisors as well.

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(Gregor Dolinar) FACULTY OF ELECTRICAL ENGINEERING, UNIVERSITY OF LJUBLJANA, TRŽAŠKA 25, SI-1000 LJUBLJANA, SLOVENIA.

E-mail address, Gregor Dolinar: gregor.dolinar@fe.uni-lj.si

(Alexander E. Guterman) FACULTY OF ALGEBRA, DEPARTMENT OF MATHEMATICS AND MECHANICS, MOSCOW STATE UNIVERSITY, GSP-1, 119991 MOSCOW, RUSSIA.

E-mail address, Alexander E. Guterman: guterman@list.ru

(Bojan Kuzma) 1UNIVERSITY OF PRIMORSKA, GLAGOLJASKA 8, SI-6000 KOPER, SLOVENIA, AND 2IMFM, JADRANSKA 19, SI-1000 LJUBLJANA, SLOVENIA.

E-mail address, Bojan Kuzma: bojan.kuzma@pef.upr.si

(Marko Orel) IMFM, JADRANSKA 19, SI-1000 LJUBLJANA, SLOVENIA.

E-mail address, Marko Orel: marko.orel@fmf.uni-lj.si